A bound for the diameter of random hyperbolic graphs

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August 14, 2014

Abstract

Random hyperbolic graphs were recently introduced by Krioukov et. al. [KPK+10] as a model for large networks. Gugelmann, Panagiotou, and Peter [GPP12] then initiated the rigorous study of random hyperbolic graphs using the following model: for \( \alpha > 1/2 \), \( C \in \mathbb{R} \), \( n \in \mathbb{N} \), set \( R = 2 \ln n + C \), and define the graph \( G = (V,E) \) with \( |V| = n \) as follows: For each \( v \in V \), i.i.d. polar coordinates \((r_v, \theta_v)\) are generated using the joint density function \( f(r, \theta) \), with \( \theta_v \) uniformly chosen from \([0, 2\pi)\) and \( r_v \) has density \( f(r) = \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R)} \) for \( 0 \leq r < R \). Two vertices are then joined by an edge if their hyperbolic distance is at most \( R \). We prove that in the range \( 1/2 < \alpha < 1 \) a.a.s. for any two vertices of the same component, their graph distance is \( O(\text{polylog}(n)) \), thus answering a question raised in [GPP12] concerning the diameter of such random graphs. As a corollary from our proof we obtain that the second largest component has size \( O(\text{polylog}(n)) \), thus answering a question of Bode, Fountoulakis and Müller [BFM13]. We also show that a.a.s. there exist isolated components forming a path of length \( \Omega(\log n) \), thus yielding a lower bound on the size of the second largest component.

1 Introduction

Building mathematical models to capture essential properties of large networks has become an important objective in order to better understand them. An interesting new proposal in this direction is the model of random hyperbolic graphs recently introduced by Krioukov et. al. [KPK+10] (see also [PKBnV10]). A good model should on the one hand replicate the characteristic properties that are observed in real world networks (e.g., power law degree distributions, high clustering and small diameter), but on the other hand it should also be susceptible to mathematical analysis. There are models that partly succeed in the first task but are hard to analyze rigorously. Other models, like the classical Erdős-Renyi \( G(n,p) \) model, can be studied mathematically, but fail to capture certain aspects observed in real-world networks. In contrast, the authors of [PKBnV10] argued empirically and via some non-rigorous methods that random hyperbolic graphs have many of the desired properties. Actually, Boguñá, Papadopoulos and Krioukov [BnP10] computed explicitly a maximum likelihood fit of the Internet graph, convincingly illustrating that this model is adequate for reproducing the structure of real networks with high accuracy. Gugelmann, Panagiotou, and

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Peter [GPP12] initiated the rigorous study of random hyperbolic graphs. They compute exact asymptotic expressions for the maximum degree, the degree distribution (confirming rigorously that the degree sequence follows a power-law distribution with controllable exponent), and also estimated the expectation of the clustering coefficient.

Formally, the random hyperbolic graph model $G_{\alpha,C}(n)$ is defined in [GPP12] as follows: for $\alpha > 1/2$, $C \in \mathbb{R}$, $n \in \mathbb{N}$, set $R = 2 \ln n + C$, and build $G = (V,E)$ with vertex set $V = [n]$ as follows:

- For each $v \in V$, polar coordinates $(r_v, \theta_v)$ are generated identically and independently distributed with joint density function $f(r, \theta)$, with $\theta_v$ chosen uniformly at random in the interval $[0,2\pi)$ and $r_v$ with density:

\[
    f(r) = \begin{cases} 
        \frac{\alpha \sinh(\alpha r)}{C(\alpha, R)}, & \text{if } 0 \leq r < R, \\
        0, & \text{otherwise},
    \end{cases}
\]

where $C(\alpha, R) = \cosh(\alpha R) - 1$ is a normalization constant.

- For $u, v \in V$, $u \neq v$, there is an edge with endpoints $u$ and $v$ provided $d(r_u, r_v, \theta_u - \theta_v) \leq R$, where $d = d(r, r', \theta - \theta')$ denotes the hyperbolic distance between two vertices whose native representation polar coordinates are $(r, \theta)$ and $(r, \theta')$, obtained by solving

\[
    \cosh(d) = \cosh(r) \cosh(r') - \sinh(r) \sinh(r') \cos(\theta - \theta').
\]

Research in random hyperbolic graphs is in a sense in its infancy. Besides the results mentioned above, very little else is known. Notable exceptions are the emergence and evolution of giant components [BFM13], connectedness [BFM], results on the clustering coefficient [CF13] and on the evolution of graphs on more general spaces with negative curvature [Fou12].

**Notation.** As typical in random graph theory, we shall consider only asymptotic properties as $n \to \infty$. We say that an event in a probability space holds asymptotically almost surely (a.a.s.) if its probability tends to one as $n$ goes to infinity. We say that $f(n) = O(g(n))$ if there exists an integer $n_0$ and a constant $c > 0$ such that $|f(n)| \leq cg(n)$ for all $n \geq n_0$, $f(n) = \Omega(g(n))$, if $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold. Also, $f(n) = \omega(g(n))$, if $\lim_{n \to \infty} |f(n)|/|g(n)| = \infty$, and $f(n) = o(g(n))$, if $g(n) = \omega(f(n))$. We will nevertheless also use $1-o(\cdot)$ when dealing with probabilities. Throughout this paper, log $n$ always denotes the natural logarithm of $n$. The constants $\alpha, C$ used in the model and the constants $C_0, \delta$ defined below have only one special meaning, other constants such as $C', C''$, $c, c_1, c_2, c_3$ change from line to line. Since we are interested in asymptotic results only, we ignore rounding issues throughout the paper.

**Results.** The main problem we address in this work is the natural question, explicitly stated in [GPP12] page 6], that asks to determine the expected diameter of the giant component of a random hyperbolic graph $G$ chosen according to $G_{\alpha,C}(n)$ for $\frac{1}{2} < \alpha < 1$. In Theorem 13 below we show that a.a.s. it is $O(polylog(n))$. By our proof we obtain as a corollary (Corollary 14) that the size of the second largest component is $O(polylog(n))$. As a complementary result, Theorem 15 shows that a.a.s. there exists a component forming an induced path of length $\Theta(log n)$. 


2 Conventions, background results and preliminaries

Henceforth, for a point \( P \) in hyperbolic space, we let \((r_P, \theta_P)\) denote its polar coordinates \((0 \leq r_P < R \text{ and } 0 \leq \theta_P < 2\pi)\). The point with polar coordinates \((0, 0)\) is called the origin and is denoted by \(O\).

By (1), the hyperbolic triangle formed by the geodesics between points \(A, B,\) and \(C\), with opposing side segments of length \(a, b,\) and \(c\) respectively, is such that the angle formed at \(C\) is:

\[
\theta_c(a,b) = \arccos\left(\frac{\cosh(a)\cosh(b) - \cosh(c)}{\sinh(a)\sinh(b)}\right).
\]

Clearly, \(\theta_c(a,b) = \theta_c(b,a)\).

It is known (see \cite{GPP12}) that for \(\alpha > 1/2\) the resulting graph has bounded average degree. In fact, although some of the proofs hold for a wider range of \(\alpha\), we will always assume \(1/2 < \alpha < 1\), as we are interested in this regime. In order to avoid unnecessary repetitions, we omit this restriction from the statement of the results that follow. First, we recall some useful estimates. A very useful approximation for \(\theta_c(\cdot, \cdot)\) is given by the following result.

**Lemma 1** \((\text{\cite{GPP12} Lemma 3.1})\). If \(0 \leq \min\{a,b\} \leq c\) and \(a+b \geq c\), then

\[
\theta_c(a,b) = 2e^{\frac{1}{2}(c-a-b)}(1 + \Theta(e^{c-a-b})).
\]

**Remark 2.** We will use the lemma also in this form: let \(A \text{ and } B\) be two points such that \(r_A, r_B > R/2, 0 \leq \min\{r_A, r_B\} \leq d := d(A,B) \leq R\). Then,

\[
|\theta_A - \theta_B| = \theta_d(r_A, r_B) = 2e^{\frac{1}{2}(d-r_A-r_B)}(1 + \Theta(e^{d-r_A-r_B})).
\]

Note also that for fixed \(r_A, r_B > R/2, |\theta_A - \theta_B|\) is increasing as a function of \(d\) (for \(d\) satisfying the constraints). When aiming for an upper bound, we will below always use \(d = R\).

Throughout, we will need estimates for measures of regions of the hyperbolic plane, and more specifically, for regions obtained by performing some set algebra involving a couple of balls. Henceforth, for a point \(P\) of the hyperbolic plane, \(\rho_P\) will be used to denote the radius of a ball centered at \(P\), and we denote by \(B_P^{\rho_P}\) the ball of radius \(\rho_P\) centered at \(P\). Also, we denote by \(\mu(S)\) the probability measure of a set \(S\), i.e.,

\[
\mu(S) = \int_S f(r, \theta)drd\theta.
\]

We collect now a few results for such measures.

**Lemma 3** \((\text{\cite{GPP12} Lemma 3.2})\). For any \(0 \leq \rho_O \leq R\), \(\mu(B_O(\rho_O)) = e^{-\alpha(R-\rho_O)}(1 + o(1))\).

**Lemma 4.** For \(r_A \leq \rho_A\) and \(\rho_O + r_A \geq \rho_A\),

\[
\mu(B_A(r_A) \cap B_O(\rho_O)) = \frac{2\alpha}{\pi(\alpha - 1/2)} \left( e^{-\alpha(R-\rho_O)-\frac{1}{2}(\rho_O-r_A)} \right) + O(e^{-\alpha(R-\rho_A+r_A)}).
\]

**Proof.** See Appendix \[A\] \(\square\)

**Lemma 5.** The following statements hold:
1. If $0 \leq \rho_O' \leq \rho_O < R$, then
\[
\mu(B_O(\rho_O) \setminus B_O(\rho'_O)) = e^{-\alpha(R-\rho_O)}(1 - e^{-\alpha(\rho_O-\rho'_O)} - o(1)).
\]

2. If $0 < c_1 < c_2$ are two positive constants, $R - c_2 \leq r_A \leq R - c_1$, and $c_3 = \Omega(1)$, then
\[
\mu((B_A(R) \setminus B_A(R - c_3)) \cap (B_O(R - c_1) \setminus B_O(R - c_2))) = \Omega(e^{-R/2}).
\]

**Proof.** See Appendix A.

**Lemma 6.** Let $0 < c_3 < c_1 < c_2$ be small positive constants so that $2e^{c_1-c_2} > e^{c_3/2}$ holds. Suppose $R - c_2 \leq r_A, r_B \leq R - c_1$ and $R - c_3 \leq d(A, B) \leq R$. Then
\[
\mu((B_B(R) \setminus B_B(R - c_3)) \cap (B_O(R - c_1) \setminus B_O(R - c_2)) \setminus B_A(R)) = \Omega(e^{-R/2}) = \Omega(1/n).
\]

**Proof.** By Lemma 5 Part 2, we know that $B_B(R) \setminus B_B(R - c_3)$ intersects the band at distance between $R - c_2$ and $R - c_1$, i.e. $B_B(R - c_1) \setminus B_B(R - c_2)$, in a region of measure $\Omega(e^{-R/2})$. Note that the intersection comprises two disconnected regions of equal measure, say $\mathcal{D}$ and $\mathcal{D}'$. We may assume that $A \in \mathcal{D}$ and also that for all vertices $A \in \mathcal{D}$ and $A' \in \mathcal{D}'$ we have $\theta_A < \theta_B < \theta_A'$. We will show that $B_A(R)$ does not intersect $\mathcal{D}'$: suppose for contradiction that $A \in \mathcal{D}$ is adjacent to a vertex $A' \in \mathcal{D}'$: then, by Lemma 1 and Remark 3 as $d(A, A') \leq R$, we would have
\[
|\theta_A - \theta_A'| \leq (2 + o(1))e^{\frac{1}{2}(R-2)} = (2 + o(1))e^{-R/2}e^{c_2}.
\]

On the other hand, for any $A \in \mathcal{D}$, since $d(A, B) \geq R - c_3 > R - c_1 > R - c_2$, we have again by Lemma 1 and Remark 2 $|\theta_B - \theta_A| \geq (2 + o(1))e^{\frac{1}{2}(R-2c_3-2R-c_1)} = (2 + o(1))e^{-R/2}e^{c_1-\frac{c_3}{2}}$, and the same bound holds for $|\theta_B - \theta_A'|$. Since $A$ and $A'$ satisfy $\theta_A < \theta_B < \theta_A'$, we have $|\theta_A - \theta_A'| = |\theta_B - \theta_A| + |\theta_B - \theta_A'|$, and thus $|\theta_A - \theta_A'| \geq (4 + o(1))e^{-R/2}e^{c_1-\frac{c_3}{2}}$. Since by assumption $2e^{c_1-\frac{c_3}{2}} > e^{c_2}$, this contradicts (2). The lemma follows.

From now on, denote $R_0 := R/2$ and $R_i := Re^{-\alpha^i/2}$ for $i \geq 1$. Observe that $R_{i-1} \leq R_i$ for all $i$.

**Lemma 7.** Let $\xi > 0$ be some constant and $i \geq 1$ be such that $R_i < R - \xi$. If $R_i < r_A \leq R_{i+1}$, then we have
\[
\mu(B_A(R) \cap (B_O(R_i) \setminus B_O(R_{i-1}))) = \Omega(\mu(B_A(R) \cap B_O(R_i))).
\]

**Proof.** Since $R_{i-1} \leq R_i$, we have that $B_O(R_{i-1}) \subseteq B_O(R_i)$, so the left hand side of the stated identity can be re-written as $\mu(B_A(R) \cap B_O(R_i)) - \mu(B_A(R) \cap B_O(R_{i-1}))$.

Now, observe that by Lemma 4 applied with $\rho_A = R$, $\rho_O = R_i$,
\[
\mu(B_A(R) \cap B_O(R_i)) = (1 + o(1)) \frac{2\alpha}{\pi(\alpha - 1/2)} \left(e^{-\alpha(R-R_i)}e^{-\frac{1}{2}(R_i-R+r_A)}\right),
\]

since $R_i + r_A > 2R_i \geq R$, and therefore the second part is of strictly smaller order. By the same argument, this time applied with $\rho_A = R$, $\rho_O = R_{i-1}$, we also have
\[
\mu(B_A(R) \cap B_O(R_{i-1})) = (1 + o(1)) \frac{2\alpha}{\pi(\alpha - 1/2)} \left(e^{-\alpha(R-R_{i-1})}e^{-\frac{1}{2}(R_{i-1}-r_A)}\right).
\]
Since $B_A(R) \cap B_O(R_{i-1}) \subseteq B_A(R) \cap B_O(R_i)$, it suffices to show that the ratio $\rho(R, i)$ between the right hand sides of the expressions in (3) and (4) is at least a constant. To do so, first note that

$$\rho(R, i) = (1 + o(1))e^{(\alpha-1/2)(R_i-R_{i-1})} = (1 + o(1))e^{(\alpha-1/2)R(e^{-\alpha/2} - e^{-\alpha-1/2})},$$

and let $i_1 = O(1)$ be large enough so that $1 - \alpha \geq \alpha^i - 1/2$. Since $1/2 < \alpha < 1$, if $i \leq i_1$, then $\beta := e^{-\alpha i/2} - e^{-\alpha i - 1/2} > 0$ is a constant so it follows that $\rho(R, i) \geq \rho(R, i_1) = (1 + o(1))e^{(\alpha-1/2)\beta R}$, and the statement clearly holds in this case.

Using now $1 - x \leq e^{-x} \leq 1 - x + x^2/2$, by our choice of $i_1$ and recalling that $\alpha < 1$,

$$R\left(e^{-\alpha i/2} - e^{-\alpha i - 1/2}\right) \geq R\frac{\alpha^{i-1}}{2}\left(1 - \alpha - \frac{\alpha^{i-1}}{4}\right) \geq R\frac{\alpha^{i-1}}{2} \cdot 1 - \alpha.$$

Since by our assumption on $R_i < R - \xi$, we also have $\alpha^i \geq (1 + o(1))\frac{\xi}{2R}$, and thus $\rho(R, i) \geq (1 + o(1))e^{(\alpha-1/2)((1-\alpha)\xi/(8\alpha)}$, finishing the proof. \(\square\)

In order to simplify our proofs, we will make use of a technique known as de-Poissonization, which has many applications in geometric probability (see [Pen03] for a detailed account of the subject). Throughout the paper we work with a Poisson point process on the hyperbolic disk of radius $R$ and denote its point set by $\mathcal{P}$. Recall that $R = 2\log n + C$ for $C \in \mathbb{R}$. The intensity function at polar coordinates $(r, \theta)$ for $0 \leq r < R$ and $0 \leq \theta < 2\pi$ is equal to

$$g(r, \theta) := \delta e^{-R/2}\frac{\alpha \sinh(\alpha r)}{2\pi C(\alpha, R)}$$

with $\delta = e^{-C/2}$. Note that \(\int_{r=0}^{R} \int_{\theta=0}^{2\pi} g(r, \theta) d\theta dr = \delta e^{-R/2} = n\), and thus $\mathbb{E}|\mathcal{P}| = n$.

The main advantage of defining $\mathcal{P}$ as a Poisson point process is motivated by the following two properties: the number of points of $\mathcal{P}$ that lie in any region $A \cap B_O(R)$ follows a Poisson distribution with mean given by $\int_A g(r, \theta) dr d\theta = n\mu(A \cap B_O(R))$; and the number of points of $\mathcal{P}$ in disjoint regions of the hyperbolic plane are independently distributed. Moreover, by conditioning $\mathcal{P}$ upon the event $|\mathcal{P}| = n$, we recover the original distribution. Therefore, since $\mathbb{P}(|\mathcal{P}| = n - k) = \Theta(1/\sqrt{n})$ for any $k = O(1)$, any event holding in $\mathcal{P}$ with probability at least $1 - o(f_n)$ must hold in the original setup with probability at least $1 - o(f_n\sqrt{n})$, and in particular, any event holding with probability at least $1 - o(1/\sqrt{n})$ holds asymptotically almost surely (a.a.s.), that is, with probability tending to 1 as $n \to \infty$, in the original model. We identify below points of $\mathcal{P}$ with vertices. We prove all results below for the Poisson model and then transform the results to the uniform model.

### 3 The upper bound on the diameter

We now turn our attention towards the upper bound. We first show that there cannot be long paths with all vertices being close to the boundary.

**Lemma 8.** Let $\xi > 0$ and let $C'' = C''(\xi, C)$ be a sufficiently large constant. With probability at least $1 - o(n^{-1})$, there is no path $v_1, \ldots, v_k$, such that $r_{v_i} \geq R - \xi$ for $i = 1, \ldots, k$, and such that there exist two vertices $v_i, v_j \in \{v_1, \ldots, v_k\}$ with $|\theta_{v_i} - \theta_{v_j}| \geq C'' \log n$.  


Lemma 9. Let \( \theta \) be two vertices belonging to the same connected component and \( v \) be a \( \theta \)-path without shortcuts to be a sequence of vertices \( \theta = v_0, v_1, \ldots, v_k = v \), such that \( d(v_i, v_{i+1}) \leq R \) for any \( 0 \leq i \leq k-1 \), and \( d(v, v_j) > R \) for any \( |i-j| > 1 \). Note that if the diameter of a component is larger than \( k \), then there must exist two vertices \( u \) and \( v \) together with a \( u \)-path without shortcuts of length at least \( k \), and note also that such a path does not expose any information about the region in which \( v_0, \ldots, v_k \) lie. Define throughout the whole section \( i_0 := \log_{1/\alpha} R/(2C_0 \log R) \) for some sufficiently large constant \( C_0 > 0 \) (in fact, below we will choose \( C_0 = \frac{2}{2-\frac{2}{\alpha+1}} \)). Observe that \( R_{i_0} = R e^{\alpha i_0/2} = R (1 - (1 + o(1)) e^{\alpha i_0/2}) = R - (1 + o(1)) C_0 \log R \).

Proof. First note that by Lemma 8 and Remark 2 for two vertices \( v_i, v_j \) with \( r_{v_i}, r_{v_j} \geq R - \xi \) and \( d(v_i, v_j) \leq R \), we must have \( |\theta_{v_i} - \theta_{v_j}| = (1 + o(1)) 2 e^{-R/2} = C' n \) for \( C' = e^{-C/2} \). Partition the disk of radius \( R \) into \( \Theta(n) \) equal sized sectors of angle \( C'/n \) and order them counterclockwise. We see that a path \( v_1, \ldots, v_k \) having two vertices \( v_i, v_j \) satisfying \( |\theta_{v_i} - \theta_{v_j}| \geq C' \log n \) must have at least one vertex in \( C''/\log n \) consecutive sectors. For each sector, the expected number of vertices inside the sector is \( C' \), and therefore this is also an upper bound for the expected number of vertices \( v \) inside this sector with the additional restriction of \( r_v \geq R - \xi \). Thus, the probability of having at least one vertex inside a sector is at most \( 1 - e^{-C'} < 1 \). The probability to have at least one vertex in \( \log n \) given consecutive sectors is thus at most \( (1 - e^{-C'})^{\Omega(n)} = o(n^{-2}) \) for \( C'' = C''(C') \) sufficiently large. By choosing the lexicographically smallest sector in the counterclockwise ordering, a set of \( \log n \) consecutive sectors is determined, and thus there are \( O(n) \) such choices. Taking a union bound over all such choices, we see that with probability at least \( 1 - o(n^{-1}) \) there is no such path with \( |\theta_{v_i} - \theta_{v_j}| \geq C'' \log n \).
within $\Theta(1)$ consecutive sectors this can happen. Also, we have to wait for at most $O(\log^{C_0-1+o(1)} n)$ sectors in this ordering, until we find a vertex $w$ with $r_w < R - \xi$. Thus, among these $\Theta(\log^{2C_0-1} n)$ sectors with probability at least $1 - o(n^{-1})$ we will find a vertex $w$ with $r_w < R - \xi$, and we let $u_1$ to be the one of these with smallest radius. Then remove all sectors at distance (again measured in terms of this ordering) at most $2 \log^{2C_0-1} n$ from $u_1$. From the remaining ones, proceed in the same way: choose again $u_2$ to be the one with smallest radial coordinate among the next $\log^{2C_0-1} n$ consecutive sectors, and so on. In this way at most $3 \log^{2C_0-1} n$ consecutive sectors get removed between two different vertices, and at least $\Omega(\log^{2C_0} n - 3 \log^{2C_0-1} n)$ sectors remain. Hence, for $C''$ large enough, this can be repeated $C'' \log n$ times, and we obtain vertices $u_i$ for $i = 1, \ldots, C'' \log n$. Note that by construction, the path cannot cycle around, and hence we have for any $i \neq j$, $|\theta_{u_i} - \theta_{u_j}| = \Omega(\log^{2C_0} n/n)$. 

\[ \square \]

For a vertex $v \in \mathcal{P}$ with $r_v > R_{i_0}$, let $\ell$ be such that $R_\ell < r_v \leq R_{\ell+1}$. Define a center path from $v$ to be a sequence of vertices $v =: w_0, w_1, \ldots, w_s$ such that $d(w_i, w_{i+1}) \leq R$, $R_\ell < r_{w_i} \leq R_{\ell+1-i}$ for $0 \leq i \leq s-1$, and $R_{i_0-1} < r_{w_s} \leq R_{i_0}$.

**Lemma 10.** Let $\xi > 0$. Let $v \in \mathcal{P}$ a vertex such that $R_\ell < r_v \leq R_{\ell+1}$ for $\ell \geq i_0$, and suppose also $R_\ell < R - \xi$. Then with probability at most $c < 1$ there exists no center path starting from $v$.

**Proof.** Let $v =: w_0$ be a vertex such that $R_\ell < r_v \leq R_{\ell+1}$ for $\ell \geq i_0$, and suppose also $R_\ell < R - \xi$. Denote the event $\mathcal{E}_0$ the event that there exists one vertex of $\mathcal{P}$ that belongs to $B_{w_0}(R) \cap (B_O(R) \setminus B_O(R_{\ell-1}))$. By Lemma \[\square\] $\mu(B_{w_0}(R) \cap (B_O(R) \setminus B_O(R_{\ell-1}))) = \Omega(\mu(B_{w_0}(R) \cap B_O(R_{\ell})))$. By Lemma \[\square\] (applied with $\rho_\ell = R$, $\rho_O = R_\ell$, and $R_\ell < r_A = r_{w_0} \leq R_{\ell+1}$), we have

\[
\mu(B_{w_0}(R) \cap B_O(R_\ell)) = \frac{2\alpha}{\pi(\alpha - 1/2)} \left( e^{-(\alpha-\frac{1}{2})(R-R_\ell)} - \frac{1}{2} r_{w_0} + O(e^{-\alpha r_{w_0}}) \right) = \Theta(e^{-(\alpha-\frac{1}{2})(R-R_\ell)} - \frac{1}{2} r_{w_0}),
\]

where the last identity follows because $r_{w_0} > R - R_\ell$. Now, since $1 - e^{-x} \leq x$, we have $R - R_\ell = R(1 - e^{-\alpha/2}) \leq R\alpha/2$, and since for $x = o(1)$, $e^{-x} = (1 + o(1))(1 - x)$, we also have $r_{w_0} \leq R_{\ell+1} = R e^{-\alpha^{\ell+1}/2} = R(1 - (1 + o(1))\alpha^{\ell+1}/2)$. Thus

\[
\mathbb{P}(\mathcal{E}_0) = \exp(-\Omega(ne^{-(\alpha-1/2)(R-R_\ell)} - \frac{1}{2} r_{w_0})) = \exp(-\Omega(ne^{-(\alpha-1/2)R\alpha/2 - \frac{\alpha}{2}(1 - (1+o(1))\alpha^{\ell+1}/2)})).
\]

Recalling that $e^{-R/2} = \Theta(1/n)$, we have

\[
\mathbb{P}(\mathcal{E}_0) = \exp(-\Omega(e^{-(\alpha-1/2)R\alpha/2 + (1+o(1))\alpha^{\ell+1}/4})).
\]

(5)

Note that if $\mathcal{E}_0$ holds, then a vertex inside the desired region is found. Assuming the existence of a vertex $w_i$ with $R_{\ell-i} < r_{w_i} \leq R_{\ell+1-i}$, we continue inductively for $i = 1, \ldots, s-1 = \ell - i_0$ in the same way: we define the event $\mathcal{E}_i$ that there exists one element of $\mathcal{P}$ that belongs to $B_{w_i}(R) \cap (B_O(R_{\ell-i}) \setminus B_O(R_{\ell-i-1}))$. By the same calculations (noting that Lemma \[\square\] can still be applied and $r_{w_i} > R - R_{\ell-i}$ still holds), we obtain

\[
\mathbb{P}(\mathcal{E}_i) = \exp(-\Omega(e^{-(\alpha-1/2)R\alpha^{\ell-i}/2 + (1+o(1))R\alpha^{\ell-i+1}/4})).
\]

Denote by $\mathcal{C}$ the event of not having a center path, we have thus by independence of the events

\[
\mathbb{P}(\mathcal{C}) = \mathbb{P}(\mathcal{E}_0) + \sum_{i=1}^{s-1} \mathbb{P}(\mathcal{E}_i|\mathcal{E}_0, \ldots, \mathcal{E}_{i-1}) = \sum_{i=0}^{s-1} \mathbb{P}(\mathcal{E}_i),
\]

7
Hence,

\[
P(\mathcal{C}) = \sum_{i=0}^{\ell-i_0} \exp(-\Omega(e^{-(\alpha-1/2)R\alpha^{\ell-i}/2+(1+o(1))R\alpha^{\ell+1-i}/4)})
\]

\[
\leq \sum_{i=0}^{\ell-i_0} \exp(-\Omega(1 - (\alpha - 1/2)R\alpha^{\ell-i}/2 + (1 + o(1))R\alpha^{\ell+1-i}/4))
\]

\[
\leq \sum_{i=0}^{\ell-i_0} \exp(-\Omega(1 + (1 + o(1))\frac{R\alpha^{\ell-i}}{4}(1 - \alpha)))
\]

\[
\leq \sum_{i=0}^{\ell-i_0} \exp(-C' - (1 + o(1))C'R\alpha^{\ell-i}/4(1 - \alpha)),
\]

where \(C' = C'(\xi) > 0\) is the constant hidden in the asymptotic notation of Lemma \[\text{7}\]. Clearly, the closer \(v\) to the boundary, the more difficult it is to find a center path, and we may thus assume that \(R_\ell < r_v \leq R_{\ell+1}\) is such that \(R_\ell = R - \xi\). Then, noting that \(R_\ell = R\alpha^{-\ell/2} = R - \xi\) implies that \(\ell = -\log_{1/\alpha}(1 + o(1))(2\xi/R)\) must hold. Plugging this into the previous sum we get

\[
P(\mathcal{C}) \leq e^{-C'} \sum_{i \geq 0} e^{-(1+o(1))C'\xi^{(1-\alpha)/2\alpha^i}},
\]

Clearly, since \((1/\alpha) > 1\), the sum converges. Note that the constant \(C'\) coming from Lemma \[\text{7}\] is nondecreasing as a function of \(\xi\). Hence, by choosing \(\xi = \xi(C', \alpha)\) big enough, the sum is less than 1, and the statement follows.

\[
\square
\]

For vertices closer to the center we have better bounds. Define \(j_0 = j_0(\alpha) \geq 1\) to be a constant sufficiently large so that \(e^{-\alpha j}/2 \leq 1 - (1 - \frac{1-\alpha}{2})\frac{\alpha^j}{2}\) for \(j \geq j_0\) (note that such \(j_0\) exists using the Taylor series of \(e^{-x}\)).

**Lemma 11.** With probability at least \(1 - o(n^{-1})\), for any vertex \(v \in P\) with \(R_j < r_v \leq R_{j+1}\) and \(j_0 \leq j \leq i_0\),

\[(B_v(R) \cap (B_O(R_j) \setminus B_O(R_{j-1})) \cap P \neq \emptyset.\]

**Proof.** As in the previous proof, by Lemma \[\text{7}\]

\[
\mu(B_v(R) \cap (B_O(R_j) \setminus B_O(R_{j-1}))) = \Omega(\mu(B_v(R) \cap B_O(R_j))),(6)
\]

and also as before, by Lemma \[\text{4}\] (applied with \(\rho_A = R, \rho_O = R_j,\) and \(R_j < r_A = r_v \leq R_{j+1}\)), we have

\[
\mu(B_v(R) \cap B_O(R_j)) = \Theta(e^{-(\alpha-1/2)(R-R_j) - \frac{1}{2}r_v}),
\]

Again, \(R - R_j = R(1 - e^{-\alpha j/2}) \leq R\alpha^j/2\), and since by assumption, \(e^{-\alpha j/2} \leq 1 - (1 - \frac{1-\alpha}{2})\frac{\alpha^j}{2}\), we have \(r_v \leq R_{j+1} = R e^{-\alpha j/2} = R(1 - (1 - \frac{1-\alpha}{2})\frac{\alpha^j}{2})\). Thus,

\[
\mu(B_v(R) \cap B_O(R_j)) = \Omega(e^{-(\alpha-1/2)R\alpha^{j}/2 - \frac{R}{2}(1-(1-\frac{1-\alpha}{2})\frac{\alpha^{j+1}}{2})}) = \Omega(e^{-R/2}R^{\alpha^{j}/2}(\frac{1}{2} - \frac{\alpha^j}{2})) = \Omega(e^{-R/2}R^{\alpha^{j}/2}(\frac{1}{2} - \frac{\alpha^j}{2})^2).
\]

\[
8
\]
Note that for $1/2 < \alpha < 1$, $\frac{1}{2} - \frac{3}{4} \alpha + \frac{\alpha^2}{4} > 0$. The previous expression is clearly decreasing in $j$. By plugging in our upper bound $j = i_0 = \log_{1/\alpha}(R/(2C_0 \log R))$, we get with our choice of $C_0 = \frac{2}{\frac{3}{4} \alpha + \frac{\alpha^2}{4}}$,\[\mu(B_v(R) \cap B_O(R_j)) = \Omega(\frac{R^2}{e^{R/2}}) = \Omega((\log n)^2/n).\]

Hence, the expected number of vertices in $B_{r_v}(R) \cap B_O(R_j)$ is $\Omega(\log^2 n)$, and by Chernoff bounds for Poisson random variables (see again Theorem A.1.15 of [AS08]), with probability $1 - e^{-\Omega(\log^2 n)}$ there are at least $\Omega(\log^2 n)$ vertices in this region. By (6) together with a union bound over all vertices, the statement follows.

Finally, for vertices $v \in \mathcal{P}$ with $R/2 < r_v \leq R_{j_0}$ we will for the next lemma modify the definition of the $R_i$: since $j_0 = O(1)$, there exists some $1/2 < c < 1$ such that $R_{j_0} = Re^{-\alpha \log 2} = cR$. Let $T \geq 1$ be the largest integer such that $c - \frac{T}{2}(1-c)(1-\alpha) > \frac{1}{2}$. Define now the new bands to be $R'_i := cR$, and for any $i = 1, \ldots, T$, define $R'_i := R(c - \frac{i}{2}(1-c)(1-\alpha))$. We have the following lemma.

**Lemma 12.** With probability at least $1 - o(n^{-1})$, for any vertex $v \in \mathcal{P}$ with $R'_j < r_v \leq R'_{j-1}$ with $1 \leq j \leq T - 1$, we have 

$$(B_v(R) \cap (B_O(R'_j) \setminus B_O(R'_{j+1}))) \cap \mathcal{P} \neq \emptyset.$$ 

Moreover, with probability at least $1 - o(n^{-1})$, for any vertex $v \in \mathcal{P}$ with $R'_T < r_v \leq R'_{T-1}$ we have 

$$(B_v(R) \cap B_O(R/2)) \cap \mathcal{P} \neq \emptyset.$$ 

**Proof.** First assume $1 \leq j \leq T - 1$. By Lemma 4 (applied with $\rho_A = R$, $\rho_O = R'_j$, and $R'_j < r_A = r_v \leq R'_j$), since $r_v > R - R'_j$ still holds, we have 

$$\mu(B_v(R) \cap B_O(R'_j)) = \Theta(e^{-(a-\frac{1}{2})(R-R'_j) - \frac{1}{2} r_v}) = \Theta(e^{-\frac{R}{2}(2a-1)(1-c+\frac{1}{2}(1-c)(1-\alpha)) - \frac{1}{2} r_v})$$

and also 

$$\mu(B_v(R) \cap B_O(R'_{j+1})) = \Theta(e^{-\frac{R}{2}(2a-1)(1-c+\frac{1}{2}(1-c)(1-\alpha)) - \frac{1}{2} r_v}).$$

Thus, $\mu(B_v(R) \cap B_O(R'_{j+1})) = \Theta(e^{-\frac{R}{2} (2a-1)(1-c)(1-\alpha)}) \mu(B_v(R) \cap B_O(R'_j))$, and we have 

$$\mu(B_v(R) \cap (B_O(R'_j) \setminus B_O(R'_{j+1}))) = (1 + o(1)) \mu(B_v(R) \cap B_O(R'_j)). \tag{7}$$

Now, since $r_v \leq R'_{j-1}$, 

$$\mu(B_v(R) \cap B_O(R'_j)) = \Omega(e^{-\frac{R}{2} (2a-1)(1-c+\frac{1}{2}(1-c)(1-\alpha)) - \frac{1}{2} R'_{j-1}}) = \Omega(e^{-R/2 ((2a-1)(1-c)+\frac{1}{2}(1-c)(1-\alpha)) + c - \frac{1}{2} (1-c)(1-\alpha)})$$

$$= \Omega(e^{-R/2 ((1-c)(1-\alpha)^2 + (1-c)(2a-1)+ \frac{1}{2} \alpha + c)})$$

Clearly, the bigger $j$, the bigger $\mu$. Thus we plug in the smallest possible value $j = 1$, and together with (7) we obtain 

$$\mu(B_v(R) \cap (B_O(R'_j) \setminus B_O(R'_{j+1}))) = \Omega(e^{-R/2 ((1-c)(1-\alpha)^2 + (1-c)(2a-1)+ \frac{1}{2} \alpha + c)})$$

$$= \Omega(e^{-R/2 ((1-c)(2a-3)-a^2)+c}).$$
Note that for $\frac{1}{2} < \alpha < 1$, we have $\frac{7n-3}{2} - \alpha^2 < 1$, and thus the constant inside the big brackets is clearly bounded by $c < 1$. Hence $\mu(B_v(R) \cap (B_O(R_j) \setminus B_O(R_{j+1}))) = \Omega(e^{-\gamma R/2})$ for some $\gamma < 1$. Hence, the expected number of vertices inside $B_v(R) \cap (B_O(R_j) \setminus B_O(R_{j+1}))$ is $n^{1-\gamma}$. By Chernoff bounds together with a union bound over all vertices the first part of the lemma follows. For the second part, by Lemma 4 (applied with $\rho_A = R$, $\rho_O = R/2$, and $R_{T'} < r_A = r_B \leq R_{T-1}$, since $r_v > R/2$ still holds, we have

$$
\mu(B_v(R) \cap B_O(R/2)) = \Theta(e^{-(\alpha-\frac{1}{2})(R/2)-\frac{1}{2}r_v}) = \Omega(e^{-(\alpha-\frac{1}{2})(R/2)-\frac{1}{2}R_{T-1}}).
$$

Note that $R_{T-1} \leq \frac{R}{2} + R(1-c)(1-\alpha)$ must hold, as otherwise $R_{T+1} = R_{T-1} - R(1-c)(1-\alpha) > R/2$ would hold, and then $c - \frac{T+1}{2}(1-c)(1-\alpha) > 1/2$, contradicting the definition of $T$. Thus,

$$
\mu(B_v(R) \cap B_O(R/2)) = \Omega(e^{-\frac{R}{2}(\alpha+(1-c)(1-\alpha))}).
$$

Since $\alpha + (1-c)(1-\alpha) < 1$, $\mu(B_v(R) \cap B_O(R/2)) = n^{1-\gamma'}$ for some $\gamma' > 0$, and by the same argument as before, the second part of the lemma follows.

We are now ready for an upper bound on the diameter. By the result of Bode, Fountoulakis and Müller [BFM13], all vertices belonging to $B_O(R/2)$ form part of a giant component; note that by Lemma 3 together with Chernoff bounds there are a.a.s. $\Theta(n^{1-\alpha})$ vertices inside $B_O(R/2)$. We call this the center giant component, and we will show that there is no other giant component.

**Theorem 13.** With probability at least $1 - o(n^{-1})$, for any two vertices $u, v \in \mathcal{P}$ belonging to the same connected component, $d_G(u,v) = O(polylog(n))$.

**Proof.** Note that by combining Lemma 11 and Lemma 12 a.a.s., any vertex $v$ with $r_v \leq R_{i_0}$ is connected to a vertex in $B_O(R/2)$. All vertices in $B_O(R/2)$ form a clique, and by [BFM13], they belong to the center giant component. Also, by the proofs of Lemma 11 and 12 with probability at least $1 - o(n^{-1})$, between any two vertices $v_i$ and $v_j$ with $r_{v_i}, r_{v_j} \leq R_{i_0}$, there is a path of length at most $O(\log R) = O(\log \log n)$ connecting them. By the proofs of these lemmas, with probability at least $1 - o(n^{-1})$, the graph distance between the first occurrence and the last occurrence of such a vertex can be bounded by $O(\log \log n)$, it suffices to show that the graph distance between any two vertices $u$ and $v$ with $r_u, r_v > R_{i_0}$ is bounded by $O(polylog(n))$. Hence we may assume that all vertices on a $uv$-path without shortcuts $u := v_1, \ldots, v_k := v$ satisfy $r_{v_i} > R_{i_0}$. If $k < 2C'' \log^{2C_0+1} n$ for some large constant $C'' > 0$, the graph distance between these two vertices is at most $2C'' \log^{2C_0+1} n$, and we are fine, so we may assume $k \geq 2C'' \log^{2C_0+1} n$. Consider the first $C'' \log^{2C_0+1} n$ vertices on this path. By Lemma 9 with probability at least $1 - o(n^{-1})$, there exist $C' \log n$ vertices $u_1, \ldots, u_{C' \log n}$ with $u_i \in \{v_1, \ldots, v_{2C'' \log^{2C_0+1} n}\}$ with $|\theta_{u_i} - \theta_{u_j}| = \Omega(\log^{2C_0} n/n)$ and $r_{u_i} < R - \xi$ for some $\xi > 0$. Let $\ell$ be such that $R_{\ell} < r_{u_i} \leq R_{\ell+1}$, and note that $\ell \geq i_0$. Suppose there exists a center path starting from $u_{i_0}$, that is, a sequence of vertices denoted by $u_{i_0} \leq u_i, u_i', \ldots, u_{i_0} \leq R_{\ell-j} < r_{u_{i_0}'} \leq R_{\ell-j+1}$ for any $0 \leq j \leq m-1$, and $R_{i_0-1} < r_{u_{i_0}'} \leq R_{i_0}$. By Lemma 1 and Remark 2 $|\theta_{u_i} - \theta_{u_{i+1}}| \leq 2e^{(R_{R_{i_0}}-R_{R_{i_0}-1})/2(1+o(1))}$, and thus, by the triangle inequality for angles,

$$
|\theta_{u_i} - \theta_{u_i'}| \leq \sum_{j=0}^{m-1} |\theta_{u_{i_j}} - \theta_{u_{i_j+1}}| \leq \sum_{i 

10
Using $R_i = R e^{-\alpha^i/2} = (1 + o(1))R(1 - \alpha^i/2)$ for $i \geq i_0$ we get

$$|\theta_{u_i} - \theta_{u_i^m}| \leq 2e^{-R/2} \sum_{i \geq i_0} e^{(1+o(1))R \alpha^{i-1}/2} = O\left(\frac{R^{C_0/0+o(1)}}{e^{R/2}}\right) = O\left((\log n)^{C_0/0+o(1)}/n\right) = o\left(\log^{2C_0} n/n\right).$$

(8)

The same clearly holds when the center path starts from $u_j$ with its corresponding terminal vertex $u_j^m$. Since $|\theta_{u_i} - \theta_{u_j}| = \Omega(\log^{2C_0} n/n)$, we have $|\theta_{u_i^m} - \theta_{u_j^m}| = \Omega(\log^{2C_0} n/n) - o(\log^{2C_0} n/n) = \Omega(\log^{2C_0} n/n)$. Again, by Lemma 1 and Remark 2, if two vertices $w, z$ with $R_{i_0-1} < r_w, r_z \leq R_{i_0}$ are adjacent, then $|\theta_w - \theta_z| \leq 2e^{(R^{-2R_{i_0}}-1)/2} = O\left(\log^{C_0/0+o(1)} n/n\right) = o\left(\log^{2C_0} n/n\right)$. Two vertices $u_i^m, u_j^m$ that correspond to such paths therefore are not adjacent, and by the same argument they cannot share a common neighbor inside $B_{O}(R_{i_0}) \setminus B_{O}(R_{i_0-1})$. Thus, for any two vertices $u_i, u_j \in \{v_1, \ldots, v_k\}$ coming from Lemma 9, the regions discovered in their center paths are disjoint. Moreover, recall that the vertices $u_i$ coming from Lemma 9 are chosen in such a way that they are the ones with smallest radius $R_\ell < r_u \leq R_{\ell+1}, \ell \geq i_0$, among all vertices of the $uv$-path within $\log^{2C_0-1} n$ consecutive sectors (one of them containing $u_i$). Thus, either to the left or to the right of the sector containing $u_i$ there are $\Theta(\log^{2C_0-1} n)$ consecutive sectors such that within them there is no vertex of the $uv$-path at radial distance less than $r_u$. We may assume that it is to the right of $u_i$. Call $S$ to be the region corresponding to these sectors, and observe thus that since the $uv$-path is a path without shortcuts, no information about $B_{O}(R_{\ell}) \cap S$ has been exposed by this path. Note also that the angle between a vertex in the first and in the last sector of $S$ is $\Theta(\log^{2C_0} n/n)$. Since by Lemma 10, $|\theta_{u_i} - \theta_{u_j}| = o(\log^{2C_0} n/n)$, all regions to the right of $u_i$ that might be explored in any center path starting at $u_i$ are unexposed. We look for a center path only within $S$: note that by symmetry, if the path has succeeded until some vertex, at the next vertex at least half of the original region is unexposed, and thus by Lemma 10 the probability that there is no center path starting at $u_i$ is still at most $c < 1$. Also, since the regions discovered by center paths starting from $u_i$ and $u_j$ are disjoint, the probability that for none of the vertices $u_i, \ldots, u_{C'/\log n}$ a center path exists, is thus at most $c^{C'/\log n} = o(n^{-3})$ for $C'$ (and therefore also for $C''$) sufficiently large. Fix for each pair of vertices $u, v$ an arbitrary $uv$-path without shortcuts. If this path is of length $k \leq 2C'' \log^{2C_0+1} n$, then the graph distance between the two vertices is $O(\log\log(n))$. Otherwise, by taking a union bound over all $O(n^2)$ pairs of vertices together with a fixed path without shortcuts, for each of them with probability $1 - o(n^{-1})$ we find among the first $C'' \log^{2C_0+1} n$ vertices and also among the last $C'' \log^{2C_0+1} n$ vertices on this path a center path. The length of a center path is $O(\log R)$. Hence, with probability at least $1 - o(n^{-1})$, the graph distance between any two vertices is at most $2C'' \log^{2C_0+1} n + O(\log R) = O(\log\log(n))$, finishing the proof of the statement. By the remarks preceding this section, the same result holds with probability at least $1 - o(n^{-1/2})$ also in the uniform model.

Corollary 14. With probability at least $1 - o(n^{-1})$, the size of the second largest component in the Poissonized model is bounded by $O(\log\log(n))$.

Proof. As in the proof of Lemma 9 partition the disk of radius $R$ centered at the origin into $\Theta(n/\log n)$ equal sized sectors of angle $C' \log n/n$ for some large constant $C' > 0$ and order them counterclockwise. Recall that in each sector the number of vertices is with probability at least $1 - o(n^{-1}), \Theta(n)$ By Lemma 11 and 12 if a component contains a vertex $v_i$ satisfying $r_{v_i} \leq R_{i_0}$, then this component with probability at least $1 - o(n^{-1})$ is equal to the center giant component. Thus we may assume that all vertices of any other component satisfy $r_{v_i} > R_{i_0}$. For any two such
vertices \( v_i, v_j \) with \( d(v_i, v_j) \leq R \), as before, by Lemma 1 and Remark 2, \( |\theta_{v_i} - \theta_{v_j}| = O((\log n)^{C_0}/n) \).
Now, if there were two vertices \( v_i \) and \( v_j \) that belong to sectors at distance \( \omega(\log^{3C_0+1}n) \) of each other, we would have \( |\theta_{v_i} - \theta_{v_j}| = \omega(\log^{3C_0+1}n) \). Hence, if they belong to the same component and they both satisfy \( r_{v_i}, r_{v_j} > R_{v_0} \), then their graph distance is \( \omega(\log^{3C_0+1-\alpha}n) = \omega(\log^{2C_0+1}n) \). By the proof of Theorem 13, however, with probability at least \( 1 - o(n^{1-\alpha}) \), for any two vertices of the same component, their graph distance is bounded by \( O(\log^{2C_0+1}n) \). Hence, two such vertices must belong to sectors at distance at most \( O(\log^{3C_0}n) \). Since in any sector there are at most \( O(\log^2n) \) vertices, the number of vertices of the second largest component is with probability at least \( 1 - o(n^{1-\alpha}) \) thus bounded by \( O(\log^{3C_0+1}n) = O(\text{polylog}(n)) \). As before, the same result holds also with probability at least \( 1 - o(n^{-1/2}) \) in the uniform model.

\[ \square \]

### 4 The existence of a path component of length \( \Theta(\log n) \)

As before, we prove the result here for a Poissonized model and then derive from it the result in the uniform model. We state the result only for the uniform model:

**Theorem 15.** A.a.s., there exists a component forming a path of length \( \Theta(\log n) \).

**Proof.** Fix throughout the proof \( c_1, c_2, c_3 \) three positive constants with \( c_3 < c_1 < c_2 \) and \( 2e^{c_1-c_2} < e^{-c_3/2} \). First, by part 1 of Lemma 5 applied with \( \rho_0 = R - c_1 \) and \( \rho'_0 = R - c_2 \) we have

\[
\mu(B_O(\rho_0) \setminus B_O(\rho'_0)) = e^{-\alpha c_1}(1 - e^{-\alpha(c_2-c_1)} + o(1)) = \Theta(1).
\]

Let \( \Theta \subseteq [0,2\pi) \) be a set of forbidden angles such that \( \mu(R_\Theta) < 1 \), where \( R_\Theta := \{(r_P, \theta_P) : 0 \leq r_P < R, \theta_P \in \Theta \} \) (for a geometric interpretation of \( R_\Theta \), note that when \( \Theta \) is an interval, \( R_\Theta \) is a cone with vertex \( O \)). As a constant fraction of the angles is still allowed, clearly,

\[
\mu((B_O(\rho_0) \setminus B_O(\rho'_0)) \setminus R_\Theta) = \Theta(1)
\]

(9) still holds. For any vertex \( A \) with \( R - c_2 \leq r_A \leq R - c_1 \), by Lemma 1

\[
\mu(B_A(R) \cap B_O(R)) = \frac{2\alpha}{\pi(\alpha - 1/2)} e^{-\frac{3}{2}r_A} + O(e^{-\alpha r_A}) = \Theta(1/n).
\]

(10)

and by part 2 of Lemma 5 together with (10),

\[
\mu((B_A(R) \setminus B_A(R - c_3)) \setminus (B_O(\rho_0) \setminus B_O(\rho'_0))) = \Theta(e^{-R/2}) = \Theta(1/n).
\]

(11)

For two vertices \( A, B \) with \( R - c_3 \leq d(A, B) \leq R \) that satisfy \( \rho'_0 \leq r_A, r_B \leq \rho_O \), by Lemma 6 together with (10):

\[
\mu((B_B(R) \setminus B_B(R - c_3)) \setminus (B_O(\rho_0) \setminus B_O(\rho'_0))) \setminus B_A(R)) = \Theta(e^{-R/2}) = \Theta(1/n).
\]

(12)

Let \( \varepsilon = \varepsilon(\alpha) \) be a constant chosen small enough so that \( 1 - \frac{1}{2\pi} - \varepsilon > 0 \). For \( 1 \leq j \leq J := \nu n^{1-\frac{1}{2\pi} - \varepsilon} \) with \( \nu \) being sufficiently small, let \( A'_j \) be a randomly chosen vertex \( v \) among all vertices with \( \rho'_0 \leq r_v \leq \rho_O \). Note that we may first run a Poisson point process of the same intensity inside this band, and then a Poisson point process on the complement of the disk of radius \( R \), and the distribution of the vertices remains unchanged; also, no information about this complement is exposed yet. Define then \( E_j \) to be the event that the following conditions hold (if one condition fails, then stop exposing and checking further conditions and proceed with the next \( j \)):
1. For any \( j \geq 1 \), let \( A^j_i \) be a randomly chosen vertex with \( \rho_O \leq r_v \leq \rho_O \). For \( j \geq 2 \), we require the polar coordinates of \( A^j_0 \) and \( A^j_j \) for any \( 1 \leq k < j \) to be sufficiently different, i.e., if \( \Theta_k = \left\{ \theta : |\theta _{A^j_0} - \theta | \leq C' n^{-(1-\frac{1}{2\alpha - 2})} \right\} \) for some large constant \( C' \), we require that \( \theta _{A^j_0} \notin \cup_{k=1}^{j-1} \Theta_k \).

2. For \( 0 \leq i \leq L := \nu' \log n \) with \( \nu' > 0 \) being a small constant, expose the region \( \left( B_{A^j_0}(R) \setminus B_{A^j_0}(R - c_3) \right) \cap \left( B_O(\rho_O) \setminus B_O(\rho_O) \right) \). We require exactly one vertex in this region, call it \( A^j_i \). Then, inductively, for \( 1 \leq i < L \), expose the region \( \left( B_{A^j_i}(R) \setminus B_{A^j_i}(R - c_3) \right) \cap \left( B_O(\rho_O) \setminus B_O(\rho_O) \right) \), and we require also exactly one vertex in this region, and name it inductively \( A^j_{i+1} \). In other words, \( A^j_{i+1} \) belongs to the \((R - c_3, R)\)-band centered at \( A^j_i \) and also to the \((\rho_O', \rho_O)\)-band centered at the origin \( O \). For \( i = L \), expose the region \( \left( B_{A^j_L}(R) \setminus B_{A^j_L}(R - c_3) \right) \cap \left( B_O(\rho_O) \setminus B_O(\rho_O) \right) \), and we require that there is no more vertex in this region.

3. For any \( 0 \leq i \leq L \), \( A^j_i \) does not have any other neighbor inside \( B_{A^j_i}(R) \cap \left( B_O(R) \setminus B_O(R(1 - \frac{1}{2\alpha - \epsilon})) \right) \) except the one(s) from the previous condition.

We will now bound from above the probability that for all \( j \) the events \( \mathcal{E}_j \) fail. Note that if for an event \( \mathcal{E}_j \) some vertex \( A^j_i \) with \( i \geq 1 \) satisfying condition 2 is found, then, since \( d(A^j_{i-1}, A^j_i) \leq R \) and \( r_{A^j_{i-1}}, r_{A^j_i} \geq R - c_2 \), by Lemma \[ \text{[1]} \] and Remark \[ \text{[2]} \] we have \( |\theta _{A^j_{i-1}} - \theta _{A^j_i}| = O(e^{-R/2}) \). Thus, if we have not failed before and we reach \( A^j_L \), we have \( |\theta _{A^j_0} - \theta _{A^j_L}| = O(e^{-R/2} \log n) = O(\log n/n) \). Thus, for \( \nu' \) sufficiently small, for \( \Theta^j := \cup_{k=1}^{j-1} \Theta_k \), it still holds that \( \mu(\mathcal{E}_{\Theta^j}) < 1 \), and \[ \text{[9]} \] can be applied to \( R_{\Theta^j} \) for any \( 1 \leq j \leq J \). Hence, for any \( j \), independently of the outcomes of previous events, there is an absolute constant \( c > 0 \) such that for every \( j \) we have \( \mathbb{P} \left( \text{condition 1 holds} \right) \geq c \). Given that condition 1 holds, by \[ \text{[11]} \] applied to \( A^j_0 \) and \[ \text{[12]} \] applied successively to \( A^j_1, \ldots, A^j_L \) we have \( \mathbb{P} \left( \text{condition 2 holds} \right) \geq c^{-L'} \) for some fixed \( 0 < c' < 1 \). Suppose then that condition 2 also holds. By a union bound and by \[ \text{[10]} \], we have

\[
\mu(\bigcup_{i=0}^{L} [B_{A^j_i}(R) \cap (B_O(R) \setminus B_O(R(1 - \frac{1}{2\alpha} - \epsilon)))] \right) \leq \sum_{i=0}^{L} \mu(B_{A^j_i}(R) \cap B_O(R)) = O(L/n),
\]

and thus for \( \nu' \) sufficiently small, \( \mathbb{P} \left( \text{condition 3 holds} \right) \geq n^{-\eta} \) for some fixed value \( \eta \) which can be made small by choosing \( \nu' \) small (in fact, part of the region might have been already exposed in condition 2, but since we know there are no other vertices in there, this only helps). Altogether, we obtain

\[
\mathbb{P}(\mathcal{E}_j) \geq n^{-\eta'}
\]

for some \( \eta' > 0 \). Again, \( \eta' \) can be made sufficiently small by making the constant \( \nu' \) (and thus \( L = \nu' \log n \)) sufficiently small.

Since \( |\theta _{A^j_0} - \theta _{A^j_L}| = O(\log n/n) \), and by construction for \( j \neq j' \) also \( |\theta _{A^j_0} - \theta _{A^{j'}_0}| = o(\log n/n) \), for \( j \neq j' \) the regions exposed in condition 2 are disjoint. Also, for any \( j \neq j' \), if the region in condition 3 containing one of the vertices \( A^j_i \) or \( A^{j'}_i \) is exposed, then \( B_{A^j_i}(R) \cap (B_O(R) \setminus (B_O(R(1 - \frac{1}{2\alpha} - \epsilon)))) \)
is disjoint from $B_{A_0'}(R) \cap (B_O(R) \setminus (B_O(R(1 - \frac{1}{2\alpha} - \varepsilon))))$: indeed, note that for any vertex $v \in B_{A_0'}(R) \cap (B_O(R) \setminus (B_O(R(1 - \frac{1}{2\alpha} - \varepsilon))))$ we have by Lemma 1 and Remark 2

$$|\theta_{A_0} - \theta_v| \leq 2e^{\frac{1}{2}(R(1 - \varepsilon) - R(1 - \frac{1}{2\alpha} - \varepsilon))}(1 + o(1)) = O(n^{-1 - \frac{1}{2\alpha} - \varepsilon}).$$

By construction, we have $|\theta_{A_0} - \theta_{A_0'}| ≥ C'n^{-1 - \frac{1}{2\alpha} - \varepsilon}$ for some large enough $C' > 0$, and thus by the triangle inequality for angles also $|\theta_{A_i} - \theta_{A_i'}| ≥ (1 + o(1))C'n^{-1 - \frac{1}{2\alpha} - \varepsilon}$ holds for any $i, i'$ and any $j ≠ j'$. Hence, the regions exposed in condition 3 are disjoint, and by the same reason, the region exposed in condition 2 for some $j$ and the one exposed in condition 3 for some $j' ≠ j$ are disjoint as well. Hence, the probabilities of the corresponding conditions to hold are thus independent. Hence, for $\nu'$ sufficiently small, $\eta'$ is small enough such that $-\eta' + 1 - \frac{1}{2\alpha} - \varepsilon > 0$, and

$$\prod_{j=1}^{J} P(\mathcal{E}_j) ≤ (1 - n^{-\eta'})^J = e^{-\Omega(n^{\varepsilon})},$$

for some positive $\xi > 0$. Thus, a.a.s. there exists one $j$, for which the event $\mathcal{E}_j$ holds. If now such an event $\mathcal{E}_j$ holds, we also have by Lemma 3 applied with $\rho_O = R(1 - \frac{1}{2\alpha} - \varepsilon)$, $\mu(B_O(\rho_O)) = O(e^{-R/2 - \alpha R}) = o(1/n)$, and thus, a.a.s. there is no such vertex inside this ball. Thus, the vertices $A_{i_0}', \ldots, A_{i_L}'$ form an induced path, and the result follows for the Poissonized model. In order to depoissonize, let $\mathcal{E}_U'$ be the event that in the Poissonized model there exists some $j$ for which the event $\mathcal{E}_j$ holds, and similarly $\mathcal{E}_U$ the corresponding event for the uniform model. Since we have shown that $P(\mathcal{E}_U') = 1 - e^{-\Omega(n^{\varepsilon})}$, we also have for some function $\omega_n$ tending to infinity arbitrarily slowly $P(\mathcal{E}_U') ≥ 1 - \omega_n \sqrt{n}e^{-\Omega(n^{\varepsilon})} = 1 - e^{-\Omega(n^{\varepsilon})}$. Denoting by $Z_U$ the random variable counting the number of vertices inside $B_O(\rho_O)$ in the uniform model, once again, by Lemma 3 $P(Z_U = 0) = 1 + o(1)$. Since

$$1 + o(1) = P(Z_U = 0) = P(Z_U = 0|\mathcal{E}_U')P(\mathcal{E}_U') + P(Z_U = 0|\mathcal{E}_U)P(\mathcal{E}_U) = (1 + o(1))P(Z_U = 0|\mathcal{E}_U') + o(1),$$

we have $P(Z_U = 0|\mathcal{E}_U') = 1 + o(1)$. Thus, $P(Z_U = 0, \mathcal{E}_U') = 1 + o(1)$, and hence also in the uniform model, a.a.s. there exist vertices $A_{i_0}', \ldots, A_{i_L}'$, such that the vertices $A_{i_0}', \ldots, A_{i_L}'$ form an induced path, and the result follows also for the uniform model. \hfill \qed

5 Conclusion

We have shown that the diameter of the giant component is $O(polylog(n))$. We have also shown that the size of the second largest component is $O(polylog(n))$, and at the same time there exists a path component of length $Θ(log n)$. In future work we will try to tighten these bounds.

References

[AS08] N. Alon and J. Spencer. The Probabilistic Method, Third Edition. John Wiley & Sons, 2008.
A Omitted proofs

Proof. [of Lemma 3] We want to bound (see Figure 1):

\[ \mu(B_A(\rho_A) \cap B_O(\rho_O)) = \mu(B_O(\rho_A - r_A)) + 2 \int_{\rho_A-r_A}^{\rho_O} \int_0^{\theta_{\rho_A(r,r_A)}} \frac{f(r)}{2\pi} d\theta dr \]
\[ = \mu(B_O(\rho_A - r_A)) + \frac{\alpha e^{2\rho_A - r_A}}{\pi C(R, \alpha)} \int_{\rho_A-r_A}^{\rho_O} (e^{(\alpha-1/2)r} - e^{-(\alpha+1/2)r}) \left(1 + \Theta(e^{\rho_A-r_A})\right) dr. \]
We will first solve the integral without the error term $\Theta(e^{\rho_A-r_A})$. For this part we obtain

$$\frac{\alpha e^{1/2}(\rho_A-r_A)}{\pi C(R, \alpha)} \left( \frac{e^{(\alpha-1/2)\rho_O} - e^{(\alpha-1/2)(\rho_A-r_A)}}{\alpha - 1/2} + O(1) \right)$$

$$= \frac{2\alpha}{\pi(\alpha - 1/2)} \left( e^{-\alpha(R-\rho_O)} - \frac{1}{2}(\rho_O-\rho_A+r_A) - e^{-\alpha(R-\rho_A+r_A)} + \Theta(e^{3/2}(\rho_A-r_A)-\alpha R) \right)$$

$$= \frac{2\alpha}{\pi(\alpha - 1/2)} \left( e^{-\alpha(R-\rho_O)} - \frac{1}{2}(\rho_O-\rho_A+r_A) + O(e^{-\alpha(R-\rho_A+r_A)}) \right),$$

where the last identity is because for $e^{1/2}(\rho_A-r_A)-\alpha R = O(e^{-\alpha(R-\rho_A+r_A)})$ for $\alpha > 1/2$. Now, for the error term we obtain

$$\frac{\alpha e^{3/2}(\rho_A-r_A)}{\pi C(R, \alpha)} \left( \frac{\Theta(e^{(3/2)\rho_O}) - \Theta(e^{(3/2)(\rho_A-r_A)})}{\alpha - 3/2} + O(1) \right) = O(e^{-\alpha(R-\rho_A+r_A)}),$$

where the last identity is due to the fact that $\alpha < 3/2$.

Applying Lemma 3 to $\mu(B_O(\rho_A - r_A))$, the desired conclusion follows.\[\square\]

Proof. [of Lemma 5] The first part of the lemma follows directly from Lemma 3. Again, relying on Lemma 1 we get that the second expression equals

$$2 \int_{R-c_2}^{R-c_1} \int_{\theta_{R-c_3}(r_A)}^{\theta_{R-c_3}(r_A)} \frac{f(r)}{2\pi} d\theta dr = 2 \frac{e^{(R-r_A)/2}(1-e^{-c_2/2})}{\pi C(R, \alpha)} \int_{R-c_2}^{R-c_1} e^{-r/2} \alpha \sinh(\alpha r) \left(1 + O(e^{-r}) \right) dr$$

$$= \frac{\Omega(1)}{C(R, \alpha)} \left( e^{(\alpha-1/2)R} + O(e^{(\alpha-3/2)R}) \right).$$

\[\square\]