The magnetic radiation of the fully-condensed states of $^{23}$Na condensates have been studied. A narrow characteristic spectral line with a wave length $\propto N^{-2/5}$ ($N$ is the number of particles) and with a probability of transition $\propto N^{17/5}$ emitted (absorbed) by the condensate was found. It implies that short wave radiation with a huge probability of transition can be obtained if numerous atoms are trapped. A new technique developed by the authors, namely, the analytical forms of the fractional parentage coefficients, was used to calculate analytically the matrix elements between the total spin-states.

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Obviously, the M1 operator keeps all the good quantum numbers completely unchanged, except \( M_S \). If spin-orbital coupling is not considered, and if there is no external magnetic field, the M1-transition can be neglected. Therefore, we shall focus on the M2-transition. During the transition, \( |L' - L| \leq 1 \), \( |S' - S| \leq 1 \) and \( M' + M_S + K = M + M_S \) are required. In particular, the parity must be changed.

Let \( m \) be the mass of the boson and \( w_o \) be the frequency of the parabolic confinement. When \( h \omega_o \) and \( \sqrt{h/mw_o} \) are used as units, the Hamiltonian reads

\[
H = \sum_{i<j} (-\nabla_i^2 + r_i^2) + \sum_{i<j} U_{ij}
\]

where \( U_{ij} = \delta(r_i - r_j)O_{ij}, \ O_{ij} = (c_0 + c_2\mathbf{F}_1 \cdot \mathbf{F}_j) \). In this paper we consider only the transitions related to the fully condensed (FC) states

\[
\Psi_{SM_S}^b = \Pi_{i=1}^N \varphi_{S}(r_i) \varphi_{SM_S}^{[N]} \]

where \( \varphi_{SM_S}^{[N]} \) is an all-symmetric normalized total spin-states with the good quantum numbers \( S \) and \( M_S \); \( S \) is allowed to range from \( N, N-2, N-4, \cdots \), to 1 or 0. [12] The single particle state \( \varphi_{S}(r) \equiv u_{S}(r)/\sqrt{4\pi r} \) has orbital angular momentum zero, and it depends on \( S \) in general. The spatial wavefunction \( \varphi_{S}(r) \) can be obtained by solving the generalized Gross-Pitaevskii equation

\[
[h_0 + (N - 1)g_S \frac{|u_S|^2}{4\pi r^2}] u_S(r) = \varepsilon_S u_S(r)
\]

where \( h_0 = \frac{p^2}{2m} + r^2 \), \( g_S = \langle \varphi_{SM_S}^{[N]} | O_{ij} | \varphi_{SM_S}^{[N]} \rangle = c_0 + c_2 \frac{S(S+1) - 2N}{N(N-1)} \)

and \( \varepsilon_S \) is the chemical potential. The eigenenergy of a FC-state depends solely on \( S \), it reads

\[
E_{S}^b = N \ h_0 + \frac{N(N-1)}{2} g_S T_S
\]

where \( h_0 = \int dr \ u_S^2 h_0 u_S \) and \( T_S = \frac{1}{4\pi} \int_0^\infty \frac{dr}{r} |u_S|^4 \). Since \( E_{S}^b \) depends on \( S \), all the FC-states with \( S \) ranged from \( N \), to 1 or 0 spread into a band, the FC-band. [12]

It is obvious that the M2-operators can excite only one \( p^- \)wave-particle. In other words, a FC-state would transit to a final state with one particle excited. From the rule of outer product \( \{1\} \otimes \{N-1\} = \{N\} + \{N-1,1\} \), the spatial wave functions of the final states may have two choices of permutation symmetry \( \lambda \). The choice \( \lambda = \{N\} \) is just an excitation of the center of mass (c.m.), the related photon energy is just \( h \omega_o \). This is a trivial case with a very low photon energy. For the choice \( \lambda = \{N-1,1\} \), the final state with an excited particle can be written as

\[
\Psi_{M,M_S}^{p} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varphi_{M,S}(r_i) \Pi_{j\neq i} \varphi_{M,S}^{[N]}(r_j) \Theta_{SM_S}^{[N],i}
\]

where \( N - 1 \) particles condense into the same \( \varphi_{S}^{[N]}(r) = \sqrt{4\pi r} u_S^b(r) \), while a particle is excited to a higher state \( \varphi_{M,M_S}^{b}(r) = \frac{1}{\sqrt{4\pi r}} u_S^b(r) Y_{LM}(\hat{r}) \). \( \Theta_{SM_S}^{[N],i} \) is a normalized total spin-state with the good quantum numbers \( S \) and \( M_S \), and with \( \lambda = \{N-1,1\} \), where \( S \) is allowed to range from 1 to \( N - 1 \). [8] 12 Obviously, the summation over \( i \) in (10) assures that \( \Psi_{M,M_S}^{b} \) is all-symmetric as required.

\[
[h_0 + g_S \frac{|u_S|^2}{4\pi r^2}] u_S = \varepsilon_a u_S^b
\]

\[
[h_1 + (N - 1)g_S \frac{|u_S|^2}{4\pi r^2}] u_S = \varepsilon_b u_S^b
\]

where \( h_1 = \frac{1}{2}[\frac{p^2}{m} + \frac{p^2}{r^2} + r^2], g_S = \langle \varphi_{SM_S}^{[N]} | O_{ij} | \varphi_{SM_S}^{[N]} \rangle \), \( (i \neq j \neq k) \), and \( g_S^b = \langle \varphi_{SM_S}^{[N]} | O_{ij} \varphi_{SM_S}^{[N]} + \varphi_{SM_S}^{[N]} \rangle \). By using the FPC, \( g_S^b \) and \( g_S^{ab} \) can be derived as

\[
g_S^a = c_0 + c_2 \frac{1}{(N-1)(N-2)} \left[ \frac{N+1+(-1)^{N-S}}{N} \right] S(S+1) - 2(N-1) \]

\[
g_S^{ab} = \frac{N-2}{N-1}(c_0 + c_2) - (1 + (-1)^{N-S}) \frac{S(S+1)}{N(N-1)} c_2
\]

Since \( g_S^a \) and \( g_S^{ab} \) are known, eigenstates can be obtained by solving (11) and (12), the associated eigenenergy reads

\[
E_{S}^{p} = \tilde{h}_1 + (N - 1)\ h_0 + (N - 1)g_S T_S
\]

\[
\frac{1}{2} + (N - 1)g_S^{ab} \frac{1}{2}
\]

where \( \tilde{h}_1 = \int dr \ u_S^2 \ h_1 u_S \), \( \tilde{h}_0 = \int dr \ u_S^2 \ h_0 u_S \), \( T_S^{ab} = \frac{1}{4\pi} \int_0^\infty \frac{dr}{r} |u_S^{ab}|^2 \), and \( T_S = \frac{1}{4\pi} \int_0^\infty \frac{dr}{r} |u_S|^4 \).

From now on let \( S \) \( (S') \) denotes the total spin of the initial (final) state. For M2 transitions we have \( S' = S \) or \( S = 1 \). From (15) and (9), the energy difference \( \langle \Delta E \rangle_{S'} = E_{S'} - E_{S} \) can be known \( (\langle |\Delta E| \rangle_{S'} \) is the photon energy).

By solving eq.(7), (11), and (12), not only the photon energies can be known, the probability of M2-transition can also be obtained. To compare with experimental observation, we define

\[
P_{f,o}^{S} = \frac{1}{2S+1} \sum_{M',M_S} P_{M',M_S}^{(2K)} f^{o}_{M',M_S,MM_S}
\]

By inserting (6) and (10) into (3), we have

\[
P_{f,o}^{S} = \frac{50}{12\pi} \frac{N}{2S+1} \beta^{(2)} \left( J_{S'}S \right)^2 Y_{S' S}
\]
where $I_{S'S} = \int dr u_{b}^{S'} r u_{S}$ and

$$\begin{align*}
Y_{S'S} &= \sum_{M_S} \left\{ |\langle \Theta^{[N,1]}_{S',M_S+1}|F_{1,1}|\theta'[N]_{SM_S}\rangle|^2 \\
&+ |\langle \Theta^{[N,1]}_{S',M_S}|F_{1,0}|\theta'[N]_{SM_S}\rangle|^2 \\
&+ |\langle \Theta^{[N,1]}_{S',M_S-1}|F_{1,-1}|\theta'[N]_{SM_S}\rangle|^2 \right\} \tag{18}
\end{align*}$$

It turns out that the quantity inside the brackets in (18) does not depend on $M_{S}$. By using the FPC and refer to the appendix of [3], we have

$$Y_{S'S} = (2S + 1)(N - S)(N + S + 1)/N^2 \tag{19}$$

and

$$Y_{S'S} = (N - S)(S + 2)/N \tag{20}$$

if $S' = S, S + 1$, and $S - 1$, respectively.

The numerical results of Na atoms are presented in the follows as examples. The interaction of Na has $c_o = 6.774 \times 10^{-4}/\sqrt{\omega_o}$ and $c_2 = 2.117 \times 10^{-7}/\sqrt{\omega_o}$, $\omega_o = 300 \text{ sec}^{-1}$ is given in this paper. $(\Delta E)_{S'S}$ is given in Fig.1. There are three points noticeable. (i) For the $S$ to $S' = S$ transition, there is a turning point appearing at $S/N = 1/\sqrt{3}$. This is associated with a sudden change of the structure of the excited particle as shown later. (ii) $(\Delta E)_{S'S}$ can be either positive or negative. It implies that, when $N$ is very large, the FC-states may not be the lowest. (iii) The magnitude of $(\Delta E)_{S'S}$ of the $S$ to $S' = S - 1$ transition is relatively small, thus the probability of this transition is negligible.

The probability $P_{S'S}^{I_{S'}}$ is given in Fig.2, where only the curves for $S' = S$ transition (solid line) and for $S' = S + 1$ transition (dashed line) can be seen. There is a striking sharp peak at $S/N = 1/\sqrt{3}$ for the $S' = S$ transition (just at the turning point of $(\Delta E)_{S'S}$), the height of the peak is $1.44 \times 10^{-34}/\text{sec}$, it is so high that only the bottom of the peak was seen in the figure. Accordingly, there would be a narrow spectral line emerged more than ten times brighter than the one contributed by the lower peak. The lower peak is much lower and much broader, it is peaked at $S/N = 0.96$ and is contributed by the $S' = S + 1$ transitions. From Fig.1 we know that the narrow peak has photon energy 127.9, while the broad peak has photon energies ranging from about 220 to 380. Thus, the feature of the spectrum is clear, namely, a bright narrow line together with a background at its violet side. The narrow line is called the characteristic line of the system thereafter.

When $N$ varies, the features of Fig.1 and 2 remain unchanged, but the magnitudes of $(\Delta E)_{S'S}$ and $P_{S'S}^{I_{S'}}$ are changed remarkably. In Fig.3 the wave lengths $\lambda_{S'S}$ associated with the photon energies of the narrow peak and the lower peak against $\log(N)$ are plotted. Where the two parallel straight lines imply that $\lambda_{S'S}$ are proportional to $N^{-2/5}$ in both cases, thus very short wave radiation can be obtained if $N$ is sufficiently large. In Fig.4 the probabilities $P_{S'S}^{I_{S'}}$ are plotted. The solid line as a straight line implies that the probability of transition of the characteristic line is proportional to $N^{17/5}$. The power $17/5$ is striking, it implies that the probability can be terribly large when $N$ is very large. The dashed line implies that the probability associated with the $S' = S + 1$ transition is much lower, and it increases with $N$ much slower. For a quantitative example, if $N > 10^{34}$, $P_{S'S}^{I_{S'}}$ of the characteristic line would be $> 10^4 \text{ sec}^{-1}$ and $\lambda_{S'S} < 150 \text{ cm}$.

Let us study the physical background of the above findings. When $N$ is very large, we can neglected the second term of (11), then $u_{b}^{S'}$ can be obtained by using further the Thomas-Fermi approximation [14] (neglecting the kinetic energy). When the $u_{b}^{S'}$ obtained in this way is inserted into (12), the equation can be rewritten as

$$\frac{1}{2}(-\frac{d^2}{dr^2} + \frac{2}{r^2}) + U_{eff} u_{b}^{S'} = \varepsilon_{b} u_{b}^{S'} \tag{22}$$

Where

$$U_{eff} = \frac{g_{b}^{S'}}{2g_{b}^{S}}(r_a^2 - r^2)\zeta(r_a - r) + \frac{r^2}{2} \tag{23}$$

where $r_a = (15Ng_{b}^{S}/4\pi)^{1/5}$ and $\zeta(r_a - r) = 1$ if $r \leq r_a$, or 0 otherwise.

It turns out that the feature of $U_{eff}$ is extremely sensitive to $g_{S}^{S'}/g_{b}^{S'}$. When $N - S'$ is even and $S'$ is close to $N/\sqrt{3}$ (or $N - S'$ is odd and $S'$ is close to $N$), $g_{S}^{S'}/g_{b}^{S'} \approx 1$, thereby $U_{eff}$ is nearly a constant in the broad domain $r < r_a$, but has a sharp border at $r_a$ as shown by the $S' = \sqrt{3}$ line of Fig.5. In this special case, $u_{b}^{S'}$ spreads widely from 0 to $r_a$, and results in having a very large $I_{S'S}$, and thereby a very large probability of transition. Alternatively, when $g_{S}^{S'}/g_{b}^{S'} > 1$, $U_{eff}$ is attractive as shown by the $S' = N$ curve of Fig.5. In this case the excited particle would be distributed close to the center. When $g_{S}^{S'}/g_{b}^{S'} > 1$, $U_{eff}$ is repulsive as shown by the $S' = 0$ curve of Fig.5. In this case the excited particle would be distributed very close to the surface of the condensate. To give quantitative data related to Fig.5, $u_{b}^{S'}$ is mainly distributed in the domain $0 < r < 11$ if $S' = N$, $0 < r < 148.5$ if $S' = N/\sqrt{3}$, and 145.5 $< r < 148.5$ if $S' = 0$. These data exhibit the great difference in the wave functions of the excited particle, which results in the dramatic variation of $I_{S'S}$. When $S' = S$ we have $I_{S'S} = 4.24, 14.07, 87.21, 0.81$, and 0.04 if $S' = 0$, $57N/100, N/\sqrt{3}, 58N/100$, and $N$, respectively. These data exhibit that, when $S'$ varies across $N/\sqrt{3}$, a great increase and a great decrease of $I_{S'S}$ occurs successively, this is the origin of the narrow peak shown in Fig.2.

For the transition with $N - S'$ odd and $S' = S + 1$, a similar increase of $I_{S'S}$ occurs if $S'$ is approaching $N$. However, in this case $Y_{S'S}$ tends to zero (refer to eq.(20)). As a result of the competition of $I_{S'S}$ and $Y_{S'S}$, the lower and broad peak appears adjacent to $S = N$ as shown in Fig.2.
In summary, we have found that the spin-dependent interaction is crucial to the magnetic radiation of spinor BEC. The effect of the interaction is embodied by the ratio $g_{S'}^b/g_{S'}^a$, which affects $U_{eff}$ seriously. Accordingly, the excited particle has three choices of structure, namely, being close to the center, close to the outer surface, and widely distributed from the center to the surface. The wide distribution leads to a great increase of $P_{f,o}^{S'}$. However, this case occurs only if $S' \approx N/\sqrt{3}$ and $S' = S$, or $S' \approx N$ and $S' = S + 1$. This leads to the appearance of the characteristic line together with a broad background as shown in Fig.2. The N-dependence of $\lambda_{S' S}$ and $P_{f,o}^{S'}$ has been studied, and a $N^{-2/5}$ and $N^{17/5}$ dependence, respectively, was found. This implies that short-wave radiation with a large probability of transition can be achieved if numerous particles are trapped.

It is mentioned that only the magnetic radiation of the FC-states has been studied in this paper, it is still a long way to understand the spectroscopy of the spinor BEC.

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FIG. 1: \( (\Delta E)_{S'S} = E_{0'}^{S'} - E_{0}^{S} \) against \( S/N \) for \(^{23}\)Na atoms with \( N = 5 \times 10^{12} \). The solid curve is for \( S' = S \), the dashed curve is for \( S' = S + 1 \), the dash-dot-dot curve is for \( S' = S - 1 \). \( w_{o} = 300 \text{ sec}^{-1} \), the unit of energy is \( \hbar w_{o} \). The notations and the value of \( w_{o} \) are the same in the following figures.
FIG. 2: The probability $P_{S'S}^{f,o}$ of the M2-transition of the FC-states of $^{23}$Na against $S/N$. $N = 5 \times 10^{12}$.

FIG. 3: The wavelengths $\lambda_{S'S}$ (in cm) of the radiation of the FC-states against $\log(N)$. The solid line is for the characteristic line which has $S = N/\sqrt{3}$ and $S' = S$, the dashed line is for the lower peak which has $S = 96N/100$ and $S' = S + 1$. 
FIG. 4: The probability of transition (in sec\(^{-1}\)) against log\((N)\). Refer to Fig.3.

FIG. 5: \(U_{\text{eff}}(r)\) for the excited particle with three cases of \(S', N - S'\) is assumed to be even. The units of energy and length are \(\hbar w_o\) and \(\sqrt{\hbar/mw_o}\), respectively.