Monomials, Binomials and Riemann-Roch

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Abstract

The Riemann-Roch theorem on a graph $G$ is related to Alexander duality in combinatorial commutative algebra. We study the lattice ideal given by chip firing on $G$ and the initial ideal whose standard monomials are the $G$-parking functions. When $G$ is a saturated graph, these ideals are generic and the Scarf complex is a minimal free resolution. Otherwise, syzygies are obtained by degeneration. We also develop a self-contained Riemann-Roch theory for artinian monomial ideals.

1 Introduction

We examine the Riemann-Roch theorem on a finite graph $G$, due to Baker and Norine [3], through the lens of combinatorial commutative algebra. Throughout this paper, $G$ is undirected and connected, has $n$ nodes, and multiple edges are allowed, but we do not allow loops. Its Laplacian is a symmetric $n \times n$-matrix $\Lambda_G$ with non-positive integer entries off the diagonal and kernel spanned by $e = (1,1,\ldots,1)$. Divisors on $G$ are identified with Laurent monomials $x^u = x_1^{u_1}x_2^{u_2}\cdots x_n^{u_n}$. The chip firing moves are binomials $x^u - x^v$ where $u, v \geq 0$ and $u - v$ is in the lattice spanned by the columns of $\Lambda_G$. The lattice ideal $I_G$ spanned by such binomials is here called the toppling ideal of the graph $G$. It was introduced by Perkinson, Perlman and Wilmes [11, 15], following an earlier study of the inhomogeneous version of $I_G$ by Cori, Rossin and Salvy [6].

For any fixed node, the toppling ideal $I_G$ has a distinguished initial monomial ideal $M_G$. This monomial ideal was studied by Postnikov and Shapiro [12], and it is characterized by the property that the standard monomials of $M_G$ are the $G$-parking functions. We construct free resolutions for both $I_G$ and $M_G$, and we study their role for Riemann-Roch theory on $G$. For an illustration, consider the complete graph on four nodes, $G = K_4$.

The chip firing moves on $K_4$ are the integer linear combinations of the columns of

$$\Lambda_G = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}. \tag{1}$$

The toppling ideal is the lattice ideal in $\mathbb{K}[x] = \mathbb{K}[x_1, x_2, x_3, x_4]$ that represents $\text{image}_\mathbb{Z}(\Lambda_G)$:

$$I_G = \langle x_1^3 - x_2x_3x_4, x_2^3 - x_1x_3x_4, x_3^3 - x_1x_2x_4, x_1x_2x_3 - x_4^3, \frac{x_1^2}{x_2^2} - \frac{x_3^2}{x_4^2}, \frac{x_2^2}{x_3^2} - \frac{x_1^2}{x_4^2}, \frac{x_3^2}{x_4^2} - \frac{x_2^2}{x_1^2} \rangle. \tag{2}$$
This ideal is *generic* in the sense of Peeva and Sturmfels [10], as each of the seven binomials contains all four variables. The minimal free resolution is given by the *Scarf complex*

\[ 0 \leftarrow \mathbb{K}[x] \leftarrow \mathbb{K}[x]^7 \leftarrow \mathbb{K}[x]^12 \leftarrow \mathbb{K}[x]^6 \leftarrow 0. \tag{3} \]

The seven binomials in (2) form a Gröbner basis of \( I_G \), with the underlined monomials generating the initial ideal \( M_G \). That monomial ideal has the irreducible decomposition

\[ M_G = \langle x_1, x_2^2, x_3^3 \rangle \cap \langle x_1, x_2^3, x_3^2 \rangle \cap \langle x_1^2, x_2, x_3^3 \rangle \cap \langle x_1^2, x_2^3, x_3^2 \rangle \cap \langle x_1^3, x_2, x_3^2 \rangle \cap \langle x_1^3, x_2^2, x_3 \rangle . \]

The ideal \( M_G \) is the *tree ideal* of [9, §4.3.4]. Its standard monomials are in bijection with the 16 spanning trees. Its Alexander dual is generated by the six socle elements

\[ x_2x_3^2, x_2^2x_3, x_1x_3^2, x_1x_2^2, x_1^2x_3, x_1^2x_2. \tag{4} \]

These correspond to the *maximal parking functions* studied in combinatorics; see [5, 12]. We claim that the duality seen in Figures 4.2 and 4.3 of [9] is the same as that expressed in the Riemann-Roch Theorem for \( G \). This will be made precise in Sections 3 and 4.

The present article is organized as follows: Section 2 is concerned with the case when \( G \) is a *saturated graph*, meaning that any two nodes \( i \) and \( j \) are connected by at least one edge. We show that here \( I_G \) is a generic lattice ideal, and we determine its minimal free resolution and its Hilbert series in the finest grading. The Scarf complex of the initial monomial ideal \( M_G \) is supported on the barycentric subdivision of the \((n-2)\)-simplex [12, §6], and this lifts to the Scarf complex of the lattice ideal \( I_G \) by [10, Corollary 5.5].

In Section 3 we revisit the Riemann-Roch formula

\[ \text{rank}(D) - \text{rank}(K-D) = \text{degree}(D) - \text{genus} + 1. \tag{5} \]

We prove this formula in an entirely new setting: the role of the curve is played by a monomial ideal, and that of the divisors \( D \) and \( K \) is played by monomials \( x^b \) and \( x^K \). The identity (5) is shown for monomial ideals that are artinian, level, and reflection-invariant. This includes the parking function ideals \( M_G \) derived from saturated graphs \( G \).

In Section 4 we extend our results to the case of graphs \( G \) that are not saturated, and we rederive Riemann-Roch for graphs as a corollary. Here \( M_G \) is still an initial ideal of \( I_G \), but the choice of term order is more delicate [11 §5]. One choice is the cost function used by Baker and Shokrieh for the integer program in [4, Theorem 4.1]. The Scarf complexes in Section 2 support cellular free resolutions of \( I_G \) and \( M_G \), but these resolutions are usually far from minimal. We conclude with several open questions.

This paper demonstrates how Riemann-Roch theory embeds into combinatorial commutative algebra. Our main results are Theorems [2, 13 and 25]. These build on earlier works, notably [11] and [12], but they go much further and are new in their current form.

When this collaboration started in the summer of 2011, both authors were unaware of the articles [11, 15] written on similar topics by David Perkinson and his students at Reed College. As our point of departure, we chose to focus on chip firing in the most classical case of undirected graphs, but with the tacit understanding that our ideals and modules generalize to directed graphs, arithmetic graphs, simplicial complexes, matroids, abelian networks, or any of the other extensions seen in the recent chip firing literature (cf. [2]).
2 Saturated graphs

In this section, we assume that the graph $G$ has $u_{ij}$ edges between node $i$ and node $j$, where $u_{ij}$ is a positive integer, for $i \neq j$. However, we do not allow loops, so that $u_{11} = u_{22} = \cdots = u_{nn} = 0$. Thus, in the language of [12], $G$ is a saturated graph.

We shall see that, under this hypothesis, the lattice ideal $I_G$ is generic, and an explicit combinatorial description of its minimal free resolution can be given. Throughout this paper we work in the polynomial ring $K[x]$.

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We begin by explicitly showing the generators of the lattice ideal $I_G$ in the case $n = 4$.

**Example 1.** If $G$ is a saturated graph on $[4] = \{1, 2, 3, 4\}$ then $I_G$ is generated by

$$x_1^{u_{12}+u_{13}+u_{14}} - x_2^{u_{12}} x_3^{u_{13}} x_4^{u_{14}}, \quad x_2^{u_{12}+u_{23}+u_{24}} - x_1^{u_{12}} x_3^{u_{23}} x_4^{u_{24}}, \quad x_3^{u_{13}+u_{23}+u_{34}} - x_1^{u_{13}} x_2^{u_{23}} x_4^{u_{34}},$$

$$x_2^{u_{14}+u_{24}+u_{34}} - x_1^{u_{14}} x_3^{u_{24}} x_4^{u_{34}}, \quad x_1^{u_{13}+u_{23}+u_{44}} - x_2^{u_{13}} x_3^{u_{23}} x_4^{u_{44}}, \quad x_3^{u_{12}+u_{23}+u_{14}+u_{34}} - x_2^{u_{12}} x_4^{u_{23}} x_3^{u_{14}+u_{34}}.$$  

Here the $u_{ij}$ are arbitrary positive integers. These binomials form a Gröbner basis. The initial ideal $M_G$ is generated by the underlined monomials. The minimal free resolution of $I_G$ has the form (3). The same holds for $M_G$, as was shown in [12, Corollary 6.9]. The minimal resolution of $M_G$ is given by the Scarf complex, which is depicted in Figure 1.

We now state our main result in this section. For disjoint subsets $I$ and $J$ of $[n]$ we set

$$x^{I \to J} := \prod_{i \in I} x_i^{\sum_{k \in J} u_{ik}}.$$

A split of the set $[n] = \{1, 2, \ldots, n\}$ is an unordered pair $(I, J)$ of non-empty disjoint subsets $I$ and $J$ whose union equals $[n]$. The number of splits equals $2^{n-1} - 1$. With each split $(I, J)$ we associate the following binomial which is well-defined up to sign:

$$x^{I \to J} - x^{J \to I}. \quad (6)$$

These are precisely the seven binomials in Example 1, one for each split $(I, J)$.

Let $\text{Cyc}_{n,k}$ denote the set of cyclically ordered partitions of the set $[n]$ into $k$ blocks. Each element of $\text{Cyc}_{n,k}$ has the form $(I_1, I_2, \ldots, I_k)$, where $I_1 \cup I_2 \cup \cdots \cup I_k = [n]$ is a partition. We regard the $(I_1, I_2, \ldots, I_k)$ as formal symbols, subject to the identifications

$$(I_1, I_2, \ldots, I_{k-1}, I_k) = (I_2, I_3, \ldots, I_k, I_1) = \cdots = (I_k, I_1, \ldots, I_{k-2}, I_{k-1}).$$

We write $\mathbb{K}[x]^{\text{Cyc}_{n,k}}$ for the free $\mathbb{K}[x]$-module generated by these symbols. The rank of this free module equals the number of cyclically ordered partitions, namely

$$|\text{Cyc}_{n,k}| = (k - 1)! \cdot S_{n,k}, \quad (7)$$

where $S_{n,k}$ is the Stirling number of the second kind, i.e., the number of partitions of the set $[n]$ into $k$ blocks. Let $\text{CYC}_G$ denote the following complex of free $\mathbb{K}[x]$-modules:

$$0 \leftarrow \mathbb{K}[x]^{\text{Cyc}_{n,1}} \leftarrow \mathbb{K}[x]^{\text{Cyc}_{n,2}} \leftarrow \mathbb{K}[x]^{\text{Cyc}_{n,3}} \leftarrow \cdots \leftarrow \mathbb{K}[x]^{\text{Cyc}_{n,n}} \leftarrow 0, \quad (8)$$
where the boundary map from $\mathbb{K}[x]_{\text{Cyc}_{n,r}}$ to $\mathbb{K}[x]_{\text{Cyc}_{n,r-1}}$ is given by the formula

$$(I_1, I_2, I_3, \ldots, I_r) \mapsto \sum_{s=1}^{r-1} (-1)^{s-1} x^{I_s \rightarrow I_{s+1}} (I_1, \ldots, I_{s-1}, I_s \cup I_{s+1}, I_{s+2}, \ldots, I_r) - x^{I_r \rightarrow I_1} (I_2, I_3, \ldots, I_{r-1}, I_1 \cup I_r).$$  \hspace{1cm} (9)$$

In this formula it is assumed that $n \in I_r$, so as to ensure that all signs are consistent.

**Theorem 2.** Let $G$ be a saturated graph. The toppling ideal $I_G$ is a generic lattice ideal. It is minimally generated by the $2^{n-1} - 1$ binomials (6), these form a reverse lexicographic Gröbner basis, the complex CYC$_G$ coincides with the Scarf complex, and this complex minimally resolves $\mathbb{K}[x]/I_G$.

**Proof.** We begin by noting that $x^{I \rightarrow J} - x^{J \rightarrow I}$ actually lies in the ideal $I_G$. To see this, let $e_I$ denote the incidence vector in $\{0,1\}^n$ that represents the subset $I$ of $[n]$. The $i$-th coordinate of the vector $\Lambda_G \cdot e_I$ is equal to $\sum_{k \in J} u_{ik}$ if $i \in I$, and it is $-\sum_{k \in I} u_{ik}$ if $i \in J$. Hence $\Lambda_G \cdot e_I$ is represented algebraically by $x^{I \rightarrow J} - x^{J \rightarrow I}$, which is hence in $I_G$.

Fix any reverse lexicographic term order on $\mathbb{K}[x]$ that has $x_n$ as the smallest variable, and let $\text{in}(I_G)$ denote the initial monomial ideal of $I_G$. Since $I_G$ is a lattice ideal, $x_n$ is a non-zerodivisor and it does not divide any of the generators of $\text{in}(I_G)$. We may thus regard $\text{in}(I_G)$ as an artinian ideal in $\mathbb{K}[x_{\setminus n}] = \mathbb{K}[x_1, \ldots, x_{n-1}]$. The index of the Laplacian lattice image $\mathbb{Z}(\Lambda_G)$ in its saturation $\{u \in \mathbb{Z}^n : u_1 + \cdots + u_n = 0\}$ equals the number $T_G$ of spanning trees of $G$. Hence $\text{in}(I_G)$ has $T_G$ standard monomials in $\mathbb{K}[x_{\setminus n}]$.

Let $M_G$ denote the ideal generated by the initial monomials of the binomials in (6):

$$M_G = \langle x^{I \rightarrow [n] \setminus I} : I \text{ non-empty subset of } [n-1] \rangle.$$  \hspace{1cm} (10)

By construction, the inclusion $M_G \subseteq \text{in}(I_G)$ holds. The monomial ideal $M_G$ was studied in [12] and shown to have precisely $T_G$ standard monomials. Indeed, the standard monomials of $M_G$ are in bijection with the $n$-reduced divisors. It is known in the chip
firing literature (cf. [3, 5, 6]) that their number equals the number $T_G$ of spanning trees. Hence $M_G$ and $\text{in}(I_G)$ are artinian of the same colength in $\mathbb{K}[x_{\setminus n}]$, so they must be equal:

$$M_G = \text{in}(I_G).$$

Therefore the binomials [6] form a Gröbner basis, and hence a generating set, of $I_G$.

The ideal $I_G$ is a generic lattice ideal, in the sense of [10], because all $n$ variables $x_1, \ldots, x_n$ occur in the binomial [6]. Here we are using that $G$ is saturated. By [10, Theorem 4.2], the Scarf complex is the (essentially unique) minimal free resolution of $I_G$.

It remains to be seen that the Scarf complex is equal to $\text{CYC}_G$. Postnikov and Shapiro [12, Corollary 6.9] showed that the Scarf complex of the initial ideal $M_G$ is supported on the barycentric subdivision of the $(n - 2)$-simplex, as shown in Figure 1. The Scarf resolution has the format (8), but with $\mathbb{K}[x]$ replaced by $\mathbb{K}[x_{\setminus n}]$. Here, we label the cells in that barycentric subdivision with ordered partitions $(I_1, I_2, \ldots, I_r)$ satisfying $n \in I_r$. The boundary maps in the Scarf resolution are then given by [9], namely, by the sum ranging from $s = 1$ to $s = r - 1$, but without the additional term $-x^{I_r \rightarrow I_1} \cdot (I_2, I_3, \ldots, I_{r-1}, I_1 \cup I_r)$.

We pass from the Scarf resolution of $M_G$ to that of $I_G$ by the combinatorial rule in [10, Theorem 5.4]. This adds precisely one term to the boundary of each Scarf simplex of $M_G$. In our case, that additional term is precisely the one above, and we get [9].

**Example 3.** Returning to Example 1, with the seven binomials in that order, here are the matrices over $\mathbb{K}[x_1, x_2, x_3, x_4]$ that represent the first and second syzygies in $\text{CYC}_4$:

\[
\begin{pmatrix}
(1,2,3,4) & (2,1,3,4) & (1,3,2,4) & (3,1,2,4) & (2,3,1,4) & (3,2,1,4) \\
\begin{pmatrix}
-x^{3 \rightarrow 3} & -x^{3 \rightarrow 2} & -x^{2 \rightarrow 3} & 0 & 0 & -x^{2 \rightarrow 4} \\
-x^{3 \rightarrow 4-1} & -x^{4 \rightarrow 1} & 0 & -x^{3 \rightarrow 14} & -x^{14 \rightarrow 3} & 0 & \cdots & 0 \\
0 & 0 & -x^{24 \rightarrow 1} & -x^{21 \rightarrow 2} & -x^{14 \rightarrow 2} & -x^{2 \rightarrow 14} & 0 & \cdots & -x^{4 \rightarrow 12} \\
-x^{1 \rightarrow 2} & 0 & 0 & 0 & 0 & 0 & x^{1 \rightarrow 23} & \cdots & x^{1 \rightarrow 12,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -x^{3 \rightarrow 4} \\
0 & 0 & 0 & 0 & 0 & x^{2 \rightarrow 3} & x^{3 \rightarrow 2} & -x^{4 \rightarrow 1} & \cdots & 0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
(1,2,3,4) & (1,3,2,4) & (1,2,3,4) & (2,1,3,4) & (2,3,1,4) & (3,1,2,4) & (3,2,1,4) & (3,2,1,4) \\
\begin{pmatrix}
x^{3 \rightarrow 4} & 0 & 0 & 0 & -x^{4 \rightarrow 3} & 0 \\
0 & 0 & x^{3 \rightarrow 4} & 0 & 0 & -x^{4 \rightarrow 3} \\
0 & x^{2 \rightarrow 4} & -x^{1 \rightarrow 2} & 0 & 0 & 0 \\
0 & 0 & 0 & -x^{4 \rightarrow 2} & x^{1 \rightarrow 4} & 0 \\
-x^{4 \rightarrow 1} & 0 & 0 & -x^{4 \rightarrow 2} & x^{1 \rightarrow 4} & 0 \\
0 & -x^{1 \rightarrow 1} & 0 & 0 & 0 & x^{1 \rightarrow 4} \\
-x^{2 \rightarrow 3} & -x^{3 \rightarrow 2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x^{2 \rightarrow 3} & 0 & x^{3 \rightarrow 2} \\
0 & 0 & -x^{1 \rightarrow 3} & -x^{3 \rightarrow 1} & 0 & 0 \\
-x^{1 \rightarrow 3} & 0 & 0 & x^{3 \rightarrow 1} & 0 \\
-x^{1 \rightarrow 2} & 0 & 0 & 0 & -x^{2 \rightarrow 1} & 0
\end{pmatrix}
\end{pmatrix}
\]

5
Note that the seven binomial generators of $I_G$ appear as the $2 \times 2$-minors of the $3 \times 2$-matrices seen in the six pairs of columns within the $7 \times 12$-matrix of first syzygies. The syzygies of the ideal $M_G$ generated by the underlined monomials in Example 1 are found by replacing with 0 all monomials that have the symbol “4” to the left of the arrow. \hfill \Box

One immediate application of our minimal free resolution is a formula for the Hilbert series of the ring $\mathbb{K}[x]/I_G$ in its natural grading by the group $\text{Div}(G) = \mathbb{Z}^n / \text{image}_x(A_G)$.

As is customary in chip firing theory [2, 3, 4, 11], we consider the decomposition $\text{Div}(G) = \mathbb{Z} \oplus \text{Div}_0(G)$, where $\mathbb{Z}$ records the degree of a divisor on $G$, and $\text{Div}_0(G)$ is the finite subgroup of divisors of degree 0. The order of $\text{Div}_0(G)$ is the number of spanning trees of $G$. Let $t$ and $q$ denote the generators of the group algebra $\mathbb{Z}[\text{Div}(G)]$ corresponding to this decomposition. The Hilbert series of $\mathbb{K}[x]/I_G$ equals $1/(1-t)$ times the Hilbert series of $\mathbb{K}[x_{\setminus n}]/M_G$, where $M_G = \text{in}(I_G)$ is the initial ideal in (10). The latter series equals

$$
\sum_u q^{\text{div}(u)}.
$$

This finite sum is over all elements $u \in \mathbb{N}^{n-1}$ that represent parking functions on $G$ with respect to the last node $n$, and $\text{div}(u)$ denotes the class of the reduced divisor of degree 0 given by the vector $(u, -\sum u_i)$. See [13] Theorem 6.14 for a nice formula, due to Merino [8], which expresses this sum with $q = 1$ in terms of the Tutte polynomial of $G$.

We fix the natural epimorphism from the semigroup algebra of $\mathbb{N}^{n-1}$ to that of $\text{Div}(G)$:

$$
\psi : \mathbb{Z}[x_{\setminus n}] \to \mathbb{Z}[\text{Div}(G)], \; x^u \mapsto t^{\text{div}(u)}.
$$

With this notation, our minimal free resolution in Theorem 2 implies the following result:

**Corollary 4.** The Hilbert series of $\mathbb{K}[x]/I_G$ in the grading by the group $\text{Div}(G)$ equals

$$
\frac{1}{1-t} \sum_u q^{\text{div}(u)} = 1 - \sum_{k=1}^{n} (-1)^k \sum_{(I_1, I_2, \ldots, I_k) \in \text{Cyc}_{n,k}} \psi(x^{I_1 \to I_2 x^{I_2 \to I_3} \cdots x^{I_{k-1} \to I_k}})
$$

\[\frac{(1-t)(1-\psi(x_1))(1-\psi(x_2)) \cdots (1-\psi(x_{n-1}))}{(1-t)(1-\psi(x_1))(1-\psi(x_2)) \cdots (1-\psi(x_{n-1}))}.
\]

**Proof.** It suffices to note that the $\mathbb{Z}^{n-1}$-degree of the basis element $(I_1, I_2, \ldots, I_k)$ of the free $\mathbb{K}[x_{\setminus n}]$-module in the $k$-th step of the resolution of $M_G$ is the exponent vector of $x^{I_1 \to I_2 x^{I_2 \to I_3} \cdots x^{I_{k-1} \to I_k}}$. This monomial does not contain $x_n$ because $n \in I_k$. \hfill \Box

We close this section with a combinatorial recipe for the socle monomials modulo $M_G$. These are the monomials $x^u$ that are not in $M_G$ but $x^u x_i \in M_G$ for $i = 1, \ldots, n-1$. Each permutation of $[n-1]$ corresponds to a flag $\mathcal{T}$ of subsets $\emptyset \subset T_1 \subset T_2 \subset T_3 \subset \cdots \subset T_{n-1}$. The flag is complete, meaning that each inclusion is strict and each $T_i \setminus T_{i-1}$ is a singleton. Let $\mathcal{T}_i$ denote the set complement of $T_i$ with respect to $[n]$. For instance, $\mathcal{T}_{n-1} = \{n\}$.

**Corollary 5.** The socle monomials of $\mathbb{K}[x_{\setminus n}]/M_G$ are precisely the $(n-1)!$ monomials

$$
s_{\mathcal{T}} = \text{lcm}(x^{T_1 \to T_1}, x^{T_2 \to T_2}, \ldots, x^{T_{n-1} \to T_{n-1}})/(x_1 x_2 \cdots x_{n-1}),
$$

where $\mathcal{T}$ runs over all complete flags of subsets of $[n-1]$.\hfill 11
Proof. The Scarf complex of $M_G$ is a minimal free resolution and it is supported on the barycentric subdivision of the $(n-2)$-simplex, by \cite{12} Corollary 6.9 and our discussion above. Each facet in that barycentric subdivision corresponds to a complete flag $T$. The vertices of that facet are labeled by $x^{T_1} \rightarrow T_1, x^{T_2} \rightarrow T_2, \ldots, x^{T_{n-1}} \rightarrow T_{n-1}$ in the Scarf complex, and the monomial label of the facet is their least common multiple. Facets of the Scarf complex are in bijection with the irreducible components of $M_G$ and also with the socle monomials modulo $M_G$. By \cite{9} Corollary 6.20, each socle monomial is multiplied by the product of all variables to give the monomial label of the corresponding facet.

Remark 6. Our results hold verbatim for all generic sublattices of finite index in the root lattice $A_n = \{ u \in \mathbb{Z}^n : \sum_{i=1}^n u_i = 0 \}$, so we recover the Voronoi theory of \cite{1, 2}. We posit that our commutative algebra derivation of their Voronoi theory is a natural and useful one, and that it opens up new and unexpected connections. For instance, Gröbner bases of lattice ideals are fundamental for integer programming \cite{14}. One original source for that application is Herbert Scarf’s seminal work on neighborhood systems in economics. A key example that motivated Scarf was the Leontief system \cite[§2A]{13}. It turns out that the lattices representing Leontief systems are precisely our generic lattices here. The Gröbner basis property stated in Theorem 2 is in fact equivalent to \cite[Theorem 2.2]{13}.

3 A Riemann-Roch Theorem for Monomial Ideals

In this section we fix an arbitrary artinian monomial ideal $M$ in a polynomial ring $K[x] = K[x_1, \ldots, x_m]$. We focus on Alexander duality \cite[§5]{9}, and we establish the Riemann-Roch formula \cite{5} in this new context. Towards the end of this section, and in the next section, we will recover the Riemann-Roch formula for graphs from the Riemann-Roch formula for monomial ideals. To begin with, we need to gather the ingredients, that is, we need to redefine the notions of divisor, genus, rank and degree.

The role of divisors on the monomial ideal $M$ is played by Laurent monomials $x^b$.

Definition 7. (Rank of a monomial) The rank of a monomial $x^b$ with respect to $M$ is one less than the minimum degree of any monomial $x^a$ that satisfies $x^b \in \langle x^a \rangle \setminus x^a M$.

This definition is restricted to honest monomials $x^b$, where $b \geq 0$. Just before the statement of Theorem \cite{13}, we shall extend the definition of rank to all Laurent monomials.

The rank measures how deeply a monomial $x^b$ sits inside the ideal $M$. We have rank($x^b$) $\geq 0$ if $x^b \in M$ and rank($x^b$) $= -1$ otherwise. Rank zero monomials form the border of $M$. Let $\text{MonSoc}(M) = \{ x^c \not\in M \mid x_i x^c \in M \ \forall i \}$ denote the set of socle monomials of $K[x]/M$. See Figure 2 for a picture of a monomial ideal. The ideal generators are the large black circles, and monomials in $M$ are labeled by their rank. The socle elements are the black squares, and other standard monomials are white squares.

Definition 8. (Reflection invariance) A monomial ideal $M$ is reflection-invariant if there exists a canonical monomial $x^K$ such that the map $\phi : x^c \mapsto x^K/x^c$ defines an involution of the set $\text{MonSoc}(M)$. This requires that every socle monomial divides $x^K$.\[7\]
Using notation as in [9, §5], we note that our artinian monomial ideal \( M \) is reflection-invariant with canonical monomial \( x^K \) if and only if the following identity holds:

\[
M^{[K+e]} = \langle \text{MonSoc}(M) \rangle.
\]

(12)

Here \( e = (1, 1, \ldots, 1) \) and \( M^{[K+e]} \) is the Alexander dual of \( M \) with respect to \( K + e \).

**Definition 9.** (Genus) The ideal \( M \) is level if all socle monomials have the same degree. If this holds then one plus that degree is called the genus of \( M \), denoted \( g = \text{genus}(M) \).

**Example 10.** Let \( M \) be the ideal generated by the seven underlined monomials in (2). Then \( M \) is level of genus \( g = 4 \), because all six socle monomials in (4) are cubics, and \( M \) is reflection-invariant. The canonical monomial \( x^K = x_1^2x_2^2x_3^2 \) has degree \( 2g - 2 = 6 \).

But, the rank of \( x^K \) is equal to \( g - 2 = 2 \), as can be seen from the following lemma.

For \( u = (u_1, \ldots, u_m) \in \mathbb{Z}^m \) we abbreviate \( \text{degree}^+(u) = \sum_{i: u_i > 0} u_i \).

**Lemma 11.** Let \( M \) be an artinian monomial ideal. Then every monomial \( x^b \) satisfies

\[
\text{rank}(x^b) = \min_{x^c \in \text{MonSoc}(M)} \text{degree}^+(b - c) - 1.
\]

(13)

**Proof.** The condition \( x^b \in \langle x^a \rangle \backslash x^a M \) in Definition 7 is equivalent to \( x^{b-a} \in K[x] \backslash M \).

Maximizing the degree of \( x^c = x^{b-a} \) subject to this condition is equivalent to minimizing the degree of \( x^a \). But, since \( M \) is artinian, the maximal degree among its finitely many standard monomials is attained by one of the socle monomials \( x^e \in \text{MonSoc}(M) \). □

**Remark 12.** Formula (13) resembles the formula in [3, Lemma 2.2] for the rank of a divisor on a finite graph. We shall exploit this resemblance at the end of this section. □

The formula in (13) can be rewritten to be reminiscent of S-pairs for Gröbner bases:

\[
\text{rank}(x^b) = \min_{x^c \in \text{MonSoc}(M)} \text{degree}^+ \left( \frac{\text{lcm}(x^b, x^c)}{x^c} \right) - 1.
\]

(14)

We now define the rank of an arbitrary Laurent monomial \( x^b \) by the formula (13). This is consistent with Definition 7 and it is the natural extension to monomials some of whose exponents are negative. The following main result gave this section its title:

**Theorem 13.** Let \( M \) be a monomial ideal that is artinian, level, and reflection-invariant. Then \( M \) satisfies the Riemann-Roch formula, i.e., every Laurent monomial \( x^b \) satisfies

\[
\text{rank}(x^b) - \text{rank}(x^K/x^b) = \text{degree}(x^b) - \text{genus}(M) + 1.
\]

(15)

**Proof.** We denote \( x^K/x^c \) by \( x^\bar{c} \). Using the formula for rank shown in Lemma 11, the left hand side of (15) equals

\[
\min_{x^c \in \text{MonSoc}(M)} \text{degree}^+(b - c) - \min_{x^c \in \text{MonSoc}(M)} \text{degree}^+(\bar{c} - b).
\]

(16)
Figure 2: The monomial ideal \( M = \langle x^9, x^6y^4, x^5y^7, x^2y^8, y^{11} \rangle \) is Riemann-Roch of genus 12, with canonical monomial \( x^9y^{13} \) and \( \text{Soc}(M) = \{x^8y^3, x^5y^6, x^4y^7, xy^{10}\} \). Generators and socle monomials highlighted in dark. Monomials in \( M \) are labeled with their rank. The square boxes correspond to the standard monomials of \( \mathbb{K}[x]/M \). The dotted lines mark the boundary of the staircase region of \( M^{[\mathbb{K}+e]} \). Note that the identity [12] holds.
For any socle monomial $x^c$ we have degree$(b - c) -$ degree$(c - b) = \text{degree}(x^b) - \text{degree}(x^c) = \text{degree}(x^b) - (\text{genus}(M) - 1)$, and hence
\[
\text{degree}^+(b - c) = \text{degree}^+(c - b) + \text{degree}(x^b) - (\text{genus}(M) - 1).
\] (17)

Taking the minimum of degree$(b - c)$ over $x^c \in \text{MonSoc}(M)$, equation (17) implies
\[
\min_{x^c \in \text{MonSoc}(M)} \text{degree}^+(b - c)
\]
(18)
\[
= \min_{x^c \in \text{MonSoc}(M)} \left( \text{degree}^+(c - b) + \text{degree}(x^b) - (\text{genus}(M) - 1) \right)
\]
\[
= \left( \min_{x^c \in \text{MonSoc}(M)} \text{degree}^+(c - b) \right) + \text{degree}(x^b) - (\text{genus}(M) - 1).
\]

Since the map $\phi$ is an involution, we can replace $\bar{c}$ by $c$ in the second row of (16). It then follows from (18) that (16) is equal to degree$(x^b) - \text{genus}(M) + 1$, as desired.

**Remark 14.** Theorem 13 can be extended to monomial ideals $M$ that are artinian and reflection invariant but not necessarily level. Such $M$ arise as initial ideals from directed regular (indegree = outdegree) graphs. Following [1, §2.3], we define genus$_{\text{min}}(M)$ as one minus the minimum degree of a socle monomial of $M$, and genus$_{\text{max}}(M)$ as one minus the maximum degree of a socle monomial of $M$. Using a technique similar to that in the proof of Theorem 13 we can derive the following Riemann-Roch inequalities:
\[
\text{genus}_{\text{min}}(M) - 1 \leq \text{degree}(x^b) - \text{rank}(x^b) + \text{rank}(x^K/x^b) \leq \text{genus}_{\text{max}}(M) - 1.
\] (19)

Of course, the above inequality generalizes the Riemann-Roch formula (15): if the ideal $M$ is also level then genus$_{\text{max}}(M) = \text{genus}_{\text{min}}(M) = \text{genus}(M)$, and the Riemann-Roch formula (15) immediately follows from the inequalities (19).

We say that a monomial ideal $M$ is Riemann-Roch if it is artinian, level, and reflection invariant. See Figure 2 for an example in two variables. In what follows we assume that $M$ is a Riemann-Roch monomial ideal. The next corollaries are formal consequences of the Riemann-Roch formula, as is the case for algebraic curves and graphs.

**Corollary 15.** If $x^b$ is a multiple of $x^K$ then rank$(x^b) = \text{degree}(x^b) - \text{genus}(M)$.

**Proof.** If $x^K$ divides $x^b$, then rank$(x^K/x^b) = -1$. Plugging this equation into the Riemann-Roch formula gives the assertion.

Note that, by definition, the degree of the canonical monomial $x^K$ equals twice the socle degree. We record the following general facts about the canonical monomial $x^K$.

**Corollary 16.** The canonical monomial of a Riemann-Roch monomial ideal $M$ satisfies
\[
\text{degree}(x^K) = 2 \cdot \text{genus}(M) - 2 \quad \text{and} \quad \text{rank}(x^K) = \text{genus}(M) - 2.
\]

Experts will note that the rank is off by one when compared to the canonical divisor of an algebraic curve or metric graph. This discrepancy will be addressed in Remark 23 below. We now prepare for an analogue of Clifford’s theorem on special divisors.
Lemma 17. The rank is superadditive for monomials $x^a$ and $x^b$ of non-negative rank:

$$\text{rank}(x^a \cdot x^b) \geq \text{rank}(x^a) + \text{rank}(x^b). \quad (20)$$

Proof. Consider an arbitrary monomial $x^c$ of degree at most $\text{rank}(x^a) + \text{rank}(x^b)$ such that $x^c$ divides $x^a \cdot x^b$. The following formulas define monomials $x^{c'}$ and $x^{c''}$ such that $x^{c'}$ divides $x^a$ and $x^{c''}$ divides $x^b$:

$$c'_i = \begin{cases} a_i & \text{if } c_i \geq a_i, \\ c_i & \text{otherwise,} \end{cases} \quad \text{and} \quad c''_i = \begin{cases} c_i - a_i & \text{if } c_i \geq a_i, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $x^{c'} \cdot x^{c''} = x^c$. This implies that either $\text{degree}(x^{c'}) \leq \text{rank}(x^a)$ or $\text{degree}(x^{c''}) \leq \text{rank}(x^b)$. In other words, either $x^a/x^{c'}$ or $x^b/x^{c''}$ is in $M$, and hence, their product $x^a \cdot x^b$ is also in $M$. From this we infer the inequality (20) as follows: Since $x^c$ is an arbitrary monomial of degree less than or equal to $\text{rank}(x^a) + \text{rank}(x^b)$, and $x^c$ divides $x^a \cdot x^b$, we know that any monomial that "defines" the rank of $x^a \cdot x^b$ (i.e., a monomial $x^d$ of minimum degree such that $x^d$ divides $x^a \cdot x^b$ and $x^d \notin M$) has degree strictly greater than $\text{rank}(x^a) + \text{rank}(x^b)$. Hence, $\text{rank}(x^a \cdot x^b) \geq \text{rank}(x^a) + \text{rank}(x^b)$. \qed

Corollary 18. (Clifford’s Theorem) Let $x^b$ be a monomial dividing $x^K$ such that both $\text{rank}(x^b)$ and $\text{rank}(x^K/x^b)$ are non-negative. Then $\text{rank}(x^b) \leq (\text{degree}(x^b) - 1)/2$.

Proof. Lemma 17 and Corollary 16 imply

$$\text{rank}(x^b) + \text{rank}(x^K/x^b) \leq \text{rank}(x^K) = \text{genus}(M) - 2.$$ 

From the Riemann-Roch formula we have

$$\text{rank}(x^b) - \text{rank}(x^K/x^b) = \text{degree}(x^b) - (\text{genus}(M) - 1).$$

The desired conclusion follows by adding these two identities and dividing by 2. \qed

The construction of all Riemann-Roch monomial ideals of genus $g$ works as follows. We first fix a monomial $x^K$ with degree $2g - 2$. Next we choose a set $M$ of monomials of degree $g - 1$ that divide $x^K$. Then there exists a unique artinian monomial ideal $M$ whose socle is spanned by the monomials in $M$ and their complements relative to $x^K$:

$$\text{MonSoc} = \mathcal{M} \cup \{ x^K/x^b : \ x^b \in \mathcal{M} \}. \quad (21)$$

Namely, the ideal $M$ is the intersection of the irreducible ideals $<x_i^{c_1+1}, \ldots, x_m^{c_1+1}>$ where $x^c$ runs over the set MonSoc. Then $M$ is artinian, level, and reflection-invariant.

We shall now make the connection to the Riemann-Roch theorem for graphs. As in Section 2, we let $G$ denote a saturated graph on $n$ nodes, with $u_{ij} > 0$ edges between nodes $i$ and $j$, and $M_G$ the initial monomial ideal in $\mathbb{K}[x_1, \ldots, x_{n-1}]$ of the toppling ideal $I_G$ with respect to a reverse lexicographic term order having $x_n$ as smallest variable.

Theorem 19. Let $G$ be a saturated graph with $n$ vertices, $e$ edges, and node $i$ having degree $d_i$. Then the monomial ideal $M_G$ is Riemann-Roch with canonical monomial

$$x^K = \prod_{i=1}^{n-1} x_i^{d_i + a_{i+1} - 2} \quad \text{and} \quad \text{genus}(M_G) = e - n + 2. \quad (22)$$
The monomial ideal $M_G$ is artinian, and it is level because all the socle monomials $s_T$ in Corollary 5 have the same degree $e - n + 1$. This quantity is the cyclotomic number (or genus) of the graph $G$, which, by [5], coincides with the common degree of all maximal parking functions. There is a natural involution $\phi$ on the set of $(n - 1)!$ maximal flags $T$ of subsets in $\{n\}$. It takes a flag $T : T_1 \subset T_2 \subset \cdots \subset T_{n-1} \subset T_n$ to the reverse flag $\phi(T) : T_{n-1} \setminus T_{n-2} \subset T_{n-1} \setminus T_{n-3} \subset \cdots \subset T_{n-1} \setminus T_1 \subset T_{n-1}$. Using the identification between flags and socle monomials in Corollary 5, we have

$$s_{\phi(T)} = x^K / s_T,$$

where $x^K$ is the monomial defined in (22). Hence, $M_G$ is also reflection-invariant. \hfill \Box

**Remark 20.** Not every Riemann-Roch monomial ideal arises as an initial monomial ideal $M_G$ for a connected graph $G$. To see this, note that the number of socle monomials of $M_G$ is at most $m!$ where $m$ is the number of variables of $M_G$. On the other hand, Riemann-Roch monomial ideals can in general have more than $m!$ socle monomials. Furthermore, the initial monomial ideal $M_G$ for a connected graph $G$ is not necessarily Riemann-Roch. To see this, consider the four-cycle $C_4$. Its initial ideal is $M_{C_4} = \langle x_1, x_2, x_3 \rangle^2$, with socle monomials $x_1, x_2$ and $x_3$, and $M_{C_4}$ is not reflection-invariant. \hfill \Box

In the remainder of this section we show how the familiar Riemann-Roch theorem for graphs is derived from Theorem 19. While the proof still assumes that $G$ is saturated, that hypothesis will be removed in the next section. The following algebraic definitions are valid for any undirected connected graph $G$ on $[n]$. Here $G$ need not be saturated.

The Laurent polynomial ring $\mathbb{K}[x^{\pm 1}] = \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a module over the polynomial ring $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$. The Laplacian lattice image $\Lambda_G$ is a sublattice of rank $n - 1$ in $\mathbb{Z}^n$, and we write $L_G$ for the corresponding *lattice module*, as in [9] Definition 9.11. Thus, $L_G$ is the $\mathbb{K}[x]$-submodule of $\mathbb{K}[x^{\pm 1}]$ generated by all Laurent monomials $x^c$ that have degree zero in the grading by the group $\text{Div}(G)$. If $G$ is saturated then $L_G$ is generic and the Scarf complex is a minimal free resolution by [9] Theorem 9.24. That Scarf complex is precisely the *Delaunay triangulation* in [1], and our point here is to redevelop the Amini-Manjunath approach in the language of commutative algebra.

Consider the set of all Laurent monomials $x^c$ that are in the socle of the module $L_G$:

$$\text{MonSoc}(L_G) = \{ x^c \notin L_G \mid x_i x^c \in L_G \ \forall i \}.$$

This socle is a set of Laurent monomials on which the lattice $L_G$ acts with finitely many orbits, so the computation of $\text{MonSoc}(L_G)$ is a finite algorithmic problem, as in [9], §9.3. The problem’s solution is given by the socle monomials [11] of our monomial ideal $M_G$.

**Lemma 21.** The socle monomials of the lattice module $L_G$ are precisely of the form $s_T \cdot x^w / x_n$, where $s_T \in \text{MonSoc}(M_G)$ and $x^w$ runs over the minimal generators of $L_G$.

**Proof.** Since $G$ is assumed to be saturated throughout this section, the lattice module $L_G$ is generic in the sense of [9], Definition 9.23, with $M_G$ being the reverse lexicographic initial ideal of the corresponding lattice ideal $I_G$. We claim that the stated characterization of the socle is valid for any generic lattice module that is artinian. Indeed, by the proof of [10] Theorem 5.2, the $\mathbb{Z}^n$-degrees of the $n$-th syzygies of $L_G$ are the vectors
\[ u + w, \text{ where } u \text{ runs over the } \mathbb{Z}^{n-1}\text{-degrees of the } (n-1)\text{-st syzygies of } M_G \text{ and } w \text{ is any vector in the lattice. The socle degrees of } M_G \text{ are the vectors } u - e_1 - \cdots - e_{n-1} \text{ in } \mathbb{Z}^{n-1}, \text{ and the socle degrees of } L_G \text{ are the vectors } u + w - e_1 - \cdots - e_{n-1} - e_n \text{ in } \mathbb{Z}^n. \]  

We now identify Laurent monomials \( x^u \) with divisors on the graph \( G \). The \( i \)-th coordinate \( u_i \) of the exponent vector \( u \) is the multiplicity of node \( i \) in the divisor \( x^u \). The degree of the divisor \( x^u \) is its total degree as a monomial, \( \deg(x^u) = u_1 + \cdots + u_n \). The rank of the divisor \( x^u \) is defined by the same formula (13) as in Lemma 11:

\[
\text{rank}(x^u) = \min_{x^c \in \text{MonSoc}(L_G)} \deg^+(u-c) - 1.
\]  

Thus, \( \text{rank}(x^u) \geq 0 \) if and only if \( x^u \) lies in \( L_G \). Our definition of rank of the divisor \( x^u \) coincides with the rank of \( u \) as in [3]. To see this, use Lemma 2.7 in [3] and note that the exponents of the socle Laurent monomials of \( L_G \) are the elements of the set \( N \) in [3].

We finally define the canonical divisor of \( G \) to be the monomial

\[
x^k = x_1^{d_1-2}x_2^{d_2-2} \cdots x_n^{d_n-2},
\]

where \( d_i = \sum_{j \neq i} u_{ij} \) is the degree of node \( i \). Finally, we recall that the genus of \( G \) is \( e - n + 1 \), where \( e \) is the number of edges. The following is precisely [3, Theorem 1.12]:

**Theorem 22** (Baker-Norine). Riemann-Roch holds for any divisor \( x^u \) on the graph \( G \):

\[
\text{rank}(x^u) - \text{rank}(x^k/x^u) = \deg(x^u) - \text{genus}(G) + 1.
\]  

**Proof.** It follows from Lemma 21 that all socle monomials of \( L_G \) have degree equal to the genus of \( G \) minus one. The lattice module \( L_G \) is also reflection-invariant, in the sense that \( x^u \in \text{MonSoc}(L_G) \) implies \( x^k/x^u \in \text{MonSoc}(L_G) \). Using the representation in Lemma 21, the resulting involution \( \phi \) on \( \text{MonSoc}(L_G) \) can be written as follows:

\[
\phi((s_T/x_n) \cdot x^w) = (s_{\phi(T)}/x_n) \cdot x^{-w} \cdot \left( \frac{x^{d_n}}{\prod_{i=1}^{n-1} x_i^{u_{in}}} \right),
\]  

where \( \phi(T) \) denotes the reverse flag as in (23). Note that the image of \( \phi \) is in \( \text{MonSoc}(L_G) \), since \( x_n^{d_n} / \prod_{i=1}^{n-1} x_i^{u_{in}} \) is in \( L_G \). The proof of Theorem 22 is now entirely analogous to that of Theorem 13. In other words, our argument for the validity of the Riemann-Roch formula for reflection-invariant artinian level monomial ideals generalizes in a straightforward manner to reflection-invariant artinian level lattice modules. \( \square \)

**Remark 23.** The rank of the canonical monomial of \( M_G \) equals the rank of the canonical divisor of the graph \( G \), but the degree of the former is two more than that of the latter. \( \square \)

## 4 Non-Saturated Graphs

We turn to graphs \( G \) that are not necessarily saturated. The binomials in (6), their syzygies in (9), and the ideals \( I_G \) and \( M_G \) are still well-defined. However, the minimality in Theorem 2 is no longer true, and the choice of term order is more subtle, as the next example shows.
Example 24. Let $G$ be the edge graph of a triangular prism, labeled so that $I_G$ equals

$$\langle a^3 - bcd, b^3 - ace, c^3 - abf, d^3 - aef, e^3 - bdf, f^3 - cde \rangle : \langle abcdef \rangle^\infty.$$ 

This toppling ideal has 22 minimal generators, and its free resolution has Betti numbers $(1, 22, 92, 147, 102, 26)$. The same holds for the ideal that represents parking functions (1, 22, 92, 147, 102, 26). The same holds for the ideal that represents parking functions (1, 22, 92, 147, 102, 26). The same holds for the ideal that represents parking functions (1, 22, 92, 147, 102, 26). The same holds for the ideal that represents parking functions (1, 22, 92, 147, 102, 26). The same holds for the ideal that represents parking functions (1, 22, 92, 147, 102, 26). The same holds for the ideal that represents parking functions (1, 22, 92, 147, 102, 26).

To explain the phenomenon in this example, we fix a spanning tree $T$ of the graph $G$ that is rooted at the node $n$, and we order the unknowns according to a linear extension of $T$. Thus, we fix an ordering of $[n]$ such that $i > j$ if the node $i$ is a descendant of the node $j$ in $T$. A term order on $\mathbb{K}[x]$ is a spanning tree order if it is a reverse lexicographic term order whose variable ordering is compatible with some spanning tree rooted at $n$. One spanning tree order is the toppling order considered in [6, Theorem 10]. See also [11, §5] for a discussion of Gröbner bases of toppling ideals in the inhomogeneous case.

Theorem 25. The toppling ideal $I_G$ is generated by the binomials $x^{I \to J} - x^{J \to I}$ where $(I,J)$ runs over splits of $[n]$ such that the subgraphs of $G$ induced on $I$ and $J$ are connected. For any spanning tree order, these binomials form a Gröbner basis of $I_G$ with initial monomial ideal $M_G$. The complexes constructed in (8) are cellular free resolutions.

Proof. The first paragraph in the proof of Theorem 2 is valid in the non-saturated case. It shows that the binomials $x^{I \to J} - x^{J \to I}$ lie in $I_G$. For the spanning tree term order, the leading monomials are $x^{I \to J}$, where $n \in J \setminus I$, and hence the initial ideal $\text{in}(I_G)$ contains the monomial ideal $M_G$ of [10]. Again, both ideals are artinian of the same colength in $\mathbb{K}[x_n]$, and hence they are equal. This establishes the Gröbner basis property. The argument in the proof of [6, Theorem 14] shows that the property that the subgraphs of $G$ induced on $I$ and $J$ are connected characterizes a minimal Gröbner basis of $I_G$ and the minimal generators of $M_G$. In particular, these binomials $x^{I \to J} - x^{J \to I}$ generate $I_G$.

Our last assertion states that (8) with differentials (9) gives a free resolution of $I_G$, and dropping the last term in (9) gives a free resolution of $M_G$. This claim is proved by deformation to generic monomial modules, as explained in [9, §6.2]. To be precise, in our situation we replace $G$ by a nearby saturated graph $G_\epsilon$ with fractional edge numbers $u_{ij}(\epsilon)$ between any pair of nodes. The monomial ideal $M_{G_\epsilon}$ is generic and degenerates to $M_G$. The lattice ideals $I_G$ and $I_{G_\epsilon}$ are represented by the corresponding lattice modules $L_G$ and $L_{G_\epsilon}$. These are submodules of the Laurent polynomial ring as in [9, Definition 9.11]. The lattice module $L_{G_\epsilon}$ is generic and degenerates to $L_G$. According to [9, Theorem 6.24], the Scarf complex of $M_G$, with labels from $G$ gives a free resolution of $M_G$. Likewise, the Scarf complex of the generic lattice module $L_{G_\epsilon}$, with labels from $G$ gives a free resolution of $L_G$. The resulting minimal free resolution of $I_G$, degenerates to a (typically non-minimal) resolution of $I_G$, using [9, Corollary 9.18]. These free resolutions of $M_G$ and $I_G$ are cellular because they are given by labeled simplicial complexes. □
Here is an example that illustrates the degeneration used in the proof above.

**Example 26.** Let \( n = 4 \) and \( G \) be the 4-cycle \( 1 - 2 - 3 - 4 - 1 \). For \( \delta, \epsilon \in \mathbb{N} \) consider the graph \( G_{\delta, \epsilon} \) that has \( \delta \) edges for every edge in \( G \) and \( \epsilon \) edges for every non-edge of \( G \). Then \( G_{\delta, \epsilon} \) is saturated for \( \delta, \epsilon > 0 \). The Scarf complex in Example 3 gives the minimal free resolution of \( M_{G_{\delta, \epsilon}} = \text{in}(I_{G_{\delta, \epsilon}}) \) and this lifts to the minimal free resolution of \( I_{G_{\delta, \epsilon}} \).

By Theorem 19, the monomial ideal \( M_{G_{\delta, \epsilon}} \) is Riemann-Roch, its genus is \( 4\delta + 2\epsilon - 2 \), and its canonical monomial equals \( x^{K} = x_{1}^{\delta+\epsilon-2}x_{2}^{2\delta+2\epsilon-2}x_{3}^{\delta+\epsilon-2} \). The involution \( x^{b} \mapsto x^{K-b} \) on its six socle monomials is given by swapping the two rows below:

\[
\text{MonSoc}(M_{G_{\delta, \epsilon}}) = \left\{ \begin{array}{ll}
x_{1}^{\delta-1}, & x_{2}^{\delta+\epsilon-1}, x_{3}^{2\delta+\epsilon-1}, \\
x_{1}^{\delta+\epsilon-1}, & x_{2}^{\delta-1}, x_{3}^{2\delta-1},
\end{array} \right\}
\] (27)

Setting \( \delta = 1 \) and \( \epsilon = 0 \), we get the parking function monomial ideal of the 4-cycle

\[ M_{G_{1, 0}} = M_{G} = \langle x_{1}, x_{2}, x_{3} \rangle^{2} = \langle x_{1}^{2}, x_{1}x_{2}, x_{1}x_{3}, x_{2}^{2}, x_{2}x_{3}, x_{3}^{2} \rangle. \]

Here, \( \text{MonSoc}(M_{G}) = \{ x_{1}, x_{2}, x_{3} \} \). This ideal is not reflection-invariant and hence not Riemann-Roch. The cellular resolution of \( M_{G} \) induced from \( M_{G_{\delta, \epsilon}} \) is not minimal. \( \square \)

We now take a closer look at the combinatorial structure of our resolutions. Let \( \text{Bary}(G) \) denote the first barycentric subdivision of the \( (n - 2) \)-simplex, whose \( 2^{n-1} - 1 \) vertices, namely the non-empty subsets \( I \) of \([n - 1]\), are labeled by the corresponding monomials \( x^{I \rightarrow [n] \setminus I} \). Thus, \( \text{Bary}(G) \) is the cellular free resolution of \( M_{G} = \text{in}(I_{G}) \) referred to in Theorem 25. Each simplex in \( \text{Bary}(G) \) is labeled by the least common multiple of the monomials that label its vertices. For any \( c \in \mathbb{N}^{n-1} \) we write \( \text{Bary}(G)_{<c} \) for the subcomplex consisting of all simplices in \( \text{Bary}(G) \) whose labels properly divide \( x^{c} \).

**Corollary 27.** The number of minimal \( i \)-th syzygies of the monomial ideal \( M_{G} \) in degree \( c \) is equal to the rank of the reduced homology group \( H_{i-1}(\text{Bary}(G)_{<c}; \mathbb{K}) \).

**Proof.** This follows immediately from Theorem 25 and [9] Theorem 4.7. \( \square \)

We next state the analogous result for the lattice module \( L_{G} \), that is, the \( \mathbb{K}[x] \)-module generated by all Laurent monomials whose exponent vector lies in the Laplacian lattice image\( _{2}(\Lambda_{G}) \). We identify this lattice with \( \mathbb{Z}^{n}/\mathbb{Z}e \) by writing its elements as \( \Lambda_{G} \cdot v \) where each \( v \in \mathbb{Z}^{n} \) is unique modulo \( \mathbb{Z}e = \ker(\Lambda_{G}) \). The tropical metric on \( \mathbb{Z}^{n}/\mathbb{Z}e \) is

\[
\text{dist}(u, v) = \max\{|u_{i} + v_{j} - u_{j} - v_{i}| : 1 \leq i < j \leq n\}.
\]

We write \( \text{Apt}(G) \) for the corresponding flag simplicial complex. Thus, \( \text{Apt}(G) \) is the simplicial complex whose simplices are subsets \( S \) of \( \mathbb{Z}^{n}/\mathbb{Z}e \) such that \( \text{dist}(u, v) \leq 1 \) for \( u, v \in S \). The notation “\( \text{Apt} \)” refers to the fact that this infinite simplicial complex is the standard apartment in the affine building of Lie type \( A_{n-1} \). It is well-known that \( \text{Apt}(G) \) is pure of dimension \( n - 1 \) and that it triangulates the \( (n-1) \)-dimensional affine space \( \mathbb{R}^{n}/\mathbb{R}e \). For more on buildings and their connection to tropical geometry, see [7].

The apartment \( \text{Apt}(G) \) is precisely the same as the Delaunay triangulation constructed in [1], and it also coincides with the Scarf complex of \( G_{\epsilon} \) that we used to
Both the toppling ideal and the ideal of parking functions are complete intersections: 
\[ L \text{ of } 12 \text{ triangles, } 28 \text{ edges and } 16 \text{ vertices, labeled by the following generators of } \text{complex } \Lambda_{G} = \langle \text{simplicial complex Bary}(G) \rangle. \]

The monomial ideal \( M \) over all \( c \) modulo image_{\Sigma}^{c}(\Lambda_{G}) \) counts the minimal \( i \)-th syzygies of the toppling ideal \( I_{G} \).

We conjecture that the ranks of the homology groups in the two corollaries coincide.

**Corollary 28.** The number of minimal \((i+1)\)-st syzygies of the lattice module \( L_{G} \) in degree \( c \) is the rank of the reduced homology \( \tilde{H}_{i}(\text{Apt}(G)_{<c}; K) \). The sum of these ranks over all \( c \) modulo image_{\Sigma}(\Lambda_{G}) \) counts the minimal \( i \)-th syzygies of the toppling ideal \( I_{G} \).

This conjecture has been verified for many graphs using the software Macaulay2. We note that the two simplicial complexes appearing in (28) are different from the complex \( \Delta_{D} \) used in Hochster’s formula for the Betti numbers of a lattice ideal [11, Theorem 7.4].

**Example 30.** The simplicial complexes \( \text{Apt}(G)_{<c} \) can be large even for small graphs. Let \( G \) be the graph on four nodes, labeled \( a, b, c, d \), with Laplacian matrix
\[
\Lambda_{G} = \begin{pmatrix}
    u_{12} + u_{13} + u_{14} & -u_{12} & -u_{13} & -u_{14} \\
    -u_{12} & u_{12} + u_{23} + u_{24} & -u_{23} & -u_{24} \\
    -u_{13} & -u_{23} & u_{13} + u_{23} + u_{34} & -u_{34} \\
    -u_{14} & -u_{24} & -u_{34} & u_{14} + u_{24} + u_{34}
\end{pmatrix} = \begin{pmatrix}
    2 & -2 & 0 & 0 \\
    -2 & 3 & -1 & 0 \\
    0 & -1 & 4 & -3 \\
    0 & 0 & -3 & 3
\end{pmatrix}.
\]
Both the toppling ideal and the ideal of parking functions are complete intersections:
\[ I_{G} = \langle a^{2} - b^{2}, b - c, c^{3} - d^{3} \rangle \text{ and } M_{G} = \langle a^{2}, b, c^{3} \rangle. \]

The monomial ideal \( M_{G} \) has one minimal first syzygy in degree \( c = (2, 0, 3, 0) \). The simplicial complex \( \text{Bary}(G)_{<c} \) consists of two isolated nodes \( a^{2} \) and \( c^{3} \). The simplicial complex \( \text{Apt}(G)_{<c} \) is two-dimensional but it has the homology of a circle. It consists of 12 triangles, 28 edges and 16 vertices, labeled by the following generators of \( L_{G} \):
\[
1, \frac{c}{b}, \frac{c^{2}}{b^{2}}, \frac{c^{3}}{b^{3}}, \frac{c^{2}}{a^{2}}, \frac{c^{3}}{a^{2}b}, \frac{c^{3}}{a^{2}b^{2}}, \frac{c^{3}}{a^{2}b^{3}}, \frac{a^{2}}{c}, \frac{a^{2}c}{b}, \frac{a^{2}c^{2}}{b^{2}}, \frac{a^{2}c^{3}}{b^{2}d}, \frac{a^{2}c^{3}}{b^{2}d^{2}}, \frac{b}{c}, \frac{b^{4}}{b^{4}}.
\]
The lattice module \( L_{G} \) has one second syzygy in this degree, translating into a first syzygy of \( I_{G} \). It is represented in \( \text{Apt}(G)_{<c} \) by the 4-cycle \( 1, a^{2}/b^{2}, a^{2}c^{3}/b^{2}d^{2}, c^{3}/d^{3} \).

At present, no explicit minimal free resolution of \( M_{G} \) is known. Finding such a resolution was stated as an open problem by Postnikov and Shapiro in [12, §6]. We do not even know whether the Betti numbers depend on the characteristic of the field \( K \).

An explicit formula for the Betti numbers of the toppling ideal \( I_{G} \) was conjectured by Wilmes in [15]. See also [11, §7.4]. Wilmes’ formula is combinatorial, and it has been verified for all graphs with \( n \leq 6 \) nodes. At present we do not know how to relate Wilmes’ conjecture to the ranks of the homology groups in Corollaries 27 and 28.

It is known, thanks to [13, Theorem 3.10], that Conjecture 29 is true for the maximal syzygies, with index \( i = n - 2 \). We have the following combinatorial characterization:
**Corollary 31.** The maximal syzygies of the parking function ideal $M_G$, or of the toppling ideal $I_G$, are in bijection with the acyclic orientations of $G$ with node $n$ as unique sink.

See [11, Theorem 7.6] for an alternative but equivalent formulation of this result.

**Proof.** The monomial ideal $M_G$ is artinian, so its maximal syzygies correspond to the socle elements. These are the maximal parking functions, and, by [5, Theorem 4.1], they correspond to cyclic orientations of $G$ with node $n$ as unique sink. Since all maximal syzygies of $M_G = \text{in}(I_G)$ lift to maximal syzygies of $I_G$, the same result holds for $I_G$. □

We now derive the Riemann-Roch theorem for non-saturated graphs $G$. Let $M_G$ be the initial ideal with respect to a spanning tree order on the variables with $x_n$ as the least. By Corollary [31], we know that the socle monomials of the Laurent monomial module $L_G$ are $s \cdot x_n^{-1} \cdot x^w$ where $s$ runs over all socle monomials of $M_G$ and $x^w$ runs over minimal generators of $L_G$. Unlike in the saturated case, the monomial ideal $M_G$ is generally not reflection-invariant. But the Laurent monomial module $L_G$ is always reflection-invariant. To see this, we use Lemma 3.2 of [3] to deduce that $s_T/x_n$, defined in (11), is not contained in $L_G$ for any complete flag $\mathcal{T}$ of $[n]$. This implies that $s_T/x_n$ is a socle element of $L_G$, since every Laurent monomial of degree greater than degree($s_T/x_n$) = genus($G$) - 1 is in $L_G$. We now immediately verify that $L_G$ is reflection-invariant with the involution on $\text{MonSoc}(L_G)$ that takes $s_Tx^w/x_n$ to $s_{\phi(\mathcal{T})}x^{-w}x_n^{d_n-1}/\prod_{i=1}^{n-1} x_i^{u_i n}$, where $\phi(\mathcal{T})$ is the reverse flag of $\mathcal{T}$ exactly as in the proof of Theorem [22]. The canonical monomial is

$$x^k = x_1^{d_1-2}x_2^{d_2-2} \cdots x_n^{d_n-2},$$

where $d_i = \sum_{j \neq i} u_{ij}$ is the degree of node $i$. Hence, $L_G$ satisfies the Riemann-Roch formula, in its monomial formulation (15), with $M = L_G$ and $\text{genus}(M) = \text{genus}(G)$.

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