UNSTABLE OPERATIONS IN ÉTALE AND MOTIVIC COHOMOLOGY

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Abstract. We classify all étale cohomology operations on \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \), showing that they were all constructed by Epstein. We also construct operations \( P^a \) on the mod-\( \ell \) motivic cohomology groups \( H^{p,q} \), differing from Voevodsky’s operations; we use them to classify all motivic cohomology operations on \( H^{p,1} \) and \( H^{1,q} \) and suggest a general classification.

In the last decade, several papers have given constructions of cohomology operations on motivic and étale cohomology, following the earlier work of Jardine [J], Kriz-May [KM] and Voevodsky [V2, V1]: see [BJ, BJ1, Jo, May1, V3, V4]. The goal of this paper is to provide, for each \( n \) and \( i \), a classification of all such operations on the étale groups \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \) and the motivic groups \( H^{p,i}_{\mathbb{A}}(\mathbb{A}, \mathbb{F}_{\ell}(q)) \), similar to Cartan’s classification of operations on singular cohomology \( H^n_{\text{top}}(\mathbb{A}, \mathbb{F}_\ell) \) in [C]. We succeed for étale operations and partially succeed for motivic operations.

We work over a fixed field \( \mathbb{A} \) and fix a prime \( \ell \) with \( 1/\ell \in \mathbb{A} \). By definition, an (unstable) étale cohomology operation on \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \) is a natural transformation \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \to H^p_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes q) \) of set-valued functors from the category of (smooth) simplicial schemes over \( \mathbb{A} \) (for some \( p \) and \( q \)). Similarly, a motivic cohomology operation on \( H^{p,i}_{\mathbb{A}}(\mathbb{A}, \mathbb{F}_{\ell}(q)) \) denotes the Nisnevich cohomology \( H^p_{\text{nis}}(\mathbb{A}, \mathbb{F}_{\ell}(q)) \), where the cochain complex \( \mathbb{F}_{\ell}(q) \) is defined in [V2] or [MVW]. Note that the set of all cohomology operations forms a ring; the product of \( \theta_1 \) and \( \theta_2 \) is the operation \( x \mapsto \theta_1(x) \cdot \theta_2(x) \).

Our classification begins with a construction of étale operations \( P^a \), due to Epstein, as a special case of the operations on sheaf cohomology he described in his 1966 paper [E]. This is carried out in Theorem 1.1 for odd \( \ell \); when \( \ell = 2 \), this was established by Jardine [J]. The coefficient ring \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \) also acts on étale cohomology; we prove in Theorem 3.5 that the ring of all étale operations on \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \) is the twisted ring \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \otimes H^*_{\text{top}}(\mathbb{A}, \mathbb{F}_{\ell}) \). (We give a precise description of Cartan’s ring in Definition 0.1 below.) This classifies all étale operations on \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \); they are \( H^n_{\text{ét}}(\mathbb{A}, \mu_{\ell}^\otimes i) \)-linear combinations of monomials in the operations \( P^i \).

A slightly different approach to cohomology operations was given by May in [May], one which produces operations in the cohomology of any \( E_{\infty} \)-algebra. In Section 2, we review this approach in the context of sheaf cohomology and show in Corollary 2.3 that the étale operations constructed in this way agree with Epstein’s. This result allows us to utilize May’s treatment of the Kudo Transgression Theorem in [May, 3.4]; see Theorem 6.5 below.

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In Section 4 we combine Epstein’s construction with the Norm Residue Theorem to define motivic operations $P^n$ (see [L3]). We show they are compatible with the étale operations, and that they are stable under simplicial suspension. The operation $P^0$ is the Frobenius $H^{n,i} \rightarrow H^{n,i}$ on motivic cohomology, induced by the $\ell$th power map $\mathbb{F}_\ell(i) \rightarrow \mathbb{F}_\ell(i\ell)$; see Proposition [L4]. One new result concerning Voevodsky’s operations is that for $n > i$ and $x \in H^{2n,i}$ we have $P^n(x) = [\zeta]^{(n-1)(i-1)}x^\ell$ (see Corollary [L10]). This extends Lemma 9.8 of [V1], which states that $P^n(x) = x^\ell$ for $x \in H^{2n,n}(X)$.

The classification of motivic cohomology operations is complicated by the presence of more operations than those constructed by Voevodsky or via Steenrod-Epstein methods. One example is that an $\ell$-torsion element $t$ in the Brauer group of $k$ gives an operation $H^{1,2} \rightarrow H^{3,3}$ by

$$H^{1,2}(X) \cong H^1_{et}(X, \mu_{\ell^2}^2) \mapright{\beta} H^3_{et}(X, \mu_{\ell^3}) \cong H^{3,3}(X).$$

Also unexpectedly, we may also use $t$ and the Bockstein $\beta$ to get an operation $H^{1,2}(X) \rightarrow H^{4,3}(X)$ (see Example [S3] below). When $k$ contains a primitive $\ell$th root of unity $\zeta$, we also have an interesting operation $H^{1,2}(X) \rightarrow H^{2,1}(X) = \mathrm{Pic}(X)/\ell t$: divide by the Bott element $[\zeta] \in H^{0,1}(k)$ and then apply the Bockstein; see Proposition [S2].

In Section 7 we determine the ring of all unstable motivic cohomology operations on $H^{n,1}$. If $\ell \neq 2$, it is the twisted ring $H^{n,1} = H^\ast \otimes H^\ast_{top}(K_\ell)$, where $H^\ast \otimes H^\ast_{top}(K_\ell)$ is the motivic cohomology of $k$ and $H^\ast_{top}(K_\ell)$ is Cartan’s ring, described in Definition 0.1 below.

In Section 8 we determine the ring of unstable cohomology operations on $H^{1,i}$. When $k$ contains the $\ell$th roots of unity, this is the graded polynomial ring over $H^{1,i}$ on operations $\gamma : H^{1,i}(X) \cong H^{1,i}(X)$ and its Bockstein, where $\gamma$ is given by the Norm Residue Theorem [L2]. For general fields, it is the Galois-invariant subring. The operations on $H^{1,2}$ referred to above arise in this way.

Finally Section 9 contains a conjecture about what the general classification might be for $H^{n,i}$ when $n, i > 1$.

Since it is the topological prototype of our classification theorem, we conclude this introduction with a description of the ring of all singular cohomology operations on $H^\ast_{top}(-, \mathbb{F}_\ell)$. Serre observed that the ring of operations from $H^n_{top}(-, \mathbb{F}_\ell)$ to $H^\ast_{top}(-, \mathbb{F}_\ell)$ is isomorphic to the cohomology $H^\ast_{top}(K_\ell)$ of the Eilenberg-MacLane space $K_\ell = K(K_\ell, n)$; the structure of this ring was determined by Serre and Cartan in [C1]. The following description is taken from [McC 6.19].

**Definition 0.1.** For $\ell > 2$, let $H^\ast_{top}(K_\ell)$ denote the free graded-commutative $\mathbb{F}_\ell$-algebra generated by the elements $P^I(t_\ell)$, where $I = (\epsilon_0, s_1, \epsilon_1, ..., s_k, \epsilon_k)$ is an admissible sequence satisfying either $e(I) < n$ or $e(I) = n$ and $e_0 = 1$.

Here the *excess* of $I$ is defined to be $e(I) = 2\sum(s_i - \ell s_{i+1} - \epsilon_i) + \sum_{i=0}^k \epsilon_i$, where $s_i = 0$ for $i > k$, and $I$ is *admissible* if $s_i \geq \ell s_{i+1} + \epsilon_i$ for all $i < k$.

When $\ell = 2$, $H^\ast_{top}(K_\ell)$ denotes the free graded-commutative $\mathbb{F}_2$-algebra generated by the elements $Sq^I(t_\ell)$, with $I = (s_1, ..., s_k)$ admissible ($s_i \geq 2s_{i+1}$) and $e(I) < n$, where the excess is $e(I) = \sum(s_i - 2s_{i+1}) = s_1 - \sum_{i=1}^k s_i$.

For example, every operation on $H^2_{top}(-, \mathbb{F}_\ell)$ is a polynomial in id, $\beta$, the $P^I \beta$ and the $\beta P^I \beta$ (where $P^I = P^{\ell^k} \cdots P^{\ell^1}$). This is because the only admissible sequences with excess $< 2$ are 0, (1) and $(0, \ell, 0, \ldots, \ell, 0, 1, 1)$. 


1. Epstein’s étale construction

Cohomology operations in étale cohomology were constructed by D. Epstein long ago in the 1966 paper [E], and (for constant coefficients) made explicit by M. Raynaud [R 4.4]. Alternative constructions were later given by L. Breen [Br III.4] and J.F. Jardine [J1 1.4], [J1 §2].

In Epstein’s approach, one starts with an $\mathbb{F}_\ell$-linear tensor abelian category $A$ (such as sheaves of $\mathbb{F}_\ell$-modules on a site), a left exact functor $H^0(X, -)$ (global sections over $X$) and a commutative associative ring object $O$ of $A$. Epstein constructs operations $Sq^a : H^n(X, O) \to H^{n+a}(X, O)$ if $\ell \neq 2$, and

$$P^a : H^n(X, O) \to H^{n+2a(\ell-1)}(X, O), \quad \ell \neq 2,$$

satisfying the usual relations: $P^a x = x^\ell$ if $n = 2a$, $P^a x = 0$ if $n < 2a$, a Cartan relation for $P^a(xy)$ and Adem relations for $P^a P^b$. Epstein also defines an operation $Q^a$ for each $a$, whose degree is one more than that of $P^a$. One subtlety is that $P^0$ is not the identity but rather the Frobenius map on the $\mathbb{F}_\ell$-algebra $H^0(X, O)$.

Now suppose that $A$ is the category of étale sheaves (on the big étale site) and that $O$ is the graded étale sheaf $\oplus_{a \geq 0} \mu_a^\otimes$. We will prove in §3 below that $Q^a = \beta P^a$. With this dictionary, Epstein’s theorem specializes to yield

**Theorem 1.1.** For each odd prime $\ell$, there are additive cohomology operations

$$P^a : H^0_{et}(X, \mu^\otimes_\ell) \to H^{n+2a(\ell-1)}_{et}(X, \mu^\otimes_\ell)$$

and a Bockstein $\beta : H^0_{et}(X, \mu^\otimes_\ell) \to H^{n+1}_{et}(X, \mu^\otimes_\ell)$ satisfying the usual relations: $P^a x = x^\ell$ if $n = 2a$, $P^a x = 0$ if $n < 2a$, the Cartan relation $P^a(xy) = \sum P^a(x)P^a(y)$ and Adem relations for both $P^a P^b$ ($a < b \ell$) and $P^a \beta P^b$ ($a \leq b \ell$).

When $\ell = 2$, there are Steenrod operations $Sq^a : H^0_{et}(X, \mu^2) \to H^{n+a}_{et}(X, \mu^2)$, or $H^0_{et}(X, \mathbb{Z}/2) \to H^{n+a}_{et}(X, \mathbb{Z}/2)$, satisfying the usual relations.

**Proof.** The existence and basic properties is given in Chapter 7 of [E]: The Adem relations are established in [E 9.7–8], using the dictionary that $P^a \beta P^b = P^a Q^b$. □

Note that, although the operations multiply the weight by $\ell$, the reindexing makes no practical difference because there are canonical isomorphisms $\mu_\ell \cong \mu^\otimes_\ell$ and $\mu^\otimes_\ell \cong \mu^\otimes_\ell $. We have emphasized the twist because of our application to motivic operations below.

**Remark 1.1.1.** The operations $P^a$ are natural in $X$: if $f : X \to Y$ is a morphism of simplicial schemes then $f^*P^a = P^a f^*$. This is immediate from the naturality of the construction of $P^a$ with respect to the left exact functor $H^0_{et}(X, -)$, and also follows from [E 11.1(8)].

If $Z$ is a closed simplicial subscheme of $X$, we get cohomology operations $P^a$ on the relative groups $H^0_{et}(X, Z; \mu^\otimes_\ell)$, by replacing $H^0(X, -)$ by the left exact functor $H^0_{et}(X, Z; -)$. The same argument shows that $P^a$ is natural in the pair $(X, Z)$.

For later use, we reproduce two key results from [E]. If $\pi$ is a finite group, we write $A[\pi]$ for the category of $\pi$-equivariant objects of $A$, i.e., objects $A$ equipped with a homomorphism $\pi \to \text{End}(A)$. If $A$ is in $A[\pi]$ then $H^0(X, A)$ is a $\pi$-module, and we define the left exact functor $H^0_{et}(X, -)$ on the category $A[\pi]$ by the formula $H^0_{et}(X, A) = H^0(X, A)^\pi$. We write $H^*_\pi(X, -)$ for the derived functors of $H^0_{et}(X, -)$. 


Theorem 1.2. Let $A$ be a bounded below cochain complex of objects of $\mathcal{A}$, on which $\pi$ acts trivially. Then there is a natural isomorphism
\[ H^*(\pi, \mathbb{F}_\ell) \otimes H^*(X, A) \to H^*_\pi(X, A). \]

Proof. (See [E 4.4.4].) Let $C_* \to \mathbb{Z}$ be the standard periodic $\mathbb{Z}[\pi]$-resolution [WH 6.2.1], with generator $e_k$ of $C_k \cong \mathbb{Z}[\pi]$, and set $C^* = \text{Hom}(C_*, \mathbb{F}_\ell)$; thus $H^*(\pi, \mathbb{F}_\ell)$ is the cohomology of $(C^*)^\pi$. Choose a quasi-isomorphism $A \to I^*$ with the $I^i$ injective in $A$. Since $\pi$ acts trivially on $A$, we have quasi-isomorphisms of complexes in $\mathcal{A}[\pi]$: $A \to I^* = \mathbb{Z} \otimes I^* \to \text{Tot}(C^* \otimes I^*)$. Since each $C^n$ is a free $\mathbb{F}_\ell[\pi]$-module of finite rank, $C^n \otimes I^i \cong \text{Hom}(C_n, I^i)$ is injective in $\mathcal{A}[\pi]$. Thus $\text{Tot}(C^* \otimes I^*)$ is an injective replacement for $A$ in $\mathcal{A}[\pi]$.

By definition, $H^*_\pi(X, A)$ is the cohomology of the total complex of $H^n_\pi(X, C^* \otimes I^*) = (C^*)^\pi \otimes I^i(X)$.

The Künneth formula tells us this is the tensor product of the cohomology of $(C^*)^\pi$ and $I^i(X)$, i.e., of $H^*(\pi, \mathbb{F}_\ell)$ and $H^*(X, A)$. □

Corollary 1.3. If $A$ is a commutative algebra object, the isomorphism of Theorem 1.2 is an algebra isomorphism.

Proof. The verification that a commutative associative product on $A$ induces an algebra structure on $H^*(X, A)$ and $H^*_\pi(X, A)$ is well known. We omit the standard proof that the isomorphism above commutes with products. □

Recall that for any complex $C$, the symmetric group $S_\ell$ acts on $C^{\otimes \ell}$ by permuting factors with the usual sign change. Now suppose that $\pi$ is the cyclic Sylow $\ell$-subgroup of $S_\ell$. Choosing an injective replacement $A^{\otimes \ell} \to J^*$ in $\mathcal{A}[\pi]$, the comparison theorem [WH 2.3.7] lifts the equivariant quasi-isomorphism $A^{\otimes \ell} \to (I^*)^{\otimes \ell}$ to an equivariant map $(I^*)^{\otimes \ell} \to J^*$, unique up to chain homotopy.

Since $H^*(X, A)$ is the cohomology of $I^*(X)$, we can represent any element of $H^n(X, A)$ by an $n$-cocycle $u \in I^n(X)$. The $n\ell$-cocycle $u \otimes \cdots \otimes u$ of $I(X)^{\otimes \ell}$ is $\pi$-invariant, because the generator of $\pi$ acts as multiplication by $(-1)^{n(\ell-1)}$, which is the identity on any $\mathbb{F}_\ell$-module. Its image $P u$ in $J^{n\ell}(X)$ is also $\pi$-invariant. Epstein shows in [E 5.1.3] that $P(u + dv) = Pu + dw$ for $v \in I^{n-1}(X)$ and $w \in J^{n\ell-1}(X)$, so the cohomology class of $Pu$ is independent of the choice of cocycle $u$.

Definition 1.4. The reduced power map is defined to be the resulting map on cohomology:
\[ P : H^n(X, A) \to H^{n\ell}(X, A^{\otimes \ell}). \]

Now suppose that there is a $\pi$-equivariant map $A^{\otimes \ell} \to B$, and that $\pi$ acts trivially on $B$. (When $A$ is a commutative ring, multiplication $A^{\otimes \ell} \to A$ is a $\pi$-equivariant map.) We write $m_*$ for the induced map $H^*_A(X, A^{\otimes \ell}) \to H^*_A(X, B)$. By Theorem 1.2 $m_* P(u) \in H^*_A(X, B)$ has an expansion $\sum w_k \otimes D_k(u)$, where $w_k \in H^k(\pi, \mathbb{F}_\ell)$ are the (dual) basis elements of [SE V.5.2]: if $\ell > 2$ then $w_0 = 1$, $w_2 = \beta w_1$, $w_4 = w_2$ and $w_{2k+1} = w_1 w_3^k$. If $\ell > 2$ and $n \geq 2a$, Epstein defines
\[ P^a : H^n(X, A) \to H^{n+2a(\ell-1)}(X, B), \quad P^a u = (-1)^n \nu_n D_{(n-2a)(\ell-1)}(u), \]
where
\[ \nu_n = (-1)^r \left( \frac{\ell-1}{2} \right)^r n! \text{ and } r = \frac{(\ell-1)(n^2 + n)}{4}. \]
(See [E] 7.1, [SE] VII.6.1 and [SErr].) If \( n < 2a \) then Epstein defines \( P^a = 0 \).

When \( \ell = 2 \), Epstein defines operations \( Sq^i \) by: \( Sq^i(u) = D_{n-i}(u) \) for \( n \geq i \), and \( Sq^i(u) = 0 \) for \( n < i \).

**Remark 1.5.1.** Epstein also defines operations \( Q^a = (-1)^{a+1} \nu_n D_{(n-2a)(\ell-1)-1}(u) \) in this setting, and establishes Adem relations for them as well.

Of course, Epstein’s construction mimics Steenrod’s construction of \( D_k, P^a \) and \( Q^a \) (see [SE], VII.3.2 and VII.6.1). In Steenrod’s setting one can lift to integral cochains; with this assumption, Steenrod proves that \( \beta D_k = -D_{2k+1} \) and hence that \( \beta P^a = Q^a \); see [SE] VII.4.6 and [SErr]. We will show that the formula \( Q^a = \beta P^a \) also holds in our setting (see Lemma 3.3 and Theorem 5.11.)

**Lemma 1.6.** For all bounded below chain complexes \( A \) and \( B \) as above, each function \( P^a : H^n(X, A) \to H^{n+2a(\ell-1)}(X, B) \) is a group homomorphism.

**Proof.** Corollary 6.7 of [E] applies in this setting. \( \square \)

**Lemma 1.7.** The \( P^a \) and \( Q^a \) are natural in the map \( A'^{\otimes \ell} \to B' \).

**Proof.** Suppose we are given a commutative diagram

\[
A_1^{\otimes \ell} \xrightarrow{m} B_1 \\
\downarrow \quad \downarrow \\
A_2^{\otimes \ell} \xrightarrow{m} B_2.
\]

Applying \( H^*_\pi(X, -) \) and composing with \( P \), which is natural in \( A \) by [E] 5.1.5, Theorem 1.2 yields the commutative diagram

\[
\begin{align*}
H^*(X, A_1) &\xrightarrow{P} H^*_\pi(X, A_1^{\otimes \ell}) \xrightarrow{m} H^*_\pi(X, B_1) \xrightarrow{\cong} H^*(\pi) \otimes H^*(X, A_1) \\
&\downarrow \quad \downarrow \quad \downarrow \\
H^*(X, A_2) &\xrightarrow{P} H^*_\pi(X, A_2^{\otimes \ell}) \xrightarrow{m} H^*_\pi(X, B_2) \xrightarrow{\cong} H^*(\pi) \otimes H^*(X, A_2).
\end{align*}
\]

The result now follows from the definition (1.3) of \( P^a \) and \( Q^a \). \( \square \)

Recall that the simplicial suspension \( SX \) of a simplicial scheme \( X \) is again a simplicial scheme. There is a canonical isomorphism \( H^{n}_{et}(X, \mu^{\otimes i}_{\ell}) \xrightarrow{\cong} H^{n+1}_{et}(SX, \mu^{\otimes i}_{\ell}) \).

**Proposition 1.8.** The operations \( P^a \) are simplicially stable in the sense that they commute with simplicial suspension: there are commutative diagrams for all \( X, n \) and \( i \), with \( N = n + 2a(\ell - 1) \):

\[
\begin{align*}
H^{n}_{et}(X, \mu^{\otimes i}_{\ell}) &\xrightarrow{P^a} H^{N}_{et}(X, \mu^{\otimes i}_{\ell}) \\
&\cong H^{n+1}_{et}(SX, \mu^{\otimes i}_{\ell}) \xrightarrow{P^a} H^{N+1}_{et}(SX, \mu^{\otimes i}_{\ell}).
\end{align*}
\]

**Proof.** The proofs of Lemmas 1.2 and 2.1 of [SE] go through, using homotopy invariance of étale cohomology and excision. \( \square \)
2. May’s adjoint construction

A somewhat different approach to constructing cohomology operations was given by Peter May in [May]. Because we will need May’s version of Kudo’s Theorem (in 6.5 below), we need to know how the two constructions compare.

First, we need a chain level version of the Steenrod-Epstein function

\[ m_* P : H^n(X, A) \to H^n_{\text{ev}}(X, A) \]

used in (1.5) to define \( P^n \). We saw in [14] that the multiplication map \( m : A^{\otimes \ell} \to A \) lifts to an equivariant map \( J^* : C^* \otimes I^*(X) \) of their injective resolutions, inducing an equivariant map

\[ \hat{m} : I^{\otimes \ell}(X) \to J^*(X) \to C^* \otimes I^*(X). \]

The cohomology function \( m_* P \) is induced by the chain-level function \( u \mapsto \hat{m}(u^{\otimes \ell}) \).

The expansion \( \hat{m}(u^{\otimes \ell}) = \sum w_k \otimes D_k(u) \) in \( C^* \otimes I^*(X) \) effectively defines functions \( D_k : I^n(X) \to I^{n-\ell-k}(X) \).

Consider the isomorphism \( \phi : C^* \otimes I^*(X) \to \text{Hom}(C_*, I^*(X)) \), defined by

\[ \phi(f \otimes x)(c) = (-1)^{|x||t|} f(c)x. \]

As a special case, \( \phi(w_1 \otimes x)(e_k) = (-1)^{k|x|} \delta_{jk} x \). The composition \( \phi \hat{m} \) sends \( I^{\otimes \ell}(X) \) to \( \text{Hom}(C_*, I^*(X)) \). It is the (signed) adjoint of the map \( \phi \hat{m} \),

\[ \theta : C_\ast \otimes I^{\otimes \ell}(X) \to I^*(X), \]

which forms the basis for May’s approach; see [May, 2.1]. May defines the function \( D_k^M : I^n(u) \to I^{n-\ell-k}(X) \) by the formula

\[ D_k^M(u) = \theta(e_k \otimes u^{\otimes \ell}). \]

May defines \( P^a_M \) and \( Q^a_M \) by

\[ P^a_M(u) = (-1)^a \nu_n D^M_{n-2a} \ell(-1)^{-1}(u) \quad \text{and} \quad Q^a_M(u) = (-1)^a \nu_n D^M_{n-2a} \ell(-1)^{-1}(u) \]

(see [May, pp. 162, 182]; his \( \nu(-n) \) is our \( \nu_n \)). The sign differences in the formulas for \( P^a \) and \( P^a_M \) (and for \( Q^a \) and \( Q^a_M \)) are explained by the following calculation.

**Proposition 2.2.** For \( u \) in \( H^n(X, A) \), \( D_k^M = (-1)^k D_k \).

**Proof.** Consider the isomorphism \( \phi : C^* \otimes I^*(X) \to \text{Hom}(C_*, I^*(X)) \), defined above. The adjoint \( \theta \) of \( \phi \hat{m} \) is the composite

\[ C_\ast \otimes I(X)^{\otimes \ell} \cong I(X)^{\otimes \ell} \otimes C_\ast \phi \hat{m} \otimes 1 \text{ Hom}(C_*, I(X)) \otimes C_\ast \eta \to I(X), \]

where the first map is the signed symmetry isomorphism and \( \eta \) is evaluation. We now compute that

\[ D_k^M(u) = \theta(e_k \otimes u^{\otimes \ell}) = (-1)^{kn\ell} \eta \left( \phi[\hat{m}(u^{\otimes \ell})] \otimes e_k \right) \]

\[ = (-1)^{kn\ell} \eta \left( \phi \sum w_j \otimes D_j(u) \right) \otimes e_k \]

\[ = (-1)^{kn\ell} \sum_j \phi [w_j \otimes D_j(u)] (e_k) \]

\[ = (-1)^{kn\ell} (-1)^{k(n\ell-k)} D_k(u) \]

\[ = (-1)^k D_k(u). \]

**Corollary 2.3.** May’s operations \( P^a_M \) and \( Q^a_M \) coincide with the \( P^a \) and \( Q^a \) of (1.5) and (7.5.1).
Lemma 2.4. Set $m = (\ell - 1)/2$. Then for each $u \in I^n(X)$:

(i) $dP^a(u) = P^a(du)$ and $dQ^a(u) = -Q^a(du)$, and

(ii) if $u$ is a cocycle representing $x \in H^n(X, A)$ then $P^a(u)$ and $Q^a(u)$ are cocycles representing $P^a(x)$ and $Q^a(x)$, respectively.

Proof. In Theorem 3.1 of [May], May shows that (i) $dP^a_M(u) = P^a_M(du)$ and $dQ^a_M(u) = -Q^a_M(du)$, and (ii) if $u$ is a cocycle representing $x \in H^n(X, A)$ then $P^a_M(u)$ and $Q^a_M(u)$ are cocycles representing $P^a_M(x)$ and $Q^a_M(x)$. The result is immediate from Corollary 2.3. 

Remark 2.5. In [May1], May gave a different approach to power operations in sheaf cohomology. If $A$ is any sheaf of $\mathbb{F}_\ell$-algebras, May shows (in 3.12) that the sections over $X$ of the Godement resolution $F^\bullet A$ of $A$ yield an algebra $C^\bullet = \text{Hom}_{\Delta}(\Lambda, F^\bullet A)$ over the Eilenberg-Zilber operad $I$ on cochains of $\mathbb{F}_\ell$-modules, which therefore inherits the structure of an $E_\infty$-algebra. Since the cohomology of $C^\bullet$ is the sheaf cohomology $H^\bullet(X, A)$, the technique in [May] produces cohomology operations.
3. The étale Steenrod algebra

In this section we determine the algebra of all étale cohomology operations \( H^\ast_{\text{et}}(-, \mu^\otimes \ell) \to H^\ast_{\text{et}}(-, \mu^\otimes \ell) \) over a field \( k \) containing \( 1/\ell \).

Recall from SGA 4 (V.2.1.2 in [SGA 4]) that if \( M \) is a (simplicial) étale sheaf of \( \mathbb{F}_\ell \)-modules then the sheaf cohomology groups \( H^\ast_\text{et}(X, M) \) are isomorphic to the (hyper) Ext-groups \( \text{Ext}^\ast(\mathbb{F}_\ell[X], M) \) in the category of étale sheaves of \( \mathbb{F}_\ell \)-modules. (Here we regard \( M \) as a cochain complex using Dold-Kan.) If \( K \) is a second simplicial étale sheaf of \( \mathbb{F}_\ell \)-modules, one writes \( H^\ast_{\text{et}}(K, M) \) for \( \text{Ext}^\ast(K, M) \).

It is well known that cohomology operations \( H^\ast_{\text{et}}(-, L) \to H^\ast_{\text{et}}(-, M) \) are in 1–1 correspondence with elements of \( H^n_{\text{et}}(K, M) \), where \( K \) denotes the standard simplicial Eilenberg-Mac Lane scheme \( K(L, n) \) associated to \( L \).

We first discuss the case of constant coefficients \( (M = \mathbb{F}_\ell) \), which is known, and due to Breen [B] and Jardine [J]. The graded ring of all unstable étale cohomology operations from \( H^0_{\text{et}}(-, \mathbb{F}_\ell) \) to \( H^0_{\text{et}}(-, \mathbb{F}_\ell) \) is isomorphic to the cohomology ring \( H^0_{\text{et}}(K_n, \mathbb{F}_\ell) \), where \( K_n = K(\mathbb{F}_\ell, n) \) is the constant simplicial sheaf classifying elements of \( H^0_{\text{et}}(-, \mathbb{F}_\ell) \). By Theorem 1.1 there is a ring homomorphism from the classical unstable Steenrod algebra \( H^\ast_{\text{top}}(K_n) \) of Definition 1.2 to \( H^\ast_{\text{et}}(K_n, \mathbb{F}_\ell) \).

There is also a ring homomorphism from \( H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \to H^\ast_{\text{top}}(K_n, \mathbb{F}_\ell) \). These two ring homomorphisms do not commute, because the Bockstein and other operations \( P^I \) may be nontrivial on \( H^n_{\text{et}}(k, \mathbb{F}_\ell) \). If \( c \in H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \) then \( cP^I \) and \( P^I(c) \) send \( x \) to \( c \cdot P^I(x) \) and to \( P^I(cx) \), respectively. Nevertheless, we can define a twisted multiplication on \( H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} H^\ast_{\text{top}}(K_n) \) using the Cartan relations \( P^a \circ \alpha = \sum P^a(\alpha)P^\beta, \alpha \in H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \). We shall refer to this non-commutative algebra as the twisted tensor algebra. It is free as a left \( H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \)-module, and a basis is given by the monomials in the Steenrod operations \( P^I \) and \( P^I \) where \( I \) has excess \( < n \), exactly as in the topological case. We summarize this:

**Theorem 3.1.** The ring of étale cohomology operations on \( H^\ast_{\text{et}}(-, \mathbb{F}_\ell) \) is the twisted tensor product \( H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \otimes H^\ast_{\text{top}}(K_n) \): every operation is a polynomial in the operations \( P^I \) with coefficients in \( H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \).

**Example 3.1.1.** When \( k = \mathbb{R} \) and \( \ell = 2 \), the ring of étale cohomology operations over \( \mathbb{R} \) is the graded polynomial ring \( H^\ast_{\text{top}}(K_n)[\sigma] \) with generator \( \sigma \) in degree 1 and all \( Sq^I(i) \) with \( I \) admissible and \( e(I) < n \). This is because \( H^\ast_{\text{et}}(\mathbb{R}, \mathbb{F}_2) = \mathbb{F}_2[\sigma] \).

**Remark 3.1.2.** When \( k = \mathbb{C} \), the action of the \( P^I \) is compatible with the canonical comparison isomorphism \( H^\ast_{\text{et}}(X, \mathbb{F}_\ell) \cong H^\ast_{\text{top}}(X(\mathbb{C}), \mathbb{F}_\ell) \). This is clear from the constructions in [B] and [J].

When \( k \) contains a primitive \( \ell \)-th root of unity, the sheaves \( \mu^\otimes \ell \) are all isomorphic. Thus the ring of operations \( H^\ast_{\text{et}}(-, \mu^\otimes \ell) \to H^\ast_{\text{et}}(-, \mu^\otimes \ell) \) is isomorphic to \( H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \otimes H^\ast_{\text{top}}(K_n) \) as a left \( H^\ast_{\text{et}}(k, \mathbb{F}_\ell) \)-module. Since this is always the case when \( \ell = 2 \), we shall restrict to the case of an odd prime \( \ell \).

Now fix \( i \) and consider cohomology operations \( H^\ast_{\text{et}}(-, \mu^\otimes i) \to H^\ast_{\text{et}}(-, \mu^\otimes i) \). As observed above, they are in 1–1 correspondence with elements of \( H^\ast_{\text{et}}(K, \mu^\otimes i) \), where \( K \) denotes the simplicial Eilenberg-Mac Lane scheme \( K(\mu^\otimes i, n) \). For example, the identity operation on \( H^\ast_{\text{et}}(-, \mu^\otimes i) \) corresponds to \( \iota_n \in H^n(\mu^\otimes i, n) \), and the étale Bockstein \( \beta : H^\ast_{\text{et}}(X, \mu^\otimes i) \to H^{n+1}_{\text{et}}(X, \mu^\otimes i) \) corresponds to \( \beta(\iota_n) \in H^{n+1}(K, \mu^\otimes i) \).

Fix a field \( k \) with \( 1/\ell \in k \), and let \( G \) be the Galois group of the extension \( k(\zeta)/k \), where \( \zeta \) denotes a primitive \( \ell \)-th root of unity. Then \( G \) is cyclic of order
etale sheaf
mand for our problem is to determine the ring  
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where In étale cohomology, Lemma 3.3. Because the Frobenius is the identity on 
now show that Epstein’s operation

\[\text{Sq} \]

Q
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Finally, we invoke the Adem relation

H
Finally, we invoke the Adem relation

\[\text{Sq} \]

H
H
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LEmMMA 3.3. In étale cohomology,  \(Q^0 = \beta\) and \(Q^a = \beta P^a\) for \(a > 0\).

Proof. We first consider the case when \(\zeta \in k\), so that \(\mu_{\ell / k}^{\otimes i} \cong \mathbb{F}_{\ell}\) for all \(i\). Jardine’s argument in [J, pp. 108–114] that Epstein’s \(\text{Sq}^1\) is the Bockstein when \(\ell = 2\) applies when \(\ell \geq 2\) as well, and proves that Epstein’s \(Q^0\) is the Bockstein operation. The identity \(Q^0 = \beta\) in the general case follows from this and the isomorphism (3.2):

\[
\begin{array}{c}
H_{et}^n(X, \mathcal{O}) \\
\cong \\
H_{et}^{n+1}(X, \mathcal{O})
\end{array}
\]

\[
\begin{array}{c}
Q^n \rightarrow \\
\cong \\
H_{et}^n(X(\zeta), \mathbb{F}_{\ell}) \\
Q^n \rightarrow \\
H_{et}^{n+1}(X(\zeta), \mathbb{F}_{\ell}).
\end{array}
\]

Finally, we invoke the Adem relation \(Q^0 P^b = Q^b P^0\) [E 9.8(4)] to get \(\beta P^b = Q^b\). □

Using the Bockstein and Epstein’s operations \(P^a\), we have operations \(P^i\) defined on \(H_{et}^*(\mu_{\ell / k}^{\otimes i})\) for every admissible sequence \(i\) in the sense of Definition 0.1.

In order to classify all operations on \(H_{et}^n\), we first consider the case \(n = 1\). In topology, the ring of operations on \(H^1(\mathbb{F}_{\ell})\) is \(H_{top}^*(K_1) \cong \mathbb{F}_v[u, v]/(u^2)\), where \(u = P^0\) is in degree 1, corresponding to the identity operation, and \(v\) is in degree 2, corresponding to the Bockstein operation. By Theorem [L] there is a canonical map from \(\mathbb{F}_v[u, v]/(u^2)\) to étale cohomology operations from \(H_{et}^1(\mu_{\ell / k}^{\otimes i})\) to \(H_{et}^*(\mu_{\ell / k}^{\otimes i})\), sending \(u\) to the identity and \(v\) to the Bockstein \(\beta: H_{et}^1(\mu_{\ell / k}^{\otimes i}) \to H_{et}^2(\mu_{\ell / k}^{\otimes i})\).

For any \(i\), the basechange \(\mu_{\ell / k}^{\otimes i}(\zeta)\) of the algebraic group \(\mu_{\ell / k}^{\otimes i}\) is isomorphic to \(\mathbb{F}_{\ell}(\zeta)\), the constant sheaf \(\mathbb{F}_\ell\) on the big étale site of \(k(\zeta)\). The induced isomorphism \((B\mu_{\ell / k}^{\otimes i})(\zeta) \cong (BF_{\ell})(\zeta)\) induces an isomorphism of cohomology groups, which immediately yields the following calculation.

**Proposition 3.4.** The graded algebra of cohomology operations from \(H_{et}^1(\mu_{\ell / k}^{\otimes i})\) to \(H_{et}^*(\mu_{\ell / k}^{\otimes i})\) is isomorphic to the \(H_{et}^*(\mu_{\ell / k}^{\otimes i} (\zeta), \mathbb{F}_{\ell})\)-module

\[
H^*(B\mu_{\ell / k}^{\otimes i}, \mathcal{O}) \cong H^*(B\mu_{\ell / k}^{\otimes i} (\zeta), \mathbb{F}_{\ell}) \cong H^*(k(\zeta), \mathbb{F}_{\ell}) \otimes \mathbb{F}_v[u, v]/(u^2), \beta(u) = v.
\]
Every operation on $H^1_{et}(\cdot, \mu^\otimes_\ell)$ is uniquely a sum of operations $\phi(x) = cx^\varepsilon x(x)^n$, where $c \in H^*_c(k, \mu^\otimes_\ell)$ and $\varepsilon \in \{0, 1\}$.

Proposition 3.4 is the case $n = 1$ of the following result.

**Theorem 3.5.** For each $i$ and $n \geq 1$, the ring of all étale cohomology operations from $H^n_{et}(\cdot, \mu^\otimes_\ell)$ to $H^*_c(k, \mathcal{O})$ is the free left $H^*_c(k(\zeta), \mathbb{F}_\ell)$-module $H^*_c(k(\zeta), \mathbb{F}_\ell) \otimes \tilde{H}_{top}(K_n)$.

The operations $P^i$ send $H^n_{et}(\cdot, \mu^\otimes_\ell)$ to $H^*_c(\cdot, \mu^\otimes_\ell)$. Thus the operations from $H^n_{et}(\cdot, \mu^\otimes_\ell)$ to $H^*_c(\cdot, \mu^\otimes_{\ell+1})$ are isomorphic to $H^*_c(\cdot, \mu^\otimes_\ell) \otimes \tilde{H}_{top}(K_n)$.

**Proof.** We first show that the basechange $K(\mu^\otimes_\ell, n) \times_k \text{Spec}(k(\zeta))$ is the space $K(\mu^\otimes_\ell, n)$ over $k(\zeta)$. This is clear for $n = 0$, and follows inductively from the construction of $K(A, n+1)$ via the bar construction on $K(A, n)$, together with the observation that $(X \times_k Y) \times_k \text{Spec}(k(\zeta))$ is $X(\zeta) \times_{k(\zeta)} Y(\zeta)$.

By (32), the cohomology of $K(\mu^\otimes_\ell, n)$ with coefficients in $\mathcal{O}$ is the same as the cohomology of $K(\mu^\otimes_\ell, n) \times_k \text{Spec}(k(\zeta))$ with coefficients in $\mathbb{F}_\ell$. The Breen-Jardine result, Theorem 3.1, shows that this is $H^*_c(k(\zeta), \mathbb{F}_\ell) \otimes \tilde{H}_{top}(K_n)$. $\square$

### 4. Motivic Steenrod operations

In this section we construct operations $P^n$ on the motivic cohomology groups $H^{n,i}(X) = H^{n,i}(X, \mathbb{F}_\ell)$, $n \geq 2a$, compatible with the operations $P^n$ in étale cohomology in the sense that there are commutative diagrams

\[
H^{n,i}(X, \mathbb{F}_\ell) \xrightarrow{P^n} H^{n+2a(\ell-1),i}(X, \mathbb{F}_\ell)
\]

(4.1)

Let $\alpha_*$ denote the direct image functor from the étale site to the Nisnevich site. If $\mathcal{F}$ is any étale sheaf then we may regard $A = R\alpha_*\mathcal{F}$ as a complex of Nisnevich sheaves such that $H^*_n(X, R\alpha_*\mathcal{F}) \cong H^*_n(X, \mathcal{F})$.

The following theorem, due to Voevodsky and Rost, is sometimes known as the Beilinson Conjecture; it is equivalent to the Norm Residue Theorem; see [SV], [V4], [Wy], [HW]). Let $\tau^{\leq i}A$ denote the good truncation of $A$ in cohomological degrees at most $i$; $H^n(\tau^{\leq i}A)$ is $H^i(A)$ for $i \leq n$, and zero for $n > i$. (cf. [WH] 1.2.7).

**Norm Residue Theorem 4.2.** For any $X$ smooth over a field of characteristic $\neq \ell$, the map $F_\ell(i) \to \tau^{\leq i}R\alpha_*\mu^\otimes_\ell$ is a quasi-isomorphism, and hence

$H^{n,i}(X, \mathbb{F}_\ell) \cong H^*_n(X, \tau^{\leq i}R\alpha_*\mu^\otimes_\ell)$.

In particular, if $n \leq i$ then $H^{n,i}(X, \mathbb{F}_\ell) \cong H^*_n(X, \mu^\otimes_\ell)$.

Explicitly, if $\mathcal{F}$ is an étale sheaf and $\mathcal{F} \to I^*$ is an injective resolution, then $\tau^{\leq i}R\alpha_*\mathcal{F}$ is represented by $\tau^{\leq i}\alpha_*I^*$. It is quasi-isomorphic to a chain complex $I^*_{\text{nis}}$ of injective Nisnevich sheaves on $X$ with $I^*_{\text{nis}} = \alpha_*I^*$ for $n \leq i$, because each $\alpha_*I^n$ is an injective Nisnevich sheaf and $Z^n(\alpha_*I)$ injects into $\alpha_*I^n$. Taking $\mathcal{F} = \mu^\otimes_\ell$, the theorem states that $H^{n,i}(X, \mathbb{F}_\ell) = H^*_n(X, \mathbb{F}_\ell(i))$ is the $n^\text{th}$ cohomology of $I^*_{\text{nis}}(X)$. 

Now the product on $\mu^\otimes_\ell$ yields a product $R\alpha_*\mu^\otimes_\ell \otimes R\alpha_*\mu^\otimes_\ell \to R\alpha_*\mu^\otimes_{i+j}$, unique up to chain homotopy, by the comparison theorem. The induced pairing $\tau^S R\alpha_*\mu^\otimes_{i} \otimes \tau^S R\alpha_*\mu^\otimes_{j} \to \tau^S R\alpha_*\mu^\otimes_{i+j}$ induces the product in motivic cohomology, by [SV 7.1]. We may choose a model for the $\ell$-fold product map $(R\alpha_*\mu^\otimes_{i})^\otimes \ell \to R\alpha_*\mu^\otimes_{i\ell}$ which is equivariant for the permutation action of the cyclic group $\pi$, by choosing a $\pi$-equivariant replacement $R\alpha_*\mu^\otimes_{i\ell} \simeq J$ and factoring through an equivariant map $(R\alpha_*\mu^\otimes_{i})^\otimes \ell \to J$. The truncation of this map is also equivariant:

$$(\tau^S R\alpha_*\mu^\otimes_{i})^\otimes \ell \to \tau^S (R\alpha_*\mu^\otimes_{i})^\otimes \ell \to \tau^S R\alpha_*\mu^\otimes_{i\ell}.$$ 

Setting $A = \oplus_{i=0}^\infty \tau^S R\alpha_*\mu^\otimes_{i}$, we have a graded map $A^\otimes \longrightarrow A$, representing the product in motivic cohomology. Composing the power map $P$ of Definition 4.3 the map $m$ and the isomorphism of Theorem 1.2 we have a graded reduced power map on $H^{n,i}(X, A) = \oplus H^{n,i}(X, F_\ell)$:

$$H^n(X, A) \overset{P^n}{\longrightarrow} H^n(X, A) \overset{m}{\longrightarrow} H^n(X, A) \cong \oplus H^{n-k}(X, A) \otimes H^k(\pi, F_\ell).$$

**Definition 4.3** ($P^n$). The function $P^n : H^{n,i}(X) \to H^{n+2a-1,i\ell}(X)$ is defined as follows. Given $u \in H^{n,i}(X)$, $m_*P(u)$ has an expansion $\sum D_k u \otimes w_k$, where $w_k \in H^k(\pi, F_\ell)$ are as before. If $\ell \neq 2$ and $n \geq 2$, we define $P^n u$ to be $(-1)^a \nu_n$ times $D_{n-2a}(\ell-1) u$, where the constant $\nu_n$ is given by the formula in (1.5). If $n < 2a$ we define $P^n u = 0$. By Lemma 1.7 each $P^n$ is in fact a homomorphism.

We call the $P^n$ **motivic cohomology operations**; they are natural in $X$, by the argument of Remark 1.11 applied to the power map on $H^{n,i}(X, A)$.

If $\ell = 2$ we define $S^{a} : H^{n,i}(X) \to H^{n+a,2i}(X)$ to be $D_{n-a}$ for $n \geq a$, and $S^{a} = 0$ for $n < a$. This follows Steenrod and Epstein. Thus $S^2a = P^n$. We will show in Theorem 5.1 below that $S^{2a+1} = \beta S^2a$.

**Remark 4.3.1.** These motivic cohomology operations are almost surely the operations defined by Kriz and May in [KM 1.7.2], and by Joshua in [Jo] §8; cf. [BJ].

**Lemma 4.4.** The motivic cohomology operations $P^n$ are compatible with the étale cohomology operations $P^n$ in the sense that the diagram (4.4) commutes.

**Proof.** By construction, the following diagram commutes:

$$(\tau^S R\alpha_*\mu^\otimes_{i\ell})^\otimes \ell \longrightarrow \tau^S (R\alpha_*\mu^\otimes_{i\ell})^\otimes \ell \longrightarrow (R\alpha_*\mu^\otimes_{i\ell})^\otimes \ell \longrightarrow m \longrightarrow R\alpha_*\mu^\otimes_{i\ell}.$$

The commutativity of (4.4) now follows from Lemma 1.7.

We now show that these operations enjoy familiar properties.

**Proposition 4.5.** If $u \in H^{2n,i}(X)$ then $P^n u = u^\ell$.

**Proof.** By [E 5.2.1], $j_* : H^2(X, R) \to H^*(X, R)$ sends $P u$ to $u \times \cdots \times u$. Since $w_0 = 1$, the proof in [E 6.3, 7.3] applies.
Lemma 4.6. Let \( \pi \) and \( \rho \) be finite subgroups of \( \Sigma_\ell \). Then their action on \( R^{\otimes \ell} \) induces a commutative diagram (whose vertical maps come from Theorem 1.2):

\[
\begin{array}{ccc}
H^*(\pi) \otimes H^*(\rho) \otimes H^*(X, R) & \otimes H^*(X, R) & \otimes H^*(X, R) \\
\downarrow & \downarrow & \downarrow \\
H^*_\pi(X, R) \otimes H^*_\rho(X, R) & \longrightarrow & H^*_\pi \times \rho(X, R) \\
\end{array}
\]

Proof. The proof of Lemma 4.4.3 of [E], which concerns the underlying chain complexes rather than the cohomology groups, goes through.

Theorem 4.7 (Cartan Formula). Let \( u \in H^{n+1}(X) \) and \( v \in H^{m+1}(Y) \). Then in \( H^{n+(i+j)\ell}(X \times Y) \) we have:

\[
P^\alpha(u \cup v) = \sum_{s+t=\alpha} P^s(u) \cup P^t(v), \quad \ell > 2,
\]

and \( Sq^\alpha(u \cup v) = \sum_{s+t=\alpha} Sq^s(u) \cup Sq^t(v) \) when \( \ell = 2 \).

Proof. Epstein’s proof in [E 7.2] carries over. In more detail, replacing [E 4.4.3] by our Lemma 4.6 in the proof of Lemma 7.2.2 in [E], we obtain the formulas

\[
D_a(u \cup v) = \sum_{s+t=\alpha} \pm D_s u \cup D_t v,
\]

where \( s \) and \( t \) cannot both be odd. By [E 6.4], if \( n \) is even (resp., odd) then \( D_s(u) = 0 \) unless \( s = m(\ell - 1) \) or \( m(\ell - 1) - 1 \) for some even (resp., odd) integer \( m \geq 0 \). The Cartan Formula follows by inspection of the signs involved.

Remark 4.7.1. Epstein also establishes a Cartan formula for the operations \( Q^a \), including the formula \( Q^0(uv) = (Q^0(u)(P^0v)) + (-1)^{|u|}(P^0u)(Q^0v) \). We omit these formulas, as they follow from Theorem 4.7 and the formulas for \( Q^a \) in Theorem 8.7 below.

We now turn to the Adem relations. Recall that by convention \( \binom{n}{k} \) is zero if \( k < 0 \). Thus the sums below run over \( \ell \leq a/\ell \).

Theorem 4.8 (Adem Relations). If \( \ell > 2 \) and \( a < b \ell \) then

\[
P^a P^b = \sum_{s+t=b} (-1)^{a+t} \binom{(\ell - 1)s - 1}{a - t\ell} P^{a+s}P^t;
\]

\[
P^a \beta P^b = \sum_{s+t=b} (-1)^{a+t} \binom{(\ell - 1)s}{a - t\ell} \beta P^{a+s}P^t + (-1)^{a+t} \binom{(\ell - 1)s - 1}{a - t\ell - 1} \beta P^{a+s} \beta P^t.
\]

Proof (Epstein). We refer to section 9 of [E], whose running assumption is that the operations \( P^a \) and \( Q^a \) are zero on \( H^{*+*} \) for \( a < 0 \). This assumption holds by [E 8.3.4], using the adjunction for sheaves in [E 11.1]. In particular, \( P^a \) and \( Q^a \) vanish on \( H^{*+*}_{\Sigma_\ell} \) by [E 9.1]. The proof of [E 9.3] only requires an equivariant map \( A^{\otimes \ell} \rightarrow A \), so 9.3 and its Corollary 9.4 of loc. cit. remain valid in the motivic setting. Since 9.2, 9.5 and 9.6 of [E] are formally true, we can now quote the proof of [E 9.8]: the proof of the Adem relations on pp. 119–122 of [SE], as amended by the Errata, carry over to this setting.
Remark 4.8.1. When \( \ell = 2 \), Epstein points out in [E, 9.7] that, given the modifications in the proof of [4.8 above, the usual Adem relations hold by the proof on p. 119 of [SE]: if \( a < 2b \) then
\[
Sq^a Sq^b = \sum_{a + s = b} \binom{s - 1}{a - 2t} Sq^{a + s} Sq^t.
\]

Remark 4.8.2. The usual Adem relations for \( Q^a Q^b \) and \( Q^a P^b \), stated in [E, 9.8], are also valid in motivic cohomology. These relations follow immediately from Theorem 4.8 given the formulas for \( Q \) in Theorem 5.11 below.

Bistable Operations 4.9. In [V1], Voevodsky defines (bistable) cohomology operations \( P^n \) on \( H^{n,i}(\cdot, F_2) \) of bidegree \( (2a\ell - 1, a(\ell - 1)) \). These satisfy: \( P^n x = x \) for all \( x \); \( P^n : x = x^s \) for \( x \in H^{2a, a}(X, F) \); \( P^n = 0 \) on \( H^{n,i} \) if \( i < 2 \). The usual Adem relations hold when \( \ell > 2 \). The cohomological degrees of \( P^a \) and \( P^n \) are the same, namely \( 2a\ell - 1 \), but the weights differ if \( a \neq i \): if \( a < i \) then \( P^a \) has lower weight, but if \( a > i \) then \( P^a \) has higher weight.

When \( \ell = 2 \), Voevodsky’s operations \( Sq^{2a} \) have bidegree \( (2i, i) \) and \( Sq^{2a+1} = \beta Sq^{2a} \). They satisfy a modified Cartan formula [V1, 9.7] which differs from our Cartan formula (Theorem 4.7) by the presence of a factor of \( c \) in some terms.

Remark 4.9.1. Brosnan and Joshua have observed in [B3, 2.1] and [B3I, 1.1(iii)] that the motivic-to-étale map sends \( P^n \) to \( P^n \) and \( Sq^a \) to \( Sq^a \). The key is to observe that Voevodsky’s total power operation [V1, 5.3] is compatible with Epstein’s reduced power map (Definition 4.3 above).

Cohomology operations on \( H^{n,0} \) are easy to describe because of the following characterization.

Lemma 4.10. Let \( A \) be any abelian group. If \( X_* \) is a smooth simplicial scheme, the motivic cohomology ring \( H^{\bullet, 0}(X_*, A) \) is isomorphic to the topological cohomology \( H^{\star, 0}(\pi_0 X_*, A) \) of the simplicial set \( \pi_0 X_* \).

Proof. For smooth connected \( X \) we have \( H^{n,0}(X, A) = H^{n}_{\text{min}}(X, A) \) for \( n > 0 \) and \( H^{0,0}(X, A) = A \), almost by definition; see [MVW, 3.4]. Hence the spectral sequence \( E^2_{p,q} = H^0(X_p, A) \Rightarrow H^{p+q, 0}(X) \) degenerates to the cohomology of the chain complex \( \text{Hom}(\pi_0 X_*, A) \), which is \( H^{\star, 0}(\pi_0 X_*, A) \). For a simplicial set \( K \) such as \( \pi_0 X_* \), the construction of the product in motivic cohomology [MVW, 3.11] shows that \( H^{\star, 0}(K) \cong H^{\star, 0}(K) \) is an isomorphism of rings.

Corollary 4.11. The ring of motivic cohomology operations on \( H^{n,0}(\cdot, F_2) \) is isomorphic to \( H^{\star, *}(k, F_2) \otimes H^{\star, 0}(K^n) \).

If \( K \) is a simplicial set, the isomorphism \( H^{\star, 0}(K, F_2) \cong H^{\star, 0}(K, F_2) \) is compatible with the action of the \( P^I \). This is clear from Lemma 4.4 and Remark 5.1.2.

Example 4.11.1. Let \( \Delta^1 \) denote the simplicial 1-simplex and \( s \in H^{1,0}(\Delta^1, \partial \Delta^1) \) the generator. By the above comparison with topology, \( P^0(s) = s \). By definition, \( P^a(s) = 0 \) for \( a > 0 \).

Recall that the simplicial suspension \( SX \) of a pointed simplicial scheme \( X \) is again a simplicial scheme. Multiplication by the element \( s \) of Example 4.11.1 induces a canonical isomorphism \( H^{a}_{et}(X, \mu_{\ell}^{\otimes i}) \cong H^{n+1}_{et}(SX, \mu_{\ell}^{\otimes i}) \). (Compare to Lemmas 1.2 and 2.1 of [SE].)
Proposition 4.12. The motivic operations $P^a$ are simplicially stable in the sense that they commute with simplicial suspension: there are commutative diagrams for all $X$, $n$ and $i$, with $N = n + 2a(\ell - 1)$:

$$
\begin{array}{ccc}
H^{n,i}(X) & \xrightarrow{P^a} & H_{et}^{N,i\ell}(X) \\
\cong & & \cong \\
H^{n+1,i}(SX) & \xrightarrow{P^a} & H^{N+1,i\ell}(SX).
\end{array}
$$

Proof. By the Cartan formula 4.7, $P^a(sx) = P^0(s)P^a(x) = s \cdot P^a(x)$. □

Example 4.13. Consider the classifying space $K_{\mathbb{F}_\ell} = K(\mathbb{F}_\ell(i), n)$ for $H^{n,i}(-, \mathbb{F}_\ell)$. If $n \geq i$, we see from [V3, 3.27] that the summands of smallest weight or degree in $F_{\ell,ir}(K_{\mathbb{F}_\ell})$ are $\mathbb{F}_\ell(i)[n]$ and $\mathbb{F}_\ell(i)[n + 1]$. It follows that: $H^{a,*}(K_{\mathbb{F}_\ell}) = H^{a,*}$ for $a < n$; $H^{n,*}(K_{\mathbb{F}_\ell}) \cong H^{0,*} \oplus H^{n,*}$ on the tautological class $i$ in $H^{n,i}(K_{\mathbb{F}_\ell})$ and constant maps to elements of $H^{n,*}(k, \mathbb{F}_\ell)$; and $H^{0,0} \oplus H^{1,*} \oplus H^{n+1,*} \xrightarrow{\cong} H^{n+1,*}(K_{\mathbb{F}_\ell})$ by the map $(c, c', c'') \mapsto c\beta(i) + c'\ell + c''$. That is, there are no cohomology operations $H^{n,i} \to H^{p,q}$ with $p \leq n$ except for constant operations and $\mathbb{F}_\ell$-linear terms when $p = n$ (multiples of the identity plus a constant), and no operations $H^{n,i} \to H^{n+1,q}$ other than the Bockstein, multiplication by elements of $H^{1,*}(k)$, and constants.

If $n < i$, this is no longer the case. In Example 5.2 below, we show that there is a weight-reducing operation $H^{1,2} \to H^{2,1}$ for all $k$, and a weight-preserving operation $H^{1,2} \to H^{3,2}$ for most $k$. For another example, suppose that $\zeta \in k$ and $n \leq i$. Then cupping with $[\zeta] \in H^{0,1}(k)$ is an isomorphism by Theorem 4.2; its inverse (defined when $n < i$) is an operation $H^{n,i} \to H^{n,i-1}$. 
5. The operations $P^0$ and $Q^0$

Sometimes we can deduce motivic operations from étale operations. For example, if $n \leq i$ (and hence $n \leq i\ell$) then the diagram 1.1 allows us to identify the motivic operation $P^0 : H^{n,i}(X) \to H^{n,i}(X)$ with the étale operation $P^0 : H^a_{\text{ét}}(X, \mu^a_{\ell}) \cong H^a_{\text{ét}}(X, \mu^a_{\ell})$, and thus conclude that $P^0$ is an isomorphism in this range. The same reasoning, using the Norm Residue Theorem 2.2 shows that if $n \leq i$ and $n + 2\ell(i - 1) \leq i\ell$, the motivic and étale operations $P^0$ agree on $H^{n,i}(X) \cong H^a_{\text{ét}}(X, \mu^a_{\ell})$, and also agree with $b^{(i-a)(i-1)/d} P^0_V$, where $b \in H^{0,d}(k)$ is defined as follows.

Fix a primitive $\ell$th root of unity, $\zeta$, in an extension field of $k$; this choice determines a canonical generator $[\zeta]$ of $H^0(k(\zeta), \mu_{\ell})$. If $[k(\zeta) : k] = d$ then $H^{0,d}(k) \cong H^a_{\text{ét}}(k, \mu^a_{\ell})$, and the element $[\zeta]^d = [\zeta^\otimes d]$ descends to a canonical “periodicity” element $b$ in $H^{0,d}(k)$. (If $d = 1$ then $b = [\zeta]$.) Note that multiplication by $b$ is a map from $H^{n,i}(X)$ to $H^{n,i+d}(X)$; by Theorem 4.2, it is an isomorphism when $i \geq n$. By construction, this is the map in cohomology induced by the change-of-truncation map

$$F(i) \cong \tau^{\leq i} R\alpha_* \mu_{\ell}^a \to \tau^{\leq i+d} R\alpha_* \mu_{\ell}^a \cong \mathcal{F}_\ell(i + d),$$

associated to the isomorphism of étale sheaves $\mu_{\ell}^a \to \mu_{\ell}^a \otimes b$ sending the generator $\zeta^\otimes$ to the generator $\zeta^\otimes + d$.

We can formulate this in the motivic derived category $\mathcal{D}M$, using the étale-to-Nisnevič change of topology map $\alpha$. Recall from [MVW] 10.2 that $H^a_{\text{ét}}(X, \mu^a_{\ell})$ is isomorphic to

$$\text{Hom}_{\mathcal{D}M}(\alpha_* \mathcal{F}_{\ell,i} X, \mathcal{F}_{\ell,i}[n]) \cong \text{Hom}_{\mathcal{D}M}(\mathcal{F}_{\ell,i} X, R\alpha_* \mu^a_{\ell}).$$

The map $\mathcal{F}_{\ell}(i) = \tau^{\leq i} R\alpha_* \mu_{\ell}^a \to R\alpha_* \mu_{\ell}^a$ is compatible with the map $\mathcal{F}_{\ell}(i) \to \mathcal{F}_{\ell}(i + d) \to R\alpha_* \mu_{\ell}^a \otimes i + d$, so it factors through a map $\mathcal{F}_{\ell}(i)[1/b] \to R\alpha_* \mu^a_{\ell}$, where $\mathcal{F}_{\ell}(i)[1/b]$ denotes the (homotopy) colimit in $\mathcal{D}M$ of

$$\mathcal{F}_{\ell}(i) \to \mathcal{F}_{\ell}(i + d) \to \mathcal{F}_{\ell}(i + 2d) \to \cdots \to \mathcal{F}_{\ell}(i + jd) \to \cdots.$$

The following calculation is originally due to Levine [L].

**Theorem 5.2.** For each $i$, $\mathcal{F}_{\ell}(i)[1/b] \to R\alpha_* \mu^a_{\ell}$ is an isomorphism in $\mathcal{D}M$.

For $X$ smooth and all $n$, $H^{n,i}(X)[1/b] \to H^a_{\text{ét}}(X, \mu^a_{\ell})$ is an isomorphism.

**Proof.** Any complex $C$ is the homotopy colimit of the change-of-truncation maps $\tau^{\leq n} C \to \tau^{\leq n+1} C$. For $C = R\alpha_* \mu^a_{\ell}$, this yields the first assertion. The second assertion is an immediate consequence of this and the fact that $\mathcal{F}_{\ell,i} X$ is a compact object in $\mathcal{D}M$, so $\text{Hom}_{\mathcal{D}M}(\mathcal{F}_{\ell,i} X, -)$ commutes with homotopy colimits. ⊓⊔
Our next goal is to compare $P^0$ to the cohomology of the change-of-truncation map $\tau^{\leq i} R\alpha_* \mu^\otimes_i \to \tau^{\leq it} R\alpha_* \mu^\otimes_i$ of (5.1).

**Lemma 5.3.** The Frobenius map $\mathbb{F}_\ell(i) \xrightarrow{\Phi} \mathbb{F}_\ell(it)$ in motivic cohomology is chain homotopic to the change-of-truncation map

$$\tau^{\leq i} R\alpha_* \mu^\otimes_i \to \tau^{\leq it} R\alpha_* \mu^\otimes_i \cong \mathbb{F}_\ell(it).$$

The Frobenius $H^{n,i}(X) \xrightarrow{\Phi} H^{n,it}(X)$ is multiplication by $b^{i(e-1)/d} = [\zeta^{\otimes i(e-1)}]$.

**Proof.** The Frobenius endomorphism is the identity on the étale sheaf of rings $O \cong \oplus_{i=1}^d \mu^\otimes_i$, so if we fix $i$ and an injective replacement $\mu^\otimes_i \to I$, the Frobenius on $\mu^\otimes_i$ lifts to a map $f_i : I \to I$ which is chain homotopic to the identity. Since the product in motivic cohomology is induced from the product on $R\alpha_* \mu^\otimes_i = \alpha_* I$, the Frobenius in motivic cohomology is represented by the good truncation in degrees at most $it$ of the composite $\tau^{\leq i} \alpha_* I \subset \alpha_* I \xrightarrow{f_i^{\otimes i}} \alpha_* I$. Since good truncation preserves chain homotopy, it is chain homotopic to the canonical map $\tau^{\leq i} \alpha_* I \subset \tau^{\leq it} \alpha_* I$.

The final assertion follows from (5.1). $\square$

**Proposition 5.4.** The map $P^0 : H^{n,i}(X) \to H^{n,it}(X)$ is multiplication by $b^{i(e-1)/d}$. Equivalently, $P^0$ is the cohomology of the change-of-truncation map

$$\tau^{\leq i} R\alpha_* \mu^\otimes_i \to \tau^{\leq it} R\alpha_* \mu^\otimes_i.$$

**Proof.** Recall that the Godement resolution $F \to S^*(F)$ is a functorial simplicial resolution of any sheaf $F$ by flasque sheaves. Letting $S^*_i$ denote the total complex of the Godement resolution of $\tau^{\leq i} R\alpha_* \mu^\otimes_i$, it follows that the product on $\oplus \tau^{\leq i} R\alpha_* \mu^\otimes_i$ induces a product pairing $S^n_i \otimes S^n_j \to S^n_{i+j}$ for all $n$. In particular, the Frobenius on $R\alpha_* \mu^\otimes_i$ induces a map $S^n_i \to S^n_{it}$.

In [E] 11.1, Epstein shows that the Godement resolution satisfies the conditions of his section 8. By functoriality, the equivariant map $(R\alpha_* \mu^\otimes_i)^{\otimes t} \to R\alpha_* \mu^\otimes_{it}$ constructed after Theorem 5.2 lifts to an equivariant map $(S^*_i)^{\otimes t} \to S^*_{it}$. This is the analogue of [E] 8.3.2, and is exactly what we need in order for the proof of [E] 8.3.4 to work. Thus if we represent $v \in H^{n,i}(X)$ by a cocycle $u$ in the algebra $H^0(X, S^n A)$, then $P^0 v$ is represented by the element $u^t$ of $H^0(X, S^n A)$. Therefore $P^0$ is represented by the Frobenius. $\square$

**Example 5.4.1.** Recall that $b \in H^{0,d}(k)$. Since $P^0(b) = b^t$ (by 5.3), the Cartan formula (4.7) yields $P^n(bx) = b^t P^n(x)$.

Recall from [V3] 2.60 that a split proper Tate motive is a direct sum of Tate motives $L^j[i]$ with $j \geq 0$. If the weights $i$ are at least $n$ then we say the motive has weight $\geq n$. Note that the cohomology of $\mathbb{L}^j[i]$ is a free bigraded $H^{*,*}$-module with a generator in bidegree $(2i + j, i)$.

It follows that we have a Künneth formula (see [W] 4.11): if $\mathbb{F}_\ell,tr(Y)$ is a split proper Tate motive then $H^{*,*}(Y)$ is a free bigraded $H^{*,*}$-module, and

$$H^{*,*}(X \times Y) \cong H^{*,*}(X) \otimes_{H^{*,*}} H^{*,*}(Y).$$

**Example 5.6.** Let $K = K(F_\ell(i), n)$ be the Eilenberg-MacLane space classifying $H^{n,i}(-, F_\ell)$; if $n \geq 2i \geq 0$ then $M = F_\ell,tr(K)$ is a split proper Tate motive of weight $\geq i$, by [V3] 3.28. It follows that $H^{*,*}(K^{\otimes p})$ is the $p$-fold tensor product of $H^{*,*}(K)$ with itself over $H^{*,*}$. 

Recall from [LMW 3.1] that \( F_{\ell}(i)[n] \) is represented by the abelian presheaf \( F_{\ell, tr}(C_m) \) so \( F_{\ell}(i)[n] \) is represented by the simplicial abelian presheaf associated to \( F_{\ell, tr}(C_m)[n - i] \) when \( n \geq i \). From the adjunction

\[
\text{Hom}_{\text{Hot}}(X, u F_{\ell}(i)[n]) \cong \text{Hom}_{DM}(F_{\ell, tr}(X), F_{\ell}(i)[n]) = H^{n, i}(X),
\]

we see that the classifying space \( K(F_{\ell}(i), n) \) of \( H^{n, i}(-, F_{\ell}) \) is the simplicial abelian presheaf \( G = u F_{\ell}(i)[n] \) underlying \( F_{\ell}(i)[n] \); see [V3 p. 5].

**Lemma 5.7.** If \( F_{\ell, tr}(Y) \) is a split proper Tate motive then multiplication by \( b^c \) is an injection from \( H^{p, q}(Y, F_{\ell}) \) into \( H^{p, q + de}(Y, F_{\ell}) \), and hence into \( H^{p,q}(Y, F_{\ell})[1/b] \).

**Proof.** It suffices to consider \( F_{\ell, tr}(Y) = L^i[j] \). There is no harm in increasing \( c \) so that \((\ell - 1)de\). Set \( p' = p - 2i - j \) and \( q' = q - i \). Since \( H^{p, q}(Y) \) is a free \( H^{*, q}(k) \)-module, the assertion for \( H^{p, q}(Y) \) amounts to the assertion that either \( H^{p', q'}(k) = 0 \) (and injectivity is obvious) or else \( 0 \leq p' \leq q' \) and \( H^{p', q'}(k) \cong H^q(k, \mu^\otimes q') \). In the latter case, we also have \( H^{p', q + de}(k) \cong H^q(k, \mu^\otimes q') \) and the isomorphism is induced from the isomorphism \( \mu^\otimes q' \cong \mu^\otimes q + de \).

In the next Proposition, we write \( K \) for \( K(F_{\ell}(i), n) \). For each \( p \) and \( q \), there is a canonical map \( H^{p, q}(K, F_{\ell}) \to H^{p, q}(K, \mu^\otimes q) \). It sends the operations \( P^n \) of Definition 4.3 to the étale operations \( P^n \) of Theorem 1.1.

**Proposition 5.8.** If \( n \geq 2i \), the canonical map is an injection, from the set \( H^{p, q}(K, F_{\ell}) \) of motivic cohomology operations \( H^{n, i} \to H^{p, q} \) to the set \( H^{p, q}(K, \mu^\otimes q) \) of étale cohomology operations \( H^{n, i}(\cdot, \mu^\otimes i) \to H^{p, q}(\cdot, \mu^\otimes q) \).

**Proof.** By the usual transfer argument, we may assume that \( \zeta \in k \). Let \( K \) denote the Eilenberg-MacLane space classifying \( H^{n, i}(-, F_{\ell}) \). By [V3 3.28], \( F_{\ell, tr}(K) \) is a split proper Tate motive. By Lemma 5.7 and Levine’s Theorem 5.2 \( H^{p, q}(K, F_{\ell}) \) injects into \( H^{p, q}(K, F_{\ell})[1/b] \cong H^{p, q}(K, \mu^\otimes q) \). Thus the group \( H^{p, q}(K, F_{\ell}) \) of motivic cohomology operations injects into the group \( H^{p, q}(K, \mu^\otimes q) \) of étale cohomology operations.

Recall that \( d = [k(\zeta) : k] \). By abuse of notation, if \( d|m \) we write \( \zeta^m \) for the element \( b^{m/d} \) of \( H^{0, m}(K) \) defined at the start of this section.

**Corollary 5.9.** Suppose that \( n \geq 2i \) and \( n \geq 2a \). Then for \( x \in H^{n, i}(X) \):

1. If \( a \leq i \), \( P^a(x) = [\zeta]^{(i-a)(\ell-1)} P_v^a(x) \);
2. If \( a \geq i \), \( P^a(x) = [\zeta]^{(a-i)(\ell-1)} P^a(x) \).

**Proof.** (Cf. [13 Thm 1.1]) The two sides have the same bidegree, and agree with \( P^a(x) \) in étale cohomology by Lemma 4.4 and Remark 4.9.1.

**Corollary 5.10.** If \( n \geq i \) and \( x \in H^{2n, i} \) then \( P^a(x) = [\zeta]^{(n-i)(\ell-1)} x^\ell \).

**Proof.** This is the case \( a = n \) of Corollary 5.9 as \( P^n(x) = x^\ell \) (Proposition 4.3).

**Theorem 5.11.** The motivic operations \( Q^a \) on \( H^{n, i} \) are related to the Bockstein \( \beta \) by \( Q^a(x) = b^{(\ell-1)/d} \beta(x) \) and \( Q^a = \beta P^a \) for \( a > 0 \).

**Proof.** Set \( K = K(F_{\ell}(i), n) \), so that motivic cohomology operations \( H^{n, i} \to H^{p,q} \) correspond to elements of \( H^{p, q}(K) \). In particular, the identity on \( H^{n, i} \) is represented by the canonical element \( \iota \) of \( H^{n, i}(K) \), and the motivic cohomology operation
\[Q^0 - b^{(\ell-1)/d} \beta \] is represented by the element \(Q^0(\iota) - b^{(\ell-1)/d} \beta(\iota)\) of \(H^{n+1,\iota}(K)\).

The map \(H^{n+1,\iota}(K) \rightarrow H^{n+1}_e(K, \mu_{\iota}^{\otimes \iota})\) is an isomorphism if \(n < \iota\) by Theorem 4.2 and is an injection if \(n \geq 2\iota\) by Proposition 5.8. By Lemma 4.4, we have a commutative diagram

\[
\begin{array}{ccc}
H^{n,\iota}(K) & \xrightarrow{\delta^0} & H^{n+1,\iota}(K) \\
\downarrow & & \downarrow \\
H^{n,\iota}_{et}(K, \mu_{\iota}^{\otimes \iota}) & \xrightarrow{\delta^0} & H^{n+1}_{et}(K, \mu_{\iota}^{\otimes \iota}).
\end{array}
\]

The bottom map is zero by Lemma 3.3. It follows that \(Q^0 = b^{(\ell-1)/d} \beta\). Similarly, if we set \(N = n + a(\ell - 1) + 1\) then we have a diagram

\[
\begin{array}{ccc}
H^{n,\iota}(K) & \xrightarrow{\delta^0} & H^{N,\iota}(K) \\
\downarrow & & \downarrow \\
H^{n,\iota}_{et}(K, \mu_{\iota}^{\otimes \iota}) & \xrightarrow{\delta^0} & H^{N}_{et}(K, \mu_{\iota}^{\otimes \iota}).
\end{array}
\]

The right vertical is an isomorphism by the Norm Residue Theorem 4.2. the upper right horizontal map is an injection by Proposition 5.4 and Lemma 5.7 and the lower left horizontal map is zero by Lemma 3.3. It follows that \(Q^\alpha = \beta P^\alpha\). \(\square\)

6. Borel’s Theorem

In order to go from \(H^{1,*}\) to \(H^{n,*}\), we need a slight generalization of Borel’s theorem \([McC\ 6.21]\), one which accounts for the coefficient ring \(H^{*,*} = H^{*,*}(\text{Spec} \, k)\).

**Definition 6.1.** Let \(H^*\) be a graded-commutative \(\mathbb{F}_\ell\)-algebra. If \(W^*\) is a graded \(H^*\)-algebra, an \(\ell\text{-simple system of generators of } W^* \) over \(H^*\) is a totally ordered set of elements \(x_i\), such that \(W^*\) is a free left \(H^*\)-module on the monomials \(\prod x_i^{m_i}\), where the \(i\)'s are in order and \(0 < m_i < \ell\) (with \(m_i \leq 1\) if \(\deg(x_i)\) is odd).

**Theorem 6.2.** Let \(H^*\) be a graded-commutative \(\mathbb{F}_\ell\)-algebra with \(H^0 = \mathbb{F}_\ell\), and suppose that \(\{E^r, d_r\}\) is a 1st-quadrant spectral sequence of graded-commutative \(H^\cdot\) -algebras converging to \(H^\cdot\). Set \(V^* = E_2^*, W^* = E_2^{0,*}\), and suppose that \((i)\) \(E_2^{*,0} \cong W^* \otimes_H V^*\) as algebras, and that \((ii)\) the \(H^\cdot\)-algebra \(W^*\) has an \(\ell\)-simple system of generators \(\{x_i\}\), each of which is transgressive.

Then \(V^*\) is the tensor product of \(H^\cdot\) and a free graded-commutative \(\mathbb{F}_\ell\)-algebra on generators \(y_i = \tau(x_i)\) and (when \(\ell \neq 2\) and \(\deg(x_i)\) is even) \(z_j = \tau(y_j \otimes x_j^{\ell-1})\). (Here \(\tau\) is the transgression.)

**Proof.** The proof of Borel’s Theorem in \([McC\ 6.21]\) goes through. \(\square\)

We use the bar construction to form the bisimplicial classifying spaces \(B_* G\) (with \(G^p\) in simplicial degree \(p\)) and \(E_* G\) (with \(G^{p+1}\) in simplicial degree \(p\)). We write \(\pi\) for the canonical projection \(E_* G \rightarrow B_* G\). The Leray spectral sequence becomes

\[
E_2^{p,q} = H^p(B_* G, R^q \pi_* A) \Rightarrow H^{p+q}(E_* G, A).
\]
Proposition 6.4. Suppose that $G$ is a connected simplicial sheaf of groups on $T$ and $A$ is a sheaf of $\mathbb{F}_l$-algebras satisfying the K"unneth condition that

$$H^*(U, A) \otimes_{H^*(T, A)} H^*(G, A) \rightarrow H^*(U \times G, A)$$

is an isomorphism for all $U$. Then the Leray spectral sequence (6.3) satisfies condition (i) of Borel’s Theorem with $E_2^{p,q} = H^p(B_*G, A) \otimes_{H^*(T, A)} H^q(G, A)$.

Proof. For simplicity of notation, let us write $\otimes_H$ for $\otimes_{H^*(T, A)}$. We first claim that the higher direct images $R^q\pi_*(A)$ are $A \otimes_H H^*(G, A)$. To see this, recall that $R^q\pi_*(A)$ is the sheafification of the presheaf that to a map $U \rightarrow B_pG$ associates $H^q(\pi^{-1}U, A)$, where $\pi^{-1}U = E_*G \times_{B_pG} U$ is $U \times G$. By hypothesis, $H^*(\pi^{-1}U, A)$ is $H^*(U, A) \otimes_H H^*(G, A)$. The claim follows, since sheafification commutes with $\otimes_H H^*(G, A)$ and the sheaf associated to $H^q(-, A)$ is $A$ if $q = 0$ and zero for $q > 0$.

Thus we have $E_2^{p,q} = H^p(B_*G, A) \otimes_H H^q(G, A)$. Since $H^0(B_*G, A) = H^0(T, A)$, we have

$$E_2^{0,q} = H^0(B_*G, A) \otimes_H H^q(G, A) = H^q(G, A).$$

Because $G$ is connected, $H^0(G, A) = H^0(T, A)$ and hence $E_2^{0,0} = H^0(B_*G, A)$. The fact that the spectral sequence is multiplicative follows from the fact that $A$ is a sheaf of algebras, and the work of Massey [1].

Kudo’s Theorem 6.5. Suppose $G$ and $A$ satisfy the hypotheses of Proposition 6.4. If $x \in H^n(G, A)$ transgresses to $y \in H^{n+1}(B_*G, A)$ then

1. $\beta(x)$ transgresses to $-\beta(y)$;
2. $P^n(x)$ transgresses to $P^n(y)$; and
3. if $n = 2a$ then $x^{\ell-1} \otimes y$ transgresses to $-Q^n(y)$.

Any simplicially stable operation commutes with the transgression; see [McC 6.5]. Hence part (2) of Theorem 6.5 is immediate whenever we know that $P^n$ is simplicially stable. This is so for the operations $P^n$ in étale and motivic cohomology (by [LS] and [14.12]).

Proof. (Cf. [May 3.4]) As in the proof of Theorem 1.2, we fix a quasi-isomorphism $A \rightarrow I^\bullet$. Let $f = \pi^*$ and $g = i^*$ be the canonical maps $I(G) \rightarrow I(E_*G) \rightarrow I(B_*G)$ coming from $G \rightarrow E_*G \rightarrow B_*G$. The assertion that $x$ transgresses to $y$ means that there is a cocycle $b$ in $H^{n+1}(B_*G)$ representing $y$, and an element $u$ in $I^n(E_*G)$, such that $f(b) = du$ and $g(u)$ is a cocycle representing $x$.

Since the Bockstein satisfies $g(\beta u) = \beta g(u)$ and $f(\beta b) = \beta(du) = -d(\beta u)$, we see that $\beta(x)$, which is represented by $g(\beta u)$, transgresses to $-\beta(y)$.

Recall from Section 2 that $b$ and $u$ determine a cocycle $P^n(b)$ in $I^*(B_*G)$ representing $P^n(y)$ and an element $P^n(u)$ in $I^*(E_*G)$ so that $P^n(x)$ is represented by $P^n g(u) = g P^n(u)$. By Lemma 2.1 we have

$$f P^n(b) = P^n f(b) = P^n(du) = d P^n(u).$$

It follows that $P^n(x)$ transgresses to $P^n(y)$.

Since $b$ is a cocycle, $Q^n(b)$ represents $Q^n(y)$, and by Lemma 2.1 we have

$$f Q^n(b) = Q^n f(b) = Q^n(du) = -d(Q^n u).$$

Thus the class of $Q^n(u)$ transgresses to $-Q^n(y)$, and it suffices to show that $Q^n(u)$ represents $x^{\ell-1} \otimes y$ under the isomorphism $E_2^{p,q} \cong H^p(B_*G) \otimes H^q(G)$ of 6.4.
Recall from (1.5) that \(\nu_n = (-1)^{r} m!^{-n}\), where \(m = (\ell - 1)/2\). We have \(\nu_n = (-1)^{n}\), because \(n = 2a\), \((m!)^2 = (-1)^{m+1}\) and \(r \equiv am \pmod{2}\). We now follow p. 167 of [Max] up to (9). Starting from \(u \in I^n(X)\), May produces elements \(t_i\) in \(I^\otimes l(X)\) and a family of elements \(\{c_a\}, \{c'_a\}\) in \(C_\ast \otimes I^\otimes l(X)\), depending naturally on \(u\), such that

\[
Q_M(u) = (-1)^n \nu(1 - n) \theta(c'_a) = m! \theta(c'_a).
\]

The proof of the terms in \(c'_a\) on top of p. 171 of [Max] shows that there is a term \(c''\) such that \(c' - d(c'')\) is \((-1)^m m! z\) plus terms mapped by \(\theta\) into lower parts of the filtration, where \(z = e_0 \otimes u \otimes \cdots \otimes u \otimes du\), and that \(\theta(z)\) represents \(x(t-1) \otimes y\). Therefore, up to terms in lower parts of the filtration we have \(Q_M(u) = m! \theta(c'_a) = (-1)^m (m!)^2 \theta(z) = -\theta(z)\). Since we saw in Corollary 2.3 that \(Q^s(u) = Q_M(u)\), the result follows.

We illustrate the use of Proposition 6.4 with the étale topology. First, consider the étale sheaf \(G = \mu_t\). If \(\mu_t\) is connected then it does not satisfy the Kunneth condition of Proposition 6.3 for \(U = \text{Spec}(k)\). Indeed, \(H^q(\mu_t, \mathbb{F}_r) = \mathbb{F}_r\) yet \(H^0(G \times \text{Spec}(k), \mathbb{F}_r) = \bigoplus_1 \mathbb{F}_r\). However, things change if we consider the étale sheaf \(\mathcal{O} = \bigoplus_{i=0}^{d-1} \mu_t^{\otimes i}\) of Section 3.

**Lemma 6.6.** \(H^s_{\text{et}}(X \times \mu_t^{\otimes i}, \mathcal{O}) \cong H^s_{\text{et}}(X, \mathcal{O}) \otimes_{H^*(k, \mathcal{O})} H^s_{\text{et}}(\mu_t^{\otimes i}, \mathcal{O})\).

**Proof.** As an étale sheaf of \(\mathbb{F}_r\)-modules, constant over \(k\), \(\mathbb{F}_r[\mu_t^{\otimes i}]\) is a direct sum of the locally constant sheaves \(\mu_t^{\otimes \alpha}\), each of which is an invertible object. Because \(\mathbb{F}_r[X \times \mu_t^{\otimes i}] \cong \mathbb{F}_r[X] \otimes \mathbb{F}_r[\mu_t^{\otimes i}], H^s_{\text{et}}(X \times \mu_t^{\otimes i}, \mathbb{F}_r)\) equals

\[
\text{Ext}^n(\mathbb{F}_r[X] \otimes \mathbb{F}_r[\mu_t^{\otimes i}], \mu_t^{\otimes q}) \cong \text{Ext}^n(\mathbb{F}_r[X], \mathcal{R}\text{Hom}(\mathbb{F}_r[\mu_t^{\otimes i}], \mu_t^{\otimes q}))
\]

\[
\cong \text{Ext}^n(\mathbb{F}_r[X], \mathcal{R}\text{Hom}(\mathbb{F}_r, \mu_t^{\otimes q-\alpha}))
\]

\[
\cong \text{Ext}^n(\mathbb{F}_r[X], \mu_t^{\otimes q-\alpha}) \cong \oplus_{\alpha} H^s_{\text{et}}(X, \mu_t^{\otimes q-\alpha}).
\]

The pairing \(H^s_{\text{et}}(X, \mathcal{O}) \otimes_{\otimes \mathcal{O}} H^s_{\text{et}}(\mu_t^{\otimes i}, \mathcal{O}) \rightarrow H^s_{\text{et}}(X \times \mu_t^{\otimes i}, \mathcal{O})\) is the direct sum over \(\alpha, s\) and \(t\) of the top row in the commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^s(\mathbb{F}_r[X], \mu_t^{\otimes i}) \otimes_{\mathbb{F}_r} \text{Ext}^s(\mu_t^{\otimes \alpha}, \mu_t^{\otimes i}) & \longrightarrow & \text{Ext}^s(\mathbb{F}_r[X] \otimes \mu_t^{\otimes \alpha}, \mu_t^{\otimes i + t}) \\
\cong & & \cong \\
\text{Ext}^s(\mathbb{F}_r[X], \mu_t^{\otimes i}) \otimes_{\mathbb{F}_r} \text{Ext}^s(\mu_t^{\otimes t-\alpha}, \mu_t^{\otimes i}) & \longrightarrow & \text{Ext}^s(\mathbb{F}_r[X], \mu_t^{\otimes t+\alpha}).
\end{array}
\]

Since \(H^s_{\text{et}}(k, \mathcal{O}) = \text{Ext}^s(\mathbb{F}_r, \mathcal{O}) \cong \oplus_{\alpha} \text{Ext}(\mathbb{F}_r, \mu_t^{\otimes t-\alpha})\) for each \(\alpha\), setting \(s = q - \alpha\) and summing over \(s\) and \(t\) yields the result.

**Corollary 6.7.** If \(Y\) is a coproduct of schemes which are finite products of \(\mu_t^{\otimes i}\), then

\[
H^s_{\text{et}}(X \times Y, \mathcal{O}) \cong H^s_{\text{et}}(X, \mathcal{O}) \otimes_{H^*(k, \mathcal{O})} H^s_{\text{et}}(Y, \mathcal{O}).
\]

**Example 6.8.** The ring of all étale cohomology operations from \(H^2_{\text{et}}(-, \mu_t^{\otimes i})\) to \(H^1_{\text{et}}(-, \mu_t^{\otimes s})\) is the free left \(H^s_{\text{et}}(k, \mathcal{O})\)-module on generators in \(H^s_{\text{et}}(K_{2}, \mu_t^{\otimes i})\): monomials in the identity \((\alpha \in H^s_{\text{et}}(K_{2}, \mu_t^{\otimes i}), \beta\), the \(P^1\) basis and the \(P^1\) basis \((P^1 = P^{1'} \cdots P^{1''})\).

This result, proven in Theorem 6.3 above, can also be obtained from the Leray spectral sequence (6.3).

Each term in the simplicial sheaf \(B_{\alpha}\mu_t^{\otimes i}\) is a coproduct of products of \(\mu_t^{\otimes i}\), so Corollary 6.7 and Proposition 6.4 imply that the Leray spectral sequence satisfies
Proposition 3.4 as \( \iota \) of generators: \( u \) condition (i) of Borel’s Theorem 6.2. The explicit description of \( H^{n+1}_c(K_2, \mathcal{O}) \) as \( \beta(u) \) satisfies, and Borel’s Theorem states that \( H^{n+1}_c(K_2, \mathcal{O}) \) is the free graded-commutative \( H^{*,*} \)-algebra on generators \( \iota \in H^{2,2}(K_2) \),

\[
y_\nu = \tau(x_\nu) \in H^{2
u+1}(K_2, \mu_\ell^{\otimes u^\nu}) \quad \text{and} \quad z_\nu = \tau(x_{\nu-1} \otimes y_\nu) \in H^{2\nu+2}(K_2, \mu_\ell^{\otimes u^{\nu+1}}).
\]

Note that \( y_0 = \beta(u) \). Since \( x_{\nu+1} = x_{\nu}^\nu = P^{\nu^\nu} x_{\nu} \), Kudo’s Theorem 6.5(2) and an inductive argument show that \( y_{\nu+1} \) is \( P^{\nu^\nu} y_\nu \) and also \( P^{\nu^\nu} \cdots P^1 \beta \). This completes the proof for \( \ell = 2 \).

For \( \ell > 2 \), it remains to show that \(-z_\nu \) is \( \beta P^{\nu^\nu} y_\nu \) = \( \beta P^{\nu^\nu} \cdots P^1 \beta \). This follows from Kudo’s Theorem 6.5(3) and Lemma 3.3.

7. Motivic operations on weight 1 cohomology

We now turn to natural operations defined on the motivic cohomology groups with weight 1, i.e., \( H^{n,1}(X) = H^{n,1}(X, \mathbb{F}_\ell) \). We begin with the case \( n = 1 \).

Let \( \mu_\ell \) be the group scheme of \( \ell^\text{th} \) roots of unity. On pp.130–131 of [MV], Morel and Voevodsky define a simplicial Nisnevich sheaf \( B_{et}\mu_\ell \) and observe that it classifies the étale cohomology group \( H^{1,1}_c(-, \mu_\ell) \), and hence the motivic group \( H^{1,1} \) by Theorem 4.2 in the sense that \( [X_+, B_{et}\mu_\ell] \cong H^{1,1}(X) \) for every smooth simplicial scheme \( X \) over \( k \).

Following [VI, p.17], we write \( B_{et}\mu_\ell \) for the geometric classifying space of \( \mu_\ell \), constructed in [MV, p.133] (where the notation \( B_{gm}\mu_\ell \) was used). By [MV, 4.2.7], \( B_{et}\mu_\ell \) is \( k \)-equivalent to \( B_{et}\mu_\ell \), so it also classifies \( H^{1,1} \).

When \( \ell = 2 \), the generator \([c]\) of \( H^{0,1}(k) = \mu_2(k) \) and its Bockstein, the element \([-1] = H^{1,1}(k) = k^* / k^{*\ell} \), play an important role.

**Proposition 7.1.** There are elements \( u \in H^{1,1}(B_{et}\mu_\ell) \), \( v \in H^{2,1}(B_{et}\mu_\ell) \) such that

\[
H^{*,*}(B_{et}\mu_\ell) \cong \begin{cases} 
H^{*,*}(k) \otimes \mathbb{F}_\ell[u, v]/(u^2), & \ell \neq 2 \\
H^{*,*}(k) \otimes \mathbb{F}_\ell[u, v]/(u^2 + [-1]u + [c]v), & \ell = 2.
\end{cases}
\]

Thus every cohomology operation on \( H^{1,1}(X) \) is uniquely a sum of the operations \( x \mapsto c x^\nu \beta(x)^m \), where \( c \in H^{p,2}(k) \), \( m \geq 0 \) and \( 0 \leq \nu \leq 1 \).

**Proof.** This is the special case \( F_* = S^0 \) in Proposition 6.10 of [VI]. Note that the operation \( c x^\nu \beta(x)^m \) has bidegree \( (s + 2m + \nu - 1, j + m + \nu - 1) \). \( \square \)

As in Example 6.8, we can use this as the starting point to describe all motivic operations on \( H^{*,*} \). For example, the motive of \( G = B_{et}\mu_\ell \) is a split proper Tate motive, so \( E_1 \) holds. By Proposition 6.2 the Leray spectral sequence has the form

\[
E_2^{p,q} = H^{p,*,(k, \mathbb{F}_\ell)} H^{q,*,(B_{et}\mu_\ell)} \Rightarrow H^{p+q,*,(E, B_{et}\mu_\ell)} = H^{p+q,*,(k)}.
\]

**Corollary 7.3.** If \( \ell \neq 2 \), the ring of cohomology operations on \( H^{2,1} \) is the tensor product of \( H^{*,*}(k, \mathbb{F}_\ell) \) and the free graded-commutative algebra generated by: the identity of \( H^{2,1} \), the Bockstein \( \beta \), the \( P^1 \beta \) and the \( \beta P^1 \beta \) where \( P^1 = P^v \cdots P^1 P^1 \).
For $\ell = 2$, the ring of cohomology operations on $H^{2,1}_r$ is the tensor product of $H^{*,*}_r(k, \mathbb{F}_2)$ and the free graded-commutative algebra generated by: the identity of $H^{2,1}_r$, $Sq^1$, ..., $Sq^1$, where $Sq^1 = Sq^2 \cdots Sq^3 Sq^1$.

Proof. Since the simplicial scheme $B_* B\mu_\ell$ is $K(\mathbb{F}_\ell(1), 2)$, we merely need to compute its motivic cohomology using (7.2).

By Proposition (7.1) $H^{*,*}_r(B\mu_\ell)$ has an $\ell$-simple system of generators over $H^{*,*}_r$ consisting of $u$ and the $x_\nu = u^{c_\nu}$ for $\nu \geq 0$. Since (5.5) holds, Proposition 8.1 implies that the hypotheses of Borel’s Theorem 6.2 and Kudo’s Theorem 6.5 hold for (7.2). Therefore $H^{*,*}_r(B\mu_\ell)$ is the tensor product of $H^{*,*}_r$ and the free graded-commutative $\mathbb{F}_\ell$-algebra on generators $\nu$, $y_\nu = \tau(x_\nu)$ and $z_\nu$ if $\ell > 2$. Since $u$ transgresses to $t$, $x_0 = v = \beta(u)$ and $x_{\nu+1} = P^{\ell} x_\nu$, Kudo’s Theorem 6.5 implies (by induction) that $y_0 = \beta(t)$, $y_{\nu+1} = P^{\ell} y_\nu = P^{\ell} \beta(t)$ and (using Theorem 5.11) that $z_\nu = -\beta P^{\ell} \beta(t)$.

To describe cohomology operations on $H^{n,1}_r$, we use the algebra $H^{n,1}_r(K_n)$, defined in (1.1). We bigrade it by giving it the weight grading that $P^I$ has weight $\ell^k - 1$, where $I = (\epsilon_0, \epsilon_1, \epsilon_2, \ldots, s_k, \epsilon_k)$.

Theorem 7.4. For each $n \geq 1$, the ring of all motivic cohomology operations on $H^{n,1}_r$ is isomorphic to the free left $H^{*,*}_r$-module $H^{*,*}_r(k) \otimes H^{n,1}_r(K)$ in which the $P^I$ are bigraded according to Definition (7.3).

Thus every cohomology operation on $H^{n,1}_r(X)$ is a sum of the operations $x \mapsto c(P^I x)(P^{I_2} x) \cdots (P^{I_\ell} x)$, where $c \in H^{*,*}_r(k)$ and each $I_j$ satisfies the excess condition of Definition (7.4).

Proof. We proceed by induction on $n$, the cases $n = 1, 2$ being given above. Set $K_n = K(\mathbb{F}_\ell(1), n)$, so $K_{n+1} = B_*(K_n)$, and suppose inductively that the algebra $H^{*,*}_r(K_n)$ is given as described in the theorem, so that it has an $\ell$-simple system of generators consisting of the $P^I(t_n)$ with $I$ admissible and $\epsilon(I) < n$ (or $\epsilon(I) = n$ and $\epsilon_1 = 1$), and $\ell^\nu$ powers of the $P^I(t_n)$ of even degree.

Since $\mathbb{F}_\ell K_n$ is a split proper Tate motive by [V3, 3.28], the Künneth condition (5.5) of Proposition 6.4 holds. Hence the hypotheses of Borel’s Theorem 6.2 are satisfied and the Leray spectral sequence (5.6) has the form

$$E_2^{p,q} = H^{p,*,*}(K_{n+1}) \otimes_{H^{*,*}_r(k)} H^{q,*,*}(K_n) \Rightarrow H^{p+q,*,*}(K_n) \cong H^{p+q,*,*}(k).$$

Therefore $H^{*,*}_r(K_{n+1})$ is the tensor product of $H^{*,*}_r$ and a free graded-commutative $\mathbb{F}_\ell$-algebra on certain generators; it remains to establish that they are the ones describe in the theorem. But, except for weight considerations, this is exactly the same as in the topological case, as presented on p. 200 of [McC]. Of course, the weight of the $x_I = P^I(t_n)$ is the same as the weight of $y_I = P^I(t_{n+1})$. Inspection of the weights of the new generators $P^{\ell^1} \cdots P^{\ell s} y_I$ (when $x_I$ has degree $2s$) shows that each additional $P^{\ell^s}$ multiplies the weight by $\ell$, as required. \qed
8. Motivic operations on degree 1 cohomology

We now turn to operations defined on $H^{1,*}$. Here we encounter new cohomology operations arising from the Norm Residue Theorem \ref{thm:nrt}, representing a negative twist. Here are a couple of examples.

**Example 8.1.** There are operations $H^{1,r(r-1)}(X) \xrightarrow{\sim} H^{1,1}(X)$, since both groups are naturally isomorphic to $H^1_{et}(X, \mu_r)$. An element $\eta \in H^1_{et}(k, \mu_r^{\otimes -1})$ determines a natural transformation $H^{1,1}(X) \to H^{2,2}(X)$.

**The case** $k = k(\zeta)$. If $k$ contains a primitive $\ell^h$ root of unity $\zeta$, the classification is immediate from Proposition \ref{prop:unit-root}. Let $[\zeta]$ be the class of $\zeta$ in $H^{0,1}(k) \cong \mu_\ell$.

**Proposition 8.2.** Suppose that $\zeta \in k$ and $i > 1$. Then there is a natural isomorphism $\gamma : H^{1,i}(X) \xrightarrow{\sim} H^{1,1}(X)$, and $[\zeta]^{i-1} \cup \gamma(x) = x$.

Every motivic cohomology operation on $H^{1,i}$ is uniquely a sum of the operations $x \mapsto (c(\gamma x))^{\beta}(\gamma x)^\ell$, where $c \in H^{*,*}(k)$, $0 \leq \varepsilon \leq 1$ and $m \geq 0$.

**Proof.** By Theorem 4.2 $H^{1,i}(X) \cong H^1_{et}(X, \mu_i^{\otimes i})$ for all $i > 0$. Since multiplication by $[\zeta]^{i-1}$ is an isomorphism between $H^1_{et}(X, \mu_i)$ and $H^2_{et}(X, \mu_i^{\otimes i})$, its inverse isomorphism $\gamma$ is natural. Via $\gamma$, operations on $H^{1,i}$ correspond to operations on $H^{1,1}$, which are described in Proposition \ref{prop:unit-root}.

For example if $i \geq 2$ and $\eta \in H^1_{et}(k, \mu_i^{\otimes 2-i})$ then the operation $H^{1,i} \to H^{2,2}$ of Example \ref{ex:unit-root} is the operation $x \mapsto c(\gamma x)$ of Proposition 8.2 where $c = \eta - [\zeta]^{i-1}$.

**Remark 8.2.1.** If $c \in H^{*,*}(k)$ then $\phi(x) = c(\gamma x)^{\varepsilon}(\beta \gamma x)^m$ is a cohomology operation of bidegree $(s + \varepsilon + 2m - 1, j + \varepsilon + m - i)$. In particular $\gamma$ is a cohomology operation of bidegree $(0, 1, i)$.

**Galois descent.** We now consider the situation in which $\mu_\ell \not\subset k$. Clearly, not all cohomology operations defined over $k(\zeta)$ are defined over $k$. However, some of these operations do descend, such as those in Example \ref{ex:unit-root}.

It is convenient to consider the étale cohomology of $k$ as being bigraded, by integers $n \geq 0$ and $i \in \mathbb{Z}$, with $H^0_{et}(k, \mu_i^{\otimes i})$ in bidegree $(n, i)$. Thus the motivic cohomology ring $H^{*,*}(k)$ is a bigraded subring of $H^*_{{et}}(k, \mu_\ell^{\otimes \ast})$.

**Definition 8.3.** For each integer $b$, let $\zeta^{-b}H^{*,*}(k)$ denote the direct sum of all $H^*_{{et}}(k, \mu_i^{\otimes b})$ with $0 \leq s \leq t + b$. This is a bigraded $H^{*,*}(k)$-submodule of $H^*_{{et}}(k, \mu_i^{\otimes \ast})$. It is a cyclic module if and only if $\zeta^b \in k$, when it is the $H^{*,*}(k)$-submodule generated by $[\zeta^b] \in H^0_{et}(k, \mu_i^{\otimes -b})$.

**Theorem 8.4.** Fix an integer $i \geq 2$. Then the ring of cohomology operations on $H^{1,i}$ is the direct sum of copies of $\zeta^{-b}H^{*,*}(k)$, $b = (i-1)(\varepsilon + m)$, over integers $m \geq 0$ and $\varepsilon \in \{0, 1\}$ If $0 \leq s \leq t + b$, the operation corresponding to $c \in H^s_{et}(k, \mu_i^{\otimes i})$, $m$ and $\varepsilon$ sends $H^{1,i}(X)$ to $H^{s+i+2m, i+s+i+m}(X)$: $\phi([\zeta]^{i-1} \cup y) = (\zeta^b \cup c)y^\varepsilon \beta(y)^m$.

**Proof.** Let $G$ denote the Galois group of $k(\zeta)/k$. Since $H^{*,*}(X)$ is the $G$-invariant summand of $H^{*,*}(X(\zeta))$, a motivic operation $H^{1,i}(X) \to H^{*,*}(X)$ is the same thing as a $G$-invariant operation $H^{1,1}(X) \to H^{*,*}(X(\zeta))$. Given $x \in H^{1,1}(X)$, there is a unique $y \in H^{1,1}(X(\zeta))$ so that $x = [\zeta]^{i-1} \cup y$, where $[\zeta] \in H^{0,1}(k(\zeta))$. By Proposition 8.2 we are reduced to determining when $G$ acts trivially on $c^\varepsilon y^\varepsilon \beta(y)^m$.
Since $g^\ell(\beta y)^m$ is in the summand of $H^{*,*}(k(\zeta))$ which is isotypical for $\mu_{\ell}^{\otimes b}$, this holds if and only if $c'$ is in the summand of $H^{*,j}(k(\zeta))$ which is isotypical for $\mu_\ell^{\otimes b}$.

By \cite{322}, there is a unique $c \in H^{*,j-b}(k)$ so that $c' = [\zeta]^b \cup c$. \hfill $\Box$

\textbf{Example 8.5} ($b = 1$). An element $c \in H_3^1(k,\mathbb{F}_\ell)$ determines operations $C : H^{1,2}(X) \to H^{2,1}(X)$ and $\phi : H^{1,2}(X) \to H^{3,2}(X)$. If $y \in H^{1,1}(X(\zeta))$ is such that $x = [\zeta] \cup y$ then, regarding $\zeta c$ as an element of $H^{1,1}(k(\zeta))$, we have $C(x) = (\zeta c) y$ and $\phi(x) = (\zeta c) \beta(y)$. Of course, we can identify $C$ with the map $H^1_\text{et}(X,\mu_\ell) \xrightarrow{\zeta \cup} H^2_\text{et}(X,\mu_\ell)$.

An element $t$ in $H_3^1(k,\mu_\ell)$ (the $\ell$-torsion subgroup of the Brauer group of $k$) determines operations $H^{1,2}(X) \to H^{3,3}(X)$ and $H^{1,2}(X) \to H^{4,3}(X)$. Writing $x = [\zeta] \cup y$ in $H^{1,2}(X(\zeta))$, the operations followed by the inclusion $H^{*,*}(X) \subset H^{*,*}(X(\zeta))$ send $x$ to $([\zeta] \cup t) y$ and $([\zeta] \cup t) \beta(y)$, respectively. As mentioned in the introduction, we can identify the first operation with $H^1_\text{et}(X,\mu_\ell^{\otimes 2}) \xrightarrow{\zeta \cup} H^3_\text{et}(X,\mu_\ell^{\otimes 3})$.

9. Conjectural matter

In the preceding two sections we have classified motivic cohomology operations on $H^{n,i}$ when $n = 1$ or $i = 1$. We have also classified operations whose targets lie inside the “étale zone” where $n \leq i$. We know little about the intermediate zone where $i < n < 2i$. In this section we make some guesses about operations in the “topological zone” where $n \geq 2i$.

\textbf{Example 9.1}. There are many operations defined on $H^{n,2}$, $n \geq 2$. Let us compare Voevodsky’s operation $P^1_V$ (landing in $H^{n+2\ell-2,\ell+1}$) with our operation $P^1$ (landing in $H^{n+2\ell-2,2\ell}$). Thus $P^1$ has the same bidegree as $[\zeta]^{\ell-1}P^1_V$, where $[\zeta] \in H^{0,1}(k)$. If $n \geq 4$, we have $P^1 = [\zeta]^{\ell-1}P^1_V$ by Corollary \cite{344}. If $n = 2$ we also have $P^1 = [\zeta]^{\ell-1}P^1_V$ because they induce the same étale operation $(P^1)$ from $H^{2,2}(X) \cong H^2_\text{et}(X,\mu_\ell^{\otimes 2})$ to $H^{2\ell,2\ell}(X) \cong H^{2\ell}_\text{et}(X,\mu_\ell^{\otimes 2\ell})$. We do not know if $P^1$ and $[\zeta]^{\ell-1}P^1_V$ agree on $H^{3,2}$.

Suppose that $\phi$ is a motivic cohomology operation on $H^{n,i}$ where $n \geq 2i$. Passing to étale cohomology sends $\phi$ to an étale operation, which by Theorem \cite{333} is a polynomial in the étale operations $P^1$. By Proposition \cite{353} some multiple of the Bott element $b$ sends $\phi$ to operations $b^n \phi$ which are in the subalgebra generated by the motivic operations $P^1$ defined in \cite{43}. It remains to determine what those powers are.

The following result of Voevodsky \cite{344} 3.6–7 shows that all non-trivial operations in the topological zone increase $n$.

\textbf{Lemma 9.2}. [Voevodsky] There are no motivic cohomology operations from $H^{2,i}$ to $H^{n,j}$ when $j < i$, or when $i = j$ and $(n,j) \neq (2i,i)$. The module of motivic cohomology operations from $H^{2,i}$ to $H^{*,*}$ is isomorphic to $\mathbb{F}_\ell$, on the identity.

\textbf{Conjecture 9.3}. Assume that $k$ contains all primitive $\ell$th roots of unity, and that $n \geq 2i$. Then the module of all motivic cohomology operations on $H^{n,i}(-,\mathbb{F}_\ell)$ is the tensor product of $H^{*,*}$ and a free graded polynomial algebra over $\mathbb{F}_\ell$ with generators all $P^1_P^J$, where $I = (\epsilon_0, s_1, \epsilon_1, \ldots, s_k, \epsilon_k)$, $J = (s_{k+1}, \epsilon_{k+1}, \ldots, s_m, \epsilon_m)$ subject to the conditions that (a) the concatenation $IJ$ is admissible with excess $e(IJ)$ either $< 4$ or else $\epsilon_0 = 1$ and $e(IJ) = 4$; and (b) for all $j > k$, $s_j < i + (\ell - 1) \sum_{j+1}^m s_i$.}
For $(n, i) = (4, 2)$ this conjecture implies that among the polynomial generators for the motivic operations on $H^{4, 2}$ we find $P^j SQ^n_i$. If $\ell = 2$, we may rewrite this as $S_{\ell}^{14} S_{\ell+1}^7 S_{\ell+3} S_{\ell+1}^1$; compare with [V3, 3.57].

**Lemma 9.4.** If Conjecture [6.3] holds for $H^{21,i}$ then it holds for all $H^{n,i}$ with $n \geq 2i$.

**Proof.** We consider the Leray spectral sequence (6.3) for $G = K(\mathbb{F}_\ell(i), n)$ and $K = B_\ast G = K(\mathbb{F}_\ell(i), n + 1)$ when $n \geq 2i$. By induction, $H^\ast (G)$ is a polynomial algebra over $H^\ast (G)$ with an $\ell$-simple system $\{x_i\}$ of generators. By [V3, 3.28], $\mathbb{F}_\ell(i, n + 1)$ is a split proper Tate motive, so the Künneth condition of Proposition [6.4] holds, and Borel’s Theorem [6.2] implies that $H^\ast (K)$ is the tensor product of $H^\ast (G)$ and a free graded-commutative $\mathbb{F}_\ell$-algebra on generators $y_i = \tau(x_i)$ and when deg$(x_i)$ is even and $\ell > 2$, $z_j = \tau(x_j^{-1} \otimes y_j)$.

We now use the fact that the transgression commutes with any $(S^\ell)$-stable cohomology operation, such as $P^j_i$; see [McC] 6.5. Since the tautological element $\iota_n$ of $H^{n,i}$ transgresses to the tautological element $\iota_{n+1}$ of $H^{n+1,i}$, the generator $x_j = P^j_i (\iota_n)$ transgresses to $y_j = P^j_i (\iota_{n+1})$ by Kudo’s Theorem [6.5]. This finishes the proof for $\ell = 2$.

If $\ell$ is odd and $x_j = P^j_i (\iota_n)$ has degree $2a$, the transgression $z_j = x_j^{-1} \otimes y_j$ is $-\beta P^a_i P^j_i (\iota_{n+1})$ by Kudo’s Theorem [6.5](3). This finishes the proof for $\ell$ odd. □

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