ON THE SPECTRUM FOR THE GENERA OF MAXIMAL CURVES OVER SMALL FIELDS

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Abstract. Motivated by previous computations in Garcia, Stichtenoth and Xing (2000) paper [11], we discuss the spectrum $\mathcal{M}(q^2)$ for the genera of maximal curves over finite fields of order $q^2$ with $7 \leq q \leq 16$. In particular, by using a result in Kudo and Harashita (2016) paper [22], the set $\mathcal{M}(7^2)$ is completely determined.

1. Introduction

Let $X$ be a (projective, nonsingular, geometrically irreducible, algebraic) curve of genus $g$ defined over a finite field $K = \mathbb{F}_\ell$ of order $\ell$. The following inequality is the so-called Hasse-Weil bound on the size $N$ of the set $X(K)$ of $K$-rational points of $X$:

$$|N - (\ell + 1)| \leq 2g \cdot \sqrt{\ell}.$$  

In Coding Theory, Cryptography, or Finite Geometry one is often interested in curves with “many points”, namely those with $N$ as big as possible. In this paper, we work out over fields of square order, $\ell = q^2$, and deal with so-called maximal curves over $K$; that is to say, those curves attaining the upper bound in (1), namely

$$N = q^2 + 1 + 2g \cdot q.$$  

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The subject matter of this note is in fact concerning the spectrum for the genera of maximal curves over $K$.

$$(3) \quad M(q^2) := \{ g \in \mathbb{N}_0 : \text{there is a maximal curve of genus } g \text{ over } K \}.$$ 

In Section 2 we subsume basic facts on a maximal curve $X$ being the key property the existence of a very ample linear series $D$ on $X$ equipped with a nice property; namely (5). We have that Castelnuovo’s genus bound (6) and Halphen’s theorem imply a nontrivial restriction on the genus $g$ of $X$, stated in (8) (see [21]); in particular, $g \leq q(q-1)/2$ is the well-known Ihara’s bound [19].

Let $r$ be the dimension of $D$. Then $r \geq 2$ by (5), and the condition $r = 2$ is equivalent to $g = q(q-1)/2$, or equivalent to $X$ being $K$-isomorphic to the Hermitian curve $g^{q+1} = x^q + x$ [30, 9, 20]. Under certain conditions, we have a similar result for $r = 3$ in Corollary 1 and Proposition 1. In fact, in Section 3 we bound $g$ via Stöhr-Voloch theory [27] applied to $D$ being the main results the aforementioned Proposition and its Corollary 2. Finally, in Section 4 we apply all these results toward the computation of $M(q^2)$ for $q = 7, 8, 9, 11, 13, 16$. In fact, here we improve [11, Sect. 6] and, in particular, we can compute $M(7^2)$ (see Corollary 3) by using Corollary 2 and a result of Kudo and Harashita [22] which asserts that there is no maximal curve of genus 4 over $\mathbb{F}_{49}$.

We recall that the approach in this paper is quite different from Danisman and Özdemir [4], where in particular the set $M(7^2)$ is missing.

**Conventions.** $\mathbb{P}^s$ is the $s$-dimensional projective space defined over the algebraic closure of the base field.

## 2. Basic facts on maximal curves

Throughout, let $X$ be a maximal curve of genus $g$ over the field $K = \mathbb{F}_{q^2}$ of order $q^2$. Let $\Phi : X \to X$ be the Frobenius morphism relative to $K$ (in particular, the set of fixed points of $\Phi$ coincides with $X(K)$). For a fixed point $P_0 \in X(K)$, let $j : X \to J, P \mapsto [P - P_0]$ be the embedding of $X$ into its Jacobian variety $J$. Then, in a natural way, $\Phi$ induces a morphism $\tilde{\Phi} : J \to J$ such that

$$(4) \quad j \circ \Phi = \tilde{\Phi} \circ j.$$ 

Now from (2) the numerator of the Zeta Function of $X$ is given by the polynomial $L(t) = (1 + qt)^{2g}$. It turns out that $h(t) := t^{2g}L(t^{-1})$ is the characteristic polynomial of $\tilde{\Phi}$; i.e., $h(\tilde{\Phi}) = 0$ on $J$. As a matter of fact, since $\tilde{\Phi}$ is semisimple and the representation of endomorphisms of $J$ on the Tate module is faithful, from (4) it follows that

$$(5) \quad (q + 1)P_0 \sim qP + \Phi(P), \quad \text{for each } P \in X.$$ 

This suggests to study the Frobenius linear series on $X$, namely the complete linear series $D := [(q + 1)P_0]$ which is in fact a $K$-invariant of $X$ by (5); see [8], [16, Ch. 10] for further information.

Moreover, $D$ is a very ample linear series in the following sense. Let $r$ be the dimension of $D$, which we refer to as the Frobenius dimension of $X$, and $\pi : X \to \mathbb{P}^r$ be a morphism related to $D$; we noticed above that $r \geq 2$ by (5). Then $\pi$ is an embedding [20, Thm. 2.5]. In particular, Castelnuovo’s genus bound applied to $\pi(X)$ gives the following constrain involving the genus $g$, $r$ and $q$ (see [16, Cor.
10.25]):
\[ g \leq F(r) = F(r, q + 1) := \begin{cases} 
   ((2q - (r - 1))^2 - 1)/8(r - 1) & \text{if } r \text{ is even}, \\
   (2q - (r - 1))^2/8(r - 1) & \text{if } r \text{ is odd}. 
\end{cases} \]

**Remark 1.** A direct computation shows that \( F(r) \leq F(s) \) provided that \( r \geq s \).

Since \( F(r) \leq F(2) = q(q - 1)/2 \), as \( r \geq 2 \), then \( g \leq q(q - 1)/2 \) which is a well-known fact on maximal curves over \( K \) due to Ihara [19]. In addition, \( F(r) \leq F(3) = (q - 1)^2/4 \) for \( r \geq 3 \), so that the genus \( g \) of a maximal curve over \( K \) satisfies the following condition (see [9])
\[ g \leq F(3) = (q - 1)^2/4 \quad \text{or} \quad g = F(2) = q(q - 1)/2. \]

As a matter of fact, the following holds true.

**Lemma 1** ([25, 9, 29]). Let \( \mathcal{X} \) be a maximal curve over \( K \) of genus \( g \) with Frobenius dimension \( r \). The following sentences are equivalent:

1. \( g = F(2) = q(q - 1)/2 \);
2. \( (q - 1)^2/4 < g \leq q(q - 1)/2 \);
3. \( r = 2 \);
4. \( \mathcal{X} \) is \( K \)-isomorphic to the Hermitian curve \( \mathcal{H} : y^{q+1} = x^q + x \).

**Corollary 1.** Let \( \mathcal{X} \) be a maximal curve over \( K \) of genus \( g \) and Frobenius dimension \( r \). Suppose that
\[ F(4) = (q - 1)(q - 2)/6 < g \leq F(3) = (q - 1)^2/4. \]

Then \( r = 3 \).

**Proof.** If \( r \geq 4 \), then \( g \leq (q - 1)(q - 2)/6 \) by (6); so \( r = 2 \) or \( r = 3 \). Thus \( r = 3 \) by Lemma 1 and the hypothesis on \( g \).

It is known that \( g = \lfloor F(3) \rfloor \) if and only if \( \mathcal{X} \) is is uniquely determined by plane models of type: \( y^{(q+1)/2} = x^q + x \) if \( q \) is odd, and \( y^{q+1} = x^{q/2} + \ldots + x \) otherwise; see [8, 2, 21].

Let us consider next an improvement on (7). Suppose that
\[ c_1(3) = c_1(q^2, 3) := \lfloor (q^2 - q + 4)/6 \rfloor < g \leq F(3). \]

Therefore Halphen’s theorem implies that \( \mathcal{X} \) is contained in a quadric surface and so \( g = \lfloor F(3) \rfloor \); see [21]. In particular, (7) improves to
\[ g \leq c_1(3), \quad \text{or} \quad g = \lfloor F(3) \rfloor, \quad \text{or} \quad g = F(2). \]

The following important remark is commonly attributed to J.P. Serre.

**Remark 2.** Any curve (nontrivially) \( K \)-covered by a maximal curve over \( K \) is also maximal over \( K \). In particular, any subcover over \( K \) of the Hermitian curve is so; see e.g. [11, 3].

**Remark 3.** We point out that there are maximal curves over \( K \) that cannot be \( K \)-covered by the Hermitian curve \( \mathcal{H} \); see [12, 28]. We observe that the examples occurring in these papers are all defined over fields of order \( q^2 = \ell^6 \) with \( \ell > 2 \).

We also point out that there are maximal curves over \( K \) that cannot be Galois covered by the Hermitian curve; see [10, 5, 28, 15, 14, 24, 13].
3. The set $\mathcal{M}(q^2)$

In this section we investigate the spectrum $\mathcal{M}(q^2)$ for the genera of maximal curves defined in (3). By using Remark 2 this set has already been computed for $q \leq 5$ [11, Sect. 6]. As a matter of fact, $\mathcal{M}(2^2) = \{0,1\}$, $\mathcal{M}(3^2) = \{0,1,3\}$, $\mathcal{M}(4^2) = \{0,1,2,6\}$, and $\mathcal{M}(5^2) = \{0,1,2,3,4,10\}$. Thus from now on we assume $q \geq 7$.

Let $F(r)$ be the function in (6). Next we complement Corollary 1. Let $X$ be a maximal curve of genus $g$ over $K$ with Frobenius dimension $r$.

**Proposition 1.** If either

(A) $q \equiv 0 \pmod{3}$, and $(3q-1)(2g-2) > (q+1)(q^2-4q-1)$; or
(B) $q \not\equiv 0 \pmod{3}$, $r = 3$, and $(4q-1)(2g-2) > (q+1)(q^2-5q-2)$,

then

$$g \geq F(4) + (q + 1)/6 = (q^2 - 2q + 3)/6.$$  

**Proof.** We shall apply Stöhr-Voloch theory [27] to the Frobenius linear series $D$ on $X$. First we notice that the hypothesis on $g$ in (A) implies $g > F(4) = (q-1)(q-2)/6$ so that $r \leq 3$. Thus by Lemma 1 we can assume $r = 3$.

Let $R = \sum r(v_p(R))P$ and $S = \sum r(v_p(S))P$ denote respectively the ramification and Frobenius divisor of $D$. Associated to each point $P \in X$, we have the sequence of possible intersection multiplicities of $X$ with hyperplanes in $\mathbb{P}^3$, namely $\mathcal{R}(P) = 0 = j_0(P) < 1 = j_1(P) = j_2(P) < j_3(P)$. From (5), $j_3(P) = q + 1$ (resp. $j_3(P) = q$) if $P \in \mathcal{X}(K)$ (resp. $P \not\in \mathcal{X}(K)$). Moreover, $\mathcal{R}(P)$ is the same for all but a finite number of points (the so-called $D$-Weierstrass points of $X$); such a generic sequence (the orders of $D$) will be denoted by $\mathcal{E} : 0 = \epsilon_0 < 1 = \epsilon_1 < \epsilon_2 < q = \epsilon_3$. The numbers $0 = \nu_0 < 1 = \nu_1 < q = \nu_2$ are the $K$-Frobenius orders of $D$ (see [27, Prop. 2.1] and [8, Thm. 1.4]). The very basic properties (1)-(6) below hold true:

1. $j_i(P) \geq \epsilon_i$ for any $i$ and $P \in X$ [27, p. 5];
2. $v_p(R) \geq \sum_{i=0}^3 j_i(P) - \epsilon_i \geq 1$ for $P \in \mathcal{X}(K)$ [27, Thm. 1.5];
3. $\deg(R) = (\epsilon_3 + \epsilon_2 + \epsilon_1)(2g-2) + 4(q+1)$ [27, p. 6];
4. $(p$-adic criterion) If $\epsilon$ is an order and $\binom{\nu}{\eta} \neq 0 \pmod{p}$ ($p$ is the characteristic of $K$), then $\eta$ is also an order [27, Cor. 1.7];
5. $v_p(S) \geq (j_1(P) - \nu_0) + (j_2(P) - \nu_1) + (j_3(P) - \nu_2) = j_2(P) + 1 \geq \epsilon_2 + 1$ for $P \in \mathcal{X}(K)$ [27, Prop. 2.4];
6. $\deg(S) = (\nu_2 + \nu_1)(2g-2) + (q+1)(q^2 + 3)$ [27, p. 9].

Now the proof of Proposition 1 is based on the following

Claim. $\epsilon_2 = 2$.

**Proof of the Claim.** Suppose that $\epsilon_2 \geq 3$. Let $q \equiv 0 \pmod{3}$. From (5), (6) and the maximal property of $X$

$$\deg(S) = (q+1)(2g-2) + (q^2 + 3)(q+1) \geq 4(q+1)^2 + 4q(2g-2)$$

so that $(q^2-4q-1)(q+1) \geq (3q-1)(2g-2)$, a contradiction. Suppose now that $q \not\equiv 0 \pmod{3}$. Then $\epsilon_2 \geq 4$ by the $p$-adic criterion. Then (5), (6) and the maximal property of $X$ gives

$$\deg(S) = (q+1)(2g-2) + (q^2 + 3)(q+1) \geq 5(q+1)^2 + 5q(2g-2)$$

so that $(q+1)(q^2 - 5q - 2) \geq (4q-1)(2g-2)$ a contradiction; the claim follows.
Finally, we use the ramification divisor $R$ of $D$; we have
\[ \deg(R) = (q + 2 + 1)(2g - 2) + 4(q + 1) \geq (q + 1)^2 + q(2g - 2) \]
and thus $g \geq (q^2 - 2q + 3)/6$. \hfill $\square$

**Remark 4.** Notation as above. There are maximal curves over $K$ with Frobenius dimension 3 and $\epsilon_2 = 3$ [6, Thm. 2].

We mainly apply Proposition 1(B) in the following form:

**Corollary 2.** Let $X$ be a maximal curve over $K$, of genus $g$, where $q \not\equiv 0 \pmod{3}$. Then
\[ g \geq (q^2 - 2q + 3)/6 \]
provided that $g > (q - 1)(q - 2)/6$.

**Proof.** As in the proof of the above proposition, we can assume that the Frobenius dimension of $X$ equals 3. Now the hypothesis on $g$ is equivalent to $(2g - 2) > (q + 1)(q - 4)/3$; thus
\[ (4q - 1)(2g - 2) > (4q - 1)(q - 1)(q - 4)/3 > (q + 1)(q^2 - 5q - 2), \]
and the result follows from Proposition 1. \hfill $\square$

4. $M(q^2)$ for $7 \leq q \leq 16$

In this section we shall improve on the following computations which follow from [11, Remark 6.1], (8), from [1] for $(q, g) \in \{(13, 1), (11, 6), (11, 8)\}$, from [11, Ex. 5.12] for $(q, g) = (13, 10)$, from [21, Thm. 2] for $22 \not\in M(13^2)$ and $35 \not\in M(16^2)$, and from [23, Remarks 7.1, 7.2] (see also [17]) for $3, 4, 5 \in M(13^2)$, $16 \in M(16^2)$.

**Proposition 2.** (1) $\{0, 1, 2, 3, 5, 7, 9, 21\} \subseteq M(7^2) \subseteq [0, 7] \cup \{9\} \cup \{21\}$; (2) $\{0, 1, 2, 3, 4, 6, 7, 9, 10, 12, 28\} \subseteq M(8^2) \subseteq [0, 10] \cup \{12\} \cup \{28\}$; (3) $\{0, 1, 2, 3, 4, 6, 8, 9, 12, 16, 36\} \subseteq M(9^2) \subseteq [0, 12] \cup \{16\} \cup \{36\}$; (4) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 18, 19, 25, 55\} \subseteq M(11^2) \subseteq [0, 19] \cup \{25\} \cup \{55\}$; (5) $\{0, 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 26, 36, 78\} \subseteq M(13^2) \subseteq [0, 21] \cup [23, 26] \cup [36] \cup \{78\}$; (6) $\{0, 1, 2, 4, 6, 8, 12, 16, 24, 28, 40, 56, 120\} \subseteq M(16^2) \subseteq [0, 34] \cup [36, 40] \cup \{56\} \cup \{120\}$.

**Proposition 3.** Let $M(q^2)$ be the spectrum for the genera of maximal curves over $K$. Then
(1) $6 \not\in M(7^2)$; (2) $8 \not\in M(8^2)$; (3) $10 \not\in M(9^2)$; (4) $16 \not\in M(11^2)$; (5) $23, 24 \not\in M(13^2)$; (6) $36, 37 \not\in M(16^2)$.

**Proof.** As above let $F(4) = (q - 1)(q - 2)/6$. If $q \equiv 0 \pmod{3}$, Corollary 2 says that $M(q^2) \cap F(4), [(q^2 - 2q + 3)/6] - 1 = \emptyset$, where $[x]$ stands for the smallest integer $\geq x$. After some computations we obtain (1), (2), (4), (5), and (6). Let $q = 9$; the hypothesis in Proposition 1(A) reads $13(q - 1) > 5 \cdot 22$, i.e. $g \geq 10$. If $10 \in M(9^2)$, $10 \geq (9^2 - 2 \cdot 9 + 3)/6 = 11$, a contradiction. \hfill $\square$

**Remark 5.** The entry for $(q^2, g) = (13^2, 24)$ in [1] must be $N_{13^2}(24) < 794$. In fact, the genus of the given curve $y^{56} = x(x + 1)^{12}$ in [1] is 26.
Corollary 3. We have
\[ M(7^2) = \{0, 1, 2, 3, 5, 7, 9, 21\}. \]

Proof. By the above propositions, it is enough to show that 4 \notin M(7^2). Indeed, this is the case as follows from a result in Kudo and Harashita paper [22, Thm. B] concerning superspecial curves.

Remark 6. To compute \( M(q^2) \) for \( q = 8, 9, 11, 13, 16 \) we need to answer the following questions:

1. Is 5 \in M(8^2)?
2. Are 5, 7, 11 \in M(9^2)?
3. Are 12, 14, 17 \in M(11^2)?
4. Are 7, 8, 11, 13, 14, 16, 17, 19, 20, 21, 25 \in M(13^2)?
5. Are 3, 5, 7, 9, 11, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 29, 30, 31, 32, 33, 34, 38, 39 \in M(16^2)?

According to Tables 9–13 in [23], in case of an affirmative answer to any of the above questions, the corresponding curve would not be a quotient of the Hermitian curve.

Example 1. Here, for the sake of completeness, we provide an example of a maximal curve of genus \( g \) for each \( g \in M(7^2) \); cf. [1], [29].

1. (\( g = 0 \)) The rational curve;
2. (\( g = 1 \)) \( y^2 = x^3 + x \);
3. (\( g = 2 \)) \( y^2 = x^2 + x \);
4. (\( g = 3 \)) \( y^2 = x^7 + x \);
5. (\( g = 5 \)) \( y^8 = x^4 - x^2 \);
6. (\( g = 7 \)) \( y^{16} = x^9 - x^{10} \);
7. (\( g = 9 \)) \( y^4 = x^7 + x \);
8. (\( g = 21 \)) \( y^8 = x^7 + x \).

Remark 7. The curves in (6), (7), and (8) above are unique up to \( F_{49} \)-isomorphism; see respectively [7, 8, 25].

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