Quantum Zeno and anti-Zeno effects on quantum and classical correlations

F. Francica,1,2 † F. Plastina,1,2 ‡ and S. Maniscalco3 ∗

1Dipartimento di Fisica, Università della Calabria, 87036 Arcavacata di Rende (CS) Italy
2INFN - Gruppo collegato di Cosenza
3Turku Centre for Quantum Physics, Department of Physics and Astronomy, University of Turku, FI-20014 Turun yliopisto, Finland

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I. INTRODUCTION

Composite quantum systems may possess correlations of a different nature with respect to classical ones. For the latter, correlations are generally measured by the mutual information, whose extension to the quantum realm, however, leads to two different quantities: the mutual information, whose extension to the quantum realm, however, leads to two different quantities: the quantum mutual information I and the classical correlations C. The quantum discord is the difference between these two quantities, i.e., \( D = I - C \). This quantity is zero only for classically correlated states, i.e., in the case of bipartite systems, for states of the form \( \rho = \sum_{k,l} p_{k,l} |k\rangle \langle l| \), with \( p_{k,l} \neq p_{k,l'} \), \( |k\rangle \) and \( |l\rangle \) being orthogonal states of the two subsystems. In this sense the quantum discord measures quantum correlations.

For mixed quantum states, the discord does not coincide with entanglement. Indeed, there exist separable states having non-zero discord. Recently, the properties of quantum and classical correlations have received a huge deal of attention in both applicative and fundamental research. In particular, it has been demonstrated that states having positive discord but zero entanglement can be used in certain models of quantum computation to achieve significant speedup with respect to the classical algorithms. Moreover, it has been shown that, under certain conditions, quantum discord is totally unaffected by non-dissipative noise for long time intervals.

In this article we study the effect of nonselective projective measurements on the dynamics of a simple bipartite quantum system. Our aim is to investigate how both quantum and classical correlations are affected by the measurements. We focus, in particular, on a system of two noninteracting qubits immersed in a common structured reservoir, such as the principal mode of a high-Q cavity. It is known that, in this case, certain types of measurements performed on either the collective state of the system or the state of the cavity, may inhibit the loss of quantum entanglement initially present in the two qubits states. The resulting measurement-induced protection of entanglement was studied in the case in which the two qubits are resonantly coupled with the cavity mode.

We now extend the results we have presented in a previous Letter in two directions. First of all we consider different cases of off-resonant interaction between the qubits and the cavity mode and we show that, under certain conditions, the same measurements that allowed to significantly suppress entanglement loss are now responsible for a much faster disentanglement, as a result of the anti-Zeno effect. Secondly, we study how measurements affect not only entanglement but also other types of correlations between the qubits, namely classical correlations and quantum discord. We show that, contrarily to what has been found in previously studied systems, the dynamics of classical and quantum correlations, in the system here considered, is qualitatively very similar to the dynamics of entanglement. Therefore, projective measurements have a very similar effect on these three types of correlations. In particular, we show that there exist working conditions for which a series of oscillations occurs between the Zeno and anti-Zeno regimes, both for the quantum and the classical correlations, and that the amount of correlations can depend on the relative phase of the initial state. Oscillations between Zeno and anti-Zeno regimes have also been predicted in the quantum Brownian motion model for certain types of structured environments.

The plan of the paper is the following. In Sec. II we describe the physical model employed in Sec. III to derive the analytic expression of classical correlation and of quantum discord. In Sec. IV, we obtain analytically...
the time evolution of the system in presence of projective non-selective measurements on the qubits, which are then used in Sec. V, where we present the quantum Zeno and anti-Zeno effects on concurrence, discord and classical correlations. Finally, Sec. VI provides a summary of the results together with some concluding remarks.

II. THE SYSTEM

Let us consider an open quantum system consisting of two qubits coupled to a common zero-temperature bosonic reservoir. The Hamiltonian of the total system is

\[ H = H_S + H_R + H_{int}, \]

where \( H_R \) is the Hamiltonian of the reservoir and \( H_S \) is the Hamiltonian of the two qubits which are coupled to the common reservoir via the interaction \( H_{int} \).

The Hamiltonian for the total system, in the dipole and the rotating-wave approximations, and in units of \( \hbar \), reads

\[ H_S = \omega_1 \sigma_1^{(1)} \sigma_1^{(1)} + \omega_2 \sigma_2^{(2)} \sigma_2^{(2)}, \]

\[ H_R = \sum_k \omega_k b_k^\dagger b_k, \]

\[ H_{int} = (\alpha_1 \sigma_1^{(1)} + \alpha_2 \sigma_2^{(2)}) \sum_k g_k b_k + \text{h.c.}, \]

where \( b_k^\dagger, b_k \) are the creation and annihilation operators of quanta of the reservoir, \( \sigma_j^{(j)} \) and \( \omega_j \) are the inversion operator and transition frequency of the \( j \)-th qubit \( (j = 1, 2) \), \( \omega_j \) are the frequencies of the reservoir \( k \)-mode, and \( \alpha_j g_k \) describe the coupling strength between the \( j \)-th qubit and the \( k \)-mode of reservoir.

The \( \alpha_j \) are dimensionless real coupling constants measuring the interaction strength of each single qubit with the reservoir. We assume that the ratio between these two constants can be varied independently. In the case of two atoms inside a cavity, e.g., this can be achieved by changing the relative position of the atoms with respect to the cavity field standing wave. We denote with \( \alpha_T = (\alpha_1^2 + \alpha_2^2)^{1/2} \) the collective coupling constant and with \( \tau_j = \alpha_j / \alpha_T \) the relative interaction strength.

We restrict ourselves to the case in which only one excitation is present in the system and the reservoir is in the vacuum. Initially the two-qubit system is assumed to be disentangled from its reservoir and the initial state for the whole system is written as

\[ |\Psi(0)\rangle = \left[ c_{01} |1_1 0_2\rangle + c_{02} |0_1 1_2\rangle \right] \otimes |0_k\rangle_R, \]

where \( c_{01} = \sqrt{(1-s)/2} \) and \( c_{02} = \sqrt{(1+s)/2} e^{i\varphi} \) are complex numbers defining the initial state for the qubits system, \(-1 \leq s \leq 1\), \( |0\rangle_j \) and \( |1\rangle_j \) \((j = 1, 2)\) are the ground and excited state of the \( j \)-th qubit, respectively, and \( |0_k\rangle_R \) is the state of the reservoir with zero excitations in the \( k \)-mode.

As a consequence of the time evolution generated by the Hamiltonian \( H \), the excitation can be shared by the qubits and the reservoir, so that

\[ |\Psi(t)\rangle = c_1(t) |1_1 0_2\rangle_R + c_2(t) |0_1 1_2\rangle_R + \sum_k c_k(t) |0_1 0_2\rangle_R |1_k\rangle_R, \]

\( |1_k\rangle_R \) being the state of the reservoir with only one excitation in the \( k \)-th mode.

In the standard basis, the reduced density matrix for the qubits, obtained from the density operator \( |\Psi(t)\rangle \langle \Psi(t)| \) after tracing over the reservoir degrees of freedom, takes the form

\[ \rho(t) = \begin{pmatrix} 1 - |c_1(t)|^2 - |c_2(t)|^2 & 0 & 0 & 0 \\ 0 & |c_2(t)|^2 & c_2^*(t) c_1(t) & 0 \\ 0 & c_1(t) c_2^*(t) & |c_1(t)|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

The two-qubit dynamics is therefore completely characterized by the amplitudes \( c_{1,2}(t) \). For certain specific structures of the reservoir, one can obtain the exact analytical expressions of \( c_{1,2}(t) \) by the Laplace transform method. In this paper we consider a structured reservoir describing the electromagnetic field inside a lossy cavity. This case can be modelled by a Lorentzian broadening of the fundamental cavity mode. The non-Markovian analytical expression for the amplitudes \( c_{1,2}(t) \), in the off-resonant regime, were presented in Ref. \[29\]. We make explicit use of these results here. As a specific example of application to a real system, this model has been shown to adequately describe the dynamics of ions trapped in an electromagnetic resonator \[30\].

III. CLASSICAL AND QUANTUM CORRELATIONS

In this section we recall the analytic expression for the entanglement dynamics, as measured by concurrence, and present the analytic formula for both the classical correlations and the discord. The entanglement dynamics for a generic initial two-qubit state containing one excitation coupled to a common structured reservoir was investigated in \[23\]. We choose the concurrence \( C_E(t) \) \[31\], ranging from 0 for separable states to 1 for maximally entangled states, to quantify the amount of entanglement encoded into the two-qubit system. This quantity can be obtained from the reduced density matrix of Eq. (7) and takes the simple form

\[ C_E(t) = 2 |c_1(t)| |c_2(t)|. \]

In Ref. \[23\] we have shown how, in the resonant regime, repeated nonselective measurements on the collective state of the qubits system induce a quantum Zeno
The analytic expression of the discord reads

\[ C_c(\rho) = \sup_{\{\Pi_k^{(2)}\}} \left[ S(\rho_1) - S(\rho|\{\Pi_k^{(2)}\}) \right], \tag{9} \]

where the maximum is taken over all projective measurements performed locally on qubit 2, described by a set of orthogonal projectors \{\Pi_k^{(2)}\} corresponding to the outcomes \(k\). In Eq. (9), \(S(\rho)\) is the von Neumann entropy, \(\rho_1\) the reduced density operator of qubit 1, and \(S(\rho|\{\Pi_k^{(2)}\})\) the conditional entropy defined as \(S(\rho|\{\Pi_k^{(2)}\}) = \sum_k p_k S(\rho_k)\), with \(p_k = \langle I(1) \otimes \Pi_k^{(2)} \rangle/\rho\) the conditional density operator of qubit 1 after qubit 2 is measured and the outcome \(k\) is obtained, with probability \(p_k = Tr[I(1) \otimes \Pi_k^{(2)} \rho(I(1) \otimes \Pi_k^{(2)})]\).

For the system considered in this paper, the optimization problem in the definition of the classical correlations can be solved exactly and a simple analytical expression for this quantity can be derived. Indeed, by calculating the action of the one-qubit projectors

\[ \Pi_k^{(2)} = I \otimes |k\rangle \langle k|, \quad \text{with} \quad k = a, b \tag{10} \]

and

\[ |a\rangle = \cos \theta |\uparrow\rangle + e^{i\phi} \sin \theta |\downarrow\rangle, \]
\[ |b\rangle = \sin \theta |\uparrow\rangle - e^{-i\phi} \cos \theta |\downarrow\rangle, \]

on the general two-qubit state given by Eq. (1), and using Eq. (2), it is straightforward to prove that the classical correlations do not explicitly depend on \(\phi\) and are maximized for \(\theta = n \pi/2\) with \(n \in \mathbb{Z}\). The analytic expression for \(C_c\) is given by

\[ C_c(\rho) = (1 - |c_1(t)|^2 - |c_2(t)|^2) \log_2 (1 - |c_1(t)|^2 - |c_2(t)|^2) - \sum_{j=1,2} (1 - |c_j(t)|^2) \log_2 (1 - |c_j(t)|^2). \tag{11} \]

The dynamics of the quantum discord is then easily calculated as \(D(t) = \mathcal{I}(t) - C_c(t)\), with

\[ \mathcal{I}(\rho) = S(\rho_1) + S(\rho_2) - S(\rho). \tag{12} \]

The analytic expression of the discord reads

\[ D(\rho) = |c_1(t)|^2 \log_2 \left( 1 + \frac{|c_2(t)|^2}{|c_1(t)|^2} \right) + |c_2(t)|^2 \log_2 \left( 1 + \frac{|c_1(t)|^2}{|c_2(t)|^2} \right). \tag{13} \]

We note that, if times \(\bar{t}\) such that \(|c_1(\bar{t})| = |c_2(\bar{t})|\) exist, then \(D(\bar{t}) = C_E(\bar{t})\), i.e., the quantum correlations as measured by the discord coincide with entanglement as measured by concurrence, although the state is not necessarily pure.

In fact, one can show in general that, for any two-qubit density matrix of the form \(\rho = (1 - \alpha)|00\rangle\langle 00| + \alpha|\psi_{me}\rangle\langle\psi_{me}|\), where \(|\psi_{me}\rangle\) is any maximally entangled state orthogonal to \(|00\rangle\), and \(\alpha \in [0, 1]\), the concurrence is equal to the discord: \(C_E = D = \alpha\). The case discussed here (with \(|c_1| = |c_2|\)) gives precisely a density matrix of the previous kind, with \(\alpha = 2|c_1|^2\).

We can conclude, therefore, that, in these cases, the system does not contain quantum correlations other than entanglement. We also note that, for \(|c_1(\bar{t})| = |c_2(\bar{t})|\), the discord recently defined by Modi et al. in terms of the distance to the closest classically correlated state \(2\), coincides with the one used in this paper.

In the next section we will study how appropriately designed nonselective projective measurements modify the dynamics of quantum and classical correlations. Contrarily to what happens in the resonant case \(2\), we will see that such type of measurements can enhance rather than protect the decay of quantum correlations, and that even classical correlations are affected in a similar way.

As peculiar specific instances in the rich parameter space of our model, we will give special attention to two possible configurations; namely, those corresponding to detunings \(\delta_1 = \pm \delta_2\) (\(\delta_i\) being the detuning of the \(i\)-th atom from the cavity mode), with equal couplings between atoms and reservoir, i.e., \(r_1 = r_2\). This choice stems from the analysis of the system dynamics in absence of measurements performed in Ref. \(29\). We have seen there that these two cases give rise to interesting dynamical behavior and therefore deserve special attention.

From the analytic expressions of the coefficients \(c_1(t)\) and \(c_2(t)\) derived in Ref. \(29\) one can prove that for \(\delta_1 = \pm \delta_2\) and \(r_1 = r_2\), \(|c_1(t)| = |c_2(t)|\) at all times \(t\). To the best of our knowledge this is the first example of open system dynamics for which during the whole time evolution the quantum correlations exactly coincide with entanglement. We recall that this is always the case for pure states (with the entanglement measured by the entropy), but for mixed states as those of our open system, this is far from being trivial.

**IV. THE EFFECT OF PROJECTIVE MEASUREMENTS**

We recall that, in order to observe the quantum Zeno effect on the entanglement, the series of nonselective measurements on the collective atomic system, performed at time intervals \(T\), must have the following properties: i) one of the possible measurement outcomes is the projection onto the collective ground state \(|\psi_0\rangle = |0\rangle_1|0\rangle_2\), and ii) the measurement cannot distinguish between the
excited-states $|\psi_1\rangle = |1\rangle_1 |0\rangle_2$ and $|\psi_2\rangle = |0\rangle_1 |1\rangle_2$.

Such measurements are described by the following two projectors:

$$\Pi_0 = |\psi_0\rangle \langle \psi_0| \otimes I_R,$$

$$\Pi_1 = (|\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|) \otimes I_R,$$

with $I_R$ the reservoir identity matrix. Projective measurements as those described by the operator $\Pi_0$ can be experimentally implemented in both cavity QED \cite{1} and in superconducting circuits with on-chip qubits and resonator \cite{3,4}.

The state of the total system, formed by the two-qubits and the electromagnetic field inside the cavity, after a series of $N$ instantaneous ideal measurements performed at time intervals $T$ is given by

$$|\Psi^{(N)}(t)\rangle = \left[ c_1^{(N)}(T) |\psi_1\rangle + c_2^{(N)}(T) |\psi_2\rangle \right] \otimes |0_k\rangle_R + \sum_k b_k^{(N)}(T) |\psi_0\rangle |1_k\rangle_R,$$

which is described by the following expressions

$$\left[ c_{12}^{(N)}(T) \right]$$

and $b_k^{(N)}(T)$ are the survival amplitudes at time $t = NT$ related to the presence of the excitation in qubit 1, qubit 2, and the cavity field, respectively.

We note that the reduced density matrix $\rho^{(N)}(t) = Tr \{ |\Psi^{(N)}(t)\rangle \langle \Psi^{(N)}(t)| \}_R$, describing the two-qubit system after $N$ measurements, has the same structure of Eq. \cite{4}, provided one changes $c_{12}(t)$ with $c_{12}^{(N)}(T)$.

Our aim is to derive simple and physically transparent equations for $c_{12}^{(N)}(T)$ able to explain the occurrence of quantum Zeno or anti-Zeno effects for classical and quantum correlations. To achieve this goal we analyze first the coarse grained evolution with $t = NT$ and then derive, under certain approximations, the time evolution of the system between successive measurements.

### A. Coarse grained dynamics

Let us introduce the matrix $E$ describing the uninterrupted evolution between two consecutive measurements in the two-qubit subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$. In the following we focus on the case of frequently observed dynamics and we assume that the interval $T$ between two successive measurements is short. The survival amplitudes $c_{12}^{(N)}(T)$ in presence of measurements can be written as

$$\left( \begin{array}{c} c_1^{(N)}(T) \\ c_2^{(N)}(T) \end{array} \right) = E^{(N)} \left( \begin{array}{c} c_1(0) \\ c_2(0) \end{array} \right),$$

where

$$E^{(N)} = \left( \begin{array}{cc} e_{11}(T) & e_{12}(T) \\ e_{21}(T) & e_{22}(T) \end{array} \right),$$

$E^{(N)}$ being the evolution matrix in presence of $N$ measurements. Note that, in general, $E^{(N)}$ is not equal to the $N$-th power of the evolution matrix $E$ between two successive measurements; therefore, the explicit expressions of the matrix elements $E_{ij}(T)$, obtained by the Laplace transform method \cite{22}, take a complicated form.

Simple forms for the survival amplitudes in presence of measurements can be found when $\omega_1 = \omega_2$, since in this case the superradiant state evolves \cite{23}. In this case, it is much more useful to express the matrix $E^{(N)}$ in the superradiant-subradiant basis, so that it has only one non-zero entry, given by the survival amplitude of the superradiant state, which takes the form

$$E(T) = e^{-(\lambda - i\delta)T/2} \left[ \cosh \left( \frac{\Omega T}{2} \right) + \frac{\lambda - i\delta}{\Omega} \sinh \left( \frac{\Omega T}{2} \right) \right],$$

where $\delta_1 = \delta_2 = \delta$, $\Omega = \sqrt{\lambda^2 - \Omega_R^2 - 24\lambda}$, and where we indicate with $\lambda$ the width of the Lorentzian spectral distribution describing the field inside the cavity. Here, $\Omega_R = \sqrt{4\gamma^2\alpha_T + \delta^2}$ is the generalized Rabi frequency, while $R = W\alpha_T$ is the vacuum Rabi frequency.

For the general case, one can prove that, if $\lambda T \ll 1$, i.e., in the limit of frequent measurements, the off-diagonal elements of the evolution matrix $E_{ij}(T)$, with $j \neq i$, are small and decrease quickly as $T$ decreases and $\delta_1, \delta_2$ increases. Hence, the dynamics for long time intervals are well described by the following expressions

$$E_{jj}^{(N)}(T) \approx E_{jj}^{(N)}(T) \left[ 1 + \theta [N - 1] \frac{E_{ij}(T)E_{ij}(T)}{E_{jj}(T)^2} \right] \times \sum_{k=0}^{N-2} (N - 1 - k) \left( \frac{E_{ii}(T)}{E_{jj}(T)} \right)^k, \quad (20)$$

$$E_{jj}^{(N)}(T) \approx E_{jj}^{(N)}(T) \frac{E_{jj}(T)}{E_{jj}(T)^2} \sum_{k=0}^{N-1} \left( \frac{E_{ii}(T)}{E_{jj}(T)} \right)^k + \theta [N - 2] \times E_{jj}^2(T)E_{ij}(T) \sum_{k=0}^{N-3} (k + 1)(N - k) \left( \frac{E_{ii}(T)}{E_{jj}(T)} \right)^k, \quad (21)$$

where $\theta[x]$ is the Heaviside step function.

Inserting Eqs. (20) - (21) into Eqs. (17) - (18) we obtain the dynamics of the two-qubits reduced density matrix $\rho^{(N)}(t)$ at time $t = NT$ in presence of measurements, which is found to retain the same structure as in Eq. (7).

As a consequence, the concurrence $C_E^{(N)}(t)$, the classical correlations $c_{ij}^{(N)}(t)$ and the discord $D^{(N)}(t)$, in presence of measurements, are simply obtained by replacing the amplitudes $c_{12}(t)$ with $c_{12}^{(N)}(T)$ in Eq. (8), Eq. (11) and Eq. (13), respectively.

The dynamics of quantum and classical correlations in presence of measurements depends on $T$, on the initial state, on the detunings $\delta_1, \delta_2$, on the relative coupling between the qubits and the cavity field, and on the quality factor of the cavity, i.e. on $\lambda$. To study in detail how
such factors influence the dynamics we need to evaluate analytic expressions for $E_{ij}(T)$. We will face this task in next subsection.

B. Evolution between two successive measurements

In the limit of frequent measurements, Eqs. (20)-(21) imply that the evolution in presence of $N$ measurements can be obtained from the matrix elements $E_{ij}(T)$. Therefore we need only to calculate, in this limit, the time evolution of the amplitudes $c_j(t)$, in the interval $0 \leq t \leq T$ between two successive measurements. To this aim we employ perturbation theory, valid when the evolution time $T$ is sufficiently short, such that $c_j(T) \approx c_j(0) \equiv c_{j0}$ with $j = 1, 2$.

The integro-differential equations for the probability amplitudes, obtained from the Schrödinger equation (21), are

$$\dot{c}_j(t) = -\alpha_j^2 \int_0^t dt' f(t') e^{i \delta_j t'} c_j(t - t') - \alpha_j \alpha_i e^{i(\delta_j - \delta_i)t} \int_0^t dt' f(t') e^{i \delta_i t'} c_i(t - t'),$$

with $j \neq i$. To first order, one gets

$$c_j(T) = c_{j0} - \alpha_j^2 \int_0^T dt f(t) e^{i \delta_j t} c_{j0} - \alpha_j \alpha_i \int_0^T dt e^{i(\delta_j - \delta_i)t} \int_0^t dt' f(t') e^{i \delta_i t'} c_{i0},$$

where $\delta_j = \omega_j - \omega c$, $\omega c$ is the fundamental frequency of the cavity, and $f(t)$ is the correlation function.

Recalling that the correlation function is the Fourier transform of the reservoir spectral density $J(\omega)$,

$$f(t) = \int d\omega J(\omega) e^{i(\omega c - \omega)t},$$

we can recast the amplitudes of Eq. (23) in the form

$$c_j(T) = c_{j0} - \alpha_j^2 \int d\omega J(\omega) F_{jj}(\omega, T) c_{j0} - \alpha_j \alpha_i \int d\omega J(\omega) F_{ji}(\omega, T) c_{i0}.$$

This equation shows how the amplitudes $c_j(T)$ depend on two form factors, $F_{jj}(\omega, T)$ and $F_{ji}(\omega, T)$, defined as

$$F_{jj}(\omega, T) = \int_0^T dt \int_0^\infty dt' \theta(t - t') e^{i \omega c t'} e^{-i \omega t'},$$

and

$$F_{ji}(\omega, T) = \int_0^T dt e^{i(\delta_j - \delta_i)t} \int_0^\infty dt' \theta(t - t') e^{i \omega c t'} e^{-i \omega t'},$$

The first term of the sum within the modulus describes the behavior of the excitation initially present in the state $|\psi_j\rangle$ (and therefore, it is proportional to $c_{j0}$). This is an excitation-trapping contribution. The second term, instead, is an excitation-transfer contribution, as it describes an excitation transfer from the state $|\psi_i\rangle$ to the state $|\psi_j\rangle$.

From Eq. (26) one immediately understands that, for $c_{i0} \neq 0$ and $c_{j0} \neq 0$, the survival probability $P_{jj}^{(N)}(T) = |c_j^{(N)}(T)|^2$ is crucially dependent on an interference term

$$E_{jj}(T) = 1 - \alpha_j^2 \int d\omega J(\omega) F_{jj}(\omega, T) \approx e^{-\alpha_j^2 \int d\omega J(\omega) F_{jj}(\omega, T)},$$

and

$$E_{ji}(T) = -\alpha_j \alpha_i \int d\omega J(\omega) F_{ji}(\omega, T), j \neq i.$$

Equations (26)-(27) show that the short-time non-exponential behavior of the survival amplitudes $c_j(T)$, and so of the decoherence, is crucially determined by the way in which the qubit-reservoir coupling is modified by the form factors, $F_{jj}(\omega, T)$ and $F_{ji}(\omega, T)$. These terms are generally functions of $\omega$ sharply peaked around $\omega_j$. We will see in next section how frequent measurements, modifying the form factors, affect the dynamics of both quantum and classical correlations.

V. ZENO AND ANTI-ZENO EFFECTS ON CLASSICAL AN QUANTUM CORRELATIONS

In this section we study the effect of measurements on the dynamics of correlations focusing on the off-resonant regime. Indeed, in these conditions, the dynamics of entanglement is much richer than in the resonant case, and we expect that measurements may cause both quantum Zeno and anti-Zeno effects. On the contrary, as shown in Ref. 23, only the Zeno effect occurs on resonance.

A. Zeno dynamics of the survival amplitude

The analytical expressions of concurrence, classical correlations and quantum discord, as given by Eq. (8), Eq. (11), and Eq. (13), respectively, all depend on the modulus of the survival amplitudes $|c_j^{(N)}(T)|$ in presence of $N$ measurements. These quantities can be written as follows

$$|c_j^{(N)}(T)| = \left| E_{jj}^{(N)}(T) c_{j0} + E_{ji}^{(N)}(T) c_{i0} \right|.$$
proportional to \( R \left[ E_j^{(N)} E_i^{(N)} \right] \). This, in turns, implies that the relative phase \( \phi \) between the two components of the initial two-qubit state plays a key role. Indeed, we will see how the correlation dynamics is strongly sensitive to the specific initial state. In particular we will show that initial pure states possessing the same degree of entanglement, such as two types of initial Bell-like states corresponding to \( \phi = 0 \) and \( \phi = \pi \), display very different qualitative dynamics of correlations in presence of measurements.

In the bad cavity limit considered in this paper, we can further approximate Eqs. (20)-(21) as follows

\[
E_j^{(N)}(T) \approx E_j^{(N)}(T),
\]

(29)

\[
E_i^{(N)}(T) \approx E_i^{(N)}(T) \left( E_j^{(N)}(T) \right) \sum_{k=0}^{N-1} \left( \frac{E_i(T)}{E_j^{(N)}(T)} \right)^k.
\]

(30)

Using Eqs. (20)-(27), Eq. (28), and Eqs. (29)-(30), a straightforward calculation allows us to recast the survival amplitudes in presence of measurements in the form

\[
\left| c_j^{(N)}(T) \right| \approx e^{-\gamma_j(T)t} \left| c_{j0} + \frac{E_j(T)}{T} \epsilon_{ji}(T,t) \right|.
\]

(31)

where

\[
\epsilon_{ji}(T,t) = \frac{e^{i[\gamma_j(T)-\gamma_i(T)+i(\phi_j(T)-\phi_i(T))]}-1}{\gamma_j(T)-\gamma_i(T)+i(\phi_j(T)-\phi_i(T))}.
\]

(32)

for \( \omega_1 \neq \omega_2 \), and

\[
\epsilon_{ji}(T,t) = \frac{t}{E_j^{(N)}(T)},
\]

(33)

for \( \omega_1 = \omega_2 \). The phase factor appearing in Eq. (32) is given by

\[
\phi_{jj}(T) = \frac{1}{T} \int_0^\infty d\omega J(\omega) F_{jj}(\omega,T),
\]

while the effective decay rate in presence of measurements is

\[
\gamma_{jj}(T) = \frac{1}{T} \int_0^\infty d\omega J(\omega) F_{jj}(\omega,T)
\]

\[
= \frac{T}{2} \int_0^\infty d\omega J(\omega) \sin^2 \left[ \frac{(\omega_1 - \omega)T}{2} \right].
\]

(34)

The effective Zeno decay rate is the overlap integral of the measurements-induced atomic level broadening of width \( \nu = 1/T \), described by \( F_{jj}(\omega,T) \) and the environmental spectrum \( J(\omega) \). Depending on the form of the reservoir spectrum, the effective decay can be enhanced or inhibited, for short \( T \), with respect to the dynamics in absence of measurements, giving rise to the anti-Zeno or quantum Zeno effects, respectively [27].

In particular, for the Lorentzian spectrum considered here, and in the near-resonant regime, one can see from Eq. (31) that the effective decay rate decreases with decreasing \( T \), since \( T \ll \lambda^{-1}, \delta_j^{-1} \) [27]. In the far off-resonant regime, on the contrary, there exist values of \( T \) short enough to satisfy the conditions of validity of our theoretical description, but at the same time such that \( T \gg \delta_j^{-1} \). In this case, one can see from Eq. (31), that the effective decay rate increases with respect to the value in absence of measurements, and it keeps increasing for increasing values of \( T \). This measurements-induced enhancement of the decay is a signature of the anti-Zeno effect. Summarizing, the Zeno decay rate is reduced in the near-resonant regime, while in the off-resonant regime an enhancement of the decay rate can occur since generally \( T \ll \delta_j^{-1} \).

The behavior of the effective decay rate of atom \( j \), therefore, strongly depends on the position of its atomic frequency with respect to the peak of \( J(\omega) \). However, Eq. (31) tells us that the dynamics of \( \left| c_j^{(N)}(T) \right| \) is influenced also by the excitation transfer contribution \( E_{ji}(T) \) which in turn depends on the position of the Bohr frequency of the other atom \( i \) with respect to \( J(\omega) \). As we will see in the following, the excitation transfer contribution is responsible for the appearance of oscillations between Zeno and anti-Zeno behavior.

B. Zeno dynamics of quantum correlations

We now focus on the dynamics of quantum correlations for equal couplings between the two atoms and the reservoir, i.e., when \( r_1 = r_2 \), and in the two regimes \( \delta_1 = \delta_2 \) and \( \delta_1 \neq \delta_2 \), with \( \delta_1, \delta_2 \neq 0 \), for which, as reminded above, the free dynamics of quantum correlations shows a peculiar behavior [24].

The two cases present distinctive features. For \( \delta_1 = \delta_2 \) the free dynamics of the concurrence is strongly dependent on the initial condition. This is due to the existence of a subradiant state. Therefore, the time evolution of the two orthogonal maximally entangled states, \( s = 0 \) and \( \phi = \pi, 0 \), is very different. The \( \phi = \pi \) initial state is subradiant and does not evolve in time, while the \( \phi = 0 \) state is coupled to the cavity field and its initial entanglement, for short initial times, monotonically decays, as one can see from Fig.1 (a). However, in the presence of
measurements, this dependence on the initial state gets substantially washed out (see below).

For \(\delta_1 = -\delta_2\), on the contrary, the free long time dynamics is independent on the initial condition as discussed in detail in Ref. [20]. However short time oscillations in the concurrence occur, and these oscillations again depend on the initial state. In Fig. 1 (b) we compare the short time entanglement dynamics of the two maximally entangled states corresponding to \(s = 0\) and \(\phi = \pi, 0\). Note that, in both cases the initial entanglement loss is non-monotonic, contrarily to the case \(\delta_1 = \delta_2\) shown in Fig. 1 (a). These initial oscillations in entanglement give rise to interesting features in the time evolution of the quantum and classical correlations in presence of measurements. Indeed, the initial short-time differences gets amplified by the measurements, giving rise to the qualitatively different behaviors of Figs. (3) and (4), where both the quantum and classical correlations in presence of measurement are compared with their functions in presence of measurements. Indeed, the initial short-time differences gets amplified by the measurements, giving rise to the qualitatively different behaviors of Figs. 3 and 4, where both the quantum and classical correlations in presence of measurement are compared with their measurement-free counterparts. In these figures show the time evolution of both both types of correlation as a function of the measurement interval \(T\) in the bad cavity limit (we have chosen the ratio between Rabi frequency and cavity line-width to be \(R = R/\lambda = 0.1\), as the effects we want to emphasize are better displayed in such a case (see below for the details).

For \(\delta_1 = \delta_2\), an anti-Zeno effect appears for values of \(T\) larger than a characteristic threshold value \(T^*\) that depends on the detuning and on the reservoir width, as shown in Fig 2(a). In particular, for increasing values of the detuning, the Zeno region becomes smaller and smaller, and the protection against the decay occurs only for very short delay between measurements.

On the other hand, for symmetric detunings \(\delta_1 = -\delta_2\), the correlations in presence of measurements show oscillations as a function of the measurements time interval \(T\), so that quantum Zeno and anti-Zeno effects for the entanglement alternatively occur for increasing values of \(T\), as shown both in Fig. 3 and Fig. 4. These two figures differ only because a different choice has been made of the relative phase \(\phi\) of the two amplitudes of the initial state. This peculiar \(\phi\)-dependence is found to occur only for symmetric detunings, \(\delta_1 = -\delta_2\), and is reminiscent of the difference in the short time behaviors shown in Fig. 1.

To understand these features, we start by writing down the concurrence in presence of measurements, \(C_{E}^{(N)}(t)\), in the form

\[
C_{E}^{(N)}(t) \approx 2 e^{-\gamma_{11}(T)t} |c_{10}|^2 + 2 |c_{10}| |c_{20}| \frac{E_{12}(T)}{T} t \cos \theta_{12} \times e^{-\gamma_{22}(T)t} |c_{20}| + 2 |c_{21}(T,t)c_{10}| e^{|c_{21}(T,t)c_{10}|} (35)
\]

In the bad cavity limit, and for \(\delta_1 = \pm \delta_2\), this expression can be further simplified so that the concurrence (coinciding with the discord) is given by

\[
C_{E}^{(N)}(t) \approx 2 \left| c_{10} \right|^2 + 2 \left| c_{10} \right| \left| c_{20} \right| \left| \frac{E_{12}(T)}{T} \right| t \cos \theta_{12} \times e^{-\gamma_{22}(T)t} \left| c_{20} \right| + 2 \left| c_{21}(T,t)c_{10} \right| (36)
\]

where \(\theta_{12} \approx \arg \left( E_{12}(T)c_{20} \right)\).

The equations above explicitly show that the appearance of oscillations on the correlations dynamics is due to the interference between excitation-trapping and excitation-transfer contributions, which is mainly determined by the terms \(E_{ji}(T)\) of Eq (27). Indeed, it is the value of \(\theta_{12}\) which is responsible for the oscillations in Eq. (30), and this angle, in turn, is crucially determined by \(E_{12}(T)\) and by the relative phase \(\phi\) between the two amplitudes of the initial state.

When the two qubits have the same frequency (\(\delta_1 = \delta_2\)), the cosine is always positive, so that the interference term decreases without showing oscillations. This implies that also the dependence on the phase \(\phi\) becomes almost irrelevant. On the contrary, for qubits symmetrically detuned from the cavity mode, the sign of the cosine changes in time, giving rise to the observed oscillations between the Zeno and anti-Zeno regimes. Furthermore, since \(E_{ji}(T)\) is divided by the measuring interval \(T\), the oscillations get amplified for frequent measurements. These features mainly depend on the 'positions' of the Bohr frequencies of the two atoms with respect to the cavity spectrum, which enter \(E_{ji}(T)\) through the off-diagonal form factor \(F_{ji}(\omega, T)\), see Eq. (27).

Since \(E_{ji}(T)\) describes an excitation-transfer contribution during time \(0 \leq t \leq T\), its behavior is determined by the effective Rabi period of the excitation exchange between the two atoms. In the dispersive regime, this is of the order of the detuning and, thus, it becomes observable in the bad cavity limit, in which one can chose a \(T\) large enough to have \(\delta^{-1} < T\), with a detuning larger than the coupling strength with the cavity.

C. Zeno dynamics of classical correlations

As one can see from Figs. 2, 3 and 4 classical correlations are affected by the measurements in essentially the same way as quantum correlations are.

In particular, \(C_c(t)\) decays in time due to the atomic relaxation processes, but frequent enough measurement can help it to survive. However, surprisingly enough, even the classical correlation can display the anti-Zeno effect and, even more surprisingly, also the oscillations between the Zeno and anti-Zeno regimes. Furthermore, when the detunings satisfy the condition \(\delta_1 = -\delta_2\), \(C_c(t)\) too shows a strong dependence on the relative phase \(\phi\) present in the initial state, which is completely attributable to the measurement as the measurement-free evolution of the classical correlations does not show any dependence on \(\phi\).
VI. CONCLUSIONS

In summary, we have analyzed the dynamics of a couple of two-level-atoms decaying in an electromagnetic resonator having a finite quality factor and discussed, in particular, the behavior of both quantum and classical correlations between the two atoms, under the effect of a series of projective measurements performed, e.g., on the cavity field. When the atoms are resonantly coupled to the cavity, a quantum Zeno effect on the entanglement occurs, but for off-resonant atoms, the anti-Zeno effect is obtained, instead. Furthermore, there is a particularly distinguished regime in which a series of oscillations occur between the Zeno and anti-Zeno effects, as a function of the time delay between successive measurements. We have investigated this behavior in details, and showed a sensitivity of the coarse grained dynamics effectively induced by the measurements to the relative phase of the initial state. This occurs, in particular, if the atoms interact dispersively with the electromagnetic field, having transition frequencies which are symmetrically displaced with respect to that of the main cavity mode. In this regime, an analogous behavior is obtained for the classical correlations between the atoms, which are affected by the measurements in a qualitatively very similar way.

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FIG. 4: (Color online) Same as Fig. 3 but with a different relative phase between the two amplitudes of the initial states. In this case, we have chosen $\phi = \pi$.

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