Electromagnetic fields in the exterior of an oscillating relativistic star – II. Electromagnetic damping

Luciano Rezzolla$^{1, 2}$ and Bobomurat J. Ahmedov$^{3, 4, 5}$

$^1$Institute for Theoretical Physics, Max-von-Laue-Str. 1, D-60438 Frankfurt, Germany
$^2$Frankfurt Institute for Advanced Studies, Ruth-Moufang-Str. 1, D-60438 Frankfurt, Germany
$^3$Institute of Nuclear Physics, Ulughbek, Tashkent 100214, Uzbekistan
$^4$Ulugh Beg Astronomical Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan
$^5$National University of Uzbekistan, Tashkent 100174, Uzbekistan

6 May 2016

ABSTRACT
An important issue in the asteroseismology of compact and magnetized stars is the determination of the dissipation mechanism which is most efficient in damping the oscillations when these are produced. In a linear regime and for low-multipolarity modes these mechanisms are confined to either gravitational-wave or electromagnetic losses. We here consider the latter and compute the energy losses in the form of Poynting fluxes, Joule heating and Ohmic dissipation in a relativistic oscillating spherical star with a dipolar magnetic field in vacuum. While this approach is not particularly realistic for rapidly rotating stars, it has the advantage that it is fully analytic and that it provides expressions for the electric and magnetic fields produced by the most common modes of oscillation both in the vicinity of the star and far away from it. In this way we revisit and extend to a relativistic context the classical estimates of McDermott et al. Overall, we find that general-relativistic corrections lead to electromagnetic damping time-scales that are at least one order of magnitude smaller than in Newtonian gravity. Furthermore, with the only exception of $g$ (gravity) modes, we find that $f$ (fundamental), $p$ (pressure), $i$ (interface) and $s$ (shear) modes are suppressed more efficiently by gravitational losses than by electromagnetic ones.

Key words: MHD – waves – stars: neutron – stars: oscillations – pulsars: general.

1 INTRODUCTION
Neutron stars are endowed with intense electromagnetic fields, but they may also subject to oscillations of various type, such as those observed as quasi-periodic oscillations (QPOs) following giant flares of soft gamma-ray repeaters (SGRs; Israel et al. 2005; Strohmayer & Watts 2005, 2006; Watts & Strohmayer 2006, 2007b,a; Huppenkothen et al. 2014; Turolla et al. 2015). The analysis of the X-ray data from SGRs has in fact revealed that the decaying part of spectrum exhibits a number of oscillations with frequencies in the range of a few tenths of Hz to a few hundreds Hz, that agree reasonably well with the expected toroidal modes of the magnetar crust (Duncan 1998). For example, QPO frequencies around 18, 26, 30, 92, 150, 590 and 1163 Hz were discovered in the outburst of SGR1806-20$^1$, while QPO frequencies around 28, 53, 84 and 155 Hz were detected for the outburst of SGR1900+14. More recently, El-Mezeini & Ibrahim (2010) claimed the discovery of 84, 103 and 648 Hz QPOs in the normal bursts of SGR1806-20. Besides providing the first evidence for oscillations in compact stars, these observations are very important because they motivate further study in neutron star oscillation modes as a powerful tool for probing the complex stellar interior.

The attempts to explain the above QPO frequencies in SGRs by using the crust toroidal oscillations has produced a vast literature in a very short time (Levin 2006; Glampedakis et al. 2006; Samuelsson & Andersson 2007; Sotani et al. 2007; Levin 2007; Sotani et al. 2008; Colaiuda et al. 2009; Cerdá-Durán et al. 2009; Sotani & Kokkotas 2009; Gabler et al. 2011, 2012, 2013, 2014). Overall, all of these works have shown that an Alfvén QPO model is a viable explanation for the observed QPOs if a number of suitable conditions are met by the

$^1$ Using a different technique, Hambaryan et al. (2011) have also found new QPOs in the data of the SGR1806-20 giant burst at frequencies 16.9, 21.4, 36.4, 59.0 and 116.3 Hz.

© 2016 RAS
magnetic-field strength and geometry. Additional observations are clearly needed to impose additional constraints on the different models. At the same time, a considerable effort has also been dedicated to the establishing how much of the vibrational modes excited in magnetar giant flares can lead to the excitation of $f$ modes and to the copious emission of gravitational waves. Also in this case, the various works have pointed out that only a small fraction of the flare's energy is converted directly into the lowest order $f$ modes and that the corresponding gravitational-wave emission is small (Ciolfi et al. 2011; Lasky et al. 2011; Ciolfi & Rezzolla 2012; Zink et al. 2012).

Crustal oscillations of nonrotating non-magnetized neutron stars have been studied in Newtonian theory (McDermott et al. 1985, 1988) as well as in general relativity (e.g., Schumaker & Thorne 1983; Finn 1990). The study of the oscillations of magnetized and rotating relativistic stars is far from being trivial or fully solved. At least in principle, in fact, it involves the solution of the coupled Einstein-Euler-Maxwell equations, as well as a detailed modelling of the stellar structure via realistic equations of state. In practice, however, the investigations have so far concentrated on two distinct but complementary approaches. The first one has focused on the study of the eigenfrequencies and eigenfunctions of selected modes of magnetized neutron stars in general relativity (e.g., Messios et al. 2001; Chugunov & Yakovlev 2005; Piro 2005; Sotani et al. 2007; Glampedakis et al. 2006; Samuelsson & Andersson 2007; Vavoulidis et al. 2007)]. The second one, instead, has focused its attention on the properties of the electromagnetic fields from near the stellar surface up to the wave-zone (e.g., McDermott et al. 1984; Muslimov & Tsygan 1986; Muslimov & Harding 1997; Konno et al. 1999, 2000; Konno & Kojima 2000; Timokhin et al. 2000; Rezzolla et al. 2001b,a; Kojima & Okita 2004; Rezzolla & Ahmedov 2004)\(^2\). The work reported in this paper belongs to the second class of studies and concentrates on calculating explicit expressions for the electromagnetic fields generated by radial, toroidal and spheroidal oscillations of the stellar surface.

A number of different motivations are behind this investigation. First, we intend to provide analytical expressions for the electric and magnetic fields produced by the most common modes of oscillation for a relativistic magnetized spherical star in vacuum, both in the vicinity of the star and far away from it. In this way we specialize the generic expressions for the vacuum electromagnetic fields produced by a relativistic magnetized spherical star presented in Rezzolla & Ahmedov (2004) to the most common oscillation modes. Secondly, we attempt to address an important issue in the asteroseismology of compact and magnetized relativistic stars, namely, the general-relativistic corrections on the dissipation mechanisms that are most efficient in damping the oscillations. Because in a linear regime, these mechanisms are confined to either gravitational-wave or electromagnetic losses, here we compute and compare, within a general-relativistic framework, the energy losses as produced by the oscillations in the form of Poynting fluxes, Joule heating and by the gravitational-wave emission. In this way, we revisit and extend to a general-relativistic context the Newtonian estimates of McDermott et al. (1988) for the damping times of oscillating magnetized neutron stars due to electromagnetic radiation.

Overall, a number of factors, such as the type of mode, the magnetic-field strength and the compactness of the star, concur in determining what is the main damping mechanism of the oscillations. However, we have found that the following results are generically true for a typical neutron star with a dipolar magnetic field of $\sim 10^{12}$ G: (i) the general-relativistic corrections to the electromagnetic fields lead to damping time-scales due to electromagnetic losses which are at least one order of magnitude smaller than their Newtonian counterparts; (ii) with the only exception of $g$ (gravity) modes, we find that $f$ (fundamental), $p$ (pressure), $i$ (interface) and $s$ (shear) modes are suppressed more efficiently by gravitational losses than by electromagnetic ones; (iii) Joule heating is not as an important damping mechanism in general relativity as it is in Newtonian gravity.

The paper is organized according to the following plan: in Section 2 we provide a detailed discussion of the electromagnetic fields produced by spheroidal and toroidal oscillations in the vicinity of the stellar surface (i.e., in the 'near zone'), while the following Section 3 is devoted to the calculation of the corresponding fields at a large distance from the star (i.e., in the 'wave zone'). In both cases, we show that in the Newtonian limit our solutions coincide with those obtained by Muslimov & Tsygan (1986). In Section 4 we instead investigate the damping due to electromagnetic emission when the star is modelled as a relativistic polytrope with infinite conductivity, as well as the damping due to Joule heating when the stellar matter has high but finite conductivity. The damping times are then compared with the corresponding damping times due to gravitational-wave emission. In spite of its general-relativistic amplification, Ohmic dissipation is shown to be almost insignificant because of the small relative motions of the stellar crust matter with respect to the field lines due to the high electric conductivity in neutron stars. Finally, Section 5 collects our results and discusses the prospects of future work.

We use in this paper a system of units in which $c = 1$, a space-like signature $(-, +, +, +)$ and a spherical coordinate system $(t, r, \theta, \phi)$. Greek indices are taken to run from 0 to 3, Latin indices from 1 to 3 and we adopt the standard convention for the summation over repeated indices. We will indicate four-vectors with bold symbols (e.g., $\mathbf{u}$) and three-vectors with an arrow (e.g., $\overline{u}$).

## 2 ELECTROMAGNETIC FIELDS IN THE NEAR-ZONE

The study of the electromagnetic fields produced by an oscillating and magnetized neutron star has a long history and a first comprehensive investigation was carried out by McDermott et al. (1984), who estimated the emission of electromagnetic radiation in the wave zone from nonradial pulsations of a neutron star with a strong dipolar magnetic field in vacuum, and who provided the first estimates of the electromagnetic damping within a Newtonian description of gravity. Subsequently, Muslimov & Tsygan (1986) derived, again in Newtonian gravity,

---

\(^2\) It is important to clarify that the solutions presented in Rezzolla et al. (2001a,b) are exact in the slow-rotation approximation; see also Pétri (2013) for a recent re-derivation of the very same equations using a different approach.
exact analytical solutions for the vacuum electromagnetic fields around an oscillating neutron star with a dipolar surface magnetic field (both in the near and wave zones), while Mc Dermott et al. (1988) completed a detailed analysis of various oscillation modes and computed the corresponding damping times. Later on, Duncan (1998) improved the results of Mc Dermott et al. (1988) with better estimates for the parameters of typical neutron stars, and has calculated the eigenfrequencies of the toroidal modes of a magnetar.

At about the same time, Timokhin et al. (2000) have pointed out another interesting aspect of oscillating magnetized stars and, in particular, that a neutron star with magnetic field \( B \) and oscillating at a frequency \( \omega \) will generate an electric field \( E \sim \omega \xi B / c \), where \( \xi \) is the characteristic (linear) amplitude of the oscillations. Interestingly, such an electric field can be strong enough to pull charged particles from the surface and create a magnetosphere even in the absence of rotation. These results were first extended to the general-relativistic context for a relativistic spherical nonrotating neutron star by Abdikamalov et al. (2009). In Morozova et al. (2010), instead, it has been shown that the electromagnetic energy losses from the polar cap region of a rotating neutron star can be significantly enhanced if oscillations are also present, and, for the mode spherical-harmonics indices \( (\ell, m) = (2, 1) \), such electromagnetic losses are a factor of \( \sim 8 \) larger than the rotational energy losses, even for a velocity oscillation amplitude at the star surface as small as \( \hat{v} = 0.05 \Omega R \), where \( \Omega \) and \( R \) are the angular velocity and the radius of the neutron star, respectively. In Morozova et al. (2012), on the other hand, the conditions for radio emission in magnetars have been considered and it has been found that, when oscillations of the magnetar are taken into account, the radio emission from the magnetosphere is generally favoured. Indeed, the major effect of the oscillations is to amplify the electric potential in the polar cap region of the magnetar magnetosphere.

In Rezzolla & Ahmedov (2004) (hereafter paper I), we have derived exact analytical expressions for the interior and exterior electromagnetic fields in a perfectly conducting relativistic star, expressing them in terms of generic velocity and magnetic fields. Starting from that work, we here specialize those expressions to the most common modes of oscillations. For simplicity we will assume star to be in vacuum and refer the discussion of the case in which a magnetosphere is present to the works of (Abdikamalov et al. 2009; Morozova et al. 2010, 2012; Zanotti et al. 2012; Morozova et al. 2014; Lin et al. 2015).

Because our analysis is essentially analytical, we simplify the treatment by considering separately the electromagnetic fields in the vicinity of the stellar surface, i.e., in the ‘near zone’, where they are almost stationary, and in regions far away from the surface, i.e., in the ‘wave zone’, where they assume the properties of electromagnetic radiation. In doing this we will assume the space-time to be that of a spherical relativistic star of mass \( M \) with a line element that in a spherical coordinate system \((t, r, \theta, \phi)\) is given by

\[
d s^2 = g_{00} d t^2 + g_{11} d r^2 + r^2 (d \theta^2 + \sin^2 \theta d \phi^2).
\]

The portion of the space-time exterior to the star (i.e., for \( r \geq R \)) is simply given by the Schwarzschild solution with \( g_{00} = N^2 := (1 - 2M/r), g_{11} = -1/g_{00} \), and \( N_2 := 1 - 2M/R \) is the redshift at the stellar surface.

In what follows, we will first discuss the generic form of the magnetic and electric fields in the near zone and then illustrate the expressions they assume in the case of radial, spheroidal and toroidal oscillations, respectively. We will then present the generic expressions for the perturbed magnetic and electric fields, and, subsequently proceed to providing the corresponding expressions for most common modes of oscillations.

### 2.1 General expressions for the magnetic field

As shown in paper I, given a background magnetic field decomposed in terms of spherical harmonics [see equations 55–57 of paper I], it is possible to express the time dependence of a linear perturbation in the magnetic field components \( \delta B^\ell, m \) in terms of new functions, \( \delta s_{\ell m}(t) \), which are linear superposition of the background ones. As a result, according to equations 67–69 of paper I, when a generic velocity perturbation of the type

\[
\delta u^\alpha \big|_{r=R} = \frac{1}{N_{\alpha}} \left(1, \delta v_{\alpha}^i \right) = \frac{1}{N_{\alpha}} \left(1, N_{\alpha} \delta v_{\alpha}^i \frac{\delta v_{\beta}^i}{R} \frac{\delta v_{\phi}^i}{R \sin \theta} \right),
\]

and

\[
\delta u_{\alpha \nu} \big|_{r=R} = \frac{1}{N_{\nu}} \left( -N_{\alpha} \frac{\delta v_{\alpha}^i}{N_{\alpha}}, R \delta v_{\beta}^i, R \sin \theta \delta v_{\phi}^i \right),
\]

(where \( \delta v_{\alpha}^i := dx^i/dt \big|_{r=R} \) is the oscillation three-velocity of the conducting stellar medium at the stellar surface) is introduced over a background dipolar magnetic field (i.e., with \( \ell = 1 \)), the complete expressions for the newly generated magnetic-field components are given by the real parts of the following complex expressions

\[
\delta B^u = \frac{1}{M^2} \left[ \ln N^2 + \frac{2M}{r} \left(1 + \frac{M}{r}\right) \right] \delta s_{1m} e^{im \phi} \cos \theta,
\]

\[
\delta B^\theta = -\frac{N}{M^2 r} \left( r \ln N^2 + \frac{1}{N^2} + 1 \right) \delta s_{1m} e^{im \phi} \sin \theta,
\]

\[
\delta B^\phi = \frac{M}{r} \ln N^2 + \frac{1}{N^2} + 1 \delta s_{1m} e^{im \phi} \cos \theta,
\]

© 2016 RAS, MNRAS 000, 1–21
where \( \Delta^i \equiv \tilde{\omega}^i_A \dot{A}^k \) are the components of an arbitrary quantity \( A^k \) in the orthonormal frame carried by static observers [see equations 6–9 of paper I for the explicit expressions of the 1-forms \( \tilde{\omega}^i_A \)]. The values of the integration constants \( \delta s_{1m}(t) \) can be calculated rather straightforwardly if the oscillation modes are assumed to have a harmonic time dependence of the type \( \exp(-i\omega_m t) \), where \( \omega_m \) is the mode frequency at the stellar surface and corresponds to the real part of the complex mode eigenfrequency. In this case, using equation (B4) from paper I and thus after requiring the continuity of the tangential electric field at the stellar surface, we obtain the following condition for the integration constant

\[
\frac{\partial}{\partial t} \delta s_{1m}(t)|_{r=R} = -\frac{3R}{8f \mu} \int d\Omega Y_{1m}^* \left\{ \left( \nabla^2 \dot{A}^i \right) \frac{1}{R \sin \theta} \left[ \partial_\theta \left( \sin \theta \delta v^\theta \right) + \partial_\phi \delta v^\phi \right] + N_\mu \left[ \left( \partial^i_{\theta \phi} \right) \partial_\theta \delta v^\theta + \frac{1}{\sin^2 \theta} \left( \partial^i_{\theta \theta} \right) \partial_\phi \delta v^\phi \right] + \left[ N_\mu \partial_i \left( \nabla^2 \dot{A}^i \right) \delta v^\phi + \frac{1}{R} \partial_\theta \left( \nabla^2 \dot{A}^i \right) \delta v^\theta + \frac{1}{R \sin \theta} \partial_\phi \left( \nabla^2 \dot{A}^i \right) \delta v^\phi \right] \right\},
\]

(7)

where \( S = S_{1m}(r)Y_{1m}, \ d\Omega = \sin \theta d\theta d\phi, \) and \( \nabla^2 \dot{A}^i \) is the angular part of the Laplacian, i.e.,

\[
\nabla^2 \dot{A}^i := \frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta \right) + \frac{1}{\sin^2 \theta} \partial_\phi^2.
\]

(8)

Note that differently from what done in paper I, here the integration constants have acquired also a time dependence as a result of the dynamics of the stellar surface across which the boundary conditions need to be imposed. However, as it will be shown in Section 2.3, these corrections are negligible at first order in the perturbation.

For a dipolar magnetic field and using equation 53 of paper I, the ‘magnetic’ scalar function \( S \) in equation (7) is given by

\[
S = \frac{r^2}{2M^2} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] s_{1m} Y_{1m},
\]

(9)

where \( Y_{1m} \) are the standard spherical harmonics and the first \( s_{1m} \) coefficients are given by

\[
s_{10} = -\frac{\sqrt{3\pi}}{4} BR^3 \cos \chi, \quad s_{11} = \frac{\sqrt{3\pi}}{2} BR^3 \sin \chi.
\]

(10)

Here \( B := 2\mu/R^3 \) is the (Newtonian) value of the surface dipolar magnetic field at the magnetic pole and \( \mu \) is magnetic moment, \( \chi \) is the inclination angle between the magnetic axis and the spherical polar axis. As a result, the integration constants defined in equation (7) have the explicit expressions

\[
\frac{\partial}{\partial t} \delta s_{1m}(t)|_{r=R} = \frac{3BR^2}{8f \mu} \int d\Omega Y_{1m}^* \left\{ \left( \cos \chi \cos \theta + \sin \chi \sin \theta e^{i\phi} \right) \left[ \frac{f_\mu}{\sin \theta} \left( \sin \theta \delta v^\theta \right) + \partial_\phi \delta v^\phi \right] - 2h_\mu \delta v^\phi \right\} - \left( \cos \chi \sin \theta - \sin \chi \cos \theta e^{i\phi} \right) \left[ h_\mu \partial_\theta \delta v^\theta + f_\mu \delta v^\phi \right] + i \sin \chi e^{i\phi} \left[ f_\mu \delta v^\phi + \frac{1}{\sin \theta} h_\mu \partial_\phi \delta v^\phi \right].
\]

(11)

In equation (11), the coefficients \( h_\mu \) and \( f_\mu \) contain the general-relativistic correction of the magnetic-field intensity at the stellar surface and are expressed as

\[
h_\mu = \frac{3R^2 N_\mu}{8M^2} \left[ \frac{R}{M} \ln N^2_e + \frac{1}{N^2_e} + 1 \right], \quad f_\mu = -\frac{3R^3}{8M^3} \left[ \ln N^2_e + \frac{2M}{R} \left( 1 + \frac{M}{R} \right) \right].
\]

(12)

so that the unperturbed magnetic-field components at the stellar surface can be written in a form that is reminiscent of the Newtonian expressions, i.e.,

\[
B^\theta_\mu = f_\mu B \left( \cos \chi \cos \theta + \sin \chi \sin \theta e^{i\phi} \right),
\]

(13)

\[
B^\phi_\mu = h_\mu B \left( \cos \chi \sin \theta - \sin \chi \cos \theta e^{i\phi} \right),
\]

(14)

\[
B^\phi_\mu = -i h_\mu B \sin \chi e^{i\phi}.
\]

(15)

Two remarks are worth making at this point. First, as a consequence of the assumption of infinite conductivity for the stellar material, the magnetic field is advected with the fluid; this is expressed by the ‘frozen-flux’ condition for the perturbed magnetic field

\[
\frac{\partial}{\partial t} \delta \vec{B} = \nabla \times (\vec{v} \times \delta \vec{B}),
\]

(16)

and is satisfied by equations (4)–(6). Secondly, expressions (4)–(6) refer to a generic velocity field and thus lead to the integration constants (11) that are totally general.

\footnote{Note that \( h_\mu = 1/2 \) and \( f_\mu = 1 \) in the Newtonian limit of \( M/R \to 0 \)}
2.2 General expressions for the electric field

The general expression for the electric fields can be obtained following the same procedure adopted for the magnetic field in the previous section. In particular, we recall that the solution for the vacuum electric field in the near zone produced via a perturbation of a dipolar magnetic field can be written as [see equations 78–80 of paper I]

$$\delta E^\varphi = \frac{M^2}{r^2} \ell^2 (\ell + 1) \left[ Q_{\ell - 1} - \left( 1 + \frac{r}{M} \ell \right) Q_{\ell} \right] \delta t_{\ell m} Y_{\ell m},$$  

\( (17) \)

$$\delta E^\vartheta = \frac{M^2}{r^2 N} \ell^2 (\ell + 1) \left[ \left( 1 - \frac{r}{M} \right) Q_{\ell} - Q_{\ell - 1} \right] \delta t_{\ell m} \partial_\theta Y_{\ell m} + \frac{r}{2N \sin \theta} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \delta x_{1m} \partial_\phi Y_{1m},$$  

\( (18) \)

$$\delta E^\varpi = \frac{M^2}{r^2 N \sin \theta} \ell^2 (\ell + 1) \left[ \left( 1 - \frac{r}{M} \right) Q_{\ell} - Q_{\ell - 1} \right] \delta t_{\ell m} \partial_\varphi Y_{\ell m} - \frac{r}{2N} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \delta x_{1m} \partial_\psi Y_{1m},$$  

\( (19) \)

where \( Q_\ell = Q_\ell (x) = Q_\ell (1 - r/M) \) is the Legendre function of the second kind (Arfken & Weber 2005). Note that, in general, the perturbed electric field will have a zero background value but a non-vanishing first-order perturbation which is proportional to the first-order velocity perturbation and to the zero-th order magnetic field, i.e., \( |\delta E| \propto |\delta v B_0| \).

As in paper I, the integration constants \( \delta t_{\ell m} \) and \( \delta x_{1m} \) can be calculated from imposing the continuity of the tangential components of the electric field, and thus through expressions involving the values of the magnetic field and of the velocities at the stellar surface as

$$\delta t_{\ell m}(t) = \frac{R^2}{\ell^3 (\ell + 1)^2 M^2} \left[ \left( 1 - \frac{R}{M} \right) Q_\ell (x_h) - Q_{\ell - 1} (x_h) \right]^{-1} \times \int d\Omega \left\{ \partial_\varphi Y^*_{\ell m} \left[ \delta v^\varphi (t) B^\varphi_{\ell m} - \delta v^\vartheta (t) B^\vartheta_{\ell m} \right] - i \frac{M Y^*_m}{r \sin \theta} \left[ \delta v^\psi (t) B^\psi_{\ell m} - \delta v^\varpi (t) B^\varpi_{\ell m} \right] \right\},$$  

\( (20) \)

$$\delta x_{1m}(t) = -\frac{3R^2}{8M^3 f_k} \int d\Omega \left\{ \partial_\varphi Y^*_{1 m} \left[ \delta v^\varphi (t) B^\varphi_{1 m} - \delta v^\vartheta (t) B^\vartheta_{1 m} \right] + i \frac{M Y^*_m}{r \sin \theta} \left[ \delta v^\psi (t) B^\psi_{1 m} - \delta v^\varpi (t) B^\varpi_{1 m} \right] \right\},$$  

\( (21) \)

where \( Q_\ell (x_h) := Q_\ell (1 - R/M) \).

Another aspect of the interconnection between electric and magnetic fields worth remarking is expressed in equations (17)–(19), where a magnetic field with multipolar components up to \( \ell \) induces a perturbed electric field with multipolar components up to \( (\ell + 1) \). As an aid for the calculations which will be presented in the following, we report the explicit expressions of the coefficients multiplying the integral in equation (20) for some relevant values of the multipoles

$$\frac{R^2}{\ell^3 (\ell + 1)^2 M^2} \left[ \left( 1 - \frac{R}{M} \right) Q_\ell (x_h) - Q_{\ell - 1} (x_h) \right]^{-1} = \begin{cases} 3R^3 / (16 M^3 N_k h_k) & \text{for } \ell = 1, \\ 1 / (54 N_k^2 g_k) & \text{for } \ell = 2, \\ 1 / (432 N_k^2 k_k) & \text{for } \ell = 3, \\ -3M^3 / (100 R^2 \gamma_k) & \text{for } \ell = 4. \end{cases}$$  

\( (22) \)

Here \( g_k \) is a constant coefficient given by [see equation 129 of paper I]

$$g_k := \left( 1 - \frac{R}{M} \right) \ln N_k^2 - \frac{2M^2}{3R^2 N_k^2} - 2 = \left( 1 - \frac{R}{M} \right) \ln \left( 1 - \frac{2M}{R} \right) - \frac{2}{3} \left( \frac{M}{R} \right)^2 \frac{R}{R - 2M} - 2,$$  

\( (23) \)

while the functions \( k_k \) and \( \gamma_k \) have explicit expressions

$$k_k := \left\{ -\frac{1}{3} \left( 1 - \frac{R}{M} \right) \left( \frac{M^2}{R^2 N_k^2} + \frac{15}{2} \right) + \left[ 1 + \frac{5R^2 N_k^2}{4M^2} \right] \ln N_k^2 \right\},$$  

\( (24) \)

$$\gamma_k := \frac{2M}{R} \left( 105N_k^2 + 95 \frac{M^2}{R^2 N_k^2} + 6 \frac{M^4}{R^2 T_k^2} \right) + N_k^2 \left( 1 - \frac{M}{R} \right) (7N_k^2 + 4) \ln N_k^2.$$  

\( (25) \)

Note that in the Newtonian limit, i.e., when \( M/R \to 0 \), the coefficients \( g_k, \gamma_k \) and \( k_k \) go to zero, but the integration constants converge to finite values.

2.3 Main properties of spheroidal oscillations

While expressions (4)–(6) and (17)–(19) are particularly effective because of their completeness and generality, they are not particularly useful if not specialized to a specific perturbation experienced by the star. In view of this, in the following sections, we concentrate on the form that these expressions attain when referred to the most common modes of oscillation and using the mode nomenclature of McDermott.
et al. (1988). Before doing that, here, we briefly recall the main properties of the typical oscillation modes of relativistic neutron stars. Much of this material is well known but we recall it here for completeness and because it will turn out useful in the subsequent discussion. More information can be found in the review by Kokkotas & Schmidt (1999).

When modelled as having a fluid core, a solid crust, and a thin surface fluid ‘ocean’, neutron stars are capable of sustaining a broad variety of normal modes of oscillation. For any star, there are two general categories of non-radial oscillations: spheroidal (or polar) modes and toroidal (or axial) modes.

Spheroidal modes, in particular, include several subclasses: the $p$, $f$, and $g$ modes, which are well known from conventional stellar pulsation theory, but also $s$ and $r$ modes, which result from crustal elasticity and play a particularly important role in neutron stars. $p$ modes have pressure gradients as the main restoring force. The eigenfunctions of a $p$ mode with quantum number $n$ has $n$ nodes and, as $n$ increases, the frequency increases and the wavelength becomes smaller. In the limit of short wavelengths, these modes represent simple acoustic waves travelling in the star. In the same limit, the frequency of the mode will tend to infinity. For a ‘canonical’ neutron star, i.e., a neutron star with mass $M \approx 1.4 M_{\odot}$ and radius $R \approx 14$ km, the typical frequency of the lowest order $p$-mode is a few kHz and are roughly proportional to the average rest-mass density, $g$, on the other hand, modes have a buoyancy force (produced, for instance, by gradients in temperature, composition or density) as the main restoring force. There are two groups of $g$ modes: the ‘core’ $g$ modes, which are displacements confined almost completely to the fluid core, and the ‘surface’ $g$ modes, which are limited primarily to the thin fluid layer overlaying the crust. The core $g$ modes have typical frequencies of $\sim 0.1$ kHz, while the surface $g$ modes have a typical frequency of $\sim 10$ Hz. The $g$-mode frequencies are roughly proportional to the internal temperature. $f$ modes have a character which is intermediate between those of $p$- and $g$ modes and are also referred to as the fundamental modes of oscillation. For each $\ell$, the frequency of this mode is between the lowest order $g$-mode (i.e., the highest frequency $g$-mode) and the lowest order $p$-mode (i.e., the lowest frequency $p$-mode). Note that for a given pair of quantum numbers $(\ell, m)$, only one $f$-mode exists and its eigenfunctions have no nodes. The typical frequency of the lowest order $f$-mode for a canonical non-rotating neutron star is $\sim 2-3$ kHz. $s$ modes are essentially normal modes of shear waves in the solid neutron star crust. These modes have quadrupole frequencies $\sim$ kHz and depend strongly on the crust thickness. Waves can also propagate on the solid-fluid (crust-core) interface, and the normal modes corresponding to such waves are the interface, or $i$ modes; these modes resemble acoustic waves scattered off a hard sphere and do not induce significant fluid motion. The frequencies of these modes depend strongly on the local density and temperature at the interface, but are normally with frequencies of a few kHz or higher.

Finally, toroidal modes are modes that, unsurprisingly, have eigenfunctions described by purely toroidal (axial) functions and hence do not have radial displacements. $r$ modes, in particular, are a well-known member of this class of modes and have the Coriolis force as restoring force; in this respect, however, $r$ modes represent more an exception than a rule. Toroidal modes (and hence $r$ modes) are, in fact, part of a larger class of modes having the Coriolis force as the main restoring force. Such modes are called inertial modes and have eigenfunctions with a mixed spheroidal and toroidal nature, approximately of the same magnitude, at least to first order in the slow-rotation expansion. In this respect, $r$ modes can be seen as inertial modes with purely toroidal eigenfunctions. The velocity eigenfunctions in the case of toroidal modes have very simple expressions and the perturbations in the density and pressure appear at orders higher than the first one in the slow-rotation expansion.

We can now start our specialization of the general expressions presented above by first considering the case of spheroidal oscillations, where the Euler velocity field is given by [see, for example, equation 13.60 of Unno et al. (1989)]

$$\delta v^i = \left( \frac{\eta(r) Y_{\ell m'}(\theta, \phi)}{\sin \theta} \partial_\theta Y_{\ell m'}(\theta, \phi) + \frac{\xi(r)}{\sin \theta} \partial_\phi Y_{\ell m'}(\theta, \phi) \right) e^{-i \omega t},$$

(26)

with $\omega$ being the real part of the oscillation frequency, while $\eta(r)$ and $\xi(r)$ are the radial eigenfunctions. We next assume that the oscillations are zero at the centre of the star, but non-vanishing at its surface, i.e., $\eta(0) = 0$, and $\eta_0 := \eta(R) \neq 0$. Note that in principle the eigenfunctions $\eta(r)$ and $\xi(r)$ should be calculated through the solution of the corresponding eigenvalue problem. Here, however, we will consider them as given functions, whose value at the stellar surface will be used to estimate the electromagnetic emission. Note also that we have used multipolar indices $\ell$ and $m'$ to distinguish the harmonic dependence of the velocity perturbations (26) from the harmonic dependence of the electromagnetic fields, which we express in terms of the indices $\ell$ and $m$.

Since the stellar surface itself is undergoing oscillations also in the radial direction, its radial position in time will be expressed as

$$R(t) \cong R_0 + \int_0^t \delta v^r (R_0, t') \, dt',$$

(27)

where $R_0$ is the position of the stellar surface at $t = 0$. We next assume that $\delta v^r (R_0, t) = 0$ is the transverse displacement, and $\eta = \eta_0 e^{-i \omega t}$ is the displacement of the stellar crust in the radial direction, with $\eta = -i \omega \eta_0 \ll 1$. From the condition of quasi-stationarity, i.e., $|\delta v^r|/c = \omega \eta_0 / c \ll 1$, one can estimate that the normalized radial displacement $\epsilon = \eta / R_0 \ll 1$ for typical oscillations in the kHz range. As a result, the radial position of the stellar surface (27) is

$$R(t) = R_0 + \frac{\eta}{R_0} R_0 = R_0 (1 + \epsilon(t)).$$

(28)

In principle, the dipolar magnetic field at the stellar surface will also change a result of the changes in the position of the stellar surface and assume a general form of type

$$B(t, R) = \frac{2 \mu(t)}{R^3(t)} \sim \frac{2 \mu_0}{R^3_0} \approx \frac{2 \mu_0}{R^3_0} [1 - 3 \epsilon(t)] = B_0 (1 - 3 \epsilon(t)),$$

(29)

© 2016 RAS, MNRAS 000, 1–21
where \( B_0 := 2\mu/R_0^3 \). However, the time derivative of the dipolar magnetic field at the dynamical boundary of the star is then
\[
\partial_t B(t, R) = -3B_0 \partial_t \epsilon = 3i\omega \frac{\eta}{R_0} B_0 = \mathcal{O}(\epsilon^2),
\]
(30)
since the terms proportional to \( \epsilon \) appear as product \( \epsilon \omega R_0 \) and are negligible in the linear approximation where \( \epsilon \ll 1 \) and \( \omega R_0 \ll 1 \).

Consider now a relativistic star having a background dipolar magnetic field (i.e., with \( \ell = 1 \)) not necessarily aligned with the polar axis (i.e., with \( \chi \neq 0 \)) and undergoing oscillations with components (26). The integration constants (11) for the spheroidal oscillations of the dipolar magnetic field then take the form
\[
\delta s_1 = 3i \left( \frac{R^2}{8\sqrt{5} \omega_k h_k} B_0 (h_k \eta_k - 3f_k \xi_k) \right) e^{-i\omega_k t} \cos \chi .
\]
(33)

As a result, the components of the perturbed magnetic field (4)–(6) in the near zone are the real parts of the following complex expressions
\[
\delta B^\ell = -i\sqrt{\frac{3}{10\pi}} \frac{3R^2}{16M^2 \omega_k h_k} \left( \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right) B_0 (h_k \eta_k - 3f_k \xi_k) e^{-i(\omega_k t - \phi)} \cos \chi \sin \theta ,
\]
(34)
\[
\delta B^\theta = -i\sqrt{\frac{3}{10\pi}} \frac{3R^2}{16M^2 \omega_k h_k} \left( \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right) B_0 (h_k \eta_k - 3f_k \xi_k) e^{-i(\omega_k t - \phi)} \cos \chi \cos \theta ,
\]
(35)
\[
\delta B^\phi = \sqrt{\frac{3}{10\pi}} \frac{3R^2}{16M^2 \omega_k h_k} \left( \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right) B_0 (h_k \eta_k - 3f_k \xi_k) e^{-i(\omega_k t - \phi)} \cos \chi ,
\]
(36)
where we recall that \( \omega_k \) is the angular frequency of oscillation measured by an observer at the stellar surface.

The oscillation frequency, in fact, is itself subject to the standard gravitational redshift, so that the frequency \( \omega(r) \) at a generic position \( r > R \) will be redshifted and given by [see equation 87 of paper I]
\[
\omega(r) = \omega_k \frac{N_k}{N} = \omega_k \sqrt{\left( \frac{R - 2M}{r - 2M} \right) \frac{r}{R}},
\]
(37)
and asymptoting to \( \omega = \omega_k \sqrt{1 - 2M/R} \) at spatial infinity. For clarity, hereafter we will indicate as \( \omega(r) \) the function given by expression (37) and as \( \omega \) the value \( \omega_k \sqrt{1 - 2M/R} \), when \( r \to \infty \).

For completeness, we also report the Newtonian limits of the time-dependent part of the perturbed magnetic field in the near zone
\[
\delta B^\ell = i\sqrt{\frac{3}{10\pi}} \frac{1}{4\omega R} \left( \frac{R}{r} \right)^3 B_0 (\eta_k - 6\xi_k) e^{-i(\omega t - \phi)} \cos \chi \sin \theta = -2t \tan \theta \delta B^\theta = 2i \sin \theta \delta B^\phi ,
\]
(38)



\[
\delta t_{21} = \frac{i}{54} \frac{h_k}{N_k^2 g_k} B_0 \eta_k \cos \chi e^{-i\omega_k t} , \quad \delta x_{11} = \frac{3R^2}{8\sqrt{5} f_k M^3} B_0 (h_k \eta_k + 2f_k \xi_k) e^{-i\omega_k t} \cos \chi ,
\]
(39)
and, consequently, the following components of the near-zone electric fields

$$\delta E^\phi = -i \sqrt{\frac{5}{6\pi}} \frac{h_k}{4\pi g_k N_k} \left[ (3 - \frac{2r}{M}) \ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4 \right] B_0 \eta_h e^{-i(\omega k t - \phi)} \cos \chi \cos \theta \sin \theta ,$$

$$\delta E^\theta = -i \left\{ \sqrt{\frac{5}{6\pi}} \frac{N}{g_k N_k} \left[ (1 - \frac{r}{M}) \ln N^2 - 2 - \frac{2M^2}{3r^2N^2} \right] h_k \eta_h \cos \theta + \sqrt{\frac{3}{10\pi}} \frac{3rR^2}{8M^3Nf_k} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] (h_k \eta_h + 2f_k \xi_k) \right\} B_0 e^{-i(\omega k t - \phi)} \cos \chi ,$$

$$\delta E^\phi = i \left\{ \sqrt{\frac{5}{6\pi}} \frac{N}{g_k N_k} \left[ (1 - \frac{r}{M}) \ln N^2 - 2 - \frac{2M^2}{3r^2N^2} \right] h_k \eta_h + \sqrt{\frac{3}{10\pi}} \frac{3rR^2}{8M^3Nf_k} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] (h_k \eta_h + 2f_k \xi_k) \right\} B_0 e^{-i(\omega k t - \phi)} \cos \chi \cos \theta .$$

Finally, we note that the Newtonian limits of the expressions for the electric fields (40) and (42) reduce to the Newtonian solutions found by Muslimov & Tsygan [cf., equation 19 with the integration constants defined by expressions A.1 and A.2 of Muslimov & Tsygan (1986)].

### 2.3.1 Radial oscillations

Next, in analogy with what done by McDermott et al. (1988), we consider the special case of spheroidal oscillations that are purely radial and thus a velocity field given by

$$\delta \hat{v}^i(r, t) := (\eta(r), 0, 0) e^{-i\omega t} ,$$

where $\eta(r)$ is the eigenfunction. Using equations (4)–(6) with the condition for the integration constant $\delta s_{\ell m}$ expressed as

$$\delta s_{10}(t) = -i \sqrt{\frac{3}{4\pi}} \frac{R^2}{f_k\omega_k} B_0 \eta_h e^{-i\omega k t} \cos \chi = -i \sqrt{\frac{1}{2\tan \chi}} \delta s_{11}(t) ,$$

the perturbed magnetic field will have components given by the expressions

$$\delta B^\phi = -i \frac{3\omega_k R^2}{4M^3f_k\omega_k} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] B_0 \eta_h e^{-i\omega k t} \left( \cos \theta \cos \chi \sin \theta \sin \chi \sin \phi \right) ,$$

$$\delta B^\theta = i \frac{3\omega_k R^2}{4M^3f_k\omega_k} \left( \frac{r}{M} \ln N^2 + 1 \right) B_0 \eta_h e^{-i\omega k t} \left( \sin \theta \cos \chi - \cos \theta \sin \chi \sin \phi \right) ,$$

$$\delta B^\phi = i \frac{3\omega_k R^2}{4M^3f_k\omega_k} \left( \frac{r}{M} \ln N^2 + 1 \right) B_0 \eta_h e^{-i\omega k t} \sin \chi \sin \phi .$$

It should be noted that the integration constants (44) could be easily obtained from the integration constants for spheroidal modes (31) if one replaced $Y_{\ell m}$ everywhere with 1; this is simply because in such a case, the spheroidal velocity field (26) coincides with the radial one (43).

It is also easy to realize, after using the expansion of the magnetic field 56–57 of paper I, the radial velocity field (43) in expression (20) and the condition $\partial_\phi Y_{\ell m} = i m Y_{\ell m}$, that the integration constants $t_{\ell m}$ are identically zero since

$$- \int d\Omega \left( \partial_\phi Y_{\ell m} \delta v^\phi - i \frac{m Y_{\ell m}}{\sin \theta} \delta v^\psi B^\phi \right) = -i \delta v^\phi \frac{N_k}{R} \partial_\phi s_{\ell m} |_{r=R} \int d\Omega \left( \partial_\phi Y_{\ell m} \frac{1}{\sin \theta} \partial_\theta Y_{\ell m} - i \frac{m Y_{\ell m}}{\sin \theta} \partial_\phi Y_{\ell m} \right) = 0 .$$

As a result, the only non-vanishing integration constant is

$$\delta x_{10} = \sqrt{\frac{3\pi}{2}} \frac{R^2}{f_k M^3} B_0 \eta_h \eta_h e^{-i\omega k t} \cos \chi = \frac{\sqrt{2}}{\tan \chi} \delta x_{11} ,$$

and the corresponding electric field has non-vanishing components

$$\delta E^\phi = -i \frac{3\omega_k R^2}{16M^3Nf_k} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] B_0 \eta_h e^{-i\omega k t} \sin \chi \sin \phi ,$$

$$\delta E^\phi = i \frac{3\omega_k R^2}{16M^3Nf_k} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] B_0 \eta_h e^{-i\omega k t} (2 \sin \theta \cos \chi + \sin \chi \cos \theta \cos \phi) .$$

It is also useful to point out that the solutions (50) and (51) guarantee the force-free condition $E^\phi B^\phi = 0$, thus expressing the fact that no particle acceleration is possible along the perturbed electromagnetic fields within the near zone. Furthermore, expressions (50) and (51)
EM fields in exterior of relativistic star – II 9

have a rather simple interpretation. Consider in fact a magnetic dipole that is aligned with the polar axis (i.e., with \( \chi = 0 \)); in this case, the magnetic field will then be only poloidal and any perturbation of this magnetic field will produce a new, oscillating electric field that will be purely toroidal (i.e., \( \delta E^\theta = 0 \)).

Also in this case, the Newtonian limits of the expressions for the electric fields (50) and (51) reduce to the Newtonian solutions of Muslimov & Tsygan (1986) [cf., equation B2 of Muslimov & Tsygan (1986)], while the corresponding Newtonian limits of the near-zone magnetic fields (45)–(47) are given by

\[
\delta B^\theta = -i \left( \frac{R}{r} \right)^3 \frac{1}{\omega R} B_0 \eta_k e^{-i\omega t} \left( \cos \theta \cos \chi + \sin \theta \sin \chi e^{i\phi} \right), \tag{52}
\]

\[
\delta B^\phi = i \left( \frac{R}{r} \right)^3 \frac{1}{2\omega R} B_0 \eta_k e^{-i\omega t} \left( \sin \theta \cos \chi - \cos \theta \sin \chi e^{i\phi} \right), \tag{53}
\]

\[
\delta B^\psi = \left( \frac{R}{r} \right)^3 \frac{1}{2\omega R} B_0 \eta_k e^{-i\omega t} \sin \chi e^{i\phi}. \tag{54}
\]

### 2.4 Toroidal oscillations

We next examine a more complex velocity field and, in particular, assess the impact that toroidal oscillations may have on the electromagnetic fields of the relativistic star. To this scope we consider a perturbative velocity field with components [cf., equation 13.71 of Unno et al. (1989)]

\[
\delta \mathbf{v} = \left( 0, \frac{1}{\sin \theta} \partial_\phi Y^*_{\ell m}, (\theta, \phi), -\partial_\theta Y^*_{\ell m}, (\theta, \phi) \right) \eta(r) e^{-i\omega t}. \tag{55}
\]

The toroidal velocity field (55) has an interest of its own and has attracted considerable attention since it corresponds to the one for \( \ell\)-mode oscillations when observed in the frame corotating with the star. In this case, in fact, it has been shown that such modes may lead to unstable oscillations (Andersson 1998; Friedman & Morsink 1998), although such an instability in a newly born neutron star has shown to be contrasted by the growth of differential rotation (Rezzolla et al. 2000; Sá & Tomé 2006) and by the amplification of magnetic fields (Rezzolla et al. 2001b,a; Sá & Tomé 2006) [see also Cuofano & Drago (2010); Cuofano et al. (2012) for the extension to low-mass X-ray binaries].

We start by computing the integration constants (11) for the toroidal oscillations, which take the form

\[
\partial_\theta \delta s_{1m}(t) = -i\omega \delta s_{1m} = -\frac{3R^2}{8 \pi M^2 \omega_k} \int d\Omega \delta Y^*_{1m} \left\{ \left( \sin \theta \cos \chi - \cos \theta \sin \chi e^{i\phi} \right) \frac{1}{\sin \theta} \partial_\phi Y_{\ell m} + i \sin \chi e^{i\phi} \partial_\theta Y_{\ell m} \right\}. \tag{56}
\]

The components of the perturbed magnetic field generated by the toroidal oscillations with \( \ell' = m' = 1 \) and when \( \chi \neq 0 \) are then given by

\[
\delta B^\theta = -\sqrt{\frac{3}{8\pi}} \frac{3R^2}{8\pi M^2 \omega_k} \left( \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right) B_0 \eta_k e^{-i(\omega t - \phi)} \cos \chi \sin \theta, \tag{57}
\]

\[
\delta B^\phi = -\sqrt{\frac{3}{8\pi}} \frac{3N^2 R^2}{8\pi M^2 \omega_k} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] B_0 \eta_k e^{-i(\omega x - \phi)} \cos \chi \cos \theta = i \delta B^\phi \cos \theta, \tag{58}
\]

where we have used the non-vanishing integration constant

\[
\delta s_{11}(t) = \frac{3R^2}{8\omega_k} B_0 \eta_k \cos \chi e^{-i(\omega t - \phi)}. \tag{59}
\]

The Newtonian limit of these near-zone magnetic fields (57)–(58) will take a form

\[
\delta B^\theta = \frac{3}{8\pi \omega} \left( \frac{R}{r} \right)^3 B_0 \eta_k e^{-i(\omega t - \phi)} \cos \chi \sin \theta = -2 \tan \theta \delta B^\theta = 2i \sin \theta \delta B^\phi. \tag{60}
\]

Similarly, the electric fields outside the oscillating magnetized star for velocity oscillation modes with \( \ell' = 1, m' = 0 \) are computed to be

\[
\delta E^\theta = -\sqrt{\frac{3}{4\pi}} \frac{f_k}{3\eta_k N^2} \left[ \left( 3 - \frac{2r}{M} \right) \ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4 \right] B_0 \eta_k (3 \cos^2 \theta - 1) \cos \chi e^{-i\omega_k t}, \tag{61}
\]

\[
\delta E^\phi = \sqrt{\frac{3}{4\pi}} \frac{N f_k}{\eta_k N^2} \left[ \left( 1 - \frac{r}{M} \right) \ln N^2 - 2 - \frac{2M^2}{3r^2 N^2} \right] B_0 \eta_k (\cos \chi \sin \theta \cos \theta) e^{-i\omega_k t}, \tag{62}
\]

where we have used the following expression for the only non-vanishing integration constant for these modes

\[
\delta t_{20} = -\frac{1}{9\sqrt{15}} \frac{f_k}{N^2 \eta_k} B_0 \eta_k \cos \chi e^{-i\omega_k t}. \tag{63}
\]
Other integration constants for modes higher than \( \ell' = 1 \) are
\[
\delta t_{31} = \frac{3}{432 \sqrt{10}} \frac{f_k}{N_k^2 R^3 k_k} B_0 \eta_k \cos \chi \cos \omega k t, \quad \delta t_{32} = \frac{3}{8 \sqrt{5}} \frac{f_k R^3}{N_k^2 \eta_k M_k} B_0 \eta_k \cos \chi \cos \omega k t, \quad \ell' = 2, m' = 1,
\]
and
\[
\delta t_{42} = -\frac{3}{25} \sqrt{\frac{3}{7}} R^3 k_k B_0 \eta_k \cos \chi \cos \omega k t, \quad \delta t_{22} = \frac{1}{9 \sqrt{3}} \frac{f_k R^3}{N_k^2 \eta_k} B_0 \eta_k \cos \chi \cos \omega k t, \quad \ell' = 3, m' = 2.
\]
Also in this case, the Newtonian limit for the electric fields (61)–(62) coincide with those reported by Muslimov & Tsygan (1986) [cf., equation (19) with the integration constants given by expressions (A.5) and (A.6) of Muslimov & Tsygan (1986)].

3 ELECTROMAGNETIC FIELDS IN THE WAVE-ZONE

In what follows we extend the work of the previous sections by providing the expressions for the electromagnetic fields in the wave zone and that will be employed when computing the expressions for the electromagnetic energy losses. We first recall that it is possible to obtain wave-zone expressions in terms of the spherical Hankel functions, which have a simple radial fall-off in the case of small arguments, i.e.,
\[
H_\ell(\omega r) \approx -i(2\ell - 1)!!(\omega r)^{-\ell - 1}, \quad DH_\ell(\omega r) \approx i(2\ell - 1)!!(\omega r)^{-\ell - 2} \omega = -i \frac{H_\ell}{r}, \quad \text{for} \quad \omega r \approx \omega_k R \ll 1,
\]
where \( DH_\ell(\omega r) := r^{-1} \partial_r [r H_\ell(\omega r)] \). On the other hand, the Hankel functions exhibit a typical oscillatory behaviour (in space) in the limit of large arguments (Arfken & Weber 2005), i.e.,
\[
H_\ell(\omega r) \approx (-i)^{\ell + 1} \frac{e^{i \omega r}}{\omega r}, \quad DH_\ell(\omega r) \approx (-i)^\ell \frac{e^{i \omega r}}{r} = \omega H_\ell, \quad \text{for} \quad \omega r \to \infty.
\]

After some rather lengthy algebra, the components of the magnetic fields in the far zone are found to be given by the general expressions [see equations 97–99 of paper I]
\[
B^\phi = \frac{e^{-i \omega t}}{\sqrt{\ell(\ell + 1)}} H_\ell(\omega r) u_{\ell m} Y_{\ell m},
\]
(68)
\[
B^\theta = \frac{e^{-i \omega t}}{\sqrt{\ell(\ell + 1)}} \left( D H_\ell(\omega r) u_{\ell m} \partial_\theta Y_{\ell m} - \omega H_\ell(\omega r) u_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} \right),
\]
(69)
\[
B^\phi = i \frac{e^{-i \omega t}}{\sqrt{\ell(\ell + 1)}} \left( D H_\ell(\omega r) u_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} - \omega H_\ell(\omega r) u_{\ell m} \partial_\theta Y_{\ell m} \right),
\]
(70)

while the electric field components are expressed as [see equations 100–102 of paper I]
\[
E^\phi = -\frac{e^{-i \omega t}}{\sqrt{\ell(\ell + 1)}} H_\ell(\omega r) v_{\ell m} Y_{\ell m},
\]
(71)
\[
E^\theta = -\frac{e^{-i \omega t}}{\sqrt{\ell(\ell + 1)}} \left( D H_\ell(\omega r) v_{\ell m} \partial_\theta Y_{\ell m} + \omega H_\ell(\omega r) u_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} \right),
\]
(72)
\[
E^\phi = -i \frac{e^{-i \omega t}}{\sqrt{\ell(\ell + 1)}} \left( D H_\ell(\omega r) v_{\ell m} \frac{m Y_{\ell m}}{\sin \theta} + \omega H_\ell(\omega r) u_{\ell m} \partial_\theta Y_{\ell m} \right).
\]
(73)

Expressions (68)–(73) have the same functional form as in the Newtonian limit, but general-relativistic corrections are introduced through the integration constants \( u_{\ell m} \) and \( v_{\ell m} \), which are specified through the matching of the electromagnetic fields (68)–(73) at the stellar surface as [see equations 105 and 106 of paper I]
\[
u_{\ell m} = \frac{1}{\sqrt{\ell(\ell + 1)}} \frac{e^{i \omega_k t}}{D H_\ell(\omega_k R) N_k} \int d\Omega \left\{ \partial_\theta Y_{\ell m}^* \left[ \delta_{\ell m} \delta_k B_\phi^\ell - \delta_{\ell m} \delta_k B_\phi^\ell \right] + i \frac{m Y_{\ell m}^*}{\sin \theta} \left[ \delta_{\ell m} B_\phi^\ell - \delta_{\ell m} B_\phi^\ell \right] \right\},
\]
(74)
\[
u_{\ell m} = \frac{1}{\sqrt{\ell(\ell + 1)}} \frac{e^{i \omega_k t}}{D H_\ell(\omega_k R) N_k \omega_k} \int d\Omega \left\{ i \partial_\theta Y_{\ell m}^* \left[ \delta_{\ell m} \delta_k B_\phi^\ell - \delta_{\ell m} \delta_k B_\phi^\ell \right] + i \frac{m Y_{\ell m}^*}{\sin \theta} \left[ \delta_{\ell m} B_\phi^\ell - \delta_{\ell m} B_\phi^\ell \right] \right\}.
\]
(75)

In what follows we will discuss the expressions for the electromagnetic fields in the wave-zone which are produced by the different velocity fields discussed in Sections 2.3 and 2.4.

3.1 Electromagnetic fields produced by spheroidal oscillations

The oscillating electric and magnetic fields will obviously produce electromagnetic waves and the outgoing electromagnetic radiation in the case of spheroidal oscillations will need to satisfy the conditions that \( m = m' \) and \( \ell = \ell', \ell' \pm 1 \) (see, e.g., Rose 1955). For typical stellar
spheroidal oscillations (Duncan 1998), it is easy to estimate that \( \omega_\nu R \ll 1 \) at least for the low-order modes, and in this limit we can calculate, for example, the electromagnetic fields radiated by an axisymmetric dipolar oscillation (i.e., with \( \ell = 1, m = 0 \)). In this specific case, the non-vanishing coefficients are given by \( u_{10}, u_{20}, v_{20} \) and have explicit expressions as

\[
u_{10} = -i \frac{3}{8 \sqrt{2}} \frac{\omega_\nu R^2}{N_R} B_0 \left( \eta_h h_K - 3 \xi_R f_K \right) \sin \chi,
\]

\[
u_{20} = \frac{1}{3} \sqrt{\frac{2 \omega_\nu R^3}{5 \sqrt{3} N_R}} B_0 \left( \eta_h h_K + \xi_R f_K \right) \cos \chi,
\]

\[v_{20} = -\frac{1}{16} \sqrt{\frac{5 \omega_\nu R^4}{2 \sqrt{5} N_R}} B_0 \eta_h h_K \sin \chi,
\]

(76)

so that the electromagnetic fields \((68) - (73)\) induced in the wave zone are given as the real parts of the following solutions

\[
B^\varphi = \left. \frac{1}{\sqrt{12 \pi}} \frac{R^2}{N_R r^2} \right] \left[ \frac{\omega_\nu R}{N_R} (\eta_h h_K + \xi_R f_K) (3 \cos^2 \theta - 1) \cos \chi + \frac{9}{8} (\eta_h h_K - 3 \xi_R f_K) \sin \chi \cos \theta \right] e^{i \omega (r - t)},
\]

(77)

\[
B^\theta = \left. \frac{1}{\sqrt{12 \pi}} \frac{\omega_\nu R^2}{N_R r} \right] B_0 \eta_h h_K \sin \chi \sin \theta \cos \theta e^{i \omega (r - t)},
\]

(78)

\[
B^\phi = \left. \frac{15}{32} \frac{1}{\sqrt{\omega_\nu R^4}} \right] B_0 \eta_h h_K \sin \chi \sin \theta \cos \theta e^{i \omega (r - t)},
\]

(79)

\[
E^\varphi = \left. \frac{15}{32} \frac{1}{\sqrt{\omega_\nu R^4}} \right] B_0 \eta_h h_K \sin \chi (3 \cos^2 \theta - 1) e^{i \omega (r - t)},
\]

(80)

\[
E^\theta = \left. \frac{15}{32} \frac{1}{\sqrt{\omega_\nu R^4}} \right] B_0 \eta_h h_K \sin \chi \sin \theta \cos \theta e^{i \omega (r - t)},
\]

(81)

\[
E^\phi = \left. \frac{1}{\sqrt{12 \pi}} \frac{\omega_\nu R^2}{N_R r} \right] \left[ \omega_\nu R (\eta_h h_K + \xi_R f_K) \sin \theta \cos \theta \cos \chi + \frac{9}{16} (\eta_h h_K - 3 \xi_R f_K) \sin \chi \sin \theta \right] e^{i \omega (r - t)}.
\]

(82)

Note that, in the expressions above, we have omitted the symbol \( \delta \) for the magnetic and electric fields in the wave zone because the fields produced there are exclusively the ones due to the stellar perturbation. Since the wave zone is located well outside the light cylinder, i.e., at \( r \gg r_* := 1 / \Omega \), expressions (77) - (82) show that, in this region, the electromagnetic fields behave essentially as radially outgoing waves, for which \( |B^\varphi|^2 / |B^\theta|^2 \sim |B^\varphi|^2 / |B^\phi|^2 \sim 1 / \omega r \ll 1 \). We postpone to Appendix A the presentation of the electromagnetic fields produced by other higher order spheroidal modes.

### 3.1.1 Electromagnetic fields produced by radial oscillations

From the asymptotic forms (66) and (67), it follows that the solutions (68) - (73) in the wave zone for radial oscillations (43) and for a dipolar perturbation are given by

\[
B^\varphi = \left. \frac{2 \eta_h R^2}{N_R r^2} \right] B_0 \eta_h e^{i \omega (r - t)} (\cos \chi \cos \theta + \sin \chi \sin \theta e^{i \phi}),
\]

(83)

\[
B^\theta = \left. \frac{-1 \eta_h \omega_\nu R^2}{N_R r} \right] B_0 \eta_h e^{i \omega (r - t)} (\sin \chi \sin \theta - \sin \chi \cos \theta e^{i \phi}),
\]

(84)

\[
B^\phi = \left. \frac{-h_\nu \omega_\nu R^2}{N_R r} \right] B_0 \eta_h e^{i \omega (r - t)} \sin \chi e^{i \phi},
\]

(85)

\[
E^\theta = \left. \frac{-h_\nu \omega_\nu R^2}{N_R r} \right] B_0 \eta_h e^{i \omega (r - t)} \sin \chi e^{i \phi},
\]

(86)

\[
E^\phi = \left. \frac{h_\nu \omega_\nu R^2}{N_R r} \right] B_0 \eta_h e^{i \omega (r - t)} (\cos \chi \sin \theta - \sin \chi \cos \theta \cos \phi),
\]

(87)

where the non-vanishing integration constants have explicit expressions

\[
u_{10} = -2 \sqrt{\frac{2 \pi}{3 \sqrt{3}}} \frac{h_\nu R^2}{N_R} B_0 \eta_h \cos \chi,
\]

\[v_{11} = 2 \sqrt{\frac{4 \pi}{3 \sqrt{3}}} \frac{\omega_\nu h_\nu R^2}{N_R} B_0 \eta_h \sin \chi.
\]

(88)

These electromagnetic field components satisfy the force-free condition \( E^\varphi B_\varphi = 0 \) and in the Newtonian limit they reduce to the results of Muslimov & Tsygan (1986) [cf., equations B3 and B4 of Muslimov & Tsygan (1986)].
3.2 Electromagnetic fields produced by toroidal oscillations

In full analogy with the case of spheroidal oscillations, the outgoing electromagnetic radiation in the case of toroidal oscillations needs to satisfy the conditions that \( m = m' \) and \( \ell = \ell', \ell' \pm 1 \) (see, e.g., Rose 1955) For typical stellar toroidal oscillations, which have only azimuthal velocities (see, e.g., Duncan 1998), \( \omega_k R \ll 1 \) for the low-order modes, and in this limit we can calculate explicitly the electromagnetic fields radiated by the mode with \( \ell' = 1, m' = 0 \) as

\[
B^\phi = \frac{i}{4\sqrt{3\pi}} \frac{f_k \omega_k^2 R^4}{N_k r} B_0 \eta \ e^{i\omega(r-t)} \sin \theta \cos \theta \cos \chi, \tag{89}
\]

\[
E^\phi = -\frac{1}{4\sqrt{3\pi}} \frac{f_k \omega_k^2 R^4}{N_k^2 r^2} B_0 \eta \ e^{i\omega(r-t)} \left( 3 \cos^2 \theta - 1 \right) \cos \chi, \tag{90}
\]

\[
E^\theta = \frac{i}{4\sqrt{3\pi}} \frac{f_k \omega_k^2 R^4}{N_k^2 r^2} B_0 \eta \ e^{i\omega(r-t)} \sin \theta \cos \theta \cos \chi, \tag{91}
\]

where the only non-vanishing integration constant is given by

\[
v_{20} = -\frac{i}{15} \sqrt{\frac{5}{2}} \frac{f_k \omega_k^2 R^4}{N_k} B_0 \eta \cos \chi. \tag{92}
\]

Also in this case, the Newtonian limits of the expressions above reduce to the expressions that can be derived from the general equations 25 and 26 of Muslimov & Tsygan (1986). We postpone to Appendix B the presentation of the electromagnetic fields produced by other higher order toroidal modes.

4 ELECTROMAGNETIC DAMPING OF OSCILLATIONS

As mentioned in the Introduction, the scope of this Section is to compute the damping times associated with the most typical modes of oscillation when the latter produce pure electromagnetic waves carrying away the energy stored in the oscillations. Of course, if an oscillation is excited in a compact star, there will be also other mechanisms, most notably gravitational-wave emission, that, together with the electromagnetic ones, will combine to damp the stellar oscillations. For simplicity, we will consider these mechanisms as acting independently and compute in the following Sections the electromagnetic radiative losses for the classes of oscillations considered so far and compare them with the losses via gravitational radiation.

The energy lost via electromagnetic radiation, \( L_{\text{EM}} \), can be readily calculated through the integral of the radial component of the Poynting vector \( \vec{P} \)

\[
L_{\text{EM}} := \int_{\partial \Sigma} P^r dS = \frac{1}{4\pi} \int_{\partial \Sigma} \left( \vec{E} \times \vec{B} \right)^r dS, \tag{93}
\]

which is integrated over the two-sphere \( \partial \Sigma \) of radius \( r \gg 1/\Omega > R \) and the surface element \( dS \) (the integration is taken on the surface \( \partial \Sigma \) at distance \( r \gg R \) where the space-time can be reasonably approximated as flat). Substituting in (93) the general expressions for the electromagnetic radiation (69)–(70) and (72)–(73), we can easily find that the oscillation energy loss due to the electromagnetic radiation is simply expressed in terms of the integration constants \( u_{\ell m} \) and \( v_{\ell m} \) [see equation 148 of paper I and its discussion for more details], i.e.,

\[
L_{\text{EM}} = \frac{1}{8\pi} \left( |u_{\ell m}|^2 + |v_{\ell m}|^2 \right), \tag{94}
\]

and thus depends on the value of the gravitational compactness parameter \( M/R \) appearing in the integration constants \( u_{\ell m} \) and \( v_{\ell m} \).

4.1 Damping due to electromagnetic radiation from toroidal oscillations

Toroidal modes, which preserve the stellar shape at the lowest order, are particularly interesting to estimate the electromagnetic losses since these modes are expected to follow from large crustal fractures and starquakes, and may be especially easy to excite on longer time-scales because the restoring force is due to the relatively weak Coulomb forces of the crustal ions. Furthermore, toroidal shear deformations require much less energy than do radial deformations of the same amplitude and the damping rate is expected to be proportional to the oscillation frequency, which is small for low-order toroidal modes.

Before considering the general-relativistic expression for the electromagnetic losses of the most interesting modes, we recall that the

---

4 Visco-damping is also expected to be present but would act mostly on very high-order modes and on longer time-scales. We recall, in fact, that in Newtonian physics, the viscous oscillation damping time-scale can be estimated as \( \tau_{\text{visc}} \sim \epsilon/\dot{\epsilon} \), where \( \epsilon \sim \rho v^2 \) is the energy density of the oscillation averaged over a period, with \( \rho \) and \( v \) being respectively the characteristic density and velocity, and \( \dot{\epsilon} \sim \eta (v/\lambda)^2 \) the viscous-loss rate, with \( \lambda \sim R/\ell \) and \( \eta \) the shear viscosity. As a result, \( \tau_{\text{visc}} \sim \lambda^2 \rho/\eta \), so that the characteristic damping time-scales decreases for higher modes.
corresponding Newtonian expression for the power $L_{EM}(\ell', m')|_{\text{Newt}}$ radiated by a toroidal oscillation mode with $(\ell', m')$ with $\ell' > 1$ is given by (McDermott et al. 1984)

$$L_{EM}(\ell', m')|_{\text{Newt}} = \frac{c}{8\pi} \left[ \frac{\omega_R}{c} \right] 2^{\ell'} \left[ \frac{B_0^2 R^2}{\omega (2\ell' - 1) (2\ell' - 3)!} \right]^2 \left[ \frac{m^2 (\ell' - 1)(\ell' + 1)}{(2\ell' - 1)(2\ell' + 1)} + \left( \ell' + 1 \right)^2 \left( \ell'^2 - m^2 \right) \right] \cos^2 \chi, \quad (95)$$

where we have assumed that $\omega_R R \ll 1$, which is surely valid for at least the lowest-order modes. Expression (95) can be extended to general relativity either by a proper calculation of the integration constants or, more simply, by considering that the general-relativistic corrections will come in the form of an amplification factor of order $f_k/N_k$ for the magnetic-field strength and of a frequency increase of the order $(1/N_k)^{2\ell'}$ to compensate for the gravitational redshift. Of course the two routes lead to the same answer and thus to a radiated power

$$L_{EM}(\ell', m') = \left( \frac{f_k}{N_k} \right) \left( \frac{\ell}{N_k} \right) 2^{\ell'} L_{EM}(\ell', m')|_{\text{Newt}}$$

$$= \frac{c}{8\pi} \left[ \frac{f_k}{N_k} \left( \frac{\omega_R}{c} \right) \right] 2^{\ell'} \left[ \frac{\omega_R^2 B_0^2 R^2}{\ell' (2\ell' - 1) (2\ell' - 3)!} \right]^2 \left[ \frac{m^2 (\ell' - 1)(\ell' + 1)}{(2\ell' - 1)(2\ell' + 1)} + \left( \ell' + 1 \right)^2 \left( \ell'^2 - m^2 \right) \right] \cos^2 \chi. \quad (96)$$

Fig. 4.1 shows the dependence on the stellar compactness of the energy emission given by expression (96) for a few representative modes and it is apparent that the corrections can be rather large and at least of one order of magnitude for typical neutron stars with a surface magnetic field $B_s = 10^{12} \text{G}$. As a reference, the electromagnetic luminosity produced by a purely quadrupolar mode (i.e., with $\ell' = 2, m' = 2$) can be expressed as

$$L_{EM}(2, 2) \approx 6 \times 10^{36} \left( \frac{\omega_R}{c} \right)^2 \frac{f_k^2}{N_k^8} B_{12}^8 R_6^8 \omega_3^6 \cos^2 \chi \ \text{erg/s}, \quad (97)$$

where $B_{12} := B_s/(10^{12} \text{G}), \omega_3 := \omega/(10^3 \text{ rad/s}), R_6 := R/(10^6 \text{ cm})$.

Given $E_\perp$ as the kinetic energy contained in the stellar oscillations and assuming that all of it is lost to the emission of electromagnetic waves, we can define the electromagnetic decay time-scale of the $(\ell, m)$ mode to be

$$\tau_{EM}^{\text{GIR}}(\ell, m) := \frac{2E_\perp}{\gamma_{EM}^{\text{GIR}}(\ell, m)}, \quad (98)$$

where the factor of 2 on the right-hand side of expression (98) is introduced because of the averaging over one oscillation period. For simplicity, we compare the electromagnetic time-scales $\tau_{EM}$ with the gravitational-radiation ones as computed by used by McDermott et al. (1988) in their model calculations denoted as NS13T8. Such a model has a mass $M = 1.326 M_\odot$, a radius $R = 7.853 \text{ km}$, and magnetic field $B = 10^{12} \text{ G}$. While this model can no longer be considered realistic (the radius is far smaller than what expected from more modern

---

5 Magnetic fields of this strength do not produce significant changes in the mode frequencies (Lee 2007), so that we can safely ignore such corrections.

© 2016 RAS, MNRAS 000, 1–21
equations of state), it has been used also quite recently (Messios et al. 2001; Yoshida & Lee 2002; Lee 2007) and it is useful here as for this model McDermott et al. (1988) have provided a complete list of eigenfrequencies, luminosities and damping times for a number of modes. As for the other models in McDermott et al. (1988), NS13T8 was obtained from the fully general-relativistic calculations of neutron stars by Richardson et al. (1982). The outer crust extended down to the neutron-drip point at $\rho = 4.3 \times 10^{14}$ g cm$^{-3}$ and were assumed to consist of bare iron nuclei embedded in a uniform, neutralizing, degenerate electron gas. The inner crust extended from the neutron-drip point to the base of the crust at $\rho = 2.4 \times 10^{14}$ g cm$^{-3}$, and it was assumed to consist of nuclei with atomic number $Z \sim 40$, degenerate electrons, and degenerate, nonrelativistic neutrons. At the greater densities, the lattice was assumed to dissolve and the core of the neutron star was taken to consist of a mixture of free and highly degenerate neutrons, protons and electrons. Finally, model NS13T8 was assumed to have a solid crust with relative thickness $\Delta r / R \sim 0.055$ and a surface ocean with relative thickness of the $\Delta r / R \sim 2.3 \times 10^{-3}$.

Table 4.1 reports the different damping times for the first toroidal modes for model NS13T8 with a total time-averaged kinetic energy in the crust defined as

$$E_k = \frac{1}{2} \omega^2 \ell (\ell + 1) \int_{\text{crust}} \rho \eta^2 r^2 dr.$$  \hfill (99)

The different columns in the table report the main characteristics of the oscillation modes, such as the frequency and kinetic energy, the power of electromagnetic radiation, the gravitational and electromagnetic damping times in the Newtonian and general-relativistic cases. The Newtonian expressions for the pulsation frequencies, the kinetic energy and the damping times are those reported in Tables 4 and 6 of McDermott et al. (1988) and reproduced in Table 4.1 in columns 1–3 and 5–6. The general-relativistic values of the electromagnetic luminosity and of the damping time reported in columns 4 and 7 are those calculated via expression (96). The last two columns represent the ratio of the gravitational and electromagnetic time-scales and the ratio of the electromagnetic time-scales in the Newtonian and general-relativistic case, respectively. Of course, the damping times $\tau_{\text{GR}}$ are much more interesting when compared with the corresponding damping times $\tau_{\text{EM}}$ calculated when the kinetic energy $E_k$ is instead lost to gravitational waves. These were computed by Schumaker & Thorne (1983) for the first-order perturbations in the displacement functions of a fully general-relativistic stellar model and are obviously reported for modes with $\ell \geq 2$, since no gravitational radiation can be produced by dipolar oscillations.

Early calculations of toroidal oscillations assumed the ‘free-slip’ condition of the solid crust over the fluid core (Hansen & Cioffi 1980; McDermott et al. 1988), but these models have been recently improved to include the gravitational redshift, the increase in the shear modulus due to magnetic pressure, more realistic models of crust composition and elasticity. In particular, more recent calculations of Duncan (1998) and Piro (2005) estimate the redshifted frequency $\nu := \omega / 2\pi$ of the fundamental $\ell = 2$ toroidal mode through the empirical expression

$$\nu(2t_0) = 29.8 \left[ \frac{1.71 - 0.71 M_{1.4} R_6 R_0^{-1}}{0.87 + 0.13 M_{1.4} R_6^{-1}} \right] \left[ 1 + \left( \frac{B}{B_c} \right)^2 \right]^{1/2} \text{Hz},$$ \hfill (100)

while the frequencies of modes of order higher than $n = 0$ are simply given by

$$\nu(n t_0) = \nu(2t_0) \left[ \frac{\ell (\ell + 1)}{6} \right]^{1/2},$$ \hfill (101)

with $M_{1.4} := M / 1.4 M_{\odot}$.

Using $B_c \approx 4 \times 10^{15}$ G, expression (100) gives, e.g., $\nu(2t_0) = 28.5$ Hz, which is the general-relativistic value of the Newtonian value of 57.7 Hz previously calculated by McDermott et al. (1988) for the slightly different NS13T8 model. The difference between the Newtonian and the general-relativistic values is partly due to the redshift factor $(1 - 2M/R)^{1/2}$, which is approximately 0.8 at the surface of the star [cf., equation (37)]. It is necessary to underline that it is reasonable to assume the oscillation modes to be independent of the magnetic field strength for $B \leq 10^{15}$ G, which is a limit that several orders of magnitude greater than the dipolar magnetic fields at the surface of the typical neutron stars (Piro 2005). Axisymmetric toroidal modes of magnetized neutron stars calculated in the general-relativistic Cowling approximation can also be found in the work of Asai & Lee (2014).

Overall, the values reported in Table 4.1 reveal that low-frequency quadrupolar modes are damped more efficiently by gravitational radiation than by electromagnetic radiation. In contrast, high-frequency quadrupolar modes are damped more efficiently by electromagnetic radiation. As two representative examples, the ratio $\tau_{\text{GR}} / \tau_{\text{EM}}$ is approximately $3.5 \times 10^{-3}$ for the lowest quadrupolar toroidal mode $2t_0$, while it grows to approximately $10^3$ for the fourth overtone $4t_2$. This scaling is very interesting as it indicates that an oscillating neutron star subject to toroidal oscillations will rapidly lose most of its kinetic energy to the lowest order quadrupolar oscillations, but will continue to emit electromagnetic energy for a longer time-scale in terms of its higher-order quadrupolar oscillations.

It is also important to underline that the damping times reported in Table 4.1 refer to a fiducial neutron star with a magnetic field $B_c = 10^{15}$ G and that the electromagnetic damping time scales like $B_c^{-2}$ [cf., equations (95) and (98)]. As a result, the damping times reported would need to be modified significantly, i.e., of about four to six orders of magnitude, if the oscillations are taking place in a magnetar with the surface magnetic field $B_c = 10^{14} - 10^{15}$ G.

---

6 More recent but also more restricted information can be found in the works of Gaertig & Kokkotas (2008, 2011), who have studied the oscillations of rapidly rotating stars.

7 A toroidal mode with $\ell = 1$ is not allowed as it would violate angular-momentum conservation.

8 Duncan (1998) also estimated the kinetic energy of the $2t_0$-mode as $E_k(2t_0) = 5 \times 10^{47}(\sigma \rho / R)^2 M_{1.4}^{-1} R_6^4 \left( 0.77 + 0.23 M_{1.4} R_6^{-2} \right)$ erg.
As a concluding remark, and to stress that the harmonic electromagnetic variability discussed above is within the range of present observations, we note that Clemens & Rosen (2004, 2008); Rosen & Clemens (2008) have proposed a detailed analysis of the evolution of the pulse shapes of radio pulsars due to high-order oscillations with multipole numbers as large as \( \ell \sim 70 \). This interpretation is difficult to conciliate with our results, which indicate that the general-relativistic redshift corrections would be very large for such high multipoles, i.e., \( N_{2\ell}^{-2\ell} = N_{2}^{-140} \sim 10^{13} \) for a typical neutron star. Furthermore, the luminosity at these high modes should be strongly suppressed as indicated by the factors \( O([2\ell - 3]!!)^2 \) appearing in the denominator of the general expression for electromagnetic radiation (96). This suppression, which is not a general-relativistic effect but is present already in Newtonian gravity, indicates that the pulsar effectively radiates energy at a vanishingly small rate for these modes, which should be unlikely to be detected in practice.

4.2 Damping due to electromagnetic radiation from spheroidal oscillations

The electromagnetic power radiated when the star is subject to spheroidal oscillations can be calculated in complete analogy to what done for toroidal oscillations. An important difference, however, is that while the electromagnetic radiation from toroidal oscillations is produced only by perturbations in the radial component of the magnetic field, spheroidal oscillations perturb all components of the magnetic field, which therefore contribute to the emitted power. As an example, the power radiated when the star is subject to dipolar and axisymmetric modes (i.e., for \( \ell = 1, \ m = 0 \)) is given by

\[
L_{EM}(1, 0) = \frac{c}{180 \pi} \left[ \left( \frac{\eta_{B}}{\omega_{B} R} \right) \frac{h_{B}}{N_{B}^3} + \left( \frac{\xi_{B}}{\omega_{B} R} \right) \frac{f_{B}}{N_{B}^3} \right]^2 \left( \frac{\omega_{B} R}{c} \right)^2 \left( B_{B} R \right)^2 \omega_{B} R \cos^2 \chi \right]^{1/2} \left( B_{12} \right)^{1/2} R_{B}^6 \omega_{B}^6 \cos^2 \chi \text{ erg s}^{-1},
\]

where and we have the following relation between the eigenfunctions [cf., equation (14.13) of Unno et al. (1989)]

\[
\eta_{B} = \xi_{B} = \frac{\omega_{B}^2 R^3}{GM}.
\]

While the expression above is strictly valid at the surface of the star, in the Cowling approximation and for a Newtonian star, it represents a reasonable approximation and highlights that the energy losses for this mode are comparable with the corresponding losses through toroidal oscillations [cf., equation (96)].

The ratio of the luminosities in the general-relativistic and Newtonian approaches is thus given by

\[
\frac{L_{EM}(\ell', m')}{L_{EM}(\ell, m)}|_{\text{Newt}} = \left( \frac{\omega_{B}^2 R^3}{GM N_{B}^3} \frac{h_{B}}{N_{B}^3} + \frac{f_{B}}{N_{B}^3} \right)^2 \left( \frac{\omega_{B}^2 R^3}{2GM} + 1 \right)^{-1}.
\]

It should be noted that in contrast with the toroidal oscillations, the electromagnetic energy losses through spheroidal modes is proportional to the additional parameter \( h_{B} / N_{B} \), which is responsible for non-radial components of the surface magnetic field. However, since the parameters \( 2h_{B} \) and \( f_{B} \) are of the same order, and since \( \omega_{B}^2 R^3 / 2GM \) is \( \sim 2.5 \times 10^{-2} \) for a typical neutron star with compactness \( M/R = 0.2 \) oscillating at a frequency \( \sim \text{kHz} \), we can neglect the contribution coming from the pulsations in the radial direction and write the amplification factor simply as \( \left( f_{B} / N_{B}^3 \right) ^2 \).

As an additional example we can consider the power emitted by a purely quadrupolar mode (i.e., with \( \ell = 2, \ m = 2 \)) and express the corresponding electromagnetic luminosity as

\[
L_{EM}(2, 2) \approx 7.2 \times 10^{30} \left( \frac{\eta_{B}}{\omega_{B} R} \right)^2 \left( \frac{5h_{B}}{3N_{B}^3} \right)^2 + \left( \frac{\xi_{B}}{\omega_{B} R} \right)^2 \left( \frac{f_{B}}{N_{B}^3} \right)^2 \left( \frac{2h_{B}}{3N_{B}^3} \right)^2 B_{12}^2 R_{B}^6 \omega_{B}^6 \cos^2 \chi \text{ erg s}^{-1},
\]

which is much smaller than the corresponding losses through toroidal oscillations [cf., equation (97)]. This is because spheroidal modes involve bulk compression and vertical motion, which have to do work against the strong degeneracy pressure of the electrons in the outer crust and the free neutrons in deeper layers and, of course, against the strong vertical gravitational field.

© 2016 RAS, MNRAS 000, 1–21
Finally, in the limit $\omega_k R \ll 1$, it is possible to derive the following general expression for the electromagnetic power $L_{EM}(l', m')$ radiated by an arbitrary spheroidal $(l', m')$ oscillation mode with $l' > 2$

$$L_{EM}(l', m') = \frac{c}{32\pi} \left[ \frac{2(l' + 1)}{(\omega_k R)^2} \left( \frac{\omega_k R}{c} \right)^{2l'} \frac{(l' - 1)(l'^2 - m'^2)}{(2l' - 3)!!(2l' + 1)(2l' - 1)} \cos^2 \chi \right].$$

In Table 4.2, we have reported the electromagnetic damping times for some typical spheroidal modes and compared them with the corresponding gravitational ones for $l' \geq 2$. In addition, the gravitational-radiation damping times $\tau_{GW}$ reported in the fifth column were calculated by McDermott et al. (1988) using the general expressions for the emission of gravitational radiation through spheroidal oscillations given by Balbinski & Schutz (1982) and with a kinetic energy

$$E_\gamma = \int_0^R \rho \left[ \eta^2 + \ell(\ell + 1)\xi^2 \right] r^2 dr.$$  

The results of our general-relativistic calculations are given in columns 4 and 7 of the Table, while the last two columns represent the ratio of the gravitational and electromagnetic time-scales. For the various nodes we have considered, this is a very good approximation over the time-scale of a few oscillations. Yet, for realistic stars, the electric conductivity will be high but finite, and thus, the magnetic-field lines are not frozen perfectly slipping slightly during oscillations. This motion of the conducting matter will have an effect on the electromagnetic damping mechanism, as well as on the gravitational damping.

4.3 Damping due to Joule heating

All of the results so far have been obtained assuming that the stellar interior has an infinite conductivity ($\sigma = \infty$). As discussed in paper I, this is a very good approximation over the time-scale of a few oscillations. Yet, for realistic stars, the electric conductivity will be high but finite, and thus, the magnetic-field lines are not frozen perfectly slipping slightly during oscillations. This motion of the conducting matter with respect to the field lines generates electric currents and Ohmic dissipation of these currents results in Joule heating which will act electromagnetic damping mechanism for the stellar interior.

To estimate the Joule heating we use the Maxwell equations, whose first pair is given by

$$3! \partial_\gamma F_\alpha\beta\gamma\delta = 2 \left( \partial_\gamma F_{\alpha\delta} + \partial_\delta F_{\gamma\alpha} + \partial_\alpha F_{\gamma\delta} \right) = 0,$$

where, for an observer with four-velocity $u^\alpha$, the covariant components of the electromagnetic (Faraday) tensor $F_{\alpha\beta}$ are given by

$$F_{\alpha\beta} = 2\mu_0 (E_\alpha + u^\alpha B^\alpha).$$

Here $T_{[\alpha\beta]} = \frac{1}{2} (T_{\alpha\beta} - T_{\beta\alpha})$ and $\eta_{\alpha\beta\gamma\delta}$ is totally antisymmetric symbol employed in the definition of the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$ (Rezzolla & Zanotti 2013)

$$\eta_{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}} \epsilon_{\alpha\beta\gamma\delta}, \quad \eta_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta}.$$  

Similarly, the second pair of Maxwell equations is given by

$$\nabla_\beta F^{\alpha\beta} = 4\pi J^\alpha = 4\pi (\rho_i u^\alpha + J^\alpha),$$

where $u^\alpha$ is the fluid four-velocity, $\rho_i$ is the charge density and $J$ the total electric-charge current, such that the divergence of electromagnetic energy-momentum tensor is

$$\nabla_\beta T^{\alpha\beta} = \frac{1}{4\pi} \nabla_\beta \left( F^\alpha\mu F^\beta_\mu - \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} F^{\mu\nu} F_{\mu\nu} \right) = F^{\alpha\beta} J_\beta.$$  

© 2016 RAS, MNRAS 000, 1–21
Note that the total electric-charge current includes the conduction current \( j^\alpha \), which is associated with electrons having electrical conductivity \( \sigma \), and that Ohm’s law can be written as
\[
j_\alpha := \sigma F_{\alpha\beta} u^\beta. \tag{113}
\]
Contracting equation (112) with the fluid four-velocity \( u_\alpha \) and using Ohm’s law (113), we obtain the change in electromagnetic energy (or Joule dissipation rate) along the worldlines of the conducting medium and hence
\[
Q_J = u_\alpha \nabla_\beta T^{\alpha\beta} = u_\alpha F^{\alpha\beta} J_\beta = \frac{1}{\sigma} j^2.
\tag{114}
\]
This expression is quadratic in the current \( j := (j^\alpha j_\alpha)^{1/2} \), linear in the resistance \( 1/\sigma \), and coincides with expression (24) derived by Page et al. (2000) through an alternative approach.

A generic misaligned dipolar magnetic field will be sustained by an electric current with components \( j^\alpha := (0, j^t, j^\theta, j^\phi) \) \(^9\) governed by the relevant components of the Maxwell equations (111)
\[
J^t = \frac{e}{4\pi} \left[ \partial_\theta \left( \sin \theta B^\phi \right) - \partial_\phi B^\theta \right],
\tag{115}
\]
\[
J^\theta = \frac{e}{4\pi \sin \theta} \left[ \partial_\phi B^\alpha - e^{-(\Phi+\Lambda)} \sin \theta \partial_t \left( r e^\Phi B^\phi \right) \right],
\tag{116}
\]
\[
J^\phi = \frac{e}{4\pi} \left[ e^{-(\Phi+\Lambda)} \partial_t \left( e^\Phi r B^\theta \right) - \partial_\theta B^\phi \right],
\tag{117}
\]
where \( g_{00} = -e^{2\Phi} \) and \( g_{11} = e^{-2\Lambda} \) are the components of the metric tensor in the interior of the neutron star and solutions of the Einstein equations.

Inserting the electric currents (115) – (117) into (114), the resulting expression for the Joule dissipation rate is
\[
Q_J = \frac{e^2}{16\pi^2\sigma r^2} \left\{ \left[ \partial_\theta \left( \sin \theta B^\phi \right) - \partial_\phi B^\theta \right]^2 + \frac{1}{\sin^2 \theta} \left[ \partial_\phi B^\alpha - e^{-(\Phi+\Lambda)} \sin \theta \partial_t \left( r e^\Phi B^\phi \right) \right]^2 + \left[ e^{-(\Phi+\Lambda)} \partial_t \left( e^\Phi r B^\theta \right) - \partial_\theta B^\phi \right]^2 \right\}.
\tag{118}
\]

Assuming that the stellar oscillations are toroidal, expanding the magnetic field \( B^\phi \) in a power series in terms of a perturbation parameter \( \tilde{\eta}/R \) (with \( \tilde{\eta}/R \ll 1 \)), and truncating at first order, i.e.,
\[
B^\phi = B_0^\phi \left[ 1 + \left( \frac{\tilde{\eta}}{R} \right) \right],
\tag{119}
\]
where \( B_0^\phi \) is the unperturbed magnetic field, the expression for Joule heating (118) will take the form
\[
Q_J = \frac{e^2 e^{-2\Lambda}}{16\pi^2\sigma} \left( \frac{\tilde{\eta}}{R} \right)^2 \left[ (B_0^\phi)^2 + (B_0^\phi)^2 \right] = e^{-2\Lambda} \left( \frac{\tilde{\eta}}{R} \right)^2 \frac{1}{4\pi} \left[ (B_0^\phi)^2 + (B_0^\phi)^2 \right] \frac{e^2}{4\pi \sigma R^2},
\tag{120}
\]
where we have used the Maxwell equations for the unperturbed magnetic field
\[
e^{-(\Phi+\Lambda)} \partial_t \left( e^\Phi r B_0^\theta \right) - \partial_\theta B_0^\phi = 0,
\tag{121}
\]
\[
\partial_\phi B_0^\alpha - e^{-(\Phi+\Lambda)} \sin \theta \partial_t \left( r e^\Phi B_0^\phi \right) = 0,
\tag{122}
\]
and have assumed \( \partial_\phi \tilde{\eta} \sim \tilde{\eta}/R \) as by McDermott et al. (1984). Note that the Joule heating (120) has two modifications with respect to its Newtonian equivalent. The first one is via the metric correction \( e^{-2\Lambda} \), while the second one is through the general-relativistic amplification of the stellar magnetic field [cf., equations (12), (13) – (15)]. We can now estimate an upper limit for the Joule dissipation by evaluating expression (120) at the surface of the star to obtain
\[
L_J = \left( \frac{\tilde{\eta}}{R} \right)^2 \frac{B_0^2}{4\pi} \left( 2 h_k N_k^2 \right)^2 \frac{4\pi}{3} \frac{R^3}{R^2} \frac{e^2}{4\pi \sigma R^2},
\tag{123}
\]
where the quantity \( \tau_\alpha := 4\pi R^2/e^2 \) is the characteristic decay time-scale. Using an approximate expression for the electrical conductivity given as (Lamb 1991; Rezzolla & Zanotti 2001)
\[
\sigma \approx 10^{23} \left( \frac{10^8 K}{T} \right)^2 \left( \frac{\rho}{10^{10} \text{ g cm}^{-3}} \right)^{3/4} \text{ s}^{-1},
\tag{124}
\]
we obtain the time-scale of magnetic-field decay to be \( \tau_\alpha \sim 10^7 - 10^{10} \) yr. For a typical conductivity of the stellar crust \( \sigma \sim 10^{23} \text{ s}^{-1} \), and assuming the oscillations to have the largest amplitude at the stellar surface, we can estimate the magnitude of the energy loss due to electromagnetic heating as given by
\[
L_J \approx 0.23 \times 10^{28} \left( \frac{\tilde{\eta}}{R} \right)^2 \left( 2 h_k N_k^2 \right)^2 \left( \frac{B_0}{10^{12} \text{ G}} \right)^2 \left( \frac{R}{10^6 \text{ cm}} \right) \left( \frac{10^{23} \text{ s}^{-1}}{\sigma} \right) \text{ erg s}^{-1}.
\tag{125}
\]
\(^9\) Note that \( J^t = 0 \) for toroidal oscillations.
Because of the intrinsically high electric conductivity, the Ohmic loss rate (125) for a canonical neutron star is several orders of magnitude smaller than the radiation loss rates given by the equation (96) for dipolar and quadrupolar toroidal modes and given by expressions (102), (105), (106) for the spheroidal modes. Furthermore, the general-relativistic contribution in the Ohmic dissipation (125) mainly arises through the change of stellar magnetic field by the factor $2\alpha_h$.

In summary, the results presented in this section via equations (96), (102)–(105) and (125), show that there are general-relativistic corrections both for the radiative electromagnetic losses produced by toroidal/spheroidal modes and by the Joule heating in the case of finite conductivity. However, while the latter can be reasonably ignored, there are situations in which the electromagnetic energy damping could be either comparable or larger than the energy losses via gravitational radiation.

5 CONCLUSION

In previous work of ours (paper I) we have developed a general-relativistic formalism describing the vacuum electrodynamics of an oscillating magnetized relativistic star. The assumptions needed to make this problem tractable analytically are those of slow rotation, infinite electric conductivity and that the electromagnetic energy does not have a feedback on to the background gravitational field. In this paper, we have applied the formalism to obtain explicit analytical expressions for the electric and magnetic fields produced by the most common modes of oscillation both in the vicinity of the star, where they are quasi-stationary, and far away from it, where they behave as electromagnetic waves.

In this way, we have revisited and extended to a general-relativistic context some of the work presented by Muslimov & Tsygan (1986) within Newtonian gravity.

In addition, we have considered the important issue in the asteroseismology of compact and magnetized stars of determining the dissipation mechanism which is most efficient in damping the low-multipolarity oscillations. More specifically, we have computed the electromagnetic radiation generated when a magnetized neutron star is subject to either toroidal or spheroidal oscillations and computed the energy losses in the form of Poynting fluxes, Joule heating and Ohmic dissipation. This has allowed us to extend to a general-relativistic context the classical Newtonian estimates of McDermott et al. (1988) for the damping times of oscillating magnetized neutron stars.

In summary, despite the fact that a number of factors concur in determining what is the main damping mechanism of the oscillations, e.g., the type of mode, the magnetic-field strength and the compactness of the star, we have found that the following results to be robust for a typical neutron star with a dipolar magnetic field of $\sim 10^{12}$ G. First, the general-relativistic corrections to the electromagnetic fields lead to damping time-scales due to electromagnetic losses which are at least one order of magnitude smaller than their Newtonian counterparts. Secondly, $j$, $p$, $i$ and $s$ modes are suppressed more efficiently by gravitational losses than by electromagnetic ones; the only exception to this behaviour is given by $g$ modes, probably because of the low typical frequencies of these modes. Finally, Joule heating is not as an important damping mechanism in general relativity as it is in Newtonian gravity.

The results obtained here could find at least two important applications. First, through a more precise characterization of the general-relativistic corrections, the electromagnetic waves emitted by an oscillating star can be used to deduce, in conjunction with the corresponding gravitational-wave signal, important constraints on the properties of matter at nuclear density. Second, by better estimating the energy loss to the emission of electromagnetic waves it is possible to determine more accurately the time-scale over which oscillations of different type can survive in a relativistic magnetized star. For example, the results presented here could be used to estimate the damping of magnetar oscillation now that the oscillation eigenfrequencies and the kinetic energy of magnetar oscillations are becoming increasingly more accurate (Cerdá-Durán et al. 2009; Sotani & Kokkotas 2009; Gabler et al. 2011; Hambaryan et al. 2011; Gabler et al. 2014).

ACKNOWLEDGEMENTS

This research was partially supported by the Volkswagen Stiftung (Grant 86 866), by the LOEWE-Program in HIC for FAIR, by “NewCompStar”, COST Action MP1304, and by the European Union’s Horizon 2020 Research and Innovation Programme under grant agreement No. 671698 (call FETHPC-1-2014, project ExaHyPE). BJA is also supported in part by the project F2-FA-F113 of the UzAS and by the ICTP through the projects OEA-NET-76, OEA-PRJ-29. BJA thanks the Institut für Theoretische Physik for warm hospitality during his stay in Frankfurt.

REFERENCES

Abdikamalov E. B., Ahmedov B. J., Miller J. C., 2009, Mon. Not. R. Astron. Soc., 395, 443
Andersson N., 1998, Astrophys. J., 502, 708
Arfken G. B., Weber H. J., 2005, Mathematical methods for physicists, 6th ed.. Elsevier, Amsterdam
Asai H., Lee U., 2014, Astrophys. J., 790, 66
Balbiski E., Schutz B. F., 1982, Mon. Not. Roy. Astr. Soc., 200, 43P
Cerdá-Durán P., Stergioulas N., Font J. A., 2009, Mon. Not. R. Astron. Soc., 397, 1607
Chugunov A. I., Yakovlev D. G., 2005, Astronomy Reports, 49, 724

© 2016 RAS, MNRAS 000, 1–21
APPENDIX A: ELECTROMAGNETIC FIELDS IN THE WAVE ZONE FOR HIGHER ORDER SPHEROIDAL MODES

When the oscillation mode is non-axisymmetric, the electromagnetic fields produced by a spheroidal oscillation mode with \( \ell' = 2, m' = 2 \) have a rather complicated form. To obtain such expressions, we start by noting that for a radiating mode with \( \ell = 3 \), the only nonzero coefficient is given by \( u_{32} \) and has explicit expression (cf., Section 3.1)

\[
u_{32} = -\frac{1}{15\sqrt{2\pi}} \frac{\omega^2 R^4}{N_k} B_0 (3\xi_n f_k - 2\eta h_k) \cos \chi.
\]

Similarly, the quadrupolar outgoing radiation defined by \( \ell = 2 \) has the only nonzero coefficient given by \( v_{22} = 0 \) and explicit form

\[
u_{22} = -\frac{1}{3} \frac{1}{\sqrt{6}} \frac{\omega^2 R^4}{N_k} B_0 \eta h_k \cos \chi.
\]

Using these results, the multipolar electromagnetic fields (68)–(73) induced in the wave-zone by the oscillation mode \( \ell' = 2, m' = 2 \) are then expressed as the real parts of the following solutions for the magnetic field

\[
B^\phi = \frac{1}{2\sqrt{30\pi} N_k r^2} B_0 (2\eta h_k - 3\xi_n f_k) \sin^2 \theta \cos \theta \cos \chi e^{i(\omega(r-t) + 2\phi)},
\]

\[
B^\theta = i \frac{1}{48\sqrt{30\pi}} \frac{\omega^2 R^4}{N_k r^2} B_0 [2\eta h_k - 3\xi_n f_k + 3(2\eta h_k - 3\xi_n f_k) \cos 2\theta] \sin \theta \cos \chi e^{i(\omega(r-t) + 2\phi)},
\]

\[
B^\phi = -\frac{1}{24\sqrt{30\pi} N_k r^2} B_0 (7\eta h_k - 3\xi_n f_k) \sin \theta \cos \theta \cos \chi e^{i(\omega(r-t) + 2\phi)},
\]

while for the electric field, they assume the form

\[
E^\phi = -\frac{i}{4} \sqrt{\frac{5}{6\pi}} \frac{\omega^2 R^4}{N_k r^2} B_0 \eta h_k \cos \chi e^{i(\omega(r-t) + 2\phi)},
\]

\[
E^\theta = -\frac{1}{24\sqrt{30\pi}} \frac{\omega^2 R^4}{N_k r^2} B_0 (7\eta h_k - 3\xi_n f_k) \sin \theta \cos \theta \cos \chi e^{i(\omega(r-t) + 2\phi)},
\]

\[
E^\phi = -\frac{i}{48\sqrt{30\pi}} \frac{\omega^2 R^4}{N_k r^2} B_0 [2\eta h_k - 3\xi_n f_k + 3(2\eta h_k - 3\xi_n f_k) \cos 2\theta] \sin \theta \cos \chi e^{i(\omega(r-t) + 2\phi)}.
\]
APPENDIX B: ELECTROMAGNETIC FIELDS IN THE WAVE ZONE FOR HIGHER ORDER TOROIDAL MODES

In similarity to what done in Appendix A, we next list the explicit wave-zone expressions for the electromagnetic fields radiated by a toroidal oscillations. We start with a mode with \( \ell' = 1, m' = 1 \), in which case the components of the magnetic field are

\[
B^\ell = \frac{3i}{2\sqrt{6\pi}} \frac{f_k R^2}{N_k r^3} B_0 \eta_k \sin \theta \cos \chi \ e^{i[\omega(r-t)+\phi]} ,
\]

(B1)

\[
B^\phi = - \frac{3}{4\sqrt{6\pi}} \frac{f_k \omega_k R^2}{N_k r} \left( 1 + \frac{i}{9} \omega_k^2 R^2 \right) B_0 \eta_k \cos \theta \cos \chi \ e^{i[\omega(r-t)+\phi]} ,
\]

(B2)

\[
B^\theta = - \frac{3i}{4\sqrt{6\pi}} \frac{f_k \omega_k R^2}{N_k r} \left( 1 + \frac{i}{9} \omega_k^2 R^2 \cos 2\theta \right) B_0 \eta_k \cos \chi \ e^{i[\omega(r-t)+\phi]} ,
\]

(B3)

while the electric field is given by

\[
E^\ell = - \frac{i}{2\sqrt{6\pi}} \frac{f_k \omega_k^2 R^4}{N_k r^2} B_0 \eta_k \sin \theta \cos \chi \ e^{i[\omega(r-t)+\phi]} ,
\]

(B4)

\[
E^\phi = - \frac{3}{4\sqrt{6\pi}} \frac{f_k \omega_k R^2}{N_k r} \left( 1 + \frac{i}{9} \omega_k^2 R^2 \cos 2\theta \right) B_0 \eta_k \cos \chi \ e^{i[\omega(r-t)+\phi]} ,
\]

(B5)

\[
E^\theta = \frac{3}{4\sqrt{6\pi}} \frac{f_k \omega_k R^2}{N_k r} \left( 1 + \frac{i}{9} \omega_k^2 R^2 \right) B_0 \eta_k \cos \theta \cos \chi \ e^{i[\omega(r-t)+\phi]} ,
\]

(B6)

where the non-vanishing integration constants are given by

\[
v_{21} = \frac{i}{3\sqrt{30}} \frac{f_k \omega_k^2 R^4}{N_k} B_0 \eta_k \cos \chi , \quad u_{11} = \frac{i}{\sqrt{2}} \frac{f_k \omega_k}{N_k} B_0 \eta_k \cos \chi .
\]

(B7)

When the mode has \( \ell' = m' = 2 \) (i.e., the prototypical \( r \) mode discussed when considering gravitational-wave losses) the non-vanishing integration constants will take the form [see also Ho & Lai (2000) for a Newtonian treatment of the electromagnetic emission from a star subject to \( r \)-mode oscillations]

\[
v_{32} = \frac{i}{15\sqrt{21}} \frac{f_k \omega_k^2 R^5}{N_k} B_0 \eta_k \cos \chi , \quad u_{22} = \frac{i}{3} \sqrt{\frac{2}{3}} \frac{f_k \omega_k^2 R^3}{N_k} B_0 \eta_k \cos \chi ,
\]

(B8)

and consequently generate the following magnetic fields in the wave zone

\[
B^\ell = - \frac{1}{2} \sqrt{\frac{5}{6\pi}} \frac{f_k \omega_k R^3}{N_k r^2} B_0 \eta_k \cos \chi \sin^2 \theta \ e^{i[\omega(r-t)+2\phi]} ,
\]

(B9)

\[
B^\phi = - \frac{i}{12} \sqrt{\frac{5}{6\pi}} \frac{f_k \omega_k^2 R^3}{N_k r} B_0 \eta_k \left( 1 + \frac{i}{10} \omega_k^2 R^2 \right) \sin \theta \cos \theta \cos \chi \ e^{i[\omega(r-t)+2\phi]} ,
\]

(B10)

\[
B^\theta = \frac{1}{12} \sqrt{\frac{5}{6\pi}} \frac{f_k \omega_k^2 R^3}{N_k r} B_0 \eta_k \left( 1 + \frac{i}{40} \omega_k^2 R^2 + \frac{3i}{40} \omega_k^2 R^2 \cos 2\theta \right) \sin \theta \cos \theta \cos \chi \ e^{i[\omega(r-t)+2\phi]} ,
\]

(B11)

while the electric fields are

\[
E^\ell = \frac{1}{2\sqrt{30\pi}} \frac{f_k \omega_k^2 R^5}{N_k r^2} B_0 \eta_k \cos \chi \sin^2 \theta \cos \theta \ e^{i[\omega(r-t)+2\phi]} ,
\]

(B12)

\[
E^\phi = \frac{1}{12} \sqrt{\frac{5}{6\pi}} \frac{f_k \omega_k^2 R^3}{N_k r} B_0 \eta_k \left( 1 + \frac{i}{40} \omega_k^2 R^2 + \frac{3i}{40} \omega_k^2 R^2 \cos 2\theta \right) \sin \theta \cos \theta \ e^{i[\omega(r-t)+2\phi]} ,
\]

(B13)

\[
E^\theta = \frac{i}{12} \sqrt{\frac{5}{6\pi}} \frac{f_k \omega_k^2 R^3}{N_k r} B_0 \eta_k \left( 1 + \frac{i}{10} \omega_k^2 R^2 \right) \sin \theta \cos \theta \ e^{i[\omega(r-t)+2\phi]} ,
\]

(B14)