CLASSIFICATION OF CONNECTING SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS

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Abstract. For a given semilinear parabolic equation with polynomial non-linearity, many solutions blow up in finite time. For a certain large class of these equations, we show that some of the solutions which do not blow up actually tend to equilibria. The characterizing property of such solutions is a finite energy constraint, which comes about from the fact that this class of equations can be written as the $L^2$ gradient of a certain functional.

1. Introduction

In this article, the global behavior of smooth solutions to the semilinear parabolic equation

\begin{equation}
\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - u^N + \sum_{i=0}^{N-1} a_i(x)u^i(t,x) = \Delta u + P(u), \text{ for } (t,x) \in \mathbb{R}^{n+1}
\end{equation}

is considered, where $N \geq 2$ and $a_i \in L^1 \cap L^\infty(\mathbb{R}^n)$ are smooth with all derivatives of all orders bounded. Since the linear portion of the right side of (1) is a sectorial operator, we can use (1) to define a nonlinear semigroup. Indeed, in [7], it is shown that short-time solutions exist to (1) when initial conditions lie in a certain subset of $L^1 \cap L^\infty$. This turns (1) into a dynamical system, the behavior of which is largely controlled by its equilibria. In particular, our main result is that solutions to (1) which connect two equilibrium solutions of (1) in a certain strong sense are characterized by finite energy (Definition 2). The study of this kind of problem is not entirely new. Blow-up behavior for equations like (1) was examined in a classic paper by Fujita. For somewhat more restricted nonlinearities, Du and Ma were able to use squeezing methods to obtain similar results to what we obtain here. In particular, they also show that certain kinds of solutions approach equilibria.

2. Finite energy constraints

It is well-known that solutions to (1) exist along strips of the form $(t,x) \in I \times \mathbb{R}^n$ for sufficiently small $t$-intervals $I$. One might hope to extend such solutions to all of $\mathbb{R}^{n+1}$, but for certain choices of initial conditions, such global solutions may fail to exist. Fujita’s classic paper [4] gives examples of this “blow-up” pathology. We will specifically avoid it by considering only global solutions to (1).

Our analysis of (1) will make considerable use of the fact that it is a gradient differential equation. That is, solutions to (1) are integral curves for the gradient of a certain functional in $L^2(\mathbb{R}^n)$.

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Definition 1. Observe that the right side of (1) is the $L^2(\mathbb{R}^n)$ gradient of the following action functional:

$$A(f) = \int \frac{1}{2} \| \nabla f \|^2 - \frac{u^{N+1}}{N+1} + \sum_{i=0}^{N-1} \frac{a_i(x)}{i+1} u^{i+1}(t,x) dx.$$ 

It is then evident that along a solution $u(t)$ to (1), $A(u(t))$ is a monotone function.

As an immediate consequence, nonconstant $t$-periodic solutions to (1) do not exist.

Definition 2. The energy functional is the following quantity defined on a dense subset of $L^2(\mathbb{R}^{n+1})$:

$$E(u) = \frac{1}{2} \int_\mathbb{R} \int \left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u + P(u)|^2 \, dx \, dt.$$ 

Calculation 3. Suppose $u \in L^2(\mathbb{R}^{n+1})$ is in the domain of definition for the energy functional, then

$$E(u) = \frac{1}{2} \int_{-T}^{T} \int \left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u + P(u)|^2 \, dx \, dt = \frac{1}{2} \int_{-T}^{T} \int \left( \frac{\partial u}{\partial t} - \Delta u - P(u) \right)^2 \, dx \, dt + \int_{-T}^{T} \left\langle \frac{\partial u}{\partial t}, \Delta u + P(u) \right\rangle \, dt = \frac{1}{2} \int_{-T}^{T} \int \left( \frac{\partial u}{\partial t} - \Delta u - P(u) \right)^2 \, dx \, dt + \int_{-T}^{T} \frac{d}{dt} A(u(t)) \, dt = \frac{1}{2} \int_{-T}^{T} \int \left( \frac{\partial u}{\partial t} - \Delta u - P(u) \right)^2 \, dx \, dt + A(u(T)) - A(u(-T)).$$

This calculation shows that finite energy solutions to (1) minimize the energy functional. If a solution to (1) connects two equilibria, then the energy functional measures the difference between the values of the action functional evaluated at the two equilibria. The main result of this article is to show the converse, so that finite energy characterizes the solutions which connect equilibria.

It is well-known that when equations like (1) exhibit the correct symmetry, they can support travelling wave solutions. A typical travelling wave solution $u$ has a symmetry like $u(t,x) = U(x-ct)$ for some $c \in \mathbb{R}$. As a result, it is immediate that travelling waves will have infinite energy. On the other hand, they also evidently connect equilibria. As a result, Calculation 3 shows that a necessary condition for travelling waves is that there exists at least one equilibrium whose action is infinite. In this article, we will consider only equilibria with finite action, and solutions with finite energy. As a result, we will not be working with travelling waves.
3. Convergence to equilibria

In this section, we show that finite energy solutions tend to equilibria as \( |t| \to \infty \). In doing this, we follow Floer in [3] which leads us through an essentially standard parabolic bootstrapping argument.

**Lemma 4.** Let \( U \subseteq \mathbb{R}^n \) and \( u \in W^{k,p}(U) \) satisfy \( \|D^j u\|_\infty \leq C < \infty \) for \( 0 \leq j \leq k \) (in particular, \( u \) is bounded). If \( P(u) = \sum_{i=1}^N a_i u^i \) with \( a_i \in L^\infty(U) \) then there exists a \( C^j \) such that \( \|P(u)\|_{k,p} \leq C^j \|u\|_{k,p} \).

**Proof.** First, using the definition of the Sobolev norm,

\[
\|P(u)\|_{k,p} = \sum_{j=0}^k \|D^j P(u)\|_p \leq \sum_{j=0}^k \sum_{i=1}^N \|D^j a_i u^i\|_p.
\]

Now \( |D^j a_i u^i| \leq P_{i,j}(u, Du, ..., D^j u) \) is a polynomial in \( j \) variables with constant coefficients, which has no constant term. (The constant coefficients is a consequence of the bounded derivatives of the \( a_i \)). Additionally,

\[
\|(D^m u)^q D^j u\|_p = \left( \int |(D^m u)^q D^j u|^p \right)^{1/p} \leq \|D^m u\|_\infty^q \left( \int |D^j u|^p \right)^{1/p} \leq C^q \|D^j u\|_p,
\]

so by collecting terms,

\[
\|P(u)\|_{k,p} \leq \sum_{j=0}^k \sum_{i=1}^N \|D^j a_i u^i\|_p \leq \sum_{j=0}^k A_j \|D^j u\|_p \leq C^j \|u\|_{k,p}.
\]

\[ \square \]

The following result is a parabolic bootstrapping argument that does most of the work. In it, we follow Floer in [3], replacing “elliptic” with “parabolic” as necessary.

**Lemma 5.** If \( u \) is a finite energy solution to \( (1) \) with \( \|D^j u\|_{L^\infty((-\infty, \infty) \times V)} \leq C < \infty \) for \( 0 \leq j \leq k \) with \( k \geq 1 \) on each compact \( V \subseteq \mathbb{R}^n \), then each of \( \lim_{t \to \pm \infty} u(t,x) \) exists, and converges with \( k \) of its first derivatives uniformly on compact subsets of \( \mathbb{R}^n \). Further, the limits are equilibrium solutions to \( (1) \).

**Proof.** Define \( u_m(t,x) = u(t+m,x) \) for \( m = 0, 1, 2, ... \). Suppose \( U \subseteq \mathbb{R}^{n+1} \) is a bounded open set and \( K \subseteq U \) is compact. Let \( \beta \) be a bump function whose support is in \( U \) and takes the value 1 on \( K \). We take \( p > 1 \) such that \( kp > n + 1 \). Then we can consider \( u_m \in W^{k,p}(U) \) (recall that \( u \) and its first \( k \) derivatives of \( u \) are bounded on the closure of \( U \)), and we have

\[
\|u_m\|_{W^{k+1,p}(K)} \leq \|\beta u_m\|_{W^{k+1,p}(U)}.
\]

Then using the standard parabolic regularity for the heat operator,

\[
\|\beta u_m\|_{W^{k+1,p}(U)} \leq C_1 \left\| \left( \frac{\partial}{\partial t} - \Delta \right) (\beta u_m) \right\|_{W^{k,p}(U)}.
\]
Let \( P'(u) = -u^N + \sum_{i=1}^{N-1} a_i u^i \), noting carefully that we have left out the \( a_0 \) term. The usual product rule, and a little work, as suggested in [S] yields the following sequence of inequalities

\[
\|u_m\|_{W^{k+1,p}(K)} \leq C_1 \left\| \beta \left( \frac{\partial}{\partial t} - \Delta \right) u_m \right\|_{W^{k,p}(U)} + C_2 \|u_m\|_{W^{k,p}(U)}
\]

\[
\leq C_1 \left\| \beta \left( \frac{\partial}{\partial t} - \Delta \right) u_m + \beta P'(u_m) - \beta P'(u_m) \right\|_{W^{k,p}(U)} + C_2 \|u_m\|_{W^{k,p}(U)}
\]

\[
\leq C_1 \|\beta a_0\|_{W^{k,p}(U)} + C_1 \|\beta P'(u_m)\|_{W^{k,p}(U)} + C_2 \|u_m\|_{W^{k,p}(U)}
\]

\[
\leq C_1 \|\beta a_0\|_{W^{k,p}(U)} + C_3 \|u_m\|_{W^{k,p}(U)},
\]

where the last inequality is a consequence of Lemma [4]. By the hypotheses on \( u \) and \( a_0 \), this implies that there is a finite bound on \( \|u_m\|_{W^{k+1,p}(K)} \), which is independent of \( m \). Now by our choice of \( p \), the general Sobolev inequalities imply that \( \|u_m\|_{C^{k+1-(n+1)/p}(K)} \) is uniformly bounded. By choosing \( p \) large enough, there is a subsequence \( \{v_{m'}\} \subset \{u_m\} \) such that \( v_{m'} \) and its first \( k \) derivatives converge uniformly on \( K \), say to \( v \). For any \( T > 0 \), we observe obtain

\[
\int_{-T}^T \int \left| \frac{\partial v}{\partial t} \right|^2 \, dx \, dt = \lim_{m' \to \infty} \int_{-T}^{m'+T} \int \left| \frac{\partial v_{m'}}{\partial t} \right|^2 \, dx \, dt
\]

\[
= \lim_{m' \to \infty} \int_{m'-T}^{m'+T} \int \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt = 0,
\]

where the last equality is by the finite energy condition. Hence \( |\frac{\partial v}{\partial t}| = 0 \) almost everywhere, which implies that \( v \) is an equilibrium and that \( \lim_{t \to \infty} u(t, x) = v(x) \). Similar reasoning works for \( t \to -\infty \).

Now we would like to relax the bounds on \( u \) and its derivatives, by showing that they are in fact consequences of the finite energy condition.

**Lemma 6.** Suppose that either \( n = 1 \) (one spatial dimension) or \( N \) is odd, then we have the following. If \( u \) is a finite energy solution to (4), then for every \( v = (t_0, x_0) \in \mathbb{R}^{n+1} \), the limits \( \lim_{t \to \pm \infty} u(t + s t_0, x + s x_0) \) exist uniformly on compact subsets of \( \mathbb{R}^{n+1} \), and additionally,

- \( u \) is bounded,
- the derivatives \( Du \) are bounded,
- and therefore the limits are continuous equilibrium solutions.

**Proof.** Note that since

\[
E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int \left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u + P(u)|^2 \, dx \, dt < \infty,
\]

we have that for any \( \epsilon > 0 \),

\[
\lim_{T \to \infty} \frac{1}{2} \int_{T-\epsilon}^{T+\epsilon} \int \left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u + P(u)|^2 \, dx \, dt = 0,
\]

whence \( \lim_{t \to -\infty} |\frac{\partial u}{\partial t}| = 0 \) for almost all \( x \). So this gives that the limit is an equilibrium almost everywhere. Of course, this argument works for \( t \to -\infty \).
Now in the case of $N$ being odd, a comparison principle shows that solutions to (1) are always bounded. So we need to consider the case with $N$ even. In that case, a comparison principle on (1) shows that $u$ is bounded from above. On the other hand, if $N$ is even we have assumed that $n = 1$ in this case, and it follows from an easy ODE phase-plane argument that unbounded equilibria are bounded from below. (Here we have used that the coefficients $a_i$ are bounded.) As a result, we must conclude that if a solution to (1) tends to any equilibrium, that equilibrium (and hence $u$ also) must be bounded.

Now observe that $\left|\frac{\partial u}{\partial t}\right| \to 0$ as $t \to \infty$ on almost all of any compact $K \subset \mathbb{R}^n$, and that $\left|\frac{\partial u}{\partial t}\right| \leq a < \infty$ for some finite $a$ on $\{(t, x)|t = 0, x \in K\}$ by the smoothness of $u$. By the compactness of $K$, this means that if $\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty((-\infty, \infty) \times K)} = \infty$, there must be a $(t^*, x^*)$ such that $\lim_{(t, x) \to (t^*, x^*)} \left|\frac{\partial u}{\partial t}\right| = \infty$. This contradicts smoothness of $u$, so we conclude $\left|\frac{\partial u}{\partial t}\right|$ is bounded on the strip $(-\infty, \infty) \times K$. On the other hand, the finite energy condition also implies that for each $v \in \mathbb{R}^n$,

$$\lim_{s \to -\infty} \int_{-\infty}^{\infty} \int_{K+sv} \left|\frac{\partial u}{\partial t}\right| \, dx \, dt = 0,$$

whence we must conclude that $\lim_{s \to -\infty} \left|\frac{\partial u(t, x+sv)}{\partial t}\right| = 0$ for almost every $t \in \mathbb{R}$ and $x \in K$. Thus the smoothness of $u$ implies that $\left|\frac{\partial u}{\partial t}\right|$ is bounded on all of $\mathbb{R}^{n+1}$.

Next, note that since $\left|\frac{\partial u}{\partial t}\right|$ and $u$ are both bounded, then so is $\Delta u$. (Use the boundedness of the coefficients of $P$.) Taken together, this implies that all the spatial first derivatives of $u$ are also bounded.

As a result, we have on $K$ a bounded equicontinuous family of functions, so Ascoli's theorem implies that they (after extracting a suitable subsequence) converge uniformly on compact subsets of $K$ to a continuous limit.

**Corollary 7.** Suppose that either $n = 1$ or $N$ is odd. A smooth global solution $u$ to (1) has finite energy if and only if for any $v = (t_0, x_0) \in \mathbb{R}^{n+1}$, each of $\lim_{s \to -\infty} u(t + st_0, x + sx_0)$ exists, and converges with its first derivatives uniformly on compact subsets of $\mathbb{R}^{n+1}$ to bounded, continuous, finite action equilibrium solutions to (1).

4. Discussion

The point of employing the bootstrapping argument of Lemma 5 is only to extract uniform convergence of the derivatives of the solution. As can be seen from the proof of Lemma 6, such regularity arguments are unneeded to obtain good convergence of the solution only.

While Corollary 7 is probably true for all spatial dimensions, the proof given here cannot be generalized to higher dimensions. In particular, Véron in [9] shows that in the case of $P(u) = -u^N$, there are solutions to the equilibrium equation $\Delta u - u^N = 0$ which are unbounded below and bounded above when the spatial dimension is greater than one. This breaks the proof of Lemma 6 that the limiting equilibria of finite energy solutions are bounded for $N$ even, since the proof requires exactly the opposite.

On the other hand, the case of $P(u) = -u|u|^{N-1} + \sum_{i=0}^{N-1} a_i u^i$ is considerably easier than what we have considered here. In particular, all solutions to (1) are then bounded. In that case, the proof of Lemma 6 works for all spatial dimensions.
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