TORELLI-TYPE THEOREMS FOR GRAVITATIONAL INSTANTONS WITH QUADRATIC VOLUME GROWTH

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Abstract. We prove Torelli-type uniqueness theorems for both ALG* gravitational instantons and ALG gravitational instantons which are of order 2. That is, the periods uniquely characterize these types of gravitational instantons up to diffeomorphism. We define a period mapping $\mathcal{P}$, which we show is surjective in the ALG cases, and open in the ALG* cases. We also construct some new degenerations of hyperkähler metrics on the K3 surface which exhibit bubbling of ALG* gravitational instantons.

1. Introduction

We begin with the following definitions.

Definition 1.1. A hyperkähler 4-manifold $(X, g, I, J, K)$ is a Riemannian 4-manifold $(X, g)$ with a triple of Kähler structures $(g, I)$, $(g, J)$, $(g, K)$ such that $IJ = K$.

We denote by $\omega = (\omega_1, \omega_2, \omega_3)$ the Kähler forms associated to $I, J, K$, respectively. It is easy to see that $\omega_i$ satisfies

$$\omega_i \wedge \omega_j = 2\delta_{ij} \, d\text{vol}_g,$$

where $d\text{vol}_g$ is the Riemannian volume element. Conversely, any triple of symplectic forms $\omega_i$ satisfying (1.1) determines a hyperkähler structure if we replace $\omega_3$ by $-\omega_3$ if necessary.

Definition 1.2. A gravitational instanton $(X, g, \omega)$ is a non-compact complete non-flat hyperkähler 4-manifold $X$ such that $|\text{Rm}_g| \in L^2(X)$.

If $X$ is a compact non-flat hyperkähler 4-manifold, then it must be the K3 surface; see [Kod64]. If X is a gravitational instanton, there are many known types of asymptotic geometry of $X$ near infinity: ALE, ALF-A_k, ALF-D_k, ALG, ALH, ALG*, ALH*. We refer the reader to [BKN89, CC21a, CC19, CC21b, Hei12, Kro89, Min10] and references therein for more background on gravitational instantons.

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There is a well-known Torelli Theorem for hyperkähler metrics on the K3 surface, and one may ask if there is an analogue for gravitational instantons. This is known to hold in several cases: such a Torelli-type theorem was proved by [Kro89] in ALE case, by [Min11] in ALF-$A_k$ case, by [CC19] in ALF-$D_k$ case and by [CC21b] in ALH case. In this paper, we are interested in an analogous result assuming that the metric is of type ALG or ALG$^\ast$. In the ALG case, it was observed in [CC21b] that the natural period map may not be injective, and a modified version of the Torelli theorem was conjectured there. In this paper, we prove the uniqueness part of this conjecture, which gives the Torelli uniqueness in the ALG case; see Theorem 1.5. We also prove a Torelli-type uniqueness theorem in the ALG$^\ast$ case; see Theorem 1.10. We note that recently, a Torelli-type uniqueness Theorem in the ALH$^\ast$ case was proved; see [CJL21]. We will also define a refined period mapping $\mathcal{P}$ in both the ALG and ALG$^\ast$ cases, which we will show to be surjective in the ALG cases, and open in the ALG$^\ast$ cases; see Theorem 1.7 and Theorem 1.12.

Previously, gravitational instantons of type ALE, ALF, ALG, ALH, ALH$^\ast$ have been shown to bubble off of the K3 surface; see [Don12, LS94, Fos19, CVZ20, CC21b, HSVZ22]. In this paper, we also show that there exist families of Ricci-flat hyperkähler metrics on the K3 surface which have ALG$^\ast$ gravitational instantons occuring as bubbles; see Theorem 1.13. These are the first known examples of this type of degeneration. These example are produced via a gluing theorem which is actually the crucial tool in proving the aforementioned Torelli uniqueness in the ALG$^\ast$ case.

1.1. ALG gravitational instantons. For background of analysis on ALG gravitational instantons, related classification results, and relations to moduli spaces of monopoles and Higgs bundles, we refer the readers to [BB04, BM11, CC21b, CK02, FMSW20, Hei12, HHM04, Maz91] and also the references therein.

In Definition 2.3 below, we will define the standard ALG model space $(C_{\beta,\tau,\varepsilon,L}(R), g^C, \omega^C)$ for parameters $L, \varepsilon, R \in \mathbb{R}_+$, and $(\beta, \tau)$ as in Table 2.1. Here we just note that $C_{\beta,\tau,L}(R)$ is diffeomorphic to $(R, \infty) \times N_{3,\beta}^3$, where $N_{3,\beta}^3$ is a torus bundle over a circle, and the metric $g^C$ as well as the induced metric on the 3-manifold $N_{3,\beta}^3$ are flat; the explicit formulae are given in Subsection 2.3. We let $r$ denote the coordinate on $(R, \infty)$.

Definition 1.3 (ALG gravitational instanton). A complete hyperkähler 4-manifold $(X, g, \omega)$ is called an ALG gravitational instanton of order $n > 0$ with parameters $(\beta, \tau)$ as in Table 2.1 and $L > 0$ if there exist $R > 0$, a compact subset $X_R \subset X$, and a diffeomorphism $\Phi : C_{\beta,\tau,L}(R) \to X \setminus X_R$, such that

\begin{align}
\left| \nabla_{g_C}^k (\Phi^* g - g^C) \right|_{g_C} &= O(r^{-k-n}), \\
\left| \nabla_{g_C}^k (\Phi^* \omega_i - \omega_i^C) \right|_{g_C} &= O(r^{-k-n}), \quad i = 1, 2, 3,
\end{align}

as $r \to \infty$, for any $k \in \mathbb{N}_0$. 


Remark 1.4. It was proved in [CC19, Theorem A] that there exist ALG coordinates so that the order $n$ is 2 in the I*, II, III, IV cases ($\beta \leq \frac{1}{2}$) and $n = 2 - \frac{1}{\beta}$ in the II*, III*, IV* cases ($\beta > \frac{1}{2}$).

It was shown in [CV21, Theorem 1.10] that any two ALG gravitational instantons with the same $\beta$ are diffeomorphic. So without loss of generality we can view any ALG gravitational instanton as living on a fixed space $X_{\beta}$. Chen-Chen proved that the naive version of the Torelli-type theorem fails when $\beta > \frac{1}{2}$; see [CC21b]. Furthermore, it was shown in [CV21, Theorem 1.12] that when $\beta > \frac{1}{2}$, each ALG gravitational instanton of order 2 corresponds to a two-parameter family of ALG gravitational instantons with the same periods $[\omega]$, with exactly one element of this family being of order 2. This reduces the general case to proving a Torelli uniqueness theorem for ALG gravitational instantons of order 2, which is our next theorem.

Theorem 1.5 (ALG Torelli uniqueness). Let $(X_{\beta}, g, \omega)$ and $(X_{\beta}, g', \omega')$ be two ALG gravitational instantons with the same $\tau$ and $L$, which are both ALG of order 2 with respect to a fixed ALG coordinate system on $X_{\beta}$. If $[\omega] = [\omega'] \in H^2_{dR}(X_{\beta}) \times H^2_{dR}(X_{\beta}) \times H^2_{dR}(X_{\beta})$, then there is a diffeomorphism $\Psi : X_{\beta} \rightarrow X_{\beta}$ which induces the identity map on $H^2_{dR}(X_{\beta})$ such that $\Psi^*g' = g$ and $\Psi^*\omega' = \omega$.

This will be proved using a modification of the gluing construction in [CVZ20] (see Theorem 6.2 below), and then invoking the Torelli Theorem for K3 surfaces. We remark that the order 2 condition is essential to control the error term in the gluing construction. Moreover, the assumption that both hyperkähler structures are ALG of order 2 in a fixed coordinate system is also crucial for the proof. However, it is superfluous in the following sense: it was proved in [CV21, Theorem 1.11] that any two ALG gravitational instantons of order 2 with the same $(\beta, \tau)$ and $L$ can be pulled-back to a fixed space $X_{\beta}$ such that they are both ALG of order 2 in a fixed ALG coordinate system $\Phi_{X_{\beta}}$ (after possibly modifying one of the ALG coordinate systems). This motivates the following definition.

Definition 1.6. Let $M_{\beta,\tau,L}$ to be the collection of all gravitational instantons on $X_{\beta}$ with parameters $\beta, \tau,$ and $L$ which are ALG of order 2 with respect to a fixed ALG coordinate system $\Phi_{X_{\beta}}$. For $(X_{\beta}, g^0, \omega^0) \in M_{\beta,\tau,L}$, the period map based at $\omega^0$, $\mathcal{P} : M_{\beta,\tau,L} \rightarrow \mathcal{H}^2 \oplus \mathcal{H}^2 \oplus \mathcal{H}^2$, is defined by

\[
\mathcal{P}(\omega) = ([\omega_1 - \omega^0_1], [\omega_2 - \omega^0_2], [\omega_3 - \omega^0_3]),
\]

where $\mathcal{H}^2 \equiv \text{Im}(H^2_{cpt}(X_{\beta}) \rightarrow H^2(X_{\beta}))$.

We will show that $\mathcal{P}$ is well-defined in Section 7. The following is our main result about the period mapping in the ALG cases.
Theorem 1.7. If \((X_\beta, g, \omega) \in \mathcal{M}_{\beta, \tau, L}\), then
\[
\omega[C] \neq (0, 0, 0) \quad \text{for all} \quad [C] \in H_2(X_\beta; \mathbb{Z}) \quad \text{satisfying} \quad [C]^2 = -2.
\]
Furthermore, the period map \(\mathcal{P}\) is surjective onto cohomology triples in \(\mathcal{H}^2 \oplus \mathcal{H}^2 \oplus \mathcal{H}^2\) satisfying (1.6).

We will prove this in Section 7. In particular, we see that the space of order 2 ALG gravitational instantons with fixed parameters \(\beta, \tau, L\) has dimension \(3(b_2(X_\beta) - 1)\), where \(b_2(X_\beta)\) is given in Table 2.1.

1.2. ALG* gravitational instantons. In Section 2, we will define the standard ALG* model space, which is denoted by
\[
(\mathfrak{M}_{2\nu}(R), g^\text{gr}_{\kappa_0, L}, \omega^\text{gr}_{\kappa_0, L}) \equiv (\mathfrak{M}_{2\nu}(R), L^2 g^\text{gr}_{\kappa_0, L}, L^2 \omega^\text{gr}_{\kappa_0, L}),
\]
which depends on parameters \(\nu \in \mathbb{Z}_+, \kappa_0 \in \mathbb{R}, R > 0\), and an overall scaling parameter \(L > 0\). Here, we just note that the manifold \(\mathfrak{M}_{2\nu}(R)\) is diffeomorphic to \((R, \infty) \times I^3_s\), where \(I^3_s\) is an infranilmanifold, which is a circle bundle of degree \(\nu\) over a Klein bottle. We will let \(r\) denote the coordinate on \((R, \infty), V\) denote the function \(\kappa_0 + \frac{2}{\nu} \log r\) and \(s\) denote the function \(rV^{1/2}\). The hyperkähler structure is obtained via a Gibbons-Hawking ansatz. See Section 2 for explicit formulae.

Definition 1.8 (ALG* gravitational instanton). A complete hyperkähler 4-manifold \((X, g, \omega)\) is called an ALG* gravitational instanton of order \(n > 0\) with parameters \(\nu \in \mathbb{Z}_+, \kappa_0 \in \mathbb{R}\) and \(L > 0\) if there exist an ALG* model space \((\mathfrak{M}_{2\nu}(R), g^\text{gr}_{\kappa_0, L}, \omega^\text{gr}_{\kappa_0, L})\) with \(R > 0\), a compact subset \(X_R \subset X\), and a diffeomorphism \(\Phi : \mathfrak{M}_{2\nu}(R) \to X \setminus X_R\) such that
\[
|\nabla^k g^\text{gr}(\Phi^* g - g^\text{gr}_{\kappa_0, L})|_{g^\text{gr}} = O(s^{-k-n}),
\]
\[
|\nabla^k g^\text{gr}(\Phi^* \omega_i - \omega^\text{gr}_{i,\kappa_0, L})|_{g^\text{gr}} = O(s^{-k-n}), \quad i = 1, 2, 3,
\]
as \(s \to \infty\) for any \(k \in \mathbb{N}_0\).

Remark 1.9. It was proved in [CVZ21, Theorem 1.9] that there exist ALG* coordinates on \(X\) so that the order satisfies \(n \geq 2\). This decay order will be crucial in the proof of Theorem 1.10 below.

It was proved in [CV21, Theorem 1.6] that any 2 ALG* gravitational instantons with the same \(\nu\), where \(1 \leq \nu \leq 4\) are diffeomorphic to each other. So without loss of generality we can view any ALG* gravitational instanton as living on a fixed space \(X_\nu\). With this understood, our next theorem is a Torelli uniqueness theorem for ALG* gravitational instantons.

Theorem 1.10 (ALG* Torelli uniqueness). Let \(1 \leq \nu \leq 4\), and \((X_\nu, g, \omega), (X'_\nu, g', \omega')\) be two ALG* gravitational instantons with the same parameters \(\kappa_0\) and \(L\), which are both ALG* of order 2 with respect to a fixed ALG* coordinate system on \(X_\nu\). If
\[
|\omega| = |\omega'| \in H^2_{\text{dR}}(X_\nu) \times H^2_{\text{dR}}(X_\nu) \times H^2_{\text{dR}}(X_\nu),
\]
then there is a diffeomorphism \( \Psi : X_\nu \to X_\nu \) which induces the identity map on \( H^2_{\text{Dir}}(X_\nu) \) such that \( \Psi^* g' = g \) and \( \Psi^* \omega' = \omega \).

This will be proved using a new gluing construction: we obtain hyperkähler metrics on the K3 surface using ALG\(^*\) gravitational instantons (see Subsection 1.3 below), and then we invoke the Torelli Theorem for K3 surfaces. In our proof, the requirement that both metrics are ALG\(^*\) of order 2 with respect to a fixed ALG\(^*\) coordinate system is crucial. However, this assumption is actually superfluous in the following sense. It was proved in [CV21, Theorem 1.7] that if \((X,g,\omega)\) and \((X',g',\omega')\) are any two ALG\(^*\) gravitational instantons of order 2 with the same parameters \(\nu,\kappa_0, L\), then after possibly changing the ALG\(^*\) coordinate system \(\Phi'\) on \(X'\), we can arrange that the diffeomorphism map commutes with \(\Phi\) and \(\Phi'\). So we can actually view any ALG\(^*\) gravitational instanton with parameters \(\nu,\kappa_0, L\) as a gravitational instanton of order 2 on a fixed space \(X_\nu\) with a fixed ALG\(^*\) coordinate system \(\Phi_{X_\nu}\). Similar to the ALG case, we make the following definition.

**Definition 1.11.** Define \( M_{\nu,\kappa_0,L} \) to be the collection of all gravitational instantons on \(X_\nu\) with parameters \(\nu,\kappa_0, L\) which are ALG\(^*\) of order 2 with respect to a fixed ALG\(^*\) coordinate system \(\Phi_{X_\nu}\). For \((X_\nu,g,\omega)\in M_{\nu,\kappa_0,L}\), the period map based at \(\omega\), \(\mathcal{P} : M_{\nu,\kappa_0,L} \to \mathcal{H}^2 \oplus \mathcal{H}^2 \oplus \mathcal{H}^2\), is defined as in (1.5), where \(\mathcal{H}^2 \equiv \text{Im}(H^{2}_{\text{cpt}}(X_\nu) \to H^2(X_\nu))\).

The following is our main result about the period mapping in the ALG\(^*\) cases.

**Theorem 1.12.** If \((X_\nu,g,\omega)\in M_{\nu,\kappa_0,L}\), then

\[ \omega[C] \neq (0,0,0) \text{ for all } [C] \in H_2(X_\nu;\mathbb{Z}) \text{ satisfying } [C]^2 = -2. \]

Furthermore, the period mapping \(\mathcal{P}\) is an open mapping into the space of cohomology triples in \(\mathcal{H}^2 \oplus \mathcal{H}^2 \oplus \mathcal{H}^2\) satisfying (1.11).

We will prove this in Section 7. In particular, we see that the space of order 2 ALG\(^*\) gravitational instantons with fixed parameters \(\nu,\kappa_0\), and \(L\) has dimension \(3(b_2(X_\nu) - 1) = 12 - 3\nu\). We conjecture that the period mapping \(\mathcal{P}\) is also surjective in the ALG\(^*\) cases.

1.3. ALG\(^*\) bubbles from the K3 surface. In [GW00], hyperkähler metrics were constructed on elliptic K3 surfaces with 24 I\(_1\) fibers, which have a 2-dimensional Gromov-Hausdorff limit \((\mathbb{P}^1, d_{\text{ML}})\), where \(d_{\text{ML}}\) is called the McLean metric. This was generalized to arbitrary elliptic K3 surfaces in [GTZ16]; see also [OO21]. Subsequently, the authors gave a new construction on arbitrary elliptic K3 surfaces in [CVZ20], which also allowed for a detailed description of the degeneration near the singular fibers, which we briefly describe next. Away from singular fibers, the degeneration is modeled by Greene-Shapere-Vafa-Yau’s semi-flat metric; see [GSVY90]. A generalization of the Ooguri-Vafa metric (see [OV98]), which we called a
multi-Ooguri-Vafa metric (with \( b \) monopole points) were used to describe the degeneration near singular fiber of type \( I_1 \). ALG metrics were used to describe the degeneration near fibers with finite monodromy. In the case of \( I_1^* \) fibers, the model used was a \( \mathbb{Z}_2 \)-quotient of certain multi-Ooguri-Vafa metrics with \( 2\nu \) monopole points, together with 4 Eguchi-Hanson metrics due to the 4 orbifold singularities of the resulting quotient. It was moreover shown in [CVZ20] that such degenerations exist for metrics which are Kähler with respect to the fixed elliptic complex structure.

In this paper, let \( K \) be an elliptic K3 surface with a singular fiber \( D^* \) of type \( I_1 \) and \( (18 - b) \) singular fibers of type \( I_1 \), where \( 1 \leq b \leq 14 \) (recall that an elliptic K3 surface can have up to an \( I_1^* \) fiber [Shi03]). Let \( X \) be an ALG* gravitational instanton of order 2 with parameters \( 1 \leq \nu \leq 4 \), \( \kappa_0 \in \mathbb{R} \) and \( L > 0 \). Near \( I_1 \) fibers, we use the Ooguri-Vafa metric as before. Near \( D^* \), we cut out a neighborhood of \( D^* \) in \( K \) and, as a new method, glue it with a neck region and a rescaling of \( X \). We call the glued manifold \( M_\lambda \).

**Theorem 1.13.** There exists a family of hyperkähler metrics \( g_\lambda \) on the K3 surface \( M_\lambda \) such that \( (M_\lambda, g_\lambda) \xrightarrow{GH} (\mathbb{P}^1, d_{ML}) \) as \( \lambda \to 0 \), and such that near \( D^* \), the rescaling limits are \( X \) together with \( b + \nu \) Taub-NUT bubbles.

In this case of an \( I_1^* \) fiber, the construction in [CVZ20] was done to preserve the elliptic complex structure. In this new gluing construction, the original elliptic complex structure is not preserved. An interesting question is to describe more precisely the complex structure degeneration of this new family. We also point out that this construction is somewhat analogous to [HSVZ22] in that we construct a neck region with nontrivial topology which interpolates between different degree infranilmanifolds (versus nilmanifolds in [HSVZ22]), and which is responsible for the Taub-NUT bubbles. The proof of Theorem 1.13 is contained in Sections 3, 4, and 5.

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2. The model hyperkähler structures

2.1. Gibbons-Hawking construction. In this subsection, we review the Gibbons-Hawking construction of the ALG* model metric. See [CVZ21] for more details. For any positive integer \( \nu \), the Heisenberg nilmanifold \( \text{Nil}^3_{2\nu} \) of degree \( 2\nu \) is the quotient of \( \mathbb{R}^3 \) by the following actions

\[
\begin{align*}
\sigma_1(\theta_1, \theta_2, \theta_3) &\equiv (\theta_1 + 2\pi, \theta_2, \theta_3), \\
\sigma_2(\theta_1, \theta_2, \theta_3) &\equiv (\theta_1, \theta_2 + 2\pi, \theta_3 + 2\pi \theta_1), \\
\sigma_3(\theta_1, \theta_2, \theta_3) &\equiv (\theta_1, \theta_2, \theta_3 + 2\pi^2 \nu^{-1}).
\end{align*}
\]
Define
\[ \Theta \equiv \frac{\nu}{\pi} (d\theta_3 - \theta_2 d\theta_1), \quad V \equiv \kappa_0 + \frac{\nu}{\pi} \log r, \]
for \( r \in (R, \infty) \), \( \kappa_0 \in \mathbb{R} \), and \( R > e^{\frac{\nu}{\pi} (1 - \kappa_0)} \) on the manifold
\[ S^1 \to \mathfrak{M}_{2\nu}(R) \equiv (R, \infty) \times \text{Nil}_2 \to \tilde{U} \equiv (\mathbb{R}^2 \setminus B_R(0)) \times S^1. \]
Then the Gibbons-Hawking metric on \( \mathfrak{M}_{2\nu}(R) \) is given by
\[
\hat{g}_{\kappa_0} = V (dr^2 + r^2 d\theta_1^2 + d\theta_2^2) + V^{-1} \frac{\nu^2}{\pi^2} (d\theta_3 - \theta_2 d\theta_1)^2 \\
= V (dx^2 + dy^2 + d\theta_2^2) + V^{-1} \Theta^2,
\]
where \( x + \sqrt{-1} y \equiv r \cdot e^{\sqrt{-1} \theta_1} \). The model hyperkähler forms on the manifold \( \mathfrak{M}_{2\nu}(R) \) are given by
\[
\omega_j = \omega_{1,\kappa_0} = E^1 \wedge E^2 + E^3 \wedge E^4 = V dx \wedge dy + d\theta_2 \wedge \Theta, \\
\omega_j = \omega_{2,\kappa_0} = E^1 \wedge E^3 - E^2 \wedge E^4 = V dx \wedge d\theta_2 - dy \wedge \Theta, \\
\omega_K = \omega_{3,\kappa_0} = E^1 \wedge E^4 + E^2 \wedge E^3 = dx \wedge \Theta + V dy \wedge d\theta_2,
\]
where
\[
\{ E^1, E^2, E^3, E^4 \} = \{ V^{1/2} dx, V^{1/2} dy, V^{1/2} d\theta_2, V^{-1/2} \Theta \}.
\]
The \( \mathbb{Z}_2 \)-action \( \iota(r, \theta_1, \theta_2, \theta_3) \equiv (r, \theta_1 + \pi, -\theta_2, -\theta_3) \) induces an automorphism of the hyperkähler structure, and we define the \( \text{ALG}_\nu^* \) model space as
\[
(\mathfrak{M}_{2\nu}(R), g_{\kappa_0}^{\mathfrak{m}}, \omega_{1,\kappa_0}^{\mathfrak{m}}, \omega_{2,\kappa_0}^{\mathfrak{m}}, \omega_{3,\kappa_0}^{\mathfrak{m}}) \equiv (\mathfrak{M}_{2\nu}(R), g_{\kappa_0}^{\mathfrak{m}}, \omega_{1,\kappa_0}^{\mathfrak{m}}, \omega_{2,\kappa_0}^{\mathfrak{m}}, \omega_{3,\kappa_0}^{\mathfrak{m}}) / (\iota).
\]
By rescaling, we have \( (\mathfrak{M}_{2\nu}(R), g_{\kappa_0, L}^{\mathfrak{m}}, \omega_{1,\kappa_0, L}^{\mathfrak{m}}, \omega_{2,\kappa_0, L}^{\mathfrak{m}}, \omega_{3,\kappa_0, L}^{\mathfrak{m}}) \) for any scaling parameter \( L > 0 \), where
\[
g_{\kappa_0, L}^{\mathfrak{m}} \equiv L^2 \cdot g_{\kappa_0}^{\mathfrak{m}}, \quad \omega_{i,\kappa_0, L}^{\mathfrak{m}} \equiv L^2 \cdot \omega_{i,\kappa_0}^{\mathfrak{m}}, \quad i = 1, 2, 3,
\]
**Remark 2.1.** The model space has the following properties. The cross-section \( r = r_0 \) is an *infranil* 3-manifold. There is a holomorphic map \( u_{\mathfrak{m}} : \mathfrak{M}_{2\nu}(R) \to \mathbb{C} \) defined as \( u_{\mathfrak{m}} = r^2 e^{2\sqrt{-1} \Theta_1} \), with torus fibers. The infinite end of the model space compactifies complex analytically by adding a singular fiber of type \( \Gamma_\nu^* \).

### 2.2. Choice of connection form
In this subsection, we make some important remarks about our choice of connection form. The connection form satisfies
\[
d\Theta = \frac{\nu}{\pi} \Theta \\
\text{and} \quad \iota^* \Theta = -\Theta.
\]
Since \( \dim(H^1_{\text{GR}}(\tilde{U})) = 2 \) and is generated by \( d\theta_1 \) and \( d\theta_2 \), more generally we could have chosen
\[
\tilde{\Theta} = \frac{\nu}{\pi} (d\theta_3 - \theta_2 d\theta_1 + df + pd\theta_1 + qd\theta_2),
\]
where \( f: \bar{U} \to \mathbb{R} \), and \( p,q \in \mathbb{R} \). Note that \( \nu^* \tilde{\Theta} = -\tilde{\Theta} \) implies that \( p = 0 \) and
\[
(2.13) \quad f(r, \theta_1, \theta_2) + f(r, \theta_1 + \pi, -\theta_2) = c
\]
for a constant \( c \in \mathbb{R} \). The mapping
\[
(2.14) \quad \varphi_f(r, \theta_1, \theta_2, \theta_3) \equiv (r, \theta_1, \theta_2, \theta_3 + \frac{c}{2} - f)
\]
commutes with \( \sigma_1, \sigma_2, \sigma_3 \) and \( \iota \). Moreover, we have
\[
(2.15) \quad \varphi_f^* \tilde{\Theta} = \frac{\nu}{\pi} \left( d\theta_3 - \theta_2 d\theta_1 + q d\theta_2 \right).
\]
Next, define the mapping
\[
(2.16) \quad \varphi_q(\theta_1, \theta_2, \theta_3) \equiv (\theta_1 - q, \theta_2, \theta_3 - q \theta_2).
\]
It is straightforward to compute that \( \varphi_q \) also commutes with \( \sigma_1, \sigma_2, \sigma_3 \) and \( \iota \). Clearly, we have \( \varphi_q^* \varphi_f^* \tilde{\Theta} = \tilde{\Theta} \). Remark that the mapping \( \varphi_f \circ \varphi_q \) is clearly an isometry of the Gibbons-Hawking metric \( \tilde{g}_{\text{GH}} \). Since the mapping \( \varphi_f \circ \varphi_q \) induces a diffeomorphism \( \varphi_f \circ \varphi_q : \tilde{M}_{2q}(R)/\iota \to \tilde{M}_{2q}(R)/\iota \), this mapping is an isometry of the quotient metric. Therefore, we may assume without loss of generality that \( f = 0 \) and \( p = q = 0 \), so any choice of connection form is equivalent to \( \Theta \) up to diffeomorphism.

**Remark 2.2.** If we replace \( \Theta \) in (2.7), (2.8), (2.9) by
\[
(2.17) \quad \tilde{\Theta} = \frac{\nu}{\pi} \left( d\theta_3 - \theta_2 d\theta_1 + df + q d\theta_2 \right),
\]
to get \( \tilde{\omega}_I, \tilde{\omega}_J, \tilde{\omega}_K \), then
\[
\varphi_q \varphi_f^*(\tilde{\omega}_I, \tilde{\omega}_J, \tilde{\omega}_K) = (\omega_I, \cos q \cdot \omega_J + \sin q \cdot \omega_K, \cos q \cdot \omega_I - \sin q \cdot \omega_J).
\]
In other words, we can use the standard \( \Theta \) after a hyperkähler rotation.

### 2.3. ALG model space

In the ALG case, we have the following definition of the model space.

**Definition 2.3** (Standard ALG model). Let \( \beta \in (0,1] \), and \( \tau \in \mathbb{H} \equiv \{ \tau \in \mathbb{C} | \text{Im} \tau > 0 \} \) be parameters in Table 2.1, and \( L > 0 \) be a scaling parameter. Consider the space
\[
(2.18) \quad \{ (\mathcal{U}, \mathcal{V}) \mid \arg \mathcal{U} \in [0, 2\pi \beta) \} \subset (\mathbb{C} \times \mathbb{C}) / (\mathbb{Z} \oplus \mathbb{Z}),
\]
where \( \mathbb{Z} \oplus \mathbb{Z} \) acts on \( \mathbb{C} \times \mathbb{C} \) by
\[
(2.19) \quad (m,n) \cdot (\mathcal{U}, \mathcal{V}) = \left( \mathcal{U}, \mathcal{V} + (m + n\tau) \cdot L \right), \quad (m,n) \in \mathbb{Z} \oplus \mathbb{Z}.
\]
We can further identify \( (\mathcal{U}, \mathcal{V}) \) with \( (e^{\sqrt{-1} \cdot 2\pi \beta} \mathcal{U}, e^{-\sqrt{-1} \cdot 2\pi \beta} \mathcal{V}) \) to obtain a manifold \( C_{\beta, \tau, L} \). Define
\[
(2.20) \quad C_{\beta, \tau, L}(R) \equiv \{ |\mathcal{U}| > R \} \subset C_{\beta, \tau, L}.
\]
Then there is a flat hyperkähler metric
\[
(2.21) \quad g^C = \frac{1}{2} (d\mathcal{U} \otimes d\mathcal{U} + d\bar{\mathcal{U}} \otimes d\mathcal{U} + d\mathcal{V} \otimes d\bar{\mathcal{V}} + d\bar{\mathcal{V}} \otimes d\mathcal{V})
\]
on $C_{\beta,\tau,L}(R)$ with a hyperkähler triple
\[
\omega^C_1 = \frac{\sqrt{-1}}{2} (dU \wedge d\bar{U} + dV \wedge d\bar{V}),
\]
\[
\omega^C_2 = \text{Re}(dU \wedge dV), \quad \omega^C_3 = \text{Im}(dU \wedge dV).
\]

Each flat space $(C_{\beta,\tau,L}(R), g^C, \omega^C)$ given as the above is called a standard ALG model.

### Table 2.1. Invariants of ALG spaces.

| $\beta \in (0, 1]$ | I$^*_0$ | II | II$^*$ | III | III$^*$ | IV | IV$^*$ |
|----------------------|--------|----|-------|-----|---------|----|-------|
| $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{5}{6}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{3}$ | $\frac{2}{3}$ |
| $\tau \in \mathbb{H}$ | Any | $e^{\sqrt{-1} \frac{\pi}{3}}$ | $e^{\sqrt{-1} \frac{2\pi}{3}}$ | $\sqrt{-1}$ | $e^{\sqrt{-1} \frac{4\pi}{3}}$ | $e^{\sqrt{-1} \frac{4\pi}{3}}$ |
| $b_2(X_\beta)$ | 5 | 9 | 1 | 8 | 2 | 7 | 3 |

**Remark 2.4.** The model space has the following properties. Letting $r = |U|$, the cross-section $\{r = r_0\}$ is a flat 3-manifold. There is a holomorphic map $u_C : C_{\beta,\tau,L}(R) \to \mathbb{C}$ defined as $u_C = \frac{U}{r^{\frac{3}{2}}}$, with torus fibers, which have area $L^2 \cdot \text{Im} \tau$. The infinite end of the model space compactifies complex analytically by adding a singular fiber of the specified type in the first row of Table 2.1.

## 3. Building blocks and approximate metrics

In this section, we will describe the construction of the “approximate” hyperkähler triple, using a gluing construction. We will divide the K3 surface into the following regions: the ALG$^*_0$ bubbling region, the Gibbons-Hawking neck transition region, the Ooguri-Vafa regions, and the collapsing semi-flat hyperkähler structure away from singular fibers.

We start with an elliptic K3 surface $\pi_K : K \to \mathbb{P}^1$ with an I$^*_0$ fiber for some $1 \leq b \leq 14$ and I$^*_1$ fibers of number $(18 - b)$. Away from all singular fibers, we choose the hyperkähler structure as $\omega^{sf}$, given by the Greene-Shapere-Vafa-Yau ansatz; see [CVZ20, Subsection 2.2]. Near the I$^*_1$ fibers, we glue in Ooguri-Vafa metrics as in [GW00, CVZ20]. These regions contribute exponentially small error terms to the weighted estimates, so in the following we will take this as understood, and will not consider those regions in any detail. We will denote this region of the K3 surface by $K^* = K \setminus D^*$, where $D^*$ is the I$^*_0$ fiber, and will continue to denote the hyperkähler triple on this region by $\omega^{sf}$, even though it is not semi-flat near the I$^*_1$ fibers.

Near the I$^*_0$ fiber, as in [CVZ20], we consider the local double cover, which is an I$^*_2$ fiber. We choose local coordinate $\mathcal{Y}$ on the base of the local double cover and local coordinate $\mathcal{X} \in \mathbb{C}/(\mathbb{Z}/(g^0) \oplus \mathbb{Z}/(g^1))$ on the fiber of the
local double cover such that $\Omega = d\mathcal{Y} \wedge d\mathcal{Y}$, and for some holomorphic function $h(\mathcal{Y})$,

\[ \tau_1(\mathcal{Y}) = 1 \quad \text{and} \quad \tau_2(\mathcal{Y}) = \frac{b}{\pi \sqrt{-1}} \log \mathcal{Y} + h(\mathcal{Y}) \]

### 3.1. ALG\(_{\nu}^*\) bubbling region

Given a fixed $\nu \in \{1,2,3,4\}$, let $(X, g^X, \omega^X)$ be an ALG\(_{\nu}^*\) gravitational instanton with parameters $\nu, \kappa, \omega$, and $L$. Without loss of generality, by scaling we can assume that $L = 1$. Recall the model space is the $\mathbb{Z}_2$-quotient of the Gibbons-Hawking model $\mathfrak{M}_{2\nu}(R)$, where the Riemannian metric $\hat{g}^\mathfrak{M}$ and hyperkähler triple $\hat{\omega}^\mathfrak{M}$ of the $\mathbb{Z}_2$-covering space $\hat{\mathfrak{M}}_{2\nu}(R)$ are given by the following explicit formulae (as in Section 2) when $r$ is sufficiently large:

\[ \hat{g}^\mathfrak{M} = V (dr^2 + r^2 d\theta_1^2 + d\theta_2^2) + V^{-1} \Theta^2, \]

\[ \hat{\omega}_1^\mathfrak{M} = V dx \wedge dy + d\theta_2 \wedge \Theta, \]

\[ \hat{\omega}_2^\mathfrak{M} = V dx \wedge d\theta_2 - dy \wedge \Theta, \quad \hat{\omega}_3^\mathfrak{M} = dx \wedge \Theta + V dy \wedge d\theta_2. \]

where $V = \frac{\nu}{\pi} \log r + \kappa_0$ and $\kappa_0 \in \mathbb{R}$. To perform the gluing construction, we will take a large region in $X$ and appropriately scale down both $(g^X, \omega^X)$ and $(g^\mathfrak{M}, \omega^\mathfrak{M})$. We will fix parameters $\lambda$ and $t$ such that

\[ \lambda \to 0, \quad t \to 0, \quad \sigma \equiv \frac{\lambda}{t} \to 0. \]

Let us consider the rescaled coordinates $\tilde{x} \equiv \lambda \cdot x$, $\tilde{y} \equiv \lambda \cdot y$ for $(x, y) \in B_{2\nu}^\Lambda(0^2) \subset \mathbb{R}^2$. Immediately, $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2)^{\frac{1}{2}} = \lambda \cdot r$. We will work with the cutoff region $X \setminus \{r > 2 \sigma^{-1}\}$ with the rescaled ALG\(_{\nu}^*\) hyperkähler structure $(\hat{g}^X, \hat{\omega}^X) = (\lambda^2, g^X, \lambda^2, \hat{\omega}^X)$. Then the rescaled metric and hyperkähler triple on the asymptotic model can be written in terms of the rescaled coordinates:

\[ \tilde{V} = T + \frac{\nu}{\pi} \cdot \log \tilde{r} + \kappa_0, \quad T \equiv \frac{\nu}{\pi} \log \left( \frac{1}{\lambda} \right) \gg 1, \]

\[ \lambda^2 \cdot \hat{g}^\mathfrak{M} = \tilde{V} (d\tilde{x}^2 + d\tilde{y}^2 + \lambda^2 d\tilde{\theta}_2^2) + \lambda^2 \cdot \tilde{V}^{-1} \cdot \Theta^2, \]

\[ \lambda^2 \cdot \hat{\omega}_1^\mathfrak{M} = \tilde{V} \cdot d\tilde{x} \wedge d\tilde{y} + \lambda^2 \cdot d\tilde{\theta}_2 \wedge \Theta, \]

\[ \lambda^2 \cdot \hat{\omega}_2^\mathfrak{M} = \lambda \cdot \tilde{V} \cdot d\tilde{x} \wedge d\tilde{\theta}_2 - \lambda \cdot d\tilde{y} \wedge \Theta, \]

\[ \lambda^2 \cdot \hat{\omega}_3^\mathfrak{M} = \lambda \cdot d\tilde{x} \wedge \Theta + \lambda \cdot \tilde{V} \cdot d\tilde{y} \wedge d\tilde{\theta}_2. \]

Note that the cutoff region becomes $O_{2\nu}(p) \equiv X \setminus \{\tilde{r} > 2 t\}$ in terms of $\tilde{r}$.

### 3.2. Neck transition region

The next building block is the neck transition region. To begin with, we take a flat product metric on $Q \equiv \mathbb{R}^2 \times S^1 = \mathbb{R}^2 \times (\mathbb{R}/2\pi \mathbb{Z})$ with $0^* \equiv (0^2, 0) \in \mathbb{R}^2 \times S^1$,

\[ g^Q = dx^2 + dy^2 + d\theta_2^2 = dr^2 + r^2 d\theta_1^2 + d\theta_2^2, \quad \theta_2 \in [0,2\pi]. \]
For fixed $\kappa_0$ in (3.2) and small parameter $\lambda \ll 1$. Let
\begin{equation}
\tilde{P} = \{ \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{2\nu+2b} \} \subset (\mathbb{R}^2 \setminus \{0^2\}) \times \{0\} \subset Q
\end{equation}
be a fixed set such that the following properties hold.

(1) (Balancing condition) Let $\tilde{d}_m \equiv d^Q(0^*, \tilde{p}_m)$ for any $1 \leq m \leq 2\nu+2b$. Then
\begin{equation}
\sum_{m=1}^{2\nu+2b} \log(1/\tilde{d}_m) + 2\pi \Im h(0) = 2\pi\kappa_0,
\end{equation}
where $h$ is the holomorphic function in (3.1).

(2) ($\mathbb{Z}_2$-invariance) $\nu(\tilde{p}_m) = \tilde{p}_{2\nu+2b+1-m}$ for any $1 \leq m \leq 2\nu+2b$.

Let $P$ be a fixed set such that the following properties hold.

(1) (Balancing condition) Let $\tilde{d}_m$ for any $1 \leq m \leq 2\nu+2b$.

Then
\begin{equation}
\sum_{m=1}^{2\nu+2b} \log(1/\tilde{d}_m) + 2\pi \Im h(0) = 2\pi\kappa_0,
\end{equation}
where $h$ is the holomorphic function in (3.1).
where \(|E(\mathfrak{g})| \leq C \cdot \lambda \cdot \tilde{r}(\mathfrak{g}) = C \cdot \lambda \cdot \tilde{r}^2(\mathfrak{g})\) for some constants \(C > 0\) independent of \(\lambda\).

2. (Near the infinity of \(Q\)) If \(\tilde{r}(\mathfrak{g}) \to \infty\), then

\[
\left| G_{\lambda}(\mathfrak{g}) - \text{Im} \tau_2(\lambda \cdot \tilde{\zeta}) \right| = \left| G_{\lambda}(\mathfrak{g}) - \left( T - \frac{b}{\pi} \cdot \log \tilde{r}(\mathfrak{g}) + \text{Im} h(\lambda \cdot \tilde{\zeta}) \right) \right| \leq \frac{C \cdot \lambda^2}{\tilde{r}(\mathfrak{g})^2},
\]

where \(\tilde{\zeta} \equiv \tilde{x} + \sqrt{-1} \tilde{y}, \ C > 0\) is independent of \(\lambda\), and \(\tau_2\) is the function in (3.1).

3. (Near a pole \(p_m \in P\)) If \(\tilde{d}^Q(\mathfrak{g}, p_m) \leq \frac{4}{\nu}\) for some \(p_m \in P\), then there exists a constant \(C > 0\) independent of \(\lambda\) such that

\[
\left| G_{\lambda}(\mathfrak{g}) - \left( G_m(\mathfrak{g}) + T^\circ + \frac{\nu}{\pi} \cdot \log \tilde{a}_m \right) \right| \leq C, \quad T^\circ \equiv \frac{2\nu + 1}{2\pi} \cdot \log \left( \frac{1}{\lambda} \right).
\]

4. (Bounded region) If there exist \(R_0 > 0\) and \(d_0 > 0\) such that \(R_0^{-1} \leq \tilde{r}(\mathfrak{g}) \leq R_0\) and \(\tilde{d}^Q(\mathfrak{g}, P) \geq \frac{4}{\nu}\), then \(|G_{\lambda}(\mathfrak{g}) - T| \leq C\), where \(C = C(R_0, d_0) > 0\) is independent of \(\lambda\).

Now we apply the Gibbons-Hawking construction using the Green’s function \(G_{\lambda}\). Let \(\mathcal{N}\) be the total space of the circle bundle \(S^1 \to \mathcal{N} \xrightarrow{\pi} Q \setminus (P \cup \{0^*\})\) with the \(S^1\)-connection form \(\Theta_{\lambda}\) that satisfies the monopole equation \(d\Theta_{\lambda} = *_{g^Q} \circ dG_{\lambda}\). Then we have a family of hyperkähler metrics \(g^{\mathcal{N}}\) and hyperkähler triples \(\omega^{\mathcal{N}}\) when \(G_{\lambda} > 0\):

\[
g^{\mathcal{N}} = \lambda^2 (G_{\lambda} \cdot g^Q + G_{\lambda}^{-1} \Theta_{\lambda}^2) = G_{\lambda}(dx^2 + d\tilde{y}^2 + \lambda^2 G_{\lambda}^{-1} \Theta_{\lambda}^2),
\]

\[
\omega_1^{\mathcal{N}} = \lambda^2 (G_{\lambda} dx \wedge dy + d\theta_2 \wedge \Theta_{\lambda}) = G_{\lambda} d\tilde{x} \wedge d\tilde{y} + \lambda^2 d\theta_2 \wedge \Theta_{\lambda},
\]

\[
\omega_2^{\mathcal{N}} = \lambda^2 (G_{\lambda} dx \wedge d\theta_2 - dy \wedge \Theta_{\lambda}) = \lambda G_{\lambda} \cdot d\tilde{x} \wedge d\theta_2 - \lambda d\tilde{y} \wedge \Theta_{\lambda},
\]

\[
\omega_3^{\mathcal{N}} = \lambda^2 (dx \wedge \Theta_{\lambda} + G_{\lambda} dy \wedge d\theta_2) = \lambda d\tilde{x} \wedge \Theta_{\lambda} + \lambda G_{\lambda} d\tilde{y} \wedge d\theta_2.
\]

It is easy to check that the completion \((\mathcal{N}, g^{\mathcal{N}}, \omega^{\mathcal{N}})\) of \((\mathcal{N}', g^{\mathcal{N}}', \omega^{\mathcal{N}}')\) along the set \(P\) of monopole points, called the neck transition region, is smooth and hyperkähler. Moreover, the neck transition region \((\mathcal{N}', g^{\mathcal{N}}', \omega^{\mathcal{N}}')\) is invariant under the involution \(\langle i \rangle \cong \mathbb{Z}_2\), and hence it descends to a hyperkähler manifold \((\mathfrak{M}, g^\mathfrak{M}, \omega^\mathfrak{M})\), where \(\mathfrak{M} \equiv \mathcal{N}/\langle i \rangle\).

3.3. Attaching the pieces. Let \((X, g^X, \omega^X)\) be an ALG\(_v^v\) gravitational instanton of order 2. We will next glue the end of \(X\) onto the neck transition region \(\mathcal{N}\) near the origin. By definition, there exists a compact subset \(X_R \subset X\), and a diffeomorphism \(\Psi : \mathfrak{M} \to X \setminus X_R\) such that for any \(k \in \mathbb{N}\),

\[
|\nabla_{g^\mathfrak{M}}^k(\Psi^* \omega^X - \omega^\mathfrak{M})| \leq C_k \cdot (r \cdot V(r)^{1/2})^{-2-k}.
\]

Thanks to the following lemma, we are able to compare the two hyperkähler triples \(\lambda^2 \cdot \omega^\mathfrak{M}\) and \(\omega^{\mathcal{N}}\) as \(\tilde{r} \to 0\).
Lemma 3.1. There exists a diffeomorphism
\begin{equation}
\Psi^N : \{ x \in N | t \leq \tilde{r}(x) \leq 2t \} \rightarrow \{ x \in \hat{\mathcal{M}} | t \leq \tilde{r}(x) \leq 2t \}
\end{equation}
such that \((\Psi^N)^* d\theta = d\theta, (\Psi^N)^* d\theta_1 = d\theta_1, (\Psi^N)^* d\theta_2 = d\theta_2,\) and
\begin{equation}
(\Psi^N)^* \Theta = \Theta_\lambda + \pi^* \zeta
\end{equation}
for some 1-form \( \zeta \) on \( \{ x \in Q | t \leq \tilde{r}(x) \leq 2t \} \) that satisfies \( \iota^* \pi^* \zeta = -\pi^* \zeta, \) and
\begin{equation}
| \nabla^k_{g^N}(\pi^* \zeta) | \leq C_k \cdot \lambda \cdot t^{1-k} \cdot V(\sigma^{-1})^{-\frac{1+k}{2}},
\end{equation}
\begin{equation}
| \nabla^k_{g^N}((\Psi^N)^*(\lambda^2 \cdot \omega^N) - \omega^N) | \leq C_k \cdot \lambda \cdot t^{1-k} \cdot V(\sigma^{-1})^{-\frac{4+k}{2}}
\end{equation}
for any \( k \in \mathbb{N}_0. \) Moreover, \( \Psi^N \) descends to a diffeomorphism
\begin{equation}
\Psi^\mathcal{M} : \{ x \in \mathcal{M} | t \leq \tilde{r}(x) \leq 2t \} \rightarrow \{ x \in \mathcal{M} | t \leq \tilde{r}(x) \leq 2t \}.
\end{equation}

Proof. The proof is the same as that of [HSVZ22, Lemma 6.1]. Here we only mention the major difference. First, both \( \mathcal{N} \) and \( \hat{\mathcal{M}} \) can be viewed as principal \( S^1 \)-bundles over \( \tilde{U} \subset \mathbb{R}^2 \times S^1 \) with the connections \( \Theta_\lambda \) and \( \Theta \) respectively, where \( \tilde{U} \equiv \mathbb{R}^2 \setminus B_R(0^2). \) One can easily check that they have the same Euler number \( 2\nu \) when \( t \leq \tilde{r}(x) \leq 2t. \)

Therefore, there exists a bundle isomorphism \( F : \mathcal{N} \rightarrow \hat{\mathcal{M}} \) which covers the identity map on \( \tilde{U} \times S^1. \) Moreover, the curvature difference is given by
\begin{equation}
F^*(d\Theta) - d\Theta_\lambda = *_{Q} d(E),
\end{equation}
where \( E \in C^\infty(Q) \) is the function given by the expansion (3.17). Applying the asymptotic estimate in (3.17), we have that
\begin{equation}
| *_{Q} d(E)|_{\mathcal{N}} \leq C \cdot \lambda \cdot V(\sigma^{-1})^{-1}.
\end{equation}

Standard Hodge theory implies that there exist a diffeomorphism \( \Psi^N, \) a flat connection \( \Theta_{\text{flat}}, \) and a 1-form \( \zeta \) on \( \tilde{U} \times S^1 \) such that
\begin{equation}
(\Psi^N)^* \Theta - \Theta_\lambda = \Theta_{\text{flat}} + \pi^* \zeta,
\end{equation}
\begin{equation}
| \nabla^k_{g^N}(\pi^* \zeta) | \leq C_k \cdot \lambda \cdot t^{1-k} \cdot V(\sigma^{-1})^{-\frac{1+k}{2}}.
\end{equation}
As discussed in Section 2.2, the flat connection \( \Theta_{\text{flat}} \) can be removed by appropriately choosing a bundle diffeomorphism. So the proof is done. \( \square \)

Lemma 3.2. There exists a triple of 1-forms \( \xi \) on \( \{ x \in \mathcal{N} | t \leq \tilde{r}(x) \leq 2t \} \) such that \( \iota^* \xi = \xi \) and
\begin{equation}
(\Psi^N)^*(\lambda^2 \cdot \omega^N) - \omega^N = d\xi.
\end{equation}
Moreover, \( \xi \) satisfies the estimate
\begin{equation}
| \nabla^k_{g^N}(\pi^* \zeta) | \leq C_k \cdot \lambda \cdot t^3 \cdot (t \cdot V(\sigma^{-1})^\frac{1}{2})^{-1-k}
\end{equation}
for any \( k \in \mathbb{N}_0. \) Moreover, \( \Psi^N \) descends to a diffeomorphism
\begin{equation}
\Psi^\mathcal{M} : \{ x \in \mathcal{M} | t \leq \tilde{r}(x) \leq 2t \} \rightarrow \{ x \in \mathcal{M} | t \leq \tilde{r}(x) \leq 2t \}.
\end{equation}
Once we have Lemma 3.1, the proof of Lemma 3.2 follows from the same arguments as in Proposition 6.2. We omit the details.

Next, we glue the cutoff region \( \{ r \leq 2\sigma^{-1} \} \subset X \) as introduced above into the neck region \( \mathcal{N} \). We define the diffeomorphism
\[
\Phi \equiv (\Psi \circ \Psi^\mathcal{N})^{-1}
\]
from \( \{ \sigma^{-1} \leq r \leq 2\sigma^{-1} \} \subset X \) to a subset \( \{ t \leq \tilde{r}(x) \leq 2t \} \subset \mathcal{N} \). Combining Lemma 3.2 and the asymptotic estimate of an ALG* gravitational instanton, we have the following.

**Lemma 3.3.** There exists a triple of 1-forms \( \eta^X \) on \( \{ x \in \mathcal{N} | t \leq \tilde{r}(x) \leq 2t \} \) such that \((\Phi^{-1})^*(\lambda^2 \cdot \omega^X) - \omega^\mathcal{N} = d\eta^X \) and satisfies the estimate
\[
|\nabla^k g^X|_{g^\mathcal{N}} \leq C_k \cdot (\lambda^2 + \tilde{\lambda} \cdot t^3) \cdot (t \cdot V(\sigma^{-1})^{\frac{3}{2}})^{1-k}
\]
for any \( k \in \mathbb{N}_0 \).

Next, we will glue a subset of \( \mathcal{K}^* \) onto the end of the neck region with \( \tilde{r} \) large. As shown in Construction 2.6 and Proposition 2.3, the hyperkähler triple \( \tilde{\lambda}^{-2} \cdot \omega^{sf} \) of the rescaled semi-flat metric on \( \mathcal{K}^* \), up to a \( \mathbb{Z}_2 \)-covering, can be written in terms of the Gibbons-Hawking ansatz by applying the harmonic function
\[
V_{sf} \equiv \text{Im } \tau_2(\tilde{\lambda} \cdot \tilde{\zeta}) = T - \frac{b}{\pi} \log \tilde{r} + \text{Im } h(\tilde{\lambda} \cdot \tilde{\zeta})
\]
which is the leading term of (3.18). Then we have the following lemma.

**Lemma 3.4.** For any sufficiently small parameter \( \lambda \ll 1 \), let \( r_\lambda \) be a large number such that \( 1 \leq G_\lambda(x) \leq 100 \) as \( r_\lambda \leq \tilde{r}(x) \leq 2r_\lambda \). There exist a triple of 1-forms \( \eta^{sf} \) on \( \{ x \in \mathcal{N} | r_\lambda \leq \tilde{r}(x) \leq 2r_\lambda \} \) and a diffeomorphism \( \Phi^\mathcal{K} \) from \( \{ x \in \mathcal{N} | r_\lambda \leq \tilde{r}(x) \leq 2r_\lambda \} \) to a subset of \( \mathcal{K}^* \) such that for all \( k \in \mathbb{N}_0 \),
\[
\omega^{\mathcal{N}} - (\Phi^\mathcal{K})^*(\tilde{\lambda}^{-2} \cdot \omega^{sf}) = d\eta^{sf},
\]
\[
|\nabla^k g^{sf}|_{g^\mathcal{N}} \leq C_k \cdot \lambda^2 \cdot \tilde{\lambda}^{1+k}.
\]

Notice that (3.38) follows from (3.18), and \( r_\lambda \) is comparable to \( \tilde{\lambda}^{-1} \).

With the above preparations, we are ready to define the closed glued manifold on which we will construct a family of collapsing hyperkähler metrics with a given ALG* gravitational instanton bubbling out. Now let us take the neck transition region \( \mathcal{N} \) equipped with the hyperkähler triple \( \omega^{\mathcal{N}} \) for any \( \lambda \ll 1 \), as constructed in Section 3.2. In the region \( \{ x \in \mathcal{N} | t \leq \tilde{r}(x) \leq 2t \} \), we glue \( \mathcal{N} \) with the finite part \( \{ r \leq 2\sigma^{-1} \} \) of an ALG* gravitational instanton \( X \) using the diffeomorphism \( \Phi \). In the region \( \{ x \in \mathcal{N} | r_\lambda \leq \tilde{r}(x) \leq 2r_\lambda \} \), we attach \( \mathcal{N} \) to \( \mathcal{K}^* \) using the diffeomorphism \( \Phi^\mathcal{K} \) as in Lemma 3.3. Using the above gluing maps, we obtain a closed smooth 4-manifold \( M_\lambda \). Now we construct a family of approximately hyperkähler triples \( \omega_\lambda \) on \( M_\lambda \).
Lemma 3.5 (Approximate hyperkähler triple). For any sufficiently small parameter $\lambda \ll 1$, let $r_\lambda$ be a large number such that $1 \leq G_\lambda(x) \leq 100$ as $r_\lambda \leq \tilde{r}(x) \leq 2r_\lambda$. Then there exist two triples of 1-forms $\eta^X$ and $\eta^{sf}$ such that the glued definite triple

$$
\omega_\lambda \equiv \begin{cases} 
\lambda^2 \cdot \omega^X, & \tilde{r} \leq t, \\
\omega^M + d\left(\varphi \cdot \eta^X - \psi \cdot \eta^{sf}\right), & t \leq \tilde{r} \leq 2r_\lambda, \\
\lambda^{-2} \cdot \omega^{sf}, & \tilde{r} \geq 2r_\lambda,
\end{cases}
$$

satisfies the following estimates with respect to associated Riemannian metric $\tilde{g}_\lambda$ for any $k \in \mathbb{N}_0$,

$$
\sup_{t \leq \tilde{r} \leq 2t} \left| \nabla^k \tilde{g}_\lambda \left( \omega_\lambda - (\Phi^{-1})^* (\lambda^2 \cdot \omega^X) \right) \right|_{\tilde{g}_\lambda} \leq C_k \cdot (\lambda^2 + \tilde{\lambda} \cdot t^3) \cdot (t \cdot V(\sigma^{-1})^{-1})^{-2-k},
$$

$$
\sup_{r_\lambda \leq \tilde{r} \leq 2r_\lambda} \left| \nabla^k \tilde{g}_\lambda \left( \omega_\lambda - (\Phi^0)^* (\tilde{\lambda}^{-2} \cdot \omega^{sf}) \right) \right|_{\tilde{g}_\lambda} \leq C_k \cdot \lambda^2 \cdot \tilde{\lambda}^{2+k},
$$

where $\tilde{\lambda} \equiv e^{-\frac{r^2}{4t^2}} = \lambda^{\tilde{r}}$, $\varphi$ and $\psi$ are smooth cut-off functions satisfying

$$
\varphi = \begin{cases} 
1, & \tilde{r} \leq t, \\
0, & \tilde{r} \geq 2t,
\end{cases}
$$

and $\psi$ is the hyperkähler triple of the semi-flat metric of Greene-Shapere-Vafa-Yau [GSVY90] with area of each fiber equal to $\lambda \cdot \lambda$ and diameter comparable to 1.

Proof. The proof is straightforward. The error estimate in the region $\{t \leq \tilde{r} \leq 2t\}$ is given by Lemma 3.3 and the error estimate in $\{r_\lambda \leq \tilde{r} \leq 2r_\lambda\}$ is due to Lemma 3.4.

It turns out that the manifold is indeed diffeomorphic to the K3 surface, but for now we do not need this fact, we only need the following calculation of the betti numbers.

Corollary 3.6. For $\lambda$ sufficiently small, the smooth 4-manifold $M_\lambda$ satisfies

$$
b^1(M_\lambda) = 0, \quad b^2_+(M_\lambda) = 3, \quad b^2_-(M_\lambda) = 19, \quad \chi(M_\lambda) = 24.
$$

Proof. This is proved using a Mayer-Vietoris argument and the estimates in Lemma 3.5 which show that $\Lambda_+^2(M_\lambda)$ is a trivial bundle if $\lambda$ is small. We omit the details which are similar to [HSVZ22, Proposition 6.6].

4. Metric geometry and regularity scales

To begin with, we will list the notations. We will always fix a small of parameter $\lambda \ll 1$.

1. Let us denote $g_\lambda \equiv \tilde{\lambda}^2 \cdot \tilde{g}_\lambda$. Then it holds that there is some constant $C_0 > 0$ independent of $\lambda$ such that $C_0^{-1} \leq \text{Diam}_{g_\lambda}(M_\lambda) \leq C_0$. 

2. Lemma 3.5, which show that $\Lambda^2_+ (M_\lambda)$ is the hyperkähler triple of the semi-flat metric of Greene-Shapere-Vafa-Yau [GSVY90] with area of each fiber equal to $\lambda \cdot \lambda$ and diameter comparable to 1. 

Proof. The proof is straightforward. The error estimate in the region $\{t \leq \tilde{r} \leq 2t\}$ is given by Lemma 3.3 and the error estimate in $\{r_\lambda \leq \tilde{r} \leq 2r_\lambda\}$ is due to Lemma 3.4.

It turns out that the manifold is indeed diffeomorphic to the K3 surface, but for now we do not need this fact, we only need the following calculation of the betti numbers.

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b^1(M_\lambda) = 0, \quad b^2_+(M_\lambda) = 3, \quad b^2_-(M_\lambda) = 19, \quad \chi(M_\lambda) = 24.
$$

Proof. This is proved using a Mayer-Vietoris argument and the estimates in Lemma 3.5 which show that $\Lambda_+^2(M_\lambda)$ is a trivial bundle if $\lambda$ is small. We omit the details which are similar to [HSVZ22, Proposition 6.6].

4. Metric geometry and regularity scales
(2) We define the smoothing function $\tau$ of the distance function $\tilde{r}$ by:

$$
\tau(x) = \begin{cases} 
\frac{\lambda \cdot R_0}{2}, & \tilde{r}(x) \leq \lambda \cdot R_0, \\
\tilde{r}(x), & 2\lambda \cdot R_0 \leq \tilde{r}(x) \leq r_\lambda, \\
2r_\lambda, & \tilde{r}(x) \geq 2r_\lambda,
\end{cases}
$$

where $R_0$ is the constant $R$ in Definition 1.8 and $r_\lambda$ is the constant in Lemma 3.3.

(3) Given $\nu \in \mathbb{Z}_+$, let $T^\nu \equiv \frac{2\nu + 1}{2\lambda} \cdot \log(\frac{1}{\lambda})$ and let $\delta$ be the following:

$$
\delta(x) = \begin{cases} 
(T^\nu)^{-\frac{1}{2}}, & d^q(x, p_m) \leq (T^\nu)^{-1} \\
&T^\nu \cdot d^q(x, p_m), & 2(T^\nu)^{-1} \leq d^q(x, p_m) \leq 1 \\
&T^\nu + \frac{1}{2\lambda} \log \frac{1}{d^q(x, p_m)} \cdot d^q(x, p_m), & 2 \leq d^q(x, p_m) \leq \frac{4}{\lambda} \cdot \lambda^{-1}
\end{cases}
$$

for some $1 \leq m \leq 2\nu + 2b$.

(4) Let us define $T \equiv \frac{b}{\pi} \log(\frac{1}{\lambda})$ and a smooth function $\Sigma_T$ that satisfies

$$
\Sigma_T(x) = \begin{cases} 
1, & \tilde{r}(x) \leq \lambda \cdot R_0, \\
T + \kappa_0 + \frac{4}{\pi} \log \tilde{r}(x), & 2\lambda \cdot R_0 \leq \tilde{r}(x) \leq \frac{4}{\lambda}, \\
T + \text{Im} h(0) - \frac{b}{\pi} \log \tilde{r}(x), & 2\lambda^{-1} \leq \tilde{r}(x) \leq r_\lambda, \\
1, & \tilde{r}(x) \geq 2r_\lambda.
\end{cases}
$$

Next, we describe the $C^{k,\alpha}$-regularity scale.

**Definition 4.1 (Local regularity).** Let $(M^n, g)$ be a Riemannian manifold. Given $r, \epsilon > 0$, $k \in \mathbb{N}$, $\alpha \in (0, 1)$, $(M^n, g)$ is said to be $(r, k + \alpha, \epsilon)$-regular at $x \in M^n$ if $g$ is at least $C^{k,\alpha}$ in $B_{2r}(x)$ such that the following holds: let $(\tilde{B}_{2r}(x), \tilde{g}, \tilde{x})$ be the Riemannian universal cover of $B_{2r}(x)$, then $B_r(\tilde{x})$ is diffeomorphic to a Euclidean disc $\mathbb{D}^n$ such that for any $1 \leq i, j \leq n$,

$$
|\tilde{g}_{ij} - \delta_{ij}|_{C^0(B_r(\tilde{x}))} + \sum_{|m| \leq k} r^{|m|} \cdot |\partial^m \tilde{g}_{ij}|_{C^0(B_r(\tilde{x}))} + r^{k+\alpha}|\tilde{g}_{ij}|_{C^{k,\alpha}(B_r(\tilde{x}))} < \epsilon,
$$

where $m$ is a multi-index, and the last term is the Hölder semi-norm.

**Definition 4.2 ($C^{k,\alpha}$-regularity scale).** Let $(M^n, g)$ be a smooth Riemannian manifold. The $C^{k,\alpha}$-regularity scale $r_{k,\alpha}(x)$ at $x \in M^n$ is defined to be the supremum of all $r > 0$ such that $M^n$ is $(r, k + \alpha, 10^{-b})$-regular at $x$.

**Remark 4.3.** Note that $r_{k,\alpha}$ is 1-Lipschitz on any Riemannian manifold $(M^n, g)$, i.e.,

$$
|r_{k,\alpha}(x) - r_{k,\alpha}(y)| \leq d_g(x, y), \quad \forall x, y \in M^n.
$$
Let $\mathcal{S}_b$ be the subset of $\mathcal{M}_\lambda$ which consists of a small annular region in $K$ centered around the $\mathbb{I}^*_b$-fiber, the neck region $\mathfrak{N}$, and the ALG* manifold $X$. The following proposition gives the regularity scale estimates and bubble limits of $g_\lambda$ on $\mathcal{S}_b$.

**Proposition 4.4.** Let $s$ be a smooth function that satisfies

\[
\begin{aligned}
    s(x) &= \begin{cases}
        \lambda \cdot (\mathcal{L}_T(x))^{\frac{1}{2}} \cdot r(x), & d^Q(x, p_m) \geq 2t_0 \\
        \lambda \cdot \mathcal{Q}(x), & d^Q(x, p_m) \leq \frac{1}{4}t_0
    \end{cases}
    \end{aligned}
\]

Then the following properties hold.

1. Given $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, there exists $v_0 = v_0(k, \alpha)$ such that for any sufficiently small parameter $\lambda \ll 1$ and $x \in \mathcal{S}_b$, the $(k, \alpha)$-regularity scale $r_{k, \alpha}$ at $x$ satisfies

\[
v_0^{-1} \leq \frac{r_{k, \alpha}(x)}{s(x)} \leq v_0.
\]

2. There is a uniform constant $C_0 > 0$ such that for every $\lambda \ll 1$ and $x \in \mathcal{S}_b$, we have

\[
C_0^{-1} \leq \frac{s(y)}{s(x)} \leq C_0, \quad y \in B_{s(x)/4}(x).
\]

3. Let $\lambda_j \to 0$ be a sequence, and let $x_j \in \mathcal{S}_b$ be a sequence of reference points. Then the rescaled spaces $(\mathcal{S}_b, s(x_j)^{-1} \cdot g_\lambda, x_j)$ converges in the Gromov-Hausdorff topology to one of the following spaces as $\lambda_j \to 0$:  
   - the Taub-NUT space $(\mathbb{C}^2, g^{TN})$ and the ALG* gravitational instanton $(X, g^X)$,
   - the flat manifolds $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$, $\mathbb{R}^2$, and the flat cone $\mathbb{R}^2 / \mathbb{Z}_2$,
   - $\mathbb{P}^1$ equipped with the McLean metric $d_{\text{ML}}$ with bounded diameter.

**Proof.** We will prove (4.3) by contradiction. Suppose that there does not exist a uniform constant $v_0$ with respect to fixed constants $k \in \mathbb{Z}_+$ and $\alpha \in (0, 1)$. That is, there are a sequence $\lambda_j \to 0$ and a sequence of points $x_j \in \mathcal{S}_b$ such that

\[
\frac{r_{k, \alpha}(x_j)}{s(x_j)} \to 0 \quad \text{or} \quad \frac{r_{k, \alpha}(x_j)}{s(x_j)} \to \infty.
\]

Let us work with the rescaled sequence $(\mathcal{S}_b, \hat{g}_{\lambda_j}, x_j)$ with $\hat{g}_{\lambda_j} \equiv s(x_j)^{-1} \cdot g_{\lambda_j}$ as $\lambda_j \to 0$. In the proof, we will show that $C^{k, \alpha}$-regularity scale at $x_j$ with respect to $\hat{g}_{\lambda_j}$ is uniformly bounded from above and below as $\lambda_j \to 0$ which contradicts (4.4). We will derive a contradiction in each of the following cases depending upon the location of $x_j$. Denote $\pi_j \equiv \pi(x_j) \in \mathcal{Q}/\mathbb{Z}_2$ for any $x_j \in \mathfrak{N}$. 

**Case (I):** there exists a constant \( \sigma_0 \geq 0 \) such that \( \tilde{r}(x_j) \cdot \lambda_j^{-1} \to \sigma_0 \) as \( j \to \infty \). Let us consider the \( \sigma_0 \leq R_0 \) case first. By definition, we have \( s(x_j) = \lambda_j \cdot \lambda_j \cdot R_0 \). We consider the rescaled metric

\[
\hat{g}_{\lambda_j} = (\lambda_j \cdot \lambda_j \cdot R_0)^{-2} g_{\lambda_j},
\]

By the gluing construction, we have that, for any \( k \in \mathbb{Z}_+ \),

\[
(S_b, \hat{g}_{\lambda_j}, x_j) \xrightarrow{C^k} (X, R_0^{-2} \cdot g^X, x_\infty).
\]

Notice that the ALG* gravitational instanton \( (X, R_0^{-2} \cdot g^X) \) is a Ricci-flat but non-flat space, which implies that for any \( k \in \mathbb{Z}_+ \) and \( \alpha \in (0, 1) \) there exists a constant \( v_0 > 0 \) such that \( \frac{v_0}{r_0} \leq r_{k, \alpha}(x_\infty) \leq \frac{v_0}{2} \). Therefore, for any \( k \in \mathbb{Z}_+ \) and \( \alpha \in (0, 1) \), \( v_0^{-1} \leq r_{k, \alpha}(x_j) \leq v_0 \) holds with respect to \( \hat{g}_{\lambda_j} \), which contradicts (4.4). Therefore, the proof in the case \( \sigma_0 \leq R_0 \) is complete. The proof in the \( \sigma_0 > R_0 \) case is the same.

**Case (II_1):** there exists some constant \( \gamma_0 > 0 \) such that

\[
\lambda_j^{-1} \cdot \tilde{d}^Q(x_j, p_m) \cdot T_j^\gamma \to \gamma_0 \quad \text{as} \quad j \to \infty,
\]

where \( T_j^\gamma = \frac{2 \gamma + 1}{2 \pi} \log(\frac{1}{\lambda_j}) \). We first assume \( \gamma_0 \leq 1 \). By definition, \( s(x_j) = \hat{\lambda}_j \cdot \lambda_j \cdot (T_j^\gamma)^{-\frac{1}{2}} \). Recall that the metric \( g_{\lambda_j} \) satisfies \( g_{\lambda_j} = \hat{\lambda}_j^2 \cdot g^N \) near \( x_j \) and by (3.20),

\[
g^N = \lambda_j^2 \cdot (G_{\lambda_j} \cdot g^Q + G_{\lambda_j}^{-1} \Theta^2),
\]

where \( |G_{\lambda_j}(x) - T_j^\gamma - \frac{1}{2 \tilde{d}^Q(x, p_m)}| \leq C \) for \( \tilde{d}^Q(x, p_m) \leq r_0 \equiv \frac{1}{4} \text{InjRad}_{g^Q}(Q) \).

Let \( (u_1, u_2, u_3) \) be a fixed coordinate system in \( B_{r_0}(p_m) \) with respect to the metric \( g^Q \). Consider the rescaled metric \( \hat{g}_{\lambda_j} \equiv s(x_j)^{-2} \cdot g_{\lambda_j} \) and the rescaled coordinates centered at \( p_m = (p_{1,m}, p_{2,m}, p_{3,m}) \in \hat{P} \),

\[
(u_1, u_2, u_3) \equiv T_j^\gamma (u_1 - p_{1,m}, u_2 - p_{2,m}, u_3 - p_{3,m}).
\]

Then by the explicit computations, \( (S_b, \hat{g}_{\lambda_j}, x_j) \) \( C^k \)-converges for any \( k \in \mathbb{Z}_+ \), to the Taub-NUT space \( (\mathbb{C}^2, g^{TN}, x_\infty) \), where the Taub-NUT metric \( g^{TN} \) can be written explicitly in terms of the Gibbons-Hawking ansatz

\[
g^{TN} = V_0 g^{\mathbb{R}^3} + V_0^{-1} \Theta^2, \quad V_0 = 1 + (2r)^{-1},
\]

and \( r \) is the Euclidean distance to the origin of \( \mathbb{R}^3 \). Therefore, there exists a constant \( v_0 \) such that \( v_0^{-1} \leq r_{k, \alpha}(x_j) \leq v_0 \) with respect to the rescaled metric \( \hat{g}_{\lambda_j} \). Rescaling back to \( g_{\lambda_j} \), we find that the above estimate contradicts (4.4). This completes the proof under the assumption \( \gamma_0 \leq 1 \). The proof in the case \( \gamma_0 > 1 \) is the same.

**Case (II_2):** for some \( p_m \in P \), the points \( x_j \) satisfy

\[
\lambda_j^{-1} \cdot \tilde{d}^Q(x_j, p_m) \cdot T_j^\gamma \to \infty \quad \text{and} \quad \lambda_j^{-1} \cdot \tilde{d}^Q(x_j, p_m) \to 0
\]
as \( j \to \infty \). In this case, by definition
\[
(4.11) \quad s(x_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot (T^b_j)^{\frac{1}{2}} \cdot d_j,
\]
where \( d_j \equiv d^Q(\mathbf{x}_j, p_m) \). We will work with \( \hat{g}_{\lambda_j} \equiv s(x_j)^{-2} \cdot g_{\lambda_j} \) and the rescaled coordinates centered at \( p_m = (p_{1, m}, p_{2, m}, p_{3, m}) \in P \),
\[
(\hat{u}_1, \hat{u}_2, \hat{u}_3) \equiv d^{-1}_j \cdot (u_1 - p_{1, m}, u_2 - p_{2, m}, u_3 - p_{3, m}),
\]
where \((u_1, u_2, u_3)\) is a fixed coordinate system in \( B_{r_0}(p_m) \). One can verify
\[
(4.12) \quad (S_b, \hat{g}_{\lambda_j}, x_j) \xrightarrow{GH} (\mathbb{R}^3, g, \mathbb{R}^3, x_\infty)
\]
with \( d^{\mathbb{R}^3}(x_\infty, 0^3) = 1 \). The detailed and explicit rescaling computations can be found in [HSVZ22] and [CVZ20]. Moreover, if we lift the metric to the local universal cover around \( x_j \), then a ball of definite size radius has uniformly bounded \( C^{k, \alpha} \)-geometry. This implies \( r_{k, \alpha}(x_j) \geq v_0 > 0 \). The upper bound for \( r_{k, \alpha}(x_j) \) follows from (4.11) and the calculation in Case (\( \Pi_1 \)).

We get a contradiction with (4.11) as rescaling back to \( g_{\lambda_j} \).

Notice that \( \mathbb{R}^3 \) is precisely the asymptotic cone of the Taub-NUT space.

**Case (\( \Pi_3 \)):** there is some constant \( d_0 \) such that for some \( p_m \in P \),
\[
(4.13) \quad \lambda_j^{-1} \cdot d^Q(\mathbf{x}_j, p_m) \to d_0 \quad \text{as} \quad j \to \infty.
\]
By the definition of \( s \), we have that
\[
(4.14) \quad s(x_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot (T^b_j)^{\frac{1}{2}} \cdot d_0 \cdot (1 + o(1)).
\]
It suffices to work with the rescaled metric \((\tilde{\lambda}_j \cdot \lambda_j \cdot (T^b_j)^{\frac{1}{2}} \cdot d_0)^{-2} \cdot g_{\lambda_j} \), still denoted by \( \hat{g}_{\lambda_j} \), and prove that the regularity scale \( r_{k, \alpha}(x_j) \) is uniform bounded from above and below. Then the contradiction arises.

Straightforward computations imply that \( \hat{g}_{\infty} = d_0^{-2} \cdot g^Q \), which is a rescaling of the flat base metric \( g^Q \) on \( \mathbb{R}^2 \times S^1 \). Since \( d^Q(P, 0^3) \geq \frac{1}{d_0} \cdot \lambda_j^{-1} \to \infty \), it follows that the origin \( 0^2 \in \mathbb{R}^2 \) translates to infinity and the \( \mathbb{Z}_2 \)-action limits to the identity. Therefore, the rescaled limit is isometric to \( \mathbb{R}^2 \times S^1 \). The collapsing keeps curvature uniformly bounded away from \( P \). Then there is some uniform constant \( v_0 > 0 \) such that \( r_{k, \alpha}(x_j) \geq v_0 > 0 \). The upper bound for \( r_{k, \alpha}(x_j) \) follows from (4.11) and the calculation in Case (\( \Pi_2 \)).

Notice that, \( \mathbb{R}^2 \times S^1 \) is the flat base of the metric \( g^N \).

**Case (\( \Pi_4 \)):** for some \( p_m \in P \), we have
\[
(4.15) \quad \lambda_j^{-1} \cdot d^Q(\mathbf{x}_j, p_m) \to \infty \quad \text{and} \quad d^Q(\mathbf{x}_j, p_m) \to 0 \quad \text{as} \quad j \to \infty.
\]
Let us denote \( d_j \equiv \lambda_j^{-1} \cdot d^Q(\mathbf{x}_j, p_m) \). In this case, the definition of \( s \) implies
\[
(4.16) \quad s(x_j) = \tilde{\lambda}_j \cdot \lambda_j \cdot (T^b_j + \frac{1}{2\pi} \log \frac{1}{d_j})^{\frac{1}{2}} \cdot d_j.
\]
We will prove that in terms of the rescaled metric \( \hat{g}_{\lambda_j} \equiv s(x_j)^{-2}g_{\lambda_j} \), the regularity scale \( r_{k,\alpha}(x_j) \) has a uniform lower bound and upper bound, which contradicts (4.1).

The flat product metric \( g^0 \) can be written as \( g^0 = dx^2 + dy^2 + d\theta^2 \) in coordinates. We also rescale the above coordinate system of \( \mathbb{R}^2 \) centered around \( x_j = (x_j, y_j) \) by letting

\[
(\hat{x}, \hat{y}) \equiv d_j^{-1} \cdot (x - x_j, y - y_j).
\]

Explicit tensorial computations show that

\[
(4.17) \quad (S_b, \hat{g}_{\lambda_j}, x_j) \xrightarrow{GH} (\mathbb{R}^2, g^{\mathbb{R}^2}, 0^2),
\]

where the Euclidean metric \( g^{\mathbb{R}^2} \) has the expression \( g^{\mathbb{R}^2} = dx^2 + dy^2 \). By assumption, \( d^2(x_j, 0^*)/d_j \to \infty \), which implies that the origin \( 0^2 \in \mathbb{R}^2 \) translates to infinity and hence the \( \mathbb{Z}_2 \)-action limits to the identity as \( j \to \infty \).

The finite set \( P \) converges to a single point \( p_0 \in \mathbb{R}^2 \) and \( d^2(p_0, 0^2) = 1 \). The above collapsing keeps curvature uniformly bounded away from the point \( p_0 \). So there is some uniform constant \( v_0 > 0 \) such that \( r_{k,\alpha}(x_j) \geq v_0 > 0 \). The upper bound for \( r_{k,\alpha}(x_j) \) follows from (4.14) and the calculation in Case (III).

**Case (III):** there exists some constant \( d_0 \) such that

\[
(4.19) \quad \hat{d}^2(x_j, P) \geq d_0 > 0, \quad \hat{r}(x_j) \cdot \lambda_j^{-1} \to \infty, \quad L_j \equiv \Sigma T_j(x_j) \to +\infty.
\]

There are the following subcases to analyze.

First, assume that \( \hat{r}(x_j) \to 0 \) as \( j \to \infty \). In this case, by definition,

\[
(4.20) \quad s(x_j) = \tilde{\lambda}_j \cdot \tilde{L}_j \cdot \hat{r}_j.
\]

We will prove that under the rescaling \( \hat{g}_{\lambda_j} = (s(x_j))^{-2} \cdot g_{\lambda_j} \) and \( (\hat{x}, \hat{y}) = \hat{r}_j^{-1} \cdot (\hat{x}, \hat{y}) \), the convergence

\[
(4.21) \quad (S_b, \hat{g}_{\lambda_j}, x_j) \xrightarrow{GH} (\mathbb{R}^2/\mathbb{Z}_2, d^{\mathbb{R}^2/\mathbb{Z}_2}(U_{\xi_j}))
\]

holds, where \( d^{\mathbb{R}^2/\mathbb{Z}_2}(U_{\xi_j}), 0^2) = 1 \), and the flat metric on \( \mathbb{R}^2/\mathbb{Z}_2 \) can be written in terms of the limit coordinate system of \( (\hat{x}, \hat{y}) \). Notice that \( \mathbb{R}^2/\mathbb{Z}_2 \) is the asymptotic cone of the ALG space \( (X, g^X, \omega^X) \). Moreover, we will show that the rescaled metrics \( \hat{g}_{\lambda_j} \) has uniformly bounded curvature away from the cone tip. This suffices to produce the desired contradiction because the upper bound on \( r_{k,\alpha}(x_j) \) follows from (4.14) and the calculation in Case (III).

To prove the above claim, let us choose a domain

\[
(4.22) \quad U_{\xi_j} \equiv \{ x \in \mathcal{U} \subset S_b \subset M_{\lambda_j} | \xi_j^{-1} \leq \hat{r}(x) \leq \xi_j \}
\]
for a sequence $\xi_j$ that satisfies $\lim_{j \to \infty} \frac{\xi_j}{L_j} = 0$. Then for any $x_j \in U_{\xi_j}$,

\begin{equation}
\frac{\tilde{L}_j}{L_j} = 1 + o(1) \quad \text{as} \quad j \to \infty.
\end{equation}

By explicit tensorial computations on the Gibbons-Hawking metric $g^N$, we can check that the $\mathbb{Z}_2$-covering of $(U_{\xi_j}, \hat{g}_{\lambda_j})$ will smoothly converge to the flat metric $dx^2 + dy^2$ on $\mathbb{R}^2$, where $(\hat{x}_\infty, \hat{y}_\infty)$ is the limit of $(\hat{x}, \hat{y})$. Also notice that the limiting reference point $x_\infty$ satisfies $d_{\mathbb{R}^2}(x_\infty, 0) = 1$. Then the $\mathbb{Z}_2$-quotient metric $\hat{g}_{\lambda_j}$ converges to the flat metric on $\mathbb{R}^2/\mathbb{Z}_2$.

Next, we consider the case $\tilde{r}_j(x_j) \to d_0' > 0$. By definition,

\begin{equation}
s(x_j) = \tilde{\lambda}_j \cdot T_j^{\frac{1}{2}} \cdot d_0' \cdot (1 + o(1))
\end{equation}

as $j \to \infty$. It suffices to work with the rescaled metric $(\tilde{\lambda}_j \cdot T_j^{\frac{1}{2}} \cdot d_0')^{-2} \cdot g_{\lambda_j}$, still denoted by $\hat{g}_{\lambda_j}$, and we can show that the regularity scale $r_{\kappa,0}(x_j)$ has a uniform lower bound. Moreover, the rescaled limit in this case is isometric to $\mathbb{R}^2/\mathbb{Z}_2$ as well. We skip the detailed computations since the arguments are the same.

In the last case, we will consider the case when $x_j$ satisfies $\tilde{r}(x_j) \to \infty$ and $L_j \to \infty$. In the proof, we still use the rescaled metric $\hat{g}_{\lambda_j} = (s(x_j))^{-2} \cdot g_{\lambda_j}$ and the rescaled coordinates $(\hat{x}, \hat{y}) = \tilde{r}_j^{-1} \cdot (x, y)$. The computations are the same. We only mention that, as $\tilde{r}_j = \tilde{r}(x_j)$ becomes very large, one can obtain the rescaled limit $\mathbb{R}^2/\mathbb{Z}_2$ as long as $L_j \to \infty$.

When $\tilde{r}_j$ is sufficiently large such that $L_j \to L_0 > 0$ as $j \to \infty$, we will obtain another rescaled limit. This becomes Case (IV).

**Case (IV):** there is some constant $L_0 > 0$ such that $L_j \to L_0 > 0$. In this case,

\begin{equation}
s(x_j) \equiv \tilde{\lambda}_j \cdot L_j^{\frac{1}{2}} \cdot \tilde{r}_j = \tilde{\lambda}_j \cdot \tilde{r}_j \cdot L_0 \cdot (1 + o(1))
\end{equation}

as $j \to \infty$. In the meantime, notice that

\begin{equation}
L_j = (T_j + \text{Im} h(0) - \frac{b}{\pi} \log \tilde{r}_j) \cdot (1 + o(1))
\end{equation}

as $j \to \infty$. It is easy to verify that $s(x_j)$ is a bounded constant. Then the rescaled limit is the McLean metric on $\mathbb{P}^1$. Moreover, the convergence keeps curvature uniformly bounded away from the singular fiber.

The above covers all the points on $S_b$ which completes the proof. □

5. Perturbation to hyperkähler metrics

In this subsection, we will glue an ALG* gravitational instanton into a region near an $I^*_b$-fiber of an elliptic K3 surface $\pi_K : K \to \mathbb{P}^1$. For our purpose, it suffices to assume that the singular fibers of $\pi_K$ consists of an $I^*_b$-fiber for some $1 \leq b \leq 14$ and $I_1$ fibers of number $(18 - b)$. Following
the notations in Section 6 of [CVZ20], we denote by \( S_b \) the subset of \( M_\lambda \) which consists of a small annular region in \( K \) centered around the \( I_1^* \)-fiber, the neck region \( \mathcal{N} \), and the ALG\(^*\) manifold \( X \). We denote by \( S_1 \), the subset of \( M_\lambda \) which consists of small annular regions in \( K \) centered around I\(_1\)fibers and Ooguri-Vafa manifolds. Let \( R_\lambda \) be the regular region in \( K \). We will prove that the glued manifold \( M_\lambda \) admits collapsing hyperkähler metrics with prescribed behaviors. In the following weighted analysis, the weight function \( \rho \) as a global smooth function on \( M_\lambda \) is defined as follows,

\[
\rho(x) = \begin{cases} 
\rho(x), & x \in S_b, \\
\rho_1(x), & x \in S_1, \\
1, & x \in R_\lambda,
\end{cases}
\]

where \( \rho_1 \) is the canonical scale function defined in Section 6.3 of [CVZ20].

With respect to the weight function \( \rho \), we will define the following weighted Hölder norms.

**Definition 5.1.** For any fixed parameter \( \lambda \ll 1 \), let \( g_\lambda \) be the approximately hyperkähler metric defined on the glued manifold \( M_\lambda \). Let \( U \subset M_\lambda \) be a compact subset. Then the weighted Hölder norm of a tensor field \( \chi \in T^{r,s}(U) \) of type \((r,s)\) is defined as follows:

1. The weighted \( C^{k,\alpha} \)-seminorm of \( \chi \) is defined by

\[
[\chi]_{C^{k,\alpha}_\mu}(x) \equiv \sup \left\{ \rho^{k+\alpha-\mu}(x) \cdot \left\| \frac{\nabla^k \hat{\chi}(\hat{x}) - \nabla^k \hat{\chi}(\hat{y})}{(d_{g_\lambda}(\hat{x}, \hat{y}))^\alpha} \right\| \left| \hat{y} \in B_{r_{k,\alpha}}(x)(\hat{x}) \right\}ight.
\]

\[
[x]_{C^{k,\alpha}_\mu}(U) \equiv \sup \left\{ [\chi]_{C^{k,\alpha}_\mu}(x) \left| x \in U \right. \right\}
\]

where \( r_{k,\alpha}(x) \) is the \( C^{k,\alpha} \)-regularity scale at \( x \). \( \hat{x} \) denotes a lift of \( x \) to the universal cover of \( B_{2r_{k,\alpha}}(x)(\hat{x}) \), the difference of the two covariant derivatives is defined in terms of parallel translation in \( B_{r_{k,\alpha}}(x)(\hat{x}) \), and \( \hat{\chi}, \hat{g}_\lambda \) are the lifts of \( \chi, g_\lambda \) respectively.

2. The weighted \( C^{k,\alpha} \)-norm of \( \chi \) is defined by

\[
\|\chi\|_{C^{k,\alpha}_\mu}(U) \equiv \sum_{m=0}^{m=k} \left\| \rho^{k+\alpha-\mu} \cdot \nabla^m \chi \right\|_{C^0(U)} + [\chi]_{C^{k,\alpha}_\mu}(U).
\]

Now let us briefly describe the perturbation scheme to produced hyperkähler triples from the approximate triples constructed in Section 3. This original characterization is due to Donaldson [Don06], which has also been used in [CC21b, CVZ20, Fos20, FLS17, HSVZ22]. Let \( M^4 \) be an oriented 4-manifold with a volume form \( \text{dvol}_0 \). A triple of closed 2-forms \( \omega = (\omega_1, \omega_2, \omega_3) \) is said to be **definite** if the matrix \( Q = (Q_{ij}) \) defined by \( \frac{1}{2} \omega_i \wedge \omega_j = Q_{ij} \text{dvol}_0 \) is positive definite. A definite triple \( \omega \) is called a **hyperkähler triple** if \( Q_{ij} = \delta_{ij} \). Given a definite triple \( \omega \), the associated volume form is defined as \( \text{dvol}_\omega \equiv (\det(Q))^\frac{1}{8} \text{dvol}_0 \), and we denote
by \( Q_\omega \equiv (\det(Q))^{-\frac{1}{3}}Q \) the normalized matrix with unit determinant. Every definite triple \( \omega \) determines a Riemannian metric \( g_\omega \) such that each \( \omega_j \), \( j \in \{1,2,3\} \), is self-dual with respect to \( g_\omega \) and \( d\text{vol}_{g_\omega} = d\text{vol}_\omega \).

Suppose that we have a closed definite triple \( \omega \) on \( \mathcal{M}_\lambda \). We want to find a triple of closed 2-forms \( \theta = (\theta_1, \theta_2, \theta_3) \) such that \( \omega \equiv \omega + \theta \) is an actual hyperkähler triple on \( \mathcal{M}_\lambda \) satisfying

\[
\frac{1}{2}(\omega_i + \theta_i) \wedge (\omega_j + \theta_j) = \delta_{ij} d\text{vol}_{\omega + \theta},
\]

which is equivalent to

\[
\frac{1}{2}(\omega_i \wedge \omega_j + \omega_i \wedge \theta_j + \omega_j \wedge \theta_i + \theta_i \wedge \theta_j) = \frac{1}{6} \delta_{ij} \sum_{k=1}^{3} (\omega_k^2 + \theta_k^2 + 2\omega_k \wedge \theta_k).
\]

Writing \( \theta = \theta^+ + \theta^- \) with \( *_{g_\omega} \theta^\pm = \pm \theta^\pm \), we define the matrices \( A = (A_{ij}) \) and \( S_{\theta^-} = (S_{ij}) \) by

\[
\theta^+_i = \sum_{j=1}^{3} A_{ij} \omega_j, \quad \frac{1}{2} \theta^-_i \wedge \theta^-_j = S_{ij} d\text{vol}_\omega, \quad 1 \leq i \leq j \leq 3.
\]

Then (5.3) is equivalent to

\[
\text{tf}(Q_\omega A^T + Q_\omega A + AQ_\omega A^T) = \text{tf}(-Q_\omega - S_{\theta^-}),
\]

where \( \text{tf}(B) \equiv B - \frac{1}{3} \text{Tr}(B) \text{Id} \) for a \( 3 \times 3 \) real matrix \( B \), and \( Q_\omega \) is the \( 3 \times 3 \) real matrix such that \( \det(Q_\omega) = 1 \) and

\[
\frac{1}{2} \omega_i \wedge \omega_j = (Q_\omega)_{ij} d\text{vol}_\omega.
\]

Then observe that a solution of

\[
d^+ \eta + \xi = \mathfrak{f}_0 \left( \text{tf}(-Q_\omega - S_{d^-} \eta) \right),
\]

\[
d^* \eta = 0, \quad \eta \in \Omega^1(\mathcal{M}_\lambda) \otimes \mathbb{R}^3, \quad \xi \in \mathcal{H}^+_{d^-}((\mathcal{M}_\lambda) \otimes \mathbb{R}^3),
\]

is also a solution of (5.5). Here \( \mathfrak{f}_0 \) denotes the local inverse near zero of

\[
\mathfrak{G}_0 : \mathcal{J}_0(\mathbb{R}^3) \rightarrow \mathcal{J}_0(\mathbb{R}^3), \quad A \mapsto \text{tf}(Q_\omega A^T + AQ_\omega + AQ_\omega A^T),
\]

on the space of trace-free symmetric \( 3 \times 3 \)-matrices \( \mathcal{J}_0(\mathbb{R}^3) \), and \( d^\pm \eta \) is the self-dual or anti-self-dual part of \( d\eta = \theta - \xi \), respectively. The linearization of the elliptic system (5.7) at \( \eta = 0 \) is given by \( \mathcal{L} = (\mathfrak{D} \oplus \text{Id}) \otimes \mathbb{R}^3 : (\Omega^1(\mathcal{M}_\lambda) \oplus \mathcal{H}^+_{d^-}((\mathcal{M}_\lambda) \otimes \mathbb{R}^3) \otimes \mathbb{R}^3 \rightarrow (\Omega^0(\mathcal{M}_\lambda) \oplus \Omega^2(\mathcal{M}_\lambda)) \otimes \mathbb{R}^3, \) where

\[
\mathfrak{D} \equiv d^* + d^+: \Omega^1(\mathcal{M}_\lambda) \rightarrow (\Omega^0(\mathcal{M}_\lambda) \oplus \Omega^2(\mathcal{M}_\lambda)).
\]

For any sufficiently small \( \lambda \ll 1 \), we will solve the elliptic system (5.7). The proof of the existence of hyperkähler triples requires the following version of implicit function theorem.
Lemma 5.2. Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be a map between two Banach spaces with
\begin{equation}
\mathcal{F}(x) = \mathcal{F}(0) + \mathcal{L}(x) + \mathcal{N}(x),
\end{equation}
where the operator $\mathcal{L}: \mathcal{A} \to \mathcal{B}$ is linear and $\mathcal{N}(0) = 0$. Assume that
\begin{enumerate}
  \item $\mathcal{L}$ is an isomorphism with $\|\mathcal{L}^{-1}\| \leq C_L$ for some $C_L > 0$.
  \item there are constants $r > 0$ and $C_N > 0$ such that:
    \begin{enumerate}
      \item $r < (10C_L \cdot C_N)^{-1}$,
      \item $\|\mathcal{N}(x) - \mathcal{N}(y)\|_{\mathcal{B}} \leq C_N \cdot (\|x\|_{\mathcal{A}} + \|y\|_{\mathcal{A}}) \cdot \|x - y\|_{\mathcal{A}}$ for all $x, y \in B_r(0) \subset \mathcal{A}$,
      \item $\|\mathcal{F}(0)\|_{\mathcal{B}} \leq \frac{r}{10C_L}$.
    \end{enumerate}
\end{enumerate}
Then $\mathcal{F}(x) = 0$ has a unique solution $x \in \mathcal{A}$ such that $\|x\|_{\mathcal{A}} \leq 2C_L\|\mathcal{F}(0)\|_{\mathcal{B}}$.

To apply the implicit function theorem, first we fix two Banach spaces
$\mathcal{A} \equiv \left( C^{1,\alpha}_{\mu} (\check{\Omega}^1(M_\lambda)) \oplus \mathcal{H}^+(M_\lambda) \right) \oplus \mathbb{R}^3$, $\mathcal{B} \equiv \left( C^{0,\alpha}_{\mu-1} (\check{\Omega}^2_+ (M_\lambda)) \right) \oplus \mathbb{R}^3$,
where $\mu \in (-1, 0)$, $\alpha \in (0, 1)$, and $\check{\Omega}^1(M_\lambda) \equiv \{ \eta \in \Omega^1(M_\lambda) \mid d^*\eta = 0 \}$. The following error estimate is an immediately corollary of Lemma 5.5

Corollary 5.3. There exists $C_0 > 0$ independent of the parameters $\lambda$ and $t$ such that
\begin{equation}
\|\mathcal{F}(0)\|_{\mathcal{B}} \leq C_0 \cdot (\lambda^2 + \bar{\lambda} \cdot t^3) \cdot \bar{\lambda}^{-\mu+1} \cdot t^{-\mu-1} \cdot V(\sigma^{-1})^{\frac{-\mu-1}{2}} + C_0 \cdot \lambda^2 \cdot \bar{\lambda}^2.
\end{equation}

Proof. Let $\mathcal{M}_\nu \equiv \{ x \in \mathcal{M} \mid r \leq \tilde{r}(x) \leq 2r \}$. Then by Lemma 5.5
\begin{align*}
\|Q_\omega - \text{Id}\|_{C^{0,\alpha}_{\mu-1}(\mathcal{M}_\nu)} &\leq C_0 \cdot (\lambda^2 + \bar{\lambda} \cdot t^3) \cdot \bar{\lambda}^{-\mu+1} \cdot t^{-\mu-1} \cdot V(\sigma^{-1})^{\frac{-\mu-1}{2}}, \\
\|Q_\omega - \text{Id}\|_{C^{0,\alpha}_{\mu-1}(\mathcal{M}_\nu)} &\leq C_0 \cdot \lambda^2 \cdot \bar{\lambda}^2.
\end{align*}

On the other hand, the error estimate near an $I_1$-fiber is much smaller. In fact, by Theorem 4.4 of [GW00] (see also [CVZ20], Proposition 8.2),
\begin{equation}
\|Q_\omega - \text{Id}\|_{C^{0,\alpha}_{\mu-1}(T_{\delta_0} S_{I_1})} \leq C_1 \cdot e^{-C_2 \bar{\lambda}^{-1}}
\end{equation}
for some constants $C_1 > 0$, $C_2 > 0$ independent of $\bar{\lambda}$ (and hence $\lambda$), where $T_{\delta_0} S_{I_1}$ is an annular neighborhood of definite size $\delta_0 > 0$ independent of $\lambda$.

We also need the weighted estimate on the nonlinear errors.

Lemma 5.4 (Nonlinear estimate). There exists some constant $K_0 > 0$ independent of $\lambda$ such that for any $v_1, v_2 \in B_1(0) \subset \mathcal{A}$, we have that
\begin{equation}
\|\mathcal{M}_\lambda(v_1) - \mathcal{M}_\lambda(v_2)\|_{\mathcal{B}} \leq K_0 \cdot (\bar{\lambda} \cdot \lambda)^{\mu-1} (T^\beta)^{\frac{\mu}{2}} (\|v_1\|_{\mathcal{A}} + \|v_2\|_{\mathcal{A}}) \cdot \|v_1 - v_2\|_{\mathcal{A}}.
\end{equation}

Proof. For any $v_1, v_2 \in B_1(0) \subset \mathcal{A}$, by explicit computations,
\begin{equation}
|\mathcal{M}_\lambda(v_1) - \mathcal{M}_\lambda(v_2)| \leq K_0 \cdot (|d^- \eta_1| + |d^- \eta_2|) \cdot |d^- (\eta_1 - \eta_2)|,
\end{equation}
where \( K_0 > 0 \) is independent of \( \lambda \). Multiplying by the weight function \( s(x)^{-\mu+1} \), we have that
\[
s(x)^{-\mu+1} \cdot |\mathcal{M}_\lambda(v_1) - \mathcal{M}_\lambda(v_2)| \leq K_0 \cdot s(x)^{-\mu+1} \cdot (|d^\nu \eta_1| + |d^\nu \eta_2|) \cdot |d^\nu (\eta_1 - \eta_2)|.
\]
By definition, the scale function \( s(x) \) achieves the minimum \( \tilde{\lambda} \cdot \lambda \cdot (T^b)^{-\frac{\mu}{2}} \) when \( d^Q(x, p_m) \leq (T^b)^{-1} \) for some \( 1 \leq m \leq 2\nu + 2b \). Then we have that
\[
|\mathcal{M}_\lambda(v_1) - \mathcal{M}_\lambda(v_2)| \leq K_0 \cdot (\tilde{\lambda} \cdot \lambda)^{-1} (T^b)^{-\frac{\mu}{2}} \left( \|v_1\|_{C_{\mu}^0(\mathcal{M}_\lambda)} + \|v_2\|_{C_{\mu}^0(\mathcal{M}_\lambda)} \right) \cdot \left( \|v_1 - v_2\|_{C_{\mu}^1(\mathcal{M}_\lambda)} \right).
\]
By similar computations, we also have the desired estimate for the \( C^{0,\alpha} \)-seminorm. This completes the proof. \( \square \)

The following is the main ingredient to carry out the perturbation.

**Proposition 5.5** (Weighted linear estimate). Let \( \mathcal{M}_\lambda \) be the glued manifold with a family of approximately hyperkähler metrics \( g_\lambda \). Then there exists \( C > 0 \), independent of \( \lambda \), such that for every self-dual 2-form \( \xi^+ \in \mathfrak{A} \), there exists a unique pair \( (\eta, \tilde{\xi}^+) \in \mathfrak{A} \) such that for some \( \mu \in (-1, 0) \) and \( \alpha \in (0, 1) \),
\[
(5.10) \quad \mathcal{L}_\lambda(\eta, \tilde{\xi}^+) = \xi^+,
\]
\[
(5.11) \quad \|\eta\|_{C_{\mu}^{0,\alpha}(\mathcal{M}_\lambda)} + \|\tilde{\xi}^+\|_{C_{\mu+1}^{0,\alpha}(\mathcal{M}_\lambda)} \leq C \|\xi^+\|_{C_{\mu+1}^{0,\alpha}(\mathcal{M}_\lambda)}.
\]

The proof is very similar to the proof of Proposition 8.7 in [CVZ20] which follows from a contradiction argument and applying various Liouville theorems on the blow-up limits. We omit the details and only mention the outline.

(1) If the blow-up limit is an ALF or ALG* gravitational instanton \((X, g, p)\) with \( p \in X \), the Liouville theorem invoked in the proof is that, any 1-form \( \omega \) that satisfies \( \Delta_H \omega = 0 \) and \( \lim_{d_\rho(x, p) \to \infty} |\omega(x)| \to 0 \) has to be vanishing everywhere, i.e., \( \omega \equiv 0 \) on \( X \).

(2) If the blow-up limit is a flat space in Proposition 4.4 namely \( \mathbb{R}^3, \mathbb{R}^2 \times S^1, \mathbb{R}^2, \mathbb{R}^2/\mathbb{Z}_2 \), then we will quote the following Liouville theorem: any harmonic function \( f \) that satisfies \( |f| \leq C \cdot r^\mu \) for any \( r \in (0, \infty) \) has to be identically zero, where \( r \) is the Euclidean distance to a fixed point.

(3) The Liouville theorem corresponding to \( (\mathbb{P}^1, d_{ML}) \) is Proposition 7.8 in [CVZ20].

Combining the above results, now we prove the perturbation theorem.

**Theorem 5.6.** Let \((X, g^X, \omega^X)\) be an order 2 ALG* gravitational instanton for some \( \nu \in \{1, 2, 3, 4\} \). Then for any integer \( 1 \leq b \leq 14 \) and for any sufficiently small parameter \( \lambda \ll 1 \), there exists a family of hyperkähler structures
(M_\lambda, h_\lambda,\omega_{h_\lambda}) on the K3 surface M_\lambda such that the following properties hold as \lambda \to 0.

(1) We have Gromov-Hausdorff convergence (M_\lambda, h_\lambda) \overset{GH}{\to} (\mathbb{P}^1, d_{ML}), where d_{ML} is the McLean metric on \mathbb{P}^1 with a finite set \mathcal{S} \equiv \{q_0, q_1, \ldots, q_{18-b}\} \subset \mathbb{P}^1. Moreover, the curvatures of h_\lambda are uniformly bounded away from \mathcal{S}, but are unbounded around \mathcal{S}.

(2) The hyperkähler structures (M_\lambda, h_\lambda,\omega_{h_\lambda}) satisfy the uniform error estimate for some positive number 0 < \epsilon \ll \min\{1, \frac{\epsilon}{b}\},

\begin{align}
&\|\omega_{h_\lambda} - \omega_0\|_{C^{0,\alpha}(M_\lambda)} \leq C \cdot (\lambda^{2-\epsilon} + \lambda^{\frac{\mu}{b}-\epsilon}), \\
&\|\omega_{h_\lambda} - \omega_{h_\lambda}\|_{C^{0,\alpha}(M_\lambda)} \leq C \cdot (\lambda^{2-\epsilon} + \lambda^{\frac{\mu}{b}-\epsilon}),
\end{align}

where g_\lambda is the metric determined by the definite triple \omega_\lambda, and C^{0,\alpha} norm is the weighted norm in Definition 5.1 when k = 0 and \mu = 0.

(3) Rescalings of (M_\lambda, h_\lambda,\omega_{h_\lambda}) around q_1 for 1 \leq i \leq 18 - b converge to a complete Taub-NUT gravitational instanton on \mathbb{C}^2.

(4) Rescalings of (M_\lambda, h_\lambda,\omega_{h_\lambda}) around q_0 converge to the given ALG^* gravitational instanton (X, g^X, \omega^X) or one of (\nu + b) copies of complete Taub-NUT gravitational instantons.

Proof of Theorem 5.6. We will apply Lemma 5.2 to perform the perturbation. Let

\begin{align}
&\mathcal{C}_{\text{err}} \equiv C_0 \cdot (\lambda^2 + \tilde{\lambda} \cdot t^3) \cdot \tilde{\lambda}^{-\mu+1} \cdot t^{-\mu-1} \cdot V(\sigma^{-1})^{-\frac{\mu-3}{2}} + C_0 \cdot \lambda^2 \cdot \tilde{\lambda}^2, \\
&\mathcal{C}_N \equiv (\tilde{\lambda} \cdot \lambda)^{\mu-1} \cdot (t^b)^{\frac{1-\mu}{2}},
\end{align}

be the constants in Corollary 5.3 and Lemma 5.4. Recall that \lambda and t are chosen such that \sigma = \frac{\lambda}{t} \to 0. To prove (5.13), we only need to fix the parameter t \equiv \lambda^\frac{b}{2} for a fixed \epsilon \ll 1 and let \mu = -1 + \frac{\epsilon}{10}. Then it is obvious that \mathcal{C}_{\text{err}} \cdot \mathcal{C}_N \to 0 as \lambda \to 0. The uniform linear estimate is given by Proposition 5.3. Then Lemma 5.2 implies that there exists a solution which satisfies the desired estimate. Moreover, (5.13) follows from (5.15). The classification of the intermediate bubbles is given by Proposition 4.4 and noticing that the solutions h_\lambda are sufficiently close to g_\lambda.

\[ \square \]

6. Proofs of Torelli uniqueness theorems

In this section, we complete the proofs of Theorem 1.10 and Theorem 1.15. We also explain the reason for the order 2 assumption in Theorem 1.15.

6.1. Proof of Theorem 1.10: ALG^* Torelli uniqueness. Let (X_\nu, g, \omega) and (X_\nu, g', \omega') be ALG^* gravitational instantons on X_\nu with the same parameters \kappa_0, L, which are both order 2 with respect to the coordinates \Phi_\nu, and which satisfy (1.10). Let \pi_K : K \to \mathbb{P}^1 be any elliptic K3 surface with a single fiber of type I_0^*, call it D^*, but has all other singular fibers of type I_1. Let U = \{x \in M_\lambda | \overline{r}(x) \geq t\} and V = \{x \in M_\lambda | \overline{r}(x) \leq 2t\}. Then
$\mathcal{M}_\lambda = U \cup V$. The gluing procedure in Section 3.3 produces approximate hyperkähler triples $\tilde{\omega}_\lambda$ and $\tilde{\omega}'_\lambda$ on $\mathcal{M}_\lambda$. Note that $U \cap V$ deformation retracts onto the 3-manifold $I^3$.

**Lemma 6.1.** The manifold $I^3_\nu = \text{Nil}^3_{2\nu}/\mathbb{Z}_2$ is an infranilmanifold, which is a circle bundle of degree $\nu$ over a Klein bottle. Furthermore, we have $b^1(I^3_\nu) = 1$, with $H^1_{\text{dR}}(I^3_\nu)$ generated by the 1-form $d\theta_1$.

**Proof.** The first statement follows since Nil$^3_{2\nu}$ is a circle bundle over a torus, and the quotient space is then clearly a circle bundle over a Klein bottle. From [HSVZ22, Proposition 2.3], we have $b^1(\text{Nil}^3_{2\nu}) = 2$, with $H^1_{\text{dR}}(\text{Nil}^3_{2\nu})$ generated by $d\theta_1$ and $d\theta_2$. These forms are harmonic with respect to any left-invariant and $\mathbb{Z}_2$-invariant metric on Nil$^3_{2\nu}$. Of these generators, only $d\theta_1$ is invariant under this action, so the lemma follows from the Hodge Theorem. □

The Mayer-Vietoris sequence in de Rham cohomology for \{U, V\} is

$$
0 \rightarrow H^1_{\text{dR}}(U) \oplus H^1_{\text{dR}}(V) \rightarrow H^1_{\text{dR}}(I^3) \rightarrow
$$

\begin{align*}
H^2_{\text{dR}}(\mathcal{M}_\lambda) \rightarrow H^2_{\text{dR}}(U) \oplus H^2_{\text{dR}}(V) \rightarrow H^2_{\text{dR}}(I^3_\nu) \rightarrow 0.
\end{align*}

From the gluing in Section 4, we have

$$(\Phi^{-1})^*(\lambda^2 \cdot \omega) - \omega^H = d\eta, \quad (\Phi^{-1})^*(\lambda^2 \cdot \omega') - \omega^H = d\eta'.$$

where $\eta$ and $\eta'$ are triples of 1-forms on $\{x \in \mathfrak{g} | t \leq \tilde{r}(x) \leq 2t\}$. From Lemma 3.5 on the region $\{x \in \mathfrak{g} | t \leq \tilde{r}(x) \leq 2t\}$, the approximate hyperkähler triples are

$$(\tilde{\omega} = \omega^H + d(\varphi \cdot \eta), \quad \tilde{\omega}' = \omega^H + d(\varphi \cdot \eta')).$$

where $\varphi$ is a cut-off function which is 1 when $\tilde{r}(x) \leq t$, and is 0 when $\tilde{r}(x) \geq 2t$. Clearly, the image of $[\tilde{\omega}]_1 \in H^2_{\text{dR}}(\mathcal{M}_\lambda)$ in $H^2_{\text{dR}}(U) \oplus H^2_{\text{dR}}(V)$ is $([\omega]_1, [\omega']_1)$. Since the two ALG* gravitational instantons have the same $[\omega]_1 = [\omega']_1$, and we also use the same $\omega^H_1$ for both, we see that the image of $[\tilde{\omega}]_1$ and $[\tilde{\omega}']_1$ are the same. So their difference is in the image of $H^1_{\text{dR}}(I^3)$. To see the image, we start with $d\theta_1 \in H^1_{\text{dR}}(I^3_\nu)$. It can be written as the difference of $\varphi d\theta_1$ on $U$ and $(\varphi - 1)d\theta_1$ on $V$. The form $d(\varphi d\theta_1) = d((\varphi - 1)d\theta_1)$ can be viewed as a two form on $\mathcal{M}_\lambda$ which is the image of $d\theta_1$ in $H^2_{\text{dR}}(\mathcal{M}_\lambda)$. Therefore, $[\tilde{\omega}]_1$ and $[\tilde{\omega}']_1$ may differ by a multiple of $[d(\varphi d\theta_1)]$. Fortunately, we can modify the 1-form $\eta_1$ by the same multiple of $d\theta_1$, and we then obtain $[\tilde{\omega}]_1 = [\tilde{\omega}']_1 \in H^2_{\text{dR}}(\mathcal{M}_\lambda)$. Remark that this modification will not affect any of the estimates in the proof of the gluing theorem.

Then we need to perturb the approximate hyperkähler triples to be actually hyperkähler. The resulting cohomology classes will not be exactly the same any more, but the span of them will remain the same. Therefore, by a rescaling and a hyperkähler rotation, we can get the same $[\omega^H_1]$ on $\mathcal{M}_\lambda$. 
Observe that the rescaling factor converges to 1 and the hyperkähler rotation matrix converges to the identity matrix as $\lambda \to 0$. By the Torelli-type theorem for K3 surfaces, there exists an isometry between them which maps the hyperkähler triples onto each other, and induces the identity mapping on $H^2(M_\lambda)$; see [Bes87, BR75, PSS71]. Therefore, the restriction of these maps to the ALG* bubbling regions will then converge to an isometry of the ALG* spaces as $\lambda \to 0$, since the isometry must map the ALG* regions to each other. Obviously, this isometry will map the hyperkähler triples onto each other. The homology class of a fiber generates $H^1(M_\lambda)$ onto each other. The homology class of a fiber generates $H^2(M_\lambda; \mathbb{R}) = \mathbb{R}$ and is nontrivial in both $H_2(U; \mathbb{R})$ and $H_2(V; \mathbb{R})$ under the natural inclusions. From the Mayer-Vietoris sequence in homology, it follows the natural mapping $H^2(M_\lambda; \mathbb{R}) \to H^2(M_\lambda; \mathbb{R})$ is injective. By duality, the restriction $H^2(M_\lambda; \mathbb{R}) \to H^2(V; \mathbb{R}) \cong H^2(X_\nu; \mathbb{R})$ is surjective, which implies that the isometry of the ALG* regions also induces the identity map in $H^2(X_\nu; \mathbb{R}) \cong H^2_\mathbb{R}(X_\nu)$, so we are done.

6.2. Proof of Theorem 1.5: ALG Torelli uniqueness. The next goal is to prove Theorem 1.5 which requires the following gluing result.

**Theorem 6.2.** Let $(X, g^X, \omega^X)$ be an ALG gravitational instanton of order 2 with $\chi(X) = \chi_0$. Then there exists a family of hyperkähler structures $(M_\lambda, h_\lambda, \omega_{h_\lambda})$ on the K3 surface $M_\lambda$ such that the following holds as $\lambda \to 0$.

1. We have Gromov-Hausdorff convergence $(M_\lambda, h_\lambda) \stackrel{GH}{\to} (\mathbb{P}^1, d_{ML})$, where $d_{ML}$ is the McLean metric on $\mathbb{P}^1$ with a finite singular set $S \equiv \{q_0, q_1, \ldots, q_{24-\chi_0}\} \subset \mathbb{P}^1$. Moreover, the curvatures of $h_\lambda$ are uniformly bounded away from $S$, but are unbounded around $S$.

2. Rescalings of $(M_\lambda, h_\lambda, \omega_{h_\lambda})$ around $q_i$ for $1 \leq i \leq 24 - \chi_0$ converge to a complete Taub-NUT gravitational instanton on $\mathbb{C}^2$.

3. Rescalings of $(M_\lambda, h_\lambda, \omega_{h_\lambda})$ around $q_0$ converge to the given ALG gravitational instanton $(X, g^X, \omega^X)$.

**Proof.** The proof is a straightforward generalization of [CVZ20, Theorem 1.1] using a general hyperkähler triple gluing argument as in Section 5. In [CVZ20], we assumed the ALG gravitational instantons were isotrivial which was necessary to preserve the complex structure. Since we are not fixing the complex structure on the K3 surface, only the order 2 assumption is necessary. For this, we just need to note that [CVZ20, Proposition 5.6] holds for any order 2 ALG space; the isotrivial condition is not necessary.  

Let $(X_\beta, g, \omega)$ and $(X_\beta, g', \omega')$ be ALG gravitational instantons on $X_\beta$ with the same parameters $\beta$, $\tau$, and $L$, which are both order 2 with respect to the coordinates $\Phi_{X_\beta}$ and which satisfy (1.4). The parameter $\beta$ determines a fiber $D$ of type $I_1^*$, $II^*$, $III^*$, $IV^*$, $IV^*$ as in Table 2.1. Let $\pi_\mathcal{K}: \mathcal{K} \to \mathbb{P}^1$ be any elliptic K3 surface with a single fiber $D^*$ of the dual type, which means $I_0^*$, $II^*$, $III^*$, $IV^*$, $II$, $III$, $IV$, respectively, but has all other singular fibers of type $I_1$. We use an attaching map $\Psi$ from $\{\lambda^{-1} \leq r \leq 2\lambda^{-1}\} \subset X_\beta$ to
a small annular region in $K$ centered around $D^*$ to obtain a manifold $M_\lambda$, where $\lambda$ is sufficiently small. Let $U$ be the subset such that $r \geq \lambda^{-1}$ and $V$ be the subset such that $r \leq 2\lambda^{-1}$. Then $M_\lambda = U \cup \Gamma$.

The gluing procedure in the proof of Theorem 6.2 produces approximate hyperkähler triples $\tilde{\omega}_\lambda$ and $\tilde{\omega}'_\lambda$ on the $M_\lambda$. Note that $U \cap V$ deformation retracts onto the 3-manifold $N^3_\beta$ which is the restriction of an elliptic fibration with a single fiber of type $D^*$ to $S^1$.

**Lemma 6.3.** The manifold $N^3_\beta$ is flat, and satisfies $b^1(N^3_\beta) = b^2(N^3_\beta) = 1$. Furthermore, a generator for $b^1(N^3_\beta)$ is the 1-form $d\theta_1$, where $\theta_1$ is the angular coordinate on the cone $C(2\pi \beta)$.

**Proof.** The 3-manifold $N^3_\beta$ is a $T^2$-fibration over $S^1$. We cover $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ by two intervals $(0, 2\pi \beta)$ and $(2\pi \beta, 3\pi \beta)$. Then we can write $N^3_\beta$ as the union of $N^3_{\beta, 1} \equiv (0, 2\pi \beta) \times T^2$ and $N^3_{\beta, 2} \equiv (2\pi \beta, 3\pi \beta) \times T^2$. The Mayer-Vietoris sequence is

$$
\begin{array}{cccc}
H^0_{\text{dr}}(N^3_\beta) & \longrightarrow & H^0_{\text{dr}}(N^3_{\beta, 1}) \oplus H^0_{\text{dr}}(N^3_{\beta, 2}) & \longrightarrow & H^0_{\text{dr}}(N^3_{\beta, 1} \cap N^3_{\beta, 2}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\text{dr}}(N^3_\beta) & \longrightarrow & H^1_{\text{dr}}(N^3_{\beta, 1}) \oplus H^1_{\text{dr}}(N^3_{\beta, 2}) & \longrightarrow & H^1_{\text{dr}}(N^3_{\beta, 1} \cap N^3_{\beta, 2}).
\end{array}
$$

If the monodromy group is $A$, then the map

$$
(6.4) \quad H^1_{\text{dr}}(N^3_{\beta, 1}) \oplus H^1_{\text{dr}}(N^3_{\beta, 2}) = \mathbb{R}^2 \oplus \mathbb{R}^2 \rightarrow H^1_{\text{dr}}(N^3_{\beta, 1} \cap N^3_{\beta, 2}) = \mathbb{R}^2 \oplus \mathbb{R}^2
$$

is given by $(C_1, C_2) \mapsto (C_1 - C_2, C_1 - AC_2)$ for $C_1, C_2 \in \mathbb{R}^2$, whose kernel is the same as $\ker(A - Id)$. For singular fibers of finite monodromy, $\ker(A - Id) = 0$. The map

$$
(6.5) \quad H^0_{\text{dr}}(N^3_{\beta, 1}) \oplus H^0_{\text{dr}}(N^3_{\beta, 2}) \rightarrow H^0_{\text{dr}}(N^3_{\beta, 1} \cap N^3_{\beta, 2})
$$

is a rank 1 map $(a, b) \mapsto (a - b, a - b)$. So $H^1_{\text{dr}}(N^3_\beta) = \mathbb{R}$ and it is generated by the image of $(2\pi \beta, 0) \in H^0_{\text{dr}}(N^3_{\beta, 1} \cap N^3_{\beta, 2})$. To see this image, we note that the difference of the function $\theta_1$ on $(\pi \beta, 3\pi \beta) \times T^2$ with the function $\theta_1$ on $(0, 2\pi \beta) \times T^2$ is exactly $2\pi \beta$ on $(0, \pi \beta) \times T^2$ and 0 on $(\pi \beta, 2\pi \beta) \times T^2$. The derivatives of them are all $d\theta_1$. So the image of $(2\pi \beta, 0)$ is $d\theta_1$. In other words, we have proved that $H^1_{\text{dr}}(N^3_\beta) = \langle d\theta_1 \rangle$. By Poincaré duality, $b^2(N^3_\beta) = b^1(N^3_\beta) = 1$. The flatness of $N^3_\beta$ is a corollary of the fact that the flat metric on $N^3_{\beta, 1}$ and $N^3_{\beta, 2}$ can be glued into a flat metric on $N^3_\beta$. □

The proof of Theorem 6.2 uses [CVZ20, Proposition 5.6], which implies

$$
(6.6) \quad \Phi^*_\beta(\omega) - \omega^C = d\eta_1, \quad \Phi^*_\beta(\omega') - \omega^C = d\eta',
$$

for some triples of 1-forms $\eta$ and $\eta'$ defined on end of the model space. On the region $U$, away from the damage zone, the approximate hyperkähler triples are exactly the same (they are semi-flat, with $I_1$ fibers resolved using
Ooguri-Vafa metrics). Using the same Mayer-Vietoris sequence (6.1), and Lemma 6.3, we can adjust the 1-form $\eta_i$ on the “damage zone” by a term of the form $d(\varphi \theta_1)$, to arrange that $[\omega_i] = [\omega'_i]$ in $H^2_{\text{dr}}(\mathcal{M}_A)$. The remainder of the proof is then exactly the same as in the ALG* case above.

7. Results on the period mapping

In this section we will use the following notation. In the ALG* case, extend $s$ to a smooth function $s$ defined on all of $X$ satisfying $s \geq 1$. In the ALG case, similarly extend $r$ to a smooth function defined on all of $X$, and again denoted the extended function by $s$.

7.1. Harmonic 2-forms of order 2. In order to properly define the period map, we begin with a proposition relating compactly supported de Rham cohomology and decaying harmonic 2-forms.

Proposition 7.1. For any ALG or ALG* gravitational instanton $(X, g, \omega)$ of order 2,

$$
\{ \omega = O(s^{-2}) \in \Omega^2(X) \mid \Delta \omega = 0 \} = \{ \omega = O(s^{-2}) \in \Omega^2(X) \mid d\omega = d^* \omega = 0 \}
$$

$$
= \{ \omega = O(s^{-2}) \in \Omega^2(X) \mid d\omega = d^* \omega = 0 \}
$$

$$
= \text{Im}(H^2_{\text{cpt}}(X) \to H^2(X)) = \{ [\omega] \in H^2(X), \int_D \omega = 0 \},
$$

where $D$ is any fiber arising from the compactification of $X$ to a rational elliptic surface.

Proof. We first consider the ALG* case. If $\omega = O(s^{-2}) \in \Omega^2(X)$, and $\Delta \omega = 0$, by standard elliptic regularity, $\omega \in W^{k,2}_\mu$ for any $k \in \mathbb{N}_0$ and $\mu > -2$. So the boundary term in

$$
\int_{r < R} \left( (\omega, \Delta \omega) - (d\omega, d\omega) - (d^* \omega, d^* \omega) \right)
$$

(7.1)

goess to 0 when $R \to \infty$, which implies $d\omega = d^* \omega = 0$. Conversely, if $d\omega = d^* \omega = 0$, then $\Delta \omega = 0$.

Then we study $\text{Im}(H^2_{\text{cpt}}(X) \to H^2(X))$. Define $U = \{ x \in X, r(x) > R \}$, then $U$ deformation retracts to the 3-manifold $\mathcal{I}^3_\nu$. By Lemma 6.1, $H^1(\mathcal{I}^3_\nu)$ is generated by $d\theta_1$. By Poincaré duality, $H_2(\mathcal{I}^3_\nu)$ is generated by $[D]$, where $D$ is any fiber, so $H_2(U)$ is also generated by $[D]$. Therefore, if $[\omega] \in H^2(X)$ and $\int_D \omega = 0$ then $[\omega]_U$ is exact, so there exists $\eta \in \Omega^1(U)$ such that $\omega = d\eta$ on $U$. Let $\chi$ be a cut-off function which is 0 when $r \leq R$ and is 1 when $r \geq 2R$. Then $\omega - d(\chi \cdot \eta)$ is compactly supported, so $[\omega] \in \text{Im}(H^2_{\text{cpt}}(X) \to H^2(X))$.

Conversely, if $\omega$ is compactly supported, then it is trivial to see that $\int_D \omega = 0$.

For any $\omega = O(s^{-2}) \in \Omega^2(X)$ such that $\Delta \omega = 0$, it is easy to see that $\int_D \omega = 0$ by choosing far enough $D$. So there is a map

$$
\{ \omega = O(s^{-2}) \in \Omega^2(X) : d\omega = d^* \omega = 0 \} \to \{ [\omega] \in H^2(X) : \int_D \omega = 0 \}.
$$
To show the surjectivity, for any compactly supported closed form $\omega$, choose an arbitrary $0 < \epsilon < 1$ and a basis $\eta_i$ of 2-forms in $W^{-1-\epsilon}_{k,2}$ such that $\Delta \eta_i = 0$. Since $(\eta_i, \eta_j)_{L^2}$ is invertible, there exist $c_i \in \mathbb{R}$ such that

$$(7.2) \quad (\omega, \eta_j)_{L^2} = \left( \sum_i c_i \eta_i, \eta_j \right)_{L^2}.$$ 

By [CVZ21, Proposition 4.2(2)], there exists $\phi \in W^{k+2,2}_{1+\epsilon}$ such that

$$(7.3) \quad \Delta \phi = \omega - \sum_i c_i \eta_i.$$ 

Since

$$\int_{r<R} \left( (\eta_i, \Delta \eta_i) - (d \eta_i, d \eta_i) - (d^* \eta_i, d^* \eta_i) \right)$$

also decays as $R \to \infty$, $\eta_i$ are closed and co-closed. So,

$$(7.5) \quad \omega - dd^* \phi = d^* d \phi + \sum_i c_i \eta_i \in W^{k,2}_{-1+\epsilon}$$

is closed and co-closed. The self-dual part is

$$\sum_{i=1}^3 f_i \omega_i$$

for decaying harmonic functions $f_i$, which must be 0. By (A.72), (A.82), (A.120), (A.121) and (A.136)-(A.142) of [CVZ21], the closed and co-closed anti-self-dual form $\omega - dd^* \phi$ must be $O(s^{-2})$, which implies the surjectivity.

To show the injectivity, assume that

$$(7.8) \quad \int_{r<R} (\omega, d \eta) - (d^* \omega, \eta)$$

also converges to 0 as $R \to \infty$. In other words, $\omega = d \eta = 0$ since $d^* \omega = 0$.

Using the same proof, and the ALG asymptotic analysis in [CC21a], a similar proof also holds for ALG gravitational instantons of order 2. See also [HHM04, Section 7.1.3] and Theorems 9.3 and 9.4 of [CVZ20].

$\square$
7.2. Definition of the period map. In this subsection, we prove the period mappings are well-defined.

**Proposition 7.2.** The period mappings $\mathcal{P}$ in Definition 1.6 and Definition 1.11 are well-defined.

**Proof.** We first consider the ALG case. If $(X_\beta, g, \omega) \in \mathcal{M}_{\beta, \tau, L}$, then it is ALG with respect to the fixed ALG coordinate system $\Phi_{X_\beta}$. Then $\omega_1$ is taken to be the Kähler form which is asymptotic to the elliptic complex structure, and the choice of $\omega_2$ and $\omega_3$ is also determined since they are asymptotic to the model Kähler forms in the $\Phi_{X_\beta}$ coordinates. The point is our Definition 1.3 removes the freedom of hyperkähler rotations, so we have a well-defined ordered choice of the 3 Kähler forms. From [CC21a, Theorem 4.14], there is a holomorphic function $u : X_\beta \to \mathbb{C}$ which is an elliptic fibration. The level sets of $u$ are tori. As $u \to \infty$, these level sets are close to the model holomorphic tori. Therefore the homology class $[D]$ of any fiber is well-defined, the same class for all elements in $\mathcal{M}_{\beta, \tau, L}$. Since the forms $\omega_2$ and $\omega_3$ are orthogonal to $\omega_1$, any torus which is holomorphic for $I$ is Lagrangian with respect to $J$ or $K$. Use Proposition 7.1 to identify $H^2$ with order 2 decaying harmonic anti-self-dual 2-forms, the classes $[\omega_2]$ and $[\omega_3]$ automatically lie in $H^2$. Finally, since the holomorphic tori for $I$ and $I_0$ are homologous, we have $\int_D (\omega_1 - \omega_1^0) = 0$ since the area of the holomorphic tori are the same. Using [CV21, Proposition 3.1], the argument in the ALG$^*$ case is exactly the same. □

7.3. The nondegeneracy condition. In this subsection, we prove the nondegeneracy condition stated in Theorems 1.7 and 1.12.

(7.9) $\omega[C] \neq (0, 0, 0)$ for all $[C] \in H_2(X; \mathbb{Z})$ satisfying $[C]^2 = -2$.

To prove this, we use the gluing construction in Theorem 6.2 in the ALG case and Theorem 5.6 in the ALG$^*$ case. A basic transversality argument shows that we can represent any $[C] \in H_2(X; \mathbb{Z})$ by an embedded surface $\iota : C \to X$. If (7.9) is not satisfied by an ALG or ALG$^*$ gravitational instanton $(X, \omega^X)$, then by choosing a small enough gluing parameter $\lambda$, we can assume that the glued closed definite triple $\omega_\lambda = \omega^X$ near $\iota(C)$. A Mayer-Vietoris argument in homology shows that $[C]$ is nontrivial in $H_2(\mathcal{M}_\lambda, \mathbb{Z})$. So, there exists $[C] \in H_2(\mathcal{M}_\lambda, \mathbb{Z})$ such that $[C]^2 = -2$ and $[\omega_\lambda] \cdot [C] = 0$. In the perturbation arguments, the span of the hyperkähler classes $[\omega_\lambda^{HK}]$ on the K3 surface $\mathcal{M}_\lambda$ is the same as the span of $[\omega_\lambda]$. Therefore, $[\omega_\lambda^{HK}] \cdot [C] = 0$, which is a contradiction with the well-known nondegeneracy condition on the K3 surface $\mathcal{M}_\lambda$.

7.4. Proofs of Theorem 1.7 and Theorem 1.12. We follow the route map of [CC21b, Section 7]. For any point in $\mathcal{H}^2 \oplus \mathcal{H}^2 \oplus \mathcal{H}^2$ whose sum with $\omega^0$ satisfies (7.9), we can connect it to $(0, 0, 0)$ by zigzags of the form

(7.10) $([\alpha_1, 0] + t[\beta_1], [\alpha_2], [\alpha_3])$. 

(7.11) \((|a_1|, |a_2,0] + t[|a_2], |a_3,0] + t[|a_3])\),

or

(7.12) \((|a_1|, |a_2,0] + t[|a_2], |a_3,0] + t[|a_3])\).

We require that the sum of \(\omega^0\) with all the points in the zigzags satisfy (7.9). Let us consider the ALG case. For the path in (7.12) we have,

(7.13) \((X, \omega_{1,0} \equiv \omega_1^0 + a_1,0, \omega_2 \equiv \omega_2^0 + a_2, \omega_3 \equiv \omega_3^0 + a_3) \in \mathcal{M}_{\beta, r, L}\).

Using Proposition 7.1, we choose the representative \(\beta_1\) in the class \([\beta_1]\) by requiring it to be closed, co-closed, and anti-self-dual with respect to the hyperkähler metric determined by \((X, \omega_{1,0}, \omega_2, \omega_3)\). Since \(\beta\) is anti-self-dual,

(7.14) \(\beta_1 \wedge \omega_{1,0} = \beta_1 \wedge \omega_2 = \beta_1 \wedge \omega_3 = 0\).

Then we choose \(s_t \in \mathbb{R}\) such that

(7.15) \(\omega_{1,t} \equiv \omega_{1,0} + t\beta_1 + s_t \sqrt{-1}\partial_t \bar{\partial}_t (\chi \cdot \log |u|)\)

satisfies

(7.16) \(\int_X (\omega_{1,t}^2 - \omega_{1,0}^2) = 0\),

where \(I, J, K\) are the hyperkähler structures determined by \((X, \omega_{1,0}, \omega_2, \omega_3)\), \(u : X \rightarrow \mathbb{C}\) is the \(I\)-holomorphic function which makes \(X\) a rational elliptic surface minus the fiber at infinity, and \(\chi\) is a cut-off function which is 0 for small \(|u|\) and is 1 for large \(|u|\). The constant \(s_t\) exists because

(7.17) \(\int_X (\omega_{1,t}^2 - \omega_{1,0}^2) = \int_X t^2 \beta_1^2 + 2s_t \int_X \omega_{1,0} \wedge \sqrt{-1}\partial_t \bar{\partial}_t (\chi \cdot \log |u|)\),

and \(\int_X \omega_{1,0} \wedge \sqrt{-1}\partial_t \bar{\partial}_t (\chi \cdot \log |u|) \neq 0\). In the (7.11) case, we use \(\sqrt{-1}\partial_t \bar{\partial}_t\) instead of \(\sqrt{-1}\partial_t \bar{\partial}_I\), and in the (7.12) case, we use \(\sqrt{-1}\partial_K \bar{\partial}_K\).

**Lemma 7.3.** Let \((X, g, p)\) be a gravitational instanton of type either ALG or ALG* of order 2, where \(p \in X\). For \(\delta \in \mathbb{R}\), there exists a constant \(C\) so that

(7.18) \(\sup_X |s^{-\delta} \varphi| \leq C\|\varphi\|_{W_{100,2}^1(X)}\),

for all \(\varphi \in W_{100,2}^1(X)\).

**Proof.** The proof is a standard rescaling argument. It suffices to prove the following inequality on any large annulus. We will show that there exists a constant \(C = C(\delta) > 0\) such that for any sufficiently large constant \(\zeta \gg 1\),

(7.19) \(\sup_{A(\zeta, 2\zeta, p)} |s^{-\delta} \varphi| \leq C \sum_{m=0}^{100} \|\nabla^m \varphi\|_{L_m^2(A(\zeta, 2\zeta, p))}, \quad \forall \varphi \in W_{100,2}^1(A(\zeta, 2\zeta, p))\),
where $A(\zeta, 2\zeta, p) \equiv \{ x \in X | \zeta \leq s(x) \leq 2\zeta \}$. Let us consider the rescaled metric $\tilde{g} \equiv \zeta^2 \cdot g$ such that $g$ is non-collapsing in the annulus $A(\zeta, 2\zeta, p)$. Then the desired inequality will follow from the standard Sobolev inequality,

$$\sup_{\tilde{A}_\zeta} |\varphi| \leq C \sum_{m=0}^{100} \frac{1}{\text{Vol}_{\tilde{g}}(\tilde{A}_\zeta)} \int_{\tilde{A}_\zeta} |\nabla^m \varphi|^2 \text{dvol}_{\tilde{g}},$$

(7.20)

where $C > 0$ is independent of $\zeta$ and $\tilde{A}_\zeta$ is the rescaled image of the annulus $A(\zeta, 2\zeta, p)$. Finally, one obtains (7.18) by a simple covering argument. □

Back to the (7.10) case, consider the collection $\mathcal{S}$ of $t \in [0, 1]$ for which there exists $\delta_t > 0$ and $\varphi_t \in W^{k,2}_{-\delta_t}(X, \omega_{1,0})$ for any $k \in \mathbb{N}$ such that

$$\omega_{1,t} + \sqrt{-1} \partial_t \overline{\partial} \varphi_t, \omega_2, \omega_3) \in M_{\beta, r, L}.$$ (7.21)

By assumption, $0 \in \mathcal{S}$. If $t_0 \in \mathcal{S}$, then by definition of $\mathcal{S}$, iterating Lemma 7.3 and using a standard elliptic regularity argument, we see that for $t$ sufficiently close to $t_0$, $\omega_{1,t} + \sqrt{-1} \partial_t \overline{\partial} \varphi_{t_0}$ will be ALG or ALG* of order 2. Furthermore,

$$\int_X (\omega_{1,t} + \sqrt{-1} \partial_t \overline{\partial} \varphi_{t_0})^2 - \omega_{1,0}^2 = \int_X (\omega_{1,t}^2 - \omega_{1,0}^2) = 0.$$ (7.22)

By [TY90] Theorem 1.1, there exists a bounded solution $\varphi_t$ of the equation

$$(\omega_{1,t} + \sqrt{-1} \partial_t \overline{\partial} \varphi_t)^2 = e^f \omega_{1,t}^2,$$ (7.23)

where

$$f \equiv \log \frac{\omega_{1,t}^2 - \omega_{1,0}^2}{\omega_{1,t}^2} = \log \frac{\omega_{1,t}^2 - \omega_{1,0}^2}{\omega_{1,t}^2} = O(r^{-4}),$$ (7.24)

and the middle equality follows from (7.14). By [Hei12] Proposition 2.6, $\int_X |\nabla \omega_{1,t} \varphi_t|^2 \omega_{1,t}^2 < \infty$. Then, by [Hei12] Proposition 2.9 (ii), there exists a $\delta_t > 0$ so that

$$\sup |\varphi_t| \leq C s^{-\delta_t},$$ (7.25)

Then, [Hei12] Proposition 2.9 (ii) implies that

$$\sup |\nabla^k \omega_{1,t} \varphi_t| \leq C_k s^{-\delta_t - k},$$ (7.26)

since these estimates are implied by Hein’s weighted H"older estimates. This implies that $\varphi_t \in W^{k,2}_{-\delta_t}(X, \omega_{1,0})$ for any $k \in \mathbb{N}$ if we slightly shrink $\delta_t$. Consequently, $\mathcal{S}$ is open. This also implies that the period mapping is an open mapping. The above arguments hold in the ALG* case (with $M_{\beta, r, L}$ replaced by $M_{\nu, \kappa_0, L}$), so this completes the proof of Theorem 1.7
To finish the proof of Theorem 1.12 we need to show that $S$ is closed in the ALG cases. So suppose that $t_i \to t_\infty$ is a sequence in $S$, then

\[
\int_X (\text{tr} \omega_i \omega_{t_i} - 2) \frac{\omega_{t_i}^2}{2} = \int_X \omega_1,0 \wedge (\omega_{t_i} - \omega_1,0) = \int_X \omega_1,0 \wedge (\omega_{t_i} - \omega_1,0) = s_{t_i} \int_X \omega_1,0 \wedge \sqrt{-1} \partial I \bar{\partial} I (\chi \cdot \log |u|) = -\frac{t_i^2}{2} \int_X \beta_1^2 \leq C
\]

for a constant independent of $t_i$, and

\[
\int_X (\text{tr} \omega_j \omega_{t_j} - 2) \frac{\omega_{t_j}^2}{2} = \int_X \omega_j \wedge (\omega_{t_j} - \omega_j) = \int_X \omega_1,t_j \wedge (\omega_{1.t_i} - \omega_1,t_j) = \int_X (\omega_{1,0} + t_j \beta_1) \wedge ((t_i - t_j) \beta_1 + (s_{t_i} - s_{t_j}) \sqrt{-1} \partial I \bar{\partial} I (\chi \cdot \log |u|)) = (s_{t_i} - s_{t_j}) \int_X \omega_1,0 \wedge \sqrt{-1} \partial I \bar{\partial} I (\chi \cdot \log |u|) + t_j(t_i - t_j) \int_X \beta_1^2
\]

\[
= \left( \frac{t_i^2}{2} + \frac{t_j^2}{2} + t_j(t_i - t_j) \right) \int_X \beta_1^2 = -\frac{(t_i - t_j)^2}{2} \int_X \beta_1^2 \to 0
\]

as $i, j \to \infty$. These bounds imply the following pointwise bound.

**Theorem 7.4.** The function $e(t_i) = \text{tr} \omega_i \omega_{t_i} = \text{tr} \omega_{t_i} \omega_0$ is uniformly bounded on $X$.

**Proof.** We use (7.27) and (7.28) to go through the arguments in [CC21b, Section 7], with some minor modifications, to get the required bound. First, cover $X$ by balls with radius 1 in the sense of the metric determined by $(X, \omega_1,0, \omega_2, \omega_3)$ such that the number of balls containing any point in $X$ is uniformly bounded. Then we use these balls to replace the sets $U_N$ in [CC21b, Theorem 7.3], which yields a bound on the diameter (in the metric $\omega_{t_i}$) of these balls. Note that the proof of [DPS93, Lemma 1.3] is valid in the ALG case, since ALG metrics are volume non-collapsed in bounded scales at infinity.

To prove the analogue of [CC21b, Theorem 7.4], we need to show that if there exists a sequence of cohomology classes $[\Sigma_i] \in H^2(X, \mathbb{Z})$ satisfying $[\Sigma_i]^2 = -2$ and $\int_{\Sigma_i} \omega_{t_i} \to 0, \int_{\Sigma_i} \omega_2 \to 0, \int_{\Sigma_i} \omega_3 \to 0$ as $i \to \infty$, then there are only finite many distinct $[\Sigma_i]$. To prove this, recall that by the assumption of Theorem 1.7 based on [CV21b, Theorem 1.10], $X$ is in particular diffeomorphic to an isotrivial ALG gravitational instanton. These compactify to an isotrivial rational elliptic surface $S$ by adding a finite monodromy fiber $D_\infty$ at infinity. By [Hei12, Section 3.1], $S \setminus D_\infty$ deformation retracts onto the dual finite monodromy fiber. Therefore the intersection form of $H^2(X, \mathbb{Z})$ is an extended Dynkin diagram. Next, for example, assume the extended Dynkin diagram is $D_4$, then $H^2(X, \mathbb{Z})$ is generated by $[E_i], i = 1, 2, \ldots, 5$, with
\[ [E_i]^2 = -2 \] for all \( i \), \([E_i] \cdot [E_j] = 1 \) for all \( \{i, j\} = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\} \), and \([E_i] \cdot [E_j] = 0 \) otherwise. The homology class of each fiber is

\[
[F] = 2[E_1] + [E_2] + [E_3] + [E_4] + [E_5].
\]

The intersection numbers of \([F] \) with all \([E_i] \) are 0. We write \([\Sigma_i] \) as

\[
[\Sigma_i] = a_i[F] + b_i[E_1] + c_i[E_2] + d_i[E_3] + e_i[E_4].
\]

Then the self-intersection number of \( b_i[E_1] + c_i[E_2] + d_i[E_3] + e_i[E_4] \) is \(-2\). The extended Dynkin diagram restricted to this subset is the unextended Dynkin diagram, which has negative definite intersection form. This implies that there are only finitely many distinct \( b_i[E_1] + c_i[E_2] + d_i[E_3] + e_i[E_4] \) with self-intersection \(-2\). Then, we use \( \int_F \omega_i = \int_F \omega_{1,0} \neq 0 \) to control \( a_i \). The proof for other extended Dynkin diagrams is similar.

This proves a uniform curvature bound, and this yields a bound on the \( \omega_i \)-holomorphic radius exactly as in \([CC21b, \text{Theorem 7.4}] \). The proof of \([CC21b, \text{Theorem 7.4}] \) relies on \([Rua99, \text{Proposition 2.1}] \), which is valid in the ALG case since these are volume non-collapsed in bounded scales at infinity. Theorem 7.6, Lemma 7.7 and Theorem 7.8 of \([CC21b] \) then go through exactly the same in the ALG cases, with \( U_N \) replaced by balls of radius 1. Note that only the hyperkähler condition is used in \([CC21b, \text{Theorem 7.6}] \).

The equation \( \omega_i = \omega_{1,0} \) and the bound on \( \int_X \omega_i \) \( \omega_{1,0} \omega_i \) imply that there exists a constant \( C \) independent of \( t \) such that

\[
C^{-1} \omega_{1,0} \leq \omega_i \leq C \omega_{1,0}.
\]

Since the difference \( \omega_{1,t} - \omega_{1,0} \) decays uniformly, there exists a constant \( R \) such that \( \frac{1}{2} \omega_{1,0} \leq \omega_{t} \leq 2 \omega_{1,0} \) for all \( s \geq R \). So

\[
|\Delta_{\omega_{1,0}} \varphi_{t}| = |\int_{X} (\omega_{1,0} - \omega_{t})| = |\int_{X} \omega_{1,0} \omega_{t} - 2s_{t} \Delta_{\omega_{1,0}} (\chi \cdot \log |u|)| \leq C
\]
on \( X \). Moreover,

\[
\int_{X} |\Delta_{\omega_{1,0}} \varphi_{t}| \omega_{1,0}^{2} \leq \int_{X} |\int_{X} (\omega_{1,0} \omega_{t} - 2s_{t} \Delta_{\omega_{1,0}} (\chi \cdot \log |u|))| \omega_{1,0}^{2} \leq C,
\]

where we have used the fact that \( \omega_{t} = \omega_{1,0}^{2} \), which implies that \( \int_{X} \omega_{1,0} \omega_{t} \geq 2 \).

So for \( \delta = \frac{1}{100} \), \( \|\Delta_{\omega_{1,0}} \varphi_{t}\|_{L_{2,1+\delta}^{2}}(X, \omega_{1,0}) \leq C \). Now we consider the operator \( \Delta_{\omega_{1,0}} : W_{2,1+\delta}^{2}(X, \omega_{1,0}) \rightarrow L_{2,1+\delta}^{2}(X, \omega_{1,0}) \). By the ALG weighted analysis in \([CC21b, CC19, CC21b, \text{and HHM04}] \), it is easy to see that any function in the kernel of this operator must be a constant, and consequently there exists
another function $\tilde{\varphi}_{t_i} \in W^{2,2}_{1,\delta}(X, \omega_{1,0})$ such that $\varphi_{t_i} - \tilde{\varphi}_{t_i}$ is a constant and

$$
||\tilde{\varphi}_{t_i}||_{W^{2,2}_{1,\delta}(X, \omega_{1,0})} \leq C||\Delta_{\omega_{1,0}} \tilde{\varphi}_{t_i}||_{L^2_{1,\delta}(X, \omega_{1,0})} = C||\Delta_{\omega_{1,0}} \varphi_{t_i}||_{L^2_{1,\delta}(X, \omega_{1,0})} \leq C.
$$

(7.34)

This implies that $||\tilde{\varphi}_{t_i}||_{W^{2,p}_{p}(\{s \leq 4R\}, \omega_{1,0})} \leq C(p)$ for any $p > 1$ using the bound on $|\Delta_{\omega_{1,0}} \varphi_{t_i}|$. For any $\alpha \in (0, 1)$, by the Evans-Krylov estimate,

$$
[\partial \bar{\partial} \tilde{\varphi}_{t_i}]^{\alpha}_{(\{s \leq 3R\}, \omega_{1,0})} \leq C(\alpha);
$$

see [Su87, Section 2.4] for instance. By standard elliptic estimates, for any $k \in \mathbb{N}$ and any $\alpha \in (0, 1)$,

$$
||\tilde{\varphi}_{t_i}||_{C^{k,\alpha}(\{s \leq 2R\}, \omega_{1,0})} \leq C(k, \alpha).
$$

(7.35)

When $s \geq R$,

$$
|\Delta_{\omega_{1,0}} \overline{\xi}_{t_i} + \omega_{t_i} \tilde{\varphi}_{t_i}| \leq C|\omega_{1,0} + \omega_{t_i}|(\omega_{1,0} - \omega_{t_i})|\omega_{1,0} + \omega_{t_i}|
$$

(7.36)

$$
= C|\omega_{1,0} - \omega_{t_i}|\omega_{1,0} + \omega_{t_i} \leq C|\omega_{1,0}^2 - \omega_{t_i}^2| \leq Cs^{-4}.
$$

(7.37)

Let $\chi_R$ be a cut-off function which is 1 when $s \geq 2R$ and is 0 when $s \leq R$. Then $|\Delta_{\omega_{1,0}} \overline{\xi}_{t_i} + \omega_{t_i} \tilde{\varphi}_{t_i}| \leq Cs^{-4}$, where $\xi_{t_i} \equiv \chi_R \cdot \tilde{\varphi}_{t_i}$.

We use the Moser iteration technique to prove that $||\xi_{t_i}||_{C^{0}} \leq C$ for a constant $C$ independent of $t_i$ and $p$. For any $j = 0, 1, 2, 3, ...$ and $p = 2^j$,

$$
p^2 \int_X \xi_{t_i}^{2p-2} |\nabla_{\omega_{1,0}} + \omega_{t_i} \xi_{t_i}|^2 \frac{2(\omega_{1,0} + \omega_{t_i})^2}{2}
$$

(7.38)

$$
= \int_X |\nabla_{\omega_{1,0}} + \omega_{t_i} (\xi_{t_i}^p)|^2 \frac{2(\omega_{1,0} + \omega_{t_i})^2}{2}
$$

$$
- p(p - 1) \int_X \xi_{t_i}^{2p-2} |\nabla_{\omega_{1,0}} + \omega_{t_i} \xi_{t_i}|^2 \frac{2(\omega_{1,0} + \omega_{t_i})^2}{2}
$$

$$
- p \int_X \xi_{t_i}^{2p-1} \Delta_{\omega_{1,0}} + \omega_{t_i} \xi_{t_i} \frac{2(\omega_{1,0} + \omega_{t_i})^2}{2}.
$$

We have used the fact that $\xi_{t_i} - \varphi_{t_i}$ is a constant when $s \geq 2R$, and there exists $\delta_{t_i} > 0$ such that $\varphi_{t_i} \in W^{k,2}_{\delta_{t_i}}(X, \omega_{1,0})$. Therefore,

$$
\int_X |\nabla_{\omega_{1,0}} + \omega_{t_i} (\xi_{t_i}^p)|^2 \frac{2(\omega_{1,0} + \omega_{t_i})^2}{2}
$$

(7.39)

$$
= - \frac{p^2}{2p - 1} \int_X \xi_{t_i}^{2p-1} \Delta_{\omega_{1,0}} + \omega_{t_i} \xi_{t_i} \frac{2(\omega_{1,0} + \omega_{t_i})^2}{2}.
$$
Recall that by Theorem 1.2(i) of [Hei11], there exist a constant $C$ and a weight function $\psi$ with $\int_X \psi^\omega_{t_0,0} = 1$ such that for any $\xi \in C^\infty_0(X)$,

$$\left( \int_X \left| \xi - \xi_0 \right|^4 s^{-4} \omega_{1,0}^2 \right)^{\frac{1}{4}} \leq C \int_X |\nabla \omega_{1,0}|^2 \omega_{1,0}^2,$$

where $\xi_0 \equiv \int_X \psi \omega_{1,0}^2$. Equation (7.40) also holds for $\xi^p_{t_i}$ because (7.40) is unchanged if we add $\xi$ by a constant and $\xi^p_{t_i}$ can be written as a constant plus a function in $W^{k,2}_{p-1}(-\delta t_i, X, \omega_{1,0})$. Then

$$|\xi_0|^2 \leq C \left( \int_{s \leq 2R} |\xi_0|^4 s^{-4} \omega_{1,0}^2 \right)^{\frac{1}{4}}$$

(7.41)

$$\leq C \left( \int_X \left| \xi - \xi_0 \right|^4 s^{-4} \omega_{1,0}^2 \right)^{\frac{1}{4}} + C \|\xi\|_{C^0(\{s \leq 2R\})}^2.$$  

So

$$\left( \int_X |\xi|^4 s^{-4} \omega_{1,0}^2 \right)^{\frac{1}{4}} \leq C \int_X |\nabla \omega_{1,0} + \xi^p_{t_i}|^2 \omega_{1,0}^2 + C \|\xi\|_{C^0(\{s \leq 2R\})}^2$$

(7.42)

for $\xi = \xi^p_{t_i}$ and a constant $C$ independent of $t_i$ and $p$. Therefore,

$$\left( \int_X |\xi_{t_i}|^4 s^{-4} \omega_{1,0}^2 \right)^{\frac{1}{4}} \leq C \int_X |\nabla \omega_{1,0} + \xi^p_{t_i}|^2 \omega_{1,0}^2 + C \|\xi_{t_i}\|_{C^0(\{s \leq 2R\})}^2$$

$$\leq C \int_X |\nabla \omega_{1,0} + \xi^p_{t_i}|^2 \omega_{1,0}^2 + C \|\xi_{t_i}\|_{C^0(\{s \leq 2R\})}^2$$

(7.43)

$$\leq \frac{Cp^2}{2p-1} \int_X |\xi_{t_i}|^{2p-1} s^{-4} \omega_{1,0}^2 + C \|\xi_{t_i}\|_{C^0(\{s \leq 2R\})}^{2p}$$

$$\leq \frac{Cp^2}{2p-1} \int_X \left( \frac{2p-1}{2p} \right) |\xi_{t_i}|^{2p} + C \|\xi_{t_i}\|_{C^0(\{s \leq 2R\})}^{2p}$$

$$\leq C \left( \int_X |\xi_{t_i}|^{2p} s^{-4} \omega_{1,0}^2 + 1 + \|\xi_{t_i}\|_{C^0(\{s \leq 2R\})}^2 \right)^{\frac{1}{2}}.$$  

For $p = 1$,

$$\int_X |\xi_{t_i}|^2 s^{-4} \omega_{1,0}^2 \leq C \left( \int_X |\xi_{t_i}|^4 s^{-4} \omega_{1,0}^2 \right)^{\frac{1}{4}}$$

(7.44)

$$\leq C \int_X |\xi_{t_i}| s^{-4} \omega_{1,0}^2 + C \|\xi_{t_i}\|_{C^0(\{s \leq 2R\})} + C$$

$$\leq C \epsilon \int_X |\xi_{t_i}|^2 s^{-4} \omega_{1,0}^2 + C \epsilon^{-1} + \|\xi_{t_i}\|_{C^0(\{s \leq 2R\})}^2 + C$$
for all $\epsilon > 0$. If we choose $\epsilon$ such that the coefficient $C\epsilon < \frac{1}{2}$, then

\begin{equation}
\int_X |\xi_{ti}|^2 s^{-4} \omega_{1,0}^2 \leq C.
\end{equation}

As in Page 715 of [CC21b], $||\xi_{ti}||_{C^0(X)} \leq C$ using (7.33). This implies that

\begin{equation}
||\varphi_{ti}||_{C^0} \leq ||\tilde{\varphi}_{ti}||_{C^0} + ||\varphi_{ti} - \tilde{\varphi}_{ti}|| \leq 2||\tilde{\varphi}_{ti}||_{C^0} \leq C
\end{equation}

because $\varphi_{ti} - \tilde{\varphi}_{ti}$ is a constant and $\varphi_{ti}$ decays. Using the Evans-Krylov estimate and standard elliptic estimates on $B(x, 1, \omega_{1,0})$ for any $x \in X$, $||\varphi_{ti}||_{C^k(X, \omega_{1,0})} \leq C(k)$ for constants $C(k)$ independent of $t_i$.

The bound on $\int_X |\xi_{ti}|^2 s^{-4} \omega_{1,0}^2$ also implies a bound on $\int_X |\nabla \omega_{1,0} \xi_{ti}|^2 \omega_{1,0}^2$ by (7.38). This implies that

\begin{equation}
\int_X |\nabla \omega_{1,0} \varphi_{ti}|^2 \omega_{1,0}^2 \leq C.
\end{equation}

Finally, we use [Hei12, Proposition 2.9] to prove that there exist a constant $\delta > 0$ and constants $C(k, \delta) > 0$ independent of $t_i$ such that

\begin{equation}
\| s^{k+\delta} \nabla \omega_{1,0} \varphi_{ti} \| \leq C(k, \delta)
\end{equation}

for all $k$. Then we use the Arzela-Ascoli Lemma, the diagonal arguments, and standard elliptic estimates to finish the proof.

7.5. Closing Remarks. There is a folklore conjecture that some examples constructed by Biquard-Boalch [BB04] are ALG and by varying parameters, achieve all possible periods satisfying (7.9). See [FMSW20] for some progress towards this conjecture. We also mention that there is a folklore conjecture that some examples constructed by Biquard-Boalch [BB04] and Cherkis-Kapustin [CK02] are ALG* and by varying parameters, achieve all possible periods satisfying (7.9).

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