Born’s Rule as Signature of a Superclassical Current Algebra

Siegfried Fussy¹, * Johannes Mesa Pascasio¹,², *, Herbert Schwabl¹, * and Gerhard Grössing¹ *

¹Austrian Institute for Nonlinear Studies, Akademiehof
Friedrichstr. 10, 1010 Vienna, Austria

²Institute for Atomic and Subatomic Physics, Vienna University of Technology
Operng. 9, 1040 Vienna, Austria and

(Dated: May 11, 2014)

Abstract

We present a new tool for calculating the interference patterns and particle trajectories of a double-, three- and N-slit system on the basis of an emergent sub-quantum theory developed by our group throughout the last years. The quantum itself is considered as an emergent system representing an off-equilibrium steady state oscillation maintained by a constant throughput of energy provided by a classical zero-point energy field. We introduce the concept of a “relational causality” which allows for evaluating structural interdependences of different systems levels, i.e. in our case of the relations between partial and total probability density currents, respectively. Combined with the application of 21st century classical physics like, e.g., modern nonequilibrium thermodynamics, we thus arrive at a “superclassical” theory. Within this framework, the proposed current algebra directly leads to a new formulation of the guiding equation which is equivalent to the original one of the de Broglie-Bohm theory. By proving the absence of third order interferences in three-path systems it is shown that Born’s rule is a natural consequence of our theory. Considering the series of one-, double-, or, generally, of N-slit systems, with the first appearance of an interference term in the double slit case, we can explain the violation of Sorkin’s first order sum rule, just as the validity of all higher order sum rules. Moreover, the Talbot patterns and Talbot distance for an arbitrary N-slit device can be reproduced exactly by our model without any quantum physics tool.

Keywords: emergent quantum mechanics, Born’s rule, multiple-slit experiments, hierarchical sum rules, Talbot effect

* E-mail: ains@chello.at; Visit: http://www.nonlinearstudies.at/
1. INTRODUCTION

In 1926, Born [1] suggested that $|\psi(x, t)|^2$ is the probability to find the particle in the time interval $[t, t + dt]$, and in the length interval $[x, x + dx]$. For different mutually excluding paths of particles between source and detector one has to sum up the $\psi$ functions of these paths coherently and then take the absolute square of the linearly summed contributions. As a direct consequence of this construction, a term appears describing the interference pattern in the double slit diffraction experiment. Born’s rule is one of the key laws in quantum mechanics and it proposes that interference occurs in pairs of possibilities, but never in triples etc. So-called multipath interference terms representing interferences of higher order are ruled out, be it in standard quantum mechanics or in the de Broglie-Bohm theory, for example. Consequently, an addition of slits or paths does not increase the complexity of the whole system, but has to be considered only quantitatively.

Although one can conclude that Born’s rule is, at least indirectly, confirmed by practically all quantum mechanical experiments within the last hundred years, there had been no explicit experiment to support this proposition until a few years ago. The experimental results hitherto seem to confirm the exact validity of Born’s rule up to the order of $10^{-4}$ [2–4]. However, these experiments were commented critically by De Raedt et al. [5]. The decomposition of a three-path wave function into its lower order interference terms might not correctly represent the experimental setup. So, it is still an open question whether or not the mathematically correct derived double slit contributions to the three slit result can be identified with the sum of the experimentally derived double and single slit contributions.

From the theoretical point of view no generally accepted derivation of Born’s rule has been given to date [6], but this does not imply that such a derivation is impossible in principle.

In the following we try to shed light on this puzzle by combining results of recently developed “Emergent Quantum Mechanics” [7] with concepts of systems theory which we denote as “relational causality” [8]. Since the physics of different scales is concerned, like, e.g., sub-quantum and classical macro physics, we denote our sub-quantum theory as “superclassical”. We consider the quantum itself as an emergent system understood as off-equilibrium steady state oscillation maintained by a constant throughput of energy provided by the (“classical“) zero-point energy field. Starting with this concept, our group was able to assess phenomena of standard quantum mechanics like Gaussian dispersion of wave packets,
superposition, double slit interference, Planck’s energy relation, or the Schrödinger equation, respectively, as the emergent property of an underlying sub-structure of the vacuum combined with diffusion processes reflecting the stochastic parts of the zero-point field.

In Section 2 we contrast the well-known physics behind the double slit with an emergent vector field representation of the observed interference field. In Section 3 the essential parts of our superclassical current algebra are presented and the velocity field (corresponding to the guiding equation of the de Broglie-Bohm theory) is derived. The crucial case testing the validity of Born’s rule by means of a three slit configuration is analyzed in Section 4, whereas the general $N$-slit setup is discussed in Section 5. In Section 6 we summarize our results and give an outlook on a possible breakdown of orthodox quantum mechanics representing the emergent mean field theory out of our sub-quantum dynamics, consequently associated with the violation of Born’s rule.

2. INTERFERENCE AND EMERGENCE AT A GAUSSIAN DOUBLE SLIT

Considering particles as oscillators (“bouncers”) coupling to regular oscillations of the vacuum’s zero-point field, which they also generate, we have shown how a quantum can be understood as an emergent system. In particular, the dynamics between the oscillator and the “bath” of its thermal environment can be made responsible not only for Gaussian diffraction at a single slit [9], but also for the well-known interference effects at a double slit [10]. We have shown that the quantum nature of the spreading of the wave packet can be exactly described by combining the convective with the orthogonal diffusive velocity fields. The close resemblance of these two different velocities with the complex velocity originally introduced by Schrödinger is discussed extensively in [9].

In Fig. 2.1 the underlying geometry for the wave vectors is sketched, both for the classical interference and the emergent case. For illustration, we show the two-dimensional setup where in the classical picture the incoming wave vector $\mathbf{k} = \frac{2\pi}{\lambda}\mathbf{\hat{k}}$ splits up at the Gaussian slits $A$ and $B$ into $\mathbf{k}_A$ and $\mathbf{k}_B$, and both are orthogonal to the particular propagating wave fronts. The respective phases for each of the beams are usually denoted as $\varphi_{A(B)} = \mathbf{k}_{A(B)} \cdot \mathbf{r}_0$. The phase difference $\varphi(x, t)$ reduces in case of coherent plane waves to a time independent variable, i.e. $\varphi = \varphi_A - \varphi_B = (\mathbf{k}_A - \mathbf{k}_B) \cdot \mathbf{r}_0$. Depending on the size of the phase difference $\varphi(x)$ at a specific point one obtains the well-known stationary interference patterns, i.e.
amplitude maxima in case of \( \varphi = 2n\pi \) and minima for \( \varphi = (2n + 1)\pi \). Examples for both cases are shown at the points \((x_1, t_1)\) and \((x_2, t_2)\), respectively.

We now take a closer look at the combined plane-wave amplitudes at an arbitrary point \((x, t)\) of the spatio-temporal plane. The amplitudes \(R_{A(B)} = \sqrt{P_{A(B)}}\) associated with each of the beams combine at \((x, t)\) to a total amplitude

\[
R = R_A \cos \varphi_A + R_B \cos \varphi_B. \tag{2.1}
\]

By defining \(\cos \varphi_{A(B)} = \hat{n}_{A(B)} \cdot \hat{k}_{A(B)}\) this can generally be written as

\[
R = \left( R_A \hat{n}_A \cdot \hat{k}_A + R_B \hat{n}_B \cdot \hat{k}_B \right). \tag{2.2}
\]

In the emergent scenario we have to treat the two slits, or beam paths respectively, as sources of two channels of a flow of probability densities. The \(N\) slits (or paths) are denoted in the following as \(A, B, C, \ldots\), whereas the emergent velocity channels are characterized by the subscript \(i = 1, 2, \ldots, N\), the reasoning for which is given below. We look again at the – in this case – stationary phase difference \(\varphi = \varphi_1 - \varphi_2\), which can also be understood as the arc cosine of two enclosed unit vectors

\[
\varphi = \arccos(\hat{s}_1 \cdot \hat{s}_2). \tag{2.3}
\]

Again, in the stationary case \(\hat{s}_1\) and \(\hat{s}_2\) depend only on \(x\) and represent two vector fields.

According to our model, we interpret one possible solution of (2.3) as the propagating wave vectors \(\kappa_i\) (also termed “convective velocity” [11]), which both develop symmetrically
to the axis given by $k_A + k_B$. The emerging wave vectors $\kappa_i$ relate to the velocities via $p_i = \hbar \kappa_i$ and thus $v_i = \frac{\hbar}{m} \kappa_i$, which generally do not coincide with the original directions of $k_A(B)$. 

Here some words of caution are appropriate. The vector fields $\kappa_i$ have emergent properties, thus the index $i$ cannot be understood as a direct link to the slits $A$ or $B$, respectively. In fact, we can only state that the setup of the double slit gives rise to two independent probability channels, which we denote with the indices $i$. Consequently, the enclosed angle $\varphi$ of the two emerging velocities $v_i$ should not be confused with the geometric angle $\delta$ of the spreading waves. Note that the differences between the incoming wave vectors $k_A(B)$ on one hand, and the emerging wave fields $\kappa_i$ on the other, are in complete analogy to those between the geometric rays and the streamlines in optical currents, as can be seen impressively from Fig. 3 in [12].

The second possible solution of (2.3) are the orthogonal osmotic velocities $u_i$, again for the two channels of probability densities. Clearly, the original amplitudes $R_A(B)$ have to be identified with those of the emergent vectors $R_i, i = 1, 2$. The osmotic velocities $u_i$ refer to diffusion processes reflecting the stochastic fluctuations of the zero-point field. For the rest of this paper, we keep the notations $v_i$ and $u_i$ to denote the emergent convective and diffusive fields, respectively.

As can be seen from Fig. 2.1, at $(x, t)$ the new, emerging wave vector or velocity field, respectively, discloses new information with regard to the amplitudes at that point: The phase difference is nonlocally encoded at each point of the plane due to the Gaussians not being truncated, as it will be discussed in the last section. Since the exemplary point $(x_1, t_1)$ lies on the central symmetrical line between slit $A$ and $B$, both paths from the slits are of equal length, with the consequence of a vanishing phase difference. Therefore, both of the emergent wave vectors $\kappa_1$ and $\kappa_2$ have to be parallel at $(x_1, t_1)$. Analogously, in the case of the destructive interference at $(x_2, t_2)$, $\kappa_1$ and $\kappa_2$ point into opposite directions.

The emergent diffusive velocities have to fulfill the condition of being unbiased w.r.t. the convective velocities, i.e. the orthogonality relation for the averaged velocities derived in [9]: $\mathbf{vu} = 0$, since any fluctuations $u = \delta (\nabla S/m)$ are shifts along the surfaces of action $S = \text{const.}$

Each point of the probability (or amplitude) landscape evolves on the spatiotemporal plane according to the emergent propagation velocities $v_i(x, t), i = 1, 2$. In addition, the
differential equations for \( u(x, t) \) describe the dispersion of the Gaussians and split up into \( u_1(x, t) \) and \( u_2(x, t) \) for each channel. The enclosed phase difference angles \( \varphi \) can be found between \( v \) and the two opposed components, of \( u_i \), \( i = 1, 2 \), respectively, denoted as \( u_{1R} \) and \( u_{1L} \) (Fig. 2.2). Since \( u_{1R(L)} \) are orthogonal to \( v \), all other enclosed angles like, e.g., the one between the unit vectors \( \hat{v}_1 \) and \( \hat{u}_{2R} \), can be expressed in terms of \( \varphi \), i.e. \( \angle(\hat{v}_1, \hat{u}_{2R}) = \frac{\pi}{2} + \varphi \), for example.

In the following we will see how the trajectories which represent averaged paths due to the averaged velocities, can be calculated with the help of a “superclassical current” algebra leading to the expressions for the total current \( J_{\text{tot}} \) and the total probability density \( P_{\text{tot}} \) at \((x, t)\).

To account for the different velocity channels \( i = 1, \ldots, 3N \), \( N \) being the number of slits, we now introduce for general cases generalized velocity vectors \( w_i \), with

\[
\begin{align*}
  w_1 &:= v_1, & w_2 &:= u_{1R}, & w_3 &:= u_{1L} \\
  w_4 &:= v_2, & w_5 &:= u_{2R}, & w_6 &:= u_{2L}
\end{align*}
\] (2.4)

for the first channel, and

\[
\begin{align*}
  w_4 &:= v_2, & w_5 &:= u_{2R}, & w_6 &:= u_{2L}
\end{align*}
\] (2.5)

for the second channel in the case of \( N = 2 \). The associated amplitudes \( R(w_i) \) for each channel are taken to be the same, i.e. \( R(w_1) = R(w_2) = R(w_3) = R_1 \), and \( R(w_4) = R(w_5) = R(w_6) = R_2 \). This renumbering procedure will turn out as an important practical bookkeeping tool.
3. A SUPERCLASSICAL CURRENT ALGEBRA

Generally, a probability density current is defined as $\mathbf{J} = P\mathbf{v}$. To calculate the total average current, we sum up all contributions of said vectors $\mathbf{v}_i(x, t)$ and $\mathbf{u}_{R(L)}(x, t)$ for each point $(x, t)$ of the plane by matching the unit velocity vector component, associated with each vector and multiplied with the corresponding amplitude $R(w_i)$, one by one with all other unit vector components together with their amplitudes. In other words, the corresponding probability density $P(w_i)$ for any channel or, respectively, for any velocity component $w_i$, is obtained by the pairwise projection on the unit vector $\hat{w}_i$ weighted by $R(w_i)$ of the totality of all amplitude weighted unit velocity vectors being operative at $(x, t)$.

In Fig. 3.1 the projection scheme generating the partial current $\mathbf{J}(a)$ is shown symbolically for a total of three velocity channels $a$, $b$, and $c$. The projections of the unit vectors $\hat{b}$ and $\hat{c}$ of the second and third velocity vectors are indicated as dashed lines. The probability density $P(a)$ for said current is built by the products $P(a) = R(a)\hat{a} \cdot [\hat{a}R(a) + \hat{b}R(b) + \hat{c}R(c)] = R^2(a) + R(a)R(b)\cos\varphi_{a,b} + R(a)R(c)\cos\varphi_{a,c}$, with $\hat{a} \cdot \hat{b} = \cos\varphi_{a,b}$, etc.

In case of only two velocities $a$ and $b$, one immediately sees the resemblance with the classical interference amplitude of Eq. (2.1): $P(a) + P(b) = R^2(a) + 2R(a)R(b)\cos\varphi_{a,b} + R^2(b)$, with the main difference consisting in the phase difference as the included angle between $\hat{a}$ and $\hat{b}$ according to Eq. (2.3).

The two-path set-up has $3N = 6$ velocity vectors at each point (cf. Figs. 2.2 and 3.1 and Eqs. (2.4) and (2.5)) and we obtain for the partial intensities and currents, i.e. for each

![Figure 3.1](image-url)
channel component $i$

\[ P(w_i) = R(w_i)\hat{w}_i \cdot \sum_{j=1}^{6} \hat{w}_j R(w_j) \]  
\[ J(w_i) = w_i P(w_i), \quad i = 1, \ldots, 6, \]  

with

\[ \cos \varphi_{i,j} := \hat{w}_i \cdot \hat{w}_j. \]  

Consequently, the total intensity and current read as

\[ P_{\text{tot}} = \sum_{i=1}^{6} P(w_i) = \left( \sum_{i=1}^{6} \hat{w}_i R(w_i) \right)^2, \]  
\[ J_{\text{tot}} = \sum_{i=1}^{6} J(w_i) = \sum_{i=1}^{6} w_i P(w_i), \]  

leading to the emergent total velocity

\[ v_{\text{tot}} = \frac{J_{\text{tot}}}{P_{\text{tot}}} = \frac{\sum_{i=1}^{6} w_i P(w_i)}{\sum_{i=1}^{6} P(w_i)}. \]

Thus we obtain phase-dependent amplitude contributions of the total system’s wave field projected on each channel’s amplitude at point $(x, t)$ via the conditional probability $P(w_i)$. The local intensity of a partial current is dependent on all other currents, and the total current itself is composed of all partial components. This mutual dependence of a current’s “totality” and its parts, we denote as “relational causality” [8], and this constitutes the essential part of what we call a “superclassical” current algebra. We denote a theory as “superclassical” if it covers a vast range of spatio-temporal scales, i.e. from a usual classical down to a hypothesized sub-quantum domain. In assuming the sub-quantum domain to be described in terms of modern classical physics, quantum theory thus appears as sandwiched between two classical regimes. “Super-classical” is introduced in close analogy to the use of the term “superstatistics”, which itself not only relates vastly disparate space-time scales, but also shows highly unexpected emergent behavior on intermediate scales [13, 14]. We thus presume that quantum phenomenology emerges from the superposition of processes on said vastly disparate scales. The term “current algebra” is borrowed from quantum field
theory, but used here in a very broad sense, with ”algebra” referring to its original meaning as ”reunion of broken parts”, i.e. in the sense of a proper combinatorics of currents. However, both of said concepts share the property of using currents as basic ingredient and not as derivation of some elementary entity like, e.g., an elementary particle.

Note that the usual symmetry (cf. the classical interference case above) between $P(w_i)$ and $R(w_i)$ is broken: $P(w_i) \neq R^2(w_i)$, i.e. although to each velocity component $w_i$ an amplitude $R(w_i)$ is associated, the partial probability density $P(w_i)$ is not the mere squared amplitude any more. The consequences of this asymmetry are discussed in the following section.

Returning now to our previous notation for the six velocity components $v_i, u_{iR}, u_{iL}; i = 1, 2$, the partial current associated with $v_1$ originates from building the scalar product of $\hat{v}_1$ with all other unit vector components and reads as

$$J(v_1) = v_1 P(v_1) = v_1 R_1 \hat{v}_1 \cdot (\hat{v}_1 R_1 + \hat{u}_{1R} R_1 + \hat{u}_{1L} R_1 + \hat{v}_2 R_2 + \hat{u}_{2R} R_2 + \hat{u}_{2L} R_2). \quad (3.7)$$

Since trivially

$$\hat{u}_{iR} R_i + \hat{u}_{iL} R_i = 0, \quad i = 1, 2, \quad (3.8)$$

Eq. (3.7) leads to

$$J(v_1) = v_1 \left( R_1^2 + R_1 R_2 \cos \varphi \right), \quad (3.9)$$

which results from the representation of the emerging velocity fields according to Eq. (2.3), since we get the cosine of the phase difference $\varphi$ as a natural result of the scalar product of the velocity vectors $v_i$. The non-zero residua of the other vector fields yield

$$J(u_{1R}) = u_{1R} P(u_{1R}) = u_{1R} (R_1 \hat{u}_{1R} \cdot \hat{v}_2 R_2) = u_{1R} R_1 R_2 \cos \left( \frac{\pi}{2} - \varphi \right) = u_{1R} R_1 R_2 \sin \varphi \quad (3.10)$$

and

$$J(u_{1L}) = u_{1L} P(u_{1L}) = u_{1L} (R_1 \hat{u}_{1L} \cdot \hat{v}_2 R_2) = u_{1L} R_1 R_2 \cos \left( \frac{\pi}{2} + \varphi \right) = -u_{1L} R_1 R_2 \sin \varphi. \quad (3.11)$$

Analogously, we obtain for the convective velocity vector field of the second channel

$$J(v_2) = v_2 P(v_2) = v_2 \left( R_2^2 + R_1 R_2 \cos \varphi \right). \quad (3.12)$$
The corresponding diffusive velocity vector fields read as

$$J(u_{2R}) = u_{2R}P(u_{2R}) = u_{2R}(R_2\hat{u}_{2R} \cdot \hat{v}_1 R_1) = u_{2R}R_1 R_2 \cos \left( \frac{\pi}{2} + \varphi \right) = -u_{2R} R_1 R_2 \sin \varphi,$$

(3.13)

$$J(u_{2L}) = u_{2L}P(u_{2L}) = u_{2L}(R_2\hat{u}_{2L} \cdot \hat{v}_1 R_1) = u_{2L}R_1 R_2 \cos \left( \frac{\pi}{2} - \varphi \right) = u_{2L} R_1 R_2 \sin \varphi.$$

(3.14)

Note that the nontrivial sine contributions to the total current stem from the projections between the diffusive velocities $u_{iR(L)}$ of the first channel on the unit vector $\hat{v}_2$ of the convective velocity of the second channel, and vice versa. Combining all terms, we obtain the result for the total current

$$J_{\text{tot}} = v_1 P(v_1) + u_{1R} P(u_{1R}) + u_{1L} P(u_{1L}) + v_2 P(v_2) + u_{2R} P(u_{2R}) + u_{2L} P(u_{2L})$$

$$= R_1^2 v_1 + R_2^2 v_2 + R_1 R_2 (v_1 + v_2) \cos \varphi + R_1 R_2 ([u_{1R} - u_{1L}] - [u_{2R} - u_{2L}]) \sin \varphi.$$

(3.15)

The resulting diffusive velocities $u_{iR} - u_{iL}$ are identified with the effective diffusive velocities $u_i$ for each channel. Note that one of those velocities, $u_{iR}$ or $u_{iL}$, respectively, is always zero, so that the product of said difference with $\sin \varphi$ guarantees the correct sign of the last term in Eq. (3.15). Thus we obtain the final expression for the total density current built from the remaining $2N = 4$ velocity components

$$J_{\text{tot}} = R_1^2 v_1 + R_2^2 v_2 + R_1 R_2 (v_1 + v_2) \cos \varphi + R_1 R_2 (u_1 - u_2) \sin \varphi.$$

(3.16)

Summing up the probabilities associated with each of the partial currents we get according to the ansatz (3.1) and the relations (3.4) and (3.8)

$$P_{\text{tot}} = (R_1 \hat{v}_1 + R_1 \hat{u}_{1R} + R_1 \hat{u}_{1L} + R_2 \hat{v}_2 + R_2 \hat{u}_{2R} + R_2 \hat{u}_{2L})^2$$

$$= (R_1 \hat{v}_1 + R_2 \hat{v}_2)^2 = R_1^2 + R_2^2 + 2R_1 R_2 \cos \varphi = P(v_1) + P(v_2).$$

(3.17)

The total velocity $v_{\text{tot}}$ according to Eq. (3.6) now reads as

$$v_{\text{tot}} = \frac{R_1^2 v_1 + R_2^2 v_2 + R_1 R_2 (v_1 + v_2) \cos \varphi + R_1 R_2 (u_1 - u_2) \sin \varphi}{R_1^2 + R_2^2 + 2R_1 R_2 \cos \varphi}.$$

(3.18)

The obtained total density current field $J_{\text{tot}}(x, t)$ spanned by the various velocity components $v_i(x, t)$ and $u_{iR(L)}(x, t)$ we have denoted as the “path excitation field” [10]. It is built by the sum of its partial currents, which themselves are built by an amplitude weighted
projection of the total current. Furthermore, we observe that the superposition principle is violated for \( J \), and, analogously for \( P \), in the following sense: Due to the “entanglement” of partial probability densities with their corresponding partial currents according to Eq. (3.1), it generally holds that

\[
J(w_i + w_j) \neq J(w_i) + J(w_j),
\]

since \((w_i + w_j)P(w_i + w_j) \neq w_iP(w_i) + w_jP(w_j)\), except for the special case of \( P(w_i + w_j) = \frac{P(w_i) + P(w_j)}{2} \), which is fulfilled either by \( P(w_i) = P(w_j) \) or \( w_i = w_j \), respectively.

This result has to be interpreted in the following way. In orthodox quantum mechanics the amplitudes of the wave function components have to be summed up coherently in the case of undisturbed paths (“superposition”), and for calculation of the probability density this sum has to be taken as absolute value squared. Or, in other words, the Schrödinger equation is linear, and observation of a state is regularized by Born’s rule. In our case, all the relevant variables, i.e. \( P(w_i) \) and \( J(w_i) \) are not linear. Consequently, to obtain the correct total probability density \( P_{\text{tot}} \) or total current \( J_{\text{tot}} \), respectively, one has to take into account all elementary, i.e. partial contributions to the corresponding variable.

Summarizing, the shift to a new representation for the emerging velocity vectors (cf. Eq. (2.3)) and the projection rule of Eq. (3.1) build the kernel for a set of relations denoted as superclassical current algebra. It is characterized by summing up the nonlinear, probability-entangled partial currents, where each of the latter ones contains information about the total field via a projection rule. This property we have characterized as “relational causality”: Any change in a local field affects the total field, and vice versa.

The trajectories or streamlines, respectively, are obtained according to \( \dot{x} = v_{\text{tot}} \) in the usual way by integration. Referring to [10], we just mention that by re-inserting the expressions for convective and diffusive velocities \( v_{i,\text{conv}} = \frac{\nabla S_i}{m} \), \( u_i = -\frac{\hbar}{m} \frac{\nabla R_i}{R_i} \), one immediately identifies Eq. (3.18) with the Bohmian guiding equation and Eq. (3.16) with the quantum mechanical pendant for the probability density current [11].

Again we have to emphasize that our result was achieved solely out of kinematic relations by applying the rules mentioned above without invoking complex \( \psi \) functions or the like. However, as opposed to the Bohmian theory, we obtain our results not in configuration space but in common coordinate space. With respect to the following discussion of the three-slit case, we can state that Eq. (3.16) reflects the canonical result for the double slit in quantum mechanics, i.e. the result for the probability density for detecting a particle passing a double
slit or a two-way interferometer undisturbed consists of the joint probabilities of having only one slit open in each case plus an interference term (in our case the sum of the sine and cosine contributions). However, the validity of Born’s rule will show only after examination of the three-slit or three-beam case, respectively.

4. THREE-SLIT INTERFERENCE AND BORN’S RULE

The extension to three slits, beams, or probability current channels, respectively, is straightforward. Analogously to Fig. 2.2 we introduce a third emergent propagation velocity $v_3$ and its corresponding diffusive velocities $u_{3L(R)}$. The phase shift of the third beam is denoted as $\chi$ and represents the angle between the second and the third beam in our geometric representation of the path excitation field. Analogously to the case of the double slit, the three slit setup can be replaced by a three path interferometer as shown in Fig. 4.1. According to Born’s rule the probability of even a single particle passing any of the three slits splits into a sum of probabilities passing the slits pairwise, i.e. going along both $A$ and $B$, $B$ and $C$, or $A$ and $C$, but never passing $A$, $B$ and $C$ simultaneously. Whether this splitting can be experimentally verified (cf. [3] and [4]) in exactly that way seems to be an open question [5], since it is argued that with the decomposition of the three-slit wave function into its pairs information about the original wave function is lost. Therefore, an experimental test of Born’s rule by measuring the outcome of blocked pathways and summing them up seems not to be physically conclusive [5].

From the theoretical point of view interference-type phenomena have been analyzed thoroughly [15], for the cases of only one open slit up to $N$ open slits. For a double slit setup the interference term is non-zero, i.e. $I_{AB} := P_{AB} - P_A - P_B \neq 0$, with $P_{A(B)}$ being the detection probability with only one slit/path $A$ or $B$, respectively, of a total of $N$ slits/paths open, and $P_{AB}$ for both slits $A$ and $B$ open. This well-known fact, representing the “heart” of quantum mechanics, is to be contrasted with Sorkin’s results for the following, so-called “second order sum rule” [15]:

$$I_{ABC} := P_{ABC} - P_{AB} - P_{AC} - P_{BC} + P_A + P_B + P_C$$

$$= P_{ABC} - (P_A + P_B + P_C + I_{AB} + I_{AC} + I_{BC}) = 0.$$

This result is remarkable insofar as it can be inferred that interference terms theoretically
always originate from pairings of paths, but never from triples etc. Any violation of this second order sum rule, i.e. \( I_{ABC} \neq 0 \), and thus of Born’s rule would have dramatic consequences for quantum theory like a modification of the Schrödinger equation, for example.

By returning to our model, the total probability density current for three paths is calculated according to the rules set up in Section 3. We adopt the notations of the two slit system also for three slits, i.e. now employing nine velocity contributions: \( \mathbf{v}_i \), \( \mathbf{u}_{iR(L)} \), \( i = 1, 2, 3 \). Analogously, the three generally different amplitudes are denoted as \( R(\mathbf{v}_i) = R(\mathbf{u}_{iR}) = R(\mathbf{u}_{iL}) = R_i, i = 1, 2, 3 \). We keep the definition of \( \varphi \) as \( \varphi := \arccos(\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2) \), and we define the second angle as \( \chi := \arccos(\hat{\mathbf{v}}_2 \cdot \hat{\mathbf{v}}_3) \). Similarly to Eq. (3.8), the diffusive velocities \( \mathbf{u}_{iR} - \mathbf{u}_{iL} \) combine to \( \mathbf{u}_i \), \( i = 1, 2, 3 \), thus ending up with \( 2N = 6 \) effective velocities.
Therefore we obtain, analogously to the calculation in the previous section,

\[ J_{\text{tot}} = R_1^2 v_1 + R_2^2 v_2 + R_3^2 v_3 + R_1 R_2 (v_1 + v_2) \cos \varphi + R_1 R_2 (u_1 - u_2) \sin \varphi \]
\[ + R_1 R_3 (v_1 + v_3) \cos (\varphi + \chi) + R_1 R_3 (u_1 - u_3) \sin (\varphi + \chi) \]
\[ + R_2 R_3 (v_2 + v_3) \cos \chi + R_2 R_3 (u_2 - u_3) \sin \chi \]

(4.2)

and

\[ P_{\text{tot}} = R_1^2 + R_2^2 + R_3^2 + 2R_1 R_2 \cos \varphi + 2R_1 R_3 \cos (\varphi + \chi) + 2R_2 R_3 \cos \chi \]

(4.3)

\[ = P(v_1) + P(v_2) + P(v_3). \]

In analogy to the double slit case (cf. Eq. (3.17)) we obtain – at first sight – a classical Kolmogorov sum rule for the probabilities on the one hand, but also the complete interference effects for the double, three- and, as will be shown in the next section, for the \( N \) -slit cases, on the other hand. However, the particular probabilities \( P(v_i) \) in Eq. (3.17) and Eq. (4.3), do not correspond to the probabilities of the assigned slits if solely opened, i.e. \( P_{AB}(v_1) = (R_1^2 + R_2 R_2 \cos \varphi) \neq P_A(v_1) = R_1^2 \). Consequently, each of the probability summands in said equations does not correspond to an independent probability of the respective slit if solely opened. Note that our result reflects an illustrative remark of Ballentine [16] stating the fact that \( I_{AB} \neq 0 \) for the double slit experiment does not mean that the classical probability sum rules are violated, since they are originally formulated for mutually exclusive states. By keeping in mind that said probabilities \( P_A, P_B, \) and \( P_{AB} \) are in fact conditional probabilities, there is no violation of any classical probability sum rule by stating the experimental observation \( P_{AB} \neq P_A + P_B \). Translated into our double slit model, we have in the case of both slits \( A \) and \( B \) open, \( P_{AB} =: P_{\text{tot}}(A \land B) = P(v_1) + P(v_2) = R_1^2 + R_2^2 + 2R_1 R_2 \cos \varphi, \) which clearly must not be confused with the mutually exclusive case \( P_{\text{tot}}(A \lor B) = P_A(v_1) + P_B(v_2) = R_1^2 + R_2^2. \)

Finally, we obtain for the cases of one (i.e. \( N = A \)), two and three open slits, respectively,

\[ I_A = P_A(v_1) = R_1^2, \]

(4.4)

\[ I_{AB} = P_{AB} - P_A(v_1) - P_B(v_2) = 2R_1 R_2 \cos \varphi, \]

(4.5)

\[ I_{ABC} = P_{ABC} - P_{AB} - P_{AC} - P_{BC} + P_A(v_1) + P_B(v_2) + P_C(v_3) = 0, \]

(4.6)

where \( P_{AB} \) is assigned to \( P_{\text{tot}} \) of Eq. (3.17) and \( P_{ABC} \) to \( P_{\text{tot}} \) of Eq. (4.3). In the double slit case, e.g., with slits \( A \) and \( B \) open, we obtain the results of (3.17). If \( B \) were closed and
$C$ were open instead, we would get the analogous result, i.e. $v_2$ and $\varphi$ replaced by $v_3$ and $\varphi_{1,3}$. If all three slits $A, B, C$ are open, we can use the pairwise permutations of the double slit case, i.e. $A \land B, A \land C,$ or $B \land C,$ respectively, with $\varphi_{1,3}$ identified with $(\varphi + \chi)$, etc. Thus we conclude that in our model the addition of “sub-probabilities” indeed works and provides the correct results.

Summarizing, with our superclassical model emerging out of a sub-quantum scenario we arrive at the same results as standard quantum mechanics fulfilling Sorkin’s sum rules [15]. Opposed to the open question in quantum mechanics of whether said decomposition of a three-slit probability term into its sum of double- and one-slit probabilities only represents a “mathematical trick” [5], we observe the following: Whereas in standard quantum mechanics Born’s rule originates from building the squared absolute values of additive $\psi$ functions representing the probability amplitudes for different paths, in our case we obtain the pairing of paths as a natural consequence of pairwise selection of unit vectors of all existing velocity components constituting the probability currents. Thus we obtain all possible pathways within an $N$-slit setup by a two-channel projection method. The sum rules, Eqs. (3.1) through (3.6), guarantee that each partial contribution, be it from the velocity contributions within a particular channel or from different channels, accounts for the final total current density for each point between source and detector. Since for only one slit open the projection rule (3.1) trivially leads to a linear relation between $P$ and $R^2$, the asymmetry between the latter quantities, due to the nonlinear projection rule, becomes effective for $N \geq 2$ slits open. Consequently, the violation of the first order sum rule (4.5), i.e. $I_{AB} \neq 0$, represents a natural result of our principle of relational causality. Moreover, as we have argued above, the opening of an additional slit solely adds pairwise path combinations. As all higher interference terms have already incorporated said asymmetry, the result can finally be reduced to the double slit case, thus yielding a zero result as in (4.6) according to Sorkin’s analysis.

This is a further hint that our model can reproduce all phenomena of standard quantum theory with the option of giving a deeper reasoning to principles like Born’s rule or the hierarchical sum-rules, respectively, with the prospect of a physics beyond quantum theory (cf. the discussion in the last section).
5. N-SLIT INTERFERENCE AND THE QUANTUM TALBOT EFFECT

We can already infer from the three-slit device that due to the pairwise selection of the velocity field components $v_i(x, t)$ and $u_{iL(R)}(x, t)$, $i = 1, \ldots, N$, the interference effect of every higher order grating can be reduced to successive double-slit algorithms. For a compact description of the $N$-slit case we return to the notation (2.4) and (2.5) of general velocity vectors $w_i$ with

$$w_1 := v_1, \quad w_2 := u_{1R}, \quad w_3 := u_{1L}, \quad w_4 := v_2, \ldots, \quad w_{3N} := u_{NL},$$

with $w_{3i-2} := v_i$ referring to the propagation velocities, $w_{3i-1} := u_{iR}$ and $w_{3i} := u_{iL}$, with $i = 1, \ldots, N$ denoting the diffusion velocities for each channel $i$. According to the Eqs. (3.1) to (3.5), now with a general number $N$ of slits, the calculation for the total probability density is straightforward, since only the contributions of the propagation velocities are non-zero:

$$P_{\text{tot}}(N) = \left( R(w_1)\tilde{w}_1 + R(w_4)\tilde{w}_4 + \ldots + R(w_{3N-2})\tilde{w}_{3N-2} \right)^2 = \left( \sum_{i=1}^{N} R(w_{3i-2})\tilde{w}_{3i-2} \right)^2$$

$$= \sum_{i=1}^{N} \left( R^2(w_{3i-2}) + \sum_{j=i+1}^{N} 2R(w_{3i-2})R(w_{3j-2}) \cos \varphi_{i,j} \right).$$

For the current density we can generalize the relations (3.8) to $\tilde{w}_{3i-1}R(w_{3i-1}) + \tilde{w}_{3i}R(w_{3i}) = 0$, and obtain

$$J_{\text{tot}} = \sum_{i=1}^{3N} J(w_i) = \sum_{i=1}^{3N} \left( R(w_i)w_i \cdot \sum_{j=1}^{N} \tilde{w}_{3i-2}R(w_{3j-2}) \right).$$

The partial current summands for the propagation velocities $w_{3i-1}$, $i = 1, \ldots, N$, read as

$$J(w_{3i-2}) = w_{3i-2} \left( R^2(w_{3i-2}) + \sum_{j \neq i=1}^{N} R(w_{3i-2})R(w_{3j-2}) \cos \varphi_{i,j} \right).$$

Since the calculations for the diffusive currents are lengthy, but straightforward, we now present the final results. We use our previous notation for the final $2N$ effective velocity contributions with $N$ slits

$$w_{3i-2} := v_i, \quad w_{3i-1} - w_{3i} := u_i, \quad R(w_{3i-2}) = R(w_{3i-1}) = R(w_{3i}) =: R_i, \quad i = 1, \ldots, N.$$
leading to

\[ P_{\text{tot}}(N) = \left( \sum_{i=1}^{N} R_i \hat{v}_i \right)^2 = \sum_{i=1}^{N} P(v_i) = \sum_{i=1}^{N} \left( R_i^2 + \sum_{j=i+1}^{N} 2R_iR_j \cos \varphi_{i,j} \right), \]  

(5.6)

\[ J_{\text{tot}}(N) = \sum_{i=1}^{N} \left( R_i^2 v_i + \sum_{j=i+1}^{N} R_iR_j \left\{ (v_i + v_j) \cos \varphi_{i,j} + (u_i - u_j) \sin \varphi_{i,j} \right\} \right). \]  

(5.7)

From these results we can clearly see that the addition of an arbitrary number of slits represents a simple inductive extension from the double slit case as we had stated in the previous section.

In well-known manner one obtains the trajectories from \( \dot{x}_{\text{tot}} = v_{\text{tot}} = J_{\text{tot}}/P_{\text{tot}} \) [11]. As opposed to this analytical procedure, we use the simulation tools disclosed in [17], which are displayed in the computer simulations of Fig. 5.1 for a 7-slit and a 27-slit setup, respectively. Already for the 7-slit case one can observe the emergence of a repetitive short range pattern until the Fraunhofer regime is reached. At the so-called Talbot distance

\[ z_T = d^2/\lambda, \]  

(5.8)

where \( d \) denotes the grating period and \( \lambda \) the wavelength of the incident plane wave, the initial patterns of the 7 vertically arranged slit openings reappear with a shift of \( d/2 \). Table 5.1 shows the results for different values of \( \lambda \) and \( d \) and compares them with the observed values \( y_T \) of the Talbot distance for the 7- and various \( N \)-slit cases. According to Table 5.1, we use the parameters for neutrons as \( d = 1.06 \text{ nm} \) and \( \lambda = 1 \text{ nm} \), with a neutron mass

\[ m_n = 1.675 \cdot 10^{-27} \text{ kg}. \]

The spatial step width is chosen as \( \Delta x = 0.0378 \text{ nm} \), the time resolution as \( \Delta t = 1.92 \cdot 10^{-14} \text{ s} \). Said shifted reappearance of the pattern occurs for the first time at time step 150, i.e. at \( t_T = 150 \cdot \Delta t = 2.88 \cdot 10^{-12} \text{ s} \). The standard transformation to the two-dimensional case [18] by re-parametrizing the \( t \)-axis via \( y = \hbar k_n \Delta t/m_n = h \Delta t/(\lambda m_n) \) leads to the observed distance \( y_T = h t_T/(\lambda m_n) = 1.14 \text{ nm} \), which matches nicely with the formula of the Talbot distance \( z_T \) of (5.8). The observed values for the Talbot distance \( y_T \) in our discretized model agree for any \( N \)-slit setup as expected in accordance with Eq. (5.8), which only depends on \( d \) and \( \lambda \). Moreover, we also obtain the correct results for any other choice of \( m \) or \( \lambda \).

For multiples of \( 2z_T \) the recurrence of the original state is observed, as it is particularly obvious in the case of 27 slits. Due to the non-crossing of all trajectories, as it has been
Figure 5.1. Intensity distributions via classical computer simulations of the Talbot carpet for a 7-slit ($d = 1.06 \text{ nm}$) and a 27-slit ($d = 0.53 \text{ nm}$) setup of Table 5.1, respectively. Averaged particle trajectories are displayed in red.

discussed in [10], the “caverns” in the middle stay confined until they are broken up by the
Table 5.1. Parameters for the Talbot carpet simulations of Fig. 5.1.

| Setup     | a   | b   | c   | d   |
|-----------|-----|-----|-----|-----|
| \( \lambda \) | 1 nm | 1 nm | 1 nm | 1 nm |
| \( d \)    | 0.53 nm | 1.06 nm | 1.59 nm | 2.12 nm |
| \( z_{T} \) | 0.28 nm | 1.13 nm | 2.53 nm | 4.5 nm |
| \( y_{T,7 \text{-slit}} \) | 0.28 nm | 1.14 nm | 2.53 nm | 4.52 nm |
| \( y_{T,N \text{-slit}} \) | 0.29 nm \((N = 27)\) | 1.13 nm \((N = 27)\) | 2.53 nm \((N = 25)\) | 4.49 nm \((N = 13)\) |

influence of the boundary area via the “light cone”. In the limit of an indefinitely extended grating the pattern clearly would be maintained \textit{ad infinitum}.

Since the averaged trajectories obtained with our superclassical current algebra are identified with the Bohmian trajectories of Sanz \textit{et al.} [19], we have thus shown that the emerging quantum carpet for \( N \) slits constituted by characteristic repetitive patterns can be reproduced without any (real or complex) quantum mechanical state function.

6. CONCLUSIONS AND OUTLOOK

It has been shown in a series of papers [9, 10, 20–24] that phenomena of standard quantum mechanics like Gaussian dispersion of wave packets, superposition, double slit interference, Planck’s energy relation, or the Schrödinger equation, can be assessed as the emergent property of an underlying sub-structure of the vacuum combined with diffusion processes reflecting also the stochastic parts of the zero-point field, i.e. the zero-point fluctuations [25]. Thus we obtain the quantum mechanical results as an averaged behavior of sub-quantum processes. The inclusion of relativistic physics has not been considered yet, but should be possible in principle.

In the present paper we have started with a “minimal set” of assumptions and results from our previous work, like the use of classical propagation velocities \( v_{i}(\mathbf{x},t) \) and diffusive velocities \( u_{i}(\mathbf{x},t) \), the orthogonality relation between them and the inclusion of the phase
angle between the propagation velocity vectors of the emerging path excitation field, the latter one spanned by said velocity vectors and additionally weighted with the corresponding probability densities.

We introduced an amplitude weighted projection rule to account for all partial velocity contributions to a partial current $J(w_i)$ per velocity channel, whereby the symmetry between amplitude and the probability density, usually given by the squared amplitude, is broken. The total wave intensity field accounting for the emerging thermal landscape consists of the sum of all ‘local’ intensities in each channel, and the ‘local’ intensity in each channel is the result of the interference with the total intensity field. The mutual dependence of a total current and its parts we denoted as “relational causality”, and this represents an essential part of the calculus which we subsumed as superclassical current algebra, combining the physics of different scales, e.g., sub-quantum and classical macro physics. The presence of all velocity channels at any point of the spatio-temporal domain between source and detector is based upon our ansatz that $N$ Gaussians (representing the wave amplitudes of a particle immediately after passing $N$ slits/paths) do not have any artificial cut-off, but actually extend across the whole slit/path system [24, 26]. This is supported by experimental evidence showing that interference can be caused by the nonlocally far-reaching action of the plane-waves of a quantum mechanical wave-function [27, 28], and by the works of Mandelis [29, 30], where diffusion wave fields are related to oscillating sources extending nonlocally across the whole domain of an experimental setup.

As an important result, and as a natural consequence of our considerations, a third order interference term violating Born’s rule in orthodox quantum mechanics is absent in our superclassical framework. We have shown, on the one hand, that the step from a one-slit/path setup to the double slit/path case introduces a new quality due to the nonlinear projection rules, which is kept in all higher extensions. On the other hand, the total probability for the $N$-slit/path case is built by the summation of “sub-probabilities”, i.e. the probabilities for $(N - 1)$-slit/path configurations. Altogether, the double slit/path case represents an exceptional system insofar as a new quality interference appears and all higher ($N > 2$) configurations are reducible to it. Consequently, the pairwise path selection, the violation of the first order sum rule, and the validity of all higher order sum rules are explained naturally within our theoretical framework.

Furthermore, we have derived general formulas for the $N$-slit current densities and thus
are able to give a micro-causal account for the kinematics of the quantum Talbot effect. The Talbot distance can be reproduced also quantitatively in our model.

Throughout the whole paper we have made use of averaged diffusive velocities emerging from billions of billions sub-quantum fluctuations. Therefore, within the framework of our theory we can tackle questions going beyond standard quantum theory. At the emerging quantum level, i.e. at times $t \gg 1/\omega$, with $\omega$ representing the familiar zitterbewegung frequency, e.g., for the electron $\omega \approx 10^{21}$ Hz, we obtain exact results strongly suggesting the validity of Born’s rule, for example. However, approaching said sub-quantum regions by increasing the time resolution to the order of $t \approx 1/\omega$ suggests a possibly gradual breakdown of said rule, since the averaging of the diffusive and convective velocities and their mutual orthogonality of the averaged velocities is not reliable any more. In principle, this should eventually be testable in experiment.

**ACKNOWLEDGMENTS**

We thank Hans De Raedt for pointing out ref. [16] to us, Jan Waldeczek for many enlightening discussions, and the Fetzer Franklin Fund for partial support of the current work.

[1] M. Born, “Zur Quantenmechanik der Stoßvorgänge,” *Z. Phys. A: Hadrons Nucl.* **37** (1926) 863–867.

[2] U. Sinha, C. Couteau, Z. Medendorp, I. Söllner, R. Laflamme, R. Sorkin, and G. Weihs, “Testing Born’s rule in quantum mechanics with a triple slit experiment,” *AIP Conf. Proc.* **1101** (2009) 200–207, arXiv:0811.2068 [quant-ph].

[3] U. Sinha, C. Couteau, T. Jennewein, R. Laflamme, and G. Weihs, “Ruling out multi-order interference in quantum mechanics,” *Science* **329** (2010) 418–421, arXiv:1007.4193 [quant-ph].

[4] I. Söllner, B. Gschösser, P. Mai, B. Pressl, Z. Vörös, and G. Weihs, “Testing Born’s rule in quantum mechanics for three mutually exclusive events,” *Found. Phys.* **42** (2012) 742–751, arXiv:1201.0195v1 [quant-ph].

21
[5] H. De Raedt, K. Michielsen, and K. Hess, “Analysis of multipath interference in three-slit experiments,” Phys. Rev. A 85 (2012) 012101, arXiv:.org/abs/1112.2665v1.
[6] N. P. Landsman, “Born rule and its interpretation,” in Compendium of Quantum Physics, D. Greenberger, K. Hentschel, and F. Weinert, eds., pp. 64–70. Springer, Berlin, Heidelberg, 2009.
[7] G. Grössing, ed., Emergent Quantum Mechanics 2011. No. 361/1. IOP Publishing, Bristol, 2012. http://iopscience.iop.org/1742-6596/361/1.
[8] J. Walleczek, “Mission statement,” 2012.
http://www.fetzer-franklin-fund.org/mission/.
[9] G. Grössing, S. Fussy, J. Mesa Pascasio, and H. Schwabl, “Emergence and collapse of quantum mechanical superposition: Orthogonality of reversible dynamics and irreversible diffusion,” Physica A 389 (2010) 4473–4484, arXiv:1004.4596 [quant-ph].
[10] G. Grössing, S. Fussy, J. Mesa Pascasio, and H. Schwabl, “An explanation of interference effects in the double slit experiment: Classical trajectories plus ballistic diffusion caused by zero-point fluctuations,” Ann. Phys. 327 (2012) 421–437, arXiv:1106.5994 [quant-ph].
[11] Á. S. Sanz and S. Miret-Artés, “A trajectory-based understanding of quantum interference,” J. Phys. A: Math. Gen. 41 (2008) 435303, arXiv:0806.2105 [quant-ph].
[12] M. V. Berry, “Optical currents,” J. Opt. A: Pure Appl. Opt. 11 (2009) 094001.
[13] P. Jizba and F. Scardigli, “Emergence of special and doubly special relativity,” Phys. Rev. D 86 (2012) 025029, arXiv:1105.3930 [hep-th].
[14] P. Jizba and F. Scardigli, “Quantum mechanics and local Lorentz symmetry violation,” J. Phys.: Conf. Ser. 361 (2012) 012026.
[15] R. D. Sorkin, “Quantum mechanics as quantum measure theory,” Mod. Phys. Lett. A 09 (1994) 3119–3127, arXiv:gr-qc/9401003.
[16] L. E. Ballentine, “Probability theory in quantum mechanics,” Am. J. Phys. 54 (1986) 883.
[17] J. Mesa Pascasio, S. Fussy, H. Schwabl, and G. Grössing, “Modeling quantum mechanical double slit interference via anomalous diffusion: Independently variable slit widths,” Physica A 392 (2013) 2718–2727, arXiv:1304.2885 [quant-ph].
[18] P. R. Holland, The Quantum Theory of Motion. Cambridge University Press, Cambridge, 1993.
[19] Á. S. Sanz and S. Miret-Artés, “A causal look into the quantum Talbot effect,” *J. Chem. Phys.* **126** (2007) 234106.

[20] G. Grössing, “The vacuum fluctuation theorem: Exact Schrödinger equation via nonequilibrium thermodynamics,” *Phys. Lett. A* **372** (2008) 4556–4563, arXiv:0711.4954 [quant-ph].

[21] G. Grössing, “On the thermodynamic origin of the quantum potential,” *Physica A* **388** (2009) 811–823, arXiv:0808.3539 [quant-ph].

[22] G. Grössing, S. Fussy, J. Mesa Pascasio, and H. Schwabl, “Elements of sub-quantum thermodynamics: Quantum motion as ballistic diffusion,” *J. Phys.: Conf. Ser.* **306** (2011) 012046, arXiv:1005.1058 [physics.gen-ph].

[23] G. Grössing, J. Mesa Pascasio, and H. Schwabl, “A classical explanation of quantization,” *Found. Phys.* **41** (2011) 1437–1453, arXiv:0812.3561 [quant-ph].

[24] G. Grössing, S. Fussy, J. Mesa Pascasio, and H. Schwabl, “‘Systemic nonlocality’ from changing constraints on sub-quantum kinematics,” *J. Phys.: Conf. Ser.* **442** (2013) 012012, arXiv:1303.2867 [quant-ph].

[25] A. M. Cetto and L. de la Peña, “Quantization as an emergent phenomenon due to matter-zeropoint field interaction,” *J. Phys.: Conf. Ser.* **361** (2012) 012013.

[26] G. Grössing, S. Fussy, J. Mesa Pascasio, and H. Schwabl, “A classical framework for nonlocality and entanglement,” *AIP Conf. Proc.* **1508** (2012) 187–196, arXiv:1210.4406 [quant-ph].

[27] H. Rauch, M. Baron, S. Filipp, Y. Hasegawa, H. Lemmel, and R. Loidl, “Hidden observables in neutron quantum interferometry,” *Physica B* **385-386, Part 2** (2006) 1359–1364.

[28] H. Rauch, “Particle and/or wave features in neutron interferometry,” *J. Phys.: Conf. Ser.* **361** (2012) 012019.

[29] A. Mandelis, “Diffusion waves and their uses,” *Phys. Today* **53** (2000) 29–34.

[30] A. Mandelis, *Diffusion-wave fields: Mathematical methods and Green functions*. Springer, New York, NY, 2001.