The IWM–CFC approximation in differentially rotating relativistic stars

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Abstract. Many astrophysically relevant systems have been studied, in recent years, by means of the spatial conformal flatness condition (IWM–CFC), yielding very satisfactory results as far as the accuracy of IWM–CFC approximation is concerned. Here, we determine the accuracy of IWM–CFC for the case of single, but strongly differentially rotating stars, modeled by a polytropic equation of state. We find that for the fastest rotating models, the deviation from full general relativity is still below 2% for most integrated quantities and reaches up to 6% for sensitive quantities, such as the angular velocity at the equator. We construct a simple error indicator and demonstrate that it correlates well with the largest errors observed in physical quantities characterizing the model.

1. Introduction

The evaluation of the accuracy of the spatial conformal flatness condition (IWM–CFC) [1, 2] in various physical systems is crucial in order to better understand the limits within which it can be applied. Cook, Shapiro and Teukolsky (CST) [3] presented a first test of the CFC approximation for the case of single, uniformly rotating, relativistic stars (see also [4]). Miller, Gressman and Suen [5] used the Cotton–York tensor (which vanishes on spacelike slices that fulfill the IWM–CFC assumption) to evaluate the accuracy of initial data for binary compact objects. Here, we study the accuracy of the CFC approximation in stars that are highly deformed due to strong differential rotation.

The spacetime metric for a stationary, axisymmetric star in equilibrium is described by a line element of the form (see [6, 7] and references therein)

$$ds^2 = -e^{\gamma} dt^2 + e^{\gamma} r^2 \sin^2 \theta (d\phi - \omega dt)^2 + e^{2\mu} (dr^2 + r^2 d\theta^2),$$

where $\gamma$, $\rho$, $\omega$ and $\mu$ are metric potentials depending only on $r$ and $\theta$. We assume that the stellar matter behaves as a perfect fluid and that the equation of state obeys the polytropic relation

$$p = K \rho^{1+\frac{1}{N}},$$

where $\rho$ is the rest mass density, $K$ the polytropic constant and $N$ the polytropic index. For the case of differential rotation we adopt the same rotation law as in Komatsu, Eriguchi and Hachisu (KEH) [8, 9]

$$u^\phi := F(\Omega) = A^2 (\Omega_c - \Omega),$$
where $A$ is a positive constant that determines the length scale over which the angular velocity $\Omega$ changes within the star and $\Omega_c$ is the angular velocity at the center of the configuration. Numerical models are obtained with the rns code by Stergioulas and Friedman [10, 11].

2. Method

In the 3+1 split, the line element is written as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

(4)

where $\alpha$ is the lapse function, $\beta^i$ is the shift vector and $\gamma_{ij}$ is the spatial metric. In the IWM–CFC approximation, one assumes that the spatial part is conformally flat, that is

$$\gamma_{ij} = \psi^4 \eta_{ij}$$

(5)

where $\psi$ is a conformal factor and $\eta_{ij}$ is the flat metric. The metric (1) takes the IWM–CFC form, if

$$\mu = \frac{\gamma - \rho}{2}$$

(6)

Imposing the above condition between the three metric functions, instead of solving the actual equation for $\mu$, allows for the construction of numerical models in the IWM–CFC approximation, without any other modification in the rns code.

For each model that is constructed, various physical quantities are calculated both in IWM–CFC and in full general relativity (GR). As one possible diagnostic of the accuracy of IWM–CFC we define the quantity

$$\Delta c = \frac{\mu - \frac{\gamma - \rho}{2}}{\mu}$$

(7)

which vanishes in IWM–CFC, but is non-vanishing for models constructed in full GR with the metric (1). In addition, relative differences between IWM–CFC and full GR are calculated for every physical quantity.

We focus on the two sequences (A and B) of differentially rotating models with $N = 1$ and $K = 100$ that are presented in Table I of Stergioulas, Apostolatos and Font [12]. For comparison, we also consider their corresponding sequences of uniformly rotating models (AU and BU). Configurations in sequences A and AU have a fixed rest mass of $M_0 = 1.506M_\odot$ and configurations in sequences B and BU have a fixed central mass density of $\rho_c = 1.28 \times 10^{-3}$ (a fixed central energy density of $\epsilon_c = 1.444 \times 10^{-3}$) in the non-dimensional units defined in [12]. We note that in addition to the models presented in [12], we constructed two models that have even higher rotation rate: model A12, with a polar to equatorial axis ratio of $r_p/r_e = 0.25$ and model B13 with $r_p/r_e = 0.34$.

3. Results

The following figures summarize our main result. Figure 1 shows the absolute value of the relative difference between CFC and full GR for three representative physical quantities (the gravitational mass $M$, the ratio $T/|W|$ of the rotational to gravitational binding energy and the Keplerian angular velocity $\Omega_K$ at the equator) as a function of $T/|W|$ along the four different sequences of equilibrium models. We observe that along sequences A and AU all relative differences saturate well below the 1% level, which is explained by the fact that higher rotation leads to smaller central densities, which counteracts the effect of larger flatness. Along sequence B, the central density is fixed and the error increases practically monotonously, reaching up to several percent.
Figure 1: Absolute value of the relative differences between full GR and CFC approximation for the gravitational mass $M$, the rotational kinetic energy over gravitational binding energy ratio $T/|W|$ and the Keplerian angular velocity $\Omega_K$. (a) sequence A, (b) sequence AU, (c) sequence B, (d) sequence BU.

Figure 2 displays the quantity $\Delta c$ in a vertical plane for the fastest rotating models of each sequence. For model A12, $\Delta c$ has a maximum value of roughly 2% inside the star. For model B13, the maximum value of $\Delta c$ becomes 6%. Thus, the maximum values of $\Delta c$ correlate well with the largest errors observed in physical quantities.

Figure 3 shows the profile of $\Delta c$ for the four fastest rotating models in the equatorial plane, as a function of the compactified radial coordinate $s = r/(r + r_e)$. We notice that in all cases the relative error peaks at $s \simeq 0.4$, or $r \simeq 2/3 r_e$.

4. Conclusions
The CFC approximation appears to be a robust method to study systems that exhibit differential rotation if the demands for accuracy are not particularly strict, i.e. if one can cope with a maximum error of several percent. For moderate rotation rates or small central densities the error can be less than the 1% level. We propose a simple error indicator, $\Delta c$, for the accuracy of the CFC approximation for isolated rotating stars, evaluated at $2/3$ of the equatorial radius and demonstrate that it correlates well with the largest errors observed in physical quantities characterizing the model.
Figure 2: Relative difference $\Delta c$ on the $x-z$ plane for the fastest rotating model of each sequence. (a) Model A12, $r_p/r_e = 0.25$, (b) Model AU5, $r_p/r_e = 0.575$, (c) Model B13, $r_p/r_e = 0.34$, (d) Model BU9, $r_p/r_e = 0.58$.

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Figure 3: Relative difference $\Delta c$ calculated on the equatorial plane for the fastest rotating model of each sequence. (a) Model A12, $r_p/r_e = 0.25$, (b) Model AU5, $r_p/r_e = 0.575$, (c) Model B13, $r_p/r_e = 0.34$, (d) Model BU9, $r_p/r_e = 0.58$.

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