Entanglement in open quantum dynamics

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Abstract
In the framework of the theory of open systems based on completely positive quantum
dynamical semigroups, we give a description of the continuous-variable entanglement for a
system consisting of two independent harmonic oscillators interacting with a general
environment. Using the Peres–Simon necessary and sufficient criterion for separability of
two-mode Gaussian states, we describe the generation and evolution of entanglement in terms
of the covariance matrix for an arbitrary Gaussian input state. For some values of diffusion and
dissipation coefficients describing the environment, the state keeps for all times its initial type:
separable or entangled. In other cases, entanglement generation or entanglement collapse
(entanglement sudden death) takes place or even a periodic collapse and revival of
entanglement takes place. We show that for certain classes of environments, the initial state
evolves asymptotically to an entangled equilibrium bipartite state, whereas for other values of
the coefficients describing the environment, the asymptotic state is separable. We calculate
also the logarithmic negativity characterizing the degree of entanglement of the asymptotic
state.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

When two systems are immersed in an environment,
beside and at the same time as the quantum decoherence
phenomenon, the environment can also generate a quantum
entanglement of the two systems [1, 2]. In certain
circumstances, the environment enhances the entanglement,
whereas in others it suppresses the entanglement and the state
describing the two systems becomes separable. The structure
of the environment may be such that not only the two systems
become entangled, but also the entanglement is maintained for
a definite time or a certain amount of entanglement survives
in the asymptotic long-time regime.

In the present paper, we investigate, in the framework
of the theory of open systems based on completely
positive quantum dynamical semigroups, the dynamics of the
continuous variable entanglement for a subsystem composed
of two identical harmonic oscillators interacting with an
environment. We are interested in discussing the correlation
effect of the environment; therefore, we assume that the two
systems are independent, i.e. they do not interact directly.
The initial state of the subsystem is taken to be of Gaussian form
and the evolution under the quantum dynamical semigroup
ensures the preservation in time of the Gaussian form of the
state.

This paper is organized as follows. In section 2, we
write and solve the equations of motion in the Heisenberg
picture for two independent harmonic oscillators interacting
with a general environment. Then, by using the Peres–Simon
necessary and sufficient condition for separability of
two-mode Gaussian states [3, 4], we investigate in section 3
the dynamics of entanglement for the considered subsystem.
In particular, with the help of the asymptotic covariance
matrix, we determine the behaviour of the entanglement in
the limit of long times. We show that for certain classes of
environments the initial state evolves asymptotically to an
equilibrium state that is entangled, whereas for other values of
the parameters describing the environment, the entanglement
is suppressed and the asymptotic state is separable. A
summary is given in section 4.

2. Equations of motion for two independent
harmonic oscillators

We study the dynamics of the subsystem composed of
two identical non-interacting oscillators in weak interaction
with an environment. In the axiomatic formalism based on completely positive quantum dynamical semigroups, the irreversible time evolution of an open system is described by the following general quantum Markovian master equation for an operator $A$ (Heisenberg representation) [5, 6]:

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[H, A(t)] + \frac{1}{2\hbar} \sum_j \left( V_j^\dagger [A(t), V_j] + [V_j^\dagger, A(t)] V_j \right).$$

(1)

Here, $H$ denotes the Hamiltonian of the open system and the operators $V_j$ and $V_j^\dagger$, defined on the Hilbert space of $H$, represent the interaction of the open system with the environment. Being interested in the set of Gaussian states, we introduce such quantum dynamical semigroups that preserve this set. Therefore $H$ is taken to be a polynomial of second degree in the coordinates $x$ and $y$ and momenta $p_x$ and $p_y$, of the two quantum oscillators, and $V_j$ and $V_j^\dagger$ are taken to be polynomials of first degree in these canonical observables. Then in the linear space spanned by the coordinates and momenta, there exist only four linearly independent operators $V_j = 1, 2, 3, 4$ [7]:

$$V_j = a_{xj} p_x + a_{yj} p_y + b_{xj} x + b_{yj} y,$$

(2)

where $a_{xj}, a_{yj}, b_{xj}, b_{yj} \in \mathbb{C}$. The Hamiltonian $H$ of the two uncoupled identical harmonic oscillators of mass $m$ and frequency $\omega$ is given by

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{m \omega^2}{2} (x^2 + y^2).$$

(3)

The fact that the evolution is given by a dynamical semigroup implies the positivity of the following matrix formed by the scalar products of the four vectors $a_x, b_x, a_y$ and $b_y$, whose entries are the components $a_{xj}, b_{xj}, a_{yj}$ and $b_{yj}$, respectively,

$$\begin{pmatrix}
(a_{x1}, a_{x2}, a_{x3}, a_{x4}) & (a_{y1}, a_{y2}, a_{y3}, a_{y4}) \\
(b_{x1}, b_{x2}, b_{x3}, b_{x4}) & (b_{y1}, b_{y2}, b_{y3}, b_{y4})
\end{pmatrix}.$$  

(4)

We take this matrix to be of the following form, where all coefficients $D_{xx}, D_{xp}, \ldots$ and $\lambda$ are real quantities (we put, for simplicity, $\hbar = 1$):

$$\begin{pmatrix}
D_{xx} & D_{xp} & D_{xy} & D_{yp} \\
-D_{xp} & -D_{xp}^2 & D_{xp} & -D_{xp}^2 \\
-D_{xp} & D_{yp} & D_{yp} & D_{yp} \\
-D_{xp} & -D_{yp} & -D_{yp} & D_{yp}^2
\end{pmatrix}.$$  

(5)

It follows that the principal minors of this matrix are positive or zero. From the Cauchy–Schwarz inequality, the following relations for the coefficients defined in equation (5) hold:

$$D_{xx} D_{pp} - D_{xp}^2 \geq \frac{\lambda^2}{4}, \quad D_{yy} D_{pp} - D_{yp}^2 \geq \frac{\lambda^2}{4},$$

$$D_{xx} D_{yy} - D_{xy}^2 \geq 0, \quad D_{xx} D_{pp} - D_{xp}^2 \geq 0,$$  

$$D_{yy} D_{pp} - D_{yp}^2 \geq 0, \quad D_{xx} D_{pp} - D_{xp}^2 \geq 0,$$  

$$D_{yy} D_{pp} - D_{yp}^2 \geq 0.$$

(6)

The matrix of the coefficients (5) can be conveniently written as

$$\begin{pmatrix}
C_1 & C_3 \\
C_3^\dagger & C_2
\end{pmatrix},$$

(7)

in terms of $2 \times 2$ matrices $C_1 = C_1^\dagger$, $C_2 = C_2^\dagger$ and $C_3$. This decomposition has a direct physical interpretation: the elements containing the diagonal matrices $C_1$ and $C_2$ represent environment coefficients corresponding to the first, respectively the second system, whereas the elements in $C_3$ represent environment generated couplings between the two, initially independent, oscillators.

We introduce the following $4 \times 4$ bimodal covariance matrix:

$$\sigma(t) = \begin{pmatrix}
\sigma_{xx} & \sigma_{xp} & \sigma_{xy} & \sigma_{yp} \\
\sigma_{xp} & \sigma_{pp} & \sigma_{yp} & \sigma_{yp} \sigma_{pp} \\
\sigma_{xy} & \sigma_{yp} & \sigma_{pp} & \sigma_{yp} \\
\sigma_{yp} & \sigma_{yp} & \sigma_{yp} & \sigma_{pp}
\end{pmatrix},$$

(8)

with the correlations of operators $A_1$ and $A_2$, defined by using the density operator $\rho$, describing the initial state of the quantum system, as follows:

$$\sigma_{A_1A_2}(t) = \frac{i}{2} \text{Tr}(\rho(A_1 A_2 + A_2 A_1(t))) - \text{Tr}(\rho(A_1(t))) \text{Tr}(\rho(A_2(t))).$$

(9)

By direct calculation, we obtain [7] (superscript T denotes the transposed matrix)

$$\frac{d\sigma(t)}{dt} = Y \sigma(t) + \sigma(t) Y^T + 2D,$$

(10)

where

$$Y = \begin{pmatrix}
-\lambda & 1/m & 0 & 0 \\
0 & -\lambda & 0 & 0 \\
0 & 0 & -\lambda & 1/m \ \\
0 & 0 & 0 & -m \omega^2
\end{pmatrix},$$

$$D = \begin{pmatrix}
D_{xx} & D_{xp} & D_{xy} & D_{yp} \\
D_{xp} & D_{pp} & D_{yp} & D_{yp} \\
D_{xy} & D_{yp} & D_{pp} & D_{yp} \\
D_{yp} & D_{yp} & D_{pp} & D_{pp}
\end{pmatrix},$$

(11)

(12)

The time-dependent solution of equation (10) is given by [7]

$$\sigma(t) = M(t)(\sigma(0) - \sigma(\infty)) M^T(t) + \sigma(\infty),$$

(13)

where the matrix $M(t)$ is $\exp(Yt)$ has to fulfill the condition $\lim_{t \to \infty} M(t) = 0$. In order that this limit exists, $Y$ must only have eigenvalues with negative real parts. The values at infinity are obtained from the equation [7]

$$Y \sigma(\infty) + \sigma(\infty) Y^T = -2D.$$

(14)

3. Dynamics of entanglement

The two-mode Gaussian state is entirely specified by its covariance matrix (8), which is a real, symmetric and positive matrix with the following block structure:

$$\sigma(t) = \begin{pmatrix}
A & C \\
C^T & B
\end{pmatrix},$$

(15)
where $A$, $B$ and $C$ are $2 \times 2$ matrices. Their entries are correlations of the canonical operators $x$, $y$, $p_x$, and $p_y$. $A$ and $B$ denote the symmetric covariance matrices for the individual reduced one-mode states, whereas the matrix $C$ contains the cross-correlations between modes. The elements of the covariance matrix depend on $Y$ and $D$ and can be calculated from equations ($13$) and ($14$). Since the two oscillators are identical, it is natural to consider environments for which the two diagonal submatrices in equation ($7$) are equal, $C_1 = C_2$, and the matrix $C_3$ is symmetric, so that in the following we take $D_{xx} = D_{yy}$, $D_{xp} = D_{yp}$, $D_{p_x p_y} = D_{p_y p_x}$, and $D_{xp} = D_{yp}$. Then both unimodal covariance matrices are equal, $A = B$, and the entanglement matrix $C$ is symmetric.

### 3.1. Asymptotic entanglement

First, we analyse the existence of entanglement in the limit of large times. With the chosen coefficients, we obtain from equation ($14$) the following elements of the asymptotic entanglement matrix $C(\infty)$:

$$
\sigma_{xy}(\infty) = \frac{m^2(2\lambda^2 + 3\omega^2)D_{sx} + 2m\lambda D_{sp} + D_{p_x p_y}}{2m^2\lambda(\lambda^2 + \omega^2)},
$$

$$
\sigma_{xp}(\infty) = \sigma_{yp}(\infty) = \frac{m^2\omega^2D_{sx} + 2m\lambda D_{sp} + D_{p_x p_y}}{2m(\lambda^2 + \omega^2)},
$$

$$
\sigma_{p_x p_y}(\infty) = \frac{m^2\lambda^2D_{sx} - 2m\lambda^2 D_{sp} + (2\lambda^2 + \omega^2)D_{p_x p_y}}{2\lambda(\lambda^2 + \omega^2)}.
$$

The elements of matrices $A(\infty)$ and $B(\infty)$ are obtained by putting $x = y$ in the previous expressions. We calculate the determinant of the entanglement matrix and obtain

$$
\text{det } C(\infty) = \frac{1}{4\lambda^2(\lambda^2 + \omega^2)}
\times \left[ \left( m\omega^2 D_{sx} + \frac{1}{m} D_{p_x p_y} \right)^2 + 4\lambda^2(D_{sx} D_{p_x p_y} - D_{sp}^2) \right].
$$

It is interesting that the general theory of open quantum systems allows couplings via the environment between uncoupled oscillators. According to the definitions of the environment parameters, the above diffusion coefficients can be different from zero and can therefore simulate an interaction between the uncoupled oscillators. Indeed, Gaussian states with $\text{det } C > 0$ are separable states, but for $\text{det } C < 0$ it is possible that the asymptotic equilibrium states are entangled, as will be shown next.

On general grounds, one expects that the effects of decoherence, counteracting entanglement production, are dominant in the long-time regime, so that no quantum correlation (entanglement) is expected to be left at infinity. Nevertheless, there are situations in which the environment allows the presence of entangled asymptotic equilibrium states. In order to investigate whether an external environment can actually entangle the two independent systems, we use the partial transposition criterion [3, 4]: a state is entangled if and only if the operation of partial transposition does not preserve its positivity. For the particular case of Gaussian states, Simon [4] obtained the following necessary and sufficient criterion for separability: $S \geq 0$, where

$$
S \equiv \text{det } A \text{ det } B + \frac{1}{4} \left[ \text{det } C \right]^2 - \text{Tr}[A J C B J C^T J] = -\frac{1}{4} \left( \text{det } A + \text{det } B \right)
$$

and $J$ is the $2 \times 2$ symplectic matrix

$$
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

In the following, we consider an environment characterized by diffusion coefficients of the form $m^2\omega^2 D_{xx} = D_{p_x p_y}$, $D_{xp} = 0$ and $m^2\omega^2 D_{yy} = D_{p_y p_x}$. This corresponds to the case when the asymptotic state is a Gibbs state [6]. Then Simon’s expression ($20$) takes the following form in the limit of large times:

$$
S(\infty) = \left( \frac{m^2\omega^2D_{sx}^2 - D_{sy}^2}{\lambda^2} + \frac{D_{sp}^2}{\lambda^2 + \omega^2} - \frac{1}{4} \right)^2 - \frac{m^2\omega^2D_{sx}^2 D_{sp}^2}{\lambda^2(\lambda^2 + \omega^2)}.
$$

For environments characterized by such coefficients that the expression $S(\infty)$ ($22$) is strictly negative, the asymptotic final state is entangled. In particular, for $D_{xy} = 0$ we obtain that $S(\infty) < 0$, i.e. the asymptotic final state is entangled, for the following range of values of the coefficient $D_{sp}$, characterizing the environment [8, 9]:

$$
\frac{m\omega D_{xx}}{\lambda} < \frac{D_{sp}}{\sqrt{\lambda^2 + \omega^2}} < \frac{m\omega D_{xx}}{\lambda} + \frac{1}{2}.
$$

where the diffusion coefficient $D_{xx}$ satisfies the condition $m\omega D_{xx}/\lambda \geq 1/2$, equivalent to the unimodal uncertainty relation. We recall that, according to inequalities ($6$), the coefficients have to fulfil also the constraint $D_{sx} \geq D_{sp}$. If the coefficients do not fulfil the inequalities ($23$), then $S(\infty) \geq 0$ and the asymptotic state of the considered system is separable.

### 3.2. Asymptotic logarithmic negativity

We apply the measure of entanglement based on negative eigenvalues of the partial transpose of the subsystem density matrix. For a Gaussian density operator, the negativity is completely defined by the symplectic spectrum of the partial transpose of the covariance matrix. The logarithmic negativity $E = -\frac{1}{4} \text{log}_2[4 f(\sigma)]$ determines the strength of entanglement for $E > 0$. If $E \leq 0$, then the state is separable. Here,

$$
f(\sigma) = \frac{1}{2} (\text{det } A + \text{det } B) - (\left\lfloor \frac{1}{2} (\text{det } A + \text{det } B) - \text{det } C \right\rfloor^{1/2} - \text{det } C).
$$

In our case the asymptotic logarithmic negativity has the form

$$
E(\infty) = -\text{log}_2 \left[ \frac{2}{\lambda} \frac{m\omega D_{xx} - D_{sp}}{\sqrt{\lambda^2 + \omega^2}} \right].
$$
It depends only on the diffusion and dissipation coefficients characterizing the environment and does not depend on the initial Gaussian state. As an example, in figure 1 we represent the asymptotic logarithmic negativity \(E(\infty)\) versus diffusion coefficients \(D_{xx} \equiv D\) and \(D_{xy} \equiv d\). We notice that for some range of these coefficients, \(E(\infty)\) takes strictly positive values, measuring the degree of entanglement of the corresponding asymptotic entangled states. Again, the coefficients have to fulfil the constraint \(D_{xx} \geq D_{xy}\), so that in figure 1 only the part of the plot corresponding to \(D \geq d\) has a physical meaning for our model.

3.3. Time evolution of entanglement

In order to describe the dynamics of entanglement, we have to analyse the time evolution of the Simon function (20). We consider separately the two cases, according to the type of the initial Gaussian state: separable or entangled.

1. To illustrate a possible generation of the entanglement, we represent in figure 2 the function \(S(t)\) versus time and diffusion coefficient \(D_{xy} \equiv d\) for a separable initial Gaussian state with initial correlations \(\sigma_{xx}(0) = 1, \sigma_{xp}(0) = 1/2\) and \(\sigma_{xy}(0) = \sigma_{xp}(0) = \sigma_{yp}(0) = 0\). We notice that, according to the Peres–Simon criterion, for relatively small absolute values of the coefficient \(d\), the initial separable state remains separable for all times, whereas for larger absolute values of \(d\), at some finite moment of time the state becomes entangled. In some cases, the entanglement is only temporarily generated and the state becomes again separable after a certain amount of time. In other cases, namely for even larger absolute values of the coefficient \(d\), the generated entangled state remains entangled forever, including the asymptotic final state.

2. The evolution of an entangled initial state is illustrated in figure 3, where we represent the function \(S(t)\) versus time and diffusion coefficient \(d\) for an initial entangled Gaussian state with initial correlations \(\sigma_{xx}(0) = 1, \sigma_{xp}(0) = 1/2, \sigma_{xp}(0) = 0, \sigma_{xy}(0) = 1/2, \sigma_{xp}(0) = -1/2\) and \(\sigma_{xx}(0) = 0\). We notice that for relatively large absolute values of the coefficient \(d\), the initial entangled state remains entangled for all times, whereas for smaller absolute values of \(d\), at some finite moment of time the state becomes separable. This is the well-known phenomenon of entanglement ‘sudden death’. Depending on the values of the coefficient \(d\), it is also possible to have a repeated collapse and revival of the entanglement.

4. Summary

In the framework of the theory of open quantum systems based on completely positive quantum dynamical semigroups, we investigated the existence of the quantum entanglement for a subsystem composed of two uncoupled identical harmonic oscillators interacting with a general environment.
By using the Peres–Simon necessary and sufficient condition for the separability of two-mode Gaussian states, we have described the generation and evolution of entanglement in terms of the covariance matrix for an arbitrary Gaussian input state. For some values of diffusion and dissipation coefficients describing the environment, the state keeps for all times its initial type: separable or entangled. In other cases, entanglement generation or entanglement collapse (entanglement sudden death) takes place, or even one can notice a repeated collapse and revival of entanglement. We have also shown that, independent of the type of the initial state, for certain classes of environments the initial state evolves asymptotically to an equilibrium state that is entangled, whereas for other values of the coefficients describing the environment, the asymptotic state is separable. We determined also the logarithmic negativity characterizing the degree of entanglement of the asymptotic state. The existence of quantum correlations between the two considered harmonic oscillators interacting with a common environment is the result of the competition between entanglement and quantum decoherence.

Due to the increased interest manifested towards the continuous variable approach to quantum information theory, these results, in particular the possibility of maintaining a bipartite entanglement in a diffusive–dissipative environment for asymptotic long times, might be useful for controlling the entanglement in open systems and also for applications in quantum information processing and communication.

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