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A. B. Mahomed , S. D. Maharaj, and R. Narain

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A family of exact models for radiating matter

A. B. Mahomed, S. D. Maharaj, and R. Narain

AFFILIATIONS
Astrophysics and Cosmology Research Unit, School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa

ABSTRACT
In this paper, the cosmological constant and electric charge are incorporated in the Einstein–Maxwell field equations. Two approaches are used to investigate the problem. First, the boundary condition is expressed as a generalized Riccati equation in one of the gravitational potentials. New classes of exact solutions are found by writing the Riccati equation in linear, Bernoulli, and inhomogeneous forms. Our solutions contain previous results in the absence of the cosmological constant and charge. Second, it is possible to preserve the form of the generalized Riccati equation by introducing a transformation called the horizon function. This transformation simplifies the generalized Riccati equation. We generate new solutions to the transformed Riccati equation when one of the metric functions serves as a generating function. We also obtain other families of new classes of exact solutions, where the horizon function serves as a generating function. Interestingly, new uncharged solutions, not contained in previous studies, arise as special cases of the inhomogeneous Riccati equation in both approaches.

I. INTRODUCTION
Radiating stellar models have been studied in many different physical situations over the years. The junction conditions at the stellar surface were completed by Santos, who showed that the presence of heat flow must be taken into account so that the interior can match to the exterior Vaidya metric at the boundary. Particular solutions of the boundary condition have been studied in astrophysical settings including recent investigations of Naidu et al., Govender et al., Sharma et al., and Pretel and da Silva. A systematic approach to investigating this problem is to use the Lie analysis of infinitesimal symmetry generators. Abebe et al. used the Lie approach to study conformally flat and geodesic stars, respectively. Stars with shear-free matter distributions were analyzed by Abebe et al. The interesting physical case of Euclidean stars, containing Newtonian stars in the limit, using the Lie analysis was studied by Govinder and Govender and Abebe et al. The results for matter, which are expanding, accelerating, and shearing, are contained in the works of Mohanlal et al. using the Lie method of symmetry generators.

A second approach to solving the boundary condition at the stellar surface is to exploit physical properties, such as the formation of horizons. This approach was first suggested by Ivanov for anisotropic spherical collapse for geodesic particles with shear. Later, Ivanov extended this method to accelerating matter with shear and radiation. Recently, generating functions in the presence of the electromagnetic field with the horizon function were considered by Ivanov. Maharaj et al. and Mohanlal et al. showed that Riccati equations arise in the presence of the horizon function using the Lie approach. Several new families of exact solutions were obtained with the horizon function. These solutions are helpful in describing gravitational collapse scenarios in radiating stars. In this process, the interior heat conducting matter loses energy across the stellar boundary in the form of null radiation to the exterior. The gravitational mass of the radiating star decreases with time.

In this paper, we consider the boundary condition at the stellar surface for a general spherically symmetric line element. The horizon function of Ivanov is utilized to reduce the Einstein–Maxwell system to generate a single evolution equation. The matter field is
We make a number of points relating to the regularity of the fields, the energy conditions at the stellar boundary, the physical relevance of exact solutions, and stability. First, the study of the junction conditions in general relativity has a long history starting with the treatment of Darmois; see the treatment of Bonnor and Vickers. A comprehensive analysis of the junction conditions for stellar surfaces and surface layers in a general relativistic setting was completed by Lake. In these treatments, there is the requirement of a maximal atlas, in which the metric is continuous. Mars and Senovilla considered the problem of matching two spacetimes across a general hypersurface. They found the conditions on the Riemann, Ricci, and Weyl tensors across the junction hypersurface of two matching spacetimes (see their Theorem 12). The particular case of matching two spherically symmetric spacetimes (containing our model) was completed by Fayos et al. In the case when the exterior spacetime is the Vaidya radiating metric, the junction conditions were explicitly generated by Santos. The spherically symmetric metric functions in the interior of the star that arise in specific models have to satisfy the Einstein field equations. Second, the energy conditions for an imperfect fluid with heat conduction were found by Kolassis and Abebe. These conditions can be applied to a collapsing sphere, which radiates energy to the exterior. Third, exact solutions to the boundary condition at the stellar surface were found which have desirable physical features. Kolassis and Rahaj have analyzed a model of a dissipating star, which is regular and contains the Friedmann dust model in the limit. Conformally flat models, which match to the Vaidya exterior, have been obtained by Herrera et al., Misnthy et al., and Herrera et al. Euclidean stars in general relativity, with equal areal and proper radii, containing Newtonian stars have been studied by Herrera et al. and Govender et al. Models with barotropic equations of state of the form \( p = \rho \) for pure radiation, have been studied by Abebe and Maharaj using the Lie analysis of differential equations; this class of models contains many earlier results. Fourth, the stability properties are difficult to study, in particular radiating stellar models. The general treatment for stability of locally anisotropic self-gravitating systems was completed by Herrera and Santos. Stability is related to the Weyl tensor in shear-free fluids as established by Herrera et al. Another approach by Reddy et al. to study stability is to use a perturbative scheme on an initially static configuration for a body undergoing collapse because of heat dissipation and anisotropic stresses. The temperature profiles in the interior of the star can be found in the Eckart theory and also in causal thermo-
$m(v) = \left[ \frac{Y}{2} \left( 1 + \frac{Y_i^2}{A^2} - \frac{Y_r^2}{B^2} \right) + \frac{\lambda Y}{2Y} - \frac{\ell^2}{6} \right] \xi$, (7d)

$\left( p_1 \right)_\Sigma = \left( q \right)_\Sigma$, (7e)

At the hypersurface $\Sigma$. Null radiation is emitted by a body with heat flow and generates a gravitational redshift given by

$$Z_\Sigma = \sqrt{\frac{L_\Sigma}{L_\infty}} - 1.$$ (8)

In Eq. (8), $L_\Sigma$ is the luminosity at the surface and $L_\infty = \frac{m}{r^2}$ is the luminosity as seen by an observer at infinity. Note the important condition in (7) that the radial pressure $p_1$ is nonvanishing at the surface $\Sigma$. Boundary condition (7e) can be written as the nonlinear differential equation

$$B_t = \frac{Y_t}{AY_r} + \frac{Y_r^2}{2AY_r} + \frac{A}{2YY_r} - \frac{A_1 Y}{A_1 Y_r} + \frac{\lambda AY}{2Y} - \frac{\ell^2 A}{2Y^2 Y_r}.$$ (9)

We can interpret (9) as a Riccati equation in $B$. The presence of cosmological constant $\lambda$ and electric charge $l$ substantially changes the form and nature of the boundary condition $p_1 = q$ on $\Sigma$. Thirukkanesh et al. found new classes of exact solutions for $\lambda = 0$ and $l = 0$ by transforming (9) to simpler forms. Ivanov also studied (9) using a particular transformation related to the formation of horizons, the so called horizon function, to obtain a generating function without the presence of the cosmological constant. For our investigation, we will consider (9) in general with $\lambda \neq 0$ and $l \neq 0$.

## III. NEW SOLUTIONS

Equation (9) has been studied before without the presence of the cosmological constant and electric charge. In our investigation, we consider how these parameters change the nature of the boundary condition by comparing previous results with new classes of exact solutions. We find that the parameters $l$ and $\lambda$ change the nature of the families of exact solutions that are possible. These parameters affect the gravitational dynamics of the collapsing star. The various solutions to the Riccati equation found are listed in Table I.

### A. Linear equation

We start by setting the coefficient of $B^2$ in (9) to zero, which gives

$$A_t = \left[ \frac{Y_t}{Y_r} + \frac{Y_r}{2Y} \right] A = \left[ \frac{1}{2YY_r} - \frac{\lambda Y}{2Y} - \frac{\ell^2}{2Y^2 Y_r} \right] A^3,$$ (10)

which is a Bernoulli equation in $A$. Solving (10) gives

$$A = \sqrt{3YY_r} \frac{\lambda Y - 3\ell^2 + 3Yf(r)}{\sqrt{Y^4 - 3Y^2 + 3Yf(r)}},$$ (11)

where $f(r)$ is the function of integration. Substituting (11) into (9) yields

$$B_t = \left( \frac{Y_t}{Y_r} - \frac{Y_r A}{YY_r} \right) B + \frac{A}{2} \left( \frac{2A_r + Y_r}{Y} \right) = 0,$$ (12)
which makes this a linear equation in $B$. The solution to (12) is given by
\[ B = Y, \exp\left[ -\int Y_A \, dt \right] \left( g(r) - f \left( \frac{Y_A}{Y_A} \right) + \frac{A}{2Y} \exp \left[ \int Y_A \, dt \right] \right), \]
where $g(r)$ is the integration constant, and we have
\[ Y = Y(r,t), \]
as an arbitrary function. When $I = \lambda = 0$, we regain the results of Thirukkanesh et al.\textsuperscript{15} A particular choice for potential $Y$ will lead to explicit forms for potentials $A$ and $B$.

### B. Bernoulli equation

By setting the inhomogeneous term in (9) to zero, we get
\[ \frac{A}{2Y} \left( \frac{2A}{A} + \frac{Y}{Y} \right) = 0, \]
which yields
\[ Y = f(t) A^2, \]
where $f(t)$ is the integration constant. With the help of (16), Eq. (9) becomes
\[ B_f = \left[ \frac{3f}{2} - f_0 \frac{4A}{A} + f_0 \frac{A}{A} \right] B = \left[ \frac{A^2}{24A} + \frac{A^8}{144A^2} + \frac{A^6}{24A^2} - \frac{A^4}{4A^2} - \frac{A^2}{A^2} - \frac{f^2}{4A^2} \right] B^2. \]
Equation (17) is a Bernoulli equation in $B$, which can be solved to give
\[ B = f^{3/2} A \]
where
\[ Q = \frac{f^{3/2} A}{4A^2} - \frac{A^8}{24A^2} + \frac{5f^{1/2} A^2}{A^2} + \sqrt{f} = f^{1/2} A + \frac{A^2}{4A^2}. \]
and the constant of integration is \( g(r) \). For this class of solutions, we have
\[
A = A(r, t),
\]
(19)
as an arbitrary function. When \( l = \lambda = 0 \), we regain the results of Thirukkanesh et al.\(^3\) as a special case. If \( A \) is specified, then we can obtain explicit forms for potentials \( B \) and \( Y \).

**C. Inhomogeneous Riccati equation**

Particular solutions to the Riccati equation [Eq. (9)] do exist, but the presence of the parameters \( f \) and \( \lambda \) leads to complications. Setting the coefficient of \( B \) to zero in (9) gives
\[
\frac{Y_{tr}}{Y_r} - \frac{Y_tA_r}{Y_A} = 0,
\]
(20)
which can be integrated to give
\[
A = f(t)Y_r,
\]
(21)
where \( f(t) \) is the constant of integration. By substituting (21) into (9), we obtain
\[
B_t = \left[ \frac{Y_t}{2YY_r} + \frac{fY_t}{2Yr} - \frac{f^2Y_t}{2Y^2r} - \frac{Y_{tr}f}{2Yr} - \frac{f_t}{f^2Y_r} \right] B^2 - \left[ fY_tY_r - fY_t \right].
\]
(22)
Equation (22) is not integrable as it is presented. If we set \( A = B \) in Eq. (22), we can regain the equations of Thirukkanesh et al.\(^3\) They presented an exact solution to Eq. (22) by assuming that the potential \( Y(r, t) \) is a separable function. Unfortunately, this approach does not work in (22). It is not possible to integrate (22) in general; particular solutions exist under certain assumptions. We demonstrate this below.

An interesting possibility arises if \( Y_t = 0 \). Then, (20) is identically satisfied but the relation in (21) does not hold. If we let \( Y_t = 0 \) in Eq. (9), then the boundary condition becomes
\[
B_t = f(r)B^2 + g(r),
\]
(24)
where
\[
f(r) = A \left[ \frac{1}{2Y} - \lambda Y - \frac{f}{Y^2} \right],
\]
(25a)
\[
g(r) = \frac{A}{2} \left[ \frac{2A_r}{A} + \frac{Y_t}{Y} \right].
\]
(25b)
An explicit solution to (24) can be found if we let
\[
af(r) = g(r),
\]
(26)
where \( a \) is the proportionality constant. Equation (26) implies
\[
A_r = \left( \frac{a}{2Y} \left[ \lambda Y + \frac{f}{Y^2} - \frac{1}{Y} \right] - \frac{Y_t}{2Y} \right) A_t.
\]
(27)
which is a separable equation in \( A \). Integrating (27) gives
\[
A = C_1 \exp \left[ \int \left( \frac{\alpha}{2Y^2} \left[ \lambda Y + \frac{f}{Y^2} - \frac{1}{Y} \right] - \frac{Y_t}{2Y} \right) dr \right].
\]
(28)
where \( C_1 \) is the integration constant. By using Eq. (26), we can integrate (24), which yields two solutions for \( B \). The first solution is given by
\[
B = \sqrt{\alpha} \tan \left( \sqrt{\alpha} \left( f(r)t + C_2(r) \right) \right), \quad \alpha > 0,
\]
(29)
where \( C_2(r) \) is the constant of integration. The second solution is
\[
B = \sqrt{-\alpha} \frac{C_3(r) \exp \left[ 2\sqrt{-\alpha} f(r)t \right] - 1}{C_3(r) \exp \left[ 2\sqrt{-\alpha} f(r)t \right] + 1}, \quad \alpha < 0,
\]
(30)
where \( C_3(r) \) is the constant of integration. For this class of solutions,
\[
Y = Y(r),
\]
(31)
which is an arbitrary function.

In the above solution, potential \( A \) is given in (28); potential \( B \) contains \( A \) and \( Y \) through the function \( f(r) \) and \( g(r) \), and \( Y(r) \) is arbitrary. This class of models obtained by integrating the Riccati equation is new. Observe that if \( \lambda = l = 0 \), then potential \( A \) becomes
\[
A = C_1 \exp \left[ - \int \left( \frac{\alpha}{2Y^2} + \frac{Y_t}{2Y} \right) dr \right].
\]
(32)
We observe that this uncharged model is also a new solution to the Riccati equation and is not contained in earlier treatments.

**IV. NEW SOLUTIONS: TRANSFORMED EQUATIONS**

Particular transformations reduce the boundary condition (9) to simpler forms. The horizon function was introduced by Ivanov,\(^1\) which gives a relation between the gravitational potentials and horizon function. The transformation is given by
\[
H = \frac{Y_t}{B} + \frac{Y_r}{A},
\]
(33)
where \( H = H(r, t) \). The horizon function is used as a transformation to simplify Eq. (9), which becomes
\[
H_t = \left[ \frac{A}{2Y} + \frac{A_r}{Y} \right] H^2 - \frac{A_A Y_t}{AY_r} + \frac{Y_t}{Y} H + \frac{A}{2Y^3} \left[ \lambda^2 Y^4 - Y^2 \right].
\]
(34)
This is also a Riccati equation in \( H \). We employ a similar method used to solve (9) in Sec. III to integrate (34) for which we find that three cases arise. The various solutions are listed in Table II.

**A. Linear equation**

We start by setting the coefficient of \( H^2 \) to zero, so that
\[
\frac{A}{2Y} + \frac{A_r}{Y} = 0.
\]
(35)
This can be solved to give
\[
A = f(t)\sqrt{Y},
\]
(36)
where $f(t)$ is the constant of integration. By substituting (36) into (34), we get

\[ H_{\lambda} + \left[ \frac{Y}{2Y} \right] H = \frac{f}{2Y^{2/2}} \left[ F + \lambda Y^4 - Y^2 \right], \tag{37} \]

which is a linear equation in $H$. We can solve this easily to give

\[ H = \frac{1}{\sqrt{Y}} \left[ \int \frac{f}{2Y^{3}} \left( F + \lambda Y^4 - Y^2 \right) dt + g(r) \right], \tag{38} \]

where $g(r)$ is the integration constant. By using Eq. (33), we can find the metric function $B$ to be

\[ B = \frac{\sqrt{YY_{\lambda}}}{f \int \frac{f}{2Y^{3}} \left( F + \lambda Y^4 - Y^2 \right) dt + g(r)} - YY_{\lambda}. \tag{39} \]

For this class of solutions,

\[ Y = Y(r, t) \tag{40} \]

is arbitrary. Note that this class of exact solutions is different from the linear solutions given in Sec. III. The horizon function

\[ \begin{array}{|c|c|c|}
\hline
\text{Cases} & \text{Gravitational potentials} & \text{Features} \\
\hline
l \neq 0 & A = \frac{f(t)}{Y} & \text{New model} \\
& B = f \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) - YY_{\lambda} & \text{Explicit solution} \\
& H = \frac{1}{\sqrt{Y}} \left[ \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) \right] & \text{Generating function $Y$} \\
\hline
l = 0 & A = \frac{f(t)}{Y} & \text{New model} \\
& B = \frac{1}{2Y^{2/2}} \left[ \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) \right] & \text{Explicit solution} \\
& H = \frac{1}{\sqrt{Y}} \left[ \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) \right] & \text{Generating function $Y$} \\
\hline
l \neq 0 & A = \frac{f(t)}{Y} & \text{New model} \\
& B = \frac{1}{2Y^{2/2}} \left[ \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) \right] & \text{Explicit solution} \\
& H = \frac{1}{\sqrt{Y}} \left[ \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) \right] & \text{Generating function $Y$} \\
\hline
l = 0 & A = \frac{f(t)}{Y} & \text{New model} \\
& B = \frac{1}{2Y^{2/2}} \left[ \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) \right] & \text{Explicit solution} \\
& H = \frac{1}{\sqrt{Y}} \left[ \int \frac{f}{2Y^{3}} (F + \lambda Y^4 - Y^2) dt + g(r) \right] & \text{Generating function $Y$} \\
\hline
\end{array} \]
transformation (33) leads to a new class of exact models to the boundary condition (9).

**B. Bernoulli equation**

We can transform (34) into a Bernoulli equation in $H$. The condition for obtaining a Bernoulli equation is that

$$
\lambda Y^4 - Y^2 + f^2 = 0, 
$$

which is a quadratic equation in $Y$. This condition is satisfied only if $Y = Y(r)$. Unfortunately, this condition is inconsistent with transformation (33), which requires $Y \neq 0$. Hence, it is not possible to generate Bernoulli equations with the horizon function. This is in contrast with the results of Sec. III. We conclude that transformation (33) restricts the number of solutions that are admissible.

**C. Inhomogeneous Riccati equation**

The Riccati equation [Eq. (34)] in $H$ cannot be solved in general. Setting the coefficient of $H$ to zero gives

$$
\frac{A_r Y_f}{A_f Y_r} + \frac{Y_f}{Y} = 0. 
$$

This is a separable equation that can be solved to yield

$$
A = \frac{f(t)}{Y},
$$

where $f(t)$ is the integration constant. Substituting (43) into (34), we get

$$
H_f = -\frac{f}{2Y^2}\left[H^2 + \frac{f}{2Y}\left[f^2 + \lambda Y^4 - Y^2\right]\right],
$$

which is a simpler Riccati equation in $H$. It is difficult to solve (44) to obtain an expression for $H$. However, it is possible to express Eq. (44) in the form

$$
(f\lambda - 2H_f)Y^4 - f(H^2 + 1)Y^2 + f^2 = 0. 
$$

We can treat (45) as an algebraic equation in variable $Y$. Four cases arise in the solution of (45). We consider these in turn. The various cases are listed in Table II.

1. **Case 1: $\lambda \neq 0, l \neq 0$**

In this case, we can solve (45) to get

$$
Y = \sqrt{\frac{f(H^2 + 1) \pm \sqrt{f^2(H^2 + 1)^2 - 4f(\lambda - 2H)Y^2}}{2f(\lambda - 2H)}}.
$$

where $H > \left[2l\sqrt{\frac{\lambda/2 - H}{\lambda}} - 1\right]^{1/2}$ and $H_f < \frac{1}{2}f\lambda$. Hence, we have a new family of exact solutions to the Riccati equation [Eq. (44)]. In this class, we observe that potentials $A$ and $Y$ are given in terms of the horizon function $H$. The function $H(r, t)$ is arbitrary. Here, the quantity $H$ plays the role of a generating function; a particular choice of $H$ will lead to analytical forms for $A$ and $Y$.

Despite the complicated form of potential $Y$ in (46), it is possible to regain potential $B$ explicitly. By substituting (46) into (43) and (33), we obtain

$$
B = \frac{2H_f(\lambda + K_1) + (f\lambda - 2H_f)K_1}{2\sqrt{2f(\lambda - 2H_f)[f(H^2 + 1) \pm K_1]}H(\lambda - 2H_f) - \frac{1}{1}K_2 \pm \frac{1}{\sqrt{4(\lambda - 2H_f)^2}}K_3},
$$

where

$$
K_1 = 2f\theta H_{ff} \pm 2f^2((H + H^3)H_f - 2\lambda f_h) + 4f^2(2lH_f + f\theta H_{ff})/K_4,
$$

$$
K_2 = (H^2 + f^2 + 2f\theta H_{ff} - \frac{1}{K_4})(2f^2H_{ff}(H^2 + 1) + 4f^2H_fH_f + f\theta((H^2 + 1)^2 - 4f^2H_f)/K_3),
$$

$$
K_3 = [K_4 + f(H^2 + 1)(f\theta - 2H_f)],
$$

$$
K_4 = \sqrt{2f^2(H^2 + 1)^2 - 4f(\lambda - 2H_f)^2}/K_2.
$$

In these expressions, the role of the horizon function $H$ as a generating function is highlighted.

2. **Case 2: $\lambda = 0, l \neq 0$**

Solving (45) gives two solutions for

$$
Y = \sqrt{\frac{f(H^2 + 1) \pm \sqrt{f^2(H^2 + 1)^2 - 8f^2H_f}}{-4H_f}},
$$

where $H > \left[2l\sqrt{\frac{-2H_f}{\lambda}} - 1\right]^{1/2}$ and $H_f < 0$. By substituting (48) into (43) and (33), the first solution, due to a positive inner root, is given by

$$
B = \frac{2f^2H^2_{ff} + [4f^2H_{ff} - 8lH_fH_f + K_1]}{K_3\sqrt{-H^2_{ff} + f(H^2 + 1) + K_3] + 8f^2H_f^2H_f - H_f(f(H^2 + 1) + K_3) + H_fK_2}. 
$$
Then, solving (45) gives

\[ K_1 = (K_3 + f(H^2 + 1))\left[H_t(H^2 + 1) - 2HH_tH_t\right], \]
\[ K_2 = f_t(H^2 + 1) + 2fHH_t + \frac{1}{K_3} \left(2f^2HH_t(H^2 + 1) + 4f^2f_tH_t + f_t(H^2 + 1)^2 + 4f^2H_t\right), \]
\[ K_3 = \sqrt{f^2(H^2 + 1)^2 + 8f^2H_t}. \]

The second solution, due to a negative inner root, is given by

\[ B = \frac{16f^2l_N\sqrt{H^2 - 8f^2\sqrt{H_t^2H_t + K_1}}}{\sqrt{K_1 - f(H^2 + 1)[f_t(H^2 + 1)K_4 - 4f^2H_t^2 + K_2 + K_3]}}, \] \hspace{1cm} (50)

where

\[ K_1 = 2f^2\sqrt{H_t}(K_4 - f(H^2 + 1)(H_t^2 + 2HH_tH_t)), \]
\[ K_2 = f^2(H^2 + 1)(H_t^2 + 2HH_t^2), \]
\[ K_3 = 10fHH_t^2K_4 - fK_4H_t(H^2 + 1) + 6fH_t[H_t^2 - f_t(H^2 + 1)^2], \]
\[ K_4 = \sqrt{f^2(H^2 + 1)^2 + 8f^2H_t}. \]

Variable \( H \) can be used as a generating function for both solutions in (49) and (50).

### 3. Case 3: \( \lambda \neq 0, l = 0 \)

Another case occurs when we set \( l = 0 \) and \( \lambda \neq 0 \) in Eq. (45). Then, solving (45) gives

\[ B = \frac{f^2\sqrt{f\lambda - 2H_t}\left[HH_t(f\lambda - 2H_t) + H_t(H^2 + 1)\right]}{\sqrt{f(H^2 + 1)[fH_t(H^2 + 1) - H_t(f_t(H^2 + 1) + 6fHH_t)]}}. \] \hspace{1cm} (52)

For this class of solutions, we can take \( H \) to be a generating function.

### 4. Case 4: \( \lambda = 0, l = 0 \)

For this case, we obtain the solution

\[ Y = \sqrt{\frac{f(H^2 + 1)}{2H_t}}, \] \hspace{1cm} (53a)
\[ B = \frac{\sqrt{2H_t}f^2[H_t(H^2 + 1) - 2HH_tH_t]}{\sqrt{f(H^2 + 1)[fH_t(H^2 + 1) - H_t(f_t(H^2 + 1) + 6fHH_t)]}}, \] \hspace{1cm} (53b)

where \( f(H^2 + 1) < 0 \) and \( H_t \neq 0 \). This solution corresponds to an uncharged model and there is no cosmological constant. This is also a new exact solution to the boundary condition (34). A simple exact explicit solution to the Riccati equation in (9) was found by Thirukkanesh et al. In their model, potentials \( A \) and \( Y \) are separable functions. In our case, the Ivanov transformation (33) leads to another family of solutions and potentials \( A = \frac{\xi(\xi)}{r} \), \( B \), and \( Y \) have different functional forms. The quantity \( H \) serves as a generating function in our case.

### V. DISCUSSION

This paper consists of two important approaches that discuss the boundary condition for a radiating star in general relativity with cosmological constant and electric charge. We first studied the boundary condition as a Riccati equation in potential \( B \). This Riccati equation is very difficult to solve in general. However, we obtained exact solutions for three cases: Linear, Bernoulli, and inhomogeneous Riccati. The linear and Bernoulli cases are extensions of the work of Thirukkanesh et al. The inhomogeneous Riccati case is a special case that allows the equation to become a separable equation after making certain assumptions for potentials \( A \) and \( B \). This allows for a simple new class of exact solutions. We next applied a transformation to the boundary condition, which was first introduced by Ivanov, called the horizon function. This transformation resulted in a simplification of the boundary condition, which was then expressed as a Riccati equation in variable \( H \). Three cases arose...
The second junction condition is obtained by requiring condition (7d) as

\[ 2M = Y\left(\frac{1}{3} - H^2\right) + \frac{1}{3} \lambda Y^2 + \frac{5}{3} \frac{P}{Y}. \]  

Note that (54) becomes Eq. (38) in Ivanov when \( \lambda = l = 0 \). We also find the time derivative to be

\[ 2M_t = Y_t\left(\frac{1}{3} - 3H^2\right) + Y_t\left(\lambda Y^2 - \frac{5}{3} \frac{P}{Y}\right). \]  

These expressions show how the cosmological constant and electric charge play a role in the mass function and its derivative. The compactness parameter \( \frac{2M}{M} \) can also be obtained using (54). These physical quantities are given in terms of potential \( H \), which serves as a generating function. Specific choices of the generating function will permit a detailed physical analysis of the evolution of the star. This is the object of further research.

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**APPENDIX A: DERIVATION OF BOUNDARY CONDITION**

In this appendix, we provide an outline of the derivation of the junction condition (9). This follows from the matching of the extrinsic curvature at the stellar boundary. We follow the approach of Santos, de Oliveira and Santos, among others. More detailed information can be found in these papers.

The boundary of a radiating star connects two distinct regions of spacetime, the interior spacetime \( \mathcal{M} \) and the exterior spacetime \( \mathcal{M}' \). Each of these regions contains \( \Sigma \) as its boundary, a timelike three-space. The interior spacetime is given by the metric (1), and the exterior spacetime is given by the metric (6). Continuity of the intrinsic metrics across \( \Sigma \) gives the first junction condition. The second junction condition is obtained by requiring

\[ K_{\alpha\beta}^+ = K_{\alpha\beta}^-, \]  

which is the continuity of the extrinsic curvature across the surface \( \Sigma \). The intrinsic metric to \( \Sigma \) is given by

\[ ds^2 = -d\tau^2 + Y^2(\dot{\theta}^2 + \sin^2 \theta d\phi^2), \]  

with coordinates \( \xi^a = (\eta, \theta, \phi) \) and \( Y = Y(\eta) \) only on the surface \( \Sigma \). The nonzero components of the extrinsic curvature \( K_{\alpha\beta}^\Sigma \) are given by

\[ K_{00}^\Sigma = -\left(\frac{1}{B} \frac{A_t}{A}\right)_\Sigma, \]  

\[ K_{0t}^\Sigma = \left(\frac{Y Y_t}{B}\right)_\Sigma, \]  

\[ K_{\phi\phi}^\Sigma = \sin^2 \theta K_{00}^\Sigma, \]  

for the interior spacetime. The defining equation for the surface \( \Sigma \) in \( \mathcal{M}' \) is given by

\[ f(r, \nu) = r - r_\Sigma(\nu) = 0. \]  

We can show that the nonvanishing components of the extrinsic curvature tensor become

\[ K_{0t}^\Sigma = \left[\dot{\nu} - \frac{m}{r^2} + \frac{Q^2}{r^2} - \frac{\lambda r \nu}{3}\right]_\Sigma, \]  

\[ K_{0\phi}^\Sigma = \left[\dot{\nu}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\lambda r^2}{3}\right) + \dot{r} r\right]_\Sigma, \]  

\[ K_{\phi\phi}^\Sigma = \sin^2 \theta K_{00}^\Sigma, \]  

for the exterior spacetime. The appropriate extrinsic curvature components (A3) and (A7) imply that

\[ \left(\frac{1}{B} \frac{A_t}{A}\right)_\Sigma = \left[\frac{\dot{\nu}}{\nu} - \frac{m}{r^2} + \frac{Q^2}{r^2} - \frac{\lambda r \nu}{3}\right]_\Sigma, \]  

\[ \left(\frac{Y Y_t}{B}\right)_\Sigma = \left[\dot{\nu}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\lambda r^2}{3}\right) + \dot{r} r\right]_\Sigma. \]  

An expression of the mass function can be found after eliminating \( r \), \( t \), and \( \nu \). This produces the result

\[ m(\nu) = \frac{Y}{2}\left[1 + \frac{Y_t^2}{A^2} - \frac{Y^2}{B^2}\right] + \frac{Q^2}{2Y} - \frac{\lambda Y^2}{6}. \]  

We can also establish the relationship

\[ \left(\frac{1}{B} \frac{A_t}{A}\right)_\Sigma = \left[\frac{B_t Y_t}{A B^2} - \frac{Y_t}{A} - \frac{Y_t Y_t}{A^2} - \frac{Y_t^2}{2A^2 Y} + \frac{Y^2}{2Y^2} - \frac{1}{2Y} + \frac{Q^2}{2Y^3} + \frac{\lambda Y}{2}\right]_\Sigma. \]  

Multiplying the above equation by \([Y_t/A] + (Y_t/B)\) and using (A3) and (A7), we obtain the result

\[ (p_1)_\Sigma = (q)_\Sigma. \]
This result shows that the pressure at the boundary of a radiating star is proportional to the heat flux. In terms of the metric potentials, we have

\[
B_t = \left( \frac{Y_{tt}}{2AY_r} + \frac{Y_r^2}{2AY_r} + \frac{AY_{tt}}{AY_r} - \frac{\lambda AY_r}{2Y_r} - \frac{Q^2A}{2Y^3Y_r} \right) B^2 + \left( \frac{Y_{tt}}{Y_r} - \frac{Y_r A_{tt}}{Y_r A} \right) B - \frac{A}{2} \left( \frac{2A Y_r}{A} + \frac{Y_r}{Y} \right). \tag{A15}
\]

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