Higher-Order Scheme-Independent Series Expansions of $\gamma_{\bar{\psi}\psi, IR}$ and $\beta'_{IR}$ in Conformal Field Theories

Thomas A. Ryttov$^a$ and Robert Shrock$^b$

$^a$ CP$^3$-Origins and Danish Institute for Advanced Study
Southern Denmark University, Campusvej 55, Odense, Denmark

$^b$ C. N. Yang Institute for Theoretical Physics and Department of Physics and Astronomy,
Stony Brook University, Stony Brook, NY 11794, USA

We study a vectorial asymptotically free gauge theory, with gauge group $G$ and $N_f$ massless fermions in a representation $R$ of this group, that exhibits an infrared (IR) zero in its beta function, $\beta$, at the coupling $\alpha = \alpha_{IR}$ in the non-Abelian Coulomb phase. For general $G$ and $R$, we calculate the scheme-independent series expansions of (i) the anomalous dimension of the fermion bilinear, $\gamma_{\bar{\psi}\psi, IR}$, to $O(\Delta_1^3)$ and (ii) the derivative $\beta' = d\beta/da$, to $O(\Delta_1^4)$, both evaluated at $\alpha_{IR}$, where $\Delta_1$ is an $N_f$-dependent expansion variable. These are the highest orders to which these expansions have been calculated. We apply these general results to theories with $G = SU(N_c)$ and $R$ equal to the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor representations. It is shown that for all of these representations, $\gamma_{\bar{\psi}\psi, IR}$, calculated to the order $\Delta_1^p$, with $1 \leq p \leq 4$, increases monotonically with decreasing $N_f$ and, for fixed $N_f$, is a monotonically increasing function of $p$. Comparisons of our scheme-independent calculations of $\gamma_{\bar{\psi}\psi, IR}$ and $\beta'_{IR}$ are made with our earlier higher-$n$-loop values of these quantities, and with lattice measurements. For $R = F$, we present results for the limit $N_c \to \infty$ and $N_f \to \infty$ with $N_f/N_c$ fixed. We also present expansions for $\alpha_{IR}$ calculated to $O(\Delta_1^4)$.

I. INTRODUCTION

An important advance in the understanding of quantum field theory was the realization that the properties of a theory depend on the Euclidean energy/momentum scale $\mu$ at which they are measured. This is of particular interest in an asymptotically free non-Abelian gauge theory, in which the running gauge coupling $g(\mu)$ and the associated quantity $\alpha(\mu) = g(\mu)^2/(4\pi)$ approach zero at large $\mu$ in the deep ultraviolet (UV). We shall consider a theory of this type, with gauge group $G$ and $N_f$ massless fermions $\psi_j$, $j = 1, \ldots, N_f$, in a representation $R$ of $G$. The dependence of $\alpha(\mu)$ on $\mu$ is described by the renormalization-group (RG)\[\] beta function, $\beta = d\alpha(\mu)/d\ln \mu$, where $dt = d\ln \mu$. The condition that the theory be asymptotically free implies that $N_f$ be less than a certain value, $N_{w}$, given below in Eq. (2.4). Since $\alpha(\mu)$ is small at large $\mu$, one can self-consistently calculate $\beta$ as a power series in $\alpha(\mu)$. As $\mu$ decreases from large values in the UV to small values in the infrared (IR), $\alpha(\mu)$ increases. A situation of special interest occurs if the beta function has a zero at some value away from the origin. For a given $G$ and $R$, this can happen for sufficiently large $N_f$, while still in the asymptotically free regime. In this case, as $\mu$ decreases from large values in the UV toward $\mu = 0$ in the IR, the coupling increases, but approaches the value of $\alpha$ at this zero in the beta function, which is thus denoted $\alpha_{IR}$. Since $\beta = 0$ at $\alpha = \alpha_{IR}$, the resultant theory in this IR limit is scale-invariant, and generically also conformally invariant\[\]. A fundamental question concerns the properties of the interacting theory at such an IR fixed point (IRFP) of the renormalization group. There is convincing evidence that if $\alpha_{IR}$ is small enough, then the IR theory is in a (deconfined) non-Abelian Coulomb phase (NACP), also called the conformal window\[\]. In terms of $N_f$, this phase occurs if $N_f$ is in the interval $N_{f,cr} < N_f < N_u$, where $N_u$ and $N_{f,cr}$ depend on $G$ and $R$. Here, $N_{f,cr}$ denotes the value of $N_f$ below which the running $\alpha(\mu)$ becomes large enough to cause spontaneous chiral symmetry breaking and dynamical fermion mass generation.

Physical quantities in the IR-limit theory at $\alpha_{IR}$ cannot depend on the scheme used for the regularization and subtraction procedure in renormalization. In conventional computations of these quantities, first, one expresses them as series expansions in powers of $\alpha$, calculated to $n$-loop order; second, one computes the IR zero of the beta function at the $n$-loop $(n\ell)$ level, denoted $\alpha_{IR,n\ell}$; and third, one sets $\alpha = \alpha_{IR,n\ell}$ in the series expansion for the given quantity to obtain its value at the IR zero of the beta function to this $n$-loop order. However, these conventional series expansions in powers of $\alpha$, calculated to a finite order, are scheme-dependent beyond the leading one or two terms. Specifically, the terms in the beta function are scheme-dependent at loop order $\ell$ $\geq$ 3 and the terms in an anomalous dimension are scheme-dependent at loop order $\ell$ $\geq$ 2\[\]. Indeed, as is well-known, the presence of scheme-dependence in higher-order perturbative calculations is a general property in quantum field theory.

It is therefore of great value to use a complementary approach in which one expresses these physical quantities at $\alpha_{IR}$ as an expansion in powers of a variable such that, at every order in this expansion, the result is scheme-independent. A very important property is that one can recast the expressions for physical quantities in a manner that is scheme-independent. A crucial point here is that, for a given gauge group $G$ and fermion representation $R$, as $N_f$ (formally generalized from non-negative integers to the real numbers) approaches the upper limit allowed
by asymptotic freedom, denoted \( N_u \) (given by Eq. \[2.1\] below), the resultant value of \( \alpha_{1R} \) approaches zero. This means that one can equivalently express a physical quantity in a scheme-independent manner as a series in powers of the variable

\[
\Delta_f = N_u - N_f = \frac{11C_A}{4T_f} - N_f,
\]

where \( C_A \) is the quadratic Casimir invariant for the adjoint representation, and \( T_f \) is the trace invariant for the fermion representation \( R \) \[6\]. Here, \( \alpha_{1R} \rightarrow 0 \rightleftharpoons \Delta_f \rightarrow 0 \). Hence, for \( N_f \) less than, but close to \( N_u \), this expansion variable \( \Delta_f \) is reasonably small, and one can envision reliable perturbative calculations of physical quantities at this IR fixed point in powers of \( \Delta_f \). Following the original calculations of the one- and two-loop coefficients of the beta function \[7\]-\[9\], some early work on this was reported in \[10\], \[11\].

In this paper we consider a vectorial, asymptotically free gauge theory and present scheme-independent calculations, for a general gauge group \( G \) and fermion representation \( R \), of two physical quantities in the IR theory at \( \alpha_{1R} \) of considerable importance, namely (i) the anomalous dimension, denoted \( \gamma_{\bar{\psi}\psi,IR} \), of the fermion bilinear \( \bar{\psi}\psi = \sum_{j=1}^{N_f} \bar{\psi}_j \psi_j \) to \( O(\Delta_f^2) \) and (ii) the derivative \( \beta'_{1R} = d\beta/da \) to \( O(\Delta_f^3) \), both evaluated at \( a = \alpha_{1R} \). These are the highest orders in powers of \( \Delta_f \) to which these quantities have been calculated. We give explicit expressions for these quantities in the special cases where \( G = SU(N_c) \) and the fermion representation \( R \) is the fundamental (\( F \)), adjoint (\( adj \)), and symmetric and antisymmetric rank-2 tensors, \( (S_2, A_2) \).

Our results extend our previous scheme-independent calculations of \( \gamma_{\bar{\psi}\psi,IR} \) to \( O(\Delta_f^j) \) in \[12\] and of the derivative \( \beta'_{1R} \) to \( O(\Delta_f^j) \) in \[13\] for general \( G \) and \( R \), and our scheme-independent calculation of \( \gamma_{\bar{\psi}\psi,IR} \) to \( O(\Delta_f^j) \) for \( G = SU(3) \) and \( R = F \) in \[14\] (see also \[15\]). A brief report on some of our results was given in \[16\].

Scheme-independent series expansions of \( \gamma_{\bar{\psi}\psi,IR} \) and \( \beta'_{1R} \) can be written as

\[
\gamma_{\bar{\psi}\psi,IR} = \sum_{j=1}^{\infty} \kappa_j \Delta_f^j
\]

and

\[
\beta'_{1R} = \sum_{j=1}^{\infty} d_j \Delta_f^j,
\]

where \( d_1 = 0 \) for all \( G \) and \( R \) \[12\], \[13\]. In general, the calculation of the coefficient \( \kappa_j \) in Eq. \[1.2\] requires, as inputs, the values of the \( b_{\ell} \) for \( 1 \leq \ell \leq j + 1 \) and the \( c_{\ell} \) for \( 1 \leq \ell \leq j \). The calculation of the coefficient \( d_j \) in Eq. \[1.3\] requires, as inputs the values of the \( b_{\ell} \) for \( 1 \leq \ell \leq j \). We refer the reader to \[12\] and \[13\] for discussions of the procedure for calculating the coefficients \( \kappa_j \) and \( d_j \). We denote the truncation of these series to maximal power \( j = p \) as \( \gamma_{\bar{\psi}\psi,IR}\Delta_f^p \) and \( \beta'_{1R}\Delta_f^p \), respectively. Where it is necessary for clarity, we will also indicate the fermion representation \( R \) in the subscript.

Our main new results here include the general expressions, for arbitrary gauge group \( G \) and fermion representation \( R \), for the coefficient, \( \kappa_4 \) in Eq. \[3.3\] below, and for the coefficient \( d_5 \), given in Eq. \[4.9\] below, as well as reductions of these formulas for special cases and, for \( R = F \), calculations in the LNN limit \[3.21\]. As will be discussed further below, the derivative \( \beta'_{1R} \) is equivalent to the anomalous dimension of the non-Abelian field strength squared, \( Tr(F_{\mu\nu}F^{\mu\nu}) \). Our present calculations make use of the newly computed five-loop coefficient in the beta function for this gauge theory for general \( G \) and \( R \) in \[17\], as our work in \[14\], \[15\] made use of the calculation of this five-loop coefficient for the case \( G = SU(3) \) and \( R = F \) in \[18\].

In addition to being of interest and value in their own right, our new scheme-independent calculations, performed to the highest order yet achieved, are useful in several ways. First, we will compare our results for \( \gamma_{\bar{\psi}\psi,IR} \) and \( \beta'_{1R} \) for various \( G \) and \( R \) with the values that we obtained at comparable order with the conventional \( n \)-loop approach in \[19\], \[21\]. Our new results have the merit of being scheme-independent at each order in \( \Delta_f \), in contrast to scheme-dependent series expansions of \( \gamma_{\bar{\psi}\psi,IR} \) and \( \beta'_{1R} \) in powers of the IR coupling. Second, there is, at present, an intensive program to study this IR behavior on the lattice \[22\]. Thus, it is of considerable interest to compare our scheme-independent results for \( \gamma_{\bar{\psi}\psi,IR} \) for various theories with values measured in lattice simulations of these theories. We have done this in \[13\], \[14\], \[16\] (as well as in our work on conventional \( n \)-loop calculations \[15\], \[19\]), and we will expand upon this comparison here. Third, we believe that our scheme-independent expansions for these physical quantities are of interest in the context of the great current resurgence of research activity on conformal field theories (CFT). Much of this current activity makes use of operator-product expansions and the associated bootstrap approach \[23\]. Our method of scheme-independent series expansions for physical quantities at an IR fixed point is complementary to this bootstrap approach in yielding information about a conformal field theory.

Our calculations rely on \( \alpha_{1R} \) being an exact zero of the beta function and thus an exact IR fixed point of the renormalization group, and this property holds in the non-Abelian Coulomb phase (conformal window). In this phase, the chiral symmetry associated with the massless fermions is preserved in the presence of the gauge interaction. However, there has also been interest in vectorial asymptotically free gauge theories that exhibit quasi-conformal behavior associated with an approximate IRFP in the phase with broken chiral symmetry, which could feature a substantial value of an effective \( \gamma_{\bar{\psi}\psi,IR} \sim O(1) \) \[24\]. Our scheme-independent calculations are also relevant to this area of research in two ways: (i) if \( N_f \lesssim N_{f,cr} \), then the effective values of quantities...
such as $\gamma_{\bar{\psi}\psi,IR}$ may be close to the values calculated via the $\Delta_f$ expansion from within the NACP; (ii) combining our calculations of $\gamma_{\bar{\psi}\psi,IR}$ with an upper bound on this anomalous dimension from conformal invariance and an assumption that this bound is saturated as $N_f \gg N_{f,cr}$ yields an estimate of the value of $N_{f,cr}$. This is useful, since the value of $N_{f,cr}$ for a given $G$ and $R$ is not known exactly at present and is the subject of current investigation, including lattice studies, as discussed further below.

Although most of our paper deals with new scheme-independent results for physical quantities, one of the outputs of our calculations is a new type of series expansion for a scheme-dependent quantity, namely $\alpha_{IR}$. The conventional procedure for calculating the IR zero of a beta function at the $n$-loop order, which we have applied in earlier work to four-loop order for arbitrary $G$ and $R$ [19–21] (see also [23]) is to examine the $n$-loop beta function, which has the form of $\alpha^2$ times a polynomial of degree $n - 1$ in $\alpha$, and then determine the $n$-loop value $\alpha_{IR,n\ell}$ as the (real, positive) root of this polynomial closest to the origin. However, in [12], we investigated the five-loop beta function for $G = SU(3)$ and $R = F$, as calculated in the standard MS scheme, and found that, over a substantial range of values of $N_f$ in the non-Abelian Coulomb phase, it does not have any positive real root. We were able to circumvent this problem in [12] by the use of Padé approximants, but nevertheless, it is a complication for this conventional approach to calculating $\alpha_{IR}$. The new calculation of $\alpha_{IR}$ as an expansion in powers of $\Delta_f$ up to $O(\Delta_f^4)$ for general $G$ and $R$ that we present here has the advantage that it always yields a physical value, in contrast to the situation with the $n$-loop beta function.

The paper is organized as follows. Some relevant background and methods are discussed in Section II. We present our calculation of $\kappa_4$ in the scheme-independent expansion of $\gamma_{\bar{\psi}\psi,IR}$ for general $G$ and $R$ in Section III together with evaluations for $G = SU(N_c)$ and $R = F, \text{adj}, S_2$, and $A_2$. These are compared with values from $n$-loop calculations and with lattice measurements. In this section we also present results for case $R = F$ in the limit $N_c \to \infty$, $N_f \to \infty$, with $N_f / N_c$ fixed, which we call the LNN limit. In Section IV we present our calculation of the coefficient $d_5$ in the scheme-independent expansion of $\beta^0_{IR}$ for general $G$ and $R$, with evaluations for the above-mentioned specific representations. Section V gives an analysis of the five-loop rescaled beta function in the LNN limit and a determination of the interval over which it exhibits a physical IR zero. Section VI is devoted to the calculation of the coefficients in an expansion of $\alpha_{IR}$ in powers of $\Delta_f$ up to $O(\Delta_f^4)$. Our conclusions are given in Section VII and some auxiliary formulas are listed in an appendix.

II. BACKGROUND AND METHODS

In this section we review some background and methods relevant for our calculations. The series expansion of $\beta$ in powers of $\alpha$ is

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \left( \frac{\alpha}{4\pi} \right)^\ell, \quad (2.1)$$

where $b_\ell$ is the $\ell$-loop coefficient. For a general operator $O$, we denote the full scaling dimension as $D_O$ and its free-field value as $D_{O,\text{free}}$. The anomalous dimension of this operator, denoted $\gamma_O$, is defined via the relation

$$D_O = D_{O,\text{free}} - \gamma_O. \quad (2.2)$$

An operator of particular interest is the (gauge-invariant) fermion bilinear, $\bar{\psi}\psi$. The expansion of the anomalous dimension of this operator, $\gamma_{\bar{\psi}\psi}$, in powers of $\alpha$ is

$$\gamma_{\bar{\psi}\psi} = \sum_{\ell=1}^{\infty} c_\ell \left( \frac{\alpha}{4\pi} \right)^\ell, \quad (2.3)$$

where $c_\ell$ is the $\ell$-loop coefficient. As noted above, the coefficients $b_1, b_2,$ and $c_3$ are scheme-independent, while the $b_\ell$ with $\ell \geq 3$ and the $c_\ell$ with $\ell \geq 2$ are scheme-dependent [3]. For a general gauge group $G$ and fermion representation $R$, the coefficients $b_1$ and $b_2$ were calculated in [7] and [8], and $b_3$ and $b_4$ were calculated in [27] and [28] (and checked in [20]) in the commonly used MS scheme. For $G = SU(3)$ and $R = F$, $b_5$ was calculated in [15] and recently, an impressive calculation of $b_5$ for general gauge group $G$ and fermion representation $R$ was presented in [17], again in the MS scheme. We also make use of the $c_\ell$ up to loop order $\ell = 4$, calculated in [31]. Although we use these coefficients as calculated in the MS scheme below, we emphasize that the main results of this paper are calculations of the quantities $\kappa_4$ and $d_5$ which, like all of the $\kappa_4$ and $d_5$, are scheme-independent. We denote the $n$-loop $\beta$, $\beta'$, and $\gamma_{\bar{\psi}\psi}$ as $\beta_{n\ell}, \beta'_{n\ell},$ and $\gamma_{\bar{\psi}\psi,n\ell}$. As discussed above, we denote the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$, and the corresponding evaluations of $\beta'_{n\ell}$ and $\gamma_{\bar{\psi}\psi,n\ell}$ at $\alpha_{IR,n\ell}$ as $\beta'_{IR,n\ell}$ and $\gamma_{\bar{\psi}\psi,IR,n\ell}$. The symbols $\alpha_{IR}, \gamma_{\bar{\psi}\psi,IR}$, and $\beta'_{IR}$ refer to the exact values of these quantities.

For a given $G$ and $R$, as $N_f$ increases, $b_1$ decreases through positive values and vanishes with sign reversal at $N_f = N_u$, with

$$N_u = \frac{11CA}{4T_f}, \quad (2.4)$$

where $C_A$ and $T_f$ are group invariants [4, 32]. Hence, the asymptotic freedom condition yields the upper bound $N_f < N_u$.

There is a range of $N_f < N_u$ where $b_2 < 0$, so the two-loop beta function has an IR zero, at the value

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}. \quad (2.5)$$
The $n$-loop beta function has a double UV zero at $\alpha = 0$ and $n - 1$ zeros away from the origin. Among the latter zeros of the beta function, the smallest (real, positive) zero, if there is such a zero, is the physical IR zero $\alpha_{IR,n\ell}$, of $\beta_{n\ell}$. As $N_f$ decreases below $N_u$, $\beta_{2\ell}$ passes through zero to positive values as $N_f$ decreases through

$$N_f = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}.$$  \hspace{1cm} (2.6)

Hence, with $N_f$ formally extended from nonnegative integers to nonnegative real numbers, $\beta_{2\ell}$ has an IR zero (IRZ) for $N_f$ in the interval

$$I_{IRZ}: \; N_\ell < N_f < N_u. \hspace{1cm} (2.7)$$

Thus, $N_\ell$ is the lower ($\ell$) end of this interval.

As $N_f$ decreases in this interval, $\alpha_{IR,2\ell}$ increases. Therefore, in order to investigate the IR zero of the beta function for $N_f$ toward the middle and lower part of $I_{IRZ}$ with reasonable accuracy, one requires higher-loop calculations. These were performed in [32, 33, 19–21, 23, 25] for $\alpha_{IR,n\ell}$ and for the anomalous dimension of the fermion bilinear operator (see also [36, 37]). Since the $b_\ell$ with $\ell \geq 3$ are scheme-dependent, it is necessary to determine the degree of sensitivity of the value obtained for $\alpha_{IR,n\ell}$ for $n \geq 3$ to the scheme used for the calculation. This was done in [38–41].

The nonanomalous global flavor symmetry of the theory is

$$G_f = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_V.$$  \hspace{1cm} (2.8)

This $G_f$ symmetry is preserved in the (deconfined) non-Abelian Coulomb phase. As in [12, 16], we focus on this phase in the present work, since both the expansion in a small $\alpha_{IR}$ and the scheme-independent expansion in powers of $\Delta_f$ start from the upper end of the interval $I_{IRZ}$ in this phase. In contrast, in the phase with confinement and spontaneous chiral symmetry breaking, the gauge interaction produces a bilinear fermion condensate, $\langle \bar{\psi} \psi \rangle$, and this breaks $G_f$ to $SU(N_f)_V \otimes U(1)_V$, where $SU(N_f)_V$ is the diagonal subgroup of $SU(N_f)_L \otimes SU(N_f)_R$.

We will consider the flavor-nonsinglet ($fns$) and flavor-singlet ($f$s) bilinear fermion operators $\sum_{j,k=1}^{N_f} \bar{\psi}_j(T_a)_{jk} \psi_k$ and $\sum_{j=1}^{N_f} \bar{\psi}_j \psi_j$, where here $T_a$ with $a = 1, \ldots, N_f^2 - 1$ is a generator of the global flavor group $SU(N_f)$. We will usually suppress the explicit flavor indices and thus write these operators as $\bar{\psi}T_a\psi$ and $\bar{\psi}\psi$. These have the same anomalous dimension (e.g., [12]), which we denote simply as the anomalous dimension for the flavor-singlet operator, $\gamma_{\bar{\psi}\psi}$. In vectorial gauge theories of the type considered here, these fermion bilinear operators are gauge-invariant, and hence the anomalous dimension $\gamma_{\bar{\psi}\psi}$ and its IR value, $\gamma_{\bar{\psi}\psi,IR}$, are physical. (In contrast, in a chiral gauge theory, fermion bilinears are generically not gauge-invariant, and hence neither are their anomalous dimensions.)

Since $\alpha_{IR}$ vanishes (linearly) with $\Delta_f$ as $\Delta_f \to 0$, we can express it as a series expansion in this variable, $\Delta_f$.

We thus write

$$\alpha_{IR} \equiv 4\pi a_{IR} = 4\pi \sum_{j=1}^{\infty} a_j \Delta_f^j.$$  \hspace{1cm} (2.9)

The calculation of the $a_j$ requires, as input, the $b_\ell$ with $\ell \leq j + 1$ [12, 13].

A basic question concerns the part of the interval $I_{IRZ}$ in which the series expansions for $\gamma_{\bar{\psi}\psi,IR}$ and $\beta_{IR}^j$ in Eqs. (2.2) and (2.3) are reliable. We analyzed this question in [12, 14, 16] and concluded that these expansions for $\gamma_{IR}$ and $\beta_{IR}^j$ should be reasonably reliable throughout much of the interval $I_{IRZ}$ and non-Abelian Coulomb phase. We will use our higher-order calculations in this paper to extend this analysis here. We recall that the properties of the theory change qualitatively as $N_f$ decreases through the value $N_{f,cr}$ and spontaneous chiral symmetry breaking occurs, with the fermions gaining dynamical masses. The (chirally symmetric) non-Abelian Coulomb phase with $N_{f,cr} < N_f < N_u$ is clearly qualitatively different from the confined phase with spontaneous chiral symmetry breaking at smaller $N_f$ below $N_{f,cr}$. Therefore, one does not, in general, expect the small-$\Delta_f$ series expansions to hold below $N_{f,cr}$. Estimating the range of applicability of this expansion is thus connected with estimating the value of $N_{f,cr}$. For general $G$ and $R$, as $N_f$ formally continued from the nonnegative integers to the nonnegative real numbers, decreases from the upper end of the interval $I_{IRZ}$ at $N_u$ to the lower end of this interval at $N_f = N_\ell$, $\Delta_f$ increases from 0 to the maximal value

$$(\Delta_f)_{max} = N_u - N_\ell = \frac{3CA(7CA + 11C_f)}{4T_f(5CA + 3C_f)} \quad \text{for } N_f \in I_{IRZ}. \hspace{1cm} (2.10)$$

Recall that for a function $f(z)$ that is analytic about $z = 0$ and has a Taylor series expansion

$$f(z) = \sum_{j=1}^{\infty} f_j z^j,$$  \hspace{1cm} (2.11)

the radius of convergence of this series, $z_c$, can be determined by the ratio test

$$z_c = \lim_{j \to \infty} \frac{|f_{j+1}|}{|f_j|}. \hspace{1cm} (2.12)$$

Of course, we cannot apply the full ratio test here, since we have only calculated the $\kappa_j$ and $d_j$ to finite order. However, we can get a rough measure of the range of applicability of the series expansions in $\Delta_f$ (and also $\Delta_t$ in the LNN limit [21] discussed below) by computing the ratios $\kappa_j/d_j$ and $d_j-1/d_j$ for the values of $j$ for which we have calculated these coefficients.

The series expansion (2.2) for $\gamma_{IR}$ starts at $\Delta_f = 0$, i.e., at the upper end of the non-Abelian Coulomb phase,
and extends downward through this phase. Given that the theory at $\alpha_{IR}$ in this phase is conformal, there is an upper bound from conformal invariance, namely

$$\gamma_{\tilde{\psi},IR} \leq 2.$$ \hfill (2.13)

We have used this in our earlier work [12, 16, 19] and we will apply it with our higher-order calculations here. As discussed in [19], in the phase with spontaneous chiral symmetry breaking (S$_{\chi}$SB), there is a similar upper bound, $\gamma_{\tilde{\psi},IR} < 2$. This follows from the requirement that if $m(k)$ is the momentum-dependent running dynamical mass generated in association with the S$_{\chi}$SB, then $\lim_{k \to \infty} m(k) = 0$ (see Eqs. (4.1)-(4.2) of [19]).

Thus, if the approximate calculation of the anomalous lower bound from conformal invariance, namely [44], we will apply it with our higher-order calculations here. We have used this in our earlier work [12–16, 19] and extends downward through this phase. Given that $\gamma_{\tilde{\psi},IR}$ decreases toward the lower part of $I_{IR}$, the ratio test already shows that higher-order terms in the $\Delta_f$ expansion are becoming increasingly non-negligible, so that the truncation of the infinite series [12] to maximal power $p = 4$ involves an increasingly great uncertainty, as does an extrapolation to $p = \infty$.

For some perspective, we note that in order to assess the accuracy of the $\Delta_f$ expansion, the coefficients $\kappa_j, SGT$ were calculated for $j = 1, 2$ in [12] and were found to be in perfect agreement with the corresponding Taylor series expansion of the exact expression (2.16). This check was carried to one higher order in [16] for the case $G = SU(N_c)$ and $R = F$ with a calculation of $\gamma_{IR,SGT,\Delta_f}$, and again, perfect agreement was found with the exact result. This agreement explicitly demonstrated the scheme independence of the $\kappa_j, SGT$, since the calculations were carried out using inputs computed in the $\overline{DR}$ scheme, while (2.16) was derived in the NSVZ scheme [15]. Furthermore, as a consequence of electric-magnetic duality [16], as $N_f \searrow N_{f,SGT}$ in the non-Abelian Coulomb phase, the physics is described by a magnetic theory with coupling strength going to zero, or equivalently, by an electric theory with divergent $\alpha_{IR}$. Therefore, this perfect agreement, order-by-order, between the $\kappa_j, SGT$ and the expansion of the exact expression (2.16) for $\gamma_{IR,SGT}$ in powers of $\Delta_f$, showed that the $\Delta_f$ expansion in this supersymmetric gauge theory is able to treat situations with strong, as well as weak, coupling. This could not be done with conventional perturbative series expansions in powers of $\alpha$ [30, 37].

III. CALCULATION OF $\gamma_{\tilde{\psi},IR}$ TO $O(\Delta_f^3)$

A. General $G$ and $R$

The coefficients $\kappa_j$ in the scheme-independent expansion of $\gamma_{\tilde{\psi},IR}$ in powers of $\Delta_f$, Eq. (1.2), contain important information about the theory. For a general asymptotically free vectorial gauge theory with gauge group $G$ and $N_f$ massless fermions in a representation $R$, the coefficients $\kappa_j$ were given in [12] up to order $j = 3$, yielding the expansion of $\gamma_{\tilde{\psi},IR}$ to order $\Delta_f^3$. It is convenient to define

$$D = 7C_A + 11C_f,$$ \hfill (3.1)

since this factor occurs repeatedly in denominators of various expressions. For reference, we list the $\kappa_j$ for $1 \leq j \leq 3$ below:

$$\kappa_1 = \frac{8C_fT_f}{CA_D}.$$ \hfill (3.2)
\[
\kappa_2 = \frac{4C_f T_f^2 (5C_A + 88C_f)(7C_A + 4C_f)}{3C_A^2 D^3}, \quad (3.3)
\]

and

\[
\kappa_3 = \frac{4C_f T_f}{3C_A^2 D^3} \left[ 3C_A T_f^2 \left( -18473 C_A^4 + 144004 C_A^3 C_f^2 + 650896 C_A^2 C_f^3 + 356928 C_A C_f^4 + 569184 C_f^5 \right) - 2560 T_f^2 D \left( \frac{d_R^{abcd} d_R^{abcd}}{d_A} \right) + 45056 C_A T_f D \left( \frac{d_R^{abcd} d_R^{abcd}}{d_A} \right) - 170368 C_A^2 T_f D \left( \frac{d_R^{abcd} d_R^{abcd}}{d_A} \right) + 33 \cdot 2^{10} D \left( 2 T_f^2 \frac{d_R^{abcd} d_R^{abcd}}{d_A} - 13 C_A T_f \frac{d_R^{abcd} d_R^{abcd}}{d_A} + 11 C_A^2 \frac{d_R^{abcd} d_R^{abcd}}{d_A} \right) \right] \zeta_3 \quad (3.4)
\]

Here, \( \zeta_3 = \sum_{n=1}^{\infty} \frac{n^{-s}}{n} \) is the Riemann zeta function, the quantities \( C_A, C_f, \) and \( T_f \) are group invariants, the contractions \( d_R^{abcd} d_R^{abcd}, d_R^{abcd} d_R^{abcd}, d_R^{abcd} d_R^{abcd} \) are additional group-theoretic quantities given in [28], and \( d_A \) is the dimension of the adjoint representation of \( G \). In [12,13], the expression for \( \kappa_3 \) was given with terms written in order of descending powers of \( C_A \). It is also useful to express this coefficient in an equivalent form that renders certain factors of \( D \) explicit and shows the simple factorization of terms multiplying \( \zeta_3 \), and we have done this in Eq. (3.4).

Our new result here for \( \kappa_4 \) for a general gauge group \( G \) and fermion representation \( R \) is

\[
\kappa_4 = \frac{T_f^2}{3 C_A^2 D^7} \left[ C_A C_f T_f^2 \left( 19515671 C_A^6 - 131455044 C_A^5 C_f + 1289299872 C_A^4 C_f^2 + 2660221312 C_A^3 C_f^3 \right) + 1058481072 C_A^2 C_f^4 + 6953709312 C_A C_f^5 + 1275715584 C_f^6 \right] + 2^{10} C_f T_f D \left( 5789 C_A^2 - 4168 C_A C_f - 6820 C_f^2 \right) \frac{d_R^{abcd} d_R^{abcd}}{d_A}\]

\[
+ 2^{14} \cdot 3 C_A T_f^2 \frac{d_R^{abcd} d_R^{abcd}}{d_R} + 2^{13} \cdot 3 C_A^2 T_f D^3 \frac{d_R^{abcd} d_R^{abcd}}{d_R} + 2^8 \frac{d_R^{abcd} d_R^{abcd}}{d_R} \right] \zeta_3 \quad (3.5)
\]

Here, \( d_R \) is the dimension of the fermion representation \( R \). As before, we have indicated the simple factors in the prefactor and, for sufficiently simple cases, also factorizations of numbers in numerator terms. We will follow the same format for indicating numerical factorizations below. We proceed to evaluate this general expression for the gauge group \( G = SU(N_c) \) and several specific fermion representations \( R \), namely the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor. As stated in the introduction, we will use the abbreviations \( F, \text{adj}, S_2, \) and \( A_2 \) to refer to these representations. It is also worthwhile to evaluate our general formulas for

other gauge groups and their representations, including orthogonal, symplectic, and exceptional groups. We will report these evaluations for other groups and their representations elsewhere. There has, indeed, been interest in conformal phases for theories with these other gauge groups [47].

The coefficients \( \kappa_1 \) and \( \kappa_2 \) are manifestly positive for all \( G \) and \( R \). For \( G = SU(N_c) \) with all physical \( N_c \), and for representations \( R = F, \text{adj}, S_2 \), we have found that \( \kappa_3 \) and \( \kappa_4 \) are also positive [12,16]. As one of the results in the present paper, we generalize this further to include \( R = A_2 \). That is, for all physical \( N_c \) and for all of these
representations, we find that $\kappa_j > 0$ for $j = 3, 4$ as well as the manifestly positive cases $j = 1, 2$. Thus, extending our previous discussion in \[12\]-\[16\], the property that, for all of these representations $R$, $\kappa_j > 0$ for $1 \leq j \leq 4$ and for all $N_c$, implies two important monotonicity results: (i) for these $R$, and with a fixed $p$ in the interval $1 \leq p \leq 4$, $\gamma_{\psi,IR,\Delta_f}$ is a monotonically increasing function of $\Delta_f$, i.e., it increases monotonically with decreasing $N_f$; and (ii) for these $R$, and with a fixed $N_f \in I_{IRZ}$, $\gamma_{\psi,IR,\Delta_f}$ is a monotonically increasing function of $p$ in the range $1 \leq p \leq 4$. In addition to the manifestly positive $\kappa_1$ and $\kappa_2$, a plausible conjecture is that, for these $R$, $\kappa_j > 0$ for all $j \geq 3$. Assuming that this conjecture is valid, then three consequences are that for these representations $R$, (iii) for fixed $N_f$, $\gamma_{\psi,IR,\Delta_f}$ is a monotonically increasing function of $p$ for all $p$; (iv) $\gamma_{\psi,IR,\Delta_f}$ is a monotonically increasing function of $\Delta_f$, i.e. it increases with decreasing $N_f$, for all $p$; and hence (v) (assuming that the infinite series \[1,2\] converges), the quantity $\gamma_{\psi,IR}$ defined by this infinite series, and equivalent to $\lim_{p \rightarrow \infty} \gamma_{\psi,IR,\Delta_f}$, is a monotonically increasing function of $\Delta_f$, i.e., it increases monotonically with decreasing $N_f$.

B. $\gamma_{\psi,IR,\Delta_f}$ for $G = SU(N_c)$ and $R = F$

An important special case is $G = SU(N_c)$ with $R$ being the fundamental representation. For this case, the general expression for the interval $I_{IRZ}$, Eq. \[2.7\], is

$$I_{IRZ} : \frac{34N_c^3}{13N_c^2 - 3} < N_f < \frac{11N_c}{2} \quad \text{for } R = F. \quad (3.6)$$

The factor $D$ in Eq. \[3.1\] has the explicit form

$$D = \frac{25N_c^2 - 11}{2N_c} \quad \text{for } R = \text{fund.} \quad (3.7)$$

The general results for $\kappa_p$ with $1 \leq p \leq 3$ in \[3.2\]-\[3.4\] from \[12\] take the following forms given in \[13\]:

$$\kappa_{1,F} = \frac{4(N_c^2 - 1)}{N_c(25N_c^2 - 11)} \quad (3.8)$$

$$\kappa_{2,F} = \frac{4(N_c^2 - 1)(9N_c^2 - 2)(49N_c^2 - 44)}{3N_c(25N_c^2 - 11)^3} \quad (3.9)$$

For $\kappa_{3,F}$, we have [11]

$$\kappa_{3,F} = \frac{8(N_c^2 - 1)}{3^3N_c^3(25N_c^2 - 11)^5} \left[ 274243N_c^8 - 455426N_c^6 - 114080N_c^4 + 47344N_c^2 + 35574 \right] - 4224N_c^2(4N_c^2 - 11)(25N_c^2 - 11)\zeta_3. \quad (3.10)$$

For $\kappa_{4,F}$, we have [16]

$$\kappa_{4,F} = \frac{4(N_c^2 - 1)}{34N_c^4(25N_c^2 - 11)^7} \left[ 263345440N_c^{12} - 673169750N_c^{10} + 256923326N_c^8 \right. \left. - 290027700N_c^6 + 557945201N_c^4 - 208345544N_c^2 + 6644352 \right] + 384(25N_c^2 - 11)\left[ 4400N_c^{10} - 123201N_c^8 + 480349N_c^6 - 486126N_c^4 + 84051N_c^2 + 1089 \right] \zeta_3 + 211200N_c^2(25N_c^2 - 11)^2(N_c^6 + 3N_c^4 - 16N_c^2 + 22)\zeta_5. \quad (3.11)$$

We have checked that when we substitute the value $N_c = 3$ in our expression for $\kappa_{4,F}$ in Eq. \[3.11\], the result agrees with our previous calculation of $\kappa_{4,F}$ for this case in Eq. (9) of Ref. \[14\].

The explicit numerical expressions for the scheme-independent series expansions of $\gamma_{\psi,IR}$ to order $\Delta_f^4$ for $R = F$ and $N_c = 2, 3, 4$ are as follows:
\[ \text{SU}(2) : \gamma_{\psi,IR,F,\Delta_f} = \Delta_f \left[ 0.067416 + \left( 0.73308 \times 10^{-2} \right) \Delta_f + \left( 0.60531 \times 10^{-3} \right) \Delta_f^2 + \left( 1.62662 \times 10^{-4} \right) \Delta_f^3 \right] \]

(3.12)

\[ \text{SU}(3) : \gamma_{\psi,IR,F,\Delta_f} = \Delta_f \left[ 0.049844 + \left( 0.37928 \times 10^{-2} \right) \Delta_f + \left( 0.23747 \times 10^{-3} \right) \Delta_f^2 + \left( 0.36789 \times 10^{-4} \right) \Delta_f^3 \right] \]

(3.13)

and

\[ \text{SU}(4) : \gamma_{\psi,IR,F,\Delta_f} = \Delta_f \left[ 0.038560 + \left( 0.22314 \times 10^{-2} \right) \Delta_f + \left( 0.11230 \times 10^{-3} \right) \Delta_f^2 + \left( 0.126505 \times 10^{-4} \right) \Delta_f^3 \right]. \]

(3.14)

In these equations,

\[ \Delta_f = \frac{11N_c}{2} - N_f \quad \text{for } R = F. \]

Plots of \( \gamma_{\psi,IR,F,\Delta_f} \) for \( N_c = 2 \) and \( N_c = 3 \) and \( 1 \leq p \leq 4 \) were given in [10]. These showed the two monotonicity properties mentioned above. For an extended comparison, we show the plots of \( \gamma_{\psi,IR,F,\Delta_f} \) for \( 2 \leq N_c \leq 4 \) and \( 1 \leq p \leq 4 \) in Figs. 1 and 2.

In Table 1 we list the values of \( \gamma_{\psi,IR,F,\Delta_f} \) for \( 1 \leq p \leq 4 \) for the \( \text{SU}(2), \text{SU}(3), \) and \( \text{SU}(4) \) theories, with \( N_f \) in the respective interval \( I_{IRZ} \) for each. For comparison, we also include the values of \( \gamma_{\psi,IR,n\ell} \) obtained with our earlier \( n \)-loop calculations in [19], using series expansions in powers of \( \alpha \) evaluated at \( \alpha = \alpha_{IR,\ell} \) for \( 1 \leq n \leq 4 \) with \( b_3 \) and \( b_4 \) and \( c_n \), \( 2 \leq n \leq 4 \) calculated in the \( \overline{\text{MS}} \) scheme. (See Table VI in [19] for a list of numerical values of values of \( \gamma_{\psi,IR,\ell} \).)

As discussed above, if, for a given \( N_c \) and \( N_f \), a calculated value of \( \gamma_{\psi,IR} \) violates the upper bound \( \gamma_{\psi,IR} \leq 2 \) in (2.13), this is unphysical (marked with a symbol “u” in Table 1) and indicates that the perturbative calculation is unreliable and hence not applicable for this \( N_f \). In the case of the \( n \)-loop values \( \gamma_{IR,\ell} \), if this occurs at the two-loop level, it also leads to caution concerning \( \gamma_{IR,\ell} \) for \( n = 3, 4 \), and this is similarly indicated with a “u”. The computations of \( \gamma_{IR,\ell} \) in [19, 25] made use of the \( b_n \) and \( c_n \) up to the \( n = 4 \) loop level, where the scheme-
dependent $b_3$, $b_4$, and $c_n$ with $2 \leq n \leq 4$ had been calculated in the widely used $\overline{\text{MS}}$ scheme [27, 29, 30] As we pointed out in [13], the five-loop beta function in the $\overline{\text{MS}}$ scheme does not exhibit a physical IR zero over a substantial lower part of $I_{IRZ}$. We discuss this further below. For compact notation, we will often leave the subscript $\psi$ implicit on these and other quantities and thus write $\gamma_{\psi,I,R,F,\Delta f}$ $\gamma_{\psi,I,R,F,\Delta f}^j$ (red), $\gamma_{\psi,I,R,F,\Delta f}^4$ (green), $\gamma_{\psi,I,R,F,\Delta f}^4$ (blue), and $\gamma_{\psi,I,R,F,\Delta f}^4$ (black).

Since the calculation of $\gamma_j$ and the resultant $\gamma_{IR,\Delta f}$ uses information from the $(j+1)$-loop beta function from [2.1] and the $j$-loop expansion of $\gamma_{\psi}$ in [2.3], it is natural to compare the (SI) $\gamma_{IR,\Delta f}$ with the (SD) $\gamma_{IR,p'\ell}$ for $p' = p$ and $p' = p + 1$. In the upper and middle part of the interval $I_{IRZ}$ for a given $N_c$, we find that $\gamma_{IR,\Delta f}^j$ is slightly larger than $\gamma_{IR,\Delta f}$, with the difference increasing as $N_f$ decreases below $N_c$, i.e., as $\Delta f$ increases.

It is important to assess the range of applicability and reliability of these results from the $\Delta f$ expansion. We did this in [12, 14] and extend our analysis here, using our new result for $\kappa_4$. Following our discussion above on the ratio test for the determination of the radius of convergence of a Taylor series, the ratios of successive coefficients, $\kappa_{j-1}/\kappa_j$, give an approximate measure of the range of applicability of the $\Delta f$ expansion for $\gamma_{IR}$.

For a given $G$ and $R$, this range may be compared with the maximum size of $\Delta f$ in the interval $I_{IRZ}$ where the scheme-independent two-loop beta function $\beta_2$ has an IR zero. For the present case of $G = SU(N_c)$ and $R = F$, the general formula (2.10) takes the form

$$R = F : \quad (\Delta f)_{\text{max}} = \frac{3N_c(25N_c^2 - 11)}{2(13N_c^2 - 3)}. \quad (3.16)$$

This has the respective values

$$(\Delta f)_{\text{max}} = 5.45, 8.45, 11.39 \quad \text{for} \quad N_c = 2, 3, 4. \quad (3.17)$$

We begin by reviewing the SU(3) theory, for which

SU(3) : \quad \frac{\kappa_{F,1}}{\kappa_{F,2}} = 13.14, \quad \frac{\kappa_{F,2}}{\kappa_{F,3}} = 15.97, \quad \frac{\kappa_{F,3}}{\kappa_{F,4}} = 6.455. \quad (3.18)

As discussed in [12, 14], these results suggest that for the SU(3) theory with $R = F$, the $\Delta f$ expansion calculated to this order should be reasonably reliable over a substantial part, including the upper and middle portions, of the interval $I_{IRZ}$ and the non-Abelian Coulomb phase.

Using our new results, we now extend this analysis to the SU(2) and SU(4) theories (and will give a further analysis in the LNN limit of Eq. (3.21)). We find

SU(2) : \quad \frac{\kappa_{F,1}}{\kappa_{F,2}} = 9.20, \quad \frac{\kappa_{F,2}}{\kappa_{F,3}} = 12.11, \quad \frac{\kappa_{F,3}}{\kappa_{F,4}} = 3.72 \quad (3.19)

and

SU(4) : \quad \frac{\kappa_{F,1}}{\kappa_{F,2}} = 17.28, \quad \frac{\kappa_{F,2}}{\kappa_{F,3}} = 19.87, \quad \frac{\kappa_{F,3}}{\kappa_{F,4}} = 8.88. \quad (3.20)

Since $(\Delta f)_{\text{max}}$ has the respective values 5.45 and 11.39 for the SU(2) and SU(4) theories, we are led to the same conclusion for these theories that we reached for the SU(3) theory, namely that the $\Delta f$ expansion should be reasonably reliable over a substantial portion of the respective intervals $I_{IRZ}$.

As discussed above, another way to assess the range of applicability of the $\Delta f$ expansion is to check to see whether the resultant values of $\gamma_{IR,\Delta f}$ obey the upper bound $\gamma_{IR} \leq 2$ in (2.13). As is evident from Table I all of our values of $\gamma_{IR,\Delta f}$ listed there obey this bound. This again shows the advantages of the scheme-independent $\Delta f$ expansion as a way of calculating $\gamma_{IR}$ to a given order, as compared with the conventional $n$-loop calculation of $\gamma_{IR,n\ell}$. As is also evident from Table I for each of the cases listed there, namely $N_c = 2, 3, 4$, one finds unphysically large values of $\gamma_{IR,n\ell}$ for values of $N_f$ in the lower portions of the respective intervals $I_{IRZ}$.
become too large for the perturbative n-loop calculations of \( \gamma_{IR,\Delta f} \) to be applicable. In contrast, the \( \Delta_f \) expansion can be applied over a considerably greater portion of the interval \( I_{1/2} \) to yield results for \( \gamma_{IR,\Delta f} \) that obey the upper bound \( \Delta_f \). We will show this further below for the LNN limit \( \gamma_{IR} \). This also demonstrates that the \( \Delta_f \) expansion for \( \gamma_{IR} \) is able to be used in situations with substantially stronger IR coupling than is the case with the conventional expansion in powers of this coupling yielding the n-loop value \( \gamma_{IR,n} \).

We proceed to compare our values in Table I with lattice measurements. The SU(3) theory with \( R = F \) and \( N_f = 12 \) has been the subject of many lattice measurements. In [14], we compared our results for this theory with lattice measurements, so we only briefly review that discussion here. We recall that there is not, at present, a consensus among all lattice groups as to whether this theory is in an IR-conformal phase or is in a chirally broken phase \[22\]. There is a considerable spread of values of \( \gamma_{IR} \) in published papers, including the values (where uncertainties in the last digits are indicated in parentheses) \( \gamma_{IR} \sim 0.414(16) \) [48], \( \gamma_{IR} \simeq 0.35 \) [49], \( \gamma_{IR} \simeq 0.45 \), \( \gamma_{IR} = 0.27(3) \) [51], \( \gamma_{IR} \simeq 0.25 \) [52] (see also [53]), \( \gamma_{IR} = 0.235(46) \) [54], and \( 0.2 \lesssim \gamma_{IR} \lesssim 0.4 \) [55]. We refer the reader to [22] and [48]–[55] for discussions of estimates of overall uncertainties in these measurements. Our value \( \gamma_{IR,\Delta f} = 0.398 \) and our extrapolated value for \( \lim_{N_f \to \infty} \gamma_{IR,\Delta f} = \gamma_{IR} \), namely \( \gamma_{IR} = 0.40 \), are consistent with this range of lattice measurements and are somewhat higher than our five-loop value \( \gamma_{IR,5} = 0.255 \) from the conventional \( \alpha \) series that we obtained in [13]. It is hoped that further work by lattice groups will lead to a consensus concerning whether this theory is IR conformal or not and concerning the value of \( \gamma_{IR} \).

The SU(3) theory with \( N_f = 10 \) has been investigated on the lattice in [56], with the result \( \gamma_{IR} \sim 1 \). While our highest-order n-loop values, namely our four-loop result, \( \gamma_{IR,4} = 0.156 \) [19], and our five-loop result, \( \gamma_{IR,5} = 0.211 \) obtained using Padé methods [15], are smaller than this lattice value, our extrapolated scheme-independent value, \( \gamma_{IR} = 0.95 \pm 0.06 \) [14], is consistent with it.

There have also been a number of lattice studies of the SU(3) theory with \( N_f = 8 \) [67]–[59], which have yielded the estimate \( \gamma_{IR} \simeq 1 \). As is evident from Fig. 4 if we were to continue the curve for \( \gamma_{IR,\Delta f} \) plotted there downward further to \( N_f = 8 \), the resultant value would be compatible with \( \gamma_{IR} \sim 1 \). We note that this theory may well be in the chirally broken phase, and there is not yet a clear consensus as to whether it is in this phase or possibly near the lower end of the IR-conformal non-Abelian Coulomb phase. In this context, one may recall that if, for a given \( G \) and \( N_{f,c}\), \( N_f < N_{f,c} \), so that there is spontaneous chiral symmetry breaking, then the IR zero of the beta function is only approximate, since the theory flows away from this value as the fermions gain dynamical mass and are integrated out, leaving a pure gluonic low-energy effective field theory. For such a theory, the quantity extracted from either continuum or lattice analyses as \( \gamma_{IR} \) is only an effective anomalous dimension that describes the renormalization-group behavior as the theory is flowing near to the approximate zero of the beta function. A general comment is that the determination of \( N_{f,c} \) relies upon effective methods to analyze the lattice data [22]; progress on this continues [45]–[61].

Theories with an SU(2) gauge group and \( N_f = 8 \) have been of interest in the context of certain ideas for physics beyond the Standard Model (SM) [62], in which the number of Dirac fermions is \( N_f = N_{w,k}(N_c + 1) = 8 \), where \( N_{w,k} = 2 \), corresponding to the SU(2) factor group in the SM and \( N_c = 3 \) colors. There have been several lattice studies of this SU(2) theory with \( N_f = 8 \), including [22, 63–64]. These are consistent with this theory being IR-conformal, and the recent study [61] has reported the measurement \( \gamma_{IR} = 0.15 \pm 0.02 \). For comparison, as listed in Table I, our previous higher n-loop values were \( \gamma_{IR,3} = 0.272 \) and \( \gamma_{IR,4} = 0.204 \) [19], and our current highest-order scheme-independent value is \( \gamma_{IR,\Delta f} = 0.298 \). These are somewhat higher than this lattice result.

There have also been a number of lattice studies of the SU(2) theory with \( N_f = 6 \) [22, 65–67]. From this work, it is not yet clear if this theory is IR-conformal or chirally broken. Ref. [66] obtained the range \( 0.26 \leq \gamma_{IR} < 0.74 \), while Ref. [67] found \( \gamma_{IR} \simeq 0.275 \). Our higher-order scheme-independent values, as listed in Table I in particular, \( \gamma_{IR,\Delta f} = 0.698 \), are in agreement with the range given in [66] and are somewhat higher than the value from [67].

### C. LNN Limit for \( G = SU(N_c) \) and \( R = F \)

For \( G = SU(N_c) \) and \( R = F \), it is of interest to consider the limit

\[
LNN : \quad N_c \to \infty , \quad N_f \to \infty
\]

with \( r \equiv \frac{N_f}{N_c} \) fixed and finite

and \( \xi(\mu) \equiv \alpha(\mu)N_c \) is a finite function of \( \mu \).

(3.21)

We will use the symbol \( \lim_{LNN} \) for this limit, where “LNN” stands for “large \( N_c \) and \( N_f \)” with the constraints in Eq. (3.21) imposed. This is also called the ’t Hooft-Veneziano limit. Anticipating our later discussion of theories with fermions in two-index representations (adjoint and symmetric and antisymmetric rank-2 tensor), we will use the symbol \( \lim_{LN} \), where “LN” stands for “large \( N_c \)”, to denote the original ’t Hooft limit

\[
L : \quad N_c \to \infty
\]

with \( \xi(\mu) \equiv \alpha(\mu)N_c \) a finite function of \( \mu \).
and $N_f$ fixed and finite.

Continuing our discussion of the LNN limit, as relevant to theories with fermions in the fundamental representation, we define the following quantities in this limit:

$$ \xi = 4\pi x = \lim_{L \to \infty} \frac{N_u}{N_c} , $$  \hspace{1cm} (3.23)

$$ r_u = \lim_{L \to \infty} \frac{N_u}{N_c} , $$  \hspace{1cm} (3.24)

and

$$ r_t = \lim_{L \to \infty} \frac{N_t}{N_c} , $$  \hspace{1cm} (3.25)

with values

$$ r_t = \frac{11}{2} = 5.5 $$  \hspace{1cm} (3.26)

and

$$ r_t = \frac{34}{13} = 2.615 . $$  \hspace{1cm} (3.27)

(to the indicated floating-point accuracy). With $I_{IRZ} : N_c < N_f < N_u$, it follows that the corresponding interval in the ratio $r$ is

$$ I_{IRZ,r} : \frac{34}{13} < r < \frac{11}{2} , \text{ i.e., } 2.615 < r < 5.5 $$  \hspace{1cm} (3.28)

The critical value of $r$ such that for $r > r_{cr}$, the LNN theory is IR-conformal and for $r < r_{cr}$, it exhibits spontaneous chiral symmetry breaking is denoted $r_{cr}$ and is defined as

$$ r_{cr} = \lim_{L \to \infty} \frac{N_{f,cr}}{N_c} . $$  \hspace{1cm} (3.29)

We define the scaled scheme-independent expansion parameter for the LNN limit

$$ \Delta_r \equiv \Delta_f = r_u - r = \frac{11}{2} - r . $$  \hspace{1cm} (3.30)

As $r$ decreases from $r_u$ to $r_t$ in the interval $I_{IRZ,r}$, $\Delta_r$ increases from 0 to a maximal value

$$ (\Delta_r)_{\text{max}} = r_u - r_t = \frac{75}{26} = 2.8846 \text{ for } r \in I_{IRZ,r} . $$  \hspace{1cm} (3.31)

We define rescaled coefficients $\hat{\kappa}_{j,F}$

$$ \hat{\kappa}_{j,F} \equiv \lim_{N_c \to \infty} N_c^j \kappa_{j,F} $$  \hspace{1cm} (3.32)

that are finite in this LNN limit. The anomalous dimension $\gamma_{IR}$ is also finite in this limit and is given by

$$ R = F : \lim_{L \to \infty} \frac{\gamma_{IR}}{\gamma_{IR,F}} = \Delta_r \left[ 0.160000 + 0.037632 \Delta_r \right. $$

$$ + \left. 0.0083207 \Delta_x^2 + 0.003649 \Delta_x^3 \right] . $$  \hspace{1cm} (3.38)

Using these results for $\gamma_{IR,F,\Delta^2}$ with $1 \leq p \leq 4$ for $R = F$ in the LNN limit, we can now carry out a polynomial extrapolation to $p = \infty$. To do this, we fit an expression for $\gamma_{IR,F,\Delta^2}$ with some subset of the $p$ terms to a polynomial in $1/p$. We denote the resultant value generically as $\gamma_{IR,F,s}$, where here $s$ denotes the subset of the $p$ terms used for the extrapolation. We shall use, as a necessary condition for $\gamma_{IR,F,s}$ to be reliable, the requirement that it not differ too much from the highest-order value, $\gamma_{IR,F,\Delta^2}$. Quantitatively, we require that for the given subset $s$, $\gamma_{IR,F,s}/\gamma_{IR,F,\Delta^2} < 1.5$. We find that this condition is satisfied if $r \in I_{IRZ,r}$ is $r \gtrsim 3.5$, but that it is not satisfied as $r$ decreases below this value toward the lower end of the interval $I_{IRZ,r}$ at $r_t = 2.615$. As an example, at $r = 4.0$, depending on the subset of terms used for the extrapolation, we obtain $\gamma_{IR,F,s}/\gamma_{IR,F,\Delta^2} \approx 1.2$, while at $r = 3.6$, this ratio increases to $\approx 1.4$. We remark that the value $r = 4.0$ corresponds to $N_f = 12$ for the SU(3) theory and $N_f = 8$ for the SU(2) theory.

Previously, in [14] we performed this analysis for the special case $G = SU(3)$ and $R = F$ and, for that work,
we studied how the extrapolated value depends on the subset of terms that one includes for the fit. We perform the corresponding analysis here for this LNN case. We study three sets of terms:

\[ \text{set}_{34} : \{ \gamma_{IR,F,\Delta^2}, \gamma_{IR,F,\Delta^4} \} \quad (3.39) \]

\[ \text{set}_{234} : \{ \gamma_{IR,F,\Delta^2}, \gamma_{IR,F,\Delta^4}, \gamma_{IR,F,\Delta^8} \} \quad (3.40) \]

\[ \text{set}_{1234} : \{ \gamma_{IR,F,\Delta^2}, \gamma_{IR,F,\Delta^4}, \gamma_{IR,F,\Delta^8}, \gamma_{IR,F,\Delta^10} \} \quad (3.41) \]

There are countervailing advantages of these sets of terms. The two-term set \[ \text{set}_{34} \] has the advantage of using the two highest-order terms, while the three-term and four-term sets have the advantage of using progressively more terms in the fit. The fits to the sets \[ \text{set}_{34}, \text{set}_{234} \] yield polynomials in the variable \( p^{-1} \) of the respective forms

\[ \text{set}_{34} \Rightarrow \gamma_{IR,F,ex_{34}} = s_{34,0} + s_{34,1}p^{-1} \quad (3.42) \]

\[ \text{set}_{234} \Rightarrow \gamma_{IR,F,ex_{234}} = s_{234,0} + s_{234,1}p^{-1} + s_{234,2}p^{-2} \quad (3.43) \]

and

\[ \text{set}_{1234} \Rightarrow \gamma_{IR,F,ex_{1234}} = s_{1234,0} + s_{1234,1}p^{-1} + s_{1234,2}p^{-2} + s_{1234,3}p^{-3}. \quad (3.44) \]

The extrapolated values in the limit \( p \to \infty \) given by these fits are, respectively, as

\[ \lim_{p \to \infty} \gamma_{IR,F,ex_{34}} = s_{34,0} = \gamma_{IR,F,ex_{34}} \quad (3.45) \]

\[ \lim_{p \to \infty} \gamma_{IR,F,ex_{234}} = s_{234,0} = \gamma_{IR,F,ex_{234}} \quad (3.46) \]

and

\[ \lim_{p \to \infty} \gamma_{IR,F,ex_{1234}} = s_{1234,0} = \gamma_{IR,F,ex_{1234}} \quad (3.47) \]

We have calculated these quantities analytically. Below, we list the corresponding expressions with coefficients given to the indicated floating-point precision:

\[ \gamma_{IR,F,ex_{34}} = 16.758754 - 11.042531r + 2.8240528r^2 - 0.32942724r^3 + 0.014595750r^4 \quad (3.48) \]

\[ \gamma_{IR,F,ex_{234}} = 27.346053 - 19.2457889r + 5.1985972r^2 - 0.63389228r^3 + 0.0291915006r^4 \quad (3.49) \]

and

\[ \gamma_{IR,F,ex_{1234}} = 33.901799 - 24.4060664r + 6.71925275r^2 - 0.832708600r^3 + 0.038922001r^4. \quad (3.50) \]

Note that there are strong cancellations between individual terms for relevant values of \( r \in I_{IRZ,r} \). Some examples will show the range of resultant values of extrapolations for these different choices of sets of terms used in the fits. As anticipated, for values of \( r \) in the upper part of the interval \( I_{IRZ,r} \), all of the different types of extrapolation give quite similar results. For example,

\[ r = 5.0 \Rightarrow \gamma_{IR,F,ex_{34}} = 0.0914, \quad \gamma_{IR,F,ex_{234}} = 0.0902, \quad \gamma_{IR,F,ex_{1234}} = 0.0905. \quad (3.51) \]

As \( r \) decreases in the interval \( I_{IRZ,r} \), the differences between the extrapolations using the different sets of terms increase slightly, e.g., for a value roughly in the middle of this interval, namely \( r = 4.0 \), we find

\[ r = 4.0 \Rightarrow \gamma_{IR,F,ex_{34}} = 0.427, \quad \gamma_{IR,F,ex_{234}} = 0.444, \quad \gamma_{IR,F,ex_{1234}} = 0.456. \quad (3.52) \]

Toward the lower part of the interval \( I_{IRZ,r} \), these differences increase further, but also, as discussed above, for a given \( r \), all of the different types of extrapolations involve greater uncertainties, since each of the extrapolated values differs more from the value of highest-order explicitly calculated quantity, \( \gamma_{IR,F,\Delta^4} \). For example, for \( r = 3.0 \),

\[ r = 3.0 \Rightarrow \gamma_{IR,F,ex_{34}} = 1.335, \quad \gamma_{IR,F,ex_{234}} = 1.645, \quad \gamma_{IR,F,ex_{1234}} = 1.826. \quad (3.53) \]

The ratios of these values divided by the highest-order explicitly calculated value, \( \gamma_{IR,F,\Delta^4} \), are

\[ r = 3.0 \Rightarrow \frac{\gamma_{IR,F,ex_{34}}}{\gamma_{IR,F,\Delta^4}} = 1.47, \quad \frac{\gamma_{IR,F,ex_{234}}}{\gamma_{IR,F,\Delta^4}} = 1.82, \quad \frac{\gamma_{IR,F,ex_{1234}}}{\gamma_{IR,F,\Delta^4}} = 2.01. \quad (3.54) \]

Given our fiducial requirement that the ratio of the extrapolated value for \( p \to \infty \) divided by the highest-order explicitly calculated value, should not be greater than 1.5 for the extrapolation to be considered reasonably reliable, it follows that we would not consider the latter two extrapolations in Eq. \[ 3.53 \] to be sufficiently reliable to meet this requirement.

It is interesting to compare these scheme-independent calculations of \( \gamma_{IR,F,\Delta^p} \) to order \( 1 \leq p \leq 4 \) with the results from the conventional \( n \)-loop calculations as truncated expansions in \( \alpha_{IR,F,n\ell} \), denoted \( \gamma_{IR,F,n\ell} \) from Table V of [21] up to \( n = 4 \) loop order. We list our scheme-independent values together with these \( n \)-loop values in Table [I]. For each value of \( r \), we also include the extrapolated value, \( \gamma_{IR,F,ex_{234}} \) for the \( p \to \infty \) limit, and the ratio \( \gamma_{IR,F,ex_{234}}/\gamma_{IR,F,\Delta^4} \). We do not include the results from the \( n = 5 \) loop conventional calculation, because of
As we did for specific SU($N_c$) theories above, here we proceed to investigate the range of applicability of the scheme-independent series expansion for $\gamma_{IR}$ in the LNN limit. As is evident from Table II all of our values of $\gamma_{IR,F,ex234}$ for $1 \leq p \leq 4$ satisfy the bound $\gamma_{IR} \leq 2$. This is also true for all of our extrapolated values, $\gamma_{IR,F,ex234}$, except for the lowest value of $r$ listed, namely $r = 2.8$, for which $\gamma_{IR,F,ex234} = 2.09$, slightly above this bound. Thus, these results in the LNN limit again demonstrate the advantage of the scheme-independent expansions, since they enable us to calculate self-consistent values of $\gamma_{IR,F,ℓ}$ over a greater range of the interval $I_{IR}$, than is the case with the conventional $n$-loop calculations. To show the latter in detail, we have explicitly listed the values of $\gamma_{IR,F,3ℓ}$ and $\gamma_{IR,F,4ℓ}$ for values of $r$ where $\gamma_{IR,F,2ℓ}$ was unphysically large.

To investigate the range of applicability of the scheme-independent expansions further, it is worthwhile to obtain an estimate of this range from ratios of successive coefficients. From the coefficients $\hat{κ}_{j,F}$ that we have calculated with $1 \leq n \leq 3$, we compute the ratios

$$\frac{\hat{κ}_{1,F}}{\hat{κ}_{2,F}} = 4.252 \quad (3.55)$$

and

$$\frac{\hat{κ}_{2,F}}{\hat{κ}_{3,F}} = 4.523 \quad (3.56)$$

Recalling that the maximal value of $\Delta_c$ in the interval $I_{IR}$ is 2.885 (Eq. (8.31), these ratios are consistent with the inference that the small-$\Delta_c$ series expansion may be reasonably accurate throughout most of this interval $I_{IR}$.

D. $\gamma_{\bar{ψ}\bar{ψ},IR,Δ_{ℓ}^j}$ for $G = SU(N_c)$ and $R = adj$

Here we present our results for the $κ_{j}$ coefficients and thus $\gamma_{\bar{ψ}\bar{ψ},IR,Δ_{ℓ}^j}$ with $1 \leq j \leq 4$ for $G = SU(N_c)$ and $N_f$ fermions in the adjoint representation, $R = adj$. We will usually denote these as $κ_{j,adj}$ and $\gamma_{\bar{ψ}\bar{ψ},IR,adj,Δ_{ℓ}^j}$ but sometimes, when no confusion will result, we will omit this adj subscript for brevity of notation.

In this theory, Eqs. (2.6) and (2.6) yield, for the upper and lower ends of the interval $I_{IR}$, the values

$$N_{u,adj} = \frac{11}{4} = 2.75 \quad (3.57)$$

and

$$N_{ℓ,adj} = \frac{17}{16} = 1.0625 \quad (3.58)$$
so this interval includes only one integral value of $N_f$, namely $N_f = 2$. We note that since the adjoint representation is self-conjugate, a theory with $N_f$ Dirac fermions with $R = \text{adj}$ is equivalent to a theory with $N_{f,\text{maj}} = 2N_f$ Majorana fermions. Hence, here, one may also allow the half-integral values $N_f = 3/2, 5/2$ corresponding to $N_{f,\text{maj}} = 3, 5$. We have

$$R = \text{adj} : \quad \Delta_f = N_u - N_f = \frac{11}{4} - N_f . \quad (3.60)$$

For this case, the factor $D$ in Eq. (3.31) is simply $D = 18$. In [13] we gave the coefficients $\kappa_{j,\text{adj}}$ for $1 \leq n \leq 3$. These are as follows:

$$\kappa_{1,\text{adj}} = \left( \frac{2}{3} \right)^2 = 0.44444 , \quad (3.61)$$

$$\kappa_{2,\text{adj}} = \frac{341}{2 \cdot 3^6} = 0.23388 , \quad (3.62)$$

and

$$\kappa_{3,\text{adj}} = \frac{61873}{25 \cdot 3^{10}} - \frac{592}{38N_c^2} .$$

In Fig. [2] we show $\gamma_{\bar{\psi}\psi,IR,adj,\Delta^f_j}$ with $1 \leq p \leq 4$ for the SU(2) theory, as a function of $N_f$, formally generalized from the nonnegative integers to the real numbers. In Table III we list values of $\gamma_{\bar{\psi}\psi,IR,adj,\Delta^f_j}$ with $1 \leq p \leq 4$ for $N_f = 2$ and $N_c = 2$ and $N_c = 3$. For comparison, we also include our $n$-loop values $\gamma_{\bar{\psi}\psi,IR,adj,n\ell}$ calculated in the conventional manner via power series in the coupling (in the MS scheme), from Table VIII of [19].

Among SU($N_c$) theories with fermions in the adjoint representation, the SU(2) theory with $N_f = 2$ (Dirac) fermions has been of particular interest [62]. In the following, for notational brevity, the subscript $\text{adj}$ is understood implicitly. For this theory, as listed in Table III, we obtain the values $\gamma_{1R,\Delta^f_j} = 0.465, \gamma_{1R,\Delta^f_j} = 0.511$, and $\gamma_{1R,\Delta^f_j} = 0.556$, which are close to our earlier higher-loop calculations in [13], namely $\gamma_{1R,\Delta^f_j} = 0.543$ and $\gamma_{1R,\Delta^f_j} = 0.500$. It is of interest to compare these values with the results of lattice studies. There have been a number of such studies, and these are consistent with the conclusion that this theory is conformal in the infrared [70]-[72]. These studies have yielded a rather large range of measured values for $\gamma_{1R}$, including the following (where the published estimated uncertainties in the last digits are indicated in parentheses): $\gamma_{1R} = 0.49(13)$ [70], $\gamma_{1R} = 0.22(6)$ [71], $\gamma_{1R} = 0.31(6)$ [72], $\gamma_{1R} = 0.17(5)$ [73], $\gamma_{1R} = 0.37(2)$ [74], $\gamma_{1R} = 0.20(3)$ [75], and $\gamma_{1R} = 0.50(26)$ [76]. (See these references and 77 for additional discussion of estimates of overall uncertainties.) Our scheme-independent calculation of $\gamma_{1R}$ to $O(\Delta^f_j)$ and our earlier $n$-loop calculations of $\gamma_{1R,n\ell}$ up to $n = 4$ loops are clearly consistent with the larger among these lattice values. Before carrying out a comparison of our results with the full set of lattice values, it will be necessary to narrow the current wide range of lattice measurements.

It is of interest to investigate the $N_c \to \infty$ limit for an SU($N_c$) gauge theory with fermions in the adjoint representation. Since in this case, the upper and lower ends of the interval $I_{1RZ}$, given by $N_u = 11/4$ in Eq. (3.55) and $N_f = 17/16$ in Eq. (2.6) are independent of $N_c$, it follows that $\Delta_f$ is also independent of $N_c$. Hence, for $R = \text{adj}$,

$$\lim_{L \to \infty} \gamma_{1R} = \sum_{j=1}^{\infty} \hat{\kappa}_{j,\text{adj}} \Delta^f_j$$

(3.66)

where

$$\hat{\kappa}_{j,\text{adj}} = \lim_{L \to \infty} \kappa_{j,\text{adj}} . \quad (3.67)$$

The values of $\hat{\kappa}_{j,\text{adj}}$ are evident from the full expressions for $\kappa_{j,\text{adj}}$ that we have given above in Eqs. (3.61)-(3.64); for example, $\hat{\kappa}_{3,\text{adj}} = 61873/(2^3 \cdot 3^{10})$. 

$$\gamma_{1R} = 0.17(5) \quad \gamma_{1R} = 0.37(2) \quad \gamma_{1R} = 0.20(3) \quad \gamma_{1R} = 0.50(26) \quad \gamma_{1R} = 0.130978 - 0.090230N_c^{-2} , \quad (3.63)$$

where, as before, we indicate the simple factorizations of the denominators. The coefficient $\kappa_{4,\text{adj}}$ is

$$\kappa_{4,\text{adj}} = \frac{53389393}{2^9 \cdot 3^{14}} + \frac{368}{3^4} \zeta_3$$

$$+ \left( - \frac{2170}{3^{10}} + \frac{33952}{3^{11}} \zeta_3 \right) N_c^{-2}$$

$$= 0.0946976 + 0.193637N_c^{-2} . \quad (3.64)$$

The coefficients $\kappa_{1,adj}$ and $\kappa_{2,adj}$ are manifestly positive, and we find that for all physical $N_c$, the coefficients $\kappa_{3,adj}$ and $\kappa_{4,adj}$ are also positive. Although $\kappa_{1,adj}$ and $\kappa_{2,adj}$ are independent of $N_c$, the coefficients $\kappa_{j,adj}$ for $j = 3, 4$ do depend on $N_c$. We find that $\kappa_{3,adj}$ and $\kappa_{4,adj}$ are, respectively, monotonically increasing and monotonically decreasing functions of $N_c$. The $N_c \to \infty$ limits of $\kappa_{3,adj}$ and $\kappa_{4,adj}$ are given by the respective first terms in Eqs. (3.63) and (3.64).

Thus, to order $\Delta^f_j$, we have

$$\gamma_{\bar{\psi}\psi,IR,adj,\Delta^f_j} = \Delta_f \left[ 0.44444 + 0.23388 \Delta_f + (0.13098 - 0.090230N_c^{-2}) \Delta_f^2 + (0.094698 + 0.193634N_c^{-2}) \Delta_f^3 \right] . \quad (3.65)$$
E. $\gamma_{\bar{\psi},IR,\Delta j}^T$ for $G = SU(N_c)$ and $R = S_2, A_2$

Here we present our results for the $\kappa_j$ coefficients and thus $\gamma_{\bar{\psi},IR,\Delta j}^T$ with $1 \leq j \leq 4$ for $G = SU(N_c)$ and $N_f$ fermions in the symmetric and antisymmetric rank-2 tensor representations of $SU(N_c)$, $S_2$ and $A_2$. Since many formulas for these two cases are simply related to each other by sign reversals in certain terms, it is convenient to treat these cases together. As before 19, we shall use the symbol $T_2$ (rank-2 tensor) to refer to these cases together. (Do not confuse this use of $T$ with our use of the symbol $T$ in Section VII of Ref. 13 for the anomalous dimension of the operators $\bar{\psi}\sigma_{\mu\nu}\psi$ and operators $\bar{\psi}T^a\sigma_{\mu\nu}\psi$, where it referred to the antisymmetric Dirac tensor $\sigma_{\mu\nu} = (i/2)([\gamma_{\mu}, \gamma_{\nu}]_+)$.)

The values of $N_u$ and $N_\ell$ for $R = T_2$ are 19

$$N_u, T_2 = \frac{11N_c}{2(N_c + 2)} \tag{3.68}$$

and

$$N_\ell, T_2 = \frac{17N_c^3}{(N_c + 2)(8N_c^2 + 3N_c - 6)} \tag{3.69}$$

so that

$$R = T_2 : \quad \Delta_j = \frac{11N_c}{2(N_c + 2)} - N_f. \tag{3.70}$$

The factor $D$ in Eq. 32 takes the explicit form

$$R = T_2 : \quad D = \frac{18N_c^2 + 11N_c - 22}{N_c} = \frac{F_\pm}{N_c} \tag{3.71}$$

whence

$$F_\pm = 18N_c^2 + 11N_c - 22. \tag{3.72}$$

Both $F_+$ and $F_-$ are positive-definite for the physical range $N_c \geq 2$. At the lower end of the interval $I_{IRZ}$, $\Delta_j$ takes on the maximum value

$$R = T_2 : \quad (\Delta_j)_{max} = \frac{3N_c F_\pm}{2(N_c + 2)(8N_c^2 + 3N_c - 6)}. \tag{3.73}$$

If $N_c = 2$, then $S_2$ is the same as the adjoint representation, so we focus on $N_c \geq 3$ here. For this $R = S_2$ theory, the illustrative values $N_c = 3$ and $N_c = 4$ yield the respective intervals $I_{IRZ} = 1.22 < N_f < 3.30$ and $1.35 < N_f < 3.67$. Hence, the physical integral values of $N_f$ in these respective intervals $I_{IRZ}$ are $N_f = 2, 3$ for both $N_c = 3$ and $N_c = 4$. Furthermore, the $A_2$ representation is the singlet if $N_c = 2$ and is the same as the conjugate fundamental, $\bar{F}$ if $N_c = 3$, so in the case of $A_2$, we restrict to $N_c \geq 3$ and focus mainly on $N_c \geq 4$. In the SU(4) theory with $R = A_2$, the interval $I_{IRZ}$ is $4.945 < N_f < 11$, including the integral values $5 \leq N_f \leq 10$.

Here, using our general results 32-35, we give explicit expressions for the $\kappa_j$ with $1 \leq j \leq 4$ for the case $G = SU(N_c)$ and fermion representation $R = T_2$. From the general expressions for $\kappa_j$ with $1 \leq j \leq 4$, Eqs. (3.2)-3.5, we calculate the following. In each expression, the + and − signs refer to the $S_2$ and $A_2$ special cases of $T_2$, respectively:

$$\kappa_{1,T_2} = \frac{4(N_c + 1)(N_c + 2)^2}{N_c F_\pm} \tag{3.74}$$

$$\kappa_{2,T_2} = \frac{(N_c + 1)(N_c + 2)^3(11N_c^3 + 4N_c - 8)(93N_c^2 + 88N_c - 176)}{3N_c^2 F_\pm} \tag{3.75}$$

$$\kappa_{3,T_2} = \frac{(N_c + 1)(N_c + 2)^3}{2 \cdot 3^3 N_c^2 F_\pm^3} \left[ (1670571N_c^9 + 7671402N_c^8 + 2181584N_c^7 + 25294256N_c^6 - 13413856N_c^5 + 17539136N_c^4 + 16707328N_c^3 + 3046912N_c^2 - 27320832N_c \pm 18213888) \right] \tag{3.76}$$

$$\kappa_{4,T_2} = \frac{(N_c + 1)(N_c + 2)^3}{2 \cdot 3^3 N_c^2 F_\pm^3} \left[ (1670571N_c^9 + 7671402N_c^8 + 2181584N_c^7 + 25294256N_c^6 - 13413856N_c^5 + 17539136N_c^4 + 16707328N_c^3 + 3046912N_c^2 - 27320832N_c \pm 18213888) \right] \tag{3.76}$$

FIG. 5: Plot of $\gamma_{\bar{\psi},IR,adj,\Delta^p}$ for $G = SU(2)$ and $1 \leq p \leq 4$ as a function of $N_f \in I_{IRZ}$ for $R = adj$ and $N_f = 2$. From bottom to top, the curves (with colors online) refer to $\gamma_{IR,adj,\Delta f}$ (red), $\gamma_{IR,adj,\Delta^2}$ (green), $\gamma_{IR,adj,\Delta^3}$ (blue), and $\gamma_{IR,adj,\Delta^4}$ (black).
and
\[ \kappa_{4,T_2} = \frac{(N_c + 1)(N_c \pm 2)^4}{24 \cdot 34N_c^4 F^4_{2\pm}} \left( 4324540833N_{c}^{13} \pm 26924228982N_{c}^{12} + 30086550336N_{c}^{11} \mp 106026091536N_{c}^{10} \
- 224952825968N_{c}^{9} \mp 105492861344N_{c}^{8} + 60058305488N_{c}^{7} \pm 45792422016N_{c}^{6} - 1067559840512N_{c}^{5} \right) \
\pm 68261028352N_{c}^{4} \pm 982655860736N_{c}^{3} \mp 38568775424N_{c}^{2} - 136076328960N_{c} \pm 54430531584 \
+ 2^9 F_{2\pm} \left( 33534N_{c}^{11} \mp 702000N_{c}^{10} + 4448403N_{c}^{9} \mp 2216812N_{c}^{8} - 38600660N_{c}^{7} \pm 2239404N_{c}^{6} \right) \
+ 124680384N_{c}^{5} \mp 9055412N_{c}^{4} \pm 64551168N_{c}^{3} - 6690816N_{c} \pm 3345408 \right) \zeta_3 \
\pm 563200N_{c}^{2} (N_c \pm 2) F^2_{2\pm} \left( 15N_{c}^{5} \pm 158N_{c}^{4} + 240N_{c}^{3} \mp 912N_{c}^{2} - 1056N_{c} \pm 2112 \right) \zeta_5 \right). \] (3.77)

We comment on some factors in these \( \kappa_{j,T_2} \) expressions. The property that the \( \kappa_{j,A_2} \) coefficients contain an overall factor of \( (N_c - 2) \) (possibly raised to a power higher than 1), and hence vanish for \( N_c = 2 \), is a consequence of the fact that for \( N_c = 2 \), the \( A_2 \) representation is a singlet, so for \( SU(2) \), fermions in the \( A_2 \) singlet representation have no gauge interactions and hence no anomalous dimensions. Clearly, this property holds in general; i.e., the coefficients \( \kappa_{j,A_2} \) for all \( j \) contain an overall factor of \( (N_c - 2) \) (as well as possible additional factors of \( (N_c - 2) \)).

As noted above, if \( N_c = 2 \), then the \( S_2 \) representation is the same as the adjoint representation, so the coefficients must satisfy the equality \( \kappa_{j,S_2} = \kappa_{j,adj} \) for this \( SU(2) \) case, and we have checked that they do. Note that this equality requires (i) that the term proportional to \( \zeta_3 \) in \( \kappa_{3,S_2} \) must be absent if \( N_c = 2 \), since \( \kappa_{3,adj} \) does not contain any \( \zeta_1 \) term, and, indeed, this is accomplished by the factor \( (N_c - 2) \) multiplying the \( \zeta_3 \) term in \( \kappa_{3,S_2} \); and (ii) the term proportional to \( \zeta_5 \) in \( \kappa_{4,S_2} \) must be absent if \( N_c = 2 \), since \( \kappa_{4,adj} \) does not contain any \( \zeta_5 \) term, and this is accomplished by the factor \( (N_c - 2) \) multiplying this \( \zeta_5 \) term in \( \kappa_{4,S_2} \). Similarly, as we observed above, if \( N_c = 3 \), then the \( A_2 \) representation is the same as the conjugate fundamental representation, \( F \), so the coefficients must satisfy the equality \( \kappa_{j,A_2} = \kappa_{j,F} \) for this \( SU(3) \) case, and we have checked that they do.

The resultant \( \Delta_f \) expansions for \( \gamma_{\bar{\psi},IR,S_2,\Delta_f} \) with \( 2 \leq N_c \leq 4 \) are

\[
SU(2): \quad \gamma_{\bar{\psi},IR,S_2,\Delta_f} = \Delta_f \left[ 0.44444 + 0.23388 \Delta_f + 0.10842 \Delta_f^2 + 0.14311 \Delta_f^3 \right] \quad (3.78)
\]

\[
SU(3): \quad \gamma_{\bar{\psi},IR,S_2,\Delta_f} = \Delta_f \left[ 0.38536 + 0.17038 \Delta_f + 0.078062 \Delta_f^2 + 0.060081 \Delta_f^3 \right] \quad (3.79)
\]

and

\[
SU(4): \quad \gamma_{\bar{\psi},IR,S_2,\Delta_f} = \Delta_f \left[ 0.34839 + 0.13875 \Delta_f + 0.059680 \Delta_f^2 + 0.38102 \Delta_f^3 \right]. \quad (3.80)
\]

For \( R = A_2 \), we give illustrative results for the \( \Delta_f \) expansion of \( \gamma_{\bar{\psi},IR} \) for \( N_c = 4, 5 \):

\[
SU(4): \quad \gamma_{\bar{\psi},IR,A_2,\Delta_f} = \Delta_f \left[ 0.090090 + (1.1114 \times 10^{-2}) \Delta_f + (1.6013 \times 10^{-3}) \Delta_f^2 + (2.9668 \times 10^{-4}) \Delta_f^3 \right] \quad (3.81)
\]

and

\[
SU(5): \quad \gamma_{\bar{\psi},IR,A_2,\Delta_f} = \Delta_f \left[ 0.11582 + (1.7570 \times 10^{-2}) \Delta_f + (2.9243 \times 10^{-3}) \Delta_f^2 + (0.59791 \times 10^{-3}) \Delta_f^3 \right]. \quad (3.82)
\]

In Fig. 1 we present a plot of \( \gamma_{\bar{\psi},S_2,IR,\Delta_f} \) for \( G = SU(3), R = S_2 \), and \( 1 \leq p \leq 4 \), as a function of \( N_f \).
FIG. 6: Plot of $\gamma_{\psi,IR,S_2,\Delta_f}^{\mu}$ for $N_c = 3$ and $1 \leq p \leq 4$ as a function of $N_f$. Here, $S_2$ denotes the symmetric rank-2 tensor representation. From bottom to top, the curves (with colors online) refer to $\gamma_{\psi,IR,S_2,\Delta_f}^{\mu}$ (red), $\gamma_{\psi,IR,S_2,\Delta_f}^{2}$ (green), $\gamma_{\psi,IR,S_2,\Delta_f}^{3}$ (blue), and $\gamma_{\psi,IR,S_2,\Delta_f}^{4}$ (black).

We list values of the $\gamma_{IR,S_2,\Delta_f}^{\nu}$ with $1 \leq p \leq 4$ for the SU(3) and SU(4) theories with $R = S_2$ in Table IV. In both of these theories, the interval $I_{IRZ}$ includes the two integer values $N_f = 2, 3$. For comparison, we also include the values $\gamma_{IR,S_2,n\ell}$ for $2 \leq n \leq 4$ calculated via the conventional power series expansion to $n$-loop order and evaluated at $\alpha = \alpha_{IR,n\ell}$ from Table XI in our previous work [19]. As evident from this table, for a given $N_c$ and $N_f$, there is reasonable agreement between the $n = 4$ loop values $\gamma_{IR,S_2,\Delta_f}^{\nu}$ and $\gamma_{IR,S_2,4\ell}$. For example, for SU(3) and $N_f = 2$, $\gamma_{IR,S_2,4\ell} = 1.12$ while $\gamma_{IR,S_2,\Delta_3} = 1.13$.

We next compare our calculation of $\gamma_{\psi,IR,S_2,\Delta_f}^{\nu}$ to order $p = 4$ with lattice measurements. A theory of particular interest is the SU(3) gauge theory with $N_f = 2$ flavors of fermions in the $S_2$ representation, and lattice studies of this theory include [78] and [79] (see also [22]). As indicated in Table IV, our higher-order scheme-independent results are $\gamma_{IR,\Delta_3} = 0.960$, and $\gamma_{IR,\Delta_4} = 1.132$. By comparison, our $n$-loop results from [19] for this theory are $\gamma_{IR,3\ell} = 0.500$ and $\gamma_{IR,4\ell} = 0.470$. The lattice study [78] concluded that this theory is IR-conformal and obtained $\gamma_{IR} < 0.45$ [78], while Ref. [79] concluded that it is not IR-conformal and got an effective $\gamma_{IR} \sim 1$ [79]. One hopes that further work by lattice groups will lead to a consensus concerning whether this theory is IR conformal or not and concerning the value of $\gamma_{IR}$.

Regarding the range of applicability of the $\Delta_f$ expansion for these cases, we compute the following ratios of successive coefficients for the $G = SU(3)$, $R = S_2$ case:

$$\frac{\kappa_{1,S_2}}{\kappa_{2,S_2}} = 2.26176 \quad (3.83)$$

and

$$\frac{\kappa_{3,S_2}}{\kappa_{4,S_2}} = 1.2993 \quad (3.85)$$

The first two ratios, [80] and [81], are slightly larger than $(\Delta_f)_{max,S_2} = 519/250 = 2.076$ in $I_{IRZ}$ for this theory. However, the third ratio is about 40% less than this maximal value of $\Delta_f$. This suggests that because of slow convergence, one must use the $\Delta_f$ expansion with caution in the lower part of the interval $I_{IRZ}$ in this theory.

We list values of the $\gamma_{IR,A_2,\Delta_f}^{\nu}$ with $1 \leq p \leq 4$ for the SU(4) theory with $R = A_2$ and $N_f \in I_{IRZ}$ for this theory in Table V. Again, for comparison, we include the values $\gamma_{IR,A_2,n\ell}$ for $2 \leq n \leq 4$ calculated via the conventional power series expansion to $n$-loop order and evaluated at $\alpha = \alpha_{IR,n\ell}$ from Table XII in our previous work [19]. As expected, the agreement between the two methods of calculation is best at the upper end of the interval $I_{IRZ}$, where the IRFP occurs at weak coupling. For example, for $N_f = 9$, $\gamma_{IR,A_2,\Delta_3} = 0.242$, while $\gamma_{IR,A_2} = 0.232$.

It is of interest to consider the $N_c \to \infty$ (LN) limit of $\gamma_{IR,Z,R}$. For the SU(4) theory with $R = S_2$ and $A_2$, in this LN limit, the upper ends of the interval $I_{IRZ}$ for the $S_2$ and $A_2$ representations approach the same limit, and similarly for the lower ends:

$$\lim_{LN} N_{u,T_2} = \frac{11}{2} = 5.5 \quad (3.86)$$

$$\lim_{LN} N_{l,T_2} = \frac{17}{8} = 2.125 \quad (3.87)$$

Hence, in this $N_c \to \infty$ limit, the interval $I_{IRZ}$ is formally $2.125 < N_f < 5.5$, including the physical integer values $3 \leq N_f \leq 5$. Similarly, in this limit, the variable $\Delta_f$ is given by $\Delta_f = (11/2) - N_f$ and reaches a maximum value, at $N_f = N_{l,T_2}$, of

$$\lim_{LN}(\Delta_f)_{max,T_2} = \frac{27}{8} = 3.375 \quad (3.88)$$

This the $N_c \to \infty$ limit of (3.73).

As with the adjoint representation, we define

$$\tilde{\kappa}_{j,T_2} = \lim_{LN} \kappa_{j,T_2} \quad (3.89)$$

We find that

$$\tilde{\kappa}_{j,S_2} = \tilde{\kappa}_{j,A_2} \quad (3.90)$$

From our general expressions for $\kappa_{j,T_2}$ with $1 \leq j \leq 4$, we calculate

$$\tilde{\kappa}_{1,T_2} = \frac{2}{3^2} = 0.2222 \quad (3.91)$$
\[ \hat{k}_{1,2} = \frac{341}{23 \cdot 36} = 0.0584705 \]  \hspace{1cm} (3.92)

\[ \hat{k}_{3,2} = \frac{61873}{29 \cdot 310} = 0.016372 \]  \hspace{1cm} (3.93)

and

\[ \hat{k}_{4,2} = \frac{53389393}{211 \cdot 314} + \frac{2334}{310} = 0.59186 \times 10^{-2} \]  \hspace{1cm} (3.94)

Hence,

\[ \lim_{L \to \infty} \gamma_{IR,S_2,\Delta f} = \lim_{L \to \infty} \gamma_{IR,A_2,\Delta f} \]

and, in the limit \( p \to \infty \),

\[ \lim_{L \to \infty} \gamma_{IR,S_2} = \lim_{L \to \infty} \gamma_{IR,A_2} \]  \hspace{1cm} (3.96)

Thus, for both \( R = S_2 \) and \( R = A_2 \),

\[ \lim_{L \to \infty} \gamma_{\psi,IR,T_2,\Delta f} = \Delta_f \left[ 0.22222 + 0.0584705 \Delta_f \right] \]

\[ + 0.016372 \Delta_f^2 + 0.0059186 \Delta_f^3 \]  \hspace{1cm} (3.97)

We observe that for all of the cases we have calculated, namely \( 1 \leq j \leq 4 \),

\[ \hat{k}_{j,2} = 2^{-j} \hat{k}_{j,adj} \]  \hspace{1cm} (3.98)

One can understand this relation from the structure of the relevant group invariants, including the fact that the trace invariant \( T(R) \) satisfies

\[ \lim_{N_\psi \to \infty} \frac{T_2}{T_{adj}} = \frac{1}{2} \]  \hspace{1cm} (3.99)

We thus infer more generally that the relation holds for all \( j \). In Table VI we list the resultant common values of \( \gamma_{IR,T_2,\Delta f} \) for \( 1 \leq p \leq 4 \) and \( N_f \in I_{IRZ} \) in the LN limit. As noted above, in this LN limit, this interval consists of the integral values \( N_f = 3, 4, 5 \).

Concerning the range of applicability of the \( \Delta_f \) expansion in this LN limit, we compute the ratios

\[ \hat{k}_{1,2} = \frac{1296}{231} = 3.8006 \]  \hspace{1cm} (3.100)

\[ \hat{k}_{2,2} = \frac{220968}{61873} = 3.5713 \]  \hspace{1cm} (3.101)

and

\[ \hat{k}_{3,2} = \frac{160374816}{53389393 + 3815424\zeta_3} = 2.76624 \]  \hspace{1cm} (3.102)

The first two ratios, \( \frac{3.100}{3.101} \), are slightly greater than the maximum value \( \Delta_f \) for general \( G \) and \( R \), and to order \( \Delta_f \) for the special case \( G = SU(3) \) and fermion representation \( R = F \), the fundamental. Here we calculate the next higher-order coefficient, namely \( d_5 \), for general \( G \) and \( R \). For this purpose, we make use of the recent computation of the five-loop beta function coefficient, \( b_5 \), in [17]. The computation in [17] was performed in the \( \overline{MS} \) scheme, so that we can combine it with the scheme-independent \( b_1 \) and \( b_2 [2,9] \) and the results for \( b_3 \) and \( b_4 \) that have also been calculated in the \( \overline{MS} \) scheme [22, 23]. However, we again stress that since the \( d_n \) coefficients are scheme-independent, it does not matter which scheme one uses to calculate them. We first recall our previous results from Ref. [13]:

\[ d_1 = 0 \]  \hspace{1cm} (4.5)

\[ d_2 = \frac{2 \sqrt{2} T_f^2}{3 C_A D} \]  \hspace{1cm} (4.6)

\[ \boxed{\text{IV. CALCULATION OF } \beta_{IR} \text{ TO } O(\Delta_f^5)} \]

A. General \( G \) and \( R \)

The derivative \( \beta'_{IR} \) is an important physical quantity characterizing the conformal field theory at \( \alpha_{IR} \). We denote the gauge field of the theory as \( A_a^\mu \) (where \( a \) is a group index), the field strength-tensor as \( F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g c_{abc} A_\mu^b A_\nu^c \) (where \( c_{abc} \) is the structure constant of the Lie algebra of \( G \)) and the rescaled field-strength tensor as \( F_{\mu\nu,r} = g F_{\mu\nu} \), so that the gauge field kinetic term in the Lagrangian is \( \mathcal{L}_g = -\frac{1}{4} g^2 F_{\mu\nu} F_r^{\mu\nu} \). The trace anomaly states that the trace of the energy-momentum tensor \( T_{\mu\nu} \) satisfies the relation

\[ T_{\mu\nu} = \frac{\beta}{16 \pi^2} F_{\mu\nu,r} F_{\rho\sigma} \]  \hspace{1cm} (4.1)

Therefore, the full scaling dimension of the operator \( F_{r,\mu\nu} F_{r}^{\mu\nu} \), which we denote as \( D_{F_2} \), satisfies

\[ D_{F_2} = 4 + \beta' - \frac{2\beta}{\alpha} \]  \hspace{1cm} (4.2)

where we use the shorthand notation \( F_2 \equiv F_{r,\mu\nu} F_{r}^{\mu\nu} \). We denote the anomalous dimension of \( F_2 \), \( \gamma_{F_2} \) via the equation

\[ D_{F_2} = D_{F_2,free} - \gamma_{F_2} = -4 - \gamma_{F_2} \]  \hspace{1cm} (4.3)

and its evaluation at \( \alpha = \alpha_{IR} \) as \( \gamma_{F_2,IR} \). From Eq. (4.2), it follows that at a zero of the beta function away from the origin, in particular, at \( \alpha_{IR} \), the derivative \( \beta'_{IR} \) is equivalent to the anomalous dimension of the operator \( F_{r,\mu\nu} F_{r}^{\mu\nu} \):

\[ \beta'_{IR} = -\gamma_{F_2,IR} \]  \hspace{1cm} (4.4)

In [13] we calculated the expansion coefficients \( d_j \) of \( \beta'_{IR} \) in Eq. (4.4) to order \( \Delta_f^4 \) for general \( G \) and \( R \), and to order \( \Delta_f^5 \) for the special case \( G = SU(3) \) and fermion representation \( R = F \), the fundamental. Here we calculate the next higher-order coefficient, namely \( d_5 \), for general \( G \) and \( R \). For this purpose, we make use of the recent computation of the five-loop beta function coefficient, \( b_5 \), in [17]. The computation in [17] was performed in the \( \overline{MS} \) scheme, so that we can combine it with the scheme-independent \( b_1 \) and \( b_2 [2,9] \) and the results for \( b_3 \) and \( b_4 \) that have also been calculated in the \( \overline{MS} \) scheme [22, 23]. However, we again stress that since the \( d_n \) coefficients are scheme-independent, it does not matter which scheme one uses to calculate them. We first recall our previous results from Ref. [13]:

\[ d_1 = 0 \]  \hspace{1cm} (4.5)

\[ d_2 = \frac{2 \sqrt{2} T_f^2}{3 C_A D} \]  \hspace{1cm} (4.6)
\[ d_3 = \frac{27T^3(5C_A + 3C_f)}{3^4C_A^2D^2}, \]  
\[ \text{and} \]
\[ d_4 = -\frac{2^3T^3}{3^6C_A^2D^5}
\left[-3C_AT_f^2\left(137445C_A^4 + 103600C_A^3C_f + 72616C_A^2C_f^2 + 951808C_AC_f^3 - 63888C_f^4\right)
- 737283360C_A^2C_f^4 + 730646400C_A^5C_f - 356750592C_f^6\right)
- 2^9T^2D\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} - 2^{9T^2D}\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} - 340736C_A^2D\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_R}
+ 8448D\left[C_A^2T_f^2\left(21C_A^2 + 12C_AC_f - 33C_f^2\right) + 16T_f^2\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} - 104C_AT_f\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} + 88C_A^2\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A}\right] \zeta_3 \right]. \]

(4.8)

In Ref. \[13\] we presented the expression for \(d_4\) with terms written in order of descending powers of \(C_A\). It is also useful to express this coefficient \(d_4\) in an equivalent form that renders certain factors of \(D\) explicit and shows the simple factorization of terms multiplying \(\zeta_3\), and we have done this in Eq. (4.8).

Here we present our calculation of \(d_5\) for arbitrary \(G\) and \(R\):

\[ d_5 = \frac{2^3T^3}{3^6C_A^2D^5}
\left[-C_AT_f^2\left(39450145C_A^6 + 235108272C_A^5C_f + 1043817726C_A^4C_f^2 + 765293216C_A^3C_f^3\right)
- 737283360C_A^2C_f^4 + 730646400C_A^5C_f - 356750592C_f^6\right)
- 2^9T^2D\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} - 2^{9T^2D}\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} - 340736C_A^2D\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_R}
+ 8448D\left[C_A^2T_f^2\left(21C_A^2 + 12C_AC_f - 33C_f^2\right) + 16T_f^2\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} - 104C_AT_f\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A} + 88C_A^2\frac{d_{A\beta\lambda\delta}d_{A\beta\lambda\delta}}{d_A}\right] \zeta_3 \right]. \]

(4.9)

We proceed to evaluate these coefficients \(d_j\) up to \(j = 5\), and hence the derivative \(\beta^j_{TR}\) up to \(O(\Delta^j_f)\) below for \(G = SU(N_c)\) and several specific representations. The coefficients \(d_2\) and \(d_3\) are manifestly positive for arbitrary \(G\) and \(R\). These signs are indicated in Table VII. We discuss the signs of \(d_4\) and \(d_5\) for various representations below.

**B. \(\beta^j_{TR,\Delta_f}\) for \(G = SU(N_c)\) and \(R = F\)**

Here we present the evaluation of our general result \[13\] for the case \(G = SU(N_c)\) and \(R = F\). For reference, we first recall our results from \[13\] for \(d_j\) with \(2 \leq j \leq 4\) (and also recall that \(d_1 = 0\) for all \(G\) and \(R\)):

\[ d_{2,F} = \frac{2^4}{3^2(25N_c^2 - 11)} , \]

(4.10)

\[ d_{3,F} = \frac{2^5(13N_c^2 - 3)}{3^4N_c(25N_c^2 - 11)^2} \]

(4.11)
\[
d_{4,F} = -\frac{2^4}{3^5} N_c^2 (25 N_c^2 - 11)^3 \left[ N_c^8 \left( -366782 + 660000 \xi_3 \right) + N_c^6 \left( 865400 - 765600 \xi_3 \right) \\
+ N_c^4 \left( -1599316 + 2241888 \xi_3 \right) + N_c^2 \left( 571516 - 894432 \xi_3 \right) + 3993 \right].
\]  

(4.12)

This coefficient can be written equivalently in a form that shows the simple factorization of the terms multiplying \(\xi_3\):

\[
d_{4,F} = -\frac{2^4}{3^5} N_c^2 (25 N_c^2 - 11)^3 \left[ -366782 N_c^8 + 865400 N_c^6 - 1599316 N_c^4 + 571516 N_c^2 + 3993 \right] \\
+ 1056 N_c^2 (25 N_c^2 - 11) (25 N_c^4 - 18 N_c^2 + 77) \xi_3.
\]  

(4.13)

In [16] we presented the expression for \(d_{5,F}\) with terms ordered as descending powers of \(N_c\). As with \(d_{4,F}\), it is also useful to display this coefficient in an equivalent form that shows the simple factorizations of the terms multiplying \(\xi_3\) and \(\xi_5\):

\[
d_{5,F} = \frac{2^5}{3^6} N_c^3 (25 N_c^2 - 11)^4 \left[ -298194551 N_c^{12} + 414681770 N_c^{10} + 80227411 N_c^8 \\
+ 210598856 N_c^6 - 442678324 N_c^4 + 129261880 N_c^2 + 3371652 \right] \\
- 96 (25 N_c^2 - 11) \left( 176375 N_c^{10} - 564526 N_c^8 + 1489367 N_c^6 - 1470392 N_c^4 + 290620 N_c^2 + 968 \right) \xi_3 \\
+ 21120 N_c^2 (25 N_c^2 - 11)^2 \left( 40 N_c^6 - 27 N_c^4 + 124 N_c^2 - 209 \right) \xi_5.
\]  

(4.14)

We have checked that when we set \(N_c = 3\) in our general result for \(d_{5,F}\) in Eq. (4.14), the result agrees with our earlier calculation of \(d_{5,F}\) in Eq. (5.20) of Ref. [13].

As observed above, the coefficients \(d_2\) and \(d_3\) are manifestly positive for any \(G\) and \(R\). We find that \(d_{4,F}\) and \(d_{5,F}\) are negative-definite for \(G = SU(N_c)\) and all physical values of \(N_c \geq 2\). These results are summarized in Table VII.

Table VII

We list below the explicit numerical expressions for \(\beta'_{IR,F}\) to order \(\Delta_f^5\), denoted \(\beta'_{IR,SU(N_c),F,\Delta_f^5}\) for the gauge groups \(SU(N_c)\) with \(N_c = 2, 3, 4\), with fermions in the fundamental representation, to the indicated floating-point precision:

\[
SU(2) : \quad \beta'_{IR,F,\Delta_f^5} = \Delta_f^2 \left[ 1.99750 \times 10^{-2} + (3.66583 \times 10^{-3}) \Delta_f - (3.57303 \times 10^{-4}) \Delta_f^2 - (2.64908 \times 10^{-5}) \Delta_f^3 \right]
\]  

(4.15)

\[
SU(3) : \quad \beta'_{IR,F,\Delta_f^5} = \Delta_f^2 \left[ 0.83074 \times 10^{-2} + (0.98343 \times 10^{-3}) \Delta_f - (0.46342 \times 10^{-4}) \Delta_f^2 - (0.56435 \times 10^{-5}) \Delta_f^3 \right]
\]  

(4.16)

and

\[
SU(4) : \quad \beta'_{IR,F,\Delta_f^5} = \Delta_f^2 \left[ 0.45701 \times 10^{-2} + (0.40140 \times 10^{-3}) \Delta_f - (0.12938 \times 10^{-4}) \Delta_f^2 - (0.15498 \times 10^{-5}) \Delta_f^3 \right].
\]  

(4.17)

In Table VIII we list the (scheme-independent) values that we calculate for \(\beta'_{IR,F,\Delta_f^5}\) with \(2 \leq p \leq 4\) for the
illustrative gauge groups \( G = SU(2), SU(3), \) and \( SU(4), \) as functions of \( N_c \) in the respective intervals \( I_{IRZ} \) given in Eq. (2.7). For comparison, we list the \( n \)-loop values of \( \beta_{IR,F,n}^' \) with the \( 2 \leq n \leq 4 \) from \[12, 20\], where \( \beta_{IR,F,3}^' \) and \( \beta_{IR,F,4}^' \) are computed in the \( MS \) scheme. Although, for completeness, we list values of \( \beta_{IR,F,2}^' \) with \( N_f \) extending down to the lower end of the respective intervals \( I_{IRZ} \) for each value of \( N_c \), we caution that in a number of cases, including \( N_f = 6 \) for \( SU(2), N_f = 9 \) for \( SU(3), \) and \( 10 \leq N_f \leq 12 \) for \( SU(4), \) the corresponding values of \( \alpha_{IR,F}^\prime \) (discussed further below) are too large for the perturbative \( n \)-loop calculations to be applicable. Moreover, since for a considerable range of values of \( N_f \in I_{IRZ} \) for each \( N_c \), the five-loop beta function \( \beta_{5} \) calculated via the conventional power series expansion has no physical IR zero, we restrict the resultant \( \beta_{IR,F,n}^' \) evaluations to \( 1 \leq n \leq 4 \) loops.

In Figs. 7-9 we plot the values of \( \beta_{IR}^' \) calculated to order \( \Delta_f^p \) with \( 2 \leq p \leq 5 \), for \( R = F \) for the gauge groups \( SU(2), SU(3), \) and \( SU(4). \) In the general calculations of \( \gamma_{IR} \) as a series in powers of \( \Delta_f \) to maximal power \( p = 3 \) (i.e., order \( \Delta_f^3 \) ) in \[12\] and, for \( G = SU(3) \) and \( R = F, \) to maximal power \( p = 4 \) in \[14\], it was found that, for a fixed value of \( N_f, \) or equivalently, \( \Delta_f, \) in the interval \( I_{IRZ}, \) these anomalous dimensions increased monotonically as a function of \( p. \) This feature motivated our extrapolation to \( p = \infty \) in \[12\] to obtain estimates for the exact \( \gamma_{IR} \). In contrast, here we find that, for a fixed value of \( N_f, \) or equivalently, \( \Delta_f, \) in \( I_{IRZ}, \) as a consequence of the fact that different coefficients \( d_n \) do not all have the same sign, \( \beta_{IR,F}^\prime \) is not a monotonic function of \( p. \) Because of this non-monotonicity, we do not attempt to extrapolate our series to \( p = \infty. \) 

A lattice measurement of \( \beta_{IR}^' \) has been reported in \[83\] for the \( SU(3) \) theory with \( R = F \) and \( N_f = 12, \) namely \( \beta_{IR}^' = 0.26(2). \) The earlier higher-order values calculated in \[20\] via \( n \)-loop expansions in the coupling are \( \beta_{IR,3}^' = 0.2955 \) and \( \beta_{IR,4}^' = 0.282, \) which agree with this lattice measurement. As indicated in Table VIII our higher-order scheme-independent values for this theory are \( \beta_{IR,F}^\prime = 0.258, \beta_{IR,F}^\prime = 0.239, \) and \( \beta_{IR,F}^\prime = 0.228. \) Given the possible contributions of higher-order terms in the \( \Delta_f \) expansion, we consider that our scheme-independent calculation of \( \beta_{IR}^' \) to this order
is also consistent with the lattice measurement from Ref. 83.
To get a rough estimate of the range of accuracy and applicability of the series expansion for $\beta'_{IR}$ for this $R = F$ case, we can compute ratios of coefficients, as discussed before. For the illustrative case of SU(3), we have

$$\frac{d_{3,F}}{d_{3,F}} = 8.447 \quad \text{for SU(3)},$$

$$\frac{d_{3,F}}{|d_{4,F}|} = 21.221 \quad \text{for SU(3)},$$

and

$$|d_{4,F}| = 8.2115 \quad \text{for SU(3)}.$$ (4.20)

Since $N_u = 16.5$ and $N_f = 153/19 = 8.053$ in this SU(3) theory, the maximal value of $\Delta_f$ in the interval $I_{IRZ}$, as given by (3.16), is

$$\left(\Delta_f\right)_{\max} = \frac{321}{38} = 8.447 \quad \text{for SU(3), } N_f \in I_{IRZ}.$$ (4.21)

Therefore, these ratios suggest that the small-$\Delta_f$ expansion may be reasonably reliable in most of this interval, $I_{IRZ}$ and the associated non-Abelian Coulomb phase.

C. $\beta'_{IR_5\Delta_f}$ in LNN Limit

The appropriately rescaled beta function that is finite in the LNN limit is

$$\beta_\xi = \frac{d\xi}{dt} = \lim_{t \to \infty} N_c \beta,$$ (4.22)

where $\xi = 4\pi x = \lim_{t \to \infty} N_c \alpha N_c$ was defined in Eq. (3.21). This has the series expansion

$$\beta_\xi = \frac{d\xi}{dt} = -8\pi x \sum_{\ell=1}^{\infty} \tilde{b}_\ell x^\ell = -2\xi \sum_{\ell=1}^{\infty} \tilde{b}_\ell \xi^\ell$$ (4.23)

where

$$\tilde{b}_\ell = \lim_{t \to \infty} \frac{b_\ell}{N_c^{\ell+1}}.$$ (4.24)

and $\tilde{b}_\ell = b_\ell/(4\pi)^\ell$. The $\tilde{b}_\ell$ with $1 \leq \ell \leq 4$ were analyzed in [20, 21] and are listed for the reader’s convenience in the Appendix.

From the recent calculation of $\tilde{b}_5$ in [17], for general $G$ and $R$, in the MS scheme [17], we calculate

$$\tilde{b}_5 = \frac{8268479}{3888} + \frac{38851}{162} \zeta_3 - \frac{121}{6} \zeta_4 - 330 \zeta_5$$

$$+ \left( - \frac{11204369}{5184} - \frac{231619}{648} \zeta_3 + \frac{77}{6} \zeta_4 + \frac{4090}{9} \zeta_5 \right) r$$

$$+ \left( - \frac{3952801}{7776} + \frac{33125}{108} \zeta_3 - \frac{241}{6} \zeta_4 - \frac{1630}{9} \zeta_5 \right) r^2$$

$$+ \left( - \frac{5173}{432} - \frac{1937}{81} \zeta_3 + 7 \zeta_4 + \frac{20}{3} \zeta_5 \right) r^3$$

$$+ \left( \frac{61}{486} - \frac{52}{81} \zeta_3 \right) r^4$$

$$= 2050.932 - 2105.880 r + 645.7474 r^2$$

$$- 26.2309 r^3 - 0.64618 r^4.$$ (4.25)

(In this expression although $\zeta_4$ could be expressed explicitly as $\zeta_4 = \pi^4/90$, we leave it in abstract form to be parallel with the $\zeta_3$ and $\zeta_5$ terms.) We find that this coefficient $\tilde{b}_5$ is positive throughout the entire asymptotically free interval $0 \leq r < 5.5$. (Considered formally as a function of $r \in \mathbb{R}$, $\tilde{b}_5$ is negative for $r < -58.609$, positive for $-58.609 < r < 14.336$, and negative for $r > 14.336$, where the numbers are quoted to the given floating-point accuracy.)

Since the derivative $d\beta_\xi/d\xi$ satisfies the relation

$$\frac{d\beta_\xi}{d\xi} = \frac{d\beta}{d\alpha} = \beta',$$ (4.26)

it follows that $\beta'$ is finite in the LNN limit [21]. In terms of the variable $x$ defined in Eq. (3.23), we have

$$\beta' = -2 \sum_{\ell=1}^{\infty} (\ell + 1) \tilde{b}_\ell x^\ell.$$ (4.27)

Because $\beta_{IR}$ is scheme-independent and is finite in the LNN limit, one is motivated to calculate the LNN limit of the scheme-independent expansion (1.3). For this purpose, in addition to the rescaled quantities $\Delta_\xi$ defined in Eq. (3.30), we define the rescaled coefficient

$$\tilde{d}_{j,F} = \lim_{s \to \infty} N_c^j \tilde{d}_{j,F},$$ (4.28)

which is finite. Then each term

$$\lim_{s \to \infty} d_{j,F} \tilde{\Delta}_j = \left( N_c^j \tilde{d}_{j,F} \right)^j \tilde{d}_{j,F} \tilde{\Delta}_j$$ (4.29)

is finite in this limit. Thus, writing $\lim_{s \to \infty} \beta'_{IR}$ as $\beta'_{IR,LNN}$ for this $R = F$ case, we have

$$\beta'_{IR,LNN} = \sum_{j=1}^{\infty} d_{j,F} \tilde{\Delta}_j = \sum_{j=1}^{\infty} \tilde{d}_{j,F} \Delta_j.$$ (4.30)

We denote the value of $\beta'_{IR,LNN}$ obtained from this series calculated to order $O(\Delta_j^2)$ as $\beta'_{IR,LNN,\Delta_j^2}$.

From Eqs. (4.13) - (4.18), we find that the approach to the LNN limits for $\tilde{d}_{j,F}$ involves correction terms that vanish like $1/N_c^2$. This is the same property that was found in [20, 21] and, in the same way, it means that
the approach to the LNN limit for finite \( N_c \) and \( N_f \) with fixed \( r = N_f/N_c \) is rather rapid, as discussed in [21]. In [13] we gave the \( \hat{d}_{i,F} \) for \( 1 \leq n \leq 4 \); in addition to \( \hat{d}_1 = 0 \) (which holds for any \( G \) and \( R \)), these are

\[
\hat{d}_{2,F} = \frac{2^4}{3^2 \cdot 5^2} = 0.0711111 , \quad (4.31)
\]

\[
\hat{d}_{3,F} = \frac{416}{3^3 \cdot 5^4} = 2.465185 \times 10^{-2} , \quad (4.32)
\]

and

\[
\hat{d}_{4,F} = \frac{5868512}{3^5 \cdot 5^{10}} - \frac{5632}{3^4 \cdot 5^6} \zeta_3 = -(2.876137 \times 10^{-3}) . \quad (4.33)
\]

Here we give the next higher coefficient:

\[
\hat{d}_{5,F} = \frac{-9542227632}{3^6 \cdot 5^{14}} - \frac{1444864}{3^5 \cdot 5^9} \zeta_3 + \frac{360448}{3^5 \cdot 5^6} \zeta_5
\]

\[
= -(1.866490 \times 10^{-3}) . \quad (4.34)
\]

In these equations we have indicated the simple factorizations of the denominators that were already evident in the general analytic expressions (4.25)-(4.27). Although the numerical coefficients in the numerators of terms in Eq. (4.34) do not, in general, have simple factorizations, they do contain various powers of 2; for example, in \( \hat{d}_{5,F} \), \( 1444864 = 2^{10} \cdot 17 \cdot 83 \), etc. Thus, numerically, to order \( \Delta_r^5 \), for the LNN limit of this theory with \( R = F \), we have

\[
\beta_{IR,LNN} = \Delta_r^2 \left[ 7.1111 \times 10^{-2} + (2.4652 \times 10^{-2}) \Delta_r \right.
\]

\[
\left. - (2.8761 \times 10^{-3}) \Delta_r^2 - (1.8665 \times 10^{-3}) \Delta_r^3 + O(\Delta_r^4) \right] , \quad (4.35)
\]

where the coefficients are given to the indicated floating-point precision. We may again calculate ratios of successive magnitudes of these coefficients to get a rough estimate of the range over which the small-\( \Delta_r \) expansion is reliable in this LNN limit. We find

\[
\frac{\hat{d}_{2,F}}{\hat{d}_{3,F}} = 2.885 , \quad (4.36)
\]

\[
\frac{\hat{d}_{3,F}}{\hat{d}_{4,F}} = 8.571 , \quad (4.37)
\]

and

\[
\frac{\hat{d}_{4,F}}{\hat{d}_{5,F}} = 1.541 . \quad (4.38)
\]

For \( r \in I_{IRZ,r} \), the maximal value of \( \Delta_r \) is \( (\Delta_r)_{\text{max}} = 75/26 = 2.885 \). The first two ratios, (4.36) and (4.37), suggest that the \( \Delta_r \) expansion for \( \beta'_{IR} \) may be reasonably reliable over a reasonable fraction of the interval \( I_{IRZ,r} \).

From the third ratio, (4.38), we infer that the expansion is expected to be more accurate in the upper portion of the interval \( I_{IRZ,r} \) than the lower portion.

In Ref. [13] we presented a comparison of these scheme-independent calculations of \( \beta'_{IR,LNN} \) calculated up to the \( \Delta_r^4 \) order with the results of conventional \( n \)-loop calculations, denoted \( \beta'_{IR,nLNN} \), computed up to the \( n = 4 \) loop order for which the \( b_n \) were known at that time. We refer the reader to [13] for details of this discussion. Here we shall extend this comparison to the \( \Delta_r^5 \) order. In Table IX we list the numerical values of these conventional \( n \)-loop calculations up to \( n = 4 \), in comparison with our scheme-independent results calculated to \( O(\Delta_r^5) \) with \( p \) up to 5. (The conventional 4-loop values \( \beta'_{IR,4F} \) for some values of \( r \) toward the lower part of \( I_{IRZ,r} \) supersede the corresponding entries in Table II of [13].) Both \( \beta'_{IR,adj} \) and \( \beta'_{IR,\Delta_r^5} \) use, as inputs, the coefficients of the beta function up to loop order \( n \), although \( \beta'_{IR,\Delta_r^5} \) does this in a scheme-independent manner. We see that, especially for \( r \) values in the upper part of the interval \( I_{IRZ,r} \), the results are rather close, and, furthermore, that, as expected, for a given \( r \), the higher the loop level \( n \) and the truncation order \( p \) in the respective calculations of \( \beta'_{IR,nF} \) in the \( \overline{\text{MS}} \) scheme and the scheme-independent \( \beta'_{IR,\Delta_r^5} \), the better the agreement between these two results. Toward the lower end of the interval \( I_{IRZ,r} \), both the conventional expansion of \( \beta'_{IR} \) and the scheme-independent expansion of \( \beta'_{IR} \) in powers of \( \Delta_r \) become less reliable, and hence it is understandable that the results differ from each other in this lower part of \( I_{IRZ,r} \).

D. \( \beta'_{IR,\Delta_r^5} \) for \( G = SU(N_c) \) and \( R = adj \)

Here we calculate the \( d_j \) and hence \( \beta'_{IR,\Delta_r^5} \) for \( j \) up to \( j = 5 \) in the \( SU(N_c) \) gauge theory with fermion representation \( R = adj \). As discussed above, in this case, the interval \( I_{IRZ} \) contains the single Dirac value, \( N_f = 2 \). For this value of \( N_f \), Eq. (3.60) yields \( \Delta_f = 3/4 \). We recall that the \( d_j \) for \( 2 \leq j \leq 4 \) are

\[
d_{2,adj} = \left( \frac{2}{3} \right)^4 = 0.19753 , \quad (4.39)
\]

\[
d_{3,adj} = \frac{9^8}{3^7} = 0.11706 , \quad (4.40)
\]

and

\[
d_{4,adj} = \frac{46871}{2^2 \cdot 3^{12}} \pm \frac{2368}{3^{10} N_c^2} = 0.022049 + 0.040102 N_c^{-2} . \quad (4.41)
\]

Here, from our new general result (4.39) for \( d_5 \), we obtain the next coefficient for this case of the adjoint represen-
thus find that
\[ \Delta \] to obtain a crude idea of the region over which the small-
and
\[ N \]
While the
d\[ N \]
and, taking the large-
\[ N \]
For example,\( \hat{\gamma}_{\psi,IR,\Delta_f} \), since many formulas for these two cases are simply related to each other by sign reversals in certain terms, it is convenient to treat these two cases together, denoting them collectively as \( T_2 \). We recall that for \( R = A_2 \), we restrict to \( N_c \geq 3 \).

From our general formulas (4.15)-(4.19), we obtain the following, where the upper and lower signs refer to the \( S_2 \) and \( A_2 \) special cases of \( T_2 \), respectively, and \( F_{\pm} \) was defined in Eq. (8.72):
\[ d_{2,T_2} = \frac{2^3(N_c \pm 2)^2}{3^2 F_{\pm}} \quad (4.46) \]
\[ d_{3,T_2} = \frac{2^4(N_c \pm 2)^3(8N_c^2 \pm 3N_c - 6)}{3^3 N_c F_{\pm}^2} \quad (4.47) \]

\[
d_{5,T_2} = \frac{(N_c \pm 2)^4}{2 \cdot 3^6 N_c^2 F_{\pm}^2} \left[ -578437605N_c^{13} + 2353001022N_c^{12} - 1643220810N_c^{11} + 168585300N_c^{10} + 12567177608N_c^9 + 29240054768N_c^8 - 75390007296N_c^7 + 70417381376N_c^6 + 243309040128N_c^5 + 27199484928N_c^4 + 228577603584N_c^3 + 143780184064N_c^2 - 38053396480N_c + 15221358592 \right] \]
\[ + 2^7 F_{\pm} \left[ 125388N_c^{11} + 372762N_c^{10} - 7324047N_c^9 + 9682414N_c^8 + 52934332N_c^7 + 12735976N_c^6 - 192234240N_c^5 + 112670976N_c^4 + 164609280N_c^3 + 111598080N_c^2 + 2973696N_c + 1486848 \right] \]
\[ + 2^{10} \cdot 55N_c^2(N_c \mp 2)F_{\pm}^2 \left[ \mp 87N_c^5 + 259N_c^4 \pm 1134N_c^3 - 3600N_c^2 \pm 5016N_c + 10032 \right] \] \[ \times \frac{\zeta_3}{5504 \cdot 3^{10} \cdot 3^{12} \zeta_3} \quad (4.48) \]
We find that, in addition to the manifestly positive $d_{2,T_2}$, the coefficient $d_{3,T_2}$ is also positive for all relevant $N_c$. Here, by “relevant $N_c$”, we mean $N_c \geq 2$ for $S_2$ and $N_c \geq 3$ for $A_2$. In contrast, while $d_{4,S_2}$ is positive for all relevant $N_c$, we find that $d_{4,A_2}$ is negative for $N_c = 3, 4, 5$, passes through zero at $N_c = 5.515$, and is positive for $N_c \geq 6$. Further, we find that $d_{5,S_2}$ and $d_{5,A_2}$ are both negative for their respective physical ranges, $N_c \geq 2$ and $N_c \geq 3$. These sign properties are listed in Table VII.

Some general comments are in order concerning these $d_{j,T_2}$ expressions. These are analogous to the comments that we made for the $\kappa_{j,T_2}$ coefficients. The property that all of the $d_{j,A_2}$ coefficients contain an overall factor of $(N_c-2)$ (possibly raised to a power higher than 1), and hence vanish for $N_c = 2$, is a consequence of the fact that for $N_c = 2$, the $A_2$ representation is a singlet, so for SU(2), fermions in the $A_2 = $ singlet representation have no gauge interactions and do not contribute to the beta function or $\beta_{1R}$.

Furthermore, if $N_c = 2$, then the $S_2$ representation is the same as the adjoint representation, so the coefficients must satisfy the equality $d_{j,S_2} = d_{j,adj}$ for this SU(2) case, and we have checked that they do. This equality requires (i) that the term proportional to $\zeta_3$ in $d_{4,S_2}$ must be absent if $N_c = 2$, since $d_{4,adj}$ does not contain any $\zeta_3$ term, and this is accomplished by the factor of $(N_c-2)$ multiplying the $\zeta_3$ term in $d_{4,S_2}$; and (ii) the term proportional to $\zeta_3$ in $d_{6,S_2}$ must be absent if $N_c = 2$, since $d_{5,adj}$ does not contain any $\zeta_3$ term, and this is accomplished by the factor $(N_c-2)$ multiplying this $\zeta_3$ term in $d_{6,S_2}$. Similarly, as observed before, if $N_c = 3$, then the $A_2$ representation is the same as the conjugate fundamental representation, $\bar{F}$, so the coefficients must satisfy the equality $d_{j,A_2} = d_{j,F}$ for this SU(3) case, and we have checked that they do.

In the LN limit (3.22), as discussed above in the case of the anomalous dimension $\gamma_{R,T_2}$, the upper ends of the interval $I_{1RZ}$ for the $S_2$ and $A_2$ theories approach the same value, $N_{t,T_2}$, given in Eq. (3.30), and similarly the lower ends of this interval for these $S_2$ and $A_2$ theories approach the same value, $N_{t,T_2}$, given in Eq. (3.87). We denote

$$\hat{d}_{j,T_2} = \lim_{N \rightarrow \infty} d_{j,T_2},$$

and we find

$$\hat{d}_{j,S_2} = \hat{d}_{j,A_2},$$

which we denote simply as $\hat{d}_{j,T_2}$. Hence,

$$\lim_{N \rightarrow \infty} \beta'_{IR,S_2} = \lim_{N \rightarrow \infty} \beta'_{IR,A_2}.$$  (4.52)

Further, again in analogy with Eq. (4.98) and for the same reasons concerning group invariants in the LN limit, we have

$$\hat{d}_{j,T_2} = 2^{-j} \hat{d}_{j,adj}$$  (4.53)

From our general expressions, we calculate

$$\hat{d}_{2,T_2} = \frac{9^2}{3^3} = 0.049383$$  (4.54)

$$\hat{d}_{3,T_2} = \frac{5^5}{3^7} = 1.46319 \times 10^{-2}$$  (4.55)

$$\hat{d}_{4,T_2} = \frac{46871}{26 \cdot 3^{12}} = 1.37806 \times 10^{-3}$$  (4.56)

and

$$\hat{d}_{5,T_2} = \frac{-7141205}{28 \cdot 3^{16} + 172 \cdot 3^{32} \zeta_3} = -(2.58981 \times 10^{-4}).$$  (4.57)

To estimate the region over which the $\Delta_f$ expansion converges, we calculate the ratios of adjacent coefficients. We have

$$\frac{d_{2,T_2}}{d_{3,T_2}} = \frac{3N_c(18N_c^2 + 11N_c - 22)}{(N_c + 2)(8N_c^2 + 3N_c - 6)}.$$  (4.58)

and similarly for the ratios $d_{j-1,T_2}/d_{j,T_2}$ for $j = 4, 5$.

For the LN limit,

$$\frac{\hat{d}_{2,T_2}}{\hat{d}_{3,T_2}} = \left(\frac{3}{2}\right)^3 = 3.375$$  (4.59)

$$\frac{\hat{d}_{4,T_2}}{\hat{d}_{5,T_2}} = \frac{497664}{46871} = 10.618$$  (4.60)

and

$$\frac{\hat{d}_{4,T_2}}{d_{5,T_2}} = 5.321.$$  (4.61)

Since formally, $(\Delta_f)_{max} = 3.375$ from Eq. 4.58, and $\Delta_f = 5.5$ for $N_f = 2$, these ratios indicate that the $\Delta_f$ expansion for the LN limit of this $R = T_2$ case should be reasonably accurate in the interval $I_{1RZ}$ for this case.

V. IR ZERO OF $\beta_\xi$ IN THE LNN LIMIT

In this section we analyze the zeros of the rescaled five-loop beta function in the LN limit. This elucidates further the result that we first found for a finite value of $N_c$, namely $N_c = 3$, in [13], namely that for SU(3), the five-loop beta function only has a physical IR zero in the upper range of the interval $I_{1RZ}$. We denote the $n$-loop rescaled beta function as $\beta'_{1RZ}$ in this LNN limit as $\beta'_{1R,nt}$, and its IR zero (if such a zero exists) as $\xi_{IR,nt} = 4\pi i R_{IR}$. The analytic expressions of $\xi_{IR,2t}$ and $\xi_{IR,3t}$ were given in [21], together with numerical values of $\xi_{IR,nt}$ for $1 \leq n \leq 4$. Here we extend these results to...
the five-loop level, using the coefficient \( \hat{b}_5 \) in Eq. (4.25). As noted before, we use the \( b_n \) with \( 3 \leq n \leq 5 \) calculated in the \( \overline{\text{MS}} \) scheme. The reader is referred to [21] for analysis of these zeros up to the four-loop level.

In general, the IR zero of the \( n \)-loop beta function, \( \beta_{\xi,n\ell} \), is the positive real root closest to the origin (if such a root exists) of the equation

\[
\sum_{\ell=1}^{n} \hat{b}_\ell x^{\ell-1} = 0 ,
\]

(5.1)
of degree \( n-1 \) in the variable \( x \). The roots of Eq. (5.1) depend on the \( n-1 \) ratios \( \hat{b}_\ell/\hat{b}_1 \) for \( 2 \leq \ell \leq n \). In particular, at the five-loop level, Eq. (5.1) is the quartic equation

\[
\hat{b}_1 + \hat{b}_2 x + \hat{b}_3 x^2 + \hat{b}_4 x^3 + \hat{b}_5 x^4 = 0 .
\]

(5.2)

To analyze the roots of this equation, it is natural to start with \( r \) in the vicinity of \( r_u = 11/2 \), where \( \hat{b}_1 \to 0 \) and hence one solution of Eq. (5.2) approaches zero, matching the behavior of \( x_{IR,n\ell} \) for \( 2 \leq n \leq 4 \) in this limit. As we reduce \( r \) from the value \( r_u \) in the interval \( I_{IRZ,r} \), we can thus calculate how the physical IR root, \( x_{IR,5\ell} = \xi_{IR,5\ell}/(4\pi) \), changes. We find that, in contrast to the behavior of the IR zero of the lower-loop beta functions \( \beta_{\xi,n\ell} \) with \( 2 \leq n \leq 4 \), here at the five-loop level, as \( r \) decreases past a certain value \( r_{cx} \), Eq. (5.2) (with \( n = 3, 4, 5 \) calculated in the \( \overline{\text{MS}} \) scheme) ceases to have a physical IR zero. We find that the value of \( r_{cx} \) is

\[
r_{cx} = 4.32264 ,
\]

(5.3)
to the indicated floating-point accuracy. This is determined as the relevant root of the discriminant of Eq. (5.2), which is a polynomial of degree 15 in the variable \( r \). (The discriminants of the corresponding equations at loop levels 3 and 4 are polynomials of degree 3 and 8 in \( r \).) For example, for the illustrative value \( r = 5 \), near to the upper end of the interval \( I_{IRZ,r} \), Eq. (5.2) has the solutions in \( x \), expressed in terms of \( \xi = 4\pi x \): \( \xi = 0.36300, 1.69540, \) and \( -1.48834 \pm 1.08446i \). Of these, we identify the first as the IR zero, \( \xi_{IR,5\ell} \). As \( r \) decreases and approaches \( r_{cx} \) from above, the two real roots approach a common value, \( \xi \simeq 1.312 \) and as \( r \) decreases below \( r_{cx} \), Eq. (5.2) has only two complex-conjugate pairs of solutions, roots, but no real positive solution. In Table X we list our new results for \( \xi_{IR,5\ell} \) in comparison with the previously calculated values of \( \xi_{IR,n\ell} \) in the LNN limit with \( 2 \leq n \leq 4 \) from Table III of [21]. Although we list \( \xi_{IR,n\ell} \) values extending to the lower part of the interval \( I_{IRZ,r} \) for completeness, it is clear that a number of these values are too large for the perturbative calculations to be reliable. For values of \( r \) where the five-loop beta function (calculated in the \( \overline{\text{MS}} \) scheme) has no physical IR zero, we denote this as unphysical (u).

We note that the absence of a physical IR zero in the five-loop beta function (calculated in the \( \overline{\text{MS}} \) scheme) for \( N_f \) values in the lower portion of the interval \( I_{IRZ} \) does not necessarily imply that higher-loop calculations would yield similarly unphysical results. We gave an example of this in Section VIII of the second paper in [38], using an illustrative exact beta function. In this example, it was shown that a certain order of truncation of the Taylor series expansion in powers of \( \alpha \) for this beta function did not yield any physical IR zero, but higher orders did converge toward this zero.

VI. \( \Delta_f \) EXPANSION FOR \( \alpha_{IR} \) TO \( O(\Delta_f^4) \)

A. General \( G \) and \( R \)

Since the exact \( \alpha_{IR} \) (and also the \( n \)-loop approximation to this exact \( \alpha_{IR} \)) vanishes as functions of \( \Delta_f \), it follows that one can expand it as a power series in this variable. This expansion was given above as Eq. (2.9), and it was noted that the calculation of the coefficient \( a_j \) requires, as input, the \( \ell \)-loop beta function coefficients \( \hat{b}_\ell \) with \( 1 \leq \ell \leq j+1 \). We denote the truncation of this infinite series (2.9) to maximal power \( j = p \) as \( \alpha_{IR,\Delta_f^p} \).

Here we present a calculation of this series to \( O(\Delta_f^4) \), which is the highest order to which it has been calculated. Since \( \alpha_{IR} \) is scheme-dependent, it follows that the \( a_j \) coefficients in Eq. (2.9) are also scheme-dependent, in contrast to the scheme-independent coefficients \( \kappa_j \) and \( d_j \) in Eqs. (1.2) and (1.3). Nevertheless, it is still worthwhile to calculate these coefficients \( a_j \) and the resultant finite-order approximations \( \alpha_{IR,\Delta_f^p} \), for several reasons. First, this method has the advantage that \( \alpha_{IR,\Delta_f^p} \) is always physical and thus avoids the problem that we found in [15] and have further studied above, that the five-loop beta function calculated in the \( \overline{\text{MS}} \) scheme does not have a physical IR zero in the lower part of the interval \( I_{IRZ} \). In [14], for the special case \( G = \text{SU}(3) \) and \( R = F \), we presented the \( a_j \) (denoted \( a_j \) there) for \( 1 \leq j \leq 4 \).

Here, as a new result, we present the expressions for the \( a_j \) for arbitrary \( G \) and \( R \), for \( 1 \leq j \leq 4 \). For this purpose, we use the \( n \)-loop beta function coefficients \( b_n \) with \( 3 \leq n \leq 5 \) calculated in the \( \overline{\text{MS}} \) scheme. In particular, our result for \( a_4 \) makes use of the recently calculated five-loop beta function for general \( G \) and \( R \) [17].

For general \( G \) and \( R \), recalling the definition of the denominator factor \( D = 7C_A + 11C_f \) in Eq. (3.1), we find

\[
a_1 = \frac{4T_f}{3C_A D} ,
\]

(6.1)

\[
a_2 = \frac{2T_f^2(-287C_A^2 + 1208C_A C_f + 924C_f^2)}{3^3C_A^2 D^3}
\]

(6.2)
We next specialize to the case $G = \text{SU}(N_c)$ and give explicit reductions of these general formulas for the representations of interest here.

\section*{B. $R = F$}

For $R = F$, our general results \eqref{eq:a3_g} - \eqref{eq:a4_g} reduce to the following expressions:

\begin{equation}
  a_{3,F} = \frac{23^3}{3^3 N_c^2 (25 N_c^2 - 11)^3} \left[ 730529 N_c^8 - 1105385 N_c^6 + 719758 N_c^4 + 389235 N_c^2 + 52272 \right] + 1584 N_c^2 (25 N_c^2 - 11) \left( 25 N_c^4 - 18 N_c^2 + 77 \right) \zeta_3
\end{equation}

\begin{equation}
  a_{2,F} = \frac{4(548 N_c^4 - 1066 N_c^2 + 231)}{3^3 N_c (25 N_c^2 - 11)^3}
\end{equation}

\begin{equation}
  a_{1,F} = \frac{4}{3(25 N_c^2 - 11)}
\end{equation}
and
\[
a_{4,F} = \frac{2^2}{3^2 N_c^2 (25 N_c^2 - 11)^2} \left[ (2783259085 N_c^{12} - 7278665930 N_c^{10} + 4578046419 N_c^8 - 1719569282 N_c^6 \\
+ 2905511455 N_c^4 - 1137735654 N_c^2 + 1341648) \\
+ 288 (25 N_c^2 - 11) (548025 N_c^{10} - 1857036 N_c^8 + 4694107 N_c^6 - 5482510 N_c^4 + 1098130 N_c^2 + 2904) \zeta_3 \\
- 190080 N_c^2 (25 N_c^2 - 11)^2 (40 N_c^6 - 27 N_c^4 + 124 N_c^2 - 209) \zeta_5 \right].
\]

(6.8)

We have checked that setting \( N_c = 3 \) in our new \( a_4 \) coefficient in Eq. (6.8) yields agreement with the value that we obtained previously for this special case in (Eq. (14) of Ref. [14].

We comment next on the signs of these coefficients. The coefficient \( a_1 \) is manifestly positive for arbitrary group \( G \) and fermion representation \( R \). We find that \( a_2,F \) and \( a_3,F \) are also positive for all physical \( N_c \geq 2 \). In contrast, we find that \( a_4,F \) is negative for \( N_c = 2 \) and positive for \( N_c \geq 3 \). With \( N_c \) generalized from positive integers to positive real numbers in the range \( N_c \geq 2 \), we calculate that as \( N_c \) increases through the value \( N_c = 2.1184 \) (given to the indicated accuracy), \( a_{4,F} \) passes through zero with positive slope.

We list below the explicit numerical expressions for \( a_{IR} \) to order \( \Delta_n^j \), for \( N_c = 2, 3, 4 \) and \( R = F \), denoted , the indicated floating-point precision:

\[
SU(2) : \quad a_{IR,F,\Delta_4^1} = \Delta_f \left[ (0.18826 + (0.62521 \times 10^{-2}) \Delta_f + (0.70548 \times 10^{-2}) \Delta_f^2 - (0.45387 \times 10^{-4}) \Delta_f^3 \right]
\]

(6.9)

\[
SU(3) : \quad a_{IR,F,\Delta_4^1} = \Delta_f \left[ (0.078295 + (2.2178 \times 10^{-3}) \Delta_f + (1.1314 \times 10^{-3}) \Delta_f^2 + (2.1932 \times 10^{-5}) \Delta_f^3 \right]
\]

(6.10)

and

\[
SU(4) : \quad a_{IR,F,\Delta_4^1} = \Delta_f \left[ (0.043072 + (0.97619 \times 10^{-3}) \Delta_f + (0.33823 \times 10^{-3}) \Delta_f^2 + (0.71999 \times 10^{-5}) \Delta_f^3 \right].
\]

(6.11)

In Figs. [10,12] we show \( a_{IR,F,\Delta_n^p} \) for \( N_c = 2, 3, 4 \) and \( 1 \leq p \leq 4 \) as a function of \( N_f \). Note that in Fig. [10] the curves for \( p = 3 \) and \( p = 4 \) are so close as to be indistinguishable for this this range of \( N_f \).

In Table [XI] we compare the values of the IR zero of the \( n \)-loop beta function for \( 1 \leq n \leq 4 \) from [19] with our values of \( a_{IR,F,\Delta_n^p} \) for \( 1 \leq p \leq 4 \) and \( N_c = 2, 3, 4 \). Since the calculation of \( a_{IR,F,\Delta_n^p} \) uses the \( \ell \)-loop beta function coefficients \( b_{\ell} \) with \( 1 \leq \ell \leq n \), while the calculation of \( a_{IR,F,\Delta_n^p} \) uses the \( b_{\ell} \) for \( 1 \leq \ell \leq p + 1 \), the closest comparison is of \( a_{IR,F,\Delta_n^p} \) with \( a_{IR,F,\Delta_n^{p+1}} \), which both use \( n \)-loop information from the beta function. Although, for completeness, we include values of \( a_{IR,F,\Delta_n^p} \) for \( N_f \) extending down to the lower end of the respective intervals \( I_{IRZ} \) for each value of \( N_c \), we caution that in a number of cases, including \( N_f = 6 \) for \( SU(2) \), \( N_f = 9 \) for \( SU(3) \), and \( 10 \leq N_f \leq 12 \) for \( SU(4) \), these values of \( a_{IR,2\ell} \) are too large for the perturbative \( n \)-loop calculations to be reliable. Concerning the comparison of the higher-order \( n \)-loop values of \( a_{IR,\Delta^p} \) with our values of \( a_{IR,F,\Delta_n^p} \), we see that for a given \( N_c \) and \( N_f \), at the upper end of the non-Abelian Coulomb phase, the values of \( a_{IR,\Delta^{n-1}} \) and \( a_{IR,\Delta^{n-1}} \) are quite close to each other, but \( N_f \) decreases in this NACP in the interval \( I_{IRZ} \), \( a_{IR,\Delta^{n-1}} \) becomes slightly larger than \( a_{IR,\Delta^{n-1}} \).

In the LNN limit, for the IR zero of the rescaled beta
function, we write

\[ \xi_{IR} = 4\pi \sum_{j=1}^{\infty} \hat{a}_{j,F} \Delta_{j}^f \]  

(LNN limit),  

(6.12)

where

\[ \hat{a}_{j,F} = \lim_{LNN} N_c^{j+1} a_{j,F} . \]  

(6.13)

From our results for \( a_{j,F} \), we calculate

\[ \hat{a}_{1,F} = \frac{4}{3 \cdot 5^2} = 0.053333 \]  

(6.14)

\[ \hat{a}_{2,F} = \frac{2192}{3^3 \cdot 5^6} = 0.519585 \times 10^{-2} \]  

(6.15)

\[ \hat{a}_{3,F} = \frac{5844232}{3^5 \cdot 5^{10}} + \frac{1408}{3^3 \cdot 5^6} \zeta_3 = 0.647460 \times 10^{-2} \]  

(6.16)

and

\[ \hat{a}_{4,F} = \frac{222660726}{3^7 \cdot 5^{13}} + \frac{935296}{3^4 \cdot 5^{10} \zeta_3} - \frac{45056}{3^4 \cdot 5^8 \zeta_5} = 0.778770 \times 10^{-3} . \]  

(6.17)

Thus, in the LNN limit, the expansion of \( \xi_{IR} \), to \( O(\Delta_{4}^f) \), is

\[ \xi_{IR,\Delta_{4}^f} = 4\pi \Delta_r \left[ 0.053333 + (0.519585 \times 10^{-2}) \Delta_r \right. \]  

\[ + \left. (0.647460 \times 10^{-2}) \Delta_r^2 + (0.778770 \times 10^{-3}) \Delta_r^3 \right] . \]  

(6.18)

C. \( R = adj \)

For \( R = adj \), our general results (6.1)-(6.4) reduce to the following expressions:

\[ a_{1,adj} = \frac{2}{3^3 N_c} = \frac{0.074747}{N_c} \]  

(6.19)
\[ a_{2,adj} = \frac{205}{2^2 \cdot 3^7 N_c} - \frac{0.023434}{N_c} \] (6.20)

\[ a_{3,adj} = \frac{49129}{2^4 \cdot 3^{11} N_c} - \frac{296}{3^3 N_c^3} \]
\[ = \frac{0.017333}{N_c} - \frac{0.015038}{N_c^3} \] (6.21)

and

\[ a_{4,adj} = \left( \frac{38811689}{2^8 \cdot 3^{15}} - \frac{40}{3^9 \zeta_3} \right) \frac{1}{N_c} \]
\[ + \left( -\frac{3157}{3^{13}} + \frac{25616}{3^{12}} \right) \frac{1}{N_c^3} \]
\[ = \frac{0.0081230}{N_c} + \frac{0.055960}{N_c^3}. \] (6.22)

The coefficients \( a_{j,adj} \) with \( j = 1, 2, 4 \) are manifestly positive, and we find that \( a_{3,adj} \) is also positive for all \( N_c \geq 2 \).

Since for the adjoint representation, \( R = adj \), the upper and lower boundaries of the interval \( I_{IRZ} = N_{a,T_2} = 11/2 \) in Eq. \( (3.55) \) and \( N_{t,adj} = 17/16 \) in \( (3.59) \), are independent of \( N_f \), it follows that \( \Delta_f = N_u - N_f \) is also independent of \( N_c \). From the general formula (2.9), in the LN limit of a theory with fermions in a two-index representation \( R_2 \), including the adjoint and symmetric and antisymmetric tensors, we can write

\[ \xi_{IR} = 4\pi \sum_{j=1}^{\infty} \hat{a}_{j,R_2} \Delta_f^j \quad \text{\( (\text{LN limit}) \).} \] (6.23)

where

\[ \hat{a}_{j,R_2} = \lim_{LN} N_c \hat{a}_{j,R_2}. \] (6.24)

From our calculations above, setting \( R_2 = adj \), we have

\[ \hat{a}_{1,adj} = \frac{2}{3^3} = 0.074747 \] (6.25)

\[ \hat{a}_{2,adj} = \frac{205}{2^2 \cdot 3^7} = 0.023434 \] (6.26)

\[ \hat{a}_{3,adj} = \frac{49129}{2^4 \cdot 3^{11}} = 0.017333 \] (6.27)

and

\[ \hat{a}_{4,adj} = \frac{38811689}{2^8 \cdot 3^{15}} - \frac{40}{3^9 \zeta_3} = 0.0081230. \] (6.28)

D. \( R = S_2, \ A_2 \)

For \( R = S_2, \ A_2 \), we give the reductions of our general results (6.1)-(6.4) next. As before, it is convenient to consider these together, since many terms differ only by sign reversal. As above, the upper and lower signs refer to the \( S_2 \) and \( A_2 \) representations, respectively.

Also, as before, for \( A_2 \), we require that \( N_c \geq 3 \). Recalling the definition of the denominator factor \( F_\pm \) in Eq. (6.29), we have

\[ a_{1,T_2} = \frac{2(N_c \pm 2)}{3F_\pm} \] (6.29)

\[ a_{2,T_2} = \frac{(N_c \pm 2)(1845N_c^4 \pm 3056N_c^3 - 5188N_c^2 \mp 3696N_c + 3696)}{2 \cdot 3^3 N_c F_\pm^3} \] (6.30)

\[ a_{3,T_2} = \frac{(N_c \pm 2)^2}{2^2 \cdot 3^9 N_c^2 F_\pm^2} \left[ \left( 3979449 N_c^9 \pm 16999002 N_c^8 + 761444 N_c^7 \mp 52233472 N_c^6 - 3099440 N_c^5 \right. \right. \\
\left. \pm 11578144 N_c^4 - 16368000 N_c^3 \mp 36440448 N_c^2 - 40144896 N_c \pm 26763264 \right) \mp 12672N_c^2(N_c \mp 2)F_\pm(12N_c^2 \mp 9N_c^2 \mp 308)\zeta_3 \] (6.31)

and

\[ a_{4,T_2} = \frac{(N_c \pm 2)^3}{2^5 \cdot 3^7 N_c^3 F_\pm^3} \left[ \left( 28293721281 N_c^{13} \pm 156860406306 N_c^{12} + 13832572748 N_c^{11} \mp 547968555432 N_c^{10} \right. \right. \\
\left. \mp 929147053664 N_c^9 \mp 428226859968 N_c^8 + 2279581786496 N_c^7 \mp 586028410624 N_c^6 - 4633121830656 N_c^5 \right. \right. \\
\left. \left. + 586028410624 N_c^6 \mp 4633121830656 N_c^5 \right) \right] \]
The same general comments that we made before concerning factors in the \(\kappa_j, T_2\) and \(\hat{a}_j, T_2\) coefficients also apply here. Thus, for arbitrary \(j\), the \(a_{j,A_2}\) coefficients contain at least one overall factor of \((N_c - 2)\) and hence vanish for \(N_c = 2\), as a result of the fact that for \(N_c = 2\), the \(A_2\) representation is a singlet, so for SU(2), fermions in the \(A_2\) = singlet representation are free fields and hence make no contribution to the beta function. Moreover, if \(N_c = 2\), then the \(S_2\) representation is the same as the adjoint representation, so the \(a_{j,S_2}\) coefficients must satisfy the equality \(a_{j,S_2} = a_{j,adj}\) for this SU(2) case, and we have checked that they do. Similarly, if \(N_c = 3\), then the \(A_2\) representation is the same as the conjugate fundamental representation, \(F\), so these coefficients must satisfy the equality \(a_{j,A_2} = a_{j,F}\) for this SU(3) case, and we have checked that they do.

We next consider the LN limit of the theory with fermions in the \(S_2\) or \(A_2\) representations. Using the definition (6.24) with \(R_2 = S_2\) and \(R_2 = A_2\), we find that

\[
\hat{a}_{j,S_2} = \hat{a}_{j,A_2} \quad (6.33)
\]

so we denote these simply as \(\hat{a}_{j,T_2}\). In general, for the same group-theoretical reasons as led to the LN relation \(\hat{\kappa}_{j,T_2} = 2^{-j} \hat{\kappa}_{j,adj}\) in Eq. (3.38) and the LN relation \(\hat{d}_{j,T_2} = 2^{-j} \hat{d}_{j,adj}\) in Eq. (4.56), we have, in the LN limit,

\[
\hat{a}_{j,T_2} = 2^{-j} \hat{a}_{j,adj} \quad . \quad (6.34)
\]

Explicitly, we calculate

\[
\hat{a}_{1,T_2} = \frac{1}{3^1} = 0.05333 \quad (6.35)
\]

\[
\hat{a}_{2,T_2} = \frac{205}{2^4 \cdot 3^7} = 5.8585 \times 10^{-2} \quad (6.36)
\]

\[
\hat{a}_{3,T_2} = \frac{49129}{2^7 \cdot 3^{11}} = 2.16668 \times 10^{-3} \quad (6.37)
\]

and

\[
\hat{a}_{4,T_2} = \frac{38811689}{2^{12} \cdot 3^{15}} - \frac{5}{2 \cdot 3^3} \zeta_3 = 0.50769 \times 10^{-3} \quad . \quad (6.38)
\]
The behavior of these coefficients \( \hat{b}_\ell \) as functions of \( r \) was discussed in \cite{21} for \( 1 \leq \ell \leq 4 \). The positivity of \( \hat{b}_1 \) is equivalent to the asymptotic freedom of the theory, and requires \( r \) to lie in the interval \( 0 \leq r < 11/2 \). The existence of an IR zero in the two-loop beta function is equivalent to the condition that \( \hat{b}_2 < 0 \), which, in turn, is equivalent to the condition that \( r \in I_{IRZ,r} \) as given in Eq. \( (3.25) \). In this interval, \( \hat{b}_3 \) is negative-definite, while \( \hat{b}_4 \) is negative for \( 2.615 < r < 3.119 \) and positive for \( 3.119 < r < 5.5 \) \cite{21}.

For the coefficients \( \hat{c}_\ell \) in Eq. \( (3.33) \), from \cite{31} and references therein, one has \cite{21}

\[
\hat{c}_1 = 3 , \tag{A5}
\]

\[
\hat{c}_3 = \frac{11413}{108} - \left( \frac{117}{54} + 12 \zeta_3 \right) r - \frac{35}{27} r^2 , \tag{A7}
\]

and

\[
\hat{c}_4 = \frac{460151}{576} - \frac{23816}{81} r + \frac{899}{162} r^2 - \frac{83}{81} r^3 \\
+ \left( \frac{115}{9} - \frac{889}{3} r + 20 r^2 + \frac{16}{9} r^3 \right) \zeta_3 \\
+ \left( \frac{66 - 12 r}{4} \right) \zeta_4 + \left( - \frac{220 + 160 r}{3} \right) \zeta_5 . \tag{A8}
\]

\[\text{Origins at SDU (T.A.R.) and by the U.S. NSF Grant NSF-PHY-16-1620628 (R.S.)}
\]

\section*{Appendix A: Series Coefficients for \( \beta_\xi \) and \( \gamma_\psi \) in the LNN Limit}

For reference, we list here the rescaled series coefficients for \( \beta_\xi \) and \( \gamma_\psi \) in the LNN limit \cite{21,28}. From the (scheme-independent) one-loop and two-loop coefficients in the beta function \cite{21,28}, it follows that in the LNN limit the \( \hat{b}_\ell \) with \( \ell = 1, 2 \) are

\[
\begin{align*}
\hat{b}_1 &= \frac{1}{3}(11 - 2r) \\
&= 3.667 - 0.667r \tag{A1}
\end{align*}
\]

and

\[
\begin{align*}
\hat{b}_2 &= \frac{1}{3}(34 - 13r) \\
&= 11.333 - 4.333r . \tag{A2}
\end{align*}
\]

The coefficients \( \hat{b}_3 \) and \( \hat{b}_4 \) have been calculated in the \( \overline{\text{MS}} \) scheme \cite{21,28}. With these inputs, one has \cite{21}

\[
\begin{align*}
\hat{b}_3 &= \frac{1}{54}(2857 - 1709r + 112r^2) \\
&= 52.907 - 31.648r + 2.074r^2 \tag{A3}
\end{align*}
\]

and

\[
\begin{align*}
\hat{b}_4 &= \left( \frac{150473}{486} + \frac{44}{9} \zeta_3 \right) - \left( \frac{485513}{1944} + \frac{20}{9} \zeta_3 \right) r \\
&\quad + \left( \frac{8654}{243} + \frac{28}{3} \zeta_3 \right) r^2 + \left( \frac{130}{243} \right) r^3
\end{align*}
\]

\[\text{\( = 315.492 - 252.421r + 46.832r^2 + 0.5350r^3 \).} \tag{A4}
\]

\[\text{\( \text{References,} \text{ and further details are given in the text.}\)
\]

[1] Some early discussions of the renormalization group in quantum field theory include E. C. G. Stueckelberg and A. Peterman, Helv. Phys. Acta 26, 499 (1953); M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954); N. N. Bogolubov and D. V. Shirkov, Doklad. Akad. Nauk SSSR 103, 391 (1955); C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970); K. Wilson, Phys. Rev. D 3, 1818 (1971).

[2] Some early analyses of connections between scale and conformal invariance include A. Salam, Ann. Phys. (NY) 53, 174 (1969); A. M. Polyakov, JETP Lett. 12, 381 (1970); D. J. Gross and J. Wess, Phys. Rev. D 2, 753 (1970); C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (NY) 59, 42 (1970). More recent works include J. Polchinski, Nucl. Phys. B 303, 226 (1988); J.-F. Fortin, B. Grinstein and A. Stergiou, JHEP 01 (2013) 184 (2013); A. Dymarsky, Z. Komargodski, A. Schwimmer, and S. Thiessen, JHEP 10, 171 (2015) and references therein.

[3] Note that our assumption that the fermions are massless does not produce any loss of generality, since if a fermion had a nonzero mass \( m_0 \), it would be integrated out of the low-energy effective field theory at scales \( \mu < m_0 \), and hence would not affect the IR limit \( \mu \to 0 \) under consideration here.

[4] Fully nonperturbative evidence for the non-Abelian Coulomb phase comes from lattice simulations, as discussed below. Furthermore, in the case where \( G = SU(N_c) \) and the fermions are in the fundamental representation, one can take the limit in Eq. \( (3.25) \), namely \( N_c \to \infty \) and \( N_f \to \infty \) with \( r = N_f/N_c \) fixed and finite. In this case, \( \alpha_{IR} \) can be made arbitrarily small so that strength of the gauge coupling at the IR fixed point approaches arbitrarily close to zero.

[5] D. J. Gross, in R. Balian and J. Zinn-Justin, eds. Methods in Field Theory, Les Houches 1975 (North Holland, Amsterdam, 1976), p. 141.

[6] The Casimir invariants \( C_2(R) \) and \( T(R) \) are defined as \( \sum_a \sum_j \mathcal{D}_R(T_a)_{ij} \mathcal{D}_R(T_a)_{jk} \equiv C_2(R) \delta_{ik} \) and \( \sum_{i,j} \mathcal{D}_R(T_a)_{ij} \mathcal{D}_R(T_b)_{ji} \equiv T(R) \delta_{ab} \), where \( R \) is the representation, \( T_a \) are the generators of \( G \), normalized ac-
See, e.g., talks in the CP3 Workshop at T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, EPFL Lec-
G. Grunberg, Phys. Rev. D 87, 065012 (2013). Our $\gamma^2 = 4\beta_{-1}$ in this paper.
[22] T. A. Ryttov, R. Shrock, Phys. Rev. D 87, 116007 (2013).
[23] Some recent reviews include S. Rychkov, EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions [arXiv:1601.05000]; D. Simmons-Duffin, 2015 TASI Lectures on the Conformal Bootstrap [arXiv:1602.07982], and D. Poland, Nature Phys. 12, 535 (2016).
[24] B. Holdom, Phys. Lett. B 150, 301 (1985); K. Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. 56, 1335 (1986); T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. 57, 957 (1986).
[25] C. Pica, F. Sannino, Phys. Rev. D 83, 035013 (2011) [arXiv:1101.5917].
[26] Some authors use the opposite sign convention for the anomalous dimension, writing $\Delta_D = \Delta_{\text{free}} + \gamma_\phi$. Our sign convention is the same as the one used in [24] and the lattice gauge theory literature.
[27] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. B 93, 429 (1980); S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303, 334 (1993).
[28] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B 400, 379 (1997).
[29] M. Czakon, Nucl. Phys. B 710, 485 (2005).
[30] W. A. Bardeen et al., Phys. Rev. D 18, 3998 (1978).
[31] K. G. Chetyrkin, Phys. Lett. B 404, 161 (1997); J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. B 405, 327 (1997).
[32] Here and below, if an expression for $N_f$ formally evaluates to a non-integral real value, it is understood implicitly that one infers an appropriate integral value from it.
[33] $N_u$ and $N_t$ were denoted $N_{f,015}$ and $N_{f,024}$ in [10] and some of our subsequent works. In principle, for an appropriate $G$ and $R$, between the confining phase with spontaneous chiral symmetry breaking at small $N_f$ and the (deconfined) non-Abelian Coulomb phase at larger $N_f \lesssim N_u$, there could be an intermediate phase with confinement but no chiral symmetry breaking, provided that the ‘t Hooft anomaly-matching conditions are satisfied (a necessary but not sufficient condition for such a phase).
[34] E. Gardi and M. Karliner, Nucl. Phys. B 529, 383 (1998); E. Gardi and G. Grunberg, JHEP 03, 024 (1999).
[35] F. A. Chishtie, V. Elias, V. A. Miransky, and T. G. Steele, Prog. Theor. Phys. 104, 603 (2000).
[36] T. A. Ryttov and R. Shrock, Phys. Rev. D 85, 076009 (2012).
[37] R. Shrock, Phys. Rev. D 91, 125039 (2015); G. Choi and R. Shrock, Phys. Rev. D 93, 065013 (2016).
[38] T. A. Ryttov and R. Shrock, Phys. Rev. D 86, 065032 (2012); T. A. Ryttov and R. Shrock, Phys. Rev. D 86, 085005 (2012).
[39] R. Shrock, Phys. Rev. D 88, 036003 (2013); R. Shrock, Phys. Rev. D 90, 045011 (2014); R. Shrock, Phys. Rev. D 91, 125039 (2015); G. Choi and R. Shrock, Phys. Rev. D 90, 125029 (2014); G. Choi and R. Shrock, Phys. Rev. D 94, 065038 (2016).
[40] T. A. Ryttov, Phys. Rev. D 89, 016013 (2014); T. A. Ryttov, Phys. Rev. D 89, 056001 (2014); T. A. Ryttov, Phys. Rev. D 90, 056007 (2014).
[41] J. A. Gracey and R. M. Simms, Phys. Rev. D 91, 085037 (2015).
[42] J. A. Gracey, Phys. Lett. B 488, 175 (2000).
[43] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, JHEP 10, 076 (2014) [arXiv:1402.6611]. Our $\alpha_\ell = 2^{1+2\ell-1}$ in this paper.
[44] G. Mack, Commun. Math. Phys. 55, 1 (1977); B. Grinstein, K. Intriligator, and I. Rothstein, Phys. Lett. B 662, 367 (2008); Y. Nakayama, Phys. Repts. 569, 1 (2015).
[45] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov (NSVZ), Nucl. Phys. B 229, 381, 407 (1983).
[46] N. Seiberg, Phys. Rev. D 49, 6857 (1994).
[47] F. Sannino, Phys. Rev. D 79, 096007 (2009); M. Mojaza, C. Pica, T. A. Ryttov, and F. Sannino, Phys. Rev. D 86, 076012 (2012).
[48] T. Appelquist et al. (LSD Collab.), Phys. Rev. D 84, 054501 (2011).
[49] T. DeGrand, Phys. Rev. D 84, 116901 (2011).
[50] Y. Aoki et al. (LatKMI Collab.), Phys. Rev. D 86, 054506 (2012).
[51] A. Hasenfratz, A. Cheng, G. Petropoulos, and D. Schaich, in Proc. Lattice-2012 [arXiv:1207.7162].
[52] A. Hasenfratz, A. Cheng, G. Petropoulos, and D. Schaich, in Proc. Lattice-2013 [arXiv:1310.1124].
[53] A. Hasenfratz and D. Schaich, arXiv:1610.10004 and private communications.
[54] M. P. Lombardo, K. Miura, T. J. Nunes da Silva, and E. Pallante, JHEP 12, 183 (2014). See also A. Deuzeman, M. P. Lombardo, T. Nunes da Silva, and E. Pallante, Phys. Lett. B 720, 358 (2013); K. Miura, M. P. Lombardo, and
E. Pallante, Phys. Lett. B 710, 676 (2012).

[55] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder, and C.H. Wong, in Proc. Lattice-2012 [arXiv:1211.4238]; Z. Fodor, K. Holland, J. Kuti, S. Mondal, D. Nogradi, and C. H. Wong, Phys. Rev. D 94 091501 (2016).

[56] T. Appelquist et al. (LSD Collab.), [arXiv:1204.6000]

[57] Z. Fodor, K. Holland, J. Kuti, S. Mondal, D. Nogradi, and C. H. Wong, Phys. Rev. D 90, 055028 (2014); R. J. Crewther and L. C. Tungstall, Phys. Rev. D 91, 034016 (2015). See also T. Nunes da Silva, E. Pallante, and L. Robrock, [arXiv:1609.06298, arXiv:1506.06396].

[58] T. Appelquist et al. (LSD Collab.), Phys. Rev. D 87, 094511 (2013); Y. Aoki et al. (LatKMI Collab.), Nucl. and Particle Physics Proc. 270-272, 242 (2016).

[59] Y. Aoki et al. (LatKMI Collab.), Phys. Rev. D 89, 114502 (2014); N. C. Christiansen and R. Shrock, Phys. Rev. D 69, 054502 (2004); N. C. Christiansen and R. Shrock, Phys. Rev. Lett. 94, 241801 (2005) and references therein.

[60] M. Hansen, K. Langæble, and F. Sannino, Phys. Rev. D 95, 036005 (2017).

[61] T. Appelquist et al. (LSD Collab.), Phys. Rev. D 94, 114502 (2014); T. Appelquist et al. (LSD Collab.), Phys. Rev. D 93, 114514 (2016) and references therein.

[62] M. Hayakawa, K.-I. Ishikawa, S. Takeda, and N. Yamada, Phys. Rev. D 88, 094504, 094506 (2013); J. M. Suorsa et al., Phys. Rev. D 88, 094504, 094506 (2013); see also V. Leino et al., [arXiv:1610.09989].

[63] T. Appelquist et al. (LSD Collab.), Phys. Rev. D 93, 054501 (2011).

[64] T. Appelquist et al. (LSD Collab.), Phys. Rev. D 94, 114502 (2014); T. Appelquist et al. (LSD Collab.), Phys. Rev. D 93, 114514 (2016) and references therein.

[65] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D 83, 074507 (2011).

[66] V. Leino et al., [arXiv:1701.04666].

[67] F. Bursa et al., Phys. Lett. B696, 374 (2011); T. Karavirta et al., JHEP 1205 (2012) 003; M. Tomii et al., arXiv:1311.0099; T. Appelquist et al. (LSD Collab.), Phys. Rev. Lett. 112, 111601 (2014).

[68] J. M. Suorsa et al., [arXiv:1611.02022] see also V. Leino et al., [arXiv:1610.09989].

[69] P. M. Stevenson, Mod. Phys. Lett. A 31, 1650226 (2016).

[70] D. D. Dietrich and F. Sannino, Phys. Rev. D 75, 034504 (2007); S. Catterall and F. Sannino, Phys. Rev. D 76, 034504 (2007).

[71] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D 84, 054501 (2011).

[72] T. Appelquist et al. (LSD Collab.), Phys. Rev. D 85, 054501 (2011).

[73] T. Appelquist et al. (LSD Collab.), Phys. Rev. D 85, 054501 (2011).

[74] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, Phys. Rev. D 82, 014510 (2010).

[75] T. DeGrand, J. Giedt, and L. Keegan, PoS Lattice2010, 057 (2010) [arXiv:1010.5909].

[76] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, Phys. Rev. D 93, 054505 (2016).

[77] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D 84, 074507 (2011).

[78] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D 87, 074507 (2013).

[79] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder, and C. H. Wong, Phys. Lett. B 718, 657 (2012).

[80] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder, and C. H. Wong, Phys. Lett. B 710, 676 (2012).

[81] Z. Fodor, K. Holland, J. Kuti, S. Mondal, D. Nogradi, and C. H. Wong, Phys. Rev. D 94 091501 (2016).

[82] J. B. Kogut and D. K. Sinclair, Phys. Rev. D 81, 114507 (2010); Phys. Rev. D 92, 054508 (2015).

[83] J. B. Kogut and D. K. Sinclair, Phys. Rev. D 101, 131601 (2008); see also M. Kurachi, S. Matsuzaki, and K. Yamawaki, Phys. Rev. D 90, 055028 (2014); R. J. Crewther and L. C. Tungstall, Phys. Rev. D 91, 034016 (2015). See also T. Nunes da Silva, E. Pallante, and L. Robrock, [arXiv:1609.06298, arXiv:1506.06396].

[84] S. Hasenfratz and D. Schaich, [arXiv:1610.10004].
TABLE I: Values of the anomalous dimension $\gamma_{\bar{\psi}\psi,IR,F}$ calculated to $O(\Delta_f^p)$, i.e., $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f^p}$, with $1 \leq p \leq 4$, for $G = SU(N_c)$, as a function of $N_c$ and $N_f$ for $2 \leq N_c \leq 4$ and $N_f$ in the respective intervals $I_{IRZ}$ for each $N_c$. For comparison, we also include the $n$-loop values $\gamma_{\bar{\psi}\psi,IR,F,n\ell}$ with $2 \leq n \leq 4$ from Table VI of [19]. Values that exceed the bound $\gamma_{\bar{\psi}\psi,IR} \leq 2$ in Eq. (2.13) are marked as unphysical (u). For notational brevity in this and successive tables, we omit the subscript $\bar{\psi}\psi$. See text for further details.

| $N_c$ | $N_f$ | $\gamma_{IR,F,2}$ | $\gamma_{IR,F,3}$ | $\gamma_{IR,F,4}$ | $\gamma_{IR,F,\Delta_f,2}$ | $\gamma_{IR,F,\Delta_f,3}$ | $\gamma_{IR,F,\Delta_f,4}$ |
|-------|-------|------------------|------------------|------------------|------------------|------------------|------------------|
| 2     | 6     | u                | u                | u                | 0.337            | 0.520            | 0.596            | 0.698            |
| 2     | 7     | u                | u                | u                | 0.270            | 0.387            | 0.426            | 0.467            |
| 2     | 8     | 0.752            | 0.272            | 0.204            | 0.202            | 0.268            | 0.285            | 0.298            |
| 2     | 9     | 0.275            | 0.161            | 0.157            | 0.135            | 0.164            | 0.169            | 0.172            |
| 2     | 10    | 0.0910           | 0.0738           | 0.0748           | 0.0674           | 0.07475          | 0.07535          | 0.0755           |
| 3     | 9     | u                | u                | u                | 0.374            | 0.587            | 0.687            | 0.804            |
| 3     | 10    | u                | u                | u                | 0.324            | 0.484            | 0.549            | 0.615            |
| 3     | 11    | 1.61             | 0.439            | 0.250            | 0.274            | 0.389            | 0.428            | 0.462            |
| 3     | 12    | 0.773            | 0.312            | 0.253            | 0.224            | 0.301            | 0.323            | 0.338            |
| 3     | 13    | 0.404            | 0.220            | 0.210            | 0.174            | 0.221            | 0.231            | 0.237            |
| 3     | 14    | 0.212            | 0.146            | 0.147            | 0.125            | 0.148            | 0.152            | 0.153            |
| 3     | 15    | 0.0997           | 0.0826           | 0.0836           | 0.0748           | 0.0833           | 0.0841           | 0.0843           |
| 3     | 16    | 0.0272           | 0.0258           | 0.0259           | 0.0249           | 0.0250           | 0.0250           | 0.0250           |
| 4     | 11    | u                | u                | u                | 0.424            | 0.694            | 0.844            | 1.029            |
| 4     | 12    | u                | u                | u                | 0.386            | 0.609            | 0.721            | 0.8475           |
| 4     | 13    | u                | u                | u                | 0.347            | 0.528            | 0.610            | 0.693            |
| 4     | 14    | u                | u                | u                | 0.308            | 0.451            | 0.509            | 0.561            |
| 4     | 15    | 1.32             | 0.420            | 0.281            | 0.270            | 0.379            | 0.418            | 0.448            |
| 4     | 16    | 0.778            | 0.325            | 0.269            | 0.231            | 0.312            | 0.336            | 0.352            |
| 4     | 17    | 0.481            | 0.251            | 0.234            | 0.193            | 0.249            | 0.263            | 0.2705           |
| 4     | 18    | 0.301            | 0.189            | 0.187            | 0.154            | 0.190            | 0.197            | 0.200            |
| 4     | 19    | 0.183            | 0.134            | 0.136            | 0.116            | 0.136            | 0.139            | 0.140            |
| 4     | 20    | 0.102            | 0.0854           | 0.0865           | 0.0771           | 0.0860           | 0.0869           | 0.0871           |
| 4     | 21    | 0.0440           | 0.0407           | 0.0409           | 0.0386           | 0.0408           | 0.0409           | 0.0409           |
TABLE II: Values of the scheme-independent \( \gamma_{IR,F,\Delta_p} \) in the LNN limit \( n=2,3,4 \) for \( 1 \leq p \leq 4 \), together with \( \gamma_{IR,F,n,t} \) with \( n=2,3,4 \) from Table V of [21] for comparison, as a function of \( r \) for \( r \in I_{IR,F} \). Values that exceed the bound \( r_\gamma \leq 2 \) are marked as unphysical (u) or placed in parentheses. We also list the extrapolated estimate \( \gamma_{IR,F,ex234} \) of \( \gamma_{IR,F,\Delta_{p}} \) and, in the last column, the ratio \( \gamma_{IR,F,ex234}/\gamma_{IR,F,\Delta_p} \).

| \( r \) | \( \gamma_{IR,F,2}\gamma \) | \( \gamma_{IR,F,3}\gamma \) | \( \gamma_{IR,F,4}\gamma \) | \( \gamma_{IR,F,\Delta_p}\gamma \) | \( \gamma_{IR,F,\Delta_p}\gamma \) | \( \gamma_{IR,F,\Delta_p}\gamma \) | \( \gamma_{IR,F,\Delta_p}\gamma \) | \( \gamma_{IR,F,\Delta_p}\gamma \) |
|--------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2.8    | 1.708           | 0.190           | 0.432           | 0.706           | 0.870           | 1.064           | (2.09)          | 1.96            |
| 3.0    | u               | 1.165           | 0.225           | 0.400           | 0.635           | 0.765           | 0.908           | 1.645           | 1.82            |
| 3.2    | u               | 0.854           | 0.264           | 0.368           | 0.567           | 0.668           | 0.770           | 1.28            | 1.66            |
| 3.4    | u               | 0.656           | 0.293           | 0.336           | 0.502           | 0.579           | 0.650           | 0.993           | 1.53            |
| 3.6    | 1.853           | 0.520           | 0.308           | 0.304           | 0.440           | 0.497           | 0.5445          | 0.763           | 1.40            |
| 3.8    | 1.178           | 0.420           | 0.306           | 0.272           | 0.381           | 0.422           | 0.452           | 0.584           | 1.29            |
| 4.0    | 0.785           | 0.341           | 0.288           | 0.240           | 0.325           | 0.353           | 0.371           | 0.444           | 1.20            |
| 4.2    | 0.537           | 0.277           | 0.257           | 0.208           | 0.272           | 0.290           | 0.300           | 0.337           | 1.12            |
| 4.4    | 0.371           | 0.222           | 0.217           | 0.176           | 0.2215          | 0.233           | 0.238           | 0.253           | 1.06            |
| 4.6    | 0.254           | 0.1735          | 0.1745          | 0.144           | 0.1745          | 0.1805          | 0.183           | 0.188           | 1.03            |
| 4.8    | 0.170           | 0.129           | 0.131           | 0.112           | 0.130           | 0.133           | 0.134           | 0.135           | 1.01            |
| 5.0    | 0.106           | 0.0889          | 0.0900          | 0.0800          | 0.0894          | 0.09045         | 0.0907          | 0.0905          | 1.00            |
| 5.2    | 0.0562          | 0.0512          | 0.0516          | 0.0480          | 0.0514          | 0.0516          | 0.0516          | 0.0516          | 1.00            |
| 5.4    | 0.0168          | 0.0164          | 0.0164          | 0.0160          | 0.0164          | 0.0164          | 0.0164          | 0.0164          | 1.00            |

TABLE III: Values of the anomalous dimension \( \gamma_{IR,adj,\Delta_p} \) with \( 1 \leq p \leq 4 \), for \( N_f = 2 \) and \( G = SU(N_c) \) with \( N_c = 2,3 \). For comparison, we also list our n-loop values, \( \gamma_{IR,adj,n,t} \) for this theory from Table VIII of Ref. [19].

| \( N_c \) | \( \gamma_{IR,adj,2} \) | \( \gamma_{IR,adj,3} \) | \( \gamma_{IR,adj,4} \) | \( \gamma_{IR,adj,\Delta_p} \) | \( \gamma_{IR,adj,\Delta_p} \) | \( \gamma_{IR,adj,\Delta_p} \) | \( \gamma_{IR,adj,\Delta_p} \) |
|----------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 2        | 0.820                    | 0.543                    | 0.500                    | 0.333                    | 0.465                    | 0.511                    | 0.556                    |
| 3        | 0.820                    | 0.543                    | 0.523                    | 0.333                    | 0.465                    | 0.516                    | 0.553                    |

TABLE IV: Values of the anomalous dimension \( \gamma_{IR,S_2,\Delta_p} \) with \( 1 \leq p \leq 4 \), for \( G = SU(N_c) \) with \( N_c = 3,4 \) and \( N_f = 2,3 \) (so \( N_f \in I_{IR,F} \)). For comparison, we also include values of \( \gamma_{IR,S_2,n,t} \) with \( 2 \leq n \leq 4 \) for this theory from Table XI in our Ref. [12]. Values that exceed the upper bound \( \gamma_{IR} < 2 \) are marked as unphysical (u).

| \( N_c \) | \( N_f \) | \( \gamma_{IR,S_2,2} \) | \( \gamma_{IR,S_2,3} \) | \( \gamma_{IR,S_2,\Delta_p} \) | \( \gamma_{IR,S_2,\Delta_p} \) | \( \gamma_{IR,S_2,\Delta_p} \) | \( \gamma_{IR,S_2,\Delta_p} \) |
|----------|----------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 3        | 2        | u                        | 1.28                     | 1.12                     | 0.501                    | 0.789                    | 0.960                    | 1.132                    |
| 3        | 3        | 0.144                    | 0.133                    | 0.133                    | 0.116                    | 0.131                    | 0.133                    | 0.1335                   |
| 4        | 2        | u                        | u                        | 1.79                     | 0.581                    | 0.966                    | 1.242                    | 1.536                    |
| 4        | 3        | 0.381                    | 0.313                    | 0.315                    | 0.232                    | 0.294                    | 0.312                    | 0.319                    |
TABLE V: Values of the anomalous dimension $\gamma_{IR,A_2,\Delta_f}$ calculated to order $1 \leq p \leq 4$, for $G = SU(4)$ and $N_f \in I_{IRZ}$. For comparison, we also include values of $\gamma_{IR,A_2,nF}$ with $2 \leq n \leq 4$ for this theory from Table XII in [19]. Values that exceed the upper bound $\gamma_{IR} < 2$ are marked as unphysical (u).

| $N_c$ | $N_f$ | $\gamma_{IR,A_2,2F}$ | $\gamma_{IR,A_2,3F}$ | $\gamma_{IR,A_2,4F}$ | $\gamma_{IR,A_2,\Delta_f}$ | $\gamma_{IR,A_2,\Delta_f}^2$ | $\gamma_{IR,A_2,\Delta_f}^3$ |
|-------|-------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 4     | 5     | u                      | u                      | 0.5405                 | 0.941                  | 1.287                  | 1.671                  |
| 4     | 6     | u                      | 1.38                   | 0.293                  | 0.450                  | 0.728                  | 0.928                  | 1.114                  |
| 4     | 7     | u                      | 0.695                  | 0.435                  | 0.360                  | 0.538                  | 0.641                  | 0.717                  |
| 4     | 8     | 0.802                  | 0.402                  | 0.368                  | 0.270                  | 0.370                  | 0.4135                 | 0.438                  |
| 4     | 9     | 0.331                  | 0.228                  | 0.232                  | 0.180                  | 0.225                  | 0.237                  | 0.242                  |
| 4     | 10    | 0.117                  | 0.101                  | 0.103                  | 0.0901                 | 0.101                  | 0.103                  | 0.103                  |

TABLE VI: Values of the anomalous dimension $\gamma_{IR,T_2,\Delta_f}$ for $T_2 = S_2$ or $T_2 = A_2$, calculated to order $1 \leq p \leq 4$, in the limit $N_c \to \infty$ with $N_f \in I_{IRZ}$ for this limit, namely $3 \leq N_f \leq 5$.

| $N_f$ | $\gamma_{IR,T_2,\Delta_f}$ | $\gamma_{IR,T_2,\Delta_f}^2$ | $\gamma_{IR,T_2,\Delta_f}^3$ | $\gamma_{IR,T_2,\Delta_f}^4$ |
|-------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 3     | 0.5555                      | 0.921                       | 1.177                       | 1.408                       |
| 4     | 0.333                       | 0.465                       | 0.520                       | 0.550                       |
| 5     | 0.111                       | 0.126                       | 0.128                       | 0.128                       |

TABLE VII: Signs of the $d_{j,R}$ coefficients for $2 \leq j \leq 5$ for gauge group $G = SU(N_c)$ and fermion representations $R$ equal to $F$ (fundamental), $adj$ (adjoint), $S_2$, and $A_2$ (symmetric and antisymmetric rank-2 tensor). Note that $d_1 = 0$ for all $G$ and $R$. In the case $R = A_2$, we restrict to $N_c \geq 3$.

| $j$ | $d_{j,F}$ | $d_{j,adj}$ | $d_{j,S_2}$ | $d_{j,A_2}$ |
|-----|-----------|-------------|-------------|-------------|
| 2   | +         | +           | +           | +           |
| 3   | +         | +           | +           | +           |
| 4   | -         | +           | -           | +           |
| 5   | -         | -           | -           | -           |
TABLE VIII: Scheme-independent values of $\beta'_{IR,F,\Delta_f}^{p}$ with $2 \leq p \leq 4$ for $G = SU(2)$, $SU(3)$, and $SU(4)$, as functions of $N_f$ in the respective intervals $I_{IRZ}$. For comparison, we list the $n$-loop values of $\beta'_{IR,F,n\ell}$ with $2 \leq n \leq 5$, where $\beta'_{IR,F,n\ell}$ with $n = 3, 4, 5$ are computed in the MS scheme. The notation $ae^n$ means $a \times 10^{-n}$.

| $N_c$ | $N_f$ | $\beta'_{IR,F,2\ell}$ | $\beta'_{IR,F,3\ell, MS}$ | $\beta'_{IR,F,4\ell, MS}$ | $\beta'_{IR,F,4\ell, MS}$ | $\beta'_{IR,F,5\ell}$ | $\beta'_{IR,F,5\ell}$ |
|-------|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| 2     | 6     | 6.061          | 1.620          | 0.975          | 0.499          | 0.957          | 0.734          | 0.6515        |
| 2     | 7     | 1.202          | 0.728          | 0.677          | 0.320          | 0.554          | 0.463          | 0.436         |
| 2     | 8     | 0.400          | 0.318          | 0.300          | 0.180          | 0.279          | 0.250          | 0.243         |
| 2     | 9     | 0.126          | 0.115          | 0.110          | 0.0799         | 0.109          | 0.1035         | 0.103         |
| 2     | 10    | 0.0245         | 0.0239         | 0.0235         | 0.0200         | 0.0236         | 0.0233         | 0.0233        |
| 3     | 9     | 4.167          | 1.475          | 1.464          | 0.467          | 0.882          | 0.7355         | 0.692         |
| 3     | 10    | 1.523          | 0.872          | 0.853          | 0.351          | 0.621          | 0.538          | 0.473         |
| 3     | 11    | 0.720          | 0.517          | 0.498          | 0.251          | 0.415          | 0.3725         | 0.344         |
| 3     | 12    | 0.360          | 0.2955         | 0.282          | 0.168          | 0.258          | 0.239          | 0.228         |
| 3     | 13    | 0.174          | 0.1556         | 0.149          | 0.102          | 0.144          | 0.137          | 0.134         |
| 3     | 14    | 0.0737         | 0.0699         | 0.0678         | 0.0519         | 0.0673         | 0.0655         | 0.0649        |
| 3     | 15    | 0.0227         | 0.0223         | 0.0220         | 0.0187         | 0.0220         | 0.0218         | 0.0217        |
| 3     | 16    | 2.21e-3        | 2.20e-3        | 2.20e-3        | 2.20e-3        | 2.20e-3        | 2.20e-3        | 2.20e-3       |
| 4     | 11    | 16.338         | 2.189          | 2.189          | 0.553          | 1.087          | 0.898          | 0.648         |
| 4     | 12    | 3.756          | 1.430          | 1.429          | 0.457          | 0.858          | 0.729          | 0.574         |
| 4     | 13    | 1.767          | 0.965          | 0.955          | 0.370          | 0.663          | 0.578          | 0.486         |
| 4     | 14    | 0.984          | 0.655          | 0.639          | 0.292          | 0.498          | 0.445          | 0.394         |
| 4     | 15    | 0.581          | 0.440          | 0.424          | 0.224          | 0.362          | 0.331          | 0.3045        |
| 4     | 16    | 0.348          | 0.288          | 0.276          | 0.1645         | 0.251          | 0.234          | 0.222         |
| 4     | 17    | 0.204          | 0.180          | 0.1725         | 0.114          | 0.164          | 0.156          | 0.1515        |
| 4     | 18    | 0.113          | 0.105          | 0.101          | 0.0731         | 0.0988         | 0.0955         | 0.0939        |
| 4     | 19    | 0.0558         | 0.0536         | 0.0522         | 0.0411         | 0.0520         | 0.0509         | 0.0505        |
| 4     | 20    | 0.0222         | 0.0218         | 0.0215         | 0.0183         | 0.0215         | 0.0213         | 0.0212        |
| 4     | 21    | 5.01e-3        | 4.99e-3        | 4.96e-3        | 4.57e-3        | 4.97e-3        | 4.96e-3        | 4.96e-3       |
TABLE IX: Scheme-independent values of $\beta'_{\IR,\Delta_l}^r$ for $2 \leq p \leq 5$ in the LNN limit \(3.21\) as functions of $r = 5.5 - \Delta_r$. For comparison, we also list the $n$-loop values $\beta'_{\IR,n\ell}^r$ with $2 \leq n \leq 5$, where $\beta'_{\IR,n\ell}^r$ with $n = 3, 4, 5$ are computed in the $\overline{\text{MS}}$ scheme. The notation aem means $a \times 10^{-m}$.

| $r$  | $\beta'_{\IR,2\ell}$ | $\beta'_{\IR,3\ell}$ | $\beta'_{\IR,4\ell}$ | $\beta'_{\IR,5\ell}$ | $\beta'_{\IR,\Delta_2}$ | $\beta'_{\IR,\Delta_3}$ | $\beta'_{\IR,\Delta_4}$ | $\beta'_{\IR,\Delta_5}$ |
|------|------------------|-------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 2.8  | 8.100            | 1.918             | 1.913            | 0.518            | 1.004            | 0.851            | 0.583            |                  |
| 3.0  | 3.333            | 1.376             | 1.379            | 0.444            | 0.830            | 0.717            | 0.535            |                  |
| 3.2  | 1.856            | 1.006             | 1.003            | 0.376            | 0.676            | 0.596            | 0.4755           |                  |
| 3.4  | 1.153            | 0.7395            | 0.729            | 0.314            | 0.542            | 0.486            | 0.410            |                  |
| 3.6  | 0.752            | 0.542             | 0.527            | 0.257            | 0.426            | 0.388            | 0.342            |                  |
| 3.8  | 0.500            | 0.393             | 0.378            | 0.2055           | 0.327            | 0.303            | 0.276            |                  |
| 4.0  | 0.333            | 0.279             | 0.267            | 0.160            | 0.243            | 0.229            | 0.214            |                  |
| 4.2  | 0.219            | 0.193             | 0.184            | 0.120            | 0.174            | 0.166            | 0.159            |                  |
| 4.4  | 0.139            | 0.128             | 0.122            | 0.0860           | 0.119            | 0.115            | 0.112            |                  |
| 4.6  | 0.0837           | 0.0792            | 0.0766           | 0.0576           | 0.0756           | 0.0737           | 0.0726           |                  |
| 4.8  | 0.0460           | 0.0445            | 0.0435           | 0.0348           | 0.0433           | 0.0426           | 0.0423           |                  |
| 5.0  | 0.0215           | 0.0212            | 0.0208           | 0.0178           | 0.0209           | 0.0207           | 0.0206           |                  |
| 5.2  | 0.714e-2         | 0.710e-2          | 0.706e-2         | 0.640e-2         | 0.707e-2         | 0.704e-2         | 0.704e-3         |                  |
| 5.4  | 0.737e-3         | 0.736e-3          | 0.7356e-3        | 0.711e-3         | 0.7358e-3        | 0.7355e-3        | 0.7355e-3        |                  |

TABLE X: Values of the IR zero $\xi^r_{\IR,n\ell}$ of the $\beta_{\xi,n\ell}$ function in the LNN limit for $2 \leq n \leq 5$ and $r \in I_r$. Notation u (unphysical) means that there is no physical IR zero $\xi^r_{\IR,5\ell}$ of the 5-loop beta function.

| $r$  | $\xi_{\IR,2\ell}$ | $\xi_{\IR,3\ell}$ | $\xi_{\IR,4\ell}$ | $\xi_{\IR,5\ell}$ |
|------|------------------|-------------------|------------------|------------------|
| 2.8  | 28.274           | 3.573             | 3.323            | u                |
| 3.0  | 12.566           | 2.938             | 2.868            | u                |
| 3.2  | 7.606            | 2.458             | 2.494            | u                |
| 3.4  | 5.174            | 2.076             | 2.168            | u                |
| 3.6  | 3.731            | 1.759             | 1.873            | u                |
| 3.8  | 2.774            | 1.489             | 1.601            | u                |
| 4.0  | 2.095            | 1.252             | 1.349            | u                |
| 4.2  | 1.586            | 1.041             | 1.115            | u                |
| 4.4  | 1.192            | 0.8490            | 0.9003           | 1.0353           |
| 4.6  | 0.8767           | 0.6725            | 0.7038           | 0.7439           |
| 4.8  | 0.6195           | 0.5083            | 0.5244           | 0.5364           |
| 5.0  | 0.4054           | 0.3538            | 0.3603           | 0.3630           |
| 5.2  | 0.2244           | 0.2074            | 0.2089           | 0.2093           |
| 5.4  | 0.06943          | 0.06769           | 0.06775          | 0.06776          |
TABLE XI: Values of $\alpha_{IR,\Delta p}$ with $1 \leq p \leq 4$ for $N_c = 2, 3, 4$ and $R = F$, as functions of $N_f \in I_{IRZ}$, together with $\alpha_{IR,2\ell}$ and $\overline{MS}$ values of $n$-loop $\alpha_{IR,n\ell}$ with $3 \leq n \leq 4$ from [14], for comparison.

| $N_c$ | $N_f$ | $\alpha_{IR,2\ell}$ | $\alpha_{IR,3\ell}$ | $\alpha_{IR,4\ell}$ | $\alpha_{IR,\Delta 2}$ | $\alpha_{IR,\Delta 3}$ | $\alpha_{IR,\Delta 4}$ |
|-------|------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 2     | 6    | 11.42               | 1.645               | 2.395               | 0.941               | 1.098               | 1.979               | 1.951               |
| 2     | 7    | 2.83                | 1.05                | 1.21                | 0.753               | 0.853               | 1.305               | 1.293               |
| 2     | 8    | 1.26                | 0.688               | 0.760               | 0.565               | 0.621               | 0.8115              | 0.808               |
| 2     | 9    | 0.595               | 0.418               | 0.444               | 0.377               | 0.402               | 0.458               | 0.457               |
| 2     | 10   | 0.231               | 0.196               | 0.200               | 0.188               | 0.1945              | 0.202               | 0.2015              |
| 3     | 9    | 5.24                | 1.028               | 1.072               | 0.587               | 0.712               | 1.19                | 1.26                |
| 3     | 10   | 2.21                | 0.764               | 0.815               | 0.509               | 0.603               | 0.913               | 0.952               |
| 3     | 11   | 1.23                | 0.578               | 0.626               | 0.431               | 0.498               | 0.686               | 0.706               |
| 3     | 12   | 0.754               | 0.435               | 0.470               | 0.352               | 0.397               | 0.500               | 0.509               |
| 3     | 13   | 0.468               | 0.317               | 0.337               | 0.327               | 0.301               | 0.350               | 0.353               |
| 3     | 14   | 0.278               | 0.215               | 0.224               | 0.196               | 0.210               | 0.227               | 0.228               |
| 3     | 15   | 0.143               | 0.123               | 0.126               | 0.117               | 0.122               | 0.126               | 0.126               |
| 3     | 16   | 0.0416              | 0.0397              | 0.0398              | 0.0391              | 0.0397              | 0.0398              | 0.0398              |
| 4     | 11   | 14.00               | 0.972               | 0.943               | 0.474               | 0.592               | 1.042               | 1.1475              |
| 4     | 12   | 3.54                | 0.754               | 0.759               | 0.431               | 0.528               | 0.867               | 0.939               |
| 4     | 13   | 1.85                | 0.6035              | 0.628               | 0.388               | 0.467               | 0.713               | 0.7605              |
| 4     | 14   | 1.16                | 0.489               | 0.521               | 0.345               | 0.407               | 0.580               | 0.610               |
| 4     | 15   | 0.783               | 0.397               | 0.428               | 0.3015              | 0.349               | 0.465               | 0.483               |
| 4     | 16   | 0.546               | 0.320               | 0.345               | 0.258               | 0.294               | 0.367               | 0.376               |
| 4     | 17   | 0.384               | 0.254               | 0.271               | 0.215               | 0.240               | 0.282               | 0.2865              |
| 4     | 18   | 0.266               | 0.194               | 0.205               | 0.172               | 0.188               | 0.210               | 0.211               |
| 4     | 19   | 0.175               | 0.140               | 0.145               | 0.129               | 0.138               | 0.147               | 0.148               |
| 4     | 20   | 0.105               | 0.091               | 0.092               | 0.0861              | 0.0905              | 0.0928              | 0.0929              |
| 4     | 21   | 0.0472              | 0.044               | 0.0444              | 0.0431              | 0.04405             | 0.0444              | 0.0444              |