ON SINGULARITIES OF THIRD SECANT VARIETIES OF VERONESE EMBEDDINGS

KANGJIN HAN

Abstract. In this paper we study singularities of third secant varieties of Veronese embedding $v_d(P^n)$, which corresponds to the variety of symmetric tensors of border rank at most three in $(\mathbb{C}^{n+1}) \otimes d$.

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1. INTRODUCTION

For a projective algebraic variety $X \subset \mathbb{P}W$, the $k$-th secant variety $\sigma_k(X)$ is defined by

$$\sigma_k(X) = \bigcup_{x_1, \ldots, x_k \in X} \mathbb{P}\langle x_1, \ldots, x_k \rangle \subset \mathbb{P}W$$

where $\langle x_1, \ldots, x_k \rangle \subset W$ denotes the linear span of the points $x_1, \ldots, x_k$ and the overline denotes Zariski closure. Let $V$ be an $(n+1)$-dimensional complex vector space and $W = S^d V$ be the subspace of symmetric $d$-way tensors in $V \otimes d$. Equivalently, we can also think of $W$ as the space of homogeneous polynomials of degree $d$ in $n+1$ variables. When $X$ is the Veronese embedding $v_d(PV)$ of rank one symmetric $d$-way tensors over $V$ in $\mathbb{P}W$, then $\sigma_k(X)$ is the variety of symmetric $d$-way tensors of border rank at most $k$ (see subsection 2.1 for terminology and details).

If $X$ is an irreducible variety and $\sigma_k(X)$ its $k$-secant variety, then it is well known that

$$\text{Sing}(\sigma_k(X)) \supseteq \sigma_{k-1}(X),$$

(e.g. see [Ad87, coro. 1.8]). Equality holds in many basic examples, like determinantal varieties defined by minors of a generic matrix, but the strict inequality also holds for some other tensors (e.g. just have a look at [MOZ12, coro. 7.17] for the case $\sigma_2(X)$ when $X$ is the Segre embedding $\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_r$ or [AOP12, figure 1, p.18] for the third secant variety of Grassmannian $G(2,6)$).

Therefore, it should be very interesting to compute more cases and to give a general treatment about singularities of secant varieties. Further, the knowledge of singular locus is known to be

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In this paper, we deal with the case of third secant variety of Veronese embeddings, decomposition (see [COV14, thm. 4.5]). It has recently been paid more attention in this context. Very crucial to the so-called identifiability problem (see [Ott09, see remarks in section 2]). Look at the table in Figure 1.

Theorem 2.1 (Singularity of $\sigma_3(v_d(\mathbb{P}^n))$). Let $X$ be the $n$-dimensional Veronese variety $v_d(\mathbb{P}^N)$ in $\mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$. Then, the following holds that the singular locus

$$\text{Sing}(\sigma_3(X)) = \sigma_2(X)$$

as a set for all $(d, n)$ with $d \geq 3$ and $n \geq 2$ unless $d = 4$ and $n \geq 3$. In the exceptional case $d = 4$, for each $n \geq 3$ the singular locus $\text{Sing}(\sigma_3(v_4(\mathbb{P}^N)))$ is $D \cup \sigma_2(v_4(\mathbb{P}^N))$, where $D$ denotes the locus of all the degenerate forms $f$ (i.e. $\dim(f) = 2$) in $\sigma_3(v_4(\mathbb{P}^N)) \setminus \sigma_2(v_4(\mathbb{P}^N))$.

Proof. Combine Corollary 2.10, Theorem 2.11 and 2.13. 

We can sum up all the relevant results into the following table:

| $(k, d, n)$ | Singular locus of $\sigma_k(v_d(\mathbb{P}^n))$ | Comment & Reference |
|-------------|---------------------|---------------------|
| $(\geq 2, \geq 2, 1)$ | $\sigma_{k-1}$ | Classical; case of binary forms, [IK99, thm. 1.45] |
| $(\geq 2, 2, \geq 1)$ | $\sigma_{k-1}$ | Symmetric matrix case, [IK99, thm. 1.26] |
| $(2, \geq 2, \geq 1)$ | $\sigma_1$ | [Kan99, thm. 3.3] |
| $(3, 3, 2)$ | $\sigma_2$ | Aronhold hypersurface, [Ott09, remarks in §2] |
| $(3, \geq 4, 2)$ | $\sigma_2$ | Thm. 2.1+Thm. 2.13 |
| $(3, 3, \geq 3)$ | $\sigma_2$ | Coro. 2.10 |
| $(3, 4, \geq 3)$ | $D \cup \sigma_2$ | Only exceptional case ($d = 4$), Thm. 2.13 |
| $(3, \geq 5, \geq 3)$ | $\sigma_2$ | Thm. 2.1+Thm. 2.13 |

Figure 1. Singular locus of $\sigma_k(v_d(\mathbb{P}^n))$. 

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2.1. Preliminaries. For the proof, we recall some preliminaries on (border) ranks and geometry of symmetric tensors and list a few known facts on them for future use.

First of all, the equations defining $\sigma_3(v_d(\mathbb{P}V))$ come from so-called symmetric flattenings. Consider the polynomial ring $S^*V = \mathbb{C}[x_0, \ldots, x_n]$ (we call this ring $S$) and consider another polynomial ring $T = S^*V^\vee = \mathbb{C}[y_0, \ldots, y_n]$, where $V^\vee$ is the dual space of $V$. Define the differential action of $T$ on $S$ as follows: for any $g \in T_{d-k}, f \in S_d$, we set
\begin{equation}
(2.1) \quad g \cdot f = g(\partial_0, \partial_1, \ldots, \partial_n)f \in S_k.
\end{equation}
Let us take bases for $S_k$ and $T_{d-k}$ as
\begin{equation}
(2.2) \quad \mathbf{X}^I = \frac{1}{i_0! \cdots i_n!}x_0^{i_0} \cdots x_n^{i_n} \quad \text{and} \quad \mathbf{Y}^J = y_0^{j_0} \cdots y_n^{j_n},
\end{equation}
with $|I| = i_0 + \cdots + i_n = k$ and $|J| = j_0 + \cdots + j_n = d - k$. For a given $f = \sum_{|I|=d} a_I \cdot \mathbf{X}^I$ in $S_d$, we have a linear map $\phi_{d-k,k}(f) : T_{d-k} \to S_k, \ g \mapsto g \cdot f$ for any $k$ with $1 \leq k \leq d - 1$, which can be represented by the following $(k+n) \times (d-k+n)$-matrix:
\begin{equation}
(2.3) \quad \left( \begin{array}{c}
a_{I,J} \end{array} \right) \quad \text{with} \quad a_{I,J} = a_{I+J},
\end{equation}
in the bases defined above. We call this the symmetric flattening (or catalecticant) of $f$. It is easy to see that the transpose $\phi_{d-k,k}(f)^T$ is equal to $\phi_{k,d-k}(f)$.

Given a homogeneous polynomial $f$ of degree $d$, the minimum number of linear forms $l_i$ needed to write $f$ as a sum of $d$-th powers is the so-called (Waring) rank of $f$ and denoted by $\text{rank}(f)$. The (Waring) border rank is this notion in the limiting sense. In other words, if there is a family \( \{ f_\epsilon \mid \epsilon > 0 \} \) of polynomials with constant rank $r$ and $\lim_{\epsilon \to 0} f_\epsilon = f$, then we say that $f$ has border rank at most $r$. The minimum such $r$ is called the border rank of $f$ and denoted by $\text{rank}^b(f)$.

Note that by definition $\sigma_k(v_d(\mathbb{P}V))$ is the variety of homogeneous polynomials $f$ of degree $d$ with border rank $\text{rank}^b(f) \leq k$.

It is obvious that if $f$ has (border) rank 1, then any symmetric flattening $\phi_{d-k,k}(f)$ has rank 1. By subadditivity of matrix rank, we also know that rank $\phi_{d-k,k}(f) \leq r$ if $\text{rank}^b(f) \leq r$. We have the following known result for the defining equations of $\sigma_3(X)$:

**Proposition 2.2** (Defining equations of $\sigma_3(v_d(\mathbb{P}V))$). Let $X$ be the $n$-dimensional Veronese variety $v_d(\mathbb{P}V)$ in $\mathbb{P}^N$ with $N = (\binom{n+d}{n} - 1)$. For any $(d,n)$ with $d \geq 3, n \geq 2$, $\sigma_3(X)$ is defined scheme-theoretically by the $4 \times 4$-minors of the two symmetric flattenings
\[ \phi_{d-1,1}(F) : S^{d-1}V^\vee \to V \quad \text{and} \quad \phi_{d-\lfloor \frac{d}{2} \rfloor,\lfloor \frac{d}{2} \rfloor}(F) : S^{d-\lfloor \frac{d}{2} \rfloor}V^\vee \to S^{\lfloor \frac{d}{2} \rfloor}V, \]
where $F$ is the form $\sum_{I \in \mathbb{N}^{n+1}} a_I \cdot \mathbf{X}^I$ of degree $d$ as considering the coefficients $a_I$’s indeterminate.

**Proof.** Aronhold invariant ($n = 2$, see e.g. [IK99, p.247]) and symmetric inheritance (Proposition 2.3.1 in [LO13]) prove the result for the case $d = 3$. For any $d \geq 4$, see Theorem 3.2.1 (1) in [LO13]. \[ \square \]

Since there is a natural $\text{SL}_{n+1}(\mathbb{C})$-group action on $\sigma_3(X)$, we may take the $\text{SL}_{n+1}(\mathbb{C})$-orbits inside $\sigma_3(X)$ into consideration for the study of singularity. And we could also regard a canonical representative of each orbit as below.

First, suppose $f \in \sigma_3(X) \setminus \sigma_2(X)$ is a degenerate form (i.e. $\dim(f) = 2$). Choose $x_0, x_1$ as the basis of $\langle f \rangle$. Then, we recall the following lemma
Lemma 2.3. For any \( d \geq 4 \) and \( n \geq 1 \), any general degenerate form \( f \in \sigma_3(v_d(\mathbb{P}^V)) \setminus \sigma_2(v_d(\mathbb{P}^V)) \) can be written as \( x_0^d + \alpha \cdot x_1^d + \beta \cdot (x_0 + x_1)^d \), up to \( \text{SL}_{n+1}(\mathbb{C}) \)-action, for some nonzero \( \alpha, \beta \in \mathbb{C} \).

Proof. Since \( \dim(f) = 2 \), let \( U := \langle f \rangle = \mathbb{C}(x_0, x_1) \), a subspace of \( V \). For such a \( f \in \sigma_3(v_d(\mathbb{P}^V)) \setminus \sigma_2(v_d(\mathbb{P}^V)) \), it is easy to see that

\[
3 = \text{rank}(f) \leq \text{rank}(f, U),
\]

where the latter is the border rank of \( f \) being considered as a polynomial in \( S^*U \). On the other hand, we also have \( \text{rank}(f, U) \leq 3 \), because the symmetric flattenings \( \phi_{d-1,1}(f, U) \) and \( \phi_{d-1,1}(f, U) \setminus \sqrt[3]{d} \) are just submatrices of \( \phi_{d-1,1}(f) \) and \( \phi_{d-1,1}(f) \setminus \sqrt[3]{d} \) respectively and therefore all their 4 \( \times \) 4-minors also vanish (so, \( f \in \sigma_3(v_d(\mathbb{P}^U)) \)). Since \( \text{rank}(f, U) \) and \( \text{rank}(f, U) \) coincide for a general \( f \) in the rational normal curve case (see e.g. [CG01]), we have \( \text{rank}(f, U) = 3 \). Thus, for some nonzero \( \lambda, \mu \in \mathbb{C} \) we can write \( f \) as

\[
f(x_0, x_1) = (a_0x_0 + a_1x_1)^d + (b_0x_0 + b_1x_1)^d + \{ \lambda(a_0x_0 + a_1x_1) + \mu(b_0x_0 + b_1x_1) \}^d
\]

\[
= X_0^d + \left( \frac{\lambda}{\mu} \right)^d \cdot X_1^d + \lambda^d \cdot (X_0 + X_1)^d,
\]

by some scaling and using a \( \text{SL}_{n+1}(\mathbb{C}) \)-change of coordinates, which proves our assertion. \( \square \)

Remark 2.4. There are some remarks related to Lemma 2.3 as follows:

(a) Note that there does not exist a degenerate form corresponding to an orbit in \( \sigma_3(v_d(\mathbb{P}^V)) \setminus \sigma_2(v_d(\mathbb{P}^V)) \) if \( d \leq 3 \). In this case, if \( f \) is degenerate, then \( f \) always belongs to \( \sigma_2(v_d(\mathbb{P}^V)) \), for the \( \phi_{d-1,1}(f) \) have at most two nonzero rows and all the \( 3 \times 3 \)-minors of \( \phi_{d-1,1}(f) \) vanish.

(b) In fact, in \( d = 4 \) case, Lemma 2.3 holds for all degenerate form \( f \in \sigma_3(v_d(\mathbb{P}^V)) \setminus \sigma_2(v_d(\mathbb{P}^V)) \), because there exist only rank 3 forms in \( \sigma_3(v_4(\mathbb{P}^1)) \setminus \sigma_2(v_4(\mathbb{P}^1)) \) (see [CG01] and also [LT10, chap.4]).

Now, let’s put all types of canonical representatives for \( \text{SL}_{n+1}(\mathbb{C}) \)-orbits together as follows:

Theorem 2.5. There are 4 types of homogeneous forms representing \( \text{SL}_{n+1}(\mathbb{C}) \)-orbits in \( \sigma_3(v_d(\mathbb{P}^V)) \setminus \sigma_2(v_d(\mathbb{P}^V)) \):

\[
x_0^d + x_1^d + x_2^d, \quad x_0^{d-1}x_1 + x_2^d, \quad x_0^{d-2}x_1^2 + x_0x_1^{d-2},
\]

which correspond to all the three non-degenerate orbits and the binary type corresponding to \( D \), the locus of all orbits represented by degenerate forms, which appears only if \( d \geq 4 \) and can be written as \( x_0^d + \alpha x_1^d + \beta(x_0 + x_1)^d \) for some nonzero \( \alpha, \beta \in \mathbb{C} \) in case of a general point of \( D \).

Proof. Combine [LT10, thm. 10.2], Lemma 2.3 and Remark 2.4. \( \square \)

Let us introduce more basic terms and facts. Let \( Z \subset \mathbb{P}^W \) be a variety and \( \check{Z} \) be its affine cone in \( W \). Consider a (closed) point \( p \in \check{Z} \) and say \([p]\) the corresponding point in \( \mathbb{P}^W \). We denote the affine tangent space to \( Z \) at \([p]\) in \( \mathbb{P}^W \) by \( \check{T}_{[p]}Z \) and we define the affine conormal space to \( Z \) at \([p]\), \( \check{N}_{[p]}^\vee Z \) as the annihilator \( (\check{T}_{[p]}Z)^\perp \subset W^\vee \). Since \( \dim \check{N}_{[p]}^\vee Z + \dim \check{T}_{[p]}Z = \dim W \) and \( \dim Z \leq \dim \check{T}_{[p]}Z - 1 \), we get that \( \dim \check{N}_{[p]}^\vee Z \leq \text{codim}(Z, \mathbb{P}^W) \) and the equality holds if and only if \( Z \) is smooth at \([p]\). This conormal space is quite useful to study the tangent space of \( Z \).

Let us recall the apolar ideal \( f^\perp \subset T \). For any given form \( f \in S^dV \), we call \( \partial \in T_f \) apolar to \( f \) if the differentiation \( \partial(f) \) gives zero (i.e. \( \partial \in \ker \phi_{d-1}(f) \)). And we define the apolar ideal \( f^\perp \subset T \) as

\[
f^\perp := \{ \partial \in T \mid \partial(f) = 0 \}.
\]

It is straightforward to see that \( f^\perp \) is indeed an ideal of \( T \). Moreover, it is well-known that the quotient ring \( T_f := T/f^\perp \) is an Artinian Gorenstein algebra with socle degree \( d \) (see e.g. [IK99]).
In our case, we have a nice description of the conormal space in terms of this apolar ideal as follows:

**Proposition 2.6.** Let \( X \) be the \( n \)-dimensional Veronese variety \( v_d(\mathbb{P}V) \) as above and \( f \) be any form in \( S^dV \). Suppose that \( f \) corresponds to a (closed) point of \( \sigma_3(X) \setminus \sigma_2(X) \) and that \( \phi_{d-1,1}(f) = 3 \), \( \text{rank } \phi_{d-\left\lfloor \frac{d}{2} \right\rfloor,\left\lfloor \frac{d}{2} \right\rfloor}(f) = 3 \). Then, for any \( (d,n) \) with \( d \geq 4, n \geq 2 \) we have

\[
\hat{N}_j^\vee \sigma_3(X) = (f^\perp)_1 \cdot (f^\perp)_{d-1} + (f^\perp)_{\left\lfloor \frac{d}{2} \right\rfloor} \cdot (f^\perp)_{d-\left\lfloor \frac{d}{2} \right\rfloor},
\]

where the sum is taken as a \( \mathbb{C} \)-subspace in \( T_d = S^dV^\vee \).

**Proof.** First, recall that \( \phi_{d-k,k}(f)^T = \phi_{k,d-k}(f) \). We also note that

\[
\ker \phi_{d-k,k}(f) = (f)_{d-k} \quad \text{and} \quad (\text{im } \phi_{d-k,k}(f))^\perp = \ker(\phi_{d-k,k}(f)^T) = \ker \phi_{k,d-k}(f) = (f)^\perp_{k}.
\]

Whenever \( \text{rank } \phi_{d-1,1}(f) = 3 \) and \( \text{rank } \phi_{d-\left\lfloor \frac{d}{2} \right\rfloor,\left\lfloor \frac{d}{2} \right\rfloor}(f) = 3 \), we have

\[
\hat{N}_j^\vee \sigma_3(X) = \langle \ker \phi_{d-1,1}(f) \cdot (\text{im } \phi_{d-1,1}(f))^\perp \rangle + \langle \ker \phi_{d-\left\lfloor \frac{d}{2} \right\rfloor,\left\lfloor \frac{d}{2} \right\rfloor}(f) \cdot (\text{im } \phi_{d-\left\lfloor \frac{d}{2} \right\rfloor,\left\lfloor \frac{d}{2} \right\rfloor}(f))^\perp \rangle
\]

(see [LO13, Proposition 2.5.1]), which proves the proposition. \( \square \)

**Remark 2.7.** Note that, in case of \( n = 2 \) or \( \text{dim}(f) = 2 \) (i.e. degenerate form), to compute conormal space \( \hat{N}_j^\vee \sigma_3(X) \) we only need to consider the symmetric flattening \( \phi_{d-\left\lfloor \frac{d}{2} \right\rfloor,\left\lfloor \frac{d}{2} \right\rfloor} \) so that we have

\[
\hat{N}_j^\vee \sigma_3(X) = (f^\perp)_{\left\lfloor \frac{d}{2} \right\rfloor} \cdot (f^\perp)_{d-\left\lfloor \frac{d}{2} \right\rfloor}.
\]

For \( n = 2 \) case, \( \phi_{d-1,1}(f) \) has only 3 rows, there is no non-trivial \( 4 \times 4 \)-minor to give a local equation of \( \sigma_3(X) \) at \( f \). In case of \( \text{dim}(f) = 2 \), we may consider \( f \in \mathbb{C}[x_0, x_1]_d \) and choose bases as (2.2). Then, we could write the matrix of \( \phi_{d-1,1} \) and its evaluation at \( f, \phi_{d-1,1}(f) \) as

\[
\phi_{d-1,1} = \begin{pmatrix}
\cdots & y_0^{d-1} & y_0^{d-2} & \cdots & y_1 \cdots & \cdots & y_n^{d-1} \\
0 & x_0 & x_1 & \cdots & a_I & \cdots & \cdots \\
1 & a_I & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 & \cdots & \cdots \\
\end{pmatrix}, \quad \phi_{d-1,1}(f) = \begin{pmatrix}
\cdots & y_0^{d-1} & y_0^{d-2} & \cdots & y_1 \cdots & \cdots & y_n^{d-1} \\
0 & x_0 & x_1 & \cdots & * & \cdots & \cdots \\
1 & * & * & \cdots & * & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots \\
\end{pmatrix}.
\]

So, each \( 4 \times 4 \)-minor of \( \phi_{d-1,1} \) (say \( D_4(\phi_{d-1,1}) \)) has at most rank 2 at \( f \). Hence, we see that all the partial derivatives in the Jacobian

\[
\frac{\partial D_4(\phi_{d-1,1})}{\partial a_I}(f) = 0
\]

for each index \( I \) with \( |I| = d \) and \( D_4(\phi_{d-1,1}) \) doesn’t contribute to span the conormal space of \( \sigma_3(X) \) at \( f \), because at least one row of \( D_4(\phi_{d-1,1}) \) (say \( (a_I a_J a_K a_L) \)) vanishes at \( f \) and the Laplace expansion of \( D_4(\phi_{d-1,1}) \) along this row

\[
D_4(\phi_{d-1,1}) = \pm \left( a_I \cdot D_3^I(\phi_{d-1,1}) - a_J \cdot D_3^J(\phi_{d-1,1}) + a_K \cdot D_3^K(\phi_{d-1,1}) - a_L \cdot D_3^K(\phi_{d-1,1}) \right)
\]

guarantees all the partials of \( D_4(\phi_{d-1,1}) \) become zero at \( f \) as follows: for example, we see that

\[
\pm \frac{\partial D_4(\phi_{d-1,1})}{\partial a_I}(f) = D_3^I(\phi_{d-1,1})(f) + a_I(f) \cdot \frac{\partial D_3^I(\phi_{d-1,1})}{\partial a_I}(f) - a_J(f) \cdot \frac{\partial D_3^J(\phi_{d-1,1})}{\partial a_I}(f) + a_K(f) \cdot \frac{\partial D_3^K(\phi_{d-1,1})}{\partial a_I}(f) - a_L(f) \cdot \frac{\partial D_3^K(\phi_{d-1,1})}{\partial a_I}(f) = 0,
\]

where
where $a_I(f) = a_J(f) = a_K(f) = a_L(f) = 0$ and $D_3^I(\phi_{d-1,1})(f) = 0$ because of rank $D_3^I(\phi_{d-1,1})$ is at most 2 at $f$.

2.2. Cases of non-degenerate orbits. For the locus of non-degenerate orbits in $\sigma_3(X) \setminus \sigma_2(X)$, we may consider a useful reduction method through the following arguments:

**Lemma 2.8.** For every $f \in \sigma_3(v_d(\mathbb{P}^n))$ ($d, n \geq 2$), there exists a linear $\mathbb{P}^2 = \mathbb{P}U \subset \mathbb{P}^n = \mathbb{P}V$ such that $f \in \sigma_3(v_d(\mathbb{P}U))$. In particular, for every $f \in \sigma_3(v_d(\mathbb{P}^n)) \setminus \sigma_2(v_d(\mathbb{P}^n))$, $2 \leq \dim(f) \leq 3$.

**Proof.** When $f \in \sigma_3(v_d(\mathbb{P}^n))$ (i.e., border rank $\leq 3$), the image of the flattening $S^{d-1}C^{n+1} = \mathbb{C}^{n+1}$ has dimension $\leq 3$ and it is contained in the required 3-dimensional subspace $U$, i.e., $\dim(f) \leq 3$. □

Recall that we denote the locus of degenerate forms in $\sigma_3(X) \setminus \sigma_2(X)$ by $D$ (see Theorem 2.1 for notation). Then, by Lemma 2.8, we have an obvious corollary as follows:

**Corollary 2.9.** For each $f \in \sigma_3(v_d(\mathbb{P}^n)) \setminus (D \cup \sigma_2(v_d(\mathbb{P}^n)))$, there exists a unique 3-dimensional subspace $U$ such that $f \in \sigma_3(v_d(\mathbb{P}U))$.

**Proof.** For those $f$, which correspond to three orbits in Theorem 2.5, the dimension of $\langle f \rangle$ is exactly 3 so that the subspace $U = \langle f \rangle$ is precisely determined in the claimed cases. □

When $d = 3$, we also have an immediate corollary as follows:

**Corollary 2.10** ($d = 3$ case). For every $n \geq 2$ and $d = 3$, $\sigma_3(v_d(\mathbb{P}^n)) \setminus \sigma_2(v_3(\mathbb{P}^n))$ is smooth.

**Proof.** By Remark 2.4 (a), there is no degenerate orbit in this case. So, it comes directly from the smoothness result on Aronhold hypersurface (i.e., $n = 2$ case in Figure 1) and using the fibration argument in the proof of Theorem 2.11 for any $n \geq 3$. □

Here is the theorem for non-degenerate orbits for any $d \geq 4$:

**Theorem 2.11** (Non-degenerate locus). For every $n \geq 2$ and $d \geq 4$, $\sigma_3(v_d(\mathbb{P}^n)) \setminus (D \cup \sigma_2(v_d(\mathbb{P}^n)))$ is smooth.

**Proof.** Let our $\mathbb{P}^n = \mathbb{P}V$ with $V = \mathbb{C}\langle x_0, x_1, \cdots, x_n \rangle$ and its dual $V^\perp = \mathbb{C}\langle y_0, y_1, \cdots, y_n \rangle$. First, we claim that one may reduce the problem to the case of $n = 2$. Construct the following map

\[
\sigma_3(v_d(\mathbb{P}^n)) \setminus (D \cup \sigma_2(v_d(\mathbb{P}^n))) \xrightarrow{\pi} \text{Gr}(\mathbb{P}U, \mathbb{P}^n) \quad \text{with dim} \mathbb{P}U = 2.
\]

This map is well defined by Corollary 2.9 and each fiber $\pi^{-1}(\mathbb{P}U)$ is isomorphic to $\sigma_3(v_d(\mathbb{P}U)) \setminus (D \cup \sigma_2(v_d(\mathbb{P}U)))$. So, if we prove our theorem for the case $n = 2$, then the fibers of $\pi$ are all isomorphic and smooth. Hence $\pi$ becomes a fibration over a smooth variety with smooth fibers. This shows that its domain $\sigma_3(v_d(\mathbb{P}^n)) \setminus (D \cup \sigma_2(v_d(\mathbb{P}^n)))$ is smooth, so proving our assertion.

So, from now on, let us assume $d \geq 4$ and $n = 2$. We can consider three different cases according to Theorem 2.5.

Case (i) $f_1 = x_0^d + x_1^d + x_2^d$ (Fermat-type). It is well-known that this Fermat-type $f_1$ becomes an almost transitive $\text{SL}_3(\mathbb{C})$-orbit, which corresponds to a general point of $\sigma_3(v_d(\mathbb{P}^2))$. Thus, $\sigma_3(v_d(\mathbb{P}^2))$ is smooth at $f_1$.

Case (ii) $f_2 = x_0^{d-1}x_1 + x_2^d$ (Unmixed-type). By Remark 2.7 (i.e., $n = 2$ case), we just need to consider $(f_2^\perp)_{C_{d-1}}(f_2^\perp)_{C_{d-2}}$ as (2.6) to compute $\dim \hat{N}_f^\perp \sigma_3(X)$. Say $s := \lfloor \frac{d}{2} \rfloor$. For $d \geq 4$, we have $2 \leq s \leq d - s \leq d - 2$. Note that $\dim \hat{N}_f^\perp \sigma_3(X) \leq \text{codim}(\sigma_3(X), \mathbb{P}U) = (\frac{d+2}{2}) - 9$. So, it is enough to show $\dim \hat{N}_f^\perp \sigma_3(X) \geq (\frac{d+2}{2}) - 9$ for proving non-singularity of $f_2$. 
Since the summands of $f_2$ separate the variables (i.e. unmixed-type), we could see that the apolar ideal $f_2^\perp$ is generated as

$$f_2^\perp = \left\{ Q_1 = y_0y_2, Q_2 = y_0^2, Q_3 = y_1y_2 \right\} \cup \{ \text{other generators in degree} \geq d \}.$$

So, we have

$$(f_2^\perp)_s = \{ h \cdot Q_i \mid \forall h \in T_{s-2}, \ i = 1, 2, 3 \} \text{ and } (f_2^\perp)_{d-s} = \{ h' \cdot Q_i \mid \forall h' \in T_{d-s-2}, \ i = 1, 2, 3 \}$$

$$\Rightarrow \quad \hat{N}^\vee_{f_2} \sigma_3(X) = (f_2^\perp)_s \cdot (f_2^\perp)_{d-s} = \{ h'' \cdot Q_i Q_j \mid \forall h'' \in T_{d-4}, \ i, j = 1, 2, 3 \}.$$

Thus, if we denote the ideal $(Q_1, Q_2, Q_3)$ by $I$, then $\dim \hat{N}^\vee_{f_2} \sigma_3(X)$ is equal to the value of Hilbert function $H(I^2, t)$ at $t = d$. But, it is easy to see that $I^2$ has a minimal free resolution as

$$0 \rightarrow T(-6) \rightarrow T(-5)^6 \rightarrow T(-4)^6 \rightarrow I^2 \rightarrow 0,$$

which shows the Hilbert function of $I^2$ can be computed as

$$H(I^2, d) = 6\binom{d - 4 + 2}{2} - 6\binom{d - 5 + 2}{2} + \binom{d - 6 + 2}{2} = \begin{cases} 0 & (d \leq 3) \\ \binom{d + 2}{2} - 9 & (d \geq 4) \end{cases}.$$

This implies that $\dim \hat{N}^\vee_{f_2} \sigma_3(X) = \binom{d + 2}{2} - 9$ for any $d \geq 4$, which means that our $\sigma_3(X)$ is smooth at $f_2$ (see also Figure 2).

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**Figure 2.** Case of $f_2 = x_0^{d-1}x_1 + x_2^d$. $P_1$ is the lattice polytope in $\mathbb{R}^3_{\geq 0}$ consisting of exponent vectors $(i, j, k)$ of the monomials $y_0^i y_1^j y_2^k$ in $(f_2^\perp)_{d-s}$ and $P_2$ is the one corresponding to $(f_2^\perp)_s$. $P_1 + P_2$ is the Minkowski sum of two polytopes whose lattice points are exactly the exponent vectors of $\hat{N}^\vee_{f_2} \sigma_3(X) = (f_2^\perp)_{d-s} \cdot (f_2^\perp)_s$, which contains all the monomial of $T_d$ but 9 monomials $y_0^d, y_0^{d-1}y_1, y_0^{d-2}y_1^2, y_0^{d-3}y_1^3, y_0^{d-1}y_2, y_0y_2^{d-1}, y_2^d, y_1y_2^{d-1}, y_0^{d-2}y_1y_2$. This also shows $\dim \hat{N}^\vee_{f_2} \sigma_3(X) = \binom{d + 2}{2} - 9$.

Case (iii) $f_3 = x_0^{d-2}x_1^2 + x_0^{d-1}x_2$ (Mixed-type). In this case, we similarly use a computation of $\dim \hat{N}^\vee_{f_3} \sigma_3(X)$ via $(f_3^\perp)_s \cdot (f_3^\perp)_{d-s}$ to show the smoothness of $f_3$ (recall $s := \left\lfloor \frac{d}{2} \right\rfloor$ and $2 \leq s \leq d - s \leq d - 2$).
Let $Q_1 := y_0 y_2 - \frac{d-1}{2} y_1^2 \in T_2$. We easily see that
\[
f^+_{i} = \left( \{Q_1, Q_2 = y_1 y_2, Q_3 = y_2^2 \} \cup \{ \text{other generators in degree } d-1 \} \right).
\]
Let $I$ be the ideal generated by three quadrics $Q_1, Q_2, Q_3$. By the same reasoning as (ii), we have
\[
\dim \mathcal{N}^\vee_{f_3} \sigma_3(X) = \dim (f^+_{i})_s \cdot (f^+_{i})_{d-s} = H(I^2, d) = \begin{cases} 0 & (d \leq 3) \\ \left( \frac{d+2}{2} \right) - 9 & (d \geq 4) \end{cases},
\]
because in this case $I^2$ also has the same minimal free resolution $0 \to T(-6) \to T(-5)^6 \to T(-4)^6 \to I^2 \to 0$. Hence, we obtain the smoothness of $\sigma_3(X)$ at $f_3$ (see also Figure 3).

**Remark 2.12.** From the viewpoint of apolarity, the three cases in Theorem 2.11 can be explained geometrically as follows: if we consider the base locus of the ideal $I$, which is generated by the three quadrics in each apolar ideal $f^+_{i}$, then case (i) corresponds to three distinct points, case (ii) to one reduced point and one non-reduced of length 2, and case (iii) to one non-reduced point of length 3 (not lying on a line).

### 2.3. Degenerate case: binary forms.
Since there is no degenerate form for $d = 3$ (see Remark 2.4 (a)), it is enough to consider the smoothness of the degenerate locus for $d \geq 4$.

**Theorem 2.13** (Degenerate locus). Let $D$ be the locus of all the degenerate forms in $\sigma_3(v_d(\mathbb{P}^n)) \setminus \sigma_2(v_d(\mathbb{P}^n))$. Then, for any $d \geq 4, n \geq 2, \sigma_3(v_d(\mathbb{P}^n))$ is singular on $D$ if and only if $d = 4$ and $n \geq 3$.

**Proof.** Let $f_D$ be any form belong to $D$. For this degenerate case, by Remark 2.7, we have
\[
\mathcal{N}^\vee_{f_D} \sigma_3(X) = (f^+_{D})_{\left\lfloor \frac{d}{2} \right\rfloor} \cdot (f^+_{D})_{d-\left\lfloor \frac{d}{2} \right\rfloor}.
\]
First of all, let us consider $f_D$ as a polynomial in $\mathbb{C}[x_0, x_1]$ (i.e. $f_D = f_D(x_0, x_1)$). Then, by Hilbert-Burch theorem (see e.g. [IK99, thm. 1.54]) we know that $T/f_D^\perp$ is an Artinian Gorenstein algebra with socle degree $d$ and that $f_D^\perp$ is a complete intersection of two homogeneous polynomials $F, G$ of each degree $a$ and $b$ with $a + b = d + 2$ as an ideal of $\mathbb{C}[y_0, y_1]$. Since rank $\phi_{d-3,3}(f_D) = 3$, there is one-dimensional kernel of $\phi_{d-3,3}(f_D)$ in $\mathbb{C}[y_0, y_1]_3$, which gives one cubic generator $F$ in $f_D^\perp$.

When $f_D$ is general, $f_D = x_0^3 + \alpha x_1^3 + \beta(x_0 + x_1)^d$ for some $\alpha, \beta \in \mathbb{C}^*$ by Lemma 2.3, so we have $F = y_0^3 y_1 - y_0 y_1^2$. Even for the case $f_D$ being not general, we have $F = y_0^3 y_1$ up to change of coordinates, because the apolar ideal of this non-general $f_D$ corresponds to the case with one multiple root on $\mathbb{P}^1$ (see [CG01] and also [LT10, chap.4]).

Therefore, we obtain that

$$f_D^\perp = (F = y_0^3 y_1 - y_0 y_1^2 \text{ or } y_0^2 y_1, G)$$

for some polynomial $G$ of degree $(d - 1)$ and that $f_D^\perp$ as an ideal in $T = \mathbb{C}[y_0, y_1, \ldots, y_n]$ has its degree parts $(f_D^\perp)|_{\frac{d}{2}}$ and $(f_D^\perp)|_{d - \frac{d}{2}}$ both of which are generated by $F, y_2, \ldots, y_n$, since $d \geq 4$ so that $\left\lfloor \frac{d}{2} \right\rfloor, d - \left\lfloor \frac{d}{2} \right\rfloor < d - 1$.

Now, let us compute the dimension of conormal space as follows:

i) $d = 4$ case (i.e. $\left\lfloor \frac{d}{2} \right\rfloor = 2$) : In this case, we have

$$\hat{N}_{f_D}^\vee \sigma_3(X) = (f_D^\perp)_2 \cdot (f_D^\perp)_2 = (y_2, \ldots, y_n)_2 \cdot (y_2, \ldots, y_n)_2 = ((y_i y_j \mid 2 \leq i, j \leq n))_4 .$$

So, we get

$$\dim \hat{N}_{f_D}^\vee \sigma_3(X) = \dim T_4 - \dim \langle y_0^4, y_0^3 y_1, \ldots, y_1^4 \rangle - \dim \langle \{y_0^3 \cdot \ell, y_0^2 y_1 \cdot \ell, y_0 y_1^2 \cdot \ell, y_1^3 \cdot \ell \mid \ell = y_2, \ldots, y_n \} \rangle$$

$$= \left( 4 + \frac{n}{4} \right) - 5 - 4(n - 1) .$$

This shows that $\sigma_3(X)$ is singular at $f_D$ if and only if $n \geq 3$, because the expected codimension is $\left( \frac{4 + n}{4} \right) - 3n - 3$.

ii) $d = 5$ case (i.e. $\left\lfloor \frac{d}{2} \right\rfloor = 2$) : Recall that $F$ is $y_0^3 y_1 - y_0 y_1^2$, the cubic generator of $f_D^\perp$.

Then,

$$\hat{N}_{f_D}^\vee \sigma_3(X) = (f_D^\perp)_2 \cdot (f_D^\perp)_3 = (y_2, \ldots, y_n)_2 \cdot (F, y_2, \ldots, y_n)_3 .$$

$$\dim \hat{N}_{f_D}^\vee \sigma_3(X) = \dim T_5 - \dim \langle y_0^5, y_0^4 y_1, \ldots, y_1^5 \rangle$$

$$- \dim \langle \{y_0^4 \cdot \ell, y_0^3 y_1 \cdot \ell, y_0^2 y_1^2 \cdot \ell, y_0 y_1^3 \cdot \ell, y_1^4 \cdot \ell \mid \ell = y_2, \ldots, y_n \} \rangle$$

$$= \left( 5 + \frac{n}{5} \right) - 6 - 3(n - 1) = \text{expected codim}(\sigma_3(X), \mathbb{P}^5 V) ,$$

which gives that $\sigma_3(X)$ is smooth at $f_D$ in this case.

ii) $d \geq 6$ case: Here we have $\hat{N}_{f_D}^\vee \sigma_3(X) = (f_D^\perp)|_{\frac{d}{2}} \cdot (f_D^\perp)|_{d - \frac{d}{2}} = (F, y_2, \ldots, y_n)|_{\frac{d}{2}} \cdot (F, y_2, \ldots, y_n)|_{d - \frac{d}{2}}$.

$$\dim \hat{N}_{f_D}^\vee \sigma_3(X) = \dim T_d - \dim \langle y_0^{d-1} \cdot \ell, y_0^{d-2} y_1 \cdot \ell, \ldots, y_1^{d-1} \cdot \ell \rangle \\setminus \{y_0^{d-4} F \cdot \ell, \ldots, y_1^{d-4} F \cdot \ell \mid \ell = y_2, \ldots, y_n \} \rangle$$

$$- \dim \langle \{y_0^d, y_0^{d-1} y_1, \ldots, y_1^d \setminus \{y_0^{d-6} F^2, y_0^{d-7} y_1 F^2, \ldots, y_1^{d-6} F^2 \} \rangle$$

$$= \left( d + \frac{n}{d} \right) - \{d - (d - 3)\}(n - 1) - \{(d + 1) - (d - 5)\}$$

$$= \left( d + \frac{n}{d} \right) - 3(n - 1) - 6 = \text{expected codim}(\sigma_3(X), \mathbb{P}^d V) ,$$

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which implies that $\sigma_3(X)$ is also smooth at $f_D$. \hfill \Box

2.4. **Defining equations of $\text{Sing}(\sigma_3(X))$.** As an immediate corollary of Theorem 2.1, we obtain defining equations of the singular locus in our third secant of Veronese embedding $\sigma_3(X)$.

**Corollary 2.14.** Let $X$ be the $n$-dimensional Veronese embedding as above. The singular locus of $\sigma_3(X)$ is cut out by $3 \times 3$-minors of the two symmetric flattenings $\phi_{d-1,1}$ and $\phi_{d-2,2}$ unless $d = 4$ and $n \geq 3$ case, in which the (set-theoretic) defining ideal of the locus is the intersection of the ideal generated by the previous $3 \times 3$-minors and the ideal generated by $3 \times 3$-minors of $\phi_{d-1,1}$ and $4 \times 4$-minors of $\phi_{d-[\frac{d}{2}],1}$.

**Proof.** It is well-known that $\sigma_2(X)$ is cut out by $3 \times 3$-minors of the two $\phi_{d-1,1}$ and $\phi_{d-2,2}$ (see [Kan99, thm. 3.3]). It is also easy to see that $D$, the locus of degenerate forms inside $\sigma_3(X)$, is cut out by $3 \times 3$-minors of $\phi_{d-1,1}$ and $4 \times 4$-minors of $\phi_{d-[\frac{d}{2}],1}$ by the argument in Remark 2.7 and Proposition 2.2. Thus, using these two facts the conclusion is straightforward by Theorem 2.1. \hfill \Box

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School of Mathematics, Korea Institute for Advanced Study (KIAS), 85 Hoegiro, Dongdaemun-gu, Seoul 130–722, Korea

E-mail address: kangjin.han@kias.re.kr