AN AXIOMATIC APPROACH TO TENSOR RANK FUNCTIONS

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Abstract. Recent work of Qi et al. [19] proposes a set of axioms for tensor rank functions. The current paper presents examples showing that their axioms allow rank functions to have some undesirable properties, and a stronger set of axioms is suggested that eliminates these properties. Two questions raised by Qi et al. involving the submax rank function are also answered.

1. Introduction

Tensors are multidimensional arrays that provide a natural generalization of matrices. The theory was originally developed in psychometrics in the work of authors including Hitchcock [12, 13], Cattell [7], Tucker [20, 21, 22], Carroll and Chang [6], and Harshman [11]. Tensors have subsequently proven to be useful in numerous other applications such as chemometrics [3], signal processing [5], numerical analysis [14, 15], computer vision [23], neuroscience [2, 12], and graph analytics [17, 18]. The concept of the canonical polyadic rank of a tensor, first proposed by Hitchcock [12, 13] in 1927, is of fundamental importance since many applications involve approximating a tensor by another tensor of low rank.

Recent work of Qi et al. [19] uses an axiomatic approach to study a more general notion of tensor rank. The authors propose a set of axioms for a tensor rank function, define a partial order on the class of all such functions, and show that there is a unique minimum rank function under this partial order. They then consider some specific rank functions, one of which we call the submax rank. They propose this function as a candidate for the minimum rank function satisfying their axioms.

The current paper continues the axiomatic approach. After reviewing terminology and fixing notation in Section 2, Section 3 studies the set of functions defined by the axioms of Qi et al. [19], which we call QZC rank functions. This section provides answers to two questions about submax rank raised by Qi et al. In particular, Proposition 3.7 leads to an example showing that the submax rank is not the minimum QZC rank function. The second question is related to the property that any matrix of rank $R$ contains an $R \times R$ submatrix of rank $R$. Qi et al. consider a similar but somewhat weaker property of some QZC rank functions, and Corollary 3.6 shows that the submax rank does have this property.

Section 3 also shows that QZC rank functions can have some properties that seem quite undesirable. For example, it is possible for a QZC rank function $r$ to satisfy $r(D) < D$ when $D$ is a diagonal tensor with $D$ nonzero entries on the diagonal. In addition, Section 3 gives an example of a QZC rank function $r$ and two tensors $X$ and $Y$ such that $Y$ is obtained from $X$ simply by appending a slab of zeros, but $r(Y) > r(X)$. To eliminate these sorts of examples, a different set of axioms for tensor rank functions is proposed in Section 4. All tensor rank functions
satisfying these axioms are QZC rank functions, but they do not have the same sort of pathological behavior.

2. Background and Notation

This section describes the notation and terminology used in the remainder of the paper. The notation is generally intended to conform to that used by Kolda and Bader [10] or to Qi et al. [19]. For simplicity all tensors considered in this paper will have entries in \( \mathbb{R} \).

The order of a tensor is the number of dimensions, which are also called modes or ways. Vectors are simply tensors of order one and are written as boldface lowercase letters such as \( \mathbf{a} \); matrices are tensors of order two and are written as boldface capital letters such as \( \mathbf{A} \); tensors of higher order or of unspecified order are written as boldface Euler script letters such as \( \mathcal{X} \). The \( i^{th} \) entry of a vector \( \mathbf{a} \) is denoted by \( a_i \), the \( (i,j) \) entry of a matrix \( \mathbf{A} \) is denoted by \( a_{ij} \), and the \( (i_1, \ldots, i_N) \) entry of a tensor \( \mathcal{X} \) of order \( N \) is denoted by \( x_{i_1 \ldots i_N} \). It will be convenient to write \( \mathcal{T} \) for the collection of all tensors over \( \mathbb{R} \). The notation \( \mathcal{S}_N \) will denote the symmetric group of all permutations of the set \( \{1, \ldots, N\} \).

When considering a tensor \( \mathcal{X} \) of order \( N \), we generally assume that the \( n^{th} \) index ranges from 1 to \( I_n \). In this case we write \( \mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \). If \( N > 1 \) and \( I_n = 1 \) for some \( n \), then \( \mathcal{X} \) has an associated tensor of order \( N - 1 \) obtained by eliminating the index corresponding to the \( n^{th} \) mode of \( \mathcal{X} \).

Let \( \mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) be a tensor. For \( 1 \leq n \leq N \) suppose that \( 1 \leq J_n \leq I_n \) and \( 1 \leq i_{n1} < \cdots < i_{nJ_n} \leq I_n \). Then the tensor \( \mathcal{Y} \in \mathbb{R}^{J_1 \times \cdots \times J_N} \) given by

\[
y_{j_1 \cdots j_N} = x_{i_{1j_1} \cdots i_{NJ_N}}
\]

is called a subtensor of \( \mathcal{X} \). When \( J_n = 1 \) for all but one or two of the dimensions \( n \) with \( 1 \leq n \leq N \), the subtensor \( \mathcal{Y} \) can be identified with either a vector \( \mathbf{y} \) or a matrix \( \mathbf{Y} \). In these cases it will be convenient to refer to either \( \mathbf{y} \) or \( \mathbf{Y} \) as a submatrix of \( \mathcal{Y} \), even though \( \mathbf{Y} \) may not itself be a matrix.

Let \( \mathbf{a}^{(n)} \in \mathbb{R}^{I_n} \) be a nonzero vector for \( 1 \leq n \leq N \). Let \( \mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) be the tensor given by

\[
x_{i_1 \cdots i_N} = a^{(1)}_{i_1} \cdots a^{(N)}_{i_N}.
\]

If \( \circ \) denotes the outer product of vectors, then this tensor is often written as

\[\mathcal{X} = \mathbf{a}^{(1)} \circ \cdots \circ \mathbf{a}^{(N)}\]

A tensor of this form is said to have rank one.

The idea of expressing a tensor as the sum of a finite number of rank-one tensors is originally due to Hitchcock [12, 13]. He proposed defining the rank of a tensor \( \mathcal{X} \) to be the minimum number of rank-one tensors having \( \mathcal{X} \) as their sum. The notion was not widely studied, however, until Kruskal [13] proposed the definition independently in 1977. This idea is now the most commonly used definition of tensor rank and is often called the canonical polyadic rank or CP rank. It is known that the CP rank of a tensor depends upon the base field. An example of a class of tensors in \( \mathbb{R}^{2 \times 2 \times 2} \) that have CP rank \( 3 \) as real tensors but CP rank \( 2 \) as complex tensors appears in work of de Silva and Lim [10, Section 7.4].

A tensor \( \mathcal{D} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) is said to be diagonal if \( i_1 = \cdots = i_N \) whenever \( d_{i_1 \cdots i_N} \neq 0 \). The entries for which \( i_1 = \cdots = i_N \) form the diagonal, which some authors call the superdiagonal. One important example of a diagonal tensor is the
It is easy to see that if 

\[ \mathbf{X} = \mathbf{A} \times_m \mathbf{B} \]

if \( m \neq n \), then

\[ \mathbf{X} \times_m \mathbf{B} \times_n \mathbf{A} = \mathbf{X} \times_n \mathbf{A} \times_m \mathbf{B} \]

if \( m = n \), then

\[ \mathbf{X} \times_n \mathbf{B} \times_n \mathbf{A} = \mathbf{X} \times_n (\mathbf{A} \mathbf{B}) \]

Suppose that \( \mathbf{A}^{(n)} \in \mathbb{R}^{J_n \times I_n} \) for \( 1 \leq n \leq N \). Then the Tucker operator \( [15] \) is defined by

\[ [\mathbf{X}; \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}] = \mathbf{X} \times_1 \mathbf{A}^{(1)} \times_2 \cdots \times_N \mathbf{A}^{(N)} \]

The following result gives a useful relationship between tensor unfoldings and the Tucker operator.

**Proposition 2.1** ([15] Proposition 3.7(c)). Suppose that \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) and \( \mathbf{A}^{(n)} \in \mathbb{R}^{J_n \times I_n} \) for \( 1 \leq n \leq N \). Then the following conditions are equivalent:

1. \( \mathbf{Y} = [\mathbf{X}; \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}] \);
2. \( \mathbf{Y}^{(n)} = \mathbf{A}^{(n)} \mathbf{X}^{(n)} \mathbf{A}^{(N)} \otimes \cdots \otimes \mathbf{A}^{(n+1)} \mathbf{A}^{(n-1)} \otimes \cdots \otimes \mathbf{A}^{(1)} \) for every \( n \) with \( 1 \leq n \leq N \);
3. \( \mathbf{Y}^{(n)} = \mathbf{A}^{(n)} \mathbf{X}^{(n)} \mathbf{A}^{(N)} \otimes \cdots \otimes \mathbf{A}^{(n+1)} \mathbf{A}^{(n-1)} \otimes \cdots \otimes \mathbf{A}^{(1)} \) for some \( n \) with \( 1 \leq n \leq N \).
3. QZC RANK FUNCTIONS AND THE SUBMAX RANK

This section is devoted to studying the class of rank functions defined by the axioms of Qi et al. [19], which we call QZC rank functions after the authors, Qi, Zhang, and Chen.

**Definition 3.1.** A function \( r : T \to \mathbb{N} \cup \{0\} \) will be called a QZC rank function if it satisfies the following axioms:

(QZC1) \( r(\mathbf{0}) = 0 \) if and only if \( \mathbf{X} = \mathbf{0} \), and \( r(\mathbf{X}) = 1 \) if and only if \( \mathbf{X} \) is a rank-one tensor.

(QZC2) If \( N \geq 2 \), then \( r(\mathbf{I}_{M,N}) = M \).

(QZC3) If \( \mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \), then \( r(\mathbf{X}) \) is equal to the matrix rank of the \( I_1 \times I_2 \) matrix corresponding to \( \mathbf{X} \).

(QZC4) \( r(\alpha \mathbf{X}) = r(\alpha \mathbf{X}) \) for all \( \alpha \in \mathbb{R} \setminus \{0\} \).

(QZC5) Suppose that \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) and \( \pi \in \mathfrak{S}_N \). Then the tensor \( \mathbf{Y} \) given by \( y_{i_1,\ldots,i_N} = x_{i_{\pi(1)},\ldots,i_{\pi(N)}} \) satisfies \( r(\mathbf{Y}) = r(\mathbf{X}) \).

(QZC6) If \( \mathbf{Y} \) is a subtensor of \( \mathbf{X} \), then \( r(\mathbf{Y}) \leq r(\mathbf{X}) \).

**Proposition 3.2 ([19, Theorem 2.2]).** Let \( r_1 \) and \( r_2 \) be QZC rank functions. Then the functions \( r, R : T \to \mathbb{N} \cup \{0\} \) given by

\[
r(\mathbf{X}) = \min\{r_1(\mathbf{X}), r_2(\mathbf{X})\}
\]

and

\[
R(\mathbf{X}) = \max\{r_1(\mathbf{X}), r_2(\mathbf{X})\}
\]

are QZC rank functions.

Define a partial ordering \( \preceq \) on the collection of all QZC rank functions by setting \( r_1 \preceq r_2 \) if and only if \( r_1(\mathbf{X}) \leq r_2(\mathbf{X}) \) for every tensor \( \mathbf{X} \). The maximum of two QZC rank functions will not be used in this work, but the minimum is interesting because of the following result.

**Proposition 3.3 ([19, Theorem 2.3]).** There is a unique minimum QZC rank function \( \mu \) given by

\[
\mu(\mathbf{X}) = \min\{r(\mathbf{X}) \mid r \text{ is a QZC rank function}\}.
\]

Suppose that \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \). For \( 1 \leq n \leq N \) let \( R_n \) denote the rank of the mode-\( n \) unfolding \( \mathbf{X}_{(n)} \) of \( \mathbf{X} \). The \( N \)-tuple \( (R_1, \ldots, R_N) \) is a special case of the multiplex rank introduced by Hitchcock [13]; it is sometimes called the multilinear rank [10] of \( \mathbf{X} \). Qi et al. [19] show that the function

\[
r(\mathbf{X}) = \max\{R_1, \ldots, R_N\}
\]

is a QZC rank function. In addition, they define \( \text{submax}\{R_1, \ldots, R_N\} \) to be the second largest value of the multiset \( \{R_1, \ldots, R_N\} \) if \( N > 1 \) and \( \text{submax}\{R_1\} = R_1 \) if \( N = 1 \); for example,

\[
\text{submax}\{1, 2, 3, 3\} = 3.
\]

They then show that the function

\[
r(\mathbf{X}) = \text{submax}\{R_1, \ldots, R_N\}
\]

is also a QZC rank function. Qi et al. call the function defined by Equation (3.1) the max Tucker rank and the function defined by Equation (3.2) the submax Tucker rank; for simplicity we refer to them as the max rank and the submax rank.
If $X \in \mathbb{R}^{I_1 \times I_2}$ is a nonzero matrix of rank $R$, then $X$ has an $R \times R$ submatrix of rank $R$. Unfortunately, the analogous property is not always satisfied for tensor rank functions. For example, let $r$ be the max rank, and consider a tensor $X \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ with $r(X) = R$. It is quite easy to construct examples in which $R > I_3$ so that $X$ has no subtensor $Y \in \mathbb{R}^{R \times R \times R}$, let alone one with $r(Y) = R$. But some QZC rank functions $r$ do have the weaker property that a nonzero tensor $X$ always has a subtensor $Y \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ such that

$$r(X) = r(Y) = J_n$$

for some $n$ with $1 \leq n \leq N$. In fact, studying QZC rank functions with this property is one of the main motivations for the work of Qi et al. They show that the max rank has this property and ask whether the submax rank does. The following lemma will lead to an answer to this question in Corollary 3.6.

**Lemma 3.4.** Let $X \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ be a nonzero tensor, and set $R = \text{rank}(X_{(n)})$ for some $n$ with $1 \leq n \leq N$. Let $j_1 < \cdots < j_R$ be the indices of $R$ linearly independent rows of $X_{(n)}$, and define $A \in \mathbb{R}^{R \times I_n}$ by

$$a_{st} = \begin{cases} 1 & \text{if } t = j_s, \\ 0 & \text{otherwise}. \end{cases}$$

Set $Y = X \times_n A$. Then $Y$ is a subtensor of $X$ with $\text{rank}(Y_{(m)}) = \text{rank}(X_{(m)})$ for $1 \leq m \leq N$.

**Proof.** By permuting the coordinates of $X$ if necessary, we may assume without loss of generality that $n = 1$. It is easy to check that $Y \in \mathbb{R}^{R \times I_2 \times \cdots \times I_N}$ is a submatrix of $X$. Row $s$ of the matrix $Y_{(1)} = AX_{(1)}$ is equal to row $j_s$ of the matrix $X_{(1)}$ for $1 \leq s \leq R$. Because $\text{rank}(X_{(1)}) = R$ and rows $j_1, \ldots, j_R$ of $X_{(1)}$ are linearly independent, $X_{(1)}$ and $Y_{(1)}$ have the same rank. Moreover, there is a matrix $B \in \mathbb{R}^{I_1 \times R}$ such that $BAX_{(1)} = X_{(1)}$, so $X = X \times_1 BA = Y \times_1 B$. By Proposition 2.1 it follows that if $2 \leq m \leq N$, then

$$X_{(m)} = Y_{(m)}(I_{I_N} \otimes \cdots \otimes I_{I_{m+1}} \otimes I_{I_{m-1}} \otimes \cdots \otimes I_2 \otimes B^T)$$

and

$$Y_{(m)} = X_{(m)}(I_{I_N} \otimes \cdots \otimes I_{I_{m+1}} \otimes I_{I_{m-1}} \otimes \cdots \otimes I_2 \otimes A^T).$$

Thus $X_{(m)}$ and $Y_{(m)}$ have the same column space, so they must have the same rank.

**Proposition 3.5.** Let $X \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ be a nonzero tensor, and set $R_n = \text{rank}(X_{(n)})$ for $1 \leq n \leq N$. Then $X$ has a subtensor $Y \in \mathbb{R}^{R_1 \times \cdots \times R_N}$ with $R_n = \text{rank}(Y_{(n)})$ for $1 \leq n \leq N$.

**Proof.** Lemma 3.4 shows that $X$ has a subtensor $X' \in \mathbb{R}^{R_1 \times I_2 \times \cdots \times I_N}$ such that $\text{rank}(X'_{(n)}) = R_n$ for all $n$ with $1 \leq n \leq N$. Applying the lemma inductively gives the desired result.

**Corollary 3.6.** Let $r$ denote either the max rank or the submax rank. If $X$ is a nonzero tensor, then there is a subtensor $Y \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ such that

$$r(X) = r(Y) = J_n$$

for some $n$ with $1 \leq n \leq N$. 
Qi et al. [19] ask whether the minimum QZC rank function \( \mu \) given by Proposition 3.2 is equal to the submax rank. To answer this question, we begin by letting \( \sim \) denote the weakest equivalence relation satisfying the following conditions on the collection \( \mathcal{T} \) of all real tensors:

1. If \( X \in \mathcal{T} \) and \( \alpha \in \mathbb{R} - \{0\}, \) then \( X \sim \alpha X. \)
2. Suppose that \( X \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) and \( \pi \in \mathcal{S}_N. \) Let \( Y \) be the tensor given by \( y_{i_1, \ldots, i_N} = x_{i_{\pi(1)}, \ldots, i_{\pi(N)}}. \) Then \( X \sim Y. \)

**Proposition 3.7.** Let \( r : \mathcal{T} \to \mathbb{N} \cup \{0\} \) be a function satisfying the following conditions:

1. \( r(X) = 0 \) if and only if \( X = 0, \) and \( r(X) = 1 \) if and only if \( X \) is a rank-one tensor.
2. \( r \) is constant on equivalence classes.
3. Suppose that \( X \neq 0 \) and \( X \) is not a rank-one tensor. Set
   \[
   S_0 = \{ M \mid J_{M,N} \text{ is a subtensor of some } Y \sim X \}
   \]
   and
   \[
   S_1 = \{ \text{rank}(A) \mid A \text{ is a submatrix of some } Y \sim X \}.
   \]
   The value \( r(X) \) is given by \( r(X) = \max(S_0 \cup S_1 \cup \{2\}). \)

Then \( r = \mu. \)

**Proof.** The first step is to show that \( r \) is a QZC rank function. If \( N \geq 2 \) and \( M = 1, \) then \( J_{M,N} \) is a rank-one tensor, so \( r(J_{M,N}) = M \) by Condition (1). If \( M > 1, \) then \( J_{M,N} \) is not a rank-one tensor. Suppose that \( Y \sim J_{M,N} + \) and \( A \) is a submatrix of \( Y. \) Then one can easily check that \( \text{rank}(A) \leq 1. \) Thus Condition (3) implies that \( r(J_{M,N}) = M. \) and Axiom (QZC2) is satisfied.

Suppose that \( X \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_2}, \) that \( \mathbf{X} \neq 0, \) and that \( X \) is not a rank-one tensor. Axiom (QZC3) is clearly satisfied unless there is a \( Y \sim X \) such that \( J_{M,N} \) is a subtensor of \( Y \) with \( M \geq 2. \) But in this case \( Y \) and \( X \) must be tensors of order \( N, \) so \( N = 2 \) and \( X \in \mathbb{R}^{I_1 \times I_2}. \) Thus the tensors \( X \) and \( Y \) are actually matrices \( X \) and \( Y. \) The identity matrix \( \mathbf{I}_M \) is a submatrix of \( Y, \) so \( \text{rank}(Y) \geq M. \) It follows that

\[
r(X) = \max\{\text{rank}(A) \mid A \text{ is a submatrix of some } Y \sim X \} = \text{rank}(X).
\]

Thus Axiom (QZC3) holds.

The condition that \( r \) is constant on equivalence classes is equivalent to Axioms (QZC4) and (QZC5), and Condition (1) is simply a restatement of Axiom (QZC1). To prove that \( r \) is a QZC rank function, therefore, it only remains to prove that Axiom (QZC6) is satisfied. Suppose that \( Y \) is a subtensor of \( X. \) To prove that \( r(Y) \leq r(X), \) we may assume that \( Y \neq 0 \) so that \( X \neq 0. \) If \( r(Y) = 1, \) then \( r(Y) \leq r(X) \) by Condition (1). Since any nonzero subtensor of a rank-one tensor is itself a rank-one tensor, we may assume by Condition (1) that \( X \neq 0, \) \( Y \neq 0, \) and that neither \( X \) nor \( Y \) is a rank-one tensor.

If \( Y' \sim Y, \) then one can easily check that there is a tensor \( X' \sim X \) such that \( Y' \) is a subtensor of \( X'. \) Thus if \( J_{M,N} \) is a subtensor of \( Y', \) then \( J_{M,N} \) is also a subtensor of \( X'. \) Similarly, if \( A \) is a submatrix of \( Y', \) then \( A \) is also a submatrix of \( X'. \) Condition (3) now implies that \( r(Y) \leq r(X), \) and \( r \) is a QZC rank function.

Every QZC rank function satisfies Conditions (1) and (2). Suppose that \( X \neq 0 \) and \( X \) is not a rank-one tensor so that \( \mu(X) \geq 2. \) If \( Y \sim X \) and \( J_{M,N} \) is a subtensor
of $\mathbf{Y}$, then $\mu$ satisfies $M = \mu(\mathbf{J}_{M,N}) \leq \mu(\mathbf{Y}) = \mu(\mathbf{X})$. Similarly, if $\mathbf{A}$ is a submatrix of $\mathbf{Y}$, then Axioms (QZC3), (QZC5), and (QZC6) imply that rank($\mathbf{A}$) $\leq \mu(\mathbf{Y}) = \mu(\mathbf{X})$. Thus $r(\mathbf{X}) \leq \mu(\mathbf{X})$, and the minimality of $\mu$ implies that $\mu = r$. \qed

We can now provide an example showing that the minimum QZC rank function $\mu$ is not equal to the submax rank.

**Example 3.8.** Let $\mathbf{D}$ denote the $3 \times 3 \times 3$ diagonal tensor with $d_{111} = d_{222} = 1$ and $d_{333} = -1$, and let $r$ denote the submax rank. It is easy to check that the three unfoldings $\mathbf{D}_{(1)}$, $\mathbf{D}_{(2)}$, and $\mathbf{D}_{(3)}$ all have rank 3, so $\sigma(\mathbf{D}) = 3$. But if $\mathbf{Y} \sim \mathbf{D}$, then every nonzero submatrix of $\mathbf{Y}$ has rank one. Moreover, exactly two of the nonzero entries of $\mathbf{Y}$ are equal, so the largest value of $M$ for which $\mathbf{J}_{M,N}$ can be a subtensor of $\mathbf{Y}$ is $M = 2$. Thus $\mu(\mathbf{D}) = 2$, and $\mu$ is not equal to $\sigma$.

The previous example shows that it is possible for a QZC rank function $r$ to have the property that $r(\mathbf{D}) < D$ even when $\mathbf{D}$ is a diagonal tensor with $D$ nonzero entries along the diagonal. This property seems undesirable, but the next result provides an example of a QZC rank function having a property that seems even less desirable.

**Proposition 3.9.** Let $r : \mathcal{T} \to \mathbb{N} \cup \{0\}$ be a function satisfying the following conditions:

1. $r(\mathbf{X}) = 0$ if and only if $\mathbf{X} = \mathbf{0}$, and $r(\mathbf{X}) = 1$ if and only if $\mathbf{X}$ is a rank-one tensor.

2. $r$ is constant on equivalence classes.

3. Suppose that $\mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and there are exactly two modes $n_1 < n_2$ such that $I_{n_1} > 1$ and $I_{n_2} > 1$. If $\mathbf{X} \in \mathbb{R}^{I_{n_1} \times I_{n_2}}$ is the matrix associated to $\mathbf{X}$, then $r(\mathbf{X}) = \text{rank}(\mathbf{X})$.

4. Suppose that $\mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ does not have rank one and $\mathbf{X} \neq \mathbf{0}$. If there are at least three modes $n$ with $I_n > 1$, then $r(\mathbf{X}) = \max\{I_1, \ldots, I_N\}$.

Then $r$ is a QZC rank function.

**Proof.** It is easy to check that the four conditions given in the statement of the proposition are consistent, so there is a unique function $r$ satisfying the conditions. Axiom (QZC1) is simply a restatement of Condition (1), and Axioms (QZC4) and (QZC5) are equivalent to Condition (2). The function $r$ satisfies Axiom (QZC3) by Condition (3).

Assume that $N \geq 2$. If $M = 1$, then $\mathbf{J}_{M,N}$ has rank one, so $r(\mathbf{J}_{M,N}) = 1$ by Condition (1). If $M > 1$ and $N = 2$, then $r(\mathbf{J}_{M,N}) = M$ by Condition (3). Finally, if $M > 1$ and $N > 2$, then Condition (4) implies that $r(\mathbf{J}_{M,N}) = M$. Thus Axiom (QZC2) is satisfied.

Finally, suppose that $\mathbf{Y} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ is a subtensor of $\mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$. If $\mathbf{Y} = \mathbf{0}$, then $r(\mathbf{Y}) \leq r(\mathbf{X})$ by Condition (1). If $\mathbf{Y}$ is a rank-one tensor, then $\mathbf{X} \neq \mathbf{0}$, so Condition (1) implies that $r(\mathbf{Y}) = 1 \leq r(\mathbf{X})$. Now suppose that $\mathbf{Y} \neq \mathbf{0}$ does not have rank one. Then at least two modes $n$ have dimension $J_n > 1$. If more than two modes have this property, then Condition (4) implies that

$$r(\mathbf{Y}) = \max\{J_1, \ldots, J_N\} \leq \max\{I_1, \ldots, I_N\} = r(\mathbf{X}).$$

Thus we may assume that there are exactly two modes $n_1 < n_2$ such that $J_{n_1} > 1$ and $J_{n_2} > 1$. If $\mathbf{Y} \in \mathbb{R}^{I_{n_1} \times I_{n_2}}$ is the matrix corresponding to $\mathbf{Y}$, then Condition (3) implies that $r(\mathbf{Y}) = \text{rank}(\mathbf{Y}) \leq \max\{J_{n_1}, J_{n_2}\}$. If exactly two modes of $\mathbf{X}$ have
dimension greater than one, then these modes must be \( n_1 \) and \( n_2 \). Let \( X \in \mathbb{R}^{I_{n_1} \times I_{n_2}} \) be the matrix corresponding to \( X \). Then \( Y \) is a submatrix of \( X \), and

\[
\text{rank}(Y) = \text{rank}(X) \leq r(X)
\]

by Condition (3). If more than two modes of \( X \) have dimension greater than one, then

\[
\text{rank}(Y) = \text{rank}(X) \leq \max \{J_{n_1}, J_{n_2}\} \leq \max \{I_1, \ldots, I_N\} = r(X)
\]

by Condition (4). Thus Axiom (QZC6) is satisfied, and \( r \) is a QZC rank function.

□

Example 3.10. Let \( r \) be the QZC rank function defined by the conditions given in Proposition 3.9. Consider the tensor \( X \in \mathbb{R}^{2 \times 2 \times 3} \) with

\[
x_{i_1i_2i_3} = \begin{cases} 
1 & \text{if } i_1 = i_2 = i_3 = 1 \text{ or } i_1 = i_2 = i_3 = 2, \\
0 & \text{otherwise.}
\end{cases}
\]

Then Condition (4) implies that \( r(X) = 3 \). In particular, \( r(X) > r(J_{2,3}) \), even though \( X \) is obtained by appending a slab of zeros to the tensor \( J_{2,3} \).

The previous example gives a second undesirable property that a QZC rank function may have. The next section gives a somewhat different set of axioms that eliminates both of these properties.

4. Axioms for tensor rank functions

The results of Section 3 show that the axioms for QZC rank functions have at least two undesirable consequences: a diagonal tensor can have a rank that is smaller than the number of nonzero diagonal entries, and the rank of a tensor may increase when a slab of zeros is appended to it. In this section the axioms of Qi et al. [19] are modified to obtain a more restrictive notion of tensor rank. The axioms proposed here rectify both of the issues discussed in Section 3, but it is still possible that they allow for other undesirable properties. Further modifications may be necessary.

Definition 4.1. A tensor rank function is a function \( r : T \to \mathbb{N} \cup \{0\} \) satisfying the following axioms:

(TR1) \( r(X) = 1 \) if and only if \( X \) is a rank-one tensor.

(TR2) If \( N \geq 2 \), then \( r(J_M \times J_N) = M \).

(TR3) If \( X \in \mathbb{R}^{I_1 \times I_2} \), then \( r(X) \) is equal to the matrix rank of \( X \).

(TR4) If \( X \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) and \( X' \) is the corresponding \((N + 1)\)-way tensor in \( \mathbb{R}^{I_1 \times \cdots \times I_N \times 1} \), then \( r(X) = r(X') \).

(TR5) Suppose that \( X \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) and \( \pi \in S_N \). Then the tensor \( Y \) given by \( y_{i_1 \cdots i_N} = x_{i_{\pi(1)} \cdots i_{\pi(N)}} \) satisfies \( r(Y) = r(X) \).

(TR6) If \( X \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) and \( A \in \mathbb{R}^{J \times I_N} \), then \( r(X \times_n A) \leq r(X) \).

It is easy to see that the definition of CP rank proposed by Hitchcock [12, 13] and Kruskal [18] satisfies the first five axioms, but it also satisfies Axiom (TR6). Indeed, suppose that \( X \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) is a nonzero tensor. Let \( \text{rank}(X) \) denote the smallest natural number \( R \) such that \( X \) can be expressed as a sum of \( R \) tensors of...
rank one. Then there are nonzero vectors \( \mathbf{a}_r^{(n)} \in \mathbb{R}^{I_n} \) for \( 1 \leq r \leq R \) and \( 1 \leq n \leq N \) such that
\[
\mathbf{X} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \odot \cdots \odot \mathbf{a}_r^{(N)}.
\]
If \( \mathbf{A} \in \mathbb{R}^{J \times I_n} \), then
\[
\mathbf{X} \times_n \mathbf{A} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \odot \cdots \odot \mathbf{a}_r^{(n-1)} \odot \mathbf{Aa}_r^{(n)} \odot \mathbf{a}_r^{(n+1)} \odot \cdots \odot \mathbf{a}_r^{(N)},
\]
so \( \text{rank}(\mathbf{X} \times_n \mathbf{A}) \leq \text{rank}(\mathbf{X}) \).

The following proposition is the first step toward showing that any tensor rank function is also a QZC rank function.

**Proposition 4.2.** Let \( r \) be a tensor rank function. If \( \mathbf{Y} \) is a subtensor of \( \mathbf{X} \), then \( r(\mathbf{Y}) \leq r(\mathbf{X}) \).

**Proof.** Suppose that \( \mathbf{Y} \in \mathbb{R}^{J_1 \times \cdots \times J_N} \) is a subtensor of \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \). For \( 1 \leq n \leq N \) there are indices \( i_{n1}, \ldots, i_{nj} \) such that \( 1 \leq i_{n1} < \cdots < i_{nj} \leq I_n \) and
\[
y_{j_1 \cdots j_N} = x_{i_{nj}1 \cdots i_{njN}}.
\]
Let \( \mathbf{A}^{(n)} \) be the \( J_n \times I_n \) matrix with a 1 in position \( (j, i_{nj}) \) for \( 1 \leq j \leq J_n \) and zeros elsewhere. Then \( \mathbf{Y} = [\mathbf{Y}; \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}] \), so Axiom (TR6) implies that \( r(\mathbf{Y}) \leq r(\mathbf{X}) \).

Example 3.8 describes a QZC rank function \( r \) and a diagonal tensor \( \mathbf{D} \) with \( D \) nonzero entries such that \( r(\mathbf{D}) < D \). Such a phenomenon cannot occur when \( r \) is a tensor rank function. Indeed, if \( D = 1 \), then \( \mathbf{D} \) is a rank-one tensor, and \( r(\mathbf{D}) = D \) by Axiom (TR1). If \( D > 1 \), then there is an invertible diagonal matrix \( \mathbf{A} \) such that \( \mathbf{D} \times_1 \mathbf{A} \) is a diagonal tensor with \( D \) ones on the diagonal. If \( \mathbf{D} \) is a tensor of order \( N \), then \( \mathbf{J}_{D,N} \) is a subtensor of \( \mathbf{D} \times_1 \mathbf{A} \), and \( r(\mathbf{J}_{D,N}) = D \) by Axiom (TR2). Because \( \mathbf{A} \) is invertible, Axiom (TR6) implies that \( r(\mathbf{D} \times_1 \mathbf{A}) = r(\mathbf{D}) \), and Proposition 4.2 shows that
\[
D = r(\mathbf{J}_{D,N}) \leq r(\mathbf{D} \times_1 \mathbf{A}) = r(\mathbf{D}).
\]

**Proposition 4.3.** Let \( r \) be a tensor rank function. Then \( r(\mathbf{X}) = 0 \) if and only if \( \mathbf{X} = 0 \).

**Proof.** Let \( \mathbf{Z} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) be the zero tensor, and let \( \mathbf{a}^{(n)} \in \mathbb{R}^{I_n} \) be any nonzero vector for \( 1 \leq n \leq N \). Then \( \mathbf{X} = \mathbf{a}^{(1)} \odot \cdots \odot \mathbf{a}^{(N)} \) is a rank-one tensor, so \( r(\mathbf{X}) = 1 \) by Axiom (TR1). Let \( \mathbf{0} \) denote the \( I_1 \times I_1 \) zero matrix. Set
\[
\mathbf{A} = \begin{bmatrix} I_{I_1} \\ 0 \end{bmatrix}
\]
and \( \mathbf{Y} = \mathbf{X} \times_1 \mathbf{A} \). Then the zero tensor \( \mathbf{Z} \) is a subtensor of \( \mathbf{Y} \in \mathbb{R}^{2I_1 \times I_2 \times \cdots \times I_N} \). Thus Proposition 4.2 and Axiom (TR6) imply that \( r(\mathbf{Z}) \leq r(\mathbf{Y}) \leq r(\mathbf{X}) = 1 \). Since \( \mathbf{Z} \) is not a rank-one tensor, it follows that \( r(\mathbf{Z}) = 0 \) by Axiom (TR1).

Conversely, suppose that \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) is nonzero. Then there are indices \( i_1, \ldots, i_N \) such that \( x_{i_1 \cdots i_N} \neq 0 \), and \( \mathbf{X} \) has a rank-one subtensor \( \mathbf{Y} \in \mathbb{R}^{1 \times \cdots \times 1} \) given by \( y_{1 \cdots 1} = x_{i_1 \cdots i_N} \). Thus \( 1 = r(\mathbf{Y}) \leq r(\mathbf{X}) \) by Axiom (TR1) and Proposition 4.2, so \( r(\mathbf{X}) \neq 0 \).
Example 3.10 gives a QZC rank function \( r \) and two tensors \( \mathbf{X} \) and \( \mathbf{Y} \) such that \( r(\mathbf{X}) < r(\mathbf{Y}) \) even though \( \mathbf{Y} \) can be obtained from \( \mathbf{X} \) simply by adding a slab of zeros. A slight variant of the idea used to prove Proposition 4.3 shows that this phenomenon cannot occur for tensor rank functions. Indeed, suppose that \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) and
\[
A = \begin{bmatrix}
I_m \\
0
\end{bmatrix},
\]
where 0 denotes the \( J_n \times I_n \) zero matrix for some \( J_n > 0 \). Then \( \mathbf{Y} = \mathbf{X} \times_n A \) is obtained by appending \( J_n \) slabs of zeros to \( \mathbf{X} \) in mode \( n \). But \( \mathbf{X} = \mathbf{Y} \times_n A^T \) because \( A^T \) is a left inverse of \( A \), and it follows from Axiom (TR6) that \( r(\mathbf{X}) = r(\mathbf{Y}) \) for any tensor rank function \( r \).

Propositions 4.2 and 4.3 imply that any tensor rank function is a QZC rank function. Many of the results for QZC rank functions generalize in a straightforward way to tensor rank functions. For example, it is easy to see that the class of all tensor rank functions contains a unique minimum function. One could also define an equivalence relation analogous to \( \sim \) and use it to prove a result for tensor rank functions similar to Proposition 3.7. We will not pursue these ideas further.

The next result shows that tensor rank functions have another property that one would expect.

**Proposition 4.4.** Suppose that \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) is a tensor and \( \pi_n \in \mathcal{S}_{I_n} \) is a permutation for \( 1 \leq n \leq N \). Let \( r \) be a tensor rank function. Then the tensor \( \mathbf{Y} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) given by
\[
y_{i_1 \cdots i_N} = x_{\pi_1(i_1) \cdots \pi_N(i_N)}
\]
satisfies \( r(\mathbf{Y}) = r(\mathbf{X}) \).

**Proof.** For \( 1 \leq n \leq N \) let \( P^{(n)} \in \mathbb{R}^{I_n \times I_n} \) be the permutation matrix corresponding to the permutation \( \pi_n \). Then \( \mathbf{Y} = [\mathbf{X}; P^{(1)}, \ldots, P^{(N)}] \), so Axiom (TR6) implies that \( r(\mathbf{Y}) \leq r(\mathbf{X}) \). But each matrix \( P^{(n)} \) is invertible, so it follows easily that \( r(\mathbf{Y}) = r(\mathbf{X}) \). \( \square \)

It is interesting to consider the axioms for tensor rank functions in the context of the higher-order singular value decomposition (HOSVD) developed by De Lathauwer et al. [9, Theorem 2]. Their work shows that any tensor \( \mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) can be written as
\[
\mathbf{X} = [\mathbf{Y}; A^{(1)}, \ldots, A^{(N)}],
\]
where \( A^{(n)} \in \mathbb{R}^{I_n \times I_n} \) is an orthogonal matrix for \( 1 \leq n \leq N \) and \( \mathbf{Y} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) is a core tensor satisfying certain orthogonality and ordering properties. Because each \( A^{(n)} \) is invertible, Axiom (TR6) implies that \( r(\mathbf{X}) = r(\mathbf{Y}) \) for any tensor rank function \( r \). Thus Axiom (TR6) implies that tensor rank functions are constant on all tensors having an HOSVD with the same core. Axiom (TR2) specifies the value of the rank when the core tensor is the identity, and Axiom (TR3) specifies its value on tensors that can be identified with matrices. But the HOSVD allows for a wide variety of core tensors, so these conditions are not actually very restrictive. It is quite possible that interesting subclasses of tensor rank functions arise by restricting the values of the functions on specific types of core tensors.

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