Conformal mapping for multivariate Cauchy families

SHOGO KATO*\(^a\) \text{ and } PETER McCULLAGH\(^b\)

\(^a\) Institute of Statistical Mathematics, Japan
\(^b\) University of Chicago, USA

Abstract

We discuss some statistical properties of the multivariate Cauchy families on the Euclidean space and on the sphere. It is seen that the two multivariate Cauchy families are closed under conformal mapping called the Möbius transformation and that, for each Cauchy family, there is a similar induced transformation on the parameter space. Some properties of a marginal distribution of the spherical Cauchy such as certain moments and a closure property associated with the real Möbius group are obtained. It is shown that the two multivariate Cauchy families are connected via stereographic projection. Maximum likelihood estimation for the two Cauchy families is considered; closed-form expressions for the maximum likelihood estimators are available when the sample size is not greater than three, and the unimodality holds for the maximized likelihood functions. A Kent-type extension of the spherical Cauchy arising from an extended Möbius subgroup is briefly considered.

1 Introduction

In this paper we discuss multivariate extensions of two univariate Cauchy families. The first multivariate family is an extension of the real Cauchy family. Let \( \mathbb{R}^d = \mathbb{R}^d \cup \{\infty\} \) be the extended Euclidean space. Then the multivariate family on \( \mathbb{R}^d \) is given by the density

\[
f(x; \mu, \sigma) = \frac{2^{d-1} \Gamma\{(d+1)/2\}}{\pi^{(d+1)/2}} \left( \frac{\sigma}{\sigma^2 + \|x - \mu\|^2} \right)^d, \quad x \in \mathbb{R}^d,
\]

where \( \mu \in \mathbb{R}^d \) is the location parameter and \( \sigma > 0 \) is the scale parameter. Here \( \| \cdot \| \) denotes the Euclidean norm. Some probabilistic properties of the family were considered by Dunau and Sénateur (1988). Clearly, the family (1) reduces to the real Cauchy family if \( d = 1 \). For convenience, we call the family (1) the Cauchy family on the Euclidean space. For \( d \geq 2 \), this model is the spherical \( t \)-distribution in \( \mathbb{R}^d \) with \( d \) degrees of freedom, not the ordinary spherical Cauchy distribution which is equivalent to the multivariate \( t \) with one degree of freedom.

\*Address for correspondence: Shogo Kato, Institute of Statistical Mathematics, 10-3 Midori-cho, Tachikawa, Tokyo 190-8562, Japan. E-mail: skato@ism.ac.jp
The second multivariate family discussed in the paper is defined on the unit sphere, $S^d$, in $\mathbb{R}^{d+1}$. The family has density
\[
f(y; \phi) = \frac{\Gamma\{(d + 1)/2\}}{2 \pi^{(d+1)/2}} \left( \frac{1 - \|\phi\|^2}{\|y - \phi\|^2} \right)^d, \quad y \in S^d,
\]
with respect to the surface area of the sphere, where $\phi \in D^{d+1} = \{\psi \in \mathbb{R}^{d+1} : \|\psi\| < 1\}$ is the parameter. The direction $\phi/\|\phi\|$ controls the mode of the density, whereas $\|\phi\|$ adjusts the concentration of the distribution. The circular case ($d = 1$) of the model (2) is known as the wrapped Cauchy or circular Cauchy family (see, e.g., Kent and Tyler, 1988; McCullagh, 1996). The family (2) is called the Cauchy family on the sphere.

McCullagh (1992, 1996) investigated some properties of the univariate Cauchy families on $\mathbb{R}$ and on $S^1$. For example, it is shown in his papers that many computations associated with both univariate Cauchy families are greatly simplified if the random variables and parameters are expressed in form of complex numbers. With this convention, it is seen that the univariate Cauchy families on $\mathbb{R}$ and on $S^1$ are closed under the Möbius transformations on $\mathbb{R}$ and on $S^1$, respectively, and that, for each Cauchy family, there is a similar induced transformation, which is a conformal mapping, on the parameter space.

Those properties of the univariate Cauchy families obtained in McCullagh (1992, 1996), especially the closure properties associated with the Möbius transformation, have been utilized in the development of statistical theory. McCullagh (1992) considered the conditional inference for the real Cauchy family. Maximum likelihood estimation for the real and circular Cauchy families is discussed in McCullagh (1996). The regression model for circular data in which the regression curve is the Möbius transformation on $S^1$ and the error distribution is the circular Cauchy has been discussed in Kato et al. (2008); the Möbius transformation was adopted as the regression curve earlier in Downs and Mardia (2002). Kato (2010) constructed a discrete time Markov process for circular data and used the closure properties of the circular Cauchy to obtain some properties. A toroidal extension of the circular Cauchy distribution was induced by applying the Möbius transformation on $S^1$ to each circular variable (Kato and Pewsey, 2015). Other works in statistics which are related to the Möbius transformation, not necessarily to the Cauchy, include flexible families of distributions on the circle (Kato and Jones, 2010; Wang and Shimizu, 2012) and a regression model with a regression curve expressed as a skew extension of the Möbius transformation (SenGupta et al., 2013).

As seen in the works described above, the closure properties of univariate Cauchy families associated with the Möbius transformation of McCullagh (1992, 1996) have been utilized in statistics. However, to our knowledge, multivariate extensions of these properties have not been given before. Dunau and Sénateur (1988) showed that the multivariate Cauchy family (1) is closed under the conformal mapping in $\mathbb{R}^d$, but the relationship between the parameters before and after the transformation, which is essential in considering statistical applications, is not discussed in their paper. Bijective mappings on the sphere or on the disc whose dimensions are greater than one have been discussed in statistical context in Chang (1986), Rivest (1989), Jones (2004) and Uesu et al. (2015). However closure properties of Cauchy models associated with these mappings have not been considered in these papers.

In this paper we investigate some statistical properties of the multivariate Cauchy families (1) and (2) and their connection via conformal mappings. First we discuss whether the closure properties of univariate Cauchy families given in McCullagh (1992, 1996) can
be extended to the multivariate Cauchy families. For that purpose, we consider two conformal mappings and the closure properties of the two multivariate Cauchy families related to these conformal mappings. Then we discuss two statistical applications of these results, i.e., maximum likelihood estimation for multivariate Cauchy families and an extension of the Cauchy family on the sphere. In addition, it is seen that the presented results enable us to propose efficient algorithms for random variate generation of the two multivariate Cauchy, calculate moments for the Cauchy on the sphere and its marginal, and obtain method of moments estimators for the Cauchy on the sphere. Some other potential applications of the closure properties of the two multivariate Cauchy are also briefly discussed.

Section 2 introduces a conformal mapping on $\mathbb{R}^d$, which is called the Möbius transformation on $\mathbb{R}^d$, and discusses a closure property related to the Möbius transformation on $\mathbb{R}^d$. In order to obtain the closure property, a key reparametrization is achieved by introducing an extension of complex number. In Section 3 a Möbius subgroup consisting of mappings which map the unit sphere onto itself is introduced. Then we show that the Cauchy family on $S^d$ is closed under the Möbius subgroup on $S^d$ and that there is a similar induced transformation on the parameter space. Also, a marginal distribution of the spherical Cauchy and its closure property related to the real Möbius group are discussed. Section 4 shows that the Cauchy families on $\mathbb{R}^d$ and on $S^d$ are related via the inverse stereographic projection. Section 5 discusses the maximum likelihood estimation for the Cauchy family on $\mathbb{R}^d$. In addition the unimodality of the maximized likelihood function of the multivariate Cauchy on $\mathbb{R}^d$ is shown, which is an extended result of Copas (1975). An extension of the Cauchy on $S^d$ in the spirit of Kent (1982) is briefly considered in Section 6. Finally, concluding remarks are given in Section 7.

2 Möbius transformation and Cauchy family on $\overline{\mathbb{R}}^d$

2.1 Möbius transformation on $\overline{\mathbb{R}}^d$

The real Cauchy family is closed under the real Möbius transformation. McCullagh (1992, 1996) showed that the expression for the induced transformation on the parameter space is greatly simplified if the parameter of the Cauchy family is represented as a complex number. The purpose of this section is to investigate whether a somewhat similar result holds for the extended Cauchy family (1).

To achieve this, we first consider the Möbius transformation by $\tilde{g} = [A, \gamma, a, b, \varepsilon]$ on the extended Euclidean space $\overline{\mathbb{R}}^d$ defined as

$$\tilde{g}(x) = A \left( \frac{x + a}{\|x + a\|\varepsilon} + b \right), \quad x \in \mathbb{R}^d \setminus \{-a\},$$

where $a, b \in \mathbb{R}^d$, $\gamma \in \mathbb{R}$, $A$ is a $d \times d$ orthogonal matrix, and $\varepsilon$ is either 0 or 2. If $x \in \{-a, \infty\}$, we define $\tilde{g}(-a) = Ab$ and $\tilde{g}(\infty) = \infty$ for $\varepsilon = 0$ and $\tilde{g}(-a) = \infty$ and $\tilde{g}(\infty) = b$ for $\varepsilon = 2$. It is known that the set of transformations (3) forms a group under composition (see Iwaniec and Martin, 2001, Section 2). Such a group is called the Möbius group acting on $\overline{\mathbb{R}}^d$ in the paper.

The Möbius group on $\overline{\mathbb{R}}^d$ is generated by composing four types of transformations,
namely, location shifts, scale multiples, orthogonal transformations, and inversions:

\[ x \mapsto x + a, \quad x \mapsto \gamma x, \quad x \mapsto Ax \quad \text{and} \quad x \mapsto x/\|x\|^2. \]

The Möbius transformation (3) is conformal in \( \mathbb{R}^d \) for \( \varepsilon = 0 \) and in \( \mathbb{R}^d \setminus \{-a\} \) for \( \varepsilon = 2 \).

For \( d = 2 \), the transformation (3) is essentially equivalent to the Möbius transformation on the complex plane

\[ h(z) = \frac{\alpha_{11} z + \alpha_{12}}{\alpha_{21} z + \alpha_{22}}, \quad z \in \overline{C}, \]

where \( \overline{C} = \mathbb{C} \cup \{\infty\} \), and complex numbers \( \alpha_{11}, \alpha_{12}, \alpha_{21} \) and \( \alpha_{22} \) satisfy \( \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0 \). This fact can be confirmed by identifying the first and second components of \( x \) and \( \tilde{g}(x) \) in (3) as the real and imaginary parts of \( z \) and \( h(z) \), respectively.

If \( d = 2 \), there are numerous conformal mappings apart from the Möbius transformation. However Liouville’s theorem states that, if \( d \geq 3 \) and if \( \tilde{g} \) is a conformal \( C^4 \)-diffeomorphism of the domain \( \mathbb{R}^d \setminus \{-a\} \), where \(-a \in \mathbb{R}^d \) is a given constant, then \( \tilde{g} \) has the form (3) (see, e.g., Iwaniec and Martin, 2001, Section 2).

### 2.2 An extension of the the Möbius transformation to \( \mathbb{R}^d \)

In order to discuss the parameter space of the Cauchy family (1), it is advantageous to consider an extension of the space from \( \mathbb{R}^d \) to \( \mathbb{R}^{d+1} \) in a manner that preserves the group action on \( \mathbb{R}^d \). First we define an extension of the complex number as

\[ \theta = \mu + i\sigma, \quad (4) \]

where \( \mu \in \mathbb{R}^d, \sigma \in \mathbb{R} \), and \( i \) is the imaginary number. For such \( \theta \) and any group element \( \tilde{g}[A, \gamma, a, b, \varepsilon] \), define the new function

\[ g(\theta) = \tilde{g}(\mu) + i\sigma g, \quad (5) \]

where \( \tilde{g}(\mu) \) and \( \sigma g (\in \mathbb{R}) \) are

\[ \begin{pmatrix} \tilde{g}(\mu) \\ \sigma g \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\theta} + \tilde{a} \\ \gamma \|\tilde{\theta} + \tilde{a}\|^\varepsilon + \tilde{b} \end{pmatrix}. \]

Here \( \tilde{\theta} = (\mu^t, \sigma^t) \in \mathbb{R}^{d+1} \setminus \{-\tilde{a}\}, \tilde{a} = (a', 0)^t \in \mathbb{R}^d \times \mathbb{R}, \tilde{b} = (b', 0)^t \in \mathbb{R}^d \times \mathbb{R}, \gamma \in \mathbb{R}, \varepsilon = 0, 2, \) and \( A \) is a \( d \times d \) orthogonal matrix. If \( \theta = -a \) or \( \infty \), define \( g(-a) = Ab \) and \( g(\infty) = \infty \) for \( \varepsilon = 0 \) and \( g(-a) = \infty \) and \( g(\infty) = b \) for \( \varepsilon = 2 \).

It is clear from the definition that the transformation (5) is essentially a Möbius transformation on \( \mathbb{R}^{d+1} \) if the imaginary part of (5) is identified as the \((d+1)\)-th component of the Möbius transformation on \( \mathbb{R}^{d+1} \). In addition, note that the transformation (5) reduces to (3) when \( \sigma = 0 \).

For simplicity, we write (5) as

\[ g(\theta) = A \left( \gamma \frac{\theta + a}{\|\theta + a\|^\varepsilon} + b \right). \quad (6) \]
This definition implies the following operations for the extended complex number (4):

\[ \theta + a = \mu + a + i\sigma, \quad \gamma\theta = \gamma\mu + i\gamma\sigma, \]

\[ \|\theta\| = \{\|\mu\|^2 + \sigma^2\}^{1/2}, \quad A\theta = A\mu + i\sigma. \]

As will be shown later, these operations simplify the expressions related to the Cauchy family (1).

2.3 Cauchy family on \( \mathbb{R}^d \)

The Cauchy family (1) has a closure property related to the transformations (3) and (6) discussed in the previous subsection. To show that, we first express the parameters of the family (1) as \( \theta = \mu + i\sigma \in \mathbb{R}^d \times i\mathbb{R}^+ \). Rather than restrict the parameter space of \( \sigma \) to be \( \mathbb{R}^+ \), it is more convenient to use the whole real line and to identify \( \sigma \) and \( -\sigma \). Then the density of the Cauchy family (1) can be expressed as

\[
f(x; \theta) = \frac{2^{d-1} \Gamma\left\{(d+1)/2\right\}}{\pi^{(d+1)/2}} \left( \frac{|\sigma|}{\|x-\theta\|^2} \right)^d, \quad x \in \mathbb{R}^d,
\]

where \( \theta = \mu + i\sigma, \mu \in \mathbb{R}^d \) and \( \sigma \neq 0 \). For \( \sigma = 0 \), we assume that the distribution is a point mass at \( \mu \). If \( \theta = \infty \), then the distribution is assumed to be a point distribution with singularity at \( \infty \). Write \( X \sim C_d(\theta) \) if an \( \mathbb{R}^d \)-valued random vector \( X \) has density (7).

Properties of the Cauchy family (7) are that if \( X \sim C_d(\theta) \), then

\[ X + a \sim C_d(\theta + a), \quad \gamma X \sim C_d(\gamma\theta), \quad AX \sim C_d(A\theta) \quad \text{and} \quad \frac{X}{\|X\|^2} \sim C_d\left(\frac{\theta}{\|\theta\|^2}\right), \]

where \( a \in \mathbb{R}^d \), \( \gamma \in \mathbb{R} \), \( A \) is a \( d \times d \) orthogonal matrix, and the operations in the arguments of \( C_d \) are defined as in (6). These results are summarized in the following theorem.

**Theorem 1.** Let \( g \) be the function (6). Then

\[ X \sim C_d(\theta) \quad \Rightarrow \quad g(X) \sim C_d\{g(\theta)\}. \]

Note that Theorem 1 is an extension of an equivalent result for the univariate Cauchy case given in McCullagh (1992, 1996).

3 Möbius transformation and Cauchy family on \( S^d \)

3.1 Möbius transformation on \( S^d \)

The goal of this section is to discuss a subgroup of the Möbius transformations (3) and investigate its association with the Cauchy family on \( S^d \). The first step to achieve this is to consider the following function.

\[
\tilde{g}(y) = R\left\{\frac{1 - \|\phi\|^2}{\|y + \phi\|^2}(y + \phi) + \phi\right\}, \quad y \in S^d,
\]

(8)
where $\phi \in \mathbb{R}^{d+1} \setminus S^d$ and $R$ is a $d \times d$ rotation matrix. The transformation (8) maps the unit sphere onto itself. For convenience, we call the transformation (8) the Möbius transformation on the sphere. As will be seen in the next subsection, a set of the transformations (8) forms a group under composition.

The Möbius transformation on $S^d$ can be derived as follows. Let $y \in S^d$ and $\tilde{g}_{R,\theta}$ be the Möbius transformation (8). Then $\tilde{g}_{I,\theta}(y)\) corresponds to the point at one of the two intersections, which is not $-y$, of the unit sphere $S^d$ and the line connecting $-y$ with $\phi$. Finally $\tilde{g}_{R,\theta}(y)$ can be obtained by rotating $\tilde{g}_{I,\theta}(y)$ using the matrix $R$.

A function which is equivalent to (8) with the restriction $R = I, \phi = (\phi_1, 0, \ldots, 0)'$ and $-1 < \phi_1 < 1$ has been discussed in McCullagh (1989, Section 10). In his paper the function is used in transforming the exit distribution for the sphere as a step to derive new families of distributions on $(-1, 1)$.

The two parameters $R$ and $\theta$ have clear interpretation. The matrix $R$ works as a rotation parameter. In order to discuss the interpretation of $\phi$, assume, without loss of generality, that $R = I$. If $\|\phi\| < 1$, $\phi$ can be interpreted as a parameter vector that attracts the points on the sphere towards $\phi/\|\phi\|$, with the concentration of the points around $\phi/\|\phi\|$ increasing as $\|\phi\|$ increases. In particular, if $\phi = 0$, then $g_{I,\phi}$ reduces to the identify mapping. As $\|\phi\| \rightarrow 1, g_{I,\phi}(y) \rightarrow \phi/\|\phi\|$ for any $y \neq -\phi/\|\phi\|$. For the case of $\|\phi\| > 1$, the transformation $g_{I,\phi}$ consists of the two types of transformations, namely, the reflection in $y = c\phi/\|\phi\|$ ($c \in \mathbb{R}$) and the transformation $g_{I,\phi/\|\phi\|^2}$.

### 3.2 A transformation related to the Möbius transformation on $S^d$

In this subsection we present a set of functions which is related to the Möbius transformation on the sphere. It is defined by

$$
g(x) = R \left\{ \frac{1 - \|\phi\|^2}{\|\tilde{x} + \phi\|^2}(\tilde{x} + \phi) + \phi \right\}, \quad x \in \mathbb{R}^{d+1} \setminus \{0, -\phi/\|\phi\|^2\}.
$$

(9)

where $\tilde{x} = x/\|x\|^2$, $\phi \in \mathbb{R}^{d+1}$, and $R$ is a $(d+1) \times (d+1)$ rotation matrix. Also, we define

$$
g(0) = R\phi, \quad g(-\phi/\|\phi\|^2) = \infty \quad \text{and} \quad g(\infty) = R\phi/\|\phi\|^2.
$$

If we restrict the domains of $x$ and $\phi$ to be $S^d$ and $\mathbb{R}^{d+1} \setminus S^d$, respectively, then $g$ reduces to the Möbius transformation on $S^d$ (8). Note that the transformation (9) can also be expressed as

$$
g(x) = RT_\phi \left\{ \frac{1 - \|\tilde{\phi}\|^2}{\|x + \tilde{\phi}\|^2}(x + \tilde{\phi}) + \tilde{\phi} \right\}, \quad x \in \mathbb{R}^{d+1} \setminus \{-\tilde{\phi}\},
$$

(10)

where $\tilde{\phi} = \phi/\|\phi\|^2$ and $T_\phi = 2\phi\tilde{\phi}/\|\phi\|^2 - I$. It is clear from this expression that the transformations (9) are a subset of the Möbius group (3). It follows then that the transformation (9) is conformal on $\mathbb{R}^d \setminus \{-\phi/\|\phi\|^2\}$.

The transformation (9) maps the unit sphere onto itself. For $\|\phi\| < 1$, it holds that $g(D^{d+1}) = D^{d+1}$ and $g(\mathbb{R}^{d+1} \setminus \{S^d \cup D^{d+1}\}) = \mathbb{R}^{d+1} \setminus \{S^d \cup D^{d+1}\}$. If $\|\phi\| > 1$, we have $g(D^{d+1}) = \mathbb{R}^{d+1} \setminus \{S^d \cup D^{d+1}\}$ and $g(\mathbb{R}^{d+1} \setminus \{S^d \cup D^{d+1}\}) = D^{d+1}$.

If $d = 2$, the transformation (9) is related to the Möbius transformation on the complex plane which is of the form

$$
h(z) = \frac{\alpha_0 z + \alpha_1}{\alpha_1 z + 1}, \quad z \in \mathbb{C},
$$

(11)
where \( \alpha_0 \) and \( \alpha_1 \) are complex numbers such that \(|\alpha_0| = 1\) and \(|\alpha_1| \neq 1\). The transformation (11) is essentially the same as (8) with \( d = 2 \) if the real and imaginary parts of (11) are identified as the first and second components of (8), respectively. This fact can be easily confirmed by expressing (11) as

\[
h(z) = \alpha_0 \frac{\alpha_2^2}{|\alpha_1|^2} \left\{ 1 - \frac{|\alpha_1|^2 - 2}{|z + \alpha_1/|\alpha_1|^2|} \left( z + \frac{\alpha_1}{|\alpha_1|^2} \right) + \frac{\alpha_1}{|\alpha_1|^2} \right\}.
\]

The set of transformations (10) has the following closure property.

**Lemma 1.** Let \( g_{R,\phi} \) be the transformation (10). Then, for \( \phi_2 \neq -R_1 \phi_1 \),

\[
g_{R_2,\phi_2} \circ g_{R_1,\phi_1} = g_{R,\phi},
\]

where \( R = R_2 T_{\phi_2} R_1 T_{\phi_1} T_{\phi} \), \( \phi = g_{I,\phi_1} (R_1 \phi_2 + \phi_1) \) and \( \beta = T_{\phi_1} (R_1 \phi_2 + \phi_1) / \| R_1 \phi_2 + \phi_1 \|^2 \). If \( \phi_2 = -R_1 \phi_1 \), then \( g_{R_2,\phi_2} \circ g_{R_1,\phi_1} = g_{R_2 R_1,0} \).

Using this lemma, the following result can be immediately obtained.

**Theorem 2.** A set of the transformations (9) forms a group under composition.

Note that the set of transformations (9) is not an abelian group. However, for fixed \( \mu \in S^d \), the subset of transformations (9) with \( R = I \) and \( \phi = a \mu \), where \( a \in \mathbb{R} \setminus \{-1, 1\} \), forms an abelian group. A similar result can be established for a subset of the Möbius transformations on the sphere (8).

### 3.3 Cauchy family on \( S^d \)

The Cauchy family on the sphere (2) has a closure property related to the Möbius subgroup (9). To see this, it is advantageous to extend the parameter space of the spherical Cauchy (2) to be \( \mathbb{R}^{d+1} \). Specifically, we write the density

\[
f(y; \phi) = \frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}} \left( \frac{|1 - \|\phi\|^2|}{\| y - \phi \|^2} \right)^d, \quad y \in S^d,
\]

where \( \phi \in \mathbb{R}^{d+1} \setminus S^d \). For \( \phi \in S^d \), we assume that the distribution is a point mass at \( \phi \). Also the density is assumed to be uniform if \( \phi = \infty \). Write \( Y \sim C_\phi^d \) if an \( S^d \)-valued random vector \( Y \) has density (12).

The parameter \( \phi \) can be clearly interpreted. The direction \( \phi/\|\phi\| \) controls the mode of the density. The concentration of the distribution is adjusted by \( \|\phi\| \). Consider the case \( \|\phi\| < 1 \). The greater the value of \( \|\phi\| \), the greater the concentration of the density (12) around the mode. In particular, as \( \|\phi\| \to 0 \), the distribution (12) converges to the uniform distribution. On the other hand, as \( \|\phi\| \) tends to 1, the distribution converges to a point distribution with singularity at \( y = \phi/\|\phi\| \). There is a similar interpretation for the case \( \|\phi\| > 1 \) because \( f(y; \phi) = f(y; \phi/\|\phi\|^2) \).

The Cauchy family (12) can be derived applying the Möbius transformation (8) to the uniform distribution on the sphere. More precisely, if \( U \) has the uniform distribution on the sphere, then \( g_{R,\phi}(U) \sim C_\phi^d (R \phi) \). Then the following result can be readily established from Lemma 1.
Theorem 3. Assume that \( g \) is the transformation (9). Then
\[
Y \sim C^*_d (\phi) \implies g(Y) \sim C^*_d \{g(\phi)\}.
\]

If \( d = 1 \), Theorem 3 is essentially the same as the result for the circular Cauchy distribution given in McCullagh (1996).

3.4 A marginal distribution and real Möbius group
Suppose \( Y = (Y_1, \ldots, Y_{\nu+1})' \sim C^*_\nu (\phi) \), where \( \phi = (\varphi, 0, \ldots, 0)' \) and \( \varphi \in \mathbb{R} \setminus \{-1, 1\} \). Then the marginal distribution of \( Y_1 \) is of the form
\[
f(y_1; \varphi, \nu) = \frac{1}{B(\nu/2, 1/2)} \left( \frac{|1 - \varphi^2|}{1 + \varphi^2 - 2\varphi y_1} \right)^\nu (1 - y_1^2)^{(\nu-2)/2}, \quad -1 < y_1 < 1,
\]
where \( B(\cdot, \cdot) \) denotes a beta function. In a similar manner as in McCullagh (1989), if we view \( \nu \) as a continuous-valued parameter with \( \nu \geq 0 \), then (13) can be considered a two-parameter family. Clearly, \( f(y_1; \varphi, \nu) = f(y_1; \varphi^{-1}, \nu) \). If \( \varphi = 0 \), then the distribution (13) reduces to the symmetric beta distribution with density
\[
f(y_1; \nu) = \frac{(1 - y_1^2)^{(\nu-2)/2}}{B(\nu/2, 1/2)}, \quad -1 < y_1 < 1.
\]

It can be readily seen from Gradshteyn and Ryzhik (2007, 8.384.5) that the family (13) with \(-1 < \varphi < 1\) is equivalent to Seshadri’s (1991) family with the parameterization given in Example 1 of his paper. As discussed there, if \( \nu = 1 \), then the family (13) reduces to the family discussed in Leipnik (1947) and McCullagh (1989) whose density is given by equation (2) of the latter paper.

It follows from the derivation and Theorem 3 that the family (13) is closed under the real Möbius transformation
\[
g(y_1) = y_1 + b, \quad -1 < y_1 < 1; \quad -1 < b < 1.
\]
Specifically, if \( Y_1 \) has the density (13), then \( g(Y_1) \) belongs to the same family with the parameter \( \varphi \) replaced by \((\varphi + \varphi')/(\varphi' + 1)\), where \( \varphi' = (1 - \sqrt{1 - b^2})/b \). This is an extension of the result given in Seshadri (1991) that the family (13) is transformed into the symmetric beta density (14) via a special case of the Möbius transformation (15) with \( b = -2\varphi/(1 + \varphi^2) \).

3.5 Moments
In this subsection we consider some moments of the multivariate Cauchy family on \( S^d \) and its marginal family (13).

First, the first and second moments of the marginal family (13) are discussed. Seshadri (1991) obtained closed-form expressions for the mean and variance of the family (13) with \( \nu = 1 \) and approximated values of these statistics with general \( \nu \). Here we discuss the moments in more general settings. Without loss of generality, assume that \(-1 < \varphi < 1\). For convenience, define \( \mu_k(\nu) = E(Y_1^k) \), where \( Y_1 \) has the density (13). It follows from the last sentence in Section 3.4 that the mean of \( Y_1 \) can be expressed as
\[
\mu_1(\nu) = E_{Y_1}(Y_1) = E_Z \left( \frac{b - Z}{bZ - 1} \right) = -\frac{1}{b} - \frac{1 - b^2}{b^2} E_Z \left( \frac{1}{Z - b^{-1}} \right),
\]
where \( b = -2\varphi/(1 + \varphi^2) \) and \( Z \) follows the symmetric beta distribution (14). Then we have

\[
\mu_1(\nu) = \frac{1 + \varphi^2}{2\varphi} \left[ 1 - \frac{(1 + \varphi)^2}{1 + \varphi^2} F \left\{ \frac{1}{2}; \nu; -\frac{4\varphi}{(1 - \varphi)^2} \right\} \right]
\]

\[
= \frac{1 + \varphi^2}{2\varphi} \left[ 1 - \frac{1 - \varphi^2}{1 + \varphi^2} F \left\{ \frac{1}{2}; \nu - 1; \frac{\nu + 1}{2}; -\frac{4\varphi^2}{(1 - \varphi^2)^2} \right\} \right].
\]

where \( F \) denotes the hypergeometric series (Gradshteyn and Ryzhik, 2007, 9.100). The second equality follows from equations (9.131.1) and (9.134.1) of Gradshteyn and Ryzhik (2007). For \( \nu = 1, \ldots, 4, \mu_1(\nu) \) is of the form

\[
\mu_1(1) = \varphi, \quad \mu_1(2) = \frac{1 + \varphi^2}{2\varphi} \left\{ 1 - \frac{(1 - \varphi^2)^2}{2\varphi(1 + \varphi^2)} \log \left( \frac{1 + \varphi}{1 - \varphi} \right) \right\},
\]

\[
\mu_1(3) = \frac{\varphi(3 - \varphi^2)}{2}, \quad \mu_1(4) = \frac{1 + \varphi^2}{2\varphi} \left\{ 1 - \frac{3(1 - \varphi^2)^2}{8\varphi^2} + \frac{3}{16\varphi^3} \left( \frac{1 - \varphi^2}{1 + \varphi^2} \right)^3 \log \left( \frac{1 + \varphi}{1 - \varphi} \right) \right\}.
\]

The integral representation of the hypergeometric series (Gradshteyn and Ryzhik, 2007, 9.111) is useful to calculate \( \mu_1(2), \mu_1(3) \) and \( \mu_1(4) \). Gauss recursion formulas (9.137.5) and (9.137.15) of Gradshteyn and Ryzhik (2007) imply that

\[
F \left( \frac{1}{2}; \frac{\nu - 1}{2}; \frac{\nu + 1}{2}; z \right) = \frac{\nu - 1}{(\nu - 2)(\nu - 3)} \left\{ \left( \nu - 2 - \frac{\nu - 3}{2} \right) F \left( \frac{1}{2}; \frac{\nu - 3}{2}; \frac{\nu - 1}{2}; z \right) + \frac{\nu - 3}{2} F \left( \frac{1}{2}; \frac{\nu - 5}{2}; \frac{\nu - 3}{2}; z \right) \right\}.
\]

It follows from these results that, for any positive integer \( \nu \), the mean of \( \mu_1(\nu) \) can be expressed in closed form.

Similarly the second moment of \( Y_1 \) can be expressed as

\[
\mu_2(\nu) = \frac{(1 + \varphi^2)^2}{4\varphi^2} \left[ 1 - 2 \frac{(1 + \varphi)^2}{1 + \varphi^2} F \left\{ \frac{1}{2}; \nu; -\frac{4\varphi}{(1 - \varphi)^2} \right\} + \frac{(1 + \varphi)^4}{(1 + \varphi^2)^2} F \left\{ 2; \nu; -\frac{4\varphi}{(1 - \varphi)^2} \right\} \right].
\]

The hypergeometric series \( F\{1, \nu/2; \nu; -(4\varphi)/(1 - \varphi^2)\} \) in the second term in the right-hand side of (16) can be obtained in the same manner as in the calculation of \( \mu_1(\nu) \). The hypergeometric series \( F\{2, \nu/2; \nu; -(4\varphi)/(1 - \varphi^2)\} \) in the third term of the right-hand side of (16) can be calculated partly using its integral representation (Gradshteyn and Ryzhik, 2007, 9.111). After some algebra, we have

\[
\mu_2(1) = \frac{1 + \varphi^2}{2}, \quad \mu_2(2) = \frac{1 + \varphi^2}{4\varphi^2} \left\{ \frac{2(1 + \varphi^4)}{1 + \varphi^2} - \frac{(1 - \varphi^2)^2}{\varphi} \log \left( \frac{1 + \varphi}{1 - \varphi} \right) \right\},
\]

\[
\mu_2(3) = \frac{1 + 6\varphi^2 - 3\varphi^4}{4},
\]

\[
\mu_2(4) = \frac{1 + \varphi^2}{16\varphi^4} \left\{ -2 \frac{(3 - 8\varphi^2 + 2\varphi^4 - 8\varphi^6 + 3\varphi^8)}{1 + \varphi^2} + \frac{3(1 - \varphi^2)^4}{\varphi} \log \left( \frac{1 + \varphi}{1 - \varphi} \right) \right\}.
\]

It follows from these results and equation (9.134.3) of Gradshteyn and Ryzhik (2007) that \( \mu_2(\nu) \) has a closed form expression for any \( \nu \in \mathbb{N} \). Thus the variance of \( Y_1 \) can also be expressed in closed form for any positive integer \( \nu \).
Some moments of the multivariate Cauchy family on $S^d$ are readily available by utilizing the moments of the marginal (13) obtained above. Let $Y$ have the multivariate spherical Cauchy $C_d^*(\phi)$. Then it is easy to see that, for $\phi \neq 0$,

$$E(Y) = \mu_1(d)\frac{\phi}{||\phi||}, \quad E(YY^*) = d^{-1}\left\{1 - \mu_2(d)\right\}I + \{(d + 1)\mu_2(d) - 1\}\frac{\phi\phi^*}{||\phi||^2}.$$  

Note that closed-form expressions for $E(Y)$ and $E(YY^*)$ are available for any $d \geq 1$. Since the spherical Cauchy (12) with $\phi = 0$ reduces to the uniform distribution on the sphere, $E(Y) = 0$ and $E(YY^*) = d^{-1}I$ for $\phi = 0$.

The method of moments estimation of the parameter $\phi$ based on $E(Y)$ is straightforward from these results.

4 Inverse stereographic projection

In this section we investigate the relationship between the two multivariate Cauchy families (7) and (12) via the inverse stereographic projection. The inverse stereographic projection $\mathbb{R}^d \rightarrow S^d$ is known to be

$$\tilde{g}(x) = \frac{2}{||x||^2 + 1}\left(x_1, \ldots, x_d, \frac{||x||^2 - 1}{2}\right)^t,$$  

where $x = (x_1, \ldots, x_d)^t \in \mathbb{R}^d$. In addition we assume $g(\infty) = e_{d+1}$, where $e_j$ is the unit vector whose $j$th component is equal to one. It is known that the inverse stereographic projection (17) maps $\mathbb{R}^d$ onto the unit sphere $S^d$. A geometrical interpretation of (17) is that $\tilde{g}(x)$ corresponds to the point at the intersection of the unit sphere $S^d$ and the line connecting the north pole $e_{d+1}$ and $(x', 0)^t$.

In order to explore the association between the two Cauchy families (7) and (12), we propose an extension of the inverse stereographic projection. Define a new function by

$$g(\theta) = \frac{2}{\|\theta + i\|^2}\left(\mu_1, \ldots, \mu_d, \frac{||\theta||^2 - 1}{2}\right)^t,$$  

where $\theta = \mu + i\sigma \in (\mathbb{R}^d + i\mathbb{R}) \setminus \{-i\}$. Also, suppose that $g(-i) = \infty$ and $g(\infty) = e_{d+1}$. Then (18) is a bijective function which maps $(\mathbb{R}^d + i\mathbb{R}) \cup \{\infty\}$ onto $\mathbb{R}^{d+1}$. Clearly, the function (18) reduces to (17) if $\sigma = 0$.

Note that the function (18) is related to the Möbius transformation on $\mathbb{R}^{d+1}$. To see this, consider the following function

$$g^*(x) = (I - 2e_{d+1}e_{d+1}^t)\left\{\frac{x + e_{d+1}}{||x + e_{d+1}||^2} - e_{d+1}\right\}, \quad x \in \mathbb{R}^{d+1} \setminus \{-e_{d+1}\}.$$  

Also, assume that $g^*(\infty) = e_{d+1}$ and $g^*(-e_{d+1}) = \infty$. Then it follows that $g(\mu + i\sigma) = g^*((\mu', \sigma'))$ for any $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}$ and $g(\infty) = g^*(\infty)$. This implies that $g$ is essentially equivalent to $g^*$, a Möbius transformation on $\mathbb{R}^{d+1}$, if the imaginary part of the argument of $g$ is viewed as the $(d + 1)$-th component of the argument of $g^*$.

Similarly, one can extend the stereographic projection $\tilde{g}^{-1}$ by considering the inverse function of (19). Since a set of the Möbius transformations forms a group under composition, it follows that the inverse of (19) is also a Möbius transformation on $\mathbb{R}^{d+1}$.

The two Cauchy families (7) and (12) are related as follows.
Theorem 4. Let $g$ be the function (18). Then

$$X \sim C_d(\theta) \quad \Rightarrow \quad g(X) \sim C_d^* \{g(\theta)\}.$$  

Equivalently,

$$Y \sim C_d^* (\phi) \quad \Rightarrow \quad g^{-1}(Y) \sim C_d \{g^{-1}(\phi)\}.$$  

Theorems 1 and 4 imply that, as is the case for random variate generation from the Cauchy on $S^d$ discussed in Section 3.3, random variates following the Cauchy family on $\mathbb{R}^d$ can also be generated from uniform random variates on the sphere. Indeed, if $U$ has the uniform distribution on $S^d$, then $g^{-1}(U)$ has the standard Cauchy on $\mathbb{R}^d$ whose general form can be obtained via location shifts and scale multiples. Practically, a uniform variate on $\mathbb{R}^d$ can be generated as follows: let $X_1, \ldots, X_{d+1}$ be a random sample from the standard normal, and then $(X_1, \ldots, X_{d+1})/\| (X_1, \ldots, X_{d+1}) \|^2$ follows the uniform distribution on $S^d$.

5 Maximum likelihood estimation

Throughout the section assume that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are iid samples from the Cauchy families $C_d(\theta)$ and $C_d^*(\phi)$, respectively. It follows from Theorem 4 that the maximum likelihood estimation for the Cauchy on $\mathbb{R}^d$ is equivalent to that for the Cauchy on $S^d$. Therefore it suffices to discuss the maximum likelihood estimation for either of the two Cauchy families (1) or (2), whichever is the more convenient.

5.1 The case $n = 1$

If $n = 1$, it is clear that the maximum likelihood estimators are $\hat{\theta} = X_1$ and $\hat{\phi} = Y_1$. The likelihood at this point is unbounded.

5.2 The case $n = 2$

Let $x_1 = e_1$ and $x_2 = -e_1$ be the observations from the Cauchy family on $\mathbb{R}^d$ (1). Then the likelihood function $L_2$ is proportional to

$$L_2(\theta) \propto \frac{\sigma^2}{(1 + 2 \mu_1 + \| \theta \|) (1 - 2 \mu_1 + \| \theta \|)}^d \left( \frac{\sigma^2}{\{(1 + \mu_1)^2 + \| \mu \|^2\} \{(1 - \mu_1)^2 + \| \mu \|^2\}} \right)^d,$$

where $\theta = \mu + i \sigma$, $\mu = (\mu_1, \tilde{\mu}', \mu_2, \ldots, \mu_d)'$, $\mu = (\mu_2, \ldots, \mu_d)'$. It follows from this expression that the maximum likelihood estimate of $\tilde{\mu}$ has to be zero. Then $L_2(\theta)$ essentially reduces to the likelihood function of the univariate real Cauchy distribution discussed in McCullagh (1996). Therefore the contour of maximum likelihood is the unit circle

$$\{(\cos \xi, 0, \ldots, 0) + i \sin \xi; -\pi \leq \xi < \pi\}.$$  

By transforming the circle via location shifts and scale multiples, the contour of maximum likelihood of $\theta$ for general $x_1$ and $x_2$ is shown to be the circle perpendicular to $\mathbb{R}^d$ with diameter $(x_1, x_2)$.  

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As for the maximum likelihood estimation for the Cauchy on the sphere, Theorem 4 implies that the contour of the maximum likelihood of \( \psi \) with the observations \( y_1 = e_1 \) and \( y_2 = -e_1 \) is given by the line

\[
L = \{(\beta, 0, \ldots, 0)'; \beta \in \mathbb{R}\}.
\]

Consider the the maximum likelihood for general \( y_1 \) and \( y_2 \) (\( y_1 \neq -y_2 \)). Let \( g \) be the function (9) with \( R = I \) and \( \phi = (0, c, 0, \ldots, 0)' \), where \( c = [(1 - (1 - y_1^2y_2^2)/2)^{1/2}]/[1 + (1 - y_1^2y_2^2)/2]^{1/2} \). Then \( g(e_1)'g(-e_1) = y_1'y_2 \). This implies that the angle between \( y_1 \) and \( y_2 \) is the same as that between \( g(e_1) \) and \( g(-e_1) \). The contour of the maximum likelihood \( L \) in (20) is transformed via \( g \) onto

\[
\{(\phi_1, \phi_2, 0, \ldots, 0)'; \phi_1^2 + \{\phi_2 - (1 + e^2)/(2c)\}^2 = ((1 - e^2)/(2c))^2 \}.
\]

Therefore the contour of the maximum likelihood for general \( y_1 \) and \( y_2 \) (\( y_1 \neq \pm y_2 \)) is the circle perpendicular to the unit sphere with chord \((y_1, y_2)\) in the two-dimensional plane spanned by \( y_1 \) and \( y_2 \). Clearly, if \( y_1 = y_2 \), the maximum likelihood estimator is \( \hat{\phi} = y_1 \).

5.3 The case \( n = 3 \)

First we consider the case \( n = 3 \) with observations \( x_1 = -e_1, \ x_2 = 0 \) and \( x_3 = e_1 \) from the Cauchy family (1). In a somewhat similar manner to Ferguson (1978) and McCullagh (1996), it can be seen that the maximum likelihood estimate of \( \theta \) is given by \( \hat{\theta} = 0 + (1/\sqrt{3})i \). Here we transform \((-e_1, 0, e_1)\) to the general \((x_1, x_2, x_3)\). To achieve this, we first set \((\tilde{x}_1, \tilde{x}_3) = R(x_1 - x_2, x_3 - x_2)\), where \( R \) is a \( d \times d \) rotation matrix such that \( \tilde{x}_j = (\tilde{x}_j1, \tilde{x}_j2, 0, \ldots, 0)' \) \((j = 1, 3)\). Note that \((x_1, x_2, x_3)\) and \((\tilde{x}_1, 0, \tilde{x}_3)\) constitute the same triangle apart from translation and rotation. Then the three points \((-e_1, 0, e_1)\) are transformed to \((\tilde{x}_1, 0, \tilde{x}_3)\) via the transformation

\[
\mathcal{M}(t) = A \left( \gamma \frac{t + a}{\|t + a\|^2} + b \right), \quad t \in \mathbb{R}^d \times i\mathbb{R}, \tag{21}
\]

where

\[
A = \begin{pmatrix} \tilde{A} & O \\ O & I \end{pmatrix}, \quad \tilde{A} = \frac{1}{|\alpha \beta|} \begin{pmatrix} -\text{Re}(\alpha \beta) & -\text{Im}(\alpha \beta) \\ -\text{Im}(\alpha \beta) & \text{Re}(\alpha \beta) \end{pmatrix},
\]

\[
a = (\text{Re}(\beta), \text{Im}(\beta), 0, \ldots, 0)', \quad b = -|\alpha/\beta| (\text{Re}(\beta), \text{Im}(\beta), 0, \ldots, 0)', \quad \gamma = |\alpha \beta|,
\]

\[
\alpha = \frac{2z_1z_3}{z_1 + z_3}, \quad \beta = \frac{z_1 - z_3}{z_1 + z_3}, \quad z_j = \tilde{x}_j1 + i\tilde{x}_j2, \quad j = 1, 3.
\]

Then \( x_1 \) and \( x_3 \) have the expression

\[
x_1 = x_2 + R'\mathcal{M}(e_1) \quad \text{and} \quad x_3 = x_2 + R'\mathcal{M}(e_1).
\]

Substituting \( t = 0 + (1/\sqrt{3})i \) into (21), the estimate of the parameter for general \((x_1, x_2, x_3)\) is given by \( \hat{\theta} = x_2 + R'\mathcal{M}(t) \). After some algebra, it can be seen that the estimate is of the form

\[
\hat{\mu} = \frac{\|x_1 - x_2\|^2 x_3 + \|x_2 - x_3\|^2 x_1 + \|x_3 - x_2\|^2 x_1}{\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_1\|^2},
\]

\[
\hat{\sigma} = \pm \sqrt{3} \frac{\|x_1 - x_2\|\|x_2 - x_3\|\|x_3 - x_1\|}{\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_1\|^2}.
\]

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5.4 The case $n = 4$

In order to discuss the case $n = 4$ for the Cauchy on $\mathbb{R}^d$, we take $s_j = g(a_j)$ ($1 \leq j \leq 4$), where $g$ is (18), $a_j$ is the configuration statistic, i.e., $a_j = (x_j - t_1)/t_2$, and $t_1$ and $t_2$ are the maximum likelihood estimates of $\mu$ and $\sigma$, respectively. Then the likelihood equation is given by

$$\sum_{j=1}^{4} s_j = 0.$$  

Note that this extended result holds for general $n$. To see the relationship among $\{s_j\}_{j=1}^{4}$ in (5.4), we first assume that $s_1 = e_1$, $s_2 = (s_{21}, s_{22}, 0, \ldots, 0)'$ and $s_3 = (s_{31}, s_{32}, s_{33}, 0, \ldots, 0)'$. Since $s_1 + s_2 + s_3 = -s_4 (\in S^d)$, the elements of $s_2$ and $s_3$ have the expression $s_{21} = \cos \xi_1$, $s_{22} = \sin \xi_1$, $s_{31} = -\cos^2(\xi_1/2) + \sin^2(\xi_1/2) \cos \xi_2$, $s_{32} = -\sin(\xi_1/2) \cos(\xi_1/2)(1 + \cos \xi_2)$, and $s_{33} = \sin(\xi_1/2) \sin \xi_2$ ($-\pi \leq \xi_1, \xi_2 < \pi$).

For general $s_1, s_2$ and $s_3$, $s_3$ is of the form

$$s_3 = -\frac{1 - \cos \xi_2}{2} s_1 - \frac{1 + \cos \xi_2}{2} s_2 + \frac{||s_2 - s_1||}{2} \sin \xi_2 \cdot b,$$

where $b \in S^d$, $b \perp s_j$, $j = 1, 2$. Using this result $a_3$ and $a_4$ can be written in terms of $a_1, a_2$ and $b$ as

$$a_3 = \begin{bmatrix} -2(||a_2||^2 + 1) \sin^2(\xi_2/2) \cdot a_1 - 2(||a_1||^2 + 1) \cos^2(\xi_2/2) \cdot a_2 \\ + \{(||a_1||^2 + 1)(||a_2||^2 + 1)(||a_1||^2 + ||a_2||^2)\}^{1/2} \sin \xi_2 \cdot \tilde{b} \end{bmatrix}$$

$$\left[2||a_1||^2||a_2||^2 + ||a_1||^2 + ||a_2||^2 - (||a_1||^2 - ||a_2||^2) \cos \xi_2 \\ - \{(||a_1||^2 + 1)(||a_2||^2 + 1)(||a_1||^2 + ||a_2||^2)\}^{1/2} \sin \xi_2 \cdot \tilde{b} \right],$$

$$a_4 = \begin{bmatrix} -2(||a_2||^2 + 1) \cos^2(\xi_2/2) \cdot a_1 - 2(||a_1||^2 + 1) \sin^2(\xi_2/2) \cdot a_2 \\ + \{(||a_1||^2 + 1)(||a_2||^2 + 1)(||a_1||^2 + ||a_2||^2)\}^{1/2} \sin \xi_2 \cdot \tilde{b} \end{bmatrix}$$

$$\left[2||a_1||^2||a_2||^2 + ||a_1||^2 + ||a_2||^2 + (||a_1||^2 - ||a_2||^2) \cos \xi_2 \\ + \{(||a_1||^2 + 1)(||a_2||^2 + 1)(||a_1||^2 + ||a_2||^2)\}^{1/2} \sin \xi_2 \cdot \tilde{b} \right].$$

where $b = (b', b_{d+1})'$ and $\tilde{b} = (b_1, \ldots, b_d)'$. The maximum likelihood estimates of $\mu$ and $\sigma$ are the solutions of (22), (23) and $a_j = (x_j - t_1)/t_2$ ($j = 1, \ldots, 4$). It does not seem clear that there are closed form expressions for $t_1$ and $t_2$.

5.5 Unimodality of the likelihood function

Copas (1975) established the unimodality of the likelihood function for the univariate Cauchy model on certain conditions. Here we consider the same topic for the multivariate Cauchy family on $\mathbb{R}^d$ (1).
First consider the log-likelihood function

\[
\ell(\mu, \sigma) = C + d \left\{ n \log \sigma - \sum_{j=1}^{n} \log \left( \sigma^2 + \|x_j - \mu\|^2 \right) \right\},
\]

(24)

where the constant \( C \) does not depend on the parameters. An extended result of Copas (1975) can be established for our multivariate Cauchy family (1) as follows. The likelihood function \( \ell \) has no saddle points, and any stationary point of the likelihood function must be a local maximum. If there is no value of \( x \) at which half or more of the observations are coincident, then, for fixed \( \mu \), there is one maximum of the likelihood function attained in the region \( \sigma > 0 \). If half or more observations take the same values at \( x = \mu_0 \), then the maximum likelihood function tends to infinity as \( \sigma \to 0 \) along the path \( \mu = \mu_0 \). If half of the observations equal \( \mu_0 \) and exactly half equal \( \mu_1 \), then the maximum likelihood estimation for \( (\mu', \sigma) \) is essentially the same as that for the case \( n = 2 \) which has been discussed in Section 5.2. As discussed there, the contour of the maximum likelihood is the circle with diameter \((\mu_0, \mu_1)\) perpendicular to the sample space. See Appendix A for the details of these results.

It can also be seen that the maximized likelihood, which is defined by

\[
\lambda(\mu) = \ell \{ \mu, \hat{\sigma}(\mu) \} = \sup_{\sigma} \ell(\mu, \sigma),
\]

is unimodal if there exist stationary points for \( (\mu', \sigma) \). See, e.g., Patefield (1977) for maximized likelihood function.

For general \( n \), it appears that closed form expression for the maximum likelihood estimator is not readily available for the Cauchy family on \( \mathbb{R}^d \). In such cases one can apply some numerical algorithms such as the EM algorithm to estimate the parameters. The same goes for the Cauchy on \( S^d \) which can be transformed into the Cauchy on \( \mathbb{R}^d \) via the stereographic projection.

6 A Kent-type extension of the Cauchy family on \( S^d \)

Densities of many existing models on the sphere, including the Cauchy on the sphere (2), have spherical contours of constant probability densities. In order to obtain greater flexibility, it appears useful to have more general distributions on \( S^d \) with oval contours. Such generalizations exist for a member of the exponential family such as Kent distribution (Kent, 1982), which is an extension of the von Mises–Fisher distribution (see, e.g., Mardia and Jupp, 1999, Section 9.3.3, for some other examples). In this section we briefly consider a Kent-type generalization of the family (2) via a mapping related to the Möbius transformation.

6.1 Definition and basic properties

Before we present a new family, we note that the Cauchy distribution on \( S^d \) can be generated as follows. Let \( U = (U_1, \ldots, U_{d+1})' \) have the uniform distribution on \( S^d \). Assume that \( g \) is the inverse stereographic projection (18) Then it follows from Theorems 1 and 4 that

\[
g^{-1} \{ \mu + \sigma g(U) \}
\]

(25)
has the Cauchy distribution on the sphere $C^*_d\{g^{-1}(\mu + i\sigma)\}$, where $\mu + i\sigma \in \mathbb{R}^d \times i\mathbb{R}$.

Here we consider an extension of the mapping (25)

$$h(u) = g^{-1}\{\mu + Lg(u)\}, \ u \in S^d,$$

where $\mu \in \mathbb{R}^d$ and $L$ is a $d \times d$ invertible matrix. If $L = \sigma I$, (26) reduces to (25). Note that the set of transformations (26) forms a group under composition.

Utilizing the mapping (26), we propose an extension of the Cauchy family on $S^d$. It is defined by the random vector

$$Y = h(U).$$

The probability density function of $Y$ is given by

$$f(y) = \frac{\Gamma((d + 1)/2)}{2\pi^{(d+1)/2} |\det(L)|} (\tilde{y}'B\tilde{y})^{-d}, \ y \in S^d,$$

where $\tilde{y} = (y - e_{d+1})/\|y - e_{d+1}\|$ and

$$A = \begin{pmatrix} (L'L)^{-1} & (L'L)^{-1}e_{d+1} \\ e_{d+1}'(L'L)^{-1} & 1 + e_{d+1}'(L'L)^{-1}e_{d+1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ e_{d+1}' & 1 \end{pmatrix} \begin{pmatrix} (L'L)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & e_{d+1} \\ 0 & 1 \end{pmatrix}.$$

Note that $\tilde{y}$ takes values in the hemisphere, i.e., $\tilde{y} \in \{\tilde{y} = (\tilde{x}_1, \ldots, \tilde{x}_{d+1})' \in S^d; \tilde{x}_{d+1} < 0\}$. Another expression for the density (27) is

$$f(y) = \frac{\Gamma((d + 1)/2)}{2\pi^{(d+1)/2} |\det(L)|} \frac{1}{(\tilde{Q} + \tilde{y}'T\tilde{y})^{d+1}},$$

where $\tilde{Q} = (\text{tr}A)/(d + 1)$ and $T = A - \tilde{Q}I$. Note that $\text{tr}(T) = 0$.

The mode and antimode of the density (27) are available. Since the matrix $A$ is symmetric, it can be decomposed as $A = P\text{diag}\{\lambda_1, \ldots, \lambda_{d+1}\}P$, where $\lambda_1, \ldots, \lambda_{d+1} (> 0)$ are the eigenvalues of $A$ and $P$ is a $(d + 1) \times (d + 1)$ orthogonal matrix. Then it follows that the density (27) takes the maximum value $f(x) = C_d\lambda_{\min}^{-d}$ at $\tilde{x} = \tilde{x}_{\max}$ or $\tilde{x} = -\tilde{x}_{\max}$, where $C_d = \Gamma((d + 1)/2)/(2\pi^{(d+1)/2} |\det(L)|)$, $\lambda_{\min} = \lambda_k = \min(\lambda_1, \ldots, \lambda_{d+1})$, and $\tilde{x}_{\max} = P'e_k$. (Note that only either of $\tilde{x}_{\max}$ or $-\tilde{x}_{\max}$ satisfies $\tilde{x}_{d+1} < 0$.) Similarly the density (27) takes the minimum value $f(\tilde{x}) = C_d\lambda_{\max}^{-d} \tilde{L} = \tilde{x}_{\min}$ or $\tilde{x} = -\tilde{x}_{\min}$, where $\lambda_{\max} = \lambda_l = \max(\lambda_1, \ldots, \lambda_{d+1})$ and $x_{\min} = P'e_l$.

It immediately follows from the derivation that the family (27) is closed under the mapping (26).

6.2 The case $d = 2$

Let $X = (X_1, X_2, X_3)'$ have the density (27) and $(X_1, X_2, X_3) = (\sin \Xi_1 \cos \Xi_2, \sin \Xi_1 \sin \Xi_2, \cos \Xi_1)$, where $0 \leq \Xi_1 \leq \pi$ and $-\pi \leq \Xi_2 < \pi$. Assume that $a_{jk}$ ($j, k = 1, 2$) is $(j, k)$th entry of $(A'A)^{-1}$. It follows that the density of $(\Xi_1, \Xi_2)$ takes the form

$$f(\xi_1, \xi_2) = C_2 \left[ 1 + \sin^2(\xi_1/2) \left( (a_{11} - 1) \cos^2 \xi_2 + 2a_{12} \cos \xi_2 \sin \xi_2 + (a_{22} - 1) \sin^2 \xi_2 \right) \right]^{-2},$$

$$= C_2 \left[ 1 + \frac{\sin^2(\xi_1/2)}{2} \left( a_{11} + a_{22} - 2 + \alpha \cos 2(\xi_2 - \beta) \right) \right]^{-2},$$

$$0 \leq \xi_1 < \pi, \ -\pi \leq \xi_2 < \pi, \ (28)$$
with respect to $\sin \xi_1 d\xi_1 d\xi_2$, where $\alpha = \{(a_{11} - a_{22})^2 + 4a_{12}^2\}^{1/2}$, $\beta = \arg\{(a_{11} - a_{22}) + 2a_{12}i\}/2$, and $C_2 = 1/(4\pi|a_{11}a_{22} - a_{12}^2|)$.

Let $a_{11} + a_{22} - 2 \geq a$. Then it can be easily seen that, as $\xi_1$ goes to 0 to $\pi$ for fixed $\xi_2$, the density (28) decreases monotonically. Thus, in this case, the density is unimodal on all great circles through the pole. A similar argument can be given for $a_{11} + a_{22} - 2 \leq -a$.

In order to plot the density, Kent (1982) used Lambert’s equal area projection

$$v_1 = \rho \cos \xi_2, \quad v_2 = \rho \sin \xi_2,$$

where $\rho = 2 \sin(\xi_1/2)$ and $0 \leq \rho \leq 2$. Note that $dv_1dv_2 = \sin \xi_1 d\xi_1 d\xi_2$. Applying this projection to the density (28), we have

$$f(v_1, v_2) = \frac{1}{4\pi|a_{11}a_{22} - a_{12}^2|} \left[1 + \frac{1}{4} \{(a_{11} - 1)v_1^2 + 2a_{12}v_1v_2 + (a_{22} - 1)v_2^2\}\right]^{-2},$$

(29)

whose functional part is an ellipse.

Figure 1 plots the density (29) for some selected values of the parameters. Its frame (a) implies that the contours are circles if the eigenvalues of $(LL^{-1})$ are equal. As can be seen in the other three frames, if the eigenvalues of $(LL^{-1})^{-1}$ differ, then the contours are not circles but ellipses. The contours displayed in frame (d) are the same as the contours in frame (c) rotated by $\pi/2$. Figure 2 displays a scatterplot of 1000 random samples generated from the density (27) with $d = 2$ for some selected combinations of the parameters. It can be seen from the figure that, if $|a_{11} - a_{22}|$ is large, then the difference in dispersions between the neighborhoods of the circle $\{x \in S^2; x_1 = 0\}$ and those of the circle $\{x \in S^2; x_2 = 0\}$ is large.

### 7 Discussion

In this paper some statistical properties of the multivariate Cauchy families on $\mathbb{R}^d$ and on $S^d$ have been investigated. It has been shown that the multivariate Cauchy families on $\mathbb{R}^d$ and on $S^d$ are closed under the Möbius transformations on $\mathbb{R}^d$ and on $S^d$, respectively; there are similar induced transformations on the parameter space. In particular...
Figure 2: Scatterplot of 1000 random variates from the density (27) with \( d = 3 \) for \( a_{23} = 0 \) and: (a) \( (a_{22}, a_{33}) = (1, 0.1^{-2}) \), (b) \( (a_{22}, a_{33}) = (0.5^{-1}, 0.1^{-2}) \), and (c) \( (a_{22}, a_{33}) = (0.1^{-2}, 0.1^{-2}) \).

the expression of the parameter for the Cauchy on \( \mathbb{R}^d \) after the transformation can be greatly simplified if the parameter is represented as an extended complex number. Some properties of a marginal distribution of the spherical Cauchy have been obtained, including its first and second moments. It has been seen that the Cauchy families on \( \mathbb{R}^d \) and on \( S^d \) can be transformed each other via the stereographic projection or its inverse and that there are similar induced transformation on the parameter space.

These properties of the two Cauchy families have been applied to statistical theory. The closure properties of the multivariate Cauchy families have been utilized for efficient algorithms for random variate generation and maximum likelihood estimation. The inverse stereographic projection enables us to show that the unimodality of the maximized likelihood function holds for the Cauchy family on the sphere as well as the Cauchy on \( \mathbb{R}^d \). The moments of the spherical Cauchy, which have been obtained via the real Möbius transformation, can be used for method of moments estimation. An extension of a Möbius transformation on \( S^d \) generates a flexible family of distributions on the sphere which has densities with oval contours in general.

Because of the mathematical tractability of the multivariate Cauchy families and Möbius transformations, some other statistical applications of the Cauchy families and/or the Möbius transformations are possible. For example it is straightforward to construct a regression model for spherical data in which the Möbius transformation on \( S^d \) is adopted as a regression curve, which is an extension of the circular regression models of Downs and Mardia (2002) and Kato et al. (2008). A Markov process for spherical data can be also obtained in which the regression curve is the Möbius transformation on \( S^d \) and the angular error is the Cauchy on \( S^d \). The Markov process is an extended Markov model for circular data presented by Kato (2010).
Appendix

A Details of Section 5.5

In order to show the unimodality of the log-likelihood function (24), it suffices to consider
\[ \tilde{\ell} = n \log \sigma - \sum_{j=1}^{n} \log \left( \sigma^2 + \|x_j - \mu\|^2 \right). \]

Setting \( \partial \tilde{\ell} / \partial \mu \) and \( \partial \tilde{\ell} / \partial \sigma \) respectively to zero gives the equations
\[
\sum_{j=1}^{n} \frac{x_j}{\sigma^2 + \|x_j\|^2} = 0, \quad (30)
\]
\[
\sum_{j=1}^{n} \frac{\sigma^2}{\sigma^2 + \|x_j\|^2} = \frac{n}{2}, \quad (31)
\]
where \( x_j = x_j - \mu \). If (31) holds, we have
\[
\frac{\partial^2 \tilde{\ell}}{\partial \sigma^2} = -4 \sum \frac{\|x_j\|^2}{(\sigma^2 + \|x_j\|^2)^2} < 0.
\]

It follows that, for fixed \( \mu \), the likelihood function for \( \sigma \) is unimodal. If the equation (31) has the solution \( \hat{\sigma} \), then the maximum likelihood estimate of \( \sigma \) is \( \hat{\sigma} \). Otherwise the maximum likelihood estimate of \( \sigma \) is equal to zero.

Consider the second derivative of \( \tilde{\ell} \) with respect to \( (\mu', \sigma) \) at the stationary points of \( (\mu', \sigma) \). Using the equality
\[
\sum_{j} \frac{x_j}{(\sigma^2 + \|x_j\|^2)^2} = -\frac{1}{2\sigma^2} \sum_{j} \frac{\|x_j\|^2 - \sigma^2}{(\sigma^2 + \|x_j\|^2)^2} x_j,
\]
which can be derived from (30), we have
\[
\frac{\partial^2 \tilde{\ell}}{\partial \sigma \partial \mu} = -4\sigma \sum_{j} \frac{x_j}{(\sigma^2 + \|x_j\|^2)^2} = \frac{2}{\sigma} \sum_{j} \frac{\|x_j\|^2 - \sigma^2}{(\sigma^2 + \|x_j\|^2)^2} x_j.
\]

It is straightforward to show that
\[
\frac{\partial^2 \tilde{\ell}}{\partial \mu \partial \mu'} = 2 \sum_{j} \frac{-(\sigma^2 + \|x_j\|^2)I + 2x_j x_j'}{(\sigma^2 + \|x_j\|^2)^2}.
\]

It follows from these results that, for \( t \in S^d \),
\[
\begin{align*}
t' \left\{ \frac{\partial^2 \tilde{\ell}}{\partial (\sigma, \mu') \partial (\sigma, \mu')} \right\} t = & \sum_{j} \frac{-4\|x_j\|^2 t_j^2 + 4\sigma^{-1}(\|x_j\|^2 - \sigma^2)y_j \cdot t_j - 2(\sigma^2 + \|x_j\|^2) + 4(\|x_j\|^2)}{(\sigma^2 + \|x_j\|^2)^2} \\
= & \sum_{j} \frac{-2\|x_j\|^2 \left( t_j - (2\sigma)^{-1}(\|x_j\|^2 - \sigma^2)(x_j \tilde{t}) \right)^2 + A_j}{(\sigma^2 + \|x_j\|^2)^2},
\end{align*}
\]

where \( A_j \) is a constant.
where \( t = (t_1, \ldots, t_{d+1})' \) and \( A_j \) can be expressed as

\[
A_j = \frac{(\|x_j\|^2 - \sigma^2)^2(t_j')}{2\|x_j\|^2\sigma^2} + 2(t_j')^2 - (\sigma^2 + \|x_j\|^2)
\]

\[
= \frac{(\|x_j\|^2 + \sigma^2)^2(t_j')^2 - 2\|x_j\|^2\sigma^2(\sigma^2 + \|x_j\|^2)}{2\|x_j\|^2\sigma^2}
\]

\[
\leq \frac{(\|x_j\|^2 + \sigma^2)^2\|x_j\|^2 - 2\|x_j\|^2\sigma^2(\sigma^2 + \|x_j\|^2)}{2\|x_j\|^2\sigma^2}
\]

\[
= \frac{(\|x_j\|^2 + \sigma^2)(\|x_j\|^2 - \sigma^2)}{2\sigma^2}.
\]

It follows from this result and (31) that

\[
\sum_j \left\{ \frac{\partial^2 \bar{\ell}}{\partial \sigma \partial \mu'} \right\} t_j \quad \leq \quad 2 \sum_j \frac{-2\|x_j\|^2 \left\{ t_1 - (2\sigma)^{-1}(\|x_j\|^2 - \sigma^2)(t_j') \right\}^2 + (2\sigma^2)^{-1}(\|x_j\|^2 + \sigma^2)(\|x_j\|^2 - \sigma^2)}{(\sigma^2 + \|x_j\|^2)^2}
\]

\[
\leq \quad \frac{1}{\sigma^2} \sum_j \left( 1 - \frac{2\sigma^2}{\sigma^2 + \|x_j\|^2} \right)
\]

\[
= \quad \frac{1}{\sigma^2} \left( n - 2 \cdot \frac{n}{2} \right)
\]

\[
= \quad 0.
\]

Both equalities in the two inequalities above hold if and only if \( x_1 = x_2 = \cdots = x_n \). Therefore the matrix \( \left\{ \frac{\partial^2 \bar{\ell}}{\partial \sigma \partial \mu'} \right\} \) is negative definite with probability one.

Next we consider the maximized likelihood function \( \lambda(\mu) \) defined by (32). The second derivate of \( \lambda(\mu) \) is given by

\[
\frac{\partial^2 \lambda}{\partial \mu \partial \mu'} = \frac{(\partial^2 \bar{\ell}/\partial \mu \partial \mu') - (\partial^2 \bar{\ell}/\partial \sigma \partial \mu)(\partial^2 \bar{\ell}/\partial \sigma \partial \mu')}{\partial^2 \bar{\ell}/\partial \sigma^2}.
\]  

(32)

We show that the second derivative is negative definite at every stationary point of \( (\mu', \sigma) \).
Clearly the denominator of the second derivative is of the right-hand side of (32) is negative.
Then it suffices to show that the numerator of the right-hand side of (32), namely,

\[
\left\{ -4 \sum_j \frac{\|x_j\|^2}{(\sigma^2 + \|x_j\|^2)} \right\} \left\{ -2 \sum_j \frac{\sigma^2 + \|x_j\|^2 - 2(t_j')^2}{(\sigma^2 + \|x_j\|^2)^2} \right\} - \frac{4}{\sigma^2} \left\{ \sum_j \frac{\|x_j\|^2 - \sigma^2}{(\sigma^2 + \|x_j\|^2)^2} (t_j')^2 \right\} ^2
\]

is positive for \( t \in S^d \). Note that the equation (31) implies that

\[
\sum_j \frac{\|x_j\|^2 - \sigma^2}{(\|x_j\|^2 + \sigma^2)^2} = -\frac{1}{2\sigma^2} \sum_j \frac{(\|x_j\|^2 - \sigma^2)^2}{(\|x_j\|^2 + \sigma^2)^2}.
\]

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It follows then that
\[
\sum_j \sigma^2 + \|\tau_j\|^2 - 2 \left( t' \tau_j \right)^2 \geq \sum_j \sigma^2 - \|\tau_j\|^2 \quad \sum_j \frac{\sigma^2 + \|\tau_j\|^2}{(\sigma^2 + \|\tau_j\|^2)}
\]
\[
= \frac{1}{2\sigma^2} \sum_j \frac{\|\tau_j\|^2 - \sigma^2}{(\|\tau_j\|^2 + \sigma^2)^2}.
\]

Then we have
\[
\left( -\frac{\partial^2 \ell}{\partial \sigma^2} \right) t' \frac{\partial^2 \lambda}{\partial \mu \partial \mu} l
\]
\[
= \frac{4}{\sigma^2} \left[ \left\{ \sum_j \frac{\|\tau_j\|^2}{(\sigma^2 + \|\tau_j\|^2)} \right\} \left\{ \sum_j \frac{\left( \|\tau_j\|^2 - \sigma^2 \right)^2}{(\|\tau_j\|^2 + \sigma^2)^2} \right\}
\]
\[
- \left\{ \sum_j \frac{\|\tau_j\|^2 - \sigma^2}{(\sigma^2 + \|\tau_j\|^2)^2} \right\} \left( t' \tau_j \right)^2 \right]
\]
\[
\geq 0.
\]

The inequality above is due to Schwarz’s inequality. Therefore \(\lambda(\mu)\) is negative at any stationary values.

The discussion in Section 5.5 related to the unimodality of the likelihood function is obtained by summarizing these results.

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