EXISTENCE AND ASYMPTOTICS OF FRONTS IN NON LOCAL COMBUSTION MODELS

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Abstract. We prove the existence and provide the asymptotics for non local fronts in homogeneous media.

Contents

1. Introduction 1
2. Truncation of the domain 3
2.1. Construction of sub- and super-solutions 3
2.2. Proof of Proposition 2.1 5
3. Proof of Theorem 1.1 7
4. Asymptotic behavior 11
References 12

1. Introduction

This paper is devoted to the study of fronts propagation in homogeneous media for a fractional reaction-diffusion equation appearing in combustion theory. More precisely, we consider the following classical scalar model for the combustion of premixed gas with ignition temperature:

\[
u_t + (-\partial_{xx})^\alpha u = f(u) \quad \text{in } \mathbb{R} \times \mathbb{R},\]

where the function \(f\) satisfies:

\[
\begin{cases}
  f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous function} \\
  f(u) \geq 0 \text{ for all } u \in \mathbb{R} \text{ and } \text{supp} \ f = [\theta, 1] \\
  f'(1) < 0
\end{cases}
\]

where \(\theta \in (0, 1)\) is a fixed number (usually referred to as the ignition temperature).

The operator \((-\partial_{xx})^\alpha\) denotes the fractional power of the Laplace operator in one dimension (with \(\alpha \in (0, 1]\)). It can be defined by the following singular integral

\[
(-\partial_{xx})^\alpha u(x) = c_\alpha \text{PV} \int_{\mathbb{R}} \frac{u(x) - u(z)}{|x - z|^{1+2\alpha}} \, dz
\]
where PV stands for the Cauchy principal value. This integral is well defined, for instance, if \( u \) belongs to \( C^2(\mathbb{R}) \) and satisfies
\[
\int_{\mathbb{R}} \frac{|u(x)|}{(1 + |x|)^{1+2\alpha}} \, dx < +\infty
\]
(in particular, smooth bounded functions are admissible). Alternatively, the fractional Laplace operator can be defined as a pseudo-differential operator with symbol \(|\xi|^{2\alpha}\). We refer the reader to the book by Landkof where an extensive study of \((-\partial_{xx})^\alpha\) is performed by means of harmonic analysis techniques (see [Lan72]).

In this paper, we will always take \( \alpha \in (1/2, 1] \), and we are interested in particular solutions of (1) which describe transition fronts between the stationary states 0 and 1 (traveling fronts). These traveling fronts are solutions of (1) that are of the form
\[
(4) \quad u(t, x) = \phi(x + ct)
\]
with
\[
\begin{cases}
\lim_{x \to -\infty} \phi(x) = 0 \\
\lim_{x \to +\infty} \phi(x) = 1.
\end{cases}
\]
The number \( c \) is the speed of propagation of the front. It is readily seen that \( \phi \) must solve
\[
(-\partial_{xx})^\alpha \phi + c \phi' = f(\phi) \quad \text{for all } x \in \mathbb{R}
\]
When \( \alpha = 1 \) (standard Laplace operator), it is well known that there exists a unique speed \( c \) and a unique profile \( \phi \) (up to translation) that correspond to a traveling front solution of (1) (see e.g. [BL90, BN92, BNS85]). The goal of this paper is to generalize these results to the case \( \alpha \in (1/2, 1) \). We are thus looking for \( \phi \) and \( c \) satisfying
\[
\begin{cases}
(-\partial_{xx})^\alpha \phi + c \phi' = f(\phi) \quad \text{for all } x \in \mathbb{R} \\
\lim_{x \to -\infty} \phi(x) = 0 \\
\lim_{x \to +\infty} \phi(x) = 1 \\
\phi(0) = \theta
\end{cases}
\]
(the last condition is a normalization condition which ensures the uniqueness of \( \phi \)). Our main theorem is the following:

**Theorem 1.1.** Let \( \alpha \in (1/2, 1] \) and assume that \( f \) satisfies (2), then there exists a unique pair \((\phi_0, c_0)\) solution of (5). Furthermore, \( c_0 > 0 \) and \( \phi_0 \) is monotone increasing.

We will also obtain the following result, which describes the asymptotic behavior of the front at \(-\infty\):
Theorem 1.2. Let \( \alpha \in (1/2, 1) \) and assume that \( f \) satisfies (2). Let \( \phi_0 \) be the unique solution of (5) provided by Theorem 1.1. Then there exist \( m, M \) such that
\[
\phi_0(x) \leq \frac{M}{|x|^{2\alpha-1}} \quad \text{for } x \leq -1
\]
and
\[
\phi'_0(x) \geq \frac{m}{|x|^{2\alpha}} \quad \text{for } x \leq -1.
\]

The proof of Theorem 1.1 follows classical arguments developed by Berestycki-Larrouturou-Lions [BLL90] (see also Berestycki-Nirenberg [BN92]): Truncation of the domain, construction of sub- and super-solutions and passage to the limit. As usual, one of the main difficulty is to make sure that we recover a finite, non trivial speed of propagation at the limit. The main novelty (compared with similar results when \( \alpha = 1 \)) is the construction of sub- and super-solutions where the classical exponential profile is replaced by power tail functions.

2. Truncation of the Domain

The first step is to truncate the domain: for some \( b > 0 \), we consider the following problem:

\[
\begin{cases}
(-\partial_{xx})^\alpha \phi_b + c_b \phi_b' = f(\phi_b) & \text{for all } x \in [-b, b] \\
\phi_b(x) = 0 & \text{for } s \leq -b \\
\phi_b(x) = 1 & \text{for } s \geq b \\
\phi_b(0) = \theta.
\end{cases}
\]

Proposition 2.1. Assume \( \alpha \in (1/2, 1) \) and that \( f \) satisfies (2). Then there exists a constant \( M \) such that if \( b > M \) the truncated problem (6) has a unique solution \( (\phi_b, c_b) \). Furthermore, the following properties hold:

(i) There exists \( K \) independent of \( b \) such that \(-K \leq c_b \leq K\).
(ii) \( \phi_b \) is non-decreasing with respect to \( x \) and satisfies \( 0 < \phi_b(x) < 1 \) for all \( x \in (-b, b) \).

Before we can prove this Proposition, we need to detail the construction of sub- and super-solutions.

2.1. Construction of sub- and super-solutions. In the proof of the existence of traveling waves for the standard Laplace operator \( (\alpha = 1) \), sub- and super-solution of the form \( e^{\gamma x} \) play a crucial role, in particular in the determination of the asymptotic behavior of the traveling waves as \( x \to -\infty \). These particular functions are replaced, in the case of the fractional Laplace operator, by functions with polynomial tail. In what follows, we will rely on two important lemmas:
Lemma 2.2. Let $\beta \in (0,1)$ and define
\[ \varphi(x) = \begin{cases} \frac{1}{|x|^\beta} & \text{if } x < -1 \\ 1 & \text{if } x > -1. \end{cases} \]
Then $\varphi$ satisfies
\[ (-\partial_{xx})^\alpha \varphi + c\varphi'(x) = -\frac{c_\alpha}{2\alpha |x|^{2\alpha}} + c \frac{\beta}{|x|^{\beta+1}} + O \left( \frac{1}{|x|^{\beta+2\alpha}} \right) \]
when $x \to -\infty$.

Lemma 2.3. Let $\beta > 1$ and define
\[ \bar{\varphi}(x) = \begin{cases} \frac{1}{|x|^\beta} & x < -1 \\ 0 & x > -1 \end{cases} \]
then
\[ (-\partial_{xx})^\alpha \bar{\varphi} + c\bar{\varphi}'(x) = -\frac{c_\alpha}{\beta-1 |x|^{2\alpha+1}} + c \frac{\beta}{|x|^{\beta+1}} + O \left( \frac{1}{|x|^{\beta+2\alpha}} \right) \]
when $x \to -\infty$.

Proof of Lemma 2.2. We want to estimate $(-\partial_{xx})^\alpha \varphi$ for $x < -1$. We have:
\[ (-\partial_{xx})^\alpha \varphi(x) = -c_\alpha \text{PV} \int_{\mathbb{R}} \frac{\varphi(x+y) - \varphi(x)}{|y|^{1+2\alpha}} dy, \]
which we decompose as follows:
\[ (-\partial_{xx})^\alpha \varphi(x) = c_\alpha \int_{-\infty}^{-1-x} \frac{\varphi(x) - \varphi(x+y)}{|y|^{1+2\alpha}} dy + c_\alpha \int_{-1-x}^{+\infty} \frac{\varphi(x) - \varphi(x+y)}{|y|^{1+2\alpha}} dy \]
\[ = I + II \]
A simple explicit computation yields:
\[ II = \left( \frac{1}{|x|^\beta} - 1 \right) \frac{c_\alpha}{2\alpha |x|^{2\alpha+1}}. \]
Performing the change of variables $y = xz$, one gets
\[ I = \frac{c_\alpha}{|x|^{\beta+2\alpha}} \int_{+\infty}^{-\frac{1}{z}-1} \frac{|z+1|^\beta - 1}{|z+1|^\beta |z|^{1+2\alpha}} dz. \]
Note that the integrand has a singularity at $z = 0$, and this integral has to be understood as a principal value. We also observe that the integrand has a singularity at $z = -1$, but since $\beta < 1$, this singularity is integrable, and thus
\[ I \sim -c_\alpha \frac{1}{|x|^{\beta+2\alpha}} \text{PV} \int_{-1}^{+\infty} \frac{|z+1|^\beta - 1}{|z+1|^\beta |z|^{1+2\alpha}} dz. \]
as $x \to -\infty$.
We deduce:
\[ (-\partial_{xx})^\alpha \varphi(x) = -\frac{c_\alpha}{2\alpha |x|^{2\alpha}} + O \left( \frac{1}{|x|^{\beta+2\alpha}} \right) \]
when \( x \to -\infty \), and the result follows. \(\square\)

**Proof of Lemma 2.3.** Again, we decompose \((-\partial_{xx})^\alpha \bar{\varphi}\) as follow:

\[
(-\partial_{xx})^\alpha \bar{\varphi}(x) = c_\alpha \int_{-\infty}^{-1-x} \frac{\bar{\varphi}(x) - \bar{\varphi}(x+y)}{|y|^{1+2\alpha}} dy + c_\alpha \int_{-1-x}^{+\infty} \frac{\bar{\varphi}(x) - \bar{\varphi}(x+y)}{|y|^{1+2\alpha}} dy
\]

Now, a simple explicit computation yields:

\[
II = \frac{c_\alpha}{|x|^{\beta} - 1} \frac{\bar{\varphi}(x) - \bar{\varphi}(x+y)}{|y|^{1+2\alpha}}
\]

And performing the change of variables \( y = xz \), one gets

\[
I = \int_{-\infty}^{-1-x} \frac{1}{|x|^{\beta+2\alpha}} \int_{z}^{0} \frac{|z| - 1}{|z|^{1+2\alpha}} dz.
\]

Note that the integrand as a singularity at \( z = 0 \), and this integral has to be understood as a principal value. We also observe that the integrand has a singularity at \( z = -1 \) and since \( \beta > 1 \), this singularity is divergent and thus

\[
I \sim -\frac{c_\alpha}{\beta - 1} |x|^{\beta - 1}.
\]

We deduce:

\[
(-\partial_{xx})^\alpha \bar{\varphi}(x) = -\frac{c_\alpha}{\beta - 1} |x|^{\beta - 1} + O \left( \frac{1}{|x|^{\beta+2\alpha}} \right)
\]

which yields the result. \(\square\)

### 2.2. Proof of Proposition 2.1

We now turn to the proof of Proposition 2.1. First, we fix \( c \in \mathbb{R} \) and consider the following problem:

\[
\begin{cases}
(-\partial_{xx})^\alpha \phi + c \phi' = f(\phi) & \text{for all } x \in [-b, b] \\
\phi(x) = 0 & \text{for } x \leq -b \\
\phi(x) = 1 & \text{for } x \geq b
\end{cases}
\]

We have:

**Lemma 2.4.** For any \( c \in \mathbb{R} \), Equation (7) has a unique solution \( \phi_c \). Furthermore \( \phi_c \) is non-decreasing with respect to \( x \) and \( c \to \phi_c \) is continuous.

**Proof.** Since 1 and 0 are respectively super- and sub-solutions, we can use Perron’s method (recall that the fractional laplacian enjoys a comparison principle) to prove the existence of a solution \( \phi_c(x) \) for any \( c \in \mathbb{R} \). By a sliding argument, we can show that \( \phi_c \) is unique and non-decreasing with respect to \( x \). The fact that the function \( c \to \phi_c \) is continuous follows from classical arguments (see [BN92] for details). \(\square\)

We now have to show that there exists a unique \( c = c_b \) such that \( \phi_{c_b}(0) = \theta \). This will be a consequence of the following lemma:
Lemma 2.5. There exist constants $M$, $K$ such that for $b > M$ the followings hold:

1. if $c > K$ then the solution of (7) satisfies $\phi_c(0) < \theta$,
2. if $c < -K$ then the solution of (7) satisfies $\phi_c(0) > \theta$.

Together with the fact that $\phi_c(0)$ is continuous with respect to $c$, Lemma 2.5 implies that there exists $c_b \in [-K, -K]$ such that $\phi_{c_b}(0) = \theta$ and is thus a solution of (6). This completes the proof of Proposition 2.1.

Proof of Lemma 2.5. We consider the function

$$\varphi(x) = \begin{cases} 
\frac{1}{|x|^{2\alpha-1}} & x < -1 \\
1 & x \geq -1
\end{cases}$$

and note that Lemma 2.2 (with $\beta = 2\alpha - 1$) yields that if $c$ is large enough ($c \geq \frac{2\alpha}{2\alpha(2\alpha-1)}$), then

$$(-\partial_{xx})^{\alpha} \varphi(x) + c\varphi'(x) \geq 0$$

for $x \leq -A$ (for some $A$ large enough). We can also assume that $\varphi(x) \leq \theta$ for $x \leq -A$, and so

$$(-\partial_{xx})^{\alpha} \varphi(x) + c\varphi'(x) \geq f(\varphi) = 0 \quad \text{for } x \leq -A.$$

Furthermore, for $-A < x < -1$, $(-\partial_{xx})^{\alpha} \varphi(x)$ is bounded while

$$c\varphi'(x) \geq c \frac{2\alpha - 1}{A^{2\alpha}}.$$

For $c$ large enough, we thus have

$$(-\partial_{xx})^{\alpha} \varphi(x) + c\varphi'(x) \geq \sup f \geq f(\varphi) \quad \text{for } -A < x < -1.$$

We deduce that there exists $K$ such that if $c \geq K$ then

$$(-\partial_{xx})^{\alpha} \varphi(x) + c\varphi'(x) \geq f(\varphi) \quad \text{for } x < -1$$

and so $\varphi$ is a supersolution for (7).

Choosing $M$ such that $\varphi(-M) < \theta$, we now see that if $c \geq K$ and $b > M$, then $\varphi(x-M)$ is a super-solution for (7). By a sliding argument, we deduce that $\phi_c(x) \leq \varphi(x-M)$ and so $\phi_c(0) \leq \varphi(-M) < \theta$.

For the lower bound, we define $\varphi_1(x) = 1 - \varphi(-x)$. Then we have, if $-c \geq K$ ($c \leq -K$) and for $x > 1$

$$(-\partial_{xx})^{\alpha} \varphi_1(x) + c\varphi_1'(x) = -[(-\partial_{xx})^{\alpha} \varphi(-x) + (-c)\varphi'(-x)] \leq 0 \leq f(\varphi).$$

Moreover, we have $\varphi_1(x) = 0$ for $x \leq 1$. Proceeding as above, we deduce that if $c \leq -K$, then $\phi_c(0) > \theta$, which concludes the proof. □
3. Proof of Theorem 1.1

In order to complete the proof of Theorem 1.1 we have to prove that we can pass to the limit $b \to \infty$ in the truncated problem. More precisely, Theorem 1.1 follows from the following proposition:

**Proposition 3.1.** Under the conditions of Proposition 2.1 there exists a subsequence $b_n \to \infty$ such that $\phi_{b_n} \to \phi_0$ and $c_{b_n} \to c_0$. Furthermore, $c_0 \in (0, K]$ and $\phi_0$ is a monotone increasing solution of (5).

**Proof of Proposition 3.1.** We recall that $c_{b_n} \in [-K, K]$, and classical elliptic estimates (see [BCP68]) yield:

$$||\phi_{b_n}||_{C^2, \gamma} \leq C$$

for some $\gamma \in (0, 1)$. Thus there exists a subsequence $b_n \to \infty$ such that

$$c_n := c_{b_n} \to c_0 \in [-K, K]$$

$$\phi_n := \phi_{b_n} \to \phi_0$$

as $n \to \infty$. It is readily seen that $\phi_0$ solves

$$(-\partial_{xx})^\alpha \phi_0 + c_0 \phi_0' = f(\phi_0) \quad \text{for all } x \in \mathbb{R}.$$  \hspace{1cm} (9)

It is also readily seen that $\phi_0(x)$ is monotone increasing, $\phi_0(0) = \theta$ and $\phi_0$ is bounded. By a standard compactness argument, there exists $\gamma_0$, $\gamma_1$ such that $\lim_{x \to -\infty} \phi_0(x) = \gamma_0$ and $\lim_{x \to +\infty} \phi_0(x) = \gamma_1$ with

$$0 \leq \gamma_0 \leq \theta \leq \gamma_1 \leq 1.$$  \hspace{1cm}

It remains to prove that $c_0 > 0$, $\gamma_0 = 0$ and $\gamma_1 = 1$. For that, we will mainly follow classical arguments (see [BL90], [BH07]).

First, we have the following lemma:

**Lemma 3.2.** The function $\phi_0$ satisfies

$$\int_{\mathbb{R}} (-\partial_{xx})^\alpha \phi_0(x) \, dx = 0.$$  \hspace{1cm}

**Proof of Lemma 3.2.** The result follows formally by integrating formula (3) with respect to $x$ and using the antisymmetry with respect to the variables $x$ and $z$. However, because of the principal value, one has to be a little bit careful with the use of Fubini’s theorem.

To avoid this difficulty, we will use instead the equivalent formula for the fractional laplacian:

$$(-\partial_{xx})^\alpha \phi_0(x) = c_0 \int_{\mathbb{R}} \frac{\phi_0(x) - \phi_0(z)}{|x - z|^{1+2\alpha}} \, dz + c_0 \int_{\mathbb{R}} \frac{\phi_0(x) - \phi_0(z) + \phi_0'(x)(z - x)}{|x - z|^{1+2\alpha}} \, dz.$$

\hspace{1cm} (10)
which is valid for all \( \varepsilon > 0 \) and does not involve singular integrals. Integrating the first term with respect to \( x \in \mathbb{R} \), and using Fubini’s theorem, we get
\[
\int_{\mathbb{R}} \int_{\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z)}{|x - z|^{1+2\alpha}} \, dx \, dz = \int_{\mathbb{R}} \int_{\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z)}{|x - z|^{1+2\alpha}} \, dx \, dz
\]
and so this integral vanishes. Using Taylor’s theorem, the second term in (10) can be rewritten as
\[
\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{|x - z|^{1+2\alpha}} \int_{x}^{z} (z - t) \phi_0''(t) \, dt \, dz = \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_{x}^{x+y} (y + x - t) \phi_0''(t) \, dt \, dy.
\]
Integrating with respect to \( x \) and using (twice) Fubini’s theorem, we deduce
\[
\int_{\mathbb{R}} \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{|x - z|^{1+2\alpha}} \int_{x}^{z} (z - t) \phi_0''(t) \, dt \, dz \, dx = \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_{x}^{x+y} (y + x - t) \phi_0''(t) \, dt \, dy \, dx
\]
where we used the fact that \( \lim_{x \to \pm \infty} \phi_0'(x) = 0 \) and so \( \int_{-\infty}^{\infty} \phi_0''(t) \, dt = 0 \). The lemma follows.

Now, we can integrate equation (9) with respect to \( x \in \mathbb{R} \), and using Lemma 3.2 we get:
\[
(11) \quad \int_{\mathbb{R}} f(\phi_0(x)) \, dx = c_0 (\gamma_1 - \gamma_0) < \infty.
\]
In particular, we observe that (11) implies that
\[
f(\gamma_0) = f(\gamma_1) = 0,
\]
otherwise the integral would be infinite.

Next, we prove:

**Lemma 3.3.** The limiting speed satisfies:
\[
c_0 > 0.
\]

**Proof.** First of all, we note that for all \( n \), there exists \( a_n \in (0, b_n) \) such that \( \phi_n(a_n) = \frac{1+\theta}{2} \). Furthermore, up to another subsequence, by elliptic
estimates, the function \( \psi_n(x) = \phi_{b_n}(a_n + x) \) converges to a function \( \psi_0 \).

Note that since \( \psi_0 \in C^\gamma \), there exists \( r > 0 \) such that
\[
\psi_0(x) \in \left[ \frac{3 + \theta}{4}, \frac{1 + 3\theta}{4} \right] \quad \text{for } x \in [-r, r]
\]
and so there exists \( \kappa_0 > 0 \) such that
\[
\int_{-\infty}^{\infty} f(\psi_0) \, dx > \kappa_0. \tag{12}
\]

Up to a subsequence, we can assume that \( b_n + a_n \) is either convergent or goes to \(+\infty\). We need to distinguish the two cases:

**Case 1:** \( b_n + a_n \to +\infty \): In that case, \( \psi_0 \) solves
\[
(-\partial_{xx})^\alpha \psi_0 + c_0 \psi'_0 = f(\psi_0) \quad \text{for all } x \in \mathbb{R}. \tag{13}
\]
Furthermore, \( \psi_0(0) = \frac{1 + \theta}{2} \) and \( \psi_0 \) is monotone increasing. In particular, it is readily seen that there exists \( \tilde{\gamma}_0 \) and \( \tilde{\gamma}_1 \) such that \( \lim_{x \to -\infty} \psi_0(x) = \tilde{\gamma}_0 \) and \( \lim_{x \to +\infty} \psi_0(x) = \tilde{\gamma}_1 \) with
\[
0 \leq \tilde{\gamma}_0 \leq \frac{1 + \theta}{2} \leq \tilde{\gamma}_1 \leq 1.
\]

Integrating (13) over \( \mathbb{R} \), and using the fact that
\[
\int_{\mathbb{R}} (-\partial_{xx})^\alpha \psi_0(x) \, dx = 0
\]
(the proof is the same as in Lemma 3.2) we deduce
\[
c_0(\tilde{\gamma}_1 - \tilde{\gamma}_0) = \int_{\mathbb{R}} f(\psi_0) \, dx < \infty \tag{14}
\]
and so
\[
f(\tilde{\gamma}_0) = f(\tilde{\gamma}_1) = 0.
\]
This implies that
\[
\tilde{\gamma}_1 = 1 \quad \text{and} \quad \tilde{\gamma}_0 \leq \theta.
\]
Finally, (14) and (12) yields
\[
c_0(1 - \theta) \geq \int_{\mathbb{R}} f(\psi_0) \, dx \geq \kappa_0
\]
which gives the result.

**Case 2:** \( a_n + b_n \to a < \infty \): In that case, \( \psi_0 \) solves
\[
(-\partial_{xx})^\alpha \psi_0 + c_0 \psi'_0 = f(\psi_0) \quad \text{for all } x \in (-\infty, a) \tag{15}
\]
and we need to modify the proof slightly. First, we notice that \( \psi_0(x) = 1 \) for \( x \geq a \), and we observe that \((-\partial_{xx})^\alpha \psi_0(x) \geq 0\) for \( x \geq a \). In particular
\[
\int_{-\infty}^{a} (-\partial_{xx})^\alpha \psi_0(x) \, dx \leq \int_{\mathbb{R}} (-\partial_{xx})^\alpha \psi_0(x) \, dx = 0
\]
Proceeding as above, we check that \( \lim_{x \to -\infty} \psi_0(x) = \gamma_0 \leq \theta \) and integrating (15) over \((-\infty, \bar{a})\), we deduce
\[
c_0(1 - \theta) \geq \int_{\mathbb{R}} f(\psi_0) \, dx > 0.
\]

\[\square\]

The positivity of the speed, together with the sub-solution constructed in Lemma 2.2 will now give \( \gamma_0 = 0 \). More precisely, we now prove:

**Lemma 3.4.** The function \( \phi_0 \) satisfies:
\[
\lim_{x \to -\infty} \phi_0(x) = 0.
\]

**Proof.** Let \( c_1 = c_0/2 > 0 \) and take \( n \) large enough so that \( c_{b_n} \geq c_1 \).

We recall that by Lemma 2.2 (see also the proof of Lemma 2.5) that the function
\[
\varphi(x) = \begin{cases} 
\frac{1}{|x|^{2\alpha}} & x < -1 \\
1 & x > -1
\end{cases}
\]
satisfies
\[
(-\partial_x)^\alpha \varphi + K \varphi' \geq 0 \quad \text{in} \{ \varphi < 1 \}
\]
for some \( K \) large enough. Introducing \( \varphi_\varepsilon(x) = \varphi(\varepsilon x) \), we deduce
\[
(-\partial_x)^\alpha \varphi_\varepsilon + \varepsilon^{2\alpha - 1} K \varphi_\varepsilon' \geq 0 \quad \text{in} \{ \varphi_\varepsilon(x) < 1 \}
\]
and taking \( \varepsilon \) small enough (recalling that \( 2\alpha > 1 \)), we get
\[
(-\partial_x)^\alpha \varphi_\varepsilon + c_1 \varphi_\varepsilon' \geq 0 \quad \text{in} \{ \varphi_\varepsilon < 1 \}.
\]
Furthermore, \( \varphi_\varepsilon = 1 \) for \( x \geq 0 \), and so by a sliding argument, we deduce \( \phi_{b_n}(x) \leq \varphi_\varepsilon(x) \) for all \( n \) such that \( c_{b_n} \geq c_1 \) and thus
\[
\phi_0(x) \leq \varphi_\varepsilon(x)
\]
which implies in particular that \( \gamma_0 = 0 \). \[\square\]

Finally, we conclude the proof of Proposition 3.1 by proving that \( \gamma_1 = 1 \):

**Lemma 3.5.** The function \( \phi_0 \) satisfies:
\[
\lim_{x \to +\infty} \phi_0(x) = 1.
\]

**Proof.** We recall that (11) implies that either \( \gamma_1 = \theta \) or \( \gamma_1 = 1 \) (otherwise the integral is infinite). Furthermore, if \( \gamma_1 = \theta \), then \( \phi_0 \leq \theta \) on \( \mathbb{R} \) and so \( \int_{\mathbb{R}} f(\phi_0(x)) \, dx = 0 \). Since \( \gamma_0 = 0 < \theta \), (11) implies \( c_0 = 0 \), which is a contradiction. Hence \( \gamma_1 = 1 \). \[\square\]
4. Asymptotic behavior

We now prove Theorem 1.2 which further characterizes the behavior of \( \phi_0 \) as \( x \to -\infty \). We recall that in the case of the regular Laplacian (\( \alpha = 1 \)), \( \phi_0 \) and its derivatives decrease exponentially fast to 0 as \( x \to -\infty \). When \( \alpha \in (1/2, 1) \), it is readily seen that the proof of Lemma 3.4 actually implies:

**Proposition 4.1** (Asymptotic behavior of \( \phi_0 \)). There exists \( M \) such that

\[
\phi_0(x) \leq \frac{M}{|x|^{2\alpha-1}} \quad \text{for} \quad x \leq -1
\]

Noticing that \( \phi'_0 > 0 \) solves

\[
(-\partial_{xx})^\alpha \phi''_0 + c_0 (\phi'_0)' = 0 \quad \text{for} \quad x \leq 0,
\]

we can also prove:

**Proposition 4.2** (Asymptotic behavior of \( \phi'_0 \)). There exists a constant \( m \) such that

\[
\phi'_0(x) \geq \frac{m}{|x|^{2\alpha}} \quad \text{for} \quad x \leq -1.
\]

**Proof.** Lemma 2.3 implies that the function

\[
\bar{\varphi}(x) = \begin{cases} 
\frac{1}{|x|^{2\alpha}} & x < -1 \\
0 & x > -1
\end{cases}
\]

satisfies

\[
(-\partial_{xx})^\alpha \bar{\varphi} + k \bar{\varphi}'(x) = -\frac{c_0}{2\alpha - 1} \frac{1}{|x|^{2\alpha+1}} + \epsilon \frac{2\alpha}{|x|^{2\alpha+1}} + O \left( \frac{1}{|x|^{4\alpha}} \right)
\]

when \( x \to \infty \), and so

\[
(-\partial_{xx})^\alpha \bar{\varphi} + k \bar{\varphi}'(x) \leq 0 \quad \text{for} \quad x \leq -A
\]

if \( k \) is small enough and \( A \) is large.

We introduce \( \varphi_\epsilon(x) = \bar{\varphi}(\epsilon x) \), which satisfies

\[
(-\partial_{xx})^\alpha \varphi_\epsilon + \epsilon^{1-2\alpha} k \varphi_\epsilon' \leq 0 \quad \text{for} \quad x < -\epsilon^{-1} A
\]

hence

\[
(-\partial_{xx})^\alpha \varphi_\epsilon + c_0 \varphi_\epsilon' \leq 0 \quad \text{for} \quad x < -\epsilon^{-1} A
\]

provided we choose \( \epsilon \) small enough.

Finally, we take \( r \) so that

\[
\phi'_0(x) \geq r \varphi_\epsilon(x) \quad \text{for} \quad -\epsilon^{-1} A < x < -\epsilon^{-1}.
\]

Proposition 4.2 now follows from the maximum principle and a sliding argument using the fact that \( \varphi_\epsilon(x) = 0 \) for \( x \geq -\epsilon^{-1} \). \( \square \)
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