Shortcuts in open quantum systems: Superadiabatic control of an open quantum oscillator

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Fast nonadiabatic control protocols known as shortcuts to adiabaticity have found a plethora of applications, but their use has been severely limited to speeding up the dynamics of isolated quantum systems. We introduce shortcuts for open quantum systems that make possible the fast control of Gaussian states in non-unitary processes. Specifically, we provide the time modulation of the trap frequency and dephasing strength that allow preparing an arbitrary thermal state in a finite time. Experimental implementation can be done via stochastic parametric driving, readily accessible in a variety of platforms.

The fast control of quantum systems with high-fidelity is broadly acknowledged as a necessity to advance quantum science and technology. In this context, techniques known as shortcuts to adiabaticity (STA) have provided an alternative to adiabatic driving with a wide variety of applications [1]. STA tailor excitations in nonadiabatic processes to prepare a given state in a finite time, without the requirement of slow driving. The experimental demonstration of STA was pioneered in a trapped thermal cloud [2], soon followed by implementations in Bose-Einstein condensates [3], cold atoms in optical lattices [4] and low-dimensional quantum fluids [5]. More recently, STA have also been applied to Fermi gases, both in the non-interacting and unitary regimes [6, 7]. Beyond the realm of cold atoms, STA have been demonstrated in quantum optical systems [8], trapped ions [9], nitrogen vacancy-centers [10], and superconducting qubits [11, 12]. Their application is not restricted to quantum systems and classical counterparts exist [13, 14], of relevance, e.g., to colloidal systems [15].

A variety of related control techniques fall under the umbrella of STA. Prominent examples include counterdiabatic or transitionless quantum driving [16–18], the fast-forward technique [19–21], reverse-engineered dynamics using Lewis-Riesenfeld invariants [22], as well as the use of dynamical scaling laws [14, 23–25], Lax pairs [26], variational methods [27] and Floquet engineering [28]. Despite the range of applications explored, the use of STA in the quantum domain is severely limited to isolated systems, in which sources of noise and decoherence are considered an unwanted perturbation [29, 30]. Applications to finite-time thermodynamics have thus been limited to the speedup of strokes in which the working substance is in isolation and decoupled from any external reservoir.

Controlling heating and cooling processes would pave the way to the realization of superadiabatic heat engines and refrigerators based, e.g., in an Otto or Carnot quantum cycle [31–36]. Hence, the possibility to speed up the dynamics of open quantum systems is highly desirable in view of applications to cooling, and more generally, in finite-time thermodynamics. In this context, the nonadiabatic control of composite and open quantum systems using STA remains an exciting open problem on which few results are available [35–40].

In this Letter, we introduce STA for open quantum systems and apply them to the superadiabatic cooling and heating of a thermal harmonic oscillator. We show that the required control protocols are local and involve only the driving of the trap frequency and the dephasing strength. They can be achieved using stochastic parametric driving, thus harnessing noise as a resource.

Model. — We shall consider a single particle in a driven harmonic trap, with Hamiltonian

$$\hat{H}_t = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega_t^2 \hat{x}^2$$

(1)

and an open quantum dynamics governed by a master equation of the form

$$\frac{d\rho_t}{dt} = -\frac{i}{\hbar}[\hat{H}_t, \rho_t] - \gamma_t [\hat{x}, [\hat{x}, \rho_t]],$$

(2)

the derivation of which will be provided below. The case with constant dephasing strength $\gamma_t$ admits the Lindblad form with position operator $\hat{x}$ as the single Hermitian Lindblad operator. This limit naturally arises as the high-temperature limit of quantum Brownian motion. The dynamics with an arbitrary time-dependence $\gamma_t$ is generally non-Markovian. We shall show how a time-inhomogeneous Markovian dynamics [41], corresponding to $\gamma_t > 0$, can be engineered by tailoring noise as a resource. The dynamics along the process is assumed to remain Gaussian, with the density matrix in coordinate space of the form

$$\rho_t(x, x') = N_t e^{-A_t(x^2+x'^2)+iB_t(x^2-x'^2)-2C_txx'},$$

(3)
where $A_t, B_t, C_t$ are time-dependent coefficients to be determined from the master equation, $N_t = \sqrt{2(A_t + C_t^2)}/\pi$ being the normalization factor.

As a relevant example, we consider the driving in a finite time $t_f$ of an initial thermal state, parameterized by the trap frequency and inverse temperature ($\omega_0, \beta_0$), to a different thermal state with ($\omega_f, \beta_f$). For the Gaussian variational Ansatz (3) to describe the exact dynamics of the master equation (2), the following consistency equations are to be satisfied [42]

$$
\dot{B}_t = \frac{2\hbar}{m}(A_t^2 - B_t^2 - C_t^2) - \frac{m\omega_f^2}{2\hbar}, \quad (4a)
$$
$$
\dot{A}_t = \gamma_t - \frac{4\hbar}{m}A_t B_t, \quad (4b)
$$
$$
\dot{C}_t = -\gamma_t - \frac{4\hbar}{m}B_tC_t. \quad (4c)
$$

The boundary conditions are given from the initial and final states, that we choose to be thermal. It follows that $A_0 = \frac{m\omega_0}{2\hbar} \coth(\hbar\omega_0\beta_0)$, $B_0 = 0$ and $C_0 = -m\omega_0/(2\hbar \sinh(\hbar\omega_0\beta_0))$. Similarly, for the final state to be thermal, the coefficients at time $t_f$ should reduce to the values $A_f = \frac{m\omega_f}{2\hbar} \coth(\hbar\omega_f\beta_f)$, $B_f = 0$ and $C_f = -m\omega_f/(2\hbar \sinh(\hbar\omega_f\beta_f))$. The initial and final states being taken as equilibrium states, are stationary. This imposes the additional boundary conditions $A_0 = \dot{B}_0 = \dot{C}_0 = 0$ and $A_f = \dot{B}_f = \dot{C}_f = 0$. We also require that $A_{0/f} = \dot{B}_{0/f} = \dot{C}_{0/f} = 0$ at initial and final time. While the latter conditions are auxiliary, they guarantee smooth variations of $\omega_t$ and $\gamma_t$. The parameter $B_t = -m/(4\hbar)(\dot{A}_t + \dot{C}_t)/(A_t + C_t)$ directly follows from the consistency equations (4).

A protocol speeding up the evolution from the thermal state characterized by $(\omega_0, \beta_0)$ to $(\omega_f, \beta_f)$ is obtained by explicitly specifying both the time-dependence of $\gamma_t$ and $\omega_t$ according to

$$
\omega_t^2 = \frac{4\hbar^2}{m^2}(A_t^2 - C_t^2) - 3 \left( \frac{\dot{A}_t + \dot{C}_t}{A_t + C_t} \right)^2 + 1 \frac{\dot{A}_t + \dot{C}_t}{2 A_t + C_t}, \quad (5)
$$
$$
\gamma_t = \frac{\dot{A}_t C_t - A_t \dot{C}_t}{A_t + C_t}. \quad (6)
$$

The engineering of a shortcut to thermalization between Gaussian states thus requires the ability to control both the frequency and dephasing. The control of the harmonic frequency $\omega_t$ is performed with routine in a variety of setups and has been demonstrated in the implementation of STA in isolated quantum systems, e.g., with ultracold trapped atomic systems [2, 3, 5–7]. The requirement of a time-dependent dephasing $\gamma_t$ makes the dynamics open. It can be experimentally implemented from the microscopic picture provided below.

**Engineering of time-dependent dephasing rates.** We propose that a modulating dephasing strength $\gamma_t > 0$ can be achieved in the laboratory by harnessing noise as a resource. To this end, consider the stochastic Hamiltonian

$$
\hat{H}_{st} = \hat{H}_s + \hbar \sqrt{2\gamma_t \xi_t \hat{x}}, \quad (7)
$$

characterized by the Wiener process defined from $W_t = W_0 + \int_0^t \xi_t dt'$ in terms of the real Gaussian process $\xi_t$. While such a stochastic process is not differentiable, all integral quantities can be defined from the Wiener increment $dW_t = \xi_t dt$. The noise-averaged expressions follow from the moments $\langle \xi_t \rangle$ and $\langle \xi_t \xi_t' \rangle$, that we choose to be zero and $\delta(t - t')$, respectively, to describe a Gaussian white-noise process [43].

The evolution of a quantum state dictated by the stochastic Hamiltonian (7) is described by a master equation that we derive below. For a small increment of time $dt$, the wave function can be written as $|\psi(t+dt)\rangle = \exp (-i(H_s dt/\hbar + \sqrt{2\gamma_t \xi_t} dW_t)) |\psi_t\rangle$, with $dW_t$ defined in the Itô sense, i.e. fulfilling $(dW_t)^2 = dt$ and $dW_t dt = 0$ [44–46]. A Taylor expansion of the exponential then gives

$$
d|\psi_t\rangle = \left(-i\hbar(\hat{H}_s dt + \hbar \sqrt{2\gamma_t \xi_t} dW_t) - \gamma_t \hat{x}^2 dt \right) |\psi_t\rangle, \quad (8)
$$

the only non-zero terms being first order in $dt$ or $(dW_t)^2$. Further, in the Itô calculus, the Leibnitz chain rule generalizes to $d(AB) = (A + dA)(B + dB) - AB = (dA)B + A(dB) + dAdB$. This gives the evolution of the density matrix $\rho_{st} = |\psi_t\rangle \langle \psi_t|$ as

$$
d\rho_{st} = -i\left[\hat{H}_s, \rho_{st}\right] dt - i \sqrt{2\gamma_t} [\hat{x}, \rho_{st}] dW_t - \gamma_t [\hat{x}, [\hat{x}, \rho_{st}]] dt, \quad (9)
$$

which preserves the norm at the level of each individual realization. We then take the average over the realizations of the noise, and denote the ensemble $\rho_t = \langle \rho_{st} \rangle$. Using the fact that the average of any function $F_t$ of the stochastic process vanishes, $\langle F_t dW_t \rangle = 0$ [44], we find the evolution for the ensemble density matrix $\rho_t$ as given by the master equation (2).

The engineering of a prescribed modulation in time of the dephasing strength $\gamma_t$ can thus be achieved via stochastic parametric driving provided that $\gamma_t > 0$. Interestingly, stochastic parametric driving allows modulating $\gamma_t$ independently from the frequency $\omega_t$, which contrasts with the time-dependent Markovian quantum master equation derived by driving the coupling of a system to a thermal bath [47].

**Characterization of the dynamics.** — The evolving density matrix can be diagonalized at all time according to $\rho_t = \sum_n P_{n,t} |\psi_{n,t}\rangle \langle \psi_{n,t}|$, the eigenvalues and eigenfunction being [42]

$$
\langle x | \psi_{n,t} \rangle = \sqrt{\frac{k_t}{2\pi n! \sqrt{\pi}}} e^{-\frac{x^2}{2k_t}} x^{2n} H_n(k_t x), \quad (10a)
$$
$$
P_{n,t} = u_{n,t}^0 (1 - u_{n,t}), \quad (10b)
$$
where $H_n$ denotes the Hermite polynomial defined from
$$\frac{d^n}{dx^n}(H_n(x)) = 2^n N!/(N-j)!H_{N-j}(x).$$
The effective inverse length $k_t$ and dimensionless constant $u_0$ that characterize the control trap are detailed in [42] and below.

Interestingly, the evolving density matrix $\rho_t$ can be interpreted as a thermal state $\sigma_t$ rotated through a unitary transformation $U_{x,t} \equiv e^{-iB_tx^2}$ by noting that $\rho_t = U_{x,t}^\dagger \sigma_t U_{x,t}$. The density matrix $\sigma_t$, with coordinate representation $\langle x|\sigma_t|x'\rangle = N_t e^{-(A_t(x^2-\tilde{x}^2)-2C_txx')}$, corresponds to the instantaneous thermal state of a harmonic oscillator with effective frequency $\tilde{\omega}_t$ and inverse temperature $\tilde{\beta}_t$ provided that
$$\tilde{\varepsilon}_t \equiv \tilde{\beta}_t \hbar \tilde{\omega}_t = \text{acosh}(-A_t/C_t),$$

$$\tilde{\omega}_t^2 = \frac{4k_t^2}{m^2}(A_t^2 - C_t^2),$$

assuming oscillators of equal mass. The effective inverse length is then explicitly given by $k_t = \sqrt{\hbar \tilde{\omega}_t / N_t}$. By construction, the two states share the same eigenvalues and $\sigma_t = \sum_n p_{n,t}|\phi_{n,t}\rangle \langle \phi_{n,t}|$, with the probability now written in terms of a thermal probability at all times, $p_{n,t} = e^{-\tilde{\beta}_t \hbar \tilde{\omega}_t / N_t}$, the partition function being $Z_t = 1/(1 - u_t)$, and $u_t = e^{-\tilde{\varepsilon}_t}$. However, the eigenvectors are different and $|\phi_{n,t}\rangle = U_{x,t}|\psi_{n,t}\rangle$ correspond to the well-known Fock states of the 'reference', time-dependent harmonic oscillator — whose parameters are distinguished with a tilde.

At all times of evolution, we have $A_t = (k_t^2/2) \text{coth} \tilde{\varepsilon}_t$ and $C_t = -k_t^2(2/\sinh \tilde{\varepsilon}_t)$. As a result, the control frequency and dephasing strength can be recast in the form
$$\tilde{\omega}_t^2 = \tilde{\omega}_0^2 - \frac{\tilde{\eta}_t}{\eta_t},$$

$$\gamma_t = k_t^2 \frac{1}{(1 - u_t)^2} \frac{\tilde{\varepsilon}_t}{\sinh^2 \tilde{\varepsilon}_t},$$

where the control parameter depends on the scaling factor $\kappa_t \equiv k_0 / k_t = \sqrt{\tilde{\omega}_0 / \tilde{\omega}_t}$ and temperatures as
$$\eta_t = N_0 / N_t = \kappa_t \sqrt{\text{coth}(\beta_0 / 2) \tanh(\tilde{\varepsilon}_t / 2)}.$$  

These are our main results. The combined modulation of the trap frequency and the dephasing strength is sufficient to engineer finite-time shortcuts to thermalization. Equation (13) gives the correction of the control of the trap frequency $\omega_t$ with respect to a reference one $\tilde{\omega}_t$ that needs to be experimentally implemented for the preparation of the thermal state in a finite, prescribed time. A comparison of these results with the ones reported for isolated systems with $\gamma_t = 0$ [25] shows that the control parameter in Eq. (15) not only depends on the scaling factor, but also accounts for the change of the temperature through an additional, non trivial term.

For illustration, we take the evolution of the parameters
$$A_t = A_0 + (A_f - A_0) f(t/t_f)$$ and
$$C_t = C_0 + (C_f - C_0) f(t/t_f)$$

according to an interpolating Ansatz $f(\tau)$ determined from the complete set of boundary conditions, and chosen as fifth-order polynomial, $f(\tau) = 10\tau^3 - 15\tau^4 + 6\tau^5$. The experimental parameters readily follow from Eqs. (11-15). Fig. 1 illustrates the control frequency and dephasing strength corresponding to phase-space compression and expansion protocols, discussed below. Short control processes require trap inversion (a negative squared frequency), which can be achieved experimentally via, e.g., a painted potential [48] or a digital micromirror device [49]. They also rely on a dephasing strength of larger amplitude, which can be experimentally more challenging to achieve. We propose to use the maximum of the dephasing strength, denoted $\gamma_{\text{max}}$ as a measure of the “cost” to implement the process by a technique such as stochastic parametric driving. We show in [42] that the maximum dephasing strength scales inversely with the process time, as illustrated in the insets of Fig. 1b.

We further use the relative entropy, defined as
$$S(\rho_t||\sigma_t) = \text{Tr}(\rho_t \ln \rho_t) - \text{Tr}(\rho_t \ln \sigma_t),$$

as a measure of the distance of the engineered state $\rho_t$ to the effective thermal state $\sigma_t$ along the dynamics. It can be written
as

$$S(\rho_t||\sigma_t) = \sum_{n=0}^{\infty} p_n \ln p_n - \sum_{m=0}^{\infty} (p_n \ln p_n) |\langle \phi_m | \psi_n \rangle|^2,$$  (17)

where the overlap of the eigenfunctions $\langle \phi_m | \psi_n \rangle$ is given explicitly in the supplementary material. Fig. 1 illustrates the relative entropy between the engineered and the thermal state. The insets show the maximum relative entropy for different process times, evidencing that the state is going further away from a thermal distribution for shorter protocols. The shape of the dephasing strength and the relative entropy is independent on the duration of the process, which only influences their maxima, $\gamma_{\text{max}}$ and $s_{\text{max}}$, respectively, as illustrated in Fig. 1b.

**Superadiabatic protocols.**— Processes satisfying $\beta_f \omega_f = \beta_0 \omega_0$ conserve the mean phonon number and are often referred to as phase-space (density) preserving. The inverse temperature and frequency can be related to two physical lengths, namely, the particle characteristic length, given by the de Broglie wavelength, $\lambda_{\text{dB}} = \hbar / (\beta_f (2m))$, and the trap characteristic length, $\lambda_{\text{HO}} = \sqrt{\hbar / (m \omega)}$. Their ratio $\lambda_{\text{dB}} / \lambda_{\text{HO}} = \sqrt{\beta \omega / 2}$ is conserved for phase-space preserving transformation. STA in closed systems are limited to phase-space preserving transformation, such as adiabatic cooling. These processes preserve the von Neumann entropy $S_I = -\text{Tr}(\rho_t \ln \rho_t)$. By contrast, cooling and heating processes alter the phase-space density and the number of populated states lead to an entropy change [50] and require an open dynamics.

STA for open processes thus allow reaching arbitrary thermal states ($\omega_f, \beta_f$) from an initial thermal state, as schematically represented in Fig. 2, along with the variation of entropy. The sign of the dephasing strength determines the variation of relative energy sign($\dot{\epsilon}_t$) and entropy change. In particular, a positive dephasing strength yields a monotonic increase of entropy. Indeed, the rate of change of the von Neumann entropy reads

$$\frac{dS_I}{dt} = -\frac{\dot{u}_t}{(1 - u_t)^2} \ln(u_t) = \frac{\gamma_t \varepsilon_t}{kT}.$$  (18)

Problems restricted to $\gamma_t \geq 0$ allow only STA for thermalization to high-temperature states (heating), with $\Delta S = S_f - S_0 > 0$. Whenever values of $\gamma_t$ can be engineered, this restriction is lifted.

The maximum dephasing strength, illustrated in Fig. 3, is specific to each scenario. It follows a different behavior when changing the trap frequency at constant temperature or vice versa. This is not surprising since the two parameters correspond to different physical phenomena, as discussed above. The plateau observed when decreasing the temperature at a fixed trap frequency (Fig. 3c) might be set by the trap size, which constrains the size of the particle. Interestingly, a given final phase-space density can be reached from different dephasing strengths, even when starting from a same initial state. In other words, processes yielding to $\beta_f \omega_f$
from \( \beta \omega_0 \) can have different implementation costs according to \( \gamma_{\text{max}} \), as illustrated in Fig. 3d.

In conclusion, we have introduced shortcuts to adiabaticity in open quantum systems and applied them to control the thermalization of an open quantum oscillator. The resulting protocols should be broadly applicable as their implementation requires only a time modulation of the harmonic frequency and the dephasing strength, accessible via stochastic parametric driving. Our results can be directly applied to non-Markovian dynamics whenever the amplitude and sign of the dephasing strength can be engineered.

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[1] E. Torrontegui, S. Ibáñez, S. Martínez-Garaot, M. Modugno, A. del Campo, D. Guéry-Odelin, A. Ruschhaupt, X. Chen, and J. G. Muga, in Advances in Atomic, Molecular, and Optical Physics, Advances In Atomic, Molecular, and Optical Physics, Vol. 62, edited by E. Arimondo, P. R. Berman, and C. C. Lin (Academic Press, 2013) pp. 117 - 169.

[2] J.-F. Schaff, X.-L. Song, P. Vignolo, and G. Labeyrie, Phys. Rev. A 82, 033430 (2010).

[3] J.-F. Schaff, X.-L. Song, P. Capuzzi, P. Vignolo, and G. Labeyrie, EPL (Europhysics Letters) 93, 23001 (2011).

[4] M. G. Bason, M. Viteau, N. Malossi, P. Huillery, E. Arimondo, R. Fazio, V. Giovannetti, R. Mannella, and O. Morsch, Nature Physics 8, 147 (2012).

[5] W. Rohringer, D. Fischer, F. Steiner, J. Mazets, I. E. and Schmiedmayer, and M. Trupke, Sci. Rep. 5, 9820 (2015).

[6] S. Deng, P. Diao, Q. Yu, A. del Campo, and H. Wu, Phys. Rev. A 97, 013628 (2018).

[7] S. Deng, A. Chenu, P. Diao, F. Li, S. Yu, I. Coulamay, A. del Campo, and H. Wu, Science Advances 4, eaar5909 (2018).

[8] Y.-X. Du, Z.-T. Liang, Y.-C. Li, X.-X. Yue, Q.-X. Lv, W. Huang, X. Chen, H. Yan, and S.-L. Zhu, Nature communications 7, 12479 (2016).

[9] S. An, D. Lv, A. Del Campo, and K. Kim, Nature Communications 7, 12999 (2016).

[10] J. Zhang, J. H. Shim, I. Niemeyer, T. Taniguchi, T. Teraj, H. Abe, S. Onoda, T. Yamamoto, T. Ohshima, J. Isoya, and D. Suter, Phys. Rev. Lett. 110, 240501 (2013).

[11] T. Wang, Z. Zhang, L. Xiang, Z. Jia, P. Duan, W. Cai, Z. Gong, Z. Zong, M. Wu, J. Wu, L. Sun, Y. Yin, and G. Guo, New Journal of Physics 20, 065003 (2018).

[12] Z. Zhang, T. Wang, L. Xiang, Z. Jia, P. Duan, W. Cai, Z. Zhan, Z. Zong, J. Wu, L. Sun, Y. Yin, and G. Guo, New Journal of Physics 20, 085001 (2018).

[13] C. Jarzynski, Phys. Rev. A 88, 040101 (2013).

[14] S. Deffner, C. Jarzynski, and A. del Campo, Phys. Rev. X 4, 021013 (2014).

[15] T. Schmiedl and U. Seifert, Phys. Rev. Lett. 98, 108301 (2007).

[16] M. Demiralp and S. A. Rice, J. Phys. Chem. A 107, 9937 (2003).

[17] M. Demiralp and S. A. Rice, J. Phys. Chem. B 109, 6838 (2005).

[18] M. V. Berry, Journal of Physics A: Mathematical and Theoretical 42, 365303 (2009).

[19] S. Masuda and K. Nakamura, Proc. R. Soc. London Ser. A 466, 1135 (2009).

[20] S. Masuda and K. Nakamura, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 466, 1135 (2010).

[21] S. Masuda, K. Nakamura, and A. del Campo, Phys. Rev. Lett. 113, 063003 (2014).

[22] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guéry-Odelin, and J. G. Muga, Phys. Rev. Lett. 104, 063002 (2010).

[23] J. G. Muga, X. Chen, A. Ruschhaupt, and D. Guéry-Odelin, Journal of Physics B: Atomic, Molecular and Optical Physics 42, 244001 (2009).

[24] A. del Campo, Phys. Rev. A 84, 031606 (2011).

[25] A. del Campo, Phys. Rev. Lett. 111, 100502 (2013).

[26] M. Okuyama and K. Takahashi, Phys. Rev. Lett. 117, 070401 (2016).

[27] D. Sels and A. Polkovnikov, Proceedings of the National Academy of Sciences 114, E3909 (2017).

[28] P. W. Claeys, M. Pandey, D. Sels, and A. Polkovnikov, Phys. Rev. Lett. 123, 090602 (2019).

[29] E. Calzetta, Phys. Rev. A 98, 032107 (2018).

[30] A. Levy, A. Kiely, J. G. Muga, R. Kosloff, and E. Torrontegui, New Journal of Physics 20, 025006 (2018).

[31] T. Feldmann and R. Kosloff, Phys. Rev. E 73, 025107 (2006).

[32] J. Deng, Q.-h. Wang, Z. Liu, P. Hänggi, and J. Gong, Phys. Rev. E 88, 062122 (2013).

[33] A. del Campo, J. Goold, and M. Paternostro, Sci. Rep. 4 (2014), 10.1038/srep06208.

[34] M. Beau, J. Jaramillo, and A. del Campo, Entropy 18, 168 (2016).

[35] T. Villazon, A. Polkovnikov, and A. Chandran, Phys. Rev. A 100, 012126 (2019).

[36] R. Dann and R. Kosloff, arXiv e-prints , arXiv:1906.06946 (2019), arXiv:1906.06946 [quant-ph].

[37] G. Vacanti, R. Fazio, S. Montangero, G. M. Palma, M. Paternostro, and V. Vedral, New Journal of Physics 16, 053017 (2014).

[38] C. W. Duncan and A. del Campo, New Journal of Physics 20, 085003 (2018).

[39] R. Dann, A. Tobalina, and R. Kosloff, Phys. Rev. Lett. 122, 250402 (2019).

[40] S. Alipour, A. Chenu, A. T. Rezakhani, and A. del Campo, arXiv e-prints , arXiv:1907.07460 (2019), arXiv:1907.07460 [quant-ph].

[41] A. Rivas, S. F. Huelga, and M. B. Plenio, Reports on Progress in Physics 77, 046001 (2014).

[42] Supplemental Material providing details of the calculations (see appendices).

[43] H. J. Carmichael, Statistical methods in quantum optics 1: master equations and Fokker-Planck equations (Springer Science & Business Media, 2013).

[44] S. L. Adler, Phys. Rev. D 67, 025007 (2003).

[45] C. Gardiner, Stochastic Methods: A Handbook for the Natural and Social Sciences, Springer Series in Synerget-
ics (Springer Berlin Heidelberg, 2009).
[46] A. Ruschhaupt, X. Chen, D. Alonso, and J. G. Muga, New J. Phys. 14, 093040 (2012).
[47] R. Dann, A. Levy, and R. Kosloff, Phys. Rev. A 98, 052129 (2018).
[48] K. Henderson, C. Ryu, C. MacCormick, and M. G. Boshier, New Journal of Physics 11, 043030 (2009).
[49] G. Gauthier, I. Lenton, N. M. Parry, M. Baker, M. J. Davis, H. Rubinsztein-Dunlop, and T. W. Neely, Optica 3, 1136 (2016).
[50] W. Ketterle and D. E. Pritchard, Phys. Rev. A 46, 4051 (1992).
[51] F. G. Mehler, J. für die Reine und Angewandte Mathematik 66, 161 (1866), cf. p 174, eqn (18) & p 173, eqn (13).
[52] A. del Campo, J. Molina-Vilaplana, and J. Sonner, Phys. Rev. D 95, 126008 (2017).
[53] A. Chenu, J. Molina-Vilaplana, and A. del Campo, Quantum 3, 127 (2019).
[54] G. E. Andrews, R. Askey, and R. Roy, Special functions, Vol. 71 (Cambridge university press, 2000) eq. (6.1.2) p.278.

A. Thermal state of a harmonic oscillator

It is well known that the thermal state of a harmonic oscillator is Gaussian in the coordinate representation. For the sake of completeness, we briefly sketch the derivation below. For a time-independent harmonic oscillator, the Hamiltonian \( \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \) reads, in second quantization \( \hbar \omega (a^\dagger a + \frac{1}{2}) \), where \( a = \sqrt{m \omega/2 \hbar} \hat{x} + i \sqrt{1/2 \hbar \omega \hat{p}} \) is the annihilation operator. The coordinate representation of the thermal operator \( e^{-\beta \hat{H}}/Z \) is easily written using the Fock states, defined as \( a^n = \sqrt{n!} n \), and reads

\[
\rho(x, x') = \langle x | e^{-\beta \hat{H}} | x' \rangle = \frac{1}{Z} e^{-\beta \hbar \omega / 2} \sum_n e^{-\beta \hbar \omega n} \langle x | n \rangle \langle n | x' \rangle.
\] (S1)

Solving the Schrödinger equation for the Fock state wave function \( \langle x | n \rangle \) gives

\[
\langle x | n \rangle = \sqrt{\frac{k}{2^n n! \sqrt{\pi}}} e^{-\frac{1}{4}(kx)^2} H_n(kx),
\] (S2)

where \( k^{-1} = \sqrt{\hbar/(m \omega)} \) denotes an effective length characteristic of the harmonic oscillator. The coordinate representation of the thermal density operator then reads

\[
\rho(x, x') = \sqrt{\frac{k}{Z \sqrt{\pi}}} e^{-\beta \hbar \omega / 2} \exp \left( -\frac{k^2}{2} (x^2 + x'^2) \right) \sum_n \left( \frac{e^{-\beta \hbar \omega}}{2} \right)^n \frac{1}{n!} H_n(kx) H_n(kx').
\] (S3)

We use Mehler’s formula [51],

\[
\sum_{n=0}^{\infty} \frac{u^n}{2^n n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1-u^2}} e^{2uxy/(1-u^2)} e^{-u^2(x^2+y^2)/(1-u^2)},
\] (S4)

to rewrite the sum with the Hermite polynomials as a Gaussian, yielding

\[
\rho(x, x') = \frac{k}{Z \sqrt{\pi}} \exp \left( -\frac{k^2}{2} (x^2 + x'^2) \right) \frac{1}{\sqrt{1-u^2}} \exp \left( -\frac{k^2}{2(1-u^2)} (u^2(x^2 + x'^2) - 2uxx') \right),
\] (S5)

where we have defined \( u = e^{-\beta \hbar \omega} \). Finally, with the explicit form of the partition function \( Z = \text{Tr}(e^{-\beta \hat{H}}) = 1/(1-u) \), this coordinate representation also takes the form

\[
\rho(x, x') = \frac{k}{\sqrt{\pi}} \sqrt{\text{tanh}(\beta \hbar \omega / 2)} \exp \left( -\frac{k^2}{2} \coth(\beta \hbar \omega)(x^2 + x'^2) + k^2 \sinh^{-1}(\beta \hbar \omega) xx' \right).
\] (S6)

The same derivation holds for a time dependent Hamiltonian, and provides the initial and final coefficients \( A \) and \( C \) given in the main text. We verify that the normalization factor is \( \frac{k}{\sqrt{\pi}} \sqrt{\text{tanh}(\beta \hbar \omega / 2)} = \sqrt{2(A+C)/\pi} \).
B. Consistency equations from the evolution of the Gaussian Ansatz

The master equation (2) in the coordinate representation reads

\[
\frac{d\rho_t(x,x')}{dt} = \left( \frac{i\hbar}{2m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) - \frac{i m \omega_t^2}{2\hbar} (x^2 - x'^2) - \gamma_t (x-x')^2 \right) \rho_t(x,x').
\]  

(S7)

For the Gaussian Ansatz given in Eq. (3) of the main text, the real and imaginary parts of the evolution equation respectively give

\[
\frac{\dot{N}_t}{N_t} + \frac{2\hbar}{m} B_t = \left( \dot{A}_t + \frac{4\hbar}{m} A_t B_t - \gamma_t \right) (x^2 + x'^2) + 2 \left( \dot{C}_t + \frac{4\hbar}{m} B_tC_t + \gamma_t \right) xx',
\]  

(S8)

\[
\dot{B}_t = \frac{2\hbar}{m} (A_t^2 - B_t^2 - C_t^2) - \frac{m \omega_t^2}{2\hbar},
\]  

(S9)

from which the consistency equations (4) directly follow.

C. Instantaneous diagonalization of the density matrix

We look for the eigenvalues \( p_{n,t} \) and eigenfunctions \( |\psi_{n,t}\rangle \) that diagonalize the density matrix \( \rho_t \) at any time. For the sake of simplicity, we omit the time dependence in the notation below. By definition, the eigenvalues fulfil \( p_n > 0 \) and \( \sum_n p_n = 1 \), so we choose to write them as \( p_n = u^n(1 - u) \), where \( u \) can be seen as an exponential \( e^{-\tilde{\varepsilon}} \). We verify below that the functions

\[
\langle x|\psi_n\rangle = \sqrt{\frac{k}{2^n n! \sqrt{\pi}}} e^{-\frac{k^2 x^2}{2}} e^{\frac{B x^2}{2}} H_n(kx)
\]  

(S10)

correspond to the eigenfunctions. Note that orthogonality of the Hermite polynomials, \( \int_{-\infty}^{\infty} dx'H_n(kx')H_m(kx')e^{-kx^2/2} = \delta_{nm} 2^n n! \sqrt{\pi}/k \), guarantees orthonormality of the wave functions, \( \langle \psi_n|\psi_m\rangle = \delta_{nm} \). To justify the choice of these Ansatz and determine the time-dependent variables \( k \) and \( u \), we start with the coordinate representation

\[
\langle x|\rho_t|x'\rangle = \sum_n u^n(1 - u) \langle x|\psi_n\rangle \langle \psi_n|x'\rangle,
\]  

(S11)

and use Mehler’s equation (S4) to get

\[
\langle x|\rho_t|x'\rangle = \frac{k}{\sqrt{\pi}} \sqrt{\frac{1 - u}{1 + u}} e^{-k^2(\frac{x^2}{2} + x'^2) + \frac{k^2}{2}} e^{\frac{B x^2}{2}} e^{\frac{B x'^2}{2}} e^{\frac{2k^2 B x x'}{2}}.
\]  

(S12)

By identification, we obtain

\[
A = \frac{k^2}{2(1 - u^2)} = \frac{k^2}{2} \coth \tilde{\varepsilon}, \quad C = -k^2 \frac{u}{1 - u^2} = -\frac{k^2}{2} \text{csch}^2 \tilde{\varepsilon},
\]  

(S13)

the reverse transformation corresponding to the physical setting being for \( u > 0 \) and \( 0 < -A/C < 1 \), and

\[
k = \left( 2\sqrt{A^2 - C^2} \right)^{1/2}, \quad u = -\frac{A}{C} - \sqrt{\left( \frac{A}{C} \right)^2 - 1}.
\]  

(S14)

The mean phonon number easily follows as \( \langle n \rangle = \sum_{n=0}^{\infty} n p_n = \frac{u}{1 - u} \), and the von Neumann entropy \( S(\rho) = -\text{Tr}(\rho \log \rho) \) reads

\[
S(\rho) = -\sum_{n=0}^{\infty} p_n \ln p_n = \frac{u \ln u}{1 - u} - \ln(1 - u).
\]  

(S15)
D. Maximum dephasing strength

We can show that the dephasing strength is inversely proportional to the time of the protocol for any polynomial Ansatz interpolating between the initial and final state. The time for which the dephasing strength is maximal is given from $\dot{\gamma}_{t_{\text{max}}} = 0$, which leads

$$\frac{d^2 f_{\tau}}{d\tau^2} \bigg|_{t_{\text{max}}} = \left( \frac{df_{\tau}}{d\tau} \bigg|_{t_{\text{max}}} \right)^2 (A_f - A_0 + C_f - C_0),$$

(S16)

where $\tau = t/t_f$ and where we have used $df_{\tau}/dt = df_{\tau}/d(1/t_f)$. This equation could be solved for a specific polynomial Ansatz $f_{\tau} = \sum_{n=0}^{N} f_n \tau^n$. Since $\gamma$ takes a zero value at initial and final time, a non-trivial solution goes through an extremum in the region $\tau \in [0 : 1]$. We denote the root corresponding to this time $r_1$. The maximum dephasing strength is reached at time $t_{\text{max}} = r_1 t_f$. We further have

$$\gamma_{t_{\text{max}}} = \frac{1}{t_f} \frac{df_{\tau}}{d\tau} \bigg|_{t_{\text{max}}} \frac{(A_f - A_0)C_{t_{\text{max}}}(C_f - C_0)}{A_{t_{\text{max}}} + C_{t_{\text{max}}}}.$$  

(S17)

The parameters $A_{t_{\text{max}}}$ and $C_{t_{\text{max}}}$ are polynomials of $r_1$ and all terms on the r.h.s, apart from $1/t_f$, depend only on the root $r_1$. This yields $\gamma_{t_{\text{max}}} \propto 1/t_f$.

E. Determine the eigenfunctions overlap in Eq. (17) to evaluate the relative entropy

We provide below the explicit form for the overlap

$$\langle \phi_m | \psi_n \rangle = \int_{-\infty}^{\infty} dx \langle \phi_m | x \rangle \langle x | \psi_n \rangle = \frac{k}{\sqrt{2^{n+m}n!m!\pi}} \int_{-\infty}^{\infty} dx e^{-k^2 x^2} e^{-iBx^2} H_n(kx) H_m(kx).$$

This overlap can be expressed as

$$\langle \phi_m | \psi_n \rangle = \frac{1}{\sqrt{2^{n+m}n!m!\pi}} I_{n,m} (1 + iB/k^2)$$

(S18)

by defining the integral

$$I_{n,m}(b) = \int_{-\infty}^{\infty} dx e^{-bx^2} H_n(x) H_m(x),$$

(S19)

where the indices $n$ and $m$ play a symmetric role. To solve this integral, we first write the exponential as $e^{-bx^2} = e^{-x^2} e^{-(b-1)x^2}$ in order to have a Gaussian for each Hermite polynomial. Then, multiple integration by parts yield $\int dx e^{-x^2} H_n(x)f(x) = \int dx e^{-x^2} D^n(g(x))$, where $D^n \equiv (d/dx)^n$, for any function $g(x)$ [52, 53]. So, for $g(x) = e^{-(b-1)x^2} H_m(x)$, and choosing $n < m$ by convention, we find

$$I_{n,m} = \int_{-\infty}^{\infty} dx e^{-x^2} D^n(e^{-(b-1)x^2}) H_m(x).$$

(S20)

We then expend the derivative $D^n$ in a binomial form, use the derivative of the Hermite polynomial, $D^j (H_N(x)) = 2^j N!/(N-j)! H_{N-j}(x)$, and the definition of the Hermite polynomial $e^{-x^2} H_N(x) = (-D)^N e^{-x^2}$ to obtain

$$I_{n,m} = \sum_{l=0}^{n} \binom{n}{l} \frac{2^{n-l}l!}{(m-n+l)!} \int_{-\infty}^{\infty} dx e^{-x^2} D^l(e^{-(b-1)x^2}) H_{m-n+l}(x)$$

$$= \sum_{l=0}^{n} \binom{n}{l} \frac{2^{n-l}l!(-1)^{m-n+l}}{(m-n+l)!} \int_{-\infty}^{\infty} dx D^l(e^{-(b-1)x^2}) D^{m-n+l}(e^{-x^2}).$$

(S21)
In order to evaluate the new integral, we write each of the derivates as a Fourier transform, using

\[ D^n(e^{-x^2}) = (2i)^n/\sqrt{\pi} \int_{-\infty}^{\infty} dt \ e^{-t^2} e^{2i\pi t x} \] [54], which gives, taking \(\alpha = (1 - b)\) and \(j = (m - n + l)\),

\[
\int_{-\infty}^{\infty} dx \ D^l(e^{-\alpha x^2}) D^j(e^{-x^2}) = (2i)^{l+j} \sqrt{\pi} \int_{-\infty}^{\infty} dt ds \ e^{t^2+s^2} t^{l}s^{j} \int_{-\infty}^{\infty} dx \ e^{2i\pi(\sqrt{\alpha} t + s)}
\]

\[
= (-1)^j(2i\sqrt{\alpha})^{l+j} \int_{-\infty}^{\infty} dt \ e^{-t^2(1+\alpha)} t^{l+j}
\]

\[
= (-1)^j(2i\alpha)^{l+j} \left(\frac{1}{2}(1+\alpha)^{-\frac{l+j+1}{2}}(1 + (-1)^{l+j})\Gamma\left(\frac{l+j+1}{2}\right)\right).
\] (S22)

This leads to

\[
I_{n,m}(\alpha) = (1 + ((-1)^{m-n})) \sum_{l=0}^{n} 2^{m+l-1} \binom{n}{l} \frac{m!}{(m-n+l)!} i^{m-n+2l} \left(1 - \frac{1}{b}\right)^{\frac{2l+m-n}{2}} \frac{1}{\sqrt{b}} \Gamma\left(\frac{1+m-n}{2} + l\right).
\] (S23)

We can further simplify this expression by noting that it is non-zero only for \((l+j) = (m-n+2l)\) even. Choosing \(m > n\) by convention, we thus find \(I_{n,n+2p+1} = 0\) for all integers \(p\), and

\[
I_{n,n+2p} = \sqrt{\pi} b \sum_{l=0}^{n} \binom{n}{l} 2^{n-l} \left(\frac{1}{b}\right)^{p+l} \frac{(2p+n)!}{(2p+l)!(l+p)!}.
\] (S24)

where we have used \(\Gamma(1/2+n) = \sqrt{\pi}(2n)!/(4^n n!)\) to explicitly write the Gamma function from Eq. (S22). Note that this sum can also be written using the hypergeometric function \(_2F_1\), specifically

\[
I_{n,n+2p} = 2^n \sqrt{\pi} b \left(\frac{1}{b}\right)^p \frac{(2p+n)!}{(2p+l)!(l+p)!} _2F_1\left(\frac{1}{2} + p, -n, 1 + 2p, 2 - \frac{2}{b}\right).
\] (S25)

This provides the overlap integral, setting \(b = 1 + iB/k\), as

\[
\langle \phi_{n,n+2p} | \psi_n \rangle = \frac{k}{2^n n! \sqrt{b}} \sum_{l=0}^{n} \binom{n}{l} 2^{n-l} \left(\frac{1}{b}\right)^{p+l} \frac{(2p+n)!}{(2p+l)!(l+p)!}.
\] (S26)