SURFACES OF COORDINATE FINITE TYPE IN THE
LORENTZ-MINKOWSKI 3-SPACE

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Abstract. In this article, we study the class of surfaces of revolution in the
3-dimensional Lorentz-Minkowski space with nonvanishing Gauss curvature
whose position vector \( x \) satisfies the condition \( \Delta^{III} x = Ax \), where \( A \) is a
square matrix of order 3 and \( \Delta^{III} \) denotes the Laplace operator of the second
fundamental form \( III \) of the surface. We show that such surfaces are either
minimal or pseudospheres of a real or imaginary radius.

1. Introduction

Let \( M^2 \) be a connected non-degenerate submanifold in the 3-dimensional Lorentz-
Minkowski space \( E^3_1 \) and \( x : M^2 \to E^3_1 \) be a parametric representation of a surface in
the Lorentz-Minkowski 3-space \( E^3_1 \) equipped with the induced metric. Let \( (x, y, z) \)
be a rectangular coordinate system of \( E^3_1 \). By saying Lorentz-Minkowski space \( E^3_1 \),
we mean the Euclidean space \( E^3 \) equipped with the standard metric given by
\[
ds^2 = -dx^2 + dy^2 + dz^2.
\]
As well known that a submanifold is called a \( k \)-type submanifold if its position
vector \( x \) can be written as a sum of eigenvectors of the Laplace-Beltrami op erator,
\( \Delta \), according to \( k \) distinct eigenvalues, i.e., \( x = y_0 + y_1 + \cdots + y_k \), for a constant
vector \( y_0 \) and smooth non-constant functions \( y_k, (i = 1, \ldots, k) \) such that \( \Delta y_i = \lambda_i y_i, \lambda_i \in \mathbb{R}. \)
The year 1966 was the beginning when Takahashi in [34] stated
that spheres and minimal surfaces are the only ones in \( E^3 \) whose position vector \( x \)
satisfies the relation
\[
\Delta I x = \lambda x, \quad \lambda \in \mathbb{R},
\]
where \( \Delta I \) is the Laplace operator associated with the 1st fundamental form \( I \) of
the surface. Since the coordinate functions of \( x \) can be denoted as \( (x_1, x_2, x_3) \), then
Takahashi’s condition (1.1) becomes

Let \( (x_1, x_2, x_3) \) be the component functions of \( x \). Then it is well-known that
\[
\Delta I x = (\Delta I x_1, \Delta I x_2, \Delta I x_3).
\]

Thus Takahashi’s condition (1.1) becomes
\[
\Delta I x_i = \lambda_i x_i, \quad i = 1, 2, 3.
\]

Later, in [25] O. Garay generalized Takahashi’s condition (1.3). Actually, he
studied surfaces of revolution in \( E^3 \), whose component functions satisfy the condition
\[
\Delta I x_i = \lambda_i x_i, \quad i = 1, 2, 3,
\]

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that is, the component functions are eigenfunctions of their Laplacian but not necessary with the same eigenvalue. Another generalization is to study surfaces whose position vector $\mathbf{x}$ satisfies a relation of the form

$$\Delta^J \mathbf{x} = A \mathbf{x},$$

where $A \in \mathbb{R}^{3 \times 3}$.

Many results concerning this can be found in ([16], [19], [20], [22], [23], [25]). This type of study can be also extended to any smooth map, not necessary for the position vector of the surface, for example, the Gauss map of a surface. Regarding this see ([8], [13], [15], [25], [17], [18], [20], [24], [26]). Similarly, another extension can be drawn by applying the conditions stated before but for the second or third fundamental form of a surface [32]. Here again, many results can be found in ([1], [2], [14], [30], [31], [33]).

On the other hand, all the ideas mentioned above can be applied in the Lorentz-Minkowski space $E_{1}^3$. So, an interesting geometric question has posed classify all the surfaces in $E_{1}^3$, which satisfy the condition

$$\Delta^J \mathbf{x} = A \mathbf{x}, \quad J = I, II, III,$$

where $A \in \mathbb{R}^{3 \times 3}$ and $\Delta^J$ is the Laplace operator, with respect to the fundamental form $J$.

Kaimakamis and Papantoniou in [28] solved the above question for the class of surfaces of revolution with respect to the second fundamental form, while in [21] Bekkar and Zoubir studied the same class of surfaces with respect to the first fundamental form satisfying

$$\Delta x^i = \lambda^i x^i, \quad \lambda^i \in \mathbb{R}.$$  

2. Basic concepts

Let $C : \mathbf{r}(s) : s \in (a, b) \subset E \rightarrow E^2$ be a curve in a plane $E^2$ of $E_{1}^3$ and $l$ be a straight line of $E^2$ which does not intersect the curve $C$. A surface of revolution $M^2$ in $E_{1}^3$ is defined to be a non-degenerate surface, revolving the curve $C$ around the axis $l$. If the axis $l$ is timelike, then we consider that the $z$-axis as axis of revolution. If the axis $l$ is spacelike, then we may assume that the $x$-axis or $y$-axis as axis of revolution. Without loss of generality, we may consider the $x$-axis as the axis of revolution. If the axis is null, then we may assume that this axis is the line spanned by the vector $(0,1,1)$ of the $yz$-plane.

We consider the axis of revolution is the $x$-axis (spacelike) and the curve $C$ is lying in the $xy$-plane. Then a parameterization of $C$ with respect to its arclength is $\mathbf{r}(s) = (f(s), g(s), 0)$ where $f, g$ are smooth functions. Without loss of generality, we may assume that $f(s) > 0, s \in (a, b)$. A surface of revolution $M^2$ in $E_{1}^3$ in a system of local curvilinear coordinates $(s, \theta)$ is given by:

$$\mathbf{x}(s, \theta) = (f(s) \cosh \theta, g(s), f(s) \sinh \theta).$$  

or

$$\mathbf{x}(s, \theta) = (f(s) \sinh \theta, g(s), f(s) \cosh \theta).$$

In the case that the axis of revolution is the $z$-axis (timelike) and the curve $C$ is given $\mathbf{r}(s) = (f(s), 0, g(s))$ and lies in the $xz$-plane, the surface of revolution $M^2$ is given by:

$$\mathbf{x}(s, \theta) = (f(s) \cos \theta, f(s) \sin \theta, g(s)).$$  

or

$$\mathbf{x}(s, \theta) = (f(s) \sin \theta, f(s) \cos \theta, g(s)).$$

Finally, if the axis of revolution is the line spanned by the vector \((0, 1, 1)\) and the curve \(C\) lies in the \(yz\)-plane, then the surface of revolution \(M^2\) can be parametrized as
\[
x(s, \theta) = (\theta h(s), g(s) + \frac{1}{2} \theta^2 h(s), f(s) + \frac{1}{2} \theta^2 h(s)),
\]
where \(h(s) = f(s) - g(s) \neq 0\).

We denote by \(g = g_{km}, b = b_{km}\) and \(e = e_{km}, k, m = 1, 2\) the first, second and third fundamental forms of \(M^2\) respectively, where we put
\[
g_{11} = E = \langle x_s, x_s \rangle, \quad g_{12} = F = \langle x_s, x_\theta \rangle, \quad g_{22} = G = \langle x_\theta, x_\theta \rangle,
\]
\[
b_{11} = L = \langle x_{ss}, n \rangle, \quad b_{12} = M = \langle x_{s\theta}, n \rangle, \quad b_{22} = N = \langle x_{\theta\theta}, n \rangle,
\]
\[
e_{11} = \frac{EM^2 - 2FLM + GL^2}{EG - F^2} = \langle n_s, n_s \rangle,
\]
\[
e_{12} = \frac{EMN - FLN + GLM - FM^2}{EG - F^2} = \langle n_s, n_\theta \rangle,
\]
\[
e_{22} = \frac{GM^2 - 2FNM + EN^2}{EG - F^2} = \langle n_\theta, n_\theta \rangle,
\]
which are the coefficients of the first, second, third fundamental form respectively, and \(\langle, \rangle\) is the Lorentzian metric.

For a sufficient differentiable function \(p(u^1, u^2)\) on \(M^2\) the second Laplace operator according to the fundamental form \(III\) of \(M^2\) is defined by [4],
\[
\Delta^{III} p = -\frac{1}{\sqrt{e}} (\sqrt{e^{km} p^l/k})_{/m},
\]
where \(p_{/k} := \frac{\partial p}{\partial u^k}\), \(e^{km}\) denote the components of the inverse tensor of \(e_{km}\) and \(e = \det(e_{km})\). After a long computation, we arrive at
\[
\Delta^{III} p \quad = \quad -\frac{\sqrt{EG - F^2}}{LN - M^2} \left( \left( \frac{(GM^2 - 2FNM + EN^2) \partial p}{(LN - M^2) \sqrt{EG - F^2}} \right)_s - \left( \frac{EMN - FLN + GLM - FM^2 \partial p}{(LN - M^2) \sqrt{EG - F^2}} \right)_s \right)
\]
\[
- \left( \frac{(EMN - FLN + GLM - FM^2 \partial p}{(LN - M^2) \sqrt{EG - F^2}} \right)_\theta \right).
\]
Here we have \(LN - M^2 \neq 0\), since the surface has no parabolic points.
3. Proof of the main results

In this paragraph we classify the surfaces of revolution $M^2$ satisfying the relation (1.5). We distinguish the following three types according to whether these surfaces are determined.

Type I. The parametric representation of $M^2$ is given by (2.1). Then
\[ f'^2(s) + g'^2(s) = 1, \] (3.1)
where $\frac{d}{ds}$, from which we obtain that
\[ E = 1, \quad F = 0, \quad G = -f^2 \] (3.2)
and
\[ L = -f'g'' + g'f'', \quad M = 0, \quad N = fg'. \] (3.3)
Denoting by $\kappa$ the curvature of the curve $C$ and $r_1, r_2$ the principal radii of curvature of $M^2$. We have
\[ r_1 = \kappa, \quad r_2 = \frac{g'}{f}, \] (3.4)
which are the Gauss and mean curvature of $M^2$ respectively. Since the relation (3.1) holds, there exists a smooth function $\varphi = \varphi(s)$ such that
\[ f' = \cos \varphi, \quad g' = \sin \varphi, \] where $\varphi = \varphi(s)$. Then $\kappa = \varphi'$ and relations (3.3), (3.4) become
\[ L = -\varphi', \quad M = 0, \quad N = f \sin \varphi, \] (3.5)
We put $r = \frac{1}{r_1} + \frac{1}{r_2} = \frac{2H}{K}$. Thus we have
\[ r = -\left( \frac{1}{\varphi'} + \frac{f}{\sin \varphi} \right). \] (3.6)
Taking the derivative of last equation, we get
\[ r' = \left( \frac{\varphi''}{\varphi'^2} + \frac{f' \cos \varphi}{\sin^2 \varphi} - \frac{\cos \varphi}{\sin \varphi} \right). \] (3.7)
From (2.4), (3.2) and (3.5) we have
\[ \Delta^III p = -\frac{1}{\varphi'^2 \varphi'^2} + \frac{1}{\sin^2 \varphi} \frac{\partial^2 p}{\partial \theta^2} + \left( \frac{\varphi''}{\varphi'^3} - \frac{\cos \varphi}{\varphi' \sin \varphi} \right) \frac{\partial p}{\partial s}. \] (3.8)
Let $(x_1, x_2, x_3)$ be the coordinate functions of the position vector $x$ of (2.3). Then according to relation (1.2), (3.8) and taking into account (3.6) and (3.7) we find that
\[ \Delta^III x_1 = \Delta^III f(s) \cosh \theta = \left( -r \sin \varphi + r' \frac{\cos \varphi}{\varphi'} \right) \cosh \theta, \] (3.9)
\[ \Delta^III x_2 = \Delta^III g(s) = r \cos \varphi + r \frac{\sin \varphi}{\varphi'}. \] (3.10)
\[ \Delta^{III} x_3 = \Delta^{III} f(s) \sinh \theta = \left( -r \sin \varphi + r' \cos \frac{\varphi'}{\varphi} \right) \sinh \theta. \] (3.11)

We denote by \( a_{ij}, i, j = 1, 2, 3 \), the entries of the matrix \( A \), where all entries are real numbers. By using (3.9), (3.10) and (3.11) condition (1.5) is found to be equivalent to the following system

\[ \left( -r \sin \varphi + r' \cos \frac{\varphi'}{\varphi} \right) \cosh \theta = a_{11} f(s) \cosh \theta + a_{12} g(s) + a_{13} f(s) \sinh \theta, \] (3.12)

\[ r \cos \varphi + r' \sin \frac{\varphi'}{\varphi'} = a_{21} f(s) \cosh \theta + a_{22} g(s) + a_{23} f(s) \sinh \theta, \] (3.13)

\[ \left( -r \sin \varphi + r' \cos \frac{\varphi'}{\varphi'} \right) \sinh \theta = a_{31} f(s) \cosh \theta + a_{32} g(s) + a_{33} f(s) \sinh \theta. \] (3.14)

From (3.13) it can be easily verified that \( a_{21} = a_{23} = 0 \). On the other hand, differentiating (3.12) and (3.14) twice with respect to \( \theta \) we get that \( a_{12} = a_{32} = 0 \). So, the system is reduced to

\[ \left( -r \sin \varphi + r' \cos \frac{\varphi'}{\varphi'} \right) \cosh \theta = a_{11} f(s) \cosh \theta + a_{13} f(s) \sinh \theta, \] (3.15)

\[ r \cos \varphi + r' \sin \frac{\varphi'}{\varphi'} = a_{22} g(s), \] (3.16)

\[ \left( -r \sin \varphi + r' \cos \frac{\varphi'}{\varphi'} \right) \sinh \theta = a_{31} f(s) \cosh \theta + a_{33} f(s) \sinh \theta. \] (3.17)

But \( \sinh \theta \) and \( \cosh \theta \) are linearly independent functions of \( \theta \), so we deduce that \( a_{13} = a_{31} = 0, a_{11} = a_{33} \). Putting \( a_{11} = a_{33} = \lambda \) and \( a_{22} = \mu \), we see that the system of equations (3.15), (3.16) and (3.17) reduces now to the following two equations

\[ -r \sin \varphi + r' \cos \frac{\varphi'}{\varphi'} = \lambda f, \] (3.18)

\[ r \cos \varphi + r' \sin \frac{\varphi'}{\varphi'} = \mu g. \] (3.19)

Hence the matrix \( A \) for which relation (1.5) is satisfied becomes

\[ A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \]

Solving the system (3.18) and (3.19) with respect to \( r \) and \( r' \), we conclude that

\[ r' = \varphi' (\lambda f \cos \varphi + \mu g \sin \varphi), \] (3.20)

\[ r = \mu g \cos \varphi - \lambda f \sin \varphi. \] (3.21)

Taking the derivative of (3.21), we find

\[ r' = \frac{1}{2} (\mu - \lambda) \cos \varphi \sin \varphi. \] (3.22)

We distinguish now the following cases:

Case I. \( \mu = \lambda = 0 \). Thus, according to (3.21) we have \( r = 0 \). Consequently \( H = 0 \). Therefore \( M^2 \) is minimal and the corresponding matrix \( A \) is the zero matrix.
Case II. \( \mu = \lambda \neq 0 \). Then from (3.22) we have \( r' = 0 \). If \( \varphi' = 0 \), then \( M^2 \) would consist only of parabolic points, which has been excluded. Therefore we find that
\[
f(s) \cos \varphi + g(s) \sin \varphi = 0,
\]
or
\[
ff' + gg' = 0.
\]
Then \( f^2 + g^2 = c^2 \), \( c \in \mathbb{R} \) and \( M^2 \) is obviously satisfies the equation \( x^2 + y^2 - z^2 = c^2 \) which is the pseudosphere \( S^2_1(c) \) of \( E^3_1 \).

Case III. \( \lambda \neq 0, \mu = 0 \). Then the system (3.18), (3.19) is equivalently reduced to
\[
-r \sin \varphi + r' \frac{\cos \varphi}{\varphi'} = \lambda f(s),
\]
\[
r \cos \varphi + r' \frac{\sin \varphi}{\varphi'} = 0.
\]
From (3.21) we have
\[
r + \lambda f \sin \varphi = 0. \tag{3.23}
\]
On differentiating (3.23) and taking into account (3.20) with \( \mu = 0 \), we obtain
\[
\lambda f \varphi' \cos \varphi + \lambda \cos \varphi \sin \varphi + \lambda f \varphi' \cos \varphi = 0
\]
or
\[
\varphi' = -\frac{\sin \varphi}{2f}.
\]
From (3.23), (3.6) and the last equation, we get
\[
\frac{f}{\sin \varphi} + \lambda f \sin \varphi = 0
\]
or
\[
f(1 + \lambda \sin^2 \varphi) = 0.
\]
A contradiction. Hence, there is no surface of revolution with parametric representation \( \mathbf{2.1} \) of \( E^3_1 \) satisfying \( \mathbf{1.5} \).

Case IV. \( \lambda = 0, \mu \neq 0 \). Then equations (3.18), (3.19) reduced to
\[
-r \sin \varphi + r' \frac{\cos \varphi}{\varphi'} = 0,
\]
\[
r \cos \varphi + r' \frac{\sin \varphi}{\varphi'} = \mu g. \tag{3.24}
\]
From (3.21) we have
\[
r - \mu g \cos \varphi = 0. \tag{3.25}
\]
Taking the derivative of (3.25) and taking into account (3.20) with \( \lambda = 0 \), we find
\[
\mu g \varphi' \sin \varphi - \mu \cos \varphi \sin \varphi + \mu g \varphi' \sin \varphi = 0
\]
or
\[
\varphi' = \frac{\cos \varphi}{2g}. \tag{3.26}
\]
Taking the derivative of (3.26), we find
\[
3 \varphi' \sin \varphi + 2g \varphi'' = 0. \tag{3.27}
\]
On account of (3.24), (3.6) and (3.7) it is easily verified that
\[
\varphi'' = \frac{\varphi'^2}{\sin \varphi} (\mu g \varphi' + 2 \cos \varphi). \tag{3.28}
\]
Inserting (3.26) and (3.28) in (3.27) we conclude

\[ 3 + \left( \frac{1}{2} \mu - 1 \right) \cos^2 \phi = 0. \]

Here we have also a contradiction.

**Case V.** \( \lambda \neq 0, \mu \neq 0 \). We write equations (3.18) and (3.19) as follows

\[
\frac{\sin \phi}{\phi'} + \frac{f}{\sin^2 \phi} + \frac{\phi'' \cos \phi}{\phi^3} - \frac{\cos^2 \phi}{\phi' \sin \phi} - \lambda f = 0, \tag{3.29}
\]

\[
\frac{\phi'' \sin \phi}{\phi'^3} - \frac{2 \cos \phi}{\phi'} - \mu g = 0. \tag{3.30}
\]

From (3.30) we have relation (3.28). By eliminating \( \phi'' \) from (3.29) we get

\[
\frac{1}{\phi'} + \frac{\mu g \cos \phi}{\sin \phi} + \frac{f}{\sin^2 \phi} = 0. \tag{3.31}
\]

On differentiating the last equation and using (3.28) we find

\[
2 \mu g \phi' \sin \phi + 2 f \phi' \cos \phi + 2 \cos \phi \phi' - (\mu - \lambda) \cos \phi = 0. \tag{3.32}
\]

Multiplying (3.31) by \( \frac{2 \phi'}{\sin \phi} \) and (3.32) by \( -\cos \phi \) we obtain

\[
\frac{2}{\sin^2 \phi} - \frac{2 \mu g \phi' \cos \phi}{\sin^2 \phi} - \frac{2 f \phi'}{\sin^3 \phi} - \frac{2 \lambda \phi' f}{\sin \phi} = 0, \tag{3.33}
\]

\[
- \frac{2 \mu g \phi' \cos \phi}{\sin^2 \phi} - \frac{2 f \phi' \cos^2 \phi}{\sin^3 \phi} - \frac{2 \cos^2 \phi}{\sin^2 \phi} + (\mu - \lambda) \cos^2 \phi = 0. \tag{3.34}
\]

Combining (3.33) and (3.34) we conclude that

\[
(\mu - \lambda) \cos^2 \phi - 2(\lambda - 1) \frac{f \phi'}{\sin \phi} + 2 = 0 \tag{3.35}
\]

or

\[
\frac{(\mu - \lambda) \cos^2 \phi}{\phi'} - 2(\lambda - 1) \frac{f}{\sin \phi} + \frac{2}{\phi'} = 0. \tag{3.36}
\]

Taking the derivative of the above equation and using (3.22) and (3.28) we find

\[
2(\mu + 1) \cos \phi + (\mu - \lambda) \mu g \phi' \cos^2 \phi - 2(\lambda - 1) \frac{f \phi' \cos \phi}{\sin \phi} + 2 \mu g \phi' = 0. \tag{3.37}
\]

Multiplying (3.35) by \( -\cos \phi \), and adding the resulting equation to (3.36) we get

\[
2 \mu \cos \phi + (2 + (\mu - \lambda) \cos^2 \phi) \mu g \phi' - (\mu - \lambda) \cos^3 \phi = 0
\]

or

\[
2 \mu \cos^2 \phi + (2 + (\mu - \lambda) \cos^2 \phi) \mu g \phi' \cos \phi - (\mu - \lambda) \cos^4 \phi = 0. \tag{3.38}
\]

On account of (3.31) we find

\[
\mu g \phi' \cos \phi = \lambda f \phi' \sin \phi - \frac{f \phi'}{\sin \phi} - 1. \tag{3.39}
\]

Eliminating \( \mu g \phi' \cos \phi \) from (3.37) by using (3.38), equation (3.37) reduces to

\[
2 \mu \cos^2 \phi - (\mu - \lambda) \cos^4 \phi + (2 + (\mu - \lambda) \cos^2 \phi) \left( (\lambda \sin^2 \phi - 1) \frac{f \phi'}{\sin \phi} - 1 \right) = 0. \tag{3.39}
\]
But from (3.35) we have

\[ \frac{f \varphi'}{\sin \varphi} = \frac{(\mu - \lambda) \cos^2 \varphi + 2}{2(\lambda - 1)}. \] (3.40)

Obviously \( \lambda \neq 1 \) because otherwise, from (3.35) we would have

\[ (\mu - \lambda) \cos^2 \varphi + 2 = 0. \]

A contradiction. Now, by inserting (3.40) in (3.39) we obtain

\[
-\lambda(\mu - \lambda)^2 \cos^4 \varphi + (\mu - \lambda)((\mu - \lambda)(\lambda - 1) - 6\lambda + 2) \cos^2 \varphi \\
+ 6\mu(\lambda - 1) - 2\lambda(\lambda + 1) = 0.
\]

This relation, however, is valid for a finite number of values of \( \varphi \). So in this case there are no surfaces of revolution with the required property. So we proved the following

**Theorem 1.** Let \( x : M^2 \to E_3 \) be a surface of revolution given by (2.1). Then \( x \) satisfies (1.5) regarding to the third fundamental form if and only if the following statements hold true

- \( M^2 \) is the pseudosphere \( S^2_1(c) \) of real radius \( c \),
- \( M^2 \) has zero mean curvature.

Type II. The parametric representation of \( M^2 \) is given by (2.2). Then the tangent vector of the revolving curve is

\[ \langle x', x' \rangle = f'^2 - g'^2 = \pm 1. \]

We assume that

\[ f'^2 - g'^2 = 1, \quad \forall s \in (a, b). \] (3.41)

Then the components of the first and second fundamental forms are respectively

\[ E = 1, \quad F = 0, \quad G = f'^2, \]

\[ L = f'g'' - g'f'', \quad M = 0, \quad N = fg'. \]

From the equation (3.41) it is obviously clear that there exist a smooth function \( \varphi = \varphi(s) \) such that

\[ f' = \cosh \varphi, \quad g' = \sinh \varphi. \]

On the other hand we have

\[ r_1 = \kappa = \varphi', \quad r_2 = \frac{g'}{f} = \frac{\sinh \varphi}{f} \]

and

\[ K = r_1r_2 = \frac{\kappa g'}{f} = \frac{f''}{f} = \frac{\varphi' \sinh \varphi}{f}, \quad 2H = r_1 + r_2 = \varphi' + \frac{\sinh \varphi}{f}. \]

Here we have

\[ r = \frac{1}{\varphi} + \frac{f}{\sinh \varphi}. \] (3.42)

Taking the derivative of last equation, we get

\[ r' = -\frac{\varphi''}{\varphi^2} - \frac{f \varphi' \cosh \varphi}{\sinh^2 \varphi} + \frac{\cosh \varphi}{\sinh \varphi}. \] (3.43)
On the other hand

\[ \Delta^{III} p = -1 - \frac{\partial^2 p}{\partial r^2} - \frac{1}{\sinh^2 \varphi} \frac{\partial^2 p}{\partial \theta^2} + \left( \frac{\varphi''}{\varphi^3} - \frac{\cosh \varphi}{\varphi''} \right) \frac{\partial p}{\partial s}. \]  

(3.44)

According to relation (2.2) and (3.44) we find that

\[ \Delta^{III} x_1 = \Delta^{III} f(s) \cos \theta = \left( - r \sinh \varphi - r' \frac{\cosh \varphi}{\varphi'} \right) \cos \theta, \]

\[ \Delta^{III} x_2 = \Delta^{III} f(s) \sin \theta = \left( - r \sinh \varphi - r' \frac{\cosh \varphi}{\varphi'} \right) \sin \theta, \]

\[ \Delta^{III} x_3 = \Delta^{III} g(s) = - r \cosh \varphi - r' \frac{\sinh \varphi}{\varphi'}. \]

Let now \( \Delta^{III} \mathbf{x} = A \mathbf{x} \). Thus, as in the former paragraph, we find

\[ \begin{pmatrix} - r \sinh \varphi - r' \frac{\cosh \varphi}{\varphi'} \end{pmatrix} \cos \theta = a_{11} f(s) \cos \theta + a_{12} f(s) \sin \theta + a_{13} g(s), \]

\[ \begin{pmatrix} - r \sinh \varphi - r' \frac{\cosh \varphi}{\varphi'} \end{pmatrix} \sin \theta = a_{21} f(s) \cos \theta + a_{22} f(s) \sin \theta + a_{23} g(s), \]

\[ - r \cosh \varphi - r' \frac{\sinh \varphi}{\varphi'} = a_{31} f(s) \cos \theta + a_{32} f(s) \sin \theta + a_{33} g(s). \]

Applying the same algebraic methods, used in the previous type, this system of equations reduced to

\[ - r \sinh \varphi - r' \frac{\cosh \varphi}{\varphi'} = \lambda f, \]  

(3.45)

\[ - r \cosh \varphi - r' \frac{\sinh \varphi}{\varphi'} = \mu g, \]  

(3.46)

where \( a_{11} = a_{22} = \lambda, a_{33} = \mu, \lambda, \mu \in R \). Solving the system (3.45) and (3.46) with respect to \( r \) and \( r' \), we conclude that

\[ r' = \varphi' (- \lambda f \cosh \varphi + \mu g \sinh \varphi), \]  

(3.47)

\[ r = \lambda f \sinh \varphi - \mu g \cosh \varphi. \]  

(3.48)

Similarly, we have the following five cases according to the values of \( \lambda, \mu \).

Case I. \( \lambda = \mu = 0 \). Thus from (3.48) we conclude that \( r = 0 \). Consequently \( H = 0 \). Therefore \( M^2 \) is minimal and the corresponding matrix \( A \) is the zero matrix.

Case II. \( \mu = \varphi \neq 0 \). Then from (3.47) we have \( r' = 0 \). If \( \varphi' = 0 \), then \( M^2 \) would consist only of parabolic points, which has been excluded. Therefore we find that

\[ - f \cosh \varphi + g \sinh \varphi = 0, \]

or

\[ - f f' + gg' = 0. \]

Then \( g^2 - f^2 = \pm c^2, c \in R \) and therefore, \( M^2 \) is obviously either the pseudosphere \( S^2_1(c) \) of real radius \( c \), given by the equation \( x^2 + y^2 - z^2 = c^2 \), or the pseudosphere \( H^2_1(c) \) with imaginary radius, given by \( x^2 + y^2 - z^2 = -c^2 \).

Case III. \( \lambda \neq 0, \mu = 0 \). Then the system (3.45), (3.46) is reduced to

\[ - r \sinh \varphi - r' \frac{\cosh \varphi}{\varphi'} = \lambda f(s), \]
\[ r \cosh \varphi + r' \sinh \varphi = 0. \]

From (3.48) we have
\[ r - \lambda f \sinh \varphi = 0. \] (3.49)

On differentiating (3.49) and taking into account (3.47) with \( \mu = 0 \), we obtain
\[ \varphi' = - \frac{\sinh \varphi}{2f}. \]

From (3.42), (3.48) and the last equation, we get
\[ f(1 - \lambda \sin^2 \varphi) = 0. \]

A contradiction. Hence, there are no surfaces of revolution with parametric representation (2.2) of \( E^3_1 \) satisfying (1.5).

Case IV. \( \lambda = 0, \mu \neq 0 \).

We write equations (3.45) and (3.46) as follows
\[ - \sinh \varphi \varphi' + \frac{f}{\sinh^2 \varphi} + \frac{\varphi'' \cosh \varphi}{\sinh^2 \varphi} - \frac{\cosh^2 \varphi}{\varphi' \sinh \varphi} - \lambda f = 0, \] (3.55)

\[ \frac{\varphi'' \sinh \varphi}{\varphi' \sinh \varphi} - \frac{2 \cosh \varphi}{\varphi' - \mu g} = 0. \] (3.56)

From (3.56) we have relation (3.54). By eliminating \( \varphi'' \) from (3.55) we get
\[ \frac{1}{\varphi' \sinh \varphi} + \frac{\mu g \cosh \varphi}{\sinh \varphi} + \frac{f}{\sinh^2 \varphi} - \lambda f = 0. \] (3.57)

On differentiating the last equation and using (3.54) we find
\[ \frac{2 \mu g \varphi'}{\sinh^3 \varphi} + \frac{2 f \varphi' \cosh \varphi}{\sinh^3 \varphi} + \frac{2 \cosh \varphi}{\sinh^2 \varphi} - (\mu - \lambda) \cosh \varphi = 0. \] (3.58)
Multiplying (3.57) by \( \frac{2\varphi'}{\sinh^2 \varphi} \) and (3.58) by \(-\cosh \varphi\) we obtain
\[
\frac{2}{\sinh^2 \varphi} + \frac{2\mu g \varphi' \cosh \varphi}{\sinh^2 \varphi} + \frac{2f \varphi' \cosh \varphi}{\sinh^2 \varphi} - \frac{2\lambda \varphi' f}{\sinh \varphi} = 0, \tag{3.59}
\]
\[-\frac{2\mu g \varphi' \cosh \varphi}{\sinh^2 \varphi} - \frac{2f \varphi' \cosh^2 \varphi}{\sinh^2 \varphi} - \frac{2 \cosh^2 \varphi}{\sinh^2 \varphi} + (\mu - \lambda) \cosh^2 \varphi = 0. \tag{3.60}\]
Combining (3.59) and (3.60) we conclude that
\[
(\mu - \lambda) \cosh^2 \varphi - 2(\lambda + 1) \frac{f \varphi'}{\sinh \varphi} - 2 = 0 \tag{3.61}
\]
or
\[
\frac{(\mu - \lambda) \cosh^2 \varphi}{\varphi'} - 2(\lambda + 1) \frac{f}{\sinh \varphi} - \frac{2}{\varphi'} = 0. \tag{3.61}
\]

Taking the derivative of the above equation and using (3.54) we find
\[
2(\mu + 1) \cosh \varphi + (\mu - \lambda) \mu g \varphi' \cosh^2 \varphi - 2(\lambda + 1) \frac{f \varphi' \cosh \varphi}{\sinh \varphi} - 2\mu g \varphi' = 0. \tag{3.62}\]
Multiplying (3.61) by \(-\cosh \varphi\), and adding the resulting equation to (3.62) we get
\[
2(\mu + 2) \cosh \varphi - (2 - (\mu - \lambda) \cos^2 \varphi) \mu g \varphi' - (\mu - \lambda) \cosh^3 \varphi = 0
\]
or
\[
2(\mu + 2) \cosh^2 \varphi - (2 - (\mu - \lambda) \cosh^2 \varphi) \mu g \varphi' \cos \varphi - (\mu - \lambda) \cos^4 \varphi = 0. \tag{3.63}\]
On account of (3.57) we find
\[
\mu g \varphi' \cosh \varphi = \lambda f \varphi' \sinh \varphi - \frac{f \varphi'}{\sinh \varphi} - 1. \tag{3.64}\]
Eliminating \(\mu g \varphi' \cosh \varphi\) from (3.63) by using (3.64), we get
\[
2(\mu + 2) \cosh^2 \varphi - (\mu - \lambda) \cosh^4 \varphi - (2 - (\mu - \lambda) \cosh^2 \varphi) ((\lambda \sinh^2 \varphi - 1) \frac{f \varphi'}{\sinh \varphi} - 1) = 0. \tag{3.65}\]
But from (3.61) we have
\[
\frac{f \varphi'}{\sinh \varphi} = \frac{2 - (\mu - \lambda) \cosh^2 \varphi}{2(\lambda + 1)}. \tag{3.66}\]
Obviously \(\lambda \neq -1\) because otherwise, from (3.61) we would have
\[
(\mu - \lambda) \cosh^2 \varphi - 2 = 0.
\]
A contradiction. Now, by inserting (3.66) in (3.65) we obtain
\[
-\lambda(\mu - \lambda)^2 \cos^6 \varphi + (\mu - \lambda)((\mu - \lambda)(\lambda - 1) + 4\lambda) \cos^4 \varphi + (6\lambda^2 - 2\lambda - 2\mu - 2\lambda\mu + 8) \cos^2 \varphi + 8(\lambda + 1) = 0.
\]
This relation, however, is valid for a finite number of values of \(\varphi\). So in this case there are no surfaces of revolution with the required property. So we proved the following

**Theorem 2.** Let \(x : M^2 \rightarrow E^3_1\) be a surface of revolution given by (2.2). Then \(x\) satisfies (1.5) regarding to the third fundamental form if and only if the following statements hold true

\[
\text{...}
\]
• $M^2$ is the pseudosphere $S^2(c)$ of real or imaginary radius $c$,
• $M^2$ has zero mean curvature.

Type III. The parametric representation of $M^2$ is given by (2.3), i.e.,
$$\mathbf{x}(s, \theta) = (f(s) + \frac{1}{2} s^2 h(s), g(s) + \frac{1}{2} s^2 h(s), \theta h(s)),$$
where $h(s) = f(s) - g(s) \neq 0$. Since $M^2$ is non-degenerate, $f'(s)^2 - g'(s)^2$ never vanishes, and so $h'(s) = f'(s) - g'(s) \neq 0$ everywhere. Now, we may take the parameter in such a way that $h(s) = -2s$.

Assume that $k(s) = g(s) - s$, then
$$f(s) = k(s) - s \quad g(s) = k(s) + s,$$
(see for example, [29]). Therefore $M^2$ can be reparametrized as follows
$$\mathbf{x}(s, \theta) = (k - s - \theta^2 s, k + s - \theta^2 s, -2s\theta),$$
with the profile curve given in (3.67) becomes
$$\mathbf{r}(s) = (0, k(s) - s, k(s) + s).$$

By using the tangent vector fields, $\mathbf{x}_s$ and $\mathbf{x}_\theta$ of $M^2$, we get the components of the first fundamental form of it as
$$E = 4k'(s), \quad F = 0, \quad G = 4s^2.$$

Now, let $M^2$ be spacelike surface, i.e., $k'(s) > 0$. Then, the timelike unit normal vector field $\mathbf{N}$ of $M^2$ is given by
$$\mathbf{N} = \frac{1}{2\sqrt{k'}}(\theta^2 + 1, \theta^2 - 1, 2\theta) + \frac{\sqrt{k'}}{2}(1, 1, 0).$$
(3.69)

Then the components of the second fundamental forms are given by
$$L = -\frac{k''}{\sqrt{k'}}, \quad M = 0, \quad N = \frac{2s}{\sqrt{k'}}.$$
Thus the relation (2.31) becomes
$$\Delta^{III}p = -\frac{4k'^2}{k''\partial^2 s^2} - k'\frac{\partial^2 p}{\partial \theta^2} + 2k'(2k'' - k'^2)\frac{\partial p}{\partial s}. \quad (3.70)$$

According to relations (2.3) and (3.70) we find that
$$\Delta^{III}x_1 = \Delta^{III}(k-s-s^2) = \frac{2k'}{k''}(2k'' - k'^2)(k' - 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk',$$
$$\Delta^{III}x_2 = \Delta^{III}(k+s-s^2) = \frac{2k'}{k''}(2k'' - k'^2)(k' + 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk',$$
$$\Delta^{III}x_3 = \Delta^{III}(-2s\theta) = -\frac{4k'}{k''}(2k'' - k'^2)\theta.$$

Let now $\Delta^{III}x = Ax$. Then
$$\frac{2k'}{k''}(2k'' - k'^2)(k' - 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk' = a_{11}(k - s - \theta^2) + a_{12}(k + s - \theta^2) + a_{13}(-2s\theta),$$
(3.71)
\[
\frac{2k'}{k'^3} \left( (2k'k''' - k''^2)(k' + 1 - \theta^2) - \frac{4k'^2}{k''} + 2sk' \right) =
\begin{align*}
a_{21}(k - s - s\theta^2) + a_{22}(k + s - s\theta^2) + a_{23}(-2s\theta), \\
- \frac{4k'}{k'^3} \left( (2k'k''' - k''^2) \theta \right) = a_{31}(k - s - s\theta^2) + a_{32}(k + s - s\theta^2) + a_{33}(-2s\theta).
\end{align*}
\] (3.72)

Regarding the above equations as polynomials in \( \theta \), so from the coefficients of (3.73) we get
\[
(a_{31} + a_{32})s = 0,
\] (3.74)
\[
\frac{2k'}{k'^3} \left( (2k'k''' - k''^2) \right) = a_{33}s,
\] (3.75)
\[
(a_{32} - a_{31})s + (a_{31} + a_{32})k = 0.
\] (3.76)

From the coefficients of (3.72) we find
\[
\frac{2k'}{k'^3} \left( (2k'k''' - k''^2) \right) = (a_{21} + a_{22})s,
\] (3.77)
\[
a_{23}s = 0,
\] (3.78)
\[
\frac{2k'}{k'^3} \left( (2k'k''' - k''^2) \right) = (a_{11} + a_{12})s,
\] (3.79)
\[
a_{13}s = 0.
\] (3.80)

From the coefficients of (3.74) we get
\[
\left( a_{23} - a_{21} \right) s + \left( a_{22} - a_{21} \right) k = 0,
\] (3.81)
\[
\left( a_{13} - a_{12} \right) s + \left( a_{12} - a_{11} \right) k = 0.
\] (3.82)

Moreover, by considering (3.75) and (3.83) in (3.79) and (3.82), respectively, we get
\[
\left( a_{23} - a_{21} \right) s + \left( a_{22} - a_{21} \right) k = 0.
\] (3.83)

From (3.83) and (3.84) we find
\[
\begin{align*}
a_{12} &= a_{33} - a_{11}, \\
a_{21} &= a_{33} - a_{22}.
\end{align*}
\] (3.87)
Taking into account relations (3.86) and (3.87), we get
\[ a_{11} + a_{22} = 2a_{33}. \]

We put \( a_{11} = \lambda \) and \( a_{22} = \mu \), so the matrix \( A \) for which relation (15) is satisfied takes finally the following form
\[
A = \begin{bmatrix}
\lambda & \frac{1}{2}(\mu - \lambda) & 0 \\
\frac{1}{2}(\lambda - \mu) & \mu & 0 \\
0 & 0 & \frac{1}{2}(\lambda + \mu)
\end{bmatrix}.
\]

Hence system of equations [(3.84), ..., (3.82)] reduces to the following two equations
\[
\frac{2k'}{k''}(2k'k''' - k''^2) = a_{33}s, \tag{3.88}
\]
\[
(a_{33} + 2)k's + 2a_{12}s - \frac{4k'^2}{k''} - a_{33}k = 0, \tag{3.89}
\]
where, as we mention before, \( a_{33} = \frac{1}{2}(\lambda + \mu) \) and \( a_{12} = \frac{1}{2}(\mu - \lambda) \).

Solving the system of equations (3.88) and (3.89) with respect to \( \lambda \) and \( \mu \) we find
\[
\lambda = \frac{k'(2s - k + sk')}{s^2k''} \left( \frac{2k'k''}{k''^2} - 1 \right) - \frac{2k'^2}{sk''} + k', \tag{3.90}
\]
\[
\mu = \frac{k'(2s + k - sk')}{s^2k''} \left( \frac{2k'k''}{k''^2} - 1 \right) + \frac{2k'^2}{sk''} - k'. \tag{3.91}
\]

**Case I.** \( \lambda = \mu = 0 \). Thus from (3.90) and (3.91) we conclude that \( k = as^3 + b \) with \( a > 0 \), \( b \) is a constant and \( s \neq 0 \). Consequently, \( H = 0 \). Therefore \( M^2 \) is minimal and the corresponding matrix \( A \) is the zero matrix.

**Case II.** \( \lambda = \mu \neq 0 \). Thus from Case I, \( k \neq as^3 + b \). Now, from (3.53) we get \( a_{23} = 0 \), and so
\[
\frac{(k - sk')(2k'k''' - k''^2)}{s^2k''} + \frac{2k'}{sk''} - 1 = 0, \tag{3.92}
\]
whose solution is \( k(s) = \pm \frac{c^2}{4s} \). By considering (3.68), we conclude \( r \) is a spherical curve and so the surface \( M^2 \) is an open piece of the pseudo-sphere \( S^2_1(0, c) \) or the hyperbolic space \( \mathbb{H}^2(0, c) \).

**Case III.** \( \lambda \neq 0, \mu = 0 \). By considering the last assumption in (3.91), i.e. \( \mu = 0 \), we have
\[
\frac{2k'}{sk''} \left( \frac{2k'k''}{k''^2} - 1 \right) = \frac{k'(-k + sk')}{s^2k''} \left( \frac{2k'k''}{k''^2} - 1 \right) - \frac{2k'^2}{sk''} + k'.
\]

By substituting this into (3.90), we get
\[
\lambda = \frac{4k'}{sk''} \left( \frac{2k'k''}{k''^2} - 1 \right),
\]
where \( \lambda \) is non-zero function. Since there is no \( k \) function to implement in both conditions, so there is no surface of revolution that fulfills these conditions.

**Case IV.** \( \lambda = 0, \mu \neq 0 \). Similarly, we get a contradiction as in Case III.

**Case V.** \( \lambda \neq \mu \) and \( \lambda \neq 0, \mu \neq 0 \). In this case, the above two relations (3.90) and (3.91) are valid only when \( \lambda \) and \( \mu \) are functions of \( s \). Thus there are no surfaces of revolution with the required property. So we proved the following:
Theorem 3. Let \( x : M^2 \rightarrow E_3^1 \) be a surface of revolution given by (2.3). Then \( x \) satisfies (1.5) regarding to the third fundamental form if and only if the following statements hold true:

- \( M^2 \) has zero mean curvature,
- \( M^2 \) is an open piece of the pseudo sphere \( S^2_1(0, c) \) of real radius \( c \),
- \( M^2 \) is an open piece of the hyperbolic space \( \mathbb{H}^2_1(0, c) \) of real radius \( c \).

Finally, as we know that the minimal surfaces of revolution with non-lightlike axis are congruent to a part of the catenoid and also with lightlike axis are congruent to a part of the surface of Enneper, (see for more details [35]). Now, by combining Theorem 1, Theorem 2, Theorem 3 and [35]:

Theorem 4. (Classification) Let \( x : M^2 \rightarrow E_3^1 \) be a surface of revolution satisfying (1.5) regarding the third fundamental form. Then \( M \) is one of the following:

- \( M^2 \) is an open part of catenoid of the 1st kind, the 2nd kind, the 3rd kind, the 4th kind or the 5th kind.
- \( M^2 \) is an open part of the surface of Enneper of the 2nd kind or the 3rd kind.
- \( M^2 \) is an open part of the pseudo sphere \( S^2_1(0, c) \) centered at the origin with radius \( c \).
- \( M^2 \) is an open part of the hyperbolic space \( \mathbb{H}^2_1(0, c) \) centered at the origin with radius \( c \).

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