We prove in this note the following result:

**Theorem.**— *A smooth hypersurface* $X$ *of dimension* $n \geq 2$ *and degree* $d \geq 3$ *admits no endomorphism of degree* $> 1$.

Since the case of quadrics is treated in [PS], this settles the question of endomorphisms of hypersurfaces. We prove the theorem in Section 1, using a simple but efficient trick devised by Amerik, Rovinsky and Van de Ven [ARV]. In Section 2 we collect some general results on endomorphisms of projective manifolds; we prove in particular that *ramified* endomorphisms occur only on varieties of Kodaira dimension $-\infty$. This leads naturally to ask the existence problem for Fano manifolds; we will settle this question for surfaces.

I am indebted to I. Dolgachev for bringing the problem to my attention.

1. Hypersurfaces

The proof of the theorem is based on the following result, which appears essentially in [ARV]:

**Proposition 1.**— *Let* $X$ *be a submanifold of* $\mathbb{P}^N$, *of dimension* $n$, *and let* $f : X \to X$ *be an endomorphism of* $X$ *such that* $f^* \mathcal{O}_X(1) = \mathcal{O}_X(m)$ *for some integer* $m \geq 2$.

Then

$$c_n(\Omega^1_X(2)) \leq 2^n \deg(X).$$

Let us sketch the proof following [ARV]. We first observe that the sheaf $\Omega^1_{\mathbb{P}^N}(2)$ is spanned by its global sections; therefore $\Omega^1_X(2)$, which is a quotient of $\Omega^1_{\mathbb{P}^N}(2)|_X$, is also spanned by its global sections. Let $\sigma$ be a general section of $\Omega^1_X(2)$; then $\sigma$ and its pull-back $f^* \sigma \in H^0(X, \Omega^1_X(2m))$ have isolated zeroes [ARV, lemma 1.1]. Counting these zeroes gives

$$c_n(\Omega^1_X(2m)) \geq \deg(f) \ c_n(\Omega^1_X(2)).$$

Since $\deg(f) = m^n$ we get $c_n(\Omega^1_X(2)) \leq m^{-n} c_n(\Omega^1_X(2m))$. Replacing $f$ by $f^k$ we obtain this inequality for $m$ arbitrarily large; therefore

$$c_n(\Omega^1_X(2)) \leq \lim_{m \to \infty} m^{-n} c_n(\Omega^1_X(2m)) = 2^n \deg(X). \blacksquare$$
Proof of the Theorem: We first discuss the case \( n \geq 3 \). Then \( b_2(X) = 1 \), so that the condition \( f^*\mathcal{O}_X(1) = \mathcal{O}_X(m) \) is automatic. In view of the Proposition we just have to prove that \( c_n(\Omega^1_X(2)) > 2^n \deg(X) \). From the exact sequences

\[
0 \to \Omega^1_{\mathbb{P}^n+1}(2)|_X \longrightarrow \mathcal{O}_X(1)^{n+2} \longrightarrow \mathcal{O}_X(2) \to 0 \\
0 \to \mathcal{O}_X(2-d) \longrightarrow \Omega^1_{\mathbb{P}^n+1}(2)|_X \longrightarrow \Omega^1_X(2) \to 0
\]

we get \( c(\Omega^1_X(2)) = (1 + h)^{n+2}(1 + 2h)^{-1}(1 + (2-d)h)^{-1} \), so that

\[
c_n(\Omega^1_X(2)) = d \text{ Res}_0 \omega \quad \text{with} \quad \omega = \frac{(1 + x)^{n+2}}{x^{n+1}(1 + 2x)(1 + (2-d)x)}.
\]

Straightforward computations give

\[
\text{Res}_\infty \omega = \frac{1}{2(d-2)} \quad \text{Res}_{-\frac{1}{2}} \omega = \frac{(-1)^{n+1}}{2d} \quad \text{Res}_{\frac{1}{d+1}} \omega = \frac{-(d-1)^{n+2}}{d(d-2)},
\]

hence, by the residue theorem,

\[
c_n(\Omega^1_X(2)) = \frac{2(d-1)^{n+2} - d + (-1)^n(d-2)}{2(d-2)}.
\]

Using \( (d-1)^2 = d(d-2) + 1 \) we get \( c_n(\Omega^1_X(2)) > d(d-1)^n \geq d2^n \), hence the result in this case.

For the case \( n = 2 \), we observe that the result is straightforward when \( K_X \) is ample or trivial (see Proposition 2 below); therefore it only remains to prove it for cubic surfaces. This can be easily done with the above method, but we will deduce it from the more general case of Del Pezzo surfaces (Proposition 3).

Remark. – The same method applies (with some work) to complete intersections of multidegree \((d_1, \ldots, d_p)\) in \( \mathbb{P}^{n+p} \), provided one of the \( d_i \) is \( \geq 3 \). On the other hand it does not work in general for complete intersection of quadrics.

2. Other manifolds

Let \( X \) be a compact manifold, and let \( f \) be an endomorphism of \( X \) degree > 1; by this we mean that \( f \) is generically finite (or equivalently surjective). If \( X \) is projective (or more generally Kähler), \( f \) is actually finite: otherwise it contracts some curve \( C \) to a point, so that the class of \([C]\) in \( H^*(X, \mathbb{Q}) \) is mapped to \( 0 \) by \( f_* \). This contradicts the following remark:

Lemma 1. – Let \( d = \deg f \). The endomorphisms \( f^* \) and \( d^{-1}f_* \) of \( H^*(X, \mathbb{Q}) \) are inverse of each other.

This follows from the formula \( f_*f^* = d \text{Id} \).
The existence of an endomorphism of degree \( > 1 \) has strong implications on the Kodaira dimension of \( X \):

**Proposition 2.** Let \( X \) be a compact manifold, with an endomorphism \( f \) of degree \( > 1 \).

a) The Kodaira dimension \( \kappa(X) \) is \( < \dim(X) \).

b) If \( \kappa(X) \geq 0 \), \( f \) is étale.

*Proof:* a) follows for instance from [KO]; let us give the proof for completeness. Consider the pluricanonical maps \( \varphi_m : X \to \mathbb{P}(H^0(X, mK_X)) \) associated to the linear systems \( |mK_X| \) \( (m \geq 1) \). The pull-back map \( f^* : H^0(X, mK_X) \to H^0(X, mK_X) \) is injective, and therefore bijective; we have a commutative diagram:

\[
\begin{array}{ccc}
X & \phi_m & \mathbb{P}(H^0(X, mK_X)) \\
\downarrow f & & \downarrow \phi(f^*) \\
X & \phi_m & \mathbb{P}(H^0(X, mK_X))
\end{array}
\]

In particular, we see that \( f \) induces an automorphism of \( \varphi_m(X) \). If \( \dim \varphi_m(X) = \dim X \) this implies \( \deg f = 1 \).

b) Let \( m \) be a positive integer such that the linear system \( |mK_X| \) is non-empty. Let \( F \) be the fixed divisor of this system, and \( |M| \) its moving part, so that \( mK_X \equiv F + M \). The Hurwitz formula reads \( K_X \equiv f^*K_X + R \), where \( R \) is the ramification divisor of \( f \); this gives

\[
F + M \equiv (f^*F + mR) + f^*M.
\]

In particular we have \( h^0(f^*M) \leq h^0(M) = h^0(mK_X) \); since the pull-back map \( f^* : H^0(X, M) \to H^0(X, f^*M) \) is injective, we get \( h^0(f^*M) = h^0(mK_X) \), which means that \( |f^*M| \) is the moving part of \( |mK_X| \) and \( f^*F + mR \) its fixed part. Thus

\[
F = f^*F + mR
\]

in the divisor group \( \text{Div}(X) \) of \( X \). Let \( \nu : \text{Div}(X) \to \mathbb{Z} \) be the homomorphism which takes the value 1 on each irreducible divisor. Since \( \nu(f^*F) \geq \nu(F) \), the above equality is possible only if \( R = 0 \); we conclude that \( f \) is étale. ■

Every Kodaira dimension \( < \dim X \) can indeed occur, as shown by the varieties \( V \times A \), where \( A \) is an abelian variety. It seems possible that all examples with \( \kappa(X) \geq 0 \) are of this type, up to an étale covering and perhaps some birational transformation. We can make this precise for surfaces:
Proposition 3. Let $S$ be a projective surface with $\kappa(S) \geq 0$, admitting an endomorphism of degree $> 1$. Then $S$ is an abelian surface or a quotient $(E \times C)/G$, where $E$ is an elliptic curve, $C$ a curve of genus $\geq 1$, and $G$ a finite group of automorphisms of $E$ and $C$ acting freely on $E \times C$.

Proof: We first observe that the surface $S$ is minimal: if $E$ was an exceptional curve on $S$, its pull back $f^{-1}(E)$ would be a disjoint union of exceptional curves $E_1, \ldots, E_d$ on $S$, with $d = \deg f$. These curves would have different classes in $H^2(S, \mathbb{Q})$ mapping to the same class $[E]$ under $f_*$, contrary to Lemma 1.

By Proposition 2, $f$ is étale; this implies that the topological Euler number $e(S)$ is zero. Also we have $\kappa(S) = 0$ or $1$. In the first case, the classification of surfaces shows that $S$ is abelian or bielliptic – that is, of the form $(E \times C)/G$ with both $E$ and $C$ elliptic. If $\kappa(S) = 1$, $S$ admits an elliptic fibration $S \rightarrow B$; since $e(S) = 0$ the fibres of $f$ are (possibly multiple) smooth elliptic curves. It is then well-known that $S$ is isomorphic to a quotient $(E \times C)/G$ (see e.g. [B], chap. VI).

Let us now turn to ramified endomorphisms. By Proposition 2 we must consider manifolds with $\kappa(X) = -\infty$; a natural place to look at is Fano manifolds. For surfaces we have a complete answer:

Proposition 4. A Del Pezzo surface $S$ admits an endomorphism of degree $> 1$ if and only if $K_S^2 \geq 6$.

Proof: a) A Del Pezzo surface of degree $\geq 6$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at some of the points $(1,0,0), (0,1,0), (0,0,1)$. The first case is trivial; in the second case, the endomorphisms $(X,Y,Z) \mapsto (X^p,Y^p,Z^p)$ of $\mathbb{P}^2$ extend to the blown up surface.

b) Let us now consider a Del Pezzo surface $S$ with an endomorphism $f : S \rightarrow S$ of degree $d > 1$. Let $E$ be an exceptional curve on $S$, $F = f(E)$, and $\delta$ the degree of $f|_E : E \rightarrow F$. We have $f_*E = \delta F$ and therefore $f^*F \equiv \frac{d}{\delta}E$ (Lemma 1). Taking squares gives $F^2 = -\frac{d^2}{\delta^2}$. Because of the genus formula $C^2 + C.K = 2g(C) - 2$, the only curves with negative square on a Del Pezzo surface are the exceptional ones. Thus $F$ is exceptional, $d = \delta^2$ and $f^*F \equiv \delta E$; since the right hand side does not move, this is an equality of divisors. It means that $f$ is ramified along $E$ with ramification index $\delta$. In other words, if we denote by $\mathcal{E}$ the (finite) set of exceptional curves on $S$ and by $R$ the ramification divisor of $f$, we have $R = \sum_{E \in \mathcal{E}} (\delta - 1)E + Z$, where $Z$ is an effective divisor. Intersecting with $-K_S$ gives

$$-K_S \cdot R \geq (\delta - 1) \text{Card}(\mathcal{E}).$$

For each $E \in \mathcal{E}$ we have $f^*K_S \cdot E = K_S \cdot f_*E = \delta K_S \cdot F = -\delta$, and therefore $(f^*K_S - \delta K_S) \cdot E = 0$. We can assume that $\mathcal{E}$ spans the Picard group of $S$.
(this holds as soon as $K_S^2 \leq 7$), thus $f^*K_S \equiv \delta K_S$. Then the Hurwitz formula $K_S \equiv f^*K_S + R$ gives $R \equiv (\delta - 1)(-K_S)$, so that the above inequality becomes $K_S^2 \geq \text{Card}(\mathcal{E})$. This is impossible for $K_S^2 \leq 5$, as the surface $S$ contains then at least 10 exceptional curves. $lacksquare$

For Fano threefolds we know the answer in the case $b_2 = 1$, as a consequence of the more general results of [A] and [ARV]: the only Fano threefold with $b_2 = 1$ admitting an endomorphism of degree $> 1$ is $\mathbb{P}^3$. Their methods apply to some other Fano threefolds, but the general case seems to require new techniques.

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