A special tiling of the rectangle

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Abstract

We count tilings of a rectangle of integer sides \(m - 1\) and \(n - 1\) by a special set of tiles. The result is obtained from the study of the kernel of the adjacency matrix of an \(m \times n\) rectangular subgraph in \(\mathbb{Z} \times \mathbb{Z}\).

1 Introduction

Let \(R = [0, m - 1] \times [0, n - 1] \subset \mathbb{R}^2\) be a rectangle with nontrivial integer sides \(m - 1\) and \(n - 1\). We consider tilings of \(R\) by tiles given in figure 1. Tiles can be rotated by integer multiples of \(\pi/2\) and their dashed sides ought to belong to \(\partial R\), the boundary of \(R\). Length of sides is indicated in the picture, angles must be \(\pi/4, \pi/2\) or \(3\pi/4\). Figure 1 shows a tiling of the \(7 \times 4\) rectangle.

![Figure 1: The tiles; \(a\) and \(b\) are integers.](image)

![Figure 2: A tiling and a polarized \(\mathbb{Z}/(2)\)-harmonic function of the \(7 \times 4\) rectangle.](image)

The main result of this paper is the following.

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Theorem 1. Let \( R = [0, m - 1] \times [0, n - 1] \) be a rectangle with nonzero integer sides \( m - 1 \) and \( n - 1 \). The number of tilings of \( R \) which make use of the tiles given in figure 1 is \( 2^\beta + 2^\omega - 2 \), where \( c = \gcd(m + 1, n + 1) - 1 \), \( 2\beta = c + (c \mod 2) \) and \( 2\omega = c - (c \mod 2) \).

The theorem is a consequence of a certain description of \( \mathbb{Z}/(2) \)-harmonic functions. More precisely, consider an \( m \times n \) rectangular mesh of points in \( \mathbb{Z} \times \mathbb{Z} \) and color the points black and white alternatively. The adjacency matrix \( M \) for the points of the mesh is a standard discretization of the Laplacian acting on functions defined on the rectangle \( R \), satisfying Dirichlet boundary conditions, up to a multiplicative factor given by the mesh spacing and to an additive factor given by four times the identity matrix.

The matrix \( M \) takes vectors supported on the set of black points (forming a subspace \( V_b \)) to vectors supported on white points (in \( V_w \)), and vice-versa. Thus, \( M \) decomposes naturally in two linear maps, \( BW : V_b \rightarrow V_w \) and \( WB : V_w \rightarrow V_b \) and the kernel of \( M \) is the direct sum of the kernels of \( BW \) and \( WB \). We assume the field of scalars for both vector spaces to be \( \mathbb{Z}/(2) \), the field of two elements: in this case, the adjacency matrix equals the discretized Laplacian. The elements on \( \ker BW \) and \( \ker WB \) are the polarized \( \mathbb{Z}/(2) \)-harmonic functions, supported respectively on black and white points. Such objects have already been considered in [DT] as a technical tool.

In order to prove the Theorem, we first construct a bijection between tilings of \( R \) and the nontrivial (i.e., nonzero) polarized \( \mathbb{Z}/(2) \)-harmonic functions: this takes most of Sections 3 and 4. We are then left with counting the dimensions of the kernels of \( BW \) and \( WB \) for rectangles. This in turn is accomplished by realizing that polarized harmonic functions on \( R \) admit very stringent symmetries: the problem then reduces to computing the kernel dimensions for squares. We conclude by listing the tilings of the \( 10 \times 4 \) rectangle, indicating the visual implications of the additive structure on tilings inherited from the domain vector spaces.

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2 Basic properties of tilings

Tiles are the polygons (topologically, closed disks) listed in figure 1. A tiling of \( R = [0, m - 1] \times [0, n - 1] \) is a covering of \( R \) by tiles which overlap at their boundaries. Points in the \( m \times n \) rectangular graph \( G = R \cap (\mathbb{Z} \times \mathbb{Z}) \) are adjacent in the obvious manner: \( (x, y) \) and \( (x', y') \) are adjacent if and only if \( x = x' \), \( y = y' \pm 1 \) or \( x = x' \pm 1, y = y' \). Notice that we call points the elements of \( G \); vertices will be the corners of tiles. Color a point \( (x, y) \in G \) black (resp. white) if \( x + y \) is even (resp. odd). Points of \( \mathbb{Z} \times \mathbb{Z} \) will be called integral points.

Proposition 1. Vertices of tiles of a tiling \( T \) are always integral. Also, with the possible exception of the four corners of \( R \), all vertices of tiles in \( T \) are of the same color.

Proof. From the shape of the tiles, if one vertex of a tile sits on an integral point, all other vertices do. Suppose now that \( T \) contains a tile with some nonintegral
vertex. Let \( N \) be the region of \( R \) covered by tiles of \( T \) having nonintegral vertices: \( N \) is a polygonal region which does not contain the corners of \( R \) — corners belong to tiles with integral vertices. Let \( v \) be a vertex of \( N \): then it must also be the vertex of some tile outside \( N \) — a contradiction. Thus, all vertices are integral.

Again from the shape of tiles, all vertices of a tile have the same color (unless the tile is a right triangle with small sides of odd length) — call this the color of the tile (for triangles, the color of the tile is the color of the vertices not sitting at the right angle). The right angle vertex of a triangle necessarily sits on a corner of \( R \), and we do not have to consider it in the sequel.

Again, with the possible exception of sides of triangles, all other horizontal or vertical sides are of even length. Thus, all tiles with a side lying on \( \partial R \) have the same color \( c \). Let \( C \) be the region covered by tiles of color \( c \), and \( D \) be its complement in \( R \). If \( D \) is not empty, take \( v \) to be one of its vertices: since \( v \in D \) is not a corner of \( R \), it must have the colors of tiles both in \( C \) and \( D \) — again a contradiction.

Thus \( T = T_b \cup T_w \), where \( T \) is the set of tilings of \( R \), and \( T_b \) and \( T_w \) are the sets of tilings whose tiles are black and white.

Let \( G \) be the rectangular graph associated to \( R \). Number black and white points from 1 to \( n_b \) and from 1 to \( n_w \), respectively. Set \( BW \) to be the \( n_w \times n_b \) black-to-white adjacency matrix, with obvious entries: \( BW_{wb} = 1 \) if point \( b \) is adjacent to point \( w \); otherwise, \( BW_{wb} = 0 \). Similarly, define \( WB \), the \( n_b \times n_w \) white-to-black adjacency matrix of \( G \).

Figure 2 describes an element in \( \ker WB \) for the \( 7 \times 4 \) rectangle (and hence, for the \( 8 \times 5 \) rectangular graph). Notice that the nonzero coordinates of this vector correspond to the points of \( G \) which belong to the non-dashed sides of the tiling in figure 2.1: kernel elements naturally induce tilings. This will be made precise in the next section: we are aiming at the following result.

**Theorem 2.** There are bijections

\[
\Phi_b : \ker BW \setminus \{0\} \rightarrow T_b \quad \text{and} \quad \Phi_w : \ker WB \setminus \{0\} \rightarrow T_w
\]

3 Defining the bijections \( \Phi_b \) and \( \Phi_w \)

Let \( u \in \ker BW \setminus \{0\} \). A black point \( b \in G \) is an active point of \( u \) if \( u(b) = 1 \). Denote by \( A_u \) the set of active points of \( u \). In \( A_u \), consider the following adjacency relation. Two points are adjacent if both conditions are satisfied: (i) both points have a common white neighbor \( w \) in \( G \); (ii) both points form a right angle with \( w \) or they are the only active neighbors of \( w \).

![Figure 3: Adjacencies in \( A_u \).](image)

A splitting of \( R \) is a decomposition in chunks, i.e., polygons with integral vertices, which are not necessarily tiles. Given a vector \( u \in \ker BW \setminus \{0\} \), let
\( \Phi_b(u) = \text{int}(R) \setminus \{b_1b_2 \mid b_1 \text{ and } b_2 \text{ are adjacent in } A_u \} \). Said differently, draw straight line segments joining adjacent points in \( A_u \). In section 3.1, we will see that this construction obtains a splitting \( \Phi_b(u) \) of \( R \). In Section 3.3, more will be proved: \( \Phi_b(u) \) is actually a tiling, i.e., its chunks are tiles.

### 3.1 Geometry of \( A_u \)

Let \( b \in A_u \) be a black point in the interior of \( R \). The star centered at \( b \) is the set of points in \( A_u \) which are adjacent to \( b \), together with the segments joining \( b \) to these points. The Lemma below follows by exhausting possibilities.

**Lemma 1.** Let \( b \in A_u \) be an interior point of \( G \). Up to rotations and reflections, the stars centered at \( b \) are listed in figure 4.

![Figure 4: Stars centered at \( b \).](image)

We point out that all stars in the list indeed occur in tilings.

**Lemma 2.** Every point of \( A_u \) which is not a corner of \( R \) has (at least) two neighbors in \( A_u \). Also, segments \( b_1b_2 \) and \( b_3b_4 \) in \( A_u \) joining four distinct points do not intercept.

**Proof.** Let \( b \) be a point in \( A_u \) which is not a corner of \( R \). Then \( b \) has two neighbors in \( G \), \( w_1 \) and \( w_2 \), so that the three points are collinear. Since \( b \) is an active neighbor of \( w_1 \) and \( w_2 \) and \( u \in \ker BW \), \( w_1 \) and \( w_2 \) have other active black neighbors in \( G \). Let \( b_1 \) (resp. \( b_2 \) ) be the active neighbor of \( w_1 \) (resp. \( w_2 \) ) closest to \( b \). Notice that \( b_1 \) and \( b_2 \) are distinct, since \( b \) is the only black neighbor common to \( w_1 \) and \( w_2 \). Thus, \( b \) is adjacent in \( A_u \) to at least two points \( b_1 \) and \( b_2 \).

We now consider the second statement. From the definition of the adjacency in \( A_u \), the possible segments joining active points are given in figure 5. Since the four points are distinct, segments may not meet at endpoints. Also, as all points have the same color, the only plausible intersection of segments are listed in figure 6. By direct inspection, each such intersection violates the definition of adjacency in \( A_u \). \( \square \)
Proposition 2. The closure of the connected components of $\Phi_b(u)$ are convex polygons with vertices in $A_u$ or in the corners of $R$. In particular, $\Phi_b(u)$ is indeed a splitting.

Proof. From Lemma 2, the juxtaposition of segments in $A_u$, together with $\partial R$, consists of closed polygonal curves and the vertices of the connected components in the statement are either in $A_u$ or in the corners of $R$. From the classification of stars, convexity follows at interior vertices: for boundary vertices, convexity is trivial.

3.2 The possible sides and angles of chunks of $\Phi_b(u)$

Sides of chunks of a splitting $\Phi_b(u)$ are active if they consist only of segments of $A_u$. Inactive sides are those which contain no segment of $A_u$. The remaining sides are called mixed. Inactive and mixed sides must lie in $\partial R$, since sides in $\Phi_b(u)$ are union of segments in $A_u$ with segments in $\partial R$.

Lemma 3. There are no mixed sides.

Proof. Suppose $e$ is a mixed side of a chunk in $\Phi_b(u)$. Without loss, suppose $e$ belongs to the upper side of $R$. The $m \times n$ rectangular graph $G$ ought to include points immediately below $e$, since $m, n > 1$. Let $b$ be a point of $e$ joining an active and an inactive segment, and let $w$ denote the white point below $b$. Figure 6 shows the possible active neighbors of $w$: any possible choice implies the existence of a segment in $A_u$ meeting $e$ at $b$: in particular, $e$ may not be a side of a chunk.

Lemma 4. Horizontal or vertical active sides of chunks in $\Phi_b(u)$ have length 2.

Proof. Consider a horizontal active side $h$ of a chunk of $\Phi_b(u)$. From the adjacency relation in $A_u$, $h$ has even length. Suppose $h$ has length greater to 2, and take $e$ in $h$ joining two active points at distance 4.

As in the previous Lemma, we may suppose that the integral points immediately under $h$ belong to $G$. The midpoint of $e$ has at least one white neighbor.
$w \in G$, as shown in figure 7. From the adjacency relation in $A_u$, the side neighbors of $w$ are inactive. There is only one possibility: the other active neighbor of $w$ must be $b$, as in figure 7. But then, the vertical segment in the same figure ought to be a segment in $A_u$, contradicting the fact that $e$ lies in a side of a chunk.

The angles of chunks are formed by the segments listed in figure 8 combined with segments in $\partial R$. Our next step will be the classification of angles according to activity of their sides. Angles defined by an active and an inactive side are mixed angles. From the adjacency relation in $A_u$, angles of $\pi/4$ in $\Phi_b(u)$ are mixed and angles of $3\pi/4$ are not. We are left with considering right angles.

Lemma 5. **Right angles are not mixed.**

Proof. Consider first the case in which the sides of the angle are horizontal and vertical, as in figure 8. The inactive side must then belong to $\partial R$. The number of active neighbors of $w$ must be even, and hence $u(b) = 1$. But then, the adjacency relation in $A_u$ implies the presence of active segments as in figure 8.

The other kind of right angle has diagonal sides: these sides may not belong to $\partial R$ and thus have to be active. Indeed, by the adjacency relation in $A_u$, this kind of right angle may only occur in square chunks of sides with length $\sqrt{2}$.

Adding up, figure 9 lists all possible angles of chunks in $\Phi_b(u)$, taking into account the activity of each side.

![Figure 9: Angles; active and inactive sides.](image)

**Lemma 6.** The number of $\pi/4$ angles in a chunk of a splitting $\Phi_b(u)$ is even.

Proof. Lemma 5 excludes the possibility of mixed sides. Clearly, every chunk ought to have an even number of mixed angles, and the only mixed angles measure $\pi/4$.  

Lemma 7. Let $C$ be a chunk in $\Phi_b(u)$ with a side $s$ common to two right angles which is active and either horizontal or vertical. Then $C$ is a square with sides of length 2.

![Figure 10](image_url)

**Figure 10:** (1) implies (2).

**Proof.** Lemma 4 implies that the side $s$ has length 2. In figure 10.1, $b$ is not active, again in accordance with the adjacency relation in $A_u$. Thus, the only active neighbors of $w$ are adjacent in $A_u$.

### 3.3 Chunks in $\Phi_b(u)$ are tiles

Since the smallest external angle of a chunk is $\pi/4$ (figure 9) chunks of $\Phi_b(u)$ have at most 8 sides — we now classify chunks.

**Octagons:** All inner angles have to measure $3\pi/4$. Horizontal and vertical sides have length 2 since they are active (Lemma 4) and thus octagons must be as in figure 1.

There are no heptagonal chunks: A heptagonal chunk should have 6 angles measuring $3\pi/4$ and one, $\pi/2$ (figure 11). Horizontal and vertical sides must be active and have length 2. From figure 11, adding coordinates on both axis, $a' - b' - c' = 0$ and $-a' - b' + c' = 0$. Thus, $a - b - c = 0$ and $-a - b + c = 0$, and hence $b = 0$: one side is gone.

![Figure 11](image_url)

**Figure 11:** There are no heptagonal chunks: $a = \sqrt{2}a'$, $b = \sqrt{2}b'$ and $c = \sqrt{2}c'$.

**Hexagons:** From Lemma 6, chunks of $\Phi_b(u)$ have an even number of angles measuring $\pi/4$. Thus, hexagons must have four $3\pi/4$ angles and two $\pi/2$ angles. Angles ought to be ordered as in the figure 1: from Lemma 7, the right angles have to be separated.

There are no pentagonal chunks: Again, from Lemma 6 a pentagon should have two $3\pi/4$ angles and three right angles. But then two consecutive right angles are inevitable, contradicting Lemma 7. Notice that the hypothesis of the Lemma is fulfilled: there are no mixed angles and hence no inactive sides.
Quadrilaterals: There are two cases. First consider rectangular chunks. At least one of its sides must be active, otherwise $u = 0$. Since right angles are not mixed, all sides must be active. But then, from Lemma 6 and the adjacency relation in $A_u$, the chunk must be one of the squares in figure 1. Second, the chunk may have two $3\pi/4$ angles and two $\pi/4$ angles and must be the trapezoidal tile in figure 1.

Triangles: The triangle must be right isosceles (figure 1). Trapezoidal and triangular chunks have dashed sides: $\pi/4$ angles are mixed. The upshot of the classification is the following — the possible chunks in a splitting $\Phi_b(u)$ are the tiles in figure 1: $\Phi_b(u)$ is a tiling. Thus, the function $\Phi_b$ indeed takes nonzero vectors in the kernel of $BW$ to the subset $T_b$ of tilings. There is an analogous function $\Phi_w$ from $ker BW \setminus \{0\}$ to $T_w$.

4 $\Phi_b$ is a bijection

We start showing that $\Phi_b : ker BW \setminus \{0\} \rightarrow T_b$ is injective. Take $u, v \in ker BW$, inducing tilings $\Phi_b(u)$ and $\Phi_b(v)$. We have to show that if $\Phi_b(u) = \Phi_b(v)$, then $A_u = A_v$ (and then, trivially, $u = v$). If $\Phi_b(u) = \Phi_b(v)$, then $A_u$ and $A_v$ may only differ on $\partial R$, since the only sides which may be active or not lie there. It is the shape of the tile which determines which side is active or not: inactive sides are the large basis of trapezoidal tiles and short sides of right triangles. Thus, sides of a tile in $\Phi_b(u) = \Phi_b(v)$ belonging to $\partial R$ must be simultaneously active or not in both $u$ and $v$, and the proof of injectivity is complete.

We now consider the surjectivity of $\Phi_b$. Given a tiling $T \in T_b$ (and in particular, the description of type of activity for each side), we have to find a nonzero vector $u \in ker BW$ for which $\Phi_b(u) = T$. The sequence of steps below is natural.

- Defining $u$ from $T$.
  $u(b) = 1$ if $b$ belongs to a non-dashed side in $T$.

- Checking that $u \in ker BW$.
  We must show that, for each white point $w$, the sum of the values $u(b)$ for its neighbors $b \in G$ is even. By checking for each kind of tile, it is clear that the sum of $u(b)$ for neighbors $b$ of $w$ in a single tile is even. Thus, if $w$ lies in the interior of a tile, we are done. Suppose now that $w$ lies in a side of a tile of $T$. First, notice that vertices of tiles belonging to active sides are black. Hence, if $w$ is such a vertex, it must belong to inactive sides, and thus, both sides containing $w$ lie in $\partial R$ — $w$ must be a corner of $R$, belonging to a right triangle. In this case, the black neighbors of $w$ are active, if the short sides of the triangle are of length 1, or inactive, for longer lengths. In both cases, the total sum of $u(b)$ over black neighbors $b$ of $w$ is even.

Finally, if $w$ belongs to a side of a tile without being a vertex, this side must be either horizontal or vertical, since diagonal sides only cross black points. If this side lies in $\partial R$, all the black neighbors of $w$ belong to the same tile, and we are done. If, instead, this side is in the interior of $R$, we may take it to be horizontal and it must be an active side. Thus, $u(b_1) = u(b_2) = 1$, where $b_1$ and $b_2$ are the side neighbors of $w$. Besides, the black neighbors of $w$ belong to two tiles, since $w$ is not a vertex. Suppose $b_1$, $b_2$ and $b_3$ in one tile and $b_1$, $b_2$ and $b_4$ in another. Then $u(b_1) + u(b_2) + u(b_3)$ and $u(b_1) + u(b_2) + u(b_4)$
are even. Since \( u(b_1) = u(b_2) = 1 \), we must have \( u(b_3) = u(b_4) = 0 \). Hence, 
\[ u(b_1) + u(b_2) + u(b_3) + u(b_4) = 0 \pmod{2}. \]

- The vector \( u \) obtained from \( T \) indeed satisfies \( \Phi_b(u) = T \).

The black points by which non-dashed sides of \( T \) and \( \Phi_b(u) \) passes are the same. Indeed, these are the active points of \( u \). We are left with checking that the segments between these points are the same in \( T \) and \( \Phi_b(u) \). Clearly, \( \partial R \) consists of segments common to both tilings. Therefore dashed sides are irrelevant: we only need to prove that \( T \) and \( \Phi_b(u) \) have the same active segments.

The active neighborhood of a point \( w \in G \) under \( u \in \ker BW \) is the set of its active black neighbors in \( G \). An active neighborhood is trivial if there are no active points. A simple check obtains all nontrivial active neighborhoods: they are listed in figure 12, up to rotations. In cases (1), (2) and (3), \( w \) is an interior point; in (4) and (5), \( w \) belongs to a side of \( \partial R \), and in (6), \( w \) is a corner of \( R \). Trivial active neighborhoods are not relevant in the forthcoming argument. In order to show that \( T \) and \( T_b \) have the same active segments, all we have to do is to show this fact for each active neighborhood.

In the case of a neighborhood of type (1), the two active points are joined by a segment in \( \Phi_b(u) \), by definition. Suppose that this segment does not belong to \( T \): \( w \) then is in the interior of a tile \( t \) of \( T \). The two active neighbors of \( w \) belong to the boundary of \( t \), by the construction of \( u \) from \( T \). Since tiles are convex, the segments of \( T \) form angles which are less than or equal to \( \pi \). Also, sides of tiles in \( T \) do not pass by points at which \( u \) is zero (we do not have to take into account boundary points). Thus, the sides of \( t \) passing by the active points of the neighborhood of \( w \) must be horizontal, as shown in figure 13; these sides extend to active neighbors at distance 2. But, by figure 13 horizontal sides have length at most 2 – contradiction. Thus \( T \) equals \( \Phi_b(u) \), when restricted to an active neighborhood of type (1).

For a neighborhood of type (2), \( T \) contains active segments joining each neighbor of \( w \) to some other neighbor. Consider the possibility of one of this segments being horizontal, as \( 1 \) in figure 14. Neighbor \( b_2 \) is either joined to
5 Counting polarized harmonic functions

To prove Theorem 1, we compute \( \#T \), the number of tilings of \( R \). Clearly,

\[
\#T = \#T_b \cup \#T_w \quad \text{(by Proposition 1)}
\]

\[
= \#T_b + \#T_w \quad \text{(since } T_b \text{ and } T_w \text{ are disjoint)}
\]

\[
= \# \ker BW \setminus \{0\} + \# \ker WB \setminus \{0\} \quad \text{(by Theorem 2)}
\]

\[
= \# \ker BW + \# \ker WB - 2.
\]

We only have to compute the dimensions \( \beta \) and \( \omega \) of the kernels of \( BW \) and \( WB \). We will exhibit two symmetries common to the kernel elements, and which are evident in figure 13. Given a vector \( v \) belonging to \( (\mathbb{Z}/(2))^{mn} \), denote by \( \sigma_{i,j} \) the sum \( \text{(mod 2)} \) of the coordinates of \( v \) at the four neighbors of \( (i,j) \) in \( G \): 

\[
\sigma_{i,j} = v_{i,j-1} + v_{i,j+1} + v_{i-1,j} + v_{i+1,j} \quad \text{(mod 2)}
\]

Here \( v_{i,j} = 0 \) if \( (i,j) \notin G \). Recall that the kernel of \( M \), the adjacency matrix of \( G \), is the direct sum of the kernels of \( BW \) and \( WB \). Without loss we will assume from now on that the rectangular graph satisfies \( m \geq n \).

**Proposition 3.** Let \( v \in \ker M \). Then

(i) \( v_{i,j} = v_{j,i} \), if both points belong to \( G \).

(ii) If the vector \( v \) equals zero along the column \( x = m_0 \) then \( v_{m_0-i,j} = v_{m_0+i,j} \), again if both points belong to \( G \).
Clearly, there are analogous symmetries across the other diagonals of the graph, and across rows of zeros of \( v \).

**Proof.** To start off the induction argument, notice that \( v_{i,j} = v_{j,i} \) if \( i = j \) (i.e., along the line \( y = x \)). Now compare values along the lines \( y = x+1 \) and \( y = x-1 \). Since \( \sigma_{0,0} = 0 \), we must have \( v_{1,0} = v_{0,1} \). Sequentially, \( \sigma_{1,1} = 0 \), \( \sigma_{2,2} = 0 \), . . . imply in turn \( v_{2,1} = v_{1,2} \), \( v_{3,2} = v_{2,3} \), . . . Compare now values along the lines \( y = x + 2 \) and \( y = x - 2 \), making use of \( \sigma_{1,0} = 0 = \sigma_{0,1} \). Proceed until the lines \( y = x \pm k \) cover the square \( [0, n-1] \times [0, n-1] \).

The proof of (ii) is a simple induction on \( i \). Case \( i = 1 \) is as hard as the general case, and follows from expanding \( 0 = \sigma_{m_0,j} = v_{m_0-1,j} + v_{m_0+1,j} \).

Consider the \( m \times n \) rectangular graph \( G \) and let \( c = \gcd(m+1, n+1) - 1 \). The grid \( G \) of \( G \) is the set of points in \( G \) belonging to the union of the horizontal lines \( y = k(c+1) - 1 \) with the vertical lines \( x = k(c+1) - 1 \), for integer \( k \). The graph \( G \), after removal of its grid \( G \), splits into \( c \times c \) squares, called fundamental squares. Figure 15 shows the grid and the fundamental squares for the 14 × 9 rectangular graph.

**Theorem 3.** Let \( G \) be the \( m \times n \) rectangular graph with grid \( G \) and adjacency matrices \( BW_{m,n} \) and \( WB_{m,n} \), with kernel dimensions \( \beta \) and \( \omega \) respectively. Let \( v \in \ker BW_{m,n} \) (resp. \( \ker WB_{m,n} \)). Then

(i) The vector \( v \) is identically zero on \( G \).

(ii) The restriction of \( v \) to each fundamental square lies in the kernel of the matrix \( BW_{c,c} \) (resp. \( WB_{c,c} \)) of this square.

(iii) Let \( s_1 \) and \( s_2 \) be two adjacent fundamental squares, bounded by a common row or column of \( G \). Then the restrictions of \( v \) to \( s_1 \) and \( s_2 \) are mirror images of each other with respect to this common row or column.

(iv) \( \beta = \dim \ker BW_{c,c} \) and \( \omega = \dim \ker WB_{c,c} \).

**Proof.** We shall prove these assertions by induction on the set of pairs \((m, n)\) for which \( \gcd(m + 1, n + 1) = c + 1 \) is fixed, ordered lexicographically. For the starting point \( m = n = c \), the result is obvious. Suppose that \( v^{m,n} \in \ker BW_{m,n} \),
in a self-explanatory notation. From the symmetry (i) of the previous proposition, the restriction of $v^{m,n}$ to column $x = n$ is zero. Indeed, for $v = v^{m,n}$, since

$$0 = \sigma_{n-1,j} = v_{n-2,j} + v_{n-1,j-1} + v_{n-1,j+1},$$

$$0 = \sigma_{j,n-1} = v_{j,n-2} + v_{j-1,n-1} + v_{j+1,n-1} + v_{j,n},$$

we must have $v_{j,n} = 0$. Removal of column $x = n$ from $G = G_{m,n}$ gives rise to two connected subgraphs: a square $G_{n,n}$ containing the first $n$ columns of $G$ and a possibly rectangular subgraph $G_{m-n-1,n}$ containing only the last $m-n-1$ columns. Also, by making use of the column of zeros and the fact that $v^{m,n} \in \ker BW_{m,n}$, we have that the restriction $v^{m-n-1,n}$ of $v^{m,n}$ to $G_{m-n-1,n}$ belongs to $\ker BW_{m-n-1,n}$. Notice that $\gcd(m - n - 1, n + 1) = c + 1$: by induction, then, $v^{m-n-1,n}$ has zeros on the grid of $G_{m-n-1,n}$. Thus, the nonzero entries of $v^{m-n-1,n}$ belong to $c \times c$ squares framed by points in which $v^{m-n-1,n}$ is zero. Now using symmetry (ii), mirroring these squares across columns $n, n - c - 1 \ldots$, we see that the square $G_{n,n}$ also splits in $c \times c$ squares with the same property. Conversely, mirroring a kernel element on the fundamental square across the grid obtains a kernel element on the full rectangle: this proves (iv).

We now compute the dimensions of the kernels of $BW$ and $WB$ for the square graph $G_{c,c}$.

**Proposition 4.** Let $G_{c,c}$ be the $c \times c$ square graph. Then

$$2\dim \ker BW_{c,c} = c + (c \mod 2) \quad \text{and} \quad 2\dim \ker WB_{c,c} = c - (c \mod 2).$$

*Proof.* It is clear that a vector in $\ker BW_{c,c}$ (or in $\ker WB_{c,c}$) is determined by its values along the first column $x = 0$ of $G_{c,c}$. The fact that any choice of values along this columns indeed gives rise to a kernel vector follows from the diagonal symmetry (i) of Proposition 3.

Certain choices of values along the first column of $G_{c,c}$ obtain especially simple kernel vectors. In figure 16, the first (resp. second) row indicates a basis for $\ker BW$ (resp. $\ker WB$).

Figure 16: Basis for $\ker BW$ and for $\ker WB$. 

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Combining the previous proposition with item (iv) of the theorem above, we complete the proof of Theorem 1. Figure 17 shows all tilings of the $10 \times 4$ rectangle: the nonzero elements in the kernel of $WB$ give rise to the first three, $A$, $B$ and $A + B$. The remaining ones correspond to nonzero elements in the kernel of $BW$.

Figure 17: All tilings of the $10 \times 4$ rectangle.

References

[DT] Deift, P. A. & Tomei, C. On the determinant of the adjacency matrix for a planar sublattice. J. Combin. Theory Ser. B 35 (1983), no. 3, 278–289.