Non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics of relativistic particles with the restriction of mass

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Abstract

The modified Dirac equations for the massive particles with the replacement of the physical mass $m$ with the help of the relation $m \rightarrow m_1 + \gamma_5 m_2$ are investigated. It is shown that for a fermion theory with a $\gamma_5$-mass term, the limiting of the mass spectrum by the value $m_{\text{max}} = m_1^2/2m_2$ takes place. In this case the different regions of the unbroken $\mathcal{PT}$ symmetry may be expressed by means of the restriction of the physical mass $m \leq m_{\text{max}}$. It should be noted that in the approach which was developed by C. Bender et al. for the $\mathcal{PT}$-symmetric version of the massive Thirring model with $\gamma_5$-mass term, the region of the unbroken $\mathcal{PT}$-symmetry was found in the form $m_1 \geq m_2$ [25]. However, on the basis of the mass limitation $m \leq m_{\text{max}}$ we obtain that the domain $m_1 \geq m_2$ consists of two different parametric sectors:

i) $0 \leq m_2 \leq m_1/\sqrt{2}$ - this values of mass parameters $m_1, m_2$ correspond to the traditional particles for which in the limit $m_{\text{max}} \rightarrow \infty$ the modified models are converting to the ordinary Dirac theory with the physical mass $m$;

ii) $m_1/\sqrt{2} \leq m_2 \leq m_1$ - this is the case of the unusual particles for which equations of motion does not have a limit, when $m_{\text{max}} \rightarrow \infty$. The presence of this possibility lets hope for that in Nature indeed there are some "exotic fermion fields".

As a matter of fact the formulated criterions may be used as a major test in the process of the division of considered models into ordinary and exotic fermion theories. It is tempting to think that the quanta of the exotic fermion field have a relation to the structure of the "dark matter".

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1 Introductory remarks

As it is well-known the idea about existence of a maximal mass in a broad spectrum of elementary particles was suggested by M.A. Markov [1]. After that a more radical approach was developed by V.G. Kadyshevsky and his colleagues. Their model contained a limiting mass $M$ as a new fundamental physical constant. Doing this condition of finiteness of the mass spectrum should be introduced by the relation:

$$m \leq M. \quad (1)$$

Really in the papers [2]-[14] the existence of mass $M$ has been understood as a new principle of Nature similar to the relativistic and quantum postulates, which was put into the grounds of the new quantum field theory. At the same time the new constant $M$ is introduced in a purely geometric way, like the velocity of light is the maximal velocity in the special relativity.

However the studying of questions of discrete symmetries of this theory were completely disregarded. That is why many long years nobody takes no notice the existence of non-Hermitian Hamiltonians which arise in the fermion sector of the theory with the maximal mass. Indeed, if one chooses a geometrical formulation of the quantum field theory, the adequate realization of the limiting mass hypothesis is reduced to the choice of the de Sitter geometry as the geometry of the 4-momentum space of the constant curvature with a radius equal to $M$.

For example, introducing the designation of $p_\mu = i\partial_\mu$ and taking into account, on the mass shell $p_5 = \sqrt{M^2 - m^2}$ we have the analogue of the Dirac equation [8],[9]:

$$\left( p_0 - \hat{\alpha} p - \hat{\beta} m_1 - \hat{\beta} \gamma^5 m_2 \right) \Psi(x,t,x_5=0) = 0 \quad (2)$$

were

$$m_1 = 2M \sin \mu/2, m_2 = 2M \sin^2 \mu/2, \sin \mu = m/M. \quad (3)$$

In the modified Dirac equation matrix $\hat{\beta} = \gamma_0, \gamma^i = \hat{\beta} \hat{\alpha}^i$. It is important to note that on the mass shell $p_5 = M \cos \mu$ there are not operators, which act on the coordinate of $x_5$, and this parameter without loss of generality can be set equal to zero [8],[9].
We see that the Hamiltonian which are associated with equations (2) can be represented in the form
\[ \hat{H} = \overrightarrow{\alpha} \overrightarrow{p} + \beta (m_1 + m_2 \gamma_5). \] (4)

It is easily checked that in a flat limit \( M \to \infty \) the (2),(4) goes into the standard Dirac expressions. It is obvious that the expression (4) is non-Hermitian due to the appearance in them of \( \gamma_5 \)-mass components \( \hat{H} \neq \hat{H}^+ \).

A. Mustafazadeh identified the necessary and sufficient requirements of reality of eigenvalues for pseudo-Hermitian and \( \mathcal{PT} \)-symmetric Hamiltonians and formalized the use these Hamiltonians in his papers [15] [16]. According to [15] and [16] we can define Hermitian operator \( \eta \), which transform non-Hermitian Hamiltonian
\[ H^+ = \alpha p + \beta (m_1 - \gamma_5 m_2) \] (5)
by means of invertible transformation to the Hermitian-conjugated one
\[ \eta H \eta^{-1} = H^+, \] (6)
where \( \eta = e^{\gamma_5 \alpha}; \alpha = \arctan(m_2/m_1) \) [14].

Now it is well-known fact the reality of the spectrum is a consequence of \( \mathcal{PT} \)-invariance of the theory, i.e. a combination of spatial and temporary parity of the total Hamiltonian: \([H, \mathcal{PT}] \psi = 0\). When the \( \mathcal{PT} \) symmetry is unbroken, the spectrum of the quantum theory is real. These results explain the growing interest in this problem. For the past a few years studied a lot of new non-Hermitian \( \mathcal{PT} \)-invariant systems (see, for example, [15]-[32]). In the literature, which was devoted to the study not Hermitian operators there are examples, with the \( \gamma_5 \) mass extension.

In particular the modified Dirac equations for the massive Thirring model in two-dimensional space-time with the replacement of the physical mass \( m \) by \( m \to m_1 + \gamma_5 m_2 \) was investigated by Bender et al. [25]. The region of the unbroken \( \mathcal{PT} \)-symmetry has been found in the form [25]
\[ m_1 \geq m_2 \] (7).

However, it is not apparent that the area with undisturbed \( \mathcal{PT} \)-symmetry \( m_1 \geq m_2 \) does not include also the sectors, corresponding to the some unusual particles, description of which radically distinguish from traditional
one. These been observed in the paper \cite{14} as "exotic particles" (see also \cite{8}, \cite{9}). Consequently the question arises: "what precisely particles are considered: exotic or traditional, when the condition \eqref{7} is executed?"

Indeed according to the analysis, carried out in the work \cite{14}, the expressions \eqref{2}, \eqref{4} can also be rewritten as

\[ \left( p_0 - \hat{\alpha} p - \hat{\beta} m_3 - \hat{\beta} \gamma^5 m_4 \right) \Psi(x, t, x_5 = 0) = 0 \]  
(8)

and

\[ \hat{H} = \overrightarrow{\hat{\alpha}} \overrightarrow{p} + \hat{\beta} (m_3 + m_4 \gamma_5). \]  
(9)

where \[ m_3 = 2M \cos \mu /2, m_4 = 2M \cos^2 \mu /2 \]  
(10)

The distinguishing feature of expressions \eqref{8}, \eqref{9} consists of the fact that they have not the limit when \( M \to \infty \), i.e. there are not values of parameters in order to obtain the ordinary Dirac expressions. Thus, one can assume that in this case we deal with a description of some new particles, properties of which have not yet been studied.

This paper has the following structure. In section II the necessary and sufficient conditions are formulated for the case of the restriction of the mass spectrum of particles in considered models. In the third section we study the basic characteristics of \( \mathcal{PT} \)-symmetry of free fermion models with \( \gamma_5 \) a massive contribution and show that the area of unbroken \( \mathcal{PT} \) symmetry in which the mass spectrum is real have an internal structure.

2 Necessary and sufficient conditions of the mass spectrum restrictions in the model with \( \gamma_5 \) mass term.

Consider now the algebraic approach developed in the numerous papers on the study of quantum non-Hermitian mechanics. It was possible to expect, that the appearance of the models described by Hamiltonians type \( \text{\ref{4}} \) is the prerogative of a purely geometric approach to the construction of a modified

\footnote{Note that similar designations for the masses \( \text{\ref{34}}, \text{\ref{35}} \) has been used also in the works \cite{34}, \cite{35}.}
theory with a maximal mass. However, in the paper [25] was considered
the $\mathcal{PT}$-symmetric massive Thirring model in $(1+1)$-dimensional space. As
the foundation of this study is assumed the a model with the density of the
Hamiltonian, which is represented in the form:

$$
H(x,t) = \bar{\psi}(x,t) \left( -i \partial \gamma + m_1 + \gamma_5 m_2 \right) \psi(x,t).
$$

(11)

The equation of motion associated with the (11), may be expressed as

$$
\left( i \partial_{\mu} \gamma^{\mu} - m_1 - \gamma_5 m_2 \right) \psi(x,t) = 0,
$$

(12)

that on the form coincides with the equation (2).

In this regard, there is interest in the study of the parameters, character-
izing of the masses, included in the $\gamma_5$ - theory. Note that the $\gamma_5$ - extension
of the mass in the Dirac equation consists of replacing $m \rightarrow m_1 + \gamma_5 m_2$ and
after that two new mass parameters: $m_1$ and $m_2$ arises. When the Dirac
equation converts to the Klein-Gordon equation:

$$
\left( \partial^2 + m^2 \right) \psi(x,t) = 0
$$

(13)

there is a relation

$$
m^2 = m_1^2 - m_2^2.
$$

(14)

It is easy to see that the physical mass $m$, appearing in the equation (13), is
real when the inequality

$$
m_1^2 \geq m_2^2.
$$

(15)

is accomplished.

The algebraic formalism developed in [25], contains no indications of the
existence of other restrictions, in which participates the physical mass of
particles $m$, in addition to (14). However, on the basis of the coincidence of
equations (2) and (12), we can assume that in this model restriction of the
type inequality (11) should also be present. It is very important fact, that
this conditions should be obtained because with its help may be established
the connection between algebraic and geometric approaches. Note also that
the existence of such restrictions, in particular, may essentially modify the
fundamental results obtained in the paper [25].

Let us consider after [25] the relativistic quantum mechanics with the
$\gamma_5$-mass term in the case of 1+1 dimensional space-time. We introduce the
following 2d representation of a \(\gamma\)-matrices [33]:

\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\] (16)

According to these definitions, \(\gamma_0^2 = 1\) and \(\gamma_1^2 = -1\). We also have the

\[
\gamma_5 = -\gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\] (17)

so that \(\gamma_5^2 = 1\).

Consider the Hamiltonians of the type (4) and show that the algebraic approach allows one to set the **sufficient condition** of the limitations of the mass spectrum of particles in relativistic quantum mechanics with \(\gamma_5\)-mass component. As noted above, the inequality (15) was considered in the paper [25] as the single requirement that determines the presence broken or unbroken \(\mathcal{PT}\) - symmetry of the Hamiltonian. However, it is easy to show that (15) may not be as the single condition.

Writing the following obvious inequality:

\[
(m - m_2)^2 \geq 0
\] (18)

and taking into account (14), we obtain

\[
m \leq \frac{m_1^2}{2m_2} = m_{\text{max}},
\] (19)

that is direct indication of the existence of the restriction of the physical mass \(m\) in the considered model.

It is interesting that (19) is obtained as the result of the simple algebraic transformation of relationships with subsidiary parameters of mass \(m_1\) and \(m_2\). It is quite natural that the value of the \(m_{\text{max}}\) is expressed through a combination of them. In particular, as the degree of deviation of the Hamiltonian \(H\) of a Hermitian forms is characterized by the mass \(m_2\), then its value can be expressed through the \(m_1\), \(m_{\text{max}}\) and, taking advantage of the symbol

\[
\frac{m_1}{2m_{\text{max}}} = \sin \theta \leq 1,
\]

rewrite (4), (9) in the resulting form

\[
H = \hat{\alpha} \hat{p} + \hat{\beta} m_1 \left( 1 + \gamma_5 \sin \theta \right),
\] (20)
where the area $0 \leq \theta < \pi/4$ - corresponds to the theory having a ”flat limit” and $\pi/4 < \theta \leq \pi/2$ - refers to the occasion when it is absent. The first condition corresponds to the description of ordinary particles and the second - to unusual or exotic particles. The limit value $\theta = \pi/4$ is responsible for the particles with the maximal mass, which was named by the maximons [1].

Thus, the limitation of the mass spectrum of particles (19), described by Hamiltonians (4), (9) and (20) is a simple consequence of the $\gamma_5$-mass extension in (2), (8). Therefor the presence of the non-Hermitian Hamiltonian (20), in essence, can be interpret as the sufficient condition of the limitation of the mass spectrum of particles in fermion models with $\gamma_5$ - massive term.

On the other hand, we have that the introduction of the new physical postulate, which is connected with the limitations of the mass spectrum, lying in the basis of the a geometric approach to the development of the modified QFT with the Maximal Mass, leads to the appearance of non-Hermitian Hamiltonians in its fermion sector (see for example [8], [9], [14]). This consequence can be interpreting as a necessary condition of the finiteness of the mass spectrum ([11]).

Thus, it is shown that the necessary and a sufficient conditions of the limitation of the physical mass of particles (19) in fermion sector are the using of the non-Hermitian $\mathcal{P}\mathcal{T}$ - symmetric quantum models with a $\gamma_5$ - massive contribution.

3 Free fermion models with $\gamma_5$-massive contributions and the areas of unbroken $\mathcal{P}\mathcal{T}$ symmetry

Conditions (14), (15) are executed automatically, if one enter the following parametrization:

$$m_1 = m \cosh(\alpha); \quad m_2 = m \sinh(\alpha).$$

Moreover, from (19) and (21) we can express the values of $m$, $m_1$, $m_2$ with parameter $\alpha$.

Fig. 1 shows the dependence of the relative values $\nu = m/m_{\text{max}}$, $\nu_1 = m_1/m_{\text{max}}$ and $\nu_2 = m_2/m_{\text{max}}$ from the parameter $\alpha$. The values of the parameters $\nu_1$, $\nu_2$ are the following $\nu \leq \nu_1 \leq 2$, $0 \leq \nu_2 \leq 2$. A particle mass
Figure 1: Dependence of \( \nu = m/m_{\text{max}} , \nu_1 = m_1/m_{\text{max}}, \nu_2 = m_2/m_{\text{max}} \) from the parameter \( \alpha \).

\( \nu \) may vary in a wide range \( 0 \leq m \leq m_{\text{max}} \). When \( \alpha_0 = 0.881 \) it reaches its maximum, which corresponds to the maximon \( (m = m_{\text{max}}) \).

Using (19) and (21), you can also get a

\[
\tanh(\alpha) = \sqrt{1 \pm \sqrt{1 - \nu^2}}.
\]

(22)

Two of the root sign in (22) are interpreted as two branches of the values of \( \nu_1(\tilde{\nu}_1) \) and \( \nu_2(\tilde{\nu}_2) \), which are multi-valued functions \( \nu \). Thus, we have

\[
\nu_1(\tilde{\nu}_1) = \sqrt{2} \sqrt{1 \mp \sqrt{1 - \nu^2}} ;
\]

(23)

\[
\nu_2(\tilde{\nu}_2) = \left( 1 \mp \sqrt{1 - \nu^2} \right).
\]

(24)

It is easy to see, that between these earlier symbols for the masses of (3), (10) and obtained here the values of (23), (24), after identification of the limiting masses \( M \) and \( m_{\text{max}} \), there are simple correlations:
\[ m_1 = m_{\text{max}} \nu_1; \quad m_2 = m_{\text{max}} \nu_2; \]  
\[ \tilde{m}_1 = m_{\text{max}} \tilde{\nu}_1 = m_{\text{max}} \nu_3 = m_3; \quad \tilde{m}_2 = m_{\text{max}} \tilde{\nu}_2 = m_{\text{max}} \nu_4 = m_4. \]

Fig. 2 demonstrates the dependence of the parameters \( \nu_1(\nu_3) \), \( \nu_2(\nu_4) \) from the variable \( \nu \). Thus, the region of the existence of unbroken \( \mathcal{PT} \) symmetry can be represented in the form \( 0 \leq \nu \leq 1 \). For these values of \( \nu \) parameters \( \nu_1 \) and \( \nu_2 \) determine the masses of the modified Dirac equation with a maximal mass \( m_{\text{max}} \), describing the particles having the actual mass \( m \leq m_{\text{max}} \). However, the new Dirac equations nonequivalent, because one of them describes ordinary particles \( (\nu_1, \nu_2) \), and the other corresponds their exotic partners \( (\tilde{\nu}_1, \tilde{\nu}_2) \). The special case of Hermiticity is on the line \( \nu = 1 \) \( (m = m_{\text{max}} \) is the case of the maximon), which is the boundary of the unbroken \( \mathcal{PT} \) - symmetry. In this point of the plot we have \( \nu_1 = \tilde{\nu}_1 = \nu_3 = \sqrt{2} \) and \( \nu_2 = \tilde{\nu}_2 = \nu_4 = 1 \).

It is easy to verify that the new values of the mass parameters \( \tilde{m}_1, \tilde{m}_2 \) still satisfy the conditions (14) and (15). We emphasize once again that, from the formulas (23), (24) in the case of the upper sign it should be \( m_1 \to m \) and \( m_2 \to 0 \) when \( m_{\text{max}} \to \infty \), i.e. there is a so-called flat limit, determined in the frame of the geometric approach [8], [9]. However, when one choose a lower sign (i.e. for the \( \tilde{m}_1 \) and \( \tilde{m}_2 \)) such a limit is absent.

In the frame of the condition \( m_1^2 \geq m_2^2 \) \footnote{Note that this inequality was considered in the paper [25] as a single defining expression} at Fig. 3 we can see three specific sectors of unbroken \( \mathcal{PT} \)-symmetry of the Hamiltonian (20) in the plane \( \nu_1, \nu_2 \). The plane \( \nu_1, \nu_2 \) may be divided by the three groups of the inequalities:

\[ I. \quad \nu_1 / \sqrt{2} \leq \nu_2 \leq \nu_1, \]
\[ II. \quad - \nu_1 / \sqrt{2} \leq \nu_2 \leq \nu_1 / \sqrt{2}, \]
\[ III. \quad - \nu_1 \leq \nu_2 \leq - \nu_1 / \sqrt{2}. \]

Only the area II. corresponds to the description of ordinary particles, then I. and III. agree with the description of some as yet unknown particles. This conclusion is not trivial, because in contrast to the geometric
Figure 2: The values of parameters $\nu_1, \nu_2, \nu_3, \nu_4$ as the function of $\nu$

approach, where the emergence of new unusual properties of particles associated with the presence in the theory a new degree of freedom (sign of the fifth component of the momentum $p_5[9]$), in the case of a simple extension of the free Dirac equation due to the additional $\gamma_5$-mass term, the satisfactory explanation is not there yet.

4 Conclusion

Starting with the researches, presented in the previous sections, we have shown that Dirac Hamiltonian of a particle with $\gamma_5$-dependent mass term is non-Hermitian, and has the unbroken $\mathcal{PT}$-symmetry in the area $m_1^2 \geq m_2^2$, which has three of a subregion. Indeed with the help of the algebraic transformations we obtain a number of the consequence of the relation (14). In particular there is the restriction of the particle mass in this model $m \leq m_{\text{max}}$, were $m_{\text{max}} = m_1^2 / 2m_2$. Outside of this area the $\mathcal{PT}$-symmetry of
The parametric areas of the unbroken $\mathcal{PT}$-symmetry $\nu_1^2 \geq \nu_2^2$ in plane $\nu_1, \nu_2$ for the Hamiltonian \[ \text{(20)} \] consists of three specific subregions. Only the shaded area $II.$ meets the ordinary particles, and the bordering with it regions $I.$ and $III.$ correspond to the description of the exotic fermions.

In addition, we have shown that the introduction of the postulate about the limitations of the mass spectrum, lying in the basis of the geometric approach to the development of the modified QFT (see, for example \cite{8}, \cite{9}), leads to the appearance of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians in the fermion sector of the model with the Maximal Mass. Thus, it is shown that using of non-Hermitian $\mathcal{PT}$-symmetric quantum theory with $\gamma_5$ mass term may be considered as necessary and sufficient conditions the appearance of the limitation of the mass particle (19) in a fermion sector of the model.

In particular, this applies to the modified Dirac equation in which produced the substitution $m \rightarrow m_1 + \gamma_5 m_2$. Into force of the ambiguity of the definition of parameters $m_1, m_2$ the inequality $m_1 \geq m_2$ describes a particle of two types. In the first case, it is about ordinary particles, and when $m_1, m_2 \geq 0$ mass parameters are limited by the terms

$$0 \leq m_2 \leq m_1/\sqrt{2}. \quad (27)$$
In the second area we are dealing with so-called exotic partners of ordinary particles, for which is still accomplished (15), but one can write

\[ m_1 / \sqrt{2} \leq m_2 \leq m_1. \] (28)

Intriguing difference between particles of the second type from traditional fermions is that they are described by the other modified Dirac equations. So, if in the first case (27), the equation of motion there has a limit transition when \( m_{\text{max}} \to \infty \) that leads to the standard Dirac equation, however in the inequality (28) such a limit is not there.

Thus, it is proved that the main progress, obtained by us the in the algebraic way of the construction of the fermion model with \( \gamma_5 \)-dependent mass term applies to the limitations of the mass spectrum. Furthermore, the possibility of describing of the exotic particles are turned out essentially the same as in the model with a maximal mass, which was investigated by V.G.Kadyshevsky with colleagues [2] - [13] on the basis of geometrical approach. It is also shown that the transition point at the scale the masses from the ordinary particles to the exotic this is mass of the maximon.

On the basis of (19) it also has been shown that the parameters \( m_1 \) and \( m_2 \) have the auxiliary nature. This fact is easily proved by means of the comparison of the ordinary and exotic fermion fields. Thus, it is the important conclusion that the description of exotic fields may be considered with the help of the algebraic approach and is not the prerogative of the geometric formalism. Note that the polarization properties of the exotic fermion fields fundamentally differ from the standard fields that with taking into consideration of interactions may be of interest in the future researches.

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