THETA CONSTANTS ASSOCIATED TO COVERINGS OF $P^1$
BRANCHING AT 8 POINTS

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Abstract. In this paper, we construct automorphic forms on the five
dimensional complex ball which give the inverse of the period map for
cyclic 4-ple coverings of the complex projective line branching at eight
points. We use theta constants associated to the Prym varieties of these
coverings.

1. Introduction

Let $\mu_1, \ldots, \mu_n$ be rational numbers such that $0 < \mu_j < 1$, \(\sum_{j=1}^n \mu_j = 2\), and
let $d$ be the common denominator of $\mu_1, \ldots, \mu_n$. For the cyclic $d$-ple covering
$C$ of $P^1$ branching at $n$ points with index $\mu = (\mu_1, \ldots, \mu_n)$ and a homology
marking $\phi$ of $C$, the period $p(C, \phi)$ of a marked curve $(C, \phi)$ can be regarded
as an element of the $(n - 3)$-dimensional complex ball $B_\mu$. The morphism
$p$ from the moduli space $M_{marked}$ of marked curves $(C, \phi)$ to $B_\mu$ is called a
period map. Since the period map $p$ is equivariant under the monodromy
group $\Gamma_\mu$, it induces the morphism $M_{marked}/\Gamma_\mu \to B_\mu/\Gamma_\mu$. According to
results of [DM] and [T], the period map $p$ is an isomorphism onto a Zariski
open set of $B_\mu/\Gamma_\mu$ if $\mu$ satisfies the condition
\[(1.1) \quad (1 - \mu_j - \mu_k)^{-1} \in \mathbb{Z} \cup \infty \text{ for } j \neq k .\]
There are finitely many such $\mu$’s for $n \geq 5$; for $n \geq 9$ there is no $\mu$, for $n = 8$
there is one, for $n = 7$ there is one, for $n = 6$ there are seven and for $n = 5$
there are 27.

It is a natural demand to describe the inverse of the period map in terms of
explicit automorphic forms with respect to $\Gamma_\mu$ for such $\mu$. In fact, for several
$\mu$ for $n = 5, 6$, the inverse of the period map is expressed in terms of theta
constants; refer to [K1], [Ma1], [Ma2] and [S]. In this paper, we construct
automorphic forms on $B_\mu$ which give the inverse of the period map for the
case $n = 8$, $\mu = (1/4)^8 = (1/4, \ldots, 1/4)$. Note that all $\mu$’s with $d = 4$
satisfying the condition (1.1) can be obtained by confluenes of the branching
index $(1/4)^8$.

Before stating our main theorem, we define a projective embedding of the
moduli space $M_{8pts}$ of 8 points on $P^1$, which is isomorphic to $M_{marked}/\Gamma_\mu$. We
divide the set \{1, \ldots, 8\} into a set of four pairs \{\{j_1, j_2\}, \ldots, \{j_7, j_8\}\}, which is called a (2, 2, 2, 2)-partition of \{1, \ldots, 8\}. We denote the set of (2, 2, 2, 2)-partitions of \{1, \ldots, 8\} by \P(2^4) which has cardinality 105. We associate a polynomial \(P_r = \prod_{k=1}^{4}(x_{j_{2k}} - x_{j_{2k-1}})\) for each (2, 2, 2, 2)-partition \(r = \{\{j_1, j_2\}, \ldots, \{j_7, j_8\}\}\). If we regard \(x_j\)’s as affine coordinates of eight points on \P^1, then \(P_r\) are relative invariants under projective transformations of \P^1. Thus the set of polynomials \(\{P_r\}_{r \in \P(2^4)}\) induces a map \(P : M_{8\text{pts}} \to \P^{104}\). It is shown in [K2] that \(P\) is an embedding.

**Theorem 1.1 (Main Theorem).** There exist 105 automorphic forms \(\mathcal{T}_r(2)\) with respect to \(\Gamma_\mu\) such that the following diagram commutes

\[
\begin{array}{ccc}
M_{8\text{pts}} & \xrightarrow{P} & \P^{104}, \\
p & & \\
B_\mu/\Gamma_\mu & \xrightarrow{\mathcal{T}(2)} & \\
\end{array}
\]

where the map \(\mathcal{T}(2)\) is given by the 105 automorphic forms \(\mathcal{T}_r(2)\). Especially, the image of \(\mathcal{T}(2)\) coincides with that of \(P\) and \(\mathcal{T}(2) \circ P^{-1}\) gives the inverse of \(p\).

Let us explain how to construct 105 automorphic forms in terms of theta constants. The curve \(C\) is of genus 9 and it can be regarded as a double cover of a hyperelliptic curve of genus 3. The period \(p(C, \phi)\) is an element of the 5-dimensional complex ball. We consider the Prym variety \(\text{Prym}(C)\) of \(C\) with respect to \(\rho^2\), which is a 6-dimensional sub-Abelian variety of the Jacobian \(J(C)\) of \(C\) obtained by the \((-1)\)-eigenspaces of \(H^0(C, \Omega^1)\) and \(H_1(C, \mathbb{Z})\) for the action of \(\rho^2\), where \(\rho\) is a generator of the group of covering transformations of \(C \to \P^1\).

Since the polarization of \(\text{Prym}(C)\) is not principal, we construct 105 principally polarized abelian varieties isogenous to \(\text{Prym}(C)\) as follows. Let \(\text{Prym}(C)_{1-\rho}\) be the group of \((1 - \rho)\)-torsion points of \(\text{Prym}(C)\), which is isomorphic to \(\mathbb{F}_2^6\) with a quadratic form. There are 105 three dimensional totally singular subspaces \(\Lambda_r\) of \(\text{Prym}(C)_{1-\rho}\). For each \(\Lambda_r\), \(A_{L_r} = \text{Prym}(C)/\Lambda_r\) is principally polarized. We show that there is an isomorphism between the automorphism groups \(\mathfrak{S}_8\) of the marking of 8 points and the orthogonal group \(O_6^+(\mathbb{F}_2)\) of \(\text{Prym}(C)_{1-\rho}\) yields a one to one correspondence between \(P(2^4)\) and the set \(\{\Lambda_r\}\).

For \(\Lambda_1\) corresponding to \(r_1 = \{\{12\}, \{34\}, \{56\}, \{78\}\}\in \P(2^4)\), we study the behavior of the pull back \(F_{m_1} = \iota^*(\vartheta_{m_1})\) of a theta function \(\vartheta_{m_1}(z)\) associated to \(A_{L_1}\) with a characteristic \(m_1\) under the composite map \(\iota : C \to A_{L_1}\) of the canonical map \(\text{jac}^- : C \to \text{Prym}(C)\) and the natural projection \(\text{Prym}(C) \to A_{L_1}\) (see Section 4). There are twelve zeros of \(F_{m_1}\) in \(C\), eight of them are in the set of fixed points of \(\rho\). There are three theta functions...
\( \vartheta_{m_j}(z) \) \((j = 2, 3, 4)\) on \( A_{L_1} \) such that the order of zero of \( F_{m_j} = i^*(\vartheta_{m_j}) \) at every fixed point of \( \rho \) is same as that of \( F_{m_1} \). By considering the four zeros of \( F_{m_1} \) not fixed by \( \rho \), we can express the cross ratio of \( x_1, x_2, x_5, x_6 \) in terms of the theta constants \( \vartheta_{m_1}, \ldots, \vartheta_{m_4} \).

The product \( \mathcal{T}_1^{(2)} = \prod_{j=1}^4 \vartheta_{m_j} \) is an automorphic form with respect to the monodromy group \( \Gamma_\mu \). The period map \( p \) induces an isomorphism of groups \( \text{Aut}(M_{8\text{pts}}) \simeq \text{Aut}(B_\mu/\Gamma_\mu) \). For each \((2, 2, 2, 2)\)-partition \( r \in P(2^4) = \text{Stab}(r_1)\backslash \mathcal{S}_8 \), we define \( \mathcal{T}_r^{(2)} \) by using the action of \( \mathcal{S}_8 \subset \text{Aut}(M_{8\text{pts}}) \simeq \text{Aut}(B_\mu/\Gamma_\mu) \), where \( \text{Stab}(r_1) \) is the stabilizer of the partition \( r_1 \). In order to prove our main theorem, we investigate the action of \( \text{Stab}(r_1) \) on the space generated by the theta constants and relations between theta functions for \( A_{L_1} \) and those for \( A_{L_r} \).

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**Notation 1.2.** In this paper, the imaginary unit is denoted by \( i \). For an element \( \alpha \in \mathbb{C} \), \( \exp(2\pi i \alpha) \) is denoted by \( e(\alpha) \). For a square matrix \( A \), the vector consisting of the diagonal elements of \( A \) is denoted by \( A_0 \). For a vector \( v = (v_1, \ldots, v_k) \), the diagonal matrix \( \begin{pmatrix} v_1 & \ldots & \cdot \cdot \cdot & \cdot \cdot \cdot & v_k \end{pmatrix} \) is denoted by \( \text{diag}(v) \).

## 2. The Prym Variety of \( C \)

### 2.1. 4-pie covering of \( \mathbb{P}^1 \) branching at 8 points.

Let \( C \) be the projective smooth model of an algebraic curve defined by

\[
(2.1) \quad w^4 = \prod_{j=1}^8 (z - x_j),
\]

where \( x_1, \ldots, x_8 \) are distinct elements of \( \mathbb{C} \). The curve \( C \) is of genus 9 since it can be regarded as a 4-pie covering of \( \mathbb{P}^1 \) branching at eight points. The automorphism

\[
(2.2) \quad \rho : C \ni (z,w) \mapsto (z, iw) \in C
\]

induces actions on \( H^1(C, \mathbb{Q}) \) and \( H_1(C, \mathbb{Q}) \). We denote the \((-1)\)-eigenspaces of \( H^1(C, \mathbb{Q}) \) and \( H_1(C, \mathbb{Q}) \) of \( \rho^2 \) by \( H^1(C, \mathbb{Q})^- \) and \( H_1(C, \mathbb{Q})^- \), respectively. We put

\[
H^1(C, \mathbb{Z})^- = H^1(C, \mathbb{Q})^- \cap H^1(C, \mathbb{Z}), \quad H_1(C, \mathbb{Z})^- = H_1(C, \mathbb{Q})^- \cap H_1(C, \mathbb{Z}).
\]

Since the action \( \rho \) preserves the polarized rational Hodge structure of \( H^1(C, \mathbb{Q}) \), the \((-1)\)-eigen subspace \( H^1(C, \mathbb{Q})^- \) of \( \rho^2 \) is a polarized rational sub-Hodge structure of \( H^1(C, \mathbb{Q}) \). The action of \( \rho \) induces an action on each factor of the Hodge decomposition

\[
H^1(C, \mathbb{Z})^- \otimes \mathbb{C} \simeq H^0(C, \Omega^1)^- \oplus \overline{H^0(C, \Omega^1)^-}.
\]
Proposition 2.1. The multiplicity of the eigenvalue $i$ (resp. $-i$) of $\rho$ on $H^0(C, \Omega^1)^-$ is 5 (resp. 1).

Proof. Differential 1-forms

$$\varphi_j = \frac{z^j dz}{w^3} \ (j = 0, \ldots, 4), \quad \varphi_5 = \frac{dz}{w}, \quad \varphi_6 = \frac{dz}{w^2}, \quad \varphi_7 = \frac{z dz}{w^2}, \quad \varphi_8 = \frac{z^2 dz}{w^2}$$

span the space $H^0(C, \Omega^1)$. We have $\rho^*(\varphi_j) = i\varphi_j$ for $j = 0, \ldots, 4$, $\rho^*(\varphi_5) = -i\varphi_5$ and $\rho^*(\varphi_k) = -\varphi_k$ for $k = 6, 7, 8$.

We study the intersection form on $H_1(C, \mathbb{Z})^-$. For the moment, we assume that $x_j \in \mathbb{R} \ (j = 1, \ldots, 8)$ and $x_1 < x_2 < \cdots < x_8$. Let $U_0$ be $\mathbb{P}^1$ cut along the eight semi-lines

$$l_j = \{x_j + it \in \mathbb{C} \subset \mathbb{P}^1 \mid t \leq 0\}, \ j = 1, \ldots, 8.$$ 

The curve $C$ can be regarded as gluing $U_0$ and the copies $U_k = \rho^k(U_0)$ ($k = 1, 2, 3$) along $l_1, \ldots, l_8$. The branch of the function $w$ is assigned as its value is in $i^k \mathbb{R}_+$ for $z \in (-\infty, x_1) \subset U_k$. Let $\alpha_j (1 \leq j \leq 7)$ be the interval $[x_j, x_{j+1}] \subset \mathbb{R}$ in the sheet $U_0$ and $\alpha_8$ be $[-\infty, x_1] \cup [x_8, \infty] \subset \mathbb{R}$ in the sheet $U_0$. The orientation of $\alpha_j$ is given in the figure.

Then the 1-chain $A_j = (1 - \rho^2)\alpha_j$ is a cycle satisfying $\rho^2(A_j) = -A_j$. We put $B_j = \rho A_j$. The above figure shows that both of $\sum_{j=1}^8 \alpha_j$ and $\sum_{j=1}^8 \rho^j \alpha_j$ are boundaries of 2-chains. Therefore we have $\sum_{j=1}^8 A_j = 0$ and $\sum_{j=1}^7 \frac{1 - \rho^j}{1 - \rho} A_j = 0$ in $H_1(C, \mathbb{Z})^-$. The intersection matrix for $\{A_j, B_j\}_{j=1, \ldots, 6}$ is given as

$$\begin{pmatrix} P & Q \\ -Q & P \end{pmatrix},$$

where

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{pmatrix}.$$

Figure 1.
Proposition 2.2. The set \( \{ A_j, B_j \}_{j=1,\ldots,6} \) is a basis of \( H_1(C, \mathbb{Z})^- \).

Proof. The determinant of the matrix (2.3) is 2^6. By the Fey’s result [F], we have this proposition. \( \square \)

2.2. Polarization of the Prym variety. The polarized Hodge structure of \( H^1(C, \mathbb{Z}) \) defines the abelian variety

\[
\text{Prym}(C) = \text{Prym}(C, \rho^2) = (H^0(C, \Omega^1)^*)/H_1(C, \mathbb{Z})^-,
\]

which is called the Prym variety of \( C \). Since the first homology group \( H_1(\text{Prym}(C), \mathbb{Z}) \) of the Prym variety \( \text{Prym}(C) \) is isomorphic to \( H_1(C, \mathbb{Z})^- \), the restriction

\[
(,): H_1(C, \mathbb{Z})^- \times H_1(C, \mathbb{Z})^- \to \mathbb{Z}
\]

of the intersection form on \( H_1(C, \mathbb{Z}) \) gives a polarization of \( \text{Prym}(C) \). Thus \( H_1(C, \mathbb{Z})^- \cong H_1(\text{Prym}(C), \mathbb{Z}) \) naturally has a polarized Hodge structure of weight \((-1)\).

Definition 2.3. Let \( H \) be a polarized \( \mathbb{Z} \)-Hodge structure of weight \((-1)\). If \( H \) has a basis whose intersection matrix is

\[
\begin{pmatrix}
0 & e \\
-e & 0
\end{pmatrix}, \quad e = \text{diag}(\varepsilon_1, \ldots, \varepsilon_d),
\]

then the type of the polarization of \( H \) is said to be \((\varepsilon_1, \ldots, \varepsilon_d)\). For a polarized abelian variety \((A, (,))\), the type of \((H_1(A, \mathbb{Z}), (,))\) is called the type of the polarized abelian variety. The polarization of type \((1, \ldots, 1)\) is called principal.

The curve \( C \) can be regarded as a double covering of a hyperelliptic curve of genus 3 branching at eight points. The results in [F] implies that the type of the polarized abelian variety \( \text{Prym}(C) \) is \((2, 2, 2, 1, 1, 1)\) (see also [Mu], §3 Cor 1). We give a symplectic basis \( \Sigma \) of \( H_1(C, \mathbb{Z})^- \).

Proposition 2.4. Let \( \Sigma = \{ \alpha'_1, \ldots, \alpha'_6, \beta'_1, \ldots, \beta'_6 \} \) be

\[
\begin{align*}
\alpha'_1 &= 2A_1 + 2A_3 + A_4 + B_4, \\
\alpha'_2 &= 2A_1 + A_2 + 2A_5 + 2A_6 + B_2 - 2B_4 - 2B_5, \\
\alpha'_3 &= A_1 + 2A_2 + A_3 + A_5 + 2A_6 - B_1 + B_3 - B_5, \\
\alpha'_4 &= A_1, \quad \alpha'_5 = A_1 + A_3, \quad \alpha'_6 = -A_1 - A_3 + B_5, \\
\beta'_1 &= 2A_1 + 2A_3 + A_5 + 2A_6 - B_5, \\
\beta'_2 &= -A_1 - 2A_2 - 2A_5 - 2A_6 + B_1 + 2B_4 + 2B_5, \\
\beta'_3 &= A_4 - A_6 + B_4 + 2B_5 + B_6, \\
\beta'_4 &= A_2, \quad \beta'_5 = A_4, \quad \beta'_6 = A_6.
\end{align*}
\]

Then \( \Sigma \) is a basis of \( H_1(C, \mathbb{Z})^- \) whose intersection matrix is given by (2.4) for \( e = \text{diag}(2, 2, 2, 1, 1, 1) \).

To simplify the notation, the half of the intersection form on \( H_1(C, \mathbb{Z})^- \) is denoted by \(<,>\). To classify the sub-principally polarized Hodge structures
of \((H_1(C, \mathbb{Z}), (,))\) of type \((2, 2, 2, 2, 2)\) is equivalent to classify principally polarized sub-Hodge structures of \((H_1(C, \mathbb{Z}), <,>)\).

For a polarized Hodge structure \(H\), the dual Hodge structure \(H^\perp\) is defined as

\[ H^\perp = \{ v \in H \otimes \mathbb{Q} | <v, w> \in \mathbb{Z} \text{ for all } w \in H \}. \]

It is easy to see that a polarized Hodge structure \(H\) is principal if and only if \(H = H^\perp\). From now on, we use the polarization <,> for sub-Hodge structures of \(H_1(C, \mathbb{Z})^-\).

**Proposition 2.5.**

1. \((H_1(C, \mathbb{Z})^-)^\perp = (1 - \rho)H_1(C, \mathbb{Z})^-\).
2. A principally polarized sub-Hodge structure \(L\) of \(H_1(C, \mathbb{Z})^-\) contains \((1 - \rho)H_1(C, \mathbb{Z})^-\) and is stable under the action of \(\rho\).

In the next section, we give a combinatorial description of the set of principally polarized sub-Hodge structures of \(H_1(C, \mathbb{Z})^-\).

We close this section by giving an example of a principally polarized sub-Hodge structure of \(H_1(C, \mathbb{Z})^-\) using the basis \(A_j, B_j, (j = 1, \ldots, 6)\) given in Proposition 2.2.

**Proposition 2.6.** The sub-Hodge structure \(L_1\) of \(H_1(C, \mathbb{Z})^-\) generated by \((1 - \rho)H_1(C, \mathbb{Z})^-\) and \(A_1, A_3, A_5\) is principal. Actually the set \(\Sigma_1 = \{a_1, \ldots, a_6, b_1, \ldots, b_6\}\), where

\[
\begin{align*}
a_1 &= A_1, \\
a_2 &= A_1 + A_2 + B_2, \\
a_3 &= A_1 + A_2 + B_2 + B_3, \\
a_4 &= A_1 + A_2 - A_4 + B_2 + B_3 + B_4, \\
a_5 &= A_1 + A_2 + A_5 + B_2 + B_3, \\
a_6 &= A_1 + A_2 + A_5 + A_6 + B_2 + B_3 + B_6, \\
b_1 &= -B_1, \\
b_2 &= A_2 - B_1 - B_2, \\
b_3 &= -A_2 - A_3 - A_4 + B_1 + B_2 - B_4, \\
b_4 &= -A_2 - A_3 + B_1 + B_2, \\
b_5 &= A_2 + A_3 - B_1 - B_2 - B_5, \\
b_6 &= A_2 + A_3 + A_6 - B_1 - B_2 - B_5 - B_6,
\end{align*}
\]

is a symplectic basis of type \((1, 1, 1, 1, 1, 1)\). The action \(\rho\) on this basis is given by

\[
(2.5) \quad t(a_1, \ldots, a_6, b_1, \ldots, b_6) \mapsto \begin{pmatrix} 0 & -U \\ U & 0 \end{pmatrix} t(a_1, \ldots, a_6, b_1, \ldots, b_6),
\]
where
\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

**Definition 2.7.** A pair \((L, \Sigma_L)\) of a principal sub-Hodge structure \(L\) of \(H_1(C, \mathbb{Z})^-\) and a symplectic basis \(\Sigma_L\) of \(L\) is called a good basis if the action \(\rho\) on \(\Sigma_L\) is given by (2.4).

**Remark 2.8.** Using a good base \(\Sigma_L\), an element of \(\frac{1}{\rho} L/L\) can be written as \(\frac{1}{2} (\mu, \mu U) \mod L \quad (\mu \in \mathbb{Z}^6)\).

We will use the lattice \(L_1\) and its symplectic basis \(\Sigma_1\) in Proposition 2.6 for explicit calculations of theta constants.

### 3. \(O_6^+ (2)\)-Level Structure and \(\mathcal{S}_8\)-Marking

#### 3.1. Total singular subspaces, length 0 elements

We define the standard lattice \(H_{\text{std}}\) as the free \(\mathbb{Z}\)-module generated by \(A_j, B_j \quad (j = 1, \ldots, 6)\). We introduce an alternating form \(<,>\) and an action of \(\rho\) on \(H_{\text{std}}\) by the half of the matrix (2.3) in (2) and \(\rho(A_j) = B_j, \rho(B_j) = -A_j\). Since the action of \(\rho^2\) on \(H_{\text{std}}\) is equal to the multiplication by \((-1), \rho_{\text{std}, R} = H_{\text{std}} \otimes \mathbb{R}\)) becomes a \(\mathbb{Z}[\rho]/(\rho^2 + 1)\)-module (resp. a vector space over \(\mathbb{R}(\rho) \cong \mathbb{C}\)). We define a bilinear form \(h(x,y)\) on \(H_{\text{std}}\) by \(h(x,y) = \langle x, \rho y \rangle = \langle x, y \rangle\). Then \(h(\cdot, \cdot)\) becomes a hermitian form of the signature \((5, 1)\) with respect to the complex structure given by \(\mathbb{R}(\rho)\). The group of isomorphisms of \(H_{\text{std}}\) preserving the alternating form \(<,>\) and the action of \(\rho\) is denoted by \(U(H_{\text{std}})\). The value of associated Hermitian metric \(q(x) = h(x,x)\) on \(H_1(C, \mathbb{Z})^-\) is integral. The class \(q(x)\) of \(q(x)\) modulo 2 defines a \(\mathbb{Z}/2\mathbb{Z}\)-valued quadratic form on \(H_1(C, \mathbb{Z})^-/(1-\rho)H_1(C, \mathbb{Z})^-\). This quadratic form \(q\) has the following simple form.

Let \((V, q)\) be a 6-dimensional vector space \(V\) over \(\mathbb{F}_2\) with the quadratic form \(q\) of Witt defect 0. Such \((V, q)\) is constructed as follows. The Hamming length of a vector \(x \in \mathbb{F}_2^8\) is defined by the number of nonzero elements. The subset \(\tilde{V}\) consisting of vectors with the even Hamming length is a subspace of \(\mathbb{F}_2^8\) containing \((1, \ldots, 1)\). Let \(V\) be the quotient space of \(\tilde{V}\) by the subspace \((1, \ldots, 1) \cdot \mathbb{F}_2\). The half of the Hamming length modulo 2 becomes the quadratic form \(q\) on \(V\) of Witt defect 0.

We have an isomorphism
\[
(H_1(C, \mathbb{Z})^-/(1-\rho)H_1(C, \mathbb{Z})^-, q) \to (V, q)
\]
by mapping the class of \(A_j\) to that of \(e_j - e_{j+1}\), where \(e_j\) is the \(j\)-th unit vector of \(\mathbb{F}_2^8\). The orthogonal group of \((V, q)\) is denoted by \(O_6^+(2)\). The symmetric group \(\mathcal{S}_8\) of degree 8 acts on the space \(V\) by \(e_j \cdot \sigma = e_{j\sigma}\) for \(\sigma \in \mathcal{S}_8\) and \(j = 1, \ldots, 8\). This action defines a group homomorphism \(\mathcal{S}_8 \to O_6^+(2)\).
Definition 3.1. For natural numbers $\varepsilon_1, \ldots, \varepsilon_d$ such that $\varepsilon_1 + \cdots + \varepsilon_d = 8$, a (unordered) partition of $\{1, \ldots, 8\}$ into sets of cardinality $\varepsilon_1, \ldots, \varepsilon_d$ is called an $(\varepsilon_1, \ldots, \varepsilon_d)$-partition. The set of $(\varepsilon_1, \ldots, \varepsilon_d)$-partitions of $\{1, \ldots, 8\}$ is denoted by $P(\varepsilon_1, \ldots, \varepsilon_d)$. The sets $P(2, 2, 2, 2)$ and $P(4, 4)$ are denoted by $P(2^4)$ and $P(4^2)$, respectively.

Note that $\#P(2^4) = 105$ and $\#P(4^2) = 35$. Therefore we have the following propositions.

Proposition 3.2. (1) The map $\mathfrak{S}_8 \to O_6^+(2)$ is an isomorphism.

(2) By mapping an element $s = \{(s_1, \ldots, s_4), \{s_5, \ldots, s_8\}\} \in P(4^2)$ to a non-zero element $v = e_{s_1} + \cdots + e_{s_4}$ with $q(v) = 0$, we have a one-to-one correspondence

$$P(4, 4) \simeq \{v \in V \mid q(v) = 0, v \neq 0\}.$$ 

Under this correspondence, the stabilizer of $s \in P(4, 4)$ is isomorphic to the stabilizer of $v$.

For an element $I = \{\{j_1, j_2\}, \ldots, \{j_7, j_8\}\}$ of $P(2^4)$, we define a subspace $V_I = \langle e_{j_1} - e_{j_2}, \ldots, e_{j_7} - e_{j_8} \rangle$ of $V$.

Proposition 3.3. Let $\psi : H_1(C, \mathbb{Z})^- \to V$ be the natural projection. Then $\psi^{-1}(V_I)$ is a principally polarized sub-Hodge structure in $H_1(C, \mathbb{Z})^-$. This gives a one to one correspondence between $P(2^4)$ and the set of sub-lattices $H$ of $H_1(C, \mathbb{Z})^-$ such that

1. $H = H^1$,
2. $H/(1 - \rho)H_1(C, \mathbb{Z})^-$ contains a vector $v$ with $q(v) \neq 0$.

Under the correspondence of Proposition 3.3, the lattice $L_1$ given in Proposition 2.6 corresponds to the partition $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$.

3.2. Moduli spaces of branched coverings of $\mathbb{P}^1$. In this section, we give analytic descriptions of moduli spaces of 4-ple coverings of $\mathbb{P}^1$ branching at 8 points. Let $H_{\text{std}}$ be the module with a symplectic form and the action of $\rho$ as in the last subsection. An isomorphism $\phi : H_{\text{std}} \to H_1(C, \mathbb{Z})^-$ compatible with the intersection pairing and with the action of $\rho$ is called a marking of $C$. A pair $(C, \phi)$ is called a marked curve. An isomorphism between two marked curves is an isomorphism between two curves which is compatible with the markings. The set of isomorphic classes of marked curves is denoted by $M_{\text{marked}}$. For an element $\sigma \in U(H_{\text{std}})$, we define $\sigma(C, \phi)$ by $(C, \phi \circ \sigma)$. By the commutative diagram

$$
\begin{array}{ccc}
H_{\text{std}} & \xrightarrow{\rho} & H_{\text{std}} \\
\downarrow & & \downarrow \\
H_{\text{std}} & \xrightarrow{\phi} & H_1(C, \mathbb{Z})^- \\
& & \downarrow \rho \\
H_{\text{std}} & & H_1(C, \mathbb{Z})^- \\
\phi & & \\
\end{array}
$$
we have $\rho(C, \sigma) = (C, \sigma)$.

We define the complex ball $B(H_{std})$ associated to $H_{std}$ as follows. The complex structure on the vector space $H_{std,R}$ is given by the action of $R(\rho)$ via the identification $R(\rho) \simeq C$ given by $\rho = -i$. In the complex projective space $P(H_{std,R}^*)$, the domain

$$B(H_{std}) = \{ h(v, *) \in P(H_{std,R}^*) \mid h(v, v) < 0 \}$$

is isomorphic to a 5-dimensional complex ball. Then an element $\sigma \in U(H_{std})$ induces an isomorphism of $B(H_{std})$.

For a marked curve $(C, \phi)$, we construct a point $p(C, \phi)$ in $B(H_{std})$ as follows. The complex structure arising from $\rho$ is denoted by $R(\rho)$ to distinguish the usual complex structure. By Proposition 2.1, the $(-i)$-eigenspace $H^0(C, \Omega^1)^{-,\rho=-i}$ of $\rho$ in the space $H^0(C, \Omega^1)^-$ is one dimensional. By the definition of a Hodge structure, we have an $R$-isomorphism

$$H_1(C, R)^- \to \text{Hom}_C(H^0(C, \Omega^1)^-, C).$$

By composing the map

$$\text{Hom}_C(H^0(C, \Omega^1)^-, C) \to \text{Hom}_C(H^0(C, \Omega^1)^{-,\rho=-i}, C),$$

and $H_{std,R} \overset{\phi}{\simeq} H_1(C, R)^-$, we have an $R$-linear map

$$(3.2) \quad H_{std,R} \to \text{Hom}(H^0(C, \Omega^1)^{-,\rho=-i}, C).$$

It is easy to see that if we consider the complex structure on $H_{std,R}$ by the action of $R(\rho)$, the $R$-linear map (3.2) is isomorphism for the complex structures via the isomorphism $R(\rho) \simeq C$ given by $\rho = -i$. This linear form defines a point in $B(H_{std})$. The corresponding point is denoted by $p(C, \phi)$. This map $p : M_{\text{marked}} \to B(H_{std})$ is called a period map. More explicitly, this map is given as follows. Let $\omega$ be a basis of one dimensional complex space of $H^0(C, \Omega^1)^{-,\rho=-i}$. Then the map (3.2) $H_{std} \otimes R \to C$ is given by $\gamma \mapsto \int_{\phi(\gamma)} \omega$. By using the dual basis $A_j^*$ of $A_j$ over $R(\rho)$, this map is expressed as $\sum_{j=1}^{6} (\int_{\phi(A_j)} \omega) A_j^*$. This period map is holomorphic with respect to the parameters $x_1, \ldots, x_8$.

**Theorem 3.4** (Terada, Deligne-Mostow). The map $p$ is an open embedding and the complement of the image is a proper analytic subset. The natural actions of $U(H_{std})$ on $M_{\text{marked}}$ and $B(H_{std})$ are compatible.

Let $M_{\text{spts}}$ (resp. $M_{\text{unord}}$) be the set of the isomorphism classes of ordered (resp. unordered) set of distinct 8 points in $P^1$. The set $M_{\text{spts}}$ (resp. $M_{\text{unord}}$) has a natural structure of an algebraic variety which is isomorphic to $((P^1)^8 - \text{Diag})/\mathfrak{S}_8 \times PGL(2, C)$ (resp. $((P^1)^8 - \text{Diag})/\mathfrak{S}_8 \times PGL(2, C)$), where $\text{Diag} = \{(x_j) \mid x_p = x_q \text{ for some } p < q\}$. By corresponding a marked curve $(C, \phi) \in M_{\text{marked}}$ to the set of branching points $\{x_1, \ldots, x_8\}$ of $C \to P^1$, we have a morphism:

$$M_{\text{marked}} \to M_{\text{unord}}.$$
Proposition 3.5. Via the open immersion $p$, $M_{\text{unord}}$ is identified with an open set of $B(H_{\text{std}})/U(H_{\text{std}})$. Moreover the covering $M_{8\text{pts}}$ of $M_{\text{unord}}$ is identified with an open set of $B(H_{\text{std}})/\Gamma(i+1)$, where

$$\Gamma(i+1) = \{g \in U(H_{\text{std}}) \mid g \equiv 1 \text{ mod } (1+\rho)H_{\text{std}}\}.$$ 

We define complex reflections $M_{p,p+1} \subset U(H_{\text{std}})$ for $p = 1, \ldots, 7$. We choose an initial point $X = (x_1, \ldots, x_8) \in M_{8\text{pts}}$ such that $x_j \in \mathbb{R}$ ($j = 1, \ldots, 8$) and $x_1 < \cdots < x_8$ (see figure in [2.1]). The image of $X$ in $M_{\text{unord}}$ is denoted by $\overline{X}$. We consider a path $M_{p,p+1} = (M_{p,p+1,j})_{j=1,\ldots,8}: [0,1] \to M_{8\text{pts}}$ by

$$M_{p,p+1,j}(t) = x_j \text{ for } j \neq p, p+1,$$

$$M_{p,p+1,p}(t) = \frac{x_{p+1} + x_p}{2} - \frac{x_{p+1} - x_p}{2}e(t/2),$$

$$M_{p,p+1,p+1}(t) = \frac{x_{p+1} + x_p}{2} + \frac{x_{p+1} - x_p}{2}e(t/2),$$

where $e(t) = \exp(2\pi it)$. Then $M_{p,p+1}$ defines a closed path in $M_{\text{unord}}$ with the base point $\overline{X}$. By fixing the point $X$, we define the monodromy action $M_{p,p+1}$ of $H_{\text{std}}$ as follows. Let $C$ be the curve defined by the equation (2.2), where $x_1, \ldots, x_8$ are the coordinates of $X$. A basis $A_1, \ldots, B_6$ of $H_1(C,\mathbb{Z})^-$ defined in [2.1] gives a marking $H_{\text{std}} \to H_1(C,\mathbb{Z}^-)$. We consider the lifting $M_{p,p+1}$ of the path $M_{p,p+1}$ beginning from the point in $M_{\text{marked}}$ corresponding to the pair $(C,\phi)$. Then the end point $(C,\phi')$ of $\tilde{M}_{p,p+1}$ is a lifting $\overline{X}$ of $X$ in $M_{\text{marked}}$. The composite map $\phi'^{-1} \circ \phi'$

$$H_{\text{std}} \xrightarrow{\phi'} H_1(C,\mathbb{Z})^- \xleftarrow{\phi} H_{\text{std}}$$

is denoted by $M_{p,p+1}$. Since the paring and the action of $\rho$ are preserved in the family $H_1(C_{M_{p,p+1}(t)},\mathbb{Z})^-$, $(t \in [0,1])$, we have $M_{p,p+1} \in U(H_{\text{std}})$. Since $\Gamma(i+1)$ is a normal subgroup of $U(H_{\text{std}})$, the covering $B(H_{\text{std}})/\Gamma(i+1) \to B(H_{\text{std}})/U(H_{\text{std}})$ is a Galois covering and

$$\text{Gal}(B(H_{\text{std}})/\Gamma(i+1) \to B(H_{\text{std}})/U(H_{\text{std}})) \simeq U(H_{\text{std}})/\Gamma(i+1) \simeq O_6^+(2).$$

By chasing the action of $M_{p,p+1}$ on $H_1(C,\mathbb{Z})^-$, we have the following lemma.

Lemma 3.6. There exists an isomorphism $O_6^+(2) \simeq \mathfrak{S}_8$ such that the image of $M_{p,p+1} \subset U(H_{\text{std}})$ is the transposition $(p,p+1)$ of $p$ and $p+1$. Under the isomorphism

$$B(H_{\text{std}})/\Gamma(i+1) \simeq M_{\text{marked}},$$

the action of $\mathfrak{S}_8 \subset \text{Aut}(M_{\text{marked}})$ is induced by $\sigma^*(x_j) = x_{j\sigma}$. Here the group $\mathfrak{S}_8$ acts on the set $\{1, \ldots, 8\}$ from the right. The action of $M_{p,p+1}$ is a complex reflection for the root $A_p$ with the eigenvalue $-\rho$, i.e., $M_{p,p+1}$ is characterized by

$$M_{p,p+1}(v) = \begin{cases} 
-\rho(A_p) & \text{if } v = A_p, \\
v & \text{if } (v,A_p) = 0.
\end{cases}$$
The isomorphism $O^+_6(2) \cong \mathcal{G}_8$ in Lemma 3.6 induces a homomorphism $\pi : U(H_{std}) \rightarrow \mathcal{G}_8$.

We define an inclusion $B(H_{std})$ to the Siegel upper half space $\mathcal{H}_6$ of degree 6 by using a good basis $(L, \Sigma_L)$ of $H_{std}$ as follows. By the Poincare duality, we have $h(y_1, y_2) = 0$ for elements $y_1 \in \ker(H_{std}, Hom(H^0(C, \Omega^1)^{-}, \chi_{\rho=-i}, C))$ and $y_2 \in \ker(H_{std}, Hom(H^0(C, \Omega^1)^{-}, \chi_{\rho=+i}, C))$, where $H^0(C, \Omega^1)^{-}$ is the $i$-eigenspace of $\rho$ in the space $H^0(C, \Omega^1)^{-}$. Let $H$ be the corresponding Hodge structure of $p \in B(H_{std})$. Let $\Sigma_L = \{a_1, \ldots, a_6, b_1, \ldots, b_6\}$ and

\[ H \otimes C \cong H^{(1,0)} \oplus H^{(0,1)} \]

be the Hodge decomposition of $H$. We choose a basis $\omega_1, \ldots, \omega_9$ of $H^{(1,0)}$ such that $\int_b \omega_k = \delta_{jk}$. We put $\tau_{jk} = \int_a \omega_k$. Then by the definition of polarized Hodge structure, $\tau = (\tau_{jk})_{j,k}$ is an element of $\mathcal{H}_6$.

3.3. Level 2 structure and exponent 2 covering of configuration space.

Let $C(x_1, \ldots, x_8)$ be the rational function field of $x_1, \ldots, x_8$ over $C$. On this field, the groups $PGL(2, C)$ and $\mathcal{G}_8$ act by

\[ g(x_j) = \frac{ax_j + b}{cx_j + d}, \quad \sigma(x_j) = x_j \sigma \]

for $g \in PGL(2, C)$ and $\sigma \in \mathcal{G}_8$, respectively. As in the last section the group $\mathcal{G}_8$ acts on the set $\{1, \ldots, 8\}$ from the right. (Note that the action of $\mathcal{G}_8$ on the space $M_{marked}$ is covariant.) Let $K$ be the fixed subfield of $M = C(x_1, \ldots, x_8)$ under the action of $PGL(2, C)$. Since the action of $\mathcal{G}_8$ commutes with that of $PGL(2, C)$, $\mathcal{G}_8$ acts on $K$. Let $L$ be the fixed subfield of $K$ under the action of $\mathcal{G}_8$. Then $K$ and $L$ are equal to the function fields of $M_{8pts}$ and $M_{unord}$, respectively. The field $K$ is generated by the cross-ratios

\[ \lambda_j = \frac{(x_3 - x_1)(x_j - x_2)}{(x_2 - x_1)(x_j - x_3)} \]

of $\{x_1, x_2, x_3, x_j\}$ for $j = 4, \ldots, 8$ over $C$.

Let $\bar{K}$ be the algebraic closure of $K$ in $\bar{M} = C(x_j, \sqrt{x_j - x_k})_{j \neq k}$. Since the extension $\bar{M}/M$ is a Galois extension, so is $\bar{K}/K$. The extensions $M$ and $\bar{K}$ are linearly independent over $K$, therefore the restriction map

\[ Gal(\bar{M}/M) \rightarrow Gal(\bar{K}/K) \]

is surjective, and $\bar{K}$ is generated by $f = \prod_{j<k} \sqrt{x_j - x_k}$ such that $f^2$ is an element of $K$. Thus $\bar{K}$ is generated by $\sqrt{\lambda_j}$, $\sqrt{\lambda_j - 1}$ for $j = 4, \ldots, 8$, and $\sqrt{\lambda_j - \lambda_k}$ for $4 \leq j < k \leq 8$ and $\bar{K}$ is a Galois extension of $L$. The inclusions of fields $L \subset K \subset \bar{K}$ imply the following exact sequence of groups:

\[ 1 \rightarrow N \rightarrow Gal(\bar{K}/L) \rightarrow \mathcal{G}_8 \rightarrow 1, \]

where $N = Gal(\bar{K}/K) \cong (\mathbb{Z}/2\mathbb{Z})^{20}$.

We compare this Galois extension with the analytic description of the corresponding moduli space of 4-pale coverings of $P^1$ branching at eight points.
Proposition 3.7. Let $\tilde{M}_{8\text{pts}}$ be the normalization of the $M_{8\text{pts}}$ in $\tilde{K}$. This variety $\tilde{M}_{8\text{pts}}$ is identified with an open subset of $B(H_{\text{std}})/\Gamma(2)$ via the period map $p$, where

$$\Gamma(2) = \{g \in U(H_{\text{std}}) \mid g \equiv 1 \text{ mod } 2H_{\text{std}}\}.$$ 

Via this isomorphism, we have

$$\text{Gal}(\tilde{K}/L) \simeq U(H_{\text{std}})/\Gamma(2) \cdot \langle i \rangle,$$

where $\langle i \rangle$ is the cyclic group generated by $i$.

Proof. We already know that

$$U(H_{\text{std}})/\Gamma(i + 1) \cdot \langle i \rangle \simeq \mathfrak{S}_8.$$ 

The group $\Gamma(i + 1) \cdot \langle i \rangle/\Gamma(2) \cdot \langle i \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{20}$ is generated by $M_{jk}^2$. By restricting the action of $U(H_{\text{std}})$ to $\tilde{K}$, we have a map

$$\Gamma(i + 1) \cdot \langle i \rangle/\Gamma(2) \cdot \langle i \rangle \to N. \quad (3.3)$$

We study the action of $M_{jk}$ on $\{\sqrt{x_p - x_q}\}$. We assign each algebraic function $\sqrt{x_j - x_k}$ on $(\mathbb{P}^1)^8 - \text{Diag}$ a branch as follows. Let $X$ be the initial point in $M_{8\text{pts}}$ as in the last section. For $j < k$, $\sqrt{x_k - x_j}$ denotes the branch of the algebraic function on $(\mathbb{P}^1)^8 - \text{Diag}$ so that it takes a positive real value at $X$. The analytic continuation of the function $\sqrt{x_k - x_j}$ along the path $M_{p,p+1}$ is given as

$$M_{p,p+1}(\sqrt{x_k - x_j}) = \begin{cases} i\sqrt{x_k - x_j} & \text{if } j = p \text{ and } p + 1 = k, \\ \sqrt{x_k - x_j} & \text{otherwise}, \end{cases}$$

where $\sigma$ is the transposition $(p, p + 1)$. As a consequence, we have

$$M_{p,p+1}^2(\sqrt{x_k - x_j}) = \begin{cases} -\sqrt{x_k - x_j} & \text{if } j = p \text{ and } p + 1 = k, \\ \sqrt{x_k - x_j} & \text{otherwise}. \end{cases}$$

By looking at the action of $M_{jk}^2$ on the set $\{\sqrt{\lambda_p}, \sqrt{1 - \lambda_p}, \sqrt{\lambda_p - \lambda_q}\}$, the homomorphism $[53]$ is an isomorphism.

\[\square\]

Let $g \in U(H_{\text{std}})$ and $r_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \in P(2^4)$. We put $r_1 \pi(g) = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}, \{j_7, j_8\}\} \in P(2^4)$, where $\pi : U(H_{\text{std}}) \to \mathfrak{S}_8$. We assume that $j_p < j_{p+1}$ for $p = 1, 3, 5, 7$. Then there exists a 4-th root of the unity $\text{arg}(g)$, called the argument of $g$, such that

$$g(\sqrt{(x_2 - x_1)(x_4 - x_3)(x_6 - x_5)(x_8 - x_7)}) = \text{arg}(g)\sqrt{(x_{j_2} - x_{j_1})(x_{j_4} - x_{j_3})(x_{j_6} - x_{j_5})(x_{j_8} - x_{j_7})}.$$
4. Theta function of standard principal sub-Hodge structure $L_1$

4.1. Abelian-Jacobi map and the order of zero. Let $p_1, \ldots, p_8$ be the ramification points of the smooth curve $C$ defined by (2.1) above $x_1, \ldots, x_8$, respectively. Let $jac$ be the Abel-Jacobi map

$$jac: C \rightarrow J(C) = \text{Hom}(H^0(C, \Omega^1), C)/H_1(C, \mathbb{Z})$$

$$p \mapsto \text{the linear function } \int_{p_1}^p \text{ on } H^0(C, \Omega^1) \text{ defined by}$$

$$\int_{p_1}^p : H^0(C, \Omega^1) \ni \omega \mapsto \int_{p_1}^p \omega \in \mathbb{C}.$$ 

The endomorphism $(1 - \rho^2) : J(C) \rightarrow J(C)$ of $J(C)$ factors through the natural inclusion $\kappa : \text{Prym}(C) \rightarrow J(C)$, i.e. there is a morphism $\alpha : J(C) \rightarrow \text{Prym}(C)$ such that $\kappa \circ \alpha = 1 - \rho^2$. Let $J(C)^+$ be the connected component of the kernel of $\alpha$. Then it is easy to see that $J(C)/J(C)^+$ is isomorphic to $\text{Prym}(C)$ and the morphism

$$J(C)/J(C)^+ \cong \text{Prym}(C) \rightarrow \text{Prym}(C)$$

induced by the morphism $\alpha$ corresponds to the index finite group $(1 - \rho^2)H_1(C, \mathbb{Z}) = (1 - \rho)H_1(C, \mathbb{Z})^- \subset H_1(C, \mathbb{Z})^-$. As a consequence, we have the following sequence of morphisms:

$$J(C) \rightarrow J(C)/J(C)^+ \cong \text{Prym}(C) \rightarrow \text{Prym}(C) \rightarrow J(C).$$

The composite map

$$C \xrightarrow{jac} J(C) \rightarrow J(C)/J(C)^+ \cong \text{Prym}(C)$$

is denoted by $jac^-$. Let $L_1$ be the principal sub-Hodge structure defined in §2.2. Then $A_{L_1} = \mathbb{C}^6/L_1$ is a principally polarized abelian variety. The inclusions

$$(1 - \rho)H_1(C, \mathbb{Z})^- \subset L_1 \subset H_1(C, \mathbb{Z})^-$$

induce homomorphisms of abelian varieties

$$\text{Prym}(C) \xrightarrow{\pi_1} A_{L_1} \xrightarrow{\pi_2} \text{Prym}(C).$$

We define the theta function $\vartheta_m(\Sigma_1, z)$ for the good basis $\Sigma_1$ defined in §2.2 with the characteristic $m = (m', m'') \in \mathbb{Q}^{12}$ by

$$\vartheta_m(\Sigma_1, z) = \sum_{\xi \in \mathbb{Z}^6} e^{\frac{1}{2}(\xi + m')^t(\xi + m') + (z + m'')^t(\xi + m')}.$$ 

where $z \in \mathbb{C}^6$ and $\tau = (\tau_{ij})_{ij} \in \mathfrak{H}_6$ is the normalized period matrix for the good basis $\Sigma_1$ defined in the last section.

Let $\iota : C \rightarrow A_{L_1}$ be the composition $\pi_1 \circ jac^-$. By Proposition 2.6 and 3.3, we have the following lemma.

**Lemma 4.1.** (1) The image of each ramification point $p_1, \ldots, p_8$ under the map $\iota$ is contained in the set of $(1 - \rho)$-torsion points of $\text{Prym}(C)$. 

(2) The image of $\pi_1$ is identified with the quotient of $\text{Prym}(C)$ by the group generated by $\text{jac}^{-}(p_2), \text{jac}^{-}(p_3) - \text{jac}^{-}(p_4), \text{jac}^{-}(p_5) - \text{jac}^{-}(p_6)$.

(3) Using the good base $\Sigma_{L_1}$ of $L_1$, we have

$$\iota(p_k) \equiv \frac{1}{2}(\xi_k, \xi_k U) \mod L_1,$$

where $\xi_1 = \xi_2 = 0$, $\xi_3 = \xi_4 = (1, 1, 0, 0, 0, 0)$, $\xi_5 = \xi_6 = (1, 1, 1, 0, 0, 0)$ and $\xi_7 = \xi_8 = (1, 1, 1, 1, 1, 1)$.

We study the order of zero of the pull back of $\vartheta_m(\Sigma_1, z)$ by $\iota$. Let $\tilde{C}$ be the universal covering of $C$ and we choose a base point $\tilde{q}_1$ of $\tilde{C}$ as a lifting of $p_1$. Then we have a lifting $\tilde{\iota} : \tilde{C} \to (H^0(C, \Omega^1)^{-})^*$ of $\iota$ by sending $\tilde{p}_1$ to 0. Let $\omega_1, \ldots, \omega_6$ be the normalized basis of $H^0(C, \Omega^1)^-$ with respect to $\Sigma_1$. Via the isomorphism

$$(H^0(C, \Omega^1)^{-})^* \ni \gamma \mapsto (\int_{\gamma} \omega_1, \ldots, \int_{\gamma} \omega_6) \in C^6,$$

$\tilde{\iota}$ is identified with $\tilde{C} \to C^6$. We define a map $F_m : \tilde{C} \to C$ by

$$\tilde{C} \ni \tilde{p} \mapsto F_m(\tilde{p}) = \vartheta_m(\Sigma_1, \iota(\tilde{p})) \in C.$$

Since $\vartheta_m(\tau, z)$ is a non-zero section of a line bundle $L_m$, the order of zero at the lifting $\tilde{p}$ of $p$ depends only on the point $p$ in $C$. It is called the order of zero at $p$ and denoted by $ord_p(F_m)$.

**Proposition 4.2.** The total sum of the order $ord_p(F_m)$ of $p$ on $C$ is 12.

**Proof.** Use a similar argument in [MT] just after Proposition 4.9. \qed

The next proposition is fundamental for determining the distribution of zeros of $F_m$.

**Proposition 4.3.** Let $m = \frac{1}{2}(\mu, \mu U)$ ($\mu \in \mathbb{Z}^6$) be an element of $\frac{1}{1-p}L_1$. (See Remark 2.8.) Let $\xi_j$ be an element of $\mathbb{Z}^6$ such that $\iota(\tilde{p}_j) \equiv \frac{1}{2}(\xi_j, \xi_j U) \mod \mathbb{Z}^{12}$, where $\tilde{p}_j$ is a lifting of $p_j$ to $\tilde{C}$. We put $q = \mu + \xi_j$. Then $ord_{p_j}(F_m)$ is equal to $-q U^{-1} q$ modulo 4.

**Proof.** Let $z$ be the coordinate for the universal covering of $A_{L_1}$. Since the point $p_j$ is fixed under the action of $\rho$, $\rho z = z + l$ ($l \in L_1$). By the transformation formula in p.85, [II], we have $F(\rho z) = u(z)F(z)$, where

$$\lim_{z \to \tilde{p}_j} u(z) = e\left(-\frac{q U q}{4}\right).$$

Therefore the order of $F_m$ at $p_j$ is congruent to $-q U q$ mod 4. \qed

Let $m_k = \frac{1}{2}(\mu_k, \mu_k U)$ ($k = 1, \ldots, 4$), where

\begin{align*}
\mu_1 &= (0, 0, 0, 0, 0, 0), & \mu_2 &= (0, 0, 1, 1, 1, 1), \\
\mu_3 &= (1, 1, 0, 0, 1, 1), & \mu_4 &= (1, 1, 1, 1, 0, 0).
\end{align*}

(4.1)
Then by Proposition 4.3, the table of \( \text{ord}_{p_j}(F_{m_k}) \) (mod 4) is given by

\[
\begin{array}{cccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2
\end{array}
\]

for \( k = 1, \ldots, 4 \).

### 4.2. Determination of extra zeros of theta functions.

Let \( \mu_1, \ldots, \mu_4 \) and \( m_1, \ldots, m_4 \) be as in §4.1. Then the sum of the known zeros of \( F_{m_k}(\bar{p}) \) is 8 for \( k = 1, \ldots, 4 \). Since the action of \( \rho \) on the curve \( \tilde{C} \) and \( H^0(C, \Omega^1) \) is compatible, the remaining \( 12 - 8 = 4 \) zeros are stable under the action of \( \rho \) and if its support contains one of \( p_j \), then its multiplicity should be 4 by the modulo 4 condition.

**Proposition 4.4.** The function \( R_{jk} = F_{m_j}(\bar{p})/F_{m_k}(\bar{p}) \) is a rational function of \( p = (z, w) \in C \) for \( 1 \leq j, k \leq 4 \). Moreover \( R_{jk}(\bar{p}) = c \cdot \frac{z - s}{z - t} \), with some constants \( s, t \) and \( c \neq 0 \).

**Proof.** Since the image of the fundamental group of \( C \) in \( L_1 \) is equal to \( (1 - \rho)H_1(C, \mathbb{Z})^- \), if the theta functions \( \vartheta_{m_j}(\tau, z) \) and \( \vartheta_{m_k}(\tau, z) \) have the same quasi-periodicity, the quotient \( F_{m_j}/F_{m_k} \) is a rational function on \( C \). By comparing the zeros of the numerator and the denominator of \( R_{jk} \) (see table (4.2)), we have the proposition. \( \square \)

**Proposition 4.5.**

1. In the expression of the rational function \( R_{13} = c \cdot \frac{z - s}{z - t} \), \( s \) and \( t \) are determined by the equation:

\[
\frac{x_1 - s}{x_1 - t} + \frac{x_2 - s}{x_2 - t} = 0, \quad \frac{x_5 - s}{x_5 - t} + \frac{x_6 - s}{x_6 - t} = 0.
\]

2. The rational function \( R_{12} \) on \( C \) is a constant.

**Proof.** By Proposition 4.4, we have

\[
c \cdot \frac{x_j - s}{x_j - t} = R_{13}(\bar{p}_j) = \frac{F_{m_1}(\bar{p}_j)}{F_{m_3}(\bar{p}_j)}.
\]

On the other hand, we have

\[
\frac{F_{m_1}(\bar{p}_1)}{F_{m_3}(\bar{p}_1)} = \frac{F_{m_1}(\bar{p}_2)}{F_{m_3}(\bar{p}_2)}
\]

by the quasi-periodicity of theta functions, thus we have the statement (1). We can prove the statement (2) similarly. \( \square \)

**Proposition 4.6.** Let \( \vartheta_k = \vartheta_{m_k}(\tau) \). Then we have

\[
\frac{(\vartheta_2 + i\vartheta_3)^2(\vartheta_1 - i\vartheta_4)^2}{4\vartheta_1^2\vartheta_3^2} = \frac{(x_1 - x_5)(x_2 - x_6)}{(x_1 - x_2)(x_5 - x_6)}.
\]
Proof. By the definition of theta constants and $R_{jk}(\tilde{p})$, we have

$$R_{13}(p_1) = \frac{\vartheta_1}{\vartheta_3} = c \cdot \frac{x_1 - s}{x_1 - t}, \quad R_{13}(p_5) = -\frac{\vartheta_4}{\vartheta_2} = c \cdot \frac{x_5 - s}{x_5 - t}.$$ 

The equality $R_{12}(p_1) = R_{12}(p_5)$ implies

$$\frac{\vartheta_1}{\vartheta_2} = \frac{\vartheta_4}{\vartheta_3}.$$ 

By computing $R_{13}(p_1)/R_{13}(p_5)$, we have

$$\frac{(x_1 - s)(x_5 - t)}{(x_1 - t)(x_5 - s)} = -\frac{\vartheta_1 \vartheta_2}{\vartheta_3 \vartheta_4} = -\frac{\vartheta_2^2}{\vartheta_3^2}$$

and

$$\left(1 + \frac{\vartheta_2^2}{\vartheta_3^2}\right)^2 = \frac{(\vartheta_2^2 + \vartheta_3^2)^2}{\vartheta_2^2 \vartheta_3^2} = \frac{\vartheta_2^4 + \vartheta_3^4}{\vartheta_2 \vartheta_3} \cdot \frac{\vartheta_1^2 + \vartheta_4^2}{\vartheta_1 \vartheta_4}$$

$$= \frac{(\vartheta_2 + i\vartheta_3)(\vartheta_2 - i\vartheta_3)(\vartheta_1 + i\vartheta_4)(\vartheta_1 - i\vartheta_4)}{\vartheta_1 \vartheta_2 \vartheta_3 \vartheta_4}$$

$$= \frac{(\vartheta_2 + i\vartheta_3)^2(\vartheta_1 - i\vartheta_4)^2}{\vartheta_1^2 \vartheta_3^2}.$$ 

Here we used the relation $(\vartheta_2 + i\vartheta_3)(\vartheta_1 + i\vartheta_4) = (\vartheta_2 + i\vartheta_3)(\vartheta_1 - i\vartheta_4)$. On the other hand, by (4.3) and (4.4), we have

$$\frac{1}{4} \left(1 + \frac{\vartheta_2^2}{\vartheta_1^2}\right)(1 + \frac{\vartheta_2^2}{\vartheta_3^2}) = \frac{(x_1 - x_5)(x_2 - x_6)}{(x_1 - x_2)(x_5 - x_6)}.$$ 

\[\square\]

4.3. An application of quadratic theta relation. In this section, we fix a principally polarized sub-Hodge structure $L = L_1$ and study quadratic relations between theta constants. We recall the quadratic relation between theta functions in [I]. For the next proposition, see p.139, [I].

Proposition 4.7. Put

$$n_1 = (n'_1, n''_1) = \frac{1}{2}(m_1 + m_2), \quad n_2 = (n'_2, n''_2) = \frac{1}{2}(m_1 - m_2),$$

for $m_1 = (m'_1, m''_1)$, $m_2 = (m'_2, m''_2) \in \mathbb{Q}^{12}$. Let $S$ be a complete set of representatives of $(\frac{1}{2}\mathbb{Z})^6/\mathbb{Z}^6$. We have

$$\vartheta_{m_1}(\tau) \vartheta_{m_2}(\tau) = \frac{1}{2^n} \sum_{a' \in S} e(-2m'_1 a'') \vartheta_{n'_1, n''_1 + a''}(\frac{\tau}{2}) \vartheta_{n'_2, n''_2 + a''}(\frac{\tau}{2}).$$

We apply this formula to

$$n_1 = \left(\frac{1}{2}v_1 - \frac{1}{4}U_0, \frac{1}{2}U_0\right), \quad n_2 = \left(-\frac{1}{4}U_0, 0\right),$$
where \( v_1 \in \mathbb{Z}^6 \), and we use Notation 1.2 for \( U_0 = (1, 1, 0, 0, 1, 1) \). Replace \( \tau \) by \( \tau + U \), then we have

(4.5)

\[
\partial_{\frac{1}{4}v_1 - \frac{1}{4}U_0, \frac{1}{4}U_0} (\tau + U) \partial_{\frac{3}{4}v_1, \frac{3}{4}U_0} (\tau + U) = \frac{1}{2^6} \sum_{a'' \in S} e(- (v_1 - U_0)^t a'') \partial_{v_1 - \frac{3}{4}U_0, \frac{1}{4}U_0 + a''} (\frac{1}{2} (\tau + U)) \partial_{- \frac{3}{4}U_0, a''} (\frac{1}{2} (\tau + U)).
\]

We assume that \( \tau \) is the normalized period matrix of the principally polarized Hodge structure \( L \) with respect to the symplectic basis \( \Sigma_1 \). Then we have

(4.6)

\( (\tau U)^2 = -I \).

By applying the transformation formula

\( \partial_{m', m''} (\tau + U) = e(- \frac{1}{2} m' U^t m' + \frac{1}{2} m'' U_0) \partial_{m', m'' + m' U + \frac{1}{2} U_0} (\tau) \)

to the left hand side of (4.5), we have

(4.7)

\[
\partial_{\frac{1}{4}v_1 - \frac{1}{4}U_0, \frac{1}{4}U_0} (\tau + U) \partial_{\frac{3}{4}v_1, \frac{3}{4}U_0} (\tau + U) = e(- \frac{1}{4} v_1^t U v_1 + \frac{3}{4} v_1^t U_0 - \frac{3}{8} U_0^t U_0) \partial_{v_1 - U_0, U_0} (\tau) \partial_{\frac{3}{4}v_1 - \frac{3}{4}U_0, \frac{1}{4}U_0} (\tau).
\]

To compute the right hand side, we apply the transformation formula in p.85, [1],

(4.8)

\( \partial_{m#} (\tau#) = \det(C\tau + D)^{1/2} \cdot u \cdot \partial_m (\tau), \)

for

\[
\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & U \\ -U & I \end{pmatrix}
\]

and \( m = (-a'' + \frac{1}{2} U_0) v_1 U + a'' + \frac{1}{2} U_0 \) (resp. \( m = (-a'' U, a'') \)). Here \( \tau# = (A\tau + B)(C\tau + D)^{-1} \) is equal to \( \frac{1}{2} (\tau + U) \) and \( \det(C\tau + D) = -8 \) by the relation (4.6). The theta characteristic

\[ m# = m \cdot \sigma^{-1} + \frac{1}{2} ((C^t D)_0, (A^t B)_0) \]

is equal to \( (v_1 - \frac{1}{2} U_0, \frac{1}{2} U_0 + a'') \) (resp. \( (-\frac{1}{2} U_0, a'') \)). We fix a branch of \( \det(C\tau + D)^{1/2} = \sqrt{8t} \) once and for all. We compute the constant \( u \) in the formula (4.8) which depends only on \( m \).

**Definition 4.8.** We define a non-zero complex number \( c(a, b) \) by

(4.9)

\[ \partial_{a,b} (U (-U\tau + I)^{-1}, z#) = c(a, b) (-8)^{1/2} \partial_{c,d + \frac{1}{2} U_0} (\tau, z), \]

where \( z# = z (-U\tau + I)^{-1} \) and \( c = -bU, d = aU + b \).

**Proposition 4.9.**

1. \( c(a, b)/c(0, 0) = e(\frac{1}{2} b U t b + a t b + \frac{1}{2} b^t U_0). \)
2. \( c(0, 0)^2 = 1. \)
Theorem 4.10. (1) Let \( v_1 \) be an element of \( \mathbb{Z}^6 \). Then we have

\[
8e(-\frac{1}{4}v_1U^tv_1 + \frac{3}{4}v_1U_0 - \frac{3}{8}U_0^tU_0) \cdot \vartheta_{\frac{1}{2}U_1-\frac{1}{2}U_0,\frac{1}{2}(v_1-U_0)U}(\tau) \vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau) = \sum_{a''} e(a''^tU + v_1^ta'') \vartheta_{a''+\frac{1}{2}U_0}U,a''(\tau) \vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau),
\]

where \( a'' \) runs over the complete set \( S \) of representatives in \( \frac{1}{2}\mathbb{Z}^6/\mathbb{Z}^6 \).

(2) Let \( v_1 \in \{0,1\}^6 \) and choose \( b'' \) as

\[
b'' \equiv \frac{1}{2}v_1 - \frac{1}{2}U_0, \quad b'' \in \{0, \frac{1}{2}\}^6.
\]

Then we have

\[
8e(-\frac{1}{4}v_1U^tv_1 + \frac{3}{4}v_1U_0 - \frac{3}{8}U_0^tU_0) \cdot e(\frac{1}{2}v_1U_0) \vartheta_{b'',b''U}(\tau) \vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau) = \sum_{a''} e(\frac{1}{2}a''^tU_0 + v_1^ta'') \vartheta_{a''+\frac{1}{2}U_0}U,a''(\tau) \vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau).
\]

Moreover if \( \frac{1}{2}v_1U^tv_1 \in \mathbb{Z} \), then we have

\[
-8e(\frac{1}{4}v_1U_0) \cdot \vartheta_{b'',b''U}(\tau) \vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau) = \sum_{a''} e(-\frac{1}{2}a''^tU_0 + v_1^ta'') \vartheta_{a''+\frac{1}{2}U_0}U,a''(\tau) \vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau).
\]

We have the following corollary of Theorem 4.10.

Corollary 4.11. Let \( v_1 \in \{0,1\}^6 \) such that \( v_1U^tv_1 \in 4\mathbb{Z} \), \( v_1 \neq 0 \), and let \( b'' \) be the element in \( \{0, \frac{1}{2}\}^6 \) defined in Theorem 4.10. Then we have

\[
e(\frac{1}{4}v_1U_0) \cdot \vartheta_{b'',b''U}(\tau) \vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau) + \vartheta_{0,0}(\tau) \vartheta_{\frac{1}{2}U_0,\frac{1}{2}U_0}(\tau) = 0.
\]
5. Comparison for theta constants of different lattices

5.1. Translation vector arising from changing lattices. Let $L$ be a principally polarized sub-Hodge structure of $H_1(C,\mathbb{Z})^-$ and 

$$\Sigma = \Sigma_L = \{a_1, \ldots, a_6, b_1, \ldots, b_6\}$$

be a good symplectic basis of $L$. Let $\{\alpha'_1, \ldots, \alpha'_6, \beta'_1, \ldots, \beta'_6\}$ be the basis of $B$ defined in Proposition 2.4. Put 

$$\alpha_j = \alpha'_j \ (\text{for } j = 1, \ldots, 6),$$

$$\beta_j = 2\beta'_j \ (\text{for } j = 1, 2, 3), \quad \beta_j = \beta'_j \ (\text{for } j = 4, 5, 6),$$

and $\Sigma_B = \{\alpha_1, \ldots, \alpha_6, \beta_1, \ldots, \beta_6\}$. Then the lattice $B$ generated by $\Sigma_B$ admits a principally polarized sub-Hodge structure of $H_1(C,\mathbb{Z})^-$. In this section we compare theta functions of $(B, \Sigma_B)$ and those of $(L, \Sigma_L)$. The elements $a_j, b_j$ in $\Sigma_L$ are linear combinations of $\Sigma_B$ as

$$a_j = \sum_{j=1}^{6} a_{jk} \alpha_k + \sum_{j=1}^{6} b_{jk} \beta_k,$$

$$b_j = \sum_{j=1}^{6} c_{jk} \alpha_k + \sum_{j=1}^{6} d_{jk} \beta_k,$$

respectively. The column vector consisting of $\alpha_j$ (resp. $\beta_j$, $a_j$ and $b_j$) for $j = 1 \ldots 6$ is denoted by $\alpha$ (resp. $\beta$, $a$ and $b$). We put $A = (a_{jk})$, $B = (b_{jk})$, $C = (c_{jk})$, $D = (d_{jk})$. Then we see that

$$\sigma = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \in Sp(6, \mathbb{Q}).$$

Let $p = (p_1, \ldots, p_6), q = (q_1, \ldots, q_6)$ be elements in $\mathbb{Q}^6$. We define two vectors $r = (r_1, \ldots, r_6)$ and $s = (s_1, \ldots, s_6)$ in $\mathbb{Q}^6$ by

$$(r, s) = (p, q) \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$ 

We have $(r, s)^t (\alpha, \beta) = (p, q)^t (a, b)$. Then $(r, s)^t (\alpha, \beta)$ is an element in $H_1(C,\mathbb{Z})^-$ (resp. $(1 - \rho)H_1(C,\mathbb{Z})^-$) if and only if $r \in \mathbb{Z}^6$ and $s \in \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3$ (resp. $r \in 2\mathbb{Z}^3 \oplus \mathbb{Z}^3$ and $s \in \mathbb{Z}^6$).

**Proposition 5.1.** Let $e = \text{diag}(2, 2, 2, 1, 1, 1)$. Then all entries of $e^t B, e^t D$ are integers.

Let $\tau$ and $\tau^\#$ be the normalized period matrix of $B$ and $L$ with respect to the symplectic bases $\Sigma_B$ and $\Sigma_L$, respectively. For an element $z \in \mathbb{C}^6$, we define $z^\# = z(C\tau + D)^{-1}$. 


For a rational vector \( n = (n', n'') \in \mathbb{Q}^{12} \), we consider the following functional equations for a function \( F \) of \( z^\# \in \mathbb{C}^6 \):

\[
\text{(Eq}_n^\#) : F(z^\# + pr^\# + q) = e\left(-\frac{1}{2}pr^\# p - p^t z^\#\right)e(n^t q - n''^t p)F(z^\#),
\]

\[
\text{(Eq}_m) : F(z + r\tau + s) = e\left(-\frac{1}{2}r\tau r - r^t z\right)e(m^t s - m''^t r)F(z).
\]

Note that the theta function \( \vartheta_n(\Sigma_L, z^\#) \) satisfies the functional equations \((\text{Eq}_n^\#)\) for \( p, q \in \mathbb{Z}^6 \).

Before studying the relation between \( \vartheta_n(\tau^\#, z^\#) \) and \( \vartheta_m(\tau, z) \), we define a translation vector \( \delta \) relative to the matrix \( \sigma \). We define \( \delta' \in \mathbb{Z}^6 \) and \( \delta'' \in \frac{1}{2} \mathbb{Z}^3 \oplus \mathbb{Z}^3 \) as

\[
\delta' = (tCA)_0, \quad \delta'' = (e^t DB e)_0 e^{-1}.
\]

The vector \((\delta', \delta'')\) is denoted by \( \delta_\Sigma \). To describe properties of \( \delta_\Sigma \), it is convenient to consider a quadratic form \( q \) on \( \frac{1}{1-\rho} H_1(C, \mathbb{Z})^- / H_1(C, \mathbb{Z})^- \) induced by the quadratic form \( \bar{q} \) on \( H_1(C, \mathbb{Z})^- / (1-\rho) H_1(C, \mathbb{Z})^- \) defined in \( \text{[3,4]} \) via the isomorphism

\[
\frac{1}{1-\rho} H_1(C, \mathbb{Z})^- / H_1(C, \mathbb{Z})^- \rightarrow H_1(C, \mathbb{Z})^- / (1-\rho) H_1(C, \mathbb{Z})^- \quad x \mapsto (1-\rho)x.
\]

Let \((L_1, \Sigma_1)\) be the good basis defined in \( \text{[2,2]} \) and \( g \in U(H_1(C, \mathbb{Z})^-) \). We put \( L_g = g(L_1) \) and \( \Sigma_g = g(\Sigma_1) \). Then it easy to see that \((L_g, \Sigma_g)\) is a good symplectic basis of \( L_g \). The translation vector \( \delta_{\Sigma_g} \) is denoted by \( \delta_g \).

**Proposition 5.2.**  
(1) The vector \( \delta_g = (\delta', \delta'') \) is contained in \( 2 \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^6 \).

(2) Let \( \bar{\bar{\Sigma}} \) and \( \frac{1}{2}\delta_g \) be \( e_1 + e_2 + e_5 + e_6 \) under the mapping \( \text{[3,4]} \) and the class of \( \frac{1}{2}\delta_g t(\alpha, \beta) \) in \( \frac{1}{1-\rho} H_1(C, \mathbb{Z})^- / H_1(C, \mathbb{Z})^- \). Then

\[
c_0 = \frac{1}{2}\delta_g - \bar{\bar{\Sigma}}
\]

is independent of \( g \). Moreover we have \( q(\bar{\bar{\Sigma}}) = 0 \) and \( \bar{\bar{\Sigma}} \neq 0 \).

**Proof.** For any \((L_g, \Sigma_g)\), we compute vectors \( \delta_g \) by definition. As a consequence, we obtain this proposition. \( \square \)

For \( m = (m', m'') \in \mathbb{Q}^{12} \), we set \( \tilde{m} = m + \frac{1}{2}\delta_\Sigma, n = \tilde{m}\sigma^{-1} \). i.e., \( n^t(a, b) = \tilde{m}^t(\alpha, \beta) \). The next proposition is fundamental for comparing theta functions for different lattices. We define \( \theta_m(z^\#) \) by

\[
\theta_m(z^\#) = e\left(\frac{1}{2}z(C\tau + D)^{-1} C^t z\right) \vartheta_m(\Sigma_B, z).
\]

In this definition, \( z \) denotes the function of \( z^\# \) by the relation \( z^\# = z(C\tau + D)^{-1} \).

**Proposition 5.3.**  
(1) The function \( \theta_m(z^\#) \) satisfies the functional equation \((\text{Eq}_n^\#)\) for \((p, q)^t(a, b) \in (1-\rho) H_1(C, \mathbb{Z})^- \).
(2) The function $\theta_{m'}(z^\#)$ satisfies the functional equation $(Eq^\#_n)$ for $m' \in Q^{12}$, $(p, q)^t(a, b) \in (1 - \rho)H_1(C, \mathbb{Z})^-$ if and only if $m - m' \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3$, i.e. $(m - m')^t(\alpha, \beta) \in H_1(C, \mathbb{Z})^-.$

Let $\Theta(\Sigma_B, m)$ and $\Theta(\Sigma_L, n)$ be the spaces of functions of $\tau$ and $\tau^\#$ satisfying the functional equations $(Eq_m)$ and $(Eq^\#_n)$ for $(r, s)^t(\alpha, \beta) \in (1 - \rho)H_1(C, \mathbb{Z})^-$ and $(p, q)^t(a, b) \in (1 - \rho)H_1(C, \mathbb{Z})^-$, respectively. Since the space $\Theta(\Sigma_B, m)$ and $\Theta(\Sigma_L, n)$ are 8 dimensional by the Riemann Roch theorem, Proposition $[5, 3]$ implies the following proposition.

**Proposition 5.4.** Let $n = (m + \frac{1}{2}\delta_\Sigma)\sigma^{-1}$. By mapping a function $f(z)$ of $z$ to a function $f^\#(z^\#) = e(\frac{1}{2}z(C\tau + D)^{-1}C^t z)f(z)$, of $z^\# = z(C\tau + D)^{-1}$, we have an isomorphism

$$\Theta(\Sigma_B, m) \rightarrow \Theta(\Sigma_L, n).$$

5.2. $\Sigma$-trace. In order to express $\vartheta_0(\Sigma_L, z^\#)$ as a linear combination of translations of $\theta_m(z^\#)$ in the last section with simple exponential coefficients, we introduce $\Sigma$-trace.

**Definition 5.5** ($\Sigma$-trace). Let $m$ be an element such that $m + \frac{1}{2}\delta_\Sigma \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3$. Let $S$ be a representative of $(p, q)$ for

$$\mathbb{Z}^{12}/\{(c, d) \mid (c, d)^t(a, b) \in (1 - \rho)H_1(C, \mathbb{Z})^\#\}.$$  

The $\Sigma$-trace $tr_\Sigma(f)(z^\#)$ is defined by

$$tr_\Sigma(f)(z^\#) = \sum_{(p, q) \in S} f(z^\# + p\tau^\# + q)e(\frac{1}{2}p\tau^#tp + p^t z^\#).$$

**Proposition 5.6.** The $\Sigma$-trace $tr_\Sigma(\vartheta_m)$ is independent of the choice of the representative $S$, and it is a constant multiple of $\vartheta_0(\Sigma_L, z^\#)$. Moreover there exists $m \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3 - \frac{1}{2}\delta_\Sigma$ such that $tr_\Sigma(\vartheta_m)$ is non-zero.

**Proof.** Note that the $\Sigma$-trace $tr_{\Sigma_L}(\vartheta_m)$ satisfies $(Eq^\#_n)$ for $(p, q)$ in a sufficiently small lattice in $L$. By Proposition $[5, 3]$ and the characterization of the space generated by theta functions for principally polarized abelian varieties, we have this proposition. □

**Definition 5.7** ($\Phi_g, \Phi_{g,n}$). (1) For each $g \in U(H_1(C, \mathbb{Z})^\#)$, we choose $m_g \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3 - \frac{1}{2}\delta_g$ such that $tr_{\Sigma_g}(\theta_{m_g})$ is non-zero. We define $\Phi_g$ by $tr_{\Sigma_g}(\theta_{m_g})$ and $c_g = \vartheta_0(\Sigma_g, z^\#) / \Phi_g(z^\#)$.

(2) For $n = (p_0, q_0) \in Q^{12}$, we define

$$\Phi_{g,n}(z^\#) = e(\frac{1}{2}p_0\tau^t q + p_0^t (z^\# + q_0))\Phi_g(z^\# + p_0\tau^# + q_0)$$

and $\Phi_{g,n} = \Phi_{g,n}(0)$.

Recall that $\Phi_{g,n}(z^\#)$ is a linear combination of translations of $\vartheta(\Sigma_B, z^\#)$. In the rest of this subsection, we compute its coefficients.
Proposition 5.8. For $n_0 = (m_0 + \frac{1}{2} \delta_g) \sigma^{-1}$, we choose representatives $S_g(n_0)$ and $S_B(m_0)$ of $n_0 + \{\tilde{n} | \tilde{n}^t(a, b) \in H_1(C, \mathbb{Z})/\mathbb{Z}^{12} \}$ and $m_0 + \mathbb{Z}^6 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3/\mathbb{Z}^{12}$, respectively. Then $\{\Phi_g(z^\#)\}_{n \in S_g(n_0)}$ and $\{\vartheta_m(S_B, z)\}_{m \in S_B(m_0)}$ are bases of the 8-dimensional vector spaces $\Theta(S_g, n_0)$ and $\Theta(S_B, m)$, respectively.

Let $(p_0, q_0) \in \mathbb{Q}^{12}$. By the definition of $\Phi_{g, (p_0, q_0)}(z^\#)$, we have the following proposition by simple calculation.

Proposition 5.9. For $(p_0, q_0) \in \mathbb{Q}^{12}$, we have

$$\Phi_{g, (p_0, q_0)}(z^\#) = \sum_{(p, q) \in S} c^{(g)}_{(p_0, q_0), (p, q)} \vartheta_{m_g + (p + p_0 + q + q_0)}(\Sigma_B, z) \cdot e\left(\frac{1}{2} z(Cr + D)^{-1} C^t z\right).$$

and

$$\Phi_{g, (p_0, q_0)} = \sum_{(p, q) \in S} c^{(g)}_{(p_0, q_0), (p, q)} \vartheta_{m_g + (p + p_0 + q + q_0)}(\Sigma_B),$$

where

$$c^{(g)}_{(p_0, q_0), (p, q)} = e(-\frac{1}{2} q^t p - \frac{1}{2} s^t r e(-r^t m''_g) e(\frac{1}{2} p_0^t q_0 - \frac{1}{2} r_0^t s_0 - r_0^t m''_g - r_0^t s).$$

Here, $m_g$ and $S$ are in the definition of $\Sigma_g$-trace and $\Phi_g(z^\#)$, $(r_0, s_0) = (p_0, q_0) \sigma_g$ and $(r, s) = (p, q) \sigma_g$.

Let $n = (p_0, q_0)$ and $(p, q)$ be elements of $S_g(n_0)$ and $\mathbb{Z}^{12}$, respectively. For $(r_0, s_0) = (p_0, q_0) \sigma_g$ and $(r, s) = (p, q) \sigma_g$, we have

$$m_g + (p + p_0 + q + q_0) \sigma_g \in m_g + n_0 \sigma_g + \mathbb{Z}^6 \oplus \frac{1}{2} \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3,$$

$$=m_g + m_0 + \frac{1}{2} \delta_g + \mathbb{Z}^6 \oplus \frac{1}{2} \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3,$$

$$=m_0 + \mathbb{Z}^6 \oplus \frac{1}{2} \mathbb{Z}^3 \oplus \mathbb{Z}^3.$$

For an element $m \in S_B(m_0)$, we put

$$I(m) = \{(p, q) \in S | m_g + (p + p_0, q + q_0) \sigma - m \in \mathbb{Z}^{12}\},$$

$$d^{(g, B)}_{n, m} = \sum_{(p, q) \in I(m)} c^{(g)}_{(p_0, q_0), (p, q)} \vartheta_{m_g + (p + q_0, q + q_0)}(\Sigma_B, z) \vartheta_m(\Sigma_B, z).$$

Then by Proposition 5.9 for $n \in S_g(n_0)$, $\Phi_{g, n}(z^\#)$ can be written as

$$\Phi_{g, n}(z^\#) = \sum_{m \in S_B(m_0)} d^{(g, B)}_{n, m} \vartheta_m(\Sigma_B, z) \cdot e\left(\frac{1}{2} z(Cr + D)^{-1} C^t z\right).$$

We put

$$D^{(g, B)} = (d^{(g, B)}_{n, m})_{n \in S_g(n_0), m \in S_B(m_0)}.$$

This is the base change matrix for $\{\vartheta_m(S_B, z)\}_{m \in S_B(m_0)}$ and $\{\Phi_{g, n}(z^\#)\}_{n \in S_g(n_0)}$ up to a constant exponential multiple.
6. Main Theorem

6.1. Action of the stabilizer of length 0 element on theta functions.

By choosing a principal lattice $L$, we get an injective homomorphism from $U(H_{std})$ to $Sp(6, \mathbb{R}) = Aut(S_6)$ and an inclusion $\iota : B(H_{std}) \to S_6$. By this inclusion, we identify $U(H_{std})$ as a subgroup of $Sp(6, \mathbb{R})$. We consider the function $\det(C_\tau + D)^{1/2}$ on $Sp(6, \mathbb{R}) \times S_6$ defined in \[4.3\]. Let $\sigma_g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$ be a matrix such that $\Sigma_{L_g} = \sigma_g(\Sigma_B)$. We put $n_0 = (m_0 + \frac{1}{2} \delta_g) \sigma_g^{-1}$ for $m_0 \in \mathbb{Q}^{12}$. We choose $S_g(n_0)$ and $S_B(m_0)$ as in the last section. For any $n \in S_L(n_0)$, there exist complex numbers $u_{n,m}^{(g,B)}$ ($m \in S_B(m_0)$) independent of $\tau$ and $z$ such that

\begin{equation}
\vartheta_n(\Sigma_g, z^\#_g) = \det(C_\tau + D_g)^{-1/2} e(z(C_\tau + D_g)C_g^{-1}z) \sum_{m \in S_B(m_0)} u_{n,m}^{(g,B)} \vartheta_m(\Sigma_B, z),
\end{equation}

where $\tau^\#_g = (A_g \tau + B_g) (C_\tau + D_g)^{-1}$ and $z^\#_g = z \cdot (C_\tau + D_g)^{-1}$ by the transformation formula in p.84, \[.I\]. Moreover this expression is unique. By comparing the right hand sides of (5.1) and (6.1), we have the following proposition.

**Proposition 6.1.** The matrix $U^{(g,B)} = (u_{n,m}^{(g,B)})_{n \in S_L(n_0), m \in S_B(m_0)}$ is a non-zero constant multiple $c_g$ of $D^{(g,B)}$. Moreover the non-zero constant $c_g$ does not depend on $\tau$. Especially for $n \in S_g(0)$, we have

\begin{equation}
\vartheta_n(\Sigma_g, z^\#_g) = \sum_{m \in S_1(0)} u_{n,m}^{(g)} \vartheta_m(\Sigma_1, z^\#_1)
= c_g \sum_{m \in S_1(0)} d_{n,m}^{(g)} \vartheta_m(\Sigma_1, z^\#_1).
\end{equation}

We define

\[U(H_{std})_{\Sigma} = \{ g \in U(H_{std}) \mid \Sigma g = \Sigma \}.
\]

We fix a representative $S_1(0)$. For example we choose $S_1(0) = \{ \frac{1}{2}(\mu_j, \mu_j U) \}$ with

\[
\mu_1 = (0, 0, 0, 0, 0, 0), \mu_2 = (0, 0, 1, 1, 1, 1), \mu_3 = (1, 1, 0, 0, 1, 1), \mu_4 = (1, 1, 1, 1, 0, 0),
\mu_5 = (1, 1, 1, 1, 0, 1), \mu_6 = (1, 0, 0, 0, 0, 0), \mu_7 = (0, 0, 1, 0, 0, 0), \mu_8 = (0, 0, 0, 0, 1, 1).
\]

If $n^t(a, b) \in H_1(C, \mathbb{Z})^\perp$, then $n^t(g(a), g(b)) = g(n^t(a, b)) \in g(H_1(C, \mathbb{Z})^\perp)$ for $g \in U(H_{std})_{\Sigma}$. Therefore we can take a representative $S_g(0)$ as $S_1(0)$. Then the vector spaces generated by $\Phi_n(z^\#_g)$ ($n \in S_g(0)$) and $\Phi_n(z^\#)$ ($n \in S_1(0)$) are isomorphic via the map defined in Proposition 5.3.

\[\Theta(\Sigma_g, 0) \overset{\sim}{\longleftrightarrow} \Theta(\Sigma_B, -\Sigma) \overset{\sim}{\longrightarrow} \Theta(\Sigma_1, 0),
\]

where $z^\#_g$ and $z^\#$ are related by $z^\# = z(C\tau + D)^{-1}$ and $z^\#_g = z(C_\tau + D_g)^{-1}$. We put $D^{(g)} = D^{(g,B)}(D^{(id,B)})^{-1}$ and $U^{(g)} = U^{(g,B)}(U^{(id,B)})^{-1}$. By the
definition of $U^{(g,B)}$, the map $g \in U(H_{std})_\Sigma \mapsto U^{(g)}$ defines a projective representation of $U(H_{std})_\Sigma$, which is denoted by $\chi$.

Since $U^{(g)} = c_g c_1^{-1} D(g)$, we have the following corollary of Proposition 6.1.

**Corollary 6.2.** The map

$$U(H_{std})_\Sigma \ni g \mapsto D^{(g,B)}(D^{(1,B)})^{-1} \in Aut(C^{S_1(0)})$$

becomes a projective representation of $U(H_{std})_\Sigma$, which is isomorphic to $\chi$.

Let $S_1(0)_{ev}$ be the subset of $S_1(0)$ consisting of $v \, t U v \in 4\mathbb{Z}$. In the example given as above, we have $S_1(0)_{ev} = \{ \frac{1}{2}(\mu_j, \mu_j U) \}$ with $j = 1, 4, 6, 7$. By evaluating (6.2) at $z^1_1 = z^1_2 = 0$, for $n_2 \in S_1(0)_{ev}$, we have

$$\vartheta_n(S_g) = \sum_{m \in S_1(0)_{ev}} u^{(g)}_{n,m} \vartheta_m(S_1)$$

$$= c_g c_1^{-1} \sum_{m \in S_1(0)_{ev}} d^{(g)}_{n,m} \vartheta_m(S_1).$$

We put $D^{(g)}_{ev} = (u^{(g,B)}_{n,m})_{n,m \in S_1(0)_{ev}}$. We define a projective representation $\chi_{const}$ as

$$\chi_{const}(g) = D^{(g)}_{ev} \in PGL(4, \mathbb{C})$$
on the space of theta constants. Note that an element $g$ in $U(H_{std})$ is in $U(H_{std})_\Sigma$ if and only if $\pi(g) \in (\mathcal{G}_4(1, 2, 5, 6) \times \mathcal{G}_4(3, 4, 7, 8)) \rtimes \mathcal{G}_2$ under the homomorphism $\pi : U(H_{std}) \to \mathcal{G}_8$ defined just after Lemma 3.6. Here $\mathcal{G}_4(1, 2, 5, 6)$ is the symmetric group of permutations of index $\{1, 2, 5, 6\}$. Let $M_{2,5}$ be the (complex) reflection corresponding to the transposition of the points $p_2$ and $p_5$. Then we have $M_{2,5} \in U(H_{std})_\Sigma$,

$$D^{(M_{2,5})}_{ev} = \left( \begin{array}{cccc} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{array} \right)$$

and

$$t(\vartheta_{m_j}(\tau^1_1))_{j=1,4,3,2} = c_{M_{2,5}} c_1^{-1} \cdot \det(\gamma_1 \tau^1_1 + \delta_1)^{1/2}$$

$$\cdot D^{(M_{2,5})}_{ev} t(\vartheta_{m_j}(\tau^1_1))_{j=1,4,3,2},$$

where

$$\sigma_g \cdot \sigma_1^{-1} = \left( \begin{array}{cc} \alpha_g & \beta_g \\ \gamma_g & \delta_g \end{array} \right) \in Sp(6, \mathbb{Q}).$$

### 6.2. Theta constants and cross-ratios of coordinates

In this section, we combine the results in [4.2] and [6.1] to get an $\mathcal{G}_8$-equivariant presentation of a projective map from the moduli space $M_{8pt}$ to $\mathbb{P}^{104}$.

We define a function $T_g(\tau)$ of $\tau \in M_{marked}$ by

$$T_g(\tau) = \det(\gamma_1 \tau + \delta_1)^{-1} \vartheta_{m_1}(\tau^1_1) \vartheta_{m_2}(\tau^1_2) \vartheta_{m_3}(\tau^1_3),$$

(6.4)
where \( \sigma_g = \begin{pmatrix} \alpha_g & \beta_g \\ \gamma_g & \delta_g \end{pmatrix} \) and \( \tau^g \) are defined as before for \( g \in U(H_{\text{std}}) \). Then we have

\[
(6.5) \quad T_g(h \cdot \tau) = T_{gh}(\tau).
\]

Since \( T_g^2 \) depends only on the image \( \pi(g) \in \mathfrak{S}_8 \), it is also denoted by \( T^2_{\pi(g)} \).

By the result of the last section, if \( g \in U(H_{\text{std}}) \), then \( T_g(\tau) \) is a homogeneous polynomial of \( \partial_{m_j}(\tau) \) with constant coefficients. For example if \( g = M_{25} \), we have

\[
T_g = -\frac{i}{2} \cdot (c_{M_{2,5}}c_1^{-1})^2(\partial_{m_1}(\tau_1^\#) - i\partial_{m_4}(\tau_1^\#)) 
\cdot (\partial_{m_3}(\tau_1^\#) - i\partial_{m_2}(\tau_1^\#)).
\]

Therefore we have

\[
(6.6) \quad \frac{T_g^2}{T_1^2} = -\frac{1}{4} \cdot (c_{M_{2,5}}c_1^{-1})^4 \frac{(\partial_{m_1}(\tau_1^\#) - i\partial_{m_4}(\tau_1^\#))^2}{\partial_{m_1}(\tau_1^\#)^2} 
\cdot \frac{(\partial_{m_3}(\tau_1^\#) - i\partial_{m_2}(\tau_1^\#))^2}{\partial_{m_3}(\tau_1^\#)}
\cdot \frac{(x_1 - x_5)(x_2 - x_6)}{(x_1 - x_2)(x_5 - x_6)},
\]

where \( c = (c_{M_{2,5}}c_1^{-1})^4 \).

Let \( R \) be a set of representatives of the composite surjection

\[
U(H_{\text{std}}) \xrightarrow{\pi} \mathfrak{S}_8 \rightarrow \text{Stab}\{\{1, 2\}, \{5, 6\}, \{3, 4\}, \{7, 8\}\} \setminus \mathfrak{S}_8,
\]

where \( \pi \) is the natural surjection and \( \text{Stab}\{\{1, 2\}, \{5, 6\}, \{3, 4\}, \{7, 8\}\} \) is the stabilizer of \{\{1, 2\}, \{5, 6\}, \{3, 4\}, \{7, 8\}\}. We fix this set \( R \) once and for all.

**Definition 6.3 (Polynomial map).** Set \( P_1 = (x_1 - x_2)(x_3 - x_4)(x_5 - x_6)(x_7 - x_8) \) and \( P_r = \pi(r)^*(P_1) \) for \( r \in R \). Since each \( P_r \) is relative invariant under the action of \( \text{PGL}(2, \mathbb{C}) \), the map \( P : (\mathbb{P}^1)^8 - \text{Diag} \rightarrow \mathbb{P}^{104} \) defined by the ratio of \( (P_r)_{r \in R} \) descends to a morphism \( M_{8 \text{pts}} \rightarrow \mathbb{P}^{104} \), which is also denoted by \( P \). The composite \( M_{\text{marked}} \rightarrow M_{8 \text{pts}} \rightarrow \mathbb{P}^{104} \) is also denoted by \( P \).

By the Definition 6.3, the last term of (6.6) is equal to \( c \cdot \frac{T_g}{T_1} \).

We have the following theorem.

**Theorem 6.4.** Let \( T(2) \) be the map from \( M_{\text{marked}} \) to \( \mathbb{P}^{104} \) defined by \( (T_r^2)_{r \in R} \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
M_{\text{marked}} & \longrightarrow & M_{8 \text{pts}} \\
p & \downarrow & \downarrow P \\
B(H_{\text{std}}) & \longrightarrow & B(H_{\text{std}})/\Gamma(1 + i)
\end{array}
\]

\[
\begin{array}{ccc}
M_{\text{marked}} & \longrightarrow & M_{8 \text{pts}} \\
P & \downarrow & \downarrow \text{Diag} \\
\mathbb{P}^{104}.
\end{array}
\]
In order to prove Theorem 6.4, we give some lemmas. Let \([r]\) denotes the class of \(r \in \mathcal{S}_8\) in \(\text{Stab}\{\{1, 2\}, \{5, 6\}, \{3, 4\}, \{7, 8\}\}\backslash \mathcal{S}_8\).

**Lemma 6.5.**

1. If \([r] = [r']\), then \(\mathcal{T}_r^2\) is a constant multiple of \(\mathcal{T}_{r'}^2\).
2. The map from \(\text{Stab}\{\{1, 2\}, \{5, 6\}, \{3, 4\}, \{7, 8\}\}\) to \(\{\pm 1\}\) defined by \(g \mapsto \frac{\mathcal{T}_g^2}{\mathcal{T}_1^2}\) is a character and this coincides with the restriction of the signature on \(\mathcal{S}_8\).

**Proof.**

1. By the equality (6.5) we have the following equation of rational functions of \(M_{8pts}\),

\[
(6.7) \quad \frac{\mathcal{T}_g^2}{\mathcal{T}_1^2}(h(\tau)) = \frac{\mathcal{T}_{gh}^2}{\mathcal{T}_1^2}(\tau)
\]

for \(h \in \mathcal{S}_8\). Thus we have only to prove the lemma for the case \(r' = 1\). If \([r] = [1]\), \(\mathcal{T}_r^2\) is a constant multiple of \(\mathcal{T}_1^2\) by Corollary 4.11 and the expression of the projective representation \(\chi_{\text{const}}\).

2. The first statement is a consequence of 1. Using the transformation formula of p.85 in [1], we have

\[
(6.8) \quad \mathcal{T}_{M_{12}} = \mathcal{T}_{M_{56}} = i\mathcal{T}_1.
\]

By applying \(M_{12}M_{56}\) to the equality (6.6), we have

\[
(6.9) \quad \mathcal{T}_{gM_{12}M_{56}}^2 = \mathcal{T}_g^2.
\]

Equalities (6.8) and (6.9) characterize the character of \(\text{Stab}\{\{1, 2\}, \{5, 6\}, \{3, 4\}, \{7, 8\}\}\) and the character \(\frac{\mathcal{T}_g^2}{\mathcal{T}_1^2}\) coincides with the restriction of the signature. \(\Box\)

**Lemma 6.6.** Let \([r]\) be an element of \(\text{Stab}\{\{1, 2\}, \{5, 6\}, \{3, 4\}, \{7, 8\}\}\backslash \mathcal{S}_8\) and \((2, 6) \in \mathcal{S}_8\) be the transposition of 2 and 6. Then there exist sequences of \(g_1, \ldots, g_{k+1}\) and \(h_1, \ldots, h_k\) of \(\mathcal{S}_8\) such that

1. \([r] = [g_1h_1 \cdots g_kh_kg_{k+1}]\),
2. \([g_1h_1 \cdots g_l] = [g_1h_1 \cdots g_lh_l]\) for \(l = 1, \ldots, k\),
3. \([\{2, 6\}] = [(2, 6)g_1], [(2, 6)g_1h_1 \cdots g_lh_l] = [(2, 6)g_1h_1 \cdots g_lh_lg_{l+1}]\) for \(l = 1, \ldots, k\).

**Proof of Theorem 6.4.** Using (6.7), we have

\[
(6.10) \quad \frac{\mathcal{T}_{M_{2,5}}^2(M_{2,6}(\tau))}{\mathcal{T}_1^2} = \frac{\mathcal{T}_{M_{2,5}M_{2,6}}^2(\tau)}{\mathcal{T}_{M_{2,6}}^2} = -\frac{\mathcal{T}_{M_{2,5}}^2(\tau)}{\mathcal{T}_{M_{2,6}}^2}.
\]

We put \(c = (c_{M_{2,5}}c_{M_{2,6}}^{-1})^4\). Since the map \(p\) is equivariant under the action of \(\mathcal{S}_8\), we have

\[
(6.11) \quad \frac{\mathcal{T}_{M_{2,5}}^2(h(\tau))}{\mathcal{T}_1^2} = c \cdot \frac{(x_{1h} - x_{5h})(x_{2h} - x_{6h})}{(x_{1h} - x_{2h})(x_{5h} - x_{6h})}
\]

for \(h \in \mathcal{S}_8\). The equations (6.10) and (6.11) yield

\[
(6.12) \quad \frac{\mathcal{T}_{(2,6)}^2}{\mathcal{T}_1^2}(\tau) = \frac{\mathcal{T}_{(2,6)}^2}{\mathcal{T}_1^2}(\tau) = \frac{P_{(2,6)}}{P_1}.
\]
For any $r \in \frak{s}$, there exist sequences $g_1, \ldots, g_{k+1}, h_1, \ldots, h_k$ such that

\[
[(2, 6)] = [(2, 6)g_1] = [(2, 6)g_1h_1] = [(2, 6)g_1h_1g_2],
\]

\[
[g_1] = [g_1h_1], \quad [g_1h_1g_2] = [g_1h_1g_2h_2]
\]

\[
\ldots \quad [(2, 6)g_1 \cdots h_k] = [(2, 6)g_1 \cdots h_k g_{k+1}]
\]

\[
\ldots \quad [g_1h_1 \cdots h_k g_{k+1}] = [r].
\]

by Lemma 6.6 By applying $g_1$ and $g_1h_1$ to the equality (6.12), we have

\[
\frac{T^2_{(2, 6)}}{T^2_{g_1}}(\tau) = \frac{T^2_{(2, 6)g_1}}{T^2_{g_1}}(\tau) = \frac{P_{(2, 6)g_1}}{P_{g_1}} = \frac{P_{(2, 6)}}{P_{g_1}}.
\]

(6.13)

\[
\frac{T^2_{(2, 6)g_1h_1}}{T^2_{g_1h_1}}(\tau) = \frac{T^2_{(2, 6)g_1h_1}}{T^2_{g_1h_1}}(\tau) = \frac{P_{(2, 6)g_1h_1}}{P_{g_1h_1}} = \frac{P_{(2, 6)g_1h_1}}{P_{g_1}}.
\]

(6.14)

From the equalities (6.12) and (6.13), we have

\[
\frac{T^2_{g_1}}{T^2_1}(\tau) = \frac{P_{g_1}}{P_1}.
\]

(6.15)

and from the equalities (6.14) and (6.15), we have

\[
\frac{T^2_{(2, 6)g_1h_1}}{T^2_1}(\tau) = \frac{P_{(2, 6)g_1h_1}}{P_1}.
\]

We continue this procedure, we get an identity $\frac{T^2_{g}}{T^2_1}(\tau) = \frac{P_{g}}{P_1}$ for all $g \in \frak{s}$, which completes the proof. \qed

6.3. Branched covering of $M_{8\text{pts}}$ corresponding to $\Gamma(2)$. In this section, we study the map from $M_{\text{marked}}$ to $M_{8\text{pts}}$ defined by the theta constants on $B(H_{\text{std}})$. As in 3.3 we choose an initial point $X = (x_1, \ldots, x_8)$ and specify the branch of the function $\sqrt{x_k - x_j}$.

Let $r_1$ be $2^4$-partition $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$ and $\pi : U(H_{\text{std}}) \to \frak{s}$ be the natural projection. Using the argument $\text{arg} (g)$ defined in 3.3 we define a multi-valued function $Q_r$ ($r \in R$), on $\mathbb{C}^8 - \text{Diag}$ by

\[
Q_r = \text{arg} (g) \sqrt{(x_{j_2} - x_{j_1})(x_{j_4} - x_{j_3})(x_{j_6} - x_{j_5})(x_{j_8} - x_{j_7})},
\]

where $r_1 \pi (r) = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}, \{j_7, j_8\}\}$ and $j_p < j_{p+1}$ for $j = 1, 3, 5, 7$. Here we chose the branch of the square root as in 3.3. Let $\tilde{N}$ be the covering of $\mathbb{C}^8 - \text{Diag}$ defined by $\sqrt{x_j - x_k}$ ($1 \leq k < j \leq 8$). Then the functions $Q_r$ on $\tilde{N}$ define a morphism $Q = (Q_r)_{r \in R} : \tilde{N} \to \mathbb{P}^{104}$. Let $pr_1 : \mathbb{P}^{104} \to \mathbb{P}^{104}$ be the morphism defined by $(y_r)_{r \in R} \mapsto (y_r^2)_{r \in R}$. Since the coordinates of the inverse image of $P(\lambda_4, \ldots, \lambda_8)$ under $pr_1$ can be expressed by polynomials of $\sqrt{\lambda_j}, \sqrt{1 - \lambda_j}, \sqrt{\lambda_j - \lambda_k}$, the morphism $M_{8\text{pts}} \to M_{8\text{pts}} \to \mathbb{P}^{104}$ factors through $pr_1$ (see the following diagram).
We define the morphism $Q$ by the above diagram.

**Theorem 6.7.** Let $\mathcal{T} = (\mathcal{T}_r)_{r \in R}$ be the morphism from $B(H_{\text{std}})$ to $\mathbb{P}^{104}$ defined by $\mathcal{T}_r$ ($r \in R$). Then the following diagram is commutative:

$$
\begin{array}{ccc}
M_{\text{marked}} & \rightarrow & \tilde{M}_{8\text{pts}} \\
\downarrow & & \downarrow \\
B(H_{\text{std}}) & \rightarrow & B(H_{\text{std}})/\Gamma(2) \\
\end{array}
\begin{array}{ccc}
\tilde{M}_{8\text{pts}} & \rightarrow & \mathbb{P}^{104} \\
P & \rightarrow & \\
P & \rightarrow & \\
\end{array}
\begin{array}{ccc}
Q & \rightarrow & \mathbb{P}^{104} \\
pr_1 & & \\
\end{array}
$$

**Proof.** We determine the branch of the square root of the last term in (6.6). Let $h = M_{1,5}, g = M_{2,5}$ be elements of $U(H_{\text{std}})$. Then we have $h(L_1) = gh(L_1)$. By the transformation formula p.85 [I], we have

$$
\mathcal{T}_{gh} = i\mathcal{T}_h.
$$

By Theorem 6.4 we have

$$
(6.16) \quad \frac{\mathcal{T}_h}{\mathcal{T}_1} = i \cdot \epsilon \cdot \sqrt{\frac{(x_5 - x_2)(x_6 - x_1)}{(x_2 - x_1)(x_6 - x_5)}}
$$

with $\epsilon = \pm 1$. By applying $g^*$ to the both sides of (6.16), and we get

$$
(6.17) \quad i \cdot \frac{\mathcal{T}_h}{\mathcal{T}_g} = \frac{\mathcal{T}_{hg}}{\mathcal{T}_g} = \frac{-\epsilon}{\arg(g)} \cdot \sqrt{\frac{(x_5 - x_2)(x_6 - x_1)}{(x_5 - x_1)(x_6 - x_2)}}.
$$

By (6.16) and (6.17), we have

$$
\frac{\mathcal{T}_g}{\mathcal{T}_1} = \frac{\mathcal{T}_g}{\mathcal{T}_h} \cdot \frac{\mathcal{T}_h}{\mathcal{T}_1} = \arg(g) \cdot \sqrt{\frac{(x_5 - x_1)(x_6 - x_2)}{(x_2 - x_1)(x_6 - x_5)}}.
$$

Using Lemma 6.6 and the same argument in the proof of Theorem 6.4 we have the theorem. \qed

**REFERENCES**

[DM] Deligne, P. and Mostow, G. D., Monodromy of hypergeometric functions and non-lattice integral monodromy, *I.H.E.S. Publ. Math.* 63 (1986), 5–89.

[F] Fay, J.D. Theta Functions on Riemann Surfaces, *LNM.* 352, Springer, 1973.
[I] Igusa, J., Theta Functions, Springer, 1972.

[K1] Koike, K., On the family of pentagonal curves of genus 6 and associated modular forms on the ball, J. Math. Soc. Japan 55 (2003) no. 1, 165–196.

[K2] Koike, K., The projective model of the configuration space $X(2, 8)$, preprint.

[MT] Matsumoto, K. and Terasoma, T., Theta constants associated to cubic three folds, to appear in J. of Alg. Geom.

[MY] Matsumoto, K. and Yoshida, M., Configuration space of 8 points on the projective line and a 5-dimensional Picard modular group, Compositio Math. 86 (1993), 265–280.

[Ma1] Matsumoto, K., On modular functions in 2 variables attached to a family of hyper-elliptic curves of genus 3, Ann. Scuola Norm. Sup. Pisa, Serie IV vol.XVI, Fasc. 4, (1989), 557–578.

[Ma2] Matsumoto, K., Theta constants associated with the cyclic triple coverings of the complex projective line branching at six points, Publ. Res. Inst. Math. Sci. 37 no. 3 (2001), 419–440.

[Mu] Mumford, D, Prym varieties I, Contributions to analysis (a collection of papers dedicated to Lipman Bers), 325–350, Academic Press, New York, 1974.

[P] E. Picard, Sur les fonctions de deux variables indépendantes analogues aux fonctions modulaires, Acta Math., 2 (1883), 114–126.

[S] Shiga, H., On the representation of Picard modular function by $\theta$ constants I-II, Publ. RIMS, Kyoto Univ. 24 (1988), 311–360.

[T] Terada, T., Fonctions hypergéométriques $F_1$ et fonctions automorphes I, II, Math. Soc. Japan 35 (1983), 451–475; 37 (1985), 173–185.

[Y] Yoshida, M., Hypergeometric Functions, My Love, Vieweg, 1997.

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