Non-integrability of a system with the Dyson Potential

Georgi Georgiev

Abstract

In this paper it is shown that the Hamiltonian system with Dyson potential is analytical non-integrable and formal non-integrable. The approach is based on the following: when a system has a family of periodic solutions around an equilibrium and if the period function is infinitely branched then the system has non additional analytic first integral. We prove formal non-integrability using Ziglin-Moralez-Ruiz-Ramis theory.

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We study the Hamiltonian system of \( n \) interacting particles of equal mass with a Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{n} y_i^2 + \sum_{i<j} V(x_i - x_j), \quad V(x) = -\log |\sin x|.
\]

(0.1)

Here \( x_1, x_2, \ldots x_n \) are the coordinates of particles, \( y_1, y_2, \ldots y_n \) are their momenta, and \( V \) is the potential energy in Dyson type. The system with this potential was studied in [1], where the statistical properties of the energy levels of one dimensional Coulomb’s gas are investigated. There is a connection between system of point vortices and regarding system in [2] by Calogero and Perelomov. The equilibria position of the system (0.1), \( x_k^0 = x_0 + \frac{\pi k}{n}, k = 1, 2, \ldots n, x_0 \in \mathbb{R} \) determines stationary collinear choreography on the sphere-point vortices are located in the equatorial plane that uniformly rotates around an axis lying in this plane.

The function \( V \) is 2\( \pi \)-periodic (and even \( \pi \)-periodic), so we can assume that the particles move in circles. The Hamiltonian system always has two integrals

\[
H, \quad F = \sum_{i=1}^{n} y_i.
\]

The question of integrability of (0.1) is studied by Calogero and Perelomov [2]. If \( V \) is a non-constant analytic periodic function without singularities, then the system (0.1) can not be integrable for \( n \geq 3 \) (Kozlov [3]). Unfortunately Dyson potential has a real logarithmic singularity. Borisov and Kozlov in [4] had proved that the system in case \( n = 3 \) is non-integrable in analytical first integrals. We consider the same case \( n = 3 \), but the approach is
different, this is the first non-trivial case and we prove that the system admits only $F$ as a holomorphic first integral in a complex domain.

Let us make in (0.1) (in our case $n = 3$) canonical change of variables

\begin{align*}
y_1 &= p_1 + p_3, \quad y_2 = -p_1 + p_2 + p_3, \quad y_3 = -p_2 + p_3, \\
q_1 &= x_1 - x_2, \quad q_2 = x_2 - x_3, \quad q_3 = x_1 + x_2 + x_3,
\end{align*}

and we obtain

\begin{equation}
H = p_1^2 - p_1 p_2 + p_2^2 + \frac{3}{2} p_3^2 - \log |\sin q_1| - \log |\sin q_2| - \log |\sin (q_1 + q_2)|. \tag{0.2}
\end{equation}

The system has another integral

\begin{equation}
F = 3 p_3 \tag{0.3}
\end{equation}

and has a stable equilibrium $p_1 = p_2 = 0, \quad q_1 = q_2 = \frac{\pi}{3}$.

The variable $q_3$ is cyclic and it reduces the system (with $p_3 = \text{const}$) to a system with 2-degrees of freedom with Hamiltonian

\begin{equation}
H_{\text{reg}} = p_1^2 - p_1 p_2 + p_2^2 - \log |\sin q_1| - \log |\sin q_2| - \log |\sin (q_1 + q_2)|. \tag{0.4}
\end{equation}

We need some basic definitions about Hamiltonian systems.

Let $H$ is a smooth real-valued function of $2n$ real variables $(p, q), \quad p, q \in \mathbb{R}^n$. Let us also assume that $dH(0) = 0$ where $0$ is an equilibrium point for the Hamiltonian system $X_H$ (with $n$ degrees of freedom), given by

\begin{equation}
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.
\end{equation}

We often write the Hamiltonian systems in the form

\begin{equation}
\dot{x} = X_H(x), \quad x \in \mathbb{R}^{2n},
\end{equation}

where $X_H$ is the flow. The system is called Liouville - Integrable near 0 if there exists $n$ functions in involution $f_1 = H, f_2, \ldots f_n$, defined around 0, are functionally independent. The Poisson bracket of $f$ and $g$ are

\begin{equation}
\{f, g\} = X_f(g) = \sum \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} = -\{g, f\}.
\end{equation}

We call that functions $f$ and $g$ are in involution if the Poisson bracket is commutative. This means that $df_1, df_2, \ldots df_n$ are linearly independent around the equilibrium 0 and $f_j = \text{const}$ for all $j$ define smooth submanifolds, these level manifolds are invariant under $X_{f_j}$. We have $X_{f_j} f_j = 0$ and $[X_{f_j}, X_{f_k}] = X_{\{f_j, f_k\}} = 0$ - these flows commute. The compact and connected component of $M_c := \{f_j = c_j, \quad j = 1, \ldots, n\}$ is diffeomorphed to a torus.
We call that the system with Hamiltonian $H$ formally integrable if there exist formal power series $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$ in involution, where $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$ are functionally independent and Taylor expansion $\tilde{H}$ of $H$ is a formal power series in $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$. An asymptotic behavior near equilibrium is like an integrable system. The formal integrability gives information about the flow. The functional independence of $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$ is stronger than the independence of smooth functions $f_1, f_2, \ldots, f_n$ of which $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n$ are the Taylor series, because formal independence leads to a functional independence of a finite part of Taylor expansions.

Here we study the formal non-integrability of the system (0.1) in case $n = 3$.

Our first aim is the following

**Theorem 1.** a) The system with Hamiltonian (0.4) is not integrable by means of analytical first integral;

b) The system with Hamiltonian (0.4) is not formal integrable.

The motivation for proving formal non-integrability I received from the remarkable paper of J. J. Duistermaat [5]. The Hamiltonian system with Dyson potential has similar structure. The proof of a) is different from [4].

**Proof a):** The proof of this is based on three propositions. The first - shows that there exists family of periodical solutions near equilibrium. The second proposition investigates the behavior on period function of these periodical solutions around the equilibrium. The third - proves that if we assume an existence of an additional first integral- it is a constant.

**Proposition 1.** On the manifold $P := \{ p_1 = p_2, q_1 = q_2 \}$, invariant under $X_H$ the system with Hamiltonian (0.4) exists a family of periodical solutions around equilibrium $p_1 = p_2 = 0$, $q_1 = q_2 = \frac{\pi}{3}$.

**Proof:** Let we take $q(t) := q_1(t) = q_2(t)$, $p(t) := p_1(t) = p_2(t)$ and $\frac{dq(t)}{dt} = p(t)$ in our system, then

$$\dot{p} = \cot q + \cot 2q,$$

and we have

$$(\dot{q})^2 = \log(\sin 2q) + 2 \log(\sin q) + E.$$ 

This is a conservative system with convex potential

$$\tilde{V}(q) = -\log(\sin 2q) - 2 \log(\sin q),$$

and we know that for these predictions the system has a periodical solution around the equilibrium for fixed energy $H_{reg} = E$ (see Figure 1).
Proposition 2. The period function has expression \( T(c) = \log \eta(c) + \Phi(c) \), where \( c = \frac{1}{2eE} \).

\[ \eta(c) = \epsilon(B(c))\delta(B(c)) \]

with

\[ B(c) = \frac{16\left(\frac{2}{3}\right)^{1/3}c^2}{(9c^2 - \sqrt{3\sqrt{27e^4 - 256c^6}})^{1/3}} + 2\left(\frac{2}{3}\right)^{1/3}(9c^2 - \sqrt{3\sqrt{27e^4 - 256c^6}})^{1/3}. \]  

(0.6)

\( \Phi(c) \) is an analytical function of the variable \( c \) and

\[ r_{1,2} = \frac{1}{2} - \frac{1}{2} \sqrt{1 + B} \pm \frac{1}{2} \sqrt{2 - B + \frac{2}{\sqrt{1 + B}}}. \]  

(0.7)

\[ \epsilon = \frac{1}{2}(\arccos r_1 - \frac{\pi}{3}), \quad \delta = \frac{1}{2}\left(\frac{\pi}{3} - \arccos r_2\right). \]

Proof: The period in this solution is

\[ T = 2 \int_{q_-}^{q_+} \frac{dq}{\sqrt{E - \tilde{V}(q) \cdot \pi}}. \]  

(0.8)

where \( q_- \) and \( q_+ \) are the roots of \( E - \tilde{V}(q) = 0 \), where \( \tilde{V}(q) \) is [0.3].
For equilibrium point \( q = \frac{\pi}{3} \) we have \( \dot{V}(\frac{\pi}{3}) = -3\log(\frac{\sqrt{3}}{2}) \) and let we fix \( q_+ = \frac{\pi}{3} + \epsilon \) and \( q_- = \frac{\pi}{3} - \delta \), for \( \epsilon > 0 \) and \( \delta > 0 \). We have

\[
A = \frac{2}{\sqrt{E - \dot{V}(q)}} = \frac{2}{\sqrt{-3\log(\frac{\sqrt{3}}{2}) + \log(\sin 2q) + 2\log(\sin q)}},
\]

The expansion of \( A \) near \( \frac{\pi}{3} \) is

\[
A = \frac{1}{|q - \frac{\pi}{3}|} - \frac{1}{3\sqrt{3}} - \frac{2}{9}|q - \frac{\pi}{3}| + O(|q - \frac{\pi}{3}|^2).
\]

For the period we obtain

\[
T = \int_{\frac{\pi}{3} - \delta}^{\frac{\pi}{3} + \epsilon} dq = \int_{\frac{\pi}{3} - \delta}^{\frac{\pi}{3} + \epsilon} \left( \frac{1}{|q - \frac{\pi}{3}|} - \frac{1}{3\sqrt{3}} - \frac{2}{9}|q - \frac{\pi}{3}| + \ldots \right) dq,
\]

where \( \eta = e\delta \) and \( \Phi(\epsilon, \delta) = \log(\epsilon \delta) + \log(\Phi(\epsilon, \delta)) = \log(\epsilon) + \log(\Phi(\epsilon, \delta)) \). We find \( \epsilon \) and \( \delta \). We have

\[
E = \dot{V}(q) = -\log(\sin 2q) - 2\log(\sin q) = -\log(2.(\sin q)^3.\cos q),
\]

and if we get \( q_+ = \frac{\pi}{3} + \epsilon \) ( or \( q_- = \frac{\pi}{3} - \delta \), then \( - (\sin(\frac{\pi}{3} + \epsilon))^3.\cos(\frac{\pi}{3} + \epsilon) = \frac{1}{2e^2} = c. \) If we put \( r = \cos(\frac{2\pi}{3} + 2\epsilon) \), we obtain equation \((1 - r)^2(1 - r^2) = 16c^2 \). The real roots of this equation are \((0.7) \) with \((0.6) \). We find \( \epsilon = \frac{1}{2}(arccos r_1 - \frac{\pi}{3}) \), \( \delta = \frac{1}{2}(\frac{\pi}{3} - arccos r_2) \).

**Proposition 3.** The system with Hamiltonian (0.4) does not possess any additional holomorphic first integral.

**Proof:** It is important for the proof that there is a family of solutions on \( P \), of the Hamiltonian system (0.4) on the hypersurface \( H = E \) are periodic. If \( T \) is the period function then complex continuation of the manifolds \( T = \text{const} \) turns out to be infinitely branched. This excludes the existence of a nontrivial analytic integral on any open subset of the complex domain where this infinite branching is true. Further, we need to show that if \( G \) is a smooth function on open set \( U \) such that \( V = U \cap (H = E) \) is \( X_H \) invariant, \( \{H, G\} = 0 \) and derivative \( d\{H, G\} = 0 \) on \( V \), then \( G \) is a function of \( H \) and \( T \) on \( V \). The function \( T(x), x \in \bar{U} \subset U \) is real analytical and it is not constant \( P \), it means that \( dT \neq 0 \) on an open dense subset \( \Delta \) of the manifold \( P \), \( G \) is invariant under flows \( X_H \) and \( X_T \). We have \( X_G T = 0 \) on \( V \). Further
we regard as Hamiltonian flow on $P$ of any smooth extension $\tilde{T}$ on $P$ on open neighborhood of $P$, the flow is independent of the choice of the extension. On the open set $\Delta \cap V$, $dT$ and $dH$ are linearly independent that is why the flows $X_H$ and $X_T$ walk around one-dimensional submanifold $P$, $H = E$ and $T = const$. Therefore on $\Delta \cap V$, $G$ is locally constant on $H = E$, and $T = const$. That is $G$ function of $H$ and $T$ on the connected component of $\Delta \cap U \cap P$.

Let suppose that $G$ is analytic on $U$ and it has complex analytic extension on $\tilde{U}$ of $U$. If $G$ is not functionally depended on $H$ on $P$ then the manifold $P$ and $G = const$ extend to closed complex analytic manifold on $\tilde{U}$, which coincides with complex analytic continuations of $H = E$ and $T = const$. If the analytic continuations have infinite branching near $x \in U$, then we have a contradiction, so $G$ has to be a function of $H$ on $P$. Now we use $H$ as a coordinate near $x$, we can write $G = G_0 + G_1 H$, $G_0$ is a function of $H$ and $G_1$ is analytic. $G_1$ commutes with $H$ that is why $G_1$, this gives us that $G_1$ is a function of $H$ on $P$. We obtain that $G$ is a function of $H$ near $x$. By the analytic continuation this is true in the connected component of $x \in U$.

**Proof b):** Let us go back to the formal non-integrability. First we will regard the case $K = H_2 + H_3$ it is Taylor expansion to a degree 3 with change of variables $\tilde{p}_1 = p_1$, $\tilde{p}_2 = p_2$, $\tilde{q}_1 = q_1 - \frac{\pi}{3}$ and $\tilde{q}_2 = q_2 - \frac{\pi}{3}$ (we move equilibrium to 0)

$$K = \tilde{p}_1^2 - \tilde{p}_1 \tilde{p}_2 + \tilde{p}_2^2 + \frac{4}{3} \tilde{q}_1^2 + \frac{4}{3} \tilde{q}_2^2 + \frac{4}{9} \sqrt{3} \tilde{q}_1^2 \tilde{q}_2 + \frac{4}{9} \sqrt{3} \tilde{q}_1 \tilde{q}_2^2. \tag{0.9}$$

Let us remove tildes and we obtain the Hamiltonian system

$$\begin{align*}
\dot{q}_1 &= 2p_1 - p_2 \\
\dot{q}_2 &= -p_1 + 2p_2 \\
\dot{p}_1 &= -\frac{8}{3} q_1 - \frac{4}{3} q_2 - \frac{8}{9} \sqrt{3} q_1 q_2 - \frac{4}{9} \sqrt{3} q_2^2 \\
\dot{p}_2 &= -\frac{4}{3} q_1 - \frac{8}{3} q_2 - \frac{8}{9} \sqrt{3} q_1 q_2 - \frac{4}{9} \sqrt{3} q_1^2
\end{align*}$$

We use the Theory of Morales-Ruiz- Ramis (see [4] for details) reducing the system to a Normal Variations Equations (NVE) near a non-trivial partial solution. We find a partial solution for $p = p_1 = p_2$, $q = q_1 = q_2$, and we have

$$\dot{q}^2 = -\frac{8}{9} \sqrt{3} q^3 - 4q^2 + h,$$

with solution $\phi(t) = -\frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} \phi(t, g_2, g_3)$, here $\phi$ is Weierstrass $p$-function with $g_2 = \frac{4}{3}$.
\[ g_3 = -\frac{4(h-2)}{27}. \] Let us put \( \eta_1 = dp_1, \eta_2 = dp_2, \xi_1 = dq_1 \) and \( \xi_2 = dq_2 \) then

\[ \begin{align*}
\dot{\xi}_1 &= 2\eta_1 - \eta_2 \\
\dot{\xi}_2 &= -\eta_1 + 2\eta_2 \\
\dot{\eta}_1 &= -\left(\frac{8}{3} + \frac{8}{9}\sqrt{3}\phi\right)\xi_1 - \left(\frac{4}{3} + \frac{16}{9}\sqrt{3}\phi\right)\xi_2 \\
\dot{\eta}_2 &= -\left(\frac{4}{3} + \frac{16}{9}\sqrt{3}\phi\right)\xi_1 - \left(\frac{8}{3} + \frac{8}{9}\sqrt{3}\phi\right)\xi_2.
\end{align*} \]

If we get \( \xi = \xi_1 - \xi_2 \), then we find an equation for NVE

\[ \dot{\xi} = (-\frac{8}{3} + 4\phi(t, g_2, g_3))\xi. \]

This is a Lame-equation with \( A = 4, B = -\frac{8}{3} \) and in this case we have non Lame-Hermite solutions \( A = n(n+1) \neq 4 \) for \( n \in \mathbb{Z} \). We have non Briochi- Halphen- Crowford solutions of its Galois group is non-commutative (see [6] for details).

The theory says that if the identity component of differential Galois group is non-commutative, then the system is not meromorphic integrable (see [7]). This proves that in the case \( H_2 + H_3 \)

there is non additional meromorphic (holomorphic) first integral.

Let us consider the case \( H_2 + H_3 + H_4 \): the Hamiltonian is

\[ \begin{align*}
\Lambda &= \ddot{p}_1^2 - \ddot{q}_1 \dddot{p}_2 + \ddot{p}_2^2 + \frac{4}{3} \ddot{q}_1^2 - \frac{4}{3} \ddot{q}_2^2 + \frac{8}{9} \ddot{q}_1^3 \dddot{q}_2 + \frac{4}{9} \dddot{q}_1^2 \dddot{q}_2 \\
&\quad + \frac{4}{9} \dddot{q}_1 \dddot{q}_2^2 + \frac{4}{9} \ddot{q}_1^4 + \frac{4}{9} \ddot{q}_2^4 - \frac{8}{9} \dddot{q}_1^3 \dddot{q}_2 + \frac{8}{9} \dddot{q}_1 \dddot{q}_2^3 + \frac{4}{3} \dddot{q}_1^2 \dddot{q}_2^2.
\end{align*} \quad (0.10) \]

We ignore the tildes and we get the system

\[ \begin{align*}
\dot{q}_1 &= 2p_1 - p_2 \\
\dot{q}_2 &= -p_1 + 2p_2 \\
\dot{p}_1 &= -\frac{8}{3} q_1 - \frac{4}{3} q_2 - \frac{8}{9} \sqrt{3} q_1 q_2 - \frac{4}{9} \sqrt{3} q_2^2 \\
&\quad - \frac{16}{9} q_1^3 - \frac{8}{3} q_1^2 q_2 - \frac{8}{3} q_1 q_2^2 - \frac{8}{9} q_2^3 \\
\dot{p}_2 &= -\frac{4}{3} q_1 - \frac{8}{3} q_2 - \frac{8}{9} \sqrt{3} q_1 q_2 - \frac{4}{9} \sqrt{3} q_1^2 \\
&\quad - \frac{16}{9} q_2^3 - \frac{8}{3} q_1^2 q_2 - \frac{8}{3} q_1 q_2^2 - \frac{8}{9} q_1^3.
\end{align*} \]

For a partial solution we get \( p = p_1 = p_2, \) \( q = q_1 = q_2 \) and we find \( \psi(t) = -\frac{3 \sqrt{3}}{\sqrt{26 \sinh (2it) + 1}}. \)

We get \( \eta_1 = dp_1, \eta_2 = dp_2, \xi_1 = dq_1, \xi_2 = dq_2, \) \( \xi = \xi_1 + \xi_2 \) and we obtain for NVE

\[ \dot{\xi} = -(4 + \frac{8}{9} \sqrt{3} \psi(t) + 24 \psi(t)^2) \xi. \quad (0.11) \]
We need to algebrize (0.11) with a standard change of variable \( w = \sqrt{26} \sinh(2i\tau) + 1 \). The result is
\[
\xi'' = r(w)\xi. \tag{0.12}
\]

Next we use the Kovacic algorithm to show that (0.12) has non Liouvillian solutions and the identity component of the Galois group of this equation is \( SL(2, \mathbb{C}) \). This means that the Hamiltonian system (0.10) is non integrable with meromorphic first integrals. This proves that in the case \( H_2 + H_3 + H_4 \) there is non additional meromorphic (holomorphic) first integral. The system \( H_3 \) is integrable, \( H_2 + H_3 \) and \( H_2 + H_3 + H_4 \) are non integrable thats why we could conclude that the system \( H_2 + H_3 + \cdots + H_k \) is non integrable for each \( k \geq 3 \). This proves B).

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Georgi Georgiev

georgiev3@fmi.uni-sofia.bg

Faculty of Mathematics and Informatics, Sofia University,
1164 Sofia, Bulgaria