ON THE EULER-POINCARÉ EQUATION WITH NON-ZERO DISPERSION

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Abstract. We consider the Euler-Poincaré equation on \(\mathbb{R}^d\), \(d \geq 2\). For a large class of smooth initial data we prove that the corresponding solution blows up in finite time. This settles an open problem raised by Chae and Liu [1]. Our analysis exhibits some new concentration mechanism and hidden monotonicity formula associated with the Euler-Poincaré flow. In particular we show the abundance of blowups emanating from smooth initial data with certain sign properties. No size restrictions are imposed on the data. We also showcase a class of initial data for which the corresponding solution exists globally in time.

1. Introduction

We consider the following Euler-Poincaré equation on \(\mathbb{R}^d\), \(d \geq 2\):

\[
\begin{aligned}
    \partial_t m + (u \cdot \nabla)m + (\nabla u)^T m + (\text{div} u)m &= 0, \quad t > 0, \quad x \in \mathbb{R}^d; \\
    m &= (1 - \alpha \Delta)u; \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d.
\end{aligned}
\]

(1.1)

Here \(u = (u_1, \cdots, u_d) : \mathbb{R}^d \to \mathbb{R}^d\) represents the velocity and \(m = (m_1, \cdots, m_d) : \mathbb{R}^d \to \mathbb{R}^d\) denotes the momentum. The parameter \(\alpha > 0\) in the second equation of (1.1) corresponds to the square of the length scale. It is sometimes called the dispersion parameter in the literature. The notation \((\nabla u)^T\) denotes the transpose of the matrix \(\nabla u\). To avoid any confusion it is useful to recast equation (1.1) in the component-wise form as

\[
\begin{aligned}
    \partial_t m_i + u_j \partial_j m_i + (\partial_i u_j) m_j + (\partial_j u_j) m_i &= 0.
\end{aligned}
\]

(1.2)

Here and throughout the rest of this paper we shall use the Einstein summation convention. By using the tensor notation, one can combine the second and the last term in (1.1) and write it more compactly as

\[
\begin{aligned}
    \partial_t m + \nabla \cdot (m \otimes u) + (\nabla u)^T m &= 0.
\end{aligned}
\]

(1.3)

The last term in (1.3) is not in conservative form. Following Chae and Liu [1] (see formula (1)–(4) on page 673 therein), one can introduce a stress-tensor \(T_{ij}\)

\[
T_{ij} = m_i u_j + \frac{1}{2} \delta_{ij} |u|^2 - \alpha \partial_i u \cdot \partial_j u + \frac{1}{2} \alpha \delta_{ij} |\nabla u|^2
\]

and rewrite (1.3) as

\[
\begin{aligned}
    \partial_t m_i + \partial_j T_{ij} &= 0.
\end{aligned}
\]

(1.4)
By the second equation in (1.1), we have
\[ m_i u_j = u_i u_j - \alpha (\partial_{kk} u_i) u_j = u_i u_j - \alpha \partial_k ((\partial_k u_i) u_j) + \alpha (\partial_k u_i \partial_k u_j). \]

Therefore the tensor \( T_{ij} \) can be rewritten as
\[ T_{ij} = u_i u_j - \alpha \partial_k ((\partial_k u_i) u_j) + \frac{1}{2} \delta_{ij}|u|^2 - \alpha \partial_i u \cdot \partial_j u + \frac{1}{2} \alpha \delta_{ij} |\nabla u|^2. \]

(1.5)

Roughly speaking, the above expressions show that the tensor \( T \) is of the form
\[ T = O(|u|^2 + |\partial u|^2 + \partial(u \partial u)). \]

Such a decomposition is very useful in deriving low frequency \( L^p \) estimates later (cf. Proposition 1.1). For smooth solutions with enough spatial decay, there are two natural conservation laws
\[ \frac{d}{dt} \int_{\mathbb{R}^d} m dx = 0, \]
\[ \frac{d}{dt} \int_{\mathbb{R}^d} (|u|^2 + \alpha |\nabla u|^2) dx = 0. \]

(1.6)

We shall only need the second one for later constructions.

The Euler-Poincaré equations were first introduced by Holm, Marsden, and Ratiu in [4, 5]. In 1D \((d = 1)\) the Euler-Poincaré equations reduce to the Camassa-Holm equations of the form
\[ \partial_t m + u \partial_x m + 2 \partial_x um = 0, \quad m = (1 - \alpha \partial_{xx}) u. \]

The well-posedness of local and global weak solutions of Camassa-Holm equations have been intensively studied (see [8] and references therein). In 2D, the Euler-Poincaré equation is known as the averaged template matching equation in the computer vision literature [2, 3, 6]. For the applications of Euler-Poincaré equations in computational anatomy, see [7, 9]. The rigorous analysis of the Euler-Poincaré equations was initiated by Chae and Liu [1] who established a fairly complete wellposedness theory for both weak and strong solutions. We summarize some of their main results (relevant to our context) as follows (here \( \alpha \) is the dispersion parameter in the second equation of (1.1)):

- Let \( \alpha \geq 0 \) and \( u_0 \in H^k(\mathbb{R}^d) \) with \( k > \frac{d}{2} + 3 \). Then, there exists \( T = T(||u_0||_{H^k}) > 0 \) and a unique classical solution \( u = u(x, t) \) to (1.1) in the space \( C([0, T), H^k(\mathbb{R}^d)) \).
- Let \( 0 < T^* \leq +\infty \) be the maximal lifespan corresponding to the solution \( u \in C^0_{t} H^k \). If \( T^* < \infty \), then
  \[ \limsup_{t \to T^*} ||u(t)||_{H^k} = \infty \iff \int_0^{T^*} ||S(t)||_{\dot{B}^0_{\infty, \infty}} dt = \infty. \]

(1.7)

Here \( S = (S_{ij}) \) is the deformation tensor of \( u \) with \( S_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \).

See (1.16) for the definition of the homogeneous Besov norm \( \| \cdot \|_{\dot{B}^0_{\infty, \infty}} \).

\[ ^1 \text{For } \alpha = 0 \text{ one only needs } k > \frac{d}{2} + 1 \text{ since the corresponding system is a symmetric hyperbolic system of conservation laws with a convex entropy, see Theorem 1 in [1] for more details.} \]
Let \( \alpha = 0 \). Let \( u_0 \in H^k(\mathbb{R}^d) \), \( k > \frac{d}{d-2} + 2 \), and has the reflection symmetry with respect to the origin, i.e.
\[
u_0(x) = -u_0(-x), \quad \forall x \in \mathbb{R}^d.
\]
If \( \text{div}u_0(0) < 0 \), then the corresponding classical solution blows up in finite time.

One should notice that the Chae-Liu blowup result stated above is only valid for \( \alpha = 0 \) in which case the Euler-Poincaré equation reduces to a version of high-dimensional Burgers system. The main idea of Chae-Liu is to consider the evolution of \( \text{div}u \) at the origin. Namely by using the reflection symmetry and (1.2), one obtains
\[
\frac{d}{dt}(\text{div}u(0,t)) = -2\left(\sum_{i,j=1}^{d}(\partial_i u_j + \partial_j u_i)\right)^2 - (\text{div}u(0,t))^2 \\
\leq -(\text{div}u(0,t))^2,
\]
and blowup follows from the assumption \( \text{div}u_0(0) < 0 \). Unfortunately for the non-degenerate case \( \alpha > 0 \), this elegant argument does not work anymore due to some extra high order terms which do not enjoy any monotonicity property. Thus Chae and Liu raised the following

**Problem:** for the Euler-Poincaré system (1.1) \((\alpha > 0)\), do there exist finite time blowups from smooth initial data?

The main purpose of this paper is to settle the above problem in the affirmative. Since we are mainly interested in the case \( \alpha > 0 \), the actual value of \( \alpha \) will play no role in our analysis. Henceforth we shall set \( \alpha = 1 \) throughout the rest of this paper. We start by considering a special class of radial flows invariant under the Euler-Poincaré dynamics. More precisely let \( m = \nabla \phi \) where \( \phi \) is a radial scalar-valued function. By (1.2) and noting that \( \partial_j m_i = \partial_i m_j \) for any \( i, j \), we have
\[
-\partial_i m_i = m_i \partial_j u_j + u_j \partial_j m_i + \partial_j u_j m_j \\
= m_i \partial_j u_j + u_j \partial_i m_j + \partial_j u_j m_j \\
= m_i (\nabla \cdot u) + \partial_i (m \cdot u).
\]
Therefore the radial function \( \phi \) satisfies
\[
-\partial_t \phi'(r,t) = -\phi(r,t)\phi'(r,t) + \phi'(r,t)((1 - \Delta)^{-1}\phi)(r,t) \\
+ \left((1 - \Delta)^{-1}\nabla \phi \cdot \nabla \phi\right)' , \quad r = |x| > 0,
\]
with initial data \( \phi(r,0) = \phi_0(r) \). Here and throughout the rest of this paper, we will slightly abuse the notation and denote any radial function \( f \) on \( \mathbb{R}^d \) as \( f(x) = f(|x|) = f(r) \) whenever there is no confusion. We also use the notation \( f' = f'(r) \) to denote the radial derivative. Assuming \( \phi \) (and its derivatives) decays sufficiently

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2By using the derivation below, it is not difficult to check that if initially \( m_0 = \nabla \phi_0 \) and \( \phi_0 \) is a smooth radial function, then for any \( t > 0 \) we can write \( m(t) = \nabla \phi(t) \) with \( \phi(t) \) being radial and smooth as well. The radial assumption here is essential. In the general case one can not expect that irrotational flows are preserved in time.
fast at infinity, we may integrate \(1.9\) on the slab \([r, \infty)\) and obtain

\[
\partial_t \phi(r, t) = \frac{1}{2} \phi(r, t)^2 + \int_r^{\infty} \phi'(s, t)((1 - \Delta)^{-1}\phi)(s, t) ds - (1 - \Delta)^{-1} \nabla \phi \cdot \nabla \phi)(r, t).
\]

At the cost of a nonlocal integration, the equation \((1.10)\) simplifies greatly the analysis and will be our main object of study in this paper. We begin with a simple proposition which in some sense justifies the validity of the equation \((1.10)\).

**Proposition 1.1.** Let the dimension \(d \geq 2\). Assume initially \(m_0 = \nabla \phi_0\) where \(\phi_0\) is a radial function on \(\mathbb{R}^d\) and \(\phi_0 \in H^k\) for some \(k > \frac{d}{2} + 4\). If \(d = 2\), we also assume \(\phi_0 \in \dot{B}^k_{1, \infty}(\mathbb{R}^2)\). Then for any \(t > 0\) the solution \(m(t) = (1 - \Delta)u(t)\) can be written as \(m(t) = \nabla \phi(t)\) where \(\phi(t)\) is radial and \(\phi(t) \in H^k(\mathbb{R}^d)\) for \(d \geq 3\), \(\phi(t) \in H^k(\mathbb{R}^2) \cap \dot{B}^k_{1, \infty}(\mathbb{R}^2)\) for \(d = 2\). Each \(\phi(t, r)\) solves \((1.10)\) in the classical sense. Moreover we have the growth estimate

\[
\|\phi(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq B_1 \cdot (1 + t)^{\frac{k}{2}}, \forall t \geq 0, \text{ if } d = 2,
\]

\[
\|\phi(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq B_2 \cdot (1 + t), \forall t \geq 0, \text{ if } d \geq 3.
\]

Here \(B_1 > 0, B_2 > 0\) are some constants depending only on the initial data \(\phi_0\).

With Proposition \(1.1\) in hand, we can control the low frequency part of the solution and express the blowup/continuation \((1.7)\) in terms of the scalar function \(\phi\) alone. Thus

**Lemma 1.2.** Let \(\phi_0\) be radial. If \(d \geq 3\), we assume \(\phi_0 \in H^k(\mathbb{R}^d)\) for some \(k > \frac{d}{2} + 4\). If \(d = 2\), we assume \(\phi_0 \in H^4(\mathbb{R}^2) \cap \dot{B}^6_{1, \infty}(\mathbb{R}^2)\) for some \(k \geq 6\). Let \(u\) be the maximal lifespan solution corresponding to initial data \(u_0 = (1 - \Delta)^{-1}m_0 = (1 - \Delta)^{-1}\nabla \phi_0\). If the maximal lifespan \(T^* < \infty\), then

\[
\limsup_{t \to T^*} \|u(t)\|_{H^k} = \infty \iff \int_0^{T^*} \|\phi(t)\|_{L^\infty(\mathbb{R}^d)} dt = \infty.
\]

We shall omit the proof of Lemma \(1.2\) since it follows directly from \((1.7), (1.11) - (1.12)\), and the embedding \(L^\infty \to \dot{B}^k_{1, \infty}\).

We now state our main results. Apart from regularity assumptions, the first result says that if initially \(\phi_0(0) \geq 0\), then the corresponding solution blows up in finite time. It is a bit surprising in that such a local condition dictates the whole nonlocal Euler-Poincaré dynamics.

**Theorem 1.3.** Let the dimension \(d \geq 2\). Let \(\phi_0\) be a radial real-valued function on \(\mathbb{R}^d\) such that \(\phi_0 \in H^k(\mathbb{R}^d)\) for some \(k > \frac{d}{2} + 4\). For \(d = 2\) we also assume \(\phi_0 \in \dot{B}^6_{1, \infty}(\mathbb{R}^2)\). Let the initial velocity \(u_0 = (1 - \Delta)^{-1}m_0 = (1 - \Delta)^{-1}\nabla \phi_0\). If \(\phi_0(0) \geq 0\) and \(\phi_0\) is not identically zero, then the corresponding solution blows up in finite time.

The next result deals with the opposite scenario \(\phi_0(0) < 0\). Under the assumption that \(\phi_0(r)\) is monotonically increasing, we show the corresponding solution exists globally in time. In some sense it reveals the nonlinear depletion mechanism hidden in the Euler-Poincaré dynamics.
Theorem 1.4 (Global regularity for a class of non-positive monotone data). Let the dimension $d \geq 2$. Let $\phi_0$ be a radial real-valued function on $\mathbb{R}^d$ such that $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2} + 4$. For $d = 2$ we also assume $\phi_0 \in B^1_{1,\infty}(\mathbb{R}^2)$. If $\phi_0(0) \leq 0$ and $\phi_0$ is monotonically increasing on $[0, \infty)$ (i.e. $\phi'_0(r) \geq 0$ for any $0 \leq r < \infty$), then the corresponding solution $u(t) = (1 - \Delta)^{-1}\nabla \phi(t)$ exists globally in time. Moreover for any $t > 0$, $\phi(t, \cdot)$ is monotonically increasing on $[0, \infty)$.

We have the following corollary which computes the asymptotics of $\phi(0, t)$ as $t \to \infty$. To allow some generality we state it as a conditional result in that we assume the corresponding solution exists globally in time.

Corollary 1.5 (Asymptotics of $\phi(0, t)$). Let the dimension $d \geq 2$. Let $\phi_0$ be a radial real-valued function on $\mathbb{R}^d$ such that $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2} + 4$. For $d = 2$ we also assume $\phi_0 \in B^1_{1,\infty}(\mathbb{R}^2)$. Assume $\phi_0(0) < 0$, $u_0 = (1 - \Delta)^{-1}\nabla \phi_0$ and the corresponding solution $u(t) = (1 - \Delta)^{-1}\nabla \phi(t)$ exists globally on $[0, \infty)$. Then $\phi(0, t)$ is strictly monotonically increasing in $t$ and $\frac{d}{dt}\phi(0, t) > 0$ for any $t \geq 0$. There are some constants $C_1 > 0$, $C_2 > 0$ such that for $d \geq 3$

$$0 < -\phi(0, t) < \frac{C_1}{1 + t}, \quad \forall t > 0;$$

and for $d = 2$

$$0 < -\phi(0, t) < \frac{C_2}{\log(10 + t)}, \quad \forall t > 0.$$ (1.13) (1.14)

In particular $\lim_{t \to \infty} \phi(0, t) = 0$.

Remark 1.6. The decay rates in (1.13)–(1.14) is probably not optimal. It is an interesting question to study the long time behavior of global solutions to such systems with no damping or dissipation.

It is very tempting to conjecture that the single condition $\phi_0(0) < 0$ may yield global wellposedness. Our last result rules out this possibility. We exhibit a family of smooth negative initial data for which the corresponding solution blows up in finite time. In particular the initial data $\phi_0$ will satisfy $\phi_0(0) < 0$.

Theorem 1.7. There exists a family $\mathcal{A}$ of smooth initial data such that the following holds:

- For each $\phi_0 \in \mathcal{A}$, we have $\phi_0(x) < 0$ for any $x \in \mathbb{R}^d$.
- The corresponding solution $u(t) = (1 - \Delta)^{-1}\nabla \phi(t)$ blows up in finite time. Moreover $\phi(0, t)$ is a monotonically increasing function of $t$ for each $t$ within the lifespan of the solution.

We conclude the introduction by setting up some

Notations. For any two quantities $X$ and $Y$, we shall write $X \lesssim Y$ if $X \leq CY$ for some harmless constant $C > 0$. Similarly we define $X \gtrsim Y$. We write $X \sim Y$ if both $X \lesssim Y$ and $X \gtrsim Y$ hold.

We will need to use the Littlewood-Paley frequency projection operators. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \leq 2$ and equal to one on the ball $|\xi| \leq 1$. For each dyadic number $N \in 2^\mathbb{Z}$ we define the Littlewood-Paley
operators
\[ \widehat{P_{\leq N}} f(\xi) := \varphi(\xi/N) \hat{f}(\xi), \]
\[ \widehat{P_{> N}} f(\xi) := [1 - \varphi(\xi/N)] \hat{f}(\xi), \]
\[ \widehat{P_{\leq M}} f(\xi) := [\varphi(\xi/N) - \varphi(2\xi/N)] \hat{f}(\xi). \] (1.15)

Similarly we can define \( P_{< N}, P_{\geq N}, \) and \( P_{M< \leq N} := P_{\leq N} - P_{\leq M} \), whenever \( M \) and \( N \) are dyadic numbers.

We recall the following standard Bernstein inequality: for any \( 1 \leq p < q \leq \infty \),
\[ \|P_{\leq N} f\|_{L^q(\mathbb{R}^d)} \lesssim N^{d(\frac{1}{p} - \frac{1}{q})} \|P_{\leq N} f\|_{L^p(\mathbb{R}^d)}. \]

Here \( P_{\leq N} \) can be replaced by \( P_{< N} \) or \( P_{\leq N} \).

For any \( 1 \leq p \leq \infty \), the homogeneous Besov norm \( \dot{B}^0_{p,\infty} \) is defined as
\[ \|f\|_{\dot{B}^0_{p,\infty}} = \sup_{M \in 2^\mathbb{Z}} \|P_M f\|_{L^p(\mathbb{R}^d)} \] (1.16)

We need the following interpolation inequality on \( \mathbb{R}^2 \):
\[ \|f\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^2)}^{\frac{1}{2}} \cdot \|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}. \] (1.17)

The proof of (1.17) is a standard exercise in Littlewood-Paley calculus. We sketch it here for the sake of completeness.

Proof of (1.17). Let \( N_0 > 0 \) be a dyadic number whose value will be chosen later. Then by Bernstein, we have
\[ \|f\|_{L^2(\mathbb{R}^2)}^2 \lesssim \sum_{N < N_0} N^2 \|P_N f\|_{L^1(\mathbb{R}^2)}^2 + \sum_{N \geq N_0} N^{-2} \|\nabla P_N f\|_{L^2(\mathbb{R}^2)}^2 \]
\[ \lesssim N_0^2 \|f\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^2)}^2 + N_0^{-2} \|\nabla f\|_{L^2(\mathbb{R}^2)}^2. \]
Choosing \( N_0 \sim \frac{\|\nabla f\|_{L^2(\mathbb{R}^2)}}{\|f\|_{\dot{B}^0_{2,\infty}}} \) then yields the result. \( \square \)

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2. Proof of Proposition 1.1 and some intermediate results

In this section we first give the proof of Proposition 1.1. After that we shall deduce several weak blowup results some of which has certain concentration and/or size restrictions on the initial data. However the proofs of these results are somewhat simpler and they serve to illustrate main difficulties in proving the sharp result Theorem 1.3.
Proof of Proposition 1.1. Since $m = (1 - \Delta)u = \nabla \phi$, we have $u = (1 - \Delta)^{-1} \nabla \phi$.

By using (1.6), we obtain
\[
\| (1 - \Delta)^{-1} \nabla \phi(t) \|_2 + \sum_{i,j=1}^d \| (1 - \Delta)^{-1} \partial_i \partial_j \phi(t) \|_2 \lesssim 1, \quad \forall t \geq 0. 
\] (2.18)

From (2.18), we have
\[
\| P_{\geq 1} \phi(t) \|_2 \lesssim 1, \quad \forall t \geq 0.
\]

By using the local theory worked out in [1], we then only need to show the persistence of negative regularity and estimate $\| P_{<1} \phi(t) \|_2$. By (1.4), we have
\[
m_i(t) = m_i(0) - \sum_{j=1}^d \int_0^t (\partial_j T_{ij})(\tau) d\tau.
\]

Therefore by (1.6) and Bernstein,
\[
\| P_{<1} \phi(t) \|_2 \lesssim \| P_{<1} \Delta^{-1} \nabla \cdot m(t) \|_2 \\
\lesssim \| \phi(0) \|_2 + \sum_{j=1}^d \int_0^t \| P_{<1} \Delta^{-1} \partial_i \partial_j T_{ij}(\tau) \|_2 d\tau \\
\lesssim \| \phi(0) \|_2 + \int_0^t \left( \| u(\tau) \|_2^2 + \| \nabla u(\tau) \|_2^2 \right) d\tau \\
\lesssim \| \phi(0) \|_2 + C_1 t, \quad \forall t \geq 0,
\]
where $C_1 > 0$ depends on $\| u_0 \|_{H^1}$, and we have used the conservation law (1.6). Hence the estimate (1.12) follows.

Similarly by using the fact that
\[
\sup_{N \in Z^d} \| P_N \Delta^{-1} \partial_i \partial_j \|_{L^1 \to L^1} < \infty,
\]
we obtain in the case $d = 2$,
\[
\| P_{<1} \phi(t) \|_{\dot{H}^1_{\infty}(\mathbb{R}^2)} \lesssim C_2 (1 + t), \quad \forall t \geq 0,
\]
where $C_2 > 0$ depends only on $\phi_0$. The growth estimate (1.11) then follows from the above estimate, the conservation law $\| \nabla P_{<1} \phi \|_2 \lesssim \| P_{<1} (1 - \Delta) u \|_2 \lesssim \| u \|_2 \lesssim 1$, and the interpolation inequality (1.17) (applied to $f = P_{<1} \phi$).

Finally we need to justify (1.10). In particular we need to show that the integral $\int_{r}^{\infty} \phi'(s) (1 - \Delta)^{-1} \phi(s,t) ds$ converges. Indeed this follows from the estimate
\[
\int_{r}^{\infty} |\phi'| (1 - \Delta)^{-1} \phi ds \lesssim \| \nabla \phi \cdot (1 - \Delta)^{-1} \phi \|_{L^1(\mathbb{R}^d)} \\
\lesssim \| \nabla \phi \|_{\infty} \cdot \| (1 - \Delta)^{-1} \phi \|_{\infty} + \| \nabla \phi \|_2 \cdot \| (1 - \Delta)^{-1} \phi \|_2 \\
< \infty.
\]

Since $\phi \in H^k$, $\phi$ is a smooth function. Since the above integral converges, it follows that (1.10) holds in the classical sense. \( \square \)

We now formulate a simple blowup result which requires three rather restrictive conditions on the initial $\phi_0$: positivity, monotonicity and sufficient concentration at the spatial origin. Due to these simplifying assumptions, the proof is much simpler.
Theorem 2.1. Let the dimension $d \geq 1$. Assume $\phi_0$ is a radial real-valued function on $\mathbb{R}^d$ and $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2} + 4$. Assume $\phi_0(r) \leq 0$ for any $r > 0$ and $\phi_0$ is not identically zero. There exists a constant $C > 0$ such that if
\[ \phi_0(0) \geq C\|\phi_0\|_{L^2(\mathbb{R}^d)} \]
and the initial velocity $u_0 = (1 - \Delta)^{-1}m_0 = (1 - \Delta)^{-1}\nabla \phi_0$, then the corresponding solution blows up in finite time.

Proof of Theorem 2.1. Note that by assumption we have $\phi_0$ attains its global maximum at $r = 0$ and $\phi_0(0) > 0$. We first show that for any $t > 0$ within the lifespan of the solution, we have $\phi'(r,t) \leq 0$ for any $r > 0$. Indeed by (1.9), we have
\[ -\partial_t \phi'(r,t) = -\phi(r,t)\phi'(r,t) + \phi'(r,t)((1 - \Delta)^{-1}\phi)(r,t) \]
\[ + \left((1 - \Delta)^{-1}\phi'\cdot\phi'\right). \]

Set $g(r,t) = \phi'(r,t)$, then by using the above equation and grouping the coefficients, we see that
\[ \partial_t g(r,t) + a_1(r,t)g(r,t) + a_2(r,t)\partial_r g(r,t) = 0, \quad \forall r \geq 0, \]
where $a_1$, $a_2$ are some smooth functions. Since $g(r,0) = \phi_0'(r) \leq 0$, a simple method of characteristics argument then yields immediately that $g(r,t) \leq 0, \forall r \geq 0$. Hence $\phi'(r,t) \leq 0$, for any $r > 0$.

Now set $r = 0$ in (1.10), we obtain
\[ \frac{d}{dt}\phi(0,t) = \frac{1}{2}\phi(0,t)^2 + \int_0^\infty \phi'(s,t)((1 - \Delta)^{-1}\phi)(s,t)ds. \tag{2.19} \]

By using an argument similar to the derivation of (1.11)–(1.12) (here we are treating all dimensions $d \geq 1$), we have for all $t \geq 0$,
\[ \|\phi(t)\|_{L^2(\mathbb{R}^d)} \lesssim \|P_{<1}\phi(t)\|_{L^2(\mathbb{R}^d)} + \|P_{\geq 1}\phi(t)\|_{L^2(\mathbb{R}^d)} \]
\[ \lesssim \|\phi_0\|_{L^2(\mathbb{R}^d)} + \|u_0\|_{H^1(\mathbb{R}^d)} \]
\[ \lesssim \|\phi_0\|_{L^2(\mathbb{R}^d)}(1 + t), \tag{2.20} \]
where we have used the relation $u_0 = (1 - \Delta)^{-1}\nabla \phi_0$.

Since $\phi'(r,t) \leq 0$ for any $r \geq 0$, we have $\|\phi(t)\|_{\infty} = \phi(0,t)$. Therefore by (2.20), we have
\[ \|(1 - \Delta)^{-1}\phi\|_{\infty} \leq C\|\phi(t)\|_{2} + \frac{1}{100}\|\phi(t)\|_{\infty} \]
\[ \leq C\|\phi_0\|_{2} + \frac{1}{100}\phi(0,0). \tag{2.21} \]

Plugging (2.21) into (2.19) and using the fact that $(1 - \Delta)^{-1}\phi \geq 0, \phi' \leq 0$, we have
\[ \frac{d}{dt}\phi(0,t) \geq \frac{1}{4}\phi(0,t)^2 - C\|\phi_0\|_{2}(1 + t) \cdot \phi(0,t). \tag{2.22} \]

Clearly for $\phi_0(0) > 0$ sufficiently large (compared to $\|\phi_0\|_{2}$), $\phi(0,t)$ will blow up in finite time. \qed

Footnote 3: In 1D, we simply require $\phi_0$ is an even function.
Our next result refines Theorem 2.1 in that it removes the size assumption on \( \phi_0 \). For some technical reasons (see Lemma 2.5), it only treats dimensions \( d \geq 3 \).

**Theorem 2.2.** Let the dimension \( d \geq 3 \). Assume \( \phi_0 \) is a radial real-valued function on \( \mathbb{R}^d \) and \( \phi_0 \in H^k(\mathbb{R}^d) \) for some \( k > \frac{d}{2} + 4 \). Assume \( \phi_0'(r) \leq 0 \) for any \( r > 0 \) and \( \phi_0 \) is not identically zero. If the initial velocity \( u_0 = (1 - \Delta)^{-1}m_0 = (1 - \Delta)^{-1}\nabla \phi_0 \), then the corresponding solution blows up in finite time.

The proof of Theorem 2.2 relies on the following lemma which can be regarded as some type of Poincaré inequality.

**Lemma 2.3.** Let the dimension \( d \geq 1 \). For any \( C_1 > 0 \), \( 1 \leq p < \infty \), there is a constant \( \epsilon_0 > 0 \) depending only on \( C_1, p \) and the dimension \( d \) such that the following holds:

Suppose \( f : \mathbb{R}^d \to \mathbb{R} \) is a function (not necessarily radial) such that

\[
0 \leq f(x) \leq f(0), \quad \forall x \in \mathbb{R}^d.
\]

If \( \|f\|_{L^p_x} \leq C_1|f(0)| < \infty \), then

\[
\left| (\frac{\Delta}{1 - \Delta}f)(0) \right| \geq \epsilon_0|f(0)|. \tag{2.23}
\]

**Proof of Lemma 2.3.** WLOG we may assume \( f(0) = 1 \). Denote the Bessel potential \( K(x) = F^{-1}((1 + |\xi|^2)^{-1})(x) \). Note that \( K \) is a positive radial function on \( \mathbb{R}^d \) and \( K \in L^1_x \cap L^\infty_x \) for any \( 1 < q < \frac{d}{d - 2} \) (for \( d = 2 \) we have \( K \in L^1_x \cap L^q_x \) for any \( q < \infty \), and for \( d = 1 \) we have \( K \in L^1_x \cap L^\infty_x \)).

Then

\[
\left( \frac{\Delta}{1 - \Delta}f \right)(0) = f(0) - \left( \frac{1}{1 - \Delta}f \right)(0) = \int_{\mathbb{R}^d} K(y)(f(0) - f(y))dy.
\]

Assume the bound (2.23) is not true. Then there exists a sequence of nonnegative functions \( f_n \) such that \( f_n(0) = 1 \), \( \|f_n\|_{L^\infty_x} \leq 1 \), \( \|f_n\|_{L^1_x} \leq C_1 \) and

\[
\int_{\mathbb{R}^d} K(y)(1 - f_n(y))dy \to 0. \tag{2.24}
\]

Now take a number \( r > p \) sufficiently large such that \( K \in L^\infty_x \). Obviously \( \|f_n\|_{L^r_x} \leq C_2 < \infty \) for some constant \( C_2 > 0 \) independent of \( n \). By passing to a subsequence in \( n \) if necessary, we have \( f_n \rightharpoonup g \) weakly in \( L^r_x \) for some \( g \in L^r_x \). Furthermore we have \( \|g\|_{L^\infty_x} \leq 1 \). By (2.24) and the fact \( K \in L^\infty_x \), we then obtain

\[
\int_{\mathbb{R}^d} K(y)(1 - g(y))dy = 0
\]

which implies \( g(y) = 1 \) for a.e. \( y \in \mathbb{R}^d \). This clearly contradicts the fact that \( g \in L^r_x \). The lemma is proved. \( \square \)

**Remark 2.4.** It is also possible to give a constructive proof of Lemma 2.3. For example in the 3D case, we have \( K(x) = \mathcal{F}^{-1}((1 + |\xi|^2)^{-1})(x) = \frac{1}{4\pi}e^{-|x|} \). Let
\( p' = \frac{p}{p-1} \) be the usual Hölder conjugate of \( p \). Let \( R > 0 \) be a number whose value will be chosen later. Then we have

\[
\int_{\mathbb{R}^3} K(y)f(y)dy = \int_{|y|<R} K(y)f(y)dy + \int_{|y|>R} K(y)f(y)dy \\
\leq \|f\|_{L^{\infty}} \int_0^R re^{-r}dr + \left( \int_{|y|>R} K^{p'}dy \right)^{1/p'} \|f\|_{L^p} \\
\leq (1 - (R+1)e^{-R})\|f\|_{L^{\infty}} \\
+ \left( \frac{1}{4\pi^{p'-1}} \int_R^\infty e^{-p'r^2-p'r}dr \right)^{1/p'} \|f\|_{L^p}. \tag{2.25}
\]

To estimate (2.25), we compute (note that \( p' > 1 \), and assume that \( R > 1 \))

\[
\int_R^\infty e^{-p'r^2-p'r}dr \leq R^{1-p'}e^{(1-p')R} \int_R^\infty e^{-r}dr \\
= R^{1-p'}e^{(1-p')R}(R+1)e^{-R} \\
\leq CR^{2-p'}e^{-p'R}.
\]

Plugging the above estimate into (2.25), we have

\[
\int_{\mathbb{R}^3} K(y)f(y)dy \leq (1 - (R+1)e^{-R})\|f\|_{L^{\infty}} + C R^{(2-p')/p'}e^{-R} \|f\|_{L^p}.
\]

Since \( p' > 1 \) we have \( (2-p')/p' < 1 \). Now if there is a constant \( C_1 \) such that \( \|f\|_{L^p} \leq C_1 \|f\|_{L^{\infty}} \) holds, then we can always choose \( R \) big enough to obtain

\[
\int_{\mathbb{R}^3} K(y)f(y)dy \leq (1 - \varepsilon_0)\|f\|_{L^{\infty}}.
\]

This then leads to (2.23).

For the proof of Theorem 2.2, we need a slightly modified version of Lemma 2.3. Note the dimension restriction \( d \geq 3 \) and see also Remark 2.6 below.

**Lemma 2.5.** Let the dimension \( d \geq 3 \). For any \( C_1 > 0, 1 \leq p < \infty \), there is a constant \( \varepsilon_0 > 0 \) depending only on \( C_1, p \) and the dimension \( d \) such that the following holds:

Suppose \( f : \mathbb{R}^d \to \mathbb{R} \) satisfies

\[
0 \leq f(x) \leq f(0), \quad \forall \ x \in \mathbb{R}^d.
\]

If \( f \in L^2_{\text{loc}}(\mathbb{R}^d) \) and

\[
\left\| \frac{\nabla}{1+|\nabla|} f \right\|_{L^2_{\text{loc}}} \leq C_1 |f(0)| < \infty,
\]

then

\[
\left| (\frac{\Delta}{1-\Delta}) f)(0) \right| \geq \varepsilon_0 |f(0)|. \tag{2.26}
\]
Remark 2.6. We stress that the dimension restriction \( d \geq 3 \) is necessary in Lemma 2.5. In dimensions \( d = 1, 2 \), there exist counterexamples which are made of approximating sequences of the constant functions. To see this, let \( t > 0 \) and define

\[
f_t(x) = e^{-t|x|^2}, \quad x \in \mathbb{R}^d.
\]

Then obviously \( f_t(0) = 1 \) and

\[
\hat{f}_t(\xi) = \text{const} \cdot t^{-\frac{d}{2}} e^{-\frac{\xi^2}{4t}}, \quad \xi \in \mathbb{R}^d.
\]

When \( d = 1, 2 \), it is not difficult to check that

\[
\left\| \frac{1}{1 + |\nabla|} \right\|_{L^2_x}^2 \lesssim \int_{\mathbb{R}^d} \frac{\xi^2}{1 + |\xi|^2} \cdot t^{-\frac{d}{2}} e^{-\frac{\xi^2}{4t}} d\xi
\]

\[
\lesssim t^{1-\frac{d}{2}} \lesssim 1, \quad \text{as } t \to 0.
\]

Similarly we have

\[
\left| \frac{\Delta}{1 - \Delta} f_t(0) \right|
\]

\[
\lesssim \int_{\mathbb{R}^d} \frac{\xi^2}{1 + |\xi|^2} \cdot t^{-\frac{d}{2}} e^{-\frac{\xi^2}{4t}} d\xi
\]

\[
\lesssim t^2 \to 0, \quad \text{as } t \to 0.
\]

Obviously (2.26) cannot hold in this case.

Proof of Lemma 2.5. Again we will argue by contradiction. Assume (2.26) does not hold. Then we can find a sequence of nonnegative functions \( f_n \in L^2_x(\mathbb{R}^d) \) with \( \|f_n\|_\infty = f_n(0) = 1 \) such that

\[
\left\| \frac{1}{1 + |\nabla|} f_n \right\|_{L^2_x} \leq C_1, \tag{2.27}
\]

and

\[
\int_{\mathbb{R}^d} K(y)(1 - f_n(y)) dy \to 0, \quad \text{as } n \to \infty. \tag{2.28}
\]

By (2.27) and passing to a subsequence in \( n \) if necessary, we can find \( g \in L^2_x(\mathbb{R}^d) \) such that

\[
\frac{1}{1 + |\nabla|} f_n \to g, \quad \text{weak in } L^2_x(\mathbb{R}^d), \quad \text{as } n \to \infty.
\]

Now for any \( \phi \in S(\mathbb{R}^d) \), observe that

\[
\frac{1 + |\nabla|}{|\nabla|} \phi \in L^2_x(\mathbb{R}^d), \quad \text{for } d \geq 3.
\]

Therefore

\[
\int_{\mathbb{R}^d} f_n \phi dx = \int_{\mathbb{R}^d} \frac{1}{1 + |\nabla|} f_n \cdot \frac{1 + |\nabla|}{|\nabla|} \phi dx
\]

\[
\to \int_{\mathbb{R}^d} g \cdot \frac{1 + |\nabla|}{|\nabla|} \phi dx
\]

\[
=: T(\phi), \quad \text{as } n \to \infty. \tag{2.29}
\]
Since $0 \leq f_n \leq 1$ and
\[
T(\phi) = \lim_{n \to \infty} \int_{\mathbb{R}^d} f_n \phi dx,
\]
it follows that for $\phi \geq 0$, we have $T(\phi) \geq 0$. Therefore by the Riesz representation theorem, we have
\[
T(\phi) = \int_{\mathbb{R}^d} \phi d\mu,
\]
for some non-negative Borel measure $d\mu$. Now since
\[
\left| \int_{\mathbb{R}^d} f_n \phi dx \right| \leq \|\phi\|_{L^1(\mathbb{R}^d)},
\]
we get
\[
\left| \int_{\mathbb{R}^d} \phi d\mu \right| \leq \|\phi\|_{L^1(\mathbb{R}^d)}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).
\]

Therefore in a standard way we can extend $d\mu \in (L^1_{\mathcal{B}})^* = L^\infty_{\mathcal{B}}$. Hence for some $f_\infty \in L^\infty_{\mathcal{B}}(\mathbb{R}^d)$ with $0 \leq f_n(x) \leq 1$, a.e. $x \in \mathbb{R}^d$, we have
\[
T(\phi) = \int_{\mathbb{R}^d} \phi(x)f_\infty(x)dx.
\]
By a density argument, we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f_n \phi dx = \int_{\mathbb{R}^d} f_\infty \phi dx, \quad \forall \phi \in L^1_{\mathcal{B}}(\mathbb{R}^d).
\]
In particular,
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} K(x)f_n(x)dx = \int_{\mathbb{R}^d} K(x)f_\infty(x)dx.
\]
Therefore by (2.28)
\[
\int_{\mathbb{R}^d} K(x)(1 - f_\infty(x))dx = 0,
\]
and obviously $f_\infty(x) = 1$ for a.e. $x \in \mathbb{R}^d$.

Plugging this back into (2.29), we obtain for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,
\[
\int_{\mathbb{R}^d} g \cdot \frac{1 + |\nabla|}{|\nabla|} \phi dx = \int_{\mathbb{R}^d} \phi dx,
\]
or on the Fourier side,
\[
\int_{\mathbb{R}^d} \hat{g}(\xi) \cdot \frac{1 + |\xi|}{|\xi|} \hat{\phi}(\xi) d\xi = \hat{\phi}(0).
\]

From this and the fact that $\hat{g} \in L^2$, it follows easily that $\hat{g}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^d$. This is obviously a contradiction.

We are now ready to complete the
Proof of Theorem 2.2. Denote $g = (1 - \Delta)^{-1}\phi$. Set $r = 0$ in (1.10), we then rewrite it as

$$
\frac{d}{dt} \phi(0, t) = \int_0^\infty \left( (1 - \Delta)g \right)' \Delta g dr
$$

$$
= \int_0^\infty g' \Delta g dr - \int_0^\infty (\Delta g)' \Delta g dr
$$

$$
= \int_0^\infty g'(g'' + \frac{d-1}{r}g') dr + \frac{1}{2} \left( (\Delta g)(0, t) \right)^2. \tag{2.30}
$$

Now note that $g$ is a radial function, so $g'(0, t) = 0$. Therefore we have

$$
\int_0^\infty g'g'' dr = 0.
$$

Therefore we obtain from (2.30) the following inequality

$$
\frac{d}{dt} \phi(0, t) \geq \frac{1}{2} \left( (\Delta g)(0, t) \right)^2
$$

$$
= \frac{1}{2} \left( (\Delta \frac{1}{1 - \Delta}\phi)(0, t) \right)^2. \tag{2.31}
$$

Now by using the energy conservation (1.6) and the relation $u = (1 - \Delta)^{-1}\nabla \phi$, we have

$$
\left\| \frac{\nabla}{1 + |\nabla|} \phi(\cdot, t) \right\|_{L^2_x} \leq C_3 < \infty,
$$

where $C_3$ is some constant independent of $t$.

Note that $\phi(0, t) \geq \phi(0, 0) > 0$. Since we assume the dimension $d \geq 3$, by Lemma 2.5, we have

$$
\left| (\Delta \frac{1}{1 - \Delta}\phi)(0, t) \right| \geq \epsilon_0 \phi(0, t),
$$

where $\epsilon_0 > 0$ is independent of $t$.

Plugging this estimate into (3.34), we obtain

$$
\frac{d}{dt} \phi(0, t) \geq \frac{1}{2} \epsilon_0^2 \phi(0, t)^2
$$

which clearly implies that $\phi(0, t)$ must blow up in finite time. \qed

3. PROOF OF MAIN THEOREMS

3.1. Proof of Theorem 1.3. Write (2.19) as

$$
\frac{d}{dt} \phi(0, t) = \int_0^\infty \phi' \left( (1 - \Delta)^{-1} - 1 \right) \phi ds
$$

$$
= \int_0^\infty \phi' \frac{\Delta}{1 - \Delta} \phi ds. \tag{3.32}
$$
Now denote \( g = (1 - \Delta)^{-1} \phi \). We then rewrite (3.32) as
\[
\frac{d}{dt} \phi(0, t) = \int_0^\infty (1 - \Delta)g' \Delta g dr = \int_0^\infty g' \Delta g dr - \int_0^\infty (\Delta g)' \Delta g dr = \int_0^\infty g'(g'' + \frac{d-1}{r} g') \Delta g dr + \frac{1}{2} \left( (\Delta g)(0, t) \right)^2. \tag{3.33}
\]
Now note that \( g'(0, t) = 0 \) and we have
\[
\int_0^\infty g' g'' dr = 0.
\]
Therefore we obtain from (3.33) the following identity
\[
\frac{d}{dt} \phi(0, t) = (d-1) \int_0^\infty \frac{(g')^2}{r} dr + \frac{1}{2} \left( (\Delta g)(0, t) \right)^2 = (d-1) \int_0^\infty \frac{(g')^2}{r} dr + \frac{1}{2} \left( \frac{\Delta}{1 - \Delta} \phi(0, t) \right)^2 = (d-1) \int_0^\infty \frac{(g')^2}{r} dr + \frac{1}{2} \left( \phi(0, t) - g(0, t) \right)^2. \tag{3.34}
\]
Since \( \phi_0(0) \geq 0 \) and \( \phi_0 \) is not identically zero, we have that for all \( t \geq t_0 \),
\[
\phi(0, t) \geq A_1, \tag{3.35}
\]
where \( t_0 > 0 \) is any fixed time and \( A_1 \) is a constant depending on \( \phi_0 \) and \( t_0 \).

Now let \( R > 1 \) be a parameter whose value will be specified later. Note that by the Fundamental Theorem of Calculus, we have
\[
|g(0, t) - g(R, t)| \leq \int_0^R |g'| dr \leq \left( \int_0^R \frac{(g')^2}{r} dr \right)^{\frac{1}{2}} \cdot R. \tag{3.36}
\]
Then clearly for dimensions \( d \geq 2 \),
\[
\geq \frac{1}{100 R^2} \left( \phi(0, t) - g(0, t) \right) + R \left( \int_0^R \frac{(g')^2}{r} dr \right)^{\frac{1}{2}} \tag{3.37}
\]
\[
\geq \frac{1}{100 R^2} \left( |\phi(0, t) - g(0, t)| + |g(0, t) - g(R, t)| \right)^2 \tag{3.38}
\]
\[
\geq \frac{1}{100 R^2} \left( \phi(0, t) - g(R, t) \right)^2. \tag{3.39}
\]
Now we discuss two cases. Consider first the case dimension \( d \geq 3 \). By radial Sobolev embedding and energy conservation (1.6), we have
\[
|g(R, t)| \leq C_d \| \nabla g \|_2 \cdot R^{-\frac{d-2}{2}} \leq C_d \| u_0 \|_{H^1} \cdot R^{-\frac{d-2}{2}}, \tag{3.40}
\]
where \( C_d \) is constant depending only on the dimension \( d \), and \( u_0 = (1 - \Delta)^{-1} \nabla \phi_0 \) is the initial velocity. By (3.40), we can choose \( R \) sufficiently large such that
\[
|g(R, t)| \leq \frac{1}{100} A_1, \tag{3.41}
\]
where $A_1$ was defined in (3.35). Therefore by (3.39), (3.40), and (3.41), we get for all $t > t_0$,

$$
\phi(0, t) - g(R, t) \geq \frac{1}{2} \phi(0, t).
$$

Plugging this estimate into (3.39), we obtain for $t > t_0$, and some constant $\epsilon_0 > 0$,

$$
\frac{d}{dt} \phi(0, t) \geq \frac{1}{2} \epsilon_0 \phi(0, t)^2
$$

which together with (3.35) clearly implies that $\phi(0, t)$ must blow up in finite time.

This finishes the case $d \geq 3$. Now we turn to the case dimension $d = 2$. We shall choose for each $g(t)$ the time-dependent parameter $R(t) = R_0(1 + t)^{-\frac{1}{2}}$ where $R_0$ will be taken sufficiently large. By (1.11) and radial Sobolev embedding, we have

$$
|g(R(t), t)| \leq C \cdot \|\phi(t)\|_2 \cdot (R(t))^{-\frac{1}{2}}
$$

$$
\leq C \cdot B_1 \cdot R_0^{-\frac{1}{2}}.
$$

Choosing $R_0$ sufficiently large gives us (3.41) and consequently

$$
\frac{d}{dt} \phi(0, t) \geq C \cdot \frac{1}{1 + t} \phi(0, t)^2.
$$

Integrating the above ODE on the interval $[t_0, \tau]$ with $\tau > t_0$, we get

$$
-\frac{1}{\phi(0, \tau)} + \frac{1}{\phi(0, t_0)} \geq \text{const} \cdot \log(1 + \tau).
$$

This implies that $\phi(0, \tau)$ become negative in finite time which obviously contradicts (3.35).

3.2. Proof of Theorem 1.4. By repeating an argument similar to the beginning part of the proof of Theorem 2.1, we have $\phi'(r, t) \geq 0$ for any $r > 0$. Set $\phi = -\psi$. Then by (2.19), we have

$$
\frac{d}{dt} \psi(0, t) = -\frac{\psi(0, t)^2}{2} - \int_0^\infty \psi'(r, t)((1 - \Delta)^{-1}\psi)(r, t)dr.
$$

By a derivation similar to (2.20), we then have for any $t > 0$,

$$
\|\psi(t)\|_{L^2(\mathbb{R}^d)} = \|\phi(t)\|_{L^2(\mathbb{R}^d)} \leq C \cdot (1 + t),
$$

where $C > 0$ depends only on $\phi_0$.

Therefore in place of (2.22), we get

$$
\frac{d}{dt} \psi(0, t) \leq -\frac{\psi(0, t)^2}{4} + C \cdot (1 + t) \cdot \psi(0, t).
$$

Since $\psi(0, 0) \geq 0$, this clearly shows that $\psi(0, t)$ is bounded for all $t > 0$. By using the blowup criteria Lemma 1.2, we conclude that the corresponding classical solution exists for all time $t > 0$. 
3.3. Proof of Corollary 1.5 The monotonicity of \( \phi(0,t) \) follows directly from the proof of Theorem 1.3 (see (3.34)). In particular we know that \( \phi(0,t) < 0 \) for any \( t \geq 0 \) (otherwise the corresponding solution will blow up). It remains to establish the estimates (1.13)–(1.14). By using the same argument as in the proof of Theorem 1.3 we obtain the inequality
\[
\frac{d}{dt} \phi(0,t) \geq \epsilon_0 \phi(0,t)^2, \quad \text{if } d \geq 3,
\]
\[
\frac{d}{dt} \phi(0,t) \geq \frac{\epsilon_1}{1 + t} \phi(0,t)^2, \quad \text{if } d = 2,
\]
where \( \epsilon_0 > 0, \epsilon_1 > 0 \) are some constants. Integrating the above inequality in time gives us the desired results.

3.4. Proof of Theorem 1.7 Let \( \psi_0 \in H^\infty(\mathbb{R}^d) \) be a smooth radial function such that \( \psi_0(0) = 0 \) and
\[
\begin{cases}
\psi'_0(x) \leq 0, & |x| \leq c_1, \\
\psi'_0(x) > 0, & |x| > c_2, \\
\psi_0(x) < 0, & c_1/2 < |x| < 2c_2.
\end{cases}
\]
(3.42)

Here \( 0 < c_1 < c_2 < \infty \) are arbitrary constants.

By local wellposedness theory, there exists \( T_0 > 0 \) and a smooth solution \( \psi = \psi(x,t) \) to (1.1) \((m = \nabla \psi)\) in the space \( C([-T_0, T_0], H^k) \) for any \( k \geq 0 \).

We make the following

Claim: there exists \( t_0 > 0 \) sufficiently small, such that \( \psi(x,-t_0) < 0 \) for any \( x \in \mathbb{R}^d \).

We now assume the claim is true and complete the proof of the theorem. Take \( \phi_0(x) := \psi(x,-t_0) \) for \( x \in \mathbb{R}^d \). Then clearly \( \phi_0(x) \) satisfies all the needed conditions. Furthermore denote the solution corresponding to the data \( \phi_0 \) as \( \phi = \phi(x,t) \). It is obvious that \( \phi(x,t) = \psi(x,t-t_0) \) for any \( t \geq 0 \). In particular we have \( \phi(0,t_0) = 0 \).

By using Theorem 1.3 it follows easily that \( \phi \) must blow up at some \( t > t_0 \). Therefore \( \phi_0 \) is the desired initial data.

It remains for us to prove the claim. Write \( \psi = \psi(r,t) = \psi(x,t) \), where \( r = |x| \). Note that \( \psi \in C^\infty([0,\infty)) \) as a function of \( r \). We can perform an even extension and regard \( \psi \in C^\infty(\mathbb{R}) \).

By (1.9), we have
\[
-\partial_t \psi' = \left( -\psi + (1 - \Delta)^{-1} \psi + (1 - \Delta)^{-1} \psi'' \right) \psi' + (1 - \Delta)^{-1} \psi'' : = c(r,t) \psi' + b(r,t) \cdot \psi''.
\]

Here \( c = c(r,t), b = b(r,t) \) are both smooth functions for \(-T_0 \leq t \leq T_0 \) and \( r \in \mathbb{R} \). Note that \( c \) is an even function and \( b \) is an odd function. Also for some constant \( B > 0 \)
\[
\sup_{|t| \leq T_0} \left( \| b(t,\cdot) \|_\infty + \| \partial_r b(t, \cdot) \|_\infty \right) \leq B < \infty.
\]
(3.43)

Denote \( f(r,t) = \psi'(r,t) \), then \( f(r,t) \) satisfies the transport equation
\[
\partial_t f + b \partial_r f + cf = 0.
\]
(3.44)
Introduce the characteristic lines:
\[
\begin{align*}
\frac{d}{dt}z(t,\alpha) &= \mathbf{b}(z(t,\alpha),t), \\
z(0,\alpha) &= \alpha \in \mathbb{R}.
\end{align*}
\]
For each $-T_0 \leq t \leq T_0$, the map $\alpha \mapsto z(t,\alpha)$ is a smooth diffeomorphism. Furthermore we have the obvious estimate
\[
|z(t,\alpha) - \alpha| \leq tB, \tag{3.45}
\]
where $B > 0$ is the same constant as in (3.43). By integrating (3.44) along the characteristic line, we have
\[
f(z(t,\alpha),t) = f(\alpha,0) \exp \left(-\int_0^t \mathbf{c}(z(s,\alpha),s)ds\right), \quad \forall \alpha \in \mathbb{R}, t \in [-T_0,T_0]. \tag{3.46}
\]

Now take $t_1$ sufficiently small such that (see (3.42) for the definition of the constant $c_1$)
\[
t_1 \leq \min\{\frac{c_1}{8B}, T_0\}.
\]
By (3.45), if $|t| \leq t_1$ and $|z(t,\alpha)| \leq \frac{c_1}{2}$, then obviously $|\alpha| \leq c_1$. By (3.46), (3.42), we conclude that
\[
\psi'(r,t) = f(r,t) \leq 0, \quad \forall |t| \leq t_1, r \leq \frac{c_1}{2}. \tag{3.47}
\]
By using a similar argument, we also obtain
\[
\psi'(r,t) > 0, \quad \forall |t| \leq t_1, r \geq 2c_2, \tag{3.48}
\]
By (3.47) and the fact that $\psi(0,0) = 0$, we obtain $\psi(0,t) < 0$ for all $t \in [-T_0,0)$. It follows from (3.47) that
\[
\psi(r,t) < 0, \quad \forall -t_1 \leq t < 0, r \leq \frac{c_1}{2}. \tag{3.49}
\]
Similarly using the fact that $\psi(\infty,t) = 0$ and (3.48), we obtain
\[
\psi(r,t) < 0, \quad \forall -t_1 \leq t < 0, r \geq 2c_2. \tag{3.50}
\]
By (3.42) and smoothness of the local solution, there exist some $t_2 > 0$ sufficiently small such that
\[
\psi(r,t) < 0, \quad \forall |t| \leq t_2, \frac{c_1}{2} \leq r \leq 2c_2.
\]
Now obviously the claim follows if we take $t_0 = \min\{t_1, t_2\}$.

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