LOGIC BLOG 2017

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• putting up results and their proofs for further research
• parking results for later use
• getting feedback before submission to a journal
• foster collaboration.

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H. Towsner, Computability of Ergodic Convergence. In André Nies (editor), Logic Blog, 2012, Part 1, Section 1, available at http://arxiv.org/abs/1302.3686.

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References

Part 1. Computability theory

1. Open questions from Capulalpan retreat, December 2016

Eight researchers met on a four-day retreat in Mexico. Here is a collection of open questions that were discussed.

1.1. Jason Rute and Rutger Kuyper.

Theorem 1.1 (Miller and Kuyper). \( A \in 2^\omega \) is K-trivial iff \( \forall X \in 2^\omega [X \text{ is MLR } \rightarrow X \triangle A \text{ is MLR}] \).

Kuyper and Rute think they can generalize this theorem to compact groups—\( \triangle \) is replaced with a group operation, MLR is for the Haar measure, and K-trivial is as in the Melnikov and Nies paper [18]. However, the following similar questions remain open.

In this next question, \( X/A \) means the bits of \( X \) selected using \( A \). A significant difference here is that selection is not invertable and therefore not a group operation or even a group action.

Open Question 1. Is the following true? A set \( A \in 2^\omega \) is K-trivial iff \( \forall X \in 2^\omega [X \text{ is MLR } \rightarrow X/A \text{ is MLR}] \).

In this next question, \( E(2) \) is the (non-Abelian) group of orientation preserving transformations of \( \mathbb{R}^2 \). \( E(2) \) is entirely composed of transformations \( T \) where \( T \) is a rotation followed by a translation, so \( E(2) \) can be thought of as the space \( T \times \mathbb{R}^2 \) (where \( T \) is the circle) with the appropriate group action. (For a similar question one could consider the compact group \( SO(2) \), the orientation preserving transformations of the sphere.) \( E(2) \) acts on \( \mathbb{R}^2 \), and the Lebesgue measure is the unique locally finite invariant Borel measure (up to scaling).

Open Question 2. Is the following true? A transformation \( A \in E(2) \) is K-trivial iff \( \forall (x, y) \in 2^\omega [(x, y) \text{ is MLR } \rightarrow A(x, y) \text{ is MLR}] \).

Since this group is a merely acting on \( \mathbb{R}^2 \) it doesn’t seem to instantly follow from Kuyper and Rute’s result mentioned above. In particular, \( E(2) \) has an extra dimension, so there are continuum-many ways to send \( (x_1, y_1) \) to \( (x_2, y_2) \). This may or may not change the answer.

There are many more questions of this type. Here is one more.

Open Question 3. Is the following true? A real \( r \in (0, \infty) \) is K-trivial iff for all MLR Brownian motions \( B \), we have that \( B(r) \) is MLR (for the Lebesgue measure).
1.2. **Brown Westrick.** Let $\mathcal{P} \subseteq 2^\omega$ be a subshift. Let $D(\mathcal{P})$ be the set of effective Hausdorff dimensions of members of $\mathcal{P}$. By Simpson paper, $D(\mathcal{P}) \subseteq [0, h(\mathcal{P})]$ where $h$ denotes the topological entropy of $\mathcal{P}$ (inf over $n$ of log of number of $n$-patterns occurring divided by $n$). If $\mathcal{P}$ is of finite type then we have equality.

Say that $x$ is $d,b$-shift-complex if $K(\sigma) \geq d|\sigma| - b$ for each pattern $\sigma$ that occurs in $x$ (see BSL 2013 survey by Khan). Say that $\mathcal{P}$ is $d, b$-shift complex if each $x \in \mathcal{P}$ is $d,b$-shift complex. In this case $d$ is a lower bound for $D(\mathcal{P})$.

**Open Question 4.** Is $D(\mathcal{P})$ necessarily closed?

**Open Question 5.** Is some $x \in \mathcal{P}$ necessarily $h(\mathcal{P}), O(1)$ shift complex?

1.3. **Denis Hirschfeldt.** For definitions see below.

**Open Question 6.** Are all 1-random sets quasiminimal in the uniform dense degrees?

**Open Question 7.** Are there minimal pairs in the [uniform or nonuniform] [generic or effective dense] degrees?

Related results on the four other reducibilities below by Igusa; Hirschfeldt, Jockusch, Kuyper and Schupp; Cholak and Igusa (contains some work joint with Hirschfeldt); Astor, Hirschfeldt and Jockusch (in preparation).

**Definition 1.2.** Let $g : \omega \to \omega$. A **partial description** of $g$ is a partial function $f : \omega \to \omega$ such that $f(n) = g(n)$ whenever $f(n)$ is defined. A **generic description** of $g$ is a partial description of $g$ with domain of density 1.

A **dense description** of a function $g : \omega \to \omega$ is a partial function $f : \omega \to \omega$ such that $f(n) \downarrow = g(n)$ on a set of density 1.

For a function $f : \omega \to \omega \cup \{\square\}$, the **strong domain** of $f$ is $f^{-1}(\omega)$. Let $g : \omega \to \omega$. A **strong partial description** of $g$ is a (total) function $f : \omega \to \omega \cup \{\square\}$ such that $f(n) = g(n)$ on the strong domain of $f$. An **effective dense description** of $g$ is a strong partial description of $f$ with strong domain of density 1.

- We say that $h$ is **nonuniformly generically reducible** to $g$, and write $h \preceq_{ug} g$, if for every generic description $f$ of $g$, there is an enumeration operator $W$ such that $W^{\text{graph}(f)}$ enumerates the graph of a generic description of $h$.
- We say that $h$ is **uniformly generically reducible** to $g$, and write $h \preceq_{ug} g$, if there is an enumeration operator $W$ such that if $f$ is a generic description of $g$, then $W^{\text{graph}(f)}$ is a generic description of $h$.
- We say that $h$ is **nonuniformly densely reducible** to $g$, and write $h \preceq_{nd} g$, if for every dense description $f$ of $g$, there is an enumeration operator $W$ such that $W^{\text{graph}(f)}$ enumerates the graph of a dense description of $h$.
- We say that $h$ is **uniformly densely reducible** to $g$, and write $h \preceq_{ud} g$, if there is an enumeration operator $W$ such that if $f$ is a dense description of $g$, then $W^{\text{graph}(f)}$ is a dense description of $h$.
- We say that $h$ is **nonuniformly effectively densely reducible** to $g$, and write $h \preceq_{ned} g$, if every effective dense description of $g$ computes an effective dense description of $h$. 

• We say that \( h \) is \textit{uniformly effectively densely reducible} to \( g \), and write \( h \leq_{ued} g \), if there is a Turing functional \( \Phi \) such that if \( f \) is an effective dense description of \( g \), then \( \Phi f \) is an effective dense description of \( h \).

Let
\[
\mathcal{R}(A) = \{2^n k : n \in A \land k \text{ odd}\}.
\]
Let \( J_n = [2^n, 2^{n+1}) \) and let
\[
\tilde{\mathcal{R}}(A) = \bigcup_{n \in A} J_n.
\]
Let \( \mathcal{E}(A) = \tilde{\mathcal{R}}(\mathcal{R}(A)) \). This operator induces embeddings of the Turing degrees into all of the degree structures arising from the above reducibilities. In any of these structures, a degree is \textit{quasiminimal} if it is not above any nontrivial degree in the image of the embedding induced by \( \mathcal{E} \).

1.4. \textbf{Andre Nies}. For \( K \)-trivial sets \( A, B \) we say that \( A \leq_{ML} B \) if every ML-random oracle \( Z \) computing \( B \) computes \( A \).

\textbf{Open Question 8.} Is \( \leq_{ML} \) arithmetical?

Note that by Gandy basis theorem, if \( A \not\leq_{ML} B \) then there is a witness \( Z \leq_T \mathcal{O} \).

A \( K \)-trivial set \( A \) is called smart if every ML-random \( Z \geq_T A \) computes all the \( K \)-trivials. Equivalently, \( A \) is ML-complete for the \( K \)-trivials. It is not even clear whether smartness is arithmetical. See Section 3 on the Logic Blog 2016 for detail.

\textbf{Open Question 9.} Can a smart \( K \)-trivial be cappable? Is there a Turing minimal pair of smart \( K \)-trivials?

\textbf{Open Question 10.} Suppose \( A \) is \( K \)-trivial. Is \( A \) Turing below each LR-hard ML-random?

\textbf{Open Question 11.} Is weak 2-randomness closed upward under \( \leq_K \)? (Miller and Yu).

\textbf{Open Question 12.} Is weak 2-randomness closed downward within the 1-randoms under \( \leq_{LR} \)?

One could also try to show that \( e + \pi \not\in \mathbb{Q} \). Or that \( e\pi \not\in \mathbb{Q} \). Good news: at least one of them holds.

2. \textbf{Khan, Nies: SNR functions versus DNR functions}

We study three closely related mass problems, and also their variants where a computable growth bound is imposed on the functions.

\textbf{Definition 2.1.}

(i) A function \( f : \omega \to \omega \) is \textit{strongly nonrecursive} (or \textit{SNR}) if for every recursive function \( g \), for all but finitely many \( n \in \omega \), \( f(n) \neq g(n) \).

(ii) A function \( f \) is \textit{strongly non-partial-recursive} (or \textit{SNPR}) if for every partial recursive function \( \psi \), for all but finitely many \( n \), if \( \psi(n) \) is defined, then \( f(n) \neq \psi(n) \).
(iii) A function $f$ is diagonally non recursive (or DNR) if $f(n) \neq J(n)$ whenever $J(n)$ is defined. Here $J$ is a fixed universal p.r. function, e.g. $J(n) \simeq \phi_n(n)$, though below we will use a different one.

Trivially SNPR implies SNR. Also, if $f$ is SNPR then a finite variant of $f$ is DNR. SNR has an analog in cardinal characteristics called $b(\neq^*)$ ([8, Section 6]). Anything in computability involving enumeration/partiality fails to have such an analog.

We will show that every non-high SNR function is SNPR and hence computes a DNR. Also, every DNR function computes an SNPR function. We can also keep track of bounds on the functions that are order functions (OF) as defined below. Theorems 3.8 and 3.10 in [16] yield a downward and an upward growth hierarchy within DNR: for every OF $g$, there is a (much faster growing) OF $h$ such that there is an $h$-bounded DNR function that computes no $g$-bounded DNR function. A similar result holds with $g$ and $h$ interchanged. Our translation between SNR and DNR can be used to obtain similar hierarchy results for SNR.

If $A$ is high then it computes a function $f$ dominating all computable functions, which is in particular SNR. On the other hand not each high set $A$ computes a DNR function (e.g., a high incomplete r.e. set $A$ doesn’t). We discuss that outside the high degrees, the degree classes of such functions are the same. We also check how potential computable bounds on the functions change when going from one class to the other. The facts suggests that SNR for the same bound is stronger. However, going from DNR to SNR the loss is still within the elementary.

By the following, highness is the only reason an SNR function can fail to compute a DNR function. The result is due to Kjos-Hansen, Merkle, and Stephan [17, Thm. 5.1 (1) → (2)]

**Proposition 2.2.** Every non-high SNR function is SNPR and hence computes a DNR.

**Proof.** Suppose that $f : \omega \to \omega$ is not high, and that $\psi$ is a partial recursive function that is infinitely often equal to it. For each $n \in \omega$, let $g(n)$ be the least stage such that $|\{x \in \omega : \psi(x)[g(n)] \downarrow = f(x)\}| \geq 2n$. Then $g$ is recursive in $f$.

Since $f$ is not high, there is a recursive function $h$ that escapes $g$ infinitely often. We define a recursive function $j$ that is infinitely often equal to $f$. Let $j_0 = \emptyset$. Given $j_n$, let

$$A = \{\langle x, \psi(x) \rangle : x \notin \text{dom}(j_n), \psi(x)[h(n)] \downarrow \}.$$ 

Let $y$ be the least such that it is not in the domain of $j_n \cup A$. Finally, let

$$j_{n+1} = j_n \cup A \cup \{\langle y, 0 \rangle\}.$$ 

Clearly $j = \bigcup_n j_n$ is recursive. To see that $j(x) = f(x)$ for infinitely many $x$, take $n$ such that $h(n) > g(n)$. Then there are $2n$ many $x$ such that we have a coincidence $f(x) = \psi(x)[h(n)]$. We have lost at most $n$ coincidences by defining $j_{k+1}(y) = 0$ at stages $k < n$. Thus $j_{n+1}(x) = f(x)$ for at least $n$ many $x$. \hfill \Box

**Definition 2.3.** An order function is a recursive, nondecreasing, and unbounded function $p : \omega \to \omega$ such that $p(0) \geq 2$. 
**Definition 2.4.** For a class \( C \) of functions from \( \omega \) to \( \omega \) and an order function \( p \), let \( C_p \) denote the subclass consisting of those functions \( f \) such that \( f(n) < p(n) \) for each \( n \).

In the following we define the universal p.r. functional by

\[
J(2^e(2x + 1)) \simeq \phi_e(x).
\]

**Proposition 2.5.** Every DNR function \( g \) (for \( J \) as above) computes an SNPR function \( h \). The reduction is fixed, i.e. DNR \( \geq_S \) SNR (Medvedev). Furthermore, if \( g \in \text{DNR}_p \) then we can arrange \( h \in \text{SNPR}_q \) where \( q(n) = \prod_{1 \leq i \leq n} p(C_i) \) for some constant \( C \).

**Proof.** We modify the argument in the proof of \cite[Thm. 7]{14}.

Given a fixed effective encoding of tuples of natural numbers by natural numbers, we let \((n)_u\) be the \( u \)-th entry of the tuple coded by \( n \), if any, and vacuously \((n)_u = 0\) otherwise. Let \( r \) be a computable function such that \( J(r(u)) \simeq J(u)_u \) for each \( u \). (Thus, \( r(u) = 2^i(2u + 1) \) where \( i \) is an index such that \( \phi_j(u) \simeq J(u)_u \).)

Let \( d \) be a computable function such that \( n = o(d(n)) \), e.g. \( d(n) = n \log n \). Now let \( h \) be a computable function such that

\[
\forall u \leq d(e) \ h(e)_u = f(r(u)).
\]

That is, \( h(e) \) encodes the initial segment of values of the function \( g \circ r \) up to length \( d(e) \).

Since \( g \) is DNR, for each \( u \leq d(e) \) we have

\[
h(e)_u = g(r(u)) \neq J(r(u)) = J(u)_u.
\]

In particular, for \( d(e) \geq u = 2^i(2e + 1) \) we have \( h(e) \neq J(u) \simeq \phi_i(e) \). Thus \( h \) is SNPR.

Suppose \( f \in \text{DNR}_p \). We can choose the encoding of initial segments \( g \circ r \downarrow_{d(e)} \) via numbers of size bounded by \( q \) (with \( C = 2^{j+2} \), \( j \) the index given above). Thus we can ensure \( h < q \). \( \square \)

**Question 2.6.** Is there an order function \( p \) and a DNR \( p \) that does not compute an SNR \( p \)?

3. **Khan, Beros, Nies, Kjos-Hanssen: Potential weakening of effective (bi-)immunity**

The researchers above discussed the following during Nies’ visit at UHM in October 2016.

\( A \subseteq \mathbb{N} \) is immune if it contains no infinite c.e. set. Starting from Post, and then Jockusch and others, people studied an effective version of this: \( A \) is effectively immune (e.i.) if there is a computable function \( h \) such that \( W_e \subseteq A \rightarrow |W_e| \leq h(e) \). Also \( A \) is effectively bi-immune if \( A, \mathbb{N} - A \) are e.i. There is a lot of work comparing the degrees of such sets with the degrees of d.n.c. functions: Jockusch 1989 show that these degrees coincide with the degrees of e.i. sets, and Lewis and Jockusch 2013 that every d.n.c. computes a bi-immune. Beros showed that not every d.n.c. computes an e.b.i.

Now let \((R_e)\) be a listing of the computable sets, say \( R_e \) is the ascending part of \( W_e \), only admitting an element if it is greater than the previously
enumerate ones. As every infinite c.e. set has an infinite computable subset, immunity doesn’t change when we restrict to computable instead of c.e. subsets. This may be different for the effective versions, which could be weaker in the sense of Muchnik reducibility. Define computably e.i., computably e.b.i. as above but using the listing \( \langle R \rangle \).

**Open Question 13.**
(i) Does every computably e.i. set compute an e.i. set?
(ii) Does every computably e.b.i. set compute an e.b.i. set?

**Part 2. Higher computability theory/effective descriptive set theory**

4. **Yu: \( \Delta^1_2 \)-degree determinacy**

This is joint work with CT Chong and Liuzhen Wu.

It is obvious (by Mansfield-Solovay’s argument) that if \( A \) is \( \Sigma^1_2 \) and not thin, then \( A \) ranges over an upper cone of \( L \)-degrees. Now it was asked by some people in the Dagstuhl workshop end of February whether it can range over an upper cone of \( \Delta^1_2 \)-degrees. The subtle thing is that if one does a Cantor-Bendixson derivation over a Suslin representation of non-thin \( \Sigma^1_2 \) set, it may go through \( (\omega_1)^L \) which is bigger than \( \delta^1_1 \), the least ordinal which cannot be represented by a \( \Delta^1_2 \) well ordering over \( \omega \). However, the answer is still yes.

**Proposition 4.1.** Suppose that there is a nonconstructible real. If \( A \) is \( \Sigma^1_2 \) and not thin, then \( A \) ranges over an upper cone of \( \Delta^1_2 \)-degrees. Actually there is a \( \Delta^1_2 \)-coded perfect set \( S \subseteq A \).

**Proof.** Since \( A \) is \( \Sigma^1_2 \), there is a \( \Pi^1_1 \) set \( B \subseteq (\omega^\omega)^2 \) so that \( \forall x(x \in A \leftrightarrow \exists y((x, y) \in B)) \). By \( \Pi^1_1 \)-uniformization, we may assume that \( \forall x(\exists y(x, y) \in B \rightarrow \exists! y(x, y) \in B) \).

By Slaman’s result, \( A \) must contain a nonconstructible real. Then so is \( B \) and so \( B \) is not thin. Now let

\[
C = \{ T \subseteq (\omega^\omega)^2 \mid [T] \subseteq B \land \forall (\sigma, \tau) \in T \exists (\sigma_0, \tau_0) \in T \exists (\sigma_1, \tau_1) \in T \\
\sigma_0 \succ \sigma \land \sigma_1 \succ \sigma \land \sigma_0[\sigma_1] \}
\]

Then \( C \) is a \( \Pi^1_1 \) nonempty set and so must contain an element \( T \in \Delta^1_2 \). Now it is easy to construct a \( \Delta^1_2 \)-coded perfect set \( S \subseteq A \) from \( T \). \( \square \)

If the assumption of Proposition 4.1 is dropped, then the first part still holds. Also note that the argument in Proposition 4.1 does not work if the assumption is dropped. For example, \( x \in L \cap \mathbb{R} \) if and only if there is a real \( y \in L_{\omega_1^y} \) so that \( x \leq_T y \). Then let \( B = \{(x, y) \mid x \leq_T y \land y \in L_{\omega_1^y} \} \) be a \( \Pi^1_1 \)-thin set. We have that \( x \in L \cap \mathbb{R} \leftrightarrow \exists y(x, y) \in B \).

**Proposition 4.2.** If \( A \) is a ZFC-provable \( \Delta^1_2 \) non-thin set, then \( A \) contains a \( \Delta^1_2 \)-perfect subset.

**Proof.** By Proposition 4.1, it is sufficient to assume that every real is constructible. So there is a perfect tree \( T \in L \) so that \( T \subseteq A \). Adding a Cohen \( g \) real to \( V \), then by Shoenfield absoluteness, \( V[g] \models T \subseteq A \) since
A is $\Delta^1_2$. Then $V[g] \models A$ contains a perfect subset. By Proposition 4.1, $V[g] \models A$ contains a $\Delta^1_2$-perfect subset $\tilde{T}$. So $\tilde{T} \in V$. By Shoenfield absoluteness again, $V \models A$ contains a $\Delta^1_2$-perfect subset $\tilde{T}$. \hfill \square

**Lemma 4.3.** If every real is constructible, then there is a co-countable $\Delta^1_2$-set $A$ having no $\Delta^1_2$-perfect subset.

**Proof.** We $L_{\omega_1}$-recursively build a set $A$ and $B$ such that $A = 2^\omega \setminus B$ as follows:

Fix an $L_{\omega_1}$-effective enumeration of $\Delta^1_1$-perfect trees $\{T_\beta\}_{\beta < \omega_1}$ (of course there are at most countably many such trees, but $L$ does not know this without using parameters).

At stage $\gamma < \omega_1$, if $\bigcup_{\gamma' < \gamma} B_{\gamma'} \cap [T_\gamma]$ is not empty, then let $A_\gamma = \bigcup_{\gamma' < \gamma} A_{\gamma'}$, $B_\gamma = \bigcup_{\gamma' < \gamma} B_{\gamma'}$, and go to next stage. Otherwise, pick up $\langle L \rangle$-least real $x \in [T_\gamma] \setminus L_\gamma$ and let $B_\gamma = \bigcup_{\gamma' < \gamma} B_{\gamma'} \cup \{x\}$. Define $A_\gamma = (L_\gamma \cap 2^\omega) \setminus B_\gamma$.

Then both $A = \bigcup_{\gamma < \omega_1} A_\gamma$ and $B = \bigcup_{\gamma < \omega_1} A_\gamma$ are r.e. in $L_{\omega_1}$ and $A = 2^\omega \setminus B$.

So $A$ is $\Delta^1_2$. By the construction, $A$ contains no $\Delta^1_2$-perfect subset. \hfill \square

So we have the following result.

**Theorem 4.4.** Every $\Sigma^1_2$ nonthin set has a $\Delta^1_2$-perfect subset if and only if there is a non-constructible real.

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Part 3. Randomness, analysis and ergodic theory

5. Turetsky: adaptive cost functions and the Main Lemma

The Main Lemma related to the Golden Run method [22, 5.5.1] states the following:

**Lemma 5.1.** If $M$ is a prefix-free oracle machine, and $A$ is a $K$-trivial set with some computable approximation $\langle A_s \rangle$, then there is a computable sequence $q_0 < q_1 < \ldots$ with

$$\sum_s \sum_\rho 2^{-|\rho|} \left[ M^A(\rho)[q_s] \downarrow \land m_s < \text{use } M^A(\rho)[q_s] \leq q_{s-1} \right] < \infty,$$

where $m_s$ is least with $A_{q_s}(m_s) \neq A_{q_{s+1}}$.

I understand this as a statement that $K$-trivial sets obey subadditive adaptive cost functions.

**Definition 5.2.** An adaptive cost function is a functional $c^X(n, s)$ such that:

- For every $X$, $c^X(n, s)$ is total;
- For every $X$, $c^X(n, s)$ is a cost function (monotonic and with limit condition);
- For every $X$, $n$ and $s$, the use of $c^X(n, s)$ is $s$.

As usual, we assume that $c^X(n, s) = 0$ for $n \geq s$.

If in addition $c^X(n, s)$ is (sub)additive for every $X$, then $c$ is a (sub)additive adaptive cost function.

**Lemma 5.3.** Every subadditive adaptive cost function is bounded by an additive adaptive cost function.
Proof. Suppose \( c^X(n, s) \) is a subadditive adaptive cost function, meaning 
\[
\frac{c^X(n, s)}{n < m < s} \geq c^X(n, m) + c^X(m, s)
\]
for every \( X \) and \( n < m < s \).

Define \( d^X(n, s) = c^X(0, s) - c^X(0, n) \). Then \( d \) is clearly an additive adaptive cost function. Further, 
\[
\frac{c^X(0, s) - c^X(0, n) \geq c^X(0, n) + c^X(n, s) - c^X(0, n) = c^X(n, s)}{\text{by subadditivity, and so } d \text{ bounds } c.}
\]

\[\square\]

Definition 5.4. If \( c \) is an adaptive cost function, and \( \langle A_s \rangle \) is a computable approximation to a \( \Delta^0_2 \) set, we say that \( \langle A_s \rangle \) obeys \( c \), written \( \langle A_s \rangle \models c \), if
\[
\sum_s c^{A_s}(n_s, s) < \infty,
\]
where \( n_s \) is least with \( A_s(n_s) \neq A_{s+1}(n_s) \).

We say that a \( \Delta^0_2 \) set \( A \) obeys \( c \), written \( A \models c \), if there is a computable approximation to \( A \) which obeys \( c \).

Proposition 5.5. If \( A \) is \( K \)-trivial and \( c \) is a subadditive adaptive cost function, then \( A \models c \).

Proof. It suffices to treat the case that \( c \) is additive. Fix a computable approximation \( \langle A_s \rangle \) to \( A \). We must construct an appropriate prefix-free oracle machine and apply the main lemma.

We may assume that \( c^X(s-1, s) \) is always a dyadic rational. By the proof of the machine existence theorem, there is a computable sequence of finite sets \( B_{\sigma} \subseteq 2^{<\omega} \) indexed by \( \sigma \in 2^{<\omega} \) such that:

- For every \( \sigma \subseteq \tau \), \( B_{\sigma} \cup B_{\tau} \) is an anti-chain;
- \( \sum_{\rho \in B_{\sigma}} 2^{-|\rho|} = c^\sigma(|\sigma - 1|, |\sigma|) \).

We define \( M^\sigma_{|\sigma|}(\rho) \downarrow \) for \( \rho \in B_{\sigma} \). By the first property above, the domain of \( M^X \) is an anti-chain for every \( X \), and so \( M \) is prefix-free. Note that \( M^\sigma_{|\sigma|} = M^\sigma_s \) for every \( s > |\sigma| \).

Now, let \( q_0 < q_1 < \ldots \) be as guaranteed by the main lemma. By pruning the first few terms if necessary, we may assume that \( q_0 \geq 2 \), and so in general \( q_s \geq s + 2 \). We claim that \( \langle D_s \rangle \models c \), where \( D_s = A_{q_s} \). For if \( n_s \) is least with
\[ D_s(n_s) \neq D_{s+1}(n_s), \text{ and letting } \sigma_k = D_s|_{k+1}, \text{ then} \]
\[ c^{D_s}(n_s, s) = \sum_{k=n_s}^{s-1} c^{\sigma_k}(k, k + 1) \]
\[ = \sum_{k=n_s}^{s-1} \sum_{\rho \in B_{\sigma_k}} 2^{-|\rho|} \left[ \rho \in \text{dom}(M^{\sigma_k}_{[\sigma_k]}_1) \& \rho \notin \text{dom}(M^{\sigma_{k-1}}_{[\sigma_k]-1}) \right] \]
\[ = \sum_{\rho} 2^{-|\rho|} \left[ \rho \in \text{dom}(M^{D_{s+1}}_s) \& \rho \notin \text{dom}(M^{D_{s}}_{n_s}) \right] \]
\[ = \sum_{\rho} 2^{-|\rho|} \left[ \rho \in \text{dom}(M^{D_{s+1}}_s) \& \rho \notin \text{dom}(M^{D_{s}}_{n_s}) \right] \]
\[ = \sum_{\rho} 2^{-|\rho|} \left[ M^{D_{s+1}}_s(\rho) \downarrow \& n_s < \text{use } M^{D_{s}}_s(\rho) \leq s + 1 \right] \]
\[ \leq \sum_{\rho} 2^{-|\rho|} \left[ M^{D_{s+1}}_s(\rho) \downarrow \& n_s < \text{use } M^{D_{s}}_s(\rho) \leq s + 1 \right] \]
\[ = \sum_{\rho} 2^{-|\rho|} \left[ M^{A}(\rho)|_{qs} \downarrow \& m_s < \text{use } M^{A}(\rho)|_{qs} \leq q_{s-1} \right] \]

So by our choice of \( q_0 < q_1 < \ldots, \sum_s c^{D_s}(n_s, s) < \infty. \) \hfill \( \square \)

6. Nies: Shannon-McMillan-Breiman theorem and its non-classical versions

This section is based on discussions with Marco Tomamichel and others, and on my talk at the Münster department of mathematics colloquium in Jan 2018 where I thank the audience for an unusually lively response during the talk.

6.1. Shannon’s work in information theory. We are given a source emitting symbols from an alphabet \( \mathcal{A} = \{a_1, \ldots, a_n\}. \) The symbol \( a_i \) has probability \( p_i. \) In Shannon’s original work the symbols are emitted independently. So this can be modelled by a sequence of i.i.d. \( \mathcal{A} \)-valued random variables.

We want to encode a string of \( n \) symbols by a bitstring, using as few bits as possible. However, it is allowed that certain strings are not encoded at all, as long as the probability of this happening goes to 0 with \( n \to \infty. \) Let \( k_n \) be the number of bits we allow for encoding \( n \) symbols. The asymptotic compression rate is \( h = \lim \inf k_n/n. \) What is the least \( h \) we can achieve?

Shannon’s source coding theorem says that \( h \) is the entropy of the probability distribution: \( -\sum_i p_i \log p_i. \) As we encode symbols strings by bit strings, the log is taken in base 2.
To prove that $h$ is an upper bound, for given $\epsilon$ one considers the set $A_{n,\epsilon}$ of $\epsilon$-typical strings, namely those $u \in A^n$ such that $\log P[u]$ (where $P[u]$ denotes the probability that $u$ happens) is within $\epsilon$ of $hn$. One shows that the probability of $A_{n,\epsilon}$ goes to 1 as $n \to \infty$, and that the size of $A_{n,\epsilon}$ is at most $2^{n(h+\epsilon)}$. So we need at most $n(h+\epsilon)$ bits to encode such a string. For $\epsilon \to 0$ we need $hn$ bits.

6.2. The Shannon-McMillan-Breiman theorem. The SMB theorem generalises the above to the case that the r.v.’s $X_n$ ($n \in \mathbb{N}$) form an ergodic process, to be defined below. It says that the entropy of the joint distribution can be seen from almost every trajectory $\omega$ as the limit of the empirical entropy $h_n(\omega)$, a random variable (r.v.) defined below.

The theorem is based on three separate papers: Claude Shannon 1948, Brockway McMillan 1953, Leo Breiman 1957. The former two worked in the spirit of information theory. Shannon only did the case of a Markov process (which includes the i.i.d. case). McMillan’s paper is long and follows the notation and terminology introduced by Shannon. He obtained the stronger $L_1$ convergence. Breiman’s paper is a short addition using an important inequality of McMillan’s; he shows a.e. convergence of the r.v.’s $h_n$ defined below. (There is also an erratum because some calculation on the last page wasn’t right.)

We follow Shields’ book [28] in the exposition of the theorem, though we adapt some notation. $A^\infty$ denotes the space one-sided infinite sequences of symbols in $A$. We can assume that this is the sample space, so that $X_n(\omega) = \omega(n)$. By $\mu$ we denote their joint distribution. A dynamics on $A^\infty$ is given by the shift operator $T$, which erases the first symbol of a sequence. A measure $\mu$ on $A^\infty$ is $T$-invariant if $\mu G = \mu T^{-1}(G)$ for each measurable $G$.

By $\omega \mid_n$ we denote the first $n$ symbols of $\omega$. Note that in ergodic theory one usually starts with 1 as an index, so they would write $\omega^n_1$ for the first $n$ symbols. Here the notation should be compatible with the one used in randomness theory.)

We consider the r.v.

$$h_n(\omega) = -\frac{1}{n} \log \mu[\omega \mid_n],$$

(recall that log is w.r.t. base 2). The main thing to prove is the following fact for general $T$-invariant measures.

**Lemma 6.1.** Let $\mu$ be an invariant measure for the shift operator $T$ on the space $A^\infty$. Then for $\mu$-a.e. $x$, $h(x) = \lim_n h_n(x \mid_n)$ exists.

Recall that $\mu$ is ergodic if every $\mu$ integrable function $f$ with $f \circ T = f$ is constant $\mu$-a.s. An equivalent condition that is easier to check is the following: for $u, v \in A^*$,

$$\lim N^{-1} \sum_{k=0}^{n-1} \mu([u] \cap T^{-k}[v]) = \mu[u] \mu[v].$$

(Don’t confuse this with the stronger property called “weakly mixing”, where one requires that the average of the absolute values of the differences goes
to 0; this happened during the talk.) It is easily seen that the Bernoulli
measure on $A^\infty$ is ergodic (and in fact, strongly mixing).

For ergodic $\mu$, the entropy $H(\mu)$ is defined as $\lim_n H_n(\mu)$, where

$$H_n(\mu) = -\frac{1}{n} \sum_{|w|=n} \mu[w] \log \mu[w].$$

One notes that $H_{n+1}(\mu) \leq H_n(\mu) \leq 1$ so that the limit exists. Also note
that $H_n(\mu) = E h_n$.

The following says that in the ergodic case, $\mu$-a.s. the empirical entropy
equals the entropy of the measure.

**Theorem 6.2** (SMB theorem). Let $\mu$ be an ergodic invariant measure
for the shift operator $T$ on the space $A^\infty$. Then for $\mu$-a.e. $\omega$ we have

$$\lim_n h_n(\omega) = H(\mu).$$

Given Lemma 6.1, this isn’t too much extra work to prove. First one checks that since $\mu$ is $T$-invariant, we have $h(Tx) \leq h(x)$ for each $x$, i.e. $h$ is subinvariant. Next, from the Poincare recurrence theorem it follows that $B = \{x : h(Tx) < q < h(x)\}$ is a null set for each $q$ (because we can’t return to $B$ outside a null set), so $h$ is actually invariant: $h(Tx) = h(x)$ for $\mu$-a.e. $x$. Also $h \in L^1(\mu)$ by the dominated convergence theorem, so if $\mu$ is ergodic then $h(x)$ has some constant value, for $\mu$-a.s. $x$. A final step is then to show
that this constant value equals $H(\mu)$.

6.3. **Proof in the i.i.d. case.** It is instructive to give a direct proof of
the SMB theorem in the i.i.d. case, that is, when $\mu$ is a Bernoulli measure.
Suppose symbol $a_i$ has probability $p_i$. By $k_{i,n}(\omega)$ we denote the number of
occurrences of the symbol $a_i$ in $\omega \upharpoonright n$. By independence, we have

$$h_n(\omega) = -\frac{1}{n} \log \prod_i p_i^{k_{i,n}(\omega)} = -\frac{1}{n} \sum_i k_{i,n}(\omega) \log p_i.$$

By the strong law of large numbers, for $\mu$-a.e. $\omega$, $k_{i,n}(\omega)/n$ converges to $p_i$.

6.4. **Algorithmic version of the SMB theorem.** We now assume that
we can compute $\mu[u]$ uniformly from a string $u$. That is, $\mu$ is a computable
measure. This is true e.g. for the Bernoulli measure when the $p_i$ are all
computable reals.

Hochman [10] and in more explicit form Hoyrup [12] have shown that the
exception set in the SMB theorem is ML-null. It is unknown at present
whether a weaker randomness notion such as Schnorr’s is sufficient, even
under the assumption that $H(\mu)$ is computable.

6.5. **Do random states satisfy the quantum SMB theorem?** Mathematically, a qubit is a unit vector in the Hilbert space $\mathbb{C}^2$. We give a brief
summary on “infinite sequences” of qubits. One considers the $C^*$ algebra $M_\infty = \lim_n M_{2^n}(\mathbb{C})$, an approximately finite (AF) $C^*$ algebra. “Quantum
Cantor space” consists of the state set $\mathcal{S}(M_\infty)$, which is a convex, compact,
connected set with a shift operator, deleting the first qubit.

Given a finite sequence of qubits, “deleting” a particular one generally
results in a statistical superposition of the remaining ones. This is why $\mathcal{S}(M_\infty)$ consists of coherent sequences of density matrices in $M_{2^n}(\mathbb{C})$ (which
formalise such superpositions) rather than just of sequences of unit vectors in $(\mathbb{C}^2)^{\otimes n}$. For more background on this, as well as an algorithmic notion of randomness for such states, see Nies and Scholz [23]. Notice that the tracial state $\tau$ is random, even though it generalises the uniform measure and hence, from a different point of view, can be considered to be computable.

Bjelakovich et al. [1] provided a quantum version of the Shannon-McMillan theorem. (They worked with bi-infinite sequences, which makes little difference here, as a stationary process is given by its marginal distributions on the places from 0 to $n$, for all $n$.) The reason they avoided the full Breiman version is that on $S(M_\infty)$ there has been so far no reasonable way to say “for almost every”. (The work in [23] introduces effective null sets, which might remedy this.) In [1], they first convert the classical SMB theorem into an equivalent form which doesn’t directly mention measure; rather, they have “chained typical sets” which generalise Shannon’s typical sets. To be chained means that they are coherent over successive lengths of symbol strings.

The von Neumann entropy of a density matrix $S$ is $H(S) = -\text{Tr}(S \log S)$. For a state $\mu$ on $M_\infty$ we let

$$h(\mu) = \lim \frac{1}{n} H(\mu | M_{2^n})$$

which exists by concavity of log. Let $\mu$ be a state on $M_\infty$. For a quantum $\Sigma_1^0$ set $G = \langle p_n \rangle_{n \in \mathbb{N}}$ we define $\mu(G) = \sup_n \text{Tr}(\mu | M_{2^n} \cdot p_n)$. A qML-test relative to a computable state $\mu$ is a uniform sequence $(G_r)$ of such sets such that $\mu(G_r) \leq 2^{-r}$. Failure and passing is defined as before. This yields qML-randomness w.r.t. $\mu$. Work in progress with Tomamichel would show the following.

**Conjecture 6.3.** Let $\mu$ be an ergodic computable state on $M_\infty$. Let $\rho$ be a state that is quantum ML-random with respect to $\mu$. Then

$$h(\mu) = -\lim \frac{1}{n} \text{Tr}(\rho | M_{2^n} \log \mu | M_{2^n}).$$

The plan is to go through more and more general cases for both $\rho$ and $\mu$. The computable state $\mu$ can be uniform (i.e. $\tau$), i.i.d. but quantum, a computable ergodic measure, and finally any computable ergodic state. The random state $\rho$ can be a bit sequence that is ML random wrt $\mu$, a $\mu$-random measure on $2^\mathbb{N}$, and finally any qML($\mu$) state. The combination that $\rho$ is a bit sequence, and $\mu$ a measure is the effective classical SMB theorem, essentially proved by Hochman [10] and in more explicit form by Hoyrup [12].

In the classical setting the case where $\mu$ is a Bernoulli measure is easy. In the quantum setting we use Chernoff bounds and some calculations to do the case for general $\rho$ but Bernoulli $\mu$.

To say that $\mu$ is i.i.d. means that for some fixed computable $V \in S(M_2)$, i.e. a 2x2 density matrix, we have $\mu | M_{2^n} = V^\otimes n$. Note that the partial trace removes the final $V$, so this “infinite tensor power” indeed can be seen as a computable state on $M_\infty$. There is a computable unitary $U \in M_2$ such that $UVU^\dagger$ is diagonal, with $p, 1 - p$ on the diagonal, $p$ is computable. Its von Neumann entropy is $h(\mu) = -p \log p - (1 - p) \log(1 - p)$. 


Note that qML($\mu$)-randomness is closed under the unitary of $M_{2^{\infty}}$ which is obtained applying conjugation by $U^\dagger$ “qubit-wise”. So replacing $\rho$ by its conjugate we may as well assume that $V$ is diagonal. Fix $\delta > 0$. Let $P_{n,\delta}$ be the projector in $M_{2^n}$ corresponding to the set of bitstrings $\{x: |x| = n \wedge |\frac{1}{n} \log \mu[x] - h(\mu)| \leq \delta\}$.

Since $\mu$ is a product measure, $\log \mu[x]$ is a sum of $n$ independent random variables looking at the bits of $x$, and the expectation of $-\frac{1}{n} \log \mu[x]$ is $h(\mu)$. The usual Chernoff bound yields $\mu(P_{n,\delta}^\perp) \leq 2 \exp(-2n\delta^2)$. Let $G_{m,\delta} = \bigcup_{n>m} P_{n,\delta}^\perp$, where these projectors are now viewed as clopen sets in Cantor space, so that $G_{m,\delta}$ determines a classical ML-test. Since $\rho$ is qML random w.r.t. $\mu$, we have $\lim_m \rho(G_{m,\delta}) = 0$.

**Theorem 6.4.** $\lim_n -\frac{1}{n} \text{Tr}(\rho \mid_{M_{2^n}} \log \mu \mid_{M_{2^n}}) = h(\mu)$.

To see this, fix $\delta > 0$, and omit it from the subscripts for now. We write $s = h(\mu)$ and write $\rho_n$ for $\rho \mid_{M_{2^n}}$ etc.

We insert the term $I_{2^n} = P_{n}^\perp + P_n$ between the two factors. We look separately at both resulting limits.

**Part 1.** We consider $-\frac{1}{n} \text{Tr}(\rho_n P_{n}^\perp \log \mu_n)$. Note that $P_{n}^\perp \log \mu_n$ is negative semidefinite as the two factors are diagonal w.r.t. the same base and therefore commute. So $-\frac{1}{n} \text{Tr}(\rho_n P_{n}^\perp \log \mu_n) \geq 0$. By cyclicity of the trace and the commutation, we have

$$\text{Tr}(\rho \mid_n P_{n}^\perp \log \mu_n) = \text{Tr}(\rho \mid_n P_{n}^\perp \log \mu_n P_{n}^\perp) = \text{Tr}(P_{n}^\perp \rho_n P_{n}^\perp \log \mu_n).$$

For positive operators $A, B$ we have $\text{Tr}(AB) \leq \|A\|_1 \cdot \|B\|_\infty$ where $\|A\|_1$ is the sum of the eigenvalues, and $\|B\|_\infty$ is their maximum. So

$$-\frac{1}{n} \text{Tr}(\rho_n P_{n}^\perp \log \mu_n) \leq \frac{1}{n} \|P_{n}^\perp \rho_n P_{n}^\perp\|_1 \log \mu_n\|_\infty.$$}

Now $\|\frac{1}{n} \log \mu_n\|_\infty$ is bounded depending only on $p$, and for large enough $n$ we have $\|P_{n}^\perp \rho_n P_{n}^\perp\|_1 \leq 2\delta$ by hypothesis.

**Part 2.** We consider $-\frac{1}{n} \text{Tr}(\rho_n P_n \log \mu_n)$. We note that $\mu_n$ is a diagonal matrix in $M_{2^n}$ where the entry in the position $(\sigma, \sigma)$ is $p^k(1-p)^{n-k}$, where the binary string $\sigma$ of length $n$ has $k$ 0s. By definition of $P_n$ it follows that $\|P_n(-\frac{1}{n} \log \mu_n) - h(\mu)P_n\|_\infty \leq \delta$. Now

$$-\frac{1}{n} \text{Tr}(\rho_n P_n \log \mu_n) = \text{Tr}\rho_n(-\frac{1}{n} P_n \log \mu_n - sP_n + sP_n)$$

$$= s\text{Tr}\rho_n P_n + \text{Tr}(\rho_n(-\frac{1}{n} P_n \log \mu_n - sP_n)).$$

Using that $\text{Tr}(AB) \leq \|A\|_1 \cdot \|B\|_\infty$ for positive $A, B$ and the definition of $P_n$, the second summand is at most $\delta$.

By hypothesis, for large $n$ we have $\text{Tr}(\rho_n P_{n}^\perp) \leq 2\delta$, and hence $\text{Tr}\rho_n P_n \geq 1 - 2\delta$. So the first summand is between $s(1 - 2\delta)$ and $s$.

To summarize, for large $n$, the quantity in Part 1 is $\leq 2\delta$ and the quantity in Part 2 is in $[s(1-2\delta), s+\delta]$. For $\delta \to 0$ their sum converges to $s$ as required.

6.6. **Random states satisfy the law of large numbers.**
Proposition 6.5. Let $\mu$ be an i.i.d computable state, and let $\rho$ be qML-random relative to $\mu$. For $i < n$ let $S_{n,i}$ be the subspace of $\mathbb{C}^{2^n}$ generated by those $\sigma$ with $\sigma_i = 1$. We have

$$\lim_n \frac{1}{n} \sum_{i<n} \text{Tr}(\rho_n S_{n,i}) = p$$

where $S_{n,i}$ is identified with its orthogonal projection.

Proof. As above, we first assume that $\mu = \left( \begin{smallmatrix} p & 0 \\ 0 & 1-p \end{smallmatrix} \right)^{\otimes \infty}$ corresponds to a classical product measure where $p \in (0,1)$ is computable.

As before we fix $\delta > 0$. Let $E_{n,\delta}$ be (the projector in $M_{2^n}$ corresponding to) the set of bitstrings $\{x: |x| = n \land |\frac{1}{n} \sum x - p| \leq \delta\}$.

The Chernoff bound now yields $\mu(E_{n,\delta}^\perp) \leq 2 \exp(-2n\delta^2)$.

Let $G_{m,\delta} = \bigcup_{n>m} E_{n,\delta}$ where these projectors are now viewed as clopen sets in Cantor space, so that $G_{m,\delta}$ determines a classical ML-test. Since $\rho$ is qML random w.r.t. $\mu$, we have $\lim_m \rho(G_{m,\delta}) = 0$. Now, $\frac{1}{n} \sum_{i<n} \rho(S_{n,i}) = \frac{1}{n} \sum_{x \in E_{n,\delta}} \sum_{i<n} \rho(S_{n,i} \cap [x]) + \frac{1}{n} \sum_{x \in E_{n,\delta}^\perp} \sum_{i<n} \rho(S_{n,i} \cap [x])$.

For large enough $n$, the second summand is $\leq 2\delta$. The first summand equals $\frac{1}{n} \sum_{x \in E_{n,\delta}} \sum \rho(x)$, where $\sum x$ is the number of 1s in a string $x$. By definition of $E_{n,\delta}$ this value is in $(p-\delta, p+\delta)\rho(E_{n,\delta})$, and $\rho(E_{n,\delta})$ tends to 1 with $n \to \infty$. Letting $\delta \to 0$ we get the value $p$.

For general i.i.d. states $\mu$, we note that the same argument works for other computable orthonormal bases $e_0, e_1$ of $\mathbb{C}^2$ instead of $|0\rangle, |1\rangle$. If $\mu$ is the infinite tensor power of $UB(p)U^\dagger$, we conjugate $\rho$ qubitwise by $U^\dagger$ and carry out the argument for $e_r = U|r\rangle U^\dagger$.

\[ \square \]

Part 4. Reverse mathematics

7. Carlucci: A variant of Hindman’s Theorem implying ACA$_0’$

The following is part of an attempt to prove (or disprove) the existence of a level-by-level combinatorial reduction from Ramsey’s Theorem to Hindman’s Theorem. Such a reduction would be a strong way of establishing that Hindman’s Theorem implies ACA$_0’$.

Let us recall some definitions.

Definition 7.1 (Hindman’s Theorem with bounded sums). For positive integers $n, \ell$, $\text{HT}_{\ell}^{\leq n}$ denotes the following principle: for every coloring $f: \mathbb{N} \to \ell$ there exists an infinite set $H$ such that $\text{FS}^{\leq n}(H)$ is monochromatic for $f$, where $\text{FS}^{\leq n}(H)$ denotes the set of all non-empty finite sums of at most $n$ distinct members of $H$.

The analogous version for sums of exactly $n$ many terms is denoted $\text{HT}_{\ell}^{=n}$.
Definition 7.2 (Apart set). A subset $X$ of the positive integers is apart if for any $n, m \in X$ such that $n < m$, the largest exponent of $n$ in base 2 is strictly smaller than the smallest exponent of $m$ in base 2.

If $P$ is an Hindman-type principle then $P$ with apartness denotes the same principle to which we add the requirement that the solution set is apart.

Recently the following results where established, where $\text{RT}^k_\ell$ denotes Ramsey’s Theorem for exponent $k$ and $\ell$ colors, $\text{IPT}^2_2$ denotes Dzhafarov and Hirst’s [6] Increasing Polarized Ramsey’s Theorem for exponent 2 and 2 colors, and $\leq_{\text{sc}}$ denotes strong computable reducibility.

1. For every positive integers $k, \ell$, $\text{RCA}_0 \vdash \text{HT}^{\leq 3}_3 \rightarrow \text{RT}^k_\ell$ (Dzhafarov et al. [7]),
2. For every positive integers $k, \ell$, $\text{RCA}_0 \vdash \text{HT}^{\leq 2}_4 \rightarrow \text{RT}^k_\ell$ (Carlucci et al. [4]),
3. For every positive integers $k, \ell$, $\text{RCA}_0 \vdash \text{HT}^{= 3}_2$ with apartness $\rightarrow \text{RT}^k_\ell$ (Carlucci et al. [4]),
4. $\text{IPT}^2_2 \leq_{\text{sc}} \text{HT}^{= 2}_4$ (Carlucci [3]),
5. $\text{IPT}^2_2 \leq_{\text{sc}} \text{HT}^{= 2}_4$ with apartness (Carlucci et al. [4]).

Despite points (1) and (2), I have not been able to lift the combinatorial reductions in points (3) and (4) to exponents higher than 2. Below I show that it is possible to do so, so as to hit Ramsey’s Theorem and not only its increasing polarized version, provided one adds an extra condition on the elements of the solution set to Hindman’s Theorem. The resulting variant of Hindman’s Theorem is then shown to be equivalent to Ramsey’s Theorem.

Definition 7.3 (Exactly large number). A positive integer $n$ is $!\alpha$-large (exactly $\alpha$-large) if the set $c(n)$ of its exponents in base 2 is $!\alpha$-large.

Recall that a set $X$ of positive integers is $!\omega$-large if $|X| = \min(X) + 1$, is exactly $\omega$-large is it has the form $X = X_1 \cup X_2$ with $\max(X_1) < \min(X_2)$ such hat $X_1$ and $X_2$ are exactly $\omega$-large, and so on for $!\omega^3$, $!\omega^4$ etc.

Definition 7.4. Let $A^k_\ell$ be the following principle: For every coloring $c$ of the positive integers $\mathbb{N}$ in $\ell$ colors there exists an infinite subset $H$ of $\mathbb{N}$ such that each $n \in H$ is $!\omega$-large, $H$ is apart, and $FS^{=k}(H)$ is monochromatic.

The principle $A^k_\ell$ is essentially $\text{HT}^k_\ell$ with apartness plus the extra (far from trivial!) constraint that the solution set is contained in the set of $!\omega$-large numbers.

Theorem 7.5. $\text{RCA}_0 \vdash \forall k > 0 \forall \ell > 0 (A^k_\ell \rightarrow \text{RT}^k_\ell)$. In fact, $\text{RT}^k_\ell \leq_{\text{sc}} A^k_\ell$.

Proof. Let $d : [\mathbb{N}]^k \rightarrow \ell$ be given. Define $c : \mathbb{N} \rightarrow \ell$ as follows. If $n$ is not $!\omega k$-large then $c(n)$ colors $n$ arbitrarily. If $n$ is $!\omega k$-large then $c(n) = d(n_1, \ldots, n_k)$, where $n_1 < \cdots < n_k$ are the unique $!\omega$-large numbers such that $n = n_1 + \cdots + n_k$. Let $H$ be a solution to $A^k_\ell$ for $c$ and let $i < \ell$ be the color of $FS^{=k}(H)$ under $c$. We claim that $[H]^k$ is monochromatic of color $i$ under $d$. Indeed, let $a_1 < \cdots < a_k$ be in $H$. Then $a_1, \ldots, a_k$ are $!\omega$-large and add in base 2 with no carry since $H$ is apart. Thus, $a = a_1 + \cdots + a_k$ is $!\omega k$-large and is in $FS^{=k}(H)$. Therefore $i = c(a) = d(a_1, \ldots, a_k)$.

$\square$
Theorem 7.6. $\text{RCA}_0 \vdash \forall k > 0 \forall \ell > 0 (\text{RT}^k_\ell \rightarrow A^k_\ell)$. In fact, $A^k_\ell \leq_{sc} \text{RT}^k_\ell$.

Proof. Let $c : \mathbb{N} \rightarrow \ell$ be given. Define $d : [\mathbb{N}]^k \rightarrow \ell$ as $d(a_1, \ldots, a_k) = c(a_1 + \cdots + a_k)$. Let $X \subseteq \mathbb{N}$ be an infinite apart set consisting of $!\omega$-large numbers. By Ramsey’s Theorem for $X$ and $d$, there exists an infinite $H \subseteq X$ such that $d$ is constant on $[H]^k$, say of color $i < \ell$. Then $H$ is a solution to $A^k_\ell$ for $c$, since, if $a \in \text{FS} = k(H)$ then $a$ is a sum of $k$ many exactly large elements of $H$, i.e., for some $a_1 < \cdots < a_k$ in $H$ we have that $a = a_1 + \cdots + a_k$. Then $i = d(a_1, \ldots, a_k) = c(a)$. □

Corollary 7.7. $\text{RCA}_0 \vdash \forall k A^k_\ell \rightarrow \forall k \text{RT}^k_\ell$.

Hence, $\forall k A^k_\ell$ implies $\text{ACA}'_0$ over $\text{RCA}_0$. The following question is then of interest:

Question 7.8. Does Hindman’s Theorem imply $\forall k A^k_\ell$?

Note that, in $A^k_\ell$, the condition that the elements of the solution set are $!\omega$-large can be replaced by various other conditions. For example we might require that all elements of the solution set have the same binary length. The argument showing that $A^k_\ell$ implies $\text{RT}^k_\ell$ is inspired by an argument attributed to Justin Moore which I learned from David Fernandez Breton (private communication), proving that a cardinal satisfying Hindman’s Finite Unions Theorem has to be weakly compact.

Part 5. Group theory and its connections to logic

8. CHIODO, NIES AND SORBI: DECIDABILITY PROBLEMS FOR F.G. GROUPS

Maurice Chiodo, Nies and Andrea Sorbi discussed decidability problems for f.g. groups in April and May, both in Siena and by Skype.

The c.e. equivalence relation of isomorphism between finitely presented (f.p.) groups. C.F. Miller [19] has shown that the c.e. equivalence relation (ceer) $\equiv_{f.p.}$ of isomorphism between finitely presented groups is $\Sigma^0_1$ complete within the ceer’s. Nies and Sorbi [24] noticed that it has a diagonal function $f$, namely $f$ is computable and $f(x)$ is not equivalent to $x$.

The ceer $\equiv_{f.p.}$ is not effectively inseparable as pointed out by Chiodo: Let $A$ be the class of f.p. groups $G$ such that $G_{ab} \cong \mathbb{Z}$. Then $A$ is computable, and separates the class of a presentation of $\mathbb{Z}$ from the class of a presentation of $\mathbb{Z} \times \mathbb{Z}$.

Recall that a group $G$ is perfect if $G' = G$, or equivalently $G_{ab}$ is trivial. We can list finite presentations of all the perfect f.p. groups by including relations that write each generator as a product of commutators in a particular way.

Question 8.1. If $\equiv_{f.p.}$ restricted to presentations of perfect groups effectively inseparable?

Problems on f.g. groups. The following questions are long standing.

Question 8.2. Is some infinite f.p. group a torsion group?

Question 8.3 (Weigold). Is each f.g. perfect group the normal closure of a single element?
Algorithmic problems.

**Question 8.4.** Is the relation among f.p. groups “B is a quotient of A” $\Sigma^0_1$-complete as a pre-order?

**Question 8.5.** Find an algorithm that on input a finite presentation $G = \langle X \mid R \rangle$, outputs a word $w$ in $X$ such that $w = 1$ in $G$ if and only if $G$ is trivial.

9. Fouche and Nies: randomness notions in computable profinite groups

Willem Fouche visited New Zealand for three weeks in October. He and Nies continued their work on the effective content of results of Jarden, Lubotzky and others. This work was started in [8, Section 16], where background is provided. We only recall here the following.

**Definition 9.1** (Smith [29]).

(i) A profinite group $G$ is called co-r.e. if it is the inverse limit of a computable inverse system $\langle G_n, p_n \rangle$ of finite groups (i.e. the groups $G_n$ and the maps $p_n$ between them are uniformly computable). Equivalently, the subgroup $U$ above is a $\Pi^0_1$ subclass of $\prod_n G_n$.

(ii) $G$ is called computable if, in addition, the maps $p_n$ can be chosen onto. In other words, the set of extendible nodes in the tree corresponding to $U$ is computable.

Each separable profinite group is equipped with a unique Haar probability measure (i.e., a probability measure that is invariant under left and under right translations). For instance, for the 2-adic integers $\mathbb{Z}_2$, the Haar measure is the usual product measure on Cantor space. If the group is computable then so is the Haar measure, using the notion of a computable probability space due to Hoyrup and Rojas [13].

The Jarden, Lubotzky et al. results are theorems of “almost everywhere” type in various profinite groups $G$: they assert a property for almost every tuple in $G^e$, for some $e$ that is fixed for the particular result. These groups are usually computable, in which case randomness notions defined via algorithmic tests (with respect to the Haar measure) can be applied in $G^e$. So, unlike the usual process of effectivizing results from analysis [2], in this case the existing “classical” results have an effective content per se, which only needs to be made explicit. To do so is our purpose.

9.1. **Computable profinite groups that are completions.** We update the information in [8, Section 16]. The definition below is taken from [27, Section 3.2]. Let $G$ be a group, $\mathcal{V}$ a set of normal subgroups of finite index in $G$ such that $U, V \in \mathcal{V}$ implies that there is $W \in \mathcal{V}$ with $W \subseteq U \cap V$. We can turn $G$ into a topological group by declaring $\mathcal{V}$ a basis of neighbourhoods (nbhds) of the identity. In other words, $M \subseteq G$ is open if for each $x \in M$ there is $U \in \mathcal{V}$ such that $xU \subseteq M$.

**Definition 9.2.** The completion of $G$ with respect to $\mathcal{V}$ is the inverse limit

$$G_{\mathcal{V}} = \lim_{U \in \mathcal{V}} G/U,$$
where \( \mathcal{V} \) is ordered under inclusion and the inverse system is equipped with the natural maps: for \( U \subseteq V \), the map \( p_{U,V} : G/U \to G/V \) is given by \( gU \mapsto gV \).

The inverse limit can be seen as a closed subgroup of the direct product \( \prod_{U \in \mathcal{V}} G/U \) (where each group \( G/U \) carries the discrete topology), consisting of the functions \( \alpha \) such that \( p_{U,V}(\alpha(gU)) = gV \) for each \( g \). Note that the map \( g \mapsto (gU)_{U \in \mathcal{V}} \) is a continuous homomorphism \( G \to G_\mathcal{V} \) with dense image; it is injective iff \( \bigcap \mathcal{V} = \{1\} \).

Suppose \( G \) is a computable group, and the class \( \mathcal{V} \) in Definition 9.2 is uniformly computable in that there is a uniformly computable sequence \( \langle R_n \rangle \) such that \( \mathcal{V} = \{ R_n : n \in \mathbb{N} \} \). Suppose further that \( W \) above can be obtained effectively from \( U,V \). Then there is a uniformly computable descending subsystem \( \langle T_k \rangle \) of \( \langle R_n \rangle \) such that \( \forall n \exists k T_k \leq R_n \). Since we can effectively find coset representatives of \( T_n \) in \( G \), the inverse system \( \langle G/T_n \rangle \) with the natural projections \( T_{n+1} \to T_n \) is computable. So \( G_\mathcal{V} \) is computable.

Suppose we are given two computable sequences \( \langle R_n \rangle \) and \( \langle S_k \rangle \) as above. If for each \( n \) we can compute \( k \) such that \( S_k \leq R_n \), and vice versa. Then the completions obtained via the two sequences are computably isomorphic.

The criterion above is satisfied by \( F_k \) and \( F_\omega \) with the systems of normal subgroups introduced in [8, Section 16]. Thus their completions \( \hat{F}_k \) and \( \hat{F}_\omega \) are computable profinite groups.

**Lemma 9.3.** Let \( G \) be \( k \)-generated (\( k \leq \omega \)). Then \( G \) is computable \([\text{-r.e.}] \) iff \( G = \hat{F}_k/N \) for some computable normal subgroup \( N \) (\( \prod_1^0 N \)).

### 9.2. Abelian free profinite groups

Let \( \hat{\mathbb{Z}} \) denote the free profinite group of rank 1. Note that \( \hat{\mathbb{Z}} \) is the inverse limit of the directed system of groups \( \mathbb{Z}/n\mathbb{Z} \) with the natural projections from \( \mathbb{Z}/n\mathbb{Z} \) to \( \mathbb{Z}/k\mathbb{Z} \) in case \( k \) divides \( n \).

By \( \langle S \rangle \) one denotes the closed subgroup generated by a subset \( S \) of a group.

**Proposition 9.4.** Let \( z \in \hat{\mathbb{Z}} \) be Kurtz random. Then \( \langle z \rangle \) has infinite index in \( \hat{\mathbb{Z}} \) and \( \langle z \rangle \cong \hat{\mathbb{Z}} \).

For tuples rather than singletons, the opposite happens: random tuples generate a subgroup of finite index.

**Proposition 9.5.** Let \( e \geq 2 \) and suppose that \( z \in (\hat{\mathbb{Z}})^e \) is Schnorr random. Then \( \langle z \rangle \) has finite index in \( \hat{\mathbb{Z}} \) and \( \langle z \rangle \cong \hat{\mathbb{Z}} \).

**Proof.** Let \( G_n = (n\hat{\mathbb{Z}})^e \). Note that \( \mu G_n = n^{-e} \) and \( G_n \) is uniformly \( \Sigma_1^0 \). Since \( \sum n^{-e} \) is finite and computable, \( \langle G_n \rangle \) is a Schnorr-Solovay test. Therefore \( z \notin G_n \) for sufficiently large \( n \).

If \( U \) is a closed subgroup of \( \hat{\mathbb{Z}} \) then \( U \) is the intersection of the groups of the form \( n\hat{\mathbb{Z}} \) containing it. So, if \( U \) has infinite index there are infinitely many such \( n \). Hence \( \langle z \rangle \) has finite index. Since it is also closed, it is open, and hence isomorphic to \( \hat{\mathbb{Z}} \). \( \square \)

**Proposition 9.6.** There is a Kurtz random \( z \in (\hat{\mathbb{Z}})^2 \) such that \( \langle z \rangle \) is of infinite index in \( \hat{\mathbb{Z}} \).

**Proof.** Let \( G_n = \mathbb{Z}/n!\mathbb{Z} \). Clearly \( \hat{\mathbb{Z}} = \text{proj lim } G_n \). Furthermore, by Subsection 9.1, the corresponding computable presentation of \( \hat{\mathbb{Z}} \) given by this
inverse limit is computably isomorphic to the standard one obtained as the completion of \( \mathbb{Z} \).

We can now view an element of \( w \in \hat{\mathbb{Z}} \) as written in factorial expansion

\[
w = \sum_{n \geq 1} a_n n!
\]

where \( 0 \leq a_n \leq n \). In this way we can think of \( w \) as a path \( f = (a_1, a_2, \ldots) \) on the tree \( T \) where every node at level \( k \) (starting at level 0 for the root) has \( k + 1 \) children. The path space \( [T] \) is homeomorphic to \( \hat{\mathbb{Z}} \) via a map turning the uniform measure on \( [T] \) into the Haar measure on \( \hat{\mathbb{Z}} \).

Given \( z = (z^0, z^1) \), let \( f^0, f^1 \) be the corresponding paths. If there are infinitely many \( n \) such that \( f^0(n) = f^1(n) = 0 \), then the subgroup of \( \hat{\mathbb{Z}} \) topologically generated by \( z \) has infinite index.

A pair \((f, g) \in [T]^2\) is called weakly 1-generic if it meets each dense \( \Sigma_0^1 \) set. In particular it meets the condition above. Any weakly 1-generic is Kurtz random. Thus the pair \( z \) corresponding to a weakly 1-generic pair of paths is Kurtz random and generates a subgroup of infinite index, as required. \( \square \)

### 9.3. Normal closure in general computable profinite groups

Let \( G \) be a topological group. For \( z \in G^\ast \), by \( [z]_G \) one denotes the topological normal closure of the tuple \( z \) in \( G \). We omit the subscript if it is clear from the context. Jarden and Lubotzky also consider almost everywhere theorems of the form \([z]_G \cong L\) for a.e. \( z \), where \( L \) is an appropriate profinite group; specifically, \( L \) can be recognized by its finite quotients among the closed normal subgroups of \( G \) (e.g., \( L \) could be free of a certain rank). We will show that weak 2-randomness of \( z \) suffices.

**Lemma 9.7.** Let \( G \) be a computable profinite group. Fix \( e \in \mathbb{N} \) and a finite group \( C \). The set \( \{ z \in G^e : [z] \text{ has } C \text{ as a quotient } \} \) is a \( \Sigma_2^0 \) subset of \( G^e \).

**Proof.** According to Definition 9.1 \( G = \text{proj lim}_n G_n \) for a computable inverse system \((G_n, p_n)\) of finite groups with onto maps \( p_n \). By \( g \mid n \), we denote the projection of \( g \in G \) into \( G_n \); similar notation applies to \( g \in G_t \) for \( t \geq n \). For \( g \in G \), we have

\[
g \in [z]_G \iff \forall n g \mid n \in [z]_n[G_n].
\]

We define the subset of \( G_n \) of elements that have a preimage in \( [z]_t[G_t] \):

\[
U_n^{z,t} = \{ v \in G_n : \exists w \in G_t [w \in [z]_t[G_t] \land w \mid n = v] \}
\]

Note that \( \bigcap_t U_n^{z,t} \) is the projection of \([z]_G \) into \( G_n \).

Note that \([z]_G \) has \( C \) as a quotient iff for some \( n \), the image of \([z]_G \) into \( G_n \) has \( C \) as a quotient. This is equivalent to the condition

\[
\exists n \exists s \geq n \forall t \geq s [U_n^{z,t} \text{ is a subgroup of } G_n \text{ with } C \text{ as a quotient}],
\]

which is in \( \Sigma_2^0 \) form as required. \( \square \)

Suppose that \( L \) is a profinite group that can be described by its finite quotients among the closed normal subgroups of \( G \). It follows that any theorem of the form \( \"[z]_G \cong L\) for almost every \( z\)\" holds for any weakly 2-random \( z \).
Corollary 9.8. Let $G = \hat{F}_\omega$. For each $e$ and each weakly 2-random $z \in G^e$, the topological normal closure $N$ of $z$ is isomorphic to $\hat{F}_\omega$.

To see this, note that by [9, Th. 25.7.3(b)], $N \cong \hat{F}_\omega$ if each finite group is a quotient of $N$. The $S$-rank function $r_N(S)$ occurring there, for a finite simple group $S$, is defined as follows (see Section 24.9): let $M_G(S)$ be the intersection of all open normal subgroups $X$ of $N$ such that $N/X \cong S$. Then $M_G(S)$ is closed normal, and $G/M_G(S) \cong S^m$ for some cardinal $m$; write $m = r_G(S)$. If $m$ is finite, it is simply the number of normal open subgroups $X$ of $N$ such that $N/X \cong S$.

Similarly, from [9, Cor. 25.7.6] we obtain a variant when $G = \hat{F}_m$ has finite rank. Note that any subgroup of finite index is open, and hence free of finite rank; see e.g. [27, Th.3.6.2], which is a profinite version of Schreier’s theorem on the rank of finite index subgroups of discrete free groups of finite rank.

Corollary 9.9. Let $G = \hat{F}_m$ where $m$ is finite. For each $e$ and each weakly 2-random $z \in G^e$, if the topological normal closure of $z$ has infinite index in $G$, then it is isomorphic to $\hat{F}_\omega$.

10. Describing a profinite structure by a single first-order sentence

The following is related to discussions of Nies with M. Aschenbrenner and T. Scanlon late 2016 at UCLA and UC Berkeley. They discussed a question that had come up during Nies’ visit at Hebrew University earlier that year when talking to Lubotzky and Meiri: can the notion of quasi-finite axiomatisability (QFA, see [21]) be meaningfully extended to the setting of topological algebra?

Suppose $C$ is a class of topological algebraic structures in a finite signature $S$. For instance, $C$ could be the class of profinite separable rings, or the class of profinite separable groups.

Definition 10.1. A first-order sentence $\phi$ in $L(S)$ describes a structure $M \in C$ if $M$ is up to topological isomorphism the unique structure in $C$ that satisfies $\phi$. A structure $M$ is finitely axiomatisable for $C$ if there is such a $\phi$.

We will show that $\text{UT}_3(\mathbb{Z}_p)$ is finitely axiomatisable within the class of separable profinite groups.

Recall the commutator $[x, y] = xyx^{-1}y^{-1}$. A group $G$ is nilpotent of class 2 (nilpotent-2 for short) if it satisfies the law $[[x, y], z] = 1$. Equivalently $G' \subseteq C(G)$. This implies distributivity $[uv, w] = [u, w][v, w]$, and $[u^n, w] = [u, w]^n$ for any $n \in \mathbb{Z}$.

10.1. Rings. Throughout let $p$ be a fixed prime number. Unless otherwise noted, rings will be commutative and with 1.

Theorem 10.2. Let $C$ be the class of profinite rings.

(i) $\mathbb{Z}_p$ is finitely axiomatisable for $C$.

(ii) $\hat{\mathbb{Z}}$ is not finitely axiomatisable for $C$. 
Proof. (i) Recall that a local ring $R$ with maximal ideal $m$ is called Henselian if Hensel’s lemma holds: if $P$ is a monic polynomial in $R[x]$, then any factorization of its image in $(R/m)[x]$ into a product of coprime monic polynomials can be lifted to a factorization of $P$ in $R[x]$. The ring $\mathbb{Z}_p$ is characterized by saying that it is a Henselian valuation ring with residue field $\mathbb{F}_p$ and a $\mathbb{Z}$-group as valuation group with $p$ generating the maximal ideal. To express this by a first order sentence, note that a compact valuation ring is complete, so being Henselian follows from completeness, and we need not include any such axioms. Likewise, we do not need to describe the value group.

(ii) This follows from Feferman-Vaught. As a ring, $\mathbb{Z} \cong \prod_p \mathbb{Z}_p$. Then for every sentence $\phi$ in the language of rings there is a finite sequence of sentences $\psi_1, \ldots, \psi_n$ in the language of rings and a formula $\theta(x_1, \ldots, x_n)$ so that for any index set $I$ and any family of rings $R_i$ indexed by $I$ if we set $X_j := \{ i \in I : R_i \models \psi_j \}$, then

$$\prod_{i \in I} R_i \models \phi \text{ if and only if } \mathcal{P}(I) \models \theta(X_1, \ldots, X_n).$$

Consider $\phi$ a supposed QFA formula. By pigeon hole principle, we can find two distinct primes $\ell \neq q$ so that for all $j \leq n$ we have $\mathbb{Z}_q \models \psi_j \iff \mathbb{Z}_q \models \psi_j$. Define $R_p := \mathbb{Z}_p$ if $p \neq \ell$ and $R_\ell := \mathbb{Z}_q$. Then $R := \prod R_p \models \phi$ but $R \neq \mathbb{Z}$ as, for example, $\ell$ is a unit in $R$ but not in $\mathbb{Z}$. \qed

10.2. QFA profinite groups. Given a ring $R$ let $UT_3(R)$ denote the set of matrices of the form $A = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$ with entries in $R$. For $R = \mathbb{Z}$ the standard generators of $UT_3(R)$ are

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$}

We will write $q = [a, b]$. Note that $q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For $R = \mathbb{Z}_p$ these are topological generators.

We will show that $UT_3(\mathbb{Z}_p)$ is finitely axiomatisable within the class of separable profinite groups. First we need some preliminaries on $UT_3(\mathbb{Z}_p)$. The pro-$p$ completion of a group $G$ is the inverse limit $\varprojlim G/N$ with the canonical projections, where $N$ ranges over normal subgroups of index a power of $p$. If $G$ is f.g. nilpotent, then we can let $N$ range over verbal subgroups $G^p$, $s \in \mathbb{N}$ (as they have index a power of $p$).

We will show that $UT_3(\mathbb{Z}_p)$ is the pro-$p$ completion of $UT_3(\mathbb{Z})$, which will imply that it is the free pro-$p$ nilpotent-2 group on free generators $a, b$.

Let now $R = \mathbb{Z}$. Every matrix $A$ as above can be uniquely written as $A = (\alpha, \beta, \gamma) = a^\beta b^\gamma q^r$ where $\alpha, \beta, \gamma \in R$. We have

$$\alpha + \beta + \gamma + \alpha' \beta' = (\alpha + \beta + \gamma + \alpha' \beta)$$

and for any $r \in \mathbb{Z}$,

$$\alpha + \beta + \gamma + \alpha' \beta = (r + \beta + \gamma + \alpha' \beta)^r = (r \alpha, r \beta, r (\gamma + \alpha \beta))$$
The following is well-known; see e.g. [15].

**Fact 10.3.** UT₃(ℤ) is the free abstract nilpotent-2 group on free generators a, b.

**Proof.** Suppose G is a nilpotent-2 group generated by u, v. Let w = [u, v]. Every element of G can be expressed (not necessarily uniquely) in the form $w^αv^βw^γ$. Since (10.1) applies also in G, $a \mapsto u, b \mapsto v$ extends to a group homomorphism.

**Fact 10.4.** UT₃(ℤₚ) is the pro-p completion of UT₃(ℤ).

**Proof.** In the setting of topological rings, $Z_p = \lim \leftarrow_s Z/p^sZ$. Therefore

$$UT₃(Z_p) = \lim \leftarrow_s UT₃(Z/p^sZ).$$

Write $G = UT₃(Z)$. Let UT₃(pⁿZ) denote the normal subgroup of G consisting of matrices with entries off the main diagonal divisible by $p^n$. Then $|G : UT₃(p^nZ)| = p^{3s}$, so it suffices to show that for the verbal subgroups $G_s = G^{p^s}$ we have

$$G_s \geq UT₃(p^nZ).$$

To this end, we are given $(α, β, γ) ∈ UT₃(p^nZ)$. Let $α' = p^{-s}α, β' = p^{-s}β$ and $γ' = p^{-s}γ - α'β'$. Then $(α', β', γ')p^s = (α, β, γ)$ by (10.2).

As a consequence, UT₃(Zₚ) can be seen as a $Zₚ$-module: for $x ∈ Zₚ$ and $g ∈ UT₃(Zₚ)$ define $g^x = lim_n g^{x|_n}$ where $x|_n$ denotes the last n digits of x. This limit exists because $p^s | x$ implies the projection of $g^{x|_n}$ in UT₃(Z/pⁿZ) vanishes.

More generally, let $\hat{Z}$ denote the free profinite group of rank 1. For any profinite group $G, g ∈ G$ and $λ ∈ \hat{Z}$ one can define exponentiation $g^λ$ as $φ(λ)$ where $φ: \hat{Z} → G$ is the unique homomorphism with $φ(1) = g$. The usual laws of exponentiation hold. See [27, Section 4.1].

The following fact should be well-known, but is somewhat hard to find in the literature.

**Proposition 10.5.** UT₃(Zₚ) is the pro-p nilpotent-2 group on free generators a, b.

**Proof.** There are two ways to see this.

1. Suppose that we are given a nilpotent-2 pro-p group $H$ topologically generated by $u, v$. Each subgroup of index a power of $p$ is open (Serre). So $H$ is its own pro-p completion.

   Let $Θ: UT₃(Z) → H$ be the abstract group homomorphism given by $a → u, b → v$ where $a, b$ are seen as standard generators of UT₃(Z). Let $\tilde{U}$ be the completion of UT₃(Z) with respect to the (restricted) system $Θ^{-1}(Hp^s)$, $s ∈ N$, where $Hp^s$ is the verbal subgroup as above. Then there are natural continuous epimorphisms UT₃(Zₚ) → $\tilde{U}$ by Fact 10.4 and $\tilde{U} → H$ since $H$ is its own pro-p completion. Their composition maps a to $u$ and $b$ to $v$.

2. Every matrix $A ∈ UT₃(Zₚ)$ can be uniquely written as $A = a^nbdgxy$ where $α, β, γ ∈ Zₚ$. Now argue as in Fact 10.3. (This second argument requires verification of some facts on exponentation in pro-p groups.)
**Theorem 10.6.** $\text{UT}_3(\mathbb{Z}_p)$ is finitely axiomatisable within the class of separable profinite groups.

In fact there is a first-order formula $\phi(r, s)$ in the language of groups such that for each separable profinite group $G$, if $G \models \phi(c, d)$ for $c, d \in G$, then $a \mapsto c, b \mapsto d$ yields a topological isomorphism $\text{UT}_3(\mathbb{Z}_p) \cong G$.

**Proof.** We follow the general outline of [26, Thm. 5.1], where it is shown that $\text{UT}_3(\mathbb{Z})$ is QFA within the class of f.g. abstract groups. As explained there in more detail, for any ring $R$ the Mal’cev formula $\mu(x, y, z; r, s)$ defines the ring operation $M_{r, s}$ on the centre $C(\text{UT}_3(R)) \cong (\mathbb{R}, +)$ when $r, s$ are assigned to the standard generators $a, b$ (also see [25]).

Sentence $\alpha_3$ expresses of a profinite group $G$ that $G$ is nilpotent-2, and the centre $C = C(G)$ equals the set of commutators (in particular, $G_{ab} = G/C$). Since $C$ is closed, by Theorem 10.2, there is a formula $\gamma(r, s)$ expressing that $(C, +, M_{r, s})$ is isomorphic to $Z_p^2$; in addition, $\gamma$ expresses that $[r, s]$ is the neutral element 1 of this ring. Finally, a sentence $\alpha_3$ expresses that $pG/C$ has index $p^2$ in $G/C$. Let $\phi(r, s) \equiv \alpha_1 \land \gamma(r, s) \land \alpha_3$.

Suppose now that $G \models \phi(c, d)$.

**Claim 10.7.** $G_{ab} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

As in [26, Thm. 5.1] since the centre $C$ is torsion free, $G_{ab}$ is torsion free: if $u \notin C$ then $[u, v] \neq 1$ for some $v \in G - \{1\}$. Then $[u^n, v] = [u, v]^n \neq 1$ so that $u^n \notin C$.

Next, since $G_{ab}$ is profinite, by the structure theorem (e.g. [27, Thm. 4.3.8]) $G_{ab} = \prod_q \mathbb{Z}_q^{m(q)}$ where $q$ ranges over the primes and $m(q)$ is a cardinal. Then $m(q) = 0$ for $q \neq p$. For otherwise we can take $v \in G - C$ such that in $G_{ab}$ we have $p^n \mid vC$ for each $n$. Choose $z \in G$ such that $[v, z] \neq 1$, and take $w \in G$ such that $p^n wC = vC$. Then $k = [w^{p^n}, z] = [w, z]^{p^n}$ so that $p^n \mid k$ for each $n$, contrary to the fact that $C \cong \mathbb{Z}_p$ as an abelian group.

Since $G \models \alpha_3$ we have $m(p) = 2$. This shows the claim.

Further, $G$ is a pro-$p$ group since the class of such groups is closed under extensions [27, Thm. 2.2.1(e)].

For a topological group $G$ and $S \subseteq G$ let $\langle S \rangle$ denote the closure of the subgroup generated by $S$.

**Claim 10.8.** $G = \langle c, d \rangle$.

**Proof.** Since $G \models \alpha_3$ we have $\langle c, d \rangle = C$. So it suffices to show that $\langle Cc, Cd \rangle = G_{ab}$. Pick $g, h \in G$ such that $\langle Cg, Ch \rangle = G_{ab}$. There are $x, y, z, w \in \mathbb{Z}_p$ and $u, v \in C$ such that $c = ug^zh^y$ and $d = vg^zh^w$. Then $[c, d] = [g, h]^{xy - zw}$. On the other hand $[c, d]^k = [g, h]$ for some $k \in \mathbb{Z}_p$. So the determinant $xy - zw$ is a unit in $\mathbb{Z}_p$, whence $\langle Cc, Cd \rangle = G_{ab}$. ⊣

By Prop. 10.5 there is a continuous group homomorphism $\Theta : \text{UT}_3(\mathbb{Z}_p) \to G$ such that $\Theta(a) = c$ and $\Theta(b) = d$. Then $\Theta([a, b]) = [c, d]$ so $\Theta$ induces an isomorphism $C(\text{UT}_3(\mathbb{Z}_p)) \to C(G)$. Also $\Theta$ induces an isomorphism $\text{UT}_3(\mathbb{Z}_p)_{ab} \to G_{ab}$. Hence $\Theta$ is an isomorphism as required. □
Part 6. Metric spaces and descriptive set theory

11. Turetsky and Nies: Scott rank of Polish metric spaces - a computability approach

We give a proof based on computability theory of Doucha’s result [5] that the Scott rank of Polish metric spaces $M$ is at most $\omega + 1$. We prove that the Scott rank of each pair of tuples $\overline{a}, \overline{b}$ of the same length is bounded by $\omega_1$. Here we view metric spaces as structures in a countable language with distance relations $R_q(x, y)$, where $q$ is a positive rational, intended to express that the distance of $x,y$ is less than $q$. William Chan announced this in 2016, giving a proof involving admissible sets.

For a structure $M$, $\overline{a}, \overline{b} \in M^n$, and a linear order $L$, the Ehrenfeucht-Fraïssé game $G^L_M(\overline{a}, \overline{b})$ is played as follows:

- On the $i$th round, Player 1 chooses a $z_i \in L$ with $z_i < L z_{i-1}$ when $i > 0$, and either chooses an element $a_{n+i} \in M$ or an element $b_{n+i} \in M$.
- Player 2 then chooses whichever of $a_{n+i}$ or $b_{n+i}$ Player 1 did not.

After round $i$, if the map from $a_0 a_1 ... a_{n+i}$ to $b_0 b_1 ... b_{n+i}$ is not a partial isomorphism, then Player 1 wins. The game ends in a win for Player 2 after either $\omega$ many rounds, or if Player 1 cannot choose a $z_{i+1} < L z_i$, and Player 1 has not already won.

We extend these games to metric spaces, replacing partial isomorphism with partial isometry.

**Fact 11.1.** If $M$ is a countable structure or a Polish metric space and $L$ is ill-founded, then Player 2 has a winning strategy in $G^L_M(\overline{a}, \overline{b})$ iff there is an automorphism or autoisometry of $M$ taking $\overline{a}$ to $\overline{b}$.

**Definition 11.2.** For a structure or metric space $M$, define $\text{rank}^M(\overline{a}, \overline{b})$ to be the least ordinal $\alpha$ for which Player 2 does not have a winning strategy in $G^\alpha_M(\overline{a}, \overline{b})$, or $\text{rank}^M(\overline{a}, \overline{b}) = \infty$ if there is no such $\alpha$.

Define $\text{rank}^M_L(\overline{a}) = \sup \{ \text{rank}^M(\overline{a}, \overline{b}) : \text{rank}^M(\overline{a}, \overline{b}) < \infty \}$.

Define $\text{rank}(M) = \sup \{ \text{rank}^M(\overline{a}) + 1 : \overline{a} \in M \}$.

Note that in some versions e.g. Doucha’s, the “$+1$” is omitted.

**Fact 11.3.** For a computable structure or Polish space $M$ and any reasonable definition of Scott Rank, $SR(M) \leq \text{rank}(M)$.

**Definition 11.4.** If $M$ is a Polish metric space and $D \subseteq M$ is dense, define $G^D_M(\overline{a}, \overline{b}, D)$ exactly as $G^L_M(\overline{a}, \overline{b})$, save that Player 1’s choice of elements is restricted to $D$. Player 2 is still allowed to choose any element from $M$.

Define $\text{rank}^D_M(\overline{a}, \overline{b}, D)$ and $\text{rank}(M, D)$ as above, using $G^D_M(\overline{a}, \overline{b}, D)$ in place of $G^\alpha_M(\overline{a}, \overline{b})$.

**Remark 11.5.** If $L$ is countable, a strategy for Player 2 is coded by a real, given some numbering of $L$ and $D$. Player 1’s possible plays at any give round are each coded by an element of $\omega$, while Player 2’s responses are given by Cauchy sequences from $D$.

**Remark 11.6.** Given numberings of $D$ and $L$ and a real, checking that this real codes a winning strategy for Player 2 is arithmetical relative to the
metric on $D$. We must check that every response is a Cauchy sequence, and that every partial play of the game results in a partial isometry.

**Remark 11.7.** Any winning strategy of Player 2’s for $G^L_M(\overline{a},\overline{b})$ restricts to a winning strategy for $G^L_M(\overline{a},\overline{b}, D)$, and so $\text{rank}^M(\overline{a},\overline{b}) \leq \text{rank}^M_{D}(\overline{a},\overline{b})$.

**Fact 11.8.** If $M$ is a Polish metric space, $D \subseteq M$ is dense and $L$ is ill-founded, then Player 2 has a winning strategy in $G^L_M(\overline{a},\overline{b}, D)$ iff there is an autoisometry of $M$ taking $\overline{a}$ to $\overline{b}$.

**Theorem 11.9.** If $M$ is a Polish metric space, $D \subseteq M$ is dense and $\text{rank}^M_{D}(\overline{a},\overline{b}) \geq \omega^1_a, b, M^\upharpoonright D$, then there is an autoisometry of $M$ taking $\overline{a}$ to $\overline{b}$, and so $\text{rank}^M(\overline{a},\overline{b}) = \omega_1$.

**Proof.** Suppose $\text{rank}^M_{D}(\overline{a},\overline{b}) \geq \omega^1_a, b, M^\upharpoonright D$. Consider the formula $\theta(e)$ stating that $\Phi_{e, M^\upharpoonright D}$ gives a total linear order $L$, and there is a real which codes a winning strategy for Player 2 in $G^L_M(\overline{a},\overline{b}, D)$. Note that $\theta$ is $\Sigma^1_1(\overline{a},\overline{b}, M^\downharpoonright D)$. By assumption, $\theta(e)$ holds for every $e$ with $\Phi_{e, M^\upharpoonright D}$ well-ordered, and so by $\Sigma^1_1$-bounding it must hold for some $e$ with $\Phi_{e, M^\upharpoonright D}$ ill-founded. As observed before, this means that there is an autoisometry of $M$ taking $\overline{a}$ to $\overline{b}$. □

We strengthen the result of William Chan that a rigid Polish metric space has countable Scott rank.

**Proposition 11.10.** Let $M$ be a Polish metric space such that the isometry relation on tuples of the same length is $\Delta^1_1$. Then the Scott rank of $M$ is computable in $M^\upharpoonright D$.

**Proof.** By hypothesis the following property of $e \in \omega$ is $\Sigma^1_1$: $\Phi^M_{e, M^\upharpoonright D}$ codes a linear order $L$ such that $\exists n \exists \overline{a}, \overline{b} \in M^n [\overline{a} \neq \overline{b} \wedge \text{Player 2 has a winning strategy in } G^L_M(\overline{a},\overline{b}, D)].$

By the argument above for each such $e$, $\Phi^M_{e, M^\upharpoonright D}$ is a well-ordering. By $\Sigma^1_1$-bounding the set of such $\Phi^M_{e, M^\upharpoonright D}$ is then bounded by an ordinal computable in $M^\upharpoonright D$. □

**Part 7. Model theory and definability**

12. SOME OPEN QUESTIONS ON COMPUTABILITY AND STRUCTURE

Noam Greenberg, Alexander Melnikov, André Nies and Dan Turetsky worked at the Research Centre Coromandel April 18-22. They discussed the following open questions.

**Question 12.1.** Let $A$ be a computable $\omega$-categorical structure in a finite signature. Show that $S_A$, the set of indices for computable structures isomorphic to $A$, is arithmetical.

Turetsky showed that $S_A$ can be arbitrarily high in the arithmetical hierarchy depending on the arity of the language. According to Melnikov, an
affirmative answer follows from Uri Andrew’s thesis around p. 42, related to
the Hrushovski construction.

**Question 12.2.** Is every computable closed subgroup \( G \) of \( S_\infty \) topologically
isomorphic to the automorphism group of a computable structure?

Such a group is given as a \( \Pi_1^0 \) class of pairs \( f, g \) of functions in Baire space
such that \( g = f^{-1} \). They can be seen as pruned subtrees of the tree \( T \) of all
pairs \((\sigma, \sigma')\) of strings of the same length \( n \) such that \( \sigma(i) = k \leftrightarrow \sigma'(k) = i \)
for each \( i, k < n \). It is well known that \( G \) is topologically isomorphic to the
automorphism group of a some countable structure, namely the one
consisting of all the \( n \)-orbits, seen as named \( n \)-ary relations, for each \( n \).
However, the orbit relation is only computable in \( \mathcal{O} \) in general, so this
structure is merely computable in \( \mathcal{O} \).

**Question 12.3.** Does computably categorical imply relatively \( \Delta_1^1 \)-categorical?

**Question 12.4.** Let \( G \) be a f.g. group with \( \Pi_1^0 \) word problem. Is \( G \) embed-
dable into the group of computable isometries of a computable metric space?

Morozov [20] showed that one cannot always choose the space discrete;
I.e., there is an example of \( G \) that is not a subgroup of the computable
permutations of \( \mathbb{N} \). Yet, one can show that this particular example can be
realised as a group of computable isometries.

Further suggestions for study (Melnikov):

- the partial order of primitive recursive presentations of a structure
  (such as \((\mathbb{Q}, <)\) under the preordering of isomorphisms that are p.r.
  (without the inverse necessarily being p.r.) For instance, do you
  get the same degree structure for \((\mathbb{Q}, <)\) and the countably atomless
  Boolean algebra?
- Does Markov computable for compact groups imply fully computable?

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