SDDs are Exponentially More Succinct than OBDDs

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Abstract

Introduced by Darwiche [7], sentential decision diagrams (SDDs) are essentially as tractable as ordered binary decision diagrams (OBDDs), but tend to be more succinct in practice. This makes SDDs a prominent representation language, with many applications in artificial intelligence and knowledge compilation.

We prove that SDDs are more succinct than OBDDs also in theory, by constructing a family of boolean functions where each member has polynomial SDD size but exponential OBDD size. This exponential separation improves a quasipolynomial separation recently established by Razgon [13], and settles an open problem in knowledge compilation [7].

1 Introduction

The idea of knowledge compilation is to deal with the intractability of certain computational tasks on a knowledge base by compiling it into a different data structure where the tasks are feasible. The choice of the target data structure involves an unavoidable trade-off between succinctness and tractability.

Darwiche and Marquis [5] systematically investigated this trade-off in the fundamental case where the knowledge bases are boolean functions and the data structures are classes of boolean circuits (representation languages).

In their setting, decomposable negation normal forms (DNNFs) and ordered binary decision diagrams (OBDDs) arise as benchmark languages for succinctness and tractability respectively [6, 5]. On the one hand, DNNFs are exponentially more succinct than OBDDs; moreover, in contrast to OBDDs, they implement efficiently conjunctive normal forms of small treewidth [6, 12, 8, 14]. On the other hand, the vast applicability of OBDDs in verification and synthesis relies on the tractability of equivalence testing (speeded up by canonicity) and boolean combinations, which DNNFs lack [5].

This gap between DNNFs (succinct but hard) and OBDDs (verbose but tractable) led to the quest for intermediate languages exponentially more succinct than, but essentially as tractable as, OBDDs.

Introduced by Darwiche [7], sentential decision diagrams (SDDs) are a most prominent candidate to narrow the gap between DNNFs and OBDDs. They are designed by strengthening the decomposability property [10] and further imposing a very strong form of determinism [11]. The resulting language can implement decisions of the form

\[ \bigvee_{i=1}^{m} P_i(X) \land S_i(Y), \]  

where \( X \) and \( Y \) are disjoint sets of variables nicely structured by an underlying variable tree, and the subcircuits \( P_1, \ldots, P_m \), called primes\(^1\), implement an exhaustive case distinction into exclusive and consistent cases\(^2\). Binary (or Shannon) decisions in OBDDs boil down to very special sentential decisions having the form

\[ (\neg x \land S_1(Y)) \lor (x \land S_2(Y)), \]

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\(^1\)The \( S_i \)'s are called subs.
\(^2\)Formally, the models of \( P_1, \ldots, P_m \) partition the set of assignments of \( X \) to \( \{0, 1\} \) into \( m \) nonempty blocks; see Section 2.
where the variable $x$ is not in the variable set $Y$.

Indeed, SDDs properly contain OBDDs, and hence are at least as succinct as OBDDs, while preserving tractability of all key tasks that are tractable on OBDDs. For this reason, they have been used in a variety of applications in artificial intelligence and probabilistic reasoning, as reported, for instance, by [2].

Not only SDDs are as tractable as OBDDs, but they also tend to be more succinct than OBDDs in practice; in fact, knowledge compilers often produce much smaller SDDs than OBDDs by heuristically leveraging the additional flexibility of variable trees in SDDs with respect to variable orderings in OBDDs [4].

Nonetheless, the basic theoretical question about the relative succinctness of OBDDs and SDDs has been open since Darwiche introduced SDDs [7]:

**Are SDDs exponentially more succinct than OBDDs?**

The results in the literature did not even exclude the possibility for OBDDs to polynomially simulate SDDs [10], until recently Razgon proved a quasipolynomial separation [13]. The above question stands, though, as for instance OBDDs could still quasipolynomially simulate SDDs.

**Contribution.** We prove in this article that SDDs are exponentially more succinct than OBDDs. Thus, in particular, OBDDs cannot quasipolynomially simulate SDDs.

More precisely, we construct an infinite family of boolean functions such that every member of the family has polynomial compressed SDD size but exponential OBDD size (Theorem 4).

Compressed SDDs contain OBDDs [3], and are regarded as a natural SDD class because of their canonicity: two compressed SDDs computing the same function are syntactically equal up to syntactic manipulations preserving polynomial size [7]. The restriction to compressed SDDs makes our result stronger, because general SDDs are believed (despite not known) to be exponentially more succinct than compressed SDDs [2].

We separate compressed SDDs and OBDDs by a function, which we call the **generalized hidden weighted bit** function because, indeed, it contains the hidden weighted bit function (HWB) as a subfunction. HWB is perhaps the simplest function known to be hard on OBDDs [3]: it computes the subsets of \( \{1, \ldots, n\} \) having size \( i \) and containing the number \( i \), for \( i = 1, \ldots, n \).

It turns out that HWB itself has small (uncompressed) SDDs (Theorem 3), which immediately separates SDDs and OBDDs. The construction, a slight variation of which gives the compressed case (Lemma 1 and Lemma 2), is based on the following two observations.

The first observation is that HWB can be expressed as a sentential decision of the form \( \top \) by distinguishing the following primes:

- for \( i = 1, \ldots, n \), the subsets of size \( i \) containing the number \( i \) (each of these \( n \) primes is taken by HWB, so their subs will be equivalent to \( \top \));
- the empty subset, and the subsets of size \( i \) not containing the number \( i \) for \( i = 1, \ldots, n - 1 \) (none of these \( n \) primes is taken by HWB, so their subs will be equivalent to \( \bot \)).

The second observation is that each of the above primes has small OBDD size under any variable ordering (Proposition 2). With these two observations it is fairly straightforward to implement the hidden weighted bit function by a small (uncompressed) SDD (Theorem 3).

A direct inspection of our construction allows to straightforwardly derive some facts about compression previously observed in the literature [2], namely that the SDD size may increase exponentially either by compressing SDDs over fixed variable trees, or by conditioning (unboundedly many variables) over fixed variable trees (see Section 4).

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3More precisely, compressed SDDs contain reduced OBDDs; see [15, Definition 1.3.2].
Organization. The article is organized as follows. In Section 2 we present the technical background, culminating in the quasipolynomial separation of SDDs and OBDDs proved by Razgon (Theorem 1). In Section 3 we separate (uncompressed) SDDs and OBDDs by the hidden weighted bit function (Theorem 3) and then modify the construction to separate compressed SDDs and OBDDs (Theorem 4). We discuss our results in Section 4.

2 Background

We collect background notions and facts from the literature [5, 10, 11, 13].

Structured Deterministic NNFs. Let $X$ be a finite set of variables. Let $C$ be a boolean circuit on input variables $X$, built using fanin 0 constant gates (labelled by $\bot$ or $\top$), fanin 1 negation gates (labelled by $\neg$), and unbounded fanin disjunction and conjunction gates (labelled by $\lor$ and $\land$). The unique sink node (outdegree 0) in the underlying directed acyclic graph (DAG) of $C$ is called the output gate of $C$; source nodes (indegree 0) are called input gates, and are labelled by constants or variables in $X$; in particular, $C$ is allowed to not read some of the variables in $X$, see Figure 1 (left).

A boolean circuit $C$ on variables $X$ is in negation normal form, in short an NNF, if the gates labelled by $\neg$ have wires only from input gates. Without loss of generality we assume that NNFs have input gates labelled by constants or literals on variables in $X$ (and no internal gates labelled by $\neg$).

As usual, an NNF $C$ on input variables $X$ computes a boolean function $f : \{0, 1\}^X \to \{0, 1\}$; in this case we also write $C \equiv f$. Two NNFs $C$ and $C'$ on the same input variables are equivalent if they compute the same boolean function; again we write $C \equiv C'$.

The size of an NNF $C$, in symbols $\text{size}(C)$, is the number of arcs in its underlying DAG. Let $f$ be a boolean function and let $\mathcal{L}$ be a class of NNFs. The size of $f$ relative to $\mathcal{L}$ (or, in short, the $\mathcal{L}$ size of $f$), denoted by $\mathcal{L}(f)$, is equal to the minimum over the sizes of all circuits in $\mathcal{L}$ computing $f$:

$$\mathcal{L}(f) = \min\{\text{size}(C) : C \in \mathcal{L}, C \equiv f\}.$$ 

Let $C$ be an NNF on input variables $X$, and let $g$ be a gate of $C$. We denote by $C_g$ the subcircuit of $C$ having $g$ as its output gate, that is, the circuit whose underlying DAG is the subgraph of the underlying DAG of $C$ induced by the nodes having a directed path to $g$ (labelled as in $C$).

An NNF $C$ on input variables $X$ is deterministic if, for every $\lor$-gate $g$ in $C$, say of the form $\lor_{i=1}^m g_i$, it holds that

$$C_{g_i} \land C_{g_j} \equiv \bot$$

for all $1 \leq i < j \leq m$, where we formally regard $C_{g_i}$, $C_{g_j}$, and $\bot$ as NNFs on input variables $X$. We denote by $\mathcal{NNF}_d$ the class of all deterministic NNFs.

Let $Y$ be a finite nonempty set of variables. A variable tree (in short, a vtree) for the variable set $Y$ is a rooted, full, ordered, binary tree $T$ whose leaves correspond bijectively to $Y$; indeed, we identify each leaf in $T$ with the variable in $Y$ it corresponds to.

Let $v$ be an internal node of the vtree $T$. We let $v_l$ and $v_r$ denote respectively the left and right child of $v$, and $T_v$ denote the subtree of $T$ rooted at $v$. We also let $Y_v \subseteq Y$ denote (the variables corresponding to) the leaves of $T_v$; clearly $T_v$ is a vtree for the variable set $Y_v$.

Let $C$ be an NNF on input variables $X$, and let $T$ be a vtree for the variable set $Y$.

We say that $C$ respects $T$ if the following holds. First, every $\land$-gate $g$ in $C$ has fanin exactly 2. Second, let $g$ be an $\land$-gate in $C$ having wires from gates $h_1$ and $h_2$. Then there exists an internal node $v$ in $T$ such that the input gates of the subcircuit $C_{h_1}$ mention only variables in $T_{h_1}$ and the input gates of the subcircuit $C_{h_2}$ mention only variables in $T_{h_2}$. In this case, we also say that $g$ respects $v$.

Note that, in particular, the sets of variables mentioned by $C_{h_1}$ and $C_{h_2}$ are disjoint; it follows that $C$ is decomposable [8]. Also note that, by definition, if an NNF reading all the variables in a set $X$ is structured by a vtree for the variable set $Y$, then $X \subseteq Y$ and the inclusion can be strict; see Figure 1. This feature is crucial in our construction (see, for instance, the proof of Theorem 3). A structured NNF is an NNF respecting some vtree. See Figure 1. We denote by $\mathcal{NNF}_s$ the class of all structured NNFs.
SDDs and OBDDs. A sentential decision diagram (SDD) \( C \) respecting a vtree \( T \) is defined inductively as follows.

- \( C \) is a single gate labelled by a literal on a variable \( x \), and \( x \) is in the variable set of \( T \).
- \( C \) is a single gate labelled by a constant, and \( T \) is any vtree.
- \( C \) is formed by an output gate \( g \) labelled by \( \lor \), with \( m \geq 2 \) wires from gates \( g_1, \ldots, g_m \) labelled by \( \land \), where each \( g_i \) has wires from two gates \( p_i \) and \( s_i \), that is,
  \[
  C = \bigvee_{i=1}^{m} C_{p_i} \land C_{s_i},
  \]
  such that for some internal node \( v \) of \( T \) the following holds (\( i = 1, \ldots, m \)):

  \( \mathbf{S1} \) \( C_{p_i} \) is an SDD respecting a subtree of \( T_v \).
  \( \mathbf{S2} \) \( C_{s_i} \) is an SDD respecting a subtree of \( T_v \).
  \( \mathbf{S3} \) \( C_{p_i} \not\equiv \perp \).
  \( \mathbf{S4} \) \( C_{p_i} \land C_{p_j} \equiv \perp \) (\( 1 \leq i < j \leq m \)).
  \( \mathbf{S5} \) \( \bigvee_{i=1}^{m} C_{p_i} \equiv \top \).

In the equivalences in (S3)-(S5), we formally regard the \( C_{p_i} \)'s, \( \perp \) and \( \top \) as NNFs on variables \( Y_v \). In words, conditions (S3)-(S5) say that the \( C_{p_i} \)'s define a partition of \( \{0, 1\}^{Y_v} \) into \( m \) nonempty blocks, where the \( i \)th block contains exactly the models of \( C_{p_i} \) (\( i = 1, \ldots, m \)).

An SDD is an SDD respecting some vtree. We let \( \text{SDD} \) denote the class of all SDDs.

An ordered binary decision diagram (OBDD) is a compressed SDD respecting a right-linear vtree \( T \); see Figure 2. We let \( \text{OBDD} \) denote the class of all OBDDs\(^4\).

Let \( C \) be an OBDD respecting a vtree \( T \), and let \( \sigma = x_1 < \cdots < x_n \) be the variable ordering induced by a left first traversal of \( T \); in this case, we also say that \( C \) respects \( \sigma \). For an ordering \( \sigma \) of a set of variables, we let \( \text{OBDD}_\sigma \) denote the class of all OBDDs respecting \( \sigma \).

\(^4\)Reduced OBDDs as usually defined in the literature [15, Definition 1.3.2] are indeed compressed SDD respecting right-linear vtrees [7, Section 6].
Quasipolynomial Separation. It follows from the definitions that

$$\text{OBDD} \subseteq \text{SDD}_c \subseteq \text{SDD} \subseteq \text{NN}_s \cap \text{NN}_d$$

which raises the natural question how OBDDs and SDDs are related in succinctness; indeed, the quest for the relative succinctness of OBDDs and SDDs has been an open problem in knowledge compilation since Darwiche introduced SDDs [7].

Recently, Razgon [13, Corollary 3] has established a quasipolynomial separation of OBDDs from compressed SDDs.

Theorem 1 (Razgon). There exists an unbounded arity class of boolean functions $\mathcal{F}$ such that every arity $n$ function $f \in \mathcal{F}$ has $\text{SDD}_c$ size in $O(n^3)$ and $\text{OBDD}$ size in $n^{\Omega(\log n)}$.

We remark that the restriction to compressed SDDs in the above statement is nontrivial; to the best of our knowledge, compressed SDDs might be exponentially more succinct than uncompressed SDDs [2]; see also the discussion in Section 4.

3 Exponential Separation

The quasipolynomial separation stated in Theorem 1 implies that OBDDs do not simulate SDDs in polynomial size, but leaves open the possibility for OBDDs to simulate SDDs in quasipolynomial size. In this section we exclude this possibility by establishing an exponential separation of OBDDs from compressed SDDs.

Hidden Weighted Bit. The separation is obtained by (a variant of) the hidden weighted bit function

$$\text{HWB}_n(x_1, \ldots, x_n),$$

that is the boolean function on $n$ inputs $x_1, \ldots, x_n$ such that, for all assignments $f: \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}$, it holds that $f$ is a model of $\text{HWB}_n$ if and only if $f(x_1) + \cdots + f(x_n) = i$ and $f(x_i) = 1 \ (i \geq 1)$.

It is well known that the hidden weighted bit function has exponential OBDD size [3].

Theorem 2 (Bryant). The OBDD size of $\text{HWB}_n$ is $2^{\Omega(n)}$.

Intuitively, a model of $\text{HWB}_n$ is a subsets of $\{1, \ldots, n\}$ of size $i$ containing the number $i$, for $i = 1, \ldots, n$. For instance, $\text{HWB}_2(1, 0) = 1$, because the set $\{1\}$ has size 1 and contains the number 1, and $\text{HWB}_2(0, 1) = 0$, because the set $\{2\}$ has size 1 but does not contain the number 1.

The simple but crucial observation underlying our construction is that the models of $\text{HWB}_n$ can be decided arguing by cases, as follows: If $S$ is a subset of $\{1, \ldots, n\}$ of size $i$, then $S$ is a model of $\text{HWB}_n$ if and only if $i \in S \ (i = 1, \ldots, n)$. With this insight it is not hard to setup an exhaustive and exclusive case distinction equivalent to $\text{HWB}_n$; the key observation is that each individual case in the distinction is computable by a small OBDD with respect to any variable ordering.

We formalize the above intuition. For $i \in \{0, 1, \ldots, n\}$, let

$$E^i_n(x_1, \ldots, x_n)$$
be the boolean function on \( n \) inputs \( x_1, \ldots, x_n \) such that, for all assignments \( f: \{x_1, \ldots, x_n\} \rightarrow \{0, 1\} \), it holds that \( f \) is a model of \( E_i^n \) if and only if \( f(x_1) + \cdots + f(x_n) = i \). Hence \( E_i^n \) computes the subsets of \( \{1, \ldots, n\} \) of size \( i \) \((i \geq 0)\). Let now

\[
\mathcal{P}_n = \{P_0, P_n\} \cup \{P_{i,0}, P_{i,1}: i = 1, \ldots, n-1\}
\]

(4)

be the family of \( 2n \) boolean functions, each over the variables \( \{x_1, \ldots, x_n\} \), defined as follows:

- \( P_0 \equiv E_0^n \)
- \( P_n \equiv E_n^n \)

and for \( i = 1, \ldots, n-1 \) let

- \( P_{i,0} \equiv E_i^n \land \neg x_i \)
- \( P_{i,1} \equiv E_i^n \land x_i \)

See Figure 4 for an illustration.

Each function in \( \mathcal{P}_n \) computes a family of subsets of \( \{1, \ldots, n\} \). Namely, \( P_0 \) computes the empty subset, \( P_n \) computes \( \{1, \ldots, n\} \), \( P_{i,0} \) computes the subsets of \( \{1, \ldots, n\} \) of size \( i \) not containing the number \( i \), and \( P_{i,1} \) computes the subsets of \( \{1, \ldots, n\} \) of size \( i \) containing the number \( i \) \((i = 1, \ldots, n-1)\).

It is readily observed that the members of \( \mathcal{P}_n \) partition the powerset of \( \{1, \ldots, n\} \) in nonempty blocks. Formally,

**Fact 1.** Let \( \mathcal{P}_n \) be as in (4), and let \( P, P' \in \mathcal{P}_n \) with \( P \neq P' \).

- \( P \neq \bot \).
- \( P \land P' \equiv \bot \).
- \( \lor_{P \in \mathcal{P}_n} P \equiv \top \).

We now establish the key property, that each member of \( \mathcal{P}_n \) is computable by a small OBDD with respect to any variable ordering.

First consider the functions \( E_i^n \). An OBDD computing \( E_i^n \) with respect to the variable ordering \( \sigma = x_1 < \cdots < x_n \) is displayed in Figure 3 for the case \( n = 4 \) and \( i = 2 \). Generalizing the construction, we have that an OBDD \( C \) computing \( E_i^n \) and respecting \( \sigma \) has at most \( 1 + 2 + \cdots + n = n(n+1)/2 \) decision nodes, each contributing 6 wires in the circuit; hence \( C \) has size \( O(n^2) \).

Since \( E_i^n \) is symmetric \([15] \) Definition 2.3.2 and Lemma 4.7.1\), the following holds.

**Proposition 1.** Let \( \sigma \) be an ordering of \( x_1, \ldots, x_n \). The OBDD_\( \sigma \) size of \( E_i^n \) is \( O(n^2) \).

It follows that every \( P \in \mathcal{P}_n \) has a small OBDD with respect to every variable ordering.
Theorem 3. The SDD size of HWB\(_n\) is \(O(n^3)\).  

**Proof.** We first define an NNF \(C\) on input variables \(X = \{x_1, \ldots, x_n\}\) computing \(E^1_n \land \neg x_1\) as follows. The output gate of \(C\) is a fanin 2\(n\) \lor-gate, with wires from 2\(n\) fanin 2 \land-gates \(g_0, g_n, \ldots, g_n\) for \(i = 1, \ldots, n-1\) and \(j = 0, 1\).  

Let \(p_0\) and \(s_0\) be the two gates wiring \(g_0\), let \(p_n\) and \(s_n\) be the two gates wiring \(g_n\), and for \(i = 1, \ldots, n-1\) and \(j = 0, 1\) let \(p_{i,j}\) and \(s_{i,j}\) be the two gates wiring \(g_{i,j}\).  

Let \(\sigma\) be any ordering of \(x_1, \ldots, x_n\). All the subcircuits of \(C\) rooted at \(p_0, s_0, p_n, s_n, p_{i,j}, \) and \(s_{i,j}\) \((i = 1, \ldots, n-1, j = 0, 1)\) are OBDDs respecting the ordering \(\sigma\). Moreover:

- \(C_{p_i}\) computes \(P_i\) for \(i \in \{1, n\}\);  
- \(C_{p_{i,j}}\) computes \(P_{i,j}\) for \(i = 1, \ldots, n-1, j = 0, 1\);  
- \(C_{s_0}\) and \(C_{s_{i,0}}\) compute \(\bot\) for \(i = 1, \ldots, n-1\);  
- \(C_{s_n}\) and \(C_{s_{i,1}}\) compute \(\top\) for \(i = 1, \ldots, n-1\).
We prove that $C$ is an SDD respecting a suitable vtree $T$ for the variable set $X \cup \{y\}$. Roughly, $T$ is a right-linear vtree with the exception of the variable $y$; see the diagram on the right in Figure 4 for the case $n = 4$ and $\sigma = x_1 < x_2 < x_3 < x_4$. Formally, $T$ is defined as follows. Let $v$ be the root of $T$. The left subtree $T_l = T_{n_0}$ of $T$ is a right-linear vtree for $\{x_1, \ldots, x_n\}$ such that the variable ordering induced by its left first traversal is $\sigma$. Similarly, the right subtree $T_r = T_{n_1}$ of $T$ is a vtree for $\{y\}$.

We check that $C$ is an SDD respecting $T$.

- The subcircuits $C_{p_0}$, $C_{p_n}$, and $C_{p_{i,j}}$ are OBDDs respecting $\sigma$, and hence SDDs respecting $T_l$ ($i = 1, \ldots, n-1$, $j = 0, 1$). This settles (S1).

- The subcircuits $C_{s_0}$, $C_{s_n}$, and $C_{s_{i,j}}$ are input gates labelled by a constant, and hence SDDs respecting $T_r$ ($i = 1, \ldots, n-1$, $j = 0, 1$). This settles (S2).

Note how the construction crucially exploits the special position of $y$ in the vtree $T$, while the circuit $C$ does not even read $y$.

The partitioning properties (S3)-(S5) follow by construction and Fact 1. Therefore, $C$ is an SDD respecting $T$. It remains to check that $C$ has size cubic in $n$.

By construction, $C$ contains the $2n$ subcircuits $C_{p_0}$, $C_{p_n}$, and $C_{p_{i,j}}$ for $i = 1, \ldots, n-1$ and $j = 0, 1$; each has size $O(n^2)$ by Proposition 2; hence, altogether, they contribute $O(n^3)$ wires in $C$. There remain $O(n)$ wires entering the output gate and the gates $g_0, g_1, \ldots, g_n$.

Combining Theorem 2 and Theorem 3, we conclude that OBDDs and SDDs are exponentially separated by the hidden weighted bit function.

**Compressed SDDs vs OBDDs.** A slight variant of the previous construction gives an exponential separation of OBDDs and compressed SDDs.

Let $y_0, y_1, \ldots, y_n$ be fresh variables. The boolean function $F_n$ of the variables $x_1, \ldots, x_n, y_0, y_1, \ldots, y_n$, called *generalized hidden weighted bit function*, is defined by

$$(P_0 \land \neg y_0) \lor (P_n \land y_n) \lor \bigvee_{i=1}^{n-1} ((P_{i,0} \land \neg y_i) \lor (P_{i,1} \land y_i)).$$

(7)

Notice that the form (7) is exactly as the form (6), except that the $n$ copies of $\bot$ and the $n$ copies of $\top$ are replaced by the $2n$ pairwise nonequivalent formulas $\neg y_0$, $y_n$, $\neg y_i$ ($i = 1, \ldots, n-1$), so that (7) has indeed a compressed SDD implementation. The details follow.

**Lemma 1.** The SDD size of $F_n$ is $O(n^3)$.

**Proof.** We construct an NNF $C$ on input variables $X = \{x_1, \ldots, x_n, y_0, y_1, \ldots, y_n\}$ computing $F_n$ along the lines of Theorem 3. The only modification is that $C_{s_0}$ is an input gate labelled $\neg y_0$, $C_{s_n}$ is an input gate labelled $y_n$, $C_{s_{i,0}}$ is an input gate labelled $\neg y_i$, and $C_{s_{i,1}}$ is an input gate labelled $y_i$ ($i = 1, \ldots, n-1$).

We claim that $C$ is a compressed SDD respecting a vtree $T$ for the variable set $X$ built exactly as in Theorem 3 except that the right subtree $T_r = T_{n_1}$ of $T$ is a right-linear vtree for $\{y_0, y_1, \ldots, y_n\}$ such that the variable ordering induced by its left first traversal is $\rho$. See Figure 5 for the case $n = 4$, $\sigma = x_1 < \cdots < x_4$, and $\rho = y_0 < y_1 < \cdots < y_4$.

To check that $C$ is a compressed SDD respecting $T$, notice that the subcircuits $C_{p_0}$ and $C_{p_{i,j}}$ are OBDDs respecting $\sigma$, and hence compressed SDDs respecting $T_l$ ($i = 1, \ldots, n$, $j = 0, 1$), and the subcircuits $C_{s_0}$ and $C_{s_{i,j}}$ are OBDDs respecting $\rho$, and hence compressed SDDs respecting $T_r$ ($i = 1, \ldots, n$, $j = 0, 1$). Moreover, it is easily verified that the output gate of $C$ is compressed as by condition (C). Hence $C$ is compressed. The rest of the proof is identical to that of Theorem 3.

We now prove that the generalized hidden weighted bit function $F_n$ needs large OBDDs.

**Lemma 2.** The OBDD size of $F_n$ is $2^{\Omega(n)}$. 

Proof. Let $N$ be the size of a smallest OBDD on variables $X = \{x_1, \ldots, x_n, y_0, y_1, \ldots, y_n\}$ computing $F_n$, and let $\rho$ be any ordering of $X$ such that $\text{OBDD}_\rho(F_n) = N$.

Let $G_n(x_1, \ldots, x_n)$ be the subfunction of $F_n$ where $y_0, y_1, \ldots, y_n$ are replaced by 1, in symbols:

$$G_n \equiv F_n(x_1, \ldots, x_n, 1, 1, \ldots, 1).$$

Since conditioning (unboundedly many variables of) an OBDD does not increase its size \cite[Theorem 2.4.1]{1}, we have that

$$\text{OBDD}_\rho(G_n) \leq \text{OBDD}_\rho(F_n).$$

We now claim that $G_n$ is the hidden weighted bit function on $n$ variables. Indeed, by construction,

$$G_n \equiv F_n(x_1, \ldots, x_n, 1, 1, \ldots, 1)$$

$$\equiv P_n \lor \bigvee_{i=1}^{n-1} P_{i,1}$$

which we already observed being equivalent to $\text{HWB}_n$. Therefore $\text{OBDD}(G_n) = 2^{\Omega(n)}$ by Theorem \ref{thm:hwb} and in particular $\text{OBDD}_\rho(G_n) \geq 2^{\Omega(n)}$. By \cite{9}, we are done.

An exponential separation of OBDDs and compressed SDDs follows.

**Theorem 4.** There exists an unbounded arity class of boolean functions $\mathcal{F}$ such that every arity $n$ function $f \in \mathcal{F}$ has SDD size in $O(n^3)$ and OBDD size in $2^{\Omega(n)}$.

**Proof.** Take $\mathcal{F} = \{F_m : m \in \mathbb{N}\}$, where $F_m$ is as in \cite{7}. Then $F_m$ has compressed SDD size $O(m^3)$ by Lemma \ref{lem:sdd} and OBDD size $2^{\Omega(m)}$ by Lemma \ref{lem:obdd}. Since $F_m$ has $n = 2m + 1$ variables, it follows that $F_m$ has SDD size in $O(n^3)$ and OBDD size in $2^{\Omega(n)}$.

Notably, the function class giving the exponential separation is as hard on compressed SDDs as the function class giving the quasipolynomial separation (cubic in both cases, see Theorem \ref{thm:hard}).

4 Discussion

We have shown that OBDDs and SDDs are exponentially separated by the hidden weighted bit function, while OBDDs and compressed SDDs are exponentially separated by the generalized hidden weighted bit function, $F_n$ in \cite{7}, that contains the hidden weighted bit function as a subfunction:

$$F_n(x_1, \ldots, x_n, 1, 1, \ldots, 1) = \text{HWB}_n(x_1, \ldots, x_n).$$

Separating OBDDs and SDDs by the hidden weighted bit function, instead of by a function designed adhoc, further corroborates the theoretical quality of SDDs. As articulated by Bollig et al. \cite{11}, any useful extension of OBDDs is expected to implement the hidden weighted bit function efficiently.
The SDD $C$ described in the proof of Theorem 3 is not compressed, because $\bot$ and $\top$ are reused $n$ times. In view of the canonical construction of an SDD over a vtree [7, Theorem 3], it is readily observed that compressing $C$ with respect to the vtree $T$ in the proof of Theorem 3 implies finding a small SDD for HWB$_n$ with respect to the left subtree of $T$, that is, a small OBDD for HWB$_n$; but this is impossible by Theorem 2. The fact that compressing an SDD over its vtree may increase the size exponentially has been observed already [2, Theorem 1]. We reiterate the observation here only because our argument is significantly shorter.

We conclude mentioning a nonobvious, and perhaps even unexpected, aspect of our separation result. An inspection of our construction shows that SDDs are already exponentially more succinct than general OBDDs even allowing only one sentential decision (and possibly many Shannon decisions); recall [6] and [7]. The construction by Xue et al. [16] already uses nested sentential decisions even to separate OBDDs over a fixed variable ordering from SDDs!

Questions. We do not know whether the hidden weighted bit function has superpolynomial compressed SDD size for all vtrees; a positive answer would separate compressed and uncompressed SDDs in succinctness and, in view of Lemma 1 and 10, would prove that compressed SDDs do not support conditioning (of unboundedly many variables) in polynomial size.

In view of Theorem 1, it is natural to ask which SDDs are quasipolynomially simulated by OBDDs. Our separating family shows that SDDs with unbounded fanin disjunctions cannot be quasipolynomially simulated by OBDDs. On the other hand, recent work by Darwiche and Oztok essentially shows that SDDs over binary disjunctions (fanin 2) admit a quasipolynomial simulation by OBDDs [9, Theorem 1]. In this light, it is tempting to conjecture that the above criterion is exact, that is, every SDD class over bounded fanin disjunctions does indeed admit a quasipolynomial simulation by OBDDs.

Finally, a natural question arising in the context of the present work is about the relative succinctness of SDDs and structured deterministic NNFs (see [3]); to the best of our knowledge, the question is open. By Theorem 3, at least we now know that the hidden weighted bit function is not a candidate to separate the two classes.

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References
[1] Beate Bollig, Martin Löbbing, Martin Sauerhoff, and Ingo Wegener. On the Complexity of the Hidden Weighted Bit Function for Various BDD Models. Theoretical Informatics and Applications 33(2):103–116, 1999.
[2] Guy van den Broek and Adnan Darwiche. On the Role of Canonicity in Knowledge Compilation. In Bonet, B., and Koenig, S., eds., Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, January 25-30, 2015, Austin, Texas, USA., 1641–1648. AAAI Press, 2015.
[3] Randal E. Bryant. Graph-Based Algorithms for Boolean Function Manipulation. IEEE Transactions on Computers 35(8):677–691, 1986.
[4] Arthur Choi and Adnan Darwiche. Dynamic Minimization of Sentential Decision Diagrams. In desJardins, M., and Littman, M. L., eds., Proceedings of the Twenty-Seventh AAAI Conference on Artificial Intelligence, July 14-18, 2013, Bellevue, Washington, USA., 187–194. AAAI Press, 2013.
[5] Adnan Darwiche and Pierre Marquis. A Knowledge Compilation Map. Journal of Artificial Intelligence Research 17:229–264, 2002.
[6] Adnan Darwiche. Decomposable Negation Normal Form. *Journal of the ACM* 48(4):608–647, 2001.

[7] Adnan Darwiche. SDD: A New Canonical Representation of Propositional Knowledge Bases. In Walsh, T., ed., *IJCAI 2011, Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence, Barcelona, Catalonia, Spain, July 16-22, 2011*, 819–826. IJCAI/AAAI, 2011.

[8] Umut Oztok and Adnan Darwiche. CV-Width: A New Complexity Parameter for CNFs. In Schaub, T.; Friedrich, G.; and O’Sullivan, B., eds., *ECAI 2014, Proceedings of the Twenty-First European Conference on Artificial Intelligence, 18-22 August 2014, Prague, Czech Republic, August 18-22, 2014*, volume 263 of *Frontiers in Artificial Intelligence and Applications*, 675–680. IOS Press, 2014.

[9] Umut Oztok and Adnan Darwiche. A Top-Down Compiler for Sentential Decision Diagrams. In Yang, Q., and Wooldridge, M., eds., *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015, Buenos Aires, Argentina, July 25-31, 2015*, 3141–3148. AAAI Press, 2015.

[10] Knot Pipatsrisawat and Adnan Darwiche. New Compilation Languages Based on Structured Decomposability. In Fox, D., and Gomes, C. P., eds., *Proceedings of the Twenty-Third AAAI Conference on Artificial Intelligence, AAAI 2008, Chicago, Illinois, USA, July 13-17, 2008*, 517–522. AAAI Press, 2008.

[11] Thammanit Pipatsrisawat and Adnan Darwiche. A Lower Bound on the Size of Decomposable Negation Normal Form. In Fox, M., and Poole, D., eds., *Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2010, Atlanta, Georgia, USA, July 11-15, 2010*. AAAI Press, 2010.

[12] Igor Razgon and Justyna Petke. Cliquewidth and Knowledge Compilation. In Järvisalo, M., and Gelder, A. V., eds., *SAT 2015, Proceedings of the Sixteenth International Conference on Theory and Applications of Satisfiability Testing, Helsinki, Finland, July 8-12, 2013*, volume 7962 of *Lecture Notes in Computer Science*, 335–350. Springer, 2013.

[13] Igor Razgon. On OBDDs for CNFs of Bounded Treewidth. *CoRR* abs/1308.3829v3, 2014.

[14] Igor Razgon. On OBDDs for CNFs of Bounded Treewidth. In Baral, C.; Giacomo, G. D.; and Eiter, T., eds., *KR 2014, Proceedings of the Fourteenth International Conference on Principles of Knowledge Representation and Reasoning, Vienna, Austria, July 20-24, 2014*. AAAI Press, 2014.

[15] Ingo Wegener. *Branching Programs and Binary Decision Diagrams*. SIAM, 2000.

[16] Yexiang Xue, Arthur Choi, and Adnan Darwiche. In Hoffmann, J., and Selman, B., eds., *Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence, July 22-26, 2012, Toronto, Ontario, Canada*. AAAI Press, 2012.