Uniqueness for inverse boundary value problems by Dirichlet-to-Neumann map on subboundaries

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Abstract

We consider inverse boundary value problems for elliptic equations of second order of determining coefficients by Dirichlet-to-Neumann map on subboundaries, that is, the mapping from Dirichlet data supported on \( \partial \Omega \setminus \Gamma^- \) to Neumann data on \( \partial \Omega \setminus \Gamma^+ \). First we prove uniqueness results in three dimensions under some conditions such as \( \Gamma^+ \cup \Gamma^- = \partial \Omega \). Next we survey uniqueness results in two dimensions for various elliptic systems for arbitrarily given \( \Gamma^- = \Gamma^+ \). Our proof is based on complex geometric optics solutions which are constructed by a Carleman estimate.

1 Introduction

Let \( \Omega \subset \mathbb{R}^n, n = 2, 3, \ldots, \) be a bounded domain with \( \partial \Omega \in C^2 \). Let \( \nu = \nu(x) \) be the outward unit normal vector to \( \partial \Omega \) at \( x \) and \( \frac{\partial v}{\partial \nu} = \nabla v \cdot \nu \). We consider the conductivity equation:

\[
\nabla \cdot (\gamma \nabla v) = 0 \quad \text{in} \quad \Omega
\]

with the Dirichlet boundary condition

\[
v = f \quad \text{on} \quad \partial \Omega.
\]

We assume suitable regularity for the conductivity \( \gamma \) and the boundary condition \( f \). For example, let \( \gamma \in \mathbb{R}^n, n = 2, 3, \ldots, \) be a strictly positive function on \( \Omega \) and \( f \in H^1(\partial \Omega) \). Then there exists a unique solution \( v \in H^1(\Omega) \) to (1) and (2) and we can define \( \gamma \frac{\partial v}{\partial \nu} \in H^{-\frac{1}{2}}(\partial \Omega) \) (e.g., Lions and Magenes [59]). We call the map \( f \mapsto \gamma \frac{\partial v}{\partial \nu} \) the Dirichlet-to-Neumann map. Since the Dirichlet-to-Neumann map depends on the conductivity \( \gamma \), we denote it by \( \Lambda_\gamma \):

\[
\Lambda_\gamma f = \gamma \frac{\partial v}{\partial \nu}, \quad \mathcal{D}(\Lambda_\gamma) = H^\frac{1}{2}(\partial \Omega).
\]

This article is concerned with the following boundary value problem:

**Inverse boundary value problem.** Determine the conductivity \( \gamma(x) \) from the Dirichlet-to-Neumann map \( \Lambda_\gamma \).

This inverse problem is a theoretical basis for example, for the electrical impedance tomography and often called Calderón’s problem. As for applications, we refer for example to Cheney, Isaacson, and Newell [15], Holder [26], Isaacson, Müller, and Siltanen [45], Jordan, Gasulla, and Pallás-Areny [51].

In this formulation, the information for the inverse problem is the operator \( \Lambda_\gamma \) itself, which is all the pairs of Dirichlet input data \( f \) and the corresponding derivatives \( \gamma \frac{\partial v}{\partial \nu} \). In other words, the formulation of the inverse problem requires infinitely many repeats of input-output manipulations.

Since the operator \( \Lambda_\gamma : H^\frac{1}{2}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega) \) is bounded for fixed \( \gamma \) and \( C^\infty(\partial \Omega) \) is dense in \( H^\frac{1}{2}(\partial \Omega) \), we can restrict \( f \in C^\infty(\partial \Omega) \) for \( \Lambda_\gamma \). More precisely, for \( \gamma_1, \gamma_2 \geq 0, \in C^2(\Omega) \), we have

\[
\{(f, \Lambda_\gamma f); f \in C^\infty(\partial \Omega)\} = \{f, \Lambda_{\gamma_2} f); f \in C^\infty(\partial \Omega)\}
\]
if and only if
\[
\{(f, \Lambda_1 f); f \in H^\frac{1}{2}(\partial \Omega)\} = \{(f, \Lambda_2 f); f \in H^\frac{1}{2}(\partial \Omega)\}.
\]
Throughout this article, we always consider that Dirichlet input data \(f\) from \(H^\frac{1}{2}(\partial \Omega)\).

**Inverse problem for stationary equations**

In general, an inverse problem of determining a spatially varying coefficient in partial differential equation in a spatially \(n\)-dimensional bounded domain, is called a coefficient inverse problem. For coefficient inverse problems for time-dependent differential equations such as parabolic equations and hyperbolic equations, one can take extra time-dependent boundary data at most finite times after changing suitable initial values to establish the stability and the uniqueness. In this case, one unknown coefficient depends on \(n\) spatial coordinates, while the extra data depend on the time variable and \(n - 1\) spatial variables (because data are limited to the boundary of the spatial domain), and so the data depend on \(n\) variables. Thus the data are not under-determinative and we can expect the uniqueness. In fact, Bukhgeim and Klibanov \cite{11} proposed a methodology for proving the uniqueness. Their method is based on a Carleman estimate which is a weighted \(L^2\)-estimate of solutions to partial differential equations. Later Imanuvilov and Yamamoto established global stability results \cite{36}, \cite{37}, \cite{38}; here we refer only to these three papers and \cite{75}, and there are many remarkable works but we do not describe because our main topics in this article are different. This formulation requires finite times (sometimes single) of repeats of observations. On the other hand, for a coefficient inverse problem for time-independent partial differential equations, there are no compensating variable such as time, and the current formulation by the inverse boundary value problem is the main theoretical setting of a coefficient inverse problem, even though it requires infinitely many times of measurements. There are various realizations for numerical computation but this is not our scope here.

**Mathematical subjects.**

For the inverse boundary value problem, we state mathematical subjects as follows:

- **Uniqueness.**
  Does \(\Lambda_1 = \Lambda_2\) imply \(\gamma_1 = \gamma_2\)?

- **Stability.**
  If \(\Lambda_1 - \Lambda_2\) is small in view of a suitable norm, then can we conclude that \(\gamma_1 - \gamma_2\) is small in some norm?

- **Reconstruction.**
  Given a data set from \(\{(f, \Lambda_\gamma f); f \in H^\frac{1}{2}(\partial \Omega)\}\), establish an algorithm for finding \(\gamma\). Here we assume that the data set really comes from some conductivity \(\gamma\), and do not discuss the existence of \(\gamma\) yielding the data set.

For the uniqueness, there is a large difference between the two dimensional case (i.e., \(n = 2\)) and the higher dimensional case (i.e., \(n \geq 3\)), in view of the degree of freedom of data. More precisely, we can explain as follows. In \(n\)-dimensional case, an unknown function depends on \(n\) number variables. On the other hand, our boundary data are the set \(\{(f, \Lambda_\gamma f); f \in H^\frac{1}{2}(\partial \Omega)\}\) and so are pairs of Dirichlet data and Neumann data which both depend on \((n - 1)\) number of variables and it can considered as data depending on \(2(n - 1)\) variables. Therefore

- **Case** \(n = 2\): we have \(2(n - 1) = n\). The inverse problem is formally determining.
- **Case** \(n \geq 2\): we have \(2(n - 1) > n\). The inverse problem is overdetermining.

Naturally the methods for cases \(n = 2\) and \(n \geq 3\) are different. Moreover, as we will explain in later sections, in the two dimensional case, one can construct a larger set of complex geometrical optics solutions, which are the key ingredient for the uniqueness of the inverse boundary value problem.

**Inverse boundary value problem for Schrödinger equations**

We consider the second order elliptic operator of the form
\[
L_q(x,D)u = \Delta u + q(x)u = 0, \quad \text{in} \quad \Omega,
\]
which is called a Schrödinger equation with potential $q$. Then we note that the uniqueness for determination of conductivity can be derived from the uniqueness for determination of some potential $q$ for the Schrödinger equation from the following Dirichlet-to-Neumann map

$$\Lambda(q)f = \frac{\partial u}{\partial \nu}|_{\partial \Omega},$$  \hspace{1cm} (3)

where

$$L_q(x,D)u = 0 \text{ in } \Omega, \quad u|_{\partial \Omega} = f, \quad u \in H^1(\Omega).$$  \hspace{1cm} (4)

We note that if 0 is not an eigenvalue of the Schrödinger equation, then the operator $\Lambda(q) : H^\frac{1}{2}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega)$ is correctly defined. Then the uniqueness in determining $\gamma$ in the conductivity equation by $\Lambda_\gamma$ follows from the uniqueness of determination for the potential $q$ by the Dirichlet-to-Neumann map $\Lambda(q)$: Indeed, if $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$ be strictly positive functions in $\overline{\Omega}$ and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then it is known that (see e.g., Alessandrini [3])

$$\gamma_1 = \gamma_2, \quad \frac{\partial \gamma_1}{\partial \nu} = \frac{\partial \gamma_2}{\partial \nu} \text{ on } \partial \Omega,$$  \hspace{1cm} (5)

Let $v$ solve the conductivity equation

$$\nabla \cdot (\gamma \nabla v) = 0 \text{ in } \Omega.$$  \hspace{1cm} (6)

Then, setting $q = -\frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}$ and $u = \sqrt{\gamma}v$, we have

$$L_q(x,D)u = \Delta u + q(x)u = 0 \text{ in } \Omega.$$  \hspace{1cm} (7)

This observation combined with (5) implies

$$\Lambda \left(-\frac{\Delta \sqrt{\gamma_1}}{\sqrt{\gamma_1}}\right) = \Lambda \left(-\frac{\Delta \sqrt{\gamma_2}}{\sqrt{\gamma_2}}\right).$$  \hspace{1cm} (8)

Therefore, by the uniqueness result of $q$ by the Dirichlet-to-Neumann map (3) and (4), we can derive the equality $\frac{\Delta \sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \frac{\Delta \sqrt{\gamma_2}}{\sqrt{\gamma_2}}$, that is,

$$\Delta w - \frac{\Delta \sqrt{\gamma_2}}{\sqrt{\gamma_2}} w = 0 \text{ in } \Omega$$  \hspace{1cm} (6)

with $w = \sqrt{\gamma_1} - \sqrt{\gamma_2}$. By (5) we have

$$w|_{\partial \Omega} = \frac{\partial w}{\partial \nu}|_{\partial \Omega} = 0.$$  \hspace{1cm} (7)

Applying to (6) and (7) the classical unique continuation for the second order elliptic operator (see e.g., Chapter XXVIII, §28.3 of [27], Corollary 2.9, Chapter XIV of [73]), we obtain $\gamma_1 \equiv \gamma_2$. Here for the inverse conductivity problem, compared to the regularity of $q$ in the Schrödinger equation, we need to increase the regularity assumptions by order 2 for unknown coefficients. Therefore in the succeeding parts, we often consider the inverse boundary value problem for Schrödinger type of equations.

**Existing results on the uniqueness.**

As for researches on the uniqueness, we can point out two main streams.

- Dirichlet-to-Neumann map on subboundaries: reduction of subboundaries where we consider Dirichlet inputs and observations of Neumann data.

- Relaxation of regularity of unknown coefficients.

Now, according to these streams, we list some important works on the uniqueness mainly before 2010. Here we do not intend any complete lists.

First of all, for the case of the Dirichlet-to-Neumann map on the whole boundary, we refer to

**Case $n \geq 3$:**
Sylvester and Uhlmann [72] proved the uniqueness for $\gamma \in C^2(\Omega)$, and Päivärinta, Panchenko and Uhlmann [66] for $\gamma \in W^{2,\infty}(\Omega)$, Brown and Torres [8] for $\gamma \in W^{2,p}(\Omega)$ with $p > 2n$. Recently Haberman and Tataru [23] proved the local uniqueness within Lipschitz conductivities $\gamma$ under the condition that $\|\nabla\gamma\|_{L^\infty(\Omega)}$ is sufficiently small.

**Case $n = 2$:**
Nachman [60] proved the uniqueness for $\gamma \in C^2(\Omega)$. Finally Astala and Päivärinta [4] established the uniqueness within $\gamma \in L^\infty(\Omega)$, which is a very sharp uniqueness result, but no corresponding results are known for the case $n \geq 3$. On the other hand, Bukhgeim [10] proves the uniqueness for the Schrödinger equations within $q \in L^\infty(\Omega)$.

In contrast to the above works, we state the formulation with Dirichlet-to-Neumann map on subboundaries. Let $\Gamma_+, \Gamma_- \subset \partial \Omega$ be subboundaries. We set

$$ \Lambda(q, \Gamma_-, \Gamma_+) f = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_+}, $$

(8)

where

$$ \Delta u + qu = 0 \text{ in } \Omega, \quad u|_{\Gamma_-} = 0, \quad u|_{\partial \Omega \setminus \Gamma_-} = f. $$

(9)

We regard $\partial \Omega \setminus \Gamma_- \text{ and } \partial \Omega \setminus \Gamma_+$ as input subboundary and output subboundary.

As for the conductivity equation, we can define the Dirichlet-to-Neumann map on subboundaries by

$$ \Gamma_+ f = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_+}, $$

where $\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega$ and $u|_{\partial \Omega \setminus \Gamma_-} = f, u|_{\Gamma_-} = 0$.

We note that the Dirichlet-to-Neumann map on the whole boundary corresponds to the case $\Gamma_+ = \Gamma_- = \emptyset$.

**Case $n \geq 3$:**
Bukhgeim and Uhlmann [12] proved the uniqueness within the class of $\gamma \in C^2(\Omega)$ if $\Gamma_- = \emptyset$ and $\Gamma_+$ is some specific part of $\partial \Omega$. Isakov [46] proved the uniqueness if $\Gamma_+$ and $\Gamma_-$ are included in planes or spheres. Knudsen [56] improved the uniqueness by Bukhgeim and Uhlmann [12] to the class of $\gamma \in C^{\alpha + \frac{1}{2}}(\Omega)$ with some $\alpha > 0$. Finally we refer to Kenig, Sjöstrand and Uhlmann [55] which established the uniqueness for some specially defined sets $\Gamma_\pm$.

**Case $n = 2$:**
Imanuvilov, Uhlmann and Yamamoto [30] first proved the uniqueness by Dirichlet-to-Neumann map on arbitrarily given subboundary provided that $\Gamma_+ = \Gamma_-$. 

The proposes of this article are
(i) to provide uniqueness results in the three dimensional case by a simpler argument and the uniqueness is sharper than e.g., Kenig et al. [55].
(ii) to simplify the existing proofs: we do not need advanced tools for example from the microlocal analysis (see [55]).
(iii) to describe uniqueness results by Dirichlet-to-Neumann map on subboundaries which have been recently obtained by the authors and colleagues.

In the succeeding sections, we will give more detailed references related to the topics of this article.

**Key to the proof of the uniqueness.**
For convenience, here we explain the key to the proof which has been an essential idea since Sylvester and Uhlmann [72].

- We consider a family $u_1 = u_1(\tau)(x), \tau > 0$ of solutions which are parameterized by $\tau > 0$:

$$ L_{q_1}(x, D)u_1 = \Delta u_1 + q_1 u_1 = 0 \text{ in } \Omega, \quad u_1|_{\Gamma_-} = 0. $$

(10)
• For \( u_1(\tau) \), we construct the solution \( u_2 = u_2(\tau) \) to

\[
L_{q_2}(x, D)u_2 = \Delta u_2 + q_2 u_2 = 0 \quad \text{in } \Omega, \quad u_2|\partial\Omega = u_1|\partial\Omega.
\]

By \( \Lambda(q_1, \Gamma_-, \Gamma_+ \rightleftharpoons \Gamma_-, \Gamma_+ \) = \( \Lambda(q_2, \Gamma_-, \Gamma_+ \rightleftharpoons \Gamma_-, \Gamma_+) \), we have \( \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \) on \( \partial\Omega \setminus \Gamma_+ \). Setting \( u = u_1 - u_2 \), we obtain

\[
L_{q_2}(x, D)u_2 = \Delta u_2 + q_2 u_2 = (q_2 - q_1)u_1 \quad \text{in } \Omega, \quad \tag{11}
\]

\[
u|\partial\Omega = 0, \quad \frac{\partial u_2}{\partial \nu}|\partial\Omega \setminus \Gamma_+ = 0. \tag{12}
\]

• We consider another family \( v = v(\tau)(x), \tau > 0 \) of solutions which are parameterized by \( \tau > 0 \):

\[
L_{q_2}(x, D)v = \Delta v + q_2 v = 0 \quad \text{in } \Omega, \quad v|\Gamma_+ = 0. \tag{13}
\]

• Multiplying (11) with \( v(\tau) \) and using (10), (12) and (13), we obtain

\[
0 = \int_{\Omega} vL_{q_2}(x, D)u_2 dx = \int_{\Omega} (q_2 - q_1)vu_1 dx,
\]

that is,

\[
\int_{\Omega} (q_1 - q_2)(x)v(\tau)(x)u_1(\tau)(x) dx = 0
\]

for all \( \tau > 0 \).

Thus the uniqueness is reduced to the completeness of products of solutions, that is, the density in \( L^2(\Omega) \) of \( \{v(\tau)u_1(\tau)\}_{\tau > 0} \). The works on the uniqueness in \( \Omega \) since [72] have relied on how to construct the families \( u_1(\tau), v(\tau), \tau > 0 \) satisfying such completeness. For it, the idea for \( u_1(\tau) \) and \( v(\tau) \) is solutions to the elliptic equations under consideration in the form:

\[
e^{\tau \Phi(x)}(a + o(1)) \quad \text{as } \tau \to \infty \tag{14}
\]

with suitably chosen phase function \( \Phi \) and solutions to the transport equation \( a \). Solution in the form of (14) are called (complex) geometric optics solutions. The paper by Kenig, Sjöstrand and Uhlmann [55] uses a Carleman estimate for constructing complex geometric optics solutions in higher dimensions and see also Bukhgeim [10]. We can understand a Carleman estimate as weighted \( L^2 \)-estimate.

Careful choices of weight functions in Carleman estimates yield the uniqueness results for various elliptic systems by Dirichlet-to-Neumann map limited on some subboundaries. We have developed such relevant Carleman estimates mainly in two dimensions, and in the succeeding sections, we clarify such Carleman estimates.

The article is composed of the following sections.

• §1. Introduction.

• §2. A key Carleman estimate for the proof of the uniqueness in §3.

• §3. Uniqueness in the three dimensional case: We demonstrate how our method can produce the uniqueness with Dirichlet-to-Neumann map on subboundaries, and the proof for the uniqueness is concise and based on Carleman estimate and the Radon transform. Moreover our uniqueness generalizes the results by Bukhgeim and Uhlmann [12], Kenig, Sjöstrand and Uhlmann [55].

• §4. Survey on two dimensional inverse boundary value problems by Dirichlet-to-Neumann map on an arbitrary subboundary.

• §5 Calderón’s problem for semilinear elliptic equations.

• §6 Uniqueness by Dirichlet-to-Neumann maps for Lamé equations and the Navier-Stokes equations.

• §7. Appendix.
This section is closed with explanations of notations which are used throughout this paper.

**Notations.** Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( x' = (x_1, x_2, \ldots, x_{n-1}) \), and \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \). Henceforth let \( N_+ = N \cup \{ 0 \} \), \( \partial^2_x = \partial^2_{x_1} \partial^2_{x_x} \cdots \partial^2_{x_n} \), \( \beta = (\beta_1, \ldots, \beta_n) \in (N_+)^n \) and \( |\beta| = \beta_1 + \cdots + \beta_n \).

We set \( i = \sqrt{-1} \), \( x_1, x_2 \in \mathbb{R}^1 \), \( z = x_1 + ix_2 \), \( \overline{z} \) denotes the complex conjugate of \( z \in \mathbb{C} \). We identify \( x = (x_1, x_2) \in \mathbb{R}^2 \) with \( z + ix_2 \in \mathbb{C} \) and \( \xi = (\xi_1, \xi_2) \) with \( \xi_1 + i\xi_2 \). \( \partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}) \), \( \partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}) \), \( D = (D_1, \ldots, D_n) = (\frac{1}{2} \partial_{x_1}, \frac{1}{2} \partial_{x_2}, \ldots, \frac{1}{2} \partial_{x_n}) \), \( D^2 = D_1^2 \cdots D_n^2 \partial_{\xi} = \frac{1}{2}(\partial_{\xi_1} + i\partial_{\xi_2}), D_{\overline{\xi}} = \frac{1}{2}(\partial_{\xi_1} - i\partial_{\xi_2}) \). Denote by \( B(x, \delta) \) a ball centered at \( x \) of radius \( \delta \).

For any strictly positive function \( \rho \) and the norm \( \| \cdot \| \), we introduce the following subsets of the boundary of domain \( \Omega \):

\[ \{ \partial_p \} \quad \{ \partial_n \} \quad \{ \partial \} \quad \{ \partial_q \} \]

Notations. This section is closed with explanations of notations which are used throughout this paper.

2 Carleman estimates for the Schrödinger equation

As we mentioned in the introduction, in order to construct the remaining term in complex geometric optics solution we will use the technique based on Carleman estimates. The Carleman estimate itself was introduced by Carleman [14] for the purpose of proving the uniqueness for the Cauchy problem for the system of elliptic equations. The Carleman estimate is some a priori estimate depending on parameter and some weight function. In this section we concentrate on the case of the second-order elliptic operator whose principal part is the Laplace operator. For the general theory of Carleman estimates see e.g. [27].

Consider the second order elliptic equation in domain \( \Omega \)

\[ P(x, D)u = \Delta u + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + cu = f \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0. \]  

(15)

The principal symbol of this operator is \( p(x, \xi) = -|\xi|^2 \) where \( \xi = (\xi_1, \ldots, \xi_n) \).

**Definition 1.** We say that the function \( \varphi \) is pseudoeconvex with respect to the principal symbol \( p \) if \( \nabla \varphi \neq 0 \) on \( \Omega \) and

\[ \{ p(x, \xi - \imath \tau \nabla \varphi(x)), p(x, \xi + \imath \tau \nabla \varphi(x)) \}/\imath \tau = 2 \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} p^{(i)}(x, \xi) p^{(k)}(x, \xi) > 0 \]

on \( \{ (x, \xi, \tau) \in \mathbb{R}^n \setminus \{ 0 \} \times \mathbb{R}_+ \times \mathbb{R} \mid p(x, \xi + \imath \tau \nabla \varphi) = 0 \}, \quad \tau = 0. \)

(16)

The construction of the pseudoeconvex function for the second-order elliptic operator is a very easy task: if \( \phi \in C^2(\Omega) \) and \( \nabla \phi(x) \neq 0 \) for any \( x \) from \( \Omega \) then for all sufficiently large positive \( \lambda \) the function \( \varphi = e^{\lambda \phi} \) is pseudoeconvex with respect to a principal symbol of this operator.

We introduce the following subsets of the boundary of domain \( \Omega \):

\[ \partial \Omega_- = \{ x \in \partial \Omega : \frac{\partial \varphi}{\partial \nu}(x) < 0 \}, \quad \partial \Omega_+ = \{ x \in \partial \Omega : \frac{\partial \varphi}{\partial \nu}(x) > 0 \}, \quad \partial \Omega_0 = \text{Int}(\{ x \in \partial \Omega : \frac{\partial \varphi}{\partial \nu}(x) = 0 \}) \]
A typical Carleman estimate for the Schrödinger equation (see e.g. [29]) is given by the following proposition:

**Proposition 1** Let a function $u$ belong to the set $\mathcal{P} \subset C^2(\Omega)$ be a solution to the complex geometric optics solutions for the operator $P(x, D)$. Then there exist constants $\tau_0$ and $C$ independent of $\tau$ such that for all $\tau \geq \tau_0$ the following estimate holds true:

\[
|\tau u^\tau|_{H^1_{\tau}(\Omega)}^2 + \int_{\partial\Omega} |\frac{\partial u}{\partial n}|^2 e^{2\tau \varphi} d\sigma + \tau \int_{\partial\Omega_{\tau} \cup \partial\Omega_0} |\frac{\partial u}{\partial \nu'}|^2 e^{2\tau \varphi} d\sigma \\
\leq C \left( \|(P(x, D)u)^\tau\|_{L^2_{\tau}(\Omega)}^2 + \tau \int_{\partial\Omega_+} |\frac{\partial u}{\partial \nu'}|^2 e^{2\tau \varphi} d\sigma \right)
\]

for $u$ satisfying (15).

For the construction of the complex geometric optics solutions for the operator $P(x, D)$ we will use as a weight function the real part of the function $\Phi$ which solves the Eikonal equation.

**Proposition 2** Let a function $\Phi(x) = \varphi + i\psi \in C^2(\Omega)$ be a solution to the Eikonal equation

\[
(\nabla \Phi, \nabla \Phi) = 0 \quad \text{on} \quad \Omega.
\]

Then $(x, \tau \nabla \psi(x), \tau)$ belongs to the set $\{(x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}_+ | P(x, \tau \nabla \varphi) = 0 \}$ and

\[
(P(x, \tau \nabla \psi - \frac{\tau}{i} \nabla \psi(x)), P(x, \tau \nabla \psi + \frac{\tau}{i} \nabla \psi(x))) = 0.
\]

**Proof.** The Eikonal equation (18) is equivalent to the following two equalities

\[
\sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_j} = \sum_{j=1}^{n} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_j} \quad \text{and} \quad \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_j} = 0.
\]

Equations (20) immediately imply

\[
(x, \tau \nabla \psi(x), \tau) \in \{(x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}_+ | P(x, \tau \nabla \varphi) = 0 \}.
\]

Differentiating equations (20) with respect to $x_k$ and then taking the sum over $k$, we have

\[
H_\varphi \nabla \varphi = H_\varphi \nabla \psi, \quad H_\varphi \nabla \psi = -H_\psi \nabla \varphi.
\]

The right-hand side of (18) can be written as

\[
(P(x, \xi - \frac{\tau}{i} \nabla \varphi(x)), P(x, \xi + \frac{\tau}{i} \nabla \varphi(x))) / i\tau = 4((H_\varphi \nabla \xi, \nabla \xi) + \tau^2(H_\varphi \nabla \varphi, \nabla \varphi)).
\]

Hence, using the equalities (22) we rewrite (23) for $\xi = \tau \nabla \psi(x)$ as

\[
\tau^2(H_\varphi \nabla \psi, \nabla \psi) + \tau^2(H_\varphi \nabla \varphi, \nabla \varphi) = -\tau^2(H_\varphi \nabla \psi, \nabla \psi) + \tau^2(H_\varphi \nabla \varphi, \nabla \varphi) = \tau^2(H_\varphi \nabla \varphi, \nabla \varphi) + \tau^2(H_\varphi \nabla \varphi, \nabla \varphi) = 0.
\]

Equalities (23) and (24) imply (19) immediately.

Proposition 2 implies that the real part of a solution of the Eikonal equation does not satisfy the psuedoconvexity condition (19). Thus we relax this pseudoconvexity condition as follows:

**Definition 2.** We say that the function $\varphi$ is weakly pseudoconvex with respect to the symbol $\tau \nabla \varphi \neq 0$ on $\Omega$ and

\[
(P(x, \xi - \frac{\tau}{i} \nabla \varphi(x)), P(x, \xi + \frac{\tau}{i} \nabla \varphi(x))) / i\tau = 2 \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} p^{(j)}(x, \xi) p^{(k)}(x, \xi) \geq 0
\]

on $\{(x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}_+ | P(x, \tau \nabla \varphi) = 0 \}$, $\zeta = \xi + \tau \nabla \varphi$.
If the real part of a solution of the Eikonal equation is weakly pseudoconvex with respect to the principal symbol of the elliptic operator, then one can construct the complex geometric optics solution for large parameter $\tau > 0$. In some cases we need to construct the complex geometric optics solutions for the large negative values of parameter $\tau$ for the same weight function as well. In [55], in order to deal with this situation, the notion of the limiting Carleman weight was introduced.

**Definition 3.** We say that the function $\varphi$ is a limiting Carleman weight for the operator $P(x, D)$ if $\nabla \varphi \neq 0$ on $\Omega$ and

\[
\left\{ p(x, \xi - i \tau \nabla \varphi(x)), p(x, \xi + i \tau \nabla \varphi(x)) \right\} / i \tau = 2 \sum_{i,j=1}^{n} \frac{\partial^{2} \varphi}{\partial x_{i} \partial x_{j}} p^{(i)}(x, \zeta) p^{(j)}(x, \zeta) = 0
\]

on $\{ (x, \xi, \tau) \in \Omega \times (\mathbb{R}^{n} \setminus \{0\}) \times \mathbb{R}_{+}, |p(x, \xi + i \tau \nabla \varphi) = 0 \}$, $\zeta = \xi + i \tau \nabla \varphi$. (26)

Obviously any limiting Carleman estimate for the operator $P(x, D)$ is weakly pseudoconvex with respect to the principal symbol of this operator.

Another important property of the limiting Carleman weight is the following.

**Proposition 3** ([55]) Let $\Omega \subset \mathbb{R}^{3}$ be a simply connected domain with the smooth boundary and $\varphi$ be a limiting Carleman weight in $\Omega$. Then there exist a family of functions $\psi$ such that the function $\Phi = \varphi + i \psi$ solves the Eikonal equation in $\Omega$.

Later we will use the following limiting Carleman weights;

**Example 1 of the limiting Carleman weights.** Let $\vec{v}_{1}, \vec{v}_{2} \in \mathbb{R}^{3}$ be to vectors such that $|\vec{v}_{1}| = |\vec{v}_{2}| \neq 0$ and $(\vec{v}_{1}, \vec{v}_{2}) = 0$. The function $\Phi = (\vec{v}_{1}, x) + i(\vec{v}_{2}, x)$ solves the Eikonal equation and the function $\varphi = (\vec{v}_{1}, x)$ is the limiting Carleman weight.

**Example 2 of the limiting Carleman weights.** Let $\Phi = \ln r + i \theta$ where $r, \varphi, \theta$ are the spherical coordinates. The function $\Phi$ is a solution to the Eikonal equation and the function $\ln(r)$ is a limiting Carleman weight, provided that the origin and the domain $\Omega$ can be separated by some plane.

In two-dimensional case we can give the complete description of the solutions of the Eikonal equations. We have

**Proposition 4** Let $\Omega \subset \mathbb{R}^{2}$ and a function $\Phi(x) = \varphi + i \psi \in C^{2}(\Omega)$ be a solution to the Eikonal equation

\[
(\nabla \Phi, \nabla \Phi) = 0 \quad \text{on } \Omega.
\]

Then the function $\Phi$ is either holomorphic or antiholomorphic in $\Omega$.

**Proof.** The short computations give the formula

\[
e^{-\tau \Phi} \Delta e^{\tau \Phi} = (\nabla \Phi, \nabla \Phi) \tau^{2} + \Delta \Phi \tau = e^{-\tau \Phi} 4 \partial_{x_{1}} \partial_{x_{1}} e^{\tau \Phi} = 4 \partial_{x_{1}} \Phi \partial_{x_{1}} \Phi \tau^{2} + \Delta \Phi \tau.
\]

Therefore, if $\Phi$ is a solution to the Eikonal equation, then we obtain form the above equality that

\[
\partial_{x_{1}} \Phi \partial_{x_{1}} \Phi = 0 \quad \text{in } \Omega.
\]

From this equality, the statement of the proposition follows immediately. ■

**Example 3 of the limiting Carleman weights.** Let $n = 2$ and $\Phi$ be a holomorphic or an antiholomorphic function such that $\nabla \Phi \neq 0$. Then the real part of the function $\Phi$ is the limiting Carleman weight.

**Example 4 of the limiting Carleman weights.** Let $\varphi$ be the limiting Carleman weight and the function $\Phi = \varphi + i \psi$ solves the Eikonal equation. Then the function $\varphi(x/|x|^{2})$ is the limiting Carleman weight and the function $\Phi \circ (\frac{x}{|x|})$ solves the Eikonal equation.

We have
Theorem 1  Let $b_j, c \in L^\infty(\Omega)$ and $\varphi$ be a weakly pseudoconvex function with respect to the principal symbol of the operator $P(x, D)$. Then there exist constants $\tau_0$ and $C$ independent of $\tau$ such that for all $\tau \geq \tau_0$ the following estimate holds true:

$$
\|ue^{\tau \varphi}\|^2_{H^1,\tau(\Omega)} + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} e^{2\tau \varphi} d\sigma + \tau \int_{\partial \Omega} \frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial \nu} e^{2\tau \varphi} d\sigma
\leq C \left( \|(P(x, D)u)e^{\tau \varphi}\|^2_{L^2(\Omega)} + \tau \int_{\partial \Omega} \frac{\partial u}{\partial \nu} e^{2\tau \varphi} d\sigma \right).
$$  

(28)

Proof. First we recall the equality

$$
\{(\mathfrak{P}(x, \xi - i\tau \nabla \varphi(x)), p(x, \xi + i\tau \nabla \varphi(x)))/i\tau = 4\tau (H_\varphi(\xi, \xi) + \tau^2 H_\varphi(\nabla \varphi, \nabla \varphi))
$$

Hence the pseudoconvexity condition (25) is equivalent to the following one:

$$
(H_\varphi(\xi, \xi) + \tau^2 H_\varphi(\nabla \varphi, \nabla \varphi)) \geq 0 \quad \text{on} \quad \{(x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \times \{0\}) \times \mathbb{R}_+ | \, p(x, \xi + i\tau \nabla \varphi) = 0\}.
$$

(29)

We show that for each $x$ from $\Omega$ the polynomial $q(x, \xi, \tau) = (H_\varphi(\xi, \xi) + \tau^2 (H_\varphi(\nabla \varphi, \nabla \varphi)$ can be represented as the sum of two homogeneous polynomials of degree two in variables $\xi, \tau$ such that

$$
q(x, \xi, \tau) = q_0(x, \xi, \tau) + q_+(x, \xi),
$$

(30)

where

$$
q_+(x, \xi) \geq 0, \quad \forall (x, \xi, \tau) \in \Omega \times \mathbb{R}^2 \times \mathbb{R}^1,
$$

(31)

and

$$
q_0(x, \xi, \tau) = 0, \quad \forall (x, \xi, \tau) \in \{(x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \times \{0\}) \times \mathbb{R}_+ | \, p(x, \xi + i\tau \nabla \varphi) = 0\}.
$$

(32)

The functions $q_+, q_0$ can be constructed in the following way. Consider the partition of unity of the domain $\Omega$:

$$
\sum_{j=1}^K e_j = 1 \quad \text{on} \quad \Omega, \quad e_j \in C_0^\infty(B(x_j, \delta)), \quad e_j(x) \geq 0, \quad \forall x \in \overline{\Omega}.
$$

(33)

Consider the symbol $r_j(x, \xi, \tau^2) = e_j((H_\varphi(\xi, \xi) + \tau^2 (H_\varphi(\nabla \varphi, \nabla \varphi))).$ Since the function $\varphi$ is assumed to be weakly pseudoconvex, taking into account (25) and (33) we obtain

$$
r_j(x, \xi, \tau^2) \geq 0 \quad \text{on} \quad \{(x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \times \{0\}) \times \mathbb{R}_+ | \, p(x, \xi + i\tau \nabla \varphi) = 0\}.
$$

(34)

Suppose that $\delta > 0$ is so small that $\frac{\partial \varphi}{\partial x_j}$ is not equal to zero on $B(x_j, \delta)$ for some $J \in \{1, \ldots, n\}$. Consider the function

$$
\bar{r}_j(x, \xi) = r_j(x, \xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n) = r_j(x, \xi_1, \ldots, \xi_{j-1}, \frac{1}{\partial x_j} \sum_{k=1, k \neq J}^n \xi_k \frac{\partial \varphi}{\partial x_k} \xi_{j+1}, \ldots, \xi_n, m_j(\xi)/|\nabla \varphi|^2),
$$

$$
m_j(\xi) = \sum_{k=1, k \neq J}^n \xi_k^2 + \left( \frac{1}{\partial x_j} \sum_{k=1, k \neq J}^n \xi_k \frac{\partial \varphi}{\partial x_k} \right)^2.
$$

Observe that if $\bar{x} \in B(x, j)$ and $(\bar{x}, \bar{\xi}, \bar{\tau}) \in \{(x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \times \{0\}) \times \mathbb{R}_+ | \, p(x, \xi + i\tau \nabla \varphi) = 0\}$, then

$$
\bar{r}_j(\bar{x}, \bar{\xi}) = r_j((H_\varphi(\bar{\xi}, \bar{\xi}) + \bar{\tau}^2 (H_\varphi(\nabla \varphi(\bar{x}), \nabla \varphi(\bar{x}))).
$$

(35)

By (33) and (33), we have

$$
\bar{r}_j(x, \xi) \geq 0 \quad \forall (x, \xi) \in \Omega \times (\mathbb{R}^n \times \{0\}).
$$

(36)

Next we set $q_+(x, \xi) = \sum_{j=1}^K \bar{r}_j(x, \xi).$ By (33) and (30), we see

$$
q_+(x, \xi) \geq 0 \quad \forall (x, \xi) \in \Omega \times (\mathbb{R}^n \times \{0\})
$$

(37)
The set of zeros of the polynomial $q(x, \xi)$ is a homogeneous polynomial of degree one in $\xi$ for all $x \in \Omega$ such that

$$q_0(x, \xi, \tau) = m(x)(|\xi|^2 - 2 \tau^2|\nabla \varphi|^2) + \ell(x, \xi, \nabla \varphi), \quad \forall (x, \xi, \tau) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}_+^1.$$ \hfill (38)

Indeed, let us fix some point $\tilde{x}$ from $\Omega$. Without loss of generality, after a possible rotation, we may assume that $\nabla \varphi$ is parallel to the vector $\hat{e}_1 = (1, 0, 0, \ldots, 0)$. Consider the polynomial $q_0(\tilde{x}, \xi, \tau)$ on the hypersurface $\{\xi_1 = 0\}$. The set of zeros of the polynomial $\sum_{k=2}^n \xi_k^2 - \tau^2|\nabla \varphi(\tilde{x})|^2$ is the subset of zeros of the quadratic polynomial $q_0(\tilde{x}, 0, \xi_2, \ldots, \xi_n, \tau)$ since

$$\{(\xi, \tau) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}_+^1|p(\tilde{x}, \xi + i\tau \nabla \varphi) = 0\}$$

The set of zeros of the polynomial $\sum_{k=2}^n \xi_k^2 - \tau^2|\nabla \varphi(\tilde{x})|^2$ forms a cone surface in $\mathbb{R}^n$. The polynomial $q(\tilde{x}, 0, \xi_2, \ldots, \xi_n, \tau)$ is a homogeneous polynomial of degree 2. There are two possibilities. First this polynomial is identically equal to zero. Then we set $m(\tilde{x}) = 0$. Second, the set of zeros of polynomials $q_0(\tilde{x}, 0, \xi_2, \ldots, \xi_n, \tau)$ and $\sum_{k=2}^n \xi_k^2 - \tau^2|\nabla \varphi(\tilde{x})|^2$ are the same. Therefore there exists $m(\tilde{x})$ such that

$$q(\tilde{x}, 0, \xi_2, \ldots, \xi_n, \tau) = m(\tilde{x})(\sum_{k=2}^n \xi_k^2 - \tau^2|\nabla \varphi(\tilde{x})|^2).$$

Hence we have

$$q_0(x, \xi, \tau) = m(x)(|\xi|^2 - 2 \tau^2|\nabla \varphi(x)|^2) \quad \text{on} \quad \{(x, \xi, \tau) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}_+^1|\xi, \nabla \varphi = 0\}. \hfill (39)$$

Since for each $x$ from $\Omega$ there exists $(\tilde{x}, \tilde{\varphi})$ such that $(x, \tilde{x}, \tilde{\varphi}) \in \{(x, \xi, \tau) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}_+^1|\xi, \nabla \varphi = 0\}$ and $|\tilde{\varphi}|^2 - \tau^2|\nabla \varphi(x)|^2 \neq 0$, by (39) the function $m(x)$ is smooth.

Consider the polynomial $d(x, \xi, \tau) = q_0(x, \xi, \tau) - m(x)(|\xi|^2 - 2 \tau^2|\nabla \varphi|^2)$. Let $A(x)$ be a smooth matrix such that the first row of $A$ is equal to $\nabla \varphi$ and $\det A(x) \neq 0$ on $\overline{\Omega}$. Then we introduce the new coordinates $\tilde{\xi} = A(x)\xi$ and set $d(\tilde{x}, \xi, \tau) = d(x, A^{-1}(x)\tilde{\xi}, \tau)$. In the new coordinates, the set $\{(x, \xi, \tau) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}_+^1|\xi, \nabla \varphi = 0\}$ is written as

$$\{(x, \xi, \tau) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}_+^1|\xi_1 = 0\}.$$  

The polynomial $\tilde{d}$ is a homogeneous polynomial of degree 2 in the variable $(\tilde{\xi}, \tau)$ for each $x \in \Omega$ and $\tilde{d}(\tilde{x}, \xi, \tau) = 0$ if $\xi_1 = 0$. Therefore we can represent this polynomial in the form

$$\tilde{d}(\tilde{x}, \xi, \tau) = \xi_1 \left( \sum_{j=1}^n b_j(x)\tilde{\xi}_j + b_{n+1}(x)\tau \right)$$

with smooth functions $b_j(x)$. Then after returning to the coordinates $\xi$ we obtain

$$d(x, \xi, \tau) = (\xi, \nabla \varphi)((\tilde{b}(x), A(x)\xi) + b_{n+1}(x)\tau), \quad \tilde{b} = (b_1, \ldots, b_n).$$

Next we need to show that the function $b_{n+1}$ is identically equal to zero in $\Omega$. Indeed the symbol $q_+$ is independent of $\tau$ and the symbol $q$ depends on $\tau^2$. Hence the symbol $q_0 = q - q_+$ depends smoothly on $\tau^2$. Since we have already proved that $q_0(x, \xi, \tau) - m(x)(|\xi|^2 - 2 \tau^2|\nabla \varphi|^2)$ is a homogeneous polynomial of degree 2 in $\xi$, $\nabla \varphi$, we observe that on the right-hand side of this equality, the $\tau$-dependent terms which are of the form $c(x)\tau^2$, $c(x) \neq 0$, $c(x) \in \mathbb{R}, \tau \in \mathbb{R}$, we get

$$d(x, \xi, \tau) = (\xi, \nabla \varphi)((\tilde{b}(x), A(x)\xi)).$$

The justification of the formula (33) is complete.
Consider a function \( f : \mathbb{R}^1 \to \mathbb{R}^1 \) such that \( f'(y) \neq 0 \) for all \( y \in \{ y | y = \varphi(x), x \in \Omega \} \). We set \( \xi = f'(\varphi) \eta \).

Then
\[
(H_{f(\varphi)} \xi, \xi) + r^2 (H_{f(\varphi)} \nabla \varphi, \nabla \varphi) = f'(\varphi)^3 ((H_{\varphi} \eta, \eta) + r^2 (H_{\varphi} \nabla \varphi, \nabla \varphi) + r^2 \frac{f''(\varphi)}{f(\varphi)} |\nabla \varphi|^4) + f''(\varphi) (\nabla \varphi, \xi) = f'(\varphi)^3 (m(x)(|\eta|^2 - r^2|\nabla \varphi|^2) + (x, \xi)(\eta, \nabla \varphi) + r^2 \frac{f''(\varphi)}{f(\varphi)} |\nabla \varphi|^4 + q(x, \eta)) + f''(\varphi) (\nabla \varphi, \xi)^2 \]

Next we take
\[ f_{N,\tau}(s) = s + \frac{Ns^2}{\tau}, \]
where \( N \) is a large positive parameter.

For the moment, assume that \( b_i = c = 0, \quad \forall i \in \{1, \ldots, n\} \).

We set
\[
P(x, D, \tau) = e^{\tau f_{N,\tau}(\varphi)} P(x, D) e^{-\tau f_{N,\tau}(\varphi)} = \Delta - 2\tau (\nabla f_{N,\tau}(\varphi), \nabla) + \tau^2 |\nabla f_{N,\tau}(\varphi)|^2 - \tau \Delta f_{N,\tau}(\varphi)
\]
and
\[
P(x, D, \tau)^* = e^{-\tau f_{N,\tau}(\varphi)} P(x, D) e^{\tau f_{N,\tau}(\varphi)} = \Delta + 2\tau (\nabla f_{N,\tau}(\varphi), \nabla) + \tau^2 |\nabla f_{N,\tau}(\varphi)|^2 + \tau \Delta f_{N,\tau}(\varphi).
\]

Using the operators \( P(x, D, \tau) \) and \( P(x, D, \tau)^* \), we construct two more operators
\[
P_+(x, D, \tau) = \frac{1}{2} (P(x, D, \tau) + P(x, D, \tau)^*) = \Delta + \tau^2 |\nabla f_{N,\tau}(\varphi)|^2
\]
and
\[
P_-(x, D, \tau) = \frac{1}{2} (P(x, D, \tau) - P(x, D, \tau)^*) = -2\tau (\nabla f_{N,\tau}(\varphi), \nabla) - \tau \Delta f_{N,\tau}(\varphi).
\]
Let \( w = e^{\tau f_{N,\tau}(\varphi)} u \). Then
\[
P_+(x, D, \tau) w + P_-(x, D, \tau) w = P(x, D, \tau) w \quad \text{in} \quad \Omega, \quad w|_{\partial \Omega} = 0.
\]
Taking the \( L^2 \)-norm of the equation \( (42) \) we obtain
\[
\|P(x, D, \tau) w\|_{L^2(\Omega)}^2 = \|P_+(x, D, \tau) w\|_{L^2(\Omega)}^2 + 2(P_+(x, D, \tau) w, P_-(x, D, \tau) w)_{L^2(\Omega)} + \|P_-(x, D, \tau) w\|_{L^2(\Omega)}^2.
\]
Integrating by parts the second term on the right-hand side of \( (43) \), we have
\[
2(P_+(x, D, \tau) w, P_-(x, D, \tau) w)_{L^2(\Omega)} = (\{P_+, P_-(x, D, \tau) w, w\}_{L^2(\Omega)} - 4\int_{\partial \Omega} \tau \frac{f_{N,\tau}(\varphi)}{\partial \nu} \frac{\partial w}{\partial \nu} \, d\sigma.
\]
The differential operator \([P_+, P_-]\) has the form
\[
-4\tau \sum_{i,j=1}^n \frac{\partial^2 f_{N,\tau}(\varphi)}{\partial x_i \partial x_j} \frac{\partial^2}{\partial x_i \partial x_j} + 4\tau^3 \sum_{i,j=1}^n \frac{\partial^2 f_{N,\tau}(\varphi)}{\partial x_i \partial x_j} \frac{\partial f_{N,\tau}(\varphi)}{\partial x_i} \frac{\partial f_{N,\tau}(\varphi)}{\partial x_j} + \tau R(x, D),
\]
where
\[
R(x, D) = -2(\nabla \Delta f_{N,\tau}(\varphi), \nabla) - \Delta^2 f_{N,\tau}(\varphi).
\]
The principal symbol of the differential operator \([P_+, P_-](x, D, \tau)\) is equal to \(4\tau ((H_{f_{N,\tau}(\varphi)} \xi, \xi) + \tau^2 (H_{f_{N,\tau}(\varphi)} \nabla \varphi, \nabla \varphi))\). Hence the representation \( (40) \) holds true.
Therefore we can write down the second term on the right-hand side of (44) as

\[
([P_+P_-](x,D)w,w)_{L^2(\Omega)} = \tau \int_\Omega (f'_{N,\tau}(\varphi)(m(x)(-\Delta - \tau^2 f_{N,\tau}(\varphi)^2)\nabla \varphi)^2 - \ell(x, \nabla)(\nabla \varphi, \nabla)) \\
+ \tau^2 f''_{N,\tau}(\varphi)(f'_{N,\tau}(\varphi))^2|\nabla \varphi|^4 - q_+(x,\nabla)) - f''_{N,\tau}(\varphi)(\nabla \varphi, \nabla)^2 w, w dx \geq -\frac{1}{4} \|P_+(x,D,\tau)w\|_{L^2(\Omega)}^2 \\
- 4\tau^2 \int_\Omega (f'_{N,\tau}(\varphi))^2 m^2 w^2 dx - \tau \int_\Omega (\nabla \varphi, \nabla)\ell(x,\nabla)w f'_{N,\tau}(\varphi) dx - \tau \int_\Omega f'_{N,\tau}(\varphi)q_+(x,\nabla)wwdx \\
+ \tau \int_\Omega \tau^3 f''_{N,\tau}(\varphi)(f'_{N,\tau}(\varphi))^2|\nabla \varphi|^4 w^2 dx + \tau \int_\Omega R(x,D)wwdx \\
- \tau \int_\Omega (\nabla \varphi, \nabla w)(\nabla \varphi, \nabla)^* (f'_{N,\tau}(\varphi)) w dx.
\] (45)

The symbol \(q_+(x,\xi)\) is a quadratic polynomial written as \(q_+(x,\xi) = \sum_{j,k=1}^n q_{jk}(x)\xi_j\xi_k\). Hence

\[
- \tau \int_\Omega f'_{N,\tau}(\varphi)q_+(x,\nabla)wwdx = -\tau \int_\Omega f'_{N,\tau}(\varphi) \sum_{j,k=1}^n q_{jk}(x) \frac{\partial^2 w}{\partial x_k \partial x_j} wdx \\
= \tau \int_\Omega f'_{N,\tau}(\varphi) \sum_{j,k=1}^n q_{jk}(x) \frac{\partial w}{\partial x_k} \frac{\partial w}{\partial x_j} dx - \frac{\tau}{2} \int_\Omega \sum_{j,k=1}^n w^2 \frac{\partial^2 (f'_{N,\tau}(\varphi)q_{jk}(x))}{\partial x_k \partial x_j} dx.
\] (46)

Let

\[
\frac{N}{\tau} \|\varphi\|_{C^0(\Omega)} \leq \frac{1}{10}.
\] (47)

Then \(f'_{N,\tau}(\varphi)\) is a nonnegative function. By (31) the function \(\sum_{j,k=1}^n q_{jk}(x)\xi_j\xi_k\) is also nonnegative on the set \(\mathbf{\Omega} \times (\mathbb{R}^n \setminus \{0\})\). Hence the integral \(\int_\Omega f'_{N,\tau}(\varphi) \sum_{j,k=1}^n q_{jk}(x) \frac{\partial w}{\partial x_k} \frac{\partial w}{\partial x_j} dx\) is nonnegative. Therefore we obtain from (46)

\[
- \tau \int_\Omega f'_{N,\tau}(\varphi)q_+(x,\nabla)wwdx \geq -C(\tau + N)\|w\|_{L^2(\Omega)}^2.
\] (48)

By the definition of the function \(f_{N,\tau}\) we can choose \(N_0\) such that for all \(N \geq N_0\) we have

\[
\int_\Omega \tau^3 f''_{N,\tau}(\varphi)(f'_{N,\tau}(\varphi))^2|\nabla \varphi|^4 w^2 dx - 4\tau^2 \int_\Omega (f'_{N,\tau}(\varphi))^2 m^2 w^2 dx \geq (N\tau^2 + N^3)\|w\|_{L^2(\Omega)}^2
\] (49)

and

\[
- \tau \int_\Omega (\nabla \varphi, \nabla w)(\nabla \varphi, \nabla)^* (f'_{N,\tau}(\varphi)) w dx = \int_\Omega 2N(\nabla \varphi, \nabla w)^2 dx \geq 0.
\] (50)

Integrating by parts we obtain

\[
\tau \int_\Omega R(x,D)wwdx = \tau \int_\Omega -(\nabla \Delta f_{N,\tau}(\varphi), \nabla w^2) - \Delta^2 f_{N,\tau}(\varphi)w^2 dx = \tau \int_\Omega ((\Delta^2 - \Delta^2)f_{N,\tau}(\varphi))w^2 dx \\
= 0.
\] (51)

Using the Cauchy inequality, we have

\[
- \tau \int_\Omega (\nabla w, \nabla \varphi)\ell(x, \nabla)^* (w f'_{N,\tau}(\varphi)) dx = -\tau \int_\Omega f'_{N,\tau}(\varphi)(\nabla w, \nabla \varphi)\ell(x, \nabla)^* wdx \\
- \tau \int_\Omega (\nabla w, \nabla \varphi)\ell(x, \nabla)^* (f'_{N,\tau}(\varphi)) w dx \geq -\frac{1}{4} \|\tau f'_{N,\tau}(\varphi)(\nabla w, \nabla \varphi)\|_{L^2(\Omega)}^2 - C\|\nabla w\|_{L^2(\Omega)}^2 - C(\tau + N)\|w\|_{L^2(\Omega)}^2.
\] (52)

Using (48)-(52) we obtain from (45)

\[
([P_+P_-](x,D)w,w)_{L^2(\Omega)} \geq -\frac{1}{4} \|P_+(x,D,\tau)w\|_{L^2(\Omega)}^2 - \frac{1}{4} \|\tau f'_{N,\tau}(\varphi)(\nabla w, \nabla \varphi)\|_{L^2(\Omega)}^2 - C\|\nabla w\|_{L^2(\Omega)}^2 \\
- C(\tau + N)\|w\|_{L^2(\Omega)}^2 + \tau^2 N\|w\|_{L^2(\Omega)}^2.
\] (53)
By (53) and (44), we obtain from (43)

\[ \|P(x, D, \tau)w\|_{L^2(\Omega)}^2 \geq \frac{1}{4}\|P_+(x, D, \tau)w\|_{L^2(\Omega)}^2 + \|P_-(x, D, \tau)w\|_{L^2(\Omega)}^2 \]

\[ -\frac{1}{4}\|\tau f'_{N, \tau}(\varphi)(\nabla w, \nabla \varphi)\|_{L^2(\Omega)}^2 - C\|\nabla w\|_{L^2(\Omega)}^2 - (\tau + N)\|w\|_{L^2(\Omega)}^2 \]

\[ + \frac{3\tau^2 N}{4} \|w\|_{L^2(\Omega)}^2 - 4 \int_{\partial\Omega} \tau \frac{\partial f_{N, \tau}(\varphi)}{\partial \nu} \frac{\partial w}{\partial \nu}^2 d\sigma. \]  

(54)

Let \( \tau_0 \) and \( N_0 \) be sufficiently large numbers. Then for all \((\tau, N)\) satisfying (47) and \( N \geq N_0 \) and \( \tau \geq \tau_0 \) we obtain from (54) that

\[ \|P(x, D, \tau)w\|_{L^2(\Omega)}^2 \geq \frac{3}{4}\|P_+(x, D, \tau)w\|_{L^2(\Omega)}^2 + \|P_-(x, D, \tau)w\|_{L^2(\Omega)}^2 \]

\[ -\frac{1}{4}\|\tau f'_{N, \tau}(\varphi)(\nabla w, \nabla \varphi)\|_{L^2(\Omega)}^2 - C\|\nabla w\|_{L^2(\Omega)}^2 + \frac{\tau^2 N}{2} \|w\|_{L^2(\Omega)}^2 - 4 \int_{\partial\Omega} \tau \frac{\partial f_{N, \tau}(\varphi)}{\partial \nu} \frac{\partial w}{\partial \nu}^2 d\sigma. \]  

(55)

Taking the scalar product in \( L^2(\Omega) \) of the functions \( P_+(x, D, \tau)w \) and \( Nw \) and integrating by parts, we obtain the inequality

\[ N\|\nabla w\|_{L^2(\Omega)}^2 \leq C\|P(x, D, \tau)w\|_{L^2(\Omega)}^2 + \tau^2 N\|w\|_{L^2(\Omega)}^2. \]  

(56)

Then increasing \( \tau_0 \) and \( N_0 \) again and using (50), from (55) we have:

\[ \|P(x, D, \tau)w\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|P_+(x, D, \tau)w\|_{L^2(\Omega)}^2 + \|P_-(x, D, \tau)w\|_{L^2(\Omega)}^2 \]

\[ -\frac{1}{4}\|\tau f'_{N, \tau}(\varphi)(\nabla w, \nabla \varphi)\|_{L^2(\Omega)}^2 + NC\|\nabla w\|_{L^2(\Omega)}^2 + \frac{\tau^2 N}{4} \|w\|_{L^2(\Omega)}^2 - 4 \int_{\partial\Omega} \tau \frac{\partial f_{N, \tau}(\varphi)}{\partial \nu} \frac{\partial w}{\partial \nu}^2 d\sigma, \]  

(57)

where \((\tau, N)\) satisfies (47) and \( \tau \geq \tau_0, N \geq N_0 \). Observe that

\[ \|\tau f'_{N, \tau}(\varphi)(\nabla w, \nabla \varphi)\|_{L^2(\Omega)}^2 = \frac{1}{4}\|P_-(x, D, \tau)w + \tau \Delta f_{N, \tau}(\varphi)w\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\|P_-(x, D, \tau)w\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\tau \Delta f_{N, \tau}(\varphi)w\|_{L^2(\Omega)}^2. \]

Using this estimate in (57), we obtain

\[ \|P(x, D, \tau)w\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|P_+(x, D, \tau)w\|_{L^2(\Omega)}^2 + \frac{1}{2}\|P_-(x, D, \tau)w\|_{L^2(\Omega)}^2 \]

\[ + NC\|\nabla w\|_{L^2(\Omega)}^2 + \frac{\tau^2 N}{8} \|w\|_{L^2(\Omega)}^2 - 4 \int_{\partial\Omega} \tau \frac{\partial f_{N, \tau}(\varphi)}{\partial \nu} \frac{\partial w}{\partial \nu}^2 d\sigma, \]  

(58)

where \((\tau, N)\) satisfies (47) and \( \tau \geq \tau_0, N \geq N_0 \).

Now we remove the assumption (41). Suppose that some coefficients of the first or zeroth order terms are not identically equal to zero. Then we have to replace the term on the right-hand side of (57) by

\[ \|P(x, D, \tau)w - \sum_{j=1}^n b_j \frac{\partial w}{\partial x_j} + (\sum_{j=1}^n \tau b_j \frac{\partial f_{N, \tau}}{\partial x_j} - c)w\|_{L^2(\Omega)}^2. \]

By

\[ \|P(x, D, \tau)w - \sum_{j=1}^n b_j \frac{\partial w}{\partial x_j} + (\sum_{j=1}^n \tau b_j \frac{\partial f_{N, \tau}}{\partial x_j} - c)w\|_{L^2(\Omega)}^2 \leq C\|P(x, D, \tau)w\|_{L^2(\Omega)}^2 + \|w\|_{H^{1, \tau}(\Omega)}^2, \]

from (58) we have

\[ C\|P(x, D, \tau)w\|_{L^2(\Omega)}^2 + \|w\|_{H^{1, \tau}(\Omega)}^2 \geq \frac{1}{2}\|P_+(x, D, \tau)w\|_{L^2(\Omega)}^2 + \frac{1}{2}\|P_-(x, D, \tau)w\|_{L^2(\Omega)}^2 \]

\[ + NC\|\nabla w\|_{L^2(\Omega)}^2 + \frac{\tau^2 N}{2} \|w\|_{L^2(\Omega)}^2 - \int_{\partial\Omega} \tau \frac{\partial f_{N, \tau}(\varphi)}{\partial \nu} \frac{\partial w}{\partial \nu}^2 d\sigma, \]  

(59)
The term $\|w\|_{H^{1,\tau}(\Omega)}^2$ on the left-hand side can be absorbed into the term $NC\|\nabla w\|_{L^2(\Omega)}^2$ on the left-hand side. Therefore even without assumption (41) we still have (58). Now we fix a parameter $N = 2N_0$ in (58).

Finally we estimate the normal derivative of the function $w$ on the boundary. Let $\rho \in C^2(\overline{\Omega})$ satisfy

$$\langle \rho, \nu \rangle > 0 \quad \text{on } \partial \Omega. \quad (60)$$

Taking the scalar product of the function $P_+(x,D,\tau)w$ and $(\nabla \rho, \nabla w)$ in $L^2(\Omega)$, we obtain

$$\int_{\Omega} P_+(x,D,\tau)w(\nabla \rho, \nabla w) dx = - \int_{\Omega} \left( \frac{1}{2} (\nabla \rho, \nabla |w|^2) + \sum_{k,j=1}^{n} \frac{\partial w}{\partial x_j} \frac{\partial^2 \rho}{\partial x_j \partial x_k} \frac{\partial w}{\partial x_k} + |\nabla f_N(\tau)\rho| (\nabla \rho, \nabla |w|^2) \right) dx$$

$$+ \int_{\partial \Omega} \frac{\partial w}{\partial \nu} (\nabla \rho, \nabla w) d\sigma = \int_{\Omega} \left( \frac{1}{2} \Delta \rho |w|^2 - \sum_{k,j=1}^{n} \frac{\partial w}{\partial x_j} \frac{\partial^2 \rho}{\partial x_j \partial x_k} \frac{\partial w}{\partial x_k} + \text{div} (|\nabla f_N(\tau)\rho| (\nabla \rho, \nabla |w|^2) \right) dx$$

$$+ \int_{\partial \Omega} \frac{\partial w}{\partial \nu} (\nabla \rho, \nabla w) d\sigma - \frac{1}{2} \int_{\partial \Omega} (\nabla \rho, \nu) |\nabla w|^2 d\sigma. \quad (61)$$

Here in order to obtain the last equality we used the equality $\frac{\partial}{\partial x_k} = \frac{\partial w}{\partial \nu}$. The equality (61) and (60) imply

$$\int_{\partial \Omega} \left| \frac{\partial w}{\partial \nu} \right|^2 d\sigma \leq C(\|P(x,D,\tau)w\|_{L^2(\Omega)}^2 + \|w\|_{H^{1,\tau}(\Omega)}^2). \quad (62)$$

From (62) and (58), the estimate (28) follows immediately.

**Remark.** Compare the Carleman estimate (28) with (17), we lose $\tau$ in front of the first term on the right-hand side. On the other hand it can be shown that the inequality (28) is sharp.

Consider a boundary value problem

$$P(x,D)u = \Delta u + qu = f_\tau e^{\tau \varphi} \quad \text{in } \Omega, \quad u|_{\partial \Omega_-} = a_\tau e^{\tau \varphi}. \quad (63)$$

For the problem (63) we can construct solutions with the following properties:

**Proposition 5** Let $b_j \in C^1(\Omega), c \in L^\infty(\Omega), f_\tau \in L^2(\Omega), a_\tau \in L^2(\partial \Omega_-)$ and a function $\varphi$ be weakly pseudoconvex with respect to the principal symbol of the operator $P(x,D)$. Then there exists a solution $u_\tau$ to problem (63) such that

$$\|u_\tau e^{-\tau \varphi}\|_{L^2(\Omega)} \leq C(\|f_\tau\|_{L^2(\Omega)} + \|a_\tau\|_{L^2(\partial \Omega_-)}) \quad \forall \tau \geq \tau_0. \quad (64)$$

If in addition $a_\tau/\|\frac{\partial \varphi}{\partial \nu}\| \in L^2(\partial \Omega_-)$, then there exists a solution to problem (63) such that

$$\|u_\tau e^{-\tau \varphi}\|_{L^2(\Omega)} \leq C(\|f_\tau\|_{L^2(\Omega)} + \|a_\tau\|_{L^2(\partial \Omega_-)}/\sqrt{\tau}) \quad \forall \tau \geq \tau_0. \quad (65)$$

Here the constants $C$ and $\tau_0$ are independent of $\tau$.

**Proof.** Let $X = \{(f,g) \in L^2(\Omega) \times L^2(\partial \Omega \setminus \partial \Omega_-)\}|P(x,D)^*w = f \quad \text{in } \Omega, \quad w|_{\partial \Omega} = 0, \quad \frac{\partial w}{\partial \nu}|_{\partial \Omega \setminus \partial \Omega_-} = g\}$ be a linear subspace of the Hilbert space $L^2_{\tau,\varphi}(\Omega) \times L^2_{\tau,\varphi}(\partial \Omega \setminus \partial \Omega_-)$ which is equipped with the norm

$$\|(f,g)\|_X^2 = \|fe^{\tau \varphi}\|_{L^2(\Omega)}^2 + \tau \|ge^{\tau \varphi}\|_{L^2(\partial \Omega \setminus \partial \Omega_-)}^2. \quad (66)$$
By \(28\), the normed space \(X\) is the closed subspace of the Hilbert space \(L^2_{2,r_0}(\Omega) \times L^2_{2,r_0}(\partial \Omega \setminus \partial \Omega_-)\). Hence \(X\) is a Hilbert space. On \(X\), we consider the linear functional

\[
\ell((f,g)) = -\int_\Omega f_\tau e^{\tau \varphi} w dx - \int_{\partial \Omega_-} a_\tau g e^{\tau \varphi} d\sigma.
\]  

(67)

In order to estimate the norm of the functional \(\ell\), observe that by Carleman estimate \(28\) we have

\[
\|we^{\tau \varphi}\|_{H^1_r(\Omega)} + \|\frac{\partial w}{\partial \nu} e^{\tau \varphi}\|_{L^2(\partial \Omega_-)} \leq \frac{C}{\tau} \|\frac{\partial \varphi}{\partial \nu} e^{\tau \varphi}\|_{L^2(\partial \Omega)} + \frac{C}{\tau^\frac{1}{2}} \|e^{\tau \varphi}\|_{L^2(\partial \Omega \setminus \partial \Omega_-)}.
\]

(68)

Then

\[
\|\ell\| = \sup_{(f,g) \in X \setminus \{0\}} \frac{\|\ell((f,g))\|}{\|f\|_X} \leq C(\|a_\tau\|_{L^2(\partial \Omega_-)} + \|f\|_{L^2(\Omega)}/\tau),
\]

(69)

and if \(a_\tau/|\frac{\partial \varphi}{\partial \nu}|^{\frac{1}{2}} \in L^2(\partial \Omega_-)\), then

\[
\|\ell\| = \sup_{(f,g) \in X \setminus \{0\}} \frac{\|\ell((f,g))\|}{\|f\|_X} \leq C(\|a_\tau\|_{L^2(\partial \Omega_-)} + \|f\|_{L^2(\Omega)}/\sqrt{\tau}) + \|f\|_{L^2(\Omega)}/\tau),
\]

(70)

where the constant \(C\) is independent of \(\tau\).

This functional is bounded on \(X\) and by the Banach theorem it can be extended on the whole space \(L^2_{2,r_0}(\Omega) \times L^2_{2,r_0}(\partial \Omega \setminus \partial \Omega_-)\) with preservation of the norm. Hence, by the Riesz theorem, there exists a pair \((w_\tau, g) \in L^2_{2,r_0}(\Omega) \times L^2_{2,r_0}(\partial \Omega \setminus \partial \Omega_-)\) such that

\[
\ell((f,g)) = (g e^{\tau \varphi}, ge^{\tau \varphi})_{L^2(\partial \Omega \setminus \partial \Omega_-)} + (e^{\tau \varphi} w_\tau, f e^{\tau \varphi})_{L^2(\Omega)}
\]

(71)

and

\[
\|\ell\| = \|(w_\tau, g)\|_{L^2_{2,r_0}(\Omega) \times L^2_{2,r_0}(\partial \Omega \setminus \partial \Omega_-)}.
\]

(72)

By \(67\) and \(71\), the function \(u_\tau = -e^{2\tau \varphi} w_\tau\) solves the problem \(63\). From \(72\) we obtain

\[
\|u_\tau e^{-\tau \varphi}\|_{L^2(\Omega)} = \|w_\tau\|_{L^2_{2,r_0}(\Omega)} \leq \|\ell\|.
\]

(73)

Hence from this estimate and \(69\), \(70\) imply \(61\) and \(63\). ■

**Corollary 2** Let \(b_j \in C^1(\Omega), c \in L^\infty(\Omega)\), the families of functions \(f_\tau\) and \(a_\tau\) be uniformly bounded in \(L^2(\Omega)\) and \(L^\infty(\partial \Omega_-)\) respectively and a function \(\varphi\) be weakly pseudoconvex with respect to the principal symbol of the operator \(P(x, D)\). Then there exist solutions \(u_\tau, \tau > 0\), to the problem \(63\) such that

\[
\|u_\tau e^{-\tau \varphi}\|_{L^2(\Omega)} = o(1) \quad \text{as} \quad \tau \to \infty.
\]

(73)

**Proof.** We set \(\mathcal{Y} = \{x \in \partial \Omega | x \in \partial (\partial \Omega_-)\}\) and \(\partial \Omega_- = \{x \in \partial \Omega_- | dist(x, \mathcal{Y}) \geq \epsilon\}\) for any positive \(\epsilon\). Obviously

\[
mes(\partial \Omega_- \setminus \Omega_-) \to 0 \quad \text{as} \quad \epsilon \to +0.
\]

(74)

We set \(g(\epsilon) = \|\frac{\partial \varphi}{\partial \nu}\|_{C^0(\Omega \setminus \Omega_-)}\). Let \(m(\tau)\) be a positive continuous function such that

\[
m(\tau) \to 0 \quad \text{as} \quad \tau \to +\infty \quad \text{and} \quad g(m(\tau)) \leq C\tau^\frac{1}{2},
\]

(75)

where the constant \(C\) is independent of \(\tau\). We look for the function \(u_\tau\) in the form \(u_\tau = u_{\tau,1} + u_{\tau,2}\) where

\[
P(x, D)u_{\tau,1} = f_\tau e^{\tau \varphi} \quad \text{in} \quad \Omega, \quad u_{\tau,1}|_{\partial \Omega_-} = 0
\]

(76)

and

\[
P(x, D)u_{\tau,2} = 0 \quad \text{in} \quad \Omega, \quad u_{\tau,2}|_{\partial \Omega_-} = a_\tau e^{\tau \varphi}.
\]

(77)
By (64) one can construct a solution to problem (76) such that
\[ \|u_{\tau_1} e^{-\tau_1 \varphi}\|_{L^2(\Omega)} = O\left(\frac{1}{\tau_1}\right) \quad \text{as} \quad \tau_1 \to +\infty. \] (78)

By (74) and (75), we have
\[ \|\chi_{\partial \Omega_{\tau(m)} \setminus \partial \Omega_{\tau}} a_{\tau}\|_{L^2(\partial \Omega)} \to 0 \quad \text{as} \quad \tau \to +0. \] (79)

Using (64) we construct a solution \(w_{\tau}\) to the boundary value problem
\[ P(x, D)w_{\tau} = 0 \quad \text{in} \quad \Omega, \quad w_{\tau}|_{\partial \Omega} = \chi_{\partial \Omega_{\tau(m)} \setminus \partial \Omega_{\tau}} a_{\tau} e^{-\tau \varphi} \] (80)
such that
\[ \|w_{\tau} e^{-\tau \varphi}\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \tau \to +\infty. \] (81)

On the other hand, we have \((1 - \chi_{\partial \Omega_{\tau(m)} \setminus \partial \Omega_{\tau}} a_{\tau}/\|\partial \varphi\|_{L^2(\partial \Omega_{\tau})}) \in L^2(\partial \Omega_{\tau})\) for all \(\tau\) sufficiently large.

Applying (65) and (74), we construct a solution \(\tilde{w}_{\tau}\) to the boundary value problem
\[ P(x, D)\tilde{w}_{\tau} = 0 \quad \text{in} \quad \Omega, \quad \tilde{w}_{\tau}|_{\partial \Omega} = (1 - \chi_{\partial \Omega_{\tau(m)} \setminus \partial \Omega_{\tau}} a_{\tau}) e^{-\tau \varphi} \] (82)
such that
\[ \|\tilde{w}_{\tau} e^{-\tau \varphi}\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\tau}} \|(1 - \chi_{\partial \Omega_{\tau(m)} \setminus \partial \Omega_{\tau}} a_{\tau})/\|\partial \varphi\|_{L^2(\partial \Omega_{\tau})}\|^{1/2} \leq g(m(\tau))\|a_{\tau}\|_{L^2(\partial \Omega_{\tau})} = O\left(\frac{1}{\tau^{1/2}}\right) \quad \text{as} \quad \tau \to +\infty. \] (83)

Finally we set \(u_{\tau_2} = w_{\tau} + \tilde{w}_{\tau}.\) By (74), (80) and (82) the function \(u_{\tau_2}\) solves the problem (85) and by (78), (81) and (83) the estimate (73) holds true. \(\blacksquare\)

In order to prove the uniqueness result in determining a potential of the Schrödinger equation in dimension \(n = 2\), we need to further relax the notion of pseudoconvex function. That is, as a solution of the Eikonal equation we should admit a holomorphic function \(\Phi\) which is degenerate at some points of domain \(\Omega\).

More precisely, let \(\Phi = \varphi + i\psi\) be a holomorphic function in \(\Omega\) such that \(\varphi, \psi\) are real-valued and
\[ \Phi \in C^2(\Omega), \quad \text{Im} \Phi|_{\Gamma_0} = 0, \quad \Gamma_0 \subset \subset \Gamma^*_0, \] (84)
where \(\Gamma^*_0\) is some open set on \(\partial \Omega\). Denote by \(\mathcal{H}\) the set of the critical points of the function \(\Phi\). Assume that
\[ \mathcal{H} \neq \emptyset, \quad \partial^2_2 \Phi(z) \neq 0 \quad \forall z \in \mathcal{H}, \quad \mathcal{H} \cap \partial \Omega \setminus \Gamma_0 = \emptyset \] (85)
and
\[ \int_{\mathcal{J}} 1 d\sigma = 0, \quad \mathcal{J} = \{x; \partial_\varphi \psi(x) = 0, x \in \partial \Omega \setminus \Gamma_0\}. \] (86)

Then \(\Phi\) has only a finite number of critical points and we can set:
\[ \mathcal{H} \setminus \Gamma_0 = \{\bar{x}_1, \ldots, \bar{x}_\ell\}, \quad \mathcal{H} \cap \Gamma_0 = \{\bar{x}_{\ell+1}, \ldots, \bar{x}_{\ell+e}\}. \] (87)

The following proposition was proved in [33].

**Proposition 6** Let \(\bar{x}\) be an arbitrary point in \(\Omega\). There exists a sequence of functions \(\{\Phi_\varepsilon\}_{\varepsilon \in (0, 1)}\) satisfying (84)-(87) such that all the critical points of \(\Phi_\varepsilon\) are nondegenerate and there exists a sequence \(\{\bar{x}_\varepsilon\}, \varepsilon \in (0, 1)\) such that
\[ \bar{x}_\varepsilon \in \mathcal{H}_\varepsilon = \{z \in \Omega; \partial_\varphi \Phi_\varepsilon(z) = 0\}, \quad \bar{x}_\varepsilon \to \bar{x} \quad \text{as} \quad \varepsilon \to +0. \]

Moreover for any \(j\) from \(\{1, \ldots, N\}\) we have
\[ \mathcal{H}_\varepsilon \cap \gamma_j = \emptyset \quad \text{if} \quad \gamma_j \cap (\partial \Omega \setminus \Gamma_0) \neq \emptyset, \]
\[ \mathcal{H}_\varepsilon \cap \gamma_j \subset \Gamma_0 \quad \text{if} \quad \gamma_j \cap (\partial \Omega \setminus \Gamma_0) = \emptyset, \]
\[ \text{Im} \Phi_\varepsilon(\bar{x}_\varepsilon) \notin \{\text{Im} \Phi_\varepsilon(x) | x \in \mathcal{H}_\varepsilon \setminus \{\bar{x}_\varepsilon\}\} \quad \text{and} \quad \text{Im} \Phi_\varepsilon(\bar{x}_\varepsilon) \neq 0. \]
Now we start the proof of the Carleman estimate for the two-dimensional Schrödinger equation. The results of Theorem 11 cannot be applied directly to this case since the weight function is allowed to have critical points. The proof of the Carleman estimate is based on the decomposition of the Laplace operator into $\partial_x$ and $\partial_\tau$.

First we establish a Carleman estimates for the operators $\partial_x$ and $\partial_\tau$.

**Proposition 7** Let $\Phi$ satisfy (87), $\tau \in \mathbb{R}^1$, and the function $C = C_1 + iC_2$ belong to $C^1(\Omega)$ where $C_1, C_2$ are real-valued. Let $f \in L^2(\Omega)$, and $\tilde{v} \in H^1(\Omega)$ be a solution to

$$2\frac{\partial}{\partial z} \tilde{v} - i\frac{\partial \Phi}{\partial z} \tilde{v} + C\tilde{v} = \tilde{f} \quad \text{in } \Omega \quad (88)$$

or let $\tilde{v}$ be a solution to

$$2\frac{\partial}{\partial z} \tilde{v} - \tau \frac{\partial \Phi}{\partial z} \tilde{v} + C\tilde{v} = \tilde{f} \quad \text{in } \Omega. \quad (89)$$

In the case (88) we have

$$\| \frac{\partial \tilde{v}}{\partial x_1} - i\text{Im}(\tau \frac{\partial \Phi}{\partial x_2} - C)\tilde{v} \|^2_{L^2(\Omega)} - \int_{\partial \Omega} \left( \tau \frac{\partial \tilde{v}}{\partial \nu} - (\nu_1 C_1 + \nu_2 C_2) \right) |\tilde{v}|^2 d\sigma - \int_{\Omega} \left( \frac{\partial C_1}{\partial x_1} + \frac{\partial C_2}{\partial x_2} \right) |\tilde{v}|^2 dx$$

$$+ \text{Re} \int_{\partial \Omega} i \frac{\partial \tilde{v}}{\partial \nu} d\sigma + \| - \frac{1}{i} \frac{\partial \tilde{v}}{\partial x_2} - \text{Re}(\tau \frac{\partial \Phi}{\partial z} - C)\tilde{v} \|^2_{L^2(\Omega)} = \| \tilde{f} \|^2_{L^2(\Omega)}. \quad (90)$$

In the case (89) we have

$$\| \frac{\partial \tilde{v}}{\partial x_1} - i\text{Im}(\tau \frac{\partial \Phi}{\partial x_2} - C)\tilde{v} \|^2_{L^2(\Omega)} - \int_{\partial \Omega} \left( \tau \frac{\partial \tilde{v}}{\partial \nu} - (\nu_1 C_1 + \nu_2 C_2) \right) |\tilde{v}|^2 d\sigma - \int_{\Omega} \left( \frac{\partial C_1}{\partial x_1} - \frac{\partial C_2}{\partial x_2} \right) |\tilde{v}|^2 dx$$

$$- \text{Re} \int_{\partial \Omega} i \frac{\partial \tilde{v}}{\partial \nu} d\sigma + \| - \frac{1}{i} \frac{\partial \tilde{v}}{\partial x_2} - \text{Re}(\tau \frac{\partial \Phi}{\partial z} - C)\tilde{v} \|^2_{L^2(\Omega)} = \| \tilde{f} \|^2_{L^2(\Omega)}. \quad (91)$$

**Proof.** We prove the equality (90). The proof of equality (91) is the same. Denote $L_-(x, D, \tau)\tilde{v} = \frac{\partial \tilde{v}}{\partial x_1} - i\text{Im}(\tau \frac{\partial \Phi}{\partial x_2} - C)\tilde{v}$ and $L_+(x, D, \tau)\tilde{v} = \frac{\partial \tilde{v}}{\partial x_2} - \text{Re}(\tau \frac{\partial \Phi}{\partial z} - C)\tilde{v}$. In the new notations we rewrite equation (89) as

$$L_-(x, D, \tau)\tilde{v} + L_+(x, D, \tau)\tilde{v} = \tilde{f} \quad \text{in } \Omega. \quad (92)$$

Taking the $L^2$- norm of the left- and right-hand sides of (92), we obtain

$$\|L_+(x, D, \tau)\tilde{v}\|^2_{L^2(\Omega)} + 2\text{Re}(L_+(x, D, \tau)\tilde{v}, L_-(x, D, \tau)\tilde{v})_{L^2(\Omega)} + \|L_-(x, D, \tau)\tilde{v}\|^2_{L^2(\Omega)} = \|\tilde{f}\|^2_{L^2(\Omega)}. \quad (93)$$

Integrating by parts the second term of (93), we obtain

$$2\text{Re}(L_+(x, D, \tau)\tilde{v}, L_-(x, D, \tau)\tilde{v})_{L^2(\Omega)} = \text{Re}\{(L_+, L_-)\tilde{v}, \tilde{v}\}_{L^2(\Omega)}$$

$$+ \int_{\partial \Omega} \{(L_-(x, D, \tau)\tilde{v})\nu_2 \tilde{v} + L_+(x, D, \tau)\nu_1 \tilde{v}\} d\sigma. \quad (94)$$

The Cauchy-Riemann equations yield

$$[L_+, L_-] = -\left( \frac{\partial C_1}{\partial x_1} + \frac{\partial C_2}{\partial x_2} \right). \quad (95)$$

Using the Cauchy-Riemann equations again, we observe

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \phi}{\partial x_1} - i \frac{\partial \phi}{\partial x_2}.$$
Therefore
\[
\Re \int_{\partial \Omega} ((L_-(x, D, \tau)v)\nu_2 \bar{v} + L_+(x, D, \tau)\nu_1 \bar{v}) d\sigma = \Re \int_{\partial \Omega} \left( \frac{\partial \bar{v}}{\partial x_1} + i \left( \frac{\partial \bar{v}}{\partial x_2} + C_2 \right)v \right) \nu_2 \bar{v} + \left( \frac{1}{i} \frac{\partial \bar{v}}{\partial x_2} - \tau \frac{\partial \bar{v}}{\partial x_1} + C_1 \bar{v} \right) \nu_1 d\sigma = - \int_{\partial \Omega} \left( \frac{\partial \bar{v}}{\partial \nu} - (\nu_1 C_1 + \nu_2 C_2) \right) |\bar{v}|^2 d\sigma + \Re \int_{\partial \Omega} \frac{i}{i} \frac{\partial \bar{v}}{\partial \nu} \bar{v} d\sigma.
\]
(96)

From (88)–(90) we obtain (50).

Consider a boundary value problem
\[
\mathcal{K}(x, D)u = \left( 4 \frac{\partial}{\partial z} + 2A \frac{\partial}{\partial z} + 2B \frac{\partial}{\partial \bar{z}} \right) u = f \text{ in } \Omega, \quad u|_{\partial \Omega} = 0.
\]

For this problem we have the following Carleman estimate with boundary terms.

**Proposition 8** (77) Suppose that \( \Phi \) satisfies (74)–(76), \( u \in H^1_0(\Omega) \) and \( \|A\|_{L^\infty(\Omega)} + \|B\|_{L^\infty(\Omega)} \leq K \). Then there exist \( \tau_0 = \tau_0(K, \Phi) \) and \( C = C(K, \Phi) \) independent of \( u \) and \( \tau \) such that for all \( |\tau| > \tau_0 \)
\[
|\tau| \|ue^{\tau \varphi}\|_{L^2(\Omega)} + \|ue^{\tau \varphi}\|_{H^1(\Omega)} + \|\frac{\partial u}{\partial \nu} e^{\tau \varphi}\|_{L^2(\Omega)} + 2\|\frac{\partial \Phi}{\partial \nu} e^{\tau \varphi}\|_{L^2(\Omega)}^2 \leq C \left( \|(\mathcal{K}(x, D)u)e^{\tau \varphi}\|_{L^2(\Omega)} + |\tau| \int_{\partial \Omega} |\frac{\partial u}{\partial \nu} e^{2\tau \varphi} d\sigma \right).
\]
(97)

**Proof.** Denote \( \bar{v} = ue^{\tau \varphi} \) and \( \mathcal{K}(x, D)u = f \). Observe that \( \varphi(x_1, x_2) = \frac{1}{2}(\Phi(z) + \bar{\Phi}(\bar{z})) \). Therefore
\[
e^{\tau \varphi} \Delta (e^{\tau \varphi} \bar{v}) = \left( \frac{\partial \Phi}{\partial \bar{z}} - \tau \frac{\partial \Phi}{\partial z} \right) e^{\tau \varphi} \bar{v} = \left( f - 2B \frac{\partial u}{\partial \bar{z}} + 2A \frac{\partial u}{\partial z} \right) e^{\tau \varphi}.
\]

Assume now that \( u \) is a real-valued function. Denote \( \bar{w} = (2 \frac{\partial u}{\partial \bar{z}} - \tau \frac{\partial u}{\partial z}) \bar{v} \).

Thanks to the zero Dirichlet boundary condition for \( u \), we have
\[
\bar{w}|_{\partial \Omega} = 2 \frac{\partial \bar{u}}{\partial z}|_{\partial \Omega} = (\nu_1 + i\nu_2) \frac{\partial \bar{u}}{\partial \nu}|_{\partial \Omega}.
\]

Let \( \mathcal{C} \) be some smooth real-valued vector function in \( \Omega \) such that
\[
2 \frac{\partial \mathcal{C}}{\partial z} = C(x) = C_1(x) + iC_2(x) \quad \text{in } \Omega, \quad \Im \mathcal{C} = 0 \quad \text{on } \Gamma_0,
\]
where \( \bar{C} = (C_1, C_2) \) is a smooth function in \( \Omega \) such that
\[
div \bar{C} = 1 \quad \text{in } \Omega, \quad (\nu, \bar{C}) = -1 \quad \text{on } \Gamma_0.
\]
(98)

By Proposition 7 we have the following integral equality:
\[
\left| \frac{\partial (\bar{w}e^{NC})}{\partial x_1} - i\Im \left( \frac{\partial \Phi}{\partial \bar{z}} + NC \right)(\bar{w}e^{NC}) \right|^2_{L^2(\Omega)} - \int_{\partial \Omega} \left( \tau \frac{\partial \Phi}{\partial \nu} + N(\nu_1 C_1 + \nu_2 C_2) \right) \left| \frac{\partial \bar{u}}{\partial \nu} e^{NC} \right|^2 d\sigma
\]
\[
+ \int_{\partial \Omega} |\bar{w}e^{NC}|^2 d\sigma + \Re \int_{\partial \Omega} i \frac{\partial (\bar{w}e^{NC})}{\partial x_2} \bar{w}e^{NC} d\sigma
\]
\[
+ \left| - \frac{1}{i} \frac{\partial (\bar{w}e^{NC})}{\partial x_2} - \Re (\frac{\partial \Phi}{\partial \nu} + NC)(\bar{w}e^{NC}) \right|^2_{L^2(\Omega)} = \|f e^{\tau \varphi + NC}\|_{L^2(\Omega)}^2.
\]
(99)

We now simplify the integral \( \Re \int_{\partial \Omega} i \frac{\partial (\bar{w}e^{NC})}{\partial x_2} \bar{w}e^{NC} d\sigma \). We recall that \( \bar{v} = ue^{\tau \varphi} \) in \( \Omega \) and \( \bar{w} = (\nu_1 + i\nu_2) \frac{\partial \bar{u}}{\partial \nu} e^{\tau \varphi} \) on \( \partial \Omega \). Denote \( (\nu_1 + i\nu_2) e^{N \Im \mathcal{C}} = R + iP \) where \( R, P \) are real-valued. Therefore
\[
\Re \int_{\partial \Omega} i \frac{\partial (\bar{w}e^{NC})}{\partial x_2} \bar{w}e^{NC} d\sigma
\]
\[
= \Re \int_{\partial \Omega} \left( (R + iP) \frac{\partial \bar{u}}{\partial \nu} e^{\tau \varphi + N \Re \mathcal{C}} \right) (R - iP) \frac{\partial \bar{u}}{\partial \nu} e^{\tau \varphi + N \Re \mathcal{C}} d\sigma
\]
\[
= \Re \int_{\partial \Omega} \frac{i}{i} \frac{\partial (\bar{u}e^{NC})}{\partial \nu} \left| \frac{\partial \bar{u}e^{NC}}{\partial \nu} \right|^2 d\sigma.
\]
(100)
Using the above formula in (99), we obtain

\[ \left\| \frac{\partial (\bar{w}e^{NC})}{\partial x_1} - \text{Im}(\tau \frac{\partial \Phi}{\partial z} + NC) e^{NC} \right\|^2_{L^2(\Omega)} = \int_{\partial \Omega} \left( \frac{\partial \Phi}{\partial \nu} + N(\nu_1 C_1 + \nu_2 C_2) \right) \frac{\partial \bar{w}e^{NC}}{\partial \nu} e^{NC} d\sigma \]

\[ + N \int_\Omega |\bar{w}e^{NC}|^2 dx + \text{Re} \int_{\partial \Omega} \left( \frac{i}{\partial \nu}(R + iP) \right) \left| \frac{\partial (\bar{w}e^{NC})}{\partial \nu} \right|^2 (R - iP) d\sigma \]

\[ + \| - \frac{1}{i} \frac{\partial (\bar{w}e^{NC})}{\partial x_2} - \text{Re}(\tau \frac{\partial \Phi}{\partial z} + NC) (\bar{w}e^{NC}) \|^2_{L^2(\Omega)} = \| \tilde{f} e^{\tau + NC} \|^2_{L^2(\Omega)}. \]

Taking a sufficiently large positive parameter \( N \) and taking into account that the function \( R + iP \) is independent of \( N \) on \( \Gamma_0 \), we conclude from (101), (98)

\[ - \int_{\partial \Omega} \left( \tau \frac{\partial \varphi}{\partial \nu} + \frac{N}{2}(\nu_1 C_1 + \nu_2 C_2) \right) \frac{\partial \bar{w}e^{NC}}{\partial \nu} e^{NC} d\sigma + N \int_\Omega |\bar{w}e^{NC}|^2 dx \leq \| \tilde{f} e^{\tau + NC} \|^2_{L^2(\Omega)} + C(N) \int_{\partial \Omega \setminus \Gamma_0} \frac{\partial \nu}{\partial \nu} e^{NC} d\sigma. \]

Simple computations give

\[ 4 \left\| \frac{\partial (\bar{w}e^{NC} R)}{\partial z} \right\|^2_{L^2(\Omega)} + \tau^2 \left\| \frac{\partial \Phi}{\partial z} (\bar{w}e^{NC} R) \right\|^2_{L^2(\Omega)} = \left\| \frac{\partial (\bar{w}e^{NC} R)}{\partial z} - \tau \frac{\partial \Phi}{\partial z} (\bar{w}e^{NC} R) \right\|^2_{L^2(\Omega)} \]

\[ \leq 2 \left\| \bar{w}e^{NC} R \right\|^2_{L^2(\Omega)} + C(N) \int_{\partial \Omega \setminus \Gamma_0} \left\| \frac{\partial \nu}{\partial \nu} e^{NC} \right\|^2 d\sigma. \]

Since the function \( \Phi \) has zeros of at most second order by assumption (85), there exists a constant \( C > 0 \) independent of \( \tau \) such that

\[ \tau \left\| \bar{w}e^{NC} R \right\|^2_{L^2(\Omega)} \leq C \left( \left\| \bar{w}e^{NC} R \right\|^2_{H^1(\Omega)} + \tau^2 \left\| \frac{\partial \Phi}{\partial z} (\bar{w}e^{NC} R) \right\|^2_{L^2(\Omega)} \right). \]

Therefore by (102), (104) there exists \( N_0 > 0 \) such that for any \( N > N_0 \) there exists \( \tau_0(N) \) such that

\[ - \int_{\partial \Omega} \left( \tau \frac{\partial \varphi}{\partial \nu} + \frac{N}{2}(\nu_1 C_1 + \nu_2 C_2) \right) \frac{\partial \bar{w}e^{NC}}{\partial \nu} e^{NC} d\sigma + \frac{N}{2} \int_\Omega |\bar{w}e^{NC}|^2 dx \]

\[ + \tau \left\| \bar{w}e^{NC} R \right\|^2_{L^2(\Omega)} + \left\| \bar{w}e^{NC} R \right\|^2_{H^1(\Omega)} + \tau^2 \left\| \frac{\partial \Phi}{\partial z} (\bar{w}e^{NC} R) \right\|^2_{L^2(\Omega)} \]

\[ \leq \tilde{f} e^{\tau \Phi + NC} \left\| \bar{w}e^{NC} \right\|^2_{L^2(\Omega)} + C(N) \int_{\partial \Omega \setminus \Gamma_0} \left\| \frac{\partial \nu}{\partial \nu} e^{NC} \right\|^2 d\sigma \]

for all \( \tau > \tau_0(N) \).

In order to remove the assumption that \( u \) is real-valued, we obtain (105) separately for the real and imaginary parts of \( u \) and combine them. This concludes the proof of the proposition. ■

As a corollary we derive a Carleman inequality for the function \( u \) which satisfies the integral equality

\[ (u, K(x, D)^* w)_{L^2(\Omega)} + (f, w)_{H^{1, \tau}(\Omega)} + (ge^{\tau \varphi}, e^{-\tau \varphi} w)_{H^{1, \tau}(\partial \Omega \setminus \Gamma_0)} = 0 \]

for all \( w \in \mathcal{X} = \{ w \in H^1(\Omega) \mid w|_{\Gamma_0} = 0 \}, K(x, D)^* w \in L^2(\Omega) \). We have

**Corollary 3** Suppose that \( \Phi \) satisfies (84)-(86), \( f \in H^1(\Omega), g \in H^{1, \tau}(\partial \Omega \setminus \Gamma_0), u \in L^2(\Omega) \) and the coefficients \( A, B \) of \( K(x, D) \) belong to \( \{ C \in C^1(\Omega) \mid \| C \|_{C^1(\Omega)} \leq K \} \). Then there exist \( \tau_0 = \tau_0(K, \Phi) \) and \( C = C(K, \Phi) \), independent of \( u \) and \( \tau \), such that

\[ \| u e^{\tau \varphi} \|^2_{L^2(\Omega)} \leq C \tau \left( \| f e^{\tau \varphi} \|^2_{H^{1, \tau}(\Omega)} + \| g e^{\tau \varphi} \|^2_{H^{1, \tau}(\partial \Omega \setminus \Gamma_0)} \right) \]

\[ \forall |\tau| \geq \tau_0 \]

for solutions of (100).
Proof. Let $\epsilon$ be some positive number and $d(x)$ be a smooth positive function on $\partial \Omega \setminus \Gamma_0$ which blows up like $\frac{1}{|x|^{\alpha}}$ for any $y \in \partial (\partial \Omega \setminus \Gamma_0)$. Consider an extremal problem

$$J_\epsilon(w) = \frac{1}{2}||we^{-\tau \varphi}||^2_{L^2(\Omega)} + \frac{1}{2\epsilon}||\mathcal{K}(x, D)^*w - \epsilon e^{2\tau \varphi}||^2_{L^2(\Omega)} + \frac{1}{2|\tau|}||we^{-\varphi}||^2_{L^2(\partial \Omega \setminus \Gamma_0)} \to \inf$$

for

$$w \in \hat{\mathcal{X}} = \{w \in H^2(\Omega) | \mathcal{K}(x, D)^*w \in L^2(\Omega), w|_{\Gamma_0} = 0\}.$$  

(108)

There exists a unique solution to (108), (109) which we denote by $\tilde{w}_\epsilon$. Using this estimate and a standard duality argument, the statement of Corollary 3 follows immediately.

(109)

Using the notation $p_\epsilon = \frac{1}{\epsilon}(\mathcal{K}(x, D)^*\tilde{w}_\epsilon - \epsilon e^{2\tau \varphi})$, we see

$$\mathcal{K}(x, D)p_\epsilon + \tilde{w}_\epsilon e^{-2\tau \varphi} = 0 \quad \forall \varphi \in \hat{\mathcal{X}}.$$  

(110)

By Proposition 5 we have

$$|\tau||p_\epsilon e^{\tau \varphi}||^2_{L^2(\Omega)} + \|p_\epsilon e^{\tau \varphi}||^2_{H^1(\Omega)} + \|\frac{\partial p_\epsilon}{\partial \nu} e^{\tau \varphi}||^2_{L^2(\partial \Omega \setminus \Gamma_0)} + \tau^2 \|\frac{\partial \Phi}{\partial \tau} p_\epsilon e^{\tau \varphi}||^2_{L^2(\Omega)} \leq C \left(\|\tilde{w}_\epsilon e^{-\tau \varphi}||^2_{L^2(\Omega)} + \frac{1}{|\tau|} \int_{\partial \Omega \setminus \Gamma_0} |\tilde{w}_\epsilon|^2 e^{-2\tau \varphi} d\sigma\right) \\ 

(112)

Substituting $\delta = \tilde{w}_\epsilon$ in (110), we obtain

$$2J_\epsilon(\tilde{w}_\epsilon) + \text{Re}(\epsilon e^{2\tau \varphi}, p_\epsilon)_{L^2(\Omega)} = 0.$$  

(113)

Applying estimate (112) to the second term of the above equality, we have

$$|\tau|J_\epsilon(\tilde{w}_\epsilon) \leq C||e^{\tau \varphi}||^2_{L^2(\Omega)}.$$  

(114)

Using this estimate, we pass to the limit in (111) as $\epsilon$ goes to zero. We obtain

$$\mathcal{K}(x, D)p + \tilde{w}_\epsilon e^{-2\tau \varphi} = 0 \quad \text{in } \Omega, \quad p|_{\partial \Omega} = 0, \quad \frac{\partial p}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = \frac{d}{|\tau|} \tilde{w}_\epsilon e^{-2\tau \varphi}.$$  

(115)

and

$$|\tau||\tilde{w}_\epsilon e^{-\tau \varphi}||^2_{L^2(\Omega)} + \|\tilde{w}_\epsilon e^{-\tau \varphi}||^2_{L^2(\partial \Omega \setminus \Gamma_0)} \leq C||e^{\tau \varphi}||^2_{L^2(\Omega)}.$$  

(116)

Since $\tilde{w} \in L^2(\Omega)$, we have $p \in H^2(\Omega)$ and $\frac{\partial p}{\partial \nu} \in H^\frac{1}{2}(\partial \Omega)$ by the trace theorem. The relation (113) implies

$$\tilde{w} \in H^\frac{1}{2}(\partial \Omega \setminus \Gamma_0).$$  

(117)

and

$$\tilde{w} \in L^2(\partial \Omega \setminus \Gamma_0).$$  

(118)

Taking the scalar product of (114) with $\tilde{w}_\epsilon e^{-2\tau \varphi}$ and using estimates (116) and (115), we obtain

$$\frac{1}{|\tau|} ||\nabla \tilde{w}_\epsilon e^{-\tau \varphi}||^2_{L^2(\Omega)} + |\tau||\tilde{w}_\epsilon e^{-\tau \varphi}||^2_{L^2(\Omega)} + \frac{1}{|\tau|} ||\tilde{w}_\epsilon e^{-\tau \varphi}||^2_{H^\frac{1}{2}(\partial \Omega \setminus \Gamma_0)} \leq C||e^{\tau \varphi}||^2_{L^2(\Omega)}.$$  

(119)

From this estimate and a standard duality argument, the statement of Corollary 3 follows immediately. 

Consider the following problem

$$L_q(x, D)u = \Delta u + qu = fe^{\tau \varphi} \quad \text{in } \Omega, \quad u|_{\Gamma_0} = ge^{\tau \varphi}.$$  

(120)

We have
**Proposition 9** Let \( q \in L^\infty(\Omega) \), \( \Phi \) satisfy (4)-(5), \( f \in L^2(\Omega) \), \( g \in H^\perp(\partial\Omega) \). There exists \( \tau_0 > 0 \) such that for all \( \tau > \tau_0 \) there exists a solution to the boundary value problem (118) such that

\[
\frac{1}{\sqrt{\tau}} \| \nabla u e^{-\tau \varphi} \|_{L^2(\Omega)} + \sqrt{\tau} \| u e^{-\tau \varphi} \|_{L^2(\Omega)} \leq C(\| f \|_{L^2(\Omega)} + \| g \|_{H^\perp(\partial\Omega)}).
\]  

**Proof.** First we reduce the problem (118) to the case \( g = 0 \). Let \( r(z) \) be a holomorphic function and \( \tilde{r}(\bar{z}) \) be an antiholomorphic function such that \( (r + \tilde{r})|_{\Gamma_0} = g \) and

\[
\| r \|_{H^1(\Omega)} + \| \tilde{r} \|_{H^1(\Omega)} \leq C \| g \|_{H^\perp(\partial\Omega)}.
\]

The existence of such functions \( r, \tilde{r} \) follows from the Fredholm theorem combined with the possibility of an arbitrary choice of the Dirichlet data on the part of the boundary.

We look for a solution \( u \) in the form

\[
u = (e^{r\Phi} + e^{\bar{r}\bar{\Phi}}) + \bar{u},
\]

where

\[
L_q(x, D)\bar{u} = \tilde{f} e^{r\varphi} \quad \text{in} \quad \Omega, \quad \bar{u}|_{\Gamma_0} = 0
\]

and \( \tilde{f} = f - qre^{r\varphi} - q\bar{e}e^{-r\varphi} \).

In order to prove (119), we consider the following extremal problem:

\[
\mathcal{I}_\epsilon(u) = \frac{1}{2} \| u e^{-\tau \varphi} \|_{H^1(\Omega)}^2 + \frac{1}{2\epsilon} \| L_q(x, D)u - \tilde{f} e^{r\varphi} \|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon} \| u e^{-\tau \varphi} \|_{H^\perp(\partial\Omega \setminus \Gamma_0)}^2 \to \inf
\]

for

\[
u \in \mathcal{Y} = \{ w \in H^1(\Omega) | L_q(x, D)w \in L^2(\Omega), \ w|_{\Gamma_0} = 0 \}.
\]

There exists a unique solution to problem (122) which we denote by \( \hat{u}_\epsilon \). By Fermat’s theorem, we have

\[
\mathcal{I}_\epsilon'(\hat{u}_\epsilon)[\delta] = 0 \quad \forall \delta \in \mathcal{Y}.
\]

Let \( p_\epsilon = \frac{1}{\epsilon}(L_q(x, D)\hat{u}_\epsilon - \tilde{f} e^{r\varphi}) \). Applying Corollary 3 we obtain from (123)

\[
\frac{1}{\| \tau \|} \| p_\epsilon e^{r\varphi} \|_{L^2(\Omega)}^2 \leq C(\| \hat{u}_\epsilon e^{-\tau \varphi} \|_{H^1(\Omega)}^2 + \| \hat{u}_\epsilon e^{-\tau \varphi} \|_{H^\perp(\partial\Omega \setminus \Gamma_0)}^2) \leq 2C\mathcal{I}_\epsilon(\hat{u}_\epsilon).
\]

Substituting in (123) with \( \delta = \hat{u}_\epsilon \), we obtain

\[
2\mathcal{I}_\epsilon(\hat{u}_\epsilon) + \text{Re}(p_\epsilon, \tilde{f} e^{r\varphi})_{L^2(\Omega)} = 0.
\]

Applying estimate (124) to this equality, we have

\[
\mathcal{I}_\epsilon(\hat{u}_\epsilon) \leq C\| \tau \| \| \tilde{f} \|_{L^2(\Omega)}^2.
\]

Using this estimate, we pass to the limit as \( \epsilon \to +0 \). We obtain

\[
L_q(x, D)u - \tilde{f} e^{r\varphi} = 0 \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0
\]

and

\[
\| u e^{-\tau \varphi} \|_{H^1(\Omega)}^2 + \| u e^{-\tau \varphi} \|_{L^2(\partial\Omega \setminus \Gamma_0)}^2 \leq C\| \tau \| \| \tilde{f} \|_{L^2(\Omega)}^2.
\]

Since \( \| \tilde{f} \|_{L^2(\Omega)} \leq C(\| f \|_{L^2(\Omega)} + \| g \|_{H^\perp(\partial\Omega)}) \), inequality (126) implies (119).

This finishes the proof of the proposition. □
3 Uniqueness in the three dimensional case by Dirichlet-to-Neumann map on subboundaries

On the basis of Carleman estimates in Section 2, we show uniqueness results in three dimensions. We recall that $\Gamma_{\pm}$ are some subsets of $\partial \Omega$ and for the Schrödinger operator with potential $q$ we consider the Dirichlet-to-Neumann map $\Lambda(q, \Gamma_-, \Gamma_+)$ on subboundaries $\Gamma_-$ and $\Gamma_+$:

$$
\Lambda(q, \Gamma_-, \Gamma_+(f) = \frac{\partial u}{\partial \nu}\big|_{\partial \Omega_{\Gamma_+}},
$$

where

$$
L_q(x, D)u = \Delta u + qu = 0 \quad \text{in} \ \Omega, \quad u|_{\Gamma_-} = 0, \quad u|_{\partial \Omega_{\Gamma_-}} = f.
$$

Consider two particular cases of the subboundaries $\Gamma_{\pm}$. Let $\vec{v}$ be a unit vector in $\mathbb{R}^3$. We introduce two subsets of the boundary $\partial \Omega$:

$$
\Gamma_+(\vec{v}) = \{x \in \partial \Omega | (\vec{v}, \vec{v}) > 0\}, \quad \Gamma_-(\vec{v}) = \{x \in \partial \Omega | (\vec{v}, \vec{v}) < 0\}.
$$

(127)

We will show three uniqueness results and the first two are concerned with determination of potentials.

**Theorem 4** Let $n = 3$, $q_1, q_2 \in L^\infty(\Omega)$, 0 be not an eigenvalue of the Schrödinger operators $L_{q_1}(x, D), L_{q_2}(x, D)$ and $\Lambda(q_1, \Gamma_-(\vec{v}), \Gamma_+(\vec{v})) = \Lambda(q_2, \Gamma_-(\vec{v}), \Gamma_+(\vec{v}))$ for some unit vector $\vec{v}$ Then $q_1 = q_2$ in $\Omega$.

Let $x^0$ be a point in $\mathbb{R}^3$ such that this point and domain $\Omega$ are separated by some plane.

We introduce the following subsets of $\partial \Omega$:

$$
\Gamma_+(x^0) = \{x \in \partial \Omega | (x - x^0, \vec{v}) > 0\}, \quad \Gamma_-(x^0) = \{x \in \partial \Omega | (x - x^0, \vec{v}) < 0\}.
$$

We have

**Theorem 5** Let $n = 3$, $q_1, q_2 \in L^2(\Omega)$, 0 be not an eigenvalue of the Schrödinger operators $L_{q_1}(x, D), L_{q_2}(x, D)$ and $\Lambda(q_1, \Gamma_-(x^0), \Gamma_+(x^0)) = \Lambda(q_2, \Gamma_-(x^0), \Gamma_+(x^0))$ for some $x^0$ which can be separated from $\Omega$ by a plane. Then $q_1 = q_2$ in $\Omega$.

**Remark 1.** Theorem 4 improved the result of Theorem 1.1 of [55]. Unlike [55], we do not need consider the neighborhoods of the sets $F(x_0)$ and $B(x_0) \cup \{x \in \partial \Omega | (x - x_0, \vec{v}) = 0\}$ (here we are using notations of [55]), but precisely these sets are sufficient as subboundaries. This in turn reduces the amount of the information which is used for the determination of a potential of the Schrödinger equation.

**Remark 2.** The assumptions of Theorems 4, 5 and the corresponding theorems from [10], [55] require the access to the whole boundary $\partial \Omega$, that is, to any point of the boundary we have to either apply the voltage or measure current. The Calderón’s problem was motivated by the search of the oil fields which are located underground, but voltage and current should be measured only on the surface. It is the most interesting and practically important that we need not apply voltage and not measure the current on the sufficiently large part of the boundary. In the three dimensional case, there are a few results results in this direction. The paper [46] treats the case when roughly speaking $\Omega$ is a half of the plane or sphere. During the preparation of this manuscript two more articles appeared: [44] and [54]. The paper [44] established the uniqueness in the case of cylindrical domain.

**Proof of Theorem 4.** Without loss of generality, performing a rotation around the origin if necessary, we can assume that $\vec{v} = \vec{e}_3 = (0, 0, 1)$ and set

$$
\Gamma_+ = \{x \in \partial \Omega | (\vec{e}_3, \vec{v}) > 0\}, \quad \Gamma_- = \{x \in \partial \Omega | (\vec{e}_3, \vec{v}) < 0\}.
$$

(128)

Let $b(s) \in C^2(\mathbb{R}^1)$ be an arbitrary function and $z \in \mathbb{C}^1, \theta \in [0, 2\pi]$ are some parameters and $g(\theta, x') = b(sin(\theta) x_1 - cos(\theta) x_2)$. We construct a complex geometric optics solution for the Schrödinger equation in the form

$$
u_1(x) = e^{(\tau + iz)\theta} g(\theta, x') + e^{ix^3} o_{L^2(\Omega)}(1) \quad \text{as} \ \tau \to +\infty,
$$

(129)
where \( \Phi(x) = x_3 + i(\cos(\theta)x_1 + \sin(\theta)x_2) \).

Indeed

\[
\Delta(e^{(\tau+iz)\Phi} g(\theta, x')) = ((\tau + iz)^2(\nabla \Phi, \nabla \Phi) + 2(\tau + iz)(\nabla \Phi, \nabla g) + \Delta g)e^{(\tau+iz)\Phi}.
\]

Observing that \((\nabla \Phi, \nabla \Phi) = (\nabla \Phi, \nabla g) = 0\), we obtain

\[
\Delta(e^{(\tau+iz)\Phi} g(\theta, x')) = (\Delta g)e^{(\tau+iz)\Phi}.
\]

Then using Corollary 2 we construct the function \( w_\tau(z, \cdot) \) which solves the boundary value problem

\[
L_{q_1}(x, D)w_\tau = -e^{(\tau+iz)\Phi}L_{q_1}(x, D)g \quad \text{in} \; \Omega, \quad w_\tau|_{\Gamma_-} = -ge^{(\tau+iz)\Phi}
\]

and satisfies the estimate

\[
\|e^{-\tau x_3}w_\tau\|_{L^2(\Omega)} = o(1) \quad \text{as} \; \tau \to +\infty.
\]

By (130) and (131), the function

\[
u_1 = ge^{(\tau+iz)\Phi} + w_\tau
\]

solves the Schrödinger equation

\[
L_{q_1}(x, D)u_1 = 0 \quad \text{in} \; \Omega, \quad u_1|_{\Gamma_-} = 0
\]

and admits the asymptotic expansion (129) by (132).

Since \( \Lambda(q_1, \Gamma_-, \Gamma_+) = \Lambda(q_2, \Gamma_-, \Gamma_+) \), there exists \( u_2 \) such that

\[
L_{q_2}(x, D)u_2 = 0 \quad \text{in} \; \Omega, \quad (u_1 - u_2)|_{\partial\Omega} = \frac{\partial(u_1 - u_2)}{\partial \nu}|_{\partial\Omega \setminus \Gamma_+} = 0.
\]

Next, in a way similar to the construction of \( u_1 \), we construct the complex geometric optics solution \( v \) to the Schrödinger operator with potential \( q_2 \)

\[
L_{q_2}(x, D)v = 0 \quad \text{in} \; \Omega, \quad v|_{\Gamma_+} = 0
\]

in the form

\[
v(x) = e^{-\tau \Phi} + e^{-\tau x_3}o_{L^2(\Omega)}(1) \quad \text{as} \; \tau \to +\infty.
\]

Setting \( u = u_1 - u_2 \), by (133) and (134), we have

\[
L_{q_2}(x, D)u = -(q_1 - q_2)u_1 \quad \text{in} \; \Omega, \quad u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial\Omega \setminus \Gamma_+} = 0.
\]

Then using (135) and (137), we obtain

\[
- \int\Omega (q_1 - q_2)u_1vdx = (L_{q_2}(x, D)u, v)_{L^2(\Omega)} = (u, L_{q_2}(x, D)v)_{L^2(\Omega)} + (\frac{\partial u}{\partial \nu}, v)_{L^2(\partial\Omega)} - (u, \frac{\partial v}{\partial \nu})_{L^2(\partial\Omega)}
\]

\[
= (v, \frac{\partial u}{\partial \nu})_{L^2(\partial\Omega \setminus \Gamma_+)} + (v, \frac{\partial u}{\partial \nu})_{L^2(\Gamma_+)} = 0.
\]

Hence

\[
\int\Omega (q_1 - q_2)e^{iz\Phi}g(\theta, x')dx = 0.
\]

Let \( \Pi = G \times [-K, K] \) be such a cylinder that \( \Omega \subset \Pi \). We extend the function \( (q_1 - q_2) \) by zero on \( \Pi \setminus \Omega \) and set

\[
r_z(x') = \int_{-K}^K (q_1 - q_2)e^{iz\cdot x_3}dx_3, \quad r_{z,k}(x') = \int_{-K}^K (q_1 - q_2)e^{iz\cdot x_3}(ix_3)^kdx_3.
\]

Therefore we have

\[
\int_G r_z(x')e^{-z(\cos(\theta)x_1 + \sin(\theta)x_2)}g(\theta, x')dx' = 0.
\]
Then for any $\omega \in \mathbb{S}^1$ and any $p \in \mathbb{R}^1$

$$\Psi(z, \omega, p) = \int_{<\omega, x'>=p} r_z e^{z(\omega^\perp x')} ds = 0. \quad (139)$$

For any fixed $(\omega, p) \in \mathbb{S}^1 \times \mathbb{R}^1$ the function $\Psi(z, \omega, p)$ is holomorphic in the variable $z$. Therefore, by (139) we obtain

$$\frac{\partial \Psi}{\partial z}(0, \omega, p) = \int_{<\omega, x'>=p} (q_1 - q_2)(ix_3 + (\omega^\perp x'))\ell ds = 0 \quad \forall \ell \in \mathbb{N}_+. \quad (140)$$

We claim that

$$r_{0,k} \equiv 0 \quad \forall k \in \mathbb{N}_+. \quad (141)$$

From (140) there exist constants $C_{k,\ell}$ such that

$$\int_{<\omega, x'>=p} r_{0,k} ds = \sum_{\ell=0}^{k-1} C_{k,\ell} \int_{<\omega, x'>=p} (\omega^\perp x')^{k-\ell} r_{0,\ell} ds. \quad (142)$$

The function $\Psi(0, \omega, p)$ is the Radon transform of the function $r_0$. By the classical uniqueness result for the Radon transform (see e.g. Theorem 5.5, p.30 in Helgason [25]), we obtain

$$r_0 = r_{0,0} \equiv 0. \quad (143)$$

Suppose that the equalities (141) are already proved for all $\ell$ less than $k$. Then equality (142) immediately implies (141) for $\ell = k$.

The function $r_z$ is holomorphic in the variable $z$ for any fixed $x'$. Equality (141) implies that derivatives of any orders with respect to $z$ of this function are equal to zero. Hence

$$r_z = 0 \quad \forall z \in \mathbb{C}^1 \text{ and } x' \in G.$$ 

Hence $q_1 = q_2$. The proof of the Theorem 4 is complete. ■

Proof of Theorem 5. Without loss of generality we can assume that $x^0 = 0$ and set $\Gamma_\pm = \Gamma_\pm(0)$. In the spherical coordinates the Laplace operator has the form:

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial}{\partial \theta} (\sin^2(\theta) \frac{\partial u}{\partial \theta}) + \frac{1}{r^4 \sin^4(\theta)} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial r^2} + \frac{2}{r^2} \frac{\partial u}{\partial r} + \cot(\theta) \frac{\partial u}{\partial \theta}. \quad (144)$$

The function $\Phi(x) = \varphi + i\psi = \ln r \pm i\theta$ satisfies the Eikonal equation. Short computations and formula (144) imply

$$\Delta \Phi = \frac{1}{r^2} (1 \pm icot(\theta)).$$

Let a function $a$ satisfy the transport equation:

$$2(\nabla a, \nabla \Phi) + \Delta \Phi a = 0.$$ 

In the spherical coordinates, the transport equation has the form

$$\frac{2}{r} \frac{\partial a}{\partial r} \pm \frac{2}{r^2} \frac{\partial a}{\partial \theta} + \frac{1}{r^2} (1 \pm icot(\theta)) a = 0. \quad (145)$$

This equation admits the following solution

$$a(r, \theta, \varphi) = \frac{1}{\sqrt{r}} e^{-\frac{i}{2} \ln(\sin(\theta))} a_0(\varphi),$$

where $a_0$ is some function from $C^2_0[0, 2\pi]$. 

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Then short computations imply
\[
\Delta (ae^{(\tau+i\zeta)\Phi}) = \left((\tau+i\zeta)^2 a \left(\frac{\partial \Phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \Phi}{\partial \theta} \right)^2 \right) + (\tau+i\zeta) \left(2 \frac{\partial \Phi \partial a}{\partial r \partial r} + 2 \frac{\partial \Phi \partial a}{\partial r \partial \theta} + \left(\frac{2 \partial \Phi}{r \partial r} + \frac{\cot(\theta) \partial \Phi}{r^2 \partial \theta} \right) a \right)
\]
\[+ \Delta a \right) e^{(\tau+i\zeta)\Phi} = (\Delta a) e^{(\tau+i\zeta)\Phi}.
\]
Then using Corollary \([2]\) we construct the function \(w_\tau(z, \cdot)\) which solves the boundary value problem
\[
L_{q_1}(x, D)w_\tau = -e^{(\tau+i\zeta)\Phi} L_{q_1}(x, D)a \quad \text{in} \quad \Omega, \quad w_\tau|_{\Gamma} = -ae^{(\tau+i\zeta)\Phi}.
\]
and satisfies the estimate
\[
\|e^{-\tau \varphi} w_\tau\|_{L^2(\Omega)} = o(1) \quad \text{as} \quad \tau \to +\infty.
\]  
(146)

The function
\[u_1 = ae^{(\tau+i\zeta)\Phi} + w_\tau\]
solves the Schrödinger equation
\[
L_{q_1}(x, D)u_1 = 0 \quad \text{in} \quad \Omega, \quad u_1|_{\Gamma} = 0
\]
and admits the asymptotic expansion
\[
u_1(x) = ae^{\tau \Phi} + e^{\tau \ln(r)\partial L^2(\Omega)}(1) \quad \text{as} \quad \tau \to +\infty
\]  
(147)
by \([146]\). Similarly we construct complex geometric optics solutions \(v\) for the Schrödinger equation with the potential \(q_2\)
\[
L_{q_2}(x, D)v = 0 \quad \text{in} \quad \Omega, \quad v|_{\Gamma} = 0
\]
in the form
\[v(x) = ae^{-\tau \Phi} + e^{-\tau \ln(r)\partial L^2(\Omega)}(1) \quad \text{as} \quad \tau \to +\infty.
\]  
(148)

Since the Dirichlet-to-Neumann maps are the same, there exists a function \(u_2\) which solves the Schrödinger equation with potential \(q_2\) in \(\Omega\) and satisfies the following equations
\[
L_{q_2}(x, D)u_2 = 0 \quad \text{in} \quad \Omega, \quad (u_1 - u_2)|_{\partial \Omega} = \frac{\partial(u_1 - u_2)}{\partial \nu}|_{\partial \Omega \\\setminus \Gamma} = 0.
\]

Setting \(u = u_1 - u_2\) we have
\[
L_{q_1}(x, D)u = -(q_1 - q_2)u_1 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial \Omega \\\setminus \Gamma} = 0.
\]  
(149)

Taking the scalar product in \(L^2(\Omega)\) of equation \([149]\) and the function \(v\), after integration by parts, we have
\[
\int_{\Omega} (q_1 - q_2)u_1 vdx = 0.
\]

Using the asymptotic formulae \([147]\) and \([148]\) for the functions \(u_1\) and \(v\), we obtain
\[
\int_{R} (q_1 - q_2)ae^{\tau \Phi}r^2 \sin(\theta)d\varphi d\psi d\theta = 0.
\]

Here \(R\) denotes the image of the domain \(\Omega\) after change of coordinates from the Cartesian to the spherical one. Taking a sequence of functions \(a_0\) converging to \(\delta(\varphi - \varphi_0)\), we obtain
\[
\int_{R \setminus \{\varphi = \varphi_0\}} (q_1 - q_2)e^{-\ln(\sin(\theta))} \sin(\theta)e^{\tau \Phi}rdrd\theta = \int_{R \setminus \{\varphi = \varphi_0\}} (q_1 - q_2)e^{\tau \Phi}rdrd\theta.
\]  
(150)
We introduce the functions $r_z, m_k : \mathbb{S}^2 \to \mathbb{R}$ as follows: for each point on the sphere we choose the ray $\ell$ starting from the origin and passing through this point. Then we set $r_z = \int (q_1 - q_2)r \exp(\pm r \ln r) dr$ and $m_k = \int (q_1 - q_2)(i \ln r)^k dr$ where $k \in \mathbb{N}_+.$

Then from (150) we obtain
\[
\int_0^\pi r_z e^{\pm z \theta} d\theta = 0, \quad \forall z \in \mathbb{C}^1 \quad \text{and} \quad \varphi = \varphi_0 \in [0, 2\pi].
\] (151)

There exists a hemisphere such that for each $z$, the support of the function $r_z$ is included in this hemisphere. Let $\Xi$ be the set of "big circles" on $\mathbb{S}^2$. By "big circle" we mean any intersection of sphere $\mathbb{S}^2$ and a plane which passes through the origin. The function $\varphi(x) = \ln r$ is invariant under rotations around the origin. Consequently (151) implies that
\[
\mathcal{H}(z, \xi) = \int_\xi r_z e^{\pm z \xi} d\sigma = 0 \quad \forall z \in \mathbb{C}^1 \quad \text{and} \quad \forall \xi \in \Xi.
\] (152)

If $z = 0$, then after proper rotation of the rectangular coordinate system around the origin, the above formula implies
\[
\int_\xi r_0 d\sigma = 0 \quad \forall \xi \in \Xi.
\] (153)

The equality (153) can be reformulated in the following way: the Minkowski-Funk transform of the function $r_0$ is identically equal to zero. Then the classical Minkowski’s result implies $r_0 = 0$ on $\mathbb{S}^2$ (see e.g. 68).

Then (152) implies that for any $k \in \mathbb{N}_+$ there exist constants $C_{k, \ell}$ such that
\[
\frac{\partial^\ell \mathcal{H}(0, \xi)}{\partial z^\ell} = \int_\xi m_{\ell} d\sigma + \sum_{k=0}^{\ell-1} C_{k, \ell} \int_\xi \theta^{\ell-k} m_k d\sigma = 0 \quad \forall \xi \in \Xi.
\] (154)

From the above formula, the induction argument yields
\[
m_{\ell} = 0 \quad \forall \ell \in \mathbb{N}_+.
\] (155)

Indeed, since $m_0 = r_0$ we have (155) for $\ell = 0$. Suppose that (155) is established for $\ell < k$. Then formula (154) implies
\[
\int_\xi m_k d\sigma = -\sum_{\ell=0}^{k-1} C_{j, k} \int_\xi \theta^{k-\ell} m_\ell d\sigma = 0 \quad \forall \xi \in \Xi.
\]

Hence the Minkowski-Funk transform of the function $m_\ell, \ell < k$ is identically equal to zero and applying the Minkowski’s result again we have $m_k = 0$.

On the other hand the function $r_z(y)$ is holomorphic in variable $z$ for any fixed $y \in \mathbb{S}^2$. Since
\[
\frac{\partial^\ell r_z}{\partial z^\ell}|_{z=0} = m_\ell, \quad \forall \ell \in \mathbb{N}_+ \quad \text{and} \quad y \in \mathbb{S}^2,
\]
we obtain
\[
r_z(y) = 0 \quad \text{on} \quad \mathbb{C}^1 \times \mathbb{S}^1.
\]

Using the definition of the function $r_z$ we obtain immediately that
\[
q_1 = q_2 \quad \text{in} \quad \Omega.
\]

The proof of Theorem 5 is complete. ■

Next we consider the Schrödinger equation with the first order terms:
\[
L_{q, A}(x, D)u = \Delta u + (A, \nabla u) + qu = 0 \quad \text{in} \quad \Omega,
\]
where $A = (A_1, A_2, A_3)$ is a regular real-valued vector field. We consider the problem of determination of the potential $q$ and the vector field $A$ from the following Dirichlet-to-Neumann map:

$$
\Lambda(q, A, \Gamma_-, \Gamma_+) = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_+},
$$

where

$$
L_{q,A}(x,D)u = 0 \quad \text{in} \ \Omega, \quad u|_{\Gamma_-} = 0, \quad u|_{\partial \Omega \setminus \Gamma_-} = f. \quad (156)
$$

We can see that a vector field $A$ and a potential $q$ cannot be determined simultaneously from the Dirichlet-to-Neumann map. More precisely we have the following proposition.

**Proposition 10** Let $\eta \in C^2(\overline{\Omega})$ be a function such that $\eta|_{\partial \Omega \setminus \Gamma_+ \cup \partial \Omega \setminus \Gamma_-} = 0$ and $\frac{\partial \eta}{\partial \nu}|_{(\partial \Omega \setminus \Gamma_+) \cup \Gamma_-} = 0$. Then the operators $L_{q,A}(x,D)$ and $e^{-\eta}L_{q,A}(x,D)e^{\eta}$ generate the same Dirichlet-to-Neumann map on $\Gamma_-$ and $\Gamma_+$.

**Proof.** Denote $\tilde{q} = q + |\nabla \eta|^2 + \Delta \eta + (A, \nabla \eta)$. If $u$ is the solution to equation (156), then $w = ue^{-\eta}$ solves the boundary value problem

$$
e^{-\eta}L_{q,A}(x,D)e^{\eta}w = L_{q,A+\nabla \eta}(x,D)w = 0 \quad \text{in} \ \Omega, \quad w|_{\Gamma_-} = 0, \quad w|_{\partial \Omega \setminus \Gamma_-} = f. \quad (157)
$$

Obviously

$$
\frac{\partial w}{\partial \nu}|_{\partial \Omega \setminus \Gamma_+ \cap \Gamma_-} = \left(\frac{\partial u}{\partial \nu}e^{-\eta}\right)|_{\partial \Omega \setminus \Gamma_+ \cap \Gamma_-} - (ue^{-\eta}\frac{\partial \eta}{\partial \nu})|_{\partial \Omega \setminus \Gamma_+ \cap \Gamma_-} = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_+ \cap \Gamma_-}
$$

and

$$
\frac{\partial w}{\partial \nu}|_{\partial \Omega \setminus (\Gamma_+ \cup \Gamma_-)} = \frac{\partial u}{\partial \nu}e^{-\eta}|_{\partial \Omega \setminus (\Gamma_+ \cup \Gamma_-)} - (ue^{-\eta}\frac{\partial \eta}{\partial \nu})|_{\partial \Omega \setminus (\Gamma_+ \cup \Gamma_-)} = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus (\Gamma_+ \cup \Gamma_-)}.
$$

The proof of the proposition is finished. \[\blacksquare\]

We have

**Theorem 6** Let $\Omega \subset \mathbb{R}^3$ be a bounded strictly convex domain with smooth boundary, $q_1, q_2 \in L^\infty(\Omega), A \in C^2(\overline{\Omega})$, $\tilde{v} \neq 0$ be an arbitrary vector from $\mathbb{R}^3$, the sets $\Gamma_+(\tilde{v})$ given by (127), and $\Lambda(q_1, A_1, \Gamma_-(\tilde{v}), \Gamma_+(\tilde{v})) = \Lambda(q_2, A_2, \Gamma_-(\tilde{v}), \Gamma_+(\tilde{v}))$. Then $rot A_1 = rot A_2$ in $\Omega$.

**Proof.** Without loss of generality, performing a rotation around the origin if necessary, we can assume that $\tilde{v} = \hat{e}_3 = (0, 0, 1)$ and sets $\Gamma_+$ are defined in (128): $z_\theta = x_3 + i(\hat{\omega}, x')$ and $\hat{\omega} = (\cos(\theta), \sin(\theta))$.

Let $\Phi(x) = x_3 - i(\cos(\theta)x_1 + \sin(\theta)x_2)$ where $\theta \in [0, 2\pi)$. We define the function $A_1(\theta)$ as a solution to differential equation

$$
4\partial_{z_\theta} A_1(\theta) + (A_1, (i\hat{\omega}, 1)) = 0 \quad \text{in} \ \Omega, \quad \text{Im} A_1(\theta)|_{\partial \Omega} = 0
$$

and set $a = \tilde{b}(\xi_\theta)|\tilde{q}(\theta, x')e^{A_1(\theta)}$ where $\tilde{q}(\theta, x') = \tilde{b}(\sin(\theta)x_1 - \cos(\theta)x_2)$ and $\tilde{b}(s) \in C^2(\mathbb{R}^1)$ be an arbitrary function. Then the function $a$ solves the differential equation:

$$
2(\nabla \Phi, \nabla a) + (A_1, \nabla \Phi)a = 0 \quad \text{in} \ \Omega. \quad (158)
$$

Let $a_{-1}$ satisfy

$$
2(\nabla \Phi, \nabla a_{-1}) + ((A_1, \nabla \Phi))a_{-1} = -(\Delta + (A_1, \nabla) + q_1)a \quad \text{in} \ \Omega. \quad (159)
$$

We construct the complex geometric optics solution to the Schrödinger equation in the form

$$
u_1(x) = e^{\tau \Phi}(a + \frac{a_{-1}}{\tau}) + e^{\tau \Phi}o_{H^1, \Gamma}(1) \quad \text{as} \ \tau \to +\infty. \quad (160)
$$

Indeed

$$
(\Delta + (A_1, \nabla) + q_1)(e^{\tau \Phi}(a + \frac{a_{-1}}{\tau})) = (\tau^2(\nabla \Phi, \nabla \Phi) + 2\tau(\nabla \Phi, \nabla (a + \frac{a_{-1}}{\tau}))) + \Delta a + \frac{\Delta a_{-1}}{\tau} + q_1(a + \frac{a_{-1}}{\tau})e^{\tau \Phi}
$$

$$
+ \tau(A_1, \nabla \Phi)(a + \frac{a_{-1}}{\tau})e^{\tau \Phi} + (A_1, \nabla (a + \frac{a_{-1}}{\tau}))e^{\tau \Phi}
$$

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Observing that \((\nabla \Phi, \nabla \Phi) = 0\) and using (158), (159), we obtain

\[
(\Delta + (A_1, \nabla) + q_1)(e^{r \Phi}(a + \frac{a-1}{\tau})) = [(\Delta + (A_1, \nabla) + q_1)a_{-1}]e^{r \Phi}.
\]  

(161)

Then using Corollary 2, we construct functions \(w_\tau(z,.)\) which solve the boundary value problem

\[
L_{q_1,A_1}(x,D)w_\tau = -\frac{e^{r \Phi}}{\tau}(\Delta + (A_1, \nabla) + q_1)a_{-1} \text{ in } \Omega, \quad w_\tau |_{\Gamma_-} = -(a + \frac{a-1}{\tau})e^{r \Phi}
\]

(162)

and satisfy the estimate

\[
\|e^{-r x^2}w_\tau\|_{H^{1,\tau}(\Omega)} = o(1) \quad \text{as } \tau \to +\infty.
\]

(163)

By (161) and (162), the function

\[
u_1 = (a + \frac{a-1}{\tau})e^{r \Phi} + w_\tau
\]

(164)

solves the Schrödinger equation

\[
L_{q_1,A_1}(x,D)u_1 = 0 \quad \text{in } \Omega, \quad u_1 |_{\Gamma_-} = 0
\]

(165)

and admits the asymptotic expansion (160) by (163).

Since \(\Lambda(q_1,\Gamma_- , \Gamma_+) = \Lambda(q_2,\Gamma_- , \Gamma_+)\), there exists \(u_2\) such that

\[
L_{q_2,A_2}(x,D)u_2 = 0 \quad \text{in } \Omega, \quad (u_1 - u_2)|_{\partial \Omega} = \frac{\partial (u_1 - u_2)}{\partial \nu}|_{\partial \Omega \backslash \Gamma_+} = 0.
\]

(166)

Next, in a way similar to the construction of \(u_1\), we construct the complex geometric optics solution \(\nu\) to the Schrödinger operator with potential \(L_{q_2,A_2}(x,D)^*:\)

\[
L_{q_2,A_2}(x,D)^* v = \Delta v - (A_2, \nabla) v - (\nabla, A_2) v + q_2 v = 0 \quad \text{in } \Omega, \quad v |_{\Gamma_+} = 0
\]

(167)

in the form

\[
v(x) = (\tilde{a} + \frac{a-1}{\tau}) e^{-r \Phi} + e^{-r x^2} \theta \|H^{1,\tau}(\Omega)\) \quad \text{as } \tau \to +\infty.
\]

(168)

Let a function \(A_2(\theta)\) solve the differential equation

\[
4 \partial_{x^*} A_2(\theta) - (A_2, (i\tilde{a},1)) = 0 \quad \text{in } \Omega, \quad \text{Im} A_2(\theta)|_{\partial \Omega} = 0.
\]

Then \(\tilde{a} = \text{e}^{A_2(\theta)}\) solves the ordinary differential equations

\[
2(\nabla \Phi, \nabla \tilde{a}) - (A_2, \nabla \Phi) \tilde{a} = 0 \quad \text{in } \Omega.
\]

(169)

A function \(\tilde{a}_{-1}\) solves the differential equation:

\[
2(\nabla \Phi, \nabla \tilde{a}_{-1}) - (A_2, \nabla \Phi) \tilde{a}_{-1} = (\Delta - (A_2, \nabla) - (\nabla, A_2) + q_2) \tilde{a} \quad \text{in } \Omega.
\]

(170)

Setting \(u = u_1 - u_2\), by (165) and (166), we have

\[
L_{q_2,A_2}(x,D)u = -(A_1 - A_2, \nabla) u_1 - (q_1 - q_2) u_1 \quad \text{in } \Omega, \quad u |_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial \Omega \backslash \Gamma_+} = 0.
\]

(171)

Then using (167) and (170), we obtain

\[
(L_{q_2,A_2}(x,D)u,v)_{L^2(\Omega)} = (u, L_{q_2,A_2}(x,D)^* v)_{L^2(\Omega)} + (\frac{\partial u}{\partial \nu}, v)_{L^2(\partial \Omega)} - (u, \frac{\partial v}{\partial \nu})_{L^2(\partial \Omega)}
\]

\[
+ \int_{\Omega} ((A_1 - A_2, \nabla) u_1 v + (q_1 - q_2) u_1 v) dx
\]

\[
= (v, \frac{\partial u}{\partial \nu})_{L^2(\partial \Omega \backslash \Gamma_+)} + (v, \frac{\partial u}{\partial \nu})_{L^2(\Gamma_+)} + \int_{\Omega} ((A_1 - A_2, \nabla u_1) v + (q_1 - q_2) u_1 v) dx
\]

\[
= \int_{\Omega} ((A_1 - A_2, \nabla u_1) v + (q_1 - q_2) u_1 v) dx = 0.
\]

(172)
We are interested in the asymptotic expansion of the right-hand side of (172). By (160) and (168), we have
\[ \int_{\Omega} ((A_1 - A_2, \nabla u_1) v + (q_1 - q_2) u_1 v) dx = \tau \mathcal{S}_1 + \mathcal{S}_0 + o(1) \quad \text{as} \quad \tau \to +\infty. \tag{173} \]
Since $\mathcal{S}_1$ is independent of $\tau$, the above asymptotic formula implies $\mathcal{S}_1 = 0$. Integrating by parts and using the equalities (158) and (169), we obtain
\[ 0 = \mathcal{S}_1 = \int_{\Omega} (A_1 - A_2, \nabla \Phi) a \tilde{a} dx = \int_{\Omega} (-2 \nabla \Phi, \nabla a) \tilde{a} \quad \text{as} \quad \tau \to +\infty. \tag{174} \]

Let $\mathcal{L}$ be the set of all planes in $\mathbb{R}^3$ orthogonal to the plane $x_3 = 0$. From (174) we have
\[ \int_{\partial \Omega \cap P} \tilde{b}(\tau_\theta) e^{A_1(\theta) + A_2(\theta) \frac{\partial \Phi}{\partial \nu}} d\sigma = 0 \quad \forall P \in \mathcal{L}. \]
By Proposition 20 there exists the antiholomorphic function $\Theta(\tau_\theta)$ such that
\[ e^{A_1(\theta) + A_2(\theta) \frac{\partial \Phi}{\partial \nu}} = \Theta \quad \text{on} \quad \partial \Omega \cap P. \]
Observe that the function $\Theta$ does not have any zeros in $\Omega$. Indeed, since the domain $\Omega$ is assumed to be convex, the two dimensional domain $\Omega \cap P$ is simply connected. Then by the well-know formula the number of zeros $N$ of the function $\Theta$ is given by formula
\[ N = \frac{1}{2\pi} \int_{\partial \Omega \cap P} \frac{\bar{\Theta}}{\Theta} d\theta = \frac{1}{2\pi} \Delta_{\partial \Omega \cap P} \arg \bar{\Theta} = \frac{1}{2\pi} \Delta_{\partial \Omega \cap P} \arg e^{A_1(\theta) + A_2(\theta)} = \frac{1}{2\pi} \Delta_{\partial \Omega \cap P} \arg e^{\Re (A_1(\theta) + A_2(\theta))} = 0. \]
Consider the form $\alpha = d\bar{\Theta}/\Theta$. This form is closed and since $\Omega \cap P$ is simply connected, the differential form $\alpha$ is exact. Hence there exists a function $a(x)$ such that $\alpha = da$. Then $\partial_{\bar{z}} a = \partial_{\bar{z}} (e^{a}/e^{\Theta})$. Consider this equality as a first-order differential equation. The general solution to this differential equation is written as $a = c(\tau_\theta)e^a$. On the other hand $\partial_{\bar{z}} \Theta = 0$. Hence $c(\tau_\theta) = const$ and since the function $a$ defined up to a constant, we have
\[ \Theta = e^a. \]
Then $a$ is a holomorphic function and we set $\ln \Theta = \bar{\Theta}$.

The function $A_1(\theta) + A_2(\theta)$ satisfies the equation
\[ 4 \partial_{\bar{z}} (A_1(\theta) + A_2(\theta)) + (A_1 - A_2, (i\bar{\omega}, 1)) = 0 \quad \text{in} \quad \Omega \tag{175} \]
and
\[ A_1(\theta) + A_2(\theta) = \ln \Theta \quad \text{on} \quad \partial \Omega \cap P. \]
Integrating the equation (166) over $\Omega \cap P$, we have
\[ \int_{\Omega \cap P} (A_1 - A_2, \nabla \Phi) dx = -\int_{\Omega \cap P} 4 \partial_{\bar{z}} (A_1(\theta) + A_2(\theta)) dx = \int_{\Omega \cap P} (\nu_3 + i(\bar{\omega}, \bar{\nu}'))(A_1(\theta) + A_2(\theta)) d\sigma \]
\[ = \int_{\partial \Omega \cap P} (\nu_3 + i(\bar{\omega}, \bar{\nu}')) \ln \Theta d\sigma = 0. \]
Since $A_1$ and $A_2$ are real-valued vector fields, from the above equality we obtain
\[ \int_{\Omega \cap P} (A_{1,3} - A_{2,3}) dx = 0, \quad \forall P \in \mathcal{L} \tag{176} \]
and
\[ \int_{\Omega \cap P} (A_1 - A_2, (\bar{\omega}, 0)) dx = 0, \quad \forall P \in \mathcal{L}. \tag{177} \]
We extend the vector fields \( A_j \) by zero outside of domain \( \Omega \). From (177) applying the uniqueness result for the Radon transform, we obtain that

\[
\int_{-K}^K (A_{1,3} - A_{2,3})dx_3 = 0, \quad \forall x' \in \mathbb{R}^2. \tag{178}
\]

By (178) there exists a function \( \Psi(x) \) such that

\[
\frac{\partial \Psi}{\partial x_3} = (A_{1,3} - A_{2,3}) \quad \text{in } \Omega, \quad \Psi|_{\partial \Omega} = 0 \quad \text{in } \Omega. \tag{179}
\]

By Proposition 10 and the assumption of strict convexity of the domain \( \Omega \), the operators \( L_{q_2, A_2}(x, D) \) and \( e^{-\Psi} L_{q_2, A_2}(x, D)e^{\Psi} \) generate the same Dirichlet-to-Neumann map. The convection terms in the operator \( e^{-\Psi} L_{q_2, A_2}(x, D)e^{\Psi} \) have the form

\[
(A_2 + \nabla \Psi, \nabla).
\]

Hence by (179) without loss of generality we can assume that

\[
A_{1,3} = A_{2,3} \quad \text{in } \Omega. \tag{180}
\]

Then from (177) we have

\[
\int_{<\omega, x'>=p} \left( \int_{-K}^K g(\bar{\omega})(A_{1,1} - A_{2,1})dx_3 \right) dx_1 + \left( \int_{-K}^K g(\bar{\omega})(A_{1,2} - A_{2,2})dx_3 \right) dx_2 = 0, \tag{181}
\]

where \( <\omega, x'> = p \) is an arbitrary line from \( \mathbb{R}^2 \). We claim that

\[
\left( \int_{-K}^K x_3^k(A_{1,1} - A_{2,1})dx_3, \int_{-K}^K x_3^k(A_{1,2} - A_{2,2})dx_3 \right) = (0,0), \quad \forall k \in \mathbb{N}_+ \quad \text{and } \forall x' \in \mathbb{R}^2. \tag{182}
\]

Our proof is by the induction method. Setting in (181) the function \( g = 1 \), we obtain (see e.g. [67], p. 78) that there exists a function \( f \) with compact support such that

\[
\nabla_{x'} f = \left( \int_{-K}^K (A_{1,1} - A_{2,1})dx_3, \int_{-K}^K (A_{1,2} - A_{2,2})dx_3 \right), \quad \forall x' \in \mathbb{R}^2. \tag{183}
\]

Setting \( g(\bar{\omega}) = \bar{\omega} \) in (181), we obtain

\[
\int_{<\omega, x'>=p} (\bar{\omega}, x') \frac{\partial f}{\partial x_1}dx_1 + (\bar{\omega}, x') \frac{\partial f}{\partial x_2}dx_2 = 0, \quad \forall p \in \mathbb{R}^1 \quad \text{and } \forall \omega \in S^1. \tag{184}
\]

Integrating by parts in this equation we obtain

\[
\int_{<\omega, x'>=p} fds = 0, \quad \forall p \in \mathbb{R}^1 \quad \text{and } \forall \omega \in S^1. \tag{185}
\]

By the uniqueness theorem for the Radon transform, we obtain \( f \equiv 0 \). Hence the beginning step of the induction method is established. Suppose that (182) is already proved for all \( k < \hat{k} \).

Setting \( g(\bar{\omega}) = (x_3 - i(\bar{\omega}, x'))^{\hat{k}} \) in (181), we obtain

\[
\int_{<\omega, x'>=p} \left( \int_{-K}^K (x_3 - i(\bar{\omega}, x'))^{\hat{k}}(A_{1,1} - A_{2,1})dx_3 \right) dx_1 + \left( \int_{-K}^K (x_3 - i(\bar{\omega}, x'))^{\hat{k}}(A_{1,2} - A_{2,2})dx_3 \right) dx_2
\]

\[= \int_{<\omega, x'>=p} \left( \int_{-K}^K x_3^{\hat{k}}(A_{1,1} - A_{2,1})dx_3 \right) dx_1 + \left( \int_{-K}^K x_3^{\hat{k}}(A_{1,2} - A_{2,2})dx_3 \right) dx_2 = 0. \tag{186}
\]
Hence there exists a function $f_\xi$ with compact support such that
\[
\nabla x' f_\xi = \left( \int_{-K}^K x_3^j (A_{1,1} - A_{2,1}) dx_3, \int_{-K}^K x_3^j (A_{1,2} - A_{2,2}) dx_3 \right), \quad \forall x' \in \mathbb{R}^2. \tag{187}
\]

Setting $g(z_\theta) = (x_3 - i(z, x'))^{\hat{k}+1}$ in (181), we obtain
\[
\int_{<\omega, x'>=p} (\bar{a}, x') \partial f_\xi / \partial x_1 + (\bar{a}, x') \partial f_\xi / \partial x_2 = 0, \quad \forall p \in \mathbb{R}^1 \quad \text{and} \quad \forall \omega \in \mathbb{S}^1. \tag{188}
\]
Integrating by parts in this equation we obtain
\[
\int_{<\omega, x'>=p} f_\xi ds = 0, \quad \forall p \in \mathbb{R}^1 \quad \text{and} \quad \forall \omega \in \mathbb{S}^1. \tag{189}
\]
By the uniqueness theorem for the Radon transform, we obtain $f_\xi \equiv 0$ and (182) is proved. On the other hand the equality (182) implies that
\[
A_{1,1} - A_{2,1} = 0 \quad \text{and} \quad A_{1,2} - A_{2,2} = 0 \quad \text{in} \ \Omega.
\]

The proof of Theorem 6 is complete. \[\square\]

For more results on recovery of coefficients of the Schrödinger equation with the first terms, see [17] where the function $\Phi = \ln \ r + i \theta$ was used for construction of the complex geometric optics solution. In the proof of Theorem 6 we used some ideas from [17].

We conclude this section with

**Proposition 11** We assume that $\Lambda(q_1, \emptyset, \emptyset) = \Lambda(q_2, \emptyset, \emptyset)$ with $q_1, q_2$ in some admissible set implies $q_1 = q_2$ in $\Omega$. If $q_1 = q_2$ near $\partial \Omega$ and $\Lambda(q_1, \Gamma_-, \Gamma_+) = \Lambda(q_2, \Gamma_-, \Gamma_+)$ with arbitrarily subboundaries $\Gamma_-, \Gamma_+$, then $q_1 = q_2$ in $\Omega$.

Thus if we can assume that the coefficients are equal near $\partial \Omega$, then the uniqueness by Dirichlet-to-Neumann map on subboundaries is trivial from the uniqueness by the Dirichlet-to-Neumann map on the whole boundary.

**Proof.** We can choose an open neighborhood $\bar{\omega}$ of $\partial \Omega$ such that $q := q_1 = q_2$ in $\omega := \bar{\omega} \cap \Omega$. Let $u_j$, $j = 1, 2$ satisfy
\[
L_{q_j}(x, D)u_j = \Delta u_j + q_j u_j = 0 \quad \text{in} \ \Omega, \quad u_j |_{\partial \Omega} = f.
\]
First we prove $\partial u_j / \partial \nu = \partial u_j / \partial \nu$ on $\partial \Omega$ if $f = 0$ on $\Gamma_-$. In fact, setting $u = u_1 - u_2$, we have
\[
L_0(x, D)u = \Delta u + qu = 0 \quad \text{in} \ \omega, \quad u |_{\partial \Omega} = 0
\]
and
\[
\partial u / \partial \nu = 0 \quad \text{on} \ \partial \Omega \setminus \Gamma_+.
\]
Therefore the unique continuation for the Schrödinger equation (e.g., Hörmander [27]) yields $u = 0$ in $\omega$, which implies $\partial u_1 / \partial \nu = \partial u_2 / \partial \nu$ on $\partial \Omega$.

Next let $f \in H^{1/2}(\partial \Omega)$ be arbitrary. Then we will prove $\partial u_j / \partial \nu = \partial u_j / \partial \nu$ on $\partial \Omega$. Let $w_j, j = 1, 2$, satisfy
\[
L_{q_j}(x, D)w_j = \Delta w_j + q_j w_j = 0 \quad \text{in} \ \Omega, \quad w_j |_{\partial \Omega} = g,
\]
where $g = 0$ on $\Gamma_-$. For $j = 1, 2$, we have
\[
0 = \int_\Omega w_j L_{q_j}(x, D)u_j dx = \int_\Omega u_j L_{q_j}(x, D)w_j dx + \int_{\partial \Omega} \left( w_j \frac{\partial u_j}{\partial \nu} - u_j \frac{\partial w_j}{\partial \nu} \right) d\sigma
\]
\[
= \int_{\partial \Omega} g \frac{\partial u_j}{\partial \nu} d\sigma - \int_{\partial \Omega} \frac{\partial w_j}{\partial \nu} d\sigma.
\]
that is,
\[ \int_{\partial \Omega \setminus Q} g \frac{\partial u_j}{\partial \nu} d\sigma = \int_{\partial \Omega} f \frac{\partial w_j}{\partial \nu} d\sigma, \quad j = 1, 2. \]

By \( \Lambda(q_1, \Gamma_-, \Gamma_+) = \Lambda(q_2, \Gamma_-, \Gamma_+) \) and the fact proved above, we see that \( \frac{\partial u_j}{\partial \nu} = \frac{\partial w_j}{\partial \nu} \) on \( \partial \Omega \). Therefore
\[ \int_{\partial \Omega \setminus Q} g \frac{\partial u_1}{\partial \nu} d\sigma = \int_{\partial \Omega \setminus Q} g \frac{\partial u_2}{\partial \nu} d\sigma. \]

Since we can choose \( g \) arbitrarily, for example, any \( g \in C^\infty_0(\partial \Omega \setminus \Gamma_-) \), we obtain \( \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \) on \( \partial \Omega \setminus \Gamma_- \).

Again setting \( u = u_1 - u_2 \), we have \( \Delta u + qu = 0 \) in \( \omega \), \( u = 0 \) on \( \partial \Omega \) and \( \frac{\partial u}{\partial \nu} = 0 \) on \( \partial \Omega \setminus \Gamma_- \). The unique continuation yields \( u = 0 \) in \( \omega \). Hence \( \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \) on \( \partial \Omega \). Hence we prove \( \Lambda(q_1, \emptyset, \emptyset) = \Lambda(q_2, \emptyset, \emptyset) \). Thus the proof of the proposition is completed. \( \blacksquare \)

4 2-D Calderón’s problem.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary such that \( \partial \Omega = \bigcup_{k=1}^N \gamma_k \), where \( \gamma_k \), \( 1 \leq k \leq N \), are smooth closed contours, and \( \gamma_N \) is the external contour. Let \( \Gamma_0 \) be an arbitrarily chosen relatively open subset of \( \partial \Omega \).

For the Schrödinger operator with potential \( q \) we consider the following Dirichlet-to-Neumann map \( \Lambda(q, \Gamma_0, \Gamma_0) \):
\[ \Lambda(q, \Gamma_0, \Gamma_0)(f) = \left. \frac{\partial u}{\partial \nu} \right|_{\partial \Omega \setminus \Gamma_0}, \]
where
\[ L_q(x, D)u = \Delta u + qu = 0 \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0, \quad u|_{\partial \Omega \setminus \Gamma_0} = f. \]

Henceforth we write \( \Lambda(q, \Gamma_0) = \Lambda(q, \Gamma_0, \Gamma_0) \).

We have

**Theorem 7 (11)** Let \( q_1, q_2 \in W^1_p(\Omega) \) for some \( p > 2 \) and \( \Lambda(q_1, \Gamma_0) = \Lambda(q_2, \Gamma_0) \). Then \( q_1 = q_2 \in \Omega \).

We modify the argument in [11] and describe the proof. Before starting the proof of the theorem we recall the classical results for the properties of the operators \( \partial_z^{-1} \) and \( \partial_{\bar{z}}^{-1} \) which are given by
\[ \partial_z^{-1} g = -\frac{1}{\pi} \int_{\Omega} \frac{g(z, \zeta)}{z - \zeta} d\zeta_2 d\zeta_1, \quad \partial_{\bar{z}}^{-1} g = \overline{\partial_z^{-1} g}. \]

The following is proved in [14] (p.47, 56, 72):

**Proposition 12 A)** Let \( m \geq 0 \) be an integer number and \( \alpha \in (0, 1) \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in \mathcal{L}(C^{m+\alpha}(\Omega), C^{m+\alpha+1}(\Omega)) \).

**B)** Let \( 1 \leq p \leq 2 \) and \( 1 < \gamma < \frac{2p}{2-p} \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega)) \).

**C)** Let \( 1 < p < \infty \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in \mathcal{L}(L^p(\Omega), W^1_p(\Omega)) \).

**Proof of Theorem 7.** We define two other operators:
\[ \mathcal{R}_r g = \frac{1}{2} e^{r(\Phi - \bar{\Phi})} \partial_z^{-1}(g e^{r(\Phi - \bar{\Phi})}), \quad \mathcal{R}_{\bar{r}} g = \frac{1}{2} e^{r(\bar{\Phi} - \Phi)} \partial_{\bar{z}}^{-1}(g e^{r(\Phi - \bar{\Phi})}), \tag{190} \]

where \( \Phi \in C^2(\Omega) \) is a holomorphic function which satisfies \( \Phi = \bar{\Phi} \). Observe that
\[ 2 \frac{\partial}{\partial z}(e^r \Phi \mathcal{R}_r g) = g e^{r \Phi}, \quad 2 \frac{\partial}{\partial \bar{z}}(e^r \Phi \mathcal{R}_{\bar{r}} g) = g e^{r \Phi} \quad \forall g \in L^2(\Omega). \tag{191} \]

Let \( a \in C^6(\Omega) \) be some holomorphic function, not identically equal to a constant on \( \Omega \), such that
\[ \text{Re } a|_{\Gamma_0} = 0, \quad \lim_{z \to \zeta} a(z)/|z - \zeta|^{100} = 0, \quad \forall \zeta \in H \cap \Gamma_0^*. \tag{192} \]
We recall that $\mathcal{H} = \{ \vec{z} \in \Omega | \partial_\vec{z} \Phi(\vec{z}) = 0 \}$ is the set of critical points of the function $\Phi$. Moreover, for some $\vec{x} \in \mathcal{H}$, we assume that

$$a(\vec{x}) \neq 0.$$  \hspace{1cm} (193)

The existence of such a function is proved in Proposition 15 in Section 7. Let polynomials $M_1(z)$ and $M_3(\tau)$ satisfy

$$(i\bar{z}^{-1}q_1 - M_1)(\vec{x}) = 0, \quad (\partial_{\bar{z}}^{-1}q_1 - M_3)(\vec{x}) = 0.$$  \hspace{1cm} (194)

We define the function $U_1$ by

$$U_1(x) = e^{r\Phi}(a + a_1/\tau) + e^{r\Phi}(b_1(\tau) - \frac{1}{2}e^{r\Phi}\bar{\tau}a(\partial_{\bar{z}}^{-1}q_1 - M_1)\} - \frac{1}{2}e^{r\Phi}\bar{\tau}\{\pi(\partial_{\bar{z}}^{-1}q_1 - M_3)\},$$

where $a_1$ is some holomorphic function and $b_1$ some antiholomorphic function. We set

$$g_\tau = q_1(e^{ir\psi}a_1/\tau + e^{-ir\psi}b_1/\tau) - \frac{e^{ir\psi}}{2}\bar{\tau}a(\partial_{\bar{z}}^{-1}q_1 - M_1)\} - \frac{e^{-ir\psi}}{2}\bar{\tau}\{\pi(\partial_{\bar{z}}^{-1}q_1 - M_3)\}).$$

After short computations, using (195), (191) and the factorization of the Laplace operator in the form $\Delta = 4\partial_{\bar{z}}\partial_z$ we reach the following equation

$$L_{q_1}(x, D)U_1 = e^{r\phi}g_\tau \quad \text{in } \Omega.$$  \hspace{1cm} (196)

We make a choice of the functions $a_1, b_1$ in such a way that

$$\|g_\tau\|_{L^2(\Omega)} = O\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty$$  \hspace{1cm} (197)

and

$$U_1|_{\Gamma_0} = e^{r\phi}O_{H^\infty(\Omega)}(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty.$$  \hspace{1cm} (198)

The holomorphic function $a_1$ and the antiholomorphic function $b_1$ are defined by $a_1(z) = a_{1,1}(z) + a_{1,2}(z)$ and $b_1(\tau) = b_{1,1}(\tau) + b_{1,2}(\tau)$ where the functions $a_{1,1}, b_{1,1} \in C^1(\overline{\Omega})$ satisfy

$$a_{1,1}(z) + b_{1,1}(\tau) = \left(\frac{a(\partial_{\bar{z}}^{-1}q_1 - M_1)}{4\partial_z \Phi} + \frac{\pi(\partial_{\bar{z}}^{-1}q_1 - M_3)}{4\partial_{\bar{z}} \Phi}\right) \quad \text{on } \Gamma_0,$$

and the functions $a_{1,2}(z, \tau), b_{1,2}(\tau, \tau) \in C^1(\overline{\Omega})$ for each $\tau$ are holomorphic and antiholomorphic function such that

$$a_{1,2}(z, \tau) = -\frac{1}{8\pi} \int_{\partial \Omega} \frac{(\nu_1 + i\nu_2)\vec{\pi}(\partial^{-1}_{\bar{z}}q_1 - M_3)e^{r(\bar{\tau} - \Phi)}}{(\zeta - z)\partial_{\bar{z}} \Phi} d\sigma$$

and

$$b_{1,2}(\tau, \tau) = -\frac{1}{8\pi} \int_{\partial \Omega} \frac{(\nu_1 - i\nu_2)a(\partial^{-1}_{\bar{z}}q_1 - M_1)e^{r(\Phi - \bar{\tau})}}{(\zeta - \tau)\partial_{\zeta} \Phi} d\sigma.$$

Here the denominators of the integrands vanish in $\mathcal{H} \cap \Gamma_0$, but thanks to the second condition in (192), the integrability is guaranteed. We represent the functions $a_{1,2}(z, \tau), b_{1,2}(\tau, \tau)$ in the form

$$a_{1,2}(z, \tau) = a_{1,2,1}(z) + a_{1,2,2}(z, \tau), \quad b_{1,2}(\tau, \tau) = b_{1,2,1}(\tau) + b_{1,2,2}(\tau, \tau),$$

where

$$a_{1,2,1}(z) = -\frac{1}{8\pi} \int_{\Gamma_3} \frac{(\nu_1 + i\nu_2)\vec{\pi}(\partial^{-1}_{\bar{z}}q_1 - M_3)}{(\zeta - z)\partial_{\bar{z}} \Phi} d\sigma, \quad b_{1,2,1}(\tau) = -\frac{1}{8\pi} \int_{\Gamma_3} \frac{(\nu_1 - i\nu_2)a(\partial^{-1}_{\bar{z}}q_1 - M_1)}{(\zeta - \tau)\partial_{\zeta} \Phi} d\sigma.$$

By (192), the functions $a_{1,2,1}, b_{1,2,1}$ belong to $C^1(\overline{\Omega})$. By (180) and Proposition 15 in Section 7, we have

$$\|b_{1,2,2}(\tau, \tau)\|_{L^2(\Omega)} + \|a_{1,2,2}(z, \tau)\|_{L^2(\Omega)} \to 0 \quad \text{as } \tau \to +\infty.$$  \hspace{1cm} (199)

In order to establish (198), we use the following proposition:
Proposition 13 The following asymptotic formula is true

\[ \left\| \int_{\Omega} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} \right\|_{H^{1/2}(\Gamma'_{\Omega})} \]

\[ + \left\| \int_{\Omega} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} \right\|_{H^{1/2}(\Gamma'_{\Omega})} = o(1) \quad \text{as} \ \tau \to +\infty. \]  

(200)

Proof. In order to prove (200), consider a function \( e \in C^{\infty}_{0}(\Omega) \) such that

\( e \equiv 1 \) in some neighborhood of the set \( \mathcal{H} \setminus \Gamma_{0}^{*} \).

The family of functions \( \int_{\Omega} e \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} \in C^{\infty}(\partial\Omega) \), are uniformly bounded in \( \tau \) in \( C^{2}(\partial\Omega) \) and by Proposition 12 in Section 7, this function converges pointwise to zero. Therefore

\[ \left\| \int_{\Omega} e \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} \right\|_{H^{1}(\partial\Omega)} = o(1) \quad \text{as} \ \tau \to +\infty. \]

(202)

Integrating by parts we obtain

\[ \int_{\Omega} (1 - e) \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} = \frac{(1 - e)}{\partial_{\zeta} \Phi} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \]

\[ - \frac{1}{\tau} \int_{\Omega} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1}. \]

Thanks to (85) and (192), we have

\[ \left\| \frac{1 - e}{\partial_{\zeta} \Phi} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \right\|_{H^{1/2}(\Gamma'_{\Omega})} = o(1) \quad \text{as} \ \tau \to +\infty. \]

(203)

By (201) and Proposition 12 the functions \( \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \) are bounded in \( L^{p}(\Omega) \)

uniformly in \( \tau \). Therefore by Proposition 12 the functions \( \int_{\Omega} \partial_{\zeta} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} \) are uniformly bounded in \( W_{p}^{1}(\Omega) \). The trace theorem yields

\[ \left\| \frac{1}{\tau} \int_{\Omega} \partial_{\zeta} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} \right\|_{H^{1/2}(\Gamma'_{\Omega})} = o(1) \quad \text{as} \ \tau \to +\infty. \]

(204)

By (202) - (204) we obtain (200). □

We note that \( \frac{a}{\partial_{\zeta} \Phi} \in C^{2}(\partial\Omega) \) by (192). Integrating by parts, we obtain the following:

\[ e^{r\Phi} \bar{R}_{\tau} \{ a(\partial_{\zeta}^{-1}q_{1} - M_{1}) \} = \frac{1}{\tau} \left( 2b_{1,2} e^{r\Phi} a(\partial_{\zeta}^{-1}q_{1} - M_{1}) \right) \]

\[ + \frac{e^{r\Phi}}{2\pi} \int_{\Omega} \partial_{\zeta} \left( \frac{a(\partial_{\zeta}^{-1}q_{1} - M_{1})}{\partial_{\zeta} \Phi} \right) e^{r(\Phi - \bar{\Phi})} \frac{e^{r(\Phi - \bar{\Phi})}}{\xi - \bar{\xi}} d\xi_{2} d\xi_{1} \]

(205)
\[ e^{\tau \Phi} R_T \{ \pi (\partial_z^{-1} q_1 - M_3) \} = \frac{1}{\tau} \left( 2a_{1,2} e^{\tau \Phi} + \frac{e^{\tau \Phi} \pi (\partial_z^{-1} q_1 - M_3)}{2\partial_z \Phi} \right) \]  \hspace{1cm} (206)

\[ + \frac{e^{\tau \Phi}}{2\pi} \int_\Omega \partial_\zeta \left( \frac{\pi (\partial_z^{-1} q_1 - M_3)}{\partial_\zeta \Phi} \right) \frac{e^{\tau (\Phi - \Phi)}}{\zeta - z} d\xi_2 d\xi_1. \]

We have

**Proposition 14** The following asymptotic formula is true:

\[ \left\| \frac{e^{-i\tau \psi}}{2\pi} \int_\Omega \partial_\zeta \left( a (\partial_z^{-1} q_1 - M_1) \right) e^{\tau (\Phi - \Phi)} \frac{d\xi_2 d\xi_1}{\zeta - \bar{\tau}} \right\|_{L^2(\Omega)} \leq C. \]  \hspace{1cm} (208)

By (86), (193) and (194), the family of these functions is bounded in \( L^p(\Omega) \) for any \( p < 2 \). Hence by Proposition 12 there exists a constant \( C \) independent of \( \tau \) such that

\[ \left\| \frac{e^{-i\tau \psi}}{2\pi} \int_\Omega \partial_\zeta \left( a (\partial_z^{-1} q_1 - M_1) \right) e^{\tau (\Phi - \Phi)} \frac{d\xi_2 d\xi_1}{\zeta - \bar{\tau}} \right\|_{L^2(\Omega)} \to 0 \quad \text{as} \ \tau \to +\infty. \]  \hspace{1cm} (207)

**Proof.** We prove the asymptotic behavior of the first term in (207). The proof for the second term is the same. Denote \( r_\tau (\xi) = \partial_\zeta \left( a (\partial_z^{-1} q_1 - M_1) \right) e^{\tau (\Phi - \Phi)} \). By (86), (193) and (194), the family of these functions is bounded in \( L^p(\Omega) \) for any \( p < 2 \). Hence by Proposition 12 there exists a constant \( C \) independent of \( \tau \) such that

\[ \left\| \frac{e^{-i\tau \psi}}{2\pi} \int_\Omega \partial_\zeta \left( a (\partial_z^{-1} q_1 - M_1) \right) e^{\tau (\Phi - \Phi)} \frac{d\xi_2 d\xi_1}{\zeta - \bar{\tau}} \right\|_{L^2(\Omega)} \leq C. \]  \hspace{1cm} (208)

By (86), (193) and (194), for any \( z \neq \bar{x}_1 + i\bar{x}_2 \), the function \( r_\tau (\xi) / (\zeta - \bar{\tau}) \) belongs to \( L^1(\Omega) \). Therefore by Proposition 17

\[ \frac{e^{-i\tau \psi}}{2\pi} \int_\Omega \partial_\zeta \left( a (\partial_z^{-1} q_1 - M_1) \right) e^{\tau (\Phi - \Phi)} \frac{d\xi_2 d\xi_1}{\zeta - \bar{\tau}} \to 0 \quad \text{a.e. in} \quad \Omega. \]  \hspace{1cm} (209)

By (208), (209) and Egorov’s theorem, the asymptotic behavior of the first term in (207) follows immediately.

The asymptotic formula (167) follows from (199), (207), (205) and (206). In order to prove (108), we set \( U_1 = I_1 + I_2 \), where

\[ I_1 = (a + a_{1,1}/\tau) e^{\tau \Phi} + (\bar{a} + b_{1,1}/\tau) e^{\tau \bar{\Psi}} = \left( \frac{a (\partial_z^{-1} q_1 - M_1)}{4\partial_z \Phi} + \frac{\pi (\partial_z^{-1} q_1 - M_3)}{4\partial_z \Phi} \right) e^{\tau \phi} \]  \hspace{1cm} (210)

and

\[ I_2 = \left( a_{1,2} e^{\tau \Phi} + b_{1,2} e^{\tau \bar{\Psi}} \right) - \frac{1}{2} \left( e^{\tau \Phi} \right) R_T \{ \pi (\partial_z^{-1} q_1 - M_1) \} \]  \hspace{1cm} (211)
Here in order to obtain the last equality, we used (200).

From (210) and (211), we obtain (198).

Finally we construct the last term of the complex geometric optics solution $e^{\tau \varphi} w_\tau$. Consider the boundary value problem

$$ L_{q_1}(x, D)(w_\tau e^{\tau \varphi}) = -q_1 e^{\tau \varphi} \quad \text{in } \Omega, \quad (w_\tau e^{\tau \varphi})|_{\Gamma_0} = -U_1. \quad (212) $$

By (197) and Proposition 9 there exists a solution to problem (212) such that

$$ \|w_\tau\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty. \quad (213) $$

Finally we set

$$ u_1 = U_1 + e^{\tau \varphi} w_\tau. \quad (214) $$

By (213), (214), (207), (205) and (206) we can represent the complex geometric optics solution $u_1$ in the form

$$ u_1(x) = e^{\tau \Phi} a + (a_{1,1} + a_{1,2,1})/\tau + e^{\tau \Phi} (\bar{a}_{1,1} + \bar{a}_{1,2,1})/\tau + e^{\tau \Phi} (\bar{a}_{1,1} + \bar{a}_{1,2,1})/\tau \quad \text{as } \tau \to +\infty. \quad (215) $$

Since the Dirichlet-to-Neumann maps for the potentials $q_1$ and $q_2$ are equal, there exists a solution $u_2$ to the Schrödinger equation with potential $q_2$ such that $\frac{\partial u_2}{\partial \nu} = \frac{\partial u_2}{\partial \nu}$ on $\partial \Omega \setminus \Gamma_0$ and $u_1 = u_2$ on $\partial \Omega \setminus \Gamma_0$. Setting $u = u_1 - u_2$, we obtain

$$ (\Delta + q_2)u = (q_2 - q_1)u_1 \quad \text{in } \Omega, \quad u|_{\partial \Omega \setminus \Gamma_0} = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = 0. \quad (216) $$

In a similar way to the construction of $u_1$, we construct a complex geometric optics solution $v$ for the Schrödinger equation with potential $q_2$. The construction of $v$ repeats the corresponding steps of the construction of $u_1$. The only difference is that instead of $q_1$ and $\tau$, we use $q_2$ and $-\tau$, respectively. We skip the details of the construction and point out that similarly to (214) it can be represented in the form

$$ v(x) = e^{\tau \Phi} a + (\bar{a}_{1,1} + \bar{a}_{1,2,1})/\tau + e^{-\tau \Phi} (\bar{a}_{1,1} + \bar{a}_{1,2,1})/\tau \quad \text{as } \tau \to +\infty, \quad v|_{\Gamma_0} = 0, \quad (217) $$

where $M_2(z)$ and $M_4(\bar{z})$ satisfy

$$ (\partial_{\bar{z}}^{-1} q_2 - M_2)(\bar{z}) = 0, \quad (\partial_{\bar{z}}^{-1} q_2 - M_4)(\bar{z}) = 0. $$

The functions $\bar{a}_1(z) = \bar{a}_{1,1}(z) + \bar{a}_{1,2}(z)$ and $\bar{b}_1(z) = \bar{b}_{1,1}(z) + \bar{b}_{1,2}(z)$ are given by

$$ \bar{a}_{1,1}(z) + \bar{b}_{1,1}(\bar{z}) = \frac{a(\partial_{\bar{z}}^{-1} q_2 - M_2)}{4\tau \partial_{\bar{z}} \Phi} + \frac{\bar{a}(\partial_{\bar{z}}^{-1} q_2 - M_4)}{4\tau \partial_{\bar{z}} \Phi} \quad \text{on } \Gamma_0, \quad \bar{a}_{1,1}, \bar{b}_{1,1} \in C^1(\bar{\Omega}). \quad (218) $$

and $\bar{a}_{1,2}(z), \bar{b}_{1,2}(\bar{z}) \in C^1(\bar{\Omega})$ are a holomorphic function and an antiholomorphic function respectively such that

$$ \bar{a}_{1,2}(z) = \frac{1}{8\pi} \int_{\Gamma_0} \frac{(\nu_1 + i\nu_2)\bar{a}(\partial_{\bar{z}}^{-1} q_2 - M_4)e^{\tau(\Phi - \bar{\Phi})}}{(\zeta - z)\partial_{\bar{z}} \Phi} d\sigma $$

and

$$ \bar{b}_{1,2}(z) = \frac{1}{8\pi} \int_{\Gamma_0} \frac{(\nu_1 - i\nu_2)a(\partial_{\bar{z}}^{-1} q_2 - M_2)e^{-\tau(\Phi - \bar{\Phi})}}{(\zeta - z)\partial_{\bar{z}} \Phi} d\sigma. $$
Denote \( q = q_1 - q_2 \). Taking the scalar product of equation (210) with the function \( v \), we have:

\[
\int_\Omega q u_1 v dx = 0. \tag{219}
\]

From formulae (215) and (217) in the construction of complex geometric optics solutions, we have

\[
0 = \int_\Omega q u_1 v dx = \int_\Omega q (a^2 + \pi^2) dx + \frac{1}{\tau} \int_\Omega q (a(a_{1,1} + a_{1,2,1} + b_{1,1} + b_{1,2,1}) + \pi(\tilde{a}_{1,1} + \tilde{a}_{1,2,1} + \tilde{b}_{1,1} + \tilde{b}_{1,2,1})) dx
\]

\[
+ \int_\Omega q(a \pi e^{2\tau i \psi} + a \pi e^{-2\tau i \psi}) dx
\]

\[
+ \frac{1}{4\tau} \int_\Omega \left( qa^2 \frac{\partial \pi^{-1} q_2 - M_2}{\partial x} + qa^2 \frac{\partial \pi^{-1} q_2 - M_2}{\partial x} \right) dx
\]

\[
- \frac{1}{4\tau} \int_\Omega \left( qa^2 \frac{\partial \pi^{-1} q_1 - M_1}{\partial x} + qa^2 \frac{\partial \pi^{-1} q_1 - M_1}{\partial x} \right) dx
\]

\[
+ o\left(\frac{1}{\tau}\right) = 0 \quad \text{as} \quad \tau \to +\infty. \tag{220}
\]

Since the potentials \( q_j \) are not necessarily from \( C^0_{\infty}(\bar{\Omega}) \), we can not directly use the stationary phase argument (see Proposition 15 in Section 7). If the function \( q \) is not identically equal to zero on \( \Omega \), then for some positive \( \alpha' \) we set \( \mathcal{X} = \{ x \in \Omega ||q(x)|| > \alpha' \} \). Since the holomorphic function \( a \) is not identically equal to the constant, this function is not equal to zero on open dense set \( V \). The set \( \mathcal{X} \cap V \) has positive measure. Let a point \( \tilde{x}_* \in \Omega \) be some point from \( \mathcal{X} \cap V \). Proposition 6 states that there exists a holomorphic function \( \Phi \) such that (211)-(216) are satisfied and a point \( \tilde{x} \in \mathcal{H} \) can be chosen arbitrarily close to any given point in \( \Omega \). Therefore such a point \( \tilde{x} \) can be chosen close to \( \tilde{x}_* \). Since \( q_j \in W^1_p(\Omega) \) with \( p > 2 \), the function \( q \) is continuous on \( \bar{\Omega} \). Therefore for the point \( \tilde{x} \in \mathcal{H} \) we have

\[
q(\tilde{x}) \neq 0 \quad \text{and} \quad a(\tilde{x}) \neq 0. \tag{221}
\]

Let \( \tilde{q} \in C^\infty(\Omega) \) satisfy \( \tilde{q}(\tilde{x}) = q(\tilde{x}) \). We have

\[
\int_\Omega q \Re (a \pi e^{2\tau i \psi}) dx = \int_\Omega \tilde{q} \Re (a \pi e^{2\tau i \psi}) dx + \int_\Omega (q - \tilde{q}) \Re (a \pi e^{2\tau i \psi}) dx. \tag{222}
\]

Using Proposition 16 and (193) we obtain

\[
\int_\Omega \tilde{q} (a \pi e^{2\tau i \psi} + a \pi e^{-2\tau i \psi}) dx = \frac{2\pi (q |a|^2)(\tilde{x}) \Re e^{2\tau i \psi} }{\tau |(\det H_{\psi})(\tilde{x})|^2} + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty. \tag{223}
\]

The second term on the right-hand side of (222) after integration by parts is written as

\[
\int_\Omega (q - \tilde{q}) (a \pi e^{2\tau i \psi} + a \pi e^{-2\tau i \psi}) dx = \int_\Omega (q - \tilde{q}) \left( a \pi (\nabla \psi, \nabla) e^{2\tau i \psi} - a \pi (\nabla \psi, \nabla) e^{-2\tau i \psi} \right) dx
\]

\[
= \int_{\partial \Omega} q \left( a \pi (\nabla \psi, \nu) e^{2\tau i \psi} - a \pi (\nabla \psi, \nu) e^{-2\tau i \psi} \right) d\sigma
\]

\[
- \frac{1}{2\tau i} \int_\Omega \left( e^{2\tau i \psi} \div (q - \tilde{q}) a \pi \frac{\nabla \psi}{|\nabla \psi|^2} - e^{-2\tau i \psi} \div (q - \tilde{q}) a \pi \frac{\nabla \psi}{|\nabla \psi|^2} \right) dx. \tag{224}
\]

Since \( \psi|_{\Gamma_0} = 0 \), we have

\[
\int_{\partial \Omega} q a \pi \left( \frac{(\nabla \psi, \nu) e^{2\tau i \psi}}{2\tau i |\nabla \psi|^2} - \frac{(\nabla \psi, \nu) e^{-2\tau i \psi}}{2\tau i |\nabla \psi|^2} \right) d\sigma = \int_{\partial \Omega \setminus \Gamma_0} \frac{q a \pi}{2\tau i |\nabla \psi|^2} (\nabla \psi, \nu)(e^{2\tau i \psi} - e^{-2\tau i \psi}) d\sigma.
\]
By [31], [86] and Proposition [18] we conclude that
\[
\int_{\partial \Omega} q \alpha \left( \frac{(\nabla \psi, \nu)e^{2\tau \psi}}{2\tau |\nabla \psi|^2} - \frac{(\nabla \psi, \nu)e^{-2\tau \psi}}{2\tau |\nabla \psi|^2} \right) d\sigma = o\left( \frac{1}{\tau} \right) \quad \text{as } \tau \to +\infty.
\]

The last integral over \( \Omega \) in formula (224) is \( o(\frac{1}{\tau}) \) by Proposition [17] and therefore
\[
\int_{\Omega} (q - \tilde{q})(a \alpha e^{2\tau \psi} + a \alpha e^{-2\tau \psi}) dx = o\left( \frac{1}{\tau} \right) \quad \text{as } \tau \to +\infty.
\]

Taking into account that \( \psi(x) \neq 0 \) and using (223), (225), we have from (221) that
\[
\frac{2\pi \sqrt{|q|}}{|(\det H_{\psi})(x)|} = 0.
\]

Hence \( q(x) = 0 \), and we have a contradiction with (221). The proof of the theorem is completed. \( \blacksquare \)

In the case \( \Gamma_0 = \emptyset \), the uniqueness in determining a potential \( q \) in the two dimensional case was proved for the conductivity equation by Nachman in [60] within \( C^\infty \) conductivities, and later in [4] within \( L^\infty \) conductivities. The case of the Schrödinger equation was solved by Bukhgeim [10] and for the improvement of regularity assumption of potential for Bukhgeim’s uniqueness result, see [43]. Theorem [7] was originally proved in [30] for \( C^{2+\alpha}(\Omega) \) potentials, and in [41], the regularity assumption on potentials was improved to up to \( W^{2,p}_{\alpha}(\Omega) \) with \( p > 2 \). The case of general second-order elliptic equation was studied in the papers [31] and [32]. See also [6], [9]. The results of [30] were extended to a Riemannian surface in [22]. Conditional stability estimates in determining a potential are obtained in [31]. As for reconstruction, see e.g., [86]. An analog of the main theorem of [30] for the Neumann-to-Dirichlet map was proved in [35].

In [40] the result of Theorem [7] was extended to the weakly coupled systems of elliptic equations. More precisely, consider the following boundary value problem:
\[
L(x, D)u = \Delta u + 2A \partial_x D u + 2B \partial_D u + Q u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0. \tag{227}
\]

Here \( u = (u_1, \ldots, u_N) \) and \( A(x), B(x), Q(x) \) be smooth complex-valued \( N \times N \) matrix-valued functions.

Consider the following Dirichlet-to-Neumann map \( \Lambda(A, B, Q, \Gamma_0): \)
\[
\Lambda(A, B, Q, \Gamma_0)(f) = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0}, \tag{228}
\]

where
\[
L(x, D)u = \Delta u + 2A \partial_x D u + 2B \partial_D u + Q u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0, \quad u|_{\partial \Omega \setminus \Gamma_0} = f. \tag{229}
\]

We have

**Theorem 8** ([17]) Let \( A_j, B_j \in C^{5+\alpha}(\Omega) \) and \( Q_j \in C^{4+\alpha}(\Omega) \) for \( j = 1, 2 \) and some \( \alpha \in (0,1) \). Suppose that \( \Lambda(A_1, B_1, Q_1, \Gamma_0) = \Lambda(A_2, B_2, Q_2, \Gamma_0) \). Then
\[
A_1 = A_2 \quad \text{and} \quad B_1 = B_2 \quad \text{on } \partial \Omega \setminus \Gamma_0, \tag{230}
\]

and
\[
2\partial_x(A_1 - A_2) + B_2(A_1 - A_2) + (B_1 - B_2)A_1 - (Q_1 - Q_2) = 0 \quad \text{in } \Omega \tag{231}
\]

and
\[
2\partial_D(B_1 - B_2) + A_2(B_1 - B_2) + (A_1 - A_2)B_1 - (Q_1 - Q_2) = 0 \quad \text{in } \Omega. \tag{232}
\]

**Remark 1.** The proof of Theorem 8 is based on the construction of the complex geometric optics solutions, which is performed in a way similar to one presented in the proof of Theorem 7. Therefore it is critically important that the principal parts of all the equations in (224) are the Laplace operator for the construction of complex geometric optics solutions. If the principal parts of the operators in (224) are different, then such a construction in general is impossible and Calderón’s problem for such a system is still open. In Section 6, we treat the Lamé system whose principal parts are different but a special structure allows us to construct complex geometric optics solutions.
The simultaneous determination of all three matrices $A,B,Q$ from the Dirichlet-to-Neumann map is impossible. Theorem \[40\] asserts that any two coefficient matrices among three are uniquely determined by Dirichlet-to-Neumann map defined by \[228\] and \[229\] for the system of elliptic differential equations. That is,

**Corollary 9** \[40\] Let $(A_j,B_j,Q_j) \in C^{5+\alpha}(\Omega) \times C^{5+\alpha}(\Omega) \times C^{4+\alpha}(\Omega)$, $j = 1, 2$ for some $\alpha \in (0,1)$ and be complex-valued. We assume either $A_1 \equiv A_2$ or $B_1 \equiv B_2$ or $Q_1 \equiv Q_2$ in $\Omega$. Then $\Lambda(A_1,B_1,Q_1,\Gamma_0) = \Lambda(A_2,B_2,Q_2,\Gamma_0)$ implies $(A_1,B_1,Q_1) = (A_2,B_2,Q_2)$ in $\Omega$.

Next we consider other form of elliptic systems:

$$L(x,D)u = \Delta u + A\partial_x u + B\partial_x u + Q u.$$  \hspace{1cm} (233)

Here $A,B,Q$ are complex-valued $N \times N$ matrices. Let us define the following Dirichlet-to-Neumann map $\tilde{\Lambda}(A,B,Q,\Gamma_0)$:

$$\tilde{\Lambda}(A,B,Q,\Gamma_0)(f) = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0},$$  \hspace{1cm} (234)

where

$$\tilde{L}(x,D)u = \Delta u + A\partial_x, u + B\partial_x u + Q u = 0 \text{ in } \Omega, \; u|_{\Gamma_0} = 0, \; u_{\partial \Omega \setminus \Gamma_0} = f, \; u \in H^1(\Omega).$$  \hspace{1cm} (235)

Then one can prove the following corollary.

**Corollary 10** \[44\] Let $Q_1,Q_2 \in C^{4+\alpha}(\Omega)$ and $(A_1,B_1),(A_2,B_2) \in C^{5+\alpha}(\Omega) \times C^{5+\alpha}(\Omega)$ for some $\alpha \in (0,1)$. We assume that $Q_1 \equiv Q_2$ in $\Omega$ and $\tilde{\Lambda}(A_1,B_1,Q_1,\Gamma_0) = \tilde{\Lambda}(A_2,B_2,Q_1,\Gamma_0)$. Then $(A_1,B_1) \equiv (A_2,B_2)$ in $\Omega$.

**Proof.** Observe that $\tilde{L}(x,D) = \Delta + A\partial_x + B\partial_x + Q$ where $A = A + iB$ and $B = A - iB$. Therefore, applying Corollary \[40\] we complete the proof. \[48\]

This corollary generalizes the result of \[48\] where for the scalar elliptic operator $\Delta + a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2}$ the uniqueness of determination of the coefficients $a,b$ was proved assuming that the measurements are made on the whole boundary.

**Remark 2.** Unlike Corollary \[40\] in the two cases of $A_1 \equiv A_2$ and $B_1 \equiv B_2$, we can not, in general, claim that $(A_1,B_1,Q_1) = (A_2,B_2,Q_2)$. We can prove only

(i) $\frac{\partial A_1}{\partial x_1} = \frac{\partial A_2}{\partial x_1}$ in $\Omega$ if $A_1 = A_2$ in $\Omega$.

(ii) $\frac{\partial A_1}{\partial x_2} = \frac{\partial A_2}{\partial x_2}$ in $\Omega$ if $B_1 = B_2$ in $\Omega$.

Moreover consider the following example

$$\Omega = (0,1) \times (0,1),$$

$$\partial \Omega \setminus \Gamma_0 = \{(x_1,x_2); \; x_2 = 0, \; 0 < x_1 < 1\} \cup \{(x_1,x_2); \; x_2 = 1, \; 0 < x_1 < 1\},$$

and let us choose $\eta(x_2) \in C_0^\infty(0,1)$. Then the operators $\tilde{L}(x,D)$ and $e^{\eta(\tilde{L}(x,D)e^{-\eta}}$ generate the same Dirichlet-to-Neumann map \[228\], \[229\], but the matrix coefficient matrices are not equal.

**General second-order elliptic operator.** We consider a general second-order elliptic operator:

$$L(x,D)u = \Delta_g u + 2A\frac{\partial u}{\partial x} + 2B\frac{\partial u}{\partial z} + Qu.$$  \hspace{1cm} (236)

Here $g = g(x) = \{g_{jk}\}_{1 \leq j,k \leq 2}$ is a positive definite symmetric matrix in $\Omega$ and $\Delta_g$ is the Laplace-Beltrami operator associated to the Riemannian metric $g$:

$$\Delta_g \equiv \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^2 \frac{\partial}{\partial x_k}(\sqrt{\det g} g^{jk} \frac{\partial}{\partial x_j}),$$

39
where we set \( \{ g^{jk} \} = g^{-1} \). Assume that \( g \in C^{7+\alpha}(\overline{\Omega}) \), \((A,B,q)\), \((A_j, B_j, q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega}) \), \( j = 1, 2 \) for some \( \alpha \in (0,1) \), are complex-valued functions. We set

\[
L_k(x, D) = \Delta_{g_k} + 2A_k \frac{\partial}{\partial z} + 2B_k \frac{\partial}{\partial \overline{z}} + q_k.
\]

We define the Dirichlet-to-Neumann map by formula:

\[
\Lambda_{g,A,B,q,\Gamma_0}(f) = \frac{\partial u}{\partial \nu_g} |_{\partial \Omega \setminus \Gamma_0},
\]

where

\[
L(x, D)u = 0 \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0, \quad u|_{\partial \Omega \setminus \Gamma_0} = f, \quad u \in H^1(\Omega)
\]

and \( \frac{\partial}{\partial \nu_g} = \sqrt{\det g} \sum_{k=1}^2 g^{jk} \frac{\partial}{\partial z} \) is the conormal derivative with respect to the metric \( g \).

Our goal is to determine the metric \( g \) and coefficients \( A, B, q \) from the Dirichlet-to-Neumann map \( \Lambda_{g,A,B,q,\Gamma_0} \) given by (237) and (238). In general, the uniqueness is impossible. There are the following main invariance properties of the Dirichlet-to-Neumann map in the problem.

- **Conformal invariance.** Let \( \beta \in C^{7+\alpha}(\overline{\Omega}) \) be a strictly positive function. Then

\[
\Lambda_{g,A,B,q,\Gamma_0} = \Lambda_{\beta g,A,\beta B,\beta q,\Gamma_0}.
\]

This follows since the Laplace-Beltrami operator is conformal invariant in two dimensions:

\[
\Delta_{\beta g} = \frac{1}{\beta} \Delta g.
\]

- **Gauge transformation.** It is easy to see that the Dirichlet-to-Neumann map for the operators \( e^{-n}L(x, D)e^n \) and \( L(x, D) \) are the same provided that \( \eta \) is a smooth complex-valued function such that

\[
\eta \in C^{6+\alpha}(\overline{\Omega}), \quad \eta|_{\partial \Omega \setminus \Gamma_0} = \frac{\partial \eta}{\partial \nu_g} |_{\partial \Omega \setminus \Gamma_0} = 0.
\]

- **Diffeomorphism invariance.** Let \( F = (F_1, F_2) : \Omega \to \overline{\Omega} \) be a diffeomorphism such that \( F|_{\partial \Omega \setminus \Gamma_0} = \text{Identity} \). The pull back of a Riemannian metric \( g \) is given as composition of matrices by

\[
F^* g = ((DF) \circ g \circ (DF)^T) \circ F^{-1}
\]

and \( DF \) denotes the differential of \( F \), \((DF)^T \) its transpose and \( \circ \) denotes the matrix composition.

Moreover we introduce the functions \( A_F = \{(A + B)(\frac{\partial F_1}{\partial z} - i \frac{\partial F_2}{\partial \overline{z}}) + i(B - A)(\frac{\partial F_1}{\partial \overline{z}} + i \frac{\partial F_2}{\partial z})\} \circ F^{-1} |\det DF^{-1}|, B_F = \{(A + B)(\frac{\partial F_1}{\partial z} + i \frac{\partial F_2}{\partial \overline{z}}) + i(B - A)(\frac{\partial F_1}{\partial \overline{z}} - i \frac{\partial F_2}{\partial z})\} \circ F^{-1} |\det DF^{-1}|, q_F = |\det DF^{-1}|(q \circ F^{-1}) \). Then we can verify

\[
\Lambda_{g,A,B,q,\Gamma_0} = \Lambda_{F^* g,A_F,B_F,q_F,\Gamma_0}.
\]

We can show the converse, that is, the above three kinds of the invariance exhaust all the possibilities.

We have

**Theorem 11** (233) Suppose that for some \( \alpha \in (0,1) \), there exists a positive function \( \tilde{\beta} \in C^{7+\alpha}(\overline{\Omega}) \) such that \((g_1 - \tilde{\beta} g_2)|_{\partial \Omega \setminus \Gamma_0} = \frac{\partial (g_1 - \tilde{\beta} g_2)}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = (A_1 - \frac{\partial A_2}{\partial \nu}) |_{\partial \Omega \setminus \Gamma_0} = (B_1 - \frac{\partial B_2}{\partial \nu}) |_{\partial \Omega \setminus \Gamma_0} = 0\). Then \( \Lambda_{g_1,A_1,B_1,q_1,\Gamma_0} = \Lambda_{g_2,A_2,B_2,q_2,\Gamma_0} \) if and only if there exist a diffeomorphism \( F \in C^{8+\alpha}(\overline{\Omega}) \), \( F : \Omega \to \overline{\Omega} \) satisfying \( F|_{\partial \Omega \setminus \Gamma_0} = \text{Id} \), a positive function \( \beta \in C^{7+\alpha}(\overline{\Omega}) \) and a complex valued function \( \eta \) satisfying (234) such that

\[
L_2(x, D) = e^{-\eta}K(x, D)e^\eta,
\]

where

\[
K(x, D) = \Delta_{\beta F^* g_1} + \frac{2}{\beta}(A_1 F \frac{\partial}{\partial z} + B_1 F \frac{\partial}{\partial \overline{z}}) + \frac{1}{\beta} q_1 F.
\]

40
Calderón’s problem for the matrix conductivity. The question proposed by Calderón [13] is whether one can uniquely determine the electrical conductivity of a medium by making voltage and current measurements at the boundary.

In the anisotropic case the conductivity depends on direction and is represented by a positive definite symmetric matrix \( \{ \sigma^{jk} \} \). The conductivity equation with voltage potential \( f \) on \( \partial \Omega \) is given by

\[
\mathcal{L}(x, D) u = \sum_{j,k=1}^{2} \frac{\partial}{\partial x_j} (\sigma^{jk} \frac{\partial u}{\partial x_k}) = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = f.
\]

The Dirichlet-to-Neumann map is defined by

\[
\Lambda_\sigma(\Gamma_0) f = \sum_{i,j=1}^{2} \sigma^{ij} \frac{\partial u}{\partial x_j} |_{\Gamma_0}, \quad \mathcal{L}(x, D) u = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega \setminus \Gamma_0} = f, \quad u|_{\Gamma_0} = 0.
\]

It has been known for a long time (e.g., [57]) that \( \Lambda_\sigma \) does not determine \( \sigma \) uniquely in the anisotropic case. Let \( F : \overline{\Omega} \to \overline{\Omega} \) be a diffeomorphism such that \( F(x) = x \) for \( x \) on \( \partial \Omega \setminus \Gamma_0 \). Then

\[
\Lambda_{F^* \sigma} = \Lambda_\sigma,
\]

where

\[
F^* \sigma = \left( \frac{(DF) \circ \sigma \circ (DF)^T}{|\det DF|} \right) \circ F^{-1}.
\] (243)

In the case of \( \Gamma_0 = \emptyset \), the question whether one can determine the conductivity up to the above obstruction has been solved in two dimensions for \( C^2 \) conductivities in [60] and merely \( L^\infty \) conductivities in [15]. See also [11]. The method of the proof in all these papers is based on the reduction to the isotropic case performed using isothermal coordinates [1].

We can prove the uniqueness in Calderón’s problem for the anisotropic conductivity:

**Theorem 12** ([33]) Let \( \sigma_1, \sigma_2 \in C^{7+\alpha}(\overline{\Omega}) \) with some \( \alpha \in (0, 1) \) be positive definite symmetric matrices on \( \overline{\Omega} \) such that \( (\sigma_1 - \sigma_2)|_{\partial \Omega \setminus \Gamma_0} = \frac{\partial}{\partial \nu} (\sigma_1 - \sigma_2)|_{\partial \Omega \setminus \Gamma_0} = 0 \). If \( \Lambda_{\sigma_1}(\Gamma_0) = \Lambda_{\sigma_2}(\Gamma_0) \), then there exists a diffeomorphism \( F : \overline{\Omega} \to \overline{\Omega} \) satisfying \( F|_{\partial \Omega \setminus \Gamma_0} = \text{Identity} \) and \( F \in C^{8+\alpha}(\overline{\Omega}) \) such that

\[
F^* \sigma_1 = \sigma_2.
\]

The uniqueness corresponding to the isotropic case was proven in [30] and in fact follows from Theorem [11] in the case where \( g = \text{Identity} \) and \( A = B = 0 \). We mention that [22] has proven a similar result for general Riemann surfaces in the case where \( g \) is not the identity but fixed.

**General case where the principal part is the Laplacian.** Assume that the principal parts of second-order elliptic operators under consideration are the Laplacian: \( g = I \). Then we can prove a bit sharper result than Theorem [11].

**Theorem 13** ([33]) The relation \( \Lambda_{L_1, A_1, B_1, q_1, \Gamma_0} = \Lambda_{L_2, A_2, B_2, q_2, \Gamma_0} \) holds true if and only if there exists a function \( \eta \in C^{6+\alpha}(\overline{\Omega}) \), \( \eta|_{\partial \Omega \setminus \Gamma_0} = \frac{\partial}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = 0 \) such that

\[
L_1(x, D) = e^{-\eta} L_2(x, D) e^\eta.
\] (244)

**Proof.** For simplicity we consider only the case when domain \( \Omega \) is simply connected. The proof for the general domain is given in [33]. By Theorem [33] we have

\[
A_1 = A_2, \quad B_1 = B_2 \quad \text{on} \quad \partial \Omega \setminus \Gamma_0,
\] (245)

and in \( \Omega \) we have

\[
-2 \frac{\partial}{\partial z} (A_1 - A_2) - A_1 B_1 + A_2 B_2 + (q_1 - q_2) = 0,
\] (246)
- \frac{\partial}{\partial z} (B_1 - B_2) - A_1 B_1 + A_2 B_2 + (q_1 - q_2) = 0. \tag{247}

We only prove the sufficiency since the necessity of the condition is easy to be checked. By (246) and (247), we have \( \frac{\partial}{\partial z} (A_1 - A_2) = \frac{\partial}{\partial z} (B_1 - B_2) \). This equality is equivalent to

\[
\frac{\partial (A - B)}{\partial x_1} = i \frac{\partial (B + A)}{\partial x_2} \quad \text{where} \quad (A, B) = (A_1 - A_2, B_1 - B_2).
\]

Since the domain \( \Omega \) simply connected, there exists a function \( \tilde{\eta} \) such that:

\[
(i(B + A), (A - B)) = \nabla \tilde{\eta}. \tag{248}
\]

By (245) we have

\[
\tilde{\eta}|_{\partial \Omega \setminus \Gamma_0} = \nabla \tilde{\eta}|_{\partial \Omega \setminus \Gamma_0} = 0.
\]

Setting \( 2\eta = -i\tilde{\eta} \) we have from (248)

\[
((B + A), i(B - A)) = 2\nabla \eta.
\]

Therefore (249) yields

\[
q_1 = q_2 + \Delta \eta + 4 \frac{\partial \eta}{\partial z} \frac{\partial \eta}{\partial z} + 2 \frac{\partial \eta}{\partial z} A_2 + 2 \frac{\partial \eta}{\partial z} B_2. \tag{249}
\]

The operator \( L_1(x, D) \) given by the right-hand side of (244) has the Laplace operator as the principal part, the coefficients of \( \frac{\partial^2}{\partial x_1^2} \) is \( A_2 + B_2 + 2 \frac{\partial \eta}{\partial z} \frac{\partial \eta}{\partial z} \), the coefficient of \( \frac{\partial}{\partial x_1} \) is \( i(B_2 - A_2) + 2 \frac{\partial \eta}{\partial z} \frac{\partial \eta}{\partial z} \), and the coefficient of the zeroth order term is given by the right-hand side of (244). The proof of the proposition is complete. \( \blacksquare \)

**The magnetic Schrödinger equation.** Denote \( \tilde{A} = (\tilde{A}_1, \tilde{A}_2) \), where \( \tilde{A}_j, j = 1, 2 \), are real-valued, \( \tilde{A} = \tilde{A}_1 - i \tilde{A}_2 \), rot \( \tilde{A} = \frac{\partial \tilde{A}_2}{\partial x_1} - \frac{\partial \tilde{A}_1}{\partial x_2} \). The magnetic Schrödinger operator is defined by

\[
L_{\tilde{A}, \tilde{q}}(x, D) = \sum_{k=1}^{2} \left( \frac{1}{i} \frac{\partial}{\partial x_k} + \tilde{A}_k \right)^2 + \tilde{q}.
\]

Let us define the following Dirichlet-to-Neumann map

\[
\Lambda_{\tilde{A}, \tilde{q}, \Gamma_0}(f) = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0},
\]

where

\[
L_{\tilde{A}, \tilde{q}}(x, D) u = 0 \text{ in } \Omega, \quad u|_{\partial \Omega \setminus \Gamma_0} = f, \quad u|_{\Gamma_0} = 0, \quad u \in H^1(\Omega).
\]

Theorem \( \blacksquare \) implies

**Corollary 14** \( \{53\} \) Let real-valued vector fields \( \tilde{A}^{(1)}, \tilde{A}^{(2)} \in C^{5+\alpha}(\Omega) \) and complex-valued potentials \( \tilde{q}^{(1)}, \tilde{q}^{(2)} \in C^{4+\alpha}(\Omega) \) with some \( \alpha \in (0, 1) \), satisfy \( \Lambda_{\tilde{A}^{(1)}, \tilde{q}^{(1)}, \Gamma_0} = \Lambda_{\tilde{A}^{(2)}, \tilde{q}^{(2)}, \Gamma_0} \). Then \( \tilde{q}^{(1)} = \tilde{q}^{(2)} \) and rot \( \tilde{A}^{(1)} = \text{rot } \tilde{A}^{(2)} \).

In the case of the Dirichlet-to-Neumann map on the whole boundary, see \{53\} and \{71\}; \{71\} proved a uniqueness result provided that both electric and magnetic potentials are small, and \{53\} proved a uniqueness result for a special case of the magnetic Schrödinger equation, namely the Pauli Hamiltonian. See also \{71\} and \{69\}.

We conclude this section with the uniqueness in the case where the subboundaries of Dirichlet input and measured Neumann data are disjoint.
Let $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_0$ where $\Gamma_1 \cap \Gamma_2 = \Gamma_0 \cap \Gamma_k = \emptyset$, $k = 1, 2$. Then we consider the unique identifiability of the conductivity by taking all pairs of Dirichlet data on the subboundary $\Gamma_1$ and the corresponding Neumann data on the subboundary $\Gamma_2$:

$$A_{\gamma}(\Gamma_1, \Gamma_2)(f) = \gamma \frac{\partial u}{\partial \nu} |_{\Gamma_2}, \ \text{div}(\gamma \nabla u) = 0 \text{ in } \Omega, \ u|_{\Gamma_0 \cup \Gamma_2} = 0, \ u|_{\Gamma_1} = f. \quad (250)$$

We consider that the input is located on $\Gamma_1$, while the output is measured on $\Gamma_2$. In the case where $\Gamma_1 = \Gamma_2$ and is an arbitrary open subset of the boundary, the global uniqueness was shown in [30] with $\gamma \in C^{3+\alpha}(\Omega)$, with some $\alpha \in (0, 1)$. See also Theorem 7.

In order to state our main result, we need the following geometric assumption on the position of the sets $\Gamma_1, \Gamma_2, \Gamma_0$ on $\partial \Omega$.

**Assumption A.** Let $\Gamma_1, \Gamma_2, \Gamma_0 \subset \partial \Omega$ be non-empty open subsets of the boundary such that $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_0$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_k = \bigcup_{j=1}^{2} \Gamma_{k,j}$, $\Gamma_0 = \bigcup_{\ell=1}^{4} \Gamma_{0,\ell}$, where $\Gamma_{k,j}, j, k = 1, 2$, $\Gamma_{0,\ell}$, $\ell = 1, 2, 3, 4$ are not empty open connected subsets of $\partial \Omega$ and mutually disjoint. Then $\partial \Omega$ is separated into $\Gamma_{0,1}, \Gamma_{2,1}, \Gamma_{0,2}, \Gamma_{1,1}, \Gamma_{0,3}, \Gamma_{2,2}, \Gamma_{0,4}, \Gamma_{1,2}$ in the clockwise order.

We note that $\Gamma_1, \Gamma_2$ can be arbitrarily small provided that the above separation condition is satisfied. Then

**Theorem 15 ([32])** We suppose Assumption A. Let $\gamma_j > 0$ on $\overline{\Omega}$ and $\gamma_j \in C^{4+\alpha}(\Omega)$, $j = 1, 2$ for some $\alpha > 0$. Assume $A_{\gamma_j}(\Gamma_1, \Gamma_2) = A_{\gamma_2}(\Gamma_1, \Gamma_2)$ and that $(\gamma_1 - \gamma_2)|_{\Gamma_1 \cup \Gamma_2} = \frac{\partial}{\partial \nu}(\gamma_1 - \gamma_2)|_{\Gamma_*} = 0$, where $\Gamma_* \subset \Gamma_1 \cup \Gamma_2$ is some open set. Then $\gamma_1 \equiv \gamma_2$ on $\overline{\Omega}$.

Next for the Schrödinger equation $L_q(x, D)u = \Delta u + qu = 0$ in $\Omega$, we consider the problem of determining a complex-valued potential $q$ by the following Dirichlet-to-Neumann map:

$$A_q(\Gamma_1, \Gamma_2)(f) = \frac{\partial u}{\partial \nu} |_{\Gamma_2}, \ \text{where } L_q(x, D)u = 0 \text{ in } \Omega, \ u|_{\Gamma_0 \cup \Gamma_2} = 0, \ u|_{\Gamma_1} = f, \ u \in H^1(\Omega). \quad (251)$$

Next we state the corresponding result for the Schrödinger equation.

**Theorem 16 ([32])** We suppose Assumption A. Let $q_j \in C^{2+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let $q_j$ be complex-valued. If

$$A_{q_j}(\Gamma_1, \Gamma_2) = A_{q_2}(\Gamma_1, \Gamma_2),$$

then we have

$$q_1 \equiv q_2 \text{ in } \Omega.$$

**Proof of Theorem 15** If $u$ is some solution to the conductivity equation then the function $u^* = u / \sqrt{\gamma}$ solves in domain $\Omega$ the Schrödinger with the potential $q = -\frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}$. We claim that the Dirichlet-to-Neumann maps $A_{\gamma_j}$ for the Schrödinger equations with potentials $q_j = -\frac{\Delta \sqrt{\gamma_j}}{\sqrt{\gamma_j}}$ are the same, provided that the Dirichlet-to-Neumann maps $A_{\gamma_j}$ are the same. Indeed let $f \in L^2(\partial \Omega), \text{supp} f \subset \Gamma_1$. Setting $\tilde{f} = f / \sqrt{\gamma_j}$ we have that $A_{\gamma_j}(\Gamma_1, \Gamma_2)(\tilde{f}) = A_{\gamma_2}(\Gamma_1, \Gamma_2)(\tilde{f})$. Denote by $\tilde{u}_j$ the corresponding solutions to the conductivity equation with the Dirichlet boundary condition $\tilde{f}$. Then $u_j = \tilde{u}_j \sqrt{\gamma_j}$ is the solution to the Schrödinger equation with the potential $q_j$ and the Dirichlet boundary condition $\tilde{f}$. Observe that

$$A_{\gamma_j}(\Gamma_1, \Gamma_2)(f) = \frac{\partial u_j}{\partial \nu} |_{\Gamma_2} = \frac{\partial (\tilde{u}_j \sqrt{\gamma_j})}{\partial \nu} |_{\Gamma_2} = \frac{\sqrt{\gamma_j} \partial \tilde{u}_j}{\partial \nu} |_{\Gamma_2} = \frac{\partial (\tilde{u}_2 \sqrt{\gamma_2})}{\partial \nu} |_{\Gamma_2} = \frac{\partial u_2}{\partial \nu} |_{\Gamma_2} = A_{\gamma_2}(\Gamma_1, \Gamma_2)(f).$$

Applying the theorem 16 we obtain that $q_1 \equiv q_2$. Then the function $w = \sqrt{\gamma_1 - \gamma_2}$ verifies

$$\Delta w - \frac{\Delta \sqrt{\gamma_1}}{\sqrt{\gamma_1}} w = 0 \text{ in } \Omega, \ w|_{\partial \Omega} = \frac{\partial w}{\partial \nu} |_{\Gamma_*} = 0.$$

Applying to the above problem the classical unique continuation for the second order elliptic operator (see e.g. Corollary 2.9 Chapter XIV of [73]), we obtain $\gamma_1 \equiv \gamma_2$. ■.
5 Calderón’s problem for semilinear elliptic equations

In this section, we assume that $\Gamma_0 \subset \partial \Omega$ is an arbitrarily fixed relatively open subset of $\partial \Omega$.

Consider the following boundary value problem:

$$P(x, D)u = \Delta u + q(x)u - f(x, u) = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0,$$

We introduce the Dirichlet-to-Neumann map $\Lambda_{q,f}$:

$$\Lambda_{q,f}(g) = \frac{\partial u}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0}, \quad \text{where } P(x, D)u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0, \quad u|_{\partial \Omega \setminus \Gamma_0} = g, \quad u \in H^1(\Omega).$$

This section is concerned with the following inverse problem: Determine a coefficient $q$ and a nonlinear term $f$ from the Dirichlet-to-Neumann map $\Lambda_{q,f}$.

In this section, we always assume that $f, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2} \in C^0(\overline{\Omega} \times \mathbb{R}^1)$. We state other conditions on semilinear terms $f$:

$$f(x, 0) = \frac{\partial f}{\partial y}(x, 0) = 0, \quad x \in \mathbb{R}^1$$

and for some positive constants $p > 1, C_1, C_2$, the following holds true:

$$f(x, y) \geq C_1|y|^{p_1} - C_2, \quad \forall (x, y) \in \Omega \times \mathbb{R}^1.$$

Moreover for some $p_1 > 0, p_2 > 0, C_3 > 0$ and $C_4 > 0$, the following inequalities holds true:

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq C_3(1 + |y|^{p_1}), \quad \left| \frac{\partial^2 f}{\partial y^2}(x, y) \right| \leq C_4(1 + |y|^{p_2}), \quad \forall (x, y) \in \Omega \times \mathbb{R}^1.$$

The first result is concerned with the uniqueness in determining a linear part, that is, a potential $q$.

Theorem 17 (42) Let functions $f_1, f_2$ satisfy (253), (254), (255) and $q_j \in C^{2+\alpha}(\overline{\Omega}), j = 1, 2$, with some $\alpha \in (0, 1)$. Suppose that $\Lambda_{q_1, f_1} = \Lambda_{q_2, f_2}$. Then $q_1 = q_2$ in $\Omega$.

Theorem 17 is concerned with the determination of potentials in spite of unknown nonlinear terms, and the proof is similar to Theorem 4.

Remark 1. Since our assumptions on the potential $q$ and nonlinear term $f$ in general do not imply the uniqueness of a solution for the boundary value problem for the elliptic operator $P(x, D)$, by the equality $\Lambda_{f_1} = \Lambda_{f_2}$, we mean the following: for any pair $(v_1, v_2)$ such that

$$P_1(x, D)w = \Delta w + q_1w - f_1(x, w) = 0, \quad w|_{\Gamma_0} = 0, \quad \frac{\partial w}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = v_2, \quad w|_{\partial \Omega \setminus \Gamma_0} = v_1$$

there exists a function $\tilde{w} \in H^1(\Omega)$ such that $P_2(x, D)\tilde{w} = \Delta \tilde{w} + q_2\tilde{w} - f_2(x, \tilde{w}) = 0, \tilde{w}|_{\Gamma_0} = 0, \tilde{w}|_{\partial \Omega \setminus \Gamma_0} = v_1$ and $\frac{\partial \tilde{w}}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = v_2$.

Remark 2. Theorem 8 is still true if condition (254) is replaced by following: there exists a continuous function $G$ such that a solution to the boundary value problem

$$P(x, D)u = 0 \quad \text{in } \Omega, \quad u|_{\partial \Omega} = g$$

satisfies the estimate

$$\|u\|_{H^1(\Omega)} \leq G(\|g\|_{H^1(\partial \Omega)}).$$

For any $F(t) \in C([0, 1]; C^{2+\alpha}(\overline{\Omega}))$ with $\alpha \in (0, 1)$, we introduce the set

$$\mathcal{O}_F = \bigcup_{0 \leq t \leq 1, x \in \Omega} \{(x, F(x, t))\}.$$
Let
\[ U_j = \{ F \in C([0,1]; C^{2+\alpha}(\overline{\Omega})); \quad F(\cdot,0) = 0 \quad u(\cdot,t) := F(t) \text{ satisfies} \]
\[ \Delta u(x,t) + q_ju(x,t) - f_j(x,u(x,t)) = 0, \quad x \in \Omega, \quad u(\cdot,t)|_{\Gamma_0} = 0, \quad j = 1,2. \]

The next theorem asserts the uniqueness for semilinear terms \( f_k, k = 1,2 \) in some range provided that the potential \( q \) is known:

**Theorem 18** (\textit{[42]}) Let \( q_1 = q_2 = q \in C^{2+\alpha}(\overline{\Omega}) \) be arbitrarily fixed. Let functions \( f_1, f_2 \in C^{3+\alpha}(\overline{\Omega} \times \mathbb{R}^1) \) for some \( \alpha \in (0,1) \), satisfy \textit{(253)}, \textit{(254)} and \( f_1(\cdot,0) = f_2(\cdot,0) = 0 \). Suppose that \( \Lambda_{q,f_1} = \Lambda_{q,f_2} \). Then
\[ f_1 - f_2 = 0 \quad \text{in} \quad \bigcup_{j \in \{1,2\}} \bigcup_{F \in U_j} O_F. \]

**Corollary 19** Let \( q_1,q_2 \in C^{2+\alpha}(\overline{\Omega}) \) and let functions \( f_1, f_2 \in C^{3+\alpha}(\overline{\Omega} \times \mathbb{R}^1) \) with some \( \alpha \in (0,1) \), satisfy \textit{(253)}, \textit{(254)} and \textit{(255)}. Suppose that \( \Lambda_{q_1,f_1} = \Lambda_{q_2,f_2} \). Then \( q_1 = q_2 \) in \( \Omega \) and
\[ f_1 - f_2 = 0 \quad \text{in} \quad \bigcup_{j \in \{1,2\}} \bigcup_{F \in U_j} O_F. \]

**Corollary 20** Let \( q_1,q_2 \in C^{2+\alpha}(\overline{\Omega}) \) and let functions \( f_1, f_2 \in C^{3+\alpha}(\mathbb{R}^1) \) be independent of the variable \( x \) with some \( \alpha \in (0,1) \), and satisfy \textit{(253)}, \textit{(254)} and \textit{(255)}. Suppose that \( \Lambda_{q_1,f_1} = \Lambda_{q_2,f_2} \). Then \( q_1 = q_2 \) and \( f_1 = f_2 \) in \( \Omega \).

In fact, since \( f_1, f_2 \) are independent of \( x \), Theorems 17 and 18 yields the conclusion. 

**Remark 3.** Under the condition of Theorem 17, we cannot completely recover the nonlinear term. Indeed, if \( \rho \in C^2(\overline{\Omega}), \rho|_{\partial\Omega} = 0, \frac{\partial \rho}{\partial n} < 0 \) on \( \partial\Omega \) and \( \rho > 0 \) in \( \Omega \), under assumptions \textit{(253)} and \textit{(254)}, we have the following a priori estimate proved in \textit{(21)}:
\[ \int_{\Omega} \rho^\kappa(|\nabla u|^2 + |u|^{p+1})dx \leq C \]
for \( u \in H^1(\Omega) \) satisfying \( P(x,D)u = 0 \) in \( \Omega \). Here a constant \( C \) is independent of \( u \) and \( \kappa \) depends on \( p \). Such a estimate immediately implies that for any \( \Omega_1 \subset \subset \Omega \), there exists a constant \( C(\Omega_1) > 0 \) such that
\[ \| u \|_{C^0(\overline{\Omega_1})} \leq C(\Omega_1). \]

This estimate and \textit{(254)} imply that for any \( x \in \Omega_1 \) a nonlinear term \( f(x,y) \) in general can not be recovered for all sufficiently large \( y \).

The uniqueness results for recovery of the nonlinear term in the semilinear elliptic equation were first obtained for the case \( \Gamma_0 = \emptyset \) in three or higher dimensional cases by Isakov and Sylvester in \textit{[50]} and in two dimensional case by Isakov and Nachman in \textit{[49]}. It should be mentioned that their papers requires the uniqueness of solution for the Dirichlet boundary problem for the operator \( P(x,D) \). Later, by Isakov in \textit{[48]}, this result was extended to the case of a system of semilinear elliptic equations with Dirichlet-to-Neumann map on a certain subboundary. Also see Kang and Nakamura \textit{[52]} for determination of coefficients of the linear and the quadratic nonlinear terms in the principal part of a quasilinear elliptic equation. As for the determination of quasilinear part, see Sun \textit{[70]}. In a special case where a nonlinear term is independent of \( x \), the uniqueness was proved in determining such a nonlinear term from partial Cauchy data \textit{[47]}. Moreover we note that in \textit{[47]} and \textit{[50]}, the monotonicity of \( f(x,u) \) with respect to \( u \) is assumed. In general, if a nonlinear term depends on \( x, u \) and the gradient of \( u \), then it is impossible to prove the uniqueness even for the linear case. This can be seen by \textit{[49]} if we consider the term \(-f(x,u,\nabla u) = A(x) \cdot \nabla u + q(x)u\).

Theorem 18 is concerned with the determination of nonlinear terms and the proof needs a different ingredient from any previous arguments. Thus for completeness, we describe the proof from \textit{[42]} with modifications.
Proof of Theorem 18. We set $P_k(x, D)u = \Delta u + q(x)u - f_k(x, u)$, $k = 1, 2$, and $u_{1,t}(x) = u(x, t) \in C([0, 1]; C^{2+\alpha} (\Omega))$ for some $\alpha \in (0, 1)$ be a function such that any $t$ function $u_{1,t}(x)$, $t \in [0, 1]$ solves the boundary value problem:

$$P_1(x, D)u_{1,t} = 0 \text{ in } \Omega, \quad u_{1,t}\vert_{\Gamma_0} = 0.$$  

Let $u_{2,t} \in H^1(\Omega)$, $t \in [0, 1]$ satisfy

$$P_2(x, D)u_{2,t} = 0 \text{ in } \Omega, \quad u_{2,t} = u_{1,t} \text{ on } \partial \Omega, \quad \forall t \in [0, 1].$$  

Then $\Lambda_{q,f_1} = \Lambda_{q,f_2}$ yields

$$\left( \frac{\partial u_{1,t}}{\partial \nu} - \frac{\partial u_{2,t}}{\partial \nu} \right)\vert_{\partial \Omega \setminus \Gamma_0} = 0, \quad \forall t \in [0, 1].$$

By (255) and the Sobolev embedding theorem, $f_2(\cdot, u_{2,t}(\cdot)) \in L^\kappa(\Omega)$ for any $\kappa > 1$. The standard solvability theory for the Dirichlet boundary value problem for the Laplace operator in Sobolev spaces implies $u_{2,t} \in H^2(\Omega)$. Hence $f_2(\cdot, u_{2,t}(\cdot)) \in C^\kappa(\Omega)$ for any $\alpha \in (0, 1)$. Then, since $u_{2,t} \in C^{2+\alpha}(\partial \Omega)$, the solvability theory for the Dirichlet boundary value problem for the Laplace operator in Hölder spaces implies $u_{2,t} \in C^{2+\alpha}(\Omega)$. By the assumption, there exists a constant $K > 0$ such that

$$\sup_{t \in [0, 1]} \|u_{1,t}\|_{C^{2+\alpha}(\Omega)} \leq K. \quad (256)$$

Next we show that

$$u_{2,t} \in C([0, 1]; C^{2+\alpha}(\Omega)). \quad (257)$$

Indeed, suppose that at some point $t_0 \in [0, 1]$ the function $u_{2,t}$ is discontinuous. Then there exists a sequence $t_j \to t_0$ such that

$$\lim_{j \to +\infty} \|u_{2,t_j} - u_{2,t_0}\|_{C^{2+\alpha}(\Omega)} \neq 0.$$  

Without loss of generality, by (250) we can assume that there exists a function $\hat{u} \in H^2(\Omega), u\vert_{\Gamma_0} = 0$ such that

$$u_{2,t_j} \to \hat{u} \text{ in } H^2(\Omega) \quad \text{as } t_j \to +\infty$$  

and

$$\hat{u} \neq u_{2,t_0}. \quad (258)$$

Obviously the function $\hat{u}$ satisfies

$$P_2(x, D)\hat{u} = 0 \text{ in } \Omega, \quad \hat{u}\vert_{\Gamma_0} = 0.$$  

In addition, since $(u_{2,t_j}, \partial u_{2,t_j}/\partial \nu) = (u_{1,t_j}, \partial u_{1,t_j}/\partial \nu) \in C([0, 1]; C^{2+\alpha}(\partial \Omega) \times C^{1+\alpha}(\partial \Omega))$, we obtain

$$(\hat{u} - u_{2,t_0})\vert_{\partial \Omega \setminus \Gamma_0} = \frac{\partial(\hat{u} - u_{2,t_0})}{\partial \nu}\vert_{\partial \Omega \setminus \Gamma_0} = 0.$$  

Therefore $\hat{w} = \hat{u} - u_{2,t_0}$ satisfies

$$L_{q,f}(x, D)\hat{w} = 0 \text{ in } \Omega, \quad \hat{w}\vert_{\partial \Omega \setminus \Gamma_0} = \frac{\partial \hat{w}}{\partial \nu}\vert_{\partial \Omega \setminus \Gamma_0} = 0.$$  

By the classical uniqueness result for the Cauchy problem for the second-order elliptic equation (see e.g., Chapter XXVIII, §28.3 of [27], Corollary 2.9, Chapter XIV of [73]) we have $\hat{w} \equiv 0$. This contradicts (258).

We claim that

$$u_{1,t} \equiv u_{2,t}, \quad \forall t \in [0, 1]. \quad (259)$$

Our proof is by contradiction. Suppose that for some $t_0 \in (0, 1]$, this equality fails. Let $t_*$ be the infimum over such $t_0$ when $u_{1,t} \equiv u_{2,t}$ holds. Since $u_{1,0} = u_{2,0}$, such infimum exists.

Setting $u_t = u_{2,t} - u_{1,t}$, we have

$$\Delta u_t - q_0(t, x)u_t = -f_1(x, u_{1,t}) + f_2(x, u_{1,t}) \text{ in } \Omega, \quad u_t\vert_{\partial \Omega} = 0, \quad \frac{\partial u_t}{\partial \nu}\vert_{\partial \Omega \setminus \Gamma_0} = 0. \quad (260)$$
where \( g_0(t, x) = -q(x) + \int_0^1 \frac{\partial f_2}{\partial y}(x, (1-s)u_{2,t}(x) + su_{1,t}(x)) \, ds \).

Let \( \phi \) be a pseudoconvex function with respect to the principal symbol of the Laplace operator. Applying the Carleman estimate (17) with boundary term to equation (260), there exists \( \tau_0 \) such that:
\[
\sqrt{\tau} \| e^{\tau \phi} u_t \|_{H^1(\Omega)} \leq C \| e^{\tau \phi} (f_1(\cdot, u_{1,t}) - f_2(\cdot, u_{1,t})) \|_{L^2(\Omega)}, \quad \forall \tau \geq \tau_0.
\]
Fixing a large \( \tau > 0 \) arbitrarily, we have
\[
\| u_t \|_{H^1(\Omega)} \leq C \| f_1(\cdot, u_{1,t}) - f_2(\cdot, u_{1,t}) \|_{L^2(\Omega)}, \quad \forall t \in [0, 1],
\]
and by the elliptic estimate, we obtain
\[
\| u_t \|_{H^2(\Omega)} \leq C \| f_1(\cdot, u_{1,t}) - f_2(\cdot, u_{1,t}) \|_{L^2(\Omega)}, \quad \forall t \in [0, 1],
\]
where the constant \( C > 0 \) depends on fixed \( \tau \).

Consider the boundary value problem
\[
\Delta v_{k,t} + q(x)v_{k,t} - \frac{\partial f_k}{\partial y}(x, u_{k,t})v_{k,t} = \tilde{f}_k(x, v_{k,t}) = \Delta v_{k,t} + q(x)v_{k,t} - f_k(x, v_{k,t} + u_{k,t}) - f_k(x, u_{k,t}) = 0 \quad \text{in } \Omega, \quad v_{k,t}|_{\Gamma_0} = 0,
\]
where \( \tilde{f}_k(x, w) = f_k(x, w + u_{k,t}) - f_k(x, u_{k,t}) - \frac{\partial f_k}{\partial y}(x, u_{k,t})w \). Obviously the functions \( \tilde{f}_k \) satisfy (253), (254) and (255). Moreover
\[
\Lambda_q - \frac{\partial f_k}{\partial y}(x, u_{k,t}, \tilde{f}_1) = \Lambda_q - \frac{\partial f_k}{\partial y}(x, u_{k,t}, \tilde{f}_2).
\]

Indeed, consider the pair \((w_1, w_2)\) such that \( w_2 = \Lambda_q - \frac{\partial f_k}{\partial y}(x, u_{k,t}, \tilde{f}_1)(w_1) \). Let \( w \in H^1(\Omega) \) be the solution to the boundary value problem
\[
\Delta w + qw - \frac{\partial f_1}{\partial y}(x, u_{1,t})w - \tilde{f}_1(x, w) = 0 \quad \text{in } \Omega, \quad w|_{\Gamma_0} = 0, \quad w|_{\partial \Omega \setminus \Gamma_0} = w_1
\]
such that \( \frac{\partial w}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = w_2 \).

On the other hand, the function \( w + u_{1,t} \) solves the boundary value problem
\[
\Delta (w + u_{1,t}) + q(w + u_{1,t}) - f_1(x, w + u_{1,t}) = 0 \quad \text{in } \Omega, \quad (w + u_{1,t})|_{\Gamma_0} = 0.
\]
Let \( \bar{u} \) satisfy
\[
\Delta \bar{u} + q\bar{u} - f_2(x, \bar{u}) = 0 \quad \text{in } \Omega, \quad \bar{u}|_{\Gamma_0} = 0
\]
and
\[
\bar{u} = w + u_{1,t} \quad \text{on } \partial \Omega \setminus \Gamma_0.
\]
In general, a solution to problem (203) and (204) is not unique, but thanks to the assumption \( \Lambda_{q, f_1} = \Lambda_{q, f_2} \), we can assume that
\[
\frac{\partial \bar{u}}{\partial \nu} = \frac{\partial (w + u_{1,t})}{\partial \nu} \quad \text{on } \partial \Omega \setminus \Gamma_0.
\]
Setting \( \bar{w} = \bar{u} - u_{2,t} \), we obtain
\[
\Delta \bar{w} + q\bar{w} - \frac{\partial f_2}{\partial y}(x, u_{2,t})\bar{w} - \tilde{f}_2(x, \bar{w}) = 0 \quad \text{in } \Omega, \quad \bar{w}|_{\Gamma_0} = 0.
\]
Then on \( \partial \Omega \setminus \Gamma_0 \) we have
\[
\bar{w} - w = (\bar{u} - u_{2,t}) - (\bar{u} - u_{1,t}) = u_{1,t} - u_{2,t} = 0
\]
and
\[
\frac{\partial \bar{w}}{\partial \nu} - \frac{\partial w}{\partial \nu} = \frac{\partial \bar{u}}{\partial \nu} - \frac{\partial u_{2,t}}{\partial \nu} = \frac{\partial w}{\partial \nu} + \frac{\partial u_{1,t}}{\partial \nu} - \frac{\partial u_{2,t}}{\partial \nu} - \frac{\partial w}{\partial \nu} = 0.
\]
Therefore \( \bar{w} = w_1 \) and \( \frac{\partial \bar{w}}{\partial t} = w_2 \) on \( \partial \Omega \setminus \Gamma_0 \). Hence for the pair \((w_1, w_2)\) we have \( w_2 = \Lambda_s - \frac{\partial f_j}{\partial y}(x,u_2,t),\bar{f}_j(w_1) \).

We can similarly prove the reverse inclusion, that is, if \((w_1, w_2)\) is a pair such that \( w_2 = \Lambda_s - \frac{\partial f_j}{\partial y}(x,u_2,t),\bar{f}_j(w_1) \), then there exists a function \( w_+ \in H^1(\Omega) \) that solves the boundary value problem

\[
\Delta w_+ + qw_+ - \frac{\partial f_2}{\partial y}(x,u_2,t)w_+ - f_2(x,w_+) = 0 \quad \text{in } \Omega, \quad w_+|_{\Gamma_0} = 0, \quad w_+|_{\partial \Omega \setminus \Gamma_0} = w_1.
\]

such that \( \frac{\partial w_+}{\partial \nu}|_{\partial \Omega \setminus \Gamma_0} = w_2 \). Therefore we have proved (262).

Therefore we can apply Theorem 17 to this equation. Hence we have the uniqueness for the potential, that is,

\[
\frac{\partial f_1}{\partial y}(x,u_1,t) = \frac{\partial f_2}{\partial y}(x,u_2,t) \quad \text{in } \Omega, \quad \forall t \in [0,1].
\] (265)

Denote \( \Xi(t) = ||u_1,t - u_1,t_+||_{C^0[\Omega]} + ||u_2,t - u_2,t_+||_{C^0[\Omega]} \). Since \( u_1,t_+ = u_2,t_+ \) in \( \Omega \), we have \( f_1(x,u_1,t_+) = \Delta u_1,t_+ = \Delta u_2,t_+ = f_2(x,u_1,t_+) \) in \( \Omega \). Therefore

\[
f_1(x,u_1,t_+) - f_2(x,u_1,t_+) = \int_{u_1,t_+(x)}^{u_1,t_+(x)} \left( \frac{\partial f_1}{\partial y}(x,u_1,t_+(s,x)) - \frac{\partial f_2}{\partial y}(x,u_1,t_+(s,x)) \right) ds.
\]

If \( s \in (u_1,t_+(x),u_1,t_+(x)) \), then, by the continuity of \( u_1,t_+(x) \) with respect to \( t \) and the intermediate value theorem, there exists \( t_0(s,x) \in [0,t] \) such that \( s = u_1,t_0(s,x)(x) \). Hence

\[
f_1(x,u_1,t_+) - f_2(x,u_1,t_+) = \int_{u_1,t_+(x)}^{u_1,t_+(x)} \left( \frac{\partial f_1}{\partial y}(x,u_1,t_0(s,x)(x)) - \frac{\partial f_2}{\partial y}(x,u_1,t_0(s,x)(x)) \right) ds.
\]

Applying (265) and (266), we have

\[
f_1(x,u_1,t_+) - f_2(x,u_1,t_+) \leq \frac{\partial^2 f_2}{\partial y^2} \sup_{\bar{t} \in (t_0,t_+)} ||u_1,\bar{t} - u_2,\bar{t}||_{C^0[\Omega]} \Xi(t).
\] (266)

In order to obtain the last inequality, we used the fact that \( u_1,\bar{t} - u_2,\bar{t} \equiv 0 \) for all \( \bar{t} \) from \( \bar{t} \in [0,t_+] \). Therefore inequality (266) implies

\[
\sup_{\bar{t} \in (t_0,t_+)} ||f_1(x,u_1,\bar{t}) - f_2(x,u_1,\bar{t})||_{L^2(\Omega)} \leq C \Xi(t) \sup_{\bar{t} \in (t_0,t_+)} ||u_1,\bar{t} - u_2,\bar{t}||_{L^2(\Omega)}.
\] (267)

From (261) and (267), we obtain

\[
\|u_t\|_{H^2(\Omega)} \leq C \Xi(t) \sup_{\bar{t} \in (t_0,t_+)} ||u_1,\bar{t} - u_2,\bar{t}||_{L^2(\Omega)}, \quad \forall \bar{t} \in (t_0,t_+).
\]

This implies that

\[
\sup_{\bar{t} \in (t_0,t_+)} ||u_1,\bar{t}||_{H^2(\Omega)} \leq C \Xi(t) \sup_{\bar{t} \in (t_0,t_+)} ||u_1,\bar{t}||_{L^2(\Omega)}.
\] (268)

From (268) and the fact that \( \Xi(t) \) goes to zero as \( t \to t_+ \), we obtain that there exists \( \hat{\bar{t}} > t_+ \) such that \( u_1,\hat{\bar{t}} = u_2,\hat{\bar{t}} \) for all \( t \) from \( (t_0,\hat{\bar{t}}) \). We reach a contradiction. Equality (259) is proved and the statement of the theorem follows from it and (265). ■
6 Uniqueness by Dirichlet-to-Neumann maps for the Lamé equations and the Navier-Stokes equations

We discussed the uniqueness for inverse boundary value problems for systems for elliptic equations with the same principal parts in Section 4. In addition to such elliptic systems, there are other important elliptic systems in mathematical physics. In this section, we survey recent results for for the Lamé equations and the Navier-Stokes equations.

6.1 Three dimensional Lamé equations

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \partial \Omega \). Let \( \nu = (\nu_1, \nu_2, \nu_3) \) be the outward unit normal vector to \( \partial \Omega \).

Assume that

\[
\mu(x) > 0, \ (3\lambda + 2\mu)(x) > 0 \quad \text{on} \ \overline{\Omega}
\]

and set

\[
C_{ijkt} = \lambda(x)\delta_{ij}\delta_{kt} + \mu(x)(\delta_{ik}\delta_{jt} + \delta_{it}\delta_{jk}),
\]

for \( 1 \leq i, j, k, \ell \leq 3 \), where \( \delta_{ij} = 0 \) if \( i \neq j \) and \( \delta_{ii} = 1 \). We call functions \( \lambda \) and \( \mu \) the Lamé coefficients, \( u(x) = (u_1(x), u_2(x), u_3(x)) \) is the displacement. We set

\[
\mathcal{L}_{\lambda, \mu}(x, D)u = \left( \sum_{j,k,\ell=1}^{3} \frac{\partial}{\partial x_j} (C_{1jkt} \frac{\partial u_k}{\partial x_\ell}), \sum_{j,k,\ell=1}^{3} \frac{\partial}{\partial x_j} (C_{2jkt} \frac{\partial u_k}{\partial x_\ell}), \sum_{j,k,\ell=1}^{3} \frac{\partial}{\partial x_j} (C_{3jkt} \frac{\partial u_k}{\partial x_\ell}) \right).
\]

Let \( \Gamma_0 \) be an arbitrarily fixed subboundary. We define the Dirichlet-to-Neumann map \( \Lambda_{\lambda, \mu, \Gamma_0} \) on \( \Gamma_0 \) as follows.

\[
\Lambda_{\lambda, \mu, \Gamma_0} f = \left( \sum_{j,k,\ell=1}^{3} \nu_j C_{1jkt} \frac{\partial u_k}{\partial x_\ell}, \sum_{j,k,\ell=1}^{3} \nu_j C_{2jkt} \frac{\partial u_k}{\partial x_\ell}, \sum_{j,k,\ell=1}^{3} \nu_j C_{3jkt} \frac{\partial u_k}{\partial x_\ell} \right) \bigg|_{\partial \Omega \setminus \Gamma_0}
\]

where

\[
\mathcal{L}_{\lambda, \mu}(x, D)u = 0 \quad \text{in} \ \Omega, \quad u|_{\Gamma_0} = 0, \quad u|_{\partial \Omega \setminus \Gamma_0} = f.
\]

We are concerned with the uniqueness in determining \( \lambda, \mu \) by \( \Lambda_{\lambda, \mu, \Gamma_0} \).

Then we can prove

**Theorem 21** Let \( \Omega \in \mathbb{R}^3 \) be a bounded domain with smooth boundary and let us assume that

\[
\mu_1, \mu_2 \text{ are some positive constants}
\]

and that \( \lambda_1, \lambda_2 \in C^\infty(\overline{\Omega}) \) satisfy \( \lambda_1 = \lambda_2 \) on \( \Gamma_0 \). Then \( \Lambda_{\lambda_1, \mu_1, \Gamma_0} = \Lambda_{\lambda_2, \mu_2, \Gamma_0} \) implies that \( \lambda_1 = \lambda_2 \) and \( \mu_1 = \mu_2 \) in \( \Omega \).

For the proof, one refers to Imanuvilov, Uhlmann and Yamamoto [34].

We can similarly formulate the two dimensional case and Imanuvilov and Yamamoto [39] recently proved a result similar to Theorem 21.

**Theorem 22** Let \( \Omega \in \mathbb{R}^2 \) be a bounded domain with smooth boundary and let us assume that

\[
\mu_1, \mu_2 \text{ are some positive constants}
\]

and that \( \lambda_1, \lambda_2 \in C^4(\overline{\Omega}) \). Then \( \Lambda_{\lambda_1, \mu_1, \Gamma_0} = \Lambda_{\lambda_2, \mu_2, \Gamma_0} \) implies that \( \lambda_1 = \lambda_2 \) and \( \mu_1 = \mu_2 \) in \( \Omega \).

The result of Theorem 22 is stronger than Theorem 21 for the three dimensional case: no information on the trace of the Lamé coefficients \( \lambda_j \) is required on \( \Gamma_0 \) and only the finite-order regularity of the Lamé coefficients is assumed.
This inverse problem has been studied since the 90’s. Ikehata [28] discussed a linearized version of this inverse problem for the Dirichlet-to-Neumann map on the whole boundary (i.e., $\Gamma_0 = \emptyset$), and in two dimensions, Akamatsu, Nakamura and Steinberg [2] proved that the Dirichlet-to-Neumann map on the whole boundary can recover the Lamé coefficients and its normal derivatives of arbitrary orders on the boundary provided that the Lamé coefficients are $C^\infty$-functions. As for higher dimensional case, see Nakamura and Uhlmann [63]. In [61] Nakamura and Uhlmann proved that the Dirichlet-to-Neumann map on the whole boundary in two dimensions uniquely determines the Lamé coefficients, assuming that they are sufficiently close to a pair of positive constants.

In the three dimensional case, Eskin and Ralston [19] proved the following uniqueness by Dirichlet-to-Neumann map on the whole boundary:

**Theorem 23** Let $\lambda_1, \mu_1, \mu_2^{-1}, j = 1, 2$, be in a bounded set $B$ in $C^k(\Omega)$ with sufficiently large $k \in \mathbb{N}$. Then there exists $\epsilon(B) > 0$ such that $\Lambda_{\lambda_1,\mu_1,0} = \Lambda_{\lambda_2,\mu_2,0}$ implies $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$ in $\Omega$ provided that $\|\nabla \mu_j\|_{C^{k-1}(\Omega)} < \epsilon(B), j = 1, 2$.

See also [20]. The proof relies on construction of complex geometric optics solutions (e.g., Eskin [18]). The proof of Theorem 21 is based on [19]. Similar attempt has been done in Nakamura and Uhlmann [62]. We note that all the above works except for [39] needs the Dirichlet-to-Neumann map on the whole boundary.

### 6.2 Navier-Stokes equations

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. We define

$$P_\mu(u,p) \equiv \left( \sum_{j=1}^2 (-2\partial_j(\mu(x)\epsilon_{1j}(u)) + u_j \partial_j u_1 + \partial_1 p, \sum_{j=1}^2 (-2\partial_j(\mu(x)\epsilon_{2j}(u)) + u_j \partial_j u_2 + \partial_2 p) \right),$$

where $\epsilon_{ij}(u) = \frac{1}{2}(\partial_3 u_i + \partial_1 u_j), 1 \leq i, j \leq 2$, and we assume that

$$\mu \in C^4(\overline{\Omega}), \quad \mu > 0 \quad \text{on} \quad \overline{\Omega}.$$

We define the Dirichlet-to-Neumann map on the whole boundary by

$$\tilde{\Lambda}_\mu f = \frac{\partial u}{\partial \nu} \quad \text{on} \quad \partial \Omega$$

where $u \in H^2(\Omega)$ and $p \in H^1(\Omega)$ satisfy $P_\mu(u,p) = 0$, div $u = 0$ in $\Omega$ and $u|_{\partial \Omega} = f$.

Then we can prove the uniqueness in determining the viscosity by the Dirichlet-to-Neumann map.

**Theorem 24** We assume that $\partial_\alpha^x \mu_1 = \partial_\alpha^x \mu_2$ on $\partial \Omega$ for each multi-index $\alpha$ with $|\alpha| \leq 1$. If $\tilde{\Lambda}_{\mu_1} = \tilde{\Lambda}_{\mu_2}$, then $\mu_1 = \mu_2$ in $\Omega$.

The proof is given in a forthcoming paper. In the three dimensional case, the uniqueness is proved in Heck, Li and Wang [24] for the Stokes system and in Li and Wang [58] for the Navier-Stokes equations.

### 7 Appendix

Here we prove several technical propositions used in the previous sections. Let $G \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, $\phi \in C^\infty(\Omega)$ be some function, and $\lambda \in \mathbb{R}^1$ be a parameter. Consider the following integral

$$I(\lambda) = \int_G ge^{i\lambda \phi(x)} dx.$$

**Definition.** Let $A$ be a symmetric $n \times n$ square matrix, $A^{-1}$ exists and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of this matrix counted with the multiplicities. Then

$$\text{sgn } A = \text{[number of positive eigenvalues]} - \text{[number of negative eigenvalues]}.$$
Let a function \( \phi \) have a finite number of critical points on \( \Omega \). We denote these points as \( \tilde{x}_1, \ldots, \tilde{x}_\ell \). Assume that
\[
\det H_\phi(x) \neq 0 \quad \forall x \in \{x_1, \ldots, x_\ell\}.
\] (269)

The following is proposition proved in [7]:

**Proposition 15** Let (269) hold true. If \( g \in C_0^\infty(G) \), then
\[
I(\lambda) = \frac{2\pi}{\lambda} \sum_{j=1}^\ell g(\tilde{x}_j)e^{\lambda \phi(\tilde{x}_j)} + \frac{\pi}{\sqrt{\det H_\phi(\tilde{x}_j)}} + o\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to +\infty.
\] (270)

If a function \( \phi \) does not have critical points on \( \partial \Omega \) and \( g \in C^\infty(\overline{G}) \), then
\[
I(\lambda) = \frac{2\pi}{\lambda} \sum_{j=1}^\ell g(\tilde{x}_j)e^{\lambda \phi(\tilde{x}_j)} + \frac{1}{\lambda} \int_{\partial G} g \frac{\partial \phi}{\partial \nu} e^{\lambda \phi(\tilde{x})} d\sigma + o\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to +\infty.
\] (271)

Using Proposition 15 we prove the following asymptotic formula.

**Proposition 16** Let \( \Phi \) satisfy (84) and (85). For every \( g \in C_0^\infty(\Omega) \), we have
\[
\int_\Omega g e^{\tau(\Phi - \Psi)} dx = \int_\Omega g e^{\tau \psi(\tilde{x})} \frac{\tau \pi(\det H_\Psi(\tilde{x}))^{1/2}}{\tau \pi(\det H_\Psi(\tilde{x}))^{1/2}} + o(1) \tau \to +\infty.
\] (272)

**Proof.** Since the function \( \Phi \) is holomorphic, the real part \( \phi \) and the imaginary part \( \psi \) of \( \Phi \) satisfy the Cauchy-Riemann equations:
\[
\frac{\partial \phi}{\partial x_1} = -\frac{\partial \psi}{\partial x_2} \quad \text{and} \quad \frac{\partial \phi}{\partial x_2} = -\frac{\partial \psi}{\partial x_1}.
\]

Hence \( \frac{\partial^2 \phi}{\partial x_1^2} = -\frac{\partial^2 \phi}{\partial x_2^2} \) and the Hessian matrix has the form
\[
H_\phi = \begin{pmatrix}
\frac{\partial^2 \phi}{\partial x_1^2} & \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \\
\frac{\partial^2 \phi}{\partial x_1 \partial x_2} & \frac{\partial^2 \phi}{\partial x_2^2}
\end{pmatrix}
\]
and \( \det H_\phi = -\left(\frac{\partial^2 \phi}{\partial x_1^2}\right)^2 - \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right)^2 \). Since all the critical points of the function \( \phi \) are nondegenerating, we have
\[
\det H_\phi(x) < 0
\]
if \( \tilde{x} \) is a critical point of the function \( \phi \). Then the eigenvalues of the matrix \( H_\phi \) are \( \pm \sqrt{-\det H_\phi} \). Hence \( \text{sgn} H_\phi = 0 \). Hence, applying formula (271) with \( \lambda = 2\tau \), we obtain (272). ■

**Proposition 17** Let \( \Phi \) satisfy (84) and (85). For every \( g \in L^1(\Omega) \), we have
\[
\int_\Omega g e^{\tau(\Phi - \Psi)} dx \to 0 \quad \text{as} \quad \tau \to +\infty.
\]

**Proof.** The space \( C_0^\infty(\Omega) \) is dense in \( L^1(\Omega) \), and so for any \( \epsilon > 0 \) there exists a function \( g_\epsilon \in C_0^\infty(\Omega) \) such that
\[
\|g - g_\epsilon\|_{L^1(\Omega)} \leq \epsilon/2.
\]
On the other hand by Proposition 16 we have
\[
\int_\Omega (g - g_\epsilon)e^{\tau(\Phi - \Psi)} dx \to 0 \quad \text{as} \quad \tau \to +\infty.
\]
Then for any positive \( \epsilon \) there exists \( \tau_\epsilon \) such that
\[
\left|\int_\Omega g e^{\tau(\Phi - \Psi)} dx\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\]

The proof of the proposition is complete. ■

We have
Proposition 18 Let $g \in C^1(\partial \Omega \setminus \Gamma_0^*)$ and a holomorphic function $\Phi$ satisfy (274)–(276). Then

$$\int_{\partial \Omega \setminus \Gamma_0^*} ge^{\tau(\Phi - \Phi)} dx = 0 \quad \text{as} \quad \tau \to +\infty.$$  \hfill (273)

**Proof.** Without loss of generality, using the partition of unity if necessary, we can assume that $\partial \Omega \setminus \Gamma_0^*$ is a segment $[c, d]$. Moreover, since $C^\infty_0(c, d)$ is dense in $L^1(c, d)$, we can assume that $g \in C^\infty_0(c, d)$. Since the function $\Phi$ belongs to $C^2(\Omega)$, the set $\mathcal{J}$ is closed on $\partial \Omega \setminus \Gamma_0^*$. For any positive $\epsilon$, consider the set $\mathcal{J}_\epsilon = \{x \in \partial \Omega \setminus \Gamma_0^* | \text{dist}(x, \mathcal{J}) \leq \epsilon\}$. Observe that

$$\lim_{\epsilon \to 0} \text{mes}(\mathcal{J}_\epsilon) = 0.$$  

Then

$$\int_{\mathcal{J}_\epsilon} ge^{\tau(\Phi - \Phi)} dx \leq \|g\|_{C^0(\partial \Omega)} \text{mes}(\mathcal{J}_\epsilon).$$

Let $\epsilon$ be sufficiently small. Consider the set $(\partial \Omega \setminus \Gamma_0^*) \setminus \mathcal{J}_{2\epsilon}$. This set is the union of non-intersecting open intervals where the distance between any two intervals is greater than or equal to $2\epsilon$. Consider an arbitrary interval $(a, b) \subset (\partial \Omega \setminus \Gamma_0^*) \setminus \mathcal{J}_{2\epsilon}$ such that

$$a \in \mathcal{J}_{2\epsilon} \cup \{x_+\} \quad \text{and} \quad b \in \mathcal{J}_{2\epsilon} \cup \{x_-\}. \quad (274)$$

Consider a function $e_{a, b} \in C_0^\infty(a - \epsilon/4, b + \epsilon/4)$ such that

$$0 \leq e_{a, b}(x) \leq 1 \quad \forall x \in (a - \epsilon/4, b + \epsilon/4), \quad e_{a, b}(a, b) = 1, \quad |e'_{a, b}| \leq K(\epsilon),$$

where $K$ is independent of $a, b$. Then we construct a function $g_\epsilon$ in the following way: for any interval $(a, b)$ which satisfies (274), we set $g_\epsilon = e_{a, b}g$. The function $g_\epsilon$ has the following properties

$$|g_\epsilon(x)| \leq |g(x)| \quad \forall x \in \partial \Omega \setminus \Gamma_0^*, \quad g = g_\epsilon \quad \text{on} \quad (\partial \Omega \setminus \Gamma_0^*) \setminus \mathcal{J}_{2\epsilon} \quad (275)$$

and

$$g_\epsilon \in C^1((\partial \Omega \setminus \Gamma_0^*) \setminus \mathcal{J}_{2\epsilon}), \quad \text{supp} g_\epsilon \subset (\partial \Omega \setminus \Gamma_0^*) \setminus \mathcal{J}_{2\epsilon}. \quad (276)$$

By (274) and (276), we have

$$\int_{\partial \Omega \setminus \Gamma_0^*} ge^{\tau(\Phi - \Phi)} dx = \int_{(\partial \Omega \setminus \Gamma_0^*) \setminus \mathcal{J}_{2\epsilon}} ge^{\tau(\Phi - \Phi)} dx + \int_{\mathcal{J}_{2\epsilon}} ge^{\tau(\Phi - \Phi)} dx \quad = \int_{(\partial \Omega \setminus \Gamma_0^*) \setminus \mathcal{J}_{2\epsilon}} ge^{\tau(\Phi - \Phi)} dx + \int_{\mathcal{J}_{2\epsilon}} e^{\tau(\Phi - \Phi)} dx \quad (277)$$

By (275) we have

$$\left| \int_{\mathcal{J}_{2\epsilon} \setminus \mathcal{J}_{4\epsilon}} ge^{\tau(\Phi - \Phi)} dx + \int_{\mathcal{J}_{4\epsilon}} ge^{\tau(\Phi - \Phi)} dx \right| \leq \int_{\mathcal{J}_{2\epsilon} \setminus \mathcal{J}_{4\epsilon}} |g| dx + \int_{\mathcal{J}_{4\epsilon}} |g| dx \leq 2 \int_{\mathcal{J}_{2\epsilon}} |g| dx \leq 2 \|g\|_{C^0(\partial \Omega)} \text{mes}(\mathcal{J}_{2\epsilon}). \quad (278)$$

Observe that by (276) we see

$$\partial \rho \left( \frac{g_\epsilon \rho \partial \rho \rho}{2\rho |\partial \rho \rho|^2} \right) \in L^1(\partial \Omega \setminus \Gamma_0^*). \quad (279)$$
Now we estimate the first term on the right-hand side of (277):

\[
\int_{(\partial \Omega \setminus \Gamma_0)} g e^{r(\phi - \overline{\phi})} \frac{\partial \bar{\varphi}}{\partial z} dx = \int_{\partial \Omega \setminus \Gamma_0} g e^{r(\phi - \overline{\phi})} \frac{\partial \bar{\varphi}}{2\tau |\bar{\varphi}|^2} dx
\]

By (279) and Proposition 17, we have

\[
\int_{(\partial \Omega \setminus \Gamma_0)} g e^{r(\phi - \overline{\phi})} dx \to 0 \quad \text{as} \quad \tau \to +\infty.
\]

From (281) and (278) we obtain (273).

**Proposition 19** There exists a holomorphic function \( w_0 \in C^{6+\alpha}(\overline{\Omega}) \) such that

\[
\lim_{x \to y} \frac{|w_0(x)|}{|x - y|^{10}} = 0, \quad \forall y \in \mathcal{H} \cap \Gamma_0, \quad w_0(\bar{x}) \neq 0.
\]

**Proof.** Let us fix a point \( \bar{x} \) from \( \mathcal{H} \). In order to prove this proposition, it suffices to construct some holomorphic function \( a(z) \in C^7(\overline{\Omega}) \) which is not identically equal to any constant, satisfies \( Im a|_{\Gamma_0} = 0 \) and vanishes at each point of the set \( \mathcal{H} \cap \Gamma_0 \). Then we set \( w_0 = a^{100} \) and this is the desired function.

Let \( b(z) \) be a holomorphic function in \( \Omega \) such that \( Re b|_{\Gamma_0} = 0 \), \( \|b - N\|_{C^0(\Gamma_0)} \ll 1 \) and \( |b(\bar{x}) - 1| \leq 14 \). Such a function exists for any positive \( N \) by Proposition 5.1 of [30]. If \( b(\bar{x} + 1) \neq 0 \), then we consider the new function \( b_1 = b - b^3/b^2(\bar{x} + 1) \). Obviously

\[
Re b_1|_{\Gamma_0} = 0, \quad b_1(\bar{x} + 1) = 0
\]

and \( b_1 \) is not identically equal to some constant. If \( b_1(\bar{x} + 1) = 0 \), then we set \( b_1 = b \).

If \( b_1(\bar{x} + 2) \neq 0 \), then we consider the new function \( b_2 = b_1 - b^4_1/b^2_1(\bar{x} + 2) \). Obviously

\[
Re b_2|_{\Gamma_0} = 0, \quad b_2(\bar{x} + 1) = b_2(\bar{x} + 2) = 0
\]

and \( b_2 \) is not identically equal to any constant. If \( b_2(\bar{x} + 2) = 0 \), then we set \( b_2 = b_1 \). Repeating this procedure \( \ell' - 2 \) times and as the result, we obtain the holomorphic function \( a \) with the prescribed properties, provided that \( N \) is sufficiently large.

**Proposition 20** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with the smooth boundary, \( \mathcal{V} \in C^0(\partial \Omega) \) satisfy

\[
\int_{\partial \Omega} \mathcal{V} P(\nu_1 - i\nu_2) d\sigma = 0, \quad \forall P(\overline{\mathcal{V}}) \in H^\frac{1}{2}(\Omega).
\]

Then there exist an antiholomorphic function \( \Theta \in H^\frac{1}{2}(\Omega) \) such that \( \Theta|_{\partial \Omega} = \mathcal{V} \).

**Proof.** Consider the extremal problem:

\[
J(\bar{\Psi}) = \|\mathcal{V} - \bar{\Psi}\|_{L^2(\partial \Omega)}^2 \to \inf,
\]

\[
\frac{\partial \bar{\Psi}}{\partial z} = 0 \quad \text{in} \ \Omega.
\]

Denote the unique solution to this extremal problem (283), (284) by \( \bar{\Psi} \). Applying Lagrange’s principle, we obtain

\[
Re(\mathcal{V} - \bar{\Psi})|_{L^2(\partial \Omega)} = 0
\]
for any \( \tilde{\delta} \) from \( H^\frac{1}{2}(\Omega) \) such that
\[
\frac{\partial \tilde{\delta}}{\partial z} = 0 \quad \text{in } \Omega
\]
and there exists a function \( \tilde{P} \in H^\frac{1}{2}(\Omega) \) such that
\[
\frac{\partial \tilde{P}}{\partial z} = 0 \quad \text{in } \Omega,
\]
(286)
\[
(\nu_1 - i \nu_2)\tilde{P} = \mathcal{V} - \hat{\Psi} \quad \text{on } \partial \Omega.
\]
(287)
From (285), taking \( \tilde{\delta} = \hat{\Psi} \), we have
\[
\text{Re}(\mathcal{V} - \hat{\Psi}, \hat{\Psi})_{L^2(\partial \Omega)} = 0.
\]
(288)
By (286), (287) and the assumption of the proposition, we obtain
\[
\text{Re}(\mathcal{V} - \hat{\Psi}, \mathcal{V})_{L^2(\partial \Omega)} = \text{Re}((\nu_1 + i \nu_2)\tilde{P}, \mathcal{V})_{L^2(\partial \Omega)} = \text{Re}(\tilde{P}, (\nu_1 - i \nu_2)\mathcal{V})_{L^2(\partial \Omega)} = 0.
\]
By (196) and (288) we see that
\[
J(\hat{\Psi}) = 0.
\]
The proof of the proposition is complete. ■

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