I. INTRODUCTION

The Pokrovsky-Talapov (PT) model describes a large variety of systems displaying a commensurate/incommensurate (C/IC) transition, ranging from vortex depinning in type-II superconductors to adsorbate layers on crystal surfaces, bromide molecules intercalated onto graphite superconductors with modulated thickness, and quantum-Hall bilayer systems under a tilted magnetic field. This model is closely related to the sine-Gordon model, with the extra feature of a characteristic driving wave vector $Q$ imposed through the cosine term. It has been studied previously using the Bethe ansatz and the replica trick in the presence of quenched disorder. In this work we present a functional renormalization-group calculation of the finite-temperature corrections to the mean-field results.

The Hamiltonian for the PT model is

$$H_{PT} = \int d^2r \left[ \frac{1}{2} \rho_t (\nabla \phi)^2 - t \cos(\phi - Qx) \right], \quad (1)$$

where $r = (x,y)$ and $d^2r = dx dy$. At mean-field level, one approximates the thermodynamic free energy by the Hamiltonian itself, evaluated at the field configuration that minimizes the energy for given boundary conditions. This neglects the contributions of all fluctuations away from the minimum, therefore becoming less accurate as the temperature is raised and the entropic contribution of fluctuations increases. In this approximation, it is straightforward to compare the free energy of a configuration with the field following the driving wave vector (commensurate phase) to that of one in which the field no longer follows the imposed $Q$. As $Q$ increases, the presence of the stiffness term, suppressing deviations of the field from uniformity, makes it more and more costly to remain in the commensurate phase, until a critical $Q_c$ is reached at which the incommensurate phase becomes energetically favorable. The aim of this paper is to investigate the effects of thermal fluctuations on this critical $Q_c$.

We shall use a functional renormalization-group (RG) scheme to study the model at finite temperatures. Our approach is as follows: we perform a simple transformation which maps the PT model to a sine-Gordon model with additional terms depending only on the total topological “charge” of the system and on the driving wave vector $Q$. The RG transformation does not couple the sine-Gordon part to the $Q$-dependent part. Taking advantage of this, we renormalize the sine-Gordon part of the Hamiltonian and obtain a long-wavelength effective action, which we subsequently use to obtain the new value of the critical $Q_c$.

The main technical complication one faces in constructing a systematic renormalization-group transformation for the sine-Gordon model is the inability to expand the cosine term in Eq. (1) in powers of the field and keep only a finite number of these. There are three main reasons for this complication: first, the periodicity of the cosine is crucial and is destroyed by any finite-order Taylor expansion. Second, we are interested in the two-dimensional case; as follows from simple power counting, polynomial interactions involving any power of the field are relevant in two dimensions. Finally, we are not interested in the fixed point of the RG transformation but in the actual values of the various parameters after integrating out the degrees of freedom that we are not interested in.

Not expanding the cosine means that, in diagrammatic language, we ought to keep track of an infinite number of diagrams to one loop. This problem has already been solved for the case of the Wilsonian RG by Wegner and Houghton in Ref. 13, where these authors derive the eponymous exact renormalization-group equation. This equation is also the limit of an approximate recursion relation first given by Wilson.14 Wilson’s approximate recursion relation has been applied to the problem of critical wetting in Refs. 15 and 16 because the effective-field theories used in studying critical wetting share with our problem its dimensionality, preventing the use of the more usual perturbation methods. In general, functional renormalization-group approaches are useful for cases where there is an effective potential with a non-trivial functional dependence on the field, such as the cosine term in Eq. (1).

The exact functional renormalization equation of Wegner and Houghton relies on a sharp, moving cutoff in momentum space. A sharp cutoff induces long-range interactions in real space and complicates the calculation of the flow of the stiffness $\rho_t$ in Eq. (1). This may be overcome by employing a smooth cutoff function, however, the resulting trajectories depend on the precise form of the cutoff. It has been shown19 that the Wilsonian RG approach (of which the
Polchinski RG is an example) suffers from strong scheme dependence even in the asymptotic regime. Various alternative formulations of RG transformations exist that do not suffer from this problem; one that has recently been applied to the sine-Gordon model is the functional renormalization of the effective average action \(^{18,23}\) in which a transformation is obtained, not for the Hamiltonian itself, but for the generating function for the one-particle irreducible (1PI) Green’s functions. This is the RG scheme that we use in this paper.

The outline of this paper is as follows: in Sec. II we describe the model and give a qualitative description of its behavior. In Sec. III we explain the basic idea behind our approach before proceeding directly to the derivation (Sec. IV) and application to the PT model (Sec. V) of the appropriate RG flow equations. Finally, we give a brief discussion of our results in Sec. VI.

II. MODEL

To qualitatively understand the features exhibited by a system described by the PT Hamiltonian given in Eq. (1), consider first the case \(Q=0\). It is then clear that, at mean-field level, \(\phi\) will simply remain at one of the minima of the potential \(V(\phi)=-t\cos(\phi)\) at \(\phi_n=2\pi n\) with \(n=0, \pm 1, \pm 2, \ldots\).

On the other hand, consider the quantity \(\langle \phi(\mathbf{r})\phi(\mathbf{r}')\rangle_c\), \(=\langle \phi(\mathbf{r})\phi(\mathbf{r}')\rangle-\langle \phi(\mathbf{r})\rangle\langle \phi(\mathbf{r}')\rangle\); for low enough temperatures, it is given by the one-loop result

\[
\langle \phi(\mathbf{r})\phi(\mathbf{r}')\rangle_c = \frac{k_BT}{\rho_t} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}\right)
\]

in which \(\xi=\sqrt{\rho_t/t}, K_0(x)\) is a modified Bessel function, and \(\langle \cdots \rangle_c\) indicates a cumulant. Since \(\xi \to \infty\) if \(t \to 0\) and \(K_0(x) = \ln(2/x) + \text{const} + O(x^2 \ln(x))\), the right-hand side of Eq. (2) diverges as \(t \to 0\), i.e., there is no long-range order in the system. This is merely an example of the Mermin-Wagner theorem.\(^{10,22}\) In addition, the \(Q=0\) system exhibits a Kosterlitz-Thouless (KT)-type transition\(^{23}\) on some line \(t(\rho_t)\), which is again completely missed by a mean-field analysis.

This transition is analogous to the roughening transition in interface problems.\(^{18,23}\) In this analogy, the phase \(\phi\) corresponds to the height of an interface above a reference level.\(^{24}\) The phase in which \(\langle \phi(\mathbf{r})\phi(\mathbf{r}')\rangle_c\) remains finite as \(|\mathbf{r}-\mathbf{r}'| \to \infty\) is called the “smooth” phase while the one in which \(\langle \phi(\mathbf{r})\phi(\mathbf{r}')\rangle_c\) diverges is called the “rough” phase. Given the value of \(\phi(0)=\phi_0\) at some arbitrary point which we take to be the origin, the value of \(\phi(\mathbf{r})\) at some other point \(\mathbf{r}'\) arbitrarily far from it either remains within a finite distance from \(\phi_0\) (smooth phase) or it does not but rather crosses over the maxima of \(V(\phi)\). Clearly, the system with \(t=0\) is in the rough phase;\(^{25}\) the usual RG analysis of the sine-Gordon model, Eq. (1) with \(Q=0\) shows that, for given temperature \(T\) and \(\rho_T/T<1/8\pi\), there exists a \(t_*\) below which the system is rough and above which it is smooth. For \(\rho_t/T>1/8\pi\) it is always smooth.

Consider now the case of finite \(Q\). In the roughening picture, this corresponds to a potential \(V\) that depends on the position \(x\); as \(x\) increases, the minima of the potential move to larger values of \(\phi_n=2\pi n+Qx\). In other words, the potential is effectively “tilted.” Thus, the potential part of the Hamiltonian tends to favor a \(\phi\) that increases with position and follows the potential, \(\phi=\phi_n\) (commensurate phase) while the gradient part favors a spatially constant \(\phi\) (incommensurate phase). The competition between \(\rho_T\) and \(t\) leads to a transition between the two states as, for example, \(t\) is varied.

Notice that there are two separate effects here: one is the roughening transition (belonging to the Kosterlitz-Thouless universality class), which is already present when \(Q=0\) and the other is the commensurate-incommensurate transition, which appears only for finite \(Q\). These two effects may be conveniently separated out as described in the next section.

III. DETERMINATION OF \(Q_c\) AT MEAN-FIELD LEVEL

A. Separation of the Hamiltonian

We begin by shifting to the new variable \(\theta=\phi-Qx\), whereupon the Hamiltonian becomes

\[
H_{\text{PT}}[\theta]=H_{\text{G}}+H_Q,
\]

with

\[
H_{\text{G}}[\theta]=\int d^2\mathbf{r} \left[ \frac{1}{2} \rho_t (\nabla \theta)^2 - t \cos(\theta) \right],
\]

\[
H_Q(n^\theta) = \frac{1}{2} \rho_t Q^2 + 2 \pi \rho_t Q n^\theta_t
\]

and

\[
n^\theta_t = \frac{1}{2\pi} \int d\theta \partial_\theta \sum_i n^\theta_i,
\]

that is, the sum of the “charges” of all solitons present (per unit length); the integer \(i\) simply indexes the soliton.

This form of the Hamiltonian is advantageous in that it consists of a sine-Gordon part, \(H_{\text{G}}\), which is independent of \(Q\), plus the two terms in \(H_Q\), which do depend on \(Q\). This last term is the essential difference from a simple sine-Gordon model. As we shall see in Sec. IV, \(H_Q\) is unaffected by an RG transformation. This will form the basis of our treatment of finite-temperature effects.

B. Determination of critical \(Q_c\)

We shall next compute the critical \(Q_c\) in the mean-field approximation; this will be straightforwardly extended to the renormalized case in Sec. V.

To obtain the critical \(Q_c\), we notice that the transition from the commensurate to the incommensurate phase is signaled by the appearance of a finite soliton density with charge \(-|n^\theta_t|\). We therefore calculate the excess energy per unit area of a configuration with a single soliton (with charge \(n^\theta_t=-1\)) over that of one with no solitons, \(\theta=0\); we denote this by \(\Delta E\). The part of the energy cost of a solitonic configuration due to \(H_{\text{G}}\) is calculated in a standard way to be

\[
E_{\text{sol}}=8\sqrt{\rho_t},
\]

hence \(\Delta E=E_{\text{sol}}-2\pi \rho_t Q\). This quantity vanishes

\[245418-2\]
at the transition point, yielding a mean-field critical $Q_c$ of
\[ Q_c = \frac{4}{\pi} \sqrt{\frac{T}{\rho_s}}. \]  \hspace{1cm} (4)

Note that $Q_c$ diverges as $\rho_s \to 0$, implying that the system remains in the commensurate state for all momenta; this is in agreement with the discussion at the end of Sec. II, according to which the creation of solitons (hence the transition to the incommensurate state) is caused by the stiffness overcoming the tendency of the phase field $\phi$ to follow the minimum of the tilted potential.

**IV. EXACT RENORMALIZATION-GROUP EQUATIONS**

Various schemes have been developed to study renormalization-group transformations of two-dimensional field theories: in the theory of critical wetting, there have been several studies using this formulation of the RG, initially to first order in the potential\(^{26}\) and then to all orders.\(^{15}\)

All these approaches have essentially used a local-potential approximation (LPA), in which the potential is allowed to change under coarse graining, while the stiffness is not. The same method was later extended to a nonlocal model.\(^{27}\) In the same context, there has also been work in which the gradient term is renormalized in an approximate way.\(^{16}\) The LPA is generally thought to be applicable to wetting problems because the anomalous dimension is expected to be zero.

In the present problem, on the other hand, it is clear that for vanishing $Q$ one should obtain Kosterlitz-Thouless behavior; furthermore, as discussed above, the quantity $Q$ only couples to the number of solitons, which is conserved under the RG flow. Thus, the appropriate RG must capture the Kosterlitz-Thouless type of behavior, which requires $\rho_s$ to flow under the transformation.

As mentioned above, we shall use the effective average functional RG scheme introduced by Wetterich\(^{20}\) and applied recently to the sine-Gordon model.\(^{21}\) For completeness, we outline the derivation of the exact flow equation for the effective action before applying it to the sine-Gordon model; for more details, see Refs. 20, 21, and 28.

From this point onwards, we will subsume the temperature into the parameters $\rho_s$ and $t$, that is, we use units in which the temperature $k_B T = 1$.

One begins by defining the bare action $S[\theta] = H[\theta]$ and adding to it a piece
\[ \Delta S_\delta[\theta] = \frac{1}{2} \sum_q R_\lambda(q) \theta(q) \theta(-q), \]

where $R_\lambda(q)$ is called a regulator function (see below). One also adds a source term $\Delta S_s[j, \theta] = \sum_q j(q) \theta(-q) + j(-q) \theta(q)$, and writes $S[j, \theta] = S[\theta] + \Delta S_\delta[\theta] + \Delta S_s[j, \theta]$. The quantity
\[ W_s[j] = \log \int D\theta \exp(-S[j, \theta]) \]
is then the generator of connected correlation functions\(^{12}\) for the action $S_\lambda$. Its Legendre transform is
\[ \bar{\theta}(q) = \frac{\delta W_s[j]}{\delta j(q)}(-q) \]
where $\bar{\theta}(q) = \delta W_s[j]/\delta j(q)$ is the average of the field. We also define a new, related function
\[ \Gamma_\lambda[\theta] = \Gamma_\lambda[\theta] - \Delta S_\delta[\theta]. \]  \hspace{1cm} (5)

Using well-known\(^{12}\) properties of $W$ and $\Gamma$, and writing $e = \ln(\Lambda_0/\Lambda)$ where $\Lambda_0$ is the initial value of the cutoff $\Lambda$, one finds
\[ \partial_q \Gamma_\lambda[\theta] = \frac{1}{2} \text{Tr} \{ \partial_q R_\lambda(q) [\Gamma_\lambda(q) + R_\lambda(q)]^{-1} \} \]  \hspace{1cm} (6)

with $\Gamma_\lambda^{(n)}(\theta)$ indicating the $n$th functional derivative of $\Gamma_\lambda$. This is an exact result.\(^{30}\) It can be shown\(^{30}\) that if $R_\lambda(q) \to \infty$ as $\Lambda \to \infty$ then $\Gamma_\lambda[\theta] \to S_\lambda[\theta]$; fluctuations about the mean-field solution are completely suppressed. Conversely, if $R_\lambda(q) \to 0$ as $\Lambda \to 0$ then $\Gamma_\lambda[\theta] \to \Gamma_\lambda[\theta]$, so that the full generator of 1PI vertices is obtained.

A full solution of Eq. (6) for $\Gamma_\lambda$ would amount to computing all 1PI functions of the system at some length scale $\sim 1/\Lambda$, including the full effects of fluctuations. This is not a simple problem, and one must resort to approximations. We will take the form of $\Gamma_\lambda[\theta]$ to be
\[ \Gamma_\lambda[\theta] = \int d^2r \left[ \frac{1}{2} \rho_s(e)(\nabla \theta)^2 + V(e, \theta) \right]. \]

A tedious but straightforward computation leads to\(^{28-30}\)
\[ \partial_q V = 2V - \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \mathcal{P}[V(q)]^2 A(q) \partial_q R_\lambda(q), \]  \hspace{1cm} (7a)
\[ \partial_q \rho_s = \frac{1}{2} \mathcal{P}[V(q)]^2 \int \frac{d^2q}{(2\pi)^2} A(q) \partial_q R_\lambda(q) \partial_q \rho_s, \]  \hspace{1cm} (7b)

where the operator $\mathcal{P}$ projects the function to its right onto the field-independent functional subspace and $A(q) = [\rho_s q^2 + R_\lambda(q) + V(2)]^{-1}$. We take the cutoff function to be
\[ R_\lambda(q) = q^{-b} \left( \frac{\Lambda^2}{q^2} \right)^b. \]  \hspace{1cm} (8)

The parameter $b$ controls the sharpness of the regulator function $R_\lambda$ in both wave vector and real space: for large $b$, $R_\lambda$ is local in wave-vector space and long range in real space; for $b \to 1$, it is instead smooth in momentum space but sharp in real space.

If $V(e, \theta)$ is restricted to its leading Fourier component $V(e, \theta) = -(e\cos(\theta))$, the flow equations are, to leading order and after rescale
\[ \partial_q e = \left( 2 - \frac{1}{4\pi \rho_s} \right) \tilde{t}(e/\Lambda^2) \]
the integrals, one obtains

$$\partial_\epsilon \rho_\epsilon(\epsilon) = \left[ \frac{\epsilon(\epsilon)}{\Lambda^2} \right]^2 \left[ \frac{\rho_\epsilon(\epsilon)}{\epsilon(\epsilon)^2} \right]$$

with $$\tau_b = b\Gamma(3-2/b)\Gamma(1+1/b)/(48\pi)$$, reproducing the well-known leading-order flow equations for the sine-Gordon model.10,18,32,33

Including higher-order terms on the right-hand side of Eqs. (7a) and (7b) is, in principle, straightforward; for example, expanding $$\mathcal{A}(\mathbf{q})$$ in powers of $$V(\epsilon, \theta)$$ and computing the integrals, one obtains

$$\partial_\epsilon V(\epsilon, \theta) = -V^{(2)} \frac{4b^3 \rho^3 \sin^2 \left( \frac{2\pi}{b} \right)}{6b^4 \rho^4 \sin \left( \frac{2\pi}{b} \right)} + \cdots$$

A similarly ungainly equation holds for $$\partial_\epsilon \rho_\epsilon(\epsilon)$$. These expressions are greatly simplified if we make the choice34 for $$b = 1$$ in the regulator function, Eq. (8). For the potential, the series on the right-hand side of Eq. (9) may be summed to yield

$$\partial_\epsilon V = \frac{\Lambda}{4\pi \rho_\epsilon} \ln \left( 1 + \frac{V^{(2)}}{\Lambda^2} \right).$$

Restricting the potential to the form $$V(\epsilon, \theta) = -t(\epsilon) \cos(\theta)$$ we obtain the flow equation21 for $$t(\epsilon)$$

$$\partial_\epsilon t(\epsilon) = 2t(\epsilon) - \frac{1}{2\pi \rho_\epsilon(\epsilon)t(\epsilon)/\Lambda^2} \left( 1 - \sqrt{1 - \frac{t(\epsilon)}{\Lambda^2} \right)^2. \)$$

In a similar way, we obtain for the flow of $$\rho_\epsilon(\epsilon)$$

$$\partial_\epsilon \rho_\epsilon(\epsilon) = \frac{t^2(\epsilon)/\Lambda^2}{24\pi(1 - t^2(\epsilon)/\Lambda^2)^{3/2}}. \)$$

The flow diagram corresponding to Eqs. (10) and (11) is shown in Fig. 1. We now turn to the application of the flow equations to the PT model.

V. APPLICATION OF RG TO THE PT MODEL

A. Calculation of $$Q_c$$ using the RG results

To apply our formalism to the PT model, we use a Ginzburg criterion.35 Using the fact that mean-field theory applies if $$\Delta \xi(\epsilon) \ll 2\pi$$, we conclude that the $$\epsilon = B\epsilon$$ at which the integration may be stopped may be located by integrating up to the point at which $$\xi(\epsilon)$$ is a minimum. At this point, the mean-field approach of Sec. III may be used to determine $$Q_c$$, with the effects of fluctuations having been taken into account by the renormalization of the parameters. The mean-field treatment of the action at this point is justified a posteriori if indeed $$\Delta \xi(\epsilon) \ll 2\pi$$. This condition is satisfied for all parameter values we have studied (see Fig. 2 for a representative example).

In Fig. 2 we show a representative plot of the evolution of $$\Delta \xi(\epsilon)$$ with $$\epsilon$$. Evidently, the minimum is well below $$2\pi$$ so that mean-field theory is applicable to the renormalized $$\Gamma$$.

Having determined the appropriate $$\epsilon_c$$, we repeat the procedure of Sec. III to determine the critical $$Q$$. How would this calculation be affected by the RG transformation? As we saw in Sec. IV, $$H_Q$$ is invariant under the RG transformation, while $$H_{\mathcal{Q}}$$ is not, i.e., the parameters in $$H_{\mathcal{Q}}$$ will change to $$\rho_\epsilon(\epsilon)$$ and $$t(\epsilon)$$, respectively. To obtain the critical $$Q_c$$, we notice that while $$H_{\mathcal{Q}}$$ is unaffected by the RG, the energy of a single soliton now depends on $$\epsilon$$: we have $$E_{\text{sol}}(\epsilon) = 8\pi \rho_\epsilon(\epsilon) \epsilon^2$$. This results in an energy difference between the phase with no solitons and the phase with a single soliton given by $$\Delta E = E_{\text{sol}}(\epsilon) - 2\pi \rho_\epsilon(0) Q$$. Setting this equal to zero and solving for $$Q$$, as in Sec. III, yields

$$\Delta \xi(\epsilon) \text{ vs. } \epsilon$$

FIG. 2. Evolution of $$\Delta \xi(\epsilon)$$ (dark full line) as a function of $$\epsilon$$. The dot-dashed line indicates the position of the minimum of $$\xi(\epsilon)$$. The steep increase in $$\xi(\epsilon)$$ after its minimum is a result of our approximations and therefore unphysical. Notice that the minimum of $$\Delta \xi(\epsilon) < 2\pi$$.

The initial values for this figure are $$\rho_\epsilon/k_B T = 0.06$$ and $$t/\Lambda^2 k_B T = 0.01$$. 

FIG. 1. Flow diagrams for the RG Eqs. (10) and (11).
The full line is \( t(\epsilon)/\Lambda^2 k_B T \), the dashed line is \( \rho_s(\epsilon)/k_B T \), and the dot-dashed line indicates the position of the minimum of \( \xi(\epsilon) \) (see Fig. 2). The steep increase in \( \xi(\epsilon) \) after its minimum is a result of our approximations and therefore unphysical (see text). The initial values for this figure are the same as for Fig. 2: \( \rho_s/k_B T=0.06 \) and \( t/\Lambda^2 k_B T=0.01 \).

\[
Q_c(\epsilon) = \exp(-\epsilon) \frac{4}{\pi \rho_s(0)} \sqrt{\rho_s(\epsilon)/t(\epsilon)}. \tag{12}
\]

The factor \( \exp(-\epsilon) \) ensures that we are using physical (as opposed to rescaled) units. Equation (12) reduces to the correct mean-field expression, Eq. (4), for \( \epsilon=0 \).

The result given in Eq. (12) is nonuniversal, involving the microscopic parameter \( \rho_s(0) \). This is expected: in calculating a nonuniversal quantity such as \( Q_c \), it is natural for microscopic, thus nonuniversal, quantities to appear. Here, the dependence on the microscopic physics arises both from the presence of \( \rho_s(0) \) and from \( \epsilon_f \), which is a function of \( \rho_s(0) \) and \( t(0) \).

Let us now discuss the scheme dependence of our calculation. In Ref. 19, it is shown that the trajectories resulting from the effective action functional RG scheme we use are scheme independent, provided that the quantities \( t \) and \( \rho_s \) flow as powers of the parameter \( \exp(\epsilon) \), i.e., \( \rho_s \sim \exp(d_s \epsilon) \) with some \( d_s \) (and similarly for \( t \)). They term the region in which this occurs the “freezing region.” Note that this scheme independence does not hold in general for the case of Wilson-type renormalization.

Figure 3 is a log plot of the evolution of \( \rho_s \) and \( t \) with \( \epsilon \). The full line is \( t(\epsilon) \), the dashed line is \( \rho_s(\epsilon) \), and the dot-dashed line indicates the position of the minimum of \( \xi(\epsilon) \) (see also Fig. 2). Evidently, both \( \rho_s(\epsilon) \) and \( t(\epsilon) \) are in the freezing region at the values of \( \epsilon \) that we are interested in. This happens for all initial values of \( \rho_s \) and \( t \) that we have checked.

### B. Results and discussion

In Figs. 4(a) and 4(b) we show the ratio \( \gamma=Q_c(\epsilon)/Q_c(0) \) determined by numerically integrating the flow equations and applying the method discussed above; this corresponds to the ratio \( \gamma=Q_c^{RG}/Q_c^{MF} \). A darker color indicates a larger decrease in the critical \( Q_c \) due to thermal effects (see the insets and notice the different scales). The plot in Fig. 4(b) is a zoomed-in part of Fig. 4(a) (notice the range of the axes and also the changed color coding).

For the purposes of this section, we will switch back to using natural units by defining \( \tilde{\rho}_s/k_B T=\rho_s \) and \( \tilde{t}/(k_B T \Lambda^2)=t \), i.e., \( \tilde{\rho}_s \) and \( \tilde{t} \) are the parameters in physical units (while \( t \) and \( \rho_s \) are in units in which \( k_B T=\Lambda=1 \)).

First let us fix \( \tilde{\rho}_s \) and \( \tilde{T} \) and vary \( \tilde{t} \), i.e., fix \( \rho_s \) and vary \( t \). For decreasing (increasing) \( t \), \( \gamma \) decreases (increases), vanishing as \( t \to 0 \). This occurs because, as described in Sec. II, for vanishing \( t \) there is no long-range order even in the absence of \( Q_c \). This effect is not present at mean-field level, hence \( \gamma \) vanishes with decreasing \( t \).

Next, fix \( \tilde{t} \) and \( \tilde{T} \) and vary \( \rho_s \). From Fig. 4, \( \gamma \) increases (decreases) for decreasing (increasing) \( \rho_s \) or \( \tilde{\rho}_s \).

Finally, consider fixing \( \tilde{t} \) and \( \tilde{T} \) and varying the temperature \( \tilde{T} \). This corresponds to fixed \( t/\Lambda^2 \rho_s \) or, in terms of Fig. 4, to moving on rays emanating from the origin with gradient.
Increasing (decreasing) the temperature $T$ corresponds to moving inward (outward) from the origin. As $T$ is increased, a $T$ will be reached at which the system will enter the rough phase at the extreme lower left corner of the diagram; thus, at high enough $T$, $Q_c$ vanishes for all $\rho_s$ and $\tilde{T}$. Equivalently, this may be viewed as a proliferation of solitons because the RG flow there is such that $t(\epsilon) \to 0$ monotonically as $\epsilon \to \infty$, so that we also have $E_{\text{ext}}(\epsilon) \to 0$. This brings up the intriguing possibility of a purely temperature-driven commensurate-incommensurate transition point of the PT model using a renormalization-group approach. Our scheme relies on splitting the PT Hamiltonian into a sine-Gordon part, $H_{\text{SG}}$, and a part depending only on the number of solitons present, $H_{\text{sol}}$. We then derive a functional RG transformation which acts on the sine-Gordon part while leaving the soliton part invariant. We are thus able to determine the critical $Q_c$ at which the incommensurate phase eventually becomes stable, taking into account thermal effects. We find a general lowering of $Q_c$ compared to the mean-field result. Furthermore, there exists a regime in the $(\rho_s, T)$ plane in which $Q_c \to 0$ even for finite $t$. This is due to the roughening transition, which is of Kosterlitz-Thouless type.

Since real experiments are performed at finite temperatures, the approach developed here may be applied to describe the CIC transition in several physical systems. An application to the case of a quantum-Hall bilayer at total filling $\nu_f = 1$ in the presence of a tilted magnetic field, motivated by recent experiments, will be described in a forthcoming publication.

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VI. CONCLUSIONS

We have studied thermal effects on the commensurate-incommensurate transition point of the PT model using a renormalization-group approach. Our scheme relies on splitting the PT Hamiltonian into a sine-Gordon part, $H_{\text{SG}}$, and a part depending only on the number of solitons present, $H_{\text{sol}}$. We then derive a functional RG transformation which acts on the sine-Gordon part while leaving the soliton part invariant. We are thus able to determine the critical $Q_c$ at which the incommensurate phase eventually becomes stable, taking into account thermal effects. We find a general lowering of $Q_c$ compared to the mean-field result. Furthermore, there exists a regime in the $(\rho_s, T)$ plane in which $Q_c \to 0$ even for finite $t$. This is due to the roughening transition, which is of Kosterlitz-Thouless type.

Since real experiments are performed at finite temperatures, the approach developed here may be applied to describe the CIC transition in several physical systems. An application to the case of a quantum-Hall bilayer at total filling $\nu_f = 1$ in the presence of a tilted magnetic field, motivated by recent experiments, will be described in a forthcoming publication.

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