Classification of $n$–th order linear ODEs up to projective transformations

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Abstract

Classification of $n$–th ($n \geq 2$) order linear ODEs is considered. The equation reduced to LaguerreForsyth form by a point transformation then, the other calculations would have done on this form. This method is due to V.A. Yumaguzhin.

Keywords: Linear ODE, symmetry, Lie algebra, projective transformations.

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Introduction

The local classification of linear ODEs up to projective transformations is obtained in this article. For $n \leq 2$, it is well known that any $n$–th order linear ODE can be transformed locally to the form $y^{(n)} = 0$ by a point transformation. For $n \geq 0$, this statement is incorrect: there is finite number of different equivalence classes of linear ODEs.

First this problem was posed by classics of the 15 century E. Laguerre, G.H. Halphen and others. They obtain results concerning classification of third and fourth orders linear ODE. Here, this problem is solved for $n \geq 0$ in a neighborhood of regular germs.

Consider a general $n$–th order ODE which is solved by the higher order derivative

$$y^{(n)} = \sum_{i=1}^{n} a_{n-i}(x)y^{(n-i)},$$

(1)

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where $y(x)$ is a smooth function of $x$.

Lie shows that the point symmetry group of a second ordinary linear differential equation has dimension at most eight, conversely the equation admits an eight-dimensional symmetry group if and only if it can be mapped, by a point transformation, to the linear equation $y'' = 0$. Thus, the main result is any linear second ordinary differential equation can mapped to the equation $y'' = 0$. So, the condition of second ordinary linear differential equation is specified.

A same result shows that for $n \geq 3$, any linear ODE admits at most an $(n + 4)$—dimensional symmetry group of point transformation, therefore, the symmetry group is $(n + 4)$—dimensional if and only if the equation is equivalent to the linear equation $y^{(n)} = 0$. In continuation we will work on the general form of linear ODE in the form of (1) once $n \geq 3$.

1 Laguerre-Forsyth form

The classification of linear differential equations is a special case of the general problem of classifying differential operators, which has a variety of important applications. Consider an $n$—th order ordinary differential operator corresponding to (1)

$$
D = a_n(x)D^n_x + a_{n-1}(x)D^{n-1}_x + \cdots + a_1(x)D_x + a_0.
$$

(2)

The aim is finding out when two operators, or two linear ODE, of type (2), can be mapped to each other by a suitable change of variables. To preserve linearity, we restrict to those of the form

$$
\bar{x} = \varphi(x), \quad \bar{y} = \psi(x)y,
$$

(3)

the chain rule action shows that $D_{\bar{x}} = (\varphi(x))^{-1}D_x$, and with a rescaling of the dependent variable by $\psi(x) = e^{\varphi(x)}$ we obtain the gauge factor. So, two differential operator $\bar{D}$ and $D$ is called gauge equivalent if they satisfy

$$
\bar{D} = \psi \cdot D \cdot \frac{1}{\psi}, \quad \bar{x} = \varphi(x).
$$

(4)
A straightforward calculation shows that the change of variables (3) is given by
\[ \bar{x} = \varphi(x) = \int \frac{dx}{\sqrt[n]{a_n(x)}}, \quad \psi(x) = \left| a_n(x) \right|^{\frac{1-n}{2n}} \exp \left\{ \int^x \frac{a_{n-1}(y)}{na_n(y)} dy \right\}, \]
thus (1) is gauge equivalent to an operator of the form
\[ \mathcal{D} = \pm D_x^n + a_{n-2}(x)D_x^{n-2} + \cdots + a_0(x). \quad (5) \]

If \( \rho(x) \) be a nonvanishing smooth function, two differential operator \( \mathcal{D} \) and \( \mathcal{D} \) is called \textit{projective equivalence} if they satisfy
\[ \mathcal{D} = \rho \cdot \psi \cdot \mathcal{D} \cdot \frac{1}{\psi}, \quad \bar{x} = \varphi(x). \quad (6) \]

A nonsingular \( n \)-th order linear operator of type (5) is projectively equivalent to one in Laguerre-Forsyth form
\[ \mathcal{D} = D_x^n + a_{n-3}(x)D_x^{n-3} + \cdots + a_0(x), \quad (7) \]
with change of variable (6) in the form of
\[ \bar{x} = \varphi(x), \quad \bar{y} = \varphi^{-\frac{n}{2}} y, \quad \rho = \varphi^{-n}, \]
where \( \varphi(x) \) is a solution of the \textit{Schwarzian} equation
\[ \frac{n(n^2 - 1)}{12} \frac{\varphi_x \varphi_{xxx} - \frac{3}{2} \varphi_x^2 \varphi_{xx}}{\varphi_x^2} = a_{n-2}(x). \]

2 Classification of linear ODEs of Laguerre-Forsyth form

A useful theorem help us to reduce the classification of ODEs up to a special transformation.

**Theorem 2.1** Let \( \Delta_1 \) and \( \Delta_2 \) be ODEs of the form (7). If there is a point transformation that takes \( \Delta_1 \) to \( \Delta_2 \), that is
\[ f(x) = \frac{ax + b}{cx + d}, \quad \hat{f}(x, y) = |f'|^{\frac{n+1}{2}} \cdot y, \quad a, b, c, d \in \mathbb{R}. \quad (8) \]
A transformation \((f, \hat{f})\) of the form (8) is generated by a projective transformation \(f\) on \(\mathbb{R}\). The isomorphisms \(f \to (f, \hat{f})\) makes a group of point transformations in the form of (8). Consider these projective transformations in a group \(G\) and denoted by all projective transformations of \(\mathbb{R}\), i.e.,

\[
G = \{f(x) = \frac{ax + b}{cx + d} \mid a, b, c, d \in \mathbb{R} \text{ and } ad \neq bc\}.
\]

It is easy to check that \(G\) has two connected component \(G_1 = \{f \in G \mid f' > 0\}\) and \(G_2 = \{f \in G \mid f' < 0\}\), thus, \(G = G_1 \cup G_2\).

### 2.1 Bundles of Laguerre-Forsyth form

Consider \(x\) as a coordinate on \(\mathbb{R}\) and \(a_{n-3}, a_{n-2}, \ldots, a_0\) coordinates on \(\mathbb{R}^{n-2}\). Then, we can construct a fiber bundle corresponding to (7) in the form of

\[
p : \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{R}.
\]

Any ODE of type (7) identifies with \(\Delta = \{p_n = a_{n-3}(x)p_{n-3} + \cdots + a_0(x)p_0\}\) is a section of (9) denoted by \(S_\Delta : x \to (x, a_{n-3}(x), \ldots, a_0(x))\), where the identification \(\Delta \to S_\Delta\) is a bijection.

Let \(\Delta_2 = \{\tilde{p}_n = \tilde{a}_{n-3}(\tilde{x})\tilde{p}_{n-3} + \cdots + \tilde{a}_0(\tilde{x})\tilde{p}_0\}\) be an ODE of the form (7). Subjecting \(\Delta_2\) to an transformation (8), the, we obtain linear ODE \(\Delta_1 = \{p_n = a_{n-3}(x)p_{n-3} + \cdots + a_0(x)p_0\}\). The coefficients \(\Delta_2\) are expressed in terms of coefficients of \(\Delta_1\) and projective transformation \(f^{-1}\) by the equation

\[
\tilde{a}_{n-i} = F_{n-i}(a_{n-3}, \ldots, a_{n-i}; \frac{df^{-1}}{d\tilde{x}}, \ldots, \frac{d^{i+1}f^{-1}}{d\tilde{x}^{i+1}}), \quad i = 3, 4, \ldots, n. \quad (10)
\]

The equation (10) is a lifting of a projective transformation \(f\) to diffeomorphism \(\tilde{f} : \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{R} \times \mathbb{R}^{n-2}\) such that \(p \circ \tilde{f} = f \circ p\).

For any \(f \in G\), a transformation of sections of \(p\) defined by the formula

\[
S \to f(S) = \tilde{f} \circ S \circ f^{-1},
\]

then, equation (10) can be represented as \(S_{\Delta_2} = f(S_{\Delta_1})\).
Lemma 2.2 Consider two equations of the form (7). Then a transformation $(f, \hat{f})$ of the form (8) maps $\Delta_1$ to $\Delta_2$ if and only if $S_{\Delta_2} = f(S_{\Delta_1})$.

The main result of the lemma (2.2) is the classification of ODEs of the form (7) up to transformation (8) reduces to classification of germs of sections of $p$ up to projective transformation on $R$.

2.2 Classification of regular germs

Let $S$ be a section of $p$ and $a$ be a point in domain of $S$. Denoted by $\{S\}_a$ the germ of $p$ at $a$. Let $\{S\}_{a_1}$ and $\{S\}_{a_2}$ be germs of sections $S_1$ and $S_2$ respectively. We say that $\{S\}_{a_1}$ and $\{S\}_{a_2}$ are $G_+\text{-equivalent}$ if there exist $f \in G_+$ such that $\{f(S_1)\}_{f(a_1)} = \{S\}_{a_2}$. A germ $\{S\}_a$ is regular of class $i$ if there exist a neighborhood $O$ of $a$ and subbundle $E_i$ such that $\text{Im}S|_O \subset E_i$. If $\{S\}_a$ is a regular germ of class $i \geq 0$, then in a neighborhood of $a$ we have $S(x) = (x, 0, ..., 0, a_i(x), ..., a_0(x))$. In the rest of the paper we will often denoted $\{S\}_a$ by $\{a_i, ..., a_0\}_a$. If $\{S\}_a$ is a regular germ, then $a$ is a regular point of $S$.

Definition 2.3 Let $S$ be a section of $p$ and $v$ be a vector field of the Lie algebra of group $G$, if $\theta_t$ be the flow of $v$, we say $v$ is a projective symmetry of $S$ if one of the following statements satisfied:

1) $\theta_t(S) = \overline{\theta_t} \circ S \circ \theta_t^{-1} = S$,

2) $\left. \frac{d}{dt} \theta_t(S) \right|_{t=0} = 0$.

Denote by $P(S)$ the Lie algebra of all projective symmetries of $S$.

Let $\Upsilon$ be the set of all regular germs at $0 \in R$ of sections of $p$. Define

$$\Upsilon_i = \left\{ \{S\}_a | \text{dim} P(S) = i \right\}, \quad i = 0, 1, 3,$$

and denote $\Upsilon = \Upsilon_0 \cup \Upsilon_1 \cup \Upsilon_3$. If $G_0$ be the isotropic subgroup of $G$ in 0, then, $\Upsilon_i$’s are $G_0$-invariant.
Define $\Upsilon_{r,i} \subset \Upsilon_r$ be the subset of all regular germs of class $i$. It follows from the invariance of subbundle $E_i$'s under $G_0$, $\Upsilon_{r,i}$ is $G_0$-invariant. Consequently we have

$$\Upsilon_r = \bigcup_{i=0}^{n-3} \Upsilon_{r,i},$$

where this union is separated invariant subsets.

Let $\mathbb{R}_+$ and $\mathbb{R}_-$ be the set of positive and negative real numbers respectively. If $\ell_{r,i} : \Upsilon_{r,i} \to (\mathbb{R}\setminus\{0\}) \times \mathbb{R}$ be a map by the formula $\{a_i,...,a_0\} \mapsto (a_i(0),a'_i(0))$ and

$$G_{0+} \times \Upsilon_{r,i} \to \Upsilon_{r,i}$$

$$ (f,\{S\}_0) \mapsto \{f(S)\}_0,$$

be the action of $G_{0+}$ on $\Upsilon_{r,i}$ then,

**Lemma 2.4** The map $\ell_{r,i}|_\Theta$ is a bijection from the orbit $\Theta$ of the action (11) either to $(\mathbb{R}_+) \times \mathbb{R}$ or to $(\mathbb{R}_-) \times \mathbb{R}$.

Let $\Omega^+_r = \ell^{-1}_{r,1}((1,0))$ and $\Omega^-_r = \ell^{-1}_{r,1}((-1,0))$. Denote by $\Gamma_{r,i}$ the subset of $\Omega^+_r \cup \Omega^-_r$ defined in the following way:

1) $\Gamma_{r,0} = \Omega^+_r \cup \Omega^-_r$ for $i = 0$,

2) if $i > 0$, then, $\Omega_{r,i}$ consists of all germs $\{a_i,...,a_0\}$ from $\Omega^+_r \cup \Omega^-_r$ satisfying one of the following conditions:

i) $a_{i-j} = 0$ for all odd numbers $j$ with $1 \leq j \leq i$,

ii) there exist an odd number $r$ with $1 \leq r \leq i$ such that $a_{i-r}(0) > 0$ and if $r > 1$, then $a_{i-j}(0) = 0$ for all odd numbers $j$ with $1 \leq j < r$.

### 2.3 Classification of regular germs from the family $\Omega_{r,i}$

Let $\mu \in G_-$ defined by $\mu(x) = -x$ for all $x \in \mathbb{R}$, then, due to lemma (2.4) and attentive to $\mu(\Omega^-_{r,i}) = \Omega^+_{r,i}$ we have:
**Theorem 2.5**  
1) The set $\Omega_{r,i}^+ \cup \Omega_{r,i}^-$ is a family of all germs from $\Upsilon_{r,i}$ nonequivalent with respect to $G_{0^+}$.

2) If $n - i$ is odd, then $\Omega_{r,i}^+$ is a family of all germs from $\Upsilon_{r,i}$ nonequivalent with respect to $G_0$.

3) If $n - i$ is even, $\Gamma_{r,i}$ is a family of all germs from $\Upsilon_{r,i}$ nonequivalent with respect to $G_0$.

An important corollary concludes this section as follows:

**Corollary 2.6** Classification of regular germs of sections of (7) is:

1) The family of germs of the form

$$\{\pm 1 + a(x)x^2, a_{i-1}(x), ..., a_0\}_0$$

is a family of all regular germs of class $i$ nonequivalent with respect to $G_{0^+}$.

2) If $n - i$ is odd, then the family of germs of the form

$$\{1 + a(x)x^2, a_{i-1}(x), ..., a_0\}_0$$

is a family of all regular germs of class $i$ nonequivalent with respect to $G_0$.

3) If $n - i$ is even, then the family of germs of the form

$$\{\pm 1 + a(x)x^2, a_{i-1}(x), ..., a_0\}_0,$$

satisfying one of the following conditions:

a) $a_{i-j}(0) = 0$ for all odd numbers $j$ with $1 \leq j \leq i$,

b) there exist an odd number $r$ with $1 \leq r \leq i$ such that $a_{i-r}(0) > 0$ and if $r > 1$, then $a_{i-j}(0) = 0$ for all odd number $j$ with $1 \leq j \leq r$,

is the family of germs of class $i$ nonequivalent with respect to $G_0$. 
3 Conclusion

This article was a qualification of classification of linear ODEs due to V.A. Yumaguzhin. First we transform the general form of ODEs to Laguerre-Forsyth form, then by a suitable change of variable up to projective transformation we reduce this classification to classification of the sections of bundles, next by construction germs and specially regular germs of this sections near identity, the classification reduced to classifying of regular germs by providing some invariant subsets of the bundles.

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