Semialgebraic decomposition
of real binary forms of a given degree’s space

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Abstract

The Waring Problem over polynomial rings asks for how to decompose an homogeneous polynomial of degree \(d\) as a finite sum of \(d^{th}\) powers of linear forms.

First, we give a constructive method to obtain a real Waring decomposition of any given real binary form with length at most its degree. Secondly, we adapt the Sylvester’s Algorithm to the real case in order to determine a Waring decomposition with minimal length and then we establish its real rank. We use bezoutian matrices to achieve a minimal decomposition.

We consider all real binary forms of a given degree and we decompose this space as a finite union of semialgebraic sets according to their real rank. Some examples are included.

Keywords:
Real binary forms, Semialgebraic sets, Real Waring rank
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1. Introduction

In the 18th century, E. Waring proposed as a conjecture (demonstrated by Hilbert in 1909) that every positive integer is the sum of $n k^{th}$ powers of positive integers, with $n$ depending on $k$. For example, four squares, nine cubic powers or nineteen fourth powers. This classical Waring Problem can be extended to polynomial decompositions in this way: any homogeneous polynomial $p$ of degree $d$ in $n$ variables over a field $K$ can be written as the sum of $r$ $d^{th}$ powers of linear forms. When we take $r$ minimal with this property, we call $r$ the Waring rank of $p$ over $K$. This expression (not necessarily unique) is known as a Waring decomposition of that polynomial, and it has many applications as much in Applied Mathematics as in Engineering (see [8] and the references therein). Applications to Theoretical Physics can be shown in [4]. Nowadays this problem is studied as the problem of decomposition of symmetric tensors. Among open problems we find the description in terms of the Waring rank of the space of tensors of given degree and dimension.

Some papers present the study of particular cases, like monomials (for instance, [6], [11] or [16]), but most authors work usually with “typical forms”, i.e., forms whose Waring rank is stable under perturbations of their coefficients. In fact, a rank $r$ is typical for a given degree $d$ if there exists an Euclidean open set in the space of real degree $d$ forms such that any $p$ in such open set has rank $r$. G. Blekherman [3] or P. Common and G. Ottaviani [14] have analyzed the “typical ranks” of general real binary forms.

The relation between the number of real linear factors and the real Waring rank of binary forms has been also studied by several authors (see [19] and the references therein). N. Tokcan in [19] has studied the real Waring rank for binary forms from the point of view of their factorization.

We study a particular case, that is $K = \mathbb{R}$ and $n = 2$. As A. Causa y R. Re afirm in [12], the real case becomes more complicated that the complex case. Also [3] emphasize the importance of the real case for the applications. This real binary case has been recently investigated by different authors (for instance, [6], [14] or [17]). It is also known that the complex Waring rank is less or equal than the real Waring rank (see [3], where a detail study of this fact is given).

In this paper we collect in Section 2 the principal definitions and notation we use hereinafter. We include Sylvester’s and Borchardt-Jacobi’s Theorems.
In Section 3 we expound on theoretical concepts that justify our Algorithm, inspired by the Sylvester’s one, for Real Waring decomposition (Algorithm 1), with little differences in odd or even cases for the rank. Using this Algorithm we can obtain different real Waring decompositions of length less or equal \( d \) choosing \( \frac{d-1}{2} \), if \( d \) is odd, or \( \frac{d}{2} + 1 \) if \( d \) is even, different parameters that satisfy certain requirements. A pair of examples of this Algorithm is shown at the end of the section.

Section 4 is dedicated to study the Real Waring rank. We present our Real rank length’s decomposition Algorithm (see Algorithm 2), that guarantees a real Waring decomposition with minimal length and then we can use it to determine the Waring rank of a real binary form. We also exhibit a step-by-step example where differences among complex and real ranks can be observed. Thus, we show how this Algorithm improves the previous one as far as Waring decomposition’s length.

In Section 5 we develop the goal of this paper, i.e., the semialgebraic decomposition of the real binary forms of a given degree’s space. We denote \( B_d \) the space of real binary forms of degree \( d \), similar to \( S^n \) or \( S_n \), used for \( \mathbb{K} \) fields in general. In order to prove that \( B_d \) is a semialgebraic set (see Corollary 5.3) we prove previously that the sets \( W^{(r)} \) (similar to \( S_{n,d} \) in the complex case) are semialgebraic sets (see Corollary 5.4, based in Theorem 5.1). Our technique to demonstrate that those sets are all of them semialgebraic is based on Borchardt-Jacobi Theorem (see Theorem 2.9). The principal minors of bezoutian matrices \( B_r(q, q') \) give us a system of conditions which determine the semialgebraic sets. The analogous decomposition in the complex case can be seen in [13]. Moreover, in order to calculate the dimension of \( W^{(r)} \) we can use the usual techniques in Real Geometry. This replies, in the real case, to the Q1 question asked by Carlini in [9] for complex binary forms. In fact, for typical rank \( r \), the dimension of \( W^{(r)} \) is \( d + 1 \).

As a by-product, in Section 6 we obtain the semialgebraic structure of the set of Waring decompositions of \( x^{d-m} y^m \) for \( 1 \leq m \leq d-1 \); the monomials are non typical but very interesting forms (see [10] for the complex case). Finally, we include the semialgebraic decomposition for \( B_3 \) and \( B_4 \) in the Section 7. In the end of this section, when we confront with degrees greater than four, we observe that the description of \( W^{(r)} \) becomes very complicated because of the length and degrees of the polynomials which define it. Therefore we restrict the decomposition for degree 5 to one of the canonical forms that P. Common and G. Ottaviani have described in [14]. In EACA 2016 [1] we
presented the semialgebraic decomposition for one of this canonical forms of degree 5. In \cite{18} we compute the semialgebraic decomposition of the second type of canonical form.

B. Reznik \cite{18} has also studied canonical forms for polynomials, although he works over $\mathbb{C}$. It is a work in progress the computation of canonical forms for typical Waring ranks.

2. Notation and Preliminaries

In this work we are going to consider real binary forms $p(x, y)$ of degree $d$ in the variables $x, y$:

$$p(x, y) = p_{\vec{c}}(x, y) = \sum_{i=0}^{d} \binom{d}{i} c_i x^i y^{d-i},$$

where $\vec{c} = (c_0, \ldots, c_d) \in \mathbb{R}^{d+1}$, except when all of them are zero.

**Definition 2.1** (Waring Decomposition over $\mathbb{K}$ of length $r$). Let $p(x, y)$ be a polynomial of degree $d$. If there exists $r > 0$ such that

$$p(x, y) = \sum_{i=1}^{r} \lambda_i (\alpha_i x + \beta_i y)^d$$

for some $\lambda_i, \alpha_i, \beta_i \in \mathbb{K}$, $i = 1, \ldots, r$, with $\alpha_i$, $\beta_i$ pairwise linearly independent, the expression (2) is a **Waring Decomposition over $\mathbb{K}$ of length $r$**.

**Definition 2.2.** Let $p(x, y)$ be a polynomial of degree $d$. If $r$ is minimal in (2), we will say than $r$ is the **$\mathbb{K}$-rank** of $p(x, y)$ and we will denote it as $rk_{\mathbb{K}}(p) = r$.

**Remark 2.3.** Two polynomials proportional with each other have the same rank.

**Definition 2.4.** We will denote $\mathcal{B}_d$ the space of real binary forms of degree $d$.

We will associate with $p_{\vec{c}}$ the projective point $\vec{c} = [c_0 : c_1 : \cdots : c_d] \in \mathbb{P}_d^d$.

For $\mathbb{K} = \mathbb{R}$ we will consider the following sets:
Definition 2.5. For $r = 1, 2, \ldots, d$, 
\[ \mathcal{W}^{(r)} = \left\{ \mathbf{c} \in \mathbb{P}_{\mathbb{R}}^d \mid r k_p(p_c(x, y)) = r \right\} \subset \mathbb{P}_{\mathbb{R}}^d. \]

Theorem 2.6 (Sylvester’s Algorithm). (See Theorem 2.1 in [8] and the references therein). A binary form of degree $d$, $p$, can be written as a finite sum of $d$th powers of complex linear forms as (2), if and only if

1. There exists a vector $\mathbf{q} = (q_0, \ldots, q_r)$ such that
\[
\begin{pmatrix}
  c_0 & c_1 & \cdots & c_r \\
  c_1 & c_2 & \cdots & c_{r+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{d-r} & c_{d-r+1} & \cdots & c_d
\end{pmatrix}
\begin{pmatrix}
  q_0 \\
  q_1 \\
  \vdots \\
  q_r
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}
\]

2. And the form $q(x, y) = \sum_{i=0}^{r} q_i x^i y^{r-i}$ factors as a product of $r$ distinct complex linear forms, i.e.,
\[ q(x, y) = \prod_{j=1}^{r} (\beta_j x - \alpha_j y). \]

Notation. We will denote $H_r$ the matrix with size $(d-r+1) \times (r+1)$ associated to $p_c(x, y)$:
\[
H_r = \begin{pmatrix}
  c_0 & c_1 & \cdots & c_r \\
  c_1 & c_2 & \cdots & c_{r+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{d-r} & c_{d-r+1} & \cdots & c_d
\end{pmatrix}
\] (3)

We denote its kernel $\text{Ker}(H_r)$, $\delta_r = \text{dim}(\text{Ker}(H_r))$ and the elements in $\text{Ker}(H_r)$ as $\mathbf{q} = (q_0, \ldots, q_r)$.

Definition 2.7. Since $\text{Ker}(H_r) \subset \mathbb{R}^{r+1}$, we have
\[ \tilde{H}_r := \mathbb{P}(\text{Ker}(H_r)) \subset \mathbb{P}(\mathbb{R}^{r+1}) \subset \mathbb{P}(\mathbb{R}^{d+1}) = \mathbb{P}_{\mathbb{R}}^d \] (4)
and also $\tilde{H}_r \hookrightarrow \tilde{H}_j$ for $r + 1 \leq j \leq d$, because $[q_0 : \ldots : q_r : 0 : \ldots : 0] \in \tilde{H}_j$, $\forall r + 1 \leq j \leq d$, for $[q_0 : \ldots : q_r] \in \tilde{H}_r$. 

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Definition 2.8 (Bezoutian matrix). Let \( u(t) = \sum_{i=0}^{n} u_i t^i \) and \( v(t) = \sum_{i=0}^{n} v_i t^i \) be two real polynomials in a variable \( t \). The Hankel’s Bezoutian or, simply, Bezoutian of \( u \) and \( v \) is the \( n \times n \) matrix

\[
B_n(u, v) = \text{Bez}_H(u, v) = (b_{ij})_{1 \leq i, j \leq n},
\]

where the \( b_{ij} \) are given by the formula

\[
B_n(u, v) = \frac{u(t)v(s) - u(s)v(t)}{t-s} = \sum_{i,j=1}^{n} b_{ij} t^{i-1}s^{j-1}
\]

Theorem 2.9 (Borchardt-Jacobi Theorem). \([7]\) The number of distinct real roots of a real polynomial \( q \) of degree \( r \) is equal to the signature of the matrix \( B_r(q, q') \), where \( q' \) stands for the usual derivative of \( q \).

Notation. For \( 1 \leq i \leq r \), we will denote \( M_B(i) \) the principal \( i^{th} \) minor of the bezoutian matrix \( B_r(q, q') \).

3. Real Waring decompositions

Let \( p(x, y) \) be

\[
p(x, y) = p_{\vec{c}}(x, y) = \sum_{i=0}^{d} \binom{d}{i} c_i x^i y^{d-i},
\]

a real binary form. Recall that the real rank of \( p(x, y) \) is at most \( d \) (see \([14]\), Prop. 2.1).

The contribution of this section is an algorithm to construct a Waring decomposition of \( p(x, y) \). We point out that our proceeding gives a family of such decompositions.

3.1. Construction for an odd number \( d \)

The procedure runs as follows:

Let it be \( d = 2\ell + 1 \). If \( p(x, y) \) is a \( d^{th} \) power of a linear form, \( p(x, y) = (\alpha x + \beta y)^d \), its rank is 1 and the process is finished. In another case, we take \( \vec{c} = (c_0, \ldots, c_d) \) the point of \( \mathbb{R}^{d+1}\setminus\{(0, \ldots, 0)\} \) associated to \( p(x, y) \) according to \([11]\).

Now, we consider the variables \( S_i, i \in \{1, \cdots, \ell\} \), and construct the matrix
and we expand its determinant by the first column:

\[ h(X_0, \ldots, X_d) = \Delta_0 X_0 + \Delta_1 X_1 + \cdots + \Delta_d X_d \in \mathbb{R}[S_1, \ldots, S_\ell, X_0, \ldots, X_d]. \]

Since the set \( \{ \Delta_d = 0 \} \subset \mathbb{R}[S_1, \ldots, S_\ell] \) is a closed set, its complementary is a Zariski open set. On the other hand, we define

\[ R = -\Delta_{d-1}/\Delta_d \in \mathbb{R}(S_1, \cdots, S_\ell), \]

and we consider the set

\[ C = \bigcup_{i=1}^\ell \{ S_i = 0 \} \cup \{ \Delta_d = 0 \} \cup \left( \bigcup_{i=1}^\ell \{ \Delta_{d-1} + S_i \Delta_d = 0 \} \right) \cup \left( \bigcup_{i<j} \{ S_i + S_j = 0 \} \right) \cup \left( \bigcup_{i<j} \{ S_i - S_j = 0 \} \right) \]

that it is an algebraic set because it is a union of algebraic sets. Therefore

\[ G_\ell = \mathbb{R}^\ell \setminus C \]

is a semialgebraic not empty open set, and it is possible to choose some \( \{ s_1, \cdots, s_\ell \} \in G_\ell. \) In these conditions, the polynomial

\[ h^*(T) = h(1, T, T^2, \ldots, T^d) = \Delta_0 + \Delta_1 T + \cdots + \Delta_d T^d \]

has \( d = 2\ell + 1 \) real roots: \( \pm s_i \in \mathbb{R} \setminus 0 \) and also \( R, \) different by choice.

Now, we will consider the matrix

\[
M = \begin{pmatrix}
X_0 & 1 & 1 & \cdots & 1 & 1 & 1 & c_d \\
X_1 & S_1 & -S_1 & \cdots & S_\ell & -S_\ell & c_{d-1} \\
: & : & : & \cdots & : & : & : & \vdots \\
X_d & S_1^d & (-1)^d S_1^d & \cdots & S_\ell^d & (-1)^d S_\ell^d & c_0 \\
\end{pmatrix}
\]

and we will find the wanted decomposition solving the system
where \( \bar{c}^t = (c_d, \ldots, c_0) \) and the vector \( \bar{\lambda}^t = (\lambda_1, \ldots, \lambda_d) \), is the solution of (9), and it gives us the coefficients to the decomposition. That is,

\[
p(x, y) = \sum_{j=1}^{d} \lambda_j L^d_j(x, y),
\]

with \( L_j(x, y) = x + s_j y \), if \( j \) is odd, \( L_j(x, y) = x - s_j y \), if \( j \) is even, when \( j < d \), and \( L_d(x, y) = x + Ry \).

Now, \( M \) is a matrix with dimension \((d + 1) \times d\), and rank \( d \). Also, \( h^*(1, R, R^2, \ldots, R^d) = 0 \), and the determinant \(|M| \bar{c}|\) evaluates to zero because of the augmented matrix \((M|\bar{c})\) has the same rank as \( M \). Thus, the system (9) can be solved, and this gives us the solution to the Waring problem in this case.

This resolution of the system provides the real solutions that we search for \( h^*(T) \). Therefore we can assert that the real rank of a binary real form is at most \( d \) (when \( d \) is odd). Observe that the rank is not necessarily \( d \). On the one hand, because it is possible that any coefficient may be zero and on the other hand because the algorithm does not guarantee that this decomposition has the minimum possible length.

### 3.2. Construction for an even number \( d \)

In this case, \( d = 2\ell \) and, analogously as above, if \( p(x, y) \) is a \( d^{th} \) power of a linear form, \( p(x, y) = (\alpha x + \beta y)^d \), its rank is 1 and the process is finished. In another case, we take the matrices

\[
V = \begin{pmatrix}
X_0 & 1 & 1 & 1 & \cdots & 1 & 1 & c_d \\
X_1 & S & S_1 & -S_1 & \cdots & S_{\ell-1} & -S_{\ell-1} & c_{d-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
X_d & S^d & S_1^d & (-1)^d S_1^d & \cdots & S_{\ell-1}^d & (-1)^d S_{\ell-1}^d & c_0
\end{pmatrix}
\]

and

\[
M = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & R \\
s & s_1 & -s_1 & \cdots & s_{\ell-1} & -s_{\ell-1} & 1 & R \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
s^d & s_1^d & (-1)^d s_1^d & \cdots & s_{\ell-1}^d & (-1)^d s_{\ell-1}^d & R^d & R^d
\end{pmatrix}
\]
and the associated system

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
s & s_1 & -s_1 & \cdots & s_{\ell-1} & -s_{\ell-1} & R \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
s^d & s_1^d & (-1)^d s_1^d & \cdots & s_{\ell-1}^d & (-1)^d s_{\ell-1}^d & R^d
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_d
\end{pmatrix}
= \begin{pmatrix}
c_d \\
c_{d-1} \\
\vdots \\
c_0
\end{pmatrix}
\]

choosing some \((s, s_1, \cdots, s_{\ell-1}) \in G_{\ell-1}^* = G_{\ell-1}\setminus C^*\) with \(G_{\ell-1}\) defined as in \([7]\) and

\[
C^* = \bigcup_{i=1}^{\ell-1}\{S + S_i = 0\} \cup \bigcup_{i=1}^{\ell-1}\{S - S_i = 0\} \subset \mathbb{R}^\ell.
\]

Now,

\[
R = \frac{-\Delta_{d-1}}{\Delta_d} - s.
\]

Observe that, in contrast with even case, the parameter \(s\) can be zero, provided that \(\Delta_d \neq 0 \neq \Delta_{d-1}\).

As in the previous case, we can affirm that the system has solution analysing the ranks of \(M\) and its augmented matrix \((M | \bar{c})\).

Therefore,

\[
p(x, y) = \lambda_1 x^d + \sum_{j=2}^{d} \lambda_j L_j^d(x, y),
\]

with \(L_j(x, y) = x + s_j y\), if \(j\) is even, \(L_j(x, y) = x - s_j y\), if \(j\) is odd, for \(1 < j < d\), and \(L_d(x, y) = x + Ry\). \qed

**Remark 3.1.** This claim seems a lot less polished than Theorem 1.1. in [18], where the length of the Waring decomposition for a binary form is given for \(\frac{d+1}{2}\) or \(\frac{d}{2} + 1\), depending on whether \(d\) is odd or even. But it is important to notice that our statement refers to “any polynomial” while Sylvester talks about “a general binary form”.

These processes yield the next algorithm:
Algorithm 1: Real Waring decomposition

\textbf{Input :} \overline{p_\ell}(x,y) = \sum_{i=0}^{d} \binom{d}{i} c_i x^i y^{d-i} \neq (\alpha x + \beta y)^d

\textbf{Output:} a real Waring decomposition to \overline{p_\ell}(x,y).

1 \text{ if } d = 2\ell + 1 \text{ then }
2 \quad \text{choose } s_1, \ldots, s_\ell, \text{ real, non zero and distinct numbers ; }
3 \quad \text{construct the matrix } V \text{ as in (6)};
4 \quad \text{if } \Delta_d = 0 \text{ then }
5 \quad \quad \text{go to step 2}
6 \quad \text{else}
7 \quad \quad \text{determine } R = -\frac{\Delta_{d-1}}{\Delta_d};
8 \quad \quad \text{if } R \text{ is the same as any } s_i \text{ or their opposite then }
9 \quad \quad \quad \text{go to step 2}
10 \quad \quad \text{else}
11 \quad \quad \quad \text{go to step 13}
12 \quad \quad \text{end}
13 \quad \quad \text{construct the matrix } M \text{ as in (8)};
14 \quad \quad \text{solve the linear system } M\vec{\lambda} = \overline{c};
15 \quad \text{end}
16 \text{The wanted decomposition is } p(x,y) = \sum_{j=1}^{d} \lambda_j L_j(x,y), \text{ with } L_j(x,y) = x + s_j y, \text{ if } j \text{ is odd, } j < d, L_j(x,y) = x - s_j y, \text{ if } j \text{ is even and } L_d(x,y) = x + Ry.
17 \text{else}
18 \quad \text{choose } s_1, \ldots, s_{\ell-1}, \text{ real, non zero and distinct numbers ; }
19 \quad \text{construct the matrix } V \text{ as in (10)};
20 \quad \text{if } \Delta_d = 0 \text{ or } \Delta_{d-1} = 0 = s \text{ then }
21 \quad \quad \text{go to step 19}
22 \quad \text{else}
23 \quad \quad \text{determine } R = -\frac{\Delta_{d-1}}{\Delta_d} - s;
24 \quad \quad \text{if } R \text{ is the same as any } s_i, \text{ their opposite, or } s \text{ then }
25 \quad \quad \quad \text{go to step 19}
26 \quad \quad \text{else}
27 \quad \quad \quad \text{go to step 30}
28 \quad \quad \text{end}
29 \quad \quad \text{construct the matrix } M \text{ as in (11)};
30 \quad \quad \text{solve the linear system } M\vec{\lambda} = \overline{c};
31 \quad \text{end}
32 \text{The wanted decomposition is } p(x,y) = \lambda_1 x^d + \sum_{j=2}^{d} \lambda_j L_j^d(x,y), \text{ with } L_j(x,y) = x + s_j y, \text{ if } j \text{ is even, } j < d, L_j(x,y) = x - s_j y, \text{ if } j \text{ is odd, } L_d(x,y) = x + Ry.
Example 3.2. Decompositions following Algorithm [7].

Example 1. Take \( p(x, y) = y^5 + \frac{1}{2}x^2y^3 - \frac{1}{2}x^4y \). With the described algorithm we obtain, for example, the following two real Waring decompositions:

\[
p(x, y) = \frac{1}{20}(x + y)^5 - \frac{1}{20}(x - y)^5 - \frac{1}{5}(x + \frac{1}{2}y)^5 + \frac{1}{5}(x - \frac{1}{2}y)^5 + \frac{73}{80}y^5 = \frac{11}{320}(x + y)^5 - \frac{11}{320}(x - y)^5 - \frac{81}{320}(x + \frac{1}{3}y)^5 + \frac{81}{320}(x - \frac{1}{3}y)^5 + \frac{14}{15}y^5.
\]

In the first decomposition we have chosen \( s_1 = 1 \) and \( s_2 = \frac{1}{2} \) and we have obtained \( R = \frac{73}{80} \). In the second one, \( s_1 = 1 \) and \( s_2 = \frac{1}{3} \), then \( R = \frac{14}{15} \).

Example 2. Take \( p(x, y) = 240y^4 + 224xy^3 + 72x^2y^2 + 8x^3y \). In this case, we have firstly chosen \( s_1 = 1 \) in the algorithm and then \( R = \frac{38}{9} \) and secondly \( s_1 = 2 \) and then \( R = 4 \). Hence, we have

\[
p(x, y) = \frac{15}{19}x^4 - \frac{40}{29}(x + y)^4 - \frac{8}{47}(x - y)^4 + \frac{19683}{25897}(x + \frac{38}{9}y)^4 = -(x + 2y)^4 + (x + 4y)^4.
\]

This last decomposition shows us an example which length is less than \( d \).

4. Real Waring rank

In this section we will show how to compute a real Waring decomposition of minimal length of a real binary form \( p(x, y) \). The method we are presenting next is effective and it points out the importance of bezoutian matrix analysis in the study of real Waring decompositions. Also we will show how to modify Algorithm 2.1. in [8] to get the real rank of \( p(x, y) \).

The key point will be to use Theorem 2.9 to guaranty the existence of a polynomial \( q(t) \) of degree \( r \) associated to the kernel of the Hankel matrix \( H_r \) such that \( q \) has \( r \) different real roots.
Algorithm 2: Real rank length’s decomposition

**Input:** $p_{\mathcal{E}}(x, y) = \sum_{i=0}^{d} \binom{d}{i} c_i x^i y^{d-i}$ or its associated point $c$.

**Output:** a Waring decomposition to $p_{\mathcal{E}}(x, y)$ with minimal length.

1. Initialize $r = 1$;
2. Define $H_r$ as (3) and determine its kernel: $H_r = \langle v_1, \ldots, v_{\delta_r} \rangle$;
3. if $\text{Ker} H_r = \{0\}$ then
   4. increment $r \leftarrow r + 1$ and go to step 2.
4. else
5. define $q = q(\mu_1, \ldots, \mu_{\delta_r}) = (q_0, \ldots, q_r) = \sum_{i=1}^{\delta_r} \mu_i v_i$, a kernel’s vector;
6. if $q_0 \neq 0$ then
   7. consider $q(t) = \sum_{i=0}^{r} q_i t^{r-i}$;
   8. calculate $B_r(q, q')$;
   9. if it is possible to find $(\mu_1^*, \ldots, \mu_{\delta_r}^*) \in \mathbb{R}_{\delta_r}^*$ such that $B_r(q, q')$ is positive definite then
      10. factorize $q(t) = \prod_{i=1}^{\delta_r} (t - \alpha_i)$;
      11. solve the linear system
          \[
          \begin{pmatrix}
          1 & 1 & \cdots & 1 \\
          \alpha_1 & \alpha_2 & \cdots & \alpha_r \\
          \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_r^2 \\
          \vdots & \vdots & \cdots & \vdots \\
          \alpha_1^d & \alpha_2^d & \cdots & \alpha_r^d \\
          \end{pmatrix}
          \begin{pmatrix}
          \lambda_1 \\
          \lambda_2 \\
          \vdots \\
          \lambda_r \\
          \end{pmatrix}
          =
          \begin{pmatrix}
          c_d \\
          c_{d-1} \\
          \vdots \\
          c_0 \\
          \end{pmatrix}
          \]
      12. else
      13. increment $r \leftarrow r + 1$ and go to step 2;
6. else
7. take $q(t) = \sum_{i=0}^{r} q_i t^i$ and go to step 9;
8. end
9. end
10. The wanted decomposition is
    \[
    p(x, y) = \sum_{i=1}^{r} \lambda_i \binom{x - \alpha_i y}{d} \quad \text{if } q \text{ was defined in the step 8}
    \]
    \[
    p(x, y) = \sum_{i=1}^{r} \lambda_i \binom{y - \alpha_i x}{d} \quad \text{in another case.}
    \]
Example 4.1. \( p(x, y) = y^5 + \frac{1}{2}x^2y^3 - \frac{1}{2}x^4y. \)

In the first case of the Examples 3.1 we have just obtained two real Waring decompositions for this form, but both of them have length 5. Now, we are going to use the Algorithm 2 to determine a Waring decomposition of length the rank of this polynomial.

1. Compute the kernel of \( H_1: \)

\[
H_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{20} & 0 & \frac{1}{10} & 0 \\
0 & 0 & \frac{1}{20} & 0 & \frac{1}{10} \\
\frac{1}{20} & 0 & 0 & \frac{1}{10} & 0
\end{pmatrix}
\Rightarrow \text{Ker}(H_1) = \{0\}
\]

2. Compute the kernel of \( H_2: \)

\[
H_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{20} & 0 & \frac{1}{10} & 0 \\
0 & 0 & \frac{1}{20} & 0 & \frac{1}{10} \\
\frac{1}{20} & 0 & 0 & \frac{1}{10} & 0
\end{pmatrix}
\Rightarrow \text{Ker}(H_2) = \{0\}
\]

3. Compute the kernel of \( H_3: \)

\[
H_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{20} & 0 & \frac{1}{10} & 0 \\
0 & 0 & \frac{1}{20} & 0 & \frac{1}{10} \\
\frac{1}{20} & 0 & 0 & \frac{1}{10} & 0
\end{pmatrix}
\Rightarrow \text{Ker}(H_3) = \{(0, 2, 0, 1)\}
\]

This vector can be associated with \( q(t) = t^3 + 2t = t(t^2 + 2), \) with three distinct solutions in \( \mathbb{C}, \) but not in \( \mathbb{R}. \) Therefore, the complex rank is 3 and we can write:

\[
p(x, y) = \frac{41}{40}y^5 - \frac{1}{80}(y + i\sqrt{2}x)^5 - \frac{1}{80}(y - i\sqrt{2}x)^5
\]

4. Compute the kernel of \( H_4: \)

\[
H_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{20} & 0 & \frac{1}{10} & 0 \\
0 & 0 & \frac{1}{20} & 0 & \frac{1}{10} \\
\frac{1}{20} & 0 & 0 & \frac{1}{10} & 0
\end{pmatrix}
\]

\[
\text{Ker}(H_4) = \{(-\frac{1}{20}, 0, 1, 0, 0), (0, 2, 0, 1, 0), (-\frac{1}{10}, 0, 0, 0, 1)\}
\]

In this kernel we take the vector \((\frac{3}{19}, 0, \frac{-22}{19}, 0, 1),\) associated with \( q(t) = t^4 - \frac{22}{19}t^2 + \frac{3}{19} = (t + 1)(t - 1)(t + \frac{1}{19}\sqrt{57})(t - \frac{1}{19}\sqrt{57}), \) with four real distinct solutions. Therefore, its real rank is 4.
5. Solve the linear system $M\vec{\lambda} = \vec{c}$, where the matrix $M$ is defined as:

$$M = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & \xi & -\xi \\
1 & 1 & \xi^2 & \xi^2 \\
1 & -1 & \xi^3 & -\xi^3 \\
1 & 1 & \xi^4 & \xi^4 \\
1 & -1 & \xi^5 & -\xi^5
\end{pmatrix}$$

with $\xi = \frac{\sqrt{57}}{19}$.

Observe that in this case we take $\vec{c}$ instead of $\bar{c}$ because we have chosen $q(t) = q_4 t^4 + q_3 t^3 + q_2 t^2 + q_1 t + q_0$.

Then, a real Waring decomposition for $p$ is

$$p(x, y) = -\frac{41}{640} (y + x)^5 - \frac{41}{640} (y - x)^5 + \frac{361}{640} (y + \xi x)^5 + \frac{361}{640} (y - \xi x)^5.$$ 

5. Semialgebraic decomposition of $B_d$

Let be $B_d$ the space of real binary forms of degree $d$. Let us define $\tilde{H}_r = P(Ker(H_r))$, where $H_r$ is defined in (3). Then we have:

$$Ker(H_r) \subset \mathbb{R}^{r+1} \Rightarrow P(Ker(H_r)) \subset P(\mathbb{R}^{r+1}) = P_{\mathbb{R}}^r.$$ 

Observe that for a point $[q_0, \ldots, q_r] \in \tilde{H}_r$ we can consider two polynomials, $q_0 t^r + q_1 t^{r-1} + \ldots + q_r$ and $q_0 + q_1 t + \ldots + q_r t^r$, to study their real roots. Let us observe that the real rank of a form $p$ depend only on the projective coordinates of $p$, i.e. $rk_{\mathbb{R}}(p) = rk_{\mathbb{R}}(ap)$, for all $a \in \mathbb{R}$, $a \neq 0$.

Following the Algorithm 2 with any $p_v(x, y)$ of degree $d$, the conditions for step 10 determine a semialgebraic set defined by $r$ inequalities with $\delta_r = \text{dim}(Ker(H_r))$ parameters: $\mu_i$, $i \in \{1, \ldots, \delta_r\}$ and $d + 1$ variables: $c_i$, $i \in \{0, \ldots, d\}$, because the requirement about $B_r(q, q')$ is positive defined is equivalent with satisfy all the inequalities $M_B(1) > 0$, $\ldots$, $M_B(r) > 0$, (recall that $M_B(i)$ describes the principal $i$th minor of the matrix $B_r(q, q')$).

Let $S^{(r)}$ be the semialgebraic set:

$$S^{(r)} = \{ q \in P_{\mathbb{R}}^{d} | M_B(i) > 0, \text{ for } 1 \leq i \leq r \} \times \{0, \ldots, d-r\} \subset P_{\mathbb{R}}^d.$$ 

Now, we consider the real algebraic set

$$A^{(r)} = \left\{ (\xi, q) \in P_{\mathbb{R}}^d \times P_{\mathbb{R}}^d \mid \sum_{i=0}^{r} c_i + \ell q_i = 0, \quad 0 \leq \ell \leq d-r \right\}.$$ 

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Finally, we define the set
\[ \mathcal{F}^{(r)} = A^{(r)} \cap (\mathbb{P}^d_R \times S^{(r)}) \subset \mathbb{P}^d_R \times \mathbb{P}^d_R \]
and then we have the following result.

**Theorem 5.1.** \( \mathcal{F}^{(r)} \) is a semialgebraic set of \( \mathbb{P}^d_R \times \mathbb{P}^d_R \).

**Proof.** The set \( \mathcal{F}^{(r)} \) is an intersection of a real algebraic set with a semialgebraic set. Hence, it is a semialgebraic set. \( \square \)

**Remark 5.2.** It is easy to verify that \( A^{(r)} \subset A^{(r+1)} \) by the formula (3), and also
\[ \mathcal{F}^{(r)} \cap U_r \subset \mathcal{F}^{(r+1)} \]
where \( U_r = \mathbb{P}^d_R \times (\mathbb{P}^d_R \setminus \{ q_0q_r = 0 \}) \subset \mathbb{P}^d_R \times \mathbb{P}^d_R \), Zariski open set.

In this way, for \( r \leq d \), we consider the set \( \mathcal{W}^{(r)} \) defined in 2.5. Observe that all these sets are cones, because if \( \mathbf{c} \in \mathcal{W}^{(r)} \) then \( a\mathbf{c} \in \mathcal{W}^{(r)} \), for any \( a \in \mathbb{R} \setminus \{ 0 \} \).

**Corollary 5.3.** The set \( \mathcal{W}^{(r)} \) of real binary forms with real coefficients which have rank \( r \) is semialgebraic.

**Proof.** We need only consider \( \mathcal{W}^{(r)} \) as the image set of \( \mathcal{F}^{(r)} \) by projection homomorphism over the first coordinates. That is, \( \mathcal{W}^{(r)} = \pi(\mathcal{F}^{(r)}) \), with
\[ \pi : \mathbb{P}^d_R \times \mathbb{P}^d_R \longrightarrow \mathbb{P}^d_R \]
\[ (\mathbf{c}, q) \longmapsto \mathbf{c} \] (13)
and, therefore, \( \mathcal{W}^{(r)} \) is a real semialgebraic set. \( \square \)

**Corollary 5.4.** The space of all real binary forms of a given degree can be decompose as a finite union of semialgebraic sets according to their real rank.

**Proof.** In fact, we can identify \( \mathcal{B}_d \) with \( \mathbb{P}^d_R \) by means of \( \mathcal{B}_d \ni p_x \rightarrow \mathbf{c} \in \mathbb{P}^d_R \) and then
\[ \mathcal{B}_d \equiv \bigcup_{r=1}^{d} \mathcal{W}^{(r)} . \]
Corollary 5.5. Moreover, in order to calculate the dimension of $W^r$ we can use the usual techniques in Real Geometry. Esto contesta, en el caso real, a la pregunta Q1 formulada por Carlini en [9] para el caso complex binary forms. De hecho, para rango $r$ tipico la dimensin de $W^r$ es $d+1$.

Some examples of this decomposition will be shown in section 7.

6. The monomials

It is known that the real rank of a non trivial degree $d$ monomial (trivially, the monomials $x^d$ or $y^d$ have rank 1) is $d$ (see [6]) but our goal is to present a constructive approach and so as not to use differential operators, neither the Apolarity Lemma. Moreover, we achieve the complete fibre of the monomial’s decomposition. Also, we will show that monomials are semialgebraically unstable.

Lemma 6.1 (Lemma 4.1. in [6]). Consider the degree $k$ polynomial $q(t) = q_0 + \cdots + q_d t^d \in \mathbb{R}[t]$. If $q_i = q_{i-1} = 0$ for some $1 \leq i \leq d$, then $q(t)$ does not have $d$ real distinct roots.

Theorem 6.2. Any monomial $x^m y^{d-m}$ has real rank of $d$, $m \geq 1$.

Proof. By symmetry of the variables, the monomial $x^m y^{d-m}$ has the same rank than $x^{d-m} y^m$. Because of this, we can restrict the proof to the case in which $m \leq \frac{d}{2}$.

Let $p_\infty$ be

$$p_\infty = p(x, y) = \binom{d}{m} c_m x^m y^{d-m}$$

with $m \leq \frac{d}{2}$, $\underline{c} = (0, 0, \ldots, 0, c_m, 0, \ldots, 0)$.

The Hankel matrix associated with the rank of $d - \ell$ is

$$H_{d-\ell} = \begin{pmatrix}
0 & \cdots & 0 & c_m & 0 & \cdots & 0 \\
0 & \cdots & c_m & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\
c_m & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}$$
and kernel’s vectors are \((0, \ldots, 0, q_{m+1}, \ldots, q_{d-\ell}, 0, \ldots, 0)\) and we can write the corresponding polynomials as \(\sum_{i=0}^{d-\ell} q_i t^{d-\ell-i}\) (or \(\sum_{i=0}^{d-\ell} q_i t^i\)), with \(q_0 = \ldots = q_m = 0\).

However, both polynomials does not have \(d-\ell\) real different roots by previous Lemma. \(\square\)

Next we will compute some explicit examples.

**Examples 6.3.** Monomial \(x^{d-2}y^2\) for degrees \(d = 5\) and \(d = 4\).

**Example 1.** The monomial \(x^3y^2\). Although the rank of a general binary form of degree 5, according to Sylvester, is 3, Proposition 3.1 in \([10]\) says us that the complex rank for this monomial is 4. For example, we can write

\[
x^3y^2 = \frac{1}{80}y^5 - \frac{1}{240}(y - 2x)^5 - \frac{1}{240} \left[ y + \left(1 + i\sqrt{3}\right) \right] x^5 - \frac{1}{240} \left[ y + \left(1 - i\sqrt{3}\right) \right] x^5,
\]

but also

\[
x^3y^2 = \frac{1}{40}(x - y)^5 + \frac{1}{40}(x + y)^5 - \frac{1}{40}(x - iy)^5 - \frac{1}{40}(x + iy)^5.
\]

However, its real rank is 5. Running the Algorithm 1, we can find the next family of decompositions for the monomial:

\[
x^3y^2 = \frac{b^2}{20a^2(b^2 - a^2)}(x + ay)^5 + \frac{b^2}{20a^2(b^2 - a^2)}(x - ay)^5 + \frac{a^2}{20b^2(a^2 - b^2)}(x + by)^5 + \frac{a^2}{20b^2(a^2 - b^2)}(x - by)^5 - \frac{a^2 + b^2}{10(a^2b^2)}x^5
\]

depending on two parameters and well defined for \(a, b\) non zero real parameters such that \(a \neq \pm b\).

However, we can find binary forms with smaller rank as near the monomial as we want. For example, in the coefficients space of the polynomials of
$5^{th}$ degree, the next polynomial belongs to any open ball centered in the monomial $x^3y^2$:

$$x^3y^2 + \frac{1}{m}xy^4 = -\frac{m}{20}x^5 + \frac{m}{40} \left( x + \frac{\sqrt{2m}}{m}y \right)^5 + \frac{m}{40} \left( x - \frac{\sqrt{2m}}{m}y \right)^5$$

and its rank is 3.

**Example 2.** The monomial $x^2y^2$. In this instance, the complex rank is 3 and we can write:

$$x^2y^2 = \frac{1}{72}(x+2y)^4 - \frac{1-i\sqrt{3}}{144}(x + (-1 + i\sqrt{3})y)^4 - \frac{1+i\sqrt{3}}{144}(x - (1 + i\sqrt{3})y)^4.$$ 

However, its real rank is 4 and a possible decomposition is

$$x^2y^2 = \frac{1}{4}(x + y)^4 + \frac{7}{108}(x-y)^4 - \frac{1}{54}(x + 2y)^4 - \frac{8}{27}(x + \frac{1}{2}y)^4.$$ 

Again we can find polynomials as near as we want with minor rank. Take for $m > 1$:

$$p_m(x, y) = \frac{1}{m}y^4 + x^2y^2 = -\frac{1}{36}x^4 + \frac{m}{72} \left( x - \frac{\sqrt{6m}}{m}y \right)^4 + \frac{m}{72} \left( x + \frac{\sqrt{6m}}{m}y \right)^4.$$ 

### 7. Examples of Semialgebraic decomposition of $B_d$

#### 7.1. Semialgebraic decomposition of $B_3$

Let $p(x, y)$ be $p(x, y) = c_0y^3 + 3c_1xy^2 + 3c_2x^2y + c_3x^3 \in \mathbb{R}[x, y]$. By direct calculation, we obtain

$$\mathcal{W}^{(1)} = \{ [1 : \alpha : \alpha^2 : \alpha^3] \mid \alpha \in \mathbb{R} \} \cup \{ [\alpha^3 : \alpha^2 : \alpha : 1] \mid \alpha \in \mathbb{R} \}.$$ 

Let $f$ be $f(c_0, c_1, c_2, c_3) = c_0^2c_3^2 - 6c_0c_1c_2c_3 + 4c_0c_2^3 + 4c_1^3c_3 - 3c_1^2c_2^2$. Then,

$$\mathcal{W}^{(2)} = \{ [c_0 : c_1 : c_2 : c_3] \mid f(c_0, c_1, c_2, c_3) > 0, c_0c_2 - c_1^2 \neq 0, c_1 \in \mathbb{R}, \} \cup$$

$$\cup \{ [c_0 : c_1 : c_2 : c_3] \mid f(c_0, c_1, c_2, c_3) > 0, c_1c_3 - c_2^2 \neq 0, c_1 \in \mathbb{R}, \} \cup$$

$$\cup \{ [1 : \alpha : \alpha^2 : \beta] \mid \beta - \alpha^3 \neq 0, \alpha, \beta \in \mathbb{R}, \alpha \neq 0 \} \cup$$

$$\cup \{ [\beta : \alpha^2 : \alpha : 1] \mid \beta - \alpha^3 \neq 0, \alpha, \beta \in \mathbb{R}, \alpha \neq 0 \}.$$ 

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Let $U_0 \subset \mathbb{P}_R$ be such that $c_0 \neq 0$, and $H_{0,\infty} = \{c_0 = 0\}$.

Let us consider

$$X_{0}^{(2)} = W^{(2)} \cap U_0$$
$$X_{0,\infty}^{(2)} = W^{(2)} \cap H_{0,\infty}$$

Then, $W^{(2)} = X_{0}^{(2)} \cup X_{0,\infty}^{(2)}$.

Finally, we get $W^{(3)}$ as the complementary of $W^{(1)} \cup W^{(2)}$. So, $W^{(3)}$ is also a real semialgebraic set of maximal dimension.

Now, we are going to analyze the border of $W^{(2)}$. In order to describe it, we set

$$u_0(c_1, c_2, c_3) = f(1, c_1, c_2, c_3)$$
$$u_\infty(c_1, c_2, c_3) = f(0, c_1, c_2, c_3) = c_1^2(4c_1c_3 - 3c_2^2)$$

Then,

$$\partial X_{0}^{(2)} = \{[1 : c_1 : c_2 : c_3] \mid u_0(c_1, c_2, c_3) = 0\}$$
$$\partial X_{0,\infty}^{(2)} = L_{0,\infty} \cup P_{0,\infty}$$
where

\[ L_{0,\infty} = \{ [0 : 0 : c_2 : c_3] \mid (c_2, c_3) \neq (0, 0) \} \text{ and} \]

\[ P_{0,\infty} = \{ [0 : 1 : c_2 : c_3] \mid 4c_3 - 3c_2^2 = 0 \}. \]

The border \( \partial X_0^{(2)} \) can be plotted using Surfer. We include its graphic for convenience of the reader. In the Figure 2, \( X_0^{(2)} \) is limited by the light surface.

Observe that the origin of the Figure 2 corresponds to the monomial \( y^3 \). The remaining monomials must be looking for in \( \partial X_{0,\infty}^{(2)} \). Let’s see them.

With the previous notations, \( x^2y \) and \( x^3 \) are in \( L_{0,\infty} \), while \( xy^2 \) is in \( P_{0,\infty} \). Observe that, in the Figure 3, \( L_{0,\infty} \) corresponds to the plane \( c_1 = 0 \). The axes, except the origin, correspond to the monomials. Both \( xy^2 \) and \( x^2y \) are in \( \mathcal{W}^{(3)} \) while \( x^3 \in \mathcal{W}^{(1)} \).

Analogously, we can also define \( X_3^{(2)} \) and \( X_{3,\infty}^{(2)} \), taking the chart for the dehomogenization of the other variable, in order to see all pieces of \( \mathcal{W}^{(2)} \).

7.2. Semialgebraic decomposition of \( \mathcal{B}_4 \).

Let \( p(x, y) = c_0 y^4 + 4c_1 xy^3 + 6c_2 x^2 y^2 + 4c_3 x^3 y + c_4 x^4 \in \mathbb{R}[x, y] \). By direct calculation, we obtain
\[ W^{(1)} = \{ [1 : \alpha : \alpha^2 : \alpha^3 : \alpha^4] \mid \alpha \in \mathbb{R} \} \cup \{ [\alpha^4 : \alpha^3 : \alpha^2 : \alpha : 1] \mid \alpha \in \mathbb{R} \}. \]

We consider the projective point \( \mathfrak{c} = [c_0 : c_1 : c_2 : c_3 : c_4], c_i \in \mathbb{R}, \) and define \( d_{ij} = c_i c_{j+1} - c_j c_{i+1}. \)

Let \( f_i, g_i, i = 1, 2, \) be

\[
\begin{align*}
    f_1(c_0, c_1, c_2, c_3) &= c_0^2 c_3^2 - 6c_0 c_1 c_2 c_3 + 4c_0 c_3^3 + 4c_2^3 c_3 - 3c_1^2 c_2^2 \\
f_2(c_1, c_2, c_3, c_4) &= c_1^2 c_4^2 - 6c_1 c_2 c_3 c_4 + 4c_1 c_3^3 + 4c_2^3 c_4 - 3c_1^2 c_2^2 \\
g_1(c_0, c_1, c_2, c_3) &= c_4 (c_0 c_2 - c_1^2) - (c_0 c_3^2 - 2c_1 c_2 c_3 + c_2^3) \\
g_2(c_1, c_2, c_3, c_4) &= c_0 (c_2 c_4 - c_3^2) - (c_1^2 c_4 - 2c_1 c_2 c_3 + c_3^2)
\end{align*}
\]

\[ W^{(2)} = \{ \mathfrak{c} \mid g_1 = 0, d_{01} \neq 0, f_1 > 0 \} \cup \{ \mathfrak{c} \mid g_2 = 0, d_{23} \neq 0, f_2 > 0 \} \cup \{ [1 : \alpha : \alpha^2 : \alpha^3 : \beta] \mid \beta \neq \alpha^4, \alpha, \beta \in \mathbb{R}, \alpha \neq 0 \} \cup \{ [\beta : \alpha^3 : \alpha^2 : \alpha : 1] \mid \beta \neq \alpha^4, \alpha, \beta \in \mathbb{R}, \alpha \neq 0 \}. \]

Let \( f_i, i = 3, \ldots, 8, \) defined as follows:
\[ f_3(c_0, c_1, c_2, c_3, c_4) = 4c_0^3c_1^3 - 12c_0^2c_1c_3c_4^2 - 9c_0c_1c_2c_4^3 + 12c_0c_2^2c_3c_4 + \\
+ 18c_0c_1c_2c_3c_4 - 9c_0c_1^2c_2^2 - 6c_0c_1^2c_2c_3c_4 - \\
- 4c_0^3c_3^2 + 9c_1^2c_2^3 \]

\[ f_4(c_0, c_1, c_2, c_3, c_4) = 4c_0^3c_4^3 - 12c_0^2c_1c_3c_4^2 - 27c_0^2c_2c_4^3 + 12c_0c_1c_2c_4^3 + \\
+ 54c_0c_1c_2c_3c_4 - 27c_0c_3^2c_4^2 + 27c_4^2c_4 - \\
54c_1^2c_2c_3c_4 - 4c_1^3c_3^2 + 27c_1^2c_2^2c_3c_4 \]

\[ f_5(c_0, c_1, c_2, c_3, c_4) = 4c_0^3c_2^3 - 12c_0^2c_1c_3c_4^2 - 9c_0c_1c_2c_3c_4 + 9c_0c_1c_2c_3c_4 + \\
+ 18c_0c_1c_2c_3c_4 - 18c_0c_1c_2c_3c_4 - \\
- 4c_1^3c_3^2 - 9c_1^2c_2c_3c_4 + 9c_1^2c_2c_3c_4 \]

\[ f_6(c_0, c_1, c_2, c_3, c_4) = 4c_0^3c_2^3 - 12c_0^2c_1c_3c_4^2 - 27c_0^2c_2c_3c_4^2 + 27c_2^2c_4^3 + \\
+ 12c_0c_1c_2c_3c_4^2 + 54c_0c_1c_2c_3c_4 - 54c_0c_1c_2c_3c_4 - \\
- 4c_1^3c_3^2 - 27c_1^2c_2^2c_3c_4 + 27c_1^2c_2^2c_3c_4 \]

\[ f_7(c_0, c_1, c_2, c_3, c_4) = 4c_0c_2c_3^2 - 4c_0c_2c_3c_4 - 3c_1c_4^3 + 2c_1c_2c_3c_4 + 4c_1c_3^3 - 3c_2c_3^3 \]

\[ f_8(c_0, c_1, c_2, c_3, c_4) = 4c_0^2c_2c_4 - 3c_0^2c_3^2 - 4c_0c_2c_4 + 2c_0c_1c_2c_3 + 4c_1c_3 - 3c_1c_2 \]

\[ \mathcal{W}^{(3)} = \{ c \mid g_1 \neq 0, d_{01} \neq 0, d_{12} \neq 0, f_1 > 0 \} \cup \\
\cup \{ c \mid g_2 \neq 0, d_{12} \neq 0, d_{23} \neq 0, f_2 > 0 \} \cup \\
\cup \{ c \mid d_{01} > 0, d_{03} > 0, d_{13} \neq 0, f_3 > 0, f_4 > 0 \} \cup \\
\cup \{ c \mid d_{01} < 0, d_{03} < 0, d_{13} \neq 0, f_3 < 0, f_4 < 0 \} \cup \\
\cup \{ c \mid d_{03} > 0, d_{23} > 0, d_{02} \neq 0, f_5 > 0, f_6 > 0 \} \cup \\
\cup \{ c \mid d_{03} < 0, d_{23} < 0, d_{02} \neq 0, f_5 < 0, f_6 < 0 \} \cup \\
\cup \{ c \mid d_{01} \neq 0, d_{12} \neq 0, f_7 \leq 0 \} \cup \\
\cup \{ c \mid d_{23} \neq 0, f_8 \leq 0 \} \cup \\
\cup \{ [1 : \alpha : \alpha^2 : \beta : \gamma] \mid \beta \neq \alpha^3, \alpha, \beta, \gamma \in \mathbb{R}, \alpha \neq 0 \} \cup \\
\cup \{ [\gamma : \beta : \alpha^2 : \alpha : 1] \mid \beta \neq \alpha^3, \alpha, \beta, \gamma \in \mathbb{R}, \alpha \neq 0 \} \cup \\
\cup \{ [1 : 0 : 0 : \beta : \gamma] \mid \beta \neq 0, \beta, \gamma \in \mathbb{R}, 27\beta^2 + 4\gamma^3 > 0 \} \cup \\
\cup \{ [\gamma : \beta : 0 : 0 : 1] \mid \beta \neq 0, \beta, \gamma \in \mathbb{R}, 27\beta^2 + 4\gamma^3 > 0 \} \cup \\
\cup \{ [0 : 0 : c_2 : c_3 : c_4] \mid c_2 \neq 0, 4c_0c_4 - 3c_3^2 > 0 \} \cup \\
\cup \{ [c_0 : c_1 : c_2 : 0 : 0] \mid c_2 \neq 0, 4c_0c_2 - 3c_1^2 > 0 \} \]

Finally \( \mathcal{W}^{(4)} = \mathcal{B} \setminus (\mathcal{W}^{(1)} \cup \mathcal{W}^{(2)} \cup \mathcal{W}^{(3)}) \); hence it is a real semialgebraic set too.

Polynomial's continuity guarantee that the sets defined by inequalities (in this case, the first four in the previous description) have maximal dimension, so \( \text{dim} \mathcal{W}^{(3)} = 5 \).
In order to study the border of $W^{(3)}$ let us proceed as for $W^{(2)}$ when $d = 3$. We take

\[ X_0^{(3)} = W^{(3)} \cap U_0 \]
\[ X_{0,\infty}^{(3)} = W^{(3)} \cap H_{0,\infty} \]

Then, $W^{(3)} = X_0^{(3)} \cup X_{0,\infty}^{(3)}$.

It's easy to observe that $W^{(2)} \subset \partial W^{(3)}$ if we pay attention to the sets in the definition of both $W^{(2)}$ and $W^{(3)}$. Besides, we set, for $i = 1, \ldots, 6$,\n\[ u_0^i(c_1, c_2, c_3, c_4) = f_i(1, c_1, c_2, c_3, c_4) \]
\[ u_\infty^i(c_1, c_2, c_3, c_4) = f_i(0, c_1, c_2, c_3, c_4) \]

Then,

\[ \partial X_0^{(3)} = W^{(2)} \cup \{ [1 : c_1 : c_2 : c_3 : c_4] \mid g_1 \neq 0, d_0_1 \neq 0, u_0^1(c_1, c_2, c_3 : c_4) = 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid g_2 \neq 0, d_0_3 \neq 0, u_0^2(c_1, c_2, c_3, c_4) = 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_1 \neq 0, d_0_3 = 0, d_1_3 \neq 0, f_3 > 0, f_4 > 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_1 \neq 0, d_0_3 \neq 0, d_1_3 = 0, f_3 > 0, f_4 > 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_1 = 0, d_0_3 \neq 0, d_1_3 \neq 0, f_3 > 0, u_0^4(c_1, c_2, c_3, c_4) = 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_1 \neq 0, d_0_3 = 0, d_2_3 \neq 0, f_5 > 0, f_6 > 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_2 \neq 0, d_0_3 \neq 0, d_2_3 = 0, f_5 > 0, f_6 > 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_2 = 0, d_0_3 = 0, d_2_3 \neq 0, f_5 > 0, f_6 > 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_2 = 0, d_0_3 \neq 0, d_2_3 \neq 0, f_5 > 0, u_0^6(c_1, c_2, c_3, c_4) = 0 \} \cup \]
\[ \{ [1 : c_1 : c_2 : c_3 : c_4] \mid d_0_2 \neq 0, d_0_3 \neq 0, d_2_3 \neq 0, f_5 > 0, u_0^6(c_1, c_2, c_3, c_4) = 0 \} \]

\[ \partial X_{0,\infty}^{(3)} = \{ [0 : 1 : c_2 : c_3 : c_4] \mid 4c_3 - 3c_2^2 > 0 \} \cup \]
\[ \{ [0 : 1 : c_2 : 1 : 0] \mid 4c_1 - 3c_2^2 > 0 \} \cup \]
\[ \{ [c_0 : c_1 : c_2 : c_3 : c_4] \} \cup \]
\[ \{ [c_0 : c_1 : c_2 : 0 : 0] \} \cup \]

Monomials $x^4$ and $y^4$ are trivially in $W^{(1)}$, while $xy^3$, $x^3y$ and $x^2y^2$ are in $\partial X_{0,\infty}^{(3)}$. And $x^2y^2$ is also the limit point of the surfaces where are, respectively, $xy^3$ and $x^3y$. Taking $c_0 = c_1 = 0$, Figure 3 is equivalent with the graphic for $x^4$, $x^3y$ and $x^2y^2$.
7.3. Decomposition of canonical forms of degree 5.

As we can appreciate in the description of $\mathcal{W}^{(3)}$ in $\mathcal{B}_4$, the study of any form of degree $d$ starts to get complicated because the space of parameters increases very quickly and this grows rapidly when $d \geq 5$.

In particular, for $d = 5$, Common and Ottaviani have proposed in [14] two canonical form families depending only on two parameters, so it is possible to visualise them on $\mathbb{R}^2$. Now, we are going to examine any polynomial of the canonical form $p(x, y) = x(x^2 - y^2)(x^2 + 2axy + by^2)$.

Observe that, in this case, $c = \left[0 : \frac{-b}{5} : \frac{-a}{5} : \frac{b-1}{10} : \frac{2a}{5} : 1\right]$ and we denote it as $c(a, b)$.

Following our Algorithm, we obtain that $\mathcal{W}^{(1)} = \emptyset$ and $\mathcal{W}^{(2)} = \emptyset$.

Let $f$ be:

$$f(a, b) = 8192a^{12} - 19712a^{10}b^2 + 7680a^8b^4 + 6560a^6b^6 + 480a^4b^8 - 77a^2b^{10} + 2b^{12} - 115712a^{10}b + 336640a^8b^3 - 287040a^6b^5 + 44400a^4b^7 - 4680a^2b^9 + 142b^{11} + 78848a^{10} + 99840a^8b^2 - 700160a^6b^4 + 700160a^4b^6 - 92940a^2b^8 + 3752b^{10} - 287488a^8b + 375552a^6b^3 + 311952a^4b^5 - 593208a^2b^7 + 43192b^9 - 4096a^8 + 392736a^6b^2 - 673952a^4b^4 + 243410a^2b^6 + 170652b^8 + 12096a^8b - 243056a^6b^3 + 348552a^4b^5 - 170652b^7 + 64a^6 - 11840a^4b^2 + 62900a^2b^4 - 43192b^5 - 144a^4b + 3960a^2b^3 - 3752b^5 + 83a^2b^2 - 142b^4 - 2b^6.$$

$\mathcal{W}^{(3)} = \{c(a, b) \mid f > 0\}$

In order to describe $\mathcal{W}^{(4)}$ we define

$$\mathcal{S}^4 = \{q \in \mathbb{P}^4_{\mathbb{R}} \mid f_1(q) > 0, f_2(q) > 0, f_3(q) > 0\} \times \{0\},$$

with

$$f_1(q) = -2q_2q_4 + q_3^2$$

$$f_2(q) = 8q_0q_2q_4^2 - 4q_0q_3^2q_4 - 9q_1^2q_4^2 + 10q_1q_2q_3q_4 - 2q_1q_3^3 - 4q_3^3q_4 + q_2q_3^2$$

$$f_3(q) = 64q_0^3q_3^2 - 72q_0^3q_1q_3^2 - 48q_0^3q_2q_4^2 + 50q_0^2q_2q_3q_4 - 9q_0^2q_3^4 + 90q_0q_1q_2q_3q_4 - 6q_0q_2q_3^2 - 4q_0q_1q_3q_4 + 10q_0q_1q_2q_4^2 + 8q_0q_2q_3^2 - 2q_0q_3^2q_4 - 27q_1^2q_4^2 + 18q_0^2q_2q_3q_4 - 4q_0^2q_3^2 - 4q_0^2q_2q_4 + q_1^2q_2q_3^2$$

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Figure 4: Shaded area corresponds with $W^{(3)}$

\[ f_4(q) = 256q_3^3q_2^3 - 192q_0q_1q_0q_4^2 - 128q_0q_2q_3^2q_4 - 27q_0q_3^4 + 144q_0q_1^2q_2q_4^2 - 6q_0q_1^2q_3^2q_4 - 80q_0q_1q_2^2q_3q_4 + 18q_0q_1q_2q_3^3 + 16q_0q_2q_3^2q_4 - 4q_0q_3^2q_3^2 - 27q_1^4q_2^2 + 18q_1^2q_3q_4 - 4q_1^2q_2q_4 + q_1^2q_2q_3^2 \]

and $A^{(4)} = \{(c, q) \in \mathbb{P}^5_\mathbb{R} \times \mathbb{P}^5_\mathbb{R} \mid q_i = q_i(a, b, \mu_1, \mu_2, \mu_3), i = 1, 2, 3, 4\}$, with

\[
\begin{align*}
q_0 &= 2(\mu_1 - 2\mu_3)a^2 + 3\mu_2ab + \mu_1b^2 + \mu_2a - (\mu_1 - 10\mu_3)b \\
q_1 &= 2(2\mu_3 - \mu_1)ab + \beta b^2 - \mu_2b \\
q_2 &= 2\mu_1b^2 \\
q_3 &= 2\mu_2b^2 \\
q_4 &= 2\mu_3b^2.
\end{align*}
\]

Then, $F^{(4)} = A^{(4)} \cap (\mathbb{P}^5_\mathbb{R} \times S^4)$ and $W^{(4)} = \pi \left(F^{(4)}\right)$, being $\pi$ the projection homomorphism over the first five coordinates.

Since $F^{(4)}$ is a semialgebraic set of $\mathbb{P}^{10}$ of maximal dimension, it is impossible to represent it in $\mathbb{R}^3$. In fact, for these canonical forms, we are working with $(a, b, \mu_1, \mu_2, \mu_3) \in \mathbb{R}^5$, and even we can specialize one of these parameters to visualize them. But, in any case, it is still a hard to obtain some graphic representatios.

This leads to emphasize how grow the sets $W^{(r)}$ when $r$ increase.

Experimental calculations allow us to determine some areas with rank 4, but not perfect. For instance, we have

\[ W^{(4)} \supset \{c(a, b) \mid F > 0\} \]
with \( F(a, b) = -1728a^4 - 1184a^2b^2 - 108b^4 + 3008a^2b + 1296b^3 - 416a^2 - 1224b^2 + 400b - 44. \)

Observe that the described set has maximal dimension in \( \mathbb{R}^2 \). In the Figure [5] this set corresponds with the lined area. \( W^{(3)} \)'s border is in \( W^{(4)} \), except the origin, which has rank 5.

![Figure 5: Lined area corresponds with \( W^{(4)} \)](image)

8. Conclusions and future work

In this paper we propose an algorithm to obtain a Waring decomposition for a real binary form \( p \) of degree \( d \). Our Algorithm 2 gives the real rank of \( p \). Using bezoutians matrices we can give a semialgebraic description of the set \( W^{(r)} \) of real binary forms of real rank \( r \). Some examples are included.

There are several questions than remain open. For instance, the dimension of \( W^{(r)} \) for non typical ranks, since for typical ranks the sets \( W^{(r)} \) are semialgebraic sets of maximal dimension. We have studied carefully degrees 3, 4 and 5. The paper contains some detailed description of several strata. Hence, it will be suitable to extend the procedure we have used in these cases to study general degrees. However, the complexity of the polynomials that describe the strata increases quickly. We are working to analyze canonical forms for generic ranks following the ideas of B. Reznick in [13].
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