Quantum Spin Systems

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1 Introduction

The theory of quantum spin systems is concerned with properties of quantum systems with an infinite number of degrees of freedom that each have a finite-dimensional state space. Occasionally, one is specifically interested in finite systems. In the most common examples one has an $n$-dimensional Hilbert space associated with each site of a $d$-dimensional lattice.

A model is normally defined by describing a Hamiltonian or a family of Hamiltonians, which are self-adjoint operators on the Hilbert space, and one studies their spectrum, the eigenstates, the equilibrium states, its dynamics, non-equilibrium stationary states etc.

More particularly, the term “quantum spin system” often refers to such models where each degree of freedom is thought of as a spin variable, i.e., there are three basic observables representing the components of the spin, $S^1, S^2, \text{ and } S^3$, and these components transform according to a unitary representation of $SU(2)$. The most commonly encountered situation is where the system consists of $N$ spins, each associated with a fixed irreducible representation of $SU(2)$. One speaks of a spin$-J$ model, if this representation is the $2J + 1$-dimensional one. The possible values of $J$ are $1/2, 1, 3/2, \ldots$.

The spins are usually thought of as each being associated with a site in a lattice, or more generally, a vertex in a graph. E.g., each spin may be associated with an ion in a crystalline lattice, which is how quantum spin models arise in condensed matter physics. Quantum spin systems are also used in quantum information theory and quantum computation, and show up as abstract mathematical objects in representation theory and quantum probability.

In this article we give a short introduction to the subject, starting with a very brief review of its history. In Section 2 we sketch the mathematical framework and give the most important definitions. Three further sections are entitled Symmetries and symmetry breaking, Phase transitions, and Dynamics, which together cover the most important aspects of quantum spin systems actively pursued today.

2 A very brief history

The introduction of quantum spin systems was the result of the marriage of two developments taking place in the 1920’s. The first was the realization that angular momentum (hence, also the magnetic moment) was quantized (Pauli, 1920; Stern and Gerlach, 1922) and that particles such as the electron have an intrinsic angular momentum called spin (Compton, 1921; Goudsmit and Uhlenbeck, 1925).

The second development was the attempt in statistical mechanics to explain ferromagnetism and the phase transition associated with it on the basis of a microscopic theory (Lenz and Ising, 1925). The fundamental interaction between spins, the so-called exchange operator which is a subtle consequence of the Pauli exclusion principle, was introduced independently by Dirac and Heisenberg in 1926. With this discovery it was realized that magnetism is a quantum effect and
that a fundamental theory of magnetism requires
the study of quantum mechanical models. This
realization and a large amount of subsequent
work notwithstanding, some of the most funda-
mental questions, such as a derivation of ferro-
magnetism from first principles, remain open.

Heisenberg gave his name to the first and most
important quantum spin model, the Heisenberg
model (see further). It has been studied intensely
ever since the early 1930’s and its study has led
to an impressive variety of new ideas in both
mathematics and physics. Here, we limit our-
selves to listing some landmark developments.

Spin waves were discovered independently by
Bloch and Slater in 1930. Spin waves continue to
play an essential role in our understanding of the
excitation spectrum of quantum spin Hamiltoni-
ans. In two papers published in 1956, Dyson ad-
vanced the theory of spin waves by showing how
interactions between spin waves can be taken
into account.

In 1931, Bethe introduced the famous Bethe
Anstaz to show how the exact eigenvectors of
the spin 1/2 Heisenberg model on the one-
dimensional lattice can be found. This exact so-
lution, directly and indirectly led to many impor-
tant developments in statistical mechanics, com-
binatorics, representation theory, quantum field
theory and more. Hulthén used Bethe’s Ansatz
to compute the ground state energy of the anti-
erromagnetic spin-1/2 Heisenberg chain in 1938.

In their famous 1961 paper Lieb, Schultz, and
Mattis showed that some quantum spin models
in one dimension can be solved exactly by map-
ing them into a problem of free Fermions. This
paper is still one of the most cited in the field.

Robinson, in 1967, laid the foundation for
the mathematical framework that we describe
in the next section. Using that framework,
Araki established the absence of phase transi-
tions at positive temperature a large class of one-
dimensional quantum spin models in 1969.

During the more recent decades the mathe-
natical and computational techniques used to
study quantum spin models have fanned out in
many directions.

When it was realized in the 1980’s that the
magnetic properties of complex materials play an
important role in high-$T_c$ superconductivity, the
variety of quantum spin models studied in the liter-
ature exploded. This motivated a large number
of theoretical and experimental studies of mate-
rials with exotic properties that are often based
on quantum effects that do not have a classical
analogue. An example of unexpected behavior
is the prediction by Haldane of the spin liquid
ground state of the spin-1 Heisenberg antiferro-
magnetic chain in 1983. In the quest for a math-
ematical proof of this prediction (a quest still on-
going today), Affleck, Kennedy, Lieb, and Tasaki
introduced the AKLT model in 1987. They were
able to prove that the ground state of this model
has all the characteristic properties predicted
by Haldane for the Heisenberg chain: a unique
ground state with exponential decay of correla-
tions and a spectral gap above the ground state.

There also are particle models that are defined
on a lattice, or more generally, a graph. Unlike
spins, particles can hop from one site to another.
These models are closely related to quantum spin
systems and in some cases mathematically equiv-
alent. The best known example of a model of
lattice fermions is the Hubbard model. We will
not further discuss such systems in this article.

3 Mathematical Framework

Quantum spin systems is an area of mathemati-
cal physics where the demands of mathematical
rigor can be fully met and in many cases this
can be done without sacrificing the ability to in-
clude all physically relevant models and phenom-
ena. This does not mean, however, that there
There are few open problems remaining. But it does mean that, in general, these open problems are precisely formulated mathematical questions.

In this section, we will review the standard mathematical framework for quantum spin systems, in which the topics discussed in the subsequent section can be given a precise mathematical formulation. It is possible, however, to skip this section and read the rest with only a physical or intuitive understanding of the notions of observable, Hamiltonian, dynamics, symmetry, ground state etc...

The most common mathematical setup is as follows. Let \( d \geq 1 \), and let \( \mathcal{L} \) denote the family of finite subsets of the \( d \)-dimensional integer lattice \( \mathbb{Z}^d \). For simplicity we will assume that the Hilbert space of the “spin” associated with each \( x \in \mathbb{Z}^d \) has the same dimension \( n \geq 2 \): \( \mathcal{H}_{\{x\}} \cong \mathbb{C}^n \). The Hilbert space associated with the finite volume \( \Lambda \in \mathcal{L} \) is then \( \mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \). The algebra of observables for the site \( x \) consists of the \( n \times n \) complex matrices: \( A_{\{x\}} \cong M_n(\mathbb{C}) \). For any \( \Lambda \in \mathcal{L} \), the algebra of observables for the system in \( \Lambda \) is given by \( \mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} A_{\{x\}} \). The primary observables for a quantum spin model are the spin-\( S \) matrices \( S^1, S^2, \) and \( S^3 \), where \( S \) is the half-integer such that \( n = 2S + 1 \). They are defined by the property that they are Hermitian matrices satisfying the \( SU(2) \) commutation relations. Instead of \( S^1 \) and \( S^2 \), one often works with the spin raising and lowering operators, \( S^+ \) and \( S^- \), defined by the relations \( S^1 = (S^+ + S^-)/2 \), and \( S^2 = (S^+ - S^-)/(2i) \). In terms of these, the \( SU(2) \) commutation relations are

\[
[S^+, S^-] = 2S^3, \quad [S^3, S^\pm] = \pm S^\pm, \quad (3.1)
\]

where we have used the standard notation for the commutator for two elements \( A \) and \( B \) in an algebra: \([A, B] = AB - BA\). In the standard basis \( S^3, S^+, \) and \( S^- \) are given by the following matrices:

\[
S^3 = \begin{pmatrix}
S & 0 & \cdots & 0 \\
0 & S-1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -S
\end{pmatrix},
\]

\[
S^- = (S^+)^*, \quad \text{and}
\]

\[
S^+ = \begin{pmatrix}
0 & c_S & 0 & \cdots & 0 \\
- c_S & 0 & c_{S-1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & c_{-S+1}
\end{pmatrix},
\]

where, for \( m = -S, -S + 1, \ldots, S \),

\[
c_m = \sqrt{S(S+1) - m(m-1)}.
\]

In the case \( n = 2 \), one often works with the Pauli matrices, \( \sigma^1, \sigma^2, \sigma^3 \), simply related to the spin matrices by \( \sigma^j = 2S^j, j = 1, 2, 3 \).

Most physical observables are expressed as finite sums and products of the spin matrices \( S^i_x \), \( j = 1, 2, 3 \), associated with the site \( x \in \Lambda \):

\[
S^i_x = \bigotimes_{y \in \Lambda} A_y
\]

with \( A_x = S^i, \) and \( A_y = 1 \) if \( y \neq x \).

The \( \mathcal{A}_\Lambda \) are finite-dimensional \( C^* \)-algebras for the usual operations of sum, product, and Hermitian conjugation of matrices and with identity \( 1_\Lambda \).

If \( \Lambda_0 \subset \Lambda_1 \), there is a natural embedding of \( \mathcal{A}_{\Lambda_0} \) into \( \mathcal{A}_{\Lambda_1} \), given by

\[
\mathcal{A}_{\Lambda_0} \cong \mathcal{A}_{\Lambda_0} \otimes 1_{\Lambda_1 \setminus \Lambda_0} \subset \mathcal{A}_{\Lambda_1}.
\]

The algebra of local observables is then defined by

\[
\mathcal{A}_{\text{loc}} = \bigcup_{\Lambda \in \mathcal{L}} \mathcal{A}_\Lambda
\]
Its completion is the $C^*$-algebra of quasi-local observables, which we will simply denote by $\mathcal{A}$.

The dynamics and symmetries of a quantum spin model are described by (groups of) automorphisms of the $C^*$-algebra $\mathcal{A}$, i.e., bijective linear transformations $\alpha$ on $\mathcal{A}$ that preserve the product and $\ast$ operations. E.g., the translation automorphisms $\tau_x, x \in \mathbb{Z}^d$, which map any subalgebra $\mathcal{A}_\Lambda$ to $\mathcal{A}_{\Lambda+x}$, in the natural way, form a representation of the additive group $\mathbb{Z}^d$ on $\mathcal{A}$.

A translation invariant interaction, or potential, defining a quantum spin model, is a map $\phi: \mathcal{L} \to \mathcal{A}$ with the following properties: for all $X \in \mathcal{L}$, we have $\phi(X) \in \mathcal{A}_X$, $\phi(X) = \phi(X)^\ast$, and for $x \in \mathbb{Z}^d$, $\phi(x) + x = \tau_x(\phi(X))$. An interaction is called finite range if there exists $R > 0$ such that $\phi(X) = 0$ whenever $\text{diam}(X) > R$. The Hamiltonian in $\Lambda$ is the self-adjoint element of $\mathcal{A}_\Lambda$ defined by

$$H_\Lambda = \sum_{X \subset \Lambda} \phi(X)$$

E.g., the Heisenberg model has

$$\phi(\{x,y\}) = -J \vec{S}_x \cdot \vec{S}_y, \quad \text{if } |x-y| = 1$$

(3.2)

and $\phi(X) = 0$ in all other cases. Here, $\vec{S}_x \cdot \vec{S}_y$ is the conventional notation for $S^1_x S^1_y + S^2_x S^2_y + S^3_x S^3_y$. The magnitude of the coupling constant $J$ sets a natural unit of energy and is irrelevant from the mathematical point of view. Its sign, however, determines whether the model is ferromagnetic ($J > 0$), or antiferromagnetic ($J < 0$).

For the classical Heisenberg model, where the role of $\vec{S}_x$ is played by a unit vector in $\mathbb{R}^3$, and which can be regarded, after rescaling by a factor $S^{-2}$, as the limit $S \to \infty$ of the quantum Heisenberg model, there is a simple transformation relating the ferro- and antiferromagnetic models (just map $\vec{S}_x$ to $-\vec{S}_x$ for all $x$ in the even sublattice of $\mathbb{Z}^d$). It is easy to see that there does not exist an automorphism of $\mathcal{A}$ mapping $\vec{S}_x$ to $-\vec{S}_x$, since that would be inconsistent with the commutation relations (3.1). Not only is there no exact mapping between the ferro- and the antiferromagnetic models, their ground states and equilibrium states have radically different properties. See below for the definitions and further discussion.

The dynamics (or time evolution), of the system in finite volume $\Lambda$ is the one-parameter group of automorphisms of $\mathcal{A}_\Lambda$ given by,

$$\alpha_t^{(\Lambda)}(A) = e^{it H_\Lambda} A e^{-it H_\Lambda}, \quad t \in \mathbb{R}.$$ 

For each $t \in \mathbb{R}$, $\alpha_t^{(\Lambda)}$ is an automorphism of $\mathcal{A}$ and the family $\{\alpha_t^{(\Lambda)} \mid t \in \mathbb{R}\}$, forms a representation of the additive group $\mathbb{R}$.

Each $\alpha_t^{(\Lambda)}$ can trivially be extended to an automorphism on $\mathcal{A}$, by tensoring with the identity map. Under quite general conditions, $\alpha_t^{(\Lambda)}$ converges strongly as $\Lambda \to \mathbb{Z}^d$ in a suitable sense, i.e., for every $A \in \mathcal{A}$, the limit

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \alpha_t^{(\Lambda)}(A) = \alpha_t(A)$$

exists in the norm in $\mathcal{A}$, and it can be shown that it defines a strongly continuous one-parameter group of automorphism of $\mathcal{A}$. $\Lambda \uparrow \mathbb{Z}^d$ stands for any sequence of $\Lambda \in \mathcal{L}$ such that $\Lambda$ eventually contains any given element of $\mathcal{L}$. A sufficient condition on the potential $\phi$ is that there exists $\lambda > 0$ such that $\|\phi\|_\lambda$ is finite, with

$$\|\phi\|_\lambda = \sum_{X \ni 0} e^{\lambda |X|} \|\phi(X)\|. \quad (3.3)$$

Here, $|\cdot|$ denotes the number of elements in $X$. One can show that under the same conditions, $\delta$ defined on $\mathcal{A}_{\text{loc}}$ by

$$\delta(A) = \lim_{\Lambda \uparrow \mathbb{Z}^d} [H_\Lambda, A]$$

is a norm-closable (unbounded) derivation on $\mathcal{A}$ and that its closure is, up to a factor $i$, the generator of $\{\alpha_t \mid t \in \mathbb{R}\}$, i.e., formally

$$\alpha_t = e^{it \delta}.$$
For the class of $\phi$ with finite $\|\Phi\|_\lambda$ for some $\lambda > 0$, $\mathcal{A}_{\text{loc}}$ is a core of analytic vectors for $\delta$. This means that for each $A \in \mathcal{A}_{\text{loc}}$, the function $t \mapsto \alpha_t(A)$, can be extended to an entire function, which will denote by $\alpha_z(A), z \in \mathbb{C}$.

A state of the quantum spin system is a linear functional on $\mathcal{A}$ such that $\omega(A^*A) \geq 0$, for all $A \in \mathcal{A}$ (positivity), and $\omega(1) = 1$ (normalization). The restriction of $\omega$ to $\mathcal{A}_\Lambda$, for each $\Lambda \in \mathcal{L}$, is uniquely determined by a density matrix, i.e., $\rho_\Lambda \in \mathcal{A}_\Lambda$, such that

$$\omega(A) = \text{Tr} \rho_\Lambda A,$$

where $\text{Tr}$ denotes the usual trace of matrices. $\rho_\Lambda$ is non-negative definite and of unit trace. If the density matrix is a one-dimensional projection, the state is called a vector state, and can be identified with a vector $\psi \in \mathcal{H}_\Lambda$, such that $\mathcal{C}\psi = \text{ran} \rho_\Lambda$.

A ground state of the quantum spin system is a state $\omega$ satisfying the local stability inequalities:

$$\omega(A^*\delta(A)) \geq 0, \text{ for all } A \in \mathcal{A}_{\text{loc}}. \quad (3.4)$$

The states describing thermal equilibrium are characterized by the Kubo-Martin-Schwinger (KMS) condition: for any $\beta \geq 0$ (related to absolute temperature by $\beta = 1/(k_B T)$, where $k_B$ is the Boltzmann constant), $\omega$ is called $\beta$-KMS if

$$\omega(A\alpha_{i\beta}(B)) = \omega(BA), \text{ for all } A, B \in \mathcal{A}_{\text{loc}}. \quad (3.5)$$

The most common way to construct ground states and equilibrium states, solutions of (3.4) and (3.5) respectively, is by taking thermodynamic limits of finite volume states with suitable boundary conditions. A ground state of the finite-volume Hamiltonian $H_\Lambda$, is a convex combination of vector states that are eigenstates of $H_\Lambda$ belonging to its smallest eigenvalue. The finite-volume equilibrium state at inverse temperature $\beta$ has density matrix $\rho_\beta$ defined by

$$\rho_\beta = Z(\Lambda, \beta)^{-1} e^{-\beta H_\Lambda},$$

where $Z(\Lambda, \beta) = \text{Tr} e^{-\beta H_\Lambda}$, is called the partition function. By considering limit points as $\Lambda \rightarrow \mathbb{Z}^d$, one can show that a quantum spin model has always at least one ground state and at least one equilibrium state for all $\beta$.

What we have discussed in this section are the basic concepts in the most standard setup. Clearly, many generalizations are possible: one can consider non-translation invariant models, models with random potentials, the state spaces at each site may have different dimensions, instead of $\mathbb{Z}^d$ one can consider other lattices or one can define models on arbitrary graphs, one can allow interactions of infinite range that satisfy weaker conditions than those imposed by the finiteness of the norm (3.3), one can restrict to subspaces of the Hilbert space by imposing symmetries or suitable hardcore conditions, and one can study models with infinite-dimensional spins. Examples of all these types of generalizations have been considered in the literature and have interesting applications.

4 Symmetries and symmetry breaking

Many interesting properties of quantum spin systems are related to symmetries and symmetry breaking. Symmetries of a quantum spin model are realized as representations of groups, Lie algebras, or quantum (group) algebras on the Hilbert space and/or the observable algebra. The symmetry property of the model is expressed by the fact that the Hamiltonian (or the dynamics) commutes with this representation. We briefly discuss the most common symmetries.

Translation invariance. We already defined the translation automorphisms $\tau_x$ on the observable algebra of infinite quantum spin systems on $\mathbb{Z}^d$. One can also define translation automor-
phisms for finite systems with periodic boundary conditions, i.e., defined on the torus $\mathbb{Z}^d/\mathbb{Z}^d$, where $T = (T_1, \ldots, T_d)$ is a positive integer vector representing the periods.

Other graph automorphisms. In general, if $G$ is a group of automorphisms of the graph $\Gamma$, and $\mathcal{H}_G = \bigotimes_{x \in \Gamma} \mathbb{C}^n$ is the Hilbert space of a system of identical spins defined on $\Gamma$, then, for each $g \in G$, one can define a unitary $U_g$ on $\mathcal{H}_G$ by linear extension of $U_g \otimes \varphi_x = \otimes \varphi_{g^{-1}(x)}$, where $\varphi_x \in \mathbb{C}^n$, for all $x \in \Gamma$. These unitaries form a representation of $G$. With the unitaries one can immediately define automorphisms of the algebra of observables: for $A \in \mathcal{A}_\Lambda$, and $U \in \mathcal{A}_\Lambda$ unitary, $\tau(A) = U^* A U$ defines an automorphism, and if $U_g$ is a group representation the corresponding $\tau_g$ will be, too. Common examples of graph automorphisms are the lattice symmetries of rotation and reflection. Translation symmetry and other graph automorphisms are often referred to collectively as spatial symmetries.

Local symmetries (also called gauge symmetries). Let $G$ be a group and $u_g, g \in G$, a unitary representation of $G$ on $\mathbb{C}^n$. Then, $U_g = \bigotimes_{x \in \Lambda} u_g$, is a representation on $\mathcal{H}_\Lambda$. E.g., the Heisenberg model commutes with such a representation of $SU(2)$. It is often convenient, and generally equivalent, to work with a representation of the Lie algebra. E.g., the $SU(2)$-invariance of the Heisenberg model is then expressed by the fact that $H_\Lambda$ commutes with the following three operators:

$$S^i = \sum_{x \in \Lambda} S^i_x, \quad i = 1, 2, 3.$$  

Note: sometimes the Hamiltonian is only symmetric under certain combinations of spatial and local symmetries. CP symmetry is an example.

For an automorphism $\tau$, we say that a state $\omega$ is $\tau$-invariant if $\omega \circ \tau = \tau$. If $\omega$ is $\tau_g$-invariant for all $g \in G$, we say that $\omega$ is $G$-invariant.

It is easy to see that if a quantum spin model has a symmetry $G$, then the set of all ground states or all $\beta$-KMS states will be $G$-invariant, meaning that if $\omega$ is in the set, then so is $\omega \circ \tau_g$, for all $g \in G$. By a suitable averaging procedure it is usually easy to establish that the sets of ground states or equilibrium states contain at least one $G$-invariant element.

An interesting situation occurs if the model is $G$-invariant, but there are ground states or KMS states that are not. I.e., for some $g \in G$, and some $\omega$ in the set (of ground states or KMS states), $\omega \circ \tau \neq \omega$. When this happens, one says that there is spontaneous symmetry breaking, a phenomenon that also plays an important role in Quantum Field Theory.

The famous Hohenberg-Mermin-Wagner Theorem, applied to quantum spin models, states that, as long as the interactions are not too long range and the dimension of the lattice is two or less, continuous symmetries cannot be spontaneously broken in a $\beta$-KMS state for any finite $\beta$.

Quantum group symmetries. We restrict ourselves to one important example: the $SU_q(2)$-invariance of the spin-1/2 XXZ Heisenberg chain with $q \in [0, 1]$, and with special boundary terms. The Hamiltonian of the $SU_q(2)$-invariant XXZ-chain of length $L$ is

$$H_L = \sum_{x=1}^{L-1} -\frac{1}{\Delta}(S^1_x S^1_{x+1} + S^2_x S^2_{x+1})$$

$$- (S^3_x S^3_{x+1} - 1/4) + \frac{1}{2} \sqrt{1 - \Delta^{-2}} (S^3_{x+1} - S^3_x),$$

where $q \in (0, 1]$ is related to the parameter $\Delta \geq 1$ by the relation $\Delta = (q + q^{-1})/2$. When $q = 0$, $H_L$ is equivalent to the Ising chain. Thus, the XXZ model interpolates between the Ising model (the primordial classical spin system) and the isotropic Heisenberg model (the most widely studied quantum spin model). In the limit of in-
finite spin \((S \to \infty)\), the model converges to the classical Heisenberg model (XXZ or isotropic). An interesting feature of the XXZ model are its non translation invariant ground states, called kink states.

In this family of models one can see how aspects of discreteness (quantized spins) and continuous symmetry (SU(2), or quantum symmetry \(SU_q(2)\)) are present at the same time in the quantum Heisenberg models, and the two classical limits \((q \to 0\) and \(S \to \infty)\), can be used as a starting point to study its properties.

Quantum group symmetry is not a special case of invariance under the action of a group. There is no group. But there is an algebra represented on the Hilbert space of each spin, for which there is a good definition of tensor product of representations, and “many” irreducible representations. In this example the representation of \(SU_q(2)\) on \(H_{[1,L]}\) commuting with \(H_L\) is generated by

\[
S^3 = \sum_{x=1}^{L} \mathbb{I}_1 \otimes \cdots \otimes S^3_x \otimes \mathbb{I}_{x+1} \otimes \cdots \mathbb{I}_L
\]

\[
S^+ = \sum_{x=1}^{L} t_1 \otimes \cdots \otimes t_{x-1} \otimes S^+_x \otimes \mathbb{I}_{x+1} \otimes \cdots \mathbb{I}_L
\]

\[
S^- = \sum_{x=1}^{L} \mathbb{I}_1 \otimes \cdots \otimes S^-_x \otimes t^{-1}_{x+1} \otimes \cdots t^{-1}_L
\]

where

\[
t = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}
\]

Quantum group symmetries were discovered in exactly solvable models, starting with the spin-1/2 XXZ chain. One can exploit their representation theory to study the spectrum of the Hamiltonian in very much the same way as ordinary symmetries. The main restriction to its applicability is that the tensor product structure of the representations is inherently one-dimensional, i.e., relying on an ordering from left to right. For the infinite XXZ chain the left-to-right and right-to-left orderings can be combined to generate an infinite-dimensional algebra, the quantum affine algebra \(U_q(\hat{sl}_2)\).

5 Phase Transitions

Quantum spin models of condensed matter physics often have interesting ground states. Not only are the ground states often a good approximation of the low-temperature behavior of the real systems that are modeled by it, and studying them is therefore useful, it is in many cases also a challenging mathematical problem. This is in contrast with classical lattice models for which the ground states are usually simple and easy to find. In more than one way ground states of quantum spin systems display behavior similar to equilibrium states of classical spin systems at positive temperature.

The spin-1/2 Heisenberg antiferromagnet on \(\Lambda \subset \mathbb{Z}^d\), with Hamiltonian

\[
H_{\Lambda} = \sum_{x,y \in \Lambda; |x-y|=1} \vec{S}_x \cdot \vec{S}_y , \tag{5.6}
\]

is a case in point. Even in the one-dimensional case \((d = 1)\), and even though the model in that case is exactly solvable by the Bethe Ansatz, its ground state is highly non-trivial. Analysis of the Bethe Ansatz solution (which is not fully rigorous) shows that spin-spin correlation function decays to zero at infinity, but slower than exponentially (roughly as inverse distance squared). For \(d = 2\), it is believed but not mathematically proved that the ground state has Néel order, i.e., long-range antiferromagnetic order, accompanied by a spontaneous breaking of the SU(2) symmetry. Using reflection positivity, Dyson, Lieb, and Simon were able to prove Néel order at sufficiently low temperature (large \(\beta\)), for \(d \geq 3\) and all \(S \geq 1/2\). This was later extended to the
ground state for \( d = 2 \) and \( S \geq 1 \), and \( d \geq 3 \) and \( S \geq 1/2 \), i.e., all cases where Néel order is expected except \( d = 2, S = 1/2 \).

In contrast, no proof of long range order in the Heisenberg ferromagnet at low temperature exists. This is rather remarkable since proving long range order in the ground states of the ferromagnet is a trivial problem.

Of particular interest are the so-called quantum phase transitions. These are phase transitions that occur as a parameter in the Hamiltonian is varied and which are driven by the competing effects of energy and quantum fluctuations, rather than the balance between energy and entropy which drives usual equilibrium phase transitions. Since entropy does not play a role, quantum phase transitions can be observed at zero temperature, i.e., in the ground states.

An important example of a quantum phase transition occurs in the two- or higher dimensional \( XY \)-model with a magnetic field in the \( Z \)-direction. It was proved by Kennedy, Lieb, and Shastry that, at zero field, this model has Off-Diagonal-Long-Range-Order (ODLRO), and can be interpreted as a hard-core boson gas at half-filling. It is also clear that if the magnetic field exceeds a critical value, \( h_c \), the model has a simple ferromagnetically ordered ground state. There are indications that there is ODLRO for all \(|h| < h_c| \). However, so far there is no proof that ODLRO exists for any \( h \neq 0 \).

What makes the ground state problem of quantum spin systems interesting and difficult at the same time is that ground states, in general, do not minimize the expectation of the interaction terms in the Hamiltonian individually although, loosely speaking, the expectation of their sum (the Hamiltonian) is minimized. However, there are interesting exceptions to this rule. Two examples are the AKLT model and the ferromagnetic XXZ model.

The wide ranging behavior of quantum spin models has required an equally wide range of mathematical approaches to study them. There is one group of methods, however, that can make a claim of substantial generality: those that start from a representation of the partition function based on the Feynman-Kac formula. Such representations turn a \( d \)-dimensional quantum spin model into a \( d + 1 \)-dimensional classical problem, albeit one with some special features. This technique was pioneered by Ginibre in 1968 and was quickly adopted by a number of authors to solve a variety of problems. Techniques borrowed from classical statistical mechanics have been adapted with great success to study ground states, the low-temperature phase diagram, or the high-temperature regime of quantum spin models that can be regarded as perturbations of a classical system. More recently, it was used to develop a quantum version of Pirogov-Sinai theory which is applicable to a large class of problems, including some with low-temperature phases not related by symmetry.

6 Dynamics

Another feature of quantum spin systems that makes them mathematically richer than their classical counterpart, is the existence of a Hamiltonian dynamics. We have seen that, quite generally, the dynamics is well-defined in the thermodynamic limit as a strongly continuous one-parameter group of automorphisms of the \( C^* \)-algebra of quasi-local observables. Strictly speaking, a quantum spin model is actually defined by its dynamics \( \alpha_t \), or by its generator \( \delta \), and not by the potential \( \phi \). Indeed, \( \phi \) is not uniquely determined by \( \alpha_t \). In particular, it is possible to incorporate various types of boundary condition into the definition of \( \phi \). This approach has proved very useful in obtaining important structural results, such as, e.g., the proof
by Araki of the uniqueness the KMS state at any finite $\beta$ in one-dimension. Another example is a characterization of equilibrium states by the Energy-Entropy Balance inequalities, which is both physically appealing and mathematically useful: $\omega$ is a $\beta$-KMS state for a quantum spin model in the setting of section 3 (and in fact also for more general quantum systems), if and only if the inequality

$$\beta \omega(X^* \delta(X)) \geq \omega(X^*X) \log \frac{\omega(X^*X)}{\omega(XX^*)}$$

is satisfied for all $X \in A_{\text{loc}}$. This characterization and several related results were proved in a series works by various authors (mainly Roepstorff, Araki, Fannes, Verbeure, and Sewell).

Detailed properties of the dynamics for specific models are generally lacking. One could point to the “immediate non-locality” of the dynamics as the main difficulty. By this, we mean that, except in trivial cases, most local observables $A \in A_{\text{loc}}$, become non-local after an arbitrarily short time, i.e., $\alpha_t(A) \not\subseteq A_{\text{loc}}$, for any $t \neq 0$. This non-locality is not totally uncontrolled however. A result by Lieb and Robinson establishes that, for models with interactions that are sufficiently short range (e.g., finite range), the non-locality propagates at a bounded speed. More precisely, under quite general conditions, there exists constants, $c, v > 0$, such that for any two local observables $A, B \in A_{\{0\}}$,

$$|||\alpha_t(A), \tau_x(B)||| \leq 2 ||A||||B||e^{-c(|x| - v|t|)}.$$

Attempts to understand the dynamics have generally been aimed at one of two issues: return to equilibrium from a perturbed state, and convergence to a non-equilibrium steady state in the presence of currents. Some interesting results have been obtained although much remains to be done.

**See Also**

Phase transitions. $C^*$-algebras. UHF-algebra. Quantum phase transitions. Reflection positivity. Hubbard model. Heisenberg model. Bethe Ansatz. Falicov-Kimball model. Symmetry breaking. Finitely Correlated States. XY model. Thermodynamic limit. Magnetism. Quantum information. $SU(2)$. $SU_q(2)$. $U_q(\hat{sl}_2)$. Density Matrix Renormalization Group. Hohenberg-Mermin-Wagner Theorem. Integrable spin chains.

**Further Reading**

A very informative overview of the early history of quantum spin systems, especially in relation to the history of the theory of magnetism and including a good bibliography, can be found in [13].

The mathematical framework briefly described in Section 3 is discussed in detail in [5]. [10] is an annotated bibliography devoted to mathematical results for the Heisenberg and related models.

Since many important results and techniques have not yet appeared in book form, we have included some seminal research papers of the field in the references.

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**Keywords**

spin matrices, Heisenberg model, phase transitions, quantum phase transitions, Bethe Ansatz, spontaneous symmetry breaking, ground state, KMS state, equilibrium state, Pauli matrices, SU(2), SUq(2), spin waves, reflection positivity, C*-algebra, Néel order, Pirogov-Sinai theory, quantum Pirogov-Sinai theory, magnetism, ferromagnet, antiferromagnet, long range order, off-diagonal long range order, kink state, XXZ model, entropy-energy inequalities, quantum spin dynamics