Endpoint Symmetries of Helicity Amplitudes

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Abstract

We investigate helicity amplitudes (HAs) of $A \to BC$-type decays for arbitrary spin towards the kinematic endpoint. We show that HAs are proportional to product of Clebsch-Gordan coefficients (CGC) and the velocity to a non-negative power. The latter can be zero in which case the HA is non-vanishing at the endpoint. At the kinematic endpoint the explicit breaking of rotational symmetry, through the external momenta, is restored and the findings can be interpreted as a special case of the Wigner-Eckart theorem. Our findings are useful for i) checking theoretical computations and ii) the case where there is a sequence of decays, say $B \to B_1B_2$ with the pair $(B_1B_2)$ not interacting (significantly) with the $C$-particle. Angular observables, which are ratios of HAs, are given by ratios of CGC at the endpoint. We briefly discuss power corrections in the velocity to the leading order.
1 Introduction

The helicity basis, as introduced by Jacob and Wick [1], has proven to be a powerful tool in describing (sequential) one-to-two decays and beyond [2, 3]. In differential angular distributions helicity amplitudes (HAs) contain the dynamic information and the angular distribution is encoded in so-called
Wigner $D$-functions. In this paper we point out that at the kinematic endpoint the helicity amplitudes are simply proportional to the Clebsch-Gordan coefficients (CGC) of $SO(3)$ and $SU(2)$ for bosons and fermions in the decays. The kinematic endpoint corresponds to the situation where the back-to-back velocity $v$ of the $B$ and $C$-particle in the restframe of $A$ is zero, i.e. all the three particles $A$, $B$ and $C$ are at rest.

Intuitively the relations can be understood as a consequence of the restoration of the spatial rotation symmetry at the kinematic endpoint. The rotational symmetry is explicitly broken by the relative momenta in the decay. Thus the rotational symmetry acts like a global internal symmetry, analogous to the isospin symmetry, and leads to simple relations amongst (helicity) amplitudes in terms of CGC. We wish to add that whereas the simplicity of HAs at the kinematic endpoint has been noted in examples, e.g. [4], we are not aware of an explicit documentation. The extension of endpoint relations to effective field theories is discussed in a follow up paper [6].

The paper is organised as follows: In section 2 we define the HAs and establish endpoint degeneracies in terms of the CGC first for integer and then for half-integer spin in sections 2.1.1 and 2.1.2 respectively. The asymptotic behaviour towards the endpoint and selection rules are discussed in sections 2.2 and 2.3 respectively. The relation to the Wigner-Eckart theorem is outlined in section 2.4. A total of eight examples are given in section 3 including selection rules in section 3.6. A brief summary of applications is given in section 4 with particular focus on the isotropicity of the angular distribution of $A \to (B \to B_1B_2)C$ at the endpoint whose derivation is deferred to appendix A. The paper ends with conclusions section 5. Conventions on polarisation vectors and remarks on higher spin polarisation tensors are deferred to appendix B.

2 Endpoint symmetries of helicity amplitudes

To begin with we consider the $1 \to 2$ decay of particles with generic spin

$$A(q_A, \lambda_A) \to B(q_B, \lambda_B)C(q_C, \lambda_C), \quad (1)$$

\footnote{Our findings have been known in terms of the $LS$-formalism in indirect ways to specialists cf. [5]. We are grateful to Jürgen Körner for making us aware it.}
where \( q_A = q_B + q_C \) are the corresponding momenta and \( \lambda_r \) \((r \in \{A, B, C\} \text{ hereafter})\) denote the helicities. Angular momentum conservation imposes

\[
\lambda_A = \lambda_B - \lambda_C = \lambda_B + \bar{\lambda}_C ,
\]

(2)

(\( \bar{\lambda} \equiv -\lambda \) hereafter). The HAs are,

\[
H_{\lambda_B \lambda_C} = \mathcal{A}(A(\lambda_A) \to B(\lambda_B)C(\lambda_C)) ,
\]

(3)

the amplitudes for fixed helicities of the \( B \) - and \( C \) -particle.

In general there are many Lorentz-invariant structures of the amplitudes

\[
H_{\lambda_B \lambda_C} = \sum_{i=1}^{n} a_i(q_A^2, q_B^2, q_C^2)P_i(\alpha, \beta^*(\lambda_B), \gamma^*(\lambda_C), q_B, q_C) ,
\]

(4)

where \( \alpha, \beta \) and \( \gamma \) (indices suppressed) are the corresponding polarisation tensors and \( n \) is usually a number well below ten. We refer to the coefficients \( a_i \) as form factors. In the case where any of the particles \( A, B \) or \( C \) are off-shell \( q_i^2 \neq m_i^2 \), such as in further sequential decay \( B \to B_1 B_2 \), the propagation of the particle \( i \) can be approximated by a Breit-Wigner ansatz. The symbols \( P_i \) denote Lorentz-invariant structures built out of the momentum vectors \( q_B, q_C \) \((q_A = q_B + q_C \text{ is dependent})\) and the corresponding polarisation tensors \( \alpha(q_A, \lambda_A), \beta(q_B, \lambda_B), \gamma(q_C, \lambda_C) \). The invariants \( P_i \) are linear in the polarisation tensors.

Before following the approach outlined in (4) we motivate or demonstrate the findings of this paper in the language of the symmetry breaking and the Wigner-Eckart theorem. The global Lorentz-symmetry is broken by the momenta of the particles \( A, B \) and \( C \), but at the kinematic endpoint\(^2\)

\[
q_A \propto q_B \propto q_C \propto \omega(t) \equiv (1, 0, 0, 0) ,
\]

(5)

where the momentum vectors are all proportional to the time direction the vectorial \( SO(3) \) and spinorial \( SU(2) \) global symmetry is restored.

\(^2\)Since particle \( C \) is moving into the opposite direction one has to flip the momenta. Intuitively it is clear that this leads to a helicity flip and thus one has to use \( \bar{\lambda}_C \) instead of \( \lambda_C \). As usual in quantum theory there can be a helicity dependent phase associated with this transformation. The important point is that in the Jacob-Wick convention, which we adapt throughout, this phase is conveniently set to unity. Some more detail is given in appendix \([3]\) and in particular section \([B.1]\)
Essentially we are left with the problem of combining $SU(2)$ representation into an invariant. The latter is non-vanishing if and only if the spin of the three particles can be added to zero which is the case if

$$|J_B - J_C| \leq J_A \leq J_B + J_C.$$  

(6)

In turn of the more familiar Kronecker product this reads,

$$(J_A \otimes J_B \otimes J_C)_{SU(2)} = 1 \cdot \mathbf{0} \oplus \ldots,$$  

(7)

where $J$ stands for the $J$-spin representation throughout and $\mathbf{0}$ is the trivial representation in this notation. An important point is that $SU(2)$ is simply reducible which means that the multiplicity of each irreducible representation is one when the Kronecker product of two irreducible representations is taken and this justify the 1 in front of the (7). In practice this implies that there’s a unique (leading) HA at the kinematic endpoint.

Denoting the state of total angular momentum $J$ and helicity $\lambda$ by $|J, \lambda\rangle$, as usual, the HA is proportional to the scalar product of $|J_A, \lambda_B + \lambda_C\rangle$ with the state $|J_B, J_C; \lambda_B \lambda_C\rangle \equiv |J_B; \lambda_B\rangle \otimes |J_C; \lambda_C\rangle$. This is precisely the definition of the CGC

$$C^{J_1 J_2}_{(\lambda_1 + \lambda_2) \lambda_1 \lambda_2} \equiv \langle J_1, J_2; \lambda_1 \lambda_2 | J, \lambda_1 + \lambda_2 \rangle.$$  

(8)

The HA at the endpoint is therefore given by

$$H_{\lambda_B \lambda_C} = a^{(0)} C^{J_A J_B J_C}_{(\lambda_B + \lambda_C) \lambda_B \lambda_C},$$  

(9)

where $a^{(0)}$ is proportional to the single form factor at the endpoint. N.B. $a^{(0)}$ generally is a linear combination out of the set $\{a_1, a_2, \ldots\}$ outlined in Eq. (4). In the following section we will discuss the case when it is proportional to a power of the velocities and selection rules through examples showing the working of Eq. (9). We will demonstrate the specifics on examples in section 3.

We wish to emphasise that the other combinations of form factors do not vanish as matrix elements per se, but their respective Lorentz structures $P_i$...
do. We comment in section 2.4 on the relation to the Wigner-Eckart theorem of (9). One immediate consequence of (9) is that (for non-vanishing 
HAs the endpoint) the uniaxial distribution, in the angle $\theta_B$ between $B_1$ and the $B$-particle in $A \rightarrow (B \rightarrow B_1B_2)C$, is isotropic in the angle if one 
sums over initial state polarisation and not both decays are parity violating 
c.f. appendix A. Intuitively this corresponds to the situation where the 
particles have lost any spatial reference point and can therefore not decay into 
a particular direction more frequently than in any other.

2.1 The invariants

In the two following subsection we discuss how the unique invariant in (7) 
works out in the case of integer and half-integer spin in sections 2.1.1 and 
2.1.2 respectively.

2.1.1 Integer spin

In the case of bosons the spatial symmetry part of the Lorentz group cor-
responds to $SO(3)$. The defining, i.e. invariant, tensors in the fundamental 
representation of $SO(3,1)$ are the metric $g_{\mu\nu}$ or the totally antisymmetric 
Levi-Cevita tensor (LCT) $\epsilon_{\mu\nu\rho\sigma}$. The former corresponds to $O(3,1)$ invari-
ance and the latter ensures unit determinant of the Lorentz transformation. 
In the following analysis we will see that these tensors effectively reduce to 
the $SO(3)$ defining tensors.

To establish this claim we note that: i) At the endpoint the $J = 1$ 
polarisation vectors, denoted by a hat, are all proportional to each other 
$\hat{\omega}(\lambda) \equiv \hat{\alpha}(\lambda) = \hat{\beta}(\lambda) = \hat{\gamma}(\bar{\lambda})$ (c.f. appendix B). N.B. the barred polarisation 
index ($\bar{\lambda}$) is used for convenience. ii) The polarisation tensors 
$\alpha_{\mu_{1}..\mu_{JA}}(q_{A},\lambda_{A}), \beta_{\mu_{1}..\mu_{JB}}(q_{B},\lambda_{B})$ and $\gamma_{\mu_{1}..\mu_{JC}}(q_{C},\lambda_{C})$ can be formed out of the 
$J = 1$ polarisation vector through appropriate CGC as described in appendix 
B.2. Armed with this knowledge the following two facts are of importance 
to establish the claim:

(a) Transversity: Any contraction of $\omega(t)$ with $\alpha, \beta, \gamma$ is zero (B.8) since 
$\omega(t)$ is, so to speak, the direction of the momentum (5).

(b) Anti-symmetric tensor: The only object with which $\omega(t)$ can be con-
tracted without vanishing is the LCT to $\epsilon_{mno} \equiv \epsilon_{mno\rho}\omega^{\rho}(t)$. Further-
more, since any product of LCT can expressed in terms of metric tensors it is sufficient to consider a single LCT.

By virtue of (a) one can safely replace the metric \( g_{\mu\nu} \to -\delta_{mn} \), where \( \delta_{mn} \) denotes the Kronecker symbol throughout this work. Roman indices run from 1, 2, 3 (spatial indices) and Greek indices run from 0, 1, 2, 3 (spacial and temporal indices). The tensors \( \delta_{mn} \) and \( \epsilon_{mno} \) are the defining tensors of \( SO(3) \) and we have thus justified our initial claim at the beginning of this section.

### 2.1.2 Extension to half-integer spin

The new element with respect to integer spin is that one can form covariants out of two half-integer spin objects. Since \((n + 1/2)\)-spinors can be formed out of integer spin and \(1/2\)-spinors (cf. section 3.4), we may restrict our attention to \(1/2\)-spinors. The important objects are the particle and anti-particle spinors \( u \) and \( v \). In the Dirac representation of the Clifford algebra, with \( \sigma^i \) as the usual \( 2 \times 2 \) Pauli matrices,

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, & \gamma_5 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

(10)

\( u \) and \( v \) (e.g. \[2\]) assume a simple form at the endpoint:

\[
\begin{align*}
u(\vec{p} = 0, \lambda) &= \begin{pmatrix} \chi^\lambda \\ 0 \end{pmatrix}, & v(\vec{p} = 0, \lambda) &= \begin{pmatrix} 0 \\ -2\lambda\chi^\lambda \end{pmatrix}.
\end{align*}
\]

(11)

The symbol \( \chi^\lambda \) denotes a 2-spinor which does not need to be specified any further for our purposes. To proceed further we need to determine the selection rules for the spinor products of the form \( L^\Gamma_{uu} = \bar{u}\Gamma u \) \( (f_1 \to f_2 b) \) and \( L^\Gamma_{vu} = \bar{v}\Gamma u \) \( (b \to \bar{f}_1 f_2) \) where \( b \) and \( f \) stand for boson and fermion respectively, and \( \Gamma \) is a combination of the Dirac matrices. For the corresponding anti-particle decay one has to exchange \( u \) and \( v \) with each other. The sets \( \Gamma_D = \{\gamma_0, 1\} \) and \( \Gamma_A = \{\gamma_i, \gamma_5\} \) induce a grading in the sense that any product of an even number of \( \Gamma_A \)-matrices is in the \( \Gamma_D \) set. Directly relevant to us is that either \( L^\Gamma_{uu} \) or \( L^\Gamma_{vu} \) are zero at the endpoint and the same is true for all other combinations of \( u \) and \( v \). This effectively reduces the \( SO(3,1) \) symmetry to \( SO(3) \) as \( \Gamma_{A,D} \) discriminate between spacial and temporal indices.

Let us illustrate this statement with one example. The decay of spin 1 boson into two spin 1/2 fermions. The Lorentz invariant is given by

\[
\bar{v}(q_B, \lambda_B)\gamma_\mu u(q_C, \lambda_C)\alpha_\nu(q_A, \lambda_A)g^{\mu\nu}.
\]

(12)
Eqns (10,11), allows to replace $g_{\mu\nu} \rightarrow -\delta_{mn}$, making the $SO(3)$-symmetry explicit. Note, for example the replacement of $\gamma_{\mu} \rightarrow \gamma_{\mu} q^{\nu} q^{\rho}$ would not bring in anything new since at the endpoint the latter reduces to $\gamma_{\mu} \gamma_0 q^r$ and differs therefore by a constant only. Furthermore, we see that $\gamma_{\mu} \rightarrow \gamma_{\mu} \gamma_5$ vanishes since $\gamma_5$ is in the other class (cf. rule (iii) in section 2.3). The case where there is half-integer spin in the initial and final state is analogous with the crucial difference that the other class has to be chosen as now the product $\bar{u}..u(\bar{v}..v)$ has to be investigated.

In conclusion, even when there are fermions amongst the particles $A$, $B$ and $C$, there is a unique form factor that enters at the kinematic endpoint and formula (9) remains correct for fermions in the decay as well. The discussion in this paragraph does have formal reminiscence with heavy quark effective theory. Although we wish to stress that the latter is a dynamic theory and that our situation is of merely kinematic nature.

2.2 Helicity amplitudes $\propto v^\Omega$ towards the endpoint

The goal of this section is to generalise (9) to the case when the condition (6) is not met. In order to do so it us useful to define $\Omega$: the number of open polarisation indices of $\alpha,\beta,\gamma$ after maximal contractions. Formally it may be written as:

$$\Omega = \Omega(J_A; J_B, J_C) = \begin{cases} 0 & |J_B - J_C| \leq J_A \leq J_B + J_C \\ p & \text{otherwise} \end{cases},$$

(13)

with $p \equiv \min\{|J_A - |J_B - J_C||, |J_A - (J_B + J_C)||\}$. For example if $(J_A; J_B, J_C) = (4; 1, 1)$ then $\Omega = 2$. If $\Omega > 0$ then necessarily all the open indices are in one polarisation tensor, say $\alpha_{\mu_1..\mu_p}$, as otherwise one would just contract the indices of different polarisation tensors. Furthermore, the indices of $\alpha_{\mu_1..\mu_p}$ cannot be contracted any further by the LCT since the polarisation tensors are totally symmetric in the indices, and neither with the metric since they are traceless (B.9). The open polarisation indices of $\alpha$ have to be contracted by the momenta $q_B$ which differs linearly from $v$ in $q_A$. Therefore the asymptotic behaviour is given by

$$H_{\lambda_B\lambda_C} = H^{(\Omega)}_{\lambda_B\lambda_C} v^{\Omega} + ..,$$

(14)

where the dots stand for higher order corrections in $v$. Eq. (9) corresponds to $\Omega = 0$. For $\Omega > 0$ one can get $H^{(\Omega)}_{\lambda_B\lambda_C}$ as follows. Let $J_A > J_{B,C}$ be the
largest of all three spins. It is useful to think of a fourth fictitious particle, sometimes called spurion in other contexts, of spin \( \Omega \) with helicity 0. And then consider the following two Kronecker products:

\[
(J_B \otimes J_C)_{SU(2)} = 1 \cdot (J_B + J_C) \oplus \ldots ,
\]

\[
(J_A \otimes \Omega)_{SU(2)} = 1 \cdot (J_B + J_C) \oplus \ldots ,
\]

which in turn can be combined into a singlet. Above the dots stands for other representations that are not of interest here. We get

\[
H^{(\Omega)}_{\lambda_B \lambda_C} = \begin{cases} 
   a^{(0)} C^{J_A J_B J_C}_{\lambda_A \lambda_B \lambda_C} & \Omega = 0 \\
   a^{(\Omega)} C^{(J_B + J_C)J_B J_C}_{\lambda_A \lambda_B \lambda_C} C^{(J_B + J_C)J_A \Omega}_{\lambda_A \lambda_A 0} & \Omega > 0
\end{cases}
\]

where we have used \( \lambda_A = \lambda_B + \tilde{\lambda}_C \) for compact notation. We note that the formula for \( \Omega > 0 \) reduces to \( \Omega = 0 \) since \( C^{J_A J_A 0}_{\lambda_A \lambda_A 0} = 1 \). The relation to the Wigner-Eckart theorem is discussed in section 2.4 and constitutes an effective proof of the formulae as opposed to the pedestrian analysis given here. In particular it also constitutes the proof for when there are fermions in the decay.

Whether or not \( a^{(\Omega)} \) vanishes for reasons other than spin is discussed in section 2.3 and illustrated in section 3.6 with examples.

### 2.2.1 Comments on power corrections

The notation of (14) is extended to include the first correction (where \( n > 0 \) and integer) as follows,

\[
H^{(\Omega)}_{\lambda_B \lambda_C} = H^{(\Omega)}_{\lambda_B \lambda_C} v^\Omega + H^{(\Omega+n)}_{\lambda_B \lambda_C} v^{\Omega+n} + \mathcal{O}(v^{\Omega+n+1})
\]

and we refer to such corrections as \( N^n \text{LO} \) power corrections. Note that in the case of parity conservation (cf. rule (ii) in section 2.3), \( n \) is even. This is of relevance as only NLO corrections \( (n = 1) \) to the asymptotic behaviour can be understood in terms of CGCs (16). However, these corrections are not necessarily governed by a unique invariant. If one of the three particles happens to be of spin 0 then the invariant is still unique such as in \( B \rightarrow K^*(\gamma^* \rightarrow \ell^+\ell^-) \) for example. If there are two \( k \) invariant at NLO then the differential distribution in the vicinity of the endpoint will be governed by \( k \) ratio of form factors. For examples cf. section 3.5 and footnote 7 in that section.
In this section we discuss selection rules beyond the spin condition \((6)\) that apply to the \(a^{(0)}\) and \(a^{(\Omega)}\)\(v\Omega\)-terms.

In the case where parity is conserved the following constraint applies to the HA (e.g. \([1, 2]\)):

\[
H_{\lambda_B \lambda_C} = \eta (-1)^{\Delta J} H_{\bar{\lambda}_B \bar{\lambda}_C},
\]

where

\[
\Delta J \equiv (J_B + J_C) - J_A, \quad \eta \equiv \eta_A \eta_B \eta_C,
\]

and \(\eta\) is the product of the intrinsic parities of the particles \(A\), \(B\) and \(C\). This has to be put into context with the CGC-symmetry property \([11]\),

\[
C_{\lambda_A \lambda_B \lambda_C}^{J_A J_B J_C} = (-1)^{\Delta J} C_{\bar{\lambda}_A \bar{\lambda}_B \bar{\lambda}_C}^{J_A J_B J_C},
\]

and Eq. \((9)\). We note that the following rules must apply:

- **if parity is conserved**
  
  (i) The total internal parity must be one, \(\eta = 1\), for the HA not to vanish at the endpoint for \((9)\), \((18)\) and \((20)\) to be consistent with each other.

  (ii) In fact (i) is implicit in a more general theorem (e.g. \([10]\)) that states that \(\eta\) equal to \(1(-1)\) implies that the S-matrix is even (odd) in powers of the external momenta. Let us write the statement in mnemonic form:

  \[
  \eta = \begin{cases} 
  +1 & \text{S even external momenta} \\
  -1 & \text{S odd external momenta}
  \end{cases}.
  \]

  A non-vanishing HA at the endpoint is constant in \(v\) and therefore an even power implying \(\eta = 1\) as in (i).

- **if parity is not conserved**

  (iii) It is easy to convince oneself, using arguments along the lines of section \(2.1.2\), that for odd and even powers of \(\Delta J + \Omega\) (with \(\Omega, \Delta J \equiv (J_B + J_C) - J_A\) as in \((13)\) and \((19)\)) there is a definite association with the endpoint amplitude and the tensorial or spinorial
structure. We use the following cryptic notation: $(\epsilon)$ and $(g)$ depending on whether the tensors contain a LCT or not. $(\gamma_5)$ or $(1)$ depending on whether the Dirac spinor product contains a $\gamma_5$ or not. With the notation $(14,16)$, $HA \propto \epsilon^{(\Omega)} a^{\Omega}$, the result can be written in mnemonic form as follows:

$$\Delta J + \Omega = \begin{cases} \text{odd} & \text{a}^{(\Omega)} \text{ contains: } \epsilon \text{ (bosons), } \gamma_5 \text{ (fermions)} \\ \text{even} & \text{a}^{(\Omega)} \text{ contains: } g \text{ (bosons), } 1 \text{ (fermions)} \end{cases}$$  (22)

- **Identical particles (Landau-Yang-type selection rule)** For identical particles the following relation holds [1]: $H_{\lambda B \lambda C} = \epsilon^\Omega H_{\lambda C \lambda B}$.

(iv) The above formula implies that for $\Delta J$ odd $H_{\lambda \lambda} = 0$. If this is the case then Eq. (9) implies that all amplitudes have to vanish at the endpoint. More precisely the form factor $a^{(0)}$ has to vanish since not all CGC are zero. An example is given in section 3.1 for $1 \rightarrow 1 + 1$ decay.

### 2.4 Relation to the Wigner-Eckart theorem

We briefly comment on the relation of the formulae (16) to the Wigner-Eckart theorem. The latter states that (e.g. [11]),

$$\langle jm|T^q_j|j'm'\rangle = C^{jj'}_{\text{mass}}\langle j||T^q||j'\rangle$$  (23)

the matrix element of a tensor operator is determined by a product of CGC and a reduced matrix element which is independent on the orientation (helicity). First we discuss the somewhat degenerate $\Omega = 0$-case. Note that the CGC, $C^{J_B J_C}_{\lambda_B \lambda_C}$ in (16), is the basis transformation from the state $|J_B, J_C; \lambda_B \lambda_C\rangle$ to $|J_A, \lambda_B + \lambda_C\rangle$. The form factor $a^{(0)}$ takes the role of the reduced matrix element, independent of the helicities, where the transition operator is a scalar (namely the Hamiltonian of the decay). The $\Omega > 0$-case is more interesting as it can be viewed as the reduced matrix element of a tensor operator with angular momentum $q = \Omega$ and helicity $k = 0$ which is then contracted by an appropriate momentum $q_r$. Thus the $C^{(J_B + J_C)}_{\lambda_A \lambda_B \lambda_C}$ in the second line of (16) corresponds to the CGC in (23).
3 Eight Examples

We illustrate our main formulae (9) and (16) by a few examples found in the literature as well as an example that we work out explicitly. In this section we frequently use the short hand notation $J_A^P \rightarrow J_B^{P'} + J_C^{P''}$ to indicate the spin ($J_i$) and the parity ($P_i$, if known) of the particles involved. In the tables given below we have not made use of the symmetry property (18) but have listed all the values explicitly.

3.1 Higgs-like (spin 0, 1, 2) decay into two $Z$-bosons

We consider $H_{J=0,1,2} \rightarrow Z^*Z$ where the particle $H$ has got spins 0, 1 or 2. By Higgs-like we mean that we are open to other spin than zero for the decaying particle. We are going to use the result in Ref. [13] for which the endpoint is given by $m_X \rightarrow m_1 + m_2$ (which in their notation implies $x \equiv [(m_X^2 - m_1^2 - m_2^2)/(2m_1m_2)]^2 - 1 \rightarrow 0$). The variables $m_X$, $m_1$ and $m_2$ denote the masses of the Higgs-like particle, the first and the second $Z$ boson respectively. The variable $x$ is proportional to the Källén-function (whose square root is proportional to the back-to-back velocity $v$ in the rest-frame of the decaying particle) and is therefore zero at the endpoint.

Note that in all the examples above $\Omega = 0$ so the HA, modulo question of parity, do not vanish at the endpoint. It is though immediate that for $J_A = J_H > 2$, $\Omega = J_H - 2$ and thus by virtue of (14) the HA $\propto v^{(J_H-2)}$.

$0 \rightarrow 1 + 1$ : The prediction [9], $H_{\lambda_B \lambda_B} = a^{(0)}C_{0\lambda_B \lambda_B}^{011}$, yields:

| $\lambda_A$ | HA | $H_{++}$ | $H_{00}$ | $H_{--}$ |
|------------|-----|--------|--------|--------|
|            | HA |        |        |        |
|            | CGC | $C_{011}^{011}$ | $C_{000}^{011}$ | $C_{011}^{011}$ |
| value      | $\sqrt{1/3}$ | $-\sqrt{1/3}$ | $\sqrt{1/3}$ |

where we have indicated the helicity of particle $A$, the CGC and its value on lines one, two and three respectively. The table above is consistent with Eq.14 [13], $m_X \rightarrow m_1 + m_2$ ($x \rightarrow 0$) and the identification $a^{(0)} = \sqrt{3}(m_1 + m_2)/v_H a_1$, denoting the Higgs vacuum expectation value by $v_H$ to avoid confusion with the velocity $v$.

This special case was noted in Ref. [14]. Our work adds the exact form (16) of the helicity dependent precoefficient.
1 → 1 + 1: The prediction \( H_{\lambda B\lambda C} = a^{(0)} C_{(\lambda B + \lambda C)\lambda B\lambda C}^{111} \), yields:

| HA | \( H_{+0} \) | \( H_{0-} \) | \( H_{++} \) | \( H_{00} \) | \( H_{--} \) | \( H_{0+} \) | \( H_{-0} \) | \( H_{-+} \) |
|----|----------|----------|----------|----------|----------|----------|----------|----------|
| \( \lambda_A \) | 1 | 1 | 0 | 0 | 0 | -1 | -1 | 0 |
| CGC | \( C_{111}^{110} \) | \( C_{111}^{101} \) | \( C_{111}^{011} \) | \( C_{111}^{100} \) | \( C_{111}^{111} \) | \( C_{111}^{011} \) | \( C_{111}^{101} \) | \( C_{111}^{110} \) |
| value | \(-\sqrt{1/2}\) | \(\sqrt{1/2}\) | \(\sqrt{1/2}\) | 0 | \(-\sqrt{1/2}\) | \(-\sqrt{1/2}\) | \(\sqrt{1/2}\) | \(\sqrt{1/2}\) |

which agrees with Eq.17 of \[13\] for \( m_X \rightarrow m_1 + m_2 \) \((x \rightarrow 0)\) and \( a^{(0)} = \sqrt{2}i(m_1 - m_2) \). In the case where \( m_1 = m_2 \) the two \( Z \) bosons are identical and the Landau-Yang theorem (rule (iv) in section 2.3), applies and implies \( H_{\lambda \lambda} = 0 \). Hence by virtue of (9) all \( H_A \)s vanish at the endpoint. This is indeed the case as \( a^{(0)} \propto m_1 - m_2 \rightarrow 0 \).

2 → 1 + 1: The prediction \( H_{\lambda B\lambda C} = a^{(0)} C_{(\lambda B + \lambda C)\lambda B\lambda C}^{211} \), yields:

| HA | \( H_{+0} \) | \( H_{0-} \) | \( H_{++} \) | \( H_{00} \) | \( H_{--} \) | \( H_{0+} \) | \( H_{-0} \) | \( H_{-+} \) |
|----|----------|----------|----------|----------|----------|----------|----------|----------|
| \( \lambda_A \) | 2 | 1 | 1 | 0 | 0 | -1 | -1 | -2 |
| CGC | \( C_{211}^{211} \) | \( C_{211}^{110} \) | \( C_{211}^{101} \) | \( C_{211}^{011} \) | \( C_{211}^{100} \) | \( C_{211}^{111} \) | \( C_{211}^{011} \) | \( C_{211}^{101} \) |
| value | 1 | \(\sqrt{1/2}\) | \(\sqrt{1/2}\) | \(\sqrt{1/2}\) | \(2/3\) | \(\sqrt{1/2}\) | \(\sqrt{1/2}\) | 1 |

which is consistent with Eq.21 in \[13\] in the limit \( m_X \rightarrow m_1 + m_2 \) \((x \rightarrow 0)\) and \( a^{(0)} \equiv c_1 m_1 m_2 / \Lambda \). The corresponding NLO power correction is worked out in section 3.5.

3.2 Higgs-like (spin 0,1,2) decay into a fermion pair

We consider \( H_{J=0,1,2} \rightarrow q \bar{q} \) where the particle \( H \) has got spin 0, 1 or 2. We are using the results of Ref. \[15\] for which at the endpoint \( m_X \rightarrow 2m_q \) (and the fermion velocity \( \beta = 0 \)) where \( m_X \) and \( m_q \) are the Higgs-like particle and the fermion mass respectively. We disagree in signs with \[15\] for some of the \( 2, 1 \rightarrow 1/2 + 1/2 \) amplitudes.\(^6\)

\(^6\)The sign differences are of no consequence for the work in \[15\], because of the (incoherent) factorisation of the production helicity amplitudes and the parton distribution functions. In essence only the absolute values, denoted by \( B_{\lambda_1 \lambda_2} \) in \[15\], of the production amplitudes enter.
0 → 1/2+ + 1/2−: One gets using (9) yields:

\[ H_{\lambda B \lambda C} = a^{(0)} C^{0 \frac{1}{2} \frac{1}{2}}_{\lambda B \lambda C} \delta_{\lambda B} \delta_{\lambda C} \].

Since \( C^{0 \frac{1}{2} \frac{1}{2}} = C^{0 \frac{1}{2} \frac{1}{2}} \), one gets \( H_{\lambda B \lambda C} = H_{\frac{1}{2} \frac{1}{2}} \) which is in accord with [15] Eq.20 with \( a^{(0)} = m_q/v_H m_X \).

1 → 1/2+ + 1/2−: For spin 1 (9) yields:

\[ H_{\lambda B \lambda C} = a^{(0)} C^{1 \frac{1}{2} \frac{1}{2}}_{\lambda B \lambda C} \delta_{\lambda B} \delta_{\lambda C} \].

which is in accord with [15] Eq.20 with \( a^{(0)} = m_q/v_H m_X \).

2 → 1/2+ + 1/2−: In this example Ω = 1 and this is a good test of the formula on the second line in Eq.(16):

\[ H_{\lambda B \lambda C} = a^{(1)} C^{1 \frac{1}{2} \frac{1}{2}}_{\lambda B \lambda C} C^{1 \frac{1}{2} \frac{1}{2}}_{\lambda B \lambda C} \delta_{\lambda B} \delta_{\lambda C} \].

Evaluating we get:

\[ H_{\frac{1}{2} \frac{1}{2}} = H_{\frac{1}{2} \frac{1}{2}} = (\sqrt{2/3}) a^{(1)} \delta_{\lambda B} \delta_{\lambda C} \],

which again agrees with Eq.(22) in [15] up to signs (|a^{(1)}| = 2m_q/\rho^{(1)}). Note though that the endpoint-relation (18) (with \( \eta = 1, \Delta J = 0 \)) reads \( H_{\lambda_1 \lambda_2} = H_{\lambda_2 \lambda_1} \) and is consistent with our results. As the statement has some degree of circularity, we have also explicitly checked our results and find agreement with (9).

3.3 \( \Lambda_b \to \Lambda_c (W \to \ell \nu) \) or 1/2+ → 1/2+ + 1

Formula (9) predicts:

\[ \frac{H_{\frac{1}{2} \frac{1}{2}}}{H_{\frac{1}{2} \frac{1}{2}}} = \frac{C^{1 \frac{1}{2} \frac{1}{2}}_{\frac{1}{2} \frac{1}{2}}}{C^{1 \frac{1}{2} \frac{1}{2}}_{\frac{1}{2} \frac{1}{2}}} = \frac{\sqrt{2/3}}{\sqrt{1/3}} = -\sqrt{2} \].
which is the result obtained in [4] in Eq 4. It is found that the axial ($J^P = 1^-$) but not the vectorial coupling is non-vanishing at the endpoint [4]. This is in accordance with our rules (section 2.3) and discussed in section 3.6.

3.4 $\Lambda_b \to \Lambda(1520)(\rho \to \ell^+\ell^-)$ or $1/2^- \to 3/2^+ + 1^-$

We are going to discuss this example by using the following interaction:

$$H_{\lambda\lambda\rho} \propto \langle \rho(q, \lambda\rho)\Lambda(p, \lambda\Lambda)|\Phi(\rho)\mu\bar{s}\gamma_\mu|\Lambda_b\rangle$$

$$= \Psi^\mu(p, \lambda\Lambda)u(p + q, \lambda\Lambda - \lambda\rho)\epsilon^*_\mu(q, \lambda\rho)f(q^2) \ ,$$

where $\Phi(\rho)$ is an interpolating operator for the $\rho$-meson. Here $\lambda\rho$ and $\lambda\Lambda$ denote polarisation indices, $\Psi_\mu$ is a Rarita-Schwinger spin $3/2$ object [19], $u$ is a Dirac spinor and $f$ is a form factor (irrelevant for our purposes as it evaluates to a constant at the endpoint). The decay of the $\rho$-mesons to leptons is not analysed, as it merely serves the possibility of an off-shell $\rho$-meson. The Rarita-Schwinger $3/2$-spinor is formed out of a Dirac spinor and a spin polarisation tensor using CGC:

$$\Psi_\mu(p, \lambda) = \sum_{s = -1/2}^{1/2} C^{\frac{3}{2}\frac{1}{2}}_{\lambda s(\lambda - s)}u(p, s)\epsilon_\mu(p, \lambda - s) \ .$$

In order to evaluate (27) we use $\bar{u}(p, \kappa)u(p, \kappa') = \delta_{\kappa\kappa'} \ e.g. \ [10]$ and $\epsilon^*(q, \kappa') = -(-)^s\kappa\kappa'$ (where $x \cdot y = x_\mu y^\mu$ throughout), which implies $s = \lambda\Lambda - \lambda\rho$. Assembling we get:

$$H_{\lambda\lambda\rho} = -(-)^{\lambda\rho}fC^{\frac{3}{2}\frac{1}{2}}_{\lambda s(\lambda\Lambda + \lambda\rho)} = \sqrt{2}fC^{\frac{3}{2}\frac{1}{2}}_{(\lambda\Lambda + \lambda\rho)\lambda\Lambda\lambda\rho} \ ,$$

which corresponds to Eq. [9] with $a^{(0)} = \sqrt{2}f$. In the second equality we have used the CGC-property $C^{Jj_1j_2}_{Mm_1m_2} = (-)^{j_2+m_2}\sqrt{(2J + 1)(2j_1 + 1)}C^{j_1j_2}_{m_1Mm_2}$ [11].

Through this concrete example we have aimed to exemplify some of the abstract statements made in section 2. Furthermore we notice that if we had chosen an axial interaction instead of the vector one then all HAs would have vanished by virtue of $\bar{u}(p, \kappa)\gamma_5u(p, \kappa') = 0$. The vanishing of the other parity interaction is in line with the arguments given in section 2.1.2 and the rule (iii) in section 2.3.
3.5 NLO power correction: $H_{J=2} \rightarrow ZZ$

In this section we wish to illustrate the discussion of the NLO power corrections in section 2.2.1 on the example of Higgs-like spin 2 particle decaying into two $Z$-bosons. The leading order discussion is given in section 3.1 under $2 \rightarrow 1 + 1$. We are left with the task of combining the $2_A \otimes 1_B \otimes 1_C \otimes 1_q$ (where the subscripts show the association with the particles and the momentum) representations into an invariant. This can be done in two different ways since:

$$(2_A \otimes 1_q)_{SU(2)} \otimes (1_B \otimes 1_C)_{SU(2)} =$$

$$(3 \oplus 2 \oplus 1)_{SU(2)} \otimes (2 \oplus 1 \oplus 0)_{SU(2)} = 2 \cdot 0 \oplus . . .$$

(30)

Note one of them vanishes. This can be seen from the fact that since $\Delta J + \Omega = 1$ is odd, according to rule (iii) section 2.3, the amplitude is formed out of a LCT. Hence we have to pick the antisymmetric part of the $(1_B \otimes 1_C)$ product which is given by the 1 representation. The $2 \otimes 2$ invariant has to vanish.$^7$ Hence there is single correction and this correction can, according to Eqs. (16,30), be computed as follows$^8$

$$H^{(1)}_{\lambda B\lambda C} = a^{(1)} C^{211}_{\lambda A\lambda B\lambda C} C^{221}_{\lambda A\lambda A\lambda 0} ,$$

(31)

where $H_{\lambda B\lambda C} = H^{(0)}_{\lambda B\lambda C} + H^{(1)}_{\lambda B\lambda C} + O(v^2)$ in the notation introduced in (17).

When evaluated we get:

$$H_{00} : H_{11} : H_{10} : H_{11} = 0 : 1 : -\sqrt{3/4} : \sqrt{3/4} : 0 .$$

(32)

All others can be obtained through $H_{\lambda C\lambda D} = -H_{\lambda C\lambda D}$ from (18) since $\Delta J = 1 + 1 - 2 = 0$ and $\eta = -1$ by rule (ii) in section 2.3. One may verify from the formulae in Eq. 21$^{13}$ that (32) indeed holds with $a_1^{(0)} \propto c_0$ in their notation.

---

$^7$ An example where there are two non-vanishing invariants at NLO is given by $\Lambda_b \rightarrow \Lambda t\nu$ $^7$. One has $\frac{1}{2} A \otimes 1_B \otimes 1_C \otimes 1_q = 2 \cdot 0 \oplus$ and the same applies when the $C$-particle is of spin $\frac{3}{2}$.

$^8$ To be more concrete we give the example of the covariants formed by the two Clebsch-Gordan products. The first one is the combination of two spin 1 into a spin 2 objects and corresponds to $X_{\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta} c_\alpha(p_B) e_\beta^*(p_C)$ and the second term is the combination of a spin 2 object and a spin 1 object into a spin two object $Y_{\gamma\delta} = \epsilon_{\gamma\lambda}(p_A) q^\lambda g_\delta$. The contracted invariant $X 
 Y$ will be linear in the velocities as one can verify using any specific combinations of non-vanishing helicity combinations. In Eq. 19 in $^{13}$ the above example corresponds to $c_0$. The $c_{5,7}$-terms correspond to Lorentz invariants of higher order in the velocities.
3.6 Parity and amplitude properties of example HAs

In this section we illustrate rules (i), (ii) and (iii) stated in section 2.3 for all eight example endpoint-HAs considered so far.

The parity quantum number of the $Z$-boson is ill-defined as it is a mixture of $J^P = 1^-, 1^+$-state even in the case of CP-conservation. Below we simply use $\eta_Z = 1$.

The same remark applies to the $W$-boson but rule (i) and (iii) imply that only the $1^+$-component couples at the endpoint and thus $\eta_W = 1$ is the outcome and strictly speaking an abuse of notation. At last we remind the reader that the parity of a Dirac fermion and an anti-Dirac fermion are opposite to each other.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{decay} & \text{section} & \eta = \eta_A\eta_B\eta_C & \Delta J (19) & \Omega (13) & \text{amp} \\
\hline
0^{0A} \rightarrow 1, 1 & 3.1 & \eta = \eta_A \Rightarrow \eta_A = 1 & 0 & g & \\
1^{1A} \rightarrow 1, 1 & \text{idem} & \eta = \eta_A \Rightarrow \eta_A = 1 & 0 & \epsilon & \\
2^{2A} \rightarrow 1, 1 & \text{idem} & \eta = \eta_A \Rightarrow \eta_A = 1 & 0 & g & \\
0^{0A} \rightarrow 1/2^+ 1/2^- & 3.2 & \eta = -\eta_A \Rightarrow \eta_A = -1 & 0 & \gamma_5 & \\
1^{1A} \rightarrow 1/2^+ 1/2^- & \text{idem} & \eta = -\eta_A \Rightarrow \eta_A = -1 & 0 & 1 & \\
2^{2A} \rightarrow 1/2^+ 1/2^- & \text{idem} & \eta = -\eta_A \Rightarrow \eta_A = 1 & 1 & 1 & \\
1/2^+ \rightarrow 1/2^+ 1_{nw} & 3.3 & \eta = \eta_W \Rightarrow \eta_W = 1 & 0 & \gamma_5 & \\
3/2^- \rightarrow 1/2^+ 1^- & 3.4 & \eta = 1 & 0 & 1 & \\
\hline
\end{array}
\]

Above amp is short for amplitude and the abbreviated notation $g, \epsilon, 1$ and $\gamma_5$ is explained in the paragraph of rule (iii). The reader might verify the statement by inspecting the explicit results in the literature and to section 3.4 for the last example. The $2 \rightarrow 1/2^+ 1/2^-$-case is special as the HA is linear in $v$ ($\Omega = 1$) at the endpoint which imposes $\eta = -1$ by rule (ii).

Note if parity is not conserved the internal parity quantum numbers cease to exist and the states become parity admixtures.

4 Brief summary of applications

HAs of the $A \rightarrow B + C$ type are widely used in particle physics in sequential decays, e.g. [3] for examples, where the decay chains do not interact with

\[9\text{N.B. of course algebraically we would find the same amplitude if say } J_B^{P_B} = 1^- \text{ and } J_C^{P_C} = 1^+ \text{ but then this would imply that } \eta_A \text{ has got the opposite parity from the one shown in the table.} \]
each other. A specific case is given by $H \rightarrow ZZ^* \rightarrow 4\ell$ where the relations might be of use in determining the Higgs quantum numbers. The relevant question in practice is whether or not the gain in the predictive power, due to the endpoint symmetries, can outweigh the loss in phase space and thus statistics. The selection rules presented in section 2.3 could potentially be used to test the parity properties (parity even, odd or admixture) of the Higgs candidate.

We wish to emphasise that it is important to form ratios since the differential rates behave schematically as $d\Gamma \propto v |\sum HA|^2d(\text{angles})$ e.g. [8], and therefore vanish at least linearly at the endpoint. The number of observables, which are often asymmetries in the context of new physics searches, one can form is vast and depends on the number of detectable final state particles.

Examples are the isotropicity (appendix A) of the angular distributions of a decay $A \rightarrow (B \rightarrow B_1B_2)C$ when all helicities summed over. It is noteworthy that for this to be true the HAs have to be non-zero at the endpoint. If $HA \propto v^n$ ($n > 0$) then the HA remembers the quantisation axis through the scalar products with other vectors. The isotropicity has implications for other well studied observables:

- Forward backwards asymmetries in that angle,
  
  $$A_{FB} = \int_{-1}^{1} d(\cos \theta_B) \text{sign}(\theta_B) \frac{d\Gamma}{\Gamma d\cos \theta_B} = 0,$$
  
  are zero at the endpoint since by definition they are sensitive to odd powers in $\cos(\theta_B)$ only.

- The longitudinal polarisation fraction corresponding to the fraction of 0-helicity (longitudinal) $B$ and $C$-particle detection. In a $(J_A = 0) \rightarrow J_B + J_B$ decay
  
  $$F_L \equiv \frac{\Gamma_L}{\Gamma_L + \Gamma_T} \equiv \frac{|H_{00}|^2}{(\sum_{\lambda_B = -J_B}^{J_B} |H_{\lambda_B \lambda_B}|^2)} = \frac{1}{(2J_B + 1)},$$
  
  at the endpoint since $|H_{\lambda_B \lambda_B}| \propto |C_{0\lambda_B \lambda_B}^{J_B}| = |(-1)^{(J_B - \lambda_B)}(2J_B + 1)^{-1/2}|$ \(9\) is independent of $\lambda_B$. \(^{10}\) This results reflects the independence of the non-relativistic limit on the spin. It contrasts with the high energy limit

\(^{10}\)For $H \rightarrow ZZ$ it was for instance noted in [17] that the longitudinal polarisation fraction approaches $F_L = 1/(2 \cdot 1 + 1) = 1/3$ at the endpoint.
where the 0-helicity component dominates, in accordance with the equivalence theorem, and \( F_L \to 1 \). This statement is easily verified in the examples treated in section 3 by inspecting the expression in the quoted references.

5 Conclusions and discussion

In this work we have discussed the simplicity of the \( A \to B + C \) HAs at the kinematic endpoint. Our main results are Eq. (9) and the refinement in Eqs. (14,16) which relate the HAs to CGCs. Parity selection rules and general statements on the structure of the HAs at the endpoint have been given in section 2.3. Our findings are illustrated in section 3 with a total of eight examples including parity selection rules. An outlook on possible applications, including \( H \to ZZ^* \to 4\ell \), has been given in the previous section.

We wish to reemphasise that it is the special kinematics which singles out a single form factor at the endpoint. As dynamic objects all the form factors exist at the endpoint; and beyond in the sense of analytic continuation and crossed processes as usual. The independence of the HAs on the spin direction has got the allure of a non-relativistic phenomenon as the velocity does indeed approach zero in the limit. Power corrections to the asymptotic behaviour were discussed in section 2.2.1. It was found that relative linear corrections (\( O(v) \)) to the leading order behaviour, which are present in the case of parity violation, can be accommodated for by CGCs (16). Relative corrections of \( O(v^2) \) are of kinematic and dynamic origin. The dynamic corrections originate from the form factors themselves. A systematic treatment of these effects, possible within some non-relativistic effective theory and logarithmic quantum corrections, is beyond the scope of this paper.

The application to include further particles, e.g. \( A \to B + C + D \) etc has not been discussed in this paper but should be possible by grouping particles together, say \( C + D \) to \((CD)\) and then proceed as before. Let us give some more detail. The main complication is that there is not a special decay axis anymore. One may single out the direction of flight of the \( B \)-particle in the \( A \)-restframe, rotate the states \( C \) and \( D \) with Wigner \( D \)-matrices onto the axis and then proceed to form invariants with CGC. This essentially reduces \( A \to B + C + D \) to \( A \to B + (CD) \). Thus we anticipate a result with two Wigner \( D \)-matrices and at least two CGC.
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A Isotropy of uniangular distribution

The uniangular distributions of unpolarised initial states ought to be isotropic (independent of the angle) at the kinematic endpoint exhibiting the underlying global SO(3)-symmetry. An important caveat, to be discussed towards the end of the section, is that the HA must be nonzero at the endpoint for isotropy to hold. We consider the \( A \to (B \to B_1B_2)C \), seen as a two times successive \( 1 \to 2 \) decays, as depicted in Fig. 1 and investigate the uniangular distribution:

\[
\frac{d\Gamma}{d\cos\theta_B}(A \to (B \to B_1B_2)C) = f(\cos\theta_B). \tag{A.1}
\]

The physical assumption is that the particle pair \( B_1B_2 \) does not interact with the \( C \)-particle in any significant way. Otherwise momentum can be exchanged and invalidate our conclusions.

The amplitude for \( A \to (B \to B_1B_2)C \) is given by (c.f. [1, 16] for similar formulae):

\[
A \propto \left( H^{(A)}_{\lambda_B\lambda_C} D_{J_A,\lambda_B+\lambda_C}^{J_A}(\Omega_A) \right) \left( H^{(B)}_{\lambda_B1,\lambda_B2} D_{\lambda_B,\lambda_B1+\lambda_B2}^{J_B}(\Omega_B) \right), \tag{A.2}
\]

where \( D_{mm'}^J(\Omega) \equiv \langle J_m | e^{-i\phi_1 J_x} e^{-i\phi_2 J_y} e^{-i\phi_2 J_z} | J_m' \rangle \) (with \( \Omega = (\phi_1, \theta, \phi_2) \)) are the Wigner D-functions. The factors of proportionality that we have dropped in (A.2) are known and have simple dependencies on \( J_{A,B,C} \) but are immaterial for our purposes. Since \( A \) is unpolarised we can choose \( \Omega_A = (0,0,0) \) which through \( D_{mm'}^J(0) = \delta_{mm'} \) implies helicity conservation on the decay axis: \( \lambda_A = \lambda_B + \lambda_C \). Further we choose \( \Omega_B = (\phi_B, \theta_B, -\phi_B) \). The decay rate is obtained by squaring the amplitude and summing incoherently over all helicity indices of final and initial state particles and coherently over internal helicity indices. Only \( \lambda_B \) is of the latter type but does effectively drop out because of the \( \lambda_A = \lambda_B + \lambda_C \) constraint. Whether or not \( C \) decays further is immaterial as long as we sum over all helicities of the final state particles.
and integrate over the associated angles. The angular distribution is then given by

\[ \frac{d\Gamma}{d\cos \theta_B} \propto \sum_{\lambda_A,\lambda_C,\lambda_{B_1},\lambda_{B_2}} |H_{\lambda_B,\lambda_C}^{(A)}|^2 |d_{\lambda_{C,\lambda_{B_1}+\lambda_{B_2}}}(\theta_B)|^2 |H_{\lambda_{B_1},\lambda_{B_2}}^{(B)}|^2 , \]  

(A.3)

where \( |d_{J_{mm'}}^J(\theta)| = |D_{J_{mm'}}^J(\Omega)| \) denotes the little Wigner \( d \)-function and is independent of the angles \( \phi_{1,2} \). Taking into account the endpoint relation \([9]\)

Figure 1: Decay \( A \rightarrow (B \rightarrow B_1B_2)C \) in the restframe of particle \( A \). Whether or not the \( C \) particle decays further is immaterial as long as we sum over all its final state helicities and integrate over all angles.

Eq. (A.3) can be rewritten as follows

\[ \frac{d\Gamma}{d\cos \theta_B} \propto \sum_{\lambda_B,\lambda_C,\lambda_{B_1},\lambda_{B_2}} |C_{(\lambda_B+\lambda_C)\lambda_B,\lambda_C}^{J_AJ_BJ_C}|^2 |d_{\lambda_{B_1,\lambda_{B_2}}}(\theta_B)|^2 |H_{\lambda_{B_1},\lambda_{B_2}}^{(B)}|^2 , \]  

(A.4)

where \( |\lambda_B - \lambda_C| \leq J_A \). First summing over \( \lambda_B \) using the second relation (A.6) and then summing over \( \lambda_C \) using the first relation (A.6), we may rewrite (A.4) as

\[ \frac{d\Gamma}{d\cos \theta_B} \propto \sum_{\lambda_{B_1},\lambda_{B_2}} |H_{\lambda_{B_1},\lambda_{B_2}}^{(B)}|^2 = \text{independent of } \cos(\theta_B) . \]  

(A.5)

\(^{11}\)For polarised or aligned states one can introduce a density matrix as for instance in \([16]\). Such situations naturally arise when the HA considered describes an intermediate process in a \( A_1A_2 \rightarrow A \rightarrow (B \rightarrow B_1B_2)C \)-type scattering for example. N.B. summing over initial state helicities is the same as integrating over all production angles. In the example above this corresponds to the angles of the inverse decay \( A \rightarrow A_1A_2 \), with respect to the \( A \rightarrow (B \rightarrow B_1B_2)C \) decay-plane in the \( A \)-restframe. Formally this is equivalent to taking the density matrix to be the unit matrix.
Thus we have shown that in the case where one sums over all initial and final state helicities the uniangular distribution in $A \to (B \to B_1B_2)C$ is isotropic; i.e. independent on the angle.

We wish to stress that this does not hold, generally, in the case where the HA vanishes like $v^\Omega$ with $\Omega > 0$. It is simple matter to use (16) and to construct counter examples. The intuitive reason is that since it scales like $v^\Omega$ the decay has a memory of the decay axis and thus can show preference for a certain direction.

Proof of:

\[
\sum_{m=J}^{J} |d_{mm'}^J|^2 = 1, \quad \sum_{m_1=-J_1}^{J_1} |C_{Mm_1m_2}^{J_1J_2}|^2 = \frac{2J+1}{2J_2+1}, \quad (A.6)
\]

where $M = (m_1 + m_2)$. The first property follows from the definition of the little Wigner $d$-functions: $d_{m,m'}^J(\theta) \equiv \langle Jm | U_R(\theta) | Jm' \rangle$ (e.g. [11]) with $U_R(\theta) = e^{-\theta J_y}$. One may write,

\[
\sum_{m} |d_{mm'}^J|^2 = \sum_{m} \langle Jm' | U_R(\theta) | Jm \rangle \langle Jm | U_R^\dagger(\theta) | Jm' \rangle = \langle Jm' | U_R(\theta) U_R^\dagger(\theta) | Jm' \rangle = \langle Jm' | Jm' \rangle = 1, \quad (A.7)
\]

to obtain the result where we have used the fact that $\sum_{m} |Jm\rangle \langle Jm| = 1_J$ is a complete set of states on the Hilbert space of angular momenta $J$. The second property follows from the so-called orthogonality relation of the CGC (e.g. [11]) from which one immediately obtains:

\[
\sum_{m_1m_2} |C_{Mm_1m_2}^{J_1J_2}|^2 = (2J+1). \quad (A.8)
\]

When only the sum over $m_1$, remains the averaging is still sufficient for the result to be independent of the direction and therefore $m_2$. Thus the sum over $m_2$ can be removed at the cost of dividing the righthand side by $(2J_2+1)$ and one therefore arrives at the second formula in (A.6).

**B Polarisation vectors**

We consider the rest frame of the $A$-particle, where $B$ and $C$ are decaying back to back. We denote vectorial ($J = 1$) polarisation vector by hatted
quantities and parameterise momenta and 0-helicity direction as follows

\[
\hat{\alpha}(0) = (0, 0, 0, 1), \quad q_A = (q_A, 0, 0, 0)
\]
\[
\hat{\beta}(0) = (v, 0, 0, (q_B)_0)/q_B, \quad q_B = ((q_B)_0, 0, 0, v)
\]
\[
\hat{\gamma}(0) = (-v, 0, 0, (q_C)_0)/q_C, \quad q_C = ((q_C)_0, 0, 0, -v),
\] (B.1)

where \(q_B(q_C)_0 \equiv \sqrt{q_B^2(q_C^2) + v^2}\) and \(q_r = \sqrt{q_r^2}\) with a slight abuse of notation.

The \(\pm\)-helicity polarisation vectors are written as

\[
\hat{\alpha}(\pm) = \hat{\beta}(\pm) = \hat{\gamma}(\mp) = (0, \pm 1, i, 0)/\sqrt{2},
\] (B.2)

This convention is consistent with the Condon-Shortly phase convention. For further important details concerning the relative phase between the \(\hat{\alpha}, \hat{\beta}\) and \(\hat{\gamma}\) we refer the reader to section B.1.

Polarisation vectors \(\omega \in \{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}\) satisfy:

\[
\omega(\lambda_1) \cdot \omega^*(\lambda_2) = -\delta_{\lambda_1\lambda_2}.
\] (B.3)

The minus sign is a remnant of the metric signature \((+, -, -, -)\). Throughout this paper \(\omega^*(\lambda) \equiv (\omega(\lambda))^*\). Furthermore we note that \(\omega^*(\lambda) = (-1)^\lambda \omega(\bar{\lambda})\) and therefore

\[
\omega(\lambda_1) \cdot \omega(\lambda_2) = -(-1)^{\lambda_1} \delta_{\lambda_1\lambda_2}.
\] (B.4)

It is instructive to write a scalar product of two polarisation vectors of non-equal type:

\[
\hat{\beta}(\lambda_B) \cdot \hat{\gamma}^*(\bar{\lambda}_C) = \begin{cases} 
-1 & \lambda_B = \lambda_C = \pm \\
-\sqrt{v^2 + q_B^2}\sqrt{v^2 + q_C^2}/(q_B q_C) & \lambda_B = \lambda_C = 0 \\
0 & \text{otherwise}
\end{cases}
\] (B.5)

**Endpoint:** At the endpoint \(v \to 0\) all polarisation vectors become proportional to each other \(^{12}\) \(\omega(\lambda) \equiv \hat{\alpha}(\lambda) = \hat{\beta}(\lambda) = \hat{\gamma}(\bar{\lambda})\) and are therefore mutually orthogonal,

\[
\hat{\alpha}(\lambda_1) \cdot \hat{\beta}^*(\lambda_2) = \hat{\alpha}(\lambda_1) \cdot \hat{\gamma}^*(\bar{\lambda}_2) = \hat{\beta}(\lambda_1) \cdot \hat{\gamma}^*(\bar{\lambda}_2) = -\delta_{\lambda_1\lambda_2}.
\] (B.6)

We observe that this can also be seen from (B.5) for \(v \to 0\).

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\(^{12}\)N.B. because the quantisation axis of the \(C\)-particle has been chosen to point into the other direction one has to flip the sign of the helicity index \((\bar{\lambda} = -\lambda)\).
B.1 Second helicity particle phase convention

It is intuitively clear that the helicity for the second spinor moving into the opposite direction as compared to the first one is just flipped. On top of that there is some freedom in redefining the phase. Let \( \chi_a(k, \lambda) \) \((a = 1, 2)\) be a two-spinor of the spinorial Lorentz group \( SL(2, \mathbb{C}) \) then \[ \chi_a(-k, \bar{\lambda}) = \xi_{\lambda} \chi_a(k, \lambda) , \] (B.7)

where \(-k\) above corresponds to \((k_0, -\vec{k})\). It is possible to choose the phase \( \xi_{\lambda} = 1 \) (Jacob-Wick phase convention \[ \[1, 2\] \]) which we shall adapt throughout.

Since any polarisation vector can be built from the spinor no phases appear for any of them. In particular for the vector: \( \omega_\mu(-k, \bar{\lambda}) = \omega_\mu(k, \lambda) \).

B.2 Higher spin polarisation tensors

Integer spin

Higher spin polarisation tensors of integer spin \( J \) denoted by \( \omega(k, \lambda)_{\mu_1..\mu_J} \) can be formed out of the \( J = 1 \) polarisation tensor \( \omega(\lambda, k)_\mu \) through appropriate Clebsch-Gordan series. This means that the transversity property \( k \cdot \omega(\lambda, k) = 0 \) is inherited:

\[ k^\mu \omega(k, \lambda)_{\mu_1..\mu_J} = 0 . \] (B.8)

Two further properties are complete symmetry under interchange of indices and tracelessness

\[ g^{\mu_i \mu_j} \omega(k, \lambda)_{\mu_1..\mu_J} = 0 , \quad 1 \leq i \neq j \leq J . \] (B.9)

Conversely by the symmetry property and the tracelessness are sufficient properties to find all irreducible representations of \( SO(3,1) \) and also of \( SL(2, \mathbb{C}) \), c.f. \[ \[9\] \] for precise statements.

Half-integer spin

A half integer spin \( J = (n + 1/2) \) polarisation tensor can be obtained from a spin \( n \) and spin \( 1/2 \)-polarisation tensor \( (n \otimes 1/2)_{SU(2)} = 1 \cdot (n + 1/2) + .., \) through a single Clebsch-Gordan series. This is the procedure of Rarita and

\[ \[13\] \] However, if there are multiple channels then the phase has to be specified cf. \[ \[18\] \].
The analogue of the tracelessness property for the spinorial index is
\[ \gamma_{\mu_i} \omega^{\mu_1 \cdots \mu_n} = 0, \]  
(B.10)
where \( \gamma_{\mu} \) is a Dirac matrix. Properties (B.8) and (B.9) remain relevant for the Lorentz indices. The contraction of the Dirac index is not shown.

References

[1] M. Jacob and G. C. Wick, “On the general theory of collisions for particles with spin,” Annals Phys. 7 (1959) 404 [Annals Phys. 281 (2000) 774].

[2] H. E. Haber, “Spin formalism and applications to new physics searches,” In *Stanford 1993, Spin structure in high energy processes* 231-272 [hep-ph/9405376].

[3] J. D. Richman, “An Experimenter’s Guide to the Helicity Formalism,” CALT-68-1148.

[4] J. G. Korner and M. Kramer, “polarisation effects in exclusive semileptonic Lambda(c) and Lambda(b) charm and bottom baryon decays,” Phys. Lett. B 275 (1992) 495.

[5] A. D. Martin and T. D. Spearman ..., “Elementary particle theory,” North-Holland, Amsterdam, and Elsevier, New York, 1970. xvi, 528 pp

[6] G. Hiller and R. Zwicky, “(A)symmetries of weak decays at and near the kinematic endpoint,” JHEP 1403 (2014) 042 arXiv:1312.1923 [hep-ph].

[7] G. Hiller and R. Zwicky, “Endpoint Relations for Baryons,” arXiv:2107.12993 [hep-ph].

[8] P. A. Zyla et al. [Particle Data Group], “Review of Particle Physics,” PTEP 2020 (2020) no.8, 083C01 doi:10.1093/ptep/ptaa104 Copy to ClipboardDownload

[9] I. L. Buchbinder and S. M. Kuzenko, “Ideas and methods of supersymmetry and supergravity: Or a walk through superspace,” Bristol, UK: IOP (1998) 656 p
N. Straumann, “Relativistic quantum theory: An introduction to quantum field theory,” Berlin, Germany: Springer (2005) 329 p

[10] S. Weinberg, “The Quantum theory of fields. Vol. 1: Foundations,” Cambridge, UK: Univ. Pr. (1995) 609 p

[11] K.T. Hecht in “Symmetry Properties of Clebsch-Gordan Coefficients,” Graduate Texts in Contemporary Physics 2000, pp 269-272 Springer New York

[12] R. Zwicky and T. Fischbacher, “On discrete Minimal Flavour Violation,” Phys. Rev. D 80 (2009) 076009 [arXiv:0908.4182 [hep-ph]].

[13] S. Bolognesi, Y. Gao, A. V. Gritsan, K. Melnikov, M. Schulze, N. V. Tran and A. Whitbeck, “On the spin and parity of a single-produced resonance at the LHC,” Phys. Rev. D 86 (2012) 095031 [arXiv:1208.4018 [hep-ph]].

[14] D. J. Miller, 2, S. Y. Choi, B. Eberle, M. M. Muhlleitner and P. M. Zerwas, “Measuring the spin of the Higgs boson,” Phys. Lett. B 505 (2001) 149 [hep-ph/0102023].

[15] Y. Gao, A. V. Gritsan, Z. Guo, K. Melnikov, M. Schulze and N. V. Tran, “Spin determination of single-produced resonances at hadron colliders,” Phys. Rev. D 81 (2010) 075022 [arXiv:1001.3396 [hep-ph]].

[16] J. R. Dell’Aquila and C. A. Nelson, “P or CP Determination by Sequential Decays: ,” Phys. Rev. D 33 (1986) 80.

[17] V. D. Barger, K.-m. Cheung, A. Djouadi, B. A. Kniehl and P. M. Zerwas, “Higgs bosons: Intermediate mass range at e+ e- colliders,” Phys. Rev. D 49 (1994) 79 [hep-ph/9306270].

[18] D. Marangotto, Adv. High Energy Phys. 2020 (2020), 6674595 doi:10.1155/2020/6674595 [arXiv:1911.10025 [hep-ph]].

[19] W. Rarita and J. Schwinger, “On a theory of particles with half integral spin,” Phys. Rev. 60 (1941) 61.