Holography in de Sitter and anti-de Sitter Spaces and Gel’fand Graev Radon transform

Samrat Bhowmick∗ 1, Koushik Ray†1, and Siddhartha Sen‡1,2

1Indian Association for the Cultivation of Science,
Calcutta 700 032. India.
2CRANN, Trinity College Dublin, Dublin – 2, Ireland

Abstract

Bulk reconstruction formulas similar to HKLL are obtained for de Sitter and anti-de Sitter spaces as the inverse Gel’fand Graev Radon transform. While these generalize our previous result on the Euclidean anti-de Sitter space, their validity in here is restricted only to odd dimensions. For the anti-de Sitter space it is shown that a reconstruction formula exists for the case of timelike boundary as well. The restriction on the domain of integration on the boundary is derived. As a special case, we point out that the formula is valid for the BTZ black hole as well.

∗email: bhowmicksamrat@gmail.com
†email: koushik@iacs.res.in
‡email: sen1941@gmail.com
1 Introduction

Holography is a duality transformation relating a pair of field theories, one living in some manifold and the other on its boundary, suitably defined. An extensively studied example of holographic duality is the AdS-CFT correspondence. It relates a theory of closed strings at weak coupling on the product of a five-dimensional sphere and a five-dimensional anti-de Sitter space to a gauge theory of three-branes on the conformal boundary of the latter. The converse problem of bulk reconstruction, which we deal with here, attempts to directly obtain a field in the bulk of the manifold from one on the boundary, usually as an integral over a portion of the boundary through a kernel. Such relations have been obtained for manifolds with constant curvature [1–14]. Determination of functions and distributions on a manifold from the knowledge of distributions on a suitable class of submanifolds is the subject of study in integral geometry. This entails specifying appropriate classes of functions on the manifold and on the submanifolds and relating those through integral transforms. In the present article, we consider scalar fields on manifolds of constant curvature, namely, the de Sitter and the anti-de Sitter spaces. Using integral geometric techniques of horospherical transform we relate such fields to the ones on the boundary. In particular, the fields in the bulk of these spaces are expressed as the inverse of a horospherical transform, called the Gel’fand-Graev-Radon (GGR) transform [15, 16]. The present article generalizes similar computations in the Euclidean anti-de Sitter space [17]. Generalization to the two-dimensional hyperbolic manifold over local fields has been worked out too [18].

The relation between the bulk and boundary fields in the anti-de Sitter space is given by the HKLL formula [3]. It expresses the bulk field in the anti-de Sitter space as an integral of the boundary field with a kernel. The domain of integration is chosen to be a spacelike region of the boundary. We find that interpreted as the integral transform the formula is valid in odd dimensions, the kernel being plagued with discontinuity of coefficients in even dimensions. We also show that in odd dimensions the inverse GGR transform allows for a similar formula with the timelike portion of the boundary as the domain of integration. The restriction of the domain of integration on the boundary is derived as a result of consistency of change of variable. We also establish a similar formula for the odd-dimensional de Sitter spaces, although, as is well-known, the time dependence of the field theories somewhat obscure the nature of holography on a de Sitter space [19–27]. Finally, as a special case, we recall that the three-dimensional anti-de Sitter space can be identified with the group manifold of $SL(2, \mathbb{R})$, a quotient of which is the BTZ black hole [28]. Through an appropriate identification of coordinates we demonstrate that the bulk reconstruction formula is also valid for the bulk of the BTZ black hole.

The strategy to derive the bulk reconstruction formula is the same as the one employed earlier [17,18]. We restrict our attention to scalar fields. The $n$-dimensional de Sitter and anti-de Sitter spaces, referred to as the bulk, are presented as quadrics in a $(n+1)$-dimensional flat space, referred to as the embedding space, with a metric of appropriate indefinite signature. A linear equation in terms of the coordinates of the embedding space and its light cone defines a horosphere. The GGR transform of a field in the bulk gives a field on the horosphere. The inverse gives a field in the bulk. By identifying the conformal boundary within the horosphere we show that if the field possesses certain
scaling properties on the light cone, then the kernel transforming it into the bulk can be defined through an integral over a portion of the boundary. The kernel comes with a constant coefficient depending on the dimension of the bulk as well as the scaling dimension of the scalar field on the light cone. Part of it is fixed by demanding consistency of the GGR transform and its inverse. The coefficient of the inverse GGR transform is usually singular for certain dimensions. This originates in the well-known ill-posedness of the inverse Radon transform. However, combined with singular terms arising from the scaling behavior of the field, the coefficient of the Kernel turns out to be non-singular for odd dimensions, but for a volume factor of hyperbolic spaces, which is to be understood in a regularized sense in each case.

In the following two sections we obtain the bulk scalar fields from the boundary using the inverse GGR transform for de Sitter and anti-de Sitter spaces, respectively. In both cases, the coefficient of the kernel, apart from the volume factor, is continuous and non-singular only in odd dimensions. Furthermore, in the anti-de Sitter space, two cases arise. The domain of integration, that is, the domain of influence on the boundary may be either spacelike or timelike. The coefficients are different in the two cases. Evaluation of the inverse GGR transform requires using Dirac delta distributions in spaces with metrics of non-Euclidean signature. We include this computation and some relevant integrals in two appendices.

2 de Sitter space

The $n$-dimensional de Sitter space, to be denoted $\mathcal{M}_{dS}$, is a hyperbolic manifold with a constant positive curvature. We consider the realization of $\mathcal{M}_{dS}$ as a quadric in the flat Minkowski space $(\mathbb{R}^{(1,n)}, \eta)$ with coordinates $\{X^a \in \mathbb{R}; a = 0, 1, \cdots, n\}$ and metric $\eta_{ab} = (\begin{smallmatrix} -1 & 0 \\ 0 & I_n \end{smallmatrix})$, where $I_n$ denotes the $n \times n$ identity matrix. Thus,

$$\mathcal{M}_{dS} = \{ X^a \in \mathbb{R} | \sum_{a,b=0}^{n} \eta_{ab} X^a X^b = 1 \}. \quad (1)$$

The light cone $\mathcal{C}_n$ in $(\mathbb{R}^{(1,n)}, \eta)$, is the set of null vectors $\xi$,

$$\mathcal{C}_n = \{ \xi^a \in \mathbb{R} | \sum_{a,b=0}^{n} \eta_{ab} \xi^a \xi^b = 0 \}. \quad (2)$$

The region of the light cone with $\xi^0 \geq 0$ is called the positive light cone, denoted $\mathcal{C}_n^+$. The metric on $\mathcal{M}_{dS}$ is the metric obtained by restriction from $\eta$. Let us consider the affine chart $\{(z, x); z \in \mathbb{R}, x \in \mathbb{R}^n\}$ on $\mathcal{M}_{dS}$, such that

$$X^0 = \frac{z}{2} \left( 1 - \frac{1 + x^2}{z^2} \right), \quad X^i = \frac{x^i}{z}, \quad X^n = \frac{z}{2} \left( 1 + \frac{1 - x^2}{z^2} \right), \quad (3)$$

$$x^2 = \sum_{i=1}^{n-1} (x^i)^2, \quad (4)$$

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$$2$$
where \( x^i \), denotes a component of \( x \). The metric on \( \mathcal{M}_{\text{dS}} \) in this chart is given by
\[
ds^2 = \frac{1}{z^2} (-dz^2 + \sum_{i=1}^{n-1} (dx^i)^2).
\]
(5)

The coordinates \( x \) are spacelike, while \( z \) is timelike. The volume element of \( \mathcal{M}_{\text{dS}} \) is
\[
dV = \frac{1}{z^n} dz d^{n-1}x,
\]
(6)

where \( d^k x \) denotes the volume element of the \( k \)-dimensional affine Euclidean space \( \mathbb{R}^k \). The light cone \( \mathcal{C}_n \) is a metric cone \( \mathbb{R}_+ \times \xi^0 S^{n-1} \) over the \((n - 1)\)-dimensional sphere \( S^{n-1} \).

The affine coordinates on the light cone commensurate with (3) are
\[
\xi^i = -\frac{2\tilde{x}^i}{1 + \tilde{x}^2} \xi^0, \quad \xi^n = -\left( \frac{1 - \tilde{x}^2}{1 + \tilde{x}^2} \right) \xi^0, \quad -\infty < \xi^0 < \infty
\]
(7)
\[
\tilde{x}^2 = \sum_{i=1}^{n-1} (\tilde{x}^i)^2, \quad -\infty < \tilde{x}^i < \infty, \quad i = 1, \ldots, n - 1.
\]
(8)

In this chart the volume element on the light cone is
\[
d\xi = \frac{d\xi^0 \cdots d\xi^{n-1}}{\xi^n} = \frac{2^{n-1}(-\xi^0)^{n-2}}{(1 + \tilde{x}^2)^{n-1}} d\xi^0 d^{n-1}\tilde{x}
\]
(9)

The inner product of a vector in the de Sitter space and one on the light cone in this chart is given by
\[
\xi \cdot X = \sum_{a,b=0}^{n} \eta_{ab}\xi^a X^b = \frac{\xi^0 (z^2 + (x - \tilde{x})^2)}{z (1 + \tilde{x}^2)}
\]
(10)

The conformal boundary is at \( \xi \cdot X = 0 \). It is situated at \( z \to 0_\pm \) and \( x \to \tilde{x} \) in the affine chart. The future and past spacelike boundaries are denoted \( \mathcal{J}^\pm \), corresponding to \( 0 \leq z < \infty \) and \( -\infty < z \leq 0 \), respectively, as sketched in Figure 1. We present expressions for the former case, the latter being similar.

Figure 1: Boundaries of the de Sitter space
2.1 GGR transform

Let us consider the horospherical GGR transform of functions on the de Sitter space. The horosphere is given by the hypersurface

$$|\xi \cdot X| - 1 = 0. \quad (11)$$

Let us point out that, the modulus, which was not required in the defining equation of the horosphere for the Euclidean case [17] arises as unlike Euclidean anti-de Sitter space, the de Sitter space does not split into two disjoint components. The GGR transform of an integrable function $f(X)$ on $M_{dS}$ is defined to be [15]

$$h(\xi) = \int_{M_{dS}} f(X) \delta(|\xi \cdot X| - 1) \, dV, \quad (12)$$

where the integration is with respect to (6). The inverse of the GGR transform is then given by

$$f(X) = c_n \int_{\epsilon_n} \frac{h(\xi)}{|\xi \cdot X| - 1}_+ \xi^n d\xi, \quad (13)$$

where we have used the abbreviation $x^a_\pm = \theta(x) x^a$, with $\theta$ denoting the Heaviside step function. Here $c_n$ is a constant which depends on the dimension of the de Sitter space.

To determine the constant we use (12) and (13) in conjunction to obtain

$$c_n I = \delta_{M_{dS}}(X - Y), \quad (14)$$

where we have defined

$$I = \int_{\epsilon_n} \frac{\delta(|\xi \cdot Y| - 1)}{|\xi \cdot X| - 1}_+ n d\xi, \quad (15)$$

and $\delta_{M_{dS}}(X - Y)$ denotes the Dirac distribution on $M_{dS}$. Performing the integration and incorporating the strength of the Dirac distribution (71) fixes $c_n$. Let us describe the computations in some detail.

In order to evaluate the integral $I$ we choose, without loss of generality, two points $X$ and $Y$ of the de Sitter space to be $Y = (1, 0, 0, \ldots, 0)$, $X = (z, 0, \ldots, 0)$ using the rotational symmetry of $M_{dS}$. This is achieved in two steps, fixing $Y$ in the first step and then using the isotropy subgroup of it to fix $X$ in the next. This corresponds to choosing $z = 1, x^i = 0$ for $Y$ and $x^i = 0$ for $X$ in (3). Using (10) the integral $I$ then simplifies to

$$I = \int_{\epsilon_n} \frac{\delta(\xi^2 - z^2)}{(\xi^2 - z^2)^{n+1}} \xi^n d\xi. \quad (16)$$

Inserting (9) and defining a new variable $\rho = \frac{\xi^2 - z^2}{x^2 + 1}$, we express $I$ as a sum of two integrals, over the domains $\rho < 0$ and $\rho > 0$. Integrating over $\rho$ then yields

$$I = (-1)^n (2)^n n! \int_{0}^{\infty} \frac{R^{n-2} dR}{(R^2 - 1)^{n-1}} \left( \frac{1}{(\frac{R^2 - z^2}{z(R^2 - 1)} - 1)}_+ + \frac{(-1)^n}{(\frac{R^2 - z^2}{z(1 - R^2)} - 1)}_+ \right), \quad (17)$$
where we have defined the positive number $R$ by $R^2 = z^2$ and denoted by $V_k$ the volume of the $k$-dimensional unit sphere. Changing variable again, to $y = \frac{(R^2 - z^2)}{z(R^2 - 1)}$, we note that, we have $(|y| - 1)_{+}^n$ and $(| - y| - 1)_{+}^n$ in the two terms of the integrand. In the domain of $z$ we have chosen, namely $0 \leq z < \infty$, we have $y > 0$. Thus, $|y| = | - y| = y$. The integral then assumes the form

$$I = (-1)^n (2)^{n-1}V_{n-2}e^{in\pi/2} \cos \frac{n\pi}{2} \int z(yz - z^2)(n-3)/2(yz - 1)^{(n-3)/2} \frac{1}{(1 - z^2)^{n-2}} dy. \quad (18)$$

The limits of integration vary depending on whether $z$ is greater or less than unity. To see this we change the variable of integration once again to $w = yz$. The integral becomes

$$I = (-2)^{n-1}V_{n-2}e^{in\pi/2} \cos \frac{n\pi}{2} \int (w - z^2)(n-3)/2(w - 1)^{(n-3)/2} \frac{1}{(w - z)^{n-2}} w^n dw. \quad (19)$$

Due to the factor $(w - z)^{n-2}$ in the denominator of the integrand the integral is to be interpreted as

$$\int_{z^2}^{1} dw = \begin{cases} \int_{z^2}^{1} dw, & \text{if } 0 \leq z \leq 1 \\ -\int_{z^2}^{1} dw, & \text{if } z \geq 1. \end{cases} \quad (20)$$

Defining a new variable $t$ as $w = t + (1 - t)z$ in the former case and $w = tz^2 + (1 - t)z$ for the latter, we arrive at

$$I = (-1)^{(n+1)/2}2^{n-1}V_{n-2}e^{in\pi/2} \cos \frac{n\pi}{2} \begin{cases} \frac{(-1)^n + 1}{(1+1z)^n} \int_{0}^{1} (1 - t)(n-3)/2(t + z)^{(n-3)/2} \frac{dt}{t}, & \text{if } 0 \leq z \leq 1 \\ \frac{1}{(1+z)^{(n-1)/2}} \int_{0}^{1} (1 - t)(n-3)/2(tz + 1)^{(n-3)/2} \frac{dt}{t}, & \text{if } z \geq 1. \end{cases} \quad (21)$$

The distance between the points $X$ and $Y$, chosen as above, is $(X - Y)^2 = -(z - 1)^2/z$ with respect to the metric $[5]$. The constant $c_n$ is given by the inverse of the coefficient of $|z - 1|^n$ in $I$ evaluated at $z = 1$. However, the above expression for $I$ shows that the coefficient as $z \rightarrow 1_{\pm}$ match only when $n$ is odd. In odd dimensions, the constant $c_n$ is given by

$$c_n = \frac{1}{c(|z - 1|^n I)_{z=1}} = \frac{e^{-in(n+1)/2} \tan \frac{n\pi}{2}}{2^{n-2}nV_{n-2}^{2}} \frac{\Gamma(n)}{\Gamma((n - 1)/2)^2}, \quad (22)$$

where the strength of $\delta_{M_{48}}(X - Y)$ is obtained in [71]. Let us emphasize that the singular $\Gamma(0)$ factors cancelled between [71] and [78].
2.2 Bulk reconstruction

Assuming that the GGR transform and its inverse are valid for fields we apply the considerations of the previous subsection to fields. We identify the function $f(X)$ in $\mathcal{M}_{4S}$ with the bulk field and denote it as $\phi(z, x) = f(X)$. We define the field on the conformal boundary from $h(\xi) = \hat{\phi}(\hat{x}) = h(\xi)$. We further assume, that on the horosphere (11) the boundary field scales as

$$\tilde{\phi}(\lambda \hat{x}) = \lambda^{-\Delta} \tilde{\phi}(\hat{x}),$$

for any function $\lambda = \lambda(\hat{x})$. In particular, this implies

$$h(\xi) = h(\xi^0, \ldots, \xi^{n-1}) = \hat{\phi} \left( \frac{2\hat{x}^1}{1 + \hat{x}^2 \xi^0}, \frac{2\hat{x}^2}{1 + \hat{x}^2 \xi^0}, \ldots, \frac{2\hat{x}^{n-2}}{1 + \hat{x}^2 \xi^0} \right)$$

$$= \left( \frac{2\xi^0}{1 + \hat{x}^2} \right)^{-\Delta} \tilde{\phi}(\hat{x}).$$

Inserting (24) and (9) in (13) yields the bulk scalar field from $\tilde{\phi}$ upon integrating over $\xi^0$. In order to perform the integration over $\xi^0$ we define a new variable of integration,

$$y = \frac{\xi^0}{z(1 + \hat{x}^2)}(-z^2 + (x - \hat{x})^2).$$

From (13) we obtain

$$\phi(z, x) = \phi_0(n, \Delta) \int_{\tilde{x}} \mathcal{K}(z, x|\hat{x}) \tilde{\phi}(\hat{x}) \, d^{n-1}\hat{x},$$

where the kernel is

$$\mathcal{K}(z, x|\hat{x}) = \left( \frac{-z^2 + (x - \hat{x})^2}{z} \right)^{\Delta+1-n}$$

and $\phi_0$ is a constant

$$\phi_0(n, \Delta) = c_n 2^{n-1-\Delta} \int_{-\infty}^{\infty} \frac{y^{n-\Delta-2}}{(|y| - 1)^n} \, dy.$$

The choice of sign in the kernel guarantees that $y$ does not change sign. The integral in the expression for $\phi_0$ is evaluated as

$$\int_{-\infty}^{\infty} \frac{y^{n-\Delta-2}}{(|y| - 1)^n} \, dy = \int_{-\infty}^{1} \frac{y^{n-\Delta-2}}{(-y - 1)^n} \, dy + \int_{1}^{\infty} \frac{y^{n-\Delta-2}}{(y - 1)^n} \, dy$$

$$= (1 + (-1)^{n-\Delta}) \int_{0}^{1} \frac{y^{\Delta}}{(1 - y)^n} \, dy$$

$$= 2\pi e^{\pi(n-\Delta)/2} \frac{\cos \frac{\pi(n-\Delta)}{2}}{\sin n\pi} \frac{\Gamma(\Delta + 1)}{\Gamma(n)\Gamma(2 + \Delta - n)^{1/2}}.$$

Using (22) this determines the constant $\phi_0$ to be

$$\phi_0(n, \Delta) = 2^{\Delta-1} e^{\pi(n-\Delta)/2} \frac{\cos \frac{(n-\Delta)\pi}{2}}{\sin n\pi} \frac{\Gamma(1 + \Delta)}{\Gamma(2 + \Delta - n)\Gamma((n-1)/2)}.$$

Let us note that the cosine factor in the denominator is non-vanishing for odd $n$ only.
3 Anti-de Sitter space

The $n$-dimensional anti-de Sitter space, to be denoted $\mathcal{M}_{\text{AdS}}$, is a hyperbolic manifold with constant negative curvature. As for the de Sitter space, we consider the realization of the anti-de Sitter space as a quadric in the flat space $(\mathbb{R}^{2,n-1}, g)$ with coordinates $\{X^a \in \mathbb{R}; a = 0, 1, \cdots, n\}$ and metric $g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \mathbb{I}_{n-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \end{pmatrix}$. Thus

$$\mathcal{M}_{\text{AdS}} = \{X^a \in \mathbb{R} | \sum_{a,b=0}^{n} g_{ab}X^aX^b = -1\}. \quad (31)$$

The light cone in $(\mathbb{R}^{2,n-1}, g)$ is the set of null vectors

$$\mathcal{C}_n = \{\xi^a \in \mathbb{R} | \sum_{a,b=0}^{n} g_{ab}\xi^a\xi^b = 0\} \quad (32)$$

defined with respect to the metric $g$. The positive light cone $\mathcal{C}^+$ is the set of null vectors with $\xi^0 \geq 0$. We work with the affine chart $\{(z, x); z \in \mathbb{R}, x \in \mathbb{R}^n\}$ on $\mathcal{M}_{\text{AdS}}$, such that

$$X^0 = \frac{z}{2}\left(1 + \frac{1+x^2}{z^2}\right), \quad X^{i+1} = \frac{x^i}{z}, \quad X^n = \frac{z}{2}\left(1 - \frac{1-x^2}{z^2}\right), \quad (33)$$

where $i = 0, 1, \cdots, n-2$ and

$$x^2 = \eta_{ij}x^ix^j, \quad -\infty < x^i < \infty. \quad (34)$$

The expressions are similar to (3), with some important difference in certain signs and the metric. Here, unlike (3), the vector $x$ can be either spacelike or timelike. We consider both cases. The metric obtained on $\mathcal{M}_{\text{AdS}}$ by restricting the $(n+1)$-dimensional flat metric $g$ is

$$ds^2 = \frac{1}{z^2}( -(dx^0)^2 + dz^2 + (dx^1)^2 + \cdots (dx^n)^2). \quad (35)$$

The volume element of $\mathcal{M}_{\text{AdS}}$ in this metric is

$$dV = \frac{1}{z^n}dzdx^{n-1}. \quad (36)$$

As before, we choose commensurate coordinates on the light cone $\mathcal{C}$ as

$$\xi^{i+1} = \frac{2\tilde{x}^i}{1 + \tilde{x}^2}\xi^0, \quad \xi^n = -\left(1 - \frac{\tilde{x}^2}{1 + \tilde{x}^2}\right)\xi^0, \quad (37)$$

with $i = 0, 1, \cdots, n-2$, and

$$\tilde{x}^2 = \sum_{i,j=0}^{n-2} \eta_{ij}\tilde{x}^i\tilde{x}^j. \quad (38)$$

The volume element on the positive light cone is

$$d\xi = \frac{d\xi^0 \cdots d\xi^{n-1}}{\xi^n} \quad (39)$$

$$= \frac{2^{n-1}(\xi^0)^{n-2}}{(1 + \tilde{x}^2)^{n-1}}d\xi^0d^{n-1}\tilde{x}, \quad (40)$$

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with \( \xi^0 \geq 0 \) and a definite sign of \( 1 + \tilde{x}^2 \). Using (33) and (37) we obtain
\[
\xi \cdot X = \sum_{a,b=0}^{n} g_{ab} \xi^a X^b = -\frac{\xi^0 (z^2 + (x - \tilde{x})^2)}{z (1 + \tilde{x}^2)},
\] (40)
where \( (x - \tilde{x})^2 = \sum_{i,j=0}^{n-2} \eta_{ij} (x^i - \tilde{x}^i)(x^j - \tilde{x}^j) \). The conformal boundary of \( \mathcal{M}_{\text{AdS}} \) is situated at \( z = 0, x^i = \tilde{x}^i \), corresponding to \( \xi \cdot X = 0 \).

### 3.1 GGR transform

The GGR transform of an integrable function in the anti-de Sitter space is defined as the integral
\[
h(\xi) = \int_{\mathcal{M}_{\text{AdS}}} f(X) \delta (|\xi \cdot X| - 1) \, dV
\] (41)
that restricts an integrable function \( f \) in \( \mathcal{M}_{\text{AdS}} \) to the horosphere
\[
|\xi \cdot X| - 1 = 0.
\] (42)
The inverse transform is given by
\[
f(X) = c_n \int_{\mathcal{V}_n^+} \frac{h(\xi)}{(|\xi \cdot X| - 1)_+^n} \, d\xi.
\] (43)
As before, \( c_n \) is a constant, dependent on the dimension of \( \mathcal{M}_{\text{AdS}} \), determined through the consistency of (41) and (43). It is determined by the consistency of the GGR transform and its inverse as
\[
c_n \mathcal{I} = \delta_{\mathcal{M}_{\text{AdS}}} (X - Y),
\] (44)
where \( \mathcal{I} \) is now defined as the integral
\[
\mathcal{I} = \int_{\mathcal{V}_n^+} \delta (|\xi \cdot Y| - 1) \frac{d\xi}{(|\xi \cdot X| - 1)_+^n}
\] (45)
for two points \( X \) and \( Y \) in \( \mathcal{M}_{\text{AdS}} \). and \( \delta_{\mathcal{M}_{\text{AdS}}} (X - Y) \) denotes the Dirac distribution on this component. In order to determine \( c_n \) we choose two points \( X = (z, x) = (z, 0) \) and \( Y = (z, x) = (1, 0) \) as before. Using (40) and (39) the integral becomes
\[
\mathcal{I} = -2^{n-1} \int_{\mathcal{V}_n^+} \frac{\delta (| - \xi^0| - 1)}{\left( \frac{-\xi^0 (z^2 + \tilde{x}^2)}{z^2 (1 + \tilde{x}^2)} - 1 \right)^n_+} \frac{(\xi^0)^{n-2}}{(1 + \tilde{x}^2)^{n-1}} \xi^0 \, d\xi \] (46)
The boundary with coordinates \( \tilde{x} \) may be either spacelike or timelike. We deal with the two cases separately.
3.1.1 Case I: spacelike boundary, \( \hat{x}^2 > 0 \)

Computations in this case are similar to that in the de Sitter case. We express the coordinates \( \hat{x} \) in terms of angular and hyperbolic coordinates, writing \( \hat{x}^2 = R^2 \), with \( R > 0 \). The integral is evaluated exactly as in the case of de Sitter space with the successive variables of integration \( y = \frac{(x^2 + R^2)}{z(1 + R^2)} \) and \( w = yz \) as before. This yields

\[
\mathcal{I} = -2^{n-2} V_{n-2} \times \begin{cases} 
\frac{(-1)^{n-1} z^{n-1}}{(1+z)^n (1-z)^n} \int_0^1 (1-t) (n-3)/2 (t + z)^{(n-3)/2} \frac{dt}{t^{n}}, \\
\frac{z^{(n-1)/2} (-1)^{n-3}}{(1+z)^{n-2} (z-1)^n} \int_0^1 (1-t) (n-3)/2 (tz+1)^{(n-3)/2} \frac{dt}{t^{n}},
\end{cases}
\]

where \( V_{n-2} \) now denotes the volume of the \( (n-2) \)-dimensional hyperboloid. The integral as a function of \( z \) is continuous at \( z = 1 \) only when \( n \) is odd.

3.1.2 Case II: timelike boundary, \( \hat{x}^2 < 0 \)

Repeating the same steps as in Case-I, with \( y = \frac{x^2 - R^2}{z(1-R^2)} \) and \( w = -yz \) leads to

\[
\mathcal{I} = (-1)^{(n+1)/2} 2^{n-2} V_{n-2} \times \begin{cases} 
\frac{(-1)^{n+1} z^{n}}{(1+z)^n (1-z)^n} \int_0^1 (1-t) (n-3)/2 (t + z)^{(n-3)/2} \frac{dt}{t^{n}}, \\
\frac{z^{(n-1)/2}}{(1+z)^{n-2} (z-1)^n} \int_0^1 (1-t) (n-3)/2 (tz+1)^{(n-3)/2} \frac{dt}{t^{n}},
\end{cases}
\]

where we have now written \( \hat{x} \) in terms of angular and hyperbolic coordinates with \( \hat{x}^2 = -R^2 \). The continuity of \( \mathcal{I} \) as a function of \( z \) again restricts \( n \) to odd numbers only. We have, for odd \( n \),

\[
c_n^I = \frac{\cos \frac{n\pi}{2} V_{n-2} \Gamma(n)}{2^{n-3}\pi e^{\pi(n-1)/2} \Gamma((n-1)/2)^2} \quad (49)
\]

\[
c_n^{II} = \frac{\sin n\pi V_{n-2} \Gamma(n)}{2^{n-2}\pi e^{\pi(n+1)/2} \Gamma((n-1)/2)^2} \quad (50)
\]

in the two cases, using (74) and (76) respectively.

3.2 Bulk reconstruction

Let us now use the inverse formula (43) for bulk reconstruction. The strategy for bulk reconstruction is the same as before. We assume that \( \hat{f}(\xi) \) is (43) has a conformal symmetry
on the null cone with conformal dimension $\Delta$,

$$
\begin{align*}
    h(\xi) &= h(\xi^0, \xi^1, \ldots, \xi^n) \\
    &= h\left( \frac{2\tilde{x}^0}{1 + \tilde{x}^2}\xi^0, \frac{2\tilde{x}^1}{1 + \tilde{x}^2}\xi^0, \ldots, \frac{2\tilde{x}^{n-2}}{1 + \tilde{x}^2}\xi^0, -\left( \frac{1 - \tilde{x}^2}{1 + \tilde{x}^2}\right) \xi^0 \right) \\
    &= \left( \frac{2\xi^0}{1 + \tilde{x}^2} \right)^{-\Delta} \tilde{\phi}(\tilde{x}),
\end{align*}
$$

(51)

where we used (37) in the second step and $\tilde{\phi}$ is a function of $\tilde{x}^0, \ldots, \tilde{x}^{n-2}$. Defining

$$
y = \frac{\xi^0(z^2 + (x - \tilde{x})^2)}{z(1 + \tilde{x}^2)}
$$

(52)

and inserting (39) and (40) in (43) we obtain

$$
f(X) = c_n 2^{n-1-\Delta} \int_1^\infty \frac{y^{n-\Delta-2}dy}{(|y| - 1)^+} \int \left( \frac{z}{z^2 + (x - \tilde{x})^2} \right)^{n-\Delta-1} \tilde{f}(\tilde{x})d\tilde{x}.
$$

(53)

The domain of integration of $y$ does not allow $y$ to vanish. Hence, the expression $z^2 + (x - \tilde{x})^2$ must be non-vanishing. We have chosen it to be positive here. Therefore, the domain of integration of $\tilde{x}$ is bounded by $z^2 + (x - \tilde{x})^2 > 0$, consistent with the HKLL formula. Had we chosen the opposite sign of $z^2 + (x - \tilde{x})^2$, the expression for $f(X)$ would have changed merely by a sign. Let us also note that this restriction did not arise in the case of de Sitter space.

Changing variable from $y$ to $t = 1/y$ yields

$$
\int_1^\infty \frac{y^{n-\Delta-2}dy}{(|y| - 1)^+} = \int_0^1 \frac{t^\Delta}{(1 - t)^n} = \frac{\Gamma(1 + \Delta)\Gamma(1 - n)}{\Gamma(2 + \Delta - n)}.
$$

(54)

In view of the sign of $1 + \tilde{x}^2$ chosen in the two cases above, this yields

$$
\phi^{I,II}(z, x) = \phi_0^{I,II}(n, \Delta) \int_{z^2+(x-\tilde{x})^2>0} \mathcal{K}(z, x|\tilde{x})\tilde{\phi}(\tilde{x})d^{n-1}\tilde{x},
$$

(55)

where we have defined $\phi^{I,II}(z, x) = f(X)$ in the two cases, along with

$$
\phi_0^I(n, \Delta) = \frac{1}{2^{\Delta-1}e^{i\pi(n-1)/2}\Gamma(n-2)\Gamma((n-1)/2)} \frac{\Gamma(1 + \Delta)}{\Gamma(2 + \Delta - n)\Gamma((n-1)/2)^2}
$$

(56)

and

$$
\phi_0^{II}(n, \Delta) = \frac{1}{2^{\Delta-1}e^{i\pi(n+1)/2}\Gamma(n-2)\Gamma((n+1)/2)} \frac{\Gamma(1 + \Delta)}{\Gamma(2 + \Delta - n)\Gamma((n-1)/2)^2}
$$

(57)

The kernel is the same in both cases, namely,

$$
\mathcal{K}(z, x|\tilde{x}) = \left( \frac{z^2 + (x - \tilde{x})^2}{z} \right)^{\Delta+1-n}.
$$

(58)
4 BTZ black hole

Let us now briefly indicate how the present formulation yields a bulk scalar field for the BTZ black hole. This is not unexpected, but the choice of chart in (33) helps bringing it out. The BTZ black hole is a quotient of $SL(2,\mathbb{R})$, corresponding to the three-dimensional anti-de Sitter space (31). In terms of the coordinates of the embedding space $SL(2,\mathbb{R})$ is parametrized as a $2 \times 2$ real unimodular matrix
\[
g = \begin{pmatrix} X^1 + X^2 & X^3 + X^0 \\ X^3 - X^0 & X^1 - X^2 \end{pmatrix} \in SL(2,\mathbb{R}).
\]
Writing
\[
X^0 = \sqrt{r^2 - r_+^2} \sinh(r_+ t - r_- \phi),
\]
\[
X^1 = \sqrt{r^2 - r_+^2} \cosh(-r_- t + r_+ \phi),
\]
\[
X^2 = \sqrt{r^2 - r_-^2} \sinh(-r_- t + r_+ \phi),
\]
\[
X^3 = \sqrt{r^2 - r_-^2} \cosh(r_+ t - r_- \phi)
\]
the BTZ black hole is given by a quotient corresponding to the periodic identification of $\phi$ as $\phi = \phi + 2\pi$. In these coordinates the metric takes the form \[28\]
\[
ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 (d\phi - \frac{r_+ r_-}{r^2} dt)^2.
\]
The coordinates $(r, t, \phi)$ are related to the coordinates (33) by
\[
x^0 = -\sqrt{\frac{r^2 - r_-^2}{r^2 - r_+^2}} e^{r_+ t - r_- \phi} \cosh(r_+ \phi - r_- t),
\]
\[
x^1 = -\sqrt{\frac{r^2 - r_-^2}{r^2 - r_+^2}} e^{r_+ t - r_- \phi} \sinh(r_+ \phi - r_- t),
\]
\[
z = -\sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{r_+ t - r_- \phi}.
\]
Similar coordinates appear in \[29\]. The periodic change $\phi \mapsto \phi + 2\pi$ then corresponds to
\[
\begin{pmatrix} x^0 \\ x^1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} x^0 \\ x^1 \\ z' \end{pmatrix} = e^{-2\pi r_-} \begin{pmatrix} \cosh 2\pi r_+ & \sinh 2\pi r_+ & 0 \\ \sinh 2\pi r_+ & \cosh 2\pi r_+ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ z \end{pmatrix}
\]
Inserting these in (55) along with the same boost as $(\tilde{x}^0, \tilde{x}^1)$ in (58) we obtain $\phi(z', x') = \phi(z, x)$. We conclude that the bulk reconstruction formula (55) is valid for the BTZ black hole as well.
5 Summary

To summarize, we have obtained bulk reconstruction formulas for the de Sitter and anti-de Sitter spaces. In both the cases, the strategy is the same as the one employed for the Euclidean version earlier [17]. We first identify the conformal boundary within the horosphere defined in the embedding flat spaces. The field on the boundary is then interpreted as the GGR transform of a bulk field and assumed to possess a conformal dimension $\Delta$. The bulk field is written as an integral with a kernel, which is the same as the smearing function of the HKLL formula with appropriate signatures of the metric. The form of the kernel is the same in the HKLL formula, as can be guessed through dimensional considerations. However, the coefficients are determined using the paraphernalia of GGR transform. The coefficients turn out to be well-defined only when the dimension of the space is odd, being discontinuous otherwise. This is in contrast with the Euclidean case, in which the formula was valid in all dimensions. Ill-posedness of the inversion of the integral transform results in singular factors in the coefficient. We show that as a consequence of the assumption of conformality of the field on the boundary these singularities are cancelled in the final formula and that too only in odd dimensions. However, there is an infinite volume factor of a hyperboloid in the case of anti-de Sitter space, which is to be understood as a regularized number. On the anti-de Sitter space, moreover, we obtain two formulas, (55) depending on whether the domain of integration on the boundary is spacelike or timelike. While the causality issue of the latter is not particularly simple, it generalizes the HKLL formula. Moreover, in the case of anti-de Sitter space the restriction on the domain of integration on the boundary present in the HKLL formula is derived by demanding consistency of change of variable (53). Finally, the BTZ black hole can be written as a quotient of the $SL(2, \mathbb{R})$ group manifold pertaining to the three-dimensional anti-de Sitter space. By relating coordinates, we show that the bulk reconstruction formula obtained here is valid for the BTZ black hole too. We hope that this formulation will be useful in revealing the structure of the bulk reconstruction problem in general.

A Dirac distribution

The Dirac delta distribution on an $n$-dimensional de Sitter or anti-de Sitter space is defined to be proportional to $1/((X - Y)^2)_{\pm}^{n/2}$, where $(X - Y)$ denotes the distance between two points $X$ and $Y$ in the space. In order to fix the constant of proportionality let us define

$$\lim_{\mu \rightarrow n/2} \frac{1}{((X - Y)^2)_{\pm}^{\mu}} = c \delta(X - Y).$$

(67)

The constant is then determined by introducing a test function $\phi(X)$ on the space and integrating over $X$ as

$$\lim_{\mu \rightarrow n/2} \int dV \frac{1}{((X - Y)^2)_{\pm}^{\mu}} \phi(X) = c \phi(Y),$$

(68)

For simplicity, we take the test function to be unity and choose $Y$ to be a special point. We consider the cases of de Sitter and anti-de Sitter spaces separately.
A.1 de Sitter space

We choose $Y = (0, \cdots, 1)$ and use the affine parametrization \[^3\] for $X$. Then

$$ (X - Y)^2 = \frac{1}{z} (x^2 - (z - 1)^2). \quad (69) $$

Using the volume element \[^4\] we have

$$ \int dV \frac{1}{((X - Y)^2)^\mu_+} = V_{n-2} \int \frac{z^{\mu-n} R^{n-2} dz dR}{(R^2 - (z - 1)^2)^{\mu_+}}, \quad (70) $$

where we have written the $x$ coordinates in terms of angular and hyperbolic coordinates such that the norm $x^2 = R^2$ with $R > 0$. Changing the variable of integration from $z$ to $t = (z - 1)/R$ then yields

$$ c = \lim_{\mu \to n/2} \int dV \frac{1}{((X - Y)^2)^{\mu_+}} = \lim_{\mu \to n/2} V_{n-2} \int_0^1 \frac{dt}{(1 - t^2)^\mu} \int_0^\infty \frac{(1 + tR)^{\mu-n} dR}{R^{2\mu-n+1}} $$

$$ = \lim_{\mu \to n/2} V_{n-2} \int_0^1 \frac{t^{n-2\mu} dt}{(1 - t^2)^\mu} \int_0^\infty \frac{(1 + \rho)^{\mu-n} d\rho}{\rho^{2\mu-n+1}} $$

$$ = V_{n-2} \sqrt{\pi} \frac{\Gamma(0) \Gamma((1-n)/2)}{2 \Gamma((3-n)/2)} $$

$$ = V_{n-2} \sqrt{\pi} \frac{\Gamma(0) \Gamma((n-1)/2)}{2 \tan \left( \frac{n\pi}{2} \right) \Gamma(n/2)}, \quad (71) $$

where $V_{n-2}$ denotes the volume of the $(n - 2)$-dimensional unit sphere. The second integral in the second line is facilitated by performing a further change of variable from $R$ to $\rho = tR$.

A.2 Anti-de Sitter space

We choose $Y = (1, \cdots, 0)$ and use the parametrization \[^5\] for $X$. Then

$$ (X - Y)^2 = \frac{1}{z} (x^2 + (z - 1)^2). \quad (72) $$

Using the volume element \[^6\] we then evaluate the integral to determine $c$. Two cases arise from the indefinite sign of $x^2$.

A.2.1 Case-I: $x^2 > 0$

If $x^2 = R^2$, with $R > 0$, then

$$ \int dV \frac{1}{((X - Y)^2)^{\mu_+}} = V_{n-2} \int \frac{z^{\mu-n} R^{n-2} dz dR}{((z - 1)^2 + R^2)^{\mu_+}}, \quad (73) $$

13
where $V_{n-2}$ denotes the unbounded volume of the unit hyperboloid. Changing variables from $z$ to $t = (z - 1)/R$ as before we obtain

\[ c = \lim_{\mu \to n/2} \int dV \frac{1}{((X - Y)^2)^\mu_+} = \lim_{\mu \to n/2} V_{n-2} \int_0^\infty \frac{dt}{(1 + t^2)^\mu} \int_0^\infty \frac{(1 + tR)^{-\mu-n} dR}{R^{2\mu-n+1}} \]

\[ = \lim_{\mu \to n/2} V_{n-2} \int_0^\infty \frac{t^{n-2\mu} dR}{(1 + t^2)^\mu} \int_0^\infty \frac{(1 + \rho)^{-\mu-n} d\rho}{\rho^{2\mu-n+1}} \]

\[ = V_{n-2} \frac{\sqrt{\pi} \Gamma(0) \Gamma((n-1)/2)}{2 \Gamma(n/2)}, \] (74)

where $\rho = tR$ in the third line is used.

### A.2.2 Case-II: $x^2 < 0$

If $x^2 = -R^2$, with $R > 0$, then

\[ \int dV \frac{1}{((X - Y)^2)^\mu_+} = V_{n-2} \int \frac{z^{\mu-n} R^{n-2} dz dR}{((z - 1)^2 - R^2)^\mu_+}, \]

where $V_{n-2}$ denotes the unbounded volume of the unit hyperboloid. Changing variables again from $R$ to $t = (z - 1)/R$ obtain

\[ c = \lim_{\mu \to n/2} \int dV \frac{1}{((X - Y)^2)^\mu_+} = \lim_{\mu \to n/2} V_{n-2} \int_1^\infty \frac{dt}{(t^2 - 1)^\mu} \int_0^\infty \frac{(1 + tR)^{-\mu-n} dR}{R^{2\mu-n+1}} \]

\[ = \lim_{\mu \to n/2} V_{n-2} \int_1^\infty \frac{t^{n-2\mu} dt}{(t^2 - 1)^\mu} \int_0^\infty \frac{(1 + \rho)^{-\mu-n} d\rho}{\rho^{2\mu-n+1}} \]

\[ = V_{n-2} \frac{\sqrt{\pi} \Gamma(0) \Gamma((n-1)/2)}{2 \Gamma(n/2)}, \] (76)

with $\rho = tR$ in the third line.

### B Two more integrals

First, the integral in $\rho$ appearing in (74) and (76) is singular. We evaluate it as follows. First we substitute $\rho' = 1 + \rho$ followed by $\tau = 1/\rho'$. This yields

\[ \lim_{\mu \to n/2} \int_0^\infty \frac{(1 + \rho)^{-\mu-n} d\rho}{\rho^{2\mu-n+1}} = \lim_{\mu \to n/2} \int_0^1 \frac{\tau^{\mu-1}(1 - \tau)^{2-2\mu-1} d\tau}{\Gamma(\mu)\Gamma(n - 2\mu)} \]

\[ = \Gamma(0). \] (77)
Next, the integrals appearing in (21), (47) and (48) when evaluated at \( z = 1 \) are singular too. With \( t^2 \) substituted with \( \tau \), the integrals become

\[
\int_0^1 (1 - t^2)^{(n-3)/2} \frac{dt}{t^n} = \frac{1}{2} \int_0^1 \tau^{-(n+1)/2} (1 - \tau)^{(n-3)/2} d\tau = \frac{\Gamma((1-n)/2)\Gamma((n-1)/2)}{\Gamma(0)} = \frac{\pi}{2 \cos \frac{n\pi}{2}} \frac{1}{\Gamma(0)}.
\]

(78)

In the formulas above we have written the singular factor \( \Gamma(0) \) without regularization to make the cancellation of singular factors in the expressions for \( c_n \) conspicuous.

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