Gauge Invariance and the Unstable Particle

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Abstract. It is shown how to construct exactly gauge-invariant $S$-matrix elements for processes involving unstable gauge particles such as the $Z^0$ boson. The results are applied to derive a physically meaningful expression for the cross-section $\sigma(e^+e^-\rightarrow Z^0Z^0)$ and thereby provide a solution to the long-standing problem of the unstable particle.

I INTRODUCTION

A resonance is fundamentally a non-perturbative object and is thus not amenable to the methods of standard perturbation theory. In order to describe physics at the $Z^0$ resonance one is forced, therefore, to employ some sort of non-perturbative procedure. Such a procedure is Dyson summation that sums strings of one-particle irreducible (1PI) self-energy diagrams as a geometric series to all orders in the coupling constant, $\alpha$ and effectively replaces the tree-level $Z^0$ propagator by a dressed propagator,

$$
\frac{1}{s - M_Z^2} \rightarrow \frac{1}{s - M_Z^2} \sum_n \left( \frac{\Pi_{ZZ}(s)}{s - M_Z^2} \right)^n = \frac{1}{s - M_Z^2 - \Pi_{ZZ}^{(1)}(s)}
$$

(1)

where $\Pi_{ZZ}(s)$ is the one-loop $Z^0$ self-energy. The problem here is that electroweak physics is described by a gauge theory. Results of calculations of physical processes must be exactly gauge-invariant but this comes about through delicate cancellations between many different Feynman diagrams each of which is separately gauge-dependent. The cancellation happens at each order in $\alpha$ when all diagrams of a given order are combined. The $Z^0$ self-energy, $\Pi_{ZZ}(s)$, is gauge-dependent at $O(\alpha)$ and hence the rhs of eq. (1) is gauge-dependent at all orders in $\alpha$.

If the dressed propagator is used in a finite-order calculation the result will be gauge-dependent at some order because the will be no diagrams available to cancel
the gauge-dependence beyond the order being calculated. This gauge-dependence should be viewed as an indicator that the approximation scheme being used is inconsistent or does not represent a physical observable.

In constructing amplitudes that represent physical observables, one must take care to respect the requirements that are laid down by analytic $S$-matrix theory [1]. These conditions are derived from general considerations such as energy conservation and causality and it will be found that appealing to them leads to a procedure for generating gauge-invariant amplitudes with no flexibility in the final result. The procedure will also make it possible to give an answer to the problem of the unstable particle that was put forward by Peierls in the early fifties [2]. Much of what appears here can be found in ref.s [3–5].

II  GAUGE-IN Variant $S$-MATRIX ELEMENTS

We will begin by reviewing what is known about $S$-matrix elements for processes containing unstable particles. An unstable particle is associated with a pole, $s_p$, lying on the second Riemann sheet below the real $s$ axis. The scattering amplitude, $A(s)$ for a process which contains an intermediate unstable particle can then be written in the form,

$$A(s) = \frac{R}{s - s_p} + B(s)$$  (2)

where $R$ and $s_p$ are complex constants and $B(s)$ is regular at $s = s_p$. The first term on the rhs of eq.(2) will be called the resonant term and the second is the non-resonant background term. It is known that $s_p$ is process-independent in the sense that any process that contains a given unstable particle as an intermediate state will have its pole at the same position. It can also be shown from Fredholm theory that the residue factorizes as, $R = R_i \cdot R_f$, into pieces that depend separately on the initial- and final-state. One can prove, by very simple arguments, that $s_p$, $R$ and $B(s)$ are separately and exactly gauge-invariant [3].

From analytic $S$-matrix theory it is known that production thresholds are associated with a branch cut. Branch points for stable particles lie on the real $s$-axis and those for unstable particles, such as $W$ bosons, lie significantly below it. Provided there are no nearby thresholds the amplitude, $A(s)$, can be adequately described in the resonance region by a Laurent expansion about the pole, $s_p$. It should be emphasized that $A(s)$ can always be written in the form (2) even when thresholds are present. In that case $B(s)$ will be an analytic function containing a branch point and the rhs of (2) continues to be an exact representation of $A(s)$. Laurent expansion refers to how the resonant term is identified. The aim is only to separate $A(s)$ into gauge-invariant resonant and background pieces and there is no necessity to perform a Laurent expansion beyond its leading term although this can provide a useful way for parameterizing electroweak data [6].
Let us consider the production process for a massless fermion pair in $e^+e^- \rightarrow f\bar{f}$. Away from the $Z^0$ resonance it is known how to calculate the amplitude to arbitrary accuracy in a gauge-invariant manner. Near resonance we must perform a Dyson summation and then separate the amplitude into its resonant and background pieces. Doing so allows expressions for the gauge-invariant quantities, $s_p$, $R$ and $B(s)$ to be identified in terms of the 1PI functions that occur in perturbation theory.

The scattering amplitude for $e^+e^- \rightarrow f\bar{f}$ is

$$A(s, t) = \frac{R_{iZ}(s_p)R_{Zf}(s_p)}{s - s_p} + \frac{R_{iZ}(s)R_{Zf}(s) - R_{iZ}(s_p)R_{Zf}(s_p)}{s - s_p} + \frac{V_{i\gamma}(s)V_{\gamma f}(s)}{s - \Pi_{\gamma\gamma}(s)} + B(s, t)$$

in which

$$R_{iZ}(s) = \left[ V_{i\gamma}(s)\frac{\Pi_{Z\gamma}(s)}{s - \Pi_{\gamma\gamma}(s)} + V_{iZ}(s) \right] F_{ZZ}^{1/2}(s),$$

$$R_{Zf}(s) = F_{ZZ}^{1/2}(s) \left[ V_{Zf}(s) + \frac{\Pi_{Z\gamma}(s)}{s - \Pi_{\gamma\gamma}(s)}V_{\gamma f}(s) \right].$$

The pole position $s_p$ is a solution of the equation

$$s - M_Z^2 - \Pi_{ZZ}(s) - \frac{\Pi_{Z\gamma}^2(s)}{s - \Pi_{\gamma\gamma}(s)} = 0$$

and $F_{ZZ}(s)$ is defined through the relation

$$s - M_Z^2 - \Pi_{ZZ}(s) - \frac{\Pi_{Z\gamma}^2(s)}{s - \Pi_{\gamma\gamma}(s)} = \frac{1}{F_{ZZ}(s)(s - s_p)}.$$

It should be emphasized that eq.(3) is exact and valid anywhere on the complex $s$-plane. The effect of $Z^0-\gamma$ mixing has been included. The quantity $\Pi_{\gamma\gamma}(s)$ and $\Pi_{Z\gamma}(s)$ are the photon self-energy and the $Z-\gamma$ mixing respectively. $V_{iZ}(q^2), V_{Zf}(q^2)$ are the initial- and final-state $Z^0$ vertices, into which the external wavefunctions have been absorbed, and $V_{i\gamma}(q^2), V_{\gamma f}(q^2)$ are the corresponding vertices for the photon. Here $B(s, t)$ denotes 1PI corrections to the matrix element that include things like as box diagrams. The first term on the rhs of eq.(3) is the resonant part of $A(s, t)$ and the three terms on the second line taken together are form the non-resonant background.

Calculations are most conveniently performed in terms of the real renormalized parameters of the theory such as the renormalized mass, $M_Z$. Eq.(6) can be solved iteratively in terms of $M_Z$ to give

$$s_p = M_Z^2 + \Pi_{ZZ}(M_Z^2) + ...$$
The rhs of eq.(8) may be substituted for \( s_p \) where it appears in eq.(3)–(5). Taylor series expansion can then be used to obtain perturbative expressions for \( s_p, R \) and \( B(s) \) in terms of Greens functions with real arguments up to any desired order. At any given order these perturbative expressions will be exactly gauge-invariant as will scattering amplitudes constructed from them.

A couple of points should be noted here. The appearance of Greens functions with complex arguments in eq.(3) is a natural consequence of the analyticity of the \( S \)-matrix. The \( S \)-matrix itself is never evaluated with complex \( s \). The arguments of \( A(s, t) \) on the lhs of eq.(3) is real as is the ‘s’ in the denominator of the first term on the rhs.

In the procedure described above, one starts by extracting the resonant term in a scattering amplitude by Laurent expansion about the exact pole position \( s_p \) and then specializes to lower orders by further expanding about the renormalized mass. Other authors [7,8] have attempted to apply the techniques described above by first expanded about the renormalized mass and then added a finite width in the denominator of the resonant part by hand. That procedure cannot be justified and leads to problems when one treats processes like \( e^+e^- \rightarrow W^+W^- \). It gives rise to spurious threshold singularities or complex scattering angles due the production threshold’s being incorrectly located on the real axis. In section IV the process \( e^+e^- \rightarrow Z^0\bar{Z}^0 \) will be treated and no threshold singularities or complex scattering angles will arise.

### III THE PROBLEM OF THE UNSTABLE PARTICLE

We have thus succeeded in our goal of producing exactly gauge-invariant scattering amplitudes to arbitrary order. One might ask whether what has been done is just a mathematical trick, in which case the gauge-invariance is accidental, or does it have some physical interpretation. In this section it will be shown that the latter is true.

Recall that the coordinate space dressed propagator for a scalar particle has an integral representation

\[
\Delta(x' - x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x' - x)}}{k^2 - m^2 - \Pi(k^2) + i\epsilon} \tag{9}
\]

The integrand has a pole at \( k^2 = s_p \) where \( s_p \) is a solution of the equation \( s - m^2 + \Pi(s) = 0 \) and as in the previous section we define \( F(s) \) via the relation \( s - m^2 + \Pi(s) = (s - s_p)/F(s) \). The dressed propagator can then be written as

\[
\Delta(x' - x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x' - x)} \left[ \frac{F(s_p)}{k^2 - s_p} + \frac{F(k^2) - F(s_p)}{k^2 - s_p} \right] \tag{10}
\]

that separates resonant and non-resonant pieces. Performing the \( k_0 \) integration resonant gives
\[ \Delta(x' - x) = -i \int \frac{d^3k}{(2\pi)^3} e^{-ik \cdot (x' - x)} \theta(t' - t) F(s_p) \]

\[ + \int \frac{d^4k}{(2\pi)^4} \frac{F(k^2) - F(s_p)}{k^2 - s_p} \]

\[ -i \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x' - x)} \theta(t - t') F(s_p) \] (11)

where \( k_0 = \sqrt{k^2 + s_p} \). The non-resonant term contributes only for \( t = t' \) and so represents a contact interaction. The resonant part splits into two terms that contribute when \( t > t' \) or \( t < t' \) and therefore connects points \( x \) and \( x' \) that are separated by a finite distance in space-time.

The problem of the unstable particle \([2]\) is may be roughly stated as follows: \( S \)-matrix theory deals with asymptotic in-states and out-states that propagate from and to infinity. Unstable particles cannot exist as asymptotic states because they decay a finite distance from the interaction region. Indeed it is known \([9]\) that the \( S \)-matrix is unitary on the Hilbert space spanned by stable particle states and hence there is no room to accommodate unstable particles as external states. How can one use the \( S \)-matrix to calculate, say, the production cross-section for an unstable particle when it cannot exist as an asymptotic state?

In the first part of this section it was shown that the resonant part of the dressed propagator connected points with a finite space-time separation. When a similar analysis is applied to a physical matrix element, such as eq.(3), one concludes that the resonant part describes a process in which there is a finite space-time separation between the initial-state vertex, \( V_i \), and the final-state vertex, \( V_f \). In other words, the resonant term describes the finite propagation of a physical \( Z^0 \) boson. The non-resonant background represents prompt production of the final state. As these two possibilities are, in principle, physically distinguishable, they must be separately gauge-invariant.

We can thus use finite propagation as a tag for identifying unstable particles without requiring that they appear in the final state. This is, after all, the way \( b \)-quarks are identified in vertex detectors. A production cross-section for an unstable particle is obtained by extracting the resonant part of the matrix element for a process containing that particle in an intermediate state and summing over all possible decay modes.

**IV THE PROCESS** \( e^+e^- \to Z^0Z^0 \)

In this section we will calculate the cross-section for \( e^+e^- \to Z^0Z^0 \). This is of both theoretical and practical importance. On the theoretical side it represents an example of a calculation of the production cross-section for unstable particles. On the practical side, at high energies \( e^+e^- \to Z^0Z^0 \) will be a dominant source of fermion pairs \((f_1\bar{f}_1)\) and \((f_2\bar{f}_2)\) due to its double resonant enhancement and hence
\(\sigma(e^+e^- \to Z^0Z^0)\) is an excellent approximation to the cross-section for 4-fermion pair production. If the experimental situation warrants it, background terms can also be included without difficulty.

In the case of \(e^+e^- \to f\bar{f}\), dealt with in section II, the invariant mass squared of the \(Z^0, s\), is fixed by the momenta of the incoming \(e^+e^-\). For the process \(e^+e^- \to Z^0Z^0\) the invariant mass of the produced \(Z^0\)'s is not constant and must be somehow included in phase space integrations. It is not immediately clear how to do this and without further guidance from S-matrix theory there would seem to be considerable flexibility in how to proceed. A new ingredient is required and that is to realize that an expression for an S-matrix element can always be divided into a part that is a Lorentz-invariant function of the kinematic invariants of the problem and Lorentz-covariant objects, such as \(\not{p}\) etc. The latter are known as \textit{standard covariants} [10–13]. It is the Lorentz invariant part that satisfies the requirements of analytic S-matrix theory and from which the resonant and non-resonant background parts are extracted while the Lorentz covariant part is untouched.

To calculate \(\sigma(e^+e^- \to Z^0Z^0)\), we begin by constructing the cross-section for \(e^+e^- \to Z^0Z^0 \to (f_1\bar{f}_1)(f_2\bar{f}_2)\), and will eventually sum over all fermion species. The part of the full matrix element that can give rise to doubly resonant contributions can be written as

\[
\mathcal{M} = \sum_i [\bar{v}_e + T_{\mu\nu}^i u_{\nu -} ] M_i(t, u, p_1^2, p_2^2) \\
\times \frac{1}{p_1^2 - M_Z^2 - \Pi_{ZZ}(p_1^2)} [\bar{u}_{f_1} \gamma^\mu (V_{ZfL}(p_1^2)\gamma_L + V_{ZfR}(p_1^2)\gamma_R) u_{f_1}] \tag{12}
\]

\[
\times \frac{1}{p_2^2 - M_Z^2 - \Pi_{ZZ}(p_2^2)} [\bar{u}_{f_2} \gamma'^\nu (V_{ZfL}(p_2^2)\gamma_L + V_{ZfR}(p_2^2)\gamma_R) u_{f_2}]
\]

where \(T_{\mu\nu}^i\) are Lorentz covariant tensors that span the tensor structure of the matrix element and \(\gamma_L, \gamma_R\) are the usual helicity projection operators. The squared invariant masses of the \(f_1\bar{f}_1\) and \(f_2\bar{f}_2\) pairs are \(p_1^2\) and \(p_2^2\). The \(M_i, \Pi_{ZZ}\) and \(V_{Zf}\) are Lorentz scalars that are analytic functions of the independent kinematic Lorentz invariants of the problem.

To extract the piece of the matrix element that corresponds to finite propagation of both \(Z^0\)'s we extract the leading term in a Laurent expansion in \(p_1^2\) and \(p_2^2\) of the analytic Lorentz-invariant part of eq.(12) leaving the Lorentz-covariant part untouched. This is the doubly-resonant term and is given by

\[
\mathcal{M} = \sum_i [\bar{v}_e + T_{\mu\nu}^i u_{\nu -} ] M_i(t, u, s_p, s_p) \\
\times \frac{F_{ZZ}(s_p)}{p_1^2 - s_p} [\bar{u}_{f_1} \gamma^\mu (V_{ZfL}(s_p)\gamma_L + V_{ZfR}(s_p)\gamma_R) u_{f_1}] \tag{13}
\]

\[
\times \frac{F_{ZZ}(s_p)}{p_2^2 - s_p} [\bar{u}_{f_2} \gamma'^\nu (V_{ZfL}(s_p)\gamma_L + V_{ZfR}(s_p)\gamma_R) u_{f_2}]
\]

where \(F_{ZZ}\) defined by a relation like (7). It should be emphasized that eq.(13) is the exact form of the doubly-resonant matrix element to all orders in perturbation
theory that we will now specialize to leading order. It is free of threshold singular-

ities noted that were found by other authors [8]. In lowest order eq. (13) becomes, up to overall multiplicative factors,

\[
M = \sum_{i=1}^{2} [\bar{v}_e T_{\mu\nu} u_e M_i
\]

\[
= \frac{1}{p_1^2 - s_p} [\bar{u}_{f_1} \gamma^\mu (V_{Zf_L} \gamma_L + V_{Zf_R} \gamma_R) u_{f_1}] 
\]

\[
= \frac{1}{p_2^2 - s_p} [\bar{u}_{f_2} \gamma^\nu (V_{Zf_L} \gamma_L + V_{Zf_R} \gamma_R) u_{f_2}].
\]

where \(T_1^{\mu\nu} = (p_e - p_1) \gamma_{\mu
u}, M_1 = t^{-1}; T_2^{\mu\nu} = (p_e - p_2) \gamma_{\mu\nu}, M_2 = u^{-1}\) and the final state vertex corrections take the form \(V_{Zf_L} = i e \beta^f_{L} \gamma_L\) and \(V_{Zf_R} = i e \beta^f_{R} \gamma_R\). The left- and right-handed couplings of the \(Z^0\) to a fermion \(f\) are

\[
\beta^f_{L} = \frac{t^f_3 - \sin^2 \theta_W Q^f}{\sin \theta_W \cos \theta_W}, \quad \beta^f_{R} = -\frac{\sin \theta_W Q^f}{\cos \theta_W}.
\]

Squaring the matrix element and integrating over the final state momenta for fixed \(p_1^2\) and \(p_2^2\) gives

\[
\frac{\partial^3 \sigma}{\partial t \partial p_1^2 \partial p_2^2} = \frac{\pi \alpha^2}{s^2} (|\beta^f_{L}|^4 + |\beta^f_{R}|^4) \rho(p_1^2) \rho(p_2^2)
\]

\[
\times \left\{ \frac{t}{u} + \frac{2(p_1^2 + p_2^2)^2}{ut} - p_1^2 p_2^2 \left( \frac{1}{t^2} + \frac{1}{u^2} \right) \right\}
\]

with

\[
\rho(p^2) = \frac{\alpha}{6 \pi} \sum_f (|\beta^f_{L}|^2 + |\beta^f_{R}|^2) \frac{p^2}{p^2 - s_p} \theta(p_0) \theta(p^2)
\]

\[
\approx \frac{1}{\pi} \frac{p^2 (\Gamma_Z/M_Z)}{(p^2 - M_Z^2)^2 + \Gamma_Z^2 M_Z^2} \theta(p_0) \theta(p^2)
\]

where the sum is over fermion species. Note that \(\rho(p^2) \to \delta(p^2 - M_Z^2) \theta(p_0)\) as \(\text{Im}(s_p) \to 0\) which is the result obtained by cutting a free propagator. The variables \(s, t, u, p_1^2\) and \(p_2^2\) in eq. (15) arise from products of standard covariants and external wave functions and therefore take real values dictated by the kinematics.

Integrating over \(t, p_1^2\) and \(p_2^2\) leads to

\[
\sigma(s) = \frac{1}{\pi} \int_0^s dp_1^2 \int_0^{(\sqrt{s} - \sqrt{p_1^2})^2} dp_2^2 \sigma(s; p_1^2, p_2^2) \rho(p_1^2) \rho(p_2^2),
\]

where
\[
\sigma(s; p_1^2, p_2^2) = \frac{2\pi \alpha^2}{s^2} (|\beta^e_L|^4 + |\beta^e_R|^4) \times \left\{ \left( 1 + \frac{(p_1^2 + p_2^2)^2}{s^2} \right) \ln \left( \frac{-s + p_1^2 + p_2^2 + \lambda}{-s + p_1^2 + p_2^2 - \lambda} - \frac{\lambda}{s} \right) \right\}
\]

and \( \lambda = \sqrt{s^2 + p_1^4 + p_2^4 - 2sp_1^2 - 2sp_2^2 - 2p_1^2p_2^2} \). For \( p_1^2 = p_2^2 = M_Z^2 \) this agrees with known results [14].

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