Quantum games on evolving random networks

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We study the advantages of quantum strategies in evolutionary social dilemmas on evolving random networks. We focus our study on the two-player games: prisoner’s dilemma, snowdrift and stag-hunt games. The obtained result show the benefits of quantum strategies for the prisoner’s dilemma game. For the other two games, we obtain regions of parameters where the quantum strategies dominate, as well as regions where the classical strategies coexist.

I. INTRODUCTION

Game theory is a widely studied branch of science with broad applications in a plethora of fields. These range from biology to social sciences and economics. It has been especially useful in the study of social dilemmas, i.e., situations where the benefit of the many should be put in front of the benefit of the individual. One of the most frequently studied approaches in this context is the evolutionary game theory [1]. The field of evolutionary games has since evolved and now studies not only games on regular grids, but also on complex graphs [2]. Recently, there are studies focused on studying social dilemmas on evolving random networks [3, 4].

In quantum game theory [5–8] we allow the agents to use quantum strategies alongside classical ones. As this is a far larger set of possible players’ moves, it offers the possibility of much more diverse behavior. The most outstanding example of this, is the fact that if only one player is aware of the quantum nature of the game, he/she will never lose in some types of games. If both players are aware of the nature of the game, one of them might still cheat by appending additional qubits to the system [9]. When we take decoherence into account, the game behavior changes. In particular, the well known Nash equilibrium of a game can shift to a different strategy [10]. Furthermore, quantum game theory allows us to solve dilemmas present in classical game theory, like the prisoner’s dilemma [11]. On top of this, there also exists a quantum version of the Parrondo’s paradox [12, 13]. Finally, there quantum pseudo-telepathy games. In these games, players utilizing quantum strategies and quantum entanglement, may seem to an outside observer as communicating telepathically [14, 15].

The combination of the fields of quantum game theory and evolutionary games has led to numerous results [17–20]. There exist cases where the quantum strategies dominate the entire network. In this work we aim to study the behavior of three quantum games on evolving random networks: prisoner’s dilemma, snowdrift and stag-hunt games. The transition between these games will be achieved by manipulating the parameters of the game.

Games on evolving networks have been studied in the classical [2, 4] as well as quantum settings [21]. The evolution of the network can be seen as aging of the agents.

This paper is organized as follows. In Sec. II we introduce quantum games. In Sec. III we discuss the model of networks used in this work. Sec. IV contains the results along with discussions. Finally, in Sec. V we draw the final conclusions.

II. QUANTUM GAMES

We call a game a quantum game if the players participating are allowed to use quantum strategies. By quantum strategies we understand moves that have no justification in classical game theory, but have a good interpretation in the realm of quantum mechanics. We will focus on two-player games. Henceforth we will call the players Alice and Bob.

A. General concepts

Formally, a two-player quantum game is a tuple \( \Gamma = (H, \rho, S_A, S_B, F_A, F_B) \). Here \( H \) is a Hilbert space of the system used in the quantum game, \( \rho \) is the system’s initial state. Note that \( \rho \) is a positive operator with unit trace, i.e., \( \rho \geq 0 \) and \( Tr \rho = 1 \). Allowed Alice’s and Bob’s strategies are given by the sets \( S_A \) and \( S_B \) respectively. Their payoff functions are given by \( F_A \) and \( F_B \). They are functions mapping players strategies to numerical values. In general the strategies \( s_A \in S_A \) and \( s_B \in S_B \) can be any quantum operations. A definition of a quantum game may contain additional rules like the ordering of players or the number of times they are allowed to make a move.

By analogy to the classical game theory we may define the following quantities in quantum game theory. We will call a strategy \( s_A \) the dominant strategy of Alice if \( P(s_A, s_B') \geq P(s_A', s_B) \) for all \( s_A \in S_A, s_B' \in S_B \). Following this pattern we may define a dominant strategy for Bob. A pair of strategies \( (s_A, s_B) \) is an equilibrium in dominant strategies if and only if \( s_A \) and \( s_B \) are Alice’s and Bob’s dominant strategies. A pair of strategies is Pareto optimal if it not possible to increase one player’s
This is shown as a quantum circuit in Fig. 1. Here, the players apply their respective strategies and the final state of the system is:

$$|\psi\rangle = J^\dagger(U_A \otimes U_B)J|\phi\rangle,$$

where $J$ is the entangling operator [22]:

$$J = \frac{1}{\sqrt{2}}(1 \otimes 1 + i\sigma_x \otimes \sigma_x).$$

(2)

Here $\sigma_x$ is the Pauli matrix:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(3)

Next, the players apply their respective strategies $U_A$ and $U_B$ and the untangling operator $J^\dagger$ is applied by the referee. Here $J^\dagger$ denotes the Hermitian conjugate of $J$. The final state of the system is:

$$|\psi\rangle = J^\dagger(U_A \otimes U_B)J|\phi\rangle.$$ 

(4)

This is shown as a quantum circuit in Fig. 1.

![Quantum circuit](image)

**FIG. 1:** Quantum circuit depicting a two-player quantum game. Here $J$ is the entangling operator and $U_A$ and $U_B$ denote Alice’s and Bob’s strategy respectively.

The payoff matrix of a two player game with cooperators $C$ and defectors $D$ is shown in Tab. 1 [3]. In the table $R$ is the reward, $P$ is the punishment for mutual defection, $S$ is known as the sucker’s payoff and finally the parameter $T$ is the defector temptation. In our analysis we set $R = 1$ and $P = 0$. The remaining two parameters range is $S \in [-1,1]$ and $T \in [0,2]$. When $T > R > P > S$ we get a social dilemma - the prisoner’s dilemma. Note that on the one hand, in this case the strategy profile $(C,C)$ is Pareto optimal, but on the other hand the profile $(D,D)$ is a Nash equilibrium. Hence, we have the dilemma. Next, when $T > R > S > P$ we get the snowdrift game. Finally, when $R > T > P > S$ we get the stag-hunt game.

In the quantum case, the payoff of Alice is determined by:

$$P_A = R(|\psi\rangle\langle 00|^2 + S(|\psi\rangle\langle 11|^2 + T(|\psi\rangle\langle 10|^2 + P(|\psi\rangle\langle 11|^2).$$

(5)

Aside from the two classical strategies, we introduce two quantum strategies. The first one is the Hadamard strategy, $H$. It introduces a miracle move [5] in the prisoners dilemma game, that is it allows the player to always win against the other player’s classical strategy. The second strategy, denoted by $Q$ will introduce a new Nash equilibrium in the prisoners dilemma game, for the strategy profile $(Q,Q)$.

We associate the player’s strategies with the following unitary matrices:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$ 

(6)

**III. SIMULATION SETUP**

We set the population size to 2500 agents located at the nodes of an Erdős–Rényi graph. The initial number of edges is set to 10000. Each agent is assigned an initial strategy at random. We study the following initial assignments:

1. The classical strategies, $S_1 = \{C,D\}$. Initial these strategies are assigned with probability 50% each.
2. Classical strategies and the miracle move, $S_2 = \{C,D,H\}$. The initial probabilities of assignment of C, D and H are 49%, 49% and 2% respectively.
3. Classical strategies and the quantum Nash equilibrium, $S_3 = \{C,D,Q\}$. The initial probabilities of assignment of C, D and Q are 49%, 49% and 2% respectively.
4. All four strategies, $S_4 = \{C,D,H,Q\}$. The initial probabilities of assignment of C, D, H and Q are 49%, 49%, 1% and 1% respectively.

The evolution of the network is performed via Monte Carlo simulation. First, we select a random agent $x$ and

| Alice: C (R, R) (S, T) | Alice: D (T, S) (P, P) |
|------------------------|------------------------|
| Bob: C Bob: D          | Bob: C Bob: D          |

**TABLE I:** Payoff matrix of a two-player game with cooperators $C$ and defectors $D$. |
his random neighbor, \( y \). Next, each of them acquires their payoff, \( p_x \) and \( p_y \) respectively by playing with all of their neighbors. Finally, if \( p_x > p_y \), agent \( y \) may adopt the strategy of agent \( x \) with probability:

\[
W = \frac{p_x - p_y}{\alpha \max(k_x, k_y)},
\]

where \( k_x \) and \( k_y \) are degrees of nodes \( x \) and \( y \) respectively and \( \alpha \) is a constant dependent on the game. We have \( \alpha = T - S \) for the prisoner’s dilemma, \( \alpha = T - P \) for the snowdrift and \( \alpha = R - S \) for the stag-hunt game. When an agent \( x \) adopts a new strategy, we remove all edges connecting him/her to other agents, except for the one connecting to the donor agent. As this scheme will quickly lead to the creation of disjoint graphs, we allow the agents to form new links at the end of each Monte Carlo step. Each agent is allowed to connect to a uniformly randomly chosen node he/she is not connected to.

To avoid creating very big graphs, we set a limit on an agent’s degree, \( k_{\text{max}} \). We set \( k_{\text{max}} = 500 \). If an agent has a degree greater or equal to \( k_{\text{max}} \), we remove all outgoing edges, except for one, randomly chosen. The agent keeps his/her strategy. This process simulates the aging and dying of agents. On average, every agent is chosen once per Monte Carlo step. We set the number of steps to \( 10^5 \). We obtain the fractions of strategies by averaging the last \( 10^5 \) steps.

**IV. RESULTS AND DISCUSSION**

First, we present the results for the classical strategies alone. These are shown in Fig. 2. These are with good agreement with previously found results. Note that in the snowdrift game we find the region, where the fractions of strategies transfer smoothly from the dominance of the \( C \) to the dominance of the \( D \) strategy. We study one line across this region, shown as the solid gray line in Fig. 2. Instead of averaging, we show only the fractions for the last iterations of the Monte Carlo process. This is shown in Fig. 3. In Fig. 4 we show full Monte Carlo history for a few selected points along the line in Fig. 3. This shows that the strategies achieve their equilibrium fractions quickly, with only minor changes after the first \( 2 \cdot 10^3 \) steps. The parameter \( r \) in Fig. 3 gives the parameters \( S \) and \( T \) as:

\[
S = 1 - r \\
T = 1 + r
\]

Note that all the figures show a very smooth transition between the \( C \) and \( D \) dominance case.

Next, we study the case where we introduce the miracle move strategy, \( H \). The fractions of strategies are shown in Fig. 5. In this case we obtain different behavior compared to the classical case. First of all, the strategy \( D \) does not dominate in the prisoner’s dilemma case. This region of parameters \( T \) and \( S \) is now dominated by the

**FIG. 2: Results for the classical strategies. Panel (a) shows the fraction of \( C \) strategy, \( \rho_C \), and panel (b) shows the fraction of strategy \( D \), \( \rho_D \). Dashed black lines mark the boundaries between the different game types. The labels correspond to prisoner’s dilemma (PD), snowdrift (SD) and stag-hunt games (SH). The solid gray lines show the regions which were examined in detail.**

**FIG. 3: Behavior of the fraction of strategy \( C \), \( \rho_C \) and the fraction of strategy \( D \), \( \rho_D \) along the line shown in Fig. 2. The parameter \( r \) gives the values of \( S = 1 - r \) and \( T = 1 + r \).**
quantum strategy $H$. In the stag-hunt game, a transition between the dominance of $D$ and $H$ emerges. In some regions the transition is smooth, while other regions show a sharp transition. This is studied in the same manner as described in the previous paragraph. We show this results in Figs. 6 and 7. We should note here, that we observe two behaviors here which depend on the parameter’s values. In the first case the system quickly converges to total dominance of the $C$ strategy. In the second case, first the fraction of $D$ strategy rises, and next starts to drop in favor of the miracle move $H$.

Furthermore, in the snowdrift game case, the region where strategies $C$ and $D$ coexist is much smaller compared to the classical case. This now occurs only in the case when $S > T - 1$. For other values of $S$ and $T$ in this region we get a sharp transition to full dominance of the strategy $H$. This is shown in more detail in Figs. 8 and 9. Studying the results across the line in the upper right corner of Fig. 4 we should note that at first we get the dominance of the cooperators. The fraction of cooperators starts to decrease with $r$ up until $r = 0.5$ when we get the dominance of the miracle move. Analyzing the detailed histories of the strategy fractions, we note that with increasing $T$ the fraction of miracle moves rises faster, but the fraction of the strategies $C$ and $D$ is almost equal to each other.

When we introduce the strategy $Q$ with the classical ones, the behavior changes slightly, compared to the case studied in the case of the miracle move, $H$. This is shown in Figs. 10, 11 and 12. We do not have the region in the stag-hunt game where there a transition
FIG. 6: Behavior of the fraction of strategy $C$, $\rho_C$, the fraction of $D$, $\rho_D$ and the fraction of strategy $H$, $\rho_H$ along the line shown in the lower left corner of Fig. 5. The parameter $r$ gives the values of $S = -\frac{3}{4} - \frac{1}{4} r$ and $T = \frac{1}{4} r$.

FIG. 7: Full Monte Carlo history of the fractions of strategies $C$, denoted $\rho_C$, $D$ denoted $\rho_D$ and $H$, denoted $\rho_H$.

FIG. 8: Behavior of the fraction of strategy $C$, denoted $\rho_C$, the fraction of $D$, denoted $\rho_D$ and the fraction of strategy $H$, denoted $\rho_H$ along the line shown in the upper right corner of Fig. 5. The parameter $r$ gives the values of $S = 1 - r$ and $T = 1 + r$.

FIG. 9: Full Monte Carlo history of the fractions of strategies $C$, denoted $\rho_C$, $D$ denoted $\rho_D$ and $H$, denoted $\rho_H$.

between $C$ and $Q$. Instead, the entire region is split in half, where one is dominated by the strategy $C$ and the other by strategy $Q$. In the snowdrift region, the transition between coexistence of $C$ and $D$ and dominance of $Q$ is much sharper compared to the miracle move case. The prisoner’s dilemma region is again dominated by the quantum strategy $Q$.

Finally, when we introduce all four strategies, we obtain results similar to the case with the $Q$ strategy only. Again, all interesting regions are dominated by the strategy $Q$. This is shown in detail in Figs. 13, 14 and 15.

V. CONCLUSIONS

To sum up, we studied evolutionary cooperation on evolving random networks with quantum agents. We identified regions of parameters where quantum strategies dominate the network as well as studied in detail regions with coexistence of strategies. These are different for the classical and quantum cases. Furthermore,
The introduction of the quantum miracle move introduces a new region where the classical strategies coexists with quantum strategies. This is consistent with previous results [17, 20].

FIG. 10: Results for the classical strategies. Panel (a) shows the fraction of strategy C, denoted $\rho_C$, the fraction of D, $\rho_D$ and the fraction of strategy Q, denoted $\rho_Q$. Dashed black lines mark the boundaries between the different game types. The labels correspond to prisoner's dilemma (PD), snowdrift (SD) and stag-hunt games (SH). The solid gray lines show the regions which were examined in detail.

FIG. 11: Behavior of the fraction of strategy C, denoted $\rho_C$, the fraction of D, $\rho_D$ and the fraction of strategy Q, denoted $\rho_Q$ along the line shown in the upper right corner of Fig. 10. The parameter $r$ gives the values of $S = 1 - r$ and $T = 1 + r$.

FIG. 12: Full Monte Carlo history of the fractions of strategies C, denoted $\rho_C$, D denoted $\rho_D$ and Q, denoted $\rho_Q$. The quantum strategies, when they are introduced. This is consistent with previous results [17, 20].

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FIG. 13: Results for the classical strategies. Panel (a) shows the fraction of $C$ strategy, $\rho_C$, panel (b) shows the fraction of strategy $D$, $\rho_D$, panel (c) shows the fraction of the strategy $H$, $\rho_H$ and panel (d) shows the fraction of the strategy $Q$, $\rho_Q$. Dashed black lines mark the boundaries between the different game types. The labels correspond to prisoner’s dilemma (PD), snowdrift (SD) and stag-hunt games (SH). The solid gray lines show the regions which were examined in detail.
FIG. 14: Behavior of the fraction of strategy $C$, denoted $\rho_C$, the fraction of $D$, $\rho_D$, fraction of strategy $H$, denoted $\rho_H$, and the fraction of strategy $Q$, denoted $\rho_Q$ along the line shown in the upper right corner of Fig. 13. The parameter $r$ gives the values of $S = 1 - r$ and $T = 1 + r$.

FIG. 15: Full Monte Carlo history of the fractions of strategies $C$, denoted $\rho_C$, $D$ denoted $\rho_D$, $H$, denoted $\rho_H$, and $Q$, denoted $\rho_Q$. 

(a) $S = 0.54$, $T = 1.46$  
(b) $S = 0.52$, $T = 1.48$  
(c) $S = 0.5$, $T = 1.5$