Partitioning into degenerate graphs in linear time

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Abstract

Let $G$ be a connected graph with maximum degree $\Delta \geq 3$ distinct from $K_{\Delta+1}$. Generalizing Brooks’ Theorem, Borodin and independently Bollobás and Manvel, proved that if $p_1, \ldots, p_s$ are non-negative integers such that $p_1 + \cdots + p_s \geq \Delta - s$, then $G$ admits a vertex partition into parts $A_1, \ldots, A_s$ such that, for $1 \leq i \leq s$, $G[A_i]$ is $p_i$-degenerate. Here we show that such a partition can be performed in time $O(n + m)$. This generalizes previous results that treated subcases of a conjecture of Abu-Khzam, Feghali and Heggernes [2] which our result settles in full.

1 Introduction

Brooks’ Theorem is a fundamental theorem in graph coloring that draws a connection between the chromatic number and the maximum degree of a graph.

Theorem 1 (Brooks’ Theorem [8]). Every connected graph with maximum degree $\Delta \geq 3$ that is distinct from $K_{\Delta+1}$ is $\Delta$-colorable.

A graph $G$ is $d$-degenerate if every non-empty subgraph of $G$ contains a vertex of degree at most $d$. Borodin [6] and, independently, Bollobás and Manvel [4] obtained the following generalization.

Theorem 2 (Borodin [6], Bollobás and Manvel [4]). Let $G$ be a non-complete connected graph with maximum degree $\Delta \geq 3$. Let $s \geq 2$ and $p_1, \ldots, p_s \geq 0$ be integers such that $\sum_{i=1}^{s} p_i \geq \Delta - s$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_s$ such that, for each $i \in \{1, \ldots, s\}$, $G[V_i]$ is (i) $p_i$-degenerate, and (ii) has maximum degree at most $p_i + 1$.

Brooks’ Theorem follows from Theorem 2 by noting that a $d$-degenerate graph is $(d + 1)$-colorable. We should also mention that similar generalizations and variants of Brooks’ Theorem exist: see [13, 15] for generalizations on hypergraphs, see [1, 9, 3] for generalizations on digraphs, and see [12] for a distributed version.

From an algorithmic perspective, a very short proof of Brooks’ Theorem due to Lovász [10] produces the coloring in linear time. The original proof of Theorem 2 and the alternative proof provided by Matamala [11] are not algorithmic. Though, another proof of Theorem 2 in [7] is algorithmic with polynomial complexity (the runtime appears to be cubic in the number of vertices). This raises the question of whether one can possibly improve its time complexity to linear. In view of this, several groups improved the complexity of such a partition algorithm, focusing on property (i) only. Bonamy et al. [5] showed that the complexity in the special case

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s = 2 with \( p_1 = 0 \) and \( p_2 = \Delta - 2 \) can be improved to quadratic for \( \Delta \geq 4 \) and to linear for \( \Delta = 3 \). Similarly, Abu-Khzam, Feghali and Heghernes [2] showed that in the special case \( p_i \leq 1 \) for all \( i \in [s] \), it can be improved to linear.

The object of this paper is to obtain a common generalization of these results in linear time.

**Theorem 3.** There exists an algorithm that, given a non-complete connected graph \( G \) with \( n \) vertices, \( m \) edges, and maximum degree \( \Delta \geq 3 \), and given a sequence \((p_1, \ldots, p_s)\) of non-negative integers such that \( s \geq 2 \) and \( \sum_{i=1}^{s} p_i \geq \Delta - s \), provides in time \( O(n + m) \) a partition of \( V(G) \) into sets \( V_1, \ldots, V_s \) such that for each \( i \in [s] \), \( G[V_i] \) is \( p_i \)-degenerate.

Theorem 3 settles a conjecture of Abu-Khzam, Feghali and Heghernes [2] and, in the special case \( s = 2 \), a problem of Bonamy et al. [5].

**Remark 4.** If the graph is not connected, we can solve the problem with the same complexity by running an algorithm for the connected case on each connected component and merging the partitions we obtain.

In what follows, we always consider connected graphs.

**Remark 5.** Note that the complexity of the algorithm does not depend on the length \( s \) of the sequence \((p_1, \ldots, p_s)\) as in fact only the first \( \Delta \) elements of this sequence will be considered by our algorithm. Indeed, those are always sufficient to fulfill the condition \( \sum_{i=1}^{s} p_i \geq \Delta - s \).

The paper is organized as follows. In Section 2 we prove Theorem 3 in the case when the constraint is loose (i.e. \( \sum_{i=1}^{s} p_i > \Delta - s \)) or when the graph is not \( \Delta \)-regular. Then, the (more difficult) regular case with \( s = 2 \) is treated in Section 3. Afterwards, in Section 4 we deduce Theorem 3 in full. In the final section, Section 5, we conclude with some remarks.

## 2 The case of non-regular graphs

In this section, we describe the algorithm for the non-regular case of Theorem 3. The proof relies on the following folklore observation which enables us to get a certificate of \( d \)-degeneracy for a graph. For completeness, we give the details.

Given a graph \( G \) and a vertex ordering \( v_1, v_2, \ldots, v_n \) of \( G \) we denote by \( N^<(v_i) \) the neighbors of \( v_i \) with lower indices, that is \( N^<(v_i) = N(v_i) \cap \{v_j \mid j < i\} \).

**Observation 6.** A graph \( G \) is \( d \)-degenerate if and only if it admits a vertex ordering \( v_1, v_2, \ldots, v_n \) such that \( |N^<(v_i)| \leq d \) for every vertex \( v_i \).

We call such an ordering a \( d \)-degenerate ordering. We now describe a greedy procedure, Algorithm 7, that can handle several cases.

**Lemma 7.** Let \( G \) be a (not necessarily connected) graph with \( n \) vertices, \( m \) edges and maximum degree \( \Delta \geq 3 \). Let \( s \geq 1 \) and \( p_1, \ldots, p_s \geq 0 \) be integers and \( P = s + \sum_{1 \leq i \leq s} p_i \). Given an ordering \( v_1, \ldots, v_n \) of \( V(G) \), if \( |N^<(v_i)| < P \) for every \( 1 \leq i \leq n \), then Algorithm 7 returns, in time \( O(n + m + s) \), a partition of \( V(G) \) into sets \( A_1, \ldots, A_s \) such that \( G[A_i] \) is \( p_i \)-degenerate for \( 1 \leq i \leq s \).

**Remark 8.** Note that as \( |N^<(v_i)| \leq \Delta \), the lemma applies if \( \Delta < P \). Note also that the lemma applies if \( G \) is \((P - 1)\)-degenerate and if the ordering \( v_1, \ldots, v_n \) is a \((P - 1)\)-degenerate ordering.

**Proof.** We first prove correctness and then analyze the runtime.

Correctness of the algorithm. We first show that the algorithm does not return ERROR. Towards a contradiction, we suppose otherwise. Then for some \( i \in \{1, \ldots, n\} \), \( |N^<(v_i)| \cap A_k \geq p_k + 1 \) for all \( 1 \leq k \leq s \). Thus, \( |N^<(v_i)| \geq s + \sum_{k=1}^{s} p_k = P \), which is a contradiction. So the algorithm terminates normally and returns sets \( A_1, \ldots, A_s \).
By doing a post-order traversal of $p$-trees, we consider non-$\Delta$-regular graphs. For every $k \leq s$ and $|N(v_i) \cap A_k| > p_k$ do
4: $k + +$
5: end while
6: if $k == s + 1$ then
7: return ERROR
8: end if
9: $A_k \leftarrow A_k \cup v_i$
10: end for
11: return $(A_1, \cdots, A_s)$

It remains to show that each $G[A_k]$ is $p_k$-degenerate. In view of Observation 6 it suffices to show that the ordering $v_1, \cdots, v_n$ restricted to $A_k$ is a $p_k$-degenerate ordering of $G[A_k]$. In other words, for every $k \in [1, \cdots, s]$ and every vertex $v_i \in A_k$, we have $|A_k \cap N^<(v_i)| \leq p_k$. This directly follows from the condition of the while loop in Algorithm 1 line 4.

Runtime analysis. Clearly, it suffices to show that the total cost of the while loop at lines 4 to 6 is $O(n + m)$. We establish this by an amortized complexity analysis, by noting that $k$ is not incremented more than $m$ times. To see this, note that $k$ is incremented because $|N(v) \cap A_k| > p_k$ for some $k \leq s$ and $v \in V(G)$. Let $w$ be a neighbor of $v$ in $A_k$ and attribute a cost of 1 to the edge $vw$. Clearly, the edge $vw$ is not attributed a cost more than once. Now, since the initialisation of the $A_i$'s takes time $O(s)$, the total complexity is $O(n + m + s)$.

Recall that Lemma 7 allows us to focus on inputs such that $\Delta = s + \sum_{i=1}^{s} p_i$. In the next algorithm, Algorithm 2 we consider non-$\Delta$-regular graphs $G$ with $\Delta = s + \sum_{i=1}^{s} p_i$.

Algorithm 2 non-$\Delta$-regular_partitioning

Input: A non-regular connected graph $G$ with maximum degree $\Delta$, and some integers $s \geq 1$ and $p_1, \cdots, p_s$ such that $p_1 + \cdots + p_s = \Delta - s$

Output: A partition of $V(G)$ into sets $A_1, \cdots, A_s$ such that each $G[A_i]$ is $p_i$-degenerate

1: $\Delta \leftarrow$ the maximum degree of $G$
2: $v \leftarrow$ a vertex of $G$ of degree less than $\Delta$
3: $T \leftarrow$ a spanning tree of $G$ rooted at $v$
4: $v_1, \cdots, v_n \leftarrow$ an ordering of $V(G)$ obtained by a post-order traversal of $T$ starting from $v$
5: $A_1, \cdots, A_s \leftarrow$ greedy_partitioning($G, v_1, \cdots, v_n, p_1, \cdots, p_s$)
6: return $A_1, \cdots, A_s$

Lemma 9. Algorithm 2 runs in time $O(n + m)$ and returns a partition of $V(G)$ into sets $A_1, \cdots, A_s$ such that for all $1 \leq i \leq s$, $G[A_i]$ is $p_i$-degenerate.

Proof. We first prove correctness and then analyze the runtime.

Correctness of the algorithm. By doing a post-order traversal of $T$ starting from $v$, the ordering $v_1, \cdots, v_n$ computed by Algorithm 2 is a $(\Delta - 1)$-degenerate ordering. In other words:

Every vertex $v_i$ has at most $\Delta - 1$ neighbors in $N^< (v_i)$. 

\[ \sum_{i=1}^{s} p_i = \Delta - s \]
Indeed, this is clear for \( v_n = v \), as \( |N^<(v)| = \deg(v) < \Delta \). This is also clear for every vertex \( v_i \neq v_n \), as its parent neighbor in \( T \) does not belong to \( N^<(v_i) \), hence \( |N^<(v_i)| \leq \deg(v_i) - 1 \leq \Delta - 1 \).

Then, given such an ordering, Lemma \ref{lemma:greedy_partitioning} guarantees that the call of \texttt{greedy_partitioning} returns a partition of \( V(G) \) with the required properties.

Runtime analysis. Clearly, computing the maximum degree of \( G \) as well as finding a vertex \( v \) of degree less than \( \Delta \) can be done in time \( O(n + m) \). Similarly, building a spanning tree rooted at \( v \) and doing a post-order traversal can be done in time \( O(n + m) \).

By Lemma \ref{lemma:greedy_partitioning} the call to \texttt{greedy_partitioning} takes time \( O(n + m + s) \). Finally, since here we have \( s \leq \Delta \), the running time is \( O(n + m) \). \qed

3 The case of regular graphs with \( s = 2 \)

We now consider the case not handled by the previous section, that is the case where \( G \) is \( \Delta \)-regular and where \( \Delta = s + \sum_{i=1}^{s} p_i \). We will see in Section \ref{section:regular} that the case with arbitrary \( s \) can be easily derived from the \( s = 2 \) case. This is why we restrict to \( s = 2 \) here. So in this section we assume that we have two integers \( p_A, p_B \geq 0 \) such that \( p_A + p_B = \Delta - 2 \).

Before giving the details, we give a sketch of the proof. Applying Algorithm \ref{algorithm:regular_partition} to \( G \) returns an ordering in which only vertex \( v_n \) has more than \( \Delta - 1 \) neighbors in \( N^<(v_n) \) (it has exactly \( \Delta \) of them). Our strategy is thus to partition \( v_n \)'s neighborhood more carefully in order to ease \( v_n \)'s coloring.

To do so, we consider a block decomposition of \( G \) and select one of its end-blocks. An easy case is when this block is a “quasi clique” (see Figure \ref{figure:quasi_clique}). Otherwise, we show that we can find a vertex \( z \) whose neighborhood \( N(z) \) has desirable properties. We can then force the coloring of almost all vertices of \( N(z) \) and call Algorithm \ref{algorithm:regular_partition} using \( z \) as the root for the spanning tree (here \( z \) plays the role of \( v_n \)). The most difficult part is to show that such a vertex with special neighborhood can be found in linear time.

**Definition 10.** For a graph \( G \) with maximum degree \( \Delta \geq 3 \), we say that a pair \((z, X)\) formed by a vertex \( z \in V(G) \) and a set \( X \subseteq N(z) \) is a special neighborhood if

a) \( |X| = \Delta - 1 \),

b) \( G[X] \) is not a complete graph, and

c) \( G \setminus X \) is connected.

Some graphs do not possess such a special neighborhood, and to deal with them we have to deal with quasi-cliques. A quasi-clique, denoted \( K^\Delta_+ \), is the graph obtained from \( K_{\Delta+1} \) by subdividing exactly one edge (see Figure \ref{figure:quasi_clique}). Note that this graph has a degree-two vertex and that all the \( \Delta + 1 \) remaining vertices have degree \( \Delta \). We can now present the algorithm for the case when \( s = 2 \) and \( G \) is \( \Delta \)-regular.

**Theorem 11.** Algorithm \ref{algorithm:regular_partition} partitions \( V(G) \) into two sets \( A \) and \( B \) such that \( G[A] \) is \( p_A \)-degenerate and \( G[B] \) is \( p_B \)-degenerate. It runs in time \( O(n + m) \).

*Proof.* We first prove the correctness and then analyze the runtime. For the properties of the algorithm \texttt{get_special_neighborhood}, we refer to the forthcoming Lemmas \ref{lemma:non-regular_partitioning} and \ref{lemma:regular_partitioning}.

Correctness of the algorithm. We have two subcases: either the considered end-block is isomorphic to \( K^\Delta_+ \), or not.

We first consider the case when the end-block \( H \) is isomorphic to \( K^\Delta_+ \) with \( v \) as its cut-vertex (so the condition of line \ref{line:isomorphic} is met).

Thanks to \texttt{non-\( \Delta \)-regular_partitioning} (see Lemma \ref{lemma:non-regular_partitioning}), we can partition the vertex set of \( G \setminus (V(H) \setminus \{v\}) \) into sets \( A \) and \( B \) with the required degeneracy properties. Assume without
Algorithm 3 $\Delta$-regular bipartitioning

Input: A $\Delta$-regular connected graph $G$, for some $\Delta \geq 3$, distinct from $K_{\Delta+1}$, and two integers $p_A \leq p_B$ such that $p_A + p_B = \Delta - 2$.

Output: A partition of $V(G)$ into sets $A$ and $B$ such that $G[A]$ and $G[B]$ are $p_A$-degenerate and $p_B$-degenerate, respectively.

1: $A, B \leftarrow \emptyset$
2: Perform a block-decomposition of $G$.
3: $H \leftarrow$ an end-block of the decomposition
4: $v \leftarrow$ the vertex linking $H$ to the rest of $G$, or any vertex if $G = H$
5: if $H$ is isomorphic to $K_-$ then
6: $A, B \leftarrow$ non-$\Delta$-regular partitioning($G \setminus (H \setminus \{v\}), p_A, p_B$)
7: Let $Y \in \{A, B\}$ be the set containing $v$ and let $Y'$ be the other set
8: Add to $Y'$ the two neighbors of $v$ in $H$, as well as $p_{Y'}$ other vertices from $H$.
9: Add to $Y$ the other $p_Y + 1$ vertices of $H$.
10: else
11: $(z, X) \leftarrow$ get_special_neighborhood($H, v$) // See Algorithm 4
12: $A \leftarrow p_A + 2$ vertices of $X$, including two that are non-adjacent
13: $B \leftarrow$ the other $p_B - 1$ vertices of $X$
14: $T \leftarrow$ a spanning tree of $G \setminus X$ rooted at $z$
15: $v_\Delta, \cdots, v_n \leftarrow$ ordering of $V \setminus X$ that is a post-order traversal of $T$ starting at $z$.
16: for $v_i$ from $v_\Delta$ to $v_n$ do
17: \hspace{1em} if $|N(v_i) \cap A| \leq p_A$ then // We consider the neighborhood in $G$, not in $G \setminus X$
18: \hspace{2em} $A \leftarrow A \cup v_i$
19: \hspace{1em} else
20: \hspace{2em} $B \leftarrow B \cup v_i$
21: \hspace{1em} end if
22: end for
23: end if
24: return $A, B$

Figure 1: The graph $K_-$ for $\Delta = 6$, and some vertex partitioning for $p_A = 1$ and $p_B = 3$ where the vertices in $A$ (which contains $v$) are represented in red and vertices in $B$ in blue.
loss of generality that $v \in A$. We extend this partial partition of $V(G)$ to the other vertices in $H$ by putting the two neighbors of $v$ in $B$, and among the $\Delta - 1$ remaining vertices of our copy of $K_\Delta^-$, we put $d_B$ of them in $B$ and the remaining $\Delta - 1 - d_B = d_A + 1$ vertices go to $A$ (see an example in Figure 1). Note that $G[A]$ is the disjoint union of two $d_A$-degenerate graphs, namely $G[(A \setminus V(H)) \setminus \{v\}]$ and $G[(A \cap V(H)) \setminus \{v\}] \simeq K_{d_A+1}$, hence $G[A]$ is $d_A$-degenerate. The same holds for $G[B]$, with $G[B] \setminus V(H)$ and $G[B \cap V(H)]$, which is isomorphic to the complete graph on $d_B + 2$ vertices minus an edge, hence $G[B]$ is $d_B$-degenerate.

It remains to consider the case when $H$ is not isomorphic to $K_\Delta^-$. Contrarily to the previous case, we will not split the graph into $H$ and $G \setminus H$. Instead, the call to get_special_neighborhood gives us a special neighborhood $(z, X)$ of $H$ such that $v \notin X$. Before proceeding with the proof, note that we postpone the proof of the existence of such a special neighborhood and of the correctness of get_special_neighborhood to Lemma 12. Note that this special neighborhood is also a special neighborhood with respect to $G$. Points a) and b) clearly hold. For c), note that the graph $G \setminus V(H)$ and $G \setminus X$ are connected, and both contain vertex $v$, therefore their union, $G \setminus X$, is also connected. Let us now show that $G[A]$ is $p_A$-degenerate and $G[B]$ is $p_B$-degenerate.

Extend the ordering $v_2, \ldots, v_n$, by assigning the vertices of $X$ to $v_1, \ldots, v_{\Delta-1}$. We impose a single constraint: $v_{\Delta-2}$ and $v_{\Delta-1}$ must be non-adjacent vertices of $X \cap A$.

The degeneracy of $G[A]$ follows from Observation 6 by considering the ordering $v_1, \ldots, v_n$ restricted to $A$. To see this, first note that the vertices of $X \cap A$ have at most $p_A$ neighbors among $\{v_1, \ldots, v_{\Delta-2}\} \cap A$. Thus, $|A \cap N^<(v_i)| \leq p_A$ for indices up to $i = \Delta - 1$. For $i \geq \Delta$, the test line 17 ensures that for every vertex $v_i \in A$ we have $|A \cap N^<(v_i)| \leq p_A$. Therefore, $G[A]$ is $p_A$-degenerate.

For $G[B]$ also, consider the ordering $v_1, \ldots, v_n$ restricted to $B$. The vertices of $X \cap B$ have at most $p_B - 2$ neighbors in $X \cap B$. Thus, $|B \cap N^<(v_i)| \leq p_B$ for indices up to $i = \Delta - 1$. For the vertices $v_i \in B$ with $\Delta \leq i < n$, we have that $N^<(v_i) \leq \Delta + 1$. This follows from the post-order traversal considered, as in the proof of Lemma 9. Then, as $\Delta$-regular_bipartitioning adds such a vertex $v_i$ to $B$ (line 20) only if $|A \cap N^<(v_i)| \geq p_A + 1$, we have

$$|B \cap N^<(v_i)| = |N^<(v_i)| - |A \cap N^<(v_i)| \leq (\Delta - 1) - (p_A + 1) = p_B.$$ 

For $v_n = z$, it is different. Since at line 12 we put $p_A + 2$ of its neighbors in $A$, it shall be put in $B$ and we have

$$|B \cap N^<(v_n)| = |N^<(v_n)| - |A \cap N^<(v_n)| \leq \Delta - (p_A + 2) = p_B.$$ 

Therefore, $G[B]$ is $p_B$-degenerate. This completes the proof of correctness.

Runtime analysis. Decomposing $G$ into blocks can be performed in linear time 14, and testing if an end-block is isomorphic to $K_\Delta^-$ (line 6) can be checked in time linear in the size of the end-block. Besides, we can detect in time $O(n + m)$, which branch of the if statement to enter. In the first case ($H$ is isomorphic to $K_\Delta^-$) identifying $v$ and running non-$\Delta$-regular_partition takes time $O(n + m)$ (by Lemma 9). Splitting the other vertices of $U$ into $A$ and $B$ can be done in the same complexity, hence the overall complexity for this subcase is $O(n + m)$. In the second case, finding a special neighborhood takes time $O(n + m)$ by Lemma 13. Then, similarly as in Section 2, the complexity of the remaining instructions is clearly $O(n + m)$, which concludes the proof. 

Lemma 12. Algorithm 4 returns a special neighborhood $(z, X)$ of $H$, such that $v \notin X$.

Proof. We must show that the returned pair has properties a), b) and c) of Definition 10. Since $H$ is different from $K_{\Delta+1}$ and all vertices of $H \setminus \{v\}$ have degree $\Delta$, the neighborhood of every vertex $z \neq v$ contains a non-edge. So finding some $(z, X)$ satisfying properties a) and b) of Definition 10 is easy. To find a special neighborhood, the difficulty thus lies in guaranteeing the connectivity of $H \setminus X$.

We begin by partitioning $V(H)$ into sets $L_0, L_1, \ldots, L_\ell$ so that a vertex belongs to $L_i$ if it is at distance $i$ from $v$. Since $H$ is not $K_{\Delta+1}$ and since vertices in $L_1$ have degree $\Delta$, at least one
Algorithm 4 get_special_neighborhood

**Input:** A 2-connected graph $H$ distinct from $K_4^-$ and from $K_{\Delta+1}$, and a vertex $v \in H$. All vertices except possibly $v$ have degree $\Delta$ in $H$.

**Output:** A special neighborhood $(z, X)$ of $H$ such that $v \not\in X$.

1: Perform a BFS on $H$ starting from $v$
2: Partition the vertices into $L_0 = \{v\}, L_1, \ldots, L_\ell$ : $L_i$ contains the vertices at distance $i$ from $v$
3: $k \leftarrow \min(|N(z) \cap L_{\ell-1}|)$ for each $z \in L_\ell$
4: if $k \geq 3$ then
5: $z \leftarrow$ a vertex in $L_\ell$ with exactly $k$ neighbors in $L_{\ell-1}$
6: $x_1, x_2 \leftarrow$ two non-adjacent neighbors of $z$.
7: $x_3 \leftarrow$ a neighbor in $N(z) \cap L_{\ell-1}$ distinct from $x_1$ and $x_2$
8: $X \leftarrow N(z) \setminus x_3$
9: return $(z, X)$
10: end if
11: for each non-marked $z \in L_\ell$ with exactly $k$ neighbors in $L_{\ell-1}$ do
12: if $\exists$ a non-edge $x_1, x_2$ in $N(z)$ such that $|\{x_1, x_2\} \cap L_{\ell-1}| < k$ then
13: $x_3 \leftarrow$ a vertex in $(N(z) \cap L_{\ell-1}) \setminus \{x_1, x_2\}$
14: $X \leftarrow N(z) \setminus x_3$
15: return $(z, X)$
16: else
17: Mark $z$ as well as all its neighbors in $L_\ell$.
18: end if
19: end for
20: if $k == 1$ then
21: $u \leftarrow$ a vertex of $L_\ell$ with exactly one neighbor in $L_{\ell-1}$
22: $C \leftarrow H[N[u] \cap L_{\ell}]$
23: $x_1 \leftarrow$ a vertex of $L_{\ell-1} \cap N(C)$ with at most $|C|/2$ neighbors in $C$
24: $z \leftarrow$ a neighbor of $x_1$ in $C$
25: $x_2, x_3 \leftarrow$ two vertices $C \setminus N(x_1)$
26: $X \leftarrow N(z) \setminus x_3$
27: return $(z, X)$
28: end if
29: else // Necessarily $k = 2$
30: $u \leftarrow$ a vertex of $L_\ell$ having exactly two neighbors in $L_{\ell-1}$
31: $x_1, x_2 \leftarrow$ the neighbors of $u$ in $L_{\ell-1}$
32: $y_1 \leftarrow$ the neighbor of $x_1$ in $L_{\ell-2}$
33: $y_2 \leftarrow$ the neighbor of $x_2$ in $L_{\ell-2}$
34: if $y_1 == v$ then
35: Exchange $x_1$ and $x_2$, also $y_1$ and $y_2$
36: end if
37: $X \leftarrow N(x_1) \setminus \{u\}$
38: return $(x_1, X)$
39: end if
40: end if
vertex in \( L_1 \) has a neighbor in \( L_2 \), hence \( \ell \geq 2 \). By the execution of line \(^3\) \( k \) is the minimum number of neighbors a vertex \( z \) has in \( L_{\ell-1} \) over all \( z \in L_\ell \). We split the proof into three subcases with respect to the value of \( k \).

**Case \( k \geq 3 \).** Since the graph contains no \( K_{\Delta+1} \), there is a non-edge in \( H[N(z)] \), say \( x_1x_2 \). Moreover, as \( k \geq 3 \), there exists a vertex \( x_3 \in N(z) \cap L_{\ell-1} \) distinct from \( x_1 \) and \( x_2 \). Setting \( X \) to \( N(z) \setminus \{x_3\} \) ensures a) and b) as \( X \) contains \( x_1 \) and \( x_2 \). Since the vertices of \( X \) belong to \( L_{\ell-1} \cup L_\ell \), we satisfy the condition that \( v \notin X \).

It remains to show condition c), that is the fact that \( H \setminus X \) is connected. It suffices to show that any vertex in \( H \setminus X \) is connected to \( v \). This follows by induction from \( L_0 \) to \( L_\ell \). Indeed, every vertex in \( L_i \) has a neighbor in \( L_{i-1} \setminus X \). For \( i < \ell \), this holds because actually \( L_{i-1} \setminus X = L_{i-1} \), as \( X \subseteq L_{\ell-1} \cup L_\ell \). For \( i = \ell \), any vertex in \( L_\ell \) has at least \( k \) neighbors in \( L_{\ell-1} \). As \( X \) contains only \( k - 1 \) vertices of \( L_{\ell-1} \), any vertex in \( L_\ell \) has a neighbor in \( L_{\ell-1} \setminus X \) hence \( L_\ell \setminus X \) is also connected to \( v \).

**Case \( k = 1 \).** In the easiest case, \((z, X)\) is returned from the first loop (line \(^{10}\) in which case \( z \) has one neighbor in \( L_{\ell-1} \), the set \( X \) has size \( \Delta - 1 \) and \( H[X] \) is not complete. As such, conditions a) and b) are verified. Moreover, \( X \) contains only vertices from \( L_\ell \) (the furthest layer from \( v \)). Proving that \( H \setminus X \) is connected is similar to the case when \( k \geq 3 \). \( L_1 \) to \( L_{\ell-1} \) are included in \( v \)’s connected component and every vertex of \( L_\ell \) has a neighbor in \( L_{\ell-1} \). We now claim that not returning at line \(^{10}\) and entering line \(^{23}\) implies the following: for every vertex \( z \in L_\ell \) having only one neighbor in \( L_{\ell-1} \) the graph \( H[N[z] \cap L_\ell] \) is isomorphic to \( K_\Delta \). This would be immediate if we checked every vertex instead of every non-marked vertex, see line \(^{12}\). So we have to show that no vertex marked line \(^{18}\) can pass the the test line \(^{13}\). Indeed, if some \( z \) fails this test (line \(^{13}\)) it means that \( H[z] \cap L_\ell \) is a clique of order \( \Delta \). No vertex in this clique has a non-edge in its neighborhood restricted to \( L_\ell \), hence all vertices of this clique can safely be marked and not examined later.

We pick some vertex \( u \in L_\ell \) with only one neighbor in \( L_{\ell-1} \) and define \( C = H[N[u] \cap L_\ell] \). Recall that \( C \) is a clique of order \( \Delta \). We select a vertex \( x_1 \in L_{\ell-1} \cap N(C) \) which is connected to at most half the vertices of \( C \) (see Figure \(^3\)). This is possible because \( C \) has at least two neighbors in \( L_{\ell-1} \) for otherwise \( C \cup N(C) \) would be a clique of order \( \Delta + 1 \). As \( C \) has order \( \Delta \geq 3 \), \( x_1 \) has at least two non-neighbors in \( C \) that we denote \( x_2 \) and \( x_3 \). At line \(^{28}\) we define \( X \) as \( N(z) \setminus x_3 \). Let us prove that this set, which we return at line \(^{29}\) has the desired properties. Property a) is true by construction. Property b) also holds because \( x_2 \) was chosen among the non-neighbors of \( x_1 \) so \( H[X] \) contains a non-edge. Finally, since \( X \subseteq L_{\ell-1} \cup L_\ell \), it follows that \( v \notin X \). It remains to show that \( H \setminus X \) is connected.

As for the case \( k \geq 3 \), the connected component of \( H \setminus X \) containing vertex \( v \) contains all the vertices of \( L_1, \ldots, L_{\ell-1} \setminus X = L_{\ell-1} \setminus \{x_1\} \). Besides, since \( x_3 \) is not connected to \( x_1 \) but must have a neighbor in \( L_{\ell-1} \), it is connected to \( v \). Therefore \( z \) is connected to \( v \) through \( x_3 \). It remains to show that the vertices of \( L_\ell \setminus (X \cup \{z, x_3\}) = L_\ell \setminus V(C) \) belong to \( v \)’s connected component. Let

![Figure 2: The layers \( L_0, L_1, \ldots, L_{\ell-1}, L_\ell \) of \( H \).](image-url)
w be a vertex in this set. Since \( H \) is 2-connected, there exist two vertex-disjoint paths \( P_1, P_2 \) from \( w \) to \( v \). We can assume then that for instance \( P_2 \) does not contain \( x_1 \). Since \( C \) is a connected component of \( H[L_\ell] \), \( P_2 \) reaches \( L_{\ell-1} \setminus \{x_1\} \) before possibly reaching \( C \). So this part of \( P_2 \) avoids \( X \) and ensures \( w \) to be in \( v \)'s connected component. This completes the case.

Case \( k = 2 \). As in the case \( k = 1 \), \((z, X)\) can be returned by the first loop (line 16). In that case, conditions a) and b) hold for the same reasons as in the case \( k = 2 \).

We are now at line 22 and consider a vertex \( u \in L_{\ell-1} \) having only two neighbors in \( L_{\ell-2} \) that we denote \( x_1 \) and \( x_2 \). We claim that not returning at line 16 and executing line 32 implies the following: \( H[N[u] \setminus \{x_1\}] \) and \( H[N[u] \setminus \{x_2\}] \) are isomorphic to \( K_\Delta \) (see Figure 2). As \( x_1 \) and \( x_2 \) have \( \Delta - 1 \) neighbors in \( L_{\ell} \), both of them necessarily have their \( \Delta^{th} \) neighbor in \( L_{\ell-2} \). We denote them by \( y_1 \) and \( y_2 \). As \( H \) is distinct from \( K_\Delta \), these vertices are distinct. Therefore, we can choose \( x_1 \) and \( y_1 \) be such that \( y_1 \neq v \) (exchanging the \( x_i \)'s and \( y_i \)'s if necessary).

Consider the set \( X = N(x_1) \setminus \{u\} \). We claim that \((x_1, X)\) is a special neighborhood, which will complete the proof. Clearly, \((x_1, X)\) satisfies a) and since \( y_1 \in L_{\ell-2} \) has no neighbors in \( L_{\ell} \), it also satisfies b). Furthermore, since \( v \neq y_1 \) and \( X \setminus \{y_1\} \subseteq L_{\ell} \), we have that \( v \notin X \). To see that c) holds, observe as before that the connected component of \( H \setminus X \) containing \( v \) contains all the vertices of \( L_1, \ldots, L_{\ell-2} \setminus \{y_1\} \). Then, consider the connected components of \( H[L_{\ell-1} \cup L_{\ell}] \). Given the maximum degree \( \Delta \), one of them corresponds to \( H[N[u]] \). As \( N[u] \setminus X = \{u, x_1, x_2\} \), the path \( x_1u \) is \( \Delta \)-connected and ensures that the vertices of \( N[u] \setminus X \) are in \( v \)'s connected component. Let any other connected component \( Q \) of \( H[L_{\ell-1} \cup L_{\ell}] \). By what precedes, \( Q \setminus X = Q \). By the 2-connectivity of \( H \), \( N(Q) \setminus \{y_1\} \) is non-empty, which connects \( Q \) to \( v \). This completes the case and therefore the proof of the lemma.

\[ \square \]

**Lemma 13.** Algorithm 4 runs in time \( O(n + m) \).

**Proof.** The first step of the algorithm, performing a BFS on \( H \) and partitioning the vertices according to their distance from \( v \), can be done in time \( O(n + m) \). In order to speed up some later operations to achieve the desired complexity, we precompute an array \( distToV \) such that for each \( u \in H \), \( distToV[u] \) contains the distance from \( v \) to \( u \). Then, computing \( k \) can be done in time \( O(n + m) \) by looping over each \( z \in L_{\ell} \), each time visiting the neighbors of \( z \) and checking with the \( distToV \) array whether they belong to \( L_{\ell-1} \). We also initialize here a boolean array \( isNeighbWithZ \) to false for each vertex of \( H \).

The case \( k \geq 3 \) takes time \( O(n + m) \). Indeed, in time \( O(\Delta) \) we first update \( isNeighbWithZ \) to assign true for each neighbor of \( z \). Thanks to this, the selection of \( x_1, x_2 \) can be done in \( O(\sum_{z} E(H[N[z]])) = O(\Delta + m) \).

The for loop at lines 12 to 20 can be done in time \( O(n + m) \). First, the list of vertices in \( L_{\ell} \) with \( k \) neighbors in \( L_{\ell-1} \) can be made in time \( O(n + m) \) thanks to \( distToV \). Second, note that the sets \( N(z) \cap L_{\ell} \) for each \( z \) we consider at line 12 are disjoint. This comes from the fact that we mark all vertices in \( N(z) \cap L_{\ell} \) for each \( z \) we investigate.

Figure 3: The case \( k = 1 \) with \( \Delta = 5 \).
We now only need to observe that lines 14 to 16 (executed at most once) take time $O(n + m)$ and that line 13 takes time $O(|E(H) \cap N(z)|)$ per iteration. Over all iterations on unmarked vertices, this sums up to $O(|E(H)| + 2\Delta |L|) = O(m)$ as the sets $N(z) \cap L$ are disjoint. The $2\Delta$ term comes from the fact that at $k \leq 2$, hence $H[N(z)]$ is made of $H[N(z) \cap L]$ plus at most two other vertices. The marking of the vertices takes time $O(m)$ in total.

The body of the “if $k == 1$” part takes time $O(n + m)$. The only non-trivial point is the selection of $x_1$. To achieve this complexity, we can first build and fill a boolean array $isInC$. We can select $x_1$ by first generating the list of neighbors of $C$ which are in $L_{\ell-1}$. Then for each such vertex we count how many of its neighbors are in $C$. This consists in visiting disjoint edges, hence it can be done in time $O(n + m)$.

Finally, it is clear that the body of the “else” part ($k = 2$) can also be done in time $O(n+m)$.

4 The proof of Theorem 3 (general case)

Consider Algorithm 5, and let us verify that it fulfills Theorem 3.

The trick we use here, to have a complexity independent of $s$, the number of $p_i$'s in input, is to restrict to the first $\Delta$ of the $p_i$'s (Algorithm 5 might not read the whole input), and output only $\min(s, \Delta)$ sets.

Proof of Theorem 3. We first prove the correctness and then analyze the runtime.

Correctness of the algorithm. If $s$ is greater than $\Delta$ we can ignore all the $p_i$ for $i > \Delta$. Indeed, in that case the $\Delta$ first $p_i$'s, even if they are all equal to zero, sum up to at least $0 \geq \Delta - s$. Thus, the correctness of the algorithm for the case $s \leq \Delta$, implies its correctness in full. As in the algorithm, let us consider the following three cases: Either $\sum_{i=1}^{s'} p_i > \Delta - s'$, $\sum_{i=1}^{s'} p_i = \Delta - s'$ and $G$ is not $\Delta$-regular, or $\sum_{i=1}^{s'} p_i = \Delta - s'$ and $G$ is $\Delta$-regular. The first two cases are handled by the calls to greedy partitioning (by Lemma 7), and to non-$\Delta$-regular partitioning (by Lemma 9), respectively. For the third case, we only prove the case $p_1 \leq p_2$, the case $p_1 > p_2$ being similar. This third case is handled in two steps, first by a call to $\Delta$-regular bipartitioning partitioning $V(G)$ into a $p_1$-degenerate graph, $G[A]$, and a $(\Delta - 2 - p_1)$-degenerate graph, $G[B]$ (by Theorem 11 since $p_1 \leq p_2 \leq \Delta - 2 - p_1$), and then by a call to greedy partitioning refining $B$ into $p_2$, $\ldots, p_{s'}$-degenerate graphs (by Remark 8 since $\Delta - 2 - p_1 < (s' - 1) + \sum_{i=2}^{s'} p_i$).

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Algorithm 5 main algorithm

| Input: A graph $G$ with maximum degree $\Delta$, and a sequence of length $s \geq 2$ of non-negative integers $p_1, \cdots, p_s$ such that $\sum_{i=1}^s p_i \geq \Delta - s$ |
| Output: A partition of $V(G)$ into sets $A_1, \cdots, A_s$ so that each $G[A_i]$ is $p_i$-degenerate |

1. $s' \leftarrow \min(s, \Delta)$
2. if $\sum_{i=1}^{s'} p_i > \Delta - s'$ then
3.     $v_1, \cdots, v_n \leftarrow$ Any ordering of $V(G)$
4.     return greedy partitioning$(G, v_1, \cdots, v_n, p_1, \cdots, p_{s'})$
5. end if
6. if $G$ is not regular then
7.     return non-$\Delta$-regular partitioning$(G, p_1, \cdots, p_{s'})$
8. end if
9. $p^- \leftarrow \min(p_1, p_2)$
10. $p^+ \leftarrow \max(p_1, p_2)$
11. $A, B \leftarrow \Delta$-regular bipartitioning$(G, p^-, \Delta - 2 - p^-)$
12. $v_1, \cdots, v_{|B|} \leftarrow$ a $(\Delta - 2 - p^-)$-degenerate ordering of $G[B]$
13. $A^+, A_3, \cdots, A_s' \leftarrow$ greedy partitioning$(G[B], v_1, \cdots, v_{|B|}, p^+, p_3, \cdots, p_{s'})$
14. if $p_1 \leq p_2$ then
15.     return $A, A^+, A_3, \cdots, A_s'$
16. else
17.     return $A^+, A, A_3, \cdots, A_s'$
18. end if

Runtime analysis. The time complexity of Algorithm 5 lies in the calls to other algorithms and in the instructions within the algorithm, the latter taking clearly only time $O(n + m + s')$. By Lemma 7, Lemma 9, and Theorem 11 these calls take time $O(n + m + s')$ and $O(n + m)$. As $s' \leq \Delta \leq n$, the overall complexity is $O(n + m)$.

5 Final remarks

5.1 Bounding the maximum degree

In [7] the authors describe an algorithm turning a partition fulfilling point (i) of Theorem 2 into one fulfilling both (i) and (ii). Their algorithm is a succession of individual vertex moves, that is, replacing two sets $V_i, V_j$ with $V_i \setminus \{u\}$ and $V_j \cup \{u\}$. Each such step diminish an energy-like function whose image lies in $[-4m, 2m]$, so we can safely bound the number of moves by $6m$. At each step, updating the partition and maintaining the list of vertices violating point (ii) needs $O(deg(u)) = O(\Delta)$ time. So the whole algorithm runs in $O(\Delta m)$ time.

5.2 Perspectives

Theorem 2 has been generalized in [7] by replacing the notion of degeneracy by the notion of variable degeneracy. This improvement was in turn recently generalized in the context of digraphs [3]. This generalization is achievable in polynomial time (quadratic or less), but it seems difficult to perform it in linear time, as it relies on finding cycles with particular properties. It would be interesting to have an algorithm performing such a partition in linear time.

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