Symmetries in particle and string theories

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Abstract: We study the space-time invariances of the bosonic relativistic particle and bosonic relativistic string using general formulations obtained by incorporating the Hamiltonian constraints into the formalism. We point out that massless particles and tensionless strings have a larger set of space-time invariances than their massive and tensionful partners, respectively. We also show that it is possible to use the reparametrization invariance of the string formulation we present to reach the classical conformal equations of motion without the use of two-dimensional Weyl scalings of the string world sheet. Finally, we show that it is possible to fix a gauge with an enlarged number of space-time invariances in which every point of the free tensionless string moves as if it were an almost-free massless particle. The existence of such a string motion agrees with what is expected from gauge theory-string duality.

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PACS numbers: 11.15.-q, 11.15.Kc, 11.25.-w, 11.30.Ly
1 Introduction

In all fundamental theories of matter, it is necessary to understand very well the free theory before trying to describe interactions. In this context, it is sometimes desirable to construct a way along which the interacting theory could be naturally linked to the free theory, thus avoiding interactions of being artificially introduced.

One way to investigate the relations between the free and the interacting theories is to examine the invariances of the corresponding actions. In interesting cases the free action has a larger number of invariances than the interacting one and in these cases the appearance of an interaction may be associated to the breaking of a specific invariance. There is also the possibility that the breaking of a symmetry causes the appearance of mass [1].

In this work we are particularly interested in space-time invariances, and how these invariances manifest themselves in the actions that describe relativistic objects embedded in the space-time. We focus our attention on relativistic particles and strings and verify the invariance of the actions that describe these objects under a general space-time transformation such as [2]

\[
\delta x^\mu = a^\mu + \omega^{\mu\nu} x_\nu + \alpha x^\mu + (2x^\mu x^\nu - \eta^{\mu\nu} x^2)b_\nu
\]  

(1)

The first two terms on the right in (1) represent a Poincaré transformation. The third is a scale transformation and the fourth is a conformal transformation. \(\alpha\) is a constant and \(b_\mu\) is a constant vector. The algebra of these space-time transformations is a closed algebra [2].

At present, the two fundamental theories of matter are Quantum Field Theory and General Relativity. By the same time, the most promising hope for a truly unified and finite description of these two fundamental theories is Superstring Theory [3,4]. This is enough reason to study the dynamics of relativistic strings. But there
is also the important fact that String Theory was originally discovered as a quantum
type, some time before a classical string action was first written [4]. String theory
has been developing backwards since then [4], with its geometric structure behind
the classical action still a subject of interest.

In this work we consider only bosonic relativistic strings. We use the reparametrisa-
tion invariance of the theory to obtain information about the classical dynamics.
Under reparametrizations, both the bosonic and fermionic degrees of freedom of the
superstring behave as scalars. So, from the reparametrization invariance point of
view, nothing substantially new is introduced by the fermionic degrees of freedom
and we can omit them for simplicity.

In our approach to the subject we treat the particle and the string as con-
strained systems [5] and this allows us to construct classically equivalent, but more
general, particle and string actions that, in the particular case of strings, have some
advantages when compared with the usual formulations.

In string theory, the most traditional approach to quantization starts by using
the reparametrization invariance of the theory, together with the two-dimensional
Weyl scaling invariance, to impose the conformal gauge. In this gauge the world
sheet metric $h_{\alpha\beta}$ is set equal to the flat two-dimensional Minkowski metric $\eta_{\alpha\beta}$. The
Weyl scaling symmetry is responsible for the tracelessness of the two-dimensional
energy-momentum tensor $T_{\alpha\beta}$. But the quantum theory gives rise to an anomaly
in the trace of $T_{\alpha\beta}$ and only under very special circumstances does this anomaly
cancel. This is related to the fact that quantum string theory seems to work only in
certain specific space-time dimensions [3]. In the string formulation we present here,
the conformal equation of motion can be reached using reparametrization invariance
only. No use is made of two-dimensional Weyl invariance.

Another advantage of the string formulation we present is that it can be used to
study the tensionless limit of string theory. The massless limit is the high-energy
limit of particle theory and, as became clear in ref. [6], the tensionless limit is the high-energy limit of string theory. As was pointed out in [6], the existence of infinite linear relations between different string quantum scattering amplitudes with the same momentum, in the high-energy limit, means that the tensionless string must be more symmetric than usual strings. But in ref. [6] no mention exists of what this larger symmetry of tensionless strings could be, or what its physical origin. This situation was further developed in ref. [7] by the use of a proposed [8] Ward identity between quantum Green functions. It was shown that the unknown larger symmetry of the tensionless string is related to the decoupling of zero-norm states from the theory [7].

Quantum Ward identities in a field theory have their origin in the local invariances of the corresponding classical action [4]. The evolution of a string defines a two-dimensional field theory, with the string coordinates $x^\mu$ as functions of the two parameters used to describe the evolution surface. On this surface, each of the four space-time transformations in (1) is a local transformation. In this work we point out that tensionless strings are invariant under the four transformations in (1), but that tensionful strings are invariant only under two of them. This is somehow related to the results in refs. [6] and [7]. We also find that free tensionless strings are invariant under another local transformation that is different from the transformations in (1). As we shall see here, these extra invariances of tensionless strings are natural extensions of extra invariances of massless particles.

Tensionless strings were originally introduced by Schild in a postumous paper [9] as strings with a singular world-surface metric tensor, $\det g^{\alpha \beta} = 0$. They are much simpler relativistic objects than usual strings and attracted attention some time ago [10,11,12, 13] because of the question if a critical dimension would emerge from its dynamics. No really definitive answer to this question was arrived at. At the same time, theoreticians were developing the idea [4] that the larger the gauge group
of a field theory, the larger the possibility that the theory is finite order by order
in a quantum perturbation expansion, and so does not need to be renormalized. This brought a new interest in the two-dimensional quantum field theory defined
by tensionless strings, which are now being investigated using different formulations
because of its richer symmetries (see, for instance, refs. [14,15]). In this work we show that it is possible to fix a gauge in which each point of the free tensionless
string moves as if it were an almost free massless particle. This was first pointed
out for tensionless superstrings in [11], with each string point moving as an almost
free massless superparticle. This particular possible motion of tensionless strings,
but in the framework of Feynman path integrals, was also mentioned in [15]. This
kind of string motion is expected from gauge theory-string duality [24,25]. It is also
shown here that this particular gauge-fixed tensionless string, when in free motion,
has one extra invariance when compared with the non gauge-fixed tensionless one,
in complete correspondence with the free massless particle case.

The paper is organized as follows: In section two we consider bosonic relativistic
particle theory. Starting from the usual square-root action we compute a particle
action that allows a transition to the massless limit of the theory. We investigate the
classical equations of motion and the space-time invariances for both the \( m \neq 0 \) and
\( m = 0 \) cases. We find that the massless action has three additional invariances which
are not present in the massive action. In section three we extend our procedure to
the case of bosonic relativistic string theory. After a brief introduction to the
subject we show how we can compute a string action that is compatible with the
tensionless limit and which allows us to reach the usual classical conformal-gauge
equations of motion without requiring two-dimensional Weyl scale invariance. We
show that in this formulation the \( T = 0 \) string action has only two additional space-
time invariances when compared with the \( T \neq 0 \) action. In section four we study
in detail the reparametrization invariance of the string formulation we present and
show that it is possible to fix a gauge in which the $T = 0$ string has three extra symmetries when compared to the $T \neq 0$ string. This creates a link between free massless relativistic particle theory and this particular sector of free tensionless relativistic string theory. We present our conclusions in section five.

2 Relativistic particles

A relativistic particle describes in space-time a one-parameter trajectory $x^\mu(\tau)$. The dynamics of the particle must be independent of the parameter choice. A possible form of the action is the one proportional to the arc length traveled by the particle and given by

$$S = -m \int ds = -m \int d\tau \sqrt{-\dot{x}^2(\tau)}$$

(2)

where $\tau$ is an arbitrary parameter, $m$ is the particle’s mass, $ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$, and $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. Our metric convention is $\eta_{\mu\nu} = (-1, +1, +1, +1)$ and we use units in which $\hbar = c = 1$.

Action (2) is invariant under Poincaré transformations

$$\delta x^\mu = a^\mu + \omega^{\mu\nu} x_\nu$$

(3)

with $\omega^{\mu\nu} = -\omega^{\nu\mu}$, and under reparametrizations of the world-line

$$\tau \rightarrow \tau' = f(\tau)$$

(4)

where $f$ is an arbitrary function of $\tau$. The classical equation of motion for $x^\mu(\tau)$ that follows from action (2) implies that

$$\ddot{x}^\mu = 0$$

(5)
Action (2) is quite awkward to deal with because of the square root, and it is also quite restricted because it does not allow a transition to the $m = 0$ limit of the theory. The formal problem of defining an action for a massless particle is naturally solved by treating the particle as a gauge system in which the gauge invariance is the reparametrization invariance (4). Nowadays, the most general treatment of a gauge system is within the constrained Hamiltonian formalism [16].

In the transition to the Hamiltonian formalism action (2) gives the canonical momentum

$$p_{\mu} = \frac{m}{\sqrt{-\dot{x}^2}} \dot{x}_{\mu}$$

and this momentum gives rise to the primary [5] constraint

$$\phi = \frac{1}{2}(p^2 + m^2) \approx 0$$

We follow Dirac’s convention that a constraint is set equal to zero only after all calculations have been performed. In this sense equation (7) means that $\phi$ “weakly” vanishes. The canonical Hamiltonian that follows from action (2), $H = p, \dot{x} - L$, identically vanishes. This is a characteristic feature of reparametrization invariant systems. Dirac’s [5] extended Hamiltonian for the particle is then

$$H_E = H + \lambda \phi = \frac{1}{2} \lambda (p^2 + m^2) \approx 0$$

where $\lambda(\tau)$ is a Lagrange multiplier enforcing the constraint $\phi$.

The canonical equations of motion obtained from (8) are

$$\dot{x}^{\mu} = \{x^{\mu}, H_E\} = \lambda p_{\mu}$$

$$\dot{p}^{\mu} = \{p^{\mu}, H_E\} = 0$$
in which \{ , \} denotes a Poisson bracket. Combining equations (9) and (10) we obtain

\[ \ddot{x}^\mu = \dot{\lambda} p^\mu + \lambda p^\mu \]

\[ = \dot{\lambda} p^\mu \]  

(11)

In order to have \( \ddot{x}^\mu = 0 \), we must impose the condition \( \dot{\lambda} = 0 \). The free particle theory obtained from Hamiltonian (8) is the one in which \( \lambda \) is a constant. But in the general theory, where \( \lambda \) has a \( \tau \)-dependence, an interaction proportional to \( \dot{\lambda} \) is present.

The Lagrangian function that corresponds to the extended Hamiltonian (8) is

\[ L = p \dot{x} - H_E \]

\[ = p \dot{x} - \frac{1}{2} \lambda (p^2 + m^2) \]  

(12)

\( L \) in equation (12) is defined on an interface between configuration space and phase space. Using equation (9) to eliminate the \( p_\mu \) we arrive at the particle action

\[ S = \frac{1}{2} \int d\tau (\lambda^{-1} \dot{x}^2 - \lambda m^2) \]  

(13)

Action (13) formally coincides with the “einbein” action [3]

\[ S = \frac{1}{2} \int d\tau (e^{-1} \dot{x}^2 - em^2) \]

in which the variable \( e(\tau) \) describes the one-dimensional geometry of the world-line. In order to clarify the role played by \( \lambda(\tau) \) in the theory defined by (13), let us briefly investigate the Hamiltonian formalism for this action. Action (13) gives the
momentum

\[ p_\mu = \frac{\dot{x}_\mu}{\lambda} \]  \hspace{1cm} (14)

and the primary constraint

\[ p_\lambda = \frac{\partial L}{\partial \dot{x}_\lambda} \approx 0 \]  \hspace{1cm} (15)

The corresponding canonical Hamiltonian is

\[ H = \frac{1}{2} \lambda (p^2 + m^2) \]  \hspace{1cm} (16)

Incorporating constraint (15) into the formalism we get the extended Hamiltonian

\[ H_E = \frac{1}{2} \lambda (p^2 + m^2) + \varsigma p_\lambda \]  \hspace{1cm} (17)

where \( \varsigma \) is a multiplier. Following Dirac’s algorithm for constrained systems we must now require that all constraints be stable. The stability condition for constraint (15), \( \dot{p}_\lambda = \{ p_\lambda, H_E \} = 0 \) is satisfied only if \( \phi = \frac{1}{2} (p^2 + m^2) \approx 0 \). Requiring now the stability of \( \phi \), \( \dot{\phi} = \{ \phi, H_E \} = 0 \), we find that it is automatically satisfied. The algorithm is completed. No condition is placed on \( \lambda(\tau) \). It remains as an arbitrary function in action (13). This is a consequence of the fact that \( \phi \), having vanishing Poisson bracket with itself, is a first-class [5] constraint. Since the \( \tau \)-derivative of \( \lambda \) does not appear in (13) its equation of motion is a constraint and \( \lambda \) can be eliminated, giving back action (2). Actions (2) and (13) are therefore classically equivalent.

Action (13) is invariant under the Poincaré transformation (3) with \( \delta \lambda = 0 \) and under the infinitesimal reparametrizations

\[ \delta x^\mu = \epsilon \dot{x}^\mu \]  \hspace{1cm} (18a)
\[ \delta \lambda = \frac{d}{d\tau} (\epsilon \lambda) \]  

(18b)

where \( \epsilon(\tau) \) is an arbitrary infinitesimal parameter. The equation of motion for \( x^\mu \) that follows from action (13) states that

\[ \frac{d}{d\tau} \left( \frac{1}{\lambda} \dot{x}^\mu \right) = 0 \]  

(19)

and again, to select the free theory, we must impose the condition that \( \dot{\lambda} = 0 \). The condition that \( \lambda \) must be a constant in the free particle theory has far-reaching consequences. For instance, fixing \( \lambda = 1 \) is a fundamental step in the functional quantization procedure based on action (13) because it precedes [4] the insertion of the Faddeev-Popov [17] term into the functional integral to get the correct measure. After exponentiation of the Faddeev-Popov term by the use of Grassmann variables, the resulting gauge-fixed effective action turns out to have an important residual global fermionic symmetry called the BRST [18,19] symmetry.

Action (13), although classically equivalent to, is more general than action (2) because it allows a transition to the \( m = 0 \) limit of the theory. A massless relativistic particle can be described by the action

\[ S = \frac{1}{2} \int d\tau \lambda^{-1} \dot{x}^2 \]  

(20)

which is the \( m = 0 \) limit of (13). Action (20) is again invariant under the Poincaré transformation (3) with \( \delta \lambda = 0 \) and under the reparametrizations (18). The equation of motion for \( x^\mu \) that follows from action (20) is identical to the equation of motion (19) for the massive particle. The \( m \neq 0 \) and \( m = 0 \) particles are governed by the same classical dynamics. In order to select the free massless theory we must again impose that \( \dot{\lambda} = 0 \). However, action (20) exhibits invariances which are not
present in action (13).

The massless particle action (20) is invariant under the scale transformation

$$\delta x^\mu = \alpha x^\mu \quad (21a)$$

$$\delta \lambda = 2\alpha \lambda \quad (21b)$$

and also invariant under the conformal transformation

$$\delta x^\mu = (2x^\mu x^\nu - \eta^\mu\nu x^2) b_\nu \quad (22a)$$

$$\delta \lambda = 4\lambda x \cdot b \quad (22b)$$

These two invariances are present in the general theory, the one in which the equation of motion is given by (19) and where there is an arbitrary \(\tau\)-dependence for \(\lambda\). They are also present if \(\lambda\) has no \(\tau\)-dependence but can suffer arbitrary variations. But there is a further invariance of action (20) which is present only in the free theory, where \(\dot{\lambda} = 0\). Using the equation for free motion, \(\ddot{x}^\mu = 0\), it can be verified that the massless action (20) is invariant under the “gauge” transformation

$$x^\mu(\tau) \rightarrow \exp\left\{\frac{1}{3} \beta(\dot{x}^2)\right\} x^\mu(\tau) \quad (23a)$$

$$\lambda \rightarrow \exp\left\{\frac{2}{3} \beta(\dot{x}^2)\right\} \lambda \quad (23b)$$

We call transformation (23) a gauge transformation because \(\beta\) is an arbitrary function of \(\dot{x}^2\). In the free theory any function \(\beta(\dot{x}^2)\) satisfies \(\dot{\beta} = 0\) and this, together with (23b), renders action (20) invariant.

The conclusions of this section on relativistic particles are clear. As a conse-
quence of the first-class property of $\phi$, the variable $\lambda(\tau)$ remains as an arbitrary function in action (13). $\lambda$ acts as a gauge function that can be used to select the free sector of relativistic particle theory. As we also saw, the massless limit of particle theory is more symmetric than the massive one, with the free $m = 0$ sector having three extra invariances if compared to the $m \neq 0$ sector.

It is well known [2] that the presence of scale invariance automatically implies the presence of conformal invariance. We are then led to the idea that the breaking of the space-time scale invariance (21), and consequently the breaking of the space-time conformal invariance (22), are related to the appearance of a non-vanishing particle mass in action (13). This is true in the general and in the free theory. Following this same reasoning, the breaking of the “gauge” invariance (23) must be related to the appearance of an interaction in the massless theory. This interaction is presumably of gravitational origin, and is related to a non-vanishing $\tau$-derivative of $\lambda$. The free massive theory does not have the “gauge” invariance (23) because the mass term spoils the invariance of action (13) under transformation (23). In the next two sections we show how this physical picture can be extended to string theory.

3 Relativistic strings

Strings are higher-dimensional extensions of the particle concept. As a consequence of its evolution, the string traces out a world sheet in space-time. In the form originally advocated by Nambu [20] and Goto [21], the action for a string is simply proportional to the area of its world sheet. Mathematically, one formula for the area of a sheet embedded in space-time is

$$S = T \int d\tau d\sigma \sqrt{\dot{x}^2 \ddot{x}^2 - (\dot{x} \cdot \ddot{x})^2}$$  \hspace{1cm} (24)
in which \( x^\mu = x^\mu(\tau, \sigma) \) are the coordinates of the string, given as \( D \) functions of the parameters that describe the world surface, and primes denote derivatives with respect to \( \sigma \). \( T \) is a constant of proportionality, required to make the action dimensionless. \( T \) must have dimension of \((\text{length})^{-2}\), or \((\text{mass})^2\), to leave a dimensionless action. It can be shown \([3, 22]\) that \( T \) is actually the tension in the string. The actual value of \( T \) may be treated as a free parameter in the theory and corresponds to an energy scale. With the action of the string taken to be the area of its world sheet, the solutions of the classical equations of the free string are the world sheets of minimal (or at least extremal) area. This generalizes the fact that the trajectories of the free particle are geodesics, or curves of minimal length.

It is difficult to work with action (24) because it is highly nonlinear and especially because of the square root. An equivalent, but more convenient form of the action can be written if we introduce in addition to \( x^\mu(\tau, \sigma) \) a new variable \( h_{\alpha\beta} \), which will be a metric tensor of the string world sheet. This more convenient action is \([23]\)

\[
S = -\frac{T}{2} \int d\tau d\sigma \sqrt{h} \hbar^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu
\]

(25)

Here \( \sqrt{h} \) is the square root of the absolute value of the determinant of \( h_{\alpha\beta} \) and \( h^{\alpha\beta} \) is the inverse of \( h_{\alpha\beta} \). Action (25) is the standard form for coupling \( D \) massless scalar fields \( x^\mu \) to \((1+1)\)-dimensional gravity \([3]\). Since the derivatives of \( h_{\alpha\beta} \) do not appear in (25) its equation of motion is a constraint and \( h_{\alpha\beta} \) can be integrated out giving back action (24).

Either (24) or (25) is invariant under general coordinate transformations of the string world sheet \( \tau \rightarrow \tau'(\tau, \sigma), \sigma \rightarrow \sigma'(\tau, \sigma) \). This reparametrization invariance is essential for solving the minimal surface equations derived from (25). The \( 2 \times 2 \) symmetric tensor \( h_{\alpha\beta} \) has three independent components. By a suitable choice of new parameters \( \tau' \) and \( \sigma' \) any two of the three independent components of \( h_{\alpha\beta} \) can
be eliminated. This leaves only one independent component. However, there is one more local two-dimensional symmetry of the string action (25). There is a local Weyl scaling of the metric

\[ h_{\alpha\beta} \to \Lambda(\tau, \sigma) h_{\alpha\beta} \]  (26)

which leaves the factor \( \sqrt{h^{\alpha\beta}} \) invariant. The reparametrization invariance together with the Weyl scaling can then be used to gauge away all the three independent \( h_{\alpha\beta} \) components by imposing the conformal gauge \( h_{\alpha\beta} = \eta_{\alpha\beta} \), where \( \eta_{\alpha\beta} \) is the two-dimensional flat space metric. In this gauge action (25) becomes

\[ S = -\frac{T}{2} \int d\tau d\sigma \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \]  (27)

The equation of motion derived from action (27) is the free two-dimensional linear wave equation

\[ \frac{\partial^2 x^\mu}{\partial \tau^2} - \frac{\partial^2 x^\mu}{\partial \sigma^2} = 0 \]  (28)

The solutions of the wave equation (28) are well-known [3]. These solutions form the basis of the “Old Covariant First Quantized” formalism for bosonic relativistic strings [3].

Now, while the string action (24) is obviously the higher-dimensional extension of the particle action (2), the string extension of the particle action (13) is still lacking. This is because action (25) can not be used to study the tensionless limit of string theory, as action (13) was used to study the massless limit of particle theory. We see that it is necessary to compute a string action that allows a transition to the \( T = 0 \) limit. To parallel our treatment of the relativistic particle as close as possible we return to the metric convention \( \eta_{\mu\nu} = -1, +1, ..., +1 \) and write the Nambu-Goto action as

\[ S = -T \int d\tau d\sigma \sqrt{-g} \]  (29a)
where

\[ g = \det g_{\alpha\beta} \quad (29b) \]

\[ g_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \quad (29c) \]

\( g_{\alpha\beta} \) is the two-dimensional metric induced on the string world sheet as a result of its embedding in space-time. In the transition to the Hamiltonian formalism the Nambu-Goto action (29) gives the canonical momentum

\[ p_\mu = -T \sqrt{-g} \, g^{0\alpha} \partial_\alpha x_\mu \quad (30) \]

and this momentum gives rise to the primary constraints

\[ \Phi = \frac{1}{2} (p^2 + T^2 \dot{x}^2) \approx 0 \quad (31a) \]

\[ \Phi_1 = p \dot{x} \approx 0 \quad (31b) \]

Due to the reparametrization invariance of the theory the canonical Hamiltonian density vanishes and Dirac’s Hamiltonian density for the Nambu-Goto string is

\[ H_E = H + \lambda \Phi + \lambda_1 \Phi_1 \]

\[ = \lambda \Phi + \lambda_1 \Phi_1 \quad (32) \]

where \( \lambda(\tau, \sigma) \) and \( \lambda_1(\tau, \sigma) \) are Lagrange multipliers for the constraints \( \Phi \) and \( \Phi_1 \).

The Lagrangian density corresponding to (32) is

\[ L = p \dot{x} - \lambda \Phi - \lambda_1 \Phi_1 \]

\[ = p \dot{x} - \frac{\lambda}{2} (p^2 + T^2 \dot{x}^2) - \lambda_1 p \dot{x} \quad (33) \]
The string Lagrangian (33) is defined in an extended configuration space. But we can return to the usual configuration space by integrating the momenta. The solution of the equation of motion for $p_\mu$ is

$$p_\mu = \frac{1}{\lambda}(\dot{x}_\mu - \lambda_1 \dot{x}_\mu)$$

(34)

Inserting this result back into (33) we obtain the string action

$$S = \frac{1}{2} \int d\tau d\sigma [\lambda^{-1} (\dot{x} - \lambda_1 \dot{x})^2 - \lambda T^2 \dot{x}^2]$$

(35)

Action (35) is the higher-dimensional extension of the particle action (13). It is Poincaré invariant with $\delta \lambda = 0$, $\delta \lambda_1 = 0$. It is also reparametrization invariant, as will be explicitly checked in the next section. If we eliminate $\lambda$ and $\lambda_1$ through their equations of motion we recover the Nambu-Goto action (24). Action (35) is then classically equivalent to the Nambu-Goto action. However, it is more general than actions (24) and (25) because it allows a transition to the $T = 0$ limit.

If we construct the Hamiltonian formulation for action (35) we will find that the two-dimensional fields $\lambda$ and $\lambda_1$ remain as arbitrary functions in the theory. This is because constraints (31) are first-class constraints obeying the Poisson bracket gauge algebra

$$\{\phi(\tau, \sigma), \phi(\tau, \sigma')\} = T^2 [\phi_1(\tau, \sigma) + \phi_1(\tau, \sigma')] \delta'(\sigma - \sigma')$$

(36a)

$$\{\phi(\tau, \sigma), \phi_1(\tau, \sigma')\} = [\phi(\tau, \sigma) + \phi(\tau, \sigma')] \delta'(\sigma - \sigma')$$

(36b)

$$\{\phi_1(\tau, \sigma), \phi_1(\tau, \sigma')\} = [\phi_1(\tau, \sigma) + \phi_1(\tau, \sigma')] \delta'(\sigma - \sigma')$$

(36c)
We may, for instance, use the arbitrariness of $\lambda$ and $\lambda_1$ to make the identification

\begin{align}
-T\sqrt{h}h^{00} &= \frac{1}{\lambda} \\
-T\sqrt{h}h^{01} &= -\frac{\lambda_1}{\lambda} \\
-T\sqrt{h}h^{11} &= \frac{\lambda_1^2}{\lambda} - \lambda T^2
\end{align}

(37a)

(37b)

(37c)

Action (35) can then be rewritten as

$$S = -\frac{T}{2} \int d\tau d\sigma \sqrt{h} \alpha^\beta \partial_\alpha x \partial_\beta x$$

which is identical to (25). In equations (37), three independent metric factors are expressed in terms of two independent arbitrary functions $\lambda$ and $\lambda_1$. This means that in the string formulation (35) the conformal-gauge equation of motion (28) can be reached by use of reparametrization invariance only. This can be explicitly verified. The classical equation of motion for $x_\mu$ that follows from action (35) is

$$\frac{\partial}{\partial \tau} \left( \frac{1}{\lambda} \dot{x}_\mu - \frac{\lambda_1}{\lambda} \dot{x}_\mu \right) + \frac{\partial}{\partial \sigma} \left[ \frac{\lambda_1}{\lambda} \dot{x}_\mu + \left( \frac{\lambda_1^2}{\lambda} - \lambda T^2 \right) \dot{x}_\mu \right] = 0$$

(38)

Now, choosing $\lambda = 1$, if we want to reproduce the conformal-gauge equation of motion, we have to satisfy the condition $\lambda_1^2 - T^2 = -1$, for which the solutions are $\lambda_1 = \pm \sqrt{T^2 - 1}$. Using the positive value of $\lambda_1$ in the first term of (38), the negative value of $\lambda_1$ in the second term, and the fact that the partial derivatives with respect to $\tau$ and $\sigma$ commute, equation of motion (38) reduces to

$$\frac{\partial^2 x_\mu}{\partial \tau^2} - \frac{\partial^2 x_\mu}{\partial \sigma^2} = 0$$

(39)

which is identical to equation (28). In the special case of equation of motion (38) reparametrization invariance allows us to make three independent gauge choices.
In the “Old Covariant First Quantized” [3] formalism based on the gauge-fixed classical action (27) the trace anomaly is an inconsistency because it implies the non-validity of the wave equation (28), upon which all the formalism is based. The anomaly must then be eliminated, which can only be completely done in $D = 26$. Here this problem does not exist because no use is made of two-dimensional Weyl scale invariance to reach the wave equation (39). The anomaly must manifest itself in another way.

Taking the limit $T = 0$ in action (35) we obtain the string action

$$ S = \frac{1}{2} \int d\tau d\sigma \lambda^{-1}(\dot{x} - \lambda \dot{x})^2 $$

(40)

With the identification

$$ g^{00} = \frac{1}{\lambda} $$

(41a)

$$ g^{01} = -\frac{\lambda_1}{\lambda} $$

(41b)

$$ g^{11} = \frac{\lambda^2_1}{\lambda} $$

(41c)

action (40) can be rewritten as

$$ S = \frac{1}{2} \int d\tau d\sigma g^{\alpha\beta} \partial_\alpha x. \partial_\beta x $$

(42)

which is identical to the one proposed by Schild [9]. Notice that using equations (41) we have $\det g^{\alpha\beta} = 0$.

Action (40) is invariant under Poincaré transformations and, as will be checked in the next section, also reparametrization invariant. The equation of motion for $x_\mu$ that follows from action (40) is the $T = 0$ limit of equation (38), namely,

$$ \frac{\partial}{\partial \tau} \left( \frac{1}{\lambda} \dot{x}_\mu - \lambda_1 \dot{x}_\mu \right) + \frac{\partial}{\partial \sigma} \left( -\frac{\lambda_1}{\lambda} \dot{x}_\mu + \frac{\lambda^2_1}{\lambda} \ddot{x}_\mu \right) = 0 $$

(43)
Equation (43) can also be reduced to the conformal wave equation. Choosing again \( \lambda = 1 \), the possible values of \( \lambda_1 \) are now \( \lambda_1 = \pm \sqrt{-1} = \pm i \). Using \( \lambda_1 = i \) in the first term, and \( \lambda_1 = -i \) in the second term, equation (43) becomes the conformal equation of motion (39). We see that in the conformal gauge both the \( T \neq 0 \) and \( T = 0 \) strings satisfy free two-dimensional linear wave equations. In this gauge they are then governed by the same classical dynamics. But, in general, the \( T \neq 0 \) and \( T = 0 \) dynamics are different because the equations of motion (38) and (43) differ by a term proportional to \( T \).

The \( T = 0 \) action (40) has invariances which are not shared by the \( T \neq 0 \) action (35). Action (40) is invariant under the scale transformation

\[
\delta x^\mu = \alpha x^\mu \tag{44a}
\]

\[
\delta \lambda = 2\alpha \lambda \tag{44b}
\]

\[
\delta \lambda_1 = 0 \tag{44c}
\]

and also invariant under the conformal transformation

\[
\delta x^\mu = (2x^\mu x^\nu - \eta^\mu\nu x^2)b_\nu \tag{45a}
\]

\[
\delta \lambda = 4\lambda x.b \tag{45b}
\]

\[
\delta \lambda_1 = 0 \tag{45c}
\]

The breaking of the \( D \)-dimensional scale invariance and conformal invariance must
then be related to the appearance of a non-vanishing tension value.

As we saw above, the $T = 0$ string action (40) has two additional space-time invariances when compared with the $T \neq 0$ action (35): scale invariance and conformal invariance. But, as we saw in the last section, the $m = 0$ particle action (20) has three additional space-time invariances when compared with the $m \neq 0$ particle action (13). Here action (40) is not invariant under the string extension of the particle transformation (23). In the next section we show that it is possible to fix a gauge in which the $T = 0$ string is invariant under a transformation that is the extension of the $m = 0$ particle transformation (23). This creates a link between these two particular relativistic objects, in the sense that they have identical extra space-time invariances when compared with their usual ($m \neq 0$ and $T \neq 0$) partners.

4 Gauge-fixed strings

The $\lambda$ variations in equations (44b) and (45b) have the same mathematical structure as the particle variations (21b) and (22b). But equations (44c) and (45c) indicate that $\lambda_1$ is useless in establishing space-time scale and conformal invariances. To clarify this situation we study the reparametrization invariance of the $T \neq 0$ string action (35).

Under the infinitesimal reparametrizations of the world-surface coordinates $\xi^\alpha = (\tau, \sigma)$ given by

$$\xi'^\alpha = \xi^\alpha + \epsilon^\alpha$$  \hspace{1cm} (46)

a generic action functional

$$S[\phi] = \int d^2 \xi L(\phi, \partial_\alpha \phi)$$  \hspace{1cm} (47)
varies as

\[ \delta S = \int d^2 \xi' L(\phi'(\xi'), \partial_{\alpha'} \phi'(\xi')) - \int d^2 \xi L(\phi(\xi), \partial_{\alpha} \phi(\xi)) \]

\[ = \int d^2 \xi \Delta L \quad (48) \]

where the variation \( \Delta L \) is given by

\[ \Delta L = \partial_{\alpha} \epsilon^\alpha L + \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \phi_{,\alpha}} \delta \phi_{,\alpha} \quad (49) \]

and

\[ \delta \phi(\xi) = \phi'(\xi') - \phi(\xi) \quad (50) \]

is the total variation of the generic field \( \phi(\xi) \). The string coordinates \( x^\mu(\tau, \sigma) \) are space-time vectors but behave as \( D \) scalar fields from the world-surface point of view. Under reparametrizations of the world-surface coordinates, their derivatives vary as

\[ \delta(\partial_{\alpha} x^\mu) = -\partial_{\alpha} \epsilon^\beta \partial_\beta x^\mu \quad (51) \]

Under these conditions, the \( T \neq 0 \) string action (35) is reparametrization invariant for

\[ \delta \lambda = (\epsilon'_1 - \dot{\epsilon}_0) \lambda - \lambda \lambda_1 \epsilon'_0 \quad (52a) \]

\[ \delta \lambda_1 = (\epsilon'_1 - \dot{\epsilon}_0) \lambda_1 + \dot{\epsilon}_1 - \epsilon'_0 \lambda_1^2 - T^2 \lambda^2 \epsilon'_0 \quad (52b) \]

Reparametrization invariance of the \( T = 0 \) string action (40) is obtained by taking this limit in equations (52) above. In particular, equation (52b) shows that there is enough reparametrization freedom to impose the gauge \( \lambda_1(\tau, \sigma) = 0 \). In this gauge
action (35) simplifies to

\[ S = \frac{1}{2} \int d\tau d\sigma (\lambda^{-1} \dot{x}^2 - \lambda T^2 \dot{x}^2) \]  

(53)

Constructing the Hamiltonian formalism for action (53) we find the constraint that the canonical momentum conjugate to \( \lambda \) must vanish. The stability of this constraint requires \( \phi \) of equation (31a) as a secondary constraint, and the stability of \( \phi \) requires \( \phi_1 \) of equation (31b) as a ternary one. The inherent string structure, together with the gauge algebra (36), are preserved in action (53).

The classical equation of motion that follows from action (53) is

\[ \frac{\partial}{\partial \tau} \left( \frac{1}{\lambda} \dot{x}^\mu \right) - T^2 \frac{\partial}{\partial \sigma} (\lambda \dot{x}^\mu) = 0 \]  

(54)

We can now use the residual reparametrization invariance to choose \( \lambda = 1 \) and get the wave equation

\[ \frac{\partial^2 x^\mu}{\partial \tau^2} - T^2 \frac{\partial^2 x^\mu}{\partial \sigma^2} = 0 \]  

(55)

Now, if we set \( T = 1 \), we return to the conformal equation of motion (39). The general equation of motion (54) shows again that when \( T = 0 \) the string dynamics is modified. In particular, equation (55) shows that when \( T = 0 \) the string is no longer described by a wave equation.

Setting \( T = 0 \) in (53), we obtain the string action

\[ S = \frac{1}{2} \int d\tau d\sigma \lambda^{-1} \dot{x}^2 \]  

(56)

To be consistent, action (56) must be complemented with the constraints

\[ \phi = p^2 \approx 0 \]  

(57a)

\[ \phi_1 = p \dot{x} \approx 0 \]  

(57b)
which are the $T = 0$ limits of constraints (31). The classical equation of motion for $x^\mu$ that follows from the string action (56) is the $T = 0$ limit of equation (54), namely

$$\frac{\partial}{\partial \tau} \left( \frac{1}{\lambda} \dot{x}^\mu \right) = 0$$

(58)

If we now choose $\lambda = \text{cons tan} t$ we find that under this condition

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = 0$$

(59)

and every point $\sigma$ of the tensionless string moves as if it were an almost free massless particle. These massless particles are not completely free but are held together in a string structure in consequence of the presence of constraint (57b). Action (56) has an extra space-time invariance when compared with the $T = 0$ action (40).

Action (56) is invariant under the scale transformation

$$\delta x^\mu = \alpha x^\mu$$

(60a)

$$\delta \lambda = 2\alpha \lambda$$

(60b)

and also invariant under the conformal transformation

$$\delta x^\mu = (2x^\mu x^\nu - \eta^\mu\nu x^2) b_\nu$$

(61a)

$$\delta \lambda = 4\lambda x.b$$

(61b)

When the $T = 0$ string satisfies the equation of motion (59), action (56) is also
invariant under the “gauge” transformation

\[ x_\mu(\tau, \sigma) \rightarrow \exp\left(\frac{1}{3}\gamma(\dot{x}^2)\right)x_\mu(\tau, \sigma) \]  
\[ \lambda \rightarrow \exp\left(\frac{2}{3}\gamma(\dot{x}^2)\right)\lambda \]  

with \( \gamma \) an arbitrary function. Transformation (62) is the string extension of the massless particle transformation (23).

5 Conclusion

In this work we treat the relativistic bosonic particle and the relativistic bosonic string as constrained Hamiltonian systems with first-class constraints only. After the elimination of the canonical momenta using their equations of motion we obtain a particle action that allows transition to the massless limit and a string action that gives the classical conformal wave equation of motion without use of two-dimensional Weyl scalings an which allows transition to the tensionless limit. It was shown that the \( m = 0 \) particle action has three extra space-time invariances if compared to the \( m \neq 0 \) action.

It was also shown that there exists a \( T = 0 \) string that can be required to obey the classical conformal wave equation and that has two extra space-time invariances if compared to his \( T \neq 0 \) partner.

Finally, we pointed out that there also exists a simpler, gauge-fixed , consistent \( T \neq 0 \) string which can be required to satisfy the classical conformal wave equation but that his corresponding \( T = 0 \) string can not. This \( T = 0 \) string obeys particle-like equations of motion which are consistent only if \( p.\dot{x} = 0 \). This particular string can have three extra space-time invariances if compared to his \( T \neq 0 \) version. We interpret it as an intermediate state between a string-like ordered system of particles and a \( T \neq 0 \) string. This particular string seems to agree with the observation of
Sundborg [24] and Witten [25], based on duality between gauge theory and string theory, that when the gauge theory is in the weak coupling regime the string tension effectively tends to zero.

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