Travelling Magnetic Waves due to Plasma Surrounding a Slow Rotating Compact Gravitational Source

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Abstract

The magnetic field due to an axially symmetric, hot and highly conducting plasma, taken as an ideal magnetohydrodynamic fluid, surrounding a slow rotating compact gravitational object is studied within the context of Einstein-Maxwell field equations. It is assumed that whereas the plasma is effected by the background spacetime it does not effect the spacetime itself. The Einstein-Maxwell equations are then solved for the magnetic field in a comoving frame with the background spacetime described by the slow rotating Kerr black hole spacetime. It is found that the solutions are magnetic waves travelling along the azimuthal angle with velocity equal to the angular velocity of a free falling inertial frame. These general solutions, when applied to various particular cases of physical interest, show that for a fixed value of the azimuthal angle the magnetic field is completely induced by the dragging of the background spacetime.

1 Introduction

The behavior of plasma in Schwarzschild or Kerr spacetime has been a subject of increasing interest in recent years. This is mainly due to the fact that in various astrophysical situations, such as in the magnetospheres of a strong gravitational source, mostly present in active galactic nuclei (AGNs), matter exists in a highly ionized state\cite{1,2}. Under such circumstances the surrounding plasma possesses a strong magnetic field which not only plays important role in various emission mechanisms but also effects the dynamics of charged particles and accretion of the surrounding plasma\cite{3,4,5}. On the other hand the presence of a strong gravitational source in such cases moulds the background spacetime
to produce effects unknown to the Newtonian theory of gravitation. Most of the attempts to treat this phenomenon in Schwarzschild or Kerr spacetime either ignore the presence of strong magnetic field present in such situations or do not discuss the problem within the frame-work of Einstein-Maxwell equations as the basic governing equations for the behavior of plasma and the associated magnetic field. Except under some very simplifying assumptions, the problem becomes not only mathematically intractable but also offers difficulties for any general physical interpretation of the results. However, it is found that it is possible to discuss the problem for the magnetic field, within the context of Einstein-Maxwell equations, at least for the case of a highly conducting, axially symmetric plasma surrounding a slow rotating compact gravitational source, such as a slow rotating neutron star or a slow rotating black hole.

In this paper we consider the phenomenon to be consisting of an axially symmetric plasma surrounding a slow rotating compact gravitational source; taking the plasma as an ideal magnetohydrodynamic (MHD) fluid. The plasma is assumed to produce no substantial perturbations to the background space-time geometry but is taken to be strongly effected by it. We then construct, in Section 2, the general form of the Einstein-Maxwell equations for the electromagnetic field outside the plasma in an axially symmetric spacetime. The formulation is based on a general definition of the electromagnetic field tensor for an ideal MHD fluid in a curved four dimensional spacetime. We then explicitly write the field equations in a comoving frame of reference in slow rotating Kerr black hole spacetime. In Section 3 we obtain analytical expressions for the magnetic field in the comoving frame under the condition of perfect conductivity. The system of coupled quasi-linear partial differential equations thus obtained is shown to admit travelling wave solutions. These solutions are obtained and discussed using the method of characteristics which is particularly useful for solving quasi-linear partial differential equations in comoving frames. The physical significance of these solutions is then discussed in Section 4. We particularly focus on those effects which result as the frame dragging of the background spacetime. It is found that outside the plasma a magnetic field is induced which for a given azimuthal is due to the dragging of the free falling inertial frames. The paper concludes, in Section 5, with a summary of these observations and suggestions for further investigations.

2 Formulation of the Basic Equations

2.1 Preliminaries

Since we are to describe the phenomenon in a curved four dimensional spacetime we first need to give a general definition of the electromagnetic field tensor for an ideal MHD fluid in such a spacetime. Throughout we assume the gravitational units in which \( G = 1 = c \).

Let the geometry of the spacetime is given by the four dimensional line element \( ds \) defined by
\[ ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2, \]  

where the metric tensor components \( g_{\alpha\beta} \) are independent of the time coordinate \( t \) and the azimuthal angle \( \varphi \). The Greek indices denote coordinates \( t, r, \theta, \varphi \).

We note that the electromagnetic field tensor \( F_{\alpha\beta} \) is a 2-form, that is, a skew-symmetric tensor field of order two in a spacetime \( V_4 \). Now if \( \eta_{\alpha\beta\gamma\delta} \) is the volume 4-form of \( V_4 \) and \( u^\alpha \) is a unit vector at an event \( x \in V_4 \) defining a time like direction at this point then the electromagnetic field tensor for the MHD fluid is give by the covariant expression \([6, 7]\):

\[ F_{\alpha\beta} = u_\alpha E_\beta - u_\beta E_\alpha + \eta_{\alpha\beta\gamma\delta}u^\gamma B^\delta, \]  

and similarly in a contravariant form

\[ F^{\alpha\beta} = u^\alpha E^\beta - u^\beta E^\alpha + \eta^{\alpha\beta\gamma\delta}u_\gamma B_\delta, \]  

where the four vectors \( E_\alpha \) and \( B_\alpha \), denoting the electric and magnetic field components in the four dimensional spacetime, are orthogonal to the velocity four vector \( u^\alpha \). Here the volume element 4-form of \( V_4 \) namely \( \eta_{\alpha\beta\gamma\delta} \) and its dual \( \eta^{\alpha\beta\gamma\delta} \) is defined as \([8]\):

\[ \eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}, \quad \eta^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}, \]  

where \( g \) represents the determinant of the metric tensor \( g_{\alpha\beta} \) and \( \epsilon_{\alpha\beta\gamma\delta} \) is the Levi-Civita symbol, which is \(+1, -1, 0\) for a cyclic, anti-cyclic, and non-cyclic permutation of \( \alpha\beta\gamma\delta \) respectively. It should be noticed that the choice of spacetime \( V_4 \) is quite general here, namely of a four dimensional vector space, or more generally of a differentiable manifold of four dimensions. Also the signature of an axially symmetric metric defined on this manifold must be either \((+ - - -)\) or \((- + + +)\).

It can be easily shown that the assumption of an everywhere finite \( J^\alpha \) leads to the fact that in a comoving frame \( E^\alpha = 0 = E_\alpha \) \([9]\). In our case, that is outside a hot and highly conducting plasma, the assumption of everywhere finite current four vector is satisfied, so the electromagnetic field tensor takes the form

\[ F_{\alpha\beta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}u^\gamma B^\delta, \]  

and in contravariant form

\[ F^{\alpha\beta} = \frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}u_\gamma B_\delta. \]  

### 2.2 The Einstein-Maxwell Field Equations for Plasma Surrounding a Rotating Kerr Black Hole

If we interpret \( g_{\alpha\beta} \) as the potentials of a gravitational field determined by the Einstein field equations then we can write the electromagnetic field equations for
the MHD fluid in a spacetime whose geometry is given by the general relativistic field equations. The resulting set of equations, called Einstein-Maxwell field equations, can be expressed as

\[ F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \quad (7) \]

\[ (\sqrt{-g}F^{\alpha\beta})_{,\beta} = 4\pi\sqrt{-g}J^\alpha, \quad (8) \]

where the electromagnetic field tensor, in covariant and contravariant form, is given by expressions (5) and (6). Also here, as usual, \( \alpha \) as a subscript denotes partial derivative with respect to the coordinate \( x^\alpha \). To discuss these equations in full we must choose the four velocity vector which means that we must specify the comoving observer. A convenient and physically meaningful choice of the comoving observer is the one belonging to the class of observers having zero angular momentum (ZAMO)\[10\]. This is an observer, circling the Kerr black hole with angular velocity \( \omega \) at a fixed \( r \) and \( \theta \), being brought into motion by the dragging of the background spacetime. Thus for a ZAMO \( u_r \) and \( u_\theta \) vanish and the velocity four vector is given by \( (u_t, 0, 0, u_\phi) \). Moreover, since the ZAMO defines a comoving frame therefore, the electric field vanishes also. Under these conditions we obtain the following set of Einstein-Maxwell field equations for the first pair

\[ (\sqrt{-g}u^t B^r),r + (\sqrt{-g}u^t B^\theta),\theta + (\sqrt{-g}u^t B^\phi),\phi = 0, \quad (9) \]

\[ (\sqrt{-g}u^\theta B^\theta),\theta + (\sqrt{-g}u^t B^t),t + (\sqrt{-g}u^\phi B^\phi),r = 0, \quad (10) \]

\[ (\sqrt{-g}u^\phi B^\phi),\phi + (\sqrt{-g}u^t B^t),t = 0, \quad (11) \]

\[ (\sqrt{-g}u^\phi B^r),\phi + (\sqrt{-g}u^t B^r),t = 0, \quad (12) \]

and similarly from expression (8) the second pair is

\[ (u_\phi B_\theta),r + (u_\phi B_r),\theta = 4\pi\sqrt{-g}J^t, \quad (13) \]

\[ - (u_\phi B_\theta),t + (u_t B_\phi),\theta - (u_t B_\theta),\phi = 4\pi\sqrt{-g}J^r, \quad (14) \]

\[ - (u_\phi B_r),t - (u_t B_\phi),r + (u_t B_r),\phi = 4\pi\sqrt{-g}J^\theta, \quad (15) \]

\[ (u_t B_\theta),r - (u_t B_r),\theta = 4\pi\sqrt{-g}J^\phi. \quad (16) \]

### 3 Travelling Wave Solutions to the Einstein-Maxwell Field Equations

Given the current four vector \( J^\alpha \), the metric tensor \( g_{\alpha\beta} \), and the velocity four vector \( u^\alpha \) the set of equations (9)-(16) can solved numerically, to give the magnetic field as a function of \( (t, r, \theta, \phi) \) provided that the problem is well posed and the solution exists. However it is very difficult to solve the system of these
quasi-linear partial differential equations analytically in most cases especially for a general definition of $J^\alpha$.

Now since plasma, as it usually exists in compact star magnetospheres, is highly conducting; it follows from the generalized Ohm’s law that

$$J^\alpha = \sigma u^\alpha,$$  (17)

where $\sigma$ is the charge density in the ZAMO. Since outside the plasma $\sigma$ is zero, therefore $J^\alpha$ vanishes outside the plasma surrounding the Kerr black hole. The case when $J^\alpha = 0$ is not only physically meaningful but can also be used as a first approximation to more general cases, such as the magnetic field inside the magnetosphere of a compact gravitational source. Further we assume that the spacetime is the slow rotating Kerr metric given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{-2\Phi(r)}dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\varphi^2 - 2\omega(r)r^2 \sin^2 \theta d\varphi dt,$$  (18)

where

$$e^{2\Phi(r)} \equiv \left(1 - \frac{2M}{r}\right),$$  (19)

$M$ is the mass of the compact star and

$$\omega(r) \equiv \frac{d\varphi}{dt} = \frac{2J}{r^3},$$  (20)

is the angular velocity of a free falling inertial frame, whereas $J$ is the total angular momentum of the Kerr black hole as measured from infinity[11]. Finally, under the assumption of slow rotation (i.e., $\omega(r)^2$ and higher powers are neglected), we take explicitly the velocity four vector in ZAMO as follows:

$$u^\alpha = (u^t, 0, 0, \omega u^t), \quad u_\alpha = (-\frac{1}{u^t}, 0, 0, 0),$$  (21)

where

$$u^t = e^{-\Phi(r)}.$$  (22)

Under these assumptions we find that the pair of equations (13)-(16) is just the condition of integrability i.e., the partial derivatives satisfy the condition $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ for $\alpha \neq \beta$. Here we assume, as usual, that at least in the region outside the plasma all relevant physical quantities are differentiable and single valued. It is worth pointing out here that when searching for shock wave solutions to Einstein-Maxwell equations this assumption may not be true[12]. When the integrability condition is satisfied the non-identical set of Einstein-Maxwell equations reduces to equations(9)-(12), which can be further simplified to give

$$\frac{1}{u^t}(\sqrt{-g}u^t B^r)_r + (\sqrt{-g}B^\theta)_\theta + \sqrt{-g}B^\varphi,\varphi = 0,$$  (23)

$$\sqrt{-g}B^\varphi, t + \frac{1}{u^t}(\sqrt{-g}\omega u^t B^r)_r + \omega(\sqrt{-g}B^\theta)_\theta = 0,$$  (24)
\[ B^\theta_{t} + \omega B^\theta_{,\varphi} = 0, \quad (25) \]
\[ B^r_{t} + \omega B^r_{,\varphi} = 0, \quad (26) \]

where \( \omega \) is given by (20) and \( u^t \) is given by (22).

Differentiating with respect to \( r \) the second term in equation (24) as a product of \( \omega(r) \) and \( \sqrt{-g}u^r \dot{B}^r \), and using equation (23) in the resulting expression we obtain a single partial differential equation

\[ B^\varphi_{t} + \omega B^\varphi_{,\varphi} = -B^r \omega_{,r}. \quad (27) \]

Now to solve equation (25), (26), and (27) we apply the method of characteristics. Suppose \( f(t, \varphi) \) represents any of the component \( (B^r, B^\theta, B^\varphi) \), then we have

\[ \frac{df}{dt} = f_{,t} + f_{,\varphi} \frac{d\varphi}{dt}. \quad (28) \]

Comparing, say equation (27) and (28), and setting \( f = B^\varphi \) we obtain two simultaneous ordinary differential equations

\[ \frac{dB^\varphi}{dt} = -B^r \omega_{,r}, \quad (29) \]

and notably

\[ \frac{d\varphi}{dt} = \omega. \quad (30) \]

This relation implies that the characteristics travel at a speed \( \omega \). Solving (30) we get the characteristics \( \varphi = \omega t + \varphi_0 \). On the other hand equation (31) has solution \( B^\varphi = -B^r(\varphi_0 t \omega_{,r} + h(\varphi_0)) \). Combining the two results we have the general solution to equation (27):

\[ B^\varphi = -B^r(\varphi - \omega t) t \omega_{,r} + h(\varphi - \omega t), \quad (31) \]

where the function \( h(t, \varphi) \) satisfies the condition

\[ h_{,t} + \omega h_{,\varphi} = 0. \quad (32) \]

A similar procedure for equations (25) and (26) shows that \( B^r \) and \( B^\theta \) indeed have general solutions of the form \( f(\varphi - \omega t) \), where \( f \) is an arbitrary function of its argument \( \varphi - \omega t \), and whose form is determined by the physical constraints.

4 Physical Interpretation of the Travelling Magnetic Wave Solutions

In the preceding Section it was shown that the magnetic field outside a plasma surrounding a slow rotating black hole has a general travelling wave solution of the type
\[ B^r = f(\varphi - \omega t), \quad B^\theta = g(\varphi - \omega t), \quad B^\varphi = -f t \omega_r + h(\varphi - \omega t) \quad (33) \]

where \( f, g, \) and \( h \) are arbitrary functions of the argument \( \varphi - \omega t \) and where \( \omega \) given by (30) can be interpreted as the velocity with which a characteristic moves or the velocity of a wave profile. To further clarify the physical significance of these solutions, let us discuss some specific cases of \( f, g, \) and \( h \) and their dependence on \( \omega \).

4.1 Plane Wave Solution

If the plasma perturbations are such that the magnetic field remains stationary outside the plasma, then it follows from (33) that the magnetic field components are given by

\[ B^r = \varphi - \omega t, \quad B^\theta = \varphi - \omega t, \quad B^\varphi = -(\varphi - \omega t) t \omega_r, \quad (34) \]

where for simplicity we have taken \( h = 0 \). Furthermore \( \varphi \) being the azimuthal angle can be given a fixed value, say zero. We notice that for a fixed value of \( \varphi \), the magnetic field components depend directly on the frame dragging frequency \( \omega \). This means that the magnetic field, in this case, is induced by the rotation of the Kerr black hole. However this induced field may not last for long as the assumption of being stationary is not valid for actual compact star magnetospheres where plasma is usually in a state of high oscillations.

4.2 Oscillatory Solution

For an oscillating plasma the functions \( f, g, \) and \( h \) have a sinusoidal form represented by \( \sin \) or \( \cos \) functions. As before we take \( h = 0 \), and \( \varphi = 0 \). Then for the magnetic field components we have from (33):

\[ B^r = \cos \omega t, \quad B^\theta = \sin \omega t, \quad B^\varphi = -\omega_r \cos \omega t. \quad (35) \]

Here we notice the frequency of oscillations is the dragging frequency. The magnetic field in this case is again induced by rotating gravitational source. We notice that the field components \( B^r \) and \( B^\theta \) remain bounded for all time, however the component \( B^\varphi \) is unbounded as \( t \) increases. For actual physical situations the magnetic field must decay with the passage of time. Again the oscillatory solutions are not valid for long periods time for a real physical situation.

4.3 Exponentially Decaying Solution

The solutions (33) to the Einstein-Maxwell equation, for \( h = 0 \) and \( \varphi = 0 \), can be written in the form of exponentially decaying functions as

\[ B^r = \exp(-\omega t), \quad B^\theta = \exp(-\omega t), \quad B^\varphi = -\omega_r \exp(-\omega t); \quad (36) \]
or in terms of complex representation

\[ B^r = \exp(-i\omega t), \quad B^\theta = \exp(-i\omega t), \quad B^\varphi = -i\omega r \exp(-i\omega t). \]  

(37)

Both of these solutions are bounded for very large time scales and each can be used to represent the magnetic field in the case of a highly conducting plasma surrounding a slow rotating Kerr black hole.

The above examples show that the magnetic field, for a given value of \( \varphi \), is induced by the rotating gravitational source and it decreases with distance from the source. Furthermore the magnetic field is bounded when it has an exponentially decaying form. We can interpret this as follows: For a compact gravitational source possessing a highly ionized atmosphere there are magnetic waves travelling along the azimuthal with velocity equal to the angular velocity of a free falling inertial frame induced by the rotating gravitational source and which remains bounded as a decaying exponential function of time.

5 Discussion

In this paper we have investigated the solutions for the magnetic field components admitted by the Einstein-Maxwell field equations in a comoving frame (ZAMO). Taking the case of plasma as an ideal MHD fluid surrounding a slow rotating Kerr black hole, we found that the magnetic field can be written as a travelling wave moving with velocity \( \omega \) in the azimuthal direction. By considering various cases of physical importance we observed that for a fixed \( \varphi \) the magnetic field depends directly on the frame dragging frequency \( \omega \), thus we can regard the magnetic field to be induced, at least partially, by the space-time dragging. There is an indication\[13\] of such induced magnetic field in the gravitomagnetic approximation to the general theory of relativity\[14\], however this is the first indication of such an effect as a consequence of the Einstein-Maxwell field equations. Since the induced magnetic field depends on the frame dragging frequency, its magnitude is extremely small to be directly measurable. In extreme astrophysical situations where such an effect is plausibly observable the correct strategy will be to study the effects of the induced magnetic field on various physical processes, such as the accretion of charge particles in a compact star magnetosphere. These and other features of the travelling wave solutions to the Einstein-Maxwell field equations are under considerations, which we hope to report in near future.

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