Treatment of the Coulomb interaction in three-nucleon reactions
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The Coulomb interaction between the two protons is included in the calculation of three-nucleon hadronic and electromagnetic reactions using screening and renormalization approach. Calculations are done using integral equations in momentum space. The reliability of the method is demonstrated. The Coulomb effect on observables is discussed.

1. Introduction

The inclusion of the Coulomb interaction in the description of the three-nucleon continuum is one of the most challenging tasks in theoretical few-body nuclear physics. The Coulomb interaction is well known, in contrast to the strong two-nucleon and three-nucleon potentials mainly studied in three-nucleon scattering. However, due to its $1/r$ behavior, the Coulomb interaction does not satisfy the mathematical properties required for the formulation of standard scattering theory. There is a long history of theoretical work on the solution of the Coulomb problem in three-particle scattering. Some of the recent suggestions \cite{1,2,3} have not matured yet into practical applications, while the others, based mostly on the configuration-space framework \cite{4,5,6}, are limited to energies below deuteron breakup threshold (DBT). Up to now only few approaches led to the results above DBT. Those are configuration-space calculations for proton-deuteron ($pd$) elastic scattering using the Kohn variational principle \cite{7} and the screening and renormalization approach in the framework of momentum-space integral equations \cite{8,9,10,11,12}; nevertheless, for the latter method only the present work published in Refs. \cite{11,12} uses realistic interactions together with fully converged calculations in terms of screening radius and two-nucleon and three-nucleon partial waves.

Section 2 shortly recalls the technical apparatus underlying the calculations and demonstrates the reliability of the method. Section 3 gives our summary.

2. Screening and renormalization approach

Our treatment of the Coulomb interaction is based on the idea of screening and renormalization proposed in Ref. \cite{12} for the scattering of two charged particles. The screened Coulomb potential of our choice in $r$-space representation is given by

$$w_R(r) = w(r) e^{-(r/R)^n},$$

\textsuperscript{*}Supported by the FCT grant SFRH/BPD/14801/2003
where \( w(r) \) is the proper Coulomb potential, \( R \) is the screening radius, and \( n \) controls the smoothness of the screening. The standard scattering theory is formally applicable to the screened Coulomb potential, e.g., the transition matrix is defined via Lippmann-Schwinger equation \( t_R(e_i + i0) = w_R + w_R g_0(e_i + i0) t_R(e_i + i0) \) and the corresponding wave function is \( |\psi_R^{(+)}(p_i)\rangle = [1 + g_0(e_i + i0) t_R(e_i + i0)] |p_i\rangle \), where \( g_0(e_i + i0) \) is the free resolvent and \(|p_i\rangle\) is the plane-wave state with momentum \( p_i \) and energy \( e_i \). Furthermore, as shown in Refs. [13,14], the on-shell screened Coulomb transition matrix \((p_f|t_R(e_i + i0)|p_i)\) with \( p_f = p_i \) and wave function diverge in \( R \to \infty \) limit, but after renormalization with also diverging phase factor \( z_R(p_i) \) they converge to the proper Coulomb amplitude \((p_f|t_C|p_i)\) and proper Coulomb wave function \(|\psi_C^{(+)}(p_i)\rangle\), respectively:

\[
\lim_{R \to \infty} (p_f|t_R(e_i + i0)|p_i) z_R^{-1}(p_i) = (p_f|t_C|p_i), \tag{2a}
\]

\[
\lim_{R \to \infty} |\psi_R^{(+)}(p_i)\rangle z_R^{-\frac{1}{2}}(p_i) = |\psi_C^{(+)}(p_i)\rangle. \tag{2b}
\]

The renormalized screened Coulomb amplitude converges to the proper Coulomb amplitude in general as a distribution. As discussed in Ref. [13], this is fully sufficient for description of real experiments and justifies the replacement of \( \lim_{R \to \infty} (p_f|t_R(e_i + i0)|p_i) z_R^{-1}(p_i) \) by \((p_f|t_C|p_i)\) in practical calculations [10][11].

The screening and renormalization approach can be applied to more complicated systems, if in the scattering amplitudes the diverging screened Coulomb contributions can be isolated in the form of two-body on-shell transition matrix and two-body wave function with known renormalization properties [2]. For the description of \((pd)\) scattering we employ Alt-Grassberger-Sandhas (AGS) three-particle scattering equations [15] in momentum space

\[
U^{(R)}_{\beta\alpha}(Z) = \tilde{\delta}_{\beta\alpha} G_0^{-1}(Z) + \sum_{\sigma} \delta_{\beta\alpha} T^{(R)}_{\sigma}(Z) G_0(Z) U^{(R)}_{\sigma\alpha}(Z), \tag{3a}
\]

\[
U^{(R)}_{0\alpha}(Z) = G_0^{-1}(Z) + \sum_{\sigma} T^{(R)}_{\sigma}(Z) G_0(Z) U^{(R)}_{\sigma\alpha}(Z), \tag{3b}
\]

with \( \tilde{\delta}_{\beta\alpha} = 1 - \delta_{\beta\alpha} \), \( G_0(Z) \) being the free resolvent, \( T^{(R)}_{\sigma}(Z) \) the two-particle transition matrix derived from nuclear plus screened Coulomb potentials, and \( U^{(R)}_{\beta\alpha}(Z) \) and \( U^{(R)}_{0\alpha}(Z) \) the three-particle transition operators for elastic/rearrangement and breakup scattering; their dependence on the screening radius \( R \) is notationally indicated. As demonstrated in Refs. [8][11][12], the three-particle transition operators can be decomposed into long-range and Coulomb-distorted short-range parts

\[
U^{(R)}_{\beta\alpha}(Z) = \delta_{\beta\alpha} T^{cm}_{\alpha R}(Z) + [1 + T^{cm}_{\beta\alpha}(Z) G^{(R)}_\beta(Z)] \tilde{U}^{(R)}_{\beta\alpha}(Z) [1 + G^{(R)}_\alpha(Z) T^{cm}_{\alpha R}(Z)], \tag{4a}
\]

\[
U^{(R)}_{0\alpha}(Z) = [1 + T_{\rho R}(Z) G_0(Z)] \tilde{U}^{(R)}_{0\alpha}(Z) [1 + G^{(R)}_\alpha(Z) T^{cm}_{\alpha R}(Z)] \tag{4b}
\]

with the channel resolvent \( G^{(R)}_\alpha(Z) \), proton-proton \((pp)\) screened Coulomb transition matrix \( T_{\rho R}(Z) \), two-body transition matrix \( T^{cm}_{\alpha R}(Z) \) derived from the screened Coulomb potential between spectator and the center of mass (c.m.) of the remaining pair, and with the reduced short-range operators \( \tilde{U}^{(R)}_{\beta\alpha}(Z) \) and \( \tilde{U}^{(R)}_{0\alpha}(Z) \). On-shell matrix elements of the
operators (14) between two- and three-body channel states $|\phi_\alpha(q_i)\nu_{\alpha_i}\rangle$ and $|\phi_0(p_f q_f)\nu_{\alpha_f}\rangle$ with discrete quantum numbers $\nu_{\alpha_i}$, Jacobi momenta $p_f$ and $q_i$, and energy $E_i$, do not have a $R \to \infty$ limit. However, the quantities diverging in that limit are already isolated in Eqs. (14) and are of two-body nature, i.e., the on-shell $T_{\alpha R}^{cm}(Z)$ and the initial/final state screened Coulomb wave functions. Those quantities, renormalized according to Eq. (2), in the $R \to \infty$ limit converge to the two-body Coulomb scattering amplitude $T_{\alpha R}^{cm}$ and to the corresponding Coulomb wave functions, respectively, thereby yielding the $pd$ scattering amplitudes in the proper Coulomb limit

$$
\langle \phi_\beta(q_f)\nu_{\beta_f}|U_{\beta\alpha}|\phi_\alpha(q_i)\nu_{\alpha_i}\rangle = \delta_{\beta\alpha}\langle \phi_\alpha(q_f)\nu_{\alpha_f}|T_{\alpha C}^{cm}|\phi_\alpha(q_i)\nu_{\alpha_i}\rangle
$$

$$
+ \lim_{R \to \infty} \left\{ Z_{\beta \alpha}^{-\frac{1}{2}}(q_f)\langle \phi_\beta(q_f)\nu_{\beta_f}|U_{\beta\alpha}^{(R)}(E_i+i0)|\phi_\alpha(q_i)\nu_{\alpha_i}\rangle Z_{\alpha \beta}^{-\frac{1}{2}}(q_i) \right\};
$$

$$
\langle \phi_0(p_f q_f)\nu_{\alpha_f}|U_{0\alpha}|\phi_\alpha(q_i)\nu_{\alpha_i}\rangle = \lim_{R \to \infty} \left\{ z_R^{-\frac{1}{2}}(p_f)\langle \phi_0(p_f q_f)\nu_{\alpha_f}|U_{0\alpha}^{(R)}(E_i+i0)|\phi_\alpha(q_i)\nu_{\alpha_i}\rangle z_R^{-\frac{1}{2}}(q_i) \right\},
$$

where the relation between Eqs. (5) and (14) is given in Refs. [13][12]. The renormalization factors $Z_{\alpha R}$ and $z_R$ are diverging phase factors defined in Ref. [13][12]. The $R \to \infty$ limit in Eqs. (5) has to be calculated numerically, but due to the short-range nature of the corresponding operators it can be reached with sufficient accuracy at rather modest $R$ if the form of the screened Coulomb potential (11), in particular the parameter $n$ controlling the smoothness of the screening, has been chosen successfully. We prefer to work with a sharper screening than the Yukawa screening ($n = 1$) of Refs. [9][10]. We want to ensure that the screened Coulomb potential $w_R(r)$ approximates well the true Coulomb one $w(r)$ for distances $r < R$ and simultaneously vanishes rapidly for $r > R$, providing a comparatively fast convergence of the partial-wave expansion. In contrast, the sharp cutoff ($n \to \infty$) yields an unpleasant oscillatory behavior in the momentum-space representation, leading to convergence problems. In Refs. [13][12] we found the values $3 \leq n \leq 6$ to provide a sufficiently smooth, but at the same time a sufficiently rapid screening around $r = R$; $n = 4$ is our standard choice. The screening radius $R$ sufficient for convergence in Eqs. (5) is considerably larger than the range of the strong interaction. As a consequence, the calculation of the three-particle transition operators for nuclear plus screened Coulomb potentials requires the inclusion of partial waves with angular momentum much higher than required for the hadronic potential alone. This problem can be solved in efficient and reliable way either by using the perturbative approach for high two-particle partial waves, developed in Ref. [16], or even without it as discussed in Ref. [17]. More details on the practical implementation of the screening and renormalization approach as well as the extension to three-nucleon electromagnetic (e.m.) reactions are presented in Refs. [13][12].

The internal criterion for the reliability of our method is the convergence of the observables with screening radius $R$ employed to calculate the Coulomb-distorted short-range part of the amplitudes in Eqs. (5). Figures (1) and (2) show characteristic examples for $pd$ elastic scattering and breakup. The hadronic interaction is the realistic coupled-channel potential CD Bonn + $\Delta$, allowing for single virtual $\Delta$-isobar excitation [18]. In most cases the convergence is impressively fast; the screening radius $R = 20$ fm is sufficient.
Figure 1. Convergence of the $pd$ elastic scattering observables with screening radius $R$. The differential cross section, proton analyzing power $A_y(N)$ and deuteron analyzing power $T_{21}$ at 3 MeV proton lab energy are shown as functions of the c.m. scattering angle. Results obtained with screening radius $R = 10$ fm (dotted curves), 20 fm (dash-dotted curves), and 30 fm (solid curves) are compared. Results without Coulomb (dashed curves) are given as reference for the size of the Coulomb effect.

Figure 2. Convergence of the $pd$ breakup observables with screening radius $R$. The differential cross section in selected kinematical configurations at 13 MeV proton lab energy is shown as function of the arclength $S$ along the kinematical curve. Notation of curves as in Fig. 1.

The exceptions requiring larger screening radii are the $pd$ elastic scattering observables at very low energies and the breakup differential cross section in kinematical situations characterized by very low $pp$ relative energy $E_{pp}$, i.e., close to the $pp$ final-state interaction ($pp$-FSI) regime, as shown in Fig. 3. In there, the $pp$ repulsion is responsible for decreasing the cross section, converting the $pp$-FSI peak obtained in the absence of Coulomb into a minimum with zero cross section at $p_f = 0$, i.e., for $E_{pp} = 0$. Such a behavior is seen in the experimental data as well [19,20]. The slow convergence close to $pp$-FSI is not surprising, since the renormalization factor $z_R(p_f = 0)$ itself is ill-defined, indicating that
Figure 3. Convergence of the $pd$ breakup observables with screening radius $R$. The differential cross section for $pd$ breakup at 13 MeV proton lab energy in the $pp$-FSI configuration is shown as function the relative $pp$ energy $E_{pp}$. Results obtained with screening radius $R = 10$ fm (dotted curve), 20 fm (dashed-double-dotted curve), 30 fm (dashed-dotted curve), 40 fm (double-dashed-dotted curve), 60 fm (solid curve), and results without Coulomb (dashed curve) are compared.

the screening and renormalization procedure cannot be applied at $p_f = 0$. Therefore an extrapolation has to be used to calculate the observables at $p_f = 0$, which works pretty well since the observables vary smoothly with $p_f$.

Furthermore, Ref. [21] makes a detailed comparison between the results for $pd$ elastic scattering obtained by the present technique and those of Ref. [7] obtained from the variational solution of the three-nucleon Schrödinger equation in configuration space with the inclusion of an unscreened Coulomb potential between the protons and imposing the proper Coulomb boundary conditions explicitly. The agreement, across the board, between the results derived from two entirely different methods, clearly indicates that both techniques for including the Coulomb interaction are reliable.

3. Summary

We have shown how the Coulomb interaction between the protons can be included into the momentum-space description of proton-deuteron scattering using old idea of screening and renormalization [13]. The theoretical framework is the AGS integral equation [15]. The calculations are done on the same level of accuracy and sophistication as for the neutron-deuteron scattering. Our practical realization differs enormously from the one of Refs. [9,10], even the form of three-particle equations including screened Coulomb is different. We use modern hadronic interactions whereas the calculations of Refs. [9,10] were based on quasiparticle equations with rank-1 separable potentials and in addition approximated the screened Coulomb transition matrix by the screened Coulomb potential.

Compared to the configuration-space treatment [7], the results for the elastic $pd$ scattering agree very well over a wide energy range [21]. Although at very low energies the coordinate-space methods remain favored, at higher energies and especially for three-body breakup reactions our momentum-space treatment is more efficient, and so far the only
one to show first results for realistic interactions [22].

A realm of applications to the rich \textit{pd} data base for elastic scattering and breakup may be seen in Refs. [11,12,19,20]. The conclusion is that in elastic \textit{pd} scattering the Coulomb effect is important at low energies for all kinematic regimes, but gets confined to the forward direction at higher energies. In \textit{pd} breakup and in three-body e.m. disintegration of \textit{3He} the Coulomb effect is extremely important in kinematical regimes close to \textit{pp}-FSI. There the \textit{pp} repulsion converts the \textit{pp}-FSI peak obtained in the absence of Coulomb into a minimum with zero cross section [20]. This significant change of the cross section behavior has important consequences in nearby configurations where one may observe instead an increase of the cross section due to Coulomb [19]. However, some of the long-standing discrepancies between experiment and theory like the space star anomaly in \textit{pd} breakup are not resolved by the inclusion of the Coulomb interaction.

Finally, the screening and renormalization approach for including the Coulomb interaction is extended to four-nucleon scattering [23].

REFERENCES

1. E. O. Alt, S. B. Levin, S. L. Yakovlev, Phys. Rev. C 69 (2004) 034002.
2. A. S. Kadyrov, I. Bray, A. M. Mukhamedzhanov, A. T. Stelbovics, Phys. Rev. A 72 (2005) 032712.
3. S. Oryu, Phys. Rev. C 73 (2006) 054001.
4. C. R. Chen, J. L. Friar, G. L. Payne, Few-Body Syst. 31 (2001) 13.
5. S. Ishikawa, Few-Body Syst. 32 (2003) 229.
6. P. Doleschall and Z. Papp, Phys. Rev. C 72 (2005) 044003.
7. A. Kievsky, M. Viviani, S. Rosati, Phys. Rev. C 64 (2001) 024002.
8. E. O. Alt, W. Sandhas, H. Ziegelmann, Phys. Rev. C 17 (1978) 1981; E. O. Alt and W. Sandhas, \textit{ibid.} 21 (1980) 1733.
9. E. O. Alt and M. Rauh, Few-Body Syst. 17 (1994) 121.
10. E. O. Alt, A. M. Mukhamedzhanov, M. M. Nishonov, A. I. Sattarov, Phys. Rev. C 65 (2002) 064613.
11. A. Deltuva, A. C. Fonseca, P. U. Sauer, Phys. Rev. C 71 (2005) 054005.
12. A. Deltuva, A. C. Fonseca, P. U. Sauer, Phys. Rev. C 72 (2005) 054004.
13. J. R. Taylor, Nuovo Cimento B23 (1974) 313; M. D. Semon and J. R. Taylor, \textit{ibid.} A26 (1975) 48.
14. V. G. Gorshkov, Sov. Phys.-JETP 13 (1961) 1037.
15. E. O. Alt, P. Grassberger, W. Sandhas, Nucl. Phys. B2 (1967) 167.
16. A. Deltuva, K. Chmielewski, P. U. Sauer, Phys. Rev. C 67, (2003) 054004.
17. A. Deltuva, A. C. Fonseca, P. U. Sauer, Phys. Rev. C 73 (2006) 057001.
18. A. Deltuva, R. Machleidt, P. U. Sauer, Phys. Rev. C 68 (2003) 024005.
19. S. Kistryn \textit{et al.}, Phys. Lett. B 641 (2006) 23.
20. K. Sagara \textit{et al.}, contribution to this conference.
21. A. Deltuva, A. C. Fonseca, A. Kievsky, S. Rosati, P. U. Sauer, M. Viviani, Phys. Rev. C 71 (2005) 064003.
22. A. Deltuva, A. C. Fonseca, P. U. Sauer, Phys. Rev. Lett. 95 (2005) 092301.
23. A. Deltuva and A. C. Fonseca, contribution to this conference.