The Representation of Numbers in Quantum Mechanics

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Abstract

Earlier work on modular arithmetic of $k$-ary representations of length $L$ of the natural numbers in quantum mechanics is extended here to $k$-ary representations of all natural numbers, and to integers and rational numbers. Since the length $L$ is indeterminate, representations of states and operators using creation and annihilation operators for bosons and fermions are defined. Emphasis is on definitions and properties of operators corresponding to the basic operations whose properties are given by the axioms for each type of number. The importance of the requirement of efficient implementability for physical models of the axioms is emphasized. Based on this, successor operations for each value of $j$ corresponding to $+k^{j-1}$ are defined. It follows from the efficient implementability of these successors, which is the case for all computers, that implementation of the addition and multiplication operators, which are defined in terms of polynomially many iterations of the successors, should be efficient. This is not the case for definitions based on just the successor for $j = 1$. This is the only successor defined in the usual axioms of arithmetic.

1 Introduction

Quantum computers are of much recent interest mainly because of their ability to implement some algorithms [1, 2] more efficiently than any known classical algorithms. Also the possibility that they can simulate other quantum systems more efficiently than is possible by classical systems [3] is of interest. Quantum robots [4] may also be of interest. These are mobile systems including a quantum computer and ancillary systems that move in and interact with an arbitrary environment of quantum systems.

A central aspect of computation is the fact that the physical states acted on by both quantum and classical computers represent numbers. This raises the question regarding exactly what are the numbers that are supposed to be represented by computer states. The viewpoint usually taken is that one knows intuitively what numbers are and how to interpret the various representations. For example in quantum mechanics the product state $|s\rangle = \otimes_{j=1}^{L} |s(j)\rangle_j$ where $s$ is a function from $1, 2, \ldots, L$ to $0, 1$ is a binary representation of numbers according to

$$s = \sum_{j=1}^{L} s(j)2^{j-1}$$

(1)

where the left hand symbol, $s$, denotes a natural number with no particular representation specified.

Another approach is to characterize numbers as models of the axioms for arithmetic or number theory [5, 6]. Any mathematical or physical system that

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satisfies (or is a model of) the axioms of arithmetic or number theory represents the natural numbers. This description can be extended to other mathematical systems. For example any mathematical or physical system that satisfies appropriate sets of axioms for integers or rational numbers is a representation or model of these types of numbers.\footnote{The intuitive base remains, though, as axioms are set up to reflect the intuitive properties of each type of number.}

This viewpoint will be taken here as it gives a precise method for characterizing the various types of numbers and discussing both mathematical and physical models of the axioms. This viewpoint also emphasizes the close connection between mathematics and physics and the relevance of mathematical logic to the development of a coherent theory of mathematics and physics. The importance of developing a coherent theory of mathematics and physics has been noted elsewhere\footnote{See\cite{15,16} for a description of some implementation conditions for microscopic systems.} and in other work\footnote{\cite{12,13,14}}. Such an approach may also help explain why mathematics is so "unreasonably effective" \cite{10} and why physics is so comprehensible \cite{11}.

A basic assumption made here is that quantum mechanics or some suitable generalization such as quantum field theory is universally applicable. One consequence of this assumption is that both microscopic and macroscopic systems must be described quantum mechanically. This includes both microscopic and macroscopic computers. The fact that macroscopic computers which are in such wide use can be described classically to a very good approximation does not invalidate a quantum mechanical description by use of pure or mixed states, no matter how complex the system may be.\footnote{The axioms of arithmetic, in common with other mathematical axiom systems, make no mention of the requirements of efficiency or implementability. These}

Because of the universality of quantum mechanics, the interest here is in quantum mechanical models of the axiom systems of various types of numbers. The approach taken here differs from that usually taken in that emphasis is placed on the operations and their properties as described by the axioms rather than on the states of a system. For example, the axioms for the natural numbers describe three basic operations, the successor (corresponding to \(+1\)), \(+\), and \(\times\) and their properties. A mathematical or physical system is considered to represent the natural numbers, or be a model of the axioms if the representations of the basic operations satisfy the axioms.

Here the main emphasis is on mathematical models based on quantum systems. The importance of physical models includes the basic requirement that each of the basic operators corresponding to the operations described by the axioms must be efficiently implementable\footnote{Such as those of microscopic quantum systems.}. In brief this requirement means that, for each basic operation \(O\), there must exist a physically implementable quantum dynamics that carries out \(O\) on the number states. The requirement of efficiency means that the space-time and thermodynamic resources needed to implement the operations on states representing a number \(N\) must be polynomial in \(\log_k N\) where \(k \geq 2\). In particular the resources required should not be polynomial in \(N\).

The requirement of efficient implementation is quite restrictive, especially for microscopic quantum systems. Quantum systems for which the basic operations are not efficiently implementable cannot serve as physical models of the axioms being considered. Examples include systems with states corresponding to unary representations of numbers (for which all arithmetic operations are exponentially hard) or in very noisy and chaotic environments.

However it is also the case that there must exist either macroscopic or microscopic physical models of the axioms of arithmetic. In particular, physical models must exist that represent the numbers 0, 1, \cdots, \(N\) where \(N\) is large, and are capable of carrying out arithmetic operations on these numbers. If such models did not exist, even for moderate values of \(N\), it would not be possible to carry out any but the most elementary calculations or to even develop a physical theory of the universe. In a more fundamental sense the requirement that there must exist physical models of the axioms must in some way place restrictions on the basic properties of the physical universe in which we live. That is, in some way it must be related to the strong anthropic principle\footnote{\cite{13,11}}.

The axioms of arithmetic, in common with other mathematical axiom systems, make no mention of the requirements of efficiency or implementability. These
are extra conditions that are essential for the existence of physical models of the axioms \[ 12 \]. They play no role in the existence of mathematical models of the axioms.

It would be desirable to expand these axiom systems to include some aspects of efficient implementability. For example, one problem that is not taken into account so far is that efficient implementability of the plus (\(+\)) and times (\(\times\)) operations does not follow from efficient implementability of the successor operation. In fact, implementation of the \(+\) or \(\times\) operations by iteration of the successor operation is not efficient in that addition or multiplication of two numbers, \(n\), \(m\) requires a number of iterations of the successor operation that is polynomial in \(m\) and \(n\) rather than in \(\log nm\). One can require that each of the three operations are efficiently implementable, but this provides no insight or relation between efficient procedures for the successor operation and the plus and times operations.

This problem is taken into account here by defining many successor operations rather than just one. For natural numbers and integers the successors \(S_j\) for \(j = 1, 2, \cdots\) correspond informally to the addition of \(k^{j-1}\) where \(k\) is an arbitrary integer \(\geq 2\). \(S_1\) is the usual successor of axiomatic arithmetic \[ 3, 4 \]. For rational numbers the indices are extended to negative \(j\) values.

The successors are required to satisfy several properties. The most important one is

\[
S_{j+1} = (S_j)^k
\]

This equation states that one iteration of \(S_{j+1}\) is equivalent to \(k\) iterations of \(S_j\). This makes quite clear why it is not sufficient to require that just \(S_1\) be efficiently implementable. Instead each of the \(S_j\) must be efficiently implementable.

The value of these successor operations is that the \(+\) and \(\times\) operations can be defined in terms of polynomially many iterations of the successor operations. It follows that if the \(S_j\) are efficiently implementable and if any operation consisting of polynomially many iterations of these operations is efficiently implementable, then the \(+\) and \(\times\) operations are efficiently implementable. This condition is in fact satisfied by all classical computers and will have to be satisfied by any quantum computer.

The definition of efficient implementability given earlier applies here. \(S_j\) is efficiently implementable if there exists a physical procedure for implementing \(S_j\) and the space-time and thermodynamic resources needed to carry out the procedure are polynomial in \(j\). The resources needed should not be exponential in \(j\). The fact that this condition applies to infinitely many \(S_j\) is not a problem, because procedures for arbitrarily large \(j\) would be needed only asymptotically. Any operation that is completed in a finite time needs only a finite number of the \(S_j\) to be efficiently implementable.

The axiom systems for the different types of numbers can be extended to all the successor operations. For natural numbers, axioms must be added to describe other conditions that the successors should satisfy. These include Eq. \[ 2 \] and the requirement that for each \(j\) there are many numbers that cannot be obtained by adding \(k^{j-1}\) to some other number. That is \(S_1(x) \neq 0\) and \(y < S_j(0) \rightarrow x \neq S_j(y)\) for all \(x, y\).

Here \(x, y\) are number variables. Other additions give the requirements that for each \(j\) \(S_j(x) = S_j(y) \rightarrow x = y\), \(x + S_j(y) = S_j(x + y)\), \(x \times S_j(y) = x \times y + x \times S_j(0)\), \(S_1(y) = x \rightarrow x \times S_j(0) = y \times S_j(0) + S_j(0)\), and \(x \times S_1(0) = x\). However the discreteness axiom \(x \leq S_1(y) \rightarrow x \leq y \vee x = S_1(y)\) holds only for \(j = 1\).

The other axioms, \[ 3, 4 \] including Peano’s induction axiom, are unchanged.

Here earlier work on modular arithmetic of \(k\)-ary representations of length \(L\) of the natural numbers \[ 12 \] will be extended to include \(k\)-ary representations of all natural numbers, not just those \(< k^L\), and the integers and rational numbers. The procedure followed here will be to give abstract quantum mechanical models of these types of number systems. These serve as a convenient common reference for discussion of physical models just as abstract representations of networks of quantum gates, as in \[ 17, 18 \] do for physical quantum gate networks.

Abstract quantum mechanical models for the natural numbers, the integers, and the rational numbers are discussed in Sections \[ 3, 4, 5 \] and \[ 6 \]. Definitions are given for operators for the successor operations for each type of number system. Operators for \(+\) and
The vacuum state is a state with at most one mode, or particle, associated with each value of the form of finite length. The vacuum state then the states represent single mode excitations of the Fermion or Boson field. These correspond to states of single particles or field systems. Multiple mode field excitations of the form \( s \) refer to such single system properties as different excitation states, spin projections, or polarization properties.

For bosons or fermions the annihilation creation (a-c) operators satisfy commutation or anticommutation relations given by

\[
[a_{\ell,j}^\dagger, a_{\ell',j'}] = \delta_{\ell \ell'} \delta_{jj'} \quad \text{for bosons}
\]

\[
\{a_{\ell,j}^\dagger, a_{\ell,j} \} = \delta_{\ell \ell'} \delta_{jj'} \quad \text{for fermions}
\]

and

\[
[a_{\ell,j}^\dagger, a_{\ell,j}^\dagger] = 0 = \{a_{\ell,j}^\dagger, a_{\ell,j} \} \quad \text{for bosons}
\]

\[
\{a_{\ell,j}^\dagger, a_{\ell,j}^\dagger \} = 0 = \{a_{\ell,j}^\dagger, a_{\ell,j} \} \quad \text{for fermions.}
\]

Here \( \{x, y\} = xy + yx \) and \( [x, y] = xy - yx \).

The basis states of interest have the form

\[
a_{\underline{s}(L), L} a_{\underline{s}(L-1), L-1, \ldots, a_{\underline{s}(1), 1}}^\dagger |0\rangle = a_{\underline{s}}^\dagger |0\rangle \tag{5}
\]

where \( \underline{s} \) is any function from 1, 2, \ldots, \( L \) to 0, 1, \ldots, \( k-1 \) with \( L \), the length of \( \underline{s} \) arbitrary. The convention used here for all states is that the component a-c operators appear in the order of increasing values of \( j \). Linear superpositions of these states have the form \( \psi = \sum_{\underline{s}} c_{\underline{s}} a_{\underline{s}}^\dagger |0\rangle \) where the sum is over all functions \( \underline{s} \) of finite length. The vacuum state \(|0\rangle\) is the state corresponding to the zero length function.

The use of fermion and boson systems to carry out quantum computation and the representation of fermions as products of Pauli operators has been the subject of some discussion in the literature. Here the change of sign associated with permutation of fermion a-c operators does not cause problems in that the order of creation operators in the states \( a_{\underline{s}}^\dagger |0\rangle \), shown in Eq. 5, will be maintained in all states considered here. Also most terms in the operators to be defined either have an even number of a-c operators that act either at the same place or on the lefthand operators in Eq. 5. For those cases where the sign change has an effect, operators for fermions will be defined differently than for bosons.

### 3 The Natural Numbers

#### 3.1 The Successor Operators

The approach taken here is to consider the states \(|\underline{s}\rangle = a_{\underline{s}}^\dagger |0\rangle\) with \( a_{\underline{s}}^\dagger |0\rangle\) given by eq. 5 as candidate natural number states. Operators for the successors,
+ and × will be defined and seen to have the properties specified by the axioms. This shows that, relative to the operator definitions, the above states represent real numbers.

It should be noted that states of the form \( a_j^\dagger |0\rangle \) give a many one representation of the numbers in that states with arbitrary extensions with 0s to the left correspond to the same number as those without. (00134 is the same number as 134). However the main interest here is in states where \( s(L) \neq 0 \); those with \( s(L) = 0 \) will play a role as intermediate states only.

Let \( \mathcal{H} \) be the Hilbert space spanned by all states of this form. Define the operators \( P_{occ,j} = \sum_{h=0}^{k-1} a_{h,j}^\dagger a_{h,j} \) and \( P_{>0,j} = \sum_{h=0}^{k-1} a_{h,j}^\dagger a_{h,j} \). These operators are the number operators for finding a particle in any state \( h \) for a fixed value of \( j \), and in any state \( h \neq 0 \). Since \( \mathcal{H} \), as a subspace of the full Fock space, is defined so that at most one component or particle can have property \( j \) for either fermions or bosons, the eigenvalues of these number operators on \( \mathcal{H} \) are just 0, 1. Because of this they are shown as projection operators. \( P_{unocc,j} = 1 - P_{occ,j} \) is the projection operator for finding the site \( j \) unoccupied.

Based on these definitions the successor operators can be defined for each value of \( j \) as

\[
V_j = N_j Z_j
\]

where

\[
N_j = \sum_{h=0}^{k-1} a_{h+1,j}^\dagger a_{h,j} + a_{j,0}^\dagger a_{j,0} P_{occ,j+1} + N_{j+1} a_{j+1,0}^\dagger a_{j-1,0} + P_{unocc,j+1} a_{j,0}^\dagger P_{unocc,j} + N_{j+1} a_{j+1,0}^\dagger a_{j-1,0} + P_{unocc,j+1} a_{j,0}^\dagger P_{unocc,j}
\]

for \( j \geq 2 \) and

\[
N_1 = \sum_{h=0}^{k-1} a_{h+1,1}^\dagger a_{h,1} + N_2 a_{0,1}^\dagger a_{k-1,1}.
\]

\( Z_j \) is defined as

\[
Z_j = P_{occ,j} + P_{unocc,j} P_{>0,j-1} + Q_{j-1}
\]

for \( j \geq 4 \). \( Z_1 = 1 = Z_2 \) and \( Z_3 \) is obtained from Eq. (6) by deleting the sum terms.

The operator \( V_j \) is a product of two operators \( N_j \) and \( Z_j \). The first three terms of \( N_j \) with the first term of \( Z_j \) act on states \( a_j^\dagger |0\rangle \) where site \( j \) is occupied. That is, \( a_j^\dagger \) includes a creation operator \( a_{h,j}^\dagger \) for some value of \( h \). The first term of \( N_j \) converts \( |h,j\rangle \) to \( |h+1,j\rangle \) if \( 1 \leq h \leq k - 2 \). The second term converts \( |0,j\rangle \) to \( |1,j\rangle \) if site \( j+1 \) is occupied, and the third term converts \( |k-1,j\rangle \) to \( |0,j\rangle \) with the carry one operation shown by the subsequent action of \( N_{j+1} \).

The last term of \( N_j \) with the remaining three terms of \( Z_j \) act on states where site \( j \) is unoccupied. The effect of the three terms of \( Z_j \) acting on \( a_j^\dagger |0\rangle \), where the length, \( L \), of \( g \) is less than \( j \), is to extend \( g \) by adding 0s so that sites \( j - 1 \) are occupied. The action of the last term of \( N_j \) on \( Z_j a_j^\dagger |0\rangle \) is to create a 1 at site \( j \) just to the left of the leftmost 0. As an example, if \( a_j^\dagger |0\rangle = |364\rangle \) and \( j = 7 \), then \( Z_7|364\rangle = |000364\rangle \) and \( N_7 Z_7|364\rangle = |1000364\rangle \).

The definition of \( Z_j \) is explicit and shows the operator to be a many system nonlocal operator. However it can also be defined recursively by

\[
Z_j = P_{occ,j} + P_{unocc,j} P_{>0,j-1} + Q_{j-1}
\]

where

\[
Q_{j-1} = a_{0,j-1}^\dagger (P_{unocc,j-1} P_{>0,j-2} + Q_{j-2})
\]

\[
Q_2 = P_{unocc,3} P_{occ,2} + a_{0,2}^\dagger P_{unocc,2}.
\]

This form shows that \( Z_j \) can also be expressed as a product of local operators. \( N_j \) is already defined in this form although the recursion is in the direction of increasing \( j \). The recursion direction does not cause a problem in that for any state \( a_j^\dagger |0\rangle \) with \( L > j \), there are at most \( L - j + 1 \) recursions with \( N_{L+1} \) being the last one.

The recursive forms of both \( N_j \) and \( Z_j \) show explicitly that these operators, and \( V_j \), are efficiently implementable, relative to that for the local a-c operators. \( Z_j \) is a sum of products of at most \( j + 1 \) local a-c operators with one term in the sum active on each number state of the form \( a_j^\dagger |0\rangle \). As such it can be implemented in polynomially \( j \) many steps as its role is to extend a number string by adding up to
for which states. For states in \( H \) spanned by states of the form \( s_j \), then they correspond to addition of \( k^{j-1} \). In this case the adjoint, \( V_j^\dagger \), corresponds to subtraction of \( k^{j-1} \) on the domain of definition. \( V_j^\dagger = Z_j^\dagger N_j^\dagger \) where

\[
N_j^\dagger = \sum_{h=0}^{k-1} a_{h,j} a_{h+1,j} + P_{occ,j+1} a_{0,j} a_{1,j} + P_{0,j} a_{0,j} N_j^\dagger + P_{unocc,j} a_{1,j} P_{unocc,j+1}
\]

and

\[
Z_j^\dagger = P_{0,j} P_{unocc,j} + P_{occ,j} + \sum_{\ell=2}^{j-2} P_{0,j} P_{unocc,\ell+1} a_{0,\ell+1}, \ldots, a_{0,j-1} + P_{unocc,2} a_{0,2}, \ldots, a_{0,j-1}.
\]

Based on the above it can be seen that, relative to the space \( H^{arith} \), each \( V_j \) is a unilateral shift. That is, \( V_j^\dagger V_j = 1 \) and \( V_j V_j^\dagger = P \) where \( P \) is a projection operator on a subspace of \( H^{arith} \). Also for each state \( a_{0,0}^\dagger = |s_j, V_j^\dagger V_j| = 0 \).

To see that \( V_j^\dagger V_j = Z_j^\dagger N_j Z_j = 1 \) one notes that \( N_j^\dagger N_j = \sum_{h=0}^{k-1} P_{h,j} + P_{occ,j+1} P_{0,j} + P_{0,j} N_{j+1} N_{j+1} + P_{unocc,j} P_{unocc,j+1} \) as only the diagonal terms are nonzero. This shows that \( |s_j N_j^\dagger N_j| = 1 \) for all \( s \) for which \( L = j - 1 \) (including \( s \) for which \( s(L) = 0 \) and \( L \geq j \) and \( s(L) > 0 \). Since \( Z_j \) passes unchanged all states \( |s_j| \) in \( H^{arith} \) for which \( L \geq j - 1 \), one has \( |s_j V_j^\dagger V_j| = 1 \) for these states. For states \( |s_j| \) in \( H^{arith} \) for which \( L < j - 1 \), \( Z_j \) and \( N_j \) are defined so that \( N_j^\dagger N_j Z_j |s_j| = Z_j |s_j| \).

Thus \( |s_j V_j^\dagger V_j| = 1 \) for these states which completes the proof.

Inspection of the terms in \( N_j Z_j \) shows directly that \( |s_j V_j^\dagger| = 0 \) for all \( s \) for which \( L \leq j - 1 \). This shows that \( V_j^\dagger V_j = 0 \) on these states. For all states \( |s_j| \) for which \( L \geq j \), \( V_j^\dagger V_j = N_j N_j^\dagger \). An argument similar to that for \( V_j^\dagger V_j = 1 \) on these states. This completes the proof that \( V_j^\dagger V_j = 1 \) is a unilateral shift.

This result and \( |s_j V_j| = 0 \) show that, if the \( V_j \) and the definitions of \( = \) and \( \times \) operators (Subsections 3.2 and 3.3) satisfy the arithmetic axioms, then the candidate number states do represent numbers. This is the reason for referring, in the foregoing, to the states \( |s_j = a_{0,0}^\dagger| \) as number states.

The most important required property of the \( V_j \) is that given by Eq. 3 or

\[
(V_j)^k = V_j + 1.
\]

To prove this one first notes that \( V_j^h|s_j| = (N_j^h Z_j)|s_j| \) for \( h \geq 1 \). To save on notation let \( V_j^h \), \( N_j^h \) denote \( V_j^h \), \( N_j^h \) respectively. First note that \( V_j^h|s_j| = N_j^h Z_j|s_j| \).

There are several cases to consider. For \( |s_j| \) where \( L < j - 1 \)

\[
Z_j|s_j| = |0_{j-1,L+1} * s_j| = a_{0,j-1}, \ldots, a_{0,L+1} a_{0,0}^\dagger|0,0|.
\]

Here \( * \) denotes concatenation and \( 0_{a,b} \) denotes a string of zeros from \( a \) to \( b \). Iteration of \( N_j \) on \( Z_j|s_j| \) gives

\[
N_j Z_j|s_j| = |1_j * 0_{j-1,L+1} * s_j|
\]

\[
N_j^{k-1} Z_j|s_j| = |k - 1_j * 0_{j-1,L+1} * s_j|
\]

\[
N_j^{k-1} Z_j|s_j| = N_{j+1}|0_j * 0_{j-1,L+1} * s_j|
\]

\[
N_{j+1} Z_j|s_j| = V_j|s_j|.
\]

Use of \( V_j^k|s_j| = N_j^k Z_j|s_j| \) completes the proof for this case.

The case of \( L = j - 1 \) is similar and is left to the reader. For \( L \geq j \) write \( |s_j| = |S_{j-1,1} * \ell_j * S_{j-2,1} \rangle \) where \( 0 \leq \ell \leq k - 1 \). Use of \( Z_j|s_j| = |s_j| \) gives

\[
V_j^{k-\ell}|s_j| = |S_{j-1,1} * k - 1_j * S_{j-1,1}\rangle.
\]
This result gives immediately that $V_j^k|\psi\rangle = V_j^{j+1}|\psi\rangle$ which is the desired result. To obtain this use was made of the fact that $N_j N_{j+1}|\psi\rangle = N_{j+1} N_j|\psi\rangle$. This holds even for fermions because the terms in $N_j$ and $N_{j+1}$ giving an odd number of creation and annihilation operators give zero contribution acting on states $|\psi\rangle$ with $L \geq j + 1$. This completes the proof of Eq. 14 as all cases have been covered.

The definitions given so far allow the representation of any state $|\psi\rangle$ by

$$|\psi\rangle = V_\ell^{(L)} \cdots V_s^{(1)} |\psi\rangle.$$  \hspace{1cm} (15)

This relation is quite useful for proving various properties of the arithmetic operators. It is a special case of addition described next. It also serves as a good illustration of the fact that, even for fermions, the $V$ operators with arbitrary subscripts commute. To see this it is sufficient to consider $V_n V_m$ acting on the state $|0\rangle$ as the argument is the same for other states. Let $n < m$. Use of Eqs. 7 and 8 gives

$$V_n V_m |0\rangle = V_n N_m a_{0,m-1} \cdots a_{0,1} |0\rangle = V_m a_{1,m} a_{0,m-1} \cdots a_{0,1} |0\rangle = V_m a_{1,m} a_{0,m-1} \cdots a_{0,1} |0\rangle.$$  \hspace{1cm} (16)

Commuting the leftmost pair of $a$ operators to $a_{0,n}$ on which they act causes no sign change. This shows that $V_n V_m |0\rangle = |0\rangle = V_m V_n |0\rangle$ where $\alpha(m) = \alpha(n) = 1, \alpha(\ell) = 0$ for all $1 \geq \ell \geq m, \ell \neq n, m$.

### 3.2 Addition

The definition of an addition operator $\widetilde{\dagger}$ is a generalization of Eq. 13. Since $\dagger$ is a binary operator, it acts on pairs of states $|\alpha\rangle \otimes |\beta\rangle = |\alpha, \beta\rangle$. The action of $\dagger$ can be defined by

$$\dagger |\alpha\rangle \otimes |\beta\rangle = |\alpha\rangle \otimes |\alpha + \beta\rangle$$ \hspace{1cm} (16)

where

$$|\alpha + \beta\rangle = V_\ell^{(L)} \cdots V_s^{(1)} |\psi\rangle$$  \hspace{1cm} (17)

As defined, $\dagger$ is an isometry. The property $\dagger \dagger = 1$ follows from the fact that the $V_j$ are unilateral shifts. That $\dagger \dagger^{-1}$ is a projection operator follows from the fact that the adjoint $\dagger^{-1}$, which corresponds to subtraction, is defined on states $|\alpha, \beta\rangle$ only if $|\alpha| \leq |eta|$. This follows from $\dagger^{-1} |\alpha, \beta\rangle = |\alpha, \beta - \alpha\rangle$ where $|\alpha - \alpha\rangle = (V_1^{(1)} \cdots V_s^{(1)})(\alpha)$ This state is defined if and only if all the iterations of the adjoints of the $V_j$ are defined on the states on which they operate.

This argument shows that $\dagger$ is the direct sum of an identity operator and a unilateral shift. It is the identity operator on the subspace spanned by all states of the form $|0\rangle |\beta\rangle$ for any $|\beta\rangle$. It is a unilateral shift on the subspace spanned by all states $|\alpha\rangle |\beta\rangle$ where $|\alpha\rangle \neq |0\rangle$.

### 3.3 Multiplication

A definition of multiplication can be given that is based on successive iterations of addition and a shift operator. The goal of the shift operator $U$ is to shift a state $|\alpha\rangle$ to a state

$$U |\alpha\rangle = |\alpha \ast 0\rangle = a_{1,0}^\dagger a_{2,0}^\dagger \cdots a_{s,0}^\dagger |0\rangle$$ \hspace{1cm} (18)

This corresponds informally to multiplying $\alpha$ by $k$.

This operator consists of two parts: shifting a state and insertion of a 0 at site 1. Because the insertion involves a single creation operator, the operator must be defined differently for fermions than for bosons. This is the first case where this distinction matters; as was noted the definitions of both the successor and addition operators were the same for both boson and fermion states in $\mathcal{H}_{arith}$. The definition of the shift operator $U^i$ for both fermions, $i = f$, and bosons, $i = b$, can be given by

$$U^i = \sum_{j=1}^{\infty} U^i_j P_{\text{unocc},j+1}$$ \hspace{1cm} (19)

where

$$U^i_j = \sum_{h=0}^{k-1} a_{h,j+1}^\dagger a_{h,j}$$
\[ U_1 = \sum_{h=0}^{k-1} a_{h,1}^\dagger a_{h,0}^\dagger a_h. \] (20)

Here \( sg(i) = -1 \) if \( i = f \) and \( sg(i) = +1 \) if \( i = b \). \( P_{\text{unocc},j} = 1 - P_{\text{occ},j} = 1 - \sum_{h=0}^{k-1} a_{h,j}^\dagger a_{h,j} \) is the projection operator for finding no component system at site \( j \). \( P_{\text{occ},j} \) is the site \( j \) number operator. It is a projection operator on \( \mathcal{H}^{\text{arith}} \) as the only possible eigenvalues are 0 and 1 for both boson and fermion states. (Recall the definition of \( \mathcal{H}^{\text{arith}} \) as the space spanned by all states of the form \(|s⟩\) with \( s(L) > 0 \) if \( L > 1 \).

The presence of \( P_{\text{unocc},j} \) in the definition gives the result that \( U_i|s⟩ = U_i^j|s⟩ \) where \( i \) denotes either \( f \) or \( b \). For fermions the presence of the minus sign in Eq. 20 means that when \( U_1 \) becomes active on the state \( a_{2(L),L+1} \cdots a_{2(2),2}^\dagger a_{1(1),1}^\dagger |0⟩ \) it is multiplied by a factor of \((-1)^{L-1}\). There are \( 3(L-1) \) commutations of the three operators in \( U_1 \) to their action on \( \cdots a_{2(1),1}^\dagger |0⟩ \) which gives a total factor of \((-1)^{L-1}\) which is positive for any \( L \). This shows that Eq. 20 is satisfied for any \(|s⟩\) for either bosons or fermions.

As defined, \( U_i \) is an isometry. That is, in \( \mathcal{H}^{\text{arith}} \) \( (U_i^\dagger)^\dagger U_i = 1 \) but \( U_i^\dagger U_i^\dagger = P_{0,1} P_{\text{occ},2} \). Here \( P_{0,1} \) is the projection operator on all states \(|s⟩\) such that \( s(1) = 0 \). This follows from the fact that \( (U_i^\dagger)|s⟩ = \cdots (U_1)^{\text{dagger}} |s⟩ = \cdots \sum_{h=0}^{3} a_{h,1}^\dagger a_{h,0}^\dagger a_{h,0} s |0⟩ = 0 \) unless \( s(1) = 0 \).

The multiplication operator \( \times \) is defined on triples of states by

\[ \times |s⟩ ⊗ |t⟩ ⊗ |x⟩ = |s⟩ ⊗ |t⟩ ⊗ |x + s × t⟩. \] (21)

Informally the operation multiplies \( s \) and \( t \) and adds the result to \( x \). Pure multiplication occurs when \(|s⟩ = a_{0,1}^\dagger |0⟩ \).

The operator \( \times \) is expressed in terms of \( U_i \) and \( + \)

\[ \times |s, t, x⟩ = \cdots \sum_{h=0}^{k} a_{h,1}^\dagger a_{h,0}^\dagger a_h |s, t, x⟩ \] (22)

In this equation \( i = f, b \) and the subscripts 2, 3 on \( +_{2,3} \) and 2 on \( U_2^\dagger \) show that \( +_{2,3} = 1 + δ \) and \( U_2^\dagger = 1 \otimes U^i \otimes 1 \). Also \(|s⟩ ⊗ |t⟩ ⊗ |x⟩ = |s, t, x⟩ \).

3.4 The Arithmetic Axioms

It is easy to show, based on the properties given above, that the operators \( V_j \), \( + \), and \( × \) satisfy the arithmetic axioms for the successors and plus. That \(|0⟩\) is the additive identity follows from Eqs. 16 and 17 and is expressed in Eq. 3. Eq. 4 has already been proved. Also \(|1⟩\) is the multiplicative identity, as can be seen from \( × |1⟩ ⊗ |1⟩ ⊗ |0⟩ = + |1⟩ ⊗ |1⟩ ⊗ |0⟩ = |1⟩ ⊗ |1⟩ ⊗ |0⟩ \). The commutativity of the \( V_j \) and \( + \), or \(+ |s⟩ \otimes V_j |d⟩ = (1 \otimes V_j) (|s⟩ ⊗ |d⟩) \) follows from the definitions of the operators involved.

Again there are no problems even for fermions because any terms with an odd number of annihilation or creation operators which give a nonzero contribution undergo an even number of permutations to arrive at the point of action (i.e. where the delta functions of Eqs. 3 and 4 apply).

Proofs of the other two axioms, \( × (V_j \otimes 1 \otimes 1) |s⟩ \otimes |1⟩ \otimes |y⟩ = |s⟩ \otimes |1⟩ \otimes |y⟩ \) where \(|y⟩ = |x + s \otimes t + U^i \otimes 1⟩ \), and distributivity of multiplication over addition, are discussed in the Appendix.

4 The Integers

As is well known the integers correspond to positive and negative natural numbers. A suitable set of axioms can be obtained by replacing the arithmetic axiom 0 \( \neq S(x) \) (0 is not a successor of any element) by

\( \forall x \exists y (x = S(y)) \) (every element is a successor).
Also an axiom stating the existence of an inverse to addition is needed. As was done for the natural numbers extension of these axioms to include all the successors \( S_1, S_2, \cdots \) is needed. Integers also satisfy the axioms for a commutative ring with identity. Here integers will be represented by states of the form

\[
|a\rangle = a_{+L+1}^a|0\rangle
\]

where \( a_{+L+1}^a|0\rangle \) is the same definition as was used for the natural numbers. By convention the integer 0 will be represented by the positive version only, or \( a_{+2}^a|0\rangle = |+0\rangle \). The Hilbert space of interest, \( \mathcal{H}^I \) is spanned by all states \( |\pm a\rangle \) where the state \( |+0\rangle \). This is a subspace of a space that includes states of the form \( |\pm a\rangle \) where \( |0\rangle \) is a possible.

As was the case for the natural numbers this definition is valid for either bosons or fermions. In the latter case the order of creation operators appearing in Eq. 23 that mirrors the ordering of the site labels for the component systems (with the sign component at the end), is taken to be fixed.

4.1 The Successor Operators

As was the case for the natural numbers, successor operators \( I_j \) are defined, one for each \( j = 1, 2, \cdots \), that are to correspond to addition of \( k^j-1 \). \( I_j \) consists of three components, one for the nonnegative integers and two for the negative integers separated on the basis of whether a sign change does or does not occur. To this end define the projection operators

\[
P_+ = \sum_{j=2}^{\infty} a_{+j}^a a_{+j},
\]

\[
P_{-\geq j} = \sum_{h=j+1}^{\infty} a_{-h}^a a_{-h},
\]

\[
P_{-<j} = \sum_{h=2}^{j} a_{-h}^a a_{-h}.
\]

These are defined as number operators. On \( \mathcal{H}^I \) they are projection operators as 0,1 are the only possible eigenvalues. Note that the subscripts \( \geq, < \) refer to the sites of the single digit number operators and do not include the sites of the signs. A sign change operator \( W \) is defined as

\[
W = \sum_{j=1}^{\infty} (a_{+j}^a a_{-j} + a_{-j}^a a_{+j}).
\]

\( W \) is unitary and \( W^2 = 1 \).

The successor operation \( I_j \) on the Hilbert space \( \mathcal{H}^I \) can be separated into two operators as

\[
I_j = I_j^+ I_j^- = I_j^+ + I_{<j}^+ I_{>j}^- + I_{>j}^+ I_{<j}^-
\]

where \( I_j^+ \) and \( I_j^- \) are defined on \( \mathcal{H}^{I+} \) and \( \mathcal{H}^{I-} \), the spaces of nonnegative and negative integer states respectively. \( I_{<j}^+ \) and \( I_{<j}^- \) are defined on the subspace \( P_{-<j}^+ \mathcal{H}^{I-} \) and \( I_{<j}^- \) is defined on \( P_{-<j}^- \mathcal{H}^{I-} \). The action of \( I_{<j}^+ \) takes negative number states into positive number states. The sign is unchanged by the action of \( I_{<j}^- \). Informally these two correspond to the addition of \( k^j-1 \) to negative numbers whose absolute value is \( < k^j-1 \) and \( \geq k^j-1 \) respectively.

The definitions of the \( I_j^+ \) are quite similar to those for the natural numbers. Corresponding to Eqs. 33 and 34 one has

\[
I_j^+ = K_j^+ Z_j^-
\]

where

\[
K_j^+ = \sum_{h=1}^{j} a_{h+1}^a a_{h,j} + a_{1,j}^a a_{0,j} P_{nocc,j+1} + a_{j+1,j}^a a_{1,j}^a a_{0,j} P_{nocc,j+1}
\]

for \( j \geq 2 \) and

\[
K_1^+ = \sum_{h=0}^{k-2} a_{h+1}^a a_{h,1} + a_{1,1}^a a_{0,1} a_{k-1,1}.
\]

For \( j \geq 4 \) \( Z_j \) is defined by

\[
Z_j^+ = P_{nocc,j} a_{+j} P_{>0,j-1} + P_{nocc,j} P_+ + \sum_{\ell=2}^{j-2} a_{0,j-1}^a a_{+\ell+1} P_{>0,\ell} + a_{0,j-1}^a a_{+2}^a a_{+2}^a.
\]
$Z_1 = P_{+}, Z_2 = P_{\text{occ},2}P_{+} + a_{+,2},$ and $Z_3$ is given by Eq. 35 by deleting the sum terms. $P_{\text{occ},j+1} = \sum_{k=0}^{j}a_{h,j+1}a_{h,j+1}$ is the projection operator for site $j+1$ occupied by a system in a single digit number state (not in a sign state). $P_{\text{unocc},j}$ is the projection operator for site $j$ to be unoccupied by a system in any state.

The operators $K_j^+$ and $Z_j^+$ serve the same function on the the Hilbert space $\mathcal{H}^+$ of nonnegative integer states as do the operators $N_j$ and $Z_j$ on $\mathcal{H}$ for the natural number states. As was done for $Z_3$ in Eq. 11, $Z_j^+$ can also be expressed recursively.

One can also prove that $I_j^+$ is a unilateral shift on $\mathcal{H}^+$ and that

$$I_j^{+1} = (I_j^+)^k.$$  (31)

The proofs will not be given here as they are quite similar to those for $V_j$ given earlier. It is also clear that, corresponding to Eq. 35 one has

$$|+\rangle = (I_L^+)\mathcal{U}(I_{L-1}^+)\cdots(I_1^+)\mathcal{U}(1) + 0 = I_1^+ + 0.$$  (32)

The adjoint $(I_j^+)^\dagger$ is given by $(I_j^+)^\dagger = (Z_j^+)^\dagger(K_j^+)^\dagger$ where

$$(K_j^+)^\dagger = \sum_{h=0}^{j-1}a_{h,j}a_{h,1,j} + P_{\text{occ},j+1}a_{0,j}a_{1,j} + a_{1,j}a_{0,j}(K_{j+1}^+) + P_{\text{unocc},j+1}a_{1,j}d_{j,1}.$$  (33)

for $j \geq 2$ and

$$(K_j^+)^\dagger = \sum_{h=0}^{j-2}a_{h,j}a_{h,1,1} + a_{1,j}a_{1,0,1}(K_2^+) + 0.$$  (34)

Also

$$(Z_j^+)^\dagger = P_{>0,j-1}a_{+,j+1,j}P_{\text{unocc},j} + P_{\text{occ},j}P_{+} + \sum_{l=0}^{j-2}P_{>0,l}a_{+,l+1}a_{0,l+1}a_{1,j} + \sum_{l=0}^{j-2}P_{>0,l}a_{+,l+1}a_{0,l+1}a_{1,j} + a_{+,j}a_{0,j}. $$  (35)

$$(Z_2^+)^\dagger = P_{+,2}a_{+,2}a_{0,2}a_{0,0}. $$  (35)

In this equation $P_{>0,j} = | + \rangle \langle + |$ is the projection operator on the state $|+\rangle$. $(I_{L-1}^+)\mathcal{U}(I_{L-2}^+)\cdots(I_1^+)\mathcal{U}(1)$ gives the state corresponding to addition of $k^j$ to 0 and subtracting $\mathcal{U}$. The sum is over all $\mathcal{U}$ whose length $L$ (excluding the sign) is less than $j$ and for which $\mathcal{U}(L) > 0$.

It is straightforward to see that $I_j$, defined by Eq. 25 is a bilateral shift. $I_j^+I_j = 1$ follows from the result that

$$I_j^+I_j = (I_j^+)^\dagger = (I_j^+)^\dagger + (I_j^-)^\dagger + (I_j^-)^\dagger I_j^- = P_{+} + P_{-} > 0 + P_{-,j-1} = 1.$$  (36)

Here Eqs. 25, 37, and 38 have been used. The sum of the projection operators gives the identity on $\mathcal{H}$. In a similar fashion one can show that $I_j^+I_j = 1.\\ (\pm \mathcal{S})I_j = 0$ for all states $|+\rangle$ and $|-\rangle$ follows directly from the definition of $I_j$.

$I_j$ also satisfies

$$I_{j+1} = (I_j)^k.$$  (39)
The definition of addition of integers is similar to that for natural numbers. In essence it is a generalization of Eq. [71]. One has

\[ (I_j)^{i} I_{j+n} = (I_j)^{i-1} \cdots (I_{j+n-1})^{i-1}. \]  

(40)

Corresponding to Eq. [32] one has

\[ | \pm s \rangle = (I_L)^{\pm (L^\dagger)} (I_{L-1})^{\pm (L-1)^\dagger} \cdots (I_1)^{\pm (1)^\dagger} + 0 \]

\[ = I_{\pm s} | \pm L \rangle. \]  

(41)

Here either the plus sign or the minus sign holds throughout. Also \((I_h)^{-1} = I_h\).

4.2 Integer Addition and Multiplication

The definition of addition of integers is similar to that given for natural numbers. In essence it is a generalization of Eq. [71]. One has \( \tilde{+} \)

\[ \tilde{+} | \pm s \rangle \otimes | \pm l \rangle = | \pm s \rangle \otimes | \pm s \pm l \rangle \]  

(42)

where

\[ | \pm l \pm s \rangle = (I_L)^{\pm (L^\dagger)} \cdots (I_1)^{\pm (1^\dagger)} | \pm l \rangle \]

\[ = I_{\pm s} | \pm l \rangle. \]  

(43)

One sees from the definitions, including Eq. [13], that \( \tilde{+} \) has the correct sign properties. Acting on the state \(| - s \rangle | \pm l \rangle\), the negative exponents in the above show that the action of \( \tilde{+} \) corresponds to a subtraction of \(| + s \rangle\) from \(| \pm l \rangle\).

\( \tilde{+} \) has been defined so that it is unitary: \( \tilde{+}^\dagger \tilde{+} = 1 = \tilde{+} \tilde{+}^\dagger \). Here \( \tilde{+}^\dagger \) corresponds to the subtraction operation on integers. Note that \( \tilde{+}^\dagger | - s \rangle | \pm l \rangle = | \pm s \pm l \rangle \) and that \( | \pm l \rangle = (I_L)^{\pm (L^\dagger)} \cdots (I_1)^{\pm (1^\dagger)} | \pm l \rangle \)

\[ = I_{\pm s} | \pm l \rangle. \]  

(44)

Since \( (I_j)^{\dagger -} = (I_j)^{-} \) and the various \( I_j \) factors can be applied in any order, this expresses the fact that subtraction of \(| - s \rangle\) corresponds to addition of \(| + s \rangle\).

The definition of multiplication for the natural numbers, Eqs. [21] and [22] can be taken over to describe integer multiplication:

\[ \tilde{\times} | \pm s, \pm L, \pm x \rangle = | \pm s, \pm L, \pm x + (\pm s \times \pm L) \rangle. \]  

(45)

More explicitly one has

\[ \tilde{\times} | \pm s, \pm L, \pm x \rangle = (U_2^{i,1} U_2^{f,1} U_2^{b,1}) U_2^{i,1} \times (U_2^{f,1} U_2^{b,1}) U_2^{i,1} \cdots U_2^{i,1} U_2^{f,1} U_2^{b,1} | \pm s, \pm L, \pm x \rangle. \]  

(46)

Here the main change in the definition is that \( \tilde{+}, \tilde{\times} \) corresponds to integer addition given by Eq. [42]. Also \( U_2^{f,1}, U_2^{b,1} \) is defined slightly differently than for the natural numbers. Eq. [19] is replaced by

\[ U_2^{i,1} = \sum_{j=1}^{\infty} U_2^{i,1} a_{L,j-1} a_{L,j-1} \]  

(47)

with the definition of \( U_2^{i,1} \) unchanged and given by Eq. [20] for \( i = f, b \). The change shown above shifts the sign qubits before the numeral qubits are shifted. As was the case for the natural numbers, \( U_2^{i,1} \) is an isometry.

It is straightforward to show that the operator \( \tilde{\times} \) is unitary. This follows from the results that \( \tilde{+}, \tilde{\times} \) is unitary and that \( U_2^{f,1} U_2^{b,1} \tilde{\times} \tilde{+} \) always acts on states on which this operator is the identity. It does not follow from this that division is defined as it is not the inverse of \( \tilde{\times} \). A correct definition of a division operator \( \tilde{\div} \) would have to satisfy the requirement that for each pair of states \(| \pm s \rangle, | \pm x \rangle\) there is a unique state \(| \pm l \rangle \) such that

\[ \tilde{\div} | \pm s, \pm L, \pm x \rangle = | \pm s, \pm L, \pm (0 \rangle. \]  

(48)

4.3 The Integer Axioms

The proofs that the operators \( I_j, \tilde{+}, \tilde{\times} \) satisfy the axioms for integers is quite similar to those for the natural numbers and will not be repeated here. The proof that every element is a \( j \)-successor, or for each integer state \(| \pm x \rangle\) there is an integer state \(| \pm y \rangle\) such that \( I_j | \pm y \rangle = | \pm x \rangle\) follows from the fact that \( I_j \) is a bilateral shift where \( | \pm y \rangle = I_j | \pm x \rangle\). Here \(| \pm x \rangle, | \pm y \rangle\) denote states of the form \(| \pm s \rangle, | \pm L \rangle\) with the sign included. The existence of an additive inverse follows immediately from the unitarity of \( \tilde{+} \).

The proof that the various ring axioms are satisfied is straightforward. It is of interest to note that proof of the commutativity and associativity of addition and multiplication for the operators implies the
corresponding properties for the numbers appearing in the exponents. This property, which was noted before, is a consequence of the string character or tensor product representation of the integers.

For example to prove that $|\pm a + \pm b| = |\pm a + \pm b|$ one uses Eqs. (12, 14) and the commutativity of the $I$ operators to obtain

$$|\pm a + \pm b| = |\pm a| \otimes (I_{L_a})^{\pm a(L_a)} \cdots (I_1)^{\pm a(1)}|\pm b|$$

$$= |\pm a| \otimes (I_{L_a})^{\pm a(L_a)} \cdots (I_{L_t+1})^{\pm a(L_t+1)}$$

$$(I_{L_t})^{\pm a(L_t)} \cdots (I_1)^{\pm a(1)} + |\pm b|$$

Here $L_t \leq L_a$ has been used. One now uses the commutativity of the numbers in the exponents to set $(I_j)^{\pm a(j)} = (I_j)^{\pm a(j)}$ for $1 \leq j \leq L_t$ and write

$$(I_{L_t})^{\pm a(L_t)} \cdots (I_1)^{\pm a(1)}|\pm b| = (I_{L_t})^{\pm a(L_t)} \cdots (I_1)^{\pm a(1)}$$

which proves commutativity.

A similar situation exists for associativity. The proof of $|\pm a + (\pm b + \pm c)| = |(\pm a + \pm b) + \pm c|$ uses the equality $(I_j)^{\pm a(j)} + \{(I_j)^{\pm b(j)} + (I_j)^{\pm c(j)}\} = \{(I_j)^{\pm b(j)} + (I_j)^{\pm c(j)}\} + (I_j)^{\pm a(j)}$. Proofs for commutativity and associativity for multiplication are more involved because of the relative complexity of the definition of the $\otimes$ operator. However the same ideas apply. These will be discussed more later on.

5 Rational Numbers

As is well known the rational numbers correspond to equivalence classes of ordered pairs of integers. Usually the class is represented by the one ordered pair $\{p, q\}$ where $p$ and $q$ are relatively prime and the rational number is represented in the form $p/q$. Rational numbers are also axiomatizable by the field axioms. These are the axioms for a commutative ring with identity plus the axiom stating the existence of a multiplicative inverse (22).

The representation of rational numbers as pairs of integers has the disadvantage that the multiplication and especially the addition operations are rather opaque and unrelated to simple physical operations. Also they are not the representation used in computers that operate on single strings of symbols as rational approximations to real numbers.

In particular, the sum of the two rational numbers $(a, b)$ and $(c, d)$ where $a, b, c, d$ are integers is the rational number $(a \times d + c \times b, b \times d) = [a \times d + c \times b]/[b \times d]$. Efficient implementation of this operation is possible, as it is based on efficient implementation of addition and multiplication of the integers. However, the use of this fairly complex combination of integer addition and multiplication to represent a basic operation of addition of rational numbers, which is simple for the string representation, is one reason the integer pair representation is not used. Also the string representation is well suited to describe rational number approximations to real numbers.

For these reasons the description of integers as tensor product states over the sites $j = 1, 2, \cdots$ will be extended here to tensor product states over the sites $j = \cdots, -1, 0, 1, \cdots$. This description has the advantage that the basic successor and addition operations already defined can be easily adapted. Also elementary multiplication operations corresponding to physical shifts are easy to define and are physically relatively easy to implement.

This representation has the obvious disadvantage that many rational numbers as infinite repeating "$k-al's" are only approximately represented. Only those rational numbers $p/q$ where all prime factors of $q$ are also factors of $k$ can be represented exactly as finite tensor product states. In spite of this the importance of the requirement of efficient physical implementation and the fact that these are used in computations as rational approximations to real numbers outweighs the disadvantages.

The corresponding tensor product states in Fock space $|\pm \nu\rangle$ have the form

$$|\pm \nu\rangle = a_{\pm, n+1}^\dagger a_{n}^\dagger \cdots a_{1,1}^\dagger a_{1}^\dagger a_{-1, -1}^\dagger \cdots a_{-m, -m}^\dagger |0\rangle.$$ (49)

Here $\nu$ is a function from the interval $[n, -m]$ to $0, 1, \cdots, k - 1$ with $\nu(0) = \nu$, the "$k-al" point. It is sometimes convenient to represent the state
\[ |\pm \ell_L \rangle = |\pm \ell_R \rangle = a_{\pm, Ls}^\dagger a_{\ell, Ls}^\dagger \cdots a_{\pm, L_1}^\dagger a_{\ell, L_1}^\dagger \cdots a_{\pm, (L_i)}^\dagger a_{\ell, (L_i)}^\dagger \cdots \cdots a_{\pm, (L_s)}^\dagger a_{\ell, (L_s)}^\dagger |0\rangle. \tag{50} \]

Here \( \ell_L, \ell_R \) are as defined before with \( L_s \) and \( L_\ell \) the lengths of \( \ell_L \) and \( \ell_R \).

As was the case for integers and natural numbers, states with leading or trailing strings of zeros will be excluded even though they represent the same rational number. To this end the Hilbert space \( \mathcal{H}_{Ra} \) of rational number states is the subspace of Fock space spanned by states of the form \( |\pm \ell_L \rangle \) where \( \ell_L(L_s) > 0 \) if \( L_s > 1 \) and \( \ell_L(-L_\ell) > 0 \) if \( L_\ell > 1 \). \( \mathcal{H}_{Ra} \) also includes the state \( |0,0\rangle = a_{\pm, 0}^\dagger a_{\ell, 0}^\dagger a_{\ell, 0}^\dagger a_{\ell, 0}^\dagger |0\rangle \) which represents the number 0. Properties of operators for basic operations will be defined relative to this space.

Here the component systems associated with each site are taken to be either bosons or fermions of the same type. Thus for each site \( j \) the there must be \( k+3 \) states available to the boson or fermion as there are the states \([+, j], [-, j], [, j] \) as well as the \( k \) number states available to each system. If desired, one can construct a representation using fermions or bosons of different types for the sign and "k-al" point states.

Also, as was the case for the natural numbers and integers, there are no problems here with the anticommutation relations for fermions provided the ordering shown in Eqs. 49 and 50 is preserved. The operators will be defined so that they do not generate any sign changes for fermion states.

### 5.1 The Successor Operators

As was the case for the integers, successor operators, \( R_j \), can be defined for rational numbers. It is quite useful to follow the definition of \( I_j \) and split the definition of \( R_j \) into two cases:

\[ R_j = R^+_j + R^-_j = R^+_j + R^-_j + R^<_j + R^>_{-j}. \tag{51} \]

Here \( R^+_j \) and \( R^-_j \) act on the subspaces \( \mathcal{H}^{Ra+} \) and \( \mathcal{H}^{Ra-} \) corresponding to the subspaces of positive and negative rational number states respectively. For \( j > 0 \) \( j < 0 \) these operators correspond informally to the addition of \( k^j-1 \) \([k^j]\). Also \( R^+_j = R^+_j P_+ \), \( R^-_j = R^-_j P_- \), and \( R^<_j = R^<_{-j} P_{0, \geq j} \). The projection operators \( P_+ \) and \( P_- \) are given by Eq. 24 and

\[
P_{0, \geq j} = \sum_{\ell=j}^{\infty} a_{h, \ell}^\dagger a_{h, \ell}; \tag{52} \]

These definitions are set up so that \( R^<_{-j} \) adds \( k^j \) to negative numbers whose magnitude is \( \geq k^j \), and \( R^>_{-j} \) adds \( k^j \) to negative numbers whose magnitude is \( < k^j \). In this last case the sign of the rational number is changed.

Two cases need to be considered: \( j > 0 \) and \( j < 0 \). For \( j > 0 \) it is clear that

\[ R_j = I_j, \quad R^+_j = I^+_j \quad R^-_j = I^-_j, \quad R^<_j = I^<_{-j}. \tag{53} \]

The reason the definitions are the same for rational numbers and integers is that for \( j > 0 \) the actions of \( R_j \) are insensitive to the presence or absence of component systems at sites \( < 1 \).

For \( j < 0 \), definitions of \( R^+_j \), \( R^<_{-j} \), \( R^>_{-j} \), can be given that are similar to those given for the corresponding \( I \) components. For \( R^+_j \) one has

\[ R^+_j = \Gamma^+_j Y^+_j \tag{54} \]

where

\[ \Gamma^+_j = \sum_{h=0}^{k-2} a_{h+1, j}^\dagger a_{h, j} + \Gamma^+_j a_{j,1}^\dagger a_{k-1,j} \]

\[ Y^+_j = P_{occ,j} + Y_{j+1}^+ a_{0,j}^\dagger P_{unocc,j}. \tag{55} \]

These equations are valid for \( j \leq -2 \) for \( \Gamma^+_j \) and \( Y^+_j \). For \( j = -1, Y^+_j = 1 \) and \( \Gamma^+_j \) is given by Eq. 25 with \( K^+_1 \) (Eq. 29) replacing \( \Gamma^+_0 \) in the definition. \( Y^+_j \) acts only on those states \([+ \ell_R \rangle \) where \( j < -L_\ell - 1 \) by adding a string of 0s to the right, as in adding \( 10^{-7} \) to 63.04. Otherwise it is the identity.

The definition of \( \Gamma^+_j \) is valid for both boson and fermion systems. This follows from the fact that all terms contain an even number of annihilation and
creation (a-c) operators with the result that anticommuting terms past other such operators to the point of action does not generate a sign change. This is not the case for \( Y^+_j \) for states in which this operator is active. These consist of states \(| + \mathbf{s} \rangle \) for which \(|j| > L_t \). (Recall that \( j < 0 \).) The problem here is that for many states \(| + \mathbf{s} \rangle \) moving \( Y^+_j \) to its point of action requires anticommuting an odd number of a-c operators past an odd number of a-c operators describing the state, giving a sign change.

This can be avoided by redefining \( Y^+_j \) for the fermion case to be

\[
Y^+_j = P_{\text{occ},j} - Y^+_{j+1} a^\dagger_{0,j} P_{\text{unocc},j}(\sum_{\ell=2}^{\infty}(-1)^\ell P_{\ell,\ell}) \times \sum_{m=-j}^{1}(-1)^{m+1}P_{m,m}P_{\text{unocc},m-1}) \tag{56}
\]

where \( P_{\ell,\ell} = a^\dagger_{\ell,\ell}a_{\ell,\ell} \). To see that sign changes are avoided one has

\[
Y^+_j + | + \mathbf{s} \rangle = (-1)^{L_j+2L_t+1} Y^+_{j+1} a^\dagger_{0,j} P_{\text{unocc},j} + | + \mathbf{s} \rangle
\]

where

\[
| + \mathbf{s} \rangle = a^\dagger_{+L_x+1}a^\dagger_{L_x} \cdots a^\dagger_{2}a^\dagger_{1}a^\dagger_{0} |0\rangle.
\]

Since the exponent of \(-1\) is even, this shows that no sign change occurs. Iterative application of \( Y^+_j \), etc., causes no sign change because the added operators all stand to the left of \( a^\dagger_{0,j} \) in order of increasing \( j \).

For \( R^-_{<j} \) one has a result similar to Eq. 33:

\[
R^-_{<j} = (P_{+0,0} + WP_{\neq 0,2})(R^+_j)WP_{\neq 0,2} \tag{57}
\]

where \( P_{\neq 0,2} = \sum_{\ell=2}^{\infty} \sum_{h=1}^{\infty} a^\dagger_{h,\ell}a_{h,\ell} \) is the projection operator for finding a qubyte in state \(|h,\ell\rangle\) with \( h \neq 0 \) and \( \ell \geq j \). \( P_- = \sum_{\ell=2}^{\infty} a^\dagger_{h,\ell}a_{h,\ell} \) is the projection operator on all negative rational number states. \( W \) is the sign change operator of Eq. 25. This equation is based on the fact that for all \( j \) \( (R^+_j) \) corresponds to subtraction of \( k^j-1 \) if \( j > 0 \) and of \( k^j \) if \( j < 0 \) over its domain of definition. It expresses the correspondence \( k^2 - x.xxxxx = -(x.xxxxx - k^2) \) for the case that \( x.xxxxx \geq k^2 \).

For \( R^-_{<j} \) for \( j < 0 \) one has an equation similar to Eq. 38:

\[
R^-_{<j} = \sum_{\ell=2}^{\infty} (R^+_1)^{\ell} R^+_{j} WP_{\ell,\ell} P_{\neq 0,2} \tag{58}
\]

This equation is based on the result that, as was the case for the integers, one sees that any rational number state \(|+\mathbf{L}\rangle\) can be written in the form

\[
| + \mathbf{L} \rangle = R^+_1 | + 0 \rangle \tag{59}
\]

where

\[
R^+_1 = (R^+_1)^{|L_1|}(R^+_2)^{|L_2|}(R^+_L)^{|L_1|} \cdots (R^+_L)^{|L_1|} \tag{60}
\]

This shows that for any state \(| - \mathbf{L},(R^+_1)^{|W|-0 \mathbf{L}} = | + 0 \rangle \). Application of \((R^+_1)^{\ell} R^+_j \) to this state gives the positive rational number state corresponding to the rational number \( k^j - t \). This sequence of operations is expressed by Eq. 38. The projection operator \( P_{\neq 0,2} \), in effect, limits the \( \mathbf{L} \) sum to states for which \( \ell(t) = 0 \) for \(-1 \geq \ell \geq j \).

The operator \( R^+_j \) has the same properties as \( I_j \) in that it is a bilateral shift, \( (R^+_j)^{1} R^+_j = 1 \), etc., and all \( a_{0,j} \) correspond to \( (R^+_j)^{1} \) to the state gives the positive rational number state corresponding to the rational number \( k^j - t \). This sequence of operations is expressed by Eq. 38. The projection operator \( P_{\neq 0,2} \), in effect, limits the \( \mathbf{L} \) sum to states for which \( \ell(t) = 0 \) for \(-1 \geq \ell \geq j \).

For positive values of \( j \) these results are immediate as \( R^+_j = I_j \) and these properties have already been proved for \( I_j \). Since all negative values of \( j \) the proof should be essentially the same as that for the positive values of \( j \) as the form and action of the operators \( R^-_{<j} \) is essentially the same as that for the corresponding integer operators.

From Eq. 33 one has results similar to Eq. 40:

\[
(R^+_j)^{1} R^+_j = (R^+_j)^{k-1} \cdots (R^+_j)^{k-1} \tag{62}
\]

This holds for all positive \( n \) and all \( j \) such that either \( j \) and \( j + n \) are both positive or both are negative. In case \( j \) is negative and \( j + n \) positive one has

\[
(R^+_j)^{1} R^+_j = (R^+_j)^{k-1} \cdots (R^+_j)^{k-1} \times (R^-_1)^{k-1} \cdots (R^-_j+n-1)^{k-1} \tag{63}
\]

If \( j, n \) are such that \( j + n = 1 \) then the subscript \( j + n - 1 \) is replaced by \( j + n - 2 \) in the above.
5.2 Rational Addition and Multiplication

The definition of addition for rationals is quite similar to that for the integers. One has
\[
\pm R| \pm p \rangle \otimes | \pm q \rangle = | \pm p \rangle \otimes | \pm q \pm p \rangle
\] (64)
where \(| p \rangle, | q \rangle\) have the form of \(| r \rangle\) of Eq. (60). Also for \(| \pm p \rangle = | \pm \pm p \rangle\)
\[
| \pm q + \pm p \rangle = (R_{L_1})^\pm \otimes \cdots (R_1)^\pm \otimes (1)\]
\[
(R_{-1})^\pm \otimes \cdots (R_{-L_t})^\pm \otimes (L_t) \pm q \rangle
\]
\[
= R_{\pm p} | \pm q \rangle.
\] (65)

\(\pm R\) has the same properties as \(\pm J\). It is unitary on \(H^\text{fr}\) and the adjoint corresponds to the subtraction operator.

The definition of multiplication has the same form as that for the integers in Eq. (66). However, the definition of the shift operator \(U_{R,j}\), corresponding to multiplication by \(k\) is more complex.

There are several ways to define \(U_{R,i}\). Here the operator \(U_{R,i}\) corresponding to multiplication by \(k\), acting on a state \(| \pm p \rangle\), first exchanges the point at site 0 with the number at site \(-1\). Then the whole state is shifted one site to the left. 0 is added to site \(-1\) if and only if the site becomes unoccupied. That is,
\[
U_{R,i} | \pm s \otimes \rho \rangle = | \pm s \otimes t(-1)_L \rho \rangle
\] (66)
where \(t^L(j) = t^L(j-1)\) for \(-1 \geq j \geq -L_t + 1\) if \(L_t > 1\) and \(t^L(-1) = 0\) if \(L_t = 1\).

The definitions are the same for bosons and fermions except for the case when 0 must be added. For bosons these operations are defined by
\[
U_{R,b} = Z \sum_{h=0}^{k-1} a_{h,0}^\dagger a_{h,-1} a_{h,-1} (P_{\text{unocc},-1} a_{h,-1} P_{\text{unocc},-1})
\] (67)
where \(Z = \sum_{j=2}^\infty Z_j,\) \(Z_j\) is given by
\[
Z_j = \left\{
\begin{array}{ll}
Z_{j-1} \sum_h a_{h,j+1}^\dagger a_{h,j} P_{\text{unocc},j+1} & \text{if } j \geq 0, \neq 2 \\
(Z_{j-1} P_{\text{occ},j+1} + P_{\text{unocc},j-1}) \times & \text{if } j \leq -2 \\
\sum_h a_{h,j+1}^\dagger a_{h,j} P_{\text{unocc},j+1} & \text{if } j \leq -2
\end{array}
\right.
\] (68)

For \(j = 2\)
\[
Z_2 = Z_1 \sum_h a_{h,3}^\dagger a_{h,2} (P_{\text{numocc},2} + P_{\text{pm},2} P_{\text{numocc},0})
\]
\[
+Z_{-1} \sum_h a_{h,0}^\dagger a_{h,1} a_{h,0} a_{h,2} P_{\text{pm},2} P_{\text{pm},0}.
\] (69)

For \(j = -1\)
\[
Z_{-1} = Z_{-2} \sum_h a_{h,0}^\dagger a_{h,-1} P_{\text{unocc},0} P_{\text{occ},-2}
\]
\[
+ \sum_h a_{h,0}^\dagger a_{h,-1} a_{h,-1} P_{\text{unocc},0} P_{\text{unocc},-1}.
\] (70)

Note that some of the \(h\) sums over 0, \ldots, \(k - 1\) also include sums over \(\ldots, +, -..\). The subscripts on the projection operators are self explanatory. \(P_{\text{numocc},2}\) is the projection operator for a qubyte state \(|\pm, 2\rangle\) where \(\pm\) denotes a number in 0, \ldots, \(k - 1\).

To understand the reason for singling out \(Z_2\) and \(Z_{-1}\) one notes that the action of \(U_{R,b}\) is given by
\[
U_{R,b} | \pm s \otimes \rho \rangle = Z | \pm s \otimes t(-1)_L \rho \rangle
\]
The cases where \(| \pm s \otimes \rho \rangle = | \pm 0 \otimes \rho \rangle\) or \(| \pm s \otimes \rho \rangle\) need special treatment in order to comply with the convention that no leading or trailing strings of 0s remain. For \(| \pm s \otimes \rho \rangle = | \pm 0 \otimes \rho \rangle\), \(Z_2\) acts on \(| \pm 2 \otimes t(-1)_L \rho \rangle\) to delete the \(0_1\) component before the shifting. For \(| \pm s \otimes \rho \rangle = | \pm 0 \otimes \rho \rangle\), \(Z_{-1}\) acts on the shifted state \(| \pm s \otimes \rho \rangle\) to add a 0, \(-1\) component to give \(| \pm s \otimes 0 \rangle\) as the final result.

For fermions one has
\[
U_{R,f} = U_{R,b} P_{\text{occ},-2} +
\]
\[
Z f \sum_{h=0}^{k-1} a_{h,0}^\dagger a_{h,-2} a_{h,-1} P_{\text{unocc},-2}
\] (71)
where \(Z f = \sum_{j=2}^\infty Z f_j,\) \(Z f_j\) is given by
\[
Z f_j = -Z f_{j-1} \sum_{h=0}^{k-1} a_{h,j+1}^\dagger a_{h,j} P_{\text{unocc},j+1}
\] (72)
if \(j \geq 0\) and
\[
Z f_{-1} = -Z f_{-1} \sum_{h=0}^{k-1} a_{h,0}^\dagger a_{h,-1} a_{h,-1} P_{\text{unocc},0}
\] (73)
if \( j = -1 \). The range of the \( h \) sums is denoted by the superscripts shown in the above.

\( U^{R,i} \) is defined to have the same action on fermion states as \( U^{R,b} \) does on boson states. The only case in which the definition of \( U^{R,b} \) and \( U^{R,i} \) differ is for the action on \( | \pm \frac{1}{2}, 0 \rangle \) when \( Z_{-1} \) is finally active at the end of the shifting. The definition is set up so that anticommuting the odd number of \( a-c \) operators to the right hand end of \( | \pm \frac{1}{2}, 0 \rangle \) to add the 0 does not change the sign. The case in which a 0 is deleted causes no problems because there is no anticommuting of an odd number of \( a-c \) operators.

\( U^{R,i} \) has the property that, for \( i = b, f \), it is a bilateral shift on \( \mathcal{H}^{R,i} \). It is clear from the definition that for all \( | \pm \frac{1}{2}, \frac{1}{2} \rangle \neq | 0, 0 \rangle \), \( (\pm \frac{1}{2}, \frac{1}{2})|U^{R,i}|(\pm \frac{1}{2}, \frac{1}{2}) = 0 \). Also \( (U^{R,i})^\dagger U^{R,i} = 1 = (U^{R,i})^\dagger (U^{R,i})^\dagger \) as \( U^{R,i} \) is a bijection on the basis \( \{ | \pm \frac{1}{2}, \frac{1}{2} \rangle \} \) spanning \( \mathcal{H}^{R,i} \).

It follows from the definition of \( U^{R,i} \) that \( U^{R,i} \) or \( (U^{R,i})^\dagger \) correspond to multiplication by \( k^{-1} \) or \( k^i \) respectively. Based on this the multiplication operator \( \tilde{x} \) is defined by

\[
\tilde{x}| \pm p, ± q \rangle = | \pm p, ± q \rangle \pm r + (|± p \times ± q \rangle). \tag{74}
\]

Here \( |± p \times ± q \rangle \) is the state denoting the result of adding \( \pm p \times \pm q \) to \( \pm r \). Following Eq. \ref{eq:46} for the integers and using \(| ± p \rangle = | ± \frac{1}{2} \rangle \) one can express \( \tilde{x} \) more explicitly as

\[
\tilde{x}| \pm \frac{1}{2}, \frac{1}{2} \rangle = | \pm \frac{1}{2}, \frac{1}{2} \rangle ((U^{R,i})^\dagger)^{L_i - 1} \times (\mp 2^{(L_i)} U^{R,i}) \times (\mp 2^{(L_i - 1)} U^{R,i}) \times \ldots \\
\times U^{R,i}_{\pm 2, 3} (\mp 2^{(L_i)}) U^{R,i}_{\pm 2, 3} (\mp 2^{(L_i - 1)}) U^{R,i}_{\pm 2, 3} \times \ldots \\
\times U^{R,i}_{\pm 2, 3} (\mp 2^{(L_i)}) U^{R,i}_{\pm 2, 3} (\mp 2^{(L_i - 1)}) U^{R,i}_{\pm 2, 3} \times \ldots \\
\times U^{R,i}_{\pm 2, 3} (\mp 2^{(L_i)}) U^{R,i}_{\pm 2, 3} (\mp 2^{(L_i - 1)}) U^{R,i}_{\pm 2, 3} \times \ldots \\
\times (|U^{R,i})^\dagger | \pm \frac{1}{2} \rangle = | \pm \frac{1}{2}, \frac{1}{2} \rangle \pm r \). \tag{75}
\]

These actions correspond to multiplying \( ± q \) by \( k^{-L_i} \), adding or subtracting \( \frac{1}{2}(L_i) \) copies of the result to the third state \( ±r \), then adding or subtracting \( \frac{1}{2} (L_i + 1) \) copies of \( k^{-L_i + 1}(± q \rangle \) to the third state, etc.

The last step recovers the original second state \( | ± q \rangle \) by multiplying by \( k^{L_i - 1} \). Whether \( ± \) carries out iterated addition or subtraction depends on the sign of the first state.

It is clear from the definition that \( \tilde{x} \) is unitary. The operator preserves orthonormality of the basis set \( \{ | ± p, ± q, ± r \rangle \} \) and all these states (and linear superpositions) are in the domain and range of the operator. As was the case for the integers, it does not follow from unitarity that the adjoint of \( \tilde{x} \) carries out division. The argument is similar to that given for the absence of a division operator for the integers in that an equation similar to Eq. \ref{eq:15} would have to be satisfied. The fact that this is not the case is a consequence of the fact that not all rational numbers are included in the representation used here.

### 5.3 The Rational Number Axioms

The axioms for rational numbers are those for a field \( \mathbb{F} \). These are the same as those for the integers with the added axiom stating the existence of an inverse to multiplication. However as was seen this is not valid for the representation used here. The proofs of the other axioms are quite similar to those for the integers and the natural numbers and will not be repeated here.

The main difference here between the rational number and integer operators is that the operator \( U^{R,i} \) is unitary whereas the corresponding operator \( U^{l,i} \) for integers, Eq. \ref{eq:17}, is not unitary.

### 6 Physical Models of the Axiom Systems

So far mathematical Hilbert space models have been constructed for the natural number, integer, and rational number axiom systems. However these models are all abstract in that nothing is implied about the existence of physical systems that can implement the operations described by the axiom systems. The ubiquitous existence of computers shows that such systems do exist, at least on a macroscopic or classical scale.

Here the emphasis is on microscopic quantum mechanical systems. These systems have the property that the switching time \( t_{sw} \) to carry out a single step is short compared to the decoherence time \( t_{dec} \), or \( t_{sw}/t_{dec} \ll 1 \). For macroscopic systems \( t_{sw}/t_{dec} \gg 1 \). The discussion will be fairly brief and will be applied here to the rational number system.
Additional details for modular arithmetic on the natural numbers are given elsewhere. [12]

Let $A, D$ be two sets of physical parameters for quantum systems. For instance $A$ could be an infinite set of space positions and $D$ a finite set of spin projections or excitation energies of the system. The physical Fock space of states $\mathcal{H}_{phy}$ for the system is spanned by states of the form $c^\dagger_{d_m, a_m} c^\dagger_{d_{m-1}, a_{m-1}} \cdots c^\dagger_{d_1, a_1} |0\rangle$. Here $m$ is an arbitrary finite number, $c^\dagger_{d, a}$ is a creation operator for a system with property $d, a$, where $a$ and $d$ are values in $A$ and $D$ respectively. The operators $c^\dagger_{d, a}$ and $c_{d', a'}$ satisfy commutation or anticommutation relations similar to Eqs. [3] and [4] if the basic physical systems are bosons or fermions.

Assume that the basic mathematical and physical systems are both either fermions or bosons. Then the a-c operators of both $\mathcal{H}_{phys} - Ra$ and $\mathcal{H}_Ra$ have the same symmetry property. Let $W$ be an arbitrary isometry from the abstract Hilbert space $\mathcal{H}_Ra$ to a subspace $\mathcal{H}_{phys} - Ra$ of $\mathcal{H}_{phys}$. Then $W$ and its adjoint $W^\dagger$ restricted to $\mathcal{H}_{phys} - Ra$ are unitary maps between $\mathcal{H}_{phys}$ and $\mathcal{H}_{phys} - Ra$.

One can then define induced annihilation and creation operators on $\mathcal{H}_{phys} - Ra$ according to

$$a^\dagger_{W, h, j} = W a^\dagger_{h, j} W^\dagger \quad a_{W, h, j} = W a_{h, j} W^\dagger. \quad (76)$$

Here $j = 1, 2, \ldots$ and $h = 0, 1, \ldots, k - 1$.

The corresponding rational number states on the physical state space are given by

$$W | \pm s, L \rangle = | \pm W \Delta_W W^L_W \rangle = a^\dagger_{W, \pm L} a^\dagger_{W, 0} a^\dagger_{W, 1} \cdots a^\dagger_{W, k-1} a^\dagger_{W, 0} |0\rangle.$$

The operators $R_{W,j}, \oplus_W, \otimes_W$ on the physical state space that correspond to the successor operators for each $j$ and the addition and multiplication operators on $\mathcal{H}_Ra$ are given by the general relation for any operator $\hat{O}$ on $\mathcal{H}_Ra$

$$\hat{O}_W = W \hat{O} W^\dagger.$$

Alternatively the physical state space operators can be obtained by replacing each creation and annihilation operator $a^\dagger_{W, j}$, $a_{W, j}$ by $a^\dagger_{W, t, j}$, $a_{W, t, j}$ in the definitions of the operators given in subsections 5.4 and 5.5.

As a very simple example of a map $W$ let $q$ and $d$ be one-one functions from the numbers $1, 2, \ldots$ to $A$ and from $\{0, 1, \ldots k - 1\}$ to $D$. Let $W$ be such that

$$a_{W, h, j} = c_{d(h), g(j)}, \quad a^\dagger_{W, h, j} = c^\dagger_{d(h), g(j)}.$$

In this case the elementary physical components of a physical quantum system correspond to the components of the abstract quantum system. This type of example was considered earlier for modular arithmetic on the natural numbers. [12]

More complex examples in which $W$ maps the abstract components onto collective degrees of freedom or multiparticle states can also be constructed. These types of examples give entangled physical states similar to those considered in some quantum error correction schemes [21, 22, 23, 24, 25] and in decoherence free subspaces [26, 27, 28]. Topological and anyonic quantum states have also been considered in the literature [29, 30, 31].

These examples also illustrate the large number of possibilities for constructing unitary maps from $\mathcal{H}_Ra$ to a physical state space for quantum systems. However it is too general in the sense that an important restriction has been left out. In particular, as is well known, there are many physical systems that are not suitable to represent or model mathematical number systems. Such systems are also not useful as quantum computers.

This feature has been realized for some time and several approaches have been discussed. Requirements discussed in the literature for quantum computers include having well characterized qubits, the ability to prepare a simple initial state, the condition that $t_{dec}/t_{sw} \gg 1$, the presence of unitary operators for a universal set of quantum gates or unitary control of suitable subsystems, and the ability to measure specific qubits or subsystem observables [25, 29].

Here the condition is expressed by the requirement that the basic operations described by the axioms of the system under consideration must be efficiently implementable. For the systems studied here this means that the successor operations for each $j$, $+$, and...
and $\tilde{\times}$ must be efficiently implementable.

This requirement means that for each of these operations there must exist a unitary time dependent operator $U(t)$ in the physical model such that the action of $U(t)$ on suitable physical system states corresponds to carrying out the operation. This can be expressed more explicitly using the rational number states and operators as an example. For each state $|\pm\rangle = a_{\pm\mathbb{L}}^\dagger|0\rangle$ let $P_{\pm\mathbb{L}}$ and $P_{O,\pm\mathbb{L}}$ be the projection operators on the states $|\pm\rangle$ and $|\tilde{O}\pm\rangle$ where $\tilde{O}$ is any of the successor operators, $R_j$, $+R$, $\times R$ defined in Section 5. Let $P^W_{\pm\mathbb{L}} = WP_{\pm\mathbb{L}} W^\dagger$ and $P^W_{\tilde{O},\pm\mathbb{L}} = WP_{\tilde{O},\pm\mathbb{L}} W^\dagger$ be the corresponding projection operators on the physical states $|\pm\rangle$ and $|\tilde{O}\pm\rangle$.

These operators are the identity on all the environmental and ancillary degrees of freedom in the overall physical system.

Let $\rho(0)$ denote the initial overall physical system density operator at time 0. Then $P^W_{\pm\mathbb{L}}\rho(0)P^W_{\pm\mathbb{L}} = \rho_{\pm\mathbb{L}}(0)$ is the initial physical system state with the model subsystem in the state corresponding to $|\pm\rangle$ under the map $W$. The time development of $\rho_{\pm\mathbb{L}}(0)$ is given by some unitary operator $U_{\tilde{O}}(t)$ with possible dependence on $\tilde{O}$ indicated. That is $\rho_{\pm\mathbb{L}}(t) = U_{\tilde{O}}(t)\rho_{\pm\mathbb{L}}(0)U_{\tilde{O}}^\dagger(t)$.

Implementability of the operator $\tilde{O}$ means that there is a unitary evolution operator $U_{\tilde{O}}(t)$ and an initial system state $\rho(0)$ such that for each $|\pm\rangle$ there is a time $t_{\pm\mathbb{L}}$ such that the components of $\rho_{\pm\mathbb{L}}(t_{\pm\mathbb{L}})$ that correspond to the state $|\tilde{O}\pm\rangle$ appear with relative probability 1. That is

$$TrP^W_{\tilde{O},\pm\mathbb{L}}\rho_{\pm\mathbb{L}}(t_{\pm\mathbb{L}}) = Tr\rho_{\pm\mathbb{L}}(0)$$

(78)

where the trace is taken over all degrees of freedom including ancillary and environmental degrees that may be present.

Implementability also means that the operator $U_{\tilde{O}}(t)$ must be physically implementable in that there must exist a physical procedure for implementing $U_{\tilde{O}}(t)$ that can actually be carried out. For Schrödinger dynamics this means there must exist a Hamiltonian $H_{\tilde{O}}$ that can be physically implemented such that $U_{\tilde{O}}(t) = e^{-iH_{\tilde{O}}t}$. Eq. (78) must also be satisfied by $H_{\tilde{O}}$.

The requirement of efficiency means that for each state $|\pm\rangle$ the time $t_{\pm\mathbb{L}}$ required to satisfy Eq. (78) must be polynomial in the length $L_r$ of $\mathbb{L}$. It cannot be exponential in $L_r$. The requirement also means that the space requirements for physical implementation must also be polynomial in $L_r$. If $H_{\tilde{O}}$ is implemented by circuits of quantum gates, as in [17, 18], then the number of gates in the circuits must be polynomial in $L_r$.

Efficiency also means that the thermodynamic resources needed to implement $H_{\tilde{O}}$ must be polynomial in $L_r$. This places limitations on the value of $k$ in that for physical systems occupying a given space-time volume it must be possible to reliably distinguish $k$ alternatives in the volume $[35]$.

As was noted in the introduction, the efficiency requirement is the reason that successor operators are defined separately for each $j$ and the efficiency requirement is applied to each operator. If the requirement applied to just one of the operators, and not to the others, then physical models could be allowed in which $t_{\pm\mathbb{L}}$ would be exponential and not polynomial in $\mathbb{L}$ for these operators. This follows from the exponential dependence of the $R_j$ on $j$ as shown in Eq. (51). The fact that efficient computation based on efficient implementation of the basic arithmetic operations is so ubiquitous shows that there are many methods of implementing the $R_j$ efficiently in classical computers at least. However this does not reduce the importance of the efficiency requirement for these operators.

The existence of the Hamiltonians $H_{\tilde{O}}$ that can be carried out is in general a nontrivial problem. For modular arithmetic on the natural numbers the existence of quantum circuits for the basic arithmetic operations $+$ and $\times$ suggests that such Hamiltonians may exist for the basic arithmetic operations on distinguishable qubits. However most physical models described to implement simple quantum compu-

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3It would be expected that these models would be excluded by the efficiency requirement applied to $\tilde{\times}$ as this operator is defined in terms of iterations of the different $R_j$. 

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tations are based on a time dependent Hamiltonian that implements a product of different unitary operators. In many of these models the computation is driven by a sequence of individually prepared laser pulses to carry out specified operations. The possibility of describing this with a time independent Hamiltonian that can be physically implemented on multiqubit systems is a question for the future.

Here the problem is more complex in that bosonic or fermionic quantum computation methods would be needed on states with an indeterminate number of degrees of freedom. Work on this problem for binary fermions using the representation of a-c operators as Pauli products of the standard spin operators [20] is a possible avenue but more needs to be done. For computations in an interactive environment one might hope that the use of decoherence free subspaces [29, 31], stabilizer codes [19, 34], or other methods of error protection [24, 25, 26, 27] will be workable.

A Appendix

A.1 Proof of Some Natural Number Axioms

It is sufficient for the proof of \( \tilde{x}(V_j \otimes \tilde{1} \otimes \tilde{1})|s\otimes|t\otimes|p\) where \(|p\) = \(|s+g \times t+ [k^{-1}]\) to set \(|p\) = 0. An expression for the product is needed for which each of the \(V_j\) are collected together. Repeating Eq. 22 for \(|p\) = 0 gives

\[
\tilde{x}|s,t,0\rangle = \left((U_2^j)^{x} \cdot \cdot \cdot (U_2^j)^{L_{s-1}}(U_2^j)^{L_{s}} U_2^j(U_3^j)^{L_{s-1}}(U_3^j)^{L_{s}} \cdot \cdot \cdot (U_3^j)^{L_{s-1}}(U_3^j)^{L_{s}} U_3^j U_2^j U_3^j \cdot \cdot \cdot \right)|s,t,0\rangle.
\]

Use of Eqs. 14, 15 and 18 gives

\[
\tilde{x}|s,t,0\rangle = (|s,t\rangle \otimes (V_{L_t}\ldots V_{L_{s-1}})U_2^j(U_2^j)^{L_{s-1}}(U_2^j)^{L_{s}} U_2^j(U_3^j)^{L_{s-1}}(U_3^j)^{L_{s}} \cdot \cdot \cdot (U_3^j)^{L_{s-1}}(U_3^j)^{L_{s}} U_3^j U_2^j U_3^j \cdot \cdot \cdot ) |s,t,0\rangle.
\]

where use was made of \((U^j)^j = |k^{m}\rangle\) and if \(k^m|n\rangle = 0\), then \(V_k|n\rangle = \tilde{1}\). These identity factors have been deleted in the above. Also \(L_s\) and \(L_t\) are the lengths of \(s\) and \(t\).

One now collects together all \(V_s\) with the same subscript value. As noted before this commuting of the \(V_s\) past one another causes no problems for either fermions or bosons. There are two cases to consider \(L_t \geq L_s\) and \(L_t - t \leq L_s\) which differ only in index labeling. Carrying this out for \(L_t \geq L_s\) and putting the \(V_s\) in order of decreasing subscript values from left to right gives

\[
(V_{L_t}+L_s-1)^k |L_s, L_{s-1}, \ldots, L_{s-2}, k_{L_{s-1}}, k_{L_{s-2}}, \ldots, k_{L_t}, k_t, L_{s-1} \rangle = (\ldots (V_2)^k U_2^j U_3^j (V_1)^k |L_{s}^{(1)} \rangle |0\rangle.
\]

A more explicit expression including the terms represented by the \(\ldots\) is

\[
(V_{L_t}+L_s-1)^E_0 \ldots (V_{L_t},t)^E_{L_s-2} \ldots (V_m)^G_{L_{m},m} \ldots (V_{L_s},t)^E_{L_s-1} \ldots (V_1)^{F_1} |0\rangle.
\]

Here

\[
E_n = \sum_{h=0}^{n} s(L_s - h) t (L_t - n + h)
\]

for \(0 \leq n \leq L_s - 2\),

\[
G_{L_s,m} = \sum_{h=0}^{L_t - 1} s(L_s - h) t (m + 1 + h - L_s)
\]

for \(L_s \leq m \leq L_t\), and

\[
F_{\ell} = \sum_{h=0}^{\ell-1} s(\ell - h) t (h + 1)
\]

for \(1 \leq \ell \leq L_s\). Note that \(F_{L_s} = G_{L_s,L_s}\) and \(G_{L_s,L_t}\) \(E_{L_{s-1}}\). Also

\[
k^{m}t(m+n) = t(m)
\]

for \(n = 0, 1, \ldots, m = 1, 2, \ldots\) was used.

In the above the values of the exponents may well be greater than \(k - 1\). Because of Eq. 3 this causes no problems provided one represents states in the form given by Eq. 13.

The desired goal is to prove that \(|s + k^{j-1} \times t\rangle = |s \times t + k^{-1}t\rangle\).
From the foregoing one has

\[
| s \times t + k^{j-1} \ell | = (V L + L + s - 1)^E_0 \ldots (V_{L+j}+1)^E_{L+j-1} \ldots (V_{L-1})^F_{L-1} \ldots (V_j)^{F_1} \\
(V_{L+j-1})^G_{L_1} \ldots (V_j)^{G_{1|0}} \\
\sum_{h=0}^{L_s-j+p} \sum_{h \neq L_s-j} s(L_s-h)\ell(L_t - L_s + j - p + h) + \ell(L_t - p)
\]

with \(0 \leq p \leq j - 2\). Similarly for the \(G\) and \(F\) exponents,

\[
G_{L_s,m} + \ell(m - j + 1) = \\
\sum_{h=0}^{L_s-j+p} s(L_s-h)\ell(m + 1 - L_s + h) + \ell(m - j + 1) \\
F_{L_s-q} + \ell(L_s - q - j + 1) = \\
\sum_{h=0}^{L_s-q-j+1} s(L_s - q - h)\ell(h + 1) + \ell(L_s - q - j + 1).
\]

Here \(L_s \geq m \geq L_t\) and \(1 \geq q \leq L_s - j\). These expressions can be rewritten as

\[
E_{L_s-j+p} + \ell(L_t - p) = \\
\sum_{h=0}^{L_s-j+p} s(L_s-h)\ell(L_t - L_s + j - p + h) + \ell(L_t - p)
\]

\[
G_{L_s,m} + \ell(m - j + 1) = \\
\sum_{h=0}^{L_s-j+p} \sum_{h \neq L_s-j} s(L_s-h)\ell(L_t - L_s + j - p + h) + \ell(L_t - p)
\]

\[
[G_L s_h^L (L_L - h) \ell(L_t - L_L + j - p + h) + \ell(L_t - p)]
\]

\[
\sum_{h=0}^{L_s-j+p} \sum_{h \neq L_s-j}^{} s(L_s-h)\ell(L_t - L_s + j - p + h) + \ell(L_t - p)
\]

where Eq. 83 was used. Again one collects \(V_s\) with the same subscripts. The explicit form of the final result depends somewhat on the magnitude of \(j\) relative to \(L_t\) and \(L_s\). Assume \(j < L_s < L_t\). Then the right-hand side of Eq. 84 can be written as

\[
(V_{L_t+j+L_s-1})^E_0 \ldots (V_{L_t+j})^{E_{L_s-j-1}} (V_{L_t+j+L_s-1})^{E_{L_s-j}} + \ell(L_t) \\
\ldots (V_{L_t+1})^{E_{L_s-j+L_t-j+2}} (V_{L_t})^{E_{L_s-j+L_t-j+1}} \\
\ldots (V_m)^{E_{L_s-j+L_t-j+1}} (V_{L_t})^{E_{L_s-j} + \ell(L_t-j)} \ldots (V_{L_s})^{E_{L_s} + \ell(1)} \ldots (V_1)^{F_1} | 0 \rangle
\]

\[
\sum_{h=0}^{L_s-j+p} | s(L_s-h)\ell(m + 1 - L_s + h) + \ell(m - j + 1) | = \\
\sum_{h=0}^{L_s-j+p} | s(L_t-h)\ell(L_t - L_s + j - p + h) + \ell(L_t - p) |
\]

Use of the axiom that is being proven for the operators \(V_j, +, \times\) for the number expressions in the square brackets in the above three equations gives

\[
[g(L_s-h)\ell(m + 1 - L_s + h) + \ell(m - j + 1)] \\
[g(L_t-h)\ell(L_t - L_s + j - p + h) + \ell(L_t - p)]
\]

\[
(g(j)\ell(m + 1 - L_s + h) + \ell(m - j + 1)] \\
(g(j)\ell(L_s - q - j + 1) + \ell(L_s - q - j + 1] \\
(g(j)\ell(L_s - q - j + 1) + \ell(L_s - q - j + 1)]
\]

for \(r = L_t - p, m - j + 1, L_s - q - j + 1\). This is the important step because by repeating the above derivation for \(\times (V_j \otimes 1 \otimes 1) | g(L_t) W_\ell \rangle\) one can show that \(| (V_j g) \times \ell \rangle\) has exactly the form given by Eqs. 83 84 with \(g(j)\ell(j) + \ell(r)\) replaced by \((g(j) + 1)\ell(r)\). This completes the proof of the axiom for \(j < L_s < L_t\). The proof for the other cases is quite similar and will not be repeated here.

A.2 Proof of \(I_{j+1} = (I_j)^k\)

For the proof of Eq. 83 one notes that

\[
I_{j+1}^k = (I_j^+)^k + \sum_{\ell=1}^{k-1} (I_j^+)^{k-1} (I_j^+ P_0 + W + I_{j+1}^+ P_{-j}) \\
\times (W(I_j^+) W)^{k-\ell} P_{-j} \geq j \\
+ (P_{0} + W + P_{-j} < 0) (W(I_j^+) W)^k P_{-j} \geq j \\
+ (I_j^+)^k - I_{j+1}^+ P_{j}.
\]

The various terms reflect the fact that \(I_{j+1}^\pm\) is either not active or is active just once (as it takes negative number states to positive number states). The \(\ell\) sum shows that \(I_{j+1}^\pm\) can be active at any iteration of \(I_j\), from the first to the \(kth\). It is preceded by iterations of \(I_{j+1}^\pm\) and succeeded by iterations of \(I_j^+\).
For the first term the result is immediate by Eq. \[2\]. Using $W^2 = 1$, the term $(P_{+,0}W + P_{-,<0})(W(I_j^+)W)^kP_{-,\geq j} = (P_{+,0}W + P_-(W(I_{j+1}^+)W)P_{-,\geq j}$. It is clear that this term gives 0 for any state $|-\underline{a}\rangle$ for which $L = j$. Thus $P_{-,\geq j}$ can be replaced by $P_{-,\geq j+1}$. This shows that this term equals $I_{<j+1}$. It remains to show that the sum of the remaining terms equals $I_{<j+1}$. For the last term, use of Eq. \[3\] gives $(I_j^+)^kI_{<j}P_{<j} = (I_j^+)^{-k+1}\sum j<k|I^+|I_j^+\rangle + 0\langle s, -\underline{a}|$. Commuting the $I_j^+$ past $(I^+_{L\underline{a}})^t$ and use of Eq. \[3\] gives $\sum j<k\langle I^+|I_{j+1}^+\rangle + 0\langle s, -\underline{a}|$ for the last term. This corresponds in the $\underline{a}$ sum for $I_{<j+1}$ to those terms for which $L < j$. The terms for which $L = j$ are contained in the $\ell$ sum. Each term can be written as $(I_j^+)^kP_{+,0}W + (I_j^+)^{l-1}\sum j<k|I^+|I_j^+\rangle + 0\langle s, -\underline{a}|W(I_j^+)^tWP_{ -, \geq j}$. The matrix element $\langle s, -\underline{a}|W(I_j^+)^tWP_{ -, \geq j}$ is nonzero if and only if the length $L$ of $\underline{a}$ is $j$ and $| - \underline{a}\rangle = a_{\underline{a}, -j+1}^t a_{\underline{a}, -j+1}^\dagger |0\rangle$. In this case the matrix element equals 1. As a result the projection operator $P_{-,\geq j}$ can be replaced by $P_{-,j}$.

Using this and replacing $\ell$ by $k - \underline{a}(j)$ gives for the second part of the $\ell$ term $\sum j<k|I^+|I_j^+\rangle + 0\langle s, -\underline{a}|WP_{ -, \geq j} = (I_j^+)^tI_{j+1}^+\langle s, -\underline{a}|WP_{ -, \geq j}$ . Here use is made of the facts that $| + \underline{a}\rangle = a_{\underline{a}, j+1}^* |j\rangle$ and $(I_j^+)^t\langle s, -\underline{a}|(I_j^+)^t = (I_j^+)^t$. Also $(I_j^+)^k-1(I_j^+)^{l+1}|0\rangle = (I_j^+)^k|0\rangle$ was used.

The definition of $I_{<j}$ is such that the $\underline{a}$ sum is over all $\underline{a}$ such that $\langle j|L\rangle > 0$. Thus states $| - \underline{a}\rangle = | - \underline{a}(j)\rangle$ are excluded. These are accounted for in the first part of the $\ell$ term: $(I_j^+)^kP_{+,0}((I_j^+)^t)^k WP_{ -, \geq j}$. Use of $\underline{a}(j) = k - \ell$ and $(I_j^+)^t = (I_j^+)^k-2\underline{a}(j) = (I_j^+)^kI_{j+1}$ completes the proof of Eq. \[3\].

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