An Oppenheim type inequality for positive definite block matrices

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Abstract

We present an Oppenheim type determinantal inequality for positive definite block matrices. Recently, Lin [Linear Algebra Appl. 452 (2014) 1–6] proved a remarkable extension of Oppenheim type inequality for block matrices, which solved a conjecture of Günther and Klotz. There is a requirement that two matrices commute in Lin’s result. The motivation of this paper is to obtain another natural and general extension of Oppenheim type inequality for block matrices to get rid of the requirement that two matrices commute.

Key words: Hadamard product; Oppenheim’s inequality; Fischer’s inequality; Positive definite; Block matrices.

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1 Introduction

We use the following standard notation. The set of $m \times n$ complex matrices is denoted by $M_{m \times n}(C)$, or simply by $M_{m \times n}$. When $m = n$, we write $M_n$ for $M_{n \times n}$. The identity matrix of order $n$ is denoted by $I_n$, or $I$ for short. Given two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ with the same order, the Hadamard product of $A$ and $B$ is defined as $A \circ B = [a_{ij}b_{ij}]$. It is easy to verify that $(A \circ B) \circ C = A \circ (B \circ C)$, so the Hadamard product of $A^{(1)}, \ldots, A^{(m)}$ could be denoted by $\prod_{i=1}^{m} A^{(i)}$. By convention, the $\mu \times \mu$ leading principal submatrix of $A$ is denoted by $A_{\mu}$, i.e., $A_{\mu} = [a_{ij}]_{i,j=1}^{\mu}$.

Let $A = [a_{ij}]_{i,j=1}^{n} \in M_n$ be positive semidefinite. The Hadamard inequality says that

$$\prod_{i=1}^{n} a_{ii} \geq \det A. \tag{1}$$

If both $A$ and $B$ are positive definite (semidefinite), it is well-known that $A \circ B$ is positive definite (semidefinite); see, e.g., [12] p. 479]. Moreover, the celebrated Oppenheim inequality

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inequality (see [19] or [12, p. 509]) states that
\[ \det(A \circ B) \geq \det A \cdot \prod_{i=1}^{n} b_{ii} \geq \det(AB). \] (2)

By setting \( B = I_n \), then (2) is reduced to (1). Note that \( A \circ B = B \circ A \), thus (2) also implies
\[ \det(A \circ B) \geq \det B \cdot \prod_{i=1}^{n} a_{ii} \geq \det(AB). \] (3)

The following inequality (4) not merely generalized Oppenheim’s result, but also presented a well connection between (2) and (3); see [21, Theorem 3.7] or [12, pp. 509–510] for more details.
\[ \det(A \circ B) + \det(AB) \geq \det A \cdot \prod_{i=1}^{n} b_{ii} + \det B \cdot \prod_{i=1}^{n} a_{ii}. \] (4)

Inequality (4) is usually called Oppenheim-Schur’s inequality. Furthermore, Chen [3] generalized (4) and proved the following implicit improvement. If \( A \) and \( B \) are \( n \times n \) positive definite matrices, then
\[ \det(A \circ B) \geq \det(AB) \prod_{\mu=2}^{n} \left( \frac{a_{\mu\mu} \det A_{\mu-1}}{\det A_{\mu}} + \frac{b_{\mu\mu} \det B_{\mu-1}}{\det B_{\mu}} - 1 \right), \] (5)

where \( A_{\mu} = [a_{ij}]_{i,j=1}^{n} \) and \( B_{\mu} = [b_{ij}]_{i,j}^{n} \) for every \( \mu = 1, 2, \ldots, n \).

The significance and applicability of Hadamard product are well known in the literature. For example, this product is used to communication and information theory in correcting codes of satellite transmissions, signal processing and pattern recognition, and is also used to discrete combinatorial geometry and graph theory in interrelations between Hadamard matrices and different combinatorial configurations of block designs and Latin square. Applications can also be found in statistical analysis. For more details and applications, we refer the reader to the survey papers [1, 10, 21, 20].

Over the years, various generalizations and extensions of (4) and (5) have been obtained in the literature. For instance, see [23, 24] for the equality cases; see [2, 15, 22, 4, 7] for the extensions of \( M \)-matrices. It is worth noting that Lin [10] recently gave a remarkable extension (Theorem 1.1) of Chen’s result [5] for positive definite block matrices. This solved a conjecture of Günther and Klotz [8]. Before stating Lin’s result, we need to introduce the definition of block Hadamard product, which was first introduced in [11].

Let \( M_n(M_k) \) be the set of \( n \times n \) block matrices with each block being a \( k \times k \) matrix. The element of \( M_n(M_k) \) is usually written as the bold letter \( \mathbf{A} = [A_{ij}]_{i,j=1}^{n} \), where \( A_{ij} \in M_k \) for all \( 1 \leq i, j \leq n \). Given \( \mathbf{A} = [A_{ij}], \mathbf{B} = [B_{ij}] \in M_n(M_k) \), the block Hadamard product of \( \mathbf{A} \) and \( \mathbf{B} \) is given as \( \mathbf{A} \boxtimes \mathbf{B} := [A_{ij}B_{ij}]_{i,j=1}^{n} \), where \( A_{ij}B_{ij} \) denotes the usual matrix product of \( A_{ij} \) and \( B_{ij} \). Clearly, when \( k = 1 \), that is, \( A \) and \( B \) are \( n \times n \) matrices with complex entries, then the block Hadamard product coincides with the classical Hadamard product; when \( n = 1 \), it is identical with the usual matrix product. Positive definite block matrices are most appealing and extensively studied over the recent years since it leads to a number of versatile and elegant matrix inequalities; see, e.g., [14, 5, 13, 9, 27, 6, 13].

Now, Lin’s result could be stated as follows.
Theorem 1.1 (see [16]) Let \( A = [A_{ij}]_{i,j=1}^n \) and \( B = [B_{ij}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k) \) be positive definite matrices such that every \( A_{ij} \) of \( A \) commutes with every \( B_{rs} \) of \( B \). Then

\[
\det(A \square B) \geq \det(AB) \prod_{\mu=2}^n \left( \frac{\det A_{\mu \mu} \det A_{\mu-1}}{\det A_{\mu}} + \frac{\det B_{\mu \mu} \det B_{\mu-1}}{\det B_{\mu}} - 1 \right),
\]

where \( A_{\mu} = [A_{ij}]_{i,j=1}^\mu \) and \( B_{\mu} = [B_{ij}]_{i,j=1}^\mu \) denote the \( \mu \times \mu \) leading principal block submatrices of \( A \) and \( B \), respectively.

Clearly, when \( k = 1 \), Theorem 1.1 reduces to Chen’s result (5).

Motivated by Theorem 1.1, we will give another natural and general extension of (5) for positive definite block matrices. The condition in Theorem 1.1 that every \( A_{ij} \) of \( A \) commutes with every \( B_{rs} \) of \( B \) is harsh and strong when the blocks are of order at least two. Our extension (Theorem 1.2) has no requirement on the commutation assumptions. It also can be viewed as a complement of Theorem 1.1.

Theorem 1.2 Let \( A^{(i)} = [A_{rs}^{(i)}]_{r,s=1}^n \in \mathbb{M}_n(\mathbb{M}_k), i = 1, \ldots, m \) be positive definite. Then

\[
\det \left( \prod_{i=1}^m \circ A^{(i)} \right) \geq \det \left( \prod_{i=1}^m A^{(i)} \right) \prod_{\mu=2}^n \left( \sum_{i=1}^m \frac{\det A^{(i)}_{\mu \mu} \det A^{(i)}_{\mu-1}}{\det A^{(i)}_{\mu}} - (m-1) \right),
\]

where \( A^{(i)}_{\mu} \) stands for the \( \mu \times \mu \) leading principal block submatrix of \( A^{(i)} \).

Additionally, based on Theorem 1.2 and a numerical inequality, we will give the following Theorem 1.3 which is an extension of Oppenheim type inequality (4).

Theorem 1.3 Let \( A^{(i)} \in \mathbb{M}_n(\mathbb{M}_k), i = 1, 2, \ldots, m \) be positive semidefinite. Then

\[
\det \left( \prod_{i=1}^m \circ A^{(i)} \right) + (m-1) \prod_{i=1}^m \left( \det A^{(i)} \right) \geq \sum_{i=1}^m \prod_{j=1, j \neq i}^m \det A^{(j)} \cdot \prod_{\mu=1}^n \det A^{(i)}_{\mu \mu}.
\]

The paper is organized as follows. Our proof of Theorem 1.2 is by induction on positive integer \( m \), so we will treat the base case \( m = 2 \) in Section 2 separately, and in this section, we will give some auxiliary lemmas and propositions to facilitate our proof. Some new determinantal inequalities for positive definite block matrices are also included. In Section 3 we will give our proof of Theorem 1.2 and Theorem 1.3.

2 The base case \( m = 2 \)

If \( X \) is positive semidefinite, we write \( X \geq 0 \). For two Hermitian matrices \( A \) and \( B \) with the same order, \( A \succeq B \) means \( A - B \succeq 0 \). It is easy to verify that \( \succeq \) is a partial ordering on the set of Hermitian matrices, referred to Löwner ordering. If \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) is a square matrix with \( A_{11} \) nonsingular, then the Schur complement of \( A_{11} \) in \( A \) is defined as \( A / A_{11} := A_{22} - A_{21} A_{11}^{-1} A_{12} \). It is obvious that \( \det(A / A_{11}) = (\det A) / (\det A_{11}) \). We refer to the integrated survey [25] for more applications of Schur complement.
Lemma 2.1 Let \( A = [a_{ij}]_{i,j=1}^n \) and \( B = [b_{ij}]_{i,j=1}^n \) be positive definite matrices. Then
\[
\frac{\det(A_p \circ B_p)}{\det(A_{p-1} \circ B_{p-1})} + \frac{\det(A_p B_p)}{\det(A_{p-1} B_{p-1})} \geq \frac{a_{pp} \det B_p}{\det B_{p-1}} + \frac{b_{pp} \det A_p}{\det A_{p-1}},
\]
where \( A_p = [a_{ij}]_{i,j=1}^p \) and \( B_p = [b_{ij}]_{i,j=1}^p \) for every \( p = 1, 2, \ldots, n \).

Proof. First, we denote \( \alpha := [a_{1p}, \ldots, a_{p-1,p}]^T \) and \( \beta := [b_{1p}, \ldots, b_{p-1,p}]^T \). Setting
\[
\hat{A}_p := \begin{bmatrix} A_{p-1} & \alpha^* \\ \alpha^* & a_{pp} - \frac{\det A_p}{\det A_{p-1}} \end{bmatrix}, \quad \hat{B}_p := \begin{bmatrix} B_{p-1} & \beta^* \\ \beta^* & b_{pp} - \frac{\det B_p}{\det B_{p-1}} \end{bmatrix}.
\]
It is easy to see that both \( \hat{A}_p \) and \( \hat{B}_p \) are singular positive semidefinite, then \( \hat{A}_p \circ \hat{B}_p \) is positive semidefinite. By taking determinant, it follows that
\[
\det(\hat{A}_p \circ \hat{B}_p) = \det \left[ \begin{bmatrix} A_{p-1} & B_{p-1} \\ \alpha^* & \beta^* \end{bmatrix} \left( a_{pp} - \frac{\det A_p}{\det A_{p-1}} \right) \left( b_{pp} - \frac{\det B_p}{\det B_{p-1}} \right) \right] \geq 0.
\]
By a direct computation, we get
\[
\det(A_p \circ B_p) + \det(A_{p-1} \circ B_{p-1}) \left( -\frac{a_{pp} \det B_p}{\det B_{p-1}} - \frac{b_{pp} \det A_p}{\det A_{p-1}} + \frac{\det(A_p B_p)}{\det(A_{p-1} B_{p-1})} \right) \geq 0.
\]
This completes the proof. \(\blacksquare\)

The following lemma first appeared in [10]. The author in [11] proved the same result under a weaker assumption \( X \geq W, X \geq Z \) and \( X + Y \geq W + Z \).

Lemma 2.2 (see [10] or [11]) Let \( X, Y, W \) and \( Z \) be positive semidefinite. If \( X \geq W \geq Y, X \geq Z \geq Y \) and \( X + Y \geq W + Z \), then
\[
\det X + \det Y \geq \det W + \det Z.
\]

Let \( X, Y, W \) and \( Z \) be diagonal matrices. Lemma 2.2 implies the following result, which will be used in the proof of Lemma 2.6.

Corollary 2.3 Let \( x_i, y_i, z_i \) and \( w_i \) be nonnegative numbers. If \( x_i \geq w_i \geq y_i, x_i \geq z_i \geq y_i \) and \( x_i + y_i \geq z_i + w_i \) for every \( i = 1, 2, \ldots, n \), then
\[
\prod_{i=1}^n x_i + \prod_{i=1}^n y_i \geq \prod_{i=1}^n w_i + \prod_{i=1}^n z_i.
\]

We next provide an extension of Lemma 2.1 for positive definite block matrices by using Lemma 2.2 Our treatment of Proposition 2.4 has its root in [10].

Proposition 2.4 Let \( A = [A_{ij}]_{i,j=1}^n \) and \( B = [B_{ij}]_{i,j=1}^n \) be positive definite. Then
\[
\det \left( \left( A_\mu \circ B_\mu \right)/ \left( A_{\mu-1} \circ B_{\mu-1} \right) \right) + \det \left( \left( A_\mu / A_{\mu-1} \right) \circ \left( B_\mu / B_{\mu-1} \right) \right) \\
\geq \det \left( A_\mu \circ \left( B_\mu / B_{\mu-1} \right) \right) + \det \left( B_\mu \circ \left( A_\mu / A_{\mu-1} \right) \right),
\]
where \( A_\mu = [A_{ij}]_{i,j=1}^\mu \) and \( B_\mu = [B_{ij}]_{i,j=1}^\mu \) for every \( \mu = 1, 2, \ldots, n \).
Proof. We first denote $X := \begin{bmatrix} A_{1\mu} \\ \vdots \\ A_{\mu-1,\mu} \end{bmatrix}$, $Y := \begin{bmatrix} B_{1\mu} \\ \vdots \\ B_{\mu-1,\mu} \end{bmatrix}$ and define

$$
\tilde{A}_\mu := \begin{bmatrix} A_{\mu-1} & X \\ X^* & X \end{bmatrix}, \quad \tilde{B}_\mu := \begin{bmatrix} B_{\mu-1} & Y \\ Y^* & Y \end{bmatrix}.
$$

It is easy to see by computing Schur complement that $\tilde{A}_\mu$ and $\tilde{B}_\mu$ are singular positive semidefinite. Therefore $\tilde{A}_\mu \circ \tilde{B}_\mu$ is positive semidefinite and so

$$(X^* A_{\mu-1}^{-1} X) \circ (Y^* B_{\mu-1}^{-1} Y) \geq (X^* \circ Y^*)(A_{\mu-1} \circ B_{\mu-1})^{-1} (X \circ Y),$$

which is equivalent to

$$(A_{\mu\mu} - (A_{\mu}/A_{\mu-1})) \circ (B_{\mu\mu} - (B_{\mu}/B_{\mu-1})) \geq A_{\mu\mu} \circ B_{\mu\mu} - (A_{\mu} \circ B_{\mu})/(A_{\mu-1} \circ B_{\mu-1}).$$

Expanding the above inequality gives

$$(A_{\mu} \circ B_{\mu})/(A_{\mu-1} \circ B_{\mu-1}) + (A_{\mu}/A_{\mu-1}) \circ (B_{\mu}/B_{\mu-1})$$

$$\geq A_{\mu\mu} \circ (B_{\mu}/B_{\mu-1}) + B_{\mu\mu} \circ (A_{\mu}/A_{\mu-1}).$$

(6)

On the other hand, since $B_{\mu\mu} \geq B_{\mu}/B_{\mu-1}$, then by (6), we have

$$(A_{\mu} \circ B_{\mu})/(A_{\mu-1} \circ B_{\mu-1}) - A_{\mu\mu} \circ (B_{\mu}/B_{\mu-1})$$

$$\geq B_{\mu\mu} \circ (A_{\mu}/A_{\mu-1}) - (A_{\mu} \circ B_{\mu})/(A_{\mu-1} \circ B_{\mu-1})$$

$$= (B_{\mu\mu} - B_{\mu}/B_{\mu-1}) \circ (A_{\mu}/A_{\mu-1}) \geq 0.$$

Therefore,

$$(A_{\mu} \circ B_{\mu})/(A_{\mu-1} \circ B_{\mu-1}) \geq A_{\mu\mu} \circ (B_{\mu}/B_{\mu-1}) \geq (A_{\mu}/A_{\mu-1}) \circ (B_{\mu}/B_{\mu-1}).$$

(7)

Similarly, we could obtain

$$(A_{\mu} \circ B_{\mu})/(A_{\mu-1} \circ B_{\mu-1}) \geq B_{\mu\mu} \circ (A_{\mu}/A_{\mu-1}) \geq (A_{\mu}/A_{\mu-1}) \circ (B_{\mu}/B_{\mu-1}).$$

(8)

Keeping (6), (7) and (8) in mind, then Lemma 2.2 yields the required inequality.

The following lemma is called Fischer’s inequality, which is an improvement as well as extension of Hadamard’s inequality (1) for positive semidefinite block matrices.

Lemma 2.5 (see [12, p. 506] or [26, p. 217]) Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be an $n \times n$ positive semidefinite matrix with diagonal blocks being square, then

$$\prod_{i=1}^{n} a_{ii} \geq \det A_{11} \det A_{22} \geq \det A.$$
The forthcoming Lemma 2.6 is similar with Proposition 2.4 in the mathematically written form. It is not only an extension of Lemma 2.1 for positive definite block matrices, but also plays a key role in our proof of Theorem 2.7.

**Lemma 2.6** Let \( A = [A_{ij}]_{i,j=1}^{n} \) and \( B = [B_{ij}]_{i,j=1}^{n} \) be positive definite. Then

\[
\frac{\det(A_{\mu} \circ B_{\mu})}{\det(A_{\mu-1} \circ B_{\mu-1})} + \frac{\det(A_{\mu}B_{\mu})}{\det(A_{\mu-1}B_{\mu-1})} \geq \frac{\det(A_{\mu}B_{\mu})}{\det(A_{\mu-1}B_{\mu-1})} + \frac{\det(B_{\mu} \circ A_{\mu})}{\det(B_{\mu-1} \circ A_{\mu})},
\]

where \( A_{\mu} = [A_{ij}]_{i,j=1}^{\mu} \) and \( B_{\mu} = [B_{ij}]_{i,j=1}^{\mu} \) for every \( \mu = 1, \ldots, n \).

**Proof.** We first can see from (7) and (8) that

\[
\frac{\det(A_{p} \circ B_{p})}{\det(A_{p-1} \circ B_{p-1})} \geq \frac{a_{pp} \det B_{p}}{\det B_{p-1}} \geq \frac{\det(A_{p}B_{p})}{\det(A_{p-1}B_{p-1})},
\]

and

\[
\frac{\det(A_{p} \circ B_{p})}{\det(A_{p-1} \circ B_{p-1})} \geq \frac{b_{pp} \det A_{p}}{\det A_{p-1}} \geq \frac{\det(A_{p}B_{p})}{\det(A_{p-1}B_{p-1})}.
\]

By Lemma 2.1 and Corollary 2.3 we have

\[
\prod_{p=(\mu-1)k+1}^{\mu k} \frac{\det(A_{p} \circ B_{p})}{\det(A_{p-1} \circ B_{p-1})} + \prod_{p=(\mu-1)k+1}^{\mu k} \frac{\det(A_{p}B_{p})}{\det(A_{p-1}B_{p-1})} \geq \prod_{p=(\mu-1)k+1}^{\mu k} \frac{a_{pp} \det B_{p}}{\det B_{p-1}} + \prod_{p=(\mu-1)k+1}^{\mu k} \frac{b_{pp} \det A_{p}}{\det A_{p-1}}.
\]

Note that \( A_{\mu k} = [a_{ij}]_{i,j=1}^{\mu k} = [A_{ij}]_{i,j=1}^{\mu} = A_{\mu} \), then the above inequality could be written as

\[
\frac{\det(A_{\mu} \circ B_{\mu})}{\det(A_{\mu-1} \circ B_{\mu-1})} + \frac{\det(A_{\mu}B_{\mu})}{\det(A_{\mu-1}B_{\mu-1})} \geq \left( \prod_{p=(\mu-1)k+1}^{\mu k} \frac{a_{pp} \det B_{p}}{\det B_{p-1}} \right) \frac{\det B_{\mu}}{\det B_{\mu-1}} + \left( \prod_{p=(\mu-1)k+1}^{\mu k} \frac{b_{pp} \det A_{p}}{\det A_{p-1}} \right) \frac{\det A_{\mu}}{\det A_{\mu-1}},
\]

which together with Lemma 2.5 leads to the desired result. \( \blacksquare \)

The following Theorem 2.7 is just the case \( m = 2 \) of Theorem 1.2.

**Theorem 2.7** Let \( A = [A_{ij}]_{i,j=1}^{n} \) and \( B = [B_{ij}]_{i,j=1}^{n} \) be positive definite. Then

\[
\det(A \circ B) \geq \det(AB) \prod_{\mu=2}^{n} \left( \frac{\det A_{\mu} \det B_{\mu}}{\det A_{\mu-1} \det B_{\mu-1}} + \frac{\det B_{\mu} \det A_{\mu-1}}{\det B_{\mu-1} \det A_{\mu}} - 1 \right),
\]

where \( A_{\mu} = [A_{ij}]_{i,j=1}^{\mu} \) and \( B_{\mu} = [B_{ij}]_{i,j=1}^{\mu} \) for every \( \mu = 1, \ldots, n \).
Proof. By Lemma 2.6 we can obtain
\[
\frac{\det(A_\mu \circ B_\mu)}{\det(A_{\mu-1} \circ B_{\mu-1})} \geq \frac{\det(A_\mu B_\mu)}{\det(A_{\mu-1} B_{\mu-1})} \times \left( \frac{\det A_\mu \det A_{\mu-1}}{\det A_\mu} + \frac{\det B_\mu \det B_{\mu-1}}{\det B_\mu} - 1 \right).
\]

Therefore, we get
\[
\prod_{\mu=2}^{n} \frac{\det(A_\mu \circ B_\mu)}{\det(A_{\mu-1} \circ B_{\mu-1})} \geq \prod_{\mu=2}^{n} \frac{\det(A_\mu B_\mu)}{\det(A_{\mu-1} B_{\mu-1})} \times \left( \frac{\det A_\mu \det A_{\mu-1}}{\det A_\mu} + \frac{\det B_\mu \det B_{\mu-1}}{\det B_\mu} - 1 \right).
\]

Note that Oppenheim’s inequality (2) leads to
\[
\det(A_1 \circ B_1) \geq \det(A_1 B_1).
\]

Thus, the required inequality now immediately follows. □

3 The General case

Now we are in a position to present a proof of our main result Theorem 1.2.

Proof of Theorem 1.2. We show the proof by induction on \(m\). When \(m = 2\), the required result is guaranteed by Theorem 2.7. Assume that the required inequality is true for the case \(m - 1\), that is
\[
\det\left( \prod_{i=1}^{m-1} A^{(i)} \circ A^{(m)} \right) \geq \det\left( \prod_{i=1}^{m-1} A^{(i)} \right) \prod_{\mu=2}^{n} \left( \sum_{i=1}^{m-1} \frac{\det A_\mu \det A_{\mu-1}}{\det A_\mu} + \frac{\det B_\mu \det B_{\mu-1}}{\det B_\mu} - (m - 2) \right).
\]

Now we consider the case \(m > 2\). Then we have
\[
\det\left( \prod_{i=1}^{m} A^{(i)} \right)
\]
\[
= \det\left( \prod_{i=1}^{m-1} A^{(i)} \circ A^{(m)} \right) \quad \text{(by Theorem 2.7)}
\]
\[
\geq \det\left( \prod_{i=1}^{m-1} A^{(i)} \right) \det A^{(m)}
\]
\[
\times \prod_{\mu=2}^{n} \left( \frac{\det\left( \prod_{i=1}^{m-1} A^{(i)} \right)_{\mu} \det\left( \prod_{i=1}^{m-1} A^{(i)} \right)_{\mu-1}}{\det\left( \prod_{i=1}^{m-1} A^{(i)} \right)_{\mu}} + \frac{\det A^{(m)}_{\mu} \det A^{(m)}_{\mu-1}}{\det A^{(m)}_{\mu}} - 1 \right).
\]

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\[
\geq \prod_{i=1}^{m} (\det A(i)) \times \prod_{\mu=2}^{n} \left( \sum_{i=1}^{m-1} \frac{\det A(\mu) \det A(i)}{\det A(\mu)} - (m - 2) \right) \\
\times \prod_{\mu=2}^{n} \left( \frac{\det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu} \det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu-1}}{\det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu}} + \frac{\det A(i) \det A(\mu-1)}{\det A(\mu)} - 1 \right).
\]

For notational convenience, we denote
\[
R_{\mu} := \sum_{i=1}^{m-1} \frac{\det A(\mu) \det A(i)}{\det A(\mu)} - (m - 2),
\]
and
\[
S_{\mu} := \frac{\det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu} \det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu-1}}{\det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu}} + \frac{\det A(i) \det A(\mu-1)}{\det A(\mu)} - 1.
\]

By Fischer’s inequality (Lemma 2.5), we can see that
\[
\det A(\mu) \det A(i) \geq \det A(i), \quad i = 1, 2, \ldots, m,
\]
which leads to
\[
R_{\mu} = \sum_{i=1}^{m-1} \frac{\det A(\mu) \det A(i)}{\det A(\mu)} - (m - 2) \geq 1. \tag{9}
\]

On the other hand, by Fischer’s inequality (Lemma 2.5) again, we have
\[
\det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu} \det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu-1} \geq \det \left( \prod_{i=1}^{m-1} \circ A(i) \right)_{\mu}.
\]

Therefore, we obtain
\[
S_{\mu} \geq \frac{\det A(i) \det A(\mu-1)}{\det A(\mu)} \geq 1. \tag{10}
\]

Since \(R_{\mu} \geq 1\) and \(S_{\mu} \geq 1\), this leads to
\[
R_{\mu}S_{\mu} \geq R_{\mu} + S_{\mu} - 1.
\]

Hence, we get from (9) and (10) that
\[
\det \left( \prod_{i=1}^{m} \circ A(i) \right) \geq \prod_{i=1}^{m} \left( \det A(i) \right) \prod_{\mu=2}^{n} \left( R_{\mu} \prod_{\mu=2}^{n} S_{\mu} \right)
\geq \prod_{i=1}^{m} \left( \det A(i) \right) \prod_{\mu=2}^{n} \left( R_{\mu} + S_{\mu} - 1 \right)
\]
\[8\]
Thus, the required inequality holds for \( n \).

**Lemma 3.1** If \((a_1^{(i)}, a_2^{(i)}, \ldots, a_m^{(i)}) \in \mathbb{R}^m, i = 1, \ldots, m \) and \( a^{(i)}_\mu \geq 1 \) for all \( i \) and \( \mu \), then

\[
\prod_{\mu=1}^{m} \left( \sum_{i=1}^{n} a^{(i)}_\mu - (m - 1) \right) \geq \sum_{i=1}^{n} \prod_{\mu=1}^{m} a^{(i)}_\mu - (m - 1).
\]

**Proof.** We apply induction on \( n \). When \( n = 1 \), there is nothing to show. Suppose that the required inequality is true for \( n = k \). Then we consider the case \( n = k + 1 \),

\[
\prod_{\mu=1}^{k+1} \left( \sum_{i=1}^{m} a^{(i)}_\mu - (m - 1) \right)
= \left( \sum_{i=1}^{m} a^{(i)}_{k+1} - (m - 1) \right) \cdot \prod_{\mu=1}^{k} \left( \sum_{i=1}^{m} a^{(i)}_\mu - (m - 1) \right)
\geq \left( \sum_{i=1}^{m} a^{(i)}_{k+1} - (m - 1) \right) \cdot \left( \sum_{\mu=1}^{k} \prod_{i=1}^{m} a^{(i)}_\mu - (m - 1) \right)
= \sum_{i=1}^{m} \left( a^{(i)}_{k+1} - 1 \right) \left( \sum_{j=1}^{m} \prod_{\mu=1}^{k} a^{(j)}_\mu - (m - 1) \right) + \sum_{i=1}^{m} \prod_{\mu=1}^{k} a^{(i)}_\mu - (m - 1)
= \sum_{i=1}^{m} \left( a^{(i)}_{k+1} - 1 \right) \left( \sum_{j=1, j \neq i}^{m} \prod_{\mu=1}^{k} a^{(j)}_\mu - (m - 1) \right) + \sum_{i=1}^{m} \left( a^{(i)}_{k+1} - 1 \right) \prod_{\mu=1}^{k} a^{(i)}_\mu + \sum_{i=1}^{m} \prod_{\mu=1}^{k} a^{(i)}_\mu - (m - 1)
= \sum_{i=1}^{m} \left( a^{(i)}_{k+1} - 1 \right) \left( \sum_{j=1, j \neq i}^{m} \prod_{\mu=1}^{k} a^{(j)}_\mu - (m - 1) \right) + \sum_{i=1}^{m} \prod_{\mu=1}^{k} a^{(i)}_\mu - (m - 1)
\geq \sum_{i=1}^{m} \prod_{\mu=1}^{k+1} a^{(i)}_\mu - (m - 1).
\]

Thus, the required inequality holds for \( n = k + 1 \), so the proof of induction step is complete.

**Remark.** When \( m = 2 \), Lemma 3.1 implies that for every \( a_\mu, b_\mu \geq 1 \), then

\[
\prod_{\mu=1}^{n} (a_\mu + b_\mu - 1) \geq \prod_{\mu=1}^{n} a_\mu + \prod_{\mu=1}^{n} b_\mu - 1. \tag{11}
\]
This inequality (11) plays an important role in [16] for deriving determinantal inequalities, and we can see from (11) that Chen’s result (5) is indeed an improvement of (4). On the other hand, (11) could be obtained from Corollary 2.3. The above proof of Lemma 3.1 is by induction on \( n \). In fact, combining (11) and by induction on \( m \), one could get another way to prove Lemma 3.1.

Now, we are ready to present a proof of Theorem 1.3.

**Proof of Theorem 1.3** Without loss of generality, we may assume by a standard perturbation argument that all \( A^{(i)} \) are positive definite. Thus, the required inequality could be rewritten as

\[
\det \left( \prod_{i=1}^{m} A^{(i)} \right) \geq \prod_{i=1}^{m} \left( \det A^{(i)} \right) \left( \sum_{i=1}^{m} \frac{\prod_{\mu=1}^{n} \det A^{(i)}_{\mu \mu}}{\det A^{(i)}} - (m - 1) \right).
\]

(12)

By Fischer’s inequality (Lemma 2.5), we have

\[
\det A^{(i)}_{\mu \mu} \det A^{(i)}_{\mu - 1} \geq \det A^{(i)}_{\mu}.
\]

Therefore, it follows from Theorem 1.2 and Lemma 3.1 that

\[
\det \left( \prod_{i=1}^{m} A^{(i)} \right) \geq \prod_{i=1}^{m} \left( \det A^{(i)} \right) \cdot \left( \sum_{i=1}^{m} \prod_{\mu=1}^{n} \frac{\det A^{(i)}_{\mu \mu} \det A^{(i)}_{\mu - 1}}{\det A^{(i)}_{\mu}} - (m - 1) \right).
\]

Observe that

\[
\prod_{\mu=2}^{n} \frac{\det A^{(i)}_{\mu \mu} \det A^{(i)}_{\mu - 1}}{\det A^{(i)}_{\mu}} = \prod_{\mu=1}^{n} \frac{\det A^{(i)}_{\mu \mu}}{\det A^{(i)}}.
\]

Hence, the proof of (12) is complete.

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