Diffusion, Influence and Best-Response Dynamics in Networks: An Action Model Approach

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Abstract. Threshold models and their dynamics may be used to model the spread of ‘behaviors’ in social networks. Regarding such from a modal logical perspective, it is shown how standard update mechanisms may be emulated using action models – graphs encoding agents’ decision rules. A small class of action models capturing the possible sets of decision rules suitable for threshold models is identified, and shown to include models characterizing best-response dynamics of both coordination and anti- coordination games played on graphs. We conclude with further aspects of the action model approach to threshold dynamics, including broader applicability and logical aspects. Hereby, new links between social network theory, game theory and dynamic ‘epistemic’ logic are drawn.

An individual’s choice of phone, language use or convictions may be influenced by the people around her [12,22,23]. How a new trend spreads through a population depends on how agents are influenced by others, which in turn depends on the structure of the population and on how easy agents are to influence.

This paper focuses on one particular account of social influence, the notion of ‘threshold influence’ [15]. Threshold influence relies on a simple imitation or conformity pressure effect: agents adopt a behavior/fashion/semantics whenever some given threshold of their social network neighbors have adopted it already. So-called threshold models, introduced by [11,19], represent diffusion dynamics under threshold influence. Threshold models have received much attention in recent literature [10,13,16,21], also from authors in the logic community [16,19,18,20,24].

Fig. 1. (Definitions below): A threshold model with 5 agents, threshold \( \theta = \frac{1}{4} \), and behavior \( B \) marked by gray. Top: agents change behavior in accordance with equation (1) and the dynamics reach a fixed point. Bottom: agents update according to equation (2). Here, the dynamics loop.
In this paper, a novel approach to threshold models is taken by constructing the dynamics using action models and product update \cite{3,4,9}. In this context, an action model may be regarded as a graph that encodes decision rules. The product of a threshold model and an action model is again a threshold model, but where each agent has now updated their behavior according to the encoded decision rules.

The paper progresses as follows. First, threshold models and two typical update rules are introduced. We then introduce a modal language interpreted over threshold models, along with action models and product update. We produce an action model for each of the two introduced update rules, and show the step-wise equivalence of the two approaches. These two action models give rise to a small class of action models, which is investigated in relation to tie-breaking rules, coordination game and anti-coordination game best-response dynamics. We conclude with a discussion of further aspects of the action model approach to threshold dynamics, including broader applicability and logical aspects.

The motivation for the work is primarily technical. The author found it interesting that threshold dynamics could so straightforwardly be encoded using action models. There is however an interesting conceptual twist: action models are not interpreted as being informational events, but as encoding decision rules of agents. Hence, the class arising from the action model encoding best-responses in coordination games may be seen as containing all possible sets of decision rules compatible with agents acting under the used notion of threshold influence. The class contains variations of tie-breaking rules, and shows a neat symmetry: for each “coordination game action model”, the class contains a “dual” version for anti-coordination games. From a logical perspective, this class is interesting as each arising dynamics may be treated in a uniform manner, using the reduction axiom method well-known from dynamic epistemic logic \cite{3,8}.

1 Threshold Models and their Dynamics

Threshold Models. A threshold model includes a network $N$ of agents $\mathcal{A}$ and a behavior $B$ (or fashion, or product, or viral video) distributed over the agents. As such, it represents the current spread of $B$ through the network. An adoption threshold prescribes how the state will evolve: agents adopt $B$ when the proportion of their neighbors who have already adopted it meets the threshold. Formally, a threshold model is a tuple $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ where $\mathcal{A}$ is a finite set of agents, $N \subseteq \mathcal{A} \times \mathcal{A}$ a irreflexive and symmetric network, $B \subseteq \mathcal{A}$ a behavior, and $\theta \in [0, 1]$ the adoption threshold. For an agent $a \in \mathcal{A}$, her neighborhood is $N(a) := \{b : (a, b) \in N\}$.

Threshold Model Dynamics. Threshold models are used to investigate the spread of a behavior over discrete time-steps $t_0, t_1, \ldots$, i.e., the dynamics of the

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\footnote{The literature contains several variations, including infinite networks \cite{16}, non-inflating behavior \cite{16}, agent-specific thresholds, non-symmetric relations, weighted links \cite{14}, and multiple behaviors \cite{1}.}
behavior. Given an initial threshold model for \( t_0 \), \( M = \{A, N, B_0, \theta\} \), several update policies for the behavior set \( B_0 \) exists. One popular such is captured by (1):

\[
B_{n+1} = B_n \cup \left\{ a : \frac{|N(a) \cap B_n|}{|N(a)|} \geq \theta \right\},
\]

(1)

I.e., \( a \) plays (adopts, follows) \( B \) at \( t_{n+1} \) iff \( a \) plays \( B \) at \( t_n \), or a proportion of \( a \)'s neighbors larger or equal to the threshold plays \( B \) at \( t_n \).

The former disjunct makes \( B \) increase over time, i.e., \( \forall n : B_n \subseteq B_{n+1} \). This guarantees that (1) reaches a fixed point. The ‘or equal to’ embeds a tie-breaking rule favoring \( B \).

Inflation may be dropped and the tie-breaking rule changed by using e.g. the policy specified by (2) instead:

\[
B_{n+1} = \left\{ a : \frac{|N(a) \cap B_n|}{|N(a)|} > \theta \right\} \cup \left\{ a : \frac{|N(a) \cap B_n|}{|N(a)|} = \theta \text{ and } a \in B_n \right\}.
\]

(2)

The second set in the union invokes a conservative tie-breaking rule: if \( \frac{|N(a) \cap B_n|}{|N(a)|} = \theta \), \( a \) will continue her behavior from \( t_n \). That (2) does not cause \( B \) to inflate implies the possibility of loops in behavior, i.e. where \( B_n = B_{n+2} \neq B_{n+1} \). Thereby (2) does not necessarily reach a fixed point.

Threshold Model Dynamics as Induced by Game Play. Threshold influence may naturally be seen as an instance of a coordination problem: given enough of an agent’s neighbors adopt behavior \( B \), the agent will seek to coordinate with that group by adopting \( B \) herself. This coordination problem may be modeled as a coordination game

\[
\begin{align*}
B & \sim B \\
B \times x, x & 0, 0 \\
\sim B & 0, 0 y, y
\end{align*}
\]

played on the network: at each time-step, each agent chooses one strategy from \{\( B, \sim B \)\} and plays this strategy against all their neighbors simultaneously. Agent \( a \)'s payoff \( t_n \) is then the sum of the payoffs of the \( |N(a)| \) coordination games that \( a \) plays at time \( t_n \). With these rules, \( B \) is a best-response for agent \( a \) at time \( t_n \) iff

\[
x \cdot \frac{|N(a) \cap B_n|}{|N(a)|} \geq \theta \cdot \frac{|N(a) \cap \sim B_n|}{|N(a)|} \iff \frac{|N(a) \cap B_n|}{|N(a)|} \geq \frac{y}{x + y}.
\]

(3)

Setting \( \theta := \frac{y}{x+y} \), the right-hand side of (3) resembles the specifications from (1) and (2). The precise correlation is that (2) captures the best-response dynamics for coordination game play on networks when using conservative tie-breaking \([10]\), while (1) captures the same with tie-breaking biased towards \( B \) and the added assumption of a (possibly irrational) ‘seed’ of agents always playing \( B \) \([10]\).

Attention is here restricted to deterministic, discrete time simultaneous updates. See e.g. \([17]\) for stochastic processes.
2 Threshold Models, Kripke Models and Action Models

A threshold model gives rise to a Kripke model [5] with $\mathcal{A}$ as domain, $N$ as relation and a valuation $\| \cdot \| : \Phi \rightarrow \mathcal{P}(\mathcal{A})$, $\Phi := \{B\}$, determining the extension of the $B$ playing agents. To describe features of agents’ neighborhoods, we use a language $L$ with suitable threshold modalities:

$\top \mid B \mid \neg \varphi \mid (\leq) \varphi \mid (\leq) \neg \varphi \mid (\leq) \varphi$.

The three operators could be parametrized by $\theta$, but to lighten notation, we leave the threshold implicit.

Intuitively, if $a$ satisfies $(\leq) \varphi$, then there exists a $\theta$ ‘large enough’ set of $a$’s neighbors that satisfy $\varphi$. E.g., if $\varphi := B$, then at least a $\theta$ fraction of $a$’s neighbors satisfy $B$. According to (1), $a$ should then change his behavior to $B$.

The operator is inspired by [2,13] and exemplified in Fig. 2. $(\leq)$ is the universal ‘box’ to the existential ‘diamond’ $(\leq)$: if $a$ satisfies $(\leq) \varphi$, then all neighbors in all $\theta$ ‘large enough’ subsets of $a$’s neighborhood satisfy $\varphi$. Finally, $(\leq)$ captures that exactly a $\theta$ fraction of the agent’s neighbors satisfy $\varphi$. In particular, if $a$ satisfies $(\leq)\varphi$, then $a$ should invoke a tie-breaking rule.

With threshold $\theta$, satisfaction in $M$ is given by standard Boolean clauses and the following:

$M, a \models B$ iff $a \in B$

$M, a \models (\leq) \varphi$ iff $\exists C : \theta \leq \frac{|C \cap N(a)|}{|N(a)|}$ and $\forall a \in C, M, a \models \varphi$

$M, a \models [\leq] \varphi$ iff $\forall C : \theta \leq \frac{|C \cap N(a)|}{|N(a)|}$ implies $\forall a \in C, M, a \models \varphi$

$M, a \models (=) \varphi$ iff $\theta = \frac{|N(a) \cap \| \varphi \|_M|}{|N(a)|}$.

The extension of $\varphi$ in $M$ is denoted $\| \varphi \|_M := \{a \in A : M, a \models \varphi\}$.

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**Fig. 2.** A threshold model $M$ with $\theta = \frac{1}{4}$ and $B$ marked by gray. $b$ satisfies $(\leq)B$, as $M, a \models B$ and $\frac{|N(a)|}{|N(a)|} \geq \frac{1}{4}$. Agent $e$ satisfies $[\leq]B$ as $\forall C \subseteq N(e) : \frac{|C \cap N(e)|}{|N(e)|} \geq \theta$ (that is, for sets $\{c\}, \{d\}, \{c, d\})$, $C \subseteq \neg B_M$. Moreover, agent $a$ satisfies $\neg B \land (\leq)\neg B$—hence, according to [2], she then should start playing $\neg B$, whereas [1] will not allow her to change.

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From $(\leq), [\leq]$ and $(\leq)$, we define strict versions of the two former. These are useful when encoding non-biased tie-breaking rules:

$(\leq)\neg \varphi := (\leq)\varphi \land (\leq)\varphi$

$[\leq]\varphi := [\leq]\varphi \land (\leq)\varphi$.
Two comments on the threshold operators are due. First, the operators do not form the basis of a normal modal logic: \((\ldots)\) distributes over neither \(\lor\) nor \(\land\), and the ‘diamond’ \(\langle \leq \rangle\) does not distribute over \(\lor\). The ‘box’ \([\leq ]\) does validate \(K:\ [\leq ](\varphi \rightarrow \psi) \rightarrow ([\leq ]\varphi \rightarrow [\leq ]\psi)\) and thus distributes over \(\land\), but it is not the dual of \((\leq )\), i.e., \([\leq ]\varphi \leftrightarrow \neg(\langle \leq \rangle \neg \varphi)\) is not valid\(^3\). If \(|\mathcal{A}| > 1\), the right-to-left direction holds, but not vice versa. Second, \((\ldots)\) does not imply that \((\ldots)\neg\), as the semantics are given w.r.t. \(\theta\). \((\ldots)\) does not imply that \([\leq ](\varphi \rightarrow \psi) \rightarrow ([\leq ]\varphi \rightarrow [\leq ]\psi)\) and thus distributes over \(\land\), but it is not the dual of \((\langle \leq \rangle)\), i.e., \([\leq ]\varphi \leftrightarrow \neg[\langle \leq \rangle \neg \varphi]\) is not valid. If \(|\mathcal{A}| > 1\), the right-to-left direction holds, but not vice versa. Second, \((\ldots)\) does not imply that \((\ldots)\neg\), as the semantics are given w.r.t. \(\theta\). \((\ldots)\) does imply that \([\leq ](\varphi \rightarrow \psi) \rightarrow ([\leq ]\varphi \rightarrow [\leq ]\psi)\) and thus distributes over \(\land\), but it is not the dual of \((\langle \leq \rangle)\), i.e., \([\leq ]\varphi \leftrightarrow \neg[\langle \leq \rangle \neg \varphi]\) is not valid.

Action Models and Product Update. Rather than updating threshold models by analyzing best responses or consulting equations like \(\ref{eq:neg-box} \) or \(\ref{eq:neg-diamond} \), they may be updated by taking the graph-theoretical product with a graph that encodes decision rules, uniformly followed by all agents. Such graphs are known as action models (with postconditions) \([3,4,9]\). To illustrate, then (cf. Proposition 1 below) \(E_1\) captures the same dynamics as those invoked by \(\ref{eq:neg-box} \): 

\[
\begin{align*}
E_1: \quad & \sigma_1: ((\langle \leq \rangle B, B) \quad \sigma_2: (-\langle \leq \rangle B, \top))
\end{align*}
\]

In the current context, it is natural to interpret each state of an action models as a decision rule\(^5\). E.g., \(\sigma_1\) encodes the rule ‘if a \(\theta\) fraction or more of your neighbors play \(B\), then play \(B\)’. State \(\sigma_2\) encodes that if the agent is not influenced to play \(B\), she should continue her current behavior.

Formally, by an action model we here refer to a tuple \(E = (|E|, R, \text{cond})\) where \(|E|\) is a non-empty domain of states, \(R \subseteq |E| \times |E|\) is a relation on \(|E|\), and \(\text{cond}\) a pre- and postcondition map \(\text{cond} : |E| \rightarrow |E| \times \{B, \neg B, \top\}\) with \(\text{cond}(\sigma) = (\varphi, \psi) =: (\text{pre}(\sigma), \text{post}(\sigma))\).

The product update \([3,9]\) of threshold model \(M\) and action model \(E\) is the threshold model \(M \odot E = (A^\uparrow, N^\uparrow, B^\uparrow, \theta)\) with \(\theta\) from \(M\), and

\[
\begin{align*}
A^\uparrow & = \{(a, \sigma) \in A \times |E| : M, a \models \text{pre}(\sigma)\}, \\
N^\uparrow & \ni \{(a, \sigma), (b, \sigma')\} \text{ iff } (a, b) \in N \text{ and } (\sigma, \sigma') \in R, \text{ and} \\
B^\uparrow & = \{(s, \sigma) : s \in B \land \text{post}(\sigma) \neq \neg B\} \cup \{(s, \sigma) : \text{post}(\sigma) = B\}.
\end{align*}
\]

By the last condition, \(B^\uparrow\) consists of 1) the agents in \(B\) minus those who change to \(\neg B\), plus 2) the agents that change to \(B\). Hence every agent will after the update again only play one strategy. If \(\text{post}(\sigma) = \top\), no change in behavior is invoked.

\(3\) The latter was pointed out by Prof. A. Baltag for a similar operator in \([2]\).

\(4\) The dual of \(\langle \leq \rangle\) would have the universal quantifier in the semantic clause of \(\langle \leq \rangle\varphi\) replaced by an existential one.

\(5\) The relation between actions is merely a technicality and is not given an interpretation. Given a re-defined product operation that ignores the relation of the action model, it could from both a conceptual and technical point be omitted.
3 Action Model Dynamics

Considering threshold models as Kripke models, it is possible to construct action models that when applied using product update will produce model sequences step-wise equivalent to those produced by (1) and (2). Moreover, the used models (in particular $E_2$ below) gives rise to a simple class of action models. This class, specified below, contains all natural variations of the decision rules emulating (1) and (2). Thus, the class specifies all the different sets of decision rules by which agents may update their behavior while still behaving in the spirit of present notion of threshold influence.

**Proposition 1.** For any threshold model $M$, the action model $E_1$ applied using product update produces model sequences step-wise equivalent to those of (1).

**Proof.** Let $M = (A, N, B, \theta)$ be arbitrary with (1)-update $(A', N', B', \theta)$ and $E_1$-update $(A^+, N^+, B^+, \theta)$. Then $f: a \mapsto (a, \sigma), \sigma \in \{\sigma_1, \sigma_2\}$ is an isomorphism from $(A, N, B)$ to $(A^+, N^+, B^+)$. 1) $|A| = |A^+|$, as the preconditions of $E_1$ partitions $A$ entailing that no agents multiply or die under product update. 2) $(a, (b, b')) \in N^+$ iff $(a, b) \in N$: $R$ from $E_1$ is the full relation, so $N$ dictates $N^+$. 3) $f(B^+) = B^+$ as

$$a \in B^+ \quad \text{iff} \quad \frac{|N(a) \cap B|}{|N(a)|} \geq \theta$$

$$\exists C \subseteq N(a) \cap B : \frac{|C|}{|N(a)|} \geq \theta \quad \text{iff}$$

$$M, a \models (\leq)B$$

$$M, a \models \text{pre}(\sigma_1) \quad \text{iff}$$

$$M \otimes E_1, (a, \sigma_1) \models B \quad \text{iff} \quad f(a) \in B^+$$

The action model $E_1$ contains only two states as (1) invokes a biased tie-breaking rule, subsumed in the state $\sigma_1$ by using the non-strict $(\leq)B$ in the precondition. (2), in contrast, invokes a conservative, unbiased tie-breaking rule. This requires an extra state to encode:

$$E_2: \sigma_1: \frac{(\leq)B}{((\leq)B, B)} \quad \sigma_2: \frac{((\leq)B, T)}{([\leq]\neg B, B)} \quad \sigma_3: \frac{([\leq]\neg B, \neg B)}{\sigma_1: \sigma_2: \sigma_3:}$$

Interpreted as decision rules, $\sigma_1$ of $E_2$ states that if strictly more than a $\theta$ fraction of an agent $a$’s neighbors plays $B$, then $a$ should do the same; $\sigma_2$ embodies the conservative tie-breaking rule: if exactly a $\theta$ fraction of $a$’s neighbors play $B$ (and hence a $1 - \theta$ fraction plays $\neg B$), then $a$ should not change her behavior; finally, for $\sigma_3$, notice that if $[\leq]\neg B$, i.e., that all $\theta$ ‘strictly large enough’ subsets of $a$’s neighbors plays $\neg B$, then there is a strictly larger than $(1 - \theta)$ fraction of
her neighbors that play $\neg B$—$\sigma_3$ states that in that case, $a$ should also play $\neg B$.

**Proposition 2.** For any threshold model $M$, the action model $E_2$ applied using product update produces model sequences step-wise equivalent to those of $[2]$. 

*Proof.* Analogous to those of Propositions 1 and 3 (see below). $\Box$

### The Class of Threshold Model Update Action Models.
For the reasons mentioned in the proof of Proposition 1, for an action model to change neither agent set nor network when applied to an arbitrary threshold model, it must be fully connected and its preconditions must form a partition on the agent set. If one further accepts only preconditions that are in the spirit of standard threshold model updates, i.e., that agents change behavior based only on the behavior of their immediate neighbors, then the class of ‘threshold model update action models’ is easy to map. For by the latter restriction, $\langle \rangle B$, $= B$ and $\rangle \neg B$ form the unique finest partition on the agent set of any threshold model. Given the three possible postconditions $B$, $\top$ and $\neg B$, the class of suitable action models contains 27 models (Table 1).

| pre: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|---|---|---|---|---|---|---|---|
| $\sigma_1$: $\langle \rangle B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $\sigma_2$: $= B$ | $B$ | $B$ | $\top$ | $\top$ | $\top$ | $\neg B$ | $\neg B$ | $\neg B$ |
| $\sigma_3$: $\rangle \neg B$ | $B$ | $\top$ | $\neg B$ | $B$ | $\top$ | $\neg B$ | $B$ | $\top$ |

| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|----|----|----|----|----|----|----|----|----|
| $\sigma_1$: $\langle \rangle B$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ | $\top$ |
| $\sigma_2$: $= B$ | $B$ | $B$ | $B$ | $\top$ | $\top$ | $\top$ | $\neg B$ | $\neg B$ |
| $\sigma_3$: $\rangle \neg B$ | $B$ | $\top$ | $\neg B$ | $B$ | $\top$ | $\neg B$ | $B$ | $\top$ |

| 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
|----|----|----|----|----|----|----|----|----|
| $\sigma_1$: $\langle \rangle B$ | $\neg B$ | $\neg B$ | $\neg B$ | $\neg B$ | $\neg B$ | $\neg B$ | $\neg B$ | $\neg B$ |
| $\sigma_2$: $= B$ | $B$ | $B$ | $B$ | $\top$ | $\top$ | $\top$ | $\neg B$ | $\neg B$ |
| $\sigma_3$: $\rangle \neg B$ | $B$ | $\top$ | $\neg B$ | $B$ | $\top$ | $\neg B$ | $B$ | $\top$ |

Table 1. Each action model contains three states with preconditions specified by pre and postconditions by columns 1 to 27.

As mention, this class of action models may be seen as containing all the possible sets of decision rules compatible with the used notion of threshold influence. Using action models it is a simple, combinatorial task to map. This is a benefit of using action models to define dynamics over the set theoretic specification.

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6 The symmetric variant $\langle \rangle \neg B$, $= \neg B$ and $\rangle B$ is ignored as it is equivalent up to interchange of $B$ and $\neg B$. 
Dynamics Induced by Action Models. Note that the action model $E_1$ is not explicitly listed in Table 1. It is not so as $E_1$ is based on a coarser partition of the agent set, containing two rather than three cells. It is however equivalent to the listed model 2: simply collapse states $\sigma_1$ and $\sigma_2$ to one.

The class include three trivial dynamics induced by models 1, 14 and 27 and seven that make little sense (4, 7, 8, 16, 17 and 24).

The best-response dynamics of coordination games are emulated by models 3, 6 and 9, capturing discriminating (3, 9) and conservative (6) tie-breaking (cf. Proposition 2), while models 2, 5, 15 and 18 capture inflating (‘seeded’) coordination game dynamics.

Proposition 3 below lends credences to the conjecture that models 19, 22 and 25 capture the best-response dynamics for anti-coordination games with discriminating (19, 25) and conservative (22) tie-breaking, and that 10, 13, 23 and 26 capture inflating dynamics of anti-coordination games.

Proposition 3. For any threshold model, the best-response dynamics of the anti-coordination game

$$
\begin{array}{c|c}
B & \neg B \\
\neg B & \begin{array}{c}
0, 0 \\
y, x
\end{array}
\end{array}
$$

played with the conservative tie-breaking rule is step-wise equivalent to applying the action model 22 ($E_{22}$) from Table 1 with $\theta = \frac{x}{y + x}$.

Proof. Let $\mathcal{M} = (A, N, B, \theta)$. Playing $B$ is a best-response in $\mathcal{M}$ for agent $a$ iff

$$
y \cdot \frac{|N(a) \cap \neg B|}{|N(a)|} \geq x \cdot \frac{|N(a) \cap B|}{|N(a)|} \Leftrightarrow \frac{|N(a) \cap \neg B|}{|N(a)|} \geq \frac{y}{y + x} = \theta
$$

Hence, given the tie-breaking rule, the next set of $B$-players will be

$$B^+ = \{a : \frac{|N(a) \cap \neg B|}{|N(a)|} > \theta\} \cup \{a : \frac{|N(a) \cap \neg B|}{|N(a)|} = \theta \text{ and } a \in B\}.$$

Let $\mathcal{M} \otimes E_{22} = (A^+, N^+, B^+, \theta)$. Then $g : a \mapsto (a, \sigma), \sigma \in |E_{22}|$, is an isomorphism from $(A, N, B^+)$ to $(A^+, N^+, B^+_\theta)$. That $(A, N) \cong g (A^+, N^+)$ follows from the proof of Proposition 1.

$$a \in B^+ \text{ iff }\frac{|N(a) \cap \neg B|}{|N(a)|} > \theta \quad \text{or} \quad \frac{|N(a) \cap \neg B|}{|N(a)|} = \theta \text{ and } a \in B$$

$$\mathcal{M}, a \models (\neg B) \quad \text{or} \quad \mathcal{M}, a \models B \land (\neg B)$$

$$\mathcal{M}, a \models pre(\sigma_1) \quad \text{or} \quad \mathcal{M}, a \models B \land pre(\sigma_2)$$

$$\mathcal{M} \otimes E_{22}, (a, \sigma_1) \models B \quad \text{or} \quad \mathcal{M} \otimes E_{22}, (a, \sigma_2) \models B$$

(as $post(\sigma_1) = B$) \quad \text{(as $post(\sigma_2) = \top$)}$$

iff $g(a) \in B^+$ \quad \square
Logics for Threshold Dynamics. Given the uniform, action model approach to the dynamics outlined, it may be conjectured that the dynamics may also be treated by a uniform logical approach, particularly the reduction axiom method well-known from dynamic epistemic logic [38].

Three things are required to obtain a complete logic for one of the dynamics:

1. A complete axiomatization for the threshold operators \( \langle \leq \rangle, [\leq] \) and \( (=) \),
2. A complete axiomatization of the network properties, and
3. Reduction laws for the used action model.

For 1, one may search for results in the literature on probabilistic modal logic. No suitable, general result is known to the author. 2 is easily obtained, though it requires a richer language, extending \( \mathcal{L} \) with a normal modal operator \( \Diamond \) and hybrid logical nominals \( \{i, j, \ldots\} \). The latter is required to express the irreflexivity of the network relation, characterized by \( i \rightarrow \neg \Diamond i \). To complete a combined logic, interaction axioms for the thresholds operators and normal modal operators should also be added. A reduction axiom-based logic for action models with post-conditions already exists (the logic \( \text{UM} \) from [9]), but the system should be modified to suit the hybrid nominals and threshold modalities. If such a combined logic is obtained for one of the dynamics, one will automatically obtain complete logics for all of the 27 dynamics induced by the action models of Table 1, with the only variation between them being the used action model in the dynamic modalities.

4 An Action Model for ‘Belief Change in the Community’

One reviewer asked whether there is a relation between the action model approach used here, and the finite state automata approach introduced in [24] for threshold influence dynamics of preferences, and in particular, whether a translation between the two approaches exist. We conjecture that this is indeed the case. To lend credence to this conjecture, we show this may be done for the slightly simpler framework of threshold influence of belief change from [15].

The basic framework of [15] investigates the dynamics of strong and weak influence of beliefs among agents in a symmetric and irreflexive network. Beliefs are represented by three mutually exclusive atoms \( Bp, B\neg p \) and \( Up \), evaluated at agents in the network, as above. \( M, a \models Bp \) reads ‘\( a \) believes \( p \)’, and being undecided about \( p, Up \), is equivalent to \( \neg Bp \land \neg B\neg p \). To describe the network, a normal box operator \( F \) is used: \( M, a \models F\varphi \) iff \( \forall b \in N(a), M, b \models \varphi \). \( F \) has dual \( \langle F \rangle \) – \( \langle F \rangle \varphi \) reads ‘I have a friend that satisfies \( \varphi \)’. Call the language \( \mathcal{L}' \).

An agent is strongly influence to believe \( \varphi \in \{p, \neg p\} \) if all her friends believe \( \varphi \), and weakly influenced to believe \( \varphi \) if no friends believe \( \neg \varphi \) while at least one friend believes \( \varphi \). With

\[
S\varphi := FB\varphi \land \langle F \rangle B\varphi \quad \text{and} \\
W\varphi := F\neg B\neg \varphi \land \langle F \rangle B\varphi,
\]


the dynamics of strong and weak influence are then characterized by the finite state automaton in Fig. 3, applied to all agents simultaneously.

Fig. 3. The automaton of [15], which characterizes agents’ belief change under weak and strong influence. If an agent is undecided about \( p \), i.e., in the state \( Up \), and is strongly influenced to believe \( p \), \( Sp \), she will change to state \( Bp \), i.e., believe \( p \). The automaton is deterministic.

Given this setup, it is no hard task to construct an action model over \( L' \) that will invoke the same dynamics. This may be done systematically by the construction: 1) for each state-transition-state triple \((s, t, s')\) from the automaton, construct an action model state \( \sigma \) with the conjunction of the labels of \( s \) and \( t \) as precondition, and the label from \( s' \) as postcondition, and 2) let the relation of the action model be the full relation. The resulting action model \( I \) is depicted in Fig. 4. It is easy to verify that the effects of the two approaches are equivalent.

Fig. 4. The action model \( I \), invoking the same dynamics as the automaton of Fig. 3 (some edges are omitted). The top-most state makes an agent change from state \( Up \) to state \( B\neg p \) if the agents also is strongly influenced to believe \( \neg p \), etc.

The construction method used defines a function from automata to action models. If one restricts attention to action models with preconditions of the form \((\varphi \land \psi)\), a function from action models to automata may be defined by the construction: 1) for each action model state \( \sigma \), \( \text{cond}(\sigma) = ((\varphi \land \psi), \chi) \), construct a automaton state with label \( \varphi \) and one with label \( \chi \), and collapse all automata states with equivalent labels, and 2) for each all automaton states with labels \( \varphi \) and \( \chi \), add a transition with label \( \psi \) between them if there exists an action model state with \( \text{cond}(\sigma) = ((\varphi \land \psi), \chi) \). Combining the two constructions provides a bijection, serving as translation.

**A Logic for Belief Change in the Community.** Given that it is possible to emulate the dynamics invoked by the finite state automaton using an action
model, finding a sound and complete logic for the dynamics should be unproblematic. In fact, as $F$ is a normal modal operator, the case is simpler than for threshold dynamics. Again some hybrid machinery is required to capture the irreflexive frame condition, but if this requirement is dropped, the reduction axiom system from [9] provides the desired result.

5 Closing Remarks

It has been argued that action models may be used to emulate the best-response dynamics on coordination and anti-coordination games played on networks by showing the product updates equivalent to the threshold model dynamics induced by game play, and that the method is applicable to the framework of threshold influence from [15]. It is conjectured that the action model approach to threshold dynamics lightens the work of finding complete logics, using methods well-known from dynamic epistemic logic, hereby providing new connections between game theory, social network theory and dynamic ‘epistemic’ logic.

Two questions present themselves. First, is it possible to rationalize the seven unaccounted for action models in the identified class, by moving from action models to game playing situations? Second, what is the extent of the applicability of action models? The present paper utilizes only a fraction of the potential of action models, as such may also be used to systematically alter the agent set and network. Changing the agent set may be used to model agent death and birth, whereby deterministic SIRS-like epidemiological dynamics [17] may be captured. Alterations to the social network may be used to model e.g. rise in popularity of information sources.

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