Long-time behavior of several point particles in a 1D viscous compressible fluid

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Abstract. We study the long-time behavior of several point particles in a 1D viscous compressible fluid. It is shown that the velocities of the point particles all obey the power law $t^{-3/2}$. This result extends author’s previous works on the long-time behavior of a single point particle. New difficulties arise in the derivation of pointwise estimates of Green’s functions due to infinite reflections of waves in-between the point particles. In particular, the differential equation technique used in previous works alone does not suffice. We overcome this by carefully analyzing the structure of Green’s functions in the Laplace variable, especially their asymptotic and analyticity properties.

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1. Introduction

1.1. Long-time behavior of a solid in a fluid

Fluid–structure interaction problems deal with phenomena caused by interaction of moving or deforming solids with fluid flows. Mathematically, this requires us to simultaneously solve PDEs for the fluids and ODEs for the solids. This brings in new aspects to mathematical analysis of fluid dynamical equations and has attracted attention of many mathematicians.

Amongst various interesting aspects of fluid–structure interaction problems, we focus here on the problem of long-time behavior of moving solids. There are several works on this problem, and we shall briefly review these results. In, Liu analyzed the motion of a point particle\(^1\) in a 1D inviscid compressible fluid: he proved that, starting from a small perturbation of a constant state, the velocity \(V(t)\) of the point particle decays at least as \(t^{-3/2}\). Afterwards, another important result was obtained in the work by Vázquez and Zuazua [25]: they considered the motion of a point particle in a 1D viscous Burgers fluid and showed that the velocity \(V(t)\) of the point particle obeys a power law \(t^{-1/2}\) (no restriction on the size of the initial data required). In multi-dimension, Ervedoza, Hillairet, and Lacave considered the motion of a disk in a 2D viscous incompressible fluid and showed that the velocity \(V(t)\) of the disk decays at least as \(t^{-(2-\epsilon)/2}\) (here \(\epsilon\) is a positive number that can be taken arbitrary small for sufficiently regular and small initial data) [3]. Moreover, this result was recently extended to the 3D case by Ervedoza, Maity, and Tucsnak [4]: they showed the decay estimate \(|V(t)| \lesssim t^{-(3-\epsilon)/2}\).

All of the works mentioned above consider the motion of a solid in an unbounded fluid domain. Of course, there are works dealing with the motion of a solid in a bounded fluid domain. We refer, for example, to [5,15,20,24] and the references therein. In this case, the velocity \(V(t)\) of the solid usually decays exponentially fast, and the interest of the research mainly lies in constructing and analyzing solutions without restrictions on the size of initial data.

One of the missing pieces in the works mentioned above was the corresponding problem for 1D viscous compressible fluids. For this problem, we recently showed in [14] that the velocity \(V(t)\) of a point particle moving in a 1D viscous compressible fluid decays at least as \(t^{-3/2}\) for small initial data. Moreover, in [13], we revealed a simple necessary and sufficient condition on the initial data guaranteeing the optimality of the decay rate \(-3/2\). These results were obtained by applying and refining the method of pointwise estimates of Green’s function developed and used, for example, in [1,2,16,17,19,27].

\(^1\)Some authors prefer to call it a piston. This is just a matter of taste. We chose the terminology a point particle to emphasise that the problem is considered in a one-dimensional setting.
1.2. Long-time behavior of several solids in a fluid

Given these basic understanding of long-time behavior of a single solid in a fluid, a natural question comes up: what happens when there are several solids? This question was addressed, for example, in [26] where the authors studied the long-time behavior of several point particles in a 1D viscous Burgers fluid. One of the interesting results of their paper is that collisions between point particles do not occur in finite time; interestingly, they also showed that the distances between the particles may asymptotically converge to zero. We also refer to [8] for a related result. These works then prompted investigations on the possibility of collisions between solids in multi-dimensions: see, e.g., [6,7,9–11,21–23] for such results.

Now what happens for the motion of several point particles in the 1D viscous compressible fluid considered in [13,14]? In this paper, we answer to this question. As it turns out, collisions between point particles do not occur; neither do they collide in finite or infinite time. This is because collisions imply unbounded growth of the fluid pressure at the contact point (see Theorem 2.3 and its proof). This part is rather easy, and the difficult part is the analysis of the decay property of the velocities of the point particles. The main technical difficulty is the derivation of sharp enough pointwise estimates of Green’s functions. The presence of several point particles makes the expression of Green’s functions in the Laplace variable quite complex, and the differential equation technique used in previous works (e.g. [1,2,14]) alone does not suffice. We overcome this difficulty by carefully analyzing the structure of Green’s functions in the Laplace variable, especially their asymptotic and analyticity properties (Sect. 3.2). As a consequence, we prove that the velocities of the point particles all decay as $t^{-3/2}$.

In the rest of this section, we give the formulation of the problem. The main theorems are presented in Sect. 2. The proof is given in Sect. 3.

1.3. Motion of two point particles in a 1D viscous compressible fluid: formulation

Let us explain the equations we consider in this paper. Although it is possible to treat three or more point particles, let us restrict ourselves to two point particles for simplicity.

Consider a one-dimensional flow in the real line $\mathbb{R}$, and let $X$ be a Cartesian coordinate on $\mathbb{R}$. Denote by $\rho = \rho(X, t)$ and $U = U(X, t)$ the density and the velocity of the fluid. We assume that the fluid is viscous with a constant viscosity coefficient $\nu > 0$ and that the fluid is barotropic, that is, the pressure $P$ is a function only of the density $\rho$: $P = P(\rho)$. For the point masses, denote by $X = h_0(t)$ and $X = h_1(t)$ the locations of point particles number 0 and 1; we assume that $h_0(t) < h_1(t)$. Their velocities are denoted by $V_0(t) = h'_0(t)$ and $V_1(t) = h'_1(t)$. For simplicity, we assume that the masses of the point particles are both unity.
With the assumptions and notations above, the fluid–solids system is governed by the following equations (cf. [13, Sect. 1.1]):

\[
\begin{align*}
\rho_t + (\rho U)_X &= 0, & X &\in \mathbb{R}\setminus[h_0(t), h_1(t)], t > 0, \\
(\rho U)_t + (\rho U^2)_X + P(\rho)_X &= \nu U_{XX}, & X &\in \mathbb{R}\setminus[h_0(t), h_1(t)], t > 0, \\
U(h_0(t)_+, t) &= V_0(t), & U(h_1(t)_+, t) &= V_1(t), & t > 0, \\
V'_0(t) &= \left[ -P(\rho) + \nu U_X \right](h_0(t), t), & t > 0, \\
V'_1(t) &= \left[ -P(\rho) + \nu U_X \right](h_1(t), t), & t > 0, \\
h_0(0) &= h_0^0, & V_0(0) &= V_0^0, & h_1(0) &= h_1^0, & V_1(0) &= V_1^0, \\
(\rho(X, 0), U(X, 0)) &= (\rho_0(X), U_0(X)), & X &\in \mathbb{R}\setminus[h_0^0, h_1^0].
\end{align*}
\]

(1)

Here, \( f(X_+, t) \) and \( f(X_-, t) \) denote \( \lim_{Y \searrow X} f(Y, t) \) and \( \lim_{Y \nearrow X} f(Y, t) \), respectively; note also that when we write \( f(X_\pm, t) = g(t) \), this means that \( f(X_+, t) = f(X_-, t) = g(t) \). The double brackets denote the jump of a function inside them: \( \left[ f \right](X, t) = f(X_+, t) - f(X_-, t) \). The first two equations are the 1D barotropic compressible Navier–Stoke equations, and the equations in the third line are boundary conditions for them. The equations in the fourth and the fifth lines are Newton’s equations of motion for the point particles. The rest are initial conditions.

Note that the equations above are posed in a time-dependent domain \( \mathbb{R}\setminus[h_0(t), h_1(t)] \). To cast the domain into a time-independent one, we introduce the Lagrangian mass coordinate. We assume for simplicity that \( \int_{h_0^0}^{h_1^0} \rho_0(X) \, dX = 1 \). Then by (1), we have \( \int_{h_0(t)}^{h_1(t)} \rho(X, t) \, dX = 1 \) for \( t \geq 0 \). Now, fix \( x \in \mathbb{R}\setminus[0, 1] \) and \( t \geq 0 \), and let \( X = X(x, t) \) be the solution to

\[
x = \int_{h_0(t)}^{X(x, t)} \rho(X', t) \, dX'.
\]

(2)

Let us assume that \( \rho(X, t) \geq \rho_0 \) for some \( \rho_0 > 0 \) (we only consider such solutions in this paper). Then (2) is uniquely solvable and determines a one-to-one map

\[
\mathbb{R}\setminus[0, 1] \ni x \mapsto X(x, t) \in \mathbb{R}\setminus[h_0(t), h_1(t)].
\]

This new coordinate \( x \) is the Lagrangian mass coordinate. Now using this, we define

\[
v(x, t) = \frac{1}{\rho(X(x, t), t)}, \quad u(x, t) = U(X(x, t), t), \quad p(v) = P \left( \frac{1}{v} \right).
\]

The quantity \( v \) is called the specific volume of the fluid. Note that by using (1), it follows that

\[
\frac{\partial X(x, t)}{\partial x} = v, \quad \frac{\partial X(x, t)}{\partial t} = u.
\]
Then we can show that, in terms of these new variables, (1) is equivalent to

\[
\begin{cases}
  v_t - u_x = 0, & x \in \mathbb{R}\setminus\{0, 1\}, t > 0, \\
  u_t + p(v)_x = v \left( \frac{u_x}{v} \right)_x, & x \in \mathbb{R}\setminus\{0, 1\}, t > 0, \\
  u(0_{\pm}, t) = V_0(t), u(1_{\pm}, t) = V_1(t), & t > 0, \\
  V'_0(t) = \left[-p(v) + vu_x/v\right](0, t), & t > 0, \\
  V'_1(t) = \left[-p(v) + vu_x/v\right](1, t), & t > 0, \\
  V_0(0) = V_0^0, V_1(0) = V_1^0; v(x, 0) = v_0(x), u(x, 0) = u_0(x), & x \in \mathbb{R}\setminus\{0, 1\}.
\end{cases}
\]

Here,

\[ v_0(x) = \frac{1}{\rho_0(X(x, 0))}, \quad u_0(x) = U_0(X(x, 0)). \]

We note that (3) does not contain \( h_0(t) \) and \( h_1(t) \), but we can recover them by

\[
h_0(t) = h_0^0 + \int_0^t V_0(s) \, ds \quad \text{and} \quad h_1(t) = h_1^0 + \int_0^t V_1(s) \, ds.
\]

2. Main theorems

The main theorems of this paper concern pointwise estimates of solutions to (3), from which results on the long-time behavior of point particles are derived as corollaries. These results are extensions of the results in [13,14] to several point particles.

2.1. Preliminaries

To state the main theorems, we start with some preliminaries. First, we study the structure of the linearized equations of the first two equations in (3) around the constant state \((v, u) = (v^*, 0)\). Here, the reference specific volume \( v = v_\ast \) can in fact be any positive number but we set \( v^* = 1 \) for simplicity. Then the linearized equations can be written as

\[
\begin{align*}
  u_t + Au_x &= Bu_{xx} + \begin{pmatrix} 0 \\ N_x \end{pmatrix}, \\
  A &= \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}, \\
  B &= \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}, \\
  N &= -p(v) + p(1) - c^2(v - 1) - \frac{v}{v - 1}u_x.
\end{align*}
\]

Here, \( c > 0 \) is the speed of sound for the state \((v, u) = (1, 0)\) defined by \( c^2 = -p'(1) \); for \( c \) to be well-defined, we assume that \( p'(1) < 0 \). The matrix \( A \) has two eigenvalues \( \lambda_1 = c \) and \( \lambda_2 = -c \), and as right and left eigenvectors of \( A \) corresponding to \( \lambda_i \), we can take \( r_i \) and \( l_i \), respectively, as follows:

\[
r_1 = \frac{2c}{p''(1)} \begin{pmatrix} -1 \\ c \end{pmatrix}, \quad r_2 = \frac{2c}{p''(1)} \begin{pmatrix} 1 \\ c \end{pmatrix}.
\]

\[
l_1 = \frac{2c}{p''(1)} \begin{pmatrix} 1 \\ c \end{pmatrix}, \quad l_2 = \frac{2c}{p''(1)} \begin{pmatrix} -1 \\ c \end{pmatrix}.
\]

\[
l_1 = \frac{2c}{p''(1)} \begin{pmatrix} 1 \\ c \end{pmatrix}, \quad l_2 = \frac{2c}{p''(1)} \begin{pmatrix} -1 \\ c \end{pmatrix}.
\]
and
\[ l_1 = \frac{p''(1)}{4c} (-1 1/c), \quad l_2 = \frac{p''(1)}{4c} (1 1/c). \]
Here and in what follows, we assume that \( p''(1) \neq 0 \).

We next decompose \( u = \ell(v - 1, u) \) with respect to the eigenbasis \((r_1, r_2)\):
\[ u = u_1 r_1 + u_2 r_2. \]
(6)
Taking into account the relation
\[ \left( \begin{array}{c} l_1 \\ l_2 \end{array} \right) \left( \begin{array}{c} r_1 \\ r_2 \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right), \]
we can calculate the component \( u_i \) by
\[ u_i = l_i u. \]

Next, we introduce diffusion waves as in [14,19]. Let
\[ M_i = \int_{-\infty}^{\infty} u_{0i}(x) \, dx + l_i \left( \begin{array}{c} 0 \\ V_0^0 \\ 0 \\ V_1^0 \end{array} \right), \]
(7)
where
\[ u_{0i} = l_i \left( \begin{array}{c} v_0 - 1 \\ u_0 \end{array} \right). \]
Then the \( i \)-th diffusion wave with mass \( M_i \) is defined as the solution \( \theta_i \) to generalized Burgers’ equation
\[ \partial_t \theta_i + \lambda_i \partial_x \theta_i + \partial_x \left( \frac{\theta_i^2}{2} \right) = \nu \partial_x^2 \theta_i, \quad x \in \mathbb{R}, \ t > 0 \]
(8)
with the initial condition
\[ \lim_{t \to -1} \theta_i(x, t) = M_i \delta(x). \]
(9)
Here, \( \delta(x) \) is the Dirac delta function. By the Cole–Hopf transformation, we can solve (8) and (9) explicitly to obtain
\[ \theta_i(x, t) = \frac{\sqrt{\nu}}{\sqrt{2(t+1)}} \left( e^{M_i t} - 1 \right) e^{- \frac{(x - \lambda_i (t+1))^2}{2\nu(t+1)}} \left[ \sqrt{\pi} + \left( e^{\frac{M_i}{\nu}} - 1 \right) \int_{\lambda_i \sqrt{\nu}(t+1)}^{\infty} e^{-y^2} \, dy \right]^{-1}. \]

Next, we introduce inter-diffusion waves as in [13]: the \( i \)-th inter-diffusion wave with mass pair \((M_1, M_2)\) is defined as the solution \( \xi_i \) to the following variable coefficient inhomogeneous convective heat equation:
\[ \partial_t \xi_i + \lambda_i \partial_x \xi_i + \partial_x (\theta_i \xi_i) + \partial_x \left( \frac{\theta_i^2}{2} \right) = \nu \partial_x^2 \xi_i, \quad x \in \mathbb{R}, \ t > 0 \]
(10)
with the initial condition

\[ \xi_i(x, 0) = 0, \quad x \in \mathbb{R}. \]

Here, \(i' = 3 - i\), i.e., \(1' = 2\) and \(2' = 1\).

We next define some auxiliary functions. First, let

\[
\psi_a(x, t; \lambda_i) = [(x - \lambda_i(t + 1))^2 + (t + 1)]^{-a/2},
\]

\[
\tilde{\psi}(x, t; \lambda_i) = [(x - \lambda_i(t + 1))^3 + (t + 1)^2]^{-1/2},
\]

\[
\bar{\psi}(x, t; \lambda_i) = [(x - \lambda_i(t + 1))^7 + (t + 1)^5]^{-1/4},
\]

and

\[
\Phi_i(x, t) = \psi_{3/2}(x, t; \lambda_i) + \tilde{\psi}(x, t; \lambda_i'),
\]

\[
\Psi_i(x, t) = \psi_{7/4}(x, t; \lambda_i) + \bar{\psi}(x, t; \lambda_i').
\]

Moreover, the following quantities are needed to state the compatibility conditions ensuring certain regularity of solutions near the space-time corners \((x, t) = (0, 0), (1, 0)\):

\[
\mathcal{C}_1(v, u) := -p(v) + \frac{v}{v} \frac{u_x}{v}, \quad \mathcal{C}_2(v, u) := -p'(v)u_x + \frac{v}{v} \mathcal{C}_1(v, u)_{xx} - \frac{v}{v^2} u_{xx}.
\]

Note that \(u_t = \mathcal{C}_1(v, u)_x, u_{tt} = \mathcal{C}_2(v, u)_x, V'_0(t) = [\mathcal{C}_1(v, u)](0, t)\), and so on. Therefore, if the solution \((v, u, V_0, V_1)\) to (3) is sufficiently regular near \((x, t) = (0, 0)\) for example, we should have \(\mathcal{C}_1(v_0, u_0)_x(0) = \lim_{x \to 0} u_t(x, 0) = V'_0(0) = [\mathcal{C}_1(v_0, u_0)](0, t)\). The compatibility conditions in Theorems 2.1 and 2.2 are derived in this way. Also, let

\[
u_{0i}^- (x) := \int_{-\infty}^{x} u_{0i}^- (y) \, dy, \quad u_{0i}^+ (x) := \int_{x}^{\infty} u_{0i}^+ (y) \, dy.
\]

Finally, let \([f]\)(x) := f(x^+) - f(x^-) and denote by \(\left\| \cdot \right\|_k (k \in \mathbb{N})\) the Sobolev \(H^k(\mathbb{R}\{0, 1\})\)-norm.

2.2. Pointwise estimates of solutions and the long-time behavior of the point particles

The first of our main theorems is the following, which is an extension of [14, Theorem 1.2].

**Theorem 2.1.** Let \(v_0 - 1, u_0 \in H^4(\mathbb{R}\{0, 1\})\) and \(V_0^0, V_1^0 \in \mathbb{R}\). Assume that they satisfy the following compatibility conditions:

\[
u_0(0_\pm) = V_0^0, \quad u_0(1_\pm) = V_1^0,
\]

\[
\mathcal{C}_1(v_0, u_0)_x(0_\pm) = [\mathcal{C}_1(v_0, u_0)](0), \quad \mathcal{C}_1(v_0, u_0)_x(1_\pm) = [\mathcal{C}_1(v_0, u_0)](1).
\]
Under these assumptions, there exist $\delta_0', C > 0$ such that if

$$\delta' := \frac{2}{2}\left\{ ||u_0||_4 + ||u_0^-||_{L^1(-\infty, 0)} + ||u_0^+||_{L^1(0, \infty)} \right\} + \sup_{x \in \mathbb{R} \setminus [0, 1]} \left[ ||x + 1||_{4/2}/u_0(x) \right] + \sup_{x > 0} \left[ ||x + 1/(u_0^-(-x)) + ||u_0^+(x)|| \right] \leq \delta_0',$$  

(11)

then (3) has a unique global-in-time solution $(v, u, V_0, V_1)$ satisfying

$$v - 1 \in C([0, \infty); H^4(\mathbb{R} \setminus [0, 1])) \cap C^1([0, \infty); H^3(\mathbb{R} \setminus [0, 1])),$$

$$u \in C([0, \infty); H^4(\mathbb{R} \setminus [0, 1])) \cap C^1([0, \infty); H^2(\mathbb{R} \setminus [0, 1]),$$

$$u_x \in L^2(0, \infty; H^4(\mathbb{R} \setminus [0, 1]),$$

$$V_0, V_1 \in C^2([0, \infty))$$

and

$$||(v-1)(t)||_4 + ||u(t)||_4 + \left( \int_0^\infty ||u_x(s)||_4^2 \, ds \right)^{1/2} + \sum_{i=0}^1 \sum_{k=0}^2 |\partial_t^k V_i(t)| \leq C \delta' (t \geq 0).$$

Moreover, this solution satisfies the following pointwise estimates:

$$|u_i - \theta_i(x, t)| \leq C \delta' \Phi_i(x, t) \quad (x \in \mathbb{R} \setminus [0, 1], t \geq 0; i = 1, 2).$$

(12)

Going back to the original variables $(v, u)$ from $(u_1, u_2)$ is made possible by the formulae

$$v - 1 = \frac{2c}{p''(1)}(-u_1 + u_2), \quad u = \frac{2c^2}{p''(1)}(u_1 + u_2),$$

which follow from (5) and (6). In particular, the pointwise estimates (12) in Theorem 2.1 imply the following decay estimates for $V_0(t)$ and $V_1(t)$ by simply noting that $V_0(t) = u(0_+, t)$ and $V_1(t) = u(1_+, t)$ (see the proof of [14, Corollary 1.2] for more details).

**Corollary 2.1.** Under the assumptions of Theorem 2.1, there exist $\delta_0', C > 0$ such that if (11) holds, then the solution $(v, u, V_0, V_1)$ to (3) satisfies

$$|V_i(t)| \leq C \delta'(t + 1)^{-3/2} \quad (t \geq 0; i = 0, 1).$$

Decay estimates of $v - 1$ and $u$ in the $L^p([0, 1])$-norm can easily be deduced from the pointwise estimates (12) in Theorem 2.1.²

²Similar decay estimates for $r - 1$ and $U$ can also be obtained by noting that the change of variable from the Lagrangian mass coordinate $x$ to the Eulerian coordinate $X$ satisfies $\partial X/\partial x = v$ and that $v$ is bounded from above and below by positive constants uniformly in time when $\delta'$ is sufficiently small.
Corollary 2.2. Let
\[ \varphi_v = \frac{2c}{p''(1)}(-\theta_1 + \theta_2), \quad \varphi_u = \frac{2c^2}{p''(1)}(\theta_1 + \theta_2). \]
Now under the assumptions of Theorem 2.1, there exist \( \delta'_0, C > 0 \) such that if \( (11) \) holds, the solution \((v, u, V_0, V_1)\) to \( (3) \) satisfies
\[ \| (v - 1 - \varphi_v, u - \varphi_u) (t) \|_{L^p(\mathbb{R}\setminus\{0, 1\})} \leq C\delta'(t + 1)^{-\frac{1}{2}\left(\frac{3}{2} - \frac{1}{p}\right)} \quad (1 \leq p \leq \infty), \]
whereas
\[ \| (\varphi_v, \varphi_u) (t) \|_{L^p(\mathbb{R}\setminus\{0, 1\})} \leq C\delta'(t + 1)^{-\frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right)} \quad (1 \leq p \leq \infty). \]

With some additional assumptions on the regularity and the spatial decay of initial data, we can obtain finer pointwise estimates corresponding to [13, Theorem 2.1].

Theorem 2.2. Let \( v_0 - 1, u_0 \in H^6(\mathbb{R}\setminus\{0, 1\}) \) and \( V_0^0, V_1^0 \in \mathbb{R} \). Assume that they satisfy the following compatibility conditions:
\[ u_0(0 \pm) = V_0^0, \quad u_0(1 \pm) = V_1^0, \]
\[ C_1(v_0, u_0)_x(0 \pm) = [C_1(v_0, u_0)](0), \quad C_1(v_0, u_0)_x(1 \pm) = [C_1(v_0, u_0)](1), \]
\[ C_2(v_0, u_0)_x(0 \pm) = [C_2(v_0, u_0)](0), \quad C_2(v_0, u_0)_x(1 \pm) = [C_2(v_0, u_0)](1). \]

Under these assumptions, there exist \( \delta_0, C > 0 \) such that if
\[ \delta := \sum_{i=1}^2 \left\{ \| u_{0i} \|_6 + \sup_{x \in \mathbb{R}\setminus\{0, 1\}} \left[ (|x| + 1)^{7/4}|u_{0i}(x)| + (|x| + 1)^{3/2}|u'_{0i}(x)| \right] \right. \]
\[ \left. + \sup_{x > 0} \left[ (|x| + 1)^{5/4}|u_{0i}^{(0, -)}(-x)| + |u_{0i}^{(0, +)}(x)| \right] \right\} \leq \delta_0, \]
then \( (3) \) has a unique global-in-time solution \((v, u, V_0, V_1)\) satisfying
\[ v - 1 \in C([0, \infty); H^6(\mathbb{R}\setminus\{0, 1\}) \cap C^1([0, \infty); H^5(\mathbb{R}\setminus\{0, 1\})), \]
\[ u \in C([0, \infty); H^6(\mathbb{R}\setminus\{0, 1\}) \cap C^1([0, \infty); H^4(\mathbb{R}\setminus\{0, 1\})), \]
\[ u_x \in L^2([0, \infty); H^6(\mathbb{R}\setminus\{0, 1\})), \]
\[ V_0, V_1 \in C^3([0, \infty)) \]
and
\[ \| (v - 1)(t) \|_6 + \| u(t) \|_6 + \left( \int_0^\infty \| u_x(s) \|^2_6 ds \right)^{1/2} + \sum_{i=0}^1 \sum_{k=0}^3 |\partial_t^k V_i(t)| \leq C\delta \quad (t \geq 0). \]

Moreover, this solution satisfies the following pointwise estimates:
\[ |(u_i - \theta_i - \xi_i - \gamma_i') \partial_x \theta_i')(x, t)| \leq C\delta \Psi_i(x, t) \quad (x \in \mathbb{R}\setminus\{0, 1\}, t \geq 0; i = 1, 2), \]
where \( i' = 3 - i \) and \( \gamma_i = (-1)^i v/(4c). \)
From this theorem, we obtain a simple necessary and sufficient condition for the optimality of the decay estimate $V_i(t) = O(t^{-3/2})$ given in Corollary 2.1 (see the proof of [13, Corollaries 2.1 and 2.2] for more details).

**Corollary 2.3.** Define $M_i$ by (7) and assume that $(M_1 + M_2)(M_1 - M_2) \neq 0$, that is,
\[
\left( \int_{-\infty}^{\infty} (v_0 - 1)(x) \, dx \right) \cdot \left( \int_{-\infty}^{\infty} u_0(x) \, dx + V_0^0 + V_1^0 \right) \neq 0.
\]

Then under the assumptions of Theorem 2.2, there exist $\delta_0 > 0$, $C > 1$, and $T(\delta) > 0$ such that if (13) holds, then the solution $(v, u, V_0, V_1)$ to (3) satisfies
\[
C^{-1}|M_1^2 - M_2^2|(t + 1)^{-3/2} \leq (\text{sgn}(M_1^2 - M_2^2))V_i(t) \leq C|M_1^2 - M_2^2|(t + 1)^{-3/2}
\]
for $t \geq T(\delta)$ and $i = 1, 2$. In particular, this implies
\[
C^{-1}|M_1^2 - M_2^2|(t + 1)^{-3/2} \leq |V_i(t)| \leq C|M_1^2 - M_2^2|(t + 1)^{-3/2} \quad (t \geq T(\delta)).
\]

**Corollary 2.4.** Define $M_i$ by (7) and assume that $(M_1 + M_2)(M_1 - M_2) = 0$, that is,
\[
\left( \int_{-\infty}^{\infty} (v_0 - 1)(x) \, dx \right) \cdot \left( \int_{-\infty}^{\infty} u_0(x) \, dx + V_0^0 + V_1^0 \right) = 0.
\]

Then under the assumptions of Theorem 2.2, there exist $\delta_0$, $C > 0$ such that if (13) holds, then the solution $(v, u, V_0, V_1)$ to (3) satisfies
\[
|V_i(t)| \leq C\delta(t + 1)^{-7/4} \quad (t \geq 0; \ i = 0, 1).
\]

A result similar to Corollary 2.2 can be obtained from Theorem 2.2 though we omit the detail for brevity.

2.3. Discussion

2.3.1. Long-time behavior of the point particles: comparison with the single point particle case

Corollary 2.1 shows that, in general, we have $V(t) = O(t^{-3/2})$. If, in addition, we assume that (15) holds, which is equivalent to
\[
\int_{-\infty}^{\infty} \rho_0(X) \, dX \neq 0, \quad \int_{-\infty}^{\infty} (\rho_0 U_0)(X) \, dX + V_0^0 + V_1^0 \neq 0
\]
in the Eulerian coordinate, Corollary 2.3 shows that the decay rate $-3/2$ is optimal. Moreover, Corollary 2.4 tells us that the condition above is a necessary and sufficient condition for the optimality of the decay rate $-3/2$.

These conclusions are similar to the ones for a single point particle [13,14]; on the other hand, their proofs require new ingredients and the novelty of this paper lies here.
What makes the analysis harder? To obtain accurate pointwise estimates of solutions, it is essential to obtain correspondingly accurate pointwise estimates of Green’s functions. And it is this part that the presence of several point particles is troublesome. Physically, the trouble is due to “infinite reflections” of waves at the point particles: when there is only one point particle, waves can reflect at most once; on the contrary, if there are two point particles, an arbitrary number of reflections can occur in-between two point particles (see also Remark 3.1).

A key finding of this paper is that Green’s functions for our problem can be written as infinite series, (19) and (20), whose terms are “Green’s functions” generated by finitely many reflections. Now although pointwise estimates of Green’s functions generated by finite reflections can be obtained by a repeated use of a previously known ODE technique (Lemma 3.1), it is difficult to obtain accurate pointwise estimates of the original Green’s functions by simply taking the infinite sums of the pointwise estimates obtained in Lemma 3.1. To deal with this problem, we found a way to directly study time-convolutions corresponding to products in the Laplace transformed side. The key ingredients of the estimates are careful analysis of the asymptotic and analyticity properties of some Laplace transformed quantities and integration by parts (Proposition 3.3). This method of analysis is robust than the ODE technique and we believe that it could be applied to other types of fluid–structure interaction problems.

2.3.2. Lack of collisions

Theorem 2.1 implies that the density $\rho = 1/v$ is bounded from above uniformly in time if $\delta'$ is sufficiently small. This excludes the possibility of collisions between the point particles (unlike viscous Burgers’ equation case considered in [26]). Physically, the pressure (or the density) of the fluid in-between the point particles prevents collisions as can be seen in the proof below; the situation can be quite different in multi-dimensional cases (see [22]).

**Theorem 2.3.** Let $h_0^0, h_1^0 \in \mathbb{R}$ and assume that $\int_{h_0^0}^{h_1^0} \rho_0(X) \, dX = 1$. Also let $h_0(t) = h_0^0 + \int_0^t V_0(s) \, ds$ and $h_1(t) = h_1^0 + \int_0^t V_1(s) \, ds$. Now under the assumptions of Theorem 2.1, there exist $\delta'_0, C > 0$ such that if (11) holds, we have

$$h_1(t) - h_0(t) \geq \inf_{t \geq 0, x \in (0,1)} v(x, t) \geq 1 - C\delta'(t + 1)^{-1/2},$$

which implies that the two particles do not collide if $\delta'$ is sufficiently small.

**Proof.** Conservation of mass implies

$$\int_{h_0(t)}^{h_1(t)} \rho(X, t) \, dX = 1.$$

From this, we obtain

$$1 \leq (h_1(t) - h_0(t)) \sup_{t \geq 0, X \in (h_0(t), h_1(t))} \rho(X, t).$$
Combining this with Corollary 2.2 (with $p = \infty$) proves (16) (note that $\rho = 1/\nu$).

\[ \square \]

**Remark 2.1.** It should be noted that global-in-time existence theorems for (3)—and also lack-of-collisions theorems like the one above—may be obtained without smallness assumptions (quantitative decay rates, however, seem difficult to obtain without smallness assumptions even with the state-of-the-art techniques). Such results could be obtained by using the methods used for example in [5, 12, 15, 20, 24] although we are not aware of a reference that treats piston problems in an unbounded fluid domain with large data.

### 3. Proofs

The basic strategy of the proof is identical to that of [14, Theorem 1.2] and [13, Theorem 2.1] and consists of three steps. Step (i): Derive integral equations satisfied by the solution, and from there, define “Green’s functions” associated to (3); Step (ii): Obtain sharp enough pointwise estimates of Green’s functions; and Step (iii): Use the preceding two Steps to conduct nonlinear estimates. The presence of several point particles makes Steps (i) and (ii) more complicated than that for a single point particle, but Step (iii) is almost identical. For this reason, we concentrate our attention on Steps (i) and (ii), and Step (iii) is only touched upon briefly; also, since the global-in-time existence parts of Theorems 2.1 and 2.2 can be proved similarly to [14, Theorem 1.1], we omit their proof.

#### 3.1. Integral equations

The first step of the proof is to derive integral equations satisfied by the solution $(v, u, V_1, V_2)$ to (3). The idea of the derivation is similar to that of [13, Proposition 3.1] and uses the Laplace transform techniques originally developed in [16, 17] and was used also, for example, in [2]. However, the resulting formulae turn out to be quite complicated. Nevertheless, we can still give a nice physical interpretation (see Remark 3.1).

To write down the integral equations, we first define Green’s functions using the Laplace transform. Let $\mathcal{L}$ be the Laplace transform in time $t$, and denote by $s$ the Laplace variable. Then, let

\[
C_{i,j} = C_{i,j}(s) = \frac{s^j \left(2\sqrt{\nu s + c^2}\right)^i}{(s + 2\sqrt{\nu s + c^2})^{i+j}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^j \quad (i \in \{0, 1, 2\}, j \in \mathbb{N}_{\geq 0}),
\]

where $\lambda = s/\sqrt{\nu s + c^2}$. Here, $\lambda$ is defined on $\mathbb{C}\setminus(-\infty, -c^2/\nu]$ and the branch of $\sqrt{\nu s + c^2}$ is chosen so that $\sqrt{\nu s + c^2} > 0$ for $s > -c^2/\nu$. Next, we introduce the
fundamental solution $G$ as the solution to the following equations:

$$
\begin{align*}
\frac{\partial_t G + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}}{\partial_x G = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}} \frac{\partial^2_x G}{, \quad x \in \mathbb{R}, \quad t > 0,} \\
G(x, 0) = \delta(x)I_2, \quad x \in \mathbb{R},
\end{align*}
$$

(17)

where $\delta(x)$ is the Dirac delta function and $I_2$ is the $2 \times 2$ identity matrix. Using these, we define

$$
G_{i, j}(x, t) = \mathcal{L}^{-1}[\mathcal{L}[G]C_{i, j}](x, t),
$$

(18)

where $\mathcal{L}^{-1}$ is the inverse Laplace transform (the Bromwich integral is taken along a vertical contour with $\text{Re} \, s \geq -\sigma_0$, where $\sigma_0$ is the positive constant in Lemma A.1). We note that $G = G_{0,0}$. Now, we define a bunch of “Green’s functions”:

$$
G_{++}(x, t) = G(x, t), \quad G^{++}(x, t) = G_{0,1}(x - 2, t) + \sum_{i=0}^{\infty} G_{2,2i+1}(x + 2i, t),
$$

$$
G_{0+}(x, t) = \sum_{i=0}^{\infty} G_{1,2i}(x + 2i, t), \quad G^{0+}(x, t) = \sum_{i=0}^{\infty} G_{1,2i+1}(x + 2i, t),
$$

$$
G_{-+}(x, t) = \sum_{i=0}^{\infty} G_{2,2i}(x + 2i, t),
$$

$$
G_{0b}^{+}(x, t) = G_{-+}(x, t), \quad G_{1b}^{+}(x, t) = G_{0+}(x - 1, t) + G^{0+}(x + 1, t)
$$

(19)

and

$$
G_{+0}(x, t) = \sum_{i=0}^{\infty} G_{1,2i}(x - 2i, t), \quad G^{+0}(x, t) = \sum_{i=0}^{\infty} G_{1,2i+1}(x + 2i, t),
$$

$$
G_{00}(x, t) = G(x, t) + \sum_{i=1}^{\infty} \left[ G_{0,2i}(x + 2i, t) + G_{0,2i}(x - 2i, t) \right],
$$

$$
G^{00}(x, t) = \sum_{i=0}^{\infty} \left[ G_{0,2i+1}(x + 2i, t) + G_{0,2i+1}(x - 2(i + 1), t) \right],
$$

$$
G_{-0}(x, t) = \sum_{i=0}^{\infty} G_{1,2i}(x + 2i, t), \quad G^{-0}(x, t) = \sum_{i=0}^{\infty} G_{1,2i+1}(x - 2(i + 1), t),
$$

$$
G_{0b}^{0}(x, t) = G_{-0}(x, t) + G^{-0}(x, t), \quad G_{1b}^{0}(x, t) = G_{+0}(x - 1, t) + G^{+0}(x + 1, t).
$$

(20)

We note that these infinite sums actually converge as we see in the proof of Proposition 3.3. Moreover, these complicated functions have a nice physical interpretation (see Remark 3.1).

By using these Green’s functions, we can write down integral equations for the solution to (3).
Proposition 3.1. Let \((v, u, V_1, V_2)\) be the global-in-time solution to (3) and let \(N\) be the nonlinear term defined in (4). Then the following integral equations hold. Case (i) \(x > 1\):
\[
\left(\frac{v - 1}{u}\right)(x, t) = \int_{-\infty}^{\infty} \left[ G_{++}(x - y, t) + G^{++}(x + y, t) \right] \left(\frac{v_0 - 1}{u_0}\right)(y) \, dy
\]
\[
+ \int_{0}^{t} \int_{-\infty}^{\infty} \left[ G_{++}(x - y, t - s) + G^{++}(x + y, t - s) \right] \left(\frac{v_0 - 1}{u_0}\right)(y) \, dy \, ds
\]
\[
+ \int_{0}^{t} \left[ G_0^+(x - y, t) + G_0^0(x + y, t) \right] \left(\frac{v_0 - 1}{u_0}\right)(y) \, dy
\]
\[
+ \int_{0}^{t} \int_{0}^{1} \left[ G_0^+(x - y, t - s) + G_0^0(x + y, t - s) \right] \left(\frac{0}{N_x}\right)(y, s) \, dy \, ds
\]
\[
+ \int_{-\infty}^{0} G_{-+}(x - y, t) \left(\frac{v_0 - 1}{u_0}\right)(y) \, dy
\]
\[
+ \int_{-\infty}^{0} \int_{-\infty}^{0} G_{-+}(x - y, t - s) \left(\frac{0}{N_x}\right)(y, s) \, dy \, ds
\]
\[
+ G_{0b}^+(x, t) \left(\frac{0}{v_0}\right) + \int_{0}^{t} G_{0b}^+(x, t - s) \left(\frac{0}{[N]}\right)(0, s) \, ds
\]
\[
+ G_{1b}^+(x, t) \left(\frac{0}{v_1}\right) + \int_{0}^{t} G_{1b}^+(x, t - s) \left(\frac{0}{[N]}\right)(1, s) \, ds.
\]
(21)

Case (iii) \(x < 0\): a similar formula holds.

Proof. We only prove Case (i) since Case (ii) can be proved similarly (although the required computations are somewhat lengthier). Let \((v_1, u_1)\) be the solution to the following Cauchy problem with initial data \((v_0, u_0)\):
\[
\begin{align*}
\partial_t v_1 - \partial_x u_1 &= 0, \quad x \in \mathbb{R}, \ t > 0, \\
\partial_t u_1 - c^2 \partial_x v_1 &= v_1 \partial_x^2 u_1 + \partial_x N, \quad x \in \mathbb{R}, \ t > 0, \\
v_1(x, 0) &= v_0(x), \quad u_1(x, 0) = u_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

Note that the nonlinear term \(N\) is defined in (4) using \((v, u)\) and not \((v_1, u_1)\). Note also that since the initial data may have discontinuity at \(x = 0\), the solution is considered in the generalized sense. In any case, it can be represented by using the fundamental solution \(G\) as follows:
\[
\left(\frac{v_1 - 1}{u_1}\right)(x, t) = \int_{-\infty}^{\infty} G(x - y, t) \left(\frac{v_0 - 1}{u_0}\right)(y) \, dy
\]
\[
+ \int_{0}^{t} \int_{-\infty}^{\infty} G(x - y, t - s) \left(\frac{0}{N_x}\right)(y, s) \, dy \, ds.
\]
(22)

Next, let
\[
(v_2, u_2) := (v - v_1, u - u_1).
\]
Then we have (see [14, p. 378–379])

\[
\begin{align*}
&\dot{v} - c_1 \dot{t} \dot{u}_2 = 0, \\
&s \ddot{u}_2 - c^2 \partial_t \dot{v}_2 = \nu \dot{v}^2 \ddot{u}_2, \\
&s \ddot{u}_2 (0, s) - (v + c^2) \ddot{u}_2 (s) = \ddot{N} (0, s) - s \ddot{u}_1 (0, s) + V_0^0, \\
&s \ddot{u}_2 (1, s) - (v + c^2) \ddot{u}_2 (1, s) = \ddot{N} (1, s) - s \ddot{u}_1 (1, s) + V_1^0,
\end{align*}
\]

(23)

where the variables with a hat are the Laplace transformed variables. General solutions to (23) have the form

\[
\begin{align*}
\left( \hat{v}_2, \hat{u}_2 \right) (x, s) &= \begin{cases} 
C_+ \left( \frac{-\lambda}{s} \right) e^{-\lambda x} & (x > 1), \\
C_0 \left( \frac{-\lambda}{s} \right) e^{^{-\lambda x}} + C_1 \left( \frac{\lambda}{s} \right) e^{\lambda x} & (0 < x < 1), \\
C_- \left( \frac{\lambda}{s} \right) e^{\lambda x} & (x < 0).
\end{cases}
\end{align*}
\]

(24)

We remind the reader that \( \lambda = s / \sqrt{v + c^2} \). Set

\[
\Psi_0 (s) = [\ddot{N}] (0, s) - s \ddot{u}_1 (0, s) + V_0^0, \quad \Psi_1 (s) = [\ddot{N}] (1, s) - s \ddot{u}_1 (1, s) + V_1^0.
\]

By the third and the fourth equations in (23), we obtain the following equations that determine the constants \( C_+, C_0, C_1, \) and \( C_- \):

\[
\begin{align*}
sc C_+ e^{-\lambda} - \sqrt{v + c^2} (-C_+ e^{-\lambda} + C_0 e^{-\lambda} - C_1 e^{\lambda}) &= \Psi_1 (s), \\
sc C_0 e^{-\lambda} + \sqrt{v + c^2} (-C_+ e^{-\lambda} + C_0 e^{-\lambda} - C_1 e^{\lambda}) &= \Psi_1 (s), \\
sc C_0 + s C_1 - \sqrt{v + c^2} (-C_0 + C_1 - C_-) &= \Psi_0 (s), \\
sc C_- - \sqrt{v + c^2} (-C_0 + C_1 - C_-) &= \Psi_0 (s).
\end{align*}
\]

Solving these equations, we obtain

\[
\begin{align*}
C_0 &= \frac{-\left( s + 2 \sqrt{v + c^2} e^{2\lambda} \Psi_0 (s) + s e^\lambda \Psi_1 (s) \right)} {\left( s + 2 \sqrt{v + c^2} e^{2\lambda} \right) [s - s + 2 \sqrt{v + c^2} e^{2\lambda}]}, \\
C_1 &= \frac{s \Psi_0 (s) - \left( s + 2 \sqrt{v + c^2} e^{2\lambda} \right) \Psi_1 (s)} {\left( s + 2 \sqrt{v + c^2} e^{2\lambda} \right) [s - s + 2 \sqrt{v + c^2} e^{2\lambda}]}, \\
C_+ &= -\frac{2 \sqrt{v + c^2} e^{2\lambda} \Psi_0 (s) + \left( s - s + 2 \sqrt{v + c^2} e^{2\lambda} \right) \Psi_1 (s)} {\left( s + 2 \sqrt{v + c^2} e^{2\lambda} \right) [s - s + 2 \sqrt{v + c^2} e^{2\lambda}]}, \\
C_- &= \frac{-s \left( s + 2 \sqrt{v + c^2} e^{2\lambda} \right) \Psi_0 (s) + \left( s - s + 2 \sqrt{v + c^2} e^{2\lambda} \right) \Psi_1 (s)} {\left( s + 2 \sqrt{v + c^2} e^{2\lambda} \right) [s - s + 2 \sqrt{v + c^2} e^{2\lambda}]},
\end{align*}
\]

Set

\[
r (s) = \frac{s^2}{(s + 2 \sqrt{v + c^2})^2} = \frac{\lambda^2}{(\lambda + 2)^2}.
\]

(26)

Then

\[
e^{2\lambda} \frac{1}{[s + (s + 2 \sqrt{v + c^2}) e^\lambda] [s - s + 2 \sqrt{v + c^2} e^\lambda]} = \frac{-1}{(s + 2 \sqrt{v + c^2})^2 (1 - r e^{-2\lambda})}.
\]
By Lemma A.1, there exists \( r_0 \in (0, 1) \) such that \(|re^{-2\lambda}| \leq r_0\) for all \( s \) with \( \text{Re } s \leq -\sigma_0\); so we have

\[
\frac{1}{1 - re^{-2\lambda}} = \sum_{i=0}^{\infty} r^i e^{-2i\lambda}.
\]

Using this series expansion, we obtain

\[
sC_+ = \sum_{i=0}^{\infty} c_{1,2i+1} e^{-2i\lambda} \Psi_0(s) - \sum_{i=0}^{\infty} c_{0,2i+2} e^{-(2i+1)\lambda} \Psi_1(s) + \sum_{i=0}^{\infty} c_{0,2i+1} e^{-(2i-1)\lambda} \Psi_1(s)
\]

\[
= \sum_{i=0}^{\infty} c_{1,2i+1} e^{-2i\lambda} \Psi_0(s) + c_{0,1} e^\lambda \Psi_1(s) + \sum_{i=1}^{\infty} (c_{0,2i+1} - c_{0,2i}) e^{-(2i-1)\lambda} \Psi_1(s)
\]

\[
= \sum_{i=0}^{\infty} c_{1,2i+1} e^{-2i\lambda} \Psi_0(s) + \left(c_{0,1} - \sum_{i=1}^{\infty} c_{1,2i} e^{-2i\lambda}\right) e^\lambda \Psi_1(s),
\]

(27)

where

\[
c_{i,j} = c_{i,j}(s) = \frac{s^j (2\sqrt{v\sigma} + c^2)^j}{(s + 2\sqrt{v\sigma} + c^2)^{i+j}} = \frac{2^i \lambda^j}{(\lambda + 2)^{i+j}}.
\]

Here, we used the formula

\[
c_{i,j} - c_{i,j+1} = c_{i+1,j}.
\]

(28)

We now recall [14, Eq. (37)]:

\[
\mathcal{L}[G](x, s) = \frac{1}{\nu s + c^2} \left( \nu \delta(x) + \frac{c^2}{2\sqrt{\nu s + c^2}} e^{-\lambda|x|} - \frac{s}{2\sqrt{\nu s + c^2}} e^{-\lambda|x|} \right).
\]

(29)

Also note that \( \hat{u}_1(0, s) \) and \( \hat{u}_1(1, s) \) in the definitions of \( \Psi_0(s) \) and \( \Psi_1(s) \), see (25), can be expressed in terms of \( \hat{G} \) using (22). Then, (24), (27), and (29) lead us to the following formula for \( x > 1 \):

\[
\begin{align*}
\hat{v}_2(x, s) &= \left( -\frac{\lambda}{s} \right) (0 - 1) e^{-\lambda x} \left( c_{0,1} - \sum_{i=1}^{\infty} c_{1,2i} e^{-2i\lambda} \right) \\
&\quad + e^\lambda \int_{-\infty}^{\infty} \hat{G}(1 - y, s) \left( \frac{(v_0 - 1)(y)}{u_0(y) + \hat{N}_x(y, s)} \right) dy \\
&\quad + \left( -\frac{\lambda}{s} \right) e^{-\lambda x} \left( c_{0,1} - \sum_{i=1}^{\infty} c_{1,2i} e^{-2i\lambda} \right) e^\lambda (V_1^0 + [\hat{N}](1, s)) \\
&\quad + \left( -\frac{\lambda}{s} \right) (0 - 1) e^{-\lambda x} \sum_{i=0}^{\infty} c_{1,2i+1} e^{-2i\lambda}
\end{align*}
\]
Finally, adding the Laplace transformed (22) to (30) and noting (18) and (28), we get

\[
\mathcal{L} \left[ \begin{pmatrix} v - 1 \\ u \end{pmatrix} \right] (x, s) = \int_{-\infty}^{\infty} \hat{G}(0, s, y) \begin{pmatrix} (v_0 - 1)(y) \\ u_0(y) + \hat{N}_x(y, s) \end{pmatrix} dy
\]

Taking into account (29) again, we obtain

\[
\begin{aligned}
\frac{\hat{v}}{\hat{u}}_2 (x, s) &= \int_1^{\infty} \hat{G}(x + y - 2, s) \left( c_{0,1} - \sum_{i=1}^{\infty} c_{1,2i} e^{-2i\lambda} \right) \begin{pmatrix} (v_0 - 1)(y) \\ -u_0(y) - \hat{N}_x(y, s) \end{pmatrix} dy \\
- \int_{-\infty}^{1} \hat{G}(x - y, s) \left( c_{0,1} - \sum_{i=1}^{\infty} c_{1,2i} e^{-2i\lambda} \right) \begin{pmatrix} (v_0 - 1)(y) \\ u_0(y) + \hat{N}_x(y, s) \end{pmatrix} dy \\
+ \hat{G}(x - 1, s) \left( c_{1,0} - \sum_{i=1}^{\infty} c_{2,i} e^{-2i\lambda} \right) \begin{pmatrix} 0 \\ V_0^0 + [\hat{N}](1, s) \end{pmatrix} \\
+ \int_{0}^{\infty} \hat{G}(x + y, s) \sum_{i=0}^{\infty} c_{1,2i+1} e^{-2i\lambda} \begin{pmatrix} (v_0 - 1)(y) \\ -u_0(y) - \hat{N}_x(y, s) \end{pmatrix} dy \\
- \int_{0}^{0} \hat{G}(x - y, s) \sum_{i=0}^{\infty} c_{1,2i+1} e^{-2i\lambda} \begin{pmatrix} (v_0 - 1)(y) \\ u_0(y) + \hat{N}_x(y, s) \end{pmatrix} dy \\
+ \hat{G}(x, s) \sum_{i=0}^{\infty} c_{2,2i} e^{-2i\lambda} \begin{pmatrix} 0 \\ V_0^0 + [\hat{N}](0, s) \end{pmatrix}.
\end{aligned}
\]
When one of these waves hits a point particle, transmission and reflection occur. We obtain (21). This ends the proof.

Remark 3.1. (Physical interpretation of Proposition 3.1)³ When a Dirac delta input $\delta(-y)I_2$ is given, the response, i.e., $G(\cdot - y, \cdot)$, propagates to the left and to the right. When one of these waves hits a point particle, transmission and reflection occur. We set the following rules: multiply $C_{1,0}$ to the wave (in the Laplace variable) when there is a transmission and multiply $C_{0,1}$ when there is a reflection. Now, suppose that a wave experiences $i$-times of transmissions and $j$-times of reflections before reaching position $x$ at time $t$, and let $\phi(x, y; i, j)$ be the total length traversed by the wave (we set $\phi(x, y; i, j)$ to be negative when the wave reaches $x$ from the right). Then the resulting wave at $(x, t)$ should be $G_{i, j}(\phi(x, y; i, j), t)$.

³The interpretation below is inspired by a closely related analysis in [18] for a heat equation with a conductivity having jumps.
Now we can give a simple interpretation of Proposition 3.1. Fix $x > 1$ and let us consider the term

$$
\int_1^\infty G^{++}(x + y, t) \left( \frac{v_0 - 1}{u_0} \right)(y) \, dy
$$

$$
= \int_1^\infty \left[ G_{0,1}(x + y - 2, t) + \sum_{i=0}^\infty G_{2,2i+1}(x + y + 2i, t) \right] \left( \frac{v_0 - 1}{u_0} \right)(y) \, dy
$$

in (21), which can be interpreted as follows: the term involving $G_{0,1}(x + y - 2, t)$ is the contribution of the wave that comes from $y > 1$ and reflects at the point particle number 1 to reach $x$ (the total distance traversed is $x + y - 2$); the term involving $G_{2,2i+1}(x + y + 2i, t)$ is the contribution of the wave that comes from $y > 1$ and transmits at the point particle number 1 to reach the point particle number 0 and then reflects $2i + 1$ times between the two point particles and finally transmits at the point particle number 1 to reach $x$ (the total distance traversed is $x + y + 2i$). All the other terms in (21) and (22) can be interpreted in this way. However, the interpretation of the boundary terms such as

$$
G_{0b}^+(x, t) \left( \begin{array}{c} 0 \\ V_0^0 \end{array} \right)
$$

is a little bit more complicated. First, note that the momentum carried by the point particle number 0 is $V_0^0$. From this, we interpret that this point particle gives an input

$$
\delta(x - \epsilon) \left( \begin{array}{c} 0 \\ V_0^0 \end{array} \right),
$$

where we take $\epsilon \searrow 0$ in the end (since the mass of the point particle does not propagate into the fluid, the first component of the input is zero). Then the corresponding response to this input should be

$$
G_{0b}^+(x, t) \left( \begin{array}{c} 0 \\ V_0^0 \end{array} \right) = \lim_{\epsilon \searrow 0} G_{-+}(x - \epsilon, t) \left( \begin{array}{c} 0 \\ V_0^0 \end{array} \right).
$$

All the other boundary terms can be interpreted in a similar manner.

Let us give a corollary of Proposition 3.1, which is just an application of integration by parts in $x$.

**Corollary 3.1.** Let $(v, u, V_1, V_2)$ be the global-in-time solution to (3) and let $N$ be the nonlinear term defined in (4). Then the following integral equations hold. Case (i) $x > 1$:

$$
\left( \begin{array}{c} v - 1 \\ u \end{array} \right)(x, t)
$$

$$
= \int_1^\infty \left[ G_{++}(x - y, t) + G^{++}(x + y, t) \right] \left( \frac{v_0 - 1}{u_0} \right)(y) \, dy
$$
Case (iii) $x < 0$: a similar formula holds.

Proof. These are simple consequences of integration by parts applied to the formulae in Proposition 3.1. For example, to show (32), it suffices to check that

$$ I = -\left[ G_{0+}(x, t - s) + G^{0+}(x, t - s) \right] \left( 0^+_N \right) (0_+, s) + G_{-+}(x, t - s) \left( 0^-_N \right) (0_-, s) + G^{0b}(x, t - s) \left( 0^0_N \right) (0, s) $$

$$ + G^{1b}(x, t) \left( 0^0_{V_1} \right) + G_{1b}(x, t) \left( 0^0_{V_1} \right). $$

(33)
and

$$II = -\left[ G_{++}(x - 1, t - s) + G^{++}(x + 1, t - s) \right] \left( \frac{0}{N} \right) (1_+, s)$$

$$+ \left[ G_{0+}(x - 1, t - s) + G^{0+}(x + 1, t - s) \right] \left( \frac{0}{N} \right) (1_-, s)$$

$$+ G^+_{1b}(x, t - s) \left( \frac{0}{\left[ N \right]} \right) (1, s)$$

vanish. For brevity, we shall only show that $I$ is zero. First, note that by (28), we have

$$\left[ G_{0+}(x, t - s) + G^{0+}(x, t - s) \right] \left( \frac{0}{N} \right) (0_+, s)$$

$$= \sum_{i=0}^{\infty} \left[ G_{1,2i}(x + 2i, t - s) + G_{1,2i+1}(x + 2i, t - s) \right] \left( \frac{0}{N} \right) (0_+, s)$$

$$= \sum_{i=0}^{\infty} G_{2,2i}(x + 2i, t - s) \left( \frac{0}{N} \right) (0_+, s)$$

$$= G_{--}(x, t - s) \left( \frac{0}{N} \right) (0_+, s).$$

Note also that $G^{+}_{0b} = G_{--}$ by definition. From these, it follows that

$$I = -G_{--}(x, t - s) \left( \frac{0}{\left[ N \right]} \right) (0, s) + G^{+}_{0b}(x, t - s) \left( \frac{0}{\left[ N \right]} \right) (0, s) = 0.$$

We can show (33) in a similar manner. This ends the proof. □

3.2. Pointwise estimates of Green’s functions

The next step of the proof is to obtain sharp enough pointwise estimates of Green’s functions defined in (19) and (20). This is the most important part in this paper. In previous works (e.g. [1,2,14]), a differential equation technique is used to obtain pointwise estimates; however, this alone is not sufficient when there are several point particles. Nonetheless, the differential equation technique is also important, so let us explain this first.

In [14], two Green’s functions $G_T$ and $G_R$ appeared, which are defined as follows:

$$G_T(x, t):=\mathcal{L}^{-1} \left[ \frac{2}{\lambda + 2} \mathcal{L}[G] \right] (x, t), \quad G_R(x, t):=(G - G_T)(x, t) \left( \frac{1}{0 - 1} \right),$$

where $\lambda = s/\sqrt{vs + c^2}$. According to (18), these are simply

$$G_T = G_{1,0}, \quad G_R = G_{0,1}.$$
Now the differential equation technique uses the relation \( \partial_x \mathcal{L}[G](x, s) = -\lambda \mathcal{L}[G](x, s) \) for \( x > 0 \), which is a consequence of (29). Because of this relation, \( G_T \) satisfies a simple first order ODE:

\[
\partial_x G_T(x, t) = 2G_T(x, t) - 2G(x, t) \quad (x > 0).
\]

Solving this, we obtain

\[
G_T(x, t) = 2 \int_{-\infty}^{0} e^{2z} G(x - z, t) \, dz \quad (x > 0).
\] (35)

The fundamental solution \( G \) in the integrand have good pointwise estimates due to the work by Liu and Zeng [19, Theorem 5.8].

**Proposition 3.2.** For any integer \( k \geq 0 \), there exists a positive constant \( C = C_k > 0 \) such that

\[
\left| \partial_x^k G(x, t) - \partial_x^k G^\star(x, t) - e^{-\frac{\nu}{2}t} \sum_{j=0}^{k} \delta^{(k-j)}(x)Q_j(t) \right| \\
\leq C(t + 1)^{-1/2} t^{-(k+1)/2} \left( e^{-\frac{(x-ct)^2}{2\nu t}} + e^{-\frac{(x+ct)^2}{2\nu t}} \right),
\]

where

\[
G^\star(x, t) = \frac{1}{2(2\pi \nu t)^{1/2}} e^{-\frac{(x-ct)^2}{2\nu t}} \begin{pmatrix} 1 & -1/c \\ -c & 1 \end{pmatrix} + \frac{1}{2(2\pi \nu t)^{1/2}} e^{-\frac{(x+ct)^2}{2\nu t}} \begin{pmatrix} 1 & 1 \\ c & 1 \end{pmatrix}.
\]

\( \delta^{(k)}(x) \) is the \( k \)-th derivative of the Dirac delta function, and \( Q_j = Q_j(t) \) is a \( 2 \times 2 \) polynomial matrix.

Combining (35) and Proposition 3.2, we can show the following [14, Appendix A]:

\[
|\partial_x^k G_T(x, t)| \leq C(t + 1)^{-1/2} t^{-k/2} \left( e^{-\frac{(x-ct)^2}{2\nu t}} + e^{-\frac{(x+ct)^2}{2\nu t}} \right) + Ce^{-\frac{|x|+t}{c}} \quad (36)
\]

for \( (x, t) \in (\mathbb{R} \setminus \{0\}) \times (0, \infty) \) and for any integer \( k \geq 0 \). On the other hand, by (34), we have the relation

\[
G_R(x, t) = (G - G_T)(x, t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{2} \partial_x G_T(x, t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (37)
\]

for \( x > 0 \) (we have a similar formula for \( x < 0 \)). From (36) and (37), we were able to obtain necessary pointwise estimates for \( G_T \) and \( G_R \). However, for Green’s functions appearing in (19) and (20), they do not seem to satisfy such simple ODEs. This is the main difficulty that needs to be resolved.

\[\text{4We only considered the case of } x > 0 \text{ above but the case of } x < 0 \text{ is similar.}\]
Our approach to this problem is as follows. First, we consider $G_{i,j}$ defined by (18) instead of their infinite sum such as $G_{0+}$. For $G_{i,j}$, we can find a simple ODE, for example,

$$\partial_x G_{2,0}(x, t) = 2G_{2,0}(x, t) - 2G_{1,0}(x, t) \quad (x > 0).$$

Since we already have pointwise estimates for $G_T = G_{1,0}$, we can obtain those for $G_{2,0}$ by solving this ODE. This then gives pointwise estimates for any finite partial sum of the infinite series in (19) and (20). Finally, to analyze the remainders, we use a simple fact: products in the Laplace transformed side are convolutions in time. And to analyze these convolutions, we look into the asymptotic and the analyticity structure in the Laplace variable and apply integration by parts in time as many time as needed to obtain necessary gain of decay (see the proof of Proposition 3.3).

Now to state concisely the main result of this section (Proposition 3.3), we first give some definitions.

**Definition 3.1.** Let $X_H \subset \mathbb{R}$ be an open set.

(i) A function $H : X_H \times (0, \infty) \to \mathbb{R}$ is said to be of Type $l$ ($l \in \mathbb{Z}_{\geq 0}$) on $X_H$ if for any integer $k \geq 0$, there exists a positive constant $C = C_k > 0$ such that

$$|\partial^k_x H(x, t)| \leq C(t + 1)^{-1/2}t^{-(k+l)/2} \left(e^{-\frac{(x-c)^2}{Ct}} + e^{-\frac{(x+c)^2}{Ct}}\right) + Ce^{-\frac{|x|+t}{c}} \quad (x \in X_H, t > 0).$$

(ii) A function $H : X_H \times (0, \infty) \to \mathbb{R}$ is said to be of Type $R$ on $X_H$ if $H$ is of Type 1 on $X_H$ and there exists a (possibly zero) matrix $C_H \in \mathbb{R}^{2 \times 2}$ such that $H - (\partial_x G^*)C_H$ is of Type 2 on $X_H$.

(iii) A function $^6 H : X_H \times (0, \infty) \to \mathbb{R}$ is said to be of Type $T$ on $X_H$ if $H - G$ is of Type $R$ on $X_H$.

The main result of this section is the following.

**Proposition 3.3.** The functions defined in (19) and (20) are of the type listed in Table 1. The table reads as follows: the function $H$ is of Type $\tau$ on $X_H$.

**Remark 3.2.** Green’s functions consisting only of $G_{i,j}$ with $j \geq 1$ are of Type $R$ and others are of Type $T$. According to Remark 3.1, this means that Green’s functions consisting only of $G_{i,j}$ resulting from one or more reflections are of Type $R$.

To prove this proposition, we first prove a lemma on pointwise estimates of $G_{i,j}$.

**Lemma 3.1.** We have

$$|\partial^k_x G_{i,j}(x, t)| \leq C(t + 1)^{-i/2}t^{-(k+1-i)/2} \left(e^{-\frac{(x-c)^2}{Ct}} + e^{-\frac{(x+c)^2}{Ct}}\right) + Ce^{-\frac{|x|+t}{c}}$$

for $(x, t) \in (\mathbb{R} \setminus \{0\}) \times (0, \infty)$ and for any integer $k \geq 0$.

---

5More precisely, it might contain a Dirac delta singularity but $H - G$ should be a usual function.
### Table 1. Types of Green’s functions

| $H$          | $\tau$ | $X_H$     | $H$          | $\tau$ | $X_H$     |
|--------------|--------|-----------|--------------|--------|-----------|
| $G_{++}$     | T      | $\mathbb{R}$ | $G_{++}$     | R      | $(2, \infty)$     |
| $G_{0+}$     | T      | $(0, \infty)$ | $G^{0+}$     | R      | $(1, \infty)$     |
| $G_{-+}$     | T      | $(1, \infty)$ | $G_{1b}^+$   | T      | $(1, \infty)$     |
| $G_{0b}^+$   | T      | $(1, \infty)$ | $G_{00}^+$   | R      | $(1, \infty)$     |
| $G_{-0}$     | T      | $(0, \infty)$ | $G_{1b}^0$   | T      | $(-\infty, 1)$     |
| $G_{0b}^0$   | T      | $(0, 1)$   | $G_{1b}^0$   | T      | $(0, 1)$     |

**Proof.** Assume that $x > 0$. The case of $x < 0$ is similar. Note first that by (18) and (29), we have

$$G_{i,j}(x,t) = 2^{-j} \partial_x^j G_{i+j,0}(x,t) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^j.$$ 

So it suffices to consider $G_{i,0}$. Note that by Proposition 3.2, the lemma is true for $G_{0,0} = G$.

We next note that $G_{i,0}$ satisfies the following ODE:

$$\partial_x G_{i,0}(x,t) = 2G_{i,0}(x,t) - 2G_{i-1,0} \quad (x > 0).$$

Solving this, we obtain

$$G_{i,0}(x,t) = 2 \int_{-\infty}^{0} e^{2z} G_{i-1,0}(x-z,t) \, dz \quad (x > 0).$$

If we now assume that the lemma is true for $G_{i-1,0}$, then [14, Lemma A.1] and a simple estimate

$$\int_{-\infty}^{0} e^{2z} e^{-|x-z|+t} \, dz \leq e^{-|x|+t} \int_{-\infty}^{0} e^z \, dz = e^{-|x|+t}$$

imply that the lemma is also true for $G_{i,0}$. This proves the lemma.  

We now move on to the proof of Proposition 3.3.

**Proof of Proposition 3.3.** We shall only prove that $G_{0+}$ is of Type $T$ on $X_H = (0, \infty)$. Others can be treated in a similar manner. Now fix $x \in X_H$. By (18), (19), (26), and (29), we have

$$G_{0+}(x,t) = G_{1,0}(x,t) + \sum_{i=1}^{\infty} G_{1,2i}(x+2i,t).$$
\[ G_1(x, t) + \sum_{i=1}^{\infty} L^{-1} \left[ \mathcal{L}[G] \frac{2}{\lambda + 2} e^{-\lambda x} \right] (x, t) \]

\[ = G_1(x, t) + L^{-1} \left[ \mathcal{L}[G] \frac{2re^{-\lambda}}{(\lambda + 2)(1 - re^{-2\lambda})} \right] (x, t) \]

\[ = G_1(x, t) + G_{1,2\infty}(x, t). \]

Next, note that
\[ (G_1 - G)(x, t) = -L^{-1} \left[ \mathcal{L}[G] \frac{\lambda}{\lambda + 2} \right] (x, t) = -G_0(x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Therefore, in order to show that \( G_{0,1} \) is of Type \( T \) on \( X_H \), it suffices to show that \( G_{0,1} \) is of Type \( R \) and \( G_{1,2\infty} \) is of Type 2 on the same set. That \( G_{0,1} \) is of Type \( R \) on \( X_H \) is proved by noting that
\[ G_{0,1}(x, t) - \frac{1}{2} \partial_x G(x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -L^{-1} \left[ \mathcal{L}[G] \frac{\lambda^2}{2(\lambda + 2)} \right] (x, t) \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = -\frac{1}{4} \partial_x^2 G_0(x, t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

and applying Proposition 3.2 and Lemma 3.1.

Let us next show that \( G_{1,2\infty} \) is of Type 2 on \( X_H \). Let
\[ \omega(t) = L^{-1} \left[ \frac{2re^{-\lambda}}{(\lambda + 2)(1 - re^{-2\lambda})} \right] (t). \]

Then, since products in the Laplace transformed side are convolutions in time, we have
\[ G_{1,2\infty}(x, t) = \int_0^t \omega(t - s) G(x, s) \, ds. \]

By Lemma A.1 (i), there exists \( \sigma_0 > 0 \) such that \( \mathcal{L}[\omega](s) \) is analytic in the half-space \( \{ \text{Re } s > -\sigma_0 \} \). Moreover, by Lemma A.1 (ii), \( \mathcal{L}[\omega](s) \) decays exponentially fast as \( |s| \to \infty \) on any vertical line with \( \text{Re } s > -c^2/\nu \). Therefore, there exist \( \alpha, C > 0 \) such that
\[ |\omega(t)| \leq Ce^{-\alpha t} \quad (t \geq 0). \]

Next, let
\[ \omega_1(t) = \int_0^t \omega(s) \, ds. \]

Then we have \( \mathcal{L}[\omega_1](s) = \mathcal{L}[\omega](s)/s \). Since \( r = O(s^2) \) as \( |s| \to 0 \), the origin \( s = 0 \) is a removable singularity of \( \mathcal{L}[\omega_1](s) \), which implies that there exist \( \alpha, C > 0 \) such that
\[ |\omega_1(t)| \leq Ce^{-\alpha t} \quad (t \geq 0). \]
Similarly, if we define

\[ \omega_2(t) = \int_0^t \omega_1(s) \, ds, \]

we have

\[ |\omega_2(t)| \leq C e^{-\alpha t} \quad (t \geq 0). \tag{40} \]

Now we divide \( G_{1,2\infty} \) as follows:

\[ G_{1,2\infty}(x, t) = I(x, t) + J(x, t) \]

\[ := \int_0^{t/2} \omega(t - s)G(x, s) \, ds + \int_{t/2}^t \omega(t - s)G(x, s) \, ds. \]

Let us first show that \( I \) is of Type 2 on \( X_H \). By Proposition 3.2 and (38), we have (note that \( x > 0 \) here)

\[ |I(x, t)| \leq C \int_0^{t/2} e^{-\alpha(t-s)} s^{-1/2} e^{-\frac{(x-cs)^2}{cs}} \, ds \]

\[ = C \int_0^{t/2} e^{-\alpha(t-s)} s^{-1/2} e^{-\frac{(x-cs)^2}{cs}} \chi_{|c(t-s)| \leq |x-ct|/4} \, ds \]

\[ + C \int_0^{t/2} e^{-\alpha(t-s)} s^{-1/2} e^{-\frac{(x-cs)^2}{cs}} \chi_{|c(t-s)| > |x-ct|/4} \, ds \]

\[ \leq Ce^{-\frac{ct}{2}} e^{-\frac{(x-cs)^2}{cs}} \int_0^{t/2} s^{-1/2} ds + Ce^{-\frac{ct}{2}} e^{-\frac{(x-ct)}{ce}} \int_0^{t/2} s^{-1/2} ds \]

\[ \leq Ce^{-\frac{ct}{2}} e^{-\frac{(x-cs)^2}{cs}} + Ce^{-\frac{|x|+t}{c}}. \]

Here, \( \chi_{[S]} \) is the indicator function of the set \( S \). The derivatives \( \partial^k_x I(x, t) \) can be treated similarly by modifying the definition of \( \omega(t) \).\(^6\) This shows that \( I \) is of Type 2 on \( X_H \).

We next show that \( J \) is of Type 2 on \( X_H \). First, assume that \( t \leq 1 \). Then by Proposition 3.2 and (38), we have

\[ |J(x, t)| \leq C \int_{t/2}^t e^{-\alpha(t-s)} s^{-1/2} e^{-\frac{(x-cs)^2}{cs}} \, ds \leq Ce^{-\frac{(x-cs)^2}{ct}}. \]

So we may now assume that \( t > 1 \). By applying integration by parts twice, we obtain

\[ J(x, t) = \int_{t/2}^t \partial_t \omega_1(t-s)G(x, s) \, ds \]

\[ = \omega_1(t/2)G(x, t/2) + \int_{t/2}^t \omega_1(t-s) \partial_t G(x, s) \, ds \]

\[ = \omega_1(t/2)G(x, t/2) + \omega_2(t/2) \partial_t G(x, t/2) + \int_{t/2}^t \omega_2(t-s) \partial^2_t G(x, s) \, ds. \tag{41} \]

\(^6\)Directly applying Proposition 3.2 for the derivatives of \( G \) results in apparently diverging integrals. We can circumvent this problem by noting that applying \( \partial^k_x \) is equivalent to multiplying \((-\lambda)^k\) in the Laplace transformed side, and the divergence of \( \lambda^k \) as \(|s| \to \infty\) is then absorbed by the exponential factor in \( \mathcal{L}[\omega](s) \).
To analyze the terms on the right-hand side, we first show that
\[ e^{-\frac{x^2}{2}(t-s)} e^{-\frac{(x-cs)^2}{Cs}} \leq e^{-\frac{(x-c)^2}{2Ct}} \]
for \( 0 < s \leq t \) if \( C \geq 2c^2/\alpha \): this inequality is equivalent to
\[ \frac{x^2 - 2cxt + c^2t^2}{Ct} \leq \frac{x^2 - 2csx + c^2s^2}{Cs} + \frac{\alpha}{2}(t-s), \]
and the right-hand side minus the left-hand side equals to
\[ \left( \frac{1}{Cs} - \frac{1}{Ct} \right) x^2 + \left( \frac{\alpha}{2} - \frac{c^2}{C} \right) (t-s) \geq 0. \]
Now using (42) with \( s = t/2 \), (39), and (40), we obtain
\[ |\omega_1(t/2)G(t/2)| + |\omega_2(t/2)\partial_t G(x, t/2)| \leq Ce^{-\frac{\alpha}{4}t} e^{-\frac{(x-c)^2}{2Ct}}. \]
Next, by (17), Proposition 3.2, (40), and (42), we obtain
\[ \int_{t/2}^{t} |\omega_2(t-s)\partial_t^2 G(x, s)| \, ds \leq C(t+1)^{-3/2} \int_{t/2}^{t} e^{-\alpha(t-s)} e^{-\frac{(x-c)^2}{2Cs}} \, ds \]
\[ \leq C(t+1)^{-3/2} e^{-\frac{(x-c)^2}{2Ct}}. \]
Then, (41), (43), and (44) imply
\[ |J(x, t)| \leq C(t+1)^{-3/2} e^{-\frac{(x-c)^2}{2Ct}}. \]
The derivatives \( \partial_x^k J(x, t) \) can be treated similarly by modifying the definition of \( \omega(t) \) (use integration by parts \( k + 2 \) times). This shows that \( J \) is of Type 2 on \( X_H \), which ends the proof of the lemma. \( \square \)

3.3. Outline of the nonlinear estimates

The final step of the proof is the nonlinear estimates. What we do is to multiply the left eigenvector \( l_i \) to (21), which gives us an integral equation for \( u_i \), and bound every term that appears on the right-hand side (this is for \( x > 1 \); other cases are similar). The terms coming from the initial data are not so difficult to handle (cf. [14, Lemma 3.1] and [13, Lemma 3.6]), and the main issue is the handling of the nonlinear terms coming from \( N \). To have a control on \( N \), we make an a priori assumption that our goal, (12) or (14), holds up to some positive time \( t = T \). Then using this ansatz, we bound the nonlinear terms to show that (12) or (14) holds for some longer time \( t = T' > T \). This then shows that (12) or (14) actually holds for any \( t > 0.7 \)

\[ \text{This is a simplified explanation and some technical arguments are needed to make it rigorous; see [14, Sect. 3.3].} \]
By virtue of Propositions 3.2 and 3.3, estimates of the nonlinear terms typically lead us to investigate space-time convolutions of the forms

\[ \mathcal{I}_i[f](x, t) = \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda_i(t-s))^2}{2(t-s)}} f_s(y, s) dy ds \]

or

\[ \mathcal{J}_i[f](x, t) = \int_{t/2}^t \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-1/2} e^{-\frac{(x-y-\lambda_i(t-s))^2}{c(t-s)}} |f(y, s)| dy ds 
+ \int_{t/2}^t \int_{-\infty}^{\infty} (t-s)^{-1/2}(t+1-s)^{-1/2} e^{-\frac{(x-y-\lambda_i(t-s))^2}{c(t-s)}} |f_s(y, s)| dy ds. \]

Now suppose we want to prove Theorem 2.1. Then it becomes necessary to show that

\[ |\mathcal{I}_i[f](x, t)|, |\mathcal{J}_i[f](x, t)| \leq C \delta^2 \Phi_i(x, t) \quad (45) \]

holds for various functions \( f = f(x, t) \). What kind of functions \( f \) do we need to consider? Since the nonlinearity \( N \) is quadratic, a typical example is \( f = u_j^2 \) (\( j = 1, 2 \)). If we set \( v_j := u_j - \theta_j \), expanding \( u_j^2 \) produces functions such as \( \theta_j^2 \) and \( \theta_j v_j \). Among these functions, \( f = \theta_j^2 \) is problematic because \( |\mathcal{I}_i[\theta_j^2](x, t)| \leq C \delta^2 \Phi(x, t) \) does not hold; but the convolution \( \mathcal{I}_i[\theta_j^2](x, t) \) is naturally connected with the generalized Burgers equation (8). This is why the pointwise estimates (12) in Theorem 2.1 are stated for \( v_i = u_i - \theta_i \) and not for \( u_i \) itself: we subtract \( \theta_i \) to eliminate the bad term \( \mathcal{I}_i[\theta_i^2](x, t) \). If we further wish to prove Theorem 2.2, we need to improve (45) to

\[ |\mathcal{I}_i[f](x, t)|, |\mathcal{J}_i[f](x, t)| \leq C \delta^2 \Psi_i(x, t). \]

Then we also have to get rid of \( \mathcal{I}_i[\theta_i^2](x, t) \) and several other terms. And this is the reason for introducing the inter-diffusion wave \( \xi_i \), which captures the interaction of waves of different speeds \( \lambda_i \) and \( \lambda_i' \); note the appearance of \( \theta_i^2 \) in (10).

Although the actual computations are important, they are almost identical to those in the proofs of [14, Lemma 3.2] and [13, Lemma 3.7]. The effect of the presence of several point particles can be handled using Proposition 3.3. As the calculations are lengthy and too much of a repetition, we omit the detail and conclude the proof here.

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A lemma on the quantity $\lambda = s/\sqrt{v s + c^2}$

**Lemma A.1.** Let $\lambda = s/\sqrt{v s + c^2}$ and $r = \lambda^2/(\lambda + 2)^2$. Then the following properties are satisfied:

(i) There exist $\sigma_0 > 0$ and $r_0 \in (0, 1)$ such that $\Re \lambda \geq -r_0$ and $|re^{-2\lambda}| \leq r_0$ for all $s \in \mathbb{C}\setminus(-\infty, -c^2/v]$ with $\Re s \geq -\sigma_0$.

(ii) We have

$$\Re \lambda \geq \sqrt{|s|}/(2\sqrt{v}) + O\left(\frac{1}{\sqrt{|s|}}\right) \quad (|s| \to \infty; \Re s > -c^2/v).$$

**Proof.** Note first that $\Re \lambda \geq 0$ for $\Re s \geq 0$. From this, it follows that for any $M > 0$, there exists $r_0 \in (0, 1)$ such that $|re^{-2\lambda}| \leq r_0$ for all $s$ with $\Re s \geq 0$ and $|s| \leq M$.

Then, if (ii) is proved, (i) follows easily.

Now we prove (ii). Let $s$ be a sufficiently large complex number with $\Re s > -c^2/v$. We can write it as $s = |s|e^{i\theta}$ with $\theta \in [-2\pi/3, 2\pi/3]$. Then, from

$$\lambda = \frac{s}{\sqrt{v s + c^2}} = \sqrt{\frac{|s|}{v}} \left[1 + O\left(\frac{1}{s}\right)\right] = \sqrt{\frac{|s|}{v}} e^{i\theta} \left[1 + O\left(\frac{1}{s}\right)\right].$$

we obtain

$$\Re \lambda = \sqrt{\frac{|s|}{v}} \cos\left(\frac{\theta}{2}\right) + O\left(\frac{1}{\sqrt{|s|}}\right) \geq \sqrt{\frac{|s|}{2v}} + O\left(\frac{1}{\sqrt{|s|}}\right).$$

This ends the proof of the lemma. □

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