Matching effective few-nucleon theories to QCD

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The emergence of complex macroscopic phenomena from a small set of parameters and microscopic concepts demonstrates the power and beauty of physical theories. A theory which relates the wealth of data and peculiarities found in nuclei to the small number of parameters and symmetries of quantum chromodynamics is by that standard of exceptional beauty.

Decade-long research on computational physics and on effective field theories facilitate the assessment of the presumption that quark masses and strong and electromagnetic coupling constants suffice to parameterize the nuclear chart. By presenting the current status of that enterprise, this article touches the methodology of predicting nuclei by simulating the constituting quarks and gluons and the development of effective field theories as appropriate representations of the fundamental theory.

While the nuclear spectra and electromagnetic responses analyzed computationally so far with lattice QCD are in close resemblance to those which intrigued experimentalists a century ago, they also test the theoretical understanding which was unavailable to guide the nuclear pioneers but developed since then. This understanding is shown to be deficient in terms of correlations amongst nuclear observables and their sensitivity to fundamental parameters. By reviewing the transition from one effective field theory to another, from QCD to pionful chiral theories to pionless and eventually to cluster theories, we identify some of those deficiencies and conceptual problems awaiting a solution before QCD can be identified as the high-energy theory from which the nuclear landscape emerges.
FIG. 1. (Color online) Range of applicability of nuclear effective theories (adapted from Ref. [4]) in a space spanned by the pion mass \( m_\pi \), a low-momentum scale \( Q \), and the nucleon number \( A \). The physical world is measured at \( m_\pi \sim 140 \text{ MeV} \) on the red sheet. LQCD assesses the few-nucleon systems of the deuteron \( d \), the triton \( t \), and the \( \alpha \)-particle on the dashed enclosed sheets at 510 and 806 MeV. EFT(\( \pi/ \)) (light green) was applied to \( A < 5 \) nuclei up to large pion masses. \( \chi\text{EFT} \) (yellow area) breaks down at some \( m_\pi \), but applies at larger momenta than its perturbative version and EFT(\( \pi/ \)). Cluster EFTs (cyan) have only been used at physical \( m_\pi \).

I. INTRODUCTION

Lattice QCD as an experiment with light baryons — It is a striking analogy that the calculation of binding energies of the lightest nuclei as compound objects, whose constituent dynamics are dictated by quantum chromodynamics (QCD), proceeds along similar paths as the early experiments in the real world. The relative determination of the deuteron mass to Helium mass [1], with its preceding preparation of a sufficiently pure beam of the respective nuclei, resembles remarkably the lattice methodology. In the latter, mass ratios are also more accessible because of a reduced statistical uncertainty. Furthermore, devising interpolating operators with a large initial overlap with the baryons of interest are the numerical analog of preparing a clean beam. The exploration of the nuclear landscape included in parallel to the study of mass spectra the response of nuclei to electromagnetic fields [2]. Again, these investigations find their resemblance in contemporary lattice calculations of the magnetic structure of light nuclei. The fact that the lattice simulates nuclei at artificially large pion masses yields more than just data points for an extrapolation to the physical pion mass. It allows for the investigation of phenomena emergent from a theory as sophisticated as the one we deem appropriate for the strong force in nature. The theory at unphysical pion masses shares symmetries and degrees of freedom and thereby can identify features that are peculiar to QCD as the theory which breaks the flavor symmetry with some specific quark-mass values or can be universally attributed to the three SU(3) color and flavor symmetric quarks.

Tools that have evolved over the century and the questions which can be investigated with the lattice technology are plenty. The basic motivation, to build a theory on the available data to predict what presumably will be measured with more effort in nuclei with a total number of neutrons and protons \( A \gtrsim 4 \), using different external probes, or looking at the systems at different scales, remains, however, invariant. Today, we start at the same point like physicists when facing the first measurements on light nuclei in the physical world.

Contemporary theoretical nuclear reality — With the increasing resolution of experimental apparatus and the ensuing discovery of the substructure of nucleons the interaction amongst them lost its status of a fundamental theory and became to be thought of as a manifestation of QCD at low energy. The current understanding of nuclei comprises a set of effective theories, each devised to describe them only up to a certain nucleon number \( A \) or momentum scale \( Q \), at which the neutrons and protons can be excited and/or deformed but their internal structure is not probed. The pioneering theories and models for the nuclear interaction, e.g., meson-exchange (Yukawa) theories and quark
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models, can be recovered from or found to be included in these effective theories.

Specifically, we understand the nuclear interaction to be given by chiral effective field theory (χEFT) which: (i) treats the pion-nucleon coupling perturbatively for very low momenta, (ii) becomes non-perturbative for higher energies, (iii) can be matched to the pionless effective field theory (EFT(π)) which is applicable in the baryon sector for momenta much smaller than the nuclear scale set by typical binding momentum in nuclei of about 100 MeV/c. The latter serves itself as basis for cluster theories which introduce composite nuclei, like α particles or deuterons, for the description of larger systems. This understanding is at present incomplete as a reconciliation of differing power-counting schemes which define χEFT is yet to be found. A fully renormalization-group invariant version of χEFT, for example, has not yet been formulated. However, given a reliable counting scheme and thereby uncertainty estimates will enable us to relate the five relevant QCD and QED parameters (three light-quark masses and the electromagnetic and strong coupling constants) via the matching of QCD and EFT amplitudes to observables, which are poorly known experimentally and would thus be an ultimate assessment of whether or not the nuclear theory has been found.

The crucial calculation of multi-hadron amplitudes with QCD became reality, recently, via the numerical solution of the QCD path integral in discrete space-time (lattice QCD or LQCD): unquenched, high-statistics LQCD measurements are available for the nuclear spectrum up to $A = 4$ using $(i) N_f = 2 + 1$, i.e., three-flavor QCD with the physical strange-quark mass and equal up, down-quark masses $m_u = m_d$ corresponding to $m_\pi \sim 300$ MeV in Ref. [5], 450 MeV in Ref. [6] ($A \leq 2$), and 510 MeV in Ref. [7], and $(ii) N_f = 3$, i.e., exact SU(3)-flavor QCD because of degenerate light-quark masses yielding $m_\pi \sim 806$ MeV in Ref. [8]. While measuring at unphysically high pion masses increases the signal-to-noise ratio in lattice simulations and allows for meaningful results, it inhibits a matching of those amplitudes to a chiral effective theory which relies on pion masses $\leq 500$ MeV. Before LQCD can probe the nuclear spectrum in this pion-mass range, its amplitudes can be matched in the region where its area of applicability overlaps with the pionless theory.

The current status of this program, to relate QCD parameters through LQCD, χEFT, and contact theories to few-nucleon observables, is reviewed in this article. A graphical outline is given in Fig. [1]. It shows areas of applicability of the various theories in the physical parameter space spanned by the pion mass, a momentum scale $Q$, and the nucleon number $A$. In this figure, QCD is shown to probe the entire space and is expected to coincide with experiment on the physical (red) sheet. Without QCD solutions for nuclear observables (dashed enclosed, transparent sheets) at physical $m_\pi$, we begin the discussion at the intersection between experiment and χEFT (yellow sheet) in Sec. [II] before showing how the overlap between χEFT and EFT(π) was utilized to reach larger nuclei (green squares indicate those where EFT(π) has been used) in Sec. [II] [II]. Before elaborating on the interface between LQCD and two contact theories (EFT(π) amongst them) in Sec. [V A] and [V B], we insert with Sec. [IV B] the status of the exploration of the two sheets at 510 and 806 MeV via the lattice method. The lattice technology is briefly touched in Sec. [IV]. We include comments on the further investigation of the $A$ axis by matching contact to cluster theories (cyan) in the outlook.

II. LATTICE QCD $\rightarrow$ χEFT

To predict LQCD observables at physical $m_\pi$, a theory is needed which is consistent with QCD at physical and some larger $m_\pi$, where it can be matched to available LQCD data. Furthermore, the power counting of that theory has to be understood for all values of the pion mass between the matching point and the physical point. The extension of chiral perturbation theory to multi-baryon systems, hereafter referred to as χEFT (concepts in Refs. [10] [11], and reviews in Refs. [12] [13]) is the only ansatz which has been used for such an extrapolation. The power counting of such an extension however, is still under development (consider Refs. [16] [20] next to a solution suggested in Refs. [21] [22]).

Given the development of such an effective field theory for few-nucleon systems and the availability of LQCD data at a $m_\pi$ where chiral perturbation theory (ChPT) converges, the algorithm (original formulation in Ref. [15]) reviewed below can be used to predict physical observables from LQCD data at unphysical $m_\pi$.

The chiral Lagrangian depends explicitly and implicitly on the pion mass. The implicit dependence resides in the chiral coupling constant $\mu^{4}$, and the nucleon mass $m_N$. Using ChPT, this dependency can be made explicit in a power expansion, e.g., for the decay constant $f_\pi$, the axial coupling constant $g_A$, and the nucleon mass $m_N$. Using ChPT, this dependency can be made explicit in a power expansion, e.g., for the decay constant $f_\pi$ with a renormalization scale $\mu$:

$$
\frac{f_\pi (m_\pi)}{f_\pi (m_\pi = 0)} = \left( 1 + \frac{m_\pi^2}{8\pi^2 f_\pi^2 (m_\pi = 0)} \left( -\ln \frac{m_\pi^2}{\mu^2} + \mathcal{O}(1) \right) + \mathcal{O}(m_\pi^4) \right).
$$

\(^1\) See Ref. [3] as an example for a derivation of a quark model from the heavy-baryon effective field theory of QCD.

\(^2\) Ref. [14] demonstrates the complexity of finding inconsistencies in a perturbative treatment of the pion exchange; Ref. [15] reviews also problems of other counting schemes.

\(^3\) Ref. [23] hints to a breakdown at $m_\pi < 500$ MeV of the chiral expansion for $f_\pi$. 
The $\chi$EFT Lagrangian contains in addition to single-nucleon, single-meson, and meson-nucleon coupling parameters pion-mass independent and dependent contact operators:

$$\mathcal{L}_{\text{four-fermi}} = C_S (N^T N)^2 + C_T (N^T \sigma_i N)^2 + D_{S1} (N^T \mathcal{M} N) \mathcal{M} (N^T N) + D_{T1} (N^T \mathcal{M} \sigma_i N) \mathcal{M} (N^T N) + D_{S2} (N^T N)^2 \mathcal{M} + D_{T2} (N^T \sigma_i N) \mathcal{M} (N^T N) \mathcal{M}.$$  \hfill (2)

The associated low-energy constants (LECs) $C_{S,T}$ and $D_i$ are pion-mass independent. They can be determined at any $m_\pi$ within the convergence radius of ChPT. From the available data at physical $m_\pi$, $D_i$ cannot be determined independently from $C_i$. One way to resolve this ambiguity are additional scattering experiments of pions on deuterons or heavier targets. Another utilizes two-nucleon observables at different pion mass and coupling strengths of natural size.

The pion-mass dependence of the effective range can be inferred from data at a single $m_\pi$ because up to NLO, operators proportional to the quark mass do not contribute. In addition to the ensuing $m_\pi$ dependence of the deuteron binding energy (as a pole of the amplitude Eq. (1)), Ref. \cite{27} derived the charge radius, the magnetic moment and polarizability, and the photodisintegration of the deuteron as a function of the pion mass. We compare the binding energies in the singlet and triplet channels of Ref. \cite{27} in Fig. 2 (thin red line) with the results obtained with a chiral potential \cite{31}. The latter neither binds the singlet state at a higher $m_\pi$, nor does it indicate $\lim_{m_\pi \to 0} B_\pi = 0$.

Before Ref. \cite{27}, and without lattice data at sufficiently low pion masses, $\chi$EFT was used, also with a modified KSW counting which uses assumptions about details of the short-range structure of the interaction, for an extrapolation in $m_\pi$ in Ref. \cite{15}. Like BKV, this Beane-Bedaque-Savage-van-Kolck (BBSvK) scheme is identical to KSW in the NN singlet channel but as an expansion about the chiral limit $m_\pi = 0$ iterates the pion exchange in the coupled triplet channel (Weinberg counting). In Ref. \cite{14}, BBSvK is used to investigate the deuteron and $^1S_0$, $^3S_1$ neutron-proton (np) scattering properties for pion masses from 0 up to 200 MeV. The two-nucleon amplitude was matched at physical $m_\pi$ to the triplet np scattering length ($a_{n\pi}$) and effective range ($r_i$) to determine the relevant contact terms in Eq. (2). With a leading order chiral expansion for $f_\pi$, $g_A$, and $m_N$, the deuteron was found to become unbound at $m_\pi \sim 100$ MeV with a corresponding divergent $a_{n\pi}$.

The extension of the method to the np-singlet channel demonstrated the relevance of the above ambiguity in the contact terms. One choice of $C_i$ and $D_i$ left the di-neutron (nn) unbound for all pion masses in [0, 300 MeV] while another yielded nn bound in the interval between 180 and 240 MeV. The problem was then addressed in Ref. \cite{33} by placing constrains on the contact terms with naive dimensional analysis. A relevant conclusion for the extrapolation from $m_\pi > 140$ MeV down was the consistency of a bound nn, i.e., singlet two-nucleon state, with data at the physical pion mass and coupling strengths of natural size.

The effect a different power counting has on the $m_\pi$ dependence of the two-nucleon system is seen when comparing the results of Refs. \cite{15, 32, 33} to Ref. \cite{34}, and to Ref. \cite{35}. Considering also two-pion-exchanges in a modified\cite{35} Weinberg counting, Ref. \cite{34} found the deuteron in the chiral limit more strongly bound than in nature. While this is in contrast to the unbound deuteron of Ref. \cite{15}, the absence of bound states in the singlet NN channel for $m_\pi \in (0, 200)$ MeV is a common result of both schemes. Pions in Ref. \cite{35} finds the singlet and deuteron state unbound in the chiral limit. This work employs composite two-nucleon spin-singlet and triplet quasi-particles (dibaryons, see Sec. \ref{sec:VIIb}) coupled to single nucleons and propagates the pion-mass dependence to a pionless theory through the analytical matching of amplitudes. These differences should concern convergence rates and precision of the EFT but must eventually lead to the same observables. This issue is unresolved. As a feature, Ref. \cite{35} demonstrates a reversed extrapolation from the physical pion mass to larger values. The LQCD results which are available (see Sec. \ref{sec:VIIb})

\begin{itemize}
  \item with $\mathcal{M} \sim \xi \mathcal{L} \mathcal{M}$, where $\xi = 1 + \frac{1}{2} \pi \mathcal{M} \pi - \frac{1}{8 \pi^2} \pi^2 + O(\pi^3)$, Pauli isospin matrices $\pi$, pion isovector $\pi$, and nucleon iso-doublet $N = \{p, n\}$
  \item The pions are dynamical in $\chi$EFT. In EFT($\sigma$), in contrast, the contact LECs do depend on $m_\pi$.
  \item The method of resonance saturation as developed in Ref. \cite{25} and applied, \textit{e.g.}, to the quark-mass sensitivity of a collection of nuclear observables relevant for big-bang nucleosynthesis\cite{26}, approaches the problem by relating $\chi$EFT contact LECs to phenomenological boson-exchange models. As it is unknown how the uncertainties of that method can be quantified, we abstain from further elaboration.
  \item The value was also found\cite{15} to be quite sensitive to the regularization of the contact interactions.
\end{itemize}
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FIG. 2. (Color online) Pion-mass dependence of the nn-singlet (top left) and deuteron (top right) binding energy, the neutron-deuteron quartet-channel scattering length (bottom right), and the neutron-deuteron(nn-singlet) (blue red, bottom left) doublet scattering length. Results for \( m_\pi < 220 \text{ MeV} \) (left y scale) represent predictions with EFT(\( \pi \)) matched to \( \chi \)EFT (bound states are red solid line \([23]\) and red circles \([24]\), unbound intervals are gray line \([31]\) and for \( m_\pi > 230 \text{ MeV} \) (right y scale) lattice data for \( B_2 \) and predictions of EFT(\( \pi \)) matched to LQCD data (see sections \([19, 20]\) and \([19, 20]\)). Box heights correspond to lattice uncertainty and the vertical black line marks \( m_\pi \sim 198 \text{ MeV} \) at which \( ^1S_0 \rightarrow \infty \). All x scales are nonlinear.

There provide support that nature as represented by experimental data at physical \( m_\pi \) “flows” into SU(3)-large-\( m_\pi \) QCD as explored via the lattice method. Regardless of the counting scheme, a critical pion mass close to the physical value was found where the two-body scattering lengths diverge. The associated scale invariance of the two-nucleon system \( w.r.t \) any coordinate transformation \( \tilde{r} \rightarrow \lambda \tilde{r} \) for \( \lambda \in \mathbb{R}_+ \) indicates an infrared fixed point of the renormalization-group (RG) flow of the QCD coupling constant \( g_\pi \). How this infrared QCD fixed point is expressed in the RG flow of \( \chi \)EFT is unknown and presents a problem intimately related to the development of a consistent power counting. In contrast, the flow is known for an effective theory that considers solely contact interactions amongst nuclei (see Sec. \([13]\) below). Although not manifest in its coupling constants, the peculiar limit-cycle trajectory of the leading three-body LEC, emergent from the fixed point, should find its signature in the \( \chi \)EFT spectrum. Part of this signature, namely remnants of an Efimov spectrum \([17]\), was found \([18]\) when the authors employed an interaction derived in Ref. \([19]\) in the three-nucleon system. The characteristic asymptotic ratio of binding energies at the deuteron-neutron accumulation point of \( \sim 515 \) was approximated, while the ground state, \( i.e. \), the triton was stable \( (B_t = 4 \pm 2 \text{ MeV}) \) in the considered pion-mass interval between 190 and 210 MeV. The \( \chi \)EFT predictions in Fig. \([8]\) were obtained at a fixed cutoff and thus do not allow an assessment of the uncertainty like the displayed LQCD and physical experiments.

In essence, all three approaches \([15, 16, 18]\) employ \( \chi \)EFT to define a Lagrangian with implicit and explicit dependence on \( m_\pi \). They differ in the power counting and thereby the dependence of observables on the parameters of \( \mathcal{L} \), \( i.e., \), \( m_\pi \). Pion-deuteron scattering experiments in the physical or lattice world separating the \( D_i \) counter terms from the \( C_i \)'s, like \( (N^T N)^2 \pi \pi \), or LQCD \( (N^T N)^2 \) amplitudes at different \( m_\pi \) will allow for a unique determination of the LECs, and thereby aid the development of the above mentioned consistent power counting.

\( ^8 \) This nontrivial inference assumes a valid approximation of QCD by \( \chi \)EFT. Only then does a statement based on a \( \chi \)EFT calculation about a QCD parameter make sense.
To that end, the EFT analysis in Ref. [27] demonstrated a match of $\chi$EFT to LQCD instead of experimental data similar to Ref. [27]. In the KSW [29] and BBSvK [33] counting schemes, this work considered the EFT convergence in the $1S_0$ and $3S_1$ scattering lengths and effective ranges up to NNLO with LQCD binding energies and phase shifts at $m_\pi = 450$ MeV as input. The convergence rate was found small in all 4 observables and different from the ones extracted from the LQCD phase shifts via an effective range formula. The latter discrepancy was explained by fitting to data beyond the natural scale of the theory.

The results at 450 MeV pion mass are entering a region where they can be analyzed with low-energy theorems as developed in Ref. [38]. These theorems provide expansions of the modified effective range parameters in $m_\pi/m_\rho$. This assumes a long-range interaction provided by $\chi$EFT, hence $m_\pi$ as the light scale, and a short-range potential of range $m_\rho^{-1}$. The uncertainty due to an unknown $m_\pi$ dependence on the short-range part of the interaction was found to increase significantly with $m_\pi$. Part of this uncertainty is, again, the isolation of the explicit pion-mass dependence in the four-fermi contact LECs. The method applied to $m_\pi \leq 400$ MeV yields results for $a/r$ which are consistent within error bars with the $m_\pi \sim 450$ MeV data (see Table II and compare Fig. 4 with Figs. 6 and 7 in Ref. [38]) regardless of being extrapolated from the physical $m_\pi$ or constrained by LQCD deuteron data directly at $m_\pi = 430$ MeV. Even a naïve continuation of the derived $m_\pi$ dependence of the ratio $a/r$ is “in line” with the peculiar value of 2 LQCD finds at 806 MeV pion mass (see discussion around Eq. (9)).

The above attempts employ effective field theories to parameterize two- and three-nucleon amplitudes, i.e., the scattering lengths in singlet and triplet S-wave channels $1/3a_{np}$ and the triton binding energy $B_\alpha$, with the pion mass. The pion mass and the theoretical uncertainty of $\chi$EFT can then be passed on to another effective theory which is more practical for larger nuclear systems. This matching between a pionful and pionless theory cited above in Refs. [27, 35], leads to the next section where we summarize work which already employed the idea to propagate the pion-mass dependence from two- into few-nucleon amplitudes with the caveat of an unknown uncertainty in those amplitudes and therefore $m_\pi$ dependence of the underlying theory which is then no longer QCD but $\chi$EFT.

III. $\chi$EFT $\rightarrow$ EFT($\pi$)

Above, we reviewed the derivation of the low-energy effective theory $\chi$EFT from its underlying theory QCD for a practical description of few-baryon systems at a momentum scale $\sim O(m_\pi)$. In this section, we summarize results of a contact effective theory without pions (EFT($\pi$)), see Sec. V B for references). Its underlying fundamental theory is $\chi$EFT from which it inherits the QCD-parameter dependence. Because of the stated unresolved problems of $\chi$EFT to quantify its theoretical uncertainty those errors are also not properly propagated to EFT($\pi$). The set of observables, thereby parameterized by couplings of QCD and the matching conditions, is smaller than that accessible with $\chi$EFT which in turn does not cover the measurable parameter space described by QCD. Matching conditions, in general, encompass: (i) The point in the QCD parameter space where amplitudes are equated, i.e., the relevant QCD interaction parameters ($m_q$, strong and electromagnetic couplings $g_\rho$ and $\alpha$); (ii) A set of consistent quantum numbers specifying the observables which are expected to be well described in either theory (e.g., the deuteron, $3a_{np}$, and/or the magnetic moment of the triton); (iii) A parametrization of the radius of convergence of the effective theory (power counting).

The resultant benefit of a practical theory to express few-nucleon observables in terms of QCD parameters was utilized for the prediction of the three-nucleon spectrum for $m_\pi \in (140, 200)$ MeV. In that work, the match to EFT($\pi$) was made to predict three-nucleon scattering observables and excited states of the triton. Prior to Ref. [31], the ultraviolet RG fixed point in the two-body and the limit cycle in the three-body sector of EFT($\pi$) was conjectured at a pion mass close to the physical point under the assumption that an amplitude relevant for the description of the three-nucleon system remains constant over the considered range of pion masses which includes the physical point from where its value was determined. This assumption was justified a posteriori in Ref. [31] by solving the three-nucleon problem with a $\chi$EFT potential and using the obtained pion-mass dependence of $B_\alpha$ to obtain the aforementioned first two states of an Efimov spectrum. The limit cycle as a periodic dependence of a coupling constant is manifest in the neutron-deuteron scattering lengths. In Ref. [31], the neutron-deuteron scattering length was found to diverge in both spin channels at a critical pion mass of $\sim 198$ MeV with EFT($\pi$) at LO. This is a noteworthy difference to the nucleon-deuteron system at physical $m_\pi$, where EFT($\pi$) predicts a limit cycle only for the doublet channel and the three-body momentum-dependent counter terms of the quartet channel appear at an

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9 The increasing effect with $m_\pi$ of the unknown $m_\pi$ dependence of the short-range interaction is visualized in Ref. [38] which in turn will also serve as a measure of future rigorous $C_i$ and $D_i$ assignments.

10 The one-dimensional pion-mass dependence will be augmented once electromagnetism and non-degenerate up and down-quark masses are considered in the lattice simulations.

11 Originally coined $\Lambda^*$ in Ref. [20].

12 Only in the doublet channel, this constant can be identified with a momentum-independent six-fermi vertex. In the quartet channel, any six-fermi counter term has to be momentum dependent or break (iso)spin symmetry to affect the quartet $S$-wave amplitude.
order expected by dimensional analysis. An ensuing NNLO analysis\cite{31} confirmed this behavior of the scattering lengths showing traces of a log-periodic behavior in the vicinity of the critical pion mass of about 198 MeV. The dependence of the nuclear landscape on $m_\pi$ ranging from 130 MeV to 806 MeV up to $A=3$, the two complementary studies of Refs. \cite{31,41} give in combination to the analyses presented below in Sec. \ref{sec:results} is shown in Figs. \ref{fig:binding} and \ref{fig:binding2}.

The LECs were deliberately chosen to that end. Singlet and triplet scattering lengths could diverge at different $m_\pi$. That magnetic fields could be tuned to realize diverging scattering lengths in all two-nucleon channel was suggested in Ref. \cite{42}.

\begin{figure}[htp]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{(Color online) Pion-mass dependence of the three-nucleon binding energy in the triton channel. For $m_\pi < 210$ MeV, the results\cite{31} utilize a match of $\chi$EFT to data at physical $m_\pi$. For $m_\pi \gtrsim 300$ MeV, LQCD data is shown with box heights representing the lattice uncertainty (see Sec. \ref{sec:results}). The vertical black line marks $m_\pi \approx 198$ MeV at which $^3a_{np} \rightarrow \infty$.}
\end{figure}
the then bound nn-singlet state in the definition of $2a_{nd}$. The latter rises from $-\infty$. In the neutron-deuteron quartet channel (right lower panel), the scattering length $4a_{nd}$ goes to $+\infty$ at the critical point and cannot be defined in the gap between $m_{\pi}^{\text{crit}}$ and some $m_{\pi}$ below 300 MeV where the $^3S_1$ state is bound, again. At present, we know of no systematic way to connect the $\chi$EFT predictions with the LQCD measurements. To understand the emergence and disappearance of bound states with $m_\pi$, LQCD calculations tracing, e.g., the two-nucleon spectrum over some range of $m_\pi$, would indicate whether the shallower state at 510 MeV corresponds to the lowered pole at 806 MeV or enters at threshold. The statistical uncertainty is the main obstacle to answer the seemingly simple question how $B_d$ reacts to an infinitesimal change in $m_\pi$. The slope of $B_d$ as a function of $m_\pi$ could be obtained for $m_\pi \gtrsim 800$ MeV and provide valuable information for the extrapolation to smaller pion mass.

We will see below (discussion around Fig. 7) hints that the nuclear interaction at large $m_\pi$ is well approximated by a SU(4) Wigner-symmetric theory. Assuming that $\chi$EFT propagates the explicitly broken SU(2) or SU(3) quark-flavor symmetry well into the interaction between nuclei, the $\chi$EFT results are not consistent with the LQCD results, which use degenerate up and down quark masses. This discrepancy highlights the significance of the symmetry breaking for an extrapolation.

In this section, we summarized results of $\chi$EFT and EFT($\pi$) as approximations of QCD. Bound and scattering properties of nucleon systems with $A \leq 3$ were found sensitive to the QCD-parameter $m_\pi$. This behavior mediated by $\chi$EFT was obtained without a systematic power counting and ambiguities due to insufficient data. How a contact theory can complement $\chi$EFT in the facilitation of scattering calculations while converging in its predictions of the three-nucleon spectrum was also shown. Until now, $\chi$EFT inherits only its symmetries from QCD. The LECs are determined by a match to experimental data at $m_\pi \sim 140$ MeV because LQCD cannot be solved for the necessary amplitudes for $m_\pi \lesssim 300$ MeV. However, the variety of nuclear behavior to expect when substituting the scattering lengths and effective ranges at physical $m_\pi$ with LQCD predictions at some $m_\pi \lesssim 200$ MeV where ChPT seems to be applicable as matching conditions, was exemplified.

The analysis of nuclei at larger $m_\pi$, where ChPT is not a reasonable approximation of QCD but where LQCD is practical, today, is the subject of the following sections.

## IV. LATTICE QCD FOR MULTI-HADRON SYSTEMS

To predict nuclear observables rigorously by solving QCD, a chain of systematic approximations has to be employed. Each component of this chain is built as an effective field theory and as such provides a prescription to recover the underlying theory. The belief in QCD as the relevant theory from which nuclei emerge can be tested by calculating observables from correlation functions of the type

$$\langle \hat{O} \rangle \equiv 2^{-1} \int DA_\mu DqD\bar{q} \hat{O}(q, \bar{q}, A_\mu) e^{-\int d^4x L_{\text{QCD}}(m_\pi, g_s)} .$$

The lattice methodology evaluates this path integral and the partition function $Z = \langle 1 \rangle$ over fermionic quark fields $(q, \bar{q} = q^\dagger \gamma^0)$ and gluon gauge fields $(A_\mu)$ by discretizing Euclidean space time. Thereby, LQCD constitutes the first approximation — or chain element in the above terminology — to relate QCD parameters, namely the quark masses $m_q$ and the QCD length scale, implicit in the coupling strength $g_s$, to nuclear physics.

For few hadron systems, in fact, it is the only approximation necessary. For spectral details beyond the ground state, larger systems, where $\hat{O}$ is comprised of $\geq 5$ baryons, and/or realistic values of $m_q$, additional EFTs are necessary. The various EFTs, ChPT, and EFT($\pi$), for instance, interface through amplitudes with $\hat{O}$ resembling an operator that can be evaluated in both, the underlying and effective theory.

### A. Methodology

First, the defining parameters of a lattice calculation, roughly taken as the spacing $b$ of the lattice, its space-time volume $L^3 \times T$, the statistical properties of the gauge-field ensemble, and the $m_q, g_s$ values affect the accuracy of the extraction. Furthermore, an appropriately chosen shape for the operator $\hat{O}$ increases the accuracy. To access spectra of few-nucleon systems, $\hat{O}$ is composed of hadronic interpolating fields with the generic structure

$$\mathcal{N} \equiv \sum_a \Gamma_h^{a_1 \ldots a_n} \bar{q}(a_1) \ldots \bar{q}(a_n) , \quad a_i \in \{\text{color, flavor, spinor, spatial } x\} .$$

A hadron is identified with a certain irreducible representation of SU(3) flavor, e.g., $n, p, \Sigma, \Xi \leftrightarrow 8$, and therefore neither the number of quark fields $n_q$ nor the structure of the tensor $\Gamma_h$ are uniquely defined by the hadron.
Spectroscopic information on hadrons is obtained through the selection of an appropriate operator $\hat{O}$, which creates a state of defined parity $\pi$, cubic angular momentum $l$, isospin, strangeness at some initial space-time point and annihilates it later. The so called smearing of the quark fields to increase the overlap of the interpolating fields with an eigenstate of the QCD Hamiltonian via

$$q(x, t) = \sum_y G(x, y, A_\mu(t))q(y, t)$$

(5)

utilizes this freedom (the calculations considered in Table I differ in the weight function $G$ and to which parts of $\hat{O}$ the technique is applied).

All lattice calculations under review, here, employ a source and sink structure for $\hat{O}$. States are created with parity $\pi$, total angular momentum $J^z$, isospin $I^z$ and strangeness, and baryon number $A$ at the same time $t_0$ to annihilate them at $t > t_0$. To access the spectrum of a three-baryon system like the triton or Helium-3, for example, correlation functions of the form

$$C_{N_1N_2N_3}\Gamma(p_2, p_3; t) = \sum_{x_1, x_2, x_3} e^{i p_2 \cdot x_1} e^{i p_3 \cdot x_2} e^{i p_3 \cdot x_3} \Gamma_{3\alpha_1\alpha_2\alpha_3} \times$$

$$\langle N_1^{\alpha_1}(x_1, t) N_2^{\alpha_2}(x_2, t) N_3^{\alpha_3}(x_3, t) \bar{N}_1_{\beta_1}(x_0, 0) \bar{N}_2_{\beta_2}(x_0, 0) \bar{N}_3_{\beta_3}(x_0, 0) \rangle$$

(6)

are used with spinor indices $\alpha, \beta$. The particle species is understood to be encoded in the $N_i$’s. The space-time points at which the source creates ($x_0, t_0 = 0$) and the sink annihilates ($x_i, t$) hadron(s) with quantum numbers selected through $\Gamma$ are choices like the projection on a state with defined momenta $p_i$. Another choice are so called wall sources, e.g., in Ref. [45], which project each quark field of zero momentum out of the vacuum viz. $q(t) = \sum_y q(x, t)$.

So far, nuclear observables have been calculated with fields $N$ resembling elements of the baryon octet, only. In Ref. [5], the proton mass is obtained with an $A = 1$ interpolating field $\bar{N}(x, t) = \epsilon_{abc}(u^{\alpha} T C_{\alpha \beta} d^\beta)u^{5\alpha}(x, t)$ corresponding to a color (indices $abc$) singlet which is combined from up ($u$) and down ($d$) quark fields defined at the same space-time point but smeared (Eq. (5)), and the charge conjugation matrix $C$ acting on the spinor components (index $\alpha$). At the hadronic level in Eq. (6) one chose to define the three sink and source operators at the same space-time point. In general, smearing these operators over the spatial coordinates is admissible, too, like shown in Eq. (6) for the individual quarks.

Back on the hadronic level this dependence on individual quark coordinates is necessary to extract not only spectral information, but also wave functions. The standard technique to access both, wave functions $\Psi(\rho_1, ..., A-1; k, t)$ and energy eigenvalues $E_n = 2\sqrt{k^2 + m_N^2}$, first translates the sink $N_s(x, t) = e^{H_{\text{QCD}} t} N_s(x, 0) e^{-H_{\text{QCD}} t}$ in time before the insertion of a complete set of states:

$$C(\{\rho_i^{s(\text{source}), si(nk)}\}_{i=1, \ldots, A_1}) \propto \langle N_{ni}(\{\rho_i^{si}\}_{i=1, \ldots, A-1, t_0 = 0}) \rangle$$

$$\times \sum_n \frac{e^{-E_n t}}{2E_n} \langle N_{ni}(\{\rho_i^{si}\}_{i=1, \ldots, A-1, t_0 = 0}) | n \rangle \langle n | N_{ni}(\{\rho_i^{si}\}_{i=1, \ldots, A-1, t_0 = 0}) \rangle$$

$$\rightarrow Z_n (\{\rho_i^{si}\}_{i=1, \ldots, A-1, k}) Z_n (\{\rho_i^{si}\}_{i=1, \ldots, A-1, k}) \Psi_0 (\rho_1, ..., A-1; k, t) e^{-E_0 t}.$$  

(7)

For $t \rightarrow \infty$, the contribution from the lightest hadronic state that couples to source and sink dominates. The overlap factors $Z_n$ depend, of course, on the sink/source structure and determine the minimal propagation time $t$ for a practical extraction of $E_n$. The residual time dependence in the sink/source overlap with the eigenstates indicates the statistical noise which is present through Monte-Carlo sampling in realistic calculations. It is noteworthy, that $n = 0$ in Eq. (7) does not need to be the ground state for a given time interval. The state of interest, ground, excited, or continuum state, guides the structure chosen for the sink and source. Smearing of quark fields in the nucleon interpolating field, for instance, is inspired by the knowledge of the extended nature of the nucleon. For scattering states (see paragraph below) in the center-of-mass frame, the correlator with a sink structure which projects hadrons with momenta of equal magnitude but opposite direction will be dominated by an excited state over some time interval, because the overlap factors $Z_n$ in Eq. (4) win over the exponential decay.

As wave functions and energies are calculated in a box of finite size $L$, scattering phases $\delta$ can be inferred from the correlation function. If the energy extracted via Eq. (7) corresponds to a scattering state with $E = 2k^2 / m_N$, Lüscher’s formula (original work in Refs. 40, 47, 48, for boosted systems see Ref. 43)

$$k \cot \delta = \frac{1}{\pi L} \sum_{i=1}^{\Lambda} \frac{1}{j^2 - \left( \frac{bL}{2\pi} \right)^2} - 4\pi \Lambda_j$$

(8)
TABLE I. Parameters of the LQCD measurements reviewed in this work. Few-nucleon systems are calculated on a $L^3 \times T$ lattice with spacing $b$ using three or two degenerate quark masses with resultant pion ($m_\pi$) and nucleon ($m_N$) masses in an ensemble comprising $N_{\text{cfg}}$ gauge configurations.

| QCD version | $L$ [fm] | $T$ [fm] | $b$ [fm] | $m_\pi$ [MeV] | $m_N$ [MeV] | $N_{\text{cfg}}$ |
|-------------|---------|---------|--------|--------------|-------------|---------------|
| SU(3)       | 3.4 → 6.7 | 6.7 → 9.0 | 0.15   | 806          | 1634        | ≥ 1905        |
| SU(2)       | 2.9     | 5.8     | 0.09   | 701          | 1583        | 390           |
| SU(2)       | 2.9 → 5.8 | 4.3 → 5.8 | 0.09   | 510          | 1320        | 200           |
| SU(2)       | 2.5     | 4.0     | 0.13   | 354,493,593  | –           | 490 → 660     |
| SU(2)       | 2.8 → 5.6 | 7.5 → 11.2 | 0.12  | 450          | 1226        | ≥ 1000        |
| SU(2)       | 4.3 & 5.8 | 4.3 & 5.8 | 0.09   | 300          | 1050        | 400 & 160     |
| SU(3)       | 3.9     | 3.9     | 0.12   | 469 → 1171   | 1161        | 2274          | 720 → 420     |

with a sum over all integer three-vectors $j$ with magnitude smaller than the lattice-momentum cutoff $\Lambda$, relates the energies to phase shifts.

The algorithm employed by the NPLQCD collaboration to calculate multi-baryon correlation functions summarizes this section.

First (hadron level), an $A$-baryon operator is defined via Eq. (4) with flavors taken as elements of the baryon octet. All baryons comprising the source in Refs. [8, 44] are defined at a single lattice site, i.e., the sum in Eq. (4) excludes spatial degrees of freedom, and the fields are created at the same $\mathbf{x}_0$. The sink is chosen to project out a state with defined momentum for each individual baryon, i.e., the sum in Eq. (4) runs over all lattice sites and the factor $\Gamma_h^{a_1 \cdots a_{nq}}$ contains a plane wave $e^{ip \cdot x}$ for each baryon. In principle, the ensuing momentum dependence of the overlap function Eq. (7) provides a rigorous assessment of the typical baryon momenta within a nucleus. Measurements like this are desirable for the construction of effective theories (see Sec. [V]) as they could provide their typical scales.

Second (quark level), the baryon operators are substituted with smeared (smeared and point) interpolating operators, again defined at a single space-time point.

From correlation functions constructed in this way, properties of nuclei with $A \leq 4$ have been calculated at the SU(3) flavor-symmetry point and for degenerate up and down quarks with the strange at its physical mass (SU(2)). We compare the parameters of those calculations in Table I using the lattice spacing $b$, the temporal and spatial lattice size $T \times L^3$, the number of gauge configurations $N_{\text{cfg}}$, and the QCD version, i.e., SU(3) or SU(2) flavor, as standard. The selected calculations supersede the pioneering studies [30, 45], which rely on the uncontrolled quenched approximation. In contrast, the analyses under review in Sec. [IV] use controlled approximations, only, in the sense that the usage of more appropriate parameters as the ones given in Table I must yield results consistent with the old predictions within error bars.

B. Data

The data selected for this review represents the most advanced LQCD extractions (judged by the standard parameter set listed in Table I) of spectra in various two, three, and four-nucleon channels which employ controlled approximations, only. The two extractions of Refs. [8, 52] use the same mass for all three light quarks, set such that $m_\pi \sim 806$ MeV. The exploratory calculation three-baryon systems in Ref. [44], Ref. [7] (510 MeV), Ref. [5] (300 MeV), Ref. [30] (550 → 590 MeV), and Ref. [6] (450 MeV) use the physical value for the strange-quark mass but degenerate up and down-quark masses corresponding to the respective $m_\pi$. At the SU(3) symmetric point, data over a wide range of states, e.g., the H-dibaryon (ΛΛ), the hyper triton ($^3\Lambda$H), or hyper Helium-4 ($^4\Lambda$He), was extracted. No comprehensive theory has been devised for few-baryon systems comprised of all elements of the $A = 1$ octet based on this data and we review the strangeness $s = 0$ sector, only. We begin with the two-body sector where we compare features of the nuclear bound and scattering systems at physical $m_\pi$ with the LQCD data. The three and four-nucleon sectors are discussed in parallel, given their correlation.

Two nucleons — Eventually, LQCD must produce the two small (relative to $\Lambda_{\text{QCD}} \sim 4\pi f_\pi \sim m_\pi \sim 1$ GeV set by $g_\rho(\Lambda_{\text{QCD}}) \sim 1$) scales characteristic for nuclear physics in their observed unnatural ratio, namely, a momentum scale associated with the poles of the low-energy two-nucleon scattering amplitude, $(3a_{np})^{-1} \sim \sqrt{m_\pi B_2(3S_1)} \equiv \gamma_3 \sim 45$ MeV in the S-wave spin-triplet and $(1a_{np})^{-1} \sim \gamma_3 \sim -8$ MeV in the S-wave spin-singlet channel, and another related to the typical range of the nuclear interaction $\sim m_\pi^{-1}$. In addition to this unnaturally large value, $m_\pi/\gamma_3 \sim 3$, only the...
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TABLE II. Few-nucleon data from physical and computational experiments. LQCD uncertainties are upper bounds of the quoted values. Binding energies, $B_A(2S^+L_J)$, are given in MeV, $\gamma_3$ is the binding momentum of the two-nucleon triplet state, and $B_{J}^{Z}$ are the binding energies corresponding to the poles of the effective-range amplitude (see Eq. 21) parameterized by the central values of the scattering length $a_{np}$ and the effective range $r_3$. Data at $m_\pi \sim 806$ MeV has been updated.\(^{(20)}\)

| $B_A(1S_0)$ | 140 | 300 | 450 | 510 | 806 |
|-------------|-----|-----|-----|-----|-----|
| $B_A(3S_1)$ | 2.22 | $14.5 \pm 3.1$ | $14.4 \pm 3.2$ | $11.5 \pm 1.3$ | $22.6 \pm 5.2$ |
| $B_A(2S_{1/2})$ | 8.48 | $21.7 \pm 14$ | $-20.3 \pm 4.5$ | $55.8 \pm 6.2$ |
| $B_4(1S_0)$ | 28.3 | $47 \pm 27$ | $-43 \pm 14$ | $100.6 \pm 17$ |
| $\frac{a_{np}}{\gamma_3}$ | 3.11 | $-3.27 \pm 14.7$ | $-2.02 \pm 0.64$ |
| $\frac{m_\pi}{\gamma_3}$ | 3.0 | $2.4$ | $3.4 \pm 0.4$ | $4.1$ | $4.2 \pm 0.5$ |
| $B_2^{+/1}(1S_0)$ | $-\infty$ | $-\infty$ | $<1$ | $-\infty$ | $12.8$ |
| $B_2^{+/1}(3S_1)$ | $2.21$ | $-\infty$ | $<1$ | $-\infty$ | $24.0$ |
| $B_2^{+/1}(3S_0)$ | $-\infty$ | $-\infty$ | $12.3$ | $-\infty$ | $25.7$ |
| $B_2^{+/1}(3S_{1/2})$ | $34.8$ | $-\infty$ | $14.2$ | $-\infty$ | $34.5$ |

TABLE III. Two-nucleon scattering lengths, $^{2S+1}a_{np}$ (in fm), and effective ranges, $r_{2S+1}$ (in fm), above the inverse physical and unphysical pion masses (in fm). The data sources are listed in Table I.

| $a_{np}$ | 140 | 353 | 450 | 493 | 593 | 806 |
|----------|-----|-----|-----|-----|-----|-----|
| $^{1}a_{np}$ | $-23.75$ | $0.63 \pm 0.50$ | $20 \pm 60$ | $0.65 \pm 0.18$ | $0.0 \pm 0.5$ | $2.3 \pm 0.46$ |
| $r_1$ | $2.75$ | $-\infty$ | $3.0 \pm 1.0$ | $-\infty$ | $-\infty$ | $1.1 \pm 0.14$ |
| $^{3}a_{np}$ | $5.42$ | $0.63 \pm 0.74$ | $-11 \pm 55$ | $0.41 \pm 0.28$ | $-0.2 \pm 1.3$ | $1.8 \pm 0.31$ |
| $r_3$ | $1.74$ | $-\infty$ | $3.4 \pm 1.8$ | $-\infty$ | $-\infty$ | $0.91 \pm 0.14$ |
| $m_\pi^{-1}$ | $1.4$ | $0.56$ | $0.44$ | $0.40$ | $0.33$ | $0.25$ |

triton binding momentum has to be known for a predictive theory whose breakdown with the nucleon number $A$ is still unknown.

Associating scales analogously at heavier $m_\pi$, this parameter, i.e., in general the scale of the system compared to the range of the interaction, and here in particular $m_\pi/\gamma_3$, is still large (Table II, 6th row) and thus suggests an unnatural EFT. At physical $m_\pi$, the scattering length and effective range provide another pair of scales that approximate the low-energy two-nucleon spectrum equally well as a consequence of the pion dominating (ex- or implicitly) the nuclear interaction up to $\sim 100$ MeV, although it cannot account for the emergence of the unnatural size of the system. Both measures, $m_\pi/\gamma_3$ and $a/r$, are therefore of similar size. This is also found at $m_\pi \sim 450$ MeV, suggesting a similar analytic structure of the amplitude. The first fully-dynamical calculation of NN scattering parameters in Ref. \(^{(30)}\) extracts scattering lengths, only. These scattering lengths are $\sim m_\pi^{-1}$ (see Table III) and thus indicate natural NN systems at 350, 490, and 590 MeV pion masses in both spin channels. Here it is assumed that the interaction range is about the inverse pion mass. This assumption fails as shown in Ref. \(^{(52)}\), where $m_\pi \sim 806$ MeV does not approximate the effective interaction range well. By taking the lattice data for $a$ and $r$ as scales, a more natural theory in which these parameters are of the same size is implied. Comparing the respective ratio with scales given by the system’s binding momentum and $m_\pi$, yields a larger ratio.

A graphical summary of available scattering-length measurements as compiled in Table III is shown in Fig. 4. There, we display the experimental $^{1}a_{np}$ (filled black circle) and $^{3}a_{np}$ (empty black circle) together with the lattice data at larger pion mass ($^{1}a_{np}$: filled red, $^{3}a_{np}$: empty blue). The continuation (transparent areas for $m_\pi \gtrsim 350$ MeV) of the allowed region of scattering-lengths (opaque blue for triplet and opaque red for singlet) as extrapolated using the BBSvK power counting of Ref. \(^{(15)}\), is a speculation which is motivated by the log-periodic running of the three-nucleon momentum-independent interaction with a regulator parameter (see next paragraph and Ref. \(^{(57)}\)). As shown in Fig. 4, the NPLQCD data indicates a critical pion mass of about 440 MeV in addition to the one found at 198 MeV at which NN scattering lengths diverge. The limit-cycle assumption of a periodic behavior of two-nucleon scattering lengths, with the pion mass instead of a cutoff parameter, leads to a nuclear interaction at a larger pion
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FIG. 4. (Color online) Pion-mass dependence of the singlet (filled circles) and triplet (empty circle) neutron-proton scattering lengths $a$. Allowed regions for $a$ for $m_\pi \lesssim 350$ MeV (opaque) were derived with KSW power counting. The transparent regions are speculations inspired by the limit cycle as observed in the running of the three-body contact interaction.

mass identical to the one found in nature. Real nuclei could thus be calculated from QCD on the lattice at large pion masses, avoiding the uncertainty associated with light quark masses. From Fig. 4 one would naively expect this copy of the real nuclear two-nucleon world at about 850 MeV.

We turn the discussion to data sets which include effective ranges. Scales are related to the analytic structure of an amplitude, i.e., the position of a pole and the radius of convergence of some expansion around that pole. In case of the effective range expansion at physical $m_\pi$, $\gamma \sim a^{-1}$ and $r \sim m_\pi^{-1}$, respectively. In that sense, $\gamma \sim \sqrt{\mathcal{B}^2 m_N} \sim k_\pm(a, r)$ (see poles of Eq. (9) below) defines a scale even at $m_\pi \sim 806$ MeV. But $\gamma \sim \sqrt{\mathcal{B}^2 m_N} \gg k_\pm(a, r = 0)$ and $m_\pi/\gamma \gg a/r$ (Table II) imply additional non-analyticities within an $m_\pi$ radius around $\gamma$ beside the existing bound-state pole.

To parameterize the scattering amplitude using effective-range theory (ERT) as

$$T(k) = \frac{4\pi}{m_N} \frac{k \cot \delta - ik}{1} = \frac{4\pi}{m_N} \frac{1}{-\frac{1}{a} + \frac{1}{2} k^2 + \mathcal{O}(k^4) - ik}$$

(9)

is justified through a pole consistent with the non-perturbative binding energy (compare the 1st(2nd) and 7th(8th) row in Table II). The amplitude Eq. (9) has two poles at momenta $k_\pm = \frac{1}{2}(1 \pm \sqrt{1 - 2r/a})$, i.e., possibly two bound states. At physical $m_\pi$, $k_\pm^2/m_N \sim 2.2$ MeV, the binding energy of the deuteron, while $k_\pm$ is beyond the range of validity of the ERT and therefore not in disagreement with the missing experimental evidence for such a deep state. At $m_\pi \sim 806$ MeV, $2r/a \sim 1$ in both S-wave channels (see also discussion in Sec. VB), which implies relatively closely

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14 Of course, the role of the nucleon mass, and the effective range has to be taken into account.
\[ B_2(3S_1) - \delta < B_2^- < B_2(3S_1) + \delta \]

![Graph showing the dependence of two-nucleon binding energies as poles of the ERT scattering amplitude parameterized at \( m_r \sim 806 \text{ MeV} \) via \( ^1a_{np} = 2.33 \pm 0.46 \text{ fm} \), \( r(^1S_0) = 1.13 \pm 0.14 \text{ fm} \), and \( ^3a_{np} = 1.83 \pm 0.31 \text{ fm} \), \( r(^3S_1) = 0.91 \pm 0.14 \text{ fm} \).](image)

The two surfaces touch at a line which marks the double pole of the amplitude.

Spaced bound states. Considering the uncertainty in the lattice extraction of the scattering length and effective range yields via Eq. (9) the following scenarios of the two-nucleon spectrum: (i) two shallow states for \( r/a = (2 + \epsilon)^{-1} \) (dark and light red areas on the energy surface in Fig. 5), (ii) one shallow state and a spurious deep state if \( r/a \to 0.3 \), i.e., the analog to physical \( m_{\pi} \) (gray energy surface and its green projection onto the \( a - r \) plane), and (iii) no bound states but resonances for \( r/a \to 0.6 \) (hatched gray area in \( a - r \) plane). The limits correspond to the smallest and largest ratios consistent with the uncertainties.

Scenarios (i) and (ii) are also consistent with the spectral measurement of \( B_2 = 22.3 \pm 5 \text{ MeV} \) albeit only one bound state has been isolated. To discriminate a second, almost degenerate state is at present impossible as the individual extraction of energy levels from the effective-mass plots suffers from significant statistical noise. A ratio measurement between a singlet and a triplet two-nucleon bound state correlator will be less affected by that noise and could hint towards a second bound state. In Sec. VI B, we comment on the implication of this peculiar analytic structure for the construction of an EFT.

Three and four nucleons — For heavier systems, LQCD provides, at present, only data on bound states, until numerical precision will suffice to identify signals corresponding to two-fragment continuum states thus enabling the calculation of, e.g., nucleon-deuteron phase shifts via a generalization of Eq. (8).

At physical \( m_\pi \), the existence of bound states in the three-nucleon system is a universal consequence of the proximity of the two-nucleon system to the unitary limit \( |1,1a_{np}| \to \infty \). The fact that there is only one such state in the \( J^P = \frac{1}{2}^+ \) channel and its exact binding energy are then peculiar to the three-nucleon system and not fixed by the two-body scattering length and effective range. Furthermore, there is no physical bound state in the \( \frac{3}{2}^+ \) channel. In contrast to this independence of \( B^+ \) of the two-body \( a \)'s and \( r \)'s, the three and four-nucleon spectra are intertwined, and the emergence of two four-body states with every three-body state, as known from unitary bosonic systems, seems to generalize to the nuclear problem. Whether or not the Coulomb repulsion transforms one of the two bound \( J^P = 0^+ \) states to a shallow resonance and the other to the \( \alpha \)-particle will be a test for EFT(\( \pi \)) and its
TABLE IV. Binding energies (in MeV) of the shallowest two, three, and four-nucleon state relative to its lowest break-up
threshold, i.e., $\Delta_{21} \equiv B_2(^1S_0) - B_2(^3S_1) \approx 140$ MeV, $\Delta_{32} \equiv B_3(^3S_1/2) - B_2(^3S_1)$, and $\Delta_{43} \equiv B_4(^3S_0) - B_3(^3S_1/2)$. Uncertainties (insignificant at $m_\pi \sim 140$ MeV) were combined in quadrature. References to LQCD data at unphysical $m_\pi$ are the same as in Table I.

| $m_\pi$ | $\Delta_{21}$ | $\Delta_{32}$ | $\Delta_{43}$ | $\Delta_{43}/\Delta_{32}$ |
|---------|---------------|---------------|---------------|--------------------------|
| 140     | 2.2           | 6.3           | 20            | 3.2                      |
| 300     | 8.5 ± 2.3     | 7.2 ± 17      | 25 ± 41       | 3.5 ± 14                 |
| 510     | 7.4 ± 1.4     | 8.8 ± 5.8     | 23 ± 19       | 2.6 ± 3.8                |
| 806     | 16 ± 5.4      | 33 ± 11       | 44 ± 24       | 1.4 ± 1.2                |

The LQCD calculations at higher $m_\pi$ identify single bound states in the triton and $\alpha$ channels, too. There is no data on other three or four-nucleon channels. In the strangeness $s = -1$ sector, the NPLQCD collaboration finds a bound $\frac{1}{2}^+$ hyper triton. At the considered SU(3) symmetric point, this state hints at a corresponding bound $s = 0$ quartet state. For $s = -2$, a second four-body $0^+$ bound state is found, reminiscent of the aforementioned universal tetramer pair associated with each three-body state in the two-body unitary limit. Given the degenerate $^3H$ and $^3\Lambda\Lambda$ three-body, and $\alpha$ and $^4\Lambda\Lambda\Lambda$ four-body states, and the above conjectured two-level structure of the two-nucleon system, LQCD investigations of possible $\frac{1}{2}^+$ three and additional $0^+$ four-nucleon bound states could reveal striking differences of few-nucleon systems at larger pion masses.

Beside the sheer existence, the relation of binding energies, i.e., thresholds, and binding energies per nucleon in nuclei of different $A$ are relevant for an effective description of heavier systems. At physical $m_\pi$, the triton is bound by $\sim 6.5$ MeV relative to the deuteron-neutron threshold and interpreted as shallow, corresponding to an Efimov state. The $\alpha$--particle is bound by $\sim 20$ MeV relative to triton-proton and considered a universal feature. Some hints to whether or not this interplay between two, three, and four-nucleon states persists at higher $m_\pi$ are found in the relative threshold positions, compiled in Table IV. For all $m_\pi > 140$ MeV, the triton is closer to the deuteron-neutron threshold than the $\alpha$--particle is to the noninteracting triton-proton system. The three-nucleon ($\Delta_{32}$) and two-nucleon break-up energies are of the same order and decrease from 806 to 510 MeV. At $m_\pi \sim 300$ MeV, a bound or an unbound three-nucleon system are within uncertainty limits.

The gap between the measured bound state energy of $\alpha$ and its lowest break-up threshold $\Delta_{43}$ is larger than $\Delta_{32}$ but of the same size or smaller than the scale set by $B_d$. The ratio $\Delta_{43}/\Delta_{32}$ decreasing with $m_\pi$ and $\Delta_{43}$ increasing simultaneously, is reminiscent of the scattering-length dependence of the two and three-body systems for $a \to \infty$. With decreasing $m_\pi$, the triton approaches threshold. The discussion below on the neutron-deuteron scattering length suggests a diverging three-nucleon amplitude at zero energy and thereby the analog of a limit cycle in the four-nucleon, three-body deuteron-neutron-nucleon system.

We summarize this section at the beginning of Sec. V B which reviews a theory trying to describe the data consistently.

V. NUCLEAR THEORIES

The interpretation of the above results as a data base for a theoretical analysis is analogous to the way early experiments on nuclei initiated theoretical nuclear physics. A theoretical analysis of lattice measurements is justified if computational resources are not expected to be available in the near future for the nuclear properties of interest. Two approaches to a systematic understanding of nuclear lattice data are available. In effect, they generalize the concept presented in Sec. III by matching a nuclear contact theory to LQCD amplitudes and the application of the ensuing theory to few-nucleon systems. The two methods differ in the matching condition and the contact theory. How exactly those differences lead to inconsistent postdictions is not known. Thus, we deem a brief summary of their respective technique and basic assumptions as useful.

15 For recent work on the assessment of the sensitivity of few-hadron LQCD results on the source structure see Ref. 62.
A. Matching wave functions

Analogous to an approach taken for the description of Kaon decays\cite{53}, a connection between QCD four-point correlation functions and non-relativistic nuclear potentials was made (the original work is Ref.\cite{45}, for a review see Ref.\cite{64}). While matching as reviewed between EFT and experiment or EFT\(\bar{g}\) identified amplitudes, here, two theories are matched through a set of wave functions. The underlying theory QCD defines a wave function which can be extracted with the lattice technology. This function satisfies a non-relativistic Schrödinger equation if the potential is chosen appropriately. The unknown is thus the potential while the wave function is input.

More specifically, a relativistic Nambu-Bethe-Salpeter wave function (introduced in Ref.\cite{53} for single hadrons) is extracted with a correlation function as in Eq. (7). Ref.\cite{45}, in particular, uses a so-called wall source for the extraction with the lattice technology. This function satisfies a non-relativistic Schrödinger equation if the potential becomes useless. This is thus the potential while the wave function is input.

The same potential can be constructed from the ground-state wave function obtained via Eq. (11) from the stationary Schrödinger equation. Therefore, one can use a rescaled correlation function \(R(x, t) \equiv C(x, t) / (e^{-m_N t})\) to define a potential \(U\) which, in general, acts between two-nucleon channels and thus shall be, like \(C\), understood as a matrix, via

\[
\left\{ -H_0 - \frac{\partial}{\partial t} + \frac{1}{4m_N} \frac{\partial^2}{\partial^2} \right\} R_{\alpha}(x, t) = \sum_{\alpha'} \int d^3y U_{\alpha\alpha'}(x, y) R_{\alpha'}(y, t). \tag{12}
\]

The same potential can be constructed from the ground-state wave function obtained via Eq. (11) from the stationary Schrödinger equation

\[
\left\{ -H_0 + \frac{p^2}{m_N} \right\} \Psi_{\alpha}(x) = \sum_{\alpha'} \int d^3y U_{\alpha\alpha'}(x, y) \Psi_{\alpha'}(y). \tag{13}
\]

A non-local, velocity-dependent ansatz

\[
U(x, y) = V(x, \nabla)\delta^{(3)}(x - y) = V_0 + V_\sigma \sigma_1 \cdot \sigma_2 + V_T S_{12} + V_{LS} L \cdot S + \mathcal{O}(\nabla^2) \tag{14}
\]

was chosen\cite{52} for the nuclear potential in a given channel. The expansion shown on the right-hand side defines a \(|x|\)-dependent coefficient functions which are constructed order by order inverting either Eq. (12) or Eq. (13).

Nuclear observables in the strangeness \(s = 0\) sector predicted with potentials as defined above relying on unquenched lattice QCD correlation functions (see Table\[\text{I}\]) include the scattering length in the two-nucleon \(^1S_0\) channel with c.m. momentum \(k\) derived in the limit

\[
a_{np}(m_\pi = 701 \text{ MeV}) = \lim_{k \to 0} k^{-1} \tan(k) = 1.6 \pm 1.1 \text{ fm}, \tag{15}
\]
Matching effective few-nucleon theories to QCD

FIG. 6. (Color online) Pion-mass dependence of momentum scales relevant for nuclear low-energy physics. The green-shaded boxes represent the range of binding momenta $\sqrt{2m_N B/A}$ for nuclei with $A = 2, 3, 4$ and binding energy $B$ as given in Table I. The pion-mass (black) and $\Delta$ (gray, see footnote 16) scales become equal for $m_\pi \sim 806$ MeV.

which was obtained from the central part in Eq. (14). Tensor and spin-orbit potentials have also been derived at $m_\pi$ up to 1.1 GeV and applied in variational calculations. Neither two nor three-nucleon bound states but indications for a shallow four-nucleon state, whose binding energy increases from $\sim 0.8$ MeV to $\sim 5.1$ MeV with $m_\pi$ decreasing over the considered range, have been found. Employing the same potentials via the Brueckner-Hartree-Fock method to $^{16}$O and $^{40}$Ca nuclei yields those nuclei bound consistent with the 5.1 MeV for the $\alpha$-particle.

Future work deriving a three-nucleon potential, as pioneered at $m_\pi \sim 1.1$ GeV in Ref. [69], will tell whether or not the above expansion of the nuclear interaction receives a significant contribution from such a force relative to higher-orders of the velocity expansion of the two-nucleon potential. In the following section IB, we will present an analysis of the role of three-nucleon force and its $m_\pi$ dependence in an EFT framework.

B. Matching correlation functions

A theory for few-nucleon systems which explains the LQCD data at $m_\pi \sim 510$ MeV and $m_\pi \sim 806$ MeV (for lattice data see Refs. [7, 8], and for the EFT Refs. [9, 43]) resembles the effective-field-theory approach to low-energy few-nucleon systems in the physical world\cite{70–72}. Compared to the method introduced in the previous section, it matches to QCD via observable binding energies instead of wave functions. The matching conditions, as part of the definition of an EFT, replace real-world data with LQCD predictions for the input — we are exploring the domain of the transparent sheets in Fig. 1.

Nucleons as eigenstates of QCD are canonically defined as an isospin doublet belonging to the lowest-mass baryon SU(3) octet. The first amendment to the most general SU(3) invariant Lagrangian, constructed solely from this octet, couples it, first, to the lowest-mass meson octet and second, to the lowest-mass baryon decuplet. A comparison between (i) the nucleon mass $m_N$, (ii) a scale associated with the excitation of a nucleon to a $\Delta$, $\sqrt{2m_N (m_\Delta - m_N)}$, (iii) the mass of the pion $m_\pi$, and (iv) the binding momenta of an $A$-body nucleus, $\sqrt{2m_N B/A}$ gives an indication whether an approximation of those couplings, specifically, pion-nucleon, $\Delta$-nucleon, and $\Delta$-pion-nucleon, with contact
interactions amongst nucleons might be useful over some energy range for nuclear amplitudes. These scales are compiled in Fig. 6 for the different pion masses and led to the ansatz of a nuclear contact theory analog to the established EFT(\(\pi/\pi\)) at physical \(m_\pi\). In essence, this analogy utilizes: (i) the much smaller typical binding momenta of nuclei relative to the nucleon mass to justify a non-relativistic treatment implying small-momentum Lorentz symmetry; (ii) with the typical momenta in bound systems — identified with \(Q\) — up to \(A = 4\) being smaller than the lightest meson and lowest baryon excitation \(m_\pi - m_\Delta\) — collectively denoted \(M\) — an (iso)spin 1/2 nucleon field \((N = (n, p))\) suffices as sole degree of freedom; (iii) the renormalized two-body LECs are assumed to be of order \(C_{2n} = \frac{4\pi}{m_N Q(M)^n}\) and hence only iterations of the two \(n = 0\) zero-derivative interactions are of the same order. The infinite iteration introduces the two-nucleon bound state poles; (iv) the three-nucleon momentum-independent interaction in the \(^2S_{1/2}\) (triton) channel is considered at the same order as the momentum-independent two-nucleon interactions in the \(^1S_0\) and \(^3S_1\) channels. This unnatural enhancement and the non-perturbative treatment assumes that the leading three-body interaction follows a limit cycle analogous to physical \(m_\pi\) in Ref. [57]. This approach was demanded by the cutoff dependence of the triton ground state with only the leading-order two-nucleon interaction. Two observations support the underlying assumption of a limit cycle: first, the parabolic increase of the triton ground state energy as a function of the cutoff in the absence of a three-body force, and second, the appearance of an additional state at some critical cutoff value.

The Lagrange density \(^17\) defining the theory together with the size estimates for the LECs in an operator basis that splits SU(4) symmetric \((C^S_0)\) and asymmetric \((C^T_0)\) components reads

\[
\mathcal{L} = \bar{N} \left( i\partial_t + \nabla^2/(2m_N) \right) N - \frac{1}{2} \left( C^S_0 \left( N^T N \right)^2 + C^T_0 \left( N^T \sigma N \right)^2 \right) + D_1 \left( N^T N \right)^3.
\]

\(^16\) The scale in Fig. 6 in contrast, considers the effect of intermediate states under natural assumptions for the LECs. Under these assumptions, \(m_\pi\) remains the lowest threshold setting the convergence rate of the theory.

\(^17\) Space-time and field coordinates are rescaled such that \(\partial_t\) and \(\nabla^2/(2m_N)\) are of the same order as in NRQCD (see comment below).
We understand $\Lambda$ as a placeholder for any cutoff introduced to regulate the theory. It is common, $\mu, \nu$.

Table V. Leading-order EFT($\pi$) post ($m_\pi$ $\sim$ 140 MeV, $B_{\alpha}$) and predictions\(^{[9,13]}\) for the quartet and doublet neutron-deuteron scattering lengths $^1a_{nd}$ and $^2a_{nd}$ and for the $\alpha$-particle binding energy $B_{\alpha}$ at three pion masses.

| $m_\pi$ [MeV] | $^1a_{nd}$ [fm] | $^2a_{nd}$ [fm] | $B_{\alpha}$ [MeV] |
|--------------|-----------------|-----------------|-------------------|
| 140          | 5.5 $\pm$ 1.3   | 0.61 $\pm$ 0.50 | 24.9 $\pm$ 4.3    |
| 510          | 2.3 $\pm$ 1.3   | 2.2 $\pm$ 2.1   | 35 $\pm$ 22       |
| 806          | 1.6 $\pm$ 1.3   | 0.62 $\pm$ 1.0  | 94 $\pm$ 45       |

The EFT and power-counting scheme implied by (i) the regularization of contact interactions with Gaussian functions on the relative coordinate $\delta(r) \rightarrow \Lambda^{-\frac{3}{2}} e^{-\frac{3}{2}r^2}$, and (ii) the calibration of the $\Lambda$-dependent LECs to $B_{\alpha}$, $B_d$, and $B_t$. The necessary calculations numerically solved the appropriate two and three-body Schrödinger equations.

An approximate Wigner SU(4) symmetry of nuclear interactions at $m_\pi$ $\sim$ 806 MeV has been noted in light of the degenerate (within lattice uncertainty) two-nucleon $^1S_0$ and $^3S_1$ ground-state energies in Ref. \(^{[75]}\) and scattering length to effective range ratios $a/r$ $\sim$ 2 in Ref. \(^{[52]}\). This independence of the nuclear force w.r.t. SU(4) spin-isospin transformations $\delta N = i\alpha_{\mu\nu}\sigma_\mu\tau_\nu$ translated into the LECs as shown in Fig. \(^{[7]}\). While the relatively small SU(4) asymmetric LEC $C_{\alpha\alpha}$ can be understood from the large scattering lengths in both channel\(^{[74]}\) at physical $m_\pi$ (lower red solid line in Fig. \(^{[7]}\)), the degeneracy $^1a_{np} \sim ^3a_{np} \sim 2$ fm is explicit at $m_\pi$ $\sim$ 806 MeV. At physical $m_\pi$, SU(4) invariance results from being close to the unitary limit, while at larger $m_\pi$, it seems to emerge as a unique feature of QCD!

The power counting is justified via RG invariance. Namely, for a cutoff EFT, predictions must converge at every order if the RG-flow parameter $\Lambda \rightarrow \infty$. Furthermore, cutoff dependence at a given order can be eliminated at some higher, not necessarily the next, order, where an LEC with the right scaling counters the dominating $\Lambda$ dependence which is not eliminated by the lower-order LECs\(^{[13]}\). At present, leading-order predictions with $\Lambda = 2 - 8$ fm$^{-1}$ are available with no sign of inconsistencies in the power counting. Specifically, the $\alpha$ binding energy $B_{\alpha}$, the nucleon-deuteron scattering lengths in doublet $^2a_{nd}$ and quartet $^4a_{nd}$ were analyzed in that light. The former as a bound-state observable can be benchmarked with the lattice data, and the latter as scattering properties which constitute predictions, LQCD will\(^{[5]}\) measure the given resources to increase its numerical accuracy. The predicted observables are compiled in Table \(^{[7]}\) and also included as gray columns in Fig. \(^{[8]}\) for $B_{\alpha}$. In contrast to physical $m_\pi$, where the uncertainty of real-world experiments is insignificant relative to the absolute observed value, EFT($\pi$) applied to lattice data has to propagate the uncertainty in the amplitudes used to renormalize the theory, namely $B_{\alpha}$, $B_d$, and $B_t$. To justify the power counting, $\Lambda$-variation suffices but was shown\(^{[9,43]}\) to converge slowly. Calculations within the dibaryon formalism, which is more flexible in its regulator, could probe the sensitivity of nuclear systems at high quark masses to short-distance structure more comprehensively.

A different power counting altogether is required for the case of degenerate ($r/a = 1/2$) or closely-spaced ($r/a = 1/2 \pm \epsilon$) two-nucleon bound states in the same spin channel. In the case of $r/a = 1/2$, the effective-range amplitude Eq. \(^{[9]}\) has a double pole which cannot result from an iteration of momentum-independent terms as in Eq. \(^{[10]}\). The breakdown of a contact as an approximate of a Yukawa theory\(^{[20]}\) with the latter sustaining a shallow and an excited state was demonstrated in Ref. \(^{[75]}\). The similarities indicated in Ref. \(^{[75]}\) of the presumed two-nucleon LQCD spectrum with more than one shallow state to non-relativistic QCD (NRQCD) become more detailed in the description of heavy quarkonium in NRQCD\(^{[70]}\), i.e., bound states of heavy (mass $M$) quarks and antiquarks with typical binding momenta $1/\sqrt{MB}$ and splitting between radial excitations of order $2B$ (note the splitting between $B_2^-$ and $B_2^+$ in Table \(^{[1]}\)). The EFT developed in Ref. \(^{[70]}\) from full QCD for quarkonium also considers $\partial_t$ and $\nabla^2/(2M)$ of the same order (compare LO EFT($\pi$) in Eq. \(^{[10]}\) which uses the same counting for a shallow nucleon state), is approximately quark-spin independent, but retains the coupling to Coulomb and transverse gluon fields. The latter as a systematic way to incorporate the model scalar field as used in Ref. \(^{[70]}\) from QCD could serve as an alternative to the EFT($\pi$) approach if the second state is found on the lattice. Another way to implement an excited state in an EFT considers it as an additional degree of freedom.

Returning to the discussion of the results obtained with the EFT appropriate for the data as available at present,

18. $\mu, \nu \in \{1, 2, 3, 4\}$, $\alpha$ parameterizes an infinitesimal SU(4) transformation with SU(2) generators $\sigma_\mu(\tau_\nu) = \{1, \sigma(\tau)\}$ acting on (iso)spin degrees of freedom.

19. We understand $\Lambda$ as a placeholder for any cutoff introduced to regulate the theory. It is common, e.g., in solving the three-body problem with the dibaryon formalism to use a combination of cutoff and dimensional regularization. In that case, the limit $\Lambda \rightarrow \infty$ must be taken in both regularization schemes.

20. This is to be understood as any mechanism that produces a Yukawa potential, e.g., the coupling of the nucleon to a scalar field.
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{(Color online) Lowest energy levels in the $^1S_0$ and $^3S_1$ two-nucleon, the $J^\pi = \frac{1}{2}^+$ three-nucleon triton and $0^+ \alpha$ channel. For each system, the physical (red) value is compared with LQCD measurements at $m_\pi \sim 300$ (blue), 450 (hatched purple), 510 (green), and 806 MeV (purple). Postdictions for $B_\alpha$ of EFT($\pi$) are in gray. The black box represents $\chi$EFT predictions and its height the considered $m_\pi$ range of $(190,210)$ MeV. The other box heights resemble the lattice uncertainties.}
\end{figure}

Addressing the breakdown of the conceived EFT with the particle number, the $^5$He and $^6$Li ground states were analyzed\cite{9}. $^5$He was found bound for small and unbound for larger cutoffs implying an improper renormalization of which signatures were also found at physical $m_\pi$ in Ref.\cite{80}. For $^6$Li, results are limited to a single cutoff value and bind the system with the same energy per nucleon as in $\alpha$. Both, the five and six-nucleon system as characteristic features of nuclei pose challenges to any nuclear theory: the former for its inability to sustain a bound isotope, the latter for its halo and borromean nature. To relate those properties to either two-body ERE and triton parameters or a larger set which includes information about the short-distance structure, e.g., P-wave $NN$ phase shifts or four-nucleon resonance parameters, poses a problem which is unresolved at physical $m_\pi$, too. The analysis of the dependence and evolution of 5 and 6-body peculiarities on $m_\pi$ drives the development of EFTs for larger, physical nuclei at those crucial nucleon numbers.

\footnote{21 We gratefully acknowledge comments by H. W. Hammer on this point.}
TABLE VI. LQCD predictions for nuclear magnetic moments, polarizabilities, scattering length \( a \), and effective range \( r \) at \( m_\pi \sim 806 \text{ MeV} \). Magnetic moments are compared with physical values. Uncertainties are naive sums of the combined energy shift extraction and magnetic-field-dependence fit uncertainties plus finite-volume effects. In the gray cells, results follow naive shell-model expectations.

| State | \( a \) [fm] | \( r \) [fm] | \( \mu_{\pi06} \) [nNM] | \( \mu_{140} \) [nNM] | \( \beta^{(M0)} \) [10^{-4} fm^3] |
|-------|--------------|--------------|----------------|----------------|----------------|
| \( n \) | \( - \) | \( - \) | \(-1.981 \pm 0.023\) | \(-1.91\) | \(1.253 \pm 0.12\) |
| \( p \) | \( - \) | \( - \) | \(+3.119 \pm 0.097\) | \(+2.80\) | \(5.22 \pm 0.89\) |
| \( nn \) | \( 2.33 \pm 0.46 \) | \( 1.13 \pm 0.14 \) | \(-\) | \(-\) | \(1.872 \pm 0.20\) |
| \( pp \) | \( 2.33 \pm 0.46 \) | \( 1.13 \pm 0.14 \) | \(-\) | \(-\) | \(5.31 \pm 2.8\) |
| \( d \) | \( 1.83 \pm 0.31 \) | \( 0.91 \pm 0.14 \) | \(+1.218 \pm 0.125\) | \(+0.857\) | \(4.4 \pm 1.8\) |
| \(^3\text{H}\) | \( - \) | \( - \) | \(+3.56 \pm 0.23\) | \(+2.98\) | \(2.6 \pm 1.8\) |
| \(^3\text{He}\) | \( - \) | \( - \) | \(-2.29 \pm 0.15\) | \(-2.13\) | \(5.4 \pm 2.4\) |
| \( \alpha \) | \( - \) | \( - \) | \(-\) | \(-\) | \(3.4 \pm 2.2\) |

VI. ELECTROMAGNETISM

While for single hadrons the effect of the U(1) gauge symmetry was calculated with the lattice methodology \( e.g. \), the proton-neutron mass splitting, the application to systems with \( A > 1 \) remains impractical. What became feasible is the response of the nuclei with \( A = 1, 2, 3, 4 \) to a static external electromagnetic field in the form of magnetic moment and polarizabilities, and the capture process of a neutron by a photon via the emission of a photon \( np \to d\) \( e.g. \). In this section, we review the interplay between LQCD data and contact theories in the extraction of those observables.

Data \( - \) Here, as sketched above in Sec. IV.A, correlation functions of operators with non-zero overlap with the states of interest provide the spectral information. In Ref. \( \text{[82]} \), a uniform magnetic background is considered with the approximation of zero-sea-quark electric charges. The approximation allows to recycle the functional determinant which results from the analytical integration over the quark degrees of freedom in Eq. \( 3 \). It includes the U(1) gauge fields in the source and sink operators \( \hat{O} \). In comparison to the spectral and scattering calculations (first row Table IV), a smaller lattice, \( L^3 \times T \sim (3.5 \text{ fm})^3 \times 5.3 \text{ fm} \), but the same \( m_\pi \sim 806 \text{ MeV} \) and three mass-degenerate quark flavors (SU(3)) with a value corresponding to the physical strange-quark mass were used.

The energy levels were obtained in this partially-quenched approximation from effective-mass plots as a function of the magnetic field strength \( |\mathbf{B}| \). By relating these levels to the expected energy eigenvalues for a hadron \( h \in \{ \text{neutron, proton, deuteron, } nn, pp, ^3\text{H}, \ (3^4)\text{He} \} \) with charge \( Q_h \), a spin \( j \leq 1 \), \( j \sim e_z \), and zero momentum occupying the Landau level \( n_L \): \( E_{h,j_L}(\mathbf{B}) = \sqrt{M_h^2 + (2n_L + 1)|Q_h\mathbf{e}_z|} \),  

\[ -\mathbf{\mu}_h \cdot \mathbf{B} - 2\pi \beta_h^{(M0)} |\mathbf{B}|^2 - 2\pi \beta_h^{(M2)} \langle \hat{T}_{ij} B_i B_j \rangle + \mathcal{O}(|\mathbf{B}|^4), \]

the magnetic moment \( \mathbf{\mu}_h \), magnetic scalar, \( \beta_h^{(M0)} \), and tensor polarizabilities, \( \beta_h^{(M2)} \), were inferred. The brackets denote the expectation value of the traceless and symmetric combination of angular momentum generators \( \hat{j} \).

We list the results in Table VI and highlight the magnetic moments of \(^3\text{H}\) and \(^3\text{He}\) as they coincide within uncertainty margins with naive expectations based on the shell model which estimates them as a sum of \( pp \) (\( nn \)) singlet and \( n \) (\( p \)) moments as recognized in Ref. \( \text{[82]} \). This behavior resembles the relations observed at physical \( m_\pi \) where the deuteron moment is the sum of \( n \) and \( p \), while triton \((^3\text{He})\) is given by the moment of \( p \) (\( n \)), approximately. In their linear response, the bound nuclei behave similar to external magnetic fields relative to each other. An inference from \( \mu = 0 \) of the \( \alpha \)-particle and the moments of the smaller nuclei on the more appropriate cluster model — relative to the shell model for the triton and \(^3\text{He} \) — with \(^3\text{H}-p, ^3\text{He}-n \), and \( d-d \) comprising \( \alpha \) is: The \( \alpha \) ground state resides predominantly in the \(^3\text{He}-n \) and \(^3\text{H}-p \) configurations with a slightly larger contribution from the former.

With this data, knowledge about the response of nuclei at large \( m_\pi \) is available. The situation is peculiar enough to be rephrased, the measurements considered the coupling of the constituents of nuclei to an external field while
disregarding the mutual interaction via the same interaction. In essence, this treatment resembles that of composite particles in gravitational fields which takes into account the external field, only, and does not concern itself with the way an atom, e.g., contributes to the bending of space-time. This scenario of large external fields was analyzed in Ref. [32]. It was shown that for strong enough fields all two-nucleon bound states at \( m_\pi \sim 806 \text{ MeV} \) and \( m_\pi \sim 450 \text{ MeV} \) become unbound. At this unbinding field strength the scattering lengths diverge and it was conjectured[22] that at some \( m_\pi \) this field strength is the same for all \( NN \) states (see assumption in Ref. [39] introduced in Sec. III).

How the spectral data in a small magnetic background field can be utilized by a match to an EFT for the prediction of reaction observables was also demonstrated and is summarized below.

**Matching to EFT(\#)** — Under the assumption that EFT(\#) is applicable at \( m_\pi \sim 806 \text{ MeV} \), a generalized form of the relation Eq. (6) was employed[22] to relate energy eigenvalues in the presence of a background field to the LECs which couple the magnetic field to nucleons up to next-to-leading order (NLO). The generalization proceeds as in Ref. [80] and requires the calculations of amplitude poles in a finite volume with the interaction with the gauge field.

The electron charge \( e \), the nucleon mass \( m_N \), and the nuclear magneton of the neutron (proton) \( \kappa_{n(p)} \), which defines the isoscalar (isovector) nucleon magnetic moment \( \kappa_{n(1)} = \frac{1}{2} (\kappa_p + \kappa_n) \), assume values as predicted by LQCD at given \( m_\pi \). The low energy constant \( L_1 \) couples \( ^1S_0 \) (projector \( T_3 \)) and \( ^3S_1 \) (projector \( P_3 \)) and thus contributes at LO to the \( np \to d_\gamma \) capture. The operator corresponding to the \( L_2 \) LEC does not induce transitions between spin states. It contributes to the magnetic dipole moment of the deuteron and is thereby relevant[87] for asymmetries in the cross sections for circularly polarized photons impinging on an unpolarized deuteron target, \( d_\gamma \to np \).

In combination with the proper EFT(\#) Lagrangian governing the nuclear interaction at NLO, the \( np \to d_\gamma \) amplitude can be evaluated[44]. The zeros of the real part of the inverse of this amplitude in a finite volume are related to \( k \cot \kappa_{\pi} \). Scattering lengths and effective ranges for the incoming singlet and outgoing triplet were taken[22] from Ref. [82] to be degenerate, \( a_{np} \sim a_{np} \) and \( r_1 \sim r_3 \). With the isovector \( \kappa_1 \) measured independently in Ref. [82], \( L_1 \) is the only parameter left in the NLO amplitude’s poles which is left undetermined by single-nucleon and scattering parameters. It is related to energy shifts between the singlet and triplet eigenstates in the presence of the background field[88].

The predictive power was demonstrated by calculating the cross section of the radiative capture with the extracted value of \( L_1 \) at \( m_\pi \sim 806 \text{ MeV} \) in Ref. [82] and \( m_\pi \sim 450 \text{ MeV} \) in Ref. [83] where only the value for 806 MeV is quoted:

\[
\sigma_{806}(np \to d_\gamma) = 17 \left( ^{+101}_{-16} \right) \text{ mb} .
\] (20)

The asymmetric uncertainty is due to the non-linear input dependence of the cross-section which, in contrast to the renormalization of \( L_1 \), uses different effective-range parameters in the singlet and triplet channel.

It is noteworthy that the LQCD results of the magnetic moments indicate a similar internal structure of the \( A = 3 \) nuclei as found in nature, i.e., a bound singlet with a single nucleon which determines the spin. Relating this fact — the three-body response being given by that of a single nucleon — to the ratio of the separation energy to the binding energy of the core singlet, at \( m_\pi \sim 806 \text{ MeV} \) this ratio is \( \sim 2.1 \) compared to \( \sim 2.9 \) at physical \( m_\pi \) (see Table VII), the structure is expected to change significantly when decreasing the pion mass where the respective ratios at 510 MeV and 300 MeV suggest a shallow triton.

**VII. SUMMARY**

The status of a unified description of particle and nuclear physics was presented. This description comprises a chain of effective field theories with QCD on the particle, EFT(\#) on the nuclear end, and a bridge through \( \chi \)EFT.

Interest in the sensitivity of nuclear observables to variations in fundamental parameters — the pion-mass, in particular — arose with the inability to solve QCD with the physical pion mass. EFT expansions around \( m_\pi = 0 \) were employed to assess how sensitive nuclei react on a variation of \( m_\pi \) up to \( \sim 200 \text{ MeV} \). Those attempts were reviewed in Secs. II and III. The introduced framework of matching a chiral EFT to data at some \( m_\pi \), and to extrapolate nuclear amplitudes to physical or larger \( m_\pi \) while tracing carefully implicit and explicit \( m_\pi \) dependences,
is presented as an ansatz constituting one link in the EFT chain which can be implemented once experimental data can be replaced by QCD amplitudes. Work related to the connection between \( \lambda \text{EFT} \) and \( \text{EFT}(\hat{q}) \), as candidate for a few-nucleon EFT, was presented in Sec. \( \text{VIII} \).

The following Secs. \( \text{V} \) through \( \text{VI} \) elaborate on the approximation of QCD in light nuclei by matching contact EFTs to lattice calculations. The methodology of LQCD is overviewed with a focus on uncertainties presumably responsible for inconsistent EFT postdictions. The available nuclear LQCD data is compiled in Tables \( \text{III} \) and \( \text{IV} \) including only results obtained with controlled approximations.

In Sec. \( \text{VA} \) and \( \text{VB} \) we introduced two methods of deriving a nuclear effective interaction. A method based on a velocity expansion of a potential consistent with a QCD wave function, and the adaption of an EFT(\( \hat{q} \)) analog to LQCD amplitudes. The latter method is included in the summary of efforts to assess the electromagnetic structure of nuclei in Sec. \( \text{VI} \).

VIII. OUTLOOK

It is one aim of this article to review research on the extrapolation of QCD solved at unphysical pion masses to physical pion masses as well as the consequences of an enlarged pion mass on nuclear systems. Future work as suggested below concerns both aspects.

a. Outlook in nucleon number — The anecdotal introductory hint to the resemblance between the contemporary numerical effort and the historical, experimental one does not apply to larger nuclei. Their properties will not be accessible with LQCD within a similar time frame as physical experiments ventured beyond the deuteron. As cluster/halo EFTs constitute the next link in the EFT chain relating QCD parameters to nuclei with \( A > 4 \) and have so far been renormalized through a match to data unavailable from LQCD, their connection to EFT(\( \hat{q} \)) is crucial. Whether or not the bulk properties of \( A > 4 \) nuclei are a universal consequence of two- and three-nucleon data and thus could already be parameterized with \( m_\pi \) is unknown. Specifically, do the unbound Helium-5, the shallow \( \alpha N \) resonances, the halo structure of Helium-6 emerge at LO EFT(\( \hat{q} \))? It is an open question if an additional renormalization condition in form of a five or six-body counter term are necessary to put those poles in the respective LO amplitudes, or if those observables are sensitive to two-body P-wave interactions as which they should not be considered before next-to-next-to-leading order. Once EFT and few-body practitioners have addressed this question at physical \( m_\pi \), the amplitudes can be matched to cluster EFTs and thereby pass the \( m_\pi \) dependence to larger nuclei, systematically.

b. Refining the interaction — The role of the electromagnetic interaction between quarks for lattice nuclei is unknown but the response of nuclei to external magnetic fields has been explored in LQCD measurements. The latter included a fascinating demonstration how this method can probe extreme conditions inaccessible by experiments. These analyses were covered in Sec. \( \text{VI} \). Next to the numerical effort to implement the electromagnetic interaction in LQCD calculations, there remain conceptual issues hampering their EFT(\( \hat{q} \)) consideration. This is an instance where EFT can make predictions by including the long-range Coulomb force in EFT(\( \hat{q} \)) at large pion masses. Contrary to nature, the proton-proton system provides both a bound and scattering specimen to study the effect resulting from the combination of a long and a presumably relatively short-ranged force. At present, we assume that the energy gap between the triton and Helium-3 is approximately an invariant w.r.t. changes in \( m_\pi \). Noting a peculiar consequence of a widening gap, namely a conceivable unbound helion in the presence of a shallowly bound triton, shall motivate work in that direction.

Of interest for understanding the difference between hyper and ordinary nuclei are LQCD measurements at a fixed pion mass for both, the SU(3) symmetric point, and with a shifted strange-quark mass \( m_s \), i.e., explicit breaking of the flavor symmetry. The effect of, e.g., an infinitesimal shift in \( m_s \) on the two states of \( \frac{3}{2}^+ \) observed at \( m_\pi \sim 806 \text{ MeV} \) by NPLQCD, could indicate the significance of the SU(3) breaking relative to that of the pion mass for the shallowness of the hyper triton w.r.t. to the ordinary triton. The LQCD data available on strange nuclei has not yet been matched to a contact theory at 450 or 806 MeV. This requires a generalization of the SU(2) isospin-symmetric Lagrangian of EFT(\( \hat{q} \)) to SU(3) and thus a comprehensive theory for the baryon octet. In combination with LQCD data, which in contrast to nature is roughly as accurate for the \( s = 0 \) as it is for the \( s = -1 \) sector, will allow for a systematic study of the peculiar differences between strange and ordinary nuclei.

The observed \( m_\pi \)-insensitivity of the approximate SU(4) symmetry of the nuclear interaction as shown in Fig. 7 allows for the investigation of the relevant QCD parameters which cause this remarkable feature. This sensitivity

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22 general framework in Ref. \[89\] ; treatment of narrow resonances in Refs. \[90,91\] ; applications: proton halo \[23\], \( \alpha \) in Ref. \[93\] , two-neutron halo \[24\], \( 7 \text{Li} + n \rightarrow 8 \text{Li} + \gamma \) in Ref. \[25\] , \( 7 \text{Be} + p \rightarrow 8 \text{B} + \gamma \) in Ref. \[96\] , \( d + t \rightarrow N + \alpha \) in Ref. \[97\] .

23 The work in Ref. \[95\] can be considered the first rigorous matching of a microscopic and a cluster EFT.

24 For recent progress in the systematic treatment of the Coulomb force see Refs. \[98–100\] .

25 We acknowledge the explanation by M. Elyahu and N. Barnea to whom this idea belongs.
analysis can be performed at large pion masses since \( m_\pi \) does not seem to be significant for the effect. We are thus able to understand the mechanism behind the breaking of Wigner’s SU(4) symmetry also at physical \( m_\pi \) — an insight that would demonstrate the beneficial interplay between LQCD and nuclear EFTs.

**Understanding \( m_\pi \) sensitivity** — To predict low-energy renormalization-group fixed points of QCD, we desire data about the smooth dependence of nuclear spectra from LQCD. Available lattice calculations investigate nuclei at isolated pion masses. For the interpolation between them, no theoretical ansatz is known for \( m_\pi \) beyond the convergence rate of ChPT. Even within that radius, the murky formulation of \( \chi \)EFT in the few-nucleon sector hampers the uncertainty quantification of those interpolations and renders them less useful. To develop or refine interpolating and extrapolating theories, knowledge of whether or not a critical RG flow trajectory is approached will be of use. LQCD analyses of energy gaps between \( nn \) singlet and the deuteron, deuteron and triton, triton and \( \alpha \), and \( \alpha \) and Helium-5 at two infinitesimally close \( m_\pi \)’s would indicate whether or not one approaches a critical \( m_\pi \) for any of those systems.

Such an analysis touches the question how parameters of constituents characterize compounds, e.g., like the infinite two-nucleon scattering lengths in combination with a shallow triton “furnish” the \( \alpha \)-particle. The aforementioned cluster EFTs work well due to a separation of scales between the excitation energy of the \( \alpha \) and the shallow \( \alpha N \) poles. The ratio between the binding energies of Helium-5 — if bound at all — and the \( \alpha \) as one characteristic of nuclei at physical \( m_\pi \), namely their shell structure, is one crucial observable which as a ratio is more accessible to LQCD than the bare values. To analyze the \( m_\pi \) dependence of it would shed light on the emergence of the shell structure, the peculiar mass gap at \( A = 5 \), and the prominent role of the \( \alpha \) as a building block for larger nuclei.

How nuclear two-body systems can exhibit a peculiar behavior like a Feshbach resonance has been shown by simulating extreme magnetic fields which reside naturally only in cosmological objects like magnetars. Similar features of larger systems, like the development of a four-nucleon Efimov spectrum due to a triton near the deuteron threshold — a scenario which is admissible within error bars at \( m_\pi = 300 \text{ MeV} \) — and thus a mass gap at \( A = 3 \) is of undeniable empirical value to identify the underlying QCD mechanisms for such characteristics of the nuclear chart. To that end, ratios of binding energies are more important for the theoretical understanding than relatively less accurately measurable absolute binding energies.

**Focal-point system** — The five-baryon system as a gateway to heavier nuclei and refined EFTs concludes this article. First, it poses a challenge for numerical techniques, for LQCD as well as traditional few-body methods. Second, because its features are neither understood as emergent or unique. Restated, whether or not conventional EFT(\( \chi \)) applies to it is unknown. Third, its amplitudes are the canonical candidates for the bridge between single-baryon and cluster EFTs. The pion-mass dependence of the dynamics of this system is therefore key to an understanding of the emergence of complex phenomena in nuclei from the interactions governing its basic building blocks.

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---

[1] K. T. Bainbridge, *Phys. Rev.* **43** (Jan 1933) 103.
[2] J. M. B. Kellogg, I. I. Rabi, N. F. Ramsey and J. R. Zacharias, *Phys. Rev.* **56** (Oct 1939) 728.
[3] L. Durand, P. Ha and G. Jaczko, *Phys. Rev.* **D64** (2001) 014008, [arXiv:hep-ph/0101267 [hep-ph]]
[4] Original graph by U. van Kolck presented at “Hadrons and Hadron Interactions in QCD” 2015 at the Yukawa Institute for Theoretical Physics, Kyoto (Japan).
[5] T. Yamazaki, K.-i. Ishikawa, Y. Kuramashi and A. Ukawa, *Phys. Rev.* **D92** (2015) 014501, [arXiv:1502.04182 [hep-lat]]
[6] K. Orginos, A. Parreno, M. J. Savage, S. R. Beane, E. Chang and W. Detmold (2015) [arXiv:1508.07583 [hep-lat]]
[7] T. Yamazaki, K.-i. Ishikawa, Y. Kuramashi and A. Ukawa, *Phys. Rev.* **D86** (2012) 074514, [arXiv:1207.4277 [hep-lat]]
[8] NPLQCD Collaboration (S. R. Beane, E. Chang, S. D. Cohen, W. Detmold, H. W. Lin, T. C. Liu, K. Orginos, A. Parreno, M. J. Savage and A. Walker-Loud), *Phys. Rev.* **D87** (2013) 034506, [arXiv:1206.5219 [hep-lat]]
[9] N. Barnea, L. Contessi, D. Gazit, F. Pederiva and U. van Kolck, *Phys. Rev. Lett.* **114** (2015) 052501, [arXiv:1311.4966 [nucl-th]]
[10] S. Weinberg, *Nuclear Physics B* **363** (1991) 3.
[11] P. F. Bedaque and U. van Kolck, *Ann. Rev. Nucl. Part. Sci.* **52** (2002) 339, [arXiv:nucl-th/0203055 [nucl-th]]
[12] R. Machleidt and D. R. Entem, *Phys. Rept.* **503** (2011) 1, [arXiv:1105.2919 [nucl-th]]
[13] E. Epelbaum and U.-G. Meißner, *Ann. Rev. Nucl. Part. Sci.* **62** (2012) 159, [arXiv:1201.2136 [nucl-th]]
[14] S. Fleming, T. Mehen and I. W. Stewart, *Nucl. Phys.* **A677** (2000) 313, [arXiv:nucl-th/9911001 [nucl-th]]
Matching effective few-nucleon theories to QCD

Takumi Iritani (HAL QCD Collaboration) at “LATTICE2015” 2015, Kobe (Japan).

Eur. Phys. J. Kirscher, H. W. Grießhammer, D. Shukla and H. M. Hofmann,

S. Bour, S. König, D. Lee, H. W. Hammer and U.-G. Meißner,

P. F. Bedaque, H. W. Hammer and U. van Kolck,

E. Epelbaum and J. Gegelia, Phys. Lett. B716 [2012] 338,

E. Epelbaum, A. M. Gasparian, J. Gegelia and H. Krebs, Eur. Phys. J. A51 [2015] 71,

S. Dürr, PoS LATTICE2014 [2015] 006,

P. Langacker and H. Pagels, Phys. Rev. Lett. 30 [1973] 630.

E. Epelbaum, U. G. Meißner, W. Glöckle and C. Elster, Phys. Rev. C65 [2002] 044001,

J. C. Berengut, E. Epelbaum, V. V. Flambaum, C. Hanhart, U. G. Meißner, J. Nebreda and J. R. Pelaez, Phys. Rev. D87 [2013] 055018,

J.-W. Chen, T.-K. Lee, C. P. Liu and Y.-S. Liu, Phys. Rev. C86 [2012] 054001,

S. R. Beane, D. B. Kaplan and A. Vuorinen, Phys. Rev. C80 [2009] 011001,

D. B. Kaplan, M. J. Savage and M. B. Wise, Phys. Lett. B424 [1998] 390,

S. R. Beane, P. F. Bedaque, K. Orginos and M. J. Savage, Phys. Rev. Lett. 97 [2006] 012001,

E. Epelbaum, H. W. Hammer, U.-G. Meißner and A. Nogga, Eur. Phys. J. C48 [2006] 169,

S. R. Beane and M. J. Savage, Nucl. Phys. A713 [2003] 148,

S. R. Beane and M. J. Savage, Nucl. Phys. A717 [2003] 91,

E. Epelbaum, U.-G. Meißner and W. Glöckle, Nucl. Phys. A714 [2003] 535,

J. Soto and J. Tarrus, Phys. Rev. C85 [2012] 044001,

E. Epelbaum, W. Glöckle and U.-G. Meißner, Nucl. Phys. A637 [1998] 107,

V. N. Efimov, Sov. J. Nucl. Phys. 12 [1971] 589.

V. Baru, E. Epelbaum, A. A. Filin and J. Gegelia, Phys. Rev. C92 [2015] 014001,

E. Braaten and H. W. Hammer, Phys. Rev. Lett. 91 [2003] 102002,

P. F. Bedaque, H. W. Hammer and U. van Kolck, Phys. Rev. Lett. 82 [1999] 463.

H. W. Hammer, D. R. Phillips and L. Platter, Eur. Phys. J. A32 [2007] 335,

W. Detmold, K. Orginos, A. Parreno, M. J. Savage, B. C. Tiburzi, S. R. Beane and E. Chang (2015)

J. Kirschner, N. Barnea, D. Gazit, F. Pederiva and U. van Kolck, Phys. Rev. C92 [2015] 054002,

S. R. Beane, W. Detmold, T. C. Luu, K. Orginos, A. Parreno, M. J. Savage, A. Torok and A. Walker-Loud, Phys. Rev. D80 [2009] 074501,

N. Ishii, S. Aoki and T. Hatsuda, Phys. Rev. Lett. 99 [2007] 022001,

M. Lüscher, Communications in Mathematical Physics 105 [1986] 153.

Z. Davoudi and M. J. Savage, Phys. Rev. D84 [2011] 114502,

PACS-CS Collaboration (T. Yamazaki, Y. Kuramashi and A. Ukawa), Phys. Rev. D81 [2010] 111504,

PACS-CS Collaboration (T. Yamazaki, Y. Kuramashi and A. Ukawa), Phys. Rev. D84 [2011] 054506,

P. de Forcrand and M. Fromm, Phys. Rev. Lett. 104 [2010] 112005,

NPLQCD Collaboration (S. Beane et al.), Phys. Rev. C88 [2013] 024003,

HAL QCD Collaboration (N. Ishii, S. Aoki, T. Doi, T. Hatsuda, Y. Ikeda, T. Inoue, K. Murano, H. Nemura and K. Sasaki), Phys. Lett. B712 [2012] 437,

HAL QCD Collaboration (T. Inoue, S. Aoki, T. Doi, T. Hatsuda, Y. Ikeda, N. Ishii, K. Murano, H. Nemura and K. Sasaki), Nucl. Phys. A881 [2012] 28,

HAL QCD Collaboration (T. Inoue, S. Aoki, B. Charron, T. Doi, T. Hatsuda, Y. Ikeda, N. Ishii, K. Murano, H. Nemura and K. Sasaki), Phys. Rev. C91 [2015] 011001,

M. J. Savage, (2015), Private communication.

P. F. Bedaque, H. W. Hammer and U. van Kolck, Nucl. Phys. A676 [2000] 357,

S. Bour, S. König, D. Lee, H. W. Hammer and U.-G. Meißner, Phys. Rev. D84 [2011] 091503,

H. W. Hammer and L. Platter, Eur. Phys. J. A32 [2007] 113,

L. Platter, H. W. Hammer and U.-G. Meißner, Phys. Lett. B607 [2005] 254,

J. Kirschner, H. W. Grießhammer, D. Shukla and H. M. Hofmann, Eur. Phys. J. A44 [2010] 239,

Takumi Iritani (HAL QCD Collaboration) at “LATTICE2015” 2015, Kobe (Japan).
Matching effective few-nucleon theories to QCD

[63] C. J. D. Lin, G. Martinelli, C. T. Sachrajda and M. Testa, Nucl. Phys. B619 (2001) 467, arXiv:hep-lat/0104006

[64] S. Aoki, Europhys. J. A49 (2013) 81, arXiv:1309.4150 [hep-lat]

[65] M.-C. Chu, M. Lissia and J. Negele, Nuclear Physics B 360 (1991) 31.

[66] M. C. Birse (2012) arXiv:1208.4807 [nucl-th]

[67] HAL QCD Collaboration (B. Charron), PoS LATTICE2013 (2014) 223, arXiv:1312.1032 [hep-lat]

[68] HAL QCD Collaboration (K. Murano et al.), Phys. Lett. B735 (2014) 19, arXiv:1305.2293 [hep-lat]

[69] HAL QCD Collaboration (T. Doi, S. Aoki, T. Hatsuda, Y. Ikeda, T. Inoue, N. Ishii, K. Murano, H. Nenmura and K. Sasaki), Prog. Theor. Phys. 127 (2012) 723, arXiv:1106.2276 [hep-lat]

[70] U. van Kolck, Nucl. Phys. A645 (1999) 273, arXiv:nucl-th/9808007 [nucl-th]

[71] J.-W. Chen, G. Rupak and M. J. Savage, Nucl. Phys. A653 (1999) 386, arXiv:nucl-th/9902056 [nucl-th]

[72] D. B. Kaplan, M. J. Savage and M. B. Wise, Nucl. Phys. B534 (1998) 329, arXiv:nucl-th/9802075 [nucl-th]

[73] M. J. Savage, Phys. Rev. C55 (1997) 2185, arXiv:nucl-th/9611022 [nucl-th]

[74] T. Mehen, I. W. Stewart and M. B. Wise, Phys. Rev. Lett. 83 (1999) 931, arXiv:hep-ph/9902370 [hep-ph]

[75] M. E. Luke and A. V. Manohar, Phys. Rev. D55 (1997) 4129, arXiv:hep-ph/9610534 [hep-ph]

[76] G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D51 (1995) 1125, arXiv:hep-ph/9407339 [hep-ph] (Erratum: Phys. Rev.D55,5853(1997))

[77] D. B. Kaplan, Nucl. Phys. B494 (1997) 471, arXiv:nucl-th/9610052 [nucl-th]

[78] S. R. Beane and M. J. Savage, Nucl. Phys. A694 (2001) 511, arXiv:nucl-th/0011067 [nucl-th]

[79] S. Weinberg, Phys. Rev. 130 (1963) 776.

[80] J. Kirscher, Phomless Effective Field Theory in Few-Nucleon Systems, PhD thesis (2015).

[81] S. Borsanyi et al., Science 347 (2015) 1452, arXiv:1406.4088 [hep-lat]

[82] NPLQCD Collaboration (E. Chang, W. Detmold, K. Orginos, A. Parreno, M. J. Savage, B. C. Tiburzi and S. R. Beane), Phys. Rev. D92 (2015) 114502, arXiv:1506.05518 [hep-lat]

[83] S. R. Beane, E. Chang, S. Cohen, W. Detmold, H. W. Lin, K. Orginos, A. Parreno, M. J. Savage and B. C. Tiburzi, Phys. Rev. Lett. 113 (2014) 252001, arXiv:1409.3556 [hep-lat]

[84] S. R. Beane, E. Chang, W. Detmold, K. Orginos, A. Parreno, M. J. Savage and B. C. Tiburzi (2015) arXiv:1505.02422 [hep-lat]

[85] P. J. Mohr, B. N. Taylor and D. B. Newell, Rev. Mod. Phys. 84 (Nov 2012) 1527.

[86] S. R. Beane, P. F. Bedaque, A. Parreno and M. J. Savage, Phys. Lett. B585 (2004) 106, arXiv:hep-lat/0312004

[87] J. Vanasse and M. R. Schindler, Phys. Rev. C90 (2014) 044001, arXiv:1404.0658 [nucl-th]

[88] W. Detmold and M. J. Savage, Nucl. Phys. A743 (2004) 170, arXiv:hep-lat/0403005 [hep-lat]

[89] C. A. Bertulani, H. W. Hammer and U. Van Kolck, Nucl. Phys. A712 (2002) 37, arXiv:nucl-th/0205063 [nucl-th]

[90] P. F. Bedaque, H. W. Hammer and U. van Kolck, Phys. Lett. B569 (2003) 159, arXiv:nucl-th/0304007 [nucl-th]

[91] B. A. Gelman, Phys. Rev. C80 (2009) 034005, arXiv:0906.5502 [nucl-th]

[92] E. Ryberg, C. Forssén, H. W. Hammer and L. Platter, Phys. Rev. C89 (2014) 014325, arXiv:1308.5975 [nucl-th]

[93] R. Higa, H. W. Hammer and U. van Kolck, Nucl. Phys. A809 (2008) 171, arXiv:0802.3426 [nucl-th]

[94] D. L. Canham and H. W. Hammer, Eur. Phys. J. A37 (2008) 367, arXiv:0807.3258 [nucl-th]

[95] X. Zhang, K. M. Nollett and D. R. Phillips, Phys. Rev. C89 (2014) 024613, arXiv:1311.6822 [nucl-th]

[96] X. Zhang, K. M. Nollett and D. R. Phillips, How well do we understand Beryllium-7 + proton → Boron-8 + photon? An Effective Field Theory perspective (2015).

[97] L. S. Brown and G. M. Hale, Phys. Rev. C89 (2014) 014622, arXiv:1308.0347 [nucl-th]

[98] S. König, H. W. Grießhammer, H. W. Hammer and U. van Kolck (2015) arXiv:1508.05085 [nucl-th]

[99] J. Kirscher and D. Gazit, Phys. Lett. B755 (2016) 253, arXiv:1510.00118 [nucl-th]

[100] J. Vanasse, D. A. Egolf, J. Kerin, S. König and R. P. Springer, Phys. Rev. C89 (2014) 064003, arXiv:1402.5441 [nucl-th]