WEAK WEAK QUENCHED LIMITS FOR THE PATH-VALUED PROCESSES OF HITTING TIMES AND POSITIONS OF A TRANSIENT, ONE-DIMENSIONAL RANDOM WALK IN A RANDOM ENVIRONMENT

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Abstract. In this article we continue the study of the quenched distributions of transient, one-dimensional random walks in a random environment. In a previous article we showed that while the quenched distributions of the hitting times do not converge to any deterministic distribution, they do have a weak weak limit in the sense that - viewed as random elements of the space of probability measures - they converge in distribution to a certain random probability measure (we refer to this as a weak weak limit because it is a weak limit in the weak topology). Here, we improve this result to the path-valued process of hitting times. As a consequence, we are able to also prove a weak weak quenched limit theorem for the path of the random walk itself.

1. Introduction and Notation

A random walk in a random environment (RWRE) is a very simple model for random motion in a non-homogeneous random medium. A nearest-neighbor RWRE on \( \mathbb{Z} \) may be described as follows. Elements of the set \( \Omega = [0,1]^\mathbb{Z} \) are called environments since they can be used to define the transition probabilities for a Markov chain. That is, for any \( \omega = \{ \omega_x \}_{x \in \mathbb{Z}} \in \Omega \) and any \( z \in \mathbb{Z} \), let \( X_n \) be a Markov chain with law \( P^z_\omega \) given by \( P^z_\omega(X_0 = z) = 1 \) and

\[
P^z_\omega(X_{n+1} = y \mid X_n = x) = \begin{cases} \omega_x & \text{if } y = x + 1 \\ 1 - \omega_x & \text{if } y = x - 1 \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( \Omega \) be endowed with the natural cylindrical \( \sigma \)-field, and let \( P \) be a probability measure on \( \Omega \). Then, if \( \omega \) is a random environment with distribution \( P \), then \( P^z_\omega \) is a random probability measure and is called the quenched law of the RWRE. By averaging over all environments we obtain the averaged law of the RWRE

\[
P^z(\cdot) = \int_{\Omega} P^z_\omega(\cdot) P(d\omega).
\]

For ease of notation, the quenched and averaged laws of the RWRE started at \( z = 0 \) will be denoted by \( P_\omega \) and \( P \), respectively. Expectations with respect to \( P \), \( P_\omega \) and \( P \) will be denoted by \( E_P \), \( E_\omega \) and \( E \), respectively.

Throughout this paper we will make the following assumptions on the distribution \( P \) on environments.

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Assumption 1. The environments are i.i.d. That is, \( \{\omega_x\}_{x \in \mathbb{Z}} \) is an i.i.d. sequence of random variables under the measure \( P \).

Assumption 2. The expectation \( E_P[\log \rho_0] \) is well defined and \( E_P[\log \rho_0] < 0 \). Here \( \rho_i = \rho_i(\omega) = \frac{1 - \omega_i}{\omega_i} \), for all \( i \in \mathbb{Z} \).

Assumption 3. The distribution of \( \log \rho_0 \) is non-lattice under \( P \), and there exists a \( \kappa > 0 \) such that \( E_P[\rho_0^\kappa] = 1 \) and \( E_P[\rho_0^\kappa \log \rho_0] < \infty \).

From Solomon’s seminal paper on RWRE [Sol75], it is well known that Assumptions 1 and 2 imply that the RWRE is transient to \( +\infty \); that is, \( \mathbb{P}(\lim_{n \to \infty} X_n = \infty) = 1 \). Moreover, Solomon showed that there exists a law of large numbers in the sense that there exists a constant \( v_P \) such that \( \lim_{n \to \infty} X_n/n = v_P, \mathbb{P}\text{-a.s.} \). Solomon also showed that the limiting velocity \( v_P \) is non-zero only if \( E_P[\rho_0^\kappa] < 1 \), which equivalent to \( \kappa > 1 \) when Assumption 3 is in effect as well. Assumption 3 was used by Kesten, Kozlov, and Spitzer in their analysis of the averaged limiting distributions for transient one-dimensional RWRE [KKS75]. The parameter \( \kappa \) in Assumption 3 determines the magnitude of centering and scaling as well as the type of distribution obtained in the limit. Define the hitting times of the RWRE by

\[
    T_x = \inf\{n \geq 0 : X_n = x\}, \quad x \in \mathbb{Z},
\]

and for \( \kappa \in (0, 2) \) define the properly centered and scaled versions of the hitting times and location of the RWRE by

\[
    t_n = \begin{cases} 
        \frac{T_n}{n^{1/\kappa}} & \kappa \in (0, 1) \\
        \frac{T_n}{n} & \kappa = 1 \\
        \frac{T_n}{n^{1/\kappa}} & \kappa \in (1, 2)
    \end{cases}
    \quad \text{and} \quad
    z_n = \begin{cases} 
        \frac{X_n}{n^{1/\kappa}} - \delta(n) & \kappa \in (0, 1) \\
        \frac{X_n}{n^{1/\kappa}} & \kappa = 1 \\
        \frac{X_n}{n^{1/\kappa}} - \delta(n) & \kappa \in (1, 2),
    \end{cases}
\]

where in the case \( \kappa = 1, A > 0 \) is a certain constant, and \( D(n) \) and \( \delta(n) \) are certain functions satisfying \( D(n) \sim A \log n \) and \( \delta(n) \sim n/(A \log n) \), respectively. Also, let \( L_{\kappa, b} \) denote the distribution function of a totally skewed to the right stable random variable with index \( \kappa \in (0, 2) \), scaling parameter \( b > 0 \), and zero shift; see [ST94]. The following averaged limiting distribution for RWRE was first proved in [KKS75].

**Theorem 1.1.** Let Assumptions 1 - 3 hold, and let \( \kappa \in (0, 2) \). Then, there exists a constant \( b > 0 \) such that for any \( x \in \mathbb{R}, \)

\[
    \lim_{n \to \infty} \mathbb{P}(t_n \leq x) = L_{\kappa, b}(x), \quad x \in \mathbb{R},
\]

and

\[
    \lim_{n \to \infty} \mathbb{P}(z_n \leq x) = \begin{cases} 
        1 - L_{\kappa, b}(x^{-1/\kappa}) & \text{for } x > 0 \text{ if } \kappa \in (0, 1) \\
        1 - L_{\kappa, b}(-x) & \text{for } x \in \mathbb{R} \text{ if } \kappa \in [1, 2).
    \end{cases}
\]

**Remark 1.2.** The cases \( \kappa = 2 \) and \( \kappa > 2 \) were also considered in [KKS75], but since our main results are for \( \kappa \in (0, 2) \) we will limit our focus to these cases. We note, however, that when \( \kappa \geq 2 \) the averaged limiting distributions for the hitting times and the location of the RWRE are Gaussian.

It is important to note that the limiting distributions in Theorem 1.1 are for the averaged measure \( \mathbb{P} \). However, for certain applications the quenched measure \( P_\omega \) may be more applicable (e.g., for repeated experiments in a fixed non-homogeneous medium), and one naturally wonders if there is a quenched analog of Theorem 1.1. Unfortunately, it was shown in [PZ09] and [Pet09] that there is no such strong quenched limiting distribution. That is, for almost every fixed environment \( \omega \), there is no centering and scaling (or even environment-dependent centering...
and scaling) for which the hitting times or location of the RWRE converge in distribution under \( P_\omega \).

The negative results of [PZ09] and [Pet09] were recently clarified by showing that quenched limiting distributions do exist in a weak sense ([PS10] [DG10] [ESTZ10]). Let \( M_1(\mathbb{R}) \) be the space of probability measures on \( \mathbb{R} \) equipped with the topology of convergence in distribution. Then, since the environment \( \omega \) is a random variable, the quenched distribution \( \mu_{n,\omega} = P_\omega (t_n \in \cdot) \) is an \( M_1(\mathbb{R}) \)-valued function of that random variable. This function can be easily shown to be measurable, hence \( \mu_{n,\omega} \) is itself a random variable, namely an \( M_1(\mathbb{R}) \)-valued random variable. It was shown in [PS10] that there exists a family of \( M_1(\mathbb{R}) \)-valued random variables \( \{ \pi_{\lambda,\kappa} \} \) such that \( \mu_{n,\omega} \Rightarrow \pi_{\lambda,\kappa} \) for some \( \lambda > 0 \), where \( \Rightarrow \) denotes weak convergence of \( M_1(\mathbb{R}) \)-valued random variables. We will refer to such limits as weak weak quenched limits since the quenched distribution converges weakly with respect to the weak topology on \( M_1(\mathbb{R}) \). Similar results were obtained independently in [DG10] and [ESTZ10].

In [PS10], this weak weak quenched limiting distribution for the hitting times was also used to obtain a result on the quenched distribution of the location of the RWRE. It was shown that

\[
P_\omega (\zeta_n \leq x) \Rightarrow \begin{cases} \pi_{\lambda,\kappa}[x^{-1/\kappa}, \infty) & \text{for } x > 0 \text{ if } \kappa \in (0, 1) \\ \pi_{\lambda,\kappa}[-x, \infty) & \text{for } x \in \mathbb{R} \text{ if } \kappa \in [1, 2) \end{cases}
\]

and here \( \Rightarrow \) denotes weak convergence of \( \mathbb{R} \)-valued random variables. Note that this is a weaker statement than the quenched limit that was obtained for the hitting times. Unfortunately, weak convergence of all one-dimensional projections of a random probability measure is not enough to specify the weak limit of the random probability measure. For example, suppose that \( \kappa \in (0, 1) \). If \( \sigma_{\lambda,\kappa} \) is the transformation of the random probability measure \( \pi_{\lambda,\kappa} \) defined by letting \( \sigma_{\lambda,\kappa}([-\infty, x]) = \pi_{\lambda,\kappa}(x^{-1/\kappa}, \infty) \), one is tempted to guess that \( P_\omega (\zeta_n \in \cdot) \Rightarrow \sigma_{\lambda,\kappa} \) in the sense of weak convergence of random probability measures. However, it can be seen from our results below that this is not true (see Corollary 1.8).

1.1. Main Results. The original goal of the current paper was to obtain a full weak limit for the random probability measure \( P_\omega, \zeta_n \in \cdot \). However, it turned out to be necessary to obtain a weak limit for not just the quenched distribution of the hitting \( T_n \) but also for the quenched distribution of the path process of the sequence of hitting times. This result, in turn leads to not only a weak limit for the quenched distribution of \( X_n \) but also to the weak limit of the quenched distribution of the entire path of the RWRE, as we will see in the sequel.

To begin, let \( D_\infty \) be the space of càdlàg functions (continuous from the right with left limits) on \([0, \infty)\). We will equip \( D_\infty \) with the \( M_1 \)-Skorohod metric \( d_M^1 \) (instead of the more standard and slightly stronger \( J_1 \)-Skorohod metric \( d_J^1 \); the definitions of the Skorohod metrics are given in Section 3). Let \( M_1(D_\infty) \) be the space of probability measures on \( D_\infty \) equipped with the topology of weak convergence induced by the \( M_1 \)-metric \( d_M^1 \) on \( D_\infty \). Since \( (D_\infty, d_M^1) \) is a Polish space, this topology is equivalent to topology induced by the Prohorov metric \( \rho_M^1 \) (see Section 3 for a precise definition).

For any realization of the random walk and \( \varepsilon > 0 \), let \( T_\varepsilon \in D_\infty \) be defined by

\[
T_\varepsilon(t) = \begin{cases} \varepsilon^{1/\kappa} T_{t/\varepsilon} & \kappa \in (0, 1) \\ \varepsilon (T_{t/\varepsilon} - t/\varepsilon D(1/\varepsilon)) & \kappa = 1 \\ \varepsilon^{1/\kappa} (T_{t/\varepsilon} - t/(\varepsilon V)) & \kappa \in (1, 2) \end{cases}
\]

(here and in the sequel we define hitting times of non-integer points by \( T_x = T_{\lfloor x \rfloor} \)). In the case \( \kappa = 1 \) the function \( D \) is the function in (1) extended to all \( x > 0 \); we will define it explicitly.

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1 Throughout the paper, if \((Z_n, Z)\) are random variables in some space \( \Psi \), then \( Z_n \Rightarrow Z \) will denote weak convergence (i.e., convergence in distribution) of \( \Psi \)-valued random variables.
in Section 4. It is easy to see that, for each environment \( \omega \) and any \( \varepsilon > 0 \), \( T_\varepsilon \) is a well-defined \( D_\infty \)-valued random variable; we denote by \( m_{\varepsilon, \omega} \) the quenched law of \( T_\varepsilon \) on \( D_\infty \). This law is a measurable function of the environment, hence a \( M_1(D_\infty) \)-valued random variable. We wish to show that this random variable converges weakly as \( \varepsilon \to 0 \). In order to identify the limit we need to introduce additional notation.

Let \( \mathcal{M}_p((0, \infty) \times [0, \infty)) \) be the space of Radon point processes on \( (0, \infty) \times [0, \infty) \). These are point processes assigning finite mass to \( [\varepsilon, \infty) \times [0, T] \) for any \( \varepsilon > 0 \) and \( T < \infty \). The topology of vague convergence on this space is metrizable, and converts \( \mathcal{M}_p((0, \infty) \times [0, \infty)) \) into a complete separable metric space; see [Res08, Proposition 3.17]. We denote by \( \mathcal{M}_p^f((0, \infty) \times [0, \infty)) \) the subset of \( \mathcal{M}_p((0, \infty) \times [0, \infty)) \) of point processes that do not put any mass on points with infinite first coordinate. Let \( \bar{\tau} = \{\tau_i\}_{i \geq 1} \) be a sequence of i.i.d. standard exponential random variables. For a point process \( \zeta = \sum_{i \geq 1} \delta_{(x_i, t_i)} \in \mathcal{M}_p^f((0, \infty) \times [0, \infty)) \) and \( \delta > 0 \) we define a stochastic process (random path) \( W_\delta(\zeta, \bar{\tau}) \) with sample paths in \( D_\infty \) by

\[
W_\delta(\zeta, \bar{\tau})(t) = \sum_{i \geq 1} x_i \tau_i 1_{\{t_i > \delta, t \leq t_i\}}.
\]

We also let

\[
W(\zeta, \bar{\tau})(t) = \begin{cases} 
\sum_{i \geq 1} x_i \tau_i 1_{\{t_i \leq t\}} & \text{if the sum is finite} \\
0 & \text{otherwise}.
\end{cases}
\]

Remark 1.3. The notation \( W_\delta(\zeta, \bar{\tau}) \) and \( W(\zeta, \bar{\tau}) \) is somewhat misleading since the actual definitions depend on the (measurable) ordering chosen for the points of \( \zeta \). Since \( \bar{\tau} \) is an i.i.d. sequence of random variables, the choice of ordering will not affect the laws of \( W_\delta(\zeta, \bar{\tau}) \) and \( W(\zeta, \bar{\tau}) \), and we are only concerned with the laws of these processes.

It is clear that \( \lim_{\delta \to 0} W_\delta(\zeta, \bar{\tau}) = W(\zeta, \bar{\tau}) \) in \( D_\infty \) for every choice of \( \bar{\tau} \) for which \( W(\zeta, \bar{\tau})(t) < \infty \) for each \( t < \infty \). We will impose assumptions on the point processes \( \zeta \) such that this holds with probability one.

For any point process \( \zeta \) such that, with probability 1, \( W(\zeta, \bar{\tau})(t) < \infty \) for each \( t < \infty \), the definitions of \( W_\delta(\zeta, \bar{\tau}) \) and \( W(\zeta, \bar{\tau}) \) induce in natural way probability measures on \( D_\infty \). Define functions \( \mathcal{H}_\delta, \mathcal{H} : \mathcal{M}_p((0, \infty) \times [0, \infty)) \to M_1(D_\infty) \) by

\[
\mathcal{H}_\delta(\zeta)(\cdot) = P_\bar{\tau}(W_\delta(\zeta, \bar{\tau}) \in \cdot), \quad \text{and} \quad \mathcal{H}(\zeta)(\cdot) = P_\bar{\tau}(W(\zeta, \bar{\tau}) \in \cdot),
\]

when \( \zeta \in \mathcal{M}_p^f((0, \infty) \times [0, \infty)) \) and (in the case of \( \mathcal{H}(\zeta) \)) when \( W(\zeta, \bar{\tau})(t) < \infty \) for each \( t < \infty \) with probability 1. Otherwise we define \( \mathcal{H}_\delta(\zeta) \) or \( \mathcal{H}(\zeta) \), respectively, to be the Dirac point mass at the zero process in \( D_\infty \). Here \( P_\bar{\tau} \) is the distribution of the i.i.d. sequence of the standard exponential random variables \( \bar{\tau} = \{\tau_i\}_{i \geq 1} \).

Before stating our theorem we need one last bit of notation. The cases \( \kappa \in \{1, 2\} \) require a centering term in the limit. Thus, for any \( m \in \mathbb{R} \) let \( \ell(m) \in \mathcal{M}_1(D_\infty) \) be the Dirac point mass concentration that is concentrated on the linear path \( t \mapsto mt \). If \( X \) is a \( D_\infty \)-valued random variable with distribution \( \mu \in \mathcal{M}_1(D_\infty) \), then \( \mu * (\ell(-m)) \) is the distribution of the path \( \{t \mapsto X(t) - mt\} \).

**Theorem 1.4.** Let \( m_\varepsilon = m_{\varepsilon, \omega} = P_\omega(T_\varepsilon \in \cdot) \) be the quenched distribution of the path \( T_\varepsilon \). For \( 0 < \kappa < 2 \) let \( \lambda = C_0 \kappa / \nu \), where \( C_0 \) and \( \nu \) are given, respectively, by (12) and (11) below. Let \( N_{\lambda, \kappa} \) be a Poisson point process on \( (0, \infty) \times [0, \infty) \) whose intensity measure puts no mass on infinite points, and is given by \( \lambda x^{-\kappa-1} dx dt \) on finite points.

1. If \( \kappa \in (0, 1) \), then \( m_\varepsilon \Rightarrow \mathcal{H}(N_{\lambda, \kappa}) \).
2. If \( \kappa = 1 \), then \( m_\varepsilon \Rightarrow \lim_{\delta \to 0} \mathcal{H}_\delta(N_{\lambda, 1}) * \ell(-\lambda \log(1/\delta)) \).
(3) If \( \kappa \in (1, 2) \), then
\[
m_\varepsilon \implies \lim_{\delta \to 0} \mathcal{H}_\delta(N_{\lambda, \kappa}) \ast \ell \left(-\lambda \delta^{\kappa+1}/(\kappa - 1)\right).
\]

**Remark 1.5.** The limits as \( \delta \to 0 \) in parts (2) and (3) of Theorem 1.4 are weak limits in \( \mathcal{M}_1(D_{\infty}) \). In fact, we will see in the sequel that these limits exist even as a.s. limits in \( \mathcal{M}_1((D_{\infty}, d_{\infty}^M)) \). Furthermore, part (1) of the theorem includes a well-known fact that for each \( t < \infty \), \( W(N_{\lambda, \kappa}, \tau)(t) < \infty \) with \( \mathbb{P}_\tau \)-probability 1 for almost every realization of the Poisson point process.

As mentioned above, we will prove a weak limit for the quenched distribution of the entire path of the RWRE. To this end, we define a centered and scaled path of the random walk \( \chi_\varepsilon \in D_{\infty} \) by
\[
\chi_\varepsilon(t) = \begin{cases} 
\varepsilon^\kappa X_{[t/\varepsilon]} - t\delta(1/\varepsilon) & \kappa \in (0, 1) \\
1/\varepsilon^{\delta(1/\varepsilon)} (X_{[t/\varepsilon]} - t\delta(1/\varepsilon)) & \kappa = 1 \\
v P_{1/\varepsilon^{1/\kappa}} \chi_{1/\kappa} (X_{[t/\varepsilon]} - t\varepsilon P/\varepsilon) & \kappa \in (1, 2),
\end{cases}
\]
where in the case \( \kappa = 1 \), \( \delta(x) \) is a function that satisfies \( \delta(x)D(\delta(x)) = x + o(1) \) as \( x \to \infty \). Here \( D \) is the same function as in (3). Note that, since \( D(x) \sim A \log x \), this implies that \( \delta(x) \sim x/(A \log x) \) so that the scaling factor in the definition of \( \chi_\varepsilon \) when \( \kappa = 1 \) is asymptotic to \( \varepsilon(A \log(1/\varepsilon))^2 \) as \( \varepsilon \to 0 \). Let \( p_{\varepsilon, \omega} = P_\omega(\chi_\varepsilon \in \cdot) \) be the quenched law of \( \chi_\varepsilon \) on \( D_{\infty} \). It is a \( \mathcal{M}_1(D_{\infty}) \)-valued random variable defined on \( \Omega \).

The weak limits of \( p_{\varepsilon, \omega} \) will be obtained by comparing the paths of the location of the RWRE \( \chi_\varepsilon \) to appropriately transformed paths of the hitting times \( T_\varepsilon \). To this end, we define two transformations of paths. Let \( D_{u, \uparrow}^+ \subset D_{\infty} \) consist of functions that are (weakly) monotone increasing, with \( x(0) \geq 0 \) and \( \lim_{t \to \infty} x(t) = \infty \). Define the time-space inversion function \( \mathfrak{T} : D_{u, \uparrow}^+ \to D_{u, \uparrow}^+ \) by
\[
\mathfrak{T}(t) = \sup \{ s \geq 0 : x(s) \leq t \}, \quad t \geq 0, \quad x \in D_{u, \uparrow}^+.
\]
Also, define the spatial reflection function \( \mathfrak{R} : D_{\infty} \to D_{\infty} \) by \( \mathfrak{R}(t) = -x(t), \quad t \geq 0, \quad x \in D_{\infty} \).

**Theorem 1.6.** (a) The following coupling results hold.

1. If \( \kappa \in (0, 1) \), then for any \( s < \infty \)
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{t \leq \varepsilon} |\chi_\varepsilon(t) - \mathfrak{T}_{\varepsilon, \kappa}(t)| \geq \eta \right) = 0, \quad \forall \eta > 0.
\]

2. If \( \kappa = 1 \), then
\[
\lim_{\varepsilon \to 0} \mathbb{P}(d_{\infty}^M(\chi_\varepsilon, -\mathfrak{T}_{1/\delta(1/\varepsilon)}) \geq \eta) = 0, \quad \forall \eta > 0.
\]

3. If \( \kappa \in (1, 2) \), then
\[
\lim_{\varepsilon \to 0} \mathbb{P}(d_{\infty}^M(\chi_\varepsilon, -\mathfrak{R}_{\varepsilon/\sqrt{\varepsilon}}) \geq \eta) = 0, \quad \forall \eta > 0.
\]

(b) Let \( p_\varepsilon = p_{\varepsilon, \omega} = P_\omega(\chi_\varepsilon \in \cdot) \) be the quenched distribution of the path \( \chi_\varepsilon \), and let \( \mu_{\lambda, \kappa} \) be the limiting element of \( \mathcal{M}_1(D_{\infty}) \) given in Theorem 1.4. Then \( p_\varepsilon \implies \mu_{\lambda, \kappa} \circ \mathfrak{T}^{-1} \) if \( \kappa \in (0, 1) \) and \( p_\varepsilon \implies \mu_{\lambda, \kappa} \circ \mathfrak{R}^{-1} \) if \( \kappa \in [1, 2) \), weakly in \( \mathcal{M}_1(D_{\infty}) \).

**Remark 1.7.** Note that the nature of conversion from time to space in the limiting random probability measure in \( \mathcal{M}_1(D_{\infty}) \) is very different in the absence of centering term \( (\kappa \in (0, 1)) \) from the case when there is a centering term \( (\kappa \in [1, 2)) \). When \( \kappa \in [1, 2) \) the conversion is accomplished by multiplying a random path distributed according to the limiting (random)
measure by -1. This is, of course, very different from the switching the time and space axes required when \( \kappa \in (0, 1) \).

Observe that, for any \( 0 \leq t < \infty \), the map \( \Phi_t : \mathcal{M}_1(D_\infty) \to \mathcal{M}_1(\mathbb{R}) \) defined by
\[
\Phi_t(\mu)(A) = \mu(\{x \in D_\infty : x(t) \in A\}), \quad \text{for any Borel } A \subset \mathbb{R},
\]
is continuous at every \( \mu \in \mathcal{M}_1(D_\infty) \) concentrated on paths continuous at \( t \). Since the limiting probability measures on \( \mathcal{M}_1(D_\infty) \) obtained in Theorem 1.6 is concentrated on \( \mu \) with this property, the continuous mapping theorem immediately implies the following weak convergence for the distributions of the location of the random walk at fixed times.

**Remark 1.11.** The statement of Corollary 1.9 in the case \( \kappa \in (0, 1) \) appeared in Remark 2.5 in [ESZ09]. To the best of our knowledge the other statements in Corollaries 1.9 and 1.10 are new.

Part (2) of Corollary 1.10 is an immediate consequence of the corresponding part of Corollary 1.9, the coupling results in parts (2) and (3) of Theorem 1.6 and Theorem 3.1 in [Bil99]. Further, [Whi02, Corollary 13.6.4] says that the operator \( \mathcal{I} \) from the subset \( D_{u,\uparrow}^+ \subset D_{u,\uparrow}^+ \) of strictly increasing, non-negative, unbounded paths endowed with the \( d_{\mathcal{M}_1} \) metric to \( D_{u,\downarrow}^+ \) endowed with the \( d_\infty \) metric, is continuous. Since, in the case \( 0 < \kappa < 1 \), a \( \kappa \)-stable subordinator is in \( D_{u,\uparrow}^+ \) with probability one, the continuous mapping theorem shows that part (1) of Corollary 1.10 also follows from the corresponding part of Corollary 1.9.

The proof of Corollary 1.9 is also rather straightforward, but, because it introduces certain key ideas and notation used later in the paper, we present the proof here.
Proof of Corollary 1.9. Let \( N_{\lambda, \kappa} \) be the Poisson point process on \((0, \infty) \times [0, \infty)\) defined in Theorem 1.4, and let \( \tau = \{\tau_i\}_{i \geq 1} \) be an i.i.d. sequence of standard exponential random variables; we assume that \( N_{\lambda, \kappa} \) and \( \tau \) are defined on two different probability spaces, with the corresponding probability measures \( P \) and \( P_\tau \). On the product probability space we define

\[
Z_{\lambda, \kappa}(t) = \begin{cases} 
W(N_{\lambda, \kappa}, \tau)(t) & \kappa \in (0, 1) \\
\lim_{\delta \to 0} W_\delta(N_{\lambda, \kappa}, \tau)(t) - \lambda t \log(1/\delta) & \kappa = 1 \\
\lim_{\delta \to 0} W_\delta(N_{\lambda, \kappa}, \tau)(t) - \lambda t^{1-\kappa}/(\kappa - 1) & \kappa \in (1, 2),
\end{cases}
\]

for \( t \geq 0 \). The definition is understood as a.s. convergence in \((D_\infty, d_{\mathcal{M}_1})\) on the product probability space. This convergence takes place by the proposition in Section 2 of [Kal74], and it is standard to see that \( Z_{\lambda, \kappa} \) is a \( \kappa \)-stable Lévy process with the required properties of Corollary 1.9. In order to show that the averaged distribution of \( T_\varepsilon \) converges to the distribution of \( Z_{\lambda, \kappa} \) under the product probability measure \( P \times P_\tau \), it is enough to show that \( P(T_\varepsilon \in A) \to P \times P_\tau(Z_{\lambda, \kappa} \in A) \) as \( \varepsilon \to 0 \) for all cylindrical sets \( A \subset D_\infty \) such that \( P \times P_\tau(Z_{\lambda, \kappa} \in \partial A) = 0 \). (Recall that the Borel \( \sigma \)-field under all the Skorohod topologies coincides with the cylindrical \( \sigma \)-field; see Theorem 11.5.2 in [Whi02].) Let \( A \) be such a set. Recall that \( P \times P_\tau(Z_{\lambda, \kappa} \in A) = E_\lambda[\mu_{\lambda, \kappa}(A)] \), where \( \mu_{\lambda, \kappa} \) is as in Theorem 1.4. By Fubini’s theorem, \( \mu_{\lambda, \kappa}(\partial A) = 0 \) almost surely. Also, the evaluation mapping mapping \( \mu \mapsto \mu(A) \) on \( \mathcal{M}_1(D_\infty) \) is continuous on the set of measures \( \{\mu \in \mathcal{M}_1(D_\infty) : \mu(\partial A) = 0\} \). Since the random measure \( \mu_{\lambda, \kappa} \) is in this set with probability one, and since Theorem 1.4 implies that \( m_{\varepsilon, \omega}(A) \to \mu_{\lambda, \kappa}(A) \) then the mapping theorem implies that \( m_{\varepsilon, \omega}(A) \) converges in distribution to \( \mu_{\lambda, \kappa}(A) \). Since these random variables are between 0 and 1, this implies that

\[
\lim_{\varepsilon \to 0} P(T_\varepsilon \in A) = \lim_{\varepsilon \to 0} E_P[m_{\varepsilon, \omega}(A)] = E[\mu_{\lambda, \kappa}(A)] = P \times P_\tau(Z_{\lambda, \kappa} \in A).
\]

\[\square\]

The limiting random probability measure \( \mu_{\lambda, \kappa} \) is a \( \kappa \)-stable random element of \( \mathcal{M}_1(D_\infty) \) under convolutions. That is, the convolution of two independent copies of this random probability measure is (after re-scaling and shifting) a random probability measure with the same law. This can be seen in the same way as the stability of the limiting random probability measures on \( \mathbb{R} \) was checked in [PS10]. Stability of random probability measures on \( D_\infty \) does not seem to have been investigated before, but a systematic description of infinitely divisible (in particular, stable) random probability measures on \( \mathbb{R} \) was given in [S106]; we recall these notions in Section 7. The latter paper introduced also a notion of \( \mathcal{M}_1(\mathbb{R}) \)-valued Lévy process. If we recall the maps \( \Phi_t, 0 \leq t < \infty \), defined in [S], then we can define a (measurable) map \( \Phi \) from \( \mathcal{M}_1(D_\infty) \) to \( D_\infty(\mathcal{M}_1(\mathbb{R})) \) by setting \( \Phi(\mu) \) to be the measure-valued path \( \{\Phi_t(\mu), t \geq 0\} \).

One would expect that a version of Theorem 1.4 would give us a convergence to a \( \mathcal{M}_1(\mathbb{R}) \)-valued Lévy process as well. The following corollary gives such convergence, but only in the sense of convergence of finite dimensional distributions.

Corollary 1.12. Let \( \mu_{\lambda, \kappa} \) and \( m_\varepsilon \) be the random probability measures on \( D_\infty \) given in Theorem 1.4, \( 0 < \kappa < 2 \) (so that \( m_\varepsilon \Rightarrow \mu_{\lambda, \kappa} \)). Then \( \Phi(m_\varepsilon) \) converges weakly to \( \Phi(\mu_{\lambda, \kappa}) \) in the sense of finite dimensional distributions. Moreover, for any \( \kappa \in (0, 2) \), \( \Phi(\mu_{\lambda, \kappa}) \) is a stable Lévy process on \( \mathcal{M}_1(\mathbb{R}) \). It is a strictly stable Lévy process if \( \kappa \neq 1 \).

Remark 1.13. One would like to improve the finite dimensional distribution convergence in Corollary 1.12 to a full convergence in distribution of \( \mathcal{M}_1(\mathbb{R}) \)-valued path processes. Such a statement seems would require setting a topology on the space \( D_\infty(\mathcal{M}_1(\mathbb{R})) \) of measure-valued path processes. Choosing an appropriate topology seems to be a difficult task as neither the Skorohod \( J_1 \)-topology nor a natural definition of the Skorohod \( M_1 \)-topology appear to be sufficient. This is complicated by the fact that the mapping \( \Phi : \mathcal{M}_\infty(D_\infty) \to D_\infty(\mathcal{M}_1(\mathbb{R})) \) is
not continuous in these topologies (even on the support of the limiting measure $\mu_{\lambda,\kappa}$). These issues are discussed further in Section 7.

2. Random environment

It will be important for us to identify sections of the environment that contribute the most to the distribution of the hitting times. To this end, we define the ladder locations $(\nu_k)$ of the environment by

$$\nu_0 = 0, \quad \text{and} \quad \nu_k = \inf \left\{ j > \nu_{k-1} : \prod_{i=\nu_{k-1}}^{j-1} \rho_i < 1 \right\} \text{ for } k \geq 1.$$  \hspace{1cm} (10)

(The ladder locations are those locations where the potential of the environment introduced by Sinai \cite{Sin83} reaches a new minimum to the right of the origin.) Occasionally we will denote $\nu_1$ by $\nu$ instead for compactness. Since the environment is i.i.d., the sections of the environment immediately to the left of $\nu_0$ are different from the environment immediately to the left of $\nu_k$ for any $k \geq 1$. Thus, the environment is not stationary under shifts of the environment by the ladder locations. To resolve this complication we define a new measure $Q$ on environments by

$$Q(\cdot) = P(\cdot | R), \quad \text{where } R = \left\{ \omega : \prod_{j=i}^{-1} \rho_j < 1, \forall i \leq -1 \right\}.$$  \hspace{1cm} (11)

It is important to note that the definition of the measure $Q$ only affects the environment to the left of the origin. Therefore, the blocks of the environment $\{\omega_x : \nu_k \leq x < \nu_{k+1}\}$ between ladder locations are i.i.d. and have the same distribution under both $P$ and $Q$. For instance

$$\bar{\nu} := E_P \nu_1 = E_Q \nu_1.$$  \hspace{1cm} (12)

For any $i \geq 1$ let $\beta_i = \beta_i(\omega)$ be defined by $\beta_i = E_\omega [T_{\nu_i} - T_{\nu_{i-1}}]$. Since the measure $Q$ is stationary under shifts of the environment by the ladder locations, the sequence $\{\beta_i\}_{i \geq 1}$ is stationary under $Q$. The following tail asymptotics of the $\beta_i$ were derived in \cite{PZ09} and will be crucial throughout this paper.

$$Q(\beta_1 > x) = Q(E_\omega T_1 > x) \sim C_0 x^{-\kappa}, \quad \text{as } x \to \infty.$$  \hspace{1cm} (13)

We conclude this section with a simple lemma that will be of use later in the paper.

\textbf{Lemma 2.1.} Let $\bar{\beta} = E_Q[\beta_1]$. If $\kappa > 1$, then $\bar{\beta} = \bar{\nu}/v_P$.

\textbf{Proof.} First, note that the sequence $\{E_\omega [T_i - T_{i-1}]\}_{i \geq 1}$ is ergodic under the measure $P$ (since it represents the shifts of a fixed function of an i.i.d., hence ergodic, sequence). Therefore, Birkhoff’s Ergodic Theorem implies that

$$\lim_{n \to \infty} \frac{E_\omega T_n}{n} = E_P [E_\omega T_1] = E T_1, \quad P\text{-a.s.}$$  \hspace{1cm} (14)

Since the measure $Q$ is defined by conditioning $P$ on an event of positive probability, we see that this holds $Q$-a.s. as well. Moreover, if $\kappa > 1$, then the limiting velocity $v_P = 1/ET_1 > 0$ (see \cite{Sol73} or \cite{Zei01} for a reference).

Secondly, note that, since the $\nu_i - \nu_{i-1}$ are i.i.d. under $Q$, it follows that $\lim_{n \to \infty} \nu_n/n = \bar{\nu}$, $Q$-a.s. This implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \beta_i = \lim_{n \to \infty} \frac{E_\omega T_{\nu_n}}{n} = \lim_{n \to \infty} \frac{E_\omega T_{\nu_n} \nu_n}{\nu_n} = \frac{\bar{\nu}}{v_P}, \quad Q\text{-a.s.}$$

Finally, since the $\beta_i$ are stationary under $Q$, it is a consequence of Birkhoff’s Ergodic Theorem that this nonrandom limit of $1/n \sum_{i=1}^{n} \beta_i$ must coincide with $E_Q[\beta_1]$. \hfill \Box
3. **Topological Generalities**

3.1. **Skorohod Topologies.** In this section we recall the definitions of the Skorohod $J_1$ and $M_1$ metrics on the space $D_\infty$, and the corresponding topologies. We also give certain technical results that will be needed in the sequel. The details that we omit can be found in [Bil99] and [Whi02].

For $0 < t < \infty$ the $J_1$ and $M_1$ Skorohod metrics on the space $D_t$ of càdlàg functions on $[0,t]$ are defined as follows. Let $\Lambda_t$ be the set of time-change functions on $[0,t]$ – functions that are strictly increasing and continuous bijections from $[0,t]$ to itself. The Skorohod $J_1$-metric (on $D_t$) is defined by

$$d_{J_1}(x,y) = \inf_{\lambda \in \Lambda_t} \max \left\{ \sup_{s \leq t} |\lambda(s) - s|, \sup_{s \leq t} |x(\lambda(s)) - y(s)| \right\}.$$ 

Next, recall that the completed graph of a càdlàg function $x \in D_t$ is the subset $\Gamma_x \subset [0,t] \times \mathbb{R}$ defined by

$$\Gamma_x = \{(u,v) : u \in [0,t], v = (1-\theta)x(u) + \theta x(u), \text{ for some } \theta \in [0,1]\}.$$ 

The natural order $\preceq_{\Gamma_x}$ on the completed graph is given by $(u_1,v_1) \preceq_{\Gamma_x} (u_2,v_2)$ if either $u_1 < u_2$ or $u_1 = u_2$ and $|v_1 - x(u_1)| \leq |v_2 - x(u_1)|$. A parametric representation of the completed graph $\Gamma_x$ is a function from $[0,1]$ onto $\Gamma_x$ that is continuous with respect to the natural order $\preceq_{\Gamma_x}$. Let $\Pi(x)$ be the set of all parametric representations of $\Gamma_x$, with each parametric representation given by a pair of functions $u$ and $v$ on $[0,1]$ such that $\Gamma_x = \{(u(s),v(s)) : s \in [0,1]\}$. The Skorohod $M_1$-metric on $D_t$ is defined by

$$d_{M_1}(x,y) = \inf_{(u,v) \in \Pi(x), (u',v') \in \Pi(y)} \max \left\{ \sup_{s \in [0,1]} |u(s) - u'(s)|, \sup_{s \in [0,1]} |v(s) - v'(s)| \right\}.$$ 

The Skorohod $J_1$ and $M_1$-metrics on $D_t$ for all finite $t$ produce corresponding metrics on the space $D_\infty$ by

$$d_{\infty}^{J_1}(x,y) = \int_0^\infty e^{-t} \left( d_{J_1}(x(t),y(t)) \wedge 1 \right) dt, \quad \text{and} \quad d_{\infty}^{M_1}(x,y) = \int_0^\infty e^{-t} \left( d_{M_1}(x(t),y(t)) \wedge 1 \right) dt.$$ 

Here, for $x \in D_\infty$, the function $x(t) \in D_t$ is the restriction of $x$ to the finite time interval $[0,t]$. Using instead the uniform metric on each $D_t$ produces the metric $d_U^{\infty}$ on the space $D_\infty$.

Using the uniform metric on each $D_t$ produces the metric $d_U^{\infty}$ on the space $D_\infty$.

The following is a list of several useful properties of the Skorohod metrics that we will use throughout the paper; see [Whi02].

- $d_{\infty}^{M_1}(x,y) \leq d_{\infty}^{J_1}(x,y)$.
- $d_{\infty}^{J_1}(x,y) \leq e^{-s} + \sup_{t \leq s} |x(t) - y(t)|$ for any $0 < s < \infty$; thus uniform convergence on compact subsets of $[0,\infty)$ implies convergence in the $J_1$-Skorohod metric.
- $d_{\infty}^{J_1}(x,y) \geq |x(t) - y(t)|$ for each $0 < t < \infty$.
- $d_{\infty}^{J_1}(x_n,x) \to 0$ if and only if $d_{\infty}^{J_1}(x_n,x) \to 0$ for all continuity points $t$ of $x$. An analogous statement is true for the $M_1$-Skorohod topology.

The $J_1$ and $M_1$-metrics generate topologies on the space of càdlàg functions. Even though the two metrics are not complete, each of them has an equivalent metric that is complete. Therefore, the $J_1$ and $M_1$ topologies are the topologies of complete separable metric spaces.

The $J_1$ and $M_1$-metrics on $D_\infty$ induce in the standard way the corresponding Prohorov’s metrics, $\rho^{J_1}$ and $\rho^{M_1}$, on the space of Borel probability measures $\mathcal{M}_1(D_\infty)$. For example, for any $\mu, \pi \in \mathcal{M}_1(D_\infty)$,

$$\rho^{M_1}(\mu, \pi) = \inf \left\{ \delta > 0 : \mu(A) \leq \pi(A^{\delta,M_1}) + \delta, \text{ for every Borel } A \subset D_\infty \right\}.$$
(recall that the $J_1$ and $M_1$-metrics generate the same Borel sets on $D_\infty$; these are also the cylindrical sets). Further,

$$A_{\delta,M_1} = \{ y : d_{\infty}^{M_1}(x,y) < \delta, \text{ for some } x \in A \}.$$

Since $(D_\infty,d_\infty^{M_1})$ is a separable metric space, convergence in the Prohorov metric $\rho^{M_1}$ is equivalent to convergence in distribution in $(D_\infty,d_\infty^{M_1})$; see Theorem 3.2.1 in [Whi02]. Moreover, the space $(M_1(D_\infty),\rho^{M_1})$ is a complete separable metric space (Theorem 6.8 in [Bil99]).

### 3.2. Continuity of functionals.

We proceed with two results on the continuity of certain functionals that we will need later. We begin, by recalling the following result from [Whi02] on the continuity of the composition map.

**Lemma 3.1** (Theorems 13.2.2, 13.2.3 in [Whi02]). The composition map $\psi : D_\infty \times D_\infty^+ \to D_\infty$ defined by $\psi(x,y) = x \circ y$ is continuous on the set $D_\infty \times C_\infty^+$, where $D_\infty^+$ is the set of all nonnegative functions in $D_\infty$, and $C_\infty^+$ is the set of continuous, non-negative, strictly increasing functions on $[0,\infty)$. The continuity holds whenever either the $J_1$-topology is used throughout, or the $M_1$-topology is used throughout.

The composition map $\psi$ induces a map $\Psi : M_1(D_\infty) \times D_\infty^+ \to M_1(D_\infty)$ by

$$\Psi(\mu,y)(\{ x : x \in A \}) = \mu(\{ x : x \circ y \in A \}).$$

 Lemma 3.1 leads to the following continuity result for $\Psi$.

**Corollary 3.2.** The map $\Psi$ is continuous on the set $M_1(D_\infty) \times C_\infty^+$, if the same topology (either $J_1$ or $M_1$) is used throughout.

**Proof.** Suppose that $(\mu_n,y_n) \to (\mu,y) \in M_1(D_\infty) \times C_\infty^+$. By the Skorohod representation theorem (e.g. Theorem 3.2.2 in [Whi02]), there are $D_\infty$-valued random elements $(X_n)_n$, $X$ defined on a common probability space such that $X_n \sim \mu_n$ for each $n$, $X \sim \mu$, and $X_n \to X$ a.s. in the corresponding Skorohod metric. By Lemma 3.1 we know that $X_n \circ y_n \to X \circ y$ in the same metric. Since a.s. convergence implies weak convergence, the claim follows. \hfill \Box

### 3.3. Deducing weak convergence of random probability measures.

In order to prove weak convergence of a sequence of random probability measures on $D_\infty$ we will often use the coupling technique which we now describe. Suppose that $\mu,\pi \in M_1(D_\infty)$. Then a coupling of $\mu$ and $\pi$ is a probability measure $\theta$ on the product space $D_\infty \times D_\infty$ with marginals $\mu$ and $\pi$, respectively. A coupling of two random probability measures on $D_\infty$ defined on a common probability space is a random element of $M_1(D_\infty \times D_\infty)$ defined on the same probability space that couples the two measures for every $\omega$. The following simple lemma, which is a path space extension of Lemma 3.1 in [PS10], is the key ingredient in our approach.

**Lemma 3.3.** Suppose that $(\mu_n),(\pi_n)$ are two sequences of random elements in $M_1(D_\infty)$ defined on a common probability space with probability measure $P$ and expectation $E$. Suppose that one of the following conditions holds.

1. $\lim_{n \to \infty} P(\rho^{M_1}(\mu_n,\pi_n) \geq \eta) = 0$, for all $\eta > 0$.
2. For each $n$ there exists a coupling $\theta_n$ of the random probability measures $\mu_n$ and $\pi_n$ such that

$$\lim_{n \to \infty} E[\theta_n(\{(x,y) : d_{\infty}^{M_1}(x,y) \geq \eta\})] = 0, \quad \text{for all } \eta > 0.$$

3. For each $n$ there exists a coupling $\theta_n$ of the random probability measures $\mu_n$ and $\pi_n$ such that

$$\lim_{n \to \infty} P(E_{\theta_n}[d_{\infty}^{M_1}(x,y) \geq \eta]) = 0, \quad \text{for all } \eta > 0,$$

where $E_{\theta_n}$ denotes expectations under the measure $\theta_n$. 

If $\mu_n \implies \mu$ weakly in $(M_1(D_{\infty}), \rho^{M_1})$, then $\pi_n \implies \mu$ weakly in $(M_1(D_{\infty}), \rho^{M_1})$.

Proof. Under condition (1) the statement follows from Theorem 3.1 in [Bil99]. Next, note that the definition of the Prohorov metric implies that if $\theta_n(\{(x, y) : d_{\infty}^{M_1}(x, y) \geq \eta\}) \leq \eta$ then $\rho^{M_1}(\mu_n, \pi_n) \leq \eta$. Therefore,

$$\mathbf{P}(\rho^{M_1}(\mu_n, \pi_n) > \eta) \leq \mathbf{P}(\theta_n(\{(x, y) : d_{\infty}^{M_1}(x, y) \geq \eta\}) > \eta) \leq \frac{1}{\eta} \mathbf{E}[\theta_n(\{(x, y) : d_{\infty}^{M_1}(x, y) \geq \eta\})].$$

Thus, condition (2) implies condition (1). Furthermore, condition (3) implies condition (2) by Chebyshev’s inequality. $\square$

The following lemma will allow us to reduce checking condition (3) in Lemma 3.3 to the finite time situation. We note that a similar reduction holds under the metrics $d^{U}$ and $d^{U}$ as well.

Lemma 3.4. Suppose that $(\mu_n), (\pi_n)$ are two sequences of random elements in $M_1(D_{\infty})$ defined on a common probability space with probability measure $\mathbf{P}$ and expectation $\mathbf{E}$. If for each $n$ there exists a coupling $\theta_n$ of the random probability measures $\mu_n$ and $\pi_n$ such that for every $0 < t < \infty$

$$\lim_{n \to \infty} \mathbf{P}(E_{\theta_n}[d_{t}^{M_1}(x(t), y(t))] \geq \eta) = 0, \quad \forall \eta > 0, \tag{14}$$

then condition (3) in Lemma 3.3 holds.

Proof. By the bounded convergence theorem, (14) implies that

$$\lim_{n \to \infty} \mathbf{E}[E_{\theta_n}[d_{t}^{M_1}(x(t), y(t))] \wedge 1] = 0, \quad \forall t < \infty.$$ 

By the definition of $d_{\infty}^{M_1}$, Fubini’s Theorem and dominated convergence theorem we immediately see that

$$\mathbf{E}[E_{\theta_n}[d_{\infty}^{M_1}(x, y)]] = \mathbf{E}[E_{\theta_n}\left[\int_0^\infty e^{-t} \left(d_{t}^{M_1}(x(t), y(t)) \wedge 1\right) dt\right]] = \int_0^\infty e^{-t} \mathbf{E}[E_{\theta_n}\left[(d_{t}^{M_1}(x(t), y(t)) \wedge 1)\right]] dt$$

vanishes as $n \to \infty$. This implies condition (3) in Lemma 3.3 $\square$

4. Comparison with sums of exponentials

The main goal of this section is to reduce the study of the hitting time process $T_\varepsilon$ to the study of a process $S_\varepsilon$ that is defined in terms of sums of exponential random variables. To this end, recall the definition of the ladder locations of the environment in (10) and the notation $\beta_i = \beta_i(\omega) = E_\omega[T_{\nu_i} - T_{\nu_{i-1}}]$ for the quenched expectation of the time to cross from $\nu_{i-1}$ to $\nu_i$. Also, we expand the measure $P_\omega$ to include an i.i.d. sequence of standard exponential random variables $(\tau_i)$; it will be used in the coupling procedure below by comparing $T_{\nu_i} - T_{\nu_{i-1}}$ with $\beta_i \tau_i$.

For any realization of the environment we construct random paths $U_\varepsilon, S_\varepsilon \in D_{\infty}$ as follows. For $t \geq 0$,

$$U_\varepsilon(t) = \begin{cases} e^{1/\kappa}T_{1/\varepsilon} & \kappa \in (0, 1), \\ \varepsilon(T_{1/\varepsilon} - t/\varepsilon D'(1/\varepsilon)) & \kappa = 1, \\ e^{1/\kappa}(T_{1/\varepsilon} - \beta t/\varepsilon) & \kappa \in (1, 2), \end{cases}$$
Let Proposition 4.2. \( M \) hold under the probability measure via the following two propositions. The first proposition establishes weak convergence of \( \kappa \) therefore we only spell out the details in the case \( m \) then \( u \). That is, in Theorem 1.1 satisfies \( A \) in Theorem 1.1 satisfies \( A \).

Remark 4.1. The proof below will show that in the case \( \kappa = 1 \), the function \( D \) in definition of \( T \) can be chosen to be \( D(x) = D'(x)/\bar{\nu} \) (recall that \( \bar{\nu} = E_Q[\nu] \)). In particular, the constant \( A \) in Theorem 1.1 satisfies \( A = C_0/\bar{\nu} \).

Let \( u_\varepsilon = u_{\varepsilon,\omega} \), \( s_\varepsilon = s_{\varepsilon,\omega} \in M_1(D_\infty) \) be the quenched distributions of \( U_\varepsilon \) and \( S_\varepsilon \), respectively. That is,

\[
u_\varepsilon = P_\omega(U_\varepsilon \in \cdot), \quad \text{and} \quad s_\varepsilon = P_\omega(S_\varepsilon \in \cdot).
\]

We view \( u_\varepsilon \) and \( s_\varepsilon \) as random elements in \( M_1(D_\infty) \). The proof of Theorem 1.4 is accomplished via the following two propositions. The first proposition establishes weak convergence of \( s_\varepsilon \) in \( M_1(D_\infty) \). The notation and the terminology are the same as in Theorem 1.4.

**Proposition 4.2.** Let \( \lambda = C_0\kappa \), where \( C_0 \) is the tail constant in \( J \). The following statements hold under the probability measure \( Q \) on the environments.

1. If \( \kappa \in (0, 1) \), then \( s_\varepsilon \Longrightarrow \mathcal{H}(N_{\lambda,\kappa}) \).
2. If \( \kappa = 1 \), then
   \[
   s_\varepsilon \Longrightarrow \lim_{\delta \to 0} \mathcal{H}_\delta(N_{\lambda,1}) \ast \ell(-\lambda\log(1/\delta)).
   \]
3. If \( \kappa \in (1, 2) \), then
   \[
   s_\varepsilon \Longrightarrow \lim_{\delta \to 0} \mathcal{H}_\delta(N_{\lambda,\kappa}) \ast \ell(-\lambda\delta^{-\kappa+1}/(\kappa-1)).
   \]

The second proposition relates a weak limit of \( s_\varepsilon \) in \( M_1(D_\infty) \) to the corresponding weak limit of \( m_\varepsilon \).

**Proposition 4.3.** Define \( \lambda_0 \in C_{\uparrow}^+ \) by \( \lambda_0(t) = t/\bar{\nu} \). If \( s_\varepsilon \Longrightarrow \mu \) weakly in \( M_1(D_\infty) \) under \( Q \), then \( m_\varepsilon \Longrightarrow \Psi(\mu, \lambda_0) \) under \( P \).

Before giving the proofs of Propositions 4.2 and 4.3 we show how they imply Theorem 1.4.

**Proof of Theorem 1.4.** The proof is essentially the same, whether \( \kappa \in (0, 1) \), \( \kappa = 1 \), or \( \kappa \in (1, 2) \), therefore we only spell out the details in the case \( \kappa \in (1, 2) \).

First, note that by Propositions 4.2 and 4.3 and Corollary 3.2 under the measure \( P \),

\[
m_\varepsilon \Longrightarrow \lim_{\delta \to 0} \Psi \left( \mathcal{H}_\delta(N_{\lambda,\kappa}) \ast \ell \left( -\frac{\lambda\delta^{-\kappa+1}}{\kappa - 1}, \lambda_0 \right) \right).
\]

Therefore, it is enough to show that, for any \( \lambda > 0 \), with \( \lambda' = \lambda/\bar{\nu} \),

\[
\Psi \left( \mathcal{H}_\delta(N_{\lambda,\kappa}) \ast \ell \left( -\frac{\lambda\delta^{-\kappa+1}}{\kappa - 1}, \lambda_0 \right) \right) \xrightarrow{\text{Law}} \mathcal{H}_\delta(N_{\lambda',\kappa}) \ast \ell \left( -\frac{\lambda'\delta^{-\kappa+1}}{\kappa - 1} \right).
\]

To see this, note that for any \( m > 0 \),

\[
\Psi \left( \mathcal{H}(\sum_{i \geq 1} \delta(x_i,t_i)) \ast \ell(-m), \lambda_0 \right) = \mathcal{H}_\delta \left( \sum_{i \geq 1} \delta(x_i,t_i) \right) \ast \ell(-m/\bar{\nu}).
\]

However, if \( \sum_{i \geq 1} \delta(x_i,t_i) \) is a Poisson point process with intensity measure \( \lambda x^{-\kappa-1} dx \) then \( \sum_{i \geq 1} \delta(x_i,t_i) \) is a Poisson point process with intensity measure \( (\lambda/\bar{\nu}) x^{-\kappa-1} dx \). This implies \( \square \).
It remains to prove Propositions 4.2 and 4.3. Proposition 4.2 will be proved in Section 5 and in the remainder of this section will focus on the proof of Proposition 4.3 which follows immediately from Lemma 3.3 and the following lemmas.

**Lemma 4.4.** There exists a coupling of $U_\varepsilon$ and $S_\varepsilon$ such that, for any $\eta > 0$,
\[
\lim_{\varepsilon \to 0} Q \left( E_\varepsilon [d^{\infty}_\varepsilon(U_\varepsilon, S_\varepsilon)] \geq \eta \right) = 0.
\]

Under the assumption of Proposition 4.3, this lemma and part (3) of Lemma 3.3 will imply that $u_\varepsilon \Rightarrow \mu$ in $M_1(D_\infty)$ under $Q$.

**Lemma 4.5.** If $u_\varepsilon \Rightarrow \mu$ in $M_1(D_\infty)$ under $Q$, then $m_\varepsilon \Rightarrow \Psi(\mu, \lambda_0)$ in $M_1(D_\infty)$ under $Q$.

Under the assumption of Proposition 4.3, this lemma will imply that $m_\varepsilon \Rightarrow \Psi(\mu, \lambda_0)$ in $M_1(D_\infty)$ under $Q$.

**Lemma 4.6.** There exists a measure $\mathcal{P}$ on pairs of environments $(\omega, \omega')$ such that the marginal distributions of $\omega$ and $\omega'$ are $P$ and $Q$, respectively, and, for each $\varepsilon > 0$, there exists a coupling $P_\varepsilon = P_{\varepsilon; \omega, \omega'}$ of the random measures $m_{\varepsilon; \omega}$ and $m_{\varepsilon; \omega'}$ such that
\[
\lim_{\varepsilon \to 0} \mathcal{P} \left( E_\varepsilon [d^J_\varepsilon(x, y)] \geq \eta \right) = 0, \quad \text{for all } \eta > 0.
\]

Under the assumption of Proposition 4.3, this lemma and another appeal to part (3) of Lemma 3.3 will imply the claim of the proposition. We proceed now to prove the three lemmas.

**Proof of Lemma 4.4.** We use the same construction as in the proof of Lemma 4.2 in [PS10]. First let $\omega$ and $\tilde{\omega}$ be independent with distributions $P$ and $Q$ respectively. Then, construct $\omega'$ by letting
\[
\omega'_x = \begin{cases} 
\tilde{\omega}_x & x \leq -1 \\
\omega_x & x \geq 0.
\end{cases}
\]

Then $\omega'$ has distribution $Q$ and is identical to $\omega$ on the non-negative integers. Let $\mathcal{P}$ be the joint law of $(\omega, \omega')$.

Given a pair of environments $(\omega, \omega')$, we construct coupled random walks $\{X_n\}$ and $\{X'_n\}$ so that the marginal laws of $\{X_n\}$ and $\{X'_n\}$ are $P_\omega$ and $P_{\omega'}$ respectively. We do that by coordinating all steps of the random walks to the right of $0$. That is, since $\omega_x = \omega'_x$ for any $x \geq 0$, we require that on the respective $i^{th}$ visits of the walks $X_n$ and $X'_n$ to site $x$ they both either move to the right or both move to the left. The details of this coupling can be found in [PS10]. Let $P_\varepsilon = P_{\varepsilon; \omega, \omega'}$ denote the joint quenched law of the two random walks coupled in this manner; the corresponding expectation is denoted by $E_\varepsilon = E_{\varepsilon; \omega, \omega'}$. Let $T_k$, $T'_k$ and $T_\varepsilon$, $T'_\varepsilon$ be the hitting times and the path processes of hitting times corresponding to the random walks $\{X_n\}$ and $\{X'_n\}$, respectively. Note that
\[
d^U_\varepsilon(T_\varepsilon, T'_\varepsilon) \leq \sup_{t < \infty} |T_\varepsilon(t) - T'_\varepsilon(t)| = \varepsilon^{1/\kappa} \sup_{n \geq 1} |T_n - T'_n|.
\]

However, it is easy to see that the coupling of $X_n$ and $X'_n$ is such that $\sup_{n \geq 1} |T_n - T'_n| = |L - L'|$, where $L$ and $L'$ are the number of steps that the walks $\{X_n\}$ and $\{X'_n\}$, respectively, spend to the left of $0$. It is easy to see (and was shown in the proof of Lemma 4.2 in [PS10]) that $E_{\varepsilon; \omega, \omega'}|L - L'| \leq E_\omega L + E_{\omega'} L' < \infty$, $\mathcal{P}$-a.s. Therefore, for any $\eta > 0$
\[
\lim_{\varepsilon \to 0} \mathcal{P} \left( E_\varepsilon \left[ d^U_\varepsilon(T_\varepsilon, T'_\varepsilon) \right] \geq \eta \right) \leq \lim_{\varepsilon \to 0} \mathcal{P} \left( \varepsilon^{1/\kappa} E_{\omega, \omega'}|L - L'| \geq \eta \right) = 0.
\]
Proof of Lemma 4.5. We start with a time change in the process $U_\varepsilon(t)$ to align its jumps with the hitting times of corresponding ladder locations in the process $T_\varepsilon$. To this end, define $\lambda_\varepsilon \in D_1^+$, the space of nonnegative non-decreasing functions in $D_\infty$, by

$$\lambda_\varepsilon(t) = \varepsilon \max\{k : \nu_k \leq t\}, \quad t \geq 0.$$ 

Then, the renewal theorem implies that $\lim_{\varepsilon \to 0} \lambda_\varepsilon(t) = \lambda_0(t)$, $Q$-a.s., for any fixed $t \geq 0$. Since $\lambda_\varepsilon$ is non-decreasing and $\lambda_0$ is continuous and non-decreasing, the convergence is uniform on compact subsets of $[0, \infty)$. Therefore, $\lim_{\varepsilon \to 0} d^M_\varepsilon(\lambda_\varepsilon, \lambda_0) = 0$, $Q$-a.s. Furthermore, it follows from the functional central limit theorem for renewal sequences with a finite variance that $\varepsilon^{-1/2}(\lambda_\varepsilon - \lambda_0)$ converges weakly in $(D_\infty, J_1)$ to a Brownian motion, as $\varepsilon \to 0$. See Theorem 7.4.1 in [Whi02].

The assumption $u_\varepsilon \implies \mu$ and Corollary 3.2 show that $\Psi(u_\varepsilon, \lambda_\varepsilon) \implies \Psi(\mu, \lambda_0)$ under $Q$, so by Lemmas 3.3 and 3.4, the claim of the present lemma will follow once we show that for every $0 < t < \infty$ and $\eta > 0$,

$$\lim_{\varepsilon \to 0} Q \left( E_\omega \left[ d^M_t(U_\varepsilon \circ \lambda_\varepsilon, T_\varepsilon) \right] \geq \eta \right) = 0. \tag{17}$$

To simplify the notation, we omit the superscripts in functions of the type $T_\varepsilon^{(t)}$. Because of the centering present when $\kappa \in (1, 2)$ but not when $\kappa \in (0, 1)$, we treat the two cases separately.

**Case I:** $\kappa \in (0, 1)$. Note that the definition of $\lambda_\varepsilon$ implies that $\mathbb{U}_\varepsilon(\lambda_\varepsilon(t)) = \varepsilon^{1/\kappa} T_{\nu_j} = T_\varepsilon(t)$ when $t = \varepsilon \nu_j$. We arrange the respective parametric representations of the completed graphs of the two random functions, $\mathbb{U}_\varepsilon \circ \lambda_\varepsilon$ and $T_\varepsilon$, so that at each $j = j/(k+1) \in [0, 1)$ both parametric representations are equal, to $(\varepsilon \nu_j, \varepsilon^{1/\kappa} T_{\nu_j})$. Here $k$ is the largest $j$ so that $\nu_j \leq t/\varepsilon$. For $s_j < s < s_{j+1}$ with $j = 0, 1, \ldots, k-1$ we arrange the two parametric representations so that the vertical $t$ coordinates always stay the same (see Figure 1). Then the distance between the corresponding points on the completed graphs on the interval in that range of $s$ is taken horizontally, and it is, at most, $\varepsilon(\nu_{j+1} - \nu_j)$. This horizontal matching cannot, generally, be performed on the interval $(s_k, 1]$ since the two functions may not be equal at time $t$. On this interval we keep horizontal $t$ coordinates the same. The distance between the corresponding points is now taken vertically, and it is, at most, $\varepsilon^{1/\kappa}(T_{\nu_{k+1}} - T_{\nu_k})$. Therefore,

$$d^M_t(U_\varepsilon \circ \lambda_\varepsilon, T_\varepsilon) \leq \max \left\{ \max_{j < k} \varepsilon (\nu_{j+1} - \nu_j), \varepsilon^{1/\kappa} (T_{\nu_{k+1}} - T_{\nu_k}) \right\}.$$ 

Since $k \leq t/\varepsilon$, we conclude, using stationarity of the sequence $(\nu_{j+1} - \nu_j)$ under $Q$ that for $0 < \varepsilon < 1$ so small that $\varepsilon(\log 1/\varepsilon)^2 \leq \eta$,

$$Q \left( E_\omega[d^M_t(U_\varepsilon \circ \lambda_\varepsilon, T_\varepsilon)] \geq \eta \right) \leq \frac{t}{\varepsilon} Q \left( \nu_1 > \log^2(1/\varepsilon) \right) + Q \left( \varepsilon^{1/\kappa} \beta_{k+1} > \eta \right. \left. \text{for } t \in [\nu_k, \nu_{k+1}] \right).$$ 

Since $\nu_1$ has some finite exponential moments (see [PZ99]), the first term on the right above vanishes as $\varepsilon \to 0$. For the second term note that $t \in [\nu_k, \nu_{k+1})$ is equivalent to $\lambda_\varepsilon(t) = \varepsilon k$, hence

$$Q \left( \varepsilon^{1/\kappa} \beta_{k+1} > \eta \right. \left. \text{for } t \in [\nu_k, \nu_{k+1}] \right) \leq Q \left( |\lambda_\varepsilon(t) - t/\nu| > \varepsilon^{1/4} \right) + Q \left( \exists k : |k - t/(\nu_\varepsilon)| \leq \varepsilon^{-3/4}, \beta_{k+1} > \eta \varepsilon^{-1/\kappa} \right) \leq Q \left( |\lambda_\varepsilon(t) - t/\nu| > \varepsilon^{1/4} \right) + 2\varepsilon^{-3/4} Q(\beta_1 > \eta \varepsilon^{-1/\kappa}),$$

using the stationarity of the $(\beta_k)$ under $Q$ in the last inequality. The functional central limit theorem for renewal sequences implies that the first probability on the right vanishes as $\varepsilon \to 0$. The second probability on the right vanishes as $\varepsilon \to 0$. Thus, we have

$$Q \left( E_\omega \left[ d^M_t(U_\varepsilon \circ \lambda_\varepsilon, T_\varepsilon) \right] \geq \eta \right) \leq 2\varepsilon^{-3/4} Q(\beta_1 > \eta \varepsilon^{-1/\kappa}).$$

This completes the proof for Case I.
The second term also vanishes as \( \varepsilon \to 0 \) by the tail decay \(^{12}\) of \( \beta_1 \). This finishes the proof of \(^{17}\) in the case \( \kappa \in (0, 1) \).

**Case II:** \( \kappa \in [1, 2) \). To overcome the difficulty of matching the centering terms of \( U_\varepsilon \circ \lambda_\varepsilon \) and \( T_\varepsilon \) we define \( V_\varepsilon \in D_\infty \) by

\[
V_\varepsilon(t) = \varepsilon^{1/\kappa} T_{\nu(t)/\varepsilon} - \begin{cases} 
D(1/\varepsilon) & \kappa = 1 \\
(t/\nu) \varepsilon^{-1+1/\kappa} & \kappa \in (1, 2).
\end{cases}
\]

\( V_\varepsilon \) is defined so that the hitting times portion is the same as in \( U_\varepsilon \circ \lambda_\varepsilon \) while the linear centering is the same as in \( T_\varepsilon \).

Since the only difference between \( U_\varepsilon \circ \lambda_\varepsilon \) and \( V_\varepsilon \) is in the centering term, we have for any \( t < \infty \)

\[
d^M_t(U_\varepsilon \circ \lambda_\varepsilon, V_\varepsilon) \leq \sup_{t' \leq t} |\lambda_\varepsilon(t') - t'/\bar{\nu}| \left\{ \begin{array}{ll} 
D'(1/\varepsilon) & \kappa = 1 \\
\varepsilon^{-1+1/\kappa} \bar{\beta} & \kappa \in (1, 2),
\end{array} \right.
\]

using \( D'(1/\varepsilon) = \bar{\nu} D(1/\varepsilon) \) when \( \kappa = 1 \) and \( \bar{\beta} = \bar{\nu} / \nu P \) when \( \kappa \in (1, 2) \). Recall that the random element of \( D_t, t' \mapsto \varepsilon^{-1/2}(\lambda_\varepsilon(t') - t'/\bar{\nu}) \), converges weakly in \( (D_t, J_1) \) to Brownian motion, which is a continuous process. Every continuous function in \( D_t \) is a continuity point of the mapping \( x \mapsto \sup_{t \leq 1} |x(t')| \) from \( D_t \) to \( \mathbb{R} \). Therefore, we can use the continuous mapping theorem to show that the term in the right hand side of \(^{18}\) converges to 0 in \( Q \)-probability as \( \varepsilon \to 0 \), by noticing that both \( D'(1/\varepsilon) \) (when \( \kappa = 1 \)) and \( \varepsilon^{-1+1/\kappa} \) (when \( \kappa \in (1, 2) \)) are \( o(e^{-1/2}) \). Therefore, in order to prove \(^{17}\), it is enough to show that for every \( 0 < t < \infty \) and \( \eta > 0 \)

\[
\lim_{\varepsilon \to 0} Q \left( \frac{d^M_t(V_\varepsilon, T_\varepsilon)}{\eta} > 0 \right) = 0.
\]

The proof of \(^{19}\) is very similar to the proof of \(^{17}\) when \( \kappa \in (0, 1) \). Indeed, note that \( V_\varepsilon(t) = T_\varepsilon(t) \) whenever \( t = \nu_j \) for some \( j \). Again, we arrange the respective parametric representations of the completed graphs of the two random functions so that, for \( k \) being the largest \( j \) so that \( \nu_j \leq t/\varepsilon \), both parametric representations are equal, to \( (\varepsilon \nu_j, \varepsilon^{1/\kappa} T_{\nu_j}) \) at \( s_j = j/(k + 1), j = 0, 1, \ldots, k \). For \( s_j < s < s_{j+1} \) with \( j = 0, 1, \ldots, k - 1 \) the two parametric representation can be chosen in such a way that the line connecting the two corresponding points is always parallel to the segment, connecting the points \( (\varepsilon \nu_j, V_\varepsilon(\varepsilon \nu_j)) \) and \( (\varepsilon \nu_{j+1}, V_\varepsilon(\varepsilon \nu_{j+1} - \nu_j)) \). See Figure 1 for a visual representation of this matching. In this case the distance between the two corresponding points does not exceed the length of the above segment, which is shorter than \( \varepsilon^{1/2}(\nu_{j+1} - \nu_j) \) for \( \varepsilon \) small enough. As in the case \( \kappa \in (0, 1) \), on the interval \( (s_k, 1] \) we keep horizontal \( (u) \) coordinates of the two parametric representations the same. Overall, we obtain

---

**Figure 1.** A demonstration of the matching of the parameterizations of the completed graphs of \( T_\varepsilon \) with the completed graphs of \( U_\varepsilon \circ \lambda_\varepsilon \) and \( V_\varepsilon \) when \( \kappa \in (0, 1) \) and \( \kappa \in [1, 2) \), respectively.
the bound
\[ d_t^{M_1}(\mathbb{V}_\varepsilon, \mathbb{V}_\varepsilon) \leq \max \left\{ \varepsilon^{1/2} \max_{j \leq k} (\nu_{j+1} - \nu_j), \varepsilon^{1/\kappa} (T_{\nu_{k+1}} - T_{\nu_k}) \right\}. \]

From here we proceed as in the case \( \kappa \in (0, 1) \) above.

**Proof of Lemma 4.4.** By Lemma 3.4 it is enough to show that for each \( 0 < s < \infty \) and \( \eta > 0 \),
\[ \lim_{\varepsilon \to 0} Q \left( E_\omega \left[ \sup_{t \leq s} |U_\varepsilon(t) - S_\varepsilon(t)| \right] \geq \eta \right) = 0. \]

Since both \( U_\varepsilon \) and \( S_\varepsilon \) are piecewise linear with the same slope between times \( t \in \varepsilon \mathbb{Z} \),
\[ \sup_{t \leq s} |U_\varepsilon(t) - S_\varepsilon(t)| = \varepsilon^{1/\kappa} \max_{k \leq [s/\varepsilon]} |T_{\nu_k} - \sum_{i=1}^{k} \beta_i \tau_i|. \]

Now, it is easy to see that \( M_k = T_{\nu_k} - \sum_{i=1}^{k} \beta_i \tau_i \) is a martingale under \( P_\omega \). Therefore, by the Cauchy-Schwartz and \( L^p \)-maximum inequalities for martingales,
\[ E_\omega \left[ \sup_{t \leq s} |U_\varepsilon(t) - S_\varepsilon(t)| \right] \leq \left( E_\omega \left[ \sup_{t \leq s} |U_\varepsilon(t) - S_\varepsilon(t)|^2 \right] \right)^{1/2} \leq \varepsilon^{1/\kappa} \left( 4 E_\omega \left[ M_{[s/\varepsilon]}^2 \right] \right)^{1/2} = 2 \varepsilon^{1/\kappa} \left( \text{Var}_\omega \left( T_{\nu_{[s/\varepsilon]}} - \sum_{i=1}^{[s/\varepsilon]} \beta_i \tau_i \right) \right)^{1/2}. \]

Therefore,
\[ Q \left( E_\omega \left[ \sup_{t \leq s} |U_\varepsilon(t) - S_\varepsilon(t)| \right] \geq \eta \right) \leq Q \left( 4 \varepsilon^{2/\kappa} \text{Var}_\omega \left( T_{\nu_{[s/\varepsilon]}} - \sum_{i=1}^{[s/\varepsilon]} \beta_i \tau_i \right) \geq \eta^2 \right). \]

In the proof of Lemma 4.4 in [PS10], a natural coupling of \( \tau_i \) and \( T_{\nu_i} - T_{\nu_{i-1}} \) was constructed so that for any \( \eta > 0 \),
\[ \lim_{n \to \infty} Q \left( n^{-2/\kappa} \text{Var}_\omega \left( T_{\nu_n} - \sum_{i=1}^{n} \beta_i \tau_i \right) \geq \eta \right) = 0. \]

Applying this to (20) completes the proof of the lemma.

5. Weak weak quenched limits for \( S_\varepsilon \)

In this section we prove Proposition 4.2. For any environment \( \omega \) and \( \varepsilon > 0 \), define a point process by
\[ N_\varepsilon = \sum_{i \geq 1} \delta_{(s_{[\varepsilon^{1/\kappa} \beta_i, \varepsilon])}}. \]

We view \( N_\varepsilon \) as a random element of \( \mathcal{M}_\nu((0, \infty) \times [0, \infty)) \). Recalling the definitions of \( \mathcal{H} \) in (5) and \( \mathcal{S}_\varepsilon \) in (15), we see that the quenched law of \( S_\varepsilon \) satisfies
\[ s_\varepsilon = \begin{cases} \mathcal{H}(N_\varepsilon) & \kappa \in (0, 1) \\ \mathcal{H}(N_\varepsilon) \ast \ell(-D'(1/\varepsilon)) & \kappa = 1 \\ \mathcal{H}(N_\varepsilon) \ast \ell(-2\varepsilon^{-1+1/\kappa}) & \kappa \in (1, 2). \end{cases} \]

The key to the proof of Proposition 4.2 is the following lemma which shows weak convergence of the point process \( N_\varepsilon \).
Lemma 5.1. Under the measure $Q$, as $\varepsilon \to 0$, the point process $N_\varepsilon$ converges weakly in the space $\mathcal{M}_p((0, \infty] \times [0, \infty))$ to a non-homogeneous Poisson point process $N_{\lambda, \kappa}$ with intensity measure $\lambda x^{-\kappa-1} dx \, dt$. Moreover, $\lambda = C_0 \kappa$, where $C_0$ is the tail constant in (12).

Proof. The idea of the proof is similar to that of the proof of Proposition 5.1 in [PS10]. It was shown in the above proof that for $0 < \varepsilon < 1$ there is a stationary under $Q$ sequence of random variables $(\beta_1^{(\varepsilon)}, i = 1, 2, \ldots)$ on $\Omega$ such that $\beta_1^{(\varepsilon)}$ and $\beta_j^{(\varepsilon)}$ are independent if $|i - j| > \varepsilon^{-1/2}$, and such that, for some $C, C' > 0$,

\begin{equation}
Q \left( \left| \beta_1^{(\varepsilon)} - \beta_j^{(\varepsilon)} \right| > e^{-\varepsilon^{-1/4}} \right) \leq C e^{-C' \varepsilon^{-1/2}}, \quad 0 < \varepsilon < 1.
\end{equation}

We define an approximating point process by

\begin{equation}
N^{(1)}_\varepsilon = \sum_{i \geq 1} \delta_{(\varepsilon^{-1/\kappa} \beta_1^{(\varepsilon)} i, \varepsilon i)}, \quad 0 < \varepsilon < 1,
\end{equation}

and proceed by proving the convergence

\begin{equation}
N^{(1)}_\varepsilon \Rightarrow N_{\lambda, \kappa} \text{ weakly in } \mathcal{M}_p((0, \infty] \times [0, \infty)) \text{ as } \varepsilon \to 0, \text{ under the measure } Q.
\end{equation}

We start by considering measurable functions $f : (0, \infty] \times [0, \infty) \to \mathbb{R}_+$ of the form

\begin{equation}
f(x, t) = \sum_{i=1}^k f_i(x) 1_{[a_{i-1}, a_i)(t)}, \quad (x, t) \in (0, \infty] \times [0, \infty),
\end{equation}

where $k = 1, 2, \ldots, f_i : (0, \infty] \to \mathbb{R}_+$, $i = 1, \ldots, k$ are continuous functions that vanish for all $0 < x < \delta$ for some $\delta > 0$, and are Lipschitz on the interval $(\delta, \infty)$, and $0 = a_0 < a_1 < \ldots < a_k < \infty$. We will prove that for such a function,

\begin{equation}
\lim_{\varepsilon \to 0} E_Q \left[ e^{-N^{(1)}_\varepsilon(f)} \right] = \exp \left\{ -\sum_{i=1}^k (a_i - a_{i-1}) \int_0^\infty (1 - e^{-f_i(x)}) \lambda x^{-\kappa-1} \, dx \right\}.
\end{equation}

To this end we define, as in [PS10], for $0 < \tau < 1$,

\begin{equation}
K_\varepsilon(\tau) = \text{card}\{i = 1, \ldots, \lfloor a_k/\varepsilon \rfloor : \text{ both } \beta_i^{(\varepsilon)} > \delta \varepsilon^{-1/\kappa} \text{ and } \beta_j^{(\varepsilon)} > \delta \varepsilon^{-1/\kappa} \text{ for some } i + 1 \leq j \leq i + \tau/\varepsilon, j \leq a_k/\varepsilon\};
\end{equation}

as in the above reference we have

\begin{equation}
\lim_{\tau \to 0} \limsup_{\varepsilon \to 0} Q(K_\varepsilon(\tau) > 0) = 0.
\end{equation}

Define random sets

\begin{equation}
D^{(j)}_\varepsilon = \{a_{j-1}/\varepsilon < i < a_j/\varepsilon : \beta_i^{(\varepsilon)} > \delta \varepsilon^{-1/\kappa}\}, \quad j = 1, \ldots, k,
\end{equation}

and proceed by proving the convergence

\begin{equation}
N^{(1)}_\varepsilon \Rightarrow \sum_{j=1}^k D^{(j)}_\varepsilon \text{ weakly in } \mathcal{M}_p((0, \infty] \times [0, \infty)) \text{ as } \varepsilon \to 0, \text{ under the measure } Q.
\end{equation}

We start by considering functions of the form

\begin{equation}
f(x, t) = \sum_{i=1}^k f_i(x), \quad (x, t) \in (0, \infty] \times [0, \infty),
\end{equation}

where $k = 1, 2, \ldots, f_i : (0, \infty] \to \mathbb{R}_+$ are continuous functions that vanish for all $0 < x < \delta$ for some $\delta > 0$, and are Lipschitz on the interval $(\delta, \infty)$. We will prove that for such a function,

\begin{equation}
\lim_{\varepsilon \to 0} E_Q \left[ e^{-N^{(1)}_\varepsilon(f)} \right] = \exp \left\{ -\sum_{i=1}^k f_i(x) \int_0^\infty (1 - e^{-f_i(x)}) \lambda x^{-\kappa-1} \, dx \right\}.
\end{equation}
so that
\[
E_Q \left[ e^{-N(1)}(f) \right] = E_Q \exp \left\{ -\sum_{j=1}^{k} \sum_{i \in D(j)} f_j \left( \epsilon^{1/\kappa} \beta_i \right) \right\} \\
= E_Q \left[ \exp \left\{ -\sum_{j=1}^{k} \sum_{i \in D(j)} f_j \left( \epsilon^{1/\kappa} \beta_i \right) \right\} 1(K_\epsilon(\tau) = 0) \right] + E_Q \left[ \exp \left\{ -\sum_{j=1}^{k} \sum_{i \in D(j)} f_j \left( \epsilon^{1/\kappa} \beta_i \right) \right\} 1(K_\epsilon(\tau) > 0) \right] \\
:= H^{(1)}_\epsilon + H^{(2)}_\epsilon.
\]

It follows from \([26]\) that the term \(H^{(2)}_\epsilon\) is negligible as \(\epsilon \to 0\) and then \(\tau \to 0\). Furthermore, given the event \(\{K_\epsilon(\tau) = 0\}\), for a fixed \(0 < \tau < 1\) and \(\epsilon\) small enough, the points in the random set \(D_\epsilon := \bigcup_j D(j)\) are separated by more than \(\epsilon^{-1/2}\), so that, given also the set \(D_\epsilon\), the random variables \(\beta_i(\epsilon)\), \(i \in D_\epsilon\) are independent, each with the conditional distribution of \(\beta_1(\epsilon)\) given \(\beta_1(\epsilon) > \delta \epsilon^{-1/\kappa}\). Since for every \(j = 1, \ldots, k\),
\[
E_Q \left( \exp \left\{ -f_j \left( \epsilon^{1/\kappa} \beta_j(\epsilon) \right) \right\} \left| \beta_j(\epsilon) > \delta \epsilon^{-1/\kappa} \right. \right) \to \int_1^{\infty} e^{-f_j(\delta \epsilon) \kappa t^{-\kappa+1}} dt,
\]
the claim \([25]\) will follow once we check that
\[
\exp \left\{ -C_0 \delta - \kappa \sum_{j=1}^{k} (a_j - a_{j-1})(1 - \alpha_j) \right\} \leq \liminf_{\tau \to 0} E_Q \left( \prod_{j=1}^{k} \alpha_j^{\text{card } D(j)} \left| K_\epsilon(\tau) = 0 \right. \right) \\
= \limsup_{\tau \to 0} E_Q \left( \prod_{j=1}^{k} \alpha_j^{\text{card } D(j)} \left| K_\epsilon(\tau) = 0 \right. \right) \leq \exp \left\{ -C_0 \delta - \kappa \sum_{j=1}^{k} (a_j - a_{j-1})(1 - \alpha_j) \right\}
\]
for any \(0 < \alpha_j < 1\), \(j = 1, \ldots, k\). This, however, can be proved in the same way as (48) was proved in \([PS10]\).

In order to prove weak convergence in \([23]\), it is enough to prove that for any Lipschitz continuous function \(f : (0, \infty) \times [0, \infty) \to \mathbb{R}_+\) with support in \([\delta, \infty) \times [0, a]\) for some \(0 < \delta, a < \infty\),
\[
(27) \quad \lim_{\epsilon \to 0} E_Q \left[ e^{-N(1)}(f) \right] = \exp \left\{ -\int_0^{\infty} \int_0^{\infty} (1 - e^{-f(x,t)}) \lambda x^{-\kappa-1} dx dt \right\};
\]
see \([Res08]\) and Remark 5.2 in \([PS10]\). To this end, for \(m = 1, 2, \ldots\) we define
\[
f_j(x) = f(x, ja/m), \quad x \in (0, \infty), \quad j = 1, \ldots, m,
\]
and
\[
\tilde{f}(x,t) = \sum_{j=1}^{k} f_j(x) 1_{[j-1]a/m, ja/m]}(t), \quad (x,t) \in (0, \infty) \times [0, \infty).
\]

Note that \(|f(x,t) - \tilde{f}(x,t)| \leq La/m\) for all finite \((x,t)\), where \(L\) is the Lipschitz constant of \(f\). Therefore,
\[
\left| E_Q \left[ e^{-N(1)}(f) \right] - E_Q \left[ e^{-N(1)}(\tilde{f}) \right] \right| \leq \frac{La}{m} E_Q \left[ N(1) \left( [\delta, \infty) \times [0, a] \right) \right].
\]
Notice that, by stationarity,
\[ E_Q \left[ N_\varepsilon^{(1)}([\delta, \infty] \times [0, a]) \right] \leq a \epsilon^{-1} Q(\beta_1^{(\epsilon)} > \delta \epsilon^{-1/\kappa}), \]
which, by (12) and (22), remains bounded as \( \epsilon \to 0 \). Since the function \( \tilde{f} \) is of the type (24), it follows from (25) that
\[
\lim_{\varepsilon \to 0} E_Q \left[ e^{-N_\varepsilon^{(1)}(\tilde{f})} \right] = \exp \left\{ - \int_0^\infty \int_0^\infty (1 - e^{-\tilde{f}(x,t)}) \lambda x^{-\kappa-1} dx dt \right\}.
\]
This proves (27) (and, hence, also (23)). It follows by (22) and the Lipschitz property that for any function \( f \) as in (27) we also have
\[
\lim_{\varepsilon \to 0} E_Q \left[ e^{-N_\varepsilon(f)} \right] = \exp \left\{ - \int_0^\infty \int_0^\infty (1 - e^{-f(x,t)}) \lambda x^{-\kappa-1} dx dt \right\}.
\]
As before, this establishes the weak convergence stated in the lemma.

We would like to use the representation (21) of \( s_\varepsilon \), and the fact that \( N_\varepsilon \Rightarrow N_{\lambda, \kappa} \) to obtain a weak limit for \( s_\varepsilon \) as a random element of \( M_1(D_\infty) \). Unfortunately, the function \( H \) is not continuous and so we need the following lemma which shows that the truncated function \( H_\delta \) is “almost continuous”.

**Lemma 5.2.** Define subsets \( C_\delta, E \subset M_p((0, \infty] \times [0, \infty)) \) by
\[ C_\delta = \{ \zeta : \zeta((0, \infty] \times [0, \infty]) = 0 \} \]
and
\[ E = \{ \zeta : \zeta((0, \infty] \times \{t\}) \leq 1, \forall t \in (0, \infty) \} \cap \{ \zeta : \zeta((0, \infty] \times \{0\}) = 0 \}. \]
Then \( H_\delta \) is continuous on \( C_\delta \cap E \).

**Proof.** Suppose that \( \zeta_n \to \zeta \in C_\delta \cap E \). We will couple the paths \( W_\delta(\zeta_n, \bar{\tau}) \) and \( W_\delta(\zeta, \bar{\tau}) \) by using the same sequence \( \bar{\tau} \) of i.i.d. standard exponential random variables. Using this coupling we will show that \( \lim_{n \to \infty} W_\delta(\zeta_n, \bar{\tau}) = W_\delta(\zeta, \bar{\tau}) \), \( \mathbb{P}_\tau \)-a.s. Since almost sure convergence implies weak convergence, \( H_\delta(\zeta_n) \to H_\delta(\zeta) \).

To prove that \( W_\delta(\zeta_n, \bar{\tau}) \) converges a.s. to \( W_\delta(\zeta, \bar{\tau}) \) it will be enough to show that for every \( 0 < s < \infty \) such that \( W_\delta(\zeta, \bar{\tau}) \) is continuous at \( s \), and for every realization \( \bar{\tau} \) with finite values,
\[
\lim_{n \to \infty} d_1^s(W_\delta(\zeta_n, \bar{\tau}), W_\delta(\zeta, \bar{\tau})) = 0.
\]
To this end, take \( s \) as above. Then \( \zeta([\delta, \infty] \times \{s\}) = 0 \). The assumption that \( \zeta \in E \) implies that we may order the atoms of \( \zeta \) in \( [\delta, \infty] \times [0, s] \) so that for \( M = \zeta([\delta, \infty] \times [0, s]) \) we have
\[
\zeta(\cdot \cap ([\delta, \infty] \times [0, s])) = \sum_{i=1}^M \delta_{(x_i, t_i)}(\cdot), \quad \text{with } 0 < t_1 < t_2 < \ldots < t_M < s.
\]
Similarly, we can order the atoms of \( \zeta_n \) in \( [\delta, \infty] \times [0, s] \) so that for \( M_n = \zeta_n([\delta, \infty] \times [0, s]) \) we have
\[
\zeta_n(\cdot \cap ([\delta, \infty] \times [0, s])) = \sum_{i=1}^{M_n} \delta_{(x_i^{(n)}, t_i^{(n)})}(\cdot), \quad \text{with } 0 \leq t_1^{(n)} \leq t_2^{(n)} \leq \ldots \leq t_{M_n}^{(n)} \leq s.
\]
The vague convergence of \( \zeta_n \) to \( \zeta \) and the fact that \( \zeta \) has no atoms on the boundary of \( [\delta, \infty] \times [0, s] \), imply that for \( n \) large enough \( M_n = M \) and
\[
\lim_{n \to \infty} \max_{1 \leq i \leq M} \left( |x_i^{(n)} - x_i| \lor |t_i^{(n)} - t_i| \right) = 0.
\]
Therefore, for \( n \) sufficiently large, \( 0 < t_1^{(n)} < t_2^{(n)} < \ldots < t_M^{(n)} < s \). For such \( n \) we define a time-change function \( \lambda_n^s \) of the interval \([0, s]\) by \( \lambda_n^s(0) = 0, \lambda_n^s(s) = s, \lambda_n^s(t_i) = t_i^{(n)} \) for all \( i \leq M \), and extend it everywhere else by linear interpolation. Then, 

\[
\sup_{t \leq s} |\lambda_n^s(t) - t| = \max_{i \leq M} |t_i^{(n)} - t_i|,
\]

and, since \( W_\delta(\zeta_n, \bar{\tau}) \) and \( W_\delta(\zeta, \bar{\tau}) \) are constant between jumps, 

\[
\sup_{t \leq s} |W_\delta(\zeta_n, \bar{\tau})(\lambda_n^s(t)) - W_\delta(\zeta, \bar{\tau})(t)| = \max_{j \leq M} \left| \sum_{i=1}^j \left( x_i^{(n)} - x_i \right) \tau_i \right| \leq \sum_{i=1}^M \left| x_i^{(n)} - x_i \right| \tau_i.
\]

Therefore, for \( n \) sufficiently large, 

\[
d_{\delta}^J(W_\delta(\zeta_n, \bar{\tau}), W_\delta(\zeta, \bar{\tau})) \leq \max \left\{ \max_{i \leq M} |t_i^{(n)} - t_i|, \sum_{i=1}^M \left| x_i^{(n)} - x_i \right| \tau_i \right\},
\]

which vanishes as \( n \to \infty \) by (29). This completes the proof of (28) and thus of the lemma. \( \square \)

The relationship between \( s_\varepsilon \) and \( N_\varepsilon \) in (21) and Lemma 5.2 will allow us now to complete the proof of Proposition 4.2.

**Proof of Proposition 4.2** For \( \delta > 0 \) we define a truncated version of \( S_\varepsilon \) by 

\[
S_{\varepsilon, \delta}(t) = \varepsilon^{1/\kappa} \sum_{i=1}^{[t/\varepsilon]} \beta_i \tau_i 1_{\{\varepsilon^{1/\kappa} \beta_i > \delta\}} - t \gamma_{\kappa, \varepsilon, \delta}, \quad t \geq 0,
\]

with 

\[
\gamma_{\kappa, \varepsilon, \delta} = \begin{cases} 
0 & \kappa \in (0, 1), \\
E_\beta \left[ \beta_1 1_{\{\varepsilon \beta_1 \in [\delta, 1]\}} \right] & \kappa = 1, \\
\varepsilon^{1/\kappa-1} E_\beta \left[ \beta_1 1_{\{\varepsilon^{1/\kappa} \beta_1 > \delta\}} \right] & \kappa \in (1, 2).
\end{cases}
\]

Then the quenched law of \( S_{\varepsilon, \delta} \) is \( \mathcal{H}_\delta(N) \star \ell \gamma_{\kappa, \varepsilon, \delta} \).

If \( N_{\lambda, \kappa} \) is the Poisson point process as in the statement of Lemma 5.1, then \( \mathbf{P}(N_{\lambda, \kappa} \in C_\delta \cap E) = 1 \) for any \( \delta > 0 \). Thus, Lemma 5.1, Lemma 5.2, and the continuous mapping theorem imply that, under the measure \( Q \), \( \mathcal{H}_\delta(N_{\varepsilon, \delta}) \Rightarrow \mathcal{H}_\delta(N_{\lambda, \kappa}) \), where \( \lambda = C_0 \kappa \). Also by (12) and Karamata’s theorem, 

\[
\lim_{\varepsilon \to 0} \gamma_{\kappa, \varepsilon, \delta} = \begin{cases} 
C_0 \ln(1/\delta) & \kappa = 1, \\
\frac{C_0 \kappa}{\kappa - 1} \delta^{-\kappa + 1} & \kappa \in (1, 2).
\end{cases}
\]

Since the mapping from \( M_1(D_\infty) \times \mathbb{R} \) to \( M_1(D_\infty) \) defined by \( (\mu, \gamma) \mapsto \mu \star \ell(\gamma) \) is continuous, we conclude that, under the measure \( Q \),

\[
s_{\varepsilon, \delta} \Rightarrow \begin{cases} 
\mathcal{H}_\delta(N_{\lambda, \kappa}) & \kappa \in (0, 1), \\
\mathcal{H}_\delta(N_{\lambda, \kappa}) \star \ell(-\lambda \ln(1/\delta)) & \kappa = 1, \\
\mathcal{H}_\delta(N_{\lambda, \kappa}) \star \ell(-\lambda \delta^{-\kappa + 1}/(\kappa - 1)) & \kappa \in (1, 2).
\end{cases}
\]

To relate (31) to a limit statement about \( s_\varepsilon \) we use [Bil99, Theorem 3.2]. To this end, it is enough to show that the limit in \( M_1((D_\infty, d_\infty^{(M_1)}) \]

\[
\lim_{\delta \to \infty} \begin{cases} 
\mathcal{H}_\delta(N_{\lambda, \kappa}) & \kappa \in (0, 1), \\
\mathcal{H}_\delta(N_{\lambda, \kappa}) \star \ell(-\lambda \ln(1/\delta)) & \kappa = 1, \\
\mathcal{H}_\delta(N_{\lambda, \kappa}) \star \ell(-\lambda \delta^{-\kappa + 1}/(\kappa - 1)) & \kappa \in (1, 2),
\end{cases}
\]

exists \( \mathbf{P}_\tau \)-a.s.
and
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{t \leq s} Q \left( \rho^{M_1}(s_{\varepsilon, \delta}, s_{\varepsilon}) \geq \eta \right) = 0, \quad \forall \eta > 0.
\]
As in the case of Lemma 3.3, this will follow from following, stronger, statement: for every \(0 < s < \infty\),
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{t \leq s} Q \left( E_x \left[ \sup_{t \leq s} |S_{\varepsilon, \delta}(t) - S_\varepsilon(t)| \right] \geq \eta \right) = 0, \quad \forall \eta > 0.
\]

Therefore, to complete the proof of Proposition 4.2, it remains only to prove (32) and (34). We divide the proof of these statements into two cases: \(\kappa \in (0, 1)\) and \(\kappa \in [1, 2)\).

5.1. Case I: \(\kappa \in (0, 1)\). To prove (32) we let \(F_1 \subset \mathcal{M}_p((0, \infty] \times [0, \infty))\) be defined by
\[
F_1 = \left\{ \zeta = \sum_{i \geq 1} \delta(x_i, t_i) : \sum_{i \geq 1} x_i 1_{\{t_i \leq t\}} < \infty, \forall t < \infty \right\}.
\]
(Note that on the set \(F_1\), the sum in the definition of \(W(\zeta, \tilde{\tau})\) is \(P_x\)-a.s. finite.) Since \(P(N_{\lambda, \kappa} \in F_1) = 1\) when \(\kappa \in (0, 1)\), it will be enough to show that \(\mathcal{H}_\delta(\zeta) \to \mathcal{H}(\zeta)\) as \(\delta \to 0\) for any \(\zeta \in F_1\). Fix \(\zeta = \sum_{i \geq 1} \delta(x_i, t_i) \in F_1\). For \(0 < s < \infty\) the obvious coupling of \(W(\zeta, \tilde{\tau})\) and \(W(\zeta, \tilde{\tau})\) gives that
\[
\sup_{t \leq s} |W(\zeta, \tilde{\tau})(t) - W(\zeta, \tilde{\tau})(t)| = \sup_{t \leq s} \sum_{i \geq 1} x_i \tau_i 1_{\{x_i \leq \delta, t_i \leq t\}} = \sum_{i \geq 1} x_i \tau_i 1_{\{x_i \leq \delta, t_i \leq s\}}.
\]
Since \(\zeta \in F_1\), finiteness of the mean of an exponential random variable shows that the sum on the right is finite with probability one for any \(\delta > 0\). Letting \(\delta \to 0\) the dominated convergence theorem shows that \(W(\zeta, \tilde{\tau})\) converges almost surely to \(W(\zeta, \tilde{\tau})\) in the space \(D_s\) in the uniform metric, hence also in the \(M_1\)-metric, for any \(0 < s < \infty\). Therefore, \(W(\zeta, \tilde{\tau})\) converges almost surely to \(W(\zeta, \tilde{\tau})\) in \(D_\infty\) as \(\delta \to 0\) and, since a.s. convergence implies convergence in distribution, \(\mathcal{H}_\delta(\zeta)\) converges to \(\mathcal{H}(\zeta)\) in the space \(\mathcal{M}_1((D_\infty, d_{M_1}))\) as \(\delta \to 0\). This proves (32). Further, since \(W(\delta(\zeta, \tilde{\tau}) = S_{\varepsilon, \delta}\) and \(W(N_{\varepsilon, \tilde{\tau}) = S_{\varepsilon}\), we have by (35) with \(\zeta = N_\varepsilon\),
\[
E_x \left[ \sup_{t \leq s} |S_{\varepsilon, \delta}(t) - S_\varepsilon(t)| \right] = E_x \left[ \sum_{i \geq 1} x_i \tau_i 1_{\{x_i \leq \delta, t_i \leq s\}} \right] = \varepsilon^{1/\kappa} \sum_{i \geq 1} \beta_i 1_{\{x_i \leq \delta\}}.
\]
By Chebyshev’s inequality and stationarity of \(\beta_i\) under \(Q\),
\[
Q \left( E_x \left[ \sup_{t \leq s} |S_{\varepsilon, \delta}(t) - S_\varepsilon(t)| \right] \geq \eta \right) \leq Q \left( \varepsilon^{1/\kappa} \sum_{i \geq 1} \beta_i 1_{\{x_i \leq \delta\}} \geq \eta \right) \leq \frac{s \varepsilon^{1/\kappa - 1}}{\eta} E_Q \left[ \beta_i 1_{\{x_i \leq \delta\}} \right].
\]
Karamata’s theorem and (12) imply that \(E_Q \left[ \beta_i 1_{\{x_i \leq \delta\}} \right] \sim C_0 \kappa \delta^{1-\kappa} \varepsilon^{1-1/\kappa}\) as \(\varepsilon \to 0\). Therefore,
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} Q \left( E_x \left[ \sup_{t \leq s} |S_{\varepsilon, \delta}(t) - S_\varepsilon(t)| \right] \geq \eta \right) \leq \lim_{\delta \to 0} \frac{s C_0 \kappa}{\eta(1-\kappa)} \delta^{1-\kappa} = 0.
\]
This proves (34).
5.2. Case II: $\kappa \in [1, 2)$. To prove (32), note that the right hand side of (32) is the law (with respect to $P_\tau$) of the random element of $D_\infty$

$$t \mapsto W_\delta(N_{\lambda, \kappa}, \tau)(t) = \begin{cases} \lambda t \log(1/\delta) & \kappa = 1 \\ \frac{\lambda_0 t}{\kappa - 1} & \kappa \in (1, 2). \end{cases}$$

It was shown in the proof of Corollary 1.9 that this random element converges almost surely, in the uniform metric, under the joint law $P \times P_\tau$ of $(N_{\lambda, \kappa}, \tau)$. Therefore, convergence takes place in the $M_1$-metric as well. Fubini’s theorem implies that the convergence also holds $P_\tau$-a.s. for almost every realization of the point process $N_{\lambda, \kappa}$. Once again, a.s. convergence implies convergence in distribution, so (32) holds.

To prove (34), note that the definition of $\gamma_{\kappa, \varepsilon, \delta}$ in (30) implies that

$$\sup_{t \leq s} |S_{\varepsilon, \delta}(t) - S_{\varepsilon}(t)|$$

$$= \sup_{t \leq s} e^{1/\kappa} \sum_{i=1}^{[t/\varepsilon]} \beta_i \tau_i 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}} - E_Q[\beta_i 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}}] e^{-1/\kappa} t$$

$$\leq \sup_{t \leq s} e^{1/\kappa} \sum_{i=1}^{[t/\varepsilon]} \beta_i (\tau_i - 1) 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}}$$

$$+ \sup_{t \leq s} e^{1/\kappa} \sum_{i=1}^{[t/\varepsilon]} \left\{ \beta_i 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}} - E_Q[\beta_i 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}}] \right\} + e^{1/\kappa} E_Q[\beta_i 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}}],$$

where the last term comes from rounding in the number of terms in the sum. This term is, clearly, bounded by $\delta$.

For $\beta_i$ fixed, the sum inside the supremum in the first term in (36) is a sum of independent, zero-mean random variables. Thus, by the Cauchy-Schwartz and $L^p$-maximum inequalities for martingales,

$$E_\omega \left[ \sup_{t \leq s} \left| e^{1/\kappa} \sum_{i=1}^{[t/\varepsilon]} \beta_i (\tau_i - 1) 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}} \right| \right] \leq \left( 4E_\omega \left| e^{1/\kappa} \sum_{i=1}^{[s/\varepsilon]} \beta_i (\tau_i - 1) 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}} \right| \right)^{1/2}$$

$$= 2e^{1/\kappa} \left( \sum_{i=1}^{[s/\varepsilon]} \beta_i^2 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}} \right)^{1/2}$$

Therefore, for $\eta > 0$ fixed and $\delta$ sufficiently small we have

$$Q \left( E_\omega \left[ \sup_{t \leq s} |S_{\varepsilon, \delta}(t) - S_{\varepsilon}(t)| \right] \geq \eta \right)$$

$$\leq Q \left( e^{2/\kappa} \sum_{i=1}^{[s/\varepsilon]} \beta_i^2 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}} \geq \eta^2/36 \right)$$

$$+ Q \left( \sup_{t \leq s} \left| e^{1/\kappa} \sum_{i=1}^{[t/\varepsilon]} \left\{ \beta_i 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}} - E_Q[\beta_i 1_{\{\varepsilon^{1/\kappa} \beta_i \leq \delta\}}] \right\} \right| \geq \eta/3 \right).$$

(37)
Notice that
\[
\limsup_{\varepsilon \to 0} Q \left( \varepsilon^{2/\kappa} \sum_{i=1}^{\lfloor s/\varepsilon \rfloor} \beta_i \mathbf{1}_{\{\varepsilon \beta_i \leq \delta \}} \geq \eta^2/36 \right) \leq \limsup_{\varepsilon \to 0} \frac{36s \varepsilon^{2/\kappa - 1}}{\eta^2} E_Q \left[ \beta_i \mathbf{1}_{\{\varepsilon \beta_i \leq \delta \}} \right] = \frac{36s}{\eta^2 - 2 - \kappa} \delta^{2 - \kappa},
\]
where the last equality follows from (12) and Karamata’s Theorem. This vanishes as \( \delta \to 0 \) since \( \kappa < 2 \). It remains only to show that the term in (37) vanishes as first \( \varepsilon \to 0 \) and then \( \delta \to 0 \). A similar statement (without the supremum inside the probability) was shown in [PS10, Lemma 5.5]. One can modify the techniques of [PS10] to give a bound on (37) that vanishes as first \( \varepsilon \to 0 \) and then \( \delta \to 0 \). Since the argument is somewhat technical, we postpone it until Appendix A. \( \square \)

6. WEAK QUENCHED LIMITS FOR THE POSITION OF THE RANDOM WALK

In this section we prove Theorem 1.6. We start by defining the running maximum version of the scaled path process of the random walk \( \chi_\varepsilon \) in (6). For \( t \geq 0 \), let \( X_t^\varepsilon = \max\{X_k : k \leq t\} \) denote the running maximum of the RWRE. The corresponding random element in \( D_\infty \) is
\[
\chi_\varepsilon^*(t) = \begin{cases} 
\varepsilon^\kappa X_{t/\varepsilon}^\kappa & \kappa \in (0, 1) \\
\frac{1}{\varepsilon^{1/\kappa}} \left( X_{t/\varepsilon}^\kappa - t \delta(1/\varepsilon) \right) & \kappa = 1 \\
\varepsilon^{1-\kappa} X_{t/\varepsilon}^{1-\kappa} - t \delta(1/\varepsilon) & \kappa \in (1, 2),
\end{cases}
\]
with the same function \( \delta \) in the case \( \kappa = 1 \) as in (6). The path \( \chi_\varepsilon^* \) is easier to compare to transforms of the hitting times path \( T_\varepsilon \) than the path \( \chi_\varepsilon \) is. The following lemma shows that the quenched distributions of \( \chi_\varepsilon \) and \( \chi_\varepsilon^* \) are asymptotically equivalent, since the distance between \( \chi_\varepsilon \) and \( \chi_\varepsilon^* \) is typically very small.

Lemma 6.1. For any \( s < \infty \) and \( \eta > 0 \),
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{t \leq s} |\chi_\varepsilon(t) - \chi_\varepsilon^*(t)| \geq \eta \right) = 0.
\]

Proof. The definitions of \( \chi_\varepsilon \) and \( \chi_\varepsilon^* \) imply that for all \( \varepsilon > 0 \) small enough,
\[
\sup_{t \leq s} |\chi_\varepsilon(t) - \chi_\varepsilon^*(t)| = \max_{k \leq s/\varepsilon} (X_k^\varepsilon - X_k) \begin{cases} 
\varepsilon^\kappa & \kappa \in (0, 1) \\
\varepsilon^{1-\kappa} & \kappa \in (1, 2)
\end{cases}
\]
If, for some \( 0 \leq k \leq s/\varepsilon \), \( X_k^\varepsilon - X_k \geq \eta \varepsilon^{-\kappa/4} \), then, for some location \( 0 \leq j \leq s/\varepsilon \) the random walk returns to \( X_j - [\eta \varepsilon^{-\kappa/4}] \) after visiting location \( j \). Thus, by the stationarity of the environment under the measure \( \mathbb{P} \),
\[
\mathbb{P} \left( \sup_{t \leq s} |\chi_\varepsilon(t) - \chi_\varepsilon^*(t)| \geq \eta \right) \leq \mathbb{P} \left( \max_{k \leq s/\varepsilon} (X_k^\varepsilon - X_k) \geq \eta \varepsilon^{-\kappa/4} \right) \leq (1 + s/\varepsilon) \mathbb{P}(T_{-[\eta \varepsilon^{-\kappa/4}]} < \infty).
\]
Since \( \mathbb{P}(T_{-x} < \infty) \) decays exponentially fast as \( x \to \infty \) (see [GS02 Lemma 3.3]), the term on the right vanishes as \( \varepsilon \to 0 \). \( \square \)

We now prove Theorem 1.6. According to Lemma 6.1, we may and will replace \( \chi_\varepsilon \) by \( \chi_\varepsilon^* \) when proving the coupling part. We consider the cases \( \kappa \in (0, 1) \), \( \kappa = 1 \), and \( \kappa \in (1, 2) \) separately.
6.1. **Case I:** $\kappa \in (0, 1)$. We wish to compare $\chi^*_\varepsilon$ with $\mathcal{J}_\varepsilon$, where $\mathcal{J}$ is the inversion operator defined in (7). To this end, note that

$$X^*_{t/\varepsilon} = \max \{ k \in \mathbb{Z} : T_k \leq t/\varepsilon \} = \sup \{ x \geq 0 : T_{[x/\varepsilon]} \leq t/\varepsilon \} \varepsilon^{-\kappa} - 1.$$

Therefore, for every $t \geq 0$,

$$\chi^*_{t/\varepsilon}(t) = \sup \{ x \geq 0 : \mathbb{T}_\varepsilon(x) \leq t\} - \varepsilon^\kappa = \mathbb{T}_\varepsilon(t) - \varepsilon^\kappa,$$

which implies that $\sup_{t < \infty} |\chi^*_{t/\varepsilon}(t) - \mathbb{T}_\varepsilon(t)| = \varepsilon^\kappa$. Hence, we obtain the stated coupling in Theorem 1.6. By Lemma 3.3, it remains only to show that, under the measure $P$,

$$m_{\varepsilon^\kappa} \circ \mathcal{J}^{-1} \implies \mathcal{H}(N_{\lambda, \kappa}) \circ \mathcal{J}^{-1}. \quad (38)$$

To this end, first note that $\mathcal{J}$ is continuous on the subset $D^+_{\lambda, \kappa} \subset D^+_{\infty}$ of strictly increasing functions, when the $M_1$ topology is used both on the domain and the range (see [Whi02, Corollary 13.6.4] for even topologically stronger statement). Therefore, the mapping theorem implies that the function $\mu \mapsto \mu \circ \mathcal{J}^{-1}$ on $M_1(D^+_{\infty})$ is continuous on the subset of measures $\{ \mu \in M_1(D^+_{\infty}) : \mu(D^+_{\lambda, \kappa}) = 1 \}$. In the notation introduced in [9], $\mathcal{H}(N_{\lambda, \kappa}) = P_\tau(Z_{\lambda, \kappa} \in \cdot)$. Since $Z_{\lambda, \kappa}$ is a $\kappa$-stable subordinator under $P \times P_\tau$, then $\mathcal{H}(N_{\lambda, \kappa})(D^+_{\lambda, \kappa}) = P_\tau(Z_{\lambda, \kappa} \in D^+_{\lambda, \kappa}) = 1$ for almost every realization of $N_{\lambda, \kappa}$, and so (38) follows from Theorem 1.4 and the continuous mapping theorem.

6.2. **Case II:** $\kappa \in (1, 2)$. We start by replacing the piecewise constant path of the hitting times in (3) by a piecewise linear and continuous version via linear interpolation. Specifically, for $x \in \mathbb{Z}$ and $\theta \in [0, 1)$ we let

$$\tilde{T}_{x+\theta} = (1 - \theta)T_x + \theta T_{x+1}.$$ 

Correspondingly, we will define $\tilde{T}_\varepsilon(t) = \varepsilon^{1/\kappa}((\tilde{T}_{t/\varepsilon} - t)/(\varepsilon \nu_P))$, $t \geq 0$. The following lemma shows that the $M_1$-distance between $T_\varepsilon$ and $\tilde{T}_\varepsilon$ is typically small.

**Lemma 6.2.** For any $\eta > 0$, $\lim_{\varepsilon \to 0} \mathbb{P}(d_{\infty}^{M_1}(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \geq \eta) = 0$.

**Proof.** As in Lemma 3.4, it is enough to prove that for every $0 < t < \infty$, $\mathbb{P}(d_t^{M_1}(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \geq \eta) \to 0$ for every $\eta > 0$. We use a matching of the kind similar to that constructed in the proof of Lemma 4.5. We will describe this matching in the case $\kappa \in (1, 2)$, but a similar argument works when $\kappa = 1$ or $\kappa \in (0, 1)$. For every $k = 0, 1, 2, \ldots$ such that $\varepsilon k \leq t$ we arrange both parametric representations to contain the point $(\varepsilon k, \varepsilon^{1/\kappa}(T_k - k/\nu_P))$. If $\varepsilon(k + 1) \leq t$, then between the points $(\varepsilon k, \varepsilon^{1/\kappa}(T_k - k/\nu_P))$ and $(\varepsilon(k + 1), \varepsilon^{1/\kappa}(T_{k+1} - (k + 1)/\nu_P))$ we keep the parametrization of $\mathbb{T}_\varepsilon$ at the former point until the parametrization of $\mathbb{T}_\varepsilon$ reaches the point $(\varepsilon(k + 1), \varepsilon^{1/\kappa}(T_k - (k + 1)/\nu_P))$, at which time we complete the two parametrizations in the interval by keeping the slope between the matched points equal to $-\varepsilon^{1/\kappa - 1}/\nu_P$. Clearly, within this interval the horizontal distance between the two parametrizations is at most $\varepsilon$ and the vertical distance is at most $\varepsilon^{1/\kappa}/\nu_P$. If $\varepsilon k < t < \varepsilon(k + 1)$, then we use the obvious vertical matching of the parameterizations, with equal horizontal components, and vertical components at most $\varepsilon^{1/\kappa}(T_{t/\varepsilon} - T_{t/\varepsilon})$ apart. Therefore, for $\varepsilon$ small enough

$$d_t^{M_1}(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \leq \max \left\{ \varepsilon^{1/\kappa}/\nu_P, \varepsilon^{1/\kappa}(T_{t/\varepsilon} + 1 - T_{t/\varepsilon}) \right\}.$$ 

Since $\mathbb{T}_{t/\varepsilon} + 1 - \mathbb{T}_{t/\varepsilon}$ has, under the measure $\mathbb{P}$, the same distribution as $T_1$ we conclude that

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon \to 0} \mathbb{P}(d_t^{M_1}(\mathbb{T}_\varepsilon, \tilde{\mathbb{T}}_\varepsilon) \geq \eta) \leq \lim_{\varepsilon \to 0} \mathbb{P} \left( \mathbb{T}_{t/\varepsilon} + 1 - \mathbb{T}_{t/\varepsilon} \geq \varepsilon^{1/\kappa - 1} \eta \right) = 0, \quad (39)$$

as required. \qed
Note that \( x \mapsto \tilde{T}_x \) is a strictly increasing and continuous function on \([0, \infty)\). Let \( \phi(t) \) be its inverse. Then \( \tilde{T}_\phi(t) = t \) for all \( t \geq 0 \). If \( T_n \leq t < T_{n+1} \) then \( X^n_t = n \leq \phi(t) < n + 1 \), so that \( \sup_{t \geq 0} |X^n_t - \phi(t)| \leq 1 \). One consequence of this comparison is that

\[
\lim_{t \to \infty} \frac{\phi(t)}{t} = \lim_{n \to \infty} \frac{X^n}{n} = v_P, \quad \mathbb{P}\text{-a.s.} \tag{40}
\]

Next define \( \phi_\varepsilon(t) = \varepsilon \phi(t/\varepsilon) \) for \( \varepsilon > 0 \) and \( \phi_0(t) = v_P t \). Then, (40) implies that \( \phi_\varepsilon \) converges pointwise to \( \phi_0 \) as \( \varepsilon \to 0 \). Moreover, since \( \phi_\varepsilon \) and \( \phi_0 \) are monotone increasing and \( \phi_0 \) is continuous, we conclude that \( \phi_\varepsilon \) converges uniformly on compact subsets to \( \phi_0 \). In particular, \( \lim_{\varepsilon \to 0} d^U(\phi_\varepsilon, \phi_0) = 0, \mathbb{P}\text{-a.s.} \)

Now, recalling the definition of \( \tilde{T}_\varepsilon \) we obtain that

\[
\tilde{T}_\varepsilon(\phi_\varepsilon(t)) = \varepsilon^{1/\kappa}((\tilde{T}_\phi(t)/\varepsilon - \phi_\varepsilon(t))/\varepsilon v_P)) = \varepsilon^{1/\kappa}((\tilde{T}_\phi(t)/\varepsilon - \phi(t)/v_P)) = -v_P^{-1} \varepsilon^{1/\kappa}(\phi(t/\varepsilon) - tv_P/\varepsilon)
\]

\[
= -v_P^{-1} \varepsilon^{1/\kappa}(X^t_{t/\varepsilon} - tv_P/\varepsilon) + v_P^{-1} \varepsilon^{1/\kappa}(X^t_{t/\varepsilon} - \phi(t/\varepsilon))
\]

\[
= -v_P^{-1} \varepsilon^{1/\kappa}(X^t_{t/\varepsilon} - \phi(t/\varepsilon)).
\]

Since \( |X^t_{t/\varepsilon} - \phi(t/\varepsilon)| \leq 1 \) for all \( t \), this implies that

\[
d^U(\phi_\varepsilon, (\tilde{T}_\varepsilon \circ \phi_\varepsilon)(t)) \leq v_P^{-1} \varepsilon^{1/\kappa} \leq v_P^{-1} \varepsilon^{1/\kappa}. \tag{41}
\]

Next, we compare \( \tilde{T}_\varepsilon \circ \phi_\varepsilon \) with \( \tilde{T}_\varepsilon \circ \phi_0 \). To this end, let \( \eta, \eta' > 0 \) be fixed. By Corollary 1.9, the laws of \( (\tilde{T}_\varepsilon) \) under \( \mathbb{P} \) are tight in \( (D_\infty, d^{M_\infty}_\infty) \). Therefore, we can choose a compact subset \( K \subset D_\infty \) so that \( \mathbb{P}(T_\varepsilon \in K) \geq 1 - \eta' \) for all \( \varepsilon \) small enough. Further, the composition function \( \psi(x, y) = x \circ y \) is continuous at any point \( (x, \phi_0) \in D_\infty \times C_{\uparrow}^+ \subset D_\infty \times D_\infty \). Therefore, it is uniformly continuous (with the \( d^{M_\infty}_\infty \) metric on each coordinate) at the points of the compact set \( K \times \{\phi_0\} \). Choose now \( \delta > 0 \) such that \( d^{M_\infty}_\infty((x' \circ \phi', x \circ \phi_0)) < \eta \) whenever \( x \in K \), \( d^{M_\infty}_\infty((x', x)) < \delta \), and \( d^{M_\infty}_\infty((\phi_0, \phi')) < \delta \). Then

\[
\lim_{\varepsilon \to 0} \mathbb{P}(d^{M_\infty}_\infty((\tilde{T}_\varepsilon \circ \phi_\varepsilon, \tilde{T}_\varepsilon \circ \phi_0) \geq \eta)) \leq \lim_{\varepsilon \to 0} \mathbb{P}((\tilde{T}_\varepsilon \notin K) + \mathbb{P}(d^{M_\infty}_\infty((\tilde{T}_\varepsilon, \tilde{T}_\varepsilon) \geq \delta) + \mathbb{P}(d^{M_\infty}_\infty((\phi_\varepsilon, \phi_0) \geq \delta)
\]

\[
\leq \eta',
\]

where the last inequality follows from our choice of the compact set \( K \), Lemma 6.2, and the almost sure convergence of \( \phi_\varepsilon \) to \( \phi_0 \). Since \( \eta' > 0 \) was arbitrary, we see that \( \mathbb{P}(d^{M_\infty}_\infty((\tilde{T}_\varepsilon \circ \phi_\varepsilon, \tilde{T}_\varepsilon \circ \phi_0) \geq \eta) \to 0 \) for any \( \eta > 0 \). Combining this with (41) we conclude that

\[
\lim_{\varepsilon \to 0} \mathbb{P}(d^{M_\infty}_\infty((\phi_\varepsilon, -v_P^{-1/\kappa}(\tilde{T}_\varepsilon \circ \phi_\varepsilon)) \geq \eta) = 0, \quad \forall \eta > 0. \tag{42}
\]

Finally, note that the definition of \( \tilde{T}_\varepsilon \) and \( \phi_0 \) imply that

\[
v_P^{-1/\kappa}(\tilde{T}_\varepsilon(\phi_0(t))) = v_P^{-1/\kappa}(\tilde{T}_\varepsilon(t/\varepsilon) - t/\varepsilon) = \tilde{T}_\varepsilon(t/v_P),
\]

so that (42) proves the coupling part of Theorem 1.6 in the case \( \kappa \in (1, 2) \). Since \( m_{\varepsilon/v_P} \circ \mathcal{R}^{-1} \) is the quenched distribution of \( -\tilde{T}_\varepsilon/v_P \), and \( \mathcal{R} \) is a continuous operator, the continuous mapping theorem implies that \( m_{\varepsilon} \circ \mathcal{R}^{-1} \implies \mu_{\lambda, \kappa} \circ \mathcal{R}^{-1} \). The coupling now implies that we also have \( p_{\varepsilon, \omega} \implies \mu_{\lambda, \kappa} \circ \mathcal{R}^{-1} \).
6.3. **Case III**: $\kappa = 1$. The proof here is similar to the proof in the case $\kappa \in (1, 2)$, so we will omit some of the details. As above, let $\hat{T}_x$ and $\phi(t)$ be as above, so that $\hat{T}_{\phi(t)} = t$. We claim that

$$
\lim_{t \to \infty} \frac{\phi(t)}{t/\log t} = \frac{1}{A}
$$

in $\mathbb{P}$-probability, where $A$ is the positive constant from Theorem 1.1. To see this, first note that by Theorem 1.1 $\lim_{n \to \infty} T_n/(n \log n) = A$ in $\mathbb{P}$-probability, hence also $\lim_{n \to \infty} \hat{T}_x/(x \log x) = A$ in $\mathbb{P}$-probability. Using $x = \phi(t)$ gives us

$$
\lim_{t \to \infty} \frac{t}{\phi(t) \log(\phi(t))} = A
$$

in $\mathbb{P}$-probability, which proves (43).

The function $\delta(x) = \sup\{u > 0 : uD(u) \leq x\}$, $x > 0$, satisfies $\delta(x) \sim x/(A \log x)$ as $x \to \infty$ and $\delta(x)D(\delta(x)) = x + o(\delta(x))$ as $x \to \infty$. We define $\delta(t) = \phi(t)/\delta(1/\varepsilon)$. Then the asymptotics of $\phi$ from (43) and the asymptotics of $\delta$ imply that for any $t \geq 0$,

$$
\lim_{\varepsilon \to 0} \phi_{\varepsilon}(t) = \lim_{\varepsilon \to 0} \frac{\phi(t/\varepsilon)}{\delta(1/\varepsilon)} = t
$$

in $\mathbb{P}$-probability.

Once again, since $\phi_{\varepsilon}$ is non-decreasing, and the identity function is continuous, $\phi_{\varepsilon}$ converges uniformly on compact subsets (and thus also in the $d_{L^2}$ metric), in $\mathbb{P}$-probability, to the identity function.

Let $\tilde{T}_{\varepsilon}(t) = \varepsilon(\hat{T}_{t/\varepsilon} - t/\varepsilon D(1/\varepsilon))$, $t \geq 0$. Then

$$
\tilde{T}_{\delta(1/\varepsilon)^{-1}}(\phi_{\varepsilon}(t)) = \delta(1/\varepsilon)^{-1}\left(\hat{T}_{\phi_{\varepsilon}(t)\delta(1/\varepsilon)} - \phi_{\varepsilon}(t)\delta(1/\varepsilon)D(\delta(1/\varepsilon))\right)
$$

$$
= \delta(1/\varepsilon)^{-1}\left(\hat{T}_{\phi(t/\varepsilon)\delta(1/\varepsilon)} - \phi(t/\varepsilon)\frac{\delta(t/\varepsilon)}{\delta(1/\varepsilon)}\left(\frac{1}{\varepsilon} + o(\delta(1/\varepsilon))\right)\right)
$$

$$
= \delta(1/\varepsilon)^{-1}\left(\frac{1}{\varepsilon}D(1/\varepsilon) + O(1)\frac{X_{t/\varepsilon}}{\delta(1/\varepsilon)} + O\left(\frac{1}{\varepsilon^2}\right)\right)
$$

$$
= -\lambda_{\varepsilon}^*(t) + o(1)\frac{X_{t/\varepsilon}}{\delta(1/\varepsilon)} + O\left(\frac{1}{\varepsilon^2}\right),
$$

where in the third equality we used that $|\phi(t) - X_{t/\varepsilon}^*| \leq 1$ for all $t$. Since $1/(\varepsilon\delta(1/\varepsilon)^2) \sim A^2\varepsilon\log^2(1/\varepsilon) \to 0$ as $\varepsilon \to 0$, while $X_{t/\varepsilon}/\delta(1/\varepsilon)$ converges in probability by Theorem 1.1, this implies that

$$
\lim_{\varepsilon \to 0} d_{L^\infty}(\lambda_{\varepsilon}^*, -\tilde{T}_{\delta(1/\varepsilon)^{-1}} \circ \phi_{\varepsilon}) = 0
$$

in $\mathbb{P}$-probability.

As in case $\kappa \in (1, 2)$ we can use the fact that $\phi_{\varepsilon}$ converges to the identity function to show that for any $\eta > 0$,

$$
\lim_{\varepsilon \to 0} \mathbb{P}\left(d_{\infty}(\tilde{T}^*_{\delta(1/\varepsilon)^{-1}} \circ \phi_{\varepsilon}, \tilde{T}_{\delta(1/\varepsilon)^{-1}}) \geq \eta\right) = 0.
$$

Combining (44), (45) and Lemma 6.1 establishes the coupling part of Theorem 1.6 and the rest is the same as in the case $\kappa \in (1, 2)$. 

7. $\mathcal{M}_1(\mathbb{R})$-valued Stable Lévy Process Limits

In this section we discuss Corollary 1.12. We begin with a short proof of the convergence of the finite dimensional distributions of $\Phi(m_\varepsilon)$. Let $m \geq 1$ and $0 \leq t_1 < t_2 < \ldots < t_m$ be given, and define $\Phi_{t_1,\ldots,t_m} : \mathcal{M}_1(D_\infty) \to \mathcal{M}_1(\mathbb{R})^m$ by

$$\Phi_{t_1,t_2,\ldots,t_m}(\mu) = (\Phi_{t_1}(\mu), \Phi_{t_2}(\mu), \ldots, \Phi_{t_m}(\mu)).$$

It is easy to see that $\Phi_{t_1,t_2,\ldots,t_m}$ is continuous at every $\mu \in \mathcal{M}_1(D_\infty)$ concentrated on paths that are continuous at $t_i$, $i = 1, 2, \ldots, m$; see p. 383 in [Whi02]. Since $m_\varepsilon \Rightarrow \mu_{\lambda,\kappa}$ and $\mu_{\lambda,\kappa}$ is, with probability, one converges on paths that are continuous at $t_i$, $i = 1, 2, \ldots, m$, then the continuous mapping theorem implies that $\Phi_{t_1,t_2,\ldots,t_m}(m_\varepsilon) \Rightarrow \Phi_{t_1,t_2,\ldots,t_m}(\mu_{\lambda,\kappa})$. This proves the convergence of finite dimensional distributions claimed in Corollary 1.12.

We now turn to the stated properties of the random measure-valued path $\Phi(t)$, namely that $\Phi(\mu_{\lambda,\kappa})$ is a stable Lévy process on $\mathcal{M}_1(\mathbb{R})$. We start by recalling the notions of stable random variables, random walks, and Lévy processes on $\mathcal{M}_1(\mathbb{R})$; the reader is referred to [ST06] for more details.

**Definition 1.** A $\mathcal{M}_1(\mathbb{R})$-valued random variable $\mu$ is a **stable random variable on** $\mathcal{M}_1(\mathbb{R})$ if for any $n \geq 2$ there exist constants $b_n \in \mathbb{R}$ and $c_n > 0$ such that

$$(\mu_1 * \mu_2 * \cdots * \mu_n)(b_n + c_n^{-1}) \overset{\text{Law}}{=} \mu(\cdot).$$

Here $\mu_1, \mu_2, \ldots, \mu_n$ are independent copies of $\mu$. Moreover, if $b_n = 0$ for every $n \geq 2$, then $\mu$ is a **strictly stable random variable on** $\mathcal{M}_1(\mathbb{R})$.

**Definition 2.** A $\mathcal{M}_1(\mathbb{R})$-valued stochastic process $\{\Xi(t)\}_{t \geq 0}$ is a **Lévy process on** $\mathcal{M}_1(\mathbb{R})$ if there exists a two parameter family of $\mathcal{M}_1(\mathbb{R})$-valued random variables $\{\Xi_{s,t}\}_{0 \leq s < t}$ such that $\Xi(t) = \Xi_{0,t}$ and

1. $\Xi(0) = \delta_0$ with probability one.
2. For any $n \geq 2$ and $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, $\{\Xi_{t_{i-1},t_i}\}_{i=1}^n$ are independent and

$$\Xi(t) = \Xi_{t_0,t_1} * \Xi_{t_1,t_2} * \cdots * \Xi_{t_{n-1},t_n},$$

almost surely.
3. For any $0 < s < t$, $\Xi(t-s) \overset{\text{Law}}{=} \Xi_{s,t}$.
4. For any fixed $t_0 \geq 0$, the process $\{\Xi(t)\}_{t \geq 0}$ is continuous at $t_0$ in probability.
5. There is an event of probability 1 on which every path $\{t \mapsto \Xi(t)\}$ is in $D_\infty(\mathcal{M}_1(\mathbb{R}))$.

**Remark 7.1.** Part 2 of Definition 2 is a version of the independent increments property for stochastic processes with values in $\mathcal{M}_1(\mathbb{R})$ with convolution of measures playing the role of addition. The version of the independent increments property used in the above definition is necessary due to absence of an inverse operation to convolution.

**Definition 3.** A Lévy process $\{\Xi(t)\}_{t \geq 0}$ on $\mathcal{M}_1(\mathbb{R})$ is a **(strictly) stable Lévy process on** $\mathcal{M}_1(\mathbb{R})$ if for every fixed $t \geq 0$, $\Xi(t)$ is a (strictly) stable random variable on $\mathcal{M}_1(\mathbb{R})$.

We are now ready to show that $\Phi(\mu_{\lambda,\kappa})$ is a stable Lévy process on $\mathcal{M}_1(\mathbb{R})$ and is strictly stable when $\kappa \neq 1$. It is easy to check that for any $t \geq 0$, $\Phi_t(\mu_{\lambda,\kappa})$ is a stable random variable on $\mathcal{M}_1(\mathbb{R})$ and is strictly stable if $\kappa \neq 1$ (see the Remark 1.5 and the paragraph following Remark 1.6 in [PS10]), so we will concentrate on showing that $\Phi(\mu_{\lambda,\kappa})$ is a Lévy process on $\mathcal{M}_1(\mathbb{R})$.

We already know that the paths of $\Phi(\mu_{\lambda,\kappa})$ are in $D_\infty(\mathcal{M}_1(\mathbb{R}))$; see the discussion before Theorem 1.12. It is also obvious that $\Phi_{0}(\mu_{\lambda,\kappa}) = \delta_0$ with probability one since $N_{\lambda_{\kappa}}((0,\infty] \times \{0\}) = 0$ with probability one. Next, recall the stochastic process $Z_{\lambda,\kappa}$ defined in [4]. Then $\Phi_t(\mu_{\lambda,\kappa})$ is the distribution of $Z_{\lambda,\kappa}(t)$ under the measure $P_\tau$, and we define for any $0 \leq s < t$

$$\Phi(\mu_{\lambda,\kappa})_{s,t} = P_\tau(Z_{\lambda,\kappa}(t) - Z_{\lambda,\kappa}(s) \in \cdot).$$
Then, the independent increments condition (2) in Definition 2 follows from the fact that \( \{ N_{\lambda, \kappa}(\cdot \cap ((0, \infty) \times (t_{i-1}, t_i])) \}_{i=1}^n \) are independent for any \( 0 \leq t_0 < t_1 < \cdots < t_n \), and the stationarity condition (3) in Definition 2 follows from the shift invariance of the Lebesgue measure governing the time component of the Poisson random measure \( N_{\lambda, \kappa} \). Finally, stochastic continuity of \( \Phi(\mu_{\lambda, \kappa}) \) at fixed points follows from the fact that for each fixed \( t_0 \), \( N_{\lambda, \kappa}((0, \infty] \times \{ t_0 \}) = 0 \) with probability 1.

7.1. **Topologies on \( D_\infty(M_1(\mathbb{R})) \).** We now give a brief discussion of the difficulties of extending Corollary 1.12 to a full weak convergence \( \Phi(m_\varepsilon) \Rightarrow \Phi(\mu_{\lambda, \kappa}) \) of \( M_1(\mathbb{R}) \)-valued path processes. It is first necessary to decide on a topology for \( \Phi(\mu_{\lambda, \kappa}) \) of \( M_1(\mathbb{R}) \)).

In the proof of Theorem 1.4, it was necessary to equip \( D_\infty(M_1(\mathbb{R})) \) with the strongest of the Skorohod \( J_1 \)-topologies, but, since convex combinations \( (1 - \theta)\mu + \theta\pi \) of two probability measures form a “line segment” between \( \mu \) and \( \pi \) in \( M_1(\mathbb{R}) \), one can define the \( M_1 \)-topology and metric on \( D_1(M_1(\mathbb{R})) \) and \( D_\infty(M_1(\mathbb{R})) \) in the natural way. Moreover, the resulting \( M_1 \)-topology on \( D_\infty(M_1(\mathbb{R})) \) defined in this way is the topology of a complete separable metric space.

Unfortunately, to this point we have been unable to prove weak convergence \( \Phi(m_\varepsilon) \Rightarrow \Phi(\mu_{\lambda, \kappa}) \) in the \( M_1 \)-topology (as defined above) on \( D_\infty(M_1(\mathbb{R})) \). In fact, some preliminary computations suggest that \( \{ \Phi(m_\varepsilon) \}_{\varepsilon > 0} \) is not a tight family of \( D_\infty(M_1(\mathbb{R})) \)-valued random variables in this topology, and thus a weaker topology on \( D_\infty(M_1(\mathbb{R})) \) may be needed. We hope to address this in a future paper.

We close this section with an example that demonstrates some of the difficulties establishing weak convergence \( \Phi(m_\varepsilon) \Rightarrow \Phi(\mu_{\lambda, \kappa}) \). A natural approach to proving \( \Phi(m_\varepsilon) \Rightarrow \Phi(\mu_{\lambda, \kappa}) \) would be to apply Theorem 1.4 and the continuous mapping theorem. Unfortunately, the mapping \( \Phi : M_1(D_\infty) \rightarrow D_\infty(M_1(\mathbb{R})) \) is not continuous. The following example demonstrates this lack of continuity even when in \( M_1(D_\infty) \) we endow the space \( D_\infty \) with the strongest of the Skorohod topologies, the \( J_1 \)-topology, and endow \( D_\infty(M_1(\mathbb{R})) \) with the weakest of the Skorohod topologies, the \( M_2 \)-topology.

We restrict everything to the interval \([0, 1]\) and consider real-valued stochastic processes \( X = (X(t), 0 \leq t \leq 1) \) and \( X_n = (X_n(t), 0 \leq t \leq 1) \), \( n = 1, 2, \ldots \), on the probability space \(([0, 1], \mathcal{B}, \text{Leb})\), defined by

\[
X(t; \omega) = \begin{cases} 
0, & 0 \leq t < \frac{1}{2}, \\
1, & \frac{1}{2} \leq t \leq 1, \quad 0 \leq \omega \leq \frac{1}{2}, \\
2, & \frac{1}{2} \leq t \leq 1, \quad \frac{1}{2} < \omega \leq 1,
\end{cases}
\]

and

\[
X_n(t; \omega) = \begin{cases} 
0, & 0 \leq t < \frac{1}{2} - \frac{1}{2^n}, \\
1, & \frac{1}{2} - \frac{1}{2^n} \leq t \leq 1, \quad 0 \leq \omega \leq \frac{1}{2}, \\
0, & \frac{1}{2} - \frac{1}{2^n} \leq t < \frac{1}{2} + \frac{1}{2^n}, \quad \frac{1}{2} < \omega \leq 1, \\
2, & \frac{1}{2} + \frac{1}{2^n} \leq t \leq 1, \quad \frac{1}{2} < \omega \leq 1.
\end{cases}
\]

\[\text{For instance, Whitt defines the Skorohod } M_1 \text{-topology on } D_1(\Psi) \text{ if } \Psi \text{ is a separable Banach space [Whi02, p. 382], but } M_1(\mathbb{R}) \text{ is not a Banach space.}\]
n = 1, 2, . . . . Clearly, each process X and X_n has its sample paths in D_1 = D[0, 1]. We denote by μ (correspondingly, μ_n) the probability measures these processes generate on the cylindrical sets in D[0, 1].

Obviously, for any ω ∈ [0, 1], d^{J_1}(X, X_n) ≤ 2^{−(n+1)}, so, with probability 1, X_n → X in D[0, 1] equipped with the J_1-topology, so that ρ^{J_1} (μ_n, μ) → 0.

Next, in the notation of (46), we have

\[ \Phi_t(\mu) = \begin{cases} \delta_0, & 0 \leq t < \frac{1}{2}, \\ \frac{1}{2} \delta_1 + \frac{1}{2} \delta_2, & \frac{1}{2} \leq t \leq 1 \end{cases} \]

and

\[ \Phi_t(\mu_n) = \begin{cases} \delta_0, & 0 \leq t < \frac{1}{2} - \frac{1}{2^{n+1}}, \\ \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1, & \frac{1}{2} - \frac{1}{2^{n+1}} \leq t < \frac{1}{2} + \frac{1}{2^{n+1}}, \\ \frac{1}{2} \delta_1 + \frac{1}{2} \delta_2, & \frac{1}{2} + \frac{1}{2^{n+1}} \leq t \leq 1, \end{cases} \]

n = 1, 2, . . . , where for x ∈ ℝ, δ_x is the point mass at x. Note that for any n, there is a point on the completed graph of the element Φ(μ_n) of D_1(M_1(ℝ)) with the second component equal to (1/2)δ_0 + (1/2)δ_1, and the distance from that point to the completed graph of the Φ(μ) has a positive lower bound that does not depend on n. Therefore, Φ(μ_n) does not converge to Φ(μ) in D_1(M_1(ℝ)) even if the latter space is endowed with the M_2-topology (see Section 11.5 in [Whi02] for the definition of the M_2-topology).

APPENDIX A. ESTIMATION OF THE TERM IN (37)

In order to finish the proof of Proposition 4.2, we need to estimate the term in (37). In this appendix we achieve that by proving the following lemma.

Lemma A.1. If κ ∈ [1, 2), then for all 0 < s < ∞ and η > 0,

\[ \lim_{\delta \to 0} \lim_{n \to \infty} \sup_{t \leq s} \left| \sup_{i=1}^{\lfloor tn \rfloor} \beta_1 \{ \beta_i \leq \delta n^{1/\kappa} \} - E_Q[\beta_i] \right| \geq \eta = 0. \]

Remark A.2. Lemma A.1 is an improvement of [PS10] Lemma 5.5, which stated that

\[ \lim_{\delta \to 0} \lim_{n \to \infty} \sup_{t \leq s} \left| \sup_{i=1}^{n} \beta_1 \{ \beta_i \leq \delta n^{1/\kappa} \} - E_Q[\beta_i] \right| \geq \eta, \quad \forall \eta > 0. \]

Before giving the proof of Lemma A.1, we introduce new notation. Recall that ρ_x = (1 − ω_x)/ω_x, and for i ≤ j let

\[ \Pi_{i,j} = \prod_{k=i}^{j} \rho_k, \quad R_{i,j} = \sum_{k=i}^{j} \Pi_{i,k}, \quad W_{i,j} = \sum_{k=i}^{j} \Pi_{k,j}, \quad W_j = \sum_{k=-\infty}^{j} \Pi_{k,j}. \]

This notation is often useful for writing certain quenched expectations or probabilities in compact form. For instance, it is easy to show that E_{ω}[T_{i+1}] = 1 + 2W_i (see [Zei04] for a reference). In particular,

\[ \beta_i = E_{ω}[T_{\nu_i} - T_{\nu_i-1}] = \sum_{j=\nu_i-1}^{\nu_i-1} E_{ω}[T_{j+1}] = \nu_i - \nu_{i-1} + 2 \sum_{j=\nu_{i-1}}^{\nu_i-1} W_j. \]

It will be important for us to be able to control the tails, under the measure Q, of the random variables of the type W_{\nu_{i-1}}. Since under Q the environment is stationary under shifts by the ladder locations, these random variables all have the same distribution under Q as W_{-1}.
Further, it was shown in [PZ09, Lemma 2.2] that $W_{-1}$ has, under $Q$, exponential tails. That is, there exist constants $C_1, C_2 > 0$ such that for any $x > 0$,

$$Q(W_{-1} > x) \leq C_1 e^{-C_2 x}. \quad (48)$$

We now proceed to prove Lemma A.1.

**Proof of Lemma A.1** First, note that $\beta_t 1_{\{\beta_i \leq \delta n^{1/\kappa}\}} = \beta_t \wedge \delta n^{1/\kappa} - \delta n^{1/\kappa} 1_{\{\beta_i > \delta n^{1/\kappa}\}}$. Thus,

$$Q \left( \sup_{t \leq s} \left| \sum_{i=1}^{\lfloor tn \rfloor} \left\{ \beta_i 1_{\{\beta_i \leq \delta n^{1/\kappa}\}} - E_Q[\beta_i 1_{\{\beta_i \leq \delta n^{1/\kappa}\}}] \right\} \right| \geq \eta \right)$$

$$\leq Q \left( \sup_{t \leq s} \left| \sum_{i=1}^{\lfloor tn \rfloor} \{ \beta_i \wedge \delta n^{1/\kappa} - E_Q[\beta_i \wedge \delta n^{1/\kappa}] \} \geq \eta/2 \right)$$

$$+ Q \left( \sup_{t \leq s} \sum_{i=1}^{\lfloor tn \rfloor} 1_{\{\beta_i > \delta n^{1/\kappa}\}} - \lfloor tn \rfloor Q(\beta_i > \delta n^{1/\kappa}) \geq \eta/2 \right). \quad (50)$$

Note that (14) implies that $\lfloor tn \rfloor Q(\beta_i > \delta n^{1/\kappa}) \to tC_0 \delta^{-\kappa}$ and, moreover, that the convergence is uniform in $t \in [0, s]$. Therefore, to bound the term in (50) it is enough to show that for all $0 < s < \infty$ and $\eta > 0$,

$$\lim_{\delta \to 0} \lim_{n \to \infty} Q \left( \sup_{t \leq s} \left| \sum_{i=1}^{\lfloor tn \rfloor} 1_{\{\beta_i > \delta n^{1/\kappa}\}} - tC_0 \delta^{-\kappa} \right| \geq \eta \right) = 0. \quad (51)$$

Now, for any $\delta > 0$ let $G_{\delta} : M_p((0, \infty] \times [0, \infty)) \to D_+^\ast$ (we equip the latter space with the $J_1$ topology) be defined by

$$G_{\delta}(\zeta) = \zeta((\delta, \infty] \times [0, t]), \quad t \geq 0.$$ 

Then $\sum_{i=1}^{\lfloor tn \rfloor} 1_{\{\beta_i > \delta n^{1/\kappa}\}} = G_{\delta}(N_{1/n}) (t)$. It is easy to see that $G_{\delta}$ is continuous on the set of point processes with no atoms on the line $\{\delta\} \times [0, \infty)$. Since $N_{\lambda, \kappa}$ belongs to this set with probability $1$, and $N_{1/n} \Rightarrow N_{\lambda, \kappa}$, the continuous mapping theorem implies that $G_{\delta}(N_{1/n}) \Rightarrow G_{\delta}(N_{\lambda, \kappa})$. Furthermore, the supremum over a compact interval is a continuous mapping from $D_+^\ast$ equipped with the $J_1$ topology to the real line. Therefore,

$$\lim_{n \to \infty} \sup_{t \leq s} Q \left( \sup_{t \leq s} \left| \sum_{i=1}^{\lfloor tn \rfloor} 1_{\{\beta_i > \delta n^{1/\kappa}\}} - tC_0 \delta^{-\kappa} \right| \geq \eta \right) \leq Q \left( \sup_{t \leq s} \left| G_{\delta}(N_{\lambda, \kappa}) (t) - tC_0 \delta^{-\kappa} \right| \geq \eta \right).$$

Note that $G_{\delta}(N_{\lambda, \kappa})$ is a homogeneous one-dimensional Poisson process with rate $\lambda/\kappa \delta^{-\kappa} = C_0 \delta^{-\kappa}$. Therefore, using once again the $L^p$-maximum inequality for martingales, we have

$$Q \left( \sup_{t \leq s} \left| G_{\delta}(N_{\lambda, \kappa}) (t) - tC_0 \delta^{-\kappa} \right| \geq \eta \right) \leq \frac{\delta^2}{\eta^2} \text{Var}_Q \left( G_{\delta}(N_{\lambda, \kappa}) (s) \right) = \frac{\lambda s \delta^{2-\kappa}}{\eta^2 \kappa}.$$ 

Since $\kappa < 2$ this last term vanishes as $\delta \to 0$ for any $\eta > 0$ and $s < \infty$. This completes the proof of (51) and, therefore, it only remains to estimate the term in (49).

To this end, we assume for (notational) simplicity that $s = 1$, in which case our task reduces to showing that for any $\eta > 0$,

$$\lim_{\delta \to 0} \lim_{n \to \infty} Q \left( \max_{k \leq n} \left| \frac{1}{n^{1/\kappa}} \sum_{i=1}^{k} \{ \beta_i \wedge \delta n^{1/\kappa} - E_Q[\beta_i \wedge \delta n^{1/\kappa}] \} \right| \geq \eta \right) = 0. \quad (52)$$
For a fixed $n$ and $\delta \in (0,1]$ denote

$$S_j = n^{-1/\kappa} \sum_{i=1}^{j} \{ \beta_i \wedge \delta n^{1/\kappa} - E_Q[\beta_1 \wedge \delta n^{1/\kappa}] \}, \quad j = 1, \ldots, n.$$  

For $\eta > 0$ let $A_m = \{ \max_{j<m} |S_j| \leq \eta < |S_m| \}$. Then,

$$Q \left( \max_{k \leq n} |n^{-1/\kappa} \sum_{i=1}^{k} \{ \beta_i \wedge \delta n^{1/\kappa} - E_Q[\beta_1 \wedge \delta n^{1/\kappa}] \} > \eta \right) = \sum_{m=1}^{n} Q(A_m)$$

$$\leq Q(|S_n| \geq \eta/2) + \sum_{m=1}^{n-1} Q(A_m \cap \{ |S_n| \leq \eta/2 \})$$

$$\leq Q(|S_n| \geq \eta/2) + \sum_{m=1}^{n-1} Q(A_m \cap \{ |S_n - S_m| > \eta/2 \}).$$  

(54)

It was shown in the proof of Lemma 5.5 in [PS10] that for some constant $C$,

$$\var{\sum_{i=1}^{n} \beta_i \wedge \delta n^{1/\kappa}} \leq C \delta^{2-\kappa} n^{2/\kappa}. \quad (55)$$

By Markov’s inequality this shows that the term $Q(|S_n| \geq \eta/2)$ does not contribute to the limit in (52). Therefore, it remains only to bound the sum on the right in (54). If the $\beta_i$ were independent, then the general term in this sum would be equal to $Q(A_m)Q(|S_n - S_m| > \eta/2)$ and the sum could be handled in the same way as the term $Q(|S_n| \geq \eta/2)$ above. While the $\beta_i$ are not independent under $Q$, they have good mixing properties and the following lemma gives an upper bound on the general term in the sum, not far off from what it would be if the $\beta_i$ were independent.

**Lemma A.3.** There are constants $C, C' > 0$ such that for any $n = 1, 2, \ldots$, $\delta \in (0,1]$, $m = 1, \ldots, n$ and $\eta > 0$,

$$Q(A_m \cap \{ |S_n - S_m| > \eta \}) \leq C e^{-C'n^{1/\kappa}/\log n} + \frac{1}{\eta^2} C \delta^{2-\kappa} (Q(A_m) + 1/n).$$

Assuming the statement of Lemma A.3, the proof of Lemma A.1 can be completed by writing (changing the constants as necessary)

$$\sum_{m=1}^{n-1} Q(A_m \cap \{ |S_n - S_m| > \eta/2 \}) \leq \sum_{m=1}^{n-1} \left\{ C e^{-C'n^{1/\kappa}/\log n} + \frac{1}{\eta^2} C \delta^{2-\kappa} (Q(A_m) + 1/n) \right\}$$

$$\leq C n e^{-C'n^{1/\kappa}/\log n} + \frac{C \delta^{2-\kappa}}{\eta^2}.$$  

Both terms vanish under the limits in (52), so we only need to prove Lemma A.3.

**Proof of Lemma A.3.** Define a (discrete time) filtration on $\Omega = [0,1]^Z$ by $\G_n = \s(\omega_i : i \leq n)$, $n = 0, 1, 2, \ldots$. Then for each $m = 0, 1, 2, \ldots$, $\nu_m - 1$ is a stopping time with respect to that filtration, and we denote $F_m = \G_{\nu_m-1}$, $m = 1, 2, \ldots$. Since each $\beta_j$ with $j \leq m$ is $F_m$-measurable, so is each $S_j$ with $j \leq m$. Therefore,

$$Q(A_m \cap \{ |S_n - S_m| > \eta \}) = E_Q \left[ 1_{\{A_m\}} Q \left( |S_n - S_m| > \eta \mid F_m \right) \right].$$  

Conditioned on $F_m$, the difference $S_n - S_m$ no longer has zero mean, but we will show that the conditional mean is typically small. We begin by comparing the conditional and unconditional means of $\beta_j \wedge \delta n^{1/\kappa}$. To this end we make explicit the dependence of $\beta_j$ on $F_m$. Recall the
definitions of $\Pi_{i,j}$, $W_{i,j}$ and $R_{i,j}$ in (46) and note that $W_i = W_{k,i} + \Pi_{k,i}W_{k-1}$ for any $k \leq i$. Therefore, for any $1 \leq m < j$ we can rewrite (47) as

$$\beta_j = \nu_j - \nu_{j-1} + 2 \sum_{i=\nu_j-1}^{\nu_j-1} (W_{\nu_m,i} + W_{\nu_m-1}\Pi_{\nu_m,i})$$

$$= \nu_j - \nu_{j-1} + 2 \sum_{i=\nu_j-1}^{\nu_j-1} W_{\nu_m,i} + 2W_{\nu_m-1}\Pi_{\nu_m,\nu_j-1} - R_{\nu_j-1,\nu_j-1}$$

$$=: \beta_{m,j} + 2W_{\nu_m-1}\Pi_{\nu_m,\nu_j-1} - R_{\nu_j-1,\nu_j-1}.$$  

Note that $\beta_{m,j}$ is independent of $\mathcal{F}_m$. We enlarge, if necessary, the probability space to define a random variable $\tilde{W}$ with the same distribution as $W_{\nu_m-1}$ and independent of all $(\omega_x)$; in particular, $\tilde{W}$ is independent of $\mathcal{F}_m$. Denote $\tilde{\beta}_j = \beta_{m,j} + 2\tilde{W}\Pi_{\nu_m,\nu_j-1} - R_{\nu_j-1,\nu_j-1}$, so that

$$E_Q[\beta_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m] - E_Q[\tilde{\beta}_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m] = E_Q[\beta_j \land \delta n^{1/\kappa} - \tilde{\beta}_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m].$$

Observe that $R_{\nu_j-1,\nu_j-1} \leq \beta_{m,j} \leq \min(\beta_j, \tilde{\beta}_j)$. Thus, if $R_{\nu_j-1,\nu_j-1} \geq \delta n^{1/\kappa}$, then both $\beta_j$ and $\tilde{\beta}_j$ are larger than $\delta n^{1/\kappa}$ as well. This implies that

$$E_Q[\beta_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m] - E_Q[\tilde{\beta}_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m]$$

$$\leq E_Q[\beta_j - \tilde{\beta}_j 1_{\{R_{\nu_j-1,\nu_j-1} \leq \delta n^{1/\kappa}\}} \mid \mathcal{F}_m]$$

$$= E_Q[2\Pi_{\nu_m,\nu_j-1} - R_{\nu_j-1,\nu_j-1} - \tilde{W} 1_{\{R_{\nu_j-1,\nu_j-1} \leq \delta n^{1/\kappa}\}} \mid \mathcal{F}_m]$$

$$\leq 2(E_Q[\Pi_{0,\nu-1}]^{j-1}E_Q[\nu_{0,\nu-1} 1_{\{\nu_{0,\nu-1} \leq \delta n^{1/\kappa}\}}] \left(E_Q[\tilde{W}] + W_{\nu_m-1}\right).$$

where in the last inequality we used the fact that the blocks of environment between ladder locations are i.i.d. under the measure $Q$. Since $R_{0,\nu-1} \leq \beta_1$, there exists a constant $C$ so that

$$Q(R_{0,\nu-1} > x) \leq Cx^{-\kappa}.$$  

This implies that $E_Q[R_{0,\nu-1} 1_{\{R_{0,\nu-1} \leq \delta n^{1/\kappa}\}}] \leq E_Q[R_{0,\nu-1}] < \infty$ when $\kappa > 1$ and $E_Q[R_{0,\nu-1} 1_{\{R_{0,\nu-1} \leq \delta n^{1/\kappa}\}}] \leq C \log n$ when $\kappa = 1$ for some other $C$. Thus, we can always bound this expectation by $C \log n$ for some $C > 0$. Also, the definition of $\nu$ implies that $r := E_Q[\Pi_{0,\nu-1}] < 1$ and (48) implies that $E_Q[\tilde{W}] = E_Q[W_{-1}] < \infty$. Thus, there exists a constant $C$ so that

$$E_Q[\beta_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m] - E_Q[\tilde{\beta}_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m] \leq C \log n r^{j-1} (1 + W_{\nu_m-1}),$$

implying that

$$|E_Q[S_n - S_m \mid \mathcal{F}_m]| \leq n^{-1/\kappa} \sum_{j=m+1}^{n} |E_Q[\beta_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m] - E_Q[\tilde{\beta}_j \land \delta n^{1/\kappa} \mid \mathcal{F}_m]|$$

$$\leq Cn^{-1/\kappa} \log n (1 + W_{\nu_m-1}).$$

Applying Chebyshev’s inequality conditionally, we obtain

$$Q(|S_n - S_m| > \eta \mid \mathcal{F}_m)$$

$$\leq 1\{|E_Q[S_n - S_m \mid \mathcal{F}_m]| > \eta/2 \} + Q(|S_n - S_m - E_Q[S_n - S_m \mid \mathcal{F}_m]| > \eta/2)$$

$$\leq 1\{1 + W_{\nu_m-1} > n^{1/\kappa} \eta/(2C \log n)\} + \frac{4}{\eta^2} \text{Var}_Q(S_n - S_m \mid \mathcal{F}_m).$$  

\[ (58) \]
To handle the conditional variance in (58) we write
\[
\Var_Q(S_n - S_m \mid \mathcal{F}_m) = m^{-2/\kappa} \sum_{j=m+1}^n \Var_Q(\beta_j \wedge \delta n^{1/\kappa} \mid \mathcal{F}_m)
\]
\[
+ 2m^{-2/\kappa} \sum_{m<j<k \leq n} \Cov_Q(\beta_j \wedge \delta n^{1/\kappa}, \beta_k \wedge \delta n^{1/\kappa} \mid \mathcal{F}_m).
\]
(59)
Upper bounds on the conditional variance and conditional covariance terms above can be obtained in a similar manner to the proof of (49) in \cite{PS10}. One adapts this approach to take into account the conditioning on \(\mathcal{F}_m\), by replacing \(\beta_j\) by \(\beta_{m,j}\), and then controlling the difference between the two similarly to what was done above when bounding \(E[S_n - S_m \mid \mathcal{F}_m]\). Doing this we obtain that there exist constants \(C > 0\) and \(r \in (0, 1)\) such that
\[
\Var_Q(\beta_j \wedge \delta n^{1/\kappa} \mid \mathcal{F}_m) \leq C \delta^{2-\kappa} n^{2/\kappa - 1} (1 + r^{j-m-1} W_{\nu_m-1}^2)
\]
and
\[
\Cov_Q(\beta_j \wedge \delta n^{1/\kappa}, \beta_k \wedge \delta n^{1/\kappa} \mid \mathcal{F}_m) \leq C \delta^{2-\kappa} n^{2/\kappa - 1} (1 + r^{j-m-1} W_{\nu_m-1}^2) \sqrt{r^{k-j-1}}.
\]
Using these bounds in (59), we see that for some \(C > 0\),
\[
\Var_Q(S_n - S_m \mid \mathcal{F}_m) \leq C \delta^{2-\kappa} \left(1 + \frac{W_{\nu_m-1}^2}{n}\right).
\]
(60)
Combining (56), (58), and (60) we obtain
\[
Q(\mathcal{A}_m \cap \{|S_n - S_m| > \eta\})
\leq Q(C'(1 + W_{\nu_m-1}) > \eta m^{1/\kappa} / \log n) + \frac{1}{\eta^2} C \delta^{2-\kappa} E_Q \left[1_{\mathcal{A}_m} (1 + W_{\nu_m-1}^2/n)\right]
\leq C e^{-C' \eta m^{1/\kappa} / \log n} + \frac{1}{\eta^2} C \delta^{2-\kappa} (Q(\mathcal{A}_m) + 1/n)
\]
where the constants \(C, C'\) may change from line to line (in the last inequality we used \(\mathcal{B}18\) and the fact that \(W_{\nu_m-1}\) has the same distribution as \(W_{-1}\) under \(Q\)). This gives us the statement of the lemma. \(\Box\)

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