GENERALIZED BALLISTIC-CONDUCTIVE HEAT CONDUCTION IN ISOTROPIC MATERIALS

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ABSTRACT. The general isotropic constitutive equations of heat conduction with second sound and ballistic propagation in isotropic materials is given using Non-Equilibrium Thermodynamics with Internal Variables (NET-IV). The consequences of Onsager reciprocity between thermodynamic fluxes and forces and positive definiteness of the entropy production is considered. The relation to theories of Extended Thermodynamics is discussed in detail.

1. Introduction

There are several generalizations of classical Fourier heat conduction that can model second sound phenomena and ballistic propagation. These theories are more and more important in nanostructures and are subjects of various challenging physical, mathematical and numerical researches see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

Second sound, the wavelike propagation of heat, is due to the inertia of internal energy. This property can be modelled by a new nonequilibrium thermodynamic state variable. A straightforward choice for this additional vectorial state variable is the heat flux [18, 19]. This choice leads to theories of Extended Thermodynamics (ET). There one requires a compatibility with kinetic theory [20, 21, 22, 23, 24, 25], and the structure of the continuum theory will be compatible with the equations derived by moment series expansion of Boltzmann equation, considering also a Callaway collision integral with two relaxation times. This compatibility with kinetic theory is a necessity for any phenomenology: a universal macroscopic approach must be valid in case of various micro- and mesostructures, in particular must be compatible with the theory of rarefied gases.

The key of universality is to introduce only general requirements, minimal number of assumptions regarding the structure of the material. In particular on must use and exploit the second law of thermodynamics and introduce a proper functional characterisation of the deviation from local equilibrium. This can be accomplished most conveniently with the help of internal variables. One can achieve the compatibility with kinetic theory if the variables have the same tensorial order than the corresponding moments, therefore their tensorial order is increasing with every new variable. However, the evolution equations of these fields are direct consequences of the second law, one can get them solving the inequality of the entropy production. This way for heat conduction one obtains the Maxwell-Cattaneo-Vernotte equation as well as the Guyer-Krumhansl one with a single vectorial internal variable [26, 27]. With an additional tensorial variable a more general theory can be derived that properly describes ballistic propagation, the propagation of heat with the speed of sound, too [28]. This approach, Non-Equilibrium Thermodynamics with Internal Variables (NET-IV), can reproduce NaF experiments quantitatively, including the correct ballistic propagation speed [29, 30]. Also the universality of the derivation indicates a broader range of validity, far beyond the validity of rarefied real or phonon gases. This broadened range of validity is a prediction, therefore one can expect non Fourier heat conduction e.g. in heterogeneous materials, too. Really, Guyer-Krumhansl type heat conduction was observed in various heterogeneous materials with heat pulse experiments at room temperature [31, 32]. Internal variables are powerful modelling concepts in other continuum theories, too [33, 34, 35, 36]. Naturally, the relation of NET-IV with theories of ET, and kinetic theory, is not straightforward and their performance should be analysed considering the complete theory, not only heat conduction [37, 38, 39].

Up to now the solutions and analyses of wave-like and ballistic propagation are mostly restricted to one spatial dimension. This approach is problematic from the point of view of experimental observations, especially considering the NaF experiments [10, 41]. In the classical experiments the setup is not one dimensional, and this fact is not considered in the modelling calculations [29, 30]. The related ET theory inherits the dimensional reduction from the particular collision integrals, e.g. the deviatoric and spherical
contributions in the evolution equation of the heat flux have the same coefficient in the usual form of the Guyer-Krumhansl equation [22], and this is preserved in nonlinear theories, too [25].

In this paper we give the complete three dimensional form of the equations of a universal theory of heat conduction in isotropic materials, including the possible reciprocity relations and second law requirements for the transport coefficients. The paper is organised as follows. In the second section the theoretical framework is outlined and the basic balances and constitutive equations are given in a strictly linear anisotropic form. Then, the isotropic form of the equations is treated in general and then with Onsager reciprocity. In the fourth section the particular special theories are introduced. Then the conclusions are formulated. A detailed matrix form of the conductivity matrix is given in the Appendix, including the transformation of the sixth order tensor to a form suitable for the calculation of the positive definiteness of the coefficients.

2. BASIC EQUATIONS OF THE HEAT CONDUCTION WITH TWO INTERNAL VARIABLES

We consider the balance equations of a rigid heat conductor, i.e. the balance of internal energy and the balance of entropy:

\[
\rho \dot{e} + q_{i,i} = 0, \quad (1)
\]

\[
\rho \dot{s} + J_{i,i} = \sigma^{(s)}, \quad (2)
\]

Here \( \rho \) is the density, \( e \) is the specific internal energy, \( q_{i} \) is the current density of the internal energy, the heat flux, \( s \) is the specific entropy and \( J_{i} \) denotes the entropy flux. The \( \sigma^{(s)} \) entropy production rate plays a central and constructive role in the theory. \( i,j,k \) are abstract spatial indices of vectors and tensors and the notation is abstract in the sense that it does not assume Descartes coordinates, but it is convenient in case of higher than second order tensors. Then comma in lower indices is for spatial derivation, and dot denotes the substantial time derivative (e.g. \( \dot{e} = \partial_{t}e + \nu e_{i,i} \), where \( \partial_{t} \) is the partial time derivative). In case of rigid conductors in rest the relative velocity of the continuum is zero, therefore the substantial time derivative is equal to the partial time derivative. Regarding the general usage of abstract indices in classical nonrelativistic continuum theories see e.g. in [42, 43]. We assume that the entropy flux is zero if \( q_{i} = 0_{i} \) and \( Q_{ij} = 0_{ij} \), therefore

\[
J_{i} = b_{ij} q_{j} + B_{ijk} Q_{jk}, \quad (3)
\]

where the \( b_{ij} \) and \( B_{ijk} \) constitutive functions are the Nyíri multipliers, that conveniently represent the deviation from the local equilibrium form of the entropy flux [44]. Then the \( K \) vector of Müller, [45], for ballistic heat conductors can be given as \( K_{i} = (b_{ij} - \delta_{ij}/T) q_{j} + B_{ijk} Q_{jk} \).

Expanding the entropy function \( s(e, q_{i}, Q_{ij}) \) up to second order approximation around an equilibrium state, we obtain

\[
s(e, q_{i}, Q_{ij}) = s^{(eq)}(e) - \frac{1}{2\rho} m_{ij} q_{i} q_{j} - \frac{1}{2\rho} M_{ijkl} Q_{ij} Q_{kl} \quad (4)
\]

We have the following symmetries

\[
m_{ij} = m_{ji}, \quad M_{ijkl} = M_{klij}.
\]

Thermodynamic stability requires that the inductivity tensors, \( m_{ij}, M_{ijkl} \) (see in [46, 47]), are positive definite. The second law of thermodynamics requires \( \sigma^{(s)} \geq 0 \), thus from [2] we have

\[
\rho \dot{s} + J_{i,i} = \rho \frac{ds^{(eq)}}{de} \dot{e} - \frac{1}{2} m_{ij} q_{i} q_{j} - \frac{1}{2} m_{ij} q_{i} q_{j} - \frac{1}{2} M_{ijkl} \dot{Q}_{ij} \dot{Q}_{kl} +
\]

\[
- \frac{1}{2} M_{ijkl} Q_{ij} \dot{Q}_{kl} + b_{ij} q_{j} + b_{ij} q_{j} + B_{ijk} Q_{jk} + B_{ij} Q_{jk, i} =
\]

\[
\left( b_{ij} - \frac{1}{T} \delta_{ij} \right) q_{i} + (b_{ij} - m_{ij} q_{j}) q_{i} +
\]

\[
+ \left( B_{kl} - M_{kl} \dot{Q}_{kl} \right) Q_{ij} + B_{ijk} Q_{jk, i} \geq 0. \quad (5)
\]

Following the procedures of non-equilibrium thermodynamics we obtain the following general tridimensional anisotropic linear relations between the thermodynamic fluxes \( b_{ij} - \frac{1}{T} \delta_{ij}, b_{ij,j} - m_{ij} q_{j}, B_{ijk}, B_{kl} -
\]
\[ M_{ijkl} \dot{Q}_{kl} \] and forces \( q_i, q_{ij}, Q_{ij}, Q_{ijk} \),

\[
\begin{align*}
b_{ji,j} - m_{ij} \dot{q}_j &= L^{(1)}_{ij} \dot{q}_j + L^{(1,2)}_{ijkl} q_{jk} + L^{(1,3)}_{ijjk} Q_{jk,l} + L^{(1,4)}_{ijkl} Q_{kl,i} \\
\quad \quad \quad &+ \frac{1}{T} \delta_{ij} = L^{(2,1)}_{ijkl} q_k + L^{(2)}_{ijkl} \dot{q}_{kl} + L^{(2,3)}_{ijkl} Q_{kl,m} + L^{(2,4)}_{ijklm} Q_{kl,m}
\end{align*}
\]

\[
\begin{align*}
B_{ij}, k - M_{ijkl} \dot{Q}_{kl} &= L^{(3,1)}_{ijkl} q_k + L^{(3,2)}_{ijkl} \dot{q}_{kl} + L^{(3)}_{ijkl} Q_{kl,m} + L^{(3,4)}_{ijklm} Q_{kl,m} \\
\quad \quad \quad &+ B_{ijkl} = L^{(4,1)}_{ijkl} q_{l} + L^{(4,2)}_{ijklm} Q_{l,m} + L^{(4,3)}_{ijklmn} Q_{lm,n} + L^{(4,4)}_{ijklmn} Q_{lmn,n}
\end{align*}
\]

Here the conductivity tensors, \( L^{\alpha, \beta} \), are restricted by material symmetries, by the second law and also reciprocity relations are to be considered.

3. **Onsager reciprocity relations**

In NET-IV we do not assume anything about the microscopic structure of the material, therefore time reversal symmetry conditions of Onsager cannot be applied [37]. Several theoretical and experimental results support this statement, e.g. in continuum mechanics [38, 39, 50]. Also for ballistic heat conduction a hyperbolic structure is unavoidable for the compatibility with kinetic theory [28, 29, 30].

In our particular approach to heat conduction we do not know anything about the symmetry or antisymmetry of the coefficients of the conductivity tensors. The antisymmetric part does not contribute to the entropy production and the symmetric part is positive definite, ensuring the nonnegativity of the bilinear form for any value of the thermodynamic forces. Therefore, the symmetric part of the conductivity tensor can be written as

\[
\begin{align*}
L^{(1)}_{ik} &= L^{(1)}_{ki}, \\
\quad \quad \quad &L^{(1,2)}_{ij} = L^{(1,2)}_{ji}, \\
L^{(1,3)}_{ijk} &= L^{(1,3)}_{kij}, \\
\quad \quad \quad &L^{(1,4)}_{ijkl} = L^{(1,4)}_{klji}, \\
L^{(2)}_{ijkl} &= L^{(2)}_{klji}, \\
\quad \quad \quad &L^{(2,3)}_{ijkl} = L^{(2,3)}_{klji}, \\
L^{(2,4)}_{ijklmn} &= L^{(2,4)}_{klmn}, \\
\quad \quad \quad &L^{(2,5)}_{ijklm} = L^{(2,5)}_{klm}, \\
L^{(3)}_{ijkl} &= L^{(3)}_{klji}, \\
\quad \quad \quad &L^{(3,2)}_{ijkl} = L^{(3,2)}_{klji}, \\
L^{(3,3)}_{ijkl} &= L^{(3,3)}_{klji}, \\
\quad \quad \quad &L^{(3,4)}_{ijklmn} = L^{(3,4)}_{klmn}, \\
L^{(4)}_{ijklmn} &= L^{(4)}_{klmn}, \\
\quad \quad \quad &L^{(4,2)}_{ijklm} = L^{(4,2)}_{klmn}, \\
L^{(4,3)}_{ijklmn} &= L^{(4,3)}_{klmn}, \\
\quad \quad \quad &L^{(4,4)}_{ijklmn} = L^{(4,4)}_{klmn}
\end{align*}
\]

4. **Perfect isotropic case**

In the perfect isotropic case, in which the symmetry properties of the body under consideration are invariant with respect to all rotations and to inversion of the frame of axes, we have [21]:

\[
\begin{align*}
m_{ij} &= m \delta_{ij}, \\
L^{(1)}_{ij} &= L^{(1)}_{ij}, \\
\quad \quad \quad &L^{(1,2)}_{ij} = L^{(1,2)}_{j}, \\
\quad \quad \quad &L^{(1,3)}_{ijkl} = L^{(1,3)}_{klji}, \\
\quad \quad \quad &L^{(1,4)}_{ijkl} = L^{(1,4)}_{klji}, \\
\quad \quad \quad &L^{(2)}_{ijkl} = L^{(2)}_{klji}, \\
\quad \quad \quad &L^{(2,3)}_{ijkl} = L^{(2,3)}_{klji}, \\
\quad \quad \quad &L^{(2,4)}_{ijklmn} = L^{(2,4)}_{klmn}, \\
\quad \quad \quad &L^{(3)}_{ijkl} = L^{(3)}_{klji}, \\
\quad \quad \quad &L^{(3,2)}_{ijkl} = L^{(3,2)}_{klji}, \\
\quad \quad \quad &L^{(3,3)}_{ijkl} = L^{(3,3)}_{klji}, \\
\quad \quad \quad &L^{(3,4)}_{ijklmn} = L^{(3,4)}_{klmn}, \\
\quad \quad \quad &L^{(4)}_{ijklmn} = L^{(4)}_{klmn}, \\
\quad \quad \quad &L^{(4,2)}_{ijklm} = L^{(4,2)}_{klmn}, \\
\quad \quad \quad &L^{(4,3)}_{ijklmn} = L^{(4,3)}_{klmn}, \\
\quad \quad \quad &L^{(4,4)}_{ijklmn} = L^{(4,4)}_{klmn}
\end{align*}
\]

Furthermore, in the isotropic case (where the symmetry properties of the considered body are invariant only with respect to all rotations of the frame of axes) the third and fifth order tensors keep the form...
\[ L_{ijk} = L \in \mathbb{R}^{ij} \quad \text{and} \quad L_{ijklm} = A_1 \in \mathbb{R}^{ij} \delta_{lm} + A_2 \in \mathbb{R}^{ij} \delta_{km} + A_3 \in \mathbb{R}^{ij} \delta_{kl} + A_4 \in \mathbb{R}^{ij} \delta_{jm} + A_5 \in \mathbb{R}^{ij} \delta_{ij} + A_6 \in \mathbb{R}^{ij} \delta_{jk} \quad \text{respectively} \]

\[ \langle \epsilon_{ijk} \rangle \] denotes the Levi-Civita tensor and the quantities \( L_{ijkl} \) and \( L_{ijklmn} \), that vanish when there is also the invariance of the properties with respect to the inversion of the axes. Thus, we obtain:

\[ L_{(1,2)}^{(1,2)} = L_{(1,3)}^{(1,3)} = L_{(2,1)}^{(2,1)} = L_{(3,1)}^{(3,1)} = 0, \tag{25} \]

\[ L_{ijklm}^{(3,4)} = L_{ijklm}^{(4,2)} = L_{ijklm}^{(4,3)} = 0. \tag{26} \]

From relations 15-29, the phenomenological equations 26-13 in the isotropic case read

\[ m \dot{q_i} - b_{ij,j} = -L_{1}^{(1,4)} q_i - L_{1}^{(1,4)} Q_{kk,k} - L_{2}^{(1,4)} Q_{bi,k} - L_{3}^{(1,4)} Q_{kk,i}, \tag{27} \]

\[ b_{ij} - \frac{1}{T} \delta_{ij} = L_{1}^{(2,3)} q_{ii} + L_{2}^{(2,3)} q_{ii} + L_{3}^{(2,3)} q_{ii} + L_{1}^{(2,3)} q_{iji}, \tag{28} \]

\[ B_{ij,k} = M_{1} \delta_{ij} Q_{kk,k} + M_{2} Q_{ij} + M_{3} Q_{ij} + L_{1}^{(3,2)} \delta_{ij} q_{kk} + L_{2}^{(3,2)} q_{ij}, \tag{29} \]

\[ B_{ij} = L_{1}^{(4,1)} \delta_{ij} q_{kk} + L_{2}^{(4,1)} \delta_{ij} q_{kk} + L_{3}^{(4,1)} \delta_{ij} q_{kk}, \tag{30} \]

\[ + \delta_{ij} \left( L_{1}^{(4)} Q_{kk,}\right) + L_{2}^{(4)} Q_{kk,} + L_{3}^{(4)} Q_{kk,} \]

Perfect isotropy reduces the number of material coefficients to 4 static and 34 conductivity parameters.

### 4.1. Onsager symmetry with isotropy.

If we require Onsager reciprocity relations, 11-14, then from 11-2 \( L_{ijkl}^{(1,4)} = L_{jkl}^{(1,4)} \) and being

\[ L_{jkl}^{(1,4)} = L_{ijkl}^{(1,4)} \]

we obtain

\[ L_{1}^{(1,4)} = L_{3}^{(1,4)}, \quad L_{2}^{(1,4)} = L_{1}^{(1,4)}, \quad L_{3}^{(1,4)} = L_{1}^{(1,4)}. \tag{32} \]

Furthermore, for each isotropic four tensor \( L_{ijkl} \) we have the following symmetry relation

\[ L_{ijkl} = L_{klji}, \tag{33} \]

because of

\[ L_{ijkl} = T_{1} \delta_{ij} \delta_{kl} + T_{2} \delta_{il} \delta_{kj} + T_{3} \delta_{il} \delta_{kj} + T_{1} \delta_{kl} \delta_{ij} + T_{2} \delta_{kl} \delta_{ij} + T_{3} \delta_{kl} \delta_{ij} = L_{klji}, \tag{34} \]

where \( T_{1}, T_{2} \) and \( T_{3} \) indicates the independent components of \( L_{ijkl} \). Taking the property into account, Onsager-Casimir relations 12-13, and 13-2 are verified in the isotropic case and from 12-2 we derive \( L_{ijkl}^{(2,3)} = L_{klji}^{(3,2)} = L_{ijkl}^{(3,2)} \), from which we have

\[ L_{i}^{(2,3)} = L_{i}^{(3,2)} \quad (i = 1, 2, 3). \tag{35} \]

Then, from 24-4 we obtain:

\[ L_{lmnijk}^{(4)} = L_{1}^{(4)} \delta_{lm} \delta_{mn} \delta_{ij} + L_{2}^{(4)} \delta_{lm} \delta_{nj} \delta_{ij} + L_{3}^{(4)} \delta_{lm} \delta_{nk} \delta_{ij} + L_{4}^{(4)} \delta_{lm} \delta_{mj} \delta_{ij} \]

\[ + L_{5}^{(4)} \delta_{lm} \delta_{mk} \delta_{ij} + L_{6}^{(4)} \delta_{lm} \delta_{mk} \delta_{ij} + L_{7}^{(4)} \delta_{lm} \delta_{mk} \delta_{ij} + L_{8}^{(4)} \delta_{lm} \delta_{mk} \delta_{ij} \]

\[ + L_{9}^{(4)} \delta_{lm} \delta_{mk} \delta_{ij} + L_{10}^{(4)} \delta_{lm} \delta_{mk} \delta_{ij} + L_{11}^{(4)} \delta_{lj} \delta_{mk} \delta_{ijn} + L_{12}^{(4)} \delta_{lj} \delta_{mk} \delta_{ijn} \]

\[ + L_{13}^{(4)} \delta_{lj} \delta_{mk} \delta_{ijn} + L_{14}^{(4)} \delta_{lj} \delta_{mk} \delta_{ijn} + L_{15}^{(4)} \delta_{lj} \delta_{mk} \delta_{ijn}. \tag{36} \]
Using (42), adding (29) and (30), and dividing by 2, we have

$$\mathcal{L}^{(4)}_{ijklmn} = C_1^{(4)} (\delta_{ij} \delta_{kl} \delta_{mn} + \delta_{in} \delta_{jk} \delta_{lm} + \delta_{im} \delta_{jkl} \delta_{ln} + \delta_{ik} \delta_{jlm} \delta_{jn})$$

$$+ C_2^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_3^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_4^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_5^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_6^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_7^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_8^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_9^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_{10}^{(4)} (\delta_{im} \delta_{jn} \delta_{kl} + \delta_{in} \delta_{jln} \delta_{km} + \delta_{il} \delta_{jmn} \delta_{kl})$$

$$+ C_{11}^{(4)} \delta_{im} \delta_{jn} \delta_{kl},$$

where

$$C_1^{(4)} = \frac{L_1^{(4)} + L_9^{(4)}}{2}, \quad C_2^{(4)} = \frac{L_2^{(4)} + L_6^{(4)}}{2}, \quad C_3^{(4)} = L_3^{(4)}$$

$$C_4^{(4)} = \frac{L_4^{(4)} + L_8^{(4)}}{2}, \quad C_5^{(4)} = L_5^{(4)}, \quad C_6^{(4)} = L_7^{(4)}, \quad C_7^{(4)} = L_6^{(4)}$$

$$C_8^{(4)} = L_8^{(4)}, \quad C_9^{(4)} = L_9^{(4)}, \quad C_{10}^{(4)} = \frac{L_5^{(4)} + L_7^{(4)}}{2}, \quad C_{11}^{(4)} = L_5^{(4)}$$

Thus, from relation \(\mathcal{L}^{(4)}_{ijklmn} = \mathcal{L}^{(4)}_{imnkij}\) the significant components of the isotropic tensor \(\mathcal{L}^{(4)}_{ijklmn}\) reduce to 11. Therefore, in case of Onsager reciprocity, the number of conductivity coefficients are reduced altogether to 24.

4.2. Entropy production. In the perfect isotropic case, with the aid of relations (27) - (30), (15) - (23), (25), (26), (55), (57), entropy production (5) can be written as

$$\sigma^{(s)} = L^{(1)}_{ik} q_{ik} + L^{(2)}_{ijkl} q_{jkl} + L^{(3)}_{ijklm} Q_{ijklm} Q_{ijklm} + L^{(4)}_{ijklmn} Q_{ijklmn} Q_{ijklmn}$$

$$+ \left( L^{(1,4)}_{ijkl} + L^{(1,4)}_{ijkl} \right) q_{ijkl} + \left( L^{(2,3)}_{ijkl} + L^{(2,3)}_{ijkl} \right) q_{ijkl} Q_{ijkl} \geq 0,$$

or in extended form:

$$\sigma^{(s)} = L^{(1)}_{ik} q_{ik} + \left( L^{(2)}_{ij} \delta_{ij} \delta_{kl} + L^{(2)}_{ij} \delta_{ik} \delta_{jl} + L^{(3)}_{ij} \delta_{ik} \delta_{jl} \right) q_{ijkl} \geq 0$$

$$+ \left( L^{(3)}_{ij} \delta_{ij} \delta_{kl} + L^{(3)}_{ik} \delta_{ik} \delta_{jl} + L^{(3)}_{ij} \delta_{ik} \delta_{jl} \right) q_{ijkl} \geq 0$$

$$+ C_1^{(4)} (\delta_{ij} \delta_{kl} \delta_{mn} + \delta_{in} \delta_{jk} \delta_{lm} + \delta_{im} \delta_{jkl} \delta_{ln} + \delta_{ik} \delta_{jlm} \delta_{jn})$$

$$+ C_2^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_3^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_4^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_5^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_6^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_7^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_8^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_9^{(4)} (\delta_{ij} \delta_{km} \delta_{ln} + \delta_{in} \delta_{jkm} \delta_{ln} + \delta_{im} \delta_{jkm} \delta_{ln})$$

$$+ C_{10}^{(4)} (\delta_{im} \delta_{jn} \delta_{kl} + \delta_{in} \delta_{jln} \delta_{km} + \delta_{il} \delta_{jmn} \delta_{kl})$$

$$+ C_{11}^{(4)} \delta_{im} \delta_{jn} \delta_{kl},$$

and also in the following form
\[ \sigma^{(s)} = L^{(1)} \delta_{ik} q_i q_k + \left( L^{(2)}_1 \delta_{ij} \delta_{kl} + L^{(2)}_2 \delta_{jk} \delta_{il} + L^{(2)}_3 \delta_{ji} \delta_{lk} \right) q_{ij} q_{kl} + \left( L^{(3)}_1 \delta_{ij} \delta_{kl} + L^{(3)}_2 \delta_{ik} \delta_{jl} + L^{(3)}_3 \delta_{il} \delta_{jk} \right) Q_{ij} q_{kl} + \left( C^{(4)}_1 \delta_{ij} \delta_{kl} \delta_{mn} + C^{(4)}_2 \delta_{jm} \delta_{kn} + C^{(4)}_3 \delta_{jm} \delta_{ln} + C^{(4)}_4 \delta_{jk} \delta_{ln} + C^{(4)}_5 \delta_{ij} \delta_{kl} \delta_{mn} \right) \]

\[ + C^{(4)}_6 \delta_{ij} \delta_{kl} \delta_{mn} + C^{(4)}_7 \delta_{jm} \delta_{kn} + C^{(4)}_8 \delta_{jm} \delta_{ln} + C^{(4)}_9 \delta_{jk} \delta_{ln} + C^{(4)}_{10} \delta_{ij} \delta_{kl} \delta_{mn} \right) Q_{ij} q_{kl}, \]

(43)

From (43) it is seen that the entropy production in a non-negative bilinear form in the components of the heat flux and its gradient, and in the components of the internal variable and its gradient (see in Appendix its matrix representation \( \sigma^{(s)} = X_\alpha X_\alpha X_\beta \), with \( X_\alpha \), \( X_\beta \) and \( L_{\alpha\beta} \) suitable matrices).

The following inequalities can be obtained for the components of the phenomenological tensors, resulting from the fact that all the elements of the main diagonal of the symbolic matrix \( \{ L_{\alpha\beta} \} \) associated to the bilinear form (43) must be non-negative (see Appendix):

\[ L^{(1)} \geq 0, \quad L^{(2)} \geq 0, \quad L^{(3)} \geq 0, \]

(44)

\[ L^{(2)}_1 + L^{(2)}_2 + L^{(2)}_3 \geq 0, \quad L^{(3)}_1 + L^{(3)}_2 + L^{(3)}_3 \geq 0, \]

(45)

\[ 2C^{(4)}_1 + 2C^{(4)}_2 + C^{(4)}_3 + 2C^{(4)}_4 + C^{(4)}_5 + C^{(4)}_6 + C^{(4)}_7 + C^{(4)}_8 + C^{(4)}_9 + 2C^{(4)}_{10} + C^{(4)}_{11} \geq 0, \]

(46)

\[ C^{(4)}_2 + C^{(4)}_8 + C^{(4)}_{10} \geq 0, \quad C^{(4)}_4 + C^{(4)}_9 + C^{(4)}_{10} \geq 0, \]

(47)

\[ C^{(4)}_{10} \geq 0, \quad C^{(4)}_1 + C^{(4)}_{10} + C^{(4)}_{11} \geq 0. \]

(48)

In particular relations (46)-(48) come from the non-negativity of the elements of the main diagonal of the sub-matrix \( L^{(4)}_{pm\mid mn} \).

Moreover, other relations can be obtained from the non-negativity of the major minors \( P_r \) \((r = 1, \ldots, 48)\) of \( \{ L_{\alpha\beta} \} \). For instance, the calculation of the major minors up to sixth order gives the relations (49), (50) and (51), the non-negativity of the seventh order major minor of \( \{ L_{\alpha\beta} \} \)

The non-negativity of the seventh order major minor of \( \{ L_{\alpha\beta} \} \)

\[ P_7 = \begin{vmatrix}
L^{(1)}_1 & 0 & 0 & 0 & 0 & 0 \\
0 & L^{(1)}_1 & 0 & 0 & 0 & 0 \\
0 & 0 & L^{(1)}_2 & 0 & 0 & 0 \\
0 & 0 & 0 & L^{(1)}_3 & 0 & 0 \\
0 & 0 & 0 & 0 & L^{(2)}_1 & 0 \\
0 & 0 & 0 & 0 & 0 & L^{(2)}_2 \\
\end{vmatrix}, \]

(49)

with \( L^{(2)}_1 \equiv L^{(2)}_1 + L^{(2)}_2 + L^{(2)}_3 \), gives the new relation

\[ L^{(2)}_2 + \left( L^{(2)}_3 \right)^2 \geq 0, \]

(50)

and so on. In the Appendix we give a two dimensional form of the conductivity matrix \( \{ L_{\alpha\beta} \} \), then the calculation of the conditions of positive definiteness is straightforward.
4.3. Rate equations for $q$ and $Q$. Changing indexes $i$ and $j$ in (28), deriving it with respect to $x_j$ and substituting it into (27), we deduce:

$$m\dot{q}_i + L^{(1)}q_i = \left( L^{(2)}_1 + L^{(2)}_2 \right) q_{k,ki} + L^{(3)}_3 q_{k,kk} + \left( L^{(2,3)}_1 - L^{(1,4)}_3 \right) Q_{kk,i}$$

$$+ \left( L^{(2,3)}_3 - L^{(1,4)}_1 \right) Q_{kk,k} + \left( L^{(2,3)}_2 - L^{(1,4)}_2 \right) Q_{ki,k} + \left( \frac{1}{T} \right) ,$$

(51)

where:

$$m > 0, \quad L^{(1)} > 0, \quad L^{(2)}_1 + L^{(2)}_2 > 0, \quad L^{(2)}_3 > 0.$$  

(52)

Equation (51) can be written as follow:

$$\tau \dot{q}_i + q_i = -\lambda T_x + l_1 q_{i,kk} + l_2 q_{k,ki} + l_{12} Q_{kk,i} + l_{13} Q_{k,k} + l_{14} Q_{k,k} ,$$

(53)

where

$$\tau = \frac{m}{L^{(1)}}, \quad \lambda = \frac{L^{(1)}T_x}{L^{(1)}}, \quad l_1 = \frac{L^{(2)}_3}{L^{(1)}}, \quad l_2 = \frac{L^{(2)}_1 + L^{(2)}_2}{L^{(1)}},$$

$$l_{12} = \frac{L^{(2,3)}_2 - L^{(1,4)}_1}{L^{(1)}}, \quad l_{13} = \frac{L^{(2,3)}_3 - L^{(1,4)}_1}{L^{(1)}}, \quad l_{14} = \frac{L^{(2,3)}_3 - L^{(1,4)}_2}{L^{(1)}},$$

(54)

(55)

being $\tau$ the relaxation time of the heat flux, that, then, has a finite velocity of propagation and $\lambda$ the heat conductivity.

In analogous way, if we change $i \rightarrow k$, $j \rightarrow i$, $k \rightarrow j$ in equation (30), deriving it with respect to $x_k$ and inserting it into (29), we have:

$$M_1 \delta_{ij} \dot{Q}_{kk} + M_2 \dot{Q}_{ij} + M_3 Q_{ji} + L^{(3)}_1 \delta_{ij} Q_{kk} + L^{(3)}_2 Q_{ij} + L^{(3)}_3 Q_{ji}$$

$$= \left( L^{(4,1)}_1 - L^{(3,2)}_2 \right) \delta_{ij} q_{k,k} + \left( L^{(4,1)}_2 - L^{(3,2)}_2 \right) \dot{q}_{ij} + \left( L^{(4,1)}_1 - L^{(3,2)}_3 \right) q_{ij},$$

$$+ \left( L^{(4)}_3 + L^{(4)}_6 \right) Q_{kk,ki} + L^{(4)}_{12} Q_{ji,ki} + L^{(4)}_{13} Q_{ji,kk} + L^{(4)}_1 + L^{(4)}_5 \right) Q_{ki,ik} +$$

$$+ \left( L^{(2)}_2 + L^{(4)}_1 \right) Q_{ij,kk} + \left( L^{(4)}_2 + L^{(4)}_1 \right) Q_{ij,ik} + \left( L^{(4)}_5 + L^{(4)}_10 \right) Q_{ki,jk} +$$

$$+ \delta_{ij} \left[ L^{(4)}_7 + L^{(4)}_8 \right] Q_{kl,ik} + L^{(4)}_9 Q_{ll,ik},$$

(56)

i.e.:

$$\tau_1 \delta_{ij} \dot{Q}_{kk} + \tau_2 \dot{Q}_{ij} + \tau_3 \dot{Q}_{ji} + \delta_{ij} Q_{kk} + L^{(3)}_1 \delta_{ij} q_{k,k} + L^{(3)}_2 q_{ij} = L^{(3)}_{21} \delta_{ij} q_{k,k} + L^{(3)}_{31} q_{ij}$$

$$+ l_{41} Q_{kk,ij} + L_{42} Q_{ij,kk} + L_{43} Q_{ji,kk} + L_{44} q_{k,k} + L_{45} Q_{kk} ,$$

(57)

where

$$\tau_1 = \frac{M_1}{L^{(3)}_1}, \quad \tau_2 = \frac{M_2}{L^{(3)}_1}, \quad \tau_3 = \frac{M_3}{L^{(3)}_1}, \quad L^{(3)}_2 = \frac{L^{(3)}_1}{L^{(3)}_1}, \quad L^{(3)}_3 = \frac{L^{(3)}_1}{L^{(3)}_1},$$

$$l_{21} = \frac{L^{(3)}_3 - L^{(3,2)}_1}{L^{(3)}_1}, \quad l_{31} = \frac{L^{(3)}_2 - L^{(3,2)}_1}{L^{(3)}_1}, \quad l_{41} = \frac{L^{(3,1)}_1 - L^{(3,2)}_1}{L^{(3)}_1},$$

(58)

(59)

$$L^{(3)}_1 = \frac{L^{(4,1)}_1 - L^{(3,2)}_2}{L^{(3)}_1}, \quad L^{(3)}_2 = \frac{L^{(4,1)}_2 - L^{(3,2)}_2}{L^{(3)}_1}, \quad L^{(3)}_3 = \frac{L^{(4,1)}_1 - L^{(3,2)}_3}{L^{(3)}_1},$$

(60)

$$L^{(3)}_4 = \frac{L^{(4)}_3 + L^{(4)}_6}{L^{(3)}_1}, \quad L^{(3)}_5 = \frac{L^{(4)}_{12} + L^{(4)}_1}{L^{(3)}_1}, \quad L^{(3)}_6 = \frac{L^{(4)}_1 + L^{(4)}_5}{L^{(3)}_1},$$

(61)

$$L^{(3)}_7 = \frac{L^{(4)}_2 + L^{(4)}_1}{L^{(3)}_1}, \quad L^{(3)}_8 = \frac{L^{(4)}_5 + L^{(4)}_10}{L^{(3)}_1}, \quad L^{(3)}_9 = \frac{L^{(4)}_7 + L^{(4)}_8}{L^{(3)}_1},$$

(62)

and $\tau_1$, $\tau_2$ and $\tau_3$ have time dimension.

Equations (51) - (57) (or (51) - (60)) are the full three dimensional versions of the one dimensional equations (12)-(13) in [28].

We split the second order tensor to orthogonal components, i.e.

$$Q_{ij} = Q_{(ij)} + Q_{[ij]} + Q_{ij} \delta_{ij},$$

(63)
where

\[ Q_{ij} = \frac{1}{2}(Q_{ij} + Q_{ji}) - Q\delta_{ij} \quad \text{(deviator of the symmetric part of \( Q_{ij} \))}, \]  
\[ Q_{[ij]} = \frac{1}{2}(Q_{ij} - Q_{ji}) \quad \text{(skew-symmetric part of \( Q_{ij} \))}, \]  
\[ Q = \frac{1}{3}Q_{kk} \quad \text{(scalar part of \( Q_{ij} \))}. \]  

From equation (57) we derive the rate equation for \( Q \) (\( i = j \)):

\[ 3(3\tau_1 + \tau_2 + \tau_3)\dot{Q} + 3(3 + l_2^2 + l_3^2)Q = (3l_{21} + l_{31} + l_{41})q_{k,k} + 3(L_1 + L_2 + L_3 + 3L_9)Q_{,kk} + (L_4 + L_5 + L_6 + L_7 + 3L_8)Q_{,kl,kl}, \]  

i.e.

\[ r^0\dot{Q} + Q = r^0q_{k,k} + L^0_1Q_{,kk} + L^0_2Q_{,kl,kl} , \]  

where

\[ r^0 = \frac{3\tau_1 + \tau_2 + \tau_3}{3 + l_2^2 + l_3^2}, \]  
\[ L^0_1 = \frac{L_1 + L_2 + L_3 + 3L_9}{3 + l_2^2 + l_3^2}, \]  
\[ L^0_2 = \frac{L_4 + L_5 + L_6 + L_7 + 3L_8}{3 + l_2^2 + l_3^2} , \]  

being \( r^0 \) the relaxation time of \( Q \);

the rate equation for \( Q_{(ij)} \):

\[ \dot{\hat{Q}}_{(ij)} + Q_{(ij)} = \hat{\dot{L}}_{q(i,j)} + \hat{L}_1Q_{kk,(i,j)} + \hat{L}_2Q_{(i,j),kk} + \hat{L}_3Q_{k(i,j),k} + \hat{L}_4Q_{(ik,kj)} , \]  

where

\[ \hat{\dot{L}} = \frac{\tau_2 + \tau_3}{l_2^2 + l_3^2}, \]  
\[ \hat{L}_1 = \frac{L_1}{l_2^2 + l_3^2}, \]  
\[ \hat{L}_2 = \frac{L_2 + L_3}{l_2^2 + l_3^2}, \]  
\[ \hat{L}_3 = \frac{L_5 + L_7}{l_2^2 + l_3^2}, \]  
\[ \hat{L}_4 = \frac{L_4 + L_6}{l_2^2 + l_3^2} ; \]  

being \( \hat{\dot{L}} \) the relaxation time of \( Q_{(ij)} \);

finally the rate equation for \( Q_{[ij]} \):

\[ \ddot{\nu}Q_{[ij]} + Q_{[ij]} = \nu\dot{L}_{q[i,j]} + \nu\dot{L}_1Q_{[i,j],kk} + \nu\dot{L}_2Q_{[i,j],kl} + \nu\dot{L}_3Q_{[ik,kj]} , \]  

where

\[ \nu = \frac{\tau_2 - \tau_3}{l_2^2 - l_3^2}, \]  
\[ \nu\dot{L}_1 = \frac{L_2 - L_3}{l_2^2 - l_3^2}, \]  
\[ \nu\dot{L}_2 = \frac{L_7 - L_5}{l_2^2 - l_3^2}, \]  
\[ \nu\dot{L}_3 = \frac{L_6 - L_4}{l_2^2 - l_3^2} ; \]  

being \( \nu \) the relaxation time of \( Q_{[ij]} \).
4.4. The rate equations for $q$ and $Q$ with Onsager reciprocity. From Onsager reciprocity relations [52]-[57], the phenomenological equations [27]-[30] become

$$m \dot{q}_i = -L^{(1)}_1 q_i - L^{(1,4)}_1 Q_{ik,k} - L^{(1,4)}_2 Q_{ki,k} - L^{(1,4)}_3 Q_{kk,i} + \frac{1}{T} \delta_{ij} \frac{\partial \bar{q}}{\partial \bar{q}} Q_{jj} \tag{77}$$

$$b_{ij} - \frac{1}{T} \delta_{ij} = L^{(2)}_1 \delta_{ij} \frac{\partial \bar{q}}{\partial \bar{q}} Q_{kk} + L^{(2)}_2 q_{ij} + L^{(2)}_3 q_{ji} + L^{(2,3)}_1 \delta_{ij} Q_{kk}$$

$$+ L^{(2,3)}_2 Q_{ij} + L^{(2,3)}_3 Q_{ji},$$

$$B_{kij,k} = M_1 \delta_{ij} Q_{kk} + M_2 Q_{ij} + M_3 \dot{Q}_{j} + L^{(2,3)}_1 \delta_{ij} q_{k} + L^{(2,3)}_2 q_{ij}$$

$$+ L^{(2,3)}_3 q_{ji} + L^{(3)}_1 \delta_{ij} Q_{kk} + L^{(3)}_2 Q_{ij} + L^{(3)}_3 Q_{ji},$$

$$B_{ijk} = L^{(1,4)}_1 \delta_{ij} q_{k} + L^{(1,4)}_2 \delta_{ik} q_{j} + L^{(1,4)}_3 \delta_{jk} q_{i}$$

$$+ \delta_{ij} \left( C_{4}^{(4)} Q_{ik,k} + C_{2}^{(4)} Q_{kk,l} + C_{3}^{(4)} Q_{ll,k} \right)$$

$$\quad + \delta_{ik} \left( C_{4}^{(4)} Q_{jl,l} + C_{2}^{(4)} Q_{ll,j} + C_{3}^{(4)} Q_{ll,l} \right)$$

$$\quad + \delta_{jk} \left( C_{4}^{(4)} Q_{il,i} + C_{2}^{(4)} Q_{ll,l} + C_{3}^{(4)} Q_{ll,i} \right)$$

$$\quad + C_{7}^{(4)} Q_{ik,k} + C_{8}^{(4)} Q_{ij,j} + C_{9}^{(4)} Q_{jk,j} + C_{10}^{(4)} Q_{kj,i} + C_{11}^{(4)} Q_{kk,k},$$

$$\quad + C_{9}^{(4)} Q_{ij,k} + C_{10}^{(4)} Q_{ji,k}.$$  \tag{80}

We observe that equations (77) and (78) are equal to (27) and (28). In (78) changing indexes $i$ in index $j$, operating the derivative with respect to $x_j$ and substituting the obtained equation into (77), we have:

$$m \ddot{q}_i + L^{(1)}_1 q_i = \left( L^{(2)}_1 + L^{(2)}_2 \right) q_{k,ki} + L^{(2)}_3 q_{kk,k} + \left( L^{(2,3)}_1 + L^{(2,3)}_2 \right) Q_{kk,i}$$

$$+ \left( L^{(2,3)}_3 - L^{(1,4)}_1 \right) Q_{ik,k} + \left( L^{(2,3)}_2 - L^{(1,4)}_3 \right) Q_{kk,k} + \left( \frac{1}{T} \right) q_{i,ki} \tag{81}$$

Equation [51] can be written as follows:

$$\tau q_i + q_i = -\lambda T_i + l_1 q_{i,k} + l_2 q_{k,ki} + l_3 Q_{kk,i} + l_4 Q_{kk,k} + l_5 Q_{kk,i},$$  \tag{82}

in which the coefficients are given by [51] and [55]. We observe that the rate equations [51] and [52] are the same to [51] and [55] in perfect isotropic case.

Furthermore, changing the indexes $i, j, k$ in the indexes $k, i, j$, respectively, in equation (80), deriving it with respect to $x_k$ and inserting the obtained equation into (78) we derive:

$$M_1 \delta_{ij} Q_{kk} + M_2 Q_{ij} + M_3 \dot{Q}_{j} + L^{(3)}_1 \delta_{ij} Q_{kk} + L^{(3)}_2 Q_{ij} + L^{(3)}_3 Q_{ji}$$

$$= \left( L^{(1,4)}_1 - L^{(2,3)}_1 \right) \delta_{ij} q_{k} + \left( L^{(2,3)}_1 - L^{(2,3)}_2 \right) q_{ij} + \left( L^{(2,3)}_3 - L^{(2,3)}_1 \right) q_{ji}$$

$$+ \left( C_{2}^{(4)} + C_{3}^{(4)} \right) Q_{kk,ij} + C_{10}^{(4)} Q_{ij,k} + C_{11}^{(4)} Q_{ji,k}$$

$$+ \left( C_{2}^{(4)} + C_{5}^{(4)} \right) Q_{kk,ik} + \left( C_{4}^{(4)} + C_{9}^{(4)} \right) Q_{ij,k} + \left( C_{4}^{(4)} + C_{7}^{(4)} \right) Q_{ij,k}$$

$$+ \delta_{ij} \left( C_{4}^{(4)} + C_{6}^{(4)} \right) Q_{kk,kl} + C_{4}^{(4)} Q_{ll,k} \tag{83}$$

i.e.:

$$\tau_1 \dot{q}_{k} + \tau_2 \dot{q}_{j} + \tau_3 \dot{q}_{ji} + \delta_{ij} Q_{kk} + l_1^{2} Q_{ij} + l_2^{3} Q_{ji} = l_2^{4} \delta_{ij} q_{k} + l_3^{1} q_{i,j}$$

$$+ l_4^{1} q_{j,i} + l_5^{1} Q_{kk,ki} + l_6^{2} Q_{ij,k} + l_7^{3} Q_{j,k,k} + l_8^{4} Q_{ji,k} + l_9^{5} Q_{kj,k}$$

$$+ l_1^{6} Q_{ij,k} + l_2^{7} Q_{k,j,k} + l_3^{8} Q_{k,k,k},$$  \tag{84}

where $\tau_1, \tau_2, \tau_3, l_1^{2}, l_2^{3}$ are given by [58], the coefficients $l_2^{4}$, $l_3^{1}$ and $l_4^{1}$ transform according to relations [52] and [53]:

$$l_2^{4} = \frac{L^{(1,4)}_1 - L^{(2,3)}_1}{L^{(3)}_1}, \quad l_3^{1} = \frac{L^{(2,3)}_1 - L^{(2,3)}_2}{L^{(3)}_1}, \quad l_4^{1} = \frac{L^{(1,4)}_1 - L^{(2,3)}_2}{L^{(3)}_1}.\tag{85}$$
and moreover

\[
C_1 = \frac{C^{(4)} + C^{(3)}}{L_1^{(3)}}, \quad C_2 = \frac{C^{(4)}_1 + C^{(3)}_1}{L_1^{(3)}}, \quad C_3 = \frac{C^{(4)}_1}{L_1^{(3)}},
\]

\[
C_4 = \frac{C^{(4)} + C^{(3)}}{L_1^{(3)}}, \quad C_5 = \frac{C^{(4)} + C^{(3)}_1}{L_1^{(3)}}, \quad C_6 = \frac{C^{(4)} + C^{(3)}_1}{L_1^{(3)}},
\]

\[
C_7 = \frac{C^{(4)}_2 + C^{(3)}_2}{L_1^{(3)}}, \quad C_8 = \frac{C^{(4)} + C^{(3)}_3}{L_1^{(3)}}, \quad C_9 = \frac{C^{(4)}_1}{L_1^{(3)}}.
\]

We observe that from relations (85) and (86) we have:

\[
l_{31} = -l_{14}, \quad l_{41} = -l_{12} - l_{13} - l_{21}.
\]

Furthermore, using (89), (87) and (88) we obtain

\[
C_9 = C_4 - C_2,
\]

so that equation (81) reads

\[
\tau \delta_{ij} Q_{kk} + \tau_2 \delta_{ij} Q_{ij} + \delta_{ij} Q_{kk} + \tau_3 \delta_{ij} Q_{ij} + \tau_4 \delta_{ij} Q_{ij} + \tau_5 \delta_{ij} Q_{ij} = l_{21} \delta_{ij} q_{kk} - l_{14} q_{kk} j
\]

\[
- (l_{12} + l_{13} + l_{21}) q_{ij,i} + C_1 Q_{kk,ij} + C_2 Q_{ij,kk} + C_3 Q_{ij,kk} + C_4 Q_{jj,ik} + C_5 Q_{jj,ik}
\]

\[
+ C_6 Q_{kk,jk} + C_7 Q_{kk,jk} + \delta_{ij} (C_8 Q_{kk,li} + C_{9}) Q_{ll,ik},
\]

As in the previous paragraph, we split the second order tensor \(Q_{ij}\) in its orthogonal components, i.e.

\[
Q_{ij} = Q_{(ij)} + Q_{(ij)} + Q_{(ij)}.
\]

From equation (91) we derive the rate equation for \(Q (i = j)\):

\[
3(\tau_1 + \tau_2 + \tau_3) \dot{Q} + 3(3 + \tau_3) Q = (2l_{21} - l_{12} - l_{13} - l_{14}) q_{kk} +
\]

\[
+ 3[C_1 - 2C_2 + C_3 + 3C_4] Q_{kk} + (C_4 + C_5 + C_6 + C_7 + 3C_8) Q_{kk,kl},
\]

i.e.

\[
\tau^0 \dot{Q} + Q = \epsilon^0 q_{kk} + C^0_{[kk]} + C^0_{[kk]},
\]

where \(\tau^0\) is given by (89) and

\[
\epsilon^0 = \frac{2l_{21} - l_{12} - l_{13} - l_{14}}{3 + \tau_3 + \tau_3}, \quad \epsilon^0_1 = \frac{C_1 - 2C_2 + C_3 + 3C_4}{3 + \tau_3 + \tau_3}, \quad \epsilon^0_2 = \frac{C_4 + C_5 + C_6 + C_7 + 3C_8}{3 + \tau_3 + \tau_3};
\]

the rate equations for \(Q_{(ij)}\):

\[
\hat{\tau} \dot{Q}_{(ij)} + Q_{(ij)} = \hat{\epsilon} q_{(ii)} + \hat{C}_1 Q_{kk,(ii)} + \hat{C}_2 Q_{ij,(ii)} + \hat{C}_3 Q_{kk,(ii)} + \hat{C}_4 Q_{kk,(ii)} + \hat{C}_5 Q_{kk,(ii)} + \hat{C}_6 Q_{kk,(ii)},
\]

where \(\hat{\tau}\) is given by (92) and

\[
\hat{\epsilon} = \frac{l_{12} + l_{13} + l_{14} + l_{21}}{\tau_3 + \tau_3}, \quad \hat{C}_1 = \frac{C_1}{\tau_3 + \tau_3}, \quad \hat{C}_2 = \frac{C_2 + C_3}{\tau_3 + \tau_3}, \quad \hat{C}_3 = \frac{C_4 + C_5 + C_6 + C_7 + 3C_8}{\tau_3 + \tau_3}, \quad \hat{C}_4 = \frac{C_4 + C_6}{\tau_3 + \tau_3};
\]

finally the rate equation for \(Q_{[ij]}\):

\[
\dot{\tau} \dot{Q}_{[ij]} + Q_{[ij]} = \ddot{\epsilon} q_{[ij]} + \ddot{C}_1 Q_{kk,[ij]} + \ddot{C}_2 Q_{ij,[ij]} + \ddot{C}_3 Q_{kk,[ij]} + \ddot{C}_4 Q_{kk,[ij]},
\]

where \(\ddot{\tau}\) is given by (94) and

\[
\ddot{\epsilon} = \frac{l_{12} + l_{13} + l_{14} + l_{21}}{\tau_3^2 - \tau_3^2}, \quad \ddot{C}_1 = \frac{C_2 - C_4}{\tau_3^2 - \tau_3^2}, \quad \ddot{C}_2 = \frac{C_7 - C_5}{\tau_3^2 - \tau_3^2}, \quad \ddot{C}_3 = \frac{C_6 - C_4}{\tau_3^2 - \tau_3^2};
\]

Therefore the number of rate equations in case of Onsager reciprocity is simplified and the number of coefficients is reduced from 38 to 32 when compared to the perfect isotropic case.
4.5. One-dimensional heat conduction. In the one-dimensional case, where
\[ q = (q, 0, 0), \quad Q = \begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \] (99)
the unique component of the third order tensor \( \mathbf{B} \) in \( B \equiv B_{111} \) and \( Q = Q_{11} \) indicates the unique component of \( Q_{ij} \), the system of equations (77) - (80) becomes
\[
\begin{align*}
mq_{t} - b_{x} &= -L^{(1)}q - L^{(1,4)}Q_{x}, \quad \text{(100)} \\
b - \frac{1}{T} &= L^{(2)}q_{x} + L^{(2,3)}Q, \quad \text{(101)} \\
MQ_{t} - b_{x} &= -L^{(2,3)}q_{x} - L^{(3)}Q, \quad \text{(102)} \\
B &= L^{(1,4)}q + C^{(4)}Q_{x}, \quad \text{(103)}
\end{align*}
\]
where
\[
\begin{align*}
L^{(1,4)} &= L_{1}^{(1,4)} + L_{2}^{(1,4)} + L_{3}^{(1,4)}, \quad L^{(2)} = L_{1}^{(2)} + L_{2}^{(2)} + L_{3}^{(2)}, \quad \text{(104)} \\
L^{(2,3)} &= L_{1}^{(2,3)} + L_{2}^{(2,3)} + L_{3}^{(2,3)}, \quad M = M_{1} + M_{2} + M_{3}, \quad \text{(105)} \\
L^{(3)} &= L_{1}^{(3)} + L_{2}^{(3)} + L_{3}^{(3)}, \quad \text{(106)} \\
C^{(4)} &= 2C_{1}^{(4)} + 2C_{2}^{(4)} + C_{3}^{(4)} + 2C_{4}^{(4)} + C_{5}^{(4)} + C_{6}^{(4)} + C_{7}^{(4)} + C_{8}^{(4)} + C_{9}^{(4)} + 2C_{10}^{(4)} + C_{11}^{(4)}, \quad \text{(107)}
\end{align*}
\]
with \((\cdot),x\) indicating the derivative of \((\cdot)\) with respect to \(x\).

In this case the entropy production \((111)\) assume the form
\[
\sigma^{(s)} = L^{(1)}q^{2} + L^{(2)}(q_{x})^{2} + L^{(3)}Q^{2} + C^{(4)}(Q_{x})^{2} + 2L^{(1,4)}qQ_{x} + 2L^{(2,3)}q_{x}Q \geq 0, \quad \text{(108)}
\]
or in symbolic matrix notation:
\[
\sigma^{(s)} = (q \quad q_{x} \quad Q \quad Q_{x}) \begin{bmatrix}
L^{(1)} & 0 & 0 & L^{(1,4)} \\
0 & L^{(2)} & L^{(2,3)} & 0 \\
0 & L^{(2,3)} & L^{(3)} & 0 \\
L^{(1,4)} & 0 & 0 & C^{(4)}
\end{bmatrix} \begin{bmatrix}
q \\
q_{x} \\
Q \\
Q_{x}
\end{bmatrix} \geq 0. \quad \text{(109)}
\]

Because of the bilinear form \((108)\) must be non-negative, the matrix \( \mathbf{B} \) associated to this form is non-negative semidefinite, so that the elements of its main diagonal and its major minors must be non-negative:
\[
L^{(1)} \geq 0, \quad L^{(2)} \geq 0, \quad L^{(3)} \geq 0, \quad C^{(4)} \geq 0, \quad \text{(110)}
\]
\[
L^{(2)}L^{(3)} - \left(L^{(2,3)}\right)^{2} \geq 0, \quad L^{(1)}C^{(4)} - \left(L^{(1,4)}\right)^{2} \geq 0. \quad \text{(111)}
\]

Using \((101)\) and \((103)\), equations \((100)\) and \((102)\) become
\[
\begin{align*}
mq_{t} + L^{(1)}q - L^{(2)}q_{xx} &= \left(\frac{1}{T}\right)_{,x} - DQ_{,x}, \quad \text{(112)} \\
MQ_{t} + L^{(3)}Q - C^{(4)}Q_{,xx} &= Dq_{,x}, \quad \text{(113)}
\end{align*}
\]
where \( D = L^{(1,4)} - L^{(2,3)} \). \( \frac{1}{M} \) is the relaxation time of the internal variable \( Q \), called in the following \( \tau^{f} \). Moreover we have supposed the the body is at rest, so that material derivative coincide with the partial time derivative and \((\cdot),t\).

Equations \((112)\) and \((113)\) are analogous to equations (12) and (13) of \([29]\).
Eliminating $Q$ from (112) and (113) we can write the following equation of heat conduction

$$mMq_{tt} + \left( ML^{(1)} + mL^{(3)} \right) q_t - \left( mC^{(4)} + ML^{(2)} \right) q_{xx} + C^{(4)}L^{(2)}q_{xxxx}$$

$$- \left( L^{(1)}C^{(4)} + H \right) q_{xx} + L^{(3)}L^{(1)}q = M \left( \frac{1}{T} \right)_{,xt} + L^{(3)} \left( \frac{1}{T} \right)_{,x} - C^{(4)} \left( \frac{1}{T} \right)_{,xxx},$$  \hspace{1cm} (114)

where

$$H = L^{(3)}L^{(2)} - D^2.$$  \hspace{1cm} (115)

Thus, we derive

$$\tau^J q_{tt} + \tau^q q_t + q - \alpha q_{xx} + \beta q_{xxxx} - \gamma q_{xx} = \nu \left( \frac{1}{T} \right)_{,xt} - \lambda \left( \frac{1}{T} \right)_{,x} - \zeta \left( \frac{1}{T} \right)_{,xxx},$$  \hspace{1cm} (116)

where

$$\tau^q = \tau + \tau^J, \hspace{0.5cm} \alpha = \frac{mC^{(4)} + ML^{(2)}}{L^{(1)}L^{(3)}}, \hspace{0.5cm} \beta = \frac{C^{(4)}L^{(2)}}{L^{(1)}L^{(3)}},$$

$$\gamma = \frac{L^{(1)}C^{(4)} + H}{L^{(1)}L^{(3)}}, \hspace{0.5cm} \nu = \frac{M}{L^{(1)}L^{(3)}}, \hspace{0.5cm} \zeta = \frac{C^{(4)}}{L^{(1)}L^{(3)}}.$$  \hspace{1cm} (117)

In (116) we see that relaxation time $\tau^q = \tau + \tau^J$ is given by two contributions: the first comes from the relaxation time of the flux and the second comes from the relaxation time of the internal variable.

4.5.1. Special case. From (114), it is possible to derive some special cases.

**Ballistic-conductive.** In the case where $C^{(4)} = L^{(2)} = 0$, the heat equation (114) becomes:

$$mMq_{tt} + \left( ML^{(1)} + mL^{(3)} \right) q_t - D^2q_{xx} + L^{(3)}L^{(1)}q = M \left( \frac{1}{T} \right)_{,xt} + L^{(3)} \left( \frac{1}{T} \right)_{,x}.$$  \hspace{1cm} (119)

Thus, we can write

$$\tau^J q_{tt} + \tau^q q_t + q - \eta q_{xx} = \nu \left( \frac{1}{T} \right)_{,xt} - \lambda T_{,x},$$  \hspace{1cm} (120)

where $\eta = \frac{D^2}{L^{(1)}L^{(3)}}$.

**Guyer-Krumhansl.** In the case where $C^{(4)} = M = 0$, the heat equation (114) becomes:

$$mL^{(3)}q_{tt} - Hq_{xx} + L^{(3)}L^{(1)}q = L^{(3)} \left( \frac{1}{T} \right)_{,x}.$$  \hspace{1cm} (121)

then we work out

$$\tau q_t - l^2q_{xx} + q = -\lambda T_{,x},$$  \hspace{1cm} (122)

where

$$l^2 = \frac{H}{L^{(1)}L^{(3)}},$$  \hspace{1cm} (123)

with $l$ the free mean path of the heat carriers i.e. the average length between successive collision amongst them. We observe that only in Guyer-Krumhansl heat equation the coefficient mulytlying the field $q_{xx}$ has the physical meaning of $l^2$.

**Cahn-Hilliard type.** In the case where $C^{(4)} = M = m = 0$, the heat equation (114) becomes:

$$L^{(3)}L^{(1)}q - Hq_{xx} = L^{(3)} \left( \frac{1}{T} \right)_{,x},$$  \hspace{1cm} (124)

from which we obtain

$$q - \gamma q_{xx} = -\lambda T_{,x}.$$  \hspace{1cm} (125)

**Jeffreys type.** In the case where $C^{(4)} = L^{(2)} = m = D = 0$, the heat equation (114) becomes:

$$ML^{(1)}q_{tt} + L^{(3)}L^{(1)}q = M \left( \frac{1}{T} \right)_{,xt} + L^{(3)} \left( \frac{1}{T} \right)_{,x},$$  \hspace{1cm} (126)

thus we derive:

$$\tau^J q_{tt} + q = \nu \left( \frac{1}{T} \right)_{,xt} - \lambda T_{,x}.$$  \hspace{1cm} (127)

We note that in this equation $\tau^J$ is the relaxation time of $q$. 
Maxwell-Cattaneo-Vernotte. In the case where $C^{(4)} = M = L^{(2)} = D = 0$, the heat equation (114) becomes:

$$mq_t + L^{(1)} q = \left( \frac{1}{T} \right)_x,$$

from which we have:

$$\tau q_t + q = -\lambda T_x.$$ 

Fourier. In the case where $C^{(4)} = M = L^{(2)} = D = m = 0$, the heat equation (114) becomes:

$$L^{(1)} q = \left( \frac{1}{T} \right)_x,$$

i.e.

$$q = -\lambda T_x.$$

5. Discussion and conclusions

In this paper ballistic heat conduction in isotropic materials was treated in the framework of Non-Equilibrium Thermodynamics with Internal Variables (NET-IV). Onsager reciprocity was considered and the consequences were derived. Two dimensional formulas, that are suitable for numerical calculations are shown in the Appendix. The conditions of positive definiteness of the corresponding conductivity matrix can be calculated directly with the help of computer algebra programs. We have obtained a complete set of equations for generalized ballistic-conductive heat conduction in isotropic rigid conductors for the variables $T, q_i, Q_{ij}$. These are the balance of internal energy (1) with the caloric equation of state $s'_{eq}(e) = 1/T$ and the balance type constitutive equations (53) and (57) in the perfect isotropic case, or (82) and (91) with Onsagerian reciprocity. (53) and (82) turned out to be identical.

There are two different aspects of ballistic heat conduction in continua. From the point of view of kinetic theory it is the propagation of phonons without collisions with the lattice. Then heat is reflected only at the boundaries of the medium. This microscopic understanding is the foundation of the so called ballistic-diffusive integro-differential model of Chen [52, 53, 54, 55, 56] and leads to two independent continuum representations. First, it is a particular boundary condition for continuum theories that can be introduced also to second sound models, like Guyer-Krunhansl equation [57]. On the other hand for ballistic phonons the speed of propagation is equal to the speed of 'first' sound, the speed of elastic waves in the medium. The speed of propagation is independent of the boundary conditions in a continuum approach and this is the meaning of ballistic in our theory, in accordance with Rational Extended Thermodynamics (RET) [58, 22]. It is also remarkable that Chen’s model is equivalent to a two component extended continuum heat conduction theory as it was shown by Lebon et al. [59, 60].

Theories of Extended Thermodynamics (ET) assume that the constitutive equations are local, and the evolution equations are balances with a characteristic coupling, i.e. the previous fluxes being the consecutive state variables in the system. These two basic assumptions are consequences of the definition of the macroscopic fields as moments of the single particle phase space probability density and of the Boltzmann equation. In our case, with internal variables, this structure is the consequence of the second law and can be observed on the left hand side of (6) and (8). Then most important aspects of ET are well represented. On the other hand NET-TIV has many material coefficients that are missing in ET, in particular in Rational Extended Thermodynamics, where only the two relaxation times of the Callaway collision integral represent the material properties. This property of RET is very attractive, but the price is not only that the validity of the theory is connected to the particularities of the microscopic model, but also the speed of the ballistic propagation, the speed of elastic waves, can be obtained exactly only when considering the complete moment series, or practically with using dozens of evolution equations (with consecutively increasing tensorial orders) [22]. The low number of material coefficients leads to a lot of evolution equations in modelling ballistic propagation of heat.

Given the three dimensional structure of ET and NET-IV for heat conduction in case of isotropic materials opens the field to build and solve realistic models of two and three dimensional experimental setups, where the two theories lead to different predictions.

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Here we give a two dimensional symmetric representation of the conductivity matrix \( \{ L_{\alpha\beta} \} \). This form is useful when the conditions of positive definiteness are to be calculated. Entropy production \( \text{(43)} \) can be also written in the symbolic matrix notation:

\[
X_\alpha L_{\alpha\beta} X_\beta \geq 0, \tag{132}
\]

where

\[
\{ X_\alpha \} = \{ q_1 ; q_{k,j} ; Q_{ij} ; Q_{ij,p} \} = \{ q_1 ; q_2 ; q_3 ; q_{1,1} ; q_{1,2} ; q_{1,3} ; q_{2,1} ; q_{2,2} ; q_{2,3} ; q_3 ; q_{3,1} ; q_{3,2} ; q_{3,3} ; Q_{11,1} ; Q_{11,2} ; Q_{11,3} ; Q_{12,1} ; Q_{12,2} ; Q_{12,3} ; Q_{13,1} ; Q_{13,2} ; Q_{13,3} ; Q_{21,1} ; Q_{21,2} ; Q_{21,3} ; Q_{22,1} ; Q_{22,2} ; Q_{22,3} ; Q_{23,1} ; Q_{23,2} ; Q_{23,3} ; Q_{31,1} ; Q_{31,2} ; Q_{31,3} ; Q_{32,1} ; Q_{32,2} ; Q_{32,3} ; Q_{33,1} ; Q_{33,2} ; Q_{33,3} \}, \quad (\alpha = 1, \ldots , 48),
\]

\[
\{ X_\beta \} = \begin{cases} 
q_k \\
q_{k,l} \\
Q_{kl} \\
Q_{lm,n}
\end{cases}, \quad (\beta = 1, \ldots , 48), \tag{134}
\]

and for \( L_{\alpha\beta} \) we introduce the following notation:

\[
\{ L_{\alpha\beta} \} = \begin{pmatrix}
\mathcal{L}^{(1)}_{ik} & \mathcal{L}^{(1)}_{ilk} & \mathcal{L}^{(1)}_{lmn} \\
0 & \mathcal{L}^{(2)}_{jikl} & \mathcal{L}^{(2)}_{ijkl} \\
0 & 0 & \mathcal{L}^{(3)}_{ijkl} \\
0 & 0 & 0 & \mathcal{L}^{(4)}_{kpijlmn}
\end{pmatrix}, \quad (\alpha, \beta = 1, \ldots , 48), \tag{135}
\]

in which \( \mathcal{L}^{(n \times m)} \) is the symbolic null matrix of dimension \( n \times m \).

In the following we have written some sub-matrixes that appear in \( \text{(135)} \), in particular

\[
\mathcal{L}^{(1)}_{ik} = \begin{pmatrix} 
L^{(1)}_{ik} & 0 \\
0 & L^{(1)}_{ik} \\
0 & 0 & L^{(1)}_{ik}
\end{pmatrix}, \tag{136}
\]
\[
L_{i\text{mn}}^{(1,4)} = \begin{pmatrix}
L_{1111}^{(1,4)} & L_{1112}^{(1,4)} & L_{1113}^{(1,4)} \\
L_{1121}^{(1,4)} & L_{1122}^{(1,4)} & L_{1123}^{(1,4)} \\
L_{1211}^{(1,4)} & L_{1212}^{(1,4)} & L_{1213}^{(1,4)} \\
\vdots & \vdots & \vdots \\
L_{3311}^{(1,4)} & L_{3321}^{(1,4)} & L_{3331}^{(1,4)} \\
L_{3312}^{(1,4)} & L_{3322}^{(1,4)} & L_{3332}^{(1,4)} \\
L_{3313}^{(1,4)} & L_{3323}^{(1,4)} & L_{3333}^{(1,4)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
L_{1}^{(1,4)} & 0 & 0 \\
0 & L_{2}^{(1,4)} & 0 \\
0 & 0 & L_{3}^{(1,4)}
\end{pmatrix}
\]

(137)

where \( L^{(1,4)} = L_{1}^{(1,4)} + L_{2}^{(1,4)} + L_{3}^{(1,4)} \).
where $L(2) \equiv L_1(2) + L_2(2) + L_3(2)$.
\[
\mathcal{L}_{ijkl}^{(3)} = \begin{pmatrix}
\mathcal{L}_{1111}^{(3)} & \mathcal{L}_{1112}^{(3)} & \mathcal{L}_{1113}^{(3)} & \mathcal{L}_{1121}^{(3)} & \mathcal{L}_{1122}^{(3)} & \mathcal{L}_{1131}^{(3)} & \mathcal{L}_{1132}^{(3)} & \mathcal{L}_{1133}^{(3)} \\
\mathcal{L}_{1211}^{(3)} & \mathcal{L}_{1212}^{(3)} & \mathcal{L}_{1213}^{(3)} & \mathcal{L}_{1221}^{(3)} & \mathcal{L}_{1222}^{(3)} & \mathcal{L}_{1231}^{(3)} & \mathcal{L}_{1232}^{(3)} & \mathcal{L}_{1233}^{(3)} \\
\mathcal{L}_{1311}^{(3)} & \mathcal{L}_{1312}^{(3)} & \mathcal{L}_{1313}^{(3)} & \mathcal{L}_{1321}^{(3)} & \mathcal{L}_{1322}^{(3)} & \mathcal{L}_{1331}^{(3)} & \mathcal{L}_{1332}^{(3)} & \mathcal{L}_{1333}^{(3)} \\
\mathcal{L}_{2111}^{(3)} & \mathcal{L}_{2112}^{(3)} & \mathcal{L}_{2113}^{(3)} & \mathcal{L}_{2121}^{(3)} & \mathcal{L}_{2122}^{(3)} & \mathcal{L}_{2131}^{(3)} & \mathcal{L}_{2132}^{(3)} & \mathcal{L}_{2133}^{(3)} \\
\mathcal{L}_{2211}^{(3)} & \mathcal{L}_{2212}^{(3)} & \mathcal{L}_{2213}^{(3)} & \mathcal{L}_{2221}^{(3)} & \mathcal{L}_{2222}^{(3)} & \mathcal{L}_{2231}^{(3)} & \mathcal{L}_{2232}^{(3)} & \mathcal{L}_{2233}^{(3)} \\
\mathcal{L}_{2311}^{(3)} & \mathcal{L}_{2312}^{(3)} & \mathcal{L}_{2313}^{(3)} & \mathcal{L}_{2321}^{(3)} & \mathcal{L}_{2322}^{(3)} & \mathcal{L}_{2331}^{(3)} & \mathcal{L}_{2332}^{(3)} & \mathcal{L}_{2333}^{(3)} \\
\mathcal{L}_{3111}^{(3)} & \mathcal{L}_{3112}^{(3)} & \mathcal{L}_{3113}^{(3)} & \mathcal{L}_{3121}^{(3)} & \mathcal{L}_{3122}^{(3)} & \mathcal{L}_{3131}^{(3)} & \mathcal{L}_{3132}^{(3)} & \mathcal{L}_{3133}^{(3)} \\
\mathcal{L}_{3211}^{(3)} & \mathcal{L}_{3212}^{(3)} & \mathcal{L}_{3213}^{(3)} & \mathcal{L}_{3221}^{(3)} & \mathcal{L}_{3222}^{(3)} & \mathcal{L}_{3231}^{(3)} & \mathcal{L}_{3232}^{(3)} & \mathcal{L}_{3233}^{(3)} \\
\mathcal{L}_{3311}^{(3)} & \mathcal{L}_{3312}^{(3)} & \mathcal{L}_{3313}^{(3)} & \mathcal{L}_{3321}^{(3)} & \mathcal{L}_{3322}^{(3)} & \mathcal{L}_{3331}^{(3)} & \mathcal{L}_{3332}^{(3)} & \mathcal{L}_{3333}^{(3)}
\end{pmatrix} = \begin{pmatrix}
L_1^{(3)} & 0 & 0 & 0 & L_1^{(3)} & 0 & 0 & 0 \\
0 & L_2^{(3)} & 0 & L_3^{(3)} & 0 & 0 & 0 & 0 \\
0 & 0 & L_2^{(3)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & L_2^{(3)} & 0 & 0 & 0 & 0 \\
L_1^{(3)} & 0 & 0 & 0 & L_1^{(3)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L_1^{(3)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L_1^{(3)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L_1^{(3)} & 0 & 0 & 0
\end{pmatrix}
\]

where \(L_1^{(3)} = L_1^{(3)} + L_2^{(3)} + L_3^{(3)}\).
where $L^{(1,4)} = L_1^{(1,4)} + L_2^{(1,4)} + L_3^{(1,4)}$. 
\[
\begin{pmatrix}
C^{(4)} & 0 & 0 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & A_3 & 0 & 0 & 0 \\
0 & A_4 & 0 & A_5 & 0 & 0 & 0 & 0 & 0 & 0 & A_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C^{(4)}_1 & 0 & 0 & C^{(4)}_9 & 0 & 0 & 0 \\
0 & 0 & A_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C^{(4)}_1 & 0 & 0 & 0 & C^{(4)}_9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Equation (143)
where

\[ C(4) = 2C_1(4) + 2C_2(4) + C_3(4) + 2C_4(4) + C_5(4) + C_6(4) + C_7(4) + C_8(4) + C_9(4) + 2C_{10}(4) + C_{11}(4), \]

\[ A_1 = C_1(4) + C_4(4) + C_6(4), \quad A_2 = C_2(4) + C_4(4) + C_5(4), \quad A_3 = C_4(4) + C_6(4) + C_8(4), \]

\[ A_4 = C_7(4) + C_{10}(4) + C_{11}(4), \quad A_5 = C_4(4) + C_9(4) + C_{10}(4), \quad A_6 = C_6(4) + C_7(4) + C_8(4), \]

\[ A_7 = C_3(4) + C_4(4) + C_7(4), \quad A_8 = C_2(4) + C_8(4) + C_{10}(4), \quad A_9 = C_3(4) + C_4(4) + C_{11}(4). \]

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