The Kähler–Ricci flow through singularities

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Abstract We prove the existence and uniqueness of the weak Kähler–Ricci flow on projective varieties with log terminal singularities. We also show that the weak Kähler–Ricci flow can be uniquely continued through divisorial contractions and flips if they exist. Finally we propose an analytic version of the minimal model program with Ricci flow.

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1 Introduction

It has been the subject of intensive study over the last few decades to understand the existence of canonical Kähler metrics of Einstein type on a compact Kähler manifold, following Yau’s solution to the Calabi conjecture (cf. [1,38,40,48,49]). The Ricci flow (cf. [8,18]) provides a canonical deformation of Kähler metrics toward such canonical metrics. Cao [4] gives an alternative proof of the existence of Kähler–Einstein metrics on a compact Kähler manifold with numerically trivial or ample canonical bundle by the Kähler–Ricci flow. However, most projective manifolds do not have a numerically definite or trivial canonical bundle. It is a natural question to ask if there exist any well-defined canonical metrics on these manifolds or varieties canonically associated to them. A projective variety is minimal if its canonical bundle is nef (numerically effective) and many results have been obtained on the Kähler–Ricci flow on minimal varieties. Tsuji [45] applies the Kähler–Ricci flow and proves the existence of a canonical singular Kähler–Einstein metric on a minimal projective manifold of general type. It is the first attempt to relate the Kähler–Ricci flow and canonical metrics to the minimal model program. Since then, many interesting results have been achieved in this direction. The long time existence of the Kähler–Ricci flow on a minimal projective manifold with any initial Kähler metric is established in [42,45]. The regularity problem of the canonical singular Kähler–Einstein metrics on minimal projective manifolds of general type is intensively studied in [13] and [50] independently. If the minimal projective manifold has positive Kodaira dimension and it is not of general type, it admits an Iitaka fibration over its canonical model. The authors define on the canonical model a new family of generalized Kähler–Einstein metrics twisted by a canonical form of Weil–Petersson type from the fibration structure ([32,33]). It is also proved by the authors that the normal-
ized Kähler–Ricci flow converges to such a canonical metric if the canonical bundle is semi-ample ([32, 33]).

Let $X$ be an $n$-dimensional projective manifold. We consider the following Kähler–Ricci flow starting with a Kähler metric $\omega_0 \in H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q})$.

$$\begin{align*}
\frac{\partial \omega}{\partial t} &= -\text{Ric}(\omega) \\
\omega|_{t=0} &= \omega_0.
\end{align*}$$

(1.1)

The Kähler–Ricci flow (1.1) has long time existence if $X$ is a minimal model, i.e., the canonical bundle $K_X$ is nef ([42]). If $K_X$ is not nef, the Kähler–Ricci flow (1.1) must become singular at certain finite time $T_0 > 0$.

At time $T_0$, the flow must develop singularities on an analytic subvariety of $X$ if the limiting cohomology class of the evolving Kähler metrics is big (cf. [42, 45]). Such a subvariety coincides with exceptional locus of the morphism induced from the limiting cohomology class. In [32, 33], we conjecture that the Ricci flow performs an analytic/geometric surgery equivalent to an algebraic surgery such as a divisorial contraction or a flip, and the flow can be continued on a new projective variety $X'$ with mild singularities. The main goal of the paper is to define the unique Kähler–Ricci flow on projective varieties with mild singularities and to partially establish the above speculation by constructing analytic surgeries for the Kähler–Ricci flow through singularities.

If the limiting cohomology class is not big, $X$ admits a Fano fibration by the minimal model theory. We conjecture that the flow (1.1) collapses onto a new projective variety $X''$ and the flow can further be continued on the base $X''$. Heuristically, we might be able to repeat the above procedures until either the flow exists for all time or it becomes extinct. If the flow exists for all time, we expect its convergence to a generalized Kähler–Einstein metric on its canonical model or a Ricci-flat metric on its minimal model after normalization, assuming the abundance conjecture. Eventually, we arrive at the final case when $X$ is Fano and it is conjectured by the second named author that the Kähler–Ricci flow becomes extinct in finite time after surgery if and only if it is birationally equivalent to a Fano variety [41]. The conjecture is proved by the first named author for smooth solutions of the flow [31]. We refer the readers to Sect. 6 for details of the above conjectural picture.

In general, varieties obtained from divisorial contractions and flips have mild singularities. Since we expect that the analytic surgeries performed by the Kähler–Ricci flow coincide with the algebraic surgeries as divisorial contractions and flips, we can not avoid singularities if the Kähler–Ricci flow can be indeed continued through surgeries. Therefore we must define the Kähler–Ricci flow on projective varieties with mild singularities.
We confine ourselves in the category of singularities considered in the minimal model program because such singularities are rather mild and they do not become worse after divisorial contractions or flips are performed. The precise definition is given in Sect. 2.3 for a $\mathbb{Q}$-factorial projective variety with log terminal singularities. Roughly speaking, let $X$ be such a projective normal variety and $\pi : \tilde{X} \to X$ be a log resolution of singularity, then the pullback of any adapted measure on $X$ is $L^p$-integrable on the nonsingular model $\tilde{X}$ for some $p > 1$. The plurisubharmonic functions and adapted measure on $X$ are introduced in [13].

We now introduce a special family of quasi-plurisubharmonic functions on a projective variety with mild singularities.

**Definition 1.1** Let $X$ be a $\mathbb{Q}$-factorial projective variety with log terminal singularities and $H$ be a big and semi-ample $\mathbb{Q}$-divisor with a birational morphism $\Phi_m : X \to \mathbb{C}P^N_m$ induced by the linear system $|mH|$ for some $m \in \mathbb{Z}^+$. Let $\omega = \frac{1}{m}(\Phi_m)^*\omega_{FS} \in [H]$ and $\Omega$ an adapted measure on $X$, where $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{C}P^N_m$ and adapted measures are defined in Sect. 4 (cf. Definition 4.1). We define for $p \in (1, \infty]$, $K_{H, p}(X) = \left\{ \omega + \sqrt{-1}\partial\bar{\partial}\varphi \mid \varphi \in PSH_p(X, \omega, \Omega) \right\}$, (1.3)

where $PSH(X, \omega)$ is the set of all $\omega$-psh functions on $X$ associated with $\omega$.

The definition of the space $K_{H, p}(X)$ does not depend on the choice of the linear system $|mH|$ and the adapted measure $\Omega$. We can now define the weak Kähler–Ricci flow on projective varieties with mild singularities.

**Definition 1.2** Let $X$ be a $\mathbb{Q}$-factorial projective variety with log terminal singularities and let $H$ be a big and semi-ample $\mathbb{Q}$-divisor such that $T_0 = \sup\{t > 0 \mid H + tK_X \text{ is ample} \} > 0$.

A family of closed semi-positive $(1, 1)$-currents $\omega(t, \cdot)$ on $X$ for $t \in [0, T_0)$ are said to be a solution of the weak Kähler–Ricci flow starting with $\omega_0 \in K_{H, p}(X)$ for some $p > 1$ if the following conditions hold.

1. $\omega \in C^\infty((0, T_0) \times X_{reg})$, where $X_{reg}$ is nonsingular part of $X$. Let $\hat{\omega}_t \in [H + tK_X]$ be the restriction of a smooth family of Kähler metrics on an
ambient projective space for a projective embedding of \( X \) for \( t \in [0, T_0) \). Then \( \omega = \hat{\omega}_t + [\sqrt{-1}] \partial \bar{\partial} \varphi \) for some \( \varphi \in C^0([0, T_0) \times X_{reg}) \cap C^\infty((0, T_0) \times X_{reg}) \) and \( \varphi(t, \cdot) \in \operatorname{PSH}(X, \hat{\omega}_t) \cap C^0(X) \) for all \( t \in [0, T_0) \). Furthermore, \( \varphi \in L^\infty([0, T] \times X) \) for all \( 0 < T < T_0 \).

The above definition of the weak Kähler–Ricci flow can be easily generalized on any \( \mathbb{Q} \)-Gorenstein variety \( X \) with log terminal singularities with a big and semi-ample Cartier \( \mathbb{Q} \)-divisor. Our first theorem establishes the existence and uniqueness for the weak Kähler–Ricci flow on projective varieties with log terminal singularities. Furthermore, it establishes a smoothing property for the Kähler–Ricci flow if the initial data is not smooth.

**Theorem 1.1** Let \( X \) be an \( n \)-dimensional \( \mathbb{Q} \)-factorial projective variety with log terminal singularities and let \( H \) be a big and semi-ample ample \( \mathbb{Q} \)-divisor on \( X \). If

\[
T_0 = \sup \{ t > 0 \mid H + t K_X \text{ is ample} \} > 0
\]

and \( \omega_0 \in \mathcal{K}_{H, p}(X) \) for some \( p > 1 \), then there exists a unique solution \( \omega \) of the weak Kähler–Ricci flow (1.1) starting with \( \omega_0 \) for \( t \in [0, T_0) \). Furthermore, if \( \Omega \) is an adapted measure on \( X \) and \( T \in (0, T_0) \), there exists \( C > 0 \) such that

\[
e^{-\frac{C}{t} \Omega} \leq \omega^n \leq e^{\frac{C}{t} \Omega}.
\]

Theorem 1.1 shows that the Kähler–Ricci flow can start with a semi-positive current with bounded local potentials and \( L^p \) Monge–Ampère mass for some \( p > 1 \). It establishes short time existence for the weak Kähler–Ricci flow and the maximal time \( T_0 \) of the flow is explicitly computed in terms of cohomology. Furthermore, the weak Kähler–Ricci flow smoothes out the initial current in the sense that the flow becomes smooth on the nonsingular part of \( X \) once \( t > 0 \) and the evolving metrics always admit bounded local potentials for any \( t \in (0, T_0) \).

In particular, the smoothing property of the Kähler–Ricci flow holds when \( X \) is a compact Kähler manifold (c.f. Theorem 3.1).

**Proposition 1.1** Let \( X \) be an \( n \)-dimensional compact Kähler manifold and \( \theta \in H^{1,1}(X, \mathbb{R}) \) a smooth Kähler metric. If \( \omega_0 \in [\theta] \) is a closed positive
(1, 1)-current with bounded local potentials and \( (\omega_0)^n \in L^p(X) \) for some \( p > 1 \), then there exist \( T_0 > 0 \) and a unique smooth solution \( \omega(t) \) of the Kähler–Ricci flow (1.1) for \( t \in (0, T_0) \) such that the local potentials of \( \omega(t) \) converges to those of \( \omega_0 \) in \( L^\infty \) as \( t \rightarrow 0^+ \).

In general, it is not clear how to define metrics on a singular variety \( X \) with reasonable regularity and curvature conditions. One natural choice is the restriction of the Fubini-Study metric \( \omega_{FS} \) for some projective embedding of \( X \), if \( X \) is normal and projective. However, even the scalar curvature of \( \omega_{FS} \) might blow up near the singularities of \( X \). More seriously, \( (\omega_{FS})^n \) is not an adapted measure on \( X \) in general, although it is the restriction of a smooth non-negative \((n, n)\)-form from an ambient projective space to \( X \) (see Sect. 4.3 for more detailed discussions). Theorem 1.1 shows that the measure of the corresponding solutions of the weak Kähler–Ricci flow becomes equivalent to an adapted measure immediately for \( t > 0 \). We believe that the weak Kähler–Ricci flow produces metrics on \( X \) with reasonably good and possibly optimal Riemannian geometric structures. For example, given a normal projective orbifold \( X \) embedded in some projective space \( \mathbb{CP}^N \), the Fubini-Study metric \( \omega_{FS} \) is in general not a smooth orbifold Kähler metric on \( X \), but if we start the Kähler–Ricci flow with \( \omega_{FS} \), the evolving metrics immediately become smooth orbifold Kähler metrics on \( X \) (cf. Theorem 4.4).

Theorem 1.1 can be generalized in various ways. For example, Theorem 1.1 still holds with little modification if the underlying projective variety is \( \mathbb{Q} \)-Gorenstein.

The first singular time \( T_0 \) is exactly when the Kähler class of the evolving metrics stops being Kähler. If \( T_0 < \infty \) and the limiting Kähler class is big, there is a contraction morphism \( \pi : X \rightarrow Y \) uniquely associated to the limiting divisor \( H + T_0K_X \) by Kawamata’s base point free theorem. Let \( \overline{NE}(X) \) be the closure of the convex cone that consists of the classes of effective curves on \( X \). If the morphism \( \pi \) contracts exactly one extremal ray of \( \overline{NE}(X) \), the recent result of [2] and [17] shows that either \( \pi \) contracts a divisor or there exists a unique flip associated to \( \pi \) (see Definition 5.4 for a flip).

Since the weak Kähler–Ricci flow cannot be continued on \( X \) at the singular time \( T_0 \), we have to replace \( X \) by another variety \( X' \) and continue the flow on \( X' \). Our next main result is to relate the finite time singularities of the Kähler–Ricci flow (1.1) to divisorial contractions and flips in the minimal model program.

**Theorem 1.2** Let \( X \) be an \( n \)-dimensional \( \mathbb{Q} \)-factorial projective variety with log terminal singularities and let \( H \) be an ample \( \mathbb{Q} \)-divisor on \( X \). Let \( \omega \) be the unique solution of the weak Kähler–Ricci flow (1.1) for \( t \in [0, T_0) \) starting with \( \omega_0 \in \mathcal{K}_{H,p}(X) \) for some \( p > 1 \), where
$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}$

is the first singular time. Suppose that $T_0 < \infty$. We let $\pi : X \to Y$ be the morphism induced by the linear system $|mH_{T_0}|$ for some $m \in \mathbb{Z}^+$, where $H_{T_0} = H + T_0K_X$ is a semi-ample $\mathbb{Q}$-divisor.

(1) If $\pi$ is a divisorial contraction, then there exists a closed semi-positive $(1, 1)$-current $\omega_Y$ with bounded local potentials on $Y$ such that
(a) $\omega_Y \in \mathcal{K}_{\pi^*H_{T_0}, p'}(Y)$ for some $p' > 1$. Therefore the weak Kähler–Ricci flow can be uniquely continued on $Y$ starting with $\omega_Y$ at $t = T_0$.
(b) $\omega(t, \cdot)$ converges to $\pi^*\omega_Y$ in $C^\infty(X_{\text{reg}} \setminus \text{Exc}(\pi))$-topology as $t \to (T_0)^-$. 
(c) Let $\omega(t, \cdot)$ be the unique weak solution of the weak Kähler–Ricci flow on $Y$ starting with $\omega_Y$ at $t = T_0$. Then $\omega(t, \cdot)$ converges to $\omega_Y$ in $C^\infty(X_{\text{reg}} \setminus \text{Exc}(\pi))$-topology as $t \to (T_0)^+$. 

(2) If $\pi$ is a small contraction and there exists a flip

$$
\begin{array}{c}
X \\
\pi \downarrow \pi' \\
\pi' \\
\pi' \\

\end{array}
\xrightarrow{\pi} Y \xrightarrow{\pi'} X^+ \xrightarrow{\pi} X^+
$$

then there exists a closed semi-positive $(1, 1)$-current $\omega_Y$ with bounded local potentials on the normal variety $Y$ such that
(a) $\pi^*\omega_Y \in \mathcal{K}_{H_{T_0}, p'}(X)$ and $(\pi^+)^*\omega_Y \in \mathcal{K}_{(\pi')^*H_{T_0}, p'}(X^+)$ for some $p' > 1$. Therefore the weak Kähler–Ricci flow can be uniquely continued on $X^+$ starting with $(\pi^+)^*\omega_Y$ at $t = T_0$.
(b) $\omega(t, \cdot)$ converges to $\pi^*\omega_Y$ in $C^\infty(X_{\text{reg}} \setminus \text{Exc}(\pi))$-topology as $t \to (T_0)^-$. 
(c) Let $\omega(t, \cdot)$ be the unique weak solution of the weak Kähler–Ricci flow on $X^+$ starting with $\omega_Y$ at $t = T_0$. Then $\omega(t, \cdot)$ converges to $(\pi^+)^*\omega_Y$ in $C^\infty((X^+)_\text{reg} \setminus \text{Exc}(\pi^+))$-topology as $t \to (T_0)^+$. 

In summary, we have the following corollary.

**Corollary 1.1** The weak Kähler–Ricci flow can be uniquely continued through divisorial contractions and flips.

In fact, the continuation of the weak Kähler–Ricci flow through flips is smooth on a Zariski open dense set of $X$ and $X^+$ through the singular time $T_0$ and the local potentials of the evolving singular Kähler metrics are uniformly bounded in $L^\infty$. We illustrate the Kähler–Ricci flow through divisorial contractions and flips by the following two examples.
Example 1.1 We denote by $X$ the blow-up of $\mathbb{C}P^2$ at one point. Let $\pi : X \rightarrow \mathbb{C}P^2$ be the blow-down map, sending the only $(-1)$-curve $E$ on $X$ to the point $y_0 \in \mathbb{C}P^2$. Kähler classes on $X$ can be written as

$$\alpha = b \pi^* [\mathcal{O}(1)] - a [E], \quad \text{for } 0 < a < b.$$ 

We consider a solution of the Kähler–Ricci flow (1.1) starting at $\omega_0$ in a Kähler class $[\omega_0] = b_0 \pi^* [\mathcal{O}(1)] - a_0 [E]$, where $a_0$ and $b_0$ satisfy the condition:

$$0 < 3a_0 < b_0.$$ 

(1.7)

It is easy to compute that the first singular time is $T_0 = a_0$ and

$$[\omega_0] + T_0 [K_X] = (b_0 - 3a_0) [\pi^* \mathcal{O}(1)].$$

Theorem 1.2 can be applied and the Kähler–Ricci flow can be continued on $\mathbb{C}P^2$. It is further shown in [34,35] that the Kähler–Ricci flow is extended through the singular time continuously in Gromov–Hausdorff topology.

Example 1.2 Let $X = \text{Proj}(\mathcal{O}_{\mathbb{C}P^m} \oplus (\mathcal{O}_{\mathbb{C}P^m} (-1))^{\oplus n})$ be the $\mathbb{C}P^n$ bundle over $\mathbb{C}P^m$ and $X^+ = \text{Proj}(\mathcal{O}_{\mathbb{C}P^n} \oplus (\mathcal{O}_{\mathbb{C}P^n} (-1))^{\oplus m})$ for $m > n$. Then $X$, $X^+$ and $Y$ satisfy the flip diagram (1.6), where $Y$ is obtained from $X$ by a birational morphism $\pi$ contracting the zero $\mathbb{C}P^m$-section. Then $H = \pi^* \mathcal{O}_{\mathbb{C}P^m} (1) - \epsilon K_X$ is ample for sufficiently small $\epsilon \in \mathbb{Q}^+$ and we consider the Kähler–Ricci flow on $X$ starting with a smooth Kähler metric in $[H]$. The limiting Kähler current at the first singular time $T_0 = \epsilon$ descends to a Kähler current $\omega_Y$ on $Y$, which is smooth on the nonsingular part of $Y$. Applying Theorem 1.2, the Kähler–Ricci flow can be uniquely continued on $X^+$ starting with the pullback of $\omega_Y$ by $\pi^+$. The geometric convergence is studied in [36].

The minimal model program is successful in dimension three by Mori’s work and recent works have (c.f. [2] and [30]) led to proving the finite generation of canonical rings. The deformation of the Kähler classes along the Kähler–Ricci flow is in line with the minimal model program with scaling (MMP with scaling) proposed in [2]. It is also proved in [2] that MMP with scaling terminates after finitely many divisorial contractions and flips if the variety $X$ is of general type.

An ample $\mathbb{Q}$-divisor $H$ is said to be a good initial divisor if there are finitely many singular times for the weak Kähler–Ricci flow (1.1) starting with $H$ and the contraction morphism at each singular time contracts exactly one extremal ray. We refer the readers to Sect. 5.1 for more details (Definition 5.3). In fact, all ample divisors are good initial when $\dim X = 2$. It is possible that a general ample $\mathbb{Q}$-divisor $H$ on $X$ is a good initial divisor.
Theorem 1.3 Let $X$ be a $\mathbb{Q}$-factorial projective variety of general type with log terminal singularities. If $H$ is a good initial divisor on $X$, then the normalized weak Kähler–Ricci flow

$$\frac{\partial \omega}{\partial t} = -\text{Ric} (\omega) - \omega$$

(1.8)

starting with any initial $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$ exists for $t \in [0, \infty)$ and it replaces $X$ by its minimal model $X_{\text{min}}$ after finitely many analytic surgeries. Furthermore, the normalized Kähler–Ricci flow converges weakly to the unique singular Kähler–Einstein metric $\omega_{\text{KE}}$ on its canonical model $X_{\text{can}}$.

Theorem 1.3 gives the general philosophy of the analytic minimal model program with Ricci Flow. The Kähler–Ricci flow deforms a given projective variety $X$ to its minimal model $X_{\text{min}}$ in finite time after finitely many metric surgeries. Then $X_{\text{min}}$ is deformed to the canonical model $X_{\text{can}}$ coupled with a generalized Kähler–Einstein metric by the flow after normalization. We also remark that the flow converges in the sense of distribution globally and in the $C^\infty$-topology away from the singularities of $X_{\text{min}}$ and the exceptional locus of the pluricanonical system. Certainly, it is desired that the convergence should also be in Gromov–Hausdorff topology. We also remark that when $X$ is a nonsingular minimal model of general type, the convergence of the normalized Kähler–Ricci flow is proved in [45] and [42].

The organization of the paper is the following. In Sect. 2, we introduce basic notations for degenerate complex Monge–Ampère equations and algebraic singularities in the minimal model theory. In Sect. 3, we solve special families of degenerate and singular parabolic complex Monge–Ampère equations on projective manifolds. In Sect. 4, we apply results in Sect. 3 to prove Theorem 1.1, establishing existence and uniqueness of the weak Kähler–Ricci flow. In Sect. 5, Theorem 1.2 and Corollary 1.1 are proved for the weak Kähler–Ricci flow through singularities. We also prove Theorem 1.3 for long time existence and convergence. Finally in Sect. 6, we propose the analytic minimal model program with Ricci flow to study classifications of projective varieties by PDEs, Riemannian geometry and transcendental methods in algebraic geometry.

2 Preliminaries

2.1 Kodaira dimension and canonical measures

Let $X$ be an $n$-dimensional compact complex projective manifold and $L \to X$ a holomorphic line bundle over $X$. Let $N(L)$ be the semi-group defined by
\[ N(L) = \{ m \in \mathbb{N} \mid H^0(X, L^m) \neq 0 \} . \]

Given any \( m \in N(L) \), the linear system \(|L^m| = \mathbb{CP}H^0(X, L^m)\) induces a rational map \( \Phi_m \)

\[ \Phi_m : X \dashrightarrow \mathbb{CP}^{d_m} \]

by any basis \( \{ \sigma_{m,0}, \sigma_{m,1}, \ldots, \sigma_{m,d_m} \} \) of \( H^0(X, L^m) \), where

\[ \Phi_m(z) = [\sigma_{m,0}, \sigma_{m,1}, \ldots, \sigma_{m,d_m}(z)] , \]

and \( d_m + 1 = \dim H^0(X, L^m) \). Let \( Y_m = \overline{\Phi_m(X)} \subset \mathbb{CP}^{d_m} \) be the closure of the image of \( \Phi_m \).

**Definition 2.1** The Iitaka dimension of \( L \) is defined to be

\[ \kappa(X, L) = \max_{m \in N(L)} \{ \dim Y_m \} \]

if \( N(L) \neq \phi \), and \( \kappa(X, L) = -\infty \) if \( N(L) = \phi \).

**Definition 2.2** Let \( X \) be a projective manifold and \( K_X \) be the canonical line bundle over \( X \). Then the Kodaira dimension \( \kappa(X) \) of \( X \) is defined to be

\[ \kappa(X) = \kappa(X, K_X) . \]

The Kodaira dimension is a birational invariant of a projective variety and the Kodaira dimension of a singular variety is equal to that of its smooth model.

**Definition 2.3** Let \( L \rightarrow X \) be a holomorphic line bundle over a compact projective manifold \( X \). \( L \) is called nef if \( L \cdot C \geq 0 \) for any curve \( C \) on \( X \) and \( L \) is called semi-ample if \( L^m \) is globally generated for some \( m > 0 \).

For any \( m \in \mathbb{N} \) such that \( L^m \) is globally generated, the linear system \(|L^m|\) induces a holomorphic map \( \Phi_m \)

\[ \Phi_m : X \rightarrow \mathbb{CP}^{d_m} \]

by any basis of \( H^0(X, L^m) \). Let \( Y_m = \overline{\Phi_m(X)} \) and so \( \Phi_m \) can be considered as

\[ \Phi_m : X \rightarrow Y_m . \]

The following theorem is well-known (cf. \([23, 46]\)).
Theorem 2.1 Let $L \to X$ be a semi-ample line bundle over an algebraic manifold $X$. Then there is an algebraic fibre space

$$\Phi_\infty : X \to Y$$

such that for any sufficiently large integer $m$ with $L^m$ being globally generated,

$$Y_m = Y \quad \text{and} \quad \Phi_m = \Phi_\infty,$$

where $Y$ is a normal projective variety. Furthermore, there exists an ample line bundle $A$ on $Y$ such that $L^m = (\Phi_\infty)^*A$.

If $L$ is semi-ample, the graded ring $R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^m)$ is finitely generated and so it is the coordinate ring of $Y$.

Let $X$ be an $n$-dimensional projective manifold. It is recently proved in [2, 17] and [30] independently that the canonical ring $R(X, K_X)$ is finitely generated. Then the canonical ring induces a rational map from $X$ to its unique canonical model $X_{\text{can}}$. The following theorem is proved in [13] and [50] when $X$ is of general type (also see [45]) and in [33] when $X$ admits an Iitaka fibration over $X_{\text{can}}$.

Theorem 2.2 Let $X$ be an $n$-dimensional projective manifold with $K_X$ being semi-ample.

1. If $\operatorname{kod}(X) = n$, then there exists a unique Kähler current $\omega_{KE} \in [K_{X_{\text{can}}}]$ on $X_{\text{can}}$ with bounded local potentials such that $\omega_{KE}$ is smooth on the regular part of $X_{\text{can}}$ and

$$\operatorname{Ric}(\omega_{KE}) = -\omega_{KE}. \quad (2.1)$$

2. If $0 < \operatorname{kod}(X) < n$, $X$ admits an algebraic fibration $\pi : X \to X_{\text{can}}$ whose general fibre is a smooth Calabi-Yau variety. There exists a unique Kähler current $\omega_{\text{can}} \in [K_{X_{\text{can}}} + L_{X/X_{\text{can}}}]$ on $X_{\text{can}}$ with bounded local potentials such that $\omega_{\text{can}}$ is smooth outside the singularities of $X$ and the singular values of $\pi$, and $\omega_{\text{can}}$ satisfies

$$\operatorname{Ric}(\omega_{\text{can}}) = -\omega_{\text{can}} + \omega_{WP}, \quad (2.2)$$

where $L_{X/X_{\text{can}}}$ is the $\mathbb{Q}$-Hodge bundle and $\omega_{WP}$ is a canonical current of Weil–Petersson type induced from the variation of the Calabi–Yau fibration.

In general, if $X$ is not minimal, the unique generalized Kähler–Einstein current on the canonical model $X_{\text{can}}$ is also constructed in [33]. Minimal projective manifolds of $\operatorname{kod}(X) = 0$ must have vanishing first Chern class, so there exists a unique smooth Ricci-flat Kähler metric in each Kähler class by Yau’s
Theorem [48]. The existence of singular Ricci-flat Kähler metrics is proved in [13] on singular Calabi–Yau varieties. Such metrics are the unique canonical metrics on projective varieties of non-negative Kodaira dimension and the generalized Kähler–Einstein equations can be viewed as an analytic version of the canonical bundle formula. They are candidates for the limiting metrics of the Kähler–Ricci flow.

2.2 Complex Monge–Ampère equations

In this section, we review some of the important results in degenerate complex Monge–Ampère equations developed by Kolodziej [20] and many others ([10, 13, 14, 50]). We start with some basic notations.

Definition 2.4 Let $X$ be an $n$-dimensional compact Kähler manifold.

1. A closed semi-positive $(1, 1)$-current $\omega$ on $X$ is said to have bounded local potentials if for any point $z \in X$, there exists an open neighborhood $U$ of $z$ such that $\omega = \sqrt{-1} \partial \bar{\partial} \varphi$ for some $\varphi \in PSH(U) \cap L^\infty(U)$.

2. A closed semi-positive $(1, 1)$-current $\omega$ with bounded local potentials is said to be big if $\int_X \omega^n > 0$.

If $\omega$ is a closed semi-positive $(1, 1)$-current with bounded local potentials on $X$, the corresponding measure $\omega^n$ is uniquely well-defined by the standard pluripotential theory.

Definition 2.5 Let $\omega$ be a closed semi-positive $(1, 1)$-current with bounded local potentials on $X$. An $\omega$-psh function is an upper semi-continuous function $\varphi : X \to [-\infty, \infty)$ such that $\omega + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0$. We denote by $PSH(X, \omega)$ the set of all $\omega$-psh functions on $X$.

In [20], Kolodziej proves the fundamental theorem on the existence of continuous solutions to the Monge–Ampère equation $(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = F \omega^n$, where $\omega$ is a Kähler form and $F \in L^p(X, \omega^n)$ for some $p > 1$ is non-negative. Its generalization was independently carried out in [11, 50] for the case when $[\omega]$ is rational, big and semi-ample and in [13, 15] for the general case when $[\omega]$ is big and semi-positive. These generalizations are summarized in the following.

Theorem 2.3 Let $X$ be an $n$-dimensional Kähler manifold and let $\omega$ be a smooth closed big and semi-positive $(1, 1)$-form on $X$. Then there exists a unique solution $\varphi \in PSH(X, \omega) \cap C^0(X)$ solving the following Monge–Ampère equation

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = F \Omega,$$
where $\Omega > 0$ is a smooth volume form on $X$ and $F \in L^p(X, \Omega)$ is a nonnegative function for some $p > 1$ satisfying $\int_X F \Omega = \int_X \omega^n$.

In [21], Kolodziej proves a stability result for solutions of the complex Monge–Ampère equations for Kähler classes. It is later improved by Dinew and Zhang [11] (also see [10] for more general cases) for big and semi-ample classes. The following is a version of their result.

**Theorem 2.4** Let $X$ be an $n$-dimensional compact Kähler manifold. Suppose $L \to X$ is a big and semi-ample line bundle. Let $\omega \in c_1(L)$ be a smooth closed big and semi-positive $(1, 1)$-form and $\Omega$ a smooth volume form on $X$. For any non-negative functions $F$ and $G \in L^p(X, \Omega)$ for some $p > 1$ with $\int_X F \Omega = \int_X G \Omega$, there exist $\varphi$ and $\psi \in PSH(X, \omega) \cap L^\infty(X)$ solving

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = F \Omega, \quad (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n = G \Omega$$

with

$$\max_X (\varphi - \psi) = \max_X (\psi - \varphi).$$

Then for any $\epsilon > 0$, there exists $C > 0$ depending on $\epsilon$ and $p$, $||F||_{L^p(X, \Omega)}$ and $||G||_{L^p(X, \Omega)}$ such that

$$||\varphi - \psi||_{L^\infty(X)} \leq C \left( ||F - G||_{L^1(X, \Omega)} \right)^{\frac{1}{p+3+\epsilon}}. \quad (2.3)$$

Theorem 2.4 can be generalized for the case where the right hand side of the Monge–Ampère equations contains terms such as $e^\varphi$ by Kolodziej’s argument in [21]. Theorem 2.4 also holds uniformly for certain families of $\omega$, such as $\omega + \epsilon \chi$ with a fixed Kähler metric $\chi$ and $\epsilon \in [0, 1]$. Also the sharper exponents are obtained in [11] and [10].

### 2.3 Singularities

We will have to study the behavior of the Kähler–Ricci flow on normal projective varieties with singularities because the original smooth manifold might be replaced by varieties with mild singularities through surgeries along the flow.

The pluripotential theory on normal varieties has been extensively studied (cf [16]). Let $X$ be a normal variety. A function $f$ on $X$ is said to be smooth if $f$ can be extended to a smooth function in a local embedding from $X$ to $\mathbb{C}^N$. A plurisubharmonic function is an upper semi-continuous function $\varphi : X \to [-\infty, \infty)$ which locally extends to a plurisubharmonic function in a local embedding from $X$ to $\mathbb{C}^N$. Any bounded plurisubharmonic function on $X_{reg}$,
the nonsingular part of $X$, can be uniquely extended to a plurisubharmonic function on $X$. Let $X$ be a normal projective variety and let $\omega$ be the restriction of a smooth semi-positive closed $(1, 1)$-form on $X$ from the ambient projective space of a projective embedding of $X$. We let $\text{PSH}(X, \omega)$ be the set of all upper semi-continuous functions $\varphi : X \to [-\infty, \infty)$ such that $\omega + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0$. In most cases, we choose the reference current $\omega$ to be the restriction of a Fubini-Study metric of the ambient projective space.

In this paper, we confine our discussions to projective varieties with mild singularities studied in the minimal model program in algebraic geometry.

**Definition 2.6** ([19]) Let $X$ be a normal projective variety such that $K_X$ is a $\mathbb{Q}$-Cartier divisor. Let $\pi : \tilde{X} \to X$ be a log resolution and $\{E_i\}_{i=1}^p$ the irreducible components of the exceptional locus $\text{Exc}(\pi)$ of $\pi$. There there exists a unique collection $a_i \in \mathbb{Q}$ such that

$$K_{\tilde{X}} = \pi^* K_X + \sum_{i=1}^p a_i E_i.$$  

Then $X$ is said to have

- terminal singularities if $a_i > 0$, for all $i$.
- canonical singularities if $a_i \geq 0$, for all $i$.
- log terminal singularities if $a_i > -1$, for all $i$.
- log canonical singularities if $a_i \geq -1$, for all $i$.

Terminal, canonical and log terminal singularities are always rational, while log canonical singularities are not necessarily rational. We can always assume that the exceptional locus is a $\mathbb{Q}$-divisor with simple normal crossings since $\pi$ is a log resolution.

**Definition 2.7** A projective variety $X$ is $\mathbb{Q}$-factorial if it is normal any $\mathbb{Q}$-Weil divisor on $X$ is $\mathbb{Q}$-Cartier.

Kodaira’s lemma states that for any big and nef line divisor $H$ on $X$, there always exists an effective divisor $E$ such that $H - \epsilon E$ is ample for any sufficiently small $\epsilon > 0$. Let $\pi : \tilde{X} \to X$ be a birational morphism between two projective varieties and $\text{Exc}(\pi)$ be the exceptional locus of $\pi$, where $\pi$ is not isomorphic. The following proposition is a special case of Kodaira’s lemma and the support of $E$ exactly coincides with $\text{Exc}(\pi)$ (see [9]).

**Proposition 2.1** If $X$ is a $\mathbb{Q}$-factorial projective variety, then for any ample $\mathbb{Q}$-divisor $H$ on $X$, there exists an effective divisor $E$ on $\tilde{X}$ whose support is $\text{Exc}(\pi)$ and $\pi^* H - \epsilon E$ is ample for sufficiently small $\epsilon > 0$. 

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It is also well-known that \( \mathbb{Q} \)-factoriality is preserved after divisorial contractions and flips in the minimal model program. \( \mathbb{Q} \)-factoriality is a sufficient condition in our discussion because we need the canonical divisor to be a Cartier \( \mathbb{Q} \)-divisor in order to define adapted measures. In fact, most results in this paper hold for \( \mathbb{Q} \)-Gorenstein projective varieties with log terminal singularities.

3 Monge–Ampère flows

3.1 Monge–Ampère flows with rough initial data

In this section, we will prove the smoothing property of the Kähler–Ricci flow with rough initial data. We will assume that \( X \) is an \( n \)-dimensional Kähler manifold.

**Definition 3.1** Suppose \( \omega_0 \) is a Kähler form and \( \Omega \) is a smooth volume form on \( X \). Then we define for \( p \in (0, \infty] \),

\[
PSH_p(X, \omega_0, \Omega) = \left\{ \varphi \in PSH(X, \omega_0) \cap L^\infty(X) \mid \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \in L^p(X) \right\}.
\]

Note that \( (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n \) is a well-defined Monge–Ampère mass for bounded \( \omega_0 \)-psh function \( \varphi \). In Definition 3.1, the Monge–Ampère mass \( (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n \) must be absolutely continuous with respect to \( \Omega \) in order to define \( \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \in L^p(X) \).

Suppose \( \varphi_0 \in PSH_p(X, \omega_0, \Omega) \) for some \( p > 1 \). Let

\[
F = \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_0)^n}{\Omega} \in L^p(X).
\]

By Kolodziej’s result [20],

\[ \varphi_0 \in C^0(X). \]

The following proposition shows that any element in \( PSH_p(X, \omega_0, \Omega) \) for \( p > 1 \) can be uniformly approximated by smooth \( \omega_0 \)-psh functions.

**Proposition 3.1** Suppose \( \varphi_0 \in PSH_p(X, \omega_0, \Omega) \) for some \( p > 1 \). There exist a sequence \( \{ \varphi_{0,j} \}_{j=1}^\infty \subset PSH(X, \omega_0) \cap C^\infty(X) \) such that

\[
\lim_{j \to \infty} ||\varphi_{0,j} - \varphi_0||_{L^\infty(X)} = 0. \quad (3.1)
\]
Proof Recall that $C^\infty(X)$ is dense in $L^p(X)$. Therefore there exists a sequence of positive functions $\{F_j\} \in C^\infty(X)$ such that $\int_X F_j \Omega = \int_X F \Omega$ and

$$\lim_{j \to \infty} ||F_j - F||_{L^p(X)} = 0.$$ 

We then consider the solutions of the following Monge–Ampère equations

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{0,j})^n = F_j \Omega. \quad (3.2)$$

Since $F_j \in C^\infty(X)$ and $F_j > 0$, $\varphi_{0,j} \in PSH(X, \omega_0) \cap C^\infty(X)$. Without loss of generality, we can assume

$$\sup_X (\varphi_{0,j} - \varphi_0) = \sup_X (\varphi_{0,j} - \varphi_0).$$

By the stability theorem of Kolodziej [21] (Theorem 2.4), we have

$$||\varphi_{0,j} - \varphi_0||_{L^\infty(X)} \leq C ||F_j - F||_{L^1(X)}^{\frac{1}{n+4}}$$

where $C$ only depends on $||F_j||_{L^p(X)}$ and $||F||_{L^p(X)}$. The proposition follows easily. \qed

Proposition 3.1 can also be proved by applying Richberg’s smoothing theorem to $(1 - \epsilon)\varphi_0$ with $\epsilon \to 0$. Alternatively, one can apply Demailly’s regularization theorem that every $\varphi \in PSH(X, \omega_0)$ can be approximated by a decreasing sequence $\varphi_j \in PSH(X, \omega_0) \cap C^\infty(X)$ and the convergence is uniform since $\varphi_0$ is continuous. More generally, Proposition 3.1 still holds when $\omega_0$ is a continuous closed semi-positive $(1, 1)$-form by a result of Blocki-Kolodziej (cf. Theorem 1 in [3]).

Now we have a sequence of smooth Kähler forms

$$\omega_{0,j} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{0,j}.$$ 

Let $\chi = \sqrt{-1} \partial \bar{\partial} \log \Omega \in [K_X]$ and $\omega_t = \omega_0 + t \chi$. It is well-known by the $\sqrt{-1} \partial \bar{\partial}$-lemma and straightforward calculations that the Kähler–Ricci flow with the initial Kähler metric $\omega_{0,j}$ can be reduced to the following Monge–Ampère flow

$$\begin{cases}
\frac{\partial \varphi_j}{\partial t} = \log \left( \frac{\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_j}{\Omega} \right), \\
\varphi_j(0, \cdot) = \varphi_{0,j}.
\end{cases} \quad (3.3)$$

The Monge–Ampère flow (3.3) corresponds to the Kähler–Ricci flow if we let $\omega_j = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_j$.
The Kähler–Ricci flow through singularities

\[ \frac{\partial \omega_j}{\partial t} = -Ric(\omega_j), \quad \omega_j|_{t=0} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{0,j}. \]

We define \( T_0 = \sup\{ t \geq 0 \mid [\omega_0] + t[K_X] \text{ is nef} \} \)
to be the first time when the Kähler class stops being positive along the Kähler–Ricci flow. It is well-known that \( T_0 \in (0, \infty) \) and by the result of [42] and the Monge–Ampère flow exists for \([0, T_0)\). The following lemma shows that the Monge–Ampère flows starting with \( \varphi_0, j \) approximate the same flow starting with \( \varphi_0 \).

**Lemma 3.1** For any \( 0 < T < T_0 \), there exists \( C > 0 \) such that for \( t \in [0, T] \) and all \( j \geq 1 \),

\[ ||\varphi_j||_{L^\infty(X)} \leq C. \] (3.4)

Furthermore, \( \{\varphi_j\} \) is a Cauchy sequence in \( L^\infty([0, T] \times X) \), i.e.,

\[ \lim_{j,k \to \infty} ||\varphi_j - \varphi_k||_{L^\infty([0,T] \times X)} = 0. \] (3.5)

**Proof** We apply the maximum principle to \( \varphi_j \). Let \( \varphi_{j,\text{max}}(t) = \max_{z \in X} \varphi_j(t, z) = \varphi_j(t, z_{j,t,\text{max}}) \) and \( \varphi_{j,\text{min}}(t) = \min_{z \in X} \varphi_j(z, t) = \varphi_j(t, z_{j,t,\text{min}}) \). Then

\[ (\sqrt{-1}\partial\bar{\partial}\varphi_j)|_{z_{j,t,\text{max}}} \leq 0, \quad (\sqrt{-1}\partial\bar{\partial}\varphi_j)|_{z_{j,t,\text{min}}} \geq 0, \]

and so

\[ \frac{\partial \varphi_{j,\text{max}}}{\partial t} \leq \max_X \left( \frac{\log \omega_t^\Omega}{\Omega} \right), \quad \frac{\partial \varphi_{j,\text{min}}}{\partial t} \geq \min_X \left( \frac{\log \omega_t^\Omega}{\Omega} \right) \]

Immediately we have

\[ \sup_{[0,T] \times X} |\varphi_j| \leq T \sup_{[0,T] \times X} \left| \frac{\omega_t^n}{\Omega} \right| + \sup_X |\varphi_{0,j}| \leq C \]

for some fixed \( C > 0 \) independent of \( j \).

Let \( \psi_{j,k} = \varphi_j - \varphi_k \). Then \( \psi_{j,k} \) satisfies the following equation

\[ \begin{aligned} \frac{\partial \psi_{j,k}}{\partial t} &= \log \left( \frac{\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_k + \sqrt{-1}\partial\bar{\partial}\psi_{j,k}}{\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_k} \right)^n, \\
\psi_{j,k}(0, \cdot) &= \varphi_{0,j} - \varphi_{0,k}. \end{aligned} \] (3.6)
By the maximum principle,
\[
\sup_{[0,T] \times X} |\varphi_j - \varphi_k| = \sup_{[0,T] \times X} |\psi_{j,k}| \leq \sup_{X} |\psi_{j,k}(0, \cdot)| \leq \sup_{X} |\varphi_{0,j} - \varphi_{0,k}|.
\]

Then
\[
\lim_{j,k \to \infty} ||\varphi_j - \varphi_k||_{L^\infty([0,T] \times X)} \leq \lim_{j,k \to \infty} ||\varphi_{0,j} - \varphi_{0,k}||_{L^\infty(X)} = 0.
\]

We also can bound the volume form along the Monge–Ampère flow, even though the initial volume form is only in \(L^p(X)\).

**Lemma 3.2** For any \(0 < T < T_0\), there exists \(C > 0\), such that for \(t \in (0, T]\) and all \(j \geq 1\),
\[
\frac{t^n}{C} \leq \frac{(\omega_t + \sqrt{-1} \bar{\partial} \partial \varphi_j)^n}{\Omega} \leq e^{\frac{C}{r}}.
\] (3.7)

**Proof** Let \(\Delta_j\) be the Laplacian operator associated to the Kähler form \(\omega_j = \omega_t + \sqrt{-1} \bar{\partial} \partial \varphi_j\). Straightforward calculations show that \((\frac{\partial}{\partial t} - \Delta_j)\varphi_j = tr_{\omega_j}(\chi)\).

Let \(H^+ = t\varphi_j - \varphi_j\). Then \(H^+(0, \cdot) = -\varphi_{0,j}\) is uniformly bounded and
\[
\left(\frac{\partial}{\partial t} - \Delta_j\right) H^+ = -tr_{\omega_j}(\omega_t - t \chi) + n = -tr_{\omega_j}(\omega_0) + n \leq n.
\]

By the maximum principle, \(H^+\) is uniformly bounded from above for \(t \in [0, T]\).

Let \(H^- = \varphi_j + A \varphi_j - n \log t\). Then \(H^-(t, \cdot)\) tends to \(\infty\) uniformly as \(t \to 0^+\) and there exist constants \(C_1, C_2\) and \(C_3 > 0\) such that
\[
\left(\frac{\partial}{\partial t} - \Delta_j\right) H^- = tr_{\omega_j}(A\omega_t + \chi) + A\varphi_j - \frac{n}{t} - An
\]
\[
\geq C_1 \left(\frac{\omega^n_0}{\omega^n_j}\right)^{\frac{1}{n}} + A \log \frac{\omega^n_j}{\omega^n_0} - \frac{n}{t} - An
\]
\[
\geq C_2 \left(\frac{\omega^n_0}{\omega^n_j}\right)^{\frac{1}{n}} - \frac{C_3}{t},
\]

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if $A$ is chosen sufficiently large such that $A\omega_t + \chi \geq \omega_0$ for $t \in [0, T]$. Then at the minimal point of $H^-$, the maximum principle gives

$$\omega^n_j \geq C_4 t^n \Omega.$$ 

It easily follows that $H^-$ is uniformly bounded from below for $t \in [0, T]$. Since $\varphi_j$ is uniformly bounded for $t \in [0, T]$, the lemma is proved. □

We now write $\omega_j = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_j$. The following smoothing lemma shows that the approximating metrics become uniformly bounded immediately along the Monge–Ampère flow.

**Lemma 3.3** For any $0 < T < T_0$, there exists $C > 0$ such that for $t \in (0, T]$ and all $j \geq 1$,

$$tr_{\omega_0}(\omega_j) \leq e^C t.$$

**Proof** This is a parabolic second order estimates similar to [22,47]. Since $\omega_j$ satisfies the Kähler–Ricci flow, the computation from [4] (cf. Proposition 2.5 in [35]) shows that for any $t \in [0, T]$, there exist uniform constants $C_1$ and $C_2 > 0$ such that

$$\left( \frac{\partial}{\partial t} - \Delta_j \right) \log tr_{\omega_0}(\omega_j) \leq C_1 tr_{\omega_j}(\omega_0) + C_2.$$

Let $H = t \log tr_{\omega_0}(\omega_j) - A\varphi_j$. Then if $A$ is sufficiently large, there exist uniform constants $C_3, C_4, \ldots, C_{10} > 0$ such that

$$\left( \frac{\partial}{\partial t} - \Delta_j \right) H \leq -tr_{\omega_j}(A\omega_t - C_1 t \omega_0) - A\varphi_j + \log tr_{\omega_0}(\omega_j) + C_3$$

$$\leq -C_4 tr_{\omega_j}(\omega_0) + C_5 \log tr_{\omega_j}(\omega_0) - C_6 \log \frac{\omega^n_j}{\omega^n_0} + C_7$$

$$\leq -C_8 \left( \frac{\omega^n_0}{\omega^n_j} \right)^{\frac{1}{n-1}} (tr_{\omega_0}(\omega_j))^{\frac{1}{n-1}} - C_9 \log t + C_{10}.$$ 

The last inequality follows from Lemma 3.2 and the inequality of arithmetic and geometric means. Suppose max$_{[0, T] \times X} H = H(t_0, z_0)$. Since $H(0, \cdot) = -\infty$, $t_0 > 0$. Then by the maximum principle, at $(t_0, z_0)$,

$$\log tr_{\omega_0}(\omega_j) \leq \log \left( \left( \log \frac{1}{t} \right)^{n-1} \left( \frac{\omega^n_j}{\omega^n_0} \right) \right) + C_{11} \leq C_{12} \frac{1}{t} + C_{13}.$$
and so $H$ is uniformly bounded from above at $(t_0, z_0)$. Since $H(0, \cdot) = 0$ and $\varphi_{0,j}$ is both uniformly bounded, $H$ is uniformly bounded for $t \in [0, T]$ and so we prove the lemma. □

Let $g^{(j)}$ be the Kähler metric associated to $\omega_j$. As in [26,48], we choose a fixed smooth background Kähler metric $\hat{g}$ and set

$$(\varphi_j)_{\rho\kappa} = \nabla_{m} \partial_{\rho} \varphi_j$$

and

$$A_{j} = \left( g^{(j)} \right)^{\rho\rho} \left( g^{(j)} \right)^{s\kappa} \left( g^{(j)} \right)^{m\iota} \left( g^{(j)} \right)_{\rho\kappa,m} \left( g^{(j)} \right)_{s\kappa,i},$$

where $\nabla$ is the covariant derivative with respect to $\hat{g}$ and $(g^{(j)})_{\rho\kappa,m}$ and $(g^{(j)})_{s\kappa,i}$ are covariant derivatives applied to $(g^{(j)})_{\rho\kappa}$ and $(g^{(j)})_{s\kappa}$ with respect to $\hat{g}$.

**Lemma 3.4** For any $0 < T < T_0$, there exist constants $\lambda > 0$ and $C > 0$ such that for $t \in [0, T]$ and all $j \geq 1$,

$$\| A_{j} \|_{C^0(X)} \leq C e^{\lambda t}.$$  \hspace{1cm} (3.9)

**Proof** We will use the techniques in [26]. We will consider a general complex Monge–Ampère flow $\frac{\partial g}{\partial t} = -Ric(g) + \alpha$, where $\alpha$ is a smooth closed $(1,1)$ form on a compact Kähler manifold $M$. Let $\hat{g}$ be a fixed smooth Kähler metric on $M$ and $A = g^{i\bar{j}} g^{\bar{k}l} g_{i\bar{j}k\bar{l}}$, where the covariant derivatives applied $g$ are calculated with respect to $\hat{g}$. In particular, $A = |\nabla h^{-1}|^2$, where $\nabla$ and tensor contractions are calculated with respect to the evolving metric $g$. The general formula for the evolution of $A$ is obtained in [26] (cf. Section 2 in [26]) as follows.

$$\frac{\partial}{\partial t} - \Delta \right) A = -|\nabla (\nabla hh^{-1})|^2 - |\nabla (\nabla hh^{-1})|^2$$

$$+ 2Re \left\{ \left( \frac{\partial}{\partial t} - \Delta \right) \nabla hh^{-1} \right\} \cdot \nabla hh^{-1} + (\nabla p hh^{-1})_i^j \left( \nabla q hh^{-1} \right)_k^l$$

$$\left\{ (h^{-1} + R)_{i\bar{j}k\bar{l}} - (h^{-1} + R)^{k\bar{l}}_{i\bar{j}g} g^{p\bar{q}} - (h^{-1} + R)^{p\bar{q}}_{i\bar{j}g} g^{k\bar{l}} \right\}.$$  

Using the evolution for $g$, we have

$$(h^{-1} + R)^{i\bar{j}}_{k} = \alpha_{k}^i, \quad \left( \frac{\partial}{\partial t} - \Delta \right) (\nabla p hh^{-1})_k^i = -\nabla^l \hat{R}_{lp}^i k + \nabla_j \alpha_{k}^i$$
and so there exist $C_1, C_2 > 0$ such that

$$
\left( \frac{\partial}{\partial t} - \Delta \right) A \leq -|\nabla(\nabla hh^{-1})|^2 - |\bar{\nabla}(\nabla hh^{-1})|^2 + C_1(1 + (tr_g(\hat{g}))^2) A + C_2(1 + tr_g(\hat{g}))^4.
$$

(3.10)

The evolution inequality for $tr_{\hat{g}}(g)$ is given by

$$
\left( \frac{\partial}{\partial t} - \Delta \right) tr_{\hat{g}}(g) \leq C_3 tr_{\hat{g}}(g) tr_g(\hat{g}) - A + C_4.
$$

(3.11)

We now can apply inequalities (3.11) to $g^{(j)}$ and $A_j$. Since there exist $C_5$ and $\lambda > 0$ such that on $X \times [0, T]$,

$$
tr_{g^{(j)}}(\hat{g}) + tr_{g^{(j)}}(\hat{g}) \leq C_5 e^{\frac{\lambda}{t}}
$$

for all $j$, there exist $C_6, C_7, C_8 > 0$ such that

$$
\left( \frac{\partial}{\partial t} - \Delta_j \right) \left( e^{-\frac{\Delta_j}{\tau}} A_j + C_6 e^{-\frac{\Delta_j}{\tau}} tr_g(\hat{g}^{(j)}) \right) \leq -C_7 e^{-\frac{\Delta_j}{\tau}} A_j + C_8.
$$

Since $e^{-\frac{\Delta_j}{\tau}} A_j = 0$ at $t = 0$, by the maximum principle and Lemma 3.3, $e^{-\frac{\Delta_j}{\tau}} A_j$ is uniformly bounded for $t \in [0, T]$ and for all $j$.

Proposition 3.2 For any $0 < \epsilon < T < T_0$ and $k \geq 0$, there exists $C_{\epsilon,T,k} > 0$ such that for all $j \geq 1$,

$$
||\varphi_j||_{C^k([\epsilon,T] \times X)} \leq C_{\epsilon,T,k}.
$$

(3.12)

Proof On $[\epsilon, T]$, the evolving Kähler metric $g_j$ and its inverse $(g_j)^{-1}$ are both uniformly bounded in $C^\alpha(X, \omega_0)$ for some $\alpha > 0$ by Lemmas 3.3 and 3.4. Let $\mathcal{L}$ be a local differential operator of order 1. Then

$$
\frac{\partial \mathcal{L}_\varphi_j}{\partial t} = \Delta_{\omega_j} \mathcal{L}_\varphi_j + tr_{\omega_j}(\eta_1)
$$

and

$$
\frac{\partial}{\partial t} \left( \frac{\partial \varphi_j}{\partial t} \right) = \Delta_{\omega_j} \frac{\partial \varphi_j}{\partial t} + tr_{\omega_j}(\eta_2)
$$

for some smooth and uniformly bounded $(1, 1)$-form $\eta_1$ and $\eta_2$. Both $\mathcal{L}_\varphi_j$ and $\frac{\partial \varphi_j}{\partial t}$ are uniformly bounded in $C^\alpha$ for $t \in [\epsilon, T]$. $\Delta_{\omega_j}$ is uniformly elliptic.
with uniform $C^\alpha$ coefficients. Therefore one can apply parabolic Schauder estimates for linear parabolic equations and the proposition is proved by the bootstrap argument (cf. Proposition 1.1 in [4]).

By Lemma 3.1, $\varphi_j$ is a Cauchy sequence in $L^\infty([0, T] \times X)$ and so $\varphi_j$ converges to $\varphi \in L^\infty([0, T] \times X)$ uniformly in $L^\infty([0, T] \times X)$. For any $0 < \delta < T$, $\varphi_j$ is uniformly bounded in $C^\infty((\delta, T] \times X)$. Therefore $\varphi_j$ converges to $\varphi$ in $C^\infty((0, T] \times X)$. Immediately we have the following consequence.

**Corollary 3.1** $\varphi \in C^0([0, T] \times X) \cap C^\infty((0, T] \times X)$.

Now we are ready to show the existence and uniqueness for the Monge–Ampère flow starting with $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$.

**Proposition 3.3** $\varphi$ is the unique solution of the following Monge–Ampère equation

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \right), & (0, T_0) \times X \\
\varphi(0, \cdot) = \varphi_0 
\end{cases}
\] (3.13)

in the space of $C^0([0, T_0) \times X) \cap C^\infty((0, T_0) \times X)$.

**Proof** It suffices to prove the uniqueness. Suppose there exists another solution $\varphi' \in C^0([0, T_0) \times X) \times C^\infty((0, T_0) \times X)$ of the Monge–Ampère flow (3.13). Let $\psi = \varphi' - \varphi$. Then

\[
\begin{cases}
\frac{\partial \psi}{\partial t} = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi + \sqrt{-1} \partial \bar{\partial} \psi)^n}{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n} \right), & (0, T_0) \times X \\
\psi(0, \cdot) = 0 
\end{cases}
\] (3.14)

By the maximum principle, $\max_X \psi(t, \cdot)$ is decreasing and $\min_X \psi(t, \cdot)$ is increasing on $(0, T_0)$. Since both of $\max_X \psi(t, \cdot)$ and $\min_X \psi(t, \cdot)$ are continuous on $[0, T_0)$ with $\max_X \psi(0, \cdot) = \min_X \psi(0, \cdot) = 0$, $\psi(t, \cdot) = 0$ for $t \in (0, T_0)$. The proposition follows immediately. \(\square\)

With the above preparations, we can show the smoothing property for the Kähler–Ricci flow with rough initial data.

**Theorem 3.1** Let $X$ be an $n$-dimensional Kähler manifold. Let $\omega_0$ be a Kähler form and $\Omega$ be a smooth volume form on $X$. Suppose that $\omega'_0 = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_0$
for some \( \varphi_0 \in \text{PSH}_p(\omega_0, \Omega) \) for some \( p > 1 \). Then there exists a unique family of smooth Kähler metrics \( \omega(t, \cdot) \in C^\infty((0, T_0) \times X) \) satisfying the following conditions.

1. \( \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega), \quad (0, T_0) \times X \).
2. There exists \( \varphi \in C^0([0, T_0) \times X) \cap C^\infty((0, T_0) \times X) \) such that \( \omega = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi \) and

\[
\lim_{t \to 0^+} ||\varphi(t, \cdot) - \varphi_0(\cdot)||_{L^\infty(X)} = 0.
\]

In particular, \( \omega(t, \cdot) \) converges in the sense of distribution to \( \omega'_0 \) as \( t \to 0^+ \).

Proof The Kähler–Ricci flow \( \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) \) is equivalent to the Monge–Ampère flow

\[
\sqrt{-1} \partial \bar{\partial} \left( \frac{\partial \varphi}{\partial t} - \log \left( \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi \right)^n \right) = 0. \tag{3.15}
\]

Then \( \frac{\partial \varphi}{\partial t} = \log \left( \frac{\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n + f(t) \) with \( \lim_{t \to 0^+} \varphi(t, 0) = \varphi_0 \) for a smooth function \( f(t) \) on \((0, T_0)\). Proposition 3.3 gives the existence of such a \( \varphi \) with \( f(t) = 0 \).

Suppose there is another solution \( \phi \in C^\infty((0, T_0) \times X) \cap C^0([0, T_0) \times X) \) to the Eq. (3.15). Then

\[
\frac{\partial \phi}{\partial t} = \log \left( \frac{\omega_t + \sqrt{-1} \partial \bar{\partial} \phi}{\Omega} \right)^n + f(t)
\]

for some smooth function \( f(t) \) on \((0, T_0)\). We can assume that \( \phi(0, \cdot) = \varphi_0 \) by subtracting a constant because \( \phi(t, \cdot) \) converges to a continuous \( \phi_0(\cdot) \) in \( C^0(X) \) as \( t \to 0 \), and \( \phi_0 \) differs from \( \varphi_0 \) by a constant. Then we consider the function \( \psi = \phi - \varphi \),

\[
\frac{\partial \psi}{\partial t} = \log \left( \frac{\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi + \sqrt{-1} \partial \bar{\partial} \psi}{\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi} \right)^n + f(t)
\]

with \( \psi(0, \cdot) = 0 \). We can now apply Proposition 3.3 for \( \psi(t, \cdot) - \int_0^t f(s) \, ds \) and conclude with \( \psi(t, \cdot) = \psi(t) = \int_0^t f(s) \, ds \). \qed

Theorem 3.1 shows that the Kähler–Ricci flow smooths out the initial semi-positive closed \((1, 1)\)-current \( \omega \) with bounded local potentials and \( \omega^n \in L^p(X) \) for some \( p > 1 \). It improves a result of Chen-Tian-Zhang [7] (also see [5,6]), where \( p > 3 \). We remark that the condition that \( p > 1 \) is essential for later estimates and geometric applications.
3.2 Monge–Ampère flows with degenerate initial data

In this section, we will investigate a family of Monge–Ampère flows with singular data on a smooth projective variety. The existence and uniqueness for the solutions will be proved.

We start with two conditions prescribing the singularity and degeneracy of the data that will be considered for certain Monge–Ampère flows. These conditions arise naturally in the geometric setting in later discussions.

**Condition A** Let $X$ be an $n$-dimensional projective manifold. Let $L_1 \to X$ be a big and semi-ample $\mathbb{Q}$-line bundle over $X$ and $L_2 \to X$ be a $\mathbb{Q}$-line bundle such that $[L_1 + \epsilon L_2]$ is still semi-ample for $\epsilon > 0$ sufficiently small.

Let $\omega_0 \in c_1(L_1)$ be a smooth semi-positive closed $(1, 1)$-form on $X$ and $\chi \in c_1(L_2)$ a smooth closed $(1, 1)$-form. Let $\omega_t = \omega_0 + t \chi$. We assume that $\omega_0$ at worst vanishes along a projective subvariety of $X$ to a finite order, that is, there exists an effective $\mathbb{Q}$-divisor $E_0$ on $X$ such that for any fixed Kähler metric $\vartheta$,

$$\omega_0 \geq C_\vartheta |S_{E_0}|^2_{h_{E_0}} \vartheta,$$

where $C_\vartheta > 0$ is a constant, $S_{E_0}$ is a defining section of $E_0$ and $h_{E_0}$ is a smooth hermitian metric on the line bundle associated to $E_0$.

Such an $\omega_0$ always exists. For example, let $m$ be sufficiently large such that $(L_1)^m$ is globally generated and let $\left\{S_j^{(m)}\right\}_{j=0}^{dm}$ be a basis of $H^0(X, (L_1)^m)$. We can then let

$$\omega_0 = \frac{1}{m} \sqrt{-1} \partial \bar{\partial} \log \sum_{j=0}^{dm} |S_j^{(m)}|^2.$$

In this case, $E_0$ can be taken as a divisor whose support contains the exceptional locus of the linear system $|mL_1|$.

**Condition B** Let $\Theta$ be a smooth volume form on $X$. Let $E = \sum_{i=1}^p a_i E_i$ and $F = \sum_{j=1}^q b_j F_j$ be effective $\mathbb{Q}$-divisors on $X$, where $E_i$ and $F_j$ are irreducible components of $E$ and $F$ with simple normal crossings. In addition, we assume $a_i \geq 0$ and $0 < b_j < 1$. Let $\Omega$ be a semi-positive $(n, n)$-form on $X$ such that $\int_X \Omega > 0$ and

$$\Omega = |S_E|_{h_E}^2 |S_F|_{h_F}^{-2} \Theta,$$ (3.16)
where $S_E$ and $S_F$ are the multi-valued holomorphic defining sections of $E$ and $F$, $h_E$ and $h_F$ are smooth hermitian metrics on the line bundles associated to $E$ and $F$.

Note that the condition $b_j \in (0, 1)$ makes $\Omega$ an integrable $(n, n)$-form on $X$. Furthermore, $\frac{\Omega}{\Theta}$ is in $L^p(X, \Theta)$ for some $p > 1$.

Since $L_1$ is big and semi-ample, by Kodaira’s lemma, there exists an effective $\mathbb{Q}$-divisor $\tilde{E}$ such that $[L_1] - \epsilon[\tilde{E}]$ is ample for any sufficiently small rational $\epsilon > 0$. Without loss of generality, we can always assume that the support of $\tilde{E}$ contains $E_0, E$ and $F$, i.e.,

$$\text{supp} E \cup \text{supp} F \cup \text{supp} E_0 \subset \text{supp} \tilde{E}.$$  

Let $S_{\tilde{E}}$ be the defining section of $\tilde{E}$ and $h_{\tilde{E}}$ a smooth hermitian metric on the line bundle associated to $[\tilde{E}]$ such that for sufficiently small $\epsilon > 0$,

$$\omega_0 - \epsilon R(h_{\tilde{E}}) > 0,$$

where $R(h_{\tilde{E}}) = -\sqrt{-1} \partial \bar{\partial} \log h_{\tilde{E}}$ is the curvature for the hermitian metric $h_{\tilde{E}}$. We can also scale $h_{\tilde{E}}$ and assume $|S_{\tilde{E}}|^2_{h_{\tilde{E}}} \leq 1$ on $X$.

Let $\omega_t = \omega_0 + t\chi$. We consider the following Monge–Ampère flow with the initial data $\varphi_0 \in PSH(X, \omega_0) \cap C^\infty(X)$.

$$\begin{align*}
\frac{\partial \varphi}{\partial t} & = \log \left( \frac{(\omega_0 + t\chi + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \right), \\
\varphi(0, \cdot) & = \varphi_0.
\end{align*}$$  \hspace{1cm} (3.17)

The Eq. (3.17) is not only degenerate in the sense that $[\omega_t]$ is not necessarily Kähler but also that $\Omega$ has zeros and poles along $E$ and $F$. The goal of the following discussion is to prove the existence and uniqueness of the solution for the Monge–Ampère flow (3.17) with appropriate assumptions. Let

$$T_0 = \sup\{ t \geq 0 \mid [L_1 + tL_2] \text{ is semi-ample} \}.$$  

If Condition A is satisfied, $T_0 > 0$ or $T_0 = \infty$. Furthermore, $L_1 + tL_2$ is big for any $t \in [0, T_0)$.

The following theorem is the main result of this section.

**Theorem 3.2** Let $X$ be an $n$-dimensional projective manifold. Suppose Condition A and Condition B are satisfied. Then for any $\varphi_0 \in PSH(X, \omega_0) \cap C^\infty(X)$, there exists a unique $\varphi \in C^\infty([0, T_0)) \times (X \setminus \tilde{E})$ with $\varphi \in$
$L^\infty([0, T] \times X)$ for any $0 < T < T_0$, satisfying the following Monge–Ampère flow

$$\begin{cases}
\frac{\partial \varphi}{\partial t} = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \right), & \text{on } [0, T_0) \times X \setminus \hat{E}, \\
\varphi(0, \cdot) = \varphi_0, & \text{on } X.
\end{cases}$$  

(3.18)

**Remark 3.3** For any fixed $T \in (0, T_0)$, we can assume that $\omega_t \geq \epsilon \omega_0$ for all $t \in [0, T]$, where $\epsilon > 0$ is a fixed sufficiently small number and it depends on $T$. Recall that $L_1 + tL_2$ is semi-ample and big for all $t \in [0, T)$. Then

$$[\omega_t] = \frac{t}{T} [\omega + T \chi] + \frac{T - t}{T} [\omega_0]$$

and $[\omega_0 + T \chi]$ is semi-positive and big. Then there exists $\phi \in \text{PSH}(X, \omega_0 + T \chi) \cap C^\infty(X)$, i.e., $\omega_0 + T \chi + \sqrt{-1} \partial \bar{\partial} \phi \geq 0$. Then for all $t \in [0, T]$,

$$\omega_t + \frac{t}{T} \sqrt{-1} \partial \bar{\partial} \phi = \frac{t}{T} (\omega_0 + T \chi) + \frac{t}{T} \sqrt{-1} \partial \bar{\partial} \phi + \frac{T - t}{T} \omega_0 \geq \frac{T - t}{T} \omega_0 \geq 0.$$

Then

$$\frac{\partial}{\partial t} \left( \varphi - \frac{t}{T} \phi \right) = \log \left( \frac{(\omega_0 + t(\chi + \frac{1}{T} \sqrt{-1} \partial \bar{\partial} \phi) + \sqrt{-1} \partial \bar{\partial} (\varphi - \frac{t}{T} \phi))^n}{\Omega} \right) - \frac{1}{T} \phi.$$

Let $\omega'_0 = \omega_0$, $\chi' = \chi + \frac{1}{T} \sqrt{-1} \partial \bar{\partial} \phi$, $\omega'_t = \omega_0 + t \chi'$, $\Omega' = \Omega e^{\frac{\phi}{T}}$ and $\varphi' = \varphi - \frac{t}{T} \phi$. We have

$$\frac{\partial}{\partial t} \varphi' = \log \left( \frac{(\omega'_t + \sqrt{-1} \partial \bar{\partial} \varphi')^n}{\Omega'} \right).$$

Then $\omega'_t \geq \frac{T - t}{T} \omega_0$ and **Condition A** and **Condition B** are still satisfied for $\omega'_0$ and $\Omega'$.

From now on, we fix $T \in (0, T_0)$ and assume without loss of generality from the previous remark that $\omega_t$ is bounded from below by $\epsilon \omega_0$ for sufficiently small $\epsilon > 0$. In order to prove Theorem 3.2, we have to perturb Eq. (3.18) in order to obtain smooth approximating solutions. Let

$$\omega_{t,s} = \omega_0 + t \chi + s \theta.$$
and

\[ \Omega_{w,r} = \frac{r + |S_E|^2_{h_E}}{w + |S_F|^2_{h_F}} \Theta \]

be the perturbations of \( \omega_t \) and \( \Omega \) for \( s, w, r \in [0, 1] \). In particular, \( \omega_{t,0} = \omega_t \) and \( \Omega_{0,0} = \Omega \). Since \( \vartheta \) is Kähler, \( \omega_{t,s} \) is Kähler for \( s > 0 \) and \( t \in [0, T_0) \).

Then we consider the following well-defined family of Monge–Ampère flows with the fixed initial data \( \varphi_0 \).

\[
\begin{cases}
\frac{\partial \varphi_{s,w,r}}{\partial t} = \log \left( \frac{\omega_{t,s} + \sqrt{-1} \partial \bar{\partial} \varphi_{s,w,r}}{\Omega_{w,r}} \right)^n, \\
\varphi_{s,w,r}(0, \cdot) = \varphi_0.
\end{cases}
\]

(3.19)

**Lemma 3.5** For any \( s, w, r \in (0, 1] \), there is a unique smooth solution \( \varphi_{s,w,r} \) of the Monge–Ampère flow (3.19) on \([0, T_0) \times X\).

**Proof** The proof follows the argument in [45] and [42] for the maximal existence time of the Kähler–Ricci flow. Notice that \([\omega_{s,t}] = [\omega_0 + t \chi] + s[\vartheta] \) stays positive on \([0, T_0) \) by the definition of \( T_0 \) for \( s > 0 \). \( \varphi_0 \) and \( \Omega_{w,r} \) are both smooth for fixed \( w, r \in (0, 1] \). For fixed \( s, w, r \in (0, 1] \), the short time existence theorem for parabolic Monge–Ampère implies that there exists \( \tilde{t} > 0 \) such that there exists a unique smooth solution \( \varphi_{s,w,r} \) for \( t \in [0, \tilde{t}) \).

Without loss of generality, we can assume \( \tilde{t} \) is the maximal existence time for the solution \( \varphi_{s,w,r} \). If \( \tilde{t} < T_0 \), then we can apply a priori estimates by the maximum principle so that for any \( k > 0 \), \( ||\varphi_{s,w,r}||_{C^k(X)} \leq C_k \) for some \( C_k = C_k(s, w, r) \). Then \( \varphi_{s,w,r} \) can be extended beyond \( \tilde{t} \), which is contradiction to the fact that \( \tilde{t} \) is the maximal existence time. \( \square \)

**Lemma 3.6** Let \( F_{w,r} = \frac{\Omega_{w,r}}{\partial^n} \). Then there exist \( p > 1 \) and \( C > 0 \) such that for \( w, r \in [0, 1] \) and

\[
||F_{w,r}||_{L^p(X, \partial^n)} \leq C.
\]

(3.20)

**Proof** There exists a constant \( C > 0 \) such that

\[
\int_X (F_{w,r})^p \partial^n = \int_X (F_{w,r})^{p-1} \Omega_{w,r}
\]

\[
= \int_X (F_{w,r})^{p-1}(r + |S_E|^2_{h_E} (w + |S_F|^2_{h_F}) \Theta
\]

\[
\leq C \int_X (F_{w,r})^{p-1}|S_F|_{h_F}^{-2} \Theta.
\]
Since $F_{w,r}$ has at worst poles along $\tilde{E}$ and the vanishing order of $|SF|_{h_F}^2$ is strictly less than 2, by choosing $p - 1 > 0$ sufficiently small, $\int_X (F_{w,r})^p \vartheta^n$ is uniformly bounded from above. □

We can apply the results for degenerate complex Monge–Ampère equations as $F_{w,r}$ is uniformly bounded in $L^p(X)$ for some $p > 1$.

**Lemma 3.7** For any $0 < T < T_0$, there exists $C > 0$ such that for all $s, w, r \in (0, 1]$,

$$||\varphi_{s,w,r}||_{L^\infty([0,T] \times X)} \leq C$$

**Proof** We first prove the uniform upper bound for $\varphi_{s,w,r}$. We define for $t \in [0, T]$

$$\alpha_{s,w,r}(t) = \frac{\int_X \Omega_{w,r}}{[\omega_{t,s}]^n}. \quad (3.21)$$

It is easy to see that $\alpha_{s,w,r}(t)$ and $(\alpha_{s,w,r}(t))^{-1}$ are uniformly bounded for $s, w, r \in (0, 1]$ and $t \in [0, T]$. Also there exists $C > 0$ such that

$$\frac{\partial}{\partial t} \int_X \varphi_{s,w,r} \Omega_{w,r} = \int_X \log \left( \frac{\omega_{t,s} + \sqrt{-1} \partial \overline{\partial} \varphi_{s,w,r}}{\Omega_{w,r}} \right)^n \Omega_{w,r} \leq \left( \int_X \Omega_{w,r} \right) \left( \log \frac{\int_X (\omega_{t,s} + \sqrt{-1} \partial \overline{\partial} \varphi_{s,w,r})^n}{\int_X \Omega_{w,r}} \right)$$

$$= - (\log \alpha_{s,w,r}(t)) \int_X \Omega_{w,r} \leq C$$

and so $\int_X \varphi_{s,w,r} \Omega_{w,r} \leq CT$.

On the other hand, $\varphi_{s,w,r} \in PSH(X, A \vartheta)$ for some fixed constant $A > 0$ satisfying $A \vartheta > \omega_{t,s}$ for all $t \in [0, T]$ and $s \in (0, 1]$. Then by Hörmander-Tian’s estimate [39], there exist $\alpha > 0$ and $C_\alpha > 0$ such that for all $s, r \in (0, 1]$ and $t \in [0, T]$,

$$\int_X e^{-\alpha(\varphi_{s,w,r} - \sup_X \varphi_{s,w,r})} \vartheta^n \leq C_\alpha.$$ 

By Hölder’s inequality, there exist $\alpha' > 0$ and $C_{\alpha'} > 0$ such that for all $s, w, r \in (0, 1]$ and $t \in [0, T]$,

$$\int_X e^{-\alpha'(\varphi_{s,w,r} - \sup_X \varphi_{s,w,r})} \Omega_{w,r} \leq C_{\alpha'}.$$
Applying Jensen’s inequality, there exists a constant \( C > 0 \) such that for all \( s, w, r \in (0, 1) \) and \( t \in [0, T] \),

\[
\sup_X \varphi_{s,w,r} - \int_X \varphi_{s,w,r} \Omega_{w,r} \leq C.
\]

It follows that \( \sup_X \varphi_{s,w,r} \) is uniformly bounded above.

Now it suffices to obtain a uniform lower bound for \( \varphi_{s,w,r} \). Let \( \kappa = \epsilon \omega_0 \), where the fixed constant \( \epsilon > 0 \) is sufficiently small such that

\[
2 \kappa \leq \omega_t
\]

for all \( t \in [0, T] \). We consider the following family of Monge–Ampère equations for \( w, r, s \in [0, 1] \).

\[
(\kappa + s \vartheta + \sqrt{-1} \partial \overline{\partial} \varphi_{s,w,r})^n = C_{s,w,r} \Omega_{w,r},
\]

with the normalization conditions \( [\kappa + s \vartheta]^n = C_{s,w,r} \int_X \Omega_{w,r} \) and \( \sup_X \varphi_{s,w,r} = 0 \).

\( C_{s,w,r} \) is uniformly bounded for all \( s, w, r \in (0, 1) \) by integrating both sides of Eq. (3.22). By Lemma 3.6, there exists \( p > 1 \) such that \( \frac{C_{s,w,r} \Omega_{w,r}}{(\kappa + s \vartheta)^n} \) is uniformly bounded in \( L^p(X, (\kappa + s \vartheta)^n) \) for all \( s, w, r \in (0, 1) \). By the results in [10,13], \( \phi_{s,w,r} \in C^\infty(X) \) satisfies the following uniform estimates: there exists a uniform constant \( C > 0 \) such that for all \( w, r, s \in (0, 1) \),

\[
||\phi_{s,w,r}||_{L^\infty(X)} \leq C.
\]

(3.23)

Let \( \psi_{s,w,r}(t, \cdot) = \varphi_{s,w,r}(t, \cdot) - \phi_{s,w,r} \). The evolution equation for \( \psi_{s,w,r} \) is given by the following formula for \( t \in [0, T] \)

\[
\frac{\partial}{\partial t} \psi_{s,w,r} = \log \frac{(\kappa + s \vartheta + \sqrt{-1} \partial \overline{\partial} \varphi_{s,w,r} + (\omega_t - \kappa) + \sqrt{-1} \partial \overline{\partial} \psi_{s,w,r})^n}{(\kappa + s \vartheta + \sqrt{-1} \partial \overline{\partial} \phi_{s,w,r})^n} + \log C_{s,w,r}.
\]

Applying the maximum principle, there exists \( C > 0 \) such that for all \( s, w, r \in (0, 1) \) and \( t \in [0, T] \),

\[
\psi_{s,w,r} \geq -C
\]

and so

\[
\varphi_{s,w,r}(t, \cdot) \geq \phi_{s,w,r}(\cdot) - C.
\]
The uniform bound for $\varphi_{s,w,r}$ immediately gives the uniform lower bound for $\varphi_{s,w,r}$. Combined with the upper bound for $\varphi_{s,w,r}$, the proof for the lemma is complete. \qed

Lemma 3.8 For any $T \in (0, T_0)$, there exist $C, \alpha > 0$ such that for all $t \in [0, T]$ and $s, w, r \in (0, 1]$

$$\left| \frac{\partial}{\partial t} \varphi_{s,w,r} \right| \leq C + \log |S_E|_{h_E}^{-2\alpha}.$$  

Proof We would like to bound $\frac{\partial \varphi_{s,w,r}}{\partial t}$ by $\varphi_{s,w,r}$ and $|S_E|_{h_E}^2$. Let $\Delta_{s,w,r}$ be the Laplace operator with respect to the Kähler metric $\omega_{s,w,r}$. Notice that

$$\frac{\partial}{\partial t} \varphi_{s,w,r} = \Delta_{s,w,r} \varphi_{s,w,r} + tr_{\omega_{s,w,r}} (\chi).$$

Let $H^+ = \left( \varphi_{s,w,r} - A^2 \varphi_{s,w,r} + A \log |S_E|_{h_E}^2 \right)$. $H^+(0, \cdot)$ is uniformly bounded from above for $A > 0$ sufficiently large. Then for $A$ sufficiently large, we have

$$\frac{\partial}{\partial t} H^+ = \Delta_{s,w,r} H^+ - tr_{\omega_{s,w,r}} \left( A^2 \omega t,s - \chi - A R (h_E) \right) - A^2 \varphi_{s,w,r} + nA^2 \leq \Delta_{s,w,r} H^+ - A^2 \varphi_{s,w,r} + nA^2$$

$$= \Delta_{s,w,r} H^+ - A^2 H^+ + A^2 \left( -A^2 \varphi_{s,w,r} + A \log |S_E|_{h_E}^2 \right) + nA^2$$

$$\leq \Delta_{s,w,r} H^+ - A^2 H^+ + C,$$

The first inequality in the above estimates follows from the fact that $\omega_0 - \epsilon R(h_E) > 0$ for all sufficiently small $\epsilon > 0$ and the last inequality follows from the uniform $L^\infty$-estimates for $\varphi_{s,w,r}$. By the maximum principle, $H^+$ is uniformly bounded above and so there exist $C_1$ and $C_2$ such that

$$\varphi_{s,w,r} \leq C_1 + C_2 \log |S_E|_{h_E}^2.$$

To estimate the lower bound of $\varphi_{s,w,r}$, we define

$$H^- = \varphi_{s,w,r} + A^2 \varphi_{s,w,r} - A \log |S_E|_{h_E}^2$$

for sufficiently large $A$. Then straightforward calculation shows that there exist constants $C_3, C_4, \ldots, C_7$ such that
\[ \frac{\partial}{\partial t} H^- = \Delta_{s, w, r} H^- + tr_{\omega_{s, w, r}} \left( A^2 \omega_{t, s} + \chi - A R \left( h_\hat{E} \right) \right) + A^2 \phi_{s, w, r} - nA^2 \]
\[ \geq \Delta_{s, w, r} H^- + C_3 \left( \frac{\omega_{t, s}^n}{\omega_{s, w, r}^n} \right)^{\frac{1}{n}} + A^2 \phi_{s, w, r} - C_4 \]
\[ = \Delta_{s, w, r} H^- + C_3 \left( \frac{\omega_{t, s}^n}{\omega_{s, w, r}^n} \right)^{\frac{1}{n}} + A^2 \log \frac{\omega_{s, w, r}^n}{\omega_{t, s}^n} + A^2 \log \frac{\omega_{t, s}^n}{\Omega_{w, r}} - C_4 \]
\[ \geq \Delta_{s, w, r} H^- - A^2 \log \frac{\omega_{s, w, r}^n}{\omega_{t, s}^n} + A^2 \log \frac{\omega_{t, s}^n}{\Omega_{w, r}} - C_5 \]
\[ = \Delta_{s, w, r} H^- - A^2 H^- - A^3 \log |S_\hat{E}|^2 h_\hat{E} + 2A^2 \log \frac{\omega_{t, s}^n}{\Omega_{w, r}} - C_6 \]
\[ \geq \Delta_{s, w, r} H^- - A^2 H^- - C_7, \]

where the second inequality in the above estimates follows from the simple inequality \( x \geq a \log x - a^2 \) for all \( a, x > 0 \).

Then a similar argument by the maximum principle gives the lower bound for \( H^- \) and \( \phi_{s, w, r} \).

**Lemma 3.9** For any \( T \in (0, T_0) \), there exist \( C, \alpha > 0 \) such that for all \( t \in [0, T] \) and \( s, r, w \in (0, 1) \)
\[ |tr_\theta (\omega_{s, w, r})| \leq C |S_\hat{E}|^{-2\alpha}. \]

**Proof** Straightforward calculations show that for some constant \( C > 0 \),
\[ \left( \frac{\partial}{\partial t} - \Delta_{s, w, r} \right) \log tr_\theta (\omega_{s, w, r}) \leq C tr_{\omega_{s, w, r}} (\theta) + \frac{tr_{\omega_{s, w, r}} (Ric (\Omega_{w, r}))}{tr_{\omega_{s, w, r}} (\theta)} + C. \]
Define
\[ H = \log tr_\theta (\omega_{s, w, r}) - A^2 \phi_{s, w, r} + A \log |S_\hat{E}|^2 h_\hat{E}. \]
Then for sufficiently large \( A > 0 \), there exist uniform constants \( C_1 \) and \( C_2 \) such that
\[ \left( \frac{\partial}{\partial t} - \Delta_{s, w, r} \right) H \leq -tr_{\omega_{s, w, r}} \left( A^2 \omega_{t, s} - AR \left( h_\hat{E} \right) \right) \]
\[ + \frac{tr_{\omega_{s, w, r}} (Ric (\Omega_{w, r}))}{tr_{\omega_{s, w, r}} (\theta)} - A^2 \log \frac{\omega_{s, w, r}^n}{\Omega_{w, r}} + C_1 \]
\[ \leq -A tr_{\omega_{s, w, r}} (\theta) + \frac{tr_{\omega_{s, w, r}} (Ric (\Omega_{w, r}))}{tr_{\omega_{s, w, r}} (\theta)} - A^2 \log \frac{\omega_{s, w, r}^n}{\Omega_{w, r}} + C_2. \]
Suppose $\max_{[0,t] \times X} H = H(t_0, z_0)$. Then $z_0 \in X \setminus \tilde{E}$ and at $(t_0, z_0)$, there exist $\alpha_1, C_3$ and $C_4$ such that

$$tr_{\omega_{s,w,r}}(\theta) \leq A^{-1} \frac{tr_{\omega_{s,w,r}}(A \text{Ric}(\Omega_{w,r}))}{tr_{\omega_{s,w,r}}(\theta)} - A \log \frac{\omega_{s,w,r}^n}{\Omega_{w,r}} + C_3 \leq C_4 |S_{\tilde{E}}|^{-2\alpha_1}. $$

Applying the mean value inequality and Lemma 3.8, there exist $\alpha_2$ and $C_5$

$$tr_\theta(\omega_{s,w,r}) \leq C_5 |S_{\tilde{E}}|^{-2\alpha_2}. $$

Therefore $H(t_0, z_0)$ is uniformly bounded from above. The lemma then follows immediately from the uniform $L^\infty$ bound for $\varphi_{s,w,r}$. \hfill \Box

Let $g_{s,w,r}$ be the Kähler metric associated to $\omega_{s,w,r}$ and $\nabla_{s,w,r}$ be the gradient operator associated to the Kähler form $\omega_{s,w,r}$. As in [26,48], we set

$$A_{s,w,r} = (g_{s,w,r})^{pv} (g_{s,w,r})^{sk} (g_{s,w,r})^{m2} (g_{s,w,r})^{pkm} (g_{s,w,r})^{fs, f_r},$$

where the covariant derivatives applied to $g_{s,w,r}$ are calculated with respect to a fixed smooth background Kähler metric $\hat{g}$.

**Lemma 3.10** For any $T \in (0, T_0)$, there exist $C$, $\alpha > 0$ such that for all $t \in [0, T]$ and $s, r, w \in (0, 1]$

$$\sup_{X \times [0, T]} A_{s,w,r} \leq C |S_{\tilde{E}}|^{-2\alpha}. $$

**Proof** We will apply the same argument in the proof of. First the evolution equation for $g_{s,w,r}$ is given by

$$\left( \frac{\partial}{\partial t} - \Delta_{s,w,r} \right) g_{s,w,r} = \alpha_{s,w,r}$$

for some closed smooth $(1, 1)$-form $\alpha_{s,w,r}$. In particular, we can choose sufficiently large $\lambda > 0$ such that $\alpha_{s,w,r}$ and its covariant derivatives of order less than 4 with respect to $\hat{g}$ are bounded by $C_1 |S_{\tilde{E}}|^{-2\lambda} \hat{g}$ for all $s, w, r \in (0, 1]$ and $t \in [0, T]$, where $C_1 > 0$ is a fixed constant. We can now derive the following differential inequality by the general calculations in the proof of Lemma 3.4

$$\left( \frac{\partial}{\partial t} - \Delta_{s,w,r} \right) A_{s,w,r} \leq - \frac{|\nabla_{s,w,r} A_{s,w,r}|^2}{A_{s,w,r}} + C_2 |S_{\tilde{E}}|^{-2\lambda} A_{s,w,r} + C_3 |S_{\tilde{E}}|^{-2\lambda}. $$

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Also the calculations in the proof of the second order estimates give
\[
\left( \frac{\partial}{\partial t} - \Delta_{s,w,r} \right) \text{tr}_{\tilde{g}}(g_{s,w,r}) \leq C_4 |S_{\tilde{E}}|_{h_{\tilde{E}}}^{-2\lambda} - A_{s,w,r}.
\]

We now let \( H_{s,w,r} = \left( |S_{\tilde{E}}|_{h_{\tilde{E}}} \right)^{4\beta} A_{s,w,r} + B \left( |S_{\tilde{E}}|_{h_{\tilde{E}}} \right)^{2\beta} \text{tr}_{\tilde{g}}(g_{s,w,r}) \) for some fixed constants \( B >> 1 \) and \( \beta >> \lambda \). Then there exists \( C_5 \) such that
\[
\left( \frac{\partial}{\partial t} - \Delta_{s,w,r} \right) H_{s,w,r} \leq -H_{s,w,r} + C_5.
\]

The maximum principle immediately implies that there exists \( C_6 > 0 \) such that \( H_{s,w,r} \leq C_6 \) and we obtain the desired uniform bound for \( A_{s,w,r} \).

The following proposition gives a uniform bound for the approximating Kähler metrics \( \omega_{s,w,r} \) away from \( \tilde{E} \).

**Proposition 3.4** For any \( T \in (0, T_0) \), \( K \subset \subset X \setminus \tilde{E} \) and \( k > 0 \), there exists \( C_{k,K,T} \) such that
\[
||\varphi_{s,w,r}||_{C^k([0,T] \times K)} \leq C_{k,K,T}.
\]

**Proof** For any \( T \in (0, T_0) \), \( K \subset \subset X \setminus \tilde{E} \), \( \omega_{s,w,r} \) and its inverse is uniformly bounded in \( C^\alpha(K) \). Then the proof of Proposition 3.2 can be immediately applied after linearizing the Monge–Ampère flow on \([0,T] \times K\). \( \square \)

We can now construct a solution of the original Monge–Ampère flow by the approximating solutions \( \varphi_{s,w,r} \).

**Lemma 3.11** The following monotonicity conditions hold for \( \varphi_{s,w,r} \).

1. For any \( 0 < r_1 \leq r_2 \leq 1 \) and \( s, w \in (0, 1] \),
\[
\varphi_{s,w,r_1} \geq \varphi_{s,w,r_2}.
\]
2. For any \( 0 < w_1 \leq w_2 \leq 1 \) and \( s, r \in (0, 1] \),
\[
\varphi_{s,w_1,r} \leq \varphi_{s,w_2,r}.
\]
3. For any \( 0 < s_1 \leq s_2 \leq 1 \) and \( w, r \in (0, 1] \),
\[
\varphi_{s_1,w,r} \leq \varphi_{s_2,w,r}.
\]
The proof is a straightforward application of the maximum principle for the following quantities \( \varphi_{s, w, r_1} - \varphi_{s, w, r_2}, \varphi_{s, w_1, r} - \varphi_{s, w_2, r}, \varphi_{s_1, w, r} - \varphi_{s_2, w, r} \).

Fix \( T \in (0, T_0) \), for each \( t \in [0, T] \), let

\[
\varphi_{s, w}(t, \cdot) = \left( \limsup_{r \to 0} \varphi_{s, w, r}(t, \cdot) \right)^*
\]

where \( f^*(z) = \lim_{\delta \to 0} \sup_{B_\delta(z)} f(\cdot) \). Then \( \varphi_{s, w} \in PSH(X, \omega_t, s) \cap L^\infty(X) \cap C^\infty(X \setminus \tilde{E}) \) and \( \varphi_{s, w, r} \) converges to \( \varphi_{s, w} \) on \( X \setminus \tilde{E} \) locally in \( C^\infty \)-topology by estimates from Lemma 3.7 and Proposition 3.4. The following monotonicity also holds and follows easily from the above results.

**Lemma 3.12** For any \( 0 < s_1 \leq s_2 \leq 1 \) and \( w \in (0, 1] \),

\[
\varphi_{s_1, w} \leq \varphi_{s_2, w}.
\]

Also for any \( 0 < w_1 \leq w_2 \leq 1 \) and \( s \in (0, 1] \),

\[
\varphi_{s, w_1} \leq \varphi_{s, w_2}.
\]

Furthermore, for any \( T \in (0, T_0) \), \( K \subset \subset X \setminus \tilde{E} \) and \( k > 0 \), there exists \( C_{K, k, T} > 0 \) such that on \( [0, T] \),

\[
||\varphi_{s, w}||_{C^k(K)} \leq C_{K, k, T}.
\]

Let

\[
\varphi_s(t, \cdot) = \lim_{w \to 0} \varphi_{s, w}(t, \cdot).
\]

Then \( \varphi_s \in PSH(X, \omega_t, s) \cap L^\infty(X) \cap C^\infty(X \setminus \tilde{E}) \) and \( \varphi_{s, w} \) converges to \( \varphi_s \) on \( X \setminus \tilde{E} \) locally in \( C^\infty \)-topology.

**Lemma 3.13** For any \( 0 < s_1 \leq s_2 \leq 1 \),

\[
\varphi_{s_1} \leq \varphi_{s_2}.
\]

Furthermore, for any \( T \in (0, T_0) \), \( K \subset \subset X \setminus \tilde{E} \) and \( k > 0 \), there exists \( C_{K, k, T} > 0 \) such that such that

\[
||\varphi_s||_{C^k([0, T] \times K)} \leq C_{K, k, T}.
\]
Let \( \varphi = \lim_{s \to 0} \varphi_s \). Since \( \varphi_s \) is decreasing as \( s \to 0 \) is and \( \varphi_s \) is bounded below uniformly, we have

\[
\varphi \in PSH(X, \omega_t, 0) \cap L^\infty(X) \cap C^\infty(X \setminus \tilde{E}).
\]

Furthermore, for any \( K \subset X \setminus \tilde{E} \),

\[
\varphi_s \to \varphi
\]

in \( C^\infty([0, T] \times K) \). The following corollary is then immediate.

**Corollary 3.2** \( \varphi \) satisfies the following Monge–Ampère flow

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = \log \left( \frac{\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n, & \text{on } [0, T] \times (X \setminus \tilde{E}) \\
\varphi(0, \cdot) = \varphi_0, & \text{on } X.
\end{cases}
\]

(3.24)

Furthermore, \( \varphi \in L^\infty([0, T] \times X) \) for any \( 0 < T < T_0 \).

Corollary 3.2 gives the existence of the solution for Theorem 3.2. In order to prove the uniqueness of the solution of the Monge–Ampère flow (3.24), we consider a family of new Monge–Ampère flows with one more parameter.

Let \( \omega_t^{(\delta)} = (1 - \delta) \omega_0 + t \chi + s \vartheta = \omega_t, s - \delta \omega_0 \). Then for fixed \( T \in [0, T_0] \), since we assume that \( \omega_t \geq \epsilon \omega_0 \) for some \( \epsilon > 0 \), there exists \( \delta_0 > 0 \) such that \( (1 - \delta) \omega_0 + t \chi \geq \omega_t - \delta \omega_0 \geq \delta_0 \omega_0 \) for all \( t \in [0, T] \) and \( \delta \in [-\delta_0, \delta_0] \). The following family of Monge–Ampère flows admit smooth solutions in \( C^\infty([0, T] \times X) \)

\[
\begin{cases}
\frac{\partial \varphi^{(\delta)}}{\partial t} = \log \left( \frac{\omega_t^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)}}{\Omega_{w, r}} \right)^n, & \text{on } [0, T] \times X \\
\varphi^{(\delta)}(0, \cdot) = (1 - \delta)\varphi_0.
\end{cases}
\]

(3.25)

If we fix \( s, w, r \in (0, 1] \), the linearization of Eq. (3.25) with respect to variation in \( \delta \) is given by

\[
\begin{cases}
\frac{\partial}{\partial t} f_{s, w, r}^{(\delta)} = \Delta_{s, w, r}^{(\delta)} \left( f_{s, w, r}^{(\delta)} \right) - tr_{\omega_{s, w, r}^{(\delta)}} (\omega_0), \\
f_{s, w, r}^{(\delta)}(0, \cdot) = -\varphi_0,
\end{cases}
\]

(3.26)

where \( \omega_{s, w, r}^{(\delta)} = \omega_t^{(\delta)} + \varphi^{(\delta)} \) and \( \Delta_{s, w, r}^{(\delta)} \) is the Laplace operator associated with \( \omega_{s, w, r}^{(\delta)} \). The linear equation (3.26) has a unique smooth solution on \([0, T] \times X\).
for fixed $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$ because $\omega_{s,w,r}^{(\delta)}$ is a smooth family of Kähler metrics on $[0, T] \times X$. Therefore we can apply the implicit function theorem for each $\delta \in [-\delta_0, \delta_0]$ after fixing $s, w, r \in (0, 1]$ and this shows that for fixed $s, w, r \in (0, 1], \varphi_{s,w,r}^{(\delta)}$ is a family of smooth functions on $[0, T] \times X$ with smooth dependence on $\delta$.

**Lemma 3.14** For any $T \in (0, T_0)$, there exist $C$ and $\delta_0 > 0$ such that for $s, w, r \in (0, 1], t \in [0, T]$ and $\delta \in [-\delta_0, \delta_0],$

$$||\varphi_{s,w,r}^{(\delta)}||_{L^\infty([0,T] \times X)} \leq C.$$ 

**Proof** The same proof of Lemma 3.7 can be applied, using the Jensen’s inequality and a barrier argument, to show that $\varphi_{s,w,r}^{(\delta)}$ is bounded in $L^\infty(X)$ uniformly in $s, w, r \in (0, 1], \delta \in [-\delta_0, \delta_0]$ and $t \in [0, T]$.

**Lemma 3.15** For any $T \in (0, T_0)$, there exist $C$ and $\delta_0 > 0$ such that for $s, w, r \in (0, 1], t \in [0, T]$ and $\delta \in [-\delta_0, \delta_0],$

$$C \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C \leq \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} \leq C.$$ (3.27)

**Proof** Let $\Delta_{s,w,r}^{(\delta)}$ be the Laplace operator associated with $\omega_{s,w,r}^{(\delta)} = \omega_{t,s}^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi_{s,w,r}^{(\delta)}. \text{ Notice that } \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} = -\varphi_0 \text{ when } t = 0 \text{ and}$

$$\left( \frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)} \right) \left( \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} \right) = -tr_{\omega_{s,w,r}^{(\delta)}} (\omega_0) \leq 0.$$ 

It is easy to see that $\frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)}$ is uniformly bounded above by the maximum principle.

Consider $H = e^{-A^2 t} \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} + A^2 \varphi_{s,w,r}^{(\delta)} - A \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2$. $H$ is uniformly bounded below when $t = 0$. There exist constants $C_1, C_2, C_3$ and $C_4 > 0$ such that

$$\left( \frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)} \right) H$$

$$= A^2 \frac{\partial}{\partial t} \varphi_{s,w,r}^{(\delta)} + tr_{\omega_{s,w,r}^{(\delta)}} \left( A^2 \omega_{t,s}^{(\delta)} - AR(h_{\tilde{E}}) - e^{-A^2 t} \omega_0 \right) - nA^2$$

$$- A^2 e^{-A^2 t} \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)}$$

$$\geq -A^2 \log \left( \frac{\Omega_{s,w,r}^{(\delta)}}{\omega_{s,w,r}^{(\delta)}} \right)^n + C_1 tr_{\omega_{s,w,r}^{(\delta)}} \left( \omega_{t,s}^{(\delta)} \right) - A^2 H - A^3 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C_2$$
\[
\geq -A^2 \log \frac{\Omega_{w,r}}{\omega^{(\delta)}} n - A^3 \log |S_{\tilde{E}}^2|_{\omega^{(\delta)}_{t,s}} - A^2 H - C_3
\]
\[
\geq -A^2 H - C_4.
\]

Here the first inequality follows from the fact that \(\omega^{(\delta)}_{t,s} \geq \delta \omega_0\) and \(A \omega_0 - R(h_{\tilde{E}})\) is a Kähler metric on \(X\) for sufficiently large \(A > 0\). The second inequality follows by inequalities of arithmetic and geometric means. Therefore \(H\) is uniformly bounded from below by the maximum principle and the lemma easily follows.

Lemma 3.15 shows that \(\varphi^{(\delta)}_{s,w,r}\) is uniformly Lipschitz in \(\delta\) away from the divisor \(\tilde{E}\).

By the same argument as for \(\delta = 0\), for fixed \(s \in (0, 1]\) and \(\delta \in [-\delta_0, \delta_0]\), there exists \(\varphi^{(\delta)}_s \in L^\infty([0, T] \times X) \cap C^\infty([0, T] \times X\setminus \tilde{E})\) such that

\[
\varphi^{(\delta)}_s = \lim_{w \to 0} \left( \limsup_{r \to 0} \varphi^{(\delta)}_{s,w,r} \right)^*,
\]

and it satisfies the following Monge–Ampère equation

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \varphi^{(\delta)}_s = \log \left( \frac{\omega^{(\delta)}_{t,s} + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)}_s}{\Omega} \right)^n, \quad \text{on } X\setminus \tilde{E} \\
\varphi^{(\delta)}_s(0, \cdot) = (1 - \delta) \varphi_0.
\end{array} \right.
\]

(3.28)

Let

\[
\varphi^{(\delta)} = \lim_{s \to 0} \varphi^{(\delta)}_s
\]

as \(\varphi^{(\delta)}_s\) is decreasing as \(s \to 0\). Then \(\varphi^{(\delta)} \in L^\infty([0, T] \times X) \cap C^\infty((0, T] \times X\setminus \tilde{E})\) solves the following Monge–Ampère equation.

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \varphi^{(\delta)} = \log \left( \frac{\omega^{(\delta)}_t + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)}}{\Omega} \right)^n, \quad \text{on } [0, T] \times X\setminus \tilde{E} \\
\varphi^{(\delta)}(0, \cdot) = (1 - \delta) \varphi_0.
\end{array} \right.
\]

(3.29)

where \(\omega^{(\delta)}_t = \omega^{(\delta)}_{t,0} = (1 - \delta) \omega_0 + t \chi\).

**Lemma 3.16** For any \(T \in (0, T_0)\), there exist \(C\) and \(\delta_0 > 0\) such that on \([0, T] \times X\), for all \(s \in (0, 1]\) and \(\delta_1, \delta_2 \in [-\delta_0, \delta_0]\).
\[|\varphi_s^{(\delta_1)} - \varphi_s^{(\delta_2)}| \leq C|\delta_1 - \delta_2|(1 - \log |S_{\hat{E}} h_{\hat{E}}|^2), \quad (3.30)\]

and so

\[|\varphi^{(\delta_1)} - \varphi^{(\delta_2)}| \leq C|\delta_1 - \delta_2|(1 - \log |S_{\hat{E}} h_{\hat{E}}|^2). \quad (3.31)\]

**Proof** This is an immediate result of Lemma 3.15 by letting \(w, r \to 0\) and then \(s \to 0\). \(\square\)

**Corollary 3.3** For any \(T \in (0, T_0)\) and \(K \subset \subset X \setminus \hat{E}\), \(\varphi^{(\delta)}\) converges to \(\varphi\) uniformly in \(L^\infty([0, T] \times K)\) as \(\delta \to 0\).

**Proof** By Lemma 3.16, \(\varphi_s^{(\delta)}\) is uniformly Lipschitz in \(\delta\) on \(K\) and \(s \in [0, 1)\). The corollary follows easily by letting \(\delta \to 0\). \(\square\)

Now we are able to prove our main result for the existence and uniqueness of the Monge–Ampère solution.

**Proof of Theorem 3.2** For any \(T \in [0, T_0)\), Corollary 3.2 gives existence of the solution for the Monge–Ampère flow (3.18). Now it suffices to prove the uniqueness for the solution on \([0, T] \times X\) for any \(0 < T < T_0\).

Suppose there is another solution \(\varphi'\) satisfying the Monge–Ampère flow (3.18) such that \(\varphi' \in C^\infty([0, T]) \times (X \setminus \hat{E}) \cap L^\infty([0, T] \times X)\).

First, we show that \(\varphi' \leq \varphi\).

Let \(\psi_{s, \epsilon} = \varphi_s - \varphi' - \epsilon s \log |S_{\hat{E}} h_{\hat{E}}|^2\) for sufficiently small \(\epsilon > 0\), where \(\varphi_s = \varphi_{s, 0, 0}\). Then \(\psi_{s, \epsilon} \in C^\infty([0, T] \times (X \setminus \hat{E}))\) and

\[\frac{\partial}{\partial t} \psi_{s, \epsilon} = \log \left(\frac{\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi' + s (\theta - \epsilon R (h_{\hat{E}})) + \sqrt{-1} \partial \bar{\partial} \psi_{s, \epsilon}}{\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi'}\right)^n.\]

Suppose \(\psi_{s, \epsilon}(t, z_{\min}) = \min_X \psi_{s, \epsilon}(t, \cdot)\). Then \(z_{\min} \in X \setminus \hat{E}\) since both \(\varphi_s\) and \(\varphi' \in L^\infty(X)\). If we choose \(\epsilon\) sufficiently small, then by the maximum principle,

\[\frac{\partial}{\partial t} \psi_{s, \epsilon}(t, z_{\min}) \geq \log \left(\frac{\omega_t (t, z_{\min}) + \sqrt{-1} \partial \bar{\partial} \varphi' (t, z_{\min}) + \sqrt{-1} \partial \bar{\partial} \psi_{s, \epsilon} (t, z_{\min})}{\omega_t (t, z_{\min}) + \sqrt{-1} \partial \bar{\partial} \varphi' (t, z_{\min})}\right)^n \geq 0.\]
Note that $\psi_{s, \epsilon}(0, \cdot) = -\epsilon s \log |S_{\tilde{E}}|_{\tilde{h}_{\tilde{E}}}^2 \geq 0$ and so

$$\psi_{s, \epsilon} \geq 0$$

for any $\epsilon$ sufficiently small. Therefore by letting $\epsilon \to 0$, we have

$$\varphi' \leq \varphi_s$$

and so

$$\varphi' \leq \varphi$$

by letting $s \to 0$.

In order to prove $\varphi \leq \varphi'$, we let

$$v_\delta = \varphi' - \varphi^{(\delta)} - \delta^2 \log |S_{\tilde{E}}|_{\tilde{h}_{\tilde{E}}}^2.$$ 

At $t = 0$, $v_\delta = \delta \varphi_0 - \delta^2 \log |S_{\tilde{E}}|_{\tilde{h}_{\tilde{E}}}^2$. Suppose $v_\delta(t, z_{\min}) = \min_X v_\delta(t, \cdot)$. Then $z_{\min} \in X \setminus \tilde{E}$ since both $\varphi^{(\delta)}$ and $\varphi' \in L^\infty(X)$. By the maximum principle, if we choose $\delta$ sufficiently small, then at $(t, z_{\min})$,

$$\frac{\partial}{\partial t} v_\delta = \log \frac{\left( \omega_t^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)} + \delta \left( \omega_0 - \delta \text{Ric} \left( h_{\tilde{E}} \right) \right) + \sqrt{-1} \partial \bar{\partial} v_\delta \right)^n}{\left( \omega_t^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)} \right)^n} \geq \log \frac{\left( \omega_t^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)} + \sqrt{-1} \partial \bar{\partial} v_\delta \right)^n}{\left( \omega_t^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)} \right)^n} \geq 0.$$ 

Therefore

$$v_\delta \geq \inf_X v_\delta (0, \cdot) \geq \delta \inf_X \left( \varphi_0 - \delta \log |S_{\tilde{E}}|_{\tilde{h}_{\tilde{E}}}^2 \right),$$

and so we have for $t \in [0, T]$,

$$\varphi^{(\delta)} + \delta \inf_X \left( \varphi_0 - \delta \log |S_{\tilde{E}}|_{\tilde{h}_{\tilde{E}}}^2 \right) + \delta^2 \log |S_{\tilde{E}}|_{\tilde{h}_{\tilde{E}}}^2 \leq \varphi' \leq \varphi.$$

For any $K \subset \subset X \setminus \tilde{E}$, there exists a constant $C_K > 0$ such that

$$|\varphi^{(\delta)} - \varphi| \leq C_K \delta$$
for sufficiently small $\delta > 0$ by Lemma 3.16. Therefore on $K$,

$$
\varphi' \geq \varphi + \delta \inf_X \left( \varphi_0 - \delta \log |S_{\tilde{E}}^2|_{h_{\tilde{E}}} \right) - C_K \delta.
$$

Letting $\delta \to 0$ and then $K \to X \setminus \tilde{E}$, we have on $X \setminus \tilde{E}$,

$$
\varphi' \geq \varphi.
$$

We now have completed the proof for the uniqueness of the solution on $[0, T] \times X$. Theorem 3.2 is proved by letting $T \to T_0$. \hfill $\square$

### 3.3 Monge–Ampère flows with rough and degenerate initial data

In this section, we will generalize Theorem 3.2 for Monge–Ampère flows with rough initial data. The main result will be applied to the Kähler–Ricci flow on singular projective varieties with surgery.

Let $X$ be an $n$-dimensional projective manifold. Let $L_1$ and $L_2$ be two holomorphic line bundles on $X$ satisfying Condition A along with $\omega_0 \in c_1(L_1)$ and $\chi \in c_1(L_2)$ being smooth closed $(1, 1)$-forms. Let $\Omega$ be a non-negative $(n, n)$-form on $X$ satisfying Condition B. Let

$$
PSH_p(X, \omega_0, \Omega) = \left\{ \varphi \in PSH(X, \omega_0) \cap L^\infty(X) \mid \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \in L^p(X, \Omega) \right\}
$$

for some $p > 1$ and

$$
T_0 = \sup\{t > 0 \mid L_1 + tL_2 \text{ is semi-ample} \} > 0.
$$

Since $L_1$ is big and semi-ample, we denote $Exc(L_1)$ be the exceptional locus for the linear system $|mL_1|$ for sufficiently large $m$. Without loss of generality, we can assume that $\tilde{E}$ as defined in Sect. 3.2 contains $Exc(L_1)$. The following lemma follows immediately from Hölder’s inequality.

**Lemma 3.17** Let $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$ for some $p > 1$ and let $\Theta$ be a smooth volume form on $X$. Then there exists $p' > 1$ such that

$$
F = \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_0)^n}{\Omega} e^{-\varphi_0} \in L^{p'}(X, \Theta). \tag{3.32}
$$
There exist a family of positive functions \( \{F_s\}_{s \in (0, 1]} \) such that \( F_s \in C^\infty(X) \) and

\[
\lim_{s \to 0} ||F_s - F||_{L^p(X, \Omega)} = 0.
\]

We let \( F_0 = F \) and then consider the following Monge–Ampère equations

\[
\left( \omega_0 + s \vartheta + \sqrt{-1} \partial \bar{\partial} \varphi_{(0,s)} \right)^n = F_s e^{\varphi_{(0,s)}} \Omega
\]

and

\[
\left( \omega_0 + s \vartheta + \sqrt{-1} \partial \bar{\partial} \hat{\varphi}_{(s,r)} \right)^n = F_{s+r} e^{\hat{\varphi}_{(s,r)}} \Omega.
\]

Obviously, \( \varphi_{(0,s)} \in C^\infty(X) \) and \( \hat{\varphi}_{(s,r)} \in C^\infty(X) \) by Yau’s theorem [48] for \( s > 0 \) and \( r > 0 \). Furthermore, \( ||\hat{\varphi}_{(s,r)}||_{L^\infty(X)} \) are uniformly bounded for \( s, r \in [0, 2] \) and \( \varphi_{(0,s)} = \hat{\varphi}_{(s,0)} \). We introduce the intermediate auxiliary function \( \hat{\varphi}_{(s,r)} \) because we cannot directly compare \( \varphi_{(0,s)} \) to \( \varphi_0 \) using the stability theorem.

**Lemma 3.18** There exists \( C > 0 \) such that for all \( s, r, \eta \in [0, 1] \),

\[
|\hat{\varphi}_{(s+\eta,r)} - \hat{\varphi}_{(s,r+\eta)}| \leq C \sqrt{\eta} \left( 2 - \log |S_E|^2_{\tilde{h}_E} \right).
\]

**Proof** We first assume that \( s, r > 0 \). Let

\[
\psi^+ = \hat{\varphi}_{(s+\eta,r)} - (1 - \eta)\hat{\varphi}_{(s,r+\eta)} - \epsilon \eta \log |S_E|^2_{\tilde{h}_E}
\]

for some fixed sufficiently small \( \epsilon > 0 \). Then \( \psi^+ \) satisfies

\[
((1 - \eta) (\omega_0 + s \vartheta + \sqrt{-1} \partial \bar{\partial} \hat{\varphi}_{(s,r+\eta)}) + \eta (\omega_0 - \epsilon R (h_{\tilde{E}}) + (1 + s) \vartheta) + \sqrt{-1} \partial \bar{\partial} \psi^+)^n
\]

\[
= e^{\psi^+ - \eta \hat{\varphi}_{(s,r+\eta)} + \epsilon \eta \log |S_E|^2_{\tilde{h}_E}}.
\]

Since \( \psi^+ \) is smooth on \( X \setminus \tilde{E} \) and tends to \( +\infty \) near \( \tilde{E} \), we can apply the maximum principle and derive the following estimate

\[
e^{\psi^+} \geq (1 - \eta)^n e^{\eta \hat{\varphi}_{(s,r+\eta)} - \epsilon \eta \log |S_E|^2_{\tilde{h}_E}} \geq e^{-C_1 \eta}
\]

for some fixed constant \( C_1 > 0 \). Then there exists \( C_2 > 0 \) such that

\[
\hat{\varphi}_{(s+\eta,r)} - \hat{\varphi}_{(s,r+\eta)} \geq -C_2 \eta \left( 2 - \log |S_E|^2_{\tilde{h}_E} \right)
\]

because \( ||\hat{\varphi}_{(s,r)}||_{L^\infty(X)} \) is uniformly bounded for all \( s, r \in [0, 2] \).
To obtain the lower bound for \( \hat{\varphi}(s+n,r) - \hat{\varphi}(s+r,n) \), we may assume that \( n \) is sufficiently small such that \( \omega_0 - \sqrt{n} \vartheta - e R(h_E) \) is strictly positive and let
\[
\psi^- = \hat{\varphi}(s+r,n) - (1 - \sqrt{n}) \hat{\varphi}(s+n,r) - e \sqrt{n} \log |S_E|^2_{h_E}.
\]
We apply the maximum principle to the equation
\[
\frac{1}{(1 - \sqrt{n}) (\omega_0 + (s + n) \vartheta + \sqrt{-1} \delta \hat{\varphi}(s+n,r)) + \sqrt{n} (\omega_0 - (\sqrt{n} - s - n) \vartheta - e R(h_E)) + \sqrt{-1} \delta \hat{\varphi}^-)^n}{(\omega_0 + (s + n) \vartheta + \sqrt{-1} \delta \hat{\varphi}(s+n,r))} = e^{\psi^- - \sqrt{n} \hat{\varphi}(s+n,r) + \sqrt{n} \log |S_E|^2_{h_E}}
\]
and conclude with the following estimate
\[
\hat{\varphi}(s+r,n) - \hat{\varphi}(s+n,r) \geq -C_3 \sqrt{n} \left( 2 - \log |S_E|^2_{h_E} \right)
\]
for some fixed constant \( C_3 > 0 \). The case when \( s = 0 \) or \( r = 0 \) is proved by applying the above uniform estimates and by letting \( s \to 0 \) or \( r \to 0 \). The lemma then immediately follows.

**Lemma 3.19** There exist \( C > 0 \) and an increasing function \( \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \lim_{x \to 0} \mu(x) = 0 \) such that for \( s, n \in [0, 1) \),
\[
|\varphi(0,s+n) - \varphi(0,s)| \leq C \mu(s+n) \left( 2 - \log |S_E|^2_{h_E} \right). \tag{3.35}
\]

**Proof** Applying the stability theorem \([11, 21]\), there exists \( C > 0 \) such that
\[
||\hat{\varphi}(0,s+n) - \hat{\varphi}(0,s)||_{L^\infty(X)} \leq C||F_{s+n} - F_s||_{L^{\frac{1}{p+3}}(X,\Omega)}^{\frac{1}{p+3}}.
\]
Since \( F_{s+n} \) and \( F_s \) both converge to \( F \) in \( L^{p'}(X) \) for some \( p' > 1 \), there exists an increasing function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \lim_{x \to 0} f(x) = 0 \) such that for \( s, n \in [0, 1) \)
\[
||\hat{\varphi}(0,s+n) - \hat{\varphi}(0,s)||_{L^\infty(X)} \leq f(s + n).
\]
Applying the estimate in Lemma 3.18, we have
\[
|\varphi(0,s+n) - \varphi(0,s)| \leq |\hat{\varphi}(s+n,0) - \hat{\varphi}(s,0)|
\]
\[
\leq |\hat{\varphi}(s+n,0) - \hat{\varphi}(s+n)| + |\hat{\varphi}(s+n) - \hat{\varphi}(s)| + |\hat{\varphi}(s) - \hat{\varphi}(s)|
\]
\[
\leq C \left( \sqrt{n} + \sqrt{s} \right) \left( 2 - \log |S_E|^2_{h_E} \right) + |\hat{\varphi}(s+n) - \hat{\varphi}(s)|
\]
\[
\leq C \left( \sqrt{n} + \sqrt{s} \right) \left( 2 - \log |S_E|^2_{h_E} \right) + Cf(s + n)
\]
for some fixed \( C > 0 \). The Lemma then follows from the above estimates. \( \square \)
The following corollary is an immediately consequence of Lemma 3.19.

**Corollary 3.4** For any $K \subset \subset X \setminus \tilde{E}$,

$$
\lim_{s \to 0} \|\varphi_{(0,s)} - \varphi_0\|_{L^\infty(K)} = 0. \tag{3.36}
$$

Consider the following family of Monge–Ampère equations.

$$
\begin{cases}
\frac{\partial \varphi_{s,w,r}^{(\delta)}}{\partial t} = \log \left( \frac{\omega_{t,s}^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi_{s,w,r}^{(\delta)}}{\Omega_{w,r}} \right)^n, \\
\varphi_{s,w,r}^{(\delta)}(0, \cdot) = (1 - \delta)\varphi_{(0,s)},
\end{cases} \tag{3.37}
$$

where $\omega_{t,s}^{(\delta)} = (1 - \delta)\omega_0 + t\chi + s\vartheta$. For any $T \in [0, T_0)$, there exists $\delta_0 > 0$ such that for $\delta \in [\delta_0, 0]$, the Eq. (3.37) admits a smooth solution on $[0, T] \times X$ as shown in Sect. 3.2. We will then fix such $T$ and $\delta_0$.

**Lemma 3.20** There exists $C > 0$ such that for $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$,

$$
\|\varphi_{s,w,r}^{(\delta)}(t, \cdot)\|_{L^\infty([0,T] \times X)} \leq C. \tag{3.38}
$$

**Proof** It can be proved by the same argument in the proof of Lemma 3.7. The uniform upper bound for $\varphi_{s,w,r}^{(\delta)}$ is achieved by using Hörmander–Tian’s estimates [39] and bounding $\int_X \varphi_{s,w,r}^{(\delta)} \Omega_{w,r}$. The lower bound is achieved by comparing $\varphi_{s,w,r}^{(\delta)}$ to a uniformly bounded barrier function as a solution of a degenerate complex Monge–Ampère equation. $\Box$

**Lemma 3.21** There exists $C > 0$ such that on $[0, T] \times X$, for all $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$,

$$
-C \leq t\varphi_{s,w,r}^{(\delta)} \leq C. \tag{3.39}
$$

**Proof** The upper bound can be proved using the same argument as that in Lemma 3.2 by applying the maximum principle on

$$
H^+ = t\varphi_{s,w,r}^{(\delta)} - \varphi_{s,w,r}^{(\delta)}.
$$

In order to prove the lower bound, we consider the following family of Monge–Ampère equations

$$
(\omega_{0,s} + \sqrt{-1} \partial \bar{\partial} \phi_{s,w,r})^n = A_{s,w,r} \Omega_{w,r}
$$
where $A_{s,w,r} = \int_X \frac{\omega_{0,s}^n}{w_{s,w,r}}$ and $\sup_X \phi_{s,w,r} = 0$. Then $A_{s,w,r}$ is uniformly bounded from above and below for $s$, $w$ and $r \in (0,1)$. As $A_{s,w,r} \Omega_{w,r}$ is uniformly bounded in $L^p(X, \Theta)$ for some $p > 1$, $\phi_{s,w,r}$ uniformly bounded in $L^\infty(X)$ for $s$, $w$, $r \in (0,1)$. Let

$$H^- = t\dot{\phi}_{s,w,r} + B^2 \phi_{s,w,r} - B\phi_{s,2,w,r}$$

for some sufficiently large $B > 0$. Let $\Delta_{s,w,r}^{(\delta)}$ be the Laplace operator associated to $\omega_{s,w,r}^{(\delta)}$. Then there exist $C_1$, $C_2$ and $C_3 > 0$ such that

$$\left( \frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)} \right) H^- = tr\left( \frac{\omega_{s,w,r}^{(\delta)}}{\omega_{s,w,r}} \right) \left( B^2 \omega_{s,w,r}^{(\delta)} + t \chi + B \sqrt{-1} \bar{\partial} \phi_{s,2,w,r}^{(\delta)} \right) + (B^2 + 1) \dot{\phi}_{s,w,r}^{(\delta)} - B^2 n$$

$$\geq tr\left( \frac{\omega_{s,w,r}^{(\delta)}}{\omega_{s,w,r}} \right) \left( B\omega_{0,s} + B \sqrt{-1} \bar{\partial} \phi_{s,2,w,r}^{(\delta)} \right) + (B^2 + 1) \dot{\phi}_{s,w,r}^{(\delta)} - B^2 n$$

$$\geq C_1 \left( \frac{(\omega_{0,s}^2 + \sqrt{-1} \bar{\partial} \phi_{s,2,w,r}^{(\delta)})^n}{\omega_{s,w,r}^{(\delta)} n} \right) + (B^2 + 1) \log \left( \frac{\omega_{s,w,r}^{(\delta)}}{\Omega_{w,r}} \right) - B^2 n$$

$$\geq C_2 \left( \frac{\Omega_{w,r}}{\omega_{s,w,r}^{(\delta)} n} \right) - C_3$$

$$\geq -C_3.$$

Applying the maximum principle, $H^-$ is uniformly bounded from below since both $\phi_{s,w,r}^{(\delta)}$ and $\phi_{s,w,r}$ are uniformly bounded in $L^\infty(X)$. This completes the proof of the lemma.

We immediately have the following volume estimate from Lemma 3.21.

**Corollary 3.5** There exists $C > 0$ such that on $[0, T] \times X$, for all $s$, $w$, $r \in (0,1)$ and $\delta \in [-\delta_0, \delta_0]$, \n
$$e^{-\frac{C}{\tau}} \leq \frac{(\omega_{s,w,r}^{(\delta)})^n}{\Omega_{w,r}} \leq e^{\frac{C}{\tau}}. \tag{3.40}$$

We also obtain the second order estimates for $\omega_{s,w,r}^{(\delta)}$. 
Lemma 3.22 There exist $\alpha, \beta > 0$ and $C > 0$ such that on $[0, T] \times X$, for all $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$,

$$
tr_{\delta} \left( \omega_{s, w, r}^{(\delta)} \right) \leq Ce^{-\frac{2\alpha}{T}} |S_{E}|_{h_E}^{-2\beta}.
$$

\textit{Proof} Let

$$
H_{s, w, r}^{(\delta)} = t \log tr_{\delta} \left( \omega_{s, w, r}^{(\delta)} \right) - A^2 \varphi_{s, w, r}^{(\delta)} + A \log |S_{E}|_{h_E}^2.
$$

The lemma follows by applying the maximum principle applied to

$$
\left( \frac{\partial}{\partial t} - \Delta_{s, w, r}^{(\delta)} \right) H_{s, w, r}^{(\delta)}
$$

and by combining the argument in Lemmas 3.3 and 3.9. $\square$

Lemma 3.23 For any compact $K \subset X \setminus \tilde{E}$, $t_0 \in (0, T)$ and $k \in \mathbb{Z}^+$, there exists $C_{K, t_0, k} > 0$ such that for all $s, w, r \in (0, 1]$, $\delta \in [-\delta_0, \delta_0]$,

$$
||\varphi_{s, w, r}^{(\delta)}||_{C^k([t_0, T] \times K)} \leq C_{K, t_0, k}.
$$

\textit{Proof} We first derive global third order estimates by combining the argument in the proof of Lemmas 3.4 and 3.10 as the following. We set

$$
A_{s, w, r}^{(\delta)} = \left( g_{s, w, r}^{(\delta)} \right)^{p_{\bar{r}}} \left( g_{s, w, r}^{(\delta)} \right)^{s_{\bar{k}}} \left( g_{s, w, r}^{(\delta)} \right)^{m_{\bar{t}}} \left( g_{s, w, r}^{(\delta)} \right)^{p_{\bar{k}, m}} \left( g_{s, w, r}^{(\delta)} \right)^{r_{\bar{s}, \bar{t}}},
$$

where the covariant derivates are computed with respect to a fixed smooth background Kähler metric $\hat{g}$ on $X$. We can apply the maximum principle to the quantity

$$
H_{s, w, r}^{(\delta)} = e^{-\frac{4\alpha}{T}} \left( |S_{E}|_{h_E} \right)^{4\beta} A_{s, w, r}^{(\delta)} + e^{-\frac{\alpha}{T}} B \left( |S_{E}|_{h_E} \right)^{2\beta} tr_{\hat{g}} \left( g_{s, w, r}^{(\delta)} \right)
$$

by choosing sufficiently large $\alpha$ and $\beta > 0$, and we can show that $A_{s, w, r}^{(\delta)}$ is uniformly bounded on $[t_0, T] \times K$ for $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$ by the same argument for Lemmas 3.4 and 3.10.

After linearizing the Monge–Ampère flow for $\varphi_{s, w, r}^{(\delta)}$, we can apply standard bootstrap techniques for the linear parabolic equations to obtain higher order estimates for $\varphi_{s, w, r}^{(\delta)}$ on $[t_0, T] \times K$ as in Proposition 3.2. $\square$

Using the same argument after the Eq. (3.25) and Lemma 3.14, we can show that for fixed $s, w, r \in (0, 1]$, $\varphi_{s, w, r}^{(\delta)}$ is a smooth family in $\delta \in [-\delta_0, \delta_0]$ and

$$
\lim_{\delta \to 0} \varphi_{s, w, r}^{(\delta)} = \varphi_{s, w, r}
$$

in $C^\infty$-topology.
Lemma 3.24 There exist constants $C$ and $\delta_0 > 0$ such that for $s, w, r \in (0, 1]$, $t \in [0, T]$ and $\delta \in [-\delta_0, \delta_0]$,

$$C \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C \leq \frac{\partial}{\partial \delta} \varphi_{s, w, r}^{(\delta)} \leq C. \quad (3.42)$$

Proof The lemma can be proved by the same argument in the proof of Lemma 3.15. \(\square\)

Fix $\delta \in [-\delta_0, \delta_0]$, we can obtain the same monotonicity for $\varphi_{s, w, r}^{(\delta)}$ in terms of the parameters $s, w, r$ as in Lemma 3.11. For each $s \in (0, 1]$, let

$$\varphi_s^{(\delta)} = \lim_{w \to 0} \left( \limsup_{r \to 0} \varphi_{s, w, r}^{(\delta)} \right)^* \quad (3.43)$$

and

$$\varphi_s = \lim_{w \to 0} \left( \limsup_{r \to 0} \varphi_{s, w, r} \right)^* \quad (3.44)$$

Both $\varphi_s^{(\delta)}$ and $\varphi_s$ are smooth on $[0, T] \times X \setminus \tilde{E}$ for $s \in (0, 1]$. We will show that $\varphi_s$ converges as $s \to 0$.

Lemma 3.25 There exist $\eta_0 > 0$ and $C > 0$ such that on $[0, T] \times X$, for $s \in [0, 1]$ and $0 < \eta < \eta_0$,

$$\varphi_s^{(n)} \leq \varphi_{s+\eta^3} - \eta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + C \mu(s + \eta) + C\eta \quad (3.45)$$

and

$$\varphi_{s+\eta^3} \leq \varphi_s - \eta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + C \mu(s + \eta) + C\eta, \quad (3.46)$$

where $\mu(\cdot)$ is the function chosen in Lemma 3.19.

Proof Let $\psi_{\eta}^+ = \varphi_{s+\eta^3} - \varphi_s^{(n)} - \eta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + A \mu(s + \eta) + A\eta$. Then

$$\frac{\partial}{\partial t} \psi_{\eta} = \log \frac{\left( \omega_{t, s}^{(n)} + \sqrt{-1} \partial \bar{\partial} \varphi_s^{(n)} + \eta(\omega_0 + \eta^2 \partial - \eta R(h_{\tilde{E}})) + \sqrt{-1} \partial \bar{\partial} \psi_{\eta}^+ \right)^n}{\left( \omega_{t, s}^{(n)} + \sqrt{-1} \partial \bar{\partial} \varphi_{s}^{(n)} \right)^n}. \quad (3.47)$$

For sufficiently large $A > 0$,

$$\psi_{\eta} |_{t=0} = \varphi_{(0, s+\eta^3)} - (1 - \eta)\varphi_{(0, s)} - \eta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + A \mu(s + \eta) + A\eta \geq 0.$$
Since $\omega_0 + \eta^2 \partial - \eta R(h_{\tilde{E}}) > 0$ for sufficiently small $\eta > 0$ and $\psi^+_{\eta}$ tends to $\infty$ near $\tilde{E}$, we can apply the maximum principle to $\psi^+_{\eta}$ and For sufficiently large $A > 0$,

$$
\psi_{\eta}|_{t=0} = \varphi_{(0,s+\eta^3)} - (1-\eta)\varphi_{(0,s)} - \eta^2 \log |S_{\tilde{E}}^2|_{h_{\tilde{E}}} + A\mu(s + \eta) + A\eta \geq 0.
$$

Therefore $\psi_{\eta} \leq 0$ on $[0, T] \times X$ by the maximum principle and this gives the estimate (3.45).

Estimate (3.46) is achieved by similar argument as above. We apply the maximum principle to the quantity $\psi^-_{\eta} = \varphi_{s+\eta^3}^\eta - \varphi_s - \eta^2 \log |S_{\tilde{E}}^2|_{h_{\tilde{E}}} + A\mu(s + \eta) + A\eta$ to show that $\psi^-_{\eta} \geq 0$ on $[0, T] \times X$ for sufficiently large $A > 0$.

Then we can show that $\{\varphi_s\}_{s \in (0,1]}$ is a Cauchy family in $L^\infty([0, T] \times K)$ for any compact subset $K$ in $X \setminus \tilde{E}$.

**Lemma 3.26** On any $K \subset \subset X \setminus \tilde{E}$,

$$
\lim_{s_1, s_2 \to 0} ||\varphi_{s_1} - \varphi_{s_2}||_{L^\infty([0,T] \times K)} = 0. \tag{3.48}
$$

**Proof** Assume $\eta = s_2 - s_1 \geq 0$. Then on $[0, T] \times K$, by Lemmas 3.24 and 3.25, there exist $C$ and $C' > 0$ such that

$$
\varphi_{s_1} \leq \varphi_{s_1}^\eta + C\eta^{1/3} \leq \varphi_{s_2} - \eta^{2/3} \log |S_{\tilde{E}}^2|_{h_{\tilde{E}}} + C\left(\mu(s_1 + \eta^{1/3}) + \eta^{1/3}\right)
$$

and

$$
\varphi_{s_1} \geq \varphi_{s_1 + \eta}^\eta + \eta^{2/3} \log |S_{\tilde{E}}^2|_{h_{\tilde{E}}} - C\left(\mu(s_1 + \eta^{1/3}) + \eta^{1/3}\right)
\geq \varphi_{s_2} + \eta^{2/3} \log |S_{\tilde{E}}^2|_{h_{\tilde{E}}} - C'\left(\mu(s_1 + \eta^{1/3}) + \eta^{1/3}\right).
$$

The lemma follows immediately by letting $s_1$ and $s_2 \to 0$ ($\eta \to 0$).

**Theorem 3.3** Let $X$ be an $n$-dimensional projective manifold. Let $L_1$ and $L_2$ be two holomorphic line bundles on $X$ satisfying Condition A. Let $\omega_0 \in c_1(L_1)$ and $\chi \in c_1(L_2)$ be two smooth closed $(1,1)$-forms and let $\Omega$ be an $(n,n)$-form on $X$ satisfying Condition B. Let

$$
T_0 = \sup\{t > 0 | L_1 + tL_2 \text{ is semi-ample}\} > 0.
$$

Then for any $\varphi_0 \in PSH_\rho(X, \omega_0, \Omega)$ for some $\rho > 1$, there exists a unique $\varphi \in C^0([0, T_0] \times (X \setminus \tilde{E})) \cap C^\infty((0, T_0) \times (X \setminus \tilde{E}))$ with $\varphi(t, \cdot) \in PSH(X, \omega_0 + t\chi) \cap L^\infty(X)$ for each $t \in [0, T_0)$ such that the following hold.
1. \[ \frac{\partial \phi}{\partial t} = \log \left( \omega_0 + t \chi + \sqrt{-1} \delta \phi \right)^n \] on \((0, T_0) \times (X \setminus \tilde{E})\).

2. \[ ||\phi||_{L^\infty([0, T] \times X)} \] is bounded for each \(T < T_0\).

3. For any \(K \subset \subset X \setminus \tilde{E}\), \(\lim_{t \to T_0^+} ||\phi(t, \cdot) - \phi_0(\cdot)||_{L^\infty(K)} = 0\).

4. For any \(T \in (0, T_0)\), there exists \(C > 0\) such that on \([0, T] \times X\),

\[ e^{-\frac{C}{T}} \leq \frac{(\omega_0 + t \chi + \sqrt{-1} \delta \phi)^n}{\Omega_1} \leq e^{\frac{C}{T}}. \tag{3.49} \]

**Proof** We first construct the solution \(\phi = \lim_{s \to 0^+} \phi_s\) by Lemma 3.26. (1) and (2) follow from the uniform estimates on \(\phi_{(\delta)}\). On any \(K \subset \subset X \setminus \tilde{E}\),

\[
||\phi(t, \cdot) - \phi_0(\cdot)||_{L^\infty(K)} \\
\leq ||\phi(t, \cdot) - \phi_s(t, \cdot)||_{L^\infty(K)} + ||\phi_s(t, \cdot) - \phi_{(0,s)}(\cdot)||_{L^\infty(K)} + ||\phi_{(0,s)}(\cdot) - \phi_0(\cdot)||_{L^\infty(K)}.
\]

For any \(\epsilon > 0\), let \(s\) be sufficiently small such that

\[
||\phi(t, \cdot) - \phi_s(\cdot)||_{L^\infty([0,T] \times K)} < \epsilon
\]

and

\[
||\phi_{(0,s)}(\cdot) - \phi_0(\cdot)||_{L^\infty(K)} < \epsilon.
\]

Fixing such \(s\), there exists \(t_0 > 0\) such that

\[
||\phi_s(t, \cdot) - \phi_{(0,s)}(\cdot)||_{L^\infty([0,t_0] \times K)} < \epsilon.
\]

Therefore \(\lim_{t \to T_0^+} ||\phi(t, \cdot) - \phi_0(\cdot)||_{L^\infty(K)} = 0\). This proves (3).

The volume estimate (3.49) follows from (3.40) by letting \(s, w, r, \delta \to 0\).

The uniqueness of the solution \(\phi\) is proved by the same argument in the proof of Theorem 3.2 using the maximum principle and barrier functions. \(\square\)

### 4 Kähler–Ricci flow on varieties with log terminal singularities

#### 4.1 Notations

Let \(X\) be a \(\mathbb{Q}\)-factorial projective variety with log terminal singularities. We denote the singular set of \(X\) by \(X_{\text{sing}}\) and let \(X_{\text{reg}} = X \setminus X_{\text{sing}}\). Let \(\pi : \tilde{X} \to X\) be a log resolution of singularities and \(K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i\) with \(a_i > -1\), where \(E_i\) is the irreducible component of the exceptional locus \(\text{Exc}(\pi)\) of \(\pi\).

We always assume that \(m K_X\) is a Cartier divisor for some positive \(m \in \mathbb{Z}\) because \(K_X\) is \(\mathbb{Q}\)-Cartier. The following measure is introduced in [13].
Definition 4.1 \( \Omega \) is said to be an adapted measure on \( X \) if \( \Omega \) is a smooth volume form on \( X_{reg} \) and for any \( z \in X \), there exists an open neighborhood \( U \) of \( z \) such that

\[
\Omega = f_U (\alpha \wedge \overline{\alpha}) \frac{1}{m},
\]

where \( f_U \) is a smooth positive function on \( U \) and \( \alpha \) is a local generator of \( mK_X \) on \( U \).

On each \( U \), \( \sqrt{-1} \partial \bar{\partial} \log (\alpha \wedge \overline{\alpha}) = 0 \) on \( U \) and \( \sqrt{-1} \partial \bar{\partial} \log f_U \) is a well-defined smooth closed \((1,1)\)-form on \( U \) by a local embedding. Then \( \chi = \sqrt{-1} \partial \bar{\partial} \log \Omega \) is a well-defined smooth closed \((1,1)\)-form on \( X \). We will use \( \sqrt{-1} \partial \bar{\partial} \)-cohomology instead of \( \partial \)-cohomology on \( X \) and \( \chi \in [K_X] \) because \( \sqrt{-1} \partial \bar{\partial} \)-lemma does not hold on singular varieties in general. If we write \( h_{\Omega} = \Omega^{-1} \), then \( h_{\Omega} \) defines a hermitian metric on \( K_X \) and \( R(h_{\Omega}) = \chi \).

After pulling back \( \Omega \) by the resolution \( \pi \), \( \pi^* \Omega \) is then a non-negative \((n,n)\)-form on \( \tilde{X} \). In particular, \( \pi^* \Omega \) has zeros or poles along the exceptional divisor \( E_i \) of order \( |a_i| \) and

\[
\pi^* \chi = Ric(\pi^* h_{\Omega}).
\]

Let \( D \) be an ample divisor on \( \tilde{X} \) such that

\[
\omega_D = R(h_D) = -\sqrt{-1} \partial \bar{\partial} \log h_D > 0,
\]

where \( h_D \) is a smooth hermitian metric equipped on the line bundle associated to \( D \).

Let \( \iota : X \to \mathbb{CP}^N \) be any projective imbedding of \( X \) and let \( \hat{\omega} \) be the pullback of a smooth Kähler metric on \( \mathbb{CP}^N \) in a rational Kähler class. Then \( \hat{\omega} \) is a smooth Kähler metric on \( X \). Since \( [\iota^* \omega_0] \) is the pullback of an ample class on \( \mathbb{CP}^N \), it is a big and semi-ample divisor on \( \tilde{X} \). By the Kodaira’s lemma, there exists an effective divisor \( \tilde{E} \) on \( \tilde{X} \) such that

\[
[t^* \hat{\omega}] - \epsilon [\tilde{E}]
\]

is ample for any \( \epsilon > 0 \) sufficiently small. We can always assume that the support of \( \tilde{E} \) contains the support of each \( E_i \). Furthermore, since \( X \) is \( \mathbb{Q} \)-factorial, we can assume by Proposition 2.1 that the support of \( \tilde{E} \) is contained in the exceptional locus of \( \pi \). There exists a smooth hermitian metric \( h_{\tilde{E}} \) equipped on the line bundle associated to \( \tilde{E} \) such that for sufficiently small \( \epsilon > 0 \),

\[
\pi^* \hat{\omega} - \epsilon R(h_{\tilde{E}}) > 0.
\]
Let $S_D$ and $S_{\tilde{E}}$ be the defining sections of $D$ and $\tilde{E}$.

The following theorem is well-known as a combination of the rationality theorem and base-point-free theorem in birational geometry (c.f. [9, 19]).

**Theorem 4.1** Let $X$ be a projective manifold such that $K_X$ is not nef. Let $H$ be an ample $\mathbb{Q}$-divisor and let

$$\lambda = \max \{ t \in \mathbb{R} \mid H + tK_X \text{ is nef} \}. \quad (4.1)$$

Then $\lambda \in \mathbb{Q}$ and $H + \lambda K_X$ is semi-ample.

### 4.2 Existence and uniqueness of the weak Kähler–Ricci flow

Let $X$ be a $\mathbb{Q}$-factorial projective variety with log terminal singularities. Let $H$ be a big and semi-ample $\mathbb{Q}$-divisor on $X$ such that $H + \epsilon K_X$ is ample for some sufficiently small $\epsilon \in \mathbb{Q}^+$. We let $\hat{\omega} \in [H]$ be the restriction of a smooth Kähler metric from a projective embedding of $X$ and let $\Omega$ be an adapted measure on $X$. We also let $\chi = \sqrt{-1} \partial \bar{\partial} \log \Omega$.

Consider the ordinary differential equation for the Kähler class defined by the Ricci flow on $X$

$$\left\{ \begin{array}{l}
\frac{\partial [\omega]}{\partial t} = [\chi] = [K_X], \\
[\omega(0, \cdot)] = [\hat{\omega}] = [H].
\end{array} \right. \quad (4.2)$$

Then

$$[\omega] = [\hat{\omega}] + t[\chi].$$

Heuristically, if the Kähler–Ricci flow exists for $t \in [0, T)$, the Kähler–Ricci flow should be equivalent to the following Monge–Ampère flow

$$\left\{ \begin{array}{l}
\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \\
\varphi(0, \cdot) = \varphi_0,
\end{array} \right. \quad (4.3)$$

where $\hat{\omega}_t = \hat{\omega} + t\chi$. Let

$$\pi : \tilde{X} \to X$$

be a log resolution of singularities as defined in Sect. 4.1. In order to define the Monge–Ampère flow on $X$, one might want to lift the flow to the nonsingular
model $\tilde{X}$ of $X$. However, $\hat{\omega}$ is not Kähler on $\tilde{X}$ and $\Omega$ in general vanishes or blows up along the exceptional divisor of $\pi$ on $\tilde{X}$ unless the resolution $\pi$ is crepant, hence the lifted flow is degenerate near the exceptional locus. So we have to perturb the Monge–Ampère flow (4.3) and obtain uniform estimates so that the flow descend to $X$. Let

$$T_0 = \sup\{t \geq 0 \mid H + tK_X \text{ is nef on } X\}.$$ 

Then for any $t \in [0, T_0)$, $[\hat{\omega}_t]$ is ample from the assumption that $H + tK_X$ is ample for sufficiently small $t > 0$. Furthermore, $T_0 \in \mathbb{Q}^+$ or $T_0 = \infty$ by Theorem 4.1 and if $T_0 < \infty$, $H + T_0K_X$ is semi-ample.

**Theorem 4.2** Let $\varphi_0 \in PSH_p(X, \hat{\omega}, \Omega)$ for some $p > 1$. Then the Monge–Ampère flow on $\tilde{X}$ defined by

$$\begin{cases} \\
\frac{\partial \hat{\varphi}(t, \cdot)}{\partial t} = \log \frac{(\pi^*\hat{\omega}_t + \sqrt{-1}\partial \bar{\partial} \hat{\varphi})^n}{\pi^*\Omega} \\
\hat{\varphi}(0, \cdot) = \pi^*\varphi_0 
\end{cases} (4.4)$$

has a unique solution $\hat{\varphi} \in C^\infty((0, T_0) \times \tilde{X}\setminus \overline{E}) \cap C^0([0, T_0) \times \tilde{X}\setminus \overline{E})$ such that for all $t \in [0, T_0)$, $\hat{\varphi}(t, \cdot) \in L^\infty(\tilde{X}) \cap PSH(\tilde{X}, \pi^*\omega_t)$. Furthermore, $\hat{\varphi}$ is constant along each fibre of $\pi$, and so $\hat{\varphi}$ descends to a unique solution $\varphi \in C^\infty((0, T_0) \times X_{reg}) \cap C^0([0, T_0) \times X_{reg})$ of the Monge–Ampère flow (4.3) such that for each $t \in [0, T_0)$, $\varphi \in PSH(X, \omega_t) \cap C^0(X)$.

**Proof** Since $[\pi^*\omega_0]$ corresponds to a big and semi-ample divisor on $\tilde{X}$ and $[\pi^*\omega] - \epsilon[\pi^*\chi]$ is also big and semi-ample for sufficiently small $\epsilon > 0$. The adjunction formula gives $K_{\tilde{X}} = \pi^*K_X + \sum_i a_i E_i + \sum_j F_j$, where $E_i$ and $F_j$ are irreducible components of the exceptional locus with $a_i \geq 0$ and $b_j > -1$. Note that $\pi^*\Omega$ vanishes only on each $E_i$ to order $a_i$ and $\pi^*\Omega$ has poles along those $F_j$ of order $b_i$. Then $\pi^*\hat{\omega}, \pi^*\chi$ and $\pi^*\Omega$ satisfy **Condition A** and **Condition B**. Furthermore, $\pi^*\varphi_0 \in PSH_p(\tilde{X}, \pi^*\hat{\omega}, \pi^*\Omega)$ and so the assumptions in Theorem 3.3 are satisfied. The first part of the theorem is then an immediate corollary of Theorem 3.3.

The singular set $\overline{E}$ can be chosen to be contained in the exceptional locus $Exc(\pi)$ of $\pi$, since $X$ is $\mathbb{Q}$-factorial. Also $\hat{\varphi}$ must be constant along each component of $Exc(\pi)$ as $[\pi^*\omega_t]$ is trivial along each component of the exceptional divisors. So it descends to a function in $PSH(X, \omega_t)$ on $X$. \[ \square \]

**Theorem 4.3** Let $X$ be a $\mathbb{Q}$-factorial projective variety with log terminal singularities and $H$ be a big and semi-ample $\mathbb{Q}$-divisor on $X$. Suppose $H + tK_X$ is ample for sufficiently small $t \in \mathbb{Q}^+$ and let

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}.$$
If $\omega_0 \in K_{H, p}(X)$ for some $p > 1$, then there exists a unique solution $\omega$ of the weak Kähler–Ricci flow as in Definition 1.2 for $t \in [0, T_0)$.

Furthermore, if $\Omega$ is an adapted measure on $X$, then for any $T \in (0, T_0)$, there exists a constant $C > 0$ such that on $[0, T] \times X$,

$$e^{-\frac{C}{T}} \Omega \leq \omega^n \leq e^{\frac{C}{T}} \Omega. \quad (4.5)$$

Proof It suffices to prove the uniqueness as the existence and the volume estimate follow easily from Theorems 4.2 3.3. Let $\omega_t = \omega_0 + t\chi$ and then

$$\omega = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi$$

with $\varphi \in L^\infty(\tilde{X}) \cap C^\infty(\tilde{X} \setminus \tilde{E})$. Then the Kähler–Ricci flow is equivalent to the following equation

$$\begin{cases}
\sqrt{-1} \partial \bar{\partial} \left( \frac{1}{\Omega} \varphi - \log \frac{\omega^n}{\Omega} \right) = 0, & \text{on } \tilde{X} \setminus \tilde{E} \\
\varphi(0, \cdot) = 0.
\end{cases} \quad (4.6)$$

Let $F = \frac{\partial}{\partial t} \varphi - \log \frac{\omega^n}{\Omega}$. Then $F \in C^\infty(\tilde{X} \setminus \tilde{E})$ and $\sqrt{-1} \partial \bar{\partial} F = 0$ on $\tilde{X} \setminus \tilde{E}$. Since $X$ is $\mathbb{Q}$-factorial, $\pi^*[\omega_0] - \epsilon[\text{Exc}(\pi)]$ is ample for $\epsilon > 0$ sufficiently small. So we can choose $\tilde{E}$ to be contained in $\text{Exc}(\pi)$. Hence $F$ descends to $X_{\text{reg}}$ and $\sqrt{-1} \partial \bar{\partial} F = 0$ on $X_{\text{reg}}$. For each $t \in (0, T_0)$, $F$ is smooth on $X_{\text{reg}}$, therefore $F$ is constant on each curve in $X$ which does not intersect $X_{\text{sing}}$. On the other hand, for any two generic points $z$ and $w$ on $X$, there exists a curve joining $z$ and $w$ without intersecting $X_{\text{sing}}$ since $\text{codim}(X_{\text{sing}}) \geq 2$. So $F(z) = F(w)$ as $F$ is constant on $C$. Then $F$ is constant on $X_{\text{reg}}$ since $F$ is continuous on $X_{\text{reg}}$. By modifying $\varphi$ by a function only in $t$, $\varphi$ would satisfy the Monge–Ampère flow $(4.4)$. The theorem follows from the uniqueness of the solution $\varphi$. \qed

We immediately have the following long time existence result generalizing the case for nonsingular minimal models due to Tian and Zhang [42].

**Corollary 4.1** Let $X$ be a minimal model with log terminal singularities and $H$ be an ample $\mathbb{Q}$-divisor on $X$. Then

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef} \} = \infty$$

and the weak Kähler–Ricci flow starting with $\omega_0 \in K_{H, p}(X)$ for some $p > 1$ exists for $t \in [0, \infty)$.
4.3 The Kähler–Ricci flow on projective varieties with orbifold singularities or with a crepant resolution

Given a normal projective variety $X$, very little is known how to construct "good" Kähler metrics on $X$ with reasonable curvature conditions. In general, the restriction of Fubini-Study metrics $\omega_{FS}$ on $X$ from ambient projective spaces behave badly near the singularities of $X$. Even the scalar curvature of $\omega_{FS}$ would have to blow up. In particular, $\omega^n$ is not necessarily equivalent to an adapted measure on $X$. For example, let $X$ be a surface containing a curve $C$ with self-intersection number $-2$ and $Y$ be the surface obtained from $X$ by contracting $C$. Then $Y$ has an isolated orbifold singularity. Let $\omega$ be the restriction of a smooth Kähler metric from a projective embedding of $Y$ and $\Omega$ a smooth orbifold volume form on $Y$. Then $\omega^n/\Omega = 0$ at the orbifold singularity. It shows that one should look at the category of smooth orbifold Kähler metrics on $Y$ instead of smooth Kähler metrics of ambient spaces from projective embedding.

As it turns out, the Kähler–Ricci flow produces singular Kähler metrics whose Monge–Ampère mass is equivalent to an adapted measure on a singular projective variety by Theorem 4.3. It is desirable that the Kähler–Ricci flow indeed improves the regularity of the initial data. In the case when $X$ has orbifold or it admits a crepant resolution, we can show that at least the scalar curvature of the singular Kähler metrics is bounded. In particular if $X$ has only orbifold singularities, the Kähler–Ricci flow immediately smoothes out the initial singular Kähler metric.

**Theorem 4.4** Let $X$ be a $\mathbb{Q}$-factorial projective normal variety with orbifold singularities. Let $H$ be an ample $\mathbb{Q}$-divisor on $X$ and

$$T_0 = \sup \{ t > 0 \mid H + tK_X \text{ is nef} \}.$$

If $\omega_0 \in \mathcal{K}_{H, p}(X)$ for some $p > 1$, then there exists a unique solution $\omega$ of the weak Kähler–Ricci flow for $t \in [0, T_0)$.

Furthermore, $\omega(t, \cdot)$ is a smooth orbifold Kähler-metric on $X$ for all $t \in (0, T_0)$ and so the weak Kähler–Ricci flow becomes the smooth orbifold Kähler–Ricci flow on $X$ immediately when $t > 0$.

**Proof** $X$ is automatically log terminal under the assumption that $X$ only admits orbifold singularities. The theorem can be proved by the same argument as in Theorem 3.1. We give a sketch of the proof below for completeness.

1. Since $X$ is an orbifold Kähler manifold, for any $z \in X$ there exist an open orbifold coordinate chart $U$ of $z$ and a uniformization $\phi : \tilde{U} \to U$, where $\tilde{U}$ is an open domain in $\mathbb{C}^n$ and $\phi$ is a holomorphic covering map. Let $\tilde{\omega} \in [H]$ be a smooth orbifold Kähler metric, i.e., $\phi^* \tilde{\omega}$ is a smooth Kähler
metric on $\tilde{U}$. Then for any smooth orbifold $\hat{\omega}$-psh function $\hat{\phi}$ with $\hat{\omega} + \sqrt{-1} \hat{\partial} \bar{\partial} \hat{\phi} > 0$, there exists a unique smooth orbifold solution of the orbifold Kähler–Ricci flow on $[0, T_0) \times X$ starting with $\hat{\omega} + \hat{\phi}$ as an immediate generalization from the smooth case because any local calculation can be lifted to the uniformization $\tilde{U}$. In particular, the maximum principle is valid in the orbifold case and Yau’s estimates for the complex Monge–Ampère equation can be generalized to the orbifold case.

2. Let $\omega_0 = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi_0$ with $\varphi_0 \in PSH_p(X, \hat{\omega})$ for some $p > 1$. We can consider the following family of complex Monge–Ampère equations on $X$

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi_{0,j})^n = F_j \omega^n, \quad \sup_X \varphi_{0,j} = \sup_X \varphi_0, \quad (4.7)$$

where $F_j$ is a sequence of positive smooth orbifold functions satisfying

$$\lim_{j \to \infty} \left\| F_j - \frac{(\omega_0)^n}{\hat{\omega}^n} \right\|_{L^p(X, \hat{\omega})} = 0, \quad \int_X F_j \omega^n = \int_X \hat{\omega}^n.$$

Since $F_j$ is a smooth positive orbifold function on $X$, $\varphi_{0,j}$ is a smooth orbifold solution of Eq. (4.7). Let $\pi : \tilde{X} \to X$ be the log resolution of $X$. Then after pulling back Eq. (4.7) to $\tilde{X}$ and applying the stability theorem in [11], we have

$$\lim_{j \to \infty} ||\varphi_{0,j} - \varphi_0||_{L^\infty(X)} = 0.$$

3. Let $\omega_{0,j} = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi_{0,j}$. We obtain the smooth orbifold solution $\omega_j(t, \cdot) = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi_j$ of the Kähler–Ricci flow on $[0, T_0) \times X$ starting with the smooth orbifold metric $\omega_{0,j}$. Lemma 3.1 holds for $\varphi_j$. We can also derive uniform second order and third order estimates for $\varphi_j$ as Lemmas 3.3 and 3.4 by applying the maximum principle on the uniformization of each orbifold coordinate chart. In particular, the limiting solution as $j \to \infty$ is a smooth family of smooth orbifold Kähler metrics on $(0, T_0) \times X$ from the uniform estimates for $\varphi_j$.

Theorem 4.4 can also be applied to the Kähler–Ricci flow on projective manifolds whose initial class is not Kähler.

**Theorem 4.5** Let $X$ be a smooth projective variety. Let $H$ be a big and semi-ample $\mathbb{Q}$-divisor on $X$. Suppose that

$$T_0 = \sup \{ t > 0 \mid H + t K_X \text{ is nef} \} > 0.$$
If $\omega_0 \in K_{H,p}(X)$ for some $p > 1$, then there exists a unique solution $\omega$ of the weak Kähler–Ricci flow for $t \in [0, T_0)$. Furthermore, there exists a positive continuous function $C(t)$ on $(0, T_0)$ such that

$$||R(\omega(t, \cdot))||_{L^{\infty}(X)} \leq C(t),$$

(4.8)

where $R(\omega(t, \cdot))$ is the scalar curvature of $\omega(t, \cdot)$.

**Proof** Kawamata’s base point free theorem implies that $H + tK_X$ is big and semi-ample for all $t \in [0, T_0)$ and $H + T_0K_X$ is semi-ample.

Let $\Omega$ be a smooth volume form on $X$ and $\chi = \sqrt{-1}\partial\bar{\partial} \log \Omega$. Let $\vartheta$ be a smooth Kähler form on $X$. Suppose that $\omega_0 = \hat{\omega} + \sqrt{-1}\partial\bar{\partial} \phi$, where $\hat{\omega} \in [H]$ is a smooth closed semi-positive $(1, 1)$-form and $\phi \in PSH_p(X, \hat{\omega})$ for some $p > 1$. We can always assume that $\hat{\omega} + t\chi$ is semi-positive for all $t \in [0, T_0]$. We consider the special case of the Monge–Ampère flow (3.37) by letting $\delta = w = r = 0$ and $\hat{\omega}_{t,s} = \hat{\omega} + s\vartheta + t\chi$.

$$\frac{\partial \varphi_s}{\partial t} = \log \frac{(\hat{\omega}_{t,s} + \sqrt{-1}\partial\bar{\partial} \varphi_s)^n}{\Omega}, \quad \varphi_s|_{t=0} = \varphi(0,s).$$

(4.9)

Using the stability theorem, we choose $\varphi(0,s) \in PSH(X, \hat{\omega} + s\vartheta) \cap C^\infty(X)$ so that $\varphi(0,s)$ converges to $\varphi_0$ in $L^\infty(X)$ as $s \to 0$. In fact, Eq. (4.9) is equivalent to the Kähler–Ricci flow on $X$ starting with $\omega_0 + s\vartheta + \sqrt{-1}\partial\bar{\partial} \varphi(0,s)$. Furthermore, $\varphi_s(t, \cdot)$ is smooth for $t \in [0, T_0)$ when $s > 0$. Let $\tilde{\omega}_s(t, \cdot) = \hat{\omega}_{t,s} + \sqrt{-1}\partial\bar{\partial} \varphi_s$. Then for $t > 0$,

$$\frac{\partial \tilde{\omega}_s}{\partial t} = -Ric(\tilde{\omega}_s)$$

and so

$$\frac{\partial}{\partial t} R(\tilde{\omega}_s) = \Delta_s R(\tilde{\omega}_s) + |Ric(\tilde{\omega}_s)|^2,$$

where $\Delta_s$ is the Laplace operator associated to $\tilde{\omega}_s$. Since

$$\left(\frac{\partial}{\partial t} - \Delta_s\right) tR(\tilde{\omega}_s) = R(\tilde{\omega}_s) + t|Ric(\tilde{\omega}_s)|^2 \geq R(\tilde{\omega}_s) + \frac{t}{n} R(\tilde{\omega}_s)^2.$$

The maximum principle immediately implies that $tR(\tilde{\omega}_s)$ is bounded from below on $[0, T_0) \times X$ uniformly in $s \in (0, 1]$. By letting $s \to 0$, $tR(\omega(t, \cdot))$ is uniformly bounded from below on $[0, T_0) \times X$.

Now we will prove the upper bound for $R(\omega)$.
Claim 1 For any $0 < t_0 < T < T_0$, there exist $A$ and $B > 0$ such that for all $s \in (0, 1]$ and on $[t_0, T] \times X$,
\[
\left( \frac{\partial}{\partial t} - \Delta_s \right) tr_{\tilde{\omega}_s} (\hat{\omega}) \leq A(tr_{\tilde{\omega}_s} (\hat{\omega}))^2 - B|\nabla_s tr_{\tilde{\omega}_s} (\hat{\omega})|^2, \tag{4.10}
\]
\[
\left( \frac{\partial}{\partial t} - \Delta_s \right) tr_{\tilde{\omega}_s} (\hat{\omega} + T \chi) \leq A(tr_{\tilde{\omega}_s} (\hat{\omega} + T \chi))^2 - B|\nabla_s tr_{\tilde{\omega}_s} (\hat{\omega} + T \chi)|^2, \tag{4.11}
\]
where $\nabla_s$ is the gradient operator associated to $\tilde{\omega}_s$.

Proof of Claim 1. Without loss of generality, we let $\pi : X \to \mathbb{CP}^{N_m}$ be the birational morphism induced by $mH$ and $m\hat{\omega}$ is the pullback of the Fubini-Study metric on $\mathbb{CP}^{N_m}$ if for some sufficiently large $m$. Notice that for $t \in [t_0, T]$, $H + TK_X$ is still semi-ample and big, so $m'(H + TK)$ induces a morphism $\pi' : X \to \mathbb{CP}^{N_m'}$. We can again assume that $\omega_0 + T \chi$ is the pullback of the Fubini-Study metric on $\mathbb{CP}^{N_m'}$. The curvature of $\hat{\omega}$ on $\mathbb{CP}^{N_m}$ and the curvature of $\hat{\omega} + T \chi$ on $\mathbb{CP}^{N_m'}$ are both bounded. Then it becomes a straightforward calculation from the parabolic Schwarz lemma in [32]. \qed

Claim 2 For any $0 < t_0 < T < T_0$, there exists $C > 0$ such that for all $s \in (0, 1]$ and on $[t_0, T] \times X$,
\[
0 \leq tr_{\tilde{\omega}_s} (\hat{\omega}_{t, 0}) < C. \tag{4.12}
\]

Proof of Claim 2. This can be proved by the parabolic Schwarz lemma from [32]. We apply the maximum principle for $t \log tr_{\tilde{\omega}_s} (\hat{\omega}) - A\varphi_s$ and $t \log tr_{\tilde{\omega}_s} (\hat{\omega} + T \chi) - A\varphi_s$ for sufficiently large $A$ so that both terms are uniformly bounded on $[0, T] \times X$ uniformly for $s \in (0, 1]$. The claim then easily follows. \qed

In particular, there exists $C > 0$ such that
\[
0 \leq tr_{\tilde{\omega}_s} (\hat{\omega}) < C, \quad -C < tr_{\tilde{\omega}_s} (\chi) < C. \tag{4.13}
\]

We will use the ideas in Perelman (unpublished data) (see [29]) to obtain a gradient estimate. Straightforward calculations show that
\[
\left( \frac{\partial}{\partial t} - \Delta_s \right) \left( \left| \nabla_s \frac{\partial \varphi_s}{\partial t} \right|^2 \right) = - \left| \nabla_s \nabla_s \frac{\partial \varphi_s}{\partial t} \right|^2 - \left| \nabla_s \frac{\partial \varphi_s}{\partial t} \right|^2
+ \left( \nabla_s tr_{\tilde{\omega}_s} (\chi) \cdot \nabla_s \frac{\partial \varphi_s}{\partial t} + \nabla_s \frac{\partial \varphi_s}{\partial t} \cdot \nabla_s tr_{\tilde{\omega}_s} (\chi) \right) \tag{4.14}
\]
and
\[
\left( \frac{\partial}{\partial t} - \Delta_s \right) \Delta_s \frac{\partial \phi_s}{\partial t} = - \left| \nabla_s \nabla_s \frac{\partial \phi_s}{\partial t} \right|^2 - g^{i\bar{j}} g^{k\bar{l}} \chi_i \bar{j} \left( \frac{\partial \phi_s}{\partial t} \right)_{k\bar{l}} \tag{4.15}
\]

Notice that \( \frac{\partial \phi_s}{\partial t} \) is uniformly bounded on \([t_0, T]\) for any \(0 < t_0 < T < T_0\) and \(s \in (0, 1]\). Then similar argument in the proof of Theorem 5.1 can be applied. Namely, one can apply the maximum principle for \((t - t_0) \mathcal{H}\) and \((t_0) \mathcal{K}\), where
\[
\mathcal{H} = \frac{\left| \nabla_s \frac{\partial \phi_s}{\partial t} \right|^2}{A - \frac{\partial \phi_s}{\partial t}} + tr_{\omega_s}(\hat{\omega}) + tr_{\omega_s}(\hat{\omega} + T\chi)
\]
and
\[
\mathcal{K} = -\frac{\Delta_s \frac{\partial \phi_s}{\partial t}}{A - u} + B \mathcal{H}.
\]

If we choose \(A > 0\) sufficiently large,
\[
\left( \frac{\partial}{\partial t} - \Delta_s \right) (t - t_0) \mathcal{H} \\
\leq -\epsilon (t - t_0) \frac{\left| \frac{\partial \phi_s}{\partial t} \right|^4}{(A - \frac{\partial \phi_s}{\partial t})^3} - \frac{2 (1 - \epsilon) (t - t_0)}{A - u} \Re \left( \nabla_s \mathcal{H} \cdot \nabla_s \frac{\partial \phi_s}{\partial t} \right) \\
+ C_1 \mathcal{H} + C_1 \\
\leq -\epsilon C_2 (t - t_0) \mathcal{H}^2 + C_1 \mathcal{H} - \frac{2 (1 - \epsilon) (t - t_0)}{A - u} \Re \left( \nabla_s \mathcal{H} \cdot \nabla_s \frac{\partial \phi_s}{\partial t} \right) + C_3.
\]

Hence \((t - t_0) \mathcal{H}\) is uniformly bounded on \([t_0, T] \times X\) for any \(s \in (0, 1]\).

If we choose \(A\) and \(B > 0\) sufficiently large,
\[
\left( \frac{\partial}{\partial t} - \Delta_s \right) (t - t_0) \mathcal{K} \\
\leq -C_4 (t - t_0) \frac{\left| \nabla_s \nabla_s \frac{\partial \phi_s}{\partial t} \right|^2}{A - \frac{\partial \phi_s}{\partial t}} - \frac{2 (t - t_0)}{A - u} \Re \left( \nabla_s \mathcal{K} \cdot \nabla_s \frac{\partial \phi_s}{\partial t} \right) + C_1 \mathcal{K} + C_1 \\
\leq -C_5 (t - t_0) \mathcal{K}^2 + C_1 \mathcal{K} - \frac{2 (t - t_0)}{A - u} \Re \left( \nabla_s \mathcal{K} \cdot \nabla_s \frac{\partial \phi_s}{\partial t} \right) + C_6.
\]

Hence \((t - t_0) \mathcal{K}\) is uniformly bounded on \([t_0, T] \times X\) for any \(s \in (0, 1]\). Here we make use of Claim 1 that
\[
T tr_{\omega_s}(\chi) = tr_{\omega_s}(\hat{\omega} + T\chi) - tr_{\omega_s}(\hat{\omega})
\]
is uniformly bounded on \([t_0, T] \times X\) uniformly for \(s \in (0, 1]\). Also the term
\[
T^2 |\nabla_s tr_{\tilde{\omega}_s}(\chi)|^2 \leq |\nabla_s tr_{\tilde{\omega}_s}(\hat{\omega} + T \chi)|^2 + |\nabla_s tr_{\tilde{\omega}_s}(\hat{\omega})|^2
\]
can be controlled by \((\frac{\partial}{\partial t} - \Delta_s)(tr_{\tilde{\omega}_s}(\hat{\omega}) + tr_{\tilde{\omega}_s}(\hat{\omega} + T \chi))\). Therefore there exists \(C > 0\) such that on \([t_0, T] \times X\)
\[
R(\tilde{\omega}_s) = -\Delta_s \frac{\partial \varphi_s}{\partial t} - tr_{\tilde{\omega}_s}(\chi) \leq C
\]
uniformly in \(s \in (0, 1]\). The theorem is then proved by letting \(s \to 0\). \(\square\)

A \(\mathbb{Q}\)-factorial projective variety \(X\) is said to admit a crepant resolution if there exists a resolution of singularities \(\pi : \tilde{X} \to X\) such that \(K_{\tilde{X}} = \pi^* K_X\). The following theorem is an immediate corollary of Theorem 4.5.

**Theorem 4.6** Let \(X\) be a \(\mathbb{Q}\)-factorial projective variety admitting a crepant resolution. Let \(H\) be an ample \(\mathbb{Q}\)-divisor on \(X\) and \(T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}\).

If \(\omega_0 \in \mathcal{K}_{H, p}(X)\) for some \(p > 1\), then there exists a unique solution \(\omega\) of the weak Kähler–Ricci flow for \(t \in [0, T_0)\). Furthermore, there exists a positive continuous function \(C(t)\) on \((0, T_0)\) such that
\[
||R(\omega(t, \cdot))||_{L^\infty(X)} \leq C(t). \quad (4.16)
\]

**Proof** Let \(\Omega\) be an adapted measure on \(X\). Let \(\pi : \tilde{X} \to X\) be a crepant resolution of \(X\). Then \(\pi^* \Omega\) is a smooth volume form on \(\tilde{X}\). Then we can directly apply Theorem 4.5 to prove the theorem. \(\square\)

It is shown in [51] that the scalar curvature is uniformly bounded along the normalized Kähler–Ricci flow on smooth manifolds of general type. On the other hand, the scalar curvature will in general blow up if the Kähler–Ricci flow develops finite time singularities (see [52]).

5 The Kähler–Ricci flow with surgery

5.1 Minimal model program with scaling

The following is a standard notion from birational geometry.

**Definition 5.1** Let \(X\) be a projective variety and \(N_1(X)_{\mathbb{Z}}\) the group of numerically equivalent 1-cycles (two 1-cycles are numerically equivalent if they have the same intersection number with every Cartier divisor). Let \(N_1(X)_{\mathbb{R}} = \)
We denote by $NE(X)$ the set of classes of effective 1-cycles. $NE(X)$ is convex and we let $\bar{NE}(X)$ be the closure of $NE(X)$ in the Euclidean topology.

A special case of the minimal model program is proposed in [2] and plays an important role for the termination of flips. We briefly explain the minimal model program with Scaling below.

**Definition 5.2 (MMP with scaling)**

1. We start with a pair $(X, H)$, where $X$ is a projective $\mathbb{Q}$-factorial variety with log terminal singularities and $H$ is a big and semi-ample $\mathbb{Q}$-divisor on $X$.
2. Let $\lambda_0 = \inf \{ \lambda > 0 \mid \lambda H + K_X \text{ is nef} \}$ be the nef threshold. If $\lambda_0 = 0$, then we stop since $K_X$ is already nef.
3. Otherwise, there is an extremal ray $R$ of the cone of curves $\bar{NE}(X)$ on which $K_X$ is negative and $\lambda_0 H + K_X$ is zero. So there exists a contraction $\pi : X \to Y$ of $R$.
   - If $\pi$ is a divisorial contraction, we replace $X$ by $Y$ and let $H_Y$ be the strict transformation of $\lambda_0 H + K_X$ by $\pi$. Then we return to 1. with $(Y, H_Y)$.
   - If $\pi$ is a small contraction, we replace $X$ by its flip $X^+$ and let $H_{X^+}$ be the strict transformation of $\lambda_0 H + K_X$ by the flip. Then we return to 1. with $(X^+, H_{X^+})$.
   - If $\dim Y < \dim X$, then $X$ is a Mori fibre space, i.e., the fibers of $\pi$ are Fano. Then we stop.

The following theorem is proved in [2].

**Theorem 5.1** If $X$ is of general type, the minimal model program with Scaling terminates in finite steps.

In general, the contraction of the extremal ray might not be the same as the contraction induced by the semi-ample divisor $\lambda_0 H + K_X$. We define the following special ample divisors so that at each step, there is only one extremal ray contracted by the morphism induced by $\lambda_0 H + K_X$.

**Definition 5.3** Let $X$ be a projective $\mathbb{Q}$-factorial variety with log terminal singularities. An ample $\mathbb{Q}$-divisor $H$ on $X$ is called a good initial divisor $H$ if the following conditions are satisfied.

1. Let $X_0 = X$ and $H_0 = H$. The MMP with scaling terminates in finite steps by replacing $(X_0, H_0)$ by $(X_1, H_1), \ldots, (X_m, H_m)$ until $X_{m+1}$ is a minimal model or $X_m$ is a Mori fibre space.
2. Let $\lambda_i$ be the nef threshold for each pair $(X_i, H_i)$ for $i = 1, \ldots, m$. Then the contraction induced by the semi-ample divisor $\lambda_i H_i + K_{X_i}$ contracts exactly one extremal ray.

\[ \odot \ Springer \]
It is possible that good initial divisors are generic if MMP with scaling holds for any pair \((X, H)\). It will be seen in the future that good initial divisors simplify the analysis for surgery along the Kähler–Ricci flow, though such an assumption is not necessary. We will explain it in detail in Sect. 5.5.

Now we relate the Kähler–Ricci flow to MMP with scaling. Consider the Kähler–Ricci flow $\frac{\partial \omega}{\partial t} = -\text{Ric} (\omega)$ on $X$ with the initial $\omega_0 \in \mathcal{K}_{H,p}(X)$ for an ample $\mathbb{Q}$-divisor $H$ on $X$ and some $p > 1$. Let $T_0 = \sup\{t \geq 0 | H + tK_X > 0\}$. By the rationality theorem 4.1, $T_0 = \infty$ or $T_0$ is a positive rational number. In particular, if $X$ is a minimal model, then $T_0 = \infty$. In fact, $T_0 = \frac{1}{\lambda_0}$ is the inverse of the nef threshold. The following theorem is a natural generalization for the long time existence theorem of Tian and Zhang [42] for the Kähler–Ricci flow on smooth minimal models.

**Theorem 5.2** Let $X$ be an $n$-dimensional $\mathbb{Q}$-factorial projective variety with log terminal singularities. If $K_X$ is nef, then for any ample $\mathbb{Q}$-divisor $H$ on $X$ and $\omega_0 \in \mathcal{K}_{H,p}(X)$ with $p > 1$, the weak Kähler–Ricci flow starting with $\omega_0$ has long time existence for $t \in [0, \infty)$.

Suppose that $X$ is not minimal and so $T_0 < \infty$. Then $H + T_0K_X$ is nef and the weak Kähler–Ricci flow exists uniquely for $t \in [0, T_0)$. By Kawamata’s base point free theorem, $H + T_0K_X$ is semi-ample and hence the ring $R(X, H + T_0K_X) = \oplus_{m=0}^{\infty} H^0(X, m(H + T_0K_X))$ is finitely generated.

- If $H + T_0K_X$ is big and hence $R(X, H + T_0K_X)$ induces a birational morphism $\pi : X \to Y$. For a generic ample divisor $H$, the morphism $\pi$ contracts exactly one extremal ray of $\overline{NE}(X)$. We discuss the following two cases according to the size of the exceptional locus of $\pi$.

1. $\pi$ is a divisorial contraction, that is, the exceptional locus $Exc(\pi)$ is a divisor whose image of $\pi$ has codimension at least two. In this case, $Y$ is still $\mathbb{Q}$-factorial and has at worst log terminal singularities.
2. $\pi$ is a small contraction, that is, the exceptional locus $Exc(\pi)$ has codimension at least two. In this case, $Y$ have rather bad singularities and $K_Y$ is no longer a Cartier $\mathbb{Q}$-divisor. The solution to such a small contraction is to replace $X$ by a birationally equivalent variety with singularities milder than those of $Y$.

**Definition 5.4** (see [19]) Let $\pi : X \to Y$ be a small contraction such that $-K_X$ is $\pi$-ample. A variety $X^+$ together with a proper birational morphism $\pi^+ : X^+ \to Y$ is called a flip of $\pi$ if $\pi^+$ is also a small contraction and $K_{X^+}$ is $\pi^+$-ample.
Here $X^+$ is again $\mathbb{Q}$-factorial and has at worst log terminal singularities.

- If $H + T_0 K_X$ is not big, then the Kodaira dimension $0 \leq \kappa = \text{kod}(H + T_0 K_X) < n$ and $X$ is a Mori fibre space admitting a Fano fibration over a normal variety $Y$ of dimension $\kappa$. In particular, $Y$ is $\mathbb{Q}$-factorial and has log terminal singularities.

We will discuss in the following sections the behavior of the Kähler–Ricci flow at the singular time $T_0$ according to the above situations.

### 5.2 Estimates

In this section, we assume that $T_0 < \infty$ and $H + T_0 K_X$ is big. Let $\hat{\omega} \in [H]$ be the restriction of a smooth Kähler metric on the ambient space of a projective embedding of $X$. Let $\Omega$ be an adapted measure on $X$ and $\chi = \sqrt{-1} \partial \bar{\partial} \log \Omega \in [K_X]$. Consider the Monge–Ampère flow associated to the Kähler–Ricci flow on $X$ with an initial Kähler form $\omega_0 = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi_0$,

\[
\frac{\partial \varphi}{\partial t} = \log \left( \frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \right), \quad [0, T_0) \times X
\]

where $p > 1$, $\hat{\omega}_t = \hat{\omega} + t \chi$ and $\omega = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi$.

Since $H + T_0 K_X$ is big and semi-ample, the linear system $|m(H + T_0 K_X)|$ for sufficiently large $m$ induces a birational morphism

$$
\pi : X \to Y \subset \mathbb{CP}^N_m.
$$

Let $\omega_Y$ be the pullback of a multiple of the Fubini-Study metric form on $\mathbb{CP}^N_m$ with $\pi^* \omega_Y \in [H + T_0 K_X]$. There exists a log resolution of singularities and the exceptional locus of $\pi$

$$
\mu : \tilde{X} \to Y
$$

satisfying the following conditions.

1. $\tilde{X}$ is smooth.
2. There exists an effective divisor $E_Y$ on $\tilde{X}$ such that $\mu^*[H + T_0 K_X] - \epsilon[E_Y]$ is ample for any sufficiently small $\epsilon > 0$ and the support of $E_Y$ coincides with the exceptional locus of $\mu$. 
Let $S_{E_Y}$ be the defining section for the line bundle associated to $[E_Y]$ and $h_{E_Y}$ the hermitian metric such that for any $\epsilon > 0,$

$$\mu^* \omega_Y - \epsilon R (h_{E_Y}) > 0.$$ 

Let $Exc(\pi)$ be the exceptional locus of $\pi.$ Then we have the following uniform estimates.

**Lemma 5.1** Let $\varphi \in C^0([0, T_0) \times X_{\text{reg}}) \cap C^\infty((0, T_0) \times X_{\text{reg}})$ with $\varphi \in L^\infty([0, T] \times X)$ for all $0 < T < T_0,$ be the solution solving the Monge–Ampère flow (5.2). There exists $C > 0$ such that

$$||\varphi||_{L^\infty([0, T_0) \times X)} \leq C. \quad (5.3)$$

Furthermore, for any $K \subset \subset X \setminus Exc(\pi)$ and $k \geq 0,$ there exists $C_{K,k} > 0$ such that

$$||\varphi||_{C^k([0, T_0) \times K)} \leq C_{K,k}. \quad (5.4)$$

**Proof** We lift the Monge–Ampère flow (5.2) on $\tilde{X}.$ The proof of the $L^\infty$-estimate proceeds in the same way as in the proof of Lemma 3.7 since $[\omega_t]$ is big and semi-ample for all $t \in [0, T_0].$ The $C^2$-estimate on $\tilde{X}$ follows the same argument as in Lemma 3.9, which is valid on $\tilde{X} \setminus E_Y.$ Since the support of $\mu(E_Y)$ is contained in $Exc(\pi),$ the $C^2$-estimate holds on $X_{\text{reg}} \setminus Exc(\pi).$ We leave the details for the readers as an exercise.

**Lemma 5.2** There exists $C > 0$ such that on $[0, T_0) \times X,$

$$\frac{\omega^n}{\Omega} \leq e^C. \quad (5.5)$$

**Proof** The weak Kähler–Ricci flow has a unique solution on $[0, T_0) \times X$ after reduced to a Monge–Ampère flow on the resolution $\tilde{X}.$ We then consider the approximating Monge–Ampère flow (3.37) with a solution $\varphi_{s,w,r} = \varphi_{s,w,r}^{(0)}.$ Let $\mathcal{H}_{s,w,r} = t \frac{\partial}{\partial t} \varphi_{s,w,r} = \varphi_{s,w,r} + \epsilon \log |S_{E_Y}|^2_{h_Y}.$ Let $\Delta_{s,w,r}$ be the Laplace operator with respect to $\omega_{s,w,r}.$ Then $\mathcal{H}_{s,w,r}$ is smooth outside $E_Y$ and

$$\left( \frac{\partial}{\partial t} - \Delta_{s,w,r} \right) \mathcal{H}_{s,w,r} = -tr \omega_{s,w,r} (\omega_0 - \epsilon R (h_{E_Y})) \leq 0.$$

As $\mathcal{H}_{s,w,r}|_{t=0} = -\varphi_{(0,s)} + \epsilon \log |S_{E_Y}|^2_{h_Y}$ is bounded from above and for each $t \in (0, T_0),$ the maximum of $\mathcal{H}_{s,w,r}$ can only be achieved on $\tilde{X} \setminus E_Y$ and
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$H_{s,w,r} \leq H_{s,w,r}|_{t=0}$ is uniformly bounded above for all $s, w, r \in (0, 1]$. By letting $\epsilon \to 0$, there exists $C > 0$ such that

$$t \dot{\psi}_{s,w,r} \leq C$$

for all $s, w, r \in (0, 1]$. This completes the proof of the lemma by letting $s, w, r \to 0$. \hfill $\square$

**Corollary 5.1** We consider the unique solution $\omega$ for the weak Kähler–Ricci flow on $X$ starting with the initial $\omega_0 \in K_{H,p}(X)$ for some $p > 1$. If $H + T_0 K_X$ is big, then $\omega(t, \cdot)$ converges to a current $\tilde{\omega}_{T_0} \in K_{H+T_0 K_X, \infty}(X)$ in $C^\infty(X_{\text{reg}} \setminus \text{Exc}(\pi))$-topology. That is, there exists $C > 0$ such that

$$(\tilde{\omega}_{T_0})^n \leq C \Omega$$

(5.6)

for a fixed adapted measure $\Omega$ on $X$.

By Corollary 5.1, $H$ is trivial over the fibres and so $\tilde{\omega}_Y$ is trivial restricted on each fibre of $\pi$. Let $\tilde{\omega}_{T_0} = \tilde{\omega}_Y + \sqrt{-1} \partial \bar{\partial} \varphi_{T_0}$, where $\varphi_{T_0} = \lim_{t \to T_0} \varphi(t, \cdot)$. Then $\varphi_{T_0}$ must be constant on each fibre as each fibre of $\pi$ is connected. Therefore $\varphi_{T_0}$ can descend onto $Y$ and $\varphi_{T_0} \in PSH(Y, \tilde{\omega}_Y) \cap L^\infty(Y)$. So the limiting current $\tilde{\omega}_{T_0}$ descends onto $Y$ as a semi-positive closed $(1, 1)$-current with bounded local potentials.

5.3 Extending the Kähler–Ricci flow through singularities by divisorial contractions

In this section, we will prove that the weak Kähler–Ricci flow can be continued through divisorial contractions.

We assume that $\pi : X \to Y$ is a divisorial contraction. It is well-known that $Y$ is again a $\mathbb{Q}$-factorial projective variety with at worst log terminal singularities if $X$ is.

**Proposition 5.1** Let $\Omega_Y$ be an adapted measure on $Y$ and $H_Y = \pi_*(H + T_0 K_X)$. Then for some $p > 1$,

$$(\pi^{-1})^* \tilde{\omega}_{T_0} \in K_{H_Y, p}(Y).$$

(5.7)

**Proof** Obviously, $\tilde{\omega}_{T_0}$ has bounded local potentia and the restriction of $\tilde{\omega}$ has constant local potentials along each fibre of $\pi$. So $\tilde{\omega}_{T_0}$ descends to $Y$ and $(\pi^{-1})^* \tilde{\omega}_{T_0}$ is well-defined and admits bounded local potentials on $Y$. Let $F = (\tilde{\omega}_{T_0})^n \Omega_Y$. It suffices to show $F \in L^p(Y, \Omega_Y)$ for some $p > 1$. There exists $C > 0$ such that

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\[
\int_Y F^p \Omega_Y = \int_Y \left( \frac{\tilde{\omega}^n_{T_0}}{\Omega_Y} \right)^{p-1} \tilde{\omega}^n_{T_0} = \int_X \left( \frac{\tilde{\omega}^n_{T_0}}{\pi^* \Omega_Y} \right)^{p-1} \tilde{\omega}^n_{T_0} \leq qC \int_X \left( \frac{\Omega}{\Omega_Y} \right)^{p-1} \Omega.
\]

Since \( \frac{\Omega}{\Omega_Y} \) has at worst poles, \( \int_Y F^p \Omega_Y < \infty \) for \( p - 1 > 0 \) sufficiently small.

\[\square\]

**Theorem 5.3** Let \( X \) be a \( \mathbb{Q} \)-factorial projective variety with log terminal singularities and \( H \) be an ample \( \mathbb{Q} \)-divisor on \( X \). Let 

\( T_0 = \sup \{ t > 0 \mid H + t K_X \text{ is nef} \} \)

be the first singular time. Suppose that the semi-ample divisor \( H + T_0 K_X \) induces a divisorial contraction \( \pi : X \to Y \).

Let \( \omega \) be the unique solution of the weak Kähler–Ricci flow for \( t \in [0, T_0) \) starting with \( \omega_0 \in K_{H, p}(X) \) for some \( p > 1 \), and let \( H_Y \) be the divisor on \( Y \) as the strict transformation of \( H + T_0 K_X \) by \( \pi \). Then there exists \( \omega_Y \in K_{H_Y, p'}(Y) \cap C^\infty(Y_{\text{reg}} \setminus \pi(\text{Exc} (\pi))) \) for some \( p' > 1 \) such that \( \omega(t, \cdot) \) converges to \( \pi^* \omega_Y \) in \( C^\infty(X_{\text{reg}} \setminus \pi(\text{Exc} (\pi))) \)-topology as \( t \to (T_0)^+ \).

Furthermore, the weak Kähler–Ricci flow can be continued on \( Y \) with the initial current \( \omega_Y \).

**Proof** Since \( H_Y \) is the strict transformation of \( H + T_0 K_X \) by \( \pi \) and \( \omega_Y \) admits bounded local potential, \( \omega_Y \in K_{H_Y, p'}(Y) \) for some \( p' > 1 \) by Proposition 5.1. Then the Kähler–Ricci flow can start with \( \omega_Y \) on \( Y \) uniquely Theorem 3.3.

\[\square\]

### 5.4 Extending the Kähler–Ricci flow through singularities by flips

In this section, we will prove that the weak Kähler–Ricci can be continued through flips.

We assume that \( \pi : X \to Y \) is a small contraction and there exists a flip 

\[ \tilde{\pi} = \pi^+ \circ \pi^{-1} : X^+ \dashrightarrow X. \]

Then \( X^+ \) is \( \mathbb{Q} \)-factorial and it has log terminal singularities. The limiting current \( \tilde{\omega}_{T_0} \) descends to a closed semi-positive \((1, 1)\)-current \( \omega_Y \) on \( Y \) with bounded local potentials and it can be then pulled back on \( X^+ \) by \( \pi^+ \). Furthermore, there exists \( C > 0 \) such that 

\[
\frac{(\pi^* \omega_Y)^n}{\Omega} \leq C. \tag{5.8}
\]
Proposition 5.2 Let $\Omega_{X^+}$ be an adapted measure on $X^+$ and $H_{X^+}$ be the strict transformation of $H + T_0K_X$ by $\hat{\pi}$. Then for some $p' > 1$,

$$\left(\pi^+\right)^*\omega_Y \in K_{H_{X^+}, p'}(X^+).$$

(5.9)

Proof Obviously $(\pi^+)^*\omega_Y$ is semi-positive closed $(1, 1)$-current on $X^+$ with bounded local potentials as well. Let $F = \frac{((\pi^+)^*\omega_Y)^n}{\Omega_{X^+}}$. It suffices to show $F \in L^{p'}(X^+, \Omega_{X^+})$ for some $p' > 1$. There exists $C > 0$ such that

$$\int_{X^+} F^p \Omega_{X^+} = \int_{X^+} \left(\frac{(\pi^+)^*\omega_Y}{\Omega_{X^+}}\right)^p = \int_{X} \left(\frac{\pi^*\omega_Y}{(\hat{\pi}^{-1})^*\Omega_{X^+}}\right)^{p-1} \left(\frac{\Omega}{(\hat{\pi}^{-1})^*\Omega_{X^+}}\right) \Omega.$$

Since $\frac{\Omega}{(\hat{\pi}^{-1})^*\Omega_{X^+}}$ has at worst poles, $\int_{X^+} F^p \Omega_{X^+} < \infty$ for $p' - 1 > 0$ sufficiently small.

Theorem 5.4 Let $X$ be a $\mathbb{Q}$-factorial projective variety with log terminal singularities and $H$ be an ample $\mathbb{Q}$-divisor on $X$. Let

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}$$

be the first singular time. Suppose that the semi-ample divisor $H_{T_0} = H + T_0K_X$ induces a small contraction $\pi : X \to Y$ and there exists a flip

$$X \xleftarrow{\pi} Y \xrightarrow{\pi^+} X^+.$$

(5.10)

Let $\omega$ be the unique solution of the weak Kähler–Ricci flow for $t \in [0, T_0)$ starting with $\omega_0 \in K_{H, p}(X)$ for some $p > 1$. Then the following hold.

1. There exists a closed semi-positive $(1, 1)$-current $\omega_Y$ on $Y$ such that $\pi^*\omega_Y \in K_{H_{T_0}, p'}(X)$ and $(\pi^+)^*\omega_Y \in K_{(\hat{\pi})^*H_{T_0}, p'}(X^+)$ for some $p' > 1$.

2. The weak Kähler–Ricci flow can be continued on $X^+$ with the initial current $(\pi^+)^*\omega_Y$ at $t = T_0$, $\omega(t)$ converges to $\pi^*\omega_Y$ in $C^\infty((X)_{\text{reg}} \setminus \text{Exc}(\pi))$-topology as $t \to (T_0)^-$ and $\omega(t)$ converges to $(\pi^+)^*\omega_Y$ in $C^\infty((X^+)_{\text{reg}} \setminus \text{Exc}(\pi^+))$-topology as $t \to (T_0)^+$. 

$\square$
Since $\omega_Y$ admits bounded local potentials, $(\pi^+)^*\omega_Y \in K_{(\tilde{\pi})^*H_{T_0}, p'}(X^+)$ for some $p' > 1$ by Proposition 5.2. Then the Kähler–Ricci flow can start with $(\pi^+)^*\omega_Y$ on $X^+$ uniquely by applying Theorem 1.1 because $H^+$ is big and semi-ample with $H^+ + \epsilon K_{X^+}$ being ample for sufficiently small $\epsilon > 0$.

\[ \square \]

### 5.5 Long time existence assuming MMP

As proved in Sects. 5.3 and 5.4, the Kähler–Ricci flow can flow through divisorial contractions and flips. In general, at the singular time $T_0$, the morphism $\pi : X \to Y$ induced by the semi-ample divisor $H + T_0K_X$ might contract more than one extremal ray, while the minimal model program with scaling usually contracts one extremal ray at one time. It simplifies the analysis to assume the existence of a good initial divisor as in Definition 5.3 as to avoid complicated contractions.

**Theorem 5.5** Let $X$ be a $\mathbb{Q}$-factorial projective variety with log terminal singularities. If there exists a good initial divisor $H$ on $X$, then either $X$ does not admit a minimal model or the weak Kähler–Ricci flow has long time existence for any current $\omega_0 \in K_{H, p}(X)$ with $p > 1$, after finitely many surgeries through divisorial contractions and flips.

**Proof** Assume $X$ admits a minimal model and let $X_0 = X$ and $H_0 = H$. Since $H$ is a good initial divisor, by MMP with scaling, at each singular time, the morphism induced by the semi-ample divisor is always a contractional contraction or flipping contraction.

More precisely, suppose the Kähler–Ricci flow performs surgeries and replaces $(X_0, H_0)$ by a finite sequence of $(X_i, H_i)$ at each singular time $T_i$, $i = 1, \ldots, m$, and $X_{m+1}$ is a minimal model of $X$. If $\lambda_i$ is the nef threshold for $(X_i, H_i)$ as in Definition 5.2, $i = 1, \ldots, m$, $\lambda_i > 0$ and

\[ T_i = T_{i-1} + \frac{1}{\lambda_i}. \]

At $T_i$, the morphism induced by the semi-ample divisor $H_i + T_iK_{X_i}$ contracts exactly one extremal ray and so it must be a divisorial contraction or a flip. By Theorem 5.3 and Theorem 5.4, the Kähler–Ricci flow with the pair $(X_i, H_i)$ is replaced by the one with the pair $(X_{i+1}, H_{i+1})$ with $H_{i+1}$ being the strict transform of $H_i + T_iK_{X_i}$ until $X$ is finally replaced by its minimal model $X_{m+1}$ and the Kähler–Ricci flow exists for all time afterwards by Theorem 4.3 as $K_{X_{m+1}}$ is nef.

If $H$ is not a good initial divisor, the surgery at the finite singular time could be complicated and a detailed speculation is given in Sect. 6.2.
5.6 Convergence on projective varieties of general type

Let \( X \) be a minimal model of general type with log terminal singularities and so \( K_X \) is big and nef. Let \( H \) be an ample \( \mathbb{Q} \)-divisor on \( X \) and \( \omega_0 \in K_{H, p}(X) \) for some \( p > 1 \). We consider the normalized Kähler–Ricci flow on \( X \).

\[
\begin{aligned}
\frac{\partial \omega}{\partial t} &= -Ric(\omega) - \omega \\
\omega|_{t=0} &= \omega_0.
\end{aligned}
\]  

(5.11)

Let \( \Omega \) be an adapted measure on \( X \) and \( \chi \in \sqrt{-1} \partial \bar{\partial} \log \Omega \in c_1(K_X) \). Let \( \hat{\omega} \in [H] \) be the restriction of a Fubini-Study metric from a birational morphism \( \phi : X \to \mathbb{C}P^N \) induced by the linear system \( |mH| \) for some sufficiently large \( m \). We can assume assume \( \chi \) is semi-positive and big because \( K_X \) is semi-ample. Then the Kähler–Ricci flow (5.11) is equivalent to the following Monge–Ampère flow

\[
\begin{aligned}
\frac{\partial \varphi}{\partial t} &= \log \left( \frac{\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n - \varphi \\
\varphi|_{t=0} &= \varphi_0,
\end{aligned}
\]  

(5.12)

where \( \hat{\omega}_t = e^{-t} \hat{\omega} + (1 - e^{-t}) \chi \) and \( \omega_0 = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi_0 \).

Now that \( K_X \) is semi-ample as \( X \) is a minimal model, the abundance conjecture holds for general type. The linear system \( |mK_X| \) for sufficiently large \( m > 0 \) induces a morphism

\[ \pi : X \to X_{can}, \]

where \( X_{can} \) is the canonical model of \( X \). Without loss of generality, we can always assume that \( \chi \geq 0 \) and \( \chi \) is big. Furthermore, we can assume \( \omega_0 \geq \epsilon \chi \) for sufficiently small \( \epsilon > 0 \) since \( H \) is ample.

The long time existence is guaranteed by Theorem 4.3 since \( K_X \) is nef and \( T_0 = \infty \).

**Proposition 5.3** The weak normalized Kähler–Ricci flow (5.11) exists on \([0, \infty) \times X\) for any initial current \( \omega_0 \in K_{H, p}(X) \) with \( p > 1 \).

**Lemma 5.3** There exists \( C > 0 \) such that

\[ ||\varphi(t, \cdot)||_{L^\infty([0, \infty) \times X)} \leq C. \]  

(5.13)
**Proof** Let $\tilde{X}$ be a nonsingular model of $X$. Without loss of generality, we can consider the Monge–Ampère flow (5.12) on $\tilde{X}$ by pullback and the following smooth approximation for the Monge–Ampère flow as discussed in Sects. 3.2 and 3.3.

$$\left\{ \begin{array}{l}
\frac{\partial \varphi_{s,w,r}}{\partial t} = \log \left( \frac{\hat{\omega}_t + \sqrt{-1} \bar{\partial} \varphi_{s,w,r}}{\Omega_{w,r}} \right)^n - \varphi_{s,w,r} \\
\varphi_{s,w,r}|_{t=0} = \varphi(0,s),
\end{array} \right. \quad (5.14)$$

where $\hat{\omega}_t = \omega_t + s \vartheta$ and $\Omega_{w,r}$ are defined as in Sect. 3.2.

Let $\phi_{s,w,r} \in C^\infty(\tilde{X})$ be the solution of the following Monge–Ampère equation

$$\left( \chi + s \vartheta + \sqrt{-1} \bar{\partial} \phi_{s,w,r} \right)^n = e^{\phi_{s,w,r}} \Omega_{w,r}. \quad (5.15)$$

There exists $C > 0$ such that for all $s, w, r \in (0, 1]$,

$$\|\phi_{s,w,r}\|_{L^\infty(\tilde{X})} \leq C.$$  

Let $\psi_\epsilon = \varphi_{s,w,r} - \phi_{s,w,r} - \epsilon \log |S_{\tilde{E}}| h_{\tilde{E}}$, where $\tilde{E}$ is a divisor whose support contains the exceptional locus of the resolution of $\tilde{X}$ over $X$ and $\chi - \epsilon R(h_{\tilde{E}}) > 0$ for sufficiently small $\epsilon > 0$. Then similar argument by the maximum principle as in Sect. 3.3 shows that $\psi_\epsilon$ is uniformly bounded from below for all $t \in [0, \infty)$ and for all sufficiently small $\epsilon > 0$. Then by letting $\epsilon \to 0$, there exists $C > 0$ such that for $t \in [0, \infty), s, w$ and $r \in (0, 1]$,

$$\varphi_{s,w,r} \geq -C.$$  

Therefore $\varphi$ is uniformly bounded from below for all $t \in [0, \infty)$ by its definition. The uniform upper bound of $\varphi$ can be obtained by similar argument.

\[\square\]

**Lemma 5.4** Let $X^\circ = X_{reg} \setminus \text{Exc}(\pi)$. For any $K \subset \subset X^\circ, t_0$ and $k > 0$, there exists $C_{K,k,t_0} > 0$ such that for $t \in [t_0, \infty)$,

$$\|\varphi(t, \cdot)\|_{C^k_{\omega_0}(K)} \leq C_{K,k,t_0}. \quad (5.16)$$

**Proof** We can assume that $X$ is nonsingular with the cost of $\omega_0$ and $\Omega$ being degenerate. We first have to show that $tr_{\omega_0}(\omega)$ is uniformly bounded on $K$. This is achieved by similar arguments for Lemma 3.9. Then the higher order estimates follow by the same argument in Sect. 3.

\[\square\]
Lemma 5.5 For any $t_0 > 0$, there exists $C > 0$ such that on $[t_0, \infty) \times X$,

$$\frac{\partial \varphi}{\partial t} \leq C t e^{-t}. \quad (5.17)$$

Proof Notice that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \frac{\partial \varphi}{\partial t} = -e^{-t} tr_\omega(\hat{\omega} - \chi) - \frac{\partial \varphi}{\partial t}.$$

Let $H = e^t \frac{\partial \varphi}{\partial t} - A \varphi + \epsilon \log |S_E|_{h_E}^2 - A t$, where $A > 0$ is sufficiently large such that $A \omega_0 \geq \chi$ and $\epsilon > 0$ is chosen to be sufficiently small. Then there exists $C > 0$ for all sufficiently small $\epsilon > 0$ such that

$$\left(\frac{\partial}{\partial t} - \Delta\right) H = -tr_\omega (A \hat{\omega} + \hat{\omega} - \chi - \epsilon Rich (h_E)) - A \frac{\partial \varphi}{\partial t}$$

$$\leq -A \frac{\partial \varphi}{\partial t}$$

$$\leq -A e^{-t} H - A^2 \varphi + A \epsilon \log |S_E|_{h_E}^2 - A t$$

$$\leq -A e^{-t} H - C.$$

Since the maximum can only be achieved on $X^0$ and $H|_{t_0=0}$ is bounded from above, by the maximum principle, there exists $C > 0$ such that on $[t_0, \infty) \times X$,

$$H \leq C(t + 1).$$

Therefore there exists $C > 0$ independent of $\epsilon$ such that on $[t_0, \infty) \times X$,

$$\frac{\partial \varphi}{\partial t} \leq C e^{-t} \left(1 + t - \epsilon \log |S_E|_{h_E}^2\right).$$

The lemma is proved by letting $\epsilon \to 0$. \hfill \Box

Corollary 5.2

$$\lim_{t \to \infty} \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^1(X)} = 0. \quad (5.18)$$

Proof There exists $T > 0$, such that $\frac{\partial \varphi}{\partial t} \leq e^{-t/2}$ for all $t \geq T$. Notice that

$$\int_{t_1}^{t_2} \frac{\partial \varphi}{\partial t}(t, z) dt = \varphi(t_2, z) - \varphi(t_1, z)$$

is uniformly bounded for all $t_1, t_2 \geq 0$ and $\frac{\partial \varphi}{\partial t} - e^{-t/2} \leq 0$ for all $t \geq T$. Then

$$\int_T^\infty \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^1(X)} dt < \infty. \quad (5.19)$$
On the other hand, $\frac{\partial}{\partial t} (\varphi + e^{-t/2}) \leq 0$ and so $\varphi + e^{-t/2}$ is decreasing in time. Since $\varphi$ is uniformly bounded, there exists $\varphi_\infty \in \text{PSH}(X, \chi) \cap L^\infty(X)$ such that $\varphi$ converges to $\varphi_\infty$ in $L^1(X)$ and $C^\infty(X^{\circ})$ as $t \to \infty$. Hence $\frac{\partial \varphi}{\partial t}$ converges to a function $F \in L^\infty(X) \cap C^\infty(X^{\circ})$ in $C^\infty(X^{\circ}) \cap L^1(X)$. Combined with (5.19), $F = 0$. Otherwise, $\int_0^\infty ||\frac{\partial \varphi}{\partial t}||_{L^1(X)} dt = \infty$. The corollary is then proved.

**Proposition 5.4** Let $\varphi_\infty \in \text{PSH}(X, \chi) \cap L^\infty(X)$ be the unique solution of the Monge–Ampère equation

$$\left(\chi + \sqrt{-1} \partial \bar{\partial} \varphi_\infty \right)^n = e^{\varphi_\infty} \Omega.$$ (5.20)

Then $\varphi$ converges to $\varphi_\infty$ in $L^1(X) \cap C^\infty(X^{\circ})$ as $t \to \infty$.

**Proof** Let $\varphi_\infty$ be the limit of $\varphi$ as $t \to \infty$. By Corollary 5.2, $\frac{\partial \varphi}{\partial t}$ converges to 0, and so $\varphi_\infty$ must satisfy Eq. (5.20). The uniqueness of $\varphi_\infty$ follows from the uniqueness of the solution to the Eq. (5.20) as $\varphi_\infty \in \text{PSH}(X, \chi) \cap L^\infty(X)$. □

The current $\chi + \sqrt{-1} \partial \bar{\partial} \varphi_\infty$ is exactly the pullback of the unique singular Kähler–Einstein metric $\omega_{KE}$ on the canonical model $X_{\text{can}}$ of $X$ in Theorem 2.2. The following theorem then follows from Proposition 5.4.

**Theorem 5.6** Let $X$ be a minimal model of general type with log terminal singularities. For any $\mathbb{Q}$-ample divisor $H$ on $X$, the normalized weak Kähler–Ricci flow converges to the unique Kähler–Eintein metric $\omega_{KE}$ on the canonical model $X_{\text{can}}$ for any initial current in $\mathcal{K}_{H,p}(X)$ with $p > 1$.

We have the following general theorem by combining Theorems 5.5 and 5.6 if the general type variety is not minimal.

**Theorem 5.7** Let $X$ be a projective $\mathbb{Q}$-factorial variety of general type with log terminal singularities. If there exists a good initial divisor $H$ on $X$, then the normalized weak Kähler–Ricci flow starting with any initial current in $\mathcal{K}_{H,p}(X)$ with $p > 1$ exists for $t \in [0, \infty)$ and replaces $X$ by its minimal model $X_{\text{min}}$ after finitely many surgeries. Furthermore, the normalized Kähler–Ricci flow converges to the unique Kähler–Eintein metric $\omega_{KE}$ on its canonical model $X_{\text{can}}$.

6 Analytic minimal model program with Ricci flow

In this section, we lay out the program relating the Kähler–Ricci flow and the classification of projective varieties based on the speculations in [32,41]. The new insight is that the Ricci flow is very likely to deform a given projective variety to its minimal model and eventually to its canonical model coupled
with a canonical metric of Einstein type, in Gromov–Hausdorff topology. We will start discussions with the case of projective surfaces.

### 6.1 Results on surfaces

A smooth projective surface is minimal if it does not contain any $(-1)$-curve. Let $X_0$ be a projective surface of non-negative Kodaira dimension. If $X_0$ is not minimal, then the Kähler–Ricci flow starting with any Kähler metric $\omega_0$ in the class of an ample divisor $H_0$ has a smooth solution until the first singular time $T_0 = \sup\{t > 0 \mid H_0 + T_0 K_{X_0} \text{ is nef}\}$. The limiting semi-ample divisor $H_0 + T_0 K_{X_0}$ induces a morphism

$$\pi_0 : X_0 \to X_1$$

by contracting finitely many $(-1)$-curves. $X_1$ is smooth and there exists a $\mathbb{Q}$-ample divisor $H_1$ on $X_1$ such that $H_0 + T_0 K_{X_0} = \pi_0^* H_1$. Then by Theorem 5.3, the Kähler–Ricci flow can be continued through the contraction $\pi_0$ at time $T_0$. Since there are finitely many $(-1)$-curves on $X$, the Kähler–Ricci flow will arrive at a minimal surface $X_{\text{min}}$ or it collapses a $\mathbb{CP}^1$ fibration in finite time after repeating the same surgery for finitely many times.

It is shown in [34,35] that the Kähler–Ricci flow on $\mathbb{CP}^2$ blow-up at one point converges to $\mathbb{CP}^2$ in Gromov–Hausdorff sense if the initial Kähler class is appropriately chosen and the initial Kähler metric satisfies the Calabi symmetry. Then the flow can be continued on $\mathbb{CP}^2$ and eventually will be contracted to a point in finite time. This shows that the Kähler–Ricci flow deforms the non-minimal surface to a minimal surface in Gromov–Hausdorff sense. Similar behavior is also shown in [35] for higher-dimensional analogues of the Hirzebruch surfaces. This leads us to propose a conjectural program in the following section for general projective varieties.

After getting rid of all $(-1)$-curves, we can focus on the minimal surfaces divided into ten classes by the Enriques-Kodaira classification.

If $\text{kod}(X_{\text{min}}) = 2$, $X_{\text{min}}$ is a minimal surface of general type and its canonical model $X_{\text{can}}$ is an orbifold surface achieved by contracting all the $(-2)$-curves on $X_{\text{min}}$. It is shown in [42] that the normalized Kähler–Ricci flow $\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) - \omega$ converges in the sense of distributions to the pullback of the orbifold Kähler–Einstein metric on the canonical model $X_{\text{can}}$.

If $\text{kod}(X_{\text{min}}) = 1$, $X_{\text{min}}$ is a minimal elliptic fibration over its canonical model $X_{\text{can}}$. It is shown in [32] that the normalized Kähler–Ricci flow $\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) - \omega$ converges in the sense of distributions to the pullback of the generalized Kähler–Einstein metric on the canonical model $X_{\text{can}}$.

If $\text{kod}(X_{\text{min}}) = 0$, $K_X$ is numerically trivial. Yau’s solution to the Calabi conjecture shows that there always exists a Ricci-flat Kähler metric in any
given Kähler class on $X_{\text{min}}$. In particular, it is shown in [4] that the Kähler–Ricci converges in the sense of distributions to the unique Ricci-flat metric in the initial Kähler class.

If $X$ is Fano, then it is proved [43] that the normalized Kähler–Ricci flow $rac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \omega$ with an appropriate initial Kähler metric will converge in Gromov–Hausdorff sense to a Kähler–Ricci soliton after normalization.

In general, the understanding of the Kähler–Ricci flow is still not completely understood for surfaces of $-\infty$ Kodaira dimension as the flow might collapse in finite time.

### 6.2 Conjectures

In this section, we discuss our program in higher dimensions. Our proposal gives new understanding of the minimal model program from the viewpoint of differential geometry. We refer it as the analytic minimal model program.

**Minimal model program with Ricci Flow**

1. We start with a triple $(X, H, \omega)$, where $X$ is a $\mathbb{Q}$-factorial projective variety with log terminal singularities, $H$ is a big semi-ample $\mathbb{Q}$-divisor on $X$ such that $H + \epsilon K_X$ is ample for sufficiently small $\epsilon > 0$, and $\omega \in \mathcal{K}_{H, p}(X)$ for some $p > 1$. Let

$$T_0 = \inf\{t > 0 \mid H + tK_X \text{ is nef}\}.$$

Let $\omega(t, \cdot)$ be the unique solution of the weak Kähler–Ricci flow for $t \in (0, T_0)$.

**Conjecture 6.1** For each $t \in (0, T_0)$, the metric completion of $X_{\text{reg}}$ by $\omega(t, \cdot)$ is homeomorphic to $X$.

2. If $T_0 = \infty$, then $X$ is a minimal model and the Kähler–Ricci flow has long time existence. The abundance conjecture predicts that $K_X$ is semi-ample and $\text{kod}(X) \geq 0$.

2.1. $\text{kod}(X) = \dim X$, i.e., $X$ is a minimal model of general type.

**Conjecture 6.2** The normalized Kähler–Ricci flow

$$\frac{\partial \tilde{\omega}}{\partial s} = -\text{Ric}(\tilde{\omega}) - \tilde{\omega}.$$

starting with $\omega$ converges to the unique Kähler–Einstein metric $\omega_{\text{KE}}$ on $X_{\text{can}}$ in Gromov–Hausdorff sense as $s \to \infty$.

The weak convergence in distribution and smooth convergence outside the exceptional locus is obtained in [45] and [42] if $X$ is nonsingular. If $X$ is
nonsingular and $K_X$ is ample, it is the classical result in [4] that the flow converges in $C^\infty(X)$-topology.

2.2. $0 < \text{kod}(X) < \dim X$.

**Conjecture 6.3** The normalized Kähler–Ricci flow

$$\frac{\partial \tilde{\omega}}{\partial s} = -\text{Ric}(\tilde{\omega}) - \tilde{\omega}.$$  

starting with $\omega$ converges to the unique generalized Kähler–Einstein metric $\omega_{\text{can}}$ on $X_{\text{can}}$ (as in Theorem 2.2) in Gromov–Hausdorff sense as $s \to \infty$.

If $K_X$ is semi-ample, $X$ admits a Calabi–Yau fibration over its canonical model $X_{\text{can}}$. The weak convergence in distribution is obtained in [32] and [33] if $X$ is nonsingular and $K_X$ is semi-ample.

2.3. $\text{kod}(X) = 0$. $K_X$ is numerically trivial.

**Conjecture 6.4** The Kähler–Ricci flow

$$\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega)$$  

converges to the unique Ricci-flat Kähler metric in $[H]$ in Gromov–Hausdorff sense as $t \to \infty$.

It is shown in [4] that the flow converges in $C^\infty(X)$-topology if $X$ is smooth. The weak convergence is obtained in [36] if $X$ has log terminal singularities.

3. If $T_0 < \infty$, then the semi-ample divisor $H + T_0 K_X$ induces a contraction

$$\pi : X \to Y.$$  

3.1 $\dim Y = \dim X$.

**Conjecture 6.5** As $t \to T_0$, $(X, \omega(t, \cdot))$ converges to a metric space $(Y, \omega_Y)$ along the Kähler–Ricci flow

$$\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega)$$  

in Gromov–Hausdorff sense. Furthermore, $\omega(t, \cdot)$ converges in $C^\infty$ outside a subvariety $S$ of $X$, and $(Y, \omega_Y)$ is the metric completion of the smooth limit of
Let \((X \setminus S, \omega(T_0, \cdot))\). In particular, \(Y\) is a normal projective variety satisfying the following diagram

\[
\begin{array}{c}
X \\
\downarrow_{\pi} \quad \downarrow_{\pi^+} \\
Y \quad X^+ \\
\end{array}
\]

where \(X^+\) is \(\pi^+\)-ample and \(\pi^+: X^+ \to Y\) is a general flip of \(X\).

Let \(H_{X^+}\) be the strict transformation of \(H + T_0K_X\) by the general flip. Then \(K_{X^+}\) and \(H_{X^+}\) are both \(\mathbb{Q}\)-Cartier with \(H_{X^+} + \epsilon K_{X^+}\) being ample for sufficiently small \(\epsilon > 0\), and \(\omega_{X^+} = (\pi^+)\ast \omega_Y \in \mathcal{K}_{H_{X^+}, p'}(X^+)\) for some \(p' > 1\).

Note that a divisorial contraction is also a general flip if we choose \(\pi^+\) to be the identity map. We then repeat Step 1 by replacing \((X, H, \omega)\) with \((X^+, H_{X^+}, \omega_{X^+})\) even though \(X^+\) is not necessarily \(\mathbb{Q}\)-factorial. We further conjecture that along the new Kähler–Ricci flow, \((X^+, \omega_{X^+}(t))\) converges in Gromov–Hausdorff sense to \((Y, \omega_Y)\) as \(t \to 0^+\).

3.2 \(0 < \dim Y < \dim X\). \(X\) then admits a Fano fibration over \(Y\).

**Conjecture 6.6** As \(t \to T_0\), \((X, \omega(t, \cdot))\) converges to a metric space \((Y', \omega_Y')\) along the Kähler–Ricci flow

\[
\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega)
\]

in Gromov–Hausdorff sense. Let \(H_{Y'}\) be the divisor where \(\omega_{Y'}\) lies. Then both \(K_{Y'}\) and \(H_{Y'}\) are \(\mathbb{Q}\)-Cartier, and \(\omega_{Y'} \in \mathcal{K}_{H_{Y'}, p'}(Y')\) for some \(p' > 1\).

We then repeat Step 1 by replacing \((X, H, \omega)\) by \((Y', H_{Y'}, \omega_{Y'})\).

3.3 If \(\dim Y = 0\), \(X\) is Fano and \(\omega \in -T_0[K_X]\). Then we have the following generalized Hamilton-Tian conjecture.

**Conjecture 6.7** Then the normalized Kähler–Ricci flow

\[
\frac{\partial \tilde{\omega}}{\partial s} = -\text{Ric}(\tilde{\omega}) + \frac{1}{T_0} \tilde{\omega}.
\]

starting with \(\omega\) converges to a Kähler–Ricci soliton \((X_\infty, \omega_{KR})\) in Gromov–Hausdorff sense as \(s \to \infty\).

A proof is given for Fano manifolds admitting a Kähler–Ricci soliton by Tian and Zhu [43].
It is conjectured by Yau [48] that the existence of a Kähler–Einstein metric on a Fano manifold is equivalent to suitable stability in the sense of geometric invariant theory. The condition of $K$-stability is proposed by Tian [38] and is refined by Donaldson [12]. The Yau–Tian–Donaldson conjecture claims that the existence of Kähler metrics with constant scalar curvature is equivalent to the $K$-stability (possibly with some additional milder conditions on holomorphic vector fields). Since the Kähler–Ricci flow provides an approach to such a conjecture for Kähler–Einstein metrics and it has attracted considerable current interest. We refer the readers to an incomplete list of literatures [24,25,27,28,37,43,44] for some recent development.

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