Monomode photon spin operators projected onto the fixed frame and
to the quantum-vacuum geometric phases of photons inside a noncoplanar optical fibre∗

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The propagation of monomode photons inside a coiled optical fibre was regarded as a time-dependent quantum evolution process, which gives rise to a geometric phase. It is well known that the investigation of non-adiabatic geometric phases ought to be performed only in the Schrödinger picture. So, the projections of photon spin operators onto the fixed frame of reference is discussed in this paper. In addition, we also treat the non-normal-order spin operators and consider the potential effects (e.g., quantum-vacuum geometric phases) of quantum fluctuation fields arising in a curved optical fibre. The quantum-vacuum geometric phase, which is of physical interest, can be deducted by using the operator normal product, and the doubt of validity and universality for the normal-normal procedure applied to time-dependent quantum systems is thus proposed. In the Appendix, the discussion of possible experimental realizations of quantum-vacuum geometric phases is briefly presented.

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I. INTRODUCTION

It is well known that geometric phases appear only in systems with time-dependent Hamiltonians (or in systems whose Hamiltonians possess some evolution parameters). We can calculate the adiabatic geometric phase (Berry’s topological phase) of time-dependent systems by means of Berry’s phase formula, an exquisite expression which opened up new opportunities for investigating the global and topological properties of quantum evolutions [3]. In order to calculate the non-cyclic non-adiabatic geometric phase one can, generally speaking, exactly solve the time-dependent Schrödinger equation by making use of the Lewis-Riesenfeld invariant theory [4].

In this paper two subjects on the wave propagation of light in a noncoplanarly (e.g., helically) curved optical fibre are discussed: the projection of photon spin in a comoving frame of reference onto the fixed frame and the geometric phases of photons at quantum-vacuum level. Although the quantum-vacuum geometric phases in the fibre, which arises from the zero-point electromagnetic fields of vacuum, has been calculated in one of our papers [2], we think this problem still deserves further detailed discussions in the present paper. In an attempt to find an experimental realization of quantum-vacuum geometric phases (i.e., extracting quantum-vacuum geometric phases from the total geometric phases of left- and right-handed light), we meet, however, with difficulties to achieve this aim, since due to the opposite signs in helicity eigenvalue of left- and right-handed circularly polarized lights, the quantal phases at vacuum level is eliminated automatically by each other and consequently we cannot easily detect them in experiments. In the Appendix, we continue to consider the problem of how to test the quantum-vacuum geometric phases of light in the noncoplanarly curved optical fibre. We hope much attention would be attracted to this subject both theoretically and experimentally.

Investigation of quantum-vacuum geometric phases due to vacuum fluctuation energies possesses the theoretical significance: in quantum field theory the infinite zero-point energy of vacuum is often cancelled (deleted) by using the normal-product procedure and we thus re-define the vacuum backgrounds. It is believed that this formalism is often valid for the time-independent quantum systems since in these cases the divergent background energies may have no observable effects and hence do not influence the physical results of the interacting quantum fields involved. This, however, may be no longer valid for the time-dependent quantum systems (such as the quantum fields in the

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time-dependent gravitational background, e.g., the expanding universe), since in these cases the time-dependent zero-point fields of vacuum will also participate in the time evolution process and therefore cannot be regarded merely as an inactive onlooker (i.e., a simple passive background). So, it is necessary to consider the validity problem of normal-order procedure in time-dependent quantum field theory. We think, in the literature, this problem gets less attention than it deserves.

II. MONOMODE PHOTON SPIN OPERATORS PROJECTED ONTO THE FIXED FRAME OF REFERENCE

It is essential to think of the non-adiabatic non-cyclic geometric phases in the Schrödinger picture. If one deals with geometric phase problem in, e.g., the Heisenberg picture, then the wavefunction will possess a time-dependent phase factor which cannot be determined by equation of motion itself. So, when considering the geometric phases [1,2,6-8] of photons inside a noncoplanarly curved optical fibre, it is of physical significance to study the projections of photon spin operators onto the fixed frame of reference. In what follows we will discuss this quantum mechanical problem.

The spin angular momentum operator of photon fields reads (in the unit $\hbar = 1$)

$$S_{ij} = -\int (\hat{A}_i A_j - \hat{A}_j A_i) d^3 x, \quad (2.1)$$

where one can expand the three-dimensional electromagnetic vector potentials $A(x,t)$ as a Fourier series [9]

$$A(x,t) = \int d^3 k \frac{1}{\sqrt{2(2\pi)^3 \omega_k}} \sum_{\lambda=1}^2 \varepsilon(k,\lambda)[a(k,\lambda) \exp(-i k \cdot x) + a\dagger(k,\lambda) \exp(i k \cdot x)], \quad (2.2)$$

where the frequencies $\omega_k = c|k|$, and $\varepsilon(k,1)$ and $\varepsilon(k,2)$ are the two mutually perpendicular real unit polarization vectors, which are also orthogonal to the wave vector $k$ of electromagnetic wave. Note that here summations with respect to $\lambda$ are over polarization states $\lambda = 1, 2$ (for each $k$).

Insertion of the expression (2.2) for $A(x,t)$ into (2.1) yields

$$S_{ij} = -i \int d^3 k [\varepsilon_i(k,1) \varepsilon_j(k,2) - \varepsilon_j(k,1) \varepsilon_i(k,2)] [a\dagger(k,1)a(k,2) - a\dagger(k,2)a(k,1)] \quad (2.3)$$

with $a\dagger(k,\lambda)$ and $a(k,\lambda)$ being the creation and annihilation operators of polarized photons, respectively, in the comoving frame of reference. Note that here the photon spin operators in (2.1) is of a normal-order form, i.e., $S_{ij} = -\int (\hat{A}_i A_j - \hat{A}_j A_i) d^3 x$; where the creation and annihilation operators appearing in normal order are so arranged: the latter is placed to the right of the former, which will theoretically cancel (or delete) the vacuum quantized fields. Thus the operator product under the normal order sign (denoted by double-dot symbols :) can also be called normal product. To discuss the quantized electromagnetic fields at quantum-vacuum level, we should consider the non-normal order of spin operators. In Sec. III, we will analyze the quantum-vacuum fluctuation contributions to geometric phases of photon fields propagating inside a curved fibre. It is shown that both left- and right-hand (LRH) circularly polarized light in the noncoplanar fibre will possess a so-called quantum-vacuum geometric phase (but the signs of these two vacuum phases are just opposite, namely, if the vacuum phase of right-handed circularly polarized light is positive, then that of left-handed polarized light has a minus sign\(^1\)). This fact will be discussed in more detail in Sec. III, where we will give a firm theoretical background for quantum-vacuum geometric phases.

Since for the planar wave, the following mathematical requirement is satisfied (i.e., the transverse nature of electromagnetic wave)

$$\varepsilon(k,1) \times \varepsilon(k,2) = \frac{k}{k}, \quad (2.4)$$

the expression (2.3) for photon spin operators may be rewritten

\(^1\)Thus the quantum-vacuum geometric phases of left- and right-hand polarized light are often cancelled with each other, and it is therefore not easy for physicists to measure them in experiments. This problem, which I have considered for three years (2000-2003), will be discussed in the Appendix.
\[ S = -i \int \frac{d^3k}{(2\pi)^3} \kappa_a^\dagger(k,1)a(k,2) - a^\dagger(k,2)a(k,1). \] (2.5)

Here the creation and annihilation operators \( a^\dagger_i(k',\lambda') \) and \( a_i(k,\lambda) \) of polarized photons agree with the commuting relation \( [a_i(k,\lambda), a^\dagger_j(k',\lambda')] = \delta^3(k-k')\delta_{ij}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda'). \) Note that for the case of discrete \( k, \) the magnetic vector potentials may be rewritten \[ \] 10

\[ A(x,t) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \sum_\lambda \varepsilon(k,\lambda)[a(k,\lambda) \exp(-ik \cdot x) + a^\dagger(k,\lambda) \exp(ik \cdot x)], \] (2.6)

where \( V \) denotes the volume of a cubic enclosure in which electromagnetic fields are present. In this case, photon spin operator (2.3) is rewritten as \( S = -i \int \frac{d^3k}{(2\pi)^3} \kappa_a^\dagger(k,1)a(k,2) - a^\dagger(k,2)a(k,1) \) (here \( k \) tends to be continuous) with the commuting relation \( [a_i(k,\lambda), a^\dagger_j(k',\lambda')] = \delta^3_{k,k'}\delta_{ij}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda') \) being satisfied. Thus the monomode photon spin operator (with discrete \( k, \)) in the comoving coordinate system is given as follows

\[ S = -i \frac{k}{k} [a^\dagger(k,1)a(k,2) - a^\dagger(k,2)a(k,1)]. \] (2.7)

Generally speaking, in an infinitely large space, the summations in (2.6) with respect to \( k \) are over all allowed momentum \( k. \) However, in a finitely large space with a finite scale length, say, \( a, \) radiation fields with \( k \) less than \( \sim \frac{\pi}{a} \) does not exist in this space.

In order to investigate the projection of spin operators in a comoving system onto the fixed frame, we first discuss the expression for the polarization vectors \( \varepsilon(k,1) \) and \( \varepsilon(k,2) \) in terms of the 3-D Cartesian orthogonal unit vectors \( i, j, k \) in the fixed frame of reference, i.e.,

\[ \varepsilon(k,1) = e_i^0 + e_j^0 + e_k^0, \quad \varepsilon(k,2) = f_i^0 + f_j^0 + f_k^0. \] (2.8)

Thus one can arrive at

\[ a(k,1) = a_i^0(k,1) + a_j^0(k,1) + a_k^0(k,1) \]

\[ = i\frac{1}{\sqrt{2(2\pi)^3}} \left\{ \int d^3x \exp(ik \cdot x)\partial_0\partial_0 A_i + \int d^3x \exp(ik \cdot x)\partial_\xi\partial_\xi A_j + \int d^3x \exp(ik \cdot x)\partial_\xi\partial_\xi A_k \right\}, \] (2.9)

where \( \partial_0 \) is defined to be \( A_\partial_0 B = A\partial_0 B - (\partial_0 A)B. \) In the same fashion,

\[ a^\dagger(k,2) = a^\dagger_i(k,2) + a^\dagger_j(k,2) + a^\dagger_k(k,2). \] (2.10)

According to Eq. (2.9) and (2.10), we obtain the following commuting relations

\[ [a_i(k,\lambda), a^\dagger_j(k',\lambda')] = \delta^3(k-k')\delta_{ij}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda'), \quad [a_i(k,\lambda), a_j(k',\lambda')] = [a^\dagger_i(k,\lambda), a^\dagger_j(k',\lambda')] = 0. \] (2.11)

Set

\[ a_i(k,\lambda) = \varepsilon_i(k,\lambda)b_i(k,\lambda), \quad a^\dagger_i(k,\lambda) = \varepsilon_i(k,\lambda)b^\dagger_i(k,\lambda), \] (2.12)

and one can readily obtain

\[ [b_i(k,\lambda), b^\dagger_j(k',\lambda')] = \delta^3(k-k')\delta_{ij}, \quad [b_i(k,\lambda), b_j(k',\lambda')] = [b^\dagger_i(k,\lambda), b^\dagger_j(k',\lambda')] = 0. \] (2.13)

Note that in (2.12) the repeated indices \( i \)'s does not imply the summations over them. By the aid of (2.8) and (2.12), one can arrive at

\[ a^\dagger(k,1)a(k,2) - a^\dagger(k,2)a(k,1) = (e_if_j - e_jf_i)b^\dagger_i(k)b_j(k) - b^\dagger_j(k)b^\dagger_i(k) b_i(k)] \]

\[ + (e_jf_k - e_kf_j)b^\dagger_i(k)b_j(k) - b^\dagger_j(k)b^\dagger_i(k) b_i(k) + (e_kf_i - e_if_k)b^\dagger_i(k)b_j(k) - b^\dagger_j(k)b^\dagger_i(k) b_i(k)]. \] (2.14)

For convenience, the photon momentum \( k \) can be expressed in terms of angle displacements \( \lambda \) and \( \gamma \) in the spherical coordinate system, i.e., \( \kappa = (\sin \lambda \cos \gamma, \sin \lambda \sin \gamma, \cos \lambda). \) So,

\[ e_if_j - e_jf_i = \cos \lambda, \quad e_jf_k - e_kf_j = \sin \lambda \cos \gamma, \quad e_kf_i - e_if_k = \sin \lambda \sin \gamma. \] (2.15)
The projection of photon spin $\mathbf{S}$ onto the photon momentum $\mathbf{k}$ is therefore written

$$I \equiv \frac{\mathbf{k} \cdot \mathbf{S}}{k} = -i[a^\dagger(k,1)a(k,2) - a^\dagger(k,2)a(k,1)] = -i\{\cos \lambda[b^\dagger_i(k)b_j(k) - b^\dagger_j(k)b_i(k)] + \sin \lambda \cos \gamma[b^\dagger_k(k)b_j(k) - b^\dagger_j(k)b_k(k)] + \sin \lambda \sin \gamma[b^\dagger_k(k)b_i(k) - b^\dagger_i(k)b_k(k)]\},$$

(2.16)

which is referred to as the photon helicity. It follows from (2.16) that the spin operator in the comoving frame projected onto $\mathbf{k}$ is just equal to the projection of the following operator vector

$$\mathbf{S}_{\text{fix}} = -ib^\dagger_j(k)b_k(k) - b^\dagger_k(k)b_j(k), \quad b^\dagger_k(k)b_j(k) - b^\dagger_j(k)b_k(k), \quad b^\dagger_i(k)b_j(k) - b^\dagger_j(k)b_i(k)$$

(2.17)

onto $\mathbf{k}$. It is thus concluded without any fear that the operator vectors $\mathbf{S}_{\text{fix}}$ can be considered the spin operator vectors in the fixed frame. Moreover, it is readily verified that $\mathbf{S}_{\text{fix}}$ agrees with the algebraic commuting relation of angular momentum operators, i.e.,

$$\mathbf{S}_{\text{fix}} \times \mathbf{S}_{\text{fix}} = i\mathbf{S}_{\text{fix}},$$

(2.18)

which confirms that $\mathbf{S}_{\text{fix}}$ in (2.17) is truly an expression for photon spin operator in the fixed frame of reference. Thus, with the help of

$$[S_k, \frac{1}{\sqrt{2}}(b^\dagger_i \pm i b^\dagger_j)] = [-ib^\dagger_jb_j(b^\dagger_i \pm i b^\dagger_j), \frac{1}{\sqrt{2}}(b^\dagger_i \pm i b^\dagger_j)] = \pm \frac{1}{\sqrt{2}}(b^\dagger_i \pm i b^\dagger_j),$$

(2.19)

the eigenstates to the eigenvalue equations $S_k|\pm, k > = \pm|\pm, k >$ of spin operator $S_k = -i(b^\dagger_ib_j - b^\dagger_jb_i)$ is obtained as follows

$$|\pm, k > = \frac{1}{\sqrt{2}}(b^\dagger_i(k) \pm ib^\dagger_j(k))|0 >.$$

(2.20)

It is worthwhile to point out that the photon helicity, $\frac{\mathbf{k}}{k} \cdot \mathbf{S}$, is a conserved operator, since it follows from (2.16) that the eigenvalue of $\frac{\mathbf{k}}{k} \cdot \mathbf{S}$ does not vary with time $t$ whether it is observed from comoving or fixed frames. This, therefore, means that the photon helicity $\frac{\mathbf{k}}{k} \cdot \mathbf{S}$ can be thought of a Lewis-Riesenfeld invariant operator [4]. Here it is also implied that if the noncoplanarly curved optical fiber is wound smoothly on a large enough diameter, then the wave vector of a photon propagating inside the fiber is always along the tangent of fiber at each point at arbitrary time.

In accordance with the Lewis-Riesenfeld theory [4], the Lewis-Riesenfeld invariant, $I(t) = \frac{\mathbf{k}}{k} \cdot \mathbf{S}$, satisfies the following Liouville-Von Neumann equation

$$\frac{\partial I(t)}{\partial t} + \frac{1}{i}[I(t), H_{\text{eff}}(t)] = 0.$$ 

(2.21)

It follows from the above Liouville-Von Neumann equation that one can obtain the effective Hamiltonian $H_{\text{eff}}(t)$ as follows

$$H_{\text{eff}}(t) = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} \cdot \mathbf{S}$$

(2.22)

with dot denoting the time derivative. Apparently, insertion of (2.22) into (2.21) yields an equation of motion of a photon

$$\dot{\mathbf{k}} + \mathbf{k} \times \left(\frac{\mathbf{k} \times \dot{\mathbf{k}}}{k^2}\right) = 0,$$

(2.23)

where $\mathbf{k} \times (\frac{\mathbf{k} \times \dot{\mathbf{k}}}{k^2})$ and $\frac{\mathbf{k} \times \dot{\mathbf{k}}}{k^2}$ may be considered the generalized Lorentz magnetic force (Coriolis force) and the generalized magnetic field strength [5], respectively. It is readily verified that Eq.(2.23) is an identity. Thus it is shown that the above mathematical treatment Eq.(2.21)-Eq.(2.23) for the propagation of photons inside a curved fibre is self-consistent.
In order to investigate the non-cyclic non-adiabatic geometric phases, we should first exactly solve the following time-dependent Schrödinger equation
\[ i \frac{\partial |\sigma, k(t)\rangle}{\partial t} = \frac{k(t) \times \dot{k}(t)}{k^2} \cdot S |\sigma, k(t)\rangle, \]  
which governs the time evolution of photon fields propagating inside the noncoplanarly curved optical fibre [1,8], where \( \sigma \) represents the helicity eigenvalue, i.e., the eigenvalue of the invariant \( I(t) = \frac{k}{c} \cdot S \).

Note that neither the Hamiltonian nor the consequent time-evolution equation similar to (2.24) existed in the Chiao and Wu’s original work [6]. Instead, there was the following eigenvalue equation
\[ \frac{k}{k} \cdot S |\sigma, k(t)\rangle = \sigma |\sigma, k(t)\rangle. \]  
The connections and differences between (2.24) and (2.25) were seen in references [1,2]. By making use of Berry’s cyclic adiabatic geometric phase formula [3] and Eq.(2.25), Chiao and Wu predicted successfully the existence of photon Berry’s phases in the fibre [6]. In our previous work [1,2], we studied Eq.(2.24) (which governs the second-quantized time-dependent spin model) by using the invariant theory and then treated the non-cyclic non-adiabatic geometric phases of photons in the curved fibre. Moreover, by considering the non-normal-order spin operators (and hence the effective Hamiltonian), we calculate the quantum-vacuum geometric phases of photon fields, which may be a new vacuum effects. It is believed that this geometric phase at quantum-vacuum level is essentially significant not only experimentally but also theoretically, which will be illustrated in the Concluding Remarks.

By complicated lengthy calculations, the solution to the time-dependent Schrödinger equation (2.24) is obtained as follows [1]
\[ |k(t)\rangle = \sum_{\sigma} C_\sigma \exp\left[\frac{1}{i} \frac{\phi^{(g)}_\sigma(t)}{\sigma} \right] V(t) |\sigma, k\rangle, \]  
where \( |\sigma, k\rangle \equiv |\sigma, k(t = 0)\rangle \) is the initial photon polarized state, and the time-independent coefficients \( C_\sigma = \langle \sigma, t = 0 | \sigma, k(t = 0) \rangle \) and \( V(t) = \exp[\beta(t) S_+ - \beta^*(t) S_-] \) with the time-dependent parameters \( \beta(t) = -\frac{\lambda(t)}{2} \exp[-i\gamma(t)], \beta^*(t) = -\frac{\lambda(t)}{2} \exp[i\gamma(t)] \). The geometric phase of photons whose initial helicity eigenvalue is \( \sigma \) can be expressed by
\[ \phi^{(g)}_\sigma(t) = \left\{ \int_0^t \dot{\gamma}(t') [1 - \cos \lambda(t')] dt' \right\} \langle \sigma, k | S_3 | \sigma, k \rangle. \]  
In the adiabatic process where both the precessional frequency \( \dot{\gamma} \) (expressed by \( \Omega \)) and \( \lambda \) (\( k \) deviating from the third axis in the fixed frame by an angle \( \lambda \)) can be regarded as constants, the adiabatic geometric phase (i.e., Berry’s topological phase) in a cycle \( (T = \frac{2\pi}{\Omega}) \) in the momentum \( k \) space is written
\[ \phi^{(g)}_\sigma(T) = 2\pi(1 - \cos \lambda) \langle \sigma, k | S_3 | \sigma, k \rangle, \]  
where \( 2\pi(1 - \cos \lambda) \) is equal to a solid angle subtended at the origin of momentum \( k \) space. This fact thus means that the geometric phase (2.27) or (2.28) carries information on the global and topological properties of time evolution of quantum systems. Geometric phases is hence of physical interest in a wide variety of fields. It should be emphasized that here the quantum-vacuum geometric phase may also be involved in \( \langle \sigma, k | S_3 | \sigma, k \rangle \) if the third component \( S_3 \) of photon spin operator \( S \) is of a non-normal-order form, which will be taken into account in the next section.

Here we omit the derivation of solving Eq.(2.24) and only quote the results. For the detailed derivation procedure the reader is directed to the reference [1].

### III. FURTHER DISCUSSING QUANTUM-VACUUM GEOMETRIC PHASES OF PHOTONS IN THE CURVED FIBRE

In this section we will further discuss the quantum-fluctuation contributions to geometric phases of photons moving inside a sufficiently perfect optical fibre. In the previous section, I deal only with the normal-order photon spin operators (2.3), which does not involve the vacuum zero-point electromagnetic fluctuation fields. So, the previous mathematical treatment cannot predict the existence of quantum-vacuum geometric phases of photons in the curved
fibre. In order to treat the so-called quantum-vacuum geometric phases, we should study the non-normal-order spin operators, where the zero-point electromagnetic fields is involved in the effective Hamiltonian (2.22) (and hence in the time-evolution equation (2.24)).

Readers may be referred to the references [6,7,11–19] for the early investigations of adiabatic cyclic geometric phases (Berry’s topological phases) of photons inside a curved fibre. Historically, the similar work has also been seen in a remarkable paper published in 1941 in which Vladimirskii had treated this global topological phase problem in an extension of an earlier paper published in 1938 by Rytov [19]. In all these researches, authors treated the photon geometric phases in the coiled optical fibre by making use of the Maxwell’s electrodynamics, differential geometry method as well as Berry’s adiabatic quantum theory. However, by constructing a second-quantized effective Hamiltonian, we considered the non-cyclic non-adiabatic (rather than cyclic adiabatic) geometric phases of photons inside a noncoplanarly curved fibre [1,2,8] based on the Lewis-Riesenfeld invariant theory [4] and the invariant-related unitary transformation formulation [20]. In the paper [1], we exactly solved some time-dependent quantum models by using the invariant theory and then studied in detail the time-evolution operator of photon wavefunction in the curved fibre (here the wavefunction time-evolution operator is an exact solution to the time-dependent Schrödinger equation, rather than that associated with the chronological product). In the paper [2], we investigated both non-adiabatic geometric phases and helicity reversals (as well as some related topics) of photons propagating inside the coiled fibre and briefly considered the potential applications of photon helicity inversion to the communication and information science. In this published work [2], we also suggested the quantum-vacuum geometric phases of photons in the fibre\(^2\), which is a physically interesting concept and might perhaps focus attention of researchers in various fields, where the quantum systems are of second quantization.

Substituting the Fourier expansion series (2.6) of \(A(x,t)\) into the expression (2.1) for photon spin operator, one can obtain the non-normal-order photon \(S\) [1], i.e.,

\[
S = \frac{i}{2} \kappa \left[a(k,1)a^\dagger(k,2) - a^\dagger(k,1)a(k,2) - a(k,2)a^\dagger(k,1) + a^\dagger(k,2)a(k,1)\right].
\]

(3.1)

In what follows we define the creation and annihilation operators, \(a_R^\dagger(k), a_L^\dagger(k), a_R(k), a_L(k)\), of right- and left-handed circularly polarized light \([9]\)

\[
a_R^\dagger(k) = \frac{1}{\sqrt{2}}[a^\dagger(k,1) + ia^\dagger(k,2)], \quad a_R(k) = \frac{1}{\sqrt{2}}[a(k,1) - ia(k,2)],
\]

\[
a_L^\dagger(k) = \frac{1}{\sqrt{2}}[a^\dagger(k,1) - ia^\dagger(k,2)], \quad a_L(k) = \frac{1}{\sqrt{2}}[a(k,1) + ia(k,2)].
\]

(3.2)

It follows that

\[
a^\dagger(k,1) = \frac{1}{\sqrt{2}}[a_R^\dagger(k) + a_L^\dagger(k)], \quad a(k,1) = \frac{1}{\sqrt{2}}[a_R(k) + a_L(k)],
\]

\[
a^\dagger(k,2) = \frac{1}{\sqrt{2}i}[a_R^\dagger(k) - a_L^\dagger(k)], \quad a(k,2) = -\frac{1}{\sqrt{2}i}[a_R(k) - a_L(k)].
\]

(3.3)

So, the monomode-photon spin operator (3.1) can be rewritten

\(^2\)As far as we are concerned, the photon propagation problem can be ascribed to a time-dependent second-quantized spin model, where the effective (phenomenological) Hamiltonian (e.g., the expression (2.22)) is of a second-quantized form. Whereas in the previous researches [6,7,11–19], this problem was treated often by using classical Maxwell’s Equations and first-quantized Schrödinger equation (and Berry’s adiabatic geometric phase formula as well [3]). Although these investigations can be said to be somewhat outstandingly successful in both predicting and studying adiabatic geometric phases of photons in the fibre, here I still want to emphasize two points: for one thing, only by using the second-quantization formulation can we investigate the photon geometric phases at quantum level; for another, only when we consider the non-normal-product second-quantized Hamiltonian can it enable us to predict the existence of geometric phases at quantum-vacuum level. Tomita and Chiao may also agree to the above first point. They held the arguments [18] that although the geometric phases in the curved fibre can also be obtained by means of classical Maxwell’s electrodynamics, they preferred to think of this phenomenon as originating at the quantum level, but surviving the correspondence-principle limit into the classical level. However, this point is not the main subject in the present paper, which will be further discussed elsewhere. Here, instead, we concentrate only on the second point, i.e., the geometric phases at quantum-vacuum level resulting from the zero-point radiation fields of vacuum, which has not been investigated in previous researches.
\[ S = \frac{1}{2k} \left\{ [a_R(k)a_R^\dagger(k) + a_L^\dagger(k)a_R(k)] - [a_L(k)a_L^\dagger(k) + a_R^\dagger(k)a_L(k)] \right\}. \] (3.4)

Thus, according to the definition of photon helicity \( I(t) \equiv \frac{\mathbf{k} \cdot \mathbf{S}}{k} \), \( I(t) \) is given by

\[ I(t) = \frac{1}{2} \left\{ [a_R(k)a_R^\dagger(k) + a_L^\dagger(k)a_R(k)] - [a_L(k)a_L^\dagger(k) + a_R^\dagger(k)a_L(k)] \right\}. \] (3.5)

Hence we can construct the left- and right- handed photon states as follows

\[ |\sigma = -1, k\rangle = a_L^\dagger(k)|0\rangle, \quad |\sigma = +1, k\rangle = a_R^\dagger(k)|0\rangle, \] (3.6)

where the two helicity eigenvalue equations are satisfied

\[ I(t)|\sigma = -1, k\rangle = -|\sigma = -1, k\rangle, \quad I(t)|\sigma = +1, k\rangle = +|\sigma = +1, k\rangle. \] (3.7)

Note that the above discussion is performed in the comoving frame of reference. In what follows, we consider the spin operator in a fixed frame. In accordance with the invariant-related unitary transformation formulation [8,1], the time-dependent invariant \( I(t) \) can be transformed into a time-independent operator

\[ I_V \equiv V(t)I(t)V(t) = S_3, \] (3.8)

where \( V(t) = \exp[\beta(t)S_+ - \beta^*(t)S_-] \) with \( \beta(t) = -\frac{\lambda(t)}{2} \exp[-i\gamma(t)], \quad \beta^*(t) = -\frac{\lambda(t)}{2} \exp[i\gamma(t)] \). Here \( \lambda(t) \) and \( \gamma(t) \) are angle displacements in the spherical coordinate system and are so defined

\[ \frac{k(t)}{\lambda(t)} = (\sin \lambda(t) \cos \gamma(t), \sin \lambda(t) \sin \gamma(t), \cos \lambda(t)) \] as mentioned in Sec. II. Since \( V(t = 0) = 1 \), we obtain \( \lambda(t = 0) = 0 \) and hence \( k_3 = k \) and \( k_1 = k_2 = 0 \). Thus in the Schrödinger picture, in which the (initial) photon momentum \( k_3 = k \)

and \( k_1 = k_2 = 0 \), the third component of photons spin operator is therefore of the form

\[ S_3 = \frac{1}{2} \left\{ [a_R(k)a_R^\dagger(k) + a_L^\dagger(k)a_R(k)] - [a_L(k)a_L^\dagger(k) + a_R^\dagger(k)a_L(k)] \right\}. \] (3.9)

It can be found from (2.22) and (3.9) that the time-dependent zero-point energy exists in the effective Hamiltonian, which will result in the quantum-vacuum geometric phases. These aspects are illustrated in the application of (3.9) to the LRH geometric phase formulae which follow. It should be noted that here \( a_R(k), a_R^\dagger(k), a_L \) and \( a_L^\dagger(k) \) are regarded as the time-independent operators.\(^3\)

The monomode multi-photon states of left- and right- handed (LRH) circularly polarized light (at \( t = 0 \)) can be defined

\[ |\sigma = -1, k, n_L\rangle = \frac{[a_L^\dagger(k)]^n}{\sqrt{n!}}|0_L\rangle, \quad |\sigma = +1, k, n_R\rangle = \frac{[a_R^\dagger(k)]^n}{\sqrt{n!}}|0_R\rangle \] (3.10)

with \( n_L \) and \( n_R \) being the LRH polarized photon occupation numbers, respectively. In the following we calculate the geometric phases of multi-photon states

\[ |\sigma = +1, k, n_R; \sigma = -1, k, n_L\rangle \equiv |\sigma = +1, k, n_R\rangle \otimes |\sigma = -1, k, n_L\rangle \] (3.11)

in the fibre. Substitution of (3.11) into (2.27) yields

\(^3\)Since in Sec. II we have discussed the photon spin operators projected onto the fixed frame of reference, it follows from (3.1) that the time-independent (i.e., in the fixed frame or Schrödinger picture) third component of spin \( S_3 \) is written as \( S_3 = \frac{1}{2} [b_1(k)b_2^\dagger(k) - b_1^\dagger(k)b_2(k)] - b_2(k)b_1^\dagger(k) + b_1^\dagger(k)b_1(k) \) (non-normal-order). So, the time-independent creation and annihilation operators of right- and left- handed circularly polarized light are respectively defined to be \( a_R^\dagger(k) = \frac{1}{\sqrt{2}}[b_1^\dagger(k) + ib_2^\dagger(k)], \quad a_R(k) = \frac{1}{\sqrt{2}}[b_1(k) - ib_2(k)] \) and \( a_L^\dagger(k) = \frac{1}{\sqrt{2}}[b_1^\dagger(k) - ib_2^\dagger(k)], \quad a_L(k) = \frac{1}{\sqrt{2}}[b_1(k) + ib_2(k)] \). The monomode-photon states corresponding to helicity eigenvalues \( \sigma = \pm 1 \) are therefore constructed in terms of \( b_1^\dagger(k), b_2^\dagger(k), b_1(k), b_2(k) \) and vacuum state \( |0\rangle \) as follows:

\[ |\sigma = \pm 1, k\rangle \equiv |a_R^\dagger(k) = \frac{1}{\sqrt{2}}[b_1^\dagger(k) \pm ib_2^\dagger(k)]|0\rangle. \] Actually, according to the analysis in Sec. II, the definition of left- and right- handed polarized photon states in (3.6) should be replaced with the above-defined \( |\sigma = \pm 1, k\rangle = \frac{1}{\sqrt{2}}[b_1^\dagger(k) \pm ib_2^\dagger(k)]|0\rangle \). But, for convenience, here we do not make a difference between them, since under the initial condition \( k_3 = k, k_1 = k_2 = 0 \), these two definitions of monomode-photon polarized states corresponding to helicity eigenvalues \( \sigma = \pm 1 \) are consistent with each other.
\[
\phi^{(g)}(t) = \sum_{\sigma = \pm 1} \left\{ \int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \right\} \langle \sigma = +1, k, n_R; \sigma = -1, k, n_L | S_3 | \sigma = +1, k, n_R; \sigma = -1, k, n_L \rangle. 
\] (3.12)

and the final result is given
\[
\phi^{(g)}(t) = (n_R - n_L) \left\{ \int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \right\}, 
\] (3.13)

which is independent of \( k \) but dependent instead on the geometric nature of the pathway (expressed in terms of \( \lambda \) and \( \gamma \)) along which the light wave propagates. This fact indicates that geometric phases possess the topological and global properties of time evolution of quantum systems. It is worth noticing that the phases \( \phi^{(g)}(t) \) in \( n_R \) and \( n_L \) are quantal in character [8]. Gao has shown why \( \phi^{(g)}(t) = (n_R - n_L)\int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \) is referred to as the quantal geometric phases [8] by taking into account the uncertainty relation between the operators \( \frac{1}{2}[a^\dagger_{R(L)}(k) + a_{R(L)}(k)] \) and \( \frac{1}{2}[a^\dagger_{R(L)}(k) - a_{R(L)}(k)] \). Although the phases \( \phi^{(g)}(t) \) in \( n_R \) are quantal geometric phases of photons, they do not belong to the geometric phases at quantum-vacuum level which arise, however, from the zero-point electromagnetic energy of vacuum quantum fluctuation.

It should be noted that the cyclic adiabatic cases of \( (3.13) \) \( (n_R\int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \) and \( -n_L\int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \) alone have been measured experimentally by Tomita and Chiao et al. [7,11–13].

According to the expression (3.9) for \( S_3 \), both the geometric phases of left- and right-handed circularly polarized photon states, \( |\sigma = -1, k, n_L \rangle \) and \( |\sigma = +1, k, n_R \rangle \), are respectively of the form
\[
\phi^{(g)}_L(t) = -(n_L + \frac{1}{2}) \left\{ \int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \right\}, \quad \phi^{(g)}_R(t) = +(n_R + \frac{1}{2}) \left\{ \int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \right\}. 
\] (3.14)

It follows that the time-dependent zero-point energy possesses physical meanings and therefore contributes to geometric phases of photon fields. Thus the non-cyclic non-adiabatic geometric phases of left- and right-handed polarized states at quantum-vacuum level are given
\[
\phi^{(\text{vacuum})}_{\sigma = \pm 1}(t) = \pm \frac{1}{2} \left\{ \int_0^t \gamma(t')[1 - \cos \lambda(t')]dt' \right\}. 
\] (3.15)

Note that the geometric phases expressed in \( (3.15) \) possess quantal belongings and, moreover, has no classical counterpart (or correspondence), namely, it cannot survive the correspondence-principle limit into the classical level.

Compared \( (3.13) \) with \( (3.15) \), it is easily seen that geometric phases \( (3.13) \) arises from the real photons (having close relation to the occupation numbers of left- and right-handed polarized photons), while the quantum-vacuum phases \( (3.15) \) results from the vacuum electromagnetic fluctuation (virtual photons). This quantum-vacuum nature in the latter case makes \( \phi^{(\text{vacuum})}_{\sigma = \pm 1}(t) \) more physically interesting and therefore the experimental observation for \( \phi^{(\text{vacuum})}_{\sigma = \pm 1}(t) \) deserves consideration.

However, it should be pointed out that, unfortunately, even at the quantum level, this observable quantum-vacuum geometric phases \( \phi^{(\text{vacuum})}_{\sigma = \pm 1}(t) \) is absent in the fibre experiment, since it follows from \( (3.14) \) and \( (3.15) \) that the signs of quantal geometric phases of left- and right-handed circularly polarized photons are just opposite to one another, and so that their quantum-vacuum geometric phases are counteracted by each other. Hence the observed geometric phases are only those expressed by \( (3.13) \), of which whose adiabatic case has been measured in the optical fibre experiment performed by Tomita and Chiao et al. [7,11–13].

It is of important significance to investigate the structures and properties of vacuum in field theory, which has attracted considerable attention of many investigators, as evidenced by a considerable amount of literature. Vacuum possesses many effects, such as Casimir’s effect [21] reflecting the zero-point energy of vacuum, vacuum polarization leading to Lamb’s shift (hyperfine structure of Hydrogen atomic spectra) [22], atomic spontaneous radiation due to the interaction between the excited atom and the zero-point electromagnetic field, and the anomalous magnetic moment of electron as well. Vacuum, as the ground states of quantum fields, has universal properties of symmetry. Increasing evidences such as the facts that Nambu and Goldstone discovered the vacuum spontaneously symmetrical breaking [23] of field theory in 1960’s, and Polyakov et al. found the topological structures of vacuum [24] in 1970’s, have demonstrated that vacuum possesses abundant properties and deserve detailed investigations. Field theory ever encounters problems such as divergent zero-point energies of quantized electromagnetic fields and infinite electric
charge density arising from the presence of electrons of negative energies. These problems can be solved by taking the normal order for the field operators, which is consistent with Lorentz covariance. The background charges and zero-point energies is thus removed, which makes the vacuum expectation values of both charge density and Hamiltonian vanish (namely, the infinite constant is harmless and easily removed by measuring all energies relative to the vacuum state). In such systems of field theory, of which whose Hamiltonian is time-independent, the same amount of zero-point energy is eliminated at different time, which is equivalent to re-defining the background energies. Hence, in these cases the normal-product procedure is both reliable and practical. However, whether the normal product is valid or not should still be taken into consideration for systems with time-dependent Hamiltonians. Since the time-dependent system is no longer Lorentz-invariant, evidences for the validity of normal product are insufficient so far. In other words, if the normal product is applied to the time-dependent systems of quantum field theory, and thus the vacuum background is so re-defined by removing different zero-point energies at different time, then some observable vacuum effects (e.g., Berry’s phase) may be cancelled theoretically and the validity of this formalism therefore deserves incredulity. In view of the above remarks, it is emphasized that investigations of vacuum states of time-dependent systems may become particularly important. So, we think the test of the above quantum-vacuum geometric phases is now of imperative necessity.

Additionally, why do we say the geometric phase (3.15) has physical meanings? The reasons are as follows: for the first, it is a time-dependent phase. It is well known that only the time-dependent phase factor in quantum mechanical wavefunction has physical meanings. In other words, the time-independent or constant phase factor lacks physical meanings; for the second, this phase (3.15) is caused by the zero-point fluctuation fields of vacuum. So, it receives our much attention.

Unfortunately, since the zero-point LRH polarized electromagnetic fields are present often accompanied by each other in the isotropic-media curved fibre, as a matter of fact the quantum-vacuum geometric phases (3.15) cannot be detected easily in the previous fiber experiments [7,11–13]. Although their total effects is vanishing (i.e., $\phi_L^{(\text{vacuum})}(t)+\phi_R^{(\text{vacuum})}(t) = 0$), the quantum-vacuum geometric phase of left- or right-handed circularly polarized photon state still truly exists (but just be cancelled by each other). The problem now we encounter is: how can we extract experimentally one of the non-vanishing quantum-vacuum geometric phases from the vanishing $\phi_L^{(\text{vacuum})}(t)+\phi_R^{(\text{vacuum})}(t)$?

We think this may prove to the physicists that the resolution of this problem is truly imperative necessary. The relevant discussions and remarks are presented in the Appendix to this paper.

IV. CONCLUDING REMARKS

In this paper, we deal with the projection problem of spin operator of photon fields in the comoving coordinate systems onto the fixed frame of reference, and discuss further the quantum-vacuum geometric phases in the optical fibre. It is shown that the non-normal order of photon spin operators yields observable effects (e.g., quantum-vacuum geometric phases) arising from zero-point vacuum energies. These two vacuum geometric phases are exactly equal but different only by a minus sign. This, therefore, implies that the total LRH quantum-vacuum geometric phases become exactly zero. So, the only retained geometric phases are those in (3.13), which result from the occupation numbers of LRH photons in polarized states.

The physical significance of quantum-vacuum geometric phases of photon fields in the curved fiber may be given as follows:

(i) the photon geometric phases at quantum-vacuum level originates from the zero-point electromagnetic fluctuations. Since geometric phases indicates topological and global properties of quantum systems in time-evolution processes, the quantum-vacuum geometric phases of photons in the helically wound fibre may contain the information

\footnote{It follows from (3.9) that the vanishing total LRH quantum-vacuum geometric phases $\phi_L^{(\text{vacuum})}(t)+\phi_R^{(\text{vacuum})}(t)$ is related close to the fact that the total angular momentum of zero-point electromagnetic fields is vanishing. Actually, the two problems have quite a lot in common. From the physical point of view, the total spin of LRH radiation fields of vacuum vanishes, so does the total LRH quantum-vacuum geometric phases.}
on the topological and global properties of time evolution of vacuum fluctuation fields and (we hope it) might capture
attention in a wide variety of fields;

(ii) quantum-vacuum geometric phases may be considered a new physically interesting vacuum effects. It is known
that Casimir’s effect is realized by changing the mode structure of zero-point electromagnetic fields between the two
parallel conducting metal plates [21] (i.e., by modifying the magnitude distribution of electromagnetic wave vector
\( k \)). However, the quantum-vacuum geometric phases of photons is produced by altering the direction of wave vector
\( k \) in the curved fibre (i.e., photon fields itself alters its wave vector \( k \) when travelling along the curved fibre);

(iii) the problem of whether the normal-product procedure is valid or not for time-dependent quantum systems
still remains unclear so far. We believe that the consideration of this problem may enable us to investigate the time-
dependent quantum field theory. If, for example, when we study the quantum field theory in the curved space-time,
where one should often deal with the particle creation problem in the time-dependent gravitational backgrounds
[25,26], we think the vacuum effects associated with evolution process of time-dependent quantized field systems may
necessarily also be taken into consideration. So, from the point of view of us, it is important to test experimentally
the quantum geometric phases so as to answer the question mentioned above. If we cannot find the existence of this
geometric phases at quantum-vacuum level (i.e., this vacuum effect does not exist), then it is believed that the normal-
product procedure in second quantization is still valid and correct for time-dependent quantum systems. But, if the
quantum-vacuum geometric phases is present experimentally, then we argue that the normal-product procedure in
second quantization may be invalid and should be discussed further when treating time-dependent quantum systems.
We think it might be a leading problem in both quantization formulation and vacuum physics. This problem is under
consideration and will be published elsewhere. We are longing to perform a relevant experiment to confirm our above
interpretations.

However, it is most unfortunate that the two quantum-vacuum geometric phases of LRH polarized photons for each
\( k \) are eliminated by each other, and furthermore no one among them appears to want to be readily extracted either.

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APPENDIX
Can We Extract the Quantum-vacuum Geometric Phases from NOTHING?

It follows from (3.15) that the total vacuum phases (i.e., \( \phi^{(\text{vacuum})}_L(t) + \phi^{(\text{vacuum})}_R(t) \)) vanishes. Here our aim
is to resolve the problem of how to extract experimentally the quantum-vacuum geometric phases \( \phi^{(\text{vacuum})}_L(t) \) or
\( \phi^{(\text{vacuum})}_R(t) \) from the vanishing total vacuum geometric phases.

Our brief history of investigating photon geometric phases in the fibre is as follows: in April 2000, Gao and I
began to consider the non-cyclic non-adiabatic geometric phases of photon fields in the curved fibre based on a
second-quantized spin model (see spin model, for example, in references [27–31]). In May 2000, Gao first proposed
the concept of quantum-vacuum geometric phases. The existence problem of quantum-vacuum geometric phases
is strongly relevant to whether the second quantization in spin model is adopted or not. Since it is in connection with
properties of quantum electromagnetic vacuum and, moreover, this geometric phase is related close to the topological
and global features of time evolution of vacuum-fluctuation fields, we think this concept is of essential significance
and therefore deserves detailed investigations. From then on, these problems gained our attention and we tried to
investigate this topological phases at quantum-vacuum level.

Since quantum-vacuum geometric phases has an important connection with vacuum energies, these experimental
realizations may be relevant to the validity problem of normal-product procedure in the time-dependent quantum
field theory (TDQFT), i.e., we also aim to re-examine the normal-product procedure in some extensions. If the
quantum-vacuum geometric phases is proved present experimentally, then it is reasonably believed that it is not
suitable for us to remove vacuum fluctuation energies and infinite charge density just by using the old formulation,
e.g., re-defining the vacuum background energies and electric charges by utilizing the normal-product procedure, since
in this re-definition, some potential physically interesting vacuum effects may also be removed theoretically. We think
only for the time-independent quantum field systems can we use safely the normal-product procedure without any fear
of introducing any new problems other than those which quantum field theory had encountered before [32]. However,
for the time-dependent quantum field systems, (e.g., photon fields propagating inside a helically curved fibre and
quantum fields in an expanding universe), the physically interesting vacuum effects will unfortunately be deduced by the second-quantization normal-order formulation. So, the normal-order technique may therefore not be applicable to the time-dependent quantum fields. To the best of our knowledge, in the literature, this normal order problem in the time-dependent quantum field theory gets less attention and interests than it deserves. To test our above theoretical viewpoints, we hope the quantum-vacuum geometric phases of photons in the curved fibre would be investigated experimentally in the near future.

Unfortunately, the left-handed polarized light due to vacuum fluctuation is often accompanied by the zero-point right-handed polarized light and their total quantum-vacuum geometric phases is therefore vanishing. So, it is not easy for physicists to investigate experimentally the quantum-vacuum geometric phases. This, therefore, means that our above theoretical remarks as to whether the normal-product procedure is valid or not for the time-dependent quantum field theory (TDQFT) cannot be examined experimentally. During the last three years, I tried my best but unfortunately failed to suggest an excellent idea of experimental realization of quantum-vacuum geometric phases. We conclude that it seems not quite satisfactory to test the quantum-vacuum geometric phases by using the optical fibre that is made of isotropic media, inhomogeneous media (e.g., photonic crystals\textsuperscript{5}), left-handed media (a kind of artificial composite metamaterial with negative refractive index), uniaxial (biaxial) crystals or chiral materials. Is it truly extremely difficult to realize such a goal? It is found finally that perhaps in the fibre composed of some anisotropic media (such as gyroelectric and gyromagnetic media, where both electric permittivity and magnetic permeability are respectively the tensors) the quantum-vacuum geometric phases may be achieved test experimentally. In these gyroelectric media, only one of the LRH polarized lights can be propagated without being absorbed by media. This result holds also for the zero-point electromagnetic fields. It is well known that people can manipulate vacuum so as to alter the zero-point mode structures of vacuum, which has been illustrated in photonic crystals and Casimir’s effect (additionally, the space between two parallel mirrors, cavity in cavity QED, etc.). If, for example, in some certain gyroelectric media one of the LRH polarized lights, say, the left-handed polarized light, dissipates due to the medium absorption and only the right-handed light is allowed to be propagated (in the meanwhile the mode structure of vacuum in these anisotropic media also alters correspondingly), then the quantum-vacuum geometric phase of right-handed polarized light can be easily tested in the fibre fabricated from these gyroelectric media.

To close this Appendix, we consider briefly the electromagnetic wave equations in a homogeneous gyroelectric medium, where the electric permittivity tensor, $(\epsilon)_{ik}$, and magnetic permeability (scalar) are given

$$(\epsilon)_{ik} = \left( \begin{array}{ccc} \epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{array} \right), \quad \mu = \mu. \tag{A1}$$

Assuming that the direction of the electromagnetic wave vector $k$ is parallel to the third component of the Cartesian coordinate system, with the help of Maxwell’s Equations, we obtain the following two planar wave equations

$$\nabla^2 \left[ \frac{1}{\sqrt{2}} (E_1 + iE_2) \right] - \epsilon_0 \mu_0 \mu (\epsilon_1 + \epsilon_2) \frac{\partial^2}{\partial t^2} \left[ \frac{1}{\sqrt{2}} (E_1 + iE_2) \right] = 0,$$

$$\nabla^2 \left[ \frac{1}{\sqrt{2}} (E_1 - iE_2) \right] - \epsilon_0 \mu_0 \mu (\epsilon_1 - \epsilon_2) \frac{\partial^2}{\partial t^2} \left[ \frac{1}{\sqrt{2}} (E_1 - iE_2) \right] = 0, \tag{A2}$$

of which whose optical refractive indices squared take the form

$$n^2_\pm = \mu (\epsilon_1 \pm \epsilon_2), \tag{A3}$$

where the sign $\pm$ corresponds to the two directions of polarization vectors of light wave.

For simplicity, without loss of generality, it is assumed that the two mutually perpendicular real unit polarization vectors $\varepsilon(k, 1)$ and $\varepsilon(k, 2)$ can be taken to be as follows: $\varepsilon_1(k, 1) = \varepsilon_2(k, 2) = 1, \varepsilon_1(k, 2) = \varepsilon_2(k, 1) = 0$ and

\textsuperscript{5}Photonic crystals are artificial materials patterned with a periodicity in dielectric constant, which can create a range of forbidden frequencies called a photonic band gap. Such dielectric structure of crystals offers the possibility of molding the flow of light (including the zero-point electromagnetic fields of vacuum). It is believed that, in the similar fashion, this effect (i.e., modifying the mode structures of vacuum electromagnetic fields) may also take place in gyroelectric media. This point will be applied to the discussion which follows.
\(\varepsilon_3(k, 1) = \varepsilon_3(k, 2) = 0\). Thus by the aid of (2.2), (3.2) and the formula \(E = -\frac{4A}{\epsilon}t\) for the electric field strength, in the second-quantization framework one can arrive at

\[
\frac{1}{\sqrt{2}}(E_1 + iE_2) = i \int d^3k \frac{\omega}{2(2\pi)^3} [a_L(k) \exp(-ik \cdot x) - a_R^\dagger(k) \exp(ik \cdot x)],
\]

\[
\frac{1}{\sqrt{2}}(E_1 - iE_2) = i \int d^3k \frac{\omega}{2(2\pi)^3} [a_R(k) \exp(-ik \cdot x) - a_L^\dagger(k) \exp(ik \cdot x)],
\]

We now discuss the wave propagation in a gyroelectric-medium optical fibre. As an illustrative example, we think of experiments. It is accepted by us that in resolving this problem (i.e., in the second-quantization framework one can arrive at negative. Thus in the former case the phase \(\phi_1\), which is expressed by \((n_\perp)^2\) in the fibre; conversely, \(\frac{1}{\sqrt{2}}(E_1 - iE_2)\) can be propagated while \(\frac{1}{\sqrt{2}}(E_1 + iE_2)\) is inhibited from being propagated (due to the imaginary propagation constant \(k_\perp\)).

Again, on the condition under which \(\varepsilon_2 > \varepsilon_1, \mu > 0\), if, for instance, \(\varepsilon_2\) is positive, then \(\frac{1}{\sqrt{2}}(E_1 + iE_2)\) can be propagated while \(\frac{1}{\sqrt{2}}(E_1 - iE_2)\) cannot be propagated (because of the negative \(n_\perp^2\) and the consequent imaginary propagation constant \(k_\perp\), which is expressed by \((n_\perp)^2\)) in the fibre; conversely, \(\frac{1}{\sqrt{2}}(E_1 - iE_2)\) can be propagated while \(\frac{1}{\sqrt{2}}(E_1 + iE_2)\) is inhibited from being propagated (due to the imaginary propagation constant \(k_\perp\)) if \(\varepsilon_2\) is negative. Thus in the former case the phase \(\phi^{(\text{vacuum})}_R(t)\) of right-handed polarized light and in the latter case, instead the phase \(\phi^{(\text{vacuum})}_L(t)\) of left-handed polarized light, may respectively be detected in the gyroelectric-medium fibre experiments. It is accepted by us that in resolving this problem (i.e., experimental realizations of quantum-vacuum geometric phases) it is necessarily of much more practical importance to think of gyrotropic media than of any other inhomogeneous and chiral materials. We hope the simple analysis (i.e., from (A1) to (A4)) would prove useful to physicists for investigating quantal geometric phases of electromagnetic fields in the fibre (of course, indeed, Eq.(A3) etc. have provided us with some insights into these problems).

We are now in a position to further investigate some new properties of this potential vacuum effect anticipated for such time-dependent zero-point fields of vacuum (or a quantum system in time-dependent environments such as a time-dependent homogeneous electric field [33], time-dependent gravitational backgrounds [34] and expanding space-time), particularly if we could perform some experiments to test the presented viewpoints in Sec. III and the Appendix. We hope this might offer some new insights into the physics of the problems mentioned above (in particular, the validity of normal product in time-dependent quantum field theories should be further clarified).

**NOTE:** In order to illustrate the content in the present paper, an outline is given as follows:

- **Chiao-Wu’s model (photons moving inside a helically curved fibre)**
  - second-quantized spin model
  - Lewis-Riesenfeld invariant theory
  - Berry’s phase formula

- **cyclic adiabatic geometric phase** \(\phi^{(g)}(T) = 2\pi \sigma (1 - \cos \lambda)\)
- **non-cyclic non-adiabatic geometric phase** \(\phi^{(g)}(t) = \sigma \int_0^t \tilde{\gamma}(t') [1 - \cos \lambda(t')] dt'\)

- **quantal geometric phase** \(\phi^{(\text{vacuum})}(t) = (n_R - n_L) \int_0^t \tilde{\gamma}(t') [1 - \cos \lambda(t')] dt'\)
- **quantum-vacuum phases** \(\phi^{(\text{vacuum})}_r(t) = \pm \frac{x}{2} \int_0^t \tilde{\gamma}(t') [1 - \cos \lambda(t')] dt'\)
- **experimental test is required**

- **unfortunately, \(\phi^{(\text{vacuum})}_L(t) + \phi^{(\text{vacuum})}_R(t) = 0\)**

by using gyrotropic-medium fibre

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