Thermo–hydrodynamics
As a Field Theory

Jacek Jezierski
Department of Mathematical Methods in Physics,
University of Warsaw, ul. Hoża 74, 00-682 Warsaw, Poland

Jerzy Kijowski
Institute for Theoretical Physics, Polish Academy of Sciences,
Aleja Lotników 32/46, 02-668 Warsaw, Poland

Abstract

The field theoretical description of thermo–hydrodynamics is given. It is based on the duality between the physical space–time and the “material space–time” which we construct here. The material space appearing in a natural way in the canonical formulation of the hydrodynamics is completed with a material time playing role of the field potential for temperature. Both Lagrangian and Hamiltonian formulations, the canonical structure, Poisson bracket, Nöther theorem and conservation laws are discussed.

1 Introduction

The simplest way to recognize the canonical structure of the electrodynamics consists in introducing the electrodynamical potentials. This way the homogeneous Maxwell equations become automatically satisfied and non–homogeneous ones can be interpreted as 2-nd order differential equations for the potentials. The equations can be derived from the first order, non–degenerate variational principle. This makes the life easy: the conservations laws can be deduced from the Nöther theorem and the canonical structure is
immediately given with “Dirac delta” Poisson brackets between the potentials and their canonical momenta. The entire system becomes in a natural way an infinite dimensional hamiltonian system. Although it is possible to reduce the above structure rewriting it in terms of the electrodynamic fields only, without the use of potentials, but the resulting structure of the “non–canonical” Poisson brackets is very complicated and – for some applications – useless.

The situation in hydrodynamics is very much similar. Introducing the material space and describing consequently the configuration of the fluid as a diffeomorphism between the material space and the physical space makes the theory formally similar to the electrodynamics. The matter continuity equation is automatically satisfied and the Euler equations can be interpreted as the second order differential equations for the field potentials. The equations can be derived from the first order, non–degenerate variational principle (see[6]). The corresponding hamiltonian theory is based on the canonical (“Dirac delta”) Poisson structure.

It is very interesting that also the thermodynamics can be formulated this way. We introduce the potential for the temperature which can be interpreted as a “material time” (see[7]) and define the temperature as a ratio between the two different times (the material one and the physical one). This way the thermo–hydrodynamics becomes a lagrangian field theory which fits also very well into the framework of the infinite dimensional hamiltonian systems.

Having at our disposal the two time variables, we are free to choose one of them as a independent variable of the theory which parameterizes the physical events, and the other one as a field potential for the temperature. These two choices correspond to the “energy picture” and the “entropy picture” in thermodynamics. The transformation between the two pictures exchanges the role of the momenta canonically conjugate to the two “time variables”, i.e. between the energy and the entropy.

The above purely phenomenological approach has also a nice microscopic interpretation which we give at the end of the paper. Some applications to the non–conservative phenomena can be found in [5].
2 Barotropic fluid: Lagrange picture

We assume that the collection of all the points of the material can be organized in a smooth, 3–dimensional differential manifold $Y$ which we call the material space. The points of $Y$ label the particles of the material. We introduce coordinates $y^a (a = 1, 2, 3)$ in the material space. Together with the physical time $t$ parameterizing the time axis $U$ they form the independent variables of our theory. It is useful to denote $y^0 := t$ and to consider coordinates $(y^a)$, with $\alpha = 0, 1, 2, 3$, in the space $Y \times U$ of independent variables of the theory.

To describe different types of materials we have to equip the material space $Y$ with different geometric structures. For example, the elastodynamics needs the metric (riemannian) structure in $Y$. For purposes of hydrodynamics, however, the only structure we need is the volume structure. This means that there is in $Y$ a differential 3–form (a scalar density) $r$ which measures the quantity of the fluid (e.g. in moles) when integrated over regular domains of $Y$. In terms of coordinates $y^a$ the volume form $r$ can be written as follows:

$$ r = h(y)dy^1 \wedge dy^2 \wedge dy^3, \quad (1) $$

where the function $h = h(y)$ is given a priori. If coordinates $y^a$ are dimensionless then $h$ has the dimension of moles. The density $h$ transforms simply when we pass from one coordinate system in $Y$ to another: it is multiplied by the Jacobian of the transformation. Of course, it is always possible to change coordinates in such a way that the numerical value of $h$ equals 1. Such coordinates will be called unimodular and we will always use them for the sake of simplicity. Such a formulation carries however a danger: some expressions have to be multiplied by $h$ which is equal to one mole. This is very easy to forget. In order to have $h$ equal to dimensionless one we will always choose dimension $\sqrt{\text{mole}}$ for the material space coordinates $y^a$. This way the factor $h$ can really be forgotten.

To describe the configuration of the material at a given instant of time $t$ we assign the position in the physical space $X$ to each particle. This means that we have a diffeomorphism

$$ F : Y \longrightarrow X \quad (2) $$

describing the configuration of the material at the time $t$. The dynamical
history of the material is given by the one parameter family $F$ of configurations:

$$F : Y \times U \mapsto X.$$ 

If the physical space is parameterized by space coordinates $x^k$ having the dimension of length, then the history can be described in terms of 3 functions depending on four parameters:

$$x^k = x^k(t, y^a) = x^k(y^a).$$

These are field variables of the theory. The laws of hydrodynamics will be formulated in terms of the second order partial differential equations for the fields. For this purpose we need to express hydrodynamic quantities (the velocity and the density of the fluid) in terms of the first derivatives of the fields. We denote

$$x^k_\alpha := \frac{\partial x^k}{\partial y^\alpha}.$$ 

The velocity vector $v$ has components

$$v^k = \frac{\partial x^k}{\partial t} = \dot{x}^k = x^k_0. \quad (3)$$

To express the density $\rho$ of the fluid we calculate the transport of the volume form $r$ from the material space to the physical space. This means that in formula (1) we have to substitute $y^a$ as functions of $x^k$, using the inverse $F^{-1} : X \mapsto Y$ of the configuration (2). Denoting

$$y^a_k = \frac{\partial y^a}{\partial x^k},$$

where $y^a = y^a(x^k)$ is the coordinate expression for the mapping $F^{-1}$, we get:

$$(F^{-1})^*(hdy^1 \wedge dy^2 \wedge dy^3) = h \det(y^a_k)dx^1 \wedge dx^2 \wedge dx^3. \quad (4)$$

We conclude that the matter density (moles per volume!) is given by the determinant:

$$\rho = h \det(y^a_k) = \det(y^a_k), \quad (5)$$

where the matrix $(y^a_k)$ is considered not as a primary quantity, but as a nonlinear function of the matrix $(x^k_a)$, namely its inverse. The inverse of the density is equal to the molar volume $V$ of the fluid:

$$V = \frac{\det (x^k_a)}{h} = \det (x^k_a) \quad (6)$$
or, in terms of the volume form:

\[ F^*(dx^1 \wedge dx^2 \wedge dx^3) = Vdy^1 \wedge dy^2 \wedge dy^3. \]  

(7)

The laws of hydrodynamics will be derived from the variational formula with the field lagrangian equal to the difference between the kinetic and the potential energy:

\[ \mathcal{L} = \frac{1}{2}Mv^2 - e(V) \]  

(8)

(if the coordinates \( y^a \) are not unimodular then the above expression has to be multiplied by \( h \)). The constant \( M \) denotes the molar mass of the fluid (it could also be \( y \)-dependent for a nonhomogeneous fluid). The internal energy \( e = e(V) \) is given for the given fluid. It determines the properties of the fluid according to the fundamental equation:

\[ de(V) = -pdV, \]  

(9)

where \( p \) denotes the pressure. Due to equations (3) and (6) the lagrangian density (8) can be considered as a function of field variables and their first derivatives together with the independent variables of the theory:

\[ \mathcal{L} = \mathcal{L}(x^k, x^k_\alpha; y^\alpha). \]  

(10)

We will show that the Euler–Lagrange equations of the above field theory

\[ \frac{\partial \mathcal{L}}{\partial x^k} - \frac{\partial}{\partial y^\alpha} \frac{\partial \mathcal{L}}{\partial x^k_\alpha} = 0 \]

are equivalent to the equations of hydrodynamics. It is useful for this purpose to analyse the canonical structure of our theory. Introduce the momenta canonically conjugate to field variables:

\[ P^\alpha_k := \frac{\partial \mathcal{L}}{\partial x^k_\alpha}. \]  

(11)

This way the Euler–Lagrange equations generated by the lagrangian \( \mathcal{L} \) can be written as follows:

\[ \frac{\partial}{\partial y^\alpha} P^\alpha_k = \frac{\partial \mathcal{L}}{\partial x^k}. \]  

(12)
Equations (11) together with (12) can be interpreted as the generating formula for the symplectic relation (see [10] and [8]):

$$\delta \mathcal{L} = \frac{\partial}{\partial y^a} \left( P^a_k \delta x^k \right) = \left( \frac{\partial}{\partial y^a} P^a_k \right) \delta x^k + P^a_k \delta x^k _a . \quad (13)$$

Using the explicit form (8) of the lagrangian function and the fundamental equation (9) we obtain the following expression for the canonical momenta:

$$P^a_0 = M v^a \quad (14)$$

and therefore $P^a_0$ represents the kinetic momentum density of the fluid,

$$P^a_k = p \frac{\partial V}{\partial y^a} = p V y^a_k . \quad (15)$$

To prove the equation (15) we used the fact that the derivative of the determinant with respect to a matrix element is equal to the corresponding element of the inverse matrix times the determinant itself.

The Euler–Lagrange equations (12) can thus be written as follows:

$$M \ddot{v}^a_k = - \frac{\partial}{\partial y^a} (p V y^a_k) = -(y^a_k \frac{\partial p}{\partial y^a}) V - p \frac{\partial}{\partial y^a} (V y^a_k) . \quad (16)$$

Observe that

$$y^a_k \frac{\partial p}{\partial y^a} = \frac{\partial y^a_k}{\partial y^a} \frac{\partial p}{\partial x^k} = \frac{\partial p}{\partial x^k} .$$

Moreover, we prove that the last term of (16) vanishes identically. Indeed, due to (7) we have:

$$0 = F^* d \left( \frac{\partial}{\partial x^k} [d x^1 \wedge d x^2 \wedge d x^3] \right) = d \left( (y^a_k \frac{\partial}{\partial y^a}) F^* (d x^1 \wedge d x^2 \wedge d x^3) \right) =$$

$$= d \left( (V y^a_k) \frac{\partial}{\partial y^a} [d y^1 \wedge d y^2 \wedge d y^3] \right) = \frac{\partial}{\partial y^a} (V y^a_k) dy^1 \wedge dy^2 \wedge dy^3 . \quad (17)$$

Equation (16) takes therefore the form of the Newton equation:

$$M \rho \ddot{v}^a_k = - \frac{\partial p}{\partial x^k} , \quad (18)$$

where the force on the right–hand side is given by the gradient of the pressure.
The dynamical equations of the theory imply the energy and momentum conservation laws via the so called Nöther theorem. We introduce for this purpose the following energy–momentum tensor:

\[ T^\alpha_\beta = \mathcal{P}_k^\alpha x_k^\beta - \delta^\alpha_\beta \mathcal{L}. \]  

(19)

To prove the Nöther theorem we observe that

\[ \frac{\partial}{\partial y^\beta} \mathcal{L}(x^k, x^k_\alpha; y^\alpha) = \frac{\partial \mathcal{L}}{\partial x^k} x^k_\beta + \frac{\partial \mathcal{L}}{\partial x^k_\alpha} x^k_\alpha + \frac{\partial \mathcal{L}}{\partial y^\beta}, \]  

where by \( x^k_\alpha \) we denote the second derivatives:

\[ x^k_\alpha = \frac{\partial}{\partial x^\beta}(x^k_\alpha) = \frac{\partial^2 x^k}{\partial x^\beta \partial x^\alpha}. \]

Combining (19), (20) and the definition (11) of the canonical momenta we obtain:

\[ \frac{\partial}{\partial y^\beta} T^\beta_\alpha + \frac{\partial \mathcal{L}}{\partial y^\alpha} = \left( \frac{\partial}{\partial y^\beta} P^\beta_k - \frac{\partial \mathcal{L}}{\partial x^k} \right)x^k_\alpha + \left( P^\beta_k - \frac{\partial \mathcal{L}}{\partial x^k_\beta} \right)x^k_\alpha = \frac{\partial}{\partial y^\beta} P^\beta_\alpha - \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}. \]  

(21)

Therefore the Euler–Lagrange equations (12) imply:

\[ \frac{\partial}{\partial y^\beta} T^\beta_\alpha + \frac{\partial \mathcal{L}}{\partial y^\alpha} = 0. \]  

(22)

The above equality is called in the field theory Nöther theorem (see [1]). We notice that from 3 equations of motion (12) we get 4 equations (22). This means that they are not independent. Indeed equations (11) and (21) imply:

\[ \frac{\partial \mathcal{L}}{\partial y^\alpha} + \frac{\partial \mathcal{L}}{\partial y^\beta} T^\beta_a y^a_0 = \left( \frac{\partial}{\partial y^\beta} \mathcal{P}^\beta_0 - \frac{\partial \mathcal{L}}{\partial x^0} \right)x^0_0 = \left( \frac{\partial}{\partial y^\beta} \mathcal{P}^\beta_0 - \frac{\partial \mathcal{L}}{\partial x^0} \right)x^0_0 = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial y^\beta} T^\beta_0, \]

so the components of the Nöther theorem corresponding to \( a = 1, 2, 3 \) imply its 0–component. Due to the equality (21) they are equivalent with the field equations.
Using the explicit form of the lagrangian $\mathcal{L}$ we obtain the following expressions for the components of the energy–momentum tensor:

$$\mathcal{T}^0_0 = \frac{1}{2}Mv^2 + e$$

and therefore $\mathcal{T}^0_0$ represents the total energy density,

$$\mathcal{T}^a_0 = pV y^a_k v^k,$$

$$\mathcal{T}^0_a = Mv_k x^k_a,$$

$$\mathcal{T}^a_b = \delta^a_b (e + pV - \frac{1}{2}Mv^2).$$

Especially interesting is the case of the lagrangians which do not depend on the independent parameters of the theory. In the case of hydrodynamics this happens e.g. when the fluid is homogeneous (both the mass and the state equation are the same at different points of the material). In this case the lagrangian depends only on $x^k$ and $x^k_\alpha$ and we obtain:

$$\frac{\partial}{\partial y^\beta} \mathcal{T}^\beta_\alpha = 0.$$  

The 0-th component of the equation (27) expresses the energy conservation:

$$\frac{\partial}{\partial t} \mathcal{T}^0_0 + \frac{\partial}{\partial y^a} \mathcal{T}^a_0 = \frac{\partial}{\partial t}(\frac{1}{2}Mv^2 + e) + \frac{\partial}{\partial y^a}(pV y^a_k v^k) = 0.$$

Integrating the above equality over a finite domain $D$ and using the Stokes theorem we can convert the volume integral of the last term into the surface integral over the boundary $\partial D$ of $D$. The integral is equal to (minus) the work performed by the pressure on the boundary of $D$. The work performed at the boundary is the only reason for the change of the total energy of the fluid contained in $D$. Using the identity (17) we can also rewrite the last term in the above equality as follows:

$$\frac{\partial}{\partial y^a}(pV y^a_k v^k) = V \frac{\partial}{\partial x^k}(p v^k).$$
Finally, we get the energy conservation law in the standard form:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} M v^2 + e \right) + V \frac{\partial}{\partial x^k} (pv^k) = 0 .$$

Using (9) the “material” components of the equation (22) take the following form:

$$\frac{\partial}{\partial t} T^b_0 + \frac{\partial}{\partial y^a} T^a_b = \frac{\partial}{\partial t} (Mv_k x^k_b) + \frac{\partial}{\partial y^a} [\delta^a_b (e + pV - \frac{1}{2} M v^2)] =$$

$$= x^k_b [M \frac{\partial v_k}{\partial t} + \frac{\partial}{\partial x^k} (e + pV)] = x^k_b (M \frac{\partial v_k}{\partial t} + V \frac{\partial}{\partial x^k} p) = 0 . \quad (29)$$

The last equality is equivalent to (18) and represents the momentum conservation law of the perfect fluid.

### 3 Euler picture

In the previous section we described the configuration of a continuous medium as a mapping $F$ from its material space $Y$ to the physical space $X$. Now we invert the role of the independent parameters and the field variables of our theory. To describe the same configuration of the material we will use the inverse mapping $G = F^{-1}$:

$$G : X \mapsto Y .$$

The time coordinate $x^0 = t$ together with coordinates $(x^k)$ form a system of spacetime coordinates $(x^\mu) \ (\mu = 0, 1, 2, 3)$ in spacetime $M := U \times X$. Now the dynamical history of the material is given by the one parameter family $G$ of configurations:

$$G : M \mapsto Y .$$

The laws of the hydrodynamics can be expressed as second order partial differential equations for the field variables $y^a(x^\mu)$ which are the functions on the spacetime $M$. For this purpose we express hydrodynamic quantities (the velocity and the density of the fluid) in terms of the first derivatives of the fields. We already know the expression for the density:

$$\rho = \text{det}(y^a_k) . \quad (30)$$
Moreover,
\[
0 = \frac{d}{dt} x^k(t, y^a(t, x')) = \frac{\partial x^k}{\partial t}(t, y^a(t, x')) + x^k_a(t, y^a(t, x')) y^a_0(t, x'),
\]
which implies:
\[
v^k = \frac{\partial x^k}{\partial t} = -x^k_a y^a.
\] (31)

The above formula expresses the velocity \( v^k \) in terms of the derivatives of the eulerian field variables \( y^a \) if we consider the matrix \( (x^k_a) \) not as a primary quantity, like in the lagrange picture, but as a nonlinear function of \( (y^a_k) \), namely its inverse.

It is important to notice that the above quantities satisfy automatically the continuity equation and therefore we do not need to postulate it independently. The situation is similar as in the classical electrodynamics, where the fields which have been derived from the electromagnetic vector–potential satisfy automatically the first pair of the Maxwell equations. The second pair becomes the dynamical 2–order equations for the potentials. We can say that our field variables \( y^a \) play the role of the “potentials” for the hydrodynamic quantities \( (v^k, \rho) \). The analogy with the electrodynamics goes further: there are gauge transformations in both theories which do not change the physical quantities. In the electrodynamics gauge transformations consist in adding a gradient to the electromagnetic vector–potential. Here, gauge transformations are given by unimodular reparameterizations of the material space \( Y \). Such a reparameterization consists only in “changing names to the particles of the material” and does not change the values of \( v^k \) and \( \rho \).

To prove the continuity equation we first introduce the matter current \( j \) as a vector density defined by the formula:
\[
j^\kappa = \epsilon^{\kappa\mu\nu\lambda} y^1_\mu y^2_\nu y^3_\lambda.
\] (32)

We have
\[
j^0 = \epsilon^{0\mu\nu\lambda} y^1_\mu y^2_\nu y^3_\lambda = \epsilon^{kmn} y^1_k y^2_m y^3_n = \det(y^a_k) = \rho.
\]

Moreover, we have:
\[
j^k y^a_k = \epsilon^{k\mu\nu\lambda} y^1_\mu y^2_\nu y^3_\lambda y^a_k = -\epsilon^{0\mu\nu\lambda} y^1_\mu y^2_\nu y^3_\lambda y^a_0 + \epsilon^{k\mu\nu\lambda} y^1_\mu y^2_\nu y^3_\lambda y^a_k.
\]
The last term vanishes being the determinant of the $4 \times 4$ matrix which has necessarily 2 columns equal. Therefore
\[ j^k y^a_k = -\rho y^a_0 , \]
which finally implies:
\[ j^k = -\rho y^a_0 x^k_a = \rho v^k . \]
Introducing now the "four–velocity" $u^\mu$ defined as follows:
\[ u^0 := 1 , \]
\[ u^k := v^k , \]
we have
\[ j^\mu = \rho u^\mu . \]

We will now show that the continuity equation for the matter current $j$ is a consequence of the fact that the volume form $r$ is closed since its external differential necessarily vanishes being a 4–form in a 3–dimensional space $Y$. For this purpose we prove that the transport of $r$ to the space–time $M$ (such a transport is a differential 3–form in the 4–dimensional manifold i.e. the vector–density in $M$) is equal to $j$. Indeed:
\[ G^*(r) = G^*(dy^1 \land dy^2 \land dy^3) = y^1_\mu y^2_\nu y^3_\lambda dx^\mu \land dx^\nu \land dx^\lambda . \]
Using the identity:
\[ dx^\mu \land dx^\nu \land dx^\lambda = \epsilon^{\kappa\mu\nu\lambda} \frac{\partial}{\partial x^\kappa} |dx^0 \land dx^1 \land dx^2 \land dx^3 \]
and definition (32) we get
\[ G^*(r) = j^\kappa \frac{\partial}{\partial x^\kappa} |dx^0 \land dx^1 \land dx^2 \land dx^3 . \quad (33) \]
Finally we get the continuity equation as a consequence of (33) and the fact that the 3–form $r$ is closed:
\[ 0 = G^*(dr) = d \left( j^\mu \frac{\partial}{\partial x^\mu} |dx^0 \land dx^1 \land dx^2 \land dx^3 \right) = \frac{\partial j^\mu}{\partial x^\mu} dx^0 \land dx^1 \land dx^2 \land dx^3 . \quad (34) \]
Observe that replacing \( r \) by \( \Mr \) we obtain the same result for the current \( M \). This means that the following identity is valid:

\[
\frac{\partial}{\partial x^\mu}(M \rho u^\mu) = 0
\]
even for the non–homogeneous fluid.

Similarly as in the Lagrange picture, the field equations can be derived from the variational principle. Obviously, the numerical value of the action integral corresponding to a given configuration has to be equal in both the Lagrange picture and the Euler picture. This means that the new lagrangian \( L \) and the old lagrangian \( \mathcal{L} \) are related via the following formula:

\[
L dx^1 \wedge dx^2 \wedge dx^3 = G^*(\mathcal{L} dy^1 \wedge dy^2 \wedge dy^3) = \rho \mathcal{L} dx^1 \wedge dx^2 \wedge dx^3.
\]

We conclude that the numerical value of the new lagrangian equals:

\[
L = \rho \left( \frac{1}{2} M v^2 - e \right), \tag{35}
\]

but now we have to express it in terms of the field variables \( y^a \) and their derivatives \( y^a_\mu \) using equations (30) and (31). Again, the Euler–Lagrange equations together with the definition of the momenta \( \Theta^\mu_a \) canonically conjugate to field variables \( y^a \) can be interpreted as a generating formula for the symplectic relation:

\[
\delta L(y^a, y^a_\mu; x^\mu) = \frac{\partial}{\partial x^\mu}(\Theta^\mu_a \delta y^a) = (\frac{\partial}{\partial x^\mu} \Theta^\mu_a) \delta y^a + \Theta^\mu_a \delta y^a_\mu \tag{36}
\]
or, equivalently:

\[
\Theta^\mu_a = \frac{\partial L}{\partial y^a_\mu}, \tag{37}
\]

and

\[
\frac{\partial}{\partial x^\mu} \Theta^\mu_a = \frac{\partial L}{\partial y^a}. \tag{38}
\]

The explicit form (35) of the lagrangian \( L \) implies the following expressions for the components of \( \Theta^\mu_a \). Using the fundamental equation (9) which – in terms of the density \( \rho \) – reads

\[
de e = \frac{p}{\rho^2} dp,
\]

12
we obtain:

\[ \Theta^a_0 = -\rho x^k_a M v_k , \]
\[ \Theta^k_a = -\rho x^l_a [M v_l v^k + \delta^k_l (e + p V - \frac{1}{2} M v^2)] . \]

Therefore, the field equations can be written as follows:

\[
\frac{\partial}{\partial x^\mu} \Theta^\mu_a = -\frac{\partial}{\partial x^\mu} (\rho x^k_a M v_k u^\mu) - \frac{\partial}{\partial x^k} [\rho x^k_a (e + p V - \frac{1}{2} M v^2) = \\
= -x^k_a \left[ \frac{\partial p}{\partial x^k} + \frac{\partial}{\partial x^\mu} (\rho u^\mu M v_k) \right] = 0 . \tag{39}
\]

Since the matrix \( x^k_a \) is non–degenerate, the equations can be rewritten in the following, equivalent form:

\[
0 = \frac{\partial p}{\partial x^k} + \frac{\partial}{\partial x^\mu} (\rho u^\mu M v_k) = \frac{\partial p}{\partial x^k} + M \rho u^\mu \frac{\partial}{\partial x^\mu} v_k , \tag{40}
\]

where we have used the continuity equation. The operator

\[
\frac{d}{dt} := u^\mu \frac{\partial}{\partial x^\mu}
\]

is called the “substantial” derivative. It is obviously equivalent to the partial time derivative in the Lagrange picture (i.e. with \( y^a \) being constant). The equations (40) are called Euler equations. They are equivalent to the field equations (18) in the Lagrange picture.

As in any other field theory we can also prove the Noether theorem for the following energy–momentum tensor:

\[ t^\mu_\nu = \Theta^\mu_a y^a_\nu - \delta^\mu_\nu L . \tag{41} \]

The theorem consists in deriving the equivalence between field equations and the (non–)conservation laws. Its proof is completely analogous to the one given in the previous section. Simple calculations lead to the identity:

\[ \frac{\partial}{\partial x^\mu} t^\mu_\nu + \frac{\partial L}{\partial x^\nu} = (\frac{\partial}{\partial x^\mu} \Theta^\mu_a - \frac{\partial L}{\partial y^a}) y^a_\nu . \tag{42} \]

Therefore, the field equations (38) imply the (non–)conservation laws:

\[ \frac{\partial}{\partial x^\mu} t^\mu_\nu + \frac{\partial L}{\partial x^\nu} = 0 . \tag{43} \]
Similarly as in the Lagrange picture only 3 among them are independent (this is due to the fact that the continuity equation has been incorporated \textit{a priori} into the structure of the theory). Moreover, they are completely equivalent to field equations (39). Indeed, equation (42) implies the following identity:

$$-(\frac{\partial}{\partial x^\mu} t^\mu_k + \frac{\partial L}{\partial x^k}) v^k = (\frac{\partial}{\partial x^\mu} \Theta^\mu_a - \frac{\partial L}{\partial y^a}) y^a_k x^k_b y^b_0 =$$

$$= (\frac{\partial}{\partial x^\mu} \Theta^\mu_a - \frac{\partial L}{\partial y^a}) y^a_0 = \frac{\partial}{\partial x^\mu} t^\mu_0 + \frac{\partial L}{\partial x^0} . \quad (44)$$

Again, the symmetry of the lagrangian with respect to the space–time translations:

$$\frac{\partial}{\partial x^\mu} L = 0 ,$$

which is physically equivalent to the absence of the external forces, implies the conservation laws:

$$\frac{\partial}{\partial x^\mu} t^\mu_\nu = 0 . \quad (45)$$

Simple calculations lead to the following formulae for the components of the energy–momentum tensor $t^\mu_\nu$ derived from the lagrangian $L$:

$$t^0_0 = \rho(\frac{1}{2} M v^2 + e)$$

$$t^k_0 = \rho v^k(\frac{1}{2} M v^2 + e + pV)$$

$$t^0_k = -\rho M v^k$$

$$t^k_0 = -p \delta^k_l - \rho M v^k v_l .$$

Equation $\partial_\mu t^\mu_0 = 0$ represents therefore the energy conservation law and equation $\partial_\mu t^\mu_k = 0$ expresses the momentum conservation law.

The following identities express relations between the objects used in the Lagrange and Euler pictures:

$$\Theta^0_a = -\rho T^0_a$$

$$\Theta^k_a = -\rho v^k T^0_a - \rho x^k_b T^b_a$$

$$P^0_k = -V t^0_k$$
We see that in the rest frame (i.e. when \( v^k = 0 \)) the canonical momentum in one picture corresponds (up to a sign) to the energy–momentum tensor in the other picture and vice versa.

4 Thermodynamics

To describe thermal properties of the fluid we need one more potential (see [7] and [3]). As we will see in the sequel, the new potential \( \tau \) can be treated as a "material time". We will describe the thermo–dynamical history of an isoentropic flow in terms of 4 field potentials depending on 4 independent variables. We already know that there are 2 possible choices as far as the “spatial” coordinates are considered: the Lagrange and the Euler picture. Similarly, in both cases we may choose the physical time \( t \) or the “material time” \( \tau \) as the independent variable and the remaining parameter as the field potential. This way we obtain 4 possible pictures of the thermo–hydrodynamics. In the present paper we are going to discuss only two of them, keeping always the physical time together with the physical space \( X \) and the “material time” together with the material space \( Y \). Remaining two pictures have no natural relativistic counterpart and are less interesting, although technically it is very easy to formulate them.

In the present section we start with the complete Euler picture, based on the choice of the physical space–time \( M \) as the space of independent parameters. To describe the space of field potentials we add new dimension \( \tau \) to the matter space \( Y \). This way we obtain the 4–dimensional matter spacetime \( Z \) with \( \tau \) playing the role of a "material time" (having a priori nothing to do with the physical time \( t = x^0 \)). Let us introduce coordinates \( z^\alpha (\alpha = 0, 1, 2, 3) \) in the material spacetime \( Z \) putting \( z^0 = \tau \) and \( z^a = y^a \). We will describe the history of the fluid in terms of 4 potentials \( z^\alpha = z^\alpha (x^\mu) \).

The potentials give the coordinate expression for the mapping:

\[
G : M \rightarrow Z.
\]

The laws of the thermo–hydrodynamics will be formulated as a system of 2–order partial differential equations for the potentials. For this purpose we have to express the thermo–dynamical quantities in terms of the derivatives
of the potentials. We already know how to do it for the hydrodynamical quantities \( j^\mu \). For the temperature \( T \) we choose the following ansatz:

\[
T = \beta (\dot{\tau} + v^k \frac{\partial \tau}{\partial x^k}) = \beta u^\mu z^0_\mu
\]

with \( \beta \) being a positive phenomenological constant. The microscopic interpretation of the above definition will be given in the sequel but in the present section we adopt a purely phenomenological point of view where \( \tau \) is merely a potential for the temperature, similarly as \( y^a \) were the potentials for the hydrodynamic quantities \( j^\mu \). We will prove that the choice of the Helmholtz free energy \( f(V, T) \) as a potential part of the Lagrangian \( L \), i.e.

\[
L = \frac{1}{V} \left[ \frac{1}{2} M v^2 - f(V, T) \right] = \rho \left[ \frac{1}{2} M v^2 - f(\rho, T) \right]
\]

leads to the field equations which are equivalent with the laws of the thermo-hydrodynamics of isoentropic flows.

Again, the Euler–Lagrange equations together with the definition of the momenta \( \Theta^\mu_\alpha \) canonically conjugate to the field variables \( z^\alpha \) can be interpreted as a generating formula for the symplectic relation:

\[
\delta L(z^\alpha, z^\alpha_\mu; v^\mu) = \frac{\partial}{\partial x^\mu}(\Theta^\mu_\alpha \delta z^\alpha) = \left( \frac{\partial}{\partial x^\mu} \Theta^\mu_\alpha \right) \delta z^\alpha + \Theta^\mu_\alpha \delta z^\alpha_\mu
\]

or, equivalently:

\[
\Theta^\mu_\alpha = \frac{\partial L}{\partial z^\alpha_\mu}
\]

and

\[
\frac{\partial}{\partial x^\mu} \Theta^\mu_\alpha = \frac{\partial L}{\partial z^\alpha}.
\]

The explicit form (47) of the lagrangian \( L \) enables us to calculate the components of \( \Theta^\mu_\alpha \) in terms of the field potentials and their derivatives. Using the fundamental equation:

\[
df = -pdV - sdT = \frac{p}{\rho^2} d\rho - sdT
\]

with \( s \) being the molar entropy of the fluid, we obtain:

\[
\Theta^\mu_0 = \beta \rho s u^\mu,
\]
\[ \Theta^0_a = -\rho x^k_a (Mv_k + \beta sz^0_k), \]
\[ \Theta^k_a = -\rho x^l_a [(Mv_l + \beta sz^0_l)v^k + \delta^k_l (f - \frac{1}{2} Mv^2 + pV)]. \]

The 0-th component of field equations gives thus the entropy conservation law:
\[ \frac{\partial}{\partial x^\mu} \Theta_0^\mu = \beta \rho \frac{\partial}{\partial x^\mu} (\rho s u^\mu) = \beta \rho v^\mu \frac{\partial}{\partial x^\mu} s = 0. \] (54)

The remaining equations read:
\[ \frac{\partial}{\partial x^\mu} \Theta_a^\mu = -\frac{\partial}{\partial x^\mu} [\rho x^k_a (Mv_k + \beta sz^0_k) u^\mu] - \frac{\partial}{\partial x^k} [\rho x^k_a (f + pV - \frac{1}{2} Mv^2)] = \]
\[ = -x^k_a [\frac{\partial p}{\partial x^k} + \frac{\partial}{\partial x^\mu} (\rho u^\mu Mv_k)] - \beta x^k_a z^0_k \frac{\partial}{\partial x^\mu} (\rho s u^\mu) = 0. \] (55)

The last term vanishes due to the entropy conservation and the above equations become simply the Euler equations discussed in the previous sections.

Similarly as in any field theory we obtain the Nöther theorem relating the translational symmetries of the system with the energy and momentum conservation laws. The energy–momentum tensor of the thermo–hydrodynamic system is given by the standard formula:
\[ t^\mu_\lambda = \Theta_\alpha^\mu z_\lambda^\alpha - \delta_\lambda^\mu L. \] (56)

Using the explicit expressions for canonical momenta \( \Theta_\alpha^\mu \) we obtain:
\[ \Theta_0^0 y^0_0 = \rho (Mv^2 + \beta sv_k z^0_k) \]
\[ \Theta_0^a y^a_k = -\rho (Mv_k + \beta sz^0_k) \]
\[ \Theta^k_a y^0_0 = \rho v^k (\frac{1}{2} Mv^2 + \beta sv_k z^0_k + f + pV) \]
\[ \Theta^k_a y^a_l = -\rho [(Mv_l + \beta sz^0_l)v^k + \delta^k_l (f + pV - \frac{1}{2} Mv^2)]. \]

Moreover
\[ \Theta_0^\mu z^0_\lambda = \beta \rho s u^\mu z^0_\lambda. \]

Finally, we obtain the formula for \( t^\mu_\lambda \):
\[ t^0_0 = \rho \left( \frac{1}{2} Mv^2 + e \right) \]
\[ t_0^k = \rho v^k \left( \frac{1}{2} M v^2 + e + pV \right) \]
\[ t_0^0 = -\rho M v^k \]
\[ t_i^k = -p \delta_{ik} - \rho M v^k v_i \]

with internal energy defined by
\[ e := f + Ts. \]

As in any other field theory we can also prove the Nöther theorem for the energy–momentum tensor \( t_\mu^\lambda \). The theorem consists in deriving the equivalence between the field equations and the (non–)conservation laws. Its proof is completely analogous to the one given in the previous sections. Simple calculations lead to the identity:

\[ \frac{\partial}{\partial x^\mu} t_\mu^\nu + \frac{\partial L}{\partial x^\nu} = \left( \frac{\partial}{\partial x^\mu} \Theta_\mu^\alpha - \frac{\partial L}{\partial y^\alpha} \right) z_\nu^\alpha. \tag{57} \]

Therefore, the field equations (50) imply the (non–)conservation laws:

\[ \frac{\partial}{\partial x^\mu} t_\mu^\nu + \frac{\partial L}{\partial x^\nu} = 0 \tag{58} \]

and vice versa because matrix \( z_\nu^\alpha \) is invertible.

We stress that now the 4 conservation laws are independent and equivalent to the 4 field equations. In particular, the energy–momentum conservation laws imply – in our formulation – the entropy conservation. This is due to the fact that the latter is incorporated \textit{implicite} in the structure of the theory via the fact that the Lagrangian \( L \) does not depend on \( z^0 \).

Again, the symmetry of the lagrangian with respect to the space–time translations:

\[ \frac{\partial}{\partial x^\mu} L = 0, \]

which is physically equivalent to the absence of the external forces, implies the energy–momentum conservation laws:

\[ \frac{\partial}{\partial x^\mu} t_\mu^\lambda = 0. \tag{59} \]

In the above formulation the phenomenological constant \( \beta \) may be chosen arbitrarily. The choice \( \beta = 1 \) is also possible. It gives us [time \times temperature]
for the dimension of the new potential \( \tau = z^0 \). As we shall see in the sequel, it is more natural to choose the dimension of \( \beta \) equal to the temperature and to measure \( \tau \) in units of time.

5 Hamiltonian formulation of thermo–hydrodynamics

The Hamiltonian description of the thermo–hydrodynamics can be obtained by the following standard, field–theoretical, Legendre transformation (see e.g. [1] or [8]). The generating formula (48) can be rewritten as follows:

\[
\delta L = \frac{\partial}{\partial x^\mu}(\Theta^\mu_\alpha \delta z^\alpha) = \frac{\partial}{\partial x^0}(\Theta^0_\alpha \delta z^\alpha) + \frac{\partial}{\partial x^k}(\Theta^k_\alpha \delta z^\alpha) = \\
\dot{\Theta}^0_\alpha \delta z^\alpha + \Theta^0_\alpha \dot{\delta z}^\alpha + \frac{\partial}{\partial x^k}(\Theta^k_\alpha \delta z^\alpha) = \\
\delta(\Theta^0_\alpha \dot{\delta z}^\alpha) - \dot{\delta z}^\alpha \delta \Theta^0_\alpha + \Theta^0_\alpha \delta z^\alpha + \frac{\partial}{\partial x^k}(\Theta^k_\alpha \delta z^\alpha). \tag{60}
\]

Introducing the Hamiltonian density:

\[
H = \Theta^0_\alpha \dot{z}^\alpha - L = t^0_0
\]

we get another generating formula:

\[
-\delta H = \dot{\Theta}^0_\alpha \delta z^\alpha - \dot{\delta z}^\alpha \delta \Theta^0_\alpha + \frac{\partial}{\partial x^k}(\Theta^k_\alpha \delta z^\alpha),
\]

where, at given time \( t \), the function \( H \) has to be expressed in terms of the "Hamiltonian variables" (or "canonical variables"), i.e. in terms of the fields \( z^\alpha \) and of the time–like component \( \Theta^0_\alpha \) of the momenta. It is useful to choose the special notation for the latter:

\[
\pi_\alpha := \Theta^0_\alpha.
\]

We have therefore:

\[
-\delta H = \dot{\pi}_\alpha \delta z^\alpha - \dot{\delta z}^\alpha \delta \pi_\alpha + \frac{\partial}{\partial x^k}(\Theta^k_\alpha \delta z^\alpha). \tag{61}
\]
As we already know, the numerical value of our hamiltonian is simply:

$$H = \rho \left( \frac{1}{2} M v^2 + e \right),$$

where now we consider the internal energy $e$ as a function of $\rho$ and $s$. It contains the entire information about the thermo–hydrodynamic properties of the fluid due to the corresponding fundamental equation:

$$de = -pdV + Tds = \frac{p}{\rho^2}d\rho + Tds.$$  \hspace{1cm} (62)

In order to complete the Legendre transformation we observe that the formulae (52) and (53) imply:

$$s = \frac{\pi_0}{\beta \rho}, \hspace{1cm} (63)$$

$$p_{k} := M v_{k} = -\frac{\pi_{\alpha} z_{\alpha k}}{\rho}. \hspace{1cm} (64)$$

Together with the formula (30), they allow us to express the hamiltonian density $H$ in terms of canonical variables $(z^\alpha, \pi_\alpha)$ and their space–like derivatives. The formula (61) can be rewritten as follows:

$$-\delta H(z^\alpha, z^\alpha_k, \pi_\alpha) = (\dot{\pi}_\alpha + \frac{\partial}{\partial x^k} \Theta^k_\alpha) \delta z^\alpha + \Theta^k_\alpha \delta z^\alpha_k - \dot{z}^\alpha \delta \pi_\alpha, \hspace{1cm} (65)$$

which is obviously equivalent to the dynamical equations:

$$\Theta^k_\alpha = -\frac{\partial H}{\partial z^\alpha_k},$$

$$\dot{\pi}_\alpha + \frac{\partial}{\partial x^k} \Theta^k_\alpha = -\frac{\partial H}{\partial z^\alpha}$$

and

$$\dot{z}^\alpha = \frac{\partial H}{\partial \pi_\alpha}.$$

The first two equations are usually written with the help of the variational derivative:

$$-\dot{\pi}_\alpha = \frac{\delta H}{\delta z^\alpha} := \frac{\partial H}{\partial z^\alpha} - \frac{\partial}{\partial x^k} \frac{\partial H}{\partial z^\alpha_k}. \hspace{1cm} (66)$$
Also the last equation can be rewritten in terms of the variational derivative since the Hamiltonian does not depend on the derivatives of \( \pi_\alpha \):

\[
\dot{z}^\alpha = \frac{\delta H}{\delta \pi_\alpha}.
\]

(67)

The theory can be also formulated in the language of infinite–dimensional Hamiltonian systems (see e.g. [2]). The infinite–dimensional phase space of the system is a functional space of 8 functions \((z^\alpha, \pi_\alpha)\) defined on a fixed domain \(D \subset X\) in the physical space \(X\) (of course, \(D\) can also be equal \(X\) if we want to describe the world filled entirely with the fluid). The dynamics of the system is governed by the Hamiltonian \(H_D\) equal to the integral of the Hamiltonian density \(H\):

\[
H_D := \int_D H(z^\alpha(x), \pi_\alpha(x)) d^3x,
\]

and the generating formula (61) becomes now:

\[
-\delta H_D = \int_D \dot{\pi}_\alpha \delta z^\alpha - \dot{z}^\alpha \delta \pi_\alpha + \int_{\partial D} \Theta^\perp_\alpha \delta z^\alpha,
\]

(68)

where \(\partial D\) denotes the boundary of \(D\) and by \(\Theta^\perp_\alpha\) we mean the transversal (with respect to \(\partial D\)) component of the momentum \(\Theta\). The above definition of our infinite dimensional phase space is not complete unless we specify some boundary conditions which enable us to annihilate the boundary integral in the above formula in order to obtain the infinite dimensional analog

\[
-\delta H_D = \int_D \dot{\pi}_\alpha \delta z^\alpha - \dot{z}^\alpha \delta \pi_\alpha
\]

(69)

of the finite dimensional generating formula:

\[-dH(q, p) = \dot{p}_a dq^a - \dot{q}^a dp_a.\]

The simplest way to remove the boundary integral is to impose the boundary conditions \(z^\alpha|_{\partial D}\) for the configuration variables. Physically, keeping the values of \(z^\alpha|_{\partial D}\) constant in time is equivalent to the condition \(v^k|_{\partial D} = 0\). Moreover, controlling the value of \(z^0|_{\partial D}\) (not necessarily constant in time) means that we control the temperature on the boundary. This means that the
system is kept in a thermal bath. Mathematically, the above boundary conditions mean that we consider the phase space \( P \) of functions \( z^\alpha \) which fulfill the Dirichlet conditions on \( \partial D \). The Hamiltonian dynamics in \( P \) generated by the Hamiltonian \( H_D \) describes the mixed Dirichlet–Cauchy problem for the field equations. Within the space \( P \) of functions fulfilling the boundary conditions we have

\[
\delta z^\alpha|_{\partial D} = 0
\]

which enables us to integrate by parts during all the calculations and to neglect all the boundary integrals.

Of course, controlling the Dirichlet data is not the only way to eliminate the boundary integrals. Using the identity

\[
\Theta^\perp_\alpha \delta z^\alpha = \delta(\Theta^\perp_\alpha z^\alpha) - z^\alpha \delta \Theta^\perp_\alpha
\]

and defining the new Hamiltonian

\[
F_D := H_D - \int_{\partial D} \Theta^\perp_\alpha z^\alpha
\]

we can rearrange the formula (68) as follows:

\[
-\delta F_D = \int_D \dot{\pi}_\alpha \delta z^\alpha - z^\alpha \delta \pi_\alpha - \int_{\partial D} z^\alpha \delta \Theta^\perp_\alpha.
\]

(70)

Now we have to complete the definition of the phase space imposing the boundary conditions on \( \Theta^\perp_\alpha|_{\partial D} \). The Hamiltonian \( F_D \) defines the dynamics within the space \( R \) defined this way. Such a dynamics corresponds to the mixed Neumann–Cauchy problem for the field equations. There are obviously other ways to control the boundary conditions for our field theory. Each of them corresponds to a different Hamiltonian but only the one corresponding to the Dirichlet control mode is equal to the “true” energy of the system.

In the present paper we limit ourselves only to the discussion of the Dirichlet control mode. In this case the generating formula (69) is equivalent to the dynamical equations (66) and (67). Using equations (63), (64) and the explicit form of the Hamiltonian density \( H \) we can easily calculate the right hand sides of these equations. We obtain

\[
\dot{z}^0 = -\frac{1}{\rho} z^0 v^k \frac{v_k}{\beta \rho} + \frac{T}{\beta \rho}
\]
\[ \dot{z}^a = -\frac{1}{\rho} z^a v^k \]

and
\[ \dot{\pi}_0 = -\frac{\partial}{\partial x^k}(\beta s v^k) \]
\[ \dot{\pi}_a = \frac{\partial}{\partial x^k}[\rho x^k_a(f + pV - \frac{1}{2}Mv^2 - v^k \pi_a)] \]

which is indeed equivalent to the field equations (55).

At the end of this section let us observe that in the rest frame (i.e. when \( v^k = 0 \)) the hamiltonian generating formula reduces to the fundamental equation (62). Indeed, in the rest frame we have \( H = \rho c, \dot{z}^a = 0 \) and therefore equation (65) takes the following form:
\[ d(\rho c) = -(\dot{\pi}_a + \frac{\partial}{\partial x^k}(\Theta^k)\delta z^a - \Theta^k \delta z^a_k + \dot{z}^a \delta \pi_a = \]
\[ = \dot{z}^0 \delta \pi_0 - \Theta^k \delta z^a_k = \frac{T}{\beta} \delta(\beta \rho s) + (f + pV)\rho x^k_a \delta y^a_k = \]
\[ = T \delta(\rho s) + (e - Ts + \frac{P}{\rho}) \delta \rho \]

which is obviously equivalent to the fundamental equation (62) with both sides multiplied by \( \rho \).

6 Poisson bracket structure of thermo–hydrodynamics

The hamiltonian dynamics of the field theory can also be equivalently formulated in terms of the Poisson bracket between the physical observables i.e. functionals over the phase space \( P \). If \( F \) and \( G \) are two such functionals, their Poisson bracket is defined by the standard formula (see [1]):
\[ \{ F, G \} = \int_D \frac{\delta F}{\delta z^a(x)} \frac{\delta G}{\delta \pi_a(x)} - \frac{\delta G}{\delta z^a(x)} \frac{\delta F}{\delta \pi_a(x)}, \quad (71) \]

since the \( z^a \) and \( \pi_a \) are canonical variables i.e. their Poisson bracket is equal to the Dirac delta distribution:
\[ \{ z^a(x), \pi_a(y) \} = \delta(x - y) . \]
The Poisson bracket of any physical observable $F$ with the hamiltonian $H_D$ gives the time derivative of the observable:

$$\dot{F} = \{F, H_D\} \quad (72)$$

and therefore also the dynamical equations can be rewritten this way:

$$\dot{z}^\alpha = \{z^\alpha, H_D\} ,$$

$$\dot{\pi}_\alpha = \{\pi_\alpha, H_D\} .$$

The reader may easily check that the above equations are equivalent to equations (67) and (66).

The phase space $\mathbf{P}$ can be factorized with respect to the following equivalence relation: we call two elements of $\mathbf{P}$ equivalent if they have the same value of the physical parameters $(v^k, \rho, s)$ or, equivalently, $(p_k, \rho, s)$ (the latter parameterization will be more suitable for some calculations). Denote by $\mathbf{Q}$ the corresponding quotient space i.e. the space of the equivalence classes. It is interesting to notice that the Poisson bracket structure can also be factorized to $\mathbf{Q}$. Indeed, the formula (71) implies the following “generalized Poisson brackets” for the quantities defined by (63), (64) and (30):

$$\{p_k(x), p_l(y)\} = \frac{2}{\rho} \left[ p_k \frac{\partial}{\partial x^l} \delta(x - y) - p_l \frac{\partial}{\partial x^k} \delta(x - y) \right]$$

$$\{p_k(x), \rho(y)\} = \frac{\partial}{\partial x^k} \delta(x - y)$$

$$\{p_k(x), s(y)\} = \frac{s}{\rho} \frac{\partial}{\partial x^k} \delta(x - y)$$

$$\{s(x), \rho(y)\} = 0 .$$

This means that the Poisson bracket of any two functionals which are constant on equivalence classes will also be constant on equivalence classes. Finally, we get the following formula for the Poisson bracket of the two functions on $\mathbf{Q}$:

$$\{F, G\} = \int_D \int_D \left( \frac{\delta F}{\delta p_k(x)} \frac{\delta G}{\delta s(y)} - \frac{\delta G}{\delta p_k(x)} \frac{\delta F}{\delta s(y)} \right) \{p_k(x), s(y)\} +$$

24
\[ \begin{align*}
+ \left( \frac{\delta F}{\delta p_k(x)} \frac{\delta G}{\delta \rho(y)} - \frac{\delta G}{\delta p_k(x)} \frac{\delta F}{\delta \rho(y)} \right) \{p_k(x), \rho(y)\} &+ \\
+ \left( \frac{\delta F}{\delta p_k(x)} \frac{\delta G}{\delta p_l(y)} - \frac{\delta G}{\delta p_k(x)} \frac{\delta F}{\delta p_l(y)} \right) \{p_k(x), p_l(y)\} = \\
= \int_D s(x) \frac{\delta F}{\delta p_k(x)} \frac{\partial}{\partial x^k} \frac{\delta G}{\delta \rho(x)} \frac{\partial}{\partial \rho(x)} \frac{\delta F}{\delta s(x)} + \\
\quad + \frac{\delta F}{\delta p_k(x)} \frac{\partial}{\partial x^k} \frac{\delta G}{\delta \rho(x)} \frac{\partial}{\partial \rho(x)} \frac{\delta F}{\delta s(x)} + \\
\quad + 4p_k(x) \frac{\partial}{\partial p_k(x)} \frac{\partial}{\partial x^l} \frac{\partial}{\partial p_l(x)} = \int_D s(x) \frac{\delta F}{\delta p_k(x)} \frac{\partial}{\partial x^k} \frac{\delta G}{\delta \rho(x)} \frac{\partial}{\partial \rho(x)} \frac{\delta F}{\delta s(x)} + \\
\quad + \frac{\delta F}{\delta p_k(x)} \frac{\partial}{\partial x^k} \frac{\delta G}{\delta \rho(x)} \frac{\partial}{\partial \rho(x)} \frac{\delta F}{\delta s(x)} + \\
\quad + 4p_k(x) \frac{\partial}{\partial p_k(x)} \frac{\partial}{\partial x^l} \frac{\partial}{\partial p_l(x)} = \int_D s(x) \frac{\delta F}{\delta p_k(x)} \frac{\partial}{\partial x^k} \frac{\delta G}{\delta \rho(x)} \frac{\partial}{\partial \rho(x)} \frac{\delta F}{\delta s(x)} + \\
\quad + \frac{\delta F}{\delta p_k(x)} \frac{\partial}{\partial x^k} \frac{\delta G}{\delta \rho(x)} \frac{\partial}{\partial \rho(x)} \frac{\delta F}{\delta s(x)} + \\
\quad + 4p_k(x) \frac{\partial}{\partial p_k(x)} \frac{\partial}{\partial x^l} \frac{\partial}{\partial p_l(x)}
\end{align*} \]

(there are no surface integrals left after the integration by parts because the elements of the configuration space satisfy the boundary conditions).

Since the Hamiltonian is also the functional on \( Q \), we can use the above “generalized Poisson structure” in order to rewrite the dynamical equations (72) in terms of the physical observables, without any reference to the original phase space \( P \) (further reduction to the barotropic case can be easily obtained if we limit ourselves to functions which do not depend on the entropy). Some authors (see e.g. \cite{11}, \cite{4} or \cite{9}) postulate the above generalized structure \textit{a priori}. Our construction shows that in fact it is the canonical structure of thermo–hydrodynamics and it does not depend upon someone’s lucky guesses.

We want to stress that “reducing” the phase space \( P \) to \( Q \) is not a symplectic reduction typical for gauge theories. The gauge transformations in the latter theories are defined as “canonically conjugate to Hamiltonian constraints”. Therefore, the future development of the system is determined by initial data up to a gauge transformation only. Fixing a gauge at any instant of time does not determine the trajectory unless we pass to the quotient phase space.

Unlike in gauge theories, there are no Hamiltonian constraints in \( P \) and therefore no symplectic reduction can be performed. Formally, we can call gauge transformations the unimodular reparameterizations of the material space and the additive reparameterizations of the material time on each world line of the fluid separately. Indeed, such a transformation does not change the physical meaning of the initial data. However, once fixed the initial gauge, the Cauchy problem for canonical variables \((z^\alpha, \pi_\alpha)\) can be uniquely...
solved. This shows that the canonical structure in $\mathbf{P}$ is well adapted to the initial value problem in the thermo-hydrodynamics and the reduction to $\mathbf{Q}$ is neither natural nor necessary.

## 7 Complete Lagrange Picture

The dynamical history of the material can also be described in terms of the mapping:

$$
\mathcal{F} : Z \mapsto M
$$

inverse to the mapping $\mathcal{G}$ used in the Euler picture.

In this picture the laws of the thermo-hydrodynamics will be expressed as the second order partial differential equations for the field variables $x^\mu(y^a)$ which are the functions on the "material" space-time $Z$. For this purpose we express thermo-hydrodynamic quantities (the temperature, the velocity and the density of the fluid) in terms of the first derivatives of the fields. We have:

$$
T = \frac{\partial \tau(t, y^a)}{\partial t} = (x_0^0)^{-1}. \quad (73)
$$

The definition of the inverse matrix element:

$$
x_0^0 = \frac{\det(z_{a0}^k)}{\det(z_{a\mu}^0)},
$$

together with equation (30), implies:

$$
V = (x_0^0)^{-1} \det(x^\mu_\alpha). \quad (74)
$$

Hence,

$$
\det(x^\mu_\alpha) = \frac{V}{T} = \frac{1}{\rho T}. \quad (75)
$$

Moreover,

$$
0 = \frac{\partial x^k}{\partial \tau}(t(\tau, y^a), y^a) = \frac{\partial x^k}{\partial t}(t, y^a) \frac{\partial t}{\partial \tau}(\tau, y^a),
$$

which implies:

$$
v^k = \frac{x_0^k}{x_0^0}. \quad (76)
$$
The above formulae express the velocity \( v^k \), the molar volume \( V \) and the temperature \( T \) in terms of derivatives of the field variables \( x^\mu \).

Obviously, the numerical value of the action integral corresponding to a given configuration is the same in both the Lagrange picture and the Euler picture. This means that the old lagrangian \( L \) and the new lagrangian \( \mathcal{L} \) are related via the following formula:

\[
\mathcal{L} dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3 = \mathcal{F}^*(L dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3) = \det(x^\mu_\alpha) L dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3 = \frac{L}{\rho T} dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3
\]

( unlike in the purely hydrodynamic case, when both the material and the physical times coincide \textit{a priori}, here the 4–dimensional integrals have to be compared!). We conclude that the numerical value of the new lagrangian equals:

\[
\mathcal{L} = x^0_0(\frac{1}{2} M v^2 - f(V, T)), \tag{77}
\]

but now we have to express it in terms of the field variables \( x^\mu \) and their derivatives \( x^\mu_\alpha \) using equations (73), (74) and (76). Again, the Euler–Lagrange equations together with the definition of the momenta \( P^\alpha_\mu \) canonically conjugate to field variables \( x^\mu \) can be interpreted as a generating formula for the symplectic relation:

\[
\delta \mathcal{L} = \frac{\partial}{\partial y_\alpha}(P^\alpha_\mu \delta x^\mu) = (\frac{\partial}{\partial y_\alpha} P^\alpha_\mu) \delta x^\mu + P^\alpha_\mu \delta x^\alpha \tag{78}
\]

Using the explicit form of the lagrangian function and the fundamental equation

\[
df = -pdV - sdT
\]

we obtain the following expressions for the canonical momenta:

\[
P^0_0 = \frac{pV}{T} z^0_0 - (e + pV + \frac{1}{2} M v^2) \tag{79}
\]

\[
P^0_k = \frac{pV}{T} z^0_k + M v_k \tag{80}
\]

\[
P^a_\mu = \frac{pV}{T} z^a_\mu, \tag{81}
\]

27
with $z^\alpha_\mu$ being a non-linear function of $x^\mu_\alpha$, namely the element of its inverse matrix.

The dynamical equations of the theory imply the energy and momentum conservation laws via the Nöther theorem. The corresponding energy–momentum tensor equals:

$$\mathcal{T}^\alpha_\beta = \mathcal{P}^\alpha_\mu x^\mu_\beta - \delta^\alpha_\beta \mathcal{L} \tag{82}$$

Using the explicit form of the lagrangian $\mathcal{L}$ we obtain the following expressions for the components of the energy–momentum tensor:

$$\mathcal{T}^0_0 = -s \tag{83}$$

and therefore $-\mathcal{T}^0_0$ represents the entropy density,

$$\mathcal{T}^a_0 = 0 \tag{84}$$

$$\mathcal{T}^0_a = Mv_k x^k_a - x^0_a (e + pV + \frac{1}{2}Mv^2) \tag{85}$$

$$\mathcal{T}^a_b = \frac{1}{T} \delta^a_b (f + pV - \frac{1}{2}Mv^2) \tag{86}$$

Similarly as in the previous section, the zero–zero component of the energy–momentum tensor plays the role of the hamiltonian of the system with $x^\mu$ and $\Pi_\mu := \mathcal{P}^0_\mu$ being canonical variables. To perform this Legendre transformation we rewrite the generating formula (78) as follows:

$$-\delta \mathcal{H}(x^\mu, x^\mu_\alpha, \Pi_\mu) = (\dot{\Pi}_\mu + \frac{\partial}{\partial z^\alpha} \mathcal{P}^a_\mu) \delta x^\mu + \mathcal{P}^a_\mu \delta x^\mu_a - \dot{x}^\mu \delta \Pi_\mu \tag{87}$$

where the hamiltonian

$$\mathcal{H} := \Pi_\mu \dot{x}^\mu - \mathcal{L} = \mathcal{T}^0_0 = -s$$

has to be expressed in terms of the canonical variables. This can be done if we solve equations (79) and (80) with respect to “velocities” $\dot{x}^\mu$. For this purpose we define:

$$\chi := \det(x^k_a)$$
and
\[ w_k := z^a_k x^0_a, \]
where by \( x^k_a \) we denote the purely space-like part of the matrix \( x^\mu_a \) and by \( z^a_k \) we denote its 3-dimensional inverse. We stress that both the scalar \( \chi \) and the vector \( w_k \) are defined by the first spacial derivatives of the configuration variables \( x^\mu \).

Due to the following identities:
\[ \chi = \frac{V}{T} z^0_0, \]
\[ \frac{z^0_k}{z^0_0} = -w_k \]
we may rewrite definitions (79) and (80) in terms of the quantities \( \chi \) and \( w_k \):
\[ \Pi_0 = p\chi - (e + pV + \frac{1}{2}Mv^2), \quad \Pi_k = Mv_k - p\chi w_k. \]

Moreover, we observe that the matrix
\[ x^k_a - \frac{x^k_0 x^0_a}{x^0_0} = x^k_a - v^k x^0_a \]
is equal to the 3-dimensional inverse of the matrix \( (y^a_k) \). Indeed:
\[ \left( x^k_a - \frac{x^k_0 x^0_a}{x^0_0} \right) y^a_l = \delta^k_l. \]
Therefore
\[ V = \det(x^k_a - v^k x^0_a) = \chi \det(\delta^k_l - v^k w_l) = \chi(1 - w_k v^k). \]

From (88), (89) and (90) we get
\[ MV = \chi(M - w_k \Pi^k - p\chi w^2) \]
and
\[ -e = \Pi_0 + \frac{\Pi^2}{2M} - \frac{p^2 w^2 \chi^2}{2M}. \]
Solving the above two equations with respect to $e$ and $V$ we can express them in terms of “legal” hamiltonian variables, i.e. $x^\mu$ (together with its first spacial derivatives) and $\Pi_\mu$.

The above equations contain only two unknowns since for given constitutive equations (i.e. for a given fluid) the pressure $p$ is a function of $e$ and $V$. The relation between those quantities is given by the following fundamental equation:

$$ds(e, V) = \frac{1}{T}de + \frac{p}{T}dV$$

(91)

or, equivalently

$$\frac{1}{T} = \frac{\partial s}{\partial e}$$

and

$$p = \frac{\partial s}{\partial V} \left( \frac{\partial s}{\partial e} \right)^{-1}.$$ 

Finally, the hamiltonian of the system is equal to $-s(e, V)$ with $e$ and $V$ expressed in terms of the canonical variables.

Let us observe that in the rest frame (i.e. when $v^k = 0$) the hamiltonian generating formula reduces to the fundamental equation (91). Indeed, in the rest frame we have $\chi = V$, $\Pi_0 = e$ and $\dot{x}^k = 0$. Therefore, the equation (87) reduces to

$$\delta s = \frac{pV}{T}z_\mu^a \delta x^\mu_a - \dot{x}^0 \delta \Pi_0 = p \left( \frac{V}{T}z_\mu^\alpha \delta x^\mu_\alpha - \frac{V}{T}z_\mu^0 \delta x^\mu_0 \right) + \frac{1}{T} \delta e =$$

$$= p \left( \delta \left( \frac{V}{T} \right) - V \delta \left( \frac{1}{T} \right) \right) + \frac{1}{T} \delta e = \frac{p}{T} \delta V + \frac{1}{T} \delta e.$$

The equation (91) has been obtained from (62) dividing both sides by $T$. Both (62) and (91) can be interpreted as symplectic relations between thermo–hydrodynamic quantities characterizing the material. In the first case the energy $e = e(s, V)$ is a generating function of a lagrangian submanifold in the symplectic space of parameters $(s, V, T, p)$, equipped with the symplectic form $dT \wedge ds + dV \wedge dp$ and with $(s, V)$ chosen as control parameters. In the second case the same physical relation is described by the generating function $s = s(e, V)$ in the symplectic space of parameters $(e, V, \frac{1}{T}, \frac{p}{T})$, equipped with the symplectic form $d(\frac{1}{T}) \wedge de + d(\frac{p}{T}) \wedge dV$ and with $(e, V)$ chosen as control parameters. The transformation between those two formulations of thermodynamics, called respectively the “energy picture”
and the “entropy picture”, is rather obscure. It has nothing to do with the Legendre transformations (e.g. the Helmholtz transformation leading from (62) to (51) and similar transformations relating the energy to the enthalpy or to the Gibbs free energy). Legendre transformations consist always in exchanging a control parameter (e.g. the entropy) with the corresponding “response parameter” (e.g. the temperature). They always refer to a fixed lagrange submanifold within a fixed phase space, which is described with respect to two different “control modes” (the comprehensive description of these phenomena can be found in [8]).

Unfortunately, the transformation mixing the phase space parameters with the generating function itself is of different nature and we do not know in the literature any convincing interpretation of such a transformation. Here, we obtain such an interpretation in a natural way: there are two possible descriptions of the dynamics depending on whether we parameterize the physical events by the physical time or by the material time. The transformation between these two parameterizations exchange also the role of the momenta canonically conjugate with both times (the energy and the entropy).

8 Relativistic theory

The relativistic version of the above theory can be found in [3]. Here we give the short review of the results.

In Euler picture the relativistic thermodynamics is a field theory with four field potentials $z^\alpha = z^\alpha(x^\mu)$ defined on the physical space–time $M$ equipped with the metric structure $g_{\mu\nu}$ (not necessarily flat!). The structure of the material space–time remains the same as in non–relativistic case. Similarly, we introduce the matter current

$$j := G^*(r)$$

which is automatically conserved since

$$dj = dG^*(r) = G^*(dr) = 0 .$$

In terms of coordinates the components of $j$ are given by the formula

$$j^\kappa := \epsilon^{\kappa\mu\nu\lambda} z^{\mu}_1 z^{\nu}_2 z^{\lambda}_3$$
which implies the continuity equation
\[ \frac{\partial}{\partial x^\mu} j^\mu = 0. \]

Decomposing now
\[ j^\mu = \sqrt{\det(g_{\mu\nu})} \rho u^\mu, \]
where the velocity vector \( u^\mu \) is normalized, we obtain the definition of the hydrodynamic quantities \((\rho, u^\mu)\) in terms of the first derivatives of the three “spatial” field potentials \( z^k \) (we stress that \( \rho \) is now the rest frame density). The fourth potential is used again to define the temperature as a ratio between the two times running along the world lines of the fluid:
\[ T := \beta u^\mu z^0_\mu. \]

We choose the rest frame free energy density as a Lagrangian of the theory:
\[ L = L(z^\alpha) = -\rho f_{\text{rel}}(\rho, T). \]

There is no kinetic energy term since the relativistic free energy \( f_{\text{rel}} \) contains also the rest mass contribution to the energy:
\[ f_{\text{rel}} := f + M \]
(we use geometric units with the speed of light being equal to 1). The Euler-Lagrange equations together with the definition of the momenta \( \Theta^\mu_\alpha \) canonically conjugate to field variables \( z^\alpha \) can be written as a generating formula for the symplectic relation:
\[ \delta L(z^\alpha, z^\alpha_\mu, x^\mu) = \frac{\partial}{\partial x^\mu}(\Theta^\mu_\alpha \delta z^\alpha) = \left( \frac{\partial}{\partial x^\mu} \Theta^\mu_\alpha \right) \delta z^\alpha + \Theta^\mu_\alpha \delta z^\alpha_\mu. \]

To check that the above field theory really describes the thermo-hydrodynamics we calculate the components of the energy momentum tensor given again by the formula (56). After some calculations we obtain:
\[ t^\mu_\nu = -\sqrt{\det(g_{\mu\nu})} \left( \rho e u^\mu u_\nu + p(\delta^\mu_\nu + u^\mu u_\nu) \right). \]

The Noether theorem now reads:
\[ \nabla_\mu t^\mu_\nu = 0. \]
which, together with the entropy equation, is really equivalent to the equations of thermo–hydrodynamics.

The hamiltonian formulation and canonical structure of the theory can be obtained from the construction very much similar to the non–relativistic one.

To describe the interaction of the hydrodynamic matter with the gravitational field it is sufficient to add the gravitational lagrangian to the above matter lagrangian and to take into account the gravitational degrees of freedom, describing the geometry of $M$. Such a theory of a self–gravitating fluid becomes especially simple in the complete Lagrange picture. The main advantage of this picture consists in the fact that the “gravitational gauge” and the “hydro–thermodynamic gauge” eliminate somehow each other and the theory can be formulated in terms of purely physical (gauge free!) quantities. This way we obtain the theory which is well adapted to the numerical analysis and can be used e.g. to computer simulations of the geometrodynamics in both cosmology and astrophysics (see again [3]).

Another advantage of our approach consists in a rather straightforward generalization of the hydrodynamics to the elastodynamics. The relativistic formulation of the latter has never obtained any satisfactory formulation. Here, the introduction of the riemannian structure into the material space $Y$ enables us to generalize the theory in such a way that it is relativistic and reduces in the rest frame to the non–relativistic, nonlinear elastodynamics. The paper containing these results will be soon published.

9 Microscopic interpretation of the material time

We suppose that the liquid is composed of molecules with mass $m$. If the temperature of the fluid equals $T$, the molecules move chaotically around the theoretical world lines of the fluid (lines tangent to the vector field $u^\mu$) and the mean kinetic energy of this motion with respect to the rest frame is equal (for the low temperature) to $\frac{3}{2} kT = \frac{1}{2} mv^2$. Due to this motion the proper time $t$ for the particles is retarded with respect to the physical time $x^0$ calculated along $u^\mu$. For velocities $v$ much smaller than the velocity of
light $c$ this retardation can be calculated from the formula:

$$t = \int \sqrt{1 - \frac{v^2}{c^2}} \, dx^0 \approx \left(1 - \frac{v^2}{2c^2}\right)x^0 = x^0 - x^0 \frac{3kT}{2mc^2}.$$ 

We identify the parameter $\tau$ with the proper time retardation:

$$\tau := x^0 - t = \frac{1}{\beta} x^0 T,$$

where the constant $\beta = \frac{2mc^2}{3k}$ has the dimension of temperature. Hence,

$$\beta \frac{\partial \tau}{\partial x^0} = T$$

similarly as in formula (46). We interpret therefore the ”material time” as the ”proper time retardation” due to the chaotic motion of the particles. This phenomenon enables us to construct (at least theoretically) a ”radium thermometer”. We inject a drop of the radioactive radium into the fluid. Due to the chaotic motion of the particles the lifetime of the radium gets lengthened proportionally to the temperature of the fluid. Therefore, measuring the lifetime we measure the temperature.

**References**

[1] N. N. Bogoliubov, D. Shirkov, *Introduction into Theory of Quantum Fields* (Science, Moscou, 1976); E. Nöther (1918)

[2] P. R. Chernoff and J. E. Marsden *Properties of infinite dimensional hamiltonian systems*, Springer Lecture Notes in Mathematics, vol. 425, 1974

[3] A. Górnicka, J. Kijowski and A. Smólski, *Hamiltonian theory of self–gravitating perfect fluid and a method of effective deparameterization of Einstein’s theory of gravitation*, to appear in Phys. Rev. D

[4] D. D. Holm; Physica **17D** (1985) p. 1–36

[5] J. Jezierski and J. Kijowski, Comptes Rendues Acad. Sci. Paris, serie II, 301 (1985) p. 221-224
[6] J. Kijowski, B. Pawlik, W. M. Tulczyjew, Bull. Acad. Polon. Sci. (math. phys. astr.) 27 (1979) p. 163; H.P. Künzle, J. M. Nester, J. Math. Phys. 25 (1984) p. 1009

[7] J. Kijowski and W. M. Tulczyjew, *Relativistic hydrodynamics of isentropic flows*, Mem. Acad. Sci. Torino, serie 5, No.6-7 (1982-83) p. 3-17

[8] J. Kijowski and W. M. Tulczyjew, *A Symplectic Framework for Field Theories*, Lecture Notes in Physics vol.107, Springer-Verlag, Berlin, (1979)

[9] D. Levis, J. Marsden, R. Montgomery; Physica, 18D (1986) p. 391–404

[10] W. M. Tulczyjew, *Hamiltonian systems, Lagrangian systems and the Legendre transformation*, Symposia matematica 14 (1974) p. 247; M. R. Menzio and W. M. Tulczyjew, *Infinitesimal symplectic relations and generalized Hamiltonian dynamics*, Ann. Inst. Henri Poincare, 28 (1978) p. 349

[11] V. E. Zakharov and E. A. Kuznetsov *Hamiltonian formalism for systems of hydrodynamic type*, Soviet Scientific Review, Section C: Mathematical Physics Review, 4 (1984) p.167–219