TIME EVOLUTION IN DYNAMICAL SPACETIMES

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ABSTRACT

We present a gauge–theoretical derivation of the notion of time, suitable to describe the Hamiltonian time evolution of gravitational systems. It is based on a nonlinear coset realization of the Poincaré group, implying the time component of the coframe to be invariant, and thus to represent a metric time. The unitary gauge fixing of the boosts gives rise to the foliation of spacetime along the time direction. The three suppressed degrees of freedom correspond to Goldstone–like fields, whereas the remaining time component is a Higgs–like boson.

1. Introduction

The notion of time is fundamental in classical Physics. The dynamical laws are understood to be the expressions of the evolution of any physical system in time. This becomes particularly evident in the Hamiltonian formulation of mechanics, since the Poisson bracket of any phase space variable with the Hamiltonian yields the time derivative. Further, Dirac’s standard quantization procedure by means of the correspondence principle rests on the Hamiltonian formalism. Thus, the requirement of disposing of a reasonable characterization of time is necessary for both, the classical and the quantum dynamical approaches. However, serious problems arise when one tries to formulate Gravitation as a Hamiltonian theory\(^{(1)}\) – as a first step to quantize it – due to the difficulty in defining a suitable general–relativistic time. Our main task will be to identify such a time with respect to which the Hamiltonian evolution of a gravitational system makes sense.

Which are the minimal features one should require from time? Rovelli\(^{(1)}\) has classified the main time properties as they manifest themselves in different physical theories. Since they are mutually contradictory, it becomes manifest that the notion of
time is not an unified one in Physics. But at least, we would like to have a criterion to judge which time features are more desirable to be maintained in the context of General Relativity. Several ones are characteristic for Newtonian mechanics and are lost in relativistic Physics. We can renounce to them without any trouble, notwithstanding the fact that we can rencounter them in particular circumstances. We mean mainly the uniqueness and the spatial globality, i.e. the possibility of measuring the same preferred time variable in all spatial points. Another characteristic of time, which is common to Newtonian and special–relativistic time, is to be external. In other words, it is supossed to exist separately, with independence of the dynamics. This point is particularly interesting. As we will see in the following, in the context of gravitational theories the status of time is different. When suitably identified, the time becomes internal, i.e. dynamically determined. Although this may constitute a source of difficulties, if one could solve them, then one had identified, from Gravity, the dynamical origin of the time variable appearing as external in any other physical theory.

The internal nature of time is thus an appealing feature we will have to deal with. However, there are other aspects of time which are more evident, and they should also be present in the physical time we are looking for. First of all, one expects the time to be one–dimensional. Further, it should be metrical, in order to make it possible to compare distinct time intervals to each other, and temporally global, i.e. such that every event goes through any value of the time variable once and only once. In addition to the sketched features that one expects from time as far as Gravitation is involved, the resulting time notion should be suitable to construct a consistent Hamiltonian formulation of the gravitational theory. Let us now look at the present situation in the development of this program.

In Newtonian mechanics, the topology of space and time is given by the Cartesian product $E^3 \times \mathbb{R}$ of the three–dimensional Euclidean space times the real line representing the time. The time intervals are invariant under Galilean transformations, and thus unique for any two events. Contrarily, the uniqueness of time is absent from relativistic Physics. In fact, in Special Relativity there exist a three parameter family of times, depending on the relative three–velocity, representable as lines filling the light cone.

The role played by time in General Relativity is considerably more confusing, due to the fact that the spacetime manifold is treated as a whole. As far as the field equations retain their original four–dimensional form, no problem seems to arise. But the general covariance of the theory avoids to identify a well defined separated time notion. In principle, only locally can one establish a soldering to the tangent spaces, locally representing inertial frames, in which the distinction between spatial and temporal directions reduces to that of Special Relativity. The time would then only posses a local meaning. Without an important additional assumption, the ideas of a cosmological time, of the age of the universe etc., i.e. all conceptions of time which presuppose its global nature, would not make any sense. The cosmological time normally used in Cosmology arises from particular solutions of the Einstein equations,
like the Friedmann–Robertson–Walker solution, in which the alluded assumption is implicitly included. We mean the following. From a general theoretical point of view, the only way to recover a global time notion in the context of General Relativity, similar in some extent to the classical or special–relativistic ones, is to perform a foliation of the four–dimensional spacetime manifold along a certain direction identifiable as the time direction itself. In order to do so, one has to impose the Frobenius foliation condition. It can be introduced in different manners, each of which leading to a particular time notion. We will analyze it in detail in the following.

For the moment, let us briefly review the three standard approaches to time in General Relativity\(^{(1)}\). All of them present theoretical difficulties which make them unsatisfactory as expressions of a well defined notion of physical time. In particular, Rovelli points out that none of them is applicable to a rigorous quantum treatment of Gravity. But their faults manifest themselves already at the classical level, as we will see immediately.

In the first place, the most naive attempt is to identify the time coordinate at any point as the general–relativistic time. This is the so called coordinate time. We mentioned above that such a time is, by its own nature, local. Further, in view of the general covariance of General Relativity, the coordinate time consists of an infinite–dimensional family of time lines which are arbitrarily rescalable. Accordingly, such a time lacks time metricity. The difference between two values of the time parameter is not really interpretable as a time interval, since no time metric is defined which guarantees this extent. On the other hand, when considered from the point of view of the gauge approach to Gravity\(^{(2)}\), it also lacks any symmetry under gauge transformations. We conclude that the coordinate time is to be disregarded as the true physical time.

A better candidate is the proper time along any time–like worldline. It has the advantage over the coordinate time of being metrical, thus allowing to determine which time intervals are of equal duration. In fact, it is invariant under arbitrary coordinate transformations. It is also time global in the sense that every event – in this case every solution of the field equations– passes through every value of the time parameter once an only once. In contrast to all pre–relativistic approaches to the notion of time, it is not external to the theory, i.e. it is not given \textit{a priori} but dynamically determined, since it depends on the whole 4×4 metric. A serious difficulty arises in the understanding of dynamics as evolution in this time variable which is itself subordinated to the dynamics. It is the following. We are confronted with the paradoxical fact that one has to formulate and solve the field equations in terms of the coordinate time, which has no metric properties, because proper time is still not at our disposal. Only after having solved the dynamical problem can proper time be defined. Consequently, the gravitational dynamics cannot be expressed as evolution with respect to the proper time variable possessing the right metric properties.

Let us look at the standard solution to this problem. It is given by a third kind of general–relativistic time, namely what is called the \textit{clock time}. Rovelli presents it as ”the only way of recovering a conventional Hamiltonian evolution in General
Relativity”. In fact, it constitutes an alternative to the previous dilemma between time as evolution parameter without metric properties (the coordinate time), and metric time (the proper time), not present at the level of the evolution equations. The clock time is the time measured by a physical clock, or more exactly, measured with respect to a dynamical variable, chosen as a clock. It allows to express the evolution of the whole system relatively to a time with metric properties. One has to consider General Relativity coupled to matter, and express the gravitational variables as functions of the matter ones. The role of physical time is then played by one of the (gauge invariant) degrees of freedom of the theory itself, standing for the clock. Being the latter a physical object, its rhythm depends on the field equations themselves. Thus it is an internal time. An example of clock time is provided by the radius of the universe, used as time variable in Cosmology. It is metrical; however, the temporal globality is absent from this clock time if the universe collapses. But the main difficulty with clock time is a more fundamental one. It has to do with its possible application to a Hamiltonian treatment. The clock time allows to recover a well defined time evolution in General Relativity, but only a parametrized evolution, not a genuine one. Unfortunately, parametrized Hamiltonian systems are not well understood as quantum systems.

The aim of the present paper is to give an answer to these troubles. We will develop an alternative notion of dynamical time which possesses the main features necessary to solve the problem of defining a standard Hamiltonian time evolution of Gravitation, and of any other system in the presence of Gravity. Further, if the gravitational effects are put off, the time to be introduced below remains as a consistent definition of time. It is an invariant time entering in a natural way the dynamical field equations. In some extent, it may be understood as a preferred clock time. In fact, it is a dynamical field of the theory, whose value is affected by the matter sources. The evolution of any physical system is evaluated with respect to it. However, it is not an arbitrarily chosen physical clock, but the time component of the coframe itself, suitably constructed to be Poincaré gauge invariant. Being invariant, it guarantees the time metricity, i.e. the comparability of time intervals. In the unitary gauge to be studied in detail in the following, the time field takes the form $\vartheta^0 = u^0 \, d\tau$, inducing a foliation of the spacetime along it.

The key to realize the program of constructing a complete characterization of the physical spacetime, including a time with the desired properties, is provided by a particular nonlinear realizations of a certain spacetime symmetry group, in particular the Poincaré group, acting on its own parameter space. The abstract group will be our unique departing point. We will not need to postulate additional mathematical structures to fulfil our scheme, such as a pre–dynamical spacetime manifold providing the coordinates, or Cartan’s repères mobiles representing the reference frames. No element exterior to the gauge group will be present. From the basic assumption that the Poincaré group is a fundamental physical symmetry, we will be able to derive simultaneously both, the differentiable coordinate manifold defining the topology of spacetime, and the dynamical fields attached to it, standing in particular for the coframes, curvature, etc.
The paper is organized as follows. In section 2, we briefly summarize the nonlinear coset realization procedure which constitutes the mathematical basis of the present work. Section 3 is devoted to the discussion of the origin of coordinates in the nonlinear approach, and some considerations are made about their physical meaning. In section 4, we apply the previous general results to the Poincaré group, which is realized nonlinearly in a particular way. As a result, a Poincaré invariant time component $\vartheta^0$ of the coframe arises, which is proposed as the candidate to play the role of the physical time. Further, in section 5 we discuss the Poincaré invariant spacetime foliation along $\vartheta^0$. After reviewing the main features of the unitary gauge in section 6, we devote section 7 to apply it in the context of Gravity, in such a way that we derive the previously studied invariant foliation condition as a result of suitably covariantly fixing the gauge.

2. Nonlinear coset realizations

The nonlinear coset approach was originally introduced by Coleman et al.\(^{(3)}\) in the context of internal symmetry groups. It was soon extended to spacetime symmetries\(^{(4)}\), and we have shown in several previous papers\(^{(5,6)}\) that it constitutes the natural framework to construct gauge theories of Gravity founded on different spacetime groups. The nonlinear realizations allow to define the coframes in terms of gauge fields. They are identified as the nonlinear connections of the translations\(^{(5,7)}\). The metric tensor does not play any dynamical role since the gravitational forces are carried exclusively by nonlinear gauge fields. Recently, we have proposed a Hamiltonian treatment of the Poincaré Gauge Theory of Gravitation\(^{(6)}\) based on a particular nonlinear realization of the Poincaré group, and we were able to derive the Einstein equations and the complete set of constraints of the theory, giving account of the Ashtekar approach. Since the departing point to all these results is constituted by the nonlinear realizations, here we will outline their essential features.

Let $G = \{g\}$ be a Lie group including a subgroup $H = \{h\}$ whose linear representations $\rho(h)$ are known, acting on functions $\psi$ belonging to a linear representation space of $H$. We distinguish between $G$ considered as a transformation group, and the group $G$ itself as a differentiable manifold. In order to define the nonlinear action of $G$ on its own group manifold, we characterize the latter as a principal fibre bundle $G(M, H)$ with base space $M = G/H$ and structure group $H$ as follows\(^{(8)}\).

Let the subgroup $H$ act freely on $G$ on the right, i.e. $\forall h \in H, \forall g \in G$, $R_h g := gh$. This action induces an equivalence relation between elements $g, g' \in G$, defined as

$$g' \equiv g \iff \exists h \in H \mid g' = R_h g , \quad (2.1)$$

which gives rise to a complete partition of the group manifold $G$ into equivalence classes $gH$, namely

$$gH := \{R_h g \mid g \in G, \forall h \in H\} . \quad (2.2)$$
The quotient space $G/H$ of $G$ by the equivalence relation induced by $H$ is taken to be the base manifold of the fibre bundle. Its elements are single representatives of each equivalence class. Since we deal with Lie groups, the elements of $G/H$ are characterized by continuous coset parameters, say $\xi$, playing the role of coordinates. We identify the canonical projection $\pi : G \to G/H$ of the fibre bundle to be the mapping from equivalent points $g$ and $R_h g$ to the same point $\xi \in G/H$. The equivalence class $\pi^{-1}(\xi) = g H$ of left cosets labeled by $\xi$ is called the fibre through $g$, and it is isomorphic to the structure group $H$. Since the latter is a subgroup of $G$ defined as

$$H := \{ h \epsilon G / \pi (R_h g) = \pi (g) , \forall g \epsilon G \} ,$$

(2.3)

alternative choices of the canonical projection allow to structurate the group manifold in different manners, each of which corresponding to a distinct structure group.

In brief, the group $G$ itself, considered as a manifold, is a differentiable principal fibre bundle $G (G/H, H)$ over the base manifold $G/H$ with structure group $H$. The fibre bundle has locally the topology of a direct product of the base space $G/H$ and the fibre $H$, in the sense that every point $\xi \in G/H$ has a neighborhood $U$ such that $\pi^{-1}(U)$ is isomorphic with $U \times H$, i.e. there exist a diffeomorphism $\chi : \pi^{-1}(U) \to U \times H$ such that $\chi (g) = (\pi (g), \varphi (g))$, where $\varphi$ satisfies the condition that $\varphi (R_h g) = R_h \varphi (g)$.

Let us now consider $G$ as a transformation group with elements parametrized as $g_t$. The left action

$$L_{g_t} g := g_t g$$

(2.4)

of $g_t \epsilon G$ on elements $g$ of the group manifold $G (G/H, H)$ constitutes the basis of the nonlinear coset realizations of the group. Let the orbit $\xi_t = L_{g_t} \xi_0$ be a curve through $\xi_0$ on $G/H$, and $\pi^{-1}(\xi_t)$ the fibres over $\xi_t$. We suppose that the projection $\pi$ is an intertwining map for the action of $g_t \epsilon G$ on $G$ and $G/H$, i.e.

$$L_{g_t} (\pi^{-1}(\xi_0)) = \pi^{-1}(\xi_t) .$$

(2.5)

Thus, the action of $G$ on the left moves from the fibre over $\xi_0$ to the fibre over $\xi_t$. Let us consider a family of sections $\{ \sigma (\xi) \} \subset \pi^{-1}(\xi)$ whose values on a given fibre are related by $H$ as $\sigma' (\xi_t) = R_h \sigma (\xi_t)$. The action of $G$ will in general change from a section to another, which is compatible with (2.5). Accordingly, we can decompose the total left action of $G$ on $\sigma (\xi_0)$ into a displacement along the section $\sigma (\xi)$ from $\sigma (\xi_0)$ to $\sigma (\xi_t)$ followed by a change along the fibre $\pi^{-1}(\xi_t)$ from $\sigma (\xi_t)$ to the new section $\sigma' (\xi_t) = R_{h(\xi_t)} \sigma (\xi_t)$, i.e.

$$L_{g_t} \sigma (\xi_0) = R_{h(g_t, \xi_0)} \sigma (\xi_t) .$$

(2.6)

In the following, we will also call the structure group $H$ the classification subgroup in order to maintain the terminology of previous papers$^{(5,6)}$. The fundamental theorem on nonlinear realizations, due to Coleman et al.$^{(3)}$ establishes that the elements $g_t$ of
the whole group $G$ considered in (2.6) act nonlinearly on the representation spaces of the classification subgroup $H$ according to

$$
\psi' = \rho (h (g_t, \xi_0)) \psi,
$$

where $\rho$, as mentioned above, is a linear representation of $H$ on the $\psi$ fields. Therefore, the action of the total group $G$ projects on the representations of the subgroup $H$ through the dependence of $h (g_t, \xi_0)$ in (2.7) on the group element $g_t$, as given by eq.(2.6). The group is realized on the couples $(\xi, \psi)$, and it reduces to the standard linear action for $H = G$.

The usual situation in Physics is that an independent spacetime differentiable manifold previous to the dynamics is postulated to exist. The remaining physical objects, including the geometrical post–topological structures, are constructed on it. In contrast, the nonlinear approach allows to derive everything, including the coordinate base manifold, from the symmetry group. This will become apparent later, when we deal with spacetime groups. The coordinates are associated to the translations, and they appear as parameters of the base space $G/H$, as far as a spacetime group $G$ including translations is taken to be the gauge group of Gravity. Thus, even the coordinate manifold is dynamically derivable in the nonlinear gauge approach to spacetime. Moreover, being the coordinates group parameters, the spacetime manifold is automatically differentiable.

The dynamical content of the physical spacetime, i.e. its post–topological structure, is provided by the connections, playing the role of gauge fields of a certain (spacetime) group. Our next step will be to introduce them in the nonlinear scheme. In terms of a suitable nonlinear connection $\Gamma$, it will be possible to define a covariant differential transforming like (2.7) under the local action of $G$. In order to facilitate calculations, let us rewrite (2.6) in the more explicit form

$$
g \sigma (\xi) = \sigma (\xi') h (g, \xi),
$$

with $g$ standing for $g_t$, and being $h (g, \xi)$ the elements of the classification subgroup $H$. The nonlinear connection relates to the ordinary linear one $\Omega$ as

$$
\Gamma = \sigma^{-1} (d + \Omega) \sigma.
$$

Since the linear connection $\Omega$ transforms as

$$
\Omega' = g \Omega g^{-1} + g d g^{-1},
$$

it is easy to check, making use of (2.8), that the nonlinear connection $\Gamma$ transforms as

$$
\Gamma' = h \Gamma h^{-1} + h d h^{-1}.
$$

The nonlinear covariant differential operator constructed in terms of (2.9) reads

$$
D := d + \Gamma.
$$
From (2.11) follows that only the components of $\Gamma$ involving the generators of $H$ behave as true connections, transforming inhomogeneously, whereas the remaining components transform as tensors with respect to the subgroup $H$, despite their nature of connections.

3. Some comments on the origin and physical meaning of coordinates

Topologically considered, the totality of events in space and time constitutes a four–dimensional manifold, since everything happens in a place at a certain instant. This fact was firstly pointed out by Minkowski\(^{(9)}\) in his early deduction of Special Relativity, alternative to that of Einstein. According to his terminology, we call any actual or possible event a worldpoint, and it is further assumed that the set of all worldpoints constitutes a differentiable manifold, so that we can assign locally to any worldpoint four coordinates $(x^0, x^1, x^2, x^3)$ of a suitable chart of $R^4$. The real coordinates characterize the continuity of the world, i.e. they describe a topological feature of the arena underlying any physical event. However, only under very particular assumptions do they become directly related to observable quantities. For instance, Cartesian coordinates posses in fact an immediate metrical meaning in classical and special relativistic mechanics. The three spatial (Cartesian) coordinates represent lengths identical with the Euclidean projections of a given event on rigid axes defined overall the space, whereas the time coordinate is measured by a clock at rest with respect to the spatial axes. Nevertheless, this is far from being the natural interpretation of coordinates in general. Moreover, the observable meaning of coordinates in classical mechanics and in Special Relativity is the result of a further theoretical development consisting in having introduced a rigid metric tensor in such a way that certain dynamical quantities, namely the coframes, become expressable directly in terms of the coordinates, see below.

As we have mentioned above, the nonlinear framework, when applied to a spacetime group including translations, suffices to yield a differentiable coordinate manifold, thus being unnecessary to postulate it \textit{a priori}, separately from the spacetime dynamics. To illustrate this point, let us consider the simple example of the affine group $A(4,R) = GL(4,R) \rtimes R^4$, which is the semidirect product of the general linear transformations and the translations. Its respective generators $\Lambda^{\alpha \beta}$ and $P_\alpha$ satisfy the commutation relations

\[
[\Lambda^{\alpha \beta}, \Lambda^{\mu \nu}] = i \left( \delta^{\alpha \nu} \Lambda^{\mu \beta} - \delta^{\mu \nu} \Lambda^{\alpha \beta} \right), \quad [\Lambda^{\alpha \beta}, P_\mu] = i \delta^\alpha_\mu P_\beta, \quad [P_\alpha, P_\beta] = 0. \quad (3.1)
\]

The (infinitesimal) group elements of the whole affine group $A(4,R)$ are parametrized as

\[
g = e^{i \epsilon^\alpha P_\alpha} e^{i \zeta_{\alpha \beta} \Lambda^{\alpha \beta}} \approx 1 + i \epsilon^\alpha P_\alpha + i \zeta_{\alpha \beta} \Lambda^{\alpha \beta}. \quad (3.2)
\]
We will realize the group action taking $H = GL(4, R)$ as the structure group. Accordingly, we choose its (infinitesimal) elements in (2.8) to be parametrized as

$$h = e^{i u_\alpha \Lambda^\alpha \beta} \approx 1 + i u_\alpha \Lambda^\alpha \beta. \quad (3.3)$$

The sections

$$\sigma = e^{-i x^\alpha P_\alpha} \quad (3.4)$$

depend on (finite) parameters $x^\alpha$ of the base space $A(4, R)/GL(4, R)$. Making use of the fundamental eq.(2.6), or equivalently of (2.8), after a little algebra we find the variation of $x^\alpha$ to be

$$\delta x^\alpha = -\zeta^\alpha_\beta x^\beta - e^\alpha, \quad u_\alpha \beta = \zeta^\alpha_\beta. \quad (3.5)$$

This shows that the parameters $x^\alpha$ associated to the translations behave in fact as coordinates. Although presented here in a particular example, the result is general.

What is the physical status one should ascribe to the underlying worldpoint manifold? Do it represent a certain ontological background –namely the spacetime– on which the events actually take place, or is it to be merely considered as a mathematical artefact? This question confronts us with the old Clarke–Leibniz disputation\(^{(10)}\), which may be translated into mathematical terms as follows.

First of all, we have to distinguish between the passive and the active interpretations of general coordinate transformations. A transformation is called passive if it represents a change from a local coordinate chart to another, whereas the described point $p$ of the manifold remains the same. Such a change is purely nominal. On the other hand, an active transformation or diffeomorphism is a bijective $C^\infty$ application between distinct open sets of the manifold, thus really moving from a point to a different one. Let us now see how both sorts of transformations are interpreted from the absolutistic and the relationalistic points of view respectively.

According to the absolute space (resp. spacetime) conception defended by Clarke, the topological worldpoint manifold constitutes a sort of actual receptacle on which the events are immerged. The points actually exist as the ultimate constituents of the space. Thus, the dependence of the physical variables on them shows that the events are actually attached to spacetime points. Accordingly, the active coordinate transformations are viewed as Essentially different from the passive ones. Although a diffeomorphism (in a diff–invariant formulation of any physical theory) preserves the reciprocal relations between events, so that no observable consequences arise, an absolutist would notwithstanding recognize the original and the transformed states as two actually different although empirically undistinguishable events, since they are distinctly located on the absolute spacetime manifold.

The relationalistic viewpoint represented by Leibniz rejects this interpretation. The points are necessary to represent the relative localizations of physical events, but neither are the events in fact attached to particular points, nor do the points really exist. The points behave as mathematical labels which allow to express spatial
relations between distinct physical variables. Nevertheless, the relations described in this way are independent from any actual points. The spatial relations and not the points of the manifold are the ultimate spatial reality. In fact, as accepted even by the absolutists, a diffeomorphism actively transforming the points involved in the description of a (diff–invariant) physical system leaves it undistinguishable from the original one. According to Leibniz’s principle of the identity of undiscernibles, both systems are identical. Thus, from the relational point of view, the empirically irrelevant diff–transformations are fictitious, *agendo nihil agere*. Not only nothing observable occurs; in fact nothing occurs absolutely. Since the spacetime points do not exist actually, whenever the relative configuration of the physical objects is not altered, the transformations from a point to another are viewed as a sort of renaming. Although mathematically different from the passive coordinate transformations, the diffeomorphisms are also considered by the relationalists as nominal changes in the sense that they *move* the physical objects through a fictitious (purely mathematical) background, not altering their real physical features.

Of course, the relationalistic interpretation is not directly readable out from the mathematical formulation. But here we are not discussing about the mathematical sintaxis of the theory but about its semantics. In this sense, the absolutistic point of view is closer to the litterality of the language employed to describe the spatial relations. In fact, it constitutes a naive realistic interpretation of the formalism. Everything which has a name (in this case the points) is supos ed to exist actually, despite its non observable nature. However, since we are concerned with Physics, we can forget the metaphysical belief in real points. We merely identify them as nominal labels whose role is that of functional arguments necessary to express relations, and we recognize the relations themselves as the only physical reality. Our nominialistic choice has the advantage of supressing non observable theoretical structures in order to deal exclusively with physically relevant objects. But at last, since the formalism (i.e. its sintaxis) and its possible predictions remain unaltered by both, the absolutistic and the relationalistic semantical interpretations, the reader is free to choose between them according to his particular taste.

The previous discussion concerns the local topology. When globally considered, the worldpoint manifold is meaningful as the domain of possible events. It tells, for instance, about the open or closed character of the universe, or about the linearity or –let us say– ciclicity of time. However, that is compatible with the fact that the points are not real by themselves. The physical irrelevance of the coordinates invites to use Cartan’s intrinsic formulation\(^\text{(11)}\), in which the geometrical inasmuch as the nongeometrical physical objects are represented in a coordinate independent manner. When expressed in the language of exterior calculus, any physical theory manifests itself as invariant under general coordinate transformations. Consequently, the differential forms are the natural mathematical objects to represent physical quantities without explicit reference to any attachement to the underlying manifold. We will make an extensive use of the intrinsic formulation in the following.
4. Nonlinear realizations of the Poincaré group, and invariant time

The physical group symmetries provide a criterion of objectivity. The objective reality is supposed to be represented by group invariants, since they are not affected by transformations leading from a reference frame to another. The covariant laws represent distinct perspectives on the (objective) group invariants. Thus, although the numerical results of the measurements depend on the reference frame, all of them describe a common reality. The choice of the suitable group is essential for any physical theory. In particular, in the dynamical approach to spacetime, the transformations under a suitable symmetry group will describe the true –i.e. relative– motions. Since the physical laws are covariant under such transformations, they describe the relative behavior of physical quantities without specifying any particular point of view. In this sense, those states which differ on an active transformation of the gauge group are physically equivalent.

In our approach to the dynamical theory of spacetime we will choose the Poincaré group as the physical spacetime symmetry. We do so for different reasons. We could appeal to Einstein’s synchronization principle relating local times, which requires to dispose of an invariant (objective) light velocity, common to all reference frames. The local validity of Special Relativity would then naturally lead to adopt the Poincaré group. But the main argument is the following. In the spirit of gauge theories, the gauge fields are derived from the local realization of the symmetry group of the sources. In particular, since the matter should determine the features of the spacetime to which it couples, it is natural to depart from the Poincaré group, since it is the classification group of the elementary particles. Certainly, we observe that this is not the only possible choice. In fact, we have shown\(^{(5)}\) that one can conciliate the existence of fermionic matter with the gauge theory of more general spacetime groups including the Poincaré group as a subgroup. But the simplest symmetry group with this feature is the Poincaré group itself. Thus, we choose it for simplicity, although the generalization to other groups, such as the affine group\(^{(12)}\), remains an open possibility which does not change the general result of the present paper on the interpretation of time.

The abstract Poincaré group \(P\) has the Minkowski metric \(\delta_{\alpha\beta}\) as its natural invariant, i.e. \(\delta_{\alpha\beta} = 0\). In addition to the choice of the symmetry group, it is important the way in which it is realized. Making use of the nonlinear procedure of section 2, the action of the group will be defined on its own parameter space. Accordingly, the spacetime manifold is provided by the Poincaré group \(P\) itself, considered as a differentiable principal fibre bundle \(P(M,H)\) over the base manifold \(M = P/\text{SO}(3)\) with structure group \(H = \text{SO}(3)\). This choice allows to single out the role of a Poincaré invariant time. In a single expression, the spacetime is represented by the mathematical object

\[
P(P/\text{SO}(3), \text{SO}(3)),
\] (4.1)
on which a nonlinear action of the Poincaré group is defined. Taking the connections into account, not only the topology, but the whole post-topological structure of spacetime is determined dynamically by the abstract gauge group. The nonlinear translational connections \( \vartheta^\alpha \) are responsible for the existence of coframes (being their dual vectors \( e_\alpha \) the reference frames), and the field strength \( R^\alpha_\beta \) of the Lorentz connections \( \Gamma^\alpha_\beta \) stands for the curvature, all of them depending on the material sources. Thus, the dynamics causes all what is physically observable about spacetime. It precedes even the topology and the kinematics. Nor the underlying manifold, neither the reference frames are given \textit{a priori}, previously to their dynamical definition from the Poincaré gauge group. The spacetime geometry is the natural interpretation of the gauge dynamics of a certain spacetime symmetry group. Since the theory of Gravity defines the dynamical spacetime, it provides the geometrical scenario for the remaining interactions. With respect to them—for instance in the context of electrodynamics, or in the standard model—, spacetime appears as externally given; but this is a consequence of having taken it from Gravitation, which explains its origin also in the absence of gravitational forces. The gauge theory of Gravity is the dynamical theory of spacetime. Relative to it, spacetime is necessarily internal, i.e. determined by field equations.

Let us now derive the main features of spacetime from the nonlinear gauge approach to the Poincaré group \( P \). Its Lorentz generators \( L^\alpha_\beta \) and the translational generators \( P^\alpha \) \((\alpha, \beta = 0, ... 3)\), satisfy the usual commutation relations as given in (A.1). In order to clarify the role played by the coframes in the nonlinear treatment of the Poincaré group, we will proceed in two steps. First we consider some aspects of the nonlinear theory with the Lorentz group as the structure group, and then we develop the theory we are here interested in, namely that with structure group \( H = SO(3) \). We do so because a simple relation between both realizations exists, which helps to understand the nature of the Poincaré invariance of time manifesting itself in the latter approach. Briefly, for \( H = \text{Lorentz} \) we choose

\[
g = e^{i \epsilon^\alpha P^\alpha} e^{i \beta^\alpha_\beta L^\alpha_\beta}, \quad \tilde{h} = e^{u^\alpha_\beta L^\alpha_\beta}, \quad \tilde{\sigma} = e^{-i \epsilon^\alpha P^\alpha},
\]

(4.2) to be substituted in (2.8). The tildes are introduced for later convenience. The nonlinear action yields the coordinate transformation

\[
\delta x^\alpha = -\beta^\beta_\alpha x^\beta - \epsilon^\alpha, \quad u^\alpha_\beta = \beta^\alpha_\beta.
\]

(4.3)

The ordinary linear Poincaré connection \( \Omega \) in (2.9), with values on the Lie algebra, reads

\[
\Omega := -i \Gamma^\alpha_\beta \alpha P^\alpha - i \Omega^\alpha_\beta L^\alpha_\beta,
\]

(4.4) defined on the base space \( P/SO(3) \) of coordinate parameters \( x^\alpha \) associated to the translations. It includes the translational and the Lorentz contributions \( \Gamma^\alpha_\beta \) and \( \Omega^\alpha_\beta \) respectively. In terms of (4.4), the nonlinear connection (2.9) reads

\[
\tilde{\Gamma} := \tilde{\sigma}^{-1} (d + \Omega) \tilde{\sigma} = -i \tilde{\beta}^\alpha P^\alpha - i \tilde{\Gamma}^\alpha_\beta L^\alpha_\beta.
\]

(4.5)
The translational nonlinear connections $\tilde{\vartheta}^\alpha$ in (4.5) are to be identified as the 1–form basis geometrically interpretable as the coframe\(^{(5–7)}\). From (4.2,4.5) we find

$$\tilde{\vartheta}^\alpha := (T)^\alpha + D x^a, \quad \tilde{\Gamma}^{\alpha\beta} = \Omega^{\alpha\beta}.$$  \hfill (4.6)

According to (2.11), whereas $\tilde{\Gamma}^{\alpha\beta}$ in (4.5) remains a true connection, the coframe $\tilde{\vartheta}^\alpha$ behaves as a Lorentz four–vector under local Poincaré transformations.

With these results at hand, we now proceed to realize the Poincaré group nonlinearly with its subgroup $H = SO(3)$ as the classification subgroup, as suggested by the Hamiltonian approach of Ref.(6). This alternative choice of the structure group automatically leads to the decomposition of the fourvector–valued coframe studied above into an $SO(3)$ triplet plus an $SO(3)$ singlet respectively, the singlet characterizing the time component of the coframe. The invariance of the time component of the coframe under $SO(3)$ transformations means in fact that it is Poincaré invariant.

Let us at the first place decompose the Lorentz generators into boosts $K_a$ and space rotations $S_a$, respectively defined as

$$K_a := 2 L_{a0} \quad , \quad S_a := -\epsilon^{bc}_{\ a} L_{bc} \quad (a = 1, 2, 3).$$  \hfill (4.7)

Their commutation relations are given in (A.5). The infinitesimal group elements of the whole Poincaré group become parametrized as

$$g = e^{i\epsilon^a P_a e^{i\beta^\alpha\beta L_{\alpha\beta}}} \approx 1 + i \left( \epsilon^0 P_0 + \epsilon^a P_a + \xi^a K_a + \theta^a S_a \right).$$  \hfill (4.8)

The difference with respect to the previous nonlinear realization given by the choice (4.2) consists in the distinct canonical projection we define in the group space. In other words, we now choose $SO(3)$ as the structure group of the Poincaré principal fibre bundle. Accordingly, the parametrization of the fibres and the sections is no more as in (4.2), but the following. The (infinitesimal) group elements of the structure group $SO(3)$ are taken to be

$$h = e^{i\Theta^a S_a} \approx 1 + i \Theta^a S_a,$$  \hfill (4.9)

and on the other hand

$$\sigma = e^{-i x^\alpha P_\alpha e^{i\lambda^a K_a}},$$  \hfill (4.10)

where $x^\alpha$ and $\lambda^a$ are the (finite) coset parameters.

According to (2.8) cum (4.8–10), the variation of the translational parameters reads

$$\delta x^0 = -\xi^a x_a - \epsilon^0,$$  \hfill (4.10)

$$\delta x^a = \epsilon^a_{\ bc} \theta^b x^c - \xi^a x^0 - \epsilon^a.$$

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which coincides exactly with (4.3) since \( \xi^a := \beta^{a0} \) and \( \theta^a := -\frac{1}{2} \epsilon^a_{bc} \beta^{bc} \), as read out from (4.8). Thus, the translational parameters still play the role of coordinates. In addition, we obtain the variations of the boost parameters of (4.10) as

\[
\delta\lambda^a = \epsilon^a_{bc} \theta^b \lambda^c + \xi^a |\lambda| \coth |\lambda| + \frac{\lambda^a \lambda^b \xi^b}{|\lambda|^2} (1 - |\lambda| \coth |\lambda|) ,
\]

being

\[
|\lambda| := \sqrt{\lambda^1^2 + \lambda^2^2 + \lambda^3^2} .
\]

The meaning of \( \lambda^a \) will be discussed later. On the other hand, according to (2.7), the infinitesimal action of the Poincaré group on arbitrary fields \( \psi \) of a given representation space of the \( SO(3) \) group reads

\[
\delta\psi = i \Theta^a \rho(S_a) \psi ,
\]

being \( \rho(S_a) \) an arbitrary representation of \( SO(3) \), and \( \Theta^a \) the nonlinear \( SO(3) \) parameter in (4.9), calculated from (2.8) to be

\[
\Theta^a = \theta^a + \epsilon^a_{bc} \frac{\lambda^b \xi^c}{|\lambda|} \tanh \left( \frac{|\lambda|}{2} \right) ,
\]

Let us now introduce the suitable gauge fields. In terms of the ordinary linear Poincaré connection (4.4), which may be rewritten as

\[
\Omega := -i \Gamma^0 \rho_a - i \Omega^a_{\alpha\beta} L_{\alpha\beta} = -i \Gamma^0 P_0 - i \Gamma^a P_a + i \Gamma^a K_a + i \Gamma^a S_a ,
\]

we define the nonlinear connection (2.9) as

\[
\Gamma := \sigma^{-1} (d + \Omega) \sigma = -i \theta^a P_a - i \Omega^a_{\alpha\beta} L_{\alpha\beta} = -i \theta^0 P_0 - i \theta^a P_a + i X^a K_a + i A^a S_a .
\]

The translational nonlinear connections \( \theta^0 \), \( \theta^a \) in (4.17) are the coframe components, whereas the vector–valued 1–forms \( X^a \equiv \Gamma^{0a} \) represent the gauge fields associated to the boosts; all of them vary as \( SO(3) \) tensors. Only \( A^a \equiv \frac{1}{2} \epsilon^a_{bc} \Gamma^{bc} \) behaves as an ordinary rotational connection. In fact, making use of (2.11) we find

\[
\begin{align*}
\delta \theta^0 &= 0 \\
\delta \theta^a &= \epsilon^a_{bc} \Theta^b \theta^c \\
\delta X^a &= \epsilon^a_{bc} \Theta^b X^c \\
\delta A^a &= -D \Theta^a := - \left( d \Theta^a + \epsilon^a_{bc} A^b \Theta^c \right) .
\end{align*}
\]

In addition, the trivial metric \( \delta_{ab} \) is a natural \( SO(3) \) invariant.
As we have repeatedly pointed out before, the time component \( \vartheta^0 \) of the coframe is invariant under local Poincaré transformations. Let us see how it happens. Making use of the four–dimensional representation (A.6,7) of the Lorentz group, the relation between the nonlinear coframe components in (4.17) and the Lorentz covector valued coframe in (4.6) may be expressed in the simple form

\[
\begin{pmatrix}
\vartheta^0 \\
\vartheta^a
\end{pmatrix} = e^{-i \lambda^a \rho(K_a)} \begin{pmatrix}
\tilde{\vartheta}^0 \\
\tilde{\vartheta}^a
\end{pmatrix},
\]

(4.19)

or, more explicitly

\[
\vartheta^0 = \tilde{\vartheta}^0 \cosh |\lambda| + \tilde{\vartheta}^a \frac{\lambda^a}{|\lambda|} \sinh |\lambda|,
\]

\[
\vartheta^a = \tilde{\vartheta}^a + \tilde{\vartheta}^b \frac{\lambda_b \lambda^a}{|\lambda|^2} (\cosh |\lambda| - 1) + \tilde{\vartheta}^0 \frac{\lambda^a}{|\lambda|} \sinh |\lambda|.
\]

(4.20)

The matrix \( e^{-i \lambda^a \rho(K_a)} \), see (A.7), performs a change of basis leading from the Lorentz covector–valued 1–forms in the r.h.s. of (4.19), varying linearly as

\[
\delta \begin{pmatrix}
\tilde{\vartheta}^0 \\
\tilde{\vartheta}^a
\end{pmatrix} = i \left[ \xi^a \rho(K_a) + \theta^a \rho(S_a) \right] \begin{pmatrix}
\tilde{\vartheta}^0 \\
\tilde{\vartheta}^a
\end{pmatrix},
\]

(4.21)

to the SO(3) quantities in the l.h.s., whose variations are specified in (4.18). In fact, taking into account the transformation properties of the coset parameter \( \lambda^a \) as given by (4.12), it is easy to verify how the nonlinear realization splits the four–dimensional representation into the SO(3) singlet \( \vartheta^0 \) plus the SO(3) triplet \( \vartheta^a \) respectively.

5. Poincaré invariant spacetime foliation.

The essential topological features of time are continuity and one–dimensionality. We will not discuss the former, which defines the vicinity of time instants in an obvious way, and neither the possible global topology of time, homeomorphic to the real line. We will be exclusively concerned with the problem of defining a one–dimensional time direction inside the original four–dimensional spacetime manifold. The possibility of recovering a suitable notion of time, which reduces to the usual one of Special Relativity in the absence of Gravity, should be a consequence of a particular foliation of the spacetime, in such a way that the resulting foliation direction makes physical sense and possesses several properties one expects from the time. In other words, according to Frobenius’ theorem\(^{(13)}\), one has to identify a certain 1–form, say \( u \), such that it satisfies the foliation condition

\[
u \wedge du = 0.
\]

(5.1)
From (5.1) follows that $u = N dt$, being $N$ the lapse function and $t$ the time parameter. The dual vector to $u$, denoted by $n$, is defined from $u$ by means of the relation $n \cdot u = 1$. It is a timelike vector field with the structure $n = \frac{1}{N} \left( \partial_t - N^A \partial_A \right)$, where $N^A$ stands for the shift functions and $\partial_A$ represent the derivatives with respect to the spatial coordinates. The time vector $n$ then defines a preferred orientation in the underlying manifold. Every value of the time parameter $t$ determines a spatial hypersurface which does not intersect the hypersurfaces corresponding to distinct time values, i.e. the spacetime is foliated into spatial sheets. Any arbitrary $p$–form $\alpha$ may thus be decomposed into a longitudinal part along the time direction plus a transversal part orthogonal to it as $\alpha = u \land (n \cdot \alpha) + n \land (u \land \alpha)$. When referred to the spatial hypersurfaces, we will call $\alpha_{\perp} := n \cdot \alpha$ the normal part and $\underline{\alpha} := n \land (u \land \alpha)$ the tangential part.

The foliability of the spacetime manifold is the fundamental requirement for a physical time to be well defined in the context of General Relativity or of other possible approaches to the theoretical description of Gravity. But this condition is by no means sufficient to uniquely characterize the time. In principle, the foliation performed with respect to the time direction $u$ is merely topological, and thus extrinsic to the dynamical aspects represented by the gauge fields. The 1–form $u$ satisfying the Frobenius’ condition (5.1) determines a topological property of the underlying manifold, but it is introduced by hand without being a priori identifiable with any dynamical object. One would expect, in a dynamical theory of spacetime, that time should be related to the metric tensor or to the vierbeine. In fact, let us consider the coframe (4.6) with linear Lorentz indices, or alternatively let us define, in the context of ordinary General Relativity, a vierbein $e^\alpha_i$ which solders the manifold to its tangential spaces at any point. The tangential spaces are Minkowskian, since Special Relativity is supposed to hold for locally inertial reference frames. Thus, the coframes defined as $\tilde{\vartheta}^\alpha := e^\alpha_i dx^i$ behave locally as Lorentz covectors. The following discussion holds for both approaches.

The time component $\tilde{\vartheta}^0$ of the coframe introduced above, what we will call the dynamical time, decomposes with respect to the topological time direction $u$ as $\tilde{\vartheta}^0 = u \land \tilde{\vartheta}^0_{\perp} + \tilde{\vartheta}^0_{\parallel}$, with a nonvanishing contribution transversal to $u$. A seemingly natural choice to define a unique physical time consists in aligning both, the topological and the dynamical time directions, by requiring $\tilde{\vartheta}^0_{\parallel} = 0$, or equivalently

$$u = \tilde{\vartheta}^0.$$  

(5.2)

The resulting coframe adapted foliation corresponds in fact to the so called time gauge introduced by Schwinger in the literature(14). Unfortunately, the assumption (5.2) breaks the local Lorentz symmetry of the theory. In fact, the foliation condition (5.1) involves not a covariant but an ordinary differential, so that it only remains an invariant condition if $u$ itself is invariant, which is not the case as far as $\tilde{\vartheta}^0$ transforms as the time component of a four–covector. Thus, apparently, the price one has to pay
to define a single physical time in the presence of Gravity is that one has to fix the time gauge, loosing the local covariance under Lorentz transformations.

Before presenting our own solution to this problem, let us summarize the desirable features one wishes to require from time. Fundamentally, one should identify a certain 1–form suitable to define a topological time direction on the underlying four-dimensional coordinate manifold, i.e. a 1–form $u$ on which one could impose the Frobenius’ foliation condition (5.1). Furthermore, the candidate to induce the spacetime foliation should preferably have the meaning of the dynamical time component $\tilde{\vartheta}^0$ of a coframe, as in (5.2), in order to define a single time with the topological and the dynamical time directions aligned, and thus interpretable as the unique physical time. On the other hand, if possible, one would expect to perform the foliation without breaking the gauge symmetry.

As discussed above, there exist neither absolute rest nor absolute motion on a topological spacetime manifold. Both, the spatial positions and motions are relative to physical references. This assert holds also for the time evolution. It cannot be merely characterized topologically, since the topological time is not directly observable. A physical evolution process has to be necessarily evaluated with respect to a physical clock, which allows to measure the relative rate of change. In our proposal, the time evolution will be referred to the natural time coframe $\vartheta^0$ in (4.17–20). The foliation condition becomes expressable in terms of dynamical objects, notwithstanding its topological nature, and its meaning and dynamical implications clarify the role played by time in Physics. The existence of the invariant time component of the coframe, see (4.18), enables us to perform an invariant foliation adapted to the nonlinear realization of the Poincaré group of previous section. The Frobenius’ foliation condition (5.1) takes the form

$$\vartheta^0 \land d \vartheta^0 = 0.$$ (5.3)

In view of (4.18a), eq.(5.3) is Poincaré invariant, thus defining an invariant foliation. Eq.(5.3) constitutes the integrability condition for $\vartheta^0$. From it follows

$$\vartheta^0 = u^0 d \tau,$$ (5.4)

with $\tau$ as a dynamical time parameter. Observe that it is absolutely different from the time coordinate. From the 1–form basis (4.19), we define its dual vector basis $e_\alpha$ such that $e_\alpha |\vartheta^\beta = \delta^\beta_\alpha$, and we identify $e_0$ as the invariant timelike vector field along which the foliation of the spacetime is defined. The Lie derivative of any arbitrary p–form $\alpha$ with respect to $e_0$ reads

$$L_{e_0} \alpha := d (e_0 |\alpha) + (e_0 | d \alpha),$$ (5.5)

representing the time evolution of $\alpha$. We remark that this evolution is not merely topological, but dynamical –being $\vartheta^0$ a gauge field– and with well defined time metricity, since $\vartheta^0$ and thus $e_0$ are invariant. This means that, on very general dynamical grounds, we have identified a physical clock time $\vartheta^0$, that is a dynamical field with
respect to which the time evolution of any system makes sense. It is meaningless to conceive $\vartheta^0$ as flowing itself. The "transcurse of time" is measured by the rate of change of any other field with respect to it. In particular, $e_0\lrcorner \vartheta^0 = 1$ is the generalization of the fact that $dt/dt = 1$ in Newtonian mechanics, so that the rate of change of time relative to itself is trivially constant and positive.

Further, $\alpha$ admits a decomposition into a longitudinal and a transversal part with respect to the invariant vector field $e_0$, namely

$$\alpha = \vartheta^0 \wedge \alpha_\perp + \bar{\alpha}, \quad (5.6)$$

with $\alpha_\perp$ and $\bar{\alpha}$ respectively defined as

$$\alpha_\perp := e_0\lrcorner \alpha, \quad \bar{\alpha} := e_0\lrcorner (\vartheta^0 \wedge \alpha). \quad (5.7)$$

Taking (5.5,6) into account, the decomposition the exterior differential of $\alpha$ reads

$$d\alpha = \vartheta^0 \wedge \left[ l_{e_0} \alpha - \frac{1}{u^0} d (u^0 \alpha_\perp) \right] + d\bar{\alpha}. \quad (5.8)$$

The invariance under time reparametrizations in the Hamiltonian approach is related to the arbitrariness in the definition (5.7) of the components of the longitudinal parts of differential forms. But we will not develop this point here.

The differential forms may describe spatial motions, but they are in fact relative ones, regulated by a certain gauge symmetry group. The physically meaningful relative behavior is expressed in terms of the coframes $\vartheta^\alpha$. These (nonlinear) gauge fields are 1–forms, i.e. intrinsic objects not attached to absolute points. Their dual vector fields $e_\alpha$ represent the reference frames, and the relative four–velocity $\tilde{u}^\alpha$ of a different coframe $\tilde{\vartheta}^\alpha$ with respect to $\vartheta^\alpha$ may be intrinsically defined to be proportional to $e_0\lrcorner \tilde{\vartheta}^\alpha$. Thus, the coframes give account of both, the relative positions and velocities.

Instead, as discussed above, the coordinates are mathematical artefacts which in general do not posses any objective meaning. In a covariant formulation, coordinate differences cannot be directly measured with the unit length. Nevertheless, under certain assumptions, in particular in the absence of gravitational effects, the differentials of the coordinates become identifyable with the Lorentz linear coframes (4.6a) themselves as

$$\tilde{\vartheta}^\alpha = \delta^\alpha_i dx^i, \quad (5.9)$$

so that they coincide with dynamical quantities. The trivialization (5.9) of the coframes yields a correspondence between coordinates and measurable quantities, as in Newtonian mechanics. However, considered from our point of view, also in this case do the coordinates be of gauge theoretical origin, since they still are parameters of the base space $G/H$ of the dynamical gauge theory of spacetime. Thus, even the special–relativistic kinematics is inseparable from the spacetime dynamics. We can
interpret the particular case (5.9) as the origin of the coordinates of Special Relativity, holding when Gravity is negligible. Moreover, let us see how, in the absence of Gravitation, the invariant time (4.20a) reduces to the proper time of Special Relativity. According to (5.9), we identify \( \tilde{\vartheta}^0 = dx^0 = c \, dt \), \( \tilde{\vartheta}^a = dx^a \). Substituting these values in (4.20a), from Fermat’s principle \( \delta \int \vartheta^0 = 0 \) we get

\[
\lambda^a = -\frac{v^a}{|v|} \arctanh \left( \frac{v}{c} \right).
\]  

(5.10)

Taking this value for \( \lambda^a \), the invariant time component of the coframe in (4.20) reduces to

\[
\vartheta^0 = c \, dt \sqrt{1 - \frac{v^2}{c^2}},
\]  

(5.11)

that is, \( c \) times the proper time, and \( \vartheta^a \) vanishes.

6. Nonlinear realizations and unitary gauge

The physically relevant fields of a dynamical theory do in general not coincide with the original degrees of freedom present in the action. To calculate the number of dynamical fields, one has to subtract the number of constraints plus the order of the symmetry group involved. Thus, in a gauge theory it is necessary to identify the complete set of constraints and fix the gauge, in order to deal only with physical degrees of freedom. The interested reader can find a detailed derivation of the former for the Poincaré Gauge Theory in Ref.(6). With respect to the gauge fixing, at least two different approaches are possible. At the first place, the ordinary gauge fixing procedure consists in establishing conditions on the fields, breaking the gauge symmetry. We will not enter technical details, but we mention that the gauge fixing should be performed after renormalization, since several critical phenomena are associated to the propagation of non–physical degrees of freedom. The second method we want to mention is the unitary gauge fixing.

The unitary gauge procedure makes use of the symmetry properties to covariantly eliminate the non–physical degrees of freedom of the theory. The fields eliminable by means of a suitable symmetry transformation are the Goldstone bosons, which are isomorphic to the group parameters. They are non–physical, and they become gauged away, embedded in the remaining fields of the theory. We point out the similitude between the absorption of the Goldstone fields in the unitary gauge mechanism, and that of the coset parameters of the nonlinear realizations, which do not explicitly appear in the theory since they are embedded in the nonlinear fields. In fact, the nonlinear approach provides the natural language to deal with the unitary gauge procedure. Let us illustrate the relation between both by examining the unitary gauge, as it commonly appears in the standard model. Accordingly, we briefly outline the nonlinear gauge approach to \( SU(2) \otimes U(1) \).
The generators $T_a, Y$, of $SU(2)$ and $U(1)$ respectively, satisfy the standard commutation relations, and the linear connection of the group reads

$$\Omega := -ig A^a T_a - ig' BY.$$  \hfill (6.1)

For further convenience we redefine the generators as

$$T^+ := \frac{1}{\sqrt{2}} (T_1 + iT_2), \quad T^- := \frac{1}{\sqrt{2}} (T_1 - iT_2), \quad Q := T_3 + \frac{Y}{2}. \hfill (6.2)$$

The generators $Q$ are those of the electromagnetic $U(1)_{el}$ group. We will perform the nonlinear realization with this group as the structure group. Thus, we apply the general formula (2.8) with the particular choices

$$g = e^{i(\epsilon^+ T^++\epsilon^- T^- + \epsilon^0 T_3 + y Q)}, \quad h = e^{i\lambda Q}, \quad \sigma = e^{i\lambda^+ T^+} e^{i\lambda^- T^-} e^{i\lambda^0 T_3}. \hfill (6.3)$$

The action (2.7) of the whole group on arbitrary fields $\psi$ of a given representation space of the classification subgroup $U(1)_{el}$ reads infinitesimally

$$\delta\psi = i\lambda \rho(Q) \psi,$$  \hfill (6.4)

being $\lambda$ the nonlinear $U(1)_{el}$ parameter, and $\rho(Q)$ a suitable representation of the $U(1)_{el}$ group. Let us show how the fields $\psi$ in (6.4), characteristic for the nonlinear approach, relate to the standard fields, say $\phi$, of the linear $SU(2) \otimes U(1)$ theory. We take in particular $\phi$ to be a complex doublet such that $\phi \phi^\dagger = \chi^2$. Its four degrees of freedom can be rearranged as follows. Let us take the corresponding $2 \times 2$ representation $T_3 = \frac{1}{2} \sigma_3, Y = I$. According to (6.2c) we get $Q = \begin{pmatrix} 10 \\ 00 \end{pmatrix}$, and thus

$$\begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

is an $U(1)_{el}$ scalar. Thus we can parametrize the linear field as

$$\phi = e^{i\xi^+ T^+} e^{i\xi^- T^-} e^{i\xi^0 T_3} \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \hfill (6.5)$$

The reader will recognize in (6.5) the usual parametrization of the Higgs multiplet with $\xi^+, \xi^-, \xi^0$ as the Goldstone bosons. The formulation of the theory in the unitary gauge requires to perform a transformation with group parameters chosen to be functions of the fields of the theory in such a way that they cancel out the physically superfluous degrees of freedom. This is equivalent to realize the theory nonlinearly with suitably chosen field-dependent coset parameters. In fact, the nonlinear fields in (6.4) relate to the linear ones as

$$\psi = \sigma^{-1} \phi,$$  \hfill (6.6)
with $\sigma^{-1}$ the inverse of $\sigma$ in (6.3c). Thus, in the present case it suffices to choose the coset parameters $\lambda^+, \lambda^-, \lambda^0$ of $\sigma$ to be respectively equal to the degrees of freedom $\xi^+, \xi^-, \xi^0$, present in (6.5), to get

$$\psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

(6.7)

in the unitary gauge. The non eliminable field $\chi$ in (6.7) is the Higgs field, which remains as the only physical degree of freedom. Simultaneously, one has to transform the linear connection (6.1) into

$$\Gamma := \sigma^{-1}(d + \Omega)\sigma$$

$$= -ig\left(\mathbf{W}^+ T^+ + \mathbf{W}^- T^-\right) + \frac{ig}{\cos \theta_w} \mathbf{Z}\left(T_3 - \sin^2 \theta_w Q\right) - ie A Q,$$

(6.8)

compare (6.8) with (2.9), being

$$\theta_w := \arctan \frac{g'}{g}, \quad e := \frac{gg'}{\sqrt{g^2 + g'^2}},$$

(6.9)

with the effective gauge fields suitably defined in terms of the linear ones (6.1) and the Goldstone bosons. Since those are equal to the coset parameters, they disappear as explicit degrees of freedom, embedded in the redefined vector fields, whose variations read

$$\delta \mathbf{W}^+ = i \lambda^+ \mathbf{W}^+, \quad \delta \mathbf{W}^- = -i \lambda^- \mathbf{W}^-, \quad \delta \mathbf{Z} = 0, \quad \delta A = \frac{1}{e} d \lambda.$$  

(6.10)

The remarkable fact is that the unitary gauge procedure consists in performing a particular transformation from the linear to a nonlinear realization of the gauge group. The tensorial character of $\mathbf{W}^+$, $\mathbf{W}^-$ and $\mathbf{Z}$ is a result of the nonlinear realization. We point out the analogy between eqs. (6.6) and (4.19), to which we will return below. The main difference resides in that the former relates a nonlinear realization to the linear one, whereas the latter establishes a relation between two different nonlinear realizations, since the translations are in both cases nonlinearly treated.

The unitary gauge fixing may be total or partial. The total one corresponds to the choice of the structure group to be $H = I$, which implies that all the group parameters are treated as Goldstone fields, and subsequently supressed as dynamical fields. On the other hand, the role of the partial unitary gauge fixing is that of restricting the number of degrees of freedom by eliminating those not corresponding to the structure group $H$. The connections and linear representations of $H$ remain unaltered. Since the gauge is fixed covariantly, the resulting nonlinear theory is formally identical to the linear one, but being the Goldstone fields absent, it depends on a fewer number of degrees of freedom. Those associated to the group parameters of $H$ may be fixed by a subsequent symmetry breaking. As in the ordinary gauge fixing, the unitary gauge
is to be introduced after the renormalization of the theory, in such a way that it does not affect the quantization. Let us now apply the unitary gauge fixing procedure to the gauge theory of spacetime outlined above.

7. The unitary gauge in Gravitation

The time component \( \vartheta^0 \) of the coframe is trivially longitudinal with respect to itself. Contrarily, \( d\vartheta^0 \) presents in principle a nonvanishing contribution transversal to \( \vartheta^0 \), such that in general \( \vartheta^0 \wedge d\vartheta^0 \neq 0 \), in disagreement with the integrability condition of \( \vartheta^0 \) represented by the Frobenius foliation condition (5.3). Thus, the presence of the transversal degrees of freedom of \( d\vartheta^0 \) in a dynamical theory would constitute a topological obstruction to the integrability of the evolution equations. The unitary gauge provides the method to eliminate the transversal degrees of freedom of \( d\vartheta^0 \) as Goldstone bosons, and thus to guarantee both, the foliation of the spacetime and the integrability of the dynamical equations.

As pointed out in the previous section, in order to fix the unitary gauge we have to choose the group parameters involved in a general nonlinear realization in such a way that they coincide with suitable functions of the fields, capable to cancel them out as non physical Goldstones. In the present case, the choice of the group parameters will be somewhat more complicated as in the example considered above, where we simply took them to be equal to the superfluous fields. Here we have to choose the \( \lambda^a \)'s in (4.19) to be certain functions of the Lorentz linear coframes in the r.h.s. of (4.19) itself in order to fix the unitary gauge in such a way that it supresses the Goldstone fields associated to the boosts. We make use of a 1–form \( \mu d\rho \) to express the dependence on the coframes \( \tilde{\vartheta}^\alpha \), which are represented by their dual vectors \( \tilde{e}_\alpha \) acting on it. For further convenience, we use the notation \( \mu d\rho \equiv u^0 d\tau \). As we will see below, this 1–form corresponds to the non eliminable Higgs–like field leaved by the unitary gauge fixing. At the first place, we define a three–velocity \( v^a \) such that

\[
\frac{v_a}{|v|} := -\frac{(\tilde{e}_a | u^0 d\tau)}{\sqrt{(\tilde{e}_0 | u^0 d\tau)^2 - 1}}. \tag{7.1}
\]

Despite it depends on the four vector fields involved, it is subjected to the constraint

\[
(\tilde{e}_a | u^0 d\tau)^2 - (\tilde{e}_a | u^0 d\tau)(\tilde{e}_a | u^0 d\tau) = 1, \tag{7.2}
\]

so that only three are in fact relevant. Now we choose the boost parameters \( \lambda^a \) in (4.19) to be

\[
\lambda^a = -\frac{v^a}{|v|} \text{arctanh} \sqrt{1 - 1/ (\tilde{e}_a | u^0 d\tau)^2}. \tag{7.3}
\]
This value of $\lambda^a$ fixes the unitary gauge. Let us see it in some detail. From (7.3) follows

$$|\lambda| = \arctanh \sqrt{1 - 1/ (\tilde{e}_o | u^0 d\tau)^2},$$

(7.4)

which yields

$$\cosh |\lambda| = (\tilde{e}_o | u^0 d\tau).$$

(7.5)

On the other hand, substituting (7.4) in (7.3) we get

$$\frac{\lambda^a}{|\lambda|} = -\frac{v^a}{|v|}.$$  

(7.6)

Making then use of definition (7.1) cum (7.5), one obtains

$$\frac{\lambda^a}{|\lambda|} \sinh |\lambda| = (\tilde{e}_a | u^0 d\tau).$$

(7.7)

Eqs.(7.5,7) as derived from (7.3) lead to the main result we were looking for. In fact, substituting them in (4.20), one obtains

$$\vartheta^0 = \tilde{\vartheta}^0 (\tilde{e}_\alpha | u^0 d\tau) = u^0 d\tau,$$

$$\vartheta^a = \tilde{\vartheta}^a + \tilde{\vartheta}^b \frac{\epsilon_{ab} v^a}{|v|^2} \left[(\tilde{e}_o | u^0 d\tau) - 1\right] + \tilde{\vartheta}^0 (\tilde{e}_a | u^0 d\tau).$$

(7.8)

The time component depends no more on four, but only on one degree of freedom. The unitary gauge leaves a unique Higgs–like time field, satisfying the Frobenius foliation condition (5.3). The remaining three degrees of freedom, associated to the boosts and corresponding to the transversal part of $d\vartheta^0$, have been gauged away as Goldstone fields. Moreover, since $\vartheta^0 = u^0 d\tau$ is invariant, being the gauge fixing condition (7.3) equivalent to the Frobenius invariant foliation condition (5.3), the right transformation properties of (7.3), namely (4.12), are guaranteed, as can be checked by explicit calculation.

In a rigorous treatment of Gravity, the unitary gauge should be fixed after renormalization –if renormalization is possible at all. The reason is that the transversal degrees of freedom of $d\vartheta^0$, eliminated as Goldstone bosons, could propagate (like the Goldstones in the standard model), and thus play a role in the quantum approach.

In our repeatedly cited Ref.(6), we have developed a Hamiltonian formalism adapted to the Poincaré Gauge Theory of Gravitation. There, the Frobenius foliation condition was introduced by hand without further justification. However, the discussion of the present paper on this subject helps understanding the meaning of such assumption. The Hamiltonian equations found by us(6) are the Einstein ones in the unitary gauge, where the Goldstone fields associated to the boosts are absent. The spacetime is foliated, and consequently the evolution equations are integrable. In the original theory both, first class constraints corresponding to the symmetries
and second class constraints, are present. After totally fixing the gauge, i.e., once we break the residual $SO(3)$ symmetry, only second class constraints will remain. This essential fact should be taken into account in order to quantize Gravitation.

8. Conclusions

We studied the gauge–theoretical foundations of the dynamical theory of spacetime. The Poincaré group $P$ realized nonlinearly on its own parameter space, structured as a principal fibre bundle $P(P/SO(3),SO(3))$, is the unique axiomatic assumption we need to derive the main features of spacetime in relativistic Physics. The differentiable manifold, the dynamical coframes and the Lorentz connections are all derived in a deductive way. The special role played by the structure group $SO(3)$ has to do with the existence of time. In fact, the dynamical time is represented by the Poincaré invariant $SO(3)$ singlet $\vartheta^0$. The unitary gauge fixing of the boost symmetry, which cancels out the Goldstone fields corresponding to the transversal degrees of freedom of $d\vartheta^0$, gives rise to a foliation of spacetime along the integrable $\vartheta^0 = u^0 d\tau$ direction. The time evolution is defined as the Lie derivative along the invariant time–like vector field $e_0$, dual to $\vartheta^0$. The Hamiltonian evolution equations of Gravity derived by us in a previous paper are to be understood as the Einstein equations in the unitary gauge which guarantees their integrability.

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APPENDIX

B.–The Poincaré group in terms of boosts, rotations and space and time translations

In the fourdimensional notation, the Lorentz generators $L_{\alpha\beta}$ and the translational generators $P_\alpha$ ($\alpha, \beta = 0...3$) of the Poincaré group satisfy the commutation relations

$$
[L_{\alpha\beta}, L_{\mu\nu}] = -i \left( o_{\alpha[\mu} L_{\nu]\beta} - o_{\beta[\mu} L_{\nu]\alpha} \right),
$$

$$
[L_{\alpha\beta}, P_\mu] = i o_{\mu[\alpha} P_{\beta]},
$$

$$
[P_\alpha, P_\beta] = 0.
$$

(A.1)
We choose the invariant metric tensor to be
\[ o_{\alpha\beta} := \text{diag}(− + + +) , \]  
and we define
\[ S_a := −\epsilon_a{}^{bc} L_{bc} , \]  
\[ K_a := 2 L_{a0} , \]
with \( a, b \) running from 1 to 3. The generators (A.3) are those of the \( SO(3) \) group, and (A.4) correspond to the boosts. In terms of them, and taking (A.2) into account, the commutation relations (A.1) transform into
\[
[S_a, S_b] = -i \epsilon_{ab}{}^c S_c ,
\]  
\[ [K_a, K_b] = i \epsilon_{ab}{}^c S_c ,
\]  
\[ [S_a, K_b] = -i \epsilon_{ab}{}^c K_c ,
\]  
\[ [S_a, P_0] = 0 ,
\]  
\[ [S_a, P_b] = -i \epsilon_{ab}{}^c P_c ,
\]  
\[ [K_a, P_0] = i P_a ,
\]  
\[ [K_a, P_b] = i \delta_{ab} P_0 ,
\]  
\[ [P_a, P_b] = [P_a, P_0] = [P_0, P_0] = 0 .
\]

In terms of the 4–dimensional representation of the Lorentz generators given by
\[
\rho(S_1) := -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \rho(S_2) := -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \rho(S_3) := -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
\[
\rho(K_1) := i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(K_2) := i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(K_3) := i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

we calculate the matrix
\[
e^{-i \lambda^a \rho(K_a)} = 1 - i \frac{\lambda^a}{|\lambda|} \rho(K_a) \sinh |\lambda| - \frac{\lambda^a \lambda^b}{|\lambda|^2} \rho(K_a) \rho(K_b) (\cosh |\lambda| - 1)
\]
\[
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ \lambda_3 & 0 & 0 & 0 \end{pmatrix} \frac{\sinh |\lambda|}{|\lambda|} + \begin{pmatrix} |\lambda|^2 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & \lambda_1 \lambda_2 & \lambda_1 \lambda_3 \\ 0 & \lambda_1 \lambda_2 & \lambda_2^2 & \lambda_2 \lambda_3 \\ 0 & \lambda_1 \lambda_3 & \lambda_2 \lambda_3 & \lambda_3^2 \end{pmatrix} \frac{\cosh |\lambda| - 1}{|\lambda|^2},
\]

which is extensively used in section 4.
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