A better lower bound for Lower-Left Anchored Rectangle Packing

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Abstract

Given any set of points \( S \) in the unit square that contains the origin, does a set of axis aligned rectangles, one for each point in \( S \), exist, such that each of them has a point in \( S \) as its lower-left corner, they are pairwise interior disjoint, and the total area that they cover is at least \( 1/2 \)? This question is also known as Freedman's conjecture (conjecturing that such a set of rectangles does exist) and has been open since Allen Freedman posed it in 1969. In this paper, we improve the best known lower bound on the total area that can be covered from 0.09121 to 0.1039. Although this step is small, we introduce new insights that push the limits of this analysis.

Our lower bound uses a greedy algorithm with a particular order of the points in \( S \). Therefore, it also implies that this greedy algorithm achieves an approximation ratio of 0.1039. We complement the result with an upper bound of \( 3/4 \) on the approximation ratio for a natural class of greedy algorithms that includes the one that achieves the lower bound.

1 Introduction

Consider the following packing problem that is best described by a two-player game between Alice and Bob. Alice is given the unit square \( U = [0, 1]^2 \) and may select any set of points \( S \subset U \) that includes the origin. After she selects this set, Bob selects a set of axis-parallel rectangles contained in the unit square, such that they are interior non-overlapping; none of them contains a point in \( S \) in their interior; and each has a point of \( S \) as their lower-left corner. We refer to such rectangles as lower-left-anchored axis-aligned empty rectangles. Bob's goal is to maximize the total area covered by his set of rectangles and Alice's goal is to minimize that same area.

In 1969, Allen Freedman [17] conjectured that no matter what set of points Alice chooses, Bob can always find a set of rectangles that satisfies the conditions and covers an area of at least half of the unit square. This conjecture remains open and it was even only in 2012 that Dumitrescu and Tóth [10] published the first result that shows that Bob can always cover any constant size area at all. Since then, the question of packing anchored rectangles in the unit-square has attracted more attention, yet, so far, no one has improved the lower bound on what Bob can achieve.

In this paper, we improve the currently best lower bound by more than 10%. Our analysis pushes the boundary of what is possible with an analysis based on the TilePacking algorithm that was suggested by Dumitrescu and Tóth [10]. We make several key improvements, which allow for the better analysis. Our new lower bound is larger than 0.1039, which we believe pushes this analysis to its limits.

The TilePacking algorithm was designed by Dumitrescu and Tóth in such a way that the GreedyPacking algorithm that treats the points in the same order covers an area that is at least as large. Therefore, the lower bound implies that the GreedyPacking algorithm covers at least a 0.1039-fraction of what Bob's optimal solution would cover. Thus implying that the GreedyPacking algorithm has a worst-case approximation guarantee for the optimization problem that Bob faces of at least 0.1039. In Section 4, we show, for a natural set of orders in which the GreedyPacking algorithm can treat the points in \( S \), that the GreedyPacking algorithm cannot have a worst-case approximation guarantee that is better than 3/4.

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1.1 Related work

The exact history of Freedman’s Conjecture is not clear. While we know that Allen Freedman posed the question in 1969 [17], it remains ambiguous if he conjectured positively or negatively about the answer. Since that time, the problem has resurfaced at least twice [12, 13] before Dumitrescu and Tóth booked the first real progress on it, showing a lower bound of 0.09121 [10]. They also show that for any set \( S \) an ordering exists such that the GREEDYPACKING algorithm that considers the points in that ordering gives the optimal solution and that there exist orderings for which the GREEDYPACKING algorithm cannot obtain any constant covered area for some sets \( S \).

From an approximation point of view, not much is known for the problem. We do not know if it is NP-hard to decide if Bob can cover at least a certain area for a given set of points, nor is there any approximation algorithm known beyond the one implied by the lower bound. For variants of the problem, however, much more is known. When, instead of arbitrary rectangles, Bob is restricted to using only squares, Balas er al. [3] show an approximation ratio of 1/3. They also show approximation guarantees for the setting where Bob may anchor rectangles at any corner, achieving a 7/12 – O(1/n)-approximation for rectangles and a 9/47-approximation for squares in that setting, as well as a QPTAS and a PTAS for rectangles and squares, respectively. Akitaya et al. [3] prove that the latter variant for squares is in fact NP-hard. Moreover, they show that for any instance with finite \( S \), the union of all feasible anchored square packings (which they call the reach) covers at least an area of 1/2. Biedl at al. [9] give a exact algorithms for rectangles or squares anchored at any corner when the points in \( S \) lie on the boundary of \( U \). Antoniadis et al. [4] introduce center anchored rectangle packing (CARP), where the point has to anchor to the center of a rectangle and the generalized \((\alpha, \beta)\)-anchored rectangle packing, where the point has to anchor to the rectangle at the relative \((\alpha, \beta)\)-point. They show that CARP is NP-hard and they show a PTAS for \((\alpha, \beta)\)-anchored rectangle packing.

From a more general point of view, Bob’s Problem is among many different problems that consider packing axis-aligned and interior-disjoint rectangles in a rectangular container. Some of the better-known problems in this category are 2D-knapsack and strip packing [1, 6], and the problem of finding a maximum-area independent set of given rectangles [2, 7, 8]. Radó and Rado also formulated a whole range of similar problems [13, 14, 15, 10].

2 Preliminaries

Consider a set \( R \) of interior-disjoint axis-aligned rectangles in the unit square \( U = [0, 1]^2 \) and a set \( S \subset U \) that contains the origin. For each point \( p \in S \) let \( x(p) \) and \( y(p) \) denote \( p \)'s \( x \)- and \( y \)-coordinate, respectively. A rectangle is lower-left anchored at a point \( p \) if \( p \) is the lower-left corner of that rectangle. We say that \( R \) is a lower-left anchored rectangle packing (LLARP) of \( S \) if each rectangle in \( R \) is an anchored rectangle, none of the rectangles contains any point in \( S \) in its interior, and there is one anchored rectangle[1] for each point.

For any shape or set of shapes, \( P \), we denote by \( \text{area}(P) \) the total area of that shape or the total area of the union of the set of shapes. We can now define Bob’s Problem as follows.

**Definition 2.1 (Bob’s Problem).** Given a set of \( n \) points \( S \subset U \) that contains the origin, find a lower-left anchored rectangle packing \( R \) that maximizes \( \text{area}(R) \).

Next we define two related algorithms, the GREEDYPACKING algorithm and the TILEPACKING algorithm. Both algorithms can be defined for any order of the points in \( S \). For this paper, we consider both only for an order \( \succ \), such that \( p \succ q \) if \( x(p) + y(p) > x(q) + y(q) \). That is, points are ordered increasingly according to the sum of their coordinates and ties are broken arbitrarily but consistently.

The GREEDYPACKING algorithm finds an anchored rectangle packing as follows. Iterate over all points in \( S \) in order of \( \succ \). Initiate \( R = \emptyset \). Consider the current point and find a maximum area rectangle anchored at that point, that is interior-disjoint with all rectangles in \( R \) and all points in \( S \). Add this rectangle to \( R \). After the last iteration, return \( R \).

The TIEPACKING algorithm first partitions the unit square into tiles, one for each point in \( S \) and then returns a set of rectangles with one largest-area rectangle within each tile. From the order \( \succ \), we obtain the

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1Here, we consider a single point or a line as a degenerate rectangle with zero area.
tiling as follows. Iterate over all points in $S$ in order of $\succ$. From the current point cast two axis-aligned rays, one upwards and one to the right until they hit either the boundary of the unit square or one of the rays of a previously considered point. See also Figure 1. Finally, from each tile, add a largest-area rectangle to $R$. Note that any rectangle with maximum area in a tile corresponding to a point $p$ has $p$ as its lower-left corner.

The tiling of the TilePacking algorithm results in tiles that are simple rectilinear polygons. Each concave corner of a tile corresponds to another point in $S$. Intuitively, the tiling is such that the tile of a point $p \in S$ contains all the area of the unit square that can never be covered by a rectangle anchored in a point $p' \in S$ such that $p' \succ p$. From this intuition it is immediately clear that the GreedyPacking algorithm always covers at least as much as what the TilePacking algorithm does. This is formalized by Lemma 2.2.

Lemma 2.2 ([10], Lemma 2.1). For each point $p \in S$, the GreedyPacking algorithm adds a rectangle anchored at $p$ with area at least as large as the rectangle anchored at $p$ that the TilePacking algorithm adds.

3 An improved lower bound for Bob’s Problem

In this section, we prove an improved lower bound of 0.10390 on the area that Bob can cover for any set of points $S$ that includes the origin by using the GreedyPacking algorithm. Our proof follows the one by Dumitrescu and Tóth [10]. Like they do, we prove that the TilePacking algorithm finds a solution that covers at least a constant fraction of the unit square. The advantage of considering the TilePacking algorithm over theGreedyPacking algorithm is that, while the GreedyPacking algorithm covers at least as much as the TilePacking algorithm, the TilePacking algorithm can be easily analyzed locally, per tile. For each tile, we can simply look at its largest contained rectangle. When this rectangle covers less than a $1/\beta$ fraction of the tile we call the tile a $\beta$-tile. We first bound the area of a $\beta$-tile in terms of the area of two specially defined parallelograms (Lemma 3.2). Then, we bound the total area of the parallelograms through a charging scheme (Lemma 3.3 and 3.5). This implies a bound on the total area of the $\beta$-tiles. The remaining area of the unit square must therefore consist of tiles which have a largest contained rectangle that has an area of at least a $1/\beta$ fraction of the area of the tile. Naturally, the set of $\beta'$-tiles includes the set of $\beta$-tiles for $\beta' \leq \beta$ and our bound on the total area of $\beta$-tiles is decreasing in $\beta$. We optimize the analysis by integrating over all possible $\beta$ that contribute positively to the area.

Our analysis improves on that of Dumitrescu and Tóth [11] in three ways. We introduce a variable $\alpha$ that varies the definition of the tips of $\beta$-tiles (to be defined later), we bound the area of such tiles by parallelograms instead of trapezoids, and we bound all $\beta$-tiles at once with the use of the inequality of arithmetic and geometric means.

We name several parts of the tiles that the TilePacking algorithm uses. We denote the tiles by $t_1, \ldots, t_n$, one for each point in $S$. A tile $t_i$ is called a $\beta$-tile if the largest possible anchored rectangle contained in $t_i$ has an area of less than $\frac{1}{\beta}$ area($t_i$) for some fixed constants $\alpha$ and $\beta \geq 3 + 2\alpha$. Let the right tip of $t_i$ be the...
smallest sub-polygon of $t_i$ that contains all points of $t_i$ to the right of a vertical line through a concave corner of $t_i$ and has an area of at least $\alpha \beta \text{area}(t_i)$. Similarly, let the upper tip of $t_i$ be the smallest sub-polygon of $t_i$ that contains all points of $t_i$ above a horizontal line through a convex corner of $t_i$ and has an area of at least $\alpha \beta \text{area}(t_i)$. We refer to the polygon that consists of all points in $t_i$ that are not part of its upper and right tip as the main body, $t'_i$. By $a_i$ and $b_i$ we denote the bottom and left edge of $t_i$, respectively. Similarly, we denote by $a'_i$ the bottom edge of the main body $t'_i$ and by $b'_i$ the left edge of $t'_i$. Figure 2 depicts a $\beta$-tile and the mentioned parts.

Next, we define a number of support shapes that let us bound the area of a $\beta$-tile. Let $\Delta_i$ be the isosceles right triangle with $a_i$ as its top edge and its $90^\circ$ angle at the bottom right point of $t_i$. Similarly, let $\Gamma_i$ be the isosceles right triangle with $b_i$ as its right edge and its $90^\circ$ angle at the top left point of $t_i$. Let $\lambda$ be a constant such that $\lambda \in (0, \alpha)$ and let $A_i$ be the parallelogram with base $a'_i$ and height $\lambda |a'_i|$ and two sides parallel to the diagonal edge of $\Delta_i$. Similarly, let $B_i$ be the parallelogram with base $b'_i$, two sides parallel to the diagonal edge of $\Gamma_i$, and height equal to $\lambda |b'_i|$. Figure 2 also depicts these support shapes.

Finally, we state a reformulation of a lemma from [10] that we need in the proof of Lemma 3.2.

**Lemma 3.1 (Lemma 3.1 in [10]).** Let $t$ be a $\beta$-tile with height $h$ and width $w$, then

$$\text{area}(t) < \frac{\beta}{e^{\beta-1}}hw.$$  

**Lemma 3.2.** The area of the $\beta$-tile $t_i$ is at most $\frac{\beta}{2e^{\beta-1}} (\text{area}(A_i) + \text{area}(B_i))$.

**Proof.** Note first that the area of the right tip is less than $\frac{1+\alpha}{\beta} \text{area}(t_i)$. If its area was larger, we could move its left boundary to the next concave corner (one 'step' on the staircase) to the right, and, since the difference is a rectangle within $t_i$, the area would be reduced by less than $\frac{1}{\beta} \text{area}(t_i)$. Therefore, the remaining area would be greater than $\frac{2}{\beta} \text{area}(t_i)$, which contradicts that the tip is the smallest such polygon. Similarly, the area of the upper tip is less than $\frac{1+\alpha}{\beta} \text{area}(t_i)$. Therefore, the area of the main body, $t'_i$, is at least $\frac{\beta - 2 - 2\alpha}{\beta} \text{area}(t_i)$, which is at least $\frac{1}{\beta} \text{area}(t_i)$, as $\beta \geq 3 + 2\alpha$. This implies that the right and upper tip do not overlap, since if they did, the main body would be a rectangle and cannot have an area of at least $\frac{1}{\beta} \text{area}(t_i)$. Furthermore, the area of the largest rectangle inside $t'_i$ has area less than

$$\frac{1}{\beta} \text{area}(t_i) \leq \frac{1}{\beta} \frac{\beta}{\beta - 2 - 2\alpha} \text{area}(t'_i) = \frac{1}{\beta - 2 - 2\alpha} \text{area}(t'_i).$$
Thus \( t'_i \) is a \( \frac{1}{\beta^2 - 2\alpha} \)-tile and from Lemma 3.1 we get that area\( (t'_i) < \frac{\beta - 2 - 2\alpha}{\beta - 2 - 2\alpha} |a'_i||b'_i| \).

Since, area\( (A_i) = \lambda |a'_i|^2 \), area\( (B_i) = \lambda |b'_i|^2 \) and area\( (t'_i) > \frac{\beta - 2 - 2\alpha}{\beta} \) area\( (t_i) \), we get

\[
\text{area}(t_i) \leq \frac{\beta}{\beta - 2 - 2\alpha} \text{area}(t'_i)
< \frac{\beta}{\beta - 2 - 2\alpha} \frac{\beta - 2 - 2\alpha}{\beta - 2 - 2\alpha} |a'_i||b'_i|
\leq \frac{\beta}{\beta - 2 - 2\alpha} |a'_i|^2 + |b'_i|^2
= \frac{\beta}{2\lambda \beta - 3 - 2\alpha} (\text{area}(A_i) + \text{area}(B_i)) ,
\]

where the last inequality is due to the arithmetic mean-geometric mean inequality. \( \square \)

Next, we bound the total area of all parallelograms \( A_i \) of \( \beta \)-tiles. To do so, we define a directed graph \( G \), where the nodes correspond to the \( A_i \) parallelograms. If a parallelogram \( A_i \) intersects some other parallelograms, then \( A_i \) gets exactly one outgoing edge to the node of the parallelogram \( A_j \) that intersects \( A_i \) and of which the corresponding line segment \( a'_j \) is the highest below \( a'_i \). Here, we assume that all points have distinct \( y \)-coordinates. If this is not the case, we introduce an ordering of the points \( \succ \) such that \( p \succ q \) if \( y(p) > y(q) \) and ties are broken arbitrarily, but consistently. Then, \( A_i \) gets exactly one outgoing edge to the node of the parallelogram \( A_j \) that intersects \( A_i \) and is first according to \( \succ \). Parallelograms that do not have an outgoing edge are at level 1. All other parallelograms are at level \( k+1 \) if they have an outgoing edge to a level \( k \) parallelogram. This constructs an acyclic graph.

We use a charging scheme for the parallelograms, where the area of each parallelogram \( A_i \) at level 2 and higher is charged to the unique parallelogram at level 1 that \( A_i \) has a directed path to. Before that, we first derive an upper bound on the total area of the level 1 parallelograms:

**Lemma 3.3.** The total area of all \( A_i \)-parallelograms at level 1 is at most \( \frac{2(1+\alpha)^2 + \lambda (2+\alpha)}{2(1+\alpha)^2} \).

**Proof.** We show that all level 1 \( A_i \)-parallelograms lie in the hexagon with corner points

\[
\{(0,0), (0,1), (1,1), (1,0), (\frac{\lambda + 1}{1+\alpha}, 1+\alpha), (\frac{\lambda}{1+\alpha}, 1), (\frac{-\lambda}{1+\alpha}, 1), (\frac{-\lambda}{1+\alpha}, 0)\}
\]

(see right half of Figure 3). This hexagon consists of the unit square and a trapezoid with height \( \frac{\lambda}{1+\alpha} \) and base lengths 1 and \( \frac{1}{1+\alpha} \), therefore it has an area of

\[
1 + \frac{1}{2} \frac{1}{1+\alpha} \left( 1 + \frac{1}{1+\alpha} \right) = \frac{2(1+\alpha)^2 + \lambda (2+\alpha)}{2(1+\alpha)^2} .
\]

Clearly, no parallelogram crosses the top or left boundary of \( U \). The largest rectangle with \( a'_i \) as its bottom edge has area at most \( \frac{1}{\beta} \) area\( (t_i) \). Moreover, since \( t'_i \) is higher than the right tip of \( t_i \), this rectangle also has a larger height than the right tip, which has area at least \( \frac{2}{\beta} \) area\( (t_i) \) (see Figure 3). The area of the right tip of \( t_i \) is at most its height times its width, \( |a_i| - |a'_i| \). Therefore, \( \alpha |a'_i| < |a_i| - |a'_i| \) and \( (1+\alpha)|a'_i| < |a_i| \). Since \( \lambda < \alpha \) and since the bottom-right corner of \( A_i \) is exactly \( \lambda |a'_i| \) to the right of \( a'_i \), the bottom-right corner of \( A_i \) has \( y \)-coordinate at most

\[
|a'_i| + \lambda |a'_i| < (1 + \alpha)|a'_i| < |a_i|
\]

larger than the \( y \)-coordinate of \( p_i \). Therefore, \( A_i \) lies completely inside \( \Delta_i \) and no parallelogram crosses the right boundary of \( U \).

Since the bottom-right corner of \( A_i \) is \( \lambda |a'_i| \) below \( a'_i \), the line through the bottom-right corner of \( A_i \) and the point at the right end of \( a_i \) has a slope of at most \( \frac{\alpha}{\lambda} \) (see the left half of Figure 3). Since the point at the right end of \( a_i \) lies inside \( U \), the parallelograms must therefore lie above the line \( y = \frac{\lambda}{\alpha} (x - 1) \).

No parallelogram crosses the line \( y = -x \), as the parallelograms have an angle of 45°. Since \( |a_i| \leq 1 \), we have \( |a'_i| < \frac{1}{1+\alpha} \), so the height of all parallelograms is less than \( \frac{\lambda}{1+\alpha} \). Therefore, all parallelograms are above the line \( y = -\frac{1}{1+\alpha} x \) and they all lie in the hexagon. Since, by construction, no two level 1 parallelograms overlap their total area is bounded by \( \frac{2(1+\alpha)^2 + \lambda (2+\alpha)}{2(1+\alpha)^2} \). \( \square \)
Now we derive an upper bound on the area of the parallelograms of level 2 and higher. We do this by charging their area to the level 1 parallelograms. In the proof, we need the following lemma from [10].

**Lemma 3.4** (Lemma 3.3 in [10]). For every \( i \in \{1, \ldots, n\} \), the interior of \( \Delta_i \) (resp., \( \Gamma_i \)) is disjoint from \( S \).

**Lemma 3.5.** For every parallelogram \( A_j \) at level 1, the total area of all parallelograms \( A_i \), with \( i \neq j \), for which there is a directed paths in \( G \) to \( A_j \), is at most \( \frac{1}{2(\alpha-\beta)} \text{area}(A_j) \).

**Proof.** For all \( i \), denote by \( l_i \) the line through \( a_i \). We consider a parallelogram \( A_i \) that intersects a parallelogram \( A_j \) where \( a_j' \) lies above \( a_j' \). That is, there is an edge from \( A_i \) to \( A_j \) in the graph \( G \). Consider the intersection of \( \Delta_i \) with \( l_i \) (see Figure 3). If the intersection of \( A_i \) with \( l_i \) contains any point of \( U \) left of \( a_j' \), the point \( p_j \) lies inside \( A_i \) and hence inside \( \Delta_i \). If it contains any point to the right of \( a_j' \), then the top left corner of the right tip of \( t_j \), which is a point in \( S \), lies inside \( \Delta_i \). By Lemma 3.4 we know that the intersection of \( S \) and the interior of \( \Delta_i \) is empty. Therefore, the intersection of \( \Delta_i \) with \( l_i \) must be a subset of \( a_j' \).

Define the triangle \( C_j \) as the triangle that is bounded by \( a_j' \), by the line with slope \(-1\) through \( p_j \), and the line with slope \(-\frac{1}{\alpha}\) through the point at the right end of \( a_j' \). We show that \( A_i \) lies completely inside \( C_j \). See the right half of Figure 3. Since the intersection of \( \Delta_i \) with \( l_i \) is a subset of \( a_j \), which is one side of \( A_j \), the parallelogram \( A_i \) is not larger than \( A_j \) and the part of \( A_i \) that lies below \( l_i \) is contained in \( A_j \). Thus, it suffices to show that \( A_i \) lies to the right of the line with slope \(-1\) through \( p_j \) and under the line with slope \(-\frac{1}{\alpha}\) through the point at the right end of \( a_j' \).

Again, since the intersection of \( \Delta_i \) with \( l_j \) is a subset of \( a_j' \), the parallelogram \( A_i \) must lie to the right of the line with slope \(-1\) through \( p_j \).

Since \((1+\alpha)|a_j'|<|a_i|\), the right edge of \( \Delta_i \) (and also the right end point of \( a_j' \)) lies at least \( \alpha|a_i'| \) to the right of the right end point of \( a_j' \). Since \( A_i \) has height \( \lambda|a_i'| \), the segment \( a_i' \) lies at most \( \lambda|a_i'| \) above \( a_j' \). Thus, the right end of \( a_i' \) lies below the line with slope \(-\frac{1}{\alpha}\) through the point at the right end of \( a_j' \). Thus,

\[
\text{if } A_i \text{ intersects } A_j \text{ and } a_i' \text{ lies above } a_j', \text{ then } A_i \text{ lies inside } C_j \cup A_j. 
\]

Now, let \( \Xi_i \) be the strip that is bounded by the two lines tangent to the left and right edge of \( A_i \) (see Figure 4). We prove by induction that the strips of parallelograms at the same level that have a directed path to the same parallelogram \( A_j \) are interior-disjoint.
As induction basis, we see that clearly the parallelograms at level 1 have interior-disjoint $\Xi_i$, as there is only one parallelogram.

Now, suppose as induction hypothesis that the strips of the parallelograms at level $k$ that have a directed path to the same level 1 parallelogram are all interior-disjoint. Since $C_j$ lies in the strip $\Xi_j$, we have from (i) that $\Xi_i \subseteq \Xi_j$ if there is an edge from $A_i$ to $A_j$ in $G$. The parallelograms at level $k + 1$ each have an outgoing edge to a level $k$ parallelogram, so it follows from the induction hypothesis that the strips $\Xi_i$ are interior-disjoint if they have outgoing edges to different level $k$ parallelograms.

Now consider two level $k + 1$ parallelograms $A_{i_1}$ and $A_{i_2}$ with outgoing edges to the same parallelogram $A_j$. Suppose that $\Xi_{i_1}$ and $\Xi_{i_2}$ are not interior-disjoint.

If $A_{i_1}$ and $A_{i_2}$ intersect, w.l.o.g., $a_{i_1}$ lies higher than $a_{i_2}$, but then $A_{i_1}$ has an outgoing edge to $A_{i_2}$ and not to $A_j$, which is a contradiction. See Figure 5.

If $A_{i_1}$ and $A_{i_2}$ do not intersect, then one parallelogram must lie completely above the other. W.l.o.g., let $A_{i_1}$ lie above $A_{i_2}$. However, then $A_{i_1}$ cannot intersect $A_j$, as its bottom edge lies above $a'_{i_2}$, which lies above the top edge of $A_j$ since $A_{i_2}$ has an edge to $A_j$. Again, a contradiction.

Thus, all parallelograms at level $k + 1$ that have directed paths to the same level 1 parallelogram have interior-disjoint $\Xi_i$.

Now, for a parallelogram $A_j$ at level 1, we apply a translation on all parallelograms $A_i$ with a directed path to $A_j$ in $G$ as follows. We translate all parallelograms per level increasing in level. When level $k$ is translated, translate all parallelograms one by one, parallel to the line $y = -x$ upwards, and translate each parallelogram exactly far enough so that their interior does not overlap with any parallelogram at a lower level. Since they also do not intersect with parallelograms in their own level, no two parallelograms overlap at the end of the translation.

We prove that after this transformation all parallelograms lie inside $C_j$. Again, we prove this by induction on the levels.

For the induction basis, no two parallelograms at level 2 can overlap after the translation. Now, since after the translation level 2 parallelograms still intersect $A_j$ with their boundary, by [1], they lie inside $C_j$.

Now, suppose as induction hypothesis that all translated parallelograms fit inside $C_j$ after level $k$ is translated. Then, any parallelogram $A_i$ at level $k + 1$ has an edge to a level $k$ parallelogram, say to $A_{i_0}$. Again by [1], after the translation $A_i$ fits inside the translated version of $C_{i_0}$. This triangle $C_{i_0}$ fits inside $C_j$, since its bottom edge lies inside $C_j$ and it is similar to $C_j$. Therefore, the parallelograms at level $k + 1$ also fit inside $C_j$.

The base of $C_j$ is $a'_{j}$ and its height is $\frac{\lambda}{\alpha - \chi}|a'_{j}|$. Therefore the area of all parallelograms with a directed
Figure 5: The strips $\Xi_{i_1}$, $\Xi_{i_2}$ and $\Xi_j$. If the strips $\Xi_{i_1}$ and $\Xi_{i_2}$ overlap, parallelograms $A_{i_1}$ and $A_{i_2}$ overlap, and if $a_{i_1}$ is above $a_{i_2}$, there cannot be an edge from $A_{i_1}$ to $A_j$.

path to $A_j$ is bounded by

$$\frac{1}{2} \frac{\lambda}{\alpha - \lambda} |a_j'|^2 = \frac{1}{2(\alpha - \lambda)} \text{area}(A_j).$$

Combining Lemmata 3.3 and 3.5 we get an upper bound on the area of the parallelograms $A_i$. By symmetry, the same bound holds for the parallelograms $B_i$. Combining these bounds with Lemma 3.2, we get

$$\sum_{t_i \text{ is a } \beta\text{-tile}} \text{area}(t_i) \leq \sum_{t_i \text{ is a } \beta\text{-tile}} \frac{\beta}{2\lambda e^{\beta - 3 - 2\alpha}} \left( \text{area}(A_i) + \text{area}(B_i) \right)$$

$$\leq \frac{\beta}{2\lambda e^{\beta - 3 - 2\alpha}} \left( \frac{1}{2(\alpha - \lambda)} \right) \left( \sum_{A_i \text{ level 1}} \text{area}(A_i) + \sum_{B_i \text{ level 1}} \text{area}(B_i) \right)$$

$$\leq \frac{\beta}{2\lambda e^{\beta - 3 - 2\alpha}} \frac{1 + 2\alpha - 2\lambda}{2\alpha - 2\lambda} \cdot 2 \cdot \frac{2(1 + \alpha)^2 + \lambda(2 + \alpha)}{2(1 + \alpha)^2}.$$

Thus, we define

$$F(\beta, \lambda, \alpha) = \frac{(1 + 2\alpha - 2\lambda)(2(1 + \alpha)^2 + \lambda(2 + \alpha))e^{3 + 2\alpha} \beta}{4\lambda(\alpha - \lambda)(1 + \alpha)^2},$$

which bounds the total area of $\beta$-tiles from above. At this point, we can bound the area that the TilePacking algorithm covers from below by realizing that all tiles that are not $\beta$-tiles contain rectangles that cover at least $1/\beta$ of their total area. Therefore, we can bound the total area that the TilePacking algorithm covers by

$$\text{area}(R) \geq \frac{1}{\beta} (1 - F(\beta, \lambda, \alpha)) = \frac{1}{\beta} - \frac{(1 + 2\alpha - 2\lambda)(2(1 + \alpha)^2 + \lambda(2 + \alpha))e^{3 + 2\alpha}}{4\lambda(\alpha - \lambda)(1 + \alpha)^2} \frac{1}{e^\beta}.$$

Optimization of the parameters through numerical methods\(^2\) has provided us with the parameter values $(\beta, \lambda, \alpha) = (11.31, 0.7696, 0.4456)$. With which we obtain a bound on the total area covered of 0.0806. This bound can be further improved by considering $\beta$ as a continuous variable and integrating over all values of $\beta$ that contribute positively to the bound. This method is explained by Dumitrescu and Tóth [10]. This provides a bound of

$$\text{area}(R) \geq \frac{1}{\beta_0} - \frac{(1 + 2\alpha - 2\lambda)(2(1 + \alpha)^2 + \lambda(2 + \alpha))e^{3 + 2\alpha}}{4\lambda(\alpha - \lambda)(1 + \alpha)^2} \int_{\beta_0}^{\infty} \frac{1}{\beta e^\beta} \, d\beta.$$

\(^2\)See Appendix B.1.
Figure 6: Left: the set $S_{1,\varepsilon}$, with the GreedyPacking LLARP with $\succ_g (r((0,0)))$ is not drawn. Right: set $S_{2,\varepsilon}$. The choice of anchored rectangles is independent for the two ‘L-shapes’, which are separated by the red lines.

We have included a proof of this bound in Appendix A.

Again, by using numerical optimization methods\textsuperscript{3} we found that the parameter values $(\beta_0, \lambda, \alpha) = (8.6142, 0.44581, 0.76975)$ yield area$(R) \geq 0.10390$.

**Theorem 3.6.** The GreedyPacking algorithm yields an anchored rectangle packing with an area of at least $0.10390$.

4 The GreedyPacking algorithm as an approximation algorithm

In this section, we treat the GreedyPacking algorithm as an approximation algorithm for Bob’s Problem. As such, we aim to bound the approximation ratio of the algorithm, the worst-case ratio over all possible instances of the area covered by a solution that the algorithm produces and the area covered by an optimal solution. Since no optimal solution can ever cover more than the whole unit square, Theorem 3.6 implies that the approximation ratio of the GreedyPacking algorithm is at least 0.10390. In this section, we prove that the approximation ratio of the GreedyPacking algorithm for a large group of orderings is at most $\frac{3}{4}$.

Before we formally state the theorem for this section, we need some definitions. We say that a function $g : U \rightarrow \mathbb{R}$ is symmetric if $g((x,y)) = g((y,x))$ for all $(x,y) \in U$ and we say that $g$ is strictly increasing in $x$, respectively $y$, if $g((x,y)) > g((x',y))$ if $x > x'$, respectively if $g((x,y)) > g((x,y'))$ if $y > y'$. For a given function $g : U \rightarrow \mathbb{R}$, we define the family of orderings $\succ_g$ as $p \succ_g q$ if $g(p) > g(q)$ for all $p, q \in U$. For two points $p$ and $q$ we say that $p$ dominates $q$ if $x(p) \geq x(q)$ and $y(p) \geq y(q)$ and at least one of the inequalities is strict. Note that the function that $\succ$ corresponds to ($g((x,y)) = x + y$) is both symmetric and strictly increasing in $x$ and $y$. Therefore, the following theorem bounds the approximation ratio of a family of GreedyPacking algorithms that include the one that we have discussed in the preceding sections.

**Theorem 4.1.** For any symmetric function $g$ that is strictly increasing in both $x$ and $y$, the worst case approximation ratio for the GreedyPacking algorithm with any ordering from $\succ_g$ is at most $\frac{3}{4}$.

**Proof.** We construct sets $S_{n,\varepsilon}$ for positive integers $n$ and small $\varepsilon > 0$. We show that, for these sets, as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$, the optimal covered area approaches 1 and the area cover by a GreedyPacking algorithm that satisfies the conditions approaches $\frac{3}{4}$.

\textsuperscript{3}See Appendix B.2
We define $S_{n, \varepsilon}$ recursively. The set $S_{n, \varepsilon}$ consists of $4n + 1$ points,

$$S_{n, \varepsilon} = \{(0,0), p_1, \ldots, p_n, q_1, \ldots, q_n, v_1, \ldots, v_n, w_1, \ldots, w_n\},$$

where the coordinates of each point (except the origin) depend on $\varepsilon$.

Before we define the recursive relation, we first look at $S_{1, \varepsilon}$. Let $p_1 = (0, \varepsilon - \varepsilon^3)$, let $q_1 = (\varepsilon, 0)$, let $v_1 = (2\varepsilon, 2\varepsilon)$, and let $w_1 = (2\varepsilon, \varepsilon)$. See also the left side of Figure 6. We see that $v_1$ dominates all other points and $w_1$ dominates all other points except $v_1$. Furthermore, the reflection of $p_1$ in the line $x = y$ is $p'_1 = (\varepsilon - \varepsilon^3, 0)$ and is dominated by $q_1$. Since $g$ is symmetric and strictly increasing, $q_1 \succ_g p_1$ and $q_1$ is treated before $p_1$ by the algorithm. Thus $v_1 \succ_g w_1 \succ_g q_1 \succ_g p_1 \succ_g (0,0)$. Denote the rectangle anchored at $p$ that the GreedyPacking algorithm adds by $r(p)$. Consider the options for $r(q_1)$, there are two. The rectangle that is bounded by $w_1$ and the right edge of $U$, and the one that is bounded by $w_1$ and the top edge of $U$. The former has an area of size

$$(1 - x(q_1)) (y(w_1) - y(q_1)) = (1 - \varepsilon) \varepsilon = \varepsilon - \varepsilon^2$$

and the latter has an area of size

$$(x(w_1) - x(q_1)) (1 - y(q_1)) = (2\varepsilon - \varepsilon) = \varepsilon.$$ 

So both have an area of $\varepsilon - O(\varepsilon^2)$, where we disregard larger powers of $\varepsilon$ in the big-O notation. Moreover, independent of the rectangle chosen for $q_1$, for $p_1$ the largest-area rectangle is bounded by the upper edge of $U$ and the left edge of $r(q_1)$ and has an area of size

$$(x(q_1) - x(p_1)) (1 - y(p_1)) = \varepsilon (1 - (\varepsilon - \varepsilon^3)) = \varepsilon - \varepsilon^2 + \varepsilon^4.$$ 

For $w_1$ the largest-area rectangle is bounded by $v_1$ and the right edge of $U$ and has an area of size

$$(1 - x(w_1)) (y(v_1) - y(w_1)) = (1 - 2\varepsilon) (2\varepsilon - \varepsilon) = \varepsilon - 2\varepsilon^2.$$ 

Again, both $r(p_1)$ and $r(w_1)$ have an area of $\varepsilon - O(\varepsilon^2)$. Moreover, the rectangle $r((0,0))$ has area $O(\varepsilon^2)$ and thus the total area covered by $r((0,0)), r(p_1), r(q_1)$, and $r(w_1)$ is $3\varepsilon - O(\varepsilon^2)$.

There exists a solution that covers $4\varepsilon - O(\varepsilon^2)$, namely by taking for $q_1$ the rectangle with upper-right point $(1, \varepsilon - \varepsilon^3)$ and for $p_1$ the rectangle with upper-right point $(2\varepsilon, 1)$. The resulting set of rectangles covers $4\varepsilon - O(\varepsilon^2)$.

By replacing $v_1$ by the same pattern in the square between $v_1$ and the edges of $U$, we obtain $S_{2, \varepsilon}$ and iteratively we obtain from $S_{k, \varepsilon}$, the set $S_{k+1, \varepsilon}$. Note that for any $S_{n, \varepsilon}$ the choices that we have for the rectangles in any two subsets $S_k, S_{k+1, \varepsilon}$, with $k \leq n - 1$, are independent. Thus, our analysis for $S_{1, \varepsilon}$ holds for all repeated patterns. See the right side of Figure 6.

Now consider $S_{n, \varepsilon}$, as $\varepsilon \to 0$ we know that the GreedyPacking algorithm covers at most $3/4$ of what an optimal solution covers in the area $U \setminus r(v_n)$. What remains is to show that as $n \to \infty$ we have $\text{area}(r(v_n)) \to 0$.

The area of $r(v_1)$ is clearly $(1 - 2\varepsilon)^2$. When we repeat the pattern for $r(v_n)$, we see that its area is $((1 - 2\varepsilon)^2)^2$. In general, we get that

$$\lim_{n \to \infty} \text{area}(r(v_n)) = \lim_{n \to \infty} (1 - 2\varepsilon)^2 = \lim_{n \to \infty} (1 - 2\varepsilon)^{2n} = 0,$$

which finishes our proof. \hfill \Box

5 Concluding remarks

When we compare our analysis for the lower bound to that of Dumițrescu and Tóth [10], we observe that our analysis allows us to bound the area for smaller beta ($\beta_0$ is that value for which $F(\beta, \lambda, \alpha)$ is equal to 1). Therefore, improving the analysis not only by having a better bound for each $\beta$, but also allowing us to integrate over slightly more values of $\beta$ in the final stage.
We believe that the most important take-away from our result is that, while we adapt the analysis in several ways that could be viewed as major, the improvement of the lower bound is minor. To us, this indicates that, indeed, as Dumitrescu and Tóth [10] have conjectured before us, significant improvements will not come from adapting this analysis, but rather from a different proof altogether. We hope that the insights that we give here can inspire someone to close the gap.

What can still be improved in the current analysis is the optimization over the variables $\lambda$ and $\alpha$. In particular, if an analytical optimum of $F(\beta, \lambda, \alpha)$ can be found for $\lambda$ and $\alpha$ for fixed $\beta$, these values can be used in the computation of (2). This should then result in a better bound since both $\lambda$ and $\alpha$ depend on $\beta$ within the integral.

With respect to an upper bound on the performance of the GREEDYPACKING algorithm, we now know have a first upper bound on the approximation ratio. Clearly, this does not tell us much about the original conjecture, since for the simple example that shows the absolute upper bound of $1/2$ (see, e.g., [10]) the optimal and GREEDYPACKING solutions coincide. In this respect, we still view the open question “is Bob’s problem NP-hard” as the most interesting one. Although it would be nice to see the approximation ratio gap closed, this may prove to be equally difficult as proving Freedman’s Conjecture.

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A Analysis of an improved lower bound through continuous $\beta$

In this section we show how to construct the lower bound on the total area covered by the TilePacking algorithm by letting $\beta$ run over all values that positively contribute to the bound. Intuitively, this means that $\beta$ runs from the value such that we can bound the total area of $\beta$-tile by less than one up to infinity. This analysis follows that of Section 3.3 in [10].

We showed that the total area of $\beta$-tiles is bounded by $F(\beta, \lambda, \alpha)$. Thus providing a lower bound on what the TilePacking algorithm can cover of

$$\frac{1 - F(\beta, \lambda, \alpha)}{\beta}. $$

If we consider, instead of one value, two different values for $\beta$, say $\beta_0$ and $\beta_1$, such that $\beta_0 > \beta_1$ and realize that the set of $\beta_1$-tiles contains all $\beta_0$ tiles as well, we get an improved lower bound of

$$\frac{1 - F(\beta_0, \lambda, \alpha)}{\beta_0} + \frac{F(\beta_0, \lambda, \alpha) - F(\beta_1, \lambda, \alpha)}{\beta_1}. $$

In the same fashion, we can improve the lower bound by considering more and more different values for $\beta$. For an arbitrary integer $k > 0$ and values $\beta_0, \ldots, \beta_k$, we obtain a lower bound of

$$\text{area}(R) \geq \frac{1 - F(\beta_0, \lambda, \alpha)}{\beta_0} + \sum_{i=1}^{k} \frac{F(\beta_{i-1}, \lambda, \alpha) - F(\beta_i, \lambda, \alpha)}{\beta_i}$$

$$= \frac{1 - F(\beta_0, \lambda, \alpha)}{\beta_0} - \sum_{i=1}^{k} \frac{1}{\beta_i} \frac{F(\beta_i, \lambda, \alpha) - F(\beta_{i-1}, \lambda, \alpha)}{\beta_i - \beta_{i-1}} (\beta_i - \beta_{i-1}).$$

In particular, we can set $\beta_i = \beta_{i-1} + \varepsilon$ to get

$$\text{area}(R) \geq \frac{1 - F(\beta_0, \lambda, \alpha)}{\beta_0} - \sum_{i=1}^{k} \frac{\varepsilon}{\beta_i} \frac{F(\beta_{i-1} + \varepsilon, \lambda, \alpha) - F(\beta_{i-1}, \lambda, \alpha)}{\varepsilon}.$$

When we now let $k \to \infty$ and $\varepsilon \to 0$, we get

$$\text{area}(R) \geq \frac{1 - F(\beta_0, \lambda, \alpha)}{\beta_0} - \int_{\beta_0}^{\infty} \frac{1}{\beta} \frac{\partial F(\beta, \lambda, \alpha)}{\partial \beta} \, d\beta.$$
Substituting (1), this is equal to
\[
\text{area}(R) \geq \frac{1}{\beta_0} - \frac{(1 + 2\alpha - 2\lambda)(2(1 + \alpha)^2 + \lambda(2 + \alpha)) e^{3+2\alpha}}{4\lambda(\alpha - \lambda)(1 + \alpha)^2}
\left( \frac{1}{e^{\beta_0}} - \int_{\beta_0}^{\infty} \left( \frac{1}{\beta} \frac{d\beta}{\beta} \right) e^{\beta} \right)
\]
\[
= \frac{1}{\beta_0} - \frac{(1 + 2\alpha - 2\lambda)(2(1 + \alpha)^2 + \lambda(2 + \alpha)) e^{3+2\alpha}}{4\lambda(\alpha - \lambda)(1 + \alpha)^2}
\left( \frac{1}{e^{\beta_0}} + \int_{\beta_0}^{\infty} \left( \frac{1}{\beta} - \frac{1}{e^{\beta}} \right) d\beta \right)
\]
\[
= \frac{1}{\beta_0} - \frac{(1 + 2\alpha - 2\lambda)(2(1 + \alpha)^2 + \lambda(2 + \alpha)) e^{3+2\alpha}}{4\lambda(\alpha - \lambda)(1 + \alpha)^2}
\int_{\beta_0}^{\infty} \frac{1}{\beta e^{\beta}} d\beta.
\]

B Matlab code

B.1 Code for 0.0806
\[
y = @(l,b,a)-(1./b-(1+2*a-2*l)./(2*a-2*l).*((2*(1+a).~2+1.*(2+a))./(1+a).~2./1.*exp(3+2*a-b).)/2);
yx = @(x) y(x(1), x(2), x(3));
A = [];
Aeq = [];
b = [];
Beq = [];
lb = [0.01 5 0.51];
ub = [0.5 100 1.5];
[x, fval] = fmincon(yx, [0.3, 6, 1], A, b, Aeq, Beq, lb, ub);
x
fval
\]

B.2 Code for 0.10390
\[
y = @(l,b,a)-(1./b-(1+2*a-2*l)./(2*a-2*l).*((2*(1+a).~2+1.*(2+a))./(1+a).~2./1.*exp(3+2*a)*expint(b).)/2);
yx = @(x) y(x(1), x(2), x(3));
A = [];
Aeq = [];
b = [];
Beq = [];
lb = [0.01 5 0.51];
ub = [0.5 100 1.5];
[x, fval] = fmincon(yx, [0.3, 6, 1], A, b, Aeq, Beq, lb, ub);
x
fval
\]