Strong continuous-variable entanglement in modulated quantum dissipative dynamics

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We investigate the creation of entangled states of bright light beams obeying the condition of strong Einstein-Podolsky-Rosen-like paradox criterion in time-modulated quantum dissipative dynamics. Having in view the generation of these states we propose a non-degenerate optical parametric oscillator (NOPO) driven by an amplitude-modulated pump field. We develop semi-classical and quantum theories of this device for all operational regimes concluding that, contrary to ordinary NOPO, the continuous-variable entanglement becomes approximately perfect for high level of modulation. Our analytical results are in well agreement with numerical simulations.

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Continuous-variable (CV) entangled states of light beams provide excellent tools for testing the foundations of quantum physics and arouse growing interest due to apparent usefulness as a promising technology in quantum information and communication protocols\textsuperscript{[1, 2]}. The efficiency of quantum information schemes and quantum measurements significantly depends on the degree of entanglement of used states. On the other hand, in the majority of real applications bright stable light beams are required. It is therefore highly desirable to elaborate reliable sources of light beams having the mentioned properties.

The recent development of CV quantum information is stipulated mainly by preparation of two-mode squeezed states, which are an approximation to EPR (Einstein-Podolsky-Rosen) entangled states and can be easily generated by a nondegenerate parametric amplifier\textsuperscript{[3, 4]}. However, up to now the generation of bright light beams with high degree of CV entanglement meets serious problems. One of these is the degradation of entanglement due to uncontrolled dissipation and decoherence processes. Nevertheless, an experimental progress in the generation of CV entangled intensive light beams from NOPO has been reported in\textsuperscript{[5]}. A source of CV entangled light has been also built using two quadrature squeezed beams combined on a beam splitter\textsuperscript{[6]}. CV entanglement of phase-locked light beams was recently studied in\textsuperscript{[7]}

Note that because the number of experimentally accessible and feasible multi-wave nonlinear interactions leading to formation of entanglement is rather limited, the class of schemes that may be practically elaborated is restricted. In this Letter we point out that the class of currently proposed schemes for generation of intensive light beams with high degree of entanglement may be significantly extended if instead of monochromatic pumping we consider periodically modulated pump fields. As a realization of this idea, we propose in this Letter a novel scheme of NOPO in a cavity, driven by an amplitude-modulated electromagnetic field and stress that this scheme provides highly effective mechanism for improvement of the degree of CV entanglement, even in the presence of dissipation and cavity induced feedback.

CV entangling resources are usually analyzed based on two-mode squeezing through the variances of the quadrature amplitudes of two generated modes. In NOPO, the two-mode integral squeezing, which characterizes the CV entanglement, reaches only 50% relative to the level of vacuum fluctuations, if the pump field intensity is close to the generation threshold.\textsuperscript{[3, 5]} We will demonstrate here, that the level of two-mode squeezing in the proposed scheme is not limited, which indicates a high degree of quadrature entanglement obeying the condition of EPR-like paradox criterion as quantified by Reid and Drummond\textsuperscript{[6]}.

It seems intuitively clear that such an achievement is due to the control of quantum dissipative dynamics through the application of suitably tailored, time-modulated driving field. Some examples of the suppression of quantum decoherence as well as an improvement of nonclassical statistics of oscillatory excitation numbers have been considered in\textsuperscript{[7, 8]}

We consider a type-II phase-matched NOPO with triply resonant cavity that supports the pump mode and two orthogonally polarized modes of subharmonics. The Hamiltonian describing intracavity interactions within the framework of rotating wave approximation and in the interaction picture is

\begin{equation}
H = \hbar f(t) \left( e^{i(\Phi L - \omega_L t)} a_3^+ - e^{-i(\Phi L - \omega_L t)} a_3 \right) + i \hbar k \left( e^{i \Phi_k} a_3 a_1 a_2^+ - e^{-i \Phi_k} a_3^+ a_1 a_2 \right),
\end{equation}

where $a_i$ are the boson operators for cavity modes at the frequencies $\omega_i$. The pump mode $a_3$ is driven by an amplitude-modulated external field at the frequency $\omega_L = \omega_3$ with amplitude $f(t)$ that is a periodic function with modulation frequency $\delta \ll \omega_L$. The constant $k e^{i \Phi_k}$ determines an efficiency of the down-conversion process in $\chi^{(2)}$ medium.
The system of interest is dissipative because the subharmonic modes suffer losses. Taking into account the cavity damping rates $\gamma_l$ of the modes we consider the case of zero detunings and high cavity losses for the pump mode ($\gamma_3 \gg \gamma$) under the assumption that $\gamma_1 = \gamma_2 = \gamma$. However, in our analysis we allow for the pump depletion effects. Following the standard procedure of quantum optics we derive in the positive-P-representation the stochastic equations for two groups of independent complex c-number variables $\alpha_{1,2}$ and $\beta_{1,2}$ corresponding to operators $a_{1,2}$ and $a_{1,2}^+$.

$$\frac{d\alpha_1}{dt} = -(\gamma + \lambda\alpha_2\beta_2)\alpha_1 + \varepsilon(t)\beta_2 + W_{\alpha_1}(t), \quad (2)$$

$$\frac{d\beta_1}{dt} = -(\gamma + \lambda\alpha_2\beta_2)\beta_1 + \varepsilon^*(t)\alpha_2 + W_{\beta_1}(t). \quad (3)$$

Here: $\varepsilon(t) = f(t)k/\gamma_3$, $\lambda = k^2/\gamma_3$ and equations for $\alpha_2, \beta_2$ are obtained from (2) by exchanging the subscripts $(1) \leftrightarrow (2)$. Our derivation is based on the Ito stochastic calculus, and the nonzero stochastic correlations are: $\langle W_{\alpha_1}(t) W_{\alpha_2}(t') \rangle = (\varepsilon(t) - \lambda\alpha_1\alpha_3)\delta(t - t')$, $\langle W_{\beta_1}(t) W_{\beta_2}(t') \rangle = (\varepsilon(t) - \lambda\beta_1\beta_3)\delta(t - t')$. Note, that while obtaining these equations we used the transformed boson operators $a_i \rightarrow a_i \exp(-i\Phi_i)$ with $\Phi_i$ being $\Phi_3 = \Phi_L, \Phi_1 = \Phi_2 = \frac{1}{2}(\Phi_L + \Phi_k)$. This leads to cancellation of phases at intermediate stages of calculation. As a result, the equations depend only on real and positive coupling constants.

The equations of motion are with time-dependent coefficients. Nevertheless, surprisingly, it is possible to find their analytical solution in the semiclassical approach for an arbitrary, but real modulation amplitude $f(t)$. The presentation of both the semiclassical and quantum theories of such a time-modulated NOPO is another important goal of this Letter.

First, we shall study the solution of stochastic equations in semiclassical treatment, neglecting the noise terms, for mean photon numbers $n_j$ and phases $\phi_j$ of the modes $(n_j = \alpha_j\beta_j, \phi_j = \ln(\alpha_j/\beta_j))$ for time-intervals exceeding the transient time, $t \gg \gamma^{-1}$. An analysis shows that similar to the standard NOPO, the considered system also exhibits threshold behavior, which is easily described through the period-averaged pump field amplitude $\bar{f}(t) = \frac{1}{T} \int_0^T f(t)dt$, where $T = 2\pi/\delta$.

The below-threshold regime with a stable trivial zero-amplitude solution is realized for $\bar{f} < f_{th}$, where $f_{th} = \gamma_3/k$ is the threshold value. When $\bar{f} > f_{th}$, the stable nontrivial solution exists with the following properties. First, the mean photon numbers for subharmonic modes $n_{\alpha} = \langle a_i^+a_i \rangle = |\alpha_i|^2$ are equal one to the other ($n_{\alpha_1} = n_{\alpha_2} = n_0$) due to the symmetry of the system, $\gamma_1 = \gamma_2 = \gamma$. As for usual NOPO, the phase difference is undefined due to the phase diffusion, while the sum of phases is equal $\phi_1 + \phi_2 = 2\pi k$. The mean photon number $n_0(t)$ satisfies the following equation

$$\frac{d}{dt} n_0(t) = 2n_0(t)(\varepsilon(t) - \gamma - \lambda n_0(t)), \quad (4)$$

which can be explicitly integrated, giving a periodic solution

$$n_0^{-1}(t) = 2\lambda \int_0^\infty \exp \left(2 \int_0^\tau (\varepsilon(t' + \tau) - \gamma) dt' \right) d\tau. \quad (5)$$

Note, that in (4) the terms corresponding to the transient dynamics are omitted for simplicity. In the absence of modulation $n_0(t)$ reaches the steady-state $n_0 = (\bar{f} - f_{th})/k$ in accordance with the theory of standard NOPO for zero detunings.

We emphasize that this model is available for experiments. It can be implemented at least for NOPO driven by polychromatic pump fields. Particularly, for the following pump field $E_{ext}(t) = T\cos(\omegaLt + \Phi_L) + (f_1/2)(\cos(\omegaLt + \delta) + \Phi_L + \Phi) + \cos(\omegaLt - \delta)(t + \Phi_L - \Phi)$ the system is described by the Hamiltonian with harmonic modulation amplitude $f(t) = \bar{f} + f_1 \cos(\delta t + \Phi)$. We present below the final results for the case of such harmonic modulation assuming without loss of generality $\bar{f} > 0$, $f_1 > 0$ and $\Phi = 0$. In this case the mean photon number (5) reads as

$$n_0^{-1}(t) = 2\lambda \int_0^\infty \exp \left(2\gamma \tau \left(\frac{T}{f_{th}} - 1\right) \right) \times \exp \left(\frac{2\gamma f_1}{\delta f_{th}} \left[\sin(\delta(t + \tau)) - \sin(\delta t)\right]\right) d\tau. \quad (6)$$

This result is illustrated in Fig. (1) for the different levels of modulation.

To characterize the CV entanglement we address to both the inseparability criterion proposed by Duan et al. and Simon and the EPR paradox criterion. These criteria could be quantified by analyzing the variances of a relevant distance $V_\bot = V(Y_1 - X_2)$ and total momentum $V_\bot = V(Y_1 + Y_2)$ in the terms of the
quadrature amplitudes of two modes \( X_k = X_k (\Theta_k) = \frac{1}{\sqrt{2}} \left( a_k^+ e^{-i\Theta_k} + a_k e^{i\Theta_k} \right) \), \( Y_k = X_k (\Theta_k - \frac{\pi}{2}) \), \((k = 1, 2)\), where \( V(x) = \langle x^2 \rangle - \langle x \rangle^2 \) is a denotation of the variance. The inseparability criterion, or weak entanglement criterion, for the sum of variances quite generally reads as \( V_+ + V_- < 2 \), and due to the mentioned symmetries is reduced to the following form \( V = V_+ + V_- < 1 \), while for the product of variances this criterion has the form \( V_+ V_- = V^2 < 1 \). The strong CV entanglement criterion shows that when the inequality \( V_+ V_- < 1/4 \) is satisfied, there arises an EPR-like paradox. Obviously, the sufficiency condition for inseparability is weaker than the EPR condition. Both criteria have been used to characterize the entanglement mainly in spectral measurements. Contrary to that, we confine ourselves to analyzing only the total intracavity variances. These variances are expressed through the stochastic variables using the relationships between normally-ordered operator averages and stochastic moments with respect to the P-function. Restoring the previous phase structure of intracavity interaction, we obtain that \( V_+ = V_- = V \) and

\[
V = 1 + \langle \alpha_1 \beta_1 \rangle + \langle \alpha_2 \beta_2 \rangle - \langle \alpha_1 \alpha_2 \rangle e^{i\vartheta} - \langle \beta_1 \beta_2 \rangle e^{-i\vartheta} ,
\]

where \( \Theta = \Theta_1 + \Theta_2 + \Phi_L + \Phi_k \).

Further, we will calculate the variance in the standard linear treatment of quantum fluctuations. To this end, it is convenient to use the following moments of quantum variables, which are given below for both operator- and P-representations: \( \langle n_+ \rangle := \langle a^+_1 a_1 \rangle + \langle a^+_2 a_2 \rangle = \langle \alpha_1 \beta_1 \rangle + \langle \alpha_2 \beta_2 \rangle \), \( \langle R \rangle := \langle a^+_1 a_1 + a^+_2 a_2 \rangle - \langle a_1 a_2 \rangle - \langle a^+_1 a^+_2 \rangle = \langle \alpha_1 - \beta_2 \rangle (\beta_1 - \alpha_2) \), \( \langle Z \rangle := \langle (a^+_1 a_1 - a^+_2 a_2)^2 \rangle = \langle (\alpha_1 \beta_1 - \alpha_2 \beta_2)^2 \rangle + \langle \alpha_1 \beta_1 \rangle + \langle \alpha_2 \beta_2 \rangle \). As can be seen, the possible minimal level of variance, realized under appropriate selection of phases \( \Theta_1 + \Theta_2 = -\Phi_L - \Phi_k \), is expressed through the moment \( \langle R \rangle \) as \( V(t) = 1 + \langle R(t) \rangle \). Using Itô rules for changing the stochastic variables, we obtain from \( \langle 2 \rangle \), \( \langle 3 \rangle \) the following equations:

\[
\frac{d}{dt} \langle n_+ \rangle = (\varepsilon (t) - 2\gamma - \lambda) \langle n_+ \rangle - \lambda \langle n^2_+ \rangle - 2\varepsilon (t) \langle R \rangle + \lambda \langle Z \rangle ,
\]

\[
\frac{d}{dt} \langle R \rangle = -(\varepsilon (t) + 2\gamma + \lambda) \langle R \rangle - \lambda \langle n_+ R \rangle - 2\varepsilon (t) + \lambda \langle Z \rangle ,
\]

\[
\frac{d}{dt} \langle Z \rangle = -2\gamma \langle Z \rangle + 2\varepsilon \langle n_+ \rangle .
\]

From Eq. \( \langle 11 \rangle \), \( \langle Z \rangle \) can be expressed as a function of \( \langle n_+ \rangle \). Substituting this expression into \( \langle 8 \rangle \), \( \langle 9 \rangle \) we get the following equations which are convenient for the perturbative analysis of quantum fluctuations

\[
\frac{d}{dt} \langle n_+ \rangle = (2\varepsilon (t) - 2\gamma - \lambda) \langle n_+ \rangle - \lambda \langle n^2_+ \rangle - 2\varepsilon (t) \langle R \rangle + 2\gamma \lambda \int_{-\infty}^{t} e^{4\gamma(t'-t)} \langle n_+ (t') \rangle \, dt' ,
\]

\[
\frac{d}{dt} \langle R \rangle = - (2\varepsilon (t) + 2\gamma + \lambda) \langle R \rangle - \lambda \langle n_+ R \rangle - 2\varepsilon (t) + 2\gamma \lambda \int_{-\infty}^{t} e^{4\gamma(t'-t)} \langle n_+ (t') \rangle \, dt' .
\]

First, we consider the above-threshold regime linearizing quantum fluctuations around the stable semiclassical solutions. In the linear treatment of quantum fluctuations we have the expansions \( \langle n_+ \rangle = n_{10} + n_{20} + \delta n_+ = 2n_0 + (\delta n_+), \langle R \rangle = R^0 + (\delta R), \langle R \rangle = 2n_0 (\delta R), \langle n^2_+ \rangle = 4n_0 (\delta n_+), \), where it was assumed that \( n_{10} = n_{20} = n_0(t), \varphi_1 + \varphi_2 = 2\pi k, \) and hence \( R^0 = 0 \). Note, that in the current experiments the ratio of nonlinearity to dumping is small, \( k/\gamma \ll 1 \) (typically \( 10^{-6} \) or less), and hence \( \lambda/\gamma = k^2 / (\gamma \gamma_\lambda) \ll 1 \) is the small parameter of the theory. Therefore, the zero order terms in the above expansion correspond to a large classical field of the order \( \gamma/\lambda \) in accordance with Eq. \( \langle 13 \rangle \), while the next terms describing the quantum fluctuations are of the order of 1. On the whole, combining the procedure of linearization with \( \lambda/\gamma \ll 1 \) approximation we get a linear equation for the variance \( V(t) = 1 + \langle \delta R \rangle \)

\[
\frac{d}{dt} V(t) = -2(2\varepsilon (t) + \lambda n_0 (t)) V(t) + 2\lambda n_0 (t)
\]

\[
+ 2\gamma + 4\gamma \int_{-\infty}^{t} e^{4\gamma(t'-t)} n_0(t') \, dt' ,
\]

with the following periodic asymptotic solution

\[
V(t) = 2 \int_{-\infty}^{t} \exp \left(-2 \int_{\tau}^{t} (\gamma + \varepsilon (t')) + \lambda n_0 (t') \right) dt' \right) \times 
\]

\[
\left[ \gamma + \lambda n_0 (\tau) + 2\gamma \lambda \int_{-\infty}^{\tau} e^{4\gamma(t'-\tau)} n_0(t') dt' \right] \, d\tau .
\]

It should be noted, that the result \( \langle 14 \rangle \) is also obtained from initial equations \( \langle 2 \rangle \), \( \langle 3 \rangle \) in the photon number and phase variables using, however, more complicated calculations. In the absence of modulation the formula \( \langle 14 \rangle \) coincides with an analogous one for ordinary NOPO. The analysis of the below-threshold regime is more simple and leads to formula \( \langle 14 \rangle \) with \( n_0 = 0 \).

Let us outline some conclusions from this result for the case of harmonic modulation, using that \( \varepsilon (t) = \frac{k}{\gamma} (\bar{f} + f_1 \cos (\delta t)) \) and the expression \( \langle 8 \rangle \). Typical results are presented in Fig. \( \langle 2 \rangle \) for the above-threshold regime, \( \bar{f}/f_{th} = 3 \). The variance is seen to show a time-dependent modulation with a period \( 2\pi/\delta \). The drastic difference between the degree of two-mode squeezing/entanglement for modulated and stationary dynamics is also clearly seen in Fig. \( \langle 2 \rangle \). The stationary variance (curve 1) near the threshold having a limiting squeezing of 0.5 (see also Fig. \( \langle 4 \rangle \), curve 1) is bounded by quantum inseparability criterion \( V < 1 \), while the variance
for the case of modulated dynamics obeys the EPR criterion $V^2 < 1/4$ of strong CV entanglement for definite time intervals. In particular, the minimum values of the variance and corresponding photon numbers of Fig. 11 at fixed time intervals $t_m = t_0 + 2\pi m/\delta$, ($m = 0, 1, 2...$) are: $n_0 \simeq 6.16 \cdot 10^7$, $V_{min} \simeq 0.27$, $t_0 = 2.64\gamma^{-1}$ (curve 3) and $n_0 \simeq 1.71 \cdot 10^8$, $V_{min} \simeq 0.56$, $t_0 = 2.51\gamma^{-1}$ (curve 2). The dependence of $V_{min}$ on the period-averaged pump field amplitude is shown in Fig. 3 for different levels of modulation. As it is expected, the degree of EPR entanglement increases with ratio $f_1/\bar{f}$. Another peculiarity here is that the stationary variance (curve 1) has a characteristic threshold behavior (see, for example Fig. 1) that disappears in case of strong modulation (curve 3). We conclude, that here nothing of the kind of principle is valid that prohibits the reaching of approximately perfect squeezing/entanglement for the case of strong modulation. The production of strong entanglement occurs for the period of modulation comparable with the characteristic time of dissipation, $\delta \approx \gamma$. For both asymptotic cases of slow ($\delta \ll \gamma$) and fast ($\delta \gg \gamma$) modulations this effect disappears.

The linearized theory is applicable only outside the critical region. As our analysis shows, the condition of the validity of linear results for the near-threshold regimes reads as $[(\bar{f}/f_{th} - 1) \gg (\lambda/\gamma) \exp [2(f_1/f_{th})(\gamma/\delta)]]$. For typical $\lambda/\gamma \ll 1$, this condition is fairly easy to satisfy even for narrow critical ranges provided that $\delta \approx \gamma$. As a rule, the quantum corrections diverge at the classical threshold, although, the variance (2) surprisingly is well defined also at the threshold. Nevertheless, in order to verify the accuracy of our analytical calculations we performed numerical simulation based on the quantum state diffusion method (13), for the case of time-modulated fields, see (14). This simulation tends to disagree with our analytical results for comparatively high values of $\lambda/\gamma$ parameter, especially at the threshold, as is shown in Fig. 4. Generally speaking, the divergence between the numerical and analytical results is typically in the order of $\lambda/\gamma$.

In conclusion, we have proposed a new approach to generation of strongly EPR entangled states of intense light beams based on the time-modulation of quantum dissipative dynamics. We have demonstrated an essential improvement of the degree of entanglement in modulated NOPO in comparison with the ordinary one, if the frequency of modulation is close to the decay rate of dissipative precesses. We have shown that surprisingly the CV entanglement becomes approximately perfect for high-level of modulation, but is only realized for a periodic sequence of time intervals synchronized with the modulation period. The analytically predicted results were confirmed in our numerical simulation. In our study we have not analyzed in detail any physical mechanisms leading to strong EPR entanglement due to the modulation. This topic is currently being explored and will be the subject of forthcoming work. We believe that the results obtained are applicable to a general class of quantum dissipative systems and can serve as a guide for further theoretical and experimental studies of intense light beams with high degree of entanglement.

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