ON THE QUANTUM COHOMOLOGY OF BLOW-UPS OF PROJECTIVE SPACES ALONG LINEAR SUBSPACES.

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Abstract. We give an explicit presentation with generators and relations of the quantum cohomology ring of the blow-up of a projective space along a linear subspace.

1. Introduction.

Let $\mathbb{P}^m$ be the complex projective space, $\Lambda \subset \mathbb{P}^m$ a linear subspace of dimension $p$ and $\alpha: \tilde{\mathbb{P}}^m \to \mathbb{P}^m$ the blow-up of $\mathbb{P}^m$ along $\Lambda$. Let $k$ be the hyperplane class on $\mathbb{P}^m$ and $\eta$ the exceptional divisor on $\tilde{\mathbb{P}}^m$. The aim of the present paper is to show a way to compute the quantum cohomology ring $\mathbb{H}_Q(\tilde{\mathbb{P}}^m)$ of $\tilde{\mathbb{P}}^m$. The (classical) cohomology ring of $\tilde{\mathbb{P}}^m$ can be expressed as (compare with lemma 2.5)

$$H^*(\tilde{\mathbb{P}}^m) = \mathbb{Z}[k, \eta]/(g_1, g_2),$$

where the two relations are

$$g_1(k, \eta) = (k - \eta)^{m-p}, \quad g_2(k, \eta) = k^{p+1}\eta.$$  

(2)

The main result of this work is the following

Theorem 1.1 (Main theorem). Suppose that $2p + 3 < m$. Then the quantum cohomology ring of $\tilde{\mathbb{P}}^m$ can be expressed as

$$H^*_Q(\tilde{\mathbb{P}}^m) = \mathbb{Z}[k, \eta]/(\tilde{g}_1, \tilde{g}_2),$$

where the two relations are

$$\tilde{g}_1(k, \eta) = (k - \eta)^{m-p} - \eta, \quad \tilde{g}_2(k, \eta) = k^{p+1}\eta - 1.$$  

(4)

The blow-up $\tilde{\mathbb{P}}^m$ can be regarded as a projective bundle on a projective space (proposition 2.1). The quantum cohomology of projective bundles on projective spaces was studied by Qin and Ruan in [8]. The relevant material from their work will be enclosed here for completeness. Other studies on quantum cohomology of blow-ups of projective spaces can be found in [5], [3] and [4].

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2. $\tilde{\mathbb{P}}^m$ as projective bundle.

Let $n := m - p - 1$, $r := p + 2$ and $V$ be the rank-$r$ vector bundle on $\mathbb{P}^n$ given by

$$V := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes r-1} \oplus \mathcal{O}_{\mathbb{P}^n}(2).$$

(5)
and consider the associate $\mathbb{P}^{r-1}$-bundle $\pi: \mathbb{P}(V) \to \mathbb{P}^n$.

**Proposition 2.1.** The two varieties $\tilde{\mathbb{P}}^m$ and $\mathbb{P}(V)$ are isomorphic.

**Proof.** Consider $\mathbb{P}^m$ as the projective space $\mathbb{P}(U)$ of lines of a vector space $U$ of dimension $m + 1$ and $\Lambda = \mathbb{P}(U_0)$, where $U_0$ is a subspace of $U$ of dimension $p + 1$ (We will use “$\mathbb{P}$” for projective spaces of hyperplanes and “$\mathbb{P}^m$” for projective spaces of lines). Two subspaces $L, W$ of $U$ of dimension 1 and $p + 2$ determine points $[L] \in \mathbb{P}(U)$ and $[W/U_0] \in \mathbb{P}(U/U_0)$ respectively. The blow-up $\tilde{\mathbb{P}}^m$ can be identified with the closed incidence subvariety $Y \subset \mathbb{P}(U) \times \mathbb{P}(U/U_0)$ defined by $Y := \{(L, [W/U_0]) : L \subset W\}$ and $\alpha: \tilde{\mathbb{P}}^m \to \mathbb{P}^m$ is the projection on the first factor restricted to $Y$:

$$
\begin{align*}
Y & \longrightarrow \mathbb{P}(U/U_0) & ([L], [W/U_0]) & \longrightarrow [W/U_0] \\
\mathbb{P}(U) & \downarrow & \downarrow & \\
& [L]
\end{align*}
$$

Note that $Y \to \mathbb{P}(U/U_0)$ is a projective bundle $\mathbb{P}(E) \to \mathbb{P}(U/U_0)$, where $E$ is the vector bundle $E = \{(w, [W]) \in U \times \mathbb{P}(U/U_0)\}$. We want to show that $\mathbb{P}(E)$ and $\mathbb{P}(V)$ are isomorphic.

The exact sequence $0 \to U_0 \to W \to U/U_0 \to 0$ gives rise to an exact sequence of vector bundles

$$ 0 \to U_0 \otimes \mathcal{O}_{\mathbb{P}^n} \to E \to \mathcal{O}_{\mathbb{P}^n}(-1) \to 0. $$

The above sequence splits $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}(-1), U_0 \otimes \mathcal{O}_{\mathbb{P}^n}) = 0$, so $E = (U_0 \otimes \mathcal{O}_{\mathbb{P}^n}) \oplus \mathcal{O}_{\mathbb{P}^n}(-1)$. Since $V = E^*(1)$, then $\mathbb{P}(V) \simeq \mathbb{P}(E^*(1)) \simeq \mathbb{P}(E^*) \simeq \mathbb{P}(E)$. \hfill \square

The exceptional locus $\alpha^{-1}(\Lambda) \simeq \Lambda \times \mathbb{P}^n$ of the blow-up $\alpha: \tilde{\mathbb{P}}^m \to \mathbb{P}^m$ corresponds to the trivial sub-bundle $\mathbb{P}(U_0 \otimes \mathcal{O}_{\mathbb{P}^n}(1))$ of $\pi: \mathbb{P}(V) \to \mathbb{P}^n$ under the isomorphism of proposition 2.1. The two restrictions $\alpha|_{\Lambda \times \mathbb{P}^n}: \Lambda \times \mathbb{P}^n \to \Lambda$ and $\pi|_{\mathbb{P}(U_0 \otimes \mathcal{O}_{\mathbb{P}^n}(1))}: \mathbb{P}(U_0 \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \to \mathbb{P}^n$ are the canonical projections of the product $\mathbb{P}(U_0) \times \mathbb{P}^n$.

Let $h$ be the hyperplane bundle on $\mathbb{P}^n$ and $\xi = \xi_V$ the tautological line bundle on $\mathbb{P}(V)$. We will make no distinction between $h$ and their corresponding pull-back $\pi^*h$ on $\mathbb{P}(V)$. Note that $\xi - h$ and $h$ are nef on $\mathbb{P}(V)$. It is well known that the classical cohomology ring of $\mathbb{P}(V)$ is generated by $h$, $\xi$ with the two relations

$$
\begin{align*}
(\xi - h)A_1 &= 1, \\
(\xi - h)A_2 &= 0, \\
hA_1 &= 0, \\
hA_2 &= 1.
\end{align*}
$$

The homology classes $A_1 := (h^na_{r-2})_*$ and $A_2 := (h^{n-1}a_{r-1} - rh^na_{r-2})_*$ form a system of generators for $H_2(\mathbb{P}(V))$ (the stars stand for the Poincare dual)
Proposition 2.2. The variety $\mathbb{P}(V)$ is Fano and the homology classes $A_1$, $A_2$ are extremal rays for $NE(\mathbb{P}^n)$. The canonical divisor is $-K = -K_{\mathbb{P}(V)} = r(\xi - h) + nh$.

Proof. In fact, $-K = r\xi + (n + 1 - c_1(V))h = r(\xi - h) + nh$. Let $C$ be an effective curve and $[C] = a_1A_1 + a_2A_2$. From (10) we have $(\xi - h) \cdot [C] = a_1$ and $h \cdot [C] = a_2$. Since $\xi - h$ and $h$ are nef, $a_1 \geq 0$ and $a_2 \geq 0$. Moreover, if $C$ is not constant, $a_1 > 0$ or $a_2 > 0$ and $-K \cdot [C] = ra_1 + na_2 > 0$. Since $h$ and $\xi - h$ are globally generated, $\mathbb{P}(V)$ is Fano.

Lemma 2.3. The classes $A_1$ and $A_2$ can be represented by rational connected curves as follows:

(i) $A_1 = [\ell]$, where $\ell$ is a line contained in a fiber of $\pi: \mathbb{P}(V) \to \mathbb{P}^n$.

(ii) $A_2 = [\tilde{\ell}]$, where $\tilde{\ell} := \left\{ [P] \times \ell \subset \mathbb{P}(U_0) \times \mathbb{P}^n \subset \mathbb{P}(V) \right\}$ with $P$ a point in $\mathbb{P}(U_0)$ and $\ell$ a line in $\mathbb{P}^n$.

Moreover, there are no other means to represent $A_1$ and $A_2$ as class of a rational connected curve beside (i) and (ii).

Proof. (i) If $\ell$ is a line in a fiber $\pi$, then $\xi \cdot [\ell] = 1$ and $h \cdot [\ell] = 0$, hence $[\ell] = A_1$ by (10). Conversely, let $\ell$ be a rational connected curve in $\mathbb{P}(V)$ such that $A_1 = [\ell]$. The curve $E$ is contained in a fiber $\pi^{-1}(Q)$ as $[\pi(\ell)] = h^n$. Since $\xi \cdot A_2 = 1$, then $\xi|_{\pi^{-1}(Q)} \cdot [\ell] = 1$ in $\pi^{-1}(Q)$, that is $\ell$ is a line in $\pi^{-1}(Q)$.

(ii) Now let $\ell \subset \mathbb{P}^n$ be a line and $P \in \mathbb{P}^n$. Note that $(\xi - h)|_{\mathbb{P}(U_0) \otimes \mathbb{P}^n(1)} = \xi V(-1)|_{\mathbb{P}(U_0) \otimes \mathbb{P}^n}$ is the pull-back of the hyperplane divisor of $\mathbb{P}(U_0)$ through the canonical projection $\mathbb{P}(U_0) \times \mathbb{P}^n \to \mathbb{P}(U_0)$. Then $(\xi - h) \cdot [\ell] = 0$ and $h \cdot [\ell] = 1$ therefore $[\ell] = A_2$ by (10). Conversely, let $\ell$ be a rational connected curve in $\mathbb{P}(V)$ such that $A_2 = [\tilde{\ell}]$ and $\ell = \pi(\tilde{\ell})$. Since $h \cdot [\tilde{\ell}] = 1$, then $\pi|_{\tilde{\ell}}: \tilde{\ell} \to \ell$ is an isomorphism and $\ell$ is a line in $\mathbb{P}^n$. The inclusion $\ell \hookrightarrow \mathbb{P}(V)$ is induced by a surjective morphism $f: V|_{\ell} \to \mathcal{O}_{\ell}(t)$, where $t = c_1(\xi|_{\ell}) = \xi \cdot [\tilde{\ell}] = 1$ (see [6] II Prop. 7.12).

Every morphism $V|_{\ell} \to \mathcal{O}_{\ell}(1)$ factors through $V|_{\ell} \to U_0 \otimes \mathcal{O}_{\ell}(1) \to \mathcal{O}_{\ell}(1)$; hence $\tilde{\ell}$ is contained in $\mathbb{P}(U_0) \times \mathbb{P}^n$ and has the form $\{P\} \times \ell$. □

So far we have described classes $A_1$ and $A_2$ in the language of projective bundles, now we are going to describe them in the language of blow-ups. Note that the preimage $\alpha^{-1}(\Lambda)$ of $\Lambda$ under the blow-up morphism $\alpha: \mathbb{P}^m \to \mathbb{P}^n$ is isomorphic to $\Lambda \times \mathbb{P}^n$ and $\alpha|_{\Lambda \times \mathbb{P}^n}: \Lambda \times \mathbb{P}^n \to \Lambda$ is the canonical projection on the first factor. Let $\tilde{\ell}_1$ be the strict transform in the blow-up of a line $\ell_1 \subset \mathbb{P}^n$ not included in $\Lambda$ and meeting $\Lambda$ in one point and $\tilde{\ell}_2 := \{P\} \times \ell_2 \subset \Lambda \times \mathbb{P}^n = \alpha^{-1}(\Lambda)$ where $P$ is a point of $\Lambda$ and $\ell_2$ is a line in $\mathbb{P}^n$.

Lemma 2.4. Under the isomorphism $\mathbb{P}^m \simeq \mathbb{P}(V)$, homology classes $B_1 = [\tilde{\ell}_1]$ and $B_2 = [\tilde{\ell}_2]$ correspond respectively to $A_1$ and $A_2$.

Proof. It is obvious that $B_2$ corresponds to $A_2$ since both are described as the class of a curve of the form $\{P\} \times \ell \subset \mathbb{P}(U_0) \times \mathbb{P}^n$.

With notations from proposition 2.1, $\ell_1$ is determined by a plane $S_{\ell_1} \subset U$ such that $S_{\ell_1} \cap U_0$ has dimension 1 and $\tilde{\ell}_1$ can be described as the subset
of $Y$ of points of the form $([L],[S_{\ell_1}])$ for all subspaces $L \subset S_{\ell_1}$ of dimension 1. Then $\ell_1$ can be regarded as a line in the fiber of the projective bundle $\pi: Y \to \mathbb{P}(U/U_0)$ over the point $[S_{\ell_1}/U_0]$, that is $B_1$ corresponds to $A_1$. □

**Corollary 2.5.** Under the isomorphism $\tilde{\mathcal{P}}^m \simeq \mathbb{P}(V)$, cohomology classes $k$, $\eta$ correspond respectively to $\xi - h$, $\xi - 2h$. The enumerative interpretation of the Gromov-Witten invariants holds, that is, if $\alpha$, $\beta$, $\gamma$ denote, as usual, the Gromov-Witten invariant. If one assume the genericity condition

\[ k \cdot B_1 = 1, \quad k \cdot B_2 = 0, \quad \eta \cdot B_1 = 1, \quad \eta \cdot B_2 = -1 \]

and from the previous lemma. □

### 3. Study of $\mathcal{M}(A,0)$.

Let $\mathcal{M}(A,0)$ be the moduli space of morphisms $f: \mathbb{P}^1 \to \mathbb{P}(V)$ with $[\text{Im } f] = A$ where $A$ is a class in $H_2(\mathbb{P}(V))$. Recall that the virtual dimension of $\mathcal{M}(A,0)$ is

\[ \text{virtdim}(\mathcal{M}(A,0)) = -K \cdot A + n + r - 1. \]

**Lemma 3.1.** The moduli space $\mathcal{M}(A_1,0)/\text{PSL}(2,\mathbb{C})$ is smooth, compact and has expected dimension.

*Proof.* By lemma 2.3, $\mathcal{M}(A_1,0)/\text{PSL}(2,\mathbb{C}) \simeq G(2,r) \times \mathbb{P}^n$, where $G(2,r)$ is the grassmannian of lines in $\mathbb{P}^{r-1}$; then it is smooth, compact and has dimension $\dim G(2,r) + \dim \mathbb{P}^n = n + 2r - 4 = -K \cdot A_1 + \dim \mathbb{P}(V) - 3$. □

**Lemma 3.2.** The moduli space $\mathcal{M}(A_2,0)/\text{PSL}(2,\mathbb{C})$ is smooth, compact and has expected dimension.

*Proof.* By lemma 2.3, $\mathcal{M}(A_2,0)/\text{PSL}(2,\mathbb{C})$ is a $\mathbb{P}(U_0)$-bundle on $G(2, n+1)$. Then it is smooth, compact and has dimension $\dim G(2, n+1) + \dim \mathbb{P}(U_0) = 2n + r - 4 = -K \cdot A_2 + \dim \mathbb{P}(V) - 3$. □

### 4. Computation of Gromov-Witten invariants.

If $A$ belongs to $H_2(\mathbb{P}(V))$ and $\alpha$, $\beta$, $\gamma$ are classes in $H^*(\mathbb{P}(V))$, $I_A(\alpha, \beta, \gamma)$ denotes, as usual, the Gromov-Witten invariant. If one assume the genericity condition

\[ (*) \text{ the moduli space } \mathcal{M}(A,0)/\text{PSL}(2,\mathbb{C}) \text{ is smooth, compact, of expected dimension } -K_{\mathbb{P}(V)} \cdot A + (n + r - 1) - 3 \]

the enumerative interpretation of the Gromov-Witten invariants holds, that is, if $B$, $C$, $D$ are three sub-varieties of $\mathbb{P}(V)$ in general position representing classes $\alpha$, $\beta$, $\gamma$, then $I_A(\alpha, \beta, \gamma)$ is the number of rational curves, counted with suitable multiplicity, that intersect $B$, $C$ and $D$.

We recall that a sufficient condition for the smoothness of the space $\mathcal{M}(A,0)/\text{PSL}(2,\mathbb{C})$ is given by the vanishing of $h^1(N_f)$ for every map $f \in \mathcal{M}(A,0)$, where $N_f$ is the normal bundle. Moreover, if $A$ can be represented only by irreducible and reduced curves, $\mathcal{M}(A,0)/\text{PSL}(2,\mathbb{C})$ is compact.
Lemma 4.1. If \( q_1 + q_2 < br \), then \( I_{\theta A_1}(h^{p_1} q_1, h^{p_2} q_2, \alpha) = 0 \).

Proof. Let \( B, C, D \) be three varieties in general position dual to \( h^{p_1} q_1, h^{p_1} q_2, h^{p_2} q_2 \) respectively. It is enough to show that there is no effective connected curve \( E \) representing \( A \) and intersecting \( B, C, D \). In fact, the genericity condition (*) can be relaxed by assuming (compare with [8] Lemma 3.7):

\[
h^{1}(N_{\ell}) = 0 \text{ for every } \ell \in \mathbb{M}(A, 0) \text{ such that } \text{Im}(f) \text{ intersects } B, C, D \text{ and there is no reducible or non reduced effective (connected) curve } E \text{ such that } |E| = A \text{ and } E \text{ intersects } B, C, D.
\]

We can choose \( \alpha \) of the form \( h^{p} q^{q} \) with \( q \leq r - 1 \) and \( p + p_1 + p_2 + q + q_1 + q_2 = n + r - 1 + br \). Notice that \( \pi(B), \pi(C), \pi(D) \) in \( \mathbb{P}^n \) are dual to \( h^{p_1}, h^{p_2}, h^{p} \). Since \( A_1 \) is an extremal ray, every irreducible component of \( E \) is a multiple of \( A_1 \) and hence contained in a fiber of \( \pi \). Therefore the whole \( E \) is contained in a fiber of \( \pi \) as \( E \) is connected. We deduce that \( \pi(B) \cap \pi(C) \cap \pi(D) \neq \emptyset \); thus \( p + p_2 + p \leq n \) and \( q_1 + q_2 = n + r - 1 + br - p_1 - p_2 - p - q \geq br \).

Lemma 4.2. We have \( I_{\theta A_1}(\xi, \xi^{-1}, h^{n} \xi^{-1}) = 1 \).

Proof. By lemma 3.1, \( \mathbb{M}(A_1, 0) \) is smooth, compact, of expected dimension. The class \( h^{n} \xi^{-1} \) is the dual of a point \( q \) in \( \mathbb{P}(V) \). Let \( p := \pi(q) \). A parametrized curve in \( \mathbb{M}(A_1, 0) \) meeting \( q \) has support on a line \( \ell \subset \pi^{-1}(q) \). Since \( \xi \mid_{\pi^{-1}(q)} \) is the cohomology class of a hyperplane in \( \pi^{-1}(q) \), then \( I_{\theta A_1}(\xi, \xi^{-1}, h^{n} \xi^{-1}) = 1 \).

Lemma 4.3. Let \( \tilde{n} \) be an integer and \( 1 \leq \tilde{n} \leq n \). Then

\[
\begin{align*}
I_{\theta A_2}(h^{\tilde{n}}, h^{n+1-\tilde{n}}, h^{n} \xi^{-2}) &= 1, \\
I_{\theta A_2}(h^{\tilde{n}}, h^{n+1-\tilde{n}}, h^{n-1} \xi^{-1}) &= r - 1.
\end{align*}
\]

Proof. By lemma 3.2, \( \mathbb{M}(A_2, 0) \) is smooth, compact, of expected dimension. As explained in lemma 2.3, \( A_2 \) is represented by curves \( \ell \) in \( \mathbb{P}(U_0) \times \mathbb{P}^n \subset \mathbb{P}(V) \) and \( \ell := \pi(\tilde{\ell}) \subset \mathbb{P}^n \) is a line.

First we prove equation (14). Let \( B, C, D \) be three sub-varieties of \( \mathbb{P}(V) \) in general position representing \( h^{\tilde{n}}, h^{n+1-\tilde{n}}, h^{n} \xi^{-2} \); then \( \pi(B), \pi(C) \) are linear subspaces of \( \mathbb{P}^n \) of dimension \( \tilde{n}, n - 1 \) and \( \pi(D) \) is a point. Assume that \( A_2 = [\tilde{\ell}] \) and \( \tilde{\ell} \) intersects \( B, C, D \). Then \( \ell \) is uniquely determined as the only line meeting \( \pi(B), \pi(C) \) and \( \pi(D) \). Since \( D \) is a line in the fiber of \( \pi : \mathbb{P}(V) \to \mathbb{P}^n \), then \( D \cap (\mathbb{P}(U_0) \times \mathbb{P}^n) \) is a set of a single point \( P \in \mathbb{P}(V) \).

It follows that \( \tilde{\ell} = \{P\} \times \ell \subset \mathbb{P}(U_0) \times \mathbb{P}^n \); hence (14) holds.

Now let us show that

\[
I_{\theta A_2}(h^{\tilde{n}}, h^{n+1-\tilde{n}}, h^{n-1} \xi^{-1} + (1 - r) h^{n} \xi^{-2}) = 0.
\]

By linearity, after the proof of equation (14) this is equivalent to equation (15). Let \( \ell' \) be a general line in \( \mathbb{P}^n \) and \( f : V|_{\ell'} \to \mathcal{O}_\ell(2) \) be a general surjective morphism of vector bundles. Let \( \tilde{\ell}' \) be the image of the induced embedding \( \mathbb{P}(f) : \mathbb{P}^1 = \mathbb{P}(\mathcal{O}_\ell(2)) \to \mathbb{P}(V|_{\ell'}) \). The curve \( \tilde{\ell}' \) represents \( h^{n-1} \xi^{-1} + (1 - r) h^{n} \xi^{-2} \) since \( \xi \cdot [\tilde{\ell}] = 2 \) and \( h \cdot [\tilde{\ell}] = 1 \). Observe that \( \tilde{\ell} \cap \tilde{\ell}' = \emptyset \), indeed, the two trivial sub-bundles \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(2)) \) and \( \mathbb{P}(U_0 \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \)
of $\mathbb{P}(V)$, which contain $\tilde{\ell}$ and $\tilde{\ell}'$ respectively, do not intersect. This implies equation (16).

5. The Quantum Cohomology Ring of $\mathbb{P}(V)$: proof of the main theorem.

As we are going to work with two ring structures, classical and quantum, on the cohomology group $H^*(\mathbb{P}(V))$, we have to fix some notation. If $\alpha, \beta$ are cohomology classes in $H^*(\mathbb{P}(V))$ then $\alpha^i \ast \beta^j$ is the quantum product of $i$ copies of $\alpha$ and $j$ copies of $\beta$. For the classical product we write $\alpha^1 \beta^1$ as usual. For any homology class $A \in H_2(\mathbb{P}(V))$ we denote by $(\alpha \ast \beta)_A$ the contribution of $A$ to the product $\alpha \ast \beta$ defined as follows. Let $d$ be the integer $d := \deg(\alpha) + \deg(\beta) + K \cdot A$. Then $(\alpha \ast \beta)_A = 0$ for $d < 0$. For $d \geq 0$, $(\alpha \ast \beta)_A$ is the class of degree $d$ in $H^*(\mathbb{P}(V))$ satisfying the property

$$(\alpha \ast \beta)_A \cdot \gamma_s = I_A(\alpha, \beta, \gamma)$$

for every $\gamma \in H^*(\mathbb{P}(V))$ with $\deg \gamma = \dim \mathbb{P}(V) - d$, being $I_A(\alpha, \beta, \gamma)$ the Gromov-Witten invariant. The quantum product is given by

$$\alpha \ast \beta = \sum_{A \in H_2(\mathbb{P}(V))} (\alpha \ast \beta)_A$$

$$= \alpha \beta + (\alpha \ast \beta)_{[E_1]} + (\alpha \ast \beta)_{[E_2]} + \ldots$$

for some non constant effective curves $[E_1], [E_2], \ldots$

**Lemma 5.1.** If $r < n$, the two relations

$$(19) \quad h^{*(n+1)} = \xi - 2h; \quad (\xi - h)^{(r-1)} \ast (\xi - 2h) = 1$$

hold in $H^*_Q(\mathbb{P}(V))$.

*Proof.* Since $r = \min(-K \cdot A_1, -K \cdot A_2)$, we have $h^{*p} \ast \xi^q = h^p \xi^q$ whenever $p+q < r$. Moreover, for $p+q < n$, the contributions to the quantum product $h^{*p} \ast \xi^q$ can only arise from curves of type $tA_1$. In particular, lemma 4.1 implies that $h^{*p} \ast \xi^q = h^p \xi^q$ for $p < n$ and $q < r$. By lemma 4.2 we have $\xi^r = 1$, then

$$(\xi - h)^{(r-1)} \ast (\xi - 2h) = (\xi - h)^{r-1}(\xi - 2h) + 1 = 1.$$

Analogously, by lemma 4.3 we have $(h^{*n} \ast h^{n+1-\tilde{n}})_{A_2} = \xi - 2h$, then

$$h^{*(n+1)} = h^{*n} \ast h^{*(n+1-\tilde{n})} = h^{*n} \ast h^{n+1-\tilde{n}}$$

$$= h^{n+1} + (h^{*n} \ast h^{n+1-\tilde{n}})_{A_2} = \xi - 2h. \square$$

We are finally ready to prove the main result of this work.

*Proof of theorem 1.1.* If $2p + 3 < m$, one has $r < n$ and lemma 5.1 applies. Relations (19) can be easily translated to the ring $H^*_Q(\mathbb{P}^m)$ with the aid of lemma 2.5, obtaining the two relations (4). Then $\tilde{g}_1, \tilde{g}_2$ are the classical relations (2) evaluated in the quantum cohomology ring. By theorem 2.2 in [9], the quantum cohomology ring of $H^*_Q(\mathbb{P}^m)$ is the ring generated by $k$, $\eta$ with the two relations $\tilde{g}_1, \tilde{g}_2$ (compare also with proposition 10 of [2]). \square
References

[1] Arnaud Beauville. Quantum cohomology of complete intersections. Mat. Fiz. Anal. Geom., 2(3-4):384–398, 1995, arXiv:alg-geom/9501008.

[2] William Fulton and Rahul Pandharipande. Notes on stable maps and quantum cohomology. In Kollár, Janos, et al., editors, Algebraic geometry., number 62 in Proc. Symp. Pure Math., pages 45–96. American Mathematical Society., July 1997, arXiv:alg-geom/9608011. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, July 9–29, 1995. Providence.

[3] Andreas Gathmann. Counting rational curves with multiple points and Gromov-Witten invariants of blow-ups, arXiv:alg-geom/9609010.

[4] Andreas Gathmann. Gromov-Witten invariants of blow-ups, arXiv:math.AG/9804043.

[5] Lothar Göttsche and Rahul Pandharipande. The quantum cohomology of blow-ups of $\mathbb{P}^2$ and enumerative geometry. J. Differ. Geom., 48(1):61–90, 1998, arXiv:alg-geom/9611012.

[6] Robin Hartshorne. Algebraic geometry. Number 53 in Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg-Berlin, 1983.

[7] Christian Okonek, Michael Schneider, and Heinz Spindler. Vector bundles on complex projective spaces. Number 3 in Progress in mathematics. Birkhäuser, Boston - Basel - Stuttgart, 1980.

[8] Zhenbo Qin and Yongbin Ruan. Quantum cohomology of projective bundles over $\mathbb{P}^n$. Trans. Am. Math. Soc., 350(9):3615–3638, 1998, arXiv:math.AG/9607223.

[9] Bernd Siebert and Gang Tian. On quantum cohomology rings of Fano manifolds and a formula of Vafa and intriligator. Asian J. Math., 1(4):679–695, 1997, arXiv:alg-geom/9403010.

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