Direct Interaction Approximation of Magnetohydrodynamic Turbulence

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In this paper we apply Kraichnan’s direct interaction approximation, which is a one loop perturbation expansion, to magnetohydrodynamic turbulence. By substituting the energy spectra both from kolmogorov-like MHD turbulence phenomenology and a generalization of Dobrowolny et al.’s model we obtain Kolmogorov’s and Kraichnan’s constant for MHD turbulence. We find that the constants depend of the Alfvén ratio and normalized cross helicity; the dependence has been studied here. We also demonstrate the inverse cascade of magnetic energy for Kolmogorov-like models. Our results are in general agreement with the earlier simulation results except for large normalized cross helicity.

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I. INTRODUCTION

Turbulence still remains primarily an unsolved problem. Among existing statistical theories of fluid turbulence, Kolmogorov’s phenomenology is the most prominent one. Kolmogorov hypothesized that the energy spectrum \( E(k) \) of fluid turbulence in the inertial range, i.e., for scales between the energy feeding and dissipation range, is given by

\[
E(k) = K_{Ko} \Pi^{2/3} k^{-5/3},
\]

where \( K_{Ko} \) is an universal constant called Kolmogorov’s constant, \( k \) is the wavenumber, and \( \Pi \) is the nonlinear cascade of energy, also equal to the dissipation rate of the fluid. This power law has been confirmed by experiments and numerical results.

Later on various theories were developed to understand fluid turbulence, the primary ones being Direct Interaction Approximation (DIA) of Kraichnan, Wyld’s field-theoretic technique, renormalization groups (RG), mode-coupling, Eddy-Damped Quasi Normal Markovian (EDQNM) closure schemes. All these theories yield results consistent with the Kolmogorov’s power law. These calculations yield \( K \approx 1.5 \) in three dimensions (3D). In two-dimensions (2D) Kraichnan, Olla, and Nandy and Bhattacharjee show that \( K_{Ko} \approx 6.4 \) in region where inverse cascade of energy occurs.

The DIA of Kraichnan is one of the fully consistent analytical turbulence theory of fluid turbulence. It essentially involves perturbation theory similar to that used in quantum field theory. In this calculation Green’s function and correlation function are calculated self consistently to first nonvanishing order. Only triad interactions are allowed to this order; for this reason it is called direct interaction approximation (refer to Figure 1). The DIA of Kraichnan yielded \( k^{-3/2} \) energy spectrum for fluid turbulence, which was found to inconsistent with the experiments. However, later on it was shown that DIA can also yield \( k^{-5/3} \) energy spectrum. The RG and mode coupling theories are similar to DIA, and they have been shown to be yield results close to those from DIA.

For magnetohydrodynamic (MHD) turbulence also there are several phenomenologies. In MHD we have velocity field \( \mathbf{u} \) and magnetic field \( \mathbf{B} = \mathbf{B}_0 + \mathbf{b} \), where \( \mathbf{B}_0 \) denotes the mean magnetic field and \( \mathbf{b} \) denotes the magnetic field fluctuations. In this paper, in place of \( \mathbf{u} \) and \( \mathbf{b} \), we use Elsässer variables \( \mathbf{z}^\pm = \mathbf{u} \pm \mathbf{b}/\sqrt{4\pi \rho} \), where \( \rho \) is the density of the plasma. The wave speed of Alfvén waves due to the mean magnetic field, called the Alfvén speed, is denoted by \( c_A \) and is equal to \( B_0/\sqrt{4\pi \rho} \). The variables \( \mathbf{z}^\pm \) denote fluctuations having positive and negative velocity-magnetic field correlations respectively. Throughout the paper we assume that \( \rho \) is constant, which implies that the fluid is incompressible. Marsch, Matthaeus and Zhou, and Zhou and Matthaeus proposed a phenomenology similar to Kolmogorov’s fluid turbulence phenomenology. We call this Komogorov-like MHD turbulence phenomenology. In this phenomenon the energy spectra \( E^\pm(k) \) of the \( \mathbf{z}^\pm \) fluctuations are

\[
E^\pm(k) = K^\pm \left( \Pi^+ \right)^{4/3} \left( \Pi^- \right)^{-2/3} k^{-5/3},
\]

where \( \Pi^\pm \) are the nonlinear energy cascades of \( \mathbf{z}^\pm \) fluctuations, and \( K^\pm \) are constants, whom we refer to as Kolmogorov’s constants for MHD turbulence. The spectral normalized cross helicity \( \sigma_c(k) \), defined as \( (E^+(k) - E^-(k))/(E^+(k) + E^-(k)) \), and spectral Alfvén ratio \( r_A(k) \), defined as \( E^u/E^b \), play an important role in MHD turbulence. Here \( E^u \) and \( E^b \) are the kinetic and magnetic energies respectively. In this paper \( \sigma_c(k) \) and \( r_A(k) \) are taken to be independent of \( k \), and we will usually drop the term spectral while referring to these constants. According to the assumptions of the model, this phenomenology is expected to be applicable when amplitudes of the fluctuations are greater than the mean magnetic field or the magnetic field of largest eddies, i.e., \( \mathbf{z}^\pm \gg B_0 \). In this paper we theoretically calculate the values of \( K^\pm \). The quantity \( E^R = (E^u - E^b) \), called the residual energy, plays an important role in MHD turbulence, as will be seen later in this paper.

Kraichnan argued that when the mean magnetic field or the magnetic field of the largest eddies \( C_A \) is much larger than the fluctuations \( \mathbf{z}^\pm \), the magnetic energy spectrum \( E^b(k) \) is given by

\[
E^b(k) = A(\Pi^b C_A)^{1/2} k^{-3/2},
\]

where \( \Pi^b \) is the magnetic energy cascade rate, and \( A \) is Kraichnan’s constant. Dobrowolny et al. generalized this model for \( \mathbf{z}^\pm \) and found that

\[
\Pi^+ = \Pi^- = A^{-2} C_A^{-1} E^+(k) E^-(k) k^3,
\]

where \( A \) is a constant. In this paper we also calculate the value of the constant \( A \) for a modified Dobrowolny et al.’s model.
Matthaeus and Zhou, and Zhou and Matthaeus [13] developed a generalization that contains both Kolmogorov-like and Dobrowolny et al.’s models as limiting cases. In their model the dissipation rates are given by

$$\Pi^\pm = \frac{A^{-2} B_0^{-1} E^+(k) E^-(k) k^3}{C_A + \sqrt{kE^+(k)}}.$$  (5)

In this paper we do not discuss this model because of its complexity.

Some of the analytical theories of fluid turbulence have been applied to MHD turbulence. For example, Pouquet et al. and Grappin et al. [13] applied the EDQNM closure schemes, and Fournier et al. and Camargo and Tasso [14] have applied the RG technique to MHD turbulence. Verma and Bhattacharjee [18] have applied DIA technique to MHD turbulence when $E^-(k) = E^+(k)$ and obtained the Kolmogorov’s constant $K$. In this paper we have generalized their formalism for cases when $E^-(k) \neq E^+(k)$ and obtained $K^\pm$. In this paper we have also theoretically obtained Kraichman’s constant $A$ for a modified Dobrowolny et al.’s model.

The solar wind is an ideal natural experiment where some of the predictions of MHD turbulence theories could be tested. The solar wind observations by the spacecraft show that energy spectra of the solar wind is closer to $k^{-5/3}$ rather than to $k^{-3/2}$ [13]. This is in spite of the fact that the mean magnetic field in the solar wind (approximately 5 nanotesla) is approximately 2 to 10 times larger than the large-scale fluctuations. However, it should be noted that the exponents $5/3$ and $3/2$ are difficult to distinguish. Recently Verma et al. [20] have performed direct numerical simulations of MHD turbulence in which they could not conclude whether the spectral index was $5/3$ or $3/2$. However, in their simulation the nonlinear energy cascade rates $\Pi^\pm$ appear to follow Eq. (5) rather than Eq. (4). Note that the EDQNM calculations of Grappin et al. [17] show that the spectral index varies between 0 and 3 depending on normalized cross helicity.

The outline of the paper is as follows: In section 2 of this paper we derive the equations for the correlations functions and flux functions that are used in subsequent sections. In section 3 we show that the energy spectra of Eq. (2) are consistent solutions of MHD equations and find the constants $K^\pm$. We find the constants for various $E^-(k)/E^+(k)$ and $E^\eta(k)/E^\nu(k)$ ratios. In section 4 we construct a generalization of Dobrowolny et al.’s model, and then calculate the Kraichman’s constant that model. Section 5 contains discussion and conclusions. Other than a brief remarks in sections 4, we do not attempt to address the important question under what ratio of $z^\pm/B_0$ the transitions from the energy spectra $k^{-5/3}$ to $k^{-3/2}$ take place; the answer to this question require further analysis which is beyond the scope of this paper.

II. MHD TURBULENCE

The MHD equations in terms of $z^\pm$ are [14]

$$\frac{\partial}{\partial t} z^\pm + C_A \cdot \nabla z^\pm + z^\mp \cdot \nabla z^\pm = -\frac{1}{\rho} \nabla p^{tot} + \nu_+ \nabla^2 z^\pm + \nu_- \nabla^2 z^\mp + f^\pm,$$  (6)

$$\nu_\pm = \frac{\nu \pm c^2/(4\pi\sigma)}{2}$$  (7)

where $C_A = B_0/\sqrt{4\pi\rho}$, $p^{tot}$ is the total pressure, i.e., thermal plus magnetic pressure, $f^\pm$ are forcing, $\nu$ is the kinematic viscosity, $c$ is the speed of light, and $\sigma$ is conductivity. The corresponding equation in Fourier space is

$$(\frac{\partial}{\partial t} + \nu_+ k^2) z^\pm_i(k, t) + \nu_- k^2 z^\mp_i(k, t) = i(C_A \cdot k) z^\pm_i(k, t) = f^\pm_i(k)e^{-ikz^{tot}(p, t)}z^\pm_i(k - p, t)$$  (8)

We can eliminate pressure $p^{tot}$ using the incompressibility condition, which is

$$k_i z^\pm_i(k, t) = 0.$$  

After the elimination of pressure the incompressible equation is

$$(\frac{\partial}{\partial t} + \nu_+ k^2) z^\pm_i(k, t) + \nu_- k^2 z^\mp_i(k, t) = i(C_A \cdot k) z^\pm_i(k, t) = f^\pm_i(k) - eM_{i,m} \int d\mathbf{p} z^\pm_j(p, t) z^\pm_m(p - k, t)$$  (9)

where
functions and their properties. Our following discussion is similar to that of Orszag. Here \( S_{ij} \)

\[
\langle f_i^+(k,t)z_j^+(k',t') \rangle = S_{ij}^{++}(k,t,t')\delta(k+k'),
\]

\[
\langle f_i^-(k,t)z_j^+(k',t') \rangle = S_{ij}^{+-}(k,t,t')\delta(k+k'),
\]

\[
\langle f_i^+(k,t)z_j^-(k',t') \rangle = S_{ij}^{-+}(k,t,t')\delta(k+k').
\]

Here \( \langle x \rangle \) denotes an ensemble average of \( x \). We assume that the turbulence is stationary, which implies that the correlations are independent of time: \( S_{ij}^{±±}(k,t,t) = S_{ij}^{±±}(k,t) \). For fluid turbulence Orszag [9] derived correlation functions and their properties. Our following discussion is similar to that of Orszag.

From Eq. (9) we can obtain equations for correlation tensors that are

\[
\left( \frac{\partial}{\partial t} + 2\nu_s k^2 \right) S_{ij}^{±±}(k,t,t) + 2\nu_s k^2 S_{ij}^{++}(k,t,t) - 2i(C_A \cdot k)S_{ij}^{±±}(k,t,t) = 2F_{ij}^+(k) - \epsilon M_{ilm}(k) \int dp T_{jlm}^{±±}(-k,p) - \epsilon M_{ilm}(-k) \int dp T_{jlm}^{±±}(k,p)
\]

where

\[
\langle f_i^+(k)z_j^+(p) \rangle = T_{jlm}^{±+}(k,p)\delta(k+p)
\]

\[
\langle f_i^+(k)z_j^-(p) \rangle = T_{jlm}^{±-}(k,p)\delta(k+p)
\]

\[
\langle f_i^-(k)z_j^-(p) \rangle = T_{jlm}^{--}(k,p)\delta(k+p)
\]

\[
\langle f_i^-(k)z_j^+(p) \rangle = T_{jlm}^{+-}(k,p)\delta(k+p)
\]

\[
\frac{1}{2} \langle f_i^+(k)z_j^-(k') + f_i^-(k)z_j^+(k') \rangle = F_{jlm}^{R}(k)\delta(k+k')
\]

For isotropic turbulence

\[
S_{ij}^{±±}(k) = P_{ij}(k)C^{±±}(k).
\]

In presence of a mean magnetic field, the correlation tensors are expected to be anisotropic. In the solar wind the fluctuations are anisotropic; the amplitudes of the fluctuations along the Parker field is one-third that of those perpendicular to the Parker field [2]. However, the analysis of anisotropic turbulence is quite complicated, and in this paper we assume isotropy for all the correlations.

We also define one-dimensional energy spectra as

\[
E^±(k) = \frac{C^{±±}(k)}{4\pi k^2}; E^R(k) = \frac{C^{+-}(k)}{4\pi k^2} : in3D
\]

\[
E^±(k) = \frac{C^{±±}(k)}{2\pi k}; E^R(k) = \frac{C^{+-}(k)}{2\pi k} : in2D
\]
For isotropic distribution in $k$ space, after substitution of $i = j$ in Eqs. (14,15) and taking the trace, we obtain

$$
\left( \frac{d}{dt} + 2\nu_+ k^2 \right) E^\pm(k, t, t) + 2\nu_- k^2 E^R(k, t, t) = T^\pm(k, t) \tag{24}
$$

$$
\left( \frac{d}{dt} + 2\nu_+ k^2 \right) E^R(k, t, t) + \nu_- k^2 \left( E^+(k, t, t) + E^-(k, t, t) \right) = T^R(k, t) \tag{25}
$$

where

$$
T^\pm(k, t) = -4\pi k^2 M_{ijm}(k) I_m \int dp T^\pm_{ijm}(k, p) \tag{26}
$$

$$
T^R(k, t) = -4\pi k^2 M_{ijm}(k) I_m \int dp \left( \frac{T^R_{ijm}(k, p) - T^{R\dagger}_{ijm}(k, p)}{2} \right). \tag{27}
$$

Note that the term $(C_A \cdot k) E^\pm(k) = C_A k_s E^\pm(k)$, where $k_s$ is the wavevector along the mean magnetic field, is zero for isotropic turbulence because it is an odd function of $k_s$.

When $\nu_+ = \nu_- = 0$, one can easily check from the above equations that the total energy $E^\pm = \int E^\pm(k) dk$ is conserved. However, total residual energy $E^R = \int E^R(k) dk$ is not conserved. In terms of $T$’s these statements of energy conservation are

$$
\int_0^\infty T^\pm(k, t) dk = 0 \tag{28}
$$

but

$$
\int_0^\infty T^R(k, t) dk \neq 0. \tag{29}
$$

We can also derive a theorem of “detailed conservation of energy $E^\pm(k)$”, i.e.,

$$
s^\pm(k, p, q) + s^\pm(k, q, p) + s^\pm(p, q, k) + s^\pm(p, k, q) + s^\pm(q, k, p) + s^\pm(q, p, k) = 0, \tag{30}
$$

where

$$
s^\pm(k, p, q) \delta(k + p + q) = -\text{Im} \left\{ \left( k \cdot z^\mp(p) \right) \left( z^\pm(k) \cdot z^\pm(q) \right) \right\}. \tag{31}
$$

The diagrammatic representation of the terms $S^\pm(\cdot, \cdot, \cdot)$ is shown in Figure 1. The solid lines represent $z^+$ and the wavy line represent $z^-$. For $S^-(-\cdot, -\cdot, -\cdot)$ we need to change the solid lines to wavy lines and vice versa. The physical interpretation of these diagrams is that the energy from any $z^\pm(k)$ mode gets transferred to modes $z^+(p)$ and $z^-(q)$ ($k + q = k$), and so does energy from $z^-(k)$. However, the quantities $E^+(k) + E^+(p) + E^+(q)$ and $E^-(k) + E^-(p) + E^-(q)$ are separately conserved.

In presence of viscosity there is dissipation in the plasma. The dissipation rates of $z^\pm$ fluctuations are given by

$$
\varepsilon^\pm = -\frac{d}{dt} \int_0^\infty E^\pm(k, t) dk = 2\nu_+ \int_0^\infty k^2 E^\pm(k, t) dk + 2\nu_- \int_0^\infty k^2 E^R(k, t) dk. \tag{32}
$$

Integrating both sides of Eqs. (24,25) over a sphere of radius $K$ in $k$ space, we obtain the energy cascade rates $\Pi^\pm, R(K, t)$ that is

$$
\Pi^\pm(K, t) = -\int_0^K T^\pm(k, t) dk = \int_K^\infty T^\pm(k, t) dk = -\frac{d}{dt} \int_0^K E^\pm(k, t) dk - 2\nu_+ \int_0^\infty k^2 E^\pm(k, t) dk - 2\nu_- \int_0^\infty k^2 E^R(k, t) dk. \tag{33}
$$

$$
\Pi^R(K, t) = -\int_0^K T^R(k, t) dk = -\frac{d}{dt} \int_0^K E^R(k, t) dk - 2\nu_+ \int_0^\infty k^2 E^R(k, t) dk - 2\nu_- \int_0^\infty k^2 E^R(k, t) dk. \tag{34}
$$

The above equations will be used in the subsequent sections. In Eqs. (24,25) we find that the $n$th-order moments involves $(n + 1)$th-order moments along with $n$th and lower order moments. Hence one cannot close these equations. This is the famous “closure problem” [11]. To circumvent this difficulty, various techniques have been devised, primary ones being renormalization groups [1,10], DIA [14], mode-coupling scheme [1], EDQNM closure schemes [17] etc. In this paper we only discuss DIA of MHD turbulence.
III. DIA OF MHD WITH \( C_A = 0 \)

In this section we assume that the mean magnetic field or the magnetic field due to the large eddies is small as compared to the fluctuations, i.e., \( C_A \ll z^\pm \). Here we set \( C_A = 0 \). In the next section we will analyse for the case when \( C_A \gg z^\pm \). Firstly, we calculate the Green’s function in the spirit of Kraichnan [4,5].

A. Computation of Green’s Function

It is convenient to derive Green’s function in Fourier space \( \hat{k} = (\mathbf{k}, \omega) \). Following Leslie [3] we define Green’s function \( G_{ij}^{s_1s_2}(\hat{k}) \) as

\[
G_{ij}^{s_1s_2}(\hat{k}) = \frac{\delta_{ij}z_{s_1}^1(\hat{k})}{\delta f_{s_2}^2}.
\]

In Fourier space the equation of Green’s function is

\[
\int_{\hat{p}+\hat{q}=\hat{k}} \frac{d\hat{p}}{2\pi} \frac{z_{m}^+ (\hat{p})}{z_{i}^1 (\hat{p})} z_{n}^+ (\hat{q}) = P_{ij} (\hat{k}) - \epsilon M_{im} (\hat{k}) \times G_{mj}^{+} (\hat{q}) G_{nj}^{-} (\hat{p})
\]

where \( \hat{p} = (p, \omega') \) and \( \hat{q} = (k - p, \omega - \omega') \). We denote

\[
\hat{G}_{in}(\mathbf{k}, \omega) = P_{in} (\hat{k}) \hat{G}(\hat{k})
\]

Now we solve the Eqs. (38) and (39) perturbatively. We expand \( z^\pm \) and Green’s function \( \hat{G} \) in series:

\[
z_i^\pm (\hat{k}) = z_i^\pm (0) (\hat{k}) + \epsilon z_i^{+ (1)} (\hat{k}) + O(\epsilon^2)
\]

\[
G_{ij}^{s_1s_2}(\hat{k}) = G_{ij}^{s_1s_2(0)}(\hat{k}) + \epsilon G_{ij}^{s_1s_2(1)}(\hat{k}) + O(\epsilon^2)
\]

In DIA we keep terms only up to first nonvanishing order [3]. By substituting these terms in Eqs. (38) and (39), and equating the terms with equal powers of \( \epsilon \), we obtain

\[
\left[\begin{array}{c}
z_i^{+ (0)} (\hat{k}) \\
z_i^{- (0)} (\hat{k})
\end{array}\right] = \hat{G}^0 (\hat{k}) \left[\begin{array}{c}
f_i^+ \\
f_i^-
\end{array}\right]
\]

\[
\left[\begin{array}{c}
z_i^{+ (1)} (\hat{k}) \\
z_i^{- (1)} (\hat{k})
\end{array}\right] = -i \hat{G}^0 (\hat{k}) M_{ijk}(\mathbf{k}) \int_{\hat{p}+\hat{q}=\hat{k}} d\hat{p} \left[\begin{array}{c}
z_j^{- (0)} (\hat{p}) z_m^+ (\hat{q}) \\
z_j^{- (0)} (\hat{p}) z_m^+ (\hat{q})
\end{array}\right] \left[\begin{array}{c}
G_{mj}^{+ (0)} (\hat{q}) G_{nj}^{- (0)} (\hat{p}) \\
G_{mj}^{+ (0)} (\hat{q}) G_{nj}^{- (0)} (\hat{p})
\end{array}\right]
\]

\[
\hat{G}^{+ (1)} (\hat{k}) = -\frac{i}{2} \hat{G}^0 (\hat{k}) M_{ijk}(\mathbf{k}) \int_{\hat{p}+\hat{q}=\hat{k}} d\hat{p} \left[\begin{array}{c}
z_j^{- (0)} (\hat{p}) z_m^+ (\hat{q}) \\
z_j^{- (0)} (\hat{p}) z_m^+ (\hat{q})
\end{array}\right] \left[\begin{array}{c}
G_{mj}^{+ (0)} (\hat{q}) G_{nj}^{- (0)} (\hat{p}) \\
G_{mj}^{+ (0)} (\hat{q}) G_{nj}^{- (0)} (\hat{p})
\end{array}\right]
\]

The factor 1/2 in the equation appears because of trace of \( P_{ij}(\hat{k}) \). In DIA the force field is assumed to have a gaussian distribution. Therefore, according to Eq. (41), \( z^b(0) \) also have a Gaussian distribution. Green’s function \( \hat{G} \) to leading order is (refer to Eq. 34).
Following Kraichnan's DIA [4,5] we substitute $\epsilon^{\eta \eta \epsilon \epsilon}$.

Here we take 
\[
\langle \hat{z}_i^{(0)} \rangle = 0, \quad \langle ZG \rangle = \langle Z(0)G^{(1)} \rangle + O(\epsilon^2).
\]

The substitution of $G^1$ of Eq. (13) in the above equation yields

\[
\hat{G}(\tilde{k}) = \hat{G}^0(\tilde{k}) - \frac{i}{2} \hat{G}^0_{\parallel}(\tilde{k}) M_{ijm}(k) \int_{\tilde{p} + \tilde{q} = k} d\tilde{p} \times \left[ \begin{array}{c} z_j^{+(0)}(\tilde{p}) z_m^{+(0)}(\tilde{q}) \\
 z_j^{-(0)}(\tilde{p}) z_m^{-(0)}(\tilde{q}) \end{array} \right] \begin{bmatrix} G_{m,+}^{+(1)}(\tilde{q}) & G_{m,-}^{-(1)}(\tilde{q}) \\
 G_{j,+}^{+(1)}(\tilde{p}) & G_{j,-}^{-(1)}(\tilde{p}) \end{bmatrix} + \ldots
\]

(44)

In the above equation we have used the fact that $\hat{G}^0(k, \omega)$ is statistically sharp [8], and $\langle \hat{z}_i^{(0)} \rangle = 0$, $\langle ZG \rangle = \langle Z(0)G^{(1)} \rangle + O(\epsilon^2)$. The substitution of $G^1$ of Eq. (13) in the above equation yields

\[
\hat{G}(\tilde{k}) = \hat{G}^0(\tilde{k}) - \frac{i}{2} \hat{G}^0_{\parallel}(\tilde{k}) M_{ijm}(k) \int_{\tilde{p} + \tilde{q} = k} d\tilde{p} \times
\]

\[
\left[ M_{sab}(\tilde{q}) G_{ms}^{+(0)}(\tilde{q}) \int_{\tilde{r} + \tilde{s} = \tilde{q}} d\tilde{r} M_{sab}(\tilde{q}) G_{ms}^{+(0)}(\tilde{q}) \int_{\tilde{r} + \tilde{s} = \tilde{q}} d\tilde{r} \right] \times
\]

\[
\left[ M_{sab}(\tilde{q}) G_{bs}^{+(0)}(\tilde{p}) \int_{\tilde{r} + \tilde{s} = \tilde{p}} d\tilde{r} M_{sab}(\tilde{q}) G_{bs}^{+(0)}(\tilde{p}) \int_{\tilde{r} + \tilde{s} = \tilde{p}} d\tilde{r} \right]
\]

(45)

Following Kraichnan’s DIA [13] we substitute $\epsilon = 1$ and replace $z^{(0)}$ by $z$ and $G^{(0)}$ in 2nd and 3rd line by $G$. Comparing the resulting equation with

\[
\hat{G}(\tilde{k}) = \hat{G}^0(\tilde{k}) - \hat{G}^0(\tilde{k}) \hat{\Sigma}(\tilde{k}) \hat{G}^0(\tilde{k})
\]

where $\Sigma$ is self-energy, we obtain

\[
\hat{\Sigma}(\tilde{k}) = \left[ \begin{array}{cc} \Sigma^{++}(\tilde{k}) & \Sigma^{+-}(\tilde{k}) \\
 \Sigma^{-+}(\tilde{k}) & \Sigma^{--}(\tilde{k}) \end{array} \right] = \frac{1}{2} \left[ \begin{array}{cc} \eta^{+}(\tilde{k}) & \eta^{-}(\tilde{k}) \\
 \eta^{+}(\tilde{k}) & \eta^{-}(\tilde{k}) \end{array} \right],
\]

(46)

(47)

\[
\eta^{+}(\tilde{k}) = \frac{i}{2} k^2 \int_{\tilde{p} + \tilde{q} = \tilde{k}} d\tilde{p} \times
\]

\[
[b_1(\vec{k}, \vec{p}, \vec{q}) (G^{+-}(\tilde{p}) + G^{-+}(\tilde{p})) C^{+-}(\tilde{q}) + b_2(\vec{k}, \vec{p}, \vec{q}) (G^{++}(\tilde{p}) + G^{-+}(\tilde{p})) C^{--}(\tilde{q})]
\]

(48)

\[
\eta^{-}(\tilde{k}) = \frac{i}{2} k^2 \int_{\tilde{p} + \tilde{q} = \tilde{k}} d\tilde{p} \times
\]

\[
[b_3(\vec{k}, \vec{p}, \vec{q}) (G^{+-}(\tilde{p}) + G^{-+}(\tilde{p})) C^{++}(\tilde{q}) + b_4(\vec{k}, \vec{p}, \vec{q}) (G^{++}(\tilde{p}) + G^{-+}(\tilde{p})) C^{--}(\tilde{q})]
\]

(49)

where $b_i(\vec{k}, \vec{p}, \vec{q}) = k^{-2} B_i(\vec{k}, \vec{p}, \vec{q})$, and $B_i$’s are

\[
B_1(\vec{k}, \vec{p}, \vec{q}) = kp(-z + z^3 + x^2 z + xy z^2)
\]

\[
B_2(\vec{k}, \vec{p}, \vec{q}) = kp(1 + z^2)(z + xy)
\]

\[
B_3(\vec{k}, \vec{p}, \vec{q}) = kp(-z + z^3 + x y + x^2 z + y^2 z + xy z^2)
\]

\[
B_4(\vec{k}, \vec{p}, \vec{q}) = kp(-z + z^3 + y^2 z + xy z^2)
\]

Here $b_i(\vec{k}, \vec{p}, \vec{q}) = k^{-2} B_i(\vec{k}, \vec{p}, \vec{q})$, and $B_i$’s are

\[
B_1(\vec{k}, \vec{p}, \vec{q}) = kp(-z + z^3 + x^2 z + xy z^2)
\]

\[
B_2(\vec{k}, \vec{p}, \vec{q}) = kp(1 + z^2)(z + xy)
\]

\[
B_3(\vec{k}, \vec{p}, \vec{q}) = kp(-z + z^3 + x y + x^2 z + y^2 z + xy z^2)
\]

\[
B_4(\vec{k}, \vec{p}, \vec{q}) = kp(-z + z^3 + y^2 z + xy z^2)
\]

(50)

where $(x, y, z)$ are the cosines of angles between $(\vec{p}, \vec{q}), (\vec{q}, \vec{k})$, and $(\vec{k}, \vec{p})$ respectively.

When $\vec{z} = \vec{z}$, we have $C^{++} = C^{--} = C^{+-} = C^{-+} = G^{++} = G^{--}$, and $G^{+-} = G^{-+}$. Under this condition we expect $\Sigma^{++} = \Sigma^{--}$, but Eqs. (17, 18, 19) do not yield this results. However, the equality can be easily achieved by symmetrizing Eqs. (18, 19) by interchange of $+$ and $-$ signs in Eq. (45) and averaging of the resulting equations with corresponding Eqs. (18, 19) and (19). The final equations for $\eta^{\pm}$ are

\[
\eta^{\pm}(\tilde{k}) = \lim_{\omega \to 0} \frac{i}{2} k^2 \int d\tilde{p} \times
\]

\[
\times [b_1(\vec{k}, \vec{p}, \vec{q}) (G^{\pm \pm}(\tilde{p}) + G^{\mp \mp}(\tilde{p})) C^{+-}(\tilde{q}) + b_2(\vec{k}, \vec{p}, \vec{q}) (G^{\pm \pm}(\tilde{p}) + G^{\mp \mp}(\tilde{p})) C^{+-}(\tilde{q})]
\]

(50)

Now when $\vec{z} = \vec{z}$, we recover the fluid limit. In this case $C^{++} = C^{--} = C^{+-} = \Sigma^{++} = \Sigma^{--} = 2S_f^{fluid}$. Here we take $\omega \to 0$ limit in our calculation; this approximation is valid for the large and inertial scale fluctuations.
After this we solve for $\Sigma$ and $G$ self consistently. A typical self-energy diagram is shown in Figure 2. Considering that $z^+ = z^-$ correspond to fluid limit, we postulate the following relaxation behaviour for the $z$–$z$ correlation functions:

$$C^{\pm\pm}(k, \omega) = \frac{1}{2\pi} \frac{C^{\pm}(k)}{(i\omega - \eta_k^\pm)}; C^{\pm\mp}(k, \omega) = \frac{1}{2\pi} \frac{C^R(k)}{(i\omega - \frac{\eta_k^+ + \eta_k^-}{2})},$$

(51)

$$C^{\pm\pm}(k, t_2 - t_1) = C^{\pm}(k) \exp(-\eta_k^+(t_2 - t_1));$$

$$C^{\pm\mp}(k, t_2 - t_1) = C^R(k) \exp(-\frac{\eta_k^+ + \eta_k^-}{2}(t_2 - t_1));$$

(52)

The Eq. (53) yields

$$G(\hat{k}) = \frac{1}{-\omega \left(\omega + i\eta_k^+ + \eta_k^-\right)} \left[-i\omega + \frac{\eta_k^+}{2} \frac{\eta_k^-}{2} \right]$$

(53)

which after substitution in Eq. (54) and integration of $\omega'$ integral \( \int \) yields

$$\eta^{\pm}(k) = \int dp \frac{2}{\eta_k^+ + \eta_k^-} \left[ \frac{b_1(k,p,q)c^R(q)\eta_k^+}{\eta_k^+ + \eta_k^- + \eta_k^+ + \eta_k^-} + \frac{b_3(k,p,q)c^R(q)\eta_k^+}{\eta_k^+ + \eta_k^- + 2\eta_k^+ + \eta_k^-} \right]$$

(54)

We have ignored the contributions from the pole at $\omega' = 0$ because it corresponds to a static situation. The quantities $\eta^{+}/2$ and $\eta^{-}/2$ correspond to the inverse of the response time of the eddies $z^+$ and $z^-$ respectively.

Our self-energy matrix is differs from that obtained by Camargo and Tasso’s using renormalization group calculations \( \hat{15} \); their $\Sigma$ is symmetrical with both the diagonal elements equal. This is due to different assumptions of the models. In Camargo and Tasso’s model cross helicity is zero which forces both the diagonal elements to be equal. However, our model can be applied to situations where cross helicity is not zero. Also, we assume independent $f^+$ and $f^-$ forcing, whereas Camargo and Tasso assume independent $f^a$ and $f^b$ forcing. Renormalization group calculation for varying normalized cross helicity is under progress.

We need to evaluate the integrals in the above equation In 3D the integral \( \int \) is

$$\int dp = \int dp dq \frac{2\pi pq}{k}$$

(55)

while in 2D it is

$$\int dp = \int dp dq \frac{1}{\sin(p, q)}$$

(56)

The integral is to be performed in the hatched region of Figure 3.

We substitute the power-law expression of $C^+(k), C^-(k),$ and $C^R(k)$ from Eqs. \( \hat{22} \). For $\eta^+(k)$, which is interpreted as the inverse of time-scale of $z^\pm$, we substitute \( \hat{3} \)

$$\eta^{\pm}(k) = \Lambda^{\pm} (\Pi^\mp)^{2/3} (\Pi^{\pm})^{-1/3} k^{2/3},$$

(57)

where $\Lambda^{\pm}$ are constants. We also assume that the inertial range spectra of the residual energy $E^R(k)$ is proportional to $k^{-5/3}$. The EDQNM calculations of Pouquet et al. and Grappin et al. \( \hat{17} \) show that $E^R(k)$ is proportional to $k^{-2}$. Marsch and Tu \( \hat{19} \) analysed the solar wind data and found that the residual energy spectra is between $k^{-3/2}$ and $k^{-5/3}$, but closer to $k^{-3/2}$. Hence, there is no consistent theory for the spectra of the residual energy. In our model, for consistency, we assume that $E^R(k) \propto k^{-5/3}$. With this power law the ratio between $E^R(k)$ and the total energy $E(k)$ is constant; we denote this constant by $\alpha$. Similarly the ratio $E^-(k)/E^+(k)$ is a constant in our model; we denote it by $\beta$.

Now the substitution of power laws for $\eta^\pm_k$ in Eq. \( \hat{24} \) yields \( \hat{3} \)

$$\left(\frac{\Lambda^{\pm}}{k}\right)^2 = \int d\xi d\varsigma x^{8/3} \times$$

\[\left[\frac{1}{1+\xi^{2/3}} \left(\frac{b_2(1,\varsigma,\varsigma)+b_3(1,\varsigma,\varsigma)}{(1+\xi^{2/3})^{\chi^2/3+2\varsigma^{2/3}}\varsigma^{2/3}}\right) + \frac{\alpha(1+\beta^{2/3})}{2(1+\xi^{1/3})} \left(\frac{b_2(1,\varsigma,\varsigma)+b_3(1,\varsigma,\varsigma)}{(1+\xi^{2/3})^{\chi^2/3+2\varsigma^{2/3}}\varsigma^{2/3}}\right)\right]$$

(58)
in three dimensions. Here $\xi = (\Lambda^{-\Pi^+})/(\Lambda^+\Pi^-)$, $p = \zeta k$ and $q = \kappa k$. In 2D the terms $\nu u^8/3$ and $w^8/3$ are replaced by $w^{-8/3}$ and $v^{-8/3}$ respectively, and the whole integrand is divided by $\pi (1 - x^2)^{1/2}$.

The integrals of Eq. (47) suffer from the well known “infrared problem” which comes from the strong dynamic coupling of fluctuations with widely differing wavenumbers. Over the years, various techniques have been developed to tackle the difficulty in the context of pure fluid turbulence. Simplest among all these methods is introduction of cutoff, which is discussed in Leslie [8]. Later on Renormalization group technique [9], the Lagrangian or semi-Lagrangian pictures [10], and self-consistent screening [8] were used to make the integral naturally finite and cut off independent. The infrared difficulties associated with Eq. (53) can be similarly resolved. Knowing that the full theory is constrained to be finite, we adopt the practical procedure of evaluating the integral in Eq. (47) with a cut off $k_0 = \lambda k$ and choosing $\lambda = 1$ so that the pure fluid value of $\Lambda^2/K$ are obtained correctly when $\alpha = \beta = 1$. Thereafter $\lambda$ is not varied.

In 2D when $E^a < E^b$, we choose $\lambda = 1$, same as $\lambda$ of 3D, but not 0.065 which yields $K = 6.6$ in fluid limit. The choice of same $\lambda$ in 2D MHD was motivated by fact that in MHD turbulence, forward cascade of energy and inverse cascade of cross helicity occur in both 2D and 3D (refer to Ting et al. [23]). Note that in 2D fluid turbulence an inverse cascade of energy occurs, hence the behaviour of fluids turbulence in 2D and in 3D are dramatically different. It appears that $\lambda = 1$ is not applicable for cases when $E^a > E^b$; therefore we have calculated the constants for $E^a > E^b$.

To obtain the numerical value of $\Lambda^\pm$ and $\Lambda^\pm$ we need two additional equation involving $K^\pm$ and $\Lambda^\pm$. The additional equations we use are the equations of cascade rates of $z^\pm$; these equations are derived in the next subsection.

B. Computation of Cascade rates

The cascade rates or the fluxes $\Pi^\pm:R$ have already been defined in terms of $T^\pm:R$ respectively (ref. Eqs. [13,14]). We evaluate $T^\pm:R$ to first order, and then, following Kraichnan [15] substitute $G^0$, $C^0$s by $G$’s and $C$’s respectively (refer to Leslie [8] for this procedure applied in fluid turbulence). We will symbolically evaluate the illustration of $T$ up to first order. For this part of the calculation we work in $(k, t)$ space (refer to Leslie ([8]) for details). The quantity $T$ is proportional to equal time triple correlations. To first order $T$ is

$$T(k, t, t) \propto \int_{p+q=k} dp \int_{t} dt \int_{0}^{t} dt' \langle z(0)(k, t)z(0)(p, t)z(1)(q, t) \rangle + \langle z(0)(k, t)z(1)(p, t)z(0)(q, t) \rangle \cdots \tag{59}$$

Note that $\langle z(0)(k, t)z(0)(p, t)z(0)(q, t) \rangle$ is zero. Now using Eq. (42) we expand $z^{(1)}$ and substitute in the above equation, which yields

$$T(k, t, t) \propto \int_{p+q=k} dp \int_{-\infty}^{t} dt' \int_{0}^{t} dt'' \langle z(0)(k, t)z(0)(p, t)G(0)(q, t-t')z(0)(r, t') \rangle + \cdots \tag{60}$$

or,

$$T(k, t, t) \propto \int_{p+q=k} dp \int_{-\infty}^{t} dt' \int_{0}^{t} dt'' \langle z(0)(k, t)z(0)(p, t)C(k, t-t')C(p, t-t') \rangle + \cdots \tag{61}$$

The terms $G$’s and $C$’s appear as $G^s_1s_2$ and $C^s_1s_2$ in the final expression. The full expression for $T^\pm$ is

$$T^\pm(k, t, t) = 4\pi k^4 \int_{p+q=k} dp \int_{-\infty}^{t} dt' \times \left\{ -b_1(k, p, q) \left\{ G^\pm_1(p, \Delta t)C^{1}_R(q, \Delta t)C^0_1(p, \Delta t) + G^\pm_1(p, \Delta t)C^{1}_R(q, \Delta t)C^0_1(p, \Delta t) \right\} \right. \tag{62}$$

where $\Delta t = t - t'$. The expression for $T^R$ is similar to $T^\pm$. Substitution of $T's$ yield the following cascade rates:

$$\Pi^\pm:R(k) = \int_k^\infty dk' \int dp dq S^{\pm:R}(k', p, q) \tag{63}$$
where the \( dpdq \) integral is over the hatched region of Figure 3. Note however that the hatched region A has \( q > p \). If we have \( p > k \), then all three vectors \( k, p, \) and \( q \) forming triangle will have magnitudes greater than \( k \); hence these triad will not contribute to the flux. Therefore, in region A the flux contribution comes from the following region of integration:

\[
\Pi^{\pm,R}(k) = \int_{p}^{k} dp \int_{q}^{k'} dq S^{\pm,R}(k', p, q)
\]

where

\[
p^* = \max(p', |p - k'|).
\]

For region B similar arguments lead to

\[
\Pi^{\pm,R}(k) = \int_{p}^{k} dp \int_{q}^{q'} dq S^{\pm,R}(k', p, q)
\]

where \( q^* = \max(q', |q - k'|) \).

After substitution of \( C^{\pm,R} \) and \( G \) in the Eq. (61) and with the change of variable (6)

\[
k' = \frac{k}{u}; p = \frac{vk}{u}; q = \frac{wk}{u}
\]

we obtain

\[
\frac{\Lambda^\pm}{K^+K^-} = \left[ \int dv \ln \frac{1}{v} \int_{v^*}^{1+v} dw + \int dw \ln \frac{1}{w} \int_{w^*}^{1+w} dv \right] \Psi^\pm(1, v, w)
\]

\[
\Pi^R = \frac{1}{2} \left[ \frac{\Pi^+K^+K^-}{\Lambda^+} \right] \left( \Psi^{R1}(1, v, w) + \Psi^{R2}(1, v, w) \right)
\]

where in 3D

\[
\Psi^\pm(1, v, w) = -\frac{vw^{-8/3}(1+\beta')}{(1+\xi^2)} \left( b_1(1, v, w) + b_3(1, v, w) \right) \times
\]

\[
\left[ \frac{1}{(1+\xi^2)(1+\xi^2+2w^2/3)} \right] + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]

\[
\frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]

\[
\Psi^{R1}(1, v, w) = -\frac{vw^{-8/3}}{(1+\xi^2)} \left( b_1(1, v, w) + b_3(1, v, w) \right) \times
\]

\[
\left[ \frac{\alpha^2(1+\beta)^2/(4\beta)}{(1+\xi^2)(1+\xi^2+2w^2/3)} \right] + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]

\[
\frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]

\[
\Psi^{R2}(1, v, w) = -\frac{vw^{-8/3}}{(1+\xi^2)} \left( b_1(1, v, w) + b_3(1, v, w) \right) \times
\]

\[
\left[ \frac{\alpha^2(1+\beta)^2/(4\beta)}{(1+\xi^2)(1+\xi^2+2w^2/3)} \right] + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]

\[
\frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]

\[
\frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]

\[
\frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)} + \frac{1}{(1+\xi^2+2w^2/3)}
\]
\begin{align}
\Psi^{R2}(1,v,w) &= -\frac{w^{-8/3}}{(\pi \xi)} (b_1(1,v,w) + b_3(1,v,w)) \\
&\times [\frac{\alpha^2(1+\beta)^2/(4\beta)}{(1+\xi)(1+w^{2/3}+w^{2/3})} + \frac{1}{2(1+\xi)e^{2/3}+2\xi w^{2/3}}] \\
&- \frac{w^{-8/3}}{w(1+\beta)} (b_2(1,w,v) + b_4(1,w,v)) \\
&\times \left[\frac{2(1+\xi)(1+w^{2/3}+w^{2/3})}{(1+\xi)} (b_1(1,v,w) + b_2(1,v,w)) \\
&+ \frac{1}{(1+\xi+2\xi)(1+w^{2/3}+w^{2/3})} \right].
\end{align}
\tag{72}

In 2D the terms $vw^{-8/3}$ and $wv^{-8/3}$ are replaced by $w^{-8/3}$ and $v^{-8/3}$ respectively, and the whole integrand is divided by $\pi(1-x^2)^{1/2}$.

C. Computation of $K^\pm$ and $\Lambda^\pm$

We numerically solve for $K^\pm$ and $\Lambda^\pm$ from Eqs. (73, 74) for given ratios of $\alpha = E^R(k)/E(k)$ and $\beta = E^-(k)/E^+(k)$. Since the integrand itself is a function of $\xi$, which itself is a function of $\Lambda^\pm, \Pi^+,$ and $\Pi^-$, we start with a guessed value of $\xi$ and iterate until the two successive values of $\xi$ are approximately equal. We perform these calculations for both 2D and 3D. The values of $K^\pm$, $\Lambda^\pm$, and $\xi$ for various $\alpha$ and $\beta$ for 3D are listed in Table 1, whereas for 2D, they are listed in Table 2.

D. Computation of $K^R$

If we postulate that

$$E^R(k) = K^{R1} (\Pi^R)^{2/3} k^{-5/3},$$
\tag{73}

then the substitution of $\Pi^R$ in Eq. (59) yields

$$K^{R1} = K^+ \left( \frac{E^R(k)}{E^+(k)} \right) \left( \frac{\Lambda^-}{K^+ \Pi^- I} \right)^{2/3} \tag{74}$$

where $I$ is the value of the integral of Eq. (59). The values of $K^{R1}$ computed for various $\alpha$ and $\beta$ are listed in Tables 1 and 2.

We can model $E^R(k)$ in yet another way. This modeling is motivated by Eq. (15). From this equation we argue that

$$\Pi^R \sim \hat{E}^R \sim k \left( \frac{z_k^+ + z_k^-}{2} \right) \left( z_k^+ z_k^- \right).$$
\tag{75}

Therefore,

$$\Pi^R = k \left( \frac{z_k^+ + z_k^-}{2} \right) \frac{E^R(k)k}{K^{R2}} \tag{76}$$

or,

$$E^R(k) = \frac{2K^{R2}k^{-5/3}\Pi^R}{\sqrt{K^+ (\Pi^-)^{-1/3} + \sqrt{K^- (\Pi^+)^{-1/3}}}} \tag{77}.$$

The constants $K^{R2}$ calculated using the above equation are listed in Tables 1 and 2.
E. Discussion

Several important conclusions can be drawn from the numbers shown in Tables 1 and 2. The Kolmogorov’s constant for MHD turbulence $K^\pm$ are not universal constants as compared to universal fluid Kolmogorov’s constant. We find that the constants $K^\pm$ depend on Alfvén ratio $r_A$ and ratio of $E^-(k)$ and $E^+(k)$. They could also depend on the mean magnetic field, but study of this effect is beyond the scope of this paper.

As reported in our earlier paper, when $\beta = E^-(k)/E^+(k) = 1$, the constants $K^+ = K^- = K$ increases monotonically from 1.43 to $\sim 4.07$ as we go from fully fluid case $r_A = \infty$ to $r_A \sim 0.2$, then it decreases and finally reaches 3.51 when the plasma is fully magnetic. Please note that the numbers here are slightly different from those in [13]; the discrepancy is due to the lower accuracy used in the earlier paper. In 2D when $\beta = 1$, $K$ is in the range of $5 - 7.5$ for $0 < r_A < 1$. In both 2D and 3D for a given $r_A$, as we decrease $\beta$ (or increase normalized cross helicity $\sigma_c$), $K^+$ (constant corresponding to larger of $E^+$ and $E^-$) decreases whereas $K^-$ increases.

The above DIA results are in qualitative agreement with the preliminary numerical results of Verma et al. [20]. Verma et al. performed direct numerical simulation of MHD turbulence in 2D (8 runs) and 3D (single run) for various initial normalized cross helicity and mean magnetic field. When $E^+ \approx E^-$, the constant $K^+$ was approximately equal to $K^-$. The mean values of the constant obtained by them were $K = 3.7$ in three-dimensions for $E^-/E^+ = 0.6$, and $K = 6.6$ in two-dimensions for $E^-/E^+ = 1.06$. However, the constant $K$ of the majority species (larger of $z^\pm$) was always marginally lower than that of minority species. Unfortunately, we do not have the inertial range Alfvén ratio $r_A(k)$ for our simulations; we will study the $r_A$ dependence in future simulations. However, we suspect the inertial range Alfvén ratio to be in the range of 0.25–1. We find that for small $\sigma_c$ the mean values of $K$ from the simulations are generally close to the DIA result for $r_A = 1$ and $\sigma_c = 0$.

However, for large $\sigma_c$ there appears to be serious discrepancies between the simulation results and our results. In simulations for large $\sigma_c$, $\Pi^+/\Pi^- \sim E^-/(E^+/k)(k)$, however, in our DIA calculations $\Pi^+/\Pi^- \ll E^+/(k)E^-(k)$ in fluid dominated cases and $\Pi^+/\Pi^- \gg E^+/(k)E^-(k)$ in equipartioned ($r_A = 1$) or magnetically dominated case. Regarding constants $K^\pm$, for $\sigma_z(k) \approx 0.9(\beta = 0.05)$ Verma et al. found that the average $K^+ \approx 2.4$ and $K^- \approx 24.0$. In DIA calculations, for magnetically dominated cases, the equations for $K^\pm$ do not yield real solutions. This indicates that our model is not fully consistent at least in magnetically dominated limit; this point is illustrated in the following paragraph. Nevertheless, there are some qualitative agreement between the simulation results and DIA results, e.g., the constant $K$ corresponding to the majority species is always greater than the $K$ corresponding to the minority species.

When we turn off the velocity field, i.e., $z^+ = -z^- = b$, the MHD equations are linear. Here the magnetic energy dissipates only due to the resistivity, and there is no turbulent cascade rate. However, in our formalism, we obtain nonzero turbulent cascade rate for this case. Also, there is no consistent solution for $K^\pm$ when $\sigma_c$ is large and $r_A \leq 1$. To circumvent this inconsistencies we may have to modify our relaxation time, Green’s functions etc., and also possibly may need to use $u$ and $b$ rather than $z^\pm$. This work is under progress.

Note that the ratio $\Pi^R/\Pi^+$ increases monotonically as $r_A$ decreases from infinity to zero. By the time $r_A = 5$, the ratio $\Pi^R/\Pi^+$ has already become large. This large ratio implies that $\Pi^R \approx -\Pi^b (\Pi^R > 0$ and $\Pi^b < 0$), and the kinetic energy is cascading from small $k$ to large $k$, while the magnetic energy is cascading from large $k$ to small $k$. This is a theoretical demonstration of inverse cascade of magnetic energy. Earlier work on amplification of magnetic energy have been done by Fyfe et al. using numerical simulations, Léorat et al. using EDQNM closure scheme, Gloguen et al. using scalar model of MHD turbulence, and Ting et al. by turbulent relaxation ([23] and references therein). Our results may be useful for modeling dynamos from turbulent energy cascade point of view. If we start with small magnetic field in a fluid dominated plasma, there will be a small flux of magnetic energy cascading to large-scales from small-scales; that will enhance the magnetic field in the system and decrease $r_A$. As $r_A$ decreases, the flux cascading inversely also increases, hence further enhancing the magnetic field of the system. After the magnetic field has become sufficiently large ($r_A = 5$), almost all the fluid energy flux goes into magnetic energy flux, which makes the enhancement of the magnetic field faster. A quantitative model based on these ideas could be useful in estimating the time-scales etc. in dynamos.

In the above discussion we have not differentiated between inertial range $r_A (E^u(k)/E^b(k))$ and the $r_A = E^u/E^b$, which may be a gross approximation. Also modeling of relaxation of $C^{+\mp}(k, \Delta t)$ as well as modeling of $E^R(k)$ is not on a strong footing. We do not get consistent answers for equipartioned and magnetically dominated cases. Numerical simulations might provide important clues that could help us modeling these quantities better.

In the next section we will apply DIA to MHD turbulence in situations when $C_A \gg z^\pm$. 

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IV. DIAGNOSTICS OF MHD WITH $C_A \gg Z^\pm$

As mentioned in the introduction, in the presence of a strong mean magnetic field, the energy spectra is expected to be proportional to $k^{-3/2}$. However, the solar wind observations [14] as well as the numerical simulations [24] do not appear to support this hypothesis at least for $C_A/z^\pm \leq 5$. It is possible that when $C_A/z^\pm \gg 1$, e.g., when the ratio is of the order of 100, the Kraichnan’s [14] or Dobrowolny et al.’s [15] arguments are valid. Unfortunately numerical simulations have not yet probed in these regimes.

There exist another possibility that $B_0$ or $C_A$ which should be substituted in Eq. (3) is not the mean magnetic field but the renormalized $C_A$. When the mean magnetic field term of Eq. (8) is comparable to the nonlinear term, simple dimensional arguments yields the following scaling for $C_A$:

$$C_A = \Pi^{1/3}k^{-1/3}. \quad (78)$$

where $\Pi$ is the total energy cascade rate. This argument is similar to that due to Bhattacharjee [23] for renormalization of sound speed in randomly stirred fluids. We conjecture that the small-scale fluctuations are affected by renormalized magnetic field rather than the mean magnetic field. Hence, we should substitute renormalized $C_A$ of Eq. (78) in Eq. (3) that immediately yields $k^{-5/3}$ energy spectra for energy. Therefore, large $C_A$ could yield $k^{-5/3}$ energy spectra rather than $k^{-3/2}$ as argued by Kraichnan [14] and Dobrowolny et al. [15]. However, in absence of a quantitative theory or definite numerical results, we will make certain assumptions and calculate the Kraichnan’s constant for a variation of Dobrowolny et al.’s model.

As argued by Kraichnan [14] and Dobrowolny et al. [15], we assume that the Alfvén time-scale, rather than the nonlinear time-scale, is the relaxation time-scale when $C_A/z^\pm \gg 1$. Using random phase approximation Veltri et al. [24] showed that Green’s function and relaxation of correlation functions are

$$\hat{G}(k, t_2 - t_1) = \begin{bmatrix} \exp[-gkC_A(t_2 - t_1)] & 0 \\ 0 & \exp[-gkC_A(t_2 - t_1)] \end{bmatrix} \quad (79)$$

$$C^{\pm, R}(k, t_2 - t_1) = C^{\pm, R}(k, t_1 = t_2) \exp[-gkC_A(t_2 - t_1)]. \quad (80)$$

We assume $g = 1$. Unfortunately, Veltri et al. [24] assumed equipartition of magnetic and fluid energy which is not the case in our model discussed in this section. In Dobrowolny et al.’s model [15] the dissipation rates $\Pi^+$ and $\Pi^-$ are equal irrespective of $E^-(k)/E^+(k)$ ratios. This has been found to be inconsistent with the numerical results [20].

To bring in some effects of $E^-(k)/E^+(k)$ we modify the Dobrowolny et al.’s model in the following manner:

$$E^{\pm}(k) = A^{\pm} (\Pi^\pm C_A)^{1/2} k^{-3/2} \quad (81)$$

We do not need to solve for self-energy in this case; Green’s function has been obtained using random phase approximation. We, however, solve for cascade rates. The equation corresponding to Eq. (82) is

$$T^{\pm}(k, t, \Delta t) = 4\pi k^4 \int^{+\pm}_{p_0} dp \int_0^{\Delta t} dt' \times$$

$$[-b_1(k, p, q)G(p, \Delta t)C^R(k, \Delta t)C^\pm(q, \Delta t) \quad (82)$$

$$-b_2(k, p, q)G(q, \Delta t)C^+(q, \Delta t)C^\mp(k, \Delta t) \quad (82)$$

$$-b_4(k, p, q)G(q, \Delta t)C^R(q, \Delta t)C^R(k, \Delta t) \quad (82)$$

$$+b_2(k, p, q)G(k, \Delta t)C^+(k, \Delta t)C^\mp(q, \Delta t) \quad (82)$$

$$+b_1(k, p, q)G(k, \Delta t)C^R(k, \Delta t)C^R(q, \Delta t)]$$

where $G(k, \Delta t) = \exp(-C_A t)$. From the Eqs. (82) and a similar equation for $T^R$ we can derive expressions $\Pi^\pm$ and $\Pi^R$.

From the equations for $\Pi^\pm$ and $\Pi^R$ we can solve for $A^{\pm}$ using manipulations similar to those used in the previous section. The values of $A^\pm$ for various values of $\alpha$ and $\beta$ are listed in Tables 1 and 2. When $\beta = 1$, the constants $A^+=A^-=A$ are approximately 2.0-2.5 in 3D and 3.5-4.0 in 2D. The constant $A$ obtained here for 3D is close to the one obtained by Matthaeus and Zhou [13]. As we vary $\beta$, the constants $A^+$ and $A^-$ begin to differ. We find that when $E^-(k) < E^+(k)$, $A^+$ is always larger than $A^-$ contrary to the Kolmogorov-like models in which $K^+ < K^-$. For $r_A < 1$, the ratio $\Pi^+/\Pi^-$ is less than one, whereas for $r_A > 1$, the ratio is greater than one. Hence $r_A$ appears to have significant effect on cascade rates in this model as well.

The ratio $\Pi^R/\Pi^+$ decreases from a very large value (2519) to nearly zero as $r_A$ varies from zero to one. This implies that there is an inverse cascade of magnetic energy for $r_A < 1$. As $r_A$ increases from 1, the ratio $\Pi^R/\Pi^+$ decreases
from zero to negative values (the magnitude of the ratio increases) till \( r_A \) reaches around 15, then it again starts increasing. It reaches nearly zero at \( r_A = 1000 \) and reaches one at \( r_A = \infty \). This is an indication of inverse cascade of kinetic energy for \( 1 < r_A < 1000 \). Qualitatively, when the magnetic energy dominates the plasma, there is an inverse cascade of magnetic energy. On the contrary, when the kinetic energy dominates (only till \( r_A = 1000 \), there is an inverse cascade of kinetic energy. These predictions differ from those in section 3. Recall that in Kolmogorov-like models of the previous section, for all \( r_A \) (except at \( \infty \)) we had an inverse cascade of magnetic energy.

Our generalized Kraichnan model is heuristic. Numerical simulations could help us in determining which of the MHD turbulence models are applicable in various parameter space. Comparisons of the numerical results with the above predictions will provide insights into these puzzles.

V. CONCLUSIONS

In this paper we have applied direct-interaction approximation to MHD turbulence and obtained the values of the Kolmogorov’s constants for MHD and Kraichnan’s constant for various \( E^-/(E^+)^{1/3} \) and \( E^u/(E^b)^{1/3} \) ratios. Ours is a field-theoretical (nonvanishing) first order calculation. We have obtained equations for the constants as well as ratios of energy cascade rates for both Kolmogorov’s and Kraichnan’s power laws. We have solved for these constants for various values of \( E^-/(E^+)^{1/3} \) and \( E^u/(E^b)^{1/3} \) ratios. When \( \sigma_c = 0 \), in three-dimensions the constants \( K^+ = K^- = K \) are close to 2 for fluid dominated cases, but between 3 and 4 for cases when \( r_A \leq 1 \). In 2D, however, \( K \) is in the range of 5 – 7.5 for \( 0 < r_A < 1 \). For a given \( r_A \) as we increase \( \sigma_c \) (or decrease \( \beta \)), \( K^+ \) (the constant corresponding to majority species) decreases whereas \( K^- \) increases.

The DIA results for Kolmogorov-like phenomenology with small \( \sigma_c \) are in good agreement with the simulation results of Verma et al. [24]. However, the DIA results for large \( \sigma_c \) are only qualitatively consistent with the numerical results. For example, the Kolmogorov’s constant for \( (K^+ \text{ here}) \) the majority species is always smaller than that for minority species, consistent with the numerical results. However, for large \( \sigma_c \) our DIA calculations yield \( \Pi^+/(\Pi^-) \ll E^+(k)/E^-(k) \) in fluid dominated cases and \( \Pi^+/(\Pi^-) \gg E^+(k)/E^-(k) \) in magnetically dominated cases; these results are in disagreement with the simulation results where \( \Pi^+/(\Pi^-) \sim E^+(k)/E^-(k) \). We believe that the inconsistency can probably be removed if we modify the relaxation time \( \epsilon_0 \) or \( E \) expressions, or possibly we may have to use \( u \) and \( b \) variables. However, note that in 3D simulation only a single run with a relatively lower resolution (128\(^3\)) was performed. We need to perform more runs to compare the DIA results with the simulations.

In this paper for the first time we have calculated Kraichnan’s constants for a generalization of Dobrowolny et al.’s model. We find that when \( \beta = 1 \), the constant \( A^+ = A^- = A \) is approximately 2.0-2.5 in 3D and 3.5-4.0 in 2D, consistent with earlier estimates of \( A \). As \( \beta \) decreases, the constants \( A^+ \) and \( A^- \) begin to differ. The DIA results for Kraichnan’s model are inconsistent with the results of the numerical simulations performed so far. It is possible that Kraichnan’s model or its generalizations becomes applicable for large \( C_A/z^\pm \), however, at present we do not know the critical \( C_A/z^+ \) where transitions from Kolmogorov-like models to Kraichnan’s model take place. Another possibility is that \( C_A \) of Kraichnan’s model is the renormalized mean magnetic field which is scale dependent \( (\propto k^{-1/3}) \), and that will yield \( k^{-5/3} \) energy spectra.

Our DIA calculations using Kolmogorov-like energy spectra exhibits an inverse cascade of magnetic energy flux. We also find that as \( r_A \) decreases, the inverse cascade rate of magnetic energy increases. This is a theoretical demonstration of inverse cascade of magnetic energy, and this result could be useful for modeling dynamos from turbulent energy cascade point of view. However, DIA calculations using the generalized Dobrowolnay et al.’s model yields an inverse cascade of magnetic energy for \( r_A < 1 \), and an inverse cascade of kinetic energy for \( 1 < r_A < 1000 \).

The results discussed in this paper is useful for various applications in the solar wind. For example, with the knowledge of the Kolmogorov’s constants for MHD, Verma et al. [24] have calculated the contribution of turbulent dissipation to the overall heating in the solar wind. We anticipate that similar theoretical calculations can be useful in understanding of inverse cascade of energy, process called dynamo mechanism, and inverse cascade of cross helicity, a process called dynamic alignment.
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Figure 1: Diagramatic representation of the nonlinear interaction terms \( s^+(\ldots) = (\mathbf{k} \cdot \mathbf{z}^+)(\mathbf{z}^- \cdot \nabla \mathbf{z}^+(\mathbf{c})) \), where \( \mathbf{a}, \mathbf{b}, \mathbf{c} = (\mathbf{k}, \mathbf{p}, \mathbf{q}) \) (refer to Eq. 30). The solid lines represent \( z^+ \) and the wavy lines represent \( z^- \).

Figure 2: Diagramatic representation of self-energy \( \Sigma(k, \omega) \). The quantities \( G \) and \( C \) in the figure are dressed quantities.

Figure 3: The region of integration \( \int_{p=q=k} d\mathbf{p} \).

| \( r_A \) | \( E^-/E^+ \) | \( \zeta \) | \( \Lambda^+ \) | \( \Lambda^- \) | \( K^+ \) | \( K^- \) | \( \Pi^+/\Pi^- \) | \( K^{R1} \) | \( K^{R2} \) | \( \Pi^R/\Pi^+ \) | \( A^+ \) | \( A^- \) | \( \Pi^+/\Pi^- \) | \( \Pi^R/\Pi^+ \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \infty \) | 1 | 1 | 0.390 | 0.390 | 1.43 | 1.43 | 1 | 1.43 | 1.20 | 1 | 2.51 | 2.51 | 1 | 1 |
| 1000 | 1 | 1 | 0.390 | 0.390 | 1.44 | 1.44 | 1 | 1.43 | 1.19 | 1 | 2.51 | 2.51 | 1 | 0.063 |
| 100 | 1 | 1 | 0.395 | 0.395 | 1.46 | 1.46 | 1 | 1.30 | 1.02 | 1.14 | 2.48 | 2.48 | 1 | -7.88 |
| 15 | 1 | 1 | 0.425 | 0.425 | 1.00 | 1.00 | 1 | 0.359 | 0.144 | 1.15 | 7.68 | 7.68 | 1 | -42.92 |
| 5 | 1 | 1 | 0.491 | 0.491 | 1.92 | 1.92 | 1 | 0.0831 | 0.015 | 0.96 | 92.1 | 92.1 | 1 | -80.0 |
| 0.5 | 0.54 | 0.75 | 0.254 | 1.69 | 2.05 | 1.55 | 1.45 | 0.96 | 92.1 | 92.1 | 1 | -80.0 |
| 0.1 | 0.21 | 0.856 | 0.064 | 1.20 | 1.34 | 3.31 | 1.42 | 0.526 | 171 | 171 | 1 | -203 |
| 2 | 1 | 1 | 0.612 | 0.612 | 2.58 | 2.58 | 1 | 0.018 | 0.016 | 345 | 1.98 | 1.98 | 1.0 | -65.3 |
| 0.5 | 0.39 | 1.20 | 0.243 | 1.92 | 3.47 | 1.93 | 0.0099 | 0.018 | 341 | 2.78 | 1.42 | 1.05 | -71.6 |
| 0.1 | 0.12 | 1.57 | 0.045 | 1.58 | 2.96 | 4.22 | 0.0055 | 0.020 | 386 | 5.82 | 0.685 | 1.39 | -167 |
| 1 | 1 | 1 | 0.748 | 0.748 | 3.38 | 3.38 | 1 | 0 | 0 | 1092 | 1.99 | 1.99 | 1.0 | -0.773 |
| 0.5 | 0.19 | 3.55 | 0.112 | 0.905 | 17.7 | 6.04 | 0 | 0 | 0 | 796 | 2.82 | 1.41 | 1.0 | -0.773 |
| 0.4 | 0.08 | 7.36 | 0.034 | 0.536 | 53.8 | 16.0 | 0 | 0 | 0 | 813 | 3.15 | 1.26 | 1.0 | -0.773 |
| 0.1 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 0.5 | 1 | 1 | 0.865 | 0.865 | 4.04 | 4.04 | 1 | -0.0075 | -0.0003 | 2391 | 2.13 | 2.13 | 1 | 148.3 |
| 0.9 | 0.66 | 1.62 | 0.475 | 2.00 | 9.11 | 2.25 | -0.0044 | -0.0007 | 1749 | 2.25 | 2.02 | 0.992 | 149.4 |
| 0.5 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 0.1 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 0.2 | 1 | 1 | 0.913 | 0.913 | 4.07 | 4.07 | 1 | -0.012 | -0.0004 | 3588 | 2.54 | 2.54 | 1 | 527.4 |
| 0 | 1 | 1 | 0.889 | 0.889 | 3.51 | 3.51 | 1 | -0.014 | -0.0005 | 3965 | 4.13 | 4.13 | 1.0 | 2519 |

Table I. For 3D MHD turbulence Kolmogorov’s constants \( K^\pm, \Lambda^\pm, A^\pm \) etc. for various values of \( r_A \) and \( E^-/E^+ \).
TABLE II. For 2D MHD turbulence Kolmogorov’s constants $K^\pm, \Lambda^\pm$ etc. for various values of $r_A$ and $E^-/E^+$. The constant $\lambda$ which appears in the lower limit of the self-energy integral is chosen to be 1.0.