Energy of boundary of spacetime

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We consider how the energy may be stored in the boundary of spacetime, in particular in a spherical bubble that can be made through quantum gravitational process. Our calculation is performed within the framework of classical Einstein gravity by identifying the Gibbons-Hawking-York term as the membrane action. We show that the energy of the bubble can be given consistently with the Schwarzschild metric. The solution of the consistency condition suggests positive membrane tension, which can explain why we do not observe such topological defects in ordinary experiences.

I. INTRODUCTION

It is widely believed that quantum gravity allows change of topology of spacetime [1–19]. However, we do not observe such topological defects in ordinary experiences, and thus we expect that such topology change only occurs at the size of the Planck length [3–6] and the topological defect is difficult to exist in larger size. One may therefore deduce that the defects of finite size carry energy [20–25] and larger defects cost larger energy. The aim of this letter is to make this picture precise by considering a framework for calculating the energy of the defect within classical Einstein gravity. Our calculation is based on the two observations: (i) Once the topological defect is settled in the form that can be described by classical gravity, the topology change adds or removes boundaries of spacetime, to which we assign the Gibbons-Hawking-York term [26, 27]. (ii) The Gibbons-Hawking-York term can be identified as the Nambu-Goto action [28, 29] describing a membrane whose tension is given by the extrinsic curvature. This identification then allows us to analyze the boundary as a dynamical membrane and calculate its energy.

As an example, we consider a spherically symmetric bubble whose interior is outside the spacetime and thus absolute nothing [30]. Our picture is however different from conventional pictures of the bubble as we consider its boundary and interpret boundary term as the membrane action, from which we calculate the energy of the defect. It should also be noted that we work in four dimension. We show that the energy of the bubble can be given consistently with the spacetime described by the Schwarzschild metric. The consistency condition equates the energy of the bubble to the Schwarzschild mass, which has a unique solution suggesting that the surface of the boundary be located outside the event horizon. This means that they are distinguishable from microscopic black holes [31–33]. Furthermore, the solution shows the tension is positive, which indicates that our universe is stable against the creation of such bubbles as it tends to shrink, not grow and swallow the whole universe [34–46]. The radius of the defect will be proportional to the stored energy, and therefore when the bubble is created by quantum fluctuation of the order of the Planck mass, its size will be the order of the Planck length. Then, since the bubbles tend to shrink, they are always smaller than the order of the Planck length. This picture thus gives a possible explanation why we do not observe such topological defects in ordinary experiences.

II. ENERGY OF THE SPHERICALLY SYMMETRIC BUBBLE

We consider an appearance of a boundary when spacetime undergoes a topology change. In the framework of the Einstein gravity, we assign the Gibbons-Hawking-York term to the spacetime boundary:

\[ S_B \equiv \frac{1}{8\pi} \int_{\partial V} d^3 y \sqrt{|\det h_{ab}|} \varepsilon K. \]  

(1)

Here \( h_{ab} \) is the induced metric on the boundary \( \partial V \):

\[ ds^2|_{\partial V} = h_{ab}dy^a dy^b, \]  

(2)

and \( K \) is the trace of the extrinsic curvature, which can be written with the unit outward normal \( n_{\mu} \):

\[ K = n^\mu ;_{\mu}. \]  

(3)

The sign factor is defined as

\[ \varepsilon \equiv n^2 = \begin{cases} -1 & (n: \text{timelike}) \\ 1 & (n: \text{spacelike}) \end{cases}. \]  

(4)

As an example, we consider the case that the induced topological defect is spherically symmetric (Fig. 1). To simplify the situation further, we consider a thought experiment in which we make the spherical bubble kept fixed with no ordinary matter present. The spacetime...
should then be static and spherically symmetric that is described by the Schwarzschild metric:

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2, \]  

(5)

where \( d\Omega \) is the line element of the unit 2-sphere. In the ordinary description, we identify \( M \) as the total energy of the matter sourcing gravity \[17, 48\]. However, in our case, there is only the boundary of spacetime in the vacuum, and thus we identify the boundary to be the source of gravity. \( S_B \) it then the action describing the corresponding degrees of freedom.

One possible interpretation of the action \[4\] is a membrane action with position-dependent tension \( T \equiv -\varepsilon K/(8\pi) \). In our case, the correspondence is more manifest because \( K \), and thus \( T \), is constant over the surface due to spherical symmetry. Since the spacetime is static with the Killing vector \( \partial_t \) where \( t \) corresponds to the Minkowski time in the asymptotically flat region, we define the potential energy \( E_{\text{pot}} \) with respect to \( t \). Using the fact that the kinetic term is zero since the membrane is held still, we have

\[ S_B = -\int dt E_{\text{pot}} = \frac{1}{8\pi} \int dt \sqrt{|g_{tt}|} \varepsilon K \Sigma, \]

(6)

where \( g_{tt} = -(1 - 2M/r) \) and \( \Sigma \) is the two-dimensional area of the membrane. We do not add a factor of \( \sqrt{|g_{tt}|} \) in the left hand side as in the case of the gravitational Hamiltonian \[49, 50\]. Again since the spacetime is static, we arrive at the following relation:

\[ E_{\text{pot}} = -\frac{1}{8\pi} \sqrt{|g_{tt}|} \varepsilon K \Sigma = \sqrt{|g_{tt}|} T \Sigma, \]

(7)

which is the desired relation giving the energy stored in the boundary of spacetime.

### III. CONSISTENCY CONDITION AND ITS SOLUTION

We denote the location of the bubble as \( r = R \). The outward normal of the bubble is

\[ n_\mu = \frac{-\varepsilon \partial_\mu r}{\sqrt{|1 - 2M| r}|_{r=R}}, \]

(8)

where \( \varepsilon = \text{sign}(1 - 2M/R) \). The overall sign of \( n_\mu \) is defined in such a way that

\[ n^\mu \partial_\mu r = -\sqrt{|1 - 2M| R} < 0, \]

(9)

reflecting the fact that \( r = 0 \) is the outside when seen from the spacetime. From eq. \[8\], we have

\[ K = -\varepsilon \frac{1}{R} \frac{1}{\sqrt{|1 - 2M| R}} \left(2 - \frac{3M}{R}\right), \]

(10)

which gives the tension:

\[ T = \frac{1}{8\pi R} \frac{1}{\sqrt{|1 - 2M| R}} \left(2 - \frac{3M}{R}\right). \]

(11)

Note that the tension \( T \) is positive for \( R > 3M/2 \).

Since \( E_{\text{pot}} \) is the energy carried by the bubble, we can identify \( E_{\text{pot}} = M \). This will give a consistency condition that relates \( M \) and \( R \). Using eq. \[7\] with \( \Sigma = 4\pi R^2 \),

\[ M = \frac{R}{2} \left(2 - \frac{3M}{R}\right), \]

(12)

i.e., \( R = 5M/2 \). This unique solution gives us important consequences: Firstly, the spherical defect is not covered by the horizon, \( R > 2M \), and thus it is distinguishable from microscopic black holes. This is important because it makes us possible to directly interact with the remnant of a quantum gravitational effect. Secondly, the tension is positive, and thus classically the bubble tends to shrink; otherwise, our universe becomes unstable as the bubble can grow until it swallows the whole universe. We thus see that the identification of the Gibbons-Hawking-York term with the membrane action gives a consistent picture of the energy stored in the boundary of spacetime.

### IV. DISCUSSION

In this letter, we considered a framework to calculate the energy of the boundary of spacetime and showed that the spherically symmetric bubble defect can be described consistently with the Schwarzschild metric. However, it is true that many assumptions are made in the argument. For example, as emphasized in the main text, we thoroughly used the properties of the Schwarzschild solution in identifying the potential energy, eq. \[4\]. The notion
of energy itself is a subtle topic in general relativity (see, e.g., [51, 55]) and we may need a more sophisticated formula to apply the argument for general cases. Another assumption we implicitly made is that the cosmological constant is irrelevant. Generalization in this direction is interesting because its small value has been related to the topological nature of quantum gravity [6, 11].

Allowing the boundary degrees of freedom to exist, there are many topics that can be explored. An interesting question is whether such bubbles can be stabilized by a quantum effect because if stable bubbles exist, they can be a candidate of dark matter. To see that this is a viable possibility, let us assume that the bubbles were created and stabilized by some quantum gravitational process before the inflation epoch. Since the relevant scale is the Planck scale $M_P$, the number density should be $n_b \sim M_P^4$ (before the inflation) and the stabilized mass $M \sim M_P$. During the inflationary expansion, the number density is exponentially diluted by the factor $\exp(-3N_c)$ with $N_c$ the number of e-foldings. After the inflation ends and the universe is reheated, the yield variable is given by $Y_b \equiv n_b/s^3 \sim M_P^4 \exp(-3N_c)/(g_s T_{RH}^4)$, where $T_{RH}$ is the reheating temperature and $g_s (\sim O(10^2))$ the effective number of relativistic degrees of freedom. The fraction of the energy density of the bubble to the critical density at the present universe then becomes

$$\Omega_b h^2 \sim Y_b \frac{M_P}{\text{GeV}} \times 2.8 \times 10^8 \sim 0.1 \left( \frac{10^6 \text{GeV}}{T_{RH}} \right)^3 \exp(151 - 3N_c) \quad (13)$$

where $h$ is the dimensionless Hubble parameter. By identifying eq. (13) with the observed relic abundance of dark matter: $\Omega_{DM} h^2 \sim 0.12$ [50] and taking the e-foldings $N_c \sim 50$ [57], we are lead to the prediction $T_{RH} \sim 10^6 \text{GeV}$, whose value is compatible with leptogenesis scenarios with low reheating temperatures (see, e.g., [58] for a review).

It is also important to consider the interaction of the bubbles with ordinary matters for detection. Note that geodesics around the bubble cannot be extended into the $r < R$ region, and in this sense the spacetime is singular. For this, it should be crucial to specify an appropriate boundary condition to determine the behavior of the particles (or the fields) at the surface of the bubbles so that we can obtain predictions for the entire physical processes.

Studies along these lines are in progress and will be reported elsewhere.

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