Lambda-conductors for group rings

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1 Introduction.

This paper is part of a project which aims to provide a method for computing the Nil groups of the group rings of finite abelian groups, by refining some of the techniques used in [1] and [2] in such a way that the allowed coefficient rings include polynomial rings. For the refinement of the $p$-adic logarithm discussed in [3] and [4] it is assumed that the rings involved have a structure of $\lambda$-ring; we refer to these papers for generalities about $\lambda$-rings. Thus it is useful to extend as much as possible of the other techniques to the context of $\lambda$-rings. In this paper we investigate how to describe a group ring of a finite abelian group as a pull back of a diagram of rings which are more accessible to calculations in algebraic K-theory.

Let be given a commutative ring $S$ and subring $R$. For each ideal $I$ of $S$ which is contained in $R$ one has a cartesian square

\[
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
R/I & \longrightarrow & S/I
\end{array}
\]

thus describing $R$ as a pull back of rings for which the $K$-theory is hopefully better understood. By taking for $I$ the sum of all such ideals one finds a diagram where the rings on the bottom row are as small as possible.

We modify this construction by assuming that $R$ has a structure of $\lambda$-ring and considering only ideals $I$ stable under the $\lambda$-operations. We call the resulting ideal the $\lambda$-conductor of $S$ into $R$. In particular we are interested in the case that $R$ is the group ring $\mathbb{Z}[G]$ of a finite abelian group, and $S$ is its normal closure in $R \otimes \mathbb{Q}$, which splits as a direct sum of rings $S_i = \mathbb{Z}[\chi_i]$ associated to equivalence classes of characters $\chi_i : G \rightarrow \mathbb{C}$.

In this situation $R$ is a $\lambda$-ring such that $\psi^n(g) = g^n$ for every $n \in \mathbb{N}$ and $g \in G$. In general $S$ is not stable under the $\lambda$-operations on $R \otimes \mathbb{Q}$, but it is stable under the associated Adams operations $\psi^n$ since they are ring homomorphisms.

We will prove that in case $G$ is a primary group its $\lambda$-conductor is precisely the intersection of the classical conductor and the augmentation ideal. We do this by exhibiting generators of the classical conductor and examining their behavior under the fundamental $\lambda$-operations.
2 The primary case

Throughout this section $G$ is a group of order $n = p^e$, where $p$ is prime. We consider representations $\rho: G \to \mathbb{C}^*$. We say that $\rho$ is of level $k$ if the image of $\rho$ has $p^k$ elements.

Two representations $\tau_1$ and $\tau_2$ are called equivalent if they have the same kernel. That means that there must be some $m \in \mathbb{Z}$ prime to $p$ such that $\tau_2(x) = \tau_1(x)^m$ for all $x \in G$. Obviously equivalent representations have the same level.

Given a representation $\tau$ of level $k > 0$ one gets a representation $\psi_\tau$ of level $k - 1$ by the formula $(\psi_\tau)(x) = \tau(x^p)$ for $x \in G$. If $\psi_\tau_1$ and $\psi_\tau_2$ are equivalent then we may replace $\tau_2$ by an equivalent representation $\tau_2'$ so that $\psi_\tau_2' = \psi_\tau_1$. So we may choose a representation in each class in such a way that $\psi_\tau$ and $\rho$ coincide if they are equivalent.

Let $\rho$ be a representation of level $k > 0$ and write $\omega = \exp(2\pi i/p)$. We define an element $b_\rho \in \mathbb{Z}[G]$ by the formula

$$b_\rho = \sum_{\rho x=1} x - \sum_{\rho \xi = \omega} \xi.$$ 

If we choose $y_\rho \in G$ such that $\rho(y) = \omega$ then we get

$$b_\rho = \left(\sum_{\rho x=1} x\right)(1 - y_\rho).$$

The only representation of level 0 is the trivial representation, which we denote by 1, and it gives rise to $b_1 = \sum_{x \in G} x$.

**Proposition 1.** If $\rho$ and $\tau$ are not equivalent then $b_\rho b_\tau = 0$. Furthermore $b_\rho^2 = p^{e-k}(1 - y_\rho) b_\rho$ for $\rho$ of level $k > 0$.

**Proof.** If $\ker(\rho) \neq \ker(\tau)$ we may assume that there is $g \in G$ with $\rho(g) = 1$ but $\tau(g) = \omega$. If $g$ has order $m$ then $\sum_{\rho(x)=1} x$ and thus $b_\rho$ is a multiple of $\sum_{j=0}^{m-1} g^j$, whereas $b_\tau$ is a multiple of $1 - g$. The product of these two factors is 0.

The second part follows from $(\sum_{\rho(\xi)=1} \xi)(\sum_{\rho(x)=1} x) = p^{n-k} \sum_{\rho(x)=1} x$ which is true because each $\xi$ gives the same contribution and there are $p^{e-k}$ of them. 

Every representation $\rho$ of level $k$ gives rise to a homomorphism $j_\rho$ from $\mathbb{Z}[G]$ to $S_\rho = \mathbb{Z}[\omega_k]$, where $\omega_k = \exp(2\pi i/p^k)$.

**Proposition 2.** If $\rho$ and $\tau$ are not equivalent then $j_\rho(b_\tau) = 0$. Furthermore $j_1(b_1) = p^e$, and $j_\rho(b_\rho) = p^{e-k}(1 - \omega)$ if $\rho$ is of level $k > 0$.

**Proof.** The second part is obvious since every $x$ in the definition of $b_\rho$ maps to 1, and $y_\rho$ maps to $\omega$. The first part follows since $j_\rho(b_\rho) j_\rho(b_\tau) = 0$ by Proposition 1 and $S_\rho$ is a domain. 


It is well known that the maps \( j_\rho \) (one from each equivalence class) combine to an embedding from \( R \) into its integral closure \( S = \oplus_\rho S_\rho \).

**Proposition 3.** The \( b_\rho \) generate the conductor ideal \( I \) of \( S \) into \( R \).

**Proof.** By the theorem of Jacobinski (Theorem 27.8 in [5]) the conductor is \( \oplus_\rho nD_\rho^{-1} \subset \oplus_\rho S_\rho = S \), where \( D_\rho^{-1} \) is the lattice in \( \mathbb{Q}[\omega_k] \) dual to \( \mathbb{Z}[\omega_k] \) under the trace form. It is a simple exercise that this fractional ideal is in fact generated by \( nD_\rho^{-1} \), which means that \( nD_\rho^{-1} = j(b_\rho)S = j(B_\rho)S_\rho = j(b_\rho R) \).

We remind the reader that in particular \( nS \) is contained in the conductor.

The \( \lambda \)-conductor \( I_\lambda \) from \( S \) into \( R \) is defined as the largest ideal of \( S \) contained in \( R \) which is stable under the fundamental \( \lambda \)-operations \( \theta_\ell \). It is of course a subset of the largest ideal of \( S \) contained in \( R \), which is the ordinary conductor \( I \) described above. Thus we have to investigate the behaviour of the operations \( \theta_\ell \) on the generators \( b_\rho \).

**Lemma 1.** If \( \rho \) is of level \( k > 0 \) and there is no \( \tau \) with \( \psi\tau = \rho \) then \( \psi^p b_\rho = 0 \).

**Proof.** Write \( G \) as a direct product of cyclic groups, with generators \( g_i \). If the order of \( \rho(g_i) \) is strictly smaller than the order of \( g_i \) for all \( i \) then one can find a suitable \( \tau \) by taking for each \( \tau(g_i) \) a \( p \)-th root of \( \rho(g_i) \). If however the orders are the same for some \( i \) then there is certainly some \( h \in G \) such that \( h^p = 1 \) and \( \rho(h) = \omega \). By definition of \( b_\rho \) we have

\[
\psi^p b_\rho = \sum_{\rho \xi = 1} \xi^p - \sum_{\rho \eta = \omega} \eta^p
\]

Here the term in the second sum associated to \( \eta = h\xi \) cancels the term in the first sum associated to \( \xi \).

**Proposition 4.** If \( \rho \) is of level \( k > 0 \) then

\[
\psi^p(b_\rho) = \sum_{\psi\tau = \rho} pb_\tau
\]

**Proof.** By definition we have

\[
\sum_{\psi\tau = \rho} b_\tau = \sum_{\psi\tau = \rho} \sum_{x = 1} x - \sum_{\psi\tau' = \rho} \sum_{\tau' \omega = \omega} x
\]

We claim that all terms with \( x \notin G^p \) cancel. To prove this assume that the class of \( x \) in \( G/G^p \) is nontrivial. Then there exists a homomorphism \( \sigma : G/G^p \to \mathbb{C}^* \) such that \( \sigma(x) = \omega \). Now the term associated to \( \tau \) in the first sum equals the term associated to \( \tau' = \tau \cdot \sigma \) in the second sum.

So we only have to consider terms of the form \( x = \xi^p \) with \( \xi \in G \). The condition \( \tau(x) = 1 \) is then independent of \( \tau \) (since it is equivalent to \( \rho(\xi) = 1 \)) and the sum over all \( \tau \) with \( \psi\tau = \rho \) reduces to a multiplication with the number
of equivalence classes of such $\tau$. By the Lemma we may assume that this number is nonzero. Now $\tau_1$ and $\tau_2$ with $\psi\tau_1 = \rho = \psi\tau_2$ are equivalent iff $\tau_2 = \tau_1^{1 + mp^k}$ for some $m$ with $0 \leq m < p$. So this number is $1/p$ times the number of homomorphisms $\sigma : G/G^p \to \mathbb{C}^*$, hence equals $p^{r-1}$, where $r$ denotes the rank of $G$.

On the other hand we have

$$\psi^p \rho = \sum_{\rho \neq 1} \xi^p - \sum_{\rho \neq \omega} \eta^p$$

Here the first sum is a certain factor times the sum over all $x \in G$ for which there exists $\xi \in G$ with $x = \xi^p$ and which satisfy $\tau(x) = 1$ for any (and thus all) $\tau$ with $\psi\tau = \rho$. The factor is the number of $\xi$ which satisfy these conditions, which equals $p^r$.

We write $h$ for the polynomial of degree $p-2$ given by

$$h(t) = \frac{1}{1-t} \left( \frac{p - 1 - t^p}{1-t} \right)$$

Proposition 5.

$$\psi^p(b_1) = b_1 + \sum_{\tau \neq 1, \psi\tau = 1} h(y_{\tau}) b_{\tau}$$

Proof. We have

$$b_{\tau} = \left( \sum_{x=1}^r x \right) (1 - y_{\tau})$$

and thus

$$h(y_{\tau}) b_{\tau} = \left( \sum_{x=1}^r x \right) \left( p - \sum_{j=0}^{p-1} y_{\tau}^j \right) = p \left( \sum_{x=1}^r x \right) - \left( \sum_{x \in G} x \right)$$

We must take the sum of $\sum_{x=1}^r x$ over all equivalence classes of $\tau \neq 1$ with $\psi\tau = 1$. Interchange the sum over $x$ and the sum over $\tau$. There are two cases:

- If $x \in G^p$ then $\tau x = 1$ for all $\tau$, and we must simply count the number of equivalence class of $\tau$. There are $p^r - 1$ of them, with $p-1$ in each class.
- If $x \not\in G^p$ then the number of $\tau$ such that $\tau x = 1$ is $p^{r-1} - 1$, with again $p-1$ in each class.

So we get

$$\sum_{\tau \neq 1, \psi\tau = 1} h(y_{\tau}) b_{\tau} = \left( \frac{p^r - 1}{p-1} \sum_{x \in G^p} x + \frac{p^{r-1} - 1}{p-1} \sum_{x \not\in G^p} x \right)$$

$$- \left( \frac{p^r - 1}{p-1} \sum_{x \in G^p} x + \frac{p^{r-1} - 1}{p-1} \sum_{x \not\in G^p} x \right)$$

$$= p^r \sum_{x \in G^p} x - \sum_{x \in G} x = \sum_{\xi \in G} \xi^p - \sum_{x \in G} x = \psi^p b_1 - b_1$$
Now we consider the effect of Adams operations $\psi^q$ for primes $q \neq p$. For any prime $q$ we write $f_q$ and $g_q$ for the polynomials given by

$$f_q(t) = \frac{1 - t^q}{1 - t}, \quad g_q(t) = \frac{(1 - t)^{q-1} - f_q(t)}{q}$$

**Proposition 6.** If $\rho$ is of level $k > 0$ then

$$\psi^q(b_\rho) = f_q(y_\rho)b_\rho$$

and $\psi^q(b_1) = b_1$.

**Proof.** We have

$$\psi^q(b_1) = \psi^q \left( \sum_{x \in G} x \right) = \sum_{\xi \in G} \xi = b_1$$

and

$$\psi^q(b_\rho) = \psi^q \left( \left( \sum_{\rho x = 1} x \right) (1 - y_\rho) \right) = \left( \sum_{\rho x = 1} a^q \right) (1 - y_\rho^q)$$

$$= \left( \sum_{\rho \xi = 1} \xi \right) (1 - y_\rho)f_q(y_\rho) = b_\rho f_q(y_\rho)$$

**Corollary 1.** For the idempotents $e_\rho \in S_\rho$ one has

$$\psi^p(e_\rho) = \sum_{\psi_\tau = p} e_\rho \quad \text{if } \rho \neq 1,$$

$$\psi^p(e_1) = e_1 + \sum_{\tau \neq 1, \psi_\tau = p} e_\tau$$

$$\psi^q(e_\rho) = e_\rho \quad \text{if } q \neq p$$

Since $R$ has no $\mathbb{Z}$-torsion the Adams operations $\psi^q$ determine the operations $\theta^q$ and we find

**Proposition 7.** If $\rho$ has level $k > 0$ then

$$\theta^p(b_\rho) = p^{e-k}(p-1)^{-1}(1 - y_\rho)^{p-1}b_\rho - \sum_{\psi_\tau = p} b_\tau \quad \text{if } k < e$$

$$\theta^p(b_1) = g_p(y_\rho)b_\rho \quad \text{if } k = e$$

$$\theta^q(b_\rho) = \left( \frac{p^{e-k}(q-1) - 1}{q} (1 - y_\rho)^{q-1} + g_q(y_\rho) \right) b_\rho \quad \text{for } q \neq p$$

Moreover

$$\theta^p(b_1) = p^{e(p-1)}b_1 - p^{-1}b_1 - p^{-1} \sum_{\tau \neq 1, \psi_\tau = 1} h(y_\tau)b_\tau$$

$$\theta^q(b_1) = \frac{p^{e(q-1)} - 1}{q} b_1 \quad \text{for } q \neq p$$
Proof. This is just a matter of combining the last three Propositions with the formula $\ell \theta^b(a) = a^\ell - \psi^\ell a$. Note that $k = e$ can only happen if $G$ is cyclic, in which case $y^a = 1$, which implies that $b^\ell = (1 - y^a)^\ell = p(1 - y^a)g_p(y^a)$. \qed

Theorem 1. The $b_p$ with $\rho \neq 1$ generate the $\lambda$-conductor ideal $I_\lambda$. In other words $I_\lambda$ is the intersection of the augmentation ideal and the ordinary conductor ideal $I$.

Proof. Write $J$ for the $R$-ideal generated by the $b_p$ with $\rho \neq 1$. From Proposition 7 one reads of that $\theta^\ell(b_p) \in J$ for $\rho \neq 1$ and for every prim $\ell$. From the identity

$$\theta^\ell(ab) = \theta^\ell(a)b^\ell + \psi^\ell(a)\theta^\ell(b)$$

it then follows that $\theta^\ell(Rb_p) \subset J$ for $\rho \neq 1$ and all $\ell$. Finally from

$$\theta^\ell(u + v) = \theta^\ell(u) + \theta^\ell(v) + \sum_{i=1}^{\ell-1} \frac{1}{\ell} \binom{\ell}{i} u^i v^{\ell-i}$$

it follows that $\theta^\ell(J) \subset J$ for every $\ell$. Since $J \subset I$ by Proposition 8 this shows that $J \subset I_\lambda$.

Suppose that $x \in I_\lambda$ and $x \notin J$. Then $x \in I$, so by Proposition 9 there are $x_p \in R$ such that $x = \sum x_p b_p$. Since $\sum_{\rho \neq 1} x_p b_p \in J \subset I_\lambda$ by the first half of the proof, it follows that $x_1 b_1 \in I_\lambda$. Since $y b_1 = b_1$ for every $g \in G$ we may assume that $x_1 \in Z$. Moreover $x_1 \neq 0$ which means that its $p$-valuation $v_p(x_1)$ is a natural number. We may assume that $x$ is chosen in such a way that $v_p(x_1)$ is minimal. Now $I_\lambda$ must also contain

$$\theta^p(x_1 b_1) = p^{-1} \left( x_1^p b^{p(p-1)} - x_1 (b_1 + \sum_{\tau \neq 1, \psi^{\tau} = 1} b_\tau) \right)$$

However the valuation of the coefficient of $b_1$ is $v_p(x_1^p b^{p(p-1)} - x_1) - 1 = v_p(x_1) - 1$, in contradiction with the way $x$ was chosen. Thus $I_\lambda \subset J$. \qed

3 Direct products of relatively prime order

Let $G_1$ be a group of order $n_1 = p^e$, and let $G_2$ a group of order $n_2 = q^f$, where $p$ and $q$ are different primes. We write $R_1 = \mathbb{Z}[G_1]$ and $R_2 = \mathbb{Z}[G_2]$, and denote their normal closures by $S_1$ and $S_2$ respectively. Finally we write $I_1$ for the conductor of $S_1$ into $R_1$ and $I_2$ for the conductor of $S_2$ into $R_2$. Since the $S_i$ are free abelian groups, the same is true for the other additive groups involved, and we can view $I_1 \otimes I_2$ as a subgroup of $R_1 \otimes I_2$ and of $R_1 \otimes R_2$.

Lemma 2.

$$I_1 \otimes I_2 = (R_1 \otimes I_2) \cap (I_1 \otimes R_2)$$
Proof. There are $m_1, m_2 \in \mathbb{Z}$ such that $m_1n_1 + m_2n_2 = 1$. If $x$ is an element of the left hand side then $x \in R_1 \otimes I_2$, so $n_1x \in n_1R_1 \otimes I_2 \subset n_1S_1 \otimes I_2 \subset I_1 \otimes I_2$ and therefore $m_1n_1x \in I_1 \otimes I_2$. Similarly $m_2n_2x \in I_1 \otimes I_2$ and thus $x = m_1n_1x + m_2n_2x \in I_1 \otimes I_2$. The other implication is obvious.

Proposition 8. The conductor $I$ of $S_1 \otimes S_2$ into $R_1 \otimes R_2$ is $I_1 \otimes I_2$.

Proof. Suppose that $x \in I$, so that $x(S_1 \otimes S_2) \subset R_1 \otimes R_2$. We write $x \in R_1 \otimes R_2$ as $\sum x_g \otimes g$, where $g$ runs trough $G_2$. For any $a \in S_1$ we have $\sum(x_ga) \otimes g = (\sum x_g \otimes g)(a \otimes 1) = x(a \otimes 1) \in R_1 \otimes R_2$. Therefore $x_ga \in I_1$ for any $a \in S_1$, which means that $a_g \in I_1$ for all $g \in G_1$. Thus $x \in I_1 \otimes I_2$. Similarly $x \in R_1 \otimes I_2$. Thus $x \in I_1 \otimes I_2$ by the Lemma. The other inclusion is obvious.

We show now that for the $\lambda$-conductor a similar theorem holds:

Theorem 2. The $\lambda$-conductor $I_\lambda$ of $S_1 \otimes S_2$ into $R_1 \otimes R_2$ is the tensor product of the $\lambda$-conductors $I_{\lambda 1}$ of $S_1$ into $R_1$ and $I_{\lambda 2}$ of $S_2$ into $R_2$.

Proof. The $\lambda$-conductor $I_\lambda$ is a subset of the classical conductor $I$, which is $I_1 \otimes I_2$. However $I_1$ is the direct sum $\mathbb{Z}b_1 \oplus I_{\lambda 1}$ by theorem $[1]$ and similarly for $I_2$. Thus any $x \in I_\lambda$ can uniquely be written as

$$x = x_0(b_1 \otimes b_1) \oplus (x_1 \otimes b_1) \oplus (b_1 \otimes x_2) \oplus y$$

with $x_0 \in \mathbb{Z}$, $x_1 \in I_{\lambda 1}$, $x_2 \in I_{\lambda 2}$, $y \in I_{\lambda 1} \otimes I_{\lambda 2}$. Since $I_{\lambda 1}$ is an ideal of $S_1 \otimes S_2$, each of these four summands must be in $I_{\lambda 1}$.

Therefore we consider the intersection of $I_{\lambda 1}$ with $b_1 \otimes I_{\lambda 2}$. Suppose that $a$ is an element of this intersection, say $a = b_1 \otimes x$ with $x \in I_{\lambda 2}$. Then $\theta^p(a) \in I_{\lambda 1}$ too. We have

$$\theta^p(a) = p^{-1}(a^p - \psi^p a) = p^{-1}(p^{e(p-1)}b_1 \otimes x^p - (b_1 + \sum_{\tau \neq 1, \psi^\tau = 1} b_\tau) \otimes \psi^p x)$$

and thus $p^{e(p-1)}b_1 \otimes x^p - p^{-1}b_1 \otimes \psi^p x$ should be in $I_{\lambda 1}$. Now the first term is a multiple of $a$ and thus in $I_{\lambda 1}$. So the other term $p^{-1}b_1 \otimes \psi^p x$ is in the aforementioned intersection. Since $\psi^p$ is an automorphism (of finite order) of $R_2$ this shows that the intersection is $p$-divisible. Since the intersection is a finitely generated abelian group this can only happen if it vanishes.

The same argument applies to the first and second summand of $x$. Thus $x = y \in I_{\lambda 1} \otimes I_{\lambda 2}$ and we have shown that $I_{\lambda 1} \subset I_{\lambda 1} \otimes I_{\lambda 2}$. The other inclusion is obvious.

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