SIGN-CHANGING TOWER OF BUBBLES FOR A SINH-POISSON EQUATION WITH ASYMMETRIC EXPONENTS

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Abstract. Motivated by the statistical mechanics description of stationary 2D-turbulence, for a sinh-Poisson type equation with asymmetric nonlinearity, we construct a concentrating solution sequence in the form of a tower of singular Liouville bubbles, each of which has a different degeneracy exponent. The asymmetry parameter \( \gamma \in (0, 1] \) corresponds to the ratio between the intensity of the negatively rotating vortices and the intensity of the positively rotating vortices. Our solutions correspond to a superposition of highly concentrated vortex configurations of alternating orientation; they extend in a nontrivial way some known results for \( \gamma = 1 \). Thus, by analyzing the case \( \gamma \neq 1 \) we emphasize specific properties of the physically relevant parameter \( \gamma \) in the vortex concentration phenomena.

1. Introduction and statement of the main result. We are interested in the existence of bubble-tower type solutions for the problem:

\[
\begin{align*}
-\Delta u &= \rho(e^u - \tau e^{-\gamma u}) & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a smooth bounded domain, \( \rho > 0 \) is a small constant, \( \gamma, \tau \in (0, 1] \).

Equation (1) arises in the statistical mechanics description of two-dimensional equilibrium turbulence, as initiated by Onsager [19]. More precisely, in an unpublished manuscript reproduced in the review article [4], Onsager derived the following equation (see also [24] for a rigorous derivation):

\[
\begin{align*}
-\Delta u &= \lambda \left( \tau_1 \frac{e^u}{\int_{\Omega} e^u \, dx} - (1 - \tau_1) \gamma \frac{e^{-\gamma u}}{\int_{\Omega} e^{-\gamma u} \, dx} \right) & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( u \) denotes the stream function of the two-dimensional flow, \( \lambda > 0 \) is a constant related to the inverse temperature, the positively rotating vortices have unit intensity, \( \gamma \in (0, 1] \) denotes the intensity of the negatively rotating vortices and
τ₁ ∈ [0, 1] determines a priori the ratio of the number of positively rotating vortices to the total number of vortices. In more recent years, a similar equation was derived by Neri [16], under the assumption that the vortex intensities are independent identically distributed random variables with probability measure \( P \), defined on the (normalized) vortex intensity range \([-1, 1]\). If such a measure is chosen in the form
\[
P(dr) = \tau₁ δ₁(dr) + (1 - \tau₁) δ₋γ(dr),
\]
where \( δ₁(dr), δ₋γ(dr) \in \mathcal{M}([-1, 1]) \) denote Dirac measures, the resulting equation reduces to:
\[
\begin{cases}
-Δu = \lambda \frac{τ₁ e^u - (1 - τ₁) e^{-γu}}{\int_{Ω} (τ₁ e^u + (1 - τ₁) e^{-γu})} & \text{in } Ω, \\
\quad u = 0 & \text{on } ∂Ω.
\end{cases}
\]
We observe that the limit case \( τ = 0 \) in (1) yields the well-known Gelfand problem
\[
-Δu = ρe^u \quad \text{in } Ω, \quad u = 0 \quad \text{on } ∂Ω
\]
and correspondingly the limit case \( τ₁ = 1 \) in (2) and (3) yields the so-called standard mean field equation
\[
-Δu = λ e^u \int_{Ω} e^u \quad \text{in } Ω, \quad u = 0 \quad \text{on } ∂Ω.
\]
There is a vast literature concerning (4)–(5), see, e.g., [1, 10, 13, 14] and the references therein.

In the special case \( γ = 1, τ = 1 \) problem (1) reduces to the sinh-Poisson problem
\[
-Δu = ρ(e^u - e^{-u}) \quad \text{in } Ω, \quad u = 0 \quad \text{on } ∂Ω
\]
while the non-local counterparts (2) and (3) of problem (1) with \( γ = 1 \) are equivalent to the problems
\[
-Δu = λ₁ e^u - λ₂ e^{-u} \quad \text{in } Ω, \quad u = 0 \quad \text{on } ∂Ω,
\]
and
\[
-Δu = λ₁ e^u - λ₂ e^{-u} \quad \text{in } Ω, \quad u = 0 \quad \text{on } ∂Ω,
\]
respectively. Problem (7) was derived in [12]–[21] by statistical mechanics arguments.

Our aim in this article is to construct a family of solutions \( u_ρ \) to problem (1) which concentrate as \( ρ \to 0^+ \) with an arbitrarily prescribed number \( k \in \mathbb{N} \) of sign-changing singular bubbles, on the line of [7].

We recall that \( m, n \in \mathbb{N} \) are coprime if they do not admit common divisors. We make the following assumptions for the domain \( Ω: 0 \in Ω \) and
\[
\begin{cases}
x ∈ Ω \Rightarrow -x ∈ Ω \quad \text{and } x e^{2π\sqrt{-1}/(m+n)} ∈ Ω, \quad \text{if } γ = \frac{m}{n}, \quad m, n \in \mathbb{N} \text{ coprime}; \\
x ∈ Ω \Rightarrow -x ∈ Ω, \quad \text{if } γ ∉ \mathbb{Q},
\end{cases}
\]
where, in complex notation, multiplication by \( e^{2π\sqrt{-1}/(m+n)} \) denotes a rotation about the origin by the angle \( 2π/(m+n) \).
Correspondingly, we define the Sobolev space

$$\mathcal{H}_\gamma = \{ \varphi \in H^1_0(\Omega) : \varphi(x)e^{2\pi\sqrt{-1/(m+n)}} = \varphi(x) \forall x \in \Omega \},$$  \hspace{1cm} (10)

if $\gamma = m/n$ and $m, n \in \mathbb{N}$ are coprime or

$$\mathcal{H}_\gamma = \{ \varphi \in H^1_0(\Omega) : \varphi(-x) = \varphi(x) \forall x \in \Omega \},$$  \hspace{1cm} (11)

if $\gamma \notin \mathbb{Q}$.

We establish the following result.

**Theorem 1.1.** Fix $\gamma \in (0,1]$. Assume that $\Omega$ satisfies the symmetry assumption [9]. For any $k \in \mathbb{N}$ there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0)$ problem (1) admits a concentrating sign-changing family of solutions $u_\rho \in \mathcal{H}_\gamma$ satisfying

$$\rho e^{u_\rho} \, dx \rightharpoonup m_+(0) \delta_0(dx), \quad \rho e^{-\gamma u_\rho} \, dx \rightharpoonup m_-(0) \delta_0(dx) \quad as \ \rho \to 0^+$$

weakly in the sense of measures, where the “blow-up masses” $m_+(0), m_-(0)$ are given by

$$\begin{align*}
\frac{m_+(0)}{4\pi} &= \frac{k+1}{2} \left( 1 + \frac{1}{\gamma} \right) k + 1 - \frac{1}{\gamma}, & \text{if } k \text{ is odd;} \\
\frac{m_-(0)}{4\pi} &= \frac{k-1}{2} \left( 1 + \frac{1}{\gamma} \right) k + 1 - \frac{1}{\gamma}, & \text{if } k \text{ is even.}
\end{align*}$$  \hspace{1cm} (12)

Moreover,

$$u_\rho(x) \to \mathcal{M}_k G(x,0)$$  \hspace{1cm} (14)

uniformly on compact subsets of $\Omega \setminus \{0\}$ as $\rho \to 0^+$, where $G(\cdot, 0)$ is Green’s function defined in [27] and $\mathcal{M}_k = m_+(0) - m_-(0)$ (see also [16]) is the “algebraic total mass” with values

$$\mathcal{M}_k = \begin{cases} 4\pi [(1 + \frac{1}{2})k - \frac{1}{\gamma} + 1], & \text{if } k \text{ is odd;} \\
-4\pi (1 + \frac{1}{2})k, & \text{if } k \text{ is even.}
\end{cases}$$  \hspace{1cm} (15)

We shall obtain the solution $u_\rho$ in the form $u_\rho = W_\rho + \phi_\rho$, where $W_\rho$ is an alternating sum of $k$ singular Liouville bubbles of the form

$$w^\rho_\alpha(x) = \ln \frac{2\alpha^2 \delta^\alpha}{(\delta^\alpha + |x|^\alpha)^2},$$

projected onto $H^1_0(\Omega)$, namely:

$$W_\rho = \sum_{i=1}^{k} (-1)^{i-1} \frac{Pw^\alpha_{\delta_i} \gamma^{\sigma(i)}}{\sigma(i)}, \text{ with } \sigma(i) = 0 \text{ if } i \text{ is odd and } \sigma(i) = 1 \text{ if } i \text{ is even,}$$

where the projection of the bubble $Pw^\alpha_{\delta_i}$ is defined in [21] and both the singularity parameters $\alpha_i \geq 2$ and the concentration parameters $\delta_i > 0$ are chosen as in [25] and [29] in order to ensure that $\| \nabla \phi_\rho \|_{L^2(\Omega)} = o(1)$ as $\rho \to 0^+$. In particular, the $\alpha_i$’s and the rate of the $\delta_i$’s with respect to $\rho$ only depend on the parameter $\gamma$, while
the parameter τ influences the ratio \( \frac{\delta_{\alpha_i}}{\tau} \) (see Lemma 3.3). The functions \( w_\beta^{\alpha_i} \) are solutions to the singular Liouville equation

\[
-\Delta w_\beta^{\alpha_i} = |x|^{\alpha_i-2}e^{w_\beta^{\alpha_i}}, \quad \int_{\mathbb{R}^2} |x|^{\alpha_i-2}e^{w_\beta^{\alpha_i}}\,dx < +\infty.
\]

Since the appropriate choice of \( \alpha_i \)'s leads to \( \alpha_i \neq \alpha_j \) for \( i \neq j \), we find that the bubble tower approximate solution \( W_\rho \) is actually the sum of solutions to different singular Liouville problems. Such new blow-up profiles were observed in [6]. Towers of concentrated solutions to different singular Liouville equations were initially introduced in the article [7], where the case \( \gamma = 1 \) is considered, and which is the main motivation to this work.

The blow-up masses \( m_+(0), m_-(0) \) satisfy the identity

\[
8\pi \left[ m_+(0) + \frac{m_-(0)}{\gamma} \right] = |m_+(0) - m_-(0)|^2 = M^2_k.
\] (16)

Moreover, in view of (9), \( 0 \in \Omega \) is a critical point for the Robin’s function. In fact, identity (16) is a general property for concentrating solution sequences for (1), and if the concentration occurs at a single point, such a point is necessarily a critical point for Robin’s function, see, e.g., Remark 3 for a proof. For \( \gamma = 1 \) identity (16) was derived in [18].

It is natural to conjecture that the blow-up mass values (12)–(13) are the only admissible values for \( m_+(0), m_-(0) \), in view of the mass quantization results for the case \( \gamma = 1 \) in [11]. In this respect, a mass quantization property for (1) was announced in [26]; a partial result in this direction concerning the minimum values for blow-up masses was obtained in [24].

From the physics interpretation point of view, the solutions \( u_\rho \) as obtained in Theorem 1.1 yield solutions to (2) with total mass

\[
\lambda = m_+(0) + \frac{m_-(0)}{\gamma} = \frac{M^2_k}{8\pi} = \begin{cases} 2\pi\left(1 + \frac{1}{\gamma}\right)k + 1 - \frac{1}{\gamma} \end{cases}, & \text{if } k \text{ is odd;} \\
2\pi\left(1 + \frac{1}{\gamma}\right)k^2, & \text{if } k \text{ is even.} \tag{17}
\]

and vortex distribution parameter

\[
\tau_1 = \frac{m_+(0)}{\lambda} = \begin{cases} \frac{k+1}{(1+\frac{1}{\gamma})k+1-\frac{1}{\gamma}}, & \text{if } k \text{ is odd;} \\
\frac{k}{(1+\frac{1}{\gamma})k^2}, & \text{if } k \text{ is even.} \tag{18}
\end{cases}
\]

They also yield solutions to (3) with total mass given by (17) and with no restriction on \( \tau_1 \). It may be interesting to note that the “total mass” \( \lambda \) is the quantity on the left hand side in the identity (16). A proof of these statements is provided in the Appendix.

As already mentioned, our approach to prove Theorem 1.1 is strongly inspired by the singular bubble-tower construction in [7], where the case \( \gamma = 1 \) is considered, in the \( L^p \)-framework introduced in [4], see also [3]. Nevertheless, the case \( \gamma \neq 1 \) turns out to be significantly more delicate to handle, and it emphasizes specific analytic and geometric properties of the asymmetry parameter. In fact, the dependence of the singularity coefficients \( \alpha_i \) and of the concentration parameters \( \delta_{\alpha_i}^{\gamma} \) on \( \gamma \) is rather subtle; in particular, unlike the case \( \gamma = 1 \), the \( \alpha_i \)'s are never monotonically increasing with respect to \( i \) and the concentration parameters \( \delta_{\alpha_i}^{\gamma} \) do not depend linearly with respect to \( i \). Consequently, new ingredients are required in several estimates. Finally, it is interesting to observe that the geometrical symmetry condition (9)
required for $\Omega$, which ensures invertibility of the linearized operator, depends in a relevant way on $\gamma$, if $\gamma \in (0, 1] \cap \mathbb{Q}$.

**Notation.** For any measurable set $A \subset \Omega$ we denote by $\chi_A$ the characteristic function of $A$. We denote by $C > 0$ a general constant whose value may vary from line to line. When the integration variable is clear from the context, we omit it. For all $\phi \in H^1_0(\Omega)$ we set $\|\phi\| := \|\nabla \phi\|_{L^2(\Omega)}$.

2. **Ansatz and idea of the proof.** We recall that the “singular Liouville bubbles” are defined for $\alpha \geq 2$ and $\delta > 0$ by

$$w_\delta^\alpha(x) = \ln \frac{2\alpha^2 \delta \alpha}{(\delta \alpha + |x|^{\alpha})^2}.$$ 

The functions $w_\delta^\alpha$ satisfy

$$- \Delta w_\delta^\alpha = |x|^{-2} e^{w_\delta^\alpha}, \quad \int_{\mathbb{R}^2} |x|^{-2} e^{w_\delta^\alpha} \, dx < +\infty. \quad (19)$$

Furthermore,

$$\int_{\mathbb{R}^2} |x|^{-2} e^{w_\delta^\alpha} \, dx = 4\pi \alpha. \quad (20)$$

The functions $w_\delta^\alpha$ are uniquely determined as radial solutions for (19), see [22]. We denote by $P w_\delta^\alpha$ the projection of $w_\delta^\alpha$ onto $H^1_0(\Omega)$, namely the solution to the problem

$$\Delta P w_\delta^\alpha = \Delta w_\delta^\alpha \text{ in } \Omega, \quad P w_\delta^\alpha = 0 \text{ on } \partial \Omega. \quad (21)$$

We define

$$\sigma(i) = \frac{1 - (-1)^{i-1}}{2} = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ 1, & \text{if } i \text{ is even,} \end{cases}$$

for $i = 1, 2, \ldots, k$.

**Ansatz.** The solutions are of the form $u_\rho = W_\rho + \varphi_\rho$, where

$$W_\rho = \sum_{i=1}^{k} (-1)^{i-1} \frac{P w_{\delta_i}^{\alpha_i}}{\gamma \sigma(i)} = \sum_{1 \leq i \leq k, \text{ i odd}} P w_{\delta_i}^{\alpha_i} - \frac{1}{\gamma} \sum_{1 \leq i \leq k, \text{ i even}} P w_{\delta_i}^{\alpha_i}, \quad (22)$$

with

$$\alpha_i = \begin{cases} 2[(1 + \frac{1}{\gamma})i - \frac{1}{2}], & \text{if } i \text{ is odd;} \\ 2[(1 + \gamma)i - 1], & \text{if } i \text{ is even;} \end{cases}$$

and

$$\delta_i = d_i \rho^{s_i} \quad \text{for some } d_i > 0, \quad i = 1, 2, \ldots, k,$$

where

$$s_i = \begin{cases} \frac{(1+\gamma)(k-i)+\gamma}{2[(1+\gamma)(k-i)+1]}, & \text{if } k \text{ is odd;} \\ \frac{(1+\gamma)(k-i)+1}{2[(1+\gamma)(k-i)+1]}, & \text{if } k \text{ is even.} \end{cases}$$

Note that in particular

$$\alpha_i \geq 2, \quad \text{for all } i = 1, 2, \ldots, k$$

$$s_i > s_{i+1} \text{ and therefore } \delta_i = o(\delta_{i+1}), \quad \text{for all } i = 1, 2, \ldots, k - 1.$$ 

See Section [3] for the precise values of $d_i$, $i = 1, 2, \ldots, k$ and for the precise power decay rate of $\delta_i/\delta_{i+1}$.

Henceforth, we denote $w_i := w_{\delta_i}^{\alpha_i}, i = 1, 2, \ldots, k$. 
The most delicate part of the construction will be to show that if \( \alpha_i, \delta_i \) are chosen according to the above definitions, then \( W_\rho \) approximates a genuine solution to (1) up to an error which vanishes as a power of \( \rho \), as \( \rho \to 0^+ \). This fact, combined with the \(|\ln \rho|\)-estimate for the norm of the linearized operator (see (26) below for the precise statement) will enable us to obtain the desired solution as the fixed point of a contraction mapping.

More precisely, let

\[
f(t) := e^t - \tau e^{-\gamma t}, \quad t \in \mathbb{R}.
\]

Then, the error term to be estimated is given by:

\[
R_\rho = \Delta W_\rho + \rho f(W_\rho)
\]

\[
= \rho e^{W_\rho} - \sum_{1 \leq i \leq k} \frac{1}{\gamma} \sum_{i \ even} |x|^{\alpha_i-2} e^{w_i}
\]

(23)

It is convenient to set:

\[
E_+ := \rho e^{W_\rho} - \sum_{1 \leq i \leq k} \frac{1}{\gamma} \sum_{i \ even} |x|^{\alpha_i-2} e^{w_i}
\]

\[
E_- := \left( \rho e^{-\gamma W_\rho} - \frac{1}{\gamma} \sum_{1 \leq i \leq k} |x|^{\alpha_i-2} e^{w_i} \right)
\]

so that

\[
R_\rho = E_+ - E_-
\]

One of the main technical issues will be to show that, provided \( \alpha_i, \delta_i \) are chosen as above, there exist \( p > 1, \beta_0 = \beta_0(k, \gamma, p) > 0 \) such that

\[
\|E_+\|_{L^p(\Omega)} + \|E_-\|_{L^p(\Omega)} = O(\rho^\beta_0).
\]

(24)

The appropriate choice of the parameters \( \alpha_i, \delta_i \) is carried out in Section 3 where some properties necessary for the subsequent estimates are also derived. Then, estimate (24) is established in Section 4 and Section 5.

In order to prove estimate (24), we define the shrinking annuli

\[
A_j = \{ x \in \Omega : \sqrt{\delta_j-1} \delta_j \leq |x| < \sqrt{\delta_j \delta_{j+1}} \}, \quad j = 1, 2, \ldots, k,
\]

where we set \( \delta_0 = 0 \) and \( \delta_{k+1} = +\infty \).

We decompose \( E_+ \):

\[
E_+ = \sum_{1 \leq j \leq k} E_+ \chi_{A_j}
\]

\[
= \sum_{1 \leq j \leq k} \left( \rho e^{W_\rho} - \sum_{i \ odd} \frac{1}{\gamma} \sum_{i \ even} |x|^{\alpha_i-2} e^{w_i} \right) \chi_{A_j} + \sum_{1 \leq j \leq k} \left( \rho e^{W_\rho} - \sum_{i \ odd} \frac{1}{\gamma} \sum_{i \ even} |x|^{\alpha_i-2} e^{w_i} \right) \chi_{A_j}
\]

\[
= \sum_{1 \leq j \leq k} \left( \rho e^{W_\rho} - \sum_{i \ odd} \frac{1}{\gamma} \sum_{i \ even} |x|^{\alpha_i-2} e^{w_i} \right) \chi_{A_j} + \sum_{1 \leq j \leq k} \rho e^{W_\rho} \chi_{A_j} - \sum_{1 \leq j \leq k} \sum_{i \ odd} |x|^{\alpha_i-2} e^{w_i} \chi_{A_j}
\]

\[
= E_1^+ + E_2^+ + E_3^+.
\]
Similarly, we decompose $E_-$:

\[
E_- = \sum_{1 \leq j \leq k} E_j \chi_{A_j}
\]

\[
= \sum_{\substack{1 \leq j \leq k \\ j \text{ even}}} \left( \rho ee^{-\gamma W_p} - \frac{1}{\gamma} \sum_{\substack{1 \leq i \leq k \\ i \text{ even}}} |x|^{\alpha_i} e^{w_i} \right) \chi_{A_j}
\]

\[
+ \sum_{\substack{1 \leq j \leq k \\ j \text{ odd}}} \left( \rho ee^{-\gamma W_p} - \frac{1}{\gamma} \sum_{\substack{1 \leq i \leq k \\ i \text{ even}}} |x|^{\alpha_i} e^{w_i} \right) \chi_{A_j}
\]

\[
= \sum_{\substack{1 \leq j \leq k \\ j \text{ even}}} \left( \rho ee^{-\gamma W_p} - |x|^{\alpha_j} e^{w_j} \right) \chi_{A_j} + \sum_{\substack{1 \leq j \leq k \\ j \text{ odd}}} \rho ee^{-\gamma W_p} \chi_{A_j}
\]

\[-\frac{1}{\gamma} \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \sum_{i \text{ even}} |x|^{\alpha_i} e^{w_i} \chi_{A_j}
\]

\[= E_-^1 + E_-^2 + E_-^3.
\]

In short, the choice of $\alpha, \delta_i$ will ensure the smallness of the error terms $E_-^1, E_-^3$, which measure the interaction in the $j$-th annulus $A_j$ between the $j$-th bubble $w_{\delta_i}^j$ and all other bubbles. Indeed, as in [7], the errors $E_-^1, E_-^3$ are small inside the $j$-th annulus $A_j$ “because” the choice of $\alpha_j$ will cancel the interaction of the $j$-th bubble and all previous (faster concentrating) bubbles, whereas the choice of $\delta_j$ will cancel the interaction of the $j$-th bubble $w_{\delta_j}^j$ and all subsequent (slower concentrating) bubbles. On the other hand, the error terms $E_-^2, E_-^2$ are estimated by some delicate recursive relations for $\alpha_j, \delta_j$. Estimation of $E_-^3, E_-^3$ follows from the fact that, outside the $j$-th annulus $A_j$, the $j$-th bubble $w_{\delta_j}^j$ is negligible, up to an error which vanishes as a power of $\rho$.

Once (24) is established, we define

\[
S_\rho := \rho f'(W_p) - \sum_{i=1}^{k} |x|^{\alpha_i} e^{w_i}
\]

\[
N_\rho(\phi) := \rho [f(W_p + \phi) - f(W_p) - f'(W_p)\phi]
\]

\[
L_\rho \phi := -\Delta \phi - \sum_{i=1}^{k} |x|^{\alpha_i} e^{w_i}.
\]

We note that

\[
S_\rho = E_+ + \gamma E_-,
\]

so that estimate (24) provides an estimate for $S_\rho$ as well. In Section 6 we show that for any $p > 1$ there exists $c > 0$ such that

\[
\|\phi\| \leq c |\ln \rho| \|L_\rho \phi\|, \quad \forall \phi \in \mathcal{H}_\gamma.
\]

At this point, we can show that there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0)$ the equation

\[
\phi = T(\phi) := (L_\rho)^{-1}(N_\rho(\phi) + S_\rho \phi + R_\rho)
\]

admits a fixed point $\phi_\rho$ satisfying $\|\phi_\rho\| \leq R \rho^p |\ln \rho|$ for some $\beta_\rho = \beta_\rho(\tau, \gamma, k) > 0$, $p > 1$ and $R > 0$. The function $u_\rho = W_p + \phi_\rho$ is the desired solution to (1). The details of the fixed point argument are contained in Section 7.
3. Definition and properties of the parameters. In this section we define the parameters $\alpha_j, \delta_j$, $j = 1, 2, \ldots, k$ and we establish some properties which will be used in order to estimate the error terms. The justification of the choice of $\alpha_j, \delta_j$ will be provided in Section 4.

We denote by $G(x, y)$, $x, y \in \Omega$, $x \neq y$, the Green’s function for $\Omega$, namely

$$-\Delta G(\cdot, y) = \delta_y \quad \text{in } \Omega, \quad G(\cdot, y) = 0 \quad \text{on } \partial \Omega.$$  

We denote by $H(x, y)$ the regular part of $G(x, y)$:

$$G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} + H(x, y). \quad (27)$$  

We set $h(x) = H(x, 0)$.

**Definition of $\alpha_j, \delta_j$.** The appropriate values for $\alpha_j, \delta_j$ are deduced from the following defining conditions.

**Odd index.** If $j \in \{1, 2, \ldots, k\}$ is odd, we define:

$$\begin{aligned}
\alpha_j &:= 2 \left(1 + \sum_{i<j} (-1)^i \gamma_{\sigma(i)} \alpha_i\right) \\
\ln(2\alpha_j^2 \delta_j^{\alpha_j}) &:= 2 \sum_{i>j} (-1)^i \gamma_{\sigma(i)} \ln \delta_i^{\alpha_i} - 4\pi h(0) \sum_{i=1}^k (-1)^i \gamma_{\sigma(i)} \alpha_i + \ln \rho.
\end{aligned} \quad (28)$$

**Even index.** If $j \in \{1, 2, \ldots, k\}$ is even, we define:

$$\begin{aligned}
\alpha_j &:= 2 \left(1 - \gamma \sum_{i<j} (-1)^i \gamma_{\sigma(i)} \alpha_i\right) \\
\ln(2\alpha_j^2 \delta_j^{\alpha_j}) &:= -2\gamma \sum_{i>j} (-1)^i \gamma_{\sigma(i)} \ln \delta_i^{\alpha_i} + 4\pi \gamma h(0) \sum_{i=1}^k (-1)^i \gamma_{\sigma(i)} \alpha_i + \ln(\rho \tau \gamma).
\end{aligned} \quad (29)$$

We note that the $\alpha_j$’s are determined by the number $k$ of bubbles only; the concentration parameters $\delta_j$ also depend on $\rho$. Moreover, [28]-[29] define $\alpha_j$ recursively in terms of $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}$ and $\delta_j$ in terms of $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $\delta_{j+1}, \delta_{j+2}, \ldots, \delta_k$.

**Remark 1.** There holds $\alpha_j \geq 2$ for all $j = 1, 2, \ldots, k$.

An explicit computation yields the following first values of the $\alpha_j$’s:

$$\begin{aligned}
\alpha_1 &:= 2, \quad \alpha_3 = 2\left(3 + \frac{2}{\gamma}\right), \quad \alpha_5 = 2\left(5 + \frac{4}{\gamma}\right), \quad \alpha_7 = 2\left(7 + \frac{6}{\gamma}\right), \ldots \\
\alpha_2 &:= 2\left(1 + 2\gamma\right), \quad \alpha_4 = 2\left(3 + 4\gamma\right), \quad \alpha_6 = 2\left(5 + 6\gamma\right), \quad \alpha_8 = 2\left(7 + 8\gamma\right) \ldots
\end{aligned} \quad (30)$$

Moreover, if $k = 2$, we obtain the following decay rates for the $\delta_j$’s:

$$\delta_1 = d_1 \rho^{(2+\gamma)/2}, \quad \delta_2 = d_2 \rho^{1/[2(1+2\gamma)]}.$$  

It will be convenient to set

$$A_k := \sum_{i=1}^k \frac{(-1)^i}{\gamma_{\sigma(i)} \alpha_i}. \quad (31)$$
Properties of the singularity coefficients $\alpha_j$. In this subsection we determine $\alpha_j$ explicitly in terms of $j$, for $j = 1, 2, \ldots, k$, and we establish the main properties of the $\alpha_j$’s which will be needed in the sequel.

**Proposition 1.** For all $j = 1, 2, \ldots, k$ we have:

$$
\alpha_j = \begin{cases}
2[(1 + \frac{1}{\gamma})j - \frac{1}{2}] & \text{if } j \text{ is odd;} \\
2[(1 + \gamma)j - 1] & \text{if } j \text{ is even.}
\end{cases}
$$

Moreover, we have

$$
\sum_{1 \leq j \leq k \atop j \text{ odd}} \alpha_j = \begin{cases}
\frac{k+1}{2}[(1 + \frac{1}{\gamma})k + 1 - \frac{1}{\gamma}] & \text{if } k \text{ is odd;} \\
\frac{k}{2}[(1 + \frac{1}{\gamma})k - \frac{1}{2}] & \text{if } k \text{ is even;}
\end{cases}
$$

and

$$
\frac{1}{\gamma} \sum_{1 \leq j \leq k \atop j \text{ even}} \alpha_j = \begin{cases}
\frac{k-1}{2}[(1 + \frac{1}{\gamma})k + 1 - \frac{1}{\gamma}] & \text{if } k \text{ is odd;} \\
\frac{k}{2}[(1 + \frac{1}{\gamma})(k + 1) + 1 - \frac{1}{\gamma}] & \text{if } k \text{ is even;}
\end{cases}
$$

In particular, the following identity holds true.

$$
A_k = \begin{cases}
-(1 + \frac{1}{\gamma})k + \frac{1}{\gamma} - 1 & \text{if } k \text{ is odd;} \\
(1 + \frac{1}{\gamma})k & \text{if } k \text{ is even,}
\end{cases}
$$

where $A_k$ is defined in (31).

The following consequence of Proposition 1 will be essential in the proof of the invertibility of the linearized operator $L_\rho$ defined in (25). Indeed, the kernel of $L_\rho$, is determined by the divisibility properties of $\alpha_j/2$, $j = 1, 2, \ldots, k$. We recall that two integers $m, n \in \mathbb{N}$ are said to be coprime if they do not admit common divisors.

**Corollary 1.** Suppose $\gamma = m/n$ with $m, n \in \mathbb{N}$, $m, n$ coprime, and suppose that $\alpha_j/2 \in \mathbb{N}$ for some $j = 1, 2, \ldots, k$. Then, there exists $k_j \in \mathbb{N} \cup \{0\}$ such that

$$
\frac{\alpha_j}{2} = \begin{cases}
(m+n)k_j + 1 & \text{if } j \text{ is odd;} \\
(m+n)k_j - 1 & \text{if } j \text{ is even.}
\end{cases}
$$

**Proof.** Suppose $j$ is odd. Then,

$$
\frac{\alpha_j}{2} = (1 + \frac{n}{m})j - \frac{n}{m} = j + \frac{n}{m}(j - 1).
$$

Since $m, n$ are coprime, it follows that $j - 1 = k_j m$ for some $k_j \in \mathbb{N} \cup \{0\}$. Consequently,

$$
\frac{\alpha_j}{2} = k_j m + 1 + nk_j = (m+n)k_j + 1,
$$

and (36) is established for odd $j$.

Similarly, suppose $j$ is even. Then,

$$
\frac{\alpha_j}{2} = (1 + \frac{m}{n})j - 1 = \frac{m}{n}j + j - 1.
$$

Since $m, n$ are coprime, it follows that $j = k_j n$ for some $k_j \in \mathbb{N}$. Consequently,

$$
\frac{\alpha_j}{2} = (1 + \frac{m}{n})k_j n - 1 = (m+n)k_j - 1.
$$

Formula (36) is completely established. 

In order to prove Proposition 1 we establish some lemmas.
Lemma 3.1. The following recursive formulae hold:

\[
\alpha_j = \begin{cases} 
\frac{1}{\gamma} \alpha_{j-1} + 2(1 + \frac{1}{\gamma}), & \text{if } j \text{ is odd} \\
\gamma \alpha_{j-1} + 2(1 + \gamma), & \text{if } j \text{ is even}, 
\end{cases}
\]  

(37)

for \( j = 2, 3, \ldots, k \).

Proof. Suppose \( j \) is odd. Then, in view of (28) and the fact that \( j - 1 \) is even, we have

\[
\alpha_j = 2 \left( 1 + \sum_{i<j} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right) = 2 \left( 1 + \frac{(-1)^{j-1}}{\gamma \sigma(j-1)} \alpha_{j-1} + \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right)
\]

\[
= 2 \left( 1 + \frac{1}{\gamma} \alpha_{j-1} + \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right).
\]

In view of (29), we have

\[
\alpha_{j-1} = 2 \left( 1 - \gamma \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right) = 2 - 2\gamma \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i
\]

and therefore

\[
\sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i = \frac{1}{\gamma} - \frac{1}{2\gamma} \alpha_{j-1}.
\]

We conclude that

\[
\alpha_j = 2 \left( 1 + \frac{1}{\gamma} \alpha_{j-1} + \frac{1}{\gamma} - \frac{1}{2\gamma} \alpha_{j-1} \right) = 2 \left( 1 + \frac{1}{\gamma} + \frac{1}{2\gamma} \alpha_{j-1} \right),
\]

and (37) is established for odd indices \( j \).

Similarly, suppose \( j \) is even. Then, in view of (29) and the fact that \( j - 1 \) is odd, we have

\[
\alpha_j = 2 \left( 1 - \gamma \sum_{i<j} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right) = 2 \left( 1 - \gamma \frac{(-1)^{j-1}}{\gamma \sigma(j-1)} \alpha_{j-1} - \gamma \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right)
\]

\[
= 2 \left( 1 + \gamma \alpha_{j-1} - \gamma \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right).
\]

Since \( j - 1 \) is odd, we have from (28) that

\[
\alpha_{j-1} = 2 \left( 1 + \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i \right) = 2 + 2 \sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i,
\]

that is,

\[
\sum_{i<j-1} \frac{(-1)^i}{\gamma \sigma(i)} \alpha_i = \frac{1}{2} \alpha_{j-1} - 1.
\]

We deduce that

\[
\alpha_j = 2 \left( 1 + \gamma \alpha_{j-1} - \frac{\gamma}{2} \alpha_{j-1} + \gamma \right) = 2(1 + \gamma)\alpha_{j-1} + \gamma \alpha_{j-1},
\]

and the recursive formula (37) is also established for all even indices \( j \). \( \Box \)

We also use the following results, whose proof is elementary.
Lemma 3.2. Let $k \in \mathbb{N}$. Then,

$$
\sum_{1 \leq j \leq k} 1 = \begin{cases} 
\frac{k+1}{2}, & \text{if } k \text{ is odd} \\
\frac{k}{2}, & \text{if } k \text{ is even}
\end{cases} \quad \sum_{1 \leq j \leq k} 1 = \begin{cases} 
\frac{k-1}{2}, & \text{if } k \text{ is odd} \\
\frac{k}{2}, & \text{if } k \text{ is even}
\end{cases}
$$

and

$$
\sum_{1 \leq j \leq k} j = \begin{cases} 
\frac{(k+1)^2}{4}, & \text{if } k \text{ is odd} \\
\frac{k^2}{4}, & \text{if } k \text{ is even}
\end{cases} \quad \sum_{1 \leq j \leq k} j = \begin{cases} 
\frac{(k-1)(k+1)}{2}, & \text{if } k \text{ is odd} \\
\frac{k(k+2)}{4}, & \text{if } k \text{ is even}.
\end{cases}
$$

Now we can provide the proof of Proposition 1.

Proof of Proposition 1. Proof of (32). We argue by induction. We already know from (30) that $\alpha_1 = 2$ and $\alpha_2 = 2(1 + 2\gamma)$.

Suppose (32) holds true for all $i < j$, with $j$ an odd index. Then, in view of (37) and the induction assumption we have:

$$
\alpha_j = \frac{1}{\gamma} \alpha_{j-1} + 2(1 + \frac{1}{\gamma}) = \frac{2}{\gamma}[(1 + \gamma)(j - 1) - 1] + 2(1 + \frac{1}{\gamma}) = 2[(1 + \frac{1}{\gamma})j - \frac{1}{\gamma}],
$$

and (32) is established in this case.

Suppose (32) holds true for all $i < j$, with $j$ an even index. Then, in view of (37) and the induction assumption we have:

$$
\alpha_j = \gamma \alpha_{j-1} + 2(1 + \gamma) = 2\gamma[(1 + \frac{1}{\gamma})(j - 1) - \frac{1}{\gamma}] + 2(1 + \gamma) = 2[(1 + \gamma)j - 1].
$$

Now (32) is completely established.

Proof of (33). Recall from (32) that if $j$ is odd, then $\alpha_j = 2\gamma[(1 + \frac{1}{\gamma})j - \frac{1}{\gamma}]$. Hence, we compute:

$$
\sum_{1 \leq j \leq k} \alpha_j = 2(1 + \frac{1}{\gamma}) \sum_{1 \leq j \leq k} j - \frac{2}{\gamma} \sum_{1 \leq j \leq k} 1.
$$

In view of Lemma 3.2 for $k$ odd we deduce that:

$$
\sum_{1 \leq j \leq k} \alpha_j = (1 + \frac{1}{\gamma})\frac{(k+1)^2}{2} - \frac{k+1}{\gamma} = k + 1 - \frac{1}{\gamma}.
$$

Similarly, for $k$ even we deduce that

$$
\sum_{1 \leq j \leq k} \alpha_j = (1 + \frac{1}{\gamma})\frac{k^2}{2} - \frac{k}{\gamma} = k\gamma[(1 + \frac{1}{\gamma})k + 1 - \frac{1}{\gamma}].
$$

Proof of (34). Recall from (32) that if $j$ is even, then $\alpha_j = 2\gamma[(1 + \gamma)j - 1]$. In view of Lemma 3.2 for $k$ odd we deduce that:

$$
\frac{1}{\gamma} \sum_{1 \leq j \leq k} \alpha_j = 2(1 + \frac{1}{\gamma}) \sum_{1 \leq j \leq k} j - \frac{2}{\gamma} \sum_{1 \leq j \leq k} 1.
$$

In view of Lemma 3.2 for $k$ odd we deduce that:

$$
\frac{1}{\gamma} \sum_{1 \leq j \leq k} \alpha_j = (1 + \frac{1}{\gamma})\frac{(k-1)(k+1)}{2} - \frac{k-1}{\gamma} = k - 1 - \frac{1}{\gamma}.
$$
Similarly, for $k$ even we deduce that

$$\frac{1}{\gamma} \sum_{j \leq k \text{ even}} \alpha_j = \left(1 + \frac{1}{\gamma}\right) \frac{k(k+2)}{2} - \frac{k}{\gamma} = \frac{k}{2} \left(1 + \frac{1}{\gamma}\right)(k+1) + 1 - \frac{1}{\gamma}. $$

Proof of (35). The proof of (35) follows from (33)–(34). However, a proof may also be derived independently from (28)–(29) and (32) with $j = k$. Indeed, suppose $k$ is odd. In view of (28) we have

$$\alpha_k = 2 \left(1 + \sum_{i<k} \frac{(-1)^i}{\gamma_{i}^{\sigma(i)} \alpha_i}\right).$$

Hence, we may write

$$\sum_{i=1}^{k} \frac{(-1)^i}{\gamma_{i}^{\sigma(i)} \alpha_i} = \sum_{i=1}^{k-1} \frac{(-1)^i}{\gamma_{i}^{\sigma(i)} \alpha_i} + \frac{(-1)^k}{\gamma_{k}^{\sigma(k)} \alpha_k} = \frac{\alpha_k}{2} - 1 - \alpha_k = -1 - \frac{\alpha_k}{2}.$$ 

In view of (32) with $j = k$ we have

$$1 + \frac{\alpha_k}{2} = 1 + \left(1 + \frac{1}{\gamma}\right)k - \frac{1}{\gamma}.$$ 

Hence, (35) is established for $k$ odd.

Similarly, suppose $k$ is even. In view of (29) we have

$$\alpha_k = 2 \left(1 - \gamma \sum_{i<k} \frac{(-1)^i}{\gamma_{i}^{\sigma(i)} \alpha_i}\right).$$

Hence, we may write

$$\sum_{i=1}^{k} \frac{(-1)^i}{\gamma_{i}^{\sigma(i)} \alpha_i} = \sum_{i=1}^{k-1} \frac{(-1)^i}{\gamma_{i}^{\sigma(i)} \alpha_i} + \frac{(-1)^k}{\gamma_{k}^{\sigma(k)} \alpha_k} = 1 - \frac{\alpha_k}{2\gamma} + \frac{\alpha_k}{\gamma} = 1 + \frac{\alpha_k}{2\gamma}.$$ 

Now, in view of (32) with $j = k$ we conclude that

$$\frac{1}{\gamma} + \frac{\alpha_k}{2\gamma} = \frac{1}{\gamma} + \frac{1}{\gamma} \left((1 + \gamma)k - 1\right) = (1 + \frac{1}{\gamma})k.$$ 

The asserted formula (35) follows.

**Remark 2.** Unlike the case $\gamma = 1$, if $0 < \gamma < 1$ the sequence $\alpha_j$ is not necessarily monotonically increasing with respect to $j = 1, 2, \ldots, k$.

Indeed, the following holds true.

**Claim.** For all $h \in \mathbb{N}$ such that $h > (1 - \gamma^2)^{-1}$ there holds $\alpha_{2h} < \alpha_{2h-1}$.

Using the explicit values of $\alpha_{2h}, \alpha_{2h-1}$ as in (32), we compute:

$$\frac{1}{2}(\alpha_{2h} - \alpha_{2h-1}) = (1 + \gamma)2h - 1 - (1 + \frac{1}{\gamma})(2h - 1) + \frac{1}{\gamma} = 2[(\gamma - 1)h + \frac{1}{\gamma}] = \frac{2}{\gamma}[1 - (1 - \gamma^2)h].$$

The claim follows.
Properties of the concentration parameters $\delta_j$. In this subsection we compute the power decay rates of the concentration parameters $\delta_j$ as $\rho \to 0^\tau$, $j = 1, 2, \ldots, k$.

Let $\kappa_j = \kappa_j(\gamma, \tau, h(0), k) > 0$, $j = 1, 2, \ldots, k$ be defined by

$$
\kappa_j = \begin{cases} 
\frac{c_{-n+h(0)A_k}}{2\alpha_k}, & \text{if } k \text{ is odd;} \\
\frac{2\gamma c_{-n+h(0)A_k}}{\alpha_k}, & \text{if } k \text{ is even;}
\end{cases}
$$

$\kappa_j$  \hspace{1cm} (38)

Let $c_j = c_j(\gamma, \tau, h(0), k) > 0$, $j = 1, 2, \ldots, k$ be defined by:

$$
c_j = \begin{cases} 
\kappa_j \kappa_{j+1} \kappa_{j+2} \cdots \kappa_k, & \text{if } k \text{ is odd, } j \text{ is odd;} \\
\kappa_j \kappa_{j+1}^{\gamma} \kappa_{j+2}^{\gamma} \cdots \kappa_k^{\gamma}, & \text{if } k \text{ is odd, } j \text{ is even;} \\
\kappa_j \kappa_{j+1} \kappa_{j+2} \kappa_{j+3} \cdots \kappa_k^{1/\gamma}, & \text{if } k \text{ is even, } j \text{ is odd;} \\
\kappa_j \kappa_{j+1}^{1/\gamma} \kappa_{j+2}^{1/\gamma} \cdots \kappa_k, & \text{if } k \text{ is even, } j \text{ is even.}
\end{cases}
$$

Let $d_j = d_j(\gamma, \tau, h(0), k) > 0$ be defined by

$$
d_j = c_j^{1/\alpha_j} = \begin{cases} 
\frac{1}{2}[(1+\frac{1}{\gamma})j-\frac{1}{4}], & \text{if } j \text{ is odd} \\
\frac{1}{2}[(1+\gamma)j-1], & \text{if } j \text{ is even.}
\end{cases}
$$

With the above definitions, we have:

**Proposition 2.** The following power decay rates hold true for all $j = 1, 2, \ldots, k$:

$$
\delta_j^{\alpha_j} = c_j \rho^j, \quad \delta_j = d_j \rho^j, \quad \text{for all } k = 1, 2, \ldots, k
$$

where $r_j = r_j(\gamma, k) > 0$ is defined by

$$
r_j = \begin{cases} 
(\gamma + 1)(k - j) + \gamma, & \text{if } k \text{ odd, } j \text{ even} \\
(1 + \frac{1}{\gamma})(k - j) + 1, & \text{if } k \text{ odd, } j \text{ odd} \\
(1 + \frac{1}{\gamma})(k - j) + \frac{1}{\gamma}, & \text{if } k \text{ even, } j \text{ odd} \\
(1 + \gamma)(k - j) + 1, & \text{if } k \text{ even, } j \text{ even.}
\end{cases}
$$

$s_j = s_j(\gamma, k) > 0$ is defined by

$$
s_j = \begin{cases} 
\frac{1+\gamma(j-k)+\gamma}{2[(1+\gamma)(j-k)+1]}, & \text{if } k \text{ odd} \\
\frac{1+\gamma(j-k)+\gamma}{2[(1+\gamma)(j-k)+1]}, & \text{if } k \text{ even.}
\end{cases}
$$

and $q_j = q_j(\gamma, k) > 0$ is defined by

$$
q_j = \begin{cases} 
\frac{1+\gamma[(1+\gamma)(k-1)+\gamma]}{2[(1+\gamma)(k-1)+\gamma]}, & \text{if } k \text{ odd;}
\frac{1+\gamma[(1+\gamma)(k-1)+\gamma]}{2[(1+\gamma)(k-1)+\gamma]}, & \text{if } k \text{ even.}
\end{cases}
$$

In order to prove Proposition 2, we first establish a recursive formula.

**Lemma 3.3.** We have $\delta_k^{\alpha_k} = \kappa_k \rho$ where $\kappa_k$ is defined in (38) and

$$
\delta_j^{\alpha_j} = \begin{cases} 
\kappa_j \rho^{1+1/\gamma} \left(\delta_{j+1}^{\alpha_{j+1}}\right)^{1/\gamma}, & \text{if } j \text{ is odd} \\
\kappa_j \rho^{1+\gamma} \left(\delta_{j+1}^{\alpha_{j+1}}\right)^{\gamma}, & \text{if } j \text{ is even, } j = 1, 2, \ldots, k - 1.
\end{cases}
$$
Proof. Suppose \( k \) is odd. Then, formula \( (28) \) takes the form
\[
\ln(2\alpha_k^2\delta_k^{\alpha_k}) = -4\pi h(0)A_k + \ln \rho.
\]
Recalling the explicit value of \( A_k \) as in \( (35) \) and of \( \alpha_k \) as in \( (28) \), we conclude the proof.

Similarly, suppose \( k \) is even. Then, formula \( (29) \) takes the form
\[
\ln(2\alpha_k^2\delta_k^{\alpha_k}) = 4\pi \gamma h(0)A_k + \ln(\rho \tau \gamma).
\]
Recalling the explicit value of \( A_k \) as in \( (35) \) and of \( \alpha_k \) as in \( (29) \), we conclude the proof.

Suppose \( j \) is odd, \( j \leq k - 1 \). Using \( (28) \) and observing that \( j + 1 \) is even we have
\[
\ln(2\alpha_j^2\delta_j^{\alpha_j}) = 2\sum_{i>j} \frac{(-1)^i}{\gamma \sigma(i)} \ln \delta_i^{\alpha_i} - 4\pi h(0)A_k + \ln \rho
\]
\[
= 2\gamma \ln \delta_{j+1}^{\alpha_{j+1}} + 2\sum_{i>j+1} \frac{(-1)^i}{\gamma \sigma(i)} \ln \delta_i^{\alpha_i} - 4\pi h(0)A_k + \ln \rho,
\]
where \( A_k \) is defined in \( (31) \). On the other hand, using \( (29) \) we have
\[
2\sum_{i>j+1} \frac{(-1)^i}{\gamma \sigma(i)} \ln \delta_i^{\alpha_i} = -\frac{1}{\gamma} \ln(2\alpha_{j+1}^2\delta_{j+1}^{\alpha_{j+1}}) + 4\pi \gamma h(0)A_k + \frac{1}{\gamma} \ln(\rho \tau \gamma).
\]
We deduce that
\[
\ln(2\alpha_j^2\delta_j^{\alpha_j}) = \frac{2}{\gamma} \ln \delta_{j+1}^{\alpha_{j+1}} - \frac{1}{\gamma} \ln(2\alpha_{j+1}^2\delta_{j+1}^{\alpha_{j+1}}) + \frac{1}{\gamma} \ln(\rho \tau \gamma) + \ln \rho
\]
and consequently
\[
2\alpha_j^2\delta_j^{\alpha_j} = \frac{(\tau \gamma)^{1/\gamma}}{(2\alpha_{j+1}^2)^{1/\gamma}} \rho^{1+1/\gamma} (\delta_{j+1}^{\alpha_{j+1}})^{1/\gamma}.
\]
Hence, the asserted recursive formula follows for \( j \) odd.

Similarly, suppose that \( j \) is even, \( j \leq k - 1 \). In view of \( (29) \) and observing that \( j + 1 \) is odd, we have
\[
\ln(2\alpha_j^2\delta_j^{\alpha_j}) = -2\gamma \sum_{i>j} \frac{(-1)^i}{\gamma \sigma(i)} \ln \delta_i^{\alpha_i} + 4\pi \gamma h(0)A_k + \ln(\rho \tau \gamma)
\]
\[
= 2\gamma \ln \delta_{j+1}^{\alpha_{j+1}} - 2\gamma \sum_{i>j+1} \frac{(-1)^i}{\gamma \sigma(i)} \ln \delta_i^{\alpha_i} + 4\pi \gamma h(0)A_k + \ln(\rho \tau \gamma).
\]
Since \( j + 1 \) is odd, we have from \( (28) \)
\[
2\sum_{i>j+1} \frac{(-1)^i}{\gamma \sigma(i)} \ln \delta_i^{\alpha_i} = \ln(2\alpha_{j+1}^2\delta_{j+1}^{\alpha_{j+1}}) + 4\pi h(0)A_k - \ln \rho.
\]
We deduce that
\[
\ln(2\alpha_j^2\delta_j^{\alpha_j}) = 2\gamma \ln \delta_{j+1}^{\alpha_{j+1}} - \gamma \ln(2\alpha_{j+1}^2\delta_{j+1}^{\alpha_{j+1}}) + \gamma \ln \rho + \ln(\rho \tau \gamma)
\]
and finally
\[
2\alpha_j^2\delta_j^{\alpha_j} = \frac{\tau \gamma}{(2\alpha_{j+1}^2)^{1/\gamma}} \rho^{1+1/\gamma} (\delta_{j+1}^{\alpha_{j+1}})^{1/\gamma}.
\]
The asserted recursive formula is now completely established. \( \square \)
Proof of Proposition A. Proof of the first decay rate in [40]. We equivalently show that

\[ \delta_{k-j}^{\alpha_k} = \begin{cases} 
  c_{k-j} \rho^{(1+\gamma)(j+\gamma)} & \text{k odd, } j \text{ odd} \\
  c_{k-j} \rho^{(1+\gamma)(j+\gamma)} & \text{k odd, } j \text{ even} \\
  c_{k-j} \rho^{(1+\gamma)(j+\gamma)} & \text{k even, } j \text{ odd} \\
  c_{k-j} \rho^{(1+\gamma)(j+\gamma)} & \text{k even, } j \text{ even}, 
\end{cases} \tag{45} \]

where \( d_{k-j} \) is defined in [39].

We argue by induction. Suppose \( k \) is odd. For \( j = 0 \) we have \( \delta_{k}^{\alpha_k} = \kappa_k \rho = c_k \rho \) and the formula holds true in this case. For \( j = 1 \) we have, since \( k-1 \) is even,

\[ \delta_{k-1}^{\alpha_{k-1}} = \kappa_{k-1} \rho^{1+\gamma}(\delta_k^{\alpha_k})^\gamma = \kappa_{k-1} \rho^{1+\gamma}(\kappa_k \rho)^\gamma = \kappa_{k-1} \kappa_k^{\gamma} \rho^{(1+\gamma)+\gamma} = c_{k-1} \rho^{(1+\gamma)+\gamma}, \]

and the formula is verified for \( j = 1 \) as well.

Hence, assume that the formula is true for all \( i \leq j < k \) with \( j \) even. Then, \( k-j-1 \) is even,

\[ \delta_{k-j-1}^{\alpha_{k-j-1}} = \kappa_{k-j-1} \rho^{1+\gamma}(\delta_{k-j}^{\alpha_k})^\gamma = \kappa_{k-j-1} \rho^{1+\gamma}(\kappa_{k-j} \kappa_j \cdot \cdots \cdot \kappa_k \rho^{(1+\gamma)(j+\gamma)})^\gamma = c_{k-j-1} \rho^{(1+\gamma)(j+1)\gamma}, \]

and the formula holds true in this case.

Suppose the formula holds true for \( j \) odd. The, \( k-j-1 \) is odd and we have:

\[ \delta_{k-j-1}^{\alpha_{k-j-1}} = \kappa_{k-j-1} \rho^{1+\gamma}(\delta_{k-j}^{\alpha_k})^\gamma = \kappa_{k-j-1} \rho^{1+\gamma}(\kappa_{k-j} \kappa_j \cdot \cdots \cdot \kappa_k \rho^{(1+\gamma)(j+\gamma)})^\gamma = c_{k-j-1} \rho^{(1+\gamma)(j+1)\gamma}, \]

By induction, the formula is established for \( k \) odd.

Now, assume that \( k \) is even. For \( j = 0 \) we have \( \delta_{k}^{\alpha_k} = \kappa_k \rho = c_k \rho \), and the statement is verified. For \( j = 1 \) we have

\[ \delta_{k-1}^{\alpha_{k-1}} = \kappa_{k-1} \rho^{1+\gamma}(\delta_{k}^{\alpha_k})^\gamma = \kappa_{k-1} \rho^{1+\gamma}(\kappa_k \rho)\gamma = \kappa_{k-1} \kappa_k^{\gamma} \rho^{(1+\gamma)+1\gamma} = c_{k-1} \rho^{(1+\gamma)+1\gamma}, \]

and the statement is verified.

Hence, suppose the statement holds true for all \( i \leq j, j \) even. The, \( j+1 \) is odd, \( k-j-1 \) is odd. We have:

\[ \delta_{k-j-1}^{\alpha_{k-j-1}} = \kappa_{k-j-1} \rho^{1+\gamma}(\delta_{k-j}^{\alpha_k})^\gamma = \kappa_{k-j-1} \rho^{1+\gamma}(\kappa_k \rho^{(1+\gamma)j+1\gamma}) = c_{k-j-1} \rho^{(1+\gamma)(j+1)\gamma}, \]

and the asserted formula follows.

Finally, suppose the statement holds true for all \( i \leq j, j \) odd. Then, \( k-j-1 \) is even. We compute:

\[ \delta_{k-j-1}^{\alpha_{k-j-1}} = \kappa_{k-j-1} \rho^{1+\gamma}(\delta_{k-j}^{\alpha_k})^\gamma = \kappa_{k-j-1} \rho^{1+\gamma}(\kappa_k \rho^{(1+\gamma)j+1\gamma}) = c_{k-j-1} \rho^{(1+\gamma)(j+1)\gamma}, \]
Proof of the second decay rate in (40). Using (40), if \( k \) is odd and \( j \) is even, we have
\[
\delta_{\alpha_j} = c_j \rho^{(1+\gamma)(k-j)+\gamma}.
\]
Recalling the explicit value of \( \alpha_j \) and the definition of \( d_j \), we derive
\[
\delta_j = d_j \rho^{\frac{(1+\gamma)(k-j)+\gamma}{2(1+\gamma)(k-j)+\gamma}}.
\]
If \( k \) is odd and \( j \) is odd, we have
\[
\delta_{\alpha_j} = c_j \rho^{(1+\frac{1}{2})(k-j)+\frac{1}{2}}.
\]
Consequently,
\[
\delta_j = d_j \rho^{\frac{(1+\gamma)(k-j)+\frac{1}{2}}{2(1+\gamma)(k-j)+\frac{1}{2}}} = \rho^{\frac{(1+\gamma)(k-j)+1}{2(1+\gamma)(k-j)+1}},
\]
and the statement follows for \( k \) odd.

If \( k \) is even and \( j \) is odd, we have
\[
\delta_{\alpha_j} = c_j \rho^{(1+\gamma)(k-j)+1}.
\]

Consequently,
\[
\delta_j = d_j \rho^{\frac{(1+\gamma)(k-j)+1}{2(1+\gamma)(k-j)+\gamma}} = \rho^{\frac{(1+\gamma)(k-j)+1}{2(1+\gamma)(k-j)+1}},
\]
and the statement follows for \( k \) odd.

Finally, if \( k \) is even and \( j \) is even, we have
\[
\delta_{\alpha_j} = c_j \rho^{(1+\gamma)(k-j)+1}.
\]

It follows that
\[
\delta_i = d_j \rho^{\frac{(1+\gamma)(k-j)+1}{2(1+\gamma)(k-j)+\gamma}}.
\]

The proof of the third decay rate in (40) is an elementary computation; for the reader’s convenience we outline it in the Appendix. \( \square \)

4. The “error function” \( \Theta_j \)

(estimated of \( \mathcal{E}_\pm^{1/2} \)). In this section we justify the choice of (28)–(29) for the parameters \( \alpha_i, \delta_i \).

We recall that the shrinking annuli \( A_j \) are defined by
\[
A_j = \{ x \in \Omega : \sqrt{\delta_j \delta_j} \leq |x| < \sqrt{\delta_j \delta_{j+1}} \}, \quad j = 1, 2, \ldots, k,
\]
where we set \( \delta_0 = 0 \) and \( \delta_{k+1} = +\infty \). With this definition, for every \( i, j = 1, 2, \ldots, k \) we have
\[
\frac{A_j}{\delta_i} = \left\{ x \in \Omega : \sqrt{\delta_j \delta_i} \leq |x| < \sqrt{\delta_j \delta_{j+1}} \right\},
\]
and it is readily checked that:
\[
A_j / \delta_i \begin{cases} 
\text{runs off to infinity,} & \text{if } i < j \\
\text{invades whole space,} & \text{if } i = j \\
\text{shrinks to the origin,} & \text{if } i > j.
\end{cases}
\]

We define the “error functions” \( \Theta_j \) in \( A_j / \delta_j \) by setting
\[
\Theta_j(x) = \begin{cases} 
W_\rho(x) - (\alpha_j - 2) \ln |x| - w_j(x) + \ln \rho, & \text{if } j \text{ is odd} \\
-\gamma W_\rho(x) - (\alpha_j - 2) \ln |x| - w_j(x) + \ln(\rho \tau \gamma), & \text{if } j \text{ is even}.
\end{cases}
\]
Corollary 2. We have, for any \( \alpha > 0 \),
\[
\begin{align*}
\rho e^{\alpha(x)} - |x|^{\alpha-2}e^{w_j} & = (e^{\Theta_j(x/\delta_j)} - 1) \chi_{A_j} & \text{if } j \text{ is odd,} \\
(\gamma \rho e^{-\gamma w_j} - |x|^{\alpha-2}e^{w_j}) & = |x|^{\alpha-2}e^{\Theta_j(x/\delta_j)} \chi_{A_j} & \text{if } j \text{ is even},
\end{align*}
\]
and consequently
\[
\begin{align*}
\mathcal{E}_1^+ & = \sum_{1 \leq j \leq k} |x|^{\alpha-2}e^{w_j}(e^{\Theta_j(x/\delta_j)} - 1) \chi_{A_j} \\
\mathcal{E}_1^- & = \sum_{1 \leq j \leq k} |x|^{\alpha-2}e^{w_j}(e^{\Theta_j(x/\delta_j)} - 1) \chi_{A_j}.
\end{align*}
\]
The key point is that \( \Theta_j \) is well estimated in the expanding annulus \( A_j/\delta_j \).

Proposition 3. For every fixed \( j = 1, 2, \ldots, k \), the error term \( \Theta_j \) satisfies:
\[
|\Theta_j(y)| = O(\delta_j/y) + O(\rho^{\beta_{j,k}}) \quad \text{for all } y \in A_j/\delta_j,
\]
with
\[
\beta_{j,k} = \min \{ r_i, q_i, \; i = 1, 2, \ldots, k \} > 0,
\]
where \( r_i, q_i \) are defined in (41)–(43). In particular,
\[
\sup_{y \in A_j/\delta_j} |\Theta_j(y)| = O(1).
\]

Proposition 3 readily implies the following \( L^p \)-estimates.

Corollary 2. We have, for any \( p \geq 1 \):
\[
\| \rho e^{\alpha(x)} - |x|^{\alpha-2}e^{w_j} \|_{L^p(A_j)} = O(\rho^{\beta_{j,k}-2s_j} \frac{p+1}{p}) \quad \text{if } j \text{ is odd,}
\]
\[
\| \rho e^{-\gamma w_j} - |x|^{\alpha-2}e^{w_j} \|_{L^p(A_j)} = O(\rho^{\beta_{j,k}-2s_j} \frac{p+1}{p}) \quad \text{if } j \text{ is even},
\]
where \( s_j > 0 \) is defined in (42) and \( \beta_{j,k} - 2s_j \frac{p+1}{p} > 0 \) for \( 0 \leq p-1 < 1 \).

In particular, we have
\[
\| \mathcal{E}_1^+ \|_{L^p(\Omega)} + \| \mathcal{E}_1^- \|_{L^p(\Omega)} = O(\rho^{\beta_{j,k}-2s_j} \frac{p+1}{p}).
\]

We devote the remaining part of this section to the proof of Proposition 3 and of Corollary 2.

We note that we may write:
\[
\Theta_j \left( \frac{x}{\delta_j} \right) = \begin{cases} 
PW_j(x) - w_j(x) - (\alpha_j - 2) \ln |x| + \ln \rho + \sum_{i \neq j} \frac{(-1)^{i-1}}{\gamma^{i+1}} PW_i(x), & \text{if } j \text{ is odd,} \\
PW_j(x) - w_j(x) - (\alpha_j - 2) \ln |x| + \ln(\rho \gamma) - \gamma \sum_{i \neq j} \frac{(-1)^{i-1}}{\gamma^{i+1}} PW_i(x), & \text{if } j \text{ is even.}
\end{cases}
\]

We recall the expansion of \( PW_0 \).

Lemma 4.1. For every \( \alpha \geq 2, \; \delta > 0 \) there holds:
\[
PW_0(x) = w_0(x) - \ln(2\alpha^2 \delta^\alpha) + 4\pi \alpha H(x, 0) + O(\delta^\alpha)
\]
\[
= -2 \ln(\delta^\alpha + |x|^{\alpha}) + 4\pi \alpha H(x, 0) + O(\delta^\alpha).
\]

Proof. The proof is a direct consequence of the maximum principle, see, e.g., [7].

The following estimates are a key ingredient.
Lemma 4.2. Let $y \in A_j/\delta_j$, $j = 1, 2, \ldots, k$. There holds:
\[
\frac{1}{|y|^\alpha_i} \left( \frac{\delta_i}{\delta_j} \right)^\alpha_i = O(\rho^{j-1}), \quad \text{if } i < j
\]
\[
|y|^\alpha_i \left( \frac{\delta_j}{\delta_i} \right)^\alpha_i = O(\rho^j), \quad \text{if } i > j,
\]
where $q_{j-1}, q_j > 0$ are the constants defined in [43].

Proof. We have, by definition of $A_j$, $\sqrt{\delta_{j-1}/\delta_j} \leq |y| < \sqrt{\delta_{j+1}/\delta_j}$, where $\delta_0 := 0$ and $\delta_{k+1} := +\infty$. For $j \geq 2$ and for $i < j$ we estimate, using (40):
\[
\frac{1}{|y|^\alpha_i} \left( \frac{\delta_i}{\delta_j} \right)^\alpha_i \leq \left( \frac{\delta_i}{\delta_{j-1}} \right)^{\alpha_i/2} \left( \frac{\delta_{j-1}}{\delta_j} \right)^{\alpha_i} \leq \left( \frac{\delta_{j-1}}{\delta_j} \right)^{\alpha_i/2} = O(\rho^{j-1}).
\]
Similarly, for $j \leq k-1$ and for $i > j$ we estimate:
\[
|y|^\alpha_i \left( \frac{\delta_j}{\delta_i} \right)^\alpha_i \leq \left( \frac{\delta_j}{\delta_{j+1}} \right)^{\alpha_i/2} \left( \frac{\delta_{j+1}}{\delta_j} \right)^{\alpha_i} \leq \left( \frac{\delta_{j+1}}{\delta_j} \right)^{\alpha_i/2} = O(\rho^j). \tag{52}
\]

The following expansion, whose proof is elementary, is useful in view of Lemma 4.2.

Lemma 4.3. Let $y \in A_j/\delta_j$. Then,
\[
\ln(\delta_i^{\alpha_i} + |\delta_j y|^{\alpha_i}) = \begin{cases} 
\alpha_i \ln(\delta_j |y|) + O \left( \frac{1}{|y|^\alpha_i} \left( \frac{\delta_i}{\delta_j} \right)^\alpha_i \right) & \text{if } i < j \\
\alpha_i \ln \delta_i + \ln(1 + |y|^{\alpha_i}) & \text{if } i = j \\
\alpha_i \ln \delta_i + O \left( |y|^{\alpha_i} \left( \frac{\delta_i}{\delta_j} \right)^\alpha_i \right) & \text{if } i > j.
\end{cases}
\]

Using these facts, together with the definition of $\alpha_i, \delta_i$, we show the following essential estimate.

Lemma 4.4. For all $y \in A_j/\delta_j$ it holds that
\[
\Theta_j(y) = O(\delta_j |y|) + \sum_{i=1}^k O(\delta_i^{\alpha_i}) + \sum_{i<j} O \left( \frac{1}{|y|^\alpha_i} \left( \frac{\delta_i}{\delta_j} \right)^\alpha_i \right) + \sum_{i>j} O(\delta_i^{\alpha_i}).
\]

Proof. Suppose $j$ is odd. Then, using the projection expansion (51) and Lemma 4.3 we have
\[
\Theta_j(y) = P w_j(\delta_j y) - w_j(\delta_j y) - (\alpha_j - 2) \ln |\delta_j y| + \ln \rho + \sum_{i \neq j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} P w_i(\delta_j y)
\]
\[
= -\ln(2\alpha_j^2 \delta_j^{\alpha_j}) + 4\pi \alpha_j h(\delta_j y) + O(\delta_j^{\alpha_j}) - (\alpha_j - 2) \ln |\delta_j y| + \ln \rho + \sum_{i \neq j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \left\{-\ln(\delta_i^{\alpha_i} + |\delta_j y|^{\alpha_i}) + 4\pi \alpha_i h(\delta_j y) + O(\delta_i^{\alpha_i})\right\}
\]
\[
= -\ln(2\alpha_j^2 \delta_j^{\alpha_j}) + 4\pi h(0) \sum_{i=1}^k \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \alpha_i + \ln \rho + \sum_{i=1}^k O(\delta_i^{\alpha_i}).
\]
The asserted expansion is completely established.

\[-(\alpha_j - 2) \ln |\delta_j y| + O(\delta_j |y|)\]
\[-2 \sum_{i<j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \left( \alpha_i \ln(\delta_j |y|) + O\left( \frac{1}{|y|^{\alpha_i} \delta_j^{\alpha_i}} \right) \right)\]
\[-2 \sum_{i>j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \left( \ln \delta_i^{\alpha_i} + O\left( |y|^{\alpha_i} \delta_j^{\alpha_i} \right) \right)\]
\[= -(\alpha_j - 2) \ln |\delta_j y| + 2 \sum_{i<j} \frac{(-1)^i}{\gamma^{\sigma(i)}} \alpha_i \ln |\delta_j y|\]
\[= 0, \text{ in view of } (29)\]
\[-\ln(2\alpha_j^2 \delta_j^{\alpha_j}) - 4\pi h(0) \sum_{i=1}^{k} \left( \frac{(-1)^i}{\gamma^{\sigma(i)}} \alpha_i \right) + \ln \rho + 2 \sum_{i>j} \frac{(-1)^i}{\gamma^{\sigma(i)}} \ln \delta_i^{\alpha_i}\]
\[= 0, \text{ in view of } (28)\]
\[+ O(\delta_j |y|) + \sum_{i<j} \gamma^{\sigma(i)} \alpha_i + \ln \rho + 2 \sum_{i>j} \gamma^{\sigma(i)} \ln \delta_i^{\alpha_i}\]
\[= 0, \text{ in view of } (28)\]
\[= O(\delta_j |y|) + \sum_{i<j} \gamma^{\sigma(i)} \alpha_i + \sum_{i>j} \gamma^{\sigma(i)} \ln \delta_i^{\alpha_i}\]
\[= O(\delta_j |y|) + \sum_{i<j} \gamma^{\sigma(i)} \alpha_i + \sum_{i>j} \gamma^{\sigma(i)} \ln \delta_i^{\alpha_i}\]

Now, suppose that \( j \) is even. We have, from (50):
\[\Theta_j(y)\]
\[= P w_j(\delta_j y) - w_j(\delta_j y) - (\alpha_j - 2) \ln |\delta_j y| + \ln(\rho \tau \gamma) - \gamma \sum_{i\neq j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \sum_{i<j} \gamma^{\sigma(i)} \alpha_i\]
\[= -\ln(2\alpha_j^2 \delta_j^{\alpha_j}) + 4\pi \alpha_j h(0) + O(|\delta_j y|) + O(\delta_j^{\alpha_j}) - (\alpha_j - 2) \ln |\delta_j y| + \ln(\rho \tau \gamma)\]
\[-\gamma \sum_{i\neq j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \left\{ -2 \ln(\delta_i^{\alpha_i}) + |\delta_j y|^{\alpha_i} + 4\pi \alpha_i h(0) + O(\delta_i^{\alpha_i}) \right\}\]
\[= -\ln(2\alpha_j^2 \delta_j^{\alpha_j}) - 4\pi \gamma h(0) \sum_{i\neq j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \alpha_i + 2\gamma \sum_{i>j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \ln \delta_i^{\alpha_i} + \ln(\rho \tau \gamma)\]
\[= 0, \text{ in view of } (29)\]
\[-(\alpha_j - 2) \ln |\delta_j y| + 2\gamma \sum_{i<j} \frac{(-1)^{i-1}}{\gamma^{\sigma(i)}} \alpha_i \ln |\delta_j y|\]
\[= 0, \text{ in view of } (29)\]
\[+ \sum_{i=1}^{k} \gamma^{\sigma(i)} \alpha_i + \sum_{i<j} \gamma^{\sigma(i)} \ln \delta_i^{\alpha_i}\]
\[= \sum_{i=1}^{k} \gamma^{\sigma(i)} \alpha_i + \sum_{i<j} \gamma^{\sigma(i)} \ln \delta_i^{\alpha_i}\]

The asserted expansion is completely established. \(\square\)

Now we can prove Proposition \(\square\)
Proof of Proposition 3. We recall from Lemma 4.4 that

$$\Theta_j(y) = O(\delta_j|y|) + \sum_{i=1}^{k} O(\delta_i^2) + \sum_{i<j} O\left(\frac{1}{|y|^\alpha}, \frac{\delta_j}{\delta_i^{\alpha_i}}\right) + \sum_{i>j} O(|y|^\alpha, (\frac{\delta_j}{\delta_i^2})^{\alpha_i}).$$

We observe that in view of (45) we have

$$\delta_i^\alpha = O(\rho^\gamma),$$

for all $i = 1, 2, \ldots, k$.

The remaining terms are estimated using Lemma 4.2. Hence, (47) and (49) are established.

Proof of Corollary 2. We begin by showing that

Claim 1. If $j$ is odd:

$$\int_{A_j} |p e^{\gamma W_p} - |x|^{\alpha_j-2} e^{w_j}|^p \leq C \delta_j^{2(1-p)} \int_{A_j/\delta_j} \frac{|y|^{(\alpha_j-2)p}}{(1+|y|^{\alpha_j})^{2p}} |\Theta_j(y)|^p \, dy. \quad (53)$$

If $j$ is even,

$$\int_{A_j} \rho^2 e^{-\gamma W_p} - \frac{1}{\gamma} |x|^{\alpha_j-2} e^{w_j}|^p \leq C \frac{\delta_j^{2(1-p)}}{\gamma^p} \int_{A_j/\delta_j} \frac{|y|^{(\alpha_j-2)p}}{(1+|y|^{\alpha_j})^{2p}} |\Theta_j(y)|^p \, dy. \quad (54)$$

Suppose $j$ is odd. Then, using (46):

$$\int_{A_j} |x|^{\alpha_j-2} e^{w_j} - p e^{w_j}|^p = \int_{A_j} |x|^{p(\alpha_j-2)} e^{pw_j} \left| 1 - e^{\Theta_j(x/\delta_j)} \right|^p \, dx$$

$$\quad = C \int_{A_j/\delta_j} |\delta_j y|^{p(\alpha_j-2)} e^{pw_j(\delta_j, y)} \left| 1 - e^{\Theta_j(y)} \right|^p \delta_j^2 \, dy$$

$$\quad \leq C \int_{A_j/\delta_j} |\delta_j y|^{p(\alpha_j-2)} \left[ \frac{\delta_j^{\alpha_j}}{(\delta_j^2 + |\delta_j y|^{\alpha_j})^2} \right]^p |\Theta_j(y)|^p \delta_j^2 \, dy$$

$$\quad = C \delta_j^{2(1-p)} \int_{A_j/\delta_j} \frac{|y|^{p(\alpha_j-2)}}{(1+|y|^{\alpha_j})^{2p}} |\Theta_j(y)|^p \, dy.$$ 

Similarly, if $j$ is even, we compute, using (46):

$$\int_{A_j} |x|^{\alpha_j-2} e^{w_j} - \rho^2 e^{-\gamma W_p}|^p \, dx = \int_{A_j} |x|^{p(\alpha_j-2)} e^{pw_j} \left| 1 - e^{\Theta_j(x/\delta_j)} \right|^p \, dx$$

$$\quad = C \int_{A_j/\delta_j} |\delta_j y|^{p(\alpha_j-2)} e^{pw_j(\delta_j, y)} \left| 1 - e^{\Theta_j(y)} \right|^p \delta_j^2 \, dy$$

$$\quad \leq C \delta_j^{2(1-p)} \int_{A_j/\delta_j} \frac{|y|^{p(\alpha_j-2)}}{(1+|y|^{\alpha_j})^{2p}} |\Theta_j(y)|^p \, dy.$$

Claim 2. The following decay estimates hold true.

$$\delta_j^{2(1-p)} \int_{A_j/\delta_j} \frac{|y|^{p(\alpha_j-2)}}{(1+|y|^{\alpha_j})^{2p}} |\Theta_j(y)|^p \, dy \leq C \delta_j^{2(1-p)+\beta_j, k} = o(\delta_k^{2(1-p)+\beta_j, k}).$$
Indeed, in view of (47) we have
\[
\delta_j^{2(1-p)} \int_{A_j/\delta_j} \frac{|y|^{p(\alpha_j-2)}}{(1+|y|^{\alpha_j})^{2p}} |\Theta_j(y)|^p \, dy \\
\leq C \delta_j^{2(1-p)} \int_{A_j/\delta_j} \frac{|y|^{p(\alpha_j-2)}}{(1+|y|^{\alpha_j})^{2p}} \left|\delta_j y + \rho^{\delta_j,k} \right|^p \\
\leq C \delta_j^{2(1-p)} \int_{A_j/\delta_j} \frac{|y|^{p(\alpha_j-2)}}{(1+|y|^{\alpha_j})^{2p}} \, dy \\
+ C \delta_j^{2(1-p)} \rho^{\delta_j,k} \int_{A_j/\delta_j} \frac{|y|^{p(\alpha_j-2)}}{(1+|y|^{\alpha_j})^{2p}} \, dy.
\]

Since the integrals appearing above are uniformly bounded, the asserted decay rates follow.

5. The error terms \( R_\rho \) and \( S_\rho \) (estimation of \( E_2^\pm, E_3^\pm \)). We recall from Section 2 that
\[
f(s) := e^{s} - \tau e^{-\gamma s}
\]
and
\[
R_\rho(x) := \rho f(W_\rho) + \Delta W_\rho = \rho f(W_\rho) - \sum_{1 \leq i \leq k} \frac{(-1)^{i-1}}{\gamma_{\sigma(i)}} |x|^{\alpha_i-2} e^{w_i} = E_+ - E_-
\]
\[
S_\rho(x) := \rho f'(W_\rho) - \sum_{1 \leq i \leq k} |x|^{\alpha_i-2} e^{w_i} = E_+ + \gamma E_-,
\]
where
\[
E_+ := \rho e^{W_\rho} - \sum_{1 \leq i \leq k, \ i \ odd} |x|^{\alpha_i-2} e^{w_i}
\]
\[
E_- := \rho e^{-\gamma W_\rho} - \frac{1}{\gamma} \sum_{1 \leq i \leq k, \ i \ even} |x|^{\alpha_i-2} e^{w_i}.
\]

Our aim in this section is to obtain power decay estimates for \( \|E_+\|_{L^p(\Omega)} \) and \( \|E_-\|_{L^p(\Omega)} \), for \( p \geq 1, \ p - 1 \ll 1 \). More precisely, we establish the following

**Proposition 4.** There exists \( p_0 > 1 \) such that for every \( p \in [1, p_0) \) there exists \( \beta_p = \beta_p(\tau, \gamma, k) > 0 \) such that:
\[
\|E_+\|_{L^p(\Omega)} + \|E_-\|_{L^p(\Omega)} = O\left(\rho^{\beta_p}\right)
\] (55)

and consequently
\[
\|R_\rho\|_{L^p(\Omega)} + \|S_\rho\|_{L^p(\Omega)} = O\left(\rho^{\beta_p}\right)
\]

By taking \( p = 1 \) in (55) and using (20) we derive from the above:

**Corollary 3.** For any \( r > 0 \) there holds, as \( \rho \to 0^+ \):
\[
\int_{B_r(0)} \rho e^{W_\rho} \, dx = \sum_{1 \leq j \leq k, j \ odd} 4\pi \alpha_j + o(1)
\]
\[
\int_{B_r(0)} \tau \rho e^{-\gamma W_\rho} \, dx = \frac{1}{\gamma} \sum_{1 \leq j \leq k, j \ odd} 4\pi \alpha_j + o(1).
\] (56)
Moreover, for any $q > 1$ we have
\[
\int_{\Omega} (\rho e^{W_p})^q \, dx = O(\rho^{2\kappa(1-q)}),
\int_{\Omega} (\tau \rho e^{-\gamma W_p})^q \, dx = O(\rho^{2\kappa(1-q)}).
\] (57)

In order to prove Proposition 4, we recall from Section 2 that $\mathcal{E}_+ = \mathcal{E}_1^+ + \mathcal{E}_2^+ + \mathcal{E}_3^+$, where
\[
E_1^+ = \sum_{1 \leq j \leq \kappa \atop j \text{ odd}} (\rho e^{W_p} - |x|^{\alpha_j} e^{\rho \tau e}) \chi_{A_j},
\]
\[
E_2^+ = \sum_{1 \leq j \leq \kappa \atop j \text{ even}} \rho e^{W_p} \chi_{A_j},
\]
\[
E_3^+ = \sum_{1 \leq j \leq \kappa \atop j \text{ even}} \sum_{1 \leq i \leq \kappa \atop i \text{ odd}} |x|^{\alpha_i} e^{\rho \tau e} \chi_{A_i},
\]
and $\mathcal{E}_- = \mathcal{E}_1^- + \mathcal{E}_2^- + \mathcal{E}_3^-$, where
\[
E_1^- = \sum_{1 \leq j \leq \kappa \atop j \text{ even}} (\rho e^{-\gamma W_p} - |x|^{\alpha_j} e^{\rho \tau e}) \chi_{A_j},
\]
\[
E_2^- = \sum_{1 \leq j \leq \kappa \atop j \text{ odd}} \rho e^{-\gamma W_p} \chi_{A_j},
\]
\[
E_3^- = \frac{1}{\gamma} \sum_{1 \leq j \leq \kappa \atop j \text{ odd}} \sum_{1 \leq i \leq \kappa \atop i \text{ even}} |x|^{\alpha_i} e^{\rho \tau e} \chi_{A_i}.
\]
The errors $E_1^+, E_1^-$ are already estimated in Corollary 2. We estimate $E_2^+, E_2^-$. To this end, we first establish the following auxiliary estimates.

**Lemma 5.1.** If $j$ is odd:
\[
\int_{A_j} |\rho e^{-\gamma W_p}| \, dy \leq C \rho^{(1+\gamma)p} \delta_j^{(1+\gamma)} \left[ \left( \frac{\delta_j+1}{\delta_j} \right)^{\frac{\alpha_j+2}{\delta_j}+1} + \left( \frac{\delta_j}{\delta_j-1} \right)^{\frac{\alpha_j-2}{\delta_j}-1} \right];
\]
if $j$ is even,
\[
\int_{A_j} |\rho e^{-\gamma W_p}| \, dy \leq C \rho^{(1+\gamma)p} (\tau \gamma)^{1+\frac{\gamma}{\delta_j}} \delta_j^{(1+\gamma)} \left[ \left( \frac{\delta_j+1}{\delta_j} \right)^{\frac{\alpha_j+2}{\delta_j}+1} + \left( \frac{\delta_j}{\delta_j-1} \right)^{\frac{\alpha_j-2}{\delta_j}-1} \right],
\]
where for the sake of simplicity it is understood that if $j = 1$ only the first term on the right hand side exists and if $j = \kappa$ only the second term on the right hand side exists.

**Proof.** We begin by showing the following.

**Claim 1.** If $j$ is odd:
\[
\int_{A_j} |\rho e^{-\gamma W_p}| \, dy \leq C \rho^{(1+\gamma)p} (\tau \gamma)^{1+\frac{\gamma}{\delta_j}} \delta_j^{(1+\gamma)} \int_{A_j/\delta_j} e^{-\rho \tau e \gamma \Theta_j(y)} (1 + |y|^{\alpha_j})^{2p \gamma} \frac{1}{|y|^{\alpha_j-2 \gamma}} \, dy.
\]
If $j$ is even,
\[
\int_{A_j} |\rho e^{-\gamma W_p}| \, dy \leq C \rho^{(1+\gamma)p} (\tau \gamma)^{1+\frac{\gamma}{\delta_j}} \delta_j^{(1+\gamma)} \int_{A_j/\delta_j} e^{-\left( \frac{\gamma}{\delta_j} \right) \Theta_j(y)} (1 + |y|^{\alpha_j})^{2p \gamma} \frac{1}{|y|^{\alpha_j-2 \gamma}} \, dy.
\]
Proof of Claim 1. Suppose \( j \) is odd. We compute:

\[
\int_{A_j} |\rho e^{-\gamma W_j}|^p \, d\rho \leq C \rho^p \int_{A_j} e^{-\frac{1}{\gamma}(\frac{\alpha_j}{\gamma} + 1) p w_i} \, d\rho
\]

\[
= C \rho^p \int_{A_j} e^{-\frac{1}{\gamma} \theta_j(x/\delta_j) \rho \gamma (-\alpha_j - \ln |x| + \ln \rho)} \, dx
\]

\[
= C \rho^{(1+\gamma)p} \int_{A_j} e^{-\frac{1}{\gamma} \theta_j(x/\delta_j)} [\frac{\gamma}{\delta_j}\theta_j(x/\delta_j)]^{p/\gamma} \, dx
\]

\[
x = \delta_j \eta \nu \delta_j^{2(1+\gamma)p} \int_{A_j/\delta_j} e^{-\frac{1}{\gamma} \theta_j(y)} \frac{(1 + |y|^{2p})}{|y|^{(\alpha_j - 2)p/\gamma}} \, dy
\]

Similarly, suppose \( j \) is even. We compute:

\[
\int_{A_j} |\rho e^{-\gamma W_j}|^p \, d\rho \leq C \rho^p \int_{A_j} e^{-\frac{1}{\gamma} \theta_j(x/\delta_j) - \frac{1}{\gamma} [w_j + (\alpha_j - 2) \ln |x| - \ln (\rho \gamma) - 1]} \, dx
\]

\[
= C \rho^{(1+\gamma)p} \int_{A_j} e^{-\frac{1}{\gamma} \theta_j(y)} \frac{(1 + |y|^{2p})}{|y|^{(\alpha_j - 2)p/\gamma}} \, dy
\]

Claim 1 is thus established.

Claim 2. For any \( \eta > 0 \) we have:

\[
\int_{A_j/\delta_j} \frac{(1 + |y|^{\alpha_j})^{2p\eta}}{|y|^{(\alpha_j - 2)p\eta}} \, dy \leq C \left[ \left( \frac{\delta_j + 1}{\delta_j} \right)^{p\eta^{\alpha_j^{2} - 2}} + \left( \frac{\delta_j}{\delta_j - 1} \right)^{p\eta^{\alpha_j^{2} - 2}} \right].
\]

Proof of Claim 2. We compute:

\[
\int_{A_j/\delta_j} \frac{(1 + |y|^{\alpha_j})^{2p\eta}}{|y|^{(\alpha_j - 2)p\eta}} \, dy
\]

\[
= \int_{\sqrt{\frac{\delta_j - 1}{\delta_j}} \leq |y| < 1} \frac{(1 + |y|^{\alpha_j})^{2p\eta}}{|y|^{(\alpha_j - 2)p\eta}} \, dy + \int_{1 \leq |y| < \sqrt{\frac{\delta_j + 1}{\delta_j}}} \frac{(1 + |y|^{\alpha_j})^{2p\eta}}{|y|^{(\alpha_j - 2)p\eta}} \, dy
\]

\[
=: I + II.
\]

We estimate, for \( j \geq 2 \):

\[
I \leq C \int_{\sqrt{\frac{\delta_j - 1}{\delta_j}}}^{1} \frac{r \, dr}{p^{(\alpha_j - 2)p\eta}} \leq C \left[ 1 + \left( \frac{\delta_j}{\delta_j - 1} \right)^{p\eta^{\alpha_j^{2} - 2}} \right].
\]

Similarly, for \( j \leq k - 1 \), we have

\[
II \leq C \int_{\sqrt{\frac{\delta_j + 1}{\delta_j}}}^{1} |y|^{2p\eta \alpha_j - p\eta (\alpha_j - 2)} \, dy
\]
Lemma 5.2. The following power decay rates hold true.

If \( j \) is odd:

\[
\rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \left( \frac{\delta_{j+1}}{\delta_j} \right)^{\frac{\alpha_j + 2}{2} + 1} = O \left( \rho^{(1+\gamma)p} \left( \frac{\delta_j}{\delta_{j+1}} \right)^{1+p\gamma} \delta_{j+1}^{2+p\gamma-p} \right)
\]

\[
\rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \left( \frac{\delta_{j-1}}{\delta_j} \right)^{\frac{\alpha_{j-2}}{2} - 1} = O \left( \rho^{(1+\gamma)p} \left( \frac{\delta_j}{\delta_{j-1}} \right)^{-p+1} \delta_{j-1}^{2+\frac{\gamma}{2}-p} \right);
\]

if \( j \) is even:

\[
\rho^{(1+\frac{1}{4})p} \delta_j^{2(1+\frac{1}{4}\gamma)} \left( \frac{\delta_{j+1}}{\delta_j} \right)^{\frac{\alpha_j + 2}{2} + 1} = O \left( \rho^{(1+\frac{1}{4})p} \left( \frac{\delta_j}{\delta_{j+1}} \right)^{\frac{1+p}{2} \gamma} \delta_{j+1}^{2+\frac{\gamma}{2}-p} \right)
\]

\[
\rho^{(1+\frac{1}{4})p} \delta_j^{2(1+p/\gamma)} \left( \frac{\delta_{j-1}}{\delta_j} \right)^{\frac{\alpha_{j-2}}{2} - 1} = O \left( \rho^{(1+\frac{1}{4})p} \left( \frac{\delta_j}{\delta_{j-1}} \right)^{-p+1} \delta_{j-1}^{2+\frac{\gamma}{2}-p} \right)
\]

Proof. Proof of the first decay rate for \( j \) odd. Since \( j + 1 \) is even, in view of the recursive formula (37) we have \( \gamma \alpha_j = \alpha_{j+1} - 2(1+\gamma) \) and therefore

\[
p \gamma \frac{\alpha_j + 2}{2} = \frac{p}{2} (\alpha_{j+1} - 2(1+\gamma)) + p \gamma = \frac{p}{2} \alpha_{j+1} - p.
\]

We deduce that

\[
\rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \left( \frac{\delta_{j+1}}{\delta_j} \right)^{\frac{\alpha_j + 2}{2} + 1} = \rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \frac{\delta_j^{\alpha_{j+1} - p + 1}}{\delta_j^{\alpha_j + 2 + 1}}
\]

\[
= \rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \frac{(\delta_{j+1})^{p/2} \delta_{j+1}^{-p+1}}{(\delta_j^{\alpha_j})^{p/2} \delta_{j+1}^{p\gamma+1}}
\]

\[
= \rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \frac{(\delta_{j+1})^{p/2} \delta_{j+1}^{-p+1}}{(\delta_j^{\alpha_j})^{p/2} \delta_{j+1}^{p\gamma+1}}
\]

Recalling from (44) that \( \delta_{j+1} = \kappa_j \rho^{1+1/\gamma} \delta_{j+1}^{\alpha_{j+1} 1/\gamma} \), in turn we derive that

\[
\rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \frac{(\delta_{j+1})^{p/2} \delta_{j+1}^{-p+1}}{(\rho^{1+1/\gamma} (\delta_{j+1}^{\alpha_{j+1} 1/\gamma})^{p/2})} \delta_{j+1}^{1-p} = \rho^{(1+\gamma)p} \left( \frac{\delta_j}{\delta_{j+1}} \right)^{1+p\gamma} \delta_{j+1}^{1+p\gamma-p+1},
\]

and the asserted estimate follows.

Proof of the second decay rate for \( j \) odd. In view of the recursive formula (37), we have \( \alpha_{j-1} = \alpha_{j-1} / \gamma + 2(1 + 1/\gamma) \) and therefore

\[
\gamma \frac{\alpha_{j-1}}{2} = \frac{p}{2} \alpha_{j-1} + p(\gamma + 1).
\]

It follows that

\[
\rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \left( \frac{\delta_j}{\delta_{j-1}} \right)^{\frac{\alpha_{j-2}}{2} - 1} = \rho^{(1+\gamma)p} \delta_j^{2(1+p\gamma)} \frac{(\delta_j^{\alpha_j})^{p/2} \delta_{j+1}^{-p\gamma-1}}{(\delta_{j-1}^{\alpha_j})^{p/2} \delta_{j-1}^{p\gamma-1}}
\]
\[ \rho^{(1+\gamma)p} \delta^2_\gamma (\delta^{\alpha_j}_j)^{p+1} = \rho^{(1+\gamma)p} \delta^2_\gamma \frac{(\delta^{\alpha_j}_j)^{p+1}}{(\rho^{1+\gamma}(\delta^{\alpha_j}_j)^{\gamma})^{p+1}} = \rho^{(1+\gamma)p} \delta_\gamma^{1+p} \frac{(\delta^{\alpha_j}_j)^{p+1}}{(\delta^{\alpha_j}_j)^{\gamma}}. \]

Since \( j-1 \) is even, in view of the recursive formula (44) that \( \delta^{\alpha_j}_{j-1} = \kappa_{j-1} \rho^{1+\gamma}(\delta^{\alpha_j}_j)^{\gamma} \), we deduce that

\[ \rho^{(1+\gamma)p} \delta^{1+p} \frac{(\delta^{\alpha_j}_j)^{p+1}}{(\rho^{1+\gamma}(\delta^{\alpha_j}_j)^{\gamma})^{p+1}} = \rho^{(1+\gamma)p} \delta^{1+p} \frac{(\delta^{\alpha_j}_j)^{p+1}}{(\delta^{\alpha_j}_j)^{\gamma}}. \]

and the asserted estimate follows.

Proof of the first estimate for \( j \) is even. Since \( j+1 \) is odd, in view of the recursive formula (37), we have: \( \alpha_j/\gamma = \alpha_{j+1} - 2(1 + 1/\gamma) \). Hence, we may write

\[ \delta^{\alpha_j}_{j+1} = \delta^{\alpha_j}_{j+1} - 2(1 + 1/\gamma) + p/\gamma + 1 = (\delta^{\alpha_j}_{j+1})^{p+1}. \]

Hence, \( \rho^{1+\gamma}(\delta^{\alpha_j}_{j+1})^{\gamma} \)

\[ \frac{(\delta^{\alpha_j}_{j+1})^{p+1}}{(\delta^{\alpha_j}_{j+1})^{\gamma}} = \rho^{1+\gamma}(\delta^{\alpha_j}_{j+1})^{\gamma} \delta^{\alpha_j}_{j+1}. \]

Since \( j \) is even, in view of the recursive formula (44) we have that \( \delta^{\alpha_j}_{j+1} = \kappa_{j+1} \rho^{1+\gamma}(\delta^{\alpha_j}_{j+1})^{\gamma} \). We deduce that

\[ \rho^{1+\gamma}(\delta^{\alpha_j}_{j+1})^{\gamma} \delta^{\alpha_j}_{j+1} = \rho^{1+\gamma}(\delta^{\alpha_j}_{j+1})^{\gamma} \delta^{\alpha_j}_{j+1}. \]

as desired.

Proof of the second estimate for \( j \) even. Since \( j \) is even, in view of the recursive formula (37) that \( \alpha_j/\gamma = \alpha_{j+1} + 2(1 + 1/\gamma) \) and therefore

\[ \delta^{\alpha_j}_{j-1} = \delta^{\alpha_j}_{j-1} + 2(1 + 1/\gamma) - \gamma - 1 = (\delta^{\alpha_j}_{j-1})^{p+1}. \]

We deduce that

\[ \frac{(\delta^{\alpha_j}_{j})^{p+1}}{(\delta^{\alpha_j}_{j})^{\gamma}} = \rho^{1+\gamma}(\delta^{\alpha_j}_{j})^{\gamma} \delta^{\alpha_j}_{j-1}. \]

Since \( j-1 \) is odd, in view of (44) we have:

\[ \delta^{\alpha_j}_{j-1} = \kappa_{j-1} \rho^{1+\gamma}(\delta^{\alpha_j}_{j-1})^{1/\gamma}. \]

It follows that

\[ \frac{(\delta^{\alpha_j}_{j})^{p+1}}{(\rho^{1+\gamma}(\delta^{\alpha_j}_{j})^{1/\gamma})^{p+1}} = \frac{(\delta^{\alpha_j}_{j})^{p+1}}{(\rho^{1+\gamma}(\delta^{\alpha_j}_{j})^{1/\gamma})^{p+1}} = \frac{(\delta^{\alpha_j}_{j})^{p+1}}{(\rho^{1+\gamma}(\delta^{\alpha_j}_{j})^{1/\gamma})^{p+1}}. \]
Consequently,

\[ \rho^{(1 + \frac{1}{p})} \delta_j^{2(1+p)/\gamma} \left( \frac{\delta_j}{\delta_{j-1}} \right)^{\frac{\alpha-2}{p} - 1} \]

\[ = \frac{\rho^{(1 + \frac{1}{p})} \delta_j^{2(1+p)/\gamma} \delta_j^{-\frac{2}{p} - 1}}{\rho_j^{2(1+p)/\gamma} \delta_j^{-1}} = \rho^{1+\frac{1}{p}} \delta_j^{\frac{2}{p} - p+1} \]

\[ = \rho^{1+\frac{1}{p}} \left( \frac{\delta_{j-1}}{\delta_j} \right)^{-p+1} \delta_j^{1+\frac{2}{p} - p+1}, \]

as desired. The asserted decay estimates are completely established. ∎

**Lemma 5.3.** There holds:

\[ \int_{A_j} |x|^{\alpha_i-2} e^{\nu_i} \, dx = \begin{cases} O \left( \delta_i^{2(1-p)-(\delta_j/\delta_{j+1})^{p^{\alpha_i-2}+1}} \right), & \text{if } i > j \\ O \left( \delta_i^{2(1-p)-(\delta_{j+1}/\delta_j)^{p^{\alpha_i}+1}} \right), & \text{if } i < j. \end{cases} \]

**Proof.** We have:

\[ \int_{A_j} |x|^{\alpha_i-2} \left[ \delta_i^{\alpha_i} \left( \delta_i + |x|^{\alpha_i} \right)^2 \right]^p \, dx = \delta_i^{p\alpha_i} \int_{A_{j}} \frac{|x|^{p\alpha_i}}{(\delta_i + |x|^{\alpha_i})^{2p}} \, dx \]

\[ \overset{x=\delta_i y}{=} \delta_i^{(1-p)} \int_{A_{j}/\delta_i} \frac{|y|^{p\alpha_i}}{(1 + |y|^{\alpha_i})^{2p}} \, dy. \]

Suppose \( i > j \) (i.e., \( i \geq j + 1 \)). Then, \( \sqrt{\delta_j \delta_{j+1}}/\delta_i = o(\sqrt{\delta_j/\delta_{j+1}}) = o(1) \) as \( \rho \to 0^+ \) and therefore

\[ \int_{A_{j}/\delta_i} \frac{|y|^{p\alpha_i}}{(1 + |y|^{\alpha_i})^{2p}} \, dy \leq C \int_{\sqrt{\delta_j/\delta_{j+1}}^{\delta_j \delta_{j+1}}} r^{p\alpha_i} \, r \, dr \leq C \left( \frac{\delta_j}{\delta_{j+1}} \right)^{\frac{\alpha-2}{p} + 1}. \]

Suppose \( i < j \) (i.e., \( i \leq j - 1 \)). Then, \( \sqrt{\delta_{j-1}\delta_j}/\delta_i \geq C^{-1} \sqrt{\delta_j/\delta_{j-1}} \to +\infty \) as \( \rho \to 0^+ \) and therefore

\[ \int_{A_{j}/\delta_i} \frac{|y|^{p\alpha_i}}{(1 + |y|^{\alpha_i})^{2p}} \, dy \leq C \int_{\sqrt{\delta_{j-1}\delta_j}^{\delta_{j-1}\delta_j}} r^{p\alpha_i-2p\alpha_i} \, r \, dr \leq C \left( \frac{\delta_{j-1}}{\delta_j} \right)^{\frac{\alpha+1}{p} + 1}. \]

The asserted decay estimates are thus established. ∎

Now we can provide the proof of Proposition 4.

**Proof of Proposition 4.** The proof is a direct consequence of Lemma 5.1, Lemma 5.2, and Lemma 5.3.

**Proof of Corollary 3.** We decompose:

\[ \int_{B_r} \rho e^{W_{\rho}} \overset{\delta_i}{=} \sum_{j=1}^{k} \int_{B_r \cap A_j} \rho e^{W_{\rho}} \]

\[ = \sum_{j=1}^{k} \int_{B_r \cap A_j} |x|^{\alpha_j-2} e^{w_j} \overset{\delta_i}{+} \sum_{j=1}^{k} \int_{B_r \cap A_j} (\rho e^{W_{\rho}} - |x|^{\alpha_j-2} e^{w_j}) \overset{\delta_i}{+} \sum_{j=1}^{k} \int_{B_r \cap A_j} \rho e^{W_{\rho}} \]
Using (55) with Proposition 5.

exists a unique $\phi$ for estimate in (57) is similar.

On the other hand, in view of (55) we have

$$\int_{B_r} \rho e^{W_\rho} = \sum_{j=1}^{k} \int_{B_r \cap A_j} |x|^{\alpha_j-2} e^{w_j} + \int_{B_r} \mathcal{E}_+ + \int_{B_r} \mathcal{E}_-.$$ 

for some $\beta_1 > 0$. On the other hand, by a standard rescaling and (20),

$$\int_{B_r \cap A_j} |x|^{\alpha_j-2} e^{w_j} = 4\pi\alpha_j + o(1).$$

Hence, (56) follows.

Proof of (57). We have:

$$\int \Omega (\rho e^{W_\rho})^q = \sum_{j=1}^{k} \int_{A_j} (\rho e^{W_\rho} \chi_{A_j})^q$$

$$\leq C \sum_{j=1}^{k} \int_{A_j} (\rho e^{W_\rho} - |x|^{\alpha_j-2} e^{w_j})^q + C \sum_{j=1}^{k} \int_{A_j} |x|^{q(\alpha_j-2)} e^{qw_j}$$

$$+ C \sum_{j=1}^{k} \int_{A_j} (\rho e^{W_\rho})^q$$

$$= C \int \Omega |\mathcal{E}_+|^q + C \sum_{j=1}^{k} \int_{A_j} \frac{|x|^{q(\alpha_j-2)} \delta_j^{q\alpha_j}}{(\delta_j^{q\alpha_j} + |x|^{\alpha_j})^{2q}} + C \int \Omega |\mathcal{E}_-|^q.$$ 

By rescaling we find

$$\int_{A_j} \frac{|x|^{q(\alpha_j-2)} \delta_j^{q\alpha_j}}{(\delta_j^{q\alpha_j} + |x|^{\alpha_j})^{2q}} = O(\delta_j^{2(1-q)}) = O(\rho^{2s_j(1-q)}) = O(\rho^{2s_k(1-q)}).$$

On the other hand, in view of (55) we have $\|\mathcal{E}_+\|_{L^q(\Omega)} = o(1)$ and $\|\mathcal{E}_-\|_{L^q(\Omega)} = o(1)$ as $\rho \to 0^+$. Hence, the first estimate in (57) is established. The proof of the second estimate in (57) is similar.

6. The linear theory. We recall from (25) that the linear operator $L_\rho$ is defined for $\phi \in W^{2,p}(\Omega)$, $p > 1$, by

$$L_\rho \phi := -\Delta \phi - \sum_{i=1}^{k} |x|^{\alpha_i-2} e^{w_i} \phi = -\Delta \phi - \sum_{i=1}^{k} 2\alpha_i^2 \delta_i^{\alpha_i} |x|^{\alpha_i-2} \phi, \quad (58)$$

where $\alpha_i, \delta_i$ are defined by (28)-(29) for $i = 1, 2, \ldots, k$.

Our aim in this section is to establish the following result.

**Proposition 5.** Let $\Omega$ satisfy the symmetry assumption (9). For any $p > 1$ there exist $\rho_0 > 0$ and $c > 0$ such that for any $\rho \in (0, \rho_0)$ and for any $\psi \in L^p(\Omega)$ there exists a unique $\phi \in W^{2,p}(\Omega) \cap \mathcal{H}_\gamma$ solution to

$$L_\rho \phi = \psi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial \Omega.$$
which satisfies
\[ \|\phi\| \leq c|\ln \rho||\psi|_p. \]

We observe that \( L_\rho \) is formally the same operator appearing in [7]. However, it actually depends significantly on the asymmetry parameter \( \gamma \in (0,1] \) via the parameters \( \alpha_i, \delta_i \). Consequently, we can follow the approach in [7] to prove Proposition 5 although some intermediate estimates require a modified proof, due to the different dependence of \( \alpha_i, \delta_i \) on \( i, \rho \). In particular, since \( \alpha_i \) does not depend monotonically on \( i \) (see Remark 2), the proof of Lemma 6.4–(iv) below differs from the proof of the corresponding estimate (4.18) in [7].

For the sake of completeness, in this section we first outline the scheme of the proof of Proposition 5, which is analogous to [7]. We then devote the remaining part of this section to prove in detail Lemma 6.4–(iv).

6.1. Outline of the proof of Proposition 5. It is convenient to extend the symmetry assumption (9) to a possibly unbounded domain \( D \subset \mathbb{R}^2 \). Let \( D \subset \mathbb{R}^2 \) be a smooth (possibly unbounded) domain. Namely, we define the following geometrical symmetry property for \( D \):
\[
0 \in D \text{ and } \begin{cases} -D = D = e^{2\pi \sqrt{-1/(m+n)}}D, & \text{if } \gamma = \frac{m}{n}, \ m,n \in \mathbb{N} \text{ coprime;} \\ -D = D, & \text{if } \gamma \not\in \mathbb{Q}, \end{cases} \quad (59)
\]

Correspondingly, we define a symmetry property for functions \( \phi : D \to \mathbb{R} \):
\[
\begin{align*}
\varphi(xe^{2\pi \sqrt{-1/(m+n)}}) &= \varphi(x) = \varphi(-x) \forall x \in D, & \text{if } \gamma = \frac{m}{n}, \ m,n \in \mathbb{N} \text{ coprime;} \\
\varphi(-x) &= \varphi(x) \forall x \in D, & \text{if } \gamma \not\in \mathbb{Q}.
\end{align*}
\quad (60)
\]

The following lemma clarifies the role of the symmetry assumption (60).

Lemma 6.1. Suppose \( \alpha \geq 2 \) is such that \( \frac{\alpha}{2} \in \mathbb{N} \) and
\[
\frac{\alpha}{2} = (1 + \frac{1}{\gamma})i - \frac{1}{\gamma} \text{ for some odd } i \in \mathbb{N}
\]
or
\[
\frac{\alpha}{2} = (1 + \gamma)i - 1 \text{ for some even } i \in \mathbb{N}.
\]
Suppose \( \phi \) is a solution to
\[
-\Delta \phi = 2\alpha^2 \frac{|y|^{\alpha-2}}{(1+|y|^\alpha)^2} \phi \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla \phi|^2 < +\infty \quad (61)
\]
satisfying (60) with \( D = \mathbb{R}^2 \). Then, there exists \( \eta \in \mathbb{R} \) such that
\[
\phi(y) = \eta \frac{1-|y|^\alpha}{1+|y|^\alpha}. \quad (62)
\]

Proof. It is shown in [7] that \( \phi \) is necessarily a bounded solution. In turn, it is shown in [2] that any bounded solution to (62) is a linear combination of the functions:
\[
\phi_0(y) = \frac{1-|y|^\alpha}{1+|y|^\alpha}, \quad \phi_1(y) = \frac{|y|^\frac{\alpha}{2} \cos \frac{\alpha}{2} \theta}{1+|y|^\alpha}, \quad \phi_2(y) = \frac{|y|^\frac{\alpha}{2} \sin \frac{\alpha}{2} \theta}{1+|y|^\alpha}.
\]

In view of Corollary 1, \( \frac{\alpha}{2} \) is of the form (36). In particular, the functions \( \phi_1, \phi_2 \) do not satisfy (60). The claim follows. \( \square \)
With these definitions, it is shown in [7] that the embedding 
\[ \text{Proposition 6.} \]
For any \( i \phi \) endowed with the norms and \( \| \cdot \| \) assume that there exist \( p > 1, \rho_n \to 0^+ \), \( \phi_n, \psi_n \) such that \( L_{\rho_n} \phi_n = \psi_n, \| \phi_n \| = 1, \| \ln \rho_n \| \psi_n \|_p \to 0 \). In particular, \( \phi_n \) satisfies
\[ - \Delta \phi_n - \sum_{i=1}^{k} 2\alpha_i^2 \frac{\delta_i^\alpha |x|^\alpha - 2}{(\delta_i^\alpha + |x|^\alpha)^2} \phi_n = \psi_n. \] (63)

Then, by (40) we obtain \( k \) sequences of scaling parameters:
\[ \delta_j \equiv d_j \rho_n^j, \quad j = 1, 2, \ldots, k. \]
For every fixed \( j = 1, 2, \ldots, k \) we set
\[ \phi_j^0(y) = \phi_n(\delta_j y), \quad y \in \Omega_n = \Omega/\delta_j. \]

For any \( \alpha \geq 2 \) we define the Banach spaces
\[ L_\alpha(\mathbb{R}^2) := \left\{ u \in L_{\text{loc}}^2(\mathbb{R}^2) : \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\} \]
and
\[ H_\alpha(\mathbb{R}^2) := \left\{ u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2) : \| \nabla u \|_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\} \]
endowed with the norms
\[ \| u \|_{L_\alpha} := \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)}, \]
\[ \| u \|_{H_\alpha} := \left( \| \nabla u \|_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\alpha-2}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} \right)^{1/2}. \]

With these definitions, it is shown in [7] that the embedding \( i_\alpha : H_\alpha(\mathbb{R}^2) \to L_\alpha(\mathbb{R}^2) \)
is compact.

**Lemma 6.2.** For every \( j = 1, 2, \ldots, k \) there exist \( \eta_j \in \mathbb{R} \) such that
\[ \phi_j^0(y) \to \phi_j(y) = \eta_j \frac{1 - |y|^{\alpha_j}}{1 + |y|^{\alpha_j}} \] (64)
weakly in \( H_{\alpha_j}(\mathbb{R}^2) \) and strongly in \( L_{\alpha_j}(\mathbb{R}^2) \).

**Proof.** Fix \( j \). There exists \( \phi_j^0 \in H_{\alpha_j} \) such that \( \phi_j^0 \to \phi_j \) weakly in \( H_{\alpha_j}(\mathbb{R}^2) \) and strongly in \( L_{\alpha_j}(\mathbb{R}^2) \). The function \( \phi_j^0 \) satisfies (61) with \( \alpha = \alpha_j \). Moreover, for every \( n \), \( \Omega_n^j \) satisfies the symmetry assumption (59) and \( \phi_j^0 \) satisfies (60) in \( \Omega_n^j \). Finally, \( \alpha_j \) is of the form (32). In view of Lemma 6.1 we conclude that \( \phi_j^0 \) is of the asserted form (64) with \( \alpha = \alpha_j \). \( \square \)

The desired contradiction will follow from the fact that, actually, \( \phi_j^0 = 0 \) for all \( i = 1, 2, \ldots, k \). Indeed, the following result holds true.

**Proposition 6.** For any \( j = 1, 2, \ldots, k \), there holds \( \eta_j = 0 \). Therefore, we have \( \phi_j^0(y) \to 0 \) weakly in \( H_{\alpha_j}(\mathbb{R}^2) \) and strongly in \( L_{\alpha_j}(\mathbb{R}^2) \) for all \( j = 1, 2, \ldots, k \).

The proof of Proposition 6 will be outlined below. Once Proposition 6 is established, it is simple to prove Proposition 5.
Proof of Proposition 6. Multiplying (63) by $\phi_n$ and integrating, we find

$$1 = \int_\Omega |\nabla \phi_n|^2 = \sum_{i=1}^k \int_\Omega 2\alpha_i^2 \frac{\phi_n^2}{(\delta_{a_i}^\alpha + |\alpha|^2)^2} \phi_n^2(x) dx + \int_\Omega \psi_n(x) \phi_n(x) dx$$

$$= \sum_{i=1}^k \int_{\Omega_n} 2\alpha_i^2 \frac{|\gamma|^{|\alpha|-2}}{(1 + |\gamma|)^2} (\phi_n^i(y))^2(y) dy + O(\|\psi_n\|_p \|\phi_n\|) = o(1),$$

because $\phi_n^i \to 0$ strongly in $L_{\alpha_i}(\mathbb{R}^2)$, for all $i = 1, 2, \ldots, k$, and $\|\psi_n\|_p = o(1)$. This is a contradiction.

In order to prove Proposition 6 for $i = 1, 2, \ldots, k$ we define the quantities

$$\sigma_i(\rho_n) = \ln \rho_n \int_{\Omega_n} 2\alpha_i^2 \frac{|\gamma|^{|\alpha|-2}}{(1 + |\gamma|)^2} \phi_n^i(y) dy.$$

**Lemma 6.3.** For any $i = 1, 2, \ldots, k$, For any $i = 1, 2, \ldots, k$ there holds

$$\sigma_i(\rho_n) = o(1), \quad \sigma_i(\rho_n) + 2 \sum_{j=1}^{i-1} \sigma_j(\rho_n) = o(1)$$

and consequently

$$\sigma_{i0} := \lim_{\rho_n \to 0} \sigma_i(\rho_n) = 0. \quad (65)$$

We first show how Lemma 6.3 implies Proposition 6. Then, we devote the remaining part of this section to the proof of Lemma 6.3.

Proof of Proposition 6. We use the following identities. For $i = 1, 2, \ldots, k$ there holds:

$$\int_{\mathbb{R}^2} 2\alpha_i^2 \frac{|\gamma|^{|\alpha|-2}}{(1 + |\gamma|)^2} \frac{1 - |\gamma|^{|\alpha|}}{1 + |\gamma|^{|\alpha|}} dy = 0$$

$$\int_{\mathbb{R}^2} 2\alpha_i^2 \frac{|\gamma|^{|\alpha|-2}}{(1 + |\gamma|)^2} \frac{1 - |\gamma|^{|\alpha|}}{1 + |\gamma|^{|\alpha|}} \ln(1 + |\gamma|) dy = -4\pi \alpha_i \quad (66)$$

Using the equations for $\phi_n$ and $Pw_{in}$, we find

$$\int_{\Omega} 2\alpha_i^2 \frac{\phi_n^2}{(\delta_{a_i}^\alpha + |\alpha|^2)^2} \phi_n = \sum_{j=1}^k \int_{\Omega} 2\alpha_j^2 \frac{\phi_n^j}{(\delta_{a_j}^\alpha + |\alpha|^2)^2} \phi_n^j Pw_{in} + \int_{\Omega} \psi Pw_{in},$$

where $w_{in} = w_{\delta_{a_i}^\alpha}$. The first term above vanishes as $\rho_n \to 0^+$, in view of the form (64) of $\phi_n^i$ and of the first integral in (66). In order to evaluate the second term, we note that similarly as in (67) we find

$$\int_{\Omega} 2\alpha_j^2 \frac{\phi_n^j}{(\delta_{a_j}^\alpha + |\alpha|^2)^2} \phi_n Pw_{in}$$
In turn, we obtain

\[-2(2(k - i) + 1)\sigma_j(\rho_n) + o(1)\]

if \( j < i \)

\[-2(2(k - i) + 1)\sigma_i(\rho_n) + \int_{\Omega_n} 2\alpha_j^2 \frac{\delta_{\alpha_j}^2}{(\delta_{\alpha_j}^2 + |x|^{\alpha_j})^2} \phi_n(y)[-2\ln(1 + |y|^{\alpha_j})] dy + o(1),\]

if \( j = i \)

\[-2(2(k - j) + 1)\sigma_j(\rho_n) + \int_{\Omega_n} 2\alpha_j^2 \frac{\delta_{\alpha_j}^2}{(\delta_{\alpha_j}^2 + |x|^{\alpha_j})^2} \phi_n(y)[-2\ln|y|] dy + o(1),\]

if \( j > i \).

Therefore, Lemma \[6.3\] and (66) yield

\[
\sum_{j=1}^{k} \int_{\Omega} 2\alpha_j^2 \frac{\delta_{\alpha_j}^2}{(\delta_{\alpha_j}^2 + |x|^{\alpha_j})^2} \phi_n P w_{in} \]

\[
= \begin{cases} 
4\pi\alpha_i(\eta_i + 2\sum_{j=i+1}^{k} \eta_j) + o(1), & \text{if } i = 1, 2, \ldots, k - 1 \\
4\pi\alpha_k\eta_k + o(1), & \text{if } i = k.
\end{cases}
\]

In turn, we obtain

\[\eta_k = 0 \quad \text{and} \quad \eta_i + 2\sum_{j=i+1}^{k} \eta_j = 0 \quad \text{for any } i = 1, 2, \ldots, k - 1.\]

Proposition \[6\] is thus established.

We are left to prove the asymptotic behavior of the quantities \(\sigma_i(\rho_n), \ i = 1, 2, \ldots, k\), as stated in Lemma \[6.3\].

6.2. Proof of Lemma \[6.3\]. Throughout this subsection, for the sake of simplicity, we omit the index \(n\). In order to establish Lemma \[6.3\], we set

\[Z_i(x) = \frac{\delta_i^{\alpha_i}}{\delta_i^{\alpha_i} + |x|^{\alpha_i}}(67)\]

and we denote by \(PZ_i\) its projection onto \(H^1_0(\Omega)\). Then, using the equations for \(\phi\) and \(PZ_i\), we find that the sequence \(\phi\) satisfies the identity

\[
\ln(1 + |x|^{\alpha_i}) Z_i(x) dx
\]

\[
= \ln \rho \sum_{j=1}^{k} \int_{\Omega} 2\alpha_j^2 \frac{\delta_{\alpha_j}^2}{(\delta_{\alpha_j}^2 + |x|^{\alpha_j})^2} \phi(x) PZ_i(x) dx + \ln(1 + |x|^{\alpha_i}) Z_i(x) dx.
\]

Equivalently, we may write

\[
\ln \rho \int_{\Omega} 2\alpha_j^2 \frac{\delta_{\alpha_j}^2}{(\delta_{\alpha_j}^2 + |x|^{\alpha_j})^2} \phi(x)(PZ_i(x) - Z_i(x)) dx
\]

\[
+ \ln \rho \sum_{j<i}^{k} \int_{\Omega} 2\alpha_j^2 \frac{\delta_{\alpha_j}^2}{(\delta_{\alpha_j}^2 + |x|^{\alpha_j})^2} \phi(x) PZ_i(x) dx + \ln \rho \int_{\Omega} \psi(x) PZ_i(x) dx = 0.
\]

The asserted identities (65) will then follow from the following facts.

Lemma 6.4. The following expansions hold.

(i) \(\ln \rho \int_{\Omega} 2\alpha_j^2 \frac{\delta_{\alpha_j}^2}{(\delta_{\alpha_j}^2 + |x|^{\alpha_j})^2} \phi(x)(PZ_i(x) - Z_i(x)) dx = \sigma_i(\rho) + o(1)\)

(ii) \(\ln \rho \int_{\Omega} \psi(x) PZ_i(x) dx = o(1)\)
(iii) If \( j > i \), then
\[
\ln \rho \int_\Omega 2 \alpha_j^2 \delta_j^{\alpha_j} |x|^{\alpha_j-2} \phi(x) PZ_i(x) \, dx = o(1)
\]

(iv) If \( j < i \), then
\[
\ln \rho \int_\Omega 2 \alpha_j^2 \delta_j^{\alpha_j} |x|^{\alpha_j-2} \phi(x) PZ_i(x) \, dx = 2 \sigma_j(\rho) + o(1)
\]

The proof of Lemma 6.4–(i)–(ii)–(iii) is completely analogous to [7]. On the other hand, the proof of Lemma 6.4–(iv) is different, due to the particular dependence on \( \gamma \) of \( \alpha_i, \delta_i \). Therefore, we provide the proof of Lemma 6.4–(iv). The underlying idea is that in order to control the integrals on the expanding domain \( \Omega_j^n \), it is convenient to decompose \( \Omega_j^n = B_{R_j} \cup (\Omega_j^n \setminus B_{R_j}) \), with \( R_j \) suitably defined as follows.

For any \( j = 1, 2, \ldots, k-1 \) we define
\[
R_j := \sqrt{\frac{\delta_{j+1}}{\delta_j}}.
\]

Then, \( R_j \to +\infty \) as \( \rho \to 0 \).

We recall some elementary facts.

**Lemma 6.5.** The following properties hold.

(i) For \( R \to +\infty \) and for any \( \beta > 0 \) there holds
\[
\int_{\mathbb{R}^2 \setminus B_R} \frac{dy}{|y|^{2+\beta}} = O\left( \frac{1}{R^\beta} \right);
\]

(ii) If \( j < i \), then for all \( y \in B_{R_j} \)
\[
|y|^{\alpha_i} \left( \frac{\delta_j}{\delta_i} \right)^{\alpha_i} \leq \left( \frac{\delta_j}{\delta_i} \right)^{\alpha_i/2} = O\left( \frac{1}{R_j^{\alpha_i}} \right);
\]

(iii) For any \( q > 1 \) there holds
\[
\|\phi^j\|_{L^q(\Omega_j^n)} = \frac{1}{\delta_j^{1/q}} \|\phi\|_{L^q(\Omega)}.
\]

**Proof.** Part (i) is elementary. Proof of (ii). See (52) or [7]. Proof of (iii). We use Hölder’s inequality. \( \square \)

**Lemma 6.6.** The following expansions hold for the function \( Z_i(x) \) defined in (67).

(i) For any \( x \in \Omega \) there holds
\[
PZ_i(x) = Z_i(x) + 1 + O(\delta_i^{\alpha_i}) = \frac{2 \delta_i^{\alpha_i}}{\delta_i^{\alpha_i} + |x|^{\alpha_i}} + O(\delta_i^{\alpha_i});
\]

In particular, \( |PZ_i(x)| \leq 2 + O(\delta_i^{\alpha_i}) \), \( i = 1, 2, \ldots, k \).

(ii) If \( j < i \) and \( y \in B_{R_j} \), then
\[
PZ_i(\delta_j y) = \frac{2 \delta_i^{\alpha_i}}{\delta_j^{\alpha_j} + \delta_j^{\alpha_i} |y|^{\alpha_i}} + O(\delta_i^{\alpha_i}) = \frac{2}{1 + (\delta_j^{\alpha_j} |y|^{\alpha_j})} + O(\delta_i^{\alpha_i}) = \frac{2}{1 + O(R_j^{\alpha_i})}
\]
and

\[ PZ_i(\delta_j y) - 2 = -2 \left( \frac{\delta_j}{\delta_j} \right)^{\alpha_i} |y|^{\alpha_i} + O(\delta_i^{\alpha_i}) = O\left( \frac{1}{R_j^{r/2}} \right) + O(\delta_i^{\alpha_i}), \]

where \( R_j \) is defined in (69).

**Proof.** The proof readily follows from the definition of \( Z_i \) in (67) and Lemma 6.5 (ii).

Rescaling the integral on the l.h.s. in (68), we have

\[ \int_\Omega \frac{\delta_j^{|x|^{\alpha_j-2}}}{\delta_j^{|x|^{\alpha_j}}} \phi(x) PZ_i(x) dx = \int_\Omega \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) PZ_i(\delta_j y) dy. \]

The proof of (68) will finally follow from the following lemmas.

**Lemma 6.7.** There exists \( \beta_1 > 0 \) such that

\[ \int_{\Omega \setminus B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) PZ_i(\delta_j y) dy = O(\rho^{\beta_1}); \quad (70) \]

\[ \int_{B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) PZ_i(\delta_j y) dy = O(\rho^{\beta_1}). \quad (71) \]

**Proof.** We have, using Lemma 6.6 (i):

\[ \int_{\Omega \setminus B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) PZ_i(\delta_j y) dy \leq \int_{\Omega \setminus B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) dy + O(\delta_i^{\alpha_i}) dy \]

\[ = O \left( \int_{\Omega \setminus B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) dy \right) \]

In view of Lemma 6.5 and Hölder's inequality we derive that for any \( r > 1 \) there holds

\[ \int_{\Omega \setminus B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) dy \leq \| \phi_j \|_{L^{r/(r-1)}(\Omega_i)} \left( \int_{\Omega \setminus B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \right)^{1/r} \]

\[ = O \left( \frac{1}{\delta_j^2 \phi(r-1)/r} \right) = O \left( \frac{1}{\delta_j^2 \phi(r-1)/r} \right). \]

By taking \( 0 < r - 1 \ll 1 \), we obtain estimate (70) for some \( \beta_1 > 0 \).

Similarly we have, using Lemma 6.6 (ii):

\[ \int_{B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) PZ_i(\delta_j y) dy \]

\[ = O \left( \int_{B_{R_j}} \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \phi_j(y) dy \left( \frac{1}{R_j^{r/2}} \right)^{1/r} \right). \]
Consequently, the form estimate (71).

Proposition 7. For we establish the following existence result.

By taking $0 < r - 1 \ll 1$ and possibly a smaller value for $\beta_1 > 0$, we deduce estimate (71).

We conclude from Lemma 6.7 that if $j < i$, then

$$\ln \rho \int_\Omega 2\sigma_j^2 \omega_j^2 \varphi \right|_{x=0}^2 \phi^i(y) dy = \ln \rho \int_{B_{R_j}} 2\sigma_j^2 \left| y \right|^{\alpha_j-2} \phi^i(y) dy + o(1).$$

(72)

Lemma 6.8. There exists $\beta_2 > 0$ such that

$$\int_{\Omega^*_i} \left| y \right|^{\alpha_j-2} dy \phi^i(y) dy = \int_{B_{R_j}} \left| y \right|^{\alpha_j-2} \phi^i(y) dy + O(\rho^{\beta_2}).$$

Consequently,

$$\ln \rho \int_\Omega 2\sigma_j^2 \omega_j^2 \varphi \right|_{x=0}^2 \phi^i(y) dy = 2\sigma_j(\rho) + o(1).$$

Proof. We have, for $r > 1$ sufficiently small,

$$\int_{\Omega^*_i \setminus B_{R_j}} \frac{\left| y \right|^{\alpha_j-2}}{(1 + \left| y \right|^{\alpha_j})^2} \phi^i(y) dy \leq \| \phi^i \|_{L^r(\Omega^*_i \setminus B_{R_j})} \left( \int_{R^2 \setminus B_{R_j}} \frac{dy}{\left| y \right|^{(\alpha_j + 2)r}} \right)^{1/r}$$

$$= O\left( \frac{1}{\delta_j^{2(r-1)/r}} R_j^{\alpha_j + 1 - 1/r} \right).$$

The statement follows by taking $0 < r - 1 \ll 1$.

Estimate (68) in Lemma 6.4 (iv) is thus completely established. In turn, the proof of (65) follows. Hence, the proof of Proposition 5 is complete.

7. The contraction mapping and the proof of Theorem 1.1. In this section we conclude the proof of Theorem 1.1 by obtaining a solution $u_\rho$ to problem (1) in the form $u_\rho = W_\rho + \phi_\rho$, with $\phi_\rho$ the fixed point of a contraction mapping. Indeed, we establish the following existence result.

Proposition 7. For $p > 1$ sufficiently close to 1 there exist $\rho_0 > 0$ and $R > 0$ such that for any $\rho \in (0, \rho_0)$ there exists a unique solution $\phi_\rho \in \mathcal{H}_\rho$ to the problem

$$-\Delta(W_\rho + \phi_\rho) = \rho f(W_\rho + \phi_\rho) \text{ in } \Omega, \quad \phi_\rho = 0 \text{ on } \partial \Omega$$

satisfying the estimate

$$\| \phi_\rho \| \leq R \rho^{\gamma_p} |\ln \rho|.$$
Here $H_\gamma$ is the space of $\gamma$-symmetric Sobolev functions defined in [10] and [11] and $\tilde{\beta}_p > 0$ is the exponent obtained in Proposition 4.

We equivalently seek a fixed point $\phi \in H_\gamma$ for the operator $T_\rho : H_\gamma \rightarrow H_\gamma$ defined by

$$T_\rho(\phi) = (L_\rho)^{-1}(N_\rho(\phi) + S_\rho\phi + R_\rho),$$

where $R_\rho, S_\rho, N_\rho$ are the operators defined in (22), (23), (25).

In the sequel we shall use the Moser-Trudinger inequality [15, 27] in the following form.

**Lemma 7.1.** There exists $c > 0$ such that for any bounded domain $\Omega \subset \mathbb{R}^2$ there holds

$$\int_{\Omega} e^{4\pi u^2/\|u\|^2} \, dx \leq c|\Omega|, \quad \forall u \in H^1_0(\Omega).$$

In particular, there exists $c > 0$ such that for any $\eta \in \mathbb{R}$

$$\int_{\Omega} e^{\eta u} \, dx \leq c|\Omega|e^{\frac{\eta^2}{16}} \|u\|^2, \quad \forall u \in H^1_0(\Omega).$$

**Lemma 7.2.** For any $p \geq 1$ and $r > 1$ there exists $\rho_0 > 0$ and $c_1, c_2 > 0$ such that for any $\rho \in (0, \rho_0)$ we have

$$\|N_\rho(\phi)\|_p \leq c_1 e^{c_2 \|\phi\|^2} \rho^{2s_k \frac{1-p}{pr}} \|\phi\|^2$$

(73)

for all $\phi \in H^1_0(\Omega)$ and

$$\|N_\rho(\phi_1) - N_\rho(\phi_2)\|_p \leq c_1 e^{c_2 (\|\phi_1\|^2 + \|\phi_2\|^2)} \rho^{2s_k \frac{1-p}{pr}} \|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|),$$

(74)

for all $\phi_1, \phi_2 \in H^1_0(\Omega)$.

Here, $s_k > 0$ is the constant defined in (42).

**Proof of Lemma 7.2.** Since (73) follows from (74) by taking $\phi_2 = 0$, it suffices to prove (74).

We readily check that

$$f(t + s) - f(t) - f'(t)s = e^t(e^s - 1 - s) - \tau e^{-\gamma t}(e^{-\gamma s} - 1 + \gamma s).$$

Consequently,

$$N_\rho(\phi_1) - N_\rho(\phi_2) = \rho e^{W_\rho} [e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)] - \tau e^{-\gamma W_\rho} [e^{-\gamma \phi_1} - e^{-\gamma \phi_2} + \gamma (\phi_1 - \phi_2)].$$

Using the Mean Value Theorem, we have

$$|e^a - e^b - a + b| \leq e^{\|a| + |b|} a - b(|a| + |b|),$$

(75)

for all $a, b \in \mathbb{R}$. Taking $a = \phi_1$, $b = \phi_2$ we derive

$$|e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)| \leq e^{\|\phi_1\| + \|\phi_2\|} |\phi_1 - \phi_2| (|\phi_1| + |\phi_2|).$$

Setting

$$I_1 := \rho e^{W_\rho} (e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)),$$

we estimate:

$$\|I_1\|_p = \left( \int_{\Omega} \rho^p e^{pW_\rho} |e^{\phi_1} - e^{\phi_2} - \phi_1 + \phi_2|^p \, dx \right)^{1/p} \leq \sum_{j=1}^2 \left( \int_{\Omega} \rho^p e^{pW_\rho} e^{p|\phi_1| + p|\phi_2|} |\phi_1 - \phi_2|^p |\phi_1|^p \, dx \right)^{1/p}. $$
By Hölder’s inequality with $r^{-1} + s^{-1} + t^{-1} = 1$ we obtain
\[
\|I_1\|_{p*} \leq C \sum_{j=1}^{2} \left( \int_{\Omega} \rho^{pr} e^{pr|W_{\rho}|} \right)^{\frac{1}{pr}} \left( \int_{\Omega} e^{ps(|\phi_1|+|\phi_2|)} \right)^{\frac{1}{ps}} \left( \int_{\Omega} |\phi_1 - \phi_2|^{pt}|\phi_j|^{pt} \right)^{\frac{1}{pt}}.
\]
In view of (57) we have
\[
\left( \int_{\Omega} (\rho^{pr} e^{pr|W_{\rho}|}) \right)^{1/(pr)} = O(\rho^{2s_k \frac{-pr}{pr}}),
\]
where $s_k > 0$ is defined in (42). Now, the Moser-Trudinger inequality as in Lemma 7.1 yields
\[
\int_{\Omega} e^{ps(|\phi_1|+|\phi_2|)} \leq C e^{(ps)^2/(8\pi)(\|\phi_1\|^2+\|\phi_2\|^2)}
\]
\[
\left( \int_{\Omega} |\phi_1 - \phi_2|^{pt}|\phi_j|^{pt} \right)^{\frac{1}{pt}} \leq C\|\phi_1 - \phi_2\| \|\phi_j\|.
\]
We conclude that
\[
\|I_1\| \leq C \rho^{2s_k \frac{-ps}{pr}} e^{(ps)^2/(8\pi)(\|\phi_1\|^2+\|\phi_2\|^2)} \|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|).
\]
Let $I_2$ be defined by
\[
I_2 := e^{-\gamma W_{\rho}} [e^{-\gamma \phi_1} - e^{-\gamma \phi_2} + \gamma (\phi_1 - \phi_2)].
\]
Taking $a = -\gamma \phi_1$, $b = -\gamma \phi_2$ in (75) we derive
\[
|e^{-\gamma \phi_1} - e^{-\gamma \phi_2} + \gamma (\phi_1 - \phi_2)| \leq \gamma^2 e^{\gamma(|\phi_1|+|\phi_2|)} |\phi_1 - \phi_2| (|\phi_1| + |\phi_2|).
\]
Hence, by analogous estimates as above, we conclude the proof of the desired estimates. \(\square\)

Now we can prove the main result of this section.

**Proof of Proposition 7.** Let
\[
B_{p,R} := \left\{ \phi \in \mathcal{H}_v : \|\phi\| \leq R \rho^{\frac{7}{p}} |\ln \rho| \right\}.
\]
We shall prove that $T_\rho$ is a contraction mapping in $B_{p,R}$, provided $R > 0$ is sufficiently large and $\rho > 0$ is sufficiently small.

**Claim 1.** $T_\rho$ maps $B_{p,R}$ into itself.

Equivalently, we claim that
\[
\|\phi\| \leq R \rho^{\frac{7}{p}} |\ln \rho| \implies \|T_\rho(\phi)\| \leq R \rho^{\frac{7}{p}} |\ln \rho|.
\]
Indeed, we have
\[
\|T_\rho(\phi)\| \leq \| \mathcal{L}_p^{-1} (|\mathcal{N}_p(\phi)|) \|_p + \| S_p^\phi \|_p + \| R_p \|_p \leq C |\ln \rho| \left( \|\phi\|^2 (2s_k \|\phi\|^2) \rho^{2s_k \frac{-ps}{pr}} + \| S_p \|_{pq} \|\phi\|_{pq'} + \rho^{\frac{5}{p'}} \right)
\]
\[
\leq C |\ln \rho| \left( \|\phi\|^2 (2s_k \|\phi\|^2) \rho^{2s_k \frac{-ps}{pr}} + \rho^{\frac{5}{p'}} \|\phi\| + \rho^{\frac{7}{p'}} \right)
\]
\[
\leq R \rho^{\frac{7}{p}} |\ln \rho|.
\]

**Claim 2.** $T_\rho$ is a contraction in $B_{p,R}$.

Equivalently, we claim that there exists $L < 1$ such that
\[
\|\phi_1\|, \|\phi_2\| \leq R \rho^{\frac{7}{p}} |\ln \rho| \implies \|T_\rho(\phi_1) - T_\rho(\phi_2)\| \leq L \|\phi_1 - \phi_2\|.
\]
Indeed, we have
\[ \| \mathcal{T}_\rho(\phi_1) - \mathcal{T}_\rho(\phi_2) \| \leq \| \mathcal{L}_\rho^{-1} \| (\| N_\rho(\phi_1) - N_\rho(\phi_2) \|_p + \| S_\rho(\phi_1 - \phi_2) \|_p) \]
\[ \leq C \ln \rho (\| N_\rho(\phi_1) - N_\rho(\phi_2) \|_p + \| S_\rho(\phi_1 - \phi_2) \|_p). \]
We estimate:
\[ \| N_\rho(\phi_1) - N_\rho(\phi_2) \|_p \leq c_1 e^{c_2 (\| \phi_1 \|^2 + \| \phi_2 \|^2) \rho^{2s_1 \frac{1-p}{r}}} \| \phi_1 - \phi_2 \|_p \]
\[ \leq C \rho^{\frac{p}{2}} \| \phi_1 - \phi_2 \| \]
provided \( p, r > 1 \) are sufficiently close to 1 and \( \rho > 0 \) is sufficiently small.

On the other hand, we have
\[ \| S_\rho(\phi_1 - \phi_2) \|_p \leq \| \mathcal{S}_\rho \|_{pq} \| \phi_1 - \phi_2 \|_{pq'} \leq C \rho^{\frac{p}{2}} \| \phi_1 - \phi_2 \| , \]
where \( \frac{1}{q} + \frac{1}{q'} = 1 \). We conclude that \( \mathcal{T}_\rho \) is a contraction in \( B_{\rho, R} \).

Now the existence of \( \phi_\rho \) follows by the Banach Contraction Principle.

\[ \square \]

**Lemma 7.3.** There holds
\[ W_\rho(x) = \mathcal{M}_k G(x, 0) + o(1), \]
uniformly on compact subsets of \( \Omega \setminus \{ 0 \} \), where \( \mathcal{M}_k \) is defined in \( (15) \).

**Proof.** In view of \( (51) \), we have
\[ P w_\alpha^\delta(x) - 4\pi \alpha G(x, 0) = 2 \ln \frac{|x|^\alpha}{\delta^\alpha + |x|^\alpha} + O(\delta^\alpha) = O(\delta^\alpha), \]
uniformly on compact subsets of \( \Omega \setminus \{ 0 \} \). It follows that
\[ W_\rho(x) = -4\pi \sum_{i=1}^k \frac{(-1)^i}{\gamma_i} \alpha_i G(x, 0) + o(1) = \mathcal{M}_k G(x, 0) + o(1), \]
as asserted. \( \square \)

Finally, we are able to provide the proof of our main result.

**Proof of Theorem 1.1.** For \( p > 1 \) sufficiently close to 1 let \( \rho_0 > 0 \) be chosen as in Proposition \( \ref{prop:existence_u_0} \). For all \( \rho \in (0, \rho_0) \) we obtain by Proposition \( \ref{prop:existence_u_0} \) a solution \( u_\rho = W_\rho + \phi_\rho \in \mathcal{H}_\gamma \) to problem \( (1) \) satisfying \( \| \phi_\rho \| \leq R \rho^{\frac{p}{2}} \| \ln \rho \| \) with \( \beta = \frac{\beta}{p} \). In view of Ansatz \( (22) \), and taking into account Lemma \( \ref{lem:w_rho} \), we conclude that \( u_\rho \) has the desired concentration properties. \( \square \)

**Appendix.** We collect in this Appendix the proof of some complementary results stated in Section \( \ref{sec:intro} \) as well as some proofs.

**Remarks on the blow-up masses.** We denote by \( \omega(x) = H(x, x) \) the Robin’s function, where \( H(x, y) \) is the regular part of the Green’s function as defined in \( (27) \). We first provide a simple proof of identity \( (16) \). We note that the proof of identity \( (16) \) may be also deduced as a special case of an identity involving probability measures established in \( (17) \).
Lemma 7.4. Let $u_\rho$ be a solution sequence for (4). Suppose

$$\rho e^{u_\rho} \rightharpoonup m_+(x_0)\delta_{x_0}, \quad \rho e^{-\gamma u_\rho} \rightharpoonup m_-(x_0)\delta_{x_0}$$

for some $x_0 \in \Omega$, weakly in the sense of measures, as $\rho \to 0^+$. Then,

$$8\pi(m_+(x_0) + \frac{m_-(x_0)}{\gamma}) = (m_+(x_0) - m_-(x_0))^2$$

and $x_0$ is a critical point for $\omega(x)$.

Proof. We adapt the argument in [28]. Without loss of generality, we may assume $x_0 = 0$. We recall that $M_k = m_+(0) - m_-(0)$. Recall that $f(t) = e^t - \tau e^{-\gamma t}$, $F(t) = e^t + \tau \gamma^{-1}e^{-\gamma t}$, so that $F'(t) = f(t)$. Then, as $\rho \to 0^+$,

$$\rho f(u_\rho) \rightharpoonup M_k\delta_0, \quad \rho F(u_\rho) \rightharpoonup (m_+(0) + \frac{m_-(0)}{\gamma})\delta_0,$$

weakly in the sense of measures. We use the standard complex notation $z = x_1 + \sqrt{-1}x_2$, $\partial z = (\partial x_1 - \sqrt{-1}\partial x_2)/2$, $\partial \bar{z} = (\partial x_1 + \sqrt{-1}\partial x_2)/2$, $\partial z\partial \bar{z} = \Delta/4$. We set

$$\mathcal{H} := \frac{u_\rho^2}{2}, \quad \mathcal{K} := N_z * [\rho F(u_\rho)\chi_{\Omega}],$$

where $N(z, \bar{z}) = (4\pi)^{-1}\ln(z\bar{z})$ is the Newtonian potential satisfying $\Delta N = \delta_0$. Then, $S_2 = 0$, i.e., $\mathcal{S} := \mathcal{H} + \mathcal{K}$ is holomorphic in $\Omega$. Hence, $\mathcal{S}$ converges uniformly to a holomorphic function $S_0$ in $\Omega$ as $\rho \to 0^+$. Since $u_\rho \to M_kG(x, 0)$ in $W^{1,q}(\Omega)$ and uniformly in $\Omega \setminus \{0\}$, we have $\mathcal{H} \to \mathcal{H}_0$, where

$$\mathcal{H}_0 = \frac{u_\rho^2}{2} = \frac{M_k^2}{2}G^2(x, 0) = \frac{M_k^2}{2}(N_z + H_z(x, 0))^2.$$

Since $N_z = (4\pi z)^{-1}$, we conclude that

$$\mathcal{H}_0 = \frac{M_k^2}{32\pi^2 z^2} + \frac{M_k^2}{4\pi^2}H_z(x, 0) + \frac{M_k^2}{2}H_z^2(x, 0).$$

On the other hand, we have

$$\mathcal{K} := N_z * [\rho F(u_\rho)\chi_{\Omega}] = -\frac{1}{4\pi z^2} * [\rho F(u_\rho)\chi_{\Omega}].$$

Taking limits, we find $\mathcal{K} \to \mathcal{K}_0$, where

$$\mathcal{K}_0 = -\frac{1}{4\pi z^2} * [(m_+(0) + \frac{m_-(0)}{\gamma})\delta_0] = -\frac{m_+(0) + \frac{m_-(0)}{\gamma}}{4\pi z^2}.$$

We conclude that

$$S_0 = \frac{1}{4\pi^2} \left[ \frac{M_k^2}{8\pi} - (m_+(0) + \frac{m_-(0)}{\gamma}) \right] + \frac{M_k^2}{4\pi^2}H_z(x, 0) + \frac{M_k^2}{2}H_z^2(x, 0).$$

Since $S_0$ is smooth in $\Omega$, we necessarily have

$$\frac{M_k^2}{8\pi} - (m_+(0) + \frac{m_-(0)}{\gamma}) = 0 \quad \text{and} \quad H_z(x, 0)|_{x_0} = 0.$$

The asserted identities follow. \hfill \Box

The following is a proof of (43) in Proposition [2].
Proof of (43). Suppose \( k \) is odd. We compute:

\[
2q_j = 2(s_j - s_{j+1}) = \frac{(1 + \gamma)(k - j) + \gamma}{(1 + \gamma)j - 1} - \frac{(1 + \gamma)(k - j - 1) + \gamma}{(1 + \gamma)(j + 1) - 1} = \frac{[(1 + \gamma)(k - j) + \gamma][(1 + \gamma)(j + 1) - 1] - [(1 + \gamma)(k - j - 1) + \gamma][(1 + \gamma)j - 1]}{[(1 + \gamma)j - 1][(1 + \gamma)(j + 1) - 1]}.
\]

Observing that \((1 + \gamma)(j + 1) - 1 = (1 + \gamma)(j + 1) + \gamma\) and \((1 + \gamma)(k - j - 1) + \gamma = (1 + \gamma)(k - j) - 1\) we find

\[
[(1 + \gamma)(k - j) + \gamma][(1 + \gamma)(j + 1) - 1] - [(1 + \gamma)(k - j - 1) + \gamma][(1 + \gamma)j - 1] = (1 + \gamma)[(1 + \gamma)k - 1 + \gamma],
\]

and the statement follows for \( k \) odd.

Suppose \( k \) is even. We compute:

\[
2q_j = 2(s_j - s_{j+1}) = \frac{(1 + \gamma)(k - j) + 1}{(1 + \gamma)j - 1} - \frac{(1 + \gamma)(k - j - 1) + 1}{(1 + \gamma)(j + 1) - 1} = \frac{[(1 + \gamma)(k - j) + 1][(1 + \gamma)(j + 1) - 1] - [(1 + \gamma)(k - j - 1) + 1][(1 + \gamma)j - 1]}{[(1 + \gamma)j - 1][(1 + \gamma)(j + 1) - 1]}.
\]

Observing that

\[
(1 + \gamma)(k - j) + 1 = (1 + \gamma)(k - j) - \gamma
\]

and

\[
(1 + \gamma)(j + 1) - 1 = (1 + \gamma)j + \gamma
\]

we find

\[
[(1 + \gamma)(k - j) + 1][(1 + \gamma)(j + 1) - 1] - [(1 + \gamma)(k - j - 1) + 1][(1 + \gamma)j - 1] = (1 + \gamma)^2 k
\]

and the asserted decay rate follows.

For the sake of completeness, we check the following fact which was stated in Section 1.

**Remark 3.** The blow-up mass values \( m_+(0), m_-(0) \), as defined in (12)–(13), satisfy the mass identity (16).

**Proof.** Throughout this proof, we set \( \tilde{m}_+ = m_+(0)/4\pi, \tilde{m}_- = m_-(0)/4\pi \). Equivalently, we check that \( \tilde{m}_+, \tilde{m}_- \) satisfy

\[
2 \left( \frac{\tilde{m}_+ + \tilde{m}_-}{\gamma} \right) = (\tilde{m}_+ - \tilde{m}_-)^2 \tag{76}
\]

Suppose \( k \) is odd. Then,

\[
\tilde{m}_+ + \frac{\tilde{m}_-}{\gamma} = \frac{1}{2} \left[ (1 + \frac{1}{\gamma})k + 1 - \frac{1}{\gamma} \right] (k + 1 + \frac{1}{\gamma}(k - 1)) = \frac{1}{2} \left[ (1 + \frac{1}{\gamma})k + 1 - \frac{1}{\gamma} \right]^2.
\]
On the other hand,
\[ \tilde{m}_+ - \tilde{m}_- = \left[ (1 + \frac{1}{\gamma})k + 1 - \frac{1}{\gamma} \right] \frac{k + 1}{2} - \frac{k - 1}{2} = (1 + \frac{1}{\gamma})k + 1 - \frac{1}{\gamma}. \]
Hence, (76) is verified for \( k \) odd.

Suppose \( k \) is even.
\[ \tilde{m}_+ + \frac{\tilde{m}_-}{\gamma} = k \left[ (1 + \frac{1}{\gamma}) \frac{k}{2} - \frac{1}{\gamma} \right] + \frac{k}{\gamma} \left[(1 + \frac{1}{\gamma}) \frac{k}{2} + 1 \right] = \frac{k^2}{2} (1 + \frac{1}{\gamma})^2. \]
On the other hand,
\[ \tilde{m}_+ - \tilde{m}_- = k \left[ (1 + \frac{1}{\gamma}) \frac{k}{2} - \frac{1}{\gamma} \right] - k \left[(1 + \frac{1}{\gamma}) \frac{k}{2} + 1 \right] = -k (1 + \frac{1}{\gamma}), \]
and (76) is verified for \( k \) even, as well. \( \square \)

**The cases of physical interest.** Finally, we compute the values of \( \lambda, \tau_1 \) for which the bubble tower construction as in Theorem 1.1 yields solutions to problem (2) and to problem (3). Let \( u_\rho, \rho \in (0, \rho_0) \) be the family of concentrating solutions as obtained in Theorem 1.1.

**Onsager’s mean field equation (2).** The solution \( u_\rho \) yields a solution to (2) with
\[ \lambda \tau_1 = m_+ (0), \quad \lambda (1 - \tau_1) \gamma = m_- (0). \]
Consequently, recalling (16),
\[ \lambda = m_+ (0) + \frac{m_- (0)}{\gamma} = \frac{M_k^2}{8 \pi}, \]
and (17)–(18) readily follow.

**Neri’s mean field equation (3).** The solution \( u_\rho \) yields a solution to (3) satisfying
\[ \frac{\lambda \tau_1}{\tau_1} \int_\Omega e^{u_\rho} + (1 - \tau_1) \int_\Omega e^{-\gamma u_\rho} = m_+ (0) + o(1) \]
\[ \frac{\lambda \gamma (1 - \tau_1)}{\tau_1} \int_\Omega e^{-\gamma u_\rho} = m_- (0) + o(1). \]
We deduce that
\[ [\lambda - m_+ (0) + o(1)] \tau_1 \int_\Omega e^{u_\rho} - [m_+ (0) + o(1)] (1 - \tau_1) \int_\Omega e^{-\gamma u_\rho} = 0 \]
\[ [m_- (0) + o(1)] \tau_1 \int_\Omega e^{u_\rho} - [\lambda \gamma - m_- (0) + o(1)] (1 - \tau_1) \int_\Omega e^{-\gamma u_\rho} = 0. \]
A non-zero solution \((\tau_1 \int_\Omega e^{u_\rho}, (1 - \tau_1) \int_\Omega e^{-\gamma u_\rho})\) to this linear system exists if and only if
\[ (\lambda - m_+ (0) + o(1)) (\lambda \gamma - m_- (0) + o(1)) - (m_+ (0) + o(1)) (m_- (0) + o(1)) = 0. \]
Taking limits and dividing by \( \gamma \) we obtain the condition
\[ (\lambda - m_+ (0)) (\lambda - \frac{m_- (0)}{\gamma}) - m_+ (0) \frac{m_- (0)}{\gamma} = 0. \]
Since \( \lambda = m_+ (0) + \gamma^{-1} m_- (0) \) is the only positive solution to the second order algebraic equation above, we deduce that (17) is satisfied for equation (3) as well.
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REFERENCES

[1] E. Caglioti, P. L. Lions, C. Marchioro and M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description, Commun. Math. Phys., 174 (1995), 229–260.
[2] M. del Pino, P. Esposito and M. Musso, Nondegeneracy of entire solutions of a singular Liouville equation, Proc. Am. Math. Soc., 140 (2012), 581–588.
[3] M. del Pino, M. Kowalczyk and M. Musso, Singular limits in Liouville-type equations, Calc. Var. Partial Differential Equations, 24 (2005), 47–81.
[4] P. Esposito, M. Grossi and A. Pistoia, On the existence of blowing-up solutions for a mean field equation, Ann. I. H. Poincaré, 22 (2005), 227–257.
[5] L. Eyink and K. R. Sreenivasan, Onsager and the theory of hydrodynamic turbulence, Reviews of Modern Physics, 78 (2006), 87–135.
[6] M. Grossi, C. Grumiau and F. Pacella, Lane Emden problems with large exponents and singular Liouville equations, J. Math. Pures Appl., 101 (2014), 735–754.
[7] M. Grossi and A. Pistoia, Multiple blow-up phenomena for the sinh-Poisson equation, Arch. Rational Mech. Anal., 209 (2013), 287–320.
[8] A. Jevnikar, An existence result for the mean-field equation on compact surfaces in a doubly supercritical regime, Proc. Roy. Soc. Edinburgh Sect. A, 143 (2013), 1021–1045.
[9] A. Jevnikar and W. Yang, Analytic aspects of the Tzitzeica equation: Blow-up analysis and existence results, Calc. Var. Partial Differential Equations, 56 (2017), Paper No. 43, 38 pp.
[10] D. D. Joseph and T. S. Lundgren, Quasilinear problems driven by positive sources, Arch. Rat. Mech. Anal., 49 (1973), 241–269.
[11] J. Jost, G. Wang, D. Ye and C. Zhou, The blow up analysis of solutions of the elliptic sinh-Gordon equation, Calc. Var. Partial Differential Equations, 31 (2008), 263–276.
[12] H. Joyce and D. Montgomery, Negative temperature states for the two-dimensional guiding centre plasma, J. Plasma Phys., 10 (1973), 107–121.
[13] C. S. Lin, An expository survey on recent development of mean field equations, Discr. Cont. Dynamical Systems, 19 (2007), 387–410.
[14] A. Malchiodi, Topological methods for an elliptic equation with exponential nonlinearities, Discr. Cont. Dynamical Systems, 21 (2008), 277–294.
[15] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1970/71), 1077–1092.
[16] C. Neri, Statistical Mechanics of the N-point vortex system with random intensities on a bounded domain, Ann. I. H. Poincaré, 21 (2004), 381–399.
[17] H. Ohtsuka, T. Ricciardi and T. Suzuki, Blow-up analysis for an elliptic equation describing stationary vortex flows with variable intensities in 2D-turbulence, J. Differential Equations, 249 (2010), 1436–1455.
[18] H. Ohtsuka and T. Suzuki, Mean field equation for the equilibrium turbulence and a related functional inequality, Adv. Differential Equations, 11 (2006), 281–304.
[19] L. Onsager, Statistical hydrodynamics, Nuovo Cimento Suppl., 6 (1949), 279–287.
[20] A. Pistoia and T. Ricciardi, Concentrating solutions for a Liouville type equation with variable intensities in 2D-turbulence, Nonlinearity, 29 (2016), 271–297.
[21] Y. B. Pointin and T. S. Lundgren, Statistical mechanics of two-dimensional vortices in a bounded container, Phys. Fluids, 19 (1976), 1459–1470.
[22] J. Prajapat and G. Tarantello, On a class of elliptic problems in R^2: Symmetry and uniqueness results, Proc. R. Soc. Edinb. Sect. A, 131 (2001), 967–985.
[23] T. Ricciardi, Mountain-pass solutions for a mean field equation from two-dimensional turbulence, Differential and Integral Equations, 20 (2007), 561–575.
[24] T. Ricciardi and G. Zocca, Minimal blow-up masses and existence of solutions for an asymmetric sinh-Poisson equation, arXiv:1805.05895
[25] K. Sawada and T. Suzuki, Derivation of the equilibrium mean field equations of point vortex and vortex filament system, *Theoret. Appl. Mech. Japan*, 56 (2008), 285–290.

[26] R. Takahashi, *Analysis Seminar*, Naples Federico II University, March 2016.

[27] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.*, 17 (1967), 473–483.

[28] D. Ye, Une remarque sur le comportement asymptotique des solutions de $-\Delta u = \lambda f(u)$, *C.R. Acad. Sci. Paris*, 325 (1997), 1279–1282.

[29] C. Zhou, Existence of solution for mean field equation for the equilibrium turbulence, *Nonlinear Anal.*, 69 (2008), 2541–2552.

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