Approximation by Operators Sequence Involving a Certain Class of Polynomials

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Abstract. This paper is devoted to the convergence properties of the operators sequence including a certain class of polynomials in the continuous functions space and weighted continuous functions space. The convergence order for the presented operators sequence is also discussed. At last, operators including Hermite polynomials of variance \( \nu \) are given as an application.

1. Introduction

Bernstein polynomials were proposed by Bernstein [5] to contribute to the proof of the Weierstrass theorem in virtue of probabilistic considerations. Quantitative properties of Bernstein polynomials together with several generalizations have been studied by many authors ([6], [15], [22], [25], [27]). Henceforth, there have been significant developments in the investigation of approximating functions which are continuous on the interval \([a, b]\) and have specific conditions on the real line. In 1930 Kantorovich [21] made a small modification to Bernstein polynomials for approximating Lebesgue integrable functions.

Szasz-Mirakjan operators which defined with the help of Poisson distribution have been introduced by Mirakjan [23] and were investigated by many authors ([9], [27]). This operator, denoted by \( S_n \), associates to any function \( \varphi \in C[0, \infty) \) the operator \( S_n(\varphi, \cdot) \), defined by

\[
S_n(\varphi; x) = \exp(-nx) \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!} \varphi\left(\frac{\nu}{n}\right).
\]

(1)

The polynomials \( q_\nu \), having the following form

\[
g(\omega) \exp(\omega x) = \sum_{\nu=0}^{\infty} q_\nu(x) \omega^\nu
\]

(2)

are called Appell polynomials, where \( g \) is a holomorphic function in the \( |\omega| < R \) \( (R > 1) \) and suppose that \( g(1) \neq 0 \). Szasz type operators containing Appell polynomials were constructed by Jakimovski and Leviatan [16]. If \( q_\nu(x) \geq 0 \) for \( x \in [0, \infty) \), the operators

\[
P_n(\varphi; x) = \frac{\exp(-nx)}{g(1)} \sum_{\nu=0}^{\infty} q_\nu(nx) \varphi\left(\frac{\nu}{n}\right)
\]

(3)

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were introduced in [16] and then the convergence properties of (3) were presented with the help of Szasz’s perspective by Jakimovski and Leviatan.

The polynomials \( q_\nu \) having the following form

\[
A(\omega) \exp(xH(\omega)) = \sum_{\nu=0}^{\infty} q_\nu(x) \omega^\nu, \ |\omega| < R, \ (R > 1)
\]

are called Sheffer polynomials, where \( A \) and \( H \) are analytic functions. By assuming \( q_\nu(x) \geq 0, \ A(1) \neq 0 \) and \( H'(1) = 1 \), the operators defined as

\[
V_n(\varphi; x) = \frac{\exp(-nxH(1))}{A(1)} \sum_{\nu=0}^{\infty} q_\nu(nx) \varphi\left(\frac{\nu}{n}\right)
\]

were presented by Ismail [12].

The polynomials \( q_\nu \) having the following form

\[
A(\omega) B(x_\omega) = \sum_{\nu=0}^{\infty} q_\nu(x) \omega^\nu
\]

are called Brenke type polynomials [7], where

\[
A(\omega) = \sum_{\nu=0}^{\infty} a_\nu \omega^\nu, \ a_0 \neq 0,
\]

\[
B(\omega) = \sum_{\nu=0}^{\infty} b_\nu \omega^\nu, \ b_\nu \neq 0 \ (\nu \geq 0)
\]

and for \( k \in \mathbb{N}_0 \)

\[
q_\nu(x) = \sum_{j=0}^{\nu} a_{\nu-j} b_j x^j.
\]

Under the some assumptions, Varma [28] presented the linear positive operators as follows

\[
L_n(\varphi; x) = \frac{1}{A(1) B(nx)} \sum_{\nu=0}^{\infty} q_\nu(nx) \varphi\left(\frac{\nu}{n}\right).
\]

There are also some other articles investigating the convergence result of these type operators sequence involving certain class of polynomials ([1], [4], [11], [14], [17], [18], [19], [20], [26]).

The following operators are the main concern of this paper

\[
T_n(f; x) = \frac{1}{\lambda(2nx + 1)} \sum_{\nu=0}^{\infty} q_\nu(nx) f\left(\frac{\nu}{n}\right),
\]

where \( q_\nu \) are polynomials [2] which have following form

\[
\lambda(2x + a^2) = \sum_{\nu=0}^{\infty} q_\nu(x) \omega^\nu
\]

and \( \lambda \) is analytic function such that

\[
\lambda(\omega) = \sum_{\nu=0}^{\infty} a_\nu \omega^\nu \ (a_\nu \neq 0).
\]
Let us confine ourselves to the polynomials (10) which have the following properties

(i) \( \lambda : \mathbb{R} \to (0, \infty) \), \( q_n(x) \geq 0 \),

(ii) For \( |\omega| < R \) \( (R > 1) \), (10) and (11) converge.

With reference to the (12), \( T_n \) operators defining by (9) are positive.

The objective of the following work is to investigate approximation of the operators defined as (9) in the continuous functions space and weighted continuous functions space. Moreover, the degree of convergence of a function is obtained.

The organization of this work is the following. Firstly, we succinctly acquire some essential knowledge about operators (9) and express our main results about convergence of operators (9) in continuous functions space and weighted continuous functions space. Furthermore, we obtain the rate of convergence through the instrument of modulus of continuity, Peetre’s K functional and second order modulus of continuity in continuous functions space and give the order of approximation under favour of weighted modulus of continuity in weighted continuous functions space. In section 3, operators including Hermite polynomials of variance \( \nu \) are given as an example.

2. Approximation Properties of \( T_n \) Operators

The following set \( E \)

\[
E = \left\{ f : \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty \right\}
\]

will be critical in our result.

We first prove two auxiliary lemmas.

Lemma 2.1. The operators (9) admit

\[
T_n(1; x) = 1,
\]

\[
T_n(s; x) = 2 \frac{\lambda'(2nx + 1)}{\lambda(2nx + 1)} x^2 + \lambda'\left(\frac{2nx + 1}{2nx + 1} \right) n',
\]

\[
T_n(s^2; x) = 4 \frac{\lambda''(2nx + 1)}{\lambda(2nx + 1)} x^3 + 2 \lambda''\left(\frac{2nx + 1}{2nx + 1} \right) + \lambda'''\left(\frac{2nx + 1}{2nx + 1} \right) x + 4 \frac{\lambda'''(2nx + 1) + \lambda''(2nx + 1)}{\lambda(2nx + 1) n^2},
\]

\[
T_n(s^3; x) = \frac{\lambda''''(2nx + 1)}{\lambda(2nx + 1)} x^4 + 12 \frac{\lambda''''(2nx + 1) + \lambda''''(2nx + 1)}{\lambda(2nx + 1) n^2} x^2 + 4 \frac{\lambda''''(2nx + 1) + \lambda''''(2nx + 1)}{\lambda(2nx + 1) n^2} x + 8 \frac{\lambda''''(2nx + 1) + \lambda''''(2nx + 1) + \lambda''''(2nx + 1)}{\lambda(2nx + 1) n^3},
\]

\[
T_n(s^4; x) = 16 \frac{\lambda^{(4)}(2nx + 1)}{\lambda(2nx + 1)} x^5 + 64 \frac{\lambda^{(4)}(2nx + 1) + 3\lambda^{(4)}(2nx + 1)}{\lambda(2nx + 1) n} x^3 + 4 \frac{24\lambda^{(4)}(2nx + 1) + 48\lambda^{(4)}(2nx + 1) + 7\lambda^{(4)}(2nx + 1)}{\lambda(2nx + 1) n^2} x^2 + 2 \frac{32\lambda^{(4)}(2nx + 1) + 120\lambda^{(4)}(2nx + 1) + 64\lambda^{(4)}(2nx + 1) + \lambda''(2nx + 1)}{\lambda(2nx + 1) n^3} x + 16 \frac{\lambda^{(4)}(2nx + 1) + 6\lambda^{(4)}(2nx + 1) + 7\lambda^{(4)}(2nx + 1) + \lambda''(2nx + 1)}{\lambda(2nx + 1) n^4},
\]

(13)
Proof. Comparing the relations (9) and (10), we establish the first equality of (13). Differentiating the statement (10) with respect to $\omega$, we get

$$\sum_{v=1}^{\infty} q_v (x) \nu \omega^{v-1} = 2\lambda' \left(2x\omega + \omega^2\right) [x + \omega],$$

$$\sum_{v=2}^{\infty} q_v (x) \nu (v-1) \omega^{v-2} = 4\lambda'' \left(2x\omega + \omega^2\right) [x + \omega] + 2\lambda' \left(2x\omega + \omega^2\right),$$

$$\sum_{v=3}^{\infty} q_v (x) \nu (v-1) (v-2) \omega^{v-3} = 8\lambda''' \left(2x\omega + \omega^2\right) [x + \omega]^3 + 12\lambda'' \left(2x\omega + \omega^2\right) [x + \omega],$$

$$\sum_{v=4}^{\infty} q_v (x) \nu (v-1) (v-2) (v-3) \omega^{v-4} = 16\lambda^{(4)} \left(2x\omega + \omega^2\right) [x + \omega]^4 + 48\lambda''' \left(2x\omega + \omega^2\right) [x + \omega]^2$$

$$+ 12\lambda'' \left(2x\omega + \omega^2\right).$$

To obtain the required equalities, we replace $x$ with $nx$ and $\omega$ with $1$ in the above statements. Finally, considering (14) in the definition of operators given by (9) will conclude the proof of lemma. \qed

Lemma 2.2. With $\lambda$ and $T_n$ as above, we have

$$T_n (s - x) = 2^{\lambda'} \frac{(2nx + 1)}{\lambda (2nx + 1)} + 1 \left[ x + nx \frac{(2nx + 1)}{\lambda (2nx + 1)} \right] + \frac{\lambda'' \left(2nx + 1\right)}{\lambda (2nx + 1) n} x$$

$$T_n \left( (s - x)^2 \right) = 4^{\lambda''} \frac{(2nx + 1)}{\lambda (2nx + 1)} + 1 \left[ x^2 + nx \frac{\lambda''' \left(2nx + 1\right)}{\lambda (2nx + 1) n} \right] + 2 \lambda' \left(2nx + 1\right) x$$

$$+ 2^{\lambda''} \frac{\lambda'' \left(2nx + 1\right)}{\lambda (2nx + 1) n^2} x^2$$

$$T_n \left( (s - x)^4 \right) = 8^{\lambda''(4)} (2nx + 1) - 4\lambda''' \left(2nx + 1\right) + 3\lambda'' \left(2nx + 1\right) - \lambda' \left(2nx + 1\right) + 1 \left[ x^4 + nx \frac{\lambda'''' \left(2nx + 1\right)}{\lambda (2nx + 1) n} \right]$$

$$+ 4 \frac{\lambda'' \left(2nx + 1\right)}{\lambda (2nx + 1) n^2} x^3$$

$$+ 2^{\lambda''(4)} (2nx + 1) + 24\lambda'''' \left(2nx + 1\right) - 23\lambda''' \left(2nx + 1\right) + 4\lambda' \left(2nx + 1\right) x^2$$

$$+ 2 \frac{32^{\lambda''(4)} \left(2nx + 1\right)}{\lambda (2nx + 1) n^3} x^3$$

$$+ 4 \frac{\lambda'' \left(2nx + 1\right)}{\lambda (2nx + 1) n^2} x^2 + 7\lambda''' \left(2nx + 1\right) + \lambda' \left(2nx + 1\right).$$

Proof. Because of the linearity of $T_n$, we can write

$$T_n (s - x) = T_n (s; x) - xT_n (1; x).$$

In a similar manner, we may obtain

$$T_n \left( (s - x)^2 \right) = T_n \left( s^2; x \right) - 2sT_n (s; x) + x^2 T_n (1; x)$$

and

$$T_n \left( (s - x)^4 \right) = T_n \left( s^4; x \right) - 4sT_n (s; x) + 6s^2 T_n \left( s^2; x \right) - 4x^3 T_n (s; x) + x^4 T_n (1; x).$$

Substituting the equalities given by (13) in the above statements, we conclude the desired result. \qed
are deduced. When we take into account the infinite series in (17), then the following inequality
\[ \lim_{y \to \infty} \frac{\lambda'(y)}{\lambda(y)} = \frac{1}{2} \quad \text{and} \quad \lim_{y \to \infty} \frac{\lambda''(y)}{\lambda(y)} = \frac{1}{4} \]  
(15)
are provided. So, for \( f \in C[0, \infty) \cap E \)
\[ \lim_{n \to \infty} T_n(f; x) = f(x) \]
converges uniformly on every bounded and closed subset of \([0, \infty)\).

Proof. Keep in view (15) in Lemma 2.1, we infer for \( i = 0, 1, 2 \)
\[ \lim_{n \to \infty} T_n(x^i; x) = x^i \]
converge uniformly on every bounded and closed subset of \([0, \infty)\). Putting into practice the Korovkin-type property from [3], the proof is completed. \( \square \)

Remark 2.4. The following examples serve to illustrate some functions which satisfy conditions (15):
(a) For \( i \in \mathbb{N} \), \( \lambda(y) = y^{i+1} \),
(b) For \( q \) is any polynomial with positive coefficients, the functions \( \lambda(y) = q(y) \exp\left(\frac{y}{2}\right) \).

If we make use of the representation \( \hat{C}[0, \infty) \) for the class of uniformly continuous functions on \([0, \infty)\), for \( \varphi \in \hat{C}[0, \infty) \)
\[ w(q; \rho) := \sup \{ |\varphi(x) - \varphi(y)| : x, y \in [0, \infty), |x - y| \leq \rho \} \]
is titled by modulus of continuity attached to the function \( \varphi \).

Theorem 2.5. Suppose that \( f \in \hat{C}[0, \infty) \cap E \), then
\[ ||T_n(f; x) - f(x)|| \leq 2w\left(f; \left[T_n\left(s-x^2; x\right)\right]^{1/2}\right). \]

Proof. In compliance with (13) and also taking advantage of the (16),
\[ ||T_n(f; x) - f(x)|| \leq \left(1 + \frac{1}{\rho} \frac{1}{\lambda(2nx+1)} \sum_{y=0}^{\infty} q_{y}(nx) \left|\frac{V}{h} - x\right|^2\right)w\left(f; \rho\right) \]  
(17)
are deduced. When we take into account the infinite series in (17), then the following inequality
\[ \sum_{y=0}^{\infty} q_{y}(nx) \left|\frac{V}{h} - x\right|^2 \leq \left\{ \sum_{y=0}^{\infty} q_{y}(nx) \right\}^{1/2} \left\{ \sum_{y=0}^{\infty} q_{y}(nx) \left|\frac{V}{h} - x\right|^2 \right\}^{1/2} \]
\[ = \lambda(2nx+1) \left\{ \left( \frac{4\lambda''(2nx+1) - \lambda'(2nx+1)}{\lambda(2nx+1)} + 1 \right) x^2 + 2\frac{4\lambda''(2nx+1) - \lambda'(2nx+1)}{\lambda(2nx+1)n} x + 4\frac{\lambda''(2nx+1) + \lambda'(2nx+1)}{\lambda(2nx+1)n^2} \right\}^{1/2} \]
is derived from well known Hölder inequality. Considering the last inequality in (17),
\[
|T_n(f; x) - f(x)| \leq \left\{ 1 + \frac{1}{\rho} \left[ T_n((s - x)^2; x) \right]^{1/2} \right\} w(f; \rho)
\]
can be composed. With choosing \( \rho = \left[ T_n((s - x)^2; x) \right]^{1/2} \) in the above expression, we get the conclusion. \( \square \)

For \( 0 < \gamma \leq 1 \) and \( M > 0 \),
\[
\text{Lip}_M(\gamma) := \{ \varphi \in C_B[0, \infty) : |\varphi(x) - \varphi(y)| \leq M|x - y|^\gamma \}
\]
is called Lipschitz class of order \( \gamma \), where \( C_B[0, \infty) \) represents the space of all continuous and bounded functions on \([0, \infty)\).

Results of Theorem 2.6 can be used to estimate for the approximation error of \( T_n \) to function \( \varphi \) in \( \text{Lip}_M(\gamma) \).

**Theorem 2.6.** Consider the function \( \varphi \) in \( \text{Lip}_M(\gamma) \), so then
\[
|T_n(\varphi; x) - \varphi(x)| \leq M \left[ T_n((s - x)^2; x) \right]^{\gamma/2}.
\]

**Proof.** Due to the fact that \( T_n \) operators are monotone,
\[
|T_n(\varphi; x) - \varphi(x)| \leq M T_n((s - x)^\gamma; x)
\]
can be expressed. If well known Hölder inequality could be applied, the above argument shows that
\[
|T_n(\varphi; x) - \varphi(x)| \leq M \frac{1}{\lambda (2nx + 1)} \left[ \sum_{v=0}^{\infty} q_v(nx) \right]^{\frac{2}{\gamma}} \left[ \sum_{v=0}^{\infty} q_v(nx) \left( \frac{v}{n} - x \right)^2 \right]^{\frac{\gamma}{2}}
\]
\[
\leq M \left[ T_n((s - x)^2; x) \right]^{\gamma/2},
\]
which proves the result. \( \square \)

Let us denote the set of all continuous and bounded functions on \([0, \infty)\) that have a continuous and bounded first and second derivative by \( C_B^2[0, \infty) \). The following functionals
\[
\|\varphi\|_{C^1[0, \infty)} = \sup_{x \in [0, \infty)} |\varphi(x)|,
\]
\[
\|\varphi\|_{C^2[0, \infty)} = \|\varphi\|_{C^1[0, \infty)} + \|\varphi'\|_{C[0, \infty)} + \|\varphi''\|_{C[0, \infty)}
\]
are a norm, respectively, on \( C_B[0, \infty) \) and \( C_B^2[0, \infty) \). Let \( \varphi \in C_B[0, \infty) \). In order to obtain the estimations we are going to define \([8]\)
\[
w_2(\varphi; \rho) = \sup_{0 < t \leq \rho} \|\varphi(\cdot + 2t) - 2\varphi(\cdot + t) + \varphi(\cdot + t)\|_{C[0, \infty)},
\]
which is called the second modulus of continuity attached to the function \( \varphi \) and
\[
\mathcal{K}(\varphi; \rho) = \inf_{\psi \in C_B^2[0, \infty)} \left\{ \|\varphi - \psi\|_{C[0, \infty)} + \rho \|\varphi''\|_{C[0, \infty)} \right\},
\]
which is called the functional of Peetre attached to the function \( \varphi \).
Theorem 2.7. Consider the function $f$ in $C_b^2 [0, \infty)$, so then
\[
|T_n (f; x) - f (x)| \leq \sigma_n (x) \| f \|_{C_b^2 [0, \infty)},
\]
where
\[
\sigma_n (x) = \left\{ \frac{4 \lambda'' (2nx + 1) - \lambda' (2nx + 1)}{2\lambda (2nx + 1)} + \lambda (2nx + 1) \right\} x^2 + \frac{4 \lambda'' (2nx + 1) + [2n - 1] \lambda' (2nx + 1) - n \lambda (2nx + 1)}{n \lambda (2nx + 1)} x
\]
\[
+ \frac{2 \lambda'' (2nx + 1) + [n + 1] \lambda' (2nx + 1)}{n^2 \lambda (2nx + 1)}. \]

Proof. To obtain the mentioned inequality, as a beginning we use the linearity of $T_n$ and then apply standard Taylor expansion of $f$
\[
|T_n (f; x) - f (x)| \leq \left| f' (x) \right| T_n (s - x; x) + \frac{|f'' (\eta)|}{2} T_n ((s - x)^2; x), \quad \eta \in (x, s).
\]
The result
\[
|T_n (f; x) - f (x)| \leq \left\{ \frac{4 \lambda'' (2nx + 1) - \lambda' (2nx + 1)}{2\lambda (2nx + 1)} + \lambda (2nx + 1) \right\} x^2 + \frac{4 \lambda'' (2nx + 1) + [2n - 1] \lambda' (2nx + 1) - n \lambda (2nx + 1)}{n \lambda (2nx + 1)} x
\]
\[
+ \frac{2 \lambda'' (2nx + 1) + [n + 1] \lambda' (2nx + 1)}{n^2 \lambda (2nx + 1)} \right\} \| f \|_{C_b^2 [0, \infty)}
\]
is obtained after the elementary calculation. \( \Box \)

Theorem 2.8. Consider the function $\varphi$ in $C_b [0, \infty)$, so then
\[
|T_n (\varphi; x) - \varphi (x)| \leq C w_2 (\varphi; \sqrt{\kappa_n (x)}) + w \left( \varphi; \left( \frac{2 \lambda' (2nx + 1)}{\lambda (2nx + 1)} - 1 \right) x + 2 \frac{\lambda' (2nx + 1)}{\lambda (2nx + 1) n} \right)
\]
where
\[
\kappa_n (x) = \left\{ T_n ((s - x)^2; x) + \left[ 2 \frac{\lambda' (2nx + 1)}{\lambda (2nx + 1)} - 1 \right] x + 2 \frac{\lambda' (2nx + 1)}{\lambda (2nx + 1) n} \right\}.
\]
and $C$ is constant.

Proof. We start with defining an operator $\mathcal{H}_n$ by
\[
\mathcal{H}_n (\varphi; x) = T_n (\varphi; x) - \varphi \left( 2 \left[ \frac{\lambda' (2nx + 1)}{\lambda (2nx + 1)} x + \frac{\lambda' (2nx + 1)}{\lambda (2nx + 1) n} \right] \right) + \varphi (x).
\]
The result
\[
\mathcal{H}_n (s - x; x) = 0
\]
can be deduced from Lemma 2.2. For this purpose, let us represent the function \( \psi \in C^2_B[0, \infty) \) in the form

\[
\psi(s) = \psi(x) + (s - x) \psi'(x) + \int_x^s (s - u) \psi''(u) \, du.
\]

Bearing in mind the above equality and (19)

\[
\left\| \mathcal{H}_n (\psi; x) - \psi(x) \right\| = \left\| \mathcal{H}_n \left( \int_x^s (s - u) \psi''(u) \, du; x \right) \right\| 
\leq \left\| \mathcal{T}_n \left( \int_x^s (s - u) \psi''(u) \, du; x \right) \right\| 
+ \left\| \mathcal{H}_n \left( \int_x^s (s - u) \psi''(u) \, du; x \right) \right\|
\leq \frac{4\kappa_n(x) ||\psi''||_{C_0[0,\infty)}}{x^2} + \left( \frac{4\kappa_n(x) ||\psi''||_{C_0[0,\infty)}}{x^2} \right)^2
\leq 4\kappa_n(x) ||\psi''||_{C_0[0,\infty)}
\]

(20)
can be written. Owing to the fact Lemma 2.1 and (20)

\[
\left\| \mathcal{T}_n (\varphi; x) - \varphi(x) \right\| \leq \left\| \mathcal{H}_n (\varphi - \psi; x) - (\varphi - \psi)(x) \right\| + \left\| \mathcal{H}_n (\psi; x) - \psi(x) \right\|
\leq \left\| \varphi \left( 2 \frac{\lambda'(2nx+1)}{\lambda(2nx+1)} x + \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n} \right) - \varphi(x) \right\|
\leq 4 \|\varphi - \psi\|_{C_0[0,\infty]} + 4\kappa_n(x) ||\psi''||_{C_0[0,\infty)}
\]

+ \psi \left( 2 \frac{\lambda'(2nx+1)}{\lambda(2nx+1)} x + \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n} \right) x + 2 \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}
\]

can be therefore hold. We can thus see in the light of the property of Peetre’s K functional

\[
\left\| \mathcal{T}_n (\varphi; x) - \varphi(x) \right\| \leq 4K (\varphi; \kappa_n(x)) + \psi \left( 2 \frac{\lambda'(2nx+1)}{\lambda(2nx+1)} x + \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n} \right) x + 2 \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}
\leq C\varphi \left( 2 \frac{\lambda'(2nx+1)}{\lambda(2nx+1)} x + \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n} \right) x + 2 \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}.
\]

This completes the proof. \( \square \)

Prior to stating and proving the weighted approximation, we first state below the family of functions

\[
B_\phi [0, \infty) = \left\{ f : \|f(x)\| \leq M_\phi \phi(x), M_\phi \text{ is a constant} \right\},
\]

and as a subset of \( B_\phi [0, \infty) \) the following class is defined by

\[
C_\phi [0, \infty) = B_\phi [0, \infty) \cap C [0, \infty),
\]

and as a subset of \( C_\phi [0, \infty) \) the following class is defined by

\[
C^k_\phi [0, \infty) = \left\{ f : f \in C_\phi [0, \infty) \text{ and } \lim_{x \to \infty} \frac{f(x)}{\phi(x)} = k \text{ (a constant)} \right\},
\]

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\]
where \( \phi(x) = 1 + x^2 \). The remarkable aspect of this family is that the following functional

\[
\|f\|_\phi = \sup_{x \geq 0} \frac{|f(x)|}{\phi(x)}
\]

is a norm on \( B_\phi[0, \infty) \).

**Theorem 2.9.** If \( f \in C^k_\phi[0, \infty) \) is the case then

\[
\lim_{n \to \infty} \|T_n(f; x) - f(x)\|_\phi = 0.
\]

**Proof.** Considering the identities proved in Lemma 2.1, it immediately follows that

\[
\lim_{n \to \infty} \|T_n(1; x) - 1\|_\phi = 0
\]

and

\[
\lim_{n \to \infty} \|T_n(s; x) - x\|_\phi = 0.
\]

A combination of Lemma 2.1 and the definition of norm yields for given \( \epsilon > 0 \) by choosing \( n \in \mathbb{N} \) sufficiently large

\[
\sup_{x \in \mathbb{R}^+} \left| T_n(s^2; x) - x^2 \right| \leq \epsilon \sup_{x \in \mathbb{R}^+} \frac{4x^2 + \frac{16}{27}x + \frac{3}{n^7}}{1 + x^2} + \frac{3}{n^2} \sup_{x \in \mathbb{R}^+} \frac{x}{1 + x^2} + \frac{3}{n^2} \sup_{x \in \mathbb{R}^+} \frac{1}{1 + x^2}.
\]

This proves \( \lim_{n \to \infty} \|T_n(s^2; x) - x^2\|_\phi = 0 \). These facts enable us to complete the proof by the weighted Korovkin-type theorem [10]. \( \square \)

We next take up the topic of quantitative result for weighted space [13]. To obtain quantitative approximation result we will need to use

\[
\Omega(f; \rho) := \sup_{x \in [0, \infty), |h| \leq \rho} \left| \frac{f(x + h) - f(x)}{(1 + h^2)(1 + x^2)} \right|, f \in C^k_\phi[0, \infty).
\]

To prove the result we require the following expression

\[
\lim_{\rho \to 0} \Omega(f; \rho) = 0,
\]

\[
\left| f(t) - f(x) \right| \leq 2 \left( 1 + \frac{|t - x|}{\rho} \right) \left( 1 + \rho^2 \right) \left( 1 + x^2 \right) \left( 1 + (t - x)^2 \right) \Omega(f; \rho), t, x \in [0, \infty). \quad (21)
\]

The following theorem, under the suitable restriction

\[
\lim_{y \to \infty} \frac{\lambda^{(k)}}{\lambda(y)} = \frac{1}{2^k}, k = 1, 2, 3, 4
\]

on \( \lambda \), provides an error estimate.

**Theorem 2.10.** Consider the function \( f \) in \( C^k_\phi[0, \infty) \), so then

\[
\sup_{x \in [0, \infty)} \left| T_n(f; x) - f(x) \right| \leq K \left( 1 + \frac{1}{n} \right) \Omega(f; \frac{1}{\sqrt{n}})
\]

holds for some constant \( K \) which is independent of \( n \).
Proof. It follows directly from Lemma 2.1 and expression (21) that

\[
\left| T_n (f; x) - f (x) \right| \leq \frac{1}{\lambda (2nx + 1)} \sum_{v=0}^{\infty} q_v (nx) \left| f \left( \frac{v}{n} \right) - f (x) \right|
\]

\[
\leq 2 \left( 1 + r^2 \right) (1 + x) \Omega (f; \rho) \frac{1}{\lambda (2nx + 1)} \sum_{v=0}^{\infty} q_v (nx) \left( 1 + \left| \frac{v}{n} - x \right| \right) \left( 1 + \left( \frac{v}{n} - x \right)^2 \right)
\]

\[
= 2 \left( 1 + r^2 \right) (1 + x) \Omega (f; \rho) \frac{1}{\lambda (2nx + 1)} \left( \sum_{v=0}^{\infty} q_v (nx) + \sum_{v=0}^{\infty} q_v (nx) \left( \frac{v}{n} - x \right)^2 \right)
\]

\[
+ \frac{1}{\rho} \sum_{v=0}^{\infty} q_v (nx) \left| \frac{v}{n} - x \right| + \frac{1}{\rho} \sum_{v=0}^{\infty} q_v (nx) \left| \frac{v}{n} - x \right| \left( \frac{v}{n} - x \right)^2 \right).
\]

If well known Hölder inequality could be applied for the above series, the mentioned relation therefore gives

\[
\left| T_n (f; x) - f (x) \right| \leq 2 \left( 1 + r^2 \right) (1 + x) \Omega (f; \rho)
\]

\[
\times \left( 1 + T_n ((s - x)^2; x) + \frac{1}{\rho} \sqrt{T_n ((s - x)^2; x)} \right)
\]

\[
+ \frac{1}{\rho} \sqrt{T_n ((s - x)^2; x)} T_n ((s - x)^4; x) \right).
\]

(22)

Taking into consideration Lemma 2.2, we readily find

\[
T_n ((s - x)^2; x) \leq O (1) (x^2 + x + 1),
\]

\[
T_n ((s - x)^4; x) \leq O \left( \frac{1}{n} \right) (x^4 + x^3 + x^2 + x + 1). \]

If these expressions are combined with (22) and by choosing \( \rho = \frac{1}{\sqrt{n}} \), we therefore have

\[
\sup_{x \in [0, \infty)} \left| T_n (f; x) - f (x) \right| \leq K \left( 1 + \frac{1}{n} \right) \Omega \left( f; \frac{1}{\sqrt{n}} \right).
\]

\( \square \)

3. Example

In the present section, our plan is to obtain an explicit example of operators (9) including certain set of polynomials ensuring all restrictions (12) and assumptions (15).

Example 3.1. Generating function of the Hermite polynomials \( H_k^{(v)} \) of variance \( v \) is given by [24]

\[
e^{-\frac{x^2}{4}} = \sum_{k=0}^{\infty} \frac{H_k^{(v)} (x)}{k!} \omega^k,
\]

where \( H_k^{(v)} \) can be calculated by the following sum

\[
H_k^{(v)} (x) = \sum_{r=0}^{\lfloor k/2 \rfloor} \left( \frac{1}{2} \right)^r \frac{k!}{r!(k-2r)!} x^{k-2r}.
\]
Let $\lambda(\omega) = e^{\omega}$. One can show that the polynomials $H^{(\nu)}_k$ satisfy the relation (10) for $\lambda$. The polynomials $H^{(\nu)}_k$ with $\nu < 0$ will satisfy the expressions (12) and (15). Hence, the explicit form of $T_n$ can be obtained by applying (10)

$$
T_n(f;x) = e^{-(nx+1)} \sum_{k=0}^{\infty} \frac{H^{(\nu)}_k}{k! (-\nu)^{\frac{k}{2}}} f\left(\frac{k}{n}\right).
$$

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