Minimal topological actions do not determine the measurable orbit equivalence class

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Abstract. We construct a minimal topological action \( \tilde{\Phi} \) of a non-amenable group on a Cantor set \( C \), which is non-uniquely ergodic and furthermore there exist ergodic invariant measures \( \mu_1 \) and \( \mu_2 \) such that \( (\tilde{\Phi}, C, \mu_1) \) and \( (\tilde{\Phi}, C, \mu_2) \) are not orbit equivalent measurable equivalence relations.

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1 Introduction

1.1 The result

In [8] Oxtoby constructed a non-uniquely ergodic minimal $\mathbb{Z}$-action on a compact metric space. It is well-known however that any two measure-preserving transformations of countably infinite amenable groups are orbit equivalent. Hence if $\mu_1$ and $\mu_2$ are ergodic invariant measures on a set $X$ under a $\mathbb{Z}$-action $\tilde{\Phi}$, then $(\tilde{\Phi}, X, \mu_1)$ and $(\tilde{\Phi}, X, \mu_2)$ are necessarily orbit equivalent. In this paper we consider the actions of non-amenable groups.

Let $\Gamma$ be a countable non-amenable group, $X$ be a compact metric space and $\tilde{\Phi} : \Gamma \to \text{Homeo}(X)$ be an injective homomorphism, in other words $\tilde{\Phi}$ is a faithful topological action of $\Gamma$ on $X$. Suppose that for any $x \in X$ the orbit $\text{Orb}(x)$ is dense, that is $\tilde{\Phi}$ is minimal. We consider such topological actions $\tilde{\Phi}$, for which there exist $\Gamma$-invariant measures on $X$ (for amenable groups such measures always exist). The goal of our paper is to construct an action $\tilde{\Phi}$ with invariant ergodic measures $\mu_1$ and $\mu_2$ such that the measurable equivalence relations $(\tilde{\Phi}, X, \mu_1)$ and $(\tilde{\Phi}, X, \mu_2)$ are not orbit equivalent.

1.2 Gromov’s graphs of dense holonomy

Let $G(V, E)$ be an infinite connected graph with edges colored by the symbols $A, B, C$ and $D$. Note that we consider only proper edge-colorings, that is for each vertex $p \in V$ the colors of the edges incident to $p$ are different. In particular, $G$ is of bounded valence 4 (each vertex has at most 4 neighbours). We call these graphs 4-colored graphs. Any such colorings define an action $\Phi$ of the group $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, the free product of four copies of the (cyclic) group of order two, on the vertices of $G$. Indeed, for any word $w = w_nw_{n-1}\ldots w_1$ in $\Gamma$ and $x \in V$, there exists a unique path $(x_0, x_1, \ldots, x_n)$ such that

- $x_0 = x$
- If $x_i$ has no incident edges colored by $w_i$, then $x_{i+1} = x_i$.
- If the edge $(x_i, y)$ is colored by $w_i$ then $y = x_{i+1}$.

Then one sets $\Phi(\gamma)(x) = x_n$. 

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Let $r > 0$. A rooted colored $r$-ball around $x \in V$, $B_r(x)$ is a spanned subgraph of $G$ with the induced 4-coloring such that

- $d_G(x, y) \leq r$, for any $y \in B_r(x)$, where $d_G$ is the usual shortest path metric.

Let $U^r_G$ be the set of rooted equivalence classes of 4-colored rooted $r$-balls in $G$. That is we consider two rooted $r$-balls $B_r(x)$ and $B_r(y)$ equivalent if there exists a graph isomorphism $\theta : B_r(x) \to B_r(y)$ preserving the edge-colorings, mapping $x$ to $y$. Clearly $U^r_G$ is a finite set. For each $x \in V$ the $r$-type of $x$ is the element $\alpha_r(x) \in U^r_G$ representing the $r$-ball around $x$.

Following Gromov [4], we call a 4-colored graph $G$ a graph of dense holonomy if, for any $\alpha \in U^r_G$, there exists an integer $m_\alpha$ such that any ball of radius $m_\alpha$ in $G$ contains at least one vertex $x$ with $r$-type $\alpha_r(x) = \alpha$. Note that in the theory of Delone-sets such graphs are called repetitive [7]. Also, we call a 4-colored graph $G$ generic if for any $x \neq y \in V$ there exists $r > 0$ such that the $r$-types of $x$ and $y$ are different.

### 1.3 Amenable graphs

Recall that an infinite, connected graph $G(V, E)$ of bounded vertex degree is amenable if for any $\epsilon > 0$, there exists a finite subset $\Omega_\epsilon \subset V$ such that

$$\frac{|\partial \Omega_\epsilon|}{|\Omega_\epsilon|} < \epsilon,$$

where

$$\partial \Omega_\epsilon := \{x \in \Omega_\epsilon \mid \text{there exists } y \in V \setminus \Omega_\epsilon \text{ such that } x \text{ and } y \text{ are adjacent}\}$$

is the boundary of $\Omega_\epsilon$. We call a graph $G(V, E)$ strongly amenable if there exists an increasing sequence of finite subsets $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$ such that

- $\bigcup_{n=1}^\infty \Omega_n = V$ (exhaustion);
- $\lim_{n \to \infty} \frac{|\partial \Omega_n|}{|\Omega_n|} = 0$.

Our first step shall be to construct strongly amenable generic 4-colored graphs with dense holonomy.
1.4 The associated topological action

For any graph $G(V,E)$ edge-colored by the set $\{A,B,C,D\}$ we have a naturally associated compact metric space $X_G$ with a topological $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$-action by the construction below. Let $\alpha = \{\alpha_1 < \alpha_2 < \ldots\}$ be an infinite chain, such that $\alpha_r \in U_r^G$ and the $r$-ball around the root of $\alpha_{r+1}$ is just $\alpha_r$ (that is $\alpha_r < \alpha_{r+1}$). The chains above form the compact metric space $X_G$, the type space, where

$$d_{X_G}(\{\alpha_1 < \alpha_2 < \ldots\}, \{\beta_1 < \beta_2 < \ldots\}) = 2^{-r}$$

where $r$ is the smallest integer for which $\alpha_r \neq \beta_r$. We shall show (Proposition 2.1) that the $\Gamma$-action on $G$ extends to $X_G$ in a natural way. For each $x \in V$, $\pi(x) = \{\alpha_1(x) < \alpha_2(x) < \ldots\}$ is the associated element in $X_G$, where $\alpha_r(x)$ is the $r$-type of $x$. Obviously, $G$ is generic if and only if $\pi : G \to X_G$ is an injective map. We prove (Proposition 2.2) that if $G$ is both generic and of dense holonomy, then $X_G$ is homeomorphic to the Cantor-set and the associated $\Gamma$-action is minimal. We shall also prove (Section 8) that for the strongly amenable generic graph we construct, this $\Gamma$-action has at least two different ergodic invariant measures. Finally (Theorem 1), we shall show that for two of these ergodic invariant measures $\mu_1$ and $\mu_2$, $c(\mu_1) \neq c(\mu_2)$, where $c(\mu_i)$ denotes the cost of the measurable equivalence relation $(\Gamma, X_G, \mu_i)$. This shows that $(\Gamma, X_G, \mu_1)$ and $(\Gamma, X_G, \mu_2)$ are not orbit equivalent relations. It is important to note that although $(\Gamma, X_G, \mu_i)$ are not hyperfinite equivalence relations, all their leaves are amenable graphs. The existence of such measurable equivalence relations was first observed by Kaimanovich [5].

2 Amenable actions on countable sets

Recall the notion of amenable actions on countable sets. Let $\Gamma$ be a countable group and $X$ be a countable set. A homomorphism $\Phi : \Gamma \to S(X)$ from $\Gamma$ into the full permutation group of $X$ is called an action. The action is amenable [2], [3] if there exists an invariant, finitely additive probability measure on $X$, that is a (positive real valued) function $\mu$ on the subsets of $X$ such that:

- $\mu(X) = 1$, 

\( \mu(A \cup B) = \mu(A) + \mu(B) \) if \( A \cap B = \emptyset \),

\( \mu(\Phi(\gamma)(A)) = \mu(A) \), for any \( \gamma \in \Gamma \), \( A \subseteq X \).

It is well-known that \( \Phi \) is amenable if and only if the following Følner-condition holds:

For any finite set \( 1 \in K \subseteq \Gamma \) and \( \epsilon > 0 \) there exists a finite subset \( \Omega_{K,\epsilon} \subseteq X \) such that

\[
\frac{|\Phi(K)\Omega_{K,\epsilon}|}{|\Omega_{K,\epsilon}|} < 1 + \epsilon.
\]

Now let \( \Gamma \) be a finitely generated group with symmetric generating system \( S \). The Schreier-graph \( G_{\Phi} = G(\Gamma, \Phi, X, S) \) of an action \( \Phi \) is given as follows:

- \( V(G_{\Phi}) = X \).
- \( (x, y) \in E(G_{\Phi}) \) if \( x = \Phi(s)(y) \) for some \( s \in S \).

Clearly, \( \Phi \) is amenable if and only if \( G_{\Phi} \) is an amenable graph. If \( \Gamma \) is amenable, than any action of \( \Gamma \) is amenable as well. On the other hand, several non-amenability groups have faithful, transitive amenable actions on countable sets (see [3] and [2]). In this paper we consider amenable \( \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \)-actions on amenable graphs with proper 4-colorings. From now on \( \Gamma \) shall always denote the group \( \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \). As we have seen in the Introduction for any \( \Gamma \)-action \( \Phi \) there exists a natural associated compact metric space of chains \( X_{\Phi} \).

**Proposition 2.1** The action \( \Phi \) extends to a topological action \( \tilde{\Phi} \) on the metric space \( X_{\Phi} \).

**Proof.** First note that, by definition, the image of \( \pi : X \to X_{\Phi} \), where, for \( x \in X \), \( \pi(x) = \{\alpha_1(x) < \alpha_2(x) < \ldots\} \) is dense in \( X_{\Phi} \). Let \( \tilde{\alpha} = \{\alpha_1 < \alpha_2 < \ldots\} \in X_{\Phi} \) and consider a word \( w = w_n w_{n-1} \cdots w_1 \in \Gamma \). For all \( i = 1, 2, \ldots \) let \( x_i \in X \) be such that \( \alpha_{i+n}(x_i) = \alpha_{i+n} \).

If \( y_i = \Phi(w)(x_i) \) setting \( \beta_i = \alpha_i(y_i) \) one has \( \beta_i < \beta_{i+1} \) and one defines

\[
\tilde{\Phi}(w)(\tilde{\alpha}) := \{\beta_1 < \beta_2 < \ldots\}.
\]

Obviously \( \tilde{\Phi}(w) \) is a continuous map for all \( w \in \Gamma \) and \( \tilde{\Phi}(w)(\pi(x)) = \pi(\Phi(w)(x)) \).

**Proposition 2.2** If \( \Phi \) is an amenable action of \( \Gamma \), then \( \tilde{\Phi} \) has invariant measure on \( X_{\Phi} \).
Proof. First fix an ultrafilter \( \omega \) on \( \mathbb{N} \) and consider the associated ultralimit

\[
\lim_{\omega} : l^\infty(\mathbb{N}) \to \mathbb{R}.
\]

Let \( \{\Omega_n\}_{n=1}^\infty \) be a Følner-sequence in \( X \) with respect to \( \Phi \). That is for any \( \gamma \in \Gamma \),

\[
\lim_{n \to \infty} \frac{|\Phi(\gamma)\Omega_n \cup \Omega_n|}{|\Omega_n|} = 1.
\]

Let \( \alpha \in U^r_G \) be an \( r \)-type of the Schreier-graph. Denote by \( \tau_n(\alpha) \) the number of vertices in the set \( \Omega_n \) which are of \( r \)-type \( \alpha \). Then let

\[
\mu(\alpha) := \lim_{\omega} \frac{\tau_n(\alpha)}{|\Omega_n|},
\]

where \( \alpha \subset X_G \) is the clopen set of chains \( \{\delta_1 \prec \delta_2 \prec \ldots\} \), with \( \delta_r = \alpha \). Now let \( \beta^1, \beta^2, \ldots, \beta^k \) be the set of \( (r+1) \)-types such that \( \alpha \prec \beta^i \). Clearly,

\[
\tau_n(\alpha) = \sum_{i=1}^k \tau_n(\beta^i) .
\]

That is \( \mu(\alpha) = \sum_{i=1}^k \mu(\beta^i) \), where \( \beta^i \subset X_G \) is the clopen set of chains with \( (r+1) \)-type \( \beta^i \). This shows immediately that \( \mu \) defines a Borel-measure on the compact metric space \( X_G \). In order to prove that \( \mu \) is \( \tilde{\Phi} \)-invariant it is enough to show that

\[
\mu(\tilde{\Phi}(w)(\alpha)) = \mu(\alpha),
\]

where \( w \) is one of the generators of \( \Gamma \), namely \( A, B, C \) or \( D \). Let \( \{s_1, s_2, \ldots, s_l\} \subset U^{r+1}_G \) be the set of \( (r+1) \)-types in the Schreier graph of the action with the following property. The \( r \)-ball around \( \Phi(w)(x_i) \) is just \( \alpha \), where \( x_i \) is the root of \( s_i \). Observe that \( \tilde{\Phi}(w)(\alpha) = \coprod_{i=1}^l s_i \). Hence we need to prove that

\[
\mu(\alpha) = \sum_{i=1}^l \mu(s_i). \quad (1)
\]

In order to check (1) it is enough to see that

\[
\lim_{n \to \infty} \frac{|\tau_n(\alpha) - \sum_{i=1}^l \tau_n(s_i)|}{|\Omega_n|} = 0 . \quad (2)
\]

Let \( \Omega_n(\alpha) \) be the set of vertices in \( \Omega_n \) of \( r \)-type \( \alpha \) and \( \Omega_n(s_i) \) be the set of vertices in \( \Omega_n \) of \( (r+1) \)-type \( s_i \). Let us observe that
• $\Omega_n(s_i) \cap \Omega_n(s_j) = \emptyset$ if $i \neq j$.

• For any $x \in \Omega_n(\alpha)$ if $\Phi(w)(x) \in \Omega_n$, then $\Phi(w)(x) \in \Omega_n(s_i)$ for some $i$.

• For any $x \in \Omega_n(s_i)$ if $\Phi(w)(x) \in \Omega_n$, then $\Phi(w)(x) \in \Omega_n(\alpha)$ (recall that $w$ is an involution, namely $w^2 = 1$).

Therefore,

$$|\tau_n(\alpha) - \sum_{i=1}^l \tau_n(s_i)| \leq 2|\partial \Omega_n|.$$

Thus (2) immediately follows from the Følner-property. 

### 3 Generic actions of dense holonomy

Again let $\Gamma$ be the free product of four cyclic groups of order two $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and $\Phi$ be an amenable $\Gamma$-action on the countable set $X$, induced by a 4-coloring of a graph $G(X, E)$. We call the action $\Phi$ generic, resp. of dense holonomy, if the 4-colored graph is generic, resp. of dense holonomy.

**Proposition 3.1** If $\Phi$ is a generic action on $X$ of dense holonomy, then $X_G$ is homeomorphic to a Cantor-set and the associated action $\Phi$ is minimal.

**Proof.** Let $\alpha \in U_G$, then there exists $\beta \neq \gamma \in U_G$ (for some $s > r$) such that $\alpha \prec \beta$ and $\alpha \prec \gamma$, that is the $r$-ball around the root of $\beta$ and $\gamma$ is of type $\alpha$. Indeed, let $x \in X$ a vertex of $r$-type $\alpha$. Then by the dense holonomy property there exists $y \in X$ of the same $r$-type. By genericity there exists some $s > r$ such that the $s$-types of $x$ and $y$ are different. Consequently, $X_G$ is a compact metric space with no isolated point, thus it is homeomorphic to the Cantor-set.

Now we prove minimality. Let $\tilde{\alpha} = \{\alpha_1 \prec \alpha_2 \prec \ldots\}$ and $\tilde{\beta} = \{\beta_1 \prec \beta_2 \prec \ldots\}$ be elements of $X_G$. It is enough to prove that for any $r > 0$ there exists $w \in \Gamma$ such that $\Phi(w)(\tilde{\alpha}) = \{\beta_1 \prec \beta_2 \prec \ldots \prec \beta_r \prec \gamma_{r+1} \ldots\}$ that is $d(\tilde{\beta}, \Phi(w)(\tilde{\alpha})) \leq 2^{-r}$. Let $m > r$ be an integer such that, by dense holonomy, any $m$-ball in the graph $G$ contains a vertex of $r$-type $\beta_r$. Consider a vertex $p \in X$ such that its $2m$-type is $\alpha_{2n}$. Let $q \in B_m(p)$ be a vertex of $r$-type $\beta_r$. Then
there exists a word $w = w_l w_{l-1} \ldots w_1$, $l \leq m$ such that $\Phi(w)(p) = q$. By the definition of
the induced topological action $\widetilde{\Phi}$ one has $\widetilde{\Phi}(w)(\tilde{\alpha}) = \{\beta_1 < \beta_2 < \ldots < \beta_r < \ldots\}$. 

Note: If the action of a group is generic but not of dense holonomy the associated action
on the type space might not be minimal. Consider the group $\Delta = \mathbb{Z}_2 \ast \mathbb{Z}_2$, the dihedral

group generated by symbols $A$ and $B$. We define an action on the positive integers the
following way:

- $\Phi(A)(2n - 1) = 2n, n \geq 1$.
- $\Phi(A)(2n) = 2n - 1, n \geq 1$.
- $\Phi(B)(2n - 1) = 2n - 2, n > 1$.
- $\Phi(B)(2n) = 2n + 1, n > 1$.
- $\Phi(B)(1) = 1$.

It is easy to check that $\Phi$ is generic. The associated compact metric space is $\{1, 2, 3, \ldots\} \cup \{\infty\}$, where

$$\widetilde{\Phi}(A)(\infty) = \infty \quad \text{and} \quad \widetilde{\Phi}(B)(\infty) = \infty.$$ 

Thus the action $\widetilde{\Phi}$ is not minimal.

4 Technicalities

For our main construction we need some simple lemmas.

**Lemma 4.1** Let $G(V, E)$ be a finite connected graph and $d$ be an integer larger than 2. Suppose that $A \subset V$ is a $2d$-net, that is if $x \neq y \in A$ then $d_G(x, y) \geq 2d$. Also suppose that $\text{diam } G \geq 2d$. Then $\frac{|A|}{|V|} \leq \frac{1}{d}$.

*Proof.* By the diameter condition, for each $x \in A$ and $i = 1, 2, \ldots, d$ there exists $z_{x,i} \in V$ such that $d_G(x, z_{x,i}) = i$. Hence $|B_d(x)| \geq d$. Since the $d$-balls around the elements of a $2d$-net are disjoint, the lemma follows. \hfill \blacksquare
Lemma 4.2 Let \( s_1 < s_2 < \ldots < s_n \) be integers, \( s_i \geq 10^{i+1} \). Suppose that \( \text{diam}(G) > 10s_n \).

Then there exists a partition

\[
V = R_1 \bigsqcup R_2 \bigsqcup \ldots \bigsqcup R_n \bigsqcup R_{n+1}
\]

such that

- For any \( 1 \leq i \leq n \), \( R_i \) is a \( 2s_i \)-net.
- For any \( x \in V \), there exists \( p \in R_i \) such that \( d_G(p, x) \leq 10s_i \).

Proof. Let \( R_1 \subset V \) be a maximal \( 2s_1 \)-net in \( V \). By the previous lemma

\[
\frac{|R_1|}{|V|} \leq \frac{1}{s_1}.
\]

Moreover, for the same reason, for any spanned subgraph \( T \subseteq G \), with \( \text{diam}(T) \geq 2s_1 \):

\[
\frac{|T \cap R_1|}{|T|} \leq \frac{1}{s_1}.
\]

By maximality, for any \( y \in V \) there exists \( p \in R_1 \) such that \( d(p, y) \leq 10s_1 \). Now let \( R_2 \subset V \) be a \( 2s_2 \)-net which is maximal with respect to the property that \( R_1 \cap R_2 = \emptyset \). Again, for any spanned subgraph \( T \subseteq G \) with \( \text{diam}(T) \geq 2s_2 \):

\[
\frac{|T \cap R_1|}{|T|} \leq \frac{1}{s_1}, \quad \frac{|T \cap R_2|}{|T|} \leq \frac{1}{s_2}.
\]

Let \( x \) be an arbitrary vertex of \( V \), we need to prove that \( B_{10s_2}(x) \cap R_2 \) is non-empty. If \( B_{10s_2}(x) \cap R_2 = \emptyset \) then by maximality, \( B_{5s_2}(x) \) is completely filled with vertices from the set \( R_1 \). However, this is in contradiction with (3). Now we can construct the partition by a simple induction. \( \square \)

5 The main construction

The goal of this section is to construct a

- faithful, transitive and amenable
- generic
dense holonomy

\(\Gamma\)-action on a countable set \(X\). In the whole section \(\Gamma\) denotes the group \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\) generated by the elements \(\{A, B, C, D\}\) of order two, and \(\tilde{\Gamma}\) denotes the subgroup generated by \(\{A, B, C\}\). We also fix a parameter \(m > 10\) to be specified later. We also need two auxiliary graph sequences. The first one is \(\{C_i\}_{i=1}^{\infty}\), where \(C_i\) is the cycle of length \(2i\) edge-colored by \(A\) and \(B\). The second graph sequence is given the following way. The group \(\tilde{\Gamma}\) is residually finite (free product of finite groups), hence one can pick a decreasing sequence of finite index normal subgroups with trivial intersection:

\[\tilde{\Gamma} \triangleright N_1 \triangleright N_2 \triangleright N_3 \triangleright \ldots, \bigcap_{i=1}^{\infty} N_i = \{1\}.\]

Then the 3-colored graph \(K_i\) is the Cayley-graph of the finite group \(\tilde{\Gamma}/N_i\). Note that \(\{K_i\}_{i=1}^{\infty}\) is a large girth sequence, that is, for any \(s > 0\), if \(i\) is large enough, then \(K_i\) does not contain cycles of length less than \(s\). Finally, we lexicographically enumerate the words in \(\Gamma\): \(A, B, C, D, AB, AC, AD, BA, \ldots\) and denote the \(n\)-th word by \(\omega_n = w_{k_n}^{(n)}w_{k_{n-1}}^{(n)}\cdots w_2^{(n)}w_1^{(n)}\); note that the word-length \(k_n\) of \(\omega_n\) is at most \(n\).

**Step (1)** Let \(G_0(V_0, E_0)\) be the graph consisting of one single vertex \(p\). Let \(G_1(V_1, E_1)\) be the graph constructed the following way. \(H_1\) is a triangle, edge-colored by \(A, B\) and \(C\). The vertex \(p\) is connected to a vertex \(q\) of \(H_1\) by an edge colored by \(D\). Finally a vertex of \(H_1\) which is not \(q\) shall be called \(r_1\). Thus we have graphs \(G_0 \subset G_1\). In the \(n\)-th step we shall have graphs \(G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n\). Finally let us set \(s_1 := 2m\), \(s_2 = m2^{[V(G_1)]}\).

**Step (2)** Let \(H_2\) be a graph \(C_i\), in such a way that \(\text{diam} \, C_i > 10s_2\). By Lemma 4.2 there exists a partition

\[V(H_2) = R_1^2 \coprod R_2^3 \coprod R_3^2\]

where \(R_1^2\) is a \(2s_1\)-net, \(R_2^3\) is a \(2s_2\)-net satisfying the properties described in the lemma. Connect each point of \(R_1^2\) to a copy of \(G_0\) (a single vertex) with an edge colored by \(D\). For one single vertex of \(R_2^3\) connect a copy of \(G_1\) (at the vertex \(r_1\) !) with an edge colored by \(D\). Let \(x_2\) be a vertex of \(R_3^2\). Consider the first word \(\omega_1 = A\). Let \(y_0^2, y_1^2\) be two new vertices (i.e. distinct from any other vertex involved in the construction till now) and connect \(x_2\) to \(y_0^2\) by an edge colored by \(D\) and connect \(y_0^2\) to \(y_1^2\) by an edge colored by \(\omega_1 = A\). Finally pick an other vertex of \(R_3^2\) and call it \(r_2\).
Thus we constructed 4-colored graphs $G_0 \subset G_1 \subset G_2$. In $G_2$ we have a spanned subgraph $H_2$ such that

$$\frac{|V(G_2)\setminus V(H_2)|}{|V(G_2)|} < \frac{1}{m}.$$ 

Set $s_3 := m^{3/2 |V(G_2)|}$.

Indeed, $|V(G_2)| = 2|R^2_1| + |R^2_2| + |R^2_3| + 2$ (the last term corresponds to $|\{y_0^2, y_1^2\}|$) and

$$\frac{|V(G_2)\setminus V(H_2)|}{|V(G_2)|} = \frac{|R^2_1| + 2}{|V(G_2)|} \leq \frac{|R^2_1| + 2}{|V(H_2)|} \leq \frac{1}{2m} + \frac{1}{10m^22^4} < 1/m.$$

**Step (n)** Suppose that we have already constructed the graphs $G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_{n-1}$ with the following properties:

- If $i > 1$, then $G_i$ contains a spanned subgraph $H_i$ such that $\frac{|V(G_i)\setminus V(H_i)|}{|V(G_i)|} < \frac{1}{m}$.
- $s_i = m^{i/2 |V(G_{i-1})|}$.
- For each $H_i$, we fixed a vertex $r_i$.

Now if $n$ is even, then let $H_n$ be isomorphic to $C_i$ for some $i$, if $n$ is odd, then let $H_n$ be isomorphic to $K_i$ for some $i$, satisfying the inequality $diam(H_n) \geq 10s_n$. We apply Lemma 4.2 again.

$$V(H_n) = R^n_1 \bigsqcup R^n_2 \bigsqcup \ldots \bigsqcup R^n_{n+1},$$

where $R^n_i$ is a $2s_i$-net and for each $x \in V(H_n)$ there exists $p \in R^n_i$ such that

$$d_{H_n}(p, x) < 10s_i.$$ 

For $i = 1, 2, \ldots, n-1$ we connect a copy of $G_{i-1}$ (at vertex $r_i$) to each vertex of $R^n_i$ by an edge colored by $D$. Then we connect one single copy of $G_{n-1}$ to $H_n$ by an edge colored by $D$ between $r_{n-1}$ and an arbitrary vertex in $R^n_{n+1}$. Now we pick a vertex $x_n$ of $R^n_{n+1}$. Consider the $n$th word $w = w_n^{(n)} w_{n-1}^{(n)} \ldots w_1^{(n)}$.

- If $w_1^{(n)} = D$, then choose vertices $y_1^n, y_2^n, \ldots, y_{k_n}^n$ disjoint from any previous vertices and connect the path $(y_{11}^n, y_{12}^n, \ldots, y_{k_n}^n)$ to $x_n$ by the edge $(x_n, y_1^n)$ colored by $D$ and let color the edge $(y_{i1}^n, y_{i1}^n)$ by $w_{i1}^{(n)}$. 

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If $w_1^{(n)} \neq D$, then choose vertices $y_0^n, y_1^n, y_2^n, \ldots, y_k^n$ disjoint from any previous vertices and connect the path $(y_0^n, y_1^n, \ldots, y_k^n)$ to $x_n$ by the edge $(x_n, y_1^n)$ colored by $D$ and let color the edge $(y_i^n, y_{i+1}^n)$ by $w_{i+1}^n$.

Finally pick an other vertex $r_n$ form $R_{n+1}$.

Thus we constructed a sequence of graphs $G_0 \subset G_1 \subset \ldots \subset G_n$. We proceed by induction. Denote the 4-colored graph $\bigcup_{n=1}^{\infty} G_n$ by $G$. In the next section we prove that $G$ is generic, has dense holonomy and carries a faithful, transitive, amenable action of the group $\Gamma$. 
6 The graph $G$

In this section we check that the graph $G$ constructed in the previous section has indeed all the required properties.

**Proposition 6.1** The action of $\Gamma$ is faithful, transitive and (strongly) amenable.

*Proof.* Transitivity is obvious from the connectivity of $G$. Faithfulness was ensured by the attachment of paths $(y_1^n, y_2^n, \ldots, y_{k_n}^n)$. The subgraphs $\{G_n\}_{n=1}^{\infty}$ are forming an increasing and exhausting Følner-sequence, since $\partial G_n = \{r_n\}$. 

**Proposition 6.2** The 4-colored graph $G$ is generic.

*Proof.* First we need a lemma.

**Lemma 6.1** The single vertex $p$ of $G_0$ has the property that for any $q \in V(G)$, $q \neq p$, there exists $r$ such that $B_r(p)$ and $B_r(q)$ are not isomorphic (as colored graphs, in other words $\alpha_r(p) \neq \alpha_r(q)$).

*Proof.* Let $q \in V(G)$ be a vertex of degree one (otherwise the statement trivially holds). If $q$ was defined in Step (k), then there exists a spanned connected subgraph of $G$ which is isomorphic to $H_k$ such that the unique path from $q$ to the subgraph does not contain any vertex of a subgraph isomorphic to $H_{k-1}$. Indeed, $H_k$ itself can be chosen as the spanned subgraph. On the other hand any path connecting $p$ to a spanned subgraph isomorphic to $H_k$ passes through such a vertex. This proves our lemma.

Now we finish the proof of Proposition 6.2. Let $x, y$ be two distinct vertices of $G$. By transitivity of the action there exists $\overline{w} \in \Gamma$ such that $\Phi(\overline{w})(x) = p$ (the single vertex of $G_0$). Set $q = \Phi(\overline{w})(y)$. By the previous lemma, there exists $r$ such that $B_r(p)$ is not colored-isomorphic to $B_r(q)$. Then $B_{r+|\overline{w}|}(x)$ is not colored-isomorphic to $B_{r+|\overline{w}|}(y)$, where $|\overline{w}|$ is the word-length of $\overline{w}$.

**Proposition 6.3** The 4-colored graph $G$ is of dense holonomy.

*Proof.* Let $t \in V(G)$ and $r > 0$. Suppose that $B_r(t) \subset G_i$. Let $M = \text{diam}(G_{i+3})$. 

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Lemma 6.2 For any \( x \in V(G) \), the ball \( B_M(x) \) contains a vertex \( y \) belonging to a spanned subgraph \( L \) isomorphic to \( H_j \) for some \( j \geq i + 3 \).

Proof. If \( x \in V(G_{i+2}) \) then the statement obviously holds. We proceed by induction. Suppose that the statement holds for any \( x \in V(G_l) \). Let \( x \in V(G_{l+1}) \setminus V(G_l) \). If \( x \in H_{l+1} \) then \( L \) can be chosen as \( H_{l+1} \) itself. If \( x \) is attached to \( H_{l+1} \) then we have two cases:

- \( B_M(x) \) intersects \( H_{l+1} \): then the statement is clearly true.
- \( B_M(x) \) does not intersect \( H_{l+1} \): then there exists a vertex \( x' \in G_l \) such that \( B_M(x) \) and \( B_M(x') \) are isomorphic, hence the statement follows by induction.

Now if \( y \in L, L \simeq H_j, j \geq i + 3 \). Then in a \((10s_{i+3})\)-neighborhood of \( y \) we attached a copy of \( G_{i+3} \). Hence \( B_{M+10s_{i+3}}(y) \) contains a vertex of the same \( r \)-type as \( t \). Therefore, \( B_{2M+10s_{i+3}}(x) \) contains a vertex of the same \( r \)-type as the vertex \( t \), in other words (recalling the definition of dense holonomy) if \( \alpha_r(t) = \alpha \), then \( m_\alpha = 2M+10s_{i+3} \), and the proposition follows.

7. Graphed equivalence relations

Let us recall the notion of Borel-equivalence relations and their \( L \)-graphings from the book of Kechris and Miller ([6], Section 17). Let \( Z \) be a standard Borel-space. Let \( \{A_i\}_{i=1}^\infty \) and \( \{B_i\}_{i=1}^\infty \) be two families of Borel subsets of \( Z \) and \( \phi_i : A_i \rightarrow B_i \) be partial Borel-isomorphisms. Then the system \( \{\phi_i\}_{i=1}^\infty \) defines a countable Borel-equivalence relation \( E \) on \( Z \) in the following way: for \( x, y \in Z \) we set \( x \equiv_E y \) if there exists a finite sequence \( x_1, x_2, \ldots, x_n \in Z \) such that

- \( x_1 = x, x_n = y \) and
- for \( i = 1, 2, \ldots, n \) one has \( x_{i+1} = \phi_j^{\pm 1}(x_i) \) for some \( j \in \mathbb{N} \).

This also means that for each \( z \in Z \) the system \( \{\phi_i\}_{i=1}^\infty \) defines a \( L \)-graphing \( G \), that is a graph structure on the equivalence classes on \( E \). The simplest examples of Borel-equivalence relations are topological actions of finitely generated groups on compact metric

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spaces: for \(x, y \in Z\), \(x \equiv_E y\) if there exists \(\gamma \in \Gamma\) such that \(x = \gamma(y)\). Moreover, if the group \(\Gamma\) is finitely generated by a symmetric generating system \(S\), then \(x\) is connected to \(z\) if \(x = s(z)\) for some \(s \in S\).

Let \(\mu\) be a Borel probability measure on \(Z\) preserved by the maps \(\{\phi_i\}_{i=1}^\infty\) e.g. an invariant measure of a \(\Gamma\)-action. Then we call \(\mu\) an \(E\)-invariant measure. Note that \(E\)-invariance does not depend on the choice of the \(L\)-graphing \(\mathcal{G}\) defining \(E\). A measurable equivalence relation is given by a triple \((Z, E, \mu)\), where \(Z\) is a standard space, \(E\) is a countable Borel-equivalence relation and \(\mu\) is an \(E\)-invariant measure \(\square\). The measurable equivalence relations \((Z_1, E_1, \mu_1)\) and \((Z_2, E_2, \mu_2)\) are orbit-equivalent if there exists a measure preserving map \(T : Z_1 \to Z_2\) such that for almost all \(z \in Z_1\) the image of the class of \(z\) is just the equivalence class of \(T(z)\). Let \(\phi : Z \to Z\) be a single homeomorphism with an ergodic measure \(\mu\). The orbit equivalence class of the relation \(E_\phi\) is called the hyperfinite ergodic relation. The ergodic actions of amenable groups always define the hyperfinite relation. In \(\text{[3]}\), the authors constructed minimal actions of certain non-amenable groups with ergodic measure still defining the hyperfinite relation. For an \(L\)-graphing \(\mathcal{G}\) and an \(E\)-invariant measure \(\mu\) the edge measure of \(\mathcal{G}\) is defined as

\[
e(\mathcal{G}, \mu) = \frac{1}{2} \int_Z \deg_{\mathcal{G}}(x) d\mu(x),
\]

where \(\deg_{\mathcal{G}}(x)\) is the degree of \(x\) in the \(L\)-graphing. The cost of the equivalence relation is defined as

\[
c_\mu(E) := \inf_{\mathcal{G}} e(\mathcal{G}, \mu),
\]

where the infimum is taken over all \(L\)-graphings defining \(E\). The cost of the hyperfinite ergodic relation is 1. Obviously, orbit equivalent relations have the same cost. Suppose that \(\tilde{\Phi}\) is the topological action of the finitely generated group \(\Gamma\) on the compact metric space \(X_G\), preserving the measure \(\mu\). Then the \(L\)-graphing of \(\tilde{\Phi}\) is just the union of the Schreier-graphs of the \(\Gamma\)-orbits. Kaimanovich \(\text{[5]}\) proved that if \(\mu\) is an ergodic invariant measure for the action then the hyperfiniteness of the induced relation implies that all the leaves (the Schreier-graphs) are amenable.

**Proposition 7.1** Let \((\tilde{\Phi}, \Gamma, X_G)\) be the associated minimal action given in Section \(\text{[2]}\) using or main construction. Then all the leaves are amenable.
Proof. First note that the action $\tilde{\Phi}$ is regular on $X_{G}$, that is if $p \in X_{G}$, $\gamma \in \Gamma$ then either $\tilde{\Phi}(\gamma)(p) \neq p$ or the homeomorphism $\tilde{\Phi}(\gamma)$ fixes a whole neighborhood of $p$. Indeed let $p = \{\alpha_{1}, < \alpha_{2} < \ldots < \alpha_{s} < \ldots\}$ and let the word-length of $\gamma$ be $s$. If $\tilde{\Phi}(\gamma)(p) = p$, then $\tilde{\Phi}(\gamma)$ also fixes all $q \in X_{G}$ of the form $q = \{\alpha_{1}, < \alpha_{2} < \ldots < \alpha_{s} < \beta_{s+1} < \ldots\}$.

We say that two 4-colored graphs $G_{1}, G_{2}$ are locally isomorphic if $U_{G_{1}}^{r} = U_{G_{2}}^{r}$ for any $r > 0$. Clearly if $G_{1}$ is amenable and of dense holonomy and $G_{2}$ is locally isomorphic to $G_{1}$, then $G_{2}$ is amenable and of dense holonomy as well [4]. Since our graph $G$ is generic, the dense leaf $\pi(G) \subset X_{G}$ is isomorphic to $G$. By minimality, regularity and dense holonomy, for any two orbits the corresponding Schreier-graphs are locally isomorphic (see also [3]) and thus our proposition follows.

As we shall see later, our $\Gamma$-action $\tilde{\Phi}$ defines a non-hyperfinite measurable equivalence relation with respect to some ergodic invariant measure. The existence of non-hyperfinite relations with amenable leaves was first observed by Kaimanovich [5].

8 The invariant measures

Notice that in our main construction we have a parameter $m$ and we have in fact constructed a graph $G^{(m)}$ for each $m > 10$. By Lemma 4.4

$$V(H_{n}^{(m)}) = R_{1}^{n,(m)} \bigg( \prod R_{2}^{n,(m)} \bigg) \ldots \bigg( \prod R_{n+1}^{n,(m)} \bigg),$$

where

$$\frac{|R_{i}^{n,(m)}|}{|V(H_{n}^{(m)})|} < \frac{1}{m^{2}|V(G_{i-1}^{(m)})|}$$

for $i = 1, 2, \ldots, n$ so that

$$\frac{|R_{n+1}^{n,(m)}|}{|V(H_{n}^{(m)})|} > 1 - \frac{1}{m}.$$ 

Here the upper index $(m)$ indicates that the parameter is chosen to be $m$. For each vertex of $R_{i}^{n}$ we attached a copy of a graph of size $|V(G_{i-1}^{(m)})|$. We also attached a path of length at most $n$ to one vertex of $R_{n+1}^{n,(m)}$. Thus we have the following inequality:

$$\frac{|V(G_{n}^{(m)}) \setminus V(H_{n}^{(m)})|}{|V(H_{n}^{(m)})|} < \frac{1}{m}. \quad (4)$$
Indeed $|V(G^{(m)}_n)| \leq 2|R^{1,(m)}_n| + |V(G^{(m)}_1)| \cdot |R^{2,(m)}_n| + \cdots + |V(G^{(m)}_{n-1})| \cdot |R^{n,(m)}_n| + n$ where the last term majorizes $k_n$, the length of the $n$-th word $w$. Thus,

$$\frac{|V(G^{(m)}_n) \setminus V(H^{(m)}_n)|}{|V(H^{(m)}_n)|} \leq \sum_{i=1}^{n} \frac{|V(G^{(m)}_{i-1})| - 1}{m^2|V(G^{(m)}_{i-1})|} < \frac{1}{m}.$$ 

Let $\Omega^{(m)}_n := V(G^{(m)}_n) \setminus V(G^{(m)}_{n-1})$. Clearly $|\partial \Omega^{(m)}_n| = 2$, thus $\{\Omega^{(m)}_n\}_{n=1}^\infty$ forms a Følner-sequence in $G^{(m)}$. We consider two Følner-sequences for $G^{(m)}$: $\{F^{1,(m)}_n\}_{n=1}^\infty$ and $\{F^{2,(m)}_n\}_{n=1}^\infty$, where $F^{1,(m)}_n = \Omega^{(m)}_{2n}$ and $F^{2,(m)}_n = \Omega^{(m)}_{2n+1}$. By the averaging process described in Section 2, we then obtain two invariant measures on $X_G$, $\mu^{(m)}_1$ and $\mu^{(m)}_2$, respectively. Thus, for each $m > 10$ we have

- A compact metric space $X_{G^{(m)}}$.
- A minimal $\Gamma$-action $\tilde{\Phi}^{(m)}$.
- An $L$-graphing $G^{(m)}$ associated with the action.
- A Borel-equivalence relation $E^{(m)}$ defined by $G^{(m)}$.
- Two $E^{(m)}$-invariant measures: $\mu^{(m)}_1$ and $\mu^{(m)}_2$.

**Proposition 8.1** $\lim_{m \to \infty} e(G^{(m)}, \mu^{(m)}_1) = 1$.

**Proof.** Let $Y^{(m)} \subset X_{G^{(m)}}$ be the set of points $x$ such that $\deg G^{(m)}(x) > 2$. It is enough to prove that $\lim_{m \to \infty} \mu^{(m)}_1(Y^{(m)}) = 0$. By the definition of the probability measure $\mu^{(m)}_1$,

$$\mu^{(m)}_1(Y^{(m)}) = \lim_{\omega} \frac{|L^{(m)}_n|}{|F^{1,(m)}_n|},$$

where $L^{(m)}_n$ is the set of vertices in $\Omega^{(m)}_{2n}$ having degree larger than 2. By (4):

$$|L^{(m)}_n| \leq \frac{2}{m} |V(H^{(m)}_{2n})| \leq \frac{2}{m} |F^{1,(m)}_n|. \quad \square$$

9 Lower estimate for the cost

A point $x \in X_{G^{(m)}}$ is called $\tilde{\Gamma}$-free if for any $\gamma \in \tilde{\Gamma} \setminus \{1\}$, $\tilde{\Phi}(\gamma)(x) \neq x$. Clearly the set of $\tilde{\Gamma}$-free points $A^{(m)} \subset X_{G^{(m)}}$ is closed.
Lemma 9.1 \( \lim_{m \to \infty} \mu_2^{(m)}(A^{(m)}) = 1. \)

**Proof.** Let \( A_k^{(m)} \) be the set of points in \( X_{G^{(m)}} \) for which \( \tilde{\Phi}(\gamma)(x) \neq x \) whenever the word-length of \( \gamma \in \tilde{\Gamma} \) is not greater than \( k \). Clearly, \( \cap_{k=1}^{\infty} A_k^{(m)} = A^{(m)} \). Observe that

\[
\mu_2^{(m)}(A_k^{(m)}) = \lim_{n \to \infty} \frac{|W_{n,k}^{(m)}|}{|F_{n}^{2,(m)}|},
\]

where \( W_{n,k}^{(m)} \) is the set of vertices \( p \) in \( F_{n}^{2,(m)} \) such that \( \Phi(\gamma)(p) \neq p \), if \( |\gamma| \leq k \). Obviously, if \( p \in H_n^{(m)} \) and \( n \) is a large odd number, then \( p \in W_{n,k}^{(m)} \). Thus \( \mu_2^{(m)}(A_k^{(m)}) \geq 1 - \frac{2}{m} \). Therefore \( \mu_2^{(m)}(A^{(m)}) \geq 1 - \frac{2}{m} \) as well. \( \square \)

Let \( E \) be a Borel-equivalence relation on a compact metric space \( Z \) and \( \mu \) be an \( E \)-invariant measure. Let \( Y \subseteq Z \) be a Borel-set of positive measure. Then \( (E \mid Y) \) is the induced equivalence relation and \( \mu_Y \) is the restriction of \( \mu \) onto \( Y \). We call \( Y \) a **complete** Borel-section if \( Y \) intersects almost every classes of \( E \). By a lemma of Gaboriau ([6], Theorem 21.1) if \( Y \) is a complete section then

\[
c_\mu(E) = c_{\mu_Y}(E \mid Y) + \mu(Z \setminus Y) \quad (5)
\]

**Lemma 9.2** The induced equivalence relation \( (E \mid A^{(m)}) \) on the set of \( \tilde{\Gamma} \)-free points is exactly the relation given by the \( \tilde{\Gamma} \)-action.

**Proof.** Notice that in the graph \( G^{(m)} \) there is no simple cycle containing an edge colored by \( D \). Hence the same holds for the orbits in \( X_{G^{(m)}} \). Thus if \( p, q \in A^{(m)} \) and \( \gamma \) contains the symbol \( D \), then \( \tilde{\Phi}(\gamma)(p) \neq q \). \( \square \)

**Lemma 9.3**

\[
c_{\mu_2^{(m)}}(E \mid A^{(m)}) = \frac{3}{2} \mu_2^{(m)}(A^{(m)}) .
\]

**Proof.** The lemma is a simple consequence of the famous theorem of Gaboriau ([11] (see also [6]) stating that the cost of an equivalence relation defined by an \( L \)-treeing \( G \) is exactly the edge measure of \( G \). Simply note that the edge measure of a free \( Z_2 * Z_2 * Z_2 \)- action on a compact metric space with an invariant measure is \( \frac{3}{2} \) times the measure of the space. \( \square \)
Lemma 9.4 $A^{(m)}$ is a complete section of $E$.

Proof. We need to prove that the set of points $p \in X_{G^{(m)}}$, such that there exists no $\gamma \in \Gamma$ with $\tilde{\Phi}(\gamma)(p) \in A^{(m)}$ has $\mu_2^{(m)}$-measure zero. We denote this set by $B^{(m)}$.

Let $S_{k,n} \subset F_n^{2^{(m)}}$ be the set of vertices $p$ such that $d_{G_n}(p, H_{2n+1}^{(m)}) \leq k$. By our construction of $G^{(m)}$ it is clear that for any $\epsilon > 0$ there exists $k_\epsilon$ such that

$$\frac{|S_{k_\epsilon,n}|}{|F_n^{2^{(m)}}|} \geq 1 - \epsilon.$$

Therefore $\mu_2^{(m)}(\Theta_{k_\epsilon}) \geq 1 - \epsilon$, where $\Theta_{k_\epsilon}$ is the set of points $x \in X_{G^{(m)}}$ for which there exists $\delta \in \Gamma$ with $|\delta| \leq k_\epsilon$ such that $\tilde{\Phi}(\delta)(x) \in A^{(m)}$. Thus $\mu_2^{(m)}(B^{(m)}) = 0$.

The following proposition is the straightforward corollary of the above lemmas.

Proposition 9.1 $\lim_{m \to \infty} c^{(m)}_{\mu_2^{(m)}}(E) = \frac{3}{2}$.

10 The main theorem

Finally we state and prove our main theorem.

Theorem 1 There exists a minimal topological action $\tilde{\Phi}$ of the group $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ on a compact metric space $Z$ for which there exist ergodic invariant measures $\mu_1$ and $\mu_2$ such that the measurable equivalence relations given by the systems $(\tilde{\Phi}, Z, \mu_1)$ and $(\tilde{\Phi}, Z, \mu_2)$ are not orbit equivalent.

Proof. By Propositions 8.1 and 9.1, for large enough $m$, the costs of the equivalence relations given by $(\tilde{\Phi}^{(m)}, X_{G^{(m)}}, \mu_1^{(m)})$ and $(\tilde{\Phi}^{(m)}, X_{G^{(m)}}, \mu_2^{(m)})$ are different. By the ergodic decomposition theorem [6, Corollary 18.6], if $E$ is a Borel-equivalence relation and $\mu$ is an $E$-invariant measure then

$$c_\mu(E) = \int_{\varepsilon I_E} c_\varepsilon(E) d\nu_\mu(e),$$

where $\varepsilon I_E$ is the space of ergodic $E$-invariant measures and $\mu = \int \varepsilon d\nu_\mu(e)$, where $\nu_\mu$ is a probability measure. Hence if $c_{\mu_1^{(m)}}(E) \neq c_{\mu_2^{(m)}}(E)$ for some invariant measures $\mu_1^{(m)}$ and $\mu_2^{(m)}$, then there exist ergodic invariant measures $\mu_1$ and $\mu_2$ for which $c_{\mu_1}(E) \neq c_{\mu_2}(E)$. Therefore our theorem follows.
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