QUASI-HAMILTONIAN GEOMETRY OF
MEROMORPHIC CONNECTIONS

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Abstract. For each connected complex reductive group G, we find a family of new
eamples of complex quasi-Hamiltonian G-spaces with G-valued moment maps. These
spaces arise naturally as moduli spaces of (suitably framed) meromorphic connections
on principal G-bundles over a disc, and they generalise the conjugacy class example of
Alekseev–Malkin–Meinrenken (which appears in the simple pole case). Using the ‘fusion
product’ in the theory this gives a finite dimensional construction of the natural symplec-
tic structures on the spaces of monodromy/Stokes data of meromorphic connections over
arbitrary genus Riemann surfaces, together with a new proof of the symplectic nature of
isomonodromic deformations of such connections.

1. Introduction

The quasi-Hamiltonian approach [2] to constructing symplectic moduli spaces of flat
connections on G-bundles over surfaces involves “fusing” together some basic pieces and
then using a reduction procedure to obtain the symplectic moduli space. Just two types
of such basic quasi-Hamiltonian G-spaces are needed to construct all the moduli spaces
considered in [2]: conjugacy classes $C \subset G$ and the internally fused double $D \cong G \times G$.
Indeed, given a genus $g$ surface $\Sigma$ with $m$ boundary components the quasi-Hamiltonian
reduction

$$\left( D \otimes \cdots \otimes D \otimes C_1 \otimes \cdots \otimes C_m \right) / G$$

of the quasi-Hamiltonian fusion of $g$ copies of $D$ and $m$ conjugacy classes $C_i$ has a symplec-
tic structure and is isomorphic to the moduli space $\text{Hom}_G(\pi_1(\Sigma), G)/G$ of representations
of the fundamental group of $\Sigma$ with holonomy around the $i$th boundary component re-
stricted to lie in $C_i$. Such symplectic moduli spaces have been much studied and in partic-
ular there are alternative finite dimensional constructions, cf. [13, 10, 4, 12]. A beautiful
feature of the approach of [2] is that the quasi-Hamiltonian moment map condition in the
reduction (1) is precisely the monodromy relation in $\text{Hom}_G(\pi_1(\Sigma), G)$.

The aim of this paper is to use the quasi-Hamiltonian approach to give a finite dimen-
sional construction of the natural symplectic structures on more general moduli spaces
where there is currently no other finite dimensional method. This is a continuation of [8]
where the Atiyah–Bott infinite dimensional approach to moduli spaces of flat connections
was extended to allow singularities in the connections, thereby constructing symplectic
structures on moduli spaces of flat singular connections. (Such moduli spaces are naturally
isomorphic both to spaces of meromorphic connections with arbitrary order poles over
Riemann surfaces, and to spaces of monodromy/Stokes data, naturally generalising the
space of fundamental group representations above.)

Due to the quasi-Hamiltonian fusion procedure (which, on the level of surfaces, amounts
to gluing two surfaces with one boundary component into two of the holes of a three-
holed sphere) we only need to understand moduli spaces of meromorphic connections on
Theorem. The manifold \( \tilde{C} = G \times (U_+ \times U_-)^{k-1} \times \mathfrak{t} \) is a complex quasi-Hamiltonian \( G \times T \)-space with action

\[
(g, t) \cdot (C, S_1, \ldots, S_{2k-2}, \Lambda) = (tCg^{-1}, tS_1t^{-1}, \ldots, tS_{2k-2}t^{-1}, \Lambda) \in \tilde{C}
\]

(where \( S_{\text{odd/even}} \in U_+/-, (g, t) \in G \times T \), moment map \( \mu : \tilde{C} \rightarrow G \times T \) where

\[
\mu : \tilde{C} \rightarrow G; \quad (C, S, \Lambda) \mapsto C^{-1}S_{2k-2} \cdots S_2S_1e^{2\pi i \Lambda} = D^{-1}E,
\]

and two-form

\[
\omega = \frac{1}{2}(\tilde{D}, \tilde{E}) + \frac{1}{2} \sum_{j=1}^{k-1} (D_j, D_{j-1}) - (E_j, E_{j-1})
\]

where \( \tilde{D} = D^*T, \tilde{E} = E^*T, D_j = D_j^*\theta, E_j = E_j^*\theta \in \Omega^1(\tilde{C}, \mathfrak{g}) \) for maps \( D_j, E_j : \tilde{C} \rightarrow G \) defined by

\[
D_j = e^{-jS_{2k-2} \cdots S_2S_1e^{2\pi i \Lambda}}, \quad E_j = e^{-\epsilon S_{2k-2} \cdots S_2S_1e^{2\pi i \Lambda}} .
\]

Moreover for each choice of \( \Lambda \in \mathfrak{t} \) the reduction

\[
\mathcal{C} := (\tilde{C}|_\Lambda)/T \cong (G \times (U_+ \times U_-)^{k-1})/T
\]

is a complex quasi-Hamiltonian \( G \)-space.

(In the body of the paper a more symmetrical, \( \epsilon \)-free, notation will be used.) Thus for each pole order there are new quasi-Hamiltonian spaces \( \mathcal{C} \) and \( \tilde{C} \), the first of which arises simply from the second upon reducing by a torus, and depends on a choice of \( \Lambda \).

For example, in the order two pole case \( k = 2 \), if we define \( b_- = e^{-\pi i \Lambda}S_{2}^{-1} \), and \( b_+ = e^{-\pi i \Lambda}S_1e^{2\pi i \Lambda} \) then

\[
\tilde{C} \cong G \times G^*, \quad \mu = C^{-1}b_-b_+, \quad \omega = \frac{1}{2}(\tilde{D}, \tilde{E}) + \frac{1}{2}(D, \gamma) - \frac{1}{2}(E, \gamma)
\]

where \( G^* \) is the simply connected Poisson Lie group dual to \( G \) and \( D = b_-C, E = b_+C, \gamma = C^*\theta \). In general the quotient \( \tilde{C}/G \) has an induced Poisson structure \([1]\) and for \( k = 2 \) this coincides with standard Poisson structure on \( G^* \). Also we will see that for \( k = 2 \) the additive analogue \( \tilde{O} \) of \( \tilde{C} \) is the cotangent bundle \( T^*G \).

To understand the geometrical origins of these spaces we remark that the Stokes multipliers \( \{S_i\} \) arise from the irregular Riemann-Hilbert correspondence; in Theorem 4 below it is explained how \( \tilde{C} \) is isomorphic to a moduli space of (framed) meromorphic connections with fixed irregular type on \( G \)-bundles over a disc.

Given a genus \( g \) compact Riemann surface \( \Sigma \) with a divisor \( D = \sum_{i=1}^{m} k_i(a_i) \) having each \( k_i \geq 1 \) the above theorem enables one to construct symplectic moduli spaces of monodromy data for meromorphic connections on \( \Sigma \) of the form

\[
(D \otimes \cdots \otimes D \otimes \tilde{C}_1 \otimes \cdots \otimes \tilde{C}_m)/G
\]
with \( g \) factors of \( D \), as constructed in [8] from an infinite dimensional viewpoint. Summing the quasi-Hamiltonian two-forms on each factor in (3) together with the fusion terms gives an explicit expression for the symplectic structure on the manifold (3). Such an expression has also been obtained directly in the recent preprint [14], however the approach here gives an algebraic proof that it is indeed a symplectic structure.

In section 5 we will recall the additive analogues \( O, \tilde{O} \) of the spaces \( C, \tilde{C} \) for each \( k \). These are symplectic manifolds of matching dimensions (\( \dim C = \dim O \), \( \dim \tilde{O} = \dim \tilde{C} \)). Indeed \( O \) is just a generic coadjoint orbit of the group \( G_k := G(\mathbb{C}[z]/z^k) \) of \( (k - 1) \)-jets of bundle automorphisms and so this nicely extends the idea that the conjugacy classes are the multiplicative analogue of coadjoint orbits of \( G \). The extended orbits \( \tilde{O} \) are larger by \( 2 \dim T \) and give rise to the orbits \( O \) upon performing a symplectic quotient by \( T \).

In the genus zero case the spaces \( O, \tilde{O} \) enable one to construct global symplectic moduli spaces of meromorphic connections on trivial \( G \)-bundles as symplectic quotients of the form \( (\tilde{O}_1 \times \cdots \times \tilde{O}_m) / G \), and in fact such moduli spaces fill out a dense subset of a component of the full moduli space. The main result of [8] then leads immediately to:

**Corollary.** The (global) irregular Riemann-Hilbert map

\[
\nu : (\tilde{O}_1 \times \cdots \times \tilde{O}_m) / G \hookrightarrow (\tilde{C}_1 \otimes \cdots \otimes \tilde{C}_m) / G
\]

associating monodromy/Stokes data to a meromorphic connection on a trivial \( G \)-bundle over \( \mathbb{P}^1 \) is a symplectic map (provided the symplectic structure on the right-hand side is divided by \( 2\pi i \)). Moreover both sides are naturally Hamiltonian \( T^{\times m} \)-spaces and \( \nu \) intertwines these actions and their moment maps.

This map \( \nu \) depends heavily on the chosen pole positions \( a_i \) (and on a choice of ‘irregular type’ at each pole). However from the formula (2) the symplectic structure on the right-hand side of (4) is manifestly independent of these choices. This shows that the isomonodromy connection is a symplectic connection, as was shown in [8] from a deRham point of view.

For example the case with two poles of order two on \( \mathbb{P}^1 \) looks as follows. Since \( \tilde{O} \cong T^*G \) for \( k = 2 \), the left-hand side of (4) is also isomorphic to the cotangent bundle \( T^*G \). The right-hand side of (4) turns out (Proposition 6) to be isomorphic as a symplectic manifold to the symplectic double groupoid \( \Gamma \) of \( G \) and \( G^* \), described in [15]. Thus in this case \( \nu \) is a (transcendental) embedding \( T^*G \hookrightarrow \Gamma \) between these (essentially algebraic) symplectic manifolds for each choice of pole positions and irregular types.

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**Notation/Conventions**

Throughout this paper \( G \) is a connected complex reductive group with maximal torus \( T \) and corresponding Lie algebras \( \mathfrak{t} \subset \mathfrak{g} \). \( B_\pm \) denote a pair of opposite Borels subgroups with \( B_+ \cap B_- = T \) and \( U_\pm \subset B_\pm \) denote their full unipotent subgroups, with corresponding Lie algebras \( \mathfrak{u}_\pm \subset \mathfrak{b}_\pm \).
Let \((,): \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}\) be a symmetric nondegenerate invariant bilinear form. (Note that invariance implies \((,\) restricts to zero on \(u_+ \otimes u_-\) and to a pairing on each of \(u_+ \otimes u_-, u_- \otimes u_+, t \otimes t\).)

\(\theta, \bar{\theta} \in \Omega^1(G, \mathfrak{g})\) denote the tautological left and right invariant \(\mathfrak{g}\)-valued holomorphic one-forms on \(G\) respectively (so in any representation \(\theta = g^{-1} dg, \bar{\theta} = (dg)g^{-1}\)).

Generally if \(A, B, C \in \Omega^1(M, \mathfrak{g})\) are \(\mathfrak{g}\)-valued holomorphic one-forms on a complex manifold \(M\) then \((A, B) \in \Omega^2(M)\) and \([A, B] \in \Omega^2(M, \mathfrak{g})\) are defined by wedging the form parts and pairing/bracketing the Lie algebra parts (so e.g. \((A\alpha, B\beta) = (A, B)\alpha \wedge \beta\) for \(A, B \in \mathfrak{g}, \alpha, \beta \in \Omega^1(M)\)).

Define \(AB := \frac{1}{2}[A, B] \in \Omega^2(M, \mathfrak{g})\) (which works out correctly in any representation of \(G\) using matrix multiplication). Then one has \(d\theta = -\theta^2, d\bar{\theta} = \bar{\theta}^2\).

Define \((ABC) = (A, BC) \in \Omega^3(M)\) (which is totally symmetric in \(A, B, C\). The canonical bi-invariant three-form on \(G\) is then \(\eta := \frac{1}{6}(\theta^3)\).

The adjoint action of \(G\) on \(\mathfrak{g}\) will be denoted \(gXg^{-1} := \text{Ad}_g X\) for any \(X \in \mathfrak{g}, g \in G\).

If \(G\) acts on \(M\), the fundamental vector field of \(X \in \mathfrak{g}\) is minus the tangent to the flow \(\langle v_X \rangle_m = -\frac{d}{dt}(e^{xt} \cdot m)\) \(t=0\), so that the map \(\mathfrak{g} \to \text{Vect}_M; X \to v_X\) is a Lie algebra homomorphism. (This sign convention differs from [2] (and agrees with [1]); this leads to sign changes in the quasi-Hamiltonian axioms and the fusion and equivalence theorems.)

### 2. Quasi-Hamiltonian \(G\)-spaces

**Definition 1** (cf. [2, 1]). A complex manifold \(M\) is a complex quasi-Hamiltonian \(G\)-space if there is an action of \(G\) on \(M\), a \(G\)-equivariant map \(\mu: M \to G\) (where \(G\) acts on itself by conjugation) and a \(G\)-invariant holomorphic two-form \(\omega \in \Omega^2(M)\) such that

\((\text{QH}1)\): The exterior derivative of \(\omega\) is the pullback of the canonical three-form on \(G\):

\[d\omega = \mu^*(\eta)\]

\((\text{QH}2)\). For all \(X \in \mathfrak{g}\)

\[\omega(v_X, \cdot) = \frac{1}{2} \mu^*(\theta + \bar{\theta}, X) \in \Omega^1(M)\]

\((\text{QH}3)\). At each point \(m \in M\) the kernel of \(\omega\) is

\[\ker \omega_m = \{(v_X)_m \mid X \in \mathfrak{g} \text{ satisfies } gXg^{-1} = -X \text{ where } g := \mu(m) \in G\}\].

These axioms are perhaps best motivated in terms of Hamiltonian loop group manifolds [2, 16], as we will sketch in Section 4.

A simple but important observation is that if \(G\) is abelian (and in particular if \(G = \{1\}\) is trivial) then these axioms imply that the two-form \(\omega\) is a symplectic form.

**Example 2** (Conjugacy classes [2]). Let \(C \subset G\) be a conjugacy class, with the conjugation action of \(G\) and moment map \(\mu\) given by the inclusion map. Then \(C\) is a quasi-Hamiltonian \(G\)-space with two-form \(\omega\) determined by

\[\omega_g(v_X, v_Y) = \frac{1}{2}((X, gYg^{-1}) - (Y, gXg^{-1}))\]

for any \(X, Y \in \mathfrak{g}, g \in C\). For later use we note that if \(g \in C\) is fixed and we define the surjective map \(\pi: G \to C: C \ni C^{-1}gC\) then one may calculate

\[\pi^*(\omega) = \frac{1}{2}(\bar{\theta}, g\bar{\theta}g^{-1}) \in \Omega^2(G)\].
Example 3 (Internally Fused Double [2]). The space $D = G \times G$ is a quasi-Hamiltonian $G$-space with $G$ acting by diagonal conjugation $(g(a, b) = (gag^{-1}, gb^{-1}))$, moment map given by the group commutator

$$
\mu(a, b) = ab^{-1}a^{-1}
$$

and two-form

$$
\omega_D = -\frac{1}{2}(a^*\theta, b^*\theta) - \frac{1}{2}(b^*\theta, a^*\theta) - \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\theta).
$$

Now let us recall the quasi-Hamiltonian reduction theorem:

**Theorem 1** ([2]). Let $M$ be a quasi-Hamiltonian $G \times H$-space with moment map $(\mu, \mu_H) : M \to G \times H$ and suppose that the quotient by $G$ of the inverse image $\mu^{-1}(1)$ of the identity under the first moment map is a manifold. Then the restriction of the two-form $\omega$ to $\mu^{-1}(1)$ descends to the reduced space $M/G := \mu^{-1}(1)/G$ and makes it into a quasi-Hamiltonian $H$-space. In particular, if $H$ is abelian (or in particular trivial) then $M/G$ is a symplectic manifold.

The fusion product, which puts a ring structure on the category quasi-Hamiltonian $G$-spaces, is defined as follows. (Also reduction at different values of the moment map may be facilitated by first fusing with a conjugacy class, analogously to the Hamiltonian case.)

**Theorem 2** ([2]). Let $M$ be a quasi-Hamiltonian $G \times G \times H$-space, with moment map $\mu = (\mu_1, \mu_2, \mu_3)$. Let $G \times H$ act by the diagonal embedding $(g, h) \to (g, g, h)$. Then $M$ with two-form

$$
\tilde{\omega} = \omega - \frac{1}{2}(\mu_1^*\theta, \mu_2^*\theta)
$$

and moment map

$$
\tilde{\mu} = (\mu_1 \cdot \mu_2, \mu_3) : M \to G \times H
$$

is a quasi-Hamiltonian $G \times H$-space.

We will refer to the extra term subtracted off in (6) as the “fusion term”. If $M_i$ is a quasi-Hamiltonian $G \times H_i$ space for $i = 1, 2$ their fusion product

$$
M_1 \circ \circ M_2
$$

is defined to be the quasi-Hamiltonian $G \times H_1 \times H_2$-space obtained from the quasi-Hamiltonian $G \times G \times H_1 \times H_2$-space $M_1 \times M_2$ by fusing the two factors of $G$.

### 3. New Examples

In this section we will describe the family of quasi-Hamiltonian spaces $\tilde{C}, \tilde{C}$ and prove directly that they are such. The motivation for, and geometrical origins of, these spaces will only become clear in Section 4 however, where their infinite dimensional counterparts are described.

Our main objects of study are the family of complex manifolds

$$
\tilde{C} := \{(C, d, e, \Lambda) \in G \times (B_+ \times B_+)^{k-1} \times t \mid \delta(d) = e = \delta(e) \text{ for all } j\},
$$

parameterised by an integer $k \geq 2$, where $d = (d_1, \ldots, d_{k-1}), e = (e_1, \ldots, e_{k-1})$ with $d_{even}, e_{odd} \in B_+$ and $d_{odd}, e_{even} \in B_-$ and where $\delta : B_\pm \to T$ is the homomorphism with
kernel $U_\pm$. This space $\tilde{\mathcal{C}}$ is isomorphic to $G \times (U_+ \times U_-)^{k-1} \times t$ but it will be more convenient to use the above definition. For the record, in terms of the Stokes multipliers: $d_j = e^{-i}S_{2k-1-j}^1 e^{i-1}$, $e_j = e^{i+2-2k}S_j e^{2k-1-j}$ where $\epsilon := e^{\frac{2\pi i}{k}}$.

In this description the action of $G$ on $\tilde{\mathcal{C}}$ given by

$$g \cdot (C, d, e, \Lambda) = (Cg^{-1}, d, e, \Lambda)$$

for $g \in G$, and the action of $T$ is given by

$$t \cdot (C, d, e, \Lambda) = (tC, td_1 t^{-1}, \ldots, td_{k-1} t^{-1}, te_1 t^{-1}, \ldots, te_{k-1} t^{-1}, \Lambda)$$

for $t \in T$. Independently these actions are both free, although the combined $G \times T$ action is not. The maps $D_i, E_i, \mu : \tilde{\mathcal{C}} \rightarrow G$ are now defined as

$$D_i(C, d, e, \Lambda) = d_i \cdots d_1 C \quad (i = 0, 1, \ldots, k-1)$$

$$E_i(C, d, e, \Lambda) = e_i \cdots e_1 C \quad (i = 0, 1, \ldots, k-1)$$

$$\mu(C, d, e, \Lambda) = C^{-1} d_1^{-1} \cdots d_{k-1}^{-1} e_{k-1} \cdots e_1 C.$$ 

To lighten the notation we will write $D = D_{k-1}, E = E_{k-1}$, so in particular $\mu = D^{-1} E$.

The main result of this section is:

**Theorem 3.** The manifold $\tilde{\mathcal{C}}$ is a quasi-Hamiltonian $G \times T$-space with the above action, moment map $(\mu, e^{-2\pi i \Lambda}) : \tilde{\mathcal{C}} \rightarrow G \times T$ and two-form:

$$(8) \quad \omega = \frac{1}{2} (\overline{\mathcal{D}}, \overline{\mathcal{E}}) + \frac{1}{2} \sum_{i=1}^{k-1} (D_i, D_{i-1}) - (\mathcal{E}_i, \mathcal{E}_{i-1}),$$

where $\overline{\mathcal{D}} = D^* (\overline{\theta}), \overline{\mathcal{E}} = E^* (\overline{\theta}), D_i = D_i^* (\theta), \mathcal{E}_i = E_i^* (\theta) \in \Omega^1 (\tilde{\mathcal{C}}, g)$.

In particular, since $T$ is abelian, this implies $\tilde{\mathcal{C}}$ is a quasi-Hamiltonian $G$-space with moment map $\mu$ and the same two-form.

**Remark 4.** Observe that $\omega$ is invariant under translations of $\Lambda$ by the lattice $L := \ker(\exp(2\pi i \cdot) : t \rightarrow T)$. The quotient $\tilde{\mathcal{C}}/L \cong G \times (U_+ \times U_-)^{k-1} \times T$ is then an algebraic quasi-Hamiltonian $G \times T$-space. Indeed all the formulae above make sense directly on the subvariety of $G \times (B_- \times B_+)^{k-1}$ cut out by the equations $\delta(d_i) \cdot \delta(e_j) = 1$, and this subvariety is a finite covering of $\tilde{\mathcal{C}}/L$ (corresponding to replacing $\exp(\frac{2\pi i \Lambda}{k})$ by $\exp(2\pi i \Lambda)$). However we prefer to keep in the choice of $\Lambda$ in order to obtain (genuine) Hamiltonian $T$-spaces upon reducing by $G$.

Also we have:

**Corollary 5.** Suppose a value of $\Lambda$ is fixed. Then the reduction

$$\mathcal{C} := (\tilde{\mathcal{C}}|_{\Lambda})/T$$

is a complex (algebraic) quasi-Hamiltonian $G$-space.

**Proof.** $\mathcal{C}$ may also be described as the quasi-Hamiltonian reduction $(\tilde{\mathcal{C}}/L)/T$ of $\tilde{\mathcal{C}}/L$ at the value $\exp(-2\pi i \Lambda)$ of the moment map for the $T$ action. □
Before proving Theorem 3 let us describe the order two pole case $k = 2$ and the specialisation to the simple pole case $k = 1$. If $k = 2$ and we define $b_\pm = e^{-\pi i \Lambda} S_1 e^{2\pi i \Lambda}$ (so that $b_- b_+ = S_2 S_1 e^{2\pi i \Lambda}$) then

$$\tilde{\mathcal{C}} \cong G \times G^*, \quad \mu = C^{-1} b_- b_+ C, \quad \omega = \frac{1}{2} (\mathcal{D}, \mathcal{E}) + \frac{1}{2} (\mathcal{D}, \gamma) - \frac{1}{2} (\mathcal{E}, \gamma)$$

where $D = b_- C, E = b_+ C, \gamma = C^* \theta$ and

$$G^* := \{(b_- , b_+, \Lambda) \in B_- \times B_+ \times t \mid \delta(b_- ) \delta(b_+) = 1, \delta(b_+) = \exp(\pi i \Lambda)\}$$

is the Poisson Lie group dual to $G$ (cf. e.g. [7], [9] Appendix B).

Considering two poles of order two on $\mathbb{P}^1$ leads to the following statement, which gives a relationship between symplectic double groupoids and meromorphic connections.

**Proposition 6.** Let $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ be two copies of $\tilde{\mathcal{C}}$ with $k = 2$. Then the quasi-Hamiltonian reduction of the fusion of $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ is isomorphic as a symplectic manifold to the symplectic double groupoid $\Gamma$ of $G$ and $G^*$ appearing in [15]:

$$\langle \tilde{\mathcal{C}}_1 \otimes \tilde{\mathcal{C}}_2 \rangle / G \cong \Gamma.$$

**Proof.** First recall $\tilde{\mathcal{C}}_1 \cong G \times G^*$ as manifolds. We will assume that the Borels chosen at the first pole are opposite to those chosen at the second (which we may since isomonodromy will give symplectic isomorphisms with the spaces arising from any other choice of Borels intersecting in $T$). Thus $\tilde{\mathcal{C}}_1 = \{(C_1 , b_-, b_+, \Lambda_1) \mid \delta(b_+) = e^{\pm \pi i \Lambda_1}\}$ and $\tilde{\mathcal{C}}_2 = \{(C_2 , c_+, c_-, \Lambda_2) \mid \delta(c_+) = e^{\mp \pi i \Lambda_2}\}$ with $b_\pm, c_\pm \in B_\pm$. The moment map on $\tilde{\mathcal{C}}_1 \otimes \tilde{\mathcal{C}}_2$ is $\mu = C_1^{-1} b_- b_+ C_1 C_2^{-1} c_- c_+ C_2$. Writing $h := C_2 C_1^{-1}$ the condition $\mu = 1$ becomes $h b_- b_+ h^{-1} c_- c_+ = 1$, and if we define $g := c_- h b_-^{-1}$ then this condition is clearly equivalent to $c_+ h = gb_+$. Thus (omitting the $\Lambda$ terms to simplify notation) we have defined a surjective map

$$\mu^{-1}(1) \rightarrow \Gamma := \{(g, b_-, b_+, h, c_+, c_-) \mid c_+ h = gb_+\} \subset (G \times G^*)^2$$

whose fibres are precisely the $G$ orbits. This is the definition of the manifold $\Gamma$ given in [15]. The symplectic structures may be shown to agree as follows.

The map $\Gamma \rightarrow G \times G; (g, b_-, b_+, h, c_+, c_-) \mapsto (gb_-, gb_+)$ expresses $\Gamma$ as the covering of a dense subset of $G \times G$. This subset is the big symplectic leaf of the Heisenberg double Poisson structure on $G \times G$ and the symplectic structure on $\Gamma$ is defined to be the pullback of the symplectic structure on this leaf. An explicit formula for this pullback (i.e. for the symplectic structure on $\Gamma$) is given in Theorem 3 of [3]. On the other hand we have an explicit formula for the symplectic structure on $\langle \tilde{\mathcal{C}}_1 \otimes \tilde{\mathcal{C}}_2 \rangle / G$ (involving seven terms, the fusion term plus three terms (9) for each factor). A straightforward calculation shows these explicit formulae on each side agree. 

In the simple pole case $k = 1$ we define $\tilde{\mathcal{C}} = G \times t_1$ where $t_1 \subset t$ is the complement of the affine root hyperplanes: $t_1 := \{\Lambda \in t \mid \alpha(\Lambda) \notin \mathbb{Z} \text{ for all roots } \alpha\}$. The correct specialisation of (2) to this case is

$$\omega = \frac{1}{2} (\mathcal{D}, \mathcal{E}) + \frac{1}{2} (\mathcal{D}, \gamma) - \frac{1}{2} (\mathcal{E}, \gamma) = 2\pi i (\gamma, d\Lambda) + \frac{1}{2} (\gamma, e^{2\pi i \Lambda} \gamma e^{-2\pi i \Lambda})$$

where $D = e^{-\pi i \Lambda} C, E = e^{\pi i \Lambda} C, \gamma = C^* \theta$. This is the restriction of the two-form (2) to the submanifold with $d_i, e_i \in T, \Lambda \in t_1$ (for any $k$) and makes $\tilde{\mathcal{C}}$ into a quasi-Hamiltonian
$G \times T$-space with moment map $(D^{-1}E, e^{-2\pi i\Lambda})$. (It also arises as a cross-section of the double in [2].) Given a fixed choice of $\Lambda$, the restriction of $\omega$ to $G \times \{\Lambda\}$ clearly agrees with (5) and so we deduce the reduction $C := (\tilde{C}|_{\Lambda})/T \cong G/T$ is isomorphic as a quasi-Hamiltonian $G$-space to the conjugacy class through $e^{2\pi i\Lambda}$.

**Proof (of Theorem 3).** To establish (QH1), that $\mu^*(\theta^3) = 6d\omega$, we observe that the expression (8) defines a two-form on $G \times (B_+ \times B_-)^{k-1}$, and working algebraically we will view (QH1) as a statement about the differential algebra generated by the symbols $C, d_i, e_j$, using only the restriction that $(d_i^*\theta^3) = (e_j^*\theta^3) = 0$ (which follow from the fact that $d_i, e_j$ live in Borel subgroups). By restriction the result for $\tilde{C}$ will then follow. This viewpoint enables us to use induction on $k$. From the definition one finds $\mu^*(\theta^3) = ((\mathcal{E} - \mathcal{D})^3)$ which expands to give

$$
\mu^*(\theta^3) = 3(\mathcal{D}\mathcal{D}\mathcal{E}) - 3(\mathcal{D}\mathcal{E}\mathcal{E}) + (\mathcal{E}^3) - (\mathcal{D}^3).
$$

On the other hand

$$
2d\omega = (\mathcal{D}\mathcal{D}\mathcal{E}) - (\mathcal{D}\mathcal{E}\mathcal{E}) + F_{k-1}
$$

where

$$
F_{k-1} := \sum_{i=1}^{k-1} (\mathcal{D}_i\mathcal{D}_{i-1}\mathcal{D}_{i-1}) - (\mathcal{D}_i\mathcal{D}_j\mathcal{D}_{i-1}) - (\mathcal{E}_i\mathcal{E}_{i-1}\mathcal{E}_{i-1}) + (\mathcal{E}\mathcal{E}\mathcal{E}_{i-1}),
$$

so that what we must prove is $(\mathcal{E}^3) - (\mathcal{D}^3) = 3F_{k-1}$ or equivalently (assuming $(\mathcal{E}_{k-2}) - (\mathcal{D}_{k-2}) = 3F_{k-2}$ inductively) that $(\mathcal{E}^3) - (\mathcal{D}^3)$ equals

$$(11) \quad (\mathcal{E}_{k-2})^3 - (\mathcal{D}_{k-2})^3 + 3((\mathcal{D}\mathcal{D}\mathcal{D}_{k-2}) - (\mathcal{D}\mathcal{D}\mathcal{D}_{k-2}) - (\mathcal{E}_{k-2}\mathcal{E}_{k-2}) + (\mathcal{E}\mathcal{E}\mathcal{E}_{k-2})).$$

To establish this, write $E = b_+ E_{k-2}, D = b_- D_{k-2}$ where $b_+ := c_{k-1}, b_- := d_{k-1}$. (Note we do not necessarily have $b_+ \in B_+$, only that they are in opposite Borels.) Thus

$$(12) \quad \mathcal{E} = E_{k-2} \theta_+ + E_{k-2} + \mathcal{E}_{k-2}, \quad \mathcal{D} = D_{k-2} \theta_- + D_{k-2} + \mathcal{D}_{k-2},$$

$$
\mathcal{E} = \theta_+ + b_+ \mathcal{E}_{k-2} b_+^{-1}, \quad \mathcal{D} = \theta_- + b_- \mathcal{D}_{k-2} b_-^{-1},
$$

where $\theta_\pm = b_\pm^*(\theta), \bar{\theta}_\pm = b_\pm^*(\bar{\theta})$ and so

$$
(\mathcal{E}^3) - (\mathcal{D}^3) = ((\theta_+ + \mathcal{E}_{k-2})^3) - ((\theta_- + \mathcal{D}_{k-2})^3) =
$$

$$(13) \quad (\mathcal{E}_{k-2})^3 - (\mathcal{D}_{k-2})^3 + 3((\theta_+ \theta_+ + \mathcal{E}_{k-2}) + (\theta_+ \mathcal{E}_{k-2} \mathcal{E}_{k-2}) - (\theta_- \mathcal{D}_{k-2} - (\theta_- \mathcal{D}_{k-2}))$$

using the fact that $(\theta_\pm^3) = 0$. Thus we must show that the coefficients of 3 in (11) and (13) are the same; this however follows easily by substituting the expressions (12) for $\mathcal{E}, \mathcal{D}$ into (11) and expanding. Finally the $k = 2$ case may be proved directly, justifying the induction; namely we must show $(\mathcal{E}^3) - (\mathcal{D}^3) = 3F_1$ and this comes about simply by expanding both sides in terms of $b_\pm$ and $C$. (The $k = 1$ case is similar.)

Next we will check (QH2) for the $G$-action. Choose $X \in \mathfrak{g}$ and an arbitrary holomorphic vector field $Y$ on $\tilde{C}$. We will denote derivatives along $v_X$ by primes and along $Y$ by dots, so e.g. $\dot{\mathcal{D}}_i = \langle Y, \mathcal{D}_i \rangle \in \mathfrak{g}$ and $\mathcal{E}'_j = \langle v_X, \mathcal{E}_j \rangle \in \mathfrak{g}$ (and in any representation of
corresponding quasi-Hamiltonian spaces. By definition of the action $\mathcal{D}'_i = \mathcal{E}'_i = X$ for all $i$, and $\mathcal{D}' = DXD^{-1}, \mathcal{E}' = EXE^{-1}$. Thus

$$2\omega(v_X, Y) = (DXD^{-1}, \mathcal{E}') - (EXE^{-1}, \mathcal{D}') + \sum_{i=1}^{k-1} (X, \dot{\mathcal{D}}_{i-1} - \dot{\mathcal{D}}_i - \dot{\mathcal{E}}_{i-1} + \dot{\mathcal{E}}_i)$$

which simplifies to $(X, DXD - E^{-1}\mathcal{D}E - \dot{\mathcal{D}} + \dot{\mathcal{E}})$. On the other hand, since $\mu = D^{-1}E$:

$$\langle (\mu \theta + \mu \tilde{\theta}, X), Y \rangle = (\mu^{-1} \hat{\mu} + \hat{\mu} \mu^{-1}, X) = (\dot{\mathcal{E}} - \dot{\mathcal{D}} + D^{-1}\mathcal{E}D - E^{-1}\mathcal{D}E, X)$$

so we have established (QH2) for the $G$-action.

For the $T$-action, if $X \in \mathfrak{t}$ then the derivatives along the corresponding fundamental vector field $v_X$ (for the $T$ action) are: $\mathcal{D}_i = \mathcal{E}_i = -X, \dot{\mathcal{D}}_i = -D_i X D_i, \dot{\mathcal{E}}_i = -E_i^{-1}X E_i$. Thus for any vector field $Y$ on $\hat{C}$

$$2\omega(v_X, Y) = (X, -\mathcal{E}' + \mathcal{D}' + \sum_{i=1}^{k-1} -D_i \mathcal{D}'_{i-1} D_i^{-1} + E_i \mathcal{E}'_{i-1} E_i^{-1} + D_i \mathcal{D}'_i D_{i-1}^{-1} - E_i^{-1} \mathcal{E}'_i E_i^{-1})$$

where the primes denote the derivatives along $Y$. Now $D_i = d_i D_{i-1}$ so that $D_i \mathcal{D}'_{i-1} D_i^{-1} = \mathcal{D}'_i - \delta_i$ and $D_{i-1} \mathcal{D}'_i D_{i-1}^{-1} = \mathcal{D}'_{i-1} + \delta'_i$ (and similarly for the $E_i$'s), where $\delta_i := d_i \theta$ etc.

Substituting thus shows

$$2\omega(v_X, Y) = (X, \sum_{i=1}^{k-1} \delta'_i + \delta_i - \epsilon_i - \epsilon'_i).$$

Since $X \in \mathfrak{t}$ we may take the $t$ component of the right-hand side yielding

$$\omega(v_X, Y) = -(2\pi i)(X, \Lambda') = -(2\pi i)(d\Lambda, X)$$

which is what appears on the right-hand side of (QH2) if the moment map is $e^{-2\pi i \Lambda}$.

The proof of the minimal degeneracy condition (QH3) is rather complicated so has been put in the appendix.

\[\square\]

4. Derivation

In this section we will explain how the quasi-Hamiltonian spaces $\mathcal{C}, \hat{\mathcal{C}}$ were found. In brief the extension of the Atiyah–Bott symplectic structure to the meromorphic case in [8] leads to new (infinite dimensional) Hamiltonian loop group manifolds and $\mathcal{C}, \hat{\mathcal{C}}$ are the corresponding quasi-Hamiltonian spaces.

In more detail recall that the equivalence theorem (Theorem 8.3) of [2] gives a correspondence between Hamiltonian $LK$-manifolds (with proper moment maps) and quasi-Hamiltonian $K$-spaces, where $K$ is a compact (connected) Lie group and $LK = C^\infty(S^1, K)$ is the corresponding loop group. The main examples of such Hamiltonian $LK$ spaces are moduli spaces of framed flat connections on principal $K$-bundles over compact two-manifolds $\Sigma$ with precisely one boundary component: Given $\Sigma$ and $K$ one defines a space of connections

$$\mathcal{A} := \{ \alpha \in \Omega^1_{C^\infty}(\Sigma, \mathfrak{k}) \}$$

on the trivial $C^\infty$ principal $K$-bundle over $\Sigma$ (where $\mathfrak{k} = \text{Lie}(K)$) and a gauge group

$$\mathcal{K} := C^\infty(\Sigma, K).$$
This has normal subgroup $\mathcal{K}_\partial := \{g \in \mathcal{K} \mid g|_{\partial \Sigma} = 1\}$ consisting of bundle automorphisms equal to the identity on the boundary circle. The quotient $\mathcal{K}/\mathcal{K}_\partial$ is thus isomorphic to the loop group $L\mathcal{K}$. Atiyah–Bott [5] define the following symplectic structure on $\mathcal{A}$:

$$\omega_{\mathcal{A}}(\phi, \psi) = \int_{\Sigma} (\phi, \psi)$$

where $(\cdot, \cdot)$ denotes a chosen pairing on $\mathfrak{k}$. Then, taking the curvature of the connections in $\mathcal{A}$ gives a moment map for the action of $\mathcal{K}_\partial$ (see [6]) and so the symplectic quotient at the zero value of the moment map is the moduli space of flat connections with a framing along the boundary circle:

$$\hat{N} := \mathcal{A}_{\text{flat}}/\mathcal{K}_\partial.$$ 

This infinite dimensional symplectic manifold is a Hamiltonian $L\mathcal{K}$-space in the sense of [2] (and such spaces constitute the main class of examples). The action of $L\mathcal{K}$ is simply the residual action of $\mathcal{K}$ and the moment map is the restriction of the connections to the boundary circle:

$$\hat{\mu} : \hat{N} \to A_{S^1}; \quad \alpha \mapsto \alpha|_{\partial \Sigma}.$$ 

(This is really a Hamiltonian $\hat{L}\mathcal{K}$-space where $\hat{L}\mathcal{K}$ is the centrally extended loop group and the central circle acts trivially on $\hat{N}$; the space $A_{S^1}$ of connections on the trivial $K$-bundle over the circle is naturally identified with the level one hyperplane in the dual of the Lie algebra of $\hat{L}\mathcal{K}$. However this complication is incorporated into the definition of Hamiltonian $L\mathcal{K}$-spaces in [2, 16].)

Now choose a point $p \in \partial \Sigma$ of the boundary circle of $\Sigma$. The equivalence theorem of [2] implies that the quotient $\mathcal{N} := \hat{N}/\Omega K$ of $\hat{N}$ by the based loop group $\Omega K = \{g \in L\mathcal{K} \mid g(p) = 1\}$ is a (finite dimensional) quasi-Hamiltonian $K$-space. In other words moduli spaces of flat connections on $\Sigma$ with a framing at one point on the boundary are naturally quasi-Hamiltonian $K$-spaces.

The two-form and moment map on $\mathcal{N}$ are constructed as follows. One has a commutative diagram:

$$\begin{array}{ccc}
\hat{N} & \xrightarrow{\hat{\mu}} & A_{S^1} \\
\downarrow{\pi} & & \downarrow{h} \\
\hat{\mathcal{N}} & \xrightarrow{\mu} & \hat{K}.
\end{array}$$

where $\pi$ is the $\Omega K$ quotient and the maps $\mu$ and $h$ take the holonomy of the connections around the boundary circle (in a positive sense starting at $p$ with initial condition $1 \in K$).

The quasi-Hamiltonian two-form $\omega_{\mathcal{N}}$ on $\mathcal{N}$ is defined by

$$-\pi^*(\omega_{\mathcal{N}}) = \omega_{\hat{\mathcal{N}}} - \hat{\mu}^*(\varpi)$$

where $\omega_{\hat{\mathcal{N}}}$ is the symplectic form on $\hat{N}$ and $\varpi$ is the following two-form on $A_{S^1}$. For each point $z \in S^1$ define a map $h_z : A_{S^1} \to K$ taking a connection $\alpha$ to its holonomy along the positive arc from $p$ to $z$, with initial condition $1 \in K$. Thus $h_z^*\theta$ is a $z$-dependent $\mathfrak{k}$-valued one-form on $A_{S^1}$ and $\varpi$ is defined to be

$$\varpi = \frac{1}{2} \int_{S^1} (h_z^*\theta, dh_z^*\theta)$$

\footnote{The signs differ from [2] as 1) we give the boundary circle the induced orientation and 2) an overall sign change has been made anyway.}
where $d$ is the exterior derivative on $S^1$. It is worth noting that this procedure of subtracting off $\hat{\mu}^*(\varpi)$ will amount simply to forgetting part of an integral in the computation below.

Remark 7. Under this map from surfaces with just one boundary component to quasi-Hamiltonian $K$-spaces, the quasi-Hamiltonian fusion operation corresponds to gluing two surfaces (each with one boundary component) into two of the holes of a three-holed sphere (so the resulting surface again has one boundary component) cf. [2, 16]. Also, quasi-Hamiltonian reduction corresponds to fixing the conjugacy class of monodromy around the boundary component and forgetting the framing, thereby giving the usual symplectic moduli space of flat connections. The upshot is that once we allow fusion, all the symplectic manifolds that arise as moduli spaces of flat connections on surfaces may be constructed from just two types of quasi-Hamiltonian $K$-spaces: conjugacy classes (one for each boundary component) and the internally fused double ($\cong K \times K$), which corresponds to the one-holed torus.

Now we will apply the above philosophy to the extension of the Atiyah–Bott symplectic structure to singular connections ($C^\infty$ connections with poles) given in [8]. First we point out that the above story may be complexified; if $\Sigma$ has just one boundary component and $G$ is a connected complex reductive group (e.g. the complexification of $K$) then the moduli space of flat connections on $G$-bundles over $\Sigma$ with framings at one point on the boundary are complex quasi-Hamiltonian $G$-spaces. In turn if $\Sigma$ has a complex structure such moduli spaces may be identified with the moduli space of holomorphic connections on holomorphic $G$-bundles over $\Sigma$ (together with a framing at one point on the boundary). (Both spaces are isomorphic to the manifold $\text{Hom}(\pi_1(\Sigma, p), G)$ of fundamental group representations.)

In a similar way the moduli spaces of flat $C^\infty$ singular connections we will define below correspond both to moduli spaces of meromorphic connections on holomorphic $G$-bundles (cf. [8] Proposition 4.5) and to spaces of monodromy/Stokes data (cf. [8] Proposition 4.8).

Due to fusion it is sufficient to consider only $C^\infty$ singular connections on a disc having just one pole. Fix an integer $k \geq 1$ (the pole order) and an irregular type

\[ \tilde{A}^0 := A_0 \frac{dz}{z^k} + \cdots + A_{k-2} \frac{dz}{z^2} \in \Omega^1[D](\Delta, g) \]

where $A_i \in t, A_0 \in t_{\text{reg}}, z$ is a coordinate on the closed unit disc $\Delta$ and $D := k(0)$ is a divisor on $\Delta$ supported at the origin. If $k = 1$ we set $\tilde{A}^0 = 0$. The spaces of $C^\infty$-singular connections we are interested in have their full infinite jets of derivatives fixed, except for the residue term:

\[ \tilde{A} := \{ \alpha \in \Omega^1_{C^\infty}[D](\Delta, g) \mid L_0(\alpha) = \tilde{A}^0 + \Lambda dz/z \text{ for some } \Lambda \in t_k \} \]

where $L_0$ takes the full $C^\infty$ Laurent expansion of $\alpha$ at the origin and $t_k = t$ if $k \geq 2$ but $t_1$ is the affine regular Cartan: $t_1 = \{ \Lambda \in t \mid \beta(\Lambda) \notin \mathbb{Z} \text{ for all roots } \beta \}$. Let

\[ G_T := \{ g \in C^\infty(\Delta, G) \mid L_0(g) \in T \subset G[z, \bar{z}] \} \]

be the group of bundle automorphisms having Taylor expansion zero at the origin except for the constant term, which should be in $T$. Clearly the tangent space to $\tilde{A}$ at a
connection $\alpha$ is

$$T_\alpha \tilde{A} = \{ \phi \in \Omega^1_{\mathbb{C}^\infty}[D](\Delta, g) \mid L_0(\phi) \in t \frac{dz}{z} \}.$$ 

Thus as in [8] we may still use the Atiyah–Bott formula in this singular situation and define a symplectic structure on $\tilde{A}$ as

$$\omega_{\tilde{A}}(\phi, \psi) = \int_{\Delta} (\phi, \psi).$$

Lemma 8. The gauge action of the subgroup $G_{1,0} := \{ g \in G_T \mid g|_{\partial \Delta} = 1, g(0) = 1 \}$ on $\tilde{A}$ is Hamiltonian with moment map given by the curvature.

Proof. See [8] Proposition 5.4. □

The symplectic quotient of $\tilde{A}$ at the zero value of the moment map is thus

$$\tilde{N} := \tilde{A}_{\text{flat}} / G_{1,0}$$

which has a residual action of $G_T / G_{1,0} \cong T \times LG$. The $T$-action is Hamiltonian with moment map

$$\alpha \mapsto -2\pi i \Lambda = -(2\pi i) \text{Res}_0 L_0(\alpha)$$

as in [8] Proposition 5.5, and (as above) the $LG$-action makes $\tilde{N}$ into a Hamiltonian $LG$-space (in the sense of [2]) with moment map

$$\tilde{\mu} : \tilde{N} \to \mathcal{A}_{S^1}; \quad \alpha \mapsto \alpha|_{\partial \Delta}.$$ 

Now fix the point $p = -1 \in \partial \Delta$. Thus (momentarily forgetting the $T$-action) the quotient $N := \tilde{N} / \Omega G$ by the based loop group should be a quasi-Hamiltonian $G$-space. First we will use the irregular Riemann-Hilbert correspondence to identify $N$ as a complex manifold. Let

$$G_{1,p} := \{ g \in G_T \mid g(p) = 1 = g(0) \}$$

so that

$$N = \tilde{N} / \Omega G = \tilde{A}_{\text{flat}} / G_{1,p}$$

which has a residual action of $G_T / G_{1,p} \cong G \times T$.

Theorem 4 ([8, 9]). The quotient $\tilde{A}_{\text{flat}} / G_{1,p}$ is isomorphic to $\tilde{C}$ as a $G \times T$-space.

Proof. As in [8] Proposition 4.5, Corollary 4.6 this quotient may be shown to be canonically isomorphic to the set of isomorphism classes of 4-tuples $(P, A, g_0, g_p)$ where $P \to \Delta$ is a holomorphic principal $G$-bundle, $A$ is a meromorphic connection on $P$ with irregular type $\tilde{A}^0$ and compatible framing $g_0$ at the origin and $g_p$ is an arbitrary framing of $P$ at $p$. Then by the irregular Riemann-Hilbert correspondence of [9] Section 2 the moduli space of such triples $(P, A, g_0)$ is analytically isomorphic to the space $(U_+ \times U_-)^{k-1} \times t$ of Stokes multipliers and exponents of formal monodromy ($\Lambda$’s). The inclusion of the framing $g_p$ in the moduli problem simply adds a factor of $G$ so the result follows. The formula for the $G$-action is immediate and for the $T$-action see [8] Corollary 3.5. □

Remark 9. The monodromy map $\tilde{\nu} : \tilde{A}_{\text{flat}} \to \tilde{C}$, whose fibres are precisely the $G_{1,p}$ orbits will be described directly in the proof of the following theorem.
Now if \( \omega_{\hat{\mathcal{N}}} \) is the symplectic structure on \( \hat{\mathcal{N}} \) and \( \varpi \) is the complex analogue of the two-form (15) on \( \mathcal{A}_{s_{1}} \) (defined exactly the same way) then, by the general theory described above, we expect the the two-form \(- \omega_{\hat{\mathcal{N}}} + \tilde{\mu}^{*}(\varpi)\) on \( \hat{\mathcal{N}} \) to be the pullback of some quasi-Hamiltonian two-form on \( \hat{\mathcal{C}} \) along the map \( \pi : \hat{\mathcal{N}} \to \mathcal{N} \cong \hat{\mathcal{C}} \). Indeed we have the following theorem.

**Theorem 5.** Let \( \omega \) be the two-form on \( \hat{\mathcal{C}} \) defined in (2). Then we have

\[
-\pi^{*}(\omega) = \omega_{\hat{\mathcal{N}}} - \tilde{\mu}^{*}(\varpi).
\]

**Proof.** Since \( \hat{\mathcal{N}} \) is the symplectic quotient of \( \hat{\mathcal{A}} \) this is equivalent to proving \( \iota^{*}\omega_{\hat{\mathcal{A}}} - \text{pr}^{*}\tilde{\mu}^{*}\varpi = -\tilde{\nu}^{*}\omega \) where \( \iota : \hat{\mathcal{A}}_{\text{flat}} \to \hat{\mathcal{A}} \) is the inclusion, \( \text{pr} : \hat{\mathcal{A}}_{\text{flat}} \to \hat{\mathcal{N}} \) is the projection and \( \tilde{\nu} : \hat{\mathcal{A}}_{\text{flat}} \to \hat{\mathcal{C}} \). To this end suppose we have a two-parameter family \( \alpha(s, t) \in \hat{\mathcal{A}}_{\text{flat}} \) of flat singular connections depending holomorphically on \( s, t \). We will evaluate the two-form \( \iota^{*}\omega_{\hat{\mathcal{A}}} - \text{pr}^{*}\tilde{\mu}^{*}\varpi \) on the pair \( \alpha', \hat{\alpha} \in \Omega^{1}_{C^{\infty}}[D](\Delta, \mathfrak{g}) \) of tangent vectors to \( \hat{\mathcal{A}}_{\text{flat}} \) at \( \alpha = \alpha(0, 0) \), where \( \alpha' = \frac{d}{ds} \alpha \big|_{s=0} \) and \( \hat{\alpha} = \frac{d}{dt} \alpha \big|_{t=0} \). If \( X = \hat{\nu}_{s}(\alpha'), Y = \tilde{\nu}_{s}(\hat{\alpha}) \in T_{\hat{\nu}(\alpha)}\hat{\mathcal{C}} \) then we should obtain \(-\omega(X, Y)\) where by definition

\[
2\omega(X, Y) = \langle \tilde{D}, \tilde{E} \rangle - \langle \tilde{D}, \mathcal{E} \rangle + \sum_{j=1}^{k-1} \langle \mathcal{D}'_{j}, \mathcal{D}_{j-1} \rangle - \langle \mathcal{D}'_{j}, \mathcal{D}_{j-1} \rangle - \langle \mathcal{E}'_{j}, \mathcal{E}_{j-1} \rangle + \langle \mathcal{E}'_{j}, \mathcal{E}_{j-1} \rangle
\]

with \( \mathcal{D}'_{j} = \langle \mathcal{D}_{j}, X \rangle \) etc.

Let \( \Delta_{r} \) denote the slit annulus obtained by cutting \( \Delta \) along the ray from 0 to \( p = -1 \) and removing the open disc of radius \( r \) centred on the origin. Denote by \( \overline{\Delta}_{r} \) the closure of \( \Delta_{r} \) in the universal cover of the punctured disc \( \Delta \setminus \{0\} \). Thus \( \overline{\Delta}_{r} \) has two straight edges \( l_{+}, l_{-} \) lying over the interval \([-1, -r] \) and has interior isomorphic to the interior of \( \Delta_{r} \subset \Delta \). In particular \( \overline{\Delta}_{r} \) is contractible. We identify the lower lip \( l_{-} \) with the interval \([-1, -r] \subset \Delta \), so that one arrives at the upper lip \( l_{+} \) by turning a full turn in a positive sense from \( l_{-} \). For each \( s, t \) let

\[
\chi(s, t) : \overline{\Delta}_{r} \to G
\]

be the fundamental solution of the connection \( \alpha(s, t) \) taking the value 1 in \( G \) at \( p \in l_{-} \).

(In other words \( \chi(s, t) \) is the map solving the differential equation \( \alpha(s, t) = \chi^{*}(\overline{\theta}) \).) Then for each \( z \in \overline{\Delta}_{r} \) let \( \chi'(z) := \frac{d}{ds}\chi(s, t, z) \big|_{s=0} \in T_{\chi(z)}G \) and so

\[
\chi^{-1}\chi' := l_{\chi^{-1}}\chi'
\]

is a \( \mathfrak{g} \)-valued function on \( \overline{\Delta}_{r} \), where \( l_{\chi^{-1}} : T_{\chi(z)}G \to \mathfrak{g} \) denotes the derivative of left multiplication by \( \chi^{-1}(z) \) in the group \( G \). Now define a one-form \( \varphi \) on \( \overline{\Delta}_{r} \) by

\[
\varphi := \frac{1}{2}(\varphi_{1} - \varphi_{2}), \quad \varphi_{1} := (\chi^{-1}\chi', d(\chi^{-1}\chi)), \quad \varphi_{2} := (d(\chi^{-1}\chi'), \chi^{-1}\chi)
\]

where \( d \) is the exterior derivative on \( \overline{\Delta}_{r} \). Thus \( d\varphi = (\alpha', \hat{\alpha}) \) as two-forms on \( \overline{\Delta}_{r} \) (since e.g. \( \alpha' = \chi d(\chi^{-1}\chi')\chi^{-1} \)). In turn since \( (\alpha', \hat{\alpha}) \) is a smooth two-form on \( \Delta \) we have

\[
\omega_{\Delta}(\alpha', \hat{\alpha}) = \int_{\Delta} (\alpha', \hat{\alpha}) = \lim_{r \to 0} \int_{\Delta_{r}} d\varphi = \lim_{r \to 0} \int_{\partial\overline{\Delta}_{r}} \varphi.
\]

This integral will be evaluated along each arc of the boundary of \( \overline{\Delta}_{r} \), neglecting any terms that vanish in the limit. A similar calculation appears in [17].
First around the outer boundary of $\Delta_r$ (the circle of radius one) we recognise that the integral of $\varphi$ is precisely $(pr^*\mu^*\varpi)(\alpha', \hat{a})$ (since on this circle $\chi$ restricts to the map $h_z$ used to define $\varpi$), which is the term to be substracting off.

For the other arcs we first need to describe directly the map $\tilde{\nu} : \tilde{\mathcal{A}}_{\text{flat}} \to \tilde{\mathcal{C}}$ associating monodromy data $(C, d, e)$ to a flat singular connection $\alpha$. The key point is that any $\alpha \in \tilde{\mathcal{A}}_{\text{flat}}$ has canonical fundamental solutions

$$\Phi_i : \text{Sect}_i \to G$$

on certain distinguished sectors $\text{Sect}_i$ defined as follows ([8] Lemma 4.7, [9] Section 2). The leading coefficient $A_0 \in t_{\text{reg}}$ of the chosen irregular type $\tilde{A}^0$ determines the anti-Stokes directions $\mathcal{A}$ at $0 \in \Delta$ defined as

$$z \in \Delta \setminus \{0\} \text{ lies on an anti-Stokes direction } \iff \frac{\beta(A_0)}{z^{k-1}} \in \mathbb{R} \text{ for some root } \beta \in \mathcal{R}.$$ 

This determines a finite set $\mathcal{A}$ of directions which is clearly invariant under rotation by $\pi/(k-1)$ and so the number $l := \#\mathcal{A}/(2k-2)$ is an integer. The sectors $\text{Sect}_i$ are just the sectors bounded by consecutive anti-Stokes directions. Without loss of generality we will assume the positive real axis $\mathbb{R}_+$ is not an anti-Stokes direction and label these sectors in a positive sense and such that $\mathbb{R}_+ \subset \text{Sect}_0$. In turn the anti-Stokes directions $a_i \in \mathcal{A}$ are labeled (modulo $\#\mathcal{A}$) such that $\text{Sect}_i = \text{Sect}(a_i, a_{i+1})$. By [9] Lemma 2.4 we know that the set of roots

$$\mathcal{R}_+ := \{ \beta \in \mathcal{R} \mid \frac{\beta(A_0)}{z^{k-1}} \in \mathbb{R}_+ \text{ for } z \text{ on one of the directions } a_1, \ldots, a_l \}$$

‘supporting’ one of the first $l$ anti-Stokes directions, is a set of positive roots, and we define $B_+$ to be the corresponding Borel subgroup containing $T$. Now to define $\Phi_i$ we recall that the Laurent expansion of $\alpha$ is

$$L_0(\alpha) = dQ + \Lambda \frac{dz}{z} =: \tilde{A}^0$$

for some $\Lambda \in \mathfrak{t}$ where $Q := \sum_{j=1}^{k-1} \frac{z^{j-k}}{j-k} A_{j-1}^0$ (so $dQ = \tilde{A}^0$). In particular the $(0, 1)$ part of $\alpha$ is nonsingular across the origin and so we may solve the $\overline{\partial}$-problem

$$(\overline{\partial}g)g^{-1} = \alpha^{0,1}$$

for a smooth map $g : U \to G$ defined in some neighbourhood $U \subset \Delta$ of the origin. Given such $g$ one observes ([8] Lemma 4.3) that the Taylor expansion $\hat{F} = L_0(g^{-1})$ is in $G[z]$ (has no $\overline{z}$ terms) and that $A := \hat{F}[\tilde{A}^0] = g^{-1}[\alpha]$ is the germ of a (convergent) meromorphic connection. In turn this implies ([9] Theorem 2.5) that there is a unique holomorphic map

$$\Sigma_i(\hat{F}) : \text{Sect}_i \to G$$

on each sector such that $\Sigma_i(\hat{F})[A^0] = A$ and that the analytic continuation of $\Sigma_i(\hat{F})$ to the supersector

$$\widehat{\text{Sect}}_i := \text{Sect} \left( a_i - \frac{\pi}{2k-2}, a_{i+1} + \frac{\pi}{2k-2} \right)$$

is asymptotic to $\hat{F}$ at $0$ in $\widehat{\text{Sect}}_i$. Now we are led to the following definition because $z^Ae^Q$ is a fundamental solution of the connection $A^0$, $\Sigma_i(\hat{F})$ is an isomorphism between $A^0$ and $A$, and $g$ is an isomorphism between $A$ and $\alpha$. (Here $z^A$ is defined on $\text{Sect}_0$ using the
branch of \( \log(z) \) that is real on \( \mathbb{R}_+ \) and by convention we extend this to the other sectors in a negative sense.)

**Definition 10.** The canonical fundamental solution of \( \alpha \in \tilde{A}_{\text{flat}} \) on \( \text{Sect}_i \) is the map 

\[
\Phi_i := g \Sigma_i (L_0 g^{-1}) z^\Lambda e^Q : \text{Sect}_i \to G
\]

for any solution \( g \) of \( (\partial g) g^{-1} = \alpha^{0.1} \).

The Stokes multipliers \( S_i \) of \( \alpha \) can now be defined (as in [9] Definition 2.6) as the elements of \( G \) relating the fundamental solutions \( \Phi_d \) and \( \Phi_{(i+1)d} \). However to define directly the elements \( d_i, e_i \) we first define new fundamental solutions \( \Psi_i, \Theta_i \) as follows:

\[
\Psi_i := \Phi_i e^{2k - 2 - i} : \text{Sect}_d \to G \quad (i = 1, \ldots, 2k - 2), \quad \Theta_i = \Psi_{2k - 2 - i}.
\]

where \( \epsilon := e^{2\pi i / 2} \). The indices of \( \Psi_i, \Theta_i \) are taken modulo \( 2k - 2 \) so \( \Psi_0 = \Theta_0 = \Phi_0 \) on \( \text{Sect}_0 \).

For \( i = 0, \ldots, k - 2 \) the sector on which \( \Psi_i \) or \( \Theta_i \) is defined intersects the slit annulus \( \Delta_i \) in a contractible set and so we may extend \( \Psi_i, \Theta_i \) uniquely (as fundamental solutions of \( \alpha \)) from maps to \( \Delta_r \) to \( G \). Now the intersection of \( \text{Sect}_{(k-1)d} \) (the sector containing \( \mathbb{R}_- \)) and \( \Delta_r \) has two components, and we extend \( \Psi_{k-1} \) from the upper component of this intersection onto \( \Delta_r \), and we extend \( \Theta_{k-1} \) from the lower component. Thus we have \( 2k \) generally distinct fundamental solutions of \( \alpha \) on \( \Delta_r \):

\[
\chi, \Phi_0 = \Psi_0 = \Theta_0, \Psi_1, \ldots, \Psi_{k-1}, \Theta_1, \ldots, \Theta_{k-1}.
\]

The monodromy data \( C, d_i, e_i \) is defined to be the set of \((z\text{-independent})\) group elements relating them, as follows:

\[
\Phi_0 C = \chi, \quad \Psi_i e_i = \Psi_{i-1}, \quad \Theta_i d_i = \Theta_{i-1} \quad (i = 1, \ldots, k - 1).
\]

If \( d_i, e_i \) are defined in this way it follows from [9] Lemma 2.7 that \( d_{\text{even}}, e_{\text{odd}} \in B_+ \), \( d_{\text{odd}}, e_{\text{even}} \in B_- \) and \( \delta(d_j)^{-1} = \epsilon = \delta(e_j) \), so we have indeed associated a point of \( \tilde{C} \) to \( \alpha \).

Note also that the maps \( D_i, E_i : \tilde{C} \to G \) arise as

\[
\Psi_i E_i = \chi, \quad \Theta_i D_i = \chi \quad (i = 0, \ldots, k - 1).
\]

It follows that \( \chi \) has holonomy \( D^{-1} E \) since \( \chi|_{l^+} = \Psi_{k-1}|_{l^+} E = \Theta_{k-1}|_{l^+} E = \chi|_{l^-} D^{-1} E \), and so this is the quasi-Hamiltonian monodromy map.

Now we return to the boundary integral. Choose a point \( q_i \) of distance \( r \) from the origin and in the intersection \( \text{Sect}_d \cap \text{Sect}_{(i-1)d} \) of two of the supersectors, for \( i = 1, \ldots, k - 1 \). Thus we know that both \( \Phi_d(q_i) \) and \( \Phi_{(i+1)d}(q_i) \) are asymptotic to \( z^\Lambda e^Q \) at 0 as \( r \to 0 \), and in turn we know the asymptotics of \( \Psi_i(q_i) \) and \( \Psi_{i-1}(q_i) \) at 0. Similarly choose \( p_i \in \text{Sect}_{-d} \cap \text{Sect}_{-(i-1)d} \) of modulus \( r \) so that we know the asymptotics of both \( \Theta_i(p_i) \) and \( \Theta_{i-1}(p_i) \) at 0 as \( r \to 0 \). Let \( p_k = -r \in l_- \) and let \( q_k \) be the point of the upper lip \( l^+ \) lying over \( -r \). Thus we may divide the inner boundary circle of \( \Delta_r \) into \( 2k - 1 \) arcs by breaking it at the points \( p_i, q_i \). Now since \( \chi = \Psi_i E_i \) and \( E_i \) is \( z \)-independent we find

\[
(16) \quad \varphi_1 = (\Psi_i^{-1} \Psi_i \cdot d(\Psi_i^{-1} \Psi_i)) + d(\varphi_i; \Psi_i^{-1} \Psi_i)
\]

where \( \varphi_i = \langle E_i^* \bar{\partial}, X \rangle \), and similarly for \( \varphi_2 \) (swapping the dot and the prime).

**Lemma 11.** The first term in (16) may be neglected in the integral from \( q_{i+1} \) to \( q_i \).
Proof. The first term of (16) and the corresponding term of \( \varphi_2 \) contribute
\[
\frac{1}{2} \int_{q_{i+1}}^{q_i} (\Psi_i^{-1} \Psi_i, d(\Psi_i^{-1} \dot{\Psi}_i)) - (\Psi_i^{-1} \dot{\Psi}_i, d(\Psi_i^{-1} \Psi_i'))
\]
to the integral of \( \varphi \). However \( \Psi_i \simeq z^A e^{Q_k 2k - i} \) at 0 uniformly in \( \text{Sect}_d \) (which contains the integration path). Substituting in this approximation gives that the integrand in (17) is zero. This implies that in the limit \( r \to 0 \) the integral (17) really is zero. \( \square \)

Thus modulo negligible terms
\[
\int_{q_{i+1}}^{q_i} \varphi_1 = (E_i, \Psi_i^{-1} \dot{\Psi}_i)|_{q_{i+1}}^{q_i}.
\]
If we sum this integral for \( i = 1, \ldots, k - 1 \) then the contribution at \( q_i \) is
\[
(E_i', \Psi_i^{-1} \dot{\Psi}_i')(q_i) - (E_{i-1}', \Psi_{i-1}^{-1} \dot{\Psi}_{i-1})(q_i)
\]
provided \( i \neq 1, k \). Now using \( \Psi_{i-1} = \Psi_i e_i \) to remove \( \Psi_{i-1} \) this becomes
\[
(E_i' - e_i E_{i-1} \dot{\Psi}_i', \Psi_i^{-1} \dot{\Psi}_i) - (E_{i-1}' - \dot{\Psi}_{i-1})(\dot{\Psi}_i)
\]
where \( \dot{\Psi}_i = (\dot{e}_i \theta, Y) \). In turn using \( E_i = e_i E_{i-1} \) this becomes
\[
(\dot{e}_i', \Psi_i^{-1} \dot{\Psi}_i) - (E_{i-1}', \dot{\Psi}_i).
\]
If we also repeat the above for \( \varphi_2 \) we get the same but with the dots and primes swapped. Now since \( \Psi_i \simeq z^A e^{Q_k 2k - i} \) and the \( T \) component of \( e_i \) is \( \epsilon \) we deduce \( (\dot{e}_i', \Psi_i^{-1} \dot{\Psi}_i) - (\dot{e}_i, \Psi_i^{-1} \dot{\Psi}_i') \to 0 \) as \( r \to 0 \). Thus the contribution at \( q_i (i \neq 1, k) \) to the integral of \( \varphi \) from \( q_k \) to \( q_1 \) is
\[
-\frac{1}{2}(\dot{e}_{i-1}', \dot{e}_i) = \frac{1}{2}(E_{i-1}, E_i)(X, Y)
\]
which is a term appearing in \( -\omega(X, Y) \). Writing \( p_0 := q_1 \) and performing the same manipulations for the \( \Theta_k \)'s, integrating \( \varphi \) from \( p_0 \) to \( p_k \) yields a contribution of
\[
\frac{1}{2}(D_{i-1}', D_i) = -\frac{1}{2}(D_{i-1}, D_i)(X, Y)
\]
at \( p_k \), provided \( i \neq 0, k \). The two left-over contributions at \( q_1 = p_0 \) combine to give the term \( \frac{1}{2}(E_1, E_0)(X, Y) \). (Thus all terms of \( -\omega \) except \( \frac{1}{2}(D, E)(X, Y) \) have been obtained so far.) The left-over contributions at \( q_k \) and \( p_k \) are:
\[
\frac{1}{2}(\dot{E}_{k-1}, \dot{E}_k) = \frac{1}{2}(\dot{E}_{k-1}, \dot{E}_k)(p_k) = \frac{1}{2}(\dot{E}_{k-1}, \dot{E}_k)(p_k) \quad (18)
\]
Now consider the two straight edges \( l_{\pm} \) of \( \Sigma_r \). Recall that \( \Theta_{k-1}|_{l_{\pm}} = \Psi_{k-1}|_{l_{\pm}} \) so that from (16)
\[
\int_{l_{\pm} + l_{\pm}} \varphi_1 = \int_{l_{\pm} + l_{\pm}} d(\dot{E}_k - E_k, \dot{E}_k - E_k) = C - E_k, \dot{E}_k - E_k, \dot{E}_k - E_k)|_{p_k}
\]
and similarly for \( \varphi_2 \). Observe that the contribution at \( p_k \) to the integral of \( \varphi \) along \( l_{\pm} \) cancels precisely with the left-over terms at \( p_k, q_k \) displayed above. Finally since \( \Theta_{k-1}(p) = \chi(p) D^{-1} = D^{-1} \) the contribution at \( p \) is
\[
-\frac{1}{2}(\dot{E}_k - E_k, \dot{E}_k - E_k) = \frac{1}{2}(D, E)(X, Y).
\]
5. Additive Analogues

Here we recall (from [8] Section 2) the symplectic manifolds $O$, $\tilde{O}$ which are the additive analogues of the quasi-Hamiltonian spaces $C, \tilde{C}$.

Fix an integer $k \geq 2$. Let $G_k := G(\mathbb{C}[z]/z^k)$ be the group of $(k - 1)$-jets of bundle automorphisms, and let $\mathfrak{g}_k = \text{Lie}(G_k)$ be its Lie algebra, which contains elements of the form $X = X_0 + X_1z + \cdots + X_{k-1}z^{k-1}$ with $X_i \in \mathfrak{g}$. Let $B_k$ be the subgroup of $G_k$ of elements having constant term 1. The group $G_k$ is the semi-direct product $G \rtimes B_k$ (where $G$ acts on $B_k$ by conjugation). Correspondingly the Lie algebra of $G_k$ decomposes as a vector space direct sum and dualising we have: $\mathfrak{g}_k^* = \mathfrak{b}_k^* \oplus \mathfrak{g}^*$. Elements of $\mathfrak{g}_k^*$ will be written as

\begin{equation}
A = A_0 \frac{dz}{z^k} + \cdots + A_{k-1} \frac{dz}{z} \tag{19}
\end{equation}

via the pairing with $\mathfrak{g}_k$ given by $\langle A, X \rangle := \text{Res}_0(A, X) = \sum_{i+j=k-1}(A_i, X_j)$. In this way $\mathfrak{b}_k^*$ is identified with the set of $A$ having zero residue and $\mathfrak{g}^*$ with those having only a residue term (zero irregular part). Let $\pi_{\text{res}} : \mathfrak{g}_k^* \rightarrow \mathfrak{g}^*$ and $\pi_{\text{irr}} : \mathfrak{g}_k^* \rightarrow \mathfrak{b}_k^*$ denote the corresponding projections.

Now choose an element $\tilde{A}^0 = A_0^0 \frac{dz}{z^k} + \cdots + A_{k-2}^0 \frac{dz}{z^2}$ of $\mathfrak{b}_k^*$ with $A_0^0 \in \mathfrak{t}$ and with regular leading coefficient $A_0^0 \in \mathfrak{t}_{\text{reg}}$. Let $O_B \subset \mathfrak{b}_k^*$ denote the $B_k$ coadjoint orbit containing $\tilde{A}^0$.

**Definition 12.** The extended orbit $\tilde{O} \subset G \times \mathfrak{g}_k^*$ associated to $O_B$ is:

$$
\tilde{O} := \{(g_0, A) \in G \times \mathfrak{g}_k^* \mid \pi_{\text{irr}}(g_0Ag_0^{-1}) \in O_B\}
$$

where $\pi_{\text{irr}} : \mathfrak{g}_k^* \rightarrow \mathfrak{b}_k^*$ is the natural projection removing the residue.

If $(g_0, A) \in \tilde{O}$ then $A$ will correspond to the principal part of a generic meromorphic connection and $g_0$ to a compatible framing.

In the simple pole case $k = 1$ we define

$$
\tilde{O} := \{(g_0, A) \in G \times \mathfrak{g}^* \mid g_0Ag_0^{-1} \in t_1\} \subset G \times \mathfrak{g}^*
$$

where $t_1 \subset t^* \cong \mathfrak{t}$ is the complement of the affine root hyperplanes. If we identify $G \times \mathfrak{g}^*$ with $T^*G$ then $\tilde{O}$ is in fact a symplectic submanifold (see [11] Theorem 26.7).

The basic properties of these extended orbits may be summarised as follows. Given $(g_0, A) \in \tilde{O}$ then by hypothesis there is some $g \in G_k$ such that $gAg^{-1} = \tilde{A}^0 + Rdz/z$ for some $R \in \mathfrak{g}$ and we define a map $\Lambda = \delta(R) : \tilde{O} \rightarrow t \cong t^*$ by taking the $t$ component of $R$ (which is independent of $g$).

**Proposition 13 ([8], 1).** The extended orbit $\tilde{O}$ is canonically isomorphic to the symplectic quotient $(T^*G_k \times O_B)/B_k$.

2). (Decoupling). The map $\tilde{O} \rightarrow (T^*G \times O_B)(g_0, A) \mapsto (g_0, \pi_{\text{res}}(A), \pi_{\text{irr}}(g_0Ag_0^{-1}))$ is a symplectic isomorphism where $T^*G \cong G \times \mathfrak{g}^*$ via the left trivialisation.

3). The map $-\Lambda$ is a moment map for the free action of $T$ on $\tilde{O}$ defined by $t(g_0, A) = (tg_0, A)$ where $t \in T$. 
4). The symplectic quotient by \( T \) at the value \( -\Lambda \) of the moment map is the \( G_k \) coadjoint orbit \( O \) through the element \( \tilde{A}^0 + \Lambda dz/z \) of \( \mathfrak{g}_k^* \).

5). The free \( G \)-action \( h(g_0, A) := (g_0 h^{-1}, h A h^{-1}) \) on \( \tilde{O} \) is Hamiltonian with moment map \( \mu_G : \tilde{O} \to \mathfrak{g}^* ; (g_0, A) \mapsto \pi_{\text{res}}(A) \).

In particular \( \tilde{O} \) is a Hamiltonian \( G \times T \)-manifold with \( T \) reductions equal to \( G_k \)-coadjoint orbits \( O \); these properties are viewed as natural analogues of those of \( \tilde{C} \) (and they do indeed match up under the Riemann-Hilbert correspondence). Note that the coadjoint orbit \( O_B \) is a point if \( k = 2 \) so that part 2) says \( \tilde{O} \cong T^* G \), the additive analogue of the fact that \( \tilde{C} \cong G \times G^* \) in this case.

Proposition 2.1 of [8] explains how the symplectic manifolds
\[
(\tilde{O}_1 \times \cdots \times \tilde{O}_m) / G
\]
for extended orbits \( \tilde{O}_i \), are isomorphic to moduli spaces of (compatibly framed) meromorphic connections on trivial \( G \)-bundles over \( \mathbb{P}^1 \) (with fixed irregular types).

**APPENDIX A. KERNEL CALCULATION**

We will establish the minimal degeneracy condition (QH3) for the two-form \( \omega \) on \( \tilde{C} \).

**Proof (of (QH3)).**

**Lemma 14.** The two-form \( 2\omega \) on \( \tilde{C} \) is also given by the formula
\[
\left( \tau, (11) \tau (11)^{-1} \right) + \sum_{i=1}^{k-1} \left( \tau, (1i) \varepsilon_i (1i)^{-1} + \{1i\}^{-1} \varepsilon_i \{1i\} - (1i)^{-1} \delta_i (1i) - [1i]^{-1} \delta_i [1i] \right)
\]
\[
+ \sum_{1 \leq i,j \leq k-1} \left( \delta_i, (ij) \varepsilon_j (ij)^{-1} \right) + \sum_{1 \leq i < j \leq k-1} \left( \delta_i, [ij] \delta_j [ij]^{-1} \right) - \left( \varepsilon_i, \{ij\} \varepsilon_j \{ij\}^{-1} \right)
\]
where \( \delta_i = d^*_i (\theta), \varepsilon_i = e^*_i (\theta), \tau = C^*(\theta), (ij) := \delta^{-1}_i d^{-1}_{i+1} \cdots d^{-1}_{k-1} e_{k-1} \cdots e_{j+1} e_j, [ij] := d_{i-1} \cdots d_j \) and \( \{ij\} := e_{i-1} \cdots e_j \).

**Proof.** This is a straight-forward direct calculation expanding each term in (8). □

Now suppose we choose a pair of tangent vectors \( X, Y \) to \( \tilde{C} \) at some point \( p \), such that \( X \) is in the kernel of \( \omega_p \) and \( Y \) is arbitrary. We will use dots/primes to denote derivatives along \( Y/X \) respectively, so e.g. \( \delta_i = \langle Y, \delta_i \rangle \in \mathfrak{g} \) and \( \varepsilon'_i = \langle X, \varepsilon_j \rangle \in \mathfrak{g} \) (and in any representation of \( G \) we have \( \delta_i = d^{-1}_i d_i \) etc). Our aim is to prove \( \delta'_i = \varepsilon'_i = 0 \) for all \( i \) (so \( X \) is tangent to the \( G \) action) and that \( \text{Ad}_{\mu(p)}(\gamma') = -\gamma' \), which is the required degeneracy condition. The equation expressing the fact that \( X \) is in the kernel of \( \omega_p \) is equivalent to
\[
2\omega(Y, X) = \langle \dot{\gamma}, \Gamma \rangle + \sum_{i=1}^{k-1} (\dot{\delta}_i, \Delta_i) + (\dot{\varepsilon}_i, \xi_i) = 0
\]
for all \( Y \) where \( \Gamma, \Delta_i, \xi_i \in g \) are the corresponding coefficients involving just \( X \) derivatives; explicitly from Lemma 14:

\[
\Delta_i = (i1)\overline{\gamma}(i1)^{-1} + [i1]\overline{\gamma}[i1]^{-1} + \sum_{j=1}^{k-1} (ij)\epsilon_j'(ij)^{-1} + \sum_{j<i} [ij]\delta_j'[ij]^{-1} - \sum_{j>i} [ji]^{-1}\delta_j'[ji],
\]

\[
\xi_i = -(i1)^{-1}\overline{\gamma}(i1) - \{i1\}\overline{\gamma}\{i1\}^{-1} - \sum_{j=1}^{k-1} (ji)^{-1}\delta_j'(ji) - \sum_{j<i} \{ij\}\epsilon_j'\{ij\}^{-1} + \sum_{j>i} \{ji\}^{-1}\epsilon_j'\{ji\},
\]

\[
\Gamma = (11)^{-1}(11) - (11)^{-1}\overline{\gamma}(11) + \sum_{j=1}^{k-1} (1j)\epsilon_j'(1j)^{-1} + \{1j\}^{-1}\epsilon_j'\{1j\}^{-1} - (1j)^{-1}\delta_j'(1j) - [1j]^{-1}\delta_j'[1j].
\]

Since \( Y \) is arbitrary (20) implies \( \Gamma = 0 \) and (since \( (,\cdot) \) pairs opposite Borels) that the piece of \( \Delta_i \) in the unipotent subalgebra opposite the Borel containing \( \dot{\delta}_i \) is zero (and similarly the piece of \( \xi_i \) in the unipotent subalgebra opposite the Borel containing \( \dot{\epsilon}_i \) is zero). The only other information about \( X \) in (20) concerns the \( t \) components as follows. Since \( Y \) is tangent to \( \mathcal{C} \) we have \( \delta(\dot{\epsilon}_j) = -\delta(\dot{\delta}_j) = \pi A \dot{\lambda} \in t \) for all \( j \), where \( \delta : g \rightarrow t \) is the projection along the root spaces. Thus, as \( \dot{\lambda} \) is arbitrary, (20) implies

\[
\sum_{i=1}^{k-1} \delta(\Delta_i) = \sum_{i=1}^{k-1} \delta(\xi_i)
\]

where \( \delta : g \rightarrow t \). Now we will proceed to deduce the required result. From the formula for \( \Delta_i \) it follows that \( d_i\Delta_i d_i^{-1} - \Delta_{i+1} = -\overline{\delta}_i - \delta_{i+1} \). Thus if we define \( T_i := d_i\Delta_i d_i^{-1} + \delta_{i+1}' = \Delta_{i+1} - \overline{\delta}_i \) (for \( i = 1, \ldots, k-2 \)) then the restrictions on the unipotent pieces of \( \Delta_i, \Delta_{i+1} \)

imply \( T_i = \delta_{i+1}' - \overline{\delta}_i + H_i \) for some \( H_i \in t \), and so in turn

\[
\Delta_i = -\delta_i' + d_i^{-1}H_id_i, \quad \Delta_{i+1} = \delta_{i+1}' + H_i.
\]

Thus \( \Delta_i = \delta_i' + H_{i-1} = -\delta_i' + d_i^{-1}H_id_i \) so that \( 2\delta_i' = d_i^{-1}H_id_i - H_{i-1} \) for \( i = 2, \ldots, k-2 \). Taking the \( t \) component of this implies \( H_{k-1} = H + H_i \) where \( H := (2\pi i)\Lambda' \) (so \( H = -2\delta(\delta_j') = 2\delta(\epsilon_j') \) for all \( j \)). If we define \( H_{k-1} := \delta(\Delta_{k-1}) - H/2 \) then \( \delta(\Delta_i) = H/2 + H_i \) for all \( i \), and since \( H_i = (k-1-i)H + H_{k-1} \) this implies

\[
\sum_{i=1}^{k-1} \delta(\Delta_i) = (k-1)H_{k-1} + (k-1)^2H/2.
\]

A similar exercise in terms of the \( \epsilon_i \) and \( \xi_i \) yields analogous formulae with some sign changes: \( Y_i := \epsilon_i\xi_i e_i^{-1} - \epsilon_i' e_i^{-1} = \epsilon_i + \epsilon_i' \) (for \( i = 1, \ldots, k-2 \)), so that \( Y_i = \xi_i - \epsilon_i' + K_i \)

for some \( K_i \in t \), and in turn \( 2\epsilon_i' = -e_i^{-1}K_ie_i + K_{i-1} \) for \( i = 2, \ldots, k-2 \). Similarly this implies \( K_{i-1} = H + K_i \) and then \( \delta(\xi_i) = H/2 + K_i \) for all \( i \) so that \( \sum_{i=1}^{k-1} \delta(\xi_i) = (k-1)K_{k-1} + (k-1)^2H/2 \). Thus equation (21) is equivalent to \( H_{k-1} = K_{k-1} \).

Now we will reconsider the equations \( \Delta_i + \delta_i' = d_i^{-1}H_id_i \) and \( \xi_i - \epsilon_i' = e_i^{-1}K_ie_i \). Using these and the initial formulae for \( \Delta_i, \xi_i \) one finds \([i1]^{-1}(\Delta_i + \delta_i'[i1]) + (i1)(\xi_i - \epsilon_i'[i1]) \) is equal to both sides of

\[
2 \sum_{j>i} ((1j)\epsilon_j'(1j)^{-1} - [ji]^{-1}\delta_j'[ji]) = [i1]^{-1}(d_i^{-1}H_id_i)[i1] + (i1)(e_i^{-1}K_i e_i)(i1)^{-1}
\]

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Conjugating by \((11)^{-1}\) this is equivalent to

\[
2 \sum_{j>i}(\{j\}^{-1} \varepsilon'_j \{j\} - (j)\{j\}^{-1} \delta'_j(j)) = (i+11)^{-1}H_i(i+11) + \{i+11\}^{-1}K_i\{i+11\}.
\]

Putting \(i = k - 2\) in (22) (so the sum has just one term) we find

\[
2\varepsilon'_{k-1} - 2\delta'_{k-1} = d_{k-1}H_{k-2}d_{k-1}^{-1} + e_{k-1}K_{k-2}e_{k-1}^{-1}.
\]

Firstly the \(t\) component of this says \(2H = H_{k-2} + K_{k-2}\) but \(H_{k-2} = H + K_{k-2} = K_{k-2}\) and thus we deduce \(H_{k-1} = K_{k-1} = 0\) (so that now \(H_i = K_i = (k - 1 - i)H\) for all \(i\)). Secondly, rewriting gives

\[
2\varepsilon'_{k-1} - e_{k-1}K_{k-2}e_{k-1}^{-1} = 2\delta'_{k-1} + d_{k-1}H_{k-2}d_{k-1}^{-1}
\]

the two sides of which live in opposite Borel subalgebras and have zero \(t\) component, and so are both zero, i.e. \(\varepsilon'_{k-1} = H/2 = -\delta'_{k-1}\).

Similarly, considering the difference \([i1]^{-1}(\Delta_i + \delta'_i)[i1] - (1i)(\xi_i - \varepsilon'_i)(1i)^{-1}\) instead and setting \(i = 1\), one obtains

\[
(24) \quad 2(11)\varepsilon'_1(11)^{-1} + 2(11)\gamma'(11)^{-1} = -2\delta'_1 - 2\gamma' + (k - 2)(d_1^{-1}Hd_1 - (12)H(12)^{-1}.
\]

Conjugating by \((11)^{-1}\) this is equivalent to

\[
(25) \quad 2(11)^{-1}\delta'_1(11) + 2(11)^{-1}\gamma'(11) = -2\varepsilon'_1 - 2\gamma' + (k - 2)((21)^{-1}H(21) - e_1^{-1}H e_1).
\]

Finally we return to the equation \(\Gamma = 0\). Observe that every term of \(2\Gamma\) appears on the left-hand side of one of the equations (22), (23), (24) or (25) (where we set \(i = 1\) in (22),(23)) except for the terms \(2\varepsilon'_1 - 2\delta'_1\). Upon substituting the right-hand sides of (22)-(25) into \(2\Gamma\) most terms cancel and we are left with:

\[
2\Gamma = 4\varepsilon'_1 - 4\delta'_1
\]

and so \(\varepsilon'_1 = \delta'_1\). Firstly taking the \(t\) component of this implies \(H = 0\) (and so \(\delta'_1 = 0 = \varepsilon'_i\) for \(i > 1\)) and secondly \(\varepsilon'_1\) and \(\delta'_1\) are in opposite Borel with zero \(t\) component and so must both be zero. Now returning to equation (24) we see

\[
(11)\gamma'(11)^{-1} = -\gamma'
\]

which says precisely that \(\text{Ad}_{\mu(p)}\gamma' = -\gamma'\) since \(\mu(p) = C^{-1}(11)C\) in the notation we are using.

\[\square\]

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