Duality of averaging of quantum states over arbitrary symmetry groups revealing Schur–Weyl duality

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Abstract

It is a well-established fact in quantum information theory, that uniform averaging over the collective action of a unitary group on a multipartite quantum state projects the state to a form equivalent to a permutation operator of the subsystems. Hence states equivalent to permutation operators are untouched by collective unitary noise. A trivial observation shows that uniform averaging over permutation operators projects the state into a form with block-diagonal structure equivalent to the one of the collective action of the unitary group. We introduce a name for this property: duality of averaging. The mathematical reason behind this duality is the fact that the collective action of the unitary group on the tensor product state space of a multipartite quantum system and the action of the permutation operations are mutual commutants when treated as matrix algebras. Such pairs of matrix algebras are known as dual reductive pairs. In this work we show, that in the case of finite dimensional quantum systems such duality of averaging holds for any pairs of symmetry groups being dual reductive pairs, regardless of whether they are compact or not, as long as the averaging operation is defined via iterated integral over the Cartan decomposition of the group action. Although our result is very general, we focus much attention on the concrete example of a dual reductive pair.
consisting of collective action of special linear matrices and permutation operations, which physically corresponds to averaging multipartite quantum states over non-unitary SLOCC-type (Stochastic Local Operations and Classical Communication) operations. In this context we show, that noiseless subsystems known from collective unitary averaging persist in the case of SLOCC averaging in a conditional way: upon postselection to specific invariant subspaces.

Keywords: Schur–Weyl duality, stochastic local operations and classical communication, averaging of quantum states, noiseless subsystems

1. Introduction

1.1. Schur–Weyl duality

Schur–Weyl duality [1–7] and the concept of noiseless subsystems [8–14] are two faces of the same coin. Schur–Weyl duality introduces pairs of matrix groups (called dual reductive pairs), which, treated as matrix algebras, maximally mutually commute, therefore quantum states spanned by operators from one of the elements of the dual pair are untouched by the quantum evolution encoded by operators from the second element of the pair [5]. There is however one important assumption in the above story: both groups have to possess finite-dimensional unitary representations. Indeed, if $U$ is a subgroup of the unitary group $U(d)$, treated here as a matrix algebra, and $S$ is a commutant of $U$, then any quantum state $\rho$, which is in the span of $S$, by definition (and linearity) commutes with arbitrary element $V \in U$, and therefore is untouched by the action of $V$:

$$V\rho V^\dagger = \rho.$$  \hspace{1cm} (1)

The canonical example of a dual reductive pair contains the following two matrix algebras:

- algebra spanned by the collective action $U^\otimes t$ of a unitary group $U(d)$ on a complex tensor space $(\mathbb{C}^d)^\otimes t$,
- algebra generated by unitary representation of permutation operators on the tensor space $(\mathbb{C}^d)^\otimes t$.

In standard quantum information processing terms, first algebra represents collective unitary noise on a system of $t d$-level quantum systems (qudits), whereas the second represents swapping of subsystems. The moral of the story is that quantum states which are equivalent to permutation operators are untouched by the collective unitary noise.

In the literature one can find other examples of dual reductive pairs, in which one of the terms is represented by collective action of some subgroup of the unitary group, whereas the second is some discrete or continuous group containing the symmetric group as a subgroup. The examples are:

- Schur–Weyl duality for collective action of the Clifford unitary operators [7],
- Schur–Weyl duality for gauge groups [5], which are defined as centralisers of some subgroup of a unitary group with respect to the unitary group.

All of the above examples share the same interpretation in the context of noiseless subsystems.
1.2. Duality of averaging

The idea of averaging quantum states over symmetry operations appears in numerous applications within the entire field of quantum information science. It is a crucial tool in analysing noisy quantum communication, both in the non-relativistic \([12–14]\) as well as relativistic context \([15–17]\). Averaging over symmetry operations plays also an important role in developing randomised quantum information processing protocols \([18, 19]\) and quantum algorithms \([20]\). Schur–Weyl duality for unitary groups and their subgroups has a crucial impact on the idea of collective uniform averaging of quantum states with respect to these groups (called twirling in the quantum information context \([12, 13]\)). Namely such a collective averaging projects a quantum state into the subspace spanned by the commutant (second counterpart of the dual reductive pair) \([12, 13]\). In more details this aspect of Schur–Weyl duality is related with the group-representation-theoretic property of a dual reductive pair, namely that irreducible representations of each elements of a pair are in one-to-one correspondence, and that their joint action has a unique block-diagonal decomposition induced by this correspondence. For example uniform collective unitary averaging of a multipartite finite-dimensional quantum state projects the state into the block-diagonal form consisting of projections of the state onto subspaces, which are irreducible with respect to the action of the permutation group. An obvious, though not already found by the authors anywhere in the literature, consequence of the Schur–Weyl duality for uniform averaging, is that the same holds if one averages quantum state uniformly over the second factor in the dual reductive pair, namely the state is projected onto irreducible subspaces with respect to the first factor. We call this property duality of averaging with respect to dual reductive pair. As a simple example, uniform averaging of a multipartite quantum state over symmetric group projects the state into a block-diagonal form consisting of projections of the state into subspaces irreducible with respect to the collective action of the unitary group.

The open problem, which we state in this work, is the question, whether described duality of averaging holds if at least one component of the reductive pair is a non-compact symmetry group. An emblematic example of such a pair is the collective action of a special linear group \(SL(d, \mathbb{C})\), which physically represents collective SLOCC-type operations (stochastic local operations and classical communication) \([21–29]\), and the symmetric group \([1, 30]\). On the one hand there seem to appear two crucial obstacles. The first one is that due to non-unitarity of the SLOCC operations, the quantum state, which commutes with the SLOCC operation still seems to be seriously affected by this transformation. Indeed, if \(L \in SL(d, \mathbb{C})\) and \([\rho, L] = 0\) we have:

\[
L_\rho L^\dagger = \rho LL^\dagger \neq \rho.
\]

In general the operator \(LL^\dagger\) is not proportional to identity, therefore the problem is more serious than just a normalisation issue. The second problem is that averaging over SLOCC operations, due to non-compact character of the transformations, cannot be performed in a uniform way, and a suppressing integration measure has to be introduced in order to assure convergent result. On the other hand, as shown in our previous work \([31]\), an important aspect of unitary averaging, namely existence of finite averaging sets (known in the quantum information community as unitary \(t\)-designs \([18, 32–34]\)) persists in the process of averaging over non-compact \(SL(d, \mathbb{C})\) group.
1.3. Main results

In this work we show, that the duality of averaging indeed persists in the context of averaging over SLOCC operations with a slight modification, namely such a process of averaging projects a quantum state into block-diagonal form consisting of projections onto subspaces irreducible with respect to the symmetric group, but with additional non-trivial weights, which correspond to the process of averaging over the non-compact part of the SLOCC operation. These weights explicitly depend on the assumed measure of integration over the non-compact part. Therefore we can say, that globally there are no noiseless subsystems under the collective action of the SLOCC operations, however we can conditionally restore them by restricting to a single irreducible subspace and performing projection to this subspace, which succeeds with probability specified by the additional weight factors.

Further on we prove a general statement that the duality of averaging in the defined sense persists for any dual reductive pair regardless of the compactness of the symmetry group under consideration, as long as one defines the generalised twirling operation using the Cartan decomposition of the group operation.

1.4. Outline

The paper is organised as follows. In section 2 we introduce the concept of duality of averaging for the well-known unitary twirling (with all the technical details moved to the appendices), in section 3 we show how one can define and understand averaging quantum states over SLOCC operations, in section 4 we derive the closed analytical form of the SLOCC-twirling map. Finally we prove the general result on duality of averaging with respect to any dual reductive pair in section 5 and present final remarks in section 6.

2. Schur–Weyl duality and duality of averaging over collective unitary operations

Schur–Weyl duality for the unitary group $U(d)$ and the symmetric group $S_t$ over the set of $t$ elements describes an interplay between the collective action of the unitary group and the symmetric group on the tensor product space $(\mathbb{C}^d)^{\otimes t}$. The collective action of the unitary group is specified by operators $U \otimes U^{\otimes (t-1)}$, whereas the action of the symmetric group $S_t$ is defined as a permutation of the factors in $(\mathbb{C}^d)^{\otimes t}$. Let

$$\ket{\psi} = \ket{v_1} \otimes \cdots \otimes \ket{v_t}$$

be arbitrary product vector in $(\mathbb{C}^d)^{\otimes t}$. Then any permutation $p \in S_t$ acts on $\ket{\psi}$ as follows:

$$p(\ket{\psi}) = \ket{v_{p^{-1}(1)}} \otimes \cdots \otimes \ket{v_{p^{-1}(t)}}.$$  

For further considerations it is necessary to describe the above action by a matrix transformation. Therefore we introduce the following matrix representation of the symmetric group on the complex vector space $(\mathbb{C}^d)^{\otimes t}$ known as tensor permutation operators $[35, 36]$. To every permutation $p \in S_t$, we associate an orthogonal matrix $O_p \in O(d^t)$ as follows. Let $\tau(p) = (\tau_1, \tau_2) = (i_1, i_2, i_3, \ldots, i_{t-1}, i_t)$ be a (in general non-unique) decomposition of $p$ into transpositions. Then the matrix $O_p$ reads:

$$O_p = O_{\tau_1} \cdots O_{\tau_{t-1}},$$

in which the orthogonal transposition matrices are defined as $[35, 36]$:  


\[ O_{(i_k, i_{k+1})} = \frac{1}{d} \mathbb{I}_d + \frac{1}{2} \sum_{j=1}^{d-1} \mathbb{I}_j \otimes \ldots \otimes \gamma_j \otimes \ldots \otimes \gamma_j \otimes \ldots \otimes \mathbb{I}, \tag{5} \]
in which \( \gamma_j \) are generalised Gell–Mann matrices (Hermitian generators of \( u(d) \) algebra) placed at positions \( i_k \) and \( i_{k+1} \) in the tensor product. Since \( \tau(p)\tau(q) = \tau(pq) \), we have due to (4) \( O_p O_q = O_{pq} \), and therefore the mapping \( p \mapsto O_p \) is indeed a group representation.

The actions on the space \( (\mathbb{C}^d)^{\otimes t} \) of the permutation group and of the unitary group mutually commute in a maximal sense, namely the matrix algebras generated by \( \{U^{\otimes t}\}_{U \in U(d)} \) and \( \{O_{\tau}\}_{\tau \in S_t} \) are mutual commutants in the algebra of endomorphisms of \( (\mathbb{C}^d)^{\otimes t} \). This implies, that there exists a basis of \( (\mathbb{C}^d)^{\otimes t} \), called Schur basis, which via outer product of vectors generates two operator bases block-diagonalising both matrix algebras \( \{U^{\otimes t}\}_{U \in U(d)} \) and \( \{O_{\tau}\}_{\tau \in S_t} \).

In order to distinguish between the vector basis and the two operator bases, we will refer to the basis on \( (\mathbb{C}^d)^{\otimes t} \) as Schur vector basis, whereas the two operator bases for operators acting on this space will be referred to as Schur operator bases.

Let us first introduce the structure of the Schur vector basis, the elements of which will be enumerated by three indices: \( |i, m, \lambda\rangle \), in which \( i \) numbers irreducible representations of unitary and symmetric groups up to isomorphism, whereas the remaining indices \( m \) and \( \lambda \) describe further structure of each \( i \)th subspace, which will be introduced later on. From representation-theoretic point of view this means that the joint action of both groups is reducible on \( (\mathbb{C}^d)^{\otimes t} \) and that moreover it decomposes into irreducible actions of both groups, which are in one-to-one correspondence. The detailed construction of the Schur vector basis is described in the appendix A.1, here we just define the crucial tools needed for further considerations.

The inequivalent irreducible representations of both groups: the unitary group \( U(d) \) and the symmetric group \( S_t \), on the tensor space \( (\mathbb{C}^d)^{\otimes t} \) are labelled by all inequivalent Young diagrams with \( t \) boxes and at most \( d \) rows, which correspond to all inequivalent partitions of \( t \) into the sum of \( d \) positive integers [37]. Let us label these representations with the index \( i \). As a side remark we point out that there exists no closed analytic formula for the range of index \( i \), namely for the number of inequivalent partitions with fixed number of terms [38]. Therefore we would not specify the range of the index \( i \). Each \( i \)th subspace of \( (\mathbb{C}^d)^{\otimes t} \) irreducible with respect to the joint action of \( U^{\otimes t} \) and \( O_\tau \) contains in general multiple subspaces, which are irreducible with respect to the separate action of symmetric or collective unitary operators. These subspaces correspond to equivalent but distinct irreducible representations of the corresponding groups, and therefore each \( i \)th subspace can be further organised in the following way:

\[
L_1^i : |i, 1, 1\rangle \quad |i, 1, 2\rangle \quad \ldots \quad |i, D_i, 1\rangle \quad |i, D_i, 2\rangle \quad \ldots \quad |i, D_i, D_i\rangle ,
\]

In the above table each vector of the form \( |i, m, \lambda\rangle \) represents one element of the Schur basis.

Subspaces \( L_1^i \), spanned by vectors \( \{ |i, m, \lambda\rangle \}_{\lambda=1}^{D_i} \) occurring in each row, are invariant and irreducible under the collective action of the unitary group \( U^{\otimes t} \), and correspond to equivalent irreducible representations of the unitary group \( U(d) \) of dimension \( D_i \) (explicit formulae for the dimensions can be found in the appendix). Similarly the subspaces \( L_2^i \), spanned by vectors \( \{ |i, m, \lambda\rangle \}_{m=1}^{D_i} \) occurring in each column are invariant and irreducible under the action of the symmetric group \( S_t \), and correspond to equivalent \( D_i \) dimensional irreducible representations of the symmetric group (they are called Specht moduli in the representation theory [30]). It can
be seen that the subspaces \( L_i^\lambda \) and \( V_m^i \) play the role of multiplicity spaces for the corresponding representations: subspaces \( V_m^i \) are multiplicity spaces for representations of the unitary group, whereas subspaces \( L_i^\lambda \) are multiplicity spaces for representations of the symmetric group [13]. Direct sum:

\[
P_i \equiv \bigoplus_\lambda L_i^\lambda = \bigoplus_m V_m^i
\]

contains all subrepresentations of \( U^{\otimes 3} \) and \( S_3 \) up to the same isomorphism type.

Before going further let us illustrate the described Schur vector basis decomposition by an example of the tensor space \((\mathbb{C}^2)^{\otimes 3}\), which physically represents a state space of three qubits (two-level systems). Under the joint action of \( U(2)^{\otimes 3} \) and \( S_3 \) the space \((\mathbb{C}^2)^{\otimes 3}\) decomposes into two irreducible subspaces corresponding to inequivalent irreducible representations of the unitary group \( U(2) \) and \( S_3 \):

- \( i = 1 \) subspace, corresponding to a Young diagram with three boxes places in one row: 4-dimensional subspace of symmetric tensors, spanned by Schur basis vectors:

\[
L_1^1 : |1, 1, 1\rangle \quad |1, 2, 1\rangle \quad |1, 3, 1\rangle \quad |1, 4, 1\rangle,
\]

on which there acts a one-dimensional trivial representation of the symmetric group \( S_3 \) (hence \( D^1_1 = 1 \)), and 4-dimensional representation of the \( U(2) \) group, namely the spin-\( \frac{1}{2} \) representation (hence \( D^1_1 = 4 \)). Note that the entire \( i = 1 \) subspace is irreducibly invariant under the action of \( U^{\otimes 3} \), whereas each one-dimensional subspace corresponding to each element of Schur vector basis is invariant under symmetric group \( S_3 \) (since each element of Schur vector basis is here fully symmetric when expressed in computational basis).

- \( i = 2 \) subspace, corresponding to a Young diagram with two boxes in first row and one box in the second: 4-dimensional subspace of mixed-symmetry tensors, spanned by Schur basis vectors:

\[
L_2^1 : |2, 1, 1\rangle \quad |2, 2, 1\rangle
\]

\[
L_2^2 : |2, 1, 2\rangle \quad |2, 2, 2\rangle,
\]

on which there act two equivalent irreducible two-dimensional representations of the unitary group \( U(2) \) (spin-\( \frac{1}{2} \) representations) with the corresponding invariant subspaces denoted by \( L_1^1 \) and \( L_2^2 \), and two equivalent two-dimensional representations of the symmetric group \( S_3 \) with the corresponding invariant subspaces \( V_1^1 \) and \( V_2^2 \).

In most common approach to Schur–Weyl duality one treats the Schur basis vectors \(|i, m, \lambda\rangle\) as virtual tensor products \(|m)_i \otimes |\lambda_i\rangle\), which correspond to tensor products of vectors from virtual subspaces \( \{|m_i\}_{m_i=1}^{D_i^1} \) and \( \{|\lambda_i\}_{\lambda_i=1}^{D_i^0} \) [8, 11, 13]. Here we introduce other approach, based on operator bases build up from outer products of Schur vector basis elements. These operator bases are extremely useful when dealing with averaging maps. Let us start from the most general operator basis build up of Schur basis vectors:

\[
\hat{T}^m_{\lambda_1, \lambda_2} = |i, m_1, \lambda_1\rangle \langle j, m_2, \lambda_2|.
\]
The above operators allow us to define two \textit{Schur operator bases}:

\[
\hat{\Pi}_{\lambda_1,\lambda_2} = \sum_{m=1}^{D_r} \hat{\Pi}_{m_{\lambda_1,\lambda_2}},
\]

\[
\hat{\Pi}_{m_1,m_2} = \sum_{\lambda=1}^{D_r} \hat{\Pi}_{m_{\lambda_1,\lambda_2}}.
\]

The set \(\{\hat{\Pi}_{\lambda_1,\lambda_2}\}\) is a basis for operators, which leave invariant the subspaces \(V_{\lambda}\) irreducible under the action of the symmetric group, whereas the set \(\{\hat{\Pi}_{m_1,m_2}\}\) spans operators, which leave invariant the subspaces \(L_{\lambda}\) irreducible under the collective action of the unitary group. Therefore the collective action of the unitary group \(U^\otimes_t\) as well as the orthogonal representation of the symmetric group \(O^\otimes\) have natural decompositions in terms of these operator bases:

\[
U^\otimes_t = \sum_{i=1}^{D_r} \frac{1}{D_r} \sum_{m_1,m_2=1}^{D_r} \text{Tr} \left( U^\otimes_t \hat{\Pi}_{m_1,m_2} \right) \hat{\Pi}_{m_1,m_2},
\]

and analogously:

\[
O^\otimes = \sum_{i=1}^{D_r} \frac{1}{D_r} \sum_{\lambda_1,\lambda_2=1}^{D_r} \text{Tr} \left( O^\otimes \hat{\Pi}_{\lambda_1,\lambda_2} \right) \hat{\Pi}_{\lambda_1,\lambda_2}.
\]

The fact that matrix algebras generated by two kinds of operators (12) and (13) are mutual commutants is reflected by the commutativity of the above defined operator bases:

\[
[\hat{\Pi}_{m_1,m_2},\hat{\Pi}_{\lambda_1,\lambda_2}] = 0.
\]

Let us now define two averaging operations with respect to both elements of the dual reductive pair:

\[
\mathcal{T}_U(\rho) = \int U^\otimes_t \rho U^\otimes_t^\dagger dU, \quad U \in U(d),
\]

\[
\mathcal{T}_{\text{sym}}(\rho) = \frac{1}{n!} \sum_{p \in S_n} O_p \rho O_p^T.
\]

The first operation, known as \textit{unitary twirling} [13], represents averaging a quantum state \(\rho\) of \(n\) \(d\)-level subsystems over the collective action of the unitary group in a uniform way: \(dU\) represents a normalised Haar measure on the unitary group. Such operation can have different physical interpretations, the two most common are the following:

- multipartite quantum system under consideration is subject to collective local unitary noise [5, 39],
- multipartite quantum system is sent between two observers that do not share a common reference frame [12, 13].

The second operation, which we call \textit{symmetric twirling} is non-local with respect to subsystems and has possible applications in the theory of randomised quantum protocols [40].
The fact that both twirling operations (15) and (16) represent averaging with respect to symmetry groups that belong to a dual reductive pair implies the following duality of averaging:

\[
T_U(\rho) = \sum_k \frac{1}{d_k} \sum_{\lambda_1, \lambda_2} \frac{\omega}{d_k} \operatorname{Tr} \left( \rho \Pi_k^{\lambda_1 \lambda_2} \right) \Pi_k^{\lambda_1 \lambda_2},
\]

(17)

\[
T_{\text{sym}}(\rho) = \sum_k \frac{1}{d_k} \sum_{m_1, m_2} \frac{\omega}{d_k} \operatorname{Tr} \left( \rho \Pi_k^{m_1 m_2} \right) \Pi_k^{m_1 m_2}.
\]

(18)

We can see that closed analytical forms of both maps are in the same relation as operators (13) and (12):

- unitary twirling \(T_U(\rho)\) projects a quantum state into block-diagonal form (with respect to block index \(k\)) consisting of subspaces which are invariant and irreducible under the action of the symmetric group;
- symmetric twirling \(T_{\text{sym}}(\rho)\) projects a quantum state into block-diagonal form consisting of subspaces which are invariant and irreducible under the collective action of the unitary group.

Both analytical formulas (17) and (18) are simple consequences of Schur’s lemma (see appendix A.2). Their explicit proofs are however a bit technical (though standard), we present them for completeness in appendices A.3 and A.4. Both twirling operations (17) and (18) are idempotent: \(T_U(T_U(\rho)) = T_U(\rho), T_{\text{sym}}(T_{\text{sym}}(\rho)) = T_{\text{sym}}(\rho)\) (see appendices A.3 and A.4), which implies that states of the form:

\[
\rho_U = \sum_k \frac{1}{d_k} \sum_{\lambda_1, \lambda_2} \frac{\omega}{d_k} \Pi_k^{\lambda_1 \lambda_2},
\]

\[
\rho_S = \sum_k \frac{1}{d_k} \sum_{m_1, m_2} \frac{\omega}{d_k} \Pi_k^{m_1 m_2},
\]

(19)

span \textit{noiseless subsystems} with respect to the twirling operations:

\[
T_U(\rho_U) = \rho_U,
\]

\[
T_{\text{sym}}(\rho_S) = \rho_S.
\]

(20)

3. Averaging multipartite quantum states over SLOCC operations

In this section we introduce the concept of averaging multipartite quantum states of a finite dimension \(d\) over the set of the most general quantum operations, namely SLOCC [21–29]. Mathematically, these operations are described by the following map [25]:

\[
S(\rho) = \bigotimes_i L_i \rho \bigotimes_i L_i^\dagger,
\]

(21)

where the matrices \(L_i\) are normalised special linear \(\text{SL}(d, \mathbb{C})\) matrices, namely:

\[
L = \frac{M}{||M||}, M \in \text{SL}(d, \mathbb{C}).
\]

(22)
Lemma 1. The map \( S \) preserves the positivity of the matrix \( \rho \), and is trace non-increasing.

Proof. The fact that \( S \) preserves positivity is a simple consequence of the tensor product commuting with Hermitian transpose. The proof that \( S \) is trace non-increasing requires some more computation and is presented for completeness in the appendix B.1.

Both the above properties stated in lemma 1 assure that the map (21) is a well-defined quantum operation. In general the trace of the matrix \( S(\rho) \) is strictly less than one. Such a state can be physically interpreted as a properly normalised quantum state \( S(\rho)/\text{Tr}(S(\rho)) \), which is prepared with success probability \( \text{Tr}(S(\rho)) \). The acceptance of a preparation of this state demands a classical communication between the local observers performing local operations \( L_i \), hence the name of the class.

Let us now introduce the concept of collective averaging over SLOCC operations, which will be done in analogy to the averaging over unitary operations. The term collective refers to the fact that all the local operations are equal, therefore we consider the maps:

\[
S_C(\rho) = L^{\otimes d} \rho L^{\otimes d}. \tag{23}
\]

In the case of unitary operations there is a unique way of defining collective averaging, see formula (15). Generalisation of the unitary twirling operation to the SLOCC case meets two problems. The first one is that the operation \( S_C(\rho) \) does not preserve the trace, the second one is that the group \( \text{SL}(d, \mathbb{C}) \) is non compact, and therefore it is not possible to average over it in a uniform way. Let us first focus on the second problem, and further, after defining our averaging process we will tackle the problem with normalisation. Our aim is to define the averaging process over SLOCC operations in a way which preserves as much uniformity as possible. Averaging by integrating over Haar measure on \( \text{SL}(d, \mathbb{C}) \) group is not possible, since this measure is not finite and the integral would be divergent. What is more, the group \( \text{SL}(d, \mathbb{C}) \) is not an amenable group [41], which means that it is not possible to define any functional on bounded functions defined on this group, which would be: normalised, non-negative definite and left or right invariant with respect to the group action. Therefore any averaging procedure for functions defined on the group \( \text{SL}(d, \mathbb{C}) \) must necessarily involve some sort of non-uniformity. In order to minimize it we should try to decompose the group action in order to separate the non-compact part. This idea is realised by the Cartan decomposition for the group \( \text{SL}(d, \mathbb{C}) \), which is in this case equivalent to a singular value decomposition (SVD) on the level of matrices. In simple terms this means that an arbitrary matrix \( M \in \text{SL}(d, \mathbb{C}) \) can be represented as a product \( M = KAK' \), in which matrices \( K, K' \in \text{SU}(d) \), whereas \( A \) is some diagonal traceless real matrix [42, 43], which represents some abelian group \( \mathbb{A} \). The special unitary group \( \text{SU}(d) \) plays the role of a maximal compact subgroup of \( \text{SL}(d, \mathbb{C}) \), whereas an abelian group \( \mathbb{A} \) is its maximal abelian subgroup. However it is important to note that the direct product group \( \text{SU}(d) \times \mathbb{A} \times \text{SU}(d) \) is not homomorphic to \( \text{SL}(d, \mathbb{C}) \). What is more, the mapping:

\[
\text{SU}(d) \times \mathbb{A} \times \text{SU}(d) \mapsto \text{SL}(d, \mathbb{C}), \tag{24}
\]

realised by \( (K, A, K') \mapsto M \) is not a diffeomorphism between the group manifolds \( \text{SU}(d) \times \mathbb{A} \times \text{SU}(d) \) and \( \text{SL}(d, \mathbb{C}) \) due to a mismatch of dimensions. Namely the group manifold corresponding to the direct product \( \text{SU}(d) \times \mathbb{A} \times \text{SU}(d) \) has higher dimension that the manifold corresponding to \( \text{SL}(d, \mathbb{C}) \). For example in the case of \( \text{SL}(2, \mathbb{C}) \), \( \text{SU}(2) \) is parametrised by 3 real parameters, whereas \( \mathbb{A} \) is one-dimensional and can be represented by matrices of the form:

\[
\mathbb{A}_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} , \ x \geq 1 \right\}. \tag{25}
\]
Therefore the product manifold $SU(2) \times \mathbb{A}_2 \times SU(2)$ has real dimension equal to 7, whereas the group manifold of $SL(2, \mathbb{C})$ is 6-dimensional. Hence on the one hand Cartan decomposition allows us to decompose the SLOCC-type operations into the maximally compact and non-compact parts, on the other hand the mismatch of dimensions in the mapping (24) complicates the averaging process. Nevertheless, since any smooth function $f$ on $SL(d, \mathbb{C})$ can be expressed by a smooth function $\tilde{f}$ on $SU(d) \times \mathbb{A} \times SU(d)$, we can define an integral over $SL(d, \mathbb{C})$ by an iterated integral over the product manifold corresponding to $SU(d) \times \mathbb{A} \times SU(d)$:

$$
\int_{SL(d, \mathbb{C})} \tilde{f} d\mu(M) \equiv \int_{SU(d) \times \mathbb{A} \times SU(d)} \tilde{f} d\mu(K) d\mu(A) d\mu(K').
$$

The above definition should be understood as follows: by fixing a measure $d\mu(K)$ on the group manifold of $SU(d)$ and $d\mu(A)$ on the group manifold of $\mathbb{A}$ we define a measure $d\mu(M)$ on the group manifold of $SL(d, \mathbb{C})$ and therefore the right-hand-side integral can be treated as an integral over $SL(d, \mathbb{C})$. Therefore we propose the following definition:

**Definition 1.** Generalisation of the unitary collective twirling to the SLOCC twirling can be defined by the following map:

$$
\mathcal{T}_{SL}(\rho) = \int \left( \frac{KAK'^t}{\|KAK'^t\|} \right)^{\otimes t} \rho \left( \frac{KAK'^t}{\|KAK'^t\|} \right)^{\dagger \otimes t} d\mu(K) d\mu(A) d\mu(K'),
$$

in which in the iterated integral the measure $d\mu(K)$ on the group manifold of $SU(d)$ is assumed to be the normalised Haar measure, whereas the measure $d\mu(A)$ is an arbitrary normalised measure on the group manifold of $\mathbb{A}$.

Due to (26) the map (27) can be equivalently represented as:

$$
\mathcal{T}_{SL}(\rho) = \int \left( \frac{M}{\|M\|} \right)^{\otimes t} \rho \left( \frac{M^t}{\|M\|} \right)^{\otimes t} d\mu(M),
$$

in which $M \in SL(d, \mathbb{C})$ and the measure $d\mu(M)$ on the group manifold of $SL(d, \mathbb{C})$ is defined via relation (26). Proposed definition assures as much uniformity of averaging over SLOCC operations as possible, namely the averaging over compact components of SLOCC operations is performed uniformly according to a Haar measure on the components. All the non-uniformity is shifted to averaging over non-compact component consisting of matrices $A \in \mathbb{A}$, interpreted physically as filtering operations [25, 44, 45]. Very similar definition has been suggested in our previous work on construction of finite averaging sets for the group $SL(d, \mathbb{C})$ [31], however with an important difference. There we utilised relation (26) in the opposite direction, namely we first defined the (divergent) averaging process by the left-hand-side with respect to the Haar measure on $SL(d, \mathbb{C})$, translated this Haar measure to the product measure in the right-hand-side iterated integral and finally added a suppressing factor in the component of the Haar measure on $\mathbb{A}$ in order to assure convergence. Here we do not impose any restriction on the measure $d\mu(A)$ other than that it has to be normalised. A similar concept of averaging to definition 1 has been proposed in [46] for defining averaging of optical states over Gaussian operations represented by symplectic matrices from a non-compact symplectic group $Sp(d, \mathbb{R})$.

Let us now make sure that defined map (28) is a properly defined quantum operation:

**Lemma 2.** The map $\mathcal{T}_{SL}$ preserves the positivity of the density matrix $\rho$, and is trace non-increasing.
**Proof.** Since $T_{SL}(\rho) = \int S_C(\rho) d\mu(M)$, and according to lemma 1, $S_C$ preserves positivity, $T_{SL}$ as a probabilistic mixture of $S_C$ also preserves positivity. The fact that it is also trace non-increasing follows simply from:

$$\text{Tr}(T_{SL}(\rho)) = \text{Tr}\left(\int S_C(\rho) d\mu(M)\right) = \int \text{Tr}(S_C(\rho)) d\mu(M) \leq \int d\mu(M) = 1,$$

in which the inequality follows from the fact that $\text{Tr}(S_C(\rho)) \leq 1$ according to lemma 1. □

The second issue is that the above defined map involves mixing of subnormalised states, therefore its physical interpretation is not clear at first glance. In order to provide such interpretation let us rewrite the map (28) as follows:

$$T_{SL}(\rho) = \int p_M(\rho) \frac{M^\dagger \rho M^\dagger}{\text{Tr}(M^\dagger \rho M^\dagger)} d\mu(M),$$

in which the probability $p_M(\rho)$ reads:

$$p_M(\rho) = \frac{\text{Tr}(M^\dagger \rho M^\dagger)}{\|M\|^2},$$

and can be interpreted as a success probability of performing the SLOCC operation [25]. Therefore the entire map $T_{SL}$ represents preparation of an ensemble of properly normalised states $M^\dagger \rho M^\dagger$ produced by the SLOCC operations $M$, for $M$ drawn randomly from the group $\text{SL}(d, \mathbb{C})$ according to a probabilistic measure $d\mu(M)$, weighted according to the success probability $p_M(\rho)$ of the respective drawn SLOCC operation $M$. The normalisation of the output state of the map (28) $\text{Tr}(T_{SL}(\rho))$ represents average success probability of the SLOCC operations:

$$\langle p_M(\rho) \rangle = \text{Tr}(T_{SL}(\rho)) = \int p_M(\rho) d\mu(M).$$

4. **Exact form of the twirling map for SLOCC operations**

In this section we will calculate closed analytic form of the proposed SLOCC twirling map (27). Our derivation utilises the closed form of the unitary twirling map (17), which is derived in appendix A.3. The main tool used there is the application of Schur’s Lemmas based on invariance of the unitary twirling procedure with respect to fixed left and right unitary rotations, see appendix A.2 for details. In the case of non-unitary averaging such invariance no longer holds, therefore we have to exploit other techniques related with the structure of the Cartan decomposition in (27), which we have successfully used in the context of finding finite averaging sets for $\text{SL}(2, \mathbb{C})$ group [31]. These techniques allow us to prove a factorisation of the SLOCC twirling map (27) into two terms, the first one being the unitary twirling map (15) applied to the input state, whereas the second being the unitary twirling map applied solely to the non-unitary part of SLOCC operations.

Let us start by rewriting the map (27) in the following way. Since $||KAK'|| = ||A||$ (which follows from the fact that the Cartan decomposition for $\text{SL}(d, \mathbb{C})$ matrices is equivalent to a singular value decomposition), we can introduce normalised filtering matrix $A_n = A/||A||$, which leads to the following form of the map:
\[ T_{SL}(\rho) = \int (KA_nK')^{\otimes r} \rho (KA_nK')^{\dagger \otimes r} d\mu(K)d\mu(A)d\mu(K'). \]  

Using distributivity of the tensor product with respect to the product of matrices the above map can be decomposed in the following form:

\[ T_{SL}(\rho) = \int K^{\otimes t}A_n^{\otimes t} \left( \int K^{\otimes r}t^k K^{\dagger \otimes r} d\mu(K) \right) A_n^{\dagger \otimes t} K^{\dagger \otimes r} d\mu(K)d\mu(A). \]  

As can be seen in the above formula a twirling operation with respect to a special unitary group \( SU(d) \) appeared in the middle integral. In our previous work [31] we have shown that such a twirling operation is equivalent to a twirling map with respect to the entire unitary group \( T_U(\rho) \) (15), therefore the map (34) can be simplified to:

\[ T_{SL}(\rho) = \int K^{\otimes t}A_n^{\otimes t} T_U(\rho) A_n^{\dagger \otimes t} K^{\dagger \otimes r} d\mu(K)d\mu(A). \]  

Now it turns out that the unitarily twirled state \( T_U(\rho) \) commutes with all other operators in the integral due to the following lemma:

**Lemma 3.** Let us take any invertible \( d \times d \) matrix \( \alpha \) and any \( d^e \times d^e \) matrix \( B \). Then the matrices \( \alpha^{\otimes t} \) and \( T_U(B) \) commute for every \( t \) and \( d \).

**Proof.** The \( r \)th tensor power of arbitrary invertible matrix \( \alpha \) can be represented in a block-diagonal form equivalent to the one of \( U^{\otimes t} \) (see formula (A14) in the appendix A.1):

\[ \alpha^{\otimes t} = \sum_k \frac{1}{D_k^t} \sum_{m_1,m_2=1}^{D_k^t} \alpha_{m_1m_2}^{k} \hat{\Pi}_{k}^{m_1m_2}, \]

in which \( \alpha_{m_1m_2}^{k} = \text{Tr} \left( \alpha^{\otimes t} \hat{\Pi}_{k}^{m_1m_2} \right) \), whereas the unitarily twirled operator \( B \) has the form (17):

\[ T_U(B) = \sum_k \frac{1}{D_k^t} \sum_{\lambda_1,\lambda_2=1}^{D_k^t} B_{\lambda_1\lambda_2}^{k} \hat{\Pi}_{k}^{\lambda_1\lambda_2}, \]

in which \( B_{\lambda_1\lambda_2}^{k} = \text{Tr} (B \hat{\Pi}_{k}^{\lambda_1\lambda_2}) \). Due to commutativity of the projection operators \( \hat{\Pi}_{k}^{m_1m_2} \) and \( \hat{\Pi}_{k}^{\lambda_1\lambda_2} \), see (A20) in appendix A.1, we have:

\[ [\alpha^{\otimes t}, T_U(B)] = \left[ \frac{1}{D_k^t} \sum_{m_1,m_2=1}^{D_k^t} \alpha_{m_1m_2}^{k} \hat{\Pi}_{k}^{m_1m_2} \frac{1}{D_L^t} B_{\lambda_1\lambda_2}^{t} \hat{\Pi}_{L}^{\lambda_1\lambda_2} \right] = \frac{1}{D_k^t D_L^t} \sum_{m_1,m_2}^{k} B_{\lambda_1\lambda_2}^{k} \left[ \hat{\Pi}_{k}^{m_1m_2}, \hat{\Pi}_{L}^{\lambda_1\lambda_2} \right] = 0, \]

in which in the above we used Einstein’s summation convention to simplify the notation. \( \square \)

Due to lemma 3 the operator \( T_U(\rho) \) commutes with both operators \( K^{\otimes t} \) and \( A_n^{\otimes t} \), therefore it can be factored out of the integral in the formula (35):

\[ T_{SL}(\rho) = T_U(\rho) \int K^{\otimes t} \left( \int A_n^{\otimes t} A_n^{\dagger \otimes t} d\mu(A) \right) K^{\dagger \otimes r} d\mu(K). \]  

Let us now introduce a compact notation for the inner integral:

\[ A = \int A_n^{\otimes t} A_n^{\dagger \otimes t} d\mu(A) = \int (A_n^{\otimes t})^2 d\mu(A), \]
in which the second equality comes from the fact that $A_n$ is a real diagonal matrix. Finally the map (39) can be presented in a factorised form:

$$\mathcal{T}_{SL}(\rho) = \mathcal{T}_U(\rho)\mathcal{T}_U(A),$$

in which the term $\mathcal{T}_U(A)$ plays the role of a \textit{correction} due to non-unitary character of SLOCC averaging. The operator $\mathcal{T}_U(A)$ has a very special form, which is explicitly provided by the following lemma:

\textbf{Lemma 4.} Let $\alpha$ be arbitrary $d \times d$ complex invertible matrix. Then the operator $\mathcal{T}_U(\alpha^{\otimes n})$ is fully diagonal in Schur operator basis:

$$\mathcal{T}_U(\alpha^{\otimes n}) = \sum_k \frac{1}{D_k} \alpha_k \Pi_k,$$

in which the coefficients read: $\alpha_k = \text{Tr}(\alpha^{\otimes n} \Pi_k^\dagger \Pi_k)$, $\Pi_k$ is a projector onto the $k$th subspace $P_k$, and $D_k$ is the dimension of the subspace $P_k$: $D_k = D^k L^k V_k$.

\textbf{Proof.} See appendix B.2.

From lemma 4 it follows that:

$$\mathcal{T}_U(A) = \sum_k \frac{1}{D_k} \beta_k \Pi_k,$$

in which $\beta_k = \text{Tr}(A \Pi_k^\dagger \Pi_k)$. This follows from linearity of integration and independence of the order of integration due to Fubini’s theorem. Indeed, we have:

$$\mathcal{T}_U(A) = \mathcal{T}_U \left( \int (A^{\otimes n})^2 d\mu(A) \right) = \int \mathcal{T}_U \left( (A^{\otimes n})^2 \right) d\mu(A),$$

$$= \int \left[ \sum_k \frac{1}{D_k} \text{Tr} \left( (A^{\otimes n})^2 \Pi_k^\dagger \Pi_k \right) \right] d\mu(A),$$

$$= \sum_k \frac{1}{D_k} \text{Tr} \left( \int (A^{\otimes n})^2 d\mu(A) \right) \Pi_k,$$

$$= \sum_k \frac{1}{D_k} \text{Tr} \left( A \Pi_k^\dagger \Pi_k \right) \Pi_k.$$

Using general form of the unitary twirling map (17) and the formula (43), we can finally simplify the map (41):

$$\mathcal{T}_{SL}(\rho) = \mathcal{T}_U(\rho)\mathcal{T}_U(A) = \sum_{k,l} \frac{1}{D_L} \sum_{\lambda_1 \lambda_2 = 1}^L \text{Tr} \left( \rho \Pi_k^{\lambda_1 \lambda_2} \right) \Pi_k^{\lambda_1 \lambda_2} \sum_l \frac{1}{D_L} \beta_l \Pi_l,$$

$$= \sum_{k,l} \frac{1}{D_L} \left( \frac{\beta_l}{D_L} \right) \sum_{\lambda_1 \lambda_2 = 1}^L \text{Tr} \left( \rho \Pi_k^{\lambda_1 \lambda_2} \right) \Pi_k^{\lambda_1 \lambda_2} \Pi_l,$$

$$= \sum_{k,l} \frac{1}{D_L} \left( \frac{\beta_l}{D_L} \right) \sum_{\lambda_1 \lambda_2 = 1}^L \text{Tr} \left( \rho \Pi_k^{\lambda_1 \lambda_2} \right) \Pi_k^{\lambda_1 \lambda_2} \delta_{kl},$$

$$= \sum_{k,l} \frac{1}{D_L} \left( \frac{\beta_k}{D_L} \right) \sum_{\lambda_1 \lambda_2 = 1}^L \text{Tr} \left( \rho \Pi_k^{\lambda_1 \lambda_2} \right) \Pi_k^{\lambda_1 \lambda_2}. \quad (45)$$
In the fourth line we utilised the property of operators $\hat{\Pi}_k^{\lambda_1\lambda_2}$ and $\hat{I}_l$ shown in appendix A.1, formulas (A23). If we express the closed form (17) of the unitary twirling map in the compact form:

$$T_U(\rho) = \sum_k T_U^{(k)}(\rho), \quad (46)$$

in which:

$$T_U^{(k)}(\rho) = \frac{1}{D_k^2} \sum_{\lambda_1\lambda_2=1}^{D_k^2} \text{Tr} \left( \rho \hat{\Pi}_k^{\lambda_1\lambda_2} \right) \hat{\Pi}_k^{\lambda_1\lambda_2}, \quad (47)$$

the SLOCC twirling map can be represented as:

$$T_{SL}(\rho) = \sum_k \left( \frac{\beta_k}{D_k} \right) T_U^{(k)}(\rho). \quad (48)$$

It can be easily seen that all the impact of averaging over non-compact part of the $\text{SL}(d, \mathbb{C})$ group is expressed in the rescaling of each of the component defined on an irreducible subspace by a factor $\left( \frac{\beta_k}{D_k} \right)$, in which $\beta_k = \text{Tr}(A \hat{\Pi}_k^\dagger)$ is a projection of an integral over the non-compact part of the SLOCC operation (40) onto the $k$th irreducible subspace. This term directly depends on the integration measure over the non-compact part $d\mu(A)$. An important difference with respect to the unitary case is that the SLOCC twirling map is not idempotent:

$$T_{SL}(T_{SL}(\rho)) = \sum_k \left( \beta_k \frac{1}{D_k} \beta_{kl} \frac{1}{D_l} \right) T^{(k)}_U T^{(l)}_U(\rho) \quad \Rightarrow \quad \sum_k \left( \beta_k \frac{1}{D_k} \right)^2 T^{(k)}_U(\rho) \quad (49)$$

By induction a general rule holds:

$$T_{SL}^m(\rho) = \sum_k \left( \frac{\beta_k}{D_k} \right)^m T^{(k)}_U(\rho), \quad (50)$$

in which:

$$T_{SL}^m = T_{SL} \circ \ldots \circ T_{SL}. \quad (51)$$

We can also easily calculate mean success probability $\langle p_M(\rho) \rangle$ of the SLOCC operations $M = KAK'$ over which we average, equal to the trace of $T_{SL}(\rho)$ (32):

$$\langle p_M(\rho) \rangle = \text{Tr}(T_{SL}(\rho)) = \sum_k \left( \frac{\beta_k}{D_k} \right) \text{Tr}(T^{(k)}_U(\rho)) \quad \Rightarrow \quad \sum_k \left( \frac{\beta_k}{D_k} \right) \text{Tr}(\rho \hat{\Pi}_k^\dagger). \quad (52)$$

This follows from the following transformations:
We can summarise all presented derivations in the following theorem:

**Theorem 1.** The collective SLOCC twirling map defined by formula (27) is represented by the following closed analytical form:

$$T_{\text{SL}}(\rho) = \sum_k \left( \frac{\beta_k}{D_k} \right) T_{\text{U}}^{(k)}(\rho),$$

(54)

in which $T_{\text{U}}^{(k)}$ (47) is the unitary twirling map acting on $k$th irreducible subspace of the tensor space $(\mathbb{C}^d)^{\otimes r}$, $D_k$ is the dimension of the subspace, whereas the coefficient

$$\beta_k = \text{Tr} \left( \left[ \int (A_n^2)^{\otimes k} \, d\mu(A) \right] \hat{\Pi}_k^1 \right)$$

(55)

is a projection of an integral over the non-compact part of the SLOCC operation (40) with respect to a normalised measure $d\mu(A)$ onto the $k$th irreducible subspace. The average success probability of the SLOCC operations drawn according to assumed measure is specified by:

$$\langle p_M(\rho) \rangle = \sum_k \left( \frac{\beta_k}{D_k} \right) \text{Tr} \left( \rho \hat{\Pi}_k^1 \right),$$

whereas $m$-times application of the map gives:

$$T_{\text{SL}}^m(\rho) = \sum_k \left( \frac{\beta_k}{D_k} \right)^m T_{\text{U}}^{(k)}(\rho).$$

The issue of noiseless subsystems in the case of SLOCC collective twirling map (27) is subtle. Formally there are no states which are invariant with respect to this map in contrast to the unitary case, on the other hand the map (48) does not change phase relations between the states within each of the irreducible subspaces. Therefore states of the form:

$$\rho_{\text{SL}}^{(k)} = \frac{1}{D_k} \sum_{\lambda_1,\lambda_2} \rho_{\lambda_1,\lambda_2}^{k} \hat{\Pi}_k^{\lambda_1 \lambda_2},$$

(56)

living within $k$th subspace are untouched by the map upon postselection to the $k$-th irreducible subspace. Indeed, due to the formula (48), and the fact that the state (56) is invariant with respect to the unitary twirling we have:

$$T_{\text{SL}} \left( \rho_{\text{SL}}^{(k)} \right) = \left( \frac{\beta_k}{D_k} \right) \rho_{\text{SL}}^{(k)},$$

(57)
What is modified in this case with respect to the unitary twirling is the projection probability $p_k$ to the subspace, which now reads:

$$p_k = \left( \frac{\beta_k}{D^k} \right),$$

and can be physically interpreted as the probability of success of postselection to the respective subspace. Note that due to the normalisation of the matrices $A_n$ we have the following relation:

$$\beta_k = \text{Tr} \left( \int (A_n^{\otimes t})^2 d\mu(A) \right) \Pi_k^\dagger \Pi_k \leq \text{Tr} \left( \Pi_k \right) = D^k,$$

hence $\beta_k \leq D^k$, which guarantees that the probability $p_k$ is well-defined. Further note that the choice of the integration measure $d\mu(A)$ determines the ordering of the coefficients $\{\beta_k\}$. The highest coefficient $\beta_1$, which corresponds to invariant subspaces $V_k^{m}$ with the dimension $V_k^m > 1$ determines the noiseless subsystems with the highest projection probabilities $p_k$. These subsystems have the slowest rate of vanishing of the projection probability $p_k$ for the number $m$ of applications of the map $T_m^{SL}(51)$ going to infinity, $m \to \infty$.

In order to make all the above considerations less abstract let us discuss an example. Let us take the four-qubit system, in which according to our notation local Hilbert space dimension is $d = 2$ and the tensor power is $t = 4$. In this case the tensor space $(\mathbb{C}^2)^{\otimes 4}$ decomposes under collective action of the unitary group $U(2)^{\otimes 4}$ into three irreducible subspaces of dimensions respectively: $D^1 = 5$, $D^2 = 9$ and $D^3 = 2$. The first subspace of dimension $D^1 = 5$ is the subspace of fully symmetric states, and corresponds to a single (of multiplicity $V_k^1 = 1$) 5-dimensional spin-2 irreducible representation of $U(2)$. The second subspace of dimension $D^2 = 9$ corresponds to three ($D_k^2 = 3$) equivalent 3-dimensional spin-1 irreducible representations of $U(2)$, whereas finally the third subspace of dimension $D^3 = 2$ corresponds to two ($D_k^3 = 2$) 1-dimensional irreducible representations. The non-compact factor of the $SL(2,\mathbb{C})$ group, which physically corresponds to a filtering operation, consists of matrices (25), with norm equal to $||A|| = x$, therefore a normalised filtering matrix in the formula (40) has the form:

$$A_n = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2} \end{pmatrix}, \ x \geq 1.$$ 

If we assume the following normalised integration measure on the non-compact component:

$$d\mu(A) = d\mu(x) = \frac{e^{-\frac{x}{2}}dx}{\int_{1}^{\infty}e^{-\frac{x}{2}}dx},$$

the coefficients $\beta_k$ in (55) read respectively:

$$\beta_1 \approx 0.30036,$$
$$\beta_2 \approx 0.14652,$$
$$\beta_3 \approx 0.12290.$$
5. Duality of averaging for generalised Schur–Weyl-dual pairs

Let us start by comparing closed formulas (54) and (18) for twirling of quantum states over SLOCC operations and over permutations of subsystems written in a specific way:

\[ T_{\text{SL}}(\rho) = \sum_k \gamma_{k}^{\text{SL}} \sum_{\lambda_1, \lambda_2 = 1}^{D_k^\rho} \text{Tr} \left( \rho \Pi_k^{\lambda_1 \lambda_2} \Pi_k^{\lambda_1 \lambda_2} \right), \]
\[ T_{\text{sym}}(\rho) = \sum_k \gamma_{k}^{\text{sym}} \sum_{m_1, m_2 = 1}^{D_k^\rho} \text{Tr} \left( \rho \Pi_k^{m_1 m_2} \Pi_k^{m_1 m_2} \right), \quad (63) \]

in which the coefficients read:

\[ \gamma_{k}^{\text{SL}} = \frac{1}{D_k^\rho} \left( \frac{\beta_k}{D_k^\rho} \right), \quad \gamma_{k}^{\text{sym}} = \frac{1}{D_k^\rho}. \quad (64) \]

These two operations correspond to averaging of quantum states with respect to elements of a dual reductive pair: \( \{ \text{SL}(d, \mathbb{C}), \mathbb{S}_t \} \) consisting of the special linear group acting collectively on \( t \) subsystems and the symmetric group of \( t \)-element set. Comparing the two formulas it can be seen that up to the coefficients these two operations project quantum states (understood as density operators) into operator subspaces the action of which leaves invariant subspaces irreducible with respect to the second counterpart of the pair. In this section we show, that this result is not specific to the already discussed pairs \( \{ \text{SL}(d, \mathbb{C}), \mathbb{S}_t \} \) and \( \{ \text{U}(d), \mathbb{S}_t \} \), but holds for arbitrary dual reductive pair. Our proof will be based on showing, that crucial derivations already presented do not depend on the specific choice of the group under consideration, but hold due to Schur–Weyl duality for the corresponding groups.

Let us start by introducing the idea of a dual reductive pair in a more formal way. The crucial concept is that of a reductive group, which is any algebraic subgroup of a general linear group, the rational \([47]\) representations of which are completely reducible, which means that they are direct sums of irreducible representations \([1]\).

**Definition 2.** Let \( G_1 \) and \( G_2 \) be two reductive subgroups of the general linear group \( \text{GL}(n, \mathbb{C}) \). A pair of groups \( \{ G_1, G_2 \} \) is called a dual reductive pair if and only if matrix algebras generated by elements of these groups are mutual commutants in the full matrix algebra \( \text{End}(\mathbb{C}^n) \).

For any dual reductive pair a Schur–Weyl duality holds (see \([1]\), Prop. 9.2.1), which implies that there exists a vector basis on \( \mathbb{C}^n \), called by us Schur vector basis, which via outer product of basis vectors defines two operator bases, called by us Schur operator bases. These two operator bases block-diagonalise actions of operators from both groups, in full analogy to the Schur operator bases \((11)\) introduced already in section \(2\) for the dual pair \( \{ \text{U}(d), \mathbb{S}_t \} \). For simplicity let us keep the entire notation from that section regarding Schur vector basis and Schur operator bases, with the convention, that the subspaces \( L_\lambda \) in \((6)\) are irreducible under the action of \( G_1 \), whereas the subspaces \( V_m^\lambda \) are irreducible under the action of \( G_2 \). All properties of the corresponding Schur operator bases \((11)\), which follow from orthogonality of Schur vector basis naturally apply in the current generalised context. Then arbitrary matrix representants \( G_1, G_2 \) of the groups \( G_1, G_2 \) have the following Schur operator bases decompositions:
Let us take any dual reductive pair \((G_1, G_2)\) acting naturally on the complex vector space \(\mathbb{C}^n\). Let us assume the notation for Schur basis decomposition of arbitrary elements \(G_1 \in G_1\) and \(G_2 \in G_2\) as presented in formulas (65). Let us also assume Cartan decompositions of these elements as \(G_1 = K^{(1)} A^{(1)} K^{(1)\dagger}\) and \(G_2 = K^{(2)} A^{(2)} K^{(2)\dagger}\). Then the generalised stochastic operation \(G\) on a state space \(\mathbb{C}^n\) of \(n\)-level quantum systems via:

\[
G(\rho) = (KA_n K') \rho (KA_n K')^\dagger,
\]

in which \(K, K'\) are matrices from a unitary representation of the maximal compact subgroup \(K\) on \(\mathbb{C}^n\) and \(A_n = A/||A||\) is a normalised matrix belonging to a representation of the maximal abelian subgroup \(A\) on \(\mathbb{C}^n\). Some comments are needed here. Firstly, in full analogy with the SLOCC map (21), the map defined by (66) is trace non-increasing (for a formal proof see remark in the appendix B.1) and preserves positivity, therefore can be interpreted as a valid quantum operation. Secondly, in the above definition we do not assume any internal structure of the state space \(\mathbb{C}^n\), so the issue whether it is a single-system space or it represents several subsystems is left free for concrete implementations. Therefore the map (66) should be thought of as an abstract interface for defining concrete quantum maps with concrete physical interpretation. In the same manner we define an abstract generalisation of the twirling map to the case of \(G\)-twirling with respect to any reductive group \(G\) by the iterated integral with respect to the Cartan decomposition, in full analogy to (33):

\[
T_G(\rho) = \int (KA_n K') \rho (KA_n K')^\dagger \, d\mu(K)d\mu(A)d\mu(K'),
\]

in which we assume, that \(d\mu(K)\) is a normalised left- and right-invariant Haar measure on the compact group \(K\) (which in the case of \(K\) being a discrete group is simply a counting measure), whereas \(d\mu(A)\) is any normalised measure on the group manifold of \(A\). The map (67) should be treated as an abstract interface for defining any meaningful twirling maps on quantum states rather than as a concrete physical operation. Note that all previously discussed twirling maps can be represented in that way.

Now we are ready to state the main result of this section:

**Theorem 2.** Let us take any dual reductive pair \((G_1, G_2)\) acting naturally on the complex vector space \(\mathbb{C}^n\). Let us assume the notation for Schur basis decomposition of arbitrary elements \(G_1 \in G_1\) and \(G_2 \in G_2\) as presented in formulas (65). Let us also assume Cartan decompositions of these elements as \(G_1 = K^{(1)} A^{(1)} K^{(1)\dagger}\) and \(G_2 = K^{(2)} A^{(2)} K^{(2)\dagger}\). Then the generalised...
Twirling maps with respect to these groups defined via iterated integral over Cartan decomposition of these groups (67) are in the following dual relation:

\[
T_{G_1}(\rho) = \sum_k \frac{1}{d_k} \left( \frac{\beta_k^{G_1}}{d_k} \right) \sum_{m_1m_2=1}^{d_k} \text{Tr} \left( \rho \Pi_k^{m_1m_2} \right) \Pi_k^{m_1m_2},
\]

\[
T_{G_2}(\rho) = \sum_k \frac{1}{d_k} \left( \frac{\beta_k^{G_1}}{d_k} \right) \sum_{\lambda_1\lambda_2=1}^{d_k} \text{Tr} \left( \rho \Pi_k^{\lambda_1\lambda_2} \right) \Pi_k^{\lambda_1\lambda_2},
\]

(68)

The coefficients \(\beta_k^{G_1}\) are in general expressed as integrals over the noncompact factors of the Cartan decomposition of both groups:

\[
\beta_k^{G_1} = \text{Tr} \left( \left[ A_n^{(i)} A_n^{(i)\dagger} \mathrm{d}\mu_n \left( A_n^{(i)} \right) \right] \Pi_k^{(i)} \right),
\]

(69)

and reduce to \(\beta_k^{G_1} = d_k\) for compact group \(G_i\).

**Proof.** The proof is actually a generalisation of the proof for the SLOCC twirling map presented in the previous section. For convenience we will proceed with the proof for the twirling map \(T_{G_1}\), the proof for the map \(T_{G_2}\) is entirely analogous so will be skipped. Firstly note that the twirling map \(T_{K_1}\) with respect to the maximally compact subgroup \(K_1\) of \(G_1\) has entirely the same form as the unitary twirling map with respect to the entire unitary group (17), namely it reads:

\[
T_{K_1}(\rho) = \sum_k \frac{1}{d_k} \sum_{m_1m_2=1}^{d_k} \text{Tr} \left( \rho \Pi_k^{m_1m_2} \right) \Pi_k^{m_1m_2}.
\]

This is because the Haar measure on any compact group is left and right invariant, therefore one can perform the entire derivation for the unitary twirling map without any modification (for alternative proof of general twirling map with respect to any compact symmetry group see [13], section II.C). Secondly note that both elements \(K^{(1)}\) and \(A^{(1)}\) of the Cartan decomposition of any matrix \(G_1\) have decompositions analogous to \(G_1\) in terms of the corresponding Schur operator basis (65), namely:

\[
K^{(1)} = \sum_k \frac{1}{d_k} \sum_{\lambda_1\lambda_2=1}^{d_k} \text{Tr} \left( K^{(1)} \Pi_k^{\lambda_1\lambda_2} \right) \Pi_k^{\lambda_1\lambda_2},
\]

\[
A^{(1)} = \sum_k \frac{1}{d_k} \sum_{\lambda_1\lambda_2=1}^{d_k} \text{Tr} \left( A^{(1)} \Pi_k^{\lambda_1\lambda_2} \right) \Pi_k^{\lambda_1\lambda_2}.
\]

(70)

Since operators \(\Pi_k^{\lambda_1\lambda_2}\) and \(\Pi_k^{m_1m_2}\) commute for any Schur operator bases (see (A20)), the generalisation of lemma 3 holds and the operator \(T_{K_1}(\rho)\) commutes with both \(K^{(1)}\) and \(A^{(1)}\). Therefore the generalised twirling map (67) factorises in analogy to the SLOCC twirling map:
implies that:

\[ \mathcal{T}_{G_k}(\rho) = \int \left( K^{(1)} A_n^{(1)} K^{(1)} \right) \rho \left( K^{(1)} A_n^{(1)} K^{(1)} \right)^\dagger d\mu_1 \left( K^{(1)} \right) d\mu_1 \left( A^{(1)} \right) \]

\[ = \int K^{(1)} A_n^{(1)} \left( \int K^{(1)} \rho K^{(1)} \dagger d\mu_1 \left( K^{(1)} \right) \right) A_n^{(1)} \left( K^{(1)} \dagger d\mu_1 \left( K^{(1)} \right) \right) d\mu_1 \left( A^{(1)} \right) \]

\[ = \int K^{(1)} A_n^{(1)} \mathcal{T}_{K_k}(\rho) A_n^{(1)} \left( K^{(1)} \dagger d\mu_1 \left( K^{(1)} \right) \right) d\mu_1 \left( A^{(1)} \right) \]

\[ = \mathcal{T}_{K_k}(\rho) \mathcal{T}_{K_k} \left( \int A_n^{(1)} A_n^{(1)} d\mu_1 \left( A^{(1)} \right) \right) \]

(71)

We have obtained a factorisation analogous to the case of twirling with respect to the SLOCC operations. Now the crucial point is to show that an analogy of lemma 4 holds also in the general case:

**Lemma 5.** Let \( G_2 \) be arbitrary element of matrix von Neumann algebra spanned by representation matrices of the group \( G_2 \). Then the operator \( \mathcal{T}_{K_k}(G_2) \) is fully diagonal in the Schur operator basis related with dual reductive pair \( \{ G_1, G_2 \} \):

\[ \mathcal{T}_{K_k}(G_2) = \sum_k \frac{1}{d^i} \alpha_k \hat{\Pi}_k, \]

(72)

in which the coefficients read: \( \alpha_k = \text{Tr}(G_2 \hat{\Pi}_k) \). \( \hat{\Pi}_k \) is a projector onto the \( i \)-th irreducible subspace (A15) and \( d^i \) is the dimension of the subspace: \( d^i = d^i d^i \).

The proof of the above lemma is entirely analogous to the one presented in appendix B.2, and holds due to the fact that the mentioned proof is based solely on orthogonality properties of the Schur basis, which do not depend on the structure of groups under consideration.

Now, since as mentioned before, the algebra generated by operators \( G_1 \) is a von Neumann algebra, therefore the operator \( \int A_n^{(1)} A_n^{(1)} d\mu_1 \left( A^{(1)} \right) \) also belongs to this algebra and consequently lemma 5 implies that:

\[ \mathcal{T}_{K_k} \left( \int A_n^{(1)} A_n^{(1)} d\mu_1 \left( A^{(1)} \right) \right) = \sum_k \frac{\beta_{G_1}^k}{d^k} \hat{\Pi}_k, \]

(73)

in which the coefficient \( \beta_{G_1}^k \) is specified by formula (69) for \( i = 1 \). Derivation of the final form of the generalised twirling map \( \mathcal{T}_{G_1} \) is now entirely analogous to the derivation performed in a sequence of formulas (45), which completes the proof.

The entire issue of noiseless subsystems can be also restated in the general case of twirling with respect to dual reductive pairs. Namely states of the form:

\[ \rho_{G_1}^{(k)} = \frac{1}{d_1^k} \sum_{m_1, m_2=1}^{d_1^k} \rho_{m_1 m_2}^{k} \hat{\Pi}_{m_1 m_2}^{k} \]

\[ \rho_{G_2}^{(k)} = \frac{1}{d_1^k} \sum_{\lambda_1, \lambda_2=1}^{d_1^k} \rho_{\lambda_1 \lambda_2}^{k} \hat{\Pi}_{\lambda_1 \lambda_2}^{k} \]

(74)

are invariant with respect to twirling operations (68) on condition that a postselection to the \( k \)-th subspace is performed, which succeeds with probability respectively \( \frac{\beta_{G_1}^k}{d^k} \) and \( \frac{\beta_{G_2}^k}{d^k} \). If any of
the groups has finite dimensional unitary representation (which holds if the group is a compact Lie group or a finite group), then $\beta_k = \delta_k$, and therefore the respective noiseless subsystem is unconditional.

6. Conclusions

In this work we introduced a very general averaging operation on the state space of any finite dimensional quantum system with respect to an arbitrary symmetry group possessing the Cartan ‘KAK’ decomposition. The averaging procedure is defined via iterated integral over the compact and non-compact components of the Cartan decomposition of the symmetry group. This very general notion of averaging of quantum states allows us to significantly generalise known results concerning averaging of quantum states over collective unitary operations.

In a typical scenario of averaging multipartite quantum states over collective unitary operations the effect of such averaging is determined by a mathematical concept called Schur–Weyl duality, which introduces the notion of a dual reductive pair, a pair of two matrix groups that as matrix algebras are mutual commutants. It turns out that the dual counterpart to the collective action of the unitary group is the symmetric group. Then Schur’s Lemmas imply, that averaging over one of the elements of the dual pair just projects the quantum state onto a subspace spanned by elements of the second counterpart. We called this relation duality of averaging. Our main result in this work is showing, that such defined duality of averaging persists in the case of arbitrary two symmetry groups, which are dual reductive pairs. We have shown this fact in three steps:

- Firstly we analysed the case of averaging multipartite quantum states over Cartan-decomposed collective SLOCC operations represented by the tensor power of the non-compact symmetry group $\text{SL}(d, \mathbb{C})$, which also forms a dual reductive pair with the symmetric group; as a result we obtained a factorisation of such an averaging map into two components, the first being a collective unitary averaging map applied to the input state, whereas the second being unitary averaging map applied to the integral over non-compact components of the SLOCC operations;
- Secondly we noticed that such a factorisation implies that the result of averaging quantum states over SLOCC operations has the same block-diagonal structure as the unitary averaging, but endowed with additional correcting coefficients; this means that averaging of quantum states with respect to SLOCC operations and with respect to permutation of subsystems also conforms to the duality of averaging;
- Finally we have shown that the proof for the SLOCC case directly generalises to the averaging procedure over any two groups forming a dual reductive pair, therefore the duality of averaging holds for averaging over any dual pair of symmetry groups.

The above points summarise the structural aspect of our work, which shows in a very general context the form of averaging maps for arbitrary finite dimensional quantum systems.

The main physical consequence of our result appears in the context of the so called noiseless subsystems. It is a well-known consequence of Schur–Weyl duality, that quantum states spanned by the operator basis of permutation operators are untouched by the collective unitary averaging. If we interpret collective unitary operations as a collective quantum noise, than such states are immune to this type of noise. Noiseless subsystems can be also used to encode quantum information in the case of lack of a common reference frame between the sender and receiver of a given state. Our result shows, that the idea of noiseless subsystems persists to
the case of averaging over arbitrary symmetry groups forming dual reductive pairs in a conditional way: if one of the groups in the pair is non-compact, then noiseless subsystems persist on condition that we make a postselection to the subsystem: all the input from the non-unitary part of the averaging process is the effect of filtering, which means that in some runs of the experiment a system will not be found in a given subspace. However all the block-diagonal structure of noiseless subsystems is solely determined by the unitary part of the averaging process.

All the discussion within this work concerns the case of finite dimensional quantum systems. In our opinion an interesting open problem would be to search for analogues of duality of averaging in the context of averaging infinite dimensional quantum systems over unitary representations of symmetry groups. A natural example would be averaging quantum states of massive and massless particles over unitary representations of Poincare group \[16\] or Galilean group \[17\].

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Direct analytical form of the unitary and symmetric twirling maps

A.1. Construction of Schur vector basis and Schur operator basis for the dual pair \{U(d)^{\otimes t}, S_t\}

In this section we provide a detailed construction of a Schur vector basis and Schur operator basis for the joint action of the product group \(U(d)^{\otimes t} \times S_t\) on the tensor space \((\mathbb{C}^d)^{\otimes t}\). The presentation below is inspired by the book of Tung [37], an alternative construction using cascaded Clebsch–Gordan decompositions can be found in the PhD thesis of Harrow [48].

Distinct irreducible representations of the action of \(U(d)^{\otimes t} \times S_t\) on \((\mathbb{C}^d)^{\otimes t}\) are labelled by distinct Young diagrams with \(t\) boxes and at most \(d\) rows. These representations can be uniquely provided by construction of a Schur vector basis, which explicitly encodes the subspaces of \((\mathbb{C}^d)^{\otimes t}\) which are irreducible with respect to the collective action of the unitary group and of the action of the symmetric group.

Before proceeding with construction of Schur bases let us briefly describe how to construct matrix irreducible representations of the symmetric group related with a given Young diagram with the numbers of boxes in subsequent rows given by \(r_1, \ldots, r_k, k \leq d\) and \(\sum_j r_j = t\). Firstly let us introduce the notion of standard Young tableau, which is a \(t\)-box Young diagram filled in with integers \(1, \ldots, t\) in such a way that the numbers are increasing row-wise from left to right and column-wise from top to bottom, but not necessarily in a strict order. With a given standard Young tableau we associate so-called Young symmetrizer, which is a building block for constructing irreducible representations of the symmetric group. Abstractly, a Young symmetrizer is an element of a group algebra of the symmetric group, that is a vector space
involving formal linear combinations of group elements endowed with natural multiplication inherited from the group composition law. A Young symmetrizer is defined as:

\[ Y_{r_1, \ldots, r_i} = \frac{1}{k!} \sum_{r=1}^k \sum_{c=1}^l \text{sgn}(p_c) p_r p_c, \]

where the sum runs over all permutations preserving all rows and columns of the standard Young tableau, weighted with the parity of the column permutations. A matrix representation of a Young symmetrizer is simply specified by:

\[ \hat{Y}_{r_1, \ldots, r_i} = \frac{1}{k!} \sum_{r=1}^k \sum_{c=1}^l \text{sgn}(p_c) O_{p_r} O_{p_c}, \]

in which the orthogonal matrices \( O_p \) are already defined in formula (4) in the main text.

Let us label all inequivalent irreducible representations of \( U(d)^{\otimes i} \) and \( S_i \) corresponding to a given Young diagram by an index \( i \). Although the index \( i \) labels uniquely all inequivalent representations of both groups, it may happen that there exist equivalent representations of both of them, which however correspond to different subspaces of \( (C^d)^{\otimes i} \). In order to describe this phenomenon let us construct a Schur basis for \( i \)-th irreducible subspace:

- let us denote a matrix representation of a Young symmetrizer related with a fixed irreducible subspace \( i \) and of a fixed allowed filling with integers \( \lambda \) as \( \hat{Y}_i^\lambda \); such Young symmetrizer corresponds to a fixed standard Young tableau with a shape uniquely specified by lengths of its rows \( r_1, \ldots, r_k \), therefore the index \( i \) stays in one-to-one correspondence with a shape of the diagram;
- for every \( \lambda \)-th standard Young tableau of the \( i \)-th type, consisting of boxes organised into \( k \) rows of lengths \( r_i, 1 \leq i \leq k \), let us take the corresponding Young symmetrizer \( \hat{Y}_i^\lambda \), and let us construct a linear subspace \( L_i^\lambda = \{ \hat{Y}_i^\lambda | \nu \} \), \( | \nu \in (C^d)^{\otimes i} \), consisting of tensors of the symmetry type \( \hat{Y}_i^\lambda \). The dimension of each space \( L_i^\lambda \) is specified by the so-called Weyl character formula:

\[ \text{dim}(L_i^\lambda) = D_L^i = \prod_{1 \leq \mu < \nu \leq k} \frac{r_\mu - r_\nu + \nu - \mu}{\nu - \mu}, \]

in which the product runs over all rows in the Young diagram of type \( i \);
- note that each subspace \( L_i^\lambda \) is invariant under the action of \( U(d)^{\otimes i} \), since: \( \{ U(d)^{\otimes i} \hat{Y}_i^\lambda | \nu \} = \{ \hat{Y}_i^\lambda | U(d)^{\otimes i} \nu \} \);  
- let us assume that \( \lambda = 1 \) corresponds to a normal Young tableau, that is a tableau with natural row-wise filling of integers from left to right. Let us denote an orthonormal basis of the subspace \( L_i^1 = \{ \hat{Y}_i^1 | \nu \} \) as \( \{|i, m, 1\} \}_{m=1}^{D_L^i} \). Since all other Young symmetrizers \( \hat{Y}_i^\lambda \) from \( i \)-th class can be obtained from \( \hat{Y}_i^1 \) by action of some permutation \( p_\lambda \), we have \( \hat{Y}_i^\lambda = O_{p_\lambda} \hat{Y}_i^1 \), and therefore orthonormal bases for other subspaces \( L_i^\lambda \) can be obtained from the basis of \( L_i^1 \):

\[ \{|i, m, \lambda\} \}_{m=1}^{D_L^i} = \{ O_{p_\lambda} |i, m, 1\} \}_{m=1}^{D_L^i}; \]

- in this way we have obtained orthonormal bases \( \{|i, m, \lambda\} \}_{m=1}^{D_L^i} \) for the subspaces \( L_i^\lambda \);
- on the other hand the subspaces \( V_n \) generated by sets of vectors \( \{|i, m, \lambda\} \lambda=1 \) are invariant under the action of the symmetric group \( S_i \); they are called Specht moduli; the dimension of
$V_m^i$ is equal to the number of inequivalent standard Young tableau’s of a fixed shape and is specified by the so-called hook-length formula:

$$\dim(V_m^i) = D_v^i = \frac{t!}{\prod_{\mu, \nu} b(\mu, \nu)}$$

in which the hook-length $h(\mu, \nu)$ corresponding to a box placed in $\mu$-th row and $\nu$-th column is defined as the number of boxes in a hook constructed by taking all the boxes to the right and to the bottom; the product is performed over hook lengths corresponding to all boxes in a tableau;

• although the vectors $\{|i, m, \lambda\}_{\lambda \in A}^{D_v^i}$ spanning the spaces $V_m^i$ are typically non-orthogonal, they can be orthogonalised, in such a way that each new vector belongs to the same symmetry type, and therefore to the same subspace $L_{\lambda_i}$; in this way we obtain an orthonormal basis for each of the symmetry type $i$, which can be presented as an array:

$$
\begin{align*}
L_1^i &: |i, 1, 1\rangle \quad |i, 2, 1\rangle \ldots |i, D_v^i, 1\rangle \\
L_2^i &: |i, 1, 2\rangle \quad |i, 2, 2\rangle \ldots |i, D_v^i, 2\rangle \\
\vdots & \\
L_{D_v^i}^i &: |i, 1, D_v^i\rangle \quad |i, 2, D_v^i\rangle \ldots |i, D_v^i, D_v^i\rangle \\
\end{align*}
$$

• the above arrangement of the Schur vector basis indicates the essence of Schur-Weyl duality: the basis vectors $|i, m, \lambda\rangle$ within each irreducible subspace $i$ can be seen as belonging to $D_v^i$ subspaces $L_{\lambda_i}$, which correspond to equivalent irreducible $D_v^i$-dimensional representations of the unitary group $U(d)$, or as belonging to $D_L^i$ subspaces $V_m^i$, which correspond to equivalent irreducible $D_L^i$-dimensional representations of the symmetric group $S_t$;

After introducing the construction of the Schur vector basis, lets come back to the block-diagonalisation of the action of unitary and symmetric group on the tensor product space $(\mathbb{C}^d)^{\otimes t}$. For this aim we introduce the following operator basis for the space $\text{End}(\mathbb{C}^d)^{\otimes t}$:

$$\hat{\Pi}^{m_1, m_2, \lambda_2}_{ij} = |i, m_1, \lambda_1\rangle \langle j, m_2, \lambda_2|.$$  \hspace{1cm} (A7)

The above operators fulfill the following property, which comes from orthonormality of the Schur vector basis (A6):

$$\hat{\Pi}^{m_1, m_1, \lambda_1}_{ij} \hat{\Pi}^{m_1, m_1, \lambda_1}_{ij} = \delta_{ij} \delta_{\lambda_1 \lambda_2} \delta_{m_1, m_2} \hat{\Pi}^{m_1, m_1, \lambda_1}_{ij}.$$ \hspace{1cm} (A8)

Note that due to the fact that operators (A7) are outer products of basis vectors, their Hermitian conjugate is equivalent to transposition of corresponding indices:

$$\hat{\Pi}^{m_1, m_2, \lambda_2}_{ij} = |j, m_2, \lambda_2\rangle \langle i, m_1, \lambda_1| = \hat{\Pi}^{m_1, m_1, \lambda_1}_{ij}.$$ \hspace{1cm} (A9)

Let us also introduce two other operators:

$$\hat{\Pi}^{\lambda_1, \lambda_2}_{ij} = \sum_{\lambda} \hat{\Pi}^{m_1, m_2, \lambda}_{ij},$$

$$\hat{\Pi}^{m_1, m_2}_{ij} = \sum_{\lambda} \hat{\Pi}^{m_1, m_2, \lambda}_{ij}.$$ \hspace{1cm} (A10)
We refer to the sets of operators \( \{ \hat{\Pi}_i^{\lambda_1 \lambda_2} \}_{\lambda_1 \lambda_2} \) and \( \{ \hat{\Pi}_i^{\nu \mu} \}_{\nu \mu} \) as Schur operator bases. They are especially important, since they represent the block-diagonalisation of the action of both groups on the tensor space, namely:

\[
U^{S\otimes t} = \sum_{i} \frac{1}{D_t} \sum_{m_1,m_2=1}^{D_t} \text{Tr} \left( U^{S\otimes t} \hat{\Pi}_i^{m_1 m_2 \dagger} \right) \hat{\Pi}_i^{m_1 m_2},
\]  
(A11)

and analogously:

\[
O_{p \in S_t} = \sum_{i} \frac{1}{D_t} \sum_{\lambda_1,\lambda_2=1}^{D_t} \text{Tr} \left( O_{p} \hat{\Pi}_i^{\lambda_1 \lambda_2 \dagger} \right) \hat{\Pi}_i^{\lambda_1 \lambda_2}.
\]  
(A12)

The normalisation factors come from the fact, that:

\[
\text{Tr}(\hat{\Pi}_i^{m_1 m_2}) = \delta_{m_1 m_2} D_t,
\]

\[
\text{Tr}(\hat{\Pi}_i^{\lambda_1 \lambda_2}) = \delta_{\lambda_1 \lambda_2} D_t.
\]  
(A13)

Both the formulas (A11) and (A12) are simple consequences of the construction of the Schur vector basis (A6), namely that the subspaces \( L_i \) correspond to equivalent irreducible representations of the unitary group, and that the subspaces \( V_m \) correspond to equivalent irreducible representations of the symmetric group. In order to see the block-diagonalisation in a matrix representation of the above operators, let us introduce two matrix bases in the space of operators \( \text{End}(\mathbb{C}^d)^{\otimes t} \):

- **U-basis**: for each irreducible subspace \( i \) we order the vectors in the table (A6) row-wise and then enumerate all such ordered vectors for all subspaces \( i \) by a single index \( \mu \); let us denote such obtained basis by \( \{ |e_\mu\rangle \}_{\mu=1}^{D_t} \); then arbitrary operator \( U^{S\otimes t} \) is block-diagonal in matrix basis \( \{ |e_\mu\rangle \langle e_\nu| \}_{\mu,\nu=1}^{D_t} \);

- **S-basis**: for each irreducible subspace \( i \) we order the vectors in the table (A6) column-wise and then enumerate all such ordered vectors for all subspaces \( i \) by a single index \( \mu \); let us denote such obtained basis by \( \{ |\hat{e}_\mu\rangle \}_{\mu=1}^{D_t} \); then arbitrary operator \( O_{p \in S_t} \) is block-diagonal in matrix basis \( \{ |\hat{e}_\mu\rangle \langle \hat{e}_\nu| \}_{\mu,\nu=1}^{D_t} \);

What is very important the block-diagonalisation of the tensor power (A11) holds as well for any invertible complex matrix \( \alpha \):

\[
\alpha^{S\otimes t} = \sum_{i} \frac{1}{D_t} \sum_{m_1,m_2=1}^{D_t} \text{Tr} \left( \alpha^{S\otimes t} \hat{\Pi}_i^{m_1 m_2 \dagger} \right) \hat{\Pi}_i^{m_1 m_2}.
\]  
(A14)

This is because the groups \( \text{GL}(d, \mathbb{C}) \) of complex invertible \( d \times d \) matrices and the symmetric group \( S_t \), treated as matrix algebras acting on the tensor space \( (\mathbb{C}^d)^{\otimes t} \), are also commutants of each other, and all the construction of Schur vector basis for their case is entirely the same. The only difference is that the subspaces \( L_i \) are now irreducibly invariant under the collective action of the general linear group \( \text{GL}(d, \mathbb{C}) \).

We can also define projectors onto the entire \( i \)th irreducible subspaces:

\[
\hat{\Pi}_i = \sum_{\lambda=1}^{D_t} \sum_{m=1}^{D_t} \hat{\Pi}_i^{m \lambda m \lambda}.
\]  
(A15)
They fulfill the following property:

\[
\hat{\Pi}_i = \sum_{m=1}^{\Omega_i} \hat{\Pi}_i^{mm} = \sum_{\lambda=1}^{\lambda_i} \hat{\Pi}_i^{\lambda\lambda}
\]

which follows directly from definition (A10).

**Lemma 6.** The operators (A10) fulfill the following important property, which will be extensively used in our considerations:

\[
\hat{\Pi}_k^{\lambda_1\lambda_2} \hat{\Pi}_l^{m_1m_2} = \delta_{kl} \hat{\Pi}_l^{m_1\lambda_1m_2\lambda_2}
\]

\[
\hat{\Pi}_j^{m_1m_2} \hat{\Pi}_k^{\lambda_1\lambda_2} = \delta_{kl} \hat{\Pi}_k^{m_1\lambda_1m_2\lambda_2}
\]

**(A17)**

**Proof.** Using definition (A10) and the orthogonality relations (A8) we obtain:

\[
\hat{\Pi}_k^{\lambda_1\lambda_2} \hat{\Pi}_l^{m_1m_2} = \sum_{\lambda,m} \hat{\Pi}_k^{\lambda m\lambda_1\lambda_2} \hat{\Pi}_l^{m_1m_2\lambda}
\]

\[
= \sum_{\lambda,m} \hat{\Pi}_l^{m_1m_2\lambda} \delta_{kl} \delta_{mm} \delta_{\lambda_1\lambda_2}
\]

\[
= \delta_{kl} \hat{\Pi}_l^{m_1\lambda_1m_2\lambda_2}
\]

**(A18)**

Similarly:

\[
\hat{\Pi}_j^{m_1m_2} \hat{\Pi}_k^{\lambda_1\lambda_2} = \sum_{m,\lambda} \hat{\Pi}_k^{m\lambda_1\lambda_2\lambda} \hat{\Pi}_l^{m_1m_2\lambda}
\]

\[
= \sum_{m,\lambda} \hat{\Pi}_l^{m_1m_2\lambda} \delta_{kl} \delta_{mm} \delta_{\lambda_1\lambda_2}
\]

\[
= \delta_{kl} \hat{\Pi}_k^{m_1\lambda_1m_2\lambda_2}
\]

**(A19)**

As a corollary we obtain the following three properties:

1. The operators (A10) commute:

\[
\left[ \hat{\Pi}_k^{\lambda_1\lambda_2}, \hat{\Pi}_l^{m_1m_2} \right] = 0
\]

**(A20)**

2. The product of the operators for the same irreducible subspace reads:

\[
\hat{\Pi}_l^{\lambda_1\lambda_2} \hat{\Pi}_l^{m_1m_2} = \hat{\Pi}_l^{m_1\lambda_1m_2\lambda_2}
\]

**(A21)**

3. Trace of their product reads:

\[
\text{Tr} \left( \hat{\Pi}_k^{\lambda_1\lambda_2} \hat{\Pi}_l^{m_1m_2} \right) = \delta_{kl} \delta_{\lambda_1\lambda_2} \delta_{m_1m_2}
\]

**(A22)**

The properties (A20) and (A21) are obvious consequences of (A17), the last one follows from (A17) and orthonormality of basis elements in the definition (A7).

As a last remark on the discussed operators let us note that the operators \(\hat{\Pi}_i\) play the role of the identity operator on the irreducible subspaces:

\[
\hat{\Pi}_i^{\lambda_1\lambda_2} \hat{\Pi}_i = \hat{\Pi}_i \hat{\Pi}_i^{\lambda_1\lambda_2} = \delta_{\lambda_1\lambda_2} \hat{\Pi}_i^{\lambda_1\lambda_2}
\]

\[
\hat{\Pi}_j^{m_1m_2} \hat{\Pi}_i = \hat{\Pi}_i \hat{\Pi}_j^{m_1m_2} = \delta_{m_1m_2} \hat{\Pi}_i^{m_1m_2}
\]

**(A23)**
The proof of the first property goes as follows, all the rest are entirely analogous:

\[ \hat{\Pi}^{\lambda_1 \lambda_2} \hat{\Pi}_i = \sum_m \hat{\Pi}^{\lambda_1 \lambda_2}_m \hat{\Pi}^{m \pi}_{i} = \delta_{ij} \sum_m \hat{\Pi}^{\lambda_1 \lambda_2}_m = \delta_{ij} \hat{\Pi}^{\lambda_1 \lambda_2}_i. \]  

(A24)

Decompositions (A11) and (A12) are often in the literature presented in the virtual tensor product form. This form is useful in the context of application of Schur’s Lemma to averaging, therefore let us for completeness briefly recall it. Let us fix \( i \)th irreducible subspace. Let \( \mathcal{I}_i \) denote trivial immersion of the matrices defined on the \( i \)th subspace into entire space of matrices on \( \mathbb{C}^d \). Let \( \{ \pi_i^{m \pi} \} \) be a standard matrix basis on the space of \( D^I_i \times D^I_i \) matrices. Then the operator \( \hat{\Pi}^{m \pi}_{i} \) has the following representation in terms of the matrices \( \pi_i^{m \pi} \):

\[
\text{U-basis: } \hat{\Pi}^{m \pi}_{i} = \mathcal{I}_i \left( \mathbf{1}_{D^I_i} \otimes \pi_i^{m \pi} \right) \\
\text{S-basis: } \hat{\Pi}^{m \pi}_{i} = \mathcal{I}_i \left( \pi_i^{m \pi} \otimes \mathbf{1}_{D^I_i} \right).
\]  

(A25)

Similarly for any operator \( \hat{\Pi}^{\lambda_1 \lambda_2} \) we have:

\[
\text{U-basis: } \hat{\Pi}^{\lambda_1 \lambda_2} = \mathcal{I}_i \left( \pi_i^{\lambda_1 \lambda_2} \otimes \mathbf{1}_{D^I_i} \right) \\
\text{S-basis: } \hat{\Pi}^{\lambda_1 \lambda_2} = \mathcal{I}_i \left( \mathbf{1}_{D^I_i} \otimes \pi_i^{\lambda_1 \lambda_2} \right),
\]  

(A26)

in which the operators \( \{ \pi_i^{\lambda_1 \lambda_2} \} \) are elements of the standard matrix basis on the space of \( D^I_i \times D^I_i \) matrices. Both sets of formulas (A25) and (A26) can be easily obtained by formally treating the elements of Schur basis \( |i, m, \lambda \rangle \) for a given fixed irreducible subspace \( i \) as elementary tensors \( |m\rangle_i \otimes |\lambda\rangle_i \), in which the vectors \( \{ m \} \) and \( \{ \lambda \} \) span virtual subspaces of dimensions \( D^I_i \) and \( D^I_i \) respectively. In this formulation the joint action of the unitary and symmetric group on the space \( \mathbb{C}^d \otimes \mathbb{C}^d \) can be represented as:

\[
\text{U-basis: } \bigoplus_i (O_p)_i \otimes U_i \\
\text{S-basis: } \bigoplus_i U_i \otimes (O_p)_i,
\]  

(A27)

in which the matrix elements of operators \( (O_p)_i \) and \( U_i \) are defined as:

\[
[(O_p)_i]_{\lambda_1 \lambda_2} = \text{Tr} \left( O_p \hat{\Pi}^{\lambda_1 \lambda_2}_i \right) \\
[U_i]_{m \pi} = \text{Tr} \left( U^{\pi} \hat{\Pi}^{m \pi}_{i} \right).
\]  

(A28)

A.2. Schur’s Lemma and its applications for averaging

Let us start from a more abstract formulation of the Schur’s Lemma. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two complex vector spaces, let \( \pi_{\mathcal{H}_1} \) and \( \pi_{\mathcal{H}_2} \) be two representations of some group \( G \) on these spaces and let \( f : \mathcal{H}_1 \to \mathcal{H}_2 \) be a \( G \)-equivariant map between the representation spaces, which means that the map commutes with the action of the group:

\[
\pi_{\mathcal{H}_2} \circ f = f \circ \pi_{\mathcal{H}_1}.
\]  

(A29)

Then the following two properties hold (Schur’s Lemmas):

**Lemma 7.** If \( \pi_{\mathcal{H}_1} \) and \( \pi_{\mathcal{H}_2} \) are irreducible representations of \( G \) then:

- if \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are non-isomorphic, then the only \( G \)-equivariant map is zero, \( f \equiv 0; \)
• if $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic and if $\pi_{\mathcal{H}_1}$ and $\pi_{\mathcal{H}_2}$ are equivalent, then the only $G$-equivariant maps are multiples of identity, $f \equiv c \mathbf{1}_{\mathcal{H}_1}$.

It is worth noticing that the second part of the above lemma holds only for representations defined on vector spaces over algebraically closed fields, as $C$.

Let us now specify $\mathcal{H}_1$ and $\mathcal{H}_2$ as $C^n$ and $C^m$ respectively and let $G = U(d)$. Then the maps $f$ are linear operators $A : C^n \to C^m$, the representations $\pi_{\mathcal{H}_1}$ and $\pi_{\mathcal{H}_2}$ are irreducible representations of the unitary group on respectively $C^n$ and $C^m$ consisting of matrices $U_1$ and $U_2$, and the $G$-equivariance means:

\[ U_1 A = AU_2 \iff U_1 A (U_2)^{-1} = A \iff U_1 A (U_2)^{\dagger} = A. \] (A30)

For discussed situation Schur Lemma’s take the following form:

**Lemma 8.** If $U_1$ and $U_2$ are irreducible representations of $U(d)$ on $C^n$ and $C^m$ then:

• for $t_1 \neq t_2$ the following implication holds:

\[ U_1 A (U_2)^\dagger = A \implies A \equiv 0; \] (A31)

• for $t_1 = t_2$ and for equivalent $U_1$ and $U_2$, the following implication holds:

\[ U_1 A (U_2)^\dagger = A \implies A \equiv c \mathbf{1}. \] (A32)

We will now apply the above version of Schur’s Lemma to averaging over unitary group. We obtain the following two further lemmas:

**Lemma 9.** Let $\{U_k\}$ be a family of non-equivalent irreducible representations of the unitary group on some complex vector spaces $\mathcal{H}_k$ and let $\mathcal{H} = \bigoplus_k \mathcal{H}_k$. Let $\rho \in \mathcal{H}$ be arbitrary linear operator on $\mathcal{H}$ and let $\mathcal{I}_k$ be trivial immersions of operators on $\mathcal{H}_k$ into operators on $\mathcal{H}$. Then the averaging of $\rho$ over representations $\{U_k\}$ is diagonal in the index $k$, namely:

\[ \sum_k \int \mathcal{I}_k(U_k) \rho \mathcal{I}_k(U_k^\dagger) \, dU = \sum_k \int \mathcal{I}_k(U_k) \rho \mathcal{I}_k(U_k^\dagger) \, dU. \] (A33)

**Proof.** Let $\mathcal{U}_k(\rho) = \int \mathcal{I}_k(U_k) \rho \mathcal{I}_k(U_k^\dagger) \, dU$. Due to the invariance of Haar measure on unitary group the operator $\mathcal{U}_k(\rho)$ is invariant with respect to left and right action by the unitaries corresponding to respective irreducible representations:

\[ \mathcal{I}_k(\hat{U}_k) \mathcal{U}_k(\rho) \mathcal{I}_k(\hat{U}_k^\dagger) = \int \mathcal{I}_k(\hat{U}_k U_k) \rho \mathcal{I}_k(U_k^\dagger) \, dU = \mathcal{U}_k(\rho). \] (A34)

Since by assumption irreps corresponding to different values of $k$ and $l$ are non-equivalent, from (A31) we obtain that $\mathcal{U}_k(\rho) = 0$ for $k \neq l$, hence the sum in (A33) contains only terms diagonal in irreps indices.

**Lemma 10.** Let $U_k$ be an irreducible representation of the unitary group on some complex vector space $\mathcal{H}_k$ of dimension $D_k$. Let $\rho \in \mathcal{H}_k$ be arbitrary linear operator on $\mathcal{H}_k$. Then the effect of averaging of $\rho$ over representation $U_k$ is proportional to the identity operator on $\mathcal{H}_k$:

\[ \int U_k \rho U_k^\dagger \, dU = \frac{1}{D_k} \text{Tr}(\rho) \mathbf{1}_{\mathcal{H}_k}. \] (A35)
Proof. Let $U_k(\rho) = \int U_k \rho U_k^\dagger \, dU$. Analogously to the previous case due to the invariance of Haar measure on unitary group the operator $U_k(\rho)$ is invariant with respect to left and right action by the unitaries corresponding to respective irreducible representations:

$$U_k U_k(\rho) U_k^\dagger = \int U_k U_k \rho U_k^\dagger dU = U_k(\rho).$$

Therefore according to Schur’s Lemma (A32):

$$U_k(\rho) = c \mathbb{1}_{\mathcal{H}_k}.$$  \hfill (A36)

In order to find the constant $c$ we calculate the trace of both sides of this property:

$$\text{Tr}(U_k(\rho)) = \text{Tr}(c \mathbb{1}_{\mathcal{H}_k})$$

$$\text{Tr}\left(\int U_k \rho U_k^\dagger dU\right) = c \text{Tr}(\mathbb{1}_{\mathcal{H}_k})$$

$$\int \text{Tr}\left(U_k \rho U_k^\dagger\right) \, dU = c D^k_{\mathcal{H}}$$

$$\int \text{Tr}(\rho) \, dU = c D^k_{\mathcal{H}}$$

$$\text{Tr}(\rho) = c D^k_{\mathcal{H}},$$

hence $c = \text{Tr}(\rho)/D^k_{\mathcal{H}}$. \hfill \blacksquare

A.3. Reduction of the unitary twirling map

The unitary twirling map:

$$\mathcal{T}(\rho) = \int U^{\otimes t} \rho U^{\otimes t} \, dU, \quad U \in U(d),$$

where $dU$ denotes the normalised Haar measure on $U(d)$, can be presented in a simple closed form using the decomposition into irreducible representations specified in (A11) and Schur’s Lemmas. Let us first introduce concise notation for this decomposition:

$$U^{\otimes t} = \sum_k \frac{1}{D^k} \sum_{m_1, m_2=1}^{D^k} \text{Tr}\left(U^{\otimes t} \Pi_{k m_1 m_2}^{m_1 m_2}\right) \Pi_{k m_1 m_2}^{m_1 m_2},$$

$$= \frac{1}{D^k} U_{m_1 m_2}^{k} \Pi_{k m_1 m_2}^{m_1 m_2},$$

in which we have $U_{m_1 m_2}^{k} = \text{Tr}(U^{\otimes t} \Pi_{k m_1 m_2}^{m_1 m_2})$ and we use the Einstein summation convention for the same indices appearing at opposite positions. In a similar way we can decompose arbitrary state $\rho$ in the Schur operator basis (A7):

$$\rho = \rho_{m_1, m_2}^{n_1, n_2, \lambda_1, \lambda_2} \Pi_{m_1 m_2}^{n_1 n_2},$$

in which the summation convention also holds. Let us now rewrite the twirling map (A38) in Schur operator basis:

$$\mathcal{T}(\rho) = \int \frac{1}{D^k} U_{m_1 m_2}^{k} \Pi_{k m_1 m_2}^{m_1 m_2} \left(\rho_{m_1, m_2, n_1, n_2, \lambda_1, \lambda_2} \Pi_{m_1 m_2}^{n_1 n_2}\right) \left(\frac{1}{D^r} (U^t)^{ij}_{n_1 n_2} \Pi_{r}^{ij}\right) \, dU,$$

in which $(U^t)^{ij}_{n_1 n_2} = \text{Tr}\left((U^t)^{\otimes t} \Pi_{r}^{ij}\right)$. The expression (A41) can be simplified in three steps:
application of Schur’s Lemma for inequivalent irreducible representations (A33), due to which averaging in (A41) is block-diagonal:

$$\mathcal{T}(\rho) = \int \left( \frac{1}{D^V_r} T^k_{m_1 m_2} \hat{\Pi}^m_{k} \right) \left( \rho^k_{n_1, n_2, \lambda_k} \hat{\Pi}^n_{k} \right) \left( \frac{1}{D^V_r} (U^\dagger)^k_{r_1 r_2} \hat{\Pi}^{r_1 r_2}_{k} \right) dU;$$  \hspace{1cm} (A42)

note that in the above equation the summation convention is applied in a way that the summation indices are unbounded and common for the entire expression; namely the transition from (A41) to (A42) relies on changing double sum $\sum_kl$ over the indices $k, l$ corresponding to in principle different irreducible subspaces into a single sum $\sum_k$ over the same irreducible subspaces for left and right action of the unitary group; in the following formulas we treat summation convention in the same way;

- utilising block-orthogonality of the corresponding operator bases:

$$\hat{\Pi}^m_{k} = \hat{\Pi}^n_{k},$$  \hspace{1cm} (A43)

which follows from orthogonality relations (A8) and (A21); due to this property we have:

$$\mathcal{T}(\rho) = \int \left( \frac{1}{D^V_r} U^k_{m_1 m_2} \hat{\Pi}^m_{k} \right) \left( \rho^k_{n_1, n_2, \lambda_k} \hat{\Pi}^n_{k} \right) \left( \frac{1}{D^V_r} (U^\dagger)^k_{r_1 r_2} \hat{\Pi}^{r_1 r_2}_{k} \right) dU = \rho^k_{n_1, n_2, \lambda_k} \int \left( \frac{1}{D^V_r} T^k_{m_1 m_2} \hat{\Pi}^m_{k} \right) \hat{\Pi}^n_{k} \left( \frac{1}{D^V_r} (U^\dagger)^k_{r_1 r_2} \hat{\Pi}^{r_1 r_2}_{k} \right) dU, \hspace{1cm} (A44)

in which the equality follows from commutativity of the operators $\hat{\Pi}^1_{\lambda_1 \lambda_2}$ and $\hat{\Pi}^n_{k}$;

- application of Schur’s Lemma for averaging over equivalent irreducible representations (A35), due to which the integral reads:

$$\int \left( \frac{1}{D^V_r} U^k_{m_1 m_2} \hat{\Pi}^m_{k} \right) \hat{\Pi}^n_{k} \left( \frac{1}{D^V_r} (U^\dagger)^k_{r_1 r_2} \hat{\Pi}^{r_1 r_2}_{k} \right) dU = \frac{1}{D^L_r} \rho^k_{n_1, n_2, \lambda_k} \hat{\Pi}^{n_1 n_2}_{k}, \hspace{1cm} (A45)

in which the tensor $\rho^k_{n_1, n_2}$ represents a Kronecker delta on the $k$th irreducible subspace, and therefore we have:

$$\mathcal{T}(\rho) = \frac{1}{D^L_r} \rho^k_{n_1, n_2, \lambda_k} \hat{\Pi}^{n_1 n_2}_{k} \hat{\Pi}^{1 \lambda_k}_{k} = \frac{1}{D^L_r} \rho^k_{\lambda_1, \lambda_2} \hat{\Pi}^{1 \lambda_k}_{k}, \hspace{1cm} (A46)

in which we used simplified notation: $\rho^k_{\lambda_1, \lambda_2} = Tr(\rho \hat{\Pi}^{1 \lambda_k}_{k})$ and used the fact that $\hat{\Pi}^1_{k}$ plays the role of an identity on the $k$th subspace, see (A23); the last equality can be easily proved by the following chain of identities:

$$\rho^k_{n_1, n_2, \lambda_k} = \sum_{n_1, n_2 = 1}^{d^k} \rho^k_{n_1, n_2, \lambda_k} \hat{\Pi}^{n_1 n_2}_{k} \hat{\Pi}^{1 \lambda_k}_{k} = \sum_{n = 1}^{d^k} \rho^k_{n, \lambda_k} \hat{\Pi}^{1 \lambda_k}_{k}, \hspace{1cm} (A47)

in which in the last step we used definition of the $\hat{\Pi}^{1 \lambda_k}_{k}$ operators (A10).
To sum up, the final form of the twirling map is given by the following expression:

$$\mathcal{T}(\rho) = \sum_k \frac{1}{D_k^2} \sum_{\lambda, \lambda_2=1}^{D_k^2} \text{Tr}(\rho \hat{\Pi}_k^{\lambda_1 \lambda_2}) \hat{\Pi}_k^{\lambda_1 \lambda_2}. \quad (A48)$$

It is equivalent to a permutation operator \((A12)\). The map \(\mathcal{T}(\rho)\) is idempotent, \(\mathcal{T}(\mathcal{T}(\rho)) = \mathcal{T}(\rho)\) (see below for a proof), hence a state of the block-diagonal form:

$$\rho_U = \sum_k \frac{1}{D_k^2} \sum_{\lambda, \lambda_2=1}^{D_k^2} \rho_k^{\lambda_1 \lambda_2} \hat{\Pi}_k^{\lambda_1 \lambda_2} \quad (A49)$$

is invariant under unitary twirling operation: \(\mathcal{T}(\rho_U) = \rho_U\). The \((D_k^1 \times D_k^2)\)-dimensional subsystems spanned by \(\{\hat{\Pi}_k^{\lambda_1 \lambda_2}\}\) are called noiseless subsystems or decoherence-free subsystems. The idempotence of the unitary twirling map follows simply from the invariance of the map with respect to the action of the unitary operations:

$$\tilde{U}_k \mathcal{T}(\rho) \tilde{U}_k^\dagger = \int \tilde{U}_k U \rho U^\dagger \tilde{U}_k^\dagger dU = \mathcal{T}(\rho). \quad (A50)$$

### A.4. Reduction of the symmetric twirling map

Derivation of a closed form \((18)\) of a symmetric twirling map \((16)\) is performed in full analogy with the one for the unitary twirling map. First, note that Schur’s Lemmas 9 and 10 for unitary averaging hold in entirely analogous form for symmetric averaging:

**Lemma 11.** Let \(\{O_k\}\) be a family of non-equivalent irreducible representations of the symmetric group \(S_n\) on some complex vector spaces \(\mathcal{H}_k\) and let \(\mathcal{H} = \bigoplus \mathcal{H}_k\). Let \(\rho \in \mathcal{H}\) be arbitrary linear operator on \(\mathcal{H}\) and let \(I_k\) be trivial immersions of operators on \(\mathcal{H}_k\) into operators on \(\mathcal{H}\). Then the averaging of \(\rho\) over representations \(\{O_k\}\) is diagonal in the index \(k\), namely:

$$\sum_k \left( \frac{1}{n} \right) \mathcal{I}_k ((O_p)_{\lambda_1}) \rho \mathcal{I}_l ((O_p)_{\lambda_2}) = \sum_k \left( \frac{1}{n} \right) \mathcal{I}_k ((O_p)_{\lambda_1}) \rho \mathcal{I}_l ((O_p)_{\lambda_2}), \quad (A51)$$

in which by \((O_p)_{\lambda}\) we mean the value of the \(k\)th irreducible representation when applied to a permutation \(p\).

**Proof.** Let \(O_{kl}(\rho) = \sum_p \left( \frac{1}{n} \right) \mathcal{I}_k ((O_p)_{\lambda_1}) \rho \mathcal{I}_l ((O_p)_{\lambda_2})\). \(O_{kl}(\rho)\) is invariant with respect to left and right action by the permutations corresponding to respective irreducible representations:

$$\mathcal{I}_k ((O_q)_{\lambda_1}) O_{kl}(\rho) \mathcal{I}_l ((O_q)_{\lambda_2}) = \sum_p \left( \frac{1}{n} \right) \mathcal{I}_k ((O_q)_{\lambda_1}) (O_p)_{\lambda_1} \rho \mathcal{I}_l ((O_q)_{\lambda_2})$$

$$= \sum_p \left( \frac{1}{n} \right) \mathcal{I}_k ((O_{qp})_{\lambda_1}) \rho \mathcal{I}_l ((O_{qp})_{\lambda_2})$$

$$= \sum_p \left( \frac{1}{n} \right) \mathcal{I}_k ((O_{qp})_{\lambda_1}) \rho \mathcal{I}_l ((O_{qp})_{\lambda_2})$$

$$= \sum_r \left( \frac{1}{n} \right) \mathcal{I}_k ((O_r)_{\lambda_1}) \rho \mathcal{I}_l ((O_r)_{\lambda_2}) = O_{kl}(\rho). \quad (A52)$$

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Since by assumption irreps corresponding to different values of \( k \) and \( l \) are non-equivalent, from (A31) we obtain that \( O_{kl}(\rho) = 0 \) for \( k \neq l \), hence the sum in (A33) contains only terms diagonal in irreps indices.

**Lemma 12.** Let \( O_k \) be an irreducible representation of the symmetric group \( S_t \) on some complex vector space \( \mathcal{H}_k \) of dimension \( D_k^t \). Let \( \rho \in \mathcal{H}_k \) be arbitrary linear operator on \( \mathcal{H}_k \). Then the effect of averaging of \( \rho \) over representation \( O_k \) is proportional to the identity operator on \( \mathcal{H}_k \):

\[
\sum_p \left( \frac{1}{D_k^t} \right) (O_p)_k \rho (O_p)_k = \frac{1}{D_k^t} \text{Tr}(\rho) \mathbb{1}_{\mathcal{H}_k}.
\]

**(Proof.** Let \( O_k(\rho) = \sum_p \left( \frac{1}{D_k^t} \right) (O_p)_k \rho (O_p)_k \). Analogously to the previous case the operator \( O_k(\rho) \) is invariant with respect to left and right action by the permutations corresponding to respective irreducible representations:

\[
(O_q)_k O_k(\rho)(O_q)_k = \sum_p \left( \frac{1}{D_k^t} \right) (O_q)_k (O_p)_k \rho (O_p)_k (O_q)_k
\]

\[
= \sum_p \left( \frac{1}{D_k^t} \right) (O_{qp})_k \rho (O_{qp})_k = O_k(\rho).
\]

Therefore according to Schur’s Lemma (A32):

\[
O_k(\rho) = c \mathbb{1}_{\mathcal{H}_k}.
\]

In order to find the constant \( c \) we calculate the trace of both sides of this property:

\[
\text{Tr}(O_k(\rho)) = \text{Tr}(c \mathbb{1}_{\mathcal{H}_k})
\]

\[
\sum_p \left( \frac{1}{D_k^t} \right) \text{Tr}(O_p)_k \rho (O_p)_k = c \text{Tr}(\mathbb{1}_{\mathcal{H}_k})
\]

\[
\sum_p \left( \frac{1}{D_k^t} \right) \text{Tr}(O_p)_k \rho (O_p)_k = c D_k^t
\]

\[
\text{Tr}(\rho) = c D_k^t,
\]

hence \( c = \text{Tr}(\rho)/D_k^t \).)

All the steps of the derivation of closed form of symmetric twirling map are analogous to the ones for unitary twirling, we however present them below for completeness. Utilising the relation (expressed for convenience in Einstein summation convention):

\[
O_p = \sum_k \frac{1}{D_k^t} \sum_{\lambda_1, \lambda_2=1} D_k^t \text{Tr}(O_p^{\lambda_1 \lambda_2}) \Pi_k^{\lambda_1 \lambda_2}
\]

\[
= \frac{1}{D_k^t} (O_p)^{\lambda_1 \lambda_2} \bar{\Pi}_k^{\lambda_1 \lambda_2}.
\]
we express the symmetric twirling map in full analogy with the expression (A41) for unitary twirling:

\[
\mathcal{T}_{\text{sym}}(\rho) = \sum_p \left( \frac{1}{n} \right) \left( \frac{1}{D_L} (O_p)_{\mu_1 \mu_2}^{k} \hat{\Pi}_k^{\mu_1 \mu_2} \right) \left( \rho_{\lambda_1 \lambda_2}^{n_1 n_2} \hat{\Pi}^{\lambda_1 \lambda_2}_{n_1 n_2} \right) \left( \frac{1}{D_L} (O^T_p)_{\eta_1 \eta_2}^{k} \hat{\Pi}^{\eta_1 \eta_2}_{k} \right).
\]

(A56)

The expression (A56) is simplified analogously:

- application of Schur’s Lemma for inequivalent irreducible representations (A51), due to which averaging in (A56) is block-diagonal:

\[
\mathcal{T}_{\text{sym}}(\rho) = \sum_p \left( \frac{1}{n} \right) \left( \frac{1}{D_L} (O_p)_{\mu_1 \mu_2}^{k} \hat{\Pi}_k^{\mu_1 \mu_2} \right) \left( \rho_{\lambda_1 \lambda_2}^{n_1 n_2} \hat{\Pi}^{\lambda_1 \lambda_2}_{n_1 n_2} \right) \left( \frac{1}{D_L} (O^T_p)_{\eta_1 \eta_2}^{k} \hat{\Pi}^{\eta_1 \eta_2}_{k} \right);
\]

(A57)

as before the entire above expression is a single sum \( \sum_n \) over the same irreducible subspaces for left and right action of the symmetric group;

- utilising block-orthogonality of the corresponding operator bases:

\[
\hat{\Pi}_k^{\mu_1 \mu_2} \hat{\Pi}_{n_1 n_2}^{\lambda_1 \lambda_2} \hat{\Pi}_{n_1 n_2}^{\eta_1 \eta_2} = \hat{\Pi}_k^{\mu_1 \mu_2} \hat{\Pi}_{n_1 n_2}^{\lambda_1 \lambda_2} \hat{\Pi}_{n_1 n_2}^{\eta_1 \eta_2},
\]

(A58)

which follows from orthogonality relations (A8) and (A21); due to this property we have:

\[
\mathcal{T}_{\text{sym}}(\rho) = \sum_p \left( \frac{1}{n} \right) \left( \frac{1}{D_L} (O_p)_{\mu_1 \mu_2}^{k} \hat{\Pi}_k^{\mu_1 \mu_2} \right) \left( \rho_{\lambda_1 \lambda_2}^{n_1 n_2} \hat{\Pi}^{\lambda_1 \lambda_2}_{n_1 n_2} \right) \left( \frac{1}{D_L} (O^T_p)_{\eta_1 \eta_2}^{k} \hat{\Pi}^{\eta_1 \eta_2}_{k} \right)
\]

\[

\rho_{\lambda_1 \lambda_2}^{n_1 n_2} \hat{\Pi}^{\lambda_1 \lambda_2}_{n_1 n_2} \hat{\Pi}^{\eta_1 \eta_2}_{n_1 n_2}.
\]

(A59)

in which the equality follows from commutativity of the operators \( \hat{\Pi}_k^{\mu_1 \mu_2} \) and \( \hat{\Pi}_k^{\eta_1 \eta_2} \); here is the crucial difference with respect to derivation of the unitary twirling: the averaging has been reduced to averaging of Schur operator basis elements spanning the irreducible representations of the symmetric group;

- application of Schur’s Lemma for averaging over equivalent irreducible representations (A53), due to which the sum over all permutations \( p \in S_n \) reads:

\[
\sum_p \left( \frac{1}{n} \right) \left( \frac{1}{D_L} (O_p)_{\mu_1 \mu_2}^{k} \hat{\Pi}_k^{\mu_1 \mu_2} \right) \hat{\Pi}^{\lambda_1 \lambda_2}_{k} \left( \frac{1}{D_L} (O^T_p)_{\eta_1 \eta_2}^{k} \hat{\Pi}^{\eta_1 \eta_2}_{k} \right) = \frac{1}{D_L} \hat{\Pi}_k \delta^{\lambda_1 \lambda_2},
\]

(A60)

therefore we finally have:

\[
\mathcal{T}_{\text{sym}}(\rho) = \frac{1}{D_L} \delta^{\lambda_1 \lambda_2} \hat{\Pi}^{\lambda_1 \lambda_2}_{k} \hat{\Pi}_k \hat{\Pi}^{\eta_1 \eta_2}_{k} = \frac{1}{D_L} \rho^{\lambda_1 \lambda_2} \hat{\Pi}^{\eta_1 \eta_2}_{k} = \sum_k \frac{1}{D_L} \sum_{n_1, n_2 = 1} \text{Tr} \left( \rho \hat{\Pi}^{\eta_1 \eta_2}_{k} \right) \hat{\Pi}^{\eta_1 \eta_2}_{k}.
\]

(A61)
in which we used simplified notation: 
\[ \rho_{n_1,n_2}^k = \text{Tr}(\rho \hat{\Pi}_{n_1,n_2}^{k\dagger}) \] and used the fact that \( \hat{\Pi}_k \) plays the role of an identity on the \( k \)-th subspace, see (A23); the last equality can be easily proved by the following chain of identities:

\[
\delta_{\lambda_1,\lambda_2} \rho_{\lambda_1,\lambda_2}^k = \sum_{\lambda_1,\lambda_2} \delta_{\lambda_1,\lambda_2} \text{Tr}(\rho \hat{\Pi}_{\lambda_1,\lambda_2}^{k\dagger}) = \sum_{\lambda_1} \text{Tr}(\rho \hat{\Pi}_{\lambda_1,\lambda_2}^{k\dagger}) = \text{Tr}(\rho \hat{\Pi}_{\lambda_1,\lambda_2}^{k\dagger}), \tag{A62}
\]

in which in the last step we used definition of the \( \hat{\Pi}_{n_1,n_2}^k \) operators (A10).

The symmetric twirling operation (A61) is also idempotent (due to invariance of the map with respect to the action of the permutation operators), and therefore the states of the form:

\[
\rho_S = \sum_k \frac{1}{d_k^2} \sum_{m_1,m_2} \rho_{m_1,m_2}^k \hat{\Pi}_{m_1,m_2}^{k\dagger}, \tag{A63}
\]

are untouched by this map: \( T_{\text{sym}}(\rho_S) = \rho_S \), and operators \( \hat{\Pi}_{m_1,m_2}^{k\dagger} \) span noiseless subsystems with respect to symmetric twirling.

**Appendix B. Some proofs connected with SLOCC twirling map**

**B.1. Proof of lemma 1**

Here we provide a direct proof that the SLOCC map \( S(\rho) \) (21) is trace non-increasing.

\[
\text{Tr}(S(\rho)) = \text{Tr} \left( \bigotimes_{i=1}^l L_i \rho \bigotimes_{i=1}^l L_i^\dagger \right) = \text{Tr} \left( \bigotimes_{i=1}^l L_i^\dagger \bigotimes_{i=1}^l L_i \rho \right) = \text{Tr} \left( \left( \bigotimes_{i=1}^l L_i^\dagger L_i \right) \rho \right), \tag{B1}
\]

in which \( L_i \) denotes normalised special linear matrix, \( L_i = M_i/\|M_i\| \) for \( M_i \in \text{SL}(d,\mathbb{C}) \). Due to the Cartan decomposition (which is in the case of \( \text{SL}(d,\mathbb{C}) \) equivalent to the SVD-decomposition) each matrix \( M_i \) can be expressed as a product \( K_i A_i K_i^\dagger \), where \( K_i, K_i^\dagger \) are unitary and \( A_i \) is diagonal with its largest entry denoted \( x_i \) [31]. Moreover we have \( \|A_i\| = x_i \), which implies, that:

\[
L_i^\dagger L_i = \frac{M_i^\dagger M_i}{\|M_i\|^2} = \frac{1}{x_i^2} K_i^\dagger A_i^2 K_i.
\]

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Further we have:

\[
\text{Tr}(S(\rho)) = \text{Tr} \left( \left( \bigotimes_{i=1}^t \frac{1}{X_i} A_i^d K_i A_i^\dagger K_i \right) \rho \right) = \text{Tr} \left( \left( \bigotimes_{i=1}^t A_i^d \right) X_i^{-2} \rho \right) \prod_{i=1}^t X_i^{t-2},
\]

in which a unitarily evolved state \( \hat{\rho} \) reads:

\[
\hat{\rho} = \bigotimes_{i=1}^t K_i \rho \bigotimes_{i=1}^t K_i^\dagger.
\]

The matrix \( A = \bigotimes_{i=1}^t A_i^d \) is diagonal with \( \|A\| = \prod_{i=1}^t \lambda_i^2 \). Therefore:

\[
\text{Tr} \left( \left( \bigotimes_{i=1}^t A_i^d \right) \hat{\rho} \right) \prod_{i=1}^t X_i^{t-2} = \sum_{i=1}^t \frac{A_{ii}}{\|A\|} \hat{\rho}_{ii}.
\]

Knowing that \( A_{ii}/\|A\| \leq 1 \), and from the fact that each positive matrix \( \rho \) has non-negative elements on the diagonal we obtain:

\[
\text{Tr}(S(\rho)) = \sum_{i=1}^t \frac{A_{ii}}{\|A\|} \hat{\rho}_{ii} \leq \sum_{i=1}^t \hat{\rho}_{ii} = \text{Tr}(\hat{\rho}) = \text{Tr}(\rho).
\]

**Remark.** Note that the above proof does not depend on the internal structure of the matrices \( A_i \) apart from the assumption that they are diagonal matrices, which holds also in the case of a Cartan decomposition of any reductive Lie group (matrices \( A \) represent maximal abelian subgroup, therefore we can always choose a basis in which the representation is diagonal). Therefore the above proof holds also for the case of general stochastic operation defined in formula (66).

**B.2. Proof of lemma 4**

Inserting the decomposition of \( \alpha^{\otimes t} \) (36) into the general formula for unitary twirling (17) we obtain:

\[
T(\alpha^{\otimes t}) = \sum_k \frac{1}{D^k} \sum_{\lambda_1, \lambda_2} \text{Tr} \left( \alpha^{\otimes t} \hat{\Pi}_k^{\lambda_1 \lambda_2} \right) \hat{\Pi}_k^{\lambda_1 \lambda_2}
\]

\[
= \sum_k \frac{1}{D^k} \sum_{\lambda_1, \lambda_2} \text{Tr} \left( \sum_{l,m_1, m_2} \frac{1}{D^l} \alpha^{l \mu_{m_1 m_2} \lambda_1 \lambda_2} \hat{\Pi}_l^{\lambda_1 \lambda_2} \hat{\Pi}_k^{\lambda_1 \lambda_2} \right)
\]

\[
= \sum_k \frac{1}{D^k} \sum_{\lambda_1, \lambda_2} \sum_{l,m_1, m_2} \frac{1}{D^l} \alpha^{l \mu_{m_1 m_2}} \text{Tr} \left( \hat{\Pi}_l^{\lambda_1 \lambda_2} \hat{\Pi}_k^{\lambda_1 \lambda_2} \right)
\]

\[
= \sum_k \frac{1}{D^k} \sum_{\lambda_1, \lambda_2} \sum_{l,m_1, m_2} \frac{1}{D^l} \alpha^{l \mu_{m_1 m_2}} \delta_{l \lambda_1 \lambda_2} \hat{\Pi}_k^{\lambda_1 \lambda_2}
\]

\[
= \sum_k \frac{1}{D^k} \sum_{\lambda, m} \alpha^{k \mu_{m m}} \hat{\Pi}_k^{\lambda \lambda} = \sum_k \frac{1}{D^k} \sum_{\lambda, m} \alpha^{k \mu_{m m}} \hat{\Pi}_k^{\lambda \lambda},
\]

(B4)
in which we used the orthogonality property for operators \((A22)\). Finally, utilising the property \((A16)\) we obtain:

\[
T(\alpha \otimes t) = \sum_k \frac{1}{D_k} \sum_{\lambda,m} \alpha_m \hat{\Pi}_k^{\lambda\lambda} = \sum_k \frac{1}{D_k} \sum_{\lambda,m} \Tr\left(\alpha \otimes \hat{\Pi}_k^{\text{mm}\dagger}\right) \hat{\Pi}_k^{\lambda\lambda} \\
= \sum_k \frac{1}{D_k} \Tr\left(\alpha \otimes \hat{\Pi}_k^{\text{mm}\dagger}\right) \sum_{\lambda} \hat{\Pi}_k^{\lambda\lambda} = \sum_k \frac{1}{D_k} \Tr\left(\alpha \otimes \hat{\Pi}_k^{\dagger}\right) \hat{\Pi}_k.
\]

\((B5)\)

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