A Binary Nature of Funding Impacts in Bilateral Contracts

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Abstract
We discuss a binary nature of funding impacts. Under some conditions, funding is either cost or benefit, i.e., one of the lending/borrowing rates does not play any role in pricing derivatives. When we price derivatives, considering different lending/borrowing rates leads to semi-linear BSDEs and PDEs, so we need to solve the equations numerically. However, once we can guarantee that only one of the rates affects pricing, we can recover linear equations and derive analytic formulae. Moreover, as a byproduct, our results explain how debt value adjustment (DVA) and funding benefits are different. It is often believed that DVA and funding benefits are overlapped but it will be shown that the two components are affected by different mathematical structures of derivative transactions. We will see later that FBA occurs where the payoff is non-increasing, but this relationship becomes weaken as the funding choices of underlying assets are transferred to repo markets.

Key words: BSDEs, Malliavin calculus, bilateral contracts, incremental cash-flows, funding benefits
AMS subject classifications: 60H07, 91G20, 91G40

Acknowledgments: We thank Stéphane Crépey for spending time to help us improve this paper and we took into account his comments in this paper. We are also grateful to Dylan Possamaï for interesting discussion at Workshop on Risk Measures, XVA Analysis, Capital Allocation and Central Counterparties (2016).

†Junbeom Lee is funded from the Singapore MOE AcRF Grant R-146-000-255-114. Email: matleej@nus.edu.sg.
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1 Introduction

The financial crisis in 2007-2008 has forced us to examine many parts of general practice to price derivatives. In the crisis, the defaults of big firms heightened default risk. Moreover, banks became more reluctant to lend money and, as a result, the gap between London inter-bank offered rate (LIBOR) and overnight indexed swap (OIS) rate was widened. By the changed market conditions, banks began to make several adjustments in derivative prices. It has long been a standard to make credit value adjustment (CVA). CVA is correcting derivative prices for the risk of counterparty’s default. On the other hand, debt value adjustment (DVA) is a deduction for the hedger’s own default. If one party defaults earlier than the contractual maturity, a promised payment should be settled as close-out amount. However, because of the default, the obligation may not fully honored. For the risk, it is general that collateral is pledged. Funding value adjustment (FVA) is the adjustment of derivative prices for the excessive cost and benefit in maintaining the hedging portfolio and posting collateral.

Considering entity-specific information in derivative prices has required many changes in classical pricing theory. First, since law of one price does not hold, when a price is given for a contract, it may be an arbitrage opportunity for a trader, but may not be for others. For the issue, entity-specific definitions of arbitrage opportunity and fair values were suggested by Bichuch et al. (2018); Bielecki et al. (2018); Bielecki & Rutkowski (2015) and those arguments were applied to various derivatives (Kim et al., 2018a,b, for example). In addition, many other pricing methodologies have been developed. A replication pricing under FVA with collateral was introduced by Piterbarg (2010). Li & Wu (2016); Wu (2015) also discussed replication pricing with CVA, FVA with collateral. For risk-neutral valuation principle, readers can refer to Brigo et al. (2011). Crépey (2015a,b) discussed min-variance pricing under default risks, funding spreads, and collateral.

Still, there have remained many puzzles among the adjustments. Especially, DVA and FVA have their own but intertwined issues. First, there is a doubt on reporting DVA, which is to say that an aggravation of own default risk can be beneficial to the shareholders. As we will see later, theoretically, the increase of own default risk can be monetized by buying back their bonds. However, it is often impossible in practice because the bank should really default to realize the benefit. Indeed, DVA is accepted by both IFRS and GAAP, but excluded from common equity tier 1 capital (CET1), which is a proxy of shareholder’s wealth. Second, it is often believed that DVA double-counts funding benefits (see Remark 2.21). FVA has two parts: funding benefit adjustment (FBA) and funding cost adjustment (FCA). Both DVA and FBA may originate from banks’ own default risk. DVA is a deduction from liabilities of a bank due to its creditworthiness. On the other hand, FBA, as a counterpart of FCA, may countervail FCA which is also from the bank’s own default risk. Thus, including both DVA and FBA may inflate the bank’s reported profit (see Cameron, 2013) as well as deflate the price charged to counterparties. Possibly due to these reasons, seller’s DVA is often excluded in derivative transactions.

FVA is more arguable. According to Modigliani-Miller (MM) theorem, which is a long-established financial principle, choices of funding should not be considered in pricing. On the other hand, in practice, with the increased interest rates offered by funding desks, it will be loss to the traders without recouping the funding costs from counterparties. Indeed, traders feel confident that funding costs are observable in derivative transactions (see Andersen et al., 2017). If the traders’ belief is true, the inclusion of FVA in derivative prices may be justified by market frictions, which is a viola-
tion of the assumptions of MM theorem (see Modigliani & Miller, 1958; Stiglitz, 1969). However, as pointed out by Hull & White (2012), if FVA is really an element to determine derivative prices, the existence of Treasury bonds is mysterious that banks buy a bond that returns less than their funding rates.

The above issues will be discussed by the main results of this paper. Our main contribution is showing a binary nature of FVA that FVA is either FCA or −FBA for many derivative contracts. For example, we shall see later that, based on our model, when buying bonds, the trader will never enter a borrowing position, i.e., FCA = 0 and FVA = −FBA. If we assume the lending rate is equal to OIS rate (for example, as in Burgard & Kjaer, 2010), we have FVA = 0. This may explain the reason why banks do not require FVA when buying Treasury bonds, while FVA is observaible in other derivative transactions.

The switching between funding costs and benefits depends on the structures of derivative payoffs, close-out conventions, and choices of funding, e.g., from a repo market or treasury department. The structure to determine the binary nature of funding impacts is whether the payoffs of derivatives increase or decrease in underlying assets. On the other hand, since DVA is a deduction on derivative payables, DVA occurs where the payoffs are positive. Therefore, DVA and FBA are affected by different mathematical structures of the payoffs. Indeed, it will be shown later by an example that DVA and FBA are clearly separated and this result is in line with a part of the conclusion in Andersen et al. (2017) that DVA should be considered in pricing.

The difference between FBA and DVA were also pointed out by Albanese & Andersen (2014). It was shown that FBA is larger than DVA mainly because of the effective discounting rate and absence of default indicator in the definition even though FBA and DVA are quite similar. In their case study, FBA was 20% larger than DVA. However, we will see later that FBA and DVA are not even close in some cases (see Table 1, for example). The values of FBA and DVA often resemble because of the funding choices commonly taken in the literature that all underlying assets can be acquired from repo markets. When available, repo markets are preferred by dealers, but it is not always possible. As we will see later, FBA occurs where the payoff is non-increasing, but the more assets are traded through repo markets, the weaker this relationship becomes. Readers may want to refer to Remark 3.2-(iv, v) in advance.

Our results are also related to accounting with FVA. In FCA/FBA accounting, which is endorsed by some leading banks, DVA is recorded in Contra-Liability (CL) account and

\[ \text{FCA} - (\text{FBA} - \text{DVA}) \]

is recorded in Contra-Asset (CA) account (see Castagna, 2011). It has been pointed out that this accounting method engenders large asset/liability asymmetry. According to the binary nature of funding impacts, the large asymmetry is inevitable since, for some derivative contracts, either FCA = 0 or FBA = 0, and DVA≠FBA, i.e., FCA does not countervail FBA and FBA is not overlapped with DVA. FVA/FDA accounting, as an alternative, was suggested to recover asset/liability symmetry as well as protect CET1 capital, which is a proxy of shareholder’s wealth (see Albanese & Andersen, 2014, for example). Readers can refer to Andersen et al. (2017) for the issue of shareholder’s wealth protection with funding spreads.

Another benefit of our results is that we can recover linear equations to price derivatives. Because different lending/borrowing rates make the pricing equations semi-linear, analytic solutions
are not generally allowed. Therefore, we need to solve the equations numerically, and sometimes the computational cost becomes expensive. For an attempt to approximate FVA of contracts with short maturity, readers may want to refer to Gobet & Pagliarani (2015). Moreover, there have been several attempts to find closed-form solutions under funding spreads (Bichuch et al., 2018; Brigo et al., 2017; Piterbarg, 2010). Still, in the arguments, the crucial assumption was the same lending/borrowing rates. This itself can be seen as a strong assumption. However, once we can guarantee that one of funding rates does not play any role in pricing derivatives, we can assume the same lending and borrowing rates without loss of generality (because the dealer will never switch from one to another). Put differently, our results allow their results widely applicable.

Even though the binary nature allows us to find analytic formulae for a large class of derivatives, depending on the close-out conventions, the analytic formulae may or may not be represented by a closed-form. In this paper, we deal with two close-out conventions: clean price and replacement cost. clean price (resp. replacement cost) is the risk-neutral price without (resp. with) with value adjustments. Under clean close-out, because pricing measures are not matched, the analytic formulae cannot be represented in a closed-form. Indeed, to avoid the inconsistency, Bichuch et al. (2018) assumed a flexibility to choose a pricing measure for calculating close-out amount, and Brigo et al. (2017) considered only un-collateralized contracts with null cash-flow at defaults. In the case of replacement close-out, the mismatch does not appear, because, in our model, the funding rates are tacitly embedded to the replacement cost. Therefore, we can provide closed-form solutions under replacement close-out.

In our model, we include incremental CVA, DVA, FVA, and variation margin. The reference filtration is generated by a Brownian motion. Then we progressively enlarge the filtration by default times of the two parties. The default times are assumed to have intensities. We do not assume that interest rates are deterministic, so our results can be applied to interest rate derivatives. In the main theorems, we assume volatility and default intensities are deterministic to avoid heavy calculations, but stochastic parameters do not necessarily change the main results. We report one example that intensities are not deterministic in Appendix.

This paper is organized as follows. In Section 2, we introduce our setup on the filtration, intensities, and construct the incremental hedging portfolio. In Section 2.4, we introduce a BSDE to price derivatives on the enlarged filtration. Instead of dealing with the BSDE, we define XVA (x-value adjustment) process in Section 2.3.2 and the XVA process is reduced to a BSDE on the reference filtration as in Crépey & Song (2015). Then our main results are provided in Section 3. For proving the main theorems, iterative transformations of the XVA process are needed and the transformations depend on the close-out conventions. The proofs for the main theorems are reported in Appendix. In Section 4, examples are examined and we provide a closed-form solution for a stock call option with replacement close-out.

2 Modeling

2.1 Mathematical Setup

We consider two parties entering a bilaterally cleared contract. We call one party a “hedger” and the other party a “counterparty”. We sometimes address the hedger (resp. counterparty) “she” (resp.
“he”). An index $H$ (resp. $C$) will be used to represent the hedger (resp. counterparty). The argument of this paper is conducted in view of the hedger. The hedger is a financial firm that holds a portfolio to hedge the exchanged cash-flows of the contract. The counterparty may or may not be a non-financial firm. Note that when we say a “dealer”, “bank”, and “trader”, it is not necessarily addressing the hedger since the counterparty can be also a bank.

Let $(\Omega, \mathcal{G}, \mathbb{Q})$ be a probability space, where $\mathbb{Q}$ is a risk-neutral probability measure. Let $\mathbb{E}$ denote the expectation under $\mathbb{Q}$. We consider random times $\tau^i$, $i \in \{H, C\}$,

$$\tau^i : (\Omega, \mathcal{G}) \rightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)),$$

which represent the default times of the hedger and counterparty. For $i \in \{H, C\}$, we assume that $\mathbb{Q}(\tau^i = 0) = 0$ and $\mathbb{Q}(\tau^i > t) > 0$, $\forall t \in \mathbb{R}_+$. We also denote

$$\tau := \tau^H \land \tau^C, \quad \bar{\tau} := \tau \land T,$$

where $T$ is the maturity of the bilateral contract.

Let $W = (W^1, \ldots, W^n)$ be a standard $n$-dimensional Brownian motion under $\mathbb{Q}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the usual natural filtration of $(W_t)_{t \geq 0}$. Then we define

$$\mathbb{G} = (\mathcal{G}_t)_{t \geq 0} := \left( \mathcal{F}_t \vee \sigma\left(\{\tau^i \leq u\} : u \leq t, i \in \{H, C\}\right) \right)_{t \geq 0}.$$

We call $\mathbb{F}$ (resp. $\mathbb{G}$) the reference filtration (resp. full filtration).

Then we consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$. Note that $\tau^i$, $i \in \{H, C\}$, are $\mathbb{G}$-stopping times but may not be $\mathbb{F}$-stopping times. As a convention, for any $\mathbb{G}$-progressively measurable process $u$, $(\mathbb{G}, \mathbb{Q})$-semimartingale $U$, and $s \leq t$,

$$\int_s^t u_s \, dU_s := \int_{(s,t]} u_s \, dU_s,$$

where the integral is well-defined. In addition, for any $\mathbb{G}$-stopping time $\theta$ and process $(\xi_t)_{t \geq 0}$, we denote

$$\xi_{\theta} := \xi_{\cdot \wedge \theta},$$

and when $\xi_{\theta-}$ exists, $\Delta \xi_{\theta} := \xi_{\theta} - \xi_{\theta-}$. In what follows, for $i \in \{H, C\}$, $t \geq 0$, we denote $G^i_t := \mathbb{Q}(\tau^i > t | \mathcal{F}_t)$, and

$$G_t := \mathbb{Q}(\tau > t | \mathcal{F}_t).$$

The following assumption stands throughout this paper.

**Assumption 2.1.** (i) $(G_t)_{t \geq 0}$ is non-increasing and absolutely continuous with respect to Lebesgue measure.
(ii) For any \(i \in \{H, C\}\), there exists a process \(h^i\), defined as
\[
h^i_t := \lim_{u \downarrow 0} \frac{1}{u} \frac{Q(t < \tau^i \leq t + u, \tau > t|\mathcal{F}_t)}{Q(\tau > t|\mathcal{F}_t)},
\]
and the process \(M^i\), given by
\[
M^i_t := 1_{\tau^i \leq t} - \int_0^{t \wedge \tau} h^i_s \, ds,
\]
is a \((G, Q)\)-martingale.

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\]
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By (i) in Assumption 2.1, there exists an \(\mathbb{F}\)-progressively measurable process \((h^0_t)_{t \geq 0}\) such that
\[
h^0_t = \lim_{u \downarrow 0} \frac{1}{u} \frac{\mathbb{P}(t < \tau^i \leq t + u, \tau > t|\mathcal{F}_t)}{\mathbb{P}(\tau > t|\mathcal{F}_t)},
\]
and
\[
M_t := 1_{\tau \leq t} - \int_0^{t \wedge \tau} h^0_s \, ds
\]
is also a \((G, \mathbb{P})\)-martingale. Let us denote \(h := h^H + h^C\). If \(\tau^H\) and \(\tau^C\) are independent on \(\mathbb{F}\), \(h^0 = h\). It is in general not the case. Moreover, by (i) in Assumption 2.1, \(\tau\) avoids any \(\mathbb{F}\)-stopping time (see Coculescu & Nikeghbali, 2012, Corollary 3.4). In other words, for any \(\mathbb{F}\)-stopping time \(\tau^F\),
\[
\mathbb{Q}(\tau = \tau^F) = 0. \tag{2.1}
\]

The nextlemma is borrowed from Bielecki et al. (2008) and Chapter 5 in Bielecki & Rutkowski (2013).

Lemma 2.2. Let \(i \in \{H, C\}\).

(i) Let \(U\) be an \(\mathcal{F}_s\)-measurable, integrable random variable for some \(s \geq 0\). Then, for any \(t \leq s\),
\[
\mathbb{E}(1_{s < t} U|\mathcal{G}_t) = 1_{s < t} G_t^{-1} \mathbb{E}(G_s U|\mathcal{F}_t),
\]
\[
\mathbb{E}(1_{s < t} U|\mathcal{G}_t) = 1_{s < t} (G_t^{-1}) \mathbb{E}(G_s^2 U|\mathcal{F}_t).
\]
(ii) Let \((U_t)_{t \geq 0}\) be a real-valued, \(\mathbb{F}\)-predictable process and \(\mathbb{E}|U_t| < \infty\). Then,

\[
\mathbb{E}(\mathbb{1}_{t \leq T} U_t | G_t) = \mathbb{1}_{t < T} \mathbb{E}\left( \int_t^T h_s G_s U_s \, ds | \mathcal{F}_t \right).
\]

We define spaces of random variables, and stochastic processes as follows.

**Definition 2.3.** Let \(m \in \mathbb{N}\) and \(p \geq 2\).

- \(\mathbb{L}^p_T\): the set of all \(\mathcal{F}_T\)-measurable random variables \(\xi\), such that

\[
\|\xi\|_p := \mathbb{E}[|\xi|^p]^{\frac{1}{p}} < \infty.
\]

- \(\mathbb{S}^p_T\): the set of all real valued, \(\mathbb{F}\)-adapted, càdlàg\(^1\) processes \((U_t)_{t \geq 0}\), such that

\[
\|U\|_{\mathbb{S}^p_T} := \mathbb{E}\left( \sup_{t \leq T} |U_t|^p \right)^{\frac{1}{p}} < \infty.
\]

- \(\mathbb{H}^{p,m}_T\): the set of all \(\mathbb{R}^m\)-valued, \(\mathbb{F}\)-predictable processes \((U_t)_{t \geq 0}\), such that

\[
\|U\|_{\mathbb{H}^{p,m}_T} := \mathbb{E}\left( \int_0^T |U_t|^p \, dt \right)^{\frac{1}{p}} < \infty.
\]

- \(\mathbb{H}^{p,m}_{T,loc}\): the set of all \(\mathbb{R}^m\)-valued, \(\mathbb{F}\)-predictable processes \((U_t)_{t \geq 0}\), such that

\[
\int_0^T |U_t|^p \, dt < \infty, \quad \text{a.s.}
\]

Moreover, we let \(D_\Theta = (D^1_\Theta, \ldots, D^n_\Theta)\) denote Malliavin derivative at \(\Theta \geq 0\), and \(\mathbb{D}^{1,2}\) denote the set of Malliavin differentiable random variables. For Malliavin calculus, readers can refer to Di Nunno et al. (2009) and Section 5.2 in El Karoui et al. (1997). In the next section, we describe hedging portfolios under incremental CVA, DVA, FVA, and collateral. These aggregated adjustment is often called XVA. For simplicity in notations, when \(n = 1\), we denote \(D_\Theta = D^1_\Theta\), \(W = W^1\), and \(\mathbb{H}^{p,1}_T := \mathbb{H}^{p,1}_T\), \(\mathbb{H}^{p,1}_{T,loc} := \mathbb{H}^{p,1}_{T,loc}\).

### 2.2 BSDEs under Incremental XVA

#### 2.2.1 Cash-flows

We consider a hedger and counterparty entering a “new” contract which exchanges promised dividends. First, we proceed our argument with the assumption that the two parties have not made any contract before the new contract. Later, this assumption will be relaxed by slightly modifying our model so that incremental effects can be considered.

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\(^1\)Right continuous and left limit
Let $D^N_t$ denote the accumulated amount of the promised dividends up to $t \geq 0$. We assume that $D^N$ is an $\mathbb{F}$-adapted càdàg process, and the value is determined by an $n$-dimensional underlying asset process $S = (S^1, \ldots, S^n)$ that follows the next stochastic differential equation (SDE):

$$dS^i_t = r^i_t S^i_t \, dt + \sigma^i_t S^i_t \, dW^i_t, \quad i \in \{1, \ldots, n\},$$

for some $\mathbb{F}$-progressively measurable processes $r^i$ and $(\sigma^i)^\top \in \mathbb{R}^n$. In addition, we assume that the $\mathbb{F}$-adapted process $D^N$ is independent of the information of defaults.

**Remark 2.4.** Note that an $\mathbb{F}$-adapted process may depend on default risks, since the default intensities are $\mathbb{F}$-adapted.

If a default occurs before the maturity of the contract $T$, two parties stop exchanging $D^N$, and the derivative contract is marked-to-market. The method to calculate the close-out amount is determined before initiation of the contract and documented in Credit Support Annex (CSA)\(^2\). Let $e^N_t$ denote the close-out amount at $t \leq T$. In this paper, we deal with two conventions for the close-out amount $e^N$: clean close-out and replacement close-out. We postpone explaining the conventions to the next section after we define the hedger’s hedging portfolio. As conventions, $dD^N_t \geq 0$, $e^N_t \geq 0$ (resp. $< 0$) means that the hedger pays to (resp. is paid by) the counterparty at $t \leq T$.

**Example 2.5.** If the hedger buys a zero coupon bond of unit notional amount, $D^N_t = -\mathbb{1}_{[T, \infty[}$.

The obligation to settle $e^N$ may not be fully honored because of the default. To mitigate the risk, the two parties post or receive collateral (often referred to as margin). The amount of the collateral posted at $t \geq 0$ (only for the new contract) is denoted by $m^N_t$. We assume that $(m^N_t)_{t \geq 0}$ is an $\mathbb{F}$-adapted process. By Assumption 2.1, $\tau^i, i \in \{H, C\}$, are totally inaccessible, which means that the defaults arrive with surprise. Margins are posted because we do not know full information of the defaults, and this is why $m^N$ is calculated on the observable information $\mathbb{F}$. The exact forms of $m^N$ will be given later after the conventions for $e^N$ are introduced. We assume that the close-out payment is settled at the moment of default without delay and $m^N$ is posted continuously. As conventions, if $m^N_t \geq 0$ (resp. $< 0$), it means that the hedger posts (receives) the collateral at $t \leq T$.

**Remark 2.6.** In practice, there exists a gap between the day of default and actual settlement. The delay is required to check whether the default really happened, collect information of the contract, find the best candidate to replace the defaulting party (Murphy, 2013). Gap risk is the risk from the gap between the close-out amount at the day of settlement and the last day that variation margin is posted. For gap risk, two parties post initial margin which is often calculated by a risk measure. Note that we ignore the gap risk and initial margin. If we consider initial margin, we encounter anticipative backward stochastic differential equations (ABSDEs) under replacement close-out. For the main result of this paper, Malliavin calculus for BSDEs will be used. However, to the best of our knowledge, Malliavin differentiability of ABSDEs has not been studied. Moreover, It is a challenging problem to solve ABSDEs numerically (see Agarwal et al., 2018). The continuous posting of variation margin can also be seen as a simplification. One may want to model $m$ as a càdlàg step process. For the discussion, readers may want to refer to Brigo, Liu, et al. (2014).

\(^2\)A part of ISDA master agreement
At default, collateral is not exchanged. Thus, we set the collateral amount posted at \( \tau \leq T \), as \( m^N_{\tau^+} \). Therefore, the cash-flow at default can be

\[
\Delta \mathcal{D}^N_{\tau^+} + e^N_{\tau^+} - m^N_{\tau^-}.
\]

However, it is immaterial whether we separate \( \Delta \mathcal{D}^N_{\tau^+} \) from \( e^N_{\tau^+} \) or not in the modeling, because jumps of \( \mathcal{F} \)-adapted càdlàg processes are exhausted by \( \mathcal{F} \)-stopping times (see He & Yan, 1992, Theorem 4.21). Thus, by (2.1),

\[
\Delta \mathcal{D}^N_{\tau^+} = 0, \ \text{a.s.}
\]

Let \( \mathcal{C}^N \) denote the accumulated cash-flows. Then, for any \( t \leq T \) a.s,

\[
\mathcal{C}^N_t := \1_{t > \tau} \mathcal{D}^N_t + \1_{t \leq \tau} (\mathcal{D}^N_t + e^N_t) - \1_{t = \tau^+} L^H(e^N_{\tau^-} - m^N_{\tau^-})^+ + \1_{t = \tau^+} L^C(e^N_{\tau^-} - m^N_{\tau^-})^-,
\]

where \( 0 \leq L^H \leq 1 \) (resp. \( 0 \leq L^C \leq 1 \)) is the loss rate of the hedger (resp. counterparty). Recall that \( \mathcal{C}^N \) is derived from the assumption that the new contract is the first contract between the hedger and counterparty. Now, we relax the assumption so that we can consider incremental cash-flows.

### 2.2.2 Incremental Cash-flows

Assume that the two parties has made contracts given by some endowed càdlàg \( \mathcal{F} \)-adapted processes \( (\mathcal{D}^E, e^E, m^E) \) before initiation of the new contract. If the two parties did not enter the contract, for the bank of the hedger, the cash-flows would be

\[
\mathcal{C}^E_t := \1_{t > \tau} \mathcal{D}^E_t + \1_{t \leq \tau} (\mathcal{D}^E_t + e^E_t) - \1_{t = \tau^+} L^H(e^E_{\tau^-} - m^E_{\tau^-})^+ + \1_{t = \tau^+} L^C(e^E_{\tau^-} - m^E_{\tau^-})^-.
\]

On the other hand, when entering the new contract, the exposure and margin would be \( e^E + e^N \) and \( m^E + m^N \), respectively. In this case, the summed cash-flows for the bank are

\[
\mathcal{C}^S_t := \1_{t > \tau} \mathcal{D}^N_t + \1_{t \leq \tau} (\mathcal{D}^N_t + e^N_t) - \1_{t = \tau^+} L^H(e^N_{\tau^-} - m^N_{\tau^-})^+ + \1_{t = \tau^+} L^C(e^N_{\tau^-} - m^N_{\tau^-})^-.
\]

Thus, the amount that should be handled by the hedger to enter the “new” contract can be given by

\[
\mathcal{C}_t := \mathcal{C}^S_t - \mathcal{C}^E_t
\]

\[
= \1_{t > \tau} \mathcal{D}^N_t + \1_{t \leq \tau} (\mathcal{D}^N_t + e^N_t) - \1_{t = \tau^+} L^H((e^N_{\tau^-} - m^N_{\tau^-})^+ - (e^E_{\tau^-} - m^E_{\tau^-})^+)
\]

\[
+ \1_{t = \tau^+} L^C((e^N_{\tau^-} - m^N_{\tau^-})^- - (e^E_{\tau^-} - m^E_{\tau^-})^-).
\]

We denote the cash-flow process after \( \tau \) by \( \Theta \), i.e.,

\[
\Theta := \mathcal{C} - (\mathcal{D}^N)^\tau.
\]

We assume that the hedger can access to (defaultable) zero coupon bonds of hedger and counterparty. Let \( S^H \) (resp. \( S^C \)) denote the defaultable bond of the hedger (resp. counterparty), where \( S^H \) and \( S^C \) follow the next SDE:

\[
dS^i_t = r_t S^i_t \, dt + \sigma^i_t S^i_t \, dW_t - S^i_t \, dM^i_t, \quad i \in \{ H, C \},
\]

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where \((\sigma^i)\top \in \mathbb{R}^n\) are \(\mathcal{F}\)-progressively measurable processes. \(S^1, \ldots, S^n\) are used to hedge \(\mathbb{N}^N\), while \(S^H\) and \(S^C\) are used to hedge \(\Theta\). We define the \((n+2) \times n\) matrix \(\sigma\) as

\[
\sigma := \begin{bmatrix}
\sigma^1 \\
\vdots \\
\sigma^n \\
\sigma^H \\
\sigma^C
\end{bmatrix}.
\]

Note that the considered financial markets are complete, since there are \(n+2\) sources of randomness, \(W, \tau^H, \tau^C\), and \(n+2\) traded assets \(S, S^H, S^C\).

**Remark 2.7.** By (2.2), underlying assets do not depend on default risk, which means that we do not deal with credit derivatives. For modeling with emphasis on contagion risk, readers can refer to Bo et al. (2017, 2018); Brigo, Capponi, & Pallavicini (2014); Jiao et al. (2013).

### 2.2.3 Accounts and Hedging Strategy

In this section, we introduce several saving accounts and the hedger’s hedging strategy. In what follows, we denote

\[
I := \{1, 2, \ldots, H, C\}.
\]

For any \(i \in I\), let \(\eta^{S,i}\) denote the number of units of \(S^i\) held by the hedger, and we assume \(\eta^{S,i}\) is \(\mathcal{G}\)-predictable. We denote

\[
\varphi := (\eta^{S,1}, \ldots, \eta^{S,n}, \eta^{S,H}, \eta^{S,C}),
\]

\[
\pi^i := \eta^{S,i} S^i, \quad i \in \{1, \ldots, n\},
\]

\[
\pi^i := \eta^{S,i} S^i, \quad i \in \{H, C\},
\]

\[
\pi := (\pi^1, \ldots, \pi^n, \pi^H, \pi^C).
\]

We call the \((n+2)\)-dimensional \(\mathcal{G}\)-predictable process \(\varphi\) the hedger’s hedging strategy.

**Remark 2.8.** \(\varphi\) is chosen to be \(\mathcal{G}\)-predictable only to describe an immediate action taken at default. We shall see later that on \([0, \tau]\), \(\varphi\) is \(\mathbb{F}\)-adapted.

If the collateral is pledged, the posting party is remunerated by the receiving party according to a certain interest rate. When \(m^N_t \geq 0\) (resp. \(< 0\)), the counterparty (resp. hedger) pays the interest rate \(R^{m,\ell}_t\) (resp. \(R^{m,b}_t\)) at \(t \leq T\). We assume that the collateral is posted as cash and the interest rate is accrued to a margin account of the hedger. We denote the lending and borrowing accounts \(B^{m,\ell}\) and \(B^{m,b}\) respectively, i.e., \(B^{m,i}, i \in \{\ell, b\}\), are given by

\[
d B^{m,i}_t = R^{m,i}_t B^{m,i}_t \, dt.
\]
Let $\eta^{m, \ell}$ (resp. $\eta^{m, b}$) denote the number of units of $B^{m, \ell}$ (resp. $B^{m, b}$). Then the following equations hold:

$$\begin{align*}
\eta^{m, \ell} &\geq 0, \quad \eta^{m, b} < 0, \quad \eta^{m, \ell} \eta^{m, b} = 0, \\
\eta^{m, \ell} B^{m, \ell} + \eta^{m, b} B^{m, b} &= mN. \tag{2.8}
\end{align*}$$

We assume that the variation margin $m$ can be rehypothecated, i.e., $mN$ is used by the hedger to maintain her portfolio.

**Remark 2.9.** The margin account for $m^E$ may be dealt with by other dealers, so it is not a part of the hedging portfolio.

Some underlying assets can be traded through repo markets. We denote the set of indices for which a repo market is available by $\rho \subseteq I \coloneqq \{1, 2, \ldots, n, H, C\}$. We assume that the borrowing and lending repo market rates are the same, and for $i \in \rho$, let $R^{\rho, i}$ denote the repo rate. Moreover, for any $i \in \rho$, let $B^{\rho, i}$ denote the account that $R^{\rho, i}$ accrues, i.e., $B^{\rho, i}$ follows

$$d B^{\rho, i}_t = R^{\rho, i} B^{\rho, i}_t \, dt. \tag{2.10}$$

For $i \in \rho$, we denote the number of units of $B^{\rho, i}$ by $\eta^{\rho, i}$. Then it follows that for any $i \in \rho$,

$$\eta^{\rho, i} B^{\rho, i} + \eta^S i S^i = 0. \tag{2.11}$$

If the hedger has any surplus cash, she can earn the lending rate $R^\ell$, while for borrowing money, she needs to pay the borrowing rate $R^b$. For $i \in \{\ell, b\}$, let $B^i$ denote the hedger’s funding account and $\eta^i$ denote the number of units of $B^i$. Therefore, it follows that

$$\begin{align*}
\eta^\ell &\geq 0, \quad \eta^b < 0, \\
\eta^\ell B^\ell + \eta^b B^b &= 0. \tag{2.12}
\end{align*}$$

For $i \in \{\ell, b\}$, let $d B^i_t = R^i B^i_t \, dt$, $i \in \{\ell, b\}$.

### 2.3 Hedger’s Incremental Hedging Portfolio

Now, we are almost ready to define dealer’s incremental portfolio. Another ingredient in defining hedging portfolios is incremental funding effects. These effects will be considered by imposing some conditions on the hedging portfolio. The detail will be followed after the next definition.

**Definition 2.10.** If $V = V(q, C)$ defined on $t \in \mathbb{R}_+$, by

$$V_t = \eta^{\ell, \ell}_t B^{\ell}_t + \eta^{b, b}_t B^{b}_t + \eta^{m, \ell}_t B^{m, \ell}_t + \eta^{m, b}_t B^{m, b}_t + \sum_{i \in I} \eta^S_i S^i_t + \sum_{i \in \rho} \eta^{\rho, i}_t B^{\rho, i}_t, \tag{2.14}$$

satisfies

$$V_t = V_0 + \sum_{i = \ell, b} \int_0^{t \wedge \tau} \eta^i_s \, dB^i_s + \sum_{i = \ell, b} \int_0^{t \wedge \tau} \eta^{m, i}_s \, dB^{m, i}_s + \sum_{i = \ell} \int_0^{t \wedge \tau} \eta^S_i \, dS^i_s$$

$$+ \sum_{i \in \rho} \int_0^{t \wedge \tau} \eta^{\rho, i}_s \, dB^{\rho, i}_s - C_{t \wedge \tau}, \tag{2.15}$$

for any $t \in \mathbb{R}_+$, then $V$ is called the hedger’s incremental hedging portfolio.
Remark 2.11. Note that by (2.3), $C_t = C_{t \wedge \bar{\tau}}, \forall t \geq 0$, and by (2.15), $V_t = V_{t \wedge \bar{\tau}}, \forall t \geq 0$.

Our goal is to find a proper price charged to the counterparty and hedging strategy $\varphi$ satisfying Definition 2.10 and a certain terminal condition. We seek to impose the terminal condition so that an incremental funding effect should be considered. The incremental funding effect means the difference between the funding cost/benefit of two choices: entering or not entering the new contract. To explain the necessity of the incremental effect briefly, consider a situation that the dealer wants to enter a new contract that makes the hedging portfolio fall in borrowing state. If there have been no business of the bank before the contract, the treasury department would finance the dealer with the borrowing rate, and the excessive borrowing cost might be charged to the counterparty. However, if the bank’s initial position was in lending state before the contract, the treasury department should consider deduction of lending profit rather than excessive borrowing cost. The mentioned activity can be seen that the dealer borrows and keeps the bank’s initial portfolio for funding the portfolio, and returns it to treasury at maturity.

To explain the mathematical detail, let $B^\epsilon$ denote the endowed bank’s portfolio without entering the new contract, for some $\epsilon \in \mathbb{R}$ such that $B^\epsilon_0 = \epsilon$. We sometimes call $B^\epsilon$ legacy portfolio. In reality, $B^\epsilon$ is a massive combination of numerous portfolios. We assume that the legacy portfolio is approximately risk-neutral and grows with respect to their funding rates. Therefore,

$$B^\epsilon_t := \epsilon \exp \left( \int_0^t R^\epsilon_s ds \right), \forall t \in \mathbb{R}_+,$$

and we denote $s^\epsilon := R^\epsilon - r$.

Now, let us think of $V$ as the bank’s summed profit/loss and consider two cases. First, if the dealer does not enter the contract exchanging $C$, the bank will have $B^\epsilon_{\bar{\tau}}$ at $\bar{\tau}$. Second, the dealer can enter the contract with a certain initial price, say $p \in \mathbb{R}$, for the contract from the counterparty. Then, the bank’s initial wealth is $\epsilon + p$, namely

$$V_0 = \epsilon + p.$$

The dealer gains from investing in risky assets and accounts. These revenues are used to pay the cash-flows $C$. Thus, in (2.15), represents the hedging error between the investment and cash-flows. Recall that we consider complete markets (there are $n + 2$ sources of randomness, $(W, \tau^H, \tau^C)$, and $n + 2$ hedging assets $\pi$). Therefore, we can expect that we can find a hedging strategy that replicates the cash-flows with null hedging error up to termination of the contract, either by the maturity or default. For example, if $\epsilon = 0$, we can expect to find $(V, \pi)$ such that $V_{\bar{\tau}} = 0$. However, since the opportunity cost for the bank is the profit/loss from legacy portfolio, the portfolio value at $\bar{\tau}$ should be compared with $B^\epsilon_{\bar{\tau}}$, i.e., we seek to find $(V, \pi)$ such that

$$V_{\bar{\tau}} = B^\epsilon_{\bar{\tau}}, \quad (2.16)$$

and the dealer may want to charge

$$p = V_0 - \epsilon \quad (2.17)$$
to the counterparty. Therefore, by (2.15) together with (2.2), (2.5), (2.7), (2.8), (2.9), (2.10), (2.11), (2.13), we need to solve the following BSDE:

\[
\begin{align*}
\text{d}V_t &= \left\{ \left( V_t - m_t^N - \sum_{i \in I_N} \pi_i^t \right)^+ R_i^f - \left( V_t - m_t^N - \sum_{i \in I_N} \pi_i^t \right)^- R_i^b + \tau_t \sum_i \pi_i^t - \sum_{i \in \rho} R_i^b \pi_i^t \right\} dt \\
&\quad + \left\{ R_{m,b} m_t^+ - R_{m,b} m_t^+ \right\} dt + \sum_{i \in I} \pi_i^t \sigma_i^t dW_t - \sum_{i \in I, i \neq C} \pi_i^t dM_t^i - d\mathcal{C}_t,
\end{align*}
\]

\[V_T = B_T.\]  

(2.18)

For now, we do not examine the existence and uniqueness of (2.18). The solvability will be examined by a reduced form of (2.18). The financial interpretation of each component in (2.18) will be provided later in the following section. Before the detail, we first discuss the incremental funding impacts by a simple example.

**Example 2.12** (Incremental FVA). Let \( n = 1, \rho = 0, \mathcal{R}^N = \mathbb{I}_{\mathbb{R}^2} \otimes \xi B_T \) for some \( \xi \in \mathbb{L}^2_T \). We ignore default risks and set \( \pi^H = \pi^C = m^N = 0 \). In this case, \((V, \pi^1)\) is given by

\[V_t = \xi B_T + B_T^\pi - \int_t^T \left( (V_s - \pi_1^s)^+ R_s^f - (V_s - \pi_1^s)^- R_s^b + r_s \pi_1^s \right) ds - \int_t^T \pi_1^s \sigma_1^s dW_s.\]

To consider a net profit/loss to the hedger without the legacy portfolio, consider

\[v := B^{-1}(V - B^\pi),\]

and we denote that \( \bar{\pi} := B^{-1} \pi^1, \bar{B}^\pi = B^{-1} B^\pi \). Then \((v, \bar{\pi})\) is given by

\[v_t = \xi - \int_t^T \left( (v_s + \bar{B}^\pi_s - \bar{\pi}_1^s)^+ s_s^f - (v_s + \bar{B}^\pi_s - \bar{\pi}_1^s)^- s_s^b - s_s \bar{B}^\pi_s \right) ds - \int_t^T \bar{\pi}_1^s \sigma_1^s dW_s.\]

Assuming there exists a unique solution \((v, \bar{\pi})\) \( \in \mathbb{R}^2_T \times \mathbb{R}^2_T \) of (2.20), for any \( t \geq 0 \), we define

\[
\text{FBA}_t^A := \mathbb{E} \left[ \int_t^T \left( (v_s + \bar{B}^\pi_s - \bar{\pi}_1^s)^+ s_s^f - (s_s \bar{B}^\pi_s)^+ \right) ds \middle| \mathcal{F}_t \right],
\]

\[
\text{FCA}_t^A := \mathbb{E} \left[ \int_t^T \left( (v_s + \bar{B}^\pi_s - \bar{\pi}_1^s)^- s_s^b - (s_s \bar{B}^\pi_s)^- \right) ds \middle| \mathcal{F}_t \right],
\]

\( \text{FBA}^A \) (resp. \( \text{FCA}^A \)) represents the incremental funding benefits (resp. costs) for entering the new contract. Notice that as \( \xi \) increases, \( v + \bar{B}^\pi - \bar{\pi} \) is more likely to be positive. Consider a case that \( v - \pi \) is negative but \( v + \bar{B}^\pi - \bar{\pi} \) is non-negative. If we ignore the incremental effect, i.e., \( \xi = 0 \), the dealer should consider the increased funding cost. However, in the view of incremental effects, instead, the deduction of funding benefit should be included in the derivative price. However, the dealer also need to consider the opportunity cost for not entering the new contract, e.g., if \( \xi \geq 0 \), the lost (discounted) benefit at \( t \leq T \) would be

\[
\bar{B}^\pi_T - \bar{B}^\pi_t = \int_t^T \text{d}\bar{B}^\pi_s = \int_t^T s_s^f \bar{B}^\pi_s ds = \int_t^T (s_s^f \bar{B}^\pi_s)^+ ds. \]

(2.21)
The difference between the two impacts is the actual net benefit and cost, $FBA^\Lambda$ and $\Delta FCA^\Lambda$, that should be charged to the counterparty. Indeed, by (2.17), (2.19) and (2.20), the dealer would want to charge

$$p = \mathbb{E}[\xi] - FBA^\Lambda_0 + FCA^\Lambda_0.$$ 

(2.22)

In addition, if $R^\ell = R^b$, (2.20) becomes

$$v_t = \xi - \int_t^T (v_s - \bar{\pi}_s^\ell) \sigma_s^\ell \, ds - \int_t^T \bar{\pi}_s^\ell \sigma_s^\ell \, dW_s.$$ 

(2.23)

Thus, under linear funding models, $v$ does not depend on $B^\ell$.

In what follows, for simplicity, we impose some realistic assumptions on interest rates, and the endowed processes, $e^E$ and $m^E$. In practice, $R^{m,i}$, $i \in \{\ell, b\}$, are chosen as Federal funds or EONIA rates, approximately $r$. In addition, the difference between OIS and repo market rates can be interpreted as the liquidity premium of the repo markets. We assume the repo markets are liquid enough for the difference to be small. Moreover, we assume that OIS rate, $(r_t)_{t \geq 0}$, is the smallest among all interest rates. In addition, $e^E$ and $m^E$ are given before the new contract, so they are chosen exogenously, i.e., they do not depend on $V$. These assumptions are summarized as follows.

Assumption 2.13. (i) $R^{b,i} = R^{m,\ell} = R^{m,b} = r$, for $i \in \rho$.

(ii) $R^\ell \geq r$ and $R^b \geq r$.

(iii) $e^E$ and $m^E$ are exogenous processes.

Remark 2.14. The assumption on repo market rates given in Assumption 2.13-(i) is merely for simplicity in representing (2.18). Mathematically, it does not play any crucial role.

Recall that we have not yet specified the amount of close-out $e^N$ in $C$. In the next section, two important close-out conventions are introduced: clean price and replacement cost. Clean price is basically the risk-neutral price of $\mathbb{D}^N$. By using the SDE that the clean price follows, we can remove $\mathbb{D}^N$ from $C$. After the elimination of $\mathbb{D}^N$, the rest of cash-flows is $\Theta$ which is exchanged only at one point $\tau$. Thus, by subtracting the clean price from $V$, we can recover a standard BSDE with $(G, Q)$-martingales, $W$ and $M^i$, $i \in \{H, C\}$. Then we further reduce the BSDE to a BSDE only with the Brownian motions by the typical argument of filtration reduction.

2.3.1 Close-out Conventions

When the contract terminates earlier than the contractual maturity by one party’s default, the defaulting party should settle the close-out amount. The close-out amount is calculated by a determining party which will act in good faith (ISDA, 2009). As stated in ISDA (2009, p.15), “the Determining Party may consider any relevant information, including, without limitation, one or more of the following types of information: quotations (either firm or indicative) for replacement transactions supplied by one or more third parties that may take into account the creditworthiness of the Determining Party at the time the quotation is provided and the terms of any relevant documentation . . . ."
The statement leaves some rooms for interpretation. ISDA recommends to consider creditworthiness of the surviving party, but it is not mandatory. Moreover, it is unclear whether funding costs should be considered in the replacement transaction. Therefore, we consider two close-out conventions: clean price and replacement cost. Clean price is the risk-neutral price without XVA. The close-out with clean price is often called clean close-out and risk-free close-out. Let \( p^N \) denote the clean price, i.e.,

\[
p^N_t := B_t \mathbb{E} \left( \int_t^T B^{-1}_s d\mathbb{N}_s \big| \mathcal{F}_t \right), \quad \forall t \in \mathbb{R}_+.
\]  

Clean close-out (or risk-free close-out) has been often chosen in literature (for example, Crépey (2015a,b)).

**Remark 2.15.** Bichuch et al. (2018) also considered expected value of discounted cash-flow in absence of default risk, but in the calculation, they assumed flexibility to choose the discounting rate and probability measure. Indeed, they chose the pricing measure as an equivalent probability measure such that \((B^i)^{-1} S_i \in \{ℓ, b\}\) are martingales. In other words, the amount is “clean price+XVA”. This choice is for avoiding mismatch of pricing measures to obtain a closed-form solution. We will explain this point later with examples.

Replacement cost (or replacement close-out) means the price under XVA. In this case, it is not clear which funding rate should be chosen. We assume that the replacing party has similar credit spreads to the hedger. Recall that \( V - B^\epsilon \) is the value for calculating the derivative price in view of the hedger. In other words, one may want to choose the value of \( V - B^\epsilon \) at the default for the replacement cost. However, at the default, there is a jump in \( V \) by \( \pi^i, i \in \{H, C\} \). Moreover, \( \pi^i \) is retained in \( V \) for the default risk, i.e., the close-out payment. Therefore, if we take

\[
e^N_t = V_t - B^\epsilon_t = V_{t-} - B^\epsilon_t + \Delta V_t
\]

\[
= V_{t-} - B^\epsilon_t - \mathbb{1}_{t=\tau_H} \pi^H_t - \mathbb{1}_{t=\tau_C} \pi^C_t,
\]

the defaulting party would pay the remainder after the deduction of the same amount, basically nothing. Hence, for replacement close-out, we should set

\[
e^N_t = V_{t-} - B^\epsilon_t.
\]  

In both conventions, we assume that the collateral is a portion of the close-out amount. More precisely, for \( 0 \leq L^m \leq 1 \) (margin loss),

\[
m^N = (1 - L^m) e^N.
\]  

Note that (2.26) is consistent with our financial modeling. Indeed, if there is a \( \mathcal{G} \)-adapted process satisfying (2.18), in our filtration setup \( \mathcal{G} \), there is an \( \mathcal{F} \)-adapted process \( V^F \) such that

\[
V = \mathbb{1}_{[0, \bar{\tau}]} V^F.
\]

Therefore, under replacement close-out, the margin process

\[
m^N = (1 - L^m) e^N = (1 - L^m)(V_{t-} - B^\epsilon)
\]

\[
= (1 - L^m)(V^F_{t-} - B^\epsilon)
\]
is $\mathbb{F}$-adapted before $\tilde{\tau}$. In what follows, we assume that the endowed margin also follows the same convention as (2.26), i.e.,

$$m^F = (1 - L^m) e^F.$$  \hfill (2.27)

**Remark 2.16.** (i) The two close-out conventions have pros and cons in financial modeling. *Clean price* may be disadvantageous to the defaulting party because the default risk of surviving party is not considered. However, the surviving party’s default risk can be negatively affected by the default, especially when defaulting party has a systemic impact. In that case, *replacement close-out* may heavily penalize the surviving party. Readers can refer to Brigo & Morini (2011) for the comparison.

(ii) A similar collateral convention was discussed by Burgard & Kjaer (2010). For BSDEs approach on general endogenous collateral, readers can refer to Nie & Rutkowski (2016).

Before further argument, we provide a lemma on properties of *clean price* $p^N$. The following lemma will be used to present an XVA process and simplify the representation of the amount of cash-flow at default $\Theta$. Readers may want to recall (2.24), the definition of $p^N$, before the following lemma.

**Lemma 2.17.** (i) $p^N_T = 0$.

(ii) $p^N_t = 1_{\tau \leq T} p^N_T$.

(iii) $dp^N_t = r_t p^N_t dt + B_t (Z^N_t)^\top dW_t - d\mathbb{D}^N_t$, $\forall t \leq T$, for some $Z^N \in \mathbb{H}^{2,n}_{T, loc}$.

(iv) $p^N_{\tau^-} = p^N_{\tau}$ almost surely.

**Proof.** (i) is from the definition and (ii) is a directly obtained from (i). For (iii), notice that $B^{-1} p^N + \int_0^T B^{-1}_s d\mathbb{D}^N_s$ is an $(\mathbb{F}, \mathbb{Q})$-local martingale. Thus, by (local) martingale representation property, there exists $Z^N \in \mathbb{H}^{2,n}_{T, loc}$ such that for any $t$,

$$B_t^{-1} p^N_t + \int_0^t B_s^{-1} d\mathbb{D}^N_s = \int_0^t (Z^N_s)^\top dW_s.$$

Therefore, $p^N_t$ follows the SDE:

$$dp^N_t = r_t p^N_t dt + B_t (Z^N_t)^\top dW_t - d\mathbb{D}^N_t \hfill (2.28)$$

By (iii), $p^N$ is an $\mathbb{F}$-adapted càdlàg process, but $\tau$ avoids any $\mathbb{F}$-stopping time. Thus, $\Delta p^N_{\tau^-} = 0$ almost surely, equivalently $p^N_{\tau^-} = p^N_{\tau}$ a.s.

**Remark 2.18.** Note that $Z^N$ is the (discounted) delta risk of *clean price*. For example, consider $n = 1$ and a stock forward contract with exercise price $K$, and assume $r, \sigma^1$ are deterministic. Then $B^{-1} p^N = B^{-1} S^1 - B^{-1}_T K$. Thus, for $t < T$,

$$Z^N_t = D_t (B_t^{-1} p^N_t) = \sigma^1_t B_t^{-1} S^1_t.$$
Recall that $c^E$ and $B^e$ are $\mathcal{F}$-adapted, so they do not jump at $\tau$, i.e., $e^F_\tau = e^F_{\tau-}$ and $B^e_\tau = B^e_{\tau-}$, a.s. Then, by (2.4) together with (iv) in Lemma 2.17, under clean close-out, a.s,

$$
\begin{align*}
\Theta_t &= \mathbb{I}_{t=\tau \leq t} \left[ p^N_{\tau-} - L^H L^m \left( (p^N_{\tau-} + e^F_{\tau-})^+ - (e^F_{\tau-})^+ \right) \right] \\
&\quad + \mathbb{I}_{t=\tau \leq t} \left[ p^N_{\tau-} + L^C L^m \left( (p^N_{\tau-} + e^F_{\tau-})^- - (e^F_{\tau-})^- \right) \right],
\end{align*}
$$

(2.29)

On the other hand, under replacement close-out,

$$
\begin{align*}
\Theta_t &= \mathbb{I}_{t=\tau \leq t} \left[ V_{\tau-} - B^e_{\tau-} - L^H L^m \left( (V_{\tau-} - B^e_{\tau-} + e^F_{\tau-})^+ - (e^F_{\tau-})^+ \right) \right] \\
&\quad + \mathbb{I}_{t=\tau \leq t} \left[ V_{\tau-} - B^e_{\tau-} + L^C L^m \left( (V_{\tau-} - B^e_{\tau-} + e^F_{\tau-})^- - (e^F_{\tau-})^- \right) \right],
\end{align*}
$$

(2.30)

### 2.3.2 Incremental XVA process

We can remove $\mathcal{D}^N$ in (2.18) by using (iii) in Lemma 2.17. To this end, we introduce an incremental XVA process. In both close-out conventions, we will deal with the XVA process instead of (2.18). The XVA process is defined by the discounted difference between $V - B^e$ and $p^N$. Let $X$ denote the (incremental) XVA process, i.e.,

$$
X := B^{-1}[V - B^e - (p^N)^\bar{\tau}].
$$

Moreover, let $\bar{p}^N := B^{-1}p^N$, $\bar{\pi} := B^{-1}\pi$, $c := B^{-1}m^N$, $\bar{\Theta} := B^{-1}\Theta$, $\bar{B}^e := B^{-1}B^e$, and

$$
s^i := R^i - r, \quad i \in \{\ell, b\}.
$$

Note that $s^\ell$ (resp. $s^b$) represents the lending (resp. borrowing) spread of the hedger. We can easily check that for $t \geq 0$, $d\bar{p}^N_{t \wedge \tau} = \mathbb{I}_{t \leq \tau} d\bar{p}^N_t$. Assuming there exists $(V, \pi)$ satisfying (2.18), by applying Itô's formula to $X$, we attain that for $t \leq \bar{\tau}(X, \bar{\pi})$ follows

$$
\begin{align*}
\left\{ 
\begin{aligned}
    dX_t &= \left( X_t + \bar{p}^N_t + \bar{B}^e_t - c_t - \sum_{i \in i \setminus \wp} \bar{\pi}^i_t \right)^{\bar{s}^\ell_t} - \left( X_t + \bar{p}^N_t + \bar{B}^e_t - c_t - \sum_{i \in i \setminus \wp} \bar{\pi}^i_t \right)^{\bar{s}^b_t} - \bar{F}^t_t \\
    X_t &= \bar{\Theta}_t - \bar{p}^N_t.
\end{aligned}
\right.
\end{align*}
$$

(2.31)

Note that under replacement close-out,

$$
\begin{align*}
\bar{\Theta}_t &= \mathbb{I}_{t=\tau \leq t} \left[ X_{\tau-} - L^H L^m \left( (X_{\tau-} + \bar{p}^N_{\tau-} + e^F_{\tau-})^+ - (e^F_{\tau-})^+ \right) \right] \\
&\quad + \mathbb{I}_{t=\tau \leq t} \left[ X_{\tau-} + L^C L^m \left( (X_{\tau-} + \bar{p}^N_{\tau-} + e^F_{\tau-})^- - (e^F_{\tau-})^- \right) \right],
\end{align*}
$$

while under clean close-out, $\bar{\Theta}$ is independent of $X$. In both cases, we denote

$$
\bar{\Theta}(X_-) := \Theta.
$$
Moreover, we define $\Theta^H(X_\cdot)$ and $\Theta^C(X_\cdot)$ such that
\[
\hat{\Theta}^H_t - \hat{p}^N_t = -1_{\tau - t \leq T}(\hat{\Theta}^H_t(X_{t-}) - L^H L^m(\tilde{e}^E_t)^+) + 1_{\tau - t \leq T}(\hat{\Theta}^C_t(X_{t-} - L^C L^m(\tilde{e}^E_t)^-)),
\]
where $\hat{\Theta}^i := B^{-1} \Theta^i$, $i \in \{H, C\}$. For example, under replacement close-out,
\[
\hat{\Theta}^H_t(X_{t-}) = -X_{t-} + L^H L^m(X_{t-} + \hat{p}^N_t + \hat{e}^E_t)^+,
\]
\[
\hat{\Theta}^C_t(X_{t-}) = X_{t-} + L^C L^m(X_{t-} + \hat{p}^N_t + \hat{e}^E_t)^-.
\]
while under clean close-out,
\[
\hat{\Theta}^H_t(X_{t-}) = L^H L^m(\tilde{p}^N_t + \tilde{e}^E_t)^+,
\]
\[
\hat{\Theta}^C_t(X_{t-}) = L^C L^m(\tilde{p}^N_t + \tilde{e}^E_t)^-.
\]
In the case of clean close-out, $\hat{\Theta}^i$, $i \in \{H, C\}$, represent the amount of breach of the contract. Recall $B$ and $p^N$ are independent of $V$. Therefore, showing the existence and uniqueness of the hedger’s hedging portfolio and hedging strategy, $(V, \bar{\pi})$, reduces to investigating the BSDE of the XVA process (2.31). Before examining the solvability, assuming the existence and integrability, we define each component in the incremental XVA and give some remarks on them.

**Definition 2.19.** Assume that there exists $(X, \bar{\pi})$ satisfying (2.31). Then for $t < \bar{t}$, we define adjustment processes: FCA, FBA, CVA, DVA, and incremental adjustment processes: $\text{FCA}^\Delta$, $\text{FBA}^\Delta$, $\text{DVA}^\Delta$, $\text{CVA}^\Delta$ processes as follows:

\[
\begin{align*}
\text{FCA}_t &:= \mathbb{E} \left[ \int_t^\bar{t} \left( X_s + \tilde{p}^N_s + \tilde{c}_s - \sum_{i \in \Gamma \setminus \Delta} \tilde{\pi}_s^i \right) s^b_s ds \bigg| \mathcal{G}_t \right], \\
\text{FBA}_t &:= \mathbb{E} \left[ \int_t^\bar{t} \left( X_s + \tilde{p}^N_s + \tilde{c}_s - \sum_{i \in \Gamma \setminus \Delta} \tilde{\pi}_s^i \right) s^f_s ds \bigg| \mathcal{G}_t \right], \\
\text{DVA}_t &:= \mathbb{E} \left[ 1_{\bar{t} = t} \hat{\Theta}^H_t(X_{t-}) \bigg| \mathcal{G}_t \right], \\
\text{CVA}_t &:= \mathbb{E} \left[ 1_{\bar{t} = t} \hat{\Theta}^C_t(X_{t-}) \bigg| \mathcal{G}_t \right],
\end{align*}
\]

and denoting $\mathcal{O}_t := \left[ \int_t^\bar{t} s^b_s \tilde{B}_s^c ds \bigg| \mathcal{G}_t \right]$,

\[
\begin{align*}
\text{FCA}^\Delta_t &:= \text{FCA}_t - \mathcal{O}_t, \\
\text{FBA}^\Delta_t &:= \text{FBA}_t - \mathcal{O}_t, \\
\text{DVA}^\Delta_t &:= \text{DVA}_t - \mathbb{E} \left[ \int_t^\bar{t} L^H L^m(\tilde{e}^E_s)^+ ds \bigg| \mathcal{G}_t \right], \\
\text{CVA}^\Delta_t &:= \text{CVA}_t - \mathbb{E} \left[ \int_t^\bar{t} L^C L^m(\tilde{e}^E_s)^- ds \bigg| \mathcal{G}_t \right],
\end{align*}
\]
where (2.34)-(2.41) are well-defined. In this case, we also define

\[
\begin{align*}
FVA & := FCA - FBA, \\
FVA^\Delta & := FCA^\Delta - FBA^\Delta.
\end{align*}
\]

**Remark 2.20.** (i) Assume that the local-martingales in (2.31) are true martingales. Then

\[
\begin{align*}
X & = FVA^\Delta - DVA^\Delta + CVA^\Delta \\
& = FCA^\Delta - FBA^\Delta - DVA^\Delta + CVA^\Delta.
\end{align*}
\]  

(ii) \( O \) is the opportunity cost of not entering the new contract. In addition, FCA and FBA is the aggregated funding cost and benefit together with the legacy portfolio. Recalling the definitions

\[
\begin{align*}
FCA^\Delta & = FCA - O^-, \\
FBA^\Delta & = FBA - O^+,
\end{align*}
\]

the incremental funding impacts are the differences between aggregated funding adjustments and the opportunity cost.

(iii) Under replacement close-out,

\[
\begin{align*}
FCA^\Delta_t & = \mathbb{E} \left[ \int_t^\tau \left( L^m X_s + L^m \tilde{p}_s^N + \tilde{B}_s^\varepsilon - \sum_{i \in I \setminus \rho} \tilde{\pi}_s^i s_s^b - (s_s^\varepsilon \tilde{B}_s^\varepsilon)^- \right) ds \mid G_t \right], \\
FBA^\Delta_t & = \mathbb{E} \left[ \int_t^\tau \left( L^m X_s + L^m \tilde{p}_s^N + \tilde{B}_s^\varepsilon - \sum_{i \in I \setminus \rho} \tilde{\pi}_s^i s_s^b + (s_s^\varepsilon \tilde{B}_s^\varepsilon)^+ \right) ds \mid G_t \right],
\end{align*}
\]  

while under clean close-out,

\[
\begin{align*}
FCA^\Delta_t & = \mathbb{E} \left[ \int_t^\tau \left( X_s + L^m \tilde{p}_s^N + \tilde{B}_s^\varepsilon - \sum_{i \in I \setminus \rho} \tilde{\pi}_s^i s_s^b - (s_s^\varepsilon \tilde{B}_s^\varepsilon)^- \right) ds \mid G_t \right], \\
FBA^\Delta_t & = \mathbb{E} \left[ \int_t^\tau \left( X_s + L^m \tilde{p}_s^N + \tilde{B}_s^\varepsilon - \sum_{i \in I \setminus \rho} \tilde{\pi}_s^i s_s^b + (s_s^\varepsilon \tilde{B}_s^\varepsilon)^+ \right) ds \mid G_t \right].
\end{align*}
\]  

It is often stated that there is no FVA when contracts are fully collateralized. Indeed, European Banking Authority (EBA) requires banks to assess FCA and FBA for derivatives that “are not strongly collateralized” (see Cameron, 2013). To examine this, assume full collateralization, i.e., \( L^m = 0, \) replacement close-out, and \( \rho = I. \) The condition of full repo markets is commonly chosen in literature. Moreover, when \( R^c \geq r \) and \( R^b \geq r, \) we can see that \( FCA^\Delta = FBA^\Delta = 0 \) from (2.43) and (2.44). Therefore, based on our model, \( FVA^\Delta \) is not necessary when the close-out amount is the replacement cost and all repo markets are fully liquid. However, even in full collateralization, \( FVA^\Delta \) still exists under clean close-out. This is one of the reasons why replacement close-out should be discussed.
(iv) Because of FVA, the BSDE of XVA (2.31) becomes semi-linear. When \( s^f \) and \( s^b \) are bounded, the generator is uniformly Lipschitz, so the existence and uniqueness are not hard to obtain. However, we need to solve the BSDE numerically, and this can be costly when a large netting set and long maturity are considered. There have been several attempts to obtain a closed-form solution. However, for the closed-form solution, it was necessary to assume \( s^f = s^b \) so that one can recover a linear equation as in Bichuch et al. (2018); Brigo et al. (2017); Piterbarg (2010).

At the stage, it was an assumption, but we will show that we do not need such assumption by proving that FVA is either FCA or -FBA, by obtaining either

\[
X_t + \tilde{p}^N_t - \tilde{B}^e_t - c_t - \sum_{i \in I^\rho} i_t \geq 0, \quad dQ \otimes dt \text{ a.s.}, \quad \text{or} \quad (2.47)
\]

\[
X_t + \tilde{p}^N_t - \tilde{B}^e_t - c_t - \sum_{i \in I^\rho} i_t \leq 0, \quad dQ \otimes dt \text{ a.s.}, \quad (2.48)
\]

for many derivative contracts. Then the BSDE becomes linear regardless of the assumption, \( s^f = s^b \), because one of the spreads does not play any role in solving the BSDE.

(v) Considering different lending/borrowing not only makes the BSDEs for replication pricing semi-linear but also makes the associated Hamiltonians non-smooth in optimal investment problems (see Bo, 2017; Bo & Capponi, 2016; Yang et al., 2017, for example).

(vi) For modeling incremental XVA with capital value adjustment and initial margin, readers may want to refer to Albanese et al. (2018).

**Remark 2.21.** It is worth mentioning how FVA and DVA related. To see this, we briefly review the results of Burgard & Kjaer (2010, 2011). Let \( n = 1, \rho = \{1, C\}, L^H = 1, e^N = p^N \). As XVA was not incremental in Burgard & Kjaer (2010, 2011), in this example, we also set \( e^N = 0 \) and \( \epsilon = 0 \).

First, we guess the amount of \( \tilde{p}^H \). At \( \tau = \tau^H \leq T \), \( X_{\tau^H} = -L^H (\tilde{p}^N_{\tau^H})^+ \). Therefore,

\[
\Delta X_{\tau^H} = -X_{\tau^H} + \tilde{p}^H_{\tau^H} - L^H (\tilde{p}^N_{\tau^H})^+.
\]

The hedger may want to hedge the jump risk at \( \tau^H \) using \( \tilde{p}^H \) so that \( \Delta X_{\tau^H} = 0 \), i.e., the hedger may choose

\[
\tilde{p}^H = X_{\tau^H} - L^H (\tilde{p}^N)^+.
\]

Indeed, it will be shown later that (2.49) is the right choice. For now, we accept (2.49).

Assuming the local martingales in (2.31) are true martingales, by Definition 2.19, for \( t < \tau \),

\[
\text{DVA}_t = \mathbb{E} \left[ \mathbf{1}_{\tau = \tau^H} L^H (\tilde{p}^N)_{\tau^H}^+ ds | \mathcal{G}_t \right],
\]

\[
\text{FCA}_t = \mathbb{E} \left[ \int_t^\tau s_{\tau}^{\rho} \left[ \tilde{p}^N_{\tau} - L^H (\tilde{p}^N)_{\tau}^+ \right] ds | \mathcal{G}_t \right],
\]

\[
\text{FBA}_t = \mathbb{E} \left[ \int_t^\tau s_{\tau}^{\rho} \left[ \tilde{p}^N_{\tau} - L^H (\tilde{p}^N)_{\tau}^+ \right] ds | \mathcal{G}_t \right].
\]
However, since $L^H \leq 1$,

$$\tilde{p}^N - L^H (\tilde{p}^N)^+ = (\tilde{p}^N)^-,$$

$$\tilde{p}^N - L^H (\tilde{p}^N)^+ = (\tilde{p}^N)^+.$$

Then by Lemma 2.2,

$$DVA_t = \mathbb{E} \left[ \int_t^T G_s h_s^H L^H (\tilde{p}_s^N)^+ \, ds \mid \mathcal{F}_t \right],$$

$$FCA_t = \mathbb{E} \left[ \int_t^T G_s s_b^b(\tilde{p}_s^N)^- \, ds \mid \mathcal{F}_t \right],$$

$$FBA_t = \mathbb{E} \left[ \int_t^T G_s s^e_s(\tilde{p}_s^N)^+ \, ds \mid \mathcal{F}_t \right].$$

If we assume that the hedger’s borrowing rate is higher than $r$ only because of the own default risk, i.e., there is no liquidity premium, then we can approximate the hedger’s borrowing spread as

$$s^b = h^H L^H.$$

With this assumption, FCA becomes a counterpart of DVA, i.e.,

$$DVA_t = \int_t^T G_s h_s^H L^H \mathbb{E}[(\tilde{p}_s^N)^+] \, ds, \quad (2.50)$$

$$FCA_t = \int_t^T G_s h_s^H L^H \mathbb{E}[(\tilde{p}_s^N)^-] \, ds. \quad (2.51)$$

The above two equations show the financial relationship between DVA and FCA. DVA is a benefit to the shareholders because the hedger may default on derivative payables. On the other hand, the bondholders will receive a partial amount of the derivative receivables, namely $(1 - L^H)(\tilde{p}^N)^-$. Therefore, the hedger should compensate the funding provider for the expected loss.

When the hedger has redundant money, she can use it to buy back loans that were already issued. In this case, we may inspect $s^e = h^H L^H$. This inspection leads to

$$DVA = FBA. \quad (2.52)$$

It is important to avoid double-counting for both pricing and accounting. However, in this case, recalling that (2.42), it seems that we have two adjustments with the approximately same value. If (2.52) is valid for many contracts, i.e., DVA is overlapped with FBA, then one of them should be ignored.

Indeed, it often believed that recording both FBA and DVA in bank’s account engenders a double-counting paradox. On the other hand, IFRS and GAAP accept DVA. To remedy this, in FCA/FBA accounting, which is endorsed by some banks, DVA is recorded in Contra-Liability (CL) account and

$$FCA - (FBA - DVA)$$
is recorded in Contra-Asset (CA) account (see Albanese & Andersen, 2014; Castagna, 2011). However, it has been pointed out that FCA/FBA accounting produces large asset/liability asymmetry. 

The large asymmetry is partly attributed to the binary nature of FVA. Based on the marginal effect of entering a contract, the FCA term in CA account does not counteract the FBA term in CL account. It will be shown later that the binary nature of FVA is related to whether the payoff functions are increasing or decreasing with respect to underlying assets. On the other hand, because DVA occurs from derivative payables, e.g., \( p^N \geq 0 \) in (2.50), DVA arises where the payoffs are positive. Therefore, FBA and DVA are affected by different mathematical structures of derivative contracts, i.e., FBA is not overlapped with DVA in CA account. Thus, FCA/FBA accounting inevitably leads to large asset/liability asymmetry. To avoid the asymmetry, in FVA/FDA accounting, FVA is recorded in CA account and funding debt value adjustment (FDA) is recorded in CL account. FDA is a benefit that the bank can default on its liability. It was named DVA \(^2\) in Hull & White (2012). If liquidity is not considered, the value of FVA can be approximated by FDA. For the detail, readers may want to refer to Albanese & Andersen (2014).

In the next section, we represent (2.31) as a standard form, and reduce it to a BSDE on the reference filtration \( \mathcal{F} \).

### 2.4 BSDE formulation

For a BSDE representation, we begin this section with considering a family of maps \((\phi_t)_{t \geq 0}\) such that

\[
\phi_t : \sum_{i \in \tilde{I}} \tilde{\pi}_i \sigma_i^1 - Z_i^N \rightarrow \sum_{i \in \tilde{I} \setminus \rho} \tilde{\pi}_i.
\]

The form of \((\phi_t)_{t \geq 0}\) varies depending on parameters and accessibility of repo markets. Before giving the general form \((\phi_t)_{t \geq 0}\), we examine some examples.

**Example 2.22.** (i) Consider \( \rho = I \), which is commonly assumed in literature. In this case,

\[
\phi_t : z \rightarrow 0. \tag{2.53}
\]

(ii) Consider \( n = 1 \), and constant parameters. Then \( S^1, S^H, S^C \) follow

\[
\begin{align*}
\text{d}S_t^1 &= rS_t^1 \text{d}t + \sigma^1 S_t^1 \text{d}W_t, \\
\text{d}S_t^i &= rS_t^i \text{d}t - S_t^i \text{d}M_t, \quad i \in \{H, C\}.
\end{align*}
\]

It follows that \( \sum_{i \in \tilde{I}} \tilde{\pi}_i \sigma^1 = \tilde{\pi}^1 \sigma^1 \). When \( \rho = \{H, C\} \),

\[
\phi_t : z \rightarrow (\sigma^1)^{-1}(z + Z_i^N) = (\sigma^1)^{-1}z + (\sigma^1)^{-1}Z_i^N. \tag{2.54}
\]

On the other hand, when \( \rho = \{1, C\} \), \( \sum_{i \in \tilde{I} \setminus \rho} \tilde{\pi}_i = \tilde{\pi}^H \). Thus,

\[
\phi_t : z \rightarrow \tilde{\pi}_i^H. \tag{2.55}
\]

This case was discussed by Burgard & Kjaer (2010).
(iii) On the other hand, let us assume that OIS rate is an \( \mathbb{F} \)-adapted process. In addition, we assume that for any \( i \in \{H, C\} \), \( (G^i_t)_{t \geq 0} \) is given by

\[
dG^i_t = -h^i_t G^i_t \, dt,
\]

where \( (h^i_t)_{t \geq 0} \) are deterministic processes. We consider non-defaultable and defaultable zero coupon bonds with the same maturity as \( T \), i.e., \( S^i, S^H, S^C \) are defined as

\[
S^i_t := B_t \mathbb{E} \left[ B^{-1}_T \mid \mathcal{F}_t \right],
\]

\[
S^i_t := B_t \mathbb{E} \left[ \mathbf{1}_{t > T} B^{-1}_T \mid \mathcal{F}_t \right], \quad i \in \{H, C\}.
\]

By Lemma 2.2, \( S^i_t = \mathbf{1}_{t < T} B_t (G^i_t)^{-1} \mathbb{E} [G^i_T B^{-1}_T | \mathcal{F}_t] \). Since \( (G^i_t)_{t \geq 0}, i \in \{H, C\} \), are deterministic,

\[
S^i_t = \mathbf{1}_{t < T} (G^i_t)^{-1} G^i_T S^1_t.
\]

It follows that for \( t < \tau \), \( \sigma^1 = \sigma^H = \sigma^C \). Recall \( \sigma = [\sigma^1 \ldots \sigma^n \sigma^H \sigma^C]^\top \). We define \( n \times n \) matrix

\[
\Sigma := \begin{bmatrix}
\sigma^1 \\
\vdots \\
\sigma^n
\end{bmatrix},
\]

and assume that \( \Sigma \) is invertible for any \( t < T \). Then,

\[
(\Sigma^\top)^{-1} \sigma^\top \bar{\pi} = \begin{bmatrix}
\bar{\pi}^1 + \bar{\pi}^H + \bar{\pi}^C \\
\bar{\pi}^2 \\
\vdots \\
\bar{\pi}^n
\end{bmatrix}.
\]

Therefore, \( \mathbf{1}^\top (\Sigma^\top)^{-1} \sigma^\top \bar{\pi} = \sum_{i \in I} \bar{\pi}_i \), where \( \mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^n \). Thus, if \( \rho = \emptyset \),

\[
\phi_1 : z \to \mathbf{1}^\top (\Sigma^\top)^{-1} (z + Z_t^N), \tag{2.56}
\]

On the other hand, consider \( \rho \neq \emptyset \) and define \( \mathbf{1}^\rho \in \mathbb{R}^n \) by

\[
(\mathbf{1}^\rho)_i := \begin{cases}
0 & i \in \rho, \\
1 & i \notin \rho,
\end{cases}
\]

where \( (\mathbf{1}^\rho)_i \) denote \( i \)-th component of \( \mathbf{1}^\rho \). When \( \rho \cap \{1, H, C\} = \emptyset \), \( \phi_1 : z \to (\mathbf{1}^\rho)_\top (\Sigma^\top)^{-1} (z + Z_t^N) \). However, if \( \rho = \{1\} \),

\[
\phi_1 : z \to (\mathbf{1}^\rho)_\top (\Sigma^\top)^{-1} (z + Z_t^N) + \bar{\pi}^1 + \bar{\pi}^H + \bar{\pi}^C. \tag{2.57}
\]

Therefore, the form of BSDE depends on the choice of model, accessibility of repo markets, etc. However, from (2.53)-(2.57), we can observe that \( \phi \) is linear in \( z, Z_t^N, \bar{\pi}^H, \bar{\pi}^C \). As we will see later that \( \bar{\pi}^H \) and \( \bar{\pi}^C \) can be dependent of a solution of the BSDE under replacement close-out. For simplicity, we assume that \( \phi \) is an independent form of \( \bar{\pi}^H \) and \( \bar{\pi}^C \) such as (2.53), (2.54), and (2.56).
Assumption 2.23. For some $n$-dimensional $\mathbb{F}$-progressively measurable process $\alpha$,

$$\phi_t(z) = \alpha_t^\top (z + Z_t^N).$$

Remark 2.24. Note that we do not exclude the convention commonly used in the literature that assets are traded from repo markets, i.e., $I = \rho$. In this case, we can set $\alpha = 0$ in Assumption 2.23.

Now, we denote the generator of (2.31) by $g^G$, i.e.,

$$g^G_t(y, z) := -(y + \tilde{p}_t^N + \tilde{B}_t^e - c_t - \phi_t(z))^+ s_t^c + (\Delta \tilde{p}_t^N + \tilde{B}_t^e - c_t - \phi_t(z))^- s_t^b + s_t^b \tilde{B}_t^e.$$

In addition, we denote the incremental exposures by $\Theta^{\Delta i}$, $i \in \{H, C\}$, more precisely,

$$\Theta^{\Delta H}(y) := \Theta^H(y) - L^H L^m(e^F)^+,$$

$$\Theta^{\Delta C}(y) := \Theta^C(y) - L^C L^m(e^F)^-. $$

Let $(Y^G, Z^G, \tilde{n}^H, \tilde{n}^C)$ denote the solution, in a certain space, of the following BSDE:

$$Y^G_t = -\mathbb{I}_{t=T=t^H} \tilde{H}^A Y^G_T + \mathbb{I}_{t=t^C} \tilde{H}^{AC}(Y^G_t) + \int_t^T g^G_s(Y^G_s, Z^G_s) ds - \int_t^T (Z^G_s)^\top dW_s + \sum_{i=H,C} \int_t^T \tilde{n}^i_s dM^i_s. \tag{2.58}$$

Then $(Y^G, Z^G, \tilde{n}^H, \tilde{n}^C)$ provides $(X, \tilde{n})$ as well as $(V, \pi)$. However, instead of directly dealing with (2.58), we will investigate a reduced BSDE on the reference filtration $\mathbb{F}$. The idea is as follows.

It is a known fact that in the progressively enlarged filtration $\mathbb{G}$, for any $\mathbb{G}$-optional (resp. predictable) process has an $\mathbb{F}$-optional (resp. predictable) reduction. Therefore, if there exists a solution of (2.58) such that $Y^G$ is $\mathbb{G}$-optional and $Z^G$ is $\mathbb{G}$-predictable, there exists an $\mathbb{F}$-adapted pair $(Y^F, Z^F)$ satisfying

$$Y^G = \mathbb{I}_{t \leq T} Y^F, \tag{2.59}$$

$$Z^G = \mathbb{I}_{t \leq T} Z^F. \tag{2.60}$$

Moreover, we guess that $(Y^F, Z^F)$ is a solution of a BSDE on the reference filtration $\mathbb{F}$, i.e., “for some” $g^F: \Omega \times [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}$,

$$Y^F_t = \int_t^T g^F_s(Y^F_s, Z^F_s) ds - \int_t^T (Z^F_s)^\top dW_s. \tag{2.61}$$

Then, by the terminal condition $Y^G_T = -\mathbb{I}_{T=T=t^H} \tilde{H}^A Y^G_T + \mathbb{I}_{T=t^C} \tilde{H}^{AC}(Y^G_T)$, together with (2.59), we can expect that

$$Y^G_t = \mathbb{I}_{t < T} Y^F_t + \mathbb{I}_{t \geq T} \left[ -\mathbb{I}_{t=t^H} \tilde{H}^A(Y^F_t) + \mathbb{I}_{t=t^C} \tilde{H}^{AC}(Y^G_t) \right]. \tag{2.62}$$

By applying Itô’s formula to (2.62), we can find what the proper $g^F$ should be. At the end, finding $(V, \pi)$ reduces to investigating the reduced BSDE (2.61). The next proposition explains the detail.
Therefore, by (H)-hypothesis holds between \( \mathbb{F} \) and \( \mathbb{G} \), i.e., any \( (\mathbb{F}, \mathbb{Q}) \)-martingale is a \( (\mathbb{G}, \mathbb{Q}) \)-martingale, then (2.65)-(2.68) is the only solution of (2.58).

**Proof.** By Itô's formula,

\[
d(1_{t \leq T} Y^\mathbb{F}_t) = 1_{t \leq t} dY^\mathbb{F}_t - \delta_t(dt) Y^\mathbb{F}_t = 1_{t \leq t} dY^\mathbb{F}_t - 1_{t \leq t} Y^\mathbb{F}_t dM_t - 1_{t \leq t} h_t Y^\mathbb{F}_t dt
\]

and

\[
d\left(1_{t \leq T} \left[ -1_{t = t = H} \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) + 1_{t = t = C} \tilde{\Theta}^\Delta C (Y^\mathbb{F}_t) \right] \right)
= -1_{t = t = H} \delta_t(dt) \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) + 1_{t = t = C} \delta_t(dt) \tilde{\Theta}^\Delta C (Y^\mathbb{F}_t)
= -1_{t \leq t} \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) dM_t - 1_{t \leq t} h_t \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) dt
+ 1_{t \leq t} \tilde{\Theta}^\Delta C (Y^\mathbb{F}_t) dM_t + 1_{t \leq t} h_t \tilde{\Theta}^\Delta C (Y^\mathbb{F}_t) dt.
\]

If we take \( Y^\mathbb{G} \) as in (2.65),

\[
dY^\mathbb{G}_t = 1_{t \leq t} dY^\mathbb{G}_t - 1_{t \leq t} Y^\mathbb{G}_t dM_t + 1_{t \leq t} Y^\mathbb{G}_t dM_t - 1_{t \leq t} h_t Y^\mathbb{G}_t dt
- 1_{t \leq t} \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) dM_t - 1_{t \leq t} h_t \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) dt
+ 1_{t \leq t} \tilde{\Theta}^\Delta C (Y^\mathbb{F}_t) dM_t + 1_{t \leq t} h_t \tilde{\Theta}^\Delta C (Y^\mathbb{F}_t) dt
= 1_{t \leq t} \left[ -1_{t \leq t} \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) - h_t Y^\mathbb{F}_t - h_t \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t) dt + 1_{t \leq t} (Z_t^\mathbb{F}_t) dW_t
- 1_{t \leq t} [Y^\mathbb{F}_t + \tilde{\Theta}^\Delta H (Y^\mathbb{F}_t)] dM_t - 1_{t \leq t} [Y^\mathbb{F}_t - \tilde{\Theta}^\Delta C (Y^\mathbb{F}_t)] dM_t
\]

Therefore, by (2.64), (2.65)-(2.68) give a solution for (2.58). Moreover, if (H)-hypothesis holds, (unique) martingale representation property holds by \( W \) and \( M^i, i \in \{H, C\} \). Therefore, by Theorem 4.1 in Crétpey & Song (2015), if \( (Y^\mathbb{G}, Z^G) \) solves (2.58), \( (Y^\mathbb{G})^\top, Z^G \mathbb{1}_{[0,1]} \) solves (2.61) as well.

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Remark 2.26. This reduction argument was also used by Bichuch et al. (2018); Brigo et al. (2016); Crépey (2015a,b).

Remark 2.27. (2.67) explains how the bank’s own default can be beneficial to the shareholders. The own default can be monetized by buying back their own bond that becomes cheaper. However, this means that banks can realize the profit only when they actually default. Indeed, DVA is often excluded from Common Equity Tier 1 capital (CET1), which is a proxy of the shareholder’s wealth.

Note that $h Y^F$ in $g^F$ is an adjustment for an early termination. To see this, let $\hat{Y} := G Y^F, \hat{Z} := G Z^F$. Then (2.63) becomes

$$\hat{Y}_t^F = \int_t^T G_s \left[ g^F_s(Y^F_s, Z^F_s) - h_s^H \left( \hat{Y}_s^F \right) - h_s^C \left( \hat{Y}_s^F \right) \right] ds - \int_t^T (\hat{Z}_s^F)^\top dW_s.$$  

Moreover, by Definition 2.19 and Lemma 2.2, if $Z^F \in \mathbb{H}^2_T$, for $t < \tau$,

$$\text{FCA}_t^A = G_t^{-1} \mathbb{E} \left[ \int_t^T G_s \left[ \left( Y^F_s + \tilde{p}_s^N + \tilde{B}_s^e - c_s - \phi_s(Z^F_s) \right)^+ - \left( \hat{Y}_s^F \right)^+ \right] ds \bigg| \mathcal{F}_t \right],$$  

$$\text{FBA}_t^A = G_t^{-1} \mathbb{E} \left[ \int_t^T G_s \left[ \left( Y^F_s + \tilde{p}_s^N + \tilde{B}_s^e - c_s - \phi_s(Z^F_s) \right)^- - \left( \hat{Y}_s^F \right)^- \right] ds \bigg| \mathcal{F}_t \right],$$  

$$\text{DVA}_t^A = G_t^{-1} \mathbb{E} \left[ \int_t^T G_s \left( \Delta^H \left( Y^F_s \right) \right) ds \bigg| \mathcal{F}_t \right],$$  

$$\text{CVA}_t^A = G_t^{-1} \mathbb{E} \left[ \int_t^T G_s \left( \Delta^C \left( Y^F_s \right) \right) ds \bigg| \mathcal{F}_t \right],$$

and $\text{XVA} = G^{-1} \hat{Y}$. Still, because of the semi-linearity in (2.69) and (2.70), we need to solve (2.61) numerically. More importantly, it will be interesting to investigate how much cancellation between FCA and FBA, and between FBA and DVA, can be expected. We answer the questions in the next section. Based on our model, the answers for the questions are both negative.

3 Main Results

In this section, we show a binary nature of FVA; FVA is either FCA or FBA. In other words, either $\text{FCA} = 0$ or $\text{FBA} = 0$. This switching property of FVA is determined by some properties of payoff structures of the derivative contract. Before the main theorem, we explain the idea by an example first.

Example 3.1 (Stock forward contract with clean close-out). Consider $n = 1, e^N = p^N$. For simplicity, we assume all parameters are constant and let the traded assets $(S^1, S^H, S^C)$ given by

$$d S^1_t = r_t S^1_t \, dt + \sigma^1 S^1_t \, dW_t,$$

$$d S^i_t = r_t S^i_t \, dt - S^i_t \, dM^i_t, \quad i \in \{H, C\}.$$
Moreover, we assume that the defaultable bonds can be traded through repo markets, i.e., \( \rho = \{ H, C \} \). Then \( \sum_{i \in I} \pi_i^1 \sigma_i = \pi_i^1 \sigma_i \) and \( \sum_{i \in I} \rho \pi_i^1 = \pi_i^1 \), therefore
\[
\phi(z) = \alpha(z + Z^N) = (\sigma^1)^{-1}(z + Z^N).
\]

Let \( \mathfrak{D}^N = \mathbb{I}_{[T, \infty]}(S^1 - K) \), for some \( K \geq 0 \). We denote \( S^1 := B^{-1} S^1 \). Recall the definition \( \tilde{p}^N = B^{-1} p^N \) and
\[
\begin{align*}
\Theta^{\Delta,H} & = L^H L^m \left( (\tilde{p}^N + \tilde{e}^F)^+ - (\tilde{e}^F)^+ \right), \\
\Theta^{\Delta,C} & = L^C L^m \left( (\tilde{p}^N + \tilde{e}^F)^- - (\tilde{e}^F)^- \right).
\end{align*}
\]

Thus, the generator \( g^F \) becomes
\[
g^F_t(y,z) = - \left[ y + L^m \tilde{p}_t^N + B_t^e - \alpha(z + Z^N_t) \right]^+ s^e + \left[ y + L^m \tilde{p}_t^N + B_t^e - \alpha(z + Z^N_t) \right]^- b + s^e B_t^e \\
- h^H L^H L^m \left( (\tilde{p}_t^N + \tilde{e}^F)^+ - (\tilde{e}^F)^+ \right) + h^C L^C L^m \left( (\tilde{p}_t^N + \tilde{e}^F)^- - (\tilde{e}^F)^- \right) - hy.
\]

To explain the idea, the hedger should pay \( S^1_T - K \) at the maturity or \( \tilde{p}_t^N = S^1_t - B_t B^{-1} K \) at an early termination \( t < T \). For the payment, she needs to retain \( S^1 \). To buy \( S^1 \), the hedger may need to borrow money, so it is expected that \( s^e \) does not play an important role in maintaining the hedging portfolio. Therefore, we guess
\[
Y^F + L^m \tilde{p}^N - \alpha(Z^F - Z^N) \leq 0, \quad d\mathbb{Q} \otimes dt \text{ a.s.} \tag{3.1}
\]

Unless the tendency of (3.1) is dominated by the legacy portfolio, we can recover a linear BSDE. For simplicity, we assume that the dealer had big enough exposure to the counterparty before the new contract and the initial exposure dominates the new exposure, i.e., we assume that
\[
\tilde{p}^N + \tilde{e}^F \geq 0, \quad \tilde{e}^F \geq 0, \quad d\mathbb{Q} \otimes dt \text{ a.s.} \tag{3.2}
\]

Then we consider \((Y^#, Z^#)\) satisfying
\[
Y_t^# = - \int_t^T \left[ \begin{array}{c}
Y_s^# + L^m \tilde{p}_s^N + B_s^e - \alpha(Z_s^# + Z_s^N) \\
\end{array} \right] s^b - s^e B_s^e + h Y_s^# \, ds \\
- \int_t^T h^H L^H L^m \tilde{p}_s^N \, ds - \int_t^T (Z_s^#)^\top \, dW_s,
\]

Then we will show that
\[
Y^# + L^m \tilde{p}^N + \tilde{B}^e - \alpha(Z^# + Z^N) \leq 0, \quad d\mathbb{Q} \otimes dt \text{ a.s.} \tag{3.3}
\]

To show this, we take another transformation, \( V^F := Y^# + L^m \tilde{p}^N + \tilde{B}^e \) and \( \Pi^F := Z^# + L^m Z^N \). Then \((V^F, \Pi^F)\) is the solution of
\[
V_t^F = L^m \zeta + \tilde{B}_t^e + \int_t^T F_t(V_s^F, \Pi_s^F) \, ds - \int_t^T (\Pi_s^F)^\top \, dW_s, \tag{3.4}
\]
where $\xi := \tilde{S}_T^1 - B_T^{-1} K$ and

$$F_t(y, z) := -(y - \alpha z)s^b - hy + \xi^b_t, \quad \xi^b := (h - h^H L^H L s^b + \alpha(1 - L^m)s^b Z^N).$$

(3.5)

Under mild conditions, we can obtain $(V^F, \Pi^F) \in L^2([0, T] : \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$, i.e., $\forall t \leq T, (V^F_t, \Pi^F_t) \in \mathbb{D}^{1,2}$ and

$$\int_0^T (\|V^F_t\|_{1,2}^2 + \|\Pi^F_t\|_{1,2}^2) dt < \infty.$$  

(3.6)

Moreover, $(D_t V^F_t)_{0 \leq t \leq T}$ is a version of $(\Pi^F_t)_{0 \leq t \leq T}$. In addition, note that (3.3) is equivalent to

$$V^F - (1 - L^m)s^b a Z^N - a \Pi^F \leq V^F - (1 - L^m)s^b a Z^N - a D V^F \leq 0, \; dQ \otimes dt - a.s.$$  

(3.7)

To show (3.7), for $\theta \leq t$, let $(V^F_{t,\theta}, \Pi^F_{t,\theta}) := (a D_\theta V^F_t, a D_\theta \Pi^F_t)$. Therefore, $(V^F_{t,\theta}, \Pi^F_{t,\theta})$ is given by

$$V^F_{t,\theta} = a L^m D_\theta \xi + \int_t^T F_{s,\theta}(V^F_{s,\theta}, \Pi^F_{s,\theta}) ds - \int_t^T (\Pi^F_{s,\theta})^{T} dW_s, \quad F_{t,\theta}(y, z) := -(y - \alpha z)s^b - h_1 y + a D_\theta \xi^b_t.$$  

(3.8)

(3.9)

Note that $F_{t,\theta}(y, z) = F_t(y, z) + a D_\theta \xi^b_t - \xi^b_t$. Namely, (3.8) can be written as

$$V^F_{t,\theta} = a L^m D_\theta \xi + \int_t^T \left[ F_s(V^F_{s,\theta}, \Pi^F_{s,\theta}) + a D_\theta \xi^b_t - \xi^b_t \right] ds - \int_t^T (\Pi^F_{s,\theta})^{T} dW_s.$$  

(3.10)

We will show $V^F_{t,\theta} \leq V^F_{t,\theta}$ by comparing (3.4) and (3.8), and it suffices to show that for $\theta \leq t$,

$$L^m \xi + \tilde{B}^e_T - a L^m D_\theta \xi \leq 0, \; a.s.$$  

(3.11)

$$\xi^b - a D_\theta \xi^b \leq 0, \; dQ \otimes dt - a.s.$$  

(3.12)

$$Z^N \geq 0, \; dQ \otimes dt - a.s.$$  

(3.13)

We will show that the above inequalities hold if $e$ is not too big. More precisely, we assume

$$\epsilon \leq \epsilon_* := K(B^e_T)^{-1} \min \left\{ L^m, 1 - \frac{h^H L^H L^m}{h} \right\}.$$  

(3.14)

It is easy to check

$$\tilde{B}^e_T + L^m \xi - a L^m D_\theta \xi = \tilde{B}^e_T + L^m (\tilde{S}^1_T - B_T^{-1} K) - (\sigma^1)^{-1} L^m D_\theta (\tilde{S}^1_T - B_T^{-1} K) = \tilde{B}^e_T + L^m \tilde{S}^1_T - B_T^{-1} K \leq 0.$$  

Moreover, by Proposition 3.12 in Di Nunno et al. (2009),

$$\tilde{p}^N_t = \epsilon E_{[\xi, F_t]} = \epsilon E_{[\xi, F_t]} = \epsilon E_{[\xi, F_t]} = \epsilon E_{[\xi, F_t]}.$$  

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Moreover, \( Z^N = D\tilde{p}^N = (\sigma^1)^{-1}\tilde{S}^1 \). Then, it follows that
\[
\xi_t^b - \alpha D_\theta \xi_t^b = (h - h^H L^H L^m)\xi_t - \alpha D_\theta \xi_t^b F + h^\theta = h(L^\xi - B\xi + B^\xi)\]
\[
= h(B^\xi - B\xi) + h^H L^H L^m B^{-1}K \tag{3.15}
\]

Therefore, by comparison principle, for \( \theta \leq t \), \( V_{i,t}^F \leq V_{i,\theta}^F \). In particular, \( V_{i,t}^F \leq V_{i,\theta}^F = \alpha D_{i,t} V_{i,\theta}^F = \alpha \Pi_{i,t}^F \). Thus, \((3.3)\) is guaranteed and \((Y^\#_t, Z^\#_t) = (Y^F_t, Z^F_t)\). Moreover, \((Y^F_t, Z^F_t)\) follows the linear BSDE \((3.4)\), so we can find an analytic form of \((Y^F_t, Z^F_t)\), namely, \((V, \pi)\) as well. In addition, FBA = 0.

**Remark 3.2.**  
(i) If \( \varepsilon \geq \varepsilon^* := K(B^b_T)^{-1} \), one may want to consider \((Y^\#, Z^\#)\) given by
\[
Y_t^\# = \int_t^T \left[ (Y_s^\# + L^m \tilde{p}_s^N + \tilde{B}^\varepsilon_s - \alpha (Z_s^\# + Z_s^N)) \xi_s + L^m \tilde{p}_s^N \right] ds
\]
\[
- \int_t^T h^H L^H L^m \tilde{p}_s^N ds - \int_t^T \left( Z_s^\# \right)^T dW_t,
\]
and obtain opposite inequalities of \((3.11)\) and \((3.12)\). In this case, FCA = 0. Thus, the binary funding impacts depend on the value of initial portfolio, \( \varepsilon \).

(ii) If inequalities such as \((3.2)\) are not satisfied, instead of \((3.6)\) in the above example, one should consider
\[
\xi_t^b := h\tilde{B} + \alpha (1 - L^m) s\tilde{p}^N + h\tilde{p}^N - \mathbb{I}_{\tilde{p}^N + \xi^b} \xi^b \varepsilon^b \geq 0, \varepsilon^b \geq 0 h^H L^H L^m \tilde{p}^N
\]
\[
- \mathbb{I}_{\tilde{p}^N + \xi^b} \varepsilon^b \geq 0, \varepsilon^b \geq 0 h^H L^H + h^C L^C \varepsilon^b
\]
\[
- \mathbb{I}_{\tilde{p}^N + \xi^b} \varepsilon^b \geq 0, \varepsilon^b \geq 0 h^H L^H + h^C L^C \varepsilon^b.
\]
If the hedger and counterparty are both major banks having similar credit risks so that we can assume \( h^H L^H = h^C L^C \), we can obtain the same result as in Example 3.1.

(iii) Note that in the above example, DVA \( \neq 0 \). Consider the same market conditions but
\[
\bar{Z}^N = \mathbb{I}_{[T, \infty]}(K - S_T^1).
\]
Namely, the hedger is in long position of the stock forward contract. We can calculate the same way for the opposite inequality of \((3.3)\). In this case, FCA = 0, and still DVA \( \neq 0 \). Note that FVA is either –FBA or FCA, so neutralization of a substantial portion between FCA and FBA is hardly expected. This binary nature of FVA is a source of asset/liability asymmetry of FCA/FBA accounting.

(iv) Example 3.1 also tells us how and when FBA and DVA are different. FBA, as a counterpart of FCA, reduces FCA which originates from the bank’s default risk. DVA is also a benefit from the bank’s default risk, but it is hard to monetize DVA before the bank actually defaults. By these reasons, it is often believed that FBA and DVA are overlapped, and DVA is not considered in derivative transactions. However, Example 3.1 shows the different mathematical structure between FBA and DVA. DVA is a benefit from the possibility that the bank may default on its
derivative payables. Thus, DVA occurs where $\xi \geq 0$. On the other hand, FBA occurs where the opposite inequality of $(3.11)$. To understand the meaning of $(3.11)$, we set $\epsilon = 0$ and consider $\xi = \psi(S_T^1)$ for some smooth function $\psi: \mathbb{R} \rightarrow \mathbb{R}_+$. Then, FBA occurs where

$$\xi - aD_\theta \xi \geq 0,$$  \hspace{1cm} (3.16)

where $a = (\sigma^1)^{-1}$ in Example 3.1. $(3.16)$ can be rewritten as

$$\psi(S_T^1) - \psi(S_T^1)S_T^1 \geq 0,$$ \hspace{1cm} (3.17)

and $(3.17)$ again can be rewritten as

$$\frac{d}{dx} \ln(\psi(x)) \leq \frac{1}{x}, \ \forall x \geq 0.$$ \hspace{1cm} (3.18)

One sufficient condition for $(3.18)$ is that $\psi$ is non-increasing. Similarly, consider $\xi = \Psi(S_T^1)$, where $\Psi(x) := a(x - k)$, for some $a \in \mathbb{R}$ and $k > 0$. In this case again, $(3.16)$ is equivalent to $a \leq 0$. In summary, DVA occurs where the payoff is positive while FBA occurs where the payoff is non-increasing.

(v) Another reason of belief that FBA is overlapped with DVA may be the convention in literature that all assets can be traded in repo markets, i.e., $\rho = I$. Recall that $\alpha = 0$ when $\rho = I$ (recall Remark 2.24). When $\alpha = 0$, by $(3.16)$, FBA also occurs where $\xi \geq 0$. However, we can not guarantee that repo markets are always available. Indeed, there are some difficulties in using equities as repo collateral. The amounts of traded equities are smaller than fixed-income securities and there is no generally accepted method to valuate equities. By these reasons, equity repo markets are often limited to equity indices and baskets including many securities.

Remark 3.3. To the best of our knowledge, Malliavin differentiability of ABSDE has not been studied. In cases of clean close-out, the absence of initial margin is merely for simplicity. However, when we consider with initial margin under replacement close-out, we can not use the same method as in Example 3.1.

3.1 Main Theorems

In what follows, for the incremental effects, we focus on funding impacts, so in what follows, we assume that

$$e^E = 0,$$

i.e., $\text{CVA}^\Delta = \text{CVA}$ and $\text{DVA}^\Delta = \text{DVA}$. However, $\epsilon$ may not be zero. Under this restriction, we give the main theorems to show that either $\text{FBA} = 0$ or $\text{FCA} = 0$. We consider deterministic default intensities, volatility, and funding spreads, but $(r_t)_{t \geq 0}$ does not need to be deterministic so that we can apply the result to cases that $(r_t)_{t \geq 0}$ is a general $\mathbb{F}$-adapted process. These assumptions are mainly for simplicity. One case of stochastic default intensities is reported in Appendix. Moreover, we consider a derivatives of European style. For derivatives that has cash-flows at multiple times, we can divide
the interval \([0, T]\) according to the time of cash-flows. For example, if \(\mathcal{D}_T = \sum_{i=1}^N 1_{T_i \leq t} \xi_i\), for \(t \in (T_{i-1}, T_i)\), we consider the following BSDE:

\[
Y_t^F = Y_T^F + \int_{T_{i-1}}^{T_i} g_s^F (Y_s^F, Z_s^F) \, ds + \int_{T_{i-1}}^{T_i} (Z_s^F)^T \, dW_s.
\]

(3.19)

Then we can apply the next theorems for each BSDE (3.19). The idea of the proofs is similar to Example 3.1, and the proofs are reported in Appendix. We start from clean close-out.

**Theorem 3.4 (Clean close-out).** We assume \(h^H, h^C, s^f, s^b\) are deterministic and bounded. Moreover, \(\alpha\) is also deterministic and \(a^f\), \(a^b\) are bounded. Consider clean close-out, i.e., \(e^N = p^N\), and \(B^{-1}\mathcal{D}_T = 1_{T \leq t} \xi\). In addition, we assume

\[
\xi \in \mathbb{L}^2_T \cap \mathbb{L}^{1,2}_T, \quad D_\theta \xi \in (\mathbb{D}^{1,2}_T)^n \quad \forall \theta \leq T, \quad \mathbb{E} \left[ \int_0^T |D_\theta \xi|^2 \, d\theta \right] < \infty, \quad \mathbb{E} \left[ \int_0^T \int_0^T |D_\theta (D_\theta \xi)|^2 \, d\theta \, dt \right] < \infty.
\]

(3.20, 3.21, 3.22)

We assume that \(1_{p^N = 0} = 0, dQ \otimes dt\) a.s. Let

\[
\xi^b := s^b (1 - L^m) \alpha^T Z^N + h^b \xi^e + (h - h^H L^H L^m)(\overline{p}^N)^+ - (h - h^C L^C L^m)(\overline{p}^N)^-, \\
\xi^f := s^f (1 - L^m) \alpha^T Z^N + h^f \xi^e + (h - h^H L^H L^m)(\overline{p}^N)^+ - (h - h^C L^C L^m)(\overline{p}^N)^-.
\]

(i) Assume that for any \(\theta \leq T\),

\[
L^m (\xi - \alpha^T D_\theta \xi) + \tilde{B}^f \leq 0, \quad \alpha^T Z^N \geq 0, \quad dQ \otimes dt - a.s.
\]

(3.23, 3.24)

then there exists \((Y^*, Z^*) \in \mathbb{S}^2_T \times \mathbb{H}^{2,n}_T\) that satisfies

\[
Y_t^* = \int_t^T \left[ - \left( Y_s^* + L^m \overline{p}^N_s + \tilde{B}^f_s - \phi_s (Z_s^*) \right) s^b_s - s^f_s \tilde{B}^e_f + h_s Y_s^* \right] \, ds \\
+ \int_t^T \left[ 1_{p^N_s \geq h_s^H L^H L^m} + 1_{p^N_s < h_s^C L^C L^m} \right] \overline{p}^N_s \, ds - \int_t^T (Z_s^*)^T \, dW_s,
\]

and \((Y^*, Z^*) = (Y^F, Z^F)\). In particular, FBA = 0.

(ii) On the other hand, assume that for any \(\theta \leq T\),

\[
L^m (\xi - \alpha^T D_\theta \xi) + \tilde{B}^f \geq 0, \quad \alpha^T Z^N \leq 0, \quad dQ \otimes dt - a.s.
\]

(3.26, 3.27, 3.28)
then there exists \((Y^\#, Z^\#) \in \mathbb{S}_T^2 \times \mathbb{H}_T^{2,n}\) that satisfies

\[
Y^\#_t = \int_t^T -\left[ \left( Y^\#_s + L^m \tilde{p}^N + \tilde{B}^c - \phi_s(Z^\#_s) \right) s^b_s - s^c_s \tilde{B}^c_s + h_s Y^\#_s \right] ds
+ \int_t^T -\left[ \mathbb{I}_{\tilde{p}_N^\# > 0} h^H_s L^H L^m + \mathbb{I}_{\tilde{p}_N^\# < 0} h^C_s L^C L^m \right] \tilde{p}^N_s ds - \int_t^T (Z^\#_s)^\top dW_s,
\]

and \((Y^\#, Z^\#) = (Y^F, Z^F)\). In particular, FCA = 0.

(iii) If the contract is un-collateralized, i.e., \(L^m = 1\), (3.24) and (3.27) are not required in (i) and (ii).

Recall that when we consider replacement close-out, \(\tilde{\Theta}^{A,i}, i \in \{H, C\}\), depend on \(Y^F\). However, \(\tilde{\Theta}^{A,i}(y)\) is not differentiable in \(y\). We can avoid the irregularity by considering contracts such that either \(\tilde{p}^N \geq 0\) or \(\tilde{p}^N \leq 0\), \(d\mathbb{Q} \otimes dt\) a.s., i.e., options.

**Theorem 3.5 (Replacement close-out).** We assume \(h^H, h^C, s^c, s^b\) are deterministic and bounded. Moreover \(\alpha\) is also deterministic and \(a_s^c, a_s^b\) are bounded. Consider replacement close-out, i.e., \(e^N = V_\_ - B^e = BY^F + p^N\), and \(B^{-1} \mathbb{D}^N = \mathbb{I}_{T \leq T'}\). In addition, we assume

\[
\xi \in \mathbb{D}_T^{1,2}, \quad \mathbb{E} \left[ \int_0^T |D_\theta \xi|^2 d\theta \right] < \infty. \tag{3.29}
\]

We assume that either \(\xi \geq 0\) or \(\xi \leq 0\) a.s., i.e., we consider options.

(i) Assume that \(e \leq 0\) and for any \(\theta \leq T\),

\[
L^m \xi - a_\theta^\top D_\theta \xi \leq 0, \quad a.s. \tag{3.31}
\]

If \(\xi \geq 0\) a.s, then there exists a solution \((Y^\#, Z^\#) \in \mathbb{S}_T^2 \times \mathbb{H}_T^{2,n}\) that satisfies

\[
Y^\#_t = \int_t^T -\left[ \left( L^m Y^\#_s + L^m \tilde{p}^N + \tilde{B}^c - \phi_s(Z^\#_s) \right) s^b_s - s^c_s \tilde{B}^c_s + h_s Y^\#_s \right] ds
+ \int_t^T -\left[ h^H_s L^H L^m (Y^\#_s + \tilde{p}^N_s) \right] ds - \int_t^T (Z^\#_s)^\top dW_s,
\]

and \((Y^\#, Z^\#) = (Y^F, Z^F)\). On the other hand, if \(\xi \leq 0\) a.s, then there exists a solution \((Y^\#, Z^\#) \in \mathbb{S}_T^2 \times \mathbb{H}_T^{2,n}\) that satisfies

\[
Y^\#_t = \int_t^T -\left[ \left( L^m Y^\#_s + L^m \tilde{p}^N + \tilde{B}^c - \phi_s(Z^\#_s) \right) s^b_s - s^c_s \tilde{B}^c_s \right] ds
+ \int_t^T -\left[ h^C_s L^C L^m (Y^\#_s + \tilde{p}^N_s) \right] ds - \int_t^T (Z^\#_s)^\top dW_s,
\]

and \((Y^\#, Z^\#) = (Y^F, Z^F)\). In particular, for both cases, FBA = 0.
(ii) Assume that $\epsilon \geq 0$ and for any $\theta \leq T$,

$$L^m \xi - a_0 \top D_0 \xi \geq 0, \text{ a.s.}$$

(3.32)

If $\xi \geq 0 \text{ a.s.}$, then there exists a solution $(Y^\#, Z^\#) \in S^2_T \times H^2_T$ that satisfies

$$Y^\#_t = \int_t^T \left[ \left( L^m y^\#_s + L^m \tilde{p}^N_s + \tilde{B}^\epsilon_s - \phi_s(Z^\#_s) \right)s^\#_s - s^\#_s \tilde{B}^\epsilon_s \right] ds$$

$$+ \int_t^T \left[ -h^H L^H L^m (y^\#_s + \tilde{p}^N_s) \right] ds - \int_t^T \left( Z^\#_s \right) \top dW_s,$$

and $(Y^\#, Z^\#) = (Y^F, Z^F)$. On the other hand, if $\xi \leq 0 \text{ a.s.}$, then there exists a solution $(Y^\#, Z^\#) \in S^2_T \times H^2_T$ that satisfies

$$Y^\#_t = \int_t^T \left[ \left( L^m y^\#_s + L^m \tilde{p}^N_s + \tilde{B}^\epsilon_s - \phi_s(Z^\#_s) \right)s^\#_s - s^\#_s \tilde{B}^\epsilon_s \right] ds$$

$$+ \int_t^T \left[ -h^C L^C L^m (y^\#_s + \tilde{p}^N_s) \right] ds - \int_t^T \left( Z^\#_s \right) \top dW_s,$$

and $(Y^\#, Z^\#) = (Y^F, Z^F)$. In particular, for both cases, $\text{FCA} = 0$.

4 Examples and a Closed-form Solution

Many standard derivatives satisfy the conditions in Theorem 3.4 and Theorem 3.5. We will apply the main theorems to several derivatives and provide a closed-form solution for a call option. In what follows, for $i \in I$, we denote

$$\tilde{S}^i := B^{-1} S^i.$$

Moreover, recall that in the main theorems, we defined $\xi$ by

$$\xi := B^{-1}_{T} \Delta D_T^N.$$

4.1 Clean close-out

Banks buy Treasury bonds that return less than their funding rate. It was insisted in Hull & White (2012) that this shows that FVA should not be considered in derivative prices. We will show that when buying bonds, $\text{FCA} = 0$ for the hedger. Therefore, if we assume

$$R^\epsilon = r,$$

as in Burgard & Kjaer (2010), the fair price for the hedger is approximately the same as the bond price derived from discounting with the Treasury rate. Recall that, in the main theorem, we only assume that $s^\ell$ and $s^b$ are deterministic. As long as the spreads are deterministic, we can apply the theorems to interest rate derivatives.
**Example 4.1** (Buying a Treasury bond). Let us consider a hedger buying a Treasury bond with unit notional amount, i.e., $D_N = -\mathbb{1}_{[T,\infty]}$. We assume that for $i \in \{H, C\}$, $dG^i_t = -h^i_t G^i_t dt$, where $(h^i_t)_{t \geq 0}$ are deterministic processes. Moreover, we assume that OIS rate $r$ is given by

$$dr_t = \kappa (\theta - r_t) dt + \Sigma dW_t,$$

for some $\kappa, \theta, \Sigma > 0$. Thus,

$$\sigma^1 = \sigma^H = \sigma^C,$$

$$\sigma^1_t = -\frac{\Sigma [1 - e^{-\kappa (T - t)}]}{\kappa}.$$

Moreover, we assume $\rho = 0$. Then $\sum_{i \in I} \tilde{p}^i \sigma^i = \sigma^1 (\sum_{i \in I} \tilde{p}^i)$. Therefore,

$$\phi(z) = \alpha(z + Z) = (\sigma^1)^{-1} (z + Z).$$

Since

$$r_t = r_0 e^{-\kappa t} + \theta (1 - e^{-\kappa t}) + \Sigma \int_0^t e^{-\kappa (t - u)} dW_u,$$

by Corollary 3.19 in Di Nunno et al. (2009), for any $\theta \leq t$, $D_0 r_t = \Sigma e^{-\kappa (t - \theta)}$, and it follows that

$$D_0 B_t^{-1} = -B_t^{-1} \int_0^t D_0 r_s ds$$

$$= -B_t^{-1} \Sigma \int_0^t e^{-\kappa (s - \theta)} ds$$

$$= -B_t^{-1} \frac{\Sigma [1 - e^{-\kappa (t - \theta)}]}{\kappa}.$$

Recalling that $\xi = -B_T^{-1}$,

$$\xi - \alpha_0 D_0 \xi = \xi - (\sigma^1)^{-1} D_0 \xi$$

$$= -B_T^{-1} + (\sigma^1_0)^{-1} B_T^{-1} \frac{\Sigma [1 - e^{-\kappa (T - \theta)}]}{\kappa} = 0.$$

It follows that for any $\theta \leq t$, $\tilde{p}^N_t - \alpha_0 D_0 \tilde{p}^N_t = 0$. Moreover,

$$\alpha_0^T Z_t^N = (\sigma^1)^{-1} D_t \tilde{p}^N_t = -S_t^1 < 0.$$

Therefore, by (ii) in Theorem 3.4, FCA = 0 where $\epsilon \geq 0$. Therefore, if the initial value of legacy portfolio is non-negative, the trader does not enter a borrowing position and there is no FVA that should be recouped.

The next example is a general form of Example 3.1.
Example 4.2 (Combination of forward contracts). Let $n = 1$, $\rho = \{H, C\}$, and the traded assets are given by
\[
dS_t^1 = rS_t^1 dt + \sigma^1_1 S_t^1 dW_t,
\]
\[
dS_t^i = rS_t^i dt + S_t^i dM_t^i, \quad i \in \{H, C\}.
\]
Since $\rho = \{H, C\}$, $\sum_{i \in I} \pi^i \sigma^i = \sigma^1 \pi^1$ and $\sum_{i \in I \setminus \rho} \pi^i = \pi^1$. Thus,
\[
\phi_t(z) = a_t(z + Z_t^N),
\]
\[
a_t = (\sigma^1)^{-1}.
\]
We consider a combination of forward contracts: $D_t^N = 1_{T \leq t} \sum_{i=1}^N \omega_i (S_{1_T}^i - K_i)$, where $\omega_i, K_i \in \mathbb{R}$. We assume that parameters are deterministic and
\[
s^b \geq 0, \quad h^H \geq 0, \quad h^C \geq 0,
\]
\[
\sum_{i=1}^N \omega_i \leq 0, \quad \sum_{i=1}^N \omega_i K_i \leq 0.
\]
By the definition of clean price,
\[
\tilde{p}_t^N = \sum_{i=1}^N \mathbb{E}[\omega_i (\tilde{S}_t - B_{1_T}^{-1} K_i) | \mathcal{F}_t]
\]
\[
= \sum_{i=1}^N \omega_i (\tilde{S}_t - B_{1_T}^{-1} K_i).
\]
Since $d\tilde{S}_t^1 = \sigma^1_1 \tilde{S}_t dW_t$, by Corollary 3.19 in Di Nunno et al. (2009), for any $\theta \leq t$,
\[
D_{\theta} \tilde{S}_t^1 = \int_\theta^t \sigma^1_0 D_{\theta} \tilde{S}_s^1 dW_s + \sigma^1_0 \tilde{S}_\theta^1.
\] (4.1)
Since $\sigma^1_0 \tilde{S}_\theta^1$ satisfies (4.1), by the uniqueness,
\[
D_{\theta} \tilde{S}_t^1 = \sigma^1_0 \tilde{S}_\theta^1.
\]
Moreover, $\omega_i Z_t^N = (\sigma^1)^{-1} Z_t^N = \tilde{S}_t^1 \sum_i \omega_i \leq 0$. By straightforward calculation, for any $\theta \leq t$,
\[
Z_t^N - \alpha_0 D_{\theta} Z_t^N = \tilde{S}_t^1 \sum_i \omega_i - \alpha_0 D_{\theta} \tilde{S}_t^1 \sum_i \omega_i
\]
\[
= \tilde{S}_t^1 \sum_i \omega_i - \tilde{S}_t^1 \sum_i \omega_i = 0,
\] (4.2)
\[
\tilde{p}_t^N - \alpha_0 D_{\theta} \tilde{p}_t^N = \sum_{i=1}^N \omega_i (\tilde{S}_t - B_{1_T}^{-1} K_i) - \alpha_0 D_{\theta} \left[ \sum_{i=1}^N \omega_i (\tilde{S}_t - B_{1_T}^{-1} K_i) \right]
\]
\[
= -B_{1_T}^{-1} \sum_{i=1}^N \omega_i K_i \leq 0.
\] (4.3)
Assume that as defined in Theorem 3.4,
\[ \xi = \sum_{i=1}^{N} \omega_i (S_i^2 - B_T^2 K_i), \]
\[ \xi^{\ell} = s^{\ell} (1 - L^m) \alpha Z^N + h \bar{B}_T^{\ell} + (h - h^L H^L L^m) (\bar{p}^N)^{\ell} - (h - h^C L^C L^m) (\bar{p}^N)^{\ell}. \]

Note that \( P_{\beta N=0} = 0, dQ \otimes dt \) a.s. Then by (4.2) and (4.3), we attain that
\[ L^m (\xi - \alpha_0 D_0 \xi) + \bar{B}_T^{\ell} = B_T^{-1} (B_T^{\ell} - L^m \sum_{i=1}^{N} \omega_i K_i) \geq 0, \]
\[ \xi^{\ell}_t - \alpha D_0 \xi^{\ell}_t = B_T^{-1} \left( hB_T^{\ell} - \sum_{i} \omega_i K_i \left[ s^{\ell} (1 - L^m) + (h - h^H H^L L^m) 1_{\rho N > 0} + (h - h^C L^C L^m) 1_{\rho N < 0} \right] \right) \geq 0. \]

Assume that
\[ \epsilon \geq (B_T^{\ell})^{-1} q \sum_{i} \omega_i K_i, \]
\[ q := \min \left\{ L^m, \frac{s^{\ell} (1 - L^m) + (h - h^H H^L L^m)}{h} \right\}. \]  

Then, by (ii) in Theorem 3.4, \((Y^F, Z^F)\) follows
\[ Y^F_t = \int_t^T - \left[ (Y^F_s + L^m \bar{p}_s^N + \bar{B}_s^{\ell} - \phi_s (Z^F_s)) s^{\ell}_s - s^{\ell}_s \bar{B}_s^{\ell} + h_s Y^F_s \right] ds \]
\[ + \int_t^T - \left[ 1_{\rho N^s > h_s H^L L^m} + 1_{\rho N^s < h_s C^L L^m} \right] \bar{p}_s^N ds - \int_t^T (Z^F_s)^\top dW_s, \]  
\[ (4.5) \]
and FCA = 0.

To find an analytic form of (4.5), let \( V^F := Y^F + \bar{p}^N, \Pi^F = Z^F + Z^N \), and let \( Q^{\ell} \) denote an equivalent measure such that \((B^{\ell})^{-1} S^\ell\) is \((Q^{\ell}, G)\)-local martingales. In particular,
\[ W^\ell_t := W_t - \int_0^t \alpha_s s^{\ell}_s ds, \]
is an \((F, Q^\ell)\)-Brownian motion. Then (4.5) becomes
\[ V^F_t = \xi + \int_t^T \left[ -(s^{\ell}_s + h_s) V^F_s + (s^{\ell}_s - s^{\ell}_s) \bar{B}_s^{\ell} + \beta_s \bar{p}_s^N \right] ds - \int_t^T \Pi^F dW^\ell_s, \]
\[ \beta_t := (1 - L^m) s^{\ell}_t + h_t - 1_{\rho N^t < h_t} h_t^L H^L L^m - 1_{\rho N^t < h_t} h_t^C L^C L^m. \]

Let \( A_t := \exp \left[ - \int_0^t (s^{\ell}_s + h_s) ds \right]. \) Then
\[ V^F_t = A_t^{-1} E^F \left[ A_T \xi + \int_t^T A_s \left[ (s^{\ell}_s - s^{\ell}_s) \bar{B}_s^{\ell} + \beta_s \bar{p}_s^N \right] ds \bigg| F_t \right], \]  
\[ (4.6) \]
where $\mathbb{E}^f$ is the expectation under $Q^f$. (4.6) reduces the computational cost because it changes from backward simulation to forward simulation. In other words, we can reduce the computational cost for calculating conditional expectations at each time step in solving BSDEs numerically. However, the advantage stops there with clean close-out convention and we cannot find a closed-form solution of $V^F$. This is because of the mismatch of the pricing measures in

$$\mathbb{E}^f[\beta_s \xi|\mathcal{F}_t] = \mathbb{E}^f[\mathbb{E}[\beta_s \xi|\mathcal{F}_s]|\mathcal{F}_t].$$

To avoid this difficulty, Brigo et al. (2017) considered un-collateralized contracts with null cash-flow at defaults. On the other hand, Bichuch et al. (2018) assumed that the close-out amount and collateral are calculated by the risk-neutral price under $Q^b$ (or $Q^b$), namely

$$e_N^t = (B^b_t - 1) \mathbb{E}^f[B^f_T - 1 \zeta|\mathcal{F}_t],$$

$$\zeta := \Delta \mathbb{D}_T^N,$$

$$m_t^N = (1 - L^N) e^N_t.$$ 

In these cases, the pricing measures are aligned and a closed-form solution is allowed. However, note that (4.7) is

"clean price + "the hedger’s FVA"."

As we will see later, when replacement close-out is assumed, the inconsistency of pricing measures does not appear and closed-form solutions are given. However, recall that the hedger’s funding information is already considered in $V^r$. Thus, the consistency of pricing measures is inherent in replacement close-out.

### 4.2 Replacement Close-out

In the next example, we deal with a non-Markovian case. This is one benefit of BSDEs and Malliavin calculus.

**Example 4.3** (Floating strike Asian call option with replacement close-out). We consider the same market condition as in Example 4.2 except

$$e^N = V^r - B^e.$$ 

Moreover, we consider a floating strike Asian call option:

$$\mathbb{D}^N = \mathbb{I}_{[T,\infty]} B_T \xi,$$

$$\xi := \left( S^1_T - B^-_T K^1_T \right)^+,$$

$$I_T := \exp \left( \frac{1}{T} \int_0^T \ln \left( S^1_u \right) du \right).$$
We will show (3.32). By Theorem 3.5 in Di Nunno et al. (2009), for $\theta \leq T$,
\[
\begin{align*}
\alpha_0 D_\theta \alpha & = \frac{\alpha_0}{T} \int_0^T D_\theta \ln(S_u^1) du \\
& = \frac{\alpha_0}{T} \int_0^T (S_u^1)^{-1} D_\theta (S_u^1) du \\
& = \frac{\alpha_0}{T} \int_0^T (S_u^1)^{-1} \sigma_u^1 S_u^1 du \\
& = T - \theta - \frac{1}{T} \int_0^T S_u^1 du.
\end{align*}
\]

Since $\alpha_0 D_\theta \tilde{S}_T^1 = \tilde{S}_T^1$, we attain that
\[
\begin{align*}
L^m \xi - \alpha_0 D_\theta \xi &= \mathbb{1}_{\tilde{S}_T^1 \geq B^{-1} K} \left[ L^m \tilde{S}_T^1 - B^T_1 K \right] - \alpha_0 D_\theta (\tilde{S}_T^1 - B^T_1 K) \\
& \leq \mathbb{1}_{\tilde{S}_T^1 \geq K_1} \left[ (L^m - 1) \tilde{S}_T^1 - B^T_1 K \right] \\
& = (L^m - 1)(\tilde{S}_T^1 - B^T_1 K)^+ \leq 0.
\end{align*}
\]

Therefore, FBA = 0, where $\epsilon \leq 0$.

As the last example, we deal with a bond option.

**Example 4.4** (Bond option with replacement close-out). We assume that the same market conditions as in Example 4.1. Let $\mathcal{D}_T = \mathbb{1}_{T \leq (S_{1,T}^1 - K)^+}$, where $K > 0$ and $S_{1,U}^1$ is a zero coupon bond with $U > T$ as its maturity. We consider two defaultable bonds with the same maturity, i.e.,
\[
\begin{align*}
S_{1,U}^1 & := B_t^1 \mathbb{E} \left[ B_{U}^{-1} \mathbb{1}_{T} \mid \mathcal{F}_t \right], \\
S_{i,U}^1 & := B_t^1 \mathbb{E} \left[ \mathbb{1}_{U < T} B_{U}^{-1} \mathbb{1}_{T} \mid \mathcal{G}_t \right], \quad i \in \{H, C\}.
\end{align*}
\]

Recall $\phi(z) = \alpha(z + Z^N) = (\sigma^1)^{-1}(z + Z^N)$, and
\[
\sigma^1 = \sigma^H = \sigma^C, \\
\sigma_t^1 = -\Sigma [1 - e^{-\kappa(U-t)}] \frac{1}{\kappa}.
\]

Moreover, recall the definition of $\xi$:
\[
\xi = \mathbb{1}_{S_{1,U}^1 \geq K} (S_{1,U}^1 - KB^{-1}_T).
\]

We can see that
\[
\begin{align*}
\alpha_0 D_\theta (KB^{-1}_T) &= (\sigma^1_0)^{-1} K D_\theta B^{-1}_T \\
& = -KB^{-1}_T (\sigma^1_0)^{-1} \Sigma [1 - e^{-\kappa(T - \theta)}] \frac{1}{\kappa} \\
& = KB^{-1}_T \frac{1 - e^{-\kappa(T - \theta)}}{1 - e^{-\kappa(U - \theta)}}.
\end{align*}
\]

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In addition,
\[ \alpha_0 D_0 S_{T,U}^1 = (\sigma^*_0)^{-1} \sigma^*_0 S_{T,U}^1 = \tilde{S}_{T,U}^1. \]

It follows that on \( \{ S_{T,U}^1 \geq K \} \),
\[
L^m \xi - \alpha_0 D_0 \xi = L^m (\tilde{S}_{T,U}^1 - KB_T^{-1}) - \alpha_0 D_0 (\tilde{S}_{T,U}^1 - KB_T^{-1}) \\
= (L^m - 1) \tilde{S}_{T,U}^1 - KB_T^{-1} \left( L^m \frac{1 - e^{-\kappa(T-\theta)}}{1 - e^{-\kappa(U-\theta)}} \right) \\
\leq (L^m - 1) \tilde{S}_{T,U}^1 - KB_T^{-1} (L^m - 1) \\
= (L^m - 1) (\tilde{S}_{T,U}^1 - KB_T^{-1}) \leq 0.
\]

(4.11)

Therefore, by (i) in Theorem 3.5, \((Y^F, Z^F)\) is given by
\[
Y^F_t = \int_t^T \left( L^m Y^F_s + L^m \tilde{P}^N_s - \phi_s(Z^F_s) \right) s^b + h^H_s L^H_s L^m (Y^F_s + \tilde{P}^N_s) ds - \int_t^T Z^F_s dW_s,
\]
and FBA= 0. Note that DVA \neq 0 because \( \xi \geq 0 \), but FBA = 0, where \( \epsilon \leq 0 \) because \( \xi \) increases in \( S_{T,U}^1 \).

Table 1 shows the effects of FBA and DVA with respect to option contracts. Only when the hedger sells a put option, FBA and DVA are both positive, but still FBA \neq DVA.

|          | FBA   | DVA  |
|----------|-------|------|
| buy/call | positive | nil   |
| sell/call | nil    | positive |
| buy/put  | nil    | nil   |
| sell/put | positive | positive |

Table 1: DVA and FBA with respect to option contracts

4.3 A Closed-form Solution of a Call Option under Replacement Close-out

Under replacement close-out, we can find a closed-form solution. As an example, we discuss the solution of a stock call option. Let \( n = 1, e^N = V_c - B^e, \rho = \{ H, C \}, \epsilon \leq 0, D^N = 1_{[T, \infty)}(S_T^1 - K)^+, \) where
\[
\begin{align*}
\mathrm{d}S_1^1 &= r S_1^1 \mathrm{d}t + \sigma^1 S_1^1 \mathrm{d}W_t, \\
\mathrm{d}S_i^1 &= r S_i^1 \mathrm{d}t + S_i^1 \mathrm{d}M_i^i, \quad i \in \{ H, C \},
\end{align*}
\]
for some constants \( r \) and \( \sigma^1 \). We also assume that \( s^b, h^H \) are constant. Recall
\[
\begin{align*}
\xi &= B_T^{-1} (S_T^1 - K)^+, \\
\phi_i(z) &= \alpha (z + Z_i^N) \\
&= (\sigma^1)^{-1} (z + Z_i^N).
\end{align*}
\]
It is easy to check that
\[ L^m \xi - a_0 D_0 \xi = L^m \xi - (s^1)^{-1} D_0 \xi = E_{S^2} \left( (L^m - 1) \bar{S}^1_t - KL^m B_T^{-1} \right) \leq 0. \]

Therefore, by (i) in Theorem 3.5, \( \text{FBA} = 0 \) and \( (Y^F, Z^F) \) follows
\[
Y^F_t = \int_t^T - \left( L^m t^F_s + L^m \bar{\rho}_s^N - \phi_s (Z^F_s) \right) s^b \, ds \\
+ \int_t^T \left[ - h_s^H L^H L^m (Y^F_s + \bar{\rho}_s^N) \right] ds - \int_t^T Z^F_s \, dW_s. \tag{4.13}
\]

Let \( V^F := Y^F + \bar{\rho}^N, \Pi^F := Z^F + Z^N \). Moreover, let \( Q^b \) denote an equivalent measure that
\[
W^b_t := W_t - \int_0^t \int_0^T \, ds \, \alpha_s \, ds \tag{4.14}
\]
is an \( (F, Q^b) \)-Brownian motion and \( E^b \) denote the expectation under \( Q^b \). Then (4.13) becomes
\[
V^F_t = \xi - \int_t^T (s^b + h^H L^H) L^m V^F_s - \int_t^T \Pi^F_s \, dW^b_s.
\]

Thus, it follows that
\[
V^F_t = A^{-1}_t E^b [A_T \xi | \mathcal{F}_t]. \tag{4.15}
\]
\[
A_t := e^{-Bt}, \tag{4.16}
\]
\[
\beta := (s^b + h^H L^H) L^m. \tag{4.17}
\]

Note that there is no inconsistency of pricing measures. This is because we considered the hedger’s funding cost and benefit in the close-out amount, \( e^N = V^- \). To represent (4.15) in an explicit form, write
\[
\xi = (B_t^b)^{-1} B_T^b B_T^{-1} B_T \xi = e^{B^T (B_t^b)^{-1} B_T^{-1} B_T} \xi.
\]

Then (4.15) becomes, for \( t < \bar{t} \),
\[
V^F_t = A^{-1}_t A_T e^{B^T (B_t^b)^{-1} B_T^b E^b [B_T^{-1} B_T^{-1} B_T ] | \mathcal{F}_t}. \tag{4.18}
\]

More explicitly,
\[
V_t = B_t V^F_t = \exp \left( \left( s^b (1 - L^m) - h^H L^H L^m \right) (T - t) \right) C^b(t, S_t^1), \tag{4.19}
\]
\[
C^b(t, S_t^1) := S_t^1 \Phi(d(t, S_t^1)) - K e^{-R^b(T - t)} \Phi(d(t, S_t^1) - \sigma^1 \sqrt{T - t}), \tag{4.20}
\]
\[
\Phi(x) := \int_{-\infty}^x e^{-y^2/2} / \sqrt{2\pi} \, dy, \tag{4.21}
\]
\[
d(t, x) := \frac{\ln(S_t^1/K) + (R^b + (\sigma^1)^2/2)(T - t)}{\sigma^1 \sqrt{T - t}}. \tag{4.21}
\]
Note that $\partial_s \mathcal{V}_t^F = 0$. Moreover, in (4.18),

$$
(1 - L^m)s^b + L^m(-h^H L^H)
$$

is a weighted sum of $s^b$ and $-h^H L^H$. It shows how the effect of DVA is transferred to FCA as $L^m$ changes. As $L^m$ increases, the effect of funding cost is weaken, since the cost of posting the collateral is more expensive than the interest rate remunerating on the collateral, namely OIS rate. In more detail,

$$
\exp\left(-h^H L^H L^m(T-t)\right)
$$

is a deduction from DVA, while

$$
\exp\left(s^b(1 - L^m)(T-t)\right)
$$

is a compensation for the hedger for posting the collateral. However, this is not the only FCA. The other part of FCA is the cost to acquire $S^1$ and it is included in $C^b$.

5 Conclusion

In summary, we discussed a binary nature of FVA. By the binary nature of FVA, we can recover linear BSDEs and analytic solutions are allowed. In cases of replacement close-out, the analytic solution can be represented as a closed-form. As a byproduct, this feature of FVA explains how FBA and DVA are different. In addition, this result provides an interpretation why banks buy Treasury bonds with presence of funding rates higher than OIS rate.

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A Proofs of the Main Theorems

*Proof of Theorem 3.4.* We only prove (i). (ii) can be proved similarly, and (iii) is an easy consequence of (i) and (ii).

(i) By (3.20) and (3.21),

\[
\mathbb{E} \left[ \int_0^T |\tilde{p}_t^N|^2 \, dt \right] = \left[ \int_0^T \mathbb{E} |\xi| F_t^2 \, dt \right] < \infty, \\
\mathbb{E} \left[ \int_0^T |Z_t^N|^2 \, dt \right] = \left[ \int_0^T \mathbb{E} |D_t \xi| F_t^2 \, dt \right] < \infty.
\]

It is easy to see that there exists a unique solution \((Y^\#, Z^\#) \in S_T^2 \times H_T^{2,n}\) of the following BSDE:

\[
Y_t^\# = \int_t^T g_s^\# (Y_s^\#, Z_s^\#) \, ds - \int_t^T (Z_s^\#)^T \, dW_s,
\]

where

\[
g_t^\# (y, z) := -(y + L^m \tilde{p}_t^N + \tilde{B} \phi(z)) s_t^\# + s_t^\# \tilde{B} - h_t y - h_t^H L^H L^m (\tilde{p}_t^N)^\# + h_t^C L^C L^m (\tilde{p}_t^N)^\# - h_t^C \tilde{B} \phi(z).
\]

We will show that

\[
(Y^\#, Z^\#) = (Y^F, Z^F), \quad (A.1)
\]

where \((Y^F, Z^F)\) is the solution of (2.63). Because \((Y^F, Z^F)\) satisfying (2.61) is unique in \(S_T^2 \times H_T^{2,n}\), to prove (A.1), it suffices to show that

\[
Y_t^\# + L^m \tilde{p}_t^N + \tilde{B} \phi(z) \leq 0, \quad d\mathbb{Q} \otimes dt - \text{a.s.} \quad (A.2)
\]

To this end, we introduce another transformation:

\[
V^F := Y^\# + L^m \tilde{p}_t^N + \tilde{B} \phi, \\
\Pi^F := Z^\# + L^m Z^N.
\]
Then, by (iii) in Lemma 2.17, \((V^F, \Pi^F)\) satisfies

\[
V^F = L^m \xi + \bar{B}_T^e + \int_t^T F_s(V^F_s, \Pi^F_s) \, ds - \int_t^T (\Pi^F_s)^T \, dW_s,
\]

where

\[
F_t(y, z) := g_t(y - L^m \bar{P}^N_t - \bar{B}_t^e, z - L^m N_t^N) \\
= - (y - \alpha_t^T z) s_t^B - h_t y \\
+ (1 - L^m s_t^B) \alpha_t^T N_t^N + h_t \bar{B}_t^e + (h_t - h_t^H H L^m)(\bar{P}_t^N)^+ - (h_t - h_t^C L^C L^m)(\bar{P}_t^N)^-
\]

Note that (A.2) is equivalent to

\[
V^F - (1 - L^m)s_t^B \alpha^T Z^N - \phi(\Pi^F) \leq 0, \quad dQ \otimes dt \text{ a.s.} \quad (A.3)
\]

To show (A.3), we use Malliavin calculus and comparison principle of BSDEs. By (3.21) and (3.22),

\[
E \left[ \int_0^T \int_0^T \left| D_\theta \bar{P}^N_t \right|^2 \, dt \, d\theta \right] = \left[ \int_0^T \int_0^T E \left[ \left| D_\theta \xi | \mathcal{F}_t \right|^2 \right] \, dt \, d\theta \right] \leq \int_0^T \int_0^T E \left[ |D_\theta \xi|^2 \right] \, dt \, d\theta < \infty,
\]

\[
E \left[ \int_0^T \int_0^T \left| D_\theta Z_t^N \right|^2 \, dt \, d\theta \right] = \left[ \int_0^T \int_0^T E \left[ \left| D_\theta (D_t \Pi^F_t) \right|^2 \right] \, dt \, d\theta \right] \leq \int_0^T \int_0^T E \left[ |D_\theta \xi|^2 \right] \, dt \, d\theta < \infty.
\]

Therefore, by Proposition 5.3 in El Karoui et al. (1997), \((V^F, \Pi^F) \in L^2([0, T] : D^{1.2} \times (D^{1.2})^n)\), and for any \(1 \leq i \leq n\), a version of \((D_{\theta}^i V^F_t, D_{\theta}^i \Pi^F_t) \mid 0 \leq \theta, t \leq T\) is given by

\[
D_\theta^i V^F_t = L^m D_{\theta}^i \xi + \int_t^T \left[ -(D_{\theta}^i V^F_s - \alpha_s^T D_{\theta}^i \Pi^F_s) s_s^B - h_s D_{\theta}^i V^F_s + D_{\theta}^i \xi_s^B \right] \, ds - \int_t^T (D_{\theta}^i \Pi^F_s)^T \, dW_s, \quad (A.4)
\]

and \((D_t V^F_t : 0 \leq t \leq T)\) is a version of \((\Pi^F_t : 0 \leq t \leq T)\). Let us denote

\[
V^F_{t, \theta} := D_{\theta}^i (D_\theta V^F_t), \\
\Pi^F_{t, \theta} := D_{\theta} (\Pi^F_t) \alpha_\theta.
\]

Then \((V^F_{t, \theta}, \Pi^F_{t, \theta})\) is given by

\[
V^F_{t, \theta} = L^m a_{\theta} D_\theta \xi + \int_t^T \left[ F_s(V^F_s, \Pi^F_s) + a_{\theta}^T D_\theta \xi_s^B - \xi_s^B \right] \, ds - \int_t^T (\Pi^F_{s, \theta})^T \, dW_s.
\]

Therefore, by (3.23) and (3.25) together with comparison principle of BSDEs, we attain that \(V^F_{t, \theta} \geq V^F_t\) for any \(\theta \leq t\). Moreover, by (3.24), for any \(\theta \leq t\),

\[
V^F_t - (1 - L^m) s_t^B Z^N_t - V^F_{t, \theta} \leq V^F_t - V^F_{t, \theta} \leq 0,
\]

and this implies (A.2) and (A.3). Moreover, by (2.70), \text{FBA} = 0.
Proof of Theorem 3.5. As in the proof of Theorem 3.4, we can check the existence, uniqueness, and Malliavin differentiability of BSDEs. We only explain the transformation and how to apply comparison principle. Without loss of generality, we assume $\xi \geq 0$. It follows that $\bar{p}^N \geq 0$, $dQ \otimes dt$ - a.s.

(i) We consider a solution $(Y^\#, Z^\#) \in S^2_T \times \mathbb{H}^2_T$ of the following BSDE:

$$
Y^\#_t = \int_t^T \left[ \left( L^m Y^\#_s + L^m \bar{p}^N_s + B^\varepsilon_s - \alpha^\varepsilon_s (Z^\#_s + Z^N_s) \right) s_b - s^b \bar{B}^\varepsilon + h^H_s L^m (Y^\#_s + \bar{p}^N_s) \right] ds
- \int_t^T (Z^\#_s)^\top dW_s.
$$

Take $V^F := Y^\# + \bar{p}^N$, $\Pi^F := Z^\# + Z^N$. Then,

$$
V^F_t = \xi + \int_t^T F_s(V^F_s, \Pi^F_s) ds - \int_t^T (\Pi^F_s)^\top dW_s,
$$

where

$$
F_s(y, z) := -(L^m y - \alpha^\varepsilon_z) s_b - h^H_s L^m y
$$

Since $(0, 0)$ is the unique solution of the following BSDE:

$$
y_t = \int_t^T \left[ \left( L^m y_s - \alpha^\varepsilon_z s_b \right) s_b + h^H_s L^m y_s \right] ds - \int_t^T z_s^\top dW_s,
$$

by comparison between (A.5) and (A.7), we can attain that $V^F \geq 0$, namely

$$
Y^\# + \bar{p}^N \geq 0.
$$

Moreover, $(V^F, \Pi^F) \in L^2([0, T]; \mathbb{D}^{1,2} \times (\mathbb{D}^{1,2})^n)$, and for any $1 \leq i \leq n$, a version of $\{(D_{t,\theta}^i V^F_t, D_{t,\theta}^i \Pi^F_t) \mid 0 \leq \theta, t \leq T\}$ is given by

$$
D_{t,\theta}^i V^F_t = D_{t,\theta}^i \xi + \int_t^T \left[ \left( L^m D_{t,\theta}^i V^F_s - \alpha^\varepsilon_z D_{t,\theta}^i \Pi^F_s \right) s_b + h^H_s L^m D_{t,\theta}^i V^F_s \right] ds - \int_t^T (D_{t,\theta}^i \Pi^F_s)^\top dW_s,
$$

and $\{D_t V^F_t \mid 0 \leq t \leq T\}$ is a version of $\{\Pi^F_t \mid 0 \leq t \leq T\}$. Let us denote

$$
V^m_t := L^m V^F_t,
$$

$$
\Pi^m_t := L^m \Pi^F_t,
$$

$$
V^F_{t,\theta} := \alpha_{\theta}^\varepsilon (D_{\theta} V^F_t),
$$

$$
\Pi^F_{t,\theta} := (D_{\theta} \Pi^F_t) \alpha_{\theta}.
$$

Then $(V^m, \Pi^m)$ and $(V^F_{t,\theta}, \Pi^F_{t,\theta})$ are given by

$$
V^F_{t,\theta} = \alpha_{\theta}^\varepsilon D_{\theta} \xi + \int_t^T F_s(V^F_{s,\theta}, \Pi^F_{s,\theta}) ds - \int_t^T (\Pi^F_{s,\theta})^\top dW_s,
$$

$$
V^m_t = L^m \xi + \int_t^T F_s(V^m_s, \Pi^m_s) ds - \int_t^T (\Pi^m_s)^\top dW_s.
$$
Therefore, by (3.31), \( V_{t_0}^m \leq V_{t_0}^F \), for any \( \theta \leq t \). It follows that
\[
L^m Y_t^\theta + L^m \bar{p}_t + \tilde{B}_t^\theta - \alpha^T (Y_t^\theta + Z_t^N) \leq L^m V_t^F - \alpha^T \Pi_t^F
\]
\[
= L^m V_t^F - V_{t_0}^F \leq 0.
\]
Hence, by uniqueness of \((Y_F^F, Z_F^F)\), we obtain \((Y_F^F, Z_F^F) = (Y^\theta, Z^\theta)\). Moreover, by (2.70), FBA = 0. The proof of (ii) is similar to (i).

B Stochastic Intensities

We assume that \( h^i, i \in \{ H, C \} \), are \( \mathcal{F} \)-adapted processes, but \( \sigma^H \) and \( \sigma^C \) are deterministic. For simplicity, we set \( e = 0 \) and \( e^E = 0 \). Let \( n = 2 \), and we consider replacement close-out. \( \xi \) is determined by an asset \( S^1 \), but the market is completed by another non-defaultable traded asset \( S^2 \). We assume
\[
\Sigma := \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}
\]
is of full rank. In this case, we can not expect that the transformation
\[
\phi_t : \sum_{i \in \mathcal{I}} (\sigma_i^T) \pi_t^i \to \sum_{i \in \mathcal{I} \setminus \rho} \tilde{\pi}_t^i
\]
is independent of \( \pi^H \) and \( \pi^C \). By (2.67) and (2.68), \( \pi^i, i \in \{ H, C \} \), are represented by \( Y^F \). Thus, we write \( \phi \) as
\[
\phi_t(y, z).
\]
We assume that \( \rho = \{ H, C \} \). Then
\[
\phi_t(y, z) = \alpha_t^T \left( z + Z_t^N - (\sigma_t^H) \tilde{\pi}_t^H - (\sigma_t^C) \tilde{\pi}_t^C \right)
\]
\[
= \mathbf{1}^T (\Sigma_t^T)^{-1} \left[ z + Z_t^N - (\alpha_t^H)^T L^H L^m (y + \bar{p}_t^N) \right].
\]
Let us consider
\[
Y_t^\theta = \int_t^T \left[ \left( L^m Y_s^\theta + L^m \bar{p}_s^N + \alpha_s^H L^H L^m (Y_s^\theta + \bar{p}_s^N) - \alpha_s^T (Z_s^\theta + Z_s^N) \right) s^f - h_s^H L^H L^m (Y_s^\theta + \bar{p}_s^N) \right] ds
\]
\[
+ \int_t^T Z_s^\theta dW_s.
\]
We will show that \((Y^\theta, Z^\theta) = (Y_F^F, Z_F^F)\). To this end, let \( V_F^F := Y^\theta + \bar{p}^N \), \( \Pi_F^F := Z^\theta + Z^N \), and \( \beta := L^m (1 + \sigma^H a L^H) \). Then \((V_F^F, \Pi_F^F)\) is given by
\[
V_t^F = \xi + \int_t^T F_s(V_s^F, \Pi_s^F) ds - \int_t^T \Pi_s^F dW_s,
\]
\[
F_t(y, z) := - (\beta_t y - \alpha_t^T z) s^f - h_t^H L^H L^m y.
\]
It suffices to show
\[
\beta V^F - \alpha^\top \Pi^F \geq 0, \quad d\mathbb{Q} \otimes dt - \text{a.s.} \tag{B.4}
\]

We denote that for \( \theta \leq t, \)
\[
V_{t, \theta}^\beta := \beta_\theta V_t^F, \quad \Pi_{t, \theta}^\beta := \beta_\theta \Pi_t^F,
\]
\[
V_{t, \theta}^D := \alpha_\theta^\top D_\theta V_t^F, \quad \Pi_{t, \theta}^D := \alpha_\theta^\top D_\theta \Pi_t^F.
\]

Then \((V_{t, \theta}^\beta, \Pi_{t, \theta}^\beta)\) and \((V_{t, \theta}^D, \Pi_{t, \theta}^D)\) are given by
\[
V_{t, \theta}^\beta = \beta_\theta \xi + \int_t^T F_s(V_{s, \theta}^\beta, \Pi_{s, \theta}^\beta) \, ds - \int_t^T \Pi_{s, \theta}^\beta \, dW_s,
\]
\[
V_{t, \theta}^D = \alpha_\theta^\top D_\theta \xi + \int_t^T \left[ F_s(V_{s, \theta}^D, \Pi_{s, \theta}^D) - \alpha_\theta^\top (D_\theta h_s) L^H L^m V_s^F \right] \, ds - \int_t^T \Pi_{s, \theta}^D \, dW_s.
\]

Recall \( \xi \geq 0. \) Thus, by (B.2), \( V^F \geq 0. \) Therefore, to show (B.4), we need
\[
\beta_\theta \xi \geq \alpha_\theta^\top D_\theta \xi, \tag{B.5}
\]
\[
D_\theta h_t^H \geq 0. \tag{B.6}
\]

If (B.5) and (B.6) are satisfied, FCA = 0.