ON THE RIGIDITY OF SOLITARY WAVES FOR THE FOCUSING MASS-CRITICAL NLS IN DIMENSIONS $d \geq 2$

DONG LI AND XIAOYI ZHANG

Abstract. For the focusing mass-critical NLS $iu_t + \Delta u = -|u|^4 u$, it is conjectured that the only global non-scattering solution with ground state mass must be a solitary wave up to symmetries of the equation. In this paper, we settle the conjecture for $H^1$ initial data in dimensions $d = 2, 3$ with spherical symmetry and $d \geq 4$ with certain splitting-spherically symmetric initial data.

CONTENTS

1. Introduction 2
1.1. Background and main results 2
1.2. Outline of the proof 8

Acknowledgements 13
2. Preliminaries 13
2.1. Some notation 13
2.2. Basic harmonic analysis 13
2.3. Strichartz estimates 15
2.4. The in-out decomposition 16
3. A non-sharp decomposition 18
4. Reduction of the proof 28
5. Proof of Proposition 4.1 Proposition 4.2 in 2,3 dimensions 32
5.1. Proof of Proposition 4.1 in 2,3 dimensions 32
5.2. Proof of Proposition 4.2 in 2,3 dimensions 40
6. 2+2 dimensions with splitting-spherical symmetry 43
6.1. Introduction and tools adapted to four dimensions 43
6.2. Proof of Proposition 4.1 46
6.3. Proof of Proposition 4.2 52
7. Higher dimensional case with admissable symmetry 58

References 65

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1. Introduction

1.1. Background and main results. We consider the focusing mass-critical nonlinear Schrödinger equation

$$iu_t + \Delta u = -|u|^4u$$

(1.1)

in dimensions $d \geq 2$; here $u(t, x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^d$. The equation is invariant under a number of symmetries,

$$u(t, x) \mapsto u(t + t_0, x + x_0), \; t_0 \in \mathbb{R}, \; x_0 \in \mathbb{R}^d,$$

(1.2)

$$u(t, x) \mapsto e^{i\theta_0}u(t, x), \; \theta_0 \in \mathbb{R},$$

(1.3)

$$u(t, x) \mapsto \lambda_0^\frac{4}{d}u(\lambda_0^2t, \lambda_0x), \; \lambda_0 > 0,$$

(1.4)

$$u(t, x) \mapsto e^{\xi_0 \cdot (x - \xi_0t)}u(t, x - 2\xi_0t), \; \xi_0 \in \mathbb{R}^d.$$ 

(1.5)

From Ehrenfest’s law, they lead to the following conserved quantities:

- **Mass:** $M(u(t)) = \int_{\mathbb{R}^d} |u(t, x)|^2 dx = M(u_0),$

- **Energy:** $E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - \frac{d}{2(d+2)}|u(t, x)|^{2(d+2)} dx$

(1.6)

$$= E(u_0),$$

- **Momentum:** $P(u(t)) = Im \int_{\mathbb{R}^d} \nabla u\bar{u}(t, x) dx = P(u_0)$

The equation is called mass-critical since mass is invariant under the scaling symmetry (1.4). It is also critical in the sense that the power of the nonlinearity is the smallest to admit finite time blowup solutions.

Equation (1.1) also preserves some other symmetries in space due to the fact the Laplacian is an isotropic operator which is invariant under orthogonal change of coordinates. For example, the spherical symmetry is preserved under the NLS flow. The Cauchy problem of (1.1) for large spherically symmetric $L^2_x(\mathbb{R}^d)$ initial data has been intensively studied in recent works [16], [17]. The advantage of using the spherical symmetry ultimately stems from the fact that the solution has to localize at the spatial origin $x = 0$ and frequency origin $\xi = 0$, and also has strong decay as $|x| \to \infty$.

A natural generalization of the radial symmetry is ”splitting-spherical symmetry” which is also preserved under the flow. To set the stage for later discussions, we now introduce:

**Definition 1.1 (Splitting-spherical symmetry).** Let $d \geq 2$. A function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is said to be splitting-spherically symmetric if there exists $k \geq 1$ and $d_1, \ldots, d_k$ with $d_i \geq 2$, $\sum_{i=1}^k d_i = d$ such that $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$ and $f$ is spherically symmetric when restricted to each of the $\mathbb{R}^{d_i}$ subspaces. To ensure the uniqueness, we require the fold number $k$ is minimal.
Clearly when $k = 1$, splitting-spherical symmetry coincides with the usual notion of spherical symmetry. In this work, we will study the Cauchy problem of (1.1) in dimensions $d \geq 2$ with general splitting-spherically symmetric initial data. We first make the notion of a solution to this Cauchy problem more precise:

**Definition 1.2** (Solution). A function $u : I \times \mathbb{R}^{d} \rightarrow \mathbb{C}$ on a non-empty time interval $I \subset \mathbb{R}$ (possibly infinite or semi-infinite) is a strong $L^{2}_{t, x}(\mathbb{R}^{d})$ solution (or solution for short) to (1.1) if it lies in the class $C^0_t L^{2}_{t, x}(K \times \mathbb{R}^{d}) \cap L^{2(d+2)/d}_{t, x}(K \times \mathbb{R}^{d})$ for all compact $K \subset I$ and obeys the Duhamel formula

$$u(t_1) = e^{i(t_1-t_0)\Delta} u(t_0) + i \int_{t_0}^{t_1} e^{i(t_1-t)\Delta} (|u|^4 u)(t) \, dt \tag{1.7}$$

for all $t_0, t_1 \in I$. We refer to the interval $I$ as the lifespan of $u$. We say that $u$ is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. We say that $u$ is a global solution if $I = \mathbb{R}$.

The condition that $u$ belongs to $L^{2(d+2)/d}_{t, x}$ locally in time is natural for several reasons. From the Strichartz estimate (see Lemma 2.7), we see that solutions to the linear equation lie in this space. Moreover, the existence of solutions that belong to this space is guaranteed by the local theory (see Theorem 1.3 below). This condition is also necessary in order to ensure uniqueness of solutions. Solutions to (1.1) in this class have been intensively studied; see, for example, [2, 6, 11, 14, 16, 17, 20, 21, 29, 33, 36] and the many references therein.

The local theory for the Cauchy problem of (1.1) in the critical $L^2_{t, x}(\mathbb{R}^{d})$ space is established by Cazenave and Weissler in [6]. We record their results in the following

**Theorem 1.3** (Local wellposedess, [6, 7]). Given $u_0 \in L^2_{x}(\mathbb{R}^{d})$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution $u$ to (1.1) with $u(t_0) = u_0$. Let $I$ denote the maximal lifespan. Then,

- (Local existence) $I$ is an open neighborhood of $t_0$.
- (Mass conservation) The solution obeys $M(u(t)) = M(u_0)$.
- (Blowup criterion) If $\sup I$ or $\inf I$ are finite, then
  $$\|u\|_{L^{2(d+2)/d}_{t, x}([t, \sup I] \times \mathbb{R}^{d})} = \infty; \quad \text{or} \quad \|u\|_{L^{2(d+2)/d}_{t, x}([\inf I, t] \times \mathbb{R}^{d})} = \infty; \quad t \in I.$$
- (Continuous dependence) The map that takes initial data to the corresponding strong solution is uniformly continuous on compact time intervals for bounded sets of initial data.
- (Scattering) If $\sup I = \infty$ and $u$ has finite spacetime norm forward in time: $\|u\|_{L^{2(d+2)/d}_{t, x}([t, \infty))} < \infty$, then $u$ scatters in that direction, that is, there exists a unique $u_+ \in L^2_{x}(\mathbb{R}^{d})$ such that
  $$\lim_{t \to \infty} \|u(t) - e^{it\Delta} u_+\|_2 = 0. \tag{1.8}$$
Conversely, given $u_+ \in L^2_x(\mathbb{R}^d)$ there is a unique solution to (1.1) in a neighborhood of infinity so that (1.8) holds. Analogous statements hold in the negative time direction.

• (Small data global existence and scatter) If $M(u_0)$ is sufficiently small depending on the dimension $d$, then $u$ is a global solution with finite $L^{2(d+2)/d}_{t,x}$-norm.

By Theorem 1.3 all solutions with sufficiently small mass are global and scatter both forward and backward in time. In that regime, the dispersion effect of the free evolution dominates the focusing nonlinearity. However for solutions with large mass, there is competition between the two and solutions may display different behaviors: they can exist globally and scatter, or blow up at finite time, or persist like a solitary wave, or even be the superposition of them [31, 30]. For this mass-critical problem (1.1), there exists soliton solutions of the type $e^{it}R(x)$, where $R(x)$ solves the elliptic equation
\[
\Delta R - R + |R|^{4d} = 0.
\]
This equation has infinitely many solutions, but only one positive, spherically symmetric solution which is Schwartz and has minimal mass within the set of solutions. This unique solution $Q(x)$ is known as the "ground state" of (1.1). It is widely believed that the mass of $Q$ serves as the borderline between the scattering solutions and possible non-scattering solutions. If the mass of the initial data is less than that of the ground state, then it is conjectured that the corresponding solution exists globally and scatters. A more precise statement is the following

**Conjecture 1.4** (Scattering conjecture). Let $u_0 \in L^2_x(\mathbb{R}^d)$ and $M(u_0) < M(Q)$. Then there exists unique global solution $u(t, x)$ such that
\[
\|u\|_{L^{2(d+2)/d}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \leq C(M(u)) < \infty.
\]

By this conjecture, non-scattering solutions must have at least ground state mass. Two examples of the non-scattering solutions are generated by the ground state. One is the solitary wave solution $e^{it}Q$ which exists globally but does not scatter on both sides. Applying pseudo-conformal transformation:
\[
u(t, x) \rightarrow \frac{1}{|t|^{\frac{d}{2}}} e^{\frac{(|x|^2}{2d}} u(\frac{1}{t}, \frac{x}{t}),
\]
which is invariant for the mass-critical NLS, one obtains a finite time blowup solution
\[
\frac{1}{|t|^{\frac{d}{2}}} e^{\frac{(|x|^2}{2d}} - 4} Q(\frac{x}{t}).
\]

1By "non-scattering" we mean the $L^{2(d+2)}_{t,x}$-norm of the solution is infinite, so the solution can blows up at finite time or exist globally but does not scatter.
These two examples are believed to be the only two obstructions to scattering when the solution has ground state mass. In particular, solitary wave is conjectured to be the only global non-scattering solution. We formulate this as the following

**Conjecture 1.5** (Solitary wave conjecture). Let $d \geq 1$. For general initial data $u_0 \in L^2_2(\mathbb{R}^d)$ with ground state mass, the corresponding non-scattering global solution must be the solitary wave up to symmetries (1.2)-(1.5).

Our purpose of the paper is to settle the solitary wave conjecture under additional assumptions on the initial data. More precisely, we will establish the conjecture for $H^1_2$ initial data in 2,3 dimensions with spherical symmetry and in dimensions $d \geq 4$ with some splitting-spherical symmetry. For simplicity, we name these ”certain symmetries” as the following

**Definition 1.6** (Admissible symmetry). In dimensions $d = 2, 3$, the admissible symmetry refers to the spherical symmetry; in dimensions $d \geq 4$, it refers to the splitting-spherical symmetry with $k = 2$, and $d_1 = [\frac{d}{2}]$ or $d_1 > \frac{d}{2}$ for sufficiently large $d$. Here $[x]$ denotes the integer part of a real number $x$.

Then our main result reads as follows.

**Theorem 1.7** (Non-scattering solutions must coincide with the solitary wave). Let $d \geq 2$. Let $u_0 \in H^1_2(\mathbb{R}^d)$ and have the admissible symmetry. Suppose also the corresponding solution exist globally, then only the following two scenarios can occur

1. The solution scatters in both time direction, i.e., $\|u\|_{L^2_tL^{2(2d+2)}_x(\mathbb{R} \times \mathbb{R}^d)} < \infty$.
2. The solution is spherically symmetric and there exist $\theta_0, \lambda_0$ such that
   \[ u(t, x) = e^{i\theta_0} \lambda_0^{\frac{d}{2}} Q(\frac{x}{\lambda_0}). \]

**Remark 1.8.** For the splitting-spherical symmetric initial data, Theorem 1.7 holds under the conditional assumption that the corresponding scattering conjecture holds for such $L^2$ initial data. In the spherical symmetric case for dimensions $d \geq 2$, the scattering conjecture has been proved in recent works [16] [17]. In a future publication we will address the scattering problem for the splitting-spherical symmetric case for dimensions $d \geq 4$.

In the following, we will first discuss the connections between this result and previous ones. Then at the end of this section, we introduce the main steps of the proof.

For this mass critical problem, there has been lots of work addressing the wellposedness theory of the solutions. We will mainly discuss the results related to the aforementioned two conjectures, with a little extension on the blowup theory. In the following discussions, we distinguish three different cases when the solution has subcritical, critical and supercritical mass respectively.
Case 1. The solution has subcritical mass $M(u) < M(Q)$.

As shown in the scattering conjecture [1.4] in this regime, the dispersion of the linear flow dominates and solutions are all believed to exist globally and scatter. The first result toward this conjecture is due to Weinstein. In [33], he established the following variational characterization of the ground state which says that the ground state $Q$ extremizes the Gagliardo-Nirenberg inequality.

**Proposition 1.9** (Sharp Gagliardo–Nirenberg inequality, [33]). For $f \in H^1_x(\mathbb{R}^d)$,

$$\|f\|_2^{2(d+2)/d} \leq \frac{d+2}{d} \left( \frac{\|f\|_2}{\|Q\|_2} \right)^{\frac{2}{d}} \|\nabla f\|_2^2,$$

(1.12)

with equality if and only if

$$f(x) = ce^{i\theta_0} \lambda_0^\frac{d}{4} Q(\lambda_0(x-x_0))$$

(1.13)

for some $\theta \in [0, 2\pi)$, $x_0 \in \mathbb{R}^d$, and $c, \lambda_0 \in (0, \infty)$. In particular, if $M(f) = M(Q)$, then $E(f) \geq 0$ with equality if and only (1.13) holds with $c = 1$.

As a consequence of the inequality (1.12), Weinstein [33] showed that if $u_0 \in H^1_x(\mathbb{R}^d)$ with $M(u_0) < M(Q)$, then the corresponding solution satisfies $\|\nabla u(t)\|_{L^2_x(\mathbb{R}^d)} < \text{Const} \cdot E(u(t))$ for all $t$ in the maximal lifespan, by which standard local theory in $H^1_x$-space yields the global wellposedness. Therefore, no finite time singularities can form in this subcritical regime at least for $H^1_x$ initial data. However, this result does not address the scattering issue of the solution.

A great breakthrough toward the scattering conjecture was recently made by Killip-Tao-Visan [16] where they settled the conjecture in two dimensions with spherical symmetry. This result was later extended to high dimensions $d \geq 3$ by Killip-Visan-Zhang [17]. The spherical symmetry in these results is used in an essential way. For example, with the spherical symmetry, the center of mass will freeze at the origin in both physical and frequency spaces. Moreover, spherical symmetry forces decay in space when $|x| \to \infty$. The techniques developed in these works take advantage of decay property and will break down when such a decay is not available.

Therefore, it is quite challenging to prove the scattering conjecture [1.4] without spherical symmetry. As a first step forward, we would like to understand the case when the solution has some symmetry which is enough to freeze the center of mass, and at the same time, provide some averaging effect. The splitting symmetry then fits this motivation. In a future publication, we will show that the scattering conjecture holds true in high dimensions under these weaker symmetries.

We make two remarks before completing this discussion. First of all, as we shall explain, with the additional $H^1_x$ assumption, the main part of the proof, which is also a necessity in proving Theorem 1.7, is to show
the localization of kinetic energy for the minimal non-scattering solutions. Secondly, it is possible to build the scattering conjecture with this splitting-spherical symmetry for $L^2$ initial data. We will address this issue in the future works. We turn now to

**Case 2.** The solution has super-critical mass $M(u) > M(Q)$.

In this case, the focusing nonlinearity dominates the dispersion effect of the linear evolution and finite time blowups may occur. The existence of finite time blowup solutions was first obtained by Glassey [10] for $\Sigma = \{ f \in H^1_x(\mathbb{R}^d), x f \in L^2_x(\mathbb{R}^d) \}$ initial data with negative energy. The argument is based on a simple application of the virial identity

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 8E(u_0), \quad (1.14)$$

This identity essentially expresses the conservation of pseudo-conformal energy which comes from the pseudo-conformal symmetry (1.10).

Concerning the quantitative theory of blowups, there has been a lot of work addressing the dynamics of the blowup solutions. For example, it is possible to show that any finite time blowup solution will concentrate at least ground state mass near the blowup time. Results in this direction were established by Merle-Tsutsumi [22] and Nawa [24], Weinstein [37] for the finite energy blowup solutions. For merely finite mass blowup solutions, such results were proved by Bourgain [3], Keraani [14], Killip-Tao-Visan [16] and Killip-Visan-Zhang [17].

To understand the structure of blowups, an important first step is to study the optimal blowup rate of an $H^1_x$ blowup solution. If an $H^1_x$ solution $u(t,x)$ blows up at a finite time $T$, then by scaling arguments, the kinetic energy blows up at least at a power like rate:

$$\|\nabla u(t)\|_{L^2_x(\mathbb{R}^d)} \gtrsim \frac{1}{|t-T|^\frac{d}{2}}.$$  

In dimensions $d = 1, 2$, the existence of blowup solution with rate $\frac{1}{|t-T|}$ was proved by Bourgain-Wang in [4]. Perelman in [25] constructed a solution blowing up at rate $\sqrt{\frac{\ln(\ln |t-T|)}{|t-T|}}$ in one dimension. In [23], Merle-Raphael proved that blowup rate has an upper bound $\sqrt{\frac{\ln(T-t)}{T-t}}$ for solution with negative energy, and having mass slightly bigger than that of the ground state.

**Case 3.** The solution has the ground state mass $M(u) = M(Q)$.

As shown in the scattering conjecture, $M(Q)$ is conjectured to be the minimal mass for all the non-scattering solutions. The ground state provides two non-scattering solutions at this minimal mass: the solitary wave and the pseudo-conformal transformation of the solitary wave. They are believed to be the only two thresholds for scattering at minimal mass. In particular,
as indicated in the solitary conjecture, the only non-scattering solution is conjectured to be the solitary wave up to symmetries of the equation.

The first work which addressed the description of the minimal non-scattering solutions is due to F. Merle in [20]. He proved that any finite time blowup solution with \( H^1_x \) initial data must be pseudo-conformal ground state solution (1.11) up to symmetries. The proof in [20], which was later simplified by Hmidi-Keraani[11] relies heavily on the finiteness of the blowup time. To see the connections between his work and the solitary wave conjecture, we simply use the pseudo-conformal transformation to transform a global non-scattering solution to a finite time blowup solution. Therefore, when the initial data belongs to \( \Sigma \) space(thus the transformed finite time blowup solution belongs to \( H^1_x(\mathbb{R}^d) \)), solitary wave conjecture follows directly from Merle’s results.

Without the strong decay assumption, the solitary wave conjecture remains totally open. In [15], Killip-Li-Visan-Zhang first proved this conjecture in dimension \( d \geq 4 \) for \( H^1_x(\mathbb{R}^d) \) initial data with spherical symmetry. In our main theorem [17] we establish the result for \( H^1_x(\mathbb{R}^d) \) initial data in 2,3-dimensions with spherical symmetry and in dimensions \( d \geq 4 \) with certain splitting-spherical symmetry. Combining our results with Merle’s results, we conclude that the solitary wave, the pseudo-conformal ground state and scattering are the only three possible states for solutions with minimal mass.

As we shall see, the techniques we are going to use in this paper rely on the fact that \( u_0 \in H^1_x \) and the splitting-spherical symmetry. For example, we need the regularity to define the energy, which then allows us to use virial-type argument. We also need this regularity to conduct the spectral analysis around the ground state \( Q \). Similar to the previous work [15, 16, 17], the decay property stemming from the splitting-spherical symmetry plays a very important role. It is not clear to us how to extend the result to the rough initial data and to the case without the splitting-spherical symmetry.

Another challenging problem is to consider the conjecture in one dimension, for example, with the evenness assumption. Since the one dimensional function does not have any spatial decay, this problem is indeed equally hard with the above mentioned ones.

We now introduce the

1.2. Outline of the proof. We first consider the case \( d = 2, 3 \). Let \( u_0 \in H^1_x, M(u_0) = M(Q) \) be spherically symmetric. If the corresponding solution \( u(t, x) \) exists globally and scatter, then Theorem [17] holds vacuously. Therefore we assume \( u(t, x) \) does not scatter at least in one time direction, for example, \( \|u\|^{2(2d+2)}_{L^2_{t,x}([0,\infty) \times \mathbb{R}^d)} = \infty \). Our goal is then to show that \( u_0 = Q \) up to phase rotation and scaling. The coincidence of the solution with the solitary wave follows from the uniqueness of the solution.

From the variational characterization of the ground state (see Proposition [19]), ground state \( Q \) minimizes the energy. Therefore, if the solution has
zero energy, we conclude the initial data coincides with the ground state \( Q \). This leaves us to consider the positive energy case and we will get a contradiction in this case.

The contradiction will follow from a suitable truncated version of the virial identity as we now explain. On the one hand, a simple computation shows that the truncated virial has a uniform upper bound. On the other hand, its second derivative in time will have a positive lower bound as long as we can show the kinetic energy concentrates at the origin uniformly in time. These two facts ultimately yields the desired contradiction.

Hence, our task is reduced to showing that the kinetic energy of the solution is uniformly localized in time. Suppose by contradiction that the kinetic energy is not localized, then there are always significant portion of kinetic energy ripples live on an ever large radii. This scenario is what we need to preclude, on the other hand, looks very "consistent" with the focusing nature of our problem. More precisely, due to the focusing nature, the solution may have asymptotic infinite kinetic energy, hence it is quite possible to shed them on the ever large radius.

To preclude this dangerous scenario, our first step is to understand how the total kinetic energy ripples are distributed away from the origin. If the kinetic energy of the solution is uniformly bounded, then so will be these total ripples. Then comes the interesting case where the kinetic energy goes to infinity along a subsequence. In this case, a qualitative argument shows that after rescaling, the solution converges to the ground state in \( H^1_2(\mathbb{R}^d) \). Therefore along this subsequence the solution can be decomposed into a rescaled copy of the ground state (ever concentrated) plus an error. A surprising result from our non-sharp decomposition Proposition 3.1 essentially gives a good control of the error. In all, we have the following

**Proposition 1.10 (Weak localization of kinetic energy).** Let \( u_0 \in H^1_2 \) have admissable symmetry and \( M(u_0) = M(Q) \). Let \( u(t,x) \) be the corresponding maximal-lifespan solution on \( I \). Then for all \( t \in I \), we have

\[
\| \phi_{\geq 1} \nabla u(t) \|_{L^2_2(\mathbb{R}^d)} \lesssim 1.
\]

(1.15)

A more precise form of Proposition 1.10 is given by Lemma 3.2 which is proved in Section 3. We still have to upgrade this weak localization to stronger one. That is to say, instead of being bounded, the ripples far away from the origin are actually very small. This fact is heavily used in the aforementioned virial argument.

The solution does enjoy some certain strong concentration property, however, not expressed in terms of the kinetic energy, but in terms of the mass. In the spherically symmetric case, this is direct result of [2, 13, 29] together with the identification of \( M(Q) \) as the minimal mass, which was done in [16, 17]. In the case with admissable symmetry, this follows from the resolution of the corresponding scattering conjecture.
We state the following result in the general dimensions $d \geq 2$ with admissible symmetry. In the general case without symmetry, we need to assume the scattering conjecture in order to identify $M(Q)$ as the minimal mass.

**Theorem 1.11 (Almost periodicity modulo symmetries).** Let $d \geq 2$. Let $u : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{C}$ be a solution to (1.1) which satisfies $M(u) = M(Q)$ and

$$
\|u\|_{L_{t,x}^{2(d+2)}([t_0,\infty) \times \mathbb{R}^d)} = \infty,
$$

where $u$ have the admissible symmetry. In the general case without symmetry, we also assume the scattering conjecture. Then $u$ is almost periodic modulo symmetries in the following sense: there exist functions $N : [t_0, \infty) \to \mathbb{R}^+$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$
\int_{|x-x(t)| \geq C(\eta)/N(t)} |u(t,x)|^2 dx \leq \eta \quad \text{and} \quad \int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 d\xi \leq \eta
$$

for all $t \in [t_0, \infty)$ and $\eta > 0$. Equivalently, the orbit $\{N(t)^{-\frac{d}{2}}e^{ix\cdot\xi(t)}u(t, \frac{x-x(t)}{N(t)})\}$ falls into a compact set in $L_x^2(\mathbb{R}^d)$. If $u$ is spherically symmetric or splitting-spherically symmetric, then $x(t) = \xi(t) = 0$.

Remark 1.12. The parameter $N(t)$ measures the frequency scale of the solution at time $t$ while $1/N(t)$ measures its spatial scale. Further properties of the function $N(t)$ are discussed in [16, 29].

One important consequence of the fact that $u$ is almost periodic modulo scaling (near positive infinity) is the following Duhamel formula, where the free evolution term disappears:

**Lemma 1.13 ([29, Section 6]).** Let $u$ be an almost periodic solution to (1.1) on $[t_0, \infty)$. Then, for all $t \in [t_0, \infty)$,

$$
u(t) = -\lim_{T \to \infty} \int_t^T e^{i(t-t')} \Delta (|u|^\frac{d}{2}u)(t') dt'
$$

as a weak limit in $L_x^2$.

We will use this Duhamel formula with the in-out decomposition technique as used in [15] to upgrade the weak localization of kinetic energy to the desired stronger one. Note that the in-out decomposition technique exploits heavily the spherical symmetry of the solution. Since Proposition 1.10 already gives us the uniform $H_x^1$ boundedness of the solution away from the spatial origin, the main obstacle is the control of the nonlinearity near the origin. In this regime due to the high degree of nonlinearity and lack of $H_x^1$ control, the contribution of this part seems difficult to handle. We shall overcome this difficulty by introducing a linear flow trick. More precisely, we decompose the solution into incoming and outgoing waves; this serves to minimize the contribution from the nonlinearity near the origin where we do not have uniform control on the kinetic energy. As was already mentioned,
the part close to the origin is the most problematic, since the kinetic energy may grow out of control as $t \to \infty$. For this regime, we decrease the power of nonlinearity by substituting the nonlinearity by the linear flow. After integration in time and using some kernel estimates, we succeed to control it by the mere mass of $u$. At large radii we can take advantage of spherical symmetry to obtain smallness. Since we have the uniform control on the kinetic energy at this regime, the high power of nonlinearity does not cause trouble. The key point of the in-out decomposition is to use the Duhamel formula into the future to control the outgoing portion of $u$ and the Duhamel formula into the past to control the incoming portion. The particular decomposition we use is taken from [16, 17]; the tool we use to exploit the spherical symmetry is a weighted Strichartz inequality Lemma 2.8 and the weighted Sobolev embedding from [16], [17] and [28].

As a consequence of the above analysis, we not only prove the frequency decay estimate Proposition 4.1, but also obtain a spatial decay estimate Proposition 4.2 with the spatial scale independent of $t$. Combining these two propositions yields a uniform kinetic energy localization result. This is very surprising comparing with the mass localization property where mass is localized with spatial scale varying with time $t$. Specifically, we have the following

**Theorem 1.14** (Kinetic energy localization in 2, 3-dimensions). Let $d = 2, 3$. Let $u_0 \in H^1_\mu(\mathbb{R}^d)$ be spherically symmetric and satisfy $M(u) = M(Q)$. In particular, as a consequence of Corollary 1.10, the solution satisfies

$$\|\phi_{\geq 1} \nabla u(t)\|_{L^2_\mu(\mathbb{R}^d)} < 1.$$

Let the corresponding solution $u(t, x)$ exists globally forward in time and satisfy the Duhamel formula (1.13). The for any $\eta > 0$, there exists $C(\eta)$ such that

$$\|\phi_{> C(\eta)} \nabla u(t)\|_{L^2_\mu(\mathbb{R}^d)} \leq \eta, \ \forall t \geq 0.$$

As explained before, this proposition together with the cheap localized virial argument establishes Theorem 1.7.

To conclude, the spherical symmetry is used in an essential way to establish the kinetic energy localization in 2,3 dimensions. First of all, it forces the solution to localize at the origin, weak localization result Proposition 1.10 cannot hold without this assumption since the center of the solution can move with time $t$. Secondly, the spherical symmetry forces decay as $|x| \to \infty$ which contributes directly to the additional smoothness (Proposition 1.1) and the additional decay estimates(Proposition 4.2), hence the kinetic energy localization Theorem 1.14 follows.

It is then interesting to consider the case when the solution is not spherically symmetric, but has some symmetry enough to localize the solution at the origin, at the same time, provides enough averaging effect. This motives us to consider the splitting-spherical symmetry(see Definition 1.1). For example, in four dimensional case, the only nontrivial splitting-spherical
symmetry simply requires the function to be spherically symmetric in each of the two 2-dimensional subspaces.

In the splitting-spherical symmetric case, by the same argument as in lower dimensions with radial assumption, all the matter is reduced to showing the kinetic energy localization. Since this symmetry also forces the solution to stay at the origin, weak localization of kinetic energy (Proposition 1.10) still holds in this case. Therefore all we have to do is to upgrade this weak localization to a stronger one. More precisely, we will prove the following

**Theorem 1.15** (Kinetic energy localization in \( d \geq 4 \) with admissible symmetry). Let \( d \geq 4 \) and \( u_0 \in H_x^1(\mathbb{R}^d) \) with \( M(u_0) \leq M(Q) \). Let \( u_0 \) have the admissible symmetry. Let \( u(t,x) \) be corresponding global solution forward in time satisfying the Duhamel formula (1.17). Assume also

\[
\|\phi_{>1} \nabla u(t)\|_{L_x^2(\mathbb{R}^d)} \lesssim 1. \tag{1.18}
\]

Then for any \( \eta > 0 \), there exists \( C(\eta) > 0 \) such that

\[
\|\phi_{>C(\eta)} \nabla u(t)\|_{L_x^2(\mathbb{R}^d)} \leq \eta.
\]

We remark that in the case when \( M(u) < M(Q) \), (1.18) is a direct consequence of the sharp Gagliardo–Nirenberg inequality; in the case when \( M(u) = M(Q) \), it is ensured by Proposition 1.10 which is a consequence of the non-sharp decomposition Proposition 3.1.

The proof of Theorem 1.15 is technically more involved. It is ultimately reduced to understanding the decay of a single frequency of the solution: \( P_N u(t) \) in both frequency and physical space. However, not like the radial case where we have only one preferred direction (namely, the radial direction), in this case, we will have two. Due to this anisotropy, the waves that travel at certain speed \( N \) may have the same speed in one direction, but stay static in the other. It is not hard to imagine that the desired smoothness and the decay will only comes from the traveling part of waves. To confirm this, we will apply the sub-dimensional in-out decomposition technique as we shall explain in Section 6 and Section 7. However, since the spatial cutoff does not necessarily induce cutoffs of the waves in the preferred direction, this decomposition is not directly useful to minimize the contribution of these waves near the origin. The outcome turns out to be a detailed discussion of various mixture terms with different sub-dimensional spatial cutoff and the sub-dimensional incoming/outgoing projection operators. Same with the 2,3 dimensional case, the contribution of the nonlinearity away from the origin is the dominate part, and the proof relies heavily on the spherical symmetry in that sub-dimension. It is here the restriction on the fold \( k \) and the minimal dimension \( d_1 \) of the splitting spherical symmetry appears. It would be interesting to extend this result to the more general splitting-spherical symmetry case.
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2. Preliminaries

2.1. Some notation. We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity $X$ such that $|X| \leq Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. The fact that these constants depend upon the dimension $d$ will be suppressed.

If $C$ depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denotes the assertion that $X \leq C_u Y$ for some $C_u$ depending on $u$. We sometimes write $C = C(Y_1, \cdots, Y_n)$ to stress that the constant $C$ depends on quantities $Y_1, \cdots, Y_n$. We denote by $X_{\pm}$ any quantity of the form $X \pm \epsilon$ for any $\epsilon > 0$.

We use the ‘Japanese bracket’ convention $\langle x \rangle := (1 + |x|^2)^{1/2}$.

We write $L^q_t L^r_x$ to denote the Banach space with norm

$$
\|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(t,x)|^r \, dx \right)^{q/r} \, dt \right)^{1/q},
$$

with the usual modifications when $q$ or $r$ are equal to infinity, or when the domain $\mathbb{R} \times \mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \mathbb{R}^d$.

When $q = r$ we abbreviate $L^q_t L^q_x$ as $L^q_{t,x}$.

Throughout this paper, we will use $\phi \in C^\infty(\mathbb{R}^d)$ be a radial bump function supported in the ball $\{ x \in \mathbb{R}^d : |x| \leq \frac{25}{24} \}$ and equal to one on the ball $\{ x \in \mathbb{R}^d : |x| \leq 1 \}$. For any constant $C > 0$, we denote $\phi_{\leq C}(x) := \phi(\frac{x}{C})$ and $\phi_{> C} := 1 - \phi_{\leq C}$.

2.2. Basic harmonic analysis. For each number $N > 0$, we define the Fourier multipliers

$$
\widehat{P}_{\leq N} f(\xi) := \phi_{\leq N}(\xi) \hat{f}(\xi),
$$

$$
\widehat{P}_{> N} f(\xi) := \phi_{> N}(\xi) \hat{f}(\xi),
$$

$$
\widehat{P}_{\sim N} f(\xi) := (\phi_{\leq N} - \phi_{\leq N/2})(\xi) \hat{f}(\xi)
$$

and similarly $P_{< N}$ and $P_{\geq N}$. We also define

$$
P_{M< \leq N} := P_{\leq N} - P_{\leq M} = \sum_{M < N' \leq N} P_{N'},
$$

whenever $M < N$. We will usually use these multipliers when $M$ and $N$ are dyadic numbers (that is, of the form $2^n$ for some integer $n$); in particular, all summations over $N$ or $M$ are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow $M$ and $N$ to not be a power of 2. As $P_N$ is not truly a projection, $P_N^2 \neq P_N$, we will occasionally need to use fattened Littlewood-Paley operators:

$$
\check{P}_N := P_{N/2} + P_N + P_{2N}. \quad (2.1)
$$
These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

Like all Fourier multipliers, the Littlewood-Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many many times, including

**Lemma 2.1** (Bernstein estimates). For $1 \leq p \leq q \leq \infty$,

$$
\| \|\nabla\|^{\pm s} P_N f \|_{L^p_x(\mathbb{R}^d)} \sim N^{\pm s} \| P_N f \|_{L^q_x(\mathbb{R}^d)},
$$

$$
\| P_N f \|_{L^1_x(\mathbb{R}^d)} \lesssim N^d \frac{d}{q} \| P_N f \|_{L^p_x(\mathbb{R}^d)},
$$

$$
\| P_N f \|_{L^\infty_x(\mathbb{R}^d)} \lesssim N^d \frac{d}{q} \| P_N f \|_{L^1_x(\mathbb{R}^d)}.
$$

While it is true that spatial cutoffs do not commute with Littlewood-Paley operators, we still have the following:

**Lemma 2.2** (Mismatch estimates in real space). Let $R, N > 0$. Then

$$
\| \phi_{>R} \nabla P_{\leq N} \phi_{\leq N} f \|_{L^p_x(\mathbb{R}^d)} \lesssim_m N^{1-m} R^{-m} \| f \|_{L^q_x(\mathbb{R}^d)}
$$

$$
\| \phi_{>R} P_{\leq N} \phi_{\leq N} f \|_{L^p_x(\mathbb{R}^d)} \lesssim_m N^{-m} R^{-m} \| f \|_{L^q_x(\mathbb{R}^d)}
$$

for any $1 \leq p \leq \infty$ and $m \geq 0$.

**Proof.** We will only prove the first inequality; the second follows similarly.

It is not hard to obtain kernel estimates for the operator $\phi_{>R} \nabla P_{\leq N} \phi_{\leq N}$. Indeed, an exercise in non-stationary phase shows

$$
\| \phi_{>R} \nabla P_{\leq N} \phi_{\leq N} f \|_{L^p_x(\mathbb{R}^d)} \lesssim N^{d+1-2k} \| f \|_{L^\infty_x(\mathbb{R}^d)}
$$

for any $k \geq 0$. An application of Young’s inequality yields the claim. \qed

Similar estimates hold when the roles of the frequency and physical spaces are interchanged. The proof is easiest when working on $L^2_x$, which is the case we will need; nevertheless, the following statement holds on $L^p_x$ for any $1 \leq p \leq \infty$.

**Lemma 2.3** (Mismatch estimates in frequency space). For $R > 0$ and $N, M > 0$ such that $\max\{N, M\} \geq 4 \min\{N, M\}$,\n
$$
\| P_N \phi_{\leq R} P_M f \|_{L^2_x(\mathbb{R}^d)} \lesssim_m \max\{N, M\}^{-m} R^{-m} \| f \|_{L^2_x(\mathbb{R}^d)}
$$

$$
\| P_N \phi_{\leq R} \nabla P_M f \|_{L^2_x(\mathbb{R}^d)} \lesssim_m M \max\{N, M\}^{-m} R^{-m} \| f \|_{L^2_x(\mathbb{R}^d)}.
$$

for any $m \geq 0$. The same estimates hold if we replace $\phi_{\leq R}$ by $\phi_{> R}$.

**Proof.** The first claim follows from Plancherel’s Theorem and Lemma 2.2 and its adjoint. To obtain the second claim from this, we write

$$
P_N \phi_{\leq R} \nabla P_M = P_N \phi_{\leq R} P_M \nabla \tilde{P}_M
$$

and note that $\| \nabla \tilde{P}_M \|_{L^2_x \to L^2_x} \lesssim M$. \qed

We will need the following radial Sobolev embedding to exploit the decay property of a radial function. For the proof and the more complete version, one refers to see [28].
Lemma 2.4 (Radial Sobolev embedding, [28]). Let dimension $d \geq 2$. Let $s > 0$, $\alpha > 0$, $1 < p, q < \infty$ obey the scaling restriction: $\alpha + s = d(\frac{1}{q} - \frac{1}{p})$. Then the following holds:

$$\| |x|^{\alpha} f\|_{L^p(\mathbb{R}^d)} \lesssim \| \nabla^s f \|_{L^q(\mathbb{R}^d)},$$

where the implicit constant depends on $s, \alpha, p, q$. When $p = \infty$, we have

$$\| |x|^{\frac{d-1}{2}} P_N f\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{\frac{1}{2}} \| P_N f \|_{L^2(\mathbb{R}^d)}.$$

We will need the following fractional chain rule lemma.

Lemma 2.5 (Fractional chain rule for a $C^1$ function, [8][26][32]). Let $F \in C^1(\mathbb{C})$, $\sigma \in (0,1)$, and $1 < r, r_1, r_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then we have

$$\| \nabla^\sigma F(u) \|_r \lesssim \| F'(u) \|_{r_1} \| \nabla^\sigma u \|_{r_2}.$$  

Proof. See [8], [26] and [32]. □

2.3. Strichartz estimates. The free Schrödinger flow has the explicit expression:

$$e^{it\Delta} f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) dy,$$

from which we can derive the kernel estimate of the frequency localized propagator. We record the following

Lemma 2.6 (Kernel estimate[16][17]). For any $m \geq 0$, we have

$$| (P_N e^{it\Delta}(x, y)) | \lesssim_m \begin{cases} |t|^{-d/2}, & |x - y| \sim Nt; \\ \frac{N^d}{|N^2|^m (N|x-y|)^m}, & \text{otherwise} \end{cases}$$

for $|t| \geq N^{-2}$ and

$$| (P_N e^{it\Delta})(x, y) | \lesssim_m N^d (N|x-y|)^{-m}$$

for $|t| \leq N^{-2}$.

We will frequently use the standard Strichartz estimate:

Lemma 2.7 (Strichartz). Let $d \geq 2$. Let $I$ be an interval, $t_0 \in I$, and let $u_0 \in L^2_x(\mathbb{R}^d)$ and $F \in L^{2(d+2)/(d+4)}(I \times \mathbb{R}^d)$. Then, the function $u$ defined by

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^t e^{i(t-t')\Delta} F(t') \, dt'$$

obeys the estimate

$$\| u \|_{L^\infty_t L^2_x} + \| u \|_{L^{\frac{2(d+2)}{d+4}}_t L^2_x} \lesssim \| u_0 \|_{L^2_x} + \| F \|_{L^{\frac{2(d+2)}{d+4}}_t L^{\frac{2(d+2)}{d+4}}_x},$$

where all spacetime norms are over $I \times \mathbb{R}^d$.

Proof. See, for example, [9][27]. For the endpoint see [12]. □
We will also need a weighted Strichartz estimate, which exploits heavily the spherical symmetry in order to obtain spatial decay.

**Lemma 2.8 (Weighted Strichartz, [16, 17]).** Let $I$ be an interval, $t_0 \in I$, and let $F : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ be spherically symmetric. Then,

$$
\left\| \int_{t_0}^{t} e^{i(t-t')\Delta} F(t') \, dt' \right\|_{L^2_x} \lesssim \|x|^{-\frac{2(d-1)}{q}} F\|_{L^{\frac{q}{q-1},2}(I \times \mathbb{R}^d)}
$$

for all $4 \leq q \leq \infty$.

2.4. **The in-out decomposition.** We will need an incoming/outgoing decomposition; we will use the one developed in [16, 17]. As there, we define operators $P^\pm$ by

$$[P^\pm f](r) := \frac{1}{2} f(r) \pm \frac{i}{2} \int_0^{\infty} \frac{r^{2-d} f(\rho) \rho^{d-1} \, d\rho}{r^2 - \rho^2},$$

where the radial function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is written as a function of radius only. We will refer to $P^+$ as the projection onto outgoing spherical waves; however, it is not a true projection as it is neither idempotent nor self-adjoint. Similarly, $P^-$ plays the role of a projection onto incoming spherical waves; its kernel is the complex conjugate of the kernel of $P^+$ as required by time-reversal symmetry.

For $N > 0$ let $P^\pm_N$ denote the product $P^\pm P_N$ where $P_N$ is the Littlewood-Paley projection. We record the following properties of $P^\pm$ from [16, 17]:

**Proposition 2.9 (Properties of $P^\pm$, [16, 17]).**

(i) $P^+ + P^-$ represents the projection from $L^2$ onto $L^2_{rad}$.

(ii) Fix $N > 0$. Then

$$\left\| \chi_{\geq \frac{1}{N}} P^\pm \right\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$$

with an $N$-independent constant.

(iii) If the dimension $d = 2$, then the $P^\pm$ are bounded on $L^2(\mathbb{R}^2)$.

(iv) For $|x| \geq N^{-1}$ and $t \geq N^{-2}$, the integral kernel obeys

$$|[P^\pm_N e^{\pm it\Delta}] (x, y)| \lesssim \begin{cases} (|x||y|)^{-\frac{d-1}{2}} |t|^{-\frac{d}{2}} & : |y| - |x| \sim Nt \\ \frac{N^d}{(N|x|)^{\frac{d-1}{2}} (N|y|)^{\frac{d-1}{2}}} (N^2 t + N|x| - N|y|)^{-m} & : \text{otherwise} \end{cases}$$

for all $m \geq 0$.

(v) For $|x| \geq N^{-1}$ and $|t| \lesssim N^{-2}$, the integral kernel obeys

$$|[P^\pm_N e^{\pm it\Delta}] (x, y)| \lesssim \frac{N^d}{(N|x|)^{\frac{d-1}{2}} (N|y|)^{\frac{d-1}{2}}} (N|x| - N|y|)^{-m}$$

for any $m \geq 0$.

We will also need the following Proposition concerning the properties of $P^\pm$ in the small $x$ regime (i.e. $|x| \lesssim N^{-1}$) where Bessel functions have logarithmic singularities. More precisely, we have
**Proposition 2.10** (Properties of \( P^\pm \), small \( x \) regime).

Let the dimension \( d = 2 \).

(i) For \( t \gtrsim N^{-2} \), \( N^{-3} \lesssim |x| \lesssim N^{-1} \), \( |y| \ll Nt \) or \( |y| \gg Nt \), the integral kernel satisfies

\[
|\left[ P_N^\pm e^{\mp it\Delta} \right](x,y) | \lesssim \frac{N^2 \log N}{(N|y|)^{1/2}} \left( N^2 t + N |y| \right)^{-m}, \quad \forall m \geq 0.
\]

(ii) For \( t \gtrsim N^{-2} \), \( N^{-3} \lesssim |x| \lesssim N^{-1} \), \( |y| \sim Nt \), the integral kernel satisfies

\[
|\left[ P_N^\pm e^{\mp it\Delta} \right](x,y) | \lesssim \frac{N^2 \log N}{(N|y|)^{1/2}}.
\]

**Proof.** We shall only provide the proof for \( P_N^+ e^{-it\Delta} \) since the other kernel is its complex conjugate. The first claim is an exercise in stationary phase.

By definition we have the following formula for the kernel

\[
\left[ P_N^+ e^{-it\Delta} \right](x,y) = \frac{1}{2} (2\pi)^{-2} \int_{\mathbb{R}^2} H_0^{(1)}(|\xi||x|) e^{it|\xi|} J_0(|\xi||y|) \psi(\frac{\xi}{N}) d\xi,
\]

and for the Hankel function \( H_0^{(1)} \) we have

\[
H_0^{(1)}(r) = J_0(r) + iY_0(r).
\]

Observe that in the regime \( |\xi| \sim N, |x| \lesssim N^{-1} \), we have \( r = |\xi| \cdot |x| \lesssim 1 \).

Since

\[
J_0(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left( \frac{r}{2} \right)^{2m},
\]

it is easy to see that

\[
\left| \frac{\partial^m J_0(r)}{\partial r^m} \right| \lesssim 1, \quad \forall m \geq 0, r \lesssim 1.
\]

For \( Y_0 \) we have

\[
Y_0(r) = \frac{2}{\pi} \left( \log \left( \frac{1}{2} r \right) + \gamma \right) J_0(r) + \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} H_k \cdot \frac{\left( \frac{1}{2} r \right)^k}{(k!)^2},
\]

where \( \gamma \) is the Euler-Masheroni constant and \( H_k \) is a harmonic number \( (H_k = \sum_{n=1}^{k} \frac{1}{n}) \). Clearly we can then write

\[
Y_0(r) = \frac{2}{\pi} (\log r) \cdot J_0(r) + b(r),
\]

where \( b(r) \) obeys

\[
\left| \frac{\partial^m b(r)}{\partial r^m} \right| \lesssim 1, \quad \forall m \geq 0, r \lesssim 1.
\]

We also need the following information about Bessel functions

\[
J_0(r) = \frac{a(r)e^{ir}}{(r)^{1/2}} + \frac{\bar{a}(r)e^{-ir}}{(r)^{1/2}},
\]

\( (2.5) \)
where \(a(r)\) obeys the estimates
\[
\left| \frac{\partial^m a(r)}{\partial r^m} \right| \lesssim |r|^{-m} \quad \text{for all } m \geq 0.
\]
Substitute (2.3), (2.4), (2.5) into (2.2), we obtain that a stationary point can only occur at \(|y| \sim Nt\). Observe that \(\log(|\xi| \cdot |x|) = \log |\xi| + \log |x|\) and \(\log |x|\) can be taken outside of the integral for \(\xi\), the logarithmic singularity of \(Y_0\) is ok for us. Now since we assume \(|y| \ll Nt\) or \(|y| \gg Nt\), the desired claim follows by integrating by parts. This establishes the first claim. Finally the second claim follows easily from a \(L^1\) estimate.

\[\square\]

Remark 2.11. In Proposition 2.10, we set the lower regime of \(|x|\) to be \(N^{-3}\) only for simplicity of presentation. The same result holds if one changes \(N^{-3}\) to be \(N^{-\alpha}\) where \(\alpha \geq 1\) (with the implied constant depending on \(\alpha\)).

3. A NON-SHARP DECOMPOSITION

In this section we establish the following non-sharp decomposition for \(H^1_x\) functions with ground state mass.

**Proposition 3.1** (Non-sharp decomposition of \(H^1_x\) functions with ground state mass). Fix the dimension \(d \geq 1\). Let \(Q\) be the ground state defined in (1.9). There exist constants \(C_1 > 0, C_2 > 0\) which depend only on the dimension \(d\) such that the following holds: for any \(u \in H^1_x(\mathbb{R}^d)\), there exists \(x_0 = x_0(u) \in \mathbb{R}^d, \theta_0 = \theta_0(u) \in \mathbb{R}, \epsilon = \epsilon(u) \in H^1_x(\mathbb{R}^d)\) for which we have
\[
u = \lambda^d e^{i\theta_0} Q(\lambda \cdot -x_0)) + \epsilon,
\]
where
\[
\frac{1}{C_2} \cdot \frac{\|\nabla u\|_{L^2}}{\|\nabla Q\|_{L^2}} \leq \lambda \leq C_2 \cdot \frac{\|\nabla u\|_{L^2}}{\|\nabla Q\|_{L^2}}, \quad \text{if } \|\nabla u\|_{L^2}^2 \geq C_1 E(u), \quad (3.1)
\]
and
\[
\lambda = 1, \quad \text{if } \|\nabla u\|_{L^2}^2 < C_1 E(u).
\]
The term \(\epsilon\) satisfies the bound:
\[
\|\epsilon\|_{H^1} \lesssim d \sqrt{E(u)} + 1. \quad (3.2)
\]
Here the energy \(E(u)\) is the same as defined in (1.6). If in addition \(u\) is an even function (i.e. \(u(x) = u(-x)\) for any \(x \in \mathbb{R}^d\)), then we can take \(x_0 = 0\).

The next lemma is essentially a corollary of Proposition 3.1. For the convenience of presentation, we postpone the proofs of both Proposition 3.1 and Lemma 3.2 till the end of this section.

**Lemma 3.2** (Uniform \(H^1_x\) boundedness away from the origin). Fix the dimension \(d \geq 1\). Let \(Q\) be the ground state in (1.9). Let \(c \geq c_0 > 0\),
$E_0 \geq 0$ be given numbers. Then for any even function $u \in H^1_x(\mathbb{R}^d)$ with $M(u) = M(Q)$, $E(u) = E_0$, we have

$$\|\phi_{>c} \nabla u\|_{L^2_x(\mathbb{R}^d)} + \|\phi_{>c} u\|_{H^1_x(\mathbb{R}^d)} \lesssim_{c_0, E_0, d} 1. \quad (3.3)$$

To establish Proposition 3.1 and Lemma 3.2 we will prepare some elementary lemmas and some general discussions. To this end, consider the $d$ dimensional focusing NLS of the form

$$i\partial_t u + \Delta u + |u|^p u = 0,$$

where we assume $p \leq \frac{4}{d}$. Let $Q$ be the associated ground state which is a positive radial Schwartz function. The linearized operators are defined by

$$L_+ = -\Delta + 1 - (p + 1)Q^p,$$

$$L_- = -\Delta + 1 - Q^p.$$

Set $w = u + iv$ and

$$L = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}.$$

We need the following fact from Weinstein [34].

**Proposition 3.3** (Conditional positivity of $L_+$, [34]). Let $p \leq \frac{4}{d}$. Then

$$\inf_{(f, Q) = 0} (L_+ f, f) = 0.$$

**Proof.** See Proposition 2.7 of [34]. \qed

**Lemma 3.4** (Spectral properties of $L$, [35], [34], [18]).

Assume $p \leq \frac{4}{d}$. Then

(i) $L_-$ is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^d)$ with the null space $N(L_-) = \text{span}\{Q\}$.

(ii) $L_+$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$ with null space $N(L_+) = \text{span}\{Q_{x_i} : 1 \leq i \leq d\}$.

(iii) $L_+$ has exactly one negative eigenvalue and its multiplicity is 1.

**Proof.** Claim (i), (ii) and first part of Claim (iii) follows directly from [34], [18] and [35]. The second part in Claim (iii) concerning the multiplicity of the negative eigenvalue is a simple consequence of Proposition 3.3. \qed

By Lemma 3.4 we can denote the negative eigenvalue of $L_+$ as $-\lambda_0$ ($\lambda_0 > 0$) and the corresponding eigenfunction with unit $L^2$ mass as $W$.

**Lemma 3.5** (Coercivity of $L$, [34], [19]).

Let $p = \frac{4}{d}$. There exist constants $\sigma_1$, $\sigma_2$ depending only on the dimension $d$ such that the following holds:

(i) For any $\epsilon \in H^1$, if $(\epsilon, W) = 0$ and $(\epsilon, Q_{x_i}) = 0$ for any $1 \leq i \leq d$, then

$$(L_+ \epsilon, \epsilon) \geq \sigma_1(\epsilon, \epsilon).$$
(ii) For any \( \epsilon \in H^1 \), if \((\epsilon, Q) = 0\), then 
\[
(L_-, \epsilon) \geq \sigma_2(\epsilon, \epsilon).
\]

Proof. See [34] and the improvement in [19] for the critical case \( p = \frac{4}{d} \). \( \square \)

We need the following compactness lemma from Hmidi and Keraani [11].

Lemma 3.6 (Compactness lemma, [11]). Let \( \{v_n\}_{n=1}^\infty \) be a bounded family of \( H^1(\mathbb{R}^d) \) such that
\[
\limsup_{n \to \infty} \|\nabla v_n\|_{L^2} \leq M \quad \text{and} \quad \limsup_{n \to \infty} \|v_n\|_{L^{2(d+2)}} \geq m.
\]
Then there exists \( \{x_n\}_{n=1}^\infty \subset \mathbb{R}^d \) such that, up to a subsequence
\[
v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly},
\]
with
\[
\|V\|_{L^2} \geq \left( \frac{d}{d+2} \right)^\frac{4}{d} \cdot \frac{m^{\frac{d}{2}+1}}{M^\frac{d}{2}} \|Q\|_{L^2}.
\]

Proof. See Theorem 1 of [11]. \( \square \)

Lemma 3.7 (Rigidity of the ground state, non-quantitative version). Let the dimension \( d \geq 1 \) and \( Q \) be the ground state corresponding to \( p = \frac{4}{d} \). Assume \( u \in H^1(\mathbb{R}^d) \) satisfies
\[
\|u\|_{L^2} = \|Q\|_{L^2} \quad \text{and} \quad \|\nabla u\|_{L^2} = \|\nabla Q\|_{L^2}.
\]
Then the energy \( E(u) \geq 0 \) and there exist \( \gamma_0 = \gamma_0(u) \in \mathbb{R}, \ x_0 = x_0(u) \in \mathbb{R}^d \) such that
\[
\|Q - e^{i\gamma_0} u(\cdot + x_0)\|_{H^1(\mathbb{R}^d)} \leq \delta(E(u)),
\]
where \( \delta(E(u)) \to 0 \) if \( E(u) \to 0 \).

Proof. That \( E(u) \) is nonnegative follows easily follow the sharp Gagliardo-Nirenberg interpolation inequality. Assume \( (3.5) \) is false. Then there exists \( \epsilon_0 > 0 \) and a sequence \( \{u_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^d) \) satisfying \( (3.4) \) such that
\[
\inf_{\gamma \in \mathbb{R}, \ x \in \mathbb{R}^d} \|Q - e^{i\gamma} u_n(\cdot + x)\|_{H^1(\mathbb{R}^d)} \geq \epsilon_0,
\]
and \( E(u_n) \to 0 \) as \( n \to \infty \). By \( (3.4) \) and the fact \( E(u_n) \to 0 \), we obtain
\[
\limsup_{n \to \infty} \|u_n\|_{L^{2(d+2)}_{x}} \geq \|\nabla Q\|_{L^2}^{\frac{d}{2(d+2)}} \cdot \left( \frac{d+2}{d} \right)^{\frac{2d}{2(d+2)}} = \|Q\|_{L^{2(d+2)}_{x}}.
\]
By Lemma 3.6, there exists \( \{x_n\}_{n=1}^\infty \subset \mathbb{R}^d \) such that up to a subsequence
\[
u_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly},
\]
\[ \|V\|_{L^2_x} \geq \|Q\|_{L^2_x}. \]

It follows that \( u_n(\cdot + x_n) \to V \) strongly in \( L^2_x \) and also in \( L^{2/(d+2)}_x \) by interpolation. We then obtain \( \|V\|_{L^2_x} = \|Q\|_{L^2_x} \), \( \|V\|_{L^{2/(d+2)}_x} = \|Q\|_{L^{2/(d+2)}_x} \) and \( \|\nabla V\|_{L^2_x} \leq \|\nabla Q\|_{L^2_x} \). By the sharp Gagliardo-Nirenberg interpolation inequality, we conclude that \( V = e^{-i\gamma_0}Q(\cdot - x_0) \) for some \( \gamma_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d \). It then follows that
\[ e^{i\gamma_0}u_n(\cdot + x_n + x_0) \to Q \text{ strongly in } H^1, \]
which is an obvious contradiction to (3.6). The lemma is proved. \( \square \)

The next lemma is central to our non-sharp decomposition.

**Lemma 3.8** (Rigidity of the ground state, quantitative version). Let \( d \geq 1 \) be the dimension, \( p = 4/d \) and \( Q \) be the corresponding ground state. There exist constants \( \eta > 0, C > 1, K > 0 \) depending only on the dimension \( d \) such that the following is true:

Let \( u \in H^1(\mathbb{R}^d) \) be such that
\[ \|u\|_{L^2_x} = \|Q\|_{L^2_x}, \quad \|\nabla u\|_{L^2_x} = \|\nabla Q\|_{L^2_x}, \]
and
\[ 0 \leq E(u) \leq \eta. \]
(The condition \( E(u) \geq 0 \) is actually unnecessary by the sharp Gagliardo-Nirenberg inequality). Then there exist \( \gamma_0 = \gamma_0(u) \in \mathbb{R}, x_0 = x_0(u) \in \mathbb{R}^d, \lambda_0 = \lambda_0(u) > 0 \) with
\[ \frac{1}{C} \leq \lambda_0 \leq C \]
(3.7)
such that
\[ \epsilon(x) = e^{i\gamma_0}d^0 \lambda_0 u(\lambda_0 x + x_0) - Q(x) \]
(3.8)
satisfies the following:

1. \( \text{Re}(\epsilon) \) is orthogonal to the negative and neutral directions of \( L_+ : \)
\[ (\text{Re}(\epsilon), W) = 0 \quad \text{and} \quad (\text{Re}(\epsilon), Q_{x_j}) = 0 \]
for any \( 1 \leq j \leq d \). \( (3.9) \)

Here \( W \) is the eigenfunction corresponding to the negative eigenvalue of the linear operator \( L_+ \) (see Lemma 3.4 and Lemma 3.5).

2. \( \text{Im}(\epsilon) \) is orthogonal to the neutral directions of \( L_- : \)
\[ (\text{Im}(\epsilon), Q) = 0. \]
(3.10)

3. The \( H^1 \) norm of \( \epsilon \) is small, more precisely:
\[ \|\epsilon\|_{H^1} \leq K\sqrt{E(u)}. \]
(3.11)

4. If \( u \) is an even function of \( x \) (i.e. \( u(x) = u(-x) \) for all \( x \in \mathbb{R}^d \)), then we can take \( x_0 = x_0(u) = 0. \)
Proof. Step 1: We show that Claim (1) and (2) holds. As in [23], the idea is to use the implicit function theorem. For any \( \alpha > 0 \), define the neighborhood

\[ U_\alpha = \{ u \in H^1(\mathbb{R}^d) : \| u - Q \|_{H^1} < \alpha \} \]

For any \( \gamma \in \mathbb{R} \), \( x \in \mathbb{R}^d \), \( u \in H^1(\mathbb{R}^d) \), define

\[ \epsilon_{\lambda, \gamma, x}(y) = e^{i\gamma} \lambda \frac{d}{2} u(\lambda y + x) - Q(y). \]

(3.12)

We first claim that there exist \( \alpha_0 > 0 \) and a unique \( C^1 \) map:

\[ U_{\alpha_0} \to \mathbb{R}_+^\lambda \times \mathbb{R}_x \times \mathbb{R}_\gamma \times \mathbb{R}_x^d \]

such that for any \( u \in U_{\alpha_0} \), there exist unique \( \lambda = \lambda(u) > 0 \), \( \gamma = \gamma(u) \in \mathbb{R} \), \( x = x(u) \in \mathbb{R}^d \) satisfying the following properties:

1. The real part of \( \epsilon_{\lambda, \gamma, x} \) is orthogonal to the negative and neutral directions of \( L^+ \):

\[ (Re(\epsilon_{\lambda, \gamma, x}), W) = 0, \quad \text{and} \quad (Re(\epsilon_{\lambda, \gamma, x}), Q_{x_i}) = 0, \quad \forall 1 \leq i \leq d. \]

(3.13)

2. The imaginary part of \( \epsilon_{\lambda, \gamma, x} \) is orthogonal to the neutral direction of \( L^- \):

\[ (Im(\epsilon_{\lambda, \gamma, x}), Q) = 0. \]

(3.14)

Furthermore there is a constant \( K_1 > 0 \) such that if \( 0 < \alpha < \bar{\alpha} \), \( u \in U_\alpha \), then

\[ \| \epsilon_{\lambda, \gamma, x} \|_{H^1} + |\lambda - 1| + |\gamma| + |x| \leq K_1 \alpha. \]

(3.15)

To establish the above claims, we define the following functionals

\[ \rho_1(u, \lambda, \gamma, x) = \int_{\mathbb{R}^d} Re(\epsilon_{\lambda, \gamma, x}(y))W(y)dy, \]

\[ \rho_j(u, \lambda, \gamma, x) = \int_{\mathbb{R}^d} Re(\epsilon_{\lambda, \gamma, x}(y))Q_{x_{j-1}}(y)dy, \quad 2 \leq j \leq d + 1, \]

\[ \rho_{d+2}(u, \lambda, \gamma, x) = \int_{\mathbb{R}^d} Im(\epsilon_{\lambda, \gamma, x}(y))Q(y)dy. \]

We then compute at \( (\lambda, \gamma, x) = (1, 0, 0) \),

\[ \frac{\partial(\epsilon_{\lambda, \gamma, x})}{\partial \lambda} = \frac{d}{2} u + y \cdot \nabla u, \]

\[ \frac{\partial(\epsilon_{\lambda, \gamma, x})}{\partial \gamma} = iu, \]

\[ \frac{\partial(\epsilon_{\lambda, \gamma, x})}{\partial x_i} = u_{x_i}, \quad \forall 1 \leq i \leq d. \]

(3.16)

(3.17)

To compute \( \frac{\partial \rho}{\partial \lambda} \) at \( (\lambda, \gamma, x, u) = (1, 0, 0, Q) \), we use the algebraic identity:

\[ L_+ \left( \frac{d}{2} Q + y \cdot \nabla Q \right) = -2Q, \]

(3.18)
which can be easily obtained by using scaling invariance. By \((3.16)\), \((3.18)\) and the fact that 
\[L + W = -\lambda_0 W,\]
we have
\[
\left. \frac{\partial \rho_1}{\partial \lambda} \right|_{(1,0,0,Q)} = \int_{\mathbb{R}^d} \left( \frac{d}{2} Q + y \cdot \nabla Q \right) \cdot W(y) dy \\
= -\frac{1}{\lambda_0} \int_{\mathbb{R}^d} \left( \frac{d}{2} Q + y \cdot \nabla Q \right) \cdot L + W(y) dy \\
= \frac{2}{\lambda_0} \int_{\mathbb{R}^d} Q(y) W(y) dy \neq 0,
\]
where the last integral does not vanish by Proposition 3.3. We then compute for \(2 \leq i \leq d + 2\),
\[
\left. \frac{\partial \rho_i}{\partial \lambda} \right|_{(1,0,0,Q)} = 0.
\]
It is easy to find that for \(1 \leq i \leq d + 1\)
\[
\left. \frac{\partial \rho_i}{\partial \gamma} \right|_{(1,0,0,Q)} = 0,
\]
and also
\[
\left. \frac{\partial \rho_{d+2}}{\partial \gamma} \right|_{(1,0,0,Q)} = \|Q\|_{L^2_x}^2.
\]
Lastly by \((3.17)\), we obtain
\[
\left. \frac{\partial \rho_i}{\partial x_j} \right|_{(1,0,0,Q)} = \delta_{j,i-1} \|Q_{x_j}\|_{L^2_x}^2, \quad \forall \ 2 \leq i \leq d + 1, \text{ and } 1 \leq j \leq d,
\]
where \(\delta_{j,i-1}\) is the usual Kronecker delta function. It follows easily that the Jacobian
\[
\left| \frac{\partial (\rho_1, \rho_{d+2}, \rho_2, \ldots, \rho_{d+1})}{\partial (\lambda, \gamma, x_1, \ldots, x_d)} \right|_{(1,0,0,Q)} \\
= \frac{2}{\lambda_0} \left( \int_{\mathbb{R}^d} Q(y) W(y) dy \right) \cdot \|Q\|_{L^2_x}^2 \cdot \prod_{j=1}^d \|Q_{x_j}\|_{L^2_x}^2 \\
\neq 0.
\]
Therefore by the implicit function theorem, there exist \(\alpha_0 > 0\), a neighborhood of \((\lambda, \gamma, x) = (1, 0, 0)\) in \(\mathbb{R}_+^\times \times \mathbb{R}_\gamma \times \mathbb{R}_x^d\), and a unique \(C^1\) map: 
\(U_{\alpha_0} \rightarrow \mathbb{R}_\Lambda^+ \times \mathbb{R}_\gamma \times \mathbb{R}_x^d\) such that \((3.13)\), \((3.14)\) holds. Furthermore by Lemma \(3.7\) and choosing \(\eta\) sufficiently small, we have \((3.15)\) holds. Finally Claim (1) and (2) follows easily from applying the previous result and Lemma \(3.7\).
Step 2: We show that Claim (3) holds. By step 1, we can choose \( \lambda_0 > 0, x_0 \in \mathbb{R}^d, \gamma_0 \in \mathbb{R} \) such that (3.7)-(3.10) holds. By Lemma 3.5 and denoting \( \sigma = \min\{\sigma_1, \sigma_2\} \), we obtain
\[
\sigma\|\epsilon\|_{L^2_x}^2 \leq (L\Re(\epsilon), \Re(\epsilon)) + (L\Im(\epsilon), \Im(\epsilon))
\]
\[
= \|\epsilon\|_{H^1}^2 - \frac{d + 4}{d} (Q\frac{4}{d}\Re(\epsilon), \Re(\epsilon)) - (Q\frac{4}{d}\Im(\epsilon), \Im(\epsilon)).
\]  
(3.19)

By conservation of mass, we have
\[
\|\epsilon\|_{L^2_x}^2 + 2(\Re(\epsilon), Q) = 0.
\]  
(3.20)

From conservation of energy, we get
\[
\lambda_0^2 E(u) = \frac{1}{2} \|\nabla Q + \nabla \epsilon\|_{L^2_x}^2 - \frac{d}{2(d + 2)} \|Q + \epsilon\|_{L^2_x}^{2(d + 2) \over d}
\]
\[
= \frac{1}{2} \|\nabla Q\|_{L^2_x}^2 - (\Delta Q, \Re(\epsilon)) + \frac{1}{2} \|\nabla \epsilon\|_{L^2_x}^2
\]
\[
- \frac{d}{2(d + 2)} \|Q + \epsilon\|_{L^2_x}^{2(d + 2) \over d}.
\]

Since the ground state \( Q \) satisfies
\[
-\Delta Q = Q^{1+\frac{4}{d}} - Q,
\]
and by the sharp Gagliardo-Nirenberg inequality
\[
\|\nabla Q\|_{L^2_x}^2 = \frac{d}{d + 2} \|Q\|_{L^2_x}^{2(d + 2) \over d},
\]
we obtain
\[
2\lambda_0^2 E(u) = \frac{d}{d + 2} \left( \|Q\|_{L^2_x}^{2(d + 2) \over d} - \|Q + \epsilon\|_{L^2_x}^{2(d + 2) \over d} \right) + \|\nabla \epsilon\|_{L^2_x}^2
\]
\[
- 2(Q, \Re(\epsilon)) + 2(Q^{1+\frac{4}{d}}, \Re(\epsilon)).
\]  
(3.21)

Adding together (3.20) and (3.21), we get
\[
2\lambda_0^2 E(u) = \frac{d}{d + 2} \left( \|Q\|_{L^2_x}^{2(d + 2) \over d} - \|Q + \epsilon\|_{L^2_x}^{2(d + 2) \over d} \right) + \|\nabla \epsilon\|_{L^2_x}^2
\]
\[
+ 2(Q^{1+\frac{4}{d}}, \Re(\epsilon)).
\]  
(3.22)

Substitute (3.22) into RHS of (3.19), we have
\[
\sigma\|\epsilon\|_{L^2_x}^2 \leq (Le, \epsilon) \leq 2\lambda_0^2 E(u) + \frac{d}{d + 2} \int_{\mathbb{R}^d} |F(\epsilon)(y)|dy,
\]  
(3.23)

where
\[
F(\epsilon) = \|Q + \epsilon\|_{L^2_x}^{2(d + 2) \over d} - Q\frac{4}{d} - \frac{d + 2}{d} Q\frac{4}{d} (\Im(\epsilon))^2
\]
\[
- \frac{2(d + 2)}{d} Q^{1+\frac{4}{d}} (\Re(\epsilon)) - \frac{d + 2}{d} \cdot \frac{d + 4}{d} \cdot Q\frac{4}{d} (\Re(\epsilon))^2.
\]
At this moment, we need the following lemma which gives a bound of $F(\epsilon)$.

**Lemma 3.9.** There exists some constant $B_1 > 0$ depending only on the dimension $d$ such that

$$
\int_{\mathbb{R}^d} |F(\epsilon)(y)| dy \leq B_1 \cdot \|\epsilon\|_{L_x^2}^{2(d+2)} \cdot \frac{d}{2(d+2)}, \quad \text{if } d \geq 4,
$$

and

$$
\int_{\mathbb{R}^d} |F(\epsilon)(y)| dy \leq B_1 \cdot \left( \|\epsilon\|_{L_x^2}^{2(d+2)} + \|\epsilon\|_{L_x^2}^3 \right), \quad \text{if } 1 \leq d \leq 3.
$$

Postponing the proof of Lemma 3.9 for the moment, we now show how to finish the proof of step 2. Let first $d \geq 4$. Then by Lemma 3.9 and (3.23), we obtain

$$
\sigma \|\epsilon\|_{L_x^2}^2 \leq (L\epsilon, \epsilon) \leq 2\lambda_0^2 E(u) + B_1 \cdot \|\epsilon\|_{L_x^2}^{2(d+2)} \cdot \frac{d}{2(d+2)}.
$$

By definition of $L_+$, $L_-$, we have

$$
\|\epsilon\|_{H^1}^2 \leq (L\epsilon, \epsilon) + \left( \frac{4}{d} + 1 \right) \int_{\mathbb{R}^d} Q^2 Re(\epsilon)^2 dy + \int_{\mathbb{R}^d} Q^2 Im(\epsilon)^2 dy
$$

$$
\leq C(d) \cdot \|\epsilon\|_{L_x^2}^2 + 2\lambda_0^2 E(u) + B_1 \cdot \|\epsilon\|_{L_x^2}^{2(d+2)} \cdot \frac{d}{2(d+2)},
$$

where $C(d)$ is a constant depending on the dimension $d$, and the last inequality follows from (3.26) together with Sobolev embedding. Clearly now (3.20) follows by taking $\eta$ sufficiently small and using Lemma 3.7. This finishes the proof of Claim 3 in the case $d \geq 4$. The case $1 \leq d \leq 3$ is similar by using Lemma 3.9. We omit the details.

**Step 3:** We show that claim (4) holds. If $u$ is even, then instead of (3.12) we define

$$
\epsilon_{\lambda, \gamma}(y) = e^{i\gamma} \lambda^{\frac{d}{2}} u(\lambda y) - Q(y).
$$

Since $u$ is even, by direct computation we have $Re(\epsilon_{\lambda, \gamma})$ is orthogonal to the neutral directions of $L_+$:

$$
(Re(\epsilon_{\lambda, \gamma}), Q_{\alpha_i}) = 0, \quad \forall 1 \leq i \leq d.
$$

Therefore we only need to take care of two directions: the negative direction of $L_+$ and the neutral direction of $L_-$. Similar to Step 1, we define two
functionals

\[ \rho_1(u, \lambda, \gamma) = \int_{\mathbb{R}^d} \text{Re}(\epsilon_{\lambda, \gamma}(y)) W(y) \, dy, \]
\[ \rho_2(u, \lambda, \gamma) = \int_{\mathbb{R}^d} \text{Im}(\epsilon_{\lambda, \gamma}(y)) Q(y) \, dy. \]

By same calculations, we find that the Jacobian

\[ \left| \frac{\partial (\rho_1, \rho_2)}{\partial (\lambda, \gamma)} \right|_{(\lambda, \gamma, u) = (1, 0, Q)} = \frac{2}{\lambda_0} \int_{\mathbb{R}^d} W(y) Q(y) \, dy \cdot \|Q\|_{L^2}^2 \neq 0. \]

The rest of the argument now follows essentially the same as in Step 1. We omit the details. This finishes the proof of Claim (4). \( \square \)

Next we complete the Proof of Lemma 3.9. We separate the integral on the LHS of (3.24) into two regions. In the first region \( \{x: |\epsilon(x)| \gtrsim Q(x)\} \), we have the pointwise estimate

\[ |F(\epsilon)(x)| \lesssim |\epsilon(x)|^{\frac{2(d+2)}{d}}, \]

and therefore

\[ \int_{|\epsilon(x)| \gtrsim Q(x)} |F(\epsilon)(x)| \, dx \lesssim \|\epsilon\|_{L^\infty}^{\frac{2(d+2)}{d(2(d+2))}} \|Q\|_{L^2} \quad (3.27) \]

which is clearly good for us.

In the second region \( \{x: |\epsilon(x)| \ll Q(x)\} \), we can write

\[ F(\epsilon)(x) = Q(x)^{\frac{2(d+2)}{d}} \cdot G(Q(x)^{-1} \epsilon(x)), \]

where

\[ G(z) = |1 + z|^{\frac{2(d+2)}{d}} - 1 - \frac{d + 2}{d} (\text{Im}(z))^2 - \frac{2(d + 2)}{d} \text{Re}(z) \]
\[ - \frac{d + 2}{d} \cdot \frac{d + 4}{d} (\text{Re}(z))^2. \]

In the region \( |z| \ll 1 \), by expanding the function \( |1 + z|^{\frac{2(d+2)}{d}} \) into power series, it is not difficult to check that

\[ |G(z)| \lesssim |z|^3, \quad \text{if } |z| \ll 1. \]

Therefore we obtain

\[ \int_{|\epsilon(x)| \ll Q(x)} |F(\epsilon)(x)| \, dx \lesssim \int_{|\epsilon(x)| \ll Q(x)} Q(x)^{\frac{2(d+2)}{d}} \cdot |Q(x)^{-1} \epsilon(x)|^3 \, dx. \quad (3.28) \]
Now we discuss two cases. If \( d \geq 4 \), then since \( \frac{2(d+2)}{d} \leq 3 \),
\[
|\text{RHS of (3.28)}| \lesssim \int_{|x| \leq Q(x)} |Q(x)^{-1}e(x)|^{3-\frac{2(d+2)}{d}} \cdot |e(x)|^{2\frac{d+2}{d}} \, dx
\]
\[
\lesssim \|e\|_{\frac{2(d+2)}{2d}},
\]
This together with (3.27) gives (3.24). Finally if \( 1 \leq d \leq 3 \), then we have
\[
|\text{RHS of (3.28)}| \lesssim \int_{|x| \leq Q(x)} Q(x)^{\frac{d}{d}-1} |e(x)|^3 \, dx
\]
\[
\lesssim \|e\|_{L^3},
\]
which together with (3.27) gives (3.25). The lemma is proved.
\[\square\]

We now complete the

Proof of Proposition 3.1. Let \( u \in H^1_x(\mathbb{R}^d) \) and \( M(u) = M(Q) \). Define the rescaled function \( \tilde{u} = \mu^{\frac{d}{2}} u(\mu x) \) where \( \mu = \|\nabla Q\|_2 / \|\nabla u\|_2 \). Clearly then
\[
\|\tilde{u}\|_2 = \|Q\|_2, \quad \text{and} \quad \|\nabla \tilde{u}\|_2 = \|\nabla Q\|_2,
\]
also
\[
E(\tilde{u}) = \frac{\|\nabla Q\|_2^2}{\|\nabla u\|_2^2} E(u).
\]

Now let the constants \( \eta = \eta(d) > 0 \), \( C = C(d) > 1 \), \( K = K(d) > 0 \) be the same as in Lemma 3.8. Define \( C_1 = \|\nabla Q\|_2^2 / \eta \). Then obviously if \( C_1 E(u) \leq \|\nabla u\|_2^2 \), then \( E(\tilde{u}) \leq \eta \). By Lemma 3.8, we can then write
\[
\tilde{e}(x) = e^{i\tilde{\gamma}_0} \tilde{\lambda}_0^\frac{d}{2} \tilde{u}(\tilde{\lambda}_0 x + \tilde{x}_0) - Q(x),
\]
where \( \frac{1}{C} \leq \tilde{\lambda}_0 \leq C \), and
\[
\|\tilde{e}\|_{H^1} \leq K \sqrt{E(\tilde{u})}.
\]

Expressing (3.29), (3.30) in terms of the original function \( u \), we then obtain
\[
u(x) = e^{-i\tilde{\gamma}_0} (\tilde{\lambda}_0 \mu)^{-\frac{d}{2}} Q(\frac{x - \mu \tilde{x}_0}{\tilde{\lambda}_0 \mu}) + e^{-i\tilde{\gamma}_0} (\tilde{\lambda}_0 \mu)^{-\frac{d}{2}} \tilde{e}(\frac{x - \mu \tilde{x}_0}{\tilde{\lambda}_0 \mu})
\]
\[
= e^{i\theta_0} \lambda^\frac{d}{2} Q(\lambda(x - x_0)) + \epsilon,
\]
where \( \theta_0 = -\tilde{\gamma}_0 \), \( x_0 = \mu \tilde{x}_0 \), \( \lambda = (\tilde{\lambda}_0 \mu)^{-1} = \tilde{\lambda}_0^{-1} \|\nabla u\|_2 / \|\nabla Q\|_2 \), and
\[
\|\epsilon\|_{H^1} \leq (1 + \lambda) K \sqrt{E(\tilde{u})}
\]
\[
\leq \frac{1 + \lambda}{\lambda} \cdot K \cdot \sqrt{E(\tilde{u})}
\]
\[
\leq K \cdot \frac{C \cdot \|\nabla Q\|_2}{\sqrt{C_1}} + K \sqrt{E(\tilde{u})}
\]
\[
\lesssim 1 + \sqrt{E(\tilde{u})}.
\]
Choose $C_2 = C$, and it is clear that (3.1) holds. Next if $\|\nabla u\|_{L^2}^2 < C_1 E(u)$, then we just set $\theta_0 = 0$, $x_0 = 0$, $\lambda = 1$ and $\epsilon = u - Q$. It is obvious that (3.2) holds in this case. Finally if $u$ is an even function of $x$, then one can repeat the previous steps with the observation that Lemma 3.8 holds in this case for $x_0 = 0$. We omit the details. □

Finally we finish the Proof of Lemma 3.2. Let $u \in H^1_x(\mathbb{R}^d)$ be an even function of $x$ and $M(u) = M(Q)$, $E(u) = E_0$. Let $C_1 = C_1(d) > 0$, $C_2 = C_2(d) > 0$ be the same constants as in Proposition 3.1. By Proposition 3.1, we can write

$$u(x) = \lambda^d e^{i\theta_0} Q(\lambda x) + \epsilon(x),$$

(3.31)

where

$$\frac{1}{C_2} \cdot \frac{\|\nabla u\|_{L^2}}{\|\nabla Q\|_{L^2}} \leq \lambda \leq C_2 \cdot \frac{\|\nabla u\|_{L^2}}{\|\nabla Q\|_{L^2}},$$

if $\|\nabla u\|_{L^2}^2 \geq C_1 E_0$,

and

$$\lambda = 1, \quad \text{if } \|\nabla u\|_{L^2}^2 < C_1 E_0,$$

also

$$\|\epsilon\|_{H^1_x} \lesssim_{d,E_0} 1.$$  

(3.32)

Now if $\|\nabla u\|_{L^2}^2 < C_1 E_0$, then (3.31) holds trivially in this case. If $\|\nabla u\|_{L^2}^2 \geq C_1 E_0$, then due to the decomposition (3.31) and the bound (3.32), we only need to estimate the LHS of (3.3) assuming $\epsilon = 0$ in (3.31), that is

LHS of (3.3) $\lesssim_{c_0,d} \int_{|x| > \frac{c_0}{100}} \lambda^{d+2} |(\nabla Q)(\lambda x)|^2 dx + \int_{|x| > \frac{c_0}{100}} \lambda^{d} |Q(\lambda x)|^2 dx$

$\lesssim_{c_0,d} 1 + (\lambda^2 + 1) \int_{|y| > \frac{\lambda c_0}{100}} (|\nabla Q(y)|^2 + |Q(y)|^2) dy$

$\lesssim_{c_0,d} 1 + \frac{\lambda^2 + 1}{\lambda^2} \int_{\mathbb{R}^d} |y|^2 (|\nabla Q(y)|^2 + |Q(y)|^2) dy$

$\lesssim_{c_0,d,E_0} 1,$

where we have used the fact that $Q$ is a Schwartz function. This finishes the proof of the lemma. □

4. REDUCTION OF THE PROOF

In this section, we first prove Theorem 1.7 by assuming the kinetic energy localization Theorem 1.14, 1.15. Then we will explain how the kinetic energy localization result Theorem 1.14, Theorem 1.15 can be derived from two Propositions: Proposition 4.2 and Proposition 4.1. Therefore our task is reduced to proving the two Propositions in all dimensions $d \geq 2$ with admissible symmetry, which will be done in next sections.
Proof of Theorem 1.7

Proof. Let $u$ be the solution of (1.1) satisfying the conditions in Theorem 1.7, that is

- $M(u) = M(Q)$;
- $u$ is a global solution and does not scatter forward in the sense that
  \[
  \|u\|_{L_t^{2(d+2)}(\mathbb{R}^d)} = \infty.
  \]
- $u_0$ verifies the admissible symmetry.

Then our aim is to show that actually, the corresponding solution has to be spherically symmetric and there exists $\lambda_0 > 0$, $\theta_0 \in \mathbb{R}$ such that

\[
  u(t, x) = \lambda_0^{-\frac{d}{2}} e^{i\theta_0} e^{it/\lambda_0^2} Q(\frac{x}{\lambda_0}).
\]

From the sharp Gagliardo-Nirenberg inequality, we observe that the energy of $u$ is always non-negative. If $E(u_0) = 0$, the variational characterization of the ground state Proposition 1.9 together with the condition $M(u) = M(Q)$ yields that: $u_0$ is spherically symmetric and there exists $\lambda_0 > 0$, $\theta_0 \in \mathbb{R}$ such that

\[
  u_0(x) = \lambda_0^{-\frac{d}{2}} e^{i\theta_0} Q(\frac{x}{\lambda_0}).
\]

The uniqueness of the solution then implies that the corresponding solution is spherically symmetric, moreover

\[
  u(t, x) = \lambda_0^{-\frac{d}{2}} e^{i\theta_0} e^{it/\lambda_0^2} Q(\frac{x}{\lambda_0}).
\]

So we only need to consider the positive energy case and try to get a contradiction in this case. First we can show the strong localization of the kinetic energy. From the results in [16, 17] for the radial case and assuming the scattering conjecture holds for the splitting-spherical symmetric case, we conclude that the ground state $M(Q)$ can be identified as the minimal mass in all dimensions $d \geq 2$ with admissible symmetry. Therefore $u$ satisfies the Duhamel formula (1.17). Moreover, from Lemma 3.2, $u$ satisfies the weak localization of the kinetic energy since $M(u) = M(Q)$ and $u \in H^1$. The strong localization of the kinetic energy then follows immediately from Theorem 1.14, Theorem 1.15. That is, $\forall \eta > 0$, there exists $C(\eta) > 0$ such that

\[
  \|\phi_{>C(\eta)} \nabla u(t)\|_{L^2_x(\mathbb{R}^d)} < \eta. \tag{4.1}
\]

Now we get the contradiction from the truncated virial argument.

Let $\phi_{\leq R}$ be the smooth cutoff function, we define the truncated virial as

\[
  V_R(t) := \int \phi_{\leq R}(x)|x|^2|u(t, x)|^2 dx.
\]
Obviously,

\[ V_R(t) \lesssim R^2, \quad \forall t \in \mathbb{R}. \tag{4.2} \]

On the other hand, we compute the second derivative of \( V_R(t) \), this gives

\[
\partial_t V_R(t) = 8E(u) \\
+ O\left( \int_{|x|>R} |\nabla u(t,x)|^2 + |u(t,x)|^{2(d+2)} \frac{2}{d} dx + \frac{1}{R^2} \int_{|x|>R} |u(t,x)|^2 dx \right). \tag{4.3}
\]

Since \( E(u) > 0 \), from (4.1) and by taking \( R \) large enough, we have

\[
\int_{|x|>R} |\nabla u(t,x)|^2 dx + \frac{1}{R^2} \int_{|x|>R} |u(t,x)|^2 dx \leq \frac{1}{100} E(u), \quad \forall t \geq 0.
\]

Moreover, from the Gagliardo-Nirenberg inequality, we have

\[
\int_{|x|>R} |u(t,x)|^{2(d+2)} \frac{2}{d} dx \lesssim \| \phi_{>R/2} u(t) \|_{L_x^\frac{2(d+2)}{d}(\mathbb{R}^d)} \| \nabla (\phi_{>R/2} u(t)) \|_{L_x^2(\mathbb{R}^d)} \lesssim \frac{1}{100} E(u).
\]

Therefore (4.3) finally gives

\[
\partial_t V_R(t) \geq 4E(u) > 0, \quad \forall t \geq 0
\]
which is a contradiction to (4.2).

The proof of Theorem 1.7 is then finished. \[\square\]

Now let us explain how the kinetic energy localization property Theorem 1.14 and Theorem 1.15 can be derived from the following two

**Proposition 4.1.** (Frequency decay estimate) Let \( d \geq 2 \). Let \( u(t,x) \in H_x^1(\mathbb{R}^d) \) be a global solution to (1.1) forward in time and have the admissible symmetry. Assume also \( u \) satisfy the Duhamel formula (1.17) and is weakly localized in \( H_x^1(\mathbb{R}^d) \) in the following sense

\[
\| \phi_{\geq R} \nabla u(t) \|_{L_x^2(\mathbb{R}^d)} \lesssim 1, \quad \forall t \geq 0. \tag{4.4}
\]

Then there exists \( \varepsilon = \varepsilon(d) \) such that, for any dyadic number \( N \geq 1 \), we have

\[
\| \phi_{>1} P_N u(t) \|_{L_x^2(\mathbb{R}^d)} \lesssim \| \tilde{P}_{\sqrt{R}} u_0 \|_{L_x^2(\mathbb{R}^d)} + N^{-1-\varepsilon}.
\]

**Proposition 4.2.** (Spatial decay estimate) Let \( u \) satisfy the same conditions as in Proposition 4.1. Let \( N_1, N_2 \) be two dyadic numbers. Then there exist \( \delta = \delta(d) \) and \( R_0 = R_0(N_1, N_2, u) \) such that for any \( N_1 < N \leq N_2 \) and \( R > R_0 \),

\[
\| \phi_{>R} P_N u(t) \|_{L_x^2(\mathbb{R}^d)} \lesssim \| \phi_{>\sqrt{R}} u_0 \|_{L_x^2(\mathbb{R}^d)} + R^{-\delta(d)}, \quad \forall t \geq 0.
\]
The proofs of Proposition 4.1, 4.2 will be presented in later sections. Assuming the two propositions hold for the moment, we now prove the strong kinetic localization. That is, for any \( \eta > 0 \), there exists \( C(\eta) > 0 \) such that
\[
\|\phi_{>\cdot} C(\eta) \nabla u(t)\|_{L^2_t(\mathbb{R}^d)} \leq \eta, \quad \forall t \geq 0.
\] (4.5)

Let \( N_1(\eta), N_2(\eta) \) be dyadic numbers and \( C(\eta) \) a large constant to be specified later. We estimate the LHS of (4.5) by splitting it into low, medium and high frequencies:

For the low frequencies, we simply discard the cutoff and use Bernstein, Lemma 2.2 and Proposition 4.2 to obtain
\[
\|\phi_{>\cdot} C(\eta) \nabla P_{\leq N_1(\eta)} u(t)\|_{L^2_t(\mathbb{R}^d)} \leq \|\phi_{>\cdot} C(\eta) \nabla P_{\leq N_1(\eta)} u(t)\|_{L^2_t(\mathbb{R}^d)}
\]
\[
+ \|\phi_{>\cdot} C(\eta) \nabla P_{N_1(\eta) < \leq N_2(\eta)} u(t)\|_{L^2_t(\mathbb{R}^d)}
\]
\[
+ \|\phi_{>\cdot} C(\eta) \nabla P_{N_2(\eta) < \cdot} u(t)\|_{L^2_t(\mathbb{R}^d)}
\]

For the medium frequencies, we simply discard the cutoff and use Bernstein,
\[
\|\phi_{>\cdot} C(\eta) \nabla P_{\leq N_1(\eta)} u(t)\|_{L^2_t(\mathbb{R}^d)} \lesssim N_1(\eta) \|u(t)\|_{L^2_t(\mathbb{R}^d)} \lesssim N_1(\eta)
\] (4.6)

To estimate the medium frequencies, we use Bernstein, mismatch estimate Lemma 2.2 and Proposition 4.2 to obtain
\[
\|\phi_{>\cdot} C(\eta) \nabla P_{N_1(\eta) < \leq N_2(\eta)} u(t)\|_{L^2_t(\mathbb{R}^d)}
\]
\[
\leq \sum_{N_1(\eta) < N \leq N_2(\eta)} \|\phi_{>\cdot} C(\eta) \nabla P_{N} u(t)\|_{L^2_t(\mathbb{R}^d)}
\]
\[
\leq C(N_1(\eta), N_2(\eta)) \max_{N_1(\eta) < N \leq N_2(\eta)} \|\phi_{>\cdot} C(\eta) \nabla P_{N} u(t)\|_{L^2_t(\mathbb{R}^d)}
\]
\[
\leq C(N_1(\eta), N_2(\eta)) \max_{N_1(\eta) < N \leq N_2(\eta)} \left( \|\phi_{>\cdot} C(\eta) \nabla P_{N} \phi_{\leq C(\eta)} P_{N} u(t)\|_{L^2_t(\mathbb{R}^d)}
\right)
\]
\[
+ \|\phi_{>\cdot} C(\eta) \nabla P_{N} \phi_{\leq C(\eta)} \tilde{P}_{N} u(t)\|_{L^2_t(\mathbb{R}^d)}
\]
\[
\leq C(N_1(\eta), N_2(\eta)) \left( \max_{N_1(\eta) < N \leq N_2(\eta)} N^{-1} C(\eta)^{-2} + N \|\phi_{\leq C(\eta)} P_{N} \phi_{\leq C(\eta)} u(t)\|_{L^2_t(\mathbb{R}^d)} + NC(\eta)^{-\delta(d)} \right)
\]
\[
\leq C(N_1(\eta), N_2(\eta)) \left( C(\eta)^{-2} + \|\phi_{\leq C(\eta)} \tilde{P}_{N} u(t)\|_{L^2_t(\mathbb{R}^d)} + C(\eta)^{-\delta(d)} \right)
\]
\[
\leq C(N_1(\eta), N_2(\eta)) \left( \|\phi_{\leq C(\eta)} \tilde{P}_{N} u(t)\|_{L^2_t(\mathbb{R}^d)} + C(\eta)^{-\delta} \right)
\]

For the high frequencies, we first use
\[
\|\phi_{>\cdot} C(\eta) \nabla P_{N_2(\eta) < \cdot} u(t)\|_{L^2_t(\mathbb{R}^d)} \leq \|\nabla (\phi_{>\cdot} C(\eta) P_{N_2(\eta) < \cdot} u(t))\|_{L^2_t(\mathbb{R}^d)} + \frac{1}{C(\eta)} \|u(t)\|_{L^2_t(\mathbb{R}^d)}
\]

to reduce matters to estimating \( \|\nabla (\phi_{>\cdot} C(\eta) P_{N_2(\eta) < \cdot} u(t))\|_{L^2_t(\mathbb{R}^d)} \), for which we use dyadic decomposition, Bernstein, and mismatch estimate Lemma 2.3.
and Proposition 4.1 to estimate
\[ \| \nabla (\phi_{>C(\eta)} P_{>N_2(\eta)} u(t)) \|_{L^2_x(\mathbb{R}^d)}^2 \]
\[ \leq \sum_{N > N_2(\eta)} \| \nabla P_N (\phi_{>C(\eta)} P_{>N_2(\eta)} u(t)) \|_{L^2_x(\mathbb{R}^d)}^2 + \| \nabla P_{\leq N_2(\eta) / 4} (\phi_{>C(\eta)} P_{>N_2(\eta)} u(t)) \|_{L^2_x(\mathbb{R}^d)}^2 \]
\[ \leq \sum_{N > N_2(\eta)} \sum_{N > N_2(\eta)} \| \nabla P_N (\phi_{>C(\eta)} P_{N < \leq 4N} u(t)) \|_{L^2_x(\mathbb{R}^d)}^2 + \| \nabla P_{N > N_2(\eta)} (\phi_{>C(\eta)} P_{>N_2(\eta)} u(t)) \|_{L^2_x(\mathbb{R}^d)}^2 \]
\[ \leq \sum_{N > N_2(\eta)} \sum_{N > N_2(\eta)} N^2 \| P_{N < 2N} u(t) \|_{L^2_x(\mathbb{R}^d)}^2 + N^{-2\varepsilon} + \sum_{N > N_2(\eta)} N(C(\eta)N)^{-10} + C(\eta)^{-10} N_2(\eta)^{-9} \]
\[ \leq \| P_{>N_2(\eta)} \nabla u_0 \|_{L^2_x(\mathbb{R}^d)}^2 + N_2(\eta)^{-2\varepsilon} + C(\eta)^{-10} N_2(\eta)^{-9}. \]

Therefore, the high frequencies give
\[ \| \phi_{>C(\eta)} \nabla P_{>N_2(\eta)} u(t) \|_{L_x^2(\mathbb{R}^d)} \lesssim \| P_{>N_2(\eta)} \nabla u_0 \|_{L^2_x(\mathbb{R}^d)} + N_2(\eta)^{-\varepsilon} + C(\eta)^{-1}. \]

Adding the estimates of three pieces together we obtain
\[ \| \phi_{>C(\eta)} \nabla u(t) \|_{L^2_x(\mathbb{R}^d)} \lesssim N_1(\eta) + C(N_1(\eta), N_2(\eta))(C(\eta)^{-\delta(d)} + \| \phi_{>\sqrt{C(\eta)}} u_0 \|_{L^2_x(\mathbb{R}^d)} + \| \phi_{>N_2(\eta)} \nabla u_0 \|_{L^2_x(\mathbb{R}^d)} + N_2(\eta)^{-\varepsilon} + C(\eta)^{-1}. \]

Now first taking \( N_1(\eta) \) sufficiently small, \( N_2(\eta) \) sufficiently large depending on \( \eta, u_0 \), then choosing \( C(\eta) \) sufficiently large depending on \( \eta, N_1(\eta), N_2(\eta) \), \( u_0 \), we obtain
\[ \| \phi_{>C(\eta)} \nabla u(t) \|_{L^2_x(\mathbb{R}^d)} \leq \eta, \]
as desired.

5. Proof of Proposition 4.1, Proposition 4.2 in 2,3 dimensions

In this Section, we prove Proposition 4.1, 4.2 in two and three dimensions. We remind the readers that the only information we need is that the solution verifies the admissible symmetry and satisfies the Duhamel formula 1.17, the weak compactness of the kinetic energy (4.4). In the proof that follows, we will use these properties many times. We first give

5.1. Proof of Proposition 4.1 in 2,3 dimensions. Let \( u \) satisfy the conditions in Proposition 4.1, we need to show \( \forall N \geq 1, \)
\[ \| \phi_{>1} P_N u(t) \|_{L^2_x(\mathbb{R}^d)} \lesssim \| P_N u_0 \|_{L^2_x(\mathbb{R}^d)} + N^{-1 - \frac{1}{4}}, \forall t \geq 0. \]

We first remark the following proof will also apply if we perturb the cutoff function by \( \phi_{>c} \) for a fixed constant \( c \) and the frequency projection \( P_N \) by the fattened one \( \tilde{P}_N \).
We begin by projecting \( u(t) \) onto incoming and outgoing waves. For the incoming wave, we use Duhamel formula backward in time, for the outgoing waves, we use Duhamel formula forward in time. We thus write
\[
\phi_1 P_N u(t) = \phi_1 P_N^\pm u(t) + \phi_1 P_N^\mp u(t) \tag{5.1}
\]
\[
= \phi_1 P_N e^{it\Delta} u_0 - i\phi_1 \int_0^t P_N^- e^{i\tau\Delta} F(u(t - \tau)) d\tau \tag{5.2}
\]
\[
+ i\phi_1 \int_0^\infty P_N^+ e^{-i\tau\Delta} F(u(t + \tau)) d\tau. \tag{5.3}
\]

The last integral should be understood in the weak \( L^2_x(\mathbb{R}^d) \) sense. We first control the linear term by Strichartz estimate:
\[
\|P_N u_0\|_{L^2_x(\mathbb{R}^d)} \lesssim \|P_N u_0\|_{L^2_x(\mathbb{R}^d)}. \tag{5.1}
\]

Now we look at the last two terms. Since the contribution from (5.2) and (5.3) will be estimated in the same way, we only give the details of the estimate of (5.3). To proceed, we first split it into different time pieces
\[
(5.3) = i\phi_1 \int_0^{\frac{1}{N}} P_N^+ e^{-i\tau\Delta} F(u(t + \tau)) d\tau 
+ i\phi_1 \int_{\frac{1}{N}}^{1} P_N^+ e^{-i\tau\Delta} F(u(t + \tau)) d\tau 
+ i \sum_{0 \leq k \leq \infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i\tau\Delta} F(u(t + \tau)) d\tau,
\]
then introducing cutoff functions in front of \( F(u) \) to further write (5.3) into
\[
(5.3) = i\phi_1 \int_0^{\frac{1}{N}} P_N^+ e^{-i\tau\Delta} \phi_{\geq \frac{1}{2}} F(u(t + \tau)) d\tau. \tag{5.4}
\]
\[
+ i\phi_1 \int_0^{\frac{1}{N}} P_N^+ e^{-i\tau\Delta} \phi_{\leq \frac{1}{2}} F(u(t + \tau)) d\tau. \tag{5.5}
\]
\[
+ i\phi_1 \int_{\frac{1}{N}}^{1} P_N^+ e^{-i\tau\Delta} \phi_{\geq \frac{1}{2}} F(u(t + \tau)) d\tau. \tag{5.6}
\]
\[
+ i\phi_1 \int_{\frac{1}{N}}^{1} P_N^+ e^{-i\tau\Delta} \phi_{\leq \frac{1}{2}} F(u(t + \tau)) d\tau. \tag{5.7}
\]
\[
+ i \sum_{0 \leq k \leq \infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i\tau\Delta} \phi_{\geq 2^k} F(u(t + \tau)) d\tau. \tag{5.8}
\]
\[
+ i \sum_{0 \leq k \leq \infty} \phi_1 \int_{2^k}^{2^{k+1}} P_N^+ e^{-i\tau\Delta} \phi_{\leq 2^k} F(u(t + \tau)) d\tau. \tag{5.9}
\]
The remaining part of the proof is devoted to estimating these six pieces.
Estimate of \((5.4)\):

For \((5.4)\), we write \(\phi'_{\frac{1}{4}}F(u) = \phi_{\frac{1}{4}}F(u\phi_{\frac{1}{4}})\), then use Strichartz estimate and mismatch estimate Lemma 2.3 to obtain,

\[
(5.4) \lesssim \|\hat{P}_N \phi_{\frac{1}{4}}F(u\phi_{\frac{1}{4}})\|_{L^1_t L^2_x([t, t+\frac{1}{N}]) \times \mathbb{R}^d} \\
\lesssim \|\hat{P}_N \phi_{\frac{1}{4}}P_{\frac{1}{8}}F(u\phi_{\frac{1}{4}})\|_{L^1_t L^2_x([t, t+\frac{1}{N}]) \times \mathbb{R}^d} + \|\hat{P}_N \phi_{\frac{1}{4}}P_{\frac{1}{2}}F(u\phi_{\frac{1}{4}})\|_{L^1_t L^2_x([0, \infty) \times \mathbb{R}^d)} \\
\lesssim \frac{1}{N}\|P_{\frac{1}{8}}F(u\phi_{\frac{1}{4}})\|_{L^\infty_t L^2_x([0, \infty) \times \mathbb{R}^d)} + \frac{1}{N}N^{-10}\|F(u\phi_{\frac{1}{4}})\|_{L^\infty_t L^2_x([0, \infty) \times \mathbb{R}^d)} \\
\lesssim N^{-\frac{2}{4}}\||\nabla|^\frac{1}{4}F(u\phi_{\frac{1}{4}})\|_{L^\infty_t L^2_x([0, \infty) \times \mathbb{R}^d)} + N^{-10}\|\phi_{\frac{1}{4}}\|_{L^\infty_t L^2_x([0, \infty) \times \mathbb{R}^d)} \lesssim \frac{d+4}{d}L^{\infty}_t L^\frac{8}{2(4d+4)}_x([0, \infty) \times \mathbb{R}^d).
\]

Note in dimensions \(d = 2, 3\), from Sobolev embedding, fractional chain rule and \((1.4)\), we have

\[
\|u\phi_{\frac{1}{4}}\|_{L^\infty_t L^\frac{2(d+4)}{4d}([0, \infty) \times \mathbb{R}^d)} \lesssim \|\phi_{\frac{1}{4}}u\|_{L^\infty_t H^1_x([0, \infty) \times \mathbb{R}^d)} \lesssim 1.
\]

\[
\||\nabla|^\frac{1}{4}F(u\phi_{\frac{1}{4}})\|_{L^\infty_t L^\frac{2}{4}([0, \infty) \times \mathbb{R}^d)} \lesssim \||\nabla|^\frac{1}{4}(\phi_{\frac{1}{4}}u)\|_{L^\infty_t L^\frac{4}{4}([0, \infty) \times \mathbb{R}^d)} \lesssim \||\phi_{\frac{1}{4}}u\|^\frac{1}{4}_{L^\infty_t L^\frac{16}{4}([0, \infty) \times \mathbb{R}^d)} \lesssim 1.
\]

We conclude the estimate of \((5.4)\) and obtain

\[
(5.10) \quad \lesssim N^{-\frac{5}{4}}.
\]

Estimate of \((5.5)\):

For \((5.5)\), since we do not have uniform control on \(\|\nabla u(t)\|_{L^2_x}\) inside the unit ball, we use the equation \((1.1)\) to replace \(F(u)\) by \((i\partial_t + \Delta)u\). We thus need the following lemma:

**Lemma 5.1.** For any \(a, b \in \mathbb{R}\), and \(C > 0\), we have

\[
\int_a^b e^{-ir\Delta}(\phi_{\leq C}(i\partial_t + \Delta)u(t + \tau))d\tau = ie^{-ib\Delta}(\phi_{\leq C}u(t + b)) - ie^{-ia\Delta}(\phi_{\leq C}u(t + a)) \quad - \quad \int_a^b e^{-ir\Delta}(u\Delta\phi_{\leq C} + 2\nabla u \cdot \nabla\phi_{\leq C})(t + \tau)d\tau.
\]

**Proof.** The proof is a simple use of Fundamental Theory of Calculus. \(\square\)
Using Lemma 5.1, we bound (5.3) as follows

$$\| (5.5) \|_{L^2_2(\mathbb{R}^d)} = \| \phi_1 \int_0^1 P_N^+ e^{-ir\Delta} (i\partial_t + \Delta) u(t + \tau) d\tau \|_{L^2_2(\mathbb{R}^d)}$$

(5.11)

$$\leq \| \phi_1 P_N^+ e^{-iN^{-1}\Delta} (\phi \leq \frac{1}{2}) u(t) \|_{L^2_2(\mathbb{R}^d)}$$

(5.12)

$$\quad + \| \phi_1 P_N^+ (\phi \leq \frac{1}{2}) u(t) \|_{L^2_2(\mathbb{R}^d)}$$

(5.13)

$$\quad + \| \phi_1 \int_0^1 P_N^+ e^{-i\Delta} (t + \tau) \Delta \phi \|_{L^2_2(\mathbb{R}^d)}$$

(5.14)

$$\quad + \| \phi_1 \int_0^1 P_N^+ e^{-i\Delta} (\nabla \phi \leq \frac{1}{2}) \cdot \nabla u(t + \tau) \|_{L^2_2(\mathbb{R}^d)}$$

(5.15)

These four terms are going to be estimated in the same manner, so we choose only to estimate (5.15) for the sake of simplicity. From Proposition 2.9, the kernel obeys the estimate

$$|(\phi_1 P_N^+ e^{-ir\Delta} \chi_{\leq \frac{1}{2}})(x, y)| \lesssim N^d \langle N|x| - N|y| \rangle^{-m} \phi_{|x| > 1} \chi_{|y| \leq \frac{1}{2}}$$

$$\lesssim N^d \langle N(x - y) \rangle^{-m}$$

$$\lesssim N^d N^{-m/2} (x - y)^{-m/2}, \tau \in [0, \frac{1}{N^2}],$$

$$|(\phi_1 P_N^+ e^{-ir\Delta} \chi_{\leq \frac{1}{2}})(x, y)| \lesssim N^d \langle N^2 \tau + N|x| - N|y| \rangle^{-m} \phi_{|x| > 1} \chi_{|y| \leq \frac{1}{2}}$$

$$\lesssim N^d \langle N^2 \tau + N|x| + N|y| \rangle^{-m} \phi_{|x| > 1} \chi_{|y| \leq \frac{1}{2}}$$

$$\lesssim N^d N^{-\frac{m}{2}} (x - y)^{-\frac{m}{2}}, \tau \in [\frac{1}{N^2}, \frac{1}{N}]$$

for any $m > 0$. We use this, Young’s inequality and (4.4) to estimate (5.15) as

$$\| (5.5) \|_{L^2_2(\mathbb{R}^d)} = \| \phi_1 \int_0^1 P_N^+ e^{-ir\Delta} \chi_{\leq \frac{1}{2}} \nabla \phi \leq \frac{1}{2} \nabla u(t + \tau) d\tau \|_{L^2_2(\mathbb{R}^d)}$$

$$\lesssim \sup_{\tau \in [0, \frac{1}{N}]} \frac{1}{N} \| \int (\phi_1 P_N^+ e^{-ir\Delta} \chi_{\leq \frac{1}{2}})(x, y) \nabla \phi \leq \frac{1}{2} \cdot \nabla u(t + \tau, y) \|_{L^2_2(\mathbb{R}^d)}$$

$$\lesssim N^{-10} \| \nabla u \phi \|_{L^2_2(\mathbb{R}^d)}$$

$$\lesssim N^{-10}.$$

The estimates of other terms give the same contribution, so finally we have

$$\| (5.5) \|_{L^2_2(\mathbb{R}^d)} \lesssim N^{-9}.$$
Now we estimate (5.16). We first write $\phi_{>N}F(u) = \phi_{>N/d}F(\phi_{>N/2}u)$, then further decompose it by introducing a frequency projection:

$$\\begin{align*}
(5.6) &= i\phi_1 \int_0^1 P_N e^{-it\Delta} \phi_{>N} P_{\leq N/8} F(\phi_{>N/2}u)(t + \tau) d\tau \\
+ i\phi_1 \int_0^1 P_N e^{-it\Delta} \phi_{>N} P_{\leq N/8} F(\phi_{>N/2})(t + \tau) d\tau.
\end{align*}$$

(5.17) can be estimated by weighted Strichartz estimate Lemma 2.8 Bern-stein and (4.4). In 2-d, we have

$$\|5.16\|_{L_x^2(\mathbb{R}^d)} \lesssim \|x^{-\frac{1}{2}} \phi_{>N} P_{\geq N/8} F(u \phi_{>N/2})(t + \tau)\|_{L_t^4 L_x^4([\frac{1}{4},1] \times \mathbb{R}^2)}$$

$$\lesssim N^{-\frac{1}{2}} \|\tau^{-\frac{1}{2}}\|_{L_\tau^4([\frac{1}{4},1])} \|P_{\geq N/8} F(u \phi_{>N/2})\|_{L_x^\infty L_t^4([0,\infty) \times \mathbb{R}^2)}$$

$$\lesssim N^{-\frac{1}{2}} N^{-1} \|\nabla (u \phi_{>N/2})\|_{L_x^\infty L_t^2([0,\infty) \times \mathbb{R}^2)} \|u \phi_{>N/2}\|_{L_t^4 L_x^4([0,\infty) \times \mathbb{R}^2)}$$

$$\lesssim N^{-\frac{3}{2}}.$$  

In three dimensions, we have

$$\|5.16\|_{L_x^2(\mathbb{R}^d)} \lesssim \|x^{-\frac{1}{2}} \phi_{>N} P_{\geq N/8} F(u \phi_{>N/2})(t + \tau)\|_{L_t^4 L_x^4([\frac{1}{4},1] \times \mathbb{R}^2)}$$

$$\lesssim N^{-\frac{1}{2}} \|\tau^{-\frac{1}{2}}\|_{L_\tau^4([\frac{1}{4},1])} \|P_{\geq N/8} F(u \phi_{>N/2})\|_{L_x^\infty L_t^4([0,\infty) \times \mathbb{R}^3)}$$

$$\lesssim N^{-\frac{1}{2}} N^{-1} \|\nabla (u \phi_{>N/2})\|_{L_x^\infty L_t^2([0,\infty) \times \mathbb{R}^3)} \|u \phi_{>N/2}\|_{L_t^4 L_x^4([0,\infty) \times \mathbb{R}^3)}$$

$$\lesssim N^{-\frac{3}{2}}.$$  

Therefore, in two and three dimensions, we all have

$$5.16 \lesssim N^{-\frac{3}{2}}.$$

To estimate (5.17), we simply use the Strichartz and mismatch estimate Lemma 2.8 and (4.4) to obtain

$$5.17 \lesssim \|\tilde{P} \phi_{>N} P_{\leq N/8} F(u \phi_{>N/2})\|_{L_t^4 L_x^2([t+\frac{1}{4},t+1] \times \mathbb{R}^d)}$$

$$\lesssim N^{-10} \|F(\phi_{>N/2} u)\|_{L_t^\infty L_x^2([0,\infty) \times \mathbb{R}^d)}$$

$$\lesssim N^{-10} \|\phi_{>N/2} u\|_{L_{\tau t}^{d+4}([t,\infty) \times \mathbb{R}^d)}$$

$$\lesssim N^{-10}. $$
This completes the estimate of (5.6) so we get

\[ \| (5.6) \|_{L^2(\mathbb{R}^d)} \lesssim N^{-\frac{3}{2}}. \] (5.26)

**Estimate of (5.7)**

Now we treat the term (5.7). The idea will be the same with that of (5.5): we replace \( F(u) \) by \((i\partial_t + \Delta)u\). However, the situation here is a bit trickier since the cutoff function also depends on \( \tau \). By counting this factor, we improve Lemma 5.1 to the following

**Lemma 5.2.** For any \( a,b \geq 0 \), we have

\[
\int_a^b e^{-i\tau \Delta} \phi_{\leq N\tau/2} (i\partial_\tau + \Delta)u(t + \tau) d\tau \\
= ie^{-ib\Delta} \phi_{\leq Nb} u(t + b) - ie^{-ia\Delta} \phi_{\leq Na} u(t + a) \\
- \int_a^b e^{-i\tau \Delta} (u \Delta \phi_{\leq N\tau} - 2\nabla u \cdot \nabla \phi_{\leq N\tau})(t + \tau) d\tau \\
+ i \int_a^b e^{-i\tau \Delta} (\frac{y}{N\tau^2} \cdot (\nabla \phi_{\leq 1})(\frac{y}{N\tau})u(t + \tau)) d\tau.
\]

Having this Lemma, we are able to write (5.7) as follows

\[ (5.7) = i\phi_1 \int_{\frac{1}{N}} \sum_{\frac{1}{N}} e^{-i\tau \Delta} \phi_{\leq N\tau/2} (i\partial_\tau + \Delta)u(t + \tau) d\tau \\
= -\phi_1 P_N^+ e^{-i\Delta} \phi_{\leq N/2} u(t + 1) \\
+ \phi_1 P_N^+ e^{-i\Delta} \phi_{\leq 1} u(t + 1) \\
- i\phi_1 \int_{\frac{1}{N}} P_N^+ e^{-i\tau \Delta} (u \Delta \phi_{\leq N\tau/2} + 2\nabla u \cdot \nabla \phi_{\leq N\tau/2})(t + \tau) d\tau \\
- \phi_1 \int_{\frac{1}{N}} P_N^+ e^{-i\tau \Delta} (\frac{y}{N\tau^2} \cdot (\nabla \phi_{\leq 1})(\frac{y}{N\tau})u(t + \tau)) d\tau. \] (5.27)

In the estimates of these terms, we will use the decay estimate of the kernel: for any \( \frac{1}{N} \leq \tau \leq 1 \), we have

\[
|\phi_1 P_N^+ e^{-i\tau \Delta} \chi_{\leq N\tau/2}(x,y)| \lesssim N^d (N^2\tau + N|x| - N|y|)^{-m} \\
\lesssim N^d N^{-\frac{m}{2}} (N(x - y))^{-\frac{m}{2}}, \ \forall m \geq 0.
\]
We only give the estimate of the (5.27) since the other terms will be the same in principle. Using this decay estimate and Young’s inequality we have

\[ \| (5.27) \|_{L^2_x(\mathbb{R}^d)} \]
\[ = \| \int_1^1 \int_{\mathbb{R}^d} (\phi_{>1} P^+ e^{-i\tau \Delta} \chi \leq N \tau / 2}(x, y) \cdot \left( \frac{y}{N \tau} \right) u(t + \tau, y) dy d\tau \|_{L^2_x(\mathbb{R}^d)} \]
\[ \lesssim N^{-11} \int_1^1 \| \frac{x}{N \tau} \phi_{|x| > \frac{1}{2} N \tau} u(t + \tau) \|_{L^2_x(\mathbb{R}^d)} d\tau \]
\[ \lesssim N^{-11} N \| u \|_{L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R}^d)} \]
\[ \lesssim N^{-10}. \]

To conclude, we have

\[ \| (5.7) \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{-10}. \] (5.28)

To deal with (5.8), we will need the following lemma which controls the weighted Strichartz norm of the solution with spatial cutoff on a unit time interval. We have

**Lemma 5.3.** Let \( d = 2, 3 \). Let \( I \) be a unit time interval, i.e., \(|I| = 1\). Let \( u \) be an \( H^1_x \) solution of (1.1) satisfying the weak compactness condition (4.4). Then we have

\[ \| |x|^{2(d-1)/q} \phi_{>1} u \|_{L^q_t L^{2d/(d-2)}_x(I \times \mathbb{R}^d)} \lesssim 1. \]

**Proof.** The case \( q = \infty \) is trivial. We only need to prove the case \( q = 4 \) since the general case follows by interpolation.

From radial Sobolev embedding Lemma 2.4 we get

\[ \| |x|^{d/2} P_n (\phi_{>1} u) \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{1/2} \| P_n (\phi_{>1} u) \|_{L^2_x(\mathbb{R}^d)} \lesssim \min(N^{1/2}, N^{-1/2}). \]

The desired estimate follows from the summation over dyadic pieces. \( \square \)

Now we treat the \( k \)-th piece in (5.8) which we denote by (5.8)\_\( k \). We put a frequency cutoff and further write it into

\[ (5.8)\_k : = i \phi_{>1} \int_{2k}^{2k+1} P^+_N e^{-i\tau \Delta} \phi_{>N2^k/4} F(\phi_{>N2^k/4} u)(t + \tau) d\tau \] (5.29)
\[ = i \phi_{>1} \int_{2k}^{2k+1} P^+_N e^{-i\tau \Delta} \phi_{>N2^k/4} P_{>N2^k/4} F(\phi_{>N2^k/4} u)(t + \tau) d\tau \] (5.30)
\[ + i \phi_{>1} \int_{2k}^{2k+1} P^+_N e^{-i\tau \Delta} \phi_{>N2^k/4} P_{\leq N2^k/4} F(\phi_{>N2^k/4} u)(t + \tau) d\tau \] (5.31)

We first look at (5.31) which contains a frequency mismatch. Hence from the \( L^2_x \) boundedness of \( \phi_{>1} P^+_N \), Strichartz and mismatch estimate Lemma
and (4.14), we have
\[
\left\| P_{N} \phi_{N} \right\|_{2} \lesssim \left\| P_{N} \phi_{N} \right\|_{2} \lesssim \left\| F(\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{d}}
\]
\[
\lesssim (N^{2}2^{k})^{-10}2^{k} \left\| F(\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(0,\infty) \times \mathbb{R}^{d}}
\]
\[
\lesssim N^{-20}2^{-8k} \left\| F(\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(0,\infty) \times \mathbb{R}^{d}} \lesssim (N^{2}2^{k})^{-2}.
\]
For the another term (5.31), we discuss the cases \(d = 2, 3\) respectively. In 2 dimensions, using weighted Strichartz Lemma 2.8, Bernstein and Lemma 5.3 we have
\[
\left\| P_{N} \phi_{N} \right\|_{2} \lesssim \left\| |x|^{-\frac{1}{2}} \phi_{N} P_{N} \right\|_{2} \lesssim \left\| F(\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{2}}
\]
\[
\lesssim N^{-\frac{1}{2}}2^{-4k} \left\| F(\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{2}}
\]
\[
\lesssim N^{-\frac{1}{2}}2^{-4k} \left\| \nabla(u\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{2}} \times \left\| u\phi_{N} / 4u \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{2}}
\]
\[
\lesssim N^{-2}2^{-4k} \left\| |x|^{-\frac{1}{2}} \phi_{N} \right\|_{2} \lesssim N^{-2}2^{-4k}.
\]
In 3 dimensions the corresponding estimates are as follows:
\[
\left\| P_{N} \phi_{N} \right\|_{3} \lesssim \left\| |x|^{-\frac{1}{2}} \phi_{N} P_{N} \right\|_{3} \lesssim \left\| F(\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{3}}
\]
\[
\lesssim N^{-\frac{1}{2}}2^{-4k} \left\| \nabla(u\phi_{N} / 4u) \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{3}} \times \left\| u\phi_{N} / 4u \right\|_{L_{t}^{\frac{d+4}{d}}L_{x}^{\alpha}(t+2k,t+2k+1) \times \mathbb{R}^{3}}
\]
\[
\lesssim N^{-\frac{1}{2}}2^{-4k} \left\| |x|^{-\frac{1}{2}} \phi_{N} \right\|_{3} \lesssim N^{-\frac{1}{2}}2^{-4k}.
\]
which is acceptable. Collecting the estimates for (5.30), (5.31), we obtain
\[
\left\| P_{N} \phi_{N} \right\|_{1} \lesssim N^{-\frac{1}{2}}2^{-4k}.
\]
Summing in \(k\) gives that
\[
\left\| P_{N} \phi_{N} \right\|_{1} \lesssim N^{-\frac{1}{2}}2^{-4k}.
\]

Estimate of the last piece (5.9).
We have only one piece (5.9) left for which we will replace \(F(u)\) by \((i\partial_{t} + \Delta)u\) and use the kernel estimate. Denote (5.30) the \(k\)-th piece, we use
Lemma 5.1 to write

\[(5.9)\]
\[k = -\phi e^{-2k\Delta} + i\phi \int_{2k}^{2k+1} \mathcal{P} e^{-i\tau \Delta} \left(\mathcal{F}_{\leq \mathcal{N}/2} (t, \tau)\right) d\tau \]

Note for any \(\tau \in [2^k, 2^{k+1}]\), from Lemma 2.9, the kernel obeys

\[|\phi| \leq N^d (N^2 \tau + N|x| + N|y|)^{-10} (N(x - y))^{10}.\]

We immediately obtain

\[\| (5.9) \|_{L^2_x} \lesssim N^{-10} 2^{-k},\]

after a simple application of Young’s inequality.

Summing in \(k\) gives us

\[\| (5.9) \|_{L^2_x} \lesssim N^{-10}.\]

We finish the estimates of all the six pieces (5.4)-(5.9). Thus we succeed to estimate \(\| (5.3) \|_{L^2_x} \lesssim N^{-1 - \frac{1}{16}}\). Collecting the estimates of (5.1) to (5.3), we establish Proposition 4.1 in 2,3 dimensions.

5.2. Proof of Proposition 4.2 in 2,3 dimensions. This subsection, we aim to prove that

\[\| \phi \mathcal{P} \|_{L^2_x} \lesssim \| \phi \mathcal{P} \|_{L^2_x} + R^{-\frac{1}{16}},\]

holds for all \(t \geq 0, N \in [N_1, N_2]\) and \(R > R_0 (N_1, N_2, u) \gg 1\). We remark that since we have the freedom to choose sufficiently large \(R\), we will no longer accurately keep track of the power of \(N\), and often denote the power of \(N\) by a constant \(C\) which can vary from one line to another.

To begin with, we use the in-out decomposition and forward, backward Duhamel formula to write

\[\phi \mathcal{P} = \phi \mathcal{P} - i\phi \int_{0}^{t} \mathcal{P} e^{-i\tau \Delta} \left(\mathcal{F}_{\leq \mathcal{N}/2} (u(t - \tau))\right) d\tau \]

Again, the last integral should be understood in weak \(L^2_x\) sense. From Proposition 2.9 the kernel obeys:

\[|\phi \mathcal{P} e^{-i\Delta} (x, y)| \leq N^d (N(x - y))^{-20} \lesssim N^C R^{-10} (N(x - y))^{10}.\]
Using this and the $L^2_t$ boundedness of $\phi_{>R}P_N^-$, we have

\[ (5.33) = \| \phi_{>R}P_N^+e^{it\Delta}u_0 \|_{L^2_t(\mathbb{R}^d)} \]

\[ \lesssim \| \phi_{>R}P_N^+e^{it\Delta}\phi_{>R/2u_0} \|_{L^2_t(\mathbb{R}^d)} + \| \phi_{>R}P_N^+e^{it\Delta}\phi_{\leq R/2u_0} \|_{L^2_t(\mathbb{R}^d)} \]

\[ \lesssim \| \phi_{>R/2u_0} \|_{L^2_t(\mathbb{R}^d)} + NC^{-10} \]

\[ \lesssim \| \phi_{>R/2u_0} \|_{L^2_t(\mathbb{R}^d)} + R^{-5}. \]

Since (5.32), (5.35) will be estimated in the same manner, we choose to estimate one of them. We first split (5.35) into time pieces:

\[ (5.35) = i\phi_{>R}\int_0^1 P^+_N e^{-i\tau\Delta} F(u(t+\tau))d\tau \]

\[ + i\phi_{>R}\sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} P^+_N e^{-i\tau\Delta} F(u(t+\tau))d\tau. \]

Then introduce cutoff to further write it into

\[ (5.36) = i\phi_{>R}\int_0^1 P^+_N e^{-i\tau\Delta} \phi_{\leq R/2} F(u(t+\tau))d\tau \]

\[ + i\phi_{>R}\int_0^1 P^+_N e^{-i\tau\Delta} \phi_{>R/2} F(u(t+\tau))d\tau \]

\[ + i\sum_{k=0}^{\infty} \phi_{>R}\int_{2^k}^{2^{k+1}} P^+_N e^{-i\tau\Delta} \phi_{\leq R/2 N^k} F(u(t+\tau))d\tau \]

\[ + i\sum_{k=0}^{\infty} \phi_{>R}\int_{2^k}^{2^{k+1}} P^+_N e^{-i\tau\Delta} \phi_{ \leq R/2 N^k} F(u(t+\tau))d\tau. \]

Here, $\gamma$ is a small constant, we can choose for example $\gamma = \frac{1}{100}$.

In the above four pieces, the main contribution comes from (5.37), (5.38) where $F(u)$ lies in large radius. We are going to estimate them by using weighted Strichartz. For the tail part (5.36), (5.39), we replace $F(u)$ by $i\partial_t u + \Delta u$ and use the decay estimate of the kernel. In what follows, we will only present the details of the last two terms. The former two are much simpler so we skip them.

We first treat the $k$-th piece in (5.38), which we denote by (5.38)$_k$. For the sake of convenience, we write $A := \gamma R^k N^{t/2} \frac{2^k}{2^k}$. In two dimensions, using weighted Strichartz Lemma (2.28), we have

\[ \| (5.38)_k \|_{L^2_t(\mathbb{R}^d)} = \| \phi_{>R} \int_{2^k}^{2^{k+1}} P^+_N e^{-i\tau\Delta} \phi_{>A} F(\phi_{>A/2u})(t+\tau)d\tau \|_{L^2_t(\mathbb{R}^d)} \]

\[ \lesssim \| |\gamma|^{-\frac{3}{2}} F(\phi_{>A/2u}) \|_{L^\infty_t L^1(\mathbb{R}^2)} \]

\[ \lesssim A^{-\frac{3}{2}} \| F(\phi_{>A/2u}) \|_{L^\infty_t L^1(\mathbb{R}^2)} \]

\[ \lesssim A^{-\frac{1}{2}} \| |\gamma|^{-\frac{1}{2}} (\phi_{>A/2u}) \|_{L^\infty_t L^\infty(\mathbb{R}^2)} \| \phi_{>A/2u} \|_{L^\infty_t L^\infty(\mathbb{R}^2)} \| \phi_{>A/2u} \|_{L^\infty_t L^\infty(\mathbb{R}^2)}^{t+2k+1} \| \phi_{>A/2u} \|_{L^\infty_t L^\infty(\mathbb{R}^2)}^{t+2k+1} \| \phi_{>A/2u} \|_{L^\infty_t L^\infty(\mathbb{R}^2)}^{t+2k+1}. \]
Using Lemma 2.8 we continue the estimate
\[
\| (5.38)_k \|_{L^2_t L^6_x} \lesssim A^{-1/2} R^{-k}
\]
\[
\lesssim R^{-1/8} 2^{-k} N^{-7/8} \lesssim R^{-1/8} 2^{-k}.
\]
In three dimensions, the corresponding computation is as follows: applying weighted Strichartz estimate Lemma 2.8 with \( q = 8 \), Lemma 5.3 and (4.4) we obtain
\[
\| (5.39)_k \|_{L^2_t L^6_x} \lesssim R^{-1/16}.
\]
This closes the estimates for (5.38).

Now let us treat the \( k \)-th piece in (5.39), we denote it by (5.39)_k. Substituting \( F(u) \) by \((i\partial_t + \Delta) u \) and using Lemma 5.1 we have
\[
(5.39)_k = -\phi_R P_N^+ e^{-i2^{k+1} \Delta} (\phi_{\leq A} u (t + 2^{k+1})) + \phi_R P_N^+ e^{-i2^k \Delta} (\phi_{\leq A} u (t + 2^k)) - i\phi_R \int_{2^k}^{2^{k+1}} P_N^+ e^{-i\Delta} (u\Delta \phi_{\leq A} + 2\nabla u \cdot \nabla \phi_{\leq A}) (t + \tau) d\tau. \tag{5.43}
\]
Note for \(|y| \leq A, |x| > R, \tau \in [2^k, 2^{k+1}]\), Young’s inequality gives that
\[
|y| \lesssim R^{-1/2} 2^k N^{-1/2} N \lesssim |x| + N \tau.
\]
we can control the kernel as: for any \( \tau \in [2^k, 2^{k+1}] \),
\[
|\phi_R P_N^+ e^{-i\Delta} \phi_{\leq A} (x, y)| \lesssim N^d (N^2 \tau + N |x| - N |y|)^{-15} \lesssim N^{-10} R^{-10} (N (x - y))^{-10}.
\]
Using this, Young’s inequality and (4.4) we immediately get
\[
\| (5.39)_k \|_{L^2_t L^6_x} \lesssim 2^{-5k} R^{-9} (\| u \|_{L^\infty_t L^6_x} + \| u \phi_{>1} \|_{L^\infty_t H_x^1}) \lesssim 2^{-5k} R^{-9}.
\]
Summing in \( k \) gives that
\[
\| (5.39) \|_{L^2_t L^6_x} \lesssim R^{-5}. \tag{5.44}
\]
Now we have estimates for all the pieces ready to conclude that
\[
\| (5.35) \|_{L^2_t L^6_x} \lesssim R^{-5}. \tag{5.45}
\]
Thus (5.32) is established and we proved Proposition 4.2 in two and three dimensions.

6. 2+2 Dimensions with Splitting-Spherical Symmetry

6.1. Introduction and tools adapted to four dimensions. In this section, we prove the frequency decay estimate Proposition 4.1 and spatial decay estimate Proposition 4.2 in four dimensions with admissible symmetry. From the definition, this requires the initial data $u_0$ to be spherically symmetric in each of the two two-dimensional subspaces. More precisely, after a possible permutation or relabelling of the coordinates, the solution $u$ satisfies

$$u(x_1, x_2, x_3, x_4, t) = u(x_1^2 + x_2^2, x_3 + x_4, t).$$

Compared with the spherically symmetrical case, the situation here is more complicated. As revealed in the 2,3-dimensional case, for any dyadic number $N \gg 1$, the additional decay $N^{-1-\varepsilon}$ is produced by those spherically symmetric waves which travels at speed $N$, and which is supported away from the origin. In our 2+2 splitting-spherically symmetric case, due to the anisotropicity, one has to look at waves which travels at the speed $N$ in one of the two subspaces in order to take advantage of the partially spherical symmetry. However, a problem one immediately encounters is that in the subspace where the waves are travelling, these waves are not supported away from the origin. To treat this problem, we will give in this section a uniform way (applying the in-out decomposition technique adapted to the subspace) to deal with the close-to-origin piece and away-from-origin piece. The price we pay is Lemma 2.10 which involves the kernel estimate close to the origin. In the next section, we present a different way to deal with this close-to-origin piece.

Besides the anisotropicity which makes the problem complicated in disguise, the actual amount of computation is slightly lighter than the low dimensional cases since in dimensions $d \geq 4$, one can always find a space so that the nonlinearity in this space can be controlled by the mere mass of $u$.

Before moving on, we introduce some notations that will be used throughout this section.

We denote $x^{12}$ to be the two dimensional vector: $(x_1, x_2)$, and $x^{34}$ as $(x_3, x_4)$. The differential operator $\nabla^{12}$, $\nabla^{34}$, $\Delta_{12}$, $\Delta_{34}$ denote the corresponding operators restricted to $(x_1, x_2)$ or $(x_3, x_4)$ plane.

By the same token, we define $P^{12}_N$, $P^{34}_N$ as two dimensional Littlewood-Paley projection operators. $P^{12\pm}_N$, $P^{34\pm}_N$ are the corresponding out-going and in-coming wave projection operator. By augmenting with the identity operator on the other two variables, they can act on a four-dimensional function.

We will also use the convention that when there is no superscript, the notation should be understood as the normal one defined on $\mathbb{R}^4$. For example,
$P_N$ is the usual four-dimensional littlewood-Paley projection operator; $\phi_{>R}$ should be understood as $\phi_{|x|>R}$ on $\mathbb{R}^4$ as well.

We need the following weighted Strichartz estimate for splitting-spherically symmetric functions. We have

**Lemma 6.1. (Weighted Strichartz estimate in splitting-spherical symmetry case).**

Let $I$ be a time interval, $t_0 \in I$. Let $u_0 \in L^2_x(\mathbb{R}^4)$, $f \in L^{\frac{2(d+2)}{d}}_{t,x}(I \times \mathbb{R}^4)$ be splitting-spherically symmetric. Then the function $u(t,x)$ defined by

$$u(t) := e^{it\Delta}u_0 - i \int_{t_0}^t e^{i(t-s)\Delta} f(s)ds$$

is also splitting-spherically symmetrical and obeys the estimate. For any $2 < q \leq \infty$,

$$\| |x|^{\frac{1}{2}} e^{it\Delta}u_0 \|_{L^q_t L^{\frac{6q}{3q-4}}_x(\mathbb{R} \times \mathbb{R}^4)} \lesssim \| u_0 \|_{L^2_x(\mathbb{R}^4)} + \| f \|_{L^{\frac{2(d+2)}{d}}_{t,x}(I \times \mathbb{R}^4)}.$$

**The same conclusion holds true if we replace $x^{12}$ by $x^{34}$.**

**Proof.** The proof is an adaptation to that in [16, 17]. By the standard Strichartz estimate Lemma 2.7 and Christ-Kiselev lemma, it suffices to prove

$$\| |x|^{\frac{1}{2}} e^{it\Delta}u_0 \|_{L^q_t L^{\frac{6q}{3q-4}}_x(\mathbb{R} \times \mathbb{R}^4)} \lesssim \| u_0 \|_{L^2_x(\mathbb{R}^4)}.$$

(6.1)

By $TT^*$ argument, (6.1) is reduced to proving the following

$$\| \int |x|^{\frac{1}{2}} e^{i(s-t)\Delta} |y|^{\frac{1}{2}} f(t)dt \|_{L^q_t L^{\frac{6q}{3q-4}}_x(\mathbb{R} \times \mathbb{R}^4)} \lesssim \| f \|_{L^q_t L^{\frac{6q}{3q-4}}_x(\mathbb{R} \times \mathbb{R}^4)}.$$

However, this is the consequence of Hardy-Littlewood-Sobolev inequality and the following dispersive estimate

$$\| |x|^{\frac{1}{2}} e^{it\Delta} |y|^{\frac{1}{2}} f \|_{L^q_t L^{\frac{6q}{3q-4}}_x(\mathbb{R} \times \mathbb{R}^4)} \lesssim |t|^\frac{-2}{q} \| f \|_{L^q_t L^{\frac{6q}{3q-4}}_x(\mathbb{R} \times \mathbb{R}^4)}.$$

(6.2)

We now prove (6.2) for all $\frac{4}{3} \leq q \leq \infty$. When $q = \infty$, this is just the trivial estimate

$$\| e^{it\Delta} f \|_{L^2_x(\mathbb{R}^4)} \lesssim \| f \|_{L^2_x(\mathbb{R}^4)}.$$

(6.3)

When $q = \frac{4}{3}$, this is the pointwise estimate

$$\| |x|^{\frac{1}{2}} e^{it\Delta} |y|^{\frac{1}{2}} f \|_{L^\infty_x(\mathbb{R}^4)} \lesssim |t|^\frac{-2}{3} \| f \|_{L^2_x(\mathbb{R}^4)}.$$

(6.4)
for all $f$ being spherically symmetric in $(x_1, x_2)$ variable. By passing to radial coordinate, we write

$$
\left( |x_1^{12}|^\frac{1}{2} e^{it\Delta} |y_1^{12}|^\frac{1}{2}\right)(x, y) = |x_1^{12}|^\frac{1}{2} |y_1^{12}|^\frac{1}{2} \frac{1}{(4\pi it)^\frac{1}{2}} e^{i((x_3-y_3)^2 + (x_4-y_4)^2)/4t} e^{i|x_2^{12}|^2 + |y_2^{12}|^2)^2/4t} \int_0^{2\pi} e^{-i|x_2^{12}|y_2^{12}|\cos \theta/2t} d\theta.
$$

A standard stationary phase yields that

$$
\left( |x_1^{12}|^\frac{1}{2} e^{it\Delta} |y_1^{12}|^\frac{1}{2}\right)(x, y) \lesssim |t|^{-\frac{3}{2}},
$$

from which (6.4) follows.

Finally we obtain (6.2) by interpolating between (6.3) and (6.4). Indeed, (6.3) and (6.4) imply that the operator

$$
|x_1^{12}|^\frac{1}{2} e^{it\Delta} : L^1_x(|x_1^{12}|^{-\frac{1}{2}} dx) \rightarrow L^\infty_x (dx),
$$

with bound $|t|^{-\frac{3}{2}}$, and

$$
|x_1^{12}|^\frac{4}{2} e^{it\Delta} : L^2_x (dx) \rightarrow L^2_x (|x_1^{12}|^{-1} dx),
$$

with bound 1. Using interpolation between $L^p$ spaces with changing of measures (see [1], page 120), we obtain (6.2). Lemma 6.1 is proved.

We need the following Lemma to control the nonlinearity when $F(u)$ is located at large radii.

**Lemma 6.2.** Let $N$ be dyadic number. Let $L > 0$ be a constant such that $LN \gtrsim 1$. Then we have

$$
\|P_{>N} F(\phi_{|x_1^{12}|>LN}u)\|_{L^\frac{8}{\beta}_x(\mathbb{R}^4)} \lesssim N^{-\frac{2}{\beta}} (NL)^{-\frac{2}{\beta}} = N^{-\frac{2}{\beta}} L^{-\frac{2}{\beta}}.
$$

(6.5)

The same estimate will still hold if we replace $x_1^{12}$ by $x_3^{34}$ and $P_{>N}$ by the fattened operator $P_{\gtrsim N}$. 

□
Proof. This is the simple use of Bernstein, Hölder, radial Sobolev embedding, and Lemma \[1.10\]. We give the estimates as follows:
\[
\|P > N F(\phi|_{|x|^{12} > LN}|u)\|_{L^2_x(\mathbb{R}^4)} \\
\leq N^{-\frac{5}{2}} \|\nabla \tilde{\phi} F(\phi|_{|x|^{12} > LN}|u)\|_{L^2_x(\mathbb{R}^4)} \\
\leq N^{-\frac{5}{2}} \|\nabla \tilde{\phi} (\phi|_{|x|^{12} > LN}|u)\|_{L^2_x(\mathbb{R}^4)} \|\phi|_{|x|^{12} > LN}|u\|_{L^{3+}_x(\mathbb{R}^4)} \\
\leq N^{-\frac{5}{2}} (LN)^{-\frac{2}{3}} \|\phi_{> 1} u\|_{H^1_x(\mathbb{R}^4)} \|x^{12} \tilde{\phi} \phi|_{|x|^{12} > LN}|u\|_{L^{3+}_{x_1, x_2}(\mathbb{R}^2)} \|L^{3+}_{x_3, x_4}(\mathbb{R}^2) \\
\leq N^{-\frac{5}{2}} L^{-\frac{2}{3}} \|\nabla x^{12} \tilde{\phi} (\phi|_{|x|^{12} > LN}|u)\|_{L^2_{x_1, x_2}(\mathbb{R}^2)} \|L^{3+}_{x_3, x_4}(\mathbb{R}^2) \\
\leq N^{-\frac{5}{2}} L^{-\frac{2}{3}} \|\nabla x^{34} \tilde{\phi} \nabla x^{12} \tilde{\phi} (\phi|_{|x|^{12} > LN}|u)\|_{L^2_{x_1, x_2}(\mathbb{R}^2)} \\
\leq N^{-\frac{5}{2}} L^{-\frac{2}{3}} \|\phi_{> LN}|u\|_{H^1_x(\mathbb{R}^4)} \\
\leq N^{-\frac{5}{2}} L^{-\frac{2}{3}}.
\]
The lemma is proved. \(\square\)

Now we are ready to give the

6.2 Proof of Proposition \[4.1\]. In this subsection, we prove Proposition \[4.1\] in four dimensions with splitting-spherical symmetry. Note that for \(|\xi| \sim N\), \(\xi \in \mathbb{R}^4\), since \(|\xi|^2 = |\xi^{12}|^2 + |\xi^{34}|^2\), we have either \(|\xi^{12}| \sim N\), or \(|\xi^{34}| \sim N\) (or both hold), therefore
\[
P_N \sim P_N P_N^{12} P_N^{34} + P_N P_N^{34} P_N^{12}.
\]
For notational simplicity, we shall drop the tilde and write instead
\[
P_N \sim P_N (P_N^{12} P_N^{34} + P_N^{34} P_N^{12}).
\]
Using triangle we obtain
\[
\|\phi_{> 10} P_N|u(t)\|_{L^2_x(\mathbb{R}^4)} \leq \|\phi_{> 10} P_N P_N^{12} P_N^{34} u(t)\|_{L^2_x(\mathbb{R}^4)} + \|\phi_{> 10} P_N P_N^{34} P_N^{12} u(t)\|_{L^2_x(\mathbb{R}^4)}
\]
By symmetry, we only need to consider the first one.
\[
\|\phi_{> 10} P_N P_N^{12} P_N^{34} u(t)\|_{L^2_x(\mathbb{R}^4)}, \tag{6.6}
\]
Commuting the spatial and frequency cutoff in \(6.6\), we have from Lemma \[2.3\] that
\[
\|\phi_{> 10} P_N P_N^{12} P_N^{34} u(t)\|_{L^2_x(\mathbb{R}^4)} \\
\leq \|\phi_{> 10} P_N \phi_{> 5} P_N^{12} P_N^{34} u(t)\|_{L^2_x(\mathbb{R}^4)} \\
+ \|\phi_{> 10} P_N \phi_{\leq 5} P_N^{12} P_N^{34} u(t)\|_{L^2_x(\mathbb{R}^4)} \\
\leq \|\phi_{> 10} P_N \phi_{> 5} P_N^{12} P_N^{34} u(t)\|_{L^2_x(\mathbb{R}^4)} + (5N)^{-10} \|u(t)\|_{L^2_x(\mathbb{R}^4)} \\
\leq N^{-9} + \|\phi_{> 5} P_N^{34} P_N^{12} u(t)\|_{L^2_x(\mathbb{R}^4)}.
\]
Therefore, Proposition 4.1 will follow if we can prove
\[ \| \phi > 5 P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \lesssim \| \tilde{P}_N u_0 \|_{L^2(\mathbb{R}^4)} + N^{-1 - \frac{1}{10}}. \] (6.7)

Like before, the additional decay factor of \( N \) will either come from the mismatch between the spatial cutoff and the linear propagator or the spherical symmetry in \((x_1, x_2)\) (resp. \((x_3, x_4)\)) variables. To this end, we introduce a spatial cutoff in \( x^{12} \) variables and further decompose the LHS of (6.7) into:
\[ \| \phi > 5 P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \leq \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \] (6.8)
\[ + \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \] (6.9)
\[ + \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| > \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \] (6.10)

As we said, the reason for doing so is that the Hankel function has logarithmic singularity at the origin. So it is necessary to isolate the singular regime where \( |x^{12}| \leq \frac{1}{N} \) and deal with the near-origin regime and away-from-origin regime separately.

(6.8) can be estimated rather directly by the Hölder and Bernstein inequalities:
\[ \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \lesssim \| \frac{1}{N^3} P_N^{12} u(t) \|_{L^2_{x^{12},x_2}(\mathbb{R}^2)} \| L^2_{x^{3,4}}(\mathbb{R}^2) \]
\[ \lesssim \frac{1}{N^2} \| P_N^{12} u(t) \|_{L^2_2(\mathbb{R}^4)} \lesssim \frac{1}{N^2}. \]

Thus the contribution due to (6.8) is acceptable. Next we estimate (6.9). Using the in-out decomposition with respect to \( x^{12} \) variable and Duhamel formula, we estimate
\[ \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \lesssim \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} e^{it \Delta} u_0 \|_{L^2_2(\mathbb{R}^4)} \] (6.11)
\[ + \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \int_0^t e^{i \tau \Delta} F(u(t - \tau)) d\tau \|_{L^2_2(\mathbb{R}^4)} \] (6.12)
\[ + \| \phi > 5 \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} u(t) \|_{L^2_2(\mathbb{R}^4)} \int_0^\infty e^{i \tau \Delta} F(u(t + \tau)) d\tau \|_{L^2_2(\mathbb{R}^4)}. \] (6.13)

The last integral should be understood to hold in the weak \( L^2_2 \) sense. For (6.11), we insert a fattened \( P_N \) and use the \( L^2 \) boundedness of \( P_N^{12} \) to get
\[ \| \phi_{\frac{1}{N^3} < |x^{12}| \leq \frac{1}{N}} P_N^{12} P_{\lesssim N}^{34} e^{it \Delta} \tilde{P}_N u_0 \|_{L^2_2(\mathbb{R}^4)} \]
\[ \lesssim \| \tilde{P}_N u_0 \|_{L^2_2(\mathbb{R}^4)}. \]
which is acceptable. 

(6.12), (6.13) will be treated in the same manner so we only provide the details of the estimate of (6.13). Splitting different time pieces and inserting spatial cutoff, we estimate 

\[
(6.13) \leq \| \phi > 5 |x| \leq 1 P_{N}^{12+} P_{N}^{34} \int_{0}^{\frac{1}{2}N} e^{-i\tau \Delta} \phi \leq 1 F(u)(t + \tau) \tau \|_{L^2_{x}(\mathbb{R}^4)}
\]  

(6.14) 

\[
+ \| \phi > 5 |x| \leq 1 P_{N}^{12+} P_{N}^{34} \int_{0}^{\frac{1}{2}N} e^{-i\tau \Delta} \phi > 1 F(u)(t + \tau) \|_{L^2_{x}(\mathbb{R}^4)}
\]  

(6.15) 

\[
+ \| \phi > 5 |x| \leq 1 P_{N}^{12+} P_{N}^{34} \int_{1}^{\infty} e^{-i\tau \Delta} \phi > N \tau / 2 F(u)(t + \tau) \|_{L^2_{x}(\mathbb{R}^4)}
\]  

(6.16) 

\[
+ \| \phi > 5 |x| \leq 1 P_{N}^{12+} P_{N}^{34} \int_{1}^{\frac{1}{2}N} e^{-i\tau \Delta} \phi > N \tau / 2 F(u)(t + \tau) \|_{L^2_{x}(\mathbb{R}^4)}
\]  

(6.17) 

\[
+ \| \phi > 5 |x| \leq 1 P_{N}^{12+} P_{N}^{34} \int_{1}^{\infty} e^{-i\tau \Delta} \phi \leq N \tau / 2 F(u)(t + \tau) \|_{L^2_{x}(\mathbb{R}^4)}
\]  

(6.18) 

Then our task is reduced to controlling the these four terms which we will do now.

**The estimate of (6.14)**

For this term, the additional decay in $N$ will come from the kernel estimate Lemma 2.6. To proceed, we first use the $L^2$ boundedness of $P_{N}^{12+}$, Bernstein
estimate, mismatch estimate Lemma 2.2 and Strichartz to obtain:

\[
\tag{6.14}
\lesssim \| \phi_{|x|>4} P_N^{12} P_N^{34} \hat{P}_N \int_0^{\frac{1}{50N}} e^{-i\tau \Delta} \phi_{\leq 1} F(u(t+\tau)) d\tau \|_{L_x^2(\mathbb{R}^4)}
\]

\[
\lesssim \| \phi_{|x|>4} P_N^{34} \hat{P}_N \int_0^{\frac{1}{50N}} e^{-i\tau \Delta} \phi_{\leq 1} F(u(t+\tau)) d\tau \|_{L_x^2(\mathbb{R}^4)}
\]

\[
\lesssim \| \phi_{|x|>4} P_N^{34} \phi_{|x|\leq 3} \hat{P}_N \int_0^{\frac{1}{50N}} e^{-i\tau \Delta} \phi_{\leq 1} F(u(t+\tau)) d\tau \|_{L_x^2(\mathbb{R}^4)}
\]

\[
+ \| \phi_{|x|>3} \hat{P}_N \int_0^{\frac{1}{50N}} e^{-i\tau \Delta} \phi_{\leq 1} F(u(t+\tau)) d\tau \|_{L_x^2(\mathbb{R}^4)}
\]

\[
\lesssim N^{-10} \| \hat{P}_N \phi_{\leq 1} F(u) \|_{L_x^\infty L_t^2}
\]

\[
+ \frac{1}{N} \sup_{0 \leq \tau \leq \frac{1}{50N}} \| \phi_{|x|>2} \hat{P}_N e^{-i\tau \Delta} \phi_{\leq 1} F(u(t+\tau)) \|_{L_x^2(\mathbb{R}^4)}
\]

Now using Bernstein and the kernel estimate: for any \( \tau \in [0, \frac{1}{50N}] \),

\[
\phi_{|x|>2} \hat{P}_N e^{-i\tau \Delta} \phi_{\leq 1} (x, y) \lesssim N^4 \langle N|x-y| \rangle^{-20} \phi_{|x|>2} \phi_{|y| \leq 1}
\]

\[
\lesssim N^{-10} \langle |x-y| \rangle^{10},
\]

we continue the estimate of (6.14) as

\[
\tag{6.14}
\lesssim N^{-9} \| F(u) \|_{L_x^\infty L_t^2} \lesssim N^{-5}.
\]

**The estimate of (6.15)**

For this term, we simply use the Strichartz estimate and weak localization of kinetic energy (4.4),

\[
\tag{6.15}
\lesssim \| P_N^{12} P_N^{34} \hat{P}_N \int_0^{\frac{1}{50N}} e^{-i\tau \Delta} \phi_{>1} F(u(t+\tau)) d\tau \|_{L_x^2(\mathbb{R}^4)}
\]

\[
\lesssim \| \hat{P}_N \phi_{>1} F(u(t+\tau)) \|_{L_x^2 L_t^4([0,\frac{1}{50N}] \times \mathbb{R}^4)}
\]

\[
\lesssim N^{-\frac{3}{4}} \| \nabla (\phi_{>1} F(u)) \|_{L_x^\infty L_t^4([0,\infty) \times \mathbb{R}^4)}
\]

\[
\lesssim N^{-\frac{3}{4}} (\| \phi_{>\frac{1}{2}} u \|_{L_x^\infty H_t^1([0,\infty) \times \mathbb{R}^4)} \| \phi_{>\frac{1}{2}} u \|_{L_x^\infty L_t^4([0,\infty) \times \mathbb{R}^4)}
\]

\[
\lesssim N^{-\frac{3}{4}}.
\]

**The estimate of (6.16)**

For this term, the decay comes from the partial spherical symmetry of the solution. To this end, we first use the \( L^2 \) boundedness of \( P_N^{12} \) and mismatch estimate Lemma 2.3 to simplify the computation, then use the weighted
Strichartz estimate Lemma 6.1, Lemma 6.2 and the weak localization of kinetic energy (4.4) to obtain

\begin{equation}
\| \tilde{P}_N \int_1^\infty e^{-i \tau \Delta} \phi_{\geq N \tau/4} F(\phi_{\geq N \tau/4} u(t + \tau)) d\tau \|_{L^2_x(\mathbb{R}^4)} \lesssim \| \tilde{P}_N \int_1^\infty e^{-i \tau \Delta} \phi_{\geq N \tau/2} (P_{< \frac{N}{8}^2} + P_{>8N}) F(u \phi_{\geq N \tau/4})(t + \tau) d\tau \|_{L^2_x(\mathbb{R}^4)}
\end{equation}

\begin{align*}
&\lesssim \| \tilde{P}_N \int_1^\infty e^{-i \tau \Delta} \phi_{\geq N \tau/2} (P_{< \frac{N}{8}^2} + P_{>8N}) F(u \phi_{\geq N \tau/4})(t + \tau) d\tau \|_{L^2_x(\mathbb{R}^4)} \\
&\quad + \| \tilde{P}_N \int_1^\infty e^{-i \tau \Delta} \phi_{\geq N \tau/2} P_{\leq 8N} F(u \phi_{\geq N \tau/4})(t + \tau) d\tau \|_{L^2_x(\mathbb{R}^4)} \\
&\lesssim \| \tilde{P}_N \phi_{\geq N \tau/2} (P_{< \frac{N}{8}^2} + P_{>8N}) F(u \phi_{\geq N \tau/4})(t + \tau) \|_{L^2_x(\mathbb{R}^4)} d\tau \\
&\quad + \| |x|^{-\frac{1}{8} +} \phi_{\geq N \tau/2} P_{\leq 8N} F(u \phi_{\geq N \tau/4}) \|_{L^2_x L^{\frac{8}{7}}(\mathbb{R}^4)} (t + \tau) d\tau \\
&\lesssim \int_1^\infty \| (N^2 \tau)^{-10} F(u \phi_{\geq N \tau/4})(t + \tau) \|_{L^2_x(\mathbb{R}^4)} d\tau \\
&\quad + \| |x|^{-\frac{1}{8} +} \phi_{\geq N \tau/2} P_{\leq 8N} F(u \phi_{\geq N \tau/4}) \|_{L^2_x L^{\frac{8}{7}}(\mathbb{R}^4)} (t + \tau) d\tau \\
&\quad \lesssim N^{-20} \int_1^\infty \tau^{-10} d\tau \| F(u \phi_{\geq N \tau/4}) \|_{L^2_x L^\infty(\mathbb{R}^4)} \\
&\quad + \| |x|^{-\frac{1}{8} +} \phi_{\geq N \tau/2} P_{\leq 8N} F(u \phi_{\geq N \tau/4}) \|_{L^2_x L^\infty(\mathbb{R}^4)} \\
&\lesssim N^{-1 - \frac{1}{10}}.
\end{align*}

The estimate of (6.17)
This term will be estimated in a similar way as (6.16). We first use $L^2$ boundedness of $P_{N}^{\frac{12}{7}+}$, the mismatch estimate 2.3, then use weighted Strichartz estimate 6.1, weak localization of kinetic energy energy (4.4) and
Bernstein to obtain

\[(6.17) \lesssim \| \tilde{P} e^{-ir\Delta} \phi_{> N\tau/2} F(\phi_{> N\tau/4} u)(t + \tau) \|_{L^2_t(\mathbb{R}^4)} \]

\[\lesssim \| \tilde{P} e^{-ir\Delta} \phi_{> N\tau/2} F(\phi_{> N\tau/4} u)(t + \tau) \|_{L^2_t(\mathbb{R}^4)} \]

\[+ \| \tilde{P} e^{-ir\Delta} \phi_{> N\tau/2} (P_{\leq \frac{N}{8}} + P_{> 8N}) F(u)(t + \tau) \|_{L^2_t(\mathbb{R}^4)} \]

\[\lesssim \int_{\frac{N}{50N}}^1 \| \tilde{P} e^{-ir\Delta} \phi_{> N\tau/2} (P_{\leq \frac{N}{8}} + P_{> 8N}) F(u)(t + \tau) \|_{L^2_t(\mathbb{R}^4)} d\tau \]

\[\lesssim \int_{\frac{N}{50N}}^1 \int_{\frac{N}{50N}}^1 \| \tilde{P} e^{-ir\Delta} \phi_{> N\tau/2} (P_{\leq \frac{N}{8}} + P_{> 8N}) F(u)(t + \tau) \|_{L^2_t(\mathbb{R}^4)} d\tau \]

\[\lesssim \int_{\frac{N}{50N}}^1 (N^2\tau)^{-20} \| F(u\phi_{> N\tau/4}) \|_{L^2_t(\mathbb{R}^4)} d\tau \]

\[+ N^{-\frac{1}{2} +} \left\| \tau^{-\frac{1}{2} +} \| P_{\leq \frac{N}{8}} \phi_{> N\tau/4} \|_{L^2_t} \right\|_{L^2_t} \]

\[\lesssim N^{-10} \| u\phi_{> N\tau/4} \|_{L^\infty_t L^2_x([0,\infty) \times \mathbb{R}^4)} \]

\[+ N^{-\frac{1}{2} +} \left\| \tau^{-\frac{1}{2} +} \| \nabla F(u\phi_{> N\tau/4}) \|_{L^2_t} \right\|_{L^2_t} \]

\[\lesssim N^{-\frac{3}{2} +} \left\| \tau^{-\frac{1}{2} +} \| \nabla (u\phi_{> N\tau/4}) \|_{L^\infty_t L^2_x([0,\infty) \times \mathbb{R}^4)} \cdot \right\] \[\| u\phi_{> N\tau/4} \|_{L^\infty_t L^2_x([0,\infty) \times \mathbb{R}^4)} + N^{-10} \]

\[\lesssim N^{-\frac{1}{4}}. \]

The estimate of \[(6.18)\]

We are left with this very last term for which we first control it as

\[(6.18) \]

\[\lesssim \| \phi_{\leq \frac{1}{N^3}} |x|^{12} \|_{L^\infty_t L^2_x} \| P_{\leq \frac{N}{8}} e^{-ir\Delta} \phi_{|y|^{12} \leq N\tau/2} P_{\leq \frac{N}{8}} \phi_{|y|^{12} \leq N\tau/2} F(u)(t + \tau) \|_{L^2_t(\mathbb{R}^4)} \]

Now we use the kernel estimate Lemma 2.11 for small \( x \) regime:

\[|\phi_{\leq \frac{1}{N^3}} |x|^{12} P_{\leq \frac{N}{8}} e^{-ir\Delta} \phi_{|y|^{12} \leq N\tau/2} (x^{12}, y^{12})| \]

\[\lesssim N^2 \log N (N^2\tau + |y|)^{-20} \phi_{\leq \frac{1}{N^3}} |x|^{12} \phi_{|y|^{12} \leq N\tau/2} \]

\[\lesssim N^2 \log N (N^2\tau + |y| + N|x|)^{-20} \]

\[\lesssim N^2 \log N |N^2\tau|^{-10} (N|x - y|)^{-10} \]
and Young’s inequality to obtain
\begin{equation}
\lesssim \int_1^{\infty} N^2 \log N (N^2 \tau)^{-10} \left\| \langle |N| \cdot \rangle^{-10} P_{n \leq N} \phi_{N \tau/2} F(u(t + \tau)) \right\|_{L^2_{x \tau}} d\tau
\end{equation}
\begin{equation}
\lesssim N^2 \log N N^{-20} \int_1^{\infty} \tau^{-10} \left\| P_{n \leq N} \phi_{n \leq N \tau/2} F(u(t + \tau)) \right\|_{L^1_{x \tau}} \left\| P_{n \leq N} \phi_{n \leq N \tau/2} \right\|_{L^2_{x \tau}} d\tau
\end{equation}
\begin{equation}
\lesssim N^{-15} \int_1^{\infty} \tau^{-10} \left\| P_{n \leq N} \phi_{n \leq N \tau/2} F(u(t + \tau)) \right\|_{L^2_{x \tau}} \left\| L^1_{x \tau} \right\|_{L^2_{x \tau}} d\tau
\end{equation}
\begin{equation}
\lesssim N^{-13} \int_1^{\infty} \tau^{-10} \left\| \phi_{n \leq N \tau/2} F(u(t + \tau)) \right\|_{L^1_{x \tau}} d\tau
\end{equation}
\begin{equation}
\lesssim N^{-2} \left\| F(u) \right\|_{L^\infty_{x \tau} L^1_{x \tau}}
\end{equation}
\begin{equation}
\lesssim N^{-2}.
\end{equation}

Collecting the estimates for (6.14) through (6.18), we conclude
\begin{equation}
(6.13) \lesssim N^{-1-\epsilon/10}.
\end{equation}

Therefore (6.9) gives the desired contribution $N^{-1-\epsilon/10}$.

Finally we remark that after minor changes, the contribution from (6.10) can be estimated in pretty much the same manner as for (6.9). More rigorously, in estimating (6.9), we will chop it into pieces like from (6.14) to (6.18). Since in the estimate of (6.15), (6.16), (6.17), we simply throw away the cutoff in $x^{12}$ variable, these three estimates will still be valid in this case. Recalling that in estimating (6.14), the decay essentially stems from the mismatch between the spatial cutoff $\phi_{\geq 5}$ and the propagator $e^{-i\tau \Delta} \phi_{\leq 1}$ which is also available in this case, we are able to handle the analogue to (6.14). Finally, the analogue to the very last term can also be treated by using the normal kernel estimate Proposition 2.9. To conclude, we prove (6.7) and the proof of Proposition 4.1 is completed.

6.3. Proof of Proposition 4.2

In this subsection, we establish Proposition 4.2 in 4 dimensions with admissible symmetry. More precisely we will show, for any dyadic number $N_1, N_2 > 0$, there exist $R_0 = R_0(u, N_1, N_2)$ such that
\begin{equation}
\left\| \phi_{> R} P_N u(t) \right\|_{L^2_{x \tau} \mathbb{R}^4} \lesssim \left\| \phi_{> \sqrt{R} u_0} \right\|_{L^2_{x \tau} \mathbb{R}^4} + R^{-1/10}, \quad \forall R > R_0.
\end{equation}

Proof. Since $P_N \sim P_N P_{n \leq N}^\perp P_{n \leq N} P_{n \leq N}^\perp + P_N P_{n \leq N}^\perp P_{n \leq N}^\perp$, it suffices for us to consider
\begin{equation}
\left\| \phi_{> R} P_N P_{n \leq N}^\perp P_{n \leq N}^\perp u(t) \right\|_{L^2_{x \tau} \mathbb{R}^4}.
\end{equation}
Commuting the spatial cut-off \( \phi \) with the projector \( P_N \), we have
\[
\| \phi R P_N P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} \\
\lesssim \| \phi R P_N \phi \frac{R}{2} P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} + \| \phi R P_N \phi \frac{R}{2} P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} \\
\lesssim \| P_N \phi \frac{R}{2} P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} + (RN)^{-10} \\
\lesssim \| \phi \frac{R}{2} P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} + R^{-5}.
\]

Let \( c \) be a small number close to but smaller than 1. By introducing spatial cutoffs, it reduces to controlling
\[
\| \phi_{|x| \leq \frac{R}{4}} \frac{R}{2} P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} \quad (6.19)
\]
\[
\| \phi_{|x| \leq \frac{R}{4}} \frac{R}{2} P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} \leq \frac{1}{R} \| P_{\leq N}^2 u(t) \|_{L^2_{x_1, x_2} (\mathbb{R}^2)} \| L^2_{x_3, x_4} (\mathbb{R}^2)} \leq \frac{N}{R} \| u(t) \|_{L^2_x (\mathbb{R}^4)} \lesssim R^{-\frac{1}{2}}.
\]

To control (6.20), we use in-out decomposition and Duhamel formula to get
\[
\| \phi_{|x| \leq \frac{R}{4}} \frac{R}{2} P_{\leq N}^2 e^{it\Delta} u_0 \|_{L^2_x (\mathbb{R}^4)} \quad (6.22)
\]
\[
\lim_{t \to \infty} e^{it\Delta} F(u)(t - \tau) d\tau \|_{L^2_x (\mathbb{R}^4)}
\]
\[
\| \phi_{|x| \leq \frac{R}{4}} \frac{R}{2} P_{\leq N}^2 u(t) \|_{L^2_x (\mathbb{R}^4)} \lesssim \| \phi \frac{R}{2} u_0 \|_{L^2_x (\mathbb{R}^4)} + \| \phi \frac{R}{2} P_{\leq N}^2 e^{it\Delta} \phi \|_{L^2_x (\mathbb{R}^4)}
\]
\[
\| \phi \frac{R}{2} u_0 \|_{L^2_x (\mathbb{R}^4)} + \| \phi \frac{R}{2} P_{\leq N}^2 e^{it\Delta} \phi \|_{L^2_x (\mathbb{R}^4)} \quad (6.24)
\]

We first estimate (6.22) by writing
\[
\| \phi_{|x| \leq \frac{R}{4}} \frac{R}{2} P_{\leq N}^2 e^{it\Delta} u_0 \|_{L^2_x (\mathbb{R}^4)} \quad (6.23)
\]
\[
\| \phi \frac{R}{2} u_0 \|_{L^2_x (\mathbb{R}^4)} + \| \phi \frac{R}{2} P_{\leq N}^2 e^{it\Delta} \phi \|_{L^2_x (\mathbb{R}^4)} \lesssim \| \phi \frac{R}{2} u_0 \|_{L^2_x (\mathbb{R}^4)} + \| \phi \frac{R}{2} P_{\leq N}^2 e^{it\Delta} \phi \|_{L^2_x (\mathbb{R}^4)}
\]

This will give desired estimate once we establish
\[
(6.24) \lesssim R^{-\frac{1}{\infty}}.
\]

We discuss two cases.
Case 1. $0 \leq t \leq \frac{R}{100N}$. We write

\begin{align}
\tag{6.24}\nonumber
&\leq \left\| \phi_{\|x^{12}\|> \frac{R}{2}} P_N^{12} P_N^{34} e^{it\Delta} \phi_{\|x^{12}\|> \frac{R}{2}} u_0 \right\|_{L^2(\mathbb{R}^4)} \\
&\lesssim \left\| \phi_{\|x^{12}\|> \frac{R}{2}} P_N^{12} P_N^{34} e^{it\Delta} \phi_{\|x^{12}\|> \frac{R}{2}} u_0 \right\|_{L^2(\mathbb{R}^4)} \\
&\lesssim \left\| \phi_{\|x^{12}\|> \frac{R}{2}} P_N^{12} e^{it\Delta} \phi_{\|x^{12}\|> \frac{R}{2}} u_0 \right\|_{L^2(\mathbb{R}^4)} \\
&\quad + (NR)^{-10} \cdot \|u_0\|_{L^2(\mathbb{R}^4)} \\
&\lesssim R^{-5},
\end{align}

where we have used the kernel estimate: \( \forall t \leq \frac{R}{100N} \)

\begin{align}
\tag{6.25}
\left|\left(\phi_{\|x^{12}\|> \frac{R}{2}} P_N^{12} e^{it\Delta} \phi_{\|x^{12}\|> \frac{R}{2}} u_0\right)(x,y)\right| \lesssim R^{-10} \langle N(x-y) \rangle^{-10}.
\end{align}

Case 2. $t > \frac{R}{100N}$. We first use triangle to split

\begin{align}
\tag{6.24}
&\lesssim \left\| \phi_{\|x^{12}\|< \frac{1}{N}} P_N^{12} e^{it\Delta} \phi_{\|x^{12}\|< \frac{1}{N}} u_0 \right\|_{L^2(\mathbb{R}^4)} \\
&\lesssim \left\| \phi_{\|x^{12}\|< \frac{1}{N}} P_N^{12} e^{it\Delta} \phi_{\|x^{12}\|< \frac{1}{N}} u_0 \right\|_{L^2(\mathbb{R}^4)} \\
&\quad + \left\| \phi_{\|x^{12}\|< \frac{1}{N}} P_N^{12} e^{it\Delta} \phi_{\|x^{12}\|< \frac{1}{N}} u_0 \right\|_{L^2(\mathbb{R}^4)}.
\end{align}

Since in both of the two terms, the kernels obey

\begin{align}
\tag{6.26}
&\left|\left(\phi_{\|x^{12}\|< \frac{1}{N}} P_N^{12} e^{it\Delta} \phi_{\|y^{12}\|< \frac{1}{N}} u_0\right)(x^{12},y^{12})\right| \\
&\lesssim (\log R)^{10} N^{C} \langle N^2 t - N|y^{12}| \rangle^{-100} \phi_{\|x^{12}\|< \frac{1}{N}} \phi_{\|y^{12}\|< \frac{1}{N}} e^{-\frac{1}{2} R^c} \\
&\lesssim (\log R)^{10} N^{C} \langle N^2 t \rangle^{-100} \langle N(x^{12} - y^{12}) \rangle^{-100} \\
&\lesssim R^{-10} \langle N^2 t \rangle^{-10} \langle N(x^{12} - y^{12}) \rangle^{-10} \lesssim R^{-10} \langle N(x^{12} - y^{12}) \rangle^{-10}
\end{align}

and

\begin{align}
\tag{6.27}
&\left|\left(\phi_{\|x^{12}\|< \frac{1}{N}} P_N^{12} e^{it\Delta} \phi_{\|y^{12}\|< \frac{1}{N}} u_0\right)(x^{12},y^{12})\right| \\
&\lesssim N^{C} \langle N^2 t + N|x^{12}| - N|y^{12}| \rangle^{-20} \phi_{\|x^{12}\|< \frac{1}{N}} \phi_{\|y^{12}\|< \frac{1}{N}} e^{-\frac{1}{2} R^c} \\
&\lesssim N^{C} \langle N^2 t + N|x^{12}| + N|y^{12}| \rangle^{-20} \\
&\lesssim N^{C} \langle N^2 t \rangle^{-10} \langle N(x^{12} - y^{12}) \rangle^{-10} \lesssim R^{-10} \langle N(x^{12} - y^{12}) \rangle^{-10}.
\end{align}

Then

\begin{align}
\tag{6.24}
&\lesssim R^{-5},
\end{align}

follows from these kernel estimate and Young’s inequality. This completes the estimate of \( \tag{6.24} \).  

Now we estimate \( \tag{6.23} \). By splitting into time pieces and putting spatial cutoff in front of \( F(u) \), our task is reduced to bounding the following four
terms

\[ \| \phi_{\frac{1}{R} \leq |x| \leq R} \phi_{\frac{1}{N} P_{N}^{12+} P_{N}^{24} \leq N} \int_{0}^{10 N} e^{-i \tau \Delta} \phi_{\leq \frac{1}{R}} F(u(t + \tau)) d\tau \|_{L_{2}^{2}(\mathbb{R}^{4})} \tag{6.28} \]

\[ \| \phi_{\frac{1}{R} \leq |x| \leq R} \phi_{\frac{1}{N} P_{N}^{12+} P_{N}^{24} \leq N} \int_{0}^{10 N} e^{-i \tau \Delta} \phi_{\leq \frac{1}{R}} F(u(t + \tau)) d\tau \|_{L_{2}^{2}(\mathbb{R}^{4})} \tag{6.29} \]

\[ \| \phi_{\frac{1}{R} \leq |x| \leq R} \phi_{\frac{1}{N} P_{N}^{12+} P_{N}^{24} \leq N} \int_{0}^{\infty} e^{-i \tau \Delta} \phi_{\leq \frac{1}{R}} F(u(t + \tau)) d\tau \|_{L_{2}^{2}(\mathbb{R}^{4})} \tag{6.30} \]

\[ \| \phi_{\frac{1}{R} \leq |x| \leq R} \phi_{\frac{1}{N} P_{N}^{12+} P_{N}^{24} \leq N} \int_{0}^{\infty} e^{-i \tau \Delta} \phi_{\leq \frac{1}{R}} F(u(t + \tau)) d\tau \|_{L_{2}^{2}(\mathbb{R}^{4})} \tag{6.31} \]

We first estimate the tail part (6.29), (6.31) where the decay comes from the decay estimate (6.25), (6.26), (6.27).

For (6.29), we use (6.25), Bernstein, the \( L^{2} \) boundedness of \( \tilde{P}_{N}^{12+} \) to obtain

\[ \| \phi_{\frac{1}{R} \leq |x| \leq R} \phi_{\frac{1}{N} P_{N}^{12+} P_{N}^{24} \leq N} \int_{0}^{10 N} e^{-i \tau \Delta} \phi_{\leq \frac{1}{R}} F(u(t + \tau)) d\tau \|_{L_{2}^{2}(\mathbb{R}^{4})} \]

\[ \lesssim \left\| \phi_{\frac{1}{R} \leq |x| \leq R} \int_{0}^{10 N} e^{-i \tau \Delta} \phi_{\leq \frac{1}{R}} F(u(t + \tau)) d\tau \|_{L_{2}^{2}(\mathbb{R}^{4})} \right\| \]

\[ + (RN)^{-10} \cdot \frac{R}{N} \cdot \| \tilde{P}_{N}(\phi_{\leq \frac{1}{R}} F(u(t + \tau)) \|_{L_{r}^{2}(\mathbb{R}^{4})} \]

\[ \lesssim R^{-5}. \]

For (6.31), we use (6.26), (6.27), Bernstein to get

\[ \| \phi_{\frac{1}{R} \leq |x| \leq R} \phi_{\frac{1}{N} P_{N}^{12+} P_{N}^{24} \leq N} e^{-i \tau \Delta} \phi_{\leq \frac{N}{200}} F(u(t + \tau)) \|_{L_{2}^{2}(\mathbb{R}^{4})} d\tau \]

\[ \lesssim \int_{0}^{\infty} \tau^{-10} N^{C} R^{-10} \left\| \int_{0}^{10 N} e^{-i \tau \Delta} \phi_{\leq \frac{N}{200}} F(u(t + \tau)) \|_{L_{2}^{2}(\mathbb{R}^{4})} d\tau \right\| \]

\[ \lesssim N^{C} R^{-10} \int_{0}^{\infty} \tau^{-10} \left\| \int_{0}^{10 N} e^{-i \tau \Delta} \phi_{\leq \frac{N}{200}} F(u(t + \tau)) \|_{L_{2}^{2}(\mathbb{R}^{4})} d\tau \right\| \]

\[ \lesssim R^{-5}. \]

Now we treat the main contribution (6.28), (6.30) for which we will use weighted Strichartz and radial Sobolev embedding. We start with (6.28).
We write

\[(6.28) \lesssim \|P_N^{12} P_{\leq N}^A \tilde{P}_N \int_0^R e^{-i \tau \Delta} \phi_{\geq \frac{1}{2} R \epsilon} F(u \phi_{> R^c/4})(t + \tau) d\tau \|_{L^2_x(R^4)} \]

\[\lesssim \|\tilde{P}_N \int_0^R e^{-i \tau \Delta} \phi_{\geq \frac{1}{2} R \epsilon} F(u \phi_{> R^c/4})(t + \tau) d\tau \|_{L^2_x(R^4)} \]

\[\lesssim \|\tilde{P}_N \int_0^R e^{-i \tau \Delta} \phi_{\geq \frac{1}{2} R \epsilon} (P_{> 16N} + P_{\leq \frac{N}{16}}) F(u \phi_{> R^c/4})(t + \tau) d\tau \|_{L^2_x(R^4)} \]

\[+ \|\tilde{P}_N \int_0^R e^{-i \tau \Delta} \phi_{\geq \frac{1}{2} R \epsilon} P_{\frac{N}{16} < \leq 16N} F(u \phi_{> R^c/4})(t + \tau) d\tau \|_{L^2_x(R^4)} \]

\[\leq \|\tilde{P}_N \phi_{> \frac{1}{2} R \epsilon} (P_{> 16N} + P_{\leq \frac{N}{16}}) F(u \phi_{> R^c/4})\|_{L^1_x L^2_t([t, t + \frac{R}{100}] \times R^4)} \]

\[+ \|\tilde{P}_N \int_0^R e^{-i \tau \Delta} \phi_{\geq \frac{1}{2} R \epsilon} P_{\frac{N}{16} < \leq 16N} F(u \phi_{> R^c/4})(t + \tau) d\tau \|_{L^2_x(R^4)} \]  

The first one is mismatched so gives us the bound

\[(NR^c)^{-10} \frac{R}{N} \|\phi_{> R^c/4} u\|^2_{L^2_t L^4([0, \infty) \times R^4)} \lesssim R^{-5}, \]

due to (4.4). The second term can be estimated as follows. Since \(\phi_{> \frac{1}{2} R \epsilon} = \phi_{> \frac{1}{2} R \epsilon} (\phi_{|x^{12}|> R^c/4} + \phi_{|x^{34}|> R^c/4})\), we decompose (6.32) into four similar terms with the following being one of the representatives

\[\|\tilde{P}_N \int_0^R e^{-i \tau \Delta} \phi_{> \frac{1}{2} R \epsilon} \phi_{|x^{12}|> R^c/4} P_{N/16 < \leq 16N} F(u \phi_{|x^{34}|> R^c/8})(t + \tau) d\tau \|_{L^2_x(R^4)} \]

(6.33)

Using weighted Strichartz and Lemma 6.24 we control it by

\[(6.33) \lesssim R^{-3 + \frac{d}{2} + \|P_{N/16 < \leq 16N} F(\phi_{|x^{34}|> R^c/8})\|_{L^6_t L^{\frac{6}{5}}([t, t + \frac{R}{100}] \times R^4)} \]

\[\lesssim R^{-3 + \frac{d}{2} + \frac{N}{2} - N^{-\frac{5}{9}} R^{-\frac{2}{9}} c}. \]

Since \(c\) is sufficiently close to 1 and \(R\) sufficiently large, we can have

\[(6.33) \lesssim R^{-\frac{3}{18}} \]

and therefore

\[(6.28) \lesssim R^{-\frac{1}{18}} \]

which is acceptable.
Now we consider (6.30). We first bound it as
\[(6.30) \lesssim \| \int_{R_{100N}}^{\infty} \tilde{P}_N e^{-i\tau \Delta} \phi > \frac{N}{200} F(u(t+\tau)) d\tau \|_{L^2_x(\mathbb{R}^4)} \]
\[(6.34)
\lesssim \| \int_{R_{100N}}^{\infty} \tilde{P}_N e^{-i\tau \Delta} \phi > \frac{N}{200} \left(P_{\leq N/16} + P_{> 16N}\right) F(u\phi > \frac{N}{200}) d\tau \|_{L^2_x(\mathbb{R}^4)} \]
\[(6.35)
+ \| \int_{R_{100N}}^{\infty} \tilde{P}_N e^{-i\tau \Delta} \phi > \frac{N}{200} P_{N/16 < \cdot \leq 16N} F(u\phi > \frac{N}{200})(t+\tau) d\tau \|_{L^2_x(\mathbb{R}^4)}. \]
\[(6.36)\]

The first term contains a mismatch. Therefore by Minkowski and mismatch estimate Lemma 2.3, we have
\[(6.35) \lesssim \int_{R_{100N}}^{\infty} \| \tilde{P}_N \phi > \frac{N}{200} \left(P_{\leq N/16} + P_{> 16N}\right) F(u\phi > \frac{N}{200})(t+\tau) d\tau \|_{L^2_x(\mathbb{R}^4)} \]
\[(6.37)\]

Estimate of (6.36) will follow the similar way as for (6.32). Again we decompose it into four terms with an example like the following:
\[\| \int_{R_{100N}}^{\infty} \tilde{P}_N e^{-i\tau \Delta} \phi > \frac{N}{200} \left| P_{\leq N/16} + P_{> 16N}\right| F(u\phi > \frac{N}{200})u(t+\tau) d\tau \|_{L^2_x(\mathbb{R}^4)}. \]
\[(6.38)\]

Using Weighted Strichartz, Minkowski, Lemma 6.2 we bound it by
\[(6.37) \lesssim N^{-\frac{1}{4}+} \| \left| P_{\leq N/16} + P_{> 16N}\right| F(\phi_{|x| > \frac{N}{200}} u)(t+\tau) \|_{L^2_x(\mathbb{R}^4)} \]
\[\lesssim N^{C} \| \tau^{-\frac{1}{4}+} \|_{L^2_{\tau} L^2_{x}((R_{100N},\infty) \times \mathbb{R}^4)} \]
\[\lesssim R^{-\frac{1}{100}}. \]

Collecting the estimates for (6.28) through (6.31), we get the bound for (6.28)
\[(6.28) \lesssim R^{-\frac{1}{100}}. \]
\[(6.38)\]

Thus (6.20) gives the desired control:
\[(6.24) \lesssim \| \phi > \sqrt{R} u_0 \|_2 + R^{-\frac{1}{100}}. \]

To finish the argument, we still have to control (6.21). However, modulo some modification, the estimate of (6.21) will be essentially a repetition of that of (6.20). For this purpose, we only briefly outline the proof.
We first use in-out decomposition to reduce matters to control
\[
\| \phi_{|x|^{12}>R^c} \|_{L^2_x(R^4)} \leq \| \sqrt{R} \phi \|_{L^2_x(R^4)} + R^{-1.100}.
\] (6.21)

Collecting the estimate of (6.19), (6.20) and (6.21), we finally prove the Proposition 4.2 in 2 + 2 dimensions is completed.

\[\square\]

**7. Higher dimensional case with admissible symmetry**

In this section, we prove Proposition 4.1 and Proposition 4.2 in high dimensions \(d \geq 5\). The proof in this case will be an adaptation of that in 2 + 2 case except for the fact the numerology is more complicated. For this reason, we will give a very brief outline of the proof, with emphasis on the important changes.

To begin with, we explain the notations we will use in this section. We denote \(P_N^{d_1}\) as a Littlewood Paley projection on \(d_1\)-dimensional space with the same explanation if we change the number \(d_1\) or replace \(N\) by \(\leq N\), etc. Differential operators \(\nabla^{d_1}, \Delta_{d_1}\) should be understood acting on \(d_1\).
dimensional functions. We also use the convention that the operator with
no subscripts is the one defined on whole \( \mathbb{R}^d \).

We first establish the following weighted Strichartz estimate for solutions
which are spherical symmetric only on subspaces of \( \mathbb{R}^d \).

**Lemma 7.1.** Let the dimension \( d \geq 5 \). Let \( u_0(x), f(t, x) \) be spherically
symmetric on the subspace \( \mathbb{R}^{d_1} \). Then the function \( u \) defined by

\[
u(t) = e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^t e^{i(t-s)\Delta} f(s) ds
\]
is also spherically symmetric on subspace \( \mathbb{R}^{d_1} \), moreover

\[
\left\| \left\| x^{d_1} \frac{2(d_1-1)}{q(d_1-d_1)} |u| \right\|_{L^q_t L^\infty_x((I \times \mathbb{R}^d))} \right\|_{L^2_t L^{q(d_1-d_1)+4}_x((\mathbb{R} \times \mathbb{R}^d))} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^d)} + \|f\|_{L^2_t L^{\frac{d+2}{d+4}}_x((I \times \mathbb{R}^d))},
\]

holds \( \forall q \) such that \( q > 2 \) and \( q \geq \frac{4}{d+1-d_1} \).

**Proof.** With several changes, the proof is in principle the same as Lemma
\ref{lem:solitary_wave_conjecture_6.1}. Here we only briefly sketch the proof.

From standard Strichartz estimate, Christ-Kiselev Lemma and \( TT^* \)
argument, it is reduced to showing

\[
\left\| \int |x^{d_1} \frac{2(d_1-1)}{q(d_1-d_1)} e^{i(t-\tau)\Delta} |y^{d_1} \frac{2(d_1-1)}{q(d_1-d_1)} f(\tau) d\tau \right\|_{L^q_t L^\infty_x((\mathbb{R} \times \mathbb{R}^d))} \lesssim \|f\|_{L^q_t L^{q(d_1-d_1)+4}_x((\mathbb{R} \times \mathbb{R}^d))}.
\]

This will be a consequence of the following decay estimate and Hardy-
Littlewood-Sobolev inequality:

\[
\left\| \left\| x^{d_1} \frac{2(d_1-1)}{q(d_1-d_1)} e^{i\tau\Delta} |y^{d_1} \frac{2(d_1-1)}{q(d_1-d_1)} f \right\|_{L^2_t L^\infty_x((\mathbb{R} \times \mathbb{R}^d))} \right\|_{L^q_t L^{\frac{d+2}{d+4}}_x((\mathbb{R} \times \mathbb{R}^d))} \lesssim |t|^{-\frac{2}{q}} \|f\|_{L^q_t L^{\frac{d(d_1-d_1)}{d(d_1-d_1)+4}}_x((\mathbb{R} \times \mathbb{R}^d))}.
\]

As in the proof of Lemma \ref{lem:solitary_wave_conjecture_6.1}, this decay estimate will follow from the interpolation between the trivial case \( q = \infty \) and the pointwise estimate

\[
\left\| \left\| x^{d_1} \frac{d_1-1}{2} e^{i\Delta} |y^{d_1} \frac{d_1-1}{2} f \right\|_{L^\infty_x(\mathbb{R}^d)} \right\|_{L^2_t(\mathbb{R}^d)} \lesssim |t|^{-\frac{d_1-1}{2}} \|f\|_{L^1_t(\mathbb{R}^d)},
\]

where \( f \) is spherically symmetric in \((x_1, \cdots, x_{d_1})\) variable. By passing to
the radial coordinate, we can write the kernel as

\[
(\left\| x^{d_1} \frac{d_1-1}{2} e^{i\Delta} |y^{d_1} \frac{d_1-1}{2} \right\|)(x, y) = |x^{d_1} \frac{d_1-1}{2} |y^{d_1} \frac{d_1-1}{2} (4\pi it) \frac{d}{2} e^{i|x^{d_1-d_1} y^{d_1-d_1}|^2/4t} e^{i|(|x^{d_1}|^2 + |y^{d_1}|^2)/4t} \int_{S^{d_1-1}} e^{iy^{d_1} |x^{d_1}|^2/2t} d\sigma(w).
\]

Stationary phase or applying the property of Bessel function then yields
the desired estimate. \( \square \)
As we can see from the calculation of the four dimensional case, the main difference in high dimensions will arise from the pieces where $F(u)$ lives on large radii. For these pieces, we will use the above lemma and radial Sobolev embedding to take advantage of the decay property of a splitting-spherically symmetric function. It is here that we need a restriction on the minimal dimension on which the solution is spherically symmetric. Technically, the restriction stems from the following

**Lemma 7.2.** Let $d_1 = \lfloor \frac{d}{2} \rfloor$ or $\frac{4}{3} < d_1 \leq \frac{d}{2}$ for sufficiently large $d$. Let $u$ be splitting-spherically symmetric with splitting subspaces $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d-d_1}$ respectively. Then there exist parameters $(\alpha, \beta, p, q)$ such that

$$
\alpha > 0, \beta > 0, 2 \leq p, q < \infty, \\
\frac{4\alpha}{d} + \frac{d_1 - 1}{d - d_1 + 1} > \frac{1}{2}, \\
\beta + \frac{4\alpha}{d} + \frac{d_1 - 1}{d - d_1 + 1} > 1, \\
\frac{1}{p} + \frac{4}{dq} = \frac{d_1 + 3}{2(d - d_1 + 1)}, \\
\beta + d(\frac{1}{2} - \frac{1}{p}) \leq 1, \\
d(\frac{1}{2} - \frac{1}{q}) - 1 \leq \alpha < d_1(\frac{1}{2} - \frac{1}{q}).
$$

Moreover, for $NL \gtrsim 1$,

$$
\|P_N F(\phi_{>NL}u)\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{-\beta} (NL)^{-\frac{4\alpha}{d}} \quad \text{(7.1)}
$$

**Proof.** We first remark the restrictions on the parameters $(\alpha, \beta, p, q)$ appear naturally when we try to control the LHS of (7.1). For example, the second and the third one correspond to the power in $L$ and $N$ being negative enough, which, consequently yield the integrability in $t$ and $1 + \varepsilon$ power of $N$ needed in later computations (cf. the proof of (7.2)). The last three conditions correspond to the possibility of using the Sobolev embedding, radial Sobolev embedding Lemma 2.4 and the estimate (4.4) to establish (7.1). In the following, we first assume these parameters can be taken and quickly prove (7.1), then verify the elementary computations at the very end.

From Bernstein, fractional chain rule Lemma 2.3, we have

$$
\text{LHS of (7.1)} \lesssim N^{-\beta} \|\nabla^\beta(\phi_{>NL}u)\|_{L^4(\mathbb{R}^d)} \|\phi_{>NL}u\|_{L^2}^{\frac{4}{3}}.
$$

Now note that by assumption $d_1 \leq \frac{d}{2}$ and $u$ is spherically symmetric when restricting to the subspaces $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d-d_1}$ respectively, the cutoff function $\phi_{>NL}$ must have nontrivial projection to a $d_1$-dimensional subspace on which the restriction of $u$ is spherically symmetric. By relabelling the coordinates
if necessary, we can assume without loss of generality that

\[ \phi_{>NL} = \phi_{>NL}\phi_{|x|>\frac{NL}{2}}. \]

Note that after picking out this \(d_1\)-dimensional subspace, \(u\) is no longer necessarily spherically symmetric on the remaining \((d - d_1)\)-dimensional subspace. Now by Sobolev embedding, radial Sobolev emebedding Lemma 2.4 and weak localization of kinetic energy estimate (4.4), we continue to estimate

\[
\text{LHS of (7.1)} \lesssim N^{-\beta} \|\nabla|^{\beta+d \left(\frac{1}{2} - \frac{1}{p}\right)}(\phi_{>NL}u)\|_{L^2} \left(\frac{4^d}{d}\right) \|\nabla|^{\alpha} \phi_{>NL}u\|_{L^2} \lesssim N^{-\beta} \|\nabla|^{\beta+d \left(\frac{1}{2} - \frac{1}{p}\right)}(\phi_{>NL}u)\|_{L^2} \left(\frac{4^d}{d}\right) \|\nabla|^{\alpha} \phi_{>NL}u\|_{L^2} \lesssim N^{-\beta} (NL)^{-\frac{4d}{d}} \|\nabla|^{\beta+d \left(\frac{1}{2} - \frac{1}{p}\right)}(\phi_{>NL}u)\|_{L^2} \left(\frac{4^d}{d}\right) \|\nabla|^{\alpha} \phi_{>NL}u\|_{L^2} \lesssim N^{-\beta} (NL)^{-\frac{4d}{d}},
\]

where the last inequality follows from the fact that the indices of differentiation are sandwiched between 0 and 1, which in turn can be controlled by interpolating the \(L^2\) mass and weak localization estimate of kinetic energy. Now we verify the existence of the set of parameters satisfying the aforementioned conditions. In the case when \(d\) is even and \(d_1 = \frac{d}{2}\), we simply take

\[
(\alpha, \beta, p, q) = \left(\frac{d - 2}{2(d + 2)}, \frac{1}{2}, \frac{2(d + 2)}{d}, +\right).
\]

In the case when \(d\) is odd and \(d_1 = \frac{d-1}{2}\), we take

\[
(\alpha, \beta, p, q) = \left(\frac{3(d-1)}{4(d+3)}, \frac{1}{2}, \frac{2(d+3)}{d}, +\right).
\]

The validity of the chosen set follows from direct computation which we omit.
Now we look at the asymptotic result for sufficiently large \(d\). Let \(d_1 = \eta d\). We rewrite the conditions equivalently as follows

\[
\frac{4\alpha}{d} > \frac{1 - 3\eta}{2(1 - \eta)} + \frac{1}{(1 - \eta)^2} \frac{1}{d} + O\left(\frac{1}{d^2}\right),
\]

\[
\beta + \frac{4\alpha}{d} > \frac{1}{2} + \frac{1 - 3\eta}{2(1 - \eta)} + \frac{1}{(1 - \eta)^2} \frac{1}{d} + O\left(\frac{1}{d^2}\right),
\]

\[
\frac{1}{p} + \frac{4}{dq} = \frac{1}{2} + \frac{1 - \eta}{1 - \eta d} - \frac{1}{(1 - \eta)^2} \frac{1}{d^2} + O\left(\frac{1}{d^3}\right),
\]

\[
\beta + d\left(\frac{1}{2} - \frac{1}{p}\right) \leq 1,
\]

\[
\beta + d\left(\frac{1}{2} - \frac{1}{q}\right) - 1 \leq \alpha < \eta d\left(\frac{1}{2} - \frac{1}{q}\right).
\]

When \(\eta > \frac{1}{3}\), the first inequality holds automatically for large \(d\). So the genuine restriction comes from the last four. Now we look at the last inequality, to make it valid, we require \(\frac{1}{2} - \frac{1}{q}\) is of order \(\frac{1}{d}\). Note from the third one,

\[
\frac{1}{q} = \left(\frac{1}{2} - \frac{1}{p}\right) d^4 + \frac{1}{4(1 - \eta)} - \frac{1}{4d(1 - \eta)^2} + O\left(\frac{1}{d^2}\right),
\]

thus this forces

\[
\left(\frac{1}{2} - \frac{1}{p}\right) d^4 + \frac{1}{4(1 - \eta)} = \frac{1}{2}.
\]

Hence, \(p, q\) are simultaneously determined:

\[
\frac{1}{q} = \frac{1}{2} - \frac{1}{4d(1 - \eta)^2} + O\left(\frac{1}{d^2}\right),
\]

\[
\frac{1}{p} = \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{4(1 - \eta)}\right) d.
\]

This in turn produce a condition on \(\beta\) from the fourth:

\[
\beta \leq \frac{\eta}{1 - \eta}.
\]

Since \(\eta > \frac{1}{3}\), by taking \(\beta = \frac{1}{2}\), this and the second hold true. Finally, by choosing

\[
\alpha = \eta d\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{\eta}{4(1 - \eta)^2} + O\left(\frac{1}{d}\right).
\]

The last one holds, hence the five conditions hold for the chosen parameters.

\[\square\]

Now we have collected enough information to prove Proposition 4.1 and Proposition 4.2. We begin with the frequency decay estimate, that is to show there exists \(\varepsilon = \varepsilon(d)\) such that, \(\forall N \geq 1\)

\[
\|\phi_{> \theta} P_N u(t)\|_{L^2_x(\mathbb{R}^d)} \leq \|\tilde{P}_0 u_0\|_{L^2_x(\mathbb{R}^d)} + N^{-1 - \varepsilon}. \tag{7.2}
\]
Proof. Note that $d_1 \leq \frac{4}{3}$. If a frequency $|\xi| \sim N$, then there must be a $d_1$ dimensional vector $|\xi^{d_1}| \sim N$, so morally we have

$$P_N \sim P_N \tilde{P}_N^{d_1}. $$

Without loss of generality, it is reduced to considering

$$\|\phi_{>10} P_N P_N^{d_1} u(t)\|_{L^2(\mathbb{R}^d)}$$

Due to the strong singularity of $P^\pm$ operators at the origin, in high dimensions, we adopt a slightly different strategy. Introducing the spatial cutoff in $x^{d_1}$ variable, we use triangle inequality to bound

$$\begin{align*}
(7.3) & \leq \|\phi_{>10} \phi_{|x^{d_1}| \leq \frac{1}{N}} P_N P_N^{d_1} u(t)\|_{L^2(\mathbb{R}^d)} \\
& + \|\phi_{>10} \phi_{|x^{d_1}| > \frac{1}{N}} P_N P_N^{d_1} u(t)\|_{L^2(\mathbb{R}^d)}
\end{align*}$$

We first estimate (7.4). Using forward Duhamel (1.17) and split it into time pieces, we write

$$\begin{align*}
(7.4) & \leq \|\phi_{>10} \phi_{|x^{d_1}| \leq \frac{1}{N}} P_N P_N^{d_1} u(t)\|_{L^2(\mathbb{R}^d)} \\
& + \|\phi_{>10} \phi_{|x^{d_1}| > \frac{1}{N}} P_N P_N^{d_1} u(t)\|_{L^2(\mathbb{R}^d)}
\end{align*}$$

We first estimate the short time piece. We write $F(u) = \phi_{>1} F(u) + \phi_{\leq 1} F(u)$ and estimate the contribution from both by Strichartz, kernel estimate and Young’s inequality as

$$\begin{align*}
(7.6) & \leq \|\phi_{>10} \phi_{|x^{d_1}| \leq \frac{1}{N}} P_N P_N^{d_1} \int_0^{\infty} e^{-i\tau \Delta} F(u(t + \tau)) d\tau\|_{L^2(\mathbb{R}^d)} \\
& + \|\phi_{>10} \phi_{|x^{d_1}| > \frac{1}{N}} P_N P_N^{d_1} \int_0^{\infty} e^{-i\tau \Delta} F(u(t + \tau)) d\tau\|_{L^2(\mathbb{R}^d)}
\end{align*}$$

$$\begin{align*}
& \leq \|\phi_{>1} F(u(t + \tau))\|_{L^\infty_t L^{\frac{2d}{d+4};2}((0,N^{-\frac{2d}{d+4}}) \times \mathbb{R}^d)} \\
& + N^{-\frac{2d}{d+4}} \sup_{0 \leq \tau \leq N^{-\frac{2d}{d+4}}} \|\phi_{>10} P_N P_N^{d_1} e^{-i\tau \Delta} \phi_{\leq 1} F(u(t + \tau))\|_{L^2(\mathbb{R}^d)}
\end{align*}$$

$$\begin{align*}
& \leq N^{-1 - \frac{3}{3+\frac{4}{d}}} \|\phi_{>1} u\|_{L^{\frac{d+4}{2d+4};2}((0,\infty) \times \mathbb{R}^d)} + N^{-10} \|F(u)\|_{L^\infty_t L^{\frac{2d}{d+4};2}((0,\infty) \times \mathbb{R}^d)} \\
& \leq N^{-1 - \frac{3}{3+\frac{4}{d}}} + N^{-10} \|u\|_{L^{\infty_t} L^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim N^{-1 - \frac{3}{3+\frac{4}{d}}}. 
\end{align*}$$
In the last line, we have used the Kernel estimate\footnote{The kernel estimate Lemma \ref{lem:kernel}} \( \forall \tau \in [0, N^{-\frac{2d}{d+1}}] \)

\[
|\phi_{>10}P_N P_N^{d_1} e^{-i\tau \Delta} \phi_{\leq 1}(x, y)| \lesssim N^{-10}(x - y)^{-10d}
\]

and Young’s inequality.

Now we estimate the contribution from the long time piece. By writing \( F(u) = \phi_{>N\tau/2} F(u) + \phi_{\leq N\tau/2} F(u) \), we further split \( \ref{eq:long_time} \) into two pieces. The estimate when \( F(u) \) is supported within a ball is given as follows. Com-muting \( P_N \) and \( \phi_{\leq N\tau/2} \) (thus adding a \( N^{-2} \) from mismatch estimate), we get

\[
\|\phi_{>10}\phi_{|x^{d_1}| \leq \frac{1}{N}} P_N P_N^{d_1} \int_{N^{-\frac{2d}{d+1}}}^{\infty} e^{-i\tau \Delta} \phi_{\leq N\tau/2} F(u(t + \tau)) d\tau\|_{L^2_x (\mathbb{R}^d)} 
\]

\[
\lesssim \|\phi_{|x^{d_1}| \leq \frac{1}{N}} P_N P_N^{d_1} \int_{N^{-\frac{2d}{d+1}}}^{\infty} e^{-i\tau \Delta} \phi_{|y^{d_1}| \leq N\tau/2} P_N^{m} \phi_{\leq \frac{8N}{\tau}} F(u(t + \tau)) d\tau\|_{L^2_x (\mathbb{R}^d)} + N^{-2}
\]

\[
\lesssim N^{-2} + \int_{N^{-\frac{2d}{d+1}}}^{\infty} \|\phi_{|x^{d_1}| \leq \frac{1}{N}} P_N^{d_1} e^{-i\tau \Delta d_1} \phi_{|y^{d_1}| \leq N\tau/2} P_N^{m} \phi_{\leq \frac{8N}{\tau}} F(u(t + \tau))\|_{L^2_x (\mathbb{R}^d)} d\tau
\]

Now using the kernel estimate (Lemma \ref{lem:kernel})

\[
|\phi_{|x^{d_1}| \leq \frac{1}{N}} P_N^{d_1} e^{-i\tau \Delta d_1} \phi_{|y^{d_1}| \leq N\tau/2}(x^{d_1}, y^{d_1})| 
\]

\[
\lesssim N^{d-2m} \tau^{-m} \langle N|x^{d_1} - y^{d_1}|\rangle^{-m},
\]

and Young’s inequality we continue to control it by

\[
N^{-2} + N^{d_1} \int_{N^{-\frac{2d}{d+1}}}^{\infty} \|N^2 \tau|^{-100d} \| P_N^{m} \phi_{\leq \frac{8N}{\tau}} F(u)\|_{L^{\frac{2d}{d+1}}_{x^{d_1}}} d\tau 
\]

\[
\lesssim N^{-2} + N^{-2} \|F(u)\|_{L^\infty_x L^{d+1}_{x^{d_1}}((0, \infty) \times \mathbb{R}^d)}
\]

\[
\lesssim N^{-2}.
\]
To estimate the contribution where $F(u)$ is supported outside the ball, we use Lemma 7.2 to obtain
\[
\|\phi > 10 \phi |_{d_1} \|_{L^2} \leq \frac{1}{N} P^*_{N} \int_{N^{-1}}^\infty e^{-i\tau \Delta} \phi > N\tau/2 F(u(t+\tau))d\tau
\]
\[
\lesssim \|P^*_{N} \int_{N^{-1}}^\infty e^{-i\tau \Delta} \phi > N\tau/2 \bar{P}_N F(\phi > N\tau/4 u)(t+\tau)d\tau \|_{L^2} + N^{-2}
\]
\[
\lesssim N^{-2} + \|\|N\tau\|^{-d_1-1} \| \bar{P}_N F(\phi > N\tau/4 u) \|_{L^2} \| \tau^{-d_1-1} \tau^{-\frac{4\alpha}{d-1}} \|_{L^2} \left(\left[\left.N^{-\frac{2d}{d+1}}, \infty\right]\right. \times \mathbb{R}^d\right)
\]
\[
\lesssim N^{-1-\epsilon}.
\]
To conclude, we have
\[
(7.7) \lesssim N^{-1-\epsilon}.
\]
Collecting the estimates for (7.6), (7.7), we obtain
\[
(7.4) \lesssim N^{-1-\epsilon}.
\]
This finishes the estimate of the piece where $|x^{d_1}| \lesssim \frac{1}{N}$. To estimate (7.5) where $|x^{d_1}|$ is large, we will have to use the in-out decomposition technique as we have done in the 2+2 case. As the details have been fully demonstrated in the 2,3 dimensions and 2+2 case, we will not repeat the argument here. We also leave the details of the spatial decay estimate Proposition 4.2 to interested readers.

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Institute for Advanced Study, Princeton, NJ

Academy of Mathematics and System Sciences, Beijing, and Institute for Advanced Study, Princeton, NJ