Finite temperature structure factor in the Haldane-Shastry spin chain

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The Haldane-Shastry spin chain can be mapped to the infinite coupling limit of the SU(2) spin Calogero-Sutherland model. We use the $gl_2$ Jack polynomials’ technology to compute the form factors of the spin operator on the multi-spinon spectrum. The spin structure factor is obtained through a form factor expansion. The expansion is proven to converge in the small momentum limit. Numerics based on two- and four-spinons contributions give an approximate result for the infinite temperature static and dynamic spin structure factor.

I. INTRODUCTION

Low-dimensional systems constitute fertile breeding grounds for exotic types of physical excitations. Fractionalization of quantum numbers like charge and spin is known to take place respectively in one-dimensional interacting electron liquids and spin chains: in cases such as these, one must forget about weakly coupled particles, and instead adopt a whole new starting point for the description of the strongly coupled physics. Obviously, the identification and proper description of this new starting point is often a very involved and risk-prone process.

In this respect, quasi-one-dimensional spin systems have provided one of the sturdiest arenas. Experimental realizations of systems with fractionalized excitations are numerous and well-documented. Probably the clearest and best studied signature comes from neutron scattering experiments on effectively one-dimensional antiferromagnetic spin-1/2 chains [1]. The excitations seen are not the naively expected spin-1 spin waves, but rather gapless “spinons”, which one could loosely present as spin-1/2 spin waves. Among many remarkable properties of these excitations are their fractional statistics, intermediate between fermions and bosons, making such a system markedly different from one obeying conventional rules.

On the theoretical side, strongly-coupled systems like spin chains have in the last few decades presented extreme, if not seemingly insurmountable difficulties. The simplest way of explaining this fact might be to say that quantum fluctuations are very strong in one-d, and cannot be tamed by perturbative approaches. Instead, excitations are strongly nonlinear, and one is faced with the seemingly impossible challenge of either providing an exact solution or risking to miss out completely on the correct physics.

The quantity of interest to experimentalists (thinking about neutron scattering experiments) is the dynamical spin structure factor (DSSF). For a chain of $N$ spins at sites $R_i$, this is defined as

$$S^{\alpha\beta}(q, \omega; T) = \frac{1}{2\pi N} \sum_{i,j} e^{iq(R_j-R_i)} \int_{-\infty}^{\infty} e^{i\omega t} \langle S^\alpha_i(0) S^\beta_j(t) \rangle_T$$

where the angular brackets denote a thermal average.

The model of choice for the description of the spin dynamics depends of course on the specifics of the experimental setup one wishes to describe. The XXZ Heisenberg model [2] often fits the bill remarkably well, at least for very low energies. The Bethe Ansatz method [3] could provide most of its thermodynamic properties, but little about its dynamics. Approximate methods have thus been used to address the computation of the DSSF. The Müller Ansatz [4] is the best conjecture for the zero temperature structure factor based on exact results and numerics. Finite temperature low-energy features were obtained by Schultz using bosonization [5].

To go beyond the field theory limit requires tackling the nonlinear nature of the original model. Considerable insight in this direction was provided by the Algebraic Bethe Ansatz method [6] and the quantum inverse scattering theory [7]. Bougourzi et al. used results from the algebraic analysis to compute the exact two-spinon contribution to the DSSF of the one-dimensional Heisenberg model [8, 9, 10]. More recently, Maillet et al proved multiple integral representations of elementary blocks of the correlation functions [11]. Nevertheless, all these approaches restrict to zero temperature and no exact thermodynamic limit is known. The computation of the DSSF at finite temperature requires a lot of further efforts.

However, one of the properties of the Heisenberg model is that the spinons, though deconfined, still suffer from a residual interaction. The spinons are thus not truly free excitations obeying fractional statistics. There exists on the other hand a very convenient alternative approach based on the Haldane-Shastry model, whose Hamiltonian is

$$H_{HS} = J \sum_{i<j} \frac{1}{[d(i-j)]^2} S_i \cdot S_j$$
where \( d(i) = \frac{N}{2} \sin \frac{\pi i}{N} \). It is in the same universality class as the Heisenberg model, and the long-distance (decaying as \( 1/R^2 \) for large distances) interaction in fact simplifies things considerably: the spinons form an ideal gas \([14]\) of particles obeying fractional exclusion statistics \([15]\). The DSSF at \( T = 0 \) can in fact be calculated exactly, and is given in the thermodynamic limit by \([16]\)

\[
S_{\mu \nu}^{\beta}(q, \omega) = \frac{\delta_{\mu \beta}}{2} \Theta(\omega_2(q_\parallel) - \omega) \Theta(\omega - \omega_1(\omega_1(-q_\parallel))) \Theta(\omega - \omega_1(q_\parallel)) \left( \frac{1}{\sqrt{(\omega - \omega_1(-q_\parallel))(\omega - \omega_1(q_\parallel))}} \right),
\]

\[
\omega_1(-q_\parallel) = \frac{J}{2} q_\parallel (\pi - q_\parallel), \quad \omega_1(q_\parallel) = \frac{J}{2} (q_\parallel - \pi) (2\pi - q_\parallel), \quad \omega_2(q_\parallel) = \frac{J}{4} q_\parallel (2\pi - q_\parallel).
\]

This \( T = 0 \) formula is made up only of contributions from the two-spinon channel: all higher channels have vanishing contributions in the zero-temperature limit.

One of the very nice features of the Haldane-Shastry model is that its dynamics turn out to be much more easily tractable than those of the Heisenberg chain. The Haldane-Shastry model is but the first representative in a wider class of solvable models dubbed the SU(N) Haldane-Shastry chains. These are in turn obtainable as a particular limit of spin Calogero-Sutherland models, for which an impressive number of exact results are known in the mathematical literature. In particular, there exists a Yangian symmetry leading to the identification of a set of eigenvectors of spin Calogero-Sutherland models, for which an impressive number of exact results are known in the mathematical literature. More details on this will be provided in the bulk of the paper.

Thus, this opens the way to the computation of the DSSF \([19]\) at nonzero temperatures for the Haldane-Shastry model. Our strategy will be to make use of the technology contained in \([17]\) to compute the form factors involved in the spin-spin correlation function needed for the DSSF. This approach has already been used for \( T = 0 \) dynamical properties of the Haldane-Shastry spin chain \([18, 19]\) and related models (spin Calogero-Sutherland model \([20]\), supersymmetric \( t-J \) model \([21, 22]\)). The present work is the first to address finite temperature dynamics.

However for a generic quantum field theory divergences appear when developing a correlation function on the Hilbert space. In the context of integrable field theories (ITF), it has been proposed that it could be rewritten as a sum free of divergences. The resulting formula, called a ‘form factor expansion’ (FFE), can be evaluated using the scattering data of the ITF. There is still on ongoing discussion about how precisely the FFE can be implemented in ITFs \([23, 24, 25, 26, 27, 28]\). For the case of conformal field theory (CFT), there has been a similar but independent proposal for writing finite-temperature correlators in a FFE \([29]\). It is based on the fractional statistics of the quasiparticles building the Hilbert space. Evidence was put forward that it converges quickly to the exact (known) result in terms of the number of excited quasi-particles \([24, 29]\). We will prove in the paper that the latter approach applies in the case of the Haldane-Shastry spin chain.

Our paper is organized as follows. First, we recall all the necessary aspects of the computation of form factors for the Haldane-Shastry model using Jack polynomials. We then set out to calculate the form factors themselves, in increasing complexity of spinon channels. The form factor expansion is then introduced and proved. We finally put the results together to provide an expression for the finite-temperature static and dynamical spin structure factors. Discussions and conclusions are amassed at the end.

II. JACK POLYNOMIAL TECHNOLOGY FOR THE SPIN CALOGERO-SUTHERLAND MODEL

Using the freezing trick \([31]\) the SU(2) Haldane-Shastry model is obtained by the strong coupling limit of the SU(2) spin Calogero-Sutherland model \([32]\). The latter describes \( N \) particles with coordinates \( \{x_i, i = 1 \ldots N\} \) moving on a circle of length \( N \) with the Hamiltonian

\[
H_{\text{spinCS}} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\pi^2}{2N^2} \sum_{i \neq j} \frac{\beta (\beta + P_{ij})}{\sin^2 \left( \frac{\pi}{N} (x_i - x_j) \right)},
\]

where \( P_{ij} \) is the SU(2) exchange between particles \( i \) and \( j \). Within the freezing trick, the interaction parameter \( \beta \) is taken to infinity. The particles are therefore pinned at \( x_i = i \) and interact via the SU(2) spin exchange.

The spin Calogero-Sutherland model is more tractable than the Haldane-Shastry model since it is continuous and all its eigenfunctions have been explicitly constructed. We recall in the following the Uglov construction \([17]\) in terms of \( gl_N \) Jack polynomials. The mathematical technology they provide to compute transition matrix elements is then introduced.
A. Yangian Gelfand-Zetlin basis

We consider the case $N$ even and $N/2$ odd, so that the ground state is unique. Uglov determined the so-called Yangian Gelfand-Zetlin basis of this model, orthogonal through the Yangian action. They are labelled by the strictly-decreasing sequences $k = \{k_i, i = 1 \ldots N\}$, $k^0 = \{N/2 + 2 - i, i = 1 \ldots N\}$ corresponding to the ground state. $k$ contains information on both momentum and spin of the excitation. Writing $k_i = 2k_i^- + k_i^-, k_i^\pm \in \mathbb{Z}$ represents momentum and $k_i^\pm \in \{1, 2\}$ color. More precisely, the momentum, the energy and spin of a state described by $k$ are

$$P_k = \frac{2\pi}{N} \sum_i k_i^-$$

$$E_k = \frac{2\pi^2}{N^2} \sum_i [k_i^- + \beta(N + 1 - 2i)/2]^2$$

$$S_k = \frac{1}{2} \sum_i [\delta_{k_i,2} - \delta_{k_i,1}]$$

For physical applications it is more convenient to work with excitations over the ground state. One identifies the state $k$ with the pair

$$k \equiv (\lambda = (k_i - k_N + i - N, i = 1 \ldots N - 1), r = k_N - k_N^0).$$

It consists of a zero mode $r$ and a partition $\lambda$ (non-increasing sequence of positive integers) of length $N - 1$. The Hilbert space is spanned by all possible pairs. We refer the reader to [17] for the expressions of the physical properties in terms of $(\lambda, r)$. They will be specified in the next section for the specific case of the Haldane-Shastry spin chain.

B. Uglov’s isomorphism

Uglov determined an isomorphism $\Omega$ between the Yangian Gelfand-Zetlin basis and the $\mathfrak{gl}_2$ Jack Polynomials defined through

$$\Omega(k) = (x_1 \ldots x_n)^\ast P^{(2\beta + 1,2)}_\lambda(\{x_i\})$$

where the Jack polynomial $P^{(\gamma,2)}_\lambda$ is the limit $q = -p$, $t = -p^\gamma$, $p \to 1$ of the Macdonald polynomial $P_\lambda(q,t)$. Then one has

$$(k,l)_{(\beta,2)} = \langle \Omega(k), \Omega(l) \rangle_{(\beta,2)}.$$  \hspace{0.5cm} (10)

$(\ldots)_{(\beta,2)}$ is the Yangian scalar product (see [17] for a definition), and $(\ldots)_{\beta,2}$ is the following scalar product in the space of symmetric Laurent polynomials

$$\langle f(\{x_i\}), g(\{x_i\}) \rangle_{\beta,2} = \frac{1}{N!} \prod_{j=1}^N \int \frac{dx_j}{2\pi i x_j} f(\{x_i\}) \left[ \prod_{1 \leq k \neq l \leq N} (1 - x_k^2x_l^{-2})^\beta (1 - x_k x_l^{-1}) \right] g(\{x_i\}).$$  \hspace{0.5cm} (11)

The physical quantity we study in this work is the action of the spin operator. Using Uglov’s isomorphism, it is given by (we dropped the scalar product indices for convenience)

$$(\lambda,r|s^\pm|\mu,r') = \frac{1}{N} \sum_{s \in \mathbb{Z}} \langle \lambda,r | p_{2s+1} | \mu,r' \rangle \delta(S_\lambda - S_\mu = \pm 1),$$

where $p_m = \sum_{i=1}^N x_i^m$ is the power sum symmetric function. One can identify $s^\pm$ with $(s^+ + s^-)/2$ and replace the $\delta$ by $1/2\delta(|S_\lambda - S_\mu| = 1)$. In this paper we only study form factors which satisfy this selection rule. By symmetry

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1 We use here a notation different than Uglov. In fact, this corresponds to the dual representation, called * in his paper. Results are not altered, as one has to sum both representations to obtain physical quantities. Our choice for this representation is mostly practical, leading to states with positive momentum when $\sigma_i$ is positive.
between the contributions of \( p_m \) and \( p_{-m} \) (or equivalently the duality relation (see [17]), one may finally only consider the form factors

\[
(\lambda, r|s^\pm|\mu, r') \equiv \frac{1}{N} \sum_{s \in \mathbb{Z}^+} \langle \lambda|p_{2s+1}|\mu \rangle \delta_{r,r'} \delta(|S_\lambda - S_\mu| = 1).
\]

(13)

C. \( \mathfrak{g}_2 \) Jack technology

To compute transition matrix elements such as (13), we need some results from the mathematical literature on Macdonald polynomials [33].

We consider a generic partition as on Fig. 1. It is a tableau made of cases labeled by their row number \( i \) and their column number \( j \). The length (the number of cases) of the row \( i \) is \( \lambda_i \) and the length of the column \( j \) is \( \lambda'_j \). One then defines

Definitions for partitions :
- Length: \( l(\lambda) = \max(\lambda'_j) \);
- Cardinal: \( |\lambda| = \sum \lambda_i \) and we note \( \lambda \vdash |\lambda| \);
- Arm-length: \( a(s) = \lambda_i - j \);
- Leg-length: \( l(s) = \lambda'_j - i \);
- Arm-colength: \( a'(s) = j - 1 \);
- Leg-colength: \( l'(s) = i - 1 \);
- Content: \( c(s) = a'(s) - l'(s) \);
- Hook-length: \( h(s) = a(s) + l(s) + 1 \);
- \( C(\lambda) = \{ s \in \lambda | c(s) \equiv 0 \text{ mod } 2 \} \);
- \( H(\lambda) = \{ s \in \lambda | h(s) \equiv 0 \text{ mod } 2 \} \).

The quantities of interest in the present work are the norm of a given state, the expansion of power sums on Jack polynomials (thus on physical states), and finally the Pieri formula. The latter gives the development on Jack polynomials of the product of a Jack polynomial and an elementary function \( e_r = P_{1r}^{(\gamma,2)} \). This is the only formula available to compute form factors in the space of symmetric polynomials.

They read

\[
\langle P_\lambda^{(\gamma,2)}|P_\lambda^{(\gamma,2)} \rangle = \langle 1|1 \rangle \prod_{C(\lambda)} \frac{a'(s) + \gamma(N - l'(s))}{a'(s) + 1 + \gamma(N - l'(s) - 1)} \prod_{H(\lambda)} \frac{a(s) + 1 + \gamma l(s)}{a(s) + \gamma(l(s) + 1)}
\]

(14)
Expansion of power sums

\[ p_{2s+1} = \sum_{\lambda \vdash 2s+1} \chi_{\lambda} P^{(\gamma,2)}_{\lambda}, \]

\[ \chi_{\lambda} = (-)^{n(\lambda)} \frac{\prod_{C(\lambda) \setminus (1,1)} (a(s') - \gamma l'(s))}{\prod_{H(\lambda)} a(s) + 1 + \gamma l(s)} \text{ for } |C(\lambda)| = |H(\lambda)| + 1 \]

Pieri formula

\[ P^{(\gamma,2)}_{\mu} c_r = \sum_{\lambda} \psi'_{\lambda/\mu} P^{(\gamma,2)}_{\lambda}, \]

\[ \psi'_{\lambda/\mu} = \prod_{C_{\lambda/\mu} \setminus R_{\lambda/\mu}} \frac{b_{\lambda}(s)}{b_{\mu}(s)}, \]

\[ b_{\lambda} = \frac{a(s) + \gamma(l(s) + 1)}{a(s) + 1 + \gamma l(s)} \text{ if } s \in H(\lambda), 1 \text{ otherwise}, \]

with \( \lambda - \mu \) being a vertical \( r \)-strip (at maximum 1 box per row, for a total of \( r \)), \( C_{\lambda/\mu} \) (resp. \( R_{\lambda/\mu} \)) being the union of columns (resp. rows) that intersect \( \lambda - \mu \).

III. TRANSITION MATRIX ELEMENTS IN THE HALDANE-SHASTRY MODEL

We now specify this mathematical background to the case of the Haldane-Shastry spin chain. The \( \beta \to \infty \) limit simplifies a great deal the algebra. The eigenstates described above can be interpreted as multi-spinon states. We give their physical properties in the following. Then a general expression for the matrix elements of the spin operator is presented along with closed analytical result in the case of few-spinon states.

A. Spinon interpretation

In the Haldane-Shastry framework, all physical quantities rewrite in terms of sums over the columns \( \lambda'_j \) of the partition \( \lambda \):

\[ P_\sigma = \pi r + \pi \sum_j \frac{2w_j}{N} \]

\[ E_\sigma = \pi^2 \left[ (1 - (-)^j/N) \frac{2w_j}{N} - \left( \frac{2w_j}{N} \right)^2 + (1 - (-)^j)/2N \right] \]

\[ S_\sigma = \sum_j [\lambda'_j - 2w_j] \]

\[ w_j = \begin{cases} \left\lfloor \frac{\lambda'_j}{N} \right\rfloor & \text{for } j + r \text{ even} \\ \left\lceil \frac{\lambda'_j}{N} \right\rceil & \text{for } j + r \text{ odd} \end{cases} \]

At the thermodynamic limit (which only is of interest), distinctions between even and odd disappear for the momentum and the energy. Defining \( x_j = \pi \lambda'_j/N \in [0, \pi] \), they are

\[ P_\sigma = \pi r + \sum_j x_j \]

\[ E_\sigma = \sum_j x_j (\pi - x_j) \]

One recognizes the dispersion relation of spinons. An excitation can then be described by a zero mode \( r \) and a set of particles called spinon defined by each column of a tableau \( \lambda \). We will for the sake of simplicity write such a state \( \{m_j = \lambda'_j\}_r \). \( r \) can be discarded in most of the physical applications.
Now we can express all the Jack polynomial technology in terms of the spinons’ quantum numbers \(m_j\). We will use the short cut notations

\[
\gamma_{\text{even}}(m) = \frac{\Gamma([m/2] + 1)\Gamma(1/2)}{\Gamma([m/2] + 1/2)} \simeq \sqrt{\frac{Nx}{2}}
\]

\[
\gamma_{\text{odd}}(m) = \frac{m}{2\gamma_{\text{even}}(m)} \simeq \frac{1}{\pi} \sqrt{\frac{Nx}{2}}
\]

The norm is

\[
N_{\{m_j, j=1...n\}} \equiv \frac{\langle P_{\{m_j\}}|P_{\{m_j\}} \rangle_{\{1|1\}}}{\langle 1|1 \rangle} = \prod_{i=1}^{n} \frac{\gamma_{i+1}(N)}{\gamma_{i+1}(N-m_i)} \prod_{1 \leq i \leq j \leq n} \frac{\gamma_{i-j}(m_i - m_j)}{\gamma_{i-j}(m_i + m_j + 1)} \simeq \prod_{i} \sqrt{\frac{2\pi}{NE_i}}
\]

One shows that the power sum operators decompose into 2-spinon states \((m_1, m_2)\) such that \(m_1 + m_2 = 2s + 1\) with

\[
\chi_{(m_1, m_2)} = (-)^{m_2} \gamma_0(m_1 - m_2).
\]

Specifying the Pieri formula to \(\mu \equiv (m_1, \ldots, m_n)\), \(\lambda \equiv (p_1, \ldots, p_{n+1})\) — with possibly \(p_{n+1} = 0\) — it gives

\[
\psi_{\lambda/\mu} = \prod_{1 \leq i \leq j \leq n} \frac{\gamma_{i-j}(p_i - p_{j+1})\gamma_{i-j}(m_i - m_j)}{\gamma_{i-j}(p_i - m_j)\gamma_{i-j}(m_i - p_{j+1})}
\]

### B. Matrix elements

Different strategies apply to the evaluation of the transition matrix elements \(\langle \lambda|p_{2s+1}|\mu \rangle\). The first one is to write \(\mu\) and \(\lambda\) into elementary functions \(e\) by inversion of the Pieri formula [27], use the decomposition [15], and apply the Pieri formula successively on the multi-\(e\) state. This solution is quite unpractical, because inverting the Pieri formula becomes increasingly difficult with the number of spinons considered. It is trivial for 1 spinon, and leads to a rather cumbersome expression already for 2 spinons. Still, we can use it to rewrite the power sums as a sum of 2-\(e\) states. The result is interestingly simple

\[
p_{2s+1} = \frac{1}{2} \sum_{r=0}^{s} (-)^{s+r+1}(2r + 1)e_{s+r+1}e_{s-r}
\]

To prove it, one uses [17] on [31], it leads to [29] thanks to the equality

\[
1 = \frac{1}{2} \sum_{r=0}^{s} \frac{1}{\gamma_0(r)\gamma_0(s-r)}
\]

It gives way to a generic strategy: use decomposition [31] and use the Pieri formula twice. The result is (with normalized states)

\[
\langle \lambda|p_{2s+1}|\mu \rangle = \frac{1}{2} \sum_{\nu} (-)^{\lambda - |\nu| - |\mu|} \sqrt{\frac{N_{\lambda}}{N_\mu}}
\]

with \(|\lambda| - |\mu| = 2s + 1\).

We will now evaluate the contribution of several channels. Eq. [33] doesn’t give a closed analytic expression for the form factors. Only in a few cases can it be so reduced. The results may be conjectural, then confirmed through the following sum rule

\[
\sum_{m > 0} \langle \mu|p_m p_{-m}|\mu \rangle \xrightarrow{N \to \infty} |\mu|
\]

which can be proved by simple Jack polynomials’ algebra.
This is the only channel present at zero temperature. It has been conjectured by Haldane, then proved using the simplectic ensemble [10], and finally Yamamato et al. [18] obtained it using Uglov’s technology.

We recall the result and give the thermodynamic limit

\[
\langle (m_1, m_2) | p_{m_1 + m_2} | 0 \rangle = \frac{\gamma_0(m_1 - m_2) \gamma_1(m_1 - m_2) \gamma_0[N] \gamma_1[N]}{\gamma_0(m_1) \gamma_0(N - m_1) \gamma_0(m_2) \gamma_1(N - m_2)} \approx \sqrt{\frac{\pi(x_1 - x_2)}{E(m_1)E(m_2)}}
\]

(35)

1 \rightarrow 1 spinon

\[
\langle (m') | p_{m' - m} | (m) \rangle = \frac{\gamma_0(m') \gamma_0(N - m)}{\gamma_0(m) \gamma_0(N - m')} \approx \left( \frac{x'(\pi - x)}{x(\pi - x')} \right)^{1/4}
\]

(36)

2 \rightarrow 2 spinons

\[
\langle (m'_1, m'_2) | p_m | (m_1, m_2) \rangle = \sqrt{\frac{\gamma_0(m'_1) \gamma_0(N - m_1) \gamma_0(N - m_2)}{\gamma_0(m'_2) \gamma_0(N - m_1) \gamma_0(N - m_2)}} \times
\]

\[
\begin{cases}
\sqrt{\frac{\gamma_0(k) \gamma_1(k'\gamma_1(k')}}{\gamma_0(k') \gamma_1(k)}} & \text{if } m'_1 = m_1, \\
\sqrt{\frac{\gamma_0(k) \gamma_1(k) \gamma_0(k') \gamma_1(k')}} & \text{if } m'_2 = m_2, \\
\sqrt{\frac{\gamma_0(k) \gamma_1(k) \gamma_0(k') \gamma_1(k')}} & \text{otherwise.}
\end{cases}
\]

(37)

\[
G(m, l, k, k') = \prod_{i=1}^{l} (-)^{(k+i+1)} \frac{\gamma_1(i)}{\gamma_0(i)} \frac{\Gamma(\lfloor \frac{m-1}{2} \rfloor)}{\Gamma(\lfloor \frac{m}{2} \rfloor + 1)} \frac{\Gamma(\lfloor \frac{k-i}{2} \rfloor + 1)}{\Gamma(\lfloor \frac{k}{2} \rfloor + 1)} \frac{\Gamma(\lfloor \frac{k+i}{2} \rfloor)}{\Gamma(\lfloor \frac{x}{2} \rfloor + 1)}
\]

(38)

with \( m = m'_1 + m'_2 - m_1 - m_2, k = m_1 - m_2, k' = m'_1 - m'_2 \). This can be interpreted as an SU(2) generalization of result (28) of [30] for \( g = 1/2 \) quasiparticles.

Higher channels

It is generally not possible to obtain a closed analytical expression for the other channels, except in a few cases. Such are, for example, \( \langle m_1 \rangle \rightarrow \langle m_1, n, n' \rangle \) and \( \langle m_1 \rangle \rightarrow \langle n, n', m_1 \rangle \) which equal \( 0 \rightarrow \langle n, n' \rangle \).

Nevertheless, formula (38) can be used to give exact numerical results. Getting the thermodynamic limit directly is a challenging but fruitful work. We leave it as an open question.

IV. CORRELATION FUNCTIONS AT FINITE TEMPERATURE

Before addressing the computation of the DSSF, general considerations on finite-temperature correlation functions are needed. For a local operator \( \mathcal{O} \), it is

\[
\langle \mathcal{O} \rangle_T = \frac{\sum_n \lambda_n \langle \mathcal{O} | \lambda_n \rangle \exp(-\beta E_{\lambda_n})}{\sum_n \sum_{\lambda_n} \exp(-\beta E_{\lambda_n})}
\]

(39)

where \( n \) is the number of quasiparticles, and \( \lambda_n \) a state with \( n \) quasiparticles.

As recalled in the introduction, divergences appear in the correlators of the right-hand side that need to be resummed. For ITFs, LeClair and Mussardo proposed such a resummation as a form factor expansion on the basis of the asymptotic particle states in the zero-temperature theory [22].

\[
\langle \mathcal{O} \rangle_T = \sum_n \sum_{\lambda_n} \langle \mathcal{O} | \lambda_n \rangle_{\text{irr}} \prod_{i=1}^{n} \tilde{u}_T(E_{\lambda_n})
\]

(40)

The irreducible form factor \( \langle \lambda_n | \mathcal{O} | \lambda_n \rangle_{\text{irr}} \) is obtained thanks to the Form Factor Bootstrap (FFB), and \( \tilde{u}_T(E) \) is the filling factor determined by the Thermodynamic Bethe Ansatz (TBA).

In the following, we show that the FFE apply also for the Haldane-Shastry spin chain in the thermodynamic limit. First we obtain the thermodynamic properties of the spinons, then give the expression of the irreducible form factor.
A. Exclusion statistics

A form factor expansion similar to (40) was proposed for CFTs [29]. It led to identify a two-body S-matrix for the CFT in the thermodynamic limit

\[ S = \exp[2i\pi(\delta - K)\Theta(\theta)] \]  

(41)

where \( K \) is the exclusion statistics’ matrix of the quasi-particles of the theory (which play the role of asymptotic states).

Fractional exclusion statistics is a tool introduced by Haldane [15] for the analysis of strongly correlated many-body systems. It is only based on the assumption that the Hilbert space is finite-dimensional and extensive, i.e. particles are excitations of the considered condensed matter system, so it is a very generic concept. The statistics are encoded in a matrix \( K = (K_{ij}) \) corresponding to the reduction of the available Hilbert space for particle of type \( i \) by filling a one-particle state by a particle of type \( j \). This is then a generalization of the Pauli principle.

For spin-1/2 spinons with species \( i = \pm \), the statistical matrix is [34]

\[ K = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \]  

(42)

As such, their 1-particle distribution functions generalize the familiar Fermi-Dirac and Bose-Einstein ones. They are derived from 1-particle grand canonical partition functions \( G_i \) given by the IOW equations [35, 36, 37]

\[ \left( \frac{G_i - 1}{G_i} \right) \prod_j G_j^{K_{ij}} = z_i \]  

(43)

where \( G_i \) depends on the generalized fugacities \( z_j = e^{\beta(\mu_j - \varepsilon)} \). The one-particle distribution functions are obtained through

\[ n_i(\varepsilon) = z_i \frac{\partial}{\partial z_i} \log \prod_j G_j \]  

(44)

In our case, where species are \( i = (+, -) \), we obtain

\[ n_\pm(\varepsilon) = \frac{z_\pm}{\sqrt{1 + \left( \frac{z_+ - z_-}{2} \right)^2}} \pm \frac{1}{\sqrt{1 + \left( \frac{z_+ - z_-}{2} \right)^2}} \mu^+ = \mu^- \frac{1}{\exp(\beta E) + 1} \]  

(45)

from which one deduces that at zero magnetic field, these distributions match the Fermi-Dirac distribution function. It means that spinons can be considered as having fermionic statistics. This is indeed what is done when they are labeled by ordered numbers within partitions, the spin degree of freedom is hidden. The norm formula [28] shows that for a multi-spinon state, the labels have to be strictly decreasing, as for spinless fermions.

B. Irreducible form factors

For ITFs, the definition of the irreducible form factors comes a priori from the FFB. Within our framework (based on [24]) it is not even necessary. Their definition is

\[ \langle \lambda_n | \mathcal{O} | \lambda_n \rangle = \langle \lambda_n | \mathcal{O} | \lambda_n \rangle_{\text{irr}} + \sum_{\lambda_{n}} \langle \lambda_{n} | \mathcal{O} | \lambda_{n} \rangle_{\text{irr}} \]  

(46)

with \( \lambda_n \) being a sub-state of \( \lambda_n \) (a substate being a state where some of the spinons have been taken out). Nonetheless, the FFB insures that irreducible form factors don’t carry divergences, which we can’t prove here. This calls for further understanding of form factors in fractional statistics’ theories.

Here follows a sketch of the proof that the irreducible form factors [40] give the correct FFE [41]. It only uses the fact that the colorless spinons are free fermions, which shows up by the strict ordering of their quantum numbers within a multi-spinon state. As such, the proof is similar to [38], but in the discrete case.
We first remark that $\langle 0 | O | 0 \rangle_{\text{irr}}$ trivially comes with the factor 1. Let us consider the factor of $D(m) = \langle (m) | O | (m) \rangle_{\text{irr}}$. The contribution $C_n$ from $n$ quasi-particles is obtained recursively, isolating the spinons whose label match with $m$:

$$C_n = \frac{1}{Z} \sum_{m_1 > \cdots > m_n} \prod_{i=1}^n D(m_i) \prod_{i=1}^{n-1} \exp(-\beta E_{m_i})$$  \hspace{1cm} (47)
$$\quad = \frac{1}{Z} \sum_{m_1 > \cdots > m_{n-1}} \prod_{i=1}^{n-1} e^{-\beta E_{m_i}} \sum_m D(m) e^{-\beta E_m} (1 - \sum_{i=1}^{n-1} \delta_{m,m_i})$$ \hspace{1cm} (48)
$$\quad = \frac{1}{Z} \left[ \sum_{m_1 > \cdots > m_{n-1}} \prod_{i=1}^{n-1} e^{-\beta E_{m_i}} \sum_m D(m) e^{-\beta E_m} \right.$$ \hspace{1cm} (49)
$$\quad \left. - \sum_{m_1 > \cdots > m_{n-2}} \prod_{i=1}^{n-2} e^{-\beta E_{m_i}} \sum_m D(m) e^{-2\beta E_m} (1 - \sum_{i=1}^{n-2} \delta_{m,m_i}) \right]$$
$$\quad = \cdots = \sum_m D(m) e^{-\beta E_m} \sum_{i=0}^{n-1} (-i)^{z_{n-1-i}} e^{-i\beta E_m}$$ \hspace{1cm} (50)

Then summing over $n$ gives

$$\sum_{n=1}^{\infty} C_n = \sum_mD(m) e^{-\beta E_m} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} (-1)^i z_{n-1-i} e^{-i\beta E_m}$$ \hspace{1cm} (52)
$$\quad = \sum_mD(m) e^{-\beta E_m} \sum_{i=0}^{\infty} (-1)^i e^{-i\beta E_m} \sum_{n=i+1}^{\infty} z_{n-1-i}$$ \hspace{1cm} (53)
$$\quad = \sum_mD(m) e^{-\beta E_m} \sum_{i=0}^{\infty} (-1)^i e^{-i\beta E_m}$$ \hspace{1cm} (54)
$$\quad = \sum_mD(m) \frac{1}{\exp(\beta E_m) + 1}$$ \hspace{1cm} (55)

The demonstration follows the same lines for the higher form factors. Eq. 40 is quite convenient. Indeed, the denominator is suppressed, so it can be seen as a perturbative series, provided that it’s converging. It also has a clear physical interpretation with quasi-particles and filling-factors. Now it can be applied to the particular case of the dynamical spin structure factor (DSSF).

V. FINITE-TEMPERATURE SPIN STRUCTURE FACTOR

Now having all the methodological ingredients we address the main subject of our paper: the computation of the finite-temperature spin structure factor of the Haldane-Shastry spin chain. We first recall the zero-temperature result. It was first obtained by Haldane and Zirnbauer [16] using the supermatrix method. It can also be accessed with $\mathfrak{g}_2$ Jack polynomials [18]. Only the $[0 \rightarrow 2]$ channel contributes so it is straightforwardly

\begin{equation}
S_0(q, \omega) = \int_0^\pi dx_1 \int_0^\pi dx_2 \frac{\pi |x_1 - x_2|}{\sqrt{E(x_1)E(x_2)}} \delta(x_1 + x_2 - q) \delta(E(x_1) + E(x_2) - \omega)
\end{equation}
\hspace{1cm} (56)

At finite temperature, all channels contribute, one needs to sum them all through a FFE. We detail in the following the static structure factor $S_T(q) = \int d\omega S_T(q, \omega)$ and the DSSF.

A. Static structure factor

The static spin-spin correlator can be described by a one-point function

\begin{equation}
S_T(q = 2\pi s/N) = \langle \frac{1}{N} p_{-(2s+1)} p_{(2s+1)} \rangle_T
\end{equation}
\hspace{1cm} (57)
To compute it, we use the 1-point FFE and the
transition matrix elements obtained before.

Due to the complexity of the form factor and the lack of FFB, it is hardly impossible to obtain analytical results
on the correlation functions. The only point for which we could obtain the correlator exactly is
$q = 0$. We compute
it as the thermodynamic limit ($N \to \infty$) of (57) at
$s = 0$. The FFE is
\[
\langle p_{-1}p_1 \rangle_T = \frac{1}{N} \sum_{\mu} n_F(E_{\mu})(\mu|p_{-1}p_1|\mu)_{\text{irr}}
\]
\[
= \sum_{n=1}^{\infty} F_n
\]
\[
F_n = \sum_{m_1 > \cdots > m_n} \prod_{i=1}^{n} n_F(E_{m_i})\langle\{m_1, \ldots, m_n\}|p_{-1}p_1|\{m_1, \ldots, m_n\}\rangle_{\text{irr}}
\]
In the appendix we show that
\[
F_n = \int_0^\pi dx [n_F(E(x))]^n
\]
from which we conclude
\[
S_T(q = 0) = \sum_{n=1}^{\infty} \int_0^\pi dx \left( \frac{1}{\exp(\beta E(x)) + 1} \right)^n
\]
\[
= \int_0^\pi dx e^{-\beta E(x)}
\]
This result brings strong evidence of the power of the FFE to obtain finite-temperature correlation functions. This is
the main achievement of the paper. Let us remark here that $F_n$ is the contribution of the $n$-spinon states in the FFE.
They appear to be of the same order in temperature, meaning that the FFE is not a low-temperature expansion
2. It really is a perturbative expansion, in the sense that the different contributions decrease exponentially with the
number of spinons involved.

Nonetheless, $S_T(q = 0)$ is a low-energy feature of the theory that can be obtained with a bosonization approach.
As in [30], we observe that the FFE is mostly powerful in regimes where other simpler methods apply. To calculate
the whole static structure factor, we then rely on numerics. We work in the infinite-temperature regime to compare
the FFE with the expected result. In this limit, correlations are only local and we expect the static structure factor
not to depend on the momentum and have the value $S_{T=\infty}(q = 0) = 1$.

To observe the FFE perturbative power, we studied the contribution from 2- and (2+4)-multispinon states. What
we understand here as 2-spinons is the sum of the contributions of the $[0 \leftrightarrow 2]$ and $[1 \leftrightarrow 1]$ channels, and as 4-spinons
the $[1 \leftrightarrow 3]$ and $[2 \leftrightarrow 2]$ channels. Results are gathered on figure 2. At first we observe the convergence to be rather
good. This strongly supports our approach. Still problems arise in the vicinity of $q = \pi$. Finite-size effects starts
appearing at the level of 4-spinons contributions, and increase with a higher number of spinons. The only way to
tackle them is to have an enormous computing time. We admit this is the most important weakness of the approach.
This emphasizes the need for a direct thermodynamic limit computation.

B. Dynamic structure factor

We show on figure 3 the sum of the contributions from 2- and 4-spinons to the infinite-temperature DSSF. As in
the previous paragraph, finite-size effects clearly appear around $q = \pi$. This makes the comparison to experiments
hardly possible. We still note that some basic features, like the double arch shape, agree qualitatively. This feature,
already present at zero temperature, comes from form factors where the spin operator acts on only one of the spinon
in a multi-spinon excitation, leading to the spinon dispersion relation. We leave more comments to the concluding
section.

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2 This would have been expected on the ground that more thermal energy is needed to excite more particles. This is not true anymore
for massless particles.
FIG. 2: Static spin structure factor. $2$ is the contribution from 2-particle form factors with $N = 10000$; $2+4$ is the contribution from 2- and 4-particle form factors with $N = 500$; the two different curves are obtained with two labellings of the spinons: for the upper curve, $0 < m < N$ (the normal one), for the lower curve, $0 \leq m \leq N$; they should match in the $N \rightarrow \infty$-limit (up to a $\delta$-peak on $q = \pi$), so we expect the thermodynamic limit to lie in-between the two curves.

VI. CONCLUSION

In this paper we studied the finite-temperature spin-spin correlation function of the Haldane-Shastry spin chain. We used the multi-spinon basis obtained as the infinite coupling limit of the Yangian basis of the spin Calogero-Sutherland model. Form factors of the spin operators were computed thanks to the $gl_2$ Jack polynomial technology. These form factors were gathered within a form factor expansion of finite-temperature correlation functions to give physical quantities directly comparable with the experiments. Though the Haldane-Shastry is not a quantitatively appropriate model to describe real spin chains such as KCuF$_3$, it should at the qualitative level. As any finite-temperature properties for the Heisenberg model are yet beyond reach, serious theoretical insights are brought by the Haldane-Shastry model. We recall that it is simpler due to the $1/r^2$ interaction responsible for the thermodynamic freedom of the spinons.

Our main analytical results are: the formal expression of any form factor of the theory, a closed analytical expression for the simplest of them, the proof of the finite-temperature FFE, and the value at zero-momentum of the static structure factor. Along with numerical results, they make evidence of the FFE technique to approximate finite-temperature correlations. As a drawback, it demands huge computation power. This calls for further theoretical refinement.

Work is on progress to provide the Haldane-Shastry model with integrability features such as the FFB. Within such a description, thermodynamic irreducible form factors would be obtained much easier using scattering properties. Problems to develop this approach come from the intrinsic discrete nature of the Haldane-Shastry spinons. But necessary effort has to be made to gather comprehensively ITFs and CFTs in a common framework. Further understanding is also needed in the link between the Haldane-Shastry model and the symplectic random matrix ensemble used in [16]. The analysis performed at zero temperature could be extended at finite temperature. It would also be interesting to treat the finite-temperature dynamics of other inverse-square interaction models and compare it to their zero-temperature exact result [20, 22]. It comes as a simple generalization of the method used in the present paper.

At the numerical level, finite size calculations can be performed. For sufficiently small sizes, the exact correlation functions are accessible using the analytical results of this paper. The FFE can not be used, for it is based on the
thermodynamic limit, rather direct application of (39) is necessary. Using directly the Yangian multiplets is another solution, as in [39] for zero temperature.

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**APPENDIX: ZERO-MOMENTUM LIMIT OF THE STATIC STRUCTURE FACTOR**

In this section we show that

$$F_n = \sum_{m_1 > \cdots > m_n} n_F(E_{m_n}) \langle \{m_1, \ldots, m_n\} | p_{-1}p_1 | \{m_1, \ldots, m_n\} \rangle_{\text{irr}} = \int_0^\pi dx [n_F(E(x))]^n$$ \hspace{1cm} (A.1)

It is clear from the conservation laws that the intermediate states for each form factor is only different from the initial state by an increase of 1 of one of the spinons’ $m$. Quite generally, we thus need

$$|\langle \{m_1, \ldots, m_k+1, \ldots, m_n\} | p_1 | \{m_1, \ldots, m_k, \ldots, m_n\} \rangle|^2 \simeq \prod_{i \neq k} \alpha_{s(gn(i-k))}^{\gamma_0(i-k)}(m_k - m_i)$$ \hspace{1cm} (A.2)

$$\alpha^+(m) = \left( \frac{\gamma_0(m)\gamma_1(m+1)}{\gamma_1(m)\gamma_0(m+1)} \right)$$ \hspace{1cm} (A.3)

$$\alpha^-(m) = \left( \frac{\gamma_1(m)\gamma_0(m-1)}{\gamma_0(m)\gamma_1(m-1)} \right)$$ \hspace{1cm} (A.4)
Explicitly developing the irreducible form factors and separating different affected \(m\)'s, we obtain

\[
F_n = \sum_{m_1 > \cdots > m_n} \prod_{i=1}^{n} n_F(E_{m_i}) \sum_k F^k_n
\]

\[
F^k_n = \sum_{J \subset \{1, \ldots, k+1, \ldots, n\}} \frac{(-1)^{-|J|}}{d_{J(k,i)}} \left| \prod_{i \in J} \alpha^{\text{sgn}(i-k)}_{J(i,k)} (m_k - m_i) \right|
\]

where \(d_{J(k,i)}\) the distance between \(k\) and \(i\) in the subset \(J\). Thus we can write \(F^k_n = F^{k>}_n F^{k<}_n\).

We will now obtain \(F^{k>}_n\) in a recursive way (the proof for \(F^{k>}_n\) follows the same lines), putting \(k = 1\) without loss of generality. We first perform the sum over \(m_n\). Writing \(I = \{2, \ldots, n-1\}\), one separates the ensemble of \(J\) as

\[
\{J \subset I \cup \{n\}\} = \{J + J \cup \{n\}, J \subset I\} = \{J + J \cup \{n\}, J \in I, |J| \text{ even}\} \cup \{J + J \cup \{n\}, J \subset I, |J| \text{ odd}\}.
\]

In the first (resp. second) subset, the distance in \(J\) between \(1\) and \(n\) is odd (resp. even). This proves that

\[
F^{k>}_n = (\alpha^+(m_1 - m_n) - 1)F^{k>\text{even}}_{n-1} + (\alpha^+(m_1 - m_n) - 1)F^{k>\text{odd}}_{n-1}
\]

where the superscripts even and odd corresponds to restrictions of the expression of \(F^{k>}_n\) to \(J\) subsets with even or odd cardinal. Repeating this recursion one ends up with

\[
F^{k>}_n = F^{k>\text{even}}_{n-2}[(\alpha^+(m_1 - m_n) - 1)\alpha^+(m_1 - m_n-1) + 1 - \alpha^+(m_1 - m_n)]
\]

\[
+ F^{k>\text{odd}}_{n-2}[(\alpha^+(m_1 - m_n) - 1)\alpha^+(m_1 - m_n-1) + 1 - \alpha^+(m_1 - m_n)]
\]

Then one easily shows that

\[
[(\alpha^+(m_1 - m_n) - 1)\alpha^+(m_1 - m_n-1) + 1 - \alpha^+(m_1 - m_n)] = \alpha^+(m_1 - m_n)\delta_{m_n-m_n-1}
\]

\[
[(\alpha^+(m_1 - m_n) - 1)\alpha^+(m_1 - m_n-1) + 1 - \alpha^+(m_1 - m_n)] = \alpha^+(m_1 - m_n)\delta_{m_n-m_n-1}
\]

thus proving the recursion

\[
F^{k>}_n = F^{k>}_{n-1} \delta_{m_n-m_n-1}
\]

As this is true only for \(n > 1\), we obtain for \(F^k_n\) the following

\[
F^1_n = (\alpha^+(m_1 - m_2) - 1)\delta_{m_2,...,m_n}
\]

\[
F^k_{n \neq 1} = (\alpha^+(m_k - m_{k+1}) - 1)(\alpha^+(m_k - m_{k-1}) - 1)\delta_{m_1,...,m_{k-1}}\delta_{m_{k+1},...,m_n}
\]

\[
F^1_n = (\alpha^+(m_n - m_{n-1}) - 1)\delta_{m_1,...,m_{n-1}}
\]

\[
F^k_n = \delta_{k,1}\delta_{m_1,...,m_n}
\]

Now, one has \(\alpha^+(m_k - m_{k+1}) - 1 \equiv \delta_{m_k,m_{k+1}}\) and \(\alpha^+(m_k - m_{k-1}) - 1 \equiv 0\) so that

\[
F^k_n = \delta_{k,1}\delta_{m_1,...,m_n}
\]

Finally

\[
F_n = \sum_{m=1}^{N-1} [n_F(E(m))]^n = \int_0^\pi dx [n_F(E(x))]^n
\]

which ends our proof.

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