The modified Lindstedt–Poincare method for solving quadratic nonlinear oscillators

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Abstract
Recently, an analytical solution of a quadratic nonlinear oscillator has been presented based on the harmonic balance method. By introducing a small parameter, a set of nonlinear algebraic equations have been solved which usually appear among unknown coefficients of several harmonic terms. But the method is not suitable for all quadratic oscillators. Earlier, introducing a small parameter to the frequency series, Cheung et al. modified the Lindstedt–Poincare method and used it to solve strong nonlinear oscillators including a quadratic oscillator. But due to some limitations of both parameters, a changed form of frequency-related parameter (introduced by Cheung et al.) has been presented for solving various quadratic oscillators.

Keywords
Modified Lindstedt–Poincare method, nonlinear oscillator, quadratic nonlinear oscillator

Introduction
Classical perturbation techniques are widely used tools for solving weak nonlinear problems in which the solutions are expanded in powers of small parameters. However, generalization or modifications of some classical perturbation techniques overcome this limitation. In particular, modifications of the Lindstedt–Poincare (LP) method, He’s homotopy perturbation method, iterative methods, the Hamiltonian approach, and linearization method are also widely used tools to solve strongly nonlinear oscillators. Cheung et al. introduced a small parameter in the frequency series and solved some strong nonlinear oscillators which are the main concern of this article.

The harmonic balance method (HBM) is another powerful tool for solving strongly nonlinear oscillators. Usually a set of nonlinear algebraic equations appear among unknown coefficients of several harmonic terms. In Hosen et al. and Yeasmin et al., such nonlinear algebraic equations are solved introducing another small parameter (different from that of Cheung et al.). In Yeasmin et al., a simple solution has been found of the quadratic oscillator

\[ \ddot{x} + x + \varepsilon x^2 = 0, \quad [x(0) = a_0, \dot{x}(0) = 0] \]  

Earlier, Hu also used HBM to solve equation (1) considering two separate solutions respectively for \( x > 0 \) and \( x < 0 \) (see also Hosen et al.). Hu’s technique can be used to solve another quadratic oscillator

\[ \ddot{x} + x + \varepsilon x^2 = 0 \]
but it requires higher approximation which increases the algebraic complexity between two solutions for \( x > 0 \) and \( x < 0 \). Although the solution is continuous when \( x = 0 \), left- and right-handed values of \( \dot{x} \) are different at \( x = 0 \) for any order of approximation. However, the difference gradually decreases as the order of approximation is increased. Moreover, a third approximation obtained in Hu\textsuperscript{14} for oscillator equation (1) does not provide good results, especially when \( \varepsilon a_0 \) is close to \( \frac{1}{2} \). So, the present method is also important for equation (1). It is noted that the method developed in Yeasmin et al.\textsuperscript{26} is not suitable for equation (2) since a small parameter related to coefficients of harmonic terms\textsuperscript{25,26} does not exist for this oscillator, but both types of parameter exist for equation (1) as well as for many nonlinear oscillators. Sometimes a changed form of proposed parameter in Cheung et al.\textsuperscript{6} helps to calculate frequency more accurately.

In this article, a single form solution of equation (2) has been determined. First, an eighth approximate solution was found by using the classical LP method and then was transformed to its modified form by utilization of some conversion formulae presented in He.\textsuperscript{7} Besides these, it has been shown that a little change of used parameter for equation (1) (presented in Cheung et al.\textsuperscript{6}) provides significantly correct frequency and solution.

**The methods**

**The LP method**

The LP perturbation method was originally developed to solve a weak nonlinear oscillator

\[
\ddot{x} + \omega_0^2 x = \varepsilon f(x)
\]  

(3)

where over dots denote differentiation with respect to \( t \), \( \omega_0 \) is a constant, \( f \) is a nonlinear function, and \( \varepsilon \) is a small parameter. By a variable transformation, \( \tau = \omega t \), equation (3) can be rewritten as

\[
\omega^2 \ddot{x} + \omega_0^2 x = \varepsilon f(x)
\]  

(4)

where primes denote differentiation with respect to \( \tau \). According to the LP method, \( x \) and \( \omega^2 \) are expanded in powers of \( \varepsilon \) as

\[
x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots
\]  

(5)

and

\[
\omega^2 = \omega_0^2 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots
\]  

(6)

Now substituting these series of \( x \) and \( \omega^2 \) into equation (4) and equating the coefficients of \( \varepsilon^0, \varepsilon^1, \varepsilon^2, \ldots \), we obtain the following linear equations for \( x_0, x_1, x_2, \ldots \)

\[
\omega_0^2 x_0'' + \omega_0^4 x_0 = 0
\]  

(7a)

\[
\omega_0^2 x_1'' + \omega_0^4 x_1 = -\omega_1 x_0'' + f(x_0)
\]  

(7b)

\[
\omega_0^2 x_2'' + \omega_0^4 x_2 = -\omega_2 x_0'' - \omega_1 x_1'' + f_2(x_0), \ldots
\]  

(7c)

The initial conditions \([x(0) = a_0, x'(0) = 0]\) are replaced by \([x_0(0) = a_0, x_0'(0) = 0]\) and \(x_1(0) = x_1'(0) = x_2(0) = x_2'(0) = \cdots = 0\). The above set of equations are solved such that \(x_1, x_2, \ldots \) exclude secular terms. This assumption helps to determine unknown coefficients \( \omega_1, \omega_2, \ldots \).

**The modified LP method**

Cheung et al.\textsuperscript{6} considered equation (6) in another form as

\[
\omega^2 = (\omega_0^2 + \varepsilon \omega_1) \left( 1 + \frac{\varepsilon^2 \omega_2}{\omega_0^2 + \varepsilon \omega_1} + \frac{\varepsilon^3 \omega_3}{\omega_0^2 + \varepsilon \omega_1} + \cdots \right)
\]  

(8)
Then a parameter, \( x(e) \) was chosen as

\[
x = \frac{\varepsilon \omega_1}{\omega_0^2 + \varepsilon \omega_1} \tag{9}
\]

Therefore \( x < 1 \) for every value of \( e \) when \( \omega_1 > 0 \) From equation (9), it is found that

\[
e = \frac{\omega_0^2 x}{(1 - x) \omega_1}, \quad \omega_0^2 + \varepsilon \omega_1 = \frac{\omega_0^2}{1 - x} \tag{10}
\]

By elimination of \( e \) from equation (8), it readily becomes

\[
\omega^2 = \frac{\omega_0^2}{1 - x}(1 + \delta_2 x^2 + \delta_3 x^3 + \cdots) \tag{11}
\]

and with the help of equations (10) and (11), equation (4) takes the form

\[
(1 + \delta_2 x^2 + \delta_3 x^3 + \cdots)\ddot{x} + (1 - x)x = x f(x)/\omega_1 \tag{12}
\]

The solution of equation (12) can be chosen in a series of \( x \) as

\[
x = x_0 + x_1 + x_2 x_2 + \cdots \tag{13}
\]

Substituting equation (13) into equation (12) and equating the coefficients of \( x^0, x^1, x^2, \ldots \), the following linear equations are obtained

\[
\ddot{x}_0 + x_0 = 0 \tag{14}
\]

\[
\ddot{x}_1 + x_1 = x_0 + \frac{f(x_0)}{\omega_1} \tag{15}
\]

\[
\ddot{x}_2 + x_2 = -\omega_1 - \delta_2 x_0'' + \dot{x}_1 + \frac{f(x_0)}{\omega_1}, \ldots \tag{16}
\]

The initial conditions are similar to those of the LP method. Cheung et al.\(^6\) utilized these equations to find an approximate solution. However, both \( x \)-series of \( x \) and \( \omega^2 \) are found easily from classical solutions using the conversion formulae presented in He.\(^7\)

**Conversion formulae**

In He,\(^7\) the following set of conversion formulae was presented which transform the classical (LP) solution together with frequency into its modified version presented in Cheung et al.\(^6\)

\[
\begin{align*}
\delta_2 &= \frac{\omega_2}{\omega_1^2}, & \delta_3 &= \frac{\omega_1 \omega_2 + \omega_3}{\omega_1^3}, & \delta_4 &= \frac{\omega_1^2 \omega_2 + 2 \omega_1 \omega_3 + \omega_4}{\omega_1^4}, \ldots \\
\end{align*}
\]

and

\[
\begin{align*}
\tilde{x}_1 &= \frac{x_1}{\omega_1^2}, & \tilde{x}_2 &= \frac{\omega_1 x_1 + x_2}{\omega_1^3}, & \tilde{x}_3 &= \frac{\omega_1^2 x_1 + 2 \omega_1 x_2 + x_3}{\omega_1^4}, \ldots \\
\end{align*}
\]

where symbols \( x_1, x_2, \ldots ; \omega_1, \omega_2, \ldots \) are used in the classical solution and \( \tilde{x}_1, \tilde{x}_2, \ldots ; \delta_2, \delta_3, \ldots \) are used in the modified solution.
Example

According to the transformation, \( \tau = \omega t, \ddot{x} + x + \varepsilon x^2 = 0 \), \([x(0) = a_0, \dot{x}(0) = 0]\) becomes

\[
\omega^2 x'' + x + \varepsilon \omega^2 x^2 = 0, \quad [x(0) = a_0, \dot{x}(0) = 0]
\]  

or

\[
x'' + \omega^{-2} x + \varepsilon x^2 = 0, \quad [x(0) = a_0, \dot{x}(0) = 0]
\]  

Let us consider:

\[
\omega^{-2} = 1 + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \cdots
\]  

Now substituting equations (5) and (21) into equation (20) and equating the coefficients of \( \sigma_0, \sigma_1, \sigma_2, \ldots \) we obtain the following equations for \( x_0, x_1, x_2, \ldots \)

\[
x_0'' + x_0 = 0 \tag{22a}
\]

\[
x_1'' + x_1 = -\sigma_1'' x_0 - \sigma_1' x_0^2 \tag{22b}
\]

\[
x_2'' + x_2 = -\sigma_1'' x_1 - \sigma_2'' x_0 - 2\sigma_2' x_1' \cdots \tag{22c}
\]

The initial conditions \([x(0) = a_0, \dot{x}(0) = 0]\) are replaced by \([x_0(0) = a_0, x_0'(0) = 0]\) and \(x_2(0) = x_4(0) = x_1'(0) = x_2'(0) = \cdots = 0\). Herein \(x_1(0), x_3(0), \ldots\) are considered nonzero. It is noted that \(x_1, x_3, \ldots\) contain only even harmonic terms as their particular solutions and the appearance of odd harmonic terms can be restricted by considering complementary functions zero (see also Cheung et al.\(^6\)). This assumption leads to \(\sigma_1' = \sigma_3' = \cdots = \sigma_{2j+1}' = 0\). The above set of equations is solved such that \(x_2, x_4, \ldots\) exclude the secular terms. The later assumption helps to determine the unknown coefficients \(\sigma_2, \sigma_4', \ldots\).

The solution of equation (22a) can be chosen as

\[
x_0 = a \cos \sigma \tag{23}
\]

Substituting the value of \(x_0\) from equation (23) into equation (22b) and simplifying, we obtain

\[
x_1'' + x_1 = -\sigma_1' a \cos \sigma - \frac{a^2(1 - \cos 2\sigma)}{2} \tag{24}
\]

Since it is restricted that \(x_1\) contains only even harmonic terms, we consider \(\sigma_1' = 0\) and the solution of equation (24) becomes

\[
x_1 = -\frac{a^2(3 + \cos 2\sigma)}{6} \tag{25}
\]

Now substituting the values of \(x_0\) and \(x_1\) into equation (22c) and simplifying, we obtain

\[
x_2'' + x_2 = -\sigma_2' a \cos \sigma + \frac{a^2(\cos \sigma - \cos 3\sigma)}{3} \tag{26}
\]

Since \(x_2\) does not contain a secular term and \(x_2(0) = 0\), we obtain

\[
\sigma_2' = \frac{a^2}{3}, \quad x_2 = \frac{a^2(-\cos \sigma + \cos 3\sigma)}{24} \tag{27}
\]
In a similar way, we have determined the following results

\[ x_3 = \frac{a^4(165 + 40\cos 2\tau - 13\cos 4\tau)}{1080} \]

\[ \omega'_4 = -\frac{7a^4}{216}, \quad x_4 = \frac{a^6(299\cos \tau - 396\cos 3\tau + 97\cos 5\tau)}{25920} \]

\[ x_5 = \frac{a^6(341040 + 53795\cos 2\tau - 32816\cos 4\tau + 6621\cos 6\tau)}{5443200} \]

Substituting the values of \( \omega'_2, \omega'_4, \omega'_6, \omega'_8 \) from equation (27) and equation (28) into equation (21), we obtain

\[ \omega^{-2} = 1 + \frac{e^2a^2}{3} + \frac{7e^4a^4}{216} + \frac{29e^6a^6}{9720} + \frac{16813e^8a^8}{65318400} - \cdots \]  

According to Cheung et al.,\(^6\) equation (29) can be transformed into a \( z \)-series where \( z \) is defined as

\[ z^2 = \frac{e^2a^2}{1 + \frac{2e^2}{3}a^2} \]

or

\[ e^2 = \frac{3z^2}{a^2(1 - z^2)} \]

By elimination of \( e \), equation (29) is transformed into the following \( z \)-series

\[ \omega^{-2} = \frac{1}{1 - z^2} \left( 1 - \frac{7z^4}{24} - \frac{19z^6}{90} - \frac{29489z^8}{268800} - \cdots \right) \]

This formula is unable to measure the desired value of \( \omega \), since the coefficients of the \( z \)-series are not small enough. According to Cheung et al.,\(^6\) an alternative formula can be found for \( \omega^{-4} \). Squaring both sides of equation (29), it becomes

\[ \omega^{-4} = 1 + \frac{2e^2a^2}{3} + \frac{5e^4a^4}{108} - \frac{19e^6a^6}{1215} + \frac{1433e^8a^8}{403200} - \cdots \]

This series is transformed into the \( z \)-series

\[ \omega^{-4} = \frac{1}{1 - z^2} \left( 1 + \frac{5z^4}{48} + \frac{37z^6}{720} + \frac{107113z^8}{6451200} + \cdots \right) \]

where

\[ z^2 = \frac{2e^2a^2}{3} + \frac{2e^2}{3}a^2 \]
The latter formula is better than the former, but it fails especially when the value \(a_0\) is close to 0.5. Comparing the coefficients of the \(x\)-series of \(\omega^{-2}\) and \(\omega^{-4}\), we expect that the said coefficients of \(\omega^{-2p}\), \(1 < p < 2\) would be much smaller. From equation (29), we obtain

\[
\omega^{-2p} = 1 + \frac{p e^2 a^2}{3} - \frac{p(129 - 19)a^4}{216} + \frac{p(60p^2 - 285p + 254)e^6 a^6}{9720} + \cdots
\]

(36)

Thus, it can be transformed into an \(x\)-series

\[
\omega^{-2p} = \frac{1}{1 - a^2} \left(1 + \frac{(12p - 19)a^4}{24p} + \frac{(120p^2 - 285p + 127)x^4}{180p^2} + \frac{(705600p^3 - 2234400p^2 + \cdots)x^8}{806400p^3} + \cdots\right)
\]

(37)

where

\[
x^2 = \frac{pe^2 a^2}{1 + pe^2 a^4/3}
\]

(38)

For \(p = \frac{3}{2}\), equation (37) becomes

\[
\omega^{-3} = \frac{1}{1 - a^2} \left(1 - \frac{x^4}{36} + \frac{61x^6}{810} - \frac{225877x^8}{2721600} + \cdots\right)
\]

(39)

where

\[
x^2 = \frac{e^2 a^4}{2 + e^2 a^4/3}
\]

(40)

At the end, we shall discuss a simple determination of \(a\). When \(\tau = \pi\), the value of \(x\) (say \(-b_0\)) becomes \(-b_0 = x(\pi)\).

It is noted that the determination of \(b_0\) is discussed in detail in Fay\(^27\) and its value is obtained from the following equation

\[
e^{2\omega_0} \left(\frac{a_0}{\bar{c}} - \frac{1}{2e^2}\right) = e^{-2\omega_0} \left(-\frac{b_0}{\bar{c}} - \frac{1}{2e^2}\right)
\]

(41)

Thus, the value of \(a\) can be calculated by

\[
a = \frac{(a_0 + b_0)}{2}
\]

**Solution obtained by Hu’s method**

According to Hu’s method,\(^14\) when \(x > 0\) then equation (2) can be written as

\[
\ddot{x} + x + \epsilon x^3 = 0, \quad x(0) = a_0, \quad \dot{x}(0) = 0
\]

(42)

and when \(x < 0\), then equation (2) can be written as

\[
\ddot{y} + y - \omega^2 = 0, \quad y(T/2) = b_0, \quad \dot{y}(T/2) = 0
\]

(43)

where \(a_0\) and \(b_0\) follow equation (41).

A third approximate solution of equation (42) can be chosen as

\[
x = a_0((1 - u - v)\cos\omega_4 t + uc\omega_4 t + v\cos5\omega_4 t)
\]

(44)
Let us consider that the Fourier series of $x^2$ is

$$b_1 \cos \omega_A t + b_3 \cos 3\omega_A t + b_5 \cos 5\omega_A t + \cdots$$  \hspace{1cm} (45)

and the unknown coefficients are calculated by

$$b_1 = \frac{4a_0(33(35 + 8u(7 + 46u)) + 264(-15 + 68u)v + 45680v^2)}{3465\pi}$$

$$b_3 = \frac{4a_0(-89232u^2 + 1976u(33 + 92v) - 11(1911 - 14872v + 28336v^2))}{45045\pi}$$

$$b_5 = \frac{4a_0(65520u^2 + 8u(-15873 + 3748v) + 13(759 + 232v + 4784v^2))}{45045\pi}$$

Substituting equation (44) and (45) into equation (42), and then equating the coefficients of the same harmonic, the following equations are obtained

$$1 - u - v + \omega_A^2 \left(-1 + u + v + \frac{4a_0}{3\pi} + \frac{32a_0u}{15\pi} + \frac{1472a_0u^2}{105\pi} - \frac{32a_0v}{7\pi} + \cdots \right) = 0$$  \hspace{1cm} (46a)

$$u + \omega_A^2 \left(-9u - \frac{28a_0}{15\pi} + \frac{608a_0u}{105\pi} - \frac{832a_0u^2}{105\pi} + \frac{4576a_0v}{315\pi} + \cdots \right) = 0$$  \hspace{1cm} (46b)

$$v + \omega_A^2 \left(-25v + \frac{92a_0}{105\pi} - \frac{1184a_0u}{105\pi} + \frac{64a_0u^2}{11\pi} + \frac{92a_0v}{3465\pi} + \cdots \right) = 0$$  \hspace{1cm} (46c)

Solving equations (46a) to (46c), the value of $u$, $v$ and $\omega_A$ is obtained.

Again, the third approximate solution of equation (43) is chosen as follows

$$y = b_0((1 - p - q) \cos \omega_B t + pcos3\omega_B t + qcos5\omega_B t)$$  \hspace{1cm} (47)

Substituting equation (47) into equation (43) and considering equation (45), then equating the coefficients of the same harmonic, we have

$$1 - p - q + \omega_B^2 \left(-1 + p + q - \frac{4b_0}{3\pi} - \frac{32b_0p}{15\pi} - \frac{1472b_0p^2}{105\pi} + \frac{32b_0q}{7\pi} - \cdots \right) = 0$$  \hspace{1cm} (48a)

$$p + \omega_B^2 \left(-9p + \frac{28b_0}{15\pi} - \frac{608b_0p}{105\pi} + \frac{832b_0p^2}{105\pi} - \frac{4576b_0q}{315\pi} + \cdots \right) = 0$$  \hspace{1cm} (48b)

$$q + \omega_B^2 \left(-25q - \frac{92b_0}{105\pi} + \frac{1184b_0p}{105\pi} - \frac{64b_0p^2}{11\pi} - \frac{92b_0q}{3465\pi} + \cdots \right) = 0$$  \hspace{1cm} (48c)

Solving equations (48a) to (c), the values of $p$, $q$ and $\omega_B$ are obtained. Both sets are nonlinear algebraic equations. First $\omega_A^2$ or $\omega_B^2$ are eliminated from three equations and the relations between two unknown coefficients are obtained by two nonlinear algebraic equations. The determination of the solutions of those equations is not a simple approach (see Wu et al.\textsuperscript{23} for details).

**Results and discussion**

In this section, the method has been compared with both classical and modified forms of the LP method. Moreover, the results obtained by the proposed method have been compared with the results obtained by a
Table 1. Comparison of the period obtained by the proposed method, numerical method, and method of Hu for \( \varepsilon = 1 \).

| \( a_0 \)  | Numerical | Present | % Error | Hu | % Error |
|-------------|-----------|---------|---------|----|---------|
| 0.45        | 7.0314    | 7.03923 | 0.108408| 7.027986 | 0.048557 |
| 0.46        | 7.1401    | 7.141282| 0.016548| 7.141136 | 0.014509 |
| 0.47        | 7.2845    | 7.285666| 0.016010| 7.292533 | 0.110278 |
| 0.48        | 7.4895    | 7.493363| 0.051581| 7.515459 | 0.346605 |
| 0.49        | 7.8455    | 7.854615| 0.116184| 7.8272   | 0.233253 |
| 0.495       | 8.198     | 8.218419| 0.249071| 8.169037 | 0.353291 |

The numerical method to verify the validity of the proposed method. In Table 1, a comparison of period obtained by the proposed method, numerical method, and the method of Hu is presented for several values of \( a_0 \) and \( \varepsilon = 1 \) (where period \( T = \frac{2\pi}{a} \) and percentage error = \( \frac{\text{numerical result} - \text{approximate result}}{\text{numerical result}} \times 100 \)). From both tables, it is seen that the period obtained by the proposed method agrees reasonably well with those obtained by the numerical method and provides better results than that obtained by the method of Hu.

In Figure 1(a) to (f), a comparison among the results obtained by both the proposed and numerical methods is presented graphically. From all the figures, it can be observed that the results from the proposed method show good agreement with those obtained by the numerical method. Besides, both solutions have been compared in phase plane (see Figure 2(a) to (f)). Again from Figure 2(a) to (f), it is also seen that the proposed solutions agree nicely with the corresponding numerical solutions.

In Figure 1(g) and (h), the results obtained by the proposed method are compared with the corresponding results obtained by the numerical method and the method of Hu. From the figure, it is observed that the result obtained by the proposed method shows better agreement with the numerical result than that is obtained by the method of Hu. Also, both solutions have been compared in phase plane (see Figure 2(g) and (h)). Again from Figure 2(g) and (h), it is also seen that the proposed solutions agree nicely with the corresponding numerical solutions and provide better results than those obtained by the method of Hu.

For quadratic oscillator equation (1), the classical solution has been derived choosing classical equation (6) and obtained the following results

\[
\begin{align*}
\omega_1 & = \omega_3 = \cdots = 0, \\
\omega_2 & = -\frac{5a^2}{6}, \\
\omega_4 & = -\frac{305a^6}{864}, \\
\omega_6 & = -\frac{4715a^8}{15552}, \\
\omega_8 & = -\frac{654985a^{10}}{1990656}, \ldots
\end{align*}
\]

and

\[
\omega^2 p = \frac{1}{1 - x^2} \left( 1 + \frac{(60p - 121)x^4}{120p} + \frac{(-600p^2 + 1815p - 1229)x^6}{900p^2} + \frac{(504000p^3 - \cdots \cdots)x^8}{576000p^3} + \cdots \right)
\]

where \( x^2 = \frac{\frac{e^2 a^2}{6}}{1 - \frac{e^2 a^2}{6}} \).

It is obvious that the first coefficient of \( x \) is much smaller than 1 when \( p = 2 \). Substituting this value of \( p \), equation (49) becomes

\[
\omega^4 = \frac{1}{1 - x^2} \left( 1 - \frac{x^4}{240} + \frac{x^6}{3600} - \frac{19x^8}{512000} + \frac{19x^{10}}{3840000} - \cdots \right), \quad x^2 = \frac{\frac{e^2 a^2}{6}}{1 - \frac{e^2 a^2}{6}}.
\]

The coefficients of \( x \) of equation (50) are not of the same sign. Therefore, the results of \( \omega \) oscillate by a little amount as the order of approximation is increased. This problem has also been found in the quintic oscillator, \( \ddot{x} + x + \epsilon x^4 = 0 \). However, the results gradually approach toward its exact value if a truncation rule is followed. When \( p = 2.001706 \cdots \) the third term of equation (49) vanishes and if the next terms are truncated, it becomes

\[
\omega^4 \approx 0.003412 = \frac{1}{1 - x^2} \left( 1 - 0.00373687 \cdots x^4 \right)
\]
Figure 1. (a): Comparison between the time versus displacement results obtained by the proposed method and a numerical method when $a_0 = 0.45$, $\varepsilon = 1$. (b): Comparison between the time versus displacement results obtained by the proposed method and a numerical method when $a_0 = 0.46$, $\varepsilon = 1$. (c): Comparison between the time versus displacement results obtained by the proposed method and a numerical method when $a_0 = 0.47$, $\varepsilon = 1$. (d): Comparison between the time versus displacement results obtained by the proposed method and a numerical method when $a_0 = 0.48$, $\varepsilon = 1$. (e): Comparison between the time versus displacement results obtained by the proposed method and a numerical method when $a_0 = 0.49$, $\varepsilon = 1$. (f): Comparison between the time versus displacement results obtained by the proposed method and a numerical method when $a_0 = 0.495$, $\varepsilon = 1$. (g): Comparison between the time versus displacement results obtained by the proposed method, a numerical method, and the method of Hu when $a_0 = 0.49$, $\varepsilon = 1$. (h): Comparison between the time versus displacement results obtained by the proposed method, a numerical method, and the method of Hu when $a_0 = 0.495$, $\varepsilon = 1$. 
Figure 2. (a): Comparison between the phase plane obtained by the proposed method and numerical method when $a_0 = 0.45$, $\varepsilon = 1$. (b): Comparison between the phase plane obtained by the proposed method and numerical method when $a_0 = 0.46$, $\varepsilon = 1$. (c): Comparison between the phase plane obtained by the proposed method and numerical method when $a_0 = 0.47$, $\varepsilon = 1$. (d): Comparison between the phase plane obtained by the proposed method and numerical method when $a_0 = 0.48$, $\varepsilon = 1$. (e): Comparison between the phase plane obtained by the proposed method and numerical method when $a_0 = 0.49$, $\varepsilon = 1$. (f): Comparison between the phase plane obtained by the proposed method and numerical method when $a_0 = 0.495$, $\varepsilon = 1$. (g): Comparison between the phase plane obtained by the proposed method, numerical method and method of Hu, when $a_0 = 0.49$, $\varepsilon = 1$. (h): Comparison between the phase plane obtained by the proposed method, numerical method, and method of Hu, when $a_0 = 0.495$, $\varepsilon = 1$. 

By equation (53): $\varepsilon = 1$. 

For both equations (50) to (53), the results are valid when $a_0 = 0.495$ which is very close to its singular point (i.e. $a = 0$). Some results measured by equations (50) to (53) have been presented in Table 2.
When \( p = 2.000049891 \ldots \) the fifth term vanishes and if the next terms are truncated, it becomes

\[
\omega^4 = \frac{1}{1 - a^2} (1 - 0.00415 \cdot a^2 + 0.000099782 \cdot a^3 - 0.000004786 \cdot a^4) \tag{52}
\]

Similarly, the next formula becomes

\[
\omega^4 = \frac{1}{1 - a^2} (1 - 0.004166 \cdot a^2 + 0.00027751 \cdot a^3 - 0.0000000791 \cdot a^6) \tag{53}
\]

The last formula (i.e. equation (53)) is very effective in evaluating frequency \( \omega \) until \( \varepsilon a_0 < 0.495 \). Some results have been presented in Table 2 measured by equations (50) to (53).

### Conclusion

A modified LP method developed by Cheung et al. has been used to solve quadratic oscillators. But a direct attempt using Cheung’s technique is not always fruitful. They used a so-called small parameter \( \varepsilon (a_0, \varepsilon) \), but it is much greater than 1 for equation (1) (see equation (50)). A changed form of this parameter has been presented to solve equation (2). A simple change of its used form (in Cheung et al.\(^6\)) significantly improves the results. For both quadratic oscillators, the results are valid when \( \varepsilon a_0 = 0.495 \) which is very close to its singular point (i.e. \( \varepsilon a_0 = 0.5 \)).

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