Elliptic Gaudin system with spin

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Abstract. The elliptic Gaudin model was obtained \cite{1, 2} in the framework of the Hitchin system on an elliptic curve with punctures. In the present paper the algebraic-geometrical structure of the system with two fixed points is clarified. We identify this system with poles dynamics of the finite gap solutions of Davey-Stewartson equation. The solutions of this system in terms of theta-functions and the action-angle variables are constructed. We also discuss the geometry of its degenerations.
Introduction

The integrable structure of the Calogero-type systems is of special interest concerning such diverse problems as hierarchies of partial differential equations [3], Seiberg-Witten solutions of N=2 SUSY [4], [5], quasi-Hopf deformations of quantum groups and many others. There are many different approaches to describe the Hamiltonian structure of such systems. One of the aims of this paper is to combine the Hitchin description and the language of the universal algebraic-geometric symplectic form. The method of the special inverse spectral problem was successfully applied in the matrix KP case in [6]. This method deals with the auxiliary linear problem coming from the Lax representation of the equation. The Baker-Akhiezer function (FBA) turns to be a function on the spectral algebraic curve, and its specific analytic properties are very restrictive and provide the reconstruction of the potential and the poles of FBA in terms of theta-functions on the Jacobian of the spectral curve. The same procedure was applied to the spin Ruijsenaars-Schneider system in [7]. But for a wide class of integrable systems of the Hitchin type the explicit solutions remain still unknown. In the first section we revisit the construction of the Hitchin system on the degenerate curve and especially the Gaudin elliptic two-point N-particle system with spin. In the second section we investigate the finite gap potentials of the matrix Davey-Stewartson equation. The system of non-linear equations of its poles dynamics is deduced and the algebraic-geometrical correspondence between the phase space of this system and the space of spectral data is constructed. In the third part the FBA is reconstructed in terms of $\theta$-functions, the explicit expressions for the potential and for the solution of the poles dynamics are given. The universal symplectic structure is constructed in the fourth section. We also demonstrate that the system of poles dynamics is hamiltonian and identify it with the Gaudin elliptic system with spin. In the last section we discuss the rational and special modular degenerations of the system.

1 Gaudin system

The Hitchin system was introduced in [8] as an integrable system on the cotangent bundle of the moduli space of stable holomorphic bundles on the algebraic curve $\Sigma$. The phase space was obtained by the Hamiltonian reduction from the space of pairs $d'_A, \Phi$, where $d'_A$ is the operator defining the holomorphic structure on the bundle $V$ and $\Phi$ is the endomorphism of this bundle, more precisely $\Phi \in \Omega^{0,1}(\Sigma, End(V))$. The reduction by the action of the loop group $GL_N(z)$ was done, i.e. the group of $GL_N$-valued functions on $\Sigma$. The invariant symplectic structure on the base space writes

$$\omega = \int_\Sigma Tr\delta\Phi \wedge \delta d'_A. \quad (1.1)$$

The zero level of the moment map is described by the condition $d'_A\Phi = 0$ which means that $\Phi$ is holomorphic with respect to the induced holomorphic structure on the bundle $End(V)$. It turns out that the system of quantities $Tr\Phi^k$, treated as vector functions on the phase space, Poisson-commute and their number is exactly half the dimension of the
phase space. This system was stated in a formal way and its explicit formulation was obtained for special cases in [4, 10].

The generalized construction of the Hitchin systems on degenerate curves [1] provides a more explicit description. One considers the moduli space of semi-stable bundles $V$ on an algebraic curve with singularities. The Higgs field in this case is a meromorphic section of the bundle $End(V)$. The holomorphic bundle on the singular curve is the holomorphic bundle on its normalization and the set of additional data characterizing the bundle over singularities.

Let us consider the elliptic curve with two simple fixed points $z_1, z_2$. The Higgs field is a meromorphic section with two poles on the curve. The moduli space $\mathcal{M}$ of semi-stable bundles on the elliptic curve with such type of singularity decomposes into the moduli space of semi-stable bundle $V$ on the normalized curve and the finite dimensional space of additional data at the fixed points: $g_i \in End(V|_{z_i})$. The cotangent bundle $\mathcal{T}^* \mathcal{M}$ has additional coordinates $p_i \in \mathcal{T}^* End(V|_{z_i})$ where $i = 1, 2$. The modified symplectic structure is

$$\omega = \int \delta \Phi \wedge \delta \Phi_A + Tr\left(\delta (g_1^{-1} p_1) \wedge \delta g_1 + \delta (g_2^{-1} p_2) \wedge \delta g_2\right).$$

(1.2)

and the zero level of the modified moment map is

$$\bar{\partial} \Phi + [A, \Phi] + p_1 \delta^2(z_1) + p_2 \delta^2(z_2) = 0.$$  

(1.3)

We restrict ourselves to the subspace of semi-stable holomorphic bundles of rank $N$ which decompose into the sum of line bundles. In the basis associated to the decomposition we can make the matrix $A$ diagonal with elements $x_1, \ldots, x_n$. The equation (1.3) becomes

$$\bar{\partial} \Phi_{ij} + (x_i - x_j) \Phi_{ij} + (p_1)_{ij} \delta^2(z_1) + (p_2)_{ij} \delta^2(z_2) = 0.$$  

(1.4)

It could be solved as in [1] in the following way

$$\Phi_{ij} = \frac{\exp((x_i - x_j)\frac{z_1 - z_2}{z_1 - z_2})}{2\pi i}\left((p_1)_{ij} \frac{\sigma(x_i - x_j + z - z_1)}{\sigma(x_i - x_j)\sigma(z - z_1)} + (p_2)_{ij} \frac{\sigma(x_i - x_j + z - z_2)}{\sigma(x_i - x_j)\sigma(z - z_2)}\right);$$

(1.5)

$$\Phi_{ii} = \omega_i + (p_1)_{ii} \zeta(z - z_1) + (p_2)_{ii} \zeta(z - z_2);$$

(1.6)

where $\tau$ is the modular parameter of the elliptic curve, $\sigma$ is the Weierstrass $\sigma$-function and $\omega_i$ are additional free parameters. Now $\text{res}_{z = z_i} 4\pi^2 Tr \Phi^2$ is equal to

$$H_1 = \sum_i \omega_i (p_1)_{ii} + \sum_i (p_1)_{ii} (p_2)_{ii} \zeta(z_1 - z_2) + \sum_{i \neq j} (p_1)_{ij} (p_2)_{ji} \frac{\sigma(x_i - x_j + z_i - z_2)}{\sigma(x_i - x_j)\sigma(z_1 - z_2)}.$$  

(1.7)

and the residue at the point $z = z_2$ gives a similar expression for the Hamiltonian $H_2$ which differs from $H_1$ by the interchanging of $z_1$ and $z_2$. These functions are called the Hamiltonians of the Gaudin elliptic system with spin. We are interested in the complete integrability of this system. On the moduli space of holomorphic bundles on the regular curve in [8] the number of coordinates on the set of quantities $Tr \Phi^k$, i.e. the number of Beltrami differentials, was proved to be exactly half the dimension of $\mathcal{T} \mathcal{M}$. Our case differs from the classical one and we calculate the dimension directly. The coefficients
of the principal part of $Tr\Phi^k$ at the vicinity of the fixed points are the integrals. The mentioned quantities are elliptic functions, so the sum of their residues is equal to zero. The number of integrals is equal to $N(N + 1)$. The number of coordinates on the phase space $x_i, \omega_i(p_1)_{ij}, (p_2)_{ij}, (g_1)_{ij}, (g_2)_{ij}$ is equal to $2N + 4N^2$. The symplectic structure \[(2.2)\]
is invariant under the action of the group $GL_N \times GL_N$ in the following way: the element $(g_1^0, g_2^0)$ acts as
\[
g_1 \mapsto g_1g_1^0, \quad g_2 \mapsto g_2g_2^0.
\]
The moment map of this action is $\mu = ((g_1)^{-1}p_1g_1, (g_2)^{-1}p_2g_2)$ and we can fix it to have a diagonal form and this kills $2N(N - 1)$ degrees of freedom. The stabilizer of this choice of moment map consists of the diagonal subgroup $Diag \times Diag \in G \times G$. It means that the choice of the representative of the stabilizer’s orbit in the moment map subspace kills $2N$ additional degrees of freedom. The dimension of the reduced phase space is equal to $2N + 4N^2 - 2N(N - 1) - 2N = 2N(N + 1)$ so that the number of integrals is exactly half the number of variables.

## 2 Davey-Stewartson equation

As was shown in \[11\], the Davey-Stewartson equation
\[
iu_t + u_{xx} - u_{yy} + u\phi = 0, \quad \phi_{xx} + \phi_{yy} = 2(|u|_{xx}^2 - |u|_{yy}^2) \tag{2.1}
\]
has a Lax representation $\hat{L} = [L, M]$ with
\[
L = \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_z \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad z = x + iy.
\]
Let us consider the $l$-dimensional generalization of the auxiliary linear problem related to this operator
\[
\left( \partial_t + \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \right) \partial_x + \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \right) \Psi = D_A\Psi = 0, \tag{2.2}
\]
where $A_1, A_2$ are $l \times l$-matrices, $d_1 = E, d_2 = -E$, and $E$ is the identity $l \times l$-matrix, and $\Psi$ is a $2l$-dimensional vector function. Looking for the quasi-periodic solutions of the matrix Davey-Stewartson equation we put the potential in the special form:
\[
A_1 = \sum_{j=1}^n a_j^2b_j^i + \frac{\sigma(x - x_j + \eta)}{\sigma(\eta)\sigma(x - x_j)}, \quad A_2 = -\sum_{j=1}^n a_j^2b_j^i + \frac{\sigma(x - x_j + \eta)}{\sigma(\eta)\sigma(x - x_j)}, \tag{2.3}
\]
where $a_j^i$ and $b_j^i$ are $l$-vectors, and $\sigma$ is a Weierstraß function. As usual we look for the quasi-periodic solutions of the linear problem. The convenient basis of such functions with simple poles on $x$ is provided by
\[
\Phi_i(z, x) = \frac{\sigma(z - z_i + x)}{\sigma(x)\sigma(z - z_i)} \exp(-\zeta(z)x), \quad \text{where} \quad z_2 - z_1 = \eta.
\]
We call $\Psi_1$ the first $l$ components of the function $\Psi$, and $\Psi_2$ the last $l$ components. The blocks of the potential have a monodromy such that the ratio of the monodromies of $\Psi_1$
and $\Psi_2$ must be equal to the monodromy of $A_1$. We look for the eigenfunction in the form:

$$
\Psi_i = \sum_{j=1}^{n} S_i^j \Phi_i(z, x - x_j) \exp(ky_+ + \frac{1}{k}y_-),
$$

(2.4)

where $y_\pm = \frac{1}{2}(t \pm x)$. We also look for the solution of the dual equation

$$
\Psi^+ D_A = 0
$$

(2.5)

in the form

$$
\Psi^+_i = \sum_{j=1}^{n} S_i^{j+} \Phi_i(z, x_j - x) \exp(-ky_+ - \frac{1}{k}y_-).
$$

(2.6)

**Theorem 1** The equation (2.3) has a solution of the form (2.4) and the equation (2.5) has a solution of the form (2.6) if and only if the poles dynamics is described by the system:

$$
\dot{a}_i^j + \lambda_i^j a_i^j + \sum_{l \neq j} a_i^l (b_1^{j+} a_2^j) \frac{\sigma(x_j - x_l + \eta)}{\sigma(\eta)\sigma(x_j - x_l)} = 0;
$$

(2.7)

$$
\dot{a}_2^j + \lambda_2^j a_2^j - \sum_{l \neq j} a_2^l (b_1^{j+} a_1^j) \frac{\sigma(x_j - x_l - \eta)}{\sigma(\eta)\sigma(x_j - x_l)} = 0;
$$

(2.8)

$$
\dot{b}_1^j - \lambda_1^j b_1^j + \sum_{l \neq j} b_1^l (b_2^{j+} a_1^j) \frac{\sigma(x_j - x_l + \eta)}{\sigma(\eta)\sigma(x_j - x_l)} = 0;
$$

(2.9)

$$
\dot{b}_2^j - \lambda_1^j b_2^j - \sum_{l \neq j} b_2^l (b_2^{j+} a_2^j) \frac{\sigma(x_j - x_l - \eta)}{\sigma(\eta)\sigma(x_j - x_l)} = 0;
$$

(2.10)

$$
\ddot{x}_j = \sum_{l \neq j} \left(\frac{(b_1^{j+} a_2^j)(b_2^{j+} a_1^j)}{\sigma(\eta)\sigma(x_j - x_l)} - \frac{(b_1^{j+} a_1^j)(b_2^{j+} a_2^j)}{\sigma(\eta)\sigma(x_j - x_l)}\right);
$$

(2.11)

$$
\ddot{\lambda}_1^j - \ddot{\lambda}_2^j = \sum_{l \neq j} \left(\frac{(b_2^{j+} a_1^j)(b_1^{j+} a_2^j)}{\sigma(\eta)\sigma(x_j - x_l)}\right)(\zeta(x_j - x_l) - \zeta(x_j - x_l + \eta) + 2\zeta(\eta) + 
\sum_{l \neq j} \left(\frac{(b_1^{j+} a_2^j)(b_2^{j+} a_1^j)}{\sigma(\eta)\sigma(x_j - x_l)}\right)(\zeta(x_j - x_l) - \zeta(x_j - x_l - \eta) - 2\zeta(\eta)).
$$

(2.12)

**Proof** As was done in [6, 7, 12], introducing the expression (2.4) into equation (2.2) and eliminating the essential singularity, we calculate the coefficients of the expansion in $x$ at the vicinity of the poles $x = x_j$. The lowest term is of order $-2$

$$
S_i^j (\dot{x}_j - 1) + a_i^j b_1^j S_i^j = 0;
$$

$$
S_i^j (\dot{x}_j + 1) + a_i^j b_1^j S_i^j = 0.
$$

The similar condition from the dual equation gives

$$
S_i^{j+} (\dot{x}_j - 1) + S_i^{j+} a_2^j b_1^{j+} = 0;
$$

$$
S_i^{j+} (\dot{x}_j + 1) + S_i^{j+} a_2^j b_1^{j+} = 0.
$$
\[ S_2^j (\dot{x}_j + 1) + S_1^j a_i^j b_i^j = 0. \]

We can fix the gauge of variables \( a_k, b_k \) such that
\[ 1 - \dot{x}_j = (b_1^j a_2^j), \quad -1 - \dot{x}_j = (b_2^j a_1^j), \]
and so
\[ S_k^j = c^j a_k^j, \quad S_k^j = c^j b_k^j \quad \bar{k} = 3 - k, \quad (2.13) \]
where \( c^j \) are scalar functions. The terms of order \(-1\) at \( x = x_j \) implying \( (2.13) \) can be represented as \( (2.7), (2.8) \) where \( \lambda_k^j \) is defined from the following equations on \( c^j \)
\[ (\partial_t + kI + L_1)C = 0; \quad (2.14) \]
\[ (\partial_t + \frac{1}{k}I + L_2)C = 0, \quad (2.15) \]
where
\[ (L_1)_{ij} = \delta_{ij}((b_1^j a_2^j)(\zeta(z - z_2) - \zeta(z) - \zeta(\eta)) - \lambda_1^j) + (1 - \delta_{ij})(b_1^j a_2^j)\Phi_2(x_i - x_j, z); \]
\[ (L_2)_{ij} = \delta_{ij}((b_2^j + a_1^j)(\zeta(z - z_1) - \zeta(z) + \zeta(\eta)) - \lambda_2^j) + (1 - \delta_{ij})(b_2^j a_1^j)\Phi_1(x_i - x_j, z). \]

Concerning the dual equation, we obtain similar equations for the variables \( c^+ \) and the equations \( (2.9), (2.10) \). Thus for the Lax equation we have
\[ \partial_t (L_1 - L_2) = [L_1, L_2], \]
which is equivalent to the equations \( (2.11), (2.12) \). \( \blacksquare \)

The standard trick in the method of algebraic-geometrical direct and inverse problem is that the spectral curve \( \Gamma \), defined by the characteristic equation \( \text{det}(k - \frac{1}{k}I + L_1 - L_2) = 0 \), has specific properties and that \( \Psi \)-function defines a holomorphic line bundle on the curve, and its essential singularity depending on time is the deformation of the fixed line bundle. The analysis is rather traditional and we venture to outline main propositions on the analytic properties of eigenfunction and properties of the spectral curve. We use the notation \( L(z) = L_1(z) - L_2(z) \) and \( \bar{k} = k - \frac{1}{k} \). First, noting that the matrix \( \text{res}_{z=z_i} L(z) \) differs from the unity matrix by a matrix of rank \( l \) we find the decomposition at the points \( z_1, z_2 \)
\[ R(\bar{k}, z) = \prod_{i=1}^{N} (\bar{k} + \nu_i^{1,2} z^{-1} + h_i^{1,2}(z)) \quad (2.16) \]
where \( h_i^{1,2}(z) \) are regular and \( \nu_i^{1,2} = 1 \) if \( i > l \). Using that \( \Gamma \) is an \( N \)-sheeted covering of the base curve \( z \) and the definition of the branch points of the curve, we count the number of branch points; it is equal to \( 4Nl - 2l(l + 1) \), and by Riemann-Hurwitz theorem the genus of \( \Gamma \) is \( g = 2Nl - l(l + 1) + 1 \). In virtue of the Lax representation the parameters of the spectral curve are integrals of motion. If, in addition, we fix \( c(t)|_{t=0} = \text{const} \) it can be deduced that the poles of vector \( c \) do not depend on time. In general position the number of such poles is half the number of branch points, and it is equal to \( 2Nl - l(l + 1) = g - 1 \). The \( \Psi \) function has essential singularities at the points over \( z = z_1, z = z_2 \) and additional \( 2l \) poles. Summarizing the analytic properties of the eigenfunction we obtain
Theorem 2 The function $\Psi$ is meromorphic on the curve $\Gamma$ of genus $g$ outside $2l$ points $P_i$, and has $g + 2l - 1$ poles $\gamma_1, ..., \gamma_{g+2l-1}$ which do not depend on $t, x$. At the points $P_i, i = 1, ..., l$ over $z = z_1$ it has the following decomposition

$$\Psi_{\alpha}(x, t, P) = (\chi_0^{\alpha i} + \sum_{s=1}^{\infty} \chi_s^{\alpha i}(x, t)z^s) \exp(\lambda_i(x + t))\Psi_1(0, 0, P)$$

and when $i = l + 1, ..., 2l$ at the points over $z = z_2$ we have

$$\Psi_{\alpha}(x, t, P) = (\chi_0^{\alpha i} + \sum_{s=1}^{\infty} \chi_s^{\alpha i}(x, t)z^s) \exp(\lambda_i(x - t))\Psi_1(0, 0, P)$$

where $\lambda_i(z) = (1 - \nu_i)z^{-1} - h_i(z)$, $h_i$ are regular functions in the vicinity of $z = 0$, $\nu_i$ are eigenvalues of the matrix $L$ at the points over $z = z_1, z = z_2$ respectively and $\chi_0^{\alpha i}$ are constants.

The similar proposition on the analytic properties of the solution of dual equation is true.

Theorem 3 The solution of the dual equation $\Psi^+$ is a meromorphic function on the same curve $\Gamma$ of genus $g$ outside the points $P_i, i = 1, ..., 2l$ It has $g + 2l - 1$ poles $\gamma_1^+, ..., \gamma_{g+2l-1}^+$ which do not depend on $t, x$. At the points $P_i, i = 1, ..., l$ over $z = z_1$ it has the following decomposition

$$\Psi^+_{\alpha}(x, t, P) = (\chi_0^{+, \alpha i} + \sum_{s=1}^{\infty} \chi_s^{+, \alpha i}(x, t)z^s) \exp(-\lambda_i(x + t))\Psi_1^+(0, 0, P),$$

and at the points $P_i$ over $z = z_2, i = l + 1, ..., 2l$ one has

$$\Psi^+_{\alpha}(x, t, P) = (\chi_0^{+, \alpha i} + \sum_{s=1}^{\infty} \chi_s^{+, \alpha i}(x, t)z^s) \exp(-\lambda_i(x - t))\Psi_1^+(0, 0, P),$$

where $\chi_0^{+, \alpha i}$ are constants.

3 Reconstruction formulas

In this section we demonstrate that the analytic properties that we found in the previous section are sufficiently restrictive and allow us to reconstruct the eigenfunction in terms of the characteristics of the curve, such as abelian differentials, $\theta$-functions and Abel transform. The standard proposition about the uniqueness of the function with the following properties

1. $\Psi_{\alpha}(x, t, P)$ has $g + 2l - 1$ poles $\gamma_1, ..., \gamma_{g+2l-1}$ on the curve $\Gamma$ of genus $g$

2. it has essential singularities at the points $P_j, j = 1, ..., l$ of the form

$$\Psi_{\alpha}(x, t, P) = e^{w_j^{-1}(x + t)}(\delta_{aj} + \sum_s F_{aj}^s(x, t)w_j^s)$$
and at the points $P_j, j = l+1, \ldots, 2l$ of the form

$$\Psi_\alpha(x, t, P) = e^{w_j^{-1}(x-t)}(\delta_{\alpha j} + \sum_s F_{\alpha j}^s(x, t)w_j^s)$$

where $w_j(P)$ is the local parameter at $P_j$ arises from the Riemann-Roch theorem. And its existence follows from the construction: let $d\Omega^1, d\Omega^2$ be unique meromorphic differentials, holomorphic outside the points $P_j$ such that:

$$d\Omega^1 = d(w_j^{-1} + O(w_j)) \text{ at the points } P_j$$

$$d\Omega^2 = \begin{cases} d(w_j^{-1} + O(w_j)) & \text{ at the points } P_j \text{ with } j = 1, \ldots, l \\ d(-w_j^{-1} + O(w_j)) & \text{ at the points } P_j \text{ with } j = l+1, \ldots, 2l \end{cases}$$

normalized by the condition $\int_{x_j} d\Omega^k = 0$. We set $U_j^{(k)} = \frac{1}{2\pi i} \int_{x_j} d\Omega^k$; then, up to a normalization constant

$$\Psi_\alpha(x, t, P) = \frac{\theta(A(P) + U(1)x + U(2)t + Z_\alpha)\theta(Z_0) \prod_{j\neq \alpha} \theta(A(P) + S_j)}{\theta(A(P) + Z_\alpha)\theta(U(1)x + U(2)t + Z_0) \prod_{i=1}^{2l} \theta(A(P) + R_i)} e^{x\Omega(1)(P) + t\Omega(2)(P)}$$

where $A(P)$ is the Abel transform, $K$ is the vector of Riemann constants,

$$R_i = -K - \sum_{s=1}^{g-1} A(\gamma_s) - A(\gamma_{g-1+i}), \quad S_i = -K - \sum_{s=1}^{g-1} A(\gamma_s) - A(P_i),$$

$$Z_0 = -K - \sum_{i=1}^{g+2l-1} A(\gamma_i) + \sum_{j=1}^{2l} A(P_j), \quad Z_\alpha = Z_0 - A(P_\alpha),$$

$$\Omega^{(k)} = \int_{P_0}^P d\Omega^k.$$ The dual eigenfunction is constructed analogously

$$\Psi_\alpha^+(x, t, P) = \frac{\theta(A(P) - U(1)x - U(2)t + Z_\alpha)\theta(Z_0^+) \prod_{j\neq \alpha} \theta(A(P) + S_j^+)}{\theta(A(P) + Z_\alpha^+)\theta(U(1)x + U(2)t - Z_0^+) \prod_{i=1}^{2l} \theta(A(P) + R_i^+)} e^{-x\Omega(1)(P) - t\Omega(2)(P)}$$

where

$$S_j^+ = -K - A(P_j) - \sum_{s=1}^{g-1} A(\gamma_s^+), \quad R_i^+ = -K - A(\gamma_{g-1+i}^+),$$

$$K_0 = \sum_{i=1}^{g+2l-1} (A(\gamma_i) + A(\gamma_i^+)) - 2\sum_{j=1}^{2l} A(P_j), \quad Z_0^+ = Z_0 - 2K - K_0, \quad Z_\alpha^+ = Z_0^+ - A(P_\alpha).$$

These functions satisfy the auxiliary linear problems (2.2), (2.3). The finite gap potential of the Dirac operator, i.e. the solution of the matrix Davey-Stewartson equation is given by $A = \sigma F^1 F - F^1$, where $\sigma$ is the diagonal matrix with diagonal elements $1, \ldots, 1, -1, \ldots, -1$

and $F_1$ is the matrix coefficient of the decomposition near the singularities of the FBA. Therefore the solution of the system of poles dynamics (2.7), ..., (2.12) is given by the formulas:
1. the poles of the eigenfunction can be found as the solutions of the equation
\[ \theta(U^{(1)}x_i(t) + U^{(2)}t + Z_0) = 0, \]

2. the spin variables are given by:
\[ \tilde{a}_{i,\alpha}(t) = Q_i^{-1}(t) \frac{\theta(U^{(1)}x_i(t) + U^{(2)}t + Z_0)}{\theta(Z_\alpha)\theta(S_\alpha)}, \]
\[ \tilde{b}_i^\alpha(t) = Q_i^{-1}(t) \frac{\theta(U^{(1)}x_i(t) + U^{(2)}t - Z_\alpha^+)}{\theta(Z_\alpha^+)\theta(S_\alpha^+)}, \]

3. the normalizing factor is:
\[ Q_i^2(t) = \frac{1}{2} \sum_{\alpha=1}^{2l} \frac{\theta(U^{(1)}x_i(t) + U^{(2)}t - Z_\alpha)\theta(U^{(1)}x_i(t) + U^{(2)}t - Z_\alpha^+)}{\theta(Z_\alpha)\theta(Z_\alpha^+)\theta(S_\alpha)\theta(S_\alpha^+)}. \]

In the formulas above we used the notation:
\[ a_{i,\alpha}^1 = 2\tilde{a}_{i,\alpha}, a_{i,\alpha}^2 = 2\tilde{a}_{i,\alpha+t} \]
\[ b_i^\alpha = \tilde{b}_i^\alpha + t, b_i^{\alpha+1} = \tilde{b}_i^{\alpha+1}. \]

4 Universal symplectic structure

The systems obtained as poles dynamics in the finite gap theory were often introduced from physical motivations. Such an explanation leaves the structural questions concerning dynamical systems apart. By structure we mean the algebraic symmetries in the broad sense. The importance of the hamiltonian structure cannot be overestimated in the fundamental problem of quantization, likewise the “action-angle” variables have a crucial meaning in the Seiberg-Witten theory. Heuristically most of the systems involved are hamiltonian and moreover integrable in the sense of Liouville. In [14, 15] was proposed the universal hamiltonian description of such systems in terms of the flows on the space of quasi-periodic operators generated by the Lax equation. First, this approach was applied to the description of the hamiltonian structure of the nonlinear equations of the soliton theory, but in [16] the efficiency of this method when applied to the finite-dimensional case was confirmed. The essential point is the algebraic-geometric correspondence which maps the coordinates of the phase space of the system of poles dynamics to the space of spectral data, namely to the set of modular coordinates, the divisor of poles of the Baker-Akhiezer function and the type of singularities, which are given by a pair of Abelian differentials on the curve, in our case by the differentials \( dk, dz \).

The coordinates on the phase space of the system \((2.7), \ldots, (2.12)\) given by \( x_i, \lambda_i, a_i^1, a_i^2, b_i^1, b_i^2 \) are invariant under the following change of variables
\[ a_i^1 \mapsto W_1^{-1}a_i^1, \quad b_i^2 \mapsto b_i^2 W_1, \]
\[ a_i^j \mapsto W_2^{-1} a_i^j, \quad b_i^{+} \mapsto b_i^{+} W_2 \]

where \( W_k, k = 1, 2 \) are constant matrices. We also have the symmetry of the system

\[ a_i^j \mapsto \mu_i a_i^j; \quad b_i^j \mapsto \mu_i^{-1} b_i^j \]

where \( \mu_i \) are constant scalars. We also recall the condition \((b_1^{+} a_2^j) - (b_2^{+} a_1^j) = 2\). The dimension of the reduced space is \( 4Nl + 2N - 2l(l - 1) - N - N = 4Nl - 2l(l - 1) \).

We fix \( W_1, W_2 \) such that the matrices \( W_k \), \( k = 1, 2 \) leave the vector \((1, \ldots, 1)\) invariant. Further we use the notation \( A_l^k, B_m^j \) for the new variables. The normalization (4.1) takes the form:

\[
\sum_i A_1^{il} B_{im}^2 = \delta_{lm} k_1^1, \\
\sum_i A_1^{il} B_{im}^1 = \delta_{lm} k_1^2.
\]

For the eigenvector of the operator \( L \) the following representation is true:

\[
c_i = (z - z_1)(c_i^j + O(z - z_1)) \exp(-x_i \zeta(z)), \text{ if } j \leq l, \\
c_i = (z - z_2)(c_i^j + O(z - z_2)) \exp(-x_i \zeta(z)), \text{ if } 2l \geq j > l, \\
c_i = (c_i^j + O(z)) \exp(-x_i \zeta(z)), \text{ if } j > 2l.
\]

The vectors \( c^j \) are proportional to the vectors \( B^j \). We chose for normalizing the following condition

\[
\sum_{i=1}^{N} A_{im} c_i^j = \delta_{jm}, j \leq 2l, \quad \sum_{i=1}^{N} A_{im} c_i^j = 0, j > 2l. \tag{4.2}
\]

It implies the customary normalizing of FBA at the points \( P_j \)

\[
\Psi_q = (\delta_{jq} + O(z)) \exp(k_j x).
\]

Now we define the “action” variables as the poles of the vector \( C \) which solves the equation \((k + L)C = 0\) and satisfies the normalizing condition

\[
\sum_{m=1}^{l} \sum_{i=1}^{N} A_{im} c_i^1 \Phi_1(-x_i, z) + \sum_{m=l+1}^{2l} \sum_{i=1}^{N} A_{im} c_i^1 \Phi_2(-x_i, z) = 1 \tag{4.3}
\]

which is completely in agreement with the previous normalization due to Riemann-Roch theorem. Indeed, the expression on the left-hand side is a meromorphic function on the spectral curve, it has \( g + 2l - 1 \) poles and it has fixed values 1 at the points \( P_i \), i.e. additional \( 2l \) conditions. This function has to be equal to 1 identically. Now all the preliminaries are done and we turn to the construction. The Lax operator and its eigenvector normalized
by the condition (4.3) are natural coordinates on the phase space and we take as $\delta L$ and $\delta C$ the corresponding cotangent vectors. We only introduce new variables $k^{\text{new}}, L^{\text{new}}$ such that $k^{\text{new}} = \hat{k} - 2\zeta(z) + \zeta(z - z_1) + \zeta(z - z_2)$ which is a well defined function on the spectral curve, new variables $\lambda_i^{\text{new}} = \lambda_i - \zeta(\eta)((B_2^i + A_1^i) + (B_1^i + A_2^i))$ and the corresponding expression for the Lax operator:

$$
L^{\text{new}}_{ij} = \delta_{ij}(\lambda_i^{\text{new}} + \zeta(z - z_2)((B_1^i + A_2^i) - 1) - \zeta(z - z_1)((B_2^i + A_1^i) + 1)) + (1 - \delta_{ij})((B_1^i + A_2^i)\Phi_2(x_i - x_j, z) - (B_1^i + A_2^i)\Phi_1(x_i - x_j, z)).
$$

(4.4)

The index $\text{new}$ is omitted in the following. We also introduce the differentials on the total space $\delta k$ and $\delta z$. To relate it with the canonical differential on the spectral curve we notice that the total space is the fibration on the specific moduli space with fiber which is the symmetric power of the spectral curve. We also notice that the sub-varieties $z = \text{const}$ are transversal to the fibers. So we could extend differentials on the fiber to the total space defining them zero along the sub-varieties $z = \text{const}$. It means that we choose the connection which is zero along such sub-varieties. For this choice of a connection we have $\delta z = dz$, where we use the notation $d$ for the extended differential (for more details see [14, 15]).

The main object of our concern is the two-form on the phase space

$$
\omega = \frac{1}{2} \sum_{i=1}^{N} \text{res}_{P_i}(< C^* (\delta L + \delta k) \wedge \delta C >) dz,
$$

(4.5)

where $C^*$ is the eigenvector of $L$ satisfying $< C^* C >= 1$.

**Theorem 4** The two-form $\omega$ is

$$
\omega = 2 \sum_s \delta z(\gamma_s) \wedge \delta k(\gamma_s).
$$

(4.6)

**Proof** We first clarify the meaning of this formula. $k$ is a well defined function on the curve and $z$ is a multi-valued function. After the choice of a branch of the function $z$, its values at the points $\gamma_s$ are functionals on the space of spectral data, i.e. on the phase space of our system. The only thing to justify is that the corresponding cotangent vector $\delta z(\gamma_s)$ does not depend on the choice of branch. Indeed, the difference between such functionals on different branches depends only on the modular parameter $\tau$ of the base elliptic curve which is a priori fixed. The differential $\Omega = < C^* \delta L \wedge \delta C > dz$ is meromorphic on the spectral curve, the essential singularities reduce. The sum of the residues at the points $P_i$ is the opposite of the sum of other residues. $\Omega$ has poles at the points $\gamma_s$. Using the condition $< C^* C >= 1$ we obtain that the residues at these points are

$$
\text{res}_{\gamma_s} \Omega = - < C^* \delta LC > \wedge \delta z(\gamma_s) = - \delta k(\gamma_s) \wedge \delta z(\gamma_s).
$$

Other possible poles of $\Omega$ are the branch points of the curve, at these points $C^*$ has poles due to the normalization, but the differential $dz$ has simple zeroes at these points.

This proposition shows that our two-form is symplectic, i.e. it is closed and non-degenerate. It is non-degenerate because the number of Darboux coordinates is equal to the dimension of the phase space.
Theorem 5  The symplectic form $\omega$ admits the representation

$$
\omega = \sum_{i=1}^{N} (\delta \lambda_i \wedge \delta x_i + \delta B_{i1} \wedge \delta A_{i2}^2 - \delta B_{i2} \wedge \delta A_{i1}).
$$

(4.7)

This is proven by a straightforward calculation. We use the notation:

$$
A_i = (A_{i1}^2, \cdots, A_{id}, -A_{i1}, \cdots, -A_{id});
B_i = (B_{i1}, \cdots, B_{id}, B_{i1}^2, \cdots, B_{id}^2).
$$

Let us decompose $L, C$ as $L = G \tilde{L} G^{-1}, C = G \hat{C}$ where $G = \text{Diag}(e^{\zeta(z)x_i})$. Then

$$
\omega = \frac{1}{2} \text{res}_{z=0} \text{Tr}[\delta \tilde{L} \wedge \delta h + \hat{C}^{-1}(\delta \tilde{L} \delta \hat{C} + [\delta h, \tilde{L}] \wedge \delta \hat{C} - \delta h \hat{C} \wedge \delta \hat{k})].
$$

Here the expression $\hat{C}$ corresponds to the section of the direct image of the line bundle on the spectral curve to the base elliptic curve and $\delta h = G^{-1} \delta G = \text{diag}(-\delta x_i \zeta(z))$. Using

$$
\text{Tr}(C^{-1}[\delta h, L] \wedge \delta C) = -\text{Tr}(C^{-1} \delta h \wedge (\delta L \hat{C} - \hat{C} \delta \hat{k})),
$$

we obtain

$$
\omega = \frac{1}{2} \text{res}_{z=0} \text{Tr}(2 \delta \tilde{L} \wedge \delta h + \hat{C}^{-1} \delta L \wedge \delta C).
$$

The first summand gives $\sum_i \delta \lambda_i \wedge \delta x_i$. Calculating the second summand we take into account the normalizing conditions

$$
\sum_{j=1}^{N} (A_{ja} \delta c_j^k + \delta A_{ja} c_j^k) = 0, \quad \sum_{j=1}^{N} (c_{kj}^* \delta B_{ja} + \delta c_{kj}^* B_{ja}^*) = 0,
$$

which demonstrates the proposition. ■

Theorem 6  The system of poles dynamics of the finite gap potentials of the Davey-Stewartson equation $\text{(2.7)}$, $\text{...}$, $\text{(2.12)}$ is hamiltonian with respect to the symplectic form $\omega$ and the Gaudin Hamiltonian

$$
H = \frac{1}{2} \sum_i \lambda_i (B_{i2}^+ A_i^1 + B_{i1}^+ A_i^2) - \sum_i \zeta(\eta) (B_{i2}^+ A_i^1 + 1)(B_{i1}^+ A_i^2 - 1) + \sum_{i \neq j} (B_{i2}^+ A_i^2)(B_{j2}^+ A_j^1) \frac{\sigma(x_j - x_i + \eta)}{\sigma(\eta) \sigma(x_j - x_i)}
$$

(4.8)

is equal to $(H_1 - H_2)/2$, defined in section 1 for the Gaudin elliptic system.

Proof  We substitute the flow defining the dynamics, i.e. the Lax equation, into the symplectic form. Our intention is to show that $i \partial_t \omega(X) = \omega(X, \partial_t) = dH(X)$. With the chosen coordinates the vector field is defined by the expressions:

$$
\partial_t (L_1 - L_2) = [L_1, L_2], \quad \partial_t k = 0;
$$
\[ \partial_t C(t, P) = L_2 C(t, P) + \mu(t, P)C(t, P), \]

where \( \mu(t, P) \) is a scalar function.

\[ i\partial_t \omega = \frac{1}{2} \sum_{i=1}^{2l} \text{res}_{P_i} (\langle C^*(\delta L - \delta k)(L_2 + \mu(t, P))C \rangle - \langle C^*[L_2, L_1]\delta C \rangle) dz = \]

\[ = -\sum_{i=1}^{2l} \text{res}_{P_i} \delta k \mu(t, P) dz. \]  \hfill (4.9)

The function \( \mu \) has the same expansion at the marked points as the coefficient at \( t \) in the exponential part of the vector \( \Psi \). It means that at \( P_i, i = 1...l \) the principal part of \( \mu \) is equal to \( \frac{1}{2} k \) and at the points \( P_i, i = l + 1, ..., 2l, \mu \approx -\frac{1}{2} k \).

We have found the Hamiltonian up to a constant. To obtain the Hamiltonian \( H_1 - H_2 \) of the Gaudin system we have to subtract \( N\zeta(\eta) \) and use the condition \( (B_1^{i^+} A_2^i) - (B_2^{i^+} A_1^i) = 2 \). We only need to note that the system (2.7),..., (2.12) is equivalent to the system of the hamiltonian dynamics with this Hamiltonian, and the equivalence is given by the formulas:

\[ a_k^i \mapsto \exp(- \int (\lambda_1^i + \lambda_2^i)/2) a_k^i; \]

\[ b_k^i \mapsto \exp(\int (\lambda_1^i + \lambda_2^i)/2) b_k^i. \]

The “action-angle” variables can be constructed in the traditional way as in [16]. We introduce the differentials \( d\Omega_j, j = 1, \ldots, 2l - 1 \) of the third kind with residues 1 and -1...
at the singular points $P_j$ and $P_{2l}$ respectively. On the fundamental domain of our curve we can define the functions $A_k(Q), k = g + 2l - 1$ such that

$$A_k(Q) = \int^Q d\omega_i, \ i = 1, \ldots, g, \quad A_{g+i}(Q) = \int^Q d\Omega_i, \ i = 1, \ldots, 2l - 1.\]$$

Now we introduce the quantities

$$\phi_i = \sum_{s=1}^{g+2l-1} A_i(\gamma_s), \quad I_k = \oint_{a_k} kdz, \quad I_{g+i} = \text{res}_{P_i} kdz.$$

The symplectic form is

$$\omega = \sum_{i=1}^{g+2l-1} \delta\phi_i \wedge \delta I_i.$$

Notice that the quantities $I_i$ are integrals of motion. So, the obtained coordinates are the “action-angle” variables for our system. For the $sl(2)$-case the separated variables was also realized in \[17\].

## 5 Degenerations

The construction of the Hitchin system on degenerate curves is well suited for the analysis of several degenerations of dynamical systems. For example the rational and trigonometric degeneration can be obtained by considering the elliptic curve with cuspidal and nodal singularities respectively instead of the regular elliptic curve. The normalization in both cases is the rational curve and the moduli of holomorphic bundles are parameterized only by the elements gluing the fibers over the singular points. For the Gaudin elliptic system such degenerations exist. The rational one is represented by the hamiltonian

$$H_{rat} = \frac{1}{2} \sum_i \lambda_i (B^+_2 A^+_1 + B^+_1 A^+_2) + \sum_{i \neq j} (B^+_1 A^+_2)(B^+_2 A^+_1) \left( \frac{1}{x_i - x_j} - \frac{1}{\eta} \right) \quad (5.1)$$

with the canonical symplectic form $\omega$ in the representation \[4\]. This system involves a rational curve with two simple points and one double point with cuspidal singularity. It also could be solved by the algebraic-geometric direct and inverse problem as it was done in \[12\], \[13\]. The expressions for the $\Psi$ function could be found in \[13\] (2.53). The “action-angle” variables for it were also constructed in \[18\]. The author deals with another Hamiltonian but the separated variables for these systems coincide. There is another type of degenerations of our original system. It is the limit $z_1 \to z_2, \eta \to 0$. This case is successfully taken care of by the direct and inverse problem as well. The FBA function can be found in \[13\] (2.60). The solution provides the double periodic potentials for the Davey-Stewartson equation. The system can be specified by the Hamiltonian

$$H_{ell} = \frac{1}{2} \sum_i \lambda_i (B^+_2 A^+_1 + B^+_1 A^+_2) + \sum_{i \neq j} (B^+_1 A^+_2)(B^+_2 A^+_1) \zeta(x_i - x_j) \quad (5.2)$$
and the standard symplectic form (4.7). A priori this system is not integrable. Proceed-
ing with the analogous construction for the “action-angle” variables, we obtain that their
number is $2l(2N-2l+1)$ which is less than the dimension of the phase space. The same
degeneracy can be observed by counting the number of Hitchin Hamiltonians. The Hamilton-
ians of the original system can be represented as $H^{k,l}_s = \text{res}_{z=z_s}((z-z_s)^{k-1}Tr\Phi^l)$, $s = 1, 2$.
When the two fixed points coincide, there is no more residues of the expressions $Tr\Phi^k$,
and the only retrievable quantities are $H^{k,l}_1 + H^{k,l}_2$. This case should be investigated with
a more delicate geometrical analysis of the jet structure of the holomorphic bundle.

Conclusion

The main result of this paper is the explicit solution of the Gaudin elliptic system with
spin and the uncovered connection between this system and the quasi periodic solutions
of the Davey-Stewartson equation. For the methodological aspects we noticed the crucial
correspondence between the Hitchin and finite gap descriptions of the dynamical systems.
Our subsequent concern is to give a more general illustration of the equivalence of these
two settings. The important technical query in this direction is the geometry of the mod-
uli space of semi-stable bundles when we vary the base algebraic curve. It could clarify
the Hitchin description of such a system as an elliptic degenerate Gaudin model. Such
an attempt was undertaken and the case of the elliptic curve was analyzed in [19]. The
construction crucially used the group structure on the elliptic curve and could not be
translated directly to the general situation. In order to realize the geometrically estab-
lished limit procedure a kind of canonical connection on the bundle over the moduli space
of algebraic curves with moduli space of holomorphic bundles as the fiber is required.
And the role of the Painlevé connection in this context is of immediate concern to us.

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