Hyperfine Effects in Ionic Orbital Electron Capture

M.A. Goñi

1Theoretical Physics Department, Euskal Herriko Unibertsitatea,
644 Posta Kutxatila, 48080 Bilbo, Spain
(Dated: March 2, 2010)

Abstract
The K-orbital electron capture in ions with one or two electrons is analyzed for a general allowed nuclear transition. For ionic hyperfine states the angular neutrino distribution and the electron capture rate are given in terms of nuclear matrix elements. A possible application towards the determination of neutrino parameters is outlined.

PACS numbers: 13.15.+g, 2340.-s, 2340-BW, 1460 pq
INTRODUCTION

Significant modifications of weak nuclear rates due to the influence of atomic electrons have been predicted and their importance emphasized. J.N. Bahcall [1] developed a general discussion and treatment of the correlations and rates of bound beta decay processes for arbitrary electronic configurations and their effects on the behavior of hot stellar plasmas. Recently, an experiment at GSI has reported [2] the measurement of the ratio of bound state to continuum state total beta decay rates for the case of bare 1/2+ 207Tl decaying into the ground state of 1/2− 207Pb. It is a nuclear first forbidden transition and the ratio is in excellent agreement with the theory [3], [4].

New facilities allow electronic capture (EC) and β+ studies on highly ionized atoms, combining novel experimental tools which use high energy accelerators, in-flight separators and heavy ions storage rings. An experiment performed recently at GSI [5] has studied the radioactive ionic decays of 1+ 140Pr (Z=59) with 0,1,2 K-orbital electrons into 0+ 140Ce (Z=58). The Lorentz factor γ is 1.43 and the results reported in [5] are EC and β+ rates in the rest frame of ions. In this new kind of experiments the atomic contribution is well known, permitting cleaner measurements of the weak nuclear parameters. EC experiments on neutral atoms were reviewed in ref. [6].

The hyperfine structure due to the coupling of the nuclear spin I and the angular momentum 1/2 of the K-electron, giving a total angular momentum \( F = I \pm 1/2 \) \((F_\pm)\), is fundamental in order to understand the EC results. The ground state (gs) of \(^{140}\text{Pr}^{58+}\) has \( F = 1/2 \), as follows from its positive magnetic moment \( \mu [7] \), whereas the weak decay of the excited \( F = 3/2 \) state is forbidden in the allowed approximation. As the relaxation time for the upper hyperfine state is much shorter than the cooling time, the ions are dominantly stored with \( F = 1/2 \) in the experiment considered in [5].

The importance and origin of the influence of the hyperfine structure on the decay rates was clearly showed by Folan and Tsifrinovich [8] in their seminal analysis of EC decays of H-like atoms. Among other results, they showed that, in a spin 1/2 mirror transition \(^{31}\text{S} \rightarrow ^{31}\text{P}\), the rate for the \( F = 0 \) states is 340 times larger than the rate for states with \( F = 1 \). They also pointed out the possibility of observing this kind of effects with trapping techniques of cold ions.

More recently, a theoretical analysis of general EC hyperfine rates has been given in [9], where the ratio of the decay rates for helium -like and hydrogen -like ions is calculated explicitly as a function of the nucleus spin \( I \) in the allowed approximation, and suitable candidates for an experimental confirmation are discussed. A more refined analysis of K-shell EC and β+ weak decays rates of \(^{140}\text{Pr}^{58+}\) and \(^{140}\text{Pr}^{57+}\), which takes into account the nuclear size and uses relativistic electron wave functions, has been presented in [10, 11]. The ratios of the decay rates computed with these refinements agree with experimental results with an accuracy better than 3 percent.

Experiments at GSI have also measured time dependence of the electron capture rate of H-like \(^{140}\text{Pr}^{58+}\), \(^{142}\text{Pm}^{60+}\) and \(^{222}\text{Th}^{22+}\) ions [12, 14]. They have found an A-dependent modulation in the capture rate \( dN_{EC}(t)/dt \) that has been interpreted as due to neutrino
oscillations [13], although this interpretation is still controversial, see for instance [16, 17].

In a different line of developments, EC experiments have been proposed, [18, 19], as sources of monoenergetic neutrino beams, aiming at the determination of the neutrino mixing angle $\theta_{13}$, and the CP violation parameter $\delta_{CP}$. The experiments would use extremely high energy heavy ions ($\gamma \approx 10^2$-$10^3$) [20] that capture one electron and produce a line neutrino spectrum which is rotationally symmetric in the CM frame. Neutrino mixing would change the nature of the neutrinos detected at a distance $L$. The fluxes of $\nu_\mu$ are predicted by well known methods, see for instance [21] and references therein. Currently, a systematic study of EC in rare-Earth nuclei, relevant for neutrino beams, is being carried out [22].

In this note a general analysis of allowed K-electronic capture, still lacking in the literature as far as we know, is presented and an application to the measurement of neutrino mixing parameters using polarized ions is outlined. The allowed transitions from the mother nucleus $i$ into the daughter $f$ give the selection rules $I \rightarrow I' = I - 1 (F_-)$, $I' = I (F_\pm)$ and $I' = I + 1 (F_+)$. The cases $I' = I \pm 1$ are pure Gamow-Teller, whereas for $I' = I$ both Fermi and Gamow-Teller contribute. In all cases, the initial and final parities are the same ($\pi_i = \pi_f$).

In what follows, we consider separately the EC decays of H-like and He-like ions.

**H-LIKE EC**

Let us consider the transition amplitude for a K-EC process from an initial hyperfine $|FM>$ state to a final state where the nucleus spin and z-component are $I', m'$ respectively and the left neutrino has momentum $\vec{q}_\nu = E_\nu \vec{n}_\nu$. This amplitude will be denoted as $T_{FM \rightarrow m'\nu}$. The general structure of gs H-like transition amplitudes $T_{FM \rightarrow m'\nu}$ in terms of reduced angular momentum Wigner-Eckart amplitudes $\overline{T}$ is

$$T_{FM \rightarrow m'\nu} = a_+(\vec{n}_\nu)\delta_{m',M'} \left[ \sqrt{\frac{I' + M' + 1/2}{2I' + 1}} < F' M' | T | FM > - \sqrt{\frac{I' - M' + 1/2}{2I' + 1}} < F' M' | T | FM > \right]$$

$$+ a_-(\vec{n}_\nu)\delta_{m',M'+1/2} \left[ \sqrt{\frac{I' + M' + 1/2}{2I' + 1}} < F' M' | T | FM > + \sqrt{\frac{I' + M' + 1/2}{2I' + 1}} < F' M' | T | FM > \right].$$  \hspace{1cm} (1)

where the fundamental property

$$< F' M' | T | FM > = \delta_{F',F}\delta_{M,M'} T_{F \rightarrow F'}$$  \hspace{1cm} (2)

incorporates angular momentum conservation and rotational invariance at the outset and the left handed neutrino spin wave function is given by

$$|\nu_L> = a_+(\vec{n}_\nu)|\uparrow> + a_-(\vec{n}_\nu)|\downarrow> = -\sin(\theta/2)e^{-i\phi}|\uparrow> + \cos(\theta/2)|\downarrow>. \hspace{1cm} (3)$$
The reduced $\widetilde{T}$ are obtained by calculating suitable weak interaction $S$-matrix elements at first order, using the standard $\Delta S = 0$ piece of the Hamiltonian with coupling constant $G_F V_{ud}/\sqrt{2}$ and renormalized neutron beta decay axial coupling $g_A$ [23]. The matrix elements factorize into a lepton factor that can be computed explicitly and the nuclear matrix elements of the Vector $V^\mu$ and Axial $A^\mu$ weak hadronic currents. In the allowed approximation these nuclear matrix elements depend only on two characteristic constants $M_\sigma, F$ (see [24] and the Appendix), that one should obtain by using a good nuclear model.

The results for the reduced $\widetilde{T}$, with the weak coupling constants and the value of the electron wave function omitted, are collected in the Appendix. The cases $I' = I \pm 1$ are pure Gamow-Teller, whereas in the case $I' = I$ both Gamow-Teller and Fermi contribute so that the hiperfine rates depend on $M_F^2, M_\sigma^2$ and the interference term $M_{F,F}$. As expected, the interference disappears when the initial spin ion is not polarized [6, 24]. The $I = 1/2$ results in [8] are easily recovered.

For fixed $M$, $|T_{F,M \rightarrow m'\nu}|^2$ depends on both the final neutrino direction and the polarization of the final nucleus. Upon summing over the unobserved polarization of the daughter nucleus the angular distribution exhibits a characteristic $\cos \theta$ linear distribution. For instance in the case $I' = I - 1$ (the remaining cases are collected in the Appendix) we find

$$\sum_{m'} |T_{F,M \rightarrow m'\nu}|^2 = \left( \frac{1}{2} - \frac{M}{2I-1} \cos \theta \right) |T_{F,M}|^2.$$  (4)

The capture rate is $M$-independent when one sums over both final nuclear polarization and neutrino momentum (rotational invariance)

$$\int d\Omega_\nu \sum_{m'} |T_{F,M \rightarrow m'\nu}|^2 = 2\pi |T|^2$$  (5)

and the hyperfine rate $W$ is

$$W = \frac{(G_F V_{ud})^2}{\pi} (g_A M_\sigma)^2 |\varphi_0|^2 Q_\nu^2 \frac{2I+1}{2I}$$  (6)

where $Q_\nu$ is the neutrino energy. The two body phase space originates the $Q_\nu^2$ factor.

In the K-EC $1^+ \rightarrow 0^+$ decay for $^{140}$Pr$^{58+}$, or in a $0^+ \rightarrow 1^+$ transition [19], the use of an effective interaction hamiltonian density in terms of effective relativistic nuclear spin 1, 0 fields $H_\mu, \phi$

$$g(1 + \gamma_5)\gamma^\mu eH_\mu \phi + h.c.$$  (7)

provides a quick way to get our results. The amplitude from initial $I = 1, I_z = m, S_z = s$ to $m' = 0, \nu$ is

$$T_{ms \rightarrow 0\nu} \propto \chi_L^+ \bar{\sigma} \chi_s \epsilon_m$$  (8)

where $\epsilon_m, m = \pm1, 0$, are the spin-1 states of the mother nucleus and $\chi_{s,L}$ are the Pauli spin function of the captured electron and the final neutrino respectively. Elaborating the hyperfine amplitudes turn out to be

$$T_{F=1/2, \lambda} \propto -\sqrt{3} \chi_L^+ \chi_\lambda, \quad T_{F=3/2, \lambda} = 0$$  (9)

in agreement [25] with the general results in the Appendix.
HE-LIKE EC

For ions with two K electrons in the ground state, S=0, the capture process changes the initial $|(ee)_{S=0}I m\rangle$ state into the state $|I\prime m'\nu e\rangle$, with a daughter nucleus, one neutrino and one bound spectator electron. The transition amplitude is

\[ T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu e} = (T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu e}^H)_{\delta e\delta e'} - (T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu e}^H)_{\delta e\delta e'} \]

(10)

and the transition probability after summing over final electron spin $W_{(ee)_{S=0}, I m\rightarrow I' m'\nu}$ is thus given by

\[ W_{(ee)_{S=0}, I m\rightarrow I' m'\nu} = \left| (T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H)_{\delta e\delta e'} \right|^2 + \left| (T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H)_{\delta e\delta e'} \right|^2 \]

(11)

In terms of the hyperfine amplitudes

\[(2I+1)W_{(ee)_{S=0}, I m\rightarrow I' m'\nu} = (I + m + 1) \left| T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H \right|^2 + (I - m) \left| T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H \right|^2 \]

\[ -2\sqrt{(I+1/2)} - M^2 Re[T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H]^* \]

\[ + (I - m + 1) \left| T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H \right|^2 + (I + m) \left| T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H \right|^2 \]

\[ +2\sqrt{(I+1/2)} - M^2 Re[T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H]^* \]

(12)

Upon summing and integrating over final $m', \nu$ the interference term disappears and the RHS of (11) becomes

\[(I + m + 1) \sum_{m'\nu} \left| T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H \right|^2 + (I - m + 1) \sum_{m'\nu} \left| T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H \right|^2 \]

\[ + (I - m) \sum_{m'\nu} \left| T_{He, (ee)_{S=0}, I m\rightarrow I' m'\nu}^H \right|^2 \].

(13)

Therefore the total H, He-like $W$ rates are related by

\[ W^{He} = 2 \frac{I+1}{2I+1} W^{He, (+)} + 2 \frac{I}{2I+1} W^{He, (-)} \]

(14)

This relation was first obtained in [9]. For $^{140}$Pr it gives good agreement with experiment [3] once the important corrections due to relativistic Coulomb effects are taken into account [10, 11].

In the case $1^+ \rightarrow 0^+$ one obtains

\[ 3W_{(ee)_{S=0}, I m\rightarrow 0\nu} = (1 - m) \left| T_{1/2, M=1/2, I m\rightarrow 0\nu}^H \right|^2 + (1 + m) \left| T_{1/2, M=-1/2, I m\rightarrow 0\nu}^H \right|^2 \]

(15)

and therefore

\[ W_{(ee)_{S=0}, I m\rightarrow 0\nu} \propto (1 - m \cos \theta), \quad m = 1, 0, -1. \]

(16)

As required by rotational invariance, the magnetic $m$-number dependence vanishes after integrating over the neutrino directions. This dependence disagrees [20] with the results of [10, 11]. In the general case $I \rightarrow I - 1$ the neutrino angular dependence is given by

\[ \frac{I - m \cos \theta}{I}, \quad m = I, I - 1, ..., -I. \]

(17)
EC AND NEUTRINO PARAMETERS

EC could be useful in order to fix the values of the yet unknown neutrino mixing parameters $\theta_{13}, \delta_{CP}$ [19, 21].

Monoenergetic – in the EC rest frame – pure electronic neutrinos are detected as $\nu_\mu$ in a long baseline neutrino experiment. Accelerated stored high gamma ions have been proposed as suitable neutrino sources (Zucchelli [20]).

J. Sato [18] and M. Rolinec and J. Sato [27] have investigated the physical potential for neutrinos coming from the EC process $e^-^{110}\text{Sn} \rightarrow ^{110}\text{In} * \nu_e, \ Q = 267 \text{ kev}$, which is an allowed $0^+ \rightarrow 1^+$ nuclear transition. H-like ions would be assembled by using two equal $\gamma$ paralell beams of bare nuclei and electrons that would be captured in fly. As $Q$ is small the lifetime should be large (recall Eq.(6), experimentally $\tau = 4.11h$) and therefore the emitted neutrino flux would be low. The proposed Setups have baseline $L = 250, 600$ km and $\gamma = 900 – 2000$.

J. Bernabeu and collaborators have independently proposed [19] EC neutrino factories and thoroughly studied the fenomenology and viability of EC neutrino experiments aiming to measure the CP violating phase $\delta_{CP}$. Recently [21] they have proposed an hybrid beta decay and EC Setup using $^{156}\text{Yb}$ ions that decay $38\%$ via EC with a $\nu$-energy of 3.46 Mev and $52\%$ via $\beta^+$ with end neutrino energy of 2.44 Mev. The daughter nucleus $^{156}\text{Tm}^*$ is an excited $1^+$ giant Gamow-Teller resonance state so that the halflife, $t_{1/2} = 26.1$ seconds, is short enough to allow EC in the decay ring.

One should note that the use of polarized H-like ions would produce a neutrino flux dependence inside the detector that could be useful in order to disentangle the values of neutrino mixing parameters. In the case of a $0^+ \rightarrow 1^+$ nuclear transition with capture of one K-electron the neutrino angular distribution from ions with polarization vector $\vec{P} = < \vec{\sigma} >$ is

$$W(\vec{P}, \vec{n}_\nu) = \frac{1}{2} + \frac{1}{3} \vec{P} \cdot \vec{n}_\nu, \quad |\vec{P}| = p$$ (18)

With $\vec{P}$ in the ion beam direction, the $\nu$-distributions in the ion rest frame (rf) show a caracteristic parity violating linear $\cos \theta_{rf}$ dependence

$$W(\theta_{rf}, p) = \frac{1}{2}(1 \pm \frac{p}{3} \cos \theta_{rf})$$ (19)

This angular modulation is observable in the Rolinec, Sato proposal, Setup II, with a large Water Cherenkov detector of radius $R = 100$ m, base $L = 250$ km. With this geometry $\tan \theta_{max} = \frac{R}{L}$ and, for a neutrino emitted at right angle $\theta_{rf} = \pi$ to be detected, a boost

$$\gamma = \frac{1}{\sin \theta_{max}} = 2500,$$ (20)

is required. Rolinec and Sato take $\gamma = 2000$ and therefore $\cos(\theta_{rf})_{max} = .22$. The flux of $\nu_\mu$ in points inside the detector would depend now, in a known way, also on the ion polarization. The requirements to achieve such a performance, i.e., very high $\gamma$, large number of isotope
decays per year, very low beam divergences for the stored isotopes [27], together with the use of very high $\gamma$ polarized electron beams are extremely demanding.

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[25] Note that the hyperfine amplitudes $T_{E=m, M}$ are not vanishing for $M = \pm \frac{1}{2}$. This disagrees with the amplitudes reported in [10, 11]. The discrepancy may be due to the fact that the amplitudes in [10, 11] seem to be referred to a frame where the variable final neutrino direction is taken as quantization axis. This does not affect the rates, which are computed by averaging over initial and summing over final spin components.
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ACKNOWLEDGMENTS

I wish to thank José Bernabeu for information on neutrino factories and to Elvira Moya de Guerra for sharing her knowledge on weak nuclear properties. I am indebted to Amalio F. Pacheco for clarifications, encouragement and very useful comments on the manuscript, and to Juan L. Mañes and Manuel Valle for invaluable criticism and suggestions.

This work is dedicated to Prof. Julio Abad, in Memoriam.

APPENDIX

A. DEFINITION OF WEAK NUCLEAR PARAMETERS \( \mathcal{M}, A \)

Let \( \Psi_{i,f} \) be the wave functions of the initial, final nuclei at rest. The operators associated to the allowed transitions are \( 1\tau_-, \sigma_m\tau_- \) summed over nucleons, where \( \tau_- \) is the Isospin lowering matrix converting a proton into a neutron and \( \sigma_m \) are the standard spherical components of the Pauli \( \vec{\sigma} \) matrices. The nuclear matrix elements \( \mathcal{M}_{F,\sigma} \) are defined as follows

\[
< \Psi(I'm')_f | 1 | \Psi(Im)_i > \equiv \mathcal{M}_F \delta_{m',m} \delta_{I,I'},
\]

\[
< \Psi(I'm')_f | \sigma_n | \Psi(Im)_i > \equiv \mathcal{M}_\sigma \sqrt{\frac{2I+1}{2I'+1}} < I1mn|I'm' >, \quad (n = \pm 1, 0).
\]

B. HYPERFINE AMPLITUDES

In that follows we use the definitions

\[
A_F = \mathcal{M}_F,
\]

\[
A_\sigma = -g_A \mathcal{M}_\sigma \sqrt{\frac{2I+1}{2I'+1}}.
\]

1. \( I' = I - 1 \) CASE

\[
T_{F_{-}M \rightarrow m'\nu} = \left[ a_+ (\vec{n}_\nu) \delta_{m',M-1/2} \sqrt{\frac{I+M-1/2}{2I-1}} + a_- (\vec{n}_\nu) \delta_{m',M+1/2} \sqrt{\frac{I-M-1/2}{2I-1}} \right] T_{F_{-}}
\]

\[
\overline{T}_{F_{-}} = \overline{T}_{F_{-} \rightarrow F'_-} = (A_\sigma) \sqrt{\frac{2I-1}{I}}
\]
2. \( I' = I \) CASE
\[
T_{F_{\pm M \rightarrow m'\nu}} = \left[ a_+(\vec{n}_\nu)\delta_{m',M-1/2}(\pm \sqrt{\frac{I \pm M + 1/2}{2I + 1}}) + a_-(\vec{n}_\nu)\delta_{m',M+1/2}\sqrt{\frac{I \mp M + 1/2}{2I + 1}} \right] T_{F_{\pm}}
\]
\[
T_{F_+} = T_{F_+} = A_F + A_\sigma \sqrt{\frac{I}{I + 1}}, \quad T_{F_-} = T_{F_-} = A_F - A_\sigma \sqrt{\frac{I + 1}{I}}
\]

3. \( I' = I + 1 \) CASE
\[
T_{F_{\pm M \rightarrow m'\nu}} = \left[ -a_+(\vec{n}_\nu)\delta_{m',M-1/2}\sqrt{\frac{I - M + 3/2}{2I + 3}} + a_-(\vec{n}_\nu)\delta_{m',M+1/2}\sqrt{\frac{I + M + 3/2}{2I + 3}} \right] T_{F_+}
\]
\[
T_{F_+} = T_{F_+} = -(A_\sigma) \sqrt{\frac{2I + 3}{I + 1}}
\]

C. NEUTRINO ANGULAR DISTRIBUTIONS, FIXED \( M \)

1. \( I' = I - 1 \)
\[
\sum_{m'} |T_{F_{- M \rightarrow m'\nu}}|^2 = \left( \frac{1}{2} - \frac{M}{2I - 1} \cos \theta \right) |T_{F_-}|^2
\]

2. \( I' = I \)
\[
\sum_{m'} |T_{F_{\pm M \rightarrow m'\nu}}|^2 = \left( \frac{1}{2} \mp \frac{M}{2I + 1} \cos \theta \right) |T_{F_{\pm}}|^2
\]

3. \( I' = I + 1 \)
\[
\sum_{m'} |T_{F_{\pm M \rightarrow m'\nu}}|^2 = \left( \frac{1}{2} \mp \frac{M}{2I + 3} \cos \theta \right) |T_{F_{\pm}}|^2
\]