DEGENERATE CAUCHY NUMBERS AND POLYNOMIALS OF
THE SECOND KIND

TAEKYUN KIM

Abstract. Recently, degenerate Cauchy numbers and polynomials are introduced in [10]. In this paper, we study the degenerate Cauchy numbers and polynomials which are different from the previous degenerate Cauchy numbers and polynomials. In addition, we give some explicit identities for these numbers and polynomials which are derived from the generating function.

1. Introduction

As is well known, the Cauchy polynomials are defined by the generating function to be
\[ \frac{t}{\log(1 + t)}(1 + t)^x = \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!}, \quad (\text{see} \ [3, 6, 8, 13]). \] (1.1)

When \( x = 0 \), \( C_n = C_n(0) \) are called the Cauchy numbers. The higher-order Bernoulli polynomials are given by the generating function to be
\[ \left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see} \ [7, 11, 12]). \] (1.2)

When \( x = 0 \), \( B_n^{(r)} = B_n^{(r)}(0) \) are called the higher-order Bernoulli numbers and \( B_n^{(1)}(x) = B_n(x) \) are called the ordinary Bernoulli polynomials.

From (1.1) and (1.2), we note that
\[ C_n(x) = B_n^{(n)}(x + 1), \quad (n \geq 0), \quad (\text{see} \ [10]). \] (1.3)

In [1,2], L. Carlitz introduced the degenerate Bernoulli polynomials which are given by the generating function to be

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\[
\frac{t}{(1 + \lambda t)\frac{1}{t} - 1} = (1 + \lambda t)^{\frac{1}{\lambda} - 1} = \sum_{n=0}^{\infty} \frac{\beta_{n,\lambda}(x)}{n!} t^n. \quad (1.4)
\]

When \( x = 0 \), \( \beta_{n,\lambda} = \beta_{n,\lambda}(0) \) are called the degenerate Bernoulli numbers.

Note that \( \lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x) \), \( (n \geq 0) \). In the viewpoint of (1.4), the degenerate Cauchy polynomials are defined by the generating function to be

\[
\int_1^1 (1 + \log(1 + \lambda t))^{x+y} dy = \frac{\log(1 + \lambda t)^\frac{1}{\lambda}}{\log(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})}(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})^x = \sum_{n=0}^{\infty} C^*_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [10])}. \quad (1.5)
\]

The falling factorial sequences are given by

\[
(x)_0 = 1, (x)_n = x(x-1) \cdots (x-n+1), \quad (n \geq 1), \quad \text{(see [4, 5, 6])}. \quad (1.6)
\]

The Stirling number of the first kind is defined as

\[
(x)_n = \sum_{l=0}^{n} S_1(n, l) x^l, \quad (n \geq 0), \quad \text{(see [8, 9, 10])}, \quad (1.7)
\]

and the Stirling number of the second kind is given by

\[
x^n = \sum_{l=0}^{n} S_2(n, l)(x)_l, \quad (n \geq 0). \quad (1.8)
\]

In [10], it were known that

\[
C_m(x) = \sum_{n=0}^{m} C^*_{m,\lambda} \lambda^{m-n} S_2(m, n), \quad \text{(1.9)}
\]

and

\[
C^*_{m,\lambda}(x) = \sum_{n=0}^{m} B^{(n)}_n (x+1) \lambda^{m-n} S_1(m, n). \quad \text{(1.10)}
\]

In this paper, we study the degenerate Cauchy numbers and polynomials of the second kind which are different from previous degenerate Cauchy numbers and polynomials. In addition, we give some explicit identities for the degenerate Cauchy numbers and polynomials of the second kind which are derived from the generating function.
2. Degenerate Cauchy numbers and polynomials of the second kind

Now, we define the degenerate Cauchy polynomials of the second kind which are given by the generating function to be

\[
\frac{t}{\lambda \log(1 + \lambda t)} (1 + \frac{1}{\lambda} \log(1 + \lambda t))^x = \sum_{n=0}^{\infty} C_{n,\lambda}(x) \frac{t^n}{n!}.
\] (2.1)

When \( x = 0 \), \( C_{n,\lambda} = C_{n,\lambda}(0) \) are called the degenerate Cauchy numbers of the second kind.

By replacing \( t \) by \( \frac{1}{\lambda} (e^{\lambda t} - 1) \) in (2.1), we get

\[
\sum_{m=0}^{\infty} C_{m,\lambda}(x) \lambda^{-m} \frac{1}{m!} (e^{\lambda t} - 1)^m = \left( \frac{1}{\lambda} \frac{t}{\log(1 + t)} \right) \left( \frac{e^{\lambda t} - 1}{t} \right) = \sum_{m=0}^{\infty} \left( \frac{1}{\lambda} \sum_{l=0}^{m} \lambda^{l+1} \frac{t^l}{l+1} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} C_{n-m}(x) \frac{\lambda^m}{m+1} \right) \frac{t^n}{n!}.
\] (2.2)

On the other hand,

\[
\sum_{m=0}^{\infty} C_{m,\lambda}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \lambda^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \frac{\lambda^{n-m} C_{m,\lambda}(x) S_2(n, m)}{m+1} \right) \frac{t^n}{n!}.
\] (2.3)

Therefore, by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
\sum_{m=0}^{n} \lambda^{n-m} C_{m,\lambda}(x) S_2(n, m) = \sum_{m=0}^{n} \binom{n}{m} C_{n-m}(x) \frac{\lambda^m}{m+1}.
\]

From Theorem 1, we note that \( \lim_{\lambda \to 0} C_{m,\lambda}(x) = C_{n-m}(x) \). Let us take \( t = \frac{1}{\lambda} \log(1 + \lambda t) \) in (1.1). Then we have
\[ \sum_{m=0}^{\infty} C_m(x) \lambda^{-m} \frac{1}{m!} (\log(1 + \lambda t))^m = \frac{\frac{1}{1} \log(1 + \lambda t) \log(1 + \frac{1}{1} \log(1 + \lambda t))}{\frac{1}{1} (1 + \frac{1}{1} \log(1 + \lambda t))^x} \]
\[ \cdot \left( \frac{\log(1 + \lambda t)}{\lambda t} \right) \left( \frac{t}{\log(1 + \frac{1}{1} \log(1 + \lambda t)) \log(1 + \frac{1}{1} \log(1 + \lambda t))} \right) \]
\[ = \left( \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} (\lambda t)^{l-1} \right) \left( \sum_{m=0}^{\infty} C_{m,\lambda}(x) \frac{t^m}{m!} \right) \]
\[ = \left( \sum_{l=0}^{\infty} \frac{\lambda^l (-1)^l}{l+1} \right) \left( \sum_{m=0}^{\infty} C_{m,\lambda}(x) \frac{t^m}{m!} \right) \]
\[ = \sum_{m=0}^{\infty} \sum_{m=0}^{n} \frac{n! \lambda^{n-m} (-1)^{n-m}}{(n-m+1)!} C_{m,\lambda}(x) \frac{t^n}{n!} \]
\[ = \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \frac{(n-m)!}{n-m+1} \lambda^{n-m} (-1)^{n-m} C_{m,\lambda}(x) \frac{t^n}{n!}. \]  \hspace{1cm} (2.4)

On the other hand

\[ \sum_{m=0}^{\infty} C_m(x) \lambda^{-m} (\log(1 + \lambda t))^m = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} C_m(x) \lambda^{-m} S_1(n, m) \frac{\lambda^n}{n!} t^n \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} C_m(x) \lambda^{-m} S_1(n, m) \frac{\lambda^n}{n!} t^n \right). \]  \hspace{1cm} (2.5)

Therefore, by (2.4) and (2.5), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have

\[ \sum_{m=0}^{n} C_m(x) \lambda^{n-m} S_1(n, m) = \sum_{m=0}^{n} \frac{(n-m)!}{n-m+1} \left( \frac{n}{m} \right) (-1)^{n-m} \lambda^{n-m} C_{m,\lambda}(x). \]

When \( x = 0 \), \( C_m = C_m(0) \) and \( C_{m,\lambda} = C_{m,\lambda}(0) \). So, by Theorem 2, we get

\[ \sum_{m=0}^{n} C_m \lambda^{n-m} S_1(n, m) = \sum_{m=0}^{n} \frac{(n-m)!}{n-m+1} \left( \frac{n}{m} \right) (-1)^{n-m} \lambda^{n-m} C_{m,\lambda}. \]

From (2.1), we note that
\[
\sum_{n=0}^{\infty} C_{n, \lambda}(x) \frac{t^n}{n!} = \left( \sum_{l=0}^{\infty} C_{l, \lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (x)_m \lambda^{-m} \frac{1}{m!} \left( \log(1 + \lambda t) \right)^m \right)
\]
\[
= \left( \sum_{l=0}^{\infty} C_{l, \lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (x)_m \lambda^{-m} \sum_{k=m}^{\infty} S_1(k, m) \lambda^k \frac{k!}{k!} \right)
\]
\[
= \left( \sum_{l=0}^{\infty} C_{l, \lambda} \frac{t^l}{l!} \right) \left( \sum_{k=0}^{\infty} \frac{k}{k!} \sum_{m=0}^{\infty} (x)_m \lambda^{-m} S_1(k, m) \right) t^k (2.6)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} C_{n-k}(x) \lambda^{k-m} S_1(k, m) \right) \frac{t^n}{n!}.
\]

Therefore, by comparing the coefficients on the both sides of (2.6), we obtain the following theorem.

**Theorem 2.3.** For \( n \geq 0 \), we have

\[
C_{n, \lambda}(x) = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} C_{n-k}(x) \lambda^{k-m} S_1(k, m).
\]

From (2.4), we note that

\[
t = \sum_{l=0}^{\infty} C_{l, \lambda} \frac{t^l}{l!} \left( \log \left(1 + \frac{1}{k} \log(1 + \lambda t)\right) \right)
\]
\[
= \sum_{l=0}^{\infty} C_{l, \lambda} \frac{t^l}{l!} \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \lambda^{-k} \left( \log(1 + \lambda t) \right)^k \right)
\]
\[
= \sum_{l=0}^{\infty} C_{l, \lambda} \frac{t^l}{l!} \left( \sum_{k=1}^{\infty} \frac{(k-1)!}{(k-1)!} (-1)^{k-1} \lambda^{-k} \sum_{m=k}^{\infty} S_1(m, k) \lambda^m \frac{m^m}{m!} \right) (2.7)
\]
\[
= \sum_{l=0}^{\infty} C_{l, \lambda} \frac{t^l}{l!} \left( \sum_{m=1}^{\infty} \frac{(m-1)!}{(m-1)!} (-1)^{k-1} \lambda^{m-k} S_1(m, k) \frac{m^m}{m!} \right)
\]
\[
= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} \sum_{k=1}^{m} \binom{n}{m} C_{n-m, \lambda}(k-1)!(-1)^{k-1} \lambda^{m-k} S_1(m, k) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients on the both sides of (2.7), we obtain the following theorem.

**Theorem 2.4.** For \( n \in \mathbb{N} \), we have

\[
C_{0, \lambda} = 1, \quad \sum_{m=1}^{n} \sum_{k=1}^{m} \binom{n}{m} C_{n-m, \lambda}(k-1)!(-1)^{k-1} \lambda^{m-k} S_1(m, k) = \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } n > 1.
\end{cases}
\]
Now, we observe that

\[
\frac{t}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} = \left( \frac{\lambda t}{\log(1 + \lambda t)} \right) \left( \frac{\frac{1}{\lambda} \log(1 + \lambda t)}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} \right) \\
= \left( \sum_{l=0}^{\infty} \lambda^l B_l \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} C_{m, \lambda}^* \frac{t^m}{m!} \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \lambda^{n-m} B_{n-m}^{(n-m)} C_{m, \lambda}^* \right) \frac{t^n}{n!}.
\]

From (2.1), we note that

\[
\frac{t}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} = \sum_{n=0}^{\infty} C_{n, \lambda} \frac{t^n}{n!}.
\]

Therefore, by (2.8) and (2.9), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 0 \), we have

\[
C_{n, \lambda} = \sum_{m=0}^{n} \binom{n}{m} \lambda^{n-m} B_{n-m}^{(n-m)} C_{m, \lambda}^*.
\]

By (1.5), we get

\[
\sum_{n=0}^{\infty} C_{n, \lambda}^* \frac{t^n}{n!} = \frac{\frac{1}{\lambda} \log(1 + \lambda t)}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} = \left( \frac{\log(1 + \lambda t)}{\lambda t} \right) \left( \frac{\frac{1}{\lambda} \log(1 + \lambda t)}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} \right) \\
= \left( \sum_{l=0}^{\infty} \lambda^l D_l \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} C_{m, \lambda} \frac{t^m}{m!} \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \lambda^{n-m} D_{n-m} C_{m, \lambda} \right) \frac{t^n}{n!}.
\]

Where \( D_n, (n \geq 0) \), are the Daehee numbers which are defined by the generating function to be

\[
\frac{\log(1 + t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad \text{see \([4, 5, 9, 11, 12]\).}
\]

Comparing the coefficients on the both sides of \(2.10\), we obtain the following theorem.
Theorem 2.6. For \( n \geq 0 \), we have

\[
C^*_n,\lambda = \sum_{m=0}^{n} \binom{n}{m} \lambda^{n-m} D_{n-m} C_m,\lambda.
\]

From (2.1), we note that

\[
\sum_{n=0}^{\infty} C_{n,\lambda}(1) \frac{t^n}{n!} = \frac{t}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} (1 + \frac{1}{\lambda} \log(1 + \lambda t)) = \sum_{n=0}^{\infty} C_{n,\lambda} \frac{t^n}{n!} + t \cdot \frac{1}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))}.
\]  

(2.11)

By (1.5) and (2.11), we get

\[
t \sum_{n=0}^{\infty} C^*_n,\lambda \frac{t^n}{n!} = t \cdot \frac{1}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} = \sum_{n=1}^{\infty} \left( C_{n,\lambda}(1) - C_{n,\lambda} \right) \frac{t^n}{n!}
\]  

(2.12)

From (2.12), we get

\[
\sum_{n=0}^{\infty} C^*_n,\lambda \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( C_{n+1,\lambda}(1) - C_{n+1,\lambda} \right) \frac{t^n}{n!}.
\]  

(2.13)

Comparing the coefficients on the both sides of (2.13), we obtain the following theorem.

Theorem 2.7. For \( n \geq 0 \), we have

\[
C^*_n,\lambda = \frac{C_{n+1,\lambda}(1) - C_{n+1,\lambda}}{n+1}.
\]

By (1.5), we get

\[
\sum_{n=0}^{\infty} C^*_n,\lambda(1) \frac{t^n}{n!} = \frac{1}{\lambda} \log(1 + \frac{1}{\lambda} \log(1 + \lambda t)) \left( 1 + \frac{1}{\lambda} \log(1 + \lambda t) \right) = \sum_{n=0}^{\infty} C^*_n,\lambda \frac{t^n}{n!} + \left( \frac{t^2}{\log (1 + \frac{1}{\lambda} \log(1 + \lambda t))} \right) \log(1 + \lambda t) \frac{\lambda t}{t^2}.
\]  

(2.14)
For $r \in \mathbb{N}$, the higher-order Daehee numbers are defined by the generating function to be
\[
\left( \frac{\log(1 + t)}{t} \right)^r = \sum_{n=0}^{\infty} D^{(r)}_n \frac{t^n}{n!}, \quad \text{(see [4, 7])}, \tag{2.15}
\]

From (2.14) and (2.15), we have
\[
\sum_{n=1}^{\infty} \left( C^*_{n,\lambda}(1) - C^*_{n,\lambda} \right) \frac{t^n}{n!} = t \left( \frac{t}{\log(1 + \frac{1}{\lambda} \log(1 + \lambda t))} \right) \left( \frac{\log(1 + \lambda t)}{\lambda t} \right)^2
\]
\[
= t \left( \sum_{l=0}^{\infty} C_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} D^{(2)}_m \frac{t^m}{m!} \right)
\]
\[
= t \left( \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} C_{l,\lambda} D^{(2)}_{n-l,\lambda} \right) \frac{t^n}{n!} \right). \tag{2.16}
\]

Thus, by (2.16), we get
\[
\sum_{n=0}^{\infty} \left( \frac{C^*_{n+1,\lambda}(1) - C^*_{n+1,\lambda}}{n+1} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} C_{l,\lambda} D^{(2)}_{n-l,\lambda} \right) \frac{t^n}{n!}. \tag{2.17}
\]
Comparing the coefficients on the both sides of (2.17), we obtain the following theorem.

**Theorem 2.8.** For $n \geq 0$, we have
\[
\frac{C^*_{n+1,\lambda}(1) - C^*_{n+1,\lambda}}{n+1} = \sum_{l=0}^{n} \binom{n}{l} C_{l,\lambda} D^{(2)}_{n-l,\lambda}.
\]

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Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

E-mail address: tkim@kw.ac.kr