Cameron–Storvick theorem associated with Gaussian paths on function space

Jae Gil Choi

School of General Education, Dankook University, Cheonan 31116, Republic of Korea

Abstract

The purpose of this paper is to provide a more general Cameron–Storvick theorem for the generalized analytic Feynman integral associated with Gaussian process \( Z_k \) on a very general Wiener space \( C_{a,b}[0,T] \). The general Wiener space \( C_{a,b}[0,T] \) can be considered as the set of all continuous sample paths of the generalized Brownian motion process determined by continuous functions \( a(t) \) and \( b(t) \) on \([0,T]\). As an interesting application, we apply this theorem to evaluate the generalized analytic Feynman integral of certain monomials in terms of Paley–Wiener–Zygmund stochastic integrals.

Keywords: Cameron–Storvick theorem, generalized analytic Feynman integral, Gaussian process, generalized Brownian motion process, Paley–Wiener–Zygmund stochastic integral.

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1. Introduction

Let \((C_0[0,T], \mathcal{W}, m)\) denote the classical Wiener space, where \( C_0[0,T] \) is the set of all \( \mathbb{R} \)-valued continuous functions \( x \) on \([0,T]\) with \( x(0) = 0 \), \( \mathcal{W} \) denotes the complete \( \sigma \)-field of all Wiener measurable subsets of \( C_0[0,T] \), and \( m \) denotes the Wiener measure characterized by

\[
m(\{x : x(t) \leq \tau\}) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\tau} \exp \left[ -\frac{u^2}{2t} \right] du.
\]

Using the Kolmogorov’s extension theorem (for instance, see [11, 19]), the Wiener space \( C_0[0,T] \) can be illustrated as the set of all (continuous) sample paths of the standard Brownian motion process (SBMP). In [1], Cameron provided an integration by parts formula for functionals on the classical Wiener space \( C_0[0,T] \). In [2], Cameron and Storvick developed the parts formula for the analytic Feynman integral of functionals on \( C_0[0,T] \). They also applied their

Email address: jgchoi@dankook.ac.kr (Jae Gil Choi)
result to establish the evaluation formula for the analytic Feynman integral of unbounded functionals on $C_0[0, T]$. The parts formula on $C_0[0, T]$ introduced in [1] also was developed in [14, 17] to establish various parts formulas for the analytic Feynman integral. The parts formula for the analytic Feynman integral is now called the Cameron–Storvick theorem.

On the other hand, the concept of the generalized Wiener integral and the generalized analytic Feynman integral on $C_0[0, T]$ were introduced in [9], and further developed and used in [3, 15, 16]. In [3, 9, 15, 16], the generalized Wiener integral was defined by the Wiener integral

$$
\int_{C_0[0, T]} F(Z_h(x, \cdot)) dm(x),
$$

where $Z_h(x, \cdot)$ is a Gaussian path defined by the Paley–Wiener–Zygmund (PWZ) stochastic integral [12, 13] as follows:

$$
Z_h(x, t) = \int_0^t h(s) dx(s) \text{ with } h \in L^2[0, T].
$$

The parts formula on the function space $C_{a,b}[0, T]$, which is a generalization of the Cameron–Storvick theorem was provided by Chang and Skoug in [8] and further developed in [6]. The function space $C_{a,b}[0, T]$ can be considered as the set of continuous sample paths of the generalized Brownian motion process (GBMP) determined by continuous functions $a(t)$ and $b(t)$ on $[0, T]$. A GBMP on a probability space $(\Omega, \mathcal{F}, P)$ and a time interval $[0, T]$ is a Gaussian process $Y \equiv \{Y_t\}_{t \in [0, T]}$ such that $Y_0 = 0$ almost surely, and for any cylinder set $I_{t_1, \ldots, t_n, B}$ having the form

$$
I_{t_1, \ldots, t_n, B} = \{\omega \in \Omega : (Y(t_1, \omega), \ldots, Y(t_n, \omega)) \in B\}
$$

with a set of time moments $0 = t_0 < t_1 < \cdots < t_n \leq T$ and a Borel set $B \subset \mathbb{R}^n$, the measure $P(I_{t_1, \ldots, t_n, B})$ of $I_{t_1, \ldots, t_n, B}$ is equal to

$$
\left(2\pi\right)^n \prod_{j=1}^n \left(b(t_j) - b(t_{j-1})\right)^{-1/2} \times \int_B \exp \left[ -\frac{1}{2} \sum_{j=1}^n \frac{[(u_j - a(t_j)) - (u_{j-1} - a(t_{j-1}))]^2}{b(t_j) - b(t_{j-1})} \right] du_1 \cdots du_n
$$

where $u_0 = 0$, $a(t)$ is a continuous real-valued function on $[0, T]$, and $b(t)$ is an increasing continuous real-valued function on $[0, T]$. For more details, see [18, 19]. Note that choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, one can see that the GBMP reduces a SBMP (or, Wiener process).

The aim of this paper is to provide a more general Cameron–Storvick theorem for the generalized analytic Feynman integral associated with Gaussian paths on the function space $C_{a,b}[0, T]$. As an application, we apply our general Cameron–Storvick theorem to evaluate the generalized analytic Feynman integral of certain monomials in terms of PWZ stochastic integrals.
In order to present our assertions, we assume that \( a(t) \) is an absolutely continuous real-valued function on \([0, T]\) such that \( a(0) = 0, a'(t) \in L^2[0, T] \), and
\[
\int_0^T |a'(t)|^2 dt < +\infty,
\]
where \(|a|()\) denotes the total variation function of the function \( a() \), and \( b(t) \) is an increasing, continuously differentiable real-valued function with \( b(0) = 0 \) and \( b'(t) > 0 \) for each \( t \in [0, T] \). We also assume familiarity with \([6, 8]\) and adopt the notation and terminologies of those papers. The basic concepts and definitions of the function space \((C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)\), which forms a complete probability space, the concept of the scale-invariant measurability on \( C_{a,b}[0, T] \), the Cameron–Martin space \( C'_{a,b}[0, T] \) and the PWZ stochastic integral on \( C_{a,b}[0, T] \) may also be found in \([4, 5]\). In particular, we refer to the reference \([5]\) for the definition and the properties of the Gaussian processes \( \mathcal{Z}_k \) used in this paper. However, in order to propose our assertions in this paper, we shall introduce the following terminologies:

(i) The Hilbert space: Let
\[
L^2_{a,b}[0, T] = \left\{ v : \int_0^T v^2(s) db(s) < +\infty \text{ and } \int_0^T v^2(s) d|a|(s) < +\infty \right\}.
\]
Then \( L^2_{a,b}[0, T] \) is a separable Hilbert space with the inner product given by
\[
(u, v)_{a,b} = \int_0^T u(t)v(t) dm_{|a,b}(t) = \int_0^T u(t)v(t) d|b(t)| + |a|(t),
\]
where \( m_{|a,b} \) denotes the Lebesgue–Stieltjes measure induced by \(|a|()\) and \( b()\).

(ii) The Cameron–Martin space in \( C_{a,b}[0, T] \): Let
\[
C'_{a,b}[0, T] = \left\{ w \in C_{a,b}[0, T] : w(t) = \int_0^t z(s) db(s) \text{ for some } z \in L^2_{a,b}[0, T] \right\}.
\]
Then \( C'_{a,b} = C'_{a,b}[0, T] \) with the inner product
\[
(w_1, w_2)_{C'_{a,b}} = \int_0^T Dw_1(t) Dw_2(t) db(t)
\]
is a separable Hilbert space, where the (homeomorphic) operator \( D : C'_{a,b}[0, T] \to L^2_{a,b}[0, T] \) is given by
\[
Dw(t) = z(t) = \frac{w'(t)}{b'(t)},
\]
\( \cdot \)

(iii) The PWZ stochastic integral: Let \( \{e_n\}_{n=1}^\infty \) be a complete orthonormal set in \((C'_{a,b}[0, T], \| \cdot \|_{C_{a,b}})\) such that the \( De_n \)'s are of bounded variation on \([0, T]\). Then for \( w \in C'_{a,b}[0, T] \) and \( x \in C_{a,b}[0, T] \), we define the PWZ stochastic integral \((w, x)\) as follows:
\[
(w, x) = \lim_{n \to \infty} \sum_{j=1}^n (w, e_j)_{C'_{a,b}} De_j(t)x(t).
\]

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if the limit exists. For each \( w \in C'_{a,b}[0, T] \), the PWZ stochastic integral \((w, x)^\sim\) exists for a.e. \( x \in C_{a,b}[0, T] \).

### 2. Gaussian processes on \( C_{a,b}[0, T] \)

In order to present our Cameron–Storvick theorem on the function space \( C_{a,b}[0, T] \), we follow the exposition of [4, 5, 7].

Let \( C_{a,b}^*[0, T] \) be the set of functions \( k \) in \( C_{a,b}'[0, T] \) such that \( Dk \) is continuous except for a finite number of finite jump discontinuities and is of bounded variation on \([0, T]\). For any \( w \in C_{a,b}'[0, T] \) and \( k \in C_{a,b}^*[0, T] \), let the operation \( \circ \) between \( C_{a,b}'[0, T] \) and \( C_{a,b}^*[0, T] \) be defined by

\[
w \circ k = D^{-1}(DwDk), \quad \text{i.e.,} \quad D(w \circ k) = DwDk,
\]

where \( DwDk \) denotes the pointwise multiplication of the functions \( Dw \) and \( Dk \). Then \( (C_{a,b}^*[0, T], \circ) \) is a commutative algebra with the identity \( b \).

For each \( t \in [0, T] \), let \( \Phi_t(\tau) = D^{-1}\chi_{[0,t]}(\tau) = \int_0^\tau \chi_{[0,t]}(u)db(u), \tau \in [0, T], \) and for \( k \in C_{a,b}^*[0, T] \) with \( Dk \neq 0 \) \( m_L \)-a.e. on \([0, T] \) (\( m_L \) denotes the Lebesgue measure on \([0, T] \)), let \( Z_k(x, t) \) be the PWZ stochastic integral

\[
Z_k(x, t) = (k \circ \Phi_t, x)^\sim. \tag{2.1}
\]

Let \( \gamma_k(t) = \int_0^t Dk(u)da(u) \) and let \( \beta_k(t) = \int_0^t (Dk(u))^2db(u) \). Then the stochastic process \( Z_k : C_{a,b}[0, T] \times [0, T] \rightarrow \mathbb{R} \) is Gaussian with mean function

\[
\int_{C_{a,b}[0, T]} Z_k(x, t)d\mu(x) = \int_0^t h(u)da(u) = \gamma_k(t)
\]

and covariance function

\[
\int_{C_{a,b}[0, T]} (Z_k(x, s) - \gamma_k(s))(Z_k(x, t) - \gamma_k(t))d\mu(x)
= \int_0^{\min\{s, t\}} \{Dk(u)^2\}db(u) = \beta_k(\min\{s, t\}).
\]

In addition, by [7, Theorem 21.1], \( Z_k(\cdot, t) \) is stochastically continuous in \( t \) on \([0, T] \). If \( Dk \) is of bounded variation on \([0, T] \), then, for all \( x \in C_{a,b}[0, T] \), \( Z_k(x, t) \) is continuous in \( t \). Of course if \( k(t) \equiv b(t) \), then \( Z_b(x, t) = x(t) \), the continuous sample paths of the GBMP \( Y \), which consist the function space \( C_{a,b}[0, T] \). Furthermore, if \( a(t) \equiv 0 \) and \( b(t) = t \) on \([0, T] \), then the function space \( C_{a,b}'[0, T] \) reduces to the classical Wiener space \( C_0[0, T] \) and the Gaussian process \( \{Z_k \} \) with \( k(t) \equiv t \) is a standard Brownian motion process.

Given any \( w \in C_{a,b}'[0, T] \) and \( k \in C_{a,b}^*[0, T] \), it follows that

\[
(w, Z_k(x, \cdot))^\sim = (w \circ k, x)^\sim \tag{2.2}
\]

for \( \mu \)-a.e \( x \in C_{a,b}[0, T] \).
In order to establish our Cameron–Storvick theorem for functionals on $C_{a,b}[0,T]$, we define a class $\text{Supp}_{C_{a,b}^*}[0,T]$ as follows:

$$\text{Supp}_{C_{a,b}^*}[0,T] = \{ k \in C_{a,b}^*[0,T] : Dk \neq 0 \text{ m}_L\text{-a.e. on } [0,T] \}.$$ 

**Remark 2.1.** (i) The space $(\text{Supp}_{C_{a,b}^*}[0,T], \odot)$ forms a monoid. The variance function $b(\cdot)$ of the GBMP $Y$ is the identity in the space $(\text{Supp}_{C_{a,b}^*}[0,T], \odot)$.

(ii) Given a function $k$ in $\text{Supp}_{C_{a,b}^*}[0,T]$, the process $Z_k$ on $C_{a,b}[0,T] \times [0,T]$ is the GBMP determined by the functions $\gamma_k$ and $\beta_k$.

For any $k \in \text{Supp}_{C_{a,b}^*}[0,T]$, the Lebesgue–Stieltjes integrals

$$\|w \odot k\|^2_{C_{a,b}'} = \int_0^T (Dw(t))^2 (Dk(t))^2 db(t),$$

and

$$(w \odot k, a)_{C_{a,b}'} = \int_0^T Dw(t)Dk(t)Da(t)db(t) = \int_0^T Dw(t)Dk(t)da(t)$$

exist for all $w \in C_{a,b}'[0,T]$. Throughout the remainder of this paper, we thus require $k$ to be in $\text{Supp}_{C_{a,b}^*}[0,T]$ for the process $Z_k$.

### 3. Parts formula for functionals in Gaussian paths

In [1], Cameron derived an integration by parts formula for functionals on the Wiener space $C_0[0,T]$. The parts formula involved the first variation (a kind of Gâteaux derivative) of functionals on $C_0[0,T]$. In this section we establish an integration by parts formula for functionals in Gaussian paths on the function space $C_{a,b}[0,T]$. To do this we first provide the definition of the first variation of functionals on the function space $C_{a,b}[0,T]$.

**Definition 3.1.** Let $F$ be a $\mathcal{W}(C_{a,b}[0,T])$-measurable functional on $C_{a,b}[0,T]$ and let $w \in C_{a,b}[0,T]$. Then given two functions $k_1$ and $k_2$ in $C_{a,b}[0,T]$, 

$$\delta_{k_1,k_2} F(x|w) \equiv \delta F(Z_{k_1}(x,\cdot)|Z_{k_2}(w,\cdot)) = \left. \frac{\partial}{\partial \alpha} F(Z_{k_1}(x,\cdot)+\alpha Z_{k_2}(w,\cdot)) \right|_{\alpha=0}$$

(if it exists) is called the first variation of $F$ in the direction $w$.

**Remark 3.2.** Setting $k_1 = k_2 \equiv b$ on $[0,T]$, our definition of the first variation reduces to the first variation studied in [4, 8]. That is,

$$\delta_{b,b} F(x|w) = \delta F(x|w).$$
Let $Z_k$ be the Gaussian process given by (2.1) on $C_{a,b}[0,T] \times [0,T]$. We define the $Z_k$-function space integral (namely, the function space integral associated with the Gaussian paths $Z_k(x, \cdot)$) for functionals $F$ on $C_{a,b}[0,T]$ by the formula

$$E_x[F(Z_k(x, \cdot))] = \int_{C_{a,b}[0,T]} F(Z_k(x, \cdot))d\mu(x)$$

whenever the integral exists.

In order to establish an integration by parts formula for functionals in Gaussian paths on $C_{a,b}[0,T]$, we need a translation theorem for the function space integral. The following translation theorem is due to Chang and Choi [4].

**Theorem 3.3.** Let $k_1$ be a function in $\text{Supp}_{C_{a,b}'}[0,T]$ and let $F$ be a functional on $C_{a,b}[0,T]$ such that $F(Z_k(x, \cdot))$ is $\mu$-integrable over $C_{a,b}[0,T]$. Then for any $\theta \in C_{a,b}'[0,T]$ and $k_2 \in \text{Supp}_{C_{a,b}'}[0,T],$

$$E_x[F(Z_{k_1}(x, \cdot) + Z_{k_2}(\theta \circ k_1, \cdot))]
= \exp \left[-\frac{1}{2}\|\theta \circ k_2\|_{C_{a,b}'}^2 - (\theta \circ k_2, a)_{C_{a,b}'} \right]
\times E_x\left[F(Z_{k_1}(x, \cdot)) \exp \left[(\theta, Z_{k_2}(x, \cdot))^\sim \right]\right].$$

We are now ready to present our integration by parts formula for functionals in Gaussian paths on $C_{a,b}[0,T]$.  

**Theorem 3.4.** Let $k_1$ and $k_2$ be functions in $\text{Supp}_{C_{a,b}'}[0,T]$, let $\theta$ be a function in $C_{a,b}'[0,T]$, and let $F$ be a functional on $C_{a,b}[0,T]$ such that $F(Z_{k_1}(x, \cdot))$ is $\mu$-integrable over $C_{a,b}[0,T]$. Furthermore assume that

$$E_x\left[|\delta F(Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \circ k_1, \cdot)|\right] < +\infty. \quad (3.3)$$

Then

$$E_x\left[\delta F(Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \circ k_1, \cdot))\right]
= E_x\left[\theta (Z_{k_2}(x, \cdot))^\sim F(Z_{k_1}(x, \cdot)) \right] - (\theta \circ k_2, a)_{C_{a,b}'} E_x\left[F(Z_{k_1}(x, \cdot))\right]. \quad (3.4)$$

**Proof.** By using (3.1) and (3.2), it follows that

$$E_x\left[\delta F(Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \circ k_1, \cdot))\right]
= E_x\left[\frac{\partial}{\partial \alpha} F(Z_{k_1}(x, \cdot) + \alpha Z_{k_2}(\theta \circ k_1, \cdot)) \right]_{\alpha=0}
= \frac{\partial}{\partial \alpha} \left(E_x\left[F(Z_{k_1}(x, \cdot) + Z_{k_2}(\theta \circ k_1, \cdot))\right]\right)_{\alpha=0}
= \frac{\partial}{\partial \alpha} \left(\exp \left[-\frac{\alpha^2}{2}\|\theta \circ k_2\|_{C_{a,b}'}^2 - \alpha(\theta \circ k_2, a)_{C_{a,b}'} \right]
\times E_x\left[F(Z_{k_1}(x, \cdot)) \exp \left[\alpha(\theta, Z_{k_2}(x, \cdot))^\sim \right]\right]\right)_{\alpha=0}
= E_x\left[(\theta, Z_{k_2}(x, \cdot))^\sim F(Z_{k_1}(x, \cdot)) - (\theta \circ k_2, a)_{C_{a,b}'} E_x\left[F(Z_{k_1}(x, \cdot))\right]\right]. \quad (3.5)$$
The second equality of (3.5) follows from (3.3) and Theorem 2.27 in [10]. □

4. Cameron–Storvick theorem for the generalized analytic Feynman integral associated with Gaussian paths

In this section, we establish the Cameron–Storvick theorem for the generalized analytic Feynman integral of functionals \( F \) on the function space \( C_{a,b} \). We begin this section with the definition of the generalized analytic Feynman integral associated with Gaussian process \( Z_k \) (Feynman integral) on \( C_{a,b} \).

Throughout the remainder of this paper, let \( \mathbb{C}_+ \) and \( \tilde{\mathbb{C}}_+ \) denote the set of complex numbers with positive real part, and non-zero complex numbers with nonnegative real part, respectively. For each \( \lambda \in \mathbb{C} \), \( \lambda^{1/2} \) denotes the principal square root of \( \lambda \); i.e., \( \lambda^{1/2} \) is always chosen to have nonnegative real part, so that \( \lambda^{-1/2} = (\lambda^{-1})^{1/2} \) is in \( \mathbb{C}_+ \) for all \( \lambda \in \tilde{\mathbb{C}}_+ \).

**Definition 4.1.** Given a function \( k \in \text{Supp}_a \), let \( Z_k \) be the Gaussian process given by (2.1) and let \( F \) be a \( \mathbb{C} \)-valued scale-invariant measurable functional on \( C_{a,b} \) such that the generalized \( Z_k \)-function space integral (namely, the function space integral associated with the Gaussian paths \( Z_k(x, \cdot) \))

\[
J_F(Z_k; \lambda) = E_x[F(\lambda^{-1/2}Z_k(x, \cdot))]
\]

exists and is finite for all \( \lambda > 0 \). If there exists a function \( J^*_F(Z_k; \lambda) \) analytic on \( \mathbb{C}_+ \) such that \( J^*_F(Z_k; \lambda) = J_F(Z_k; \lambda) \) for all \( \lambda \in (0, +\infty) \), then \( J^*_F(Z_k; \lambda) \) is defined to be the analytic \( Z_k \)-function space integral (namely, the analytic function space integral associated with the Gaussian paths \( Z_k(x, \cdot) \)) of \( F \) over \( C_{a,b} \) with parameter \( \lambda \), and for \( \lambda \in \mathbb{C}_+ \) we write

\[
E^{an}_x[F(Z_k(x, \cdot))] = \int_{C_{a,b}[0,T]} F(Z_k(x, \cdot))d\mu(x) = J^*_F(Z_k; \lambda).
\]

Let \( q \) be a non-zero real number and let \( F \) be a scale-invariant measurable functional whose analytic \( Z_k \)-function space integral, \( E^{an}_x[F(Z_k(x, \cdot))] \), exists for all \( \lambda \in \mathbb{C}_+ \). If the following limit exists, we call it the generalized analytic \( Z_k \)-Feynman integral of \( F \) with parameter \( q \), and we write

\[
E^{anf}_x[F(Z_k(x, \cdot))] = \int_{C_{a,b}[0,T]} F(Z_k(x, \cdot))d\mu(x) = \lim_{\lambda \to -iq} E^{an}_x[F(Z_k(x, \cdot))].
\] (4.1)

We are now ready to establish a Cameron–Storvick type theorem for our generalized analytic Feynman integral. It will be helpful to establish the following lemma before giving the main theorem.
Lemma 4.2. Let $k_1$, $k_2$, $\theta$, and $F$ be as in Theorem 3.4. For each $\rho > 0$, assume that $F(\rho Z_{k_1}(x, \cdot))$ is $\mu$-integrable over $C_{a,b}[0,T]$. Furthermore assume that for each $\rho > 0$,

$$E_x[\delta F(\rho Z_{k_1}(x, \cdot)|\rho Z_{k_2}(\theta \circ k_1, \cdot))] < +\infty.$$  

Then

$$E_x[\delta F(\rho Z_{k_1}(x, \cdot)|\rho Z_{k_2}(\theta \circ k_1, \cdot))]$$

$$= E_x[(\theta, Z_{k_2}(x, \cdot))^\sim F(\rho Z_{k_1}(x, \cdot))] - (\theta \circ k_2, a)_{C_{a,b}} E_x[F(\rho Z_{k_1}(x, \cdot))].$$  

(4.2)

Proof. Let $G(x) = F(\rho x)$. Then

$$G(Z_{k_1}(x, \cdot) + \alpha Z_{k_2}(w, \cdot)) = F(\rho Z_{k_1}(x, \cdot) + \rho \alpha Z_{k_2}(w, \cdot))$$

and

$$\frac{\partial}{\partial \alpha} G(Z_{k_1}(x, \cdot) + \alpha Z_{k_2}(w, \cdot)) \bigg|_{\alpha = 0} = \frac{\partial}{\partial \alpha} F(\rho Z_{k_1}(x, \cdot) + \rho \alpha Z_{k_2}(w, \cdot)) \bigg|_{\alpha = 0}.$$  

Thus $\delta G(Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \circ k_1, \cdot)) = \delta F(\rho Z_{k_1}(x, \cdot)|\rho Z_{k_2}(\theta \circ k_1, \cdot))$. Hence by equation (4.1) with $F$ replaced with $G$, we have

$$E_x[\delta F(\rho Z_{k_1}(x, \cdot)|\rho Z_{k_2}(\theta \circ k_1, \cdot))]$$

$$= E_x[\delta G(Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \circ k_1, \cdot))]$$

$$= E_x[(\theta, Z_{k_2}(x, \cdot))^\sim G(Z_{k_1}(x, \cdot))] - (\theta \circ k_2, a)_{C_{a,b}} E_x[G(Z_{k_1}(x, \cdot))]$$

$$= E_x[(\theta, Z_{k_2}(x, \cdot))^\sim F(\rho Z_{k_1}(x, \cdot))] - (\theta \circ k_2, a)_{C_{a,b}} E_x[F(\rho Z_{k_1}(x, \cdot))]$$

which establishes (4.2). \qed

Next we provide the Cameron–Storvick theorem for the generalized analytic $Z_t$-Feynman integral on the function space $C_{a,b}[0,T]$.

Theorem 4.3. Let $k_1$, $k_2$, $\theta$, and $F$ be as in Lemma 4.2. Then if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:

$$E_{x}^{\text{ant}}\left[\delta F(Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \circ k_1, \cdot))\right]$$

$$= -iqE_{x}^{\text{ant}}\left[(\theta, Z_{k_2}(x, \cdot))^\sim F(Z_{k_1}(x, \cdot))\right]$$

$$- (-iq)^{1/2}(\theta \circ k_2, a)_{C_{a,b}}^{\text{ant}} E_{x}^{\text{ant}}\left[F(Z_{k_1}(x, \cdot))\right].$$  

(4.3)

Proof. Given $\rho > 0$ and $\theta \in C_{a,b}^{r}[0,T]$, let $\theta_\rho = \frac{1}{\rho} \theta$. Then $\theta_\rho$ is a function in $C_{a,b}^{r}[0,T]$, and $\theta \circ k_1 = \rho \theta_\rho \circ k_1$. By equation (4.2) with $\theta$ replaced with $\theta_\rho$,

$$E_{x}\left[\delta F(\rho Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \circ k_1, \cdot))\right]$$

$$= E_{x}\left[\delta F(\rho Z_{k_1}(x, \cdot)|\rho Z_{k_2}(\theta_\rho \circ k_1, \cdot))\right]$$

$$= E_{x}\left[(\theta_\rho, Z_{k_2}(x, \cdot))^\sim F(\rho Z_{k_1}(x, \cdot))\right] - (\theta_\rho \circ k_2, a)_{C_{a,b}} E_{x}[F(\rho Z_{k_1}(x, \cdot))]$$

$$= \rho^{-2} E_{x}\left[(\theta_\rho, Z_{k_2}(x, \cdot))^\sim F(\rho Z_{k_1}(x, \cdot))\right]$$

$$- \rho^{-1}(\theta \circ k_2, a)_{C_{a,b}} E_{x}[F(\rho Z_{k_1}(x, \cdot))].$$  

(4.3)
Now let $\rho = \lambda^{-1/2}$. Then equation (4.3) becomes
\[
E_x[\delta F(\lambda^{-1/2}Z_{k_1}(x, \cdot)|Z_{k_2}(w, \cdot))] \\
= \lambda E_x[(\theta, \lambda^{-1/2}Z_{k_2}(x, \cdot)\sim F(\lambda^{-1/2}Z_{k_1}(x, \cdot))] \\
- \lambda^{1/2}(\theta \otimes k_2, a)_{C_{a,b} E_x}[F(\lambda^{-1/2}Z_{k_1}(x, \cdot))].
\] (4.4)

Since $\rho > 0$ was arbitrary, we have that equation (4.4) holds for all $\lambda > 0$. We now use Definition 4.1 to obtain our desired conclusions. □

**Corollary 4.4.** Under the assumptions as given in Theorem 4.3, it follows that if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:
\[
E^\text{anf}_x[(\theta, Z_{k_2}(x, \cdot))\sim F(Z_{k_1}(x, \cdot))] \\
= \frac{i}{q} E^\text{anf}_x[\delta F(Z_{k_1}(x, \cdot)|Z_{k_2}(\theta \otimes k_1, \cdot))] \\
+ (-iq)^{-1/2}(\theta \otimes k_2, a)_{C_{a,b} E_x}[F(Z_{k_1}(x, \cdot))].
\] (4.5)

**Remark 4.5.** As commented in Section 2 above, if $k \equiv b$ on $[0, T]$, then $Z_b(x, t) = x(t)$ for each $x \in C_{a,b}[0, T]$. In this case the generalized analytic $Z_b$-Feynman integral $E^\text{anf}_x[F(Z_b(x, \cdot))]$ agrees with the previous definition of the generalized analytic Feynman integral $E^\text{anf}_x[F(x)]$, see [2, 3].

In view of Remarks 3.2 and 4.5, we have the following corollary.

**Corollary 4.6 (6).** Let $\theta$ be a function in $C'_{a,b}[0, T]$, and let $F$ be a functional on $C_{a,b}[0, T]$ such that for each $\rho > 0$, $F(\rho x)$ is $\mu$-integrable over $C_{a,b}[0, T]$. Furthermore assume that for each $\rho > 0$,
\[
E_x[|\delta F(\rho Z_b(x, \cdot)|\rho Z_b(\theta \otimes b, \cdot))|] \equiv E_x[|\delta F(\rho x)|] < +\infty.
\]
Then if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:
\[
E^\text{anf}_x[\delta F(x|\theta)] = -iq E^\text{anf}_x[(\theta, x)\sim F(x)] - (-iq)^{1/2}(\theta, a)_{C_{a,b}E_x}[F(x)].
\]

The formulas and results in this paper are more complicated than the corresponding formulas and results in [1, 2, 13, 17] because the Gaussian process used in this paper is neither centered nor stationary in time. However, by choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the function space $C_{a,b}[0, T]$ reduces to the Wiener space $C_0[0, T]$, and so the expected results on $C_0[0, T]$ are immediate corollaries of the results in this paper.
5. Generalized analytic Feynman integral of monomials in terms of PWZ stochastic integrals

When we evaluate the following generalized analytic Feynman integral

\[
E_{x}^{\text{anal}} \left[ \prod_{j=1}^{m} (\theta \circ k_{j}, x) \right]
\]  

we might not be able to use the change of variables theorem of the usual measure theory, because the set of Gaussian random variables \((\theta \circ k_{j}, x)\), \(j = 1, \ldots, m\), are generally not independent. In this case, to apply the change of variables theorem for the calculation of \((5.1)\), we might apply the Gram–Schmidt process for the set of functions \(\{\theta \circ k_{1}, \ldots, \theta \circ k_{m}\}\).

Using equation (4.1), we indeed see that the generalized Feynman integral of functionals having the form \((5.1)\) can be calculated very explicitly. In this section we present interesting examples to which equation \((5.1)\) can be applied.

**Example 5.1.** Let \(k_{1}\) and \(k_{2}\) be functions in \(\text{Supp}_{\mathcal{C}^{\prime}_{a,b}}[0, T]\), and given a function \(\theta\) in \(\mathcal{C}^{\prime}_{a,b}[0, T]\), set \(F(x) = (\theta, x)\) for \(x \in \mathcal{C}_{a,b}[0, T]\). Then using equations (4.1) and (2.2), it follows that for any \(w\) in \(\mathcal{C}_{a,b}[0, T]\),

\[
\delta F(Z_{k_{1}}(x, \cdot)|Z_{k_{2}}(w, \cdot)) = \frac{\partial}{\partial \alpha} \{(\theta, Z_{k_{1}}(x, \cdot)) - \alpha(\theta, Z_{k_{2}}(w, \cdot))\} \bigg|_{\alpha=0} = (\theta, Z_{k_{2}}(w, \cdot)) = (\theta, k_{2}, w) = (\theta \circ k_{2}, w). \]

From this, we see that

\[
\delta F(Z_{k_{1}}(x, \cdot)|Z_{k_{2}}(\theta \circ k_{1}, \cdot)) = (\theta \circ k_{2}, \theta \circ k_{1})_{\mathcal{C}^{\prime}_{a,b}}. \tag{5.2}
\]

Also using (4.1), it follows that

\[
E_{x}^{\text{anal}}[F(Z_{k_{1}}(x, \cdot))] = E_{x}^{\text{anal}}[(\theta \circ k_{1}, x)]
\]

\[
= E_{x}^{\text{anal}}[(\theta \circ k_{1}, x) - i(q^{-1}2(\theta \circ k_{1}, a))_{\mathcal{C}^{\prime}_{a,b}}. \tag{5.3}
\]

Next using equations (4.1), (5.2), and (5.3), we obtain the formula

\[
E_{x}^{\text{anal}}[(\theta \circ k_{2}, \theta \circ k_{1}, x)]
\]

\[
= \int_{\mathcal{C}_{a,b}[0, T]} \delta F(Z_{k_{1}}(x, \cdot)|Z_{k_{2}}(\theta \circ k_{1}, \cdot)) \, dm(x)
\]

\[
= E_{x}^{\text{anal}}[(\theta \circ k_{2}, \theta \circ k_{1}, x)]
\]

\[
= \frac{i}{q} E_{x}^{\text{anal}}[F(Z_{k_{1}}(x, \cdot)] F(Z_{k_{2}}(\theta \circ k_{1}, \cdot))]
\]

\[
+ (-iq)^{-1}2(\theta \circ k_{2}, a)_{\mathcal{C}^{\prime}_{a,b}} E_{x}^{\text{anal}}[F(Z_{k_{1}}(x, \cdot)] F(Z_{k_{2}}(\theta \circ k_{1}, \cdot))]
\]

\[
= \frac{i}{q} (\theta \circ k_{2}, \theta \circ k_{1})_{\mathcal{C}^{\prime}_{a,b}} + \frac{i}{q} (\theta \circ k_{2}, a)_{\mathcal{C}^{\prime}_{a,b}} (\theta \circ k_{1}, a)_{\mathcal{C}^{\prime}_{a,b}}. \tag{5.4}
\]
In our next example, for any positive integer \( m \in \{3, 4, \ldots \} \), we obtain a recurrence relation for the generalized analytic Feynman integral

\[
E_{anf}^\ast \left[ \prod_{j=1}^{m} (\theta \circ k_j, x) \sim \right].
\]

**Example 5.2.** For a positive integer \( m \geq 3 \), let \( \{k_1, \ldots, k_{m-1}, k_m\} \) be a finite set of functions in \( \text{Supp}_{C_{a,b}^\ast}[0,T] \), and given a function \( \theta \in C_{a,b}^\prime[0,T] \), set

\[
F(x) = \prod_{j=1}^{m-1} (\theta \circ k_j, x) \sim = \prod_{j=1}^{m-1} (\theta \circ k_j, Z_b(x, \cdot)) \sim.
\]

First, using equation (3.1), it follows that for all \( w \in C_{a,b}^\prime[0,T] \),

\[
\delta F(z_{k_m}(w, \cdot)) = \delta F(Z_b(x, \cdot)|Z_{k_m}(w, \cdot))
\]

\[
= \left. \frac{\partial}{\partial \alpha} \prod_{j=1}^{m-1} \left\{ (\theta \circ k_j, Z_b(x, \cdot)) \sim + \alpha(\theta \circ k_j, Z_{k_m}(w, \cdot)) \sim \right\} \right|_{\alpha=0}
\]

\[
= \sum_{l=1}^{m-1} \left( \prod_{j=1}^{m-1} (\theta \circ k_j, Z_b(x, \cdot)) \sim \right) (\theta \circ k_l, Z_{k_m}(w, \cdot)) \sim.
\]

Then, in particular, it follows that

\[
\delta F(z_b(x, \cdot)|Z_{k_m}(\theta \circ b, \cdot)) = \delta F(Z_b(x, \cdot)|Z_{k_m}(\theta, \cdot))
\]

\[
= \sum_{l=1}^{m-1} \left( \prod_{j=1}^{m-1} (\theta \circ k_j, Z_b(x, \cdot)) \sim \right) (\theta \circ k_l, Z_{k_m}(\theta, \cdot)) \sim
\]

\[
= \sum_{l=1}^{m-1} \left( \prod_{j=1}^{m-1} (\theta \circ k_j, Z_b(x, \cdot)) \sim \right) (\theta \circ k_l, \theta \circ k_m)_{C_{a,b}}.
\]

Next, using (4.5) with \( k_1 \) and \( k_2 \) replaced with \( b \) and \( k_m \), respectively, and with \( F(x) = \prod_{j=1}^{m-1} (\theta \circ k_j, x) \sim \), \((5.5)\), and the equation \( Z_b(x, \cdot) = x \), it follows
that
\[ E_{x}^\text{anf}_q \left[ \prod_{j=1}^{m} (\theta \odot k_j, x)^\sim \right] \]
\[ = E_{x}^\text{anf}_q \left[ (\theta, Z_{k_m}(x, \cdot))^{\sim} \prod_{j=1}^{m-1} (\theta \odot k_j, Z_b(x, \cdot))^{\sim} \right] \]
\[ = E_{x}^\text{anf}_q \left[ (\theta, Z_{k_m}(x, \cdot))^{\sim} F(Z_b(x, \cdot)) \right] \]
\[ = \frac{i}{q} E_{x}^\text{anf}_q \left[ \delta F(Z_b(x, \cdot)|Z_{k_m}(\theta \odot b, \cdot)) \right] \]
\[ + (-iq)^{-1/2}(\theta \odot k_m, a)_{C^r_{u,b}} E_{x}^\text{anf}_q \left[ F(Z_b(x, \cdot)) \right] \]
\[ = \frac{i}{q} \sum_{l=1}^{m-1} (\theta \odot k_l, \theta \odot k_m)_{C^r_{u,b}} E_{x}^\text{anf}_q \left[ \prod_{j=1}^{m-1} (\theta \odot k_j, x)^\sim \right] \]
\[ + (-iq)^{-1/2}(\theta \odot k_m, a)_{C^r_{u,b}} E_{x}^\text{anf}_q \left[ \prod_{j=1}^{m-1} (\theta \odot k_j, x)^\sim \right]. \]

Remark 5.3. Letting \( m = 3 \) in equation (5.6) and applying equations (5.3) and (5.4) allows us to easily and completely calculate the generalized Feynman integral
\[ E_{x}^\text{anf}_q \left[ (\theta \odot k_1, x)^\sim (\theta \odot k_2, x)^\sim (\theta \odot k_3, x)^\sim \right] \]
\[ = E_{x}^\text{anf}_q \left[ (\theta, Z_{k_1}(x, \cdot))^{\sim} (\theta, Z_{k_2}(x, \cdot))^{\sim} (\theta, Z_{k_3}(x, \cdot))^{\sim} \right]. \]

Then setting \( m = 4 \) in equation (5.6) allows us to completely evaluate the generalized Feynman integral
\[ E_{x}^\text{anf}_q \left[ (\theta \odot k_1, x)^\sim (\theta \odot k_2, x)^\sim (\theta \odot k_3, x)^\sim (\theta \odot k_4, x)^\sim \right] \]
\[ = E_{x}^\text{anf}_q \left[ (\theta, Z_{k_1}(x, \cdot))^{\sim} (\theta, Z_{k_2}(x, \cdot))^{\sim} (\theta, Z_{k_3}(x, \cdot))^{\sim} (\theta, Z_{k_4}(x, \cdot))^{\sim} \right], \]

since we already have complete evaluation formulas for
\[ E_{x}^\text{anf}_q \left[ \prod_{j=1}^{l} (\theta \odot k_j, x)^\sim \right], \quad l = 1, 2, 3. \]

Then we can evaluate
\[ E_{x}^\text{anf}_q \left[ \prod_{j=1}^{l} (\theta \odot k_j, x)^\sim \right], \]

since we have already evaluated
\[ E_{x}^\text{anf}_q \left[ \prod_{j=1}^{l} (\theta \odot k_j, x)^\sim \right] \]

for \( l = 1, 2, 3 \) and 4; etc.
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