Dynamical Chiral Symmetry Breaking in QED in a Magnetic Field: Toward Exact Results

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We describe a (first, to the best of our knowledge) essentially soluble example of dynamical symmetry breaking phenomenon in a 3+1 dimensional gauge theory without fundamental scalar fields: QED in a constant magnetic field.

Recently the magnetic catalysis of dynamical chiral symmetry breaking has been established as a universal phenomenon in 2+1 and 3+1 dimensions: a constant magnetic field leads to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions. The essence of this effect is the dimensional reduction $D \to D - 2$ in the dynamics of fermion pairing in a magnetic field: at weak coupling, this dynamics is dominated by the lowest Landau level (LLL) which is essentially $(D - 2)$-dimensional. The effect may have interesting applications in condensed matter physics and cosmology.

In particular, this phenomenon was considered in 3+1 dimensional QED. Since the dynamics of the LLL is long-range (infrared), and the QED coupling constant is weak in the infrared region, one may think that the rainbow (ladder) approximation is reliable in this problem. As was shown in Refs. [2,3], the dynamical mass of fermions in this approximation is

$$ m_{\text{dyn}} = C \sqrt{|eB|} \exp \left[ \frac{\pi}{2} \left( \frac{\pi}{2\alpha} \right)^{1/2} \right],$$

where $C$ is the renormalized coupling constant related to the scale $\mu^2 \sim |eB|$.

Are higher order contributions indeed suppressed in this problem? The answer is “no”. As was shown in Ref. 3, because of the (1+1)-dimensional form of the fermion propagator of the LLL fermions, there are relevant higher order contributions. In particular, considering this problem in the improved rainbow approximation (when the vertex is bare, and the polarization operator is calculated in one-loop approximation), it was shown that, in all covariant gauges, the fermion mass $m_{\text{dyn}}$ is given by Eq. (1) but with $\alpha \to \alpha/2$.

As we wrote in the paper [3], “it is a challenge to define the class of all those diagrams in QED in a magnetic field that give a relevant contribution in this problem”. The aim of this letter is to solve the problem. We will show that there exists a (non-covariant) gauge in which the Schwinger-Dyson equations written in the improved rainbow approximation are reliable. The expression for $m_{\text{dyn}}$ takes the following form:

$$ m_{\text{dyn}} = \tilde{C} \sqrt{|eB|} F(\alpha) \exp \left[ \frac{\pi}{\alpha \ln (C_1/N\alpha)} \right],$$

where $N$ is the number of fermion flavors, $F(\alpha) \simeq (N\alpha)^{1/3}$, $C_1 \simeq 1.82 \pm 0.06$ and $\tilde{C} \sim O(1)$. This expression for $m_{\text{dyn}}$ is essentially different from that in the rainbow approximation (1). As we will see, this reflects rather rich and sophisticated dynamics in this problem.

The lagrangian density of massless QED in a magnetic field is

$$ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} [\bar{\psi}, (i\gamma^\mu D_\mu)\psi],$$

where the covariant derivative $D_\mu$ is

$$ D_\mu = \partial_\mu - ie(A^\text{ext}_\mu + A_\mu),$$

$$ A^\text{ext}_\mu = \left( 0, -\frac{B}{2} x_2, \frac{B}{2} x_1, 0 \right),$$

i.e. we use the so-called symmetric gauge for $A^\text{ext}_\mu$. The magnetic field $B$ is in the $+x_3$ direction.

Besides the Dirac index $(n)$, the fermion field carries an additional flavor index $a = 1, 2, \ldots, N$. Then the Lagrangian density in Eq. (3) is invariant under the chiral $SU_L(N) \times SU_R(N)$ symmetry.

The Schwinger-Dyson (SD) equations in QED in external fields were derived by Schwinger and Fradkin (for a review, see Ref. [3]). The equation for the fermion propagator $G(x, y)$ is

$$ G(x, y) = S(x, y) - 4\pi \alpha \int d^4 u d^4 u' d^4 z d^4 z' S(x, u) \gamma^\mu$$
$$ \times G(u, z) \Gamma^\nu(z, u', z') G(u', y) D_{\mu\nu}(z', u).$$

Here $S(x, y)$ is the bare fermion propagator in the external field $A^\text{ext}_\mu$, and $D_{\mu\nu}(x, y)$, $\Gamma^\nu(x, y, z)$ are the full photon propagator and the amputated vertex.

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The full photon propagator satisfies the equations:
\[
D_{\mu\nu}^{-1}(x, y) = D_{\mu\nu}^{\text{bare}}^{-1}(x, y) + \Pi_{\mu\nu}(x, y),
\]
where \(D_{\mu\nu}(x, y)\) is the free photon propagator and \(\Pi_{\mu\nu}(x, y)\) is the polarization operator.

The bare fermion propagator \(S(x, y)\) in a constant magnetic field was calculated by Schwinger \[1,2\]. In the symmetric gauge \(x = 0\), it has the form:
\[
S(x, y) = \exp \left(i e x^\mu A^{\text{ext}}_{\mu}(y)\right) \tilde{S}(x - y).
\]

Then, it is not difficult to show directly from the SD equations that
\[
G(x, y) = \exp \left(i e x^\mu A^{\text{ext}}_{\mu}(y)\right) \tilde{G}(x - y),
\]
\[
\Gamma(x, y, z) = \exp \left(i e x^\mu A^{\text{ext}}_{\mu}(y)\right) \tilde{\Gamma}(x - z, y - z),
\]
\[
D_{\mu\nu}(x, y) = \tilde{D}_{\mu\nu}(x, y),
\]
\[
\Pi_{\mu\nu}(x, y) = \tilde{\Pi}_{\mu\nu}(x, y).
\]

In other words, in a constant magnetic field, the Schwinger phase is universal for Green functions containing one fermion field, one antifermion field, and any number of photon fields, and the full photon propagator is translation invariant.

Our aim is to show that there exists a gauge in which the approximation with a bare vertex,\[10\] is reliable for the description of spontaneous chiral symmetry breaking in a magnetic field.

The bare vertex \(\tilde{v}(p)\) of fermions from the LLL is\[2\]
\[
\tilde{v}(p) = 2 ie^{-|eB|/2} \hat{p}_4^\dagger p_4 + \frac{m}{p_4^2 - m^2} O(-),
\]
where the magnetic length \(l = \frac{|eB|^{-1}}{2}, p_4 = (p^1, p^2, p_4), \hat{p}_4 = p_4^0 - p_3^3\gamma^3\). The operator \(O(-) = \left[1 - i\gamma^1\gamma^2\text{sgn}(eB)\right]/2\) is the projection operator on the fermion states with the spin polarized along the magnetic field. This point and Eq. \[11\] clearly reflect the \((1+1)\)-dimensional character of the dynamics of fermions in the LLL.

3. In the one-loop approximation, with fermions from the LLL, the photon propagator takes the following form in covariant gauges \[11,3\]:
\[
D_{\mu\nu}(k) = -i \left[ \frac{1}{k^2} g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \frac{1}{k^2 + k_\perp^2} \Pi(k_\perp, k_\perp' \to k_\perp') \right],
\]

where \(\lambda\) is a gauge parameter. The explicit expression for \(\Pi(k_\perp^2, k_\perp^2') = \exp[-(k_\perp l)^2/2]\Pi(k_\perp^2)\) is given in Refs. \[11,3\]. For our purposes, it is sufficient to know its asymptotes,
\[
\Pi(k_\perp^2) \approx -\frac{\bar{\alpha}}{3\pi} \frac{|eB|}{m^2} \quad \text{as} \quad |k_\perp^2| \ll m^2, \quad (13)
\]
\[
\Pi(k_\perp^2) \approx -\frac{2\bar{\alpha}}{\pi} \frac{|eB|}{k_\perp^2} \quad \text{as} \quad |k_\perp^2| \gg m^2, \quad (14)
\]
where \(\bar{\alpha} = N\alpha\). The polarization effects are absent in the transverse components of \(D_{\mu\nu}(k)\). This is because the bare vertex for fermions from the LLL is \(O(-)\gamma^\mu O(-)\equiv O(-)\gamma^\mu\). Therefore the LLL fermions couple only to the longitudinal \((0, 3)\) components of the photon field. Then, there is a strong screening effect in the \(g_{\mu\nu} - k_\mu k_\nu/k^2\) component of the photon propagator. For \(m^2 \ll |k_\perp^2| \ll |eB|\) and \(|k_\perp^2| \ll |eB|\), Eq. \[13\] implies that
\[
1 \lesssim \frac{k^2 + k_\perp^2 \Pi(k_\perp^2, k_\perp^2')}{k_\perp^2 - M_\gamma^2}, \quad (15)
\]
with \(M_\gamma^2 = 2\bar{\alpha}|eB|/\pi\). This is reminiscent of the Higgs effect in the \((1+1)\)-dimensional QED (Schwinger model) \[12,13\].

We emphasize that infrared dynamics in this problem is very different from that in the Schwinger model: since photon is neutral, there is the four-dimensional \(k^2 = k_\perp^2 - k_3^2\) in the denominator of the photon propagator. However, the tensor and the spinor structure of this dynamics is exactly the same as in the Schwinger model. This point will be crucial for finding a gauge in which the improved rainbow approximation [with the bare vertex \(\tilde{v}(p)\)] is reliable.\[4\]

\[1\] Since an external magnetic field does not lead to confinement of fermions, their mass is gauge invariant in QED in a magnetic field. Therefore any gauge can be used for the calculations of the mass if either the calculations provide the exact result or a good approximation is used, i.e., one can show that corrections to the obtained result are small. Below we will define such a gauge in this model.
We recall that, as was shown in Ref. [3], despite the smallness of \( \alpha \), the expansion in \( \alpha \) is broken in covariant gauges in this problem. The reason is that, because of the smallness of \( m_{\text{dyn}} \) in Eq. (10), as compared to \( \sqrt{\epsilon B} \), there are mass singularities, in \( \epsilon B/m_{\text{dyn}}^2 \sim \alpha^{-1/2} \), in infrared dynamics. In particular, calculating the one-loop correction to the vertex in covariant gauges with the photon propagator [3], one finds that, when external momenta are of order \( m_{\text{dyn}} \) or less, there are contributions of order \( \alpha \ln^2(\epsilon B/m_{\text{dyn}}^2) \sim O(1) \). They come from the term \( k_\perp k_\parallel/k^2 \) in \( D_{\mu\nu}(k) \) in Eq. (12).

How can one avoid such mass singularities? A solution is suggested by the Schwinger model. It is known that there is a gauge in which the full vertex is just the bare one [1]. It is the gauge with a bare photon propagator

\[
D_{\alpha\beta}(k) = -\frac{1}{k^2} \left( g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) - i\delta(k^2) \frac{k_\alpha k_\beta}{(k^2)^2}, \tag{16}
\]

with the (non-local) gauge function \( d = 1/(1 + \Pi) \), where the polarization function \( \Pi(k^2) = -\epsilon^2/\pi k^2 \) in the Schwinger model (of course, here \( \alpha, \beta = 1, 0 \)). Then, the full propagator is proportional to \( g_{\alpha\beta} \):

\[
D_{\mu\nu}(k) = D_{\alpha\beta}(k) + i \left( g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) \frac{\Pi(k^2)}{k^2(1 + \Pi(k^2))} = -i \frac{g_{\alpha\beta}}{k^2(1 + \Pi(k^2))}. \tag{17}
\]

The point is that since now \( D_{\alpha\beta}(k) \sim g_{\alpha\beta} \) and since the fermion mass \( m = 0 \) in the Schwinger model, all loop contributions to the vertex are proportional to \( P_{2n+1} \equiv \gamma_0 \gamma_1 \cdots \gamma_{2n+1} \gamma^\alpha = 0 \) in this gauge and, therefore, disappear [3].

Let us return to the present problem. As it was emphasized above, the tensor and the spinor structure of the LLL dynamics is (1+1)-dimensional. Now, take the bare propagator

\[
D_{\mu\nu}(k) = -\frac{1}{k^2} \left( \frac{\delta_{\mu\nu} - k_\mu k_\nu}{k^2} \right) - i\delta(k^2) \frac{k_\mu k_\nu}{(k^2)^2}, \tag{18}
\]

with \( d = -k^2_\parallel/[k^2 + k^2_\parallel] + k^2_\parallel/k^2 \). Then, the full propagator is

\[
D_{\mu\nu}(k) = D_{\mu\nu}(k) + i \left( \frac{\delta_{\mu\nu} - k_\mu k_\nu}{k^2} \right) \frac{\Pi(k^2)}{k^2(1 + \Pi(k^2))} \times \frac{g_{\alpha\beta}}{k^2} = -i \frac{g_{\alpha\beta}}{k^2 + k^2_\parallel \Pi(k^2, k^2_\parallel)}, \tag{19}
\]

The crucial point is that, as was pointed out above, the transverse degrees of freedom decouple from the LLL dynamics. Therefore only the first term in \( D_{\mu\nu} \), proportional to \( g_{\parallel\parallel} \), is relevant.

Notice now that mass singularities in loop corrections to the vertex might potentially occur only in the terms containing \( q_\parallel^2 = q_\parallel^0 - q_\parallel^3 \) from a numerator \( (q_\parallel^2 + m_{\text{dyn}}) \) of each fermion propagator in a diagram (all other terms contain positive powers of \( m_{\text{dyn}} \), coming from at least some of the numerators and, therefore, are harmless [4]). However, because of the same reasons as in the gauge [1] in the Schwinger model, all those potentially dangerous terms disappear in the gauge [1]. Therefore all the loop corrections to the vertex are suppressed by positive powers of \( \alpha \) in this gauge. This in turn implies that those loop corrections may result only in a change \( \tilde{C} \sim O(1) \to \tilde{C}' \sim O(1) \) in Eq. (2), i.e., this expression yields the exact singularity at \( \alpha = 0 \) for the fermion mass. In other words, in gauge [1] there exists a consistent truncation of the SD equations and the problem is essentially soluble in this gauge [1].

As a result, in this gauge, the SD equations (10), (11), and (12) with the bare vertex are reliable. They form a closed system of integral equations. Using Eqs. (9a)–(14) and the bare propagator (11) of massless \( (m = 0) \) fermions from the LLL, one finds the SD equations for the full fermion propagator

\[
\tilde{G}(p) = 2ie^{-(p_\perp l)^2} \frac{A(p_\perp^2)p_\perp + B(p_\perp^2)}{A^2(p_\perp^2)p_\perp^2 - B^2(p_\perp^2)} \delta\Gamma \tag{20}
\]

[compare with \( \tilde{S}(p) \) in Eq. (11)]. In Euclidean space they are: \( A(p_\perp^2) = 1 \) and

\[
B(p_\perp^2) = \frac{\alpha}{2\pi^2} \int \frac{d^2q_\perp B((p_\perp - q_\perp)^2)}{(p_\perp - q_\perp)^2 + B^2((p_\perp - q_\perp)^2)} \times \int_0^\infty \frac{dx \exp(-x l^2/2)}{x + q_\perp^0 + q_\perp^3 \Pi_E(x, q_\perp^3)}, \tag{21}
\]

where the polarization function \( \Pi \) is defined from Eq. (7) with a bare vertex.

A detailed analysis of these equations will be presented elsewhere. Here we just indicate the crucial points in the analysis.

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\(^2\) \( P_{2n+1} = 0 \) follows from the two identities for the two-dimensional Dirac matrices: \( \gamma_\alpha \gamma_\lambda \gamma^\alpha = 0 \) and \( \gamma_\alpha \gamma_\lambda \gamma_{n+1} = g_\lambda \gamma_\lambda + \varepsilon_\lambda \gamma_{n+1} \gamma_5 \) [5] (\( \gamma_5 = \gamma_0 \gamma_1 \), \( \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \varepsilon_{01} = 1 \)).

\(^3\) For example, one can show that the contribution of the term with one-loop vertex correction in the SD equation is suppressed as \( \alpha \ln \alpha \) with respect to the leading term.

\(^4\) The gauge [1] is unique in that. In other gauges, there is an infinite set of diagrams giving relevant contributions to the vertex. Therefore, in other gauges, one needs to sum up an infinite set of diagrams to recover the same result for the fermion mass.
The polarization function \( \Pi \) is a complicated functional of the fermion mass function \( B(p_f^2) \). However, one can show that the leading singularity, \( 1/\alpha \ln(\alpha) \), in \( \ln(m_{dyn}^2) \) in Eq. (\ref{eq:effective_mass}) is induced in the kinematic region with \( m_{dyn}^2 \ll |q_f^2| \ll |eB| \) and \( m_{dyn}^2 \ll M_{\pi}^2 \ll q_f^2 \ll |eB| \). In that region, the fermions can be treated as massless, and therefore the polarization function is \( \Pi \propto 2\alpha/m_{dyn}^2 \) [see Eqs. (\ref{eq:propagator}) and (\ref{eq:sd_equation})]. Therefore, in this approximation, the photon propagator is a propagator of a free massive boson with \( M_{\pi}^2 \). Such an equation of motion is the SD equation (\ref{eq:sd_equation}) with \( \Pi \). This fit corresponds to expression (\ref{eq:fit_function}) with \( C_1 = \alpha \pi \approx 1.82 \pm 0.06 \). The analytical solution yields a similar result.

The magnetic catalysis of chiral symmetry breaking in QED yields an essentially soluble, and quite non-trivial, example of the phenomenon of dynamical symmetry breaking in a (3+1)-dimensional gauge theory without scalar fields. It may provide insight into the deep infrared dynamics with \( |q| \ll m_{dyn} \). It is noticeable as an example for a possibility discussed in QCD: chiral symmetry breaking might be independent of the dynamics of confinement with \( |q| \ll \Lambda_{QCD} \sim m_{dyn} \). Another noticeable point is the dimensional reduction in the present model: there are arguments in support of a similar reduction in the dynamics of chiral symmetry breaking in QCD (\ref{eq:sd_equation}).

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5 The solution shows that the function \( B(p_f^2) \) is essentially constant for \( p_f^2 \ll |eB| \), \( B(p_f^2) = m_{dyn}^2 \), and rapidly decreases for \( p_f^2 \gg |eB| \). Therefore this approximation is self-consistent: the Ward identity for the vertex is satisfied in the relevant kinematic region and the fermion pole is at \( p_f^2 = m_{dyn}^2 \). Symmetry breaking in this model is generated in the region of intermediate momenta, i.e. it is independent of the deep infrared dynamics with \( |q| \ll m_{dyn} \). It is noticeable as an example for a possibility discussed in QCD: chiral symmetry breaking might be independent of the dynamics of confinement with \( |q| \ll \Lambda_{QCD} \sim m_{dyn} \). Another noticeable point is the dimensional reduction in the present model: there are arguments in support of a similar reduction in the dynamics of chiral symmetry breaking in QCD (\ref{eq:sd_equation}).