NEAR-ISOMETRIC DUALITY OF HARDY NORMS WITH APPLICATIONS TO HARMONIC MAPPINGS

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Abstract. Hardy spaces in the complex plane and in higher dimensions have natural finite-dimensional subspaces formed by polynomials or by linear maps. We use the restriction of Hardy norms to such subspaces to describe the set of possible derivatives of harmonic self-maps of a ball, providing a version of the Schwarz lemma for harmonic maps. These restricted Hardy norms display unexpected near-isometric duality between the exponents 1 and 4, which we use to give an explicit form of harmonic Schwarz lemma.

1. Introduction

This paper connects two seemingly distant subjects: the geometry of Hardy norms on finite-dimensional spaces and the gradient of a harmonic map of the unit ball. Specifically, writing $H^1_*$ for the dual of the Hardy norm $H^1$ on complex-linear functions (defined in §2), we obtain the following description of the possible gradients of harmonic maps of the unit disk $D$.

Theorem 1.1. A vector $(\alpha, \beta) \in \mathbb{C}^2$ is the Wirtinger derivative at 0 of some harmonic map $f: D \to D$ if and only if $\|(\alpha, \beta)\|_{H^1_*} \leq 1$.

Theorem 1.1 can be compared to the behavior of holomorphic maps $f: \mathbb{D} \to \mathbb{D}$ for which the set of all possible values of $f'(0)$ is simply $\mathbb{D}$. The appearance of $H^1_*$ norm here leads one to look for a concrete description of this norm. It is well known that the duality of holomorphic Hardy spaces $H^p$ is not isometric, and in particular the dual of $H^1$ norm is quite different from $H^\infty$ norm even on finite dimensional subspaces (see (3.4)). However, it has a striking similarity to $H^4$ norm.

Theorem 1.2. For all $\xi \in \mathbb{C}^2 \setminus \{(0, 0)\}$, $1 \leq \|\xi\|_{H^4} / \|\xi\|_{H^1} \leq 1.01$.

Since the $H^4$ norm can be expressed as $\|(\xi_1, \xi_2)\|_4 = (|\xi_1|^4 + 4|\xi_1\xi_2|^2 + |\xi_2|^4)^{1/4}$, Theorem 1.2 supplements Theorem 1.1 with an explicit estimate.

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In general, Hardy norms are merely quasinorms when \( p < 1 \), as the triangle inequality fails. However, their restrictions to the subspaces of degree 1 complex polynomials or of \( 2 \times 2 \) real matrices are actual norms (Theorem 2.1 and Corollary 5.2). We do not know if this property holds for \( n \times n \) matrices with \( n > 2 \).

The paper is organized as follows. Section 2 introduces Hardy norms on polynomials. In Section 3 we prove Theorem 1.2. Section 4 concerns the Schwarz lemma for planar harmonic maps, Theorem 1.1. In section 5 we consider higher dimensional analogues of these results.

2. HARDY NORMS ON POLYNOMIALS

For a polynomial \( f \in \mathbb{C}[z] \), the Hardy space \( (H^p) \) quasinorm is defined by

\[
\|f\|_{H^p} = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p \, dt \right)^{1/p}
\]

where \( 0 < p < \infty \). There are two limiting cases: \( p \to \infty \) yields the supremum norm

\[
\|f\|_{H^\infty} = \max_{t \in \mathbb{R}} |f(e^{it})|
\]

and the limit \( p \to 0 \) yields the Mahler measure of \( f \):

\[
\|f\|_{H^0} = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt \right).
\]

An overview of the properties of these quasinorms can be found in [12, Chapter 13] and in [11]. In general they satisfy the definition of a norm only when \( p \geq 1 \).

The Hardy quasinorms on vector spaces \( \mathbb{C}^n \) are defined by

\[
\|(a_1, \ldots, a_n)\|_{H^p} = \|f\|_{H^p}, \quad f(z) = \sum_{k=1}^n a_k z^{k-1}.
\]

We will focus on the case \( n = 2 \), which corresponds to the \( H^p \) quasinorm of degree 1 polynomials \( a_1 + a_2 z \). These quantities appear as multiplicative constants in sharp inequalities for polynomials of general degree: see Theorems 13.2.12 and 14.6.5 in [12], or Theorem 5 in [11]. In general, \( H^p \) quasinorms cannot be expressed in elementary functions even on \( \mathbb{C}^2 \). Notable exceptions include

\[
\|(a_1, a_2)\|_{H^0} = \max(|a_1|, |a_2|),
\]

\[
\|(a_1, a_2)\|_{H^2} = (|a_1|^2 + |a_2|^2)^{1/2},
\]

\[
\|(a_1, a_2)\|_{H^4} = (|a_1|^4 + 4|a_1|^2|a_2|^2 + |a_2|^4)^{1/4},
\]

\[
\|(a_1, a_2)\|_{H^\infty} = |a_1| + |a_2|.
\]

Another easy evaluation is

\[
\|(1, 1)\|_{H^1} = \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{it}| \, dt = \frac{1}{2\pi} \int_0^{2\pi} 2|\cos(t/2)| \, dt = \frac{4}{\pi}.
\]
However, the general formula for the $H^1$ norm on $\mathbb{C}^2$ involves the complete elliptic integral of the second kind $E$. Indeed, writing $k = |a_2/a_1|$, we have

$$
\|(a_1, a_2)\|_{H^1} = |a_1| \|(1, k)\|_{H^1} = \frac{|a_1|}{2\pi} \int_0^{2\pi} |1 + ke^{2it}| \, dt \\
= |a_1| \frac{2(k + 1)}{\pi} \int_0^{\pi/2} \sqrt{1 - \left(\frac{2\sqrt{k}}{k + 1}\right)^2 \sin^2 t} \, dt \\
= |a_1| \frac{2(k + 1)}{\pi} E\left(\frac{2\sqrt{k}}{k + 1}\right).
$$

(2.3)

Perhaps surprisingly, the Hardy quasinorm on $\mathbb{C}^2$ is a norm (i.e., it satisfies the triangle inequality) even when $p < 1$.

**Theorem 2.1.** The Hardy quasinorm on $\mathbb{C}^2$ is a norm for all $0 \leq p \leq \infty$. In addition, it has the symmetry properties

$$
\|(a_1, a_2)\|_{H^p} = \|(a_2, a_1)\|_{H^p} = \|(\|a_1\|, \|a_2\|)\|_{H^p}.
$$

(2.4)

**Proof.** For $p = 0, \infty$ all these statements follow from (2.1), so we assume $0 < p < \infty$. The identities

$$
\int_0^{2\pi} |a_1 + a_2 e^{it}|^p \, dt = \int_0^{2\pi} |a_1 e^{-it} + a_2|^p \, dt = \int_0^{2\pi} |a_2 + a_1 e^{it}|^p \, dt
$$

(2.5)

imply the first part of (2.4). Furthermore, the first integral in (2.5) is independent of the argument of $a_2$ while the last integral is independent of the argument of $a_1$. This completes the proof of (2.4).

It remains to prove the triangle inequality in the case $0 < p < 1$. To this end, consider the following function of $\lambda \in \mathbb{R}$.

$$
G(\lambda) := \|(1, \lambda)\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |1 + \lambda e^{it}|^p \, dt\right)^{1/p}.
$$

(2.6)

We claim that $G$ is convex on $\mathbb{R}$. If $|\lambda| < 1$, the binomial series

$$
(1 + \lambda e^{it})^{p/2} = \sum_{n=0}^{\infty} \binom{p/2}{n} \lambda^n e^{nit}
$$

together with Parseval’s identity imply

$$
G(\lambda) = \left(\sum_{n=0}^{\infty} \binom{p/2}{n} \lambda^{2n}\right)^{1/p}.
$$

(2.7)

Since every term of the series is a convex function of $\lambda$, it follows that $G$ is convex on $[-1, 1]$. The power series also shows that $G$ is $C^\infty$ smooth on $(0, 1)$. For $\lambda > 1$ the symmetry property (2.4) yields $G(\lambda) = \lambda G(1/\lambda)$ which is a convex function by virtue of the
identity \( G''(\lambda) = \lambda^{-3}G''(1/\lambda) \). The piecewise convexity of \( G \) on \([0, 1]\) and \([1, \infty)\) will imply its convexity on \([0, \infty)\) (hence on \(\mathbb{R}\)) as soon as we show that \( G \) is differentiable at \( \lambda = 1 \). Note that \(|1 + \lambda e^{it}|^p \) is differentiable with respect to \( \lambda \) when \( e^{it} \neq -1 \) and that for \( \lambda \) close to 1,

\[
\frac{\partial}{\partial \lambda} |1 + \lambda e^{it}|^p \leq p|1 + \lambda e^{it}|^{p-1} \leq C|t - \pi|^{p-1}
\]

for all \( t \in [0, 2\pi] \setminus \{\pi\} \), with \( C \) independent of \( \lambda, t \). The integrability of the right hand side of (2.8) justifies differentiation under the integral sign:

\[
\frac{d}{d\lambda} G(\lambda)^p = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \lambda} |1 + \lambda e^{it}|^p dt.
\]

Thus \( G'(1) \) exists.

Now that \( G \) is known to be convex, the convexity of the function \( F(x, y) := \|(x, y)\|_{H^p} = xG(y/x) \) on the halfplane \((x, y) \in \mathbb{R}^2, x > 0\), follows by computing its Hessian, which exists when \(|y| \neq x\):

\[
H_F = G''(y/x) \begin{pmatrix} x^{-3}y^2 & -x^{-2}y \\ -x^{-2}y & x^{-1} \end{pmatrix}.
\]

Since \( H_F \) is positive semidefinite, and \( F \) is \( C^1 \) smooth even on the lines \(|y| = |x|\), the function \( F \) is convex on the halfplane \( x > 0\). By symmetry, convexity holds on other coordinate halfplanes as well, and thus on all of \( \mathbb{R}^2 \). The fact that \( G \) is an increasing function on \([0, \infty)\) also shows that \( F \) is an increasing function of each of its variables in the first quadrant \( x, y \geq 0 \).

Finally, for any two points \((a_1, a_2)\) and \((b_1, b_2)\) in \( \mathbb{C}^2 \) we have

\[
\|(a_1 + b_1, a_2 + b_2)\|_{H^p} = F(|a_1 + b_1|, |a_2 + b_2|) \leq F(|a_1| + |b_1|, |a_2| + |b_2|) \\
\leq F(|a_1|, |a_2|) + F(|b_1|, |b_2|) = \|(a_1, a_2)\|_{H^p} + \|(b_1, b_2)\|_{H^p}
\]

using (2.4) and the monotonicity and convexity of \( F \).

\[\Box\]

Remark 2.2. In view of Theorem 2.1 one might guess that the restriction of \( H^p \) quasinorm to the polynomials of degree at most \( n \) should satisfy the triangle inequality provided that \( p > p_n \) for some \( p_n < 1 \). This is not so: the triangle inequality fails for any \( p < 1 \) even when the quasinorm is restricted to quadratic polynomials. Indeed, for small \( \lambda \in \mathbb{R} \) we have

\[
\|(\lambda, 1, \lambda)\|^p_{H^p} = \frac{1}{2\pi} \int_0^{2\pi} (1 + 2\lambda \cos t)^p dt \\
= \frac{1}{2\pi} \int_0^{2\pi} \left( 1 + 2\lambda p \cos t + 2\lambda^2 p(p-1) \cos^2 t + O(\lambda^3) \right) dt \\
= 1 + \lambda^2 p(p-1) + O(\lambda^3)
\]

and this quantity has a strict local maximum at \( \lambda = 0 \) provided that \( 0 < p < 1 \).
3. Dual Hardy norms on polynomials

The space $\mathbb{C}^n$ is equipped with the inner product $\langle \xi, \eta \rangle = \sum_{k=1}^{n} \xi_k \bar{\eta}_k$. Let $H_p^*$ be the norm on $\mathbb{C}^n$ dual to $H_p$, that is

$$\|\xi\|_{H_p^*} = \sup_{\|\eta\|_{H_p} \leq 1} \{ |\langle \xi, \eta \rangle| \}.$$  

One cannot expect the $H_p^*$ norm to agree with $H_q$ for $q = p/(p - 1)$ (unless $p = 2$), as the duality of Hardy spaces is not isometric [5, Section 7.2]. However, on the space $\mathbb{C}^2$ the $H_1^*$ norm turns out to be surprisingly close to $H_4^*$, indicating that $H_1$ and $H_4$ have nearly isometric duality in this setting. The following is a restatement of Theorem 1.2 in the form that is convenient for the proof.

**Theorem 3.1.** For all $\xi \in \mathbb{C}^2$ we have

$$\|\xi\|_{H_1} \leq \|\xi\|_{H_4^*} \leq 1.01 \|\xi\|_{H_1}$$

and consequently

$$\|\xi\|_{H_4} \leq \|\xi\|_{H_4^*} \leq 1.01 \|\xi\|_{H_4}.$$  

It should be noted that while the $H_1$ norm on $\mathbb{C}^2$ is a non-elementary function (2.3), the $H_4$ norm has a simple algebraic form (2.1). To see that having the exponent $p = 4$, rather than the expected $p = \infty$, is essential in Theorem 3.1, compare the following:

$$\|(1,1)\|_{H_1} = \frac{2}{\|(1,1)\|_{H_1}} = \frac{\pi}{2} \approx 1.57,$$

$$\|(1,1)\|_{H_4} = 2,$$

$$\|(1,1)\|_{H_4^*} = 6^{1/4} \approx 1.57.$$

The proof of Theorem 3.1 requires an elementary lemma from analytic geometry.

**Lemma 3.2.** If $0 < r < a$ and $b \in \mathbb{R}$, then

$$\sup_{\theta \in \mathbb{R}} \frac{b - r \sin \theta}{a - r \cos \theta} = \frac{ab + r \sqrt{a^2 + b^2 - r^2}}{a^2 - r^2}.$$  

**Proof.** The quantity being maximized is the slope of a line through $(a, b)$ and a point on the circle $x^2 + y^2 = r^2$. The slope is maximized by one of two tangent lines to the circle passing through $(a, b)$. Let $\tan \alpha = b/a$ be the slope of the line $L$ through $(0, 0)$ and $(a, b)$. This line makes angle $\beta$ with the tangents, where $\tan \beta = r/\sqrt{a^2 + b^2 - r^2}$. Thus, the slope of the tangent of interest is

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{b \sqrt{a^2 + b^2 - r^2} + ar}{a \sqrt{a^2 + b^2 - r^2} - br},$$

which simplifies to (3.5). \qed
Proof of Theorem 3.1. Because of the symmetry properties (2.4) and the homogeneity of norms, it suffices to consider \( \xi = (1, \lambda) \) with \( 0 \leq \lambda \leq 1 \). This restriction on \( \lambda \) will remain in force throughout this proof.

The function
\[
G(\lambda) := \| (1, \lambda) \|_{H^1} = \frac{1}{2\pi} \int_0^{2\pi} |1 + \lambda e^{it}| dt
\]
has been intensely studied due to its relation with the arclength of the ellipse and the complete elliptic integral [1, 3]. It can be written as
\[
(3.6) \quad G(\lambda) = \frac{L(x, y)}{\pi(x + y)} = 2F_1(-1/2, -1/2; 1; \lambda^2) = \sum_{n=0}^{\infty} \left( \frac{(-1/2)_n}{n!} \right)^2 \lambda^{2n}
\]
where \( L \) is the length of the ellipse with semi-axes \( x, y \) and \( \lambda = (x - y)/(x + y) \). The Pochhammer symbol \((z)_n = z(z + 1) \cdots (z + n - 1)\) and the hypergeometric function \(2F_1\) are involved in (3.6) as well. A direct way to obtain the Taylor series (3.6) for \( G \) is to use the binomial series as in (2.7).

As noted in (2.1), the \( H^4 \) norm of \((1, \lambda)\) is an elementary function:
\[
F(\lambda) := \| (1, \lambda) \|_{H^4} = (1 + 4\lambda^2 + \lambda^4)^{1/4}.
\]
The dual norm \( H^4_\ast \) can be expressed as
\[
(3.7) \quad F^\ast(\lambda) := \| (1, \lambda) \|_{H^4_\ast} = \sup_{t \in \mathbb{R}} \frac{1 + \lambda t}{(1 + 4t^2 + t^4)^{1/4}}
\]
where the second equality follows from (3.1) by letting \( b = (1, t) \). Similarly, the \( H^1_\ast \) norm of \((1, \lambda)\) is
\[
(3.8) \quad G^\ast(\lambda) := \| (1, \lambda) \|_{H^1_\ast} = \sup_{t \in \mathbb{R}} \frac{1 + \lambda t}{G(t)}.
\]

Our first goal is to prove that
\[
(3.9) \quad G^\ast(\lambda) \leq 1.01 F(\lambda).
\]
The proof of (3.9) is based on Ramanujan’s approximation \( G(\lambda) \approx 3 - \sqrt{4 - \lambda^2} \) which originally appeared in [13]; see [1] for a discussion of the history of this and several other approximations to \( G \). Barnard, Pearce, and Richards [3, Proposition 2.3] proved that Ramanujan’s approximation gives a lower bound for \( G \):
\[
(3.10) \quad G(\lambda) \geq 3 - \sqrt{4 - \lambda^2}.
\]
We will use this estimate to obtain an upper bound for \( G^\ast \).

The supremum in (3.8) only needs to be taken over \( t \geq 0 \) since the denominator is an even function. Furthermore, it can be restricted to \( t \in [0, 1] \) because for \( t > 1 \) the homogeneity
and symmetry properties of $H^1$ norm imply
\[
\frac{1 + \lambda t}{\|(1, t)\|_{H^1}} = \frac{t^{-1} + \lambda}{\|(1, t^{-1})\|_{H^1}} < \frac{1 + \lambda t^{-1}}{\|((1, t^{-1})\|_{H^1}}.
\]
Restricting $t$ to $[0, 1]$ in (3.8) allows us to use inequality (3.10):
\[
(3.11) \quad G^*(\lambda) \leq \sup_{t \in [0, 1]} \frac{1 + \lambda t}{3 - \sqrt{4 - t^2}}.
\]
Writing $t = -2 \sin \theta$ and applying Lemma 3.5 we obtain
\[
G^*(\lambda) \leq \lambda \sup_{\theta \in [-\pi/6, 0]} \frac{\lambda^{-1} - 2 \sin \theta}{3 - 2 \cos \theta} \leq \lambda \frac{3\lambda^{-1} + 2\sqrt{5 + \lambda^2}}{5}
\]
\[
= \frac{3 + 2\sqrt{1 + 5\lambda^2}}{5}.
\]
The function
\[
f(s) := \frac{3 + 2\sqrt{1 + 5s}}{(1 + 4s + s^2)^{1/4}}
\]
is increasing on $[0, 1]$. Indeed,
\[
f'(s) = \frac{3(6s + 2 - (s + 2)\sqrt{1 + 5s})}{2\sqrt{1 + 5s}(1 + 4s + s^2)^{5/4}}
\]
which is positive on $(0, 1)$ because
\[
(6s + 2)^2 - (s + 2)^2(1 + 5s) = 5s^2(3 - s) > 0.
\]
Since $f$ is increasing, the estimate (3.12) implies
\[
\frac{G^*(\lambda)}{F(\lambda)} \leq \frac{1}{5} f(\lambda^2) \leq \frac{1}{5} f(1) = \frac{3 + 2\sqrt{6}}{5 \cdot 6^{1/4}} < 1.01.
\]
This completes the proof of (3.9).

Our second goal is the following comparison of $F^*$ and $G$ with a polynomial:
\[
(3.13) \quad G(\lambda) \leq 1 + \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4 + \frac{1}{128}\lambda^6 \leq F^*(\lambda).
\]
To prove the left hand side of (3.13), let $T_4(\lambda) = 1 + \lambda^2/4 + \lambda^4/64$ be the Taylor polynomial of $G$ of degree 4. Since all Taylor coefficients of $G$ are nonnegative (3.6), the function
\[
\phi(\lambda) := \frac{G(\lambda) - T_4(\lambda)}{\lambda^6} - \frac{1}{128}
\]
is increasing on $(0, 1]$. At $\lambda = 1$, in view of (2.2), it evaluates to
\[
G(1) - 1 - \frac{1}{4} - \frac{1}{64} - \frac{1}{128} = \frac{4}{\pi} - \frac{163}{128}
\]
which is negative because $512/163 = 3.1411 \ldots < \pi$. Thus $\phi(\lambda) < 0$ for $0 < \lambda \leq 1$, proving the left hand side of (3.13).
The right hand side of (3.13) amounts to the claim that for every \( \lambda \) there exists \( t \in \mathbb{R} \) such that
\[
\frac{1 + \lambda t}{(1 + 4t^2 + t^4)^{1/4}} \geq 1 + \frac{1}{4} \lambda^2 + \frac{1}{64} \lambda^4 + \frac{1}{128} \lambda^6.
\]
This is equivalent to proving that the polynomial
\[
\Phi(\lambda, t) := (1 + \lambda t)^4 - (1 + 4t^2 + t^4) \left( 1 + \frac{1}{4} \lambda^2 + \frac{1}{64} \lambda^4 + \frac{1}{128} \lambda^6 \right)^4
\]
satisfies \( \Phi(\lambda, t) \geq 0 \) for some \( t \) depending on \( \lambda \). We will do so by choosing \( t = 4\lambda/(8 - 3\lambda^2) \).

The function
\[
\Psi(\lambda) := (8 - 3\lambda^2)^4 \Phi(\lambda, 4\lambda/(8 - 3\lambda^2))
\]
is a polynomial in \( \lambda \) with rational coefficients. Specifically,
\[
(3.14) \quad \frac{\Psi(\lambda)}{\lambda^8} = 50 + \lambda^2 - \frac{149}{24} \lambda^4 - \frac{209}{26} \lambda^6 - \frac{5375}{212} \lambda^8 - \frac{3069}{213} \lambda^{10} - \frac{8963}{217} \lambda^{12} - \frac{7837}{219} \lambda^{14} - \frac{36209}{224} \lambda^{16} - \frac{2049}{223} \lambda^{18} - \frac{1331}{225} \lambda^{20} - \frac{45}{228} \lambda^{22} - \frac{81}{225} \lambda^{24}
\]
which any computer algebra system will readily confirm. On the right hand side of (3.14), the coefficients of \( \lambda^4, \lambda^6, \lambda^8 \) are less than 10 in absolute value, while the coefficients of higher powers are less than 1 in absolute value. Thanks to the constant term of 50, the expression (3.14) is positive as long as \( 0 < \lambda \leq 1 \). This completes the proof of (3.13).

In conclusion, we have \( G(\lambda) \leq F^*(\lambda) \) from (3.13) and \( G^*(\lambda) \leq 1.01 F(\lambda) \) from (3.9). This proves the first half of (3.2) and the second half of (3.3). The other parts of (3.2)–(3.3) follow by duality.

\[\square\]

### 4. Schwarz lemma for harmonic maps

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk in the complex plane. The classical Schwarz lemma concerns holomorphic maps \( f : \mathbb{D} \to \mathbb{D} \) normalized by \( f(0) = 0 \). It asserts in part that \( |f'(0)| \leq 1 \) for such maps. This inequality is best possible in the sense that for any complex number \( \alpha \) such that \( |\alpha| \leq 1 \) there exists \( f \) as above with \( f'(0) = \alpha \). Indeed, \( f(z) = \alpha z \) works.

The story of the Schwarz lemma for harmonic maps \( f : \mathbb{D} \to \mathbb{D} \), still normalized by \( f(0) = 0 \), is more complicated. Such maps satisfy the Laplace equation \( \partial \bar{\partial} f = 0 \) written here in terms of Wirtinger’s derivatives
\[
\partial f = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \bar{\partial} f = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).
\]
The estimate \( |f(z)| \leq \frac{4}{\pi} \tan^{-1} |z| \) (see [6] or [4, p. 77]) implies that
\[
(4.1) \quad |\partial f(0)| + |\bar{\partial} f(0)| \leq \frac{4}{\pi}.
\]
Numerous generalizations and refinements of the harmonic Schwarz lemma appeared in recent years [8, 10]. An important difference with the holomorphic case is that (4.1) does not completely describe the possible values of the derivative \( \partial f(0), \bar{\partial} f(0) \). Indeed, an application of Parseval’s identity shows that

\[
|\partial f(0)|^2 + |\bar{\partial} f(0)|^2 \leq 1
\]

and neither of (4.1) and (4.2) imply each other. It turns out that the complete description of possible derivatives at 0 requires the dual Hardy norm from (3.1). The following is a refined form of Theorem 1.1 from the introduction.

**Theorem 4.1.** For a vector \((\alpha, \beta) \in \mathbb{C}^2\) the following are equivalent:

(i) there exists a harmonic map \( f: \mathbb{D} \to \mathbb{D} \) with \( f(0) = 0 \), \( \partial f(0) = \alpha \), and \( \bar{\partial} f(0) = \beta \);

(ii) there exists a harmonic map \( f: \mathbb{D} \to \mathbb{D} \) with \( \partial f(0) = \alpha \) and \( \bar{\partial} f(0) = \beta \);

(iii) \( \| (\alpha, \beta) \|_{H^1_1} \leq 1 \).

**Remark 4.2.** Both (4.1) and (4.2) easily follow from Theorem 4.1. To obtain (4.1), use the definition of \( H^1_1 \) together with the fact that \( \| (a_1, a_2) \|_{H^1} = 4/\pi \) whenever \( |a_1| = |a_2| = 1 \) (see (2.2), (2.4)). To obtain (4.2), use the comparison of Hardy norms: \( \| \cdot \|_{H^1} \leq \| \cdot \|_{H^2} \), hence \( \| \cdot \|_{H^1} \geq \| \cdot \|_{H^2} = \| \cdot \|_{H^2} \).

**Remark 4.3.** Combining Theorem 4.1 with Theorem 3.1 we obtain (4.3)

\[
\| (\partial f(0), \bar{\partial} f(0)) \|_{H^4} \leq 1
\]

for any harmonic map \( f: \mathbb{D} \to \mathbb{D} \). In view of (2.1) this means \( |\partial f(0)|^4 + 4|\partial f(0)\bar{\partial} f(0)|^2 + |\bar{\partial} f(0)|^4 \leq 1 \).

**Proof of Theorem 4.1.** (i) \( \implies \) (ii) is trivial. Suppose that (ii) holds. To prove (iii), we must show that

\[
|\alpha \bar{\gamma} + \beta \bar{\delta}| \leq \| (\gamma, \delta) \|_{H^1}
\]

for every vector \((\gamma, \delta) \in \mathbb{C}^2\). Let \( g(z) = \gamma z + \bar{\delta} \bar{z} \). Expanding \( f \) into the Taylor series \( f(z) = f(0) + \alpha z + \beta \bar{z} + \ldots \) and using the orthogonality of monomials on every circle \( |z| = r, 0 < r < 1 \), we obtain

\[
|\alpha \bar{\gamma} + \beta \bar{\delta}| = \frac{1}{2\pi r^2} \left| \int_0^{2\pi} f(re^{it})g(re^{it}) \, dt \right| \leq \frac{1}{2\pi r^2} \int_0^{2\pi} |g(re^{it})| \, dt.
\]

Letting \( r \to 1 \) and observing that

\[
\frac{1}{2\pi} \int_0^{2\pi} |\gamma e^{it} + \delta e^{-it}| \, dt = \frac{1}{2\pi} \int_0^{2\pi} |\gamma + \delta e^{2it}| \, dt = \frac{1}{2\pi} \int_0^{2\pi} |\gamma + \delta e^{it}| \, dt = \| (\gamma, \delta) \|_{H^1}
\]

completes the proof of (4.3).
It remains to prove the implication (iii) \(\Rightarrow\) (i). Let \(\mathcal{F}_0\) be the set of harmonic maps \(f: \mathbb{D} \to \mathbb{D}\) such that \(f(0) = 0\), and let \(\mathcal{D} = \{ (\partial f(0), \bar{\partial} f(0)) : f \in \mathcal{F}_0 \}\). Since \(\mathcal{F}_0\) is closed under convex combinations, the set \(\mathcal{D}\) is convex. Since the function \(f(z) = \alpha z + \beta \bar{z}\) belongs to \(\mathcal{F}_0\) when \(|\alpha| + |\beta| \leq 1\), the point \((0, 0)\) is an interior point of \(\mathcal{D}\). The estimate (4.2) shows that \(\mathcal{D}\) is bounded. Furthermore, \(c\mathcal{D} \subset \mathcal{D}\) for any complex number \(c\) with \(|c| \leq 1\), because \(\mathcal{F}_0\) has the same property. We claim that \(\mathcal{D}\) is also a closed subset of \(\mathbb{C}^2\). Indeed, suppose that a sequence of vectors \((\alpha_n, \beta_n) \in \mathcal{D}\) converges to \((\alpha, \beta) \in \mathbb{C}^2\). Pick a corresponding sequence of maps \(f_n \in \mathcal{F}_0\). Being uniformly bounded, the maps \(\{f_n\}\) form a normal family [2, Theorem 2.6]. Hence there exists a subsequence \(\{f_{n_k}\}\) which converges uniformly on compact subsets of \(\mathbb{D}\). The limit of this subsequence is a map \(f \in \mathcal{F}_0\) with \(\partial f(0) = \alpha\) and \(\bar{\partial} f(0) = \beta\).

The preceding paragraph shows that \(\mathcal{D}\) is the closed unit ball for some norm \(\| \cdot \|_\mathcal{D}\) on \(\mathbb{C}^2\). The implication (iii) \(\Rightarrow\) (i) amounts to the statement that \(\| \cdot \|_\mathcal{D} \leq \| \cdot \|_{H^1}\) for all \((\gamma, \delta) \in \mathbb{C}^2\). Since norms are continuous functions, it suffices to consider \((\gamma, \delta) \in \mathbb{C}^2\) with \(|\gamma| \neq |\delta|\). Let \(g: \mathbb{D} \to \mathbb{D}\) be the harmonic map with boundary values

\[
g(z) = \frac{\gamma z + \delta \bar{z}}{|\gamma z + \delta \bar{z}|}, \quad |z| = 1.
\]

Note that \(g(-z) = -g(z)\) on the boundary, and therefore everywhere in \(\mathbb{D}\). In particular, \(g(0) = 0\), which shows \(g \in \mathcal{F}_0\). Let \((\alpha, \beta) = (\partial g(0), \bar{\partial} g(0)) \in \mathcal{D}\). A computation similar to (4.5) shows that

\[
\gamma \bar{\alpha} + \bar{\delta} \beta = \frac{1}{2\pi} \int_0^{2\pi} (\gamma e^{it} + \delta e^{-it}) \overline{g(e^{it})} dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} (\gamma e^{it} + \delta e^{-it}) \frac{\gamma e^{it} + \delta e^{-it}}{|\gamma e^{it} + \delta e^{-it}|} dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} |\gamma e^{it} + \delta e^{-it}| dt = \| (\gamma, \delta) \|_{H^1}
\]

where the last step uses (4.6). This proves (4.7) and completes the proof of Theorem 4.1. \(\square\)

5. Higher Dimensions

A version of the Schwarz lemma is also available for harmonic maps of the (Euclidean) unit ball \(\mathbb{B}\) in \(\mathbb{R}^n\). Let \(\mathcal{S} = \partial \mathbb{B}\). For a square matrix \(A \in \mathbb{R}^{n \times n}\), define its Hardy quasinorm by

\[
\| A \|_{H^p} = \left( \int_{\mathcal{S}} \| Ax \|^p d\mu(x) \right)^{1/p}
\]
where the integral is taken with respect to normalized surface measure \( \mu \) on \( S \) and the vector norm \( \|Ax\| \) is the Euclidean norm. In the limit \( p \to \infty \) we recover the spectral norm of \( A \), while the special case \( p = 2 \) yields the Frobenius norm of \( A \) divided by \( \sqrt{n} \). The case \( p = 1 \) corresponds to “expected value norms” studied by Howe and Johnson in [7]. Also, letting \( p \to 0 \) leads to

\[
\|A\|_{H^0} = \exp \left( \int_S \log \|Ax\| \, d\mu(x) \right)
\]

In general, \( H^p \) quasinorms on matrices are not submultiplicative. However, they have another desirable feature, which follows directly from (5.1):

\[
\|UAV\|_{H^p} = \|A\|_{H^p} = \|D\|_{H^p}
\]

where \( D \) is the diagonal matrix with the singular values of \( A \) on its diagonal.

Let us consider the matrix inner product \( \langle A, B \rangle = \frac{1}{n} \text{tr}(B^T A) \), which is normalized so that \( \langle I, I \rangle = 1 \). This inner product can be expressed by an integral involving the standard inner product on \( \mathbb{R}^n \) as follows:

\[
\langle A, B \rangle = \int_S \langle Ax, Bx \rangle \, d\mu(x).
\]

Indeed, the right hand side of (5.3) is the average of the numerical values \( \langle B^T Ax, x \rangle \), which is known to be the normalized trace of \( B^T A \), see [9].

The dual norms \( H^p_* \) are defined on \( \mathbb{R}^{n \times n} \) by

\[
\|A\|_{H^p_*} = \sup \{ \langle A, B \rangle : \|B\|_{H^p} \leq 1 \} = \sup_{B \in \mathbb{R}^{n \times n} \setminus \{0\}} \frac{\langle A, B \rangle}{\|B\|_{H^p}}
\]

Applying Hölder’s inequality to (5.3) yields \( \langle A, B \rangle \leq \|A\|_{H^q} \|B\|_{H^p} \) when \( p^{-1} + q^{-1} = 1 \). Hence \( \|A\|_{H^p_*} \leq \|A\|_{H^q} \) but in general the inequality is strict. As an exception, we have \( \|A\|_{H^2} = \|A\|_{H^2} \) because \( \langle A, A \rangle = \|A\|_{H^2}^2 \). As in the case of polynomials, our interest in dual Hardy norms is driven by their relation to harmonic maps.

**Theorem 5.1.** For a matrix \( A \in \mathbb{R}^{n \times n} \) the following are equivalent:

(i) there exists a harmonic map \( f : \mathbb{B} \to \mathbb{B} \) with \( f(0) = 0 \) and \( Df(0) = A \);
(ii) there exists a harmonic map \( f : \mathbb{B} \to \mathbb{B} \) with \( Df(0) = A \);
(iii) \( \|A\|_{H^1} \leq 1 \).

**Proof.** Since the proof is essentially the same as of Theorem 4.1, we only highlight some notational differences. Suppose (ii) holds. Expand \( f \) into a series of spherical harmonics, \( f(x) = \sum_{d=0}^{\infty} p_d(x) \) where \( p_d : \mathbb{R}^n \to \mathbb{R}^n \) is a harmonic polynomial map that is homogeneous of degree \( d \). Note that \( p_1(x) = Ax \). For any \( n \times n \) matrix \( B \) the orthogonality of spherical
harmonics \[2\] Proposition 5.9] yields
\[
\langle A, B \rangle = \lim_{r \to 1} \int_{S} \langle f(rx), Bx \rangle \, d\mu(x) \leq \|B\|_1
\]
which proves (iii).

The proof of (iii) \(\implies\) (i) is based on considering, for any nonsingular matrix \(B\), a harmonic map \(g\): \(B \to B\) with boundary values \(g(x) = \frac{(Bx)/\|Bx\|}{\|Bx\|}\). Its derivative \(A = Dg(0)\) satisfies
\[
\langle B, A \rangle = \int_{S} \langle Bx, g(x) \rangle \, d\mu(x) = \int_{S} \frac{(Bx, Bx)}{\|Bx\|} \, d\mu(x) = \|B\|_{H^1}
\]
and (i) follows by the same duality argument as in Theorem 4.1. □

As an indication that the near-isometric duality of \(H^1\) and \(H^4\) norms (Theorem 3.1) may also hold in higher dimensions, we compute the relevant norms of the matrix of an orthogonal projection of rank \(k\) in \(\mathbb{R}^3\). For rank 1 projection, the norms are
\[
\|P_1\|_{H^1} = \int_{0}^{1} r \, dr = \frac{1}{2},
\]
\[
\|P_1\|_{H^4} = \left( \int_{0}^{1} r^4 \, dr \right)^{1/4} = \frac{1}{\sqrt[4]{5}} \approx 0.67,
\]
\[
\|P_1\|_{H^1}^* = \frac{\langle P_1, P_1 \rangle}{\|P_1\|_1} = \frac{1/3}{1/2} = \frac{2}{3} \approx 0.67.
\]
For rank 2 projection, they are
\[
\|P_2\|_{H^1} = \int_{0}^{1} \sqrt{1 - r^2} \, dr = \frac{\pi}{4},
\]
\[
\|P_2\|_{H^4} = \left( \int_{0}^{1} (1 - r^2)^2 \, dr \right)^{1/4} = \left( \frac{8}{15} \right)^{1/4} \approx 0.85,
\]
\[
\|P_2\|_{H^1}^* = \frac{\langle P_2, P_2 \rangle}{\|P_2\|_1} = \frac{2/3}{\pi/4} = \frac{8}{3\pi} \approx 0.85.
\]
This numerical agreement does not appear to be merely a coincidence, as numerical experiments with random \(3 \times 3\) indicate that the ratio \(\|A\|_{H^1} / \|A\|_{H^4}\) is always near 1. However, we do not have a proof of this.

As in the case of polynomials, there is an explicit formula for the \(H^4\) norm of matrices. Writing \(\sigma_1, \ldots, \sigma_n\) for the singular values of \(A\), we find
\[
\langle A, B \rangle = \lim_{r \to 1} \int_{S} \langle f(rx), Bx \rangle \, d\mu(x) \leq \|B\|_1
\]
(5.5) \[\|A\|^4_{H^1} = \alpha \sum_{k=1}^{n} \sigma_k^4 + 2\beta \sum_{k<l} \sigma_k^2 \sigma_l^2\]
where \( \alpha = \int_S x_1^4 d\mu(x) \) and \( \beta = \int_S x_1^2 x_2^2 d\mu(x) \). For example, if \( n = 3 \), the expression \([5.5]\)
evaluates to
\[
\|A\|_{H^4}^4 = \frac{1}{5} \sum_{k=1}^{3} \sigma_k^4 + \frac{2}{15} \sum_{k<l} \sigma_k^2 \sigma_l^2.
\]

Theorem \([2.1]\) has a corollary for \( 2 \times 2 \) matrices.

**Corollary 5.2.** The \( H^p \) quasinorm on the space of \( 2 \times 2 \) matrices satisfies the triangle inequality even when \( 0 \leq p < 1 \).

**Proof.** A real linear map \( x \mapsto Ax \) in \( \mathbb{R}^2 \) can be written in complex notation as \( z \mapsto az + b\bar{z} \) for some \( (a, b) \in \mathbb{C}^2 \). A change of variable yields
\[
\int_{|z|=1} |az + b\bar{z}|^p = \int_{|z|=1} |a + bz|^p
\]
which implies \( \|A\|_{H^p} = \|(a, b)\|_{H^p} \) for \( p > 0 \). The latter is a norm on \( \mathbb{C}^2 \) by Theorem \([2.1]\). The case \( p = 0 \) is treated in the same way. \( \square \)

The aforementioned relation between a \( 2 \times 2 \) matrix \( A \) and a complex vector \( (a, b) \) also shows that the singular values of \( A \) are \( \sigma_1 = |a| + |b| \) and \( \sigma_2 = ||a| - |b|| \). It then follows from \([2.1]\) that
\[
\|A\|_{H^0} = \max(|a|, |b|) = \frac{\sigma_1 + \sigma_2}{2},
\]
which is, up to scaling, the trace norm of \( A \). Unfortunately, this relation breaks down in dimensions \( n > 2 \): for example, rank 1 projection \( P_1 \) in \( \mathbb{R}^3 \) has \( \|P_1\|_{H^0} = 1/e \) while the average of its singular values is \( 1/3 \).

We do not know whether \( H^p \) quasinorms with \( 0 \leq p < 1 \) satisfy the triangle inequality for \( n \times n \) matrices when \( n \geq 3 \).

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