Local thermodynamical equilibrium and relativistic dissipation

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(Dated: October 20, 2022)

We introduce a class of relativistic fluid states satisfying the relativistic local thermodynamical equilibrium postulate (abbreviated as relativistic (LTE) postulate). States satisfying this postulate, are states "near equilibrium" (a term defined precisely in the course of the paper) and permit us to attach a fictitious "local thermodynamical equilibrium" state that fits event by event the actual fluid state. They single out an admissible class of rest frames relative to which thermodynamical variables like the energy density, thermodynamical pressure, stresses, particle number density (or densities) measured by observers at rest relative to these frames are becoming frame independent provided second (or higher) order deviations from the fictitious state of "local thermodynamical equilibrium" are ignored. We have verified this property for a large class of theories of relativistic dissipation that include the Hiscock-Lindblom class of first order theories, the Eckart and Landau-Lifshitz theories, the Israel-Stewart transient thermodynamics, the Liu-Müller-Ruggeri theory, fluids of divergence type and the latest developed (BDNK) theory. Moreover, the phenomenological equations describing first order deviations from the fictitious "local thermodynamical equilibrium" state satisfy equations that remain form invariant under change of frame within the class of admissible frames. We proved this property for the Hiscock-Lindblom class of first order theories the Eckart and Landau-Lifshitz theories, the Israel-Stewart transient thermodynamics and the Liu-Müller-Ruggeri theory of relativistic dissipation the (BDNK) theory and we expect that the same property to hold for the class of relativistic fluids of divergence type.

I. INTRODUCTION

We begin this paper, by recalling the essential features of the first theory of Classical Irreversible Thermodynamics (CIT), developed by Onsager \cite{1, 2} and independently by Eckart \cite{3, 4} in the decade of 1930s-1940s. The theory deals with states of continuous Newtonian media that obey (or are compatible with) the Local Thermodynamical Equilibrium (LTE) postulate. This postulate primary provides a recipe that assigns an entropy to off-equilibrium states and presupposes that thermodynamical equilibrium holds but only locally. The latter means that for any short time interval of the Newtonian time $t$ and for any point within the medium, there exists a subsystem of volume $V$ so that within this $V$ thermodynamical equilibrium prevails. Evidence supporting the existence of such states comes from the diversity that continuous media appear in nature. For instance, whenever states admit an inter-particle collision time scale and this scale is much shorter than any time scale that determines the evolution of macroscopic parameters, then space time inhomogeneities in the macroscopic variables are ironed out by collisions so that locally the state appears to be uniform implying that (locally) thermodynamical equilibrium holds.

The hypothesis that within each subsystem $V$ thermodynamical equilibrium prevails has important consequences. It asserts the existence of local extensive variables $(X_1, X_2, ..., X_n)$ and a macroscopic entropy $S = S(X_1, X_2, ..., X_n)$ having the same functional form as the fundamental equation of state describing states in global thermodynamical equilibrium. The additivity property of $S(X_1, X_2, ..., X_n)$ implies that one can define an entropy density $s(x_1, x_2, ..., x_n)$ function of $x_i = X_i/V$, $i \in (1, 2, ..., n)$ and this $s(x_1, x_2, ..., x_n)$ plays a key role within the Onsager-Eckart theory of (CIT). The (LTE) postulate asserts that this $s(x_1, x_2, ..., x_n)$ represents the physical entropy density of the underlying off-equilibrium state and this property combined with a set of balance laws, constitutive relations and by

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\footnote{This volume $V$ is considered to be macroscopically small but microscopically large.}
imposing the second law, closes the system of equations for these off equilibrium states.

To get more insights into the implications of the (LTE) postulate and for later needs of the paper, we take the underlying medium to be a collection of (classical) electrically neutral particles, so that within each subsystem of volume \( V \) labelled by \((t, \vec{x})\), an inter-particle collision time scale \( \tau_C(t, \vec{x}) \) is defined obeying \( \tau_C(t, \vec{x}) \ll \tau_M(t, \vec{x}) \) where \( \tau_M(t, \vec{x}) \) is any macroscopic time scale where the system change appreciably. The inequality \( \tau_C(t, \vec{x}) \ll \tau_M(t, \vec{x}) \) asserts that within each \( V \), local thermodynamical equilibrium prevails and identifies the regime where the framework of fluid dynamics \( {\text{[5]}} \) is applicable\(^2\). In that regime, the subsystems of volume \( V \) are the familiar fluid elements or fluid cells and they are labeled by a continuous real variables \( \vec{x} = (x^1, x^2, x^3) \) so that \((t, \vec{x})\) labels the fluid element at time \( t \) centered around \( \vec{x} \). If \( K \) is an inertial frame chosen so that the \((t, \vec{x})\) cell is momentarily at rest, then relative to this frame one can introduce (local) extensive variables such as the internal energy \( U(t, \vec{x}) \), the total particle number \( N(t, \vec{x}) \) and an additive entropy \( S(t, \vec{x}) \) function of \((U(t, \vec{x}), V, N(t, \vec{x}))\). Via these variables, the (LTE) postulate assigns a physical entropy to the underlying state and as a consequence of this property, these variables satisfy a host of thermodynamical relations that we now briefly mention.

For their derivation, let us for the moment consider a (global) equilibrium fluid state at rest relative to a global inertial frame, and denote by \((S, U, V, N)\) the spatially and temporally homogenous total entropy \( S \), internal energy \( U \), and particle number \( N \) within a macroscopic volume \( V \). These variables, under an infinitesimal reversible transformation, satisfy the Gibbs equilibrium relation

\[
dS = \frac{1}{T}dU + \frac{P}{T}dV - \frac{\mu}{T}dN
\]  

(1)

where \((T, P, \mu)\) are the temperature \( T \), pressure \( P \) and chemical potential \( \mu \) of the particle species. The homogeneity property of \( S(U, V, N) \) under rescaling implies that

\[
S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N), \quad \lambda > 0
\]  

(2)

and by choosing \( \lambda = 1 + \epsilon \) and expanding the identity

\[
S(U + \epsilon U, V + \epsilon V, N + \epsilon N) = (1 + \epsilon)S(U, V, N)
\]

then one gets the fundamental relation:

\[
U = TS - PV + \mu N,
\]  

(3)

which implies the Gibbs-Duhem relation:

\[
SdT - VdP + Nd\mu = 0.
\]  

(4)

Dividing (3) by \( V \) and introducing the internal energy density \( e \), entropy density \( s \) and particle density \( n \), then (3) yields

\[
e = Ts - P + \mu n
\]  

(5)

which via differentiation and in combination with the densitized form of the Gibbs-Duhem relation (4) i.e. \( sdT - dP + nd\mu = 0 \), implies

\[
de = Tds + \mu dn.
\]  

(6)

Thus knowledge of the equation of state \( e = e(s, n) \) determines via differentiation the equilibrium temperature and chemical potential \( \mu \) of the underlying equilibrium fluid state.

Formulas (1-6) are the well known relations that fluid states in global thermodynamical equilibrium must satisfy and we derive then here just to illustrate the implications that the (LTE) postulate has upon the description of hydrodynamical states. Whenever a state is compatible with this postulate, then \( U(t, \vec{x}), N(t, \vec{x}), S(t, \vec{x}) \)

\(^2\) Systems obeying \( \tau_C(t, \vec{x}) \gg \tau_M(t, \vec{x}) \) are described by the one particle distribution function and in that case one enters into the province of the kinetic theory of gases (for an introduction see for instance \( {\text{[6]}} \)). We also take the opportunity to clarify the difference between the term local thermodynamical equilibrium and the (LTE)-postulate employed within (CTT). Validity of the first simply requires \( \tau_C(t, \vec{x}) \ll \tau_M(t, \vec{x}) \) while validity of (LTE)-postulate at first presupposes \( \tau_C(t, \vec{x}) \ll \tau_M(t, \vec{x}) \) but most importantly, the postulate assigns the non equilibrium entropy to the underlying state.
and the corresponding densities \((e(t, \vec{x}), n(t, \vec{x}), s(t, \vec{x}))\) satisfy relations \(1\) i.e. the same relations as if the state was in global thermodynamical equilibrium. However a major difference should be noted: the variables \((U(t, \vec{x}), N(t, \vec{x}), e(t, \vec{x}), n(t, \vec{x}), s(t, \vec{x}))\) obey \(1\) only locally i.e. at the interior of (any) fluid cell.

The (CIT) of Onsager-Eckart has been proven to be a successful theory. It leads to the Fourier-Navier-Stokes fluid system which is the standard theory that describes terrestrial and a large class of astrophysical fluid flows and the classical book by de Groot and Mazur 
\[\text{[ref.]}\] deals extensively with the development and applications of the Onsager-Eckart theory of (CIT) to linear thermodynamics of irreversible processes (the interest reader is referred to that book for further discussion and additional references).

However, despite these successes, years of efforts and experience lead to the realization that not all goes that well with the dynamics of states compatible with the (LTE) postulate within the (CIT). The postulate dictates a very rigid functional dependance of the physical entropy density \(s\) upon the local thermodynamical variables, and this leads to the prediction that disturbances in temperature and stresses propagate with an unbounded speed, a highly unexpected and counterintuitive prediction.

Müller \[3\] in a fundamental paper written in (1967), suggested a way out of this problematic issue. He proposed that the physical entropy density \(s\) of non equilibrium fluid states should receive contributions from dissipative variables such as heat flux, bulk and shear stresses, a proposition which is in a blatant contradiction to the spirit of the (LTE) postulate.

Müller (1967) suggestion lead to the development of extended theories of irreversible thermodynamics and pave the way to the formulation of the entropy principle introduced by Coleman and Noll \[4\] \[5\], Müller \[6\] \[7\] and others, as well as lead to the realization that systems of symmetric-hyperbolic equations should be the appropriate set of dynamical equations describing irreversible thermodynamics (for a historical development, references and current status of extended theories, consult refs \[8\] \[9\] \[10\]).

Although the developments of extended theories remove the assertion of the (LTE) postulate that dictates the dependance of the physical entropy density \(s\) upon the local thermodynamical variables, nevertheless in these extended theories the validity of the hydrodynamical regime is left intact and thus also the validity of local thermodynamical equilibrium is preserved. This in turn implies the existence of (local) hydrodynamical variables \((X_1, X_2, ..., X_n)\) associated with the underlying fluid state. Using these variables, one may still employ \(s(x_1, x_2, ..., x_n)\) as an "equilibrium equation of state" and via the Gibbs formalism, attaches a "fictitious" "local thermodynamical equilibrium" state to the underlying fluid state. This fictitious "local thermodynamical equilibrium state" it is not always of physical relevance but nevertheless it can be very convenient. As we shall see further ahead, upon passing to the relativistic description, the idea of attaching a fictitious local "thermodynamical equilibrium state" to a fluid state, will be proven essential in the development of this work.

The present paper discuss an adaptation of the (LTE) postulate to the relativistic regime so that the postulate becomes a tool for the analysis of relativistic fluids states. The relativistic framework offers the possibility to introduce a special class of relativistic fluid states refereed as states compatible with the relativistic (LTE) postulate and their precise definition will be introduced in the following sections. Here, we offer a few comments regarding the motivations that lead us to introduce this special class of states.

For this, it is of relevance to recall the observational breakthroughs that took place within the last ten years or so, confirming the minute predictions of Einstein’s general relativity or of relativistic physics. It is sufficient to recall the monumental detection of gravitational waves by the LIGO observatory, the binary neutron star inspiral and the formation of the terrestrial “mini bing-bangs” at the center of our galaxy, the ongoing experiments at the Relativistic Heavy Ion Collision at BNL and at Large Hadron Collider at CERN, and the formation of the terrestrial “mini bing-bangs” in the form of quark-gluon plasma.

Although presently, the scientific community is in the process of digesting these observational breakthroughs, nevertheless one message comes across clearly: relativists and high energy physicists alike need to develop reliable theories of relativistic continuous medium in order to confront the new observational realities. It has been however realized and in fact long long ago, that relativistic dissipation is a tough mathematical problem which is still wide open (refs \[11\] \[12\] \[13\] \[14\] \[15\] offers an update on the latest developments). As we shall discuss further ahead, all so far

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3 We refer to this state as a "fictitious" state because the entropy that is defined by \(s(x_1, x_2, ..., x_n)\) bears no relation to the physical entropy of the underlying state.

4 Traces of this fictitious "local thermodynamical equilibrium state" can be seen in the introductory treatments of extended thermodynamics (see for example \[16\] \[17\]), where one often encounters terms like the equilibrium temperature, pressure, etc. These equilibrium quantities are generated by employing the equilibrium equation of state \(s(x_1, x_2, ..., x_n)\) alluded above.
proposed theories are highly non-linear and they are plagued by conceptual issues such as causality violation [36], instability of equilibrium states (for a recent discussion see [17], [18], [19]) lack of a sensible definition of a fluid four-velocity (see for instance [20], [21], [22] and the next two sections of this paper) and often the dynamical evolution equations fail to constitute a symmetric-hyperbolic and causal set of equations (see for example [38], [40]).

In this rough territory, progress has accomplished by restricting attention to equilibrium states and the analysis of their perturbations. The work in [36] studies the stability properties of first order theories, while in refs. [36], [32] the stability properties of transient thermodynamics has been addressed. The stability of equilibrium states and causality properties of Carter’s theory of heat conducting relativistic fluids (for an introduction to this theory see [23], [24], [25]) has been the focus of refs [26], [27], [29].

Even though this brief survey, shows a sizable amount of scientific activity on equilibrium states and their properties, still others aspects of relativistic dissipation needed to be investigated. The aim of the present work is to introduce a family of states as an alternative to equilibrium states. This class has been motivated by the properties of states satisfying the (LTE) postulate within the (CIT) and they will be referred as states satisfying the relativistic (LTE) postulate. They are states that on the one hand, go beyond the class of the equilibrium states and on the other hand, avoid (partially) the complexities associated with the non-linear nature of the underlying fluid equations. The central idea that underlies this special class of states is their property that they are states "near equilibrium" a term introduced by Israel long time ago and used as the building block in the development of transient thermodynamics. Although the term states "near equilibrium" at a first side suggest that such states could be just perturbations of equilibrium states, that is not the case and that will become clear in the course of the paper. The identification of states satisfying the relativistic (LTE) postulate is almost independent upon the underlying fluid theory. Their definition requires only that within the the underlying theory, states to be described by a conserved symmetric energy momentum tensor and a collection of conserved particle currents (this last requirement is optional).

The key ingredient that allows us to introduce this special class of states is the possibility to attach a fictitious "local thermodynamical equilibrium" state to an underlying fluid state and this attachment proceeds along the same reason as for the Newtonian framework. However, in the relativistic domain local observers and their measurements in combination to an "equilibrium equation of state" are the key elements that lead to this fictitious "local thermodynamical equilibrium" state. This fictitious state in combination with states near thermodynamical equilibrium (the precise coordinate free definition of such states is discussed in latter sections) yield the class of fluid states compatible with the relativistic (LTE) postulate and details of their construction are discussed in the following sections.

II. ON DISSIPATIVE RELATIVISTIC FLUID THEORIES

In this section, we prepare the ground for the formulation of the relativistic version of the (LTE) postulate and for this purpose, at first, we discuss briefly the salient features of a few theories of dissipative relativistic fluids developed so far.

Early attempts to construct viable theories initiated with the work of Eckart [34] in the (1940)s, the work by Landau and Lifshitz [35] in the (1950)s and culminated in the formulation of the class of first order theories developed by Hiscock and Lindblom [36] in the (1980)s. At around 1970s, Israel [21] and Israel and Stewart [22] formulated the transient thermodynamics which is a second order theory a term that will become clear further ahead. Within these theories, states of a relativistic fluid are described by a symmetric energy momentum tensor $T^{\mu\nu}$ a particle current $J^{\mu}$ obeying the conservation laws

$$\nabla_{\mu}T^{\mu\nu} = \nabla_{\mu}J^{\mu} = 0, \quad (7)$$

and accompanied by an entropy flux four vector $S^{\alpha}$ obeying

$$\nabla_{\alpha}S^{\alpha} = \sigma, \quad (8)$$

where $\sigma \geq 0$ is the entropy production scalar.

With the development of Rational Extended Irreversible Thermodynamics in the late 1980s, by Müller, Liu, Ruggeri and collaborators, a new trend in the field of relativistic dissipative fluids opened up (for and introduction consult

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5 For simplicity, in this work we treat the case of a simple fluid. For a fluid mixture, one may introduce $n$ conserved particle currents $J_i^{\mu}$, $i \in (1, 2, \ldots n)$ and proceeds in a similar manner.
ref.\[13, 33\]). New theories have been developed whose states are described by a collection of tensor fields obeying a manifestly symmetric-hyperbolic (or causal) systems of dynamical equations. Such theories include the theory developed by Liu, Müller and Ruggeri (see \[13, 38\]), the theory of relativistic fluids of divergence type developed by Pennisi \[39\], and independently by Geroch and Lindblom \[40\]. Carter’s theory of heat conducting fluids \[28, 29\], the unified Extended Irreversible Thermodynamics (UEIT) developed by Gavasino and Antonelli \[29, 31\]. Finally, we should mention the enormous impact the recent experiments on heavy ion collisions had upon the development of theories of relativistic dissipation (for details on this connection consult for instance \[28\]). Here the unexpected realization that models of relativistic viscous hydrodynamics describe reliably observational data acted as a stimulus for the development of new theories. Such theories include the Denicol-Niemi-Molnar-Rischke (DNMR) theory \[42\], the Baier-Romatschke-Son-Starinets-Stephanov theory \[42\], or anisotropic hydrodynamics theories (for an introduction see see \[43, 45\]) etc. These theories are second order theories derived either as limits of the relativistic Boltzmann equation case of (DNMR) theory, or as effective theories obtained from the gradient expansion within the quantum field theory, truncated at second order, case of Baier-Romatschke-Son-Starinets-Stephanov (BRSSS) theory.

The quest to develop theories with physically relevant characteristics, for instance theories that their equilibrium states are stable, respect causality, the corresponding dynamical equations admit a well posed initial value formulation, lead to the development of the (BDNK) formalism (for an introduction see \[46, 47, 48, 49\]) which is a formalism that lead to the development of first order theories with remarkable properties and in section (IV) we shall have the opportunity to comment more on this formalism.

It is not the purpose of this paper to provide a detailed description of the above mentioned theories neither to discuss their virtues and failures (for recent reviews see for instance \[49, 33, 51, 52, 54, 16, 15, 50\]). Our purpose is to adapt the (LTE) postulate from its natural habitat i.e. the regime of Newtonian continuous media, to the realm of relativistic fluids and our approach has been motivated by the development of the Israel-Stewart transient thermodynamics although the framework that we shall develop is applicable to arbitrary theories of relativistic dissipation\[6\].

As we have already mentioned, the description of relativistic dissipation presents a few conceptual challenges that are absent, for instance, from states of the Newtonian Fourier-Navier-Stokes system. For the later, the field equations are balance laws expressing conservation of mass, energy and linear momentum and involve the velocity field, mass density, the components of a stress and the components of the heat flux. By invoking the entropy principle, constitutive relations are proposed so that one gets a closed system of equations for the components of the velocity field, energy density and temperature, all of them defined relative to a fixed global inertial frame.

Unfortunately, attempts to interpret dissipative states of a relativistic fluid in a similar manner run into difficulties. Terms like particle and energy densities, stresses, temperature etc. are in general observer dependent quantities and more dramatic is the realization that the fluid’s four velocity is either non existent or if it is defined loses its prominence\[7\]. For fluid theories whose states are described partially\[8\] by a conserved particle current \(J^\mu\) and a conserved energy momentum tensor \(T^{\mu\nu}\), one has the liberty to define the fluid’s four velocity as the unique, future directed timelike eigenvector \(u_E^\mu\) of the energy momentum tensor \(T^{\mu\nu}\) i.e. identify the fluid’s four velocity with the flow of energy\[9\] or one may identify the fluid four velocity with the 4-velocity \(u_N^\mu\) defined via \(J^\mu = nu_N^\mu\) where \(n\) is the particle density measured by an observer comoving with \(u_N^\mu\) and thus identify the fluid four velocity with the particle motion. Should one identify the fluid’s four velocity with \(u_E^\mu\), \(u_N^\mu\) or may one employ an altogether different four velocity field? This ambiguity regarding the choice of the fluid’s four velocity is a common feature of all so far proposed theories of relativistic dissipation and to this day, there is not a satisfactory resolutions of this dilemma\[10\].

It is however worth noticing that although the notion of a fluid four velocity may be ambiguous or even ill defined,

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6 The only restriction we shall impose on the structure of the underlying theory is that amongst the variables that describe fluid states, includes a conserved particle current \(J^\mu\) and a symmetric energy momentum tensor \(T^{\mu\nu}\) which satisfies the weak energy condition.

7 An exception is provided by states describing a simple perfect fluid. Here a four velocity field \(u^\mu\) is assigned dynamically by the theory and perfect fluid states are described by the future-directed timelike velocity field \(u^\mu\) and two spacetime scalar fields \((n, \rho)\) representing the particle density and energy density measured by a comoving with the flow observer. The particle number and energy momentum conservation laws supplemented by an equation of state fix these field uniquely.

8 The use of the world “partially” reflects the fact that we have not specified the underlying fluid theory. It may happen that for an arbitrary theory, one may need besides \((T^{\mu\nu}, J^\mu)\) additional variables to specify a fluid state for instance the entropy flux \(S^\mu\) or other additional fields.

9 This \(u_E^\mu\) is well defined provided the energy momentum tensor obeys the weak energy condition.

10 Viewing relativistic fluids as arising from a microscopic descriptions either from relativistic kinetic theory or from the expectation values of quantum observables via suitable averaging, it is hoped that this transitions may single out a unique fluid flow field. Unfortunately up to to now there is no such a satisfactory prediction.
the general relativistic framework allows to introduce a smooth, timelike future directed "velocity field" \( u^\mu \) in the spacetime region occupied by the fluid and view this \( u^\mu \) as providing a field of "rest frames" employed by the \( u^\mu \)-observers who probe the state of the fluid. This liberty in the choice of a potential "velocity field" \( u^\mu \), combined with a particular class of relativistic fluid states defined below, are the building blocks for the formulation of states satisfying the relativistic (LTE) postulate and in the next section we address this construction.

III. ON STATES COMPATIBLE WITH THE RELATIVISTIC (LTE)-POSTULATE

In this section, we consider a test relativistic fluid propagating on a smooth background spacetime \((M, g)\) and within the region occupied by the fluid, we introduce a smooth, non singular, future pointing velocity field \( u^\mu \) whose sole purpose is to provide a family of local observes who perform measurements upon the fluid’s state. At any event \( p \) in the fluids interior and relative to the local rest frame whose time axis at \( p \) coincides with \( u^\mu(p) \), we consider an infinitesimal element of a spacelike 3-volume \( V \), orthogonal a \( u^\mu(p) \) and view this \( V \) as the local fluid cell. Furthermore, we assume that it is possible to choose this \( u^\mu \) so that relative to the rest frame defined by \( u^\mu(p) \), a collision time scale \( \tau_C(p) \) is defined that satisfies \( \tau_C(p) \ll \tau_M(p) \) where \( \tau_M(p) \) is another time scale\(^1\) determined by the velocity field, for instance by its expansion, shear, rotation, etc. As for the Newtonian case, this inequality implies that the \( u^\mu(p)\)-observer will conclude that local thermodynamical equilibrium prevails within \( V \) and using \( T^{\mu\nu} \), \( J^\mu \) and \( S^\mu \), assigns at \( p \) a particle number \( n(p) \), energy density \( \rho(p) \) and an entropy density \( s(p) \) via

\[
\begin{align*}
n(p) &= -J_\mu u^\mu, \\
\rho(p) &= T_{\mu\nu} u^\mu u^\nu, \\
s(p) &= -S_\mu u^\mu.
\end{align*}
\]

Using this \( u^\mu \) one in general decomposes\(^2\) \( T^{\mu\nu} \) and \( J^\mu \) according to:

\[
T^{\mu\nu} = \rho(u) u^\mu u^\nu + P(u) \Delta(u)^{\mu\nu} + h(u)^\mu u^\nu + h(u)^\nu u^\mu + \tau(u)^{\mu\nu},
\]

\[J^\mu = n(u) u^\mu + n(u)^\mu,
\]

where \( h(u)^\mu, n(u)^\mu, \tau(u)^{\mu\nu} \) stand for the energy flow vector, the particle "drift" and the spatial symmetric pressure tensor \( \tau(u)^{\mu\nu} \) respectively all of them measured by the \( u^\mu \) observer. In these expansions, the pressure tensor \( \tau(u)^{\mu\nu} \) defines the bulk pressure \( \pi(u) \) and shear stresses \( \pi(u)^{\mu\nu} \) according to

\[
\tau(u)^{\mu\nu} = \pi(u) \Delta(u)^{\mu\nu} + \pi(u)^{\mu\nu}, \\
\pi(u)^\mu_\mu = 0,
\]

the fields \( (h^\mu, n^\mu) \) and \( \tau(u)^{\mu\nu} \) satisfy

\[
h(u)^\mu u_\mu = n(u)^\mu u_\mu = u_\mu \tau(u)^{\mu\nu} = 0,
\]

while \( \Delta^{\mu\nu}(u) = g^{\mu\nu} + u^\mu u^\nu \) stands for the projection tensor.

Since as we have already mentioned, the inequality \( \tau_C(p) \ll \tau_M(p) \) implies validity of the local thermodynamical equilibrium within \( V \), by appealing to Gibbs formulation\(^3\) of equilibrium thermodynamics, this \( u^\mu \) observer will conclude that \( n(p) \) and \( \rho(p) \) must satisfy an "equilibrium equation of state" of the form \( s = s(\rho, n) \). In turn this \( s = s(\rho, n) \) implies

\[
ds = \frac{1}{T} d\rho - \Theta dn,
\]

\(^1\) These time scales are identical to those that we defined earlier on for Newtonian fluids, although there is however a pronounced difference between the Newtonian and relativistic case. While in the former case the inequality \( \tau_C(p) \ll \tau_M(p) \) is frame independent, in the relativistic regime and due to time dilatation effects in general this inequality is frame dependent. States compatible with the relativistic version of (LTE) postulate that we are about to introduce, is a class of states that allows to introduce a preferable class of frames, refereed as admissible frames, having the property that if \( \tau_C(p) \ll \tau_M(p) \) holds in one frame within this admissible class, then frame changes within this admissible class, leaves this inequality intact.

\(^2\) In \((10,12)\), and in the sequel, we write \( \rho(u), P(u), h(u), \tau(u)^{\mu\nu} \), etc., in order to remind the reader that these variables are measured by the \( u^\mu \)-observer. This notation further signifies that the decompositions in \((10,12)\) are frame dependent. Often we write \( n(p), \rho(p) \) etc to denote results of measurements by the \( u^\mu \)-observer at a particular event \( p \).

\(^3\) For an introduction to this formulation see for instance \([27]\).
thus defining the local temperature \( T(u) \) and the thermal potential \( \Theta(u) \) as measured by the \( u^\mu \)-observer, while the fundamental relation \( s(u) = (\rho(u) + P(u))T^{-1}(u) - \Theta(u)n(u) \) with \( s(u) := s(\rho(u), n(u)) \) defines the (local) equilibrium pressure \( P(u) \). Therefore, once a velocity field \( u^\mu \) and an equation of state \( s = s(\rho, n) \) have been specified, then \( (T^{\mu\nu}, J^\mu) \) define at \( p \) a "local thermodynamical equilibrium" state specified by

\[
(\nu(p), \rho(p), P(p), s(p), T(p), \Theta(p)).
\]

As long as the velocity field \( u^\mu \) and the fluid state allow the inequality \( \tau_C(p) << \tau_M(p) \) to hold at any event \( p \) within the fluid’s interior, then the variables in (15) are defined over the entire spacetime region occupied by the fluid and combined with \( u^\mu \) give rise to a (local) particle current14 \( J_0^\mu \), a tensor \( T^{\mu\nu}_0 \) and an "entropy flux" vector \( S_0^\mu \), via

\[
J_0^\mu = nu^\mu, \quad T_0^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}, \quad S_0^\mu = s(\rho, n)u^\mu.
\]

These fields, in the terminology of Israel-Stewart [21, 22], define a “local equilibrium reference state” attached to the physical state described by the variables \( (T^{\mu\nu}, J^\mu, S^\mu) \). This "local equilibrium reference state" depends upon the (almost arbitrarily) chosen velocity field \( u^\mu \) and thus at a first sight seem to be of limited utility. Any other \( \hat{u}^\alpha(p) \)-observer and as long as at \( p \) local equilibrium prevails relative to her/his frame, will measure

\[
(\hat{n}(p), \hat{\rho}(p), \hat{P}(p), \hat{s}(p), \hat{T}(p), \hat{\Theta}(p))
\]

which are related to those in (15) via complicated transformations formulas induced by a local Lorentz transformation that relates the frames defined by \( u^\alpha \) and \( \hat{u}^\alpha \). However, for a particular states, the fields in (15) transform under local Lorentz transformations in a simple manner so they become useful tools in describing properties of the underlying fluid state.

In order to identify these states, we consider again the unique timelike eigenvector \( u^\mu_N \) of the energy momentum tensor \( T^{\mu\nu} \) which specifies the energy frame, and the timelike and future directed vector \( u^\mu_N \) that specifies the particle frame. For an arbitrary fluid state determined (partially) by \( (T^{\mu\nu}, J^\mu) \) the fields \( (u^\mu, u^\mu_N) \) are in general distinct and there is not any obvious relation between the two. However, by appealing to an idea introduced by Israel long time ago15 and employed in the formulation of the transient thermodynamics, we introduce states referred as "states near equilibrium" (or "states close to equilibrium") in the following manner. At an event \( p \) within the fluid’s region, one sets up an orthonormal frame with a time axis parallel to \( u_E \) accompanied by a triad \( e_i, \ i \in \{1, 2, 3\} \) of spacelike vectors so that \( (u_E, e_i) \) constitutes an orthonormal tetrad. Since there is freedom in the choice of the triad \( e_i, \ i \in \{1, 2, 3\} \), without loss of generality, one may assume:

\[
u^\mu_N = \left[ 1 - \frac{v^2}{c^2} \right] \frac{1}{4} u^\mu_E + \frac{v}{c} \left[ 1 - \frac{v^2}{c^2} \right] \frac{1}{4} e_i^\mu \]

\[
= \cosh \epsilon \ u^\mu_E + \sinh \epsilon \ e_i^\mu, \quad g(u_E, e_1) = 0,
\]

where \( \tilde{v} = ve_1 \) is the "relative velocity" of the particle frame relative to the energy frame. This relation defines a pseudo angle \( \epsilon \) between \( u_E \) and \( u_N \) according to:

\[
cosh \epsilon = -g(u_E, u_N) = \left[ 1 - \left( \frac{v^2}{c^2} \right) \right]^{-\frac{1}{2}},
\]

and this angle plays an important role. It is suffice to mention that for all proposed theories of relativistic dissipation, it holds that for states in a global equilibrium the fields \( u_E \) and \( u_N \) coincide and thus \( \epsilon = 0 \).

Motivated by this property and in the spirit of transient thermodynamics, we defines for a simple fluid states near16

---

14 It is understood that in (10), the fields \( (n, \rho, P, s) \) stand for \( (\rho(u), \rho(u), P(u), s(u(n(u), \rho(u))) \).

15 Israel suggested that for states near equilibrium, a theory can be developed that describes the first order deviations from a local "equilibrium reference state" and this theory is invariant under particular change of rest frames within the class of admissible frames. This idea was elaborated further in a M.Sc. thesis written by Aitken [23] then student of Israel.

16 For a fluid mixture consisting of \( n \)-particle currents \( (J_1, J_2, ..., J_n) \), one may define \( n \)-four velocities \( (u_1, u_2, ..., u_n) \) and thus introduce \( n \)-pseudo angles \( (\epsilon_1, \epsilon_2, ..., \epsilon_n) \) between \( u_E \) and the corresponding \( (u_1, u_2, ..., u_n) \). A state then is close to equilibrium, whenever \( (\epsilon_1, \epsilon_2, ..., \epsilon_n) \) obey \( \epsilon_i \leq 1 \) for all \( i \in \{1, 2, ..., n\} \).
equilibrium\footnote{Further below, we shall introduce their close relatives named states satisfying with the relativistic (LTE) postulate.} as those states that have the property that the pseudo-angle $\epsilon$ in \eqref{19} satisfies everywhere within the region occupied by the fluid the condition $\epsilon = \frac{\lambda}{c} << 1$. For such states, one notices from

$$J^\mu = n_N u_N^\mu = n_N (\cosh \epsilon \ u_E^\mu + \sinh \epsilon \ e_1^\mu) = n_E u_E^\mu + n^\mu, \quad (20)$$

and thus the densities $n_N$ and $n_E$ measured by the $u_E^\mu$ respectively the $u_N^\mu$, observers satisfy

$$n_E = n_N \cosh \epsilon = n_N + O(\epsilon^3), \quad n \geq 2 \quad (21)$$

implying that $n_N$ and $n_E$ are considered to be frame independent provided terms of order $\epsilon^2 << 1$ and higher are neglected\footnote{Here after we follow the notation of ref. [21] often we set $\epsilon := O_1$ while terms like $O_2, O_3, \ldots$ signify terms of first, second, third order... deviations.}. This (approximately) “invariance property” of the particle density holds also for other fields that appear in \eqref{15} and \eqref{17} and in order to investigate in a systematic manner the transformation properties of the thermodynamical variables under frame change, let $(u^\mu, \hat{u}^\mu)$ be two (future pointing) unit timelike vectors lying within the “cone” of opening angle $\epsilon << 1$. These vectors define the time axis of the two rest frames\footnote{If relative to these frames the inequality $\tau_R(p) << \tau_C(p)$ holds for all $p$ within the fluids interior, then these class of rest frames is refereed as the admissible class of rest frames, a term adapted from the terminology employed in transient thermodynamics. Local Lorentz transformations between such frames are approximately described by \eqref{24}.} and by complementing them by two triads of spacelike unit vectors $e_i, i \in (1, 2, 3)$ and $\hat{e}_i, i \in (1, 2, 3)$, then $(u, e_i)$ respectively $(\hat{u}, \hat{e}_i)$ constitute an orthonormal bases at the event under consideration. Accordingly, we have

$$\hat{u} = \frac{u}{(1 - \frac{\hat{e}^2}{c^2})^{1/2}} \frac{v^i e_i}{c (1 - \frac{\hat{e}^2}{c^2})^{1/2}}, \quad v^2 = v^i v_i, \quad (22)$$

where $v^i$ are the components of the three velocity of the frame $u$ relative to the $\hat{u}$ one. Following Israel [21], we write this transformation law in the equivalent form

$$\hat{u}^\mu = (1 + \hat{\epsilon}^2)^{1/2} u^\mu + \hat{\epsilon}^\mu, \quad \hat{\epsilon}^2 = \hat{\epsilon}^a \hat{\epsilon}_a, \quad \hat{\epsilon}^a u_a = 0. \quad (23)$$

and assume $\hat{\epsilon} \leq \epsilon = O_1$ so that \eqref{20} is approximated by

$$\hat{u}^\mu = u^\mu + \hat{\epsilon}^\mu + O(\hat{\epsilon}^2), \quad \hat{\epsilon}^\mu \leq O_1, \quad (24)$$

showing that $\hat{\epsilon}$ is a measure of the relative three velocity of $u^\mu$ relative to $\hat{u}^\mu$ frame.

Let now the local ”thermodynamical equilibrium state” associated with $(u^\mu, s(n, \rho))$ constructed according to \eqref{14} [16]. If $Z(u)$ stands for any of the thermodynamical variable in \eqref{15} (as measured by the $u$ observer) and we consider the frame change described in \eqref{24}, of relevance for the following analysis is the variation $\delta Z := Z(\hat{u}) - Z(u)$ that suffers the variable $Z$ under such frame change. Variations of these type worked out by Israel in [21] and also reworked in [53] and below we present a summary of such variations:

$$\delta \rho \equiv \rho(\hat{u}) - \rho(u) = \hat{\epsilon} O_1 \quad (25)$$

$$\delta h^\alpha \equiv h^\alpha(\hat{u}) - h^\alpha(u) = -(\rho + P) \hat{\epsilon}^\alpha \quad (26)$$

$$\delta n \equiv n(\hat{u}) - n(u) = \hat{\epsilon} O_1, \quad (27)$$

$$\delta P \equiv P(\hat{u}) - P(u) = \hat{\epsilon} O_1, \quad (28)$$

$$\delta \tau^{\alpha \beta} \equiv \tau^{\alpha \beta}(\hat{u}) - \tau^{\alpha \beta}(u) = \hat{\epsilon} O_1. \quad (29)$$

$$\delta n^\alpha \equiv n^\alpha(\hat{u}) - n^\alpha(u) = -n \hat{\epsilon}^\alpha, \quad (30)$$
\[ \delta s \equiv s(\dot{u}) - s(u) = \epsilon O_1, \]  
\[ \delta T \equiv T(\dot{u}) - T(u) = \epsilon O_1, \]  
\[ \delta \Theta \equiv \Theta(\dot{u}) - \Theta(u) = \epsilon O_1. \]

These transformations show that the variables like \((n, \rho, P, \text{etc})\) behave as frame independent quantities to an accuracy \(\epsilon O_1 \leq O_2\) while others variables like \(h^\alpha, u^\alpha\) are independent only to \(\epsilon \leq O_1\) accuracy\(^{20}\).

For later use, we mention that the following 4-vector

\[ q^\alpha(u) = h^\alpha(u) - \frac{\rho(u) + P(u)}{n(u)} n^\alpha(u), \]  

is frame independent i.e. its variation obeys:

\[ \delta q^\alpha \equiv q^\alpha(\dot{u}) - q^\alpha(u) = \epsilon^\alpha O_1. \]

and is referred as the the invariant heat flux vector.

The formulas in \((25-33)\) express the transformation laws of the various thermodynamical variables under a frame change described in \((24)\) and there will be used frequently in the following analysis. Under such frame change, they show that most of the thermodynamical variables remain practically frame independent as long as second order and higher order deviations from the state of "local thermodynamical equilibrium" specified by \((u^\mu, s(\rho, n))\), are neglected. Notice that the assumption \(\epsilon << 1\) in \((19)\) implies that the inequality \(\tau_C(p) << \tau_M(p)\), becomes observer independent in the sense that as long as \((u^\mu, \ddot{u}^\mu)\) in \((24)\) are chosen to lie within the "cone" of the opening pseudo-angle \(\epsilon = O_1 << 1\) then time dilatation and length contraction are effects considered as been inessential. Moreover, the "invariance" of the inequality \(\tau_C(p) << \tau_M(p)\), under the frame change in \((24)\) in combination to \((25-33)\), permit us to simply refer to a state of "local thermodynamical equilibrium" without any further reference to which particular \((u^\mu, s(\rho, n))\) this fictitious state’ is associated with.

The so far analysis used only the property that fluid states are described (partially) by the fields \((T^{\mu\nu}, J^\mu)\) and all of the above conclusions are independent of any underlying fluid theory. This observation allows us to group together states characterized by common properties. To do so, let us begin with a state described by the fields \((T^{\mu\nu}, J^\mu)\) subject to the restriction that the pseudo-angle \(\epsilon \in (19)\) satisfies \(\epsilon << 1\) everywhere within the fluid region and let a four velocity \(\dot{u}^\mu\) lying within the cone of opening angle \(\epsilon << 1\) chosen so that \(\tau_C(p) << \tau_M(p)\) holds. Any such \(u^\mu\) combined with \((T^{\mu\nu}, J^\mu)\) defines a particular fluid state which is described by the fields defined in the expansions \((10-11)\). We refer to the collection of such states, as states satisfying the relativistic (LTE) postulate (or often as states near equilibrium).

One notices that for any two states within this class, specified by \(u^\mu, \ddot{u}^\mu\) it holds \(\ddot{u}^\mu - u^\mu \leq \epsilon^\mu\) and thus under a frame change described by \((24)\), the fields in \((15-17)\) transform according to \((25-33)\). Therefore even though technically one deals with the two distinct fluid states, as long as one is interested only in the physics of first order deviations from the state of "local thermodynamical equilibrium" one really is dealing with the same state. Moreover in the next sections, we show that for a large class of fluid theories, states compatible with this relativistic (LTE) postulate, the phenomenological equations that describe the dynamics of first order deviations from the "local thermodynamical equilibrium" state are equivalent in the sense that from one solution one generates solutions of the other equation (or equivalently from one state in (LTE) one generates all the other states within this class).

\(^{20}\) For the particular case where \((u^\mu, \ddot{u}^\mu)\) are identified as \((u^\mu_E, u^\mu_N)\), the transformation law in \((25)\) take the form:

\[ u^\mu_N \rightarrow u^\mu_E = u^\mu_N + \epsilon^\mu + O(\epsilon^2) \]  

then one replaces in the right hand sides of \((25-33)\) \(\epsilon\) by \(\epsilon = O_1\) and the resulting formulas describes frame change from the particle to the energy frame.
As a preparatory step to analyze further properties of states satisfying the relativistic (LTE) postulate, let us consider one of them described by \((T^{\mu \nu}, J^\mu)\) and let us introduce also a physical entropy current \(S^\mu\) associated to this state. If and in accordance of the above discussion, this state is specified by \((u^\mu, s(\rho, n))\), we define the fields,

\[
\delta S^\mu = S^\mu - S_0, \quad \delta T^{\mu \nu} = T^{\mu \nu} - T_0^{\mu \nu}, \quad \delta J^\mu = J^\mu - J_0^\mu
\]  

which describe the deviations of the physical state \((S^\mu, T^{\mu \nu}, J^\mu)\) away from the state of "local thermodynamical equilibrium" state defined by \((u^\mu, s(\rho, n))\) or by \((S_0^\mu, T_0^{\mu \nu}, J_0^\mu)\) as defined in according to \([10]\). Since by construction, \((T_0^{\mu \nu}, J_0^\mu)\) satisfy the fitting conditions

\[
(J^\mu - J(0)^\mu)u_\mu = (T^{\mu \nu} - T(0)^{\mu \nu})u_\mu u_\nu = 0,
\]  

the perturbations \(\delta J^\mu\) and \(\delta T^{\mu \nu}\) are described by

\[
\delta J^\mu = n(u)^\mu
\]

\[
\delta T^{\mu \nu} = h(u)^\mu u^\nu + h(u)^\nu u^\mu + \tau(u)^{\mu \nu} = \pi(u) \Delta(u)^{\mu \nu} + \pi(u)^{\mu \nu} + h(u)^\mu u^\nu + h(u)^\nu u^\mu,
\]  

and these forms are actually independent of the dynamics of the underlying fluid theory. However that is not any longer the case for the entropy perturbation \(\delta S^\mu = S^\mu - S_0^\mu\), the structure of \(S^\mu\) plays an important role in specifying he underlying theory.

Before we leave this section, it is worth stressing a point. For states satisfying the relativistic (LTE) postulate, specified by \((u^\mu, s(\rho, n))\) even though the equilibrium equation of state \(s(\rho, n)\) combined with the velocity field \(u^\mu\) defines an entropy like current \(S_0^\mu = s(\rho, n)u^\mu\) as defined in \([10]\) this current is formal and has nothing to do with the physical entropy \(S^\mu\) of the underlying state. Of relevance in the analysis of relativistic fluid states is the dependence of the physical entropy current \(S^\mu\) upon other fluid variables and below, we shall have the opportunity to see how this dependance leads to alternative theories of relativistic dissipation.

**IV. ON STATES SATISFYING THE RELATIVISTIC (LTE) POSTULATE AND FIRST ORDER THEORIES**

In this section, we study states compatible with the relativistic (LTE) postulate within the context of the Hiscock-Lindblom class of first order theories\(^{21}\). We recall first that this class has been introduced in \([36, 37]\) and states within this class, are described by the fields \((T^{\mu \nu}, J^\mu, S^\mu)\) satisfying \((18)\). Even though states within the theory do not make no reference to any fluid four velocity, nevertheless, in all treatments of this theory, a velocity field \(u^\mu\) defines a "equilibrium equation of state" \(s\) which combined with \((T^{\mu \nu}, J^\mu, S^\mu)\) defines a "local thermodynamical equilibrium" state associated to this \((u^\mu, s(\rho, n))\) in the manner discussed in the previous section.

A key ingredient that characterizes the Hiscock-Lindblom class is the structure of the physical entropy current \(S^\mu\) which is postulated to have the form (see discussion in \([36]\))

\[
S^\mu = su^\mu + \hat{\beta}h^\mu(u) - \hat{\Theta}n^\mu(u) = S_0^\mu + \hat{\beta}h^\mu(u) - \hat{\Theta}n^\mu(u),
\]  

\(^{21}\) The term "first order theories" coined by Hiscock and Lindblom in \([36, 37]\) and describes theories where the entropy current \(S^\mu\) receives only first order contributions from a suitably defined state of "local thermodynamical equilibrium". Initially in this section we shall be concerned with the Hiscock-Lindblom class, but at the end of the section we shall introduce the (BDNK) class of first order theories that differs from the Hiscock-Lindblom class.

\(^{22}\) Just to avoid confusion, we stress that the choice of this velocity field \(u^\mu\) is arbitrary, (except that it is restricted so that relative to the family of the rest frames that it defines, the inequality \(\tau_C(p) < < \tau_M(p)\) holds) and this \(u^\mu\) bears no relation to the four velocity field entering in the specification of states compatible with the (LTE) postulate. For instance, we have not yet introduced the pseudo angle \(\epsilon\) defined in \([19]\) restricted to obey \(\epsilon < < 1\).
where \((\hat{\beta}, \hat{\Theta})\) are undetermined functions. The choice \((u^\mu = u^\mu_E, h^\mu(u) = 0)\) generates the Landau-Lifshitz theory, while \((u^\mu = u^\mu_N, n^\mu = 0)\) generates the Eckart theory.

With reference to the velocity field \(u^\mu\) in (41), the expansions of \((T^{\mu\nu}, J^\mu)\) in (10)-(11), introduce the fields \((\rho(u), n(u), P(u), h^\mu(u), \tau^{\mu\nu}(u), n^\mu(u))\) which are required to obey:

\[
\nabla_\mu T^{\mu\nu} = \nabla_\mu [\rho(u)u^\mu u^\nu + P(u)\Delta(u)_{\mu\nu} + h(u)^\mu u^\nu + h(u)^\nu u^\mu + \tau(u)^{\mu\nu}] = 0, \\
\nabla_\mu J^\mu = \nabla_\mu [n(u)u^\mu + n(u)^\mu] = 0.
\]

Moreover, by imposing the second law \(\nabla_\mu S^\mu \geq 0\) and recalling the decomposition \(\tau(u)^{\mu\nu} = \pi(u)\Delta(u)^{\mu\nu} + \pi(u)^{\mu\nu}\), \(\pi(u)^{\mu\nu}_0 = 0\) in (12), a calculation shows that this law can be fulfilled whenever the following equations hold (for a derivations see (39)):

\[
h^\mu(u) = -kT(u)\Delta^{\mu\nu}(u)\left[\frac{1}{T(u)}\nabla_\mu T(u) + u^\alpha \nabla_\alpha u_\nu\right],
\]

\[
n^\mu(u) = -\sigma T^2(u)\Delta(u)^{\mu\nu}\nabla_\nu \Theta(u),
\]

\[
\pi^{\mu\nu}(u) = -2\eta <\nabla^\mu u^\nu>,
\]

\[
\pi(u) = -\zeta \nabla_\mu u^\mu,
\]

with the coefficients \((\hat{\beta}, \hat{\Theta})\) in (41) given by:

\[
\hat{\beta}(u) = T^{-1}(u) := \beta(u), \quad \hat{\Theta}(u) = \frac{\rho(u) + P(u)}{n(u)T(u)} - s(u) := \Theta(u),
\]

and the angular bracket in (40), and here after, signifies symmetric, purely spatial, trace free part of the enclosed tensor. From these equations one gets

\[
T \nabla_\mu S^\mu = \frac{\pi^2}{\zeta} + \frac{h^\mu h_\mu}{kT} + \frac{n^\mu n_\mu}{\sigma T} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta} \geq 0,
\]

implying that the entropy production is manifestly non negative provided the four coefficients \((\zeta, \eta, k, \sigma)\) are chosen to be positive and these coefficients are identified as the bulk viscosity \(\zeta\), the thermal conductivity \(k\), a particle diffusion constant \(\sigma\) and the shear viscosity \(\eta\).

The system (12)-(18) constitutes a closed system of equations whose solutions describe arbitrary fluid states within the Hiscock-Lindblom class of first order theories. From the mathematical viewpoint, this system is a mixed parabolic-hyperbolic-elliptic system and as we shall discuss further ahead, its solutions are characterized by a number of undesirable properties. Nevertheless, viewed as a closed system, and given suitable initial data, determines the unknowns variables and solutions possessing distinct velocity fields are considered to be distinct solutions.

However, matters differ when emphasis is restricted to states satisfying the relativistic (LTE) postulate. Primary for such states, the fields \(T^{\mu\nu}\) and \(J^\mu\) are restricted so that the pseudo-angle \(\epsilon\) between \(u_E\) and \(u_N\) (see (19)) satisfies everywhere within the region occupied by the fluid the condition \(\epsilon << 1\) and moreover exists a (highly non unique) velocity field \(u^\mu\) that lies within the cone of the opening angle \(\epsilon << 1\) that generates a state of the "local thermodynamical equilibrium" state specified by \((u^\mu, s(\rho, n))\) in the manner discussed in the previous section. Clearly there exist infinitely many states satisfying the (LTE) postulate, with each one of them is specified by the same pair \((T^{\mu\nu}, J^\mu)\) but by a different velocity field.
We show below, that all such states are in effect equivalent to each other as long as quadratic and higher order deviations from the state of the "local thermodynamical equilibrium" specified by \((u^\mu, s(\rho, n))\) are omitted.

To establish this equivalence, let us begin with two velocity fields \((u^\mu, \hat{u}^\mu)\) that satisfy two states satisfying the relativistic (LTE) postulate. Using \(u^\mu\), let

\[
(n(u), \rho(u), P(u), h(u)^\mu, n(u)^\mu, \tau(u)^{\mu\nu})
\]

be the fields obtained by expanding \((T^{\mu\nu}, J^\mu)\) according to (10) and let us assume that these fields satisfy the exact equations (12) for a set on non negative coefficients \((\zeta, \eta, k, \sigma)\). Using this solution as a reference, we shall generate all other states compatible with the relativistic (LTE) postulate. For this, let the frame change

\[
u \rightarrow \tilde{\nu} = u^\mu + \tilde{\nu}^\mu, \quad \tilde{\nu} \leq \epsilon,
\]

then (25-33) combined with the fields in (50), generate

\[
(\tilde{\nu}^\mu, \rho(\tilde{\nu}), n(\tilde{\nu}), P(\tilde{\nu}), h(\tilde{\nu})^\mu, n(\tilde{\nu})^\mu, \tau(\tilde{\nu})^{\mu\nu}).
\]

The claim is, that this new state satisfy (52), as long as deviations from a the state of "local thermodynamical equilibrium" specified by \((u^\mu, s(\rho, n))\) are omitted.

In order to show this, we first consider the special case where \(\hat{u}^\mu\) is chosen to be the four velocity \(u^\mu_E\) of the energy frame so that (51) takes the form

\[
u \rightarrow \nu^E = u^\mu + \nu^\mu, \quad \nu^\mu \leq \epsilon^\mu.
\]

Using (25-33), let

\[
u^E, n(u_E), \rho(u_E), P(u_E), h(u_E)^\mu, n(u_E)^\mu, \tau(u_E)^{\mu\nu}
\]

defined according to

\[
n(u_E) = n(u) + \epsilon O_1, \quad \rho(u_E) = \rho(u) + \epsilon O_1, \quad etc.,
\]

and in this generation process, it is worth noticing that the condition \(h^\mu(u_E) = 0\) demands that \(\nu^\mu\) to satisfy

\[
h^\mu(u) = (\rho + P)\nu^\mu,
\]

while \(n^\mu(u_E)\) has the the value

\[
n^\mu(u_E) = n^\mu(u) - n\nu^\mu = n^\mu(u) - \frac{n(u)h^\mu(u)}{(\rho + P)}.
\]

The state in (53), approximatively satisfy (54). Indeed, starting from eqs. (12) and replacing \(u^\mu\) by \(u^\mu_E\) and \((n(u), \rho(u), etc\) by the fields in (54), we find

\[
0 = \nabla_\mu T^{\mu\nu}(u)
= \nabla_\mu [\rho(u)u^\mu u^\nu + P(u)\Delta(u)^{\mu\nu} + h(u)^\mu u^\nu + h(u)^\nu u^\mu + \tau(u)^{\mu\nu}]
= \nabla_\mu [\rho(u_E)\nu^\mu \nu^\nu + P(u_E)\Delta(u_E)^{\mu\nu} + \tau(u_E)^{\mu\nu} + O(\epsilon O_1)],
\]

where we arrived at the last equality using \(u^\mu = u^\mu_E - \nu^\mu + O(\epsilon)^n, n \geq 2\) and \(h^\mu(u) = (\rho + P)\nu^\mu + O(\epsilon)^n, n \geq 2\). Thus the state in (53) satisfy the conservation equation \(\nabla_\mu T^{\mu\nu} = 0\), provided terms of order \(\epsilon O_1\) and their gradients are throwed away. Via a similar reasoning, but now starting from eq. (43), one finds

\[
0 = \nabla_\mu J^\mu = \nabla_\mu [n(u)u^\mu + n(u)^\mu]
= \nabla_\mu [n(u_E)\nu^\mu + n^\mu(u_E) + O(\epsilon O_1)],
\]

where we used \(n^\mu(u_E) = n^\mu(u) - n\nu^\mu + O(\epsilon)^n, n \geq 2\) and thus to an \(O_1\) accuracy the state in (53) indeed satisfy the conservation law in (43).
We now examine whether the fields \((n(u_E)^\mu, \tau(u_E)^{\mu\nu})\) satisfy to linear order equations \([15,17]\). The easiest way to prove this assertion is to start from the entropy current \(S^\mu\) in \([11]\) and eliminate \(u^\mu, h^\mu(u)\) and \(n^\mu(u)\) in favor of the fields measured relative to the energy frame. Using \(u^\mu = u_E^\mu - \hat{\epsilon}^\mu + O(\hat{\epsilon})^n, h^\mu(u) = (\rho + P)\hat{\epsilon}^\mu + O(\hat{\epsilon})^n, n \geq 2,\) it follows that \(S^\mu\) takes the form

\[
S^\mu = s(u_E^\mu - \hat{\epsilon}^\mu) + \beta(\rho + P)\hat{\epsilon}^\mu - \Theta(n^\mu(u_E) + n\hat{\epsilon}^\mu) \\
= su_E^\mu - \Theta(u)n_E^\mu - [s - \beta(\rho + P) + \Theta n]\hat{\epsilon}^\mu + O(\hat{\epsilon})^2 \\
= s(u_E)u_E^\mu - \Theta(u)\hat{\epsilon}^\mu + O(\hat{\epsilon}^2)
\]  

(58)

where \(s(u_E) := s(\rho(u_E), n(u_E))\) and we used the fundamental relation \(s = (\rho + P)T^{-1} - \Theta n.\) By imposing the second law on this form of \(S^\mu\) but now written relative to the energy frame, one arrives at

\[
n^\mu(u_E) = -\sigma T^2 \Delta(u_E)^{\mu\nu} \nabla_\nu \Theta(u_E) + O(\hat{\epsilon})^2; \tag{59}\]

\[
\pi^{\mu\nu}(u_E) = -2n < \nabla^\mu u_E^\nu + O(\hat{\epsilon})^2 >, \tag{60}\]

\[
\pi(u_E) = \zeta \nabla_\mu u_E^\mu + O(\hat{\epsilon})^2. \tag{61}\]

Thus the state in \([68]\) satisfy the equations of the Landau-Lifshitz theory provided that non linear terms in the deviations from the state of ”local thermodynamical equilibrium” and gradients of \(h^\mu, \pi, \) and \(\pi^{\mu\nu}\) have been depreciated\(^{23}\).

It is worth noticing that if in this analysis, we replace \(u^\mu\) by \(u_N^\mu\) and set \(n^\mu(u_N) := 0\) in \([50]\) and \(\sigma := 0\) in \([15]\), so that

\[
u^\mu_N, n(u_N), \rho(u_N), P(u_N), h^\mu(u_N), \tau^{\mu\nu}(u_N),
\]

satisfy the exact equations \([12,18]\) for the Eckart theory, then the state \(u_E^\mu, n(u_E), \rho(u_E), P(u_E), n(u_E)^\mu, \tau(u_E)^{\mu\nu}\) satisfy approximately the equations for the Landau-Lifshitz theory. This process can be also reversed i.e. starting from a state satisfying the the Landau-Lifshitz theory, one can generate a state that to \(O_1\) accuracy satisfy the Eckart theory.

Actually a more general statement holds. Within the context of the Hiscock-Lindblom class of first order theories, states compatible with the relativistic (LTE) postulate are to linear order equivalent states. To verify this claim, we return to the original fields \((u^\mu, \hat{u}^\mu)\) and consider the frame change in \([72]\). We assume that \(u^\mu, \) and the fields in \([50]\) satisfy \([10,11]\) and generate \(n(\hat{u}), \rho(\hat{u}), P(\hat{u}), h(\hat{u})^\mu, n(\hat{u})^\mu, \tau(\hat{u})^{\mu\nu}\) according to

\[
n(\hat{u}) = n(u) + \hat{\epsilon}O_1, \quad \rho(\hat{u}) = \rho(u) + \hat{\epsilon}O_1, \quad \text{etc.}
\]

By similar arguments that lead us to \([50,57]\), the conservation eqs yield

\[
0 = \nabla_\mu T^{\mu\nu} = \nabla_\mu [(\rho + P)\hat{u}\hat{u}^\nu + Pg^{\mu\nu} + h(\hat{u})^\nu\hat{u}^\mu + h(\hat{u})^\mu\hat{u}^\nu] + \tau(\hat{u})^{\mu\nu} + \hat{\epsilon}O_1 - \nabla_\mu [h^\mu(\hat{u})\hat{e}^\nu + h^\nu(\hat{u})\hat{e}^\mu]
\]

(62)

\[
0 = \nabla_\mu J^\mu = \nabla_\mu [n(u)u^\mu + n(u)^\mu] = \nabla_\mu [n(u)u^\mu + n(u)^\mu + \hat{\epsilon}O_1],
\]

(63)

and under the assumptions that gradients of \(h^\mu\) and of \(\epsilon^\mu\) are neglected in comparison to the velocity gradients, it follows that the terms linear in \(\epsilon^\mu\) in the right hand side of \([62,63]\) can be neglected and thus the conservation laws

\(^{23}\) Notice that equations \([60,61]\) could be derived directly from \([60,67]\) by replacing \(u^\mu\) by \(\hat{u}^\mu = \hat{u}^\mu - \epsilon^\mu\) and treating \((\nabla_\mu \epsilon^\mu, \nabla_\mu \epsilon_\nu)\) as \(\epsilon O_1\) and thus depreciated them in comparison to the gradients of the velocity field (this estimate will be employed further ahead)
to an $O_1$ accuracy holds true for the state $(\hat{u}^\mu, n(\hat{u}), \rho(\hat{u}), P(\hat{u}), h(\hat{u})^\mu, n(\hat{u})^\mu, \tau(\hat{u})^\mu\nu)$. We now indicate that equations (44-47) remain valid and for this we observe that the entropy flux $S^\mu$ in (11) that under the frame change described in (62) implies:

\[
S^\mu = s u^\mu + \beta(u) h^\mu(u) - \Theta(u) n^\mu(u) \\
= s \hat{u}^\mu + \beta(\hat{u}) h^\mu(\hat{u}) - \hat{\Theta}(u) n^\mu(\hat{u}) + O(\epsilon^2)
\]

and this $O_1$-invariance property of $S^\mu$ implies that the fields $h(\hat{u})^\mu, n(\hat{u})^\mu, \tau(\hat{u})^\mu\nu$ satisfy the analogues of (44-47) to an $O_1$ accuracy i.e to linear order in the deviations from the state of “local thermodynamical equilibrium” specified by $(u^\mu, s(\rho, n))$.

Before we continue, we offer a few comments regarding the constitutive relations that states satisfying the relativistic (LTE) postulate obey. We recall first the derivation of the exact relations (44-47). For this derivation, one begins by imposing the second law on the the entropy current $S^\mu$ in (11) and using the exact eqs $\nabla_\mu T^{\mu\nu} = \nabla_\mu J^\mu = 0$ one arrives at the exact relations (44-47) with the positive coefficients $(\zeta, \eta, k, \sigma)$ putting in by hand. However for states satisfying the (LTE) postulate, the constitutive relations obeyed by $(h(u), n^\mu(u), \pi^{\mu\nu}(u), \pi(u))$ hold approximately, since in their derivation we used the approximate expressions for the entropy currents $S^\mu$ shown in the last lines of (63) and (64). It is worth however, to probe this approximation and in particularly to probe the transition from the constitutive relations of the Eckart theory to the corresponding relations for the Landau-Lifshitz theory. To do so, we start from the easily verifiable identity (see for instance eq.(34) in (57))

\[
nT \n_\alpha \Theta = \n_\alpha P - \frac{\rho + P}{T} \n_\alpha T
\]

and let for generality purpose assume that the fluid is described by a $T^{\mu\nu}$ shown in (10). For this case the following identity holds

\[
\Delta^{\mu\nu} \left[ \frac{\n_\nu T}{T} + a_\nu \right] = -\frac{nT}{\rho + P} \Delta^{\mu\nu} \n_\nu \Theta + \frac{F^\mu}{\rho + P}, \quad a^\mu = u^\nu \n_\nu u^\mu
\]

where $F^\mu$ is defined from the conservation equation $\Delta^{\mu\nu} \n_\nu T^{\mu\nu} = 0$ which can be written in the equivalent form

\[
\Delta^{\mu\nu} \n_\nu P + (\rho + P) a^\mu = F^\mu
\]

where the precise form of $F^\mu$ is easily derivable, but it is not really needed since it contributes terms of $O(\epsilon^2)$ or terms that make negligible contribution like $\epsilon^\mu \n_\sigma u^\sigma$ etc. It is understood that in (65,67), the terms $T, \Theta, P, \Delta^{\mu\nu}$ stand for $T(u), \Theta(u), P(u), \Delta^{\mu\nu}(u)$ etc.

Suppose now we start from the Eckart theory so that the following relations hols

\[
h^\mu(u_N) = -kT(u_N) \Delta^{\mu\nu}(u_N) \left[ \frac{\n_\nu T(u_N)}{T(u_N)} + a_\nu \right], \quad \pi^{\mu\nu}(u_N) = -2n(\n_\nu u^\nu), \quad \pi(u_N) = -\zeta \n_\nu u^\nu_N
\]

for some non vanishing $(k, \zeta, \eta)$. Let us now perform a frame change described by

\[u^\mu_N \rightarrow u^\mu_E = u^\mu_N + \epsilon^\mu
\]

and for this frame change, we evaluate the identity (65) at $u^\mu_N$ and combine it with

\[
h^\mu(u_N) = -kT(u_N) \Delta^{\mu\nu}(u_N) \left[ \frac{\n_\nu T(u_N)}{T(u_N)} + a_\nu \right].
\]

Using $h^\mu(u_N) = (\rho + P) \epsilon^\mu$, $n^\mu(u_E) = -n \epsilon^\mu$ writing $\Delta^{\mu\nu}(u_N) = \Delta^{\mu\nu}(u_E) - u^\mu_E \epsilon^\nu - u^\nu_E \epsilon^\mu + O(\epsilon^2)$, $T(u_N) = T(u_E) + O(\epsilon^2)$ etc, one finds after some algebra that (68) transforms into

\[
n^\mu(u_E) = -\sigma T^2(u_E) \Delta^{\mu\nu}(u_E) \n_\nu \Theta(u_E) + O(\epsilon)
\]

where $O(\epsilon)$ denote terms involving products of $\epsilon$ with other other terms which in general considered as negligible in comparison to the leading term $\sigma T^2(u_E) \Delta^{\mu\nu}(u_E) \n_\nu \Theta(u_E)$ and in arriving at (69) we introduced the coefficient $\sigma$ via

\[
\sigma = k \frac{n^2}{(\rho + P)^2}
\]
Similarly, starting from

\[ \pi^{\mu\nu}(u_N) = -2n(\nabla^\mu u^\nu), \quad \pi(u_N) = -\zeta \nabla_\mu u^\mu_N, \]

then under the frame change \( u^\mu_N \to u^\mu_E = u^\mu_N + \epsilon^\mu \), it can be easily seen that they transform into

\[ \pi^{\mu\nu}(u_E) = -2n(\nabla^\mu u^\nu_E), \quad \pi(u_E) = -\zeta \nabla_\mu u^\mu_E, \]

where we have dropped terms linear in \( \epsilon^\mu \) and its derivatives. One can also reverse this process i.e. one can start from the constitutive relation in the Landau-Lifshitz theory to generate the constitutive relations for the Eckart theory. More generally one can consider frame changes described for instance by [31] and starting from the constitutive relations relative to the \( u^\mu \) frame work out the constitutive relations relative to the \( \hat{u}^\mu \). In general one finds to lowest order \([14,17]\) hold but they are modified by corrections terms of order \( O(\epsilon^\mu) \), etc. and these correction terms are neglected as long as we are in the regime where \( \epsilon << 1 \) (see also a discussion at this approximation, in the review article by Israel in [51] page (179)).

In summary, therefore within the Hiscock-Lindblom class of first order theories, states that satisfy the relativistic (LTE) postulate are equivalent states as long as one neglects terms of \( \epsilon^2 \) and higher order in the deviations from the local equilibrium field and gradients of \( h^\mu, \pi, \) and \( \pi^{\mu\nu} \) are small compared to thermal and velocity gradients and this conclusion holds also for the Landau-Lifshitz and Eckart theories.

In the so far analysis, we used the Hiscock-Lindblom class of first order theories as a test bed, to get insights on the properties of states satisfying the relativistic (LTE) postulate. However, ought to be mentioned that this class of theories, including the Eckart and Landau-Lifshitz theories, are pathological. According to the results in [36, 37], they do not respect causality and they are unstable in the sense that linear perturbations of their global equilibrium states become unbounded on a very short time scale. These important conclusions have been derived in [36, 37] by analyzing linearized perturbations in the form of exponentially plane waves off a globally homogeneous equilibrium state propagating on a Minkowski spacetime. As long as one of the coefficients \((\zeta, \eta, k, \sigma)\) in \([19]\) is different than zero, there exist transversal and longitudinal exponentially growing modes. Based on this property, other physically acceptable solutions of the perturbations equations exhibit also this type of instability (for more details, consult [30]).

The results in [36, 37] revealed the following highly counter-intuitive property: an equilibrium state appear to be stable when it is observed from a particular frame, but becomes unstable when is observed from a frame related to the first one by a Lorentz boost. This behavior can be seen but analyzing the behavior of the perturbing plane waves modes constructed in [36]. For instance, whenever the equilibrium state is observed from the comoving frame, then for the Eckart theory (or more generally for any theory within the Hiscock-Lindblom class) where \( k \neq 0 \), these modes contain exponentially growing modes while for the Landau-Lifshitz theory (where \( k = 0 \)) these plane waves contains only decaying modes. However, when the state is viewed from a Lorentz frame where the state is in motion, then for both theories i.e. Eckart and also the Landau-Lifshitz theory (or more generally for all of first order theories) contain exponentially growing modes and thus exhibit instability. Here one sees that within the Landau-Lifshitz theory the equilibrium state is stable when is observed from the comoving with the state frame, but is unstable when is observed from any other boosted frame.

This counter intuitive behavior has triggered a sizable amount of research activity (see for example [17, 19, 49, 58, 63]) and these efforts lead to powerful statements linking causality to stability of equilibrium states. One of the strongest results describing this interplay is Gavassino’s criterion [17] which asserts that if for a theory causality holds, then stability of equilibrium states is a Lorentz-invariance property. In more intuitive terms, the criterion asserts that if for a causal theory one is able to prove stability of an equilibrium state relative to one reference frame, then there cannot be any growing Fourier mode in any other boosted frame. A slightly different statement has proven also in [49] which restricts slightly however the nature of the field equations. In view of these developments, an interesting interpretation regarding the origin of the instability in the Hiscock-Lindblom class seen in [36, 37], has been put forward in ref. [19]. It was shown in that reference that for these theories the total entropy as a function of the accessible states fail to have upper bound a situation which is in sharp contrast to what occurs for instance for the case of the Israel-Stewart transient thermodynamics where the total entropy exhibits an absolute maximum (for the stability properties of that theory consult [57, 37, 32]). Finally very recently, a connection between causality and thermodynamical stability has been discussed in [18]. It was shown in that work for any theory that is thermodynamically stable i.e. the total entropy is maximized at equilibrium, it is
also causal at least close to equilibrium, a conclusion that indicates clearly that causality is tied up thermodynamic stability (and not hydrodynamic stability) that is commonly believed (see illuminating discussion in [18]).

In view of this perplexing state of affairs centered on causality, stability and the Hiscock-Lindblom class of first order theories, below we discuss briefly a few properties of equilibrium states perturbed by states satisfying the relativistic (LTE) postulate. For this, let again the class of states satisfying the relativistic (LTE) postulate assume here that the fields $T^{\mu \nu}$ and $J^\mu$ are defined on a Minkowski spacetime restricted so that the pseudo-angle $\epsilon$ in [19] satisfies $\epsilon << 1$ everywhere within the fluid region. Let a four velocity $u^\mu_o$ is chosen within the cone of opening angle $\epsilon << 1$ and let the fields in the expansion (11) satisfy

$$h^\mu = n^\mu = \pi = \pi^{\mu \nu} = 0,$$

while $(\rho_o, n_o, P_o)$ are chosen to be homogeneous and isotropic and relative to a global inertial coordinates $(t, x, y, z)$ the velocity field $u^\mu_o$ takes the form $u^\mu_o = \delta^\mu_t$ i.e. the state is at rest relative to this global rest frame. The so defined state satisfies the relativistic (LTE) postulate but it also is an equilibrium state.

Following the notation in [36], the (Eulerian) perturbations of the thermodynamical variables are denoted by $\delta \rho, \delta n, \delta u^\mu$ etc and represent the difference between the value of the non equilibrium variable and the corresponding equilibrium one evaluated at the same spacetime point. It is not difficult to show that the equations for these perturbations are obtained by linearizing (12, 18) around a background equilibrium state are given by (see also [36])

$$\nabla_\mu \delta T^{\mu \nu} = 0, \quad \nabla_\mu \delta J^\mu = 0, \quad \delta h^\mu = -kT \Delta^{\mu \nu} [\nabla_\nu (\delta T^T) + u^\lambda \nabla_\lambda u_\nu + \delta u^\lambda \nabla_\lambda u_\nu], \quad \delta \pi^\mu = -\sigma T^2 \Delta^{\mu \nu} \nabla_\nu \delta \Theta \quad (72)$$

$$\delta \pi^{\mu \nu} = -2\eta <\nabla^\mu \delta u^\nu + \delta u^\mu u^\lambda \nabla_\lambda u^\nu >, \quad \delta \pi = -\zeta \nabla_\mu \delta u^\mu, \quad (73)$$

with the perturbations $\delta T^{\mu \nu}$ and $\delta J^\mu$ defined by

$$\delta T^{\mu \nu} = (\rho + P) (\delta u^\mu u^\nu + u^\mu \delta u^\nu + \delta \rho u^\mu u^\nu + (\delta P + \delta \pi)) \Delta^{\mu \nu} + u^\mu \delta h^\nu + u^\nu \delta h^\mu + \delta \pi^{\mu \nu}, \quad (74)$$

$$\delta J^\mu = \delta n u^\mu + n \delta u^\mu + \delta \pi u^\mu. \quad (75)$$

and in these equations and hereafter, variables without the prefix $\delta$, denote equilibrium values. Notice also that as a consequence of the constraints $u^\mu u^\mu = -1, \, h^\mu u^\mu = 0$ etc, the perturbations variables are subject to the constraints (see also [36]):

$$u^\mu \delta u^\mu = u^\mu \delta h^\mu = u^\mu \delta n^\mu = u^\mu \delta \pi^\mu = 0 \quad (76)$$

Since in the derivation of equations (72, 73), are actually independent of whether the perturbing state is an arbitrary state or a state compatible with the (LTE) postulate, the results in [36] holds true for our problem as well. Thus for small (Eulerian) perturbations of exponential plane waves of the form

$$\delta Q = \delta Q_0 e^{ik ax + \Gamma t} \quad (77)$$

propagating along the $x$-axis with constant "frequency" $\Gamma$, the results in [36] show that for a non vanishing $k_x$, exist exponentially growing (and decreasing) transverse modes with real "frequencies" $\Gamma_+ \pm \Gamma_-$ given by

$$2k T \Gamma_\pm = (\rho + P) \pm [(\rho + P)^2 + 4\zeta kT k_x^2]^{1/2} \quad (78)$$

24 Here after thermodynamical variables describing this equilibrium state are denoted by subscript $(o)$. In particularly, for emphasis we write $(T^{\mu \nu}_o, J^\mu_o)$ instead of writing the correct expressions $(T^{\mu \nu}, J^\mu)$.

25 For typographical convenience we have written $(T, u^\mu, \Delta^{\mu \nu}, P,...)$ instead of $(T_o, u^\mu_o, \Delta^{\mu \nu}_o, P_o,...)$. 
while for the case of the Landau-Lifshitz theory there exist decreasing transverse modes with

$$\Gamma = -\frac{\eta k^2}{\rho + P},$$  \hspace{1cm} (79)$$

and since this $\Gamma$ is purely real and negative, it follows that the Landau-Lifshitz theory escapes the instability that is manifest in the Eckart (or any other first order theory subject to $k > 0$). However, as was shown in \[36, 37\] this property disappears once the equilibrium state is viewed from a Lorentz frame where the equilibrium state is in motion.

The results expressed in \[78, 79\] hold also for the equilibrium states perturbed by states satisfying the relativistic (LTE) postulate and thus these equilibrium states exhibit the same instabilities as the equilibrium states within the full class of first order theories.

Suppose however, we consider another state satisfying the relativistic (LTE) postulate specified by $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$, which is initially close to the equilibrium state in \[71\]. We consider again the linear perturbations denoted here after by a prime i.e. $\delta'\rho, \delta'\pi, \delta'\mathcal{R}$, etc induced by $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ on the equilibrium state in \[71\]. Clearly these new perturbations satisfy again \[72-76\] (obtained by linearizing the system \[42-48\]). As we have seen in this section although, the states $(u^\mu, s(\rho, n))$ and $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ are in fact equivalent and formulas \[25-33\], shows that equivalence, below we show that the perturbations induced by $(u^\mu, s(\rho, n))$ and $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ on the background equilibrium state in \[71\] fail to be equivalent.

Indeed starting from the relation

$$\hat{u}^\mu - u^\mu = \hat{\epsilon}^\mu + O(\hat{\epsilon})^2$$

the following relations hold between the primed and unprimed perturbations:

$$\delta u^\mu := u^\mu - u^\mu = \hat{u}^\mu - u^\mu + \hat{\epsilon}^\mu = \delta' u^\mu - \hat{\epsilon}^\mu + O(\hat{\epsilon})^2$$ \hspace{1cm} (80)$$

$$\delta h^\mu = h^\mu(u) = \delta' h^\mu + (\rho + P)\hat{\epsilon}^\mu + O(\hat{\epsilon})^2, \hspace{1cm} \delta n^\mu = \delta' n^\mu + n\hat{\epsilon}^\mu + O(\hat{\epsilon})^2$$ \hspace{1cm} (81)$$

$$\delta \rho = \delta' \rho + O(\hat{\epsilon})^2, \hspace{1cm} \delta n = \delta' n + O(\hat{\epsilon})^2, \hspace{1cm} \delta P = \delta' P + O(\hat{\epsilon})^2, \hspace{1cm} \delta \pi = \delta' \pi + O(\hat{\epsilon})^2, \hspace{1cm} \delta \pi^{\mu\nu} = \delta' \pi^{\mu\nu} + O(\hat{\epsilon})^2$$ \hspace{1cm} (82)$$

Moreover it can be easily seen that if the unprimed perturbations satisfy equations \[72, 76\]. This means that the perturbations induced on the background equilibrium state by $(u^\mu, s(\rho, n))$ and $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ states are in fact related by formulas \[25-33\] that are analogous to \[25-33\]. Since however, these relations hold for arbitrary solutions of the perturbations equations, they also hold for the theory of exponential plane waves solutions described in \[77\] i.e. propagating along the $x$-axis and of constant ”frequency” $\Gamma$, as in \[32\]. However, here we lead into an impasse. If for instance we choose $(u^\mu, s(\rho, n))$ to be described by an Eckart state ( or more generally an arbitrary first order state with $k > 0$) and $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ say a Landau-Lifshitz state (thus $k = 0$), then formulas \[25-33\] describe an incompatibility. One perturbation is exponentially growing while the other is exponentially decreasing. This in turn implies that the equivalence between the two states $(u^\mu, s(\rho, n))$ and $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ that satisfy the (LTE) postulate as expressed by formulas \[25-33\], breaks down for the first order perturbations of an equilibrium state. To put it differently, if the states $(u^\mu, s(\rho, n))$ and $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ were truly equivalent one would have expected that the induced perturbations on a background equilibrium state of the theory to in fact identical solutions.

Interestingly this kind of equivalence holds for the Israel-Stewart theory. If we replace the Hiscock-Libdbloom class of first order theories by the Israel-Stewart theory and the $(u^\mu, s(\rho, n))$, $(\hat{u}^\mu, s(\hat{\rho}, \hat{n}))$ states by the Israel-Stewart theory expressed in the Eckart frame respectively Landau-Lifshitz frame, one would find that the resulting induced perturbations are indeed equivalent and this equivalence has been demonstrated\[26\] in ref. \[31\] (we shall return to this point at the end of the section (V)). Here we only mentioned that this pronounced difference between the two theories can be traced in thermodynamical reasoning. Gavasino, Antonelli and Haskell in \[19\] have shown that the total entropy between the two theories behaves differently. Sable point behavior for the Hiscock-Libdbloom class of

\[26\] Our thanks to an anonymous referee who pointed out to us ref. \[31\] and suggested to probe the interconnection between Hiscock-Libdbloom class of first order theories and the Israel-Stewart theory.
first order theories versus an absolute maximum for the Israel-Stewart theory. We shall discuss this point further at the end of the next section.

The so far analysis demonstrates that states satisfying the relativistic (LTE) postulate with the Hiscock-Lindblom class of first order theories exhibit the same pathologies as arbitrary states within this class and thus their utility is very limited.

We shall leave this section, by briefly discussing the connection between states obeying the relativistic (LTE) postulate and the (BDNK) theory. As we have already mentioned in section (II) the (BDNK) theory is branded as a first order theory\textsuperscript{27} and it has some remarkable properties. It was shown in (40, 47, 48, 49) that under suitable choice of frame, the theory respects causality and admits stable equilibrium states and moreover in (48) it was shown that the Cauchy problem for the theory is locally well posed, and strongly hyperbolic and this locally well posedness and strong hyperbolicity remain intact even when the fluid is dynamically coupled to Einstein’s equations (see ref 49 for details and references).

For a simple, electrically neutral, fluid the dynamical equations for this theory are $\nabla_\mu T^{\mu\nu} = 0$ and fluid states are specified by assigning a "hydrodynamical frame" or simply a frame\textsuperscript{28} which signifies a specification of a fluid four velocity $u^\mu$ a temperature $T$ and a chemical potential $\mu$ (both of them measured by the $u^\mu$-observer). The theory targets states that are near equilibrium, although a precise definition of what exactly constitutes a state near thermal equilibrium has not been addressed in the so far development of the theory\textsuperscript{29}. Within this theory, the fields $(T^{\mu\nu}, J^\mu)$ depend upon $(u^\mu, T, \mu)$ and the theory utilizes (or better adapts) the gradient expansion technique from quantum field theory to the hydrodynamical regime (for an introduction to this technique within the hydrodynamical regime, consult (52, 60) and references therein). The fields $(T^{\mu\nu}, J^\mu)$ are expanded according to

\[ T^{\mu\nu} = E u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} (u) + (Q^\mu u^\nu + Q^\nu u^\mu) + T^{\mu\nu} \]

\[ J^\mu = N u^\mu + J^\mu \]

which are the same expansions as in (10 11) except that new symbols\textsuperscript{30} $(E, \mathcal{P}, Q^\mu, T^{\mu\nu}, N, J^\mu)$ are used instead of $(\rho, T, u^\mu, T^{\mu\nu}, n, n^\mu)$ entering in (10 11). However, within the (BDNK) formalism, it is postulated that the fields $(E, \mathcal{P}, Q^\mu, T^{\mu\nu}, N, J^\mu)$ admit a (convergent) derivative series expansion formed from the derivatives of $(T, u^\mu, \mu)$. The series employs the scalars $(T, \mu)$, their derivatives along the flow of $u^\mu$ i.e. $u^\mu \nabla_\mu T := \dot{T}$, $\nabla_\mu u^\mu$ and $u^\mu \nabla_\alpha \mu := \dot{\mu}$, utilizes also the transverse vectors $\Delta^{\mu\nu} \nabla_\mu T$, $a^\mu := u^\mu \nabla_\nu u^\alpha$, $\Delta^{\mu\nu} \nabla_\nu \mu$, and the transverse second rank, traceless symmetric tensor, $\sigma_{\mu\nu}$ and in a standard notation, the series expansions have the form (for details consult (40, 47, 48, 49)):

\[ E = \epsilon + \epsilon_1 \frac{\dot{T}}{T} + \epsilon_2 \nabla_\mu u^\mu + \epsilon_3 u^\mu \nabla_\mu (\frac{\dot{\mu}}{T}) + O(\partial^2) \]

\textsuperscript{27} It ought to be stressed that while in the Hiscock-Lindblom terminology, a first order theory is a theory where the entropy flux $S^\mu$ receives contributions only from first order deviations from a fictitious ”local thermodynamical equilibrium” state, the term ”first order theories” within the (BDNK) formalism has a very different meaning as it will become clear further ahead. A suitably defined series is approximated only by the first term.

\textsuperscript{28} Here we are warning the reader of a potential confusion arising from the use of the term ”frame”. While in this work the term ”frame” signifies an orthonormal tetrad (or often a collections of orthonormal tetrads defined along an integral curve (or on all integral curves)) of a smooth velocity field $u^\mu$ and the term ”change of frame” is a transition to a new tetrad induced by the action of a point wise Lorentz transformation (or a family of such point wise transformations), within the (BDNK) formalism, a ”hydrodynamical frame” or simply ”frame” signifies a specification of a triplet $(T, \mu, u^\mu)$ and a change of the ”hydrodynamical frame” (or a field redefinition), is a passage to a new ”hydrodynamical frame” i.e. to a new set of hydrodynamical variables $(\dot{T}, \dot{\mu}, \dot{u}^\mu)$. This change of frame advocated in the (BDNK) formalism is in general different than the change of frame induced by the action of a local Lorentz transformation in our terminology, although the two are related as we shall see further ahead (in reference (49) a complete set of references are compiled where the reader can find the diverse meaning assigned to the term frames and change of frames).

\textsuperscript{29} An attempt to define precisely the term equilibrium and states near equilibrium is pursued in refs 48, 49 via the employment of relativistic kinetic theory. Although a Chapman-Enskog expansion, or a Grad like or any other expansion yield fluid states that capture the spirit of the relativistic (LTE) postulate, still we feel that an independent definition of states compatible with a relativistic (LTE) postulate build entirely within the field of Relativistic hydrodynamics viewed as a discipline in its own right, is worth having (and this is precisely the main purpose of the present work).

\textsuperscript{30} The usage of $(E, \mathcal{P}, Q^\mu, T^{\mu\nu}, N, J^\mu)$ is the standard notation for the practioners of the (BDNK) formalism.
\[ P = p + \pi_1 \frac{T}{T} + \pi_2 \nabla_\mu u^\mu + \pi_3 u^\mu \nabla_\mu (\frac{\mu}{T}) + O(\partial^2) \]  
(86)

\[ Q^\mu = \theta_1 \dot{u}^\mu + \frac{\theta_2}{T} \Delta^\mu_\nu \nabla_\nu T + \theta_3 \Delta^\mu_\nu \nabla_\nu (\frac{\mu}{T}) + O(\partial^2) \]  
(87)

\[ T^\mu_\nu = -\eta \sigma^\mu_\nu + O(\partial^2) \]  
(88)

\[ N = n + \nu_1 \frac{T}{T} + \nu_2 \nabla_\mu u^\mu + \nu_3 u^\mu \nabla_\mu (\frac{\mu}{T}) + O(\partial^2) \]  
(89)

\[ J^\mu = \gamma_1 \dot{u}^\mu + \frac{\gamma_2}{T} \Delta^\mu_\nu \nabla_\nu (\frac{\mu}{T}) + \gamma_3 \Delta^\mu_\nu \nabla_\nu \frac{\mu}{T} + O(\partial^2) \]  
(90)

where \( O(\partial^2) \) (or more generally \( O(\partial^k) \)) signifies terms of second order (or \( k \) order) derivatives in the variables \( (T, u^\mu, \mu) \), while \( (\epsilon_i, \pi_i, \theta_i, \nu_i), \ (\gamma_i, i \in (1, 2, 3)) \) and \( \eta \) are considered to be transport coefficients depending upon \( (T, \mu) \). Moreover, it is postulated that the basic fields \( (T(x), u^\mu(x), \mu(x)) \) are non unique and can be transformed to new fields \( (T'(x), u'^\mu(x), \mu'(x)) \) according to:

\[ (T(x), u^\mu(x), \mu(x)) \rightarrow (T'(x), u'^\mu(x), \mu'(x)) = (T(x) + \delta T(x), u^\mu(x) + \delta u^\mu(x), \mu(x) + \delta \mu(x)) \]  
(91)

where the variations \( (\delta T(x), \delta u^\mu(x), \delta \mu(x)) \) are defined via:

\[ \delta T = \alpha_1 \frac{T}{T} + \alpha_2 \nabla_\mu u^\mu + \alpha_3 u^\mu \nabla_\mu (\frac{\mu}{T}) + O(\partial^2) \]  
(92)

\[ \delta u^\mu = b_1 \dot{u}^\mu + b_2 \Delta^\mu_\nu \nabla_\nu T + b_3 \Delta^\mu_\nu \nabla_\nu (\frac{\mu}{T}) + O(\partial^2) \]  
(93)

\[ \delta \mu = c_1 \frac{T}{T} + c_2 \nabla_\mu u^\mu + c_3 u^\mu \nabla_\mu (\frac{\mu}{T}) + O(\partial^2) \]  
(94)

where \( (a_i, b_i, c_i, i \in (1, 2, 3)) \) are arbitrary functions of the \( (T, \mu) \) (for a discussion leading to the above transformation see [40], [47]). The transformation in [91] describes a change of the hydrodynamical frame and plays an important role within the theory.

We shall not discuss any further properties of this theory, but for the rest of this section, we shall link states obeying the relativistic (LTE) postulate to the (BDNK) theory and for this let us begin by choosing a state compatible with the relativistic (LTE) postulate within however the (BDNK) theory. Accordingly we choose \( T'^\mu_\nu \) and \( J^\mu \) such that the angle \( \epsilon \) formed by \( u'^\mu_\nu \) and \( u^\mu_\nu \) (see [19]) satisfies \( \epsilon << 1 \) everywhere within the fluid region and choose a velocity field \( u^\mu \) within the cone of the angle \( \epsilon << 1 \). We denote by \( (u^\mu, s(\rho, n)) \) be the state of “local thermodynamical equilibrium” and let \( T(u) \) the local temperature and \( \mu(u) \) the local chemical potential, defined from derivatives of \( s(\rho, n, n(u)) \) (see [14]) and remember that \( \Theta(u) = \frac{u^\mu(\mu)}{T(u)} \). Thus any state satisfying the relativistic (LTE) postulate leads to a hydrodynamical frame (in the (BDNK)-sense) specified by \( (T(u), u^\mu, \mu(u)) \).

Under an admissible frame change described by

\[ u^\mu \rightarrow \dot{u}^\mu = u^\mu + \dot{\epsilon}^\mu + O(\dot{\epsilon})^2, \quad \dot{\epsilon}^\mu \leq O_1. \]  
(95)

formulas [25, 33], imply

\[ T' := T(\dot{u}) = T(u) + \dot{\epsilon}O_1, \quad \mu' := \mu'(\dot{u}) = \mu(u) + \dot{\epsilon}O_1, \quad u'^\mu := \dot{u}^\mu = u^\mu + \dot{\epsilon}^\mu + (\dot{\epsilon})^2 \]  
(96)
where \((T(\hat{u}), \mu(\hat{u}))\) are the local temperatures and local chemical potentials measured by the \(\hat{u}\)-observer. Thus any frame change as in \((95)\), induces a particular field redefinition described in \((96)\) and this field redefinition can be seen as a particular case of the field redefinition in \((91)-(94)\). Indeed it can be generated by choosing \(\alpha_i = c_i = 0, i \in (1, 2, 3)\), taking \(b_1 = b_2 = b_3 = b << 1\) in \((92)-(94)\), and identify \(\hat{\varepsilon}^\mu\) in \((95)\) via

\[
\hat{\varepsilon}^\mu := b[\hat{u}^\mu + \Delta^{\mu\nu}\nabla_\nu T + \Delta^{\mu\nu}\nabla_\nu (\frac{\mu}{T}) + O(\beta^2)].
\]

Under the field redefinition in \((93)\), the series expansion in \((85)-(90)\) remain form invariant, since terms like \(\nabla_\mu \hat{\varepsilon}_\nu\) and \(\nabla_\mu \hat{\varepsilon}^\mu\) are considered as second order in the deviation from the state of ”local thermodinamical equilibrium” and the contributions of terms like \(u^\mu \hat{\varepsilon}_\nu T\), \(\hat{\varepsilon}^\mu \hat{\varepsilon}^\nu T\) etc are neglected in comparison to terms \(u^\mu u^\nu \nabla_\nu T\), \(u^\mu u^\nu \nabla_\nu T\) etc. The invariance of the series in \((85)-(90)\) implies that the dynamical equations satisfied by \((T, \mu, u^\mu)\) remain form invariant under the transformation of frame induced by \((95)\) as long as second order deviations from a ”local thermodinamical equilibrium state are neglected. Thus within the (BDNK) formalism states compatible to the relativistic (LTE) postulate are equivalent provided second order deviations from from a ”local thermodinamical equilibrium” state are thrown away.

However ought to be stressed that the real power of the the (BDNK) formalism lies in the freedom of frame change described in \((91)-(94)\). This liberty, allows to select particular hydrodynamical frames where the second order equations \(\nabla_\mu T^{\mu\nu} = \nabla_\mu J^\mu = 0\) for the basic variables \((u^\mu, T, \mu)\) (or an equivalent set) exhibit causality and stability of equilibrium states (for details consult \((48, 47, 48, 49)\)). Thus within the (BDNK) framework, states satisfying the relativistic (LTE) postulate are becoming relevant.

V. TRANSIENT THERMODYNAMICS AND (LTE) STATES

In this section, we discuss states satisfying the relativistic (LTE) postulate within the Israel-Stewart \([21,22]\) transient thermodynamics. We ought to bear in mind however, that transient thermodynamics, by design deals with states that are near equilibrium (or in our terminology with states satisfying the relativistic (LTE) postulate\(^{31}\)). It is worth remembering this restriction since as pointed in \([61]\) for states with large deviations from equilibrium the theory it is not always well behaved. In that regard, is worth remembering some recent results derived in ref. \([30]\) where sufficient conditions, in the form of algebraic inequalities derived that ensure that the theory is causal away from equilibrium and a theorem established on the local existence and uniqueness of solutions of the theory and these results raise the confidence on the theory.

With this comments in mind, and since the structure of the theory is discussed in the original articles, by Israel \([21]\) and Israel and Stewart \([22]\) below, we shall only provide a brief overview of the principles of transient thermodynamics and shall offer a few comments regarding the equivalence of the phenomenological equations as expressed relative to distinct frames.

We begin by considering a fluid state compatible with the relativistic (LTE) postulate described by \((J^\mu, T^{\mu\nu}, S^\mu)\) obeying \((7)-(8)\) with the entropy flux \(S^\mu\) to be defined shortly. We choose a four velocity \(u^\mu\) defined within the fluid region and lying within the familiar cone of opening angle \(\epsilon << 1\) and let \(s(\rho, n)\) be the corresponding equilibrium equation of state. To this \((u^\mu, s(\rho, n))\), we attach the ”local thermodinamical equilibrium” state \((S_0^\mu, T_0^{\mu\nu}, J_0^\mu)\) in the manner discussed in section \((III)\) (see discussion leading to \((16)\)). The velocity \(u^\mu\) and the expansions in \((10)-(11)\), define the fields

\[
(n(u), \rho(u), P(u), h(u)^\mu, n(u)^\mu, \sigma(u)^{\mu\nu})
\]

while the thermal potential \(\Theta(u)\) and the local inverse temperature \(\beta(u) = T(u)^{-1}\) measured by the \(u\)-observer are defined by the fundamental Gibbs relation \((14)\). The velocity \(u^\mu\) and the fields in \((98)\) satisfy the eqs of transient thermodynamics provided they obey:

\(^{31}\) As we have already mentioned, the definition of states compatible with the relativistic (LTE) postulate proposed in this work, has been motivated by the properties of states satisfying the equations of transient thermodynamics. Also we mention here that a state near equilibrium it is not necessary a state satisfying the relativistic (LTE) postulate. The four velocity that identifies the former does not necessarily leads to a state where ”local thermodinamical equilibrium” prevails.
\[ \nabla_{\mu}T^{\mu\nu} = \nabla_{\mu}[\rho(u)u^{\mu}u^{\nu} + P(u)\Delta(u)u^{\mu}u^{\nu} + h(u)^{\mu}u^{\nu} + \tau(u)^{\mu\nu}] = 0, \]  

(99)

\[ \nabla_{\mu}J^{\mu} = \nabla_{\mu}[n(u)u^{\mu} + n(u)^{\mu}] = 0, \]  

(100)

and additional equations arising by imposing the second law. The entropy flux \( S^{\mu} \) is postulated to have the form (see [21, 22] a discussion leading to this choice):

\[ S^{\mu} = P(u)\beta^{\mu}(u) - \Theta(u)J^{\mu} - \beta_{\nu}(u)T^{\mu\nu} - Q^{\mu}(\delta J^{\alpha}, \delta T^{\alpha\beta}, X_{(i)}^{\alpha\beta\gamma\cdots}) \]  

(101)

where \( \beta_{\lambda} = \beta(u)u_{\lambda} = \frac{\lambda_{\lambda}}{T(u)} \) and \( Q^{\mu} \) is a vector field that depends quadratically upon the deviations \( (\delta J^{\alpha}, \delta T^{\alpha\beta}) \) from the state of a "local thermodynamical equilibrium" specified by \( (u^{\mu}, s(\rho, u)) \). For the following analysis, it is convenient to transform the right hand side of (101) in a form so that its relation to the entropy flux \( S^{\mu} \) for first order theories, becomes apparent. For this, starting from \( s(u) = \beta(u)(\rho(u) + P(u)) - \Theta(u)n(u), \) and the fundamental Gibbs relation \( ds = \beta d\rho - \Theta dn, \) one gets their covariant versions

\[ S_{0}^{\mu} = P\beta^{\mu} - \Theta J_{0}^{\mu} - \beta_{\lambda}T_{0}^{\lambda\mu}, \quad dS_{0}^{\mu} = -\Theta dJ_{0}^{\mu} - \beta_{\lambda}dT_{0}^{\lambda\mu}, \]  

(102)

where for their derivations we used: \( S_{0}^{\mu} = s u^{\mu}, \rho u^{\mu} = -u_{\lambda}T_{0}^{\lambda\mu} \) and \( (J_{0}^{\mu}, T_{0}^{\mu\nu}, S_{0}^{\mu}) \) are the fields introduced in (16). Eliminating the contribution of \( P(u)\beta^{\mu} \) from the right hand side of (101) one gets

\[ S^{\mu} = S_{0}^{\mu} - \Theta(J^{\mu} - J_{0}^{\mu}) - \beta_{\lambda}(T^{\lambda\mu} - T_{0}^{\lambda\mu}) - Q^{\mu} \]

\[ = S_{0}^{\mu} - \Theta\delta J^{\mu} - \beta_{\lambda}\delta T^{\lambda\mu} - Q^{\mu}(\delta J^{\alpha}, \delta T^{\alpha\beta}) \]

\[ = su^{\mu} - \Theta(u)n^{\mu}(u) + \beta(u)h^{\mu}(u) - Q^{\mu}(\delta J^{\alpha}, \delta T^{\alpha\beta}), \]  

(103)

which shows that transient thermodynamics is an extension of the Hiscock-Lindblom class first order theories by incorporating in \( S^{\mu} \) second order deviations from the state of "local thermodynamical equilibrium" manifesting themselves in the non vanishing of the \( Q^{\mu}(\delta J^{\alpha}, \delta T^{\alpha\beta}) \) term.

Motivated from the relativistic kinetic theory of diluted gases, Israel and Stewart [21, 22] proposed that \( S^{\mu} \) (and thus \( Q^{\mu} \)) should be independent of the gradients of \( J^{\mu} \) and \( T^{\mu\nu} \) and should be quadratic in the deviations from the state of "local thermodynamical equilibrium". Within the hydrodynamical approximation 33, they proposed that for an electrically neutral, simple fluid the term \( Q^{\mu}(u) \) should have the form:

\[ Q^{\mu}(u) = \frac{1}{2}u^{\mu}[\beta_{0}(\rho) + \beta_{1}(\rho)q_{\nu} + \beta_{2}(\rho)\pi_{\lambda\nu}] - \alpha_{0}\pi q^{\mu} - \alpha_{1}\pi^{\mu\nu}q_{\nu} + R^{\mu}(u), \]  

(104)

where \( R^{\mu}(u) \) stands for

\[ R^{\mu}(u) = \frac{1}{T(\rho + P)} \left[ \frac{1}{2}u^{\mu}h^{\nu}h_{\nu} + \tau^{\mu\nu}h_{\nu} \right], \]  

(105)

and in above \( q^{\mu} \) stands for the invariant heat flux defined in (25). The coefficients \( \alpha_{j}, j \in (0, 1), \beta_{i}, i \in (0, 1, 2), \) are undetermined depending upon \( (\rho, n) \) and \( (T, \rho, P, \pi, etc) \) are the local thermodynamical variables measured by the \( u^{\mu} \)-observer (for simplicity of the representation we suppressed their implicit dependence upon \( u^{\mu} \)).

As we mentioned, the dynamical equations for transient thermodynamics, are those in (99)-(100) augmented by the phenomenological equations that follow by imposing the second law on the entropy flux \( S^{\mu} \) in (101) and \( Q^{\mu}(u) \) as in (104)-(105). Their forms and the detail calculations leading to their derivations can be found in [21, 22]. Here we shall

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32 The variables \( X_{(i)}^{\mu\lambda\cdots}, i \in (1, 2, ...) \) appearing in \( Q^{\mu} \), are additional variables needed to completely specify the non equilibrium state. In this work, and in fact within transient thermodynamics, are assumed to be zero and thus will be disregarded hereafter.

33 The name derives itself from the assumption that \( S^{\mu} \) is specified only by the hydrodynamical variables \( (J^{\mu}, T^{\mu\nu}) \).
offer a few comments regarding their behavior under a frame change. For this, we consider another velocity field \( \hat{u}^\mu \) lying within the cone of the opening pseudo-angle \( \epsilon << 1 \) and let the frame change

\[
u^\mu \rightarrow \hat{u}^\mu = u^\mu + \epsilon^\mu, \quad \epsilon^\mu \leq \epsilon^\mu.
\]

For this frame change, we introduce the new fields \( n(\hat{u}), \rho(\hat{u}), P(\hat{u}), h(\hat{u})^\mu, n(\hat{u})^\mu, \tau(\hat{u})^{\mu
u} \) defined according to

\[n(\hat{u}) = n(u) + \epsilon O_1, \quad \rho(\hat{u}) = \rho(u) + \epsilon O_1, \quad \text{etc},
\]

and at first we show that these new fields satisfy to an \( O_1 \) accuracy the conservation laws (99, 100). Indeed a repetition of the calculation that lead us to (92,93) for the case of first order theories, yields to the same conclusion. To an \( O_1 \) accuracy in the deviations from the state of ”local thermodynamical equilibrium” the conservation laws (99, 100) remain valid.

However proving that the same property holds for the rest of the equations that follow by imposing the second law on \( S^\mu \) in (103) becomes more subtle. One may for instance, start from the equations written say relative to the \( u^\mu \) frame and subsequently via (25-33), rewrite them relative to the \( \hat{u}^\mu \) frame keeping only terms describing first order deviations. Unfortunately this procedure is long and time consuming, but there is an alternative method that avoids this route. For this, we return to the entropy flux \( S^\mu \) in (103) and examine its variation under the frame change in (106). Denoting by \( \delta \) the resulting variation, and noting that \( \delta S^\mu = \delta J^\mu = \delta T^{\mu\nu} = 0 \), we find that \( \delta S^\mu \) in (103) satisfies:

\[0 = \delta S^\mu = u^\mu \delta s + s \delta u^\mu - (J^\mu - J_0^\mu) \delta \Theta - \Theta \delta (J^\mu - J_0^\mu) - \delta \beta_\lambda (T^{\mu\lambda} - T_0^{\mu\lambda}) - \beta_\lambda \delta (T^{\mu\lambda} - T_0^{\mu\lambda}) - \delta Q^\mu.
\]

Due to the identity \( s(u) = \beta(u) (\rho(u) + P(u)) - \Theta(u) n(u) \), one finds that \( s \delta u^\mu - \Theta \delta (J^\mu - J_0^\mu) - \beta_\lambda \delta (T^{\mu\lambda} - T_0^{\mu\lambda}) \) is vanishing to linear order in \( \epsilon^\mu \) and thus (107) implies

\[\delta Q^\mu = -(J^\mu - J_0^\mu) \delta \Theta - (T^{\mu\lambda} - T_0^{\mu\lambda}) \delta \beta_\lambda + \epsilon O_1 + \epsilon O_2.
\]

Taking into account that

\[\delta \Theta \equiv \Theta(\hat{u}) - \Theta(u) = \epsilon O_1,
\]

\[\delta \beta^\mu = \frac{\epsilon^\mu}{T(u)} + u^\mu \delta \left( \frac{1}{T(u)} \right) = \frac{\epsilon^\mu}{T(u)} + \epsilon O_1,
\]

results

\[\delta Q^\mu = -(T^{\mu\lambda} - T_0^{\mu\lambda}) \delta \beta_\lambda + \epsilon O_1 = \frac{1}{T(u)} [u^\mu h^\lambda + \tau^{\mu\lambda}] \epsilon_\lambda + \epsilon O_2.
\]

On the other hand, working out the variation of the \( Q^\mu(u) \) term arising (104, 105) one finds that the majority of the terms yield an \( \epsilon O_2 \) contribution except the variation of the \( R^\mu(u) \) terms. Working out \( \delta R^\mu(u) \) using formulas (25-33), one finds an expression identical to that in (110) which implies that \( Q^\mu(u) \) has been chosen consistently in (104) and (105) and this consistency in the choice of the \( Q^\mu(u) \) plays an important role in proving the frame invariance property of the phenomenological equations. Under the frame change in (106), a calculation shows that \( S^\mu \) in (103) satisfies the identity:

\[S^\mu = su^\mu + \beta(u) h^\mu(u) - \Theta(u) n^\mu(u) - Q^\mu(u)
\]

\[= \hat{s} \hat{u}^\mu + \beta(\hat{u}) h^\mu(\hat{u}) - \Theta(\hat{u}) n^\mu(\hat{u}) - (T^{\mu\lambda} - T_0^{\mu\lambda}) \delta \beta_\lambda
\]

\[- Q^\mu(\hat{u}) - \delta R^\mu + \epsilon O_2 + \epsilon O_1.
\]

However the term \( -(T^{\mu\lambda} - T_0^{\mu\lambda}) \delta \beta_\lambda - \delta R^\mu \) cancel out leaving

\[S^\mu = su^\mu + \beta(u) h^\mu(u) - \Theta(u) n^\mu(u) - Q^\mu(u)
\]

\[= \hat{s} \hat{u}^\mu + \beta(\hat{u}) h^\mu(\hat{u}) - \Theta(\hat{u}) n^\mu(\hat{u}) - Q^\mu(\hat{u}) + \epsilon O_2 + \epsilon O_1.
\]
which establishes the $O_1$ invariance of $S^\mu$ under frame change described by (105). This property of $S^\mu$ implies that the resulting phenomenological equations are equivalent under the frame changes described in (106).

To get insights into the implications of this invariance property of $S^\mu$ shown in (112), let us evaluate the $Q^\mu(u)$ in the energy frame. Since relative to this frame

$$h^\mu(u_E) = 0, \quad q^\mu(u_E) = -\frac{\rho + P}{n}n^\mu(u_E)$$

using (104) and (105), one finds

$$Q^\mu(u_E) = \frac{1}{2}u_E^\mu \left[ \beta_0 \pi^2 + \beta_1 n_{E,\nu} n_{E,\nu} + \beta_2 \pi^\mu \pi_{\nu} \right] - \alpha_0 \pi n^\mu_E - \alpha_1 \pi^\mu \pi_{\nu} n_{E,\nu}$$

(113)

where we arrived in this expression by normalizing the $\beta_1$, $\alpha_0$ and $\alpha_1$ coefficient by absorbing terms like $(\rho + p)n^{-1}$ into their definition. If on the other hand, we evaluate the same term relative to the particle frame where

$$q^\mu = n^\mu_N, \quad n^\nu_N = 0$$

and one introduce new coefficients $\bar{\beta}_i$, $j \in (0, 1)$, $\bar{\beta}_i$, $i \in (0, 1, 2)$, one finds

$$Q^\mu(u_N) = \frac{u^\mu_N}{2} \left[ \bar{\beta}_0 \pi^2 + \left( \bar{\beta}_1 + \frac{1}{T(\rho + P)} \right) h^\mu h^\nu + \bar{\beta}_2 \pi^\mu \pi_{\nu} + \bar{\beta}_0 \pi^\mu h^\nu - \bar{\beta}_1 \pi_{\nu} h^\mu \right] - \left( \bar{\alpha}_0 - \frac{1}{T(\rho + P)} \right) \pi n^\mu_N - \left( \bar{\alpha}_1 - \frac{1}{T(\rho + P)} \right) \pi^\mu \pi_{\nu} h^\nu$$

(114)

A comparison between (113) and (114) show identical structures provided one defines

$$\beta_0 = \bar{\beta}_0, \quad \beta_1 = \bar{\beta}_1 + \frac{1}{T(\rho + P)} \beta_2 = \bar{\beta}_2, \quad \alpha_0 = \bar{\alpha}_0 - \frac{1}{T(\rho + P)} \alpha_1 = \bar{\alpha}_1 - \frac{1}{T(\rho + P)}$$

(115)

and these relations have been also derived in Israel [21] and Israel and Stewart [22] using different means that the ones employed in the present work. If formally one introduce $q^\mu = n^\mu_E$ and $q^\mu = h^\mu_N$ as fields variables in (113) respectively and use (112) to evaluate the phenomenological equations in view of (115), one obtains identical forms provided terms of $\epsilon O_1, O_2$ have been neglected, showing the $O_1$ equivalence of the phenomenological equations relative to the energy or particle frame. For arbitrary frame changes generated by ($u^\mu, \bar{u}^\mu$) in (106), the equivalence of the phenomenological equations can also established but the situation is more complex. For this case, the number of thermodynamical variables increase and either one can treat ($q^\mu(u), h^\mu(u)$) as independent variables or treat $h^\mu(u)$ and the particle drift $n^\nu(u)$ as independent variables. In either case (112) shows their equivalence.

We shall leave this section by offering a few comments regarding the causality and stability properties of the Israel-Stewart transient thermodynamics.

The classical works by Hiscock and Lindblom in [57], [37] and Olson in [32], established that linear perturbations about a spatially homogeneous equilibrium state are bounded and propagate causally as long as suitable restrictions upon the parameters and the equations of state are imposed. This conclusion is in sharp contrast to the causality and stability behavior of the Hiscock-Lindblom class of first order theories as we have already discussed in sec. (IV). Gavassino, Antonelli and Haskell in [10] convincingly argue that this difference should be traced in thermodynamics and it is due to the different structure that the entropy current $S^\mu$ has in the two theories (compare $S^\mu$ in (11) to that of Israel-Stewart theory shown in (103). It was shown in [10] that the total entropy based on (111) has no upper bound, while the inclusion of the quadratic contributions in (103) for the Israel-Stewart theory implies that the total entropy has an absolute maximum presumably representing the equilibrium state. Finally in a recent work, Gavassino, Antonelli and Haskell in [31] showed for the Israel-Stewart transient thermodynamics, perturbations

\[\text{References}\]
VI. ON STATES SATISFYING THE (LTE) POSTULATE AND THE LIU-MÜLLER-RUGGERI THEORY

We finish this paper by turning attention to fluid states compatible with the relativistic (LTE) postulate and their relation to Liu-Müller-Ruggeri theory (for an introduction to this theory see [13, 38]) and the theory of relativistic fluids of divergence type developed by Pennisi [39], and independently by Geroch and Lindblom in [40]. Of these two closely related theories, we shall consider only the relation between states compatible with the relativistic (LTE) postulate and the Liu-Müller-Ruggeri theory since their relation to the theory of fluids of divergence type requires to introduce the generating function and the field of Lagrange multipliers and this needs a more elaborate treatment that will be presented elsewhere [70].

We recall that within the Liu-Müller-Ruggeri theory [13, 38], arbitrary fluid states are described by the 10 components of the symmetric energy momentum tensor \( T^{\mu \nu} \) and the 4 components of the particle current \( J^{\mu} \) satisfying:

\[
\nabla_{\mu} T^{\mu \nu} = \nabla_{\mu} J^{\mu} = 0,
\]

\[
\nabla_{\mu} A^{\mu \nu \lambda} = I^{\nu \lambda},
\]

where \( A^{\mu \nu \lambda} \) is a completely symmetric tensor field subject to \( A^{\mu \nu} = m^2 c^2 J^{\mu} \) and thus necessarily \( \nabla_{\mu} A^{\mu \nu} = 0 \), while \( I^{\mu \nu} \) is symmetric and traceless \( I^{\mu \mu} = 0 \). The fields \( A^{\mu \nu \lambda} \) and \( I^{\mu \nu} \) are considered to be constitutive relations i.e. \( A^{\mu \nu \lambda} = A^{\mu \nu \lambda}(J^{\mu}, T^{\mu \nu}) \), \( I^{\mu \nu} = I^{\mu \nu}(J^{\mu}, T^{\mu \nu}) \) and this is a special feature of the theory. In addition to the balance laws \([116, 117]\), the theory employs an entropy flux vector \( S^{\mu} \) which is also a constitutive function i.e. \( S^{\mu} = S^{\mu}(J^{\mu}, T^{\mu \nu}) \) and for any solution \((J^{\mu}, T^{\mu \nu})\) of \([116, 117]\) obeys:

\[
\nabla_{\mu} S^{\mu}(J^{\mu}, T^{\mu \nu}) \geq 0.
\]

A major issue in this theory is the specification of the constitutive functions \( A^{\mu \nu \lambda}(J^{\mu}, T^{\mu \nu}) \), \( I^{\mu \nu} = I^{\mu \nu}(J^{\mu}, T^{\mu \nu}) \) and \( S^{\mu} = S^{\mu}(J^{\mu}, T^{\mu \nu}) \) as well as the implementation of the entropy inequality in \([118]\). Liu, Müller and Ruggeri in ref. [38] (see also ref. [13]) were able to construct representations of \( A^{\mu \nu \lambda}(J^{\mu}, T^{\mu \nu}), I^{\mu \nu} = I^{\mu \nu}(J^{\mu}, T^{\mu \nu}) \) and \( S^{\mu} = S^{\mu}(J^{\mu}, T^{\mu \nu}) \) by invoking three principles: the Entropy Principle, the Principle of Relativity, and the requirement of the Hyperbolic nature of the dynamical equations. Subsequently, by employing the Eckart frame\(^{37}\) they worked out predictions of their theory and compared its predictions of those of transient theory (for details of this comparison see refs. [13, 38]).

The purpose in this section is to discuss states compatible with the relativistic (LTE) postulate within this theory. For this, let a state satisfying the (LTE) postulate i.e. it is described by \((J^{\mu}, T^{\mu \nu})\) obeying \([116, 117]\) subject to the restriction that the pseudo-angle \( \epsilon \) between \((u^E_\mu, u^L_\mu)\) satisfies everywhere within the fluid region the condition \( \epsilon << 1 \). Let \( u^a \) be a velocity field chosen in the region occupied by the fluid and lying within the cone formed by this \( \epsilon \), so that local thermodynamical equilibrium prevails. If \( s(n, \rho) \) denotes the corresponding equilibrium equation

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\(^{35}\) Our thanks to an anonymous referee who suggested to probe this interconnection between the two classes of theories.

\(^{36}\) The structure of eqs. \([116, 117]\), the trace free property of \( I^{\mu \nu} \) and the relation \( A^{\mu \nu} = m^2 c^2 J^{\mu} \) are motivated by relativistic kinetic theory of a simple gas. For an introduction to this theory see refs. [42, 68].

\(^{37}\) The analysis in refs. [38, 13] has been performed relative to the Eckart frame. In principle however, one could have chosen an arbitrary velocity field \( u^a \) and decompose \( J^{\mu} \) and \( T^{\mu \nu} \) relative to this new \( u^a \) and carry out the analysis relative to this new frame. We are not aware of any related work on this issue.
of state, then \((u^\mu, s(\rho, n))\) introduce the "local thermodynamical equilibrium state" specified by \((S^\mu_0, T^\mu_\nu, J^\mu_0)\) (see discussion leading to (16)). This \(u^\mu\) and the decompositions in (10,11) define the fields

\[ (n(u), \rho(u), P(u), h(u)^\mu, n(u)^\mu, \tau(u)^{\mu\nu}) \]  

and these fields describe a state within the Liu-Müller-Ruggeri theory, provided they satisfy (116, 117). However, in order to impose these equations, we need a representation of \(A^{\mu\nu\lambda}(J^\mu, T^{\mu\nu})\), \(I^{\mu\nu} = I^{\mu\nu}(J^\mu, T^{\mu\nu})\) and their specification is a delicate problem. Since in this work, we are interested only in the behavior of small deviations away from the local "thermodynamical equilibrium state" specified by \((u^\mu, s(\rho, n))\) (or equivalently by \((S^\mu_0, T^\mu_\nu, J^\mu_0)\)), we set

\[ T^{\mu\nu} = T^\mu_0 + \delta T^{\mu\nu}, \quad J^\mu = J^\mu_0 + \delta J^\mu. \]  

and take advantage of the constitutive nature of \(A^{\mu\nu\lambda}(J^\mu, T^{\mu\nu})\), \(I^{\mu\nu} = I^{\mu\nu}(J^\mu, T^{\mu\nu})\). This later property allows us to write:

\begin{align}
A^{\mu\nu\lambda}(T^{\mu\nu}, J^\mu) &= A^{\mu\nu\lambda}(T^\mu_0 + \delta T^{\mu\nu}, J^\mu_0 + \delta J^\mu) \\
&= A^{\mu\nu\lambda}(T^\mu_0, J^\mu_0) + \frac{\delta A^{\mu\nu\lambda}}{\delta T^{\mu\nu}} \delta T^{\alpha\beta} \\
&\quad + \frac{\delta A^{\mu\nu\lambda}}{\delta J^{\alpha}} \delta J^{\alpha} + O(\delta J^{\mu})^2 + O(\delta T^{\mu\nu})^2,
\end{align}

\[ I^{\nu\lambda}(T^{\mu\nu}, J^\mu) = I^{\nu\lambda}(T^\mu_0 + \delta T^{\mu\nu}, J^\mu_0 + \delta J^\mu) \\
= I^{\nu\lambda}(T^\mu_0, J^\mu_0) + \frac{\delta I^{\nu\lambda}}{\delta T^{\mu\nu}} \delta T^{\alpha\beta} \\
&\quad + \frac{\delta I^{\nu\lambda}}{\delta J^{\alpha}} \delta J^{\alpha} + O(\delta J^{\mu})^2 + O(\delta T^{\mu\nu})^2,
\]

where in these expansions we kept terms describing only first order deviations from the state of the local equilibrium field.

The symmetries of \(A^{\mu\nu\lambda}\) and the condition \(A^{\mu\nu\lambda} = m^2 c^2 J^\mu\) allows us to propose a representation of \(A^{\mu\nu\lambda}(T^\mu_0, J^\mu_0) := A^{\mu\nu\lambda}_0(u)\) involving only terms that appear in the representation of the local equilibrium field \((S^\mu_0, T^\mu_\nu, J^\mu_0)\). These symmetries and the trace condition dictate that \(A^{\mu\nu\lambda}_0(u)\) should be of the form

\[ A^{\mu\nu\lambda}_0(u) = 2\gamma_1 u^{(\mu} u^{\nu)} u^{\lambda)} + (nm^2 + \gamma_1) g^{(\nu\lambda} u^{\mu)} , \]  

where \(\gamma_1\) is a function of \((n, \rho)\) to be determined and \(m\) is a mass scale. Similarly the symmetry requirements upon \(I^{\mu\nu}\) permit us to propose that the leading contributions of the \(I^{\nu\lambda}(T^{\mu\nu}, J^\mu)\) to be vanishing i.e. \(I^{\nu\lambda}(T^\mu_0, J^\mu_0) := 0\) and thus to set the expansion of \(I^{\nu\lambda}\) in (124) to have the form

\begin{align}
I^{\nu\lambda}(T^{\mu\nu}, J^\mu) &= \frac{\delta I^{\nu\lambda}}{\delta T^{\mu\nu}} \delta T^{\alpha\beta} + \frac{\delta I^{\nu\lambda}}{\delta J^{\alpha}} \delta J^{\alpha} \\
&= \delta_1 \pi (q^{\mu\nu} + 4u^{\mu} u^{\nu}) + \delta_2 \pi^{\mu\nu} \\
&\quad + \delta_3 (q^{\mu\nu} u^\nu + q^{\nu} u^\mu),
\end{align}

where \(q^{\mu}\) is the invariant heat flux defined in (55) and the coefficients \(\delta_i, i \in (1, 2, 3)\) are unknown functions of \((n, \rho)\). With these choices, the fields in (119) satisfy

\[ \nabla_{\mu} T^{\mu\nu} = \nabla_{\mu} [\rho(u) u^{\mu} u^{\nu} + P(u) \Delta(u) u^{\mu} u^{\nu} + h(u)^{\mu} u^{\nu}] + h(u)^{\nu} u^\mu + \tau(u)^{\mu\nu} = 0, \]  

---

38 For this section indices enclosed in a parenthesis indicate symmetrization, thus \(u^{(\mu} u^{\nu)} u^{\lambda)} = \frac{1}{2} (u^{\mu} u^{\nu} u^{\lambda} + u^{\nu} u^{\lambda} u^{\mu} + u^{\lambda} u^{\mu} u^{\nu}).\)

39 The right hand-side of (123) as well as of (124) are indeed functions of \(T^{\mu\nu}\) and \(J^\mu\). One can see that by expressing the fields in (119) in terms of \(T^{\mu\nu}\) and \(J^\mu\).
\[ \nabla_{\mu}J^\mu = \nabla_{\mu}[n(u)u^\mu + n(u)^\mu] = 0 \]  
(126)

\[ \nabla_{\mu}A_{0}^{\mu\nu\lambda}(u) = \delta_1 \pi(g^{\mu\nu} + 4u^\nu u^\mu) + \delta_2 \pi^{\mu\nu} + \delta_3 (q^\mu u^\nu + q^\nu u^\mu). \]  
(127)

We leave aside for the moment the implementation of the entropy law in (118) since as we have already mentioned, its implementation requires to introduce the generating function, the field of Lagrange multipliers and apply of Liu’s procedure [69] for the implementation of of the second law. Since these issue need altogether a different machinery they will be discussed in a separate account in connection to fluids of divergence type [70]. For the rest of this section we shall treat (125-127) as the equations that fluid states compatible to the relativistic (LTE) postulate are required to satisfy.

Let us suppose now that the system (125-127) admits a solution \((u^\mu, n(u), \rho(u), P(u), h(u)^\mu, n(u)^\mu, \tau(u)^{\mu\nu})\) and let us consider a frame change described by

\[ u^\mu \rightarrow \hat{u}^\mu = u^\mu + \epsilon^\mu, \quad \epsilon \ll \epsilon, \]  
(128)

where \(\hat{u}^\mu\) has been chosen to lie within the cone of the opening pseudo-angle \(\epsilon \ll 1\) and relative to this new frame, we define fields \((\rho(\hat{u}), n(\hat{u}), P(\hat{u}), h(\hat{u})^\mu, \tau(\hat{u})^{\mu\nu}, n(\hat{u})^\mu)\) by applying the same procedure as for the case of first order theory and the case of transient theory. It is easily seen that the transformed fields \((\hat{u}^\mu, \rho(\hat{u}), n(\hat{u}), P(\hat{u}), h(\hat{u})^\mu, \tau(\hat{u})^{\mu\nu}, n(\hat{u})^\mu)\) satisfy to an \(O_1\) accuracy the conservation laws in (119). However matters complicate in proving that the transformed fields satisfy to an \(O_1\) accuracy equation (125-127). To facilitate matters, we note that (127) imply

\[ u_{\nu\alpha} \nabla_{\mu}A_{0}^{\mu\nu\alpha}(u) = 3\delta_1 \pi(u), \]  
(129)

\[ \Delta_{\rho\nu}(u)u_{\lambda} \nabla_{\mu}A_{0}^{\mu\nu\lambda}(u) = 3\delta q_\rho(u), \]  
(130)

\[ \left( \Delta_{\alpha\nu}(u)\Delta_{\beta\lambda}(u) - \frac{1}{3}\Delta_{\alpha\beta}(u)\Delta_{\nu\lambda}(u) \right) \nabla_{\mu}A_{0}^{\mu\nu\lambda}(u) = 2\delta_2 \pi_{\alpha\beta}(u), \]  
(131)

and the idea is to show that these equations remain form \(O_1\) invariant under a change of the rest frame described by (128). Based on the representation of \(A_{0}^{\mu\nu\lambda}(u)\) in (123), then a calculus shows that that under the frame change in (128) the following formula holds:

\[ \nabla_{\mu}A_{0}^{\mu\nu\lambda}(u) = \nabla_{\mu}A_{0}^{\mu\nu\lambda}(\hat{u}) - \nabla_{\mu}[2\gamma_1 \epsilon^{(\mu\nu\lambda)}\hat{u}^\lambda] - (nm^2 + \gamma_1)g^{(\nu\lambda)}\hat{u}^\mu + \epsilon O_1]. \]  
(132)

Using this formula, it is straightforward to verify that (129) (130) (131) remains form invariant to \(O_1\) deviation from the state of local equilibrium taking into account that \(\nabla_{\mu}u^\mu = O(1), \nabla_{\mu}e^\mu = O(1)\) and always \(u^\mu e_\mu = 0\). In summary within the class of the Liu-Müller-Ruggeri theory, states satisfying the relativistic (LTE) postulate satisfy eqs that remain invariant under a frame change described in (131) and thus exhibit the same behavior as states near equilibrium for the first order theories and the transient thermodynamics. We shall discuss further properties of states satisfying the relativistic (LTE) postulate in light of some recent advances on the theory of divergence type fluids reported lately in refs. [31] (see also [29]).

VII. DISCUSSION

In this article, we have introduced a special class of fluid states satisfying (or been compatible to) the relativistic (LTE) postulate. Our motivation to introduce this class has been partially induced by the role of the (LTE) postulate within the Onsager-Eckart theory and largely by Israel’s ideas on the description of relativistic fluid states that are considered to be near equilibrium within the transient thermodynamics.

We started the article by pointing out the implications of the (LTE) postulate within the Onsager-Eckart theory and stressed that states compatible with the (LTE) postulate are special. Firstly they permit to introduce local
thermodynamical variables and secondly their physical entropy is determined by these local variables\textsuperscript{40}. The extension of the classical (LTE) postulate to the relativistic regime proposed in this paper, define fluid states compatible with the relativistic (LTE) postulate as states subject to two restrictions. The first one requires that the state should allow the attachment of a "local thermodynamical equilibrium" state and this attachment requires that the four velocity $u^\mu$ ought to be chosen so that this $u$-observer, relative to her/his local rest frame, detects a collision time scale which is much shorter than any other macroscopically defined time scale. Under this condition, an "equilibrium equation of state" of the form $s = s(\rho, n)$ may be postulated to exist and thus "thermal" aspects such as local temperature and local chemical or thermal potential appear\textsuperscript{41}. The second restriction requires the state to be a state near equilibrium in Israel's sense, meaning that everywhere within the region occupied by the fluid, the "angle $\epsilon$" formed by the four velocity that defines the energy frame and the four velocity that defines the particle frame, satisfies $\epsilon << 1$. Whenever any $u^\mu$ lies within the cone of opening angle $\epsilon$, then the appearance of the class of admissible frames and the treatment of deviations of the physical state, from the state of "local thermodynamical equilibrium" follows the root that we presented in sections (III, VI).

Parenthetically we add, that in order that a fluid state be compatible with the relativistic (LTE) postulate, it is necessary that it satisfies simultaneously both of the above mentioned restrictions. One can imagine fluid states where either the attachment of a "local thermodynamical equilibrium" it is not possible or the state it is not a state near equilibrium in Israel's sense. In other case, one does not get the benefits arising from the existence of admissible frames.

Although in this paper, we examined properties of states satisfying the relativistic (LTE) postulate within the context of the Hiscock-Lindblom class of first order theories, transient thermodynamics, the (BDNK) theory, and the Liu-Müller-Ruggeri theory, it would be of interest to include in this list, the class of relativistic fluids of divergence type in view on some recent advances in this theory reported by Gavassino, Antonelli and Haskell in refs \textsuperscript{31}, \textsuperscript{29}. Overall the analysis in this paper shows that that within the Hiscock-Lindblom class of first order theories, states satisfying the relativistic (LTE)-postulate they do not seem to offer any advantage. Like arbitrary states within this theory, they violate causality and equilibrium states are unstable. However, maters are different within the (BDNK) theory and transient thermodynamics. As far as the latter theory is concerned, we have argued in section \textit{V}, that the theory deals exclusively with states obeying the relativistic (LTE)-postulate and in reaching this conclusion, refs \textsuperscript{64}, \textsuperscript{29}, \textsuperscript{31} are very relevant. As far as the former theory is concerned, in section (IV), we presented evidences suggesting that states obeying the relativistic (LTE)-postulate are states referred as states near equilibrium within the (BDNK) formalism. In fact we believe, that all states within the (BDNK) theory where the gradient series expansion is meaningful, are in fact states compatible to the relativistic (LTE)-postulate, although at this point this is a conjectural claim. In that regard, it would be of interest, starting from the relativistic Boltzmann equation and via suitable expansion investigate whether one may generate fluid states where the pseudo-angle $\epsilon$ formed by $u_E^\mu$ and $u^\mu_s$ satisfies everywhere the condition $\epsilon << 1$. So far the relativistic Boltzmann equation has offered considerable insights in the derivation of relativistic hydrodynamic equation (see for instance \textsuperscript{22}, \textsuperscript{48}, \textsuperscript{72} and references therein) and may be further efforts one gets insights into the nature of the geometry of possible rest frames may be helpful. Currently we have that issue under active investigation.

We finish this paper by mentioning that in the literature there have been advanced alternative definitions of states obeying the relativistic (LTE) postulate and these definitions either originate from the quantum statistical framework (see for instance \textsuperscript{71} and reference therein) It would be of some interest to compare these definitions and their predictions to the one proposed in this work.

\section*{VIII. ACKNOWLEDGMENTS}

Our warm thanks to the members of the relativity group at the IFM Univ. Michoacana for multiple discussions. Special thanks to O. Sarbach whose continuing poking lead to the development of this work. The research of T.Z was supported by a Grant from CIC-UMSNH. During the completion of this work J.F.S was funded by CONACYT by a

\textsuperscript{40}Because of these properties, often in the literature and even within the context of relativistic fluids, one encounter the term "theories based on the local equilibrium hypothesis". With this term one understands theories whose independent variables are those that determine equilibrium states. Clearly this definition is not wide enough, for instance transient thermodynamics, incorporates fluxes as independent variables and these variables do not enter in the description of equilibrium states.

\textsuperscript{41}Notice, that even though any observer with an arbitrary four velocity $u^\mu$, using the state variables ($J^\mu$, $T^{\mu\nu}$) can always define along his/her world line particle density $n$ energy density $\rho$ energy flux $h^\mu$ y stresses $\tau^{\mu\nu}$, still "thermal" aspects do not appear unless an "equilibrium equation of state" $s = s(\rho, n)$ is postulated to exist.
post doctoral fellowship through the project A1 -S-31269.

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