Spike-and-Slab Approximate Message-Passing for High-Dimensional Piecewise-Constant Recovery

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Abstract—One of the challenges in Big Data is efficient handling of high-dimensional data or signals. This paper proposes a novel AMP algorithm for solving high-dimensional linear systems $\mathbf{Y} = \mathbf{H} \mathbf{X} + \mathbf{W}$ which has a piecewise-constant solution $\mathbf{X} \in \mathbb{R}^N$, under a compressed sensing framework ($M \leq N$). We refer to the proposed AMP as ssAMP. This ssAMP algorithm is derived from the classical message-passing rule over a bipartite graph which includes spike-and-slab potential functions to encourage the piecewise-constant nature of $\mathbf{X}$. The ssAMP iteration includes a novel scalarwise denoiser satisfying the Lipschitz continuity, generating an approximate MMSE estimate of the signal. The Lipschitz continuity of our denoiser enables the ssAMP to use the state evolution framework, given by the works [16],[19], for MSE prediction. In addition, we empirically show that ssAMP has better phase transition characteristic than TV-AMP [22] and GrAMPA [26] which are the existing AMPs for piecewise-constant recovery. We also discuss computational efficiency, empirically showing that ssAMP has computational advantage over the other recent algorithms under a high-dimensional setting.

Index Terms—Compressed sensing, piecewise-constant signals, approximate message-passing (AMP), TV-AMP, total variation denoising

I. INTRODUCTION

In an era of “Big Data”, one of the technical challenges is developing fast processing techniques to handle high-dimensional data in real time. One fundamental approach to the challenge is to find and utilize low-dimensionality structures of the high-dimensional data. The compressed sensing (CS) theory can provide a solution to the approach [1].

This paper considers a high-dimensional CS problem for the Big Data scenario where we are particularly interested in cases that the signal $\mathbf{X} \in \mathbb{R}^N$ consists of $K + 1 (<< N)$ different constant values. Concretely, our aim is fast solving of an underdetermined system which is $\mathbf{Y} = \mathbf{H} \mathbf{X} + \mathbf{W}$, where $\mathbf{W} \in \mathbb{R}^M$ represents noise distributed with $\mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$, $\mathbf{Y} \in \mathbb{R}^M$ is compressive linear measurements, and $\mathbf{H} \in \mathbb{R}^{M \times N}$ is a fat random measurement matrix ($M \leq N$). Such a piecewise-constant recovery problem has been intensively applied to video and image processing [8]-[13] and also investigated in the other areas such as big-signal recognition [2],[3] and wideband spectrum sensing of communication systems [4],[5].

The most popular approach to the problem is to use total variation (TV) regularization [6],[8] also called Fused lasso [7]. This TV approach encourages sparsity in difference of the signal elements; hence, which can effectively capture the blocky feature of the signal $\mathbf{X}$. For the TV solvers, iterative fixed-point algorithms have been mostly considered [11]-[15]. As recent fast solvers, the Augmented-Lagrangian-and-alternating-direction algorithms [11],[12], the Chambolle-Pock’s primal-dual algorithm [13],[14], and the Efficient-fused-lasso algorithm [15] are well known.

Approximate message-passing (AMP) iteration recently has got attention as an approach to solve high-dimensional linear systems [16]-[29]. Main advantages of the AMP solvers are from (i) its mean-square-error (MSE) performance is deterministically predictable via the state evolution (SE) framework [16],[19], and (ii) the AMP solvers operate with low-computational cost [16],[18]. Motivated by such excellent properties, several AMP-type solvers have been developed for various types of sparse signals [21]-[26].

A. Motivation

The present work is motivated by TV-AMP [22] which is an extension of the standard AMP [16] for the piecewise-constant recovery. TV-AMP operates iteratively according to:

$$\mu^{(t+1)} = \eta_{TV}(\mathbf{H}^T \mathbf{z}^{(t)} + \mu^{(t)}) \quad (1)$$

$$\mathbf{z}^{(t)} = \mathbf{y} - \mathbf{H} \mu^{(t)} + \mathbf{z}^{(t-1)} - \frac{N}{2M} \left( \eta_{TV}(\mathbf{H}^T \mathbf{z}^{(t-1)} + \mu^{(t-1)}) \right) \quad (2)$$

where $\mu^{(t)} \in \mathbb{R}^N$ and $\mathbf{z}^{(t)} \in \mathbb{R}^M$ are a signal estimate and a residual vector at the $t$-th iteration respectively. However, TV-AMP has shown limited success in practice because the authors utilized an external TV denoising package in the implementation of its denoiser function, given as

$$\eta_{TV}(\rho) = \arg \min_{\mathbf{x}} \frac{1}{2} \| \rho - \mathbf{x} \|_2^2 + \lambda \sum_{i=2}^{N} |X_i - X_{i-1}|, \quad (2)$$

where the input $\rho$ is a noisy observation of the signal $\mathbf{X}$, and $\lambda \geq 0$ is a control parameter for the TV penalty. Hence, the denoiser of TV-AMP is not scalarwise, leading to difficulty in the MSE prediction by SE [22]. In addition, we note that complexity of TV-AMP highly depends upon that of the external package for [2].

Most recently, Borgerding et. al suggested a very good alternative of (2) in the work of GrAMPA [26]. GrAMPA is a variation of the generalized-AMP [21] for the analysis CS problem [29]. In the work, the authors set the analysis matrix to a finite-difference matrix, i.e., $\Omega = \mathbf{D}^T$ for the piecewise-constant recovery. This makes the GrAMPA solver to include

\footnote{Here, we follow the notation used in [26].}
a scalarwise MMSE denoiser selecting a sparse support of the difference $X_i - X_{i-1}$ with a Bernoulli-Uniform prior, which is given as
\begin{equation}
\hat{\eta}_{\text{GrAMPA}}(\rho_i; \nu_i, \tau) \equiv \mathbb{E}[X_i - X_{i-1}|\rho_i, \nu_i, \tau] = \frac{\rho_i}{1 + \tau N(0; \rho_i, \nu_i)} \quad (3)
\end{equation}
\[ \forall i \in \{2, ..., N\}, \] where the input $\rho_i$ is an observation of $X_i - X_{i-1}$ corrupted by noise $W'$ following $N(u'; 0, \nu_i)$, and $\tau > 0$ is a tunable design parameter. Therefore, GrAMPA seeks the piecewise-constancy by denoising with (3). Although the authors of the paper [26] have reported an excellent result with GrAMPA, there still remains room for improvement in performance. Especially, during the GrAMPA iteration, its residual for $X_i$ is calculated based on not real measurements but the estimate by (3). This may lead to ineffective convergence of the recovery particularly when the sampling rate $\frac{1}{N}$ is very low.

B. Contribution

In this paper, we propose an alternative AMP solver for the piecewise-constant recovery. The proposed AMP solver is referred to as Spike-and-Slab Approximate Message-Passing (ssAMP). The development of the ssAMP algorithm starts from a bipartite graphical modeling of the linear system which has a piecewise-constant solution $X$. Then, we construct a classical message-passing (MP) rule from the joint distribution over the bipartite graph, where each relation of two consecutive elements $X_{i-1}$ and $X_i$ is described by a spike-and-slab potential function. The ssAMP iteration rule is derived from the classical MP rule by applying the central limit theorem (CLT) and the first-order approximation, analogously to the approaches introduced in [7], [13], [3]. The novelty of the proposed ssAMP is mainly from the following statements:

1) The ssAMP algorithm includes a novel scalarwise denoiser which satisfies the Lipschitz continuity. Therefore, this proposed denoiser can be an alternative of the external TV denoising, given by (2), being easily applied to the SE framework for the MSE prediction of the ssAMP iteration.

2) The ssAMP shows phase transition curve (PTC) covering that of the two existing AMPs for piecewise-constant recovery: TV-AMP [22] and GrAMPA [26].

3) Computational advantage of the ssAMP is remarkable compared to the other recent algorithms for piecewise-constant recovery in a high-dimensional setting.

The present work is independent of the work of GrAMPA [26] in that ssAMP applies an MMSE denoiser to each signal scalar $X_i$, not to each difference $X_i - X_{i-1}$ as GrAMPA, for seeking the piecewise-constancy. For the scalarwise denoising at each $X_i$, information on the neighboring elements $X_{i-1}, X_{i+1}$ should be provided to the denoiser. Therefore, the proposed ssAMP includes Right-toward passing (R2P) / Left-toward passing (L2P) update to exchanges the neighbor information, in addition to the conventional AMP update rule. Such an independent point makes significant improvement in phase transition characteristic from GrAMPA. We empirically validate this claim in Section V-A.

C. Organization and Notation

The remainder of the paper is organized as follows. In Section II, we describe our approach to construct the ssAMP algorithm. Section III characterizes the ssAMP algorithm. Section IV describes the SE framework for the ssAMP iteration. Section V provides extensive empirical results to validate several aspects of the ssAMP algorithm, compared to the existing AMPs for piecewise-constant recovery, TV-AMP [22] and GrAMPA [26], as well as recent TV solvers, such as TVAL3 [11], CP [13]. Finally, we conclude this paper in Section VI.

Throughout the paper, we use the following notation. We use underlined letter like $\underline{l}$ to denote vectors, boldface capital letters like $\underline{F}$ to indicate set symbols. The vectors $\underline{1} \equiv [1, ..., 1]^T$ and $\underline{0} \equiv [0, ..., 0]^T$ denote an one vector and a zero vector respectively. In addition, $f_{X_i}(x_i)$ is a probability density function (PDF) of a random variable $X_i \sim f_{X_i}(x_i)$ and its realization is denoted by small letters like $x_i$. We use $\mathbb{E}[f_{X_i}(x_i)]$ and $\text{Var}[f_{X_i}(x_i)]$ to denote the expectation and the variance with respect to the PDF $f_{X_i}(x_i)$, respectively. We define notation for sample mean as $\langle \cdot \rangle_N \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \cdot \rangle_i$ and that for the first derivative of a function $\eta(\rho)$ as $\eta'(\rho) \equiv \frac{d}{d\rho} \eta(\rho)$. Finally, $N(x; \mu, \sigma^2) \equiv \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ denote a Gaussian PDF with mean $\mu$ and variance $\sigma^2$, and $\mathcal{U}(x; \underline{0}, \frac{1}{N})$ is a discrete uniform PDF with $N$ points.

II. APPROACH FOR ALGORITHM CONSTRUCTION

In this section, we discuss our approach to construct the proposed algorithm, ssAMP. First, we construct a classical message-passing (MP) rule in our approach for the piecewise-constant recovery using the spike-and-slab potential function. Then, we move the discussion to AMP rule construction from the classical MP rule.

A. Message-Passing Rule for Piecewise-Constant Recovery

The statistical dependency of linear systems can be effectively modeled using factor graphs. Let $\mathcal{V} \equiv \{1, ..., N\}$ be a variable set whose element $i \in \mathcal{V}$ corresponds to $X_i$, and $\mathcal{F}_m \equiv \{1, ..., M\}$ is a factor set whose element $j \in \mathcal{F}_m$ corresponds to $Y_j$. To the problem, we include another factor set, defined as $\mathcal{F}_s \equiv \{2, ..., N\}$, to describe statistical connections between adjacent elements $X_{i-1}$ and $X_i$. In order to clarify two different factors, we name the set $\mathcal{F}_m$ as m-factor set, and the set $\mathcal{F}_s$ as s-factor set. In addition, we define a neighbor set of $i \in \mathcal{V}$ to the s-factors as $\mathcal{N}(i) \equiv \{k_1, k_2 \in \mathcal{F}_s|k_1 = i - 1, k_2 = i\}$, which is a subset of the s-factor set $\mathcal{F}_s$. Then, a factor graph $\mathcal{G}(\mathcal{V}, \mathcal{F}_m, \mathcal{F}_s)$ fully models the linear system in the present work, where we omit definitions for the graph edges to avoid abuse of notation.

Based on the graph model, the joint PDF is represented as
\begin{equation}
\rho_{\mathcal{X}} \rho_{\mathcal{Y}}(\underline{x}, \underline{y}) = \frac{1}{Z} \prod_{i=2}^{N} f_{X_{i-1},X_{i}}(x_{i-1}, x_i) \prod_{j=1}^{M} f_{Y_j}(y_j) \rho_{\mathcal{X}} \rho_{\mathcal{Y}}(\underline{x}, \underline{y}) (4)
\end{equation}
where $Z \in \mathbb{R}$ is a partition function to valid $\int f_{X_{i-1},X_{i}}(x_{i-1}, x_i) \rho_{\mathcal{X}} \rho_{\mathcal{Y}}(\underline{x}) dx_i = 1$. In the joint PDF, the potential
function $f_{Y_j|X}(y_j|x)$ corresponding to each $j \in F_m$ is a Gaussian PDF since $Y_j \sim \mathcal{N}(y_j; (H_{S_j})_j, \Delta)$ from the noise model. For the potential function $f_{X_i, x_i}(x_{i-1}, x_i)$ to each s-factor $k \in F_s$, we consider a spike-and-slab PDF which is a special case of the Bernoulli-Gaussian PDF, given as

$$f_{X_i, x_i}(x_{i-1}, x_i) = (1-q)\delta(x_i-x_{i-1}) + q\mathcal{N}(x_i-x_{i-1}; 0, \sigma^2_q)$$  \hspace{1cm} (5)$$

where $\delta(x_i-x_{i-1})$ denote a Dirac function peaked at $x_i-x_{i-1} = 0$, $q \in [0, 1]$ is a probability weight, and $\sigma^2_q \in (0, \infty)$ is the variance of the Gaussian PDF. In (5), the spike part, represented by the Dirac PDF $\delta(x_i-x_{i-1})$, corresponds to the case of $X_i = X_{i-1}$, and the slab part, expressed by the Gaussian PDF $\mathcal{N}(x_i-x_{i-1}; 0, \sigma^2_q)$, corresponds to the case of $X_i \neq X_{i-1}$. The number of the constant changes in the signal $X$ is Binomial random, i.e., $K \sim \mathcal{B}(k; N-1,q)$. Such spike-and-slab PDFs have been used as sparsifying prior PDFs in CS literature with respect to Bayesian algorithms [18],[31],[23],[32] because i) the PDFs showed very good sparsifying ability in various domains, ii) the PDF can be simply parameterized with the weight $q$ and the Gaussian variance $\sigma^2_q$, and iii) integration with respect to the PDF is tractable. In the present work, the product of the spike-and-slab PDFs in the joint PDF (4) behaves as a prior PDF of $X$ to encourage the blocky feature of the piecewise-constancy.

The MP algorithm provides an computationally efficient finding of marginal posteriors with respect to Bayesian algorithms [33]. Based on the factor graph modeled above, four types of message update are required for the classical MP as illustrated in Fig 1 whose expressions are given by

- **a)** m-factor to variable (mF2V) update:
  $$m_{j \rightarrow i}^{(t)}(x_i) = \int_{X \setminus x_i} f_{Y_j|X}(y_j|X) \prod_{x_{i'} \neq i} v^{(t)}_{i' \rightarrow j}(x_{i'})dx_{i'},$$  
- **b)** s-factor to variable (sF2V) update:
  $$s_{k \rightarrow i}^{(t)}(x_i) = \int_{X \setminus x_i} f_{X_i, x_i}(x_{i-1}, x_i) \prod_{x_{i} \neq i} \tilde{v}_{k \rightarrow i}(x_{i})dx_{i'} \quad (i' \neq i),$$  
- **c)** Variable to m-factor (V2mF) update:
  $$v_{i \rightarrow j}^{(t+1)}(x_i) = \frac{1}{Z_{i \rightarrow j}} \prod_{k \in \mathcal{S}_i(i)} s_{k \rightarrow i}^{(t)}(x_i) \prod_{j' \neq j} m_{j' \rightarrow i}^{(t)}(x_i),$$  
- **d)** Variable to s-factor (V2sF) update:
  $$v_{i \rightarrow k}^{(t+1)}(x_i) = \frac{1}{Z_{i \rightarrow k}} s_{k \rightarrow i}^{(t)}(x_i) \prod_{j \neq k} m_{j \rightarrow i}^{(t)}(x_i) \quad (k' \neq k),$$

where $Z_{i \rightarrow j}$ and $Z_{i \rightarrow k}$ are partition functions. It is noteworthy from (6) that by using our graph model, we can decouple tasks of pursuing the piecewise-constancy of $X$ and seeking the measurement fidelity with $Y$, where the task for the piecewise-constancy pursuit is taken by the V2sF and sF2V updates; the measurement fidelity task is by the V2mF and mF2V updates. Then, the marginal posterior of $X_i$ can be obtained at the fixed point of the MP iteration $t = t^*$, i.e.,

$$f_{X_i|x_i, Y} = \frac{1}{Z} \prod_{k \in \mathcal{S}_i(i)} s_{k \rightarrow i}^{(t^*)}(x_i) \prod_{j} m_{j \rightarrow i}^{(t^*)}(x_i).$$  \hspace{1cm} (7)$$

As claimed in literature [17],[18], the update rule in (6) is infeasible in practice because i) the messages are density function over the real line, and ii) $2MN + 4(N-1)$ message exchanges are required per iteration.

![Factor graphical representation of message update](image)

Fig. 1. Factor graphical representation of message update: (a) m-factor to variable (mF2V) update, (b) s-factor to variable (sF2V) update, (c) variable to m-factor (V2mF) update, (d) variable to s-factor (V2sF) update.

### B. Message-Passing to Approximate Message-Passing

The AMP approach can provide a solution for the infeasibility of the classical MP iteration. The development of the AMP rule from the classical MP rule is generally divided into two steps: i) Parameterization step and ii) First-order approximation step. In addition to these two steps, the proposed algorithm includes the third step, iii) Simplification step, to handle the sF2V and V2sF update. We refer to Appendix I for a detailed description constructing the ssAMP rule from the classical MP rule of [6]. Analogous approaches for the AMP construction were used in works of [17],[18],[21]. However, their works are different from our work in that they are with respect to directly sparse signals. Algorithm 1 summarizes the ssAMP iteration in a vector form. We note from Algorithm 1 that the ssAMP algorithm handles $O(N + M)$ messages per iteration.

### III. CHARACTERISTIC OF ALGORITHM

As shown in Algorithm 1, the ssAMP iteration consists of three updates. In the V2mF update, we update the mean vector $\hat{\mu} \in \mathbb{R}^N$, which is an actual ssAMP estimate of $X$, with the corresponding variance $\sigma^2 \in \mathbb{R}^N$. In the mF2V update, we update the residual vector $\rho \in \mathbb{R}^M$ and calculate the vector $\vartheta \in \mathbb{R}^N$, which is a noisy estimate of $X$, with its variance $\eta \in \mathbb{R}^N$. In addition, the ssAMP iteration includes the R2P/L2P update handling the parameters $\mu_{R2P}, \sigma^2_{R2P}, \mu_{L2P}, \sigma^2_{L2P} \in \mathbb{R}^N$, differently from the conventional AMP solvers. For the expression of the functions, $\eta(\cdot), \gamma(\cdot), \phi(\cdot), \zeta(\cdot)$, built in the ssAMP iteration, we refer to Table VI in Appendix I. In the remaining of this section, we discuss the detail of the R2P/L2P update and our scalarwise denoiser $\eta(\cdot)$ which are the most distinguishing parts of the ssAMP algorithm.

#### A. The R2P/L2P Update

The R2P/L2P update in the ssAMP iteration originates from the sF2V and V2sF message updates in (6). By evaluating the
Algorithm 1 ssAMP iteration

Inputs: Measurements $y$, a measurement matrix $H$, parameters from prior knowledge $q, \sigma_0$, and the variance of additive noise $\Delta$

Outputs: A ssAMP estimate $\hat{x}_{\text{ssAMP}} = \mu^{(t)}$

Initialization:

1. mF2V update

\[
\begin{align*}
\tilde{y}^{(t)} & = \mu_2^{(t-1)} - \frac{1}{M} \sum_{i \in V} \theta_i^{(t-1)} \left[ \phi_2^{(i)} \left( \phi_1^{(i)}(v_i^{(t-1)}) \right) \right] \\
\hat{y}^{(t)} & = \mu_2^{(t-1)} + \frac{1}{M} \sum_{i \in V} \theta_i^{(t-1)} \left[ \phi_2^{(i)} \left( \phi_1^{(i)}(v_i^{(t-1)}) \right) \right]
\end{align*}
\]

2. R2P/L2P update

\[
\begin{align*}
\mu_{\text{R2P}}, & \sigma_{\text{R2P}}^{(t)} = \left( \phi(\rho_{\text{R2P}}), \phi(\rho_{\text{R2P}}) \right) \\
\sigma_{\text{L2P}}, & \sigma_{\text{L2P}}^{(t)} = \left( \phi(\rho_{\text{L2P}}), \phi(\rho_{\text{L2P}}) \right)
\end{align*}
\]

3. V2mF update

\[
\begin{align*}
\phi_2^{(i)}(v_i^{(t-1)}) & = \frac{1}{N} \sum_{i \in V} \theta_i^{(t-1)} \left[ \phi_2^{(i)} \left( \phi_1^{(i)}(v_i^{(t-1)}) \right) \right] \\
\phi_2^{(i,t)}(v_i^{(t-1)}) & = \phi_2^{(i,t)}(v_i^{(t-1)})
\end{align*}
\]

i. Rightward passing (R2P)

\[
\begin{align*}
\mu_{\text{R2P}}, & \sigma_{\text{R2P}}^{(t+1)} = \left( \phi(\rho_{\text{R2P}}), \phi(\rho_{\text{R2P}}) \right) \\
\sigma_{\text{L2P}}, & \sigma_{\text{L2P}}^{(t+1)} = \left( \phi(\rho_{\text{L2P}}), \phi(\rho_{\text{L2P}}) \right)
\end{align*}
\]

ii. Leftward passing (L2P)

\[
\begin{align*}
\mu_{\text{L2P}}, & \sigma_{\text{L2P}}^{(t+1)} = \left( \phi(\rho_{\text{L2P}}), \phi(\rho_{\text{L2P}}) \right) \\
\sigma_{\text{R2P}}, & \sigma_{\text{R2P}}^{(t+1)} = \left( \phi(\rho_{\text{R2P}}), \phi(\rho_{\text{R2P}}) \right)
\end{align*}
\]

B. Denoiser for Piecewise-Constancy

The denoiser $\eta(\rho)$ of the ssAMP algorithm scalarwisely calculates the mean of each scalar $X_i$ given an approximate marginal posterior $p_{X_i}(x_i | Y = y)$ in the V2mF update of every iteration $t \geq 0$:

\[
\mu_i^{(t)} = E_{p_{X_i}(x_i | Y = y)} [X_i | Y = y] = \eta(\rho_1^{(t)}; \theta^{(t)}; \{a_{i,l}^{(t)}\}, \{b_{i,l}^{(t)}\}, \{c_{i,l}^{(t)}\})
\]

for all $i \in V$. Thus, the mean vector $\mu$ is an approximate MMSE estimate of the piecewise-constant signal $X$.

The detailed expression of $\eta(\rho)$ is provided in Table [V] of Appendix I. Note that this estimation of $X_i$ is a main distinctive point of ssAMP from the existing AMPs: TV-AMP [22] and GrAMPA [26]. For TV-AMP, the signal estimate is obtained by solving the non-scalarwise optimization in [2]. In the GrAMPA case, the scalar $X_i$ is estimated in a maximum-likelihood sense using the messages from the s- and m-factors since any prior is not imposed for $X_i$.

In ssAMP, two types of the information are required for the scalar estimation of $X_i$: i) the information on the adjacent elements $X_{i-1}, X_{i+1}$, which is given by the parameter set $\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}$, and ii) the information on denoising threshold, which is brought by the parameter $\theta$. In the remaining of this section, we discuss the behavior of the ssAMP denoiser $\eta(\rho)$, given in [5], according to its input parameters, $\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}$ and $\theta$.

1) Denoising by Parameters Sets $\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}$

The input parameters set $\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}$ are associated with the sF2V product in the marginal posterior of [7] which results in a four-state Gaussian mixture PDF.

\[
\prod_{k \in ne(i)} s_{k \rightarrow i}(x_i) = \sum_{l=0}^{4} c_{i,l} N(x_i; a_{i,l}, b_{i,l}),
\]

where the four-state is caused by the spike-and-slab functions, $f_{X_{i-1}, X_i}(\cdot)$ and $f_{X_i, X_{i+1}}(\cdot)$, imposed to two neighboring

\begin{itemize}
  \item Only, the interval information on $X_i$ can be given to the estimator [3].
\end{itemize}

integration of (b)-(b), the sF2V message is represented as a two-state Gaussian mixture PDF which is parameterized by two parameters $\mu_{k \rightarrow i}, \sigma_{k \rightarrow i}^2$ (see (21)). When the s-factor is $k = k_i(i \in ne(i))$ such that the sF2V message is toward right, as shown in Fig[2](a), the sF2V parameters $(\mu_{k \rightarrow i}, \sigma_{k \rightarrow i}^2)$ are equivalent to the mean and variance $(\mu_{k \rightarrow i-1}, \sigma_{k \rightarrow i-1}^2)$ of the V2sF message sent by the leftside node $i - 1 \in V \setminus \{i\}$. Likewise, when $k = k_{i+1}(i \in ne(i))$ such that the sF2V message is toward left, as shown in Fig[2](b), the sF2V parameters are given by the V2sF parameters $(\mu_{i+1 \rightarrow k}, \sigma_{i+1 \rightarrow k}^2)$ from the rightside node $i + 1 \in V \setminus \{i\}$. Therefore, what we only need is to keep track the V2sF update according to the direction of the sF2V message-passing. We combine and simplify those two updates to the R2P/L2P update. The detailed discussion about this simplification is provided in Appendix I-C. To each $X_i$, the R2P parameter brings the information on the leftside neighbor $X_{i-1}$ and the L2P provides the information on rightside neighbor $X_{i+1}$. Then, these R2P/L2P parameters are used to compute the parameters $\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}$ by [39] in the V2mF update.

Worth mentioning here is that ssAMP generates full information on the two neighbors $X_{i-1}, X_{i+1}$ from the s-factors, whereas the s-factors of GrAMPA [26] provide residual information on the sparse support of the signal difference $X_i - X_{i-1}$. This difference of the two algorithms leads to a significant performance gap when the sampling rate $\frac{1}{N}$ is low. The detailed discussion is provided in Section V-A.
s-factors $k_1, k_2 \in \text{nc}(i)$. In the sF2V product, the sets \(\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}\) are the mean, variance, and mixing rate of each Gaussian kernel describing the different situations on $X_{i-1}, X_i, X_{i+1}$, respectively. Specifically, these situations are following:

- When $l = 1$, the kernel is for $X_{i-1} \approx X_i \approx X_{i+1}$.
- When $l = 2$, the kernel is for $X_{i-1} \neq X_i \approx X_{i+1}$.
- When $l = 3$, the kernel is for $X_{i-1} \approx X_i \neq X_{i+1}$.
- When $l = 4$, the kernel is for $X_{i-1} \neq X_i \neq X_{i+1}$.

Therefore, the highest mixing rate $c_{i,l} / \sum c_{i,l'}$ decides the effective kernels which dominate the behavior of the denoising among the four Gaussian kernels. The number of the effective Gaussian kernels can be at most two due to $c_{i,2} = c_{i,3}$ from $(39)$. This is intuitively reasonable because the information on the sets $\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}$ is basically about the two neighbors $X_{i-1}, X_{i+1}$. The mean $a_{i,l}$ of the effective kernels decides the base values for the denoising and the corresponding variance $b_{i,l}$ regulates the sharpness of the denoising at each base value. Fig. $3^2$ supports our explanations by showing function plots of $\eta(\rho)$ for a variety of the sets $\{a_{i,l}\}, \{b_{i,l}\}, \{c_{i,l}\}$ listed in Table $1$, where we set the prior parameters to $q = 0.1, \sigma_0 = 10$ and provide the normalized rate $c_{i,l} / \sum c_{i,l'}$ instead of $\{c_{i,l}\}$. The cases of Table $1$ include the most representative situations on the two neighbors $X_{i-1}, X_{i+1}$ where the situations on $X_{i-1}$ and $X_{i+1}$ are indicated by the corresponding R2P and L2P parameters respectively.

2) Denoising by Parameter $\theta$: The input parameters $\theta$ is related to the mF2V product in the marginal posterior of $(7)$ which is a Gaussian PDF with the parameters $\rho_i$ and $\theta$.

$$\prod_j m_{j\rightarrow i}(x_i) \propto \mathcal{N}(x_i; \rho_i, \theta). \quad (10)$$

Therefore, the parameter $\theta$ can be interpreted as the variance of the noisy estimate of $X_i$, determining the threshold level in the denoising $\eta(\rho; \theta)$ of $X_i$. It is noteworthy that how the denoising behavior changes in the extreme cases of $\theta$, i.e.,

$$\lim_{\theta \to \infty} \eta(\rho; \theta) = \frac{\sum c_{i,l}}{\sum_i c_{i}}, \quad (11)$$

$$\lim_{\theta \to 0} \eta(\rho; \theta) = \rho. \quad (12)$$

Interpretation for the result of $(11)$ and $(12)$ is that for SNR improving, $i$) the denoiser should generate a constant output for all inputs $\rho$ when $\theta \to \infty$, and $ii)$ the denoiser simply passes the input $\rho$ such that the denoising is not necessary when $\theta \to 0$. The function plots in Fig.$3^3$ validates such function behaviors of $\eta(\rho)$ for a variety of $\theta$.

IV. STATE EVOLUTION FOR sSAMP ITERATION

State evolution (SE) is a framework to track MSE performance of the AMP iteration under a high-dimensional assumption $(N, M \to \infty)$, which has been used in many analyses of AMP algorithms $[16]-[22]$. The essence of the SE framework is that at every iteration $t$, the MSE of the AMP iteration is deterministically predictable in the high-dimensional setting. Namely, by Theorem 1 of $[19]$, we have

$$\begin{align*}
\sigma_{SE}^2(t) &= \mathbb{E}_{f_{x_i \rightarrow v}(x_i, v)} \left[ \left( \eta(x_i + \sqrt{\frac{\tau(t)}{N}} V) - x_i \right)^2 \right] \\
\mathbb{E}_{\mathcal{X}_0} \lim_{N \to \infty} \text{eMSE}(\mathcal{X}_0, t) &= \left| \mathbb{E}_{\mathcal{X}_0} \right| \lim_{N \to \infty} \text{eMSE}(\mathcal{X}_0, t)
\end{align*} \quad (13)$$

where $x_i \in \mathbb{R}^N$ a deterministic realization of $X$ and $V \sim \mathcal{N}(0, 1)$ is the noise term; we define an empirical MSE at $t$-th iteration of AMP as $\text{eMSE}(\mathcal{X}_0, t) \equiv \frac{1}{N} \| \mathbf{H}^{(t)} - \mathbf{W}_0 \|^2$.

Then, we can predict the MSE of the AMP iteration using the SE given by $(13)$ and a recursive equation $\tau_{SE}^{(t)} = \Delta + \frac{N}{\sigma_{SE}^2(t)} \tau_{SE}^{(t-1)} \forall t \geq 0$, where the initial value is set to $\tau_{SE}^{(0)} = \Delta + \frac{N}{\sigma_0^2}$. In the present work, we basically assume that the conditions presented in Theorem 1 of the paper $[19]$ are satisfied$^4$. One thing we need to clarify for the SE for ssAMP iteration is whether the scalarwise denoiser, given in $(8)$, has the Lipschitz continuity. We refer to Appendix II for an validation justifying the Lipschitz continuity of our scalarwise denoiser. Thus, the ssAMP can adopt the easy SE framework of $[19]$ to predict MSE, differently from TV-AMP which uses a more challenging approach $[22]$ due to its the non-scalarwise denoiser given in $(2)$.

We numerically compare the empirical MSE of ssAMP, given in $(14)$, with finite $N$ and the predicted MSE by the SE recursion, given in $(15)$. This comparison is important for the practical use of the SE framework for ssAMP. For the MSE

$^4$The conditions for Theorem 1 of $[19]$ are given as: 1) Each entry of the measurement matrix $\mathbf{H} \in \mathbb{R}^{M \times N}$ is drawn according to $h_{ij} \sim \mathcal{N}(0, 1/M)$, 2) An empirical distribution of a scalar $x_{0,i}$ converges weakly to $f_{x_i}(x)$ as $N, M \to \infty$, 3) An empirical second moment of $x_{0,i}$ and $w_i$ is bounded by its finite second moment as $N, M \to \infty$, and 4) The denoiser $\eta(\cdot)$ is a Lipschitz continuous function.
Conditional prior PDFs $q$ hence, which has the probability weight each piecewise-constancy is described by conditional PDFs $X$-wise-constancy of $X$ cannot be defined without conditioning on the adjacent elements $X_{i-1}, X_{i+1}$. In this comparison, we investigate three types of the piecewise-constancy: Gaussian, Ternary, Binary; each piecewise-constancy is described by conditional PDFs $f_{X_i}(x_i|X_{i-1}, X_{i+1})$ listed in Table II. In the conditional PDFs, the first term is for the case of $x_i \neq \{x_{i-1}, x_{i+1}\}$; hence, which has the probability weight $q^2$ according to the potential function given in (5).

The conditional expectation of (15) can be calculated using a Monte Carlo integration since the ssAMP denoiser $\eta(\rho)$ is a nonlinear function which is analytically intractable. That is,

$$
(\sigma^2_{SE}(t))^i = \mathbb{E}_{f_{X_i, V}(x_i, v_i)} \left[ \left( \eta(X_i + \sqrt{\text{SE}(t)}V) - X_i \right)^2 \right| X_{i-1}, X_{i+1}]
$$

(15)

since the piecewise-constancy of $X_i$ cannot be defined without conditioning on the adjacent elements $X_{i-1}, X_{i+1}$. In this comparison, we investigate three types of the piecewise-constancy of $X_i$ (Gaussian, Ternary, Binary); each piecewise-constancy is described by conditional PDFs $f_{X_i}(x_i|X_{i-1}, X_{i+1})$ listed in Table II. In the conditional PDFs, the first term is for the case of $x_i \neq \{x_{i-1}, x_{i+1}\}$; hence, which has the probability weight $q^2$ according to the potential function given in (5).

The conditional expectation of (15) can be calculated using a Monte Carlo integration since the ssAMP denoiser $\eta(\rho)$ is a nonlinear function which is analytically intractable. That is,

$$
(\sigma^2_{SE}(t))^i \approx \sum_{u=1}^{U} \left( \eta(x_i^{(u)} + \sqrt{\text{SE}(t)}v^{(u)}) - x_i^{(u)} \right)^2 \tilde{f}_{X_i, V}(x_i^{(u)}, v^{(u)}),
$$

(16)

where $\tilde{f}_{X_i, V}(x_i^{(u)}, v^{(u)})$ denotes the normalized joint distribution of $X_i, V$ by sampling, $u$ is the sample index, and $U$ is the number of the samples. The signal samples $\{x_i^{(u)}\}$ used in the calculation of (16) are generated by Gibbs sampling based on the conditional PDFs in Table II and the noise samples $\{v^{(u)}\}$ are i.i.d. drawn from $f_V(v) = \mathcal{N}(v; 0, 1)$, where we set the prior parameters to $q = 0.1, \sigma_0 = 10$. For the evaluation of $\eta(\rho)$, we numerically chose its input parameters in an MMSE sense. Specifically, at each iteration $t \geq 1$, we chose $\theta(t)$ by

$$
\theta(t) = \arg\min_{\theta} \frac{1}{U} \sum_{u} (\eta(x_i^{(u)} + \sqrt{\text{SE}(t)}v^{(u)}; \theta) - x_i^{(u)})^2,
$$

and the parameter sets $\{a_i^{(t)}\}, \{b_{i, l}^{(t)}\}, \{c_{i, l}^{(t)}\}$ of $\eta(\cdot)$ are set by the R2P/L2P parameters using (39). And, these R2P/L2P parameters are obtained from $(\mu_{R2P/L2P})^{\ast}(t)^2(t) = \arg\min_{(\mu, \sigma^2)} \frac{1}{U} \sum_{u} (\phi(x_i^{(u)} + \sqrt{\text{SE}(t-1)}v^{(u)}; \theta(t-1), \mu, \sigma^2) - x_i^{(u)})^2$

where the signal samples $\{x_i^{(u)}\}$ are drawn from a conditional PDF $f_{X_i}(x_i|X_{i-1} = x_{i-1})$ for the R2P case and $f_{X_i}(x_i|X_{i+1} = x_{i+1})$ for the L2P case. In addition, we consider a noisy case with the noise variance $\Delta = 0.01$.

We provide the comparison result at the fixed point $(t = t^*)$, which consists of three items:

1) the predicted MSE $(\sigma^2_{SE}(t^*))$,
2) the sample mean of the empirical MSE, denoted by $\langle \text{eMSE}(t^*) \rangle$,
3) a standard deviation of the empirical MSE, denoted by $\text{std}(\text{eMSE}(t^*))$.

These results are summarized in Table III for $N = 200$ and in Table IV for $N = 2000$, for various sampling ratios, $M/N = 0.25, 0.50, 0.75$. In addition, Fig. 4 shows a SE plot for the ternary piecewise-constancy case for a variety of $N$. On the comparison results, we make a remark in Note 1.

**Note 1 (Remarks on SE of the ssAMP iteration)**

In the ssAMP iteration,

1) the empirical MSE coincides with the predicted MSE by
Thus, we conclude that the SE framework can be used in the phase transition characteristic and the predicted MSE at the fixed point becomes lower as the size of $N$. This comparison results confirm that the empirical MSE coincides with the predicted MSE by SE, and its accuracy is improved as $N$ increases.

### Comparison results for $N = 200$, $q = 0.1$, $\sigma_0 = 10$, $\Delta = 0.01$.

| Type    | $N$  | $\sigma_0 [10^{-3}]$ | $\langle eMSE(t) \rangle$ | std of $\langle eMSE(t) \rangle$ |
|---------|------|-----------------------|-----------------------------|----------------------------------|
| Gaussian| 0.25 | 1.14 $\times 10^{-4}$ | 1.32 $\times 10^{-4}$       | 1.11 $\times 10^{-4}$           |
|         | 0.50 | 8.56 $\times 10^{-4}$ | 1.12 $\times 10^{-4}$       | 5.76 $\times 10^{-4}$           |
|         | 0.75 | 7.95 $\times 10^{-4}$ | 1.11 $\times 10^{-4}$       | 4.47 $\times 10^{-4}$           |
| Ternary | 0.25 | 7.01 $\times 10^{-4}$ | 8.18 $\times 10^{-4}$       | 4.81 $\times 10^{-4}$           |
|         | 0.50 | 6.31 $\times 10^{-4}$ | 8.12 $\times 10^{-4}$       | 4.20 $\times 10^{-4}$           |
|         | 0.75 | 5.87 $\times 10^{-4}$ | 7.12 $\times 10^{-4}$       | 2.78 $\times 10^{-4}$           |
| Binary  | 0.25 | 5.94 $\times 10^{-4}$ | 5.30 $\times 10^{-4}$       | 3.16 $\times 10^{-4}$           |
|         | 0.50 | 5.62 $\times 10^{-4}$ | 6.56 $\times 10^{-4}$       | 3.32 $\times 10^{-4}$           |
|         | 0.75 | 5.05 $\times 10^{-4}$ | 6.08 $\times 10^{-4}$       | 2.78 $\times 10^{-4}$           |

### Comparison results for $N = 2000$, $q = 0.1$, $\sigma_0 = 10$, $\Delta = 0.01$.

| Type    | $N$  | $\sigma_0 [10^{-3}]$ | $\langle eMSE(t) \rangle$ | std of $\langle eMSE(t) \rangle$ |
|---------|------|-----------------------|-----------------------------|----------------------------------|
| Gaussian| 0.25 | 1.13 $\times 10^{-4}$ | 1.26 $\times 10^{-4}$       | 5.54 $\times 10^{-4}$           |
|         | 0.50 | 8.30 $\times 10^{-4}$ | 1.13 $\times 10^{-4}$       | 2.07 $\times 10^{-4}$           |
|         | 0.75 | 7.88 $\times 10^{-4}$ | 1.04 $\times 10^{-4}$       | 1.35 $\times 10^{-4}$           |
| Ternary | 0.25 | 6.93 $\times 10^{-4}$ | 8.32 $\times 10^{-4}$       | 1.79 $\times 10^{-4}$           |
|         | 0.50 | 5.92 $\times 10^{-4}$ | 7.13 $\times 10^{-4}$       | 9.85 $\times 10^{-5}$           |
|         | 0.75 | 5.61 $\times 10^{-4}$ | 7.09 $\times 10^{-4}$       | 1.13 $\times 10^{-4}$           |
| Binary  | 0.25 | 5.71 $\times 10^{-4}$ | 6.44 $\times 10^{-4}$       | 1.33 $\times 10^{-4}$           |
|         | 0.50 | 5.02 $\times 10^{-4}$ | 5.82 $\times 10^{-4}$       | 9.13 $\times 10^{-5}$           |
|         | 0.75 | 4.91 $\times 10^{-4}$ | 5.45 $\times 10^{-4}$       | 8.39 $\times 10^{-5}$           |

SE, and the gap between the prediction and the empirical result is reduced as $N$ increases,
2) the predicted MSE at the fixed point becomes lower as $M/N$ increases,
3) the predicted MSE is low in the order of Binary < Ternary < Gaussian given the same $N$ and $M/N$.
Thus, we conclude that the SE framework can be used in the MSE prediction of the ssAMP iteration in practice.

### V. Performance Validation

In this section, we empirically validate performance of the ssAMP algorithm from three viewpoints: 1) phase transition curve (PTC), and 2) algorithm runtime over the signal dimension $N$, 3) MSE convergence over iterations.

#### A. Phase Transition Characteristic

We provide an empirical PTC of ssAMP compared to that of the two existing AMPs algorithms, TV-AMP [22] and GrAMPA [26], for three types of piecewise-constant signals listed in Table II. For each empirical PTC, we fixed $N = 500$, and considered a 96 x 96 grid where we uniformly divided the range $\frac{M}{N} \in [0.05, 0.1]$ as the x-axis and the range $\frac{\Delta}{\sigma_0} \in [0.05, 1.0]$ as the y-axis with the stepsize 0.01. These empirical PTCs are connection of experimental points having 0.5 success rate of the signal recovery where the recovery success is declared when the normalized mean-square-error (NMSE) has $\|\hat{\mathbf{e}}_N - \mathbf{e}\|_2^2 \leq 10^{-2}$. For all the algorithms, we limited the number of maximum iterations to $t = 500$ and the iteration stopping tolerance to $\|\mu^{(t)} - \mu^{(t+1)}\|_2^2 < \|\mu^{(t)}\|_2^2 \leq 10^{-6}$.

Fig 4 and Fig 6 and Fig 7 show the empirical PTC comparison for Gaussian, Ternary, and Binary cases, respectively.

From the figures, we note that the empirical PTC of ssAMP fully covers that of GrAMPA as well as that of TV-AMP. These PTCs display that the recovery via GrAMPA is not successful when the sampling rate is low, i.e., approximately $0 < \frac{M}{N} < 0.1$, for all types of the signals. This is partially reported in Fig 1 of the paper [26] where the piecewise-constancy is set to

---

5 The TV-AMP solver was implemented by the authors in MATLAB where we used the fixed-lasso-signal-approximator [15] for the calculation of $\mathbf{A}$.  
6 The MATLAB codes of GrAMPA was obtained from the website [28], and we ran the GrAMPA solver under the guideline from the website.
This unsuccessful recovery of GrAMPA is caused by its residual update for the signal difference $X_i - X_{i-1}$ from the s-factors. GrAMPA generates the residual $r_k^{(t)}$, whose expression can be simplified to

$$r_k^{(t+1)} = \eta_{\text{GrAMP}}(\mu_i^{(t)} - \mu_{i-1}^{(t)}) - (\mu_i^{(t)} - \mu_{i-1}^{(t)}) + \text{Onsager},$$

(17)

where $k \in \{e(i) \in \mathcal{F}_s\}$. This residual update has two weakpoints. i) First, the residual relies on not real measurements but the estimate of $X_i - X_{i-1}$ by the denoiser $\eta_{\text{GrAMP}}$ given in [3]. Therefore, if the measurements is not sufficient to improve the estimate, $\eta_{\text{GrAMP}}(\mu_i^{(t)} - \mu_{i-1}^{(t)})$, over iterations, the residual does not decline and the recovery may not converge. ii) The second weakpoint is from the Onsager term. We discuss about the onsager term in Appendix I-B and refer further interested readers to the paper [19]. The role of the Onsager term is cancellation of the correlation between the matrix $A$ and the finite-difference matrix $D$ since each s-factor is connected to only two variables with $D$ such that the law of large numbers does not work. When the sampling rate is very low, the correlation becomes a main cause to hinder the recovery convergence since the characteristic of $A$ is dominated by $D$. For these two reasons, we argue that ineffective convergence is observable over the region of the low sampling rate in the GrAMPA recovery.

The ssAMP outperforms GrAMPA in the region of $0 < \frac{M}{N} < 0.1$ for the all signal cases. Also, we observe from the PTC figures that ssAMP shows improvement from GrAMPA even when the sampling rate is high, where the region is approximately $0.6 < \frac{M}{N} \leq 1.0$. The main reason for this improvement is from that ssAMP does not depends on the approximation by the law of large numbers for handling the messages over the s-factors. TV-AMP generally underperforms ssAMP and GrAMPA in terms of PTC except the region where GrAMPA does not work as discussed above. One thing noteworthy is that for the Gaussian case, TV-AMP has better PTC characteristic than GrAMPA in the region of $0.65 < \frac{M}{N} < 1.0$.

### B. Algorithm Runtime over Signal Dimension $N$

We validate the low-computationality of the ssAMP algorithm over the signal dimension $N$, compared to recent fast algorithms: TVAL3 [11], the Chambolle-Pock algorithm (CP) [13], TV-AMP [22], and GrAMPA [26].

For fair comparison, we limited the number of maximum iterations to $t = 300$ and the iteration stopping tolerance to $\frac{||\mu_i^{(t)} - \mu_{i-1}^{(t)}||_2^2}{||\mu_i^{(t)}||_2^2} \leq 10^{-6}$ for all the solvers. For TVVAL3, we set all the other parameters by following the recommendation of [11], and for the CP solver, we set

$^7$Here, we follow the definition of the matrix $A$ and the finite-difference matrix used in [26].

$^8$Please see the supplementary material 1 to support our argument here.

$^9$The TV-AMP and GrAMPA solvers, considered here, are the same ones that we used in Section V-A. We ran the TVAL3 solver using the MATLAB code obtained from the website of [11], and we implemented the CP solver in MATLAB codes by referring to the papers [13][14].
the regularization parameter to $\lambda = 0.01$. For the runtime measuring over $N$, we fixed the conditions of $\frac{K}{M}$ and $\frac{M}{N}$, and varied the signal dimension from $N = 20^2$ to $N = 160^2$. In addition, we used MATLAB with a 2.93-GHz Intel Quad Core Xeon processor for this experiment.

Fig. 8 shows runtimes over $N$ for noisy recovery of the ternary signals with the noise variance $\Delta = 10^{-5}$, for four cases of fixed $\frac{K}{M}$ and $\frac{M}{N}$. In Fig. 8, it is visible that complexity scaling of all the algorithms is basically the same. This is reasonable since complexity of all the algorithms are dominated by cost of matrix multiplication with $H$ or $H^T$; therefore, total complexity of all the algorithms scales as $O(MN)$. In such a situation, ssAMP retains its place as the fastest algorithm when $N \geq 10^3$ for all the cases of $\frac{K}{M}$ and $\frac{M}{N}$. This computational advantage of ssAMP becomes more remarkable as the signal dimension $N$ is higher. When $N = 160^2$, the ssAMP runs more than 10 times faster than CP and GrAMPA.

We note from Fig. 8 that TV-AMP shows faster runtime than GrAMPA. However, such a result does not refute the argument in the paper [26] that GrAMPA offers advantage over TV-AMP in runtime. This is because the denoiser of TV-AMP used in this comparison was implemented using a recent fast TV package, called fused-lasso-signal-approximator [15], in C language. Hence, the TV-AMP solver in the present work may be faster than the solver used in the work of [26]. Such a fact also clarifies the statement that the complexity of TV-AMP highly relies on the external denoising package.

C. MSE Convergence over Iterations

In order to clarify the computational advantage of ssAMP discussed in Section V-B, we examined MSE convergence over iterations in Fig. 9, also providing per-iteration runtime in Table V where we consider $N = 100^2$, $\Delta = 10^{-5}$. Among the four cases in Fig. 8 of Section V-B, we only consider two recovery conditions in this experiment: $i)$ the severe condition ($\frac{K}{N} = 0.3$, $\frac{M}{N} = 0.25$), which was in Fig. 8 (a), and $ii)$ the rich condition ($\frac{K}{N} = 0.1$, $\frac{M}{N} = 0.5$), which was in Fig. 8 (d). Parameter settings for all the algorithms are the same as the previous section except that we removed the iteration stopping tolerance.

From the results of Fig. 9 and Table V we remark that the computational advantage of ssAMP is mainly from its fast convergence. In the both cases of Fig. 9 we observe that ssAMP achieves its convergence most rapidly. TV-AMP is the most competitive with respect to per-iteration runtime. However, TV-AMP cannot overtake ssAMP in convergence speed, and also cannot achieve deep convergence in MSE when the recovery condition is severe. In the case of GrAMPA,

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Conditions & TVAL3 & ssAMP & TV-AMP & CP & GrAMPA \\
\hline
(a) Severe condition & variable & 0.17 & 0.16 & 2.23 & 0.82 \\
(b) Rich condition & variable & 0.32 & 0.31 & 2.37 & 0.99 \\
\hline
\end{tabular}
\caption{Runtime (in seconds) per iteration in the experiment of Fig. 9 ($N = 100^2$, $\Delta = 10^{-5}$)}
\end{table}

even though it achieves lower MSE at its fixed-point than TV-AMP in the both conditions, its slow convergence and high per-iteration cost leads to its long runtime. We can move up the convergence of GrAMPA by controlling its stepsize increasing rate, but it still runs slower than ssAMP and TV-
AMP due to its per-iteration cost\textsuperscript{11} The CP algorithm has slow convergence speed, but the main reason why its runtime is slow, is due to its high per-iteration cost. Indeed, CP includes more matrix multiplications per iteration, compared to the others. One interesting algorithm is TVAL3, which has variable per-iteration runtime according to situations of its inner iteration loop at every outer iteration\textsuperscript{12} For this reason, the runtime of TVAL3 can be shorter than that of CP and GrAMPA although its convergence is the last among the algorithms.

VI. CONCLUSIONS AND FURTHER WORKS

The ssAMP algorithm which has been constructed and analyzed in the present work, aims to find a piecewise-constant solution $\mathbf{X}$ of high-dimensional linear systems under a CS framework. It has been discussed that the ssAMP algorithm includes a scalarwise denoiser $\eta(\rho)$ which generates an approximate MMSE estimate of piecewise-constant signals $\mathbf{X}$. With this scalarwise denoiser, we have argued that the ssAMP algorithm is a very good alternative of TV-AMP\textsuperscript{22} which uses external optimization for denoising of $\mathbf{X}$. In addition, we have validated that the SE framework\textsuperscript{16},\textsuperscript{19} holds for the ssAMP. Particularly, we have noted that the ssAMP has advantage in PTC when sampling ratio $M/N$ is low ($0 < M/N \leq 0.1$) and high ($0.6 \leq M/N \leq 1$). Furthermore, we have demonstrated with empirical evidences that the ssAMP has computational advantage over not only TV-AMP but also the other recent algorithms for the piecewise-constant recovery, such as TVAL3\textsuperscript{11}, CP\textsuperscript{13}, and GrAMPA\textsuperscript{26}, and its advantage becomes more notable as the signal dimension $N$ increases.

An important further work is simplification of our scalarwise denoiser and the functions given in Table VI. Following the philosophy of Occams razor\textsuperscript{36}, we believe that these functions have too complex expression for their behavior. Another further work is two-dimensional extension of the ssAMP algorithm. This work is very essential in order to apply the ssAMP algorithm to image denoising applications. In addition, we need to consider the tuning problem of the prior parameters $\sigma$ and $\sigma_0$ in practice even though we assumed oracle tuning in the present work. For this problem, we refer interested readers to the recent works\textsuperscript{23},\textsuperscript{35} investigating parameterless AMP iteration.

APPENDIX I

CONSTRUCTION OF SSAMP FROM CLASSICAL MESSAGE-PASSING

A. Parameterization Step

During the MP iteration, we only keep track of the means and variance of the V2mF messages $\{v_{i \rightarrow j}(x_i)\}$ and the V2sF messages $\{\hat{v}_{i \rightarrow k}(x_i)\}$. Therefore, we first define the below parameters:

$$
\mu_{i \rightarrow j} = \mathbb{E}_{v_{i \rightarrow j}(x_i) | X_i|} x_i, \quad \sigma_{i \rightarrow j}^2 = \text{Var}_{v_{i \rightarrow j}(x_i) | X_i|} x_i, \\
\mu_{i \rightarrow k} = \mathbb{E}_{\hat{v}_{i \rightarrow k}(x_i) | X_i|} x_i, \quad \sigma_{i \rightarrow k}^2 = \text{Var}_{\hat{v}_{i \rightarrow k}(x_i) | X_i|} x_i.
$$

We consider parameterization of the mF2V message $m_{j \rightarrow i}(x_j)$ for the V2mF messages $\{v_{i \rightarrow j}(x_i)\}_{i \neq j}$ are Gaussian PDFs from our assumption, we can calculate the integration in the mF2V update of (6)-a) through the Gaussian moment generating function on the basis of the CLT. Then, the mF2V message is rewritten from (6)-a) as

$$
m_{j \rightarrow i}(x_j) \propto \exp \left( -\frac{1}{2\alpha_{j \rightarrow i}} (x_j - \beta_{j \rightarrow i})^2 \right)
$$

where the parameters $\alpha_{j \rightarrow i}, \beta_{j \rightarrow i}$ are given by

$$
\alpha_{j \rightarrow i} = \frac{h_{ij}}{\Delta + \sum_{i \neq j} h_{ij}^2 \sigma_{i \rightarrow j}^2}, \quad \beta_{j \rightarrow i} = \frac{h_{ij} (y_j - \sum_{i \neq j} h_{ij} \mu_{i \rightarrow j})}{\Delta + \sum_{i \neq j} h_{ij}^2 \sigma_{i \rightarrow j}^2},
$$

where $h_{ij}$ is the $(j, i)$-th element of the measurement matrix $H \in \mathbb{R}^{M \times N}$. Therefore, the mF2V message takes the form of a scaled Gaussian PDF with the mean $\frac{\alpha_{j \rightarrow i} \mu_{j \rightarrow i}}{\alpha_{j \rightarrow i} + \sigma_{j \rightarrow i}^2}$ and variance $\frac{1}{\alpha_{j \rightarrow i} + \sigma_{j \rightarrow i}^2}$, and can be parameterized.

The sF2V message $s_{k \rightarrow i}(x_k)$ is represented as a two-state Gaussian mixture PDF by evaluating the integration in (6)-b) with the spike-and-slab potential function given in (5), i.e.,

$$
s_{k \rightarrow i}(x_k) = (1 - q)N(x_k; \mu_{k \rightarrow i}, \sigma_{k \rightarrow i}^2) + qN(x_k; \mu_{k \rightarrow i}, \sigma_{k \rightarrow i}^2 + \sigma_{k \rightarrow i}^2).
$$

Therefore, the sF2V message can be parameterized by the mean and variance $\mu_{k \rightarrow i}, \sigma_{k \rightarrow i}^2$, which are equivalent to those of the V2sF message $\hat{v}_{k \rightarrow i}(x_k)$ where $i' = i - 1$ when $k = k_1$ and $i' = i$ when $k = k_2$ and $i \in \mathcal{V}\{1\}$; $i' = i + 1$ when $k = k_2$ and $i \in \mathcal{V}\{N\}$. Namely, we have

$$
(\mu_{k \rightarrow i}, \sigma_{k \rightarrow i}^2) = 
\begin{cases} 
(\mu_{i-1 \rightarrow k}, \sigma_{i-1 \rightarrow k}^2), & \text{when } k = k_1 \\
(\mu_{i+1 \rightarrow k}, \sigma_{i+1 \rightarrow k}^2), & \text{when } k = k_2.
\end{cases}
$$

In the V2mF message calculation of (6)-c), the product of the mF2V messages results in a scaled Gaussian PDF, and the product of the sF2V messages is a four-state Gaussian mixture PDF. Hence, the V2mF message of (6)-c) is expressed as

$$
u_{i \rightarrow j}(x_i) = 
\sum_{l=1}^{4} \frac{c_{i,l}}{\sum_{l'} c_{i,l'}} N(x_i; \alpha_{i,l}, b_{i,l}) \exp \left( -\frac{(x_i - \mu_{i \rightarrow j})^2}{2\theta_{i \rightarrow j}} \right)
$$

where the parameters $\mu_{i \rightarrow j}, \theta_{i \rightarrow j}$ are from the mF2V product, which are defined as

$$
\theta_{i \rightarrow j} = \sum_{l \neq j} \frac{1}{\alpha_{l \rightarrow i}}, \quad \rho_{i \rightarrow j} = \sum_{l \neq j} \frac{\beta_{l \rightarrow i}}{\alpha_{l \rightarrow i}}.
$$
and the parameter sets \( \{a_{i,i}\}, \{b_{i,i}\}, \{c_{i,i}\} \) are associated with the sF2V product; therefore, these sets are functions of the sF2V parameters \( \{\mu_{k \rightarrow i}\}_{k \in ne(i)}, \{\sigma_{k' \rightarrow i}^2\}_{k \in ne(i)} \).

The V2sF message calculation of (6)-(d) is analogous to the V2mF message calculation in (23), but it includes the \( j \)-th mF2V messages and excludes the \( k \)-th sF2V message where \( j \in F_m, k \in ne(i) \). Namely,

\[
\hat{v}_{i \rightarrow k}(x_i) \propto s_{k' \rightarrow i}(x_i; \mu_{k' \rightarrow i}, \sigma_{k' \rightarrow i}^2) \exp \left( -\frac{(x_i - \rho_i)^2}{2\sigma_i} \right),
\]

where \( \rho_i, \theta_i \) are calculated by including \( \alpha_{i \rightarrow i}, \beta_{i \rightarrow i} \) from (24), and \( \mu_{k' \rightarrow i}, \sigma_{k' \rightarrow i}^2 \) are assigned by the sF2V parameters coming from another s-factor in the opposite side, i.e., \( k' \neq k \), and \( k' \in ne(i) \).

To establish a recursive MP iteration, we formulate the calculation of the mean and variance given in (18) with the \( k \)-th mF2V messages and excludes the \( j \)-th V2mF message calculation in (23), but it includes the \( m \)-th mF2V message, is to establish a recursion of the residual term, \( \{sF2V \text{ parameters}\} \), therefore, these sets are functions of the \( \{sF2V \text{ parameters}\} \) designed for a scalar input \( \mu \).

Then, through (20)-(22), the MP update rule given in (6) is fully parameterized. We refer to this parameterized update rule as \textit{parametric MP}. At the fixed point \( (t = t^*) \) of the parametric MP iterations, we calculate the mean and variance given the marginal posteriors \( f_{X_i}(x_i|\Sigma) \) using the functions \( \eta(\cdot) \) and \( \gamma(\cdot) \) given in (26), i.e., \( \forall i \)

\[
\mu_i = \eta(\rho_i; \theta_i, \{a_{i,i}\}, \{b_{i,i}\}, \{c_{i,i}\}), \quad \sigma_i^2 = \gamma(\rho_i; \theta_i, \{a_{i,i}\}, \{b_{i,i}\}, \{c_{i,i}\}).
\]

where \( k' \neq k \) and \( k' \in ne(i) \). Detailed expression of the functions \( \eta(\cdot), \gamma(\cdot), \phi(\cdot), \zeta(\cdot) \) in (26)-(27) are given in Table VII. Here, we note that all these functions are basically designed for a scalar input \( \rho_i \).

Then, through (20)-(22), the MP update rule given in (6) is fully parameterized. We refer to this parameterized update rule as \textit{parametric MP}. At the fixed point \( (t = t^*) \) of the parametric MP iterations, we calculate the mean and variance given the marginal posteriors \( f_{X_i}(x_i|\Sigma) \) using the functions \( \eta(\cdot) \) and \( \gamma(\cdot) \) given in (26), i.e., \( \forall i \)

\[
\mu_i = \eta(\rho_i; \theta_i, \{a_{i,i}\}, \{b_{i,i}\}, \{c_{i,i}\}), \quad \sigma_i^2 = \gamma(\rho_i; \theta_i, \{a_{i,i}\}, \{b_{i,i}\}, \{c_{i,i}\}).
\]

\[
2) \text{Applying the first-order Taylor approximation to the calculation, } \mu_{i \rightarrow j} = \eta(\rho_i + \Delta \rho_{i \rightarrow j}; \theta_i + \Delta \theta_{i \rightarrow j}).
\]

\[
3) \text{Substituting the result of the second step to (29).}
\]

\[
4) \text{Applying the high dimensional assumption } (N, M \to \infty) \text{ to the result of third step with the mF2V parameter expressions given in (20).}
\]

Analogous approaches from the parametric MP to the AMP have been introduced in [17],[18],[21], where the authors verified that although approximation errors are induced in the manipulation, the errors are negligible with the high-dimensional assumption. Here, we omit detailed manipulation for Note 2 by referring the reader to [17],[18],[21]. Finally, we obtain the recursion for a residual \( r_j \) given as

\[
r_j \equiv y_j - \sum_i h_{ji} \mu_i + \frac{N}{M} \langle \rho_i(\theta_i) \rangle.
\]

Also, we have from (24) that

\[
\rho_i = \sum_j h_{ji} r_j + \mu_i, \quad \theta_i = \Delta + \frac{1}{M} \sum_i \sigma_i^2.
\]

We note that the last term of (30) corresponds to the term \( \sum_i h_{ji} \Delta \mu_{i \rightarrow j} \) in (29), correcting the dependency on the index \( j \) of \( \mu_{i \rightarrow j} \) in the residual calculation. This correcting term have been called \textit{Onsager term} in the AMP literature [16]-[25], which is known as a key for convergence of the AMP iterations [19].

C. Simplification Step for sF2V and V2sF Update

The sF2V update \( (k \rightarrow i) \) in (22) is simple assignment of the V2sF parameters \( (i' \rightarrow k) \) according to the direction of message-passing. This direction is decided by placement of the s-factor \( k \in ne(i) \) (see Fig 2). i) When \( k = k_1 \) such that the s-factor is placed on the leftside of the variable node \( i \in V \{1\} \), we have \( i' = i - 1 \); hence, the sF2V parameters is toward right. ii) When \( k = k_2 \) such that the s-factor is on the rightside of the node \( i \in V \{N\} \), we have \( i' = i + 1 \); hence, the sF2V parameters is left-toward. Therefore, we replace the sF2V parameters by defining R2P and L2P parameters, given as

\[
1) \text{R2P parameters:}
\]

\[
\mu_{R2P,i} = \begin{cases} \mu_{k \rightarrow i|k=k_1}, & \text{when } 2 \leq i \leq N, \\ 0, & \text{when } i = 1, \\ \sigma_{R2P,i}^2 = \begin{cases} \sigma_{k \rightarrow i|k=k_1}^2, & \text{when } 2 \leq i \leq N, \\ \sigma_0^2, & \text{when } i = 1. \end{cases} \end{cases}
\]

\[
2) \text{L2P parameters:}
\]

\[
\mu_{L2P,i} = \begin{cases} \mu_{k \rightarrow i|k=k_2}, & \text{when } 1 \leq i \leq N - 1, \\ 0, & \text{when } i = N. \end{cases} \quad \sigma_{L2P,i}^2 = \begin{cases} \sigma_{k \rightarrow i|k=k_2}^2, & \text{when } 1 \leq i \leq N - 1, \\ \sigma_0^2, & \text{when } i = N. \end{cases}
\]

respectively, then updating these R2P/L2P parameters using the V2sF update functions \( \phi(\rho), \zeta(\rho) \) by (22) and (27). Finally,
Next, we show that the first derivative iteration for the piecewise-constant recovery is constructed.

For the first statement of Note 3, we confirm that
\[ \eta(p; \theta, \{a_i\}, \{b_i\}, \{c_i\}) = \frac{b_i}{\theta + b_i} \] which satisfies \( 0 < \frac{b_4}{\theta + b_4} \leq 1 \) since \( b_4 \) is a variance parameter. When \( \theta \to \infty \), we have \( \frac{b_4}{\theta + b_4} \to 0 \), which corresponds to the result of \((41)\) where the denoiser \( \eta(p) \) generates a constant output. When \( \theta \to 0 \), we have \( \frac{b_4}{\theta + b_4} \to 1 \), which also coincides with the result of \((42)\) where the denoising is not performed, i.e., \( \eta(p) = \rho \). In the practical ssAMP iterations, the value of \( \frac{b_4}{\theta + b_4} \) approach to ‘1’ within 5 iterations given the sufficient number of the measurements. This is interpreted that actual denoising of \( \eta(p) \) is performed early in the iterations and the denoiser \( \eta(p) \) precisely refines the signal in the remaining iterations.

The second statement can be proved by showing that boundness of the first derivative \( \eta'(p) \) for finite \( \rho \). In order to prove the boundness, we take the absolute value on both sides of \((34)\). Then, we give a string of inequalities:

\[ |\eta'(p)| \leq \sum_{i=1}^{4} \frac{1}{\theta + b_i} \sum_{l=1}^{\frac{d}{\theta + b_i}} \frac{l}{\theta + b_i} \sum_{a_i, b_i} N(p; a_i, b_i) \] where the inequality of \((a)\) holds from the fundamental properties of the absolute value such as \( |a| = |a|, |a| \geq |b|, \sum a_i \leq \sum |a_i|, \sum |a_i| \leq \sum |a_i|, \forall a_i, b_i \in \mathbb{R} \), and the inequality of \((b)\) holds from the properties of the absolute value and the fact that values of \( \{c_i\}, N(p; a_i, b_i, \theta) \), \( b_i, \theta \), are nonnegative. Now, we rewire the function \( \eta(p) \) as

\[ \eta(p) = \sum_{i=1}^{4} \frac{a_i + bp_i}{\theta + b_i} m_i \] which is a linear combination of \( \frac{a_i + bp_i}{\theta + b_i} \) and \( m_i \), where we define the weights \( m_i = \frac{c_i N(p; a_i, \theta + b_i)}{\sum_{i=1}^{4} c_i N(p; a_i, \theta + b_i)} \). The weights of \( \{m_i\} \) have two properties: \( i) 0 \leq m_i \leq 1 \), and \( ii) \) the summation of the weights is equal to 1, i.e., \( \sum_{i=1}^{4} m_i = 1 \). By substituting \((37)\) to \((36)\), we continue the manipulation on
the inequalities of $|\eta'(\rho)|$, i.e.,

$$|\eta'(\rho)| \leq \sum_{i=1}^{4} b_i |\theta + b_i| + \sum_{i=1}^{4} a_i |\theta + b_i| \left| \frac{\theta + b_i}{\theta + b_i} \right|,$$

where the inequality of (a) holds true from the properties of $0 \leq m_i \leq 1$. The upper bound obtained from (38) is a quadratic polynomial function of $\rho$. Thus, the upper bound of $|\eta'(\rho)|$ is finite for finite $\rho$.

In conclusion, the function $\eta(\rho)$ is Lipschitz continuous because its first derivative $\eta'(\rho)$ is bounded for finite $\rho$, as well as $\eta'(\rho)$ approaches to a finite value as $\rho \to \infty$.

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