STRUCTURE PRESERVING MODEL ORDER REDUCTION
OF PORT-HAMILTONIAN SYSTEMS

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Abstract. This work proposes a structure-preserving model reduction method
for linear, time-invariant port-Hamiltonian systems. We show that a low
order system of the same type can be constructed which interpolates the
original transfer function in a given set of frequencies.

Keywords: Structure preserving MOR, Symplectic MOR, Port-Hamiltonian sys-
tem.
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1. Introduction

We study structure preserving model reduction (MOR) for port-Hamiltonian
linear, time-invariant (LTI) systems. These are systems of the following form
\[
\dot{x}(t) = (J - R)Hx(t) + Bu(t), \quad y(t) = B^T Hx(t)
\]
where \(x, u, y\) denote the state, input and output, respectively, and the matrices
satisfy \(-J = J^T, R = R^T \geq 0,\) and \(H = H^T > 0.\) The aim is to find a system
of the same form, but with a small state space dimension, such that the two
transfer functions are close to each other.

In many control design methods, like \(LQ\) and \(H^\infty,\) the controller will be of
the same order as the original system, and that is why there is a need for a
reduced model capturing the essentials of the original system. Model reduction
is a well-established field with a long history, \[8, 15,\] but model reduction
for pH-systems is more recent, see e.g. \[1,\] Although the term pH-system
is not used in that overview paper, since many pH-systems have a positive
real transfer function, the result presented there can be used for pH-systems.
Although the first results on model reduction of pH-system/passive system goes
back almost two decades, recently there has been many new approaches to this
question. Moment matching model reduction for pH-systems is done in \[17, 18,\]
Approximation of the underlying Dirac structure is the approach taken in \[19,\]
In \[6\] the authors use the spectral factorisation of the Popov function, i.e.,
\(G(s) + G(-s)^T\) to and the singular values of the solutions to the Lur’e equations
to construct a reduced pH model. In \[2\] the authors design a reduced model
based on LQG balancing.

In this article we combine the techniques of \[16\] with the transfer function
interpolation of \[2,\] Although the transfer function of our reduce model equals

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2. Preliminaries and problem setting

In this section we summarize the fundamentals of MOR and discuss the conventional approach to MOR.

2.1. Model Order Reduction by Projection. Consider the finite-dimensional linear time-invariant (LTI) dynamical system described by the standard state-space model

\[
\Sigma : \begin{cases} 
\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0; \\
y(t) = Cx(t) + Du(t),
\end{cases}
\]

(2)

where \(x \in \mathbb{X} = \mathbb{R}^n\) is the state variable, and \(u \in \mathbb{U} = \mathbb{R}^p\) and \(y \in \mathbb{Y} = \mathbb{R}^p\) are the inputs and outputs of the system, respectively. The matrices \(A, B, C, D\) are constant matrices of appropriate dimensions and called the system matrix, the input matrix, the output matrix and the direct (input-output) matrix, respectively. It is often useful to analyse a dynamical system in frequency domain. By applying Laplace transform to (2) we can obtain its transfer function

\[
G(s) = C(sI_n - A)^{-1}B + D
\]

which is a \(p \times p\)-matrix-valued rational function, describing the input-output mapping in frequency domain.

The model reduction problem can be described as to find a reduced-order LTI system of the similar form as (2), i.e.,

\[
\Sigma_r : \begin{cases} 
\dot{x}_r(t) = A_rx_r(t) + B_ru(t); \quad x_r(0) = x_r0; \\
y_r(t) = C_rx_r(t) + D_ru(t),
\end{cases}
\]

(4)

where \(x_r \in \mathbb{R}^k\) is the state variable of the reduced system with dimension \(k\), \(k \ll n\), such that the reduced-order model’s output \(y_r\) approximates the full-order model’s output \(y\) very closely. In particular, the transfer function of the reduced-model

\[
G_r(s) = C_r(sI_k - A_r)^{-1}B_r + D_r
\]

(5)

should behave similar to that of the full-order model.

We now briefly recall the most standard strategy to project the full equations of the system dynamics onto a lower-dimensional subspace, namely Petrov-Galerkin projective approximation. In this technique, we need to choose two \(r\)-dimensional subspaces \(V_r \subset \mathbb{R}^n\) and \(W_r \subset \mathbb{R}^n\) associated with two basis matrices \(V, W \in \mathbb{R}^{n \times k}\) such that \(V_r = \text{ran}(V)\) and \(W_r = \text{ran}(W)\), respectively. Then the full order state is approximated as \(x(t) \approx Vx_r(t)\) and the residual is constrained to be orthonormal to \(W\) i.e., \(W^T(x(t) - Vx_r(t)) = 0\). We obtain the \(r\)-dimensional reduced model (4) with

\[
A_r = W^TAV, \quad B_r = W^TB, \quad C_r = CV, \quad D_r = D.
\]

(6)

The proper choice of this basis determines the accuracy and effectiveness of the reduced system.
2.1.1. Rational Interpolation. In rational interpolation the projection matrices \( V \) and \( W \) are chosen so that the transfer function \( G(s) = C(sI_n - A)^{-1}B + D \) of the original system (2) and the transfer function \( G_r(s) = C_r(sI_k - A_r)^{-1}B_r + D_r \) of the reduced system (4) (and some of their derivatives) coincide at certain interpolation points \( s \in \mathbb{C} \cup \{\infty\} \). Rational interpolation is a powerful method: almost every reduced LTI system (4) can be obtained via rational interpolation from (2).

Given a full-order system (1), the tangential interpolation problem seeks a reduced system, \( G_r(s) \), that interpolates \( G(s) \) at selected points \( \{s_m\}_{m=1}^M \subseteq \mathbb{C} \) along selected tangent directions \( \{u_m\}_{m=1}^M \subseteq \mathbb{C}^p \):

\[
G(s_m)u_m = G_r(s_m)u_m, \quad \text{for} \quad m = 1, \ldots, M. \tag{7}
\]

Conditions forcing (7) to be satisfied by a reduced system of the form (4) are provided by Theorem 2.1.

**Theorem 2.1** (2). Suppose \( G(s) = C(sI_n - A)^{-1}B + D \). Given a set of distinct interpolation points \( \{s_m\}_{m=1}^M \subseteq \mathbb{C} \) and right tangent directions \( \{u_m\}_{m=1}^M \subseteq \mathbb{C}^p \), define \( V \in \mathbb{C}^{n \times k} \) as

\[
V = [(s_1I_n - A)^{-1}Bu_1, \ldots, (s_kI_n - A)^{-1}Bu_k] \tag{8}
\]

Then for any \( W \in \mathbb{C}^{n \times k} \), the reduced systems \( G_r(s) = C_r(sI_k - A_r)^{-1}B_r + D_r \) defined via (2) satisfies (7), provided that \( s_1I_n - A \) and \( s_kI_k - A_r \) are all invertible.

\[\square\]

2.2. Review of the port-Hamiltonian Representation. The port Hamiltonian (pH) modelling is a generalisation of the classical Hamiltonian framework. Among many other reasons, the pH framework is very useful in engineering problems since it combines the classical Hamiltonian approach with a powerful framework for modelling and simulation of many classes of open physical systems, the network modelling. In this section, we provide a brief introduction to pH modelling, for the details see, e.g., [7], [13], [20], and [24]. We start with recalling the classical Hamiltonian equations of motion.

The standard Hamiltonian equations for a mechanical system are defined by

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}(q, p); \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p), \tag{9}
\]

for \( i = 1, \ldots, n \), where \( H \) is the Hamiltonian of the system that corresponds, for instance to its total energy, \( q = [q_1, q_2, \ldots, q_n]^T \in \mathbb{R}^n \) and \( p = [p_1, p_2, \ldots, p_n]^T \in \mathbb{R}^n \) are position states and momentum states, respectively, and \((q_i, p_i)\)-space refers to the generalized phase space of the system. One can easily check that the Hamiltonian is constant i.e., \( \frac{dH}{dt}(q, p) = 0 \) expressing that the Hamiltonian/energy is conserved within the system.

Now, if we define the state variable \( x = [q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n]^T \in \mathbb{R}^{2n} \) for the phase space, then (9) takes the form

\[
\dot{x}(t) = J_{2n} \frac{\partial H}{\partial x}(x(t)), \tag{10}
\]
where $\frac{\partial H}{\partial x}(x)$ is the gradient of the Hamiltonian function $H$, and the matrix $J_{2n} \in \mathbb{R}^{2n \times 2n}$, defined as
\begin{equation}
J_{2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix},
\end{equation}
is the well-known Poisson matrix, with $0_n \in \mathbb{R}^{n \times n}$ and $I_n \in \mathbb{R}^{n \times n}$ denoting the zero and the identity matrices, respectively.

Adding external forces, the mechanical system becomes
\begin{equation}
\dot{q}_i = \frac{\partial H}{\partial p_i}(q,p); \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q,p) + F_i;
\end{equation}
where $F = [F_1, F_2, \cdots, F_n]^T$ is the vector of generalised forces. One can immediately derive the following balance equation:
\begin{equation}
\frac{d}{dt}H = \frac{\partial}{\partial q} (q,p) \dot{q} + \frac{\partial}{\partial p} (q,p) \dot{p} = \dot{q}^T F,
\end{equation}
expressing the balance between internal and external power. This also motivates to define the output of the system as $y = \dot{q}$, and thus the power balance becomes $\frac{dH}{dt} = y^T F$. With $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$, we formulate a such system as
\begin{equation}
\dot{x}(t) = J_{2n}(x(t)) + Bu(t) \\
y(t) = B^T \frac{\partial H}{\partial x}(x(t)),
\end{equation}
where $u = F$. The above is a port-Hamiltonian (pH) system. Generalisations are possible by inclusion of energy-dissipating elements, and by representing the interconnection with a skew-symmetric matrix rather than the Poisson matrix. This brings us to a formal definition of a general pH-system model.

A time-invariant pH-system without feed-through is a standard explicit input-state-output (ISO) model of the form
\begin{equation}
\Sigma_{\text{pHS}} : \begin{cases}
\dot{x}(t) = (J_{2n}(x) - R(x)) \frac{\partial H}{\partial x}(x) + B(x)u(t), \\
y(t) = B^T(x) \frac{\partial H}{\partial x}(x),
\end{cases}
\end{equation}
with state $x \in \mathbb{X} \subseteq \mathbb{R}^{2n}$, input $u \in \mathbb{U} \subseteq \mathbb{R}^p$ and output $y \in \mathbb{Y} \subseteq \mathbb{R}^p$. The Hamiltonian function $H : \mathbb{X} \rightarrow \mathbb{R}$ characterizes a generalized energy. The skew-symmetric matrix $J_{2n}(x) \in \mathbb{R}^{2n \times 2n}$, $J_{2n}(x) = -J_{2n}(x)^T$ represents the exchange of energy between the storage elements i.e., states $x$ of the system while the dissipation is characterized by the symmetric and positive semi-definite matrix $R(x) \in \mathbb{R}^{2n \times 2n}$. The input matrix $B(x) \in \mathbb{R}^{2n \times p}$ and the gradient $\frac{\partial H}{\partial x}(x)$ of the energy function define the collocated output, which together with the input constitutes the power port $(u, y)$ of the system. When $R = 0$, then $\frac{dH}{dt} = y^T u$, see also [15].

Note that the restriction to even-dimensional spaces is not necessary for a pH-system in general, but is assumed for the symplectic MOR that will be described in the later section. In the case of linear pH-systems, the Hamiltonian equals $H(x) = \frac{1}{2} x^T \mathcal{H} x$ with $\mathcal{H} \in \mathbb{R}^{2n \times 2n}$ symmetric and positive definite. The gradient of the Hamiltonian equals
\begin{equation}
\frac{\partial H}{\partial x}(x) = \mathcal{H} x.
\end{equation}
and the state-space representation (15) becomes

$$
\Sigma_{\text{LpHS}} : \begin{cases}
\dot{x}(t) = (J_{2n} - R)Hx(t) + Bu(t) \\
y(t) = B^T H x(t),
\end{cases}
$$

(17)

where the constant matrices $J_{2n}, R,$ and $B$ are independent of the state vector $x$. The rate of change of energy satisfies the following inequalities:

$$
\frac{d}{dt}(H(x(t))) = \frac{1}{2}(\langle \dot{x}(t), Hx(t) \rangle + \langle Hx(t), \dot{x}(t) \rangle)
= \langle u(t), y(t) \rangle - \langle RHx(t), Hx(t) \rangle \leq \langle u(t), y(t) \rangle
$$

which guarantees the passivity of the system and implies the following dissipation inequality:

$$
H(x(t_1)) - H(x(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt}H(x)dt \leq \int_{t_0}^{t_1} y(t)^T u(t)dt.
$$

(18)

In system theory, the inequality (18) implies that the dynamical system described in (17) is a passive system.

Note that a passive system can be thought of as a system which only stores or releases energy which was provided to the system. It can be analyzed by studying their input-output relationships. The transfer function of the system (17) equals

$$
G(s) = B^T H(sI_{2n} - (J_{2n} - R)H)^{-1} B.
$$

(19)

Note that the transfer function in (19) of the pH-system (17) found as the solution of the following equation:

$$
\begin{bmatrix}
sI_{2n} - (J_{2n} - R)H & -B \\
B^T H & 0
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}
= \begin{bmatrix}
0 \\
y
\end{bmatrix}.
$$

(20)

Namely, for given $u$ find $x$ and $y$ such that (20) is satisfied. When $(sI_{2n} - (J_{2n} - R)H)$ is invertible this is always possible and the (unique) $y$ equals $G(s)u$. A related concept of passive system is the positive realness of its transfer function. The detailed account of the relationship between the passivity property of a system and the positive-realness of its transfer function can be found in [14], [4], [6]. Here we briefly recall the definition of the positive-realness of a transfer function.

**Definition 2.2.** An $m \times m$-matrix valued function $G : \mathbb{C} \mapsto (\mathbb{C} \cup \infty)^{p \times p}$ is **positive real (strictly)** if:

1. $G$ is analytic in $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \text{Re}(s) > 0\}$;
2. $G(s^*) = G(s)^*$ for all $s \in \mathbb{C}_+$;
3. $G(s) + G(s)^* \geq 0$ (> for strict positive realness) for all $s \in \mathbb{C}_+$.

In the case of any matrix-valued transfer function (3) associated to the system (2) with realization $(A, B, C, D)$, the analyticity in $\mathbb{C}_+$, property 1, is related to the stability and causality of the system. The second property implies that the transfer matrix is real for any $s \in \mathbb{R}$, and poles must be real or if complex must appear in conjugate pairs. This condition is always satisfied for a real realization i.e., $(A, B, C, D)$ are constant and real-valued. The positivity condition on $G(s)$, property 3, can be reformulated by using the maximum modulus theorem when
$G(s)$ is assumed to be real-rational. It turns out that the non-negativeness of $G(s) + G(s)^*$ can be checked just along the imaginary axis rather than in the whole right half complex plane.

**Theorem 2.3.** A real rational $m \times m$-matrix valued transfer function $G : \mathbb{C} \mapsto (\mathbb{C} \cup \infty)^{m \times m}$ is positive real(strictly) iff

1. $G(s)$ has no poles in $\text{Re}(s) \geq 0$;
2. $G(j\omega) + G^T(-j\omega) \geq 0$ for all $\omega \in \mathbb{R}$ (> for strict positive realness) s.t. $j\omega$ is not a pole in $G(s)$.

Note that the positive-realness of a transfer function of a passive, stable, LTI system can be determined graphically since the Nyquist plot for such system is always in the right half of the complex plane. We can now formulate the relationship among the pH-system (17), a passive system representation (2), and its transfer function (3).

**Theorem 2.4.** Consider a linear time-invariant system in state-space form (2) and the associated input-output transfer matrix (3). Consider, furthermore, a linear pH-system of the form (17). Then the following statements hold:

1. If $H$ is bounded from below, then the linear pH-system (17) is passive.
2. If the system (2) is passive with a quadratic storage function $\frac{1}{2}x^T H x$ with $H \geq 0$, then (2) can be rewritten into the pH form (17).
3. If the system (2) is passive, then the transfer function (3) is positive real.

**Proof:** Statements 1 & 2 are in [7], and statement 3 is in [14].

2.3. **Problem setting.** It is evident from the previous discussion that the symmetry and definiteness property of the matrices $J, R,$ and $H$ define the pH structure. A structure-preserving MOR technique, therefore, is one that preserves the symmetry and definiteness property of the matrices in the reduced-order system.

As a full-order system, we consider the linear pH-system of the form (17). The MOR problem is to find a reduced-order system of the form:

$$
\Sigma_{rLpHS} : \begin{cases}
\dot{z}(t) = (J_{2k} - \hat{R})\hat{H}z(t) + \hat{B}u(t) \\
\hat{y}(t) = \hat{B}^T\hat{H}z(t),
\end{cases}
$$

(21)

where, $J_{2k}, \hat{R}, \hat{H} \in \mathbb{R}^{2k \times 2k}$ and $\hat{B} \in \mathbb{R}^{2k \times p}$ with $k < n$. Furthermore, $J_{2k}$ is invertible, $J_{2k}^T = -J_{2k}$, $\hat{R}^T = \hat{R} \geq 0$, and $\hat{H}^T = \hat{H} > 0$. Similar to the full-order system, the transfer function of the reduced ordered pH-system equals:

$$
G_r(s) = \hat{B}^T\hat{H}(sI_{2k} - (J_{2k} - \hat{R})\hat{H})^{-1}\hat{B};
$$

(22)

and $u \mapsto G_r(s)u = y$ is the solution (for some $z$) of the equation:

$$
\begin{bmatrix}
    sI_{2k} - (J_{2k} - \hat{R})\hat{H} & -\hat{B} \\
    \hat{B}^T\hat{H} & 0
\end{bmatrix}
\begin{bmatrix}
    z \\
    u
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    y
\end{bmatrix}.
$$

(23)
3. Symplectic MOR

This section provides a structure preserving MOR technique namely symplectic MOR which can be applied for a linear pH-system $\Sigma_{\text{LpHS}}$ in (17). The symplectic MOR was first introduced in [16] for autonomous Hamiltonian systems. This work can be viewed as an extension of the work [16] to pH-system with symplectic-Hamiltonian structure i.e., $J_{2n} \neq J_{2n}$ and non-zero dissipation. We start by recalling some essentials of symplectic geometry that are necessary to understand the methods used, and, then, we formulate some definitions and propositions that are necessary to prove the main result Theorem 3.9. Since we interpolate at frequency points, which are complex, we need to introduce symplectic form on complex vector spaces.

3.1. Symplectic Geometry.

Definition 3.1. Let $V$ be a vector space over $\mathbb{C}$ of even dimension. A symplectic form on $V$ is a mapping $\Omega : V \times V \to \mathbb{C}$ linear in the first component, anti-linear in the second, and satisfying

1. Antisymmetric: $\Omega(v_1, v_2) = -\Omega(v_2, v_1)$ for all $v_1, v_2 \in V$;
2. Non-degenerate: $\Omega(v_1, v_2) = 0$ for all $v_1 \in V$ if and only if $v_2 = 0$.

The pair $(V, \Omega)$ is called a symplectic vector space. On finite-dimensional vector spaces any full rank skew-symmetric matrix induces a symplectic form.

Proposition 3.2. Let $J_{2n} \in \mathbb{C}^{2n \times 2n}$ be a full-rank skew-symmetric structure matrix i.e., $J_{2n}^* = -J_{2n}$. Then $\Omega_{J_{2n}}$ defined as

$$\Omega_{J_{2n}}(u, v) = \langle J_{2n}u, v \rangle = v^*J_{2n}u, \quad u, v \in \mathbb{C}^{2n}$$

is a symplectic form on $\mathbb{C}^{2n}$.

Proof: The (anti)linearity of $\Omega_{J_{2n}}$ follows directly from the (anti)linearity of the inner product. The antisymmetry is a direct consequence of the skew-symmetry of $J_{2n}$. So it remains to show that the form is non-degenerate.

If $\Omega_{J_{2n}}(u, v) = 0$ for all $v \in \mathbb{C}^{2n}$, then $J_{2n}u$ must equal the zero vector. Since $J_{2n}$ is full rank, this implies that $u = 0$, and thus the form is non-degenerated.

Next we introduce mappings that keep the symplectic structures.

Definition 3.3. Let $(\mathbb{C}^{2n}, \Omega_{J_{2n}})$ and $(\mathbb{C}^{2k}, \Omega_{J_{2k}})$ be two symplectic vector spaces with $k, n \in \mathbb{N}$ and $k < n$ where the symplectic form $\Omega_{J_{2n}}$ and $\Omega_{J_{2k}}$ are defined as in (21) and $Q : \mathbb{C}^{2k} \to \mathbb{C}^{2n}$, $y \mapsto Qy$, $Q \in \mathbb{C}^{2n \times 2k}$ be a linear mapping. The matrix $Q$ is called a $(J_{2n}, J_{2k})$-symplectic if

$$Q^*J_{2n}Q = J_{2k}. \quad (25)$$

Given a $(J_{2n}, J_{2k})$-symplectic matrix $Q$, it is easy to see that

$$\Omega_{J_{2n}}(Qu, Qv) = \langle J_{2n}Qu, Qv \rangle = \langle Q^*J_{2n}Qu, v \rangle = \langle J_{2k}u, v \rangle = \Omega_{J_{2k}}(u, v),$$

and thus it preserves the symplectic structure. An important object related to a symplectic mapping is the following.
### Definition 3.4.
For a \((J_{2n}, J_{2k})\)-symplectic matrix \(Q \in \mathbb{C}^{2n \times 2k}\), the symplectic inverse, denoted by \(Q^{-L}\), is defined by
\[
Q^{-L} := K_{2k}Q^*J_{2n} \in \mathbb{C}^{2k \times 2n}
\]
where \(K_{2k} = J_{2k}^{-1}\).

Among others the symplectic inverse is a left inverse, as is shown next. This lemma is an extension of [16, Lemma 3.3 and 3.4].

### Lemma 3.5.
Let \(Q \in \mathbb{C}^{2n \times 2k}\) be a \((J_{2n}, J_{2k})\)-symplectic matrix and let \(Q^{-L}\) be the symplectic inverse from Definition 3.4. By \(K_{2n}\) and \(K_{2k}\) we denote the inverses of \(J_{2n}\) and \(J_{2k}\), respectively. The following holds:
1. \(Q^{-L}Q = I_{2k}\);
2. \(Q^{-L}K_{2n} = K_{2k}Q^*\);
3. \((Q^{-L})^*\) is \((K_{2n}, K_{2k})\)-symplectic;
4. If \(v \in \text{ran}(Q)\), then \(v = QQ^{-L}v\).

**Proof:**
1. Using the expression of \(Q^{-L}\) of (26) we see that \(Q^{-L}Q = K_{2k}Q^*J_{2n}Q = K_{2k}J_{2k} = I_{2k}\).

2. Similarly as above we have \(Q^{-L}K_{2n} = K_{2k}Q^*J_{2n}K_{2k} = K_{2k}Q^*\).

3. To show that \((Q^{-L})^*\) is \((K_{2n}, K_{2k})\)-symplectic, we have to show that, see (26),
\[
(Q^{-L})K_{2n}(Q^{-L})^* = K_{2k}.
\]
Note that \(K_{2n}\), and \(K_{2k}\) are full rank, skew-symmetric matrices, since the corresponding \(J\)'s are. Using the expression on \(Q^{-L}\), we find
\[
Q^{-L}K_{2n}(Q^{-L})^* = K_{2k}Q^*J_{2n}K_{2n}(K_{2k}Q^*J_{2n})^* = (-1)^2K_{2k}Q^*J_{2n}K_{2n}J_{2n}QK_{2k} = K_{2k}J_{2k}K_{2k} = K_{2k}.
\]
4. If \(v \in \text{ran}(Q)\) there there exists a \(w \in \mathbb{C}^{2n}\) such that \(v = Qw\). By item 1. this implies
\[
Q^{-L}v = Q^{-L}Qw = w
\]
and so \(v = Qw = QQ^{-L}v\).

For the canonical symplectic matrices \(J_{2n}\) and \(J_{2k}\), the above expressions become simpler, since
\[
J_{2n}^{-1} = -J_{2n} = J_{2n}^T = J_{2n}^*.
\]
In that case, we don’t call \(Q\) a \((J_{2n}, J_{2k})\)-symplectic matrix, but just \(J_{2n}\)-symplectic.

### 3.2. Symplectic MOR for Port-Hamiltonian System.
In this section, we derive symplectic MOR techniques preserving the port-Hamiltonian structure.
We consider two cases:
1. Without dissipation i.e., \(R = 0\);
2. With dissipation i.e., \(R \neq 0\).
We start with system without dissipation.
3.2.1. pH system without dissipation. Consider the pH-system \( \Sigma_{\text{pHS}} \) on the symplectic vector space \( \mathbb{R}^{2n}, \Omega_{f_{2n}} \) with \( R = 0 \). In the following theorem we give conditions under which we can find a reduced order system \( \Sigma_{\text{pHS}} \), without dissipation, such that the reduced and original transfer functions are equal at prescribed frequencies.

**Theorem 3.6.** Consider the pH-system without dissipation, i.e. \( \Sigma_{\text{pHS}} \) of \( \Sigma_{\text{pHS}} \) with \( R = 0 \).

Let \( Q \in \mathbb{C}^{2n \times 2k} \) be a \((J_{2n}, J_{2k})\)-symplectic matrix, and let \( Q^{-1} \in \mathbb{C}^{2k \times 2n} \) be its symplectic inverse, where \( J_{2k} \) is the skew-symmetric matrix of the reduced system.

Finally, let \( u_1, \ldots, u_M \in \mathbb{R}^p \), and \( s_1, \ldots, s_M \in \mathbb{C} \) be given.

If \( \text{ran}(B) \subseteq \text{ran}(Q) \), and \( (s_m I - J_{2n} \mathcal{H})^{-1} Bu_m \in \text{ran}(Q), m = 1, \ldots, M \), then the reduced pH-system \( \Sigma_{\text{pHS}} \) with \( \tilde{B} = Q^{-1} B \) and \( \tilde{\mathcal{H}} = Q^* \mathcal{H} Q \) satisfies \( G(s_m)u_m = G_r(s_m)u_m, m = 1, \ldots, M \) i.e., the reduced pH-system equals the full pH-system at the given frequencies.

**Proof:** Let us denote \((s_m I - J_{2n} \mathcal{H})^{-1} Bu_m \) by \( x_m \). Since \( x_m \in \text{ran}(Q) \), we have by Lemma 3.5.4 that \( x_m = QQ^{-1} x_m \). Now,

\[
\begin{align*}
s_m x_m - J_{2n} \mathcal{H} x_m - Bu_m &= 0 \Rightarrow (27) \\
\end{align*}
\]

\[
\begin{align*}
s_m Q^{-1} x_m - Q^{-1} J_{2n} \mathcal{H} x_m - Q^{-1} Bu_m &= 0 \Rightarrow (27) \\
\end{align*}
\]

\[
\begin{align*}
s_m Q^{-1} x_m - J_{2k} Q^* \mathcal{H} QQ^{-1} x_m - Q^{-1} Bu_m &= 0 \Rightarrow (28) \\
\end{align*}
\]

with \( z_m = Q^{-1} x_m, \mathcal{H} = Q^* \mathcal{H} Q, \) and \( \tilde{B} = Q^{-1} B \), and where the third equality follows from the Lemma 3.5.3. Furthermore,

\[
G(s_m)u_m = y_m = B^T \mathcal{H} x_m = B^T \mathcal{H} QQ^{-1} x_m. \tag{29}
\]

Since \( \text{ran}(B) \subseteq \text{ran}(Q) \), we have by Lemma 3.5.4 that \( Bu = QQ^{-1} Bu \) for all \( u \). Equivalently, \( B^T = B^* (Q^{-1})^* Q^* \). Therefore, the output of the reduced system \( \Sigma_{\text{pHS}} \) is

\[
y_m = B^T (Q^{-1})^* Q^* \mathcal{H} QQ^{-1} x_m = \tilde{B}^T \tilde{\mathcal{H}} z_m. \tag{30}
\]

From \( (27), (30) \) and \( (28), (31) \), it follows that \( G_r(s_m)u_m = G(s_m)u_m, m = 1, \ldots, M \). It is easy to check that \( \tilde{\mathcal{H}} \) is symmetric and positive semi-definite, i.e., \( \tilde{\mathcal{H}}^T = \mathcal{H} \geq 0 \). Hence, the reduced system preserves the pH-structure.

3.2.2. pH system with dissipation. It is easy to see that the pH-system \( \Sigma_{\text{pHS}} \) with dissipation does not have a strictly symplectic-Hamiltonian structure. As a result, we can’t apply the Theorem 3.6 directly to this class of systems. To show that the symplectic MOR technique also preserves the structure of the dissipative pH-system, we first reformulate the system \( \Sigma_{\text{pHS}} \) as follows:

\[
\begin{align*}
\begin{bmatrix} \dot{x}(t) \\ f_2(t) \end{bmatrix} &= \begin{bmatrix} J_{2n} & -I_{2n} \\ I_{2n} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{H} & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ f_2(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t); \\
y(t) &= \begin{bmatrix} B^T \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{H} & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ f_2(t) \end{bmatrix};
\end{align*}
\]
Lemma 3.8. Let $J_k$ be skew-symmetric matrices, and let $H$ be a possible choice for $Q$ 
into a non-block diagonal one, which would not translate back to a dissipative pH-system. So we have to choose symplectic mappings which preserve this block diagonal structure.

Proposition 3.7. Let $J_{4n} := \begin{bmatrix} J_{2n} & -I_{2n} \\ I_{2n} & 0 \end{bmatrix}$, $J_{4k} := \begin{bmatrix} J_{2k} & -I_{2k} \\ I_{2k} & 0 \end{bmatrix}$, where $J_{2n}, J_{2k}$ are skew-symmetric matrices. Then the following holds

1. $J_{4n}$ and $J_{4k}$ are skew-symmetric and full rank. $\tilde{K}_{4n} := J_{4n}^{-1} = \begin{bmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{bmatrix}$ 

2. The matrix $Q_E := \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$ is $(\tilde{K}_{4n}, \tilde{K}_{4k})$-symplectic if and only if $Q_2$ is a left inverse of $Q_2$. 

Proof: Part 1. is easy to show, and so we concentrate on part 2.

By definition, we have that $Q_E$ is $(\tilde{K}_{4n}, \tilde{K}_{4k})$-symplectic if and only if $Q_E^{-1} \tilde{K}_{4n} Q_E = \tilde{K}_{4k}$, see (25). This becomes

$$\begin{bmatrix} 0 & I_{2k} \\ -I_{2k} & 0 \end{bmatrix} = \tilde{K}_{4k} = Q_E^{-1} \tilde{K}_{4n} Q_E = \begin{bmatrix} Q_1^* & 0 \\ 0 & Q_2^* \end{bmatrix} \begin{bmatrix} 0 & J_{2n} \\ -I_{2n} & 0 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} 0 & Q_1^* Q_2 \\ -Q_2^* Q_1 & Q_2 J_{2n} Q_2 \end{bmatrix}.$$ 

We see that this holds if and only if $Q_2$ is $(J_{2n}, J_{2k})$-symplectic and $Q_1^*$ is a left inverse of $Q_2$. Note that when $Q_1^*$ is a left inverse of $Q_2$, then $Q_2^* Q_1 = (Q_1^* Q_2)^* = I_{2k} = I_{2k}$.

Since the symplectic inverse is always a left inverse, we see from the above result that a possible choice for $Q_1$ is to take it the transpose of $Q_2^{-1}$.

Any symplectic matrix has a symplectic inverse, and so we calculate the symplectic inverse of $Q_E$ next.

Lemma 3.8. Let $J_{4n} := \begin{bmatrix} J_{2n} & -I_{2n} \\ I_{2n} & 0 \end{bmatrix}$, $J_{4k} := \begin{bmatrix} J_{2k} & -I_{2k} \\ I_{2k} & 0 \end{bmatrix}$, with $J_{2n}, J_{2k}$ skew-symmetric matrices, and let $\tilde{K}_{4n}$ and $\tilde{K}_{4k}$ denote their inverses. If $Q_E := \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in \mathbb{C}^{4n \times 4k}$ is $(\tilde{K}_{4n}, \tilde{K}_{4k})$-symplectic, then

$$Q_E^{-1} = \begin{bmatrix} Q_2^* & J_{2k} Q_1^* - Q_2^* J_{2n} \\ 0 & Q_1^* \end{bmatrix}.$$
When additionally, \( Q_1^* = Q_2^{-L} \), then this relation becomes
\[
Q_{E}^{-L} = \begin{bmatrix} Q_2^* & 0 \\ 0 & Q_1^* \end{bmatrix}.
\] (34)

In general, \( J_{2k}Q_1^* - Q_2^*J_{2n} \) equals zero on the range of \( Q_2 \).

Proof: By (26) the symplectic inverse is given by
\[
Q_{E}^{-L} = \tilde{K}_4^{-1}Q_2^*\tilde{K}_4 = J_{4k}Q_{E}^*\tilde{K}_4 = \begin{bmatrix} J_{2k} & -I_{2k} \\ I_{2k} & 0 \end{bmatrix} \begin{bmatrix} Q_1^* & 0 \\ 0 & Q_2^* \end{bmatrix} \begin{bmatrix} 0 & I_{2n} \\ -I_{2n} & J_{2n} \end{bmatrix} = \begin{bmatrix} Q_2^* & J_{2k}Q_1^* - Q_2^*J_{2n} \\ 0 & Q_1^* \end{bmatrix}
\]
which proves (34).

By Proposition 3.7.2, \( Q_{E} \) being \((\tilde{K}_4, \tilde{K}_4)-\)symplectic implies that \( Q_1^* \) is a left inverse of \( Q_2 \) and \( Q_2 \) is \((J_{2n}, J_{2k})\)-symplectic. So for \( x \in \mathbb{C}^{2k} \) we find
\[
(J_{2k}Q_1^* - Q_2^*J_{2n})Q_{2}x = J_{2k}x - J_{2k}x = 0,
\]
which shows the last assertion.

If \( Q_1^* \) is chosen as \( Q_2^{-L} \), then see (26)
\[
J_{2k}Q_1^* - Q_2^*J_{2n} = J_{2k}K_{2k}Q_2^*J_{2n} - Q_2^*J_{2n} = Q_2^*J_{2n} - Q_2^*J_{2n} = 0,
\]
which proves (34).

Using these results we can extend Theorem 3.6 to the case in which there is dissipation.

**Theorem 3.9.** Consider the full-order pH-system with dissipation (17), given in frequency domain by (32). Take
\[
Q_{E} := \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in \mathbb{C}^{2n \times 2k},
\]
where \( k > n \) with \( Q_2 \in \mathbb{C}^{2n \times 2k} \) is \((J_{2n}, J_{2k})\)-symplectic, and \( Q_1^* \) is a left inverse of \( Q_2 \). Here \( J_{2k} \) is the skew-symmetric matrix of the reduced pH-system (21).

Let \( u_1, \ldots, u_M \in \mathbb{C}^{p} \), and \( s_1, \ldots, s_M \in \mathbb{C} \) be given. If \( \text{ran}(B) \subseteq \text{ran}(Q_1) \), \( (s_mI - J_{2n}\mathcal{H})^{-1}B_{u_m} \in \text{ran}(Q_1) \), and \( \mathcal{H}(s_mI - J_{2n}\mathcal{H})^{-1}B_{u_m} \in \text{ran}(Q_2) \), \( m = 1, \ldots, M \), then the (reduced) pH-system (21) with \( \tilde{B} = Q_2^*B, \mathcal{H} = Q_1^*\mathcal{H}Q_1 \), and \( \tilde{R} = Q_2^*RQ_2 \) satisfies \( G(s_m)u_m = G_r(s_m)u_m, m = 1, \ldots, M \), i.e., the reduced pH-system equals the full pH-system at the given frequencies.

Proof: From Proposition 3.7 we know that the conditions on \( Q_2 \) and \( Q_1 \) give that \( Q_{E} \) is \((\tilde{K}_4, \tilde{K}_4)\)-symplectic.

By (32) we know that \( y_m = G(s_m)u_m \) satisfies
\[
\begin{bmatrix} s_m & x_m \\ f_{2m} & \end{bmatrix} = \begin{bmatrix} J_{2n} & -I_{2n} \\ I_{2n} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{H} & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x_m \\ f_{2m} \end{bmatrix} + \begin{bmatrix} B_0 & \end{bmatrix} u_m \]
\]
and
\[
y_m = \begin{bmatrix} B_0^T & \mathcal{H} & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x_m \\ f_{2m} \end{bmatrix}.
\]
where we have used Lemma 3.5.2 and the expression of $Q$ as given in (33).

Similarly, by applying Lemma 3.5.2, we can write the above expression as

$$Q_L^{-1} \begin{bmatrix} s_m x_m \\ f_{2m} \end{bmatrix} = J_{4k} Q_E^{-1} \begin{bmatrix} \mathcal{H} \\ 0 \end{bmatrix} \begin{bmatrix} f_{2m} \\ x_m \end{bmatrix} + Q_L^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} u_m$$

where $\hat{Q}$ is positive definite i.e., $\hat{Q} > 0$. Similarly, we have that $\begin{bmatrix} f_{2m} \\ x_m \end{bmatrix} \in \text{ran}(Q_E)$.

Combining this with (35) gives

$$\begin{bmatrix} s_m z_m \\ \phi_{2m} \end{bmatrix} = J_{4k} \begin{bmatrix} \mathcal{H} \\ 0 \end{bmatrix} \begin{bmatrix} z_m \\ \phi_{2m} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_m,$$

where $\hat{H} = Q_1^* \mathcal{H} Q_1$, $\hat{R} = Q_2^* \mathcal{R} Q_2$, and $\hat{B} = Q_2^* B$.

Since $\begin{bmatrix} f_{2m} \\ x_m \end{bmatrix} \in \text{ran}(Q_E)$, we have that $\begin{bmatrix} B \end{bmatrix} \in \text{ran}(Q_E)$, and so $\begin{bmatrix} B \end{bmatrix} = Q_E \begin{bmatrix} \hat{B} \end{bmatrix}$. Similarly, $\begin{bmatrix} f_{2m} \\ x_m \end{bmatrix} = Q_E \begin{bmatrix} f_{2m} \end{bmatrix}$. Combining this gives

$$G(s_m) u_m = \begin{bmatrix} B \\ 0 \end{bmatrix}^T \begin{bmatrix} \mathcal{H} \\ 0 \end{bmatrix} \begin{bmatrix} f_{2m} \\ x_m \end{bmatrix} = \hat{B}^* Q_E^{-1} \begin{bmatrix} \mathcal{H} \\ 0 \end{bmatrix} Q_E \begin{bmatrix} z_m \\ \phi_{2m} \end{bmatrix}$$

Combining (37) and (38) shows that $G(s_m) u_m = G_r(s_m) u_m$.

It is straightforward to check that $Q$ has full rank, and so $\hat{H}$ is symmetric and positive definite i.e., $\hat{H}^* = \hat{H} > 0$. $\hat{R}$ is symmetric and positive semi-definite i.e., $\hat{R}^T = \hat{R} \geq 0$ and $J_{4k}$ preserves the symplectic structure. Hence, the reduced system preserves the pH structure.

**Remark 3.10.** As is clear from the proof, if instead of interpolating $G(s_m) u_m$, we want to interpolate $y_m^T G(s_m) u_m$, then we only have to assume that $B y_m \in \text{ran}(Q_1)$, $m = 1, \ldots, M$. 

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4. Existence and construction of the symplectic matrices

In the previous section we have shown that model reduction for pH-system was possible, provided we can construct symplectic matrices satisfying range conditions. In this section we show that this is indeed possible. Since the construction of the symplectic matrix $Q_E$ is the most elaborated one, see Proposition 3.7 and Theorem 3.9 we treat that case. Since there is a choice in the $J_{2k}$ we choose the canonical form, namely $J_{2k} = \mathbb{J}_{2k}$.

Lemma 4.1. Let $J_{2n}$ be given, and let $V$ and $W$ be two matrices in $\mathbb{C}^{2n \times 2k}$. Then there exist matrices $Q_1, Q_2 \in \mathbb{C}^{2n \times 2k}$ such that

- $\text{ran}(V) \subseteq \text{ran}(Q_1), \text{ran}(W) \subseteq \text{ran}(Q_2)$;
- $Q_1^* \text{ is the left inverse of } Q_2$, and $Q_2$ is $(J_{2n}, \mathbb{J}_{2k})$-symplectic,

provided $i \cdot W^* J_{2n} W$ has positive and negative index of inertia equal to $k$ and $V^* W$ has rank $2k$.

Proof: Since $Q_1$ and $Q_2$ must have the same range as $V$ and $W$, respectively, we have that $Q_1 = V A_1$ and $Q_2 = W A_2$ with $A_1$ and $A_2$ invertible matrices to be designed such that the other two conditions hold.

$Q_1^*$ being the left inverse of $Q_2$ is equivalent to

$$A_1^* V^* W A_2 = I_{2k},$$  \hspace{1cm} (39)

whereas $Q_2$ is $(J_{2n}, \mathbb{J}_{2k})$-symplectic is equivalent to

$$A_2^* W^* J_{2n} W A_2 = \mathbb{J}_{2k}.$$  \hspace{1cm} (40)

Now $W^* J_{2n} W$ is skew-symmetric, and hence $i \cdot W^* J_{2n} W$ is symmetric. By assumption this matrix positive and negative index of inertia equal to $k$, which is the same as that of $i \cdot \mathbb{J}_{2k}$. Hence by Sylvester law of inertia an invertible $A_2 \in \mathbb{C}^{2k \times 2k}$ exists such that (40) holds. Since $V^* W$ has rank $2k$, and since $A_2$ is invertible, we can choose $A_1$ as the inverse of $A_2^* W^* V$, and thus (39) holds.

In the lemma the condition on the inertia of the matrix $i \cdot W^* J_{2n} W$ may seem very restrictive. However, from (40) we see that the condition that $Q_2$ is $(J_{2n}, \mathbb{J}_{2k})$-symplectic implies this condition on $i \cdot W^* J_{2n} W$. If we don’t choose $J_{2k} = \mathbb{J}_{2k}$, then this condition is replaced by the inertia of the matrix $i \cdot W^* J_{2n} W$ equals the inertia of $i \cdot \mathbb{J}_{2k}$. Hence given $J_{2n}$ and $W$, we can choose $J_{2k}$ such that the inertia match.

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