Redundancy in string cone inequalities and multiplicities in potential functions on cluster varieties

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Abstract
We study defining inequalities of string cones via a potential function on a reduced double Bruhat cell. We give a necessary criterion for the potential function to provide a minimal set of inequalities via tropicalization and conjecture an equivalence.

Keywords Cluster algebras · Quantum groups · Canonical bases · String polytopes

Introduction
Let \( g \) be simple simply laced complex Lie algebra. To every reduced expression \( i \) of the longest element \( w_0 \) of the Weyl group of \( g \) is associated a polyhedral cone \( C_i \subset \mathbb{R}^N \) called the string cone. Here \( N \) is the length of \( w_0 \). The string cones arise in many different contexts, e.g. they are closely connected to the dual canonical basis of the universal enveloping algebra of the negative part of \( g \) [1], may be seen as generalizations of Gelfand-Tsetlin cones [18], play a great role in toric degenerations of flag varieties [6] and are closely related to tensor product multiplicities of representations of \( g \) [2].

This paper deals with the problem of determining a minimal set of inequalities for \( C_i \), i.e. describing the facets of \( C_i \). The way we approach this is as follows. If \( j \) is another reduced word of \( w_0 \), there is a piecewise linear bijection \( \Psi^j_i : C_i \rightarrow C_j \). The cone \( C_i \) consists of all \( t \in \mathbb{R}^N \) such that for any reduced word \( j \) the last coordinate of \( \Psi^j_i(t) \) is non-negative. This fact was used in [13] to define a function \( \varsigma_1 \in \mathbb{C}[x_j^\pm \mid j \in \{1, 2, \ldots, N\}] \) (see Definition 3) such that \( C_i \) is given by all \( t \in \mathbb{R}^N \)
with \([ς_1]_{\text{trop}}(t) \geq 0\). Here trop means the tropicalization as defined in Sect. 1.3. We prove the following sufficient criterion.

**Theorem 1** (Theorem 2) *If the exponents of all variables in \(ς_1\) have absolute value less than or equal to 1, then the set of inequalities given by \([ς_1]_{\text{trop}}\) is non-redundant.*

We can refine the above thanks to the observation that the Laurent-polynomial \(ς_1\) naturally splits into a sum of Laurent polynomials \(ς_1 = \sum_{i \in I} ς_i\), where \(I\) is the index set of simple roots of \(g\).

**Proposition 1** (Proposition 3) *The inequalities arising from \([ς_i,i]_{\text{trop}}\) and \([ς_j,i]_{\text{trop}}\) are independent, i.e. the set of inequalities given by \([ς_1]_{\text{trop}}\) is non-redundant if and only if the sets inequalities given by \([ς_i,i]_{\text{trop}}\) are non-redundant for all \(i \in I\).*

Proposition 3 allows us to study each summand \(ς_i\) independently. We call \(ς_i\) *multiplicity-free* if the exponents of all variables have absolute value less than or equal to 1 and thus the monomials of \(ς_i\) correspond to facets of the string cone via tropicalization.

We provide several classes of examples of reduced words \(i\) where \(ς_{i,i}\) is multiplicity-free. In Theorem 4 we show that this is the case if \(i\) is simply braided for \(i\). This notion was studied in [21] and means that there is a particularly simple sequence of braid moves changing \(i\) to a reduced word which ends with \(i\) (see Definition 11). Furthermore \(ς_{i,i}\) is multiplicity-free for all \(i \in I\) if \(i\) is a nice word in the sense of Littelmann [18] by Theorem 5.

In the case that \(ω_i\) is a minuscule weight of \(g\) we show that \(ς_{i,i}\) is given via Berenstein-Zelevinsky’s \(i\)-trails in Sect. 6.2. Theorem 6 proves that \(ς_{i,i}\) is again multiplicity-free in that case. Thus, in particular, \(ς_{i,i}\) is always multiplicity-free for \(g\) of type \(A\).

We conjecture that Theorem 2 is indeed also a necessary criterion.

**Conjecture 1** (Conjecture 3) *The inequalities arising from \([ς_{i,i}]_{\text{trop}}\) are non-redundant if and only if \(ς_{i,i}\) is multiplicity-free.*

Finally in Sect. 7 we provide an example for which \(ς_{i,i}\) is not multiplicity-free. In this example there is exactly one Laurent monomial of \(ς_{i,i}\) whose tropicalization leads to a redundant inequality. We note that this monomial is the only monomial of \(ς_{i,i}\) with a coefficient greater than 1. This leads us to the final conjecture given a criterion to determine the facets of the string cone.

**Conjecture 2** (Conjecture 4) *An inequality arising from the tropicalization of a monomial of \(ς_{i,i}\) is redundant if and only if the coefficient of this monomial is greater than 1.*

Note that this criterion cannot be seen merely in the tropicalization of \(ς_{i,i}\) since coefficients do not play any role there. However, it is visible in the tropicalization whether \(ς_{i,i}\) is multiplicity-free or not. It would be very interesting to find a conceptual explanation of our result. In Remark 5 we suggest a relation to \(F\)-polynomials of cluster variables.
The proofs of Theorem 2 and Proposition 3 are making use of the following fact proven in [13]. The function $\varphi$ is the pullback of the potential function of [11] on the big reduced double Bruhat cell (see Proposition 5) by an isomorphism of tori. An important point here is that the potential function (expressed in appropriate torus coordinates) only has non-positive exponents (Proposition 4). The results may be obtained from this using cluster combinatorics.

The paper is organized as follows. Section 1 introduces important notion related to reduced words and tropicalization. Section 2 deals with string cones and their defining inequalities and states our main results. Section 3 recalls the notion of $\mathcal{A}$- and $\mathcal{X}$-cluster varieties and potential functions in the sense of [11]. In Sect. 4 we restrict ourself to the big reduced double Bruhat cell as specific cluster variety and recall how to obtain string cone inequalities via potential functions. The proofs of Theorem 2 and Proposition 3 are obtained in Sect. 5. Section 6 provides examples of reduced words for which our system of inequalities is non-redundant. In the final Sect. 7 we provide an example of a redundancy corresponding to a multiplicity in $\varphi$.

1 Background

1.1 Notation

For a positive integer $m \in \mathbb{Z}_{\geq 0}$ we denote by $[m]$ the set $\{1, 2, \ldots, m\}$. Let $\mathfrak{g}$ be simple simply laced complex Lie algebra of rank $n$, $I := [n]$, $C = (b_{i,j})_{i,j \in I}$ its Cartan matrix and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. We choose simple roots $\{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ and simple coroots $\{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$ with $\alpha_i(\alpha_j^\vee) = b_{i,j}$. We denote by $\Delta^+ \subset \mathfrak{h}^*$ the set of positive roots and by $\Delta$ the set of roots associated to $\{\alpha_i \mid i \in I\}$.

The fundamental weights $\{\omega_i\}_{i \in I} \subset \mathfrak{h}^*$ of $\mathfrak{g}$ are given by $\omega_i(\alpha_j^\vee) = \delta_{i,j}$. We denote by $P = \langle \omega_i \mid i \in [n] \rangle_{\mathbb{Z}}$ the weight lattice of $\mathfrak{g}$ and by $P^+ = \langle \omega_i \mid i \in [n] \rangle_{\mathbb{Z}_{\geq 0}} \subset P$ the set of dominant weights.

The Langlands dual Lie algebra $^L\mathfrak{g}$ of $\mathfrak{g}$ is the simple, simply laced complex Lie algebra with Cartan matrix $C$, Cartan subalgebra $\mathfrak{h}^*$, simple roots $\{\alpha_i^\vee\}_{i \in I}$, simple coroots $\{\alpha_i\}_{i \in I}$ and $\alpha_i^\vee(\alpha_j) = b_{i,j}$. The fundamental weights of $^L\mathfrak{g}$ are $\{\omega_i^\vee\}_{i \in I} \subset \mathfrak{h}$ where $\alpha_i(\omega_j^\vee) = \delta_{i,j}$.

1.2 Weyl groups and reduced words

The Weyl group $W$ of $\mathfrak{g}$ is a Coxeter group generated by the simple reflections $s_i$ ($i \in I$) with relations

\[
s_i^2 = id, \quad s_i s_j s_i = s_j s_i s_j \quad \text{if } b_{i_1,i_2} = 0 \quad (2\text{-term relation}),
\]

\[
s_i s_j s_i = s_j s_i s_j \quad \text{if } b_{i_1,i_2} = -1 \quad (3\text{-term relation}).
\]

We sometimes call a 2-term relation also a commutation relation.
The group $W$ has a unique longest element $w_0$ of length $N = \#\Delta^+$, where the length is given by the minimal number of generators in an expression. For a reduced expression $s_{i_1} \cdots s_{i_N}$ of $w_0$, i.e., an expression of minimal length, we write $i := (i_1, \ldots, i_N)$ and call $i$ a reduced word (for $w_0$). The set of reduced words for $w_0$ is denoted by $R(w_0)$.

The group $W$ acts on $P$ as $s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$ for $\lambda \in P$ and $i \in I$. We denote by $i^\ast \in I$ the unique element such that $w_0 \omega_i = -\omega_{i^\ast}$.

We have two operations on the set of reduced words $R(w_0)$.

A reduced word $j = (j_1, \ldots, j_N)$ is defined to be obtained from $i = (i_1, \ldots, i_N) \in R(w_0)$ by a 2-term move at position $k \in [N - 1]$ if $i_\ell = j_\ell$ for all $\ell \notin \{k, k + 1\}$, $(i_k, i_{k+1}) = (j_k, j_{k+1})$ and $b_{i_k, i_{k+1}} = 0$.

A reduced word $j = (j_1, \ldots, j_N)$ is defined to be obtained from $i = (i_1, \ldots, i_N) \in R(w_0)$ by a 3-term move at position $k \in [N - 1]$ if $i_\ell = j_\ell$ for all $\ell \notin \{k - 1, k, k + 1\}$, $j_{k-1} = j_{k+1} = i_k$, $j_k = i_{k-1} = i_{k+1}$ and $b_{i_k, i_{k+1}} = -1$.

By the Tits theorem every two reduced words for an element $w \in W$ can be obtained from each other by a sequence of 2-term and 3-term moves.

We call a total order $\leq$ on $\Delta^+$ convex if for any two positive roots $\beta_1, \beta_2$ such that $\beta_1 + \beta_2 \in \Delta^+$, we either have $\beta_1 < \beta_1 + \beta_2 < \beta_2$ or $\beta_2 < \beta_1 + \beta_2 < \beta_1$. By [20, Theorem p. 662] the set of reduced words is in bijection with the set of convex orders as follows. For a reduced word $i = (i_1, \ldots, i_N) \in R(w_0)$ the total order

$$\alpha_{i_1} \prec i s_{i_1}(\alpha_{i_2}) \prec \cdots \prec i s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$$

on $\Delta^+$ is convex and every convex order on $\Delta^+$ arises that way. We write $\Delta^+_i = \{\beta_1, \beta_2, \ldots, \beta_N\}$ for the set of positive roots ordered with respect to the convex order $\prec_i$ and identify $\Delta^+_i$ with $[N]$ via

$$\beta_k \mapsto k.$$  

### 1.3 Tropicalization

We recall the notion of tropicalization from [11]. Let $G_m$ be the multiplicative group. Let $T = G_m^k$ be an algebraic torus. We denote by $[\mathbb{T}]_{trop} = Hom(G_m, \mathbb{T}) = \mathbb{Z}^k$ its cocharacter lattice. A positive (i.e. subtraction-free) rational map $f$ on $\mathbb{T}$, $f(x) = \sum_{u \in I} a_u x^u$ with $a_u, b_u \in \mathbb{R}_+$, gives rise to a piecewise-linear map

$$[f]_{trop} : [\mathbb{T}]_{trop} \to [G_m]_{trop} = \mathbb{Z}, \ x \mapsto \min_{u \in I} (x, u) - \min_{u \in J} (x, u),$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{Z}^k$. We call $[f]_{trop}$ the tropicalization of $f$.

For a positive rational map

$$f = (f_1, \ldots, f_t) : G_m^k \to G_m^\ell$$
we define its tropicalization as

\[ [f]_{\text{trop}} := ([f_1]_{\text{trop}}, \ldots, [f_\ell]_{\text{trop}}) : [G^k_m]_{\text{trop}} \to [G^\ell_m]_{\text{trop}}. \]

## 2 String cones

### 2.1 String parametrization of the canonical basis

Let \( B(\infty) \) be the crystal basis of \( U_q^{-} \) in the sense of [16] with partially inverse crystal operators \( \tilde{e}_i, \tilde{f}_i \) for \( i \in I \). For each reduced word \( \mathbf{i} = (i_1, i_2, \ldots, i_N) \in R(w_0) \) we define the \( \mathbf{i} \)-string datum of an element of \( B(\infty) \) as follows. Let \( \mathbb{R}^{\Delta_1^+} = \{(x_j)_{\beta_j \in \Delta^+} | \beta_1 \leq \beta_2 \leq \cdots \leq \beta_N, x_j \in \mathbb{R} \forall j \in [N] \}\). Let further \( \mathbb{Z}^{\Delta_1^+}_{\geq 0} \) be the set of all \( (x_j)_{\beta_j \in \Delta^+} \in \mathbb{R}^{\Delta_1^+} \) such that \( x_j \in \mathbb{Z}_{\geq 0} \) for all \( j \in [N] \).

**Definition 1** An \( \mathbf{i} \)-string datum \( \text{str}_\mathbf{i}(b) \) of \( b \in B(\infty) \) is defined by a tuple \( (x_1, x_2, \ldots, x_N) \in \mathbb{Z}^{\Delta_1^+}_{\geq 0} \) determined inductively by

\[
\begin{align*}
  x_1 &= \max_{k \in \mathbb{Z}_{\geq 0}} \{ \tilde{e}_i b \in B(\infty) \}, \\
  x_2 &= \max_{k \in \mathbb{Z}_{\geq 0}} \{ \tilde{e}_j \tilde{e}_i b \in B(\infty) \}, \\
  & \vdots \\
  x_N &= \max_{k \in \mathbb{Z}_{\geq 0}} \{ \tilde{e}_j \tilde{e}_{j-1} \cdots \tilde{e}_i b \in B(\infty) \}.
\end{align*}
\]

By [2, 18] the subset

\[ C_\mathbf{i} := \{ \text{str}_\mathbf{i}(b) \mid b \in B(\infty) \} \subseteq \mathbb{Z}^{\Delta_1^+}_{\geq 0} \]

is a polyhedral cone called the **string cone associated to** \( \mathbf{i} \).

In the following we explain the piecewise-linear map between string cones associated to different reduced words.

**Definition 2** Let \( \mathbf{j} \in R(w_0) \) be obtained from \( \mathbf{i} \in R(w_0) \) by a 3-term move at position \( k \). We have a piecewise linear bijection

\[
\Phi_\mathbf{j} : \mathbb{R}^{\Delta_1^+} \to \mathbb{R}^{\Delta_1^+}
\]

\[
(x_\ell)_{\beta_\ell \in \Delta^+} \mapsto (x_\ell')_{\beta_\ell \in \Delta^+}
\]

given by

\[
x_{k-1}' = \max(x_{k+1}, x_k - x_{k-1}), \quad x_k' = x_{k-1} + x_{k+1}, \quad x_{k+1}' = \min(x_{k-1}, x_k - x_{k+1}).
\]

and \( x_\ell' = x_\ell \) \( \forall \ell \notin \{k - 1, k, k + 1\} \).
Let \( j \in R(w_0) \) be obtained from \( i \in R(w_0) \) by a 2-term move at position \( k \). We have a linear bijection

\[
\Psi^i_j : \mathbb{R}^{\Delta^+} \rightarrow \mathbb{R}^{\Delta^+}
\]

given by

\[
x'_k = x_{k+1}, \quad x'_{k+1} = x_k, \quad x'_\ell = x_\ell \quad \forall \ell \notin \{k - 1, k\}.
\]

Let \( j, i \in R(w_0) \) be two arbitrary reduced words. We define \( \Psi^i_j : \mathbb{R}^{\Delta^+} \rightarrow \mathbb{R}^{\Delta^+} \) to be the composition of the above defined bijections corresponding to a sequence of 2— and 3—moves transforming \( i \) into \( j \).

Let \( j \in R(w_0) \) be obtained from \( i \in R(w_0) \) by a 3-term move at position \( k \). By [18, Proposition 2.3.] we get for \( b \in B(\infty) \) that \( \text{str}_j(b) = \Psi^i_j(\text{str}_i(b)) \). Therefore, inductively, we obtain for any \( i, j \in R(w_0) \) that \( \Psi^i_j \) restricts to a bijection from \( C_i \) to \( C_j \) and is independent of the chosen sequence of 2— and 3—moves transforming \( i \) into \( j \) for any \( i, j \in R(w_0) \).

### 2.2 Inequalities of string cones

**Definition 3** Let \( i, j \in R(w_0) \) and \( i \in I \). We define \( \varsigma_{i, i} \) to be the rational function on \( \mathcal{T}_i = (\mathbb{C}^\ast)^{\Delta^+} \) uniquely determined by the following two conditions.

1. If \( i_N = i \), then \( \varsigma_{i, i}(x_N) = x_N \).
2. We have \([\varsigma_{i, i}]_{\text{trop}} \circ \Psi^i_j = [\varsigma_{j, i}]_{\text{trop}}\).

Note that \( \varsigma_{i, i} \) is well-defined since \( \Psi^i_j \) does not depend on the chosen sequence of 2— and 3—moves transforming \( i \) into \( j \). In [13] we have shown that the tropicalization of this functions gives rise to the string cone inequalities.

**Proposition 2** [13, Proposition 3.5] For \( i \in R(w_0) \), we have

\[
C_i = \{ x \in [\mathcal{T}_i]_{\text{trop}} \mid [\varsigma_{i, i}]_{\text{trop}}(x) \geq 0 \text{ for all } i \in I \}.
\]

The explicit form of the function \([\varsigma_{i, i}]_{\text{trop}}\) is not known. Explicit string cone inequalities are obtained in [18] for a special class of reduced words called nice words (see Sect. 6.1.2 for the definition of nice words) and in [14] for all reduced words in type \( A \) (also in [2] for arbitrary reduced words but in a less explicit form). In [13] we show that the functions \([\varsigma_{i, i}]_{\text{trop}}, i \in I \) recover the string cone inequalities from [14] in type \( A \), an analogous result has been found independently in [4]. In Sect. 6.1.2 we further show that the functions \([\varsigma_{i, i}]_{\text{trop}}, i \in I \), recover the string cone inequalities from [18].
2.3 (Non-)redundancy of string cone inequalities

We fix \( i \in R(w_0) \) and \( i \in I \). In this section we present a sufficient criterion for the non-redundancy of string cone inequalities (Theorem 2). The missing proofs are provided in Sect. 5.

First we note that the function \( \varsigma_{i,i} \) is a Laurent polynomial with non-negative integer coefficients.

Lemma 1 We have \( \varsigma_{i,i} \in \mathbb{Z}_{\geq 0}[x_j^{\pm 1} | j \in N] \).

Proof The fact that \( \varsigma_{i,i} \) is a Laurent polynomial follows from [13, Corollary 7.6.]. The positivity of the coefficients is guaranteed by the definition of \( \varsigma_{i,i} \) since the function is positive provided \( i \) ends on the letter \( i \) and positivity is preserved by \( \Psi_i \) for all \( j \in R(w_0) \). \( \square \)

We denote by \( M_{i,i}(\varsigma) \) the set of Laurent monomials of \( \varsigma_{i,i} \). Our first observation is that the inequalities arising from \( \varsigma_{i,j}, j \neq i \), cannot be expressed in terms of the inequalities of \( \varsigma_{i,i} \).

Proposition 3 Let \( m_0 \in M_{i,i}(\varsigma) \) and assume that the inequality arising from \( m_0 \) is redundant, i.e.

\[
[m_0]_{\text{trop}} = \sum_{j \in I} \sum_{m \in M_{i,j}(\varsigma)} r_m [m]_{\text{trop}}
\]

with \( r_m \in \mathbb{R}_{>0} \). Then \( m \in M_{i,i}(\varsigma) \) with \( i \neq j \) implies that \( r_m = 0 \).

The proof can be found in Sect. 5. We define the following notion.

Definition 4 Let \( i \in R(w_0) \) and \( i \in I \). We say that \( \varsigma_{i,i} = \sum_{k \in \mathbb{Z}^N} a_k x_1^{k_1} \cdots x_N^{k_N} \) is multiplicity-free if for all \( k \in \mathbb{Z}^N \) such that \( a_k \neq 0 \) we have \( |k_j| \leq 1 \) for all \( j \in [N] \).

The notion of multiplicity-free gives us a sufficient criterion for the non-redundancy of string cones inequalities as the following theorem shows.

Theorem 2 If \( \varsigma_{i,i} \) is multiplicity-free for all \( i \in I \), then the set of inequalities given in (2) is non-redundant.

Remark 1 The Laurent-polynomial \( \varsigma_{i,i} \) is multiplicity-free if and only if it is a square-free polynomial in the polynomial ring \( \mathbb{R}[x_1^{-1}, \ldots, x_N^{-1}] \).

We conjecture that the sufficient criterion of Theorem 2 is indeed also necessary.

Conjecture 3 The set of inequalities given in (2) is non-redundant if and only if \( \varsigma_{i,i} \) is multiplicity-free.

In Sect. 7 we provide an evidence of Conjecture 3 in a case which is not multiplicity-free.
3 Cluster varieties and potential functions

We first define the notion of a seed.

**Definition 5**  Let $M$ be a finite index set and $M_0 \subset M$. We associate a quiver $\Gamma_\Sigma$ to a datum $\Sigma = (\Lambda, \langle , \rangle_\Sigma, \{e_k\}_{k \in M})$, where

(i) $\Lambda$ is a lattice,
(ii) $\langle , \rangle_\Sigma$ is a skew-symmetric $\mathbb{Z}$-valued bilinear form on $\Lambda$,
(iii) $\{e_k\}_{k \in M}$ is a basis of $\Lambda$.

The set of vertices $\{v_k\}_{k \in M}$ of the quiver $\Gamma_\Sigma$ is indexed by the set $M$. The set $\{v_k\}_{k \in M_0}$ is called the subset of frozen vertices and the set $\{v_k\}_{k \in M \setminus M_0}$ is called the subset of mutable vertices. Two vertices $v_k$ and $v_\ell$ of $\Gamma_\Sigma$ are connected by $\langle e_k, e_\ell \rangle_\Sigma$ arrows in $\Gamma_\Sigma$ with source $v_k$ and target $v_\ell$ if and only if $\langle e_k, e_\ell \rangle_\Sigma \geq 0$. The datum $\Sigma$ is called a seed.

**Example 1**  The quiver $\Gamma_\Sigma = v_1 \rightarrow v_2$ with frozen vertex $v_2$ and mutable vertex $v_1$ is associated to the seed $\Sigma = (\mathbb{Z}^2, \langle , \rangle_\Sigma, \{e_1, e_2\})$, where $\langle e_1, e_1 \rangle_\Sigma = \langle e_2, e_2 \rangle_\Sigma = 0$ and $\langle e_1, e_2 \rangle_\Sigma = \langle e_2, e_1 \rangle_\Sigma = 1$.

Let $\Sigma = (\Lambda, \langle , \rangle_\Sigma, \{e_k\}_{k \in M})$ be a seed. For each $k \in M \setminus M_0$ we define the seed $\mu_k(\Gamma_\Sigma)$, called the mutation of $\Sigma$ at $k$, by replacing the basis $\{e_k\}_{k \in M}$ by the new basis $\{e'_k\}_{k \in M}$ defined as

$$e'_j = \begin{cases} e_j + \max\{0, \langle e_j, e_k \rangle_\Sigma\}e_k & \text{if } j \neq k \\ -e_k & \text{if } j = k. \end{cases}$$

The quiver $\Gamma_{\mu_k(\Sigma)} = \mu_k(\Gamma_\Sigma)$ has the same vertex set as $\Gamma_\Sigma$. The mutable vertices of $\mu_k(\Gamma_\Sigma)$ equal the mutable vertices of $\Gamma_\Sigma$. The arrow set of $\mu_k(\Gamma_\Sigma)$ equals the arrow set of $\Gamma_\Sigma$ with the following changes:

(i) All arrows of $\Gamma_\Sigma$ with source or target $v_k$ gets replaced in $\mu_k(\Gamma_\Sigma)$ by the reversed arrow.
(ii) For every pair of arrows $(h_1, h_2) \in \Gamma_\Sigma \times \Gamma_\Sigma$ with

$$v_k = \text{target of } h_1 = \text{source of } h_2$$

we add to $\mu_k(\Gamma_\Sigma)$ an arrow with source the source of $h_1$ and target the target of $h_2$.

(iii) If a 2-cycles was obtained during (i) and (ii), the arrows of this 2-cycle get erased in $\mu_k(\Gamma_\Sigma)$.
(iv) Finally we erase all arrows between frozen vertices.

The quiver $\mu_k(\Gamma_\Sigma)$ is called the mutation of $\Gamma_\Sigma$ at $k$.

**Example 2**  For the seed $\Sigma$ given in Example 1 we have only one mutable vertex. Hence the only seed which can be obtained from $\Sigma$ by a sequence of mutations is the seed $\mu_1(\Sigma) = \Sigma' = (\mathbb{Z}^2, \langle , \rangle_{\Sigma'}, \{-e_1, e_2\})$ which associated quiver $\Gamma_{\Sigma'} = v_2 \rightarrow v_1$ with frozen vertex $v_2$ and mutable vertex $v_1$. Here $\langle -e_1, -e_1 \rangle_{\Sigma'} = \langle e_2, e_2 \rangle_{\Sigma'} = 0$ and $\langle -e_1, e_2 \rangle_{\Sigma'} = -\langle e_2, -e_1 \rangle_{\Sigma'} = -1$.
To each seed $\Sigma = (\Lambda, \langle, \rangle_\Sigma, \{e_k\}_{k \in M})$ we assign a collection of $A$-cluster variables $\{A_k(\Sigma)\}_{k \in M}$ and $X$-cluster variables $\{X_k(\Sigma)\}_{k \in M}$ and tori

$$A_\Sigma := \text{Spec} \mathbb{C}[A_k^{\pm 1}(\Sigma) \mid k \in M], \quad X_\Sigma := \text{Spec} \mathbb{C}[X_k^{\pm 1}(\Sigma) \mid k \in M],$$

called the $A$-cluster torus and the $X$-cluster torus associated to $\Sigma$, respectively. We call $\{A_k(\Sigma)\}_{k \in M}$ and $\{X_k(\Sigma)\}_{k \in M}$ the frozen $A$- and $X$-cluster variables respectively.

Assume that the quiver $\Gamma_{\Sigma'}$ of the seed $\Sigma'$ is obtained from the quiver $\Gamma_\Sigma$ of the seed $\Sigma$ by mutation at the vertex $k$. We define birational transition maps (see [8, Equations (13) and (14)])

$$\mu_k^* A_i(\Sigma') = \begin{cases} A_k^{-1}(\Sigma) \left( \prod_{j : \langle e_j, e_k \rangle_\Sigma > 0} A_j(\Sigma)^{\langle e_j, e_k \rangle_\Sigma} + \prod_{j : \langle e_j, e_k \rangle_\Sigma < 0} A_j(\Sigma)^{-\langle e_j, e_k \rangle_\Sigma} \right) & \text{if } i = k, \\ A_i(\Sigma) & \text{else,} \end{cases}$$

$$\tilde{\mu}_k^* X_i(\Sigma') = \begin{cases} X_k(\Sigma)^{-1} & \text{if } i = k, \\ X_i(\Sigma) \left( 1 + X_k(\Sigma)^{-\text{sgn}(\langle e_i, e_k \rangle_\Sigma)} \right)^{-\langle e_i, e_k \rangle_\Sigma} & \text{else,} \end{cases}$$

which we call $A$- and $X$-cluster mutation, respectively.

**Definition 6** Given a fixed initial seed $\Sigma_0$ the $A$- and $X$-cluster variety, respectively, is defined as the scheme

$$A := \bigcup_\Sigma A_\Sigma \quad X := \bigcup_\Sigma X_\Sigma,$$

obtained by gluing the tori $A_\Sigma$ and $X_\Sigma$ along the $A$- and $X$-cluster mutation, respectively. Here $\Sigma$ varies over all seeds which can be obtained from $\Sigma_0$ by a finite sequence of mutations.

Furthermore we define a partial compactification $\overline{A}$ of $A$ by adding the divisors corresponding to the vanishing locus of the frozen cluster variables.

In [11] a Landau-Ginzburg potential $W$ on $X$ is defined as the sum of certain global monomials attached to the frozen cluster variables. We only give the definition in the case that every frozen cluster variable has an optimized seed.

**Definition 7** Let $\Sigma$ be a seed and $k \in M_0$ a frozen vertex of $\Gamma_\Sigma$. We say that $\Sigma$ is optimized for $k$ if whenever $k' \in M \setminus M_0$ is adjacent to $k$, the source of the corresponding arrow is $k'$.

Let $X$ be the cluster variety obtained from the initial seed $\Sigma_0$. If for every $k \in M_0$ there exists a seed $\Sigma_k$ of $X$ which is optimized for $v_k$, we define

$$W = \sum_{k \in M_0} W_k \in \mathbb{C}[X]$$

by

$$W_k|_{X_{\Sigma_k}} = X_k^{-1}(\Sigma_k).$$
In the remainder of this section we prove that for any seed $\Sigma$ the Laurent polynomial $W|_{A_{\Sigma}}$ only has non-positive exponents. To show this we first introduce notions related to cluster algebras of geometric type following [17].

Given a fixed initial seed $\Sigma_0$, let $\Sigma$ be a seed obtained from $\Sigma_0$ by a finite sequence of mutations and $Q := \Gamma_{\Sigma}$ the corresponding quiver as given by Definition 5. The principal extension of $Q_{pr}$ of $Q$ is the quiver obtained from $Q$ by adding new vertices \{$w_k \mid k \in M$\} and a new arrow $w_k \to v_k$ for each vertex $v_k \in Q$.

The $c$-vector $(c_1, i(\Sigma), \ldots, c_m, i(\Sigma))^t$ of the cluster variable $A_i(\Sigma)$ has as entries $c_{k,i}(\Sigma)$ equals the difference of the number of arrows $a$ of $Q_{pr}$ with source $w_k$ and target $v_i$ and the number of arrows $a$ of $Q_{pr}'$ with source $v_i$ and target $w_k$, where $Q_{pr}'$ is obtained from $Q_{pr}$ by applying the sequence of mutations from $\Sigma_0$ to $\Sigma$.

By [10, Proposition 3.6] the cluster variable $A_j(\Sigma)$ of $A_{Q_{pr}}$ belongs to the ring $\mathbb{Z}[A_k(\Sigma_0)^{\pm 1}, A_{\ell}(\Sigma_0) \mid k \in M]$. The F-polynomial $F_j(t) \in \mathbb{Z}[A_k(\Sigma_0) \mid k \in M]$ of the cluster variable $A_j(\Sigma)$ is defined as the specialization of $A_j(\Sigma)$ to $\mathbb{Z}[A_k(\Sigma_0)^{\pm 1}, A_{\ell}(\Sigma_0) \mid k \in M]$ at $A_k(\Sigma_0) = 1$ for all $k \in M$.

The following separation formula due to Fomin-Zelevinsky expresses any $X$-cluster variable of a seed $\Sigma$ obtained from $\Sigma_0$ by a finite sequence of mutations as a rational function of in the $X$-cluster variables of the seed $\Sigma_0$.

**Theorem 3** ([10, 17, Theorem 5.7]) Given a fixed initial seed $\Sigma_0$ whose associated quiver $Q$ has no frozen vertices. Let $\Sigma$ be a seed obtained from $\Sigma_0$ by a finite sequence of mutations and $M = \{1, \ldots, m\}$. We have

$$X_j^{-1}(\Sigma) = X_1^{-1}(\Sigma_0)^{c_{1,j}(\Sigma)} \cdots X_m^{-1}(\Sigma_0)^{c_{m,j}(\Sigma)} \prod_{i \in M} F_i(\Sigma)(X_1^{-1}(\Sigma_0), \ldots, X_m^{-1}(\Sigma_0))^{\langle e_i, e_k \rangle},$$

where $F_i(\Sigma)(X_1^{-1}(\Sigma_0), \ldots, X_m^{-1}(\Sigma_0))$ is obtained by substituting for $A_k(\Sigma_0)$ the variable $X_k^{-1}(\Sigma_0)$ for all $k \in M$.

**Remark 2** Theorem 3 is stated in [9, 17] in terms of $Y$-variables defined therein. However, comparing the mutation rule of $Y$-variables (see [17, Equation 22]) with the one for $X$-cluster variables given in (3), the claim as stated here follows immediately.

**Proposition 4** Let $\Sigma_0 = (\Lambda, \langle \cdot, \cdot \rangle, \{e_k\}_{k \in M})$ be a seed and $X$ be the cluster variety obtained from the initial datum $\Sigma_0$. Assume furthermore that every frozen vertex has an optimized seed. For any seed $\Sigma$ of $X$ we have

$$W|_{A_{\Sigma}} \in \mathbb{C}[X_k^{-1}(\Sigma) \mid k \in M].$$

**Proof** We proof the statement for an arbitrary $k \in M_0$. Let $\Sigma_k$ be a seed of $X$ which is optimized for $k$. Assume that $\Sigma_k$ can be obtained from $\Sigma_0$ by the sequence of mutations at $v_{j_1}, v_{j_2}, \ldots, v_{j_m}$ and assume that $\Sigma$ can be obtained from $\Sigma_0$ by the sequence of mutations at $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_p}$.

Let $\tilde{\Sigma}_0 = (\tilde{\Lambda}, \langle \cdot, \cdot \rangle_{\tilde{\Sigma}_0}, \{e_j\}_{j \in \tilde{M}})$ where $\tilde{M} = (M \setminus M_0) \cup k$, $\tilde{\Lambda}$ is the sublattice of $\Lambda$ spanned by $e_j$ with $j \in \tilde{M}$ and $\langle \cdot, \cdot \rangle_{\tilde{\Sigma}_0}$ is the restriction of $\langle \cdot, \cdot \rangle_{\Sigma_0}$. We define all

\[ \tilde{\Sigma}_0 \] Springer
vertices $v_j$ of $\Gamma_1$ to be mutable. Hence the cluster variety $\tilde{X}$ obtained from this initial datum has no frozen cluster variables. Moreover we denote by $\tilde{\Sigma}$ the seed of $\tilde{X}$ which is obtained from the initial seed $\tilde{\Sigma}_0$ by mutations at the sequence $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_p}$ and by $\tilde{\Sigma}_k$ the seed of $\tilde{X}$ which is obtained from the initial seed $\tilde{\Sigma}_0$ by mutations at the sequence $v_{j_1}, v_{j_2}, \ldots, v_{j_m}$. In other words $\tilde{\Sigma} (\tilde{\Sigma}_k)$ are obtained from the same sequence of mutation at vertices $v_j, j \in \tilde{M} \subset M$ as $\tilde{\Sigma} (\tilde{\Sigma}_k), \text{respectively,}$ is obtained from $\tilde{\Sigma}_0$.

By Theorem 3 and the definition of $W_k$ (Definition 7), we have

$$W_k |_{\chi_{\Sigma}} = W_k |_{\chi_{\tilde{\Sigma}}} = \prod_{i \in \tilde{M}} X_i (\Sigma)^{-c_{i,k}(\tilde{\Sigma}_k)} \prod_{j \in \tilde{M}} F_j (\tilde{\Sigma}_k) (X_1^{-1} (\Sigma), \ldots, X_n^{-1} (\Sigma))^{(e_j, e_k)_{\tilde{\Sigma}_k}}. \quad (4)$$

Since the $F-$polynomial is an honest polynomial and $(e_i, e_k)_{\tilde{\Sigma}_k} \geq 0$ for all $i$ due to the fact that $\tilde{\Sigma}_k$ is optimized for the vertex $v_k$, it remains to show that $c_{i,k} \geq 0$ for all $i$. This follows from the fact that in the sequence of mutations from $\tilde{\Sigma}$ to $\tilde{\Sigma}_k$ we have never mutated at $k$. \hfill \Box

4 String cones and potential functions

Let $G$ be a simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $H$ be a maximal torus with Lie algebra $\mathfrak{h}$. Let $B, B_-$ denote a pair of opposite Borel subgroups of $G$ with $B \cap B_- = H$ and $N \subset B$ be the unipotent radical of $B$. The reduced double Bruhat cell $L_{e, w_0}$ associated to $e$ and $w_0$ is defined as:

$$L_{e, w_0} = N \cap B_- w_0 B_-.$$

Following Berenstein-Fomin-Zelevinsky, Fomin-Zelevinsky and Fock-Goncharov [3, 8–10] we endow $L_{e, w_0}$ with the structure of a cluster variety.

Definition 8 Following [3] we associate to every reduced word $i = (i_1, i_2, \ldots, i_N) \in R(w_0)$ a seed $\Sigma_i$ of $L_{e, w_0}$ and the corresponding quiver $\Gamma_i = \Gamma_{\Sigma_i}$ with vertex set \( \{v_k \mid k \in N\} \). For an index $k \in N$ we denote by $k^+ = k_1^+$ the smallest index $\ell \in M$ such that $k < \ell$ and $i_\ell = i_k$. If no such $\ell$ exists, we set $k^+ = N + 1$. Two vertices $v_k, v_\ell, k < \ell$ are connected by an edge in $\Gamma_i$ if and only if $\{k^+, \ell^+\} \cap [N] \neq \emptyset$ and one of the two conditions are satisfied

(i) $\ell = k^+$,
(ii) $\ell < k^+ < \ell^+$.

An edge of type (i) is directed from $k$ to $\ell$ and an edge of type (ii) is directed from $\ell$ to $k$. The set of frozen vertices of $\Gamma_i$ is given by all $v_k$ such that $k^+ = N + 1$. 
**Example 3** Let \( g = \text{sl}_3(\mathbb{C}) \). In this example we have \( R(w_0) = \{i, j\} \) with \( i = (1, 2, 1) \) and \( j = (2, 1, 2) \). We have

\[
\Gamma_i = \begin{array}{ccc}
  & v_2 & \\
v_1 & & v_3,
\end{array}
\]

where the set of frozen vertices is given by \( \{v_2, v_3\} \). Moreover

\[
\Gamma_j = \begin{array}{ccc}
  & v_2 & \\
v_1 & & v_3,
\end{array}
\]

where the set of frozen vertices is again given by \( \{v_2, v_3\} \).

Throughout, to simplify notation, we abbreviate \( \mathcal{X}_i := \mathcal{X}_{\Sigma_i} \). We recall the relation between seeds associated to different reduced words.

**Lemma 2** ([13, Lemma 4.6]) Let \( j \in R(w_0) \) be obtained from \( i \in R(w_0) \) by a 3-term move in position \( k \). Then the swapping of vertex \( v_k \) with vertex \( v_{k+1} \) is an isomorphism of quivers \( \Gamma_j \cong \mu_{k-1}\Gamma_i \).

**Example 4** Let \( i \) and \( j \) be as in Example 3. Swapping \( v_2 \) and \( v_3 \) is an isomorphism between \( \Gamma_j \) and

\[
\mu_1\Gamma_i = \begin{array}{ccc}
  & v_2 & \\
v_1 & & v_3.
\end{array}
\]

We get the following lemma as a consequence.

**Lemma 3** Let \( i = (i_1, \ldots, i_N) \in R(w_0) \). Then \( \Gamma_i \) is optimized for the frozen vertex \( v_N \). Moreover, for any \( j \in R(w_0) \) and any frozen cluster variable there exists a sequence of mutations to a seed which is optimized for this cluster variable such that all seeds appearing in this sequence are of the form \( \Gamma_i \) for some \( i' \in R(w_0) \).

**Proof** The fact that \( \Gamma_i \) is optimized for the frozen vertex \( v_N \) follows from Definition 8. Let \( i, j \in R(w_0) \) be arbitrary. It is well-known (see e.g. [19, Theorem 1.9]) that \( j \) may be obtained from \( i \) by a finite sequence of 2-term and 3-term moves. Therefore one may find a sequence of 2-term and 3-term moves transforming \( j \in R(w_0) \) into a reduced word ending in \( i \) for any \( i \in I \). By Lemma 2 we thus get the sequence of mutations as required. \( \square \)
We fix a reduced word $i \in R(w_0)$. In the following we explain the relation between the string cone inequalities and the potential function on $X'$ with initial datum given by Definition 8.

**Definition 9** We define $\widehat{CA}_i \in \text{Hom}((\mathbb{C}^*)^{\Delta_i^+}, X_i)$ as follows

$$\widehat{CA}_i(x)_k = \prod_{\ell \in \mathbb{N}} x^{[k, \ell]}_{\ell},$$

where $[k, \ell] := -b_{i_k, i_\ell}$,

$$\begin{cases} 
1 & \text{if } k < \ell < k^+, \\
\frac{1}{2} & \text{if } \ell = k \text{ or } \ell = k^+, \\
0 & \text{else}.
\end{cases}$$

**Proposition 5** The map $\widehat{CA}_i \in \text{Hom}((\mathbb{C}^*)^{\Delta_i^+}, X_i)$ is an isomorphism of algebraic tori and satisfies $\varsigma_{i, i} = W_i|_{X_i} \circ \widehat{CA}_i$.

**Proof** The second part is proved analogously as [13, Theorem 7.5] noting that the right diagram in Lemma 7.4 of op. cit. also commutes if we specialize the variables $X_k$ of $X_i$ with $k < 0$ and the variables $\lambda_j$, $j \in I$ of gr$S_i$ to 1.

The first part follows by Lemma 8.1 of op. cit. applying the same specialization of variables. \(\square\)

### 5 Proof of the sufficient criterion for non-redundancy of inequalities

Let $\mathcal{M}_{i, i}$ be the set of monomials in $W_i|_{X_i}$. We abbreviate the cluster variable $X_k(\Sigma_i)$ by $X_k(i)$. For a fixed $i \in I$ we write $\{\beta_\ell \in \Delta_i^+ \mid i_\ell = i\} = \{\beta_{i, 1}, \ldots, \beta_{i, m_i}\}$ with $m_i = m_{i, i} \in \mathbb{N}$ and $\beta_{i, 1} <_i \cdots <_i \beta_{i, m_i}$. We denote a vertex $v_k$ of $\Gamma_i$ by $v_{i, \ell}$ if $\beta_{i, \ell} = \beta_k \in \Delta_i^+$. If we want to stress that we fix the reduced word $i \in R(w_0)$, we also write $v_{i, \ell}(i)$.

**Lemma 4** (1) For every $m \in \mathcal{M}_{i, i}$, we have $m = X_{i, m_i}^{-1}(i)m'$, where $m'$ is a Laurent monomial in which $X_{i, m_i}^{-1}$ does not appear with a strictly positive exponent.

(2) Let $j \in I$ and $m_0 \in \mathcal{M}_{i, j}$. Assume that the inequality arising from $m_0$ is redundant, i.e.

$$[m_0]_{trop} = \sum_{i \in I} \sum_{m \in \mathcal{M}_{i, i}} r_m[m]_{trop}$$

with $r_m \in \mathbb{R}_{>0}$. Then $m \in \mathcal{M}_{i, i}$ with $i \neq j$ implies that $r_m = 0$, thus redundancies occur separately within the summands of the potential of the form $W_i$ for an $i \in I$.

**Proof** Let $i' = (i'_1, \ldots, i'_{N}) \in R(w_0)$ be such that $i'_{N} = i$. Since $i'$ can be obtained from $i$ by a finite sequence of 2-term and 3-term moves, we can find by Lemma 2 a sequence of mutations at vertices $(k_1, \ldots, k_t)$ corresponding to these braid moves,
transforming (up to relabelling coordinates) $\Sigma_i$ to $\Sigma_i'$. By Definition 7 and Lemma 3 we have

$$W_i|_{X_i} = \tilde{\mu}_{k_1}^* \circ \ldots \circ \tilde{\mu}_{k_t}^* X_{i,m_i}^{-1}(i').$$  \hfill (5)

We prove (1) by induction over $t$. The case $t = 0$ is trivial. Let $j$ be the reduced word we get after applying the sequence of mutations at the vertices $(k_2, \ldots, k_t)$ to $i'$. We have

$$W_i|_{X_i} = \tilde{\mu}_{k_1}^* \sum_{i \in I} \sum_{m \in M_{j,i}} m.$$  

By induction hypothesis, we have for every $m \in M_{j,i}$ that $m = X_{i,m_i}^{-1}(j)m'$, where $m'$ is a monomial in which $X_{i,m_i}^{-1}(j)$ does not appear with a strictly positive exponent. Thus

$$\tilde{\mu}_{k_1}^* m = \tilde{\mu}_{k_1}^* X_{i,m_i}^{-1}(j)m'.$$

Since $X_{i,m_i}(j)$ is frozen and does not appear with strictly negative exponent in $m'$, the cluster variable $X_{i,m_i}(i)$ does not appear with strictly negative exponent in any monomial of $\tilde{\mu}_{k_1}^* m'$.

Let $\ell \in \mathbb{N}$ be such that $X_{i,m_i}(j) = X_{\ell}(j)$. Then we have (again since $X_{i,m_i}(j)$ is frozen)

$$\tilde{\mu}_{k_1}^* X_{\ell}^{-1}(j) = X_{\ell}^{-1}(i) (1 + X_{\ell}(i)^{-\text{sgn}(e_{\ell}, e_k)i})^{-\langle e_{\ell}, e_k \rangle_i}.$$  

The first claim follows.

To prove (2) note that by the $\mathcal{X}$-cluster mutation rule (3) and (5) there exists a monomial of $\tilde{\mu}_{k_s}^* \circ \ldots \circ \tilde{\mu}_{k_1}^* X_{i,m_i}^{-1}(i)$ divisible by the variable $X_{\ell}$ if and only if there exists a monomial of $\tilde{\mu}_{k_s}^* \circ \ldots \circ \tilde{\mu}_{k_t}^* X_{i,m_i}^{-1}(i)$ divisible by the variable $X_{\ell}$ or $s = \ell$.

This implies in particular that $X_{i,m_i}(i)$ does not divide any element of $M_{k_i,j}$.

On the other hand, we have have, by the first part of this lemma, that $X_{i,m_i}(i)$ divides any element of $M_{k_i,j}$ which proves the second part. \hfill $\Box$

**Proof of Proposition 3** Assume that

$$[m_0]_{\text{ trop}} = \sum_{j \in I} \sum_{m \in M_{k_j,j}(\xi)} r_m[m]_{\text{ trop}}$$

with $r_m \in \mathbb{R}_{>0}$. This implies by Proposition 5

$$[m_0 \circ \hat{C}_{k_i}^{-1}]_{\text{ trop}} = \sum_{j \in I} \sum_{m \in M_{k_j,j}(\xi)} r_m[m \circ \hat{C}_{k_i}^{-1}]_{\text{ trop}}.$$ 

Now Lemma 4 yields the claim. \hfill $\Box$
Definition 10 Let $i \in I$ and $i \in R(w_0)$. We say that $W_i|_{X_i}$ is multiplicity-free, if for all $m \in \mathcal{M}_{i,i}$, $m = \prod_{k=1}^{N} a_k X_{jk}^k$, we have $|j_k| \leq 1$.

We say that $W$ is multiplicity-free for $i$ if $W_i$ is multiplicity-free for $i$ for all $i \in I$.

Note that by Proposition 4 $|j_k| \leq 1$ is equivalent to $j_k \geq -1$ in Definition 10.

Proposition 6 Let $i \in I$ and $i \in R(w_0)$ and $W_i|_{X_i}$ be multiplicity-free. Then the set of inequalities

$$
\{[m]_{trop}(x) \geq 0 \mid m \in \mathcal{M}_{i,i}\}
$$

is non-redundant.

Proof Let $W_i|_{X_i}$ be multiplicity-free and assume that the inequality arising from $m_0 \in \mathcal{M}_{i,i}$ is redundant, i.e. there exists $\emptyset \neq J \subset \mathcal{M}_{i,i} \setminus \{m_0\}$ such that

$$
[m_0]_{trop} = \sum_{m \in J} r_m [m]_{trop}
$$

with $r_m \in \mathbb{R}_{>0}$. Let $k$ be such that $i_k = i$ and $i_k^+ = N + 1$ and let $e_k \in \mathbb{R}^N$ be defined as $(e_k)_j = \delta_{k,j}$. By the first part of Lemma 4 and the fact that $W_i|_{X_i}$ is multiplicity-free, we get by plugging in $-e_k$ into (6):

$$
1 = \sum_{m \in J} r_m.
$$

Since $\emptyset \neq J \subset \mathcal{M}_{i,i} \setminus \{m_0\}$, we can find $m_0 \neq m_1 \in J$ and $1 \leq s \leq N$ such that either (1) $[m_0](e_s) = 0$ and $[m_1](e_s) \neq 0$ or (2) $[m_0](e_s) \neq 0$ and $[m_1](e_s) = 0$.

In the first case we get by Proposition 4 and the assumption that $W_i|_{X_i}$ is multiplicity-free by plugging in $-e_s$ into (6):

$$
0 = 1 + \sum_{m \in J \setminus m_1} r_m [m]_{trop}(-e_s).
$$

Since $[m]_{trop}(-e_s) \geq 0$ by Proposition 4, we obtain a contradiction.

In the second case, we get by by plugging in $-e_s$ into (6) again using Proposition 4 and the assumption that $W_i|_{X_i}$ is multiplicity-free:

$$
1 = \sum_{m \in J \setminus m_1} r_m [m]_{trop}(-e_s)
$$

and $0 \leq [m]_{trop}(-e_s) \leq 1$. Therefore

$$
\sum_{m \in J \setminus m_1} r_m [m]_{trop}(-e_s) < \sum_{m \in J} r_m = 1.
$$

Again a contradiction.
Proof of Theorem 2.} Note that, by Proposition 5, \( \varsigma_{i,i} \) is multiplicity-free if and only if \( W_i \rvert_{X_i} \) is multiplicity-free. Moreover, the set of inequalities given in (2) is redundant if and only if the set of inequalities \( \{ \lceil m \rceil_{trop}(x) \geq 0 \mid m \in \mathcal{M}_{i,i} \} \) is redundant by Proposition 3. Therefore the claim follows from Proposition 6.

\[ \square \]

6 Multiplicity-free examples

6.1 Nice words and simply braided words

6.1.1 Simply braided words

Following [21] we define

Definition 11 Let \( i \in I \). We call \( i \in R(w_0) \) simply braided for \( i \) if one can perform a sequence of braid moves changing \( i \) to a reduced word \( i' = (i'_1, \ldots, i'_{\ell}) \in R(w_0) \) with \( i'_N = i \), and each move in the sequence is either

- a 2-move, or
- a 3-move at position \( k \) such that \( \alpha_i = \beta_{k-1} \) in the \( \prec_i \)-order on \( \Phi^+ \), i.e. \( \alpha_i \) is the leftmost root affected.

We call \( i \) simply braided if it is simply braided for all \( i \in I \). We call the above sequence from \( i \) to \( i' \) of 2- and 3-moves a simply braided sequence.

Remark 3 Note that we take a slightly different convention than [21] here. If \( i \) satisfies Definition 11 this implies that the reverse word is simply braided in the sense of [21].

Proposition 7 Let \( i \in R(w_0) \) be simply braided and fix a simply braided sequence. Let \( \beta_{k_1}, \ldots, \beta_{k_s} \in \Delta^+ \) be indexed w.r.t. the \( \prec_i \)-order such that in the \( j \)-th 3-term move in the simply braided sequence \( \beta_{k_j} \) is the middle root affected. Then

\[
W_i \rvert_{X_i} = X_{\alpha_i}^{-1}(i) \left( \sum_{\ell=1}^{s} \prod_{j=1}^{\ell} X_{k_j}^{-1}(i') + 1 \right).
\]

Proof We prove the claim by induction over \( s \). If \( s = 0 \), then \( \Gamma_i \) is optimized for \( i \), \( W_i \rvert_{X_i} = X_{\alpha_i}^{-1} \) and our claim is true. Assume now that \( s > 0 \) and let \( \beta_{t} = \alpha_k \). Since \( i \) is by assumption simply braided, we get (up to 2-term moves) that \( (i_t, i_{t+1}, i_{t+2}) \) is the first 3-term move in the simply braided sequence. Let \( i' \) be the reduced word obtained from \( i \) by a 3-term move at position \( t + 1 \). Clearly \( i' \) is still simply braided with a simply braided sequence such that \( \beta_{k_j} \) is the middle root affected in the \( j + 1 \)-th 3-term move for all \( j \in \{2, \ldots, s\} \). Hence, by induction hypothesis,

\[
W_i \rvert_{X'_i} = \mu_i \circ W_i \rvert_{X_i} = X_{\alpha_i}^{-1}(i')(\sum_{\ell=1}^{s} \prod_{j=2}^{\ell} X_{k_j}^{-1}(i') + 1).
\]
By the definition of the graph $\Gamma_{\mathcal{I}}$, it looks locally around $v_t$ as follows:

$$v_t(i') \xrightarrow{v_{t+1}} v_{t+1}(i') \xleftarrow{v_{t+1}^{-1}} v_t \xrightarrow{v_t(i')} v_t+1(i')$$

Since $\beta_{k_j} \leq \alpha_i$ for all $j < s$ and $X_t(i') = X_{\alpha_i}(i')$, $X_{i'}(i') = X_{k_i}(i')$, we get the claim by the $\mathcal{X}$-cluster mutation rule (3) and Lemma 2.

We are ready to prove that the string cone inequalities are non-redundant for simply braided $i$.

**Theorem 4** If $i \in R(w_0)$ is simply braided, $\mathcal{S}_{1,i}$ is multiplicity-free. In particular, the inequalities from (2) are non-redundant.

**Proof** We have for all $i \in I$ that $W_i|_{\mathcal{X}_i}$ be multiplicity-free by Proposition 7. Hence the the inequalities are non-redundant by Propsition 5 and Theorem 6.

### 6.1.2 Nice words

We call a fundamental weight $\omega_i$ minuscule if $\beta(\omega_i^\vee) \in \{-1, 0, 1\}$ for all $\beta \in \Delta$. We recall the following notions from [18].

**Definition 12**

1. We call an enumeration $\{\alpha_1, \ldots, \alpha_n\}$ of $\Delta$ a good enumeration if for all $i \in \{1, \ldots, n\}$ the fundamental weight $\omega_i$ is minuscule for $G'$ corresponding to the Dynkin diagram of $G$ with the nodes labeled by $\{1, \ldots, i-1\}$ removed.

2. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a good enumeration. For $j \in \{1, \ldots, n\}$ we denote by $W_j$ the subgroup of $W$ generated by $s_1, \ldots, s_j$. We call a reduced word $i = (i_1, \ldots, i_N) \in R(w_0)$ a nice word if $s_1 s_2 \cdots s_N = \tau_1 \tau_2 \cdots \tau_n$, where $\tau_j$ is the longest word in the set of minimal representative in $W_j$ of $W_j - 1 \setminus W_j$.

**Remark 4** Nice words exist for all $\mathfrak{g}$ not of type $E_8$ by [18]. Moreover, each $\tau_j$ in Definition 12 is unique up to 2–term braid moves by [18, Lemma 3.2].

By [21, Lemma 4.17] every nice word $i \in R(w_0)$ is simply braided for all $i \in I$. Hence we get as a direct consequence of Propositions 7 and 5.

**Theorem 5** Let $i \in R(w_0)$ be a nice word. Then $W_i|_{\mathcal{X}_i}$ is multiplicity-free and the inequalities of $C_i$ from (2) are non-redundant. Moreover, the inequalities 2 are explicitly given by $t_k \geq t_{k'}$ for any $k, k' \in [N]$ such that $\beta_k \leq_1 \beta_{k'}$ and thus recover the inequalities from [18].

### 6.2 Minuscule weights and $i$-trails

We recall the notion of an $i$-trail from [2].
For a finite dimensional representation $V$ of $\mathfrak{g}$, two weights $\gamma, \delta$ of $V$ and $i \in R(w_0)$, we say that a sequence of weights $\pi = (\gamma = \gamma_0, \gamma_1, \ldots, \gamma_N = \delta)$ is an $i$-trail from $\gamma$ to $\delta$ if

- $\gamma_{s-1} - \gamma_k = c_k \alpha_i$ for some $c_k \in \mathbb{Z}_{\geq 0}$.
- $e_i^{c_1} e_i^{c_2} \cdots e_i^{c_t}$ is a non-zero map from $V_\delta$ to $V_\gamma$. Here $V_\delta$ ($V_\gamma$) denotes the weight space of $V$ corresponding to the weight $\delta$ ($\gamma$, respectively).

We further define for any $i$-trail $\pi = (\gamma_0, \gamma_1, \ldots, \gamma_{\ell})$ in a $\mathfrak{g}$-module and every $k \in [\ell]$ the value

$$d_k = d_k(\pi) = \frac{\gamma_{k-1} + \gamma_k}{2} (\alpha_i^*)\delta.$$  

By [2, Theorem 3.10] the string cone $C_i$ is the cone in $\mathbb{R}^N$ given by all $(t_1, \ldots, t_N)$ such that $\sum_k d_k(\pi)t_k \geq 0$ for any $i \in I$ and any $i$-trail $\pi$ from $\omega_i^\vee$ to $w_0 \sigma_i \omega_i^\vee$ in the $L_\mathfrak{g}$-module $V(\omega_i^\vee)$.

The aim of this section is to prove that the defining inequalities of $C_i(i)$ are non-redundant provided $\omega_i$ is minuscule. The essential argument relies on extremal $i$-trails introduced in [2]. We first give the relation to our function $s_{l_i}$ from (2).

Recall for $i \in I$ that we denote by $i^* \in I$ the unique element such that $w_0 \omega_i = \omega_i^*$. Note that $\omega_i$ is minuscule if and only if $\omega_i^*$ is minuscule.

**Proposition 8** Let $i \in I$ be such that $\omega_i$ is minuscule and $i \in R(w_0)$. We have for $t \in \mathbb{R}^N$

$$[s_{l_i}]_{trop}(t) = \min_\pi \sum_k d_k(\pi)t_k \geq 0,$$

where the minimum is taken over all $i$-trails $\pi$ from $\omega_i^\vee$ to $w_0 \sigma_i \omega_i^\vee$ in $V(\omega_i^\vee)$.

**Proof** For $i \in I$ let us denote by $t_i : \mathbb{Z}^N \to \mathbb{Z}$ the piecewise-linear function such that $t_i(t) = \min_\pi \sum_k d_k(\pi)t_k$ where the minimum is taken over all $i$-trails $\pi$ from $\omega_i^\vee$ to $w_0 \sigma_i \omega_i^\vee$ in $V(\omega_i^\vee)$. To prove the claim we need to show that $[s_{l_i}]_{trop} = t_i$. By the proof of [2, Theorem 3.10], we have for any $i' \in R(w_0)$,

$$t_i \circ \Psi_{l_i}^{i'} = t_i^{'*}.$$  

(7)

By Definition 3 it suffices to prove the claim for one fixed $i \in R(w_0)$. Let $w_0'$ be the longest element in the maximal parabolic subgroup of $W$ generated by $\{s_j \mid j \in I \setminus \{i\}\}$. Let $\tau_i$ be as in Definition 12. Let $i_0$ be a reduced word such that $i_0 = (i_0', i_0''') = (i_1', \ldots, i_N')$ where $i_0'$ is a reduced word for $w_0'$ and $i_0''$ is a reduced word for $\tau_i$. By [18, Lemma 3.2], the word $i_0'$ is unique up to 2-term braid moves hence we are in the situation of [2, Proof of Theorem 3.13]. From this we conclude that $t_{i_0}(t) = t_N$ and that $i_N' = (i^*)^* = i$. We conclude, by Definition 3, that $[s_{i_0}, i]_{trop}(t) = t_N$ which proves the claim. \hfill \Box

We are ready to prove the main result of this section.
Theorem 6 Let \( i \in I \) be such that \( \omega_i \) is minuscule and \( i \in R(w_0) \). Then \( \zeta_{1,i} \) is multiplicity-free and the inequalities from (2) are non-redundant.

Proof Let \( W_i \) be the maximal parabolic subgroup of \( W \) generated by all \( s_j \) with \( j \neq i \) and let \( u(i) \) be the minimal representative of the coset \( W_i s_i w_0 \) in \( W \). Let \( t_i(t) \) be as in the proof of Proposition 8. With the convention that \( k(0) = 0 \) and \( k(p + 1) = N + 1 \) we have by [2, Proposition 9.2, Theorem 3.10] that

\[
t_i(t) = \min_{(i_{k(1)}, \ldots, i_{k(p)})} \sum_{j=0}^{p} \sum_{k(j) < k < k(j+1)} s_{i_{k(1)}} \cdots s_{i_{k(j+1)}} \alpha_{i_{k_j}}(\omega_i^\vee) \cdot t_k,
\]

where the minimum ranges over all subwords \((i_{k(1)}, \ldots, i_{k(p)})\) of \( i \) which are a reduced word for \( u(i) \). The claim now follows from the assumption that \( \omega_i \) is minuscule and Proposition 8. \( \square \)

We get as a direct corollary.

Corollary 1 Let \( g = sl_{n+1}(\mathbb{C}) \). Then \( \zeta_{1,i} \) is multiplicity-free and the inequalities from (2) are non-redundant for all \( i \in I \).

Corollary 1 was already proven in [5, Proposition 4.5.]. Using the notation of [5] we note that by [12]

\[
W|_{\chi_i} = \sum_P \prod_{C_j \text{is in a chamber enclosed by } P} X_j(i)^{-u_j},
\]

where \( P \) varies over all rigorous paths.

We end this section by remarking that an algorithm for the computation of all \( i \)-trails from \( \omega_i^\vee \) to \( w_0 s_i \omega_i^\vee \) for minuscule \( \omega_i \) was given recently in [15] by computing the monomials in the Berenstein-Kazhdan decoration functions as defined in op. cit. From this one may deduce an alternative proof of Theorem 6 by combining [15, Theorem 2.5 and the argument below, Lemma 2] and [13, Theorem 7.5].

7 Beyond the multiplicity-free case

In this subsection we study the following example. Let \( \mathfrak{g} = so_8(\mathbb{C}) \). We fix a labelling of the Dynkin diagram as follows

[Diagram]

1 —— 2 —— 4

3

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Fix $i = (2, 1, 3, 4, 2, 1, 3, 4, 2, 1, 3, 4) \in R(w_0)$. The quiver $\Gamma_1$ looks as follows:

![Quiver Diagram]

The vertices $v_{12}, v_{11}, v_9$ and $v_{10}$ are frozen. Let $\mathcal{X}$ be the cluster variety obtained from the initial datum $\Sigma_i$. Note that $\Sigma_i$ is optimized for the frozen vertices $v_{10}, v_{11}$ and $v_{12}$. Thus, by Definition 7, we have

$$W_1 \big|_{\mathcal{X}_i} = X_{10}^{-1}(i), \quad W_3 \big|_{\mathcal{X}_i} = X_{11}^{-1}(i), \quad W_4 \big|_{\mathcal{X}_i} = X_{12}^{-1}(i).$$

It remains to compute $W_2 \big|_{\mathcal{X}_i}$. Note that $\omega_2$ is not minuscule and $i$ is not simply braided for 2. One checks that the following sequence of mutations (read from left to right) leads to a seed $\Sigma'$ which is optimized for $v_0$:

$$(6, 3, 5, 4, 3, 1, 2, 7, 6, 8).$$

We use this sequence to compute:

$$W_2 \big|_{\mathcal{X}_i} = X_9^{-1}X_1^{-1}X_2^{-1}X_3^{-1}X_4^{-1}X_5^{-2}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_3^{-2}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_2^{-1}X_4^{-1}X_5^{-2}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_3^{-1}X_4^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_3^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_4^{-1}X_5^{-2}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_2^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_3^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_2^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_2^{-1}X_3^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_4^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_4^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_3^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_2^{-1}X_3^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_3^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + 2X_9^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_6^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_7^{-1}X_8^{-1} + X_9^{-1}X_8^{-1} + X_9^{-1}X_7^{-1}X_8^{-1}. $$
One checks that the cone given by all \( t \in \mathbb{R}^{12} \) such that \( [W_2|_{\mathcal{X}_i}]_{\text{trop}}(t) \geq 0 \) has 26 facets but \( W_2|_{\mathcal{X}_i} \) has 27 monomials. Hence there must be a redundancy. This is expected by Conjecture 3 since \( W|_{\mathcal{X}_i} \) is not multiplicity-free for \( i = 2 \) by the above calculation.

Ones notes further that the redundancy is given by the following equality:

\[
[2X_9^{-1}X_5^{-1}X_6^{-1}X_7^{-1}X_8^{-1}]_{\text{trop}} = \frac{1}{2}(\{X_9^{-1}X_5^{-2}X_6^{-1}X_7^{-1}X_8^{-1}\}_{\text{trop}} + [X_9^{-1}X_6^{-1}X_7^{-1}X_8^{-1}]_{\text{trop}}).
\] (8)

Hence the redundant inequality corresponds precisely to the monomial with coefficient 2. Indeed, let \( m_1 \) and \( m_2 \) be monomials in the polynomial ring \( \mathbb{Z}[X^{-1}, \ldots, X_N^{-1}] \). Then the identity \( [(m_1 + m_2)a]_{\text{trop}} = [m_1^a + m_2^a]_{\text{trop}} \), known in the literature as Freshman’s dream, holds for every \( a \in \mathbb{N} \). We expect every redundancy to occur due to this identity. This inspires the following stronger version of Conjecture 3 in the sense that is not only gives a criterion for the existence of redundancies but also spots the monomials which lead to redundant inequalities under tropicalization.

**Conjecture 4** Let \( i \in I \) and \( \mathfrak{d} \in R(w_0) \). The inequality \( [\mathfrak{d}]_{\text{trop}}(x) \geq 0 \) for an \( \mathfrak{d} \in \mathcal{M}_{I,i} \) is redundant if and only if \( W|_{\mathcal{X}_i} \) is not multiplicity-free for \( i \) and the coefficient \( a \in \mathbb{Q} \) of \( \mathfrak{d} \) satisfies \( a > 1 \).

**Remark 5** Note that the coefficients of monomials of \( W|_{\mathcal{X}_i} \) are not visible in its tropicalization. Hence Conjecture 4 suggests a criterion to determine facets of string cones which is not visible in the tropical version of the string cone inequalities. It would be very interesting to give a conceptual explanation of this. We suggest the following relation.

Recently Fei has showed in [7] for acyclic cluster algebras that the exponent vector of a monomial of the \( F \)-polynomial of any cluster variable gives rise to a vertex of its Newton polytope if and only if its coefficient is equal to 1. He furthermore conjectures this to be true for any cluster algebras. Recall from the proof of Proposition 4 for \( k \in I \)

\[
W_k|_{\mathcal{X}_i} = \prod_{i \in \tilde{M}} X_i(i)^{-c_{i,k}(\Sigma_k)} \prod_{j \in \tilde{M}} F_j(\tilde{\Sigma}_k)(X_1^{-1}(i), \ldots, X_n^{-1}(i)) \langle e_j, e_k \rangle \tilde{\Sigma}_k,
\]

where \( \tilde{\Sigma}_k \) is an optimized seed for the cluster variety obtained by forgetting all frozen vertices except for \( v_k \). For our setup we may pick a seed \( \Sigma_j \) such that \( j \in W(w_0) \) and \( j_N = k \). In this case the only arrow with target \( v_k \) in \( \Gamma_j \) has source \( v_\ell \) where \( \ell^+ = k \). Hence

\[
W_k|_{\mathcal{X}_i} = X_k(i)^{-1}F_j(\tilde{\Sigma}_j)(X_1^{-1}(i), \ldots, X_n^{-1}(i)),
\]

where \( X_j(j) = X_\ell(j) \).

Therefore the set of inequalities \( [W_k|_{\mathcal{X}_i}]_{\text{trop}}(t) \geq 0 \) is non-redundant if and only if the exponent vector of every monomial of \( F_j(\tilde{\Sigma}_j) \) is a vertex of its Newton polytope.
However, we are not aware of a relation between the exponents and the coefficients of the $F$-polynomial.

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**References**

1. Berenstein, A., Zelevinsky, A.: String bases for quantum groups of type $A_r$. In: I. M. Gelfand Seminar, volume 16 of Adv. Soviet Math., pp. 51–89. Amer. Math. Soc., Providence, RI (1993)
2. Berenstein, A., Zelevinsky, A.: Tensor product multiplicities, canonical bases and totally positive varieties. Invent. Math. 143(1), 77–128 (2001)
3. Berenstein, A., Fomin, S., Zelevinsky, A.: Cluster algebras. III. Upper bounds and double Bruhat cells. Duke Math. J. 126(1), 1–52 (2005)
4.Bossinger, L., Fourier, G.: String cone and superpotential combinatorics for flag and Schubert varieties in type $A$. J. Combin. Theory Ser. A 167, 213–256 (2019)
5. Cho, Y., Kim, Y., Lee, E., Park, K.-D.: On the combinatorics of string polytopes. J. Comb. Theory Ser. A 184, 105508 (2021)
6. Fang, X., Fourier, G., Littelmann, P.: On toric degenerations of flag varieties. Representation Theory - Current Trends and Perspectives, edited by H. Krause et al, Series of Congress Reports, EMS (2017)
7. Fei, J.: Combinatorics of $F$-polynomials Preprint. arXiv:1909.10151
8. Fock, V.V., Goncharov, A.B.: Cluster ensembles, quantization and the dilogarithm. Ann. Sci. de l’Ecole Norm. Sup. 42, 865–930 (2009)
9. Fomin, S., Zelevinsky, A.: Cluster algebras I: foundations. J. Amer. Math. Soc. 15, 497–529 (2002)
10. Fomin, S., Zelevinsky, A.: Cluster algebras. IV. Coefficients. Compos. Math. 143, 112–164 (2007)
11. Gross, M., Hacking, P., Keel, S., Kontsevich, M.: Canonical bases for cluster algebras. J. Amer. Math. Soc. 31, 497–608 (2018)
12. Genz, V., Koshevoy, G., Schumann, B.: Combinatorics of canonical bases revisited: type A. Sel. Math. (NS) 27(4), 45 (2021)
13. Genz, V., Koshevoy, G., Schumann, B.: Polyhedral parametrizations of canonical bases & cluster duality. Adv. Math. 369 (2020)
14. Gleizer, O., Postnikov, A.: Littelwood–Richardson coefficients via Yang-Baxter equation. Int. Math. Res. Not. 14, 741–774 (2000)
15. Kanakubo, Y., Koshevoy, G., Nakashima, T.: An algorithm for Berenstein–Kazhdan decoration functions and trails for minuscule representations. Preprint. arXiv:2109.01997
16. Kashiwara, M.: On crystal bases. *Representations of groups (Banff, AB, 1994)*, volume 16 of CMS Conf. Proc., pp. 155–197. Amer. Math. Soc., Providence, RI (1995)
17. Keller, B.: Cluster algebras and derived categories. Derived categories in algebraic geometry, 2013, EMS Ser. Congr. Rep., pp. 123–183. Eur. Math. Soc., Zürich
18. Littelmann, P.: Cones, crystals, and patterns. Transform. Groups 3(2), 145–179 (1998)
19. Lusztig, G.: Hecke algebras with unequal parameters. arXiv:math/0208154
20. Papi, P.: A characterization of a special ordering in a root system. Proc. Amer. Math. Soc. 120(3), 661–665 (1994)
21. Salisbury, B., Schultze, A., Tingley, P.: Combinatorial descriptions of the crystal structure on certain PBW bases. Transform. Groups 23, 501–525 (2018)

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