ON SMALL HOMOTOPIES OF LOOPS

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Abstract. Two natural questions are answered in the negative:

• “If a space has the property that small nullhomotopic loops bound small
nullhomotopies, then are loops which are limits of nullhomotopic loops
themselves nullhomotopic?”
• “Can adding arcs to a space cause an essential curve to become nullho-
mutopic?”

The answer to the first question clarifies the relationship between the notions
of a space being homotopically Hausdorff and \( \pi_1 \)-shape injective.

1. Introduction

Anomalous behavior in homotopy theory arises when an essential map is the
uniform limit of inessential maps. Such behavior manifests itself in such oddities
as pointed unions of contractible spaces being non-contractible, and (infinite) con-
catenations of nullhomotopic loops being essential [CC1].

![Figure 1. The “surface” portion of example A](image)

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suggestions and comments.
Often topologists attempt to control such behavior by requiring “small” maps to be nulhomotopic. This is the flavor of the k-ULC property from geometric topology.

In the current article there are two natural notions of “small” curves which we shall study – curves which can be homotoped into arbitrarily small neighborhoods of a point, and curves which can be uniformly approximated by nulhomotopic curves. This article describes how various embodiments of these notions are related.

In several settings one is led to ask the following

**Question 1.1.** If $X$ is a space in which small nulhomotopic loops bound small homotopies, then is a loop which is the uniform limit of a family of nulhomotopic loops necessarily nulhomotopic?

Informally, this article is meant to clarify the above question and answer it in the negative. Formally, this article studies two relatively new and subtly different separation axioms: homotopically Hausdorff and $\pi_1$-shape injective.

The underlying notions were introduced in a number of papers including [CC1, CL, Z, CF, CC2] and were put to good use in [FZ1] and [FZ2]. The intuition behind a space being homotopically Hausdorff is that curves which can be homotoped into arbitrarily small neighborhoods of a point are nulhomotopic, whereas in a $\pi_1$-shape injective space one intuits that curves which can be homotoped arbitrarily close to a nulhomotopic curve are themselves nulhomotopic. Lemma 4.1 shows that the property of small nulhomotopic loops bounding small nulhomotopies implies homotopically Hausdorff.

This motivates the more formal

**Question 1.2.** Does the homotopically Hausdorff property imply $\pi_1$-shape injectivity?

Section 3 constructs two examples $A$ and $B$, neither of which is $\pi_1$-shape injective, by rotating a topologist’s sine curve in $\mathbb{R}^3$ to create a “surface” and adding a null sequence of arcs to make the space locally path connected (see the schematic diagrams for the space $A$ above). Both spaces are homotopically Hausdorff and $B$ is strongly homotopically Hausdorff.

For the sake of completeness, Lemma 2.1 shows shape injective implies strongly homotopically Hausdorff, and strongly homotopically Hausdorff implies homotopically Hausdorff.

The proofs that $A$ and $B$ have the desired properties require Lemma 4.3 which answers, in the negative, the following natural
Question 1.3. Can adding arcs to a space turn an essential loop into a nullhomotopic loop?

1.1. Historical perspective. In [CC2] the property of being homotopically Hausdorff is described and is shown to be equivalent to the path space of the space being Hausdorff. This same notion was independently considered in [Z] under the name weak \( \pi_1 \)-continuity. The definition of shape injectivity was introduced in [CC2] as the injectivity of the natural map from the fundamental group of a space into the shape group of the space. Previously [CF] studied this property extensively but it was not given a name there. In [FZ1] it is shown that shape injectivity of a space implies unique path lifting from the space to its path space and the path space is a type of generalized covering space.

Recently there has been renewed interest in the notion of a path space of a separable metric space [B, BS, CC2, FZ1]. The underlying desire is to find a suitable replacement for covering spaces in situations where appropriate covering spaces do not exist. To be suitable, this replacement should have unique path lifting and so must be Hausdorff. In [CC2], the path space is briefly described and its topology described. In short, one uses the definition of the universal covering space in [M], but does not require the base space to be semilocally simply connected. There are other weaker topologies which can be put on this space, see [B] for instance. However, if the path space as defined in [CC2] fails to be Hausdorff or has non-unique path lifting, then any weaker topology on the space will also suffer from these same deficiencies. There is a long history of generalizations of covering spaces ranging from the work of Fox on overlays [F1, F2] up to the present [MM, BS, B, CC2, FZ1].

2. Definitions

This section briefly recalls a number of standard definitions and introduces the definition of (strongly) homotopically Hausdorff.

A curve or path \( \gamma \) in the space \( X \) is a continuous function from the interval \([0,1]\) into \( X \), the base or initial point of \( \gamma \) is \( \gamma(0) \), the end or terminal point of \( \gamma \) is \( \gamma(1) \), \( \gamma \) emanates from \( \gamma(0) \) and terminates at \( \gamma(1) \). Furthermore, \( \gamma \) is closed (and is
called a *loop* if $\gamma(0) = \gamma(1)$, $\gamma$ is *simple* if it is injective and *simple closed* if it is closed and injective except at $\{0, 1\}$. Closed curves are often considered as having domain $S^1$, in the obvious way.

A *free homotopy* between loops in $X$ is a continuous map from the closed annulus $[1, 2] \times S^1$ to $X$ whose restriction to the boundary components of the annulus are the given loops. A *based homotopy* between loops $\gamma, \gamma'$ with the same base point is a free homotopy with the additional property that the interval $[1, 2] \times \{0\}$ maps to the base point. Two loops are *(freely) homotopic* if there is a (free) homotopy between them. A loop is *nullhomotopic* or *inessential* if it is homotopic to a constant map and is *essential* otherwise. A loop is nullhomotopic if and only if, when considered as a map from $S^1$ into $X$, it can be completed to a map from $B^2$ into $X$.

In [CC2] the *path space of the space $X$ based at $x_0$, $\Omega(X, x_0)$*, is defined to be the space of homotopy classes rel$\{0, 1\}$ of paths in the space $X$ based at the point $x_0 \in X$. The path space is given the following topology: if $p$ is a path in $X$ emanating from $x_0$, and $U$ is an open neighborhood of $p(1)$, define $O(p, U)$ to be the collection of homotopy classes of paths rel$\{0, 1\}$ containing representatives of the form $p \cdot \alpha$ where $\alpha$ is a path in $U$ emanating from $p(1)$, and take $\{O(p, U)\}$ as a basis for the topology of $\Omega(X, x_0)$. If $X$ is semilocally simply connected then $\Omega(X, x_0)$ is the universal covering space of $X$ [M]. See [CC1, FZ2] for discussions of the path space of the Hawaiian earring.

A space $X$ is *$\pi_1$-shape injective* (or just *shape injective*) if there is an absolute retract $R$ which contains $X$ as a closed subspace so that whenever $\gamma$ is an essential closed curve in $X$ then there is a neighborhood $V$ of $X$ in $R$ such that $\gamma$ essential in $V$. If the above condition holds for $X$ as a closed subspace of the absolute retract $R$, and $X$ is a closed subspace of the absolute retract $S$, then $X$ also satisfies the above condition for $S$. For the purposes of this paper, $R$ will always be $\mathbb{R}^3$.

If $X$ is connected, locally path connected and compact, the above definition is equivalent to the following: $X$ is $\pi_1$-shape injective if given any essential loop $\gamma$ in $\pi_1(X)$ there is a finite open cover $\mathcal{U}$ of $X$ so the natural image of $\gamma$ in $\pi_1(N(\mathcal{U}))$ is essential, where $N(\mathcal{U})$ denotes the nerve of $\mathcal{U}$. This is, furthermore, equivalent to the property of the natural map from $\pi_1(X)$ to *shape group* of $X$,

$$\lim_{\mathcal{U} \text{ a finite open cover of } X} \pi_1(N(\mathcal{U})),$$

is an injective homomorphism. Thus, the name *shape injective* is somewhat natural.

A space $X$ is *homotopically Hausdorff at a point $x_0 \in X$* if for all essential loops $\gamma$ based at $x_0$ there exists a neighborhood $U$ of $x_0$ such that no loop in $U$ is homotopic (in $X$) to $\gamma$ rel $x_0$. Furthermore, $X$ is *homotopically Hausdorff* if $X$ is homotopically Hausdorff at every point.

A space $X$ is *strongly homotopically Hausdorff at $x_0 \in X$* if for each essential closed curve $\gamma \in X$ there is a neighborhood of $x_0$ which contains no closed curve freely homotopic (in $X$) to $\gamma$. We say that a space $X$ is *strongly homotopically Hausdorff* if $X$ is strongly homotopically Hausdorff at each of its points.

Intuitively, a space is homotopically Hausdorff if loops which can be made (homotopically) arbitrarily small are in fact nullhomotopic, where the modifier *strongly* allows the homotopies involved to be free homotopies. The article [CC2] mentions that the name *homotopically Hausdorff* was motivated by the fact that $\Omega(X, x_0)$ is Hausdorff if and only if $X$ is homotopically Hausdorff.
Care is needed when defining these properties for non-compact spaces since we wish the notions to be topological invariants. For instance, a punctured plane is strongly homotopically Hausdorff (being strongly homotopically Hausdorff at each of its points), but contains an essential loop which can be homotoped to be arbitrarily small. On the other hand, the punctured plane is homeomorphic to $S^1 \times \mathbb{R}$, endowed with its natural metric, in which no essential loop has small diameter.

Lemma 2.1.

(1) If $X$ is shape injective, then $X$ is strongly homotopically Hausdorff.

(2) If $X$ is strongly homotopically Hausdorff at $x_0 \in X$ then $X$ is homotopically Hausdorff at $x_0$.

Proof. For part (1), suppose $X$ is a closed subspace of the absolute retract $R$. Let $\gamma$ be a loop in $X$ which can be freely homotoped in $X$ into an arbitrary neighborhood of $x_0$ in $X$. Then $\gamma$ is nullhomotopic in any neighborhood of $X$ in $R$, and since $X$ is shape injective, $\gamma$ must be nullhomotopic. Therefore $X$ is strongly homotopically Hausdorff at each of its points.

Part (2) follows immediately, since a loop which is nullhomotopic rel its base point is freely nullhomotopic.

□

The reverse implications do not hold, not even if the space is required to be a Peano continuum. This article constructs two Peano continua which are subspaces of $\mathbb{R}^3$, $A$ and $B$, and shows neither is shape injective while both are homotopically Hausdorff and one is even strongly homotopically Hausdorff. Both spaces will be formed by rotating a topologist’s sine curve and adding a null sequence of arcs to make the space locally path connected.

3. Examples

The first example, $A$, is obtained by taking the “surface” obtained by rotating the topologist’s sine curve about its limiting arc—a space which is not locally connected at its central arc—and then adding a null sequence of arcs on a countable dense set of radial cross sections to make the space locally path connected at the central arc. See Figures 1 and 2.

The left half of Figure 2 above shows a radial projection of the space $A$, where the horizontal lines are the connecting arcs which have been added to the various radial cross sections. The right half of the diagram shows a top view of the space, with the concentric circles denoting the crests of the rotated sine curve, and the line segments depicting the added arcs. We will refer to the various pieces of $A$ as the surface (the rotated $\sin(1/x)$ curve), the central limit arc, and the connecting arcs. Let $\Gamma$ be the union of the interiors of the connecting arcs.

Lemma 3.1. A loop $b$ of constant radius in the surface of $A$ is not freely nullhomotopic unless it is nullhomotopic in its image.

Proof. If $b$ were nullhomotopic, then by Lemma 4.3 there is a nullhomotopy of $b$ whose image does not intersect the interior of any of the connecting arcs. Thus the image of this homotopy lies in the path component of $b$ in $A - \Gamma$ (the complement of the connecting arcs). This path component is the surface of $A$, which is a punctured
disc, but \( b \) is not nulhomotopic in the punctured disc unless it is nulhomotopic in its image.

**Corollary 3.2.** The Peano continuum \( A \) is not shape injective.

*Proof.* Let \( b \) be a simple closed curve of constant radius in the surface, and let \( a \) be a path from the base point to the initial point of \( b \). Let \( \gamma = aba^{-1} \). Then \( \gamma \) is nulhomotopic if and only if \( b = a^{-1}\gamma a \). Then, by Lemma 3.1, \( \gamma \) is not nulhomotopic, since it is not even freely nulhomotopic.

Every neighborhood of \( A \) in \( \mathbb{R}^3 \) contains a neighborhood of the surface union the central arc, which is just a 3-ball. Since \( \gamma \) is conjugate to a loop in the surface, \( \gamma \) is then nulhomotopic in any neighborhood of \( A \), and thus \( A \) is not shape injective.

**Theorem 3.3.** The space \( A \) is homotopically Hausdorff, but not strongly homotopically Hausdorff.

*Proof.* Clearly \( A \) cannot be strongly homotopically Hausdorff because of the loop \( b \), mentioned above, which is not nulhomotopic but can be freely homotoped, in the surface, into any neighborhood of any point on the central limit arc.

Now, \( A \) is homotopically Hausdorff at every point not in the central arc since \( A \) is locally contractible at any such point. In the following section, sufficient conditions for being homotopically Hausdorff are proven in Lemma 4.1. Thus it remains to be shown that \( A \) satisfies the hypothesis of Lemma 4.1 for a base point \( x_0 \) in the central arc. Let \( \varepsilon > 0 \) be given. Choose \( 0 < \delta < \varepsilon \), and let \( \gamma \) be a nulhomotopic loop contained in the ball \( B(x_0, \delta) \). Since there are only countably many connecting arcs, one can find a closed ball \( C \) of radius slightly greater than \( \delta \) so \( \partial C \) does not intersect any of the end points of the connecting arcs, and so \( C \subset B(x_0, \varepsilon) \).

Let \( h \colon B^2 \to A \) be a nulhomotopy of \( \gamma \). One can alter \( h \) to obtain a nulhomotopy whose image stays in the ball of radius \( \varepsilon \) centered at \( x_0 \). To do this, consider the places where the image of \( h \) intersects \( \partial C \). The following describes how to modify the homotopy so its image remains in a small neighborhood of \( C \), which is contained in \( B(x_0, \varepsilon) \).

First, by Lemma 4.3, one can assume the image of \( h \) does not intersect any of the connecting arcs which \( \gamma \) does not intersect. In particular, the intersection of the image of \( h \) with \( \partial C \) does not intersect any connecting arc.

Let \( \ell \) denote the intersection of \( C \) with the central limit arc. Consider \( h^{-1}(\ell) \) in \( B^2 \). Let \( K \) be the closure of the union of all components of \( B^2 - h^{-1}(\ell) \) which intersect \( \partial B^2 \), and let \( O \) be the open set \( K^c \). Since \( h(\partial O) \subset \ell \), and since \( \ell \) is an absolute retract, one may adjust \( h \) on \( O \) leaving \( h\bigr|_K \) fixed and sending \( \overline{O} \) into \( \ell \).

Thus the image of (the modified) \( h \) will not pass through \( \partial C \) to exit \( B(x_0, \varepsilon) \) along the central arc.

Since \( C \) is a ball centered at a point on the central arc, the intersection of \( \partial C \) with the surface is a discrete collection of circles \( \{c_i\} \). Let \( n \) be a component of \( h^{-1}(c_i) \). Since \( n \) is a component of the closed set \( h^{-1}(c_i) \), it is closed. By the way \( C \) was chosen, \( \gamma = h(\partial B^2) \) does not intersect \( \partial C \), so there exists a simple closed curve \( s \) which separates \( n \) from \( \partial B^2 \). One can choose \( s \) to be close enough to \( n \) so \( s \cap h^{-1}(c_j) = \emptyset \) for all \( j \neq i \), and also so \( h(s) \) is contained in \( B(x_0, \varepsilon) \). Since \( s \) is a simple closed curve in the disc \( B^2 \), the Schoenflies theorem says \( s \) bounds a disc \( D \), and then \( h|_D \) is a nulhomotopy for \( h(s) \). Because of the way \( s \) was chosen, \( h(s) \) is a loop in the surface, contained in a small neighborhood of the circle \( c_i \) which is an annulus \( a_i \) contained in \( B(x_0, \varepsilon) \). Now, \( c_i \) is a deformation retract of \( a_i \), and every
nonzero multiple of \( c_i \) is essential in \( A \) by Lemma 3.1. If \( h(s) \) were essential in \( a_i \) then it would be homotopic in \( A \) to a non-zero multiple of \( c_i \) and would thus be essential in \( A \). Thus \( h(s) \) is nullhomotopic in \( a_i \). Choose a nullhomotopy \( g_i \) for \( h(s) \) which lies in \( a_i \) and whose image has diameter no larger than that of \( h(D) \); if \( h|_D \) already lies in \( a_i \), then let \( g_i = h|_D \). Adjust the homotopy \( h \) on the interior of the disc \( D \) to be \( g_i \). Repeat this for all components \( n \) of the various preimages \( h^{-1}(c_i) \), to ensure the image of \( h \) does not intersect \( \partial B(x_0, \varepsilon) \).

The modified homotopy is still continuous, since whenever the function was altered on a subset of \( B^2 \), the new image had diameter less than or equal to the original diameter. Thus the modified homotopy \( h \) has image contained in \( B(x_0, \varepsilon) \), and so Lemma 4.1 applies. Therefore the space \( A \) is homotopically Hausdorff at any base point \( x_0 \) contained in the central arc.

The second example is similar to the space \( A \) but is constructed by rotating the topologist’s sine curve about its initial point \((r_0, \sin(1/r_0))\) along a vertical axis, instead of rotating about the limit arc as depicted in Figure 3. To be precise, one can express the space \( B \) in cylindrical coordinates in terms of the space \( A \): \( B = \{(r_0 - r, \phi, z) \mid (r, \phi, z) \in A, 0 \leq r \leq r_0\} \). The central limiting arc of \( A \) corresponds to the limiting outer annulus in \( B \), and the connecting arcs of \( B \) limit on every point of this annulus. The surface of \( B \) is homeomorphic to \( \mathbb{R}^2 \). Again, let \( \Gamma \) denote the union of the interiors of the connecting arcs.

**Theorem 3.4.** The Peano continuum \( B \) is not shape injective.

**Proof.** Consider a loop about the limiting outer annulus. As in the proof of Lemma 3.1, one applies Lemma 4.3 to see that this loop cannot be nullhomotopic, as it is not nullhomotopic in the path component of \( B - \Gamma \) containing it, which is an annulus. Any neighborhood of \( B \) in \( \mathbb{R}^2 \) contains a neighborhood of the annulus which intersects the rotated surface, and this curve can then be homotoped into the surface. Thus this curve is nullhomotopic in any neighborhood of \( B \). Once again, one conjugates by a path to the base point. Since the original path is essential it follows that \( B \) is not shape injective.

**Theorem 3.5.** The space \( B \) is strongly homotopically Hausdorff.

**Proof.** This proof will be remarkably similar to the proof of Theorem 3.3. Evidently \( B \) is strongly homotopically Hausdorff at any point outside the outer annulus, since it is locally contractible there. Let \( x_0 \) be a point on the outer annulus.

Let \( \varepsilon > 0 \) and choose \( 0 < \delta < \varepsilon \). Let \( \gamma, \gamma' \) be homotopic essential loops contained in the ball \( B(x_0, \delta) \). One can find a closed ball \( C \) contained in \( B(x_0, \varepsilon) \) and containing \( B(x_0, \delta) \) so \( \partial C \) does not intersect any of the end points of the connecting arcs.

Let \( h: (S^1 \times I) \to B \) be a homotopy between \( \gamma \) and \( \gamma' \). By Lemma 4.3, one may assume the image of \( h \) does not intersect any of the connecting arcs which do not intersect either \( \gamma \) or \( \gamma' \). In particular, the intersection of the image of \( h \) with \( \partial C \) does not intersect any connecting arc.

Let \( d \) denote the disc which is the intersection of \( C \) with the outer annulus. Similar to the proof of Theorem 3.3, since \( d \) is an absolute retract (\( d \) plays the role of \( \ell \) in that proof), \( h \) may be altered so every component of \( (S^1 \times [0, 1]) - h^{-1}(d) \)
intersects the boundary of $S^1 \times [0, 1]$. Thus the image of (the modified) $h$ cannot pass through $\partial C$ to exit $B(x_0, \varepsilon)$ along the outer annulus.

Since $C$ is a ball centered on the outer annulus, the intersection of $\partial C$ with the surface is a discrete collection of circles $\{c_i\}$, each of which bounds a disc $d_i$ in the surface. Let $n$ be a component of $h^{-1}(c_i)$. Since $h^{-1}(c_i)$ is closed, $n$ is closed. Because of the way $C$ was chosen, $\gamma \cup \gamma' = h(S^1 \times \{0, 1\})$ does not intersect $\partial C$. If $h^{-1}(c_i)$ separates $S^1 \times \{0\}$ from $S^1 \times \{1\}$, then both $\gamma, \gamma'$ will be nulhomotopic, since they are both homotopic to a power of $c_i$ which bounds the disc $d_i$, contradicting the choice of an essential curve $\gamma$.

Since $h^{-1}(c_i)$ does not separate $S^1 \times \{0, 1\}$, there exists a simple closed curve $s$ which separates $n$ from $S^1 \times \{0, 1\}$. Choose $s$ to be close enough to $n$ so $s \cap h^{-1}(c_j) = \emptyset$ for all $j \neq i$, and also so $h(s)$ is contained in $B(x_0, \varepsilon)$.

By the way $s$ was chosen, $h(s)$ is a loop in the surface, which lies in a small neighborhood of $c_i$, and thus bounds a disc in a small neighborhood of the disc $d_i$. Let $g$ denote a nulhomotopy of $h(s)$ whose image lies in this disk and has diameter no larger than that of $h(s)$. By the Schoenflies theorem, the component of the complement of $s$ which does not contain the boundary $S^1 \times \{0, 1\}$ is a disc. Adjust the homotopy $h$ on this disc to be the nulhomotopy $g$; $h$ need not be adjusted in the case that its image is already sufficiently small. Carry out this adjustment for all components $n$ of the various preimages $h^{-1}(c_i)$, to ensure the image of $h$ does not intersect $\partial B(x_0, \varepsilon)$.

As in the proof of Theorem 3.3, the modified homotopy is still continuous, since whenever the function was altered on a subset of $S^1 \times [0, 1]$ the new image had diameter less than or equal to the original diameter. Thus the modified homotopy $h$ has image contained in $B(x_0, \varepsilon)$, and so Lemma 4.2 applies. Therefore the space $B$ is strongly homotopically Hausdorff at any point $x_0$ contained in the outer annulus.

\[\square\]

4. Technical Lemmas

First, a condition on metric spaces is given which implies the condition homotopically Hausdorff at a point. The basic idea is that for every small nulhomotopic curve, there is a nulhomotopy of small diameter. This is similar to 1-ULC, but the condition is only required to hold for nulhomotopic loops.

**Lemma 4.1.** Suppose the metric space $X$ contains a point $x_0$ enjoying the property that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every continuous function $f : B^2 \to X$ with $f(S^1) \subset B(x_0, \delta)$, there is a continuous function $g : B^2 \to X$ such that $g|_{S^1} = f|_{S^1}$, and $g(B^2) \subset B(x_0, \varepsilon)$. Then $X$ is homotopically Hausdorff at $x_0$.

**Proof.** Let $\gamma_i$ be a null sequence of loops based at $x_0$ representing the same homotopy class in $\pi_1(X, x_0)$. Construct a nulhomotopy $f$ of $\gamma_1$ as follows: Consider a Hawaiian earring in the disc $B^2$ as in Figure 4 below. Define $f$ on each of the arcs $c_i$ to be the loops $\gamma_i$ in $X$. Then each portion $D_i$ of the disc where $f$ is not yet defined is bounded by a curve $\gamma_i \gamma_{i+1}$, which is nulhomotopic. Thus $f$ can be defined on the entire disc so that it is continuous at every point except possibly at the base point of the Hawaiian earring.

One carefully chooses nulhomotopies $f|_{D_i}$ of $\gamma_i \gamma_{i+1}$ to ensure the continuity of $f$ at the base point. Let $(\varepsilon_n)$ be a sequence of positive numbers decreasing to 0. Then by hypothesis there exists a sequence $(\delta_n)$ such that any nulhomotopic loop
Lemma 4.2. Let $X$ be a metric space and $x_0 \in X$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every essential map of an annulus $f : S^1 \times [0, 1] \to X$ with $f(S^1 \times \{0, 1\}) \subset B(x_0, \delta)$, there is a map of an annulus $g : S^1 \times [0, 1] \to X$ such that $g|_{S^1 \times \{0, 1\}} = f|_{S^1 \times \{0, 1\}}$, and $g(S^1 \times [0, 1]) \subset B(x_0, \varepsilon)$. Then $X$ is strongly homotopically Hausdorff at $x_0$. 

Proof. Let $\gamma_i$ be a null sequence of loops which are freely homotopic to each other and which converge to $x_0$. It must be shown that $\gamma_1$ is freely nullhomotopic. Suppose not. Then each $\gamma_i$ is essential. By way of contradiction, one constructs a nullhomotopy $f$ of $\gamma_1$ as follows. In the unit disc $B^2$, specify concentric circles $c_i$ of radius $1/i$. Define $f|_{c_i} = \gamma_i$ in $X$. Let $A_i$ be the annulus in $B^2$ bounded by $c_i \cup c_{i+1}$. Since $f|_{c_i} = \gamma_i$, and since $\gamma_i$ is homotopic to $\gamma_{i+1}$, one can extend $f$ to each $A_i$ in such a

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{hawaiian_earring}
\caption{The Hawaiian earring}
\end{figure}

contained in the ball $B(x_0, \delta_n)$ has a nullhomotopy whose image is contained in the ball $B(x_0, \varepsilon_n)$. Without loss of generality, assume $\delta_n \geq \delta_{n+1}$. Since the loops $\gamma_i$ form a null sequence limiting to a point, choose $k_n$ to be the minimal index so $\gamma_i$, $i \geq k_n$ has diameter less than $\delta_n$ for all $i \geq k_n$. Then since $\delta_n \geq \delta_{n+1}$, it follows that $k_n \leq k_{n+1}$. Then for all $k_n \leq i < k_{n+1}$, define $f|_{D_i}$ to be a nullhomotopy of $\gamma_i$ with diameter less than $\varepsilon_n$, which exists by hypothesis.

To see this defines $f$ as a continuous function at the base point $y$ of the Hawaiian earring, let $\varepsilon > 0$ be given. Then there is some $n$ such that $\varepsilon_n < \varepsilon$. Then by the construction, the arc $c_k$, in the disc (which maps to $\gamma_{k_n}$) bounds a disc whose image is contained in $B(x_0, \varepsilon_n) \subset B(x_0, \varepsilon)$. Since there are only finitely many discs $D_i$, for $i < n$, one can find a $\delta > 0$ such that $f$ maps $\bigcup_{i=1}^{n-1} D_i \cap B(y, \delta)$ into $B(x_0, \varepsilon)$. Then since $f$ maps $\bigcup_{i=1}^{\infty} D_i$ into $B(x_0, \varepsilon)$, it follows that $f(B(y, \delta)) \subset B(x_0, \varepsilon)$, and thus $f$ is continuous.

Thus $\gamma_1$ is nullhomotopic and consequently all of the curves $\gamma_i$ are nullhomotopic. Therefore $X$ is homotopically Hausdorff at $x_0$. 

While this condition is sufficient, it is not necessary. Consider the cone over the Hawaiian earring in Figure 5, which is contractible, hence homotopically Hausdorff. The loops of the base Hawaiian earring are nullhomotopic, yet they require nullhomotopies of large diameter (passing over the cone point).

We now describe a condition which is sufficient to imply strongly homotopically Hausdorff at a point; it guarantees a homotopy of small diameter between every pair of essential homotopic curves nearby a point.
way that, by an argument similar to the end of Lemma 4.1, we may extend $f$ to a map which is also continuous at the center point of $B^2$. Thus $f$ is a nulhomotopy of the curve $\gamma_1$, and hence $X$ is strongly homotopically Hausdorff at $x_0$. \hfill \Box

The next lemma can be thought of as a general position result for arcs and nulhomotopies. It says that if a nulhomotopic loop does not meet the interiors of a collection of arcs, then there is nulhomotopy for the loop which does not meet the interiors of the arcs.

**Lemma 4.3.** Let $X$ be a topological space. Let $\Xi$ be a disjoint union of open sets in $X$, each of which is homeomorphic to an open arc, and let $Z = X - \Xi$.

1. Let $g : B^2 \to X$ be a nulhomotopy such that $g(\partial B^2) \subseteq Z$. Then there is a nulhomotopy $g' : B^2 \to Z$ with $g |_{\partial B^2} = g' |_{\partial B^2}$.

2. Let $h : (S^1 \times [0, 1]) \to X$ be a homotopy between two essential curves $\gamma$ and $\gamma'$ in $Z$. Then there is a homotopy $h'$ between $\gamma$ and $\gamma'$ such that the image of $h'$ lies in $Z$.

An alternate way of stating the conclusion of this theorem would be to say the natural map $i_* : \pi_1(Z) \to \pi_1(X)$ induced by inclusion is injective.

**Proof.** For each arc $\xi$ in $\Xi$, let $a_\xi$ be an open arc in $\xi$ whose closure is contained in $\xi$, and let $A$ be the union of the arcs $a_\xi$. The subspace $Z$ is a strong deformation retract of $X - A$, so it suffices to show that the maps described above take values in $X - A$.

For the moment we proceed with the proof of (2). Let $K$ be the boundary of the component of $S^1 \times [0, 1] - h^{-1}(A)$ containing $S^1 \times \{0\}$. Now, $h$ is constant on each component of $K$ since the boundary of $A$ is totally disconnected.

Suppose $K$ separates $S^1 \times \{0, 1\}$, the boundary of the annulus. Since $\mathbb{R}^2$ is unicoherent, the interior of the annulus $S^1 \times (0, 1)$ has the following property: if a compact subspace contained in the interior of the annulus separates two points of the closed annulus, then one of the components of the subspace separates those two points. Consequently, one may choose a component, $T$, of $K$ which separates $S^1 \times \{0\}$ from $S^1 \times \{1\}$. One now creates a new map which is equal to $h$ everywhere except for the component of the complement of $T$ which contains $S^1 \times \{1\}$ and
defines it to be the constant $h(T)$ on that component. Furthermore, one may adjoin a disk to $S^1 \times \{0, 1\}$ along $S^1 \times \{1\}$ to obtain $B^2$ and extend the new map by defining it to be the constant $h(T)$ on this disk also. The result is a nullhomotopy of $\gamma$, contradicting the hypothesis that $\gamma$ is essential.

Thus assume $K$ does not separate $S^1 \times \{0, 1\}$. Recall $h$ is constant on each component of $K$. Hence we may define $h'$ by having it agree with $h$ on the component of the complement $K$ which contains $S^1 \times \{0, 1\}$ and defining it to be constant on the other components of the complement of $K$, thus proving (2).

To prove (1) it is enough to mention that if $L$ is the boundary of the component of $B^2 - g^{-1}(A)$ containing $\partial B^2$, then, as in the argument above, $g$ is constant on each component of $L$. Define $g'$ to be equal to $g$ on the component of the complement of $L$ containing $\partial B^2$ and to be constant on the other components of the complement of $L$. \hfill $\Box$

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