Existence and uniqueness results for the Boussinesq system with data in Lorentz spaces

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Abstract

This paper is devoted to the study of the Cauchy problem for the Boussinesq system with partial viscosity in dimension $N \geq 3$. First we prove a global existence result for data in Lorentz spaces satisfying a smallness condition which is at the scaling of the equations. Second, we get a uniqueness result in Besov spaces with negative indices of regularity (despite the fact that there is no smoothing effect on the temperature). The proof relies on a priori estimates with loss of regularity for the nonstationary Stokes system with convection. As a corollary, we obtain a global existence and uniqueness result for small data in Lorentz spaces.

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1 Introduction and main results

The present paper is devoted to the mathematical study of the so-called Boussinesq system with partial viscosity:

\begin{equation}
\begin{aligned}
\partial_t \theta + u \cdot \nabla \theta &= 0 \\
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi &= \theta e_N \\
\text{div } u &= 0.
\end{aligned}
\end{equation}

Above, $\theta = \theta(t, x)$ and $\Pi = \Pi(t, x)$ are real valued functions, and $u = u(t, x)$ is a time dependent vector field. We denote by $e_N$ the vertical unit vector of $\mathbb{R}^N$. It is assumed that the space variable $x$ belongs to $\mathbb{R}^N$. We supplement the system with Cauchy conditions $(\theta_0, u_0)$ at time $t = 0$ and address the question of solvability for $t \geq 0$.

Boussinesq system arises in simplified models for geophysics in which case $u$ stands for the velocity field and the forcing term $\theta e_N$ is proportional either to the temperature, or to the salinity or to the density (see [17]). Here, we shall call $\theta$ the temperature.

Remark that the standard incompressible Navier-Stokes equations arise as a particular case of (1) (just take $\theta \equiv 0$). Hence, it is tempting to study whether the classical theory

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for the Navier-Stokes equations may be extended to those more general fluids. As far as one is concerned with global results, the main difficulty that we have to face is that \( \theta \) is transported by the flow of \( u \). Hence it has no time decay nor gain of smoothness whatsoever and the standard approach for solving Navier-Stokes equations with a (given) source term is bound to fail. Nevertheless, in a recent paper (see [11]), we have stated that system (1) has global finite energy weak solutions for any data \((\theta_0, u_0)\) in \( L^2 \) and that uniqueness holds true in dimension two. That result may be compared with Leray’s theorem [16] for the Navier-Stokes equations.

In [11], we have also obtained a global existence result (for small data) in the spirit of Fujita and Kato’s theorem (see [12] and [14]) for the Navier-Stokes equations:

**Theorem 1** Let \( N \geq 3 \). Let \( \theta_0 \in L^\infty(\mathbb{R}^N) \cap \dot{B}^0_{N,1}(\mathbb{R}^N) \) and \( u_0 \in L^{N,\infty}(\mathbb{R}^N) \cap \dot{B}^{-1+\frac{N}{p}}_{p,1}(\mathbb{R}^N) \) for some \( p \in [N, \infty] \). There exists a positive \( c \) depending only on \( N \) and such that if

\[
\|u_0\|_{L^{N,\infty}} + \nu^{-1}\|\theta_0\|_{L^\infty} \leq c \nu
\]

then Boussinesq system (1) admits a unique global solution

\[
(\theta, u, \nabla \Pi) \in C(\mathbb{R}^+; \dot{B}^0_{N,1}) \times \left( C(\mathbb{R}^+; \dot{B}^{-\frac{N}{p}+1}_{p,1}) \cap L^1_{loc}(\mathbb{R}^+; \dot{B}^{-\frac{N}{p}+1}_{p,1}) \right)^N \times \left( L^1_{loc}(\mathbb{R}^+; \dot{B}^{-\frac{N}{p}+1}_{p,1}) \right)^N.
\]

Besides, from dimension \( N = 4 \) on, space \( L^\infty(\mathbb{R}^N) \) may be replaced by \( L^{\frac{N}{2}}(\mathbb{R}^N) \).

The reader is referred to section 2 for the definition of Lorentz spaces \( L^{p,\infty} \) and Besov spaces \( \dot{B}^s_{p,r} \) which have been used above. Remind that we have the (continuous) inclusions

\[
\dot{B}^0_{N,1} \subset L^N \subset L^{N,\infty}.
\]

Theorem 1 may be interpreted in terms of scaling of Boussinesq system. Indeed, (1) is obviously invariant under the transform

\[
u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad \theta(t, x) \mapsto \lambda^3 \theta(\lambda^2 t, \lambda x) \quad \text{for all} \quad \lambda > 0.
\]

Hence data \((\theta_0, u_0)\) belong to a functional space \( E \) which is at the scaling of the system if and only if the norm of \( E \) is invariant by

\[
\|u_0\|_{L^N} + \nu^{-1}\|\theta_0\|_{L^\infty} \leq c \nu
\]

In dimension \( N \geq 3 \), the spaces \( L^\infty \times (L^{N,\infty})^N \) and \( L^{\frac{N}{2},\infty} \times (L^{N,\infty})^N \) satisfy (2). Hence, Theorem 1 is a global well-posedness result for suitably smooth data satisfying a scaling invariant smallness condition. In fact, prescribing additional regularity in homogeneous Besov spaces ensures that the velocity belongs to the set \( L^1_{loc}(\mathbb{R}^+; \text{Lip}) \) of locally integrable functions in \( t \) with values in the set of Lipschitz vector fields. This property was needed for uniqueness. Note in passing that the additional condition required for the initial velocity still satisfies (2).
The present paper aims at weakening as much as possible the Besov space assumption. Ideally, we would like to consider general (small) data \((\theta_0, u_0)\) in the scaling invariant space \(L^{N,\infty}_N \times (L^{N,\infty})^N\).

For proving existence, the strategy is the following. We smooth out the data so that Theorem 1 provides a sequence \((\theta^n, u^n)\) of global solutions which is bounded in 

\[ L^\infty(\mathbb{R}_+; L^{N,\infty}_N \times (L^{N,\infty})^N). \]

Obviously, those bounds are insufficient to pass to the limit in the nonlinear terms. However, the parabolicity of the second equation of (1) provides additional regularity so that there is some chance to pass to the limit anyway provided \(\theta_0\) belongs also to \(L^{p,\infty}\) for some large enough \(p\). Those basic considerations will enable us to prove the following statement:

**Theorem 2** Let \(u_0\) be a solenoidal vector field with coefficients in \(L^{N,\infty}\). Assume that \(\theta_0\) belongs to

- \(L^1 \cap L^{p,\infty}\) for some \(p > 3/2\) if \(N = 3\),
- \(L^{4/3,\infty} \cap L^{p,\infty}\) for some \(p > 4/3\) if \(N = 4\),
- \(L^{N/3,\infty} \cap L^{p,\infty}\) for some \(p \geq N/3\) if \(N \geq 5\).

There exists a constant \(c > 0\) depending only on \(N\), and such that if

\[ \|u_0\|_{L^{3,\infty}} + \|\theta_0\|_{L^1} \leq c\nu \quad \text{if} \quad N = 3, \quad \|u_0\|_{L^{N,\infty}} + \|\theta_0\|_{L^{N/3,\infty}} \leq c\nu \quad \text{if} \quad N \geq 4 \]

then system (1) has a global solution \((\theta, u, \nabla\Pi)\) with \(u \in L^\infty(\mathbb{R}_+; L^{N,\infty})\) and

\[ \theta \in L^\infty(\mathbb{R}_+; L^{1 \cap L^{p,\infty}}) \quad \text{if} \quad N = 3, \quad \theta \in L^\infty(\mathbb{R}_+; L^{N/3,\infty} \cap L^{p,\infty}) \quad \text{if} \quad N \geq 4. \]

**Remark:** In fact, the heat semi-group supplies some additional regularity properties for \(u\) (that we shall use in the proof of Theorem 2).

Note that from dimension 5 on, one can take \(p = N/3\) so that the above statement is a global existence result in a scaling invariant space for the system. The scaling of the system may still be almost achieved in dimension 4. In dimension 3 however, it is very unlikely that an existence result may be proved for \(\theta_0 \in L^1\) and \(u_0 \in L^{N,\infty}\) (see Remark 4 for further explanations).

As regards uniqueness, let us stress that for general \(u_0\) in \(L^{N,\infty}\) the function \(e^{t\Delta}u_0\) (where \((e^{t\Delta})_{t>0}\) denotes the heat semi-group) need not be in \(L^1_{loc}(\mathbb{R}_+; \text{Lip})\). Therefore, the velocity field of the solution constructed in Theorem 2 need not be in \(L^1_{loc}(\mathbb{R}_+; \text{Lip})\) either which precludes us from proving stability estimates for system (1) by mean of standard arguments. Indeed, we have to deal with a transport equation associated to a vector field which is not in \(L^1_{loc}(\mathbb{R}_+; \text{Lip})\). This difficulty has been overcome in [11] in the framework of two-dimensional finite energy solutions. In the present paper, we shall see that similar arguments may be used to state uniqueness in dimension \(N \geq 3\).
Theorem 3 Let \((\theta_1, u_1, \nabla \Pi_1)\) and \((\theta_2, u_2, \nabla \Pi_2)\) satisfy (1) with the same data. Assume that for some \(p \in [1, 2N]\) and \(i = 1, 2\), we have

\[
\theta_i \in L_T^\infty(\dot{B}_{p,\infty}^{-1+\frac{N}{p}}) \quad \text{and} \quad u_i \in L_T^\infty(\dot{B}_{p,\infty}^{-1+\frac{N}{p}}) \cap L_T^1(\dot{B}_{p,\infty}^{1+\frac{N}{p}}).
\]

There exists a constant \(c > 0\) depending only on \(N\) and on \(p\) such that if in addition

\[
\|u_1\|_{L_T^1(\dot{B}_{p,\infty}^{1+\frac{N}{p}})} + \nu^{-1}\|u_2\|_{L_T^\infty(\dot{B}_{p,\infty}^{-1+\frac{N}{p}})} \leq c
\]

then \((\theta_1, u_1, \nabla \Pi_1) \equiv (\theta_2, u_2, \nabla \Pi_2)\).

In the above statement, the space \(\tilde{L}_T^1(\dot{B}_{p,\infty}^{1+\frac{N}{q}})\) is slightly larger than the set \(L_T^1(\dot{B}_{p,\infty}^{1+\frac{N}{p}})\) of integrable functions over \([0, T]\) with values in the Besov space \(\dot{B}_{p,\infty}^{1+\frac{N}{p}}\) (see section 2).

Remark: The limit case \(p = 2N\) may be considered if the velocity field belongs to a Besov space with third index 1 (see Theorem 5).

Finally, putting together the embedding \(L^{N,\infty} \hookrightarrow \dot{B}_{q,\infty}^{1+\frac{N}{q}}\) for \(q > N\) (see Lemma 2 below), the existence and uniqueness theorems, and the further regularity properties for the velocity given by the heat semi-group, one ends up with the following global well-posedness result:

Theorem 4 Assume that \((\theta_0, u_0)\) satisfies the assumptions of Theorem 2 with \(p = N\). Then system (1) has a unique solution \((\theta, u, \nabla \Pi)\) such that \(u \in L^\infty(\mathbb{R}_+; L^{N,\infty})\), \(\theta \in L^\infty(\mathbb{R}_+; L^{\frac{N}{N-1},\infty})\) if \(N \geq 4\) and \(\theta \in L^1(\mathbb{R}_+; L^1)\) if \(N = 3\), with moreover

\[
u \in \tilde{L}_T^{1+\frac{N}{q}}(\dot{B}_{q,\infty}^0) \quad \text{for all} \quad q > N.
\]

The paper is structured as follows. In the next section, we present a few tools borrowed from harmonic and functional analysis. Section 3 is devoted to the proof of existence. The study of uniqueness is postponed in section 4.

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2 Tools and functional spaces

2.1 Lorentz spaces

To start with, let us recall the definition of weak \(L^p\) spaces (denoted by \(L^{p,\infty}\)):

Definition 1 For \(1 \leq p < \infty\), we denote by \(L^{p,\infty}(\mathbb{R}^N)\) (or simply \(L^{p,\infty}\)) the space of all real valued measurable functions over \(\mathbb{R}^N\) such that

\[
\|f\|_{L^{p,\infty}} := \sup_{\lambda > 0} \lambda \left| \{ x \in \mathbb{R}^N / |f(x)| > \lambda \} \right|^\frac{1}{p} < \infty.
\]
Remark 1 The space $L^{p, \infty}$ may be alternatively defined by real interpolation:

$$L^{p, \infty} = (L^{\infty}, L^1)_{\left(\frac{1}{p}, \infty\right)}.$$ 

In other words, a function $f$ belongs to $L^{p, \infty}$ if and only if, for all $A > 0$, one may write $f = f^A + f_A$ for some functions $f_A \in L^1$ and $f^A$ in $L^\infty$ such that

$$\|f_A\|_{L^1} \leq CA^{1-1/p} \quad \text{and} \quad \|f^A\|_{L^\infty} \leq CA^{-1/p}. $$

The “best constant” $C$ defines a norm which is equivalent to $\|f\|_{L^{p, \infty}}$.

The set $C^\infty_c$ of smooth compactly supported functions is not dense in spaces $L^{p, \infty}$. It turns out however that $C^\infty_c$ is locally dense in $L^{p, \infty}$ (despite the fact that $L^{p, \infty}$ is not separable). More details are given in the following proposition.

Proposition 1 For all $p \in ]1, \infty[$ and $f \in L^{p, \infty}$, there exists a family $(f_\varepsilon)_{\varepsilon > 0}$ of $C^\infty_c$ functions and a constant $C$ so that

$$\sup_{\varepsilon > 0} \|f_\varepsilon\|_{L^{p, \infty}} \leq C \|f\|_{L^{p, \infty}} \quad \text{and} \quad f_\varepsilon \to f \quad \text{in} \quad L^1 + L^\infty. \quad (5)$$

Proof: Let $(\varphi_\varepsilon)_{\varepsilon > 0}$ be a family of mollifiers and $(\chi_\varepsilon)_{\varepsilon > 0}$, a family of cut-off functions with values in $[0, 1]$, supported in $B(0, 2\varepsilon^{-1})$ and equal to 1 in a neighborhood of $B(0, \varepsilon^{-1})$.

Let $f \in L^{p, \infty}$. For all $\varepsilon > 0$, set $f_\varepsilon := \varphi_\varepsilon \ast (\chi_\varepsilon f)$ and fix some $\eta > 0$. From the above definition and remark, it is obvious that

$$\|f_\varepsilon\|_{L^{p, \infty}} \leq C \|\chi_\varepsilon f\|_{L^{p, \infty}} \leq C \|f\|_{L^{p, \infty}} \quad \text{for all} \quad \varepsilon > 0,$$

and that one can find two functions $g \in L^1$ and $h \in L^\infty$ such that $f = g + h$ and $\|h\|_{L^\infty} \leq \eta/4$ (just take $A$ large enough).

Let us split $f_\varepsilon - f$ as follows:

$$f_\varepsilon - f = \left\{ \varphi_\varepsilon \ast (\chi_\varepsilon g) - \varphi_\varepsilon \ast g \right\} + \left\{ \varphi_\varepsilon \ast g - g \right\} + \left\{ \varphi_\varepsilon \ast (\chi_\varepsilon h) - h \right\}.$$

On the one hand, Lebesgue dominated convergence theorem and standard results on convolution ensure that the first two terms between curly brackets have $L^1$ norm less than $\eta/2$ for small enough $\varepsilon$. On the other hand, the $L^\infty$ norm of the last term is obviously less than $\eta/2$ for any $\varepsilon > 0$.

It is now easy to complete the proof of the proposition. \[\blacksquare\]

2.2 Besov spaces

In this subsection we define the Littlewood-Paley decomposition and the Besov spaces that we are going to work with. The reader is referred to the monographs [5, 15] or [18] for a more detailed presentation.
To start with, fix a smooth nonnegative radial function $\chi$ with support in the ball $\{ |\xi| \leq \frac{3}{4} \}$, value 1 over $\{ |\xi| \leq \frac{3}{4} \}$, and such that $r \mapsto \chi(\xi r)$ be nonincreasing over $\mathbb{R}_+$. Let $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. We obviously have

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^N \setminus \{0\}. \quad (6)$$

We define the spectral localization operators $\hat{\Delta}_q$ and $\hat{\Delta}_q$ ($q \in \mathbb{Z}$) by

$$\hat{\Delta}_q u := \varphi(2^{-q}D)u \quad \text{and} \quad \hat{\Delta}_q u := \chi(2^{-q}D)u.$$ 

For any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^N)$, functions $\hat{\Delta}_q u$ and $\hat{\Delta}_q u$ are analytic with at most polynomial growth and $u = \sum_{q \in \mathbb{Z}} \hat{\Delta}_q u$ modulo polynomials (see e.g. [15]).

We shall often use the following quasi-orthogonality property:

$$\hat{\Delta}_k \hat{\Delta}_q u \equiv 0 \quad \text{if} \quad |k - q| \geq 2 \quad \text{and} \quad \hat{\Delta}_k (\hat{\Delta}_q u \hat{\Delta}_q v) \equiv 0 \quad \text{if} \quad |k - q| \geq 5. \quad (7)$$

Let us now recall the definition of homogeneous Besov spaces.

**Definition 2** Let $s \in \mathbb{R}$, $(p, r) \in [1, \infty] \times [1, \infty]$ and $u \in \mathcal{S}'(\mathbb{R}^N)$. We set

$$\| u \|_{\dot{B}^s_{p, r}} := \left( \sum_{q \in \mathbb{Z}} 2^{qs} \| \hat{\Delta}_q u \|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if} \quad r < \infty, \quad \text{and} \quad \| u \|_{\dot{B}^s_{p, \infty}} := \sup_{q \in \mathbb{Z}} 2^{qs} \| \hat{\Delta}_q u \|_{L^p}.$$

- If $s < \frac{N}{p}$ or $s = \frac{N}{p}$ and $r = 1$ then the homogeneous Besov space $\dot{B}^s_{p, r} := \dot{B}^s_{p, r}(\mathbb{R}^N)$ is defined as the set of those tempered distributions $u$ such that $\| u \|_{\dot{B}^s_{p, r}} < \infty$.

- If $\frac{N}{p} + k \leq s < \frac{N}{p} + k + 1$ (or $s = \frac{N}{p} + k + 1$ and $r = 1$) for some $k \in \mathbb{N}$ then $\dot{B}^s_{p, r}$ is the set of tempered distributions $u$ so that $\partial^a u \in \dot{B}^{s-k-1}_{p, r}$ for all multi-index $a$ of length $k + 1$.

**Remark:** Let us all also recall in passing that the nonhomogeneous Besov space $B^s_{p, r}$ is the set of tempered distributions $u$ such that

$$\| u \|_{B^s_{p, r}} < \infty \quad \text{with} \quad \| u \|_{B^s_{p, r}} := \| \hat{S}_0 u \|_{L^p} + \| 2^{qs} \| \hat{\Delta}_q u \|_{L^p} \|_{L^r(\mathbb{R}^N)}.$$ 

**Remark:** For all $s \in \mathbb{R}$, the Besov space $\dot{B}^s_{2, 2}$ coincides with the homogeneous Sobolev space $H^s$, and, if $s \in \mathbb{R}_+ \setminus \mathbb{N}$ then $\dot{B}^s_{\infty, \infty}$ coincides with the homogeneous Hölder space $C^s$.

The following Bernstein inequality will be of constant use in the paper.

**Lemma 1** Let $k \in \mathbb{N}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $\psi \in C^\infty_c(\mathbb{R}^N)$. There exists a constant $C$ depending only on $k, \ N$ and $\text{Supp} \ \psi$ such that

$$\| D^k \psi(2^{-q}D)u \|_{L^{p_2}} \leq C 2^{q(k + N(\frac{1}{r_1} - \frac{1}{r_2}))} \| \psi(2^{-q}D)u \|_{L^{p_1}}.$$ 

\footnote{Note that the definition is somewhat simpler than in the homogeneous framework since low frequencies cannot experience divergence.}
We shall also use the following fundamental properties of Besov spaces.

**Proposition 2** (i) The set $\dot{B}_{p,r}^s(\mathbb{R}^N)$ is a complete subspace of $S'(\mathbb{R}^N)$ if and only if $s < N/p$ or $s = N/p$ and $r = 1$.

(ii) There exists a positive constant $c$ so that
\[
c^{-1} \|u\|_{\dot{B}_{p,r}^s} \leq \|\nabla u\|_{\dot{B}_{p,r}^{s-1}} \leq c \|u\|_{\dot{B}_{p,r}^s}.
\] (8)

(iii) For all $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, we have $\dot{B}_{p_1,r_1}^s \hookrightarrow \dot{B}_{p_2,r_2}^{s-N\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}$.

(iv) If $p \in [1, \infty]$ then $\dot{B}_{p,1}^N \hookrightarrow \dot{B}_{p,\infty}^N \cap L^\infty$. If $p$ is finite then space $\dot{B}_{p,1}^N$ is an algebra.

(v) Real interpolation: $(\dot{B}_{p,r}^{s_1}, \dot{B}_{p,r}^{s_2})_{\theta, r'} = \dot{B}_{p,r'}^{\theta s_2 + (1-\theta)s_1}$ whenever $0 < \theta < 1$ and $1 \leq p, r, r' \leq \infty$.

We shall often use the fact that Lorentz spaces are embedded in Besov spaces (see the proof in [11]).

**Lemma 2** For any $1 < p < q \leq \infty$, we have
\[
L^{p,\infty}(\mathbb{R}^N) \hookrightarrow \dot{B}_{q,\infty}^{N - \frac{N}{p}}(\mathbb{R}^N).
\]

In order to pass to the limit in the nonlinear terms of System (1), the following compactness result in nonhomogeneous Besov spaces will be most useful.

**Lemma 3** For any $1 < p < q \leq \infty$, $\varepsilon > 0$ and $\psi \in C^\infty_c$, the map $u \mapsto \psi u$ is compact from $L^{p,\infty}(\mathbb{R}^N)$ to $B_{q,\infty}^{N - \frac{N}{p} - \varepsilon}(\mathbb{R}^N)$.

**Proof:** It is well known that $u \mapsto \psi u$ is a compact mapping from $B_{p,r}^s$ to $B_{p,r}^{s'}$ whenever $s' < s$ (see e.g. [13] and the references therein). So it suffices to combine the previous lemma with the fact that, owing to $N/q - N/p < 0$, we have $\dot{B}_{q,\infty}^{N - \frac{N}{p}} \hookrightarrow \dot{B}_{q,\infty}^{N - \frac{N}{p}}$. $\blacksquare$

### 2.3 Product and paraproduct in Besov spaces

In order to bound the nonlinear terms in system (1), it will be useful to know how the product of two functions operates in Besov spaces. In fact, optimal results may be achieved by taking advantage of (basic) paradifferential calculus, a tool which was introduced by J.-M. Bony in [4]. More precisely, the product of two functions $f$ and $g$ may be decomposed according to
\[
fg = \hat{T}_fg + \hat{T}_gf + \hat{R}(f,g)
\] (9)

where the paraproduct operator $\hat{T}$ is defined by the formula
\[
\hat{T}_fg := \sum_q \hat{S}_{q-1}f \hat{\Delta}_q g,
\]

and the remainder operator, $\hat{R}$, by
\[
\hat{R}(f,g) := \sum_q \hat{\Delta}_q f \bar{\Delta}_q g \quad \text{with} \quad \bar{\Delta}_q := \hat{\Delta}_{q-1} + \hat{\Delta}_q + \hat{\Delta}_{q+1}.
\]
We shall make an extensive use of the following results of continuity for operators \( \dot{T} \) and \( \dot{R} \) (see the proof in e.g. \[18\]):

**Proposition 3** Let \( 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty \) so that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \).

Operator \( \dot{T} \) is continuous:

- from \( L^\infty \times \dot{B}^t_{p,r} \) to \( \dot{B}^t_{p,r} \) for all \( t \in \mathbb{R} \),
- from \( \dot{B}^{-s}_{p_1,r_1} \times \dot{B}^t_{p_2,r_2} \) to \( \dot{B}^{-s}_{p,r} \) for all \( t \in \mathbb{R} \) and \( s > 0 \).

Operator \( \dot{R} \) is continuous:

- from \( \dot{B}^{s}_{p_1,r_1} \times \dot{B}^t_{p_2,r_2} \) to \( \dot{B}^{s+t}_{p,r} \) for all \((s, t) \in \mathbb{R}^2 \) such that \( s + t > 0 \),
- from \( \dot{B}^{s}_{p_1,r_1} \times \dot{B}^{-s}_{p_2,r_2} \) to \( \dot{B}^{0}_{p,\infty} \) if \( s \in \mathbb{R} \) and \( \frac{1}{r_1} + \frac{1}{r_2} \geq 1 \).

Combining the above continuity results with Bony’s decomposition \( [9] \), we get:

**Proposition 4** Let \((p, r) \in [1, \infty]^2 \) and \((s_1, s_2) \in \mathbb{R}^2 \). The following inequalities hold true:

- If \( s_1 + s_2 + N \inf(0, 1 - \frac{2}{p}) > 0 \), \( s_1 < \frac{N}{p} \) and \( s_2 < \frac{N}{p} \) then
  \[
  \|uv\|_{\dot{B}^{s_1+s_2-N\frac{2}{p}}_{p,r}} \leq C\|u\|_{\dot{B}^{s_1}_{p,r}}\|v\|_{\dot{B}^{s_2}_{p,\infty}}.
  \]
  (10)

  If \( s_1 = \frac{N}{p} \) then \( \|u\|_{\dot{B}^{s_1}_{p,r}} \) (resp. \( \|v\|_{\dot{B}^{s_2}_{p,\infty}} \)) has to be replaced with \( \|u\|_{\dot{B}^{s_1}_{p,r} \cap L^\infty} \) (resp. \( \|v\|_{\dot{B}^{s_2}_{p,\infty} \cap L^\infty} \)). If \( s_2 = \frac{N}{p} \) then \( \|v\|_{\dot{B}^{s_2}_{p,\infty}} \) has to be replaced with \( \|v\|_{\dot{B}^{s_2}_{p,\infty} \cap L^\infty} \).

- If \( s_1 + s_2 = 0 \), \( s_1 \in (-\frac{N}{p}, \frac{N}{p}) \) and \( p \geq 2 \) then
  \[
  \|uv\|_{\dot{B}^{s_1}_{p,\infty}} \leq C\|u\|_{\dot{B}^{s_1}_{p,\infty}}\|v\|_{\dot{B}^{s_2}_{p,\infty}}.
  \]
  (11)

- If \( p \geq 2 \) then
  \[
  \|uv\|_{\dot{B}^{s}_{p,\infty}} \leq C\|u\|_{\dot{B}^{s}_{p,\infty}}\|v\|_{\dot{B}^{s}_{p,\infty} \cap L^\infty}.
  \]
  (12)

- If \( |s| < \frac{N}{p} \) and \( p \geq 2 \) or \( -\frac{N}{p'} < s < \frac{N}{p} \) and \( p < 2 \) then
  \[
  \|uv\|_{\dot{B}^{s}_{p,r}} \leq C\|u\|_{\dot{B}^{s}_{p,r}}\|v\|_{\dot{B}^{s}_{p,\infty} \cap L^\infty}.
  \]
  (13)
2.4 Estimates for the heat and Stokes equations

We shall often use the following smoothing property for the heat equation which has been stated by J.-Y. Chemin in [6].

**Proposition 5** Let \( s \in \mathbb{R}, \; 1 \leq p, r, \rho_1 \leq \infty \). Let \( u_0 \in \dot{B}_p^{s_\rho} \) and \( f \in \dot{L}_T^{\rho} (\dot{B}_p^{s_\rho} \dot{\dot{B}}_r^{q_\rho}) \). Then the heat equation
\[
\partial_t u - \nu \Delta u = f, \quad u|_{t=0} = u_0
\]
admits a unique solution \( u \in \dot{L}_T^{\infty} (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho}) \) and there exists a constant \( C \) depending only on the dimension \( N \) so that for all \( t \in [0, T] \) and \( \rho \geq \rho_1 \), we have:
\[
\nu^\frac{1}{p} \| u \|_{L^p_t (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho})} \leq C \left( \| u_0 \|_{\dot{B}_p^{s_\rho}} + \nu^{\frac{1}{p} - 1} \| f \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho})} \right). \tag{14}
\]
In the above statement, spaces \( \dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho}) \) are defined along the lines of definition 2 with
\[
\| u \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho})} := \left\| 2^{sn} \| \hat{u} \|_{L^p_t (\dot{L}^r)} \right\|_{\ell^n (\mathbb{Z})}. \tag{15}
\]
Note that by virtue of Minkowski inequality, we have
\[
\| u \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho})} \leq \| u \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho})} \quad \text{for} \quad \rho \geq r, \tag{16}
\]
and the opposite inequality if \( \rho \leq r \).

We shall set \( \dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho}) = \bigcap_{T>0} \dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}) \).

**Remark 2** The above results of continuity for the paraproduct, remainder and product may be easily carried out to \( \dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}) \) spaces. The time exponents just behave according to Hölder inequality.

**Remark 3** Since the Leray projector \( \mathcal{P} \) over solenoidal vector fields maps \( \dot{B}_p^{s_\rho} \) to \( \dot{B}_p^{s_\rho} \) for every \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), Proposition 5 may be extended to the nonstationary Stokes system
\[
\begin{aligned}
\partial_t u - \nu \Delta u + \nabla \Pi &= f, & \text{div} u &= 0,
\end{aligned}
\tag{17}
\]
with divergence free initial data \( u_0 \in \dot{B}_p^{s_\rho} \), and source term \( f \) in \( \dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}) \). In particular, denoting \( Q = \text{Id} - \mathcal{P} \), we have the following a priori estimates for all \( \rho \geq 1 \):
\[
\nu^\frac{1}{p} \| u \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho}, \dot{\dot{B}}_r^{q_\rho})} \leq C \left( \| u_0 \|_{\dot{B}_p^{s_\rho}} + \| Pf \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho})} \right) \quad \text{and} \quad \| \nabla \Pi \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho})} \leq C \| Qf \|_{\dot{L}_T^{\rho} (\dot{B}_p^{s_\rho})}. \]

3 Existence of solutions with infinite energy

This section is devoted to the proof of Theorem 2. The principle of the proof is standard:

1. approximate the data \( (\theta_0, u_0) \) by a sequence \( (\theta_0^n, u_0^n)_{n \in \mathbb{N}} \) of smooth solutions;
2. solve (globally) the Boussinesq system with data \( (\theta_0^n, u_0^n) \);
3. resort to compactness arguments to prove the convergence of a subsequence;
4. pass to the limit in the system.
3.1 Global existence: the smooth case

As a warm up, let us first consider the case $p = N$ which is easier to deal with. For notational simplicity, we agree that $L^\frac{N}{q},\infty$ stands for $L^1$ if $N = 3$.

Let $(\theta_0,u_0)$ be in $\left(L^\frac{N}{q},\infty \cap L^{N,\infty}\right) \times \left(L^{N,\infty}\right)^N$ with $\text{div} u_0 = 0$. According to (a slight generalization of) Proposition 1, one can find a sequence $(\theta^n_0,u^n_0)\in C^\infty$ functions which tends to $(\theta_0,u_0)$ in the sense of the distributions and such that in addition

$$\|\theta^n_0\|_{L^\frac{N}{q},\infty} \leq C\|\theta_0\|_{L^\frac{N}{q},\infty}, \quad \|\theta^n_0\|_{L^{N,\infty}} \leq C\|\theta_0\|_{L^{N,\infty}}, \quad \|u^n_0\|_{L^{N,\infty}} \leq C\|u_0\|_{L^{N,\infty}}.$$

Note that projecting on the set of solenoidal vector fields is needed to ensure that $\text{div} u^n_0 = 0$. This operation is harmless since $\mathcal{P}$ maps $L^{N,\infty}$ in $L^{N,\infty}$.

According to [11] Theorem 1.5, we deduce that there exists a positive constant $c$ so that if condition (3) is satisfied then system (1) admits a unique global solution $(\theta^n,u^n)$ in $C(\mathbb{R};\dot{B}^N_{q,1})$ with in addition

$$\|u^n(t)\|_{L^{N,\infty}} \leq C(\nu^{-1}\|\theta_0\|_{L^\frac{N}{q},\infty} + \|u_0\|_{L^{N,\infty}}) \quad \text{for all} \quad t \geq 0.$$

As $\theta_0 \in L^\frac{N}{q},\infty \cap L^{N,\infty}$ and $\text{div} u^n = 0$ one can assert (see e.g Proposition 4.6 of [11]) that

$$\|\theta^n(t)\|_{L^\frac{N}{q},\infty} \leq C\|\theta_0\|_{L^\frac{N}{q},\infty} \quad \text{and} \quad \|\theta^n(t)\|_{L^{N,\infty}} \leq C\|\theta_0\|_{L^{N,\infty}} \quad \text{for all} \quad t \geq 0.$$

Note that the above estimates are not sufficient to pass to the limit in the nonlinear terms of system (1). So we shall take advantage of the smoothing properties given by the Stokes operator in the equation for the velocity. Let $q$ be in $[N,\infty[$. Putting together Remark 3 and the embedding $L^{N,\infty} \hookrightarrow \dot{B}^\frac{N}{q+1}_{q,\infty}$, we discover that

$$\nu\|u^n\|_{L^1(\dot{B}^\frac{N}{q+1}_{q,\infty})} \leq C(\|u_0\|_{L^{N,\infty}} + \|u^n \otimes u^n\|_{L^1(\dot{B}^\frac{N}{q,\infty})} + \|\theta^n\|_{L^1(L^{N,\infty})}). \quad (18)$$

Using Bony’s decomposition (9) followed by Proposition 3, Remark 2 and Lemma 2 we can write that

$$\|u^n \otimes u^n\|_{L^1(\dot{B}^\frac{N}{q+1}_{q,\infty})} \leq C\|u^n\|_{L^\infty(\dot{B}^\frac{N}{q+1}_{q,\infty})}\|u^n\|_{L^\infty(\dot{B}^\frac{N}{q+1}_{q,\infty})},$$

$$\leq C\|u^n\|_{L^\infty(L^{N,\infty})}\|u^n\|_{L^1(\dot{B}^\frac{N}{q+1}_{q,\infty})}.$$

Plugging this latter inequality in (18), and assuming that $c$ is suitably small, we thus get

$$\nu\|u^n\|_{L^1(\dot{B}^\frac{N}{q+1}_{q,\infty})} \leq C(\|u_0\|_{L^{N,\infty}} + t\|\theta_0\|_{L^{N,\infty}}). \quad (19)$$

Now, compactness arguments will enable us to pass to the limit. Indeed, we have proved that sequence $(\theta^n,u^n)_{n \in \mathbb{N}}$ is bounded in

$$L^\infty(\mathbb{R};\dot{B}^\frac{N}{q+1}_{q,\infty} \cap L^{N,\infty}) \times \left(L^\infty(\mathbb{R};\dot{B}^\frac{N}{q,\infty}) \cap \bar{L}^1_{(\dot{B}^\frac{N}{q+1}_{q,\infty})}\right)^N \quad \text{for all} \quad q > N.$$
So it is easy to show that sequence \((\partial_t \theta^n)\) is bounded in the set of space derivatives of functions of \(L^\infty(\mathbb{R}^+; L^{N^q}_{N,\infty})\), which is embedded in \(L^\infty(\mathbb{R}^+; B^{-3+\frac{N}{q}}_{q,\infty})\). Now, according to Lemma 3 for all \(\psi \in C_c^\infty\) the map \(f \mapsto \psi f\) is compact from \(L^{N,\infty}\) to \(B^{-3+\frac{N}{q}}_{q,\infty}\). So one may conclude by combining Ascoli theorem, Cantor diagonal process and interpolation that there exists some function \(\theta \in L^\infty(\mathbb{R}^+; L^N_{N,\infty} \cap L^{N,\infty})\) such that, up to extraction,

\[
\psi \theta^n \to \psi \theta \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}^+; B^{-3+\frac{N}{q}-\varepsilon}_{q,\infty}) \quad \text{for all} \quad \psi \in C^\infty_c, \quad \varepsilon \in [0, 1] \quad \text{and} \quad q > N.
\]

Similar arguments show that \((u^n \otimes u^n)\) is bounded in the space \(L^\infty(\mathbb{R}^+; L^{N^q}_{N,\infty})\) and that \((\theta^n)\) is bounded in \(L^\infty(\mathbb{R}^+; L^{N^q}_{N,\infty})\) so that, using embeddings, we conclude that \((\partial_t u^n)\) is bounded in \(L^\infty(\mathbb{R}^+; B^{-3+\frac{N}{q}}_{q,\infty})\). Repeating the above compactness argument, we get some distribution \(u \in L^\infty(\mathbb{R}^+; L^{N,\infty}) \cap \tilde{L}^1_{loc}(\mathbb{R}^+; \tilde{B}^{-\frac{N}{q}+1}_{q,\infty})\) so that, up to extraction,

\[
\psi u^n \to \psi u \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}^+; B^{-3+\frac{N}{q}-\varepsilon}_{q,\infty}) \quad \text{for all} \quad \psi \in C^\infty_c, \quad \varepsilon \in [0, 1] \quad \text{and} \quad q > N.
\]

Interpolating with the uniform bounds in \(\tilde{L}^1_{loc}(\mathbb{R}^+; \tilde{B}^{1+\frac{N}{q}}_{q,\infty})\), we discover that convergence for \((\psi u^n)\) also holds in every space \(\tilde{L}^r_{loc}(\mathbb{R}^+; \tilde{B}^{\frac{N}{q}+1-\varepsilon}_{q,\infty})\) with \(r > 1\) and \(q > N\), which suffices to pass to the limit in all the nonlinear terms. So \((\theta, u)\) is a (weak) solution to system (1).

**3.2 Global existence: the general case**

Let us now prove the existence part of Theorem 2 in the general case. Let \(u_0 \in L^{N,\infty}\) and \(\theta_0 \in L^{N^q,\infty} \cap L^{p,\infty}\) for some \(p\) satisfying the conditions of Theorem 2. Assume that the smallness condition (3) is satisfied.

As before, we solve system (1) with smoothed out data and obtain a solution \((\theta^n, u^n)\) in \(C(\mathbb{R}^+; \tilde{B}^{N}_{q,1})\) satisfying

\[
\|u^n(t)\|_{L^{N,\infty}} \leq c(N^{-1}\|\theta_0\|_{L^{N^q,\infty}} + \|u_0\|_{L^{N,\infty}}),
\]

\[
\|\theta^n(t)\|_{L^{N^q,\infty}} \leq \|\theta_0\|_{L^{N^q,\infty}} \quad \text{and} \quad \|\theta^n(t)\|_{L^{p,\infty}} \leq \|\theta_0\|_{L^{p,\infty}}.
\]

Note that \(u^n\) satisfies

\[
u^n(t) = e^{\nu \Delta} u^n_0 - \int_0^t e^{(t-\tau)\nu \Delta} \mathcal{P} \text{ div}(u^n \otimes u^n) d\tau + \int_0^t e^{(t-\tau)\nu \Delta} \mathcal{P}(\theta^n e_N) d\tau.
\]

By virtue of the embedding \(L^{N,\infty} \hookrightarrow \tilde{B}^{N-1}_{q_1,\infty}\) for \(q_1 > N\) and of Proposition 5 we get

\[
u^n_1 \in \tilde{L}^1_{loc}(\mathbb{R}^+; \tilde{B}^{N-1}_{q_1,\infty}) \quad \text{uniformly with respect to} \quad n \quad \text{for} \quad q_1 > N.
\]
Next, we notice that \((u^n \otimes u^n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty(\mathbb{R}_+; L^{\frac{N}{q}, \infty})\) so that, combining the embedding \(L^\frac{N}{2}, \infty \hookrightarrow \dot{B}^{\frac{N-4}{2}, 2}_{q_3, \infty}\) for \(q_2 > N/2\) and Proposition 5,

\[
u^n \in L^\infty_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N-4}{2}, 2}_{q_3, \infty}) \quad \text{uniformly with respect to } n \quad \text{for } q_2 > N/2.
\] (21)

Finally, \((\theta^n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty(\mathbb{R}_+; L^{p, \infty})\) and \(L^{p, \infty} \hookrightarrow \dot{B}^{\frac{N}{q}, p}_{0, \infty}\) for \(q > p\) hence

\[
u^n \in L^\infty_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N}{q}, p+2}_{q_3, \infty}) \quad \text{uniformly with respect to } n \quad \text{for } q_3 > p.
\] (22)

Therefore, compactness arguments similar to those which have been used in the previous section enable us to show that, up to extraction, \((\theta^n, u^n)_{n \in \mathbb{N}}\) tends in the sense of distributions to some \((\theta, u)\) such that \(\theta \in L^\infty(\mathbb{R}_+; L^{\frac{N}{q}, \infty} \cap L^{p, \infty})\) and, for all \(q_1 > N, q_2 > N/2\) and \(q_3 > p,\)

\[
u \in L^\infty(\mathbb{R}_+; L^{N, \infty}) \cap \left( L^1_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N+1}{q_1}, \infty}) + L^\infty_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N-4}{2}, 2}_{q_3, \infty}) \right).
\]

By interpolation with the uniform bounds stated above, we deduce that convergence holds true for all \(\epsilon \in ]0, 1[, \quad q > p, \quad q_1 > N, \quad q_2 > N/2\) and \(q_3 > p,\)

- locally in \(L^\infty_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N-4}{2}, \frac{p}{4}}_{q_3, \infty})\) for the temperature,
- locally in \(L^r(\mathbb{R}_+; \dot{B}^{\frac{N-1}{q+1}, \frac{2}{p-\epsilon}}_{q_1, \infty}) + L^\infty_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N}{q_3}, \frac{p}{4}}_{q_3, \infty})\) for the velocity.

Taking advantage of continuity properties for the paraproduct and the remainder, it is then possible to pass to the limit in \(u^n \theta^n, \quad u^n \theta^n\) and \(u^n \theta^n\) whenever there exist some exponents \(q > p, \quad q_1 > N, \quad q_2 > N/2\) and \(q_3 > p\) satisfying

\[
\frac{N}{q} - \frac{N}{p} + \frac{N}{q_1} + 1 > N \max \left(0, \frac{1}{q} + \frac{1}{q_1} - 1\right),
\] (23)

\[
\frac{N}{q} - \frac{N}{p} + \frac{N}{q_2} - 1 > N \max \left(0, \frac{1}{q} + \frac{1}{q_2} - 1\right),
\] (24)

\[
\frac{N}{q} - \frac{N}{p} + \frac{N}{q_3} - 2 > N \max \left(0, \frac{1}{q} + \frac{1}{q_3} - 1\right).
\] (25)

It is clear that the first condition is satisfied for \(q\) close enough to \(p\). Next, second condition is verified if \(q_2\) is close enough to \(N/2\) provided \(p > N/(N - 1)\). Finally, one can find some \(q_3\) so that the third condition be fulfilled if and only if \(p > \frac{2N}{N+2}\).

Note that in dimension three, this implies that \(p > 3/2\), and that in dimension four, we must have \(p > 4/3\). From dimension five on, one may find some \(q, q_1, q_2\) and \(q_3\) such that conditions (23), (24) and (25) are satisfied with \(p = N/3\). That no further condition is needed to pass to the limit in \(u^n \otimes u^n\) is left to the reader. As a matter of fact, because

\[
u = u^n \otimes (u^n_1 + u^n_2 + u^n_3),
\]

it suffices to put together the fact that \((u^n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty_{loc}(\mathbb{R}_+; L^{N, \infty})\) and the properties which have been stated above for sequences \((u^n_1)_{n \in \mathbb{N}}, \quad (u^n_2)_{n \in \mathbb{N}}\) and \((u^n_3)_{n \in \mathbb{N}}\).
Remark 4 In dimension three, one may prove global existence under the weaker condition that \( p > \frac{6}{5} \). This may be achieved by combining the bounds satisfied by \((u^n_1)_{n \in \mathbb{N}}, (u^n_2)_{n \in \mathbb{N}}\) and \((u^n_3)_{n \in \mathbb{N}}\) with a bootstrap argument.

It is not clear however that one can take \( p \leq \frac{6}{5} \). Indeed, having \( \theta^n \in L^\infty(\mathbb{R}_+; L^{p,\infty}) \) implies that \( u^n_3 \) is no better than \( L^\infty_{\text{loc}}(\mathbb{R}_+; \dot{B}^{1+\frac{N}{p}}_{p,\infty}) \) for all \( q > p \). Hence \( p > \frac{6}{5} \) is needed to pass to the limit in \( \text{div} (\theta^n u^n_3) \).

4 Uniqueness

Let us first give the heuristics of the proof of Theorem 3.

Consider two solutions \((\theta_1, u_1, \nabla \Pi_1)\) and \((\theta_2, u_2, \nabla \Pi_2)\) of system (1) corresponding to the same initial data. Assume that for some \( 1 \leq p < 2N \),

\[
(\theta_i, u_i) \in L^\infty_T(\dot{B}^{1+\frac{N}{p}}_{p,\infty}) \times \left( L^\infty_T(\dot{B}^{1+\frac{N}{p}}_{p,\infty}) \cap \dot{L}^1_T(\dot{B}^{1+\frac{N}{p}}_{p,\infty}) \right)^N \quad \text{for } i = 1, 2.
\]

The system satisfied by the difference \((\delta \theta, \delta u, \nabla \delta \Pi)\) between the two solutions reads

\[
\begin{align*}
\partial_t \delta \theta + \text{div} (u_1 \delta \theta) &= -\text{div} (\theta_2 \delta u), \\
\partial_t \delta u + \text{div} (u_1 \otimes \delta u) - \nu \Delta \delta u + \nabla \delta \Pi &= -\text{div} (\delta u \otimes u_2) + \delta \theta \epsilon_N.
\end{align*}
\]

Note that the right-hand side of the first equation (which is a transport equation associated to the vector field \( u_1 \)) is (at least) one derivative less regular than \( \theta_2 \). Because no smoothing property may be expected for such an equation, this obliges us to perform estimates in \( L^\infty_T(\dot{B}^{-2+\frac{N}{p}}_{p,\infty}) \) for \( \delta \theta \) rather than in \( L^\infty_T(\dot{B}^{1+\frac{N}{p}}_{p,\infty}) \). Now, due to the coupling between the equations for \( \delta u \) and \( \delta \theta \), this loss of one derivative also occurs in the estimates for \( \delta u \). This yields the constraint \( p < 2N \) when bounding the quadratic terms (see Proposition 4).

The second difficulty that we have to face is much more serious: as the space \( \dot{L}^1_T(\dot{B}^{1+\frac{N}{p}}_{p,\infty}) \) fails to be embedded in \( L^1_T(\text{Lip}) \), the vector field \( u_1 \) is not in \( L^1_T(\text{Lip}) \). Therefore, the initial regularity of \( \delta \theta \) need not be preserved during the evolution. It turns out however that \( \dot{B}^{1+\frac{N}{p}}_{p,\infty} \) is embedded in the set \( \text{Loglip} \) of Log-Lipschitz functions so that one may resort to arguments similar to those used by H. Bahouri and J.-Y. Chemin in [3] to prove estimates with (small) loss of regularity. Of course, we will have to cope with the fact that, due to the “tilde”, the space \( \dot{L}^1_T(\dot{B}^{1+\frac{N}{p}}_{p,\infty}) \) is not quite embedded in \( L^1_T(\text{Loglip}) \). Overcoming this ultimate difficulty is the purpose of the next subsection.

4.1 A priori estimates with loss of regularity

This section is devoted to proving a priori estimates with loss of regularity for transport-diffusion equations of the type

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div} (\rho u) - \nu \Delta \rho &= f, \\
\rho|_{t=0} &= \rho_0
\end{align*}
\]
with $u$ a given solenoidal vector field with coefficients in $\bar{L}^{1+\frac{N}{p'}}_T(\dot{B}^{\frac{1+\frac{N}{p}}{p}}_{p,\infty})$ or, more generally, for the nonstationary Stokes system

$$\begin{cases}
\partial_t v + \text{div} (u \otimes v) - \nu \Delta v + \nabla \Pi = f, \\
\text{div} u = 0.
\end{cases}$$

Let us first state a result for equation (26) which is a trifle easier to deal with.

**Proposition 6** Let $1 < p < \infty$ and $s \in \big[-1 - \min\left(\frac{N}{p}, \frac{N}{p'}\right), 1 + \frac{N}{p}\big]$. Let $\rho$ be a solution to the transport-diffusion equation (26). There exists some $N_0 \in \mathbb{N}$ depending only on the choice of the Littlewood-Paley decomposition, a universal constant $C_0$ and two constants $c$ and $C$ depending only on $s$, $p$ and $N$ so that

$$||\nabla u||_{\bar{L}^{s+\frac{N}{p}}_T(\dot{B}^{s+\frac{N}{p}}_{p,\infty})} \leq c$$

(28)

then $\varepsilon_q(t) := C \sum_{q' \leq q} 2^{q'(1+\frac{N}{p'})} \int_0^t ||\tilde{\Delta}_q u||_{L^p} d\tau$ with $\tilde{\Delta}_q = \sum_{|q| \leq N_0} \hat{\Delta}_q$ satisfies

$$\varepsilon_q(T) - \varepsilon_q(t) \leq \frac{1}{2} \left(1 + s + \min\left(\frac{N}{p}, \frac{N}{p'}\right)\right)(q - q') \quad \text{for all } q \geq q'$$

and the following a priori estimate is satisfied for all $t \in [0, T]$:

$$\sup_{q \in \mathbb{Z}} 2^{qs-\varepsilon_q(\tau)} ||\tilde{\Delta}_q \rho(\tau)||_{L^p} \leq C_0 \left(||\rho_0||_{\bar{L}^{s+\frac{N}{p}}_T(\dot{B}^{s+\frac{N}{p}}_{p,\infty})} + \sup_{q \in \mathbb{Z}} \int_0^t 2^{qs-\varepsilon_q(\tau)} ||\tilde{\Delta}_q f(\tau)||_{L^p} d\tau\right).$$

**Proof:** The proof is similar to that of Proposition 4.9 in [11] (see also Theorem 3.12 in [9]). First, we localize equation (26) in the Fourier space by mean of the operator $\tilde{\Delta}_q$. We get

$$\partial_t \tilde{\Delta}_q \rho + \dot{S}_{q-1} u \cdot \nabla \tilde{\Delta}_q \rho - \nu \Delta \dot{\Delta}_q \rho = \dot{\Delta}_q f + F_q$$

with $F_q = F^1_q + F^2_q + F^3_q + F^4_q$ and

$$F^1_q := \sum_{|q' - q| \leq 4} \bar{S}_{q'-1} u \cdot \nabla \Delta_\rho, \quad F^2_q := \sum_{|q' - q| \leq 1} \left(\dot{S}_{q-1} - \dot{S}_{q'-1}\right) u \cdot \nabla \Delta_\rho,$$

$$F^3_q := -\Delta_\rho \left(\sum_{|q' - q| \leq 4} \dot{S}_{q'-1} \partial_i \Delta_\rho \Delta_\rho u^i\right), \quad F^4_q := -\sum_{q' \geq 3} \partial_i \Delta_\rho \left(\sum_{|\alpha| \leq 1} \Delta_{q' + \alpha} u^i\right).$$

Multiply both sides by $|\Delta_\rho|^p \text{sgn}(\Delta_\rho)$, integrate over $\mathbb{R}^N$ and apply Hölder inequality. Owing to $\text{div} \dot{S}_{q-1} u = 0$, we get:

$$\frac{1}{p} \frac{d}{dt} ||\Delta_\rho||_{L^p}^p - \nu \int_{\mathbb{R}^N} \Delta_\rho \text{sgn}(\Delta_\rho) \cdot |\Delta_\rho|^{p-1} \Delta_\rho dx \leq \left(||\Delta_\rho||_{L^p} + ||F_q||_{L^p}\right) ||\Delta_\rho||_{L^p}^{p-1}.$$
Therefore, we end up with

$$- \int_{\mathbb{R}^N} \Delta \dot{\Delta}_q \rho |\dot{\Delta}_q \rho|^p \frac{dx}{dx} \geq \kappa 2^{2q} \| \dot{\Delta}_q \rho \|_{L^p}^p.$$  

Therefore, we end up with

$$\frac{1}{p} \frac{d}{dt} \| \dot{\Delta}_q \rho \|_{L^p}^p + \kappa 2^{2q} \| \dot{\Delta}_q \rho \|_{L^p}^p \leq \left( \| \dot{\Delta}_q f \|_{L^p} + \sum_{i=1}^{4} \| E_i^q \|_{L^p} \right) \| \dot{\Delta}_q \rho \|_{L^p}^{p-1}. \quad (29)$$

In the following calculations, assume that $2 \leq p < \infty$. Using a standard commutation estimate for bounding $F_q^1$ (see e.g. [3]) and the definition of operators $\dot{\Delta}_q$ and $\dot{S}_q$ yields

$$\| F_q^1 \|_{L^p} \lesssim \sum_{|q'-q| \leq 1} \| \nabla \dot{S}_{q',-1} u \|_{L^\infty} \| \dot{\Delta}_q \rho \|_{L^p}, \quad \| F_q^2 \|_{L^p} \leq \sum_{|q'-q| \leq 1} 2^q \| \dot{\Delta}_q u \|_{L^\infty} \| \dot{\Delta}_q \rho \|_{L^p},$$

$$\| F_q^3 \|_{L^p} \lesssim \sum_{q' \leq q+2} 2^q \| \dot{\Delta}_q \rho \|_{L^\infty} \| \dot{\Delta}_q u \|_{L^p}, \quad \| F_q^4 \|_{L^p} \lesssim \sum_{q' \leq q-3} 2^q \| \dot{\Delta}_q \rho \|_{L^p} \| \dot{\Delta}_q u \|_{L^p},$$

with $\dot{\Delta}_q := \sum_{|\alpha| \leq N_0} \dot{\Delta}_q \rho'$ for some large enough positive integer $N_0$.

Note that

$$\| \nabla \dot{S}_{q',-1} u \|_{L^\infty} \leq \sum_{q'' < q' - N_0} 2^{q''(1+\frac{N}{p})} \| \Delta_{q''} \rho u \|_{L^p},$$

$$\| \dot{\Delta}_q u \|_{L^\infty} \leq C 2^{q \frac{N}{p}} \| \dot{\Delta}_q \rho \|_{L^p}, \quad \| \dot{\Delta}_q \rho \|_{L^\infty} \leq C 2^{q \frac{N}{p}} \| \dot{\Delta}_q \rho \|_{L^p},$$

so that plugging the above inequalities in (29), we get

$$\frac{1}{p} \frac{d}{dt} \| \dot{\Delta}_q \rho \|_{L^p}^p + \kappa \nu 2^{2q} \| \dot{\Delta}_q \rho \|_{L^p}^p \leq \left( \| \dot{\Delta}_q f \|_{L^p} + \sum_{i=1}^{4} \| E_i^q \|_{L^p} \right) \| \dot{\Delta}_q \rho \|_{L^p}^{p-1}$$

with $\varepsilon_r(t) := \int_0^t \sum_{r' \leq r} 2^{q'+(1+\frac{N}{p})} \| \dot{\Delta}_{q'} u \|_{L^p} \, dt$.

Let $\lambda > 0$ be a large enough positive parameter (to be fixed hereafter). We set

$$\rho_\lambda(t) := 2^{qs} e^{-\lambda \varepsilon_q(t)} \| \dot{\Delta}_q \rho(t) \|_{L^p} \quad \text{and} \quad f_\lambda(t) := 2^{qs} e^{-\lambda \varepsilon_q(t)} \| \dot{\Delta}_q f(t) \|_{L^p}.$$

Obviously, the above inequality rewrites

$$\frac{1}{p} \frac{d}{dt} (\rho_\lambda)^p + \lambda \varepsilon'_\lambda (\rho_\lambda)^p + \kappa \nu 2^{2q} (\rho_\lambda)^p \leq (\rho_\lambda)^{p-1}$$

$$\times \left( f_\lambda + C 2^{qs} e^{-\lambda \varepsilon_q} \left( \sum_{q' \leq q-4} 2^{q'-q}(1+\frac{N}{p}) \varepsilon'_q \| \dot{\Delta}_{q'} \rho \|_{L^p} + \sum_{q' \leq q} 2^{q'-q}(1+\frac{N}{p}) \varepsilon'_q \| \dot{\Delta}_{q'} \rho \|_{L^p} \right) \right).$$
so that, performing a time integration, we eventually get:

\[
\rho^\lambda_q(t) + \kappa \nu 2^q \int_0^t \rho^\lambda_q(\tau') d\tau' + \lambda \int_0^t \varepsilon'_q(\tau') \rho^\lambda_q(\tau') d\tau' \leq \rho^\lambda_q(0) + \int_0^t f^\lambda_q(\tau) d\tau + C \sum_{q' \geq q} 2^{(q-q')(1+\frac{N}{p}+s)} \int_0^t \varepsilon'_q(\tau) \rho^\lambda_q(\tau') d\tau' + C \sum_{q' \geq q} 2^{(q-q')(1+\frac{N}{p}+s)} \int_0^t \varepsilon'_q(\tau) \rho^\lambda_q(\tau') d\tau'.
\]

Using the decomposition \( \varepsilon'_q(\tau) = \varepsilon'_q(\tau) + (\varepsilon'_q - \varepsilon'_q)(\tau) \) in the last term and the fact that sequence \((\varepsilon_n)_{n \in \mathbb{Z}}\) is nonnegative and nondecreasing, we gather that for all \( q \in \mathbb{Z} \),

\[
\rho^\lambda_q(t) + \kappa \nu 2^q \int_0^t \rho^\lambda_q(\tau') d\tau' + \lambda \int_0^t \varepsilon'_q(\tau') \rho^\lambda_q(\tau') d\tau' \leq \rho^\lambda_q(0) + \int_0^t f^\lambda_q(\tau) d\tau + C \sum_{q' \geq q} 2^{(q-q')(1+\frac{N}{p}+s)} \int_0^t \varepsilon'_q(\tau) \rho^\lambda_q(\tau') d\tau' + C \sum_{q' \geq q} 2^{(q-q')(1+\frac{N}{p}+s)} \sup_{\tau \in [0,t]} \rho^\lambda_q(\tau).
\]

Suppose now that the following condition is satisfied:

\[
\sup_{q \in \mathbb{Z}} 2^{q(1+\frac{N}{p})} \| \Delta_q u \|_{L^1(L^p)} \leq \varepsilon \log 2 \quad \text{for some } \varepsilon \text{ such that } \lambda \varepsilon \leq \frac{1}{2} \left( \frac{N}{p} + 1 + s \right). \tag{30}
\]

This ensures that for all \( q' \geq q \), we have

\[
2^{(q-q')(1+\frac{N}{p}+s)} \leq 2^{2^{\frac{N}{2}}(1+\frac{N}{p}+s)}.
\]

Taking the supremum with respect to \( q \) in the equation preceding (30), we thus get

\[
\sup_{q \in \mathbb{Z}} \left( \rho^\lambda_q(\tau) + \kappa \nu 2^q \int_0^t \rho^\lambda_q(\tau') d\tau' + \lambda \int_0^t \varepsilon'_q(\tau') \rho^\lambda_q(\tau') d\tau' \leq \sup_{q \in \mathbb{Z}} \rho^\lambda_q(0) + \int_0^t f^\lambda_q(\tau) d\tau + C \sup_{q \in \mathbb{Z}} \int_0^t \varepsilon'_q(\tau) \rho^\lambda_q(\tau') d\tau' + C \sum_{q' \geq q} 2^{(q-q')(1+\frac{N}{p}+s)} \sup_{\tau \in [0,t]} \rho^\lambda_q(\tau).
\]

Therefore,

\[
\sup_{q \in \mathbb{Z}} \rho^\lambda_q(\tau) + \nu \sup_{q \in \mathbb{Z}} 2^q \int_0^t \rho^\lambda_q(\tau) d\tau + \lambda \sup_{q \in \mathbb{Z}} \int_0^t \varepsilon'_q(\tau) \rho^\lambda_q(\tau) d\tau \leq \| \rho_0 \|_{B_{p,\infty}} + 3 \sup_{q \in \mathbb{Z}} \int_0^t f^\lambda_q(\tau) d\tau + 3C \sup_{q \in \mathbb{Z}} \int_0^t \varepsilon'_q(\tau) \rho^\lambda_q(\tau) d\tau + \frac{3C}{\lambda} \sup_{\tau \in [0,t]} \rho^\lambda_q(\tau).
\]

\]
In order to conclude, it is only a matter of choosing $\lambda = 6C$. We get

$$\sup_{q \in \mathbb{Z}} \rho_q^\lambda(\tau) + \nu \sup_{q \in \mathbb{Z}} 2^{q \lambda} \int_0^t \rho_q^\lambda(\tau) d\tau \leq C_0 \left( \|\rho_0\|_{B_{p,\infty}} + \sup_{q \in \mathbb{Z}} \int_0^t f_q^\lambda(\tau) d\tau \right).$$

This is exactly what we wanted.

In the case $p < 2$, the above bound for $F_q^1$ turns out to be wrong. It may be replaced however by the following inequality

$$\|F_q^1\|_{L^p} \leq C \sum_{q \geq q_0 - 3} 2^{q \lambda} \|\dot{\Delta}_q \rho\|_{L^p} \|\dot{\Delta}_q u\|_{L^p}.$$

Hence, knowing that $\|\dot{\Delta}_q \rho\|_{L^p} \leq C 2^{q' \lambda} \|\dot{\Delta}_q \rho\|_{L^p}$, the term $2^{q'(q-q') \lambda} \|\dot{\Delta}_q \rho\|_{L^p}$ has to replaced by $2^{q' \lambda} \|\dot{\Delta}_q \rho\|_{L^p}$ in all the summations over indices $(q, q')$ such that $q' \geq q - 4$. Again, this leads to the desired inequality.

Let us now extend the previous estimate to the nonstationary Stokes system (27).

**Proposition 7** Let $u$ be as in Proposition 6 and $v$ satisfy the nonstationary Stokes system (27). Then

$$\sup_{q \in \mathbb{Z}} \int_0^t 2^{q \varepsilon_q(\tau)} \|\dot{\Delta}_q v(\tau)\|_{L^p} d\tau + \nu \sup_{q \in \mathbb{Z}} \int_0^t 2^{q \varepsilon_q(\tau)} \|\dot{\Delta}_q v(\tau)\|_{L^p} d\tau$$

$$\leq C_0 \left( \|v_0\|_{B_{p,\infty}} + \sup_{q \in \mathbb{Z}} \int_0^t 2^{q \varepsilon_q(\tau)} \|\dot{\Delta}_q F_q(\tau)\|_{L^p} d\tau \right)$$

where $\mathcal{P}$ stands for the Leray projector over solenoidal vector fields.

**Proof:** Applying operator $\mathcal{P}$ to the identity

$$\partial_t \dot{\Delta}_q v + \dot{S}_{q-1} u \cdot \nabla \dot{\Delta}_q v - \nu \Delta \dot{\Delta}_q v = \dot{\Delta}_q f + F_q,$$

and using that $\mathcal{P} \dot{\Delta}_q v = \dot{\Delta}_q v$, we get

$$\partial_t \dot{\Delta}_q v + \dot{S}_{q-1} u \cdot \nabla \dot{\Delta}_q v - \nu \Delta \dot{\Delta}_q v = \mathcal{P} \dot{\Delta}_q f + \mathcal{P} F_q + [S_{q-1} u, \mathcal{P}] \cdot \nabla \dot{\Delta}_q v.$$

On the one hand, because $\mathcal{P} F_q$ is supported in some annulus $2^q C(0, r_1, r_2)$, there exists a constant $C > 0$ such that

$$\|\mathcal{P} F_q\|_{L^p} \leq C \|F_q\|_{L^p}.$$

On the other hand, standard commutator estimates (see e.g. [5]) ensure that the new term $[S_{q-1} u, \mathcal{P}] \cdot \nabla \dot{\Delta}_q v$ satisfies the same inequalities as $F_q^1$. Arguing as for system (26), it is then easy to complete the proof.
4.2 Proof of Theorem 3

Remind that the system satisfied by the difference between the two solutions reads
\[
\begin{align*}
\partial_t \delta \theta + \text{div} (u_1 \delta \theta) &= - \text{div} (\theta_2 \delta u), \\
\partial_t \delta u + \text{div} (u_1 \otimes \delta u) - \nu \Delta \delta u + \nabla \Pi &= - \text{div} (\delta u \otimes u_2) + \delta \theta \varepsilon_N.
\end{align*}
\]
We aim at proving that \((\delta \theta, \delta u) \equiv 0\). To achieve it, we shall apply proposition \(\ref{prop:delta-0}\) or \(\ref{prop:delta-0-2}\) to the two equations of the above system with \(s = -2 + \frac{\alpha}{p} - \eta\) (for some positive \(\eta\) such that \(\eta < -1 + N \min(\frac{1}{p}, 1)\), which is consistent with the assumption \(p < 2N\).

So we first have to justify that \(\delta \theta\) and \(\delta u\) belong to \(L_T^\infty(\tilde{B}_{p,\infty}^{\frac{\alpha}{p} + \varepsilon})\) for all \(\varepsilon \in [0, 1]\). Combining interpolation with the assumptions on the solutions \((u_1, \theta_1)\) and \((u_2, \theta_2)\), we see that \(u_i\) belongs to every space \(\tilde{L}_T^{r}(\tilde{B}_{p,\infty}^{\frac{\alpha}{p} - 1 + \frac{\eta}{2}})\) with \(1 \leq r \leq \infty\). Because
\[
\partial_t \delta \theta = - \text{div} (\theta_2 u_2 - \theta_1 u_1), \quad \delta \theta(0) = 0,
\]
Proposition \(\ref{prop:delta-0}\) and Hölder inequality enable us to get \(\delta \theta \in C([0, T]; \tilde{B}_{p,\infty}^{\frac{\alpha}{p} + \varepsilon})\) for all \(\varepsilon \in [0, 1]\). Plugging this new information in the equation for \(\delta u\) and using again Proposition \(\ref{prop:delta-0}\) it is then easy to justify that \(\delta u\) is also in \(L_T^\infty(\tilde{B}_{p,\infty}^{\frac{\alpha}{p} + \varepsilon})\).

One can now tackle the proof of uniqueness. Assume that the constant \(c\) has been chosen so small as condition (28) to be satisfied by the vector field \(u_1\). Denoting
\[
\varepsilon_q(t) = C \sum_{q' \leq q} 2^{q(1 + \frac{\alpha}{p})} \|\dot{\Delta}_q u_1\|_{L_q^1(L^p)},
\]
\[
\delta \Theta(t) := \sup_{\tau \in [0, t]} \sup_{q \in \mathbb{Z}} 2^{-q(-2 + \frac{\alpha}{p} - \eta) - \varepsilon_q(\tau)} \|\dot{\Delta}_q \delta \theta(\tau)\|_{L^p},
\]
\[
\delta U(t) := \sup_{\tau \in [0, t]} \sup_{q \in \mathbb{Z}} 2^{-q(-2 + \frac{\alpha}{p} - \eta) - \varepsilon_q(\tau)} \|\dot{\Delta}_q \delta u(\tau)\|_{L^p} + \nu \sup_{q \in \mathbb{Z}} \int_0^t 2^{q(\frac{\alpha}{p} - \eta) - \varepsilon_q(\tau)} \|\dot{\Delta}_q \delta u\|_{L^p} \, d\tau,
\]
we thus get according to Propositions \(\ref{prop:delta-0}\) and \(\ref{prop:delta-0-2}\)
\[
\delta \Theta(t) \leq C \sup_{q} \int_0^t 2^{q(-2 + \frac{\alpha}{p} - \eta) - \varepsilon_q(\tau)} \|\dot{\Delta}_q \text{div} (\theta_2 \delta u)\|_{L^p} \, d\tau,
\]
\[
\delta U(t) \leq C \sup_{q} \int_0^t 2^{q(-2 + \frac{\alpha}{p} - \eta) - \varepsilon_q(\tau)} \|\dot{\Delta}_q \text{div} (\delta u \otimes u_2)\|_{L^p} + \|\dot{\Delta}_q \delta \theta\|_{L^p} \, d\tau.
\]
Let us admit that the nonlinear terms may be bounded as follows (see the proof in the appendix):
\[
\sup_{q} \int_0^t 2^{q(-2 + \frac{\alpha}{p} - \eta) - \varepsilon_q(\tau)} \|\dot{\Delta}_q \text{div} (\theta_2 \delta u)\|_{L^p} \, d\tau \leq C \|\theta_2\|_{L_T^\infty(\tilde{B}_{p,\infty}^{\frac{\alpha}{p} + \varepsilon})} \sup_{q} \int_0^t 2^{q(\frac{\alpha}{p} - \eta) - \varepsilon_q(\tau)} \|\dot{\Delta}_q \delta u\|_{L^p} \, d\tau,
\]
We eventually get
\[ \text{when, if} \]
\[ \sum \text{of indices of regularity is negative.} \]
Carrying out the method which has been used in the previous section to the limit case \( \theta \), \( \delta U \) Gronwall Lemma thus ensures that
\[ L \] so as to have a velocity field in
\[ \text{that which has been used in [8] and [10]) will be required. Let us state the result.} \]
\[ \text{in Proposition 4 so that a logarithmic interpolation argument (similar to} \]
\[ \text{that has been used in [8 and 10] will be required. Let us state the result.} \]
\[ \text{which} \]
\[ \text{on} \]
\[ \text{if} \]
\[ \text{a priori estimate is needed. On} \]
\[ \text{for} \]
\[ \text{one may resort to Besov spaces with third index} 1 \]
\[ \text{so as to have a velocity field in} L^1(\text{Lip). Thus no losing a priori estimate is needed. On} \]
\[ \text{the other hand, due to the weak regularity assumptions, we shall be in the limit case for} \]
\[ \text{laws in Proposition 4 so that a logarithmic interpolation argument (similar to} \]
\[ \text{that which has been used in [8 and 10] will be required. Let us state the result.} \]
\[ \text{with the same data. Assume that} \]
\[ \text{for} i = 1, 2. \]
\[ \text{Then} (\theta, u_1, \nabla \Pi_1) \equiv (\theta_2, u_2, \nabla \Pi_2) \text{ on} [0, T]. \]
\[ \text{Proof:} \]
\[ \text{We omit the proof of the fact that} (\partial \theta, \delta u, \nabla \Pi) \text{ belongs to the space} \]
\[ \text{G}_T := L^\infty \left( \dot{B}^{\frac{3}{2}}_{2, N, \infty} \right) \times \left( \dot{C}([0, T]; \dot{B}^{\frac{3}{2}}_{2, N, \infty}) \cap \dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty}) \right)^N \times \left( \dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty}) \right)^N. \]
\[ \text{In the following computations, the space} G_T \text{ will be endowed with the norm} \]
\[ \| (\theta, u, \nabla \Pi) \|_{G_T} := \| \theta \|_{L^\infty \left( \dot{B}^{\frac{3}{2}}_{2, N, \infty} \right)} + \| u \|_{L^\infty \left( \dot{B}^{\frac{3}{2}}_{2, N, \infty} \right)} + \| u \|_{\dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty})} + \| \nabla \Pi \|_{\dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty})}. \]
\[ \text{Theorem 5 Let} (\theta_1, u_1, \nabla \Pi_1) \text{ and} (\theta_2, u_2, \nabla \Pi_2) \text{ satisfy (1) with the same data. Assume that} \]
\[ \text{that which has been used in [8 and 10] will be required. Let us state the result.} \]
\[ \text{with} \]
\[ \text{on} \]
\[ \text{if} \]
\[ \text{a priori estimate is needed. On} \]
\[ \text{for} \]
\[ \text{one may resort to Besov spaces with third index} 1 \]
\[ \text{so as to have a velocity field in} L^1(\text{Lip). Thus no losing a priori estimate is needed. On} \]
\[ \text{the other hand, due to the weak regularity assumptions, we shall be in the limit case for} \]
\[ \text{laws in Proposition 4 so that a logarithmic interpolation argument (similar to} \]
\[ \text{that which has been used in [8 and 10] will be required. Let us state the result.} \]
\[ \text{with the same data. Assume that} \]
\[ \text{for} i = 1, 2. \]
\[ \text{Then} (\theta_1, u_1, \nabla \Pi_1) \equiv (\theta_2, u_2, \nabla \Pi_2) \text{ on} [0, T]. \]
\[ \text{Proof:} \]
\[ \text{We omit the proof of the fact that} (\partial \theta, \delta u, \nabla \Pi) \text{ belongs to the space} \]
\[ \text{G}_T := L^\infty \left( \dot{B}^{\frac{3}{2}}_{2, N, \infty} \right) \times \left( \dot{C}([0, T]; \dot{B}^{\frac{3}{2}}_{2, N, \infty}) \cap \dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty}) \right)^N \times \left( \dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty}) \right)^N. \]
\[ \text{In the following computations, the space} G_T \text{ will be endowed with the norm} \]
\[ \| (\theta, u, \nabla \Pi) \|_{G_T} := \| \theta \|_{L^\infty \left( \dot{B}^{\frac{3}{2}}_{2, N, \infty} \right)} + \| u \|_{L^\infty \left( \dot{B}^{\frac{3}{2}}_{2, N, \infty} \right)} + \| u \|_{\dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty})} + \| \nabla \Pi \|_{\dot{L}^1_T(\dot{B}^{\frac{3}{2}}_{2, N, \infty})}. \]
A priori estimates for the transport equation (see e.g. the limit case in Proposition 4.7 of [11]) and inequality (11) guarantee that for all \( t \leq T \)

\[
\| \delta u \|_{L^\infty_t(B_{2N,\infty}^{\frac{3}{2}})} \leq C \exp \left( C \| \nabla u_1 \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})} \right) \int_0^t \| \delta u \|_{L^1(B_{2N,1}^{\frac{3}{2}})} \| \theta_2 \|_{L^1(B_{2N,\infty}^{\frac{1}{2}})} \ d\tau. \tag{31}
\]

Next, we have according to Proposition 3.2 of [2] and Inequality (16),

\[
\| \delta u \|_{L^\infty_t(B_{2N,\infty}^{\frac{3}{2}})} + \nu \| \delta u \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})} + \| \nabla \delta \Pi \|_{L^1_t(B_{2N,\infty}^{\frac{3}{2}})} \\
\leq C e^{C \| u_1 \|_{L^1(B_{2N,1}^{\frac{3}{2}})}} \int_0^t \| \delta e_N - \text{div} (\delta u \otimes u_2) \|_{B_{2N,\infty}^{\frac{3}{2}}} \ d\tau.
\]

Because \( \text{div} \delta u = 0 \), we have for \( 1 \leq i \leq N \),

\[
(\text{div} (\delta u \otimes u_2))^i = \partial_j (T_{\delta u}^i u_2^j + R(\delta u^j, u_2^j)) + T_{\partial_i u_2} \delta u^j.
\]

So using Proposition 3, we find that

\[
\| \text{div} (\delta u \otimes u_2) \|_{B_{2N,\infty}^{\frac{3}{2}}} \leq C \| \delta u \|_{B_{2N,\infty}^{\frac{3}{2}}} \| u_2 \|_{B_{2N,1}^{\frac{3}{2}}}.\]

Now, taking advantage of Proposition 1.8 in [8], one may write

\[
\| \delta u \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})} \leq C \| \delta u \|_{L^1(B_{2N,\infty}^{\frac{3}{2}})} \log \left( e + \frac{\| \delta u \|_{L^1(B_{2N,1}^{\frac{3}{2}})} + \| \delta u \|_{L^1(B_{2N,\infty}^{\frac{3}{2}})}}{\| \delta u \|_{L^1(B_{2N,\infty}^{\frac{3}{2}})}} \right).
\]

Let us introduce the notation

\[
V(t) = t \left( \| u_1 \|_{L^\infty_t(B_{2N,1}^{\frac{3}{2}})} + \| u_2 \|_{L^\infty_t(B_{2N,1}^{\frac{3}{2}})} \right) + \| u_1 \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})} + \| u_2 \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})},
\]

\[
W(t) = \| \delta u \|_{L^\infty_t(B_{2N,\infty}^{\frac{3}{2}})} + \| \delta u \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})} + \| \nabla \delta \Pi \|_{L^1_t(B_{2N,\infty}^{\frac{3}{2}})}.
\]

Because

\[
\| \delta u \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})} + \| \delta u \|_{L^1_t(B_{2N,1}^{\frac{3}{2}})} \leq V(t),
\]

and the map \( x \mapsto x \ln(e + \frac{y}{x}) \) (for fixed \( y \geq 0 \)) is nondecreasing over \( \mathbb{R}_+ \), we end up with

\[
W(t) \leq C e^{C \| u_1 \|_{L^1(B_{2N,1}^{\frac{3}{2}})}} \int_0^t \left( \| u_2(s) \|_{B_{2N,1}^{\frac{3}{2}}} + \| \theta_2(s) \|_{B_{2N,\infty}^{\frac{1}{2}}} \right) W(s) \log \left( e + \frac{V(s)}{W(s)} \right) ds.
\]

Applying Osgood lemma (see e.g. [5]) thus yields \( W \equiv 0 \) on \([0, T] \) whence also \( \delta \theta = 0 \) according to inequality (31). This completes the proof. \( \blacksquare \)
5 Appendix

This appendix is devoted to proving the estimates for the convection terms that we used in section 4.2.

**Lemma 4** Let \((\alpha_q)_{q \in \mathbb{Z}}\) be a sequence of nonnegative functions over \([0,T]\). Let \(s_1, s_2, p\) satisfy

\[
1 \leq p \leq \infty, \quad \frac{N}{p} + 1 > s_1, \quad \frac{N}{p} > s_2 \quad \text{and} \quad s_1 + s_2 > N \max \left(0, \frac{2}{p} - 1\right).
\]

Assume that for all \(q' \geq q\) and \(t \in [0,T]\), we have

\[
0 \leq \alpha_q(t) - \alpha_q(t) \leq \frac{1}{2} \left(s_1 + s_2 + N \min \left(0, 1 - \frac{2}{p}\right)\right)(q' - q).
\]

Then for all \(r \in [1,\infty]\), there exists a constant \(C\) depending only on \(s_1, s_2, N\) and \(p\) such that for all function \(b\) and solenoidal vector field \(a\) over \(\mathbb{R}^N\), the following estimate holds true for all \(t \in [0,T]\):

\[
\sup_{q \in \mathbb{Z}} \int_0^t 2^{q(s_1+s_2-1-\frac{N}{p})-\alpha_q(\tau)} \|\dot{\Delta}_q \text{div}(ab)\|_{L^p} \, d\tau \leq C\|b\|_{\mathcal{L}_1(B^{s_1}_{p,\infty})} \sup_{q \in \mathbb{Z}} 2^{qs_2-\alpha_q} \|\dot{\Delta}_qa\|_{L^p} \big|_{L^t_1}.
\]

**Proof:** The proof relies on Bony’s decomposition. Knowing that \(\text{div} b = 0\), we have (with the usual summation convention over repeated indices):

\[
\dot{\Delta}_q \text{div} (ab) = \dot{\Delta}_q (\mathring{T}_{\partial_j} ba^j) + \dot{\Delta}_q (\mathring{T}_{a} \partial_j b) + \dot{\Delta}_q \partial_j \mathcal{R}(a^j, b).
\]

By virtue of (7), one may write

\[
\dot{\Delta}_q (\mathring{T}_{\partial_j} ba^j) = \sum_{|q' - q| \leq 4} \dot{\Delta}_q (\mathring{S}_{q' - 1 \partial_j} b \Delta_{q'} a^j).
\]

For the sake of simplicity, let us proceed as if \(\dot{\Delta}_q (\mathring{T}_{\partial_j} ba^j) = \mathring{S}_{q - 1 \partial_j} b \dot{\Delta}_qa^j\) (having (32) justifies this approximation). Using the definition of \(\mathring{S}_{q - 1}\), we get

\[
2^{q(s_1-1-\frac{N}{p})} \|\mathring{S}_{q - 1 \partial_j} b \Delta_{q'} a^j\|_{L^p} \leq \|\dot{\Delta}_qa\|_{L^p} \sum_{q' \leq q - 2} 2^{q(s_1-1-\frac{N}{p})} \|\dot{\Delta}_q \nabla b\|_{L^\infty} 2^{(q-q')(s_1-1-\frac{N}{p})}.
\]

In consequence, we have for all \(0 \leq t \leq T\),

\[
\int_0^t 2^{q(s_1+s_2-1-\frac{N}{p})-\alpha_q(\tau)} \|\mathring{S}_{q - 1 \partial_j} b \Delta_{q'} a^j\|_{L^p} \, d\tau \\
\leq \|2^{qs_2-\alpha_q} \|\dot{\Delta}_qa\|_{L^p} \big|_{L^t_1} \sum_{q' \leq q - 2} 2^{q(s_1-1-\frac{N}{p})} \|\dot{\Delta}_q \nabla b\|_{L^t_1(\mathcal{L}^{\infty})} 2^{(q-q')(s_1-1-\frac{N}{p})}.
\]

Since \(s_1 - 1 - \frac{N}{p} < 0\), we thus get

\[
\sup_{q \in \mathbb{Z}} \int_0^t 2^{q(s_1+s_2-1-\frac{N}{p})-\alpha_q(\tau)} \|\dot{\Delta}_q \mathring{T}_{\partial_j} ba^j\|_{L^p} \, d\tau \\
\leq C\|\nabla b\|_{\mathcal{L}_1(B^{s_1-1-\frac{N}{p}}_{p,\infty})} \sup_{q \in \mathbb{Z}} 2^{qs_2-\alpha_q} \|\dot{\Delta}_qa\|_{L^p} \big|_{L^t_1}.
\]

(34)
Likewise, in order to bound the second term of \((33)\), one may proceed as if
\[
\hat{\Delta}_q \left( \hat{T}_{a^j} \partial_j b \right) = \hat{S}_{q-1} a^j \hat{\Delta}_q \partial_j b.
\]
Now, for all \(0 \leq \tau \leq T\), we have
\[
2^q\left( s_1 + s_2 - 1 - \frac{N}{N_p} \right) - \alpha_q(\tau) \| \hat{S}_{q-1} a^j \hat{\Delta}_q \partial_j b \|_{L^p} \leq 2^q\left( s_1 - 1 \right) \| \hat{\Delta}_q \nabla b \|_{L^p} \sum_{q' \leq q-2} 2^{-q'\left( \frac{N}{N_p} - s_2 \right) - \alpha_q(\tau)} \| \hat{\Delta}_{q'} a \|_{L^\infty} 2^{q'-q'\left( \frac{N}{N_p} - s_2 \right)}.
\]
Knowing that \(\alpha_q \geq \alpha_{q'}\) for \(q \geq q'\), we deduce that
\[
\int_0^t 2^q\left( s_1 + s_2 - 1 - \frac{N}{N_p} \right) - \alpha_q(\tau) \| \hat{S}_{q-1} a^j \hat{\Delta}_q \partial_j b \|_{L^p} dt \leq 2^q\left( s_1 - 1 \right) \| \hat{\Delta}_q \nabla b \|_{L^p} \sum_{q' \leq q-2} 2^{-q'\left( \frac{N}{N_p} - s_2 \right) - \alpha_{q'}(\tau)} \| \hat{\Delta}_{q'} a \|_{L^\infty} \| \hat{\Delta}_{q'} b \|_{L^\infty} 2^{q'-q'}\left( \frac{N}{N_p} - s_2 \right).
\]
Combining Bernstein inequality and the fact that \(s_2 < \frac{N}{N_p}\), we thus conclude that
\[
\sup_{q \in \mathbb{Z}} \int_0^t 2^q\left( s_1 + s_2 - 1 - \frac{N}{N_p} \right) - \alpha_q(\tau) \| \hat{S}_{q-1} a^j \hat{\Delta}_q \partial_j b \|_{L^p} dt \leq C \| \nabla b \|_{\dot{L}^{\infty}_t(B^s_{p,\infty})} \sup_{q \in \mathbb{Z}} 2^{q_2 - \alpha_q} \| \hat{\Delta}_q a \|_{L^p} \| \hat{\Delta}_q b \|_{L^p}.
\]
In order to treat the remainder term, we shall consider the cases \(p \geq 2\) and \(p < 2\). Let us start with the case \(p \geq 2\) which is slightly easier. We have
\[
\hat{\Delta}_q \partial_j \hat{R}(a^j, b) = \sum_{q' \geq q-3} \partial_j \hat{\Delta}_q \left( \hat{\Delta}_{q'} a^j \hat{\Delta}_{q'} b \right) \quad \text{with} \quad \hat{\Delta}_{q'} = \hat{\Delta}_{q'-1} + \hat{\Delta}_{q'} + \hat{\Delta}_{q'+1}.
\]
Because \(\mathcal{F}(\partial_j \hat{\Delta}_q(\hat{\Delta}_{q'} a^j \hat{\Delta}_{q'} b))\) is supported in a ball of size \(2^q\), Bernstein inequality ensures that
\[
2^q\left( s_1 + s_2 - 1 - \frac{N}{N_p} \right) - \alpha_q(\tau) \| \hat{\Delta}_q \partial_j \hat{R}(a^j, b) \|_{L^p} \leq C \sum_{q' \geq q-3} 2^q\left( s_1 + s_2 \right) - \alpha_q(\tau) \| \hat{\Delta}_{q'} a \|_{L^p} \| \hat{\Delta}_{q'} b \|_{L^p},
\]
whence
\[
\int_0^t 2^q\left( s_1 + s_2 - 1 - \frac{N}{N_p} \right) - \alpha_q(\tau) \| \hat{\Delta}_q \partial_j \hat{R}(a^j, b) \|_{L^p} dt \leq C \sum_{q' \geq q-3} \int_0^t 2^{q_2 - \alpha_{q'}(\tau)} \| \hat{\Delta}_{q'} a \|_{L^p} \| \hat{\Delta}_{q'} b \|_{L^p} \times 2^{(q - q')(s_1 + s_2)}.
\]
Thanks to assumption \((32)\), we have
\[
(\alpha_{q'} - \alpha_q)(\tau) + (q - q')(s_1 + s_2) \leq (q - q')\left( \frac{s_1 + s_2}{2} \right) \quad \text{for all} \quad q' \geq q.
\]
As \(s_1 + s_2 > 0\), we end up for all \(q \in \mathbb{Z}\) with
\[
\int_0^t 2^q\left( s_1 + s_2 - 1 - \frac{N}{N_p} \right) - \alpha_q(\tau) \| \hat{\Delta}_q \partial_j \hat{R}(a^j, b) \|_{L^p} dt \lesssim \| b \|_{\dot{L}^{\infty}_t(B^s_{p,\infty})} \sup_{q \in \mathbb{Z}} 2^{q_2 - \alpha_q} \| \hat{\Delta}_q a \|_{L^p} \| \hat{\Delta}_q b \|_{L^p},
\]
which, together with \((33)\), \((34)\) and \((35)\), completes the proof in the case \(p \geq 2\).
If $p < 2$, one may write (use Bernstein inequality)

$$2^{q(s_1+s_2-1+\frac{N}{p})-\alpha_q(\tau)} \| \tilde{\Delta} q \partial_j R(a', b) \|_{L^p} \leq C \sum_{q' \geq q-3} 2^{q(s_1+s_2+\frac{N}{p})-\alpha_q(\tau)} \| \tilde{\Delta} q a \|_{L^{p'}} \| \tilde{\Delta} q b \|_{L^p}$$

and use that $\| \tilde{\Delta} q a \|_{L^{p'}} \leq C 2^{q'\left(\frac{N}{p} - \frac{N}{p'}\right)} \| \tilde{\Delta} q a \|_{L^p}$. Therefore, one may go along the lines of the case $p \geq 2$. It is only a matter of changing the term $2^{q-q'}(s_1+s_2)$ into $2^{(q-q')(s_1+s_2+\frac{N}{p} - \frac{N}{p'})}$.

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