Planar anti-Ramsey numbers of matchings

Gang Chen\textsuperscript{1}, Yongxin Lan\textsuperscript{2} and Zi-Xia Song\textsuperscript{3*}
\textsuperscript{1}School of mathematics and statistics
Ningxia University, China
\textsuperscript{2}Center for Combinatorics and LPMC
Nankai University, Tianjin, 300071, China
\textsuperscript{3}Department of Mathematics
University of Central Florida
Orlando, FL 32816, USA

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Abstract

Given a positive integer \( n \) and a planar graph \( H \), let \( T_n(H) \) be the family of all plane triangulations \( T \) on \( n \) vertices such that \( T \) contains a subgraph isomorphic to \( H \). The planar anti-Ramsey number of \( H \), denoted \( \text{ar}_p(n, H) \), is the maximum number of colors in an edge-coloring of a plane triangulation \( T \in T_n(H) \) such that \( T \) contains no rainbow copy of \( H \). In this paper we study planar anti-Ramsey numbers of matchings. For all \( t \geq 1 \), let \( M_t \) denote a matching of size \( t \). We prove that for all \( t \geq 6 \) and \( n \geq 3t - 6 \),

\[ 2n + 3t - 15 \leq \text{ar}_p(n, M_t) \leq 2n + 4t - 14, \]

which significantly improves the existing lower and upper bounds for \( \text{ar}_p(n, M_t) \). It seems that for each \( t \geq 6 \), the lower bound we obtained is the exact value of \( \text{ar}_p(n, M_t) \) for sufficiently large \( n \). This is indeed the case for \( M_6 \). We prove that \( \text{ar}_p(n, M_6) = 2n + 3 \) for all \( n \geq 30 \).

1 Introduction

All graphs considered in this paper are finite and simple. For a graph \( G \) we use \( |G| \) and \( e(G) \) to the number of vertices and number of edges of \( G \), respectively. For a vertex \( x \in V(G) \), we will use \( N_G(x) \) to denote the set of vertices in \( G \) which are adjacent to \( x \). We define \( d_G(x) = |N_G(x)| \). For any \( A \subseteq V(G) \), the subgraph of \( G \) induced by \( A \), denoted \( G[A] \), is the graph with vertex set \( A \) and edge set \( \{xy \in E(G) : x, y \in A\} \). We denote \( G \setminus A \) the subgraph of \( G \) induced on \( V(G) \setminus A \). If \( A = \{a\} \), we simply write \( G \setminus a \). For disjoint subsets \( A, B \) of \( V(G) \), we use \( e_G(A, B) \) to denote the number of edges in \( G \) with one end in \( A \) and the other in \( B \). Since every planar bipartite graph on \( n \geq 3 \) vertices has at most \( 2n - 4 \) edges, we will frequently use the fact that \( e_G(A, B) \leq 2(|A| + |B|) - 4 \) when \( G \) is planar and \( |A \cup B| \geq 3 \). Given two isomorphic graphs \( G \) and \( H \), we may (with a slight but common abuse of notation) write \( G = H \). For any positive integer \( k \), let \([k] := \{1, 2, \ldots, k\} \). We use the convention that “\( A := \)” means that \( A \) is defined to be the right-hand side of the relation.

*Corresponding Author. Email address: Zixia.Song@ucf.edu
Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós [4] in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. A subgraph of an edge-colored graph is \textit{rainbow} if all of its edges have different colors. Given a planar graph $H$ and a positive integer $n \geq |H|$, let $T_n(H)$ be the family of all plane triangulations $T$ on $n$ vertices such that $T$ contains a subgraph isomorphic to $H$. The \textit{planar anti-Ramsey number} of $H$, denoted $ar_p(n, H)$, is the maximum number of colors in an edge-coloring of a plane triangulation $T \in T_n(H)$ such that $T$ contains no rainbow copy of $H$. Analogous to the relation between anti-Ramsey numbers and Turán numbers proved in [4], planar anti-Ramsey numbers are closely related to planar Turán numbers [13], where the \textit{planar Turán number} of $H$, denoted $ex_p(n, H)$, is the maximum number of edges of a planar graph on $n$ vertices that contains no subgraph isomorphic to $H$.

**Proposition 1.1 ([13])** Given a planar graph $H$ and a positive integer $n \geq |H|$, 

$$1 + ex_p(n, H) \leq ar_p(n, H) \leq ex_p(n, H),$$

where $H = \{H - e : e \in E(H)\}$.

Dowden [3] began the study of planar Turán numbers (under the name of “extremal” planar graphs). Results on planar Turán numbers of paths and cycles can be found in [3, 12]. The study of planar anti-Ramsey numbers was initiated by Horňák, Jendrol', Schiermeyer and Soták [6] (under the name of rainbow numbers). Results on planar anti-Ramsey numbers of paths and cycles can be found in [6, 13]. Colorings of plane graphs that avoid rainbow faces have also been studied, see, e.g., [5, 7, 17, 18]. Various results on anti-Ramsey numbers can be found in: [1, 2, 9, 10, 11, 14, 16] to name a few.

Finding exact values of $ar_p(n, H)$ is far from trivial. As observed in [6], an induction argument in general cannot be applied to compute $ar_p(n, H)$ because deleting a vertex from a plane triangulation may result in a graph that is no longer a plane triangulation. In this paper, we study planar anti-Ramsey numbers of matchings. For all $t \geq 1$, let $M_t$ denote a matching of size $t$. In [8], the exact value of $ar_p(n, M_t)$ when $t \leq 4$ was determined, and lower and upper bounds for $ar_p(n, M_t)$ were also established for all $t \geq 5$ and $n \geq 2t$. Recently, the exact value of $ar_p(n, M_5)$ was determined in [15] and an improved upper bound for $ar_p(n, M_t)$ was also obtained in [15]. We summarize the results in [8, 15] below.

**Theorem 1.2 ([8])** Let $n$ and $t$ be positive integers. Then 

(a) for all $n \geq 7$, $ar_p(n, M_3) = n$.

(b) for all $n \geq 8$, $ar_p(n, M_4) = 2n - 2$.

(c) for all $t \geq 5$ and $n \geq 2t$, $2n + 2t - 10 \leq ar_p(n, M_t) \leq 2n + 2k - 7 + 2(2^{t-2})$.

**Theorem 1.3 ([15])** Let $n$ and $t$ be positive integers. Then 

(a) for all $n \geq 11$, $ar_p(n, M_5) = 2n$. 

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(b) for all \( t \geq 5 \) and \( n \geq 2t \), \( ar_p(n, M_t) \leq 2n + 6t - 17 \).

In this paper, we further improve the existing lower and upper bounds for \( ar_p(n, M_t) \).

**Theorem 1.4** For all \( t \geq 6 \) and \( n \geq 3t - 6 \), \( 2n + 3t - 15 \leq ar_p(n, M_t) \leq 2n + 4t - 14 \).

Theorem 1.4 significantly improves the lower bound in Theorem 1.2(c) and the new upper bound in Theorem 1.3(b). We believe that for each \( t \geq 6 \), the lower bound we obtained in Theorem 1.4 is the exact value of \( ar_p(n, M_t) \) for sufficiently large \( n \). This is indeed the case for \( M_6 \).

**Theorem 1.5** For all \( n \geq 30 \), \( ar_p(n, M_6) = 2n + 3 \).

It seems that the method we developed in the proof of Theorem 1.5 can be applied to close the gap in Theorem 1.4. We prove Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3.

## 2 Proof of Theorem 1.4

We are ready to prove Theorem 1.4. Let \( t, n \) be given as in the statement. We first prove that \( ar_p(n, M_t) \geq 2n + 3t - 15 \). Let \( P \) be a path with vertices \( v_1, v_2, \ldots, v_{t-4} \) in order. Let \( H \) be the plane triangulation obtained from \( P \) by adding two adjacent vertices \( x, y \) and joining each of \( x \) and \( y \) to all the vertices on \( P \) with the outer face of \( H \) having \( x, y, v_1 \) on its boundary. Then \( |H| = t - 2 \geq 4 \) and \( H \) is hamiltonian. Let \( T_H \) be the plane triangulation obtained from \( H \) by adding a new vertex to each face \( F \) of \( H \) and then joining it to all vertices on the boundary of \( F \). Then \( T_H \) is a plane triangulation on \((t - 2) + (2(t - 2) - 4) = 3t - 10 \) vertices. Let \( w \) be the new vertex added to the outer-face of \( H \). Let \( T \) be the plane triangulation on \( n \) vertices obtained from \( T_H \) by adding \( n - (3t - 10) \geq 4 \) vertices, say \( w_1, w_2, \ldots, w_{n-3t+10} \), to the face of \( T_H \) containing \( x, y, w \), such that \( w_i \in E(T) \), and for all \( i \in [n - 3t + 9] \), \( w_i \) is adjacent to \( x, y, w_{i+1} \) in \( G \). The construction of \( T \) when \( t = 6 \) and \( n = 13 \) is depicted in Figure 1. Clearly, \( T \in T_n(M_t) \). Let \( c \) be an edge-coloring of \( T \) by first coloring all the edges \( ww_1, w_1w_2, \ldots, w_{n-3t+9}w_{n-3t+10} \) by color 1 and then all the remaining edges of \( T \) by distinct colors other than 1. It can be easily checked that \( T \) has no rainbow spanning subgraph of \( T \) with the total number of colors used by \( c \) is \((3n - 6) - (n - 3t + 10) + 1 = 2n + 3t - 15 \). This proves that \( ar_p(n, M_t) \geq 2n + 3t - 15 \), as desired.

It remains to prove that \( ar_p(n, M_t) \leq 2n + 4t - 13 \). Suppose \( ar_p(n, M_t) \geq 2n + 4t - 13 \) for some \( t \geq 6 \) and \( n \geq 3t - 6 \). Then there exists a \( T \in T_n(M_t) \) such that \( T \) has no rainbow copy of \( M_t \) under some onto mapping \( c : E(T) \to [k] \), where \( k \geq 2n + 4t - 13 \). We choose such a \( T \) with \( t \) minimum. Let \( G \) be a rainbow spanning subgraph of \( T \) with \( k \) edges. Then \( G \) does not contain \( M_t \) because \( T \) has no rainbow copy of \( M_t \). By minimality of \( t \) and Theorem 1.3(a) (when \( t = 6 \)), \( G \) contains a copy of \( M_{t-1} \). Let \( M := \{u_iw_i \in E(G) : i \in [t - 1] \} \) be a matching of size \( t - 1 \) in \( G \), and let \( V(M) := \{u_1, \ldots, u_{t-1}, w_1, \ldots, w_{t-1} \} \). Let \( R := V(G) \setminus V(M) \). For each \( i \in [t - 1] \), we may assume that \( |N_G(u_i) \cap R| \leq |N_G(w_i) \cap R| \). Since \( M \) is the largest matching in \( G \), we see that \( G \) has no \( M \)-augmenting path. It follows that \( G[R] \) has no edges, and for each \( i \in [t - 1] \), either \( |N_G(u_i) \cap R| = 0 \) or \( |N_G(u_i) \cap R| = 1 \) with \( N_G(u_i) \cap R = N_G(w_i) \cap R \). We may further assume that \( u_1w_1, \ldots, u_{t-1}w_{t-1} \) are such that \( |N_G(u_i) \cap R| = 1 \) for all \( 0 \leq i \leq \ell \), and \( |N_G(u_j) \cap R| = 0 \) for all \( j \in \{\ell + 1, \ldots, t - 1\} \),
Figure 1: The construction of $T$ when $t = 6$ and $n = 13$

where $0 \leq \ell \leq t - 1$. Then $e_G(\{u_1, \ldots, u_\ell, w_1, \ldots, w_\ell\}, R) = 2\ell$. Let $L := \{w_{\ell+1}, \ldots, w_{t-1}\}$. Then $e_G(L, R) = 0 \leq 2n - 2t - 2\ell - 2$ when $\ell = t - 1$, and $e_G(L, R) \leq 2(n - (t - 1) - \ell) - 4 = 2n - 2t - 2\ell - 2$ when $\ell \leq t - 2$ because $G[L \cup R] \setminus E(G[L])$ is a planar bipartite graph on $n - (t - 1) - \ell \geq 3$ vertices.

Since $G$ is planar and $|V(M)| = 2(t - 1) > 3$, we have $e(G[V(M)]) \leq 3(2t - 2) - 6 = 6t - 12$. Thus, $e(G) = e(G[V(M)]) + e_G(\{u_1, \ldots, u_\ell, w_1, \ldots, w_\ell\}, R) + e_G(L, R) \leq (6t - 12) + 2\ell + (2n - 2t - 2\ell - 2) = 2n + 4t - 14$, contrary to $e(G) = k \geq 2n + 4t - 13$. This completes the proof of Theorem 1.4.

**Remark.** For $t \geq 7$, the condition “$n \geq 3t - 6$” in the statement of Theorem 1.4 can be replaced by “$n \geq 3t - 7$”.

### 3 Proof of Theorem 1.5

We need to introduce more notation that shall be used in this section only. For $n \geq 3$, let $\mathcal{T}_n$ be the set of all plane triangulations on $n$ vertices, and let $\mathcal{T}_n^-$ be the set of all planar graphs with $n$ vertices and $3n - 7$ edges. Clearly, every graph in $\mathcal{T}_n^-$ is isomorphic to a plane triangulation on $n$ vertices with one edge removed. By abusing notation, let $e(\mathcal{T}_n) := 3n - 6$ and $e(\mathcal{T}_n^-) := 3n - 7$. It is known that every plane triangulation on $n \geq 4$ vertices is 3-connected. It is also known that every plane triangulation on $n \leq 10$ vertices has a Hamilton cycle and every plane triangulation on $n \geq 11$ vertices does not necessarily have a Hamilton cycle. We summarize these facts as follows.

**Observation 3.1** Let $T$ be a planar triangulation on $n \geq 4$ vertices. Then

(a) $T$ is 3-connected.

(b) for every $n \leq 10$, $T$ has a Hamilton cycle.

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1 The third author would like to thank Jason Bentley, a Ph.D. student at the University of Central Florida, for his help in carefully verifying these facts with her.
(c) for every $n \geq 11$, $T$ does not necessarily have a Hamilton cycle.

Let $o(H)$ denote the number of odd components in a graph $H$. We shall make use of the following theorem in the proof of Theorem 3.5.

**Theorem 3.2 (Berge-Tutte Formula)** Let $G$ be a graph on $n$ vertices and let $d$ be the size of a maximum matching of $G$. Then there exists an $S \subseteq V(G)$ with $|S| \leq d$ such that

$$2d = n - o(G \setminus S) + |S|.$$  

Moreover, each odd component of $G \setminus S$ is factor-critical.

**Proof of Theorem 3.5** Let $n \geq 30$ be an integer. By Theorem 1.3, $ar_p(n, M_6) \geq 2n+3$. We next show that $ar_p(n, M_6) \leq 2n+3$. Suppose $ar_p(n, M_6) \geq 2n+4$. Then there exists a $T \in \mathcal{T}_n(M_6)$ such that $T$ has no rainbow $M_6$ under some onto mapping $c : E(T) \to [k]$, where $k \geq 2n+4$. Let $G$ be a rainbow spanning subgraph of $T$ with $k$ edges. By Theorem 1.3(a), $G$ has a copy of $M_5$. Clearly, $G$ has no copy of $M_6$ because $T$ has no rainbow copy of $M_6$ under $c$. By Theorem 3.2 there exists an $S \subseteq V(G)$ with $s := |S| \leq 5$ such that $q := o(G \setminus S) = n + s - 10$. Let $H_1, H_2, \ldots, H_q$ be all the odd components of $G \setminus S$. We may assume that $|H_1| \leq |H_2| \leq \cdots \leq |H_q|$. Let $r := \max \{i : |H_i| = 1\}$. Then $n = |G| \geq |S| + (|H_1| + \cdots + |H_r|) + (|H_{r+1}| + \cdots + |H_q|) \geq s + r + 3(q-r) = s - 2r + 3(n + s - 10)$. It follows that $r \geq n + 2s - 15 \geq 15$. Let $S := \{v_1, \ldots, v_s\}$ when $s \geq 1$ and $V(H_i) = \{u_i\}$ for all $i \in [r]$. We may further assume that $d_G(u_1) \geq d_G(u_2) \geq \cdots \geq d_G(u_r)$. Let $U := \{u_1, \ldots, u_r\}$ and $W := V(G) \setminus (S \cup U)$. Then $w := |W| = n - s - r$ and $e_G(U, S) \leq 2(r + s) - 4$ when $r + s \geq 3$. We next prove several claims.

**Claim 1.** If $G$ has two edge-disjoint matchings of size 5, say $M'$ and $M''$, then $T[V(G) \setminus V(M' \cup M'')]$ has no edges.

**Proof.** Suppose $T[V(G) \setminus V(M' \cup M'')]$ has an edge $e$. We may assume that $c(e) \neq c(e')$ for all $e' \in M'$. But then $M' \cup \{e\}$ is a rainbow $M_6$ in $T$ under the coloring $c$, a contradiction. 

**Claim 2.** If $w + s \leq 9$, then $H := G[W \cup S \cup \{u_1, \ldots, u_{10-w-s}\}] \notin \mathcal{T}_{10}$.

**Proof.** Suppose $H \in \mathcal{T}_{10}$. Then $H$ has a Hamilton cycle by Observation 3.1(b), and thus has two edge-disjoint matchings of size 5. By Claim 1, $T[\{u_{11-w-s}, \ldots, u_r\}]$ has no edges. But then

$$e(T) = e(T[W \cup S \cup \{u_1, \ldots, u_{10-w-s}\}]) + e_T(\{u_{11-w-s}, \ldots, u_r\}, W \cup S \cup \{u_1, \ldots, u_{10-w-s}\}) \leq e(T_{10}) + (2n - 4) = 24 + (2n - 4),$$

which implies that $n \leq 26$ because $e(T) = 3n - 6$, contrary to $n \geq 30$.

**Claim 3.** $|H_q| \geq 3$. 


Proof. Suppose $|H_q| < 3$. Then $r = q$ and so $w + s = n - q = 10 - s$. It follows that
\[
2n + 4 \leq e(G) = e(G[W \cup S]) + e_G(U, S) \leq (3(w + s) - 6) + (2(n - w) - 4)
= 2n + w + 3s - 10 = 2n + (10 - 2s) + 3s - 10 = 2n + s,
\]
which implies that $s \geq 4$. If $s = 4$, then $w = 2$. But then
\[
2n + 4 \leq e(G) = e(G[S]) + e(G[W]) + e_G(S, U \cup W) \leq e(T_4) + 1 + 2n - 4 = 2n + 3,
\]
which is impossible. Thus $s = 5$, and so $w + s = 5$ and $r = q = n - 5$. By Claim 2, $e(H) \leq 23$. Since $2n + 4 \leq e(G) = e(G[S]) + e_G(U, S) \leq e(G[S]) + 2n - 4$, we see that $e(G[S]) \geq 8$. Thus $G[S] \in T_5$ or $G[S] \in T_5^\perp$. Then $d_G(u_1) \leq 12 - e(G[S])$, else $e(G[S \cup \{u_1\}) \geq 13 > e(T_6) = 12$, a contradiction. Suppose $G[S] \in T_5$. Then $d_G(u_1) \leq 3$ and $e_G(U, S) = 2n - 5$. It follows that $d_G(u_5) = 3$, else $e_G(U, S) = e_G(\{u_1, u_2, u_3, u_4\}, S) + e_G(\{u_5, \ldots, u_r\}, S) \leq 12 + 2(n - 9) = 2n - 6$. But then $e(H) = e(G[S]) + e_G(\{u_1, \ldots, u_5\}, S) = 9 + 3 \times 5 = 24$, contrary to $e(H) \leq 23$. This proves that $G[S] \in T_5^\perp$. We may assume that $v_1v_2 \notin E(G[S])$. Then $d_G(u_1) \leq 12 - e(G[S]) = 4$, and $e_G(U, S) = e(G) - e(G[S]) \geq (2n + 4) - 8 = 2n - 4$, which implies that $e_G(U, S) = 2n - 4$. If $d_G(u_2) \geq 4$, then $e(G[S \cup \{u_1, u_2\}) = 8 + 8 = 16 > e(T_7)$, a contradiction. Thus $d_G(u_2) \leq 3$. Furthermore, $d_G(u_5) = 3$, else $e_G(U, S) = e_G(\{u_1, u_2, u_3, u_4\}, S) + e_G(\{u_5, \ldots, u_r\}, S) \leq (4 + 9) + (2(n - 9) = 2n - 5$, contrary to $e_G(U, S) = 2n - 4$. Since $e(H) \leq 23$, we see that $d_G(u_1) = 3$. Then $d_G(u_6) = 3$, else $e_G(U, S) \leq 15 + 2(n - 10) = 2n - 5$, a contradiction. Since $G[S \cup \{u_1, \ldots, u_6\}]$ does not contain $K_{3,3}$ as a subgraph, we may assume that $v_1u_6 \in E(G)$. Then $H \in T_\infty$ because $e(H) = e(G[S]) + e_G(\{u_1, \ldots, u_5\}, S) = 8 + 15 = e(T_{10})$. Note that $H + v_1v_2 \in T_{10}$. By Observation 3.1(b), $H$ has a hamiltonian path with $v_1$ as an end. Since $v_1u_6 \in E(G)$, we see that $G[S \cup \{u_1, \ldots, u_6\}]$ has two edge-disjoint matchings of size 5. By Claim 1, $T[\{u_7, \ldots, u_r\}]$ has no edges. But then
\[
3n - 6 = e(T) = e(T[S \cup \{u_1, \ldots, u_6\}]) + e_T(\{u_7, \ldots, u_r\}, S \cup \{u_1, \ldots, u_6\})
\leq e(T_{11}) + (2n - 4) = 27 + (2n - 4),
\]
which implies that $n \leq 29$, contrary to $n \geq 30$.

By Claim 3, $w \geq 3$ and $r \leq q - 1$. Then $n \geq s + |H_1| + \cdots + |H_q| \geq s + q + 2 = n + 2s - 8$, which implies that $s \leq 4$, with $s = 4$ only when $w = |H_q| = 3$.

Claim 4. $|H_{q-1}| \geq 3$.

Proof. Suppose $|H_{q-1}| = 1$. By Claim 3, $r = q - 1$. Thus $w + s = n - r = n - (q - 1) = 11 - s$. It follows that
\[
2n + 4 \leq e(G) = e(G[W \cup S]) + e_G(U, S) \leq (3(w + s) - 6) + (2(n - w) - 4)
= 2n + w + 3s - 10 = 2n + (11 - 2s) + 3s - 10 = 2n + s + 1,
\]
which implies that $s \geq 3$. If $s = 3$, then $w + s = 8$ and so $d_G(u_1) = 3$, else $e(G) \leq e(G[W \cup S]) + e_G(U, S) \leq e(T_8) + 2(n - 8) = 2n + 2$, a contradiction. Since $G$ does not contain $K_{3,3}$ as a subgraph,
we see that \( d_G(u_3) \leq 2 \). By Claim 2, \( e(H) \leq 23 \). Thus \( e(G[W \cup S]) = e(H) - d_G(u_1) - d_G(u_2) \leq 20 - d_G(u_2) \). Note that \( e_G(U, S) \leq 3 + d_G(u_2) + 2(n - w - s - 2) = 2n - 17 + d_G(u_2) \). But then

\[
2n + 4 \leq e(G) = e(G[W \cup S]) + e_G(U, S) \leq (20 - d_G(u_2)) + (2n - 17 + d_G(u_2)) = 2n + 3,
\]

which is impossible. Thus \( s = 4 \). Then \( w = 3 \). It follows that \( W = V(H_q) \) and \( G[W] = K_3 \) because \( G[W] \) is factor-critical. Then \( d_G(u_2) \geq 3 \), else \( e_G(U, S) \leq 4 + 2(n - 8) = 2n - 12 \) and so \( 2n + 4 \leq e(G) = e(G[W \cup S]) + e_G(U, S) \leq (2n - 12) = 2n + 3 \), a contradiction. Since \( e_G(U, S) \leq 2(n - 3) - 4 = 2n - 10 \), we see that \( e(G[W \cup S]) = e(G) - e_G(U, S) \geq (2n + 4) - (2n - 10) \geq 14 \). Thus \( G[W \cup S] \in \mathcal{T}_7 \) or \( G[W \cup S] \in \mathcal{T}_7^- \). Suppose \( G[W \cup S] \in \mathcal{T}_7^- \). Then \( d_G(u_i) \leq 3 \) for all \( i \in [r] \). Since \( e_G(U, S) = e(G) - e(T_7) \geq 2(n - 7) + 3 \), we see that \( d_G(u_1) = d_G(u_2) = d_G(u_3) = 3 \). But then \( e(H) = 24 \), contrary to \( e(H) \leq 23 \). This proves that \( G[W \cup S] \in \mathcal{T}_7^- \). Then \( d_G(u_2) \leq 3 \), else \( e(G[W \cup S \cup \{u_1, u_2\}]) = e(T_7^-) + 8 \leq 14 + 8 \geq e(T_9) \), a contradiction. Since \( e_G(U, S) = e(G) - e(T_7^-) \geq 2(n - 7) + 4 \), we see that \( d_G(u_2) = d_G(u_3) = 3 \). Then \( d_G(u_1) = 3 \) because \( e(H) \leq 23 \). Since \( G[W \cup S] \in \mathcal{T}_7^- \) and \( G[W] = K_3 \), we may assume that \( G[W \cup S] + v_1v \in \mathcal{E}_v \) for some \( v \in W \cup S \) with \( v \neq v_1 \). Note that \( G[S \cup \{u_2, u_3, u_4\}] \) does not contain \( K_{3,3} \) as a subgraph. We may further assume that \( v_1u_4 \in E(G) \). Then \( H + v_1v \in \mathcal{T}_{10} \). By Observation 3.4(b), \( H \) has a hamiltonian path with \( v_1 \) as an end. Since \( v_1u_4 \in E(G) \), we see that \( G[W \cup S \cup \{u_1, u_2, u_3, u_4\}] \) has two edge-disjoint matchings of size 5. By Claim 1, \( T[\{u_5, \ldots, u_r\}] \) has no edges. But then

\[
3n - 6 = e(T) = e(T[W \cup S \cup \{u_1, u_2, u_3, u_4\}]) + e_T(\{u_5, \ldots, u_r\}, W \cup S \cup \{u_1, u_2, u_3, u_4\}) \leq e(T_11) + (2n - 4) = 27 + (2n - 4),
\]

which implies that \( n \leq 29 \), contrary to \( n \geq 30 \).
3(w + s) - 8 = 3(n - r) - 8. But then

\[ 2n + 4 \leq e(G) = e_G(U, S) + e(G[S \cup W]) \]
\[ \leq rs + 3(n - r) - 8 \]
\[ = 3n - (3 - s)r - 8 \]
\[ \leq 3n - (3 - s)(n + 2s - 15) - 8, \]

which is impossible because \( s \leq 2 \) and \( n \geq 30 \). This completes the proof of Theorem 1.5.

**Remark.** In the proof of Theorem 1.5, Claim 1 is applied to two vertex-disjoint matchings, instead of edge-disjoint matchings. It seems that the method we developed in the proof of Theorem 1.5 can be used to close the gap in Theorem 1.4.

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