COEFFICIENT ESTIMATES FOR SCHWARZ FUNCTIONS

HITOSHI SHIRAISHI AND TOSHIO HAYAMI

Abstract. Let $B$ be the class of functions $w(z)$ of the form $w(z) = \sum_{k=1}^{\infty} b_k z^k$ which are analytic and satisfy the condition $|w(z)| < 1$ in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Then we call $w(z) \in B$ the Schwarz function. In this paper, we discuss new coefficient estimates for Schwarz functions by applying the lemma due to Livingston (Proc. Amer. Math. Soc. 21 (1969), 545–552).

1. Introduction

Let $B$ be the class of functions $w(z)$ of the form

$$w(z) = \sum_{k=1}^{\infty} b_k z^k$$

which are analytic and satisfy the condition $|w(z)| < 1$ in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Also, let $P$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic and satisfy the condition $\text{Re}(p(z)) > 0$ in $U$. Then we call $w(z) \in B$ and $p(z) \in P$ the Schwarz function and the Carathéodory function, respectively.

The following results are well-known for the class $B$.

Lemma 1 (Schwarz lemma). If $w(z) \in B$, then

$$|w(z)| \leq |z| \quad (z \in U) \quad \text{and} \quad |b_1| \leq 1$$

are obtained. In particular, $|w(z_0)| = |z_0|$ for some $z_0 \in U \setminus \{0\}$ or $|b_1| = 1$ if and only if $w(z) = e^{i\theta} z$ for some $\theta \ (0 \leq \theta < 2\pi)$.

By the subordination principle, we establish the following coefficient bounds.
Lemma 2. If \( w(z) \in B \), then
\[
|b_k| \leq 1 \quad (k = 1, 2, 3, \ldots).
\]
Furthermore, \( |b_k| = 1 \) for some \( k \) \((k = 1, 2, 3, \ldots)\) if and only if \( w(z) = e^{i\theta} z^k \).

Applying the Schwarz–Pick lemma (see, for example, [6]), we derive the next coefficient estimate.

Theorem 1. If \( w(z) \in B \), then
\[
|b_2| \leq 1 - |b_1|^2
\]
with equality for
\[
w(z) = \begin{cases} 
e^{i\theta} z & (|b_1| = 1) \\ \frac{b_1 z + e^{i\theta} z^2}{1 + e^{i\theta} b_1 z} = b_1 z + (1 - |b_1|^2) e^{i\theta} z^2 + \ldots & (|b_1| < 1) \end{cases}
\]
for each \( \theta \) \((0 \leq \theta < 2\pi)\).

In this paper, we discuss new coefficient estimates for Schwarz functions by using the following lemma due to Livingston [5].

Lemma 3. If \( p(z) \in P \), then
\[
|c_s - c_t c_{s-t}| \leq 2
\]
for any positive integers \( s \) and \( t \) \((1 \leq t < s)\). For all \( s \) and \( t \), the equality is attained by the function
\[
p(z) = \frac{1+z}{1-z} = 1 + \sum_{k=1}^{\infty} 2z^k.
\]

2. Main results

Our first result is contained in the following theorem by use of Lemma 3.

Theorem 2. If \( p(z) \in P \) with \( c_k = 2e^{i\theta} \) \((0 \leq \theta < 2\pi)\) for some \( k \) \((k = 1, 2, 3, \ldots)\), then
\[
c_{nk} = 2e^{in\theta}
\]
for each \( n \) \((n = 1, 2, 3, \ldots)\).

Proof. Taking \( s = 2k \) and \( t = k \) in Lemma 3 we see that
\[
|c_{2k} - c_k^2| = |c_{2k} - 4e^{i2\theta}| \leq 2.
\]
On the other hand, we know that \( |c_{2k}| \leq 2 \), and therefore it follows that \( c_{2k} = 2e^{i2\theta} \). Similarly, since
\[
|c_{3k} - c_k c_{2k}| = |c_{3k} - 4e^{i3\theta}| \leq 2 \quad \text{and} \quad |c_{3k}| \leq 2,
\]
we have that $c_{3k} = 2e^{i3\theta}$. In the same manner, for all $n (n = 1, 2, 3, \ldots)$, we conclude that $c_{nk} = 2e^{in\theta}$.

By virtue of the above theorem, we obtain

**Corollary 1.** If $p(z) \in \mathcal{P}$ with $c_1 = 2e^{i\theta} (0 \leq \theta < 2\pi)$, then we can declare that

$$p(z) = \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} = 1 + \sum_{k=1}^{\infty} 2e^{ik\theta} z^k.$$ 

Moreover, applying Lemma 3, we have a new coefficient bound for Schwarz functions.

**Theorem 3.** If $w(z) \in \mathcal{B}$, then

$$|b_3| \leq 1 - |b_1|^3.$$ 

**Proof.** We first note that if a function $w(z) \in \mathcal{B}$ then

$$e^{i\theta}w(z) \in \mathcal{B}$$

for all $\theta (0 \leq \theta < 2\pi)$. Then, we know that the function $p(z)$ defined by

$$p(z) = \frac{1 + e^{i\theta}w(z)}{1 - e^{i\theta}w(z)} = 1 + 2e^{i\theta}b_1z + 2(e^{i2\theta}b_1^2 + e^{i\theta}b_2)z^2 + 2(e^{i3\theta}b_1^3 + 2e^{i2\theta}b_1b_2 + e^{i\theta}b_3)z^3 + 2(e^{i4\theta}b_1^4 + 3e^{i3\theta}b_1^2b_2 + 2e^{i2\theta}b_1b_3 + e^{i\theta}b_4)z^4 + \ldots$$

$$= 1 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \ldots \quad (z \in \mathbb{U}), \quad (1)$$

belongs to the class $\mathcal{P}$. In view of Lemma 3, we obtain that

$$|c_3 - c_1c_2| = |2(e^{i3\theta}b_1^3 + 2e^{i2\theta}b_1b_2 + e^{i\theta}b_3) - 2e^{i\theta}b_12(e^{i2\theta}b_1^2 + e^{i\theta}b_2)|$$

$$\leq 2$$

which gives us that

$$|b_3 - e^{i2\theta}b_1^3| \leq 1.$$ 

Thus, $b_3$ is in the region

$$\bigcap_{\theta} \{ b_3 : |b_3 - e^{i2\theta}b_1^3| \leq 1 \} = \{ b_3 : |b_3| \leq 1 - |b_1|^3 \}.$$ 

This completes the proof of the theorem.

The same process in the proof of Theorem 3 leads us another proof of Theorem 1.
Proof. Applying Lemma 3 to the function (1) with \( s = 2 \) and \( t = 1 \), we deduce that
\[
|c_2 - c_1^2| = |2(e^{i2\theta}b_1^2 + e^{i\theta}b_2) - (2e^{i\theta}b_1)^2|
\]
\[
= |2e^{i\theta}(b_2 - e^{i\theta}b_1^2)|
\]
\[
\leq 2.
\]
This implies that for all \( \theta \) \((0 \leq \theta < 2\pi)\)
\[
|b_2 - e^{i\theta}b_1^2| \leq 1
\]
which means that
\[
|b_2| \leq 1 - |b_1|^2.
\]
\[\square\]

But, using the same process in the proof of Theorem 3 we have no good result for coefficients \( b_4, b_5 \) and so on. Because, for example, applying Lemma 3 to the function (1) with \( s = 4 \) and \( t = 1 \) to find the estimate of \( b_4 \), we obtain the next inequality.
\[
|c_4 - c_1c_3| = |2(e^{i4\theta}b_1^4 + 3e^{i3\theta}b_1^2b_2 + 2e^{i2\theta}b_1b_3 + e^{i\theta}b_4)
- 4e^{i\theta}b_1(e^{i3\theta}b_1^2 + 2e^{i2\theta}b_1b_2 + e^{i\theta}b_3)|
\]
\[
= |2e^{i\theta}(b_4 + e^{i\theta}b_2^2 - e^{i2\theta}b_1^2b_2 - e^{i3\theta}b_1^4)|
\]
\[
\leq 2.
\]
Calculating this inequality, we have
\[
|b_4 + e^{i\theta}b_2^2 - e^{i2\theta}b_1^2b_2 - e^{i3\theta}b_1^4| \leq 1.
\]
(2)

Also, if we apply Lemma 3 to the function (1) with \( s = 4 \) and \( t = 2 \), then we obtain another inequality as follows:
\[
|c_4 - c_2^2| = |2(e^{i4\theta}b_1^4 + 3e^{i3\theta}b_1^2b_2 + 2e^{i2\theta}b_1b_3 + e^{i\theta}b_4)
- 2(e^{i2\theta}b_1^2 + e^{i\theta}b_2)^2|
\]
\[
= |2e^{i\theta}(b_4 + 2e^{i\theta}b_1b_3 - e^{i2\theta}b_2^2 - e^{i3\theta}b_1^2)|
\]
\[
\leq 2.
\]
Calculating this inequality, we have
\[
|b_4 + 2e^{i\theta}b_1b_3 - e^{i\theta}b_2^2 - e^{i2\theta}b_1^2b_2 - e^{i3\theta}b_1^4| \leq 1.
\]
(3)

We don’t know the region to which both inequalities (2) and (3) point.

References
[1] C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen, Math. Ann. 64(1907), 95–115.
[2] P. L. Duren, Univalent Functions, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
[3] A. W. Goodman, Univalent Functions, Vol. I and Vol. II, Mariner Publishing Company, Tampa, Florida (1983).
[4] T. Hayami and S. Owa, The Fekete-Szegő problem for p-valently Janowski starlike and convex functions, Int. J. Math. Math. Sci. 2011, Article ID 583972, 1–11.
[5] A. E. Livingston, The coefficients of multivalent close-to-convex functions, Proc. Amer. Math. Soc. 21(1969), 545–552.
[6] Z. Nehari, *Conformal Mapping*, McGraw Hill Company, New York (1952).
[7] W. Rogosinski, *On subordinate functions*, Proc. Cambridge Philos. Soc. 35 (1939), 1–26.
[8] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc. 48 (1943), 48–82.
[9] J. Sokół, *Coefficient estimates in a class of strongly starlike functions*, Kyungpook Math. J. 49 (2009), 349–353.

Hitoshi Shiraishi  
Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577-8502, Japan  
E-mail address: step625@hotmail.com

Toshio Hayami  
Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577-8502, Japan  
E-mail address: hayato112@hotmail.com