Squeezing spectra from $s$-ordered quasiprobability distributions.  
Application to dispersive optical bistability

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Abstract

It is well known that the squeezing spectrum of the field exiting a nonlinear cavity can be directly obtained from the fluctuation spectrum of normally ordered products of creation and annihilation operators of the cavity mode. In this article we show that the output field squeezing spectrum can be derived also by combining the fluctuation spectra of any pair of $s$-ordered products of creation and annihilation operators. The interesting result is that the spectrum obtained in this way from the linearized Langevin equations is exact, and this occurs in spite of the fact that no $s$-ordered quasiprobability distribution verifies a true Fokker–Planck equation, i.e., the Langevin equations used for deriving the squeezing spectrum are not exact. The (linearized) intracavity squeezing obtained from any $s$-ordered distribution is also exact. These results are exemplified in the problem of dispersive optical bistability.
I. INTRODUCTION

An appropriate tool for studying the fluctuations of a quantum light field $\hat{E}(t)$ is their spectrum, which is defined as the Fourier transform of the field correlations $\langle \hat{E}(t+\tau), \hat{E}(t) \rangle$, where $\langle U,V \rangle = \langle UV \rangle - \langle U \rangle \langle V \rangle$. Here we are concerned with the study of the amount of squeezing provided by nonlinear cavities, i.e., optical cavities containing a nonlinear medium and pumped by some input field. Nonlinear cavities are known to produce large amounts of squeezed light for a particular frequency or band of frequencies in the field exiting the cavity, as a result of the interference at the cavity output mirror between the partially squeezed intracavity mode and the reservoir modes [1, 2, 3, 4, 5].

When calculating the fluctuations of the output field quadratures, one uses the squeezing spectrum [6, 7] defined as

$$S_{\varphi}^{\text{out}}(\omega) \equiv \frac{1}{4} + \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \langle : \hat{X}_{\varphi}^{\text{out}}(t+\tau), \hat{X}_{\varphi}^{\text{out}}(t) : \rangle,$$

(1)

where the field quadrature

$$\hat{X}_{\varphi}^{\text{out}}(t) \equiv \frac{\hat{a}_{\text{out}}(t)e^{-i\varphi} + \hat{a}_{\text{out}}^\dagger(t)e^{i\varphi}}{2},$$

(2)

being $\hat{a}_{\text{out}}^\dagger$ and $\hat{a}_{\text{out}}$ the creation and annihilation operators of the output field that verify $[\hat{a}_{\text{out}}(t), \hat{a}_{\text{out}}^\dagger(t')] = \delta(t-t')$, and $\varphi$ an arbitrary phase. In Eq. (1) the label “out” refers to the field exiting the nonlinear cavity, $::$ denotes normal and time ordering, and the term $\frac{1}{4}$ corresponds to the shot noise level.

A. Squeezing spectrum from the generalized $P$ distribution

The use of Eq. (1) requires relating the correlations of the field outside the cavity—which are the ones that are actually detected— with those of the intracavity field, which are readily calculated by solving the master equation for the particular system. This relation is given by the input–output theory [5], which, when the nonlinear cavity is fed with a coherent or vacuum field, states that

$$\langle : \hat{a}_{\text{out}}(t+\tau), \hat{a}_{\text{out}}(t) : \rangle = \gamma_{\text{out}} \langle : \hat{a}(t+\tau), \hat{a}(t) : \rangle,$$

(3)

$$\langle : \hat{a}_{\text{out}}^\dagger(t+\tau), \hat{a}_{\text{out}}(t) : \rangle = \gamma_{\text{out}} \langle : \hat{a}_{\text{out}}^\dagger(t+\tau), \hat{a}_{\text{out}}(t) : \rangle,$$

(4)
where $\gamma_{\text{out}}$ represents the cavity loss rate of the field intensity at the output mirror, and $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the intracavity field, which verify $\left[ \hat{a}(t), \hat{a}^\dagger(t) \right] = 1$. We recall that no simple relations like Eq. (3,4) exist that relate correlations between outgoing fields and intracavity fields unless those are calculated in normal order.

The dynamics of the nonlinear cavity is usually described, although not necessarily, through Langevin equations obtained from a Fokker–Planck equation. Then the form of Eqs. (3,4) suggests the use of the generalized $P$ representation [8], which we denote by $\mathcal{P}$, as in this case

$$\langle \hat{a}^\dagger(t + \tau), \hat{a}(t) \rangle = \langle \alpha(t + \tau), \alpha(t) \rangle_{\mathcal{P}}, \quad (5)$$

$$\langle \hat{a}^\dagger(t + \tau), \hat{a}(t) \rangle = \langle \beta(t + \tau), \alpha(t) \rangle_{\mathcal{P}}, \quad (6)$$

where $\alpha$ and $\beta$ are independent $c$-numbers associated to $\hat{a}$ and $\hat{a}^\dagger$ respectively that verify $\langle \beta \rangle_{\mathcal{P}} = \langle \alpha \rangle_{\mathcal{P}}^*$, and $\langle f \rangle_{\mathcal{P}}$ denotes the average value of any function $f(\alpha, \beta)$ calculated in the $\mathcal{P}$ representation. Then, making use of the above equations one can write the squeezing spectrum as

$$S_{\text{out}}^\varphi(\omega) = \frac{1}{4} + \gamma_{\text{out}} \mathcal{V}_{\mathcal{P}}(\omega, \varphi), \quad (7)$$

$$\mathcal{V}_{\mathcal{P}}(\omega, \varphi) \equiv \int d\tau e^{-i\omega\tau} \langle X_{\varphi}(t + \tau), X_{\varphi}(t) \rangle_{\mathcal{P}},$$

with $X_{\varphi}(t) \equiv [\alpha(t) e^{-i\varphi} + \beta(t) e^{i\varphi}] / 2$, which is a well known and widely used result.

B. Squeezing spectrum from other quasiprobability distributions

The above presentation suggests that the use of the $\mathcal{P}$ distribution is mandatory in order to derive the squeezing properties of the output field. (The usual Glauber–Sudarshan $P$ distribution presents well known problems that will be recalled below.) Nevertheless, and this is at the heart of our work, a suitable combination of two $s$-ordered [9] two–time correlations, for example antinormally ($s = -1$) and symmetrically ($s = 0$) ordered as obtained by using the $Q$ (Husimi) and $W$ (Wigner) representations [3, 5, 10], leads to the same result.
Consider the obvious property
\[ 2 \langle \alpha(t + \tau), \alpha(t) \rangle_W = \langle \alpha(t + \tau), \alpha(t) \rangle_P + \langle \alpha(t + \tau), \alpha(t) \rangle_Q , \quad (8) \]
where the subscript indicates the quasiprobability distribution used for obtaining the correlations. Then it follows that \[ \langle \hat{a}(t + \tau), \hat{a}(t) \rangle = 2 \langle \alpha(t + \tau), \alpha(t) \rangle_W - \langle \alpha(t + \tau), \alpha(t) \rangle_Q \]
and then \[ \hat{X}_\phi^{\text{out}}(t + \tau), \hat{X}_\phi^{\text{out}}(t) \rangle = \gamma_{\text{out}} \left[ 2 \langle X_\phi(t + \tau), X_\phi(t) \rangle_W - \langle X_\phi(t + \tau), X_\phi(t) \rangle_Q \right] , \]
so that the squeezing spectrum \( S_{\phi}^{\text{out}}(\omega) \) can be written as
\[ S_{\phi}^{\text{out}}(\omega) = \frac{1}{4} + \gamma_{\text{out}} \left[ 2 \mathcal{V}_W(\omega, \varphi) - \mathcal{V}_Q(\omega, \varphi) \right] , \quad (9) \]
where the notation is self-explanatory, see Eq. \( \text{(7)} \). This can be easily generalized to any pair of \( s \)-ordered two-time correlations. Taking into account that the \( s \)-ordered two-time correlation is nothing but
\[ \langle \alpha(t + \tau), \alpha(t) \rangle_s = \frac{1 + s}{2} \langle \alpha(t + \tau), \alpha(t) \rangle_P + \frac{1 - s}{2} \langle \alpha(t + \tau), \alpha(t) \rangle_Q , \quad (10) \]
with \( s \in [-1, 1] \), it follows that
\[ \langle \alpha(t + \tau), \alpha(t) \rangle_P = \frac{1 - s'}{s - s'} \langle \alpha(t + \tau), \alpha(t) \rangle_s + \frac{1 - s}{s' - s} \langle \alpha(t + \tau), \alpha(t) \rangle_s' , \quad (11) \]
for any \( s \neq s' \). Notice that Eq. \( \text{(8)} \) is retrieved from Eq. \( \text{(11)} \) when \( s = 0 \) (symmetric ordering, which is obtained with the \( W \) distribution) and \( s' = -1 \) (antinormal ordering, which is obtained with the \( Q \) distribution). Now, following the same arguments that lead to Eq. \( \text{(9)} \) one gets
\[ S_{\phi}^{\text{out}}(\omega) = \frac{1}{4} + \gamma_{\text{out}} \left[ \frac{1 - s'}{s - s'} \mathcal{V}_s(\omega, \varphi) + \frac{1 - s}{s' - s} \mathcal{V}_{s'}(\omega, \varphi) \right] . \quad (12) \]
We see that the use of the \( P \) distribution is equivalent to the combined use of a pair of \( s \)-ordered distributions.

The interest of this approach is that Eqs. \( \text{(7)} \) and \( \text{(12)} \) provide a way for comparing the predictions of a pair of \( s \)-ordered distributions, which we denote by \( W_s \) (\( W_1 \equiv P, W_0 \equiv W, W_{-1} \equiv Q \)), with that of the \( P \) distribution. This is interesting because the equation of evolution for a particular \( W_s \) needs not be of the Fokker–Planck type. For example, in the case we treat along this article (dispersive optical bistability \( [11, 12, 13] \)), the equation of \( W_s \) includes additional terms (namely, third order derivatives) but for \( s = \pm 1 \), and,
in general, the diffusion matrix is not positive semidefinite [12, 13], but for \( s = 0 \). But these limitations do not necessarily prevent the use of these distributions as under some reasonable approximations their equations of evolution can be approximated to a Fokker–Planck equation (by neglecting the higher order derivatives in the Wigner case [12, 13] or by limiting the study to a parameter domain where the diffusion matrix is well behaved in the Husimi case [14]). The point is that after making these approximations, Eq. (12) should provide not an exact but an approximate result. Then, by comparing the predictions of Eq. (12) to that of Eq. (7) one could evaluate the influence of these approximations.

In this article we shall make use of these approximations for the special case of dispersive optical bistability [11, 12, 13]. We then derive the fluctuation spectra from the linearized Langevin equations coming from \( s \)-ordered quasiprobability distributions, \( W_s \), and compute the (linearized) squeezing spectrum. The main result we obtain is that, although any \( W_s \) obeys an approximate Fokker–Planck equation, and thus approximated Langevin equations can be obtained, the linearized squeezing spectrum given by Eq.(12) is identical to that given by Eq. (7). In other words, the approximations made in deriving Langevin equations from the approximated equations of evolution do not manifest in the linearized fluctuations spectra. We show further that the predictions for the (linearized) intracavity squeezing from any \( W_s \) is also exact.

II. MODEL FOR DISPERSE OPTICAL BISTABILITY

A. Master equation

We shall adopt the model for dispersive optical bistability studied by Drummond and Walls [11], consisting of a single–ended optical cavity containing a purely dispersive and isotropic \( \chi^{(3)} \) medium and pumped by a coherent field of frequency \( \omega \) close to that of a cavity mode, \( \omega_c \). The system Hamiltonian in the interaction picture reads

\[
H = \hbar \left[ (\theta - g) \hat{a}^{\dagger} \hat{a} + i E_0 (\hat{a}^{\dagger} - \hat{a}) - \frac{g}{2} \hat{a}^{\dagger 2} \hat{a}^2 \right],
\]

where \( E_0 \) is proportional to the amplitude of the injected field, \( \theta = \omega_c - \omega \) is a detuning, and \( g \equiv 3\varepsilon_0 \hbar \omega_c^2 \chi / (\varepsilon^2 V) \) is the coupling constant, with \( V \) the quantization volume, \( \varepsilon \) the medium dielectric constant and \( \chi = \chi^{(3)}_{ii} \) \((i = 1, 2, 3)\) the nonlinear susceptibility [15]. We note that we used a symmetrized Hamiltonian and this is the reason why the detuning is not \( \theta \) but
\((\theta - g)\): Had it been calculated in, say, normal or antinormal order, the detuning would have been \(\theta\) and \((\theta - 2g)\), respectively. Then, the correction \(g\) to the detuning is nothing but the modification of the cavity frequency \(\omega_c\) due to vacuum fluctuations as described with the different ordering choices.

The intracavity field mode exits the cavity through the output mirror. Assuming weak coupling between the field mode and the rest of vacuum modes, which are treated as a reservoir (see, e.g., [10]), the master equation of the system at zero temperature reads [11]

\[
\dot{\rho} = E_0 \left( a^\dagger \rho - \rho a^\dagger + \rho a - a \rho \right) - i \left( \theta - g \right) \left( a^\dagger a \rho - \rho a^\dagger a \right) + i \frac{g}{2} \left( a \rho a^\dagger - \rho a^\dagger a - a^\dagger a \rho \right),
\]

where \(\gamma\) represents the cavity losses of the field intensity and, as the cavity is single-ended, \(\gamma_{out} = \gamma\).

**B. Quasiprobability distributions**

The equation of evolution for the \(s\)-ordered quasiprobability distribution for dispersive optical bistability was first derived by Vogel and Risken [13]. With our notation

\[
\frac{\partial}{\partial t} W_s(\alpha, \alpha^*) = \left[ - \sum_i \frac{\partial}{\partial \alpha_i} A^{(s)}_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} D^{(s)}_{ij} \right] W_s + ig \frac{1 - s^2}{4} \left( \frac{\partial^3}{\partial \alpha^2 \partial \alpha^*} \alpha - \frac{\partial^3}{\partial \alpha^3 \partial \alpha^2} \alpha^* \right) W_s,
\]

where \(\alpha_1 \equiv \alpha, \alpha_2 \equiv \alpha^*\), and

\[
A^{(s)}_1 \equiv E_0 - \left[ \frac{\gamma}{2} + i \left( \theta - sg \right) \right] \alpha + ig \alpha^2 \alpha^*, \quad \text{(16)}
\]

\[
A^{(s)}_2 \equiv E_0 - \left[ \frac{\gamma}{2} - i \left( \theta - sg \right) \right] \alpha^* - ig \alpha (\alpha^*)^2, \quad \text{(17)}
\]

\[
D^{(s)} \equiv \begin{pmatrix}
  isg \alpha^2 & \frac{1 - s}{2} \gamma \\
  \frac{1 - s}{2} \gamma & -isg (\alpha^*)^2
\end{pmatrix}.
\]

\([D^{(s)}]_{ij} \equiv D^{(s)}_{ij}\). Making use of Eq. (18) we obtain

\[
D^{(1)} = \frac{1 - s'}{s - s'} D^{(s)} + \frac{1 - s}{s' - s} D^{(s')}, \quad \text{(19)}
\]

We shall make use of this property later.
For $s = 1, 0, -1$, $W_s$ corresponds to the $P$ (Glauber–Sudarshan), $W$ (Wigner), and $Q$ (Husimi) distributions respectively as commented. An alternative quasiprobability distribution to $W_s$ is the so–called generalized $P$ distribution \[\mathcal{P}\], which we have already denoted by $\mathcal{P}$. Its equation of evolution has been derived by Drummond and Walls \[11\]. It is given by Eq. (15) with $s = 1$ after changing $\alpha^*$ by the complex variable $\beta$, which is independent of $\alpha$ and verifies $\langle \alpha \rangle = \langle \beta \rangle^*$ \[8\]. Hence in this representation the phase space is doubled with respect to the $W_s$ representation.

C. Fokker–Planck equation for the $W_s$ distribution

By construction all three $\mathcal{P}$, $P$, and $Q$ quasiprobability distributions formally obey a Fokker–Planck equation \[3, 10\] as they do not contain derivatives of order higher than 2. This is not the case for any $W_s$ with $s \neq \pm 1$. We show next that, in spite of this fact, any $W_s$ distribution verifies an approximate Fokker–Planck equation. This statement is equivalent to saying that, in some limit, the third order derivatives in Eq.(15), existing unless $s = \pm 1$, can be neglected. For that we make a system size expansion \[10, 13\], which is based on the very large value attained by the mean number of intracavity photons $\langle a^\dagger a \rangle \sim |\alpha|^2 \sim \gamma/|g|$ \[12, 13\]. For example, by taking $V = 1\text{cm}^3$, $|\chi| = 5 \cdot 10^{-23}\text{m}^2\text{Volt}^{-2}$, $\varepsilon = 4\varepsilon_0$, $\omega_c = 3 \cdot 10^{15}\text{s}^{-1}$, one obtains $|g| \sim 10^{-9}\text{s}^{-1}$ and, taking $\gamma = 10^9\text{s}^{-1}$, one has $\gamma/|g| \sim 10^{18}$. Now, normalizing time to $\gamma$ and $\alpha$ to $\sqrt{\gamma/|g|}$, one obtains an equation equivalent to Eq. (15) in which the third order derivatives are multiplied by $(g/\gamma)^2$, whilst the second order derivatives are multiplied by $g/\gamma$ and the first order derivatives are of order one. Then, the neglect of the third order derivatives looks like a very accurate approximation. Notice, however, that the predictions of such a truncated equation may differ significantly from their correct values, as it occurs with the tunneling times \[12, 13\]. Finally note that, given the smallest value of $g$, all $A_i(s)$ [Eqs. (16,17)] can be approximated by $A_i(s=0)$ \[12, 13\], which we denote just by $A_i$. Once the system size expansion has been performed Eq. (15) becomes

$$\frac{\partial}{\partial t} W_s \simeq \left[ -\sum_i \frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} D_{ij}^{(s)} \right] W_s, \quad (20)$$

$$A_1 \equiv E_0 - \left(\frac{\gamma}{2} + i\theta\right) \alpha + i g \alpha^2 \alpha^*, \quad (21)$$

$$A_2 \equiv E_0 - \left(\frac{\gamma}{2} - i\theta\right) \alpha^* - i g (\alpha^*)^2, \quad (22)$$
\( \alpha_1 \equiv \alpha, \alpha_2 \equiv \alpha^*, \) and \( \mathbb{D}^{(s)} \) is given by Eq. (18). The symbol \( \simeq \) is used instead of the equality symbol in order to stress that Eq. (20) is approximate (but for \( s = \pm 1 \)).

As announced Eq. (20) is a Fokker–Planck equation. However it is in fact a pseudo Fokker–Plank equation \([12, 13]\) as the diffusion matrix \( \mathbb{D}^{(s)} \) is not positive semidefinite in general and then the equation cannot be interpreted as describing a generalized Brownian motion. That \( \mathbb{D}^{(s)} \) is not positive semidefinite is easy to see by writing Eq. (20) in terms of the real variables \( x = \text{Re} \alpha, y = \text{Im} \alpha. \) For these new variables an equation similar to Eq. (20) is obtained with a diffusion matrix \( \mathbb{D}_{xy}^{(s)} \) given by \([12]\)

\[
\mathbb{D}_{xy}^{(s)} \equiv \begin{pmatrix}
\frac{1-s}{4} \gamma - sgxy & \frac{sg}{2} \left( x^2 - y^2 \right) \\
\frac{sg}{2} \left( x^2 - y^2 \right) & \frac{1-s}{4} \gamma + sgxy
\end{pmatrix},
\]

whose eigenvalues \( d^{(s)}_{\pm} \) read

\[
d_{\pm}^{(s)} = \frac{1-s}{4} \gamma \pm \frac{|sg|}{2} \left( x^2 + y^2 \right).
\]

The positive semidefiniteness of \( \mathbb{D}^{(s)} \) then requires that \( d_{-}^{(s)} \geq 0: \)

\[
|\alpha|^2 \leq \frac{\gamma}{|g|} \frac{1-s}{2|s|}.
\] (Remind that \( |\alpha|^2 = x^2 + y^2. \)) Clearly, only for \( s = 0 \) (Wigner distribution) condition (23) is fulfilled for any \( \alpha. \) On the other hand for \( s = +1 \) (\( P \) distribution) condition (23) is never satisfied. In general, for any \( s \neq 0,1 \) that condition is verified inside a bounded region of the phase space and thus, \( \mathbb{D}^{(s)} \) is never, strictly speaking, positive semidefinite. Nevertheless if \( \alpha \) is replaced by its classical steady value \( \bar{\alpha} \), what is done for calculating linearized spectra as we do here (see below), \( \mathbb{D}^{(s)} \) will be positive semidefinite whenever condition (23) holds when applied to the classical steady state. This restricted condition is in fact verified in a bounded region of the parameter space. We note that this approximation was done for the case of second–harmonic generation by Savage \([14]\). It can be understood in the sense that one assumes that \( W_s \) is peaked around the steady state value and that the parameters of the system are such that a negligible part of the distribution violates the condition \( d_{-}^{(s)} \geq 0. \) Under this approximation \( \mathbb{D}^{(s)} \) is a well behaved diffusion matrix and the equation of evolution of \( W_s \) is a true Fokker–Planck equation for any \( s (\neq +1). \)

The requirement of positive semidefiniteness of the diffusion matrix \( \mathbb{D}^{(s)} \) comes from the fact that \( \alpha \) and \( \alpha^* \) are complex-conjugate variables, and the noise terms in the Langevin
equations equivalent to the Fokker–Planck equation (see below) will not be complex-conjugated if this requirement is not fulfilled. Notice that this requirement is lifted in the case of the $P$ distribution as $\alpha$ and $\beta$ are not complex-conjugate variables, but in their mean.

III. LINEARIZED FLUCTUATIONS SPECTRA AND SQUEEZING

The Ito stochastic differential equations that are equivalent to the Fokker–Planck equation \[ \frac{d\alpha}{dt} \simeq A(\alpha) + B(s)(\alpha) \cdot \xi(t) , \] (24) where $\alpha \equiv (\alpha, \alpha^*)^T$, $A \equiv (A_1, A_2)^T$, $D(s)(\alpha) \equiv B(s)(\alpha)[B(s)(\alpha)]^T$, and the white Gaussian noise term $\xi \equiv (\xi_1, \xi_2)^T$ verifies $\langle \xi_i(t) \rangle = 0$, and $\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t-t')$. (We note that $\alpha$ should contain the label $s$, e.g. $\alpha(s)$, as that stochastic variable is representation dependent. We avoid this labeling in order to not overburden the notation.)

In this article we shall limit ourselves to the study of fluctuations around the classical steady state $\alpha = \bar{\alpha}$ (the solution to $A(\bar{\alpha}) = 0$) in the linear approximation. The linearized Langevin equations read

\[ \frac{d}{dt}\delta\alpha \simeq \bar{A} \cdot \delta\alpha + \bar{B}(s) \cdot \xi(t) , \] (25)

\[ [\bar{A}]_{ij} \equiv \left( \frac{\partial A_i}{\partial \alpha_j} \right)_{\alpha=\bar{\alpha}} , \quad \bar{B}(s) \equiv B(s)(\alpha = \bar{\alpha}) , \] (26)

$\alpha_1 \equiv \alpha$ and $\alpha_2 \equiv \alpha^*$.

We are concerned with the calculation of the spectral matrix of fluctuations, $S^{(s)}(\omega)$, of elements

\[ S^{(s)}_{ij}(\omega) \equiv \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \langle \alpha_i(t+\tau), \alpha_j(t) \rangle_s , \] (27)

(not to be confused with the squeezing spectrum) which, from the linearized Langevin equations \[ 25 \], can be easily obtained by making use of \[ 16 \]

\[ S^{(s)}(\omega) \simeq S^{(s)}_{\text{approx}}(\omega) , \] (28)

\[ S^{(s)}_{\text{approx}}(\omega) \equiv (\bar{A} + i\omega \mathbb{I})^{-1} \bar{D}(s)(\bar{A}^T - i\omega \mathbb{I})^{-1} , \] (29)

where $\mathbb{I}$ denotes the $2 \times 2$ identity matrix and $\bar{D}(s) = D(s)(\alpha = \bar{\alpha})$ and again the symbol $\simeq$ stresses that the result is an approximation. The result for the different $s$–orderings is given in Appendix A.
We remind that the Fokker–Planck equation obeyed by $P$ is given by Eq. (20) for $s = 1$ with the replacement $\alpha_2 = \alpha^* \to \beta$. Then the full and linearized Langevin equations in the $P$ representation are given by Eqs. (24) and (25) respectively, under the previous replacement, and the spectral matrix corresponding to the $P$ distribution, $S^P$, is given by Eq. (29) with $s = 1$ (note that $\bar{\beta} = \bar{\alpha}^*$):

$$S^P(\omega) = (\bar{A} + i\omega I)^{-1} \bar{D}^{(1)} (\bar{A}^T - i\omega I)^{-1}. \quad (30)$$

We also remind that Eq. (20) for $s = 1$ is exact as the original Eq. (15) does not contain third order derivatives in this case; hence all symbols $\simeq$ must be replaced by $=$ in this case, as in Eq. (30). On the other hand the normally ordered spectral matrix of fluctuations defined as

$$:S(\omega): \equiv \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \left( \langle \hat{a} (t + \tau), \hat{a} (t) \rangle, \langle \hat{a}^\dagger (t + \tau), \hat{a}^\dagger (t) \rangle \right);$$

equals, by definition, $S^P(\omega)$. Thus Eq. (30) yields the exact normally ordered spectral matrix of fluctuations,

$$:S(\omega): = S^P(\omega). \quad (31)$$

We now recall property (11) that, together with definition (27), allows to state that

$$S^P(\omega) = \frac{1 - s'}{s - s'} S^{(s)}(\omega) + \frac{1 - s}{s' - s} S^{(s')}_{\text{approx}}(\omega), \quad (32)$$

which, making use of Eq. (28), can be approximated as

$$S^P(\omega) \simeq \frac{1 - s'}{s - s'} S^{(s)}_{\text{approx}}(\omega) + \frac{1 - s}{s' - s} S^{(s')}_{\text{approx}}(\omega). \quad (33)$$

Now, substituting Eq. (29) into (33), and recalling property (19) and Eq. (30), we observe that the approximate equality (33) is a true equality indeed.

This means that, although the used Langevin equations come from approximated (truncated in general) Fokker–Planck equations, the spectral matrices $S^{(s)}_{\text{approx}}(\omega)$ given by Eq. (29) provide the correct result. In other words, the third order derivatives present in the original pseudo Fokker–Planck equation (15) seem to play no role on the correlations between fluctuations in a linearized theory. Moreover, the approximated Langevin equations (24) ignore that the diffusion matrices $D^{(s)}$ can be non positive semidefinite, as discussed above. Nevertheless, as relation (33) is not approximate but exact, and it holds for any.
parameter set, even where $D^{(s)}$ is not positive semidefinite, we conclude that the positive semidefinite condition on $D^{(s)}$ is irrelevant in the following sense. We recall that the same occurs with the Glauber–Sudarshan $P$ distribution: even if the pseudo Fokker–Planck equation governing its evolution has a non positive semidefinite diffusion matrix $D^{(1)}$, one can nevertheless write down a corresponding Langevin equation, which yields the correct result for the spectral matrix $S$. The explanation for this was given by Drummond by introducing the generalized $P$ representation, which operationally amounts to substitute the complex conjugated variable $\alpha^*$ in the pseudo Fokker–Planck equation verified by the Glauber–Sudarshan $P$ by an independent complex variable $\beta$. This suggests that our result can be understood in terms of ”generalized $W_s$ distributions”, call them $W_s$: Should we substitute $\alpha^*$ by an independent complex variable $\beta$ in the original pseudo Fokker–Planck equation $D^{(1)}$ a positive semidefinite diffusion matrix would be not needed in order to derive corresponding Langevin equations (once the third order derivatives had been neglected). These Langevin equations for $W_s$ would read as those for $W_s$ but with $\alpha^* \rightarrow \beta$, and the final expression for the spectral matrix, which would be exact in this generalized representation, would be given by Eq. in the linear approximation, just as it happens in our case. The possibility of defining ”generalized $W_s$ distributions” should be studied, probably by defining $W_s$ in terms of $P$, as $W_s$ is defined in terms of $P$. We leave this discussion open as it is out of the scope of the present work.

The above discussion implies that the squeezing spectra of the output field, given by Eqs. and with

$$V_P (\omega, \varphi) = \frac{1}{4} \left[ S_{11}^P (\omega) e^{-2i\varphi} + S_{22}^P (\omega) e^{+2i\varphi} + S_{12}^P (\omega) + S_{21}^P (\omega) \right],$$

$$V_s (\omega, \varphi) = \frac{1}{4} \left[ S_{11}^{(s)} (\omega) e^{-2i\varphi} + S_{22}^{(s)} (\omega) e^{+2i\varphi} + S_{12}^{(s)} (\omega) + S_{21}^{(s)} (\omega) \right],$$

are, obviously, the same. We shall not analyze here the properties of this squeezing spectrum as this analysis can be found in [3]. We just quote in Appendix B the expression for $V_s (\omega, \varphi)$. Nevertheless we want to make a comment on the amount of squeezing attainable inside the nonlinear cavity. This can be calculated by integrating the spectrum of fluctuations of the field quadratures, i.e.

$$V_s (\varphi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega V_s (\omega, \varphi).$$

We must take into account how the different $V_s$ are related with the squeezing $V (\varphi) \equiv \langle \hat{X}_\varphi (t) , \hat{X}_\varphi (t) \rangle$ where $\hat{X}_\varphi$ is defined as $\hat{X}_\varphi^{\text{out}}$, Eq. , but replacing $(\hat{a}_{\text{out}}, \hat{a}_{\text{out}}^\dagger)$ with
Performing the calculation one easily obtains, with the help of the commutator \[ [\hat{a}(t), \hat{a}^\dagger(t)] = 1 \] the result
\[ V(\varphi) = V_s(\varphi) + \frac{s}{4}. \] (37)

The point is that, as any \( V_s(\omega, \varphi), \) Eq. (35), can be computed from \( S_{\text{approx}}^{(s)}(\omega), \) Eq. (28), and \( S_{\text{approx}}^{(s)}(\omega) \) yields the correct result, any single \( W_s \) is useful for computing the intracavity squeezing. We note that \( V(\varphi) \) is minimum for a particular \( \varphi \) and is \( V_{\text{min}} = \frac{s}{4} \). This is a well known result: The maximum degree of squeezing attainable inside a nonlinear cavity, which happens at the bifurcation points, is half that of a coherent state.

IV. CONCLUSIONS

In this article we have discussed how the spectrum of squeezing of the field outgoing a nonlinear cavity can be derived from a combination of the spectra of intracavity fluctuations obtained from Langevin equations derived from \( W_s \) distributions. We have illustrated this for the special case of dispersive optical bistability. The interesting result is that the linearized spectrum of squeezing obtained this way is exact in spite of the fact that no \( W_s \) quasiprobability distribution verifies Fokker–Planck equations but only approximate ones. We have also shown that the predictions for the (linearized) squeezing attainable inside the nonlinear cavity is correct when calculated with any \( W_s \). The conclusion is that the linearized Langevin equations corresponding to a \( W_s \) representation are correct or, in other words, that the approximations made for converting the equation of evolution of \( W_s \) into Fokker–Planck equations do not manifest in the linearized theory, even if the diffusion matrix of the Fokker–Planck equation is not positive semidefinite. The latter has allowed us to conjecture the definition of ”generalized \( W_s \) distributions”, following the spirit of the generalized \( P \) distributions of Drummond. Of course, when going to the nonlinear regime, as in the calculation of e.g. tunneling times, one must be cautious about the truncation of pseudo Fokker–Planck equations containing third (or higher order) derivatives.

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VI. APPENDIX A

In this Appendix the expression for the spectral matrix $S_{\text{approx}}^{(s)}(\omega)$ defined in Eq. (29) is given for any $s$-ordered quasidistribution $W_s$. The calculation needs the computation of the classical steady state ($\alpha = \bar{\alpha}, \alpha^* = \bar{\alpha}^*$), which is given by $A_1 = A_2 = 0$, Eqs. (21) and (22). After introducing the quantities

$$
\Delta = \frac{2\eta\theta}{\gamma}, \quad \mu = \left(\frac{2}{\gamma}\right)^3 |g|E_0^2, \quad \sqrt{I}e^{i\phi} = \left(\frac{2|g|}{\gamma}\right)^{\frac{1}{2}} \bar{\alpha}, \quad \eta = \text{sign} (g),
$$

the classical steady state is given by

$$
\mu = I \left[1 + (I - \Delta)^2\right], \quad \text{(39)}
$$

$$
e^{i\phi} = \frac{1 + i\eta(I - \Delta)}{\sqrt{1 + (I - \Delta)^2}}, \quad \text{(40)}
$$

The characteristic $I$ vs. $\mu$ displays bistable behaviour for $\Delta > \sqrt{3}$ as is well known, and the values of the intensity $I$ at the turning points of the characteristic, $I = I_{\pm}$, are given by

$$
I_{\pm} \equiv \frac{2\Delta \pm \sqrt{\Delta^2 - 3}}{3}.
$$

(41)

For $I_- < I < I_+$ the steady state is unstable; otherwise it is linearly stable. As the state equation (39) implies that the pump power $\mu$ is univocally determined by $I$ one can use the latter as the control parameter, and this is more convenient mathematically.

Making use of Eq. (29) one readily obtains ($\Omega \equiv 2\omega/\gamma$):

$$
S_{11}^{(s)}(\omega) = \frac{2}{\gamma} I e^{2i\phi} 2(\Delta - 2I) + i\eta [2 + s(\Omega^2 - I)] \frac{2|g|E_0^2}{(\Omega^2 - I)^2 + 4\Omega^2},
$$

(42)

$$
S_{12}^{(s)}(\omega) = \frac{2}{\gamma} 2I^2 + (1 - s) \frac{[\Omega^2 + I - 2\eta\Omega(2I - \Delta)]}{(\Omega^2 - I)^2 + 4\Omega^2},
$$

(43)

$$
S_{22}^{(s)}(\omega) = \left[S_{11}^{(s)}(\omega)\right]^*, \quad S_{21}^{(s)}(\omega) = S_{12}^{(s)}(-\omega),
$$

(44)

where

$$
\mathcal{I} = 3(I - I_+)(I - I_-).
$$

(45)

Note that at the turning points of the characteristic ($I = I_+$ or $I = I_-$) $\mathcal{I} = 0$ and all $S_{ij}^{(s)}(\omega)$ diverge at $\omega = 0$. 

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VII. APPENDIX B

The expression for $V_s(\omega, \varphi)$, Eq. (35), making use of Eqs. (42)–(45) in Appendix A, reads

$$\gamma V_s(\omega, \varphi) = \frac{2I (\Delta - 2I) \cos \psi - \eta I [2 + s (\Omega^2 - I)] \sin \psi + 2I^2 + (1 - s) (\Omega^2 + I)}{\left(\Omega^2 - I\right)^2 + 4\Omega^2} \quad (46)$$

where $\psi \equiv 2 (\phi - \varphi)$.

As discussed $V_s(\omega, \varphi)$ for $s = +1$ coincides with the corresponding expression calculated in the $P$ representation:

$$\gamma V_P(\omega, \varphi) = \frac{2I (\Delta - 2I) \cos \psi - \eta I [2 + (\Omega^2 - I)] \sin \psi + 2I^2}{\left(\Omega^2 - I\right)^2 + 4\Omega^2} \quad (47)$$

Note that this quantity is just $S_{\varphi}^{\text{out}}(\omega) - \frac{1}{4}$, Eq. (17) (remind that $\gamma_{\text{out}} = \gamma$).

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