A Note on Generalized Majority Colorings

Marcin Anholcer
Institute of Informatics and Electronic Economics, Poznań University of Economics and Business, Poznań, Poland

Bartłomiej Bosek
Institute of Theoretical Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

Jarosław Grytczuk
Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland

Grzegorz Gutowski
Institute of Theoretical Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

Jakub Przybyło
AGH University of Science and Technology, Faculty of Applied Mathematics, al. A. Mickiewicza 30, 30-059 Kraków, Poland

Mariusz Zając
Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland

Abstract

A majority coloring of a directed graph is a vertex coloring in which each vertex has the same color as at most half of its out-neighbors. In this note we simplify some proof techniques and generalize previously known results on various variants of majority coloring. In particular, our unified and simple approach gives the best known results for:

- directed and undirected graphs,
- $\frac{1}{k}$-majority colorings (each vertex has the same color as at most $\frac{1}{k}$ of its out-neighbors),
- weighted edges,
- list colorings (choosability),
- on-line list colorings (paintability),
- non-uniform list lengths,
- ranked colors.

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1 Introduction

Let $D = (V,E)$ be a directed graph. Let $d^+(v)$ denote the number of out-neighbors of a vertex $v$. A coloring $c$ of the vertices of $D$ is called a majority coloring if for every vertex $v$ the number of its out-neighbors in color $c(v)$ is at most $\frac{1}{2}d^+(v)$. This concept was studied
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by Kreutzer, Oum, Seymour, van der Zypen, and Wood [13]. It is proved there, among other results, that every directed graph is majority 4-colorable. It is conjectured that 3 colors are sufficient for majority coloring of any directed graph and this would be the best possible.

This problem was initially considered for undirected graphs, where the majority condition states that every vertex needs at least half of its neighbors to have a different color than its own. Lovász [14] proved that every finite undirected graph is majority 2-colorable (although the theorem was not stated explicitly).

We can naturally generalize the majority coloring constraint for weighted graphs. Let $\sigma$ be a vertex weighting that assigns a real weight $0 \leq \sigma(v) \leq 1$ to every vertex $v$, and $\tau$ be an edge weighting that assigns a non-negative real weight $\tau(e)$ to every edge $e \in E$. From now on we will assume that for every vertex $v$, $\sum_{e \in E} \tau(ve) > 0$, where $ve$ denotes edge incident with $v$ in the undirected case and are emanating from $v$ in the directed case. Coloring $c$ is a $(\sigma, \tau)$-majority coloring if for every vertex $v \in V$ we have that at most $\sigma(v)$-fraction of $\tau$-weighted out-edges (edges in the undirected case) of $v$ are monochromatic:

$$\frac{\sum_{vw \in E, c(v) = c(w)} \tau(vw)}{\sum_{vw \in E} \tau(vw)} \leq \sigma(v).$$

When $\sigma$ assigns value $s$ uniformly for every vertex $v$, or $\tau$ assigns value $t$ uniformly for every edge $e$, we can say that a $(\sigma, \tau)$-majority coloring is an $(s, t)$-majority coloring. Observe that the regular majority coloring corresponds to $(\frac{1}{2}, 1)$-majority coloring.

Suppose that each vertex $v$ of a graph $G$ is assigned with a list of colors $L(v)$. Then $G$ is $(\sigma, \tau)$-majority colorable from these lists if there is a $(\sigma, \tau)$-majority coloring $c$ of $G$ with $c(v) \in L(v)$ for every vertex $v$. If $G$ is $(\sigma, \tau)$-majority colorable from any lists of size $k$, then we say that $G$ is $(\sigma, \tau)$-majority $k$-choosable. Anholcer, Bosek and Grytczuk [4] showed that every directed graph is $(\frac{1}{2}, 1)$-majority 4-choosable. Their techniques can also give that for every integer $k \geq 1$, every directed graph is $(\frac{1}{2}, 1)$-majority $k^2$-choosable. Later, Girão, Kittipassorn and Popielarz [9], and independently Knox and Sámal [12] showed that every directed graph is $(\frac{1}{2}, 1)$-majority 2$k$-choosable. It is possible that every directed graph is $(\frac{1}{2}, 1)$-majority $(2k - 1)$-choosable. This would be the best possible and it is unknown if it holds even for $k = 2$. Some evidence supporting this conjecture is given by Anastos, Lamaison, Steiner and Szabó [3]. In the case of undirected graphs, the proof of Lovász [14] mentioned above extends easily on the list version of the problem (i.e., every finite graph is $(\frac{1}{2}, 1)$-majority 2-choosable).

The problems of majority coloring and majority list coloring were also considered for infinite graphs. The famous Unfriendly Partition Conjecture by Cowan and Emerson ([5], see [11]) states that every countable undirected graph is majority 2-colorable. It was proved for graphs with finitely many vertices of infinite degree by Aharoni, Milner and Prikry [11], for rayless graphs by Bruhn, Diestel, Georgakopoulos, and Sprüssel [17] and for graphs not containing an infinite clique subdivision by Berger [6]. On the other hand, Shelah and Milner [16] showed that every infinite undirected graph is majority 3-colorable and that there are uncountable graphs for which 3 colors are necessary. Anholcer, Bosek and Grytczuk [5] proved that every countable directed graph is $(\frac{1}{2}, 1)$-majority 4-choosable. Recently Haslegrave [11] showed that every infinite undirected graph is $(\frac{1}{2}, 1)$-majority 3-choosable and that the same holds for countable directed acyclic graphs.

A natural generalization of list colorings is the concept of paintability (also called on-line list coloring, see [15, 17]). Let $\lambda$ be a function that assigns a positive integer $\lambda(v)$ to every vertex $v$. The $(\sigma, \tau)$-majority $\lambda$-painting game on $G$ is a game played in rounds by two players: Lister and Painter. The $i$-th round, for $i = 1, 2, \ldots$, starts with Lister presenting a
non-empty subset \( X_i \) of vertices of \( G \). Then, Painter responds by selecting a subset \( Y_i \subseteq X_i \) and assigns color \( i \) to all the vertices in \( Y_i \). The vertices in \( Y_i \) must satisfy the \((\sigma, \tau)\)-majority constraints, that is, for every \( v \in V \):

\[
\sum_{v \in E, w \in Y_i} \tau(vw) \leq \sigma(v).
\]

Painter wins the game if after some round all the vertices are colored, i.e. \( \bigcup_i Y_i = V \). Lister wins the game if after some round there is a vertex \( v \) that was presented \( \lambda(v) \) many times, and is not colored, i.e.:

\[
|I_v| = \lambda(v) \text{ and } v \notin \bigcup_i Y_i
\]

for some \( v \in V \), where \( I_v = \{ i : v \in X_i \} \). We say that \( G \) is \((\sigma, \tau)\)-majority \( \lambda \)-paintable if Painter has a winning strategy in the corresponding painting game on \( G \). When \( \lambda \) assigns value \( k \) uniformly to every vertex \( v \), then a winning strategy for Painter constructs a \((\sigma, \tau)\)-majority coloring from any lists of size \( k \) and implies \((\sigma, \tau)\)-majority \( k \)-choosability.

There is another generalization that we are going to consider in this paper. It was initially used by Anholcer, Bosek and Grytczuk [4]. Here the vertex weighting \( \sigma \) depends not only on \( v \) itself, but also on the color \( i \). To be more specific, for every vertex \( v \in V \) and for every \( i \in I_v \) we define the weight \( 0 \leq \sigma'(v, i) \leq 1 \). Then the respective majority constraint takes the form

\[
\sum_{v \in E, w \in Y_i} \tau(vw) \leq \sigma'(v, i).
\]

As before, we say that \( G \) is \((\sigma', \tau)\)-majority \( \lambda \)-paintable if Painter has a winning strategy in the corresponding game.

Note that actually the values \( \sigma'(v, i) \) can be arbitrary. In particular, if \( \sigma'(v, i) < 0 \), then \( v \) can never be the member of \( Y_i \).

## 2 Results

Our main contribution is the following lemma (of kernel flavor) that allows for a construction of easy, yet effective, strategies for Painter in \((\sigma, \tau)\)-majority \( \lambda \)-painting games on undirected graphs. Intuitively, the lemma gives a good Painter response \( Y \) to any Lister move \( X \). For every vertex \( v \) in \( X \) we have that: either \( v \) is in \( Y \) (and gets colored instantly), or many (a reasonable fraction of the weighted edges) neighbors of \( v \) are in \( Y \) (and this can happen only limited number of times). The proof bases on a similar idea as the argument of Lovász [14]: we choose the set maximizing certain function and then prove that moving a vertex into or out of this set would lead to a contradiction.

\[\textbf{Lemma 1.}\] \( \text{Let } G = (V, E) \text{ be an undirected graph, and } \tau \text{ be an edge weighting that assigns non-negative real weight } \tau(e) \text{ to every edge } e \in E. \text{ Let } X \subseteq V \text{ be a subset of vertices and } \rho \text{ be a vertex weighting that assigns a real rank } \rho(v) \text{ to every vertex } v \in X. \text{ Then, there exists a subset } Y \subseteq X \text{ such that for every } v \in X \)

\[v \in Y \iff \rho(v) \geq \sum_{v \in E, w \in Y} \tau(vw).
\]

\[\text{Proof.} \text{ Given the set } X, \text{ for any vertex } v \in X \text{ we define the cost of } v \text{ in the following way:}
\]

\[
\text{total}_v = \sum_{v \in E, w \in X} \tau(vw),
\]

\[
\text{cost}_v = 2\rho(v) - \text{total}_v.
\]
Now, given any subset $Y \subseteq X$ we can define the cost of $Y$ as the sum of the costs of vertices in $Y$ plus the sum of weights of all the edges in the cut $(Y, X \setminus Y)$:

$$\text{cost}_Y = \sum_{v \in Y} \text{cost}_v + \sum_{vw \in E, v \in Y, w \in X \setminus Y} \tau(vw).$$

Now, let $Y$ be such that it maximizes the value of cost among all subsets of $X$ and maximizes the size of $Y$ among all subsets of $X$ that maximize the value of cost. We shall prove that $Y$ satisfies the conditions of the lemma. Suppose to the contrary that for some vertex $v \in X$ the condition does not hold. Define:

$$\text{left}_v = \sum_{vw \in E, w \in Y} \tau(vw),$$
$$\text{right}_v = \sum_{vw \in E, w \in X \setminus Y} \tau(vw),$$

and observe that $\text{left}_v + \text{right}_v = \text{total}_v$. We distinguish two cases, depending on whether $v \in Y$ or not.

For the first case, suppose that $v \in Y$ and $\varrho(v) < \text{left}_v$. Consider the set $Y' = Y \setminus \{v\}$ and observe that:

$$\text{cost}_{Y'} = \text{cost}_Y - \text{cost}_v - \text{right}_v + \text{left}_v =$$
$$= \text{cost}_Y - (2\varrho(v) - \text{total}_v) - (\text{total}_v - \text{left}_v) + \text{left}_v =$$
$$= \text{cost}_Y - 2\varrho(v) + 2\text{left}_v >$$
$$> \text{cost}_Y - 2\varrho(v) + 2\varrho(v) =$$
$$= \text{cost}_Y,$$

which contradicts the choice of $Y$, as we have $\text{cost}_{Y'} > \text{cost}_Y$.

Similarly for the second case, suppose that $v \notin Y$ and $\varrho(v) \geq \text{left}_v$. Consider the set $Y' = Y \cup \{v\}$ and observe that:

$$\text{cost}_{Y'} = \text{cost}_Y + \text{cost}_v + \text{right}_v - \text{left}_v =$$
$$= \text{cost}_Y + (2\varrho(v) - \text{total}_v) + (\text{total}_v - \text{left}_v) - \text{left}_v =$$
$$= \text{cost}_Y + 2\varrho(v) - 2\text{left}_v \geq$$
$$\geq \text{cost}_Y + 2\varrho(v) - 2\varrho(v) =$$
$$= \text{cost}_Y,$$

which contradicts the choice of $Y$, as we have $\text{cost}_{Y'} \geq \text{cost}_Y$ and $|Y'| > |Y|$.

Applying Lemma 1 directly in every round of the painting game gives the following.

**Theorem 2 (Paintability With Ranked Colors).** Let $G$ be an undirected graph with any non-negative edge weighting $\tau$. Suppose that each vertex $v$ is assigned with a set of real weights $\sigma'(v, i), i \in I_v$, where $I_v$ is the set of indices of the sets $X_i$ for which $v \in X_i$. Assume that for every vertex $v$, the weights $\sigma'(v, i)$ satisfy the following condition:

$$\sum_{i \in I_v} \sigma'(v, i) \geq 1.$$

Then $G$ is $(\sigma', \tau)$-majority $\lambda$-paintable, where for every $v$, $\lambda(v) = |I_v|$.
Then there is a vertex coloring of $D$ (edge weighting $\sum v \in E \tau(v)$ in step $i$). If some vertex $v$ is not painted in the end of the painting game, then for every $i \in I_v$ we have

$$\sum_{v \in E, w \in Y_i} \tau(vw) > \sigma'(v, i) \sum_{v \in E} \tau(vw).$$

This implies

$$\sum_{v \in E} \tau(vw) \geq \sum_{i \in I_v} \sum_{v \in E, w \in Y_i} \tau(vw)$$

$$> \sum_{i \in I_v} \sigma'(v, i) \sum_{v \in E} \tau(vw) \geq \sum_{v \in E} \tau(vw),$$

a contradiction. Thus, when the game ends, every vertex is painted. Obviously, applying Lemma 1 in every step guarantees that the majority condition is satisfied. \hfill \blacksquare

Theorem 2 in particular immediately easily improves on the ideas from Anholcer, Bosek and Grytczuk [4].

- **Corollary 3 (Ranked List Coloring).** Let $G$ be an undirected graph with any non-negative edge weighting $\tau$. Suppose that each vertex $v$ is assigned with a list $L(v)$ of colors. Suppose further that for each vertex $v$, each color $x$ in $L(v)$ is assigned a real number $r_v(x)$, the rank of color $x$ in $L(v)$. Assume that for every vertex $v$, the color ranks $r_v(x)$ satisfy the following condition:

$$\sum_{x \in L(v)} r_v(x) \geq \sum_{v \in E} \tau(vw).$$

Then there is a vertex coloring of $G$ from lists $L(v)$ satisfying the following constraint: If $x$ is a color assigned to $v$, then the sum of weights of edges connecting $v$ to a neighbor in color $x$ is at most $r_v(x)$.

We can also prove the following result. It is enough to set $\sigma'(v, i) = \frac{1}{\lambda(v)}$ for every $v \in V$ and every $i \in I_v$ to get the following.

- **Corollary 4 (Non-uniform List Lengths).** Let $\lambda(v)$ be a positive integer for each vertex $v$ of an undirected graph $G$. Set $\sigma(v) = \frac{1}{\lambda(v)}$. For any non-negative edge weighting $\tau$, $G$ is $(\sigma, \tau)$-majority $\lambda$-paintable.

Moreover, by setting $\lambda(v) = k$ for every $v \in V$ we obtain the following generalization of the results previously known from the literature.

- **Corollary 5 (Undirected Paintability).** Every undirected graph with any non-negative edge weighting $\tau$ is $\left(\frac{1}{k}, \tau\right)$-majority $k$-paintable.

Lemma 1 combined with some of the ideas from Girão, Kittipassorn and Popielarz [9], and Knox and Šámal [12] allows to derive a strengthening of their results. Let us start with the following lemma.

- **Lemma 6.** Let $D = (V, E)$ be a strongly connected digraph with any positive edge weighting $\tau$. Suppose that each vertex $v$ is assigned with a set of weights $\sigma'(v, i), i \in I_v, \text{where } I_v$ is the set of indices of the sets $X_i$ for which $v \in X_i$. Assume that for every vertex $v$, the weights $\sigma'(v, i)$ satisfy the following condition:

$$\sum_{i \in I_v} \sigma'(v, i) \geq 2.$$

Then $D$ is $(\sigma', \tau)$-majority $\lambda$-paintable, where for every $v$, $\lambda(v) = |I_v|$. 


Proof. Observe that, since in the directed case the incoming edges do not influence the majority condition, it holds for the original weights $\tau(e)$ if and only if it is true for the weights normalized in the following way:

$$\tau(vw) := \frac{\tau(vw)}{\sum_{u : vu \in E} \tau(vu)}.$$  

For that reason, from now on, we will assume that for every vertex the sum of the weights on its outgoing edges equals to 1. Let us define the $n \times n$ matrix $T$ so that

$$T_{vw} = \tau(vw),$$

for every out-neighbor $w$ of $v$ and 0 for all other entries. We have that $T_{vw}$ is nonnegative and for every $v \in V$

$$\sum_{w \in V} T_{vw} = 1.$$  

If $Ty = cy$ for some vector $y$, then by choosing $v$ with maximum value of $|y_v|$ we get

$$|cy_v| = \sum_{w \in V} T_{vw} y_w \leq \sum_{w \in V} |T_{vw}| |y_w| = \sum_{w \in V} T_{vw} |y_v| = |y_v|,$$

so $|c| \leq 1$, which means that the spectral radius $\rho(T) = 1$.

Since $D$ is strongly connected, $T$ is irreducible and we can apply the Perron-Frobenius Theorem (see e.g. [10, Theorem 8.8.1]). According to it there exists a positive left eigenvector of $T$ with eigenvalue 1, that is a vector $x$ such that for every $v \in V$ we have $x_v > 0$ and

$$\sum_{w \in V} T_{vw} x_w = x_v.$$

In particular, we have the following:

$$\sum_{w \in V} T_{vw} x_w = x_v \sum_{w \in V} T_{vw}.$$  

Let us define the new labels for all the edges:

$$\tau'(vw) = x_v \tau(vw) = x_v T_{vw},$$

where $vw \in E$. It means that for every vertex, the sum of the new weights of the outgoing edges is the same as the sum of the weights of the incoming edges:

$$\sum_{w \in V} \tau'(vw) = \sum_{w \in V} x_w T_{vw} = \sum_{w \in V} x_v T_{vw} = \sum_{w \in V} \tau'(vw).$$

Consider now the underlying graph $G$ of $D$, with edge weights $\tau'$. For every $v \in V$ and every $i \in I_v$, let $\sigma''(v,i) = \frac{\sigma'(v,i)}{2}$. By the assumptions we have

$$\sum_{i \in I_v} \sigma''(v,i) \geq 1,$$

so Theorem 2 guarantees that $G$ is $(\sigma'', \tau')$-majority $\lambda$-paintable, where for every $v \in V$, $\lambda(v) = |I_v|$, that is, there is a strategy for the Painter such that after performing all steps, every vertex belongs to some set $Y_i$ and for every $v \in Y_i$

$$\sum_{vw \in E(G), w \in Y_i} \tau'(vw) \leq \sigma''(v,i) \sum_{vw \in E(G)} \tau'(vw).$$
This implies that
\[
\sum_{vw \in E, w \in Y_i} \tau'(vw) \leq \sum_{vw \in E(G), w \in Y_i} \tau'(vw) \\
\leq \sigma''(v, i) \sum_{vw \in E(G)} \tau'(vw) = \sigma'(v, i) \frac{1}{2} \sum_{vw \in E} \tau'(vw) \\
= \sigma'(v, i) \frac{1}{2} \left( \sum_{vw \in E} \tau'(vw) + \sum_{vw \in E} \tau'(vw) \right) = \sigma'(v, i) \sum_{vw \in E} \tau'(vw).
\]

By dividing both sides by \(x_v\) we obtain
\[
\sum_{vw \in E, w \in Y_i} \frac{\tau'(vw)}{x_v} \leq \sigma'(v, i) \sum_{vw \in E} \frac{\tau'(vw)}{x_v},
\]
that is
\[
\sum_{vw \in E, w \in Y_i} \tau(vw) \leq \sigma'(v, i) \sum_{vw \in E} \tau(vw),
\]
which concludes the proof.

Lemma 7 (Rank Reduction). Let \(D = (V, E)\) be a digraph on a vertex set \(V = S \cup T\) with a nonnegative edge weighting \(\tau\) such that \(D[S]\) is strongly connected digraph and the restriction of \(\tau\) to \(E(D[S])\) is positive. Suppose that each vertex \(v \in S\) is assigned with a set of color ranks \(\varrho'(v, i)\), for every \(i \in I_v\), where \(I_v\) is the set of indices of the sets \(X_i\) for which \(v \in X_i\). Assume that for every vertex \(v\), the ranks \(\varrho'(v, i)\) satisfy the following condition:
\[
\sum_{i \in I_v} \varrho'(v, i) \geq 2 \sum_{vw \in E} \tau(vw).
\]

Suppose finally that during the painting game, at step \(i\) the Lister shows not only the set \(X_i\), but also the set \(T \cap Y_i\) and that it is guaranteed that in the end of the game \(\bigcup_i (T \cap Y_i) = T\) (that is, every vertex of \(T\) will be assigned to some \(Y_i\)). Then for every \(i\) one can extend the set \(Y_i\) on \(V\) so that in the end \(\bigcup_i Y_i = V\) and for every \(v \in S\)
\[
\sum_{vw \in E, w \in Y_i} \tau(vw) \leq \varrho'(v, i).
\]

Proof. During the painting game, at step \(i\), let
\[
\varrho''(v, i) = \varrho'(v, i) - \sum_{vw \in E, w \in T \cap Y_i} \tau(vw)
\]
for every \(v \in S\). This guarantees that
\[
\sum_{vw \in E, w \in Y_i} \tau(vw) \leq \varrho'(v, i)
\]
if and only if
\[
\sum_{vw \in E(G), w \in S \cap Y_i} \tau(vw) \leq \varrho''(v, i).
\]
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On the other hand for every \( v \in S \) we have that
\[
\sum_{i \in I_v} \vartheta''(v, i) = \sum_{i \in I_v} \vartheta'(v, i) - \sum_{i \in I_v} \sum_{vw \in E, w \in T \cap Y_i} \tau(vw) = \sum_{i \in I_v} \vartheta'(v, i) - \sum_{vw \in E, w \in T} \tau(vw)
\gilh\geq 2 \sum_{vw \in E} \tau(vw) - \sum_{vw \in E, w \in S \cap T} \tau(vw) = 2 \sum_{vw \in E, w \in S} \tau(vw) - \sum_{vw \in E, w \in S} \tau(vw)
\gilh\geq 2 \sum_{vw \in E, w \in S} \tau(vw).
\]

From Lemma 6 it follows that \( D[S] \) is \((\sigma', \tau)\)-majority \( \lambda \)-paintable, where for each \( v \in S \) and \( i \in I_v \)
\[
\sigma'(v, i) = \frac{\vartheta'(v, i)}{\sum_{vw \in E(D[S])} \tau(vw)}
\]
and for every \( v \in S \), \( \lambda(v) = |I_v| \). It means that we can choose the sets \( Y_i \) so that in the end of the painting game \( \bigcup_i (S \cap Y_i) = S \) and for every \( v \in S \)
\[
\sum_{vw \in E(D[S]), w \in S \cap Y_i} \tau(vw) \leq \vartheta''(v, i).
\]
This concludes the proof. \(\blacktriangleleft\)

Lemma 7 allows us to extend the result from Lemma 6 to all digraphs, not necessarily strongly connected.

**Theorem 8** (Directed Ranked Colors). Let \( D \) be a digraph with any non-negative edge weighting \( \tau \). Suppose that each vertex \( v \) is assigned with a set of weights \( \varrho \sigma'(v, i) \), where \( i \) is the set of indices of the sets \( X_i \) for which \( v \in X_i \). Assume that for every vertex \( v \), the weights \( \sigma'(v, i) \) satisfy the following condition:
\[
\sum_{i \in I_v} \sigma'(v, i) \geq 2.
\]
Then \( D \) is \((\sigma', \tau)\)-majority \( \lambda \)-paintable, where for every \( v \in V \), \( \lambda(v) = |I_v| \).

**Proof.** Let us start with removing all the edges with weight 0, since they do not influence the values of the respective weighted sums.

Now, decompose \( D \) into strongly connected components \( D_1, D_2, \ldots, D_r \) such that there are no edges from \( D_j \) to \( D_k \) for \( j < k \).

Assume that at some step \( i \), the set \( X_i \) has been presented. We construct the set \( Y_i \) component-wise, by applying Lemma 7 with initial ranks
\[
\vartheta'(v, i) = \sigma'(v, i) \sum_{vw \in E} \tau(vw).
\]
We start with component \( D_1 \), where \( S = V(D_1) \) and \( T = \emptyset \) and then for \( k \geq 2 \), we apply Lemma 7 with \( S = V(D_k) \) and \( T = \bigcup_{j=1}^{k-1} V(D_j) \). Since Lemma 7 guarantees the paintability of \( D \left[ \bigcup_{j=1}^{k-1} V(D_j) \right] \) if and only if \( D \left[ \bigcup_{j=1}^{k-1} V(D_j) \right] \) is paintable, and \( D_1 \) is paintable by Lemma 7, we arrive at the thesis. \(\blacktriangleleft\)

Theorem 8 immediately implies the following analogues of Corollaries 3, 4 and 5.
\textbf{Corollary 9 (Directed Ranked List Coloring).} Let $D$ be a directed graph with any non-negative edge weighting $\tau$. Suppose that each vertex $v$ is assigned with a list $L(v)$ of colors. Suppose further that for each vertex $v$, each color $x$ in $L(v)$ is assigned a real number $r_v(x)$, the rank of color $x$ in $L(v)$. Assume that for every vertex $v$, the color ranks $r_v(x)$ satisfy the following condition:

$$\sum_{x \in L(v)} r_v(x) \geq 2 \sum_{vw \in E} \tau(vw).$$

Then there is a vertex coloring of $G$ from lists $L(v)$ satisfying the following constraint: If $x$ is a color assigned to $v$, then the sum of weights of edges connecting $v$ to an out-neighbor in color $x$ is at most $r_v(x)$.

\textbf{Corollary 10 (Directed Non-uniform List Lengths).} Let $\lambda(v)$ be a positive integer for each vertex $v$ of a directed graph $D$. Set $\sigma(v) = 2\lambda(v)$. For any non-negative edge weighting $\tau$, $D$ is $(\sigma, \tau)$-majority $\lambda$-paintable.

\textbf{Corollary 11 (Directed Paintability).} Every directed graph with any non-negative edge weighting $\tau$ is $(\frac{1}{k}, \tau)$-majority $2k$-paintable.

3 Discussion

The proof for digraphs presented in this paper consists of three main parts. The first one is Lemma 1 developing the idea of Lovász [14]. The second one is the use of Perron-Frobenius Theorem to transfer the results from undirected graphs to strongly connected digraphs with positive weights (Lemma 6). Finally, the usage of ranks and their reduction (Lemma 7) that bases on the idea of Anholcer, Bosek and Grytczuk [4] allowed us to extend the result on all digraphs with non-negative weightings.

As already mentioned, the idea of choosing a set minimizing or maximizing some function comes from Lovász, but it was exploited many times in similar context. Our extension relied on using weights on edges and then appropriate encoding of the problem (in particular the choice of the appropriate cost function was essential). It would be useful to strengthen this technique and apply it directly to the digraphs (without applying the Perron-Frobenius Theorem), which could finally allow to prove the conjecture that every digraph is $(\frac{1}{2}, 1)$-majority 3-choosable, or maybe even $(\frac{1}{2}, 1)$-majority 3-paintable. Unfortunately we were not able to do this.

The Perron-Frobenius Theorem is a powerful tool used in various disciplines, like Computer Science (search engines), Demography (population growth models), Economy (growth and equilibrium models) or Statistics (Markov chains, random walks). We applied it to the context of directed graphs, using the fact that it is true for non-reducible matrices, which are precisely the adjacency matrices of weighted strongly connected digraphs. For that reason we extended rather the ideas from the paper of Knox and Šámal [12] than those by Girão, Kittipassorn and Popielarz [9]. The argument presented in the latter paper allows to avoid the decomposition of digraphs into strongly connected components, however we decided not to use this approach. Note also that in fact the main tool used in [9] (Lemma 9) is a special case of the result formulated previously by Alon [2, Lemma 3.1].

The results presented in this paper generalize in several ways all the majority results for directed and undirected graphs, $\frac{1}{k}$-majority colorings, weighted edges, choosability, paintability, non-uniform list lengths and ranked colors. The question is, to which extent it could be applied to infinite graphs and digraphs.
A Note on Generalized Majority Colorings

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