Abelian Categories of Modules over a (Lax) Monoidal Functor

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Abstract: In [CY98] Crane and Yetter introduced a deformation theory for monoidal categories. The related deformation theory for monoidal functors introduced by Yetter in [Yet98] is a proper generalization of Gerstenhaber’s deformation theory for associative algebras [Ger63, Ger64, GS88]. In the present paper we solidify the analogy between lax monoidal functors and associative algebras by showing that under suitable conditions, categories of functors with an action of a lax monoidal functor are abelian categories. The deformation complex of a monoidal functor is generalized to an analogue of the Hochschild complex with coefficients in a bimodule, and the deformation complex of a monoidal natural transformation is shown to be a special case. It is shown further that the cohomology of a monoidal functor $F$ with coefficients in an $F,F$-bimodule is given by right derived functors.
1 Introduction

In [CY98] Crane and Yetter introduced an infinitesimal deformation theory for monoidal categories. Yetter [Yet98] introduced related deformation theories for monoidal functors and braided monoidal categories. In [Yet98, Yet01] Yetter showed that the rich structure present in Gerstenhaber’s deformation theory of associative algebras is again present in these new theories and gave the hitherto missing proof that all obstructions are closed. For monoidal functors, as for associative algebras, first order deformations are classified by $H^2$, while obstructions to higher order deformations lie in $H^3$. For monoidal categories, the first order deformations lie in $H^3$, while obstructions lie in $H^4$.

One problem with categorical deformation theories not found in algebraic deformation theories is the fact that the natural definition of the cochain groups as module of natural transformations does not live comfortably in a natural setting for a calculus of derived functors: the source category for the functors between which natural transformations are taken changes from degree to degree. The structures and results of this paper arose from the need to remedy this apparent defect.

In sections 2 through 6 we follow closely [Yet01], except that we treat strong monoidal functors as special cases of lax rather than oplax monoidal functors.
2 Review of Categorical Deformations

Throughout, we work in a setting in which all categories are $R$-linear for some fixed commutative ring $R$ and in which all functors are linear (in each variable). As was observed in [Yet92a, Yet01], given an $R$-linear category $C$ and an $R$-algebra $A$, we can form a category $C \otimes A$ by extension of scalars:

$$\text{Ob}(C \otimes A) = \text{Ob}(C)$$

and

$$\text{Hom}_{C \otimes A}(X, Y) = \text{Hom}_C(X, Y) \otimes_R A$$

extending both composition and any structural functors by (multi)linearity. If the algebra is an $m$-adically complete local ring, we can also form an $m$-adic completion $\hat{C} \otimes A$, by $m$-adically completing all of the homsets and extending composition and any structural functors by continuity.

As in [Yet92a, Yet01], we denote $C \otimes R[\epsilon]/<\epsilon^{n+1}>$ by $C^{(n)}$, and $C \otimes R[[x]]$ by $C^{(\infty)}$.

Similarly, if we have a functor $F : C \to D$, we may obtain a functor from $C \otimes A$ to $D \otimes A$ by extending the definition of $F$ on maps by linearity, likewise in the $m$-adically complete setting. In the case of $A = R[\epsilon]/<\epsilon^{n+1}>$ or $A = R[[x]]$, we will denote the resulting functor by $F^{(n)}$ or $F^{(\infty)}$, respectively.

We will always work in a context where we have a specified category $C$, a specified functor $F : C \to D$, or a specified natural transformation $\theta : F \Rightarrow G$ for $F, G : C \to D$. The category(ies) will be equipped with some structure functors and structural natural transformations between them and satisfying specified coherence conditions. The functor(s) will be equipped with some structural natural transformations relating them and the structure functors of $C$ and $D$ and specified coherence conditions relating these with the structural natural transformations of $C$ and $D$. The natural transformation $\theta$ will be equipped with specified coherence conditions relating $\theta$ and the structural natural transformations of $F, G, C$ and $D$. We call such a category (resp. functor, natural transformation) a category with structure (resp. functor with structure, natural transformation with structure).

In the case of a natural transformation with structure, we consider the natural transformation itself to be a “structural natural transformation”. We then make:

**Definition 1** An $n^{th}$ order deformation of a category with structure $C$ (resp. a functor with structure $F : C \to D$, a natural transformation with structure $\theta : F \Rightarrow G$) is an assignment to each structural natural transformation $\phi : \Phi \Rightarrow \Psi$ of $C$ (resp. of $C$, $D$, and $F$) of a structural natural transformation

$$\phi^{(n)} = \phi + \phi_1 \epsilon + \ldots + \phi_n \epsilon^n$$

such that $\phi^{(n)} : \Phi^{(n)} \Rightarrow \Psi^{(n)}$ and $C^{(n)}$ (resp. $F^{(n)}$, $\theta^{(n)}$) is a category with structure (resp. functor with structure, natural transformation with structure) satisfying the same coherence conditions as $C$ (resp. $F$, $\theta$).

A formal deformation is defined similarly by a formal power series.

Among these we can distinguish one which always exists:
Definition 2 The trivial deformation of $C$ (resp. $F : C \to D$, $\theta : F \Rightarrow G$) is given by letting $\phi^{(n)} = \phi$ for all structural natural transformations.

In terms of this strict notion of triviality, we can distinguish important special cases of deformations of a functor or natural transformation. Observe that a deformation of a functor with structure (resp. natural transformation with structure) induces “forgetful” deformations of its source and target (resp. its source and target and their source and target). We can thus consider cases in which certain of the induced “forgetful” deformations are trivial:

Definition 3 A deformation of a functor with structure $F : C \to D$ between two categories with structure is purely functorial (resp. fibred, cofibred) if the induced deformations on the source and target (resp. target, source) are trivial.

A deformation of a natural transformation with structure $\theta : F \Rightarrow G$ between two functors with structure is purely transformational (resp. 1-fibred, 1-cofibred, purely functorial, 2-fibred, 2-cofibred) if the source and target (resp. target, source, source and target of the parallel functors, target of the parallel functors, source of the parallel functors) are trivial.

In the case to which we will apply this very general notion — semigroupal and monoidal categories, and the various types of monoidal and semigroupal functors — there is always a good notion of equivalence between deformations. Let us review the relevant definitions and some theorems and lemmas which will be needed:

Definition 4 A monoidal category $C$ is a category $C$ equipped with a functor $\otimes : C \times C \to C$ and an object $I$, together with natural isomorphisms $\alpha : \otimes(\otimes \times 1_C) \Rightarrow \otimes(1_C \times \otimes)$, $\rho : \otimes I \Rightarrow 1_C$ and $\lambda : I \otimes \Rightarrow 1_C$, satisfying the pentagon and triangle coherence conditions of Figure 1 and the bigon $(\rho_1 = \lambda_1)$ coherence condition (cf. [ML98]). Similarly, a semigroupal category is a category equipped with only $\otimes$ and $\alpha$, satisfying the pentagon of Figure 1.

Definition 5 A lax monoidal functor $F : C \to D$ between two monoidal categories $C$ and $D$ is a functor $F$ between the underlying categories, equipped with a natural transformation

$$\tilde{F} : F(-) \otimes F(-) \to F(- \otimes -)$$

and a map $F_I : I \to F(I)$, satisfying the hexagon and two squares of Figure 2.

If $\tilde{F}$ and $F_I$ are isomorphisms, $F$ is a strong monoidal functor. If $F_I$ and all components of $\tilde{F}$ are identity maps, $F$ is a strict monoidal functor.

Lax, strong and strict semigroupal functors are defined similarly.

We will have no cause to consider oplax monoidal functors in this work.

We refer to the components of the natural transformations and maps specified in these definitions, and to their inverses (if any), as structure maps. Likewise, a map which is obtained from some other map $f$ by forming an iterated monoidal product of $f$ with identity maps for various objects is called a prolongation of $f$. Sometimes by abuse of terminology prolongations of structure maps are themselves referred to as structure maps.

We will also refer to a diagram obtained by applying the same iterated monoidal product with identity maps to every map of a given diagram as a prolongation of the given diagram.

\[1\] It can be shown that the bigon condition is redundant.
Figure 1: Coherence Conditions for Monoidal Categories

It is, of course, a matter of taste whether one defines strong monoidal functors as lax monoidal functors with invertible structure maps, as here, or as oplax monoidal functors with invertible structure maps as in [Yet98, Yet01].

Crucial to the construction of our deformation theories are the coherence theorem of Mac Lane [ML63] and a non-symmetric variant of the coherence theorem of Epstein [Eps66], which we will soon state in the most convenient form for our purposes.

**Definition 6** For any set \( S \), \( S \downarrow \text{MonCat} \) (resp. \( S \downarrow \text{SGCat} \)) is the category whose objects are (small) monoidal (resp. semigroupal) categories equipped with a map from \( S \) to their set of objects, and whose arrows are strict monoidal functors whose object maps commute with the map from \( S \).

\( S \downarrow \text{LaxSGFun} \) (resp. \( S \downarrow \text{StrongSGFun} \)) is the category whose objects are lax (resp. strong) semigroupal functors between a pair of semigroupal categories, the source of which is equipped with a map from \( S \) to its set of arrows, and whose arrows are pairs of strict monoidal functors forming commuting squares and commuting with the map from \( S \).

Observe that \( S \downarrow \text{MonCat} \), (resp. \( S \downarrow \text{SGCat} \), \( S \downarrow \text{LaxSGFun} \), \( S \downarrow \text{StrongSGFun} \)) is a category of models of an essentially algebraic theory, and thus by general principles has an initial
Definition 7 A formal diagram in the theory of monoidal categories (resp. semigroupal categories) is a diagram in the free monoidal (resp. semigroupal) category on $S$ for some set $S$.

A formal diagram in the theory of lax (resp. strong) semigroupal functors is a diagram in the target category of the free lax (resp. strong) semigroupal functor on $S$ for some set $S$.

The coherence theorem of Mac Lane [ML63] may then be stated as

**Theorem 8** Every formal diagram in the theory of monoidal categories commutes. Consequently, any diagram which is the image of a formal diagram under a (strict monoidal) functor commutes.

The same proof carries the weaker result:
Theorem 9

Every formal diagram in the theory of semigroupal categories commutes. Consequently, any diagram which is the image of a formal diagram under a (strict semigroupal) functor commutes.

Epstein [Eps66] proved a coherence theorem only for lax semigroupal functors between symmetric semigroupal categories, but the same proof will carry the result:

Theorem 10

Every formal diagram in the theory of lax (resp. strong) semigroupal functors commutes. Every formal diagram in the theory of strong monoidal functors commutes. Consequently, any diagram which is a functorial image of such a formal diagram under a (strict monoidal) functor commutes.

These coherence theorems are the basis for a very useful notion and notational convention: throughout our discussion of categorical deformation theory we will use padded composition operators $⌈ ⌉$. These operators are an embodiment of the coherence theorems of Mac Lane [ML63] and Epstein [Eps66].

Definition 11

Given a monoidal category $C$ (resp. a semigroupal or strong monoidal functor $F : \mathcal{X} \to C$), and a sequence of maps $f_1, \ldots, f_n$ in $C$ such that the source of $f_{i+1}$ is isomorphic (resp. maps) to the target of $f_i$ by a composition of prolongations of structure maps (i.e. by a formal diagram with underlying diagram a chain of composable maps), we let

$⌈ f_1, \ldots, f_n ⌉$

denote the composite $a_0f_1a_1f_2 \ldots a_{n-1}f_na_n$, where the $a_i$’s are composites of prolongations of structure maps and the following hold:

1. The source of $a_0$ is reduced (no tensorands of $I$) and completely left-parenthesized (resp. reduced and completely left-parenthesized and free from images of monoidal products under $F$).

2. The target of $a_n$ is reduced and completely right-parenthesized (resp. reduced and completely right-parenthesized and free from products both of whose factors are images under $F$).

3. The composite is well-defined.

The fact that this defines a well-defined map will be a consequence of the coherence theorems. However, if the $f_i$ are simply maps, $⌈ ⌉$ may not be well-defined in the event that there are “accidental coincidences”. In our circumstance, the maps in the sequences to which the padded composition operator is applied will always be components of natural transformations with a particular structure:

Definition 12

Given a monoidal category $C$ (resp. a monoidal functor $F : C \to D$), a natural transformation is $C$-paracoherent (resp. $F$-paracoherent) if its source and target functors are iterated prolongations of the structure functors $\otimes, I$, and $1_C$ (resp. $\otimes, F, I, 1_C$, and $1_D$), where $I$ is regarded as a functor from the trivial one object category.

In the case where the maps in the sequence are specified not merely as maps, but as components of particular paracoherent natural transformations, their sources and targets are given an explicit structure as images of iterated prolongations of structure functors. Thus we may require that the “padding” maps given in terms of the structural natural transformations be (components of) natural transformations between the appropriate functors.

A number of elementary lemmas hold for these operators. Proofs are left to the reader.
Lemma 13
\[ [f_1 \ldots f_n] = [[f_1 \ldots f_k][f_{k+1} \ldots f_n]]. \]

Lemma 14 If \([ \ ]\) is applied in the case of a monoidal category or strong monoidal functor
\[ [f_1 \ldots g \otimes I \ldots f_n] = [f_1 \ldots g \ldots f_n] = [f_1 \otimes g \ldots f_n]. \]

Lemma 15
\[ [f_1 \ldots f_n] = [f_1 \ldots [f_k \ldots f_l] \ldots f_n]. \]

Lemma 16
\[ [f_1 \ldots g \otimes h \ldots f_n] = [f_1 \ldots [g \otimes h \ldots f_n] = [f_1 \ldots g \otimes [h] \ldots f_n]. \]

Lemma 17 If \(\phi_{X_1,\ldots,X_n}\) is a \(C\)-paracoherent natural transformation (resp. \(F\)-paracoherent natural transformation, for \(F\) a strong monoidal functor), then so is \(\phi_{X_1,\ldots,I,\ldots,X_n}\), where \(I\) is inserted in the \(i^{th}\) position, and similarly if \(I\) is inserted in the \(i^{th}\) position for all \(i \in T \subset \{1,\ldots,n\}\). Moreover, in this latter case \([\phi_{\ldots}]\) is a paracoherent natural transformation from the fully left-parenthesized product of \(X_{i_1} \ldots X_{i_k}\) (resp. the fully left-parenthesized product of \(F(X_{i_1}) \ldots, F(X_{i_k})\)) to the fully right-parenthesized product of (resp. \(F\) of the fully right-parenthesized product) of \(X_{i_1} \ldots X_{i_k}\), where \(\{i_1,\ldots,i_k\} = \{1,\ldots,n\} \setminus T\)
and \(i_1 < i_2 < \ldots < i_n\).

From these lemmas we deduce a final lemma:

Lemma 18 If \(\psi_{A,B,C}, \phi_{A,B,C} : [A \otimes B] \otimes C \to A \otimes [B \otimes C]\) are natural transformations, then
\[ [[\phi_{A,I,I} \otimes B]\psi_{A,I,B}] = [\psi_{A,I,B}[\phi_{A,I,I} \otimes B]] \]
and
\[ [[A \otimes \phi_{I,I,B}]\psi_{A,I,B}] = [\psi_{A,I,B}[A \otimes \phi_{I,I,B}]]. \]

proof: First, apply Lemma 14. Then use the naturality of \([\psi_{A,I,B}] : A \otimes B \to A \otimes B\) and the source and target data for \(\phi_{A,I,I}\) and \(\phi_{I,I,B}\), as given by Lemma 17. \(\square\)

Finally, we make

Definition 19 A monoidal natural transformation is a natural transformation \(\phi : F \Rightarrow G\) between monoidal functors which satisfies
\[ \tilde{G}_{A,B}(\phi_{A \otimes B}) = \phi_A \otimes \phi_B(\tilde{F}_{A,B}) \]
and \(F_0 = G_0(\phi_I)\). A semigroupal natural transformation between semigroupal functors is defined similarly.
Definition 20 A monoidal equivalence between monoidal categories \( \mathcal{C} \) and \( \mathcal{D} \) is an equivalence of categories in which the functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) are equipped with the structure of monoidal functors, and the natural isomorphisms \( \phi : FG \Rightarrow \text{Id}_{\mathcal{C}} \) and \( \psi : GF \Rightarrow \text{Id}_{\mathcal{D}} \) are both monoidal natural transformations. If there exists a monoidal equivalence between \( \mathcal{C} \) and \( \mathcal{D} \), we say that \( \mathcal{C} \) and \( \mathcal{D} \) are monoidally equivalent.

Using this, we can now define equivalences between deformations of monoidal and semigroupal categories and functors.

Definition 21 Two \( n \)th order deformations of a monoidal (semigroupal) category \( \mathcal{C} \) are equivalent if there exists a monoidal (semigroupal) functor between them, whose underlying functor is \( \text{Id}_{\mathcal{C}(n)} \) and whose structural natural transformations reduces modulo \( \epsilon \) to the identity natural transformations.

Definition 22 Two \( n \)th order purely functorial deformations \( \tilde{F} \) and \( \hat{F} \) of a monoidal (semigroupal) functor \( F : \mathcal{C} \to \mathcal{D} \) are equivalent if there exists a monoidal (semigroupal) natural isomorphism \( \vartheta : \tilde{F} \Rightarrow \hat{F} \) which reduces modulo \( \epsilon \) to the identity natural transformation.

Observe that in this last case, it is a matter of indifference whether we are considering strong or lax monoidal functors.

For purely transformational monoidal (semigroupal) natural transformations equivalence of deformations is simply equality.
3 Deformation Complexes of Semigroupal Categories and Functors

We define a cochain complex associated to any semigroupal category or semigroupal functor:

**Definition 23** The deformation complex of a strong or lax semigroupal functor \((F : C \to D, \tilde{F})\) is the cochain complex

\[ (X^\bullet(F), \delta), \]

where

\[ X^n(F) = \text{Nat}(n \otimes (F^n), F(\otimes^n)) \]

and

\[
\delta(\phi)_{A_0,\ldots,A_n} = \left[ F(A_0) \otimes \phi_{A_1,\ldots,A_n} \right] \\
+ \sum_{i=1}^{n} (-1)^i \left[ \phi_{A_0,\ldots,A_{i-1} \otimes A_i,\ldots,A_n} \right] \\
+ (-1)^{n+1} \left[ \phi_{A_0,\ldots,A_{n-1} \otimes F(A_n)} \right].
\]

A similar definition applies for oplax semigroupal functors. Observe that here we regard strong semigroupal and monoidal functors as special cases of lax semigroupal functors, while in [Yet01] they are treated as special cases of oplax functors.

**Definition 24** The deformation complex of a semigroupal category \((C, \otimes, \alpha)\) is \((X^\bullet(Id_C), \delta)\). We also denote this by \((X^\bullet(C), \delta)\).

The motivation for these definitions is given in [CY98] and [Yet98], or can be readily discovered by the reader by computing by hand the conditions on the term \(\alpha^{(1)}\) or \(\tilde{F}^{(1)}\) in a first order deformation.

We also have

**Theorem 25** The first-order deformations of a semigroupal category \(C\) are classified up to equivalence by \(H^3(C)\).

**Sketch of Proof:** Consider two first-order deformations \(\tilde{\alpha} = \alpha + \alpha^{(1)}\epsilon\) and \(\hat{\alpha} = \alpha + a^{(1)}\epsilon\) of \(C\). Consider also a semigroupal functor whose underlying functor is the identity functor and whose structural transformation is of the form

\[ 1_{A \otimes B} + \phi_{A,B}\epsilon : A \otimes B \to A \otimes B. \]

Now, write out the coherence condition for semigroupal functors in this case, and look at the degree 1 terms. The resulting equation is nothing more than

\[ \alpha^{(1)} - a^{(1)} = \delta(\phi). \]

\[ \square \]

In [Yet98] it is shown that...
Theorem 26  The purely functorial first-order deformations of a semigrou pal functor $F : C \rightarrow D$ are classified up to equivalence by $H^2(F)$.

The proof, similar to that of the previous theorem, is given in [Yet98].

Obstructions to extending $n^{th}$ order deformations to $(n+1)^{st}$ order deformations will be discussed below.

First, however, we will show that the deformation complexes already defined together with standard machinery from homological algebra suffice to handle the cases of fibred and total deformations and deformations of braided monoidal categories.

Compositions of natural transformations with functors induce two cochain maps whenever we have a semigroupal functor $F : C \rightarrow D$ (whether lax, oplax, or strong):

\[ [F(-)] : X^\bullet(C) \rightarrow X^\bullet(F) \]

and

\[ [(-)F^\bullet] : X^\bullet(D) \rightarrow X^\bullet(F). \]

Recalling that $X^\bullet(C) = X^\bullet(Id_C)$, we see that the cochain maps just defined are, in fact, special cases of more general families defined for composable pairs of functors

\[ C \xrightarrow{F} D \xrightarrow{G} E \]

namely,

\[ [G(-)] : X^\bullet(F) \rightarrow X^\bullet(G(F)) \]

and

\[ [(-)F^\bullet] : X^\bullet(G) \rightarrow X^\bullet(G(F)). \]

To consider deformations of braided monoidal categories the following is also useful: given any $K$-linear semigroupal category $C$, there is a "diagonal" cochain map

\[ \Delta : X^\bullet(C) \rightarrow X^\bullet(C \otimes C) \]

given by:

\[ \Delta(\phi) = \phi \otimes [Id] + [Id] \otimes \phi. \]

These cochain maps allow us to assemble the simpler deformation complexes for semigroupal categories and semigroupal functors into complexes whose cohomology is related to more general types of deformations.

Recall the construction of a cone over a cochain map:

Definition 27  Given a map of cochain complexes $u^\bullet : A^\bullet \rightarrow B^\bullet$, the cone on $u^\bullet$ is the cochain complex

\[ (C^\bullet_{u^\bullet}, d_u) = \left( B^\bullet \oplus A^{\bullet+1}, \begin{bmatrix} d_B & 0 \\ u & -d_A \end{bmatrix} \right). \]

Here we adopt the convention that elements of direct sums are written as row vectors with entries in the summands, and that arrays of maps act on the right by matrix multiplication (with the action of maps in lieu of scalar multiplication). Note that this is consistent with our notational convention: maps act on the right on elements (improperly) thought of as maps, unless parentheses denoting application intervene.

In the next section use cones on the cochain maps discussed above to classify first order deformations more general than those classified by Theorems 25 and 26.
4 First Order Deformations

Let us now consider the problem of classifying first order fibred deformations of semigroupal functors. If we have a lax (resp. oplax, strong) semigroupal functor \([F, \tilde{F}] : (\mathcal{C}, \otimes, \alpha) \to (\mathcal{D}, \otimes, a)\), and we replace \(\tilde{F}(0) = \tilde{F}\) with \(\tilde{F}(0) + \tilde{F}(1)\) and \(\alpha(0) = \alpha\) with \(\alpha(0) + \alpha(1)\) for \(\epsilon^2 = 0\), the conditions for the new coherence diagrams to commute become

\[\delta(\alpha(1)) = 0\]

and

\[\delta(\tilde{F}(1)) + [F(\alpha)] = 0,\]

as can be verified readily by computing the \(\epsilon\)-degree 1 terms going around the pentagon and hexagon coherence diagrams.

It then follows directly that the pair \([\tilde{F}(1), \alpha(1)]\) is a 2-cocycle in

\[(C^\bullet_{[F(-)], d_{-[F(-)]}}).\]

Now, consider the condition that two such 2-cocycles \([\tilde{F}(1)_1, \alpha(1)_1]\) and \([\tilde{F}(1)_2, \alpha(1)_2]\) are equivalent. Let \(F_1 : \mathcal{C}_1 \to \mathcal{D}\) and \(F_2 : \mathcal{C}_2 \to \mathcal{D}\) denote the semigroupal functors from the corresponding deformations (suppressing here the naming of structural maps). In particular, there is a structure map which makes \(Id_{\mathcal{C}(2)}\) into a (necessarily strong) semigroupal functor and which reduces modulo \(\epsilon\) to the identity natural transformation. Second, there is a semigroupal natural isomorphism \(\psi\) from \(F_1\) to \(F_2(I)\) which reduces modulo \(\epsilon\) to the identity, where \(I\) is the identity functor on \(\mathcal{C}(2)\) made into a semigroupal functor by given structure map.

Denoting the structural map for \(Id_{\mathcal{C}(2)}\) by \(id + \iota(1)\) and letting \(\psi = id + \psi(1)\), the coherence conditions become

\[[\tilde{F} + \tilde{F}_{2A,B}\epsilon][[id + \iota(1)_{A,B}\epsilon][id_{F(A\otimes B)} + \psi(1)_{A\otimes B}\epsilon]] =

[[id_{F(A)} + \psi(1)_A\epsilon] \otimes [id_{F(B)} + \psi(1)_B\epsilon]](\tilde{F} + \tilde{F}_{2A,B}\epsilon)\]

and

\[[id_{F(A)} \otimes [id_{F(B\otimes C)} + \iota(1)_{B,C}\epsilon]]

([id_{F(A\otimes[B\otimes C])} + \iota(1)_{A,B\otimes C}\epsilon](F(\alpha + \alpha_{1A,B,C}\epsilon))) =

[F(\alpha + \alpha_{2A,B,C}\epsilon)][[id_{F(A\otimes B)} + \iota(1)_{A,B}\epsilon] \otimes id_{F(C)}]

([id_{F([A\otimes B]\otimes C)} + \iota(1)_{A\otimes B,C}\epsilon])] .

Using the bilinearity of composition and \(\otimes\), the coherence conditions on the original maps, and the condition \(\epsilon^2 = 0\), these readily reduce to

\[\tilde{F}_1(1) - \tilde{F}_2(1) = \iota(1) - \delta(\psi(1))\]

and

\[\alpha_1(1) - \alpha_2(1) = \delta(\iota(1)) .\]

We have thus demonstrated
Theorem 28  The first order fibred deformations of a semigroupal functor $F : C \to D$ are classified up to equivalence by the third cohomology of the cone $C_{[F(-)]} = X^\bullet_{\text{fibred}}(F)$.

A similar analysis shows

Theorem 29  The first order total deformations of a semigroupal functor $F : C \to D$ are classified up to equivalence by the third cohomology of the cone $C \cdot \bigl\lfloor F(p_1) \bigr\rfloor \被视为[\lfloor (p_2) \rfloor] = X^\bullet_{\text{total}}(F)$.

The case of total deformations of a multiplication (or equivalently, deformations of a braided monoidal category) presents another subtlety: the source and target must be deformed in tandem.

Proposition 30  If $C(n), \otimes, \alpha(0) + \alpha(1) \epsilon + \ldots + \alpha(n) \epsilon^n$ is an $n$th-order deformation of $(C, \otimes, \alpha)$ and

$$\beta(k) = \sum_{i=0}^{k} \alpha^{(i)} \otimes \alpha^{(k-i)},$$

then $([C \otimes C]^{(n)}, \otimes \otimes, \beta(0) + \beta(1) \epsilon + \ldots + \beta(n) \epsilon^n)$ is an $n$th-order deformation of $(C \otimes C, \otimes \otimes, \alpha \otimes \alpha)$. We call this deformation the diagonal deformation of $C \otimes C$.

proof: Observe first that $C \otimes C$ is defined with respect to the commutative ring $R$, and that $[C \otimes R C]^{(n)}$ is canonically isomorphic to $C^{(n)} \otimes_R [R_{(\epsilon)}]/<\epsilon^n> C^{(n)}$.

The diagonal deformation is then simply the $R[\epsilon]/<\epsilon^n>$-linearized version of the diagonal semigroupal structure induced on $C^{(n)} \times C^{(n)}$ by the (deformed) semigroupal structure on $C^{(n)}$. The formula for the $\beta(k)'s$ is derived by simply collecting terms according to their degree in $\epsilon$. \qed

Definition 31  A coarse deformation of a multiplication is a total deformation of the semigroupal functor such that the deformation of the source $C \otimes C$ is the diagonal deformation induced by the deformation of the target. A deformation of a multiplication (and thus of a braided monoidal category) is a coarse deformation which is equipped with natural isomorphisms as required to make it into a multiplication.

We will consider the behavior of units in general in Section 6, so we here confine ourselves to consider the appropriate deformation complex for coarse deformations of multiplications:

Consider the composite cochain map

$$\phi : X^\bullet(C) \xrightarrow{[\Delta,Id]} X^\bullet(C \otimes C) \oplus X^\bullet(C) \xrightarrow{[\Phi(p_1)]-[\Phi(p_2)\cdot]} X^\bullet(\Phi).$$

An argument similar to that given above for fibred deformations shows that:

Theorem 32  The first order coarse deformations of a multiplication $\Phi : C \otimes C \to C$ are classified up to equivalence by the third cohomology of the cone $C_{\Phi}^\bullet = X^\bullet_{\text{coarse}}(\Phi)$.

Since $\phi$ is defined as a composite, something more remains to be said: if we consider our cochain complexes as objects in the homotopy category $K^+(R)$ or the derived category $D^+(R)$, the octahedral property ensures the existence of an exact triangle relating $X^\bullet_{\text{coarse}}(\Phi)$, $X^\bullet_{\text{total}}(\Phi)$ and $C_{(\Delta,Id)}$, and thus of a long-exact sequence in cohomology.
5 Obstructions and the Cup Product and Pre-Lie Structures on $X^\bullet(F)$

The cochain complex associated to any of the types of semigroupal functors shares many of the properties of the Hochschild complex of an associative algebra $A$ with coefficients in $A$, which were described by Gerstenhaber [Ger63, Ger64, GS88]. In particular, we have two products defined on cochains. The first, the cup product,

$$- \cup - : X^n(F) \times X^m(F) \to X^{n+m}(F),$$

is given by

$$G \cup H_{A_1,\ldots,A_{n+m}} = [G_{A_1,\ldots,A_n} \otimes H_{A_{n+1},\ldots,A_{n+m}}].$$

The second, the composition product,

$$\langle - , - \rangle : X^n(F) \times X^m(F) \to X^{n+m-1}(F)$$

is given by

$$\langle G, H \rangle_{A_1,\ldots,A_{n+m-1}} = \sum (-1)^{m_i} [ (G_{A_1,\ldots,A_{i+1},A_{i+2},\ldots,A_{i+n},A_{i+n+1},\ldots,A_{n+m-1}})$$

$$F(A_1) \otimes \ldots \otimes F(A_i) \otimes H_{A_{i+1},\ldots,A_{i+n}} \otimes F(A_{i+n+1}) \otimes$$

$$\ldots \otimes F(A_{n+m-1})]$$

in the case of strong and lax semigroupal functors. A similar formula is given in [Yet01] for the oplax case.

**Proposition 33** The product $\langle - , - \rangle$ comes from a “pre-Lie system”, in the terminology of Gerstenhaber [Ger63], given by

$$\langle G, H \rangle_{A_1,\ldots,A_{n+m-1}}^{(i)} =$$

$$\sum (-1)^{m_i} [(G_{A_1,\ldots,A_{i+1},A_{i+2},\ldots,A_{i+n},A_{i+n+1},\ldots,A_{n+m-1}})F(A_1) \otimes \ldots$$

$$F(A_i) \otimes H_{A_{i+1},\ldots,A_{i+n}} \otimes F(A_{i+n+1}) \otimes \ldots \otimes F(A_{n+m-1})]$$

in the case of strong and lax semigroupal functors, where $X^n(F)$ has degree $n - 1$.

Again in [Yet01] the oplax case is treated as well.

**proof:** First, note that the ambiguities of parenthesization in the semigroupal products in this definition are rendered irrelevant by the $[ ]$ on each term, by virtue of the coherence theorems for semigroupal functors.

It is obvious that the product is given by a sum of these terms with the correct signs for the construction of a Lie bracket from a pre-Lie system, so actually the content of the proposition is that the $\langle - , - \rangle^{(i)}$’s satisfy the definition of a pre-Lie system. That is, for $G \in X^m(F)$, $H \in X^n(F)$ and $K \in X^p(F)$, we have
\[
\langle (G,H)^{(i)}, K^{(j)} \rangle =\begin{cases} 
\langle (G,K)^{(j)}, H^{(i+p-1)} \rangle & \text{if } 0 \leq j \leq i-1 \\
\langle (G,(H,K)^{(j-i)})^{(i)} \rangle & \text{if } i \leq j \leq n
\end{cases}
\]

(recall that a \(k\)-chain has degree \(k-1\)).

This is a simple computational check. In verifying the first case, naturality will allow the prolongations of \(K\) and \(H\) to commute. \(\square\)

Now, suppose we have an \(M-1\)st order deformation
\[
\tilde{\alpha} = \alpha^{(0)} + \alpha^{(1)} \epsilon + \ldots + \alpha^{(M-1)} \epsilon^{M-1}.
\]

As was shown in \([CY98]\), the obstruction to extending this to an \(M\)th order deformation is the 4-cochain
\[
\omega^{(M)}_{A,B,C,D} = \sum_{i+j=M \atop 0 \leq i, j < M} \left[ \alpha^{(i)}_{A\otimes B,C,D} \alpha^{(j)}_{A,B,C} \right] - \sum_{i+j+k=M \atop 0 \leq i, j, k < M} \left[ \left[ \alpha_{A,B,C} \otimes D \right] \alpha^{(j)}_{A,B,C,D} \right] \alpha^{(k)}_{A,B,C,D}.
\]

The deformation extends precisely when this cochain is a coboundary, in which case \(\alpha^{(M)}\) may be any solution to \(\delta(\alpha^{(M)}) = \omega^{(M)}\).

In \([Yet01]\) it is shown that

**Theorem 34** For all \(M\), the obstruction \(\omega^{(M)}\) is a 4-cocycle. Thus, an \((M-1)\)st order deformation extends to an \(M\)th order deformation if and only if the cohomology class \([\omega^{(M)}] \in H^4(C)\) vanishes.

**proof:** The proof is essentially computational. Following \([Yet01]\) we introduce notation for the summands of the coboundary of a 3-cochain \(\phi_{A,B,C}\) —

\[
\begin{align*}
\partial_0 \phi_{A,B,C,D} & = A \otimes \phi_{B,C,D} \\
\partial_1 \phi_{A,B,C,D} & = \phi_{A \otimes B,C,D} \\
\partial_2 \phi_{A,B,C,D} & = \phi_{A,B \otimes C,D} \\
\partial_3 \phi_{A,B,C,D} & = \phi_{A,B,C \otimes D} \\
\partial_4 \phi_{A,B,C,D} & = \phi_{A,B,C} \otimes D
\end{align*}
\]

— and of a 4-cochain \(\psi_{A,B,C,D}\) —

\[
\begin{align*}
\partial_0 \psi_{A,B,C,D,E} & = A \otimes \psi_{B,C,D,E} \\
\partial_1 \psi_{A,B,C,D,E} & = \psi_{A \otimes B,C,D,E} \\
\partial_2 \psi_{A,B,C,D,E} & = \psi_{A,B \otimes C,D,E} \\
\partial_3 \psi_{A,B,C,D,E} & = \psi_{A,B,C \otimes D,E} \\
\partial_4 \psi_{A,B,C,D,E} & = \psi_{A,B,C,D} \otimes E
\end{align*}
\]
We then have
\[ \delta(\phi)_{A,B,C,D} = \sum_{i=0}^{4} (-1)^{i+1} \partial_i \phi_{A,B,C,D} \]
for 3-cochains \( \phi \), and
\[ \delta(\psi)_{A,B,C,D,E} = \sum_{i=0}^{5} (-1)^{i+1} \partial_i \psi_{A,B,C,D,E} \]
for 4-cochains \( \psi \).

In this notation the obstruction cochain \( \omega^M \) becomes
\[ \omega^{(M)} = \sum_{i+j=M} \left[ \partial_1 \alpha^{(i)} \partial_3 \alpha^{(j)} \right] - \sum_{i+j+k=M} \left[ \partial_4 \alpha^{(i)} \partial_2 \alpha^{(j)} \partial_0 \alpha^{(k)} \right] , \]
while the vanishing of the obstruction \( \omega^{(N)} \) (for \( N < M \)) becomes
\[ 0 = \delta \alpha^{(N)} + \omega^{(N)} \]
\[ = \sum_{i+j=N} \left[ \partial_1 \alpha^{(i)} \partial_3 \alpha^{(j)} \right] - \sum_{i+j+k=N} \left[ \partial_4 \alpha^{(i)} \partial_2 \alpha^{(j)} \partial_0 \alpha^{(k)} \right] . \]

We wish to show that
\[ \sum_{i=0}^{5} (-1)^{i+1} \partial_i \omega^{(M)}_{A,B,C,D,E} = 0 . \]

Observe that \( \omega^{(1)} = 0 \) and \( \delta(\alpha^{(1)}) = 0 \), so we may proceed by induction under the assumption that \( \omega^{(N)} \) and \( \alpha^{(N)} \) satisfy
\[ \delta(\alpha^{(N)}) + \omega^{(N)} = 0 \]
for \( N < M \).

It is convenient to picture the summands of the left-hand side in terms of compositions of maps along the boundaries of faces of the “associahedron” (or 3-dimensional Stasheff polytope) \[Sta63\] given in Figure 3.

Suppose we have an \((M-1)\)st order deformation of a semigroupal category with structure map. Observe that each summand
\[ \partial_i \omega^{(M)}_{A,B,C,D,E} \]
essentially represents the sum of all composites with total degree \( M \) along the three-edge directed path minus the sum of all composites with total degree \( M \) along the two-edge directed path on the boundary of one of the pentagonal faces. (Here, degree refers to the power of \( \epsilon \) whose coefficient is given by the composite.)

This is “essentially” the content of each summand, but one must remember that the context \( [\ ] \) is not contentless—the summands are actually composites of the differences just described with various structure maps (prolongations of \( \alpha^{(0)} \)) with the property that all sources are
Figure 3: The Associahedron

\[ \left[ [A \otimes B] \otimes C \otimes D \right] \otimes E \]

and all targets are

\[ A \otimes [B \otimes [C \otimes [D \otimes E]]]. \]

The odd-index summands correspond to the pentagonal faces on the bottom of the associahedron as shown in Figure 3, while the even-index summands correspond to those on the top. The square faces of the associahedron correspond to families of naturality squares, one for each possible pair of degrees.

In fact, it will suffice to compute \([\partial_1 + \partial_3 + \partial_5](\omega^{(M)})\):

**Lemma 35** Suppose for all \(N < M\) we have \(\delta(\alpha^{(N)}) + \omega^{(N)} = 0\). Then

\[
[\partial_1 + \partial_3 + \partial_5](\omega^{(M)}) = \sum_{i + j + k = M} \left[ \partial_1 \partial_1 \alpha^{(i)} \partial_3 \partial_3 \alpha^{(j)} \partial_4 \partial_4 \alpha^{(k)} \right] \\
- \sum_{i + j + k + l + m + n = M} \left[ \partial_5 \partial_5 \alpha^{(i)} \partial_5 \partial_5 \alpha^{(j)} \partial_5 \partial_5 \alpha^{(k)} \partial_5 \partial_5 \alpha^{(l)} \partial_5 \partial_5 \alpha^{(m)} \partial_5 \partial_5 \alpha^{(n)} \right].
\]

This lemma suffices to complete the proof of the theorem, since the lemma and calculation by which it is derived are precisely dual to a corresponding statement and derivation concerning \([\partial_0 + \partial_2 + \partial_4](\omega^{(M)})\). The value derived for this last expression is

\[
\sum_{i + j + k = M} \left[ \partial_1 \partial_1 \alpha^{(i)} \partial_4 \partial_4 \alpha^{(j)} \partial_4 \partial_4 \alpha^{(k)} \right]
\]
\[ - \sum_{i+j+k+l+m+n = M} \left[ \partial_5 \partial_4 \alpha^{(i)} \partial_2 \partial_2 \alpha^{(j)} \partial_0 \partial_4 \alpha^{(k)} \partial_0 \partial_0 \partial_2 \alpha^{(m)} \partial_0 \partial_0 \alpha^{(n)} \right]. \]

Once coincidences of different names for the same map (all of which may be read off from the associahedron) are taken into account, this expression differs from that computed in the lemma only in the third and fourth factors of the composites in the second summation. The terms, however, may be matched one-to-one by swapping the indices \( k \) and \( l \) into pairs that are equal by virtue of naturality, thus completing the proof.

Thus, it suffices to prove Lemma 35. The ambitious reader may reconstruct the proof by realizing that the vanishing of earlier obstructions is just what is needed to “fuse” the summands corresponding to paths round two adjacent faces of the associahedron into a similar expression corresponding to paths round the union of the two faces. The less ambitious reader is referred to [Yet01], where complete details are given. \( \Box \)

We now turn to the question of obstructions for fibred and total deformations of monoidal functors, and for deformations of multiplications on monoidal categories (or equivalently, of braided monoidal categories).

Since fibred deformations and deformations of multiplications are special cases of total deformations, defined by restricting the deformation of the target to be trivial or the deformation of the source to be the diagonal deformation induced by the deformation of the target, respectively, it suffices to consider obstructions in the case of total deformations. We begin by giving an explicit formula for these obstructions, and then show that they are closed.

Recall that the appropriate deformation complex for total deformations of a strong (or lax) semigroupal functor \( F : \mathcal{C} \to \mathcal{D}, \phi : F(- \otimes -) \Rightarrow F(-) \otimes F(-) \) is

\[ X_{\text{total}}^\bullet(F) = C_{[F(p_1)] - [p_2], p_\bullet}^\bullet = X^\bullet(F) \oplus X^{\bullet+1}(\mathcal{C}) \oplus X^{\bullet+1}(\mathcal{D}) \]

with coboundary given by

\[
\begin{bmatrix}
\delta_F & 0 & 0 \\
[F(-)] & -\delta_C & 0 \\
-[-(F)_{\mathcal{C}}] & 0 & -\delta_D
\end{bmatrix}.
\]

Thus, a cochain will have coboundary which vanishes in each of the second and third coordinates if and only if its second and third coordinates are cocycles in \( X^{\bullet+1}(\mathcal{C}) \) and \( X^{\bullet+1}(\mathcal{D}) \), respectively. Similarly, it is easy to see that the obstruction cochain for a total deformation must have as second and third coordinates the obstructions for the deformations of the source and target category, respectively.

Thus, we are left to consider the value of the first coordinate of the obstruction, and the value of the first coordinate of the coboundary. Consider the hexagonal coherence diagram for oplax monoidal functors given in Figure 2, with the maps replaced by their deformed versions.

Calculating the difference of the degree \( n \) terms of the two directions around the diagram gives

\[
\sum_{i+j+k=n} \left[ a_{F(A),F(B),F(C)}^{(i)} [F(A) \otimes \Phi_{B,C}^{(j)}] \Phi_{A,B\otimes C}^{(k)} \right] - \\
\sum_{i+j+k=n} \left[ \Phi_{A,B}^{(i)} \otimes F(C) \Phi_{A\otimes B,C}^{(j)} F(\alpha_{A,B,C}^{(k)}) \right],
\]

18
where $\alpha$ and $a$ are the associators for $C$ and $D$, respectively. This must vanish for $n = 1$ for first order total deformations: the vanishing is simply the cocycle condition in $X_{\text{total}}^n(F)$. For a deformation to extend to an $N$th order deformation this quantity must vanish for all $n \leq N$, and indeed in addition to the vanishing of the corresponding second and third coördinates, this condition is sufficient. Separating out the terms in which the index $^{(n)}$ occurs, we find that the vanishing conditions are precisely the condition that $[\phi^{(n)}, \alpha^{(n)}, a^{(n)}]$ cobounds $[\Omega^{(n)}, \omega^{(n)}, o^{(n)}]$, where

$$
\Omega^{(n)} = \sum_{\begin{subarray}{c}i, j, k < n \\
i + j + k = n \end{subarray}} [d^{(i)}_{F(A), F(B), F(C)}[F(A) \otimes \Phi^{(j)}_{B,C}]\Phi^{(k)}_{A,B\otimes C}]
$$

and $\omega^{(n)}$ and $o^{(n)}$ are the obstructions to the extension of the deformations of the source and target categories, respectively.

All that remains to show is that the first coördinate of the coboundary of $[\Omega^{(n)}, \omega^{(n)}, o^{(n)}]$ vanishes. We leave the details of the proof to the reader. The method is identical to that applied in the case of the obstructions for deformations of a semigroupal category, except that the associahedron must be replaced with the diagram given in Figure $4$. In Figure $4$ we have suppressed all object and arrow labels except for the objects on the inner pentagon which are written with the null infix in place of $\otimes$ to save space. The labels can be recovered by labeling all radial maps with prolongations of $\Phi$ and all maps parallel to those between the labeled objects with prolongations of (functorial images of) $\alpha$. All hexagons are prolongations of the coherence hexagon for semigroupal functors, and all squares, except the diamond-shaped one in the top center, are naturality squares. The diamond is a functoriality square.

---

2The diagram of Figure $4$ was given the name “the Chinese lantern” due to its resemblance to a paper lantern when it is drawn in perspective as the edges of a 3-dimensional polytope with the innermost and outermost pentagons a parallel horizontal faces. The diagram had occurred previously in work of Stasheff, who did not name it, and has been dubbed the “multiplicahedron” by other recent researchers.
Figure 4: The “Chinese Lantern”
6 Units

Thus far we have discussed deformations of semigroupal categories and functors. In [Yet01] it is shown that there is very little additional content to the deformation theory of monoidal categories and functors. We recall the relevant results from [Yet01], and refer the reader there for proofs:

**Theorem 36** Every semigroupal deformation of a monoidal category

\((\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)\)

becomes a monoidal category when equipped with unit transformations \(\tilde{\nu}\) and \(\tilde{\lambda}\) given by

\[
\tilde{\nu} = \sum_i \nu^{(i)} e^i
\]

\[
\tilde{\lambda}_A = \sum_i \lambda^{(i)}_A e^i
\]

\[
\tilde{\rho}_A = \sum_i \rho^{(i)}_A e^i
\]

where

\[
\lambda^{(n)}_A = \sum_{i+j=n} [\alpha^{(i)}_{A,I,I} [A \otimes \nu^{(j)}]]
\]

\[
\rho^{(n)}_B = \sum_{i+j=n} [\beta^{(i)}_{I,I,B} [\nu^{(j)} \otimes B]],
\]

\(\nu^{(i)} : I \to I\) is any family of maps satisfying \(\nu^{(0)} = \lambda_I = \rho_I\), and \(\tilde{\alpha}^{-1} = \sum_i \beta^{(i)} e^i\).

For strong monoidal functors, we have

**Theorem 37** If \((F : \mathcal{C} \to \mathcal{D}, \Phi, F_\alpha)\) is a strong monoidal functor, then every semigroupal deformation of \(F\) extends uniquely to a deformation as a monoidal functor.

The final conditions involving units are the conditions in the definition of a multiplication on a monoidal category. This condition, however, is trivially satisfied by any deformation, since the isomorphism giving the structure of the given multiplication still provides the necessary structure after deformation.
7 Modules over Monoidal Categories and Functors

The key to the construction of the deformation complexes for monoidal categories and monoidal functors was the fact that both can serve in different ways as analogues of rings. In order to move the deformation complexes of Section 2 into the realm of classical homological algebra, we need to consider the analogues of modules in each case. In the case of categories, the author introduced the required notion in [Yet92b].

**Definition 38** If $C$ and $D$ are monoidal categories, a (strong) left $C$-module $M$ is a category equipped with a functor $\rhd : C \times M \to M$ (written as an infix) and natural isomorphisms

\[
ar_l : (A \otimes B) \rhd X \to A \rhd (B \rhd X)
\]

and

\[
\ell : I \rhd X \to X
\]

satisfying

\[
\begin{array}{ccc}
((A \otimes B) \otimes C) \rhd X & \xrightarrow{\alpha} & A \rhd (B \rhd (C \rhd X)) \\
(A \otimes (B \otimes C)) \rhd X & \xrightarrow{a_l} & A \rhd ((B \otimes C) \rhd X)
\end{array}
\]

\[
\begin{array}{ccc}
(A \otimes I) \rhd X & \xrightarrow{a_l} & A \rhd (I \rhd X) \\
\rho \rhd X & \xrightarrow{\rho} & A \rhd \ell
\end{array}
\]
A (strong) right $D$-module $\mathcal{M}$ is a category equipped with a functor $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ (also as an infix) and natural isomorphisms $a_r : (X \triangleleft C) \triangleleft D \to X \triangleleft (C \otimes D)$ and $r : X \triangleleft I \to X$ satisfying

A (strong) $C, D$-bimodule is a category $\mathcal{M}$ equipped with both a left $C$-module structure and a right $D$-module structure and a natural isomorphism $a_m : (A \triangleright X) \triangleleft C \to A \triangleright (X \triangleleft C)$ and satisfying
We will be primarily concerned with the special case where $C = D = \mathcal{M}$ with $\triangleright = \triangleleft = \otimes$ and the obvious identification of structure maps with the structure maps of the monoidal structure, but most of our results hold more generally.

Other examples may be found in [Yet92b].

More important for our purposes are the corresponding notions for (lax) monoidal functors:

**Definition 39** Suppose $\mathcal{C}, \mathcal{D},$ and $\mathcal{E}$ are monoidal categories and that

$$(F, \tilde{F}, F_0) : \mathcal{C} \to \mathcal{E}$$

and

$$(G, \tilde{G}, G_0) : \mathcal{D} \to \mathcal{E}$$

are monoidal functors. Suppose, moreover, that $\mathcal{M}$ is a left $\mathcal{C}$-module (resp. right $\mathcal{D}$-module, $\mathcal{C}$, $\mathcal{D}$-bimodule). A left $F$-module over $\mathcal{M}$ (resp. a right $G$-module over $\mathcal{M}$, an $F,G$-bimodule over $\mathcal{M}$) is then a functor $M : \mathcal{M} \to \mathcal{E}$ equipped with natural transformations $\mu_l : F(-) \otimes M(-) \Rightarrow M(- \triangleright -)$ (resp. $\mu_r : M(-) \otimes G(-) \Rightarrow M(- \triangleleft -)$, both $\mu_l$ and $\mu_r$) and satisfying
Definition 40 A left (resp. right, bi-) module map from $M$ to $N$ is a natural transformation $f : M \Rightarrow N$ satisfying
Henceforth by abuse of notation, we will denote monoidal functors by the name of the functor only.

There are a number of obvious examples, and some not-so-obvious:

First, it is clear that any monoidal functor \( F : \mathcal{C} \to \mathcal{D} \) equipped with \( \mu_l = \mu_r = \tilde{F} \) is an \( F,F \)-bimodule over \( \mathcal{C} \).

Second, suppose we have a second monoidal functor \( G : \mathcal{C} \to \mathcal{D} \) and a natural transformation \( \phi : F \Rightarrow G \). \( G \) then becomes an \( F,F \)-bimodule over \( \mathcal{C} \), when equipped with the structure \( \mu_l = \tilde{G}(\phi \otimes \text{Id}_G) \) and \( \mu_r = \tilde{G}(\text{Id}_G \otimes \phi) \).

We now turn to the main theorem of this section:

**Theorem 41** If \( \mathcal{E} \) is an abelian category equipped with a monoidal structure, and \( F : \mathcal{C} \to \mathcal{E} \) and \( G : \mathcal{D} \to \mathcal{E} \) are monoidal functors such that for all \( A \in \text{Ob}(\mathcal{C}) \) \( F(A) \otimes - \) is exact (resp. for all \( C \in \text{Ob}(\mathcal{D}) \) \( - \otimes G(C) \) is exact, both), then for any left \( \mathcal{C} \)-module (resp. right \( \mathcal{D} \)-module, \( \mathcal{C},\mathcal{D} \)-bimodule) \( \mathcal{M} \), the category of left \( F \)-modules (resp. right \( G \)-modules, \( F,G \)-bimodules) over \( \mathcal{M} \) is an abelian category.

**proof:** We proceed by showing that the forgetful functor to the functor category \( \mathcal{E}^\mathcal{M} \) induces an additive structure and all necessary limits and colimits, then verify the additional conditions for abelianess in the form in terms of the “parallel” of a map as given in Popescu [Pop73].

Now, for the null object, we can take the constant functor 0, since by the exactness hypotheses \( F(A) \otimes - \) and \( - \otimes g(C) \) preserve 0, and thus the zero map is the unique action. Initial and terminal conditions all follow from the uniqueness of the zero map and consideration of the diagrams which assert that it is a left \( F \)-module map (resp. right \( G \)-module map, both).

For kernels, consider first the case of a left \( F \)-module map \( f : M \Rightarrow N \), we claim that the kernel in \( \mathcal{E}^\mathcal{M} \) has a unique left \( F \)-module structure such that the inclusion is an \( F \)-module map, as are all canonical maps induced by \( F \)-module maps annihilated by post-composition with \( f \).

Now, for each object \( X \in \text{Ob}(\mathcal{M}) \) we have the exact sequence

\[
0 \to \ker(f_X) \to M(C) \to N(C)
\]

but \( F(A) \otimes - \) is exact, and \( M \) and \( N \) are left \( F \)-modules. Thus we have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & F(A) \otimes \ker(f_X) & \longrightarrow & F(A) \otimes M(X) & \longrightarrow & F(A) \otimes N(X) \\
\downarrow \kappa_l & & \downarrow \mu_l & & \downarrow \nu_l & & \\
0 & \longrightarrow & \ker(f_{A \bowtie X}) & \longrightarrow & M(A \triangleright X) & \longrightarrow & N(A \triangleright X)
\end{array}
\]
in which the left vertical exists uniquely by the 5-lemma [ML98].

Coherence conditions follow from the uniqueness condition in the 5-lemma by applying in the hexagon of exact sequences of Figure 5, and a similar triangle of exact sequences for the unit condition.

![Diagram](image)

**Figure 5:** Coherence for the left action on a kernel

Thus, the kernel has a unique left $F$-module structure such that the inclusion is an $F$-module map.

Applying the same type of argument to the right action and its hexagon and triangles, and to the two additional hexagons for a bimodule shows that the kernel of a right $G$-module map (resp. $F,G$-bimodule map) admits a unique $G$-module structure (resp. $F,G$-bimodule structure) such that the inclusion is a $G$-module map (resp. $F,G$-bimodule map).

Now, given an $F$-module map $e : L \Rightarrow M$ such that $f(e) = 0$, we claim that the induced map in $\mathcal{E}^M$ is an $F$-module map.

Now, for each $X \in \text{Ob}(\mathcal{M})$ we have a diagram

![Diagram](image)

with an exact top row.

As before, we form a diagram relating the image of this diagram under $F(A) \otimes -$ and its instance for $A \triangleright X$ as in Figure 6. Here the “back” rows are both exact, and the diagram commutes.
Figure 6: Canonical maps are module maps

by construction except possibly from $F(A) \otimes L(X)$ to $\ker(f_{A\triangleright X})$, the commutativity of which is precisely what is to be shown.

Now, $f_{A\triangleright X}(e_{A\triangleright X}(\lambda)) = 0$ (since $f_{A\triangleright X}(e_{A\triangleright X}) = 0$). Thus by the universal property of $\ker(f_{A\triangleright X})$, there exists a unique map $\phi : F(A) \otimes L(X) \to \ker(f_{A\triangleright X})$ (in $D$) such that $\iota_{A\triangleright X}(\phi) = e_{A\triangleright X}(\lambda)$.

On the one hand, it is clear that $\phi = can_{A\triangleright X}(\lambda)$ since $\iota_{A\triangleright X}(can_{A\triangleright X}) = e_{A\triangleright X}$.

On the other, we have

$$\iota_{A\triangleright X}(\kappa(Id \otimes can_X)) = \mu(Id \otimes \iota_X(Id \otimes can_X)) = \mu(Id \otimes e_X) = e_{A\triangleright X}(\lambda)$$

where the first equality holds by the construction of the kernel action $\kappa$, the second by the definition of $can_X$, and the third by the fact that $e$ is an $F$-module map. Thus $\phi = \kappa(Id \otimes can_X)$ and the required square commutes by the uniqueness of $\phi$.

An essentially identical proof gives the corresponding result for right $G$-modules, and thus for $F,G$-bimodules.

The proof that cokernels of (bi)module maps have (bi)module structures and that the quotient map and canonical maps are (bi)module maps may be obtained by dualizing the rows (only) in the proof for kernels.

As regards biproducts, observe that since the universal properties follow from the equational conditions on the projections and inclusions, it suffices to show that given two left $F$-modules $M$ and $N$, the biproduct of their underlying objects in $E^M$ admit a unique left $F$-module structure such that the inclusions and projections are left module maps.

Now, since biproducts are equationally defined $F(A) \otimes -$ preserves biproducts up to a canonical isomorphism commuting with both projections and inclusions. Thus, if we let

$$d : F(A) \otimes [M(X) \oplus N(X)] \to [F(A) \otimes M(X)] \oplus [F(A) \otimes N(X)]$$

denote the canonical arrow, the diagram obtained from that of Figure 3 by omitting either projections or inclusions commutes. Thus $[\mu_l \oplus \nu_l](d)$ give the unique left module structure on $M \oplus N$.

Coherence follows from the universal properties of biproducts.

The construction of the right module structure for biproducts of right modules (or bimodules) is entirely similar.
Finally, we must show that the parallel of a map (cf. Popescu [Pop73]) is invertible in the category of (bi)modules. It already is invertible in $E^M$, so it suffices to show that a left $F$-module map which is invertible in $E^M$ is invertible in the category of left $F$-modules over $M$. (Right actions will follow by an essentially identical proof.)

Now, observe that if $f : M \Rightarrow N$ is invertible in $E^M$, we must show that the diagram

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\mu_l} & N(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f^{-1}_X & \xrightarrow{\mu_l} & N(A \triangleright X)
\end{array}
\]

commutes. However, since both $F(A) \otimes f_X$ and $f_{A \triangleright X}$ are invertible in $E$, it suffices to see that the diagram

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\mu_l} & N(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f_X & \xrightarrow{\mu_l} & N(A \triangleright X)
\end{array}
\]

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\mu_l} & N(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f^{-1}_X & \xrightarrow{\mu_l} & N(A \triangleright X)
\end{array}
\]

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\nu_l} & M(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f_X & \xrightarrow{\nu_l} & M(A \triangleright X)
\end{array}
\]

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\nu_l} & M(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f^{-1}_X & \xrightarrow{\nu_l} & M(A \triangleright X)
\end{array}
\]

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\nu_l} & M(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f_X & \xrightarrow{\nu_l} & M(A \triangleright X)
\end{array}
\]

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\nu_l} & M(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f^{-1}_X & \xrightarrow{\nu_l} & M(A \triangleright X)
\end{array}
\]

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\nu_l} & M(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f_X & \xrightarrow{\nu_l} & M(A \triangleright X)
\end{array}
\]

\[
\begin{array}{cccc}
F(A) \otimes M(X) & \xrightarrow{\nu_l} & M(A \triangleright X) \\
\downarrow & & \downarrow \\
F(A) \otimes f^{-1}_X & \xrightarrow{\nu_l} & M(A \triangleright X)
\end{array}
\]
commutes from $F(A) \otimes M(X)$ to $N(A \triangleright X)$. But calculating the two paths, we see that this is just the coherence diagram for $f$ as a left $F$-module map.

A similar technique will show

**Theorem 42** If $\mathcal{E}$ is a cocomplete abelian category equipped with a monoidal structure, and $F: \mathcal{C} \to \mathcal{E}$ and $G: \mathcal{D} \to \mathcal{E}$ are monoidal functors such that for all $A \in \text{Ob}(\mathcal{C})$ $F(A) \otimes -$ is exact and cocontinuous (resp. for all $C \in \text{Ob}(\mathcal{D})$ $- \otimes G(C)$ is exact and cocontinuous, both), then for any left $\mathcal{C}$-module (resp. right $\mathcal{D}$-module, $\mathcal{C},\mathcal{D}$-bimodule) $M$, the category of left $F$-modules (resp. right $G$-modules, $F,G$-bimodules) over $M$ is a cocomplete abelian category.

Although the hypotheses of Theorems 41 and 42 may seem rather restrictive, it should be noted that they hold whenever the target category $\mathcal{E}$ is of the form $k$-$\text{v.s.}$ for $k$ a field.

With the category of $F,F$-bimodules as a setting, we can generalize the deformation complex of $F$ to give an analogue of the Hochschild cohomology of an algebra with coefficients in a bimodule:

Let

$$X^n(F,M) = \text{Nat}(^n \otimes (F^n), M(\otimes^n))$$

with coboundary given by the the obvious generalization of the formula for the deformation complex.

Deformation complexes for semigroupal categories and functors are then the special case for $F = M = \text{Id}_C$ and $M = F$ respectively. There is also another special case of interest. If $M$ is a monoidal functor $G$ made into an $F,F$-bimodule with the structure induced by a monoidal natural transformation $\phi: F \Rightarrow G$ as described above, it is easy to show that the first order deformations of $\phi$ as a monoidal natural transformation are classified by $H^1(F,G)$, and the obstructions to higher order deformations are classes in $H^2(F,G)$. 
In this section, we turn to the principal use of Theorems 41 and 42: the reduction of the cohomology of a (cocomplete) abelian monoidal category (with exact cocontinuous $\otimes$), and of the cohomology of a monoidal functor targetted at such a category, with coefficients in a module to a calculus of derived functors. We will need some auxiliary hypotheses, these, however, will be satisfied both by algebras viewed as lax monoidal functors, and by strong monoidal functors targetted at categories of vector-spaces, provided the source category satisfies the mild size restriction that there exists a small subcategory such that all objects are colimits of diagrams in the small subcategory.

It should be observed that even when it is possible, this reduction does not in itself solve the problems of categorical deformations: the resulting expression for the cohomology groups does not shed very much light on the behavior of the obstruction cocycles, which are governed by the pre-Lie structure. Nonetheless, the result is important, in that it places categorical deformation theory comfortably within the realm of classical homological algebra.

Recall that the $n^{th}$ cochain group associated to a lax monoidal functor $F : C \to E$ with coefficients in the $F$-module $M : C \to E$ is defined by

$$X^n(F, M) = \text{Nat}(n \otimes (F^n), M(\otimes^n)).$$

Note that as $n$ varies, the source category for the functors varies. This rather uncomfortable circumstance from the point of view of classical homological algebra can be rectified if the functors $n \otimes (F^n)$ admit left Kan extensions along $\otimes^n$ for all $n$. In this case we can redefine $X^n(F, M)$ by

$$X^n(F, M) = \text{Nat}(\text{Lan}_{\otimes^n}(n \otimes (F^n)), M).$$

Here, regardless of $n$ we have the abelian group of natural transformations between functors from $C$ to $E$.

Our first goal, then, is to show under suitable hypotheses that for $n \geq 2$ the functors

$$\text{Lan}_{\otimes^n}(n \otimes (F^n)) : C \to E$$

are in fact $F, F$-bimodules over $C$ (with $\triangleright = \triangleleft = \otimes$) in a natural way, and are projective as such. Our second goal is to show that the cochain complex $(X^\bullet(F, M), \delta^\bullet)$ arises by applying $\text{Nat}[-, M]$ to a projective resolution made up of these Kan extensions.

Before embarking on the construction of the desired lifts, let us show that the Kan extensions admit an $F, F$-bimodule structure, and examine which maps from iterated tensor products of $F$ correspond to bimodule maps from the corresponding Kan extension.

Assume that $F(X) \otimes -$ and $- \otimes F(X)$ are cocontinuous for all $X$. Now, this being so, we have

$$F(X) \otimes \text{Lan}_{\otimes^n}(n \otimes (F^n)) = \text{Lan}_{\otimes^n}(F(X) \otimes^n \otimes (F^n)).$$

Thus a left action corresponds to a natural transformation

$$\text{Lan}_{\otimes^n}(F(X) \otimes^n \otimes (F^n)) \Rightarrow \text{Lan}_{\otimes^n}(n \otimes (F^n))(X \otimes -).$$

By the universal property of the target, this in turn corresponds to a natural filling of the rectangle
But, we have

\[
\begin{align*}
C &\cong D \\
X &\cong Lan_\otimes(\otimes^n(F^n))
\end{align*}
\]

The coherence for the left action follows from the coherence of \(\tilde{F}\) and universality. The construction of the right action is completely analogous. The additional coherence condition for the \(F,F\)-bimodule structure follows trivially from the separation of the actions. (Note: it fails for \(n=1\).)

We can now consider what condition on a natural transformation \(\tilde{\phi} : n \otimes (F^n) \Rightarrow M(\otimes^n)\) implies that the induced natural transformation \(\phi : Lan_\otimes(n \otimes (F^n)) \Rightarrow M\) is a left (resp. right, bi-) module map.

We need all instances of

\[
\begin{align*}
F(A) \otimes Lan_\otimes(\otimes^n(F^n))(B) &\to Lan_\otimes(\otimes^n(F^n))(A \otimes B) \\
F(A) \otimes \phi_B &\to \phi_{A \otimes B} \\
F(A) \otimes M(B) &\to M(A \otimes B)
\end{align*}
\]

The top path round this diagram is induced by
while the bottom path is induced by

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F^n} & \mathcal{D} \\
\mathbb{D} & \xrightarrow{F(A) \otimes - \otimes \mathcal{D}^{\otimes n-1}} & \mathbb{D}
\end{array}
\]

where in each case \(\alpha\) represents the appropriate unique natural transformation given by Mac Lane’s coherence theorem [ML63, ML98].

The equality between the two natural fillers is given on objects by an equation between

\[
\begin{align*}
\otimes^n (F(A) \otimes (F(X_1), F(X_2), \ldots, F(X_n)) &
\end{align*}
\]

and

\[
\begin{align*}
\otimes^n (F(A) \otimes [\otimes^n (F(X_1), \ldots, F(X_n))] &
\end{align*}
\]

Suppressing the \(\alpha\) and parenthesizations by invoking Mac Lane’s coherence theorem, this becomes

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This diagram, then, is the condition required for a map from $F(X_1) \otimes \ldots \otimes F(X_n)$ to $M(X_1 \otimes \ldots \otimes X_n)$ to induce a left-module map from $\text{Lan}_{\otimes n}(^n \otimes (F^n))$ to $M$. The condition for inducing right-module and bimodule maps are entirely similar.

Consideration of the classical case of associative algebras shows that one wants to begin with some projectivity assumptions about the underlying objects. In particular we will assume throughout the following discussion that $F(I)$ is projective, that for all $A \in \text{Ob}(\mathcal{C})$ the functors $F(A) \otimes -$ and $- \otimes F(A)$ preserve epis, and moreover that $F(I) \otimes -$ and $- \otimes F(I)$ have epi-preserving right adjoints. We will then proceed by attempting to construct the requisite lifts to show that the various $\text{Lan}_{\otimes n}(^n \otimes (F^n))$ are projective. Along the way we will discover what other hypotheses will be needed. At each point, we will look only for hypotheses satisfied in the case of $\mathcal{E} = k\text{-v.s.}$ once the size restriction mentioned above has been placed on the source category.

We begin with the case $n = 2$.

Now observe that a map from $\text{Lan}_{\otimes 2}(^2 \otimes (F^2)) = \text{Lan}_{\otimes}(F \otimes F)$ to a functor $M : \mathcal{C} \to \mathcal{E}$ corresponds canonically to a natural transformation from $F(-) \otimes F(-) : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{E}$ to $M(- \otimes -) : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{E}$.

Thus a lift of a map $f : \text{Lan}_{\otimes 2}(^2 \otimes (F^2)) \to N$ along an epi $q : M \to N$ corresponds to a lift of a map $\phi : F(-) \otimes F(-) \to N(- \otimes -)$ along $q_{-\otimes -}$, as in Figure 8. Thus, in particular, one must have a lift of the map $\phi_{1,1} : F(I) \otimes F(I) \to N(I \otimes I)$ along $q_{1\otimes 1} : M(I \otimes I) \to N(I \otimes I)$.

The existence of the particular lift with $A = B = I$ follows trivially from the hypotheses that $F(I)$ be projective and that $- \otimes F(I)$ (or $F(I) \otimes -$) admit an epi-preserving right adjoint. Now, recall that we want a lift in the category of $F,F$-bimodules over $\mathcal{C}$, so that $M$ and $N$ are bimodules and $q$ is a bimodule map. We will now try to use the actions to extend the particular case to the general case.

Tensoring the lifting diagram of in the upper corner of Figure 8 with $F(A)$ on the left and $F(B)$ on the right, applying both actions, and adjoining obvious isomorphisms gives the lower diagram of Figure 8.

This, however, is not quite what we want: the map between $F(A \otimes I) \otimes F(I \otimes B)$

and

$F(A) \otimes F(I) \otimes F(I) \otimes F(B)$

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runs in the wrong direction. If $F$ is a strong monoidal functor this is easy to remedy, since we can invert the map. Taking this as a hypothesis would, however, exclude the classical case of associative algebras. We therefore assume the weaker hypothesis that $	ilde{F}_{I,A}$ and $	ilde{F}_{A,I}$ are epi for all $A$ and admits a natural splitting $\tilde{F}_{I,A}$ (resp. $\tilde{F}_{I,A}$).

Adjoining the splitting to the diagram almost completes the construction. We still must verify that the resulting composite from $F(A) \otimes F(B)$ to $N(A \otimes B)$ is $\phi_{A,B}$. Now, consider the diagram

---

**Figure 8:** Lifting for $n = 2$
From this it follows that the square beginning at $F_n$ is the action on $M$ the original lift satisfied the same condition. The latter follows from the coherence property of the conditions to induce left and right module maps. The former follows from the hypothesis that

\[ \text{Theorem 43} \]

If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor with a small source, targetted in a cocomplete abelian category $\mathcal{D}$ such that for all $X \in \text{Ob}(\mathcal{C})$, $F(X) \otimes -$ and $- \otimes F(X)$ are exact and cocontinuous, $F(I)$ is projective, $F(I) \otimes -$ and $- \otimes F(I)$ have epi-preserving right adjoints, and $\tilde{F}_{I,A}$ and $\tilde{F}_{A,I}$ are split epis with splitting natural in $A$, then for all $n \geq 2$, $\text{Lan}_{\otimes n}(\text{id}(F^n))$ is a projective $F,F$-bimodule.

First, let us observe that the rather baroque seeming technical hypotheses are satisfied in two very natural cases: strong monoidal functors targetted at a category of vectorspaces and algebras over a field, regarded as monoidal functors.

One annoying feature of this theorem is the size restriction on the source, which makes the theorem inapplicable to identity functors on large categories. This size restriction can be relaxed somewhat by considering the construction of left Kan extensions as colimits. In particular, recall
from [ML98] that the left Kan extension $\text{Lan}_K(T)$, for $K : \mathcal{M} \to \mathcal{C}$ and $T : \mathcal{M} \to \mathcal{A}$ is given on objects by $\text{Lan}_K(T)(c) = \text{colim}(K \downarrow c) \xrightarrow{P_{\mathcal{M}}} \mathcal{M} \xrightarrow{T} \mathcal{A}$, where $P$ is the obvious projection functor from the comma category to $\mathcal{M}$. It thus suffices for all of the comma categories $\otimes^n \downarrow c$ arising in the Kan extensions used to admit small final subcategories and for $\mathcal{D}$ to admit colimits over all diagrams of no greater than the supremum of the sizes of these final subcategories.

It is easy to see that any semisimple category with a small set of simple objects satisfies the small-final subcategory condition on all of the relevant comma categories.

Thus we state the seemingly more technical

**Theorem 44** Suppose $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor such that for all $n \in \mathbb{N}$ and all $c \in \text{Ob}(\mathcal{C})$, the comma category $\otimes^n \downarrow c$ admits a small final subcategory $\mathcal{J}_{n,c}$, and the target is an abelian category $\mathcal{D}$, which admits all colimits of diagrams with cardinality less than $\sup |\text{Arr}(\mathcal{J}_{n,c})|$ or all small diagrams if no supremum of the cardinalities exists, and such that for all $X \in \text{Ob}(\mathcal{C})$ $F(X) \otimes -$ and $- \otimes F(X)$ are exact and cocontinuous, $F(I)$ is projective, $F(I) \otimes -$ and $- \otimes F(I)$ have epim-preserved right adjoints, and $\tilde{F}_{I,A}$ and $\tilde{F}_{A,I}$ are split epis with splitting natural in $A$. In this case $\text{Lan}_{\otimes^n}(\otimes^n(F^n))$ is a projective $F, F$-bimodule for all $n \geq 2$.

Despite the technical nature of the theorem, it now applies to identity functors on many abelian monoidal categories of interest.

The point of this result is, of course, to show that in many cases the cohomology of a monoidal functor with coefficients in a bimodule, and in particular the deformation cohomology of a monoidal functor or monoidal category, is given by right derived functors.

To show this, we must see that the complex $X^\bullet(F, M)$ is obtained by applying $\text{Nat}[-, M]$ to a projective resolution whose objects are the $\text{Lan}_{\otimes^n}(\otimes^n(F^n))$.

First, we show that the coboundary maps are induced by maps between the Kan extensions:

**Theorem 45** The coboundary of the complex $X^\bullet(F, M)$ is induced by a map of $F, F$-bimodules

$$\partial : \text{Lan}_{\otimes^{n+1}}(\otimes^{n+1}(F^{n+1})) \to \text{Lan}_{\otimes^n}(\otimes^n(F^n)).$$

**proof:** The key is to consider the universal property defining $\text{Lan}_{\otimes^n}(\otimes^n(F^n))$, and to find natural transformations filling the square

$$
\begin{array}{ccc}
\mathcal{C}^{\otimes n+1} & \xrightarrow{F^{n+1}} & \mathcal{D}^{\otimes n+1} \\
\otimes^{n+1} \downarrow & & \downarrow \otimes^{n+1} \\
\mathcal{C} & \xrightarrow{\text{Lan}_{\otimes^n}(\otimes^n(F^n))} & \mathcal{D}
\end{array}
$$

in such a way that their composition with any $\phi : \text{Lan}_{\otimes^n}(\otimes^n(F^n)) \to M$ is induced by each term of $\delta(\tilde{\phi})$, where $\tilde{\phi} : \otimes^n \to M(\otimes^n)$ is the corresponding map.

All but the first and last terms have fillers of the form
The first has the filler

while the last has a similar one with left and right reversed. In both of the diagrams above, each $\alpha$, denotes an appropriate map given by Mac Lane’s coherence theorem.

It is immediate that the composition of these with $\phi$ has the desired property. Therefore we can let the boundary map be the map between Kan extensions induced by their alternating sum.

Finally, we must show that the sequence of these maps actually forms a resolution in the category of $F,F$-bimodules over $C$.

To do this, we use

**Lemma 46** In any abelian category $A$, given a complex

$$A \xrightarrow{\beta} B \xrightarrow{\gamma} C$$

such that for all $M \in \text{Ob}(A)$

$$\text{Hom}(C, M) \xrightarrow{\text{Hom}(\gamma, M)} \text{Hom}(B, M) \xrightarrow{\text{Hom}(\beta, M)} \text{Hom}(A, M)$$

is exact in $\text{Ab}$, the original complex is itself exact in $A$.
proof: Suppose the original sequence is not exact, that is, \( \text{Im}(\beta) \) is a proper subobject of \( \text{ker}(\gamma) \). Now, let \( M = \text{coker}(\beta) \). The canonical map \( B \to M \) is in the kernel of \( \text{Hom}(\beta, M) \), but is not in the image of \( \text{Hom}(\gamma, M) \). All such maps have kernels containing \( \text{ker}(\gamma) \). \( \square \)

Now, we apply this lemma to the complex \( \text{Lan}_\otimes(\bullet \otimes (F^\bullet)), \partial^\bullet \) to obtain

**Theorem 47** Under the hypotheses of Theorem 44, the complex

\[ \text{Lan}_\otimes(\bullet \otimes (F^\bullet)), \partial^\bullet \]

is a projective resolution of \( F \) as an \( F,F \)-bimodule.

proof: Now by the lemma, it suffices to show that the complex

\[ \text{Hom}_{F,F-\text{bimod}}(\text{Lan}_\otimes(\bullet \otimes (F^\bullet)), M), \text{Hom}_{F,F-\text{bimod}}(\partial^\bullet, M) \]

For general \( n \), observe that by the bimodule coherence conditions, the first and second and the last and penultimate terms in the expression for the coboundary cancel in pairs. Using this, it is easy to see that any \( n + 1 \)-cocycle \( \phi_{X_1,X_2,\ldots,X_{n+1}} \) is the coboundary of \( \phi_{X_1,I,X_2,\ldots,X_{n+1}} \).

The cases of 1, 2 and 3-cocycles must be handled separately: For 1-cocycles, use the coherence condition to cancel the first two terms of the coboundary. The remaining term is 0 if and only if \( \phi \) is 0, since tensoring with \( F(X) \) is exact.

For 2-cocycles, first observe that any 2-cochain is a cocycle (by the pairwise cancellation noted above). But application of the coherence conditions to cancel two terms and rewrite the remaining term shows that any \( \phi_{X,Y} \) is the coboundary of the 1-cochain \( \phi_{X,I} \).

For 3-cocycles, the pairwise cancellation of terms in the coboundary formula reduces the cocycle condition to \([\phi_{X,Y} \otimes Z,W] = 0 \) for all \( X,Y,Z,W \in \text{Ob}(C) \). Specializing to \( Y = I \) then shows that \( \phi = 0 \). But this taken with the previous observation that all 2-cochains are cocycles completes the proof of the exactness of the sequence of \( \text{Hom}(-, M) \)'s, and by the previous lemma, of the theorem. \( \square \)

As a consequence we now have

**Theorem 48** If \( H^\bullet(F,M) \) denotes the cohomology of \( (X^\bullet(F,M), \delta) \), then

\[ H \bullet (F,M) = R \text{Nat}[-,M](F). \]

Although we have no cause to pursue the matter, this last result allows us to generalize the deformation cohomology of a monoidal functor to a special cohomology of an \( F,F \)-bimodule \( N \) with coefficients in another \( F,F \)-bimodule \( M \), given by right derived functors of \( \text{Nat}[-,M] \). Notice that this is not simply \( \text{Ext} \) for the abelian category of \( F,F \)-bimodules, since it involves natural transformations which are not bimodule maps.
9 Conclusions

The preceding results do not, in and of themselves, provide much help in calculating deformations of monoidal categories, monoidal functors or braided monoidal categories. They do, however, move the subject into the realm of classical homological algebra.

As such, it may be hoped that they will further the development of categorical deformation theory, both as a subject in its own right, and in its applications to the theory of Vassiliev invariants (cf. [Vas90, Yet98]) and to the quest of non-trivial Hopf categories (cf. [CF94, CY98]) which was its original motivation.
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