Development of the mathematical model for the tilt-rotor aircraft

K S Lelkov¹, D V Ulyanov², D A Surkov² and A N Ushakov²

¹Department 305 “Automated complexes of orientation and navigation systems”, Moscow Aviation Institute (National Research University), 4 Volokolamskoe Highway, 125993, Moscow, Russian Federation
²Experiment Automatization Department, Moscow Aviation Institute (National Research University), 4 Volokolamskoe Highway, 125993, Moscow, Russian Federation

*E-mail: lelkovks@mai.ru
**E-mail: UlyanovDV@mai.ru

Abstract. In this paper, main challenges of the urban air mobility concept are discussed and comparison of the different aircraft types, capable of performing vertical take-off and landing for compliance to the urban aerial mobility concept is provided. The mathematical model for the tilt-rotor aircraft is derived by the Newton-Euler method. This model takes into account all the forces and moments applied to the aircraft. Aerodynamic characteristics for the developed aircraft were obtained using virtual aerodynamic blowing method. Simulation results of the tilt-rotor aircraft motion on the various flight stages, including vertical take-off and landing, horizontal flight and transition stages are presented.

1. Introduction

In recent years urban aerial vehicle applications have become trending worldwide. In this light, National Aeronautics and Space Administration (NASA, USA) has developed the Urban Air Mobility (UAM) concept – the system for air passenger and cargo transportation within an urban area, inclusive of small package delivery and other urban unmanned aircraft system services [1]. This concept has been approved by many countries and a huge number of research and developments have been already made to support it. In order to bring UAM to our everyday lives, a number of challenges must be solved [2,3].

The biggest challenge is safety. The new aerial mobility systems should contain very high safety threshold to ensure that the air transportations is the safest method of transportation there is.

The second challenge is collaboration. To bring these ideas and technologies to life would require the whole aviation industry working together to make this a success.

Administration challenges include airspace management and certification. Airspace diversity is increasing every year and maintaining it while keeping all air traffic moving safely and efficiently will be challenging on its own, but to do so while certifying an entirely new type of aircraft vehicles using a regulatory framework developed for traditional fixed-wing aircraft will be the significant challenge.

Other challenges related directly to aircraft development are autonomy and accessibility. Autonomy is the ability of the aircraft control system to control a vehicle and to respond to various unexpected events and hazards. Today there are two different approaches to achieve a fully autonomous aircraft.
First one, fall-back pilot is a safety pilot system that is always online and is ready to take control of the vehicle at any time for any reason. The second one – full autonomy from the beginning of the development process. Using the fall-back pilot approach enables the aircraft to become airborne more quickly (decreases the development time), but it requires significant investment in systems that will become obsolete in the near future. Development of the fully autonomous aircraft eliminates the need for human-machine interfaces, but obtaining public acceptance and certification for this kind of vehicle can be a challenge. Accessibility challenge refers to the ability of the users (individual passengers and transportation companies) to access aerial vehicle framework. In most cases, this requires the vehicle to be able to perform the vertical take-off and landing (VTOL) within the city area.

In the department 305 (automated complexes of orientation and navigation systems) of the Moscow Aviation Institute (MAI) in collaboration with Experiment Automatization Department, the research is being carried to develop the unmanned aerial vehicle (UAV), capable of transporting passengers within the city area. During the research, we analyzed different types of vehicles capable of performing VTOL.

The first candidates were wingless multirotor vehicles [4]. The key advantages were simple vehicle structure and low efforts required to develop a control system. However, the disadvantages were severe: limited flight speeds, limited energy efficiency in cruise flight and no gliding capabilities (that also results in lesser energy efficiency and safety penalty).

The second candidates were fixed-wing lift & cruise vehicles [5]. These vehicles have medium energy efficiency since there is no need for all the rotors to be working in the same time, the control system is relatively easy to develop and gliding possibility allows to meet additional UAM requirements. On the other hand, extra weight from carrying temporarily unused engines results in ineffective use of propulsion. And additional rotors lead to increased drag in horizontal flight (if not retractable).

The third candidate was tilt-rotor aircraft [6]. Tilt-rotor structure brings in good energy efficiency with effective use of propulsion systems and gliding possibility, good cargo lift parameters due to no extra weight for carrying temporarily unused engines. However the hardware structure provides a certain challenge to develop the rotating engine's mechanism, and flight control system is relatively hard to develop.

**Figure 1.** The tilt-rotor aircraft with four rotating engines.

Summarizing these results, tilt-rotor aircraft structure with four rotating engines (figure 1) was chosen since it is the most suited for the UAM applications with VTOL capability, huge cargo capacity and high energy efficiency. The purpose of this research is to develop the mathematical model for the described tilt-rotor aircraft. The development of this model is an essential requirement for conducting further research on the autonomous control systems for tilt-rotor aircraft.

2. **Mathematical model**

The definitions of the main aircraft wing-body reference framed and corresponding conversion matrices describing rotation from one frame to another are well known [7]. In this paper, there are two coordinate
systems. The body-fixed frame \( O_{b}x_{b}y_{b}z_{b} \) is attached to the centre of gravity of the aircraft. The origin point \( O_{g} \) of the inertial earth-fixed frame \( O_{g}x_{g}y_{g}z_{g} \) is located at aircraft’s starting point. Axis \( O_{g}x_{g} \) aligns with the north direction and axis \( O_{g}z_{g} \) - with the east. Described reference coordinate frames are shown in figure 2. According to the principle of coordinate transfer [8], the rotation matrix describing the rotation from the earth-fixed frame to the body-fixed frame is given by

\[
R_{g}^{b} = \begin{pmatrix}
\cos \psi \cos \vartheta \cos \gamma - \sin \psi \sin \gamma \\
- \cos \psi \cos \vartheta \sin \gamma - \sin \psi \cos \gamma \\
\sin \vartheta \cos \gamma & \cos \vartheta \cos \gamma & - \sin \vartheta
\end{pmatrix}.
\]  

Figure 2. Aircraft orientation angles and the tilting angle of the engines: (a) the relation between the earth-fixed coordinate frame and the body-fixed frame; (b) the positive direction of engines rotation.

The nonlinear dynamic model of the tilt-rotor aircraft motion can be derived in the form of a Newton-Euler formulation [9]. For the body-fixed frame it can be expressed as follows:

\[
mV_{b} + \omega_{b} \times V_{b} = F_{b};
\]

\[
I_{b} \ddot{\omega}_{b} + \omega_{b} \times I_{b} \dot{\omega}_{b} = M_{b},
\]

where \( m \) is the mass of the aircraft, \( F_{b} = [F_{xb} F_{yb} F_{zb}] \) and \( M_{b} = [M_{xb} M_{yb} M_{zb}] \) represent the total force and moment vectors in the body-fixed frame, \( V_{b} = [V_{xb} V_{yb} V_{zb}] \) is the velocity vector in the body-fixed frame, \( \omega_{b} = [\omega_{xb} \omega_{yb} \omega_{zb}] \) is the vector of angular rotations in the body-fixed frame, and \( I_{b} \) is the inertia tensor of the rigid body and can be expressed as

\[
I_{b} = \begin{pmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{pmatrix}.
\]

The forces and moments in (2) are contributed by gravity, aerodynamics and the propulsion system. The can be expressed as
\[
E_b = E_b^g + E_b^a + E_b^p;
\]
\[
M_b = M_b^g + M_b^a + M_b^p,
\]
where \(E_b^g, E_b^a, E_b^p, M_b^g, M_b^a, M_b^p\) are the component forces and moments described below.

The gravity force in (4) can be expressed as
\[
E_b^g = \begin{pmatrix}
-mg \sin \theta \\
-mg \cos \theta \cos \gamma \\
mg \cos \theta \sin \gamma
\end{pmatrix},
\]
where \(g\) is the gravity acceleration constant.

Aerodynamic forces in (4) are presented as follows [10]:
\[
E_b^a = \begin{pmatrix}
c_x \frac{\rho V^2}{2} S_{y} \\
c_y \frac{\rho V^2}{2} S_{x} \\
c_z \frac{\rho V^2}{2} S_{z}
\end{pmatrix},
\]
where \(c_x, c_y, c_z\) are aerodynamic parameters of the aircraft described in section 3 of this paper, \(\rho\) is the air density, \(S_{x}\) is the characteristic area of the aircraft, \(V\) is the module of the velocity vector.

Propulsion forces in (4) can be decomposed as the projections of the thrust vector along the body-fixed frame axis:
\[
E_b^p = \begin{pmatrix}
(P_1 + P_2 + P_3 + P_4) \sin \varepsilon \\
(P_1 + P_2 + P_3 + P_4) \cos \varepsilon \\
0
\end{pmatrix},
\]
where \(P_1, P_2, P_3, P_4\) are the thrust force generated by each of the aircraft’s engines, \(\varepsilon\) is the tilting angle of the engines.

Aerodynamic moments in (4) are presented as follows [10]:
\[
M_b^a = \begin{pmatrix}
m_x \frac{\rho V^2}{2} S_{l_x} \\
m_y \frac{\rho V^2}{2} S_{l_y} \\
m_z \frac{\rho V^2}{2} S_{l_z}
\end{pmatrix},
\]
where \(m_x, m_y, m_z\) are aerodynamic parameters of the aircraft described in section 3 of this paper, \(l_x\) is the wingspan of the aircraft, \(l_y\) is the aircraft’s length.

The moment vector generated by the propulsion systems can be expressed as
\[
M_b^p = \begin{pmatrix}
l_{w1} (P_1 - P_2 - P_3 + P_4) \cos \varepsilon \\
l_{w1} (-P_1 + P_2 + P_3 - P_4) \sin \varepsilon + k_{r} (-P_1 + P_2 - P_3 + P_4) \cos \varepsilon \\
l_{w1} (P_1 + P_2 - l_{w2} (P_3 + P_4)) \cos \varepsilon + (l_{w1} (P_1 + P_2) - l_{w2} (P_3 + P_4)) \sin \varepsilon
\end{pmatrix},
\]
where \(l_{w1}\) is the distance from \(O_{b}x_{b}\) axis to the point of application of the thrust force shown in figure 3a, \(k_{r}\) is the reactive torque coefficient, \(l_{w1}, l_{w2}\) are the distance from \(O_{b}z_{b}\) axis to the points of application of the thrust force shown in figure 3b.
Figure 3. Constructive dimensions of the tilt-rotor aircraft.

Taking into consideration all forces and moments described in (4)-(9), the full mathematical model for the tilt-rotor aircraft motion in the body-fixed frame can be derived:

\[
\frac{dV_{sh}}{dt} = \frac{1}{m} \left( P_1 + P_2 + P_3 + P_4 \right) \sin \theta + c_z \frac{\rho V^2}{2} S - mg \sin \theta \right) - (\omega_{sh} V_{sh} - \omega_{hV_{sh}}); \\
\frac{dV_{sh}}{dt} = \frac{1}{m} \left( P_1 + P_2 + P_3 + P_4 \right) \cos \theta + c_y \frac{\rho V^2}{2} S - mg \cos \theta \cos \gamma \right) - (\omega_{sh} V_{sh} - \omega_{hV_{sh}}); \\
\frac{dV_{sh}}{dt} = \frac{1}{m} \left( mg \cos \theta \sin \gamma + c_z \frac{\rho V^2}{2} S \right) - (\omega_{sh} V_{sh} - \omega_{hV_{sh}}); \\
\frac{d\omega_{sh}}{dt} = \frac{1}{I_x} \left( l_x \left( P_1 - P_2 - P_3 + P_4 \right) \cos \theta + m_x \frac{\rho V^2}{2} S l_x + (I_{yy} - I_{zz}) \omega_{sh} \right); \\
\frac{d\omega_{sh}}{dt} = \frac{1}{I_y} \left( l_y \left( -P_1 + P_2 + P_3 - P_4 \right) \sin \theta + m_y \frac{\rho V^2}{2} S l_y + (I_{zz} - I_{xx}) \omega_{sh} \right); \\
\frac{d\omega_{sh}}{dt} = \frac{1}{I_z} \left( l_z \left( P_1 + P_2 - P_3 + P_4 \right) \cos \theta + \\
\left( I_{yx} - I_{xz} \right) \omega_{sh} \right); \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]; \\
\frac{d\omega_{sh}}{dt} = \frac{1}{I_z} \left( l_z \left( P_1 + P_2 - P_3 + P_4 \right) \cos \theta + \\
\left( I_{xz} - I_{yz} \right) \omega_{sh} \right); \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]; \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\frac{d\omega_{sh}}{dt} = \frac{1}{I_z} \left( l_z \left( P_1 + P_2 - P_3 + P_4 \right) \cos \theta + \\
\left( I_{xz} - I_{yz} \right) \omega_{sh} \right); \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]; \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)]. \\
\sin \gamma + m_z \frac{\rho V^2}{2} S l_z - \left( I_{yy} - I_{xx} \right) \omega_{sh} \omega_{sh} \right)].
\]

To transform these equations to the earth-fixed frame aircraft’s orientation angles must be found from translation and rotation kinematic equations derived according to the transformation relationship between coordinate systems.

\[
\frac{d\psi}{dt} = \frac{1}{\cos \gamma} \left( \omega_{sh} \cos \gamma - \omega_{hsh} \sin \gamma \right); \\
\frac{d\theta}{dt} = \omega_{sh} \cos \gamma + \omega_{hsh} \sin \gamma; \\
\frac{d\phi}{dt} = \omega_{sh} - \left( \omega_{sh} \cos \gamma - \omega_{hsh} \sin \gamma \right) \tan \theta.
\]

Given (11), the velocity of the aircraft in the earth-fixed frame can be expressed as
The coordinates and orientation angles of the tilt-rotor aircraft can be found by integrating expressions (11)-(12). In conclusion, the six degrees of freedom nonlinear dynamic model of the tilt-rotor aircraft can be described by the equations (10)-(12).

3. Aerodynamics research

To determine aerodynamic coefficients of the tilt-rotor aircraft we used flow 5 software pack [11], as it provides various in-depth methods for aerodynamic analysis, such as Lifting-Line Theory, Vortex Lattice Method, and Uniform Density Triangular Panel method which we used in this research. Advantage of using this software is the ability to quickly and conveniently obtain results for the analysis of the developed preliminary designs of wings, fuselages and other aerodynamic elements of aircraft.

In order to analyze aerodynamic effects, rear wing profile design, shown in figure 4 was developed [12]. The characteristics of the wing control surface (aileron) are as follows: deflection angle \( \delta \in [-25,25] \); aileron’s linear dimension along \( X_f \) axis equals 20% of the wing profile’s side chord.

![Figure 4. Rear wing profile design with CLARK Y aileron.](image)

For front wings, NACA0012H profile design was chosen. Wing control surface for this wing profile has the following parameters: deflection angle \( \delta \in [-25,25] \); aileron’s linear dimension along \( X_f \) axis equals 25% of the wing profile’s side chord. Front wing profile design is shown in figure 5.

![Figure 5. Front wing profile design with NACA0012H aileron.](image)

When designing the surface of the UAV fuselage we used the ability of flow5 software to import a surface from a CAD file as shown in figure 6.

The results of simulation results for designed profiles allowed us to obtain aerodynamic coefficients as functions of the current attack angle and aileron deflection angle. In this research, we assumed that the slippage angle equals zero, and the ailerons of both wings (front and rear) are deflected symmetrically. We also didn’t take into account the aerodynamic interference from UAV’s motors.

For the simulation purposes attack angle \( \alpha \) and slippage angle \( \beta \) can be defined as follows:
\[ \alpha = \arctan \frac{-V_{ph}}{V_{sb}}; \]
\[ \beta = \arcsin \frac{V_{sb}}{V}. \]  

(13)

Figure 6. The simulation of aerodynamic forces for complete UAV model.

The results of aerodynamic research for the developed preliminary designs of wings and fuselage are presented in figure 7. These functions will be used in the simulation to determine aerodynamic forces and moment affecting the aircraft. The real aerodynamic parameters may differ, however. This will require further research in the aerodynamic tube once the prototype of the tilt-rotor aircraft is developed.

Figure 7. Aerodynamic coefficients of the aircraft: a) lift dependency; b) drag dependency.

4. Simulation

In this section, the results of the tilt-rotor aircraft motion according to the described model are presented. The proposed mathematical model was simulated on all stages of the aircraft motion: vertical take-off in helicopter mode, transition to plane mode, horizontal flight in plane mode with a coordinated turn (without slippage), transition to helicopter mode and vertical landing in helicopter mode. The simulation
parameters are: \( l_x = 3.129; l_{xw} = 1.75; l_{yw} = 0.036; l_{zw} = 1.0; l_{zw} = 0.84; \) \( l_z = 2.8; \)
\( m = 108; S_x = 2.18; J_{xx} = 143.27; J_{yy} = 212.57; J_{zz} = 81; \) \( \rho = 1.22; g = 9.81; k_r = 0.05. \) Initial conditions were set as: \( x_0 = 0; y_0 = 0; z_0 = 0; \psi_0 = 0; \delta_0 = 0; \gamma_0 = 0. \)

Figure 8 shows that velocity of the aircraft in the earth-fixed frame changes according to the motion stage. During vertical take-off \( V_{yg} \) increases and slowly falls down as UAV is approaching target safe attitude. During the transition from helicopter mode to plane mode, horizontal velocity \( V_{xg} \) increases to provide sufficient lift force for the UAV to stay in the air. During horizontal flight stage, vertical velocity is essentially zero and the velocity vector only has projections on axis \( O_g x_g \) and axis \( O_g z_g \), once a coordinated turn is made. During the transition to the helicopter mode, all velocities come to a halt as the aircraft performs aerodynamic deceleration. During vertical landing \( V_{yg} \) has negative values and slowly decreases as UAV is approaching the landing pad.

In figure 9 the horizontal trajectory of the aircraft is presented. Transition stages are marked with black circles. Simulation time took 420 seconds and UAV translocation during this time equals to 9450 meters with an average velocity of 22.5 m/s.

The simulation results proved that the developed mathematical model of the tilt-rotor aircraft is adequate on all flight stages and in both flight modes. The analysis shows that it is possible to maintain stable flight during the transition process from one flight mode to another and the development of the corresponding control system is subject to subsequent research. Derived mathematical model correlates with the results of similar research for slightly different UAV designs [6,7], but the proposed method for accounting for aerodynamic forces appears to be more accurate as it is based on full-scale aerodynamic analysis for the given tilt-rotor aircraft design.

![Figure 8. Velocities of the UAV in the earth-fixed frame. Blue line indicates velocity along \( O_g x_g \) axis; green line – velocity along \( O_g z_g \) axis; red line – velocity along \( O_g y_g \) axis.](image)

![Figure 9. The horizontal trajectory of the UAV in the earth-fixed frame.](image)

5. Conclusion
This paper described the main challenges of the UAM concept and provides a comparison of the different aircraft types to be used as an autonomous transportation vehicle, capable of performing VTOL. The development process of the mathematical model of the tilt-rotor aircraft is presented along with the final results – differential motion equations. Aerodynamic research for the developed aircraft model was carried out with flow5 software and approximate functions for aerodynamic coefficients were obtained. Simulation results of the tilt-rotor aircraft motion on every flight stage proved the adequacy of the developed mathematical model.
The proposed mathematical model plays a significant role in the tilt-rotor aircraft design. It provides a baseline for further research and control system development. In particular, it is shown, that it is possible to design a single state point controller to control aircraft’s motion in both flight modes as well as during the transition from one mode to another.

Acknowledgements
This work was carried out in collaboration with Central Aerodynamic Institute (TsAGI). The authors would like to acknowledge the financial support from TsAGI and the assistance from Krugov A.A.

References
[1] The Future of Transportation: White paper on urban air mobility, available at: https://www.ehang.com/app/en/EHang%20White%20Paper%20on%20Urban%20Air%20Mobility%20Systems.pdf
[2] Postorino M N and Sarne G M L 2020 Reinventing mobility paradigms: flying car scenarios and challenges for urban mobility. *MDPI Sustainability* **12** 3581 doi: 10.3390/su12093581
[3] Cokorilo O 2020 Urban air mobility: safety challenges. *Transport. Res. Proc.* **45** 21 doi: 10.1016/j.trpro.2020.02.058.
[4] Pradeep P, Lauderdale T, Chatterji G, Sheth K, Lai C, Sridhar B, Edholm K M and Erzberger H 2020 Wind-Optimal trajectories for multirotor eVTOL aircraft on UAM missions. *Proc. of AIAA AVIATION 2020 FORUM* (Reston, USA: AIAA) p 14 doi: 10.2514/6.2020-3271.
[5] Yayli et al 2017 Design optimization of a fixed wing aircraft *Advances in Aircraft and Spacecraft Science* **4** 65-80 doi: 10.12989/aas.2017.4.1.065.
[6] Zaibin C and Hongguang J 2020 Design of flight control system for a novel tilt-rotor UAV. *Complexity* **1-14** doi: 10.1155/2020/4757381.
[7] Zhiwei K and Qiang L 2018 Mathematical modeling and modal switching control of a novel tiltrotor UAV. *J. of Robotics* 12 doi: 10.1155/2018/8641731.
[8] Fern F, Füßl R, Eichfelder G, Manske E and Kühnel M 2020 Coordinate transformation and its uncertainty under consideration of a non orthogonal coordinate base. *Meas. Sci. Technol.* **32** 045001 doi: 10.1088/1361-6501/aba3f5.
[9] Bascetta L, Ferretti G and Scaglioni B 2017 Closed form Newton–Euler dynamic model of flexible manipulators. *Robotica* **35**(5) 1006 doi: 10.1017/S0263574715000934
[10] Jameson A 1989 Computational aerodynamics for aircraft design. *Science* **245** 361 doi: 10.1126/science.245.4916.361.
[11] An analysis tool for planes and sails operating at low Reynolds numbers, available at: https://flow5.tech/
[12] Askar M, Hameed A, Suffer K and Mohamad R 2018 Numerical simulation of a new spoiler on upper surface of Clark Y14 wing. *IOP Conf. Ser.-Mat. Sci.* **429** 012080 doi: 10.1088/1757-899X/429/1/012080.