The mixed problem for the Laplacian in Lipschitz domains

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Abstract

We consider the mixed boundary value problem, or Zaremba’s problem, for the Laplacian in a bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$, $n \geq 2$. We decompose the boundary $\partial \Omega = D \cup N$ with $D$ and $N$ disjoint. The boundary between $D$ and $N$ is assumed to be a Lipschitz surface in $\partial \Omega$. We find an exponent $q_0 > 1$ so that for $p$ between 1 and $q_0$ we may solve the mixed problem for $L^p$. Thus, if we specify Dirichlet data on $D$ in the Sobolev space $W^{1,p}(D)$ and Neumann data on $N$ in $L^p(N)$, the mixed problem with data $f_N$ and $f_D$ has a unique solution and the non-tangential maximal function of the gradient lies in $L^p(\partial \Omega)$. We also obtain results for $p = 1$ when the data comes from Hardy spaces.

Keywords: Mixed boundary value problem, Laplacian

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1 Introduction

Over the past thirty years, there has been a great deal of interest in studying boundary value problems for the Laplacian in Lipschitz domains. A fundamental paper of Dahlberg [9] treated the Dirichlet problem. Jerison and Kenig [19] treated the Neumann problem and provided a regularity result for the Dirichlet problem. Another boundary value problem of interest is the mixed problem or Zaremba’s problem where we specify Dirichlet data on part of the boundary and Neumann data on the remainder of the boundary. To state this boundary value problem, we let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and suppose that we have written $\partial \Omega = D \cup N$ where $D$ is an open subset of the boundary.

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and \( N = \partial \Omega \setminus D \). We consider the \( L^p \)-mixed problem by which we mean the boundary value problem

\[
\begin{align*}
\Delta u &= 0, & \text{in } \Omega \\
u &= f_D, & \text{on } D \\
\frac{\partial u}{\partial \nu} &= f_N, & \text{on } N \\
(\nabla u)^* &\in L^p(\partial \Omega).
\end{align*}
\] (1.1)

Here, we are using \((\nabla u)^*\) to denote the non-tangential maximal function of \(\nabla u\) and the restriction to the boundary of \(u\) and \(\nabla u\) are defined using non-tangential limits. See section 2 for details. The normal derivative at the boundary \(\partial u/\partial \nu\) is defined as \(\nabla u \cdot \nu\) where \(\nu\) is the outer unit normal defined a.e. on the boundary.

Our goals are to find conditions on \(\Omega, N, D\) which allow us to show that (1.1) has at most one solution and to find conditions on \(\Omega, N, D, f_N\) and \(f_D\) which guarantee the existence of solutions.

The study of the mixed problem in Lipschitz domains is listed as an open problem in Kenig’s CBMS lecture notes [20, Problem 3.2.15]. Recall that simple examples show that we cannot expect to find solutions whose gradient lies in \(L^2\) of the boundary. For example, the function \(\text{Re} \sqrt{z}\) on the upper half-plane has zero Neumann data on the positive real axis and zero Dirichlet data on the negative real axis but the gradient is not locally square integrable on the boundary of the upper half-space. This appears to present a technical problem as the standard technique for studying boundary value problems has been the Rellich identity which produces estimates in \(L^2\).

In 1994, one of the authors observed that the Rellich identity could be used to study the mixed problem in a restricted class of Lipschitz domains [3]. Roughly speaking, this work requires that the sets \(N\) and \(D\) meet at an angle less than \(\pi\). Based on this work and the methods used by Dahlberg and Kenig to study the Neumann problem [10], J. Sykes [36, 37] established results for the mixed problem in a restricted class of Lipschitz graph domains. I. Mitrea and M. Mitrea [31] have studied the mixed problem for the Laplacian with data taken from a large family of function spaces, but with a restriction on the class of domains. Brown and I. Mitrea have studied the mixed problem for the Lamé system [5] and Brown, I. Mitrea, M. Mitrea and Wright have considered a mixed problem for the Stokes system [2]. More recently, Lanzani, Capogna and Brown [24] used a variant of the Rellich identity to establish an estimate for the mixed problem in two-dimensional graph domains when the data comes from weighted \(L^2\) spaces and the Lipschitz constant is less than one. The present work also relies on weighted estimates, but uses a simpler, more flexible approach that applies to all Lipschitz domains.

Several other authors have treated the mixed problem in various settings. Verchota and Venouziou [38] treat a large class of three dimensional polyhedral domains under the condition that the Neumann and Dirichlet faces meet at an angle of less than \(\pi\). Maz’ya and Rossman [26, 28, 27] have studied the Stokes system in polyhedral domains. Finally, we note that Savaré [34] has shown
that on smooth domains, we may find solutions in the Besov space $B^{2,\infty}_{3/2}$. This result seems to be very close to optimal. The example $\text{Re} \sqrt{z}$ described above shows that we cannot hope to obtain an estimate in the Besov space $B^{2,2}_{3/2}$.

We outline the rest of the paper and describe the main tools of the proof. Our first main result is an existence result for the mixed problem when the Neumann data is an atom for a Hardy space. We begin with the weak solution of the mixed problem and use Jerison and Kenig’s results for the Dirichlet problem and Neumann problem [19] to obtain estimates for the gradient of the solution on the interior of $D$ or $N$. This leads to a weighted estimate where the weight is a power of the distance to the common boundary between $D$ and $N$. The estimate involves a term in the interior of the domain $\Omega$. We handle this term by showing that the gradient of a weak solution lies in $L^p(\Omega)$ for some $p > 2$. The $L^p(\Omega)$ estimates for the gradient of a weak solution are proved in section 3 using the reverse Hölder technique of Gehring [14] and Giaquinta and Modica [17]. Using this weighted estimate for solutions of the mixed problem, we obtain existence for solutions with Hardy space data by extending the methods of Dahlberg and Kenig [10]. Uniqueness of solutions is proven in section 5.

With the Hardy space results in hand, we establish the existence of solutions to the mixed problem when the Neumann data is in $L^p(N)$ and the Dirichlet data is in the Sobolev space $W^{1,p}(D)$. This is done in sections 6 and 7 by adapting the reverse Hölder technique used by Shen to study boundary value problems for elliptic systems [35]. The novel feature in our work is that we are able to use the estimates in Hardy spaces proven in section 4, whereas Shen’s work begins with existence in $L^2$.

2 Definitions and preliminaries

We say that a bounded, connected open set $\Omega$ is a Lipschitz domain if the boundary is locally the graph of a Lipschitz function. To make this precise, for $M > 0$, $x \in \partial \Omega$ and $r > 0$, we define a coordinate cylinder $Z_r(x)$ to be $Z_r(x) = \{y : |y' - x'| < r, |y_n - x_n| < (1 + M)r\}$. We use coordinates $(x', x_n) = (x_1, x''', x_n) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$ and assume that this coordinate system is a translation and rotation of the standard coordinates. We say that $\Omega$ is a Lipschitz domain if for each $x \in \partial \Omega$, we may find a coordinate cylinder and a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ with Lipschitz constant $M$ so that

$$\Omega \cap Z_r(x) = \{(y', y_n) : y_n > \phi(y')\} \cap Z_r(x)$$
$$\partial \Omega \cap Z_r(x) = \{(y', y_n) : y_n = \phi(y')\} \cap Z_r(x).$$

For a Lipschitz domain $\Omega$, we define a decomposition of the boundary for the mixed problem, $\partial \Omega = D \cup N$, as follows. We assume that $D$ is a relatively open subset of $\partial \Omega$, $N = \partial \Omega \setminus D$ and let $\Lambda$ be the boundary (relative to $\partial \Omega$) of $D$. For each $x$ in $\Lambda$, we require that a coordinate cylinder centered at $x$ have some additional properties. We ask that there be a coordinate system $(x_1, x''', x_n)$, a
coordinate cylinder $Z_r(x)$, a function $\phi$ as above and also a Lipschitz function $\psi : \mathbb{R}^{n-2} \to \mathbb{R}$ with Lipschitz constant $M$ so that

$$Z_r(x) \cap D = \{(y_1, y''_n, y_n) : y_1 > \psi(y''), \; y_n = \phi(y')\} \cap Z_r(x)$$

$$Z_r(x) \cap N = \{(y_1, y''_n, y_n) : y_1 \leq \psi(y''), \; y_n = \phi(y')\} \cap Z_r(x).$$

We fix a covering of the boundary by coordinate cylinders $\{Z_r(x_i)\}_{i=1}^L$ so that each $Z_{100r_i}(x_i)$ is also a coordinate cylinder. We assume that for each $i$, the cylinder $Z_{100r_i}(x_i) \cap \partial \Omega \subset D$, $Z_{100r_i}(x_i) \cap \partial \Omega \subset N$ or $Z_{100r_i}(x_i)$ is one of the coordinate cylinders from the definition of the boundary decomposition for the mixed problem. We let $r_0 = \min\{r_i : i = 1, \ldots, L\}$ be the smallest radius in the collection.

We will call a Lipschitz domain $\Omega$ and a decomposition of the boundary $\partial \Omega = N \cup D$ satisfying the above properties a standard domain for the mixed problem.

We will use $\delta(y) = \text{dist}(y, \Lambda)$ to denote the distance from a point $y$ to $\Lambda$. We will let $B_r(x) = \{y : |y - x| < r\}$ denote the standard ball in $\mathbb{R}^n$ and then $\Delta_r(x) = B_r(x) \cap \partial \Omega$ will denote a surface ball. Throughout this paper we will need to be careful of several points. The surface balls may not be connected and we will use the notation $\Delta_r(x)$ where $x$ may not be on the boundary. We use $\Psi_r(x)$ to stand for $B_r(x) \cap \Omega$. Since $\Lambda$ is a Lipschitz graph, we may find a constant $c = c(n, M) > 0$ so that we have the property

$$\text{If } x \in \Lambda \text{ and } 0 < r < r_0, \text{ then } \sigma(\Delta_r(x) \cap D) > cr^{n-1}. \quad (2.1)$$

Here and throughout this paper, we use $\sigma$ for surface measure.

Our main tool for estimating solutions will be the non-tangential maximal function. We fix $\alpha > 0$ and for $x \in \partial \Omega$ we define a non-tangential approach region by

$$\Gamma(x) = \{y \in \Omega : |x - y| \leq (1 + \alpha) \text{dist}(y, \partial \Omega)\}.$$

Given a function $u$ defined on $\Omega$, we define the non-tangential maximal function by

$$u^*(x) = \sup_{y \in \Gamma(x)} |u(y)|, \quad x \in \partial \Omega.$$ 

It is well-known that for different values of $\alpha$, the non-tangential maximal functions have comparable $L^p$-norms. Thus, the dependence on $\alpha$ is not important for our purposes and we suppress the value of $\alpha$ in our notation. In (2.1), we define the restriction of $u$ and $\nabla u$ to the boundary using non-tangential limits. Thus, for a function $v$ defined in $\Omega$ and $x \in \partial \Omega$, we define

$$v(x) = \lim_{\Gamma(x) \ni y \to x} v(y)$$

provided the limit exists. It is well-known that if $v$ is harmonic in a Lipschitz domain, then the non-tangential limits exist at almost every point where the non-tangential maximal function is finite. In addition, if the non-tangential
 maximal function of \( \nabla u \) lies in \( L^p(\partial \Omega) \), then according to the argument in [4, Lemma 2.2], as corrected in Wright [41], the non-tangential maximal function of \( u \) lies in an \( L^p \)-space and hence has non-tangential limits a.e..

Many of our estimates will be of a local, scale invariant form and hold on a scale \( r \) that is less than \( r_0 \). The constants in these local estimates will depend on the constant \( M \), the dimension \( n \), and any \( L^p \)-indices that appear in the estimate. If a constant depends on \( M \), \( n \), any \( L^p \)-indices and also depends on the collection of coordinate cylinders which cover \( \partial \Omega \) and the constant in the coercivity condition (3.2), then we say that the constant depends on the global character of \( \Omega \).

We will use \( L^p(E) \) to denote \( L^p \)-spaces. If \( E \subset \partial \Omega \), then we use the \((n-1)\)-dimensional measure on the boundary to define the \( L^p \)-space. Otherwise, the \( L^p \)-norm is taken with respect to \( n \)-dimensional Lebesgue measure. For \( \Omega \) an open subset of \( \mathbb{R}^n \), \( k = 1, 2, \ldots \) and \( 1 \leq p \leq \infty \), we use \( W^{k,p}(\Omega) \) to denote the Sobolev space of functions having \( k \) derivatives in \( L^p(\Omega) \). We introduce notation for the tangential gradient of a function defined on the boundary, \( \nabla_{t,u} \). If \( u \) is a smooth function defined in a neighborhood of \( \partial \Omega \), then we have that \( \nabla_{t,u} = \nabla u - (\nabla u \cdot \nu) \nu \). See [40, p. 580] for more details. For \( D \) an open subset of \( \partial \Omega \), we use \( W^{1,p}(D) \) to denote the Sobolev space of functions defined on \( D \) and having one derivative in \( L^p(D) \). The norm in this space is given by

\[
\|f\|_{W^{1,p}(D)} = \|f\|_{L^p(D)} + \|\nabla_{t,f}\|_{L^p(D)}.
\]

Before stating the main theorem, we recall the definitions of atoms and atomic Hardy spaces. We say that \( a \) is an atom for the boundary \( \partial \Omega \) if \( a \) is supported in a surface ball \( \Delta_r(x) \) for some \( x \) in \( \partial \Omega \), \( \|a\|_{L^\infty(\partial \Omega)} \leq 1/\sigma(\Delta_r(x)) \) and \( \int_{\partial \Omega} a \, d\sigma = 0 \).

When we consider the mixed problem, we will want to consider atoms for the subset \( N \). We say that \( a \) is an atom for \( N \) if \( a \) is the restriction to \( N \) of a function \( \tilde{a} \) which is an atom for \( \partial \Omega \). For \( N \) a subset of \( \partial \Omega \), the Hardy space \( H^1(N) \) is the collection of functions \( f \) which can be represented as \( \sum \lambda_j a_j \) where each \( a_j \) is an atom for \( N \) and the coefficients satisfy \( \sum |\lambda_j| < \infty \). This includes, of course, the case where \( N = \partial \Omega \) and then we obtain the standard definition. It is easy to see that the Hardy space \( H^1(N) \) is the restriction to \( N \) of elements of the Hardy space \( H^1(\partial \Omega) \).

We give a similar definition for the Hardy-Sobolev space \( H^{1,1} \). We say that \( A \) is an atom for \( H^{1,1}(\partial \Omega) \) if \( A \) is supported in a surface ball \( \Delta_r(x) \) for some \( x \in \partial \Omega \) and \( \|\nabla_{t,A}\|_{L^\infty(\partial \Omega)} \leq 1/\sigma(\Delta_r(x)) \). We say that \( A \) is an atom for \( H^{1,1}(D) \) if \( A \) is the restriction to \( D \) of an atom \( \tilde{A} \) for \( \partial \Omega \). Again, the space \( H^{1,1}(D) \) is the collection generated by taking sums of \( H^{1,1}(D) \) atoms with coefficients in \( \ell^1 \). See the article of Coifman and Weiss [8] for more information about Hardy spaces.

We are now ready to state our main theorem.

**Theorem 2.2** Let \( \Omega \), \( N \) and \( D \) be a standard domain for the mixed problem.

\( a) \) For \( p \geq 1 \), the \( L^p \)-mixed problem has at most one solution.
b) If $f_N$ lies in $H^1(N)$ and $f_D$ lies in $H^{1,1}(D)$, the $L^1$-mixed problem has a solution which satisfies the estimate

$$\| (\nabla u)^* \|_{L^1(\partial \Omega)} \leq C(\| f_N \|_{H^1(N)} + \| f_D \|_{H^{1,1}(D)}).$$

The constants in the estimates depend on the global character of the domain and the index $p$.

The rest of the paper is devoted to the proof of this theorem. We outline the main steps of the proof.

Outline of the proof. We begin by recalling that for the Dirichlet problem with data from a Sobolev space, we obtain non-tangential maximal function estimates for the gradient of the solution. This is treated for $p = 2$ by Jerison and Kenig [19] and for $1 < p < 2$ by Verchota [39, 40]. The Hardy space problem was studied by Dahlberg and Kenig [10] and by D. Mitrea in two dimensions [30, Theorem 3.6]. Using these results, it suffices to prove Theorem 2.2 in the case when the Dirichlet data is zero.

The existence when the Neumann data is taken from the atomic Hardy space and the Dirichlet data is zero is given in Theorem 4.14. The existence for $L^p$ data appears in section 7. It suffices to establish uniqueness when $p = 1$ and this is treated in Theorem 5.1.

3 Higher integrability of the gradient of a weak solution

It is well-known that one can obtain higher integrability of the gradient of weak solutions of an elliptic equation. An early result of this type is due to Meyers [29]. Meyers’s result has been extended to the mixed problem by Gröger [18]. However, we choose to obtain our estimates using the reverse Hölder technique introduced by Gehring [14] and Giaquinta and Modica [17] (we use the formulation from Giaquinta [15, p. 122]). This approach allows us to include non-zero boundary data and obtain local, scale-invariant results. At a few points of the proof it will be simpler if we are working in a coordinate cylinder $Z$ where we have that $\partial \Omega \cap Z$ lies in a hyperplane. Thus, we will establish results for divergence form elliptic operators with bounded measurable coefficients as this class is preserved by a change of variable that will flatten part of the boundary.

We will consider several formulations of the mixed problem. Our goal is to obtain solutions whose gradient lies in $L^p(\partial \Omega)$ for $p$ near 1. Our argument
begins with a weak solution whose gradient lies in $L^2(\Omega)$. We will show that under appropriate assumptions on the data, this solution will have a gradient in $L^p(\partial \Omega)$.

We describe a weak formulation of the mixed boundary value problem. Some of the results of this section will hold for solutions of divergence form operators. Thus, we define weak solutions in this more general setting. For $D$ a subset of the boundary, we let $W^{1,2}_D(\Omega)$ be the closure in $W^{1,2}(\Omega)$ of functions in $C^\infty_0(\mathbb{R}^n)$ for which $\text{supp } u \cap \overline{D} = \emptyset$. We let $W^{1/2,2}_D(\partial \Omega)$ be the restrictions to $\partial \Omega$ of the space $W^{1,2}_D(\Omega)$. We define $W^{-1/2,2}_D(\partial \Omega)$ to be the dual of $W^{1/2,2}_D(\partial \Omega)$. The Neumann data $f_N$ will be taken from the space $W^{-1/2,2}_D(\partial \Omega)$. If $A(x)$ is a symmetric matrix with bounded, measurable entries and satisfies the ellipticity condition

$$\lambda |\xi|^2 \geq A(x)\xi \cdot \xi \geq \lambda^{-1} |\xi|^2$$

for some $\lambda > 0$ and all $\xi \in \mathbb{R}^n$, we consider the problem

$$\begin{cases}
\text{div} A\nabla u = 0, & \text{in } \Omega \\
u &= 0, & \text{on } D \\
A\nabla u \cdot \nu = f_N, & \text{on } N.
\end{cases} \quad (3.1)$$

We say that $u$ is a weak solution of this problem if $u \in W^{1,2}_D(\Omega)$ and we have

$$\int_\Omega A\nabla u \cdot \nabla v \, dy = \langle f_N, v \rangle_{\partial \Omega}, \quad \text{for all } v \in W^{1,2}_D(\Omega).$$

Here, we are using $\langle \cdot, \cdot \rangle_{\partial \Omega}$ to denote the pairing between $W^{-1/2,2}_D(\partial \Omega)$ and the dual $W^{1/2,2}_D(\partial \Omega)$. To establish existence of weak solutions of the mixed problem, we assume the coercivity condition

$$\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega)}, \quad u \in W^{1,2}_D(\Omega). \quad (3.2)$$

Under this assumption, the existence and uniqueness of weak solutions to (3.1) is a consequence of the Lax-Milgram theorem. It is easy to see that (3.2) holds when $\Omega$, $N$ and $D$ is a standard domain for the mixed problem.

If $f_N$ is a function on $N$, then we may identify $f_N$ with an element of the space $W^{-1/2,2}_D(\partial \Omega)$ by

$$\langle f_N, \phi \rangle_{\partial \Omega} = \int_N f_N \phi \, d\sigma, \quad \text{for all } \phi \in W^{1/2,2}_D(\partial \Omega).$$

From Sobolev embedding we have $W^{1/2,2}_D(\partial \Omega) \subset L^p(\partial \Omega)$, where $p = 2(n-1)/(n-2)$ if $n \geq 3$ or $p < \infty$ when $n = 2$. Thus the integral on the right-hand side will be well-defined if we have $f_N$ in $L^{2(n-1)/n}(N)$ when $n \geq 3$ or $L^p(N)$ for any $p > 1$ when $n = 2$.

We define a sub-linear operator $P$ which takes functions on $\partial \Omega$ to functions in $\Omega$ by

$$Pf(x) = \sup_{s > 0} \frac{1}{s^{n-1}} \int_{\Delta_s(x)} |f| \, d\sigma, \quad x \in \Omega$$
and a local version of $P$ by

$$P_r f(x) = \sup_{r > s > 0} \frac{1}{s^{n-1}} \int_{\Delta_s(x)} |f| d\sigma, \quad x \in \Omega.$$ 

On the boundary, we have that $P f$ is the Hardy-Littlewood maximal function

$$M f(x) = P f(x) = \sup_{s > 0} \frac{1}{s^{n-1}} \int_{\Delta_s(x)} |f| d\sigma, \quad x \in \partial \Omega.$$ 

The following result is probably well-known, but we could not find a reference.

**Lemma 3.3** For $1 < p < \infty$, $1 \leq q \leq pn/(n-1)$, $x \in \partial \Omega$ and $r < r_0$, we have

$$\left( \int_{y} |P_r f|^q dy \right)^{1/q} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta_{2r}(x)} |f|^p d\sigma \right)^{1/p}. \quad (3.4)$$

The constant in this estimate depends only on the Lipschitz constant $M$ and the dimension.

**Proof.** We begin by considering the case where $\Omega = \{(y', y_n) : y_n > 0\}$ is a half-space. We use coordinates $y = (y', y_n)$ and we claim that

$$P f(y', y_n) \leq M f(y', 0) \quad (3.5)$$

$$P f(y) \leq C \|f\|_{L^p(\partial \Omega)} y_n^{(1-n)/p}, \quad y_n > 0. \quad (3.6)$$

The estimate (3.5) follows easily since $\Delta_s((y', y_n)) \subset \Delta_s((y', 0))$. To establish the second estimate, we observe that if $s < y_n$, then $\Delta_s(y) = \emptyset$ and hence

$$P f(y) = \sup_{s \geq y_n} \frac{1}{s^{n-1}} \int_{\Delta_s(y)} |f| d\sigma \leq C y_n^{1-n/p} \|f\|_{L^p(\partial \Omega)}.$$ 

We claim that we have the following weak-type estimate for $P f$,

$$|\{x \in \Omega : P f(x) > \lambda\}| \leq C \|f\|_{L^p(\partial \Omega)}^p \lambda^{-pn/(n-1)}, \quad \lambda > 0. \quad (3.7)$$

To prove (3.7), we may assume $\|f\|_{L^p(\partial \Omega)} = 1$. With this normalization, the observation (3.6) implies that $\{y' : P f(y', y_n) > \lambda\} = \emptyset$ if $y_n > c\lambda^{-p/(n-1)}$. Thus, we may use Fubini’s theorem to write

$$|\{x \in \Omega : P f(x) > \lambda\}| = \int_0^{c\lambda^{-p/(n-1)}} \sigma(\{y' : P f(y', y_n) > \lambda\}) dy_n \leq C \int_0^{c\lambda^{-p/(n-1)}} \sigma(\{y' : M f(y', 0) > c\lambda\}) dy_n = C\lambda^{-pn/(n-1)}$$

where we used (3.5), the weak-type $(p, p)$ inequality for the maximal operator on $\mathbb{R}^{n-1}$ and our normalization of the $L^p$-norm of $f$. 

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From the weak-type estimate (3.7) and the Marcinkiewicz interpolation theorem we obtain that there is a constant C depending on p and n so that for p > 1,

\[
\|Pf\|_{L^{p(n-1)}(\Omega)} \leq C\|f\|_{L^p(\mathbb{R}^{n-1})}.
\]  
(3.8)

To obtain the estimate (3.4), we observe that if \( y \in B_r(x) \) then \( B_r(y) \subset B_{2r}(x) \) and hence

\[
P_r f(y) \leq P_r(\chi_{\Delta r(x)} f)(y), \quad y \in B_r(x).
\]

Thus in the case where \( \Omega \) is a half-space, the result (3.4) follows from (3.8) and Hölder’s inequality.

Finally, to obtain the local result on a general Lipschitz domain, one may change variables so that the boundary is flat near \( \Delta r(x) \). This introduces the dependence on the constant \( M \).

We recall several versions of the Poincaré and Sobolev inequalities.

**Lemma 3.9** Let \( \Omega \) be a convex domain of diameter \( d \). Suppose that \( S \) is a subset of \( \bar{\Omega} \) that satisfies: a) for some \( r \) with \( 0 < r < d \) we have \( \sigma(S \cap B_r(x)) = r^{n-1} \) and b) there is a constant \( A \) so that \( \sigma(S \cap B_t(x)) \leq At^{n-1} \) for \( t > 0 \). Let \( u \) be a function in \( W^1p(\Omega) \) and suppose that \( u \) vanishes on \( S \). Then for \( 1 < p < n \), we have a constant \( C \)

\[
\left( \int_{\Omega} |u|^p \, dy \right)^{1/p} \leq \frac{Cd^n}{|\Omega|^{1/p}} r^{1-n/p} \left( \int_{\Omega} |\nabla u|^p \, dy \right)^{1/p}.
\]

The constant \( C \) depends on \( p \), the dimension \( n \) and \( A \).

**Proof.** It suffices to consider functions \( u \) which are smooth in \( \bar{\Omega} \) and vanish on \( S \). We follow the proof of Corollary 8.2.2 in the book of Adams and Hedberg [1], except that we substitute the Riesz capacity for the Bessel capacity in order to obtain a scale-invariant estimate. Following their arguments, we obtain that if \( u \) vanishes on \( S \), then

\[
|u(x)| \leq \frac{d^n}{|\Omega|} (I_1(|\nabla u|)(x) + \|\nabla u\|_{L^p(\Omega)} \|I_1(\mu)\|_{L^{p'}(\Omega)}).
\]  
(3.10)

Here \( I_1(f)(x) = \int_{\Omega} f(y)|x-y|^{1-n} \, dy \) is the first-order fractional integral and \( \mu \) is any non-negative measure on \( S \) normalized so that \( \mu(S) = 1 \). To estimate \( \|I_1(\mu)\|_{L^{p'}(\Omega)} \) we use Theorem 4.5.4 of Adams and Hedberg [1] which gives that

\[
\int_{\mathbb{R}^n} (I_1(\mu))^{p'} \, d\mu \leq C \int_{\mathbb{R}^n} \hat{W}_{1,p}^{\mu} \, d\mu
\]

where \( \hat{W}_{1,p}^{\mu}(x) \) is the Wolff potential of \( \mu \) defined by

\[
\hat{W}_{1,p}^{\mu}(x) = \int_0^\infty (\mu(B_t(x))t^{p-n})^{1/(p-1)} \, dt/t.
\]

Our assumptions imply that with \( \mu = r^{1-n}\sigma \) denoting normalized surface measure on \( S \), we have \( I_1(\mu)(x) \leq Cr^{(p-n)/(p-1)} \) where \( C \) depends only on \( A \). Using this estimate for the Wolff potential and Young’s convolution inequality to estimate \( I_1(|\nabla u|) \), the Lemma follows from (3.10).  

\[\square\]
The next inequality is also taken from Adams and Hedberg [1, Corollary 8.1.4]. Let \( 1/q + 1/n < 1 \) and assume that \( \Omega \) is a convex domain of diameter \( d \). We let \( \tilde{u} = \frac{1}{\Omega} \int \Omega u \, dy \) and then we may find a constant \( C = C_{q,n} \) depending only on \( q \) and \( n \) so that

\[
\int_{\Omega} |u - \tilde{u}|^q \, dy \leq C \frac{d^n}{|\Omega|} \left( \int_{\Omega} |\nabla u|^{nq/(n+q)} \, dy \right)^{(n+q)/n}. \tag{3.11}
\]

Finally, we suppose that \( \Omega \) is a domain and \( \Psi_r(x) \) lies in a coordinate cylinder \( Z \) so that \( \partial \Omega \cap Z \) lies in a hyperplane and let \( \tilde{u} = \frac{1}{\Psi_r(x)} \int \Psi_r(x) u \, dy \). Provided \( \Psi_r(x) \subset Z \), we have

\[
\left( \int_{\Delta_r(x)} |u - \tilde{u}|^q \, d\sigma \right)^{1/q} \leq C \left( \int_{\Psi_r(x)} |\nabla u|^p \, dy \right)^{1/p}. \tag{3.12}
\]

In the inequality (3.12), \( p \) and \( q \) are related by \( 1/q = 1/p - (1 - 1/p)/(n - 1) \) and \( p > 1 \).

**Lemma 3.13** Let \( \Omega, N \) and \( D \) be a standard domain for the mixed problem. Suppose that (2.7) holds, let \( x \in \Omega \) and \( 0 < r < r_0 \). Let \( u \) be a weak solution of the mixed problem for a divergence form elliptic operator with zero Dirichlet data and Neumann data \( f_N \). We have the estimate

\[
\left( \int_{\Psi_r(x)} |\nabla u|^2 \, dy \right)^{1/2} \leq C \left[ \int_{\Psi_r(x)} |\nabla u|^2 \, dy + \left( \frac{1}{r^{n-1}} \int_{N \cap \Delta_r(x)} |f_N|^p \, d\sigma \right)^{1/p} \right].
\]

Here, \( p = 2 \) if \( n = 2 \) and \( p = 2(n - 1)/(n - 2) \) for \( n \geq 3 \). The constant \( C \) depends only on \( M \) and the dimension \( n \).

**Proof.** Changing variables to flatten the boundary of a Lipschitz domain preserves the class of elliptic operators with bounded measurable coefficients, thus it suffices to consider the case where the ball \( \Delta_r(x) \) lies in a hyperplane. We may rescale to set \( r = 1 \). We claim that we can find an exponent \( a \) so that for \( s \) and \( t \) which satisfy \( 1/2 \leq s < t \leq 1 \), we have

\[
\left( \int_{\Psi_t(x)} |\nabla u|^2 \, dy \right)^{1/2} \leq \frac{C}{(t - s)^a} \left( \int_{\Psi_s(x)} |\nabla u|^q \, dy \right)^{1/q} + \left( \int_{N \cap \Delta_1(x)} |f_N|^p \, d\sigma \right)^{1/p} \tag{3.14}
\]

where we may choose the exponents \( p = 2(n - 1)/(n - 2) \) and \( q = 2n/(2n + 2) \) if \( n \geq 3 \) or \( p = 2 \) and \( q = 4/3 \) if \( n = 2 \).

We give the details when \( n \geq 3 \). In the argument that follows, let \( \epsilon = (t - s)/2 \) and choose \( \eta \) to be a cut-off function which is one on \( B_{s}(x) \), supported
in $B_{s+\epsilon}(x)$ and satisfies $|\nabla \eta| \leq C/\epsilon$. We let $v = \eta^2(u - E)$ where $E$ is a constant. If we choose $E$ so that $v \in W^{1,2}_D(\Omega)$, the weak formulation of the mixed problem and Hölder’s inequality gives for $1 < p < \infty$

$$
\int_{\Omega} |\nabla u|^2 \eta^2 \, dy \leq C \left[ \int_{\Omega} |u - E|^2 |\nabla \eta|^2 \, dy + \left( \int_{(N \cap \Delta_{s+\epsilon}(x))} |f_N|^p \, d\sigma \right)^{2/p'} \right].
$$

(3.15)

We consider two cases: a) $B_{s+\epsilon}(x) \cap D = \emptyset$ and b) $B_{s+\epsilon}(x) \cap D \neq \emptyset$. In case a) we may choose $E = \bar{u} = \frac{1}{\Psi(x)} \int_{\Psi(x)} u \, dy$. We use the Poincaré-Sobolev inequality (3.11) and the inequality (3.12) to estimate the first two terms on the right of (3.15) and conclude that

$$
\int_{\Psi(x)} |\nabla u|^2 \, dy \leq C \left[ \int_{\Psi(x)} \frac{1}{(t-s)^2} \left( \int_{\Psi(x)} |\nabla u|^{\frac{2p}{n+2}} \, dy \right)^{2(np-n+1)} \, dy + \left( \int_{(N \cap \Delta_{1}(x))} |f_N|^p \, d\sigma \right)^{\frac{1}{p}} \right].
$$

If $n \geq 3$, we may choose $p = 2(n-1)/(n-2)$ and then we have that $np/(np - n + 1) = 2n/(n + 2)$ to obtain the claim.

We now turn to case b). Since $B_{s+\epsilon}(x)$ meets the set $D$, we cannot subtract a constant from $u$ and remain in the space of test functions, $W^{1,2}_D(\Omega)$. Thus, we let $E = 0$ in (3.15). We let $\bar{u}$ be the average value of $u$ on $\Psi_{s+\epsilon}(x)$ and obtain

$$
\int_{\Psi_{s+\epsilon}(x)} u^2 |\nabla \eta|^2 \, dy \leq C \epsilon^2 \left[ \int_{\Psi_{s+\epsilon}(x)} |u - \bar{u}|^2 \, dy + \bar{u}^2 \right].
$$

Since $B_{s+\epsilon}(x) \cap D \neq \emptyset$, our assumption (2.1) on the set $D$ implies that we may find a point $\tilde{x} \in \Lambda$ so that $B_{\epsilon}(\tilde{x}) \subset B_{t}(x)$ and so that $\sigma(B_{\epsilon}(\tilde{x}) \cap D) \geq c \epsilon^{n-1}$. As $c$ depends on $M$ our final constant may be taken to depend on $M$. Using (3.11) and the Poincaré inequality of Lemma 3.9 we conclude that

$$
\int_{\Psi_{s+\epsilon}(x)} u^2 |\nabla \eta|^2 \, dy \leq C \epsilon^2 \left[ \int_{\Psi_{s+\epsilon}(x)} |\nabla u|^{2n/(n+2)} \, dy \right]^{(n+2)/n}
$$

$$
+ \frac{1}{\epsilon^{2n/q}} \left( \int_{\Psi_{s+\epsilon}(x)} |\nabla u|^q \, dy \right)^{2/q}
$$

(3.16)
for $1 < q < n$. A similar argument using (3.12) and Lemma 3.9 gives us

$$\left( \int_{\Delta_{s+2\epsilon}(x)} |u|^{p'} d\sigma \right)^{1/p'} \leq \left( \int_{\Delta_{s+2\epsilon}(x)} |u - \bar{u}|^{p'} d\sigma \right)^{1/p'} + |\bar{u}|$$

$$\leq C \left[ \left( \int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^{np/(np-n+1)} dy \right)^{(np-n+1)/(np)} \right]^{(np-n+1)/(np)}$$

$$+ \epsilon^{1-n/q} \left( \int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^q dy \right)^{1/q} \right]$$

(3.17)

where the use of Lemma 3.9 requires that we have $1 < q < n$. We use (3.16) and (3.17) in (3.15) and choose $q = 2n/(n+2)$ and $p = 2(n-2)/(n-1)$ if $n \geq 3$. Once we recall that $t - s = 2\epsilon$, we obtain (3.14).

Finally, we may use the techniques given in [16, pp. 80-82] or [13, pp. 1004–1005] to see that the claim (3.14) implies the estimate

$$\left( \int_{\Psi_{1/2}(x)} |\nabla u|^2 dy \right)^{1/2} \leq C \left[ \int_{\Psi_{1}(x)} |\nabla u| dy + \left( \int_{N \cap \Delta_1(x)} |f_N|^p d\sigma \right)^{1/p} \right]$$

(3.18)

with $p$ as in (3.14).

When the dimension $n = 2$, the exponent $2n/(n+2)$ is 1 and it is not clear that we have (3.1) as used to obtain (3.10). However, from (3.11) and Hölder’s inequality we can show

$$\left( \int_{\Psi_{s+2\epsilon}(x)} |u - \bar{u}|^2 dy \right)^{1/2} \leq C \left( \int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^{4/3} dy \right)^{3/4}$$

This may be substituted for (3.11) in the above argument to obtain (3.14) when $n = 2$.

**Lemma 3.19** Let $\Omega$, $D$ and $N$ be a standard domain for the mixed problem. Let $x \in \Omega$ and suppose that $r$ satisfies $0 < r < r_0$. Let $u$ be a weak solution of the mixed problem (3.1) with zero Dirichlet data and Neumann data $f$ in $L^p(N)$ which is supported in $N \cap \Delta_r(x)$. There exists $p_0 = p_0(n, M) > 2$ so that for $t$ in $[2, p_0]$ if $n \geq 3$ or $t$ in $(2, p_0)$ if $n = 2$, we have the estimate

$$\left( \int_{\Psi_{r}(x)} |\nabla u|^t dy \right)^{1/t} \leq C \left[ \int_{\Psi_{2r}(x)} |\nabla u| dy + \left( \frac{1}{r^{n-1}} \int_{\Delta_{2r}(x) \cap N} |f|^t(n-1)/n d\sigma \right)^{n/(t(n-1))} \right]$$

The constant in this estimate depends on $t$, $M$ and $n$. 

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Proof. According to Lemma 3.13, \( \nabla u \) satisfies a reverse H"older inequality and thus we may apply a result of Giaquinta [15, p. 122] to conclude that there exists \( p_0 > 2 \) so that we have

\[
\left( \int_{\Psi_r(x)} |\nabla u|^t \, dy \right)^{1/t} \leq C \left[ \int_{\Psi_{2r}(x)} |\nabla u| \, dy + \left( \int_{\Psi_{2r}(x)} (P_{2r}|f|^p)^{t/p} \, dy \right)^{1/t} \right]
\]

for \( t \in [2, p_0) \) and \( p \) as in Lemma 3.13. From this, we may use Lemma 3.3 to obtain

\[
\left( \int_{\Psi_r(x)} |\nabla u|^t \, dy \right)^{1/t} \leq C \left[ \int_{\Psi_{2r}(x)} |\nabla u| \, dy + \left( \int_{\Delta_{4r}(x)} |f|^{\frac{(n-1)/n}{d\sigma}} \, d\sigma \right) \right]^{n/t(n-1)}
\]

when \( n \geq 3 \) and \( t \) is in \( [2, p_0) \). If \( n = 2 \) we need \( t > 2 \) so that \( f \) is raised to a power larger than 1. Now a simple argument that involves covering \( \Delta_r(x) \) by surface balls of radius \( r/4 \) allows us to conclude the estimate of the Lemma.

\[
\textbf{4 Estimates for solutions with atomic data}
\]

We establish an estimate for the solution of the mixed problem when the Neumann data is an atom for \( H^1 \) and the Dirichlet data is zero. The key step is to establish decay of the solution as we move away from the support of the atom. We will measure the decay by taking \( L^q \)-norms in dyadic rings around the support of the atom. Thus, given a surface ball \( \Delta_r(x), x \in \partial\Omega \), we define \( \Sigma_k = \Delta_{2^kr}(x) \setminus \Delta_{2^{k-1}r}(x) \) and define \( S_k = \Psi_{2^kr}(x) \setminus \Psi_{2^{k-1}r}(x) \).

Theorem 4.1 Let \( \Omega, N \) and \( D \) be a standard domain for the mixed problem. Let \( u \) be a weak solution of the mixed problem with Neumann data \( a \) which is an atom which is supported in \( N \cap \Delta_r(x) \) and zero Dirichlet data.

If \( p_0 \) is as in Lemma 3.19 and \( 1 < q < p_0/2 \), then we have \( \nabla u \in L^q(\partial\Omega), \)

\[
\left( \int_{\Delta_r(x)} |\nabla u|^q \, d\sigma \right)^{1/q} \leq C \sigma(\Delta_{8r}(x))^{-1/q'}
\]

(4.2)

and for \( k \geq 4, \)

\[
\left( \int_{\Sigma_k} |\nabla u|^q \, d\sigma \right)^{1/q} \leq C 2^{-\beta k} \sigma(\Sigma_k)^{-1/q'}
\]

(4.3)

Here, \( \beta \) is as in Lemma 4.9 and the constant \( C \) in the estimates (4.2) and (4.3) depends on \( q \) and the global character of the domain.

If \( r < r_0 \) and \( x \) is in \( \partial\Omega \), then we may construct a star-shaped Lipschitz domain \( \Omega_r(x) = Z_r(x) \cap \Omega \) where \( Z_r(x) \) is the coordinate cylinder defined above. Given a function \( v \) defined in \( \Omega, x \in \partial\Omega, \) and \( r > 0, \) we define a truncated non-tangential maximal function \( v_r^* \) by

\[
v_r^*(x) = \sup_{y \in \Gamma(x) \cap B_r(x)} |v(y)|.
\]
Lemma 4.4 Let Ω be a Lipschitz domain. Suppose that \( x \in \partial \Omega \) and \( 0 < r < r_0 \). Let \( u \) be a harmonic function in \( \Omega_r(x) \). If \( \nabla u \in L^2(\Omega_r(x)) \) and \( \partial u / \partial \nu \) is in \( L^2(\partial \Omega \cap \partial \Omega_{4r}(x)) \), then we have \( \nabla u \in L^2(\Delta_r(x)) \) and

\[
\int_{\Delta_r(x)} ((\nabla u)^*)^2 d\sigma \leq C \left( \int_{\partial \Omega \cap \partial \Omega_{4r}(x)} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma + \frac{1}{r} \int_{\Omega_{4r}(x)} |\nabla u|^2 dy \right).
\]

The constant \( C \) depends only on the dimension \( n \) and \( M \).

Proof. Since the estimate only involves \( \nabla u \), we may subtract a constant from \( u \) so that \( \int_{\Omega_r(x)} u dy = 0 \). We pick a smooth cut-off function \( \eta \) which is one on \( Z_{3r}(x) \) and zero outside \( Z_{4r}(x) \). Since we assume that \( \nabla u \) is in \( L^2(\Omega_{4r}(x)) \), it follows that \( \Delta(\eta u) = u \Delta \eta + 2 \nabla u \cdot \nabla \eta \) is in \( L^2(\Omega_{4r}(x)) \). Thus, with \( \Xi \) the usual fundamental solution of the Laplacian, \( w = \Xi * (\Delta(\eta u)) \) will be in the Sobolev space \( W^{2,2}(\mathbb{R}^n) \). We have defined \( \Delta(\eta u) \) to be zero outside \( \Omega_{4r}(x) \) in order to make sense of the convolution in the definition of \( w \). Next, we let \( v \) be the solution of the Neumann problem

\[
\begin{align*}
\Delta v &= 0, & & \text{in } \Omega_{4r}(x) \\
\frac{\partial v}{\partial \nu} &= \frac{\partial (\eta u)}{\partial \nu} - \frac{\partial w}{\partial \nu}, & & \text{on } \partial \Omega_{4r}(x).
\end{align*}
\]

According to Jerison and Kenig [19], the solution \( v \) will have non-tangential maximal function in \( L^2(\partial \Omega_{4r}(x)) \). By uniqueness of weak solutions to the Neumann problem, we may add a constant to \( v \) so that we have \( \eta u = v + w \). As \( w \) and all its derivatives are bounded in \( \Omega_{2r}(x) \) and the non-tangential maximal function of \( \nabla v \) is in \( L^2(\partial \Omega_{4r}(x)) \), we obtain the Lemma.

The proof of the following Lemma for the regularity problem is identical to the proof of Lemma 4.4.

Lemma 4.5 Let \( \Omega \) be a Lipschitz domain. Suppose that \( x \in \partial \Omega \) and \( 0 < r < r_0 \). Let \( u \) be a harmonic function in \( \Omega_r(x) \). If \( \nabla u \in L^2(\Omega_{4r}(x)) \) and \( \nabla u \) is in \( L^2(\partial \Omega \cap \partial \Omega_{4r}(x)) \), then we have \( \nabla u \in L^2(\Delta_r(x)) \) and

\[
\int_{\Delta_r(x)} ((\nabla u)^*)^2 d\sigma \leq C \left( \int_{\partial \Omega \cap \partial \Omega_{4r}(x)} |\nabla u|^2 d\sigma + \frac{1}{r} \int_{\Omega_{4r}(x)} |\nabla u|^2 dy \right).
\]

The constant \( C \) depends only on the dimension \( n \) and \( M \).

The following weighted estimate will be an intermediate step towards our estimates for solutions with atomic data. In the next lemma, \( \Omega \) is a bounded Lipschitz domain and the boundary is written \( \partial \Omega = D \cup N \). Recall that \( \delta(x) \) denotes the distance from \( x \) to the set \( \Lambda \).
Lemma 4.6 Let \( \Omega, D \) and \( N \) be a standard domain for the mixed problem.

Let \( u \) be a weak solution of the mixed problem (3.1) with Neumann data \( f_N \in L^2(N) \) and zero Dirichlet data.

Let \( \epsilon \in \mathbb{R}, x \in \partial \Omega \) and \( 0 < r < r_0 \) and assume that for some \( A > 0 \), \( \delta(x) \leq Ar \). Then we have

\[
\int_{\Delta_r(x)} ((\nabla u)_{\partial \delta})^2 \delta^{1-\epsilon} \, d\sigma \leq C \left( \int_{\Delta_{2r}(x)} |f_N|^2 \delta^{1-\epsilon} \, d\sigma + \int_{\Psi_{2r}(x)} |\nabla u|^2 \delta^{-\epsilon} \, dy \right).
\]

The constant in this estimate depends on \( M, n, \epsilon \) and \( A \).

Proof. We may assume that \( \Psi_{2r}(x) \) is contained in a coordinate cylinder \( Z_{2r_0} \).

If \( Z_{100r_0} \cap \partial \Omega \subset N \) or \( Z_{100r_0} \cap \partial \Omega \subset D \), then the estimate of the Lemma follows easily from Lemma 4.4 or Lemma 4.5 since we have that \( \delta(y) \) is equivalent to \( r \) for \( y \in \Psi_{2r}(x) \). This equivalency follows from our assumption that \( \delta(x) < Ar \) and that \( Z_{100r_0} \) does not intersect \( \Lambda \).

If \( Z_{100r_0} \) meets both \( D \) and \( N \), we begin by finding a decomposition of \((\partial \Omega \cap Z_{4r_0}) \setminus \Lambda\) into non-overlapping surface cubes \( \{ Q_j \} \) which satisfy: 1) For each cube \( Q_j \), we have constants \( c'' \) and \( c' \) so that \( c'' \delta(y) \leq \text{diam}(Q_j) \leq c' \delta(y) \) for \( y \in Q_j \). The constant \( c' \) may be chosen as small as we like. 2) We let \( T(Q) = \{ y \in \Omega : \text{dist}(y, Q) < \text{diam} Q \} \). Then the family \( \{ T(2Q_j) \} \) has bounded overlaps and thus

\[
\sum \chi_{T(2Q_j)} \leq C(n, M, c'').
\]

To construct the family of surface cubes, begin with a Whitney decomposition of \( \mathbb{R}^{n-1} \setminus \{ (x'', x') : x'' \in \mathbb{R}^{n-2} \} \) and then map the cubes onto the boundary with the map \( x' \to (x', \phi(x')) \). Here, \( \phi \) and \( \psi \) are the functions used to describe \( \partial \Omega \) and \( \Lambda \) in the coordinate cylinder \( Z_{r_0} \).

As the surface cubes \( Q_j \) are connected and \( \delta \) never vanishes on \( Q_j \), we have that either \( Q_j \subset N \) or that \( Q_j \subset D \). We choose the constant \( c' \) small so that \( Q_j \cap \Delta_r(x) \neq \emptyset \) implies that \( T(2Q_j) \subset \Psi_{2r}(x) \). Let \( r_j \) be the diameter of the cube \( r_j \). Applying Lemma 4.4 or Lemma 4.5 we conclude that

\[
\int_{Q_j} |\nabla u|^2 \, d\sigma \leq C \left( \int_{2Q_j \cap N} \frac{|\partial u|}{\partial \nu}^2 \, d\sigma + \frac{1}{r_j} \int_{T(2Q_j)} |\nabla u|^2 \, dy \right). \tag{4.7}
\]

We multiply equation (4.7) by \( r_j^{1-\epsilon} \), choose \( c' \) small so that \( r_j \) is equivalent to \( \delta(y) \) in \( T(2Q_j) \) and obtain

\[
\int_{Q_j} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma \leq C \left( \int_{2Q_j \cap N} \frac{|\partial u|}{\partial \nu}^2 \delta^{1-\epsilon} \, d\sigma + \int_{T(2Q_j)} |\nabla u|^2 \delta^{-\epsilon} \, dy \right). \tag{4.8}
\]

We sum over \( j \) such that \( Q_j \cap \Delta_r(x) \neq \emptyset \) and use that the family \( \{ T(2Q_j) \} \) has bounded overlaps to obtain the Lemma.
An important part of the proof of our estimate for the mixed problem is to show that a solution with Neumann data an atom will decay as we move away from the support of the atom. This decay is encoded in estimates for the Green function for the mixed problem. These estimates rely in large part on the work of de Giorgi [11], Moser [32] and Nash [33], on Hölder continuity of weak solutions of elliptic equations with bounded and measurable coefficients, and the work of Littman, Stampacchia and Weinberger [25] who constructed the fundamental solution of such operators. Also, see Kenig and Ni [21] for the construction of a global fundamental solution in two dimensions. Given the free space fundamental solution, the Green function may be constructed by reflection in a manner similar to the construction given for graph domains in [24]. A similar argument was used by Dahlberg and Kenig [10] and by Kenig and Pipher [22] in their studies of the Neumann problem. Once we have a Green function which satisfies the correct boundary conditions in a coordinate cylinder, we may solve a weak version of the mixed problem to obtain a Green function in all of Ω.

**Lemma 4.9** Let Ω, N and D be a standard domain for the mixed problem. There exists a Green function $G(x, y)$ for the mixed problem which satisfies: 1) If $G_x(y) = G(x, y)$, then $G_x$ is in $W^{1, 2}_D(Ω \setminus B_r(x))$ for all $r > 0$, 2) $ΔG_x = δ_x$, the Dirac δ-measure at x, 3) If $f_N$ lies in $W^{-1/2, 2}_D(∂Ω)$, then the solution of the mixed problem with $f_D = 0$ can be represented by

$$u(x) = -⟨f_N, G_x⟩_{∂Ω},$$

4) The Green function is Hölder continuous away from the pole and satisfies the estimates

$$|G(x, y) - G(x, y')| \leq \frac{C|y - y'|^β}{|x - y|^{n-2+β}}, \quad |x - y| > 2|y - y'|,$$

$$|G(x, y)| \leq \frac{C}{|x - y|^{n-2}}, \quad n \geq 3,$$

and with $d = \text{diam}(Ω)$,

$$|G(x, y)| \leq C(1 + \log(d/|x - y|)), \quad n = 2.$$

**Lemma 4.10** Let $u$ be a weak solution of the mixed problem (3.1) with Neumann data $f$ in $L^p(N)$ where $p = (2n - 2)/n$ for $n \geq 3$. Then we have the estimate

$$\int_Ω |∇u|^2 dy \leq C\|f\|^2_{L^p(N)}.$$

If $n = 2$, we have

$$\int_Ω |∇u|^2 dy \leq C\|f\|^2_{H^1(N)}.$$

In each case, the constant $C$ depends on $Ω$ and the constant in (3.2).
Proof. When \( n \geq 3 \), we use that \( W^{1/2,2}_D(\partial \Omega) \subset L^{2(n-1)/(n-2)}(\partial \Omega) \). By duality, we see that \( L^{2(n-1)/n}(\partial \Omega) \subset W^{-1/2,2}_D(\partial \Omega) \) and since the weak solution of the mixed problem satisfies

\[
\int_{\Omega} |\nabla u|^2 \, dy \leq C\|f\|_{W^{-1/2,2}_D(\partial \Omega)}^2
\]

the Lemma follows.

When \( n = 2 \), the proof above fails since we do not have \( W^{1/2,2}_D(\partial \Omega) \subset L^{\infty}(\partial \Omega) \). However, we do have the embedding \( W^{1/2,2}_D(\partial \Omega) \subset BMO(\partial \Omega) \). Since \( \phi \in W^{1/2,2}_D(\partial \Omega) \) vanishes on \( D \) and \( f \in H^1(N) \) has an extension \( \tilde{f} \) which lies in \( H^1(\partial \Omega) \), we obtain the result for \( n = 2 \).

Finally, we give a technical lemma that will be used below.

**Lemma 4.11** Let \( \Omega, N \) and \( D \) be a standard domain for the mixed problem and suppose that \( 0 < r < r_0, x \in \partial \Omega \) and \( \delta(x) > r\sqrt{1 + M^2} \). Then we have \( \Delta_r(x) \subset N \) or \( \Delta_r(x) \subset D \).

**Proof.** We fix \( y \in \Delta_r(x) \). Since \( r < r_0 \), we may find a coordinate cylinder \( Z \) which contains \( \Delta_r(x) \). We let \( \phi \) be the function whose graph gives \( \partial \Omega \) near \( Z \). Since \( y \in \Delta_r(x) \), we have \( |x' - y'| < r \). We let \( x'(t) = (1 - t)x' + ty' \) and then \( \gamma(t) = (x'(t), \phi(x'(t))) \) gives a path in \( \partial \Omega \) joining \( x \) to \( y \) and of length at most \( r\sqrt{1 + M^2} \). Since \( \delta(x) > r\sqrt{1 + M^2} \) and \( \delta \) is Lipschitz with constant one, we have that \( \delta(\gamma(t)) > 0 \) for \( 0 \leq t \leq 1 \). Since \( \gamma(t) \) does not pass through \( \Lambda \) we must have \( x \) and \( y \) both lie in \( D \) or both lie in \( N \). As \( y \) is an arbitrary point in \( \Delta_r(x) \), it follows that \( \Delta_r(x) \) lies entirely in \( D \) or entirely in \( N \).

**Proof of 4.11.** It suffices to restrict attention to atoms which are supported in a surface ball \( \Delta_r(x) \), with \( x \in \partial \Omega \) and \( 0 < r < r_0 \) since an atom which is supported in a larger surface ball can be sub-divided into a finite number of atoms which are supported in balls of the form \( \Delta_{r_0}(x) \). The increase in the constant due to this step will depend on the global character of the domain.

Thus, we fix an atom \( a \) that is supported in the set \( \Delta_r(x) \cap N \) and begin the proof of (4.2). We consider two cases: a) \( \delta(x) \leq 16r\sqrt{1 + M^2} \), and b) \( \delta(x) > 16r\sqrt{1 + M^2} \).

In case a) we fix \( q \) between 1 and 2 and use Hölder’s inequality with exponents \( 2/q \) and \( 2/(2 - q) \) to find

\[
\left( \int_{\Delta_{2r}(x)} |\nabla u|^q \, d\sigma \right)^{1/q} \leq \left( \int_{\Delta_{2r}(x)} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma \right)^{1/2} \left( \int_{\Delta_{2r}(x)} \delta^{\frac{q(\epsilon - 1)}{2-q}} \, d\sigma \right)^{\frac{2-q}{2\epsilon}}
\]

\[
\leq C r^{(n-1)(\frac{1}{q} - \frac{1}{2}) + \frac{\epsilon - 1}{2}} \left( \int_{\Delta_{2r}(x)} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma \right)^{1/2}.
\]
The second inequality requires that $q$ and $\epsilon$ satisfy $q(\epsilon - 1)/(2 - q) > -1$ or $q < 1/(1 - \epsilon/2)$. Next, we use Lemma 4.6 and our assumption that $\delta(x) \leq 16r\sqrt{1 + M^2}$ to bound the weighted $L^2(\delta^{1-\epsilon}d\sigma)$ norm of $\nabla u$. This gives us

$$\left( \int_{\Delta_{8r}(x)} |\nabla u|^q d\sigma \right)^{1/q} \leq C \left[ \left( \int_{\Delta_{r}(x) \cap N} |a|^2 \delta^{-\epsilon} d\sigma \right)^{1/2} + \left( \int_{\Psi_{16r}(x)} |\nabla u|^2 \delta^{-\epsilon} dy \right)^{1/2} \right] r^{(n-1)(\frac{1}{q} - \frac{1}{2}) + \frac{1}{2}}.$$ 

We estimate the integral over $\Psi_{16r}(x)$ in this last expression with Hölder’s inequality and obtain

$$\left( \int_{\Psi_{16r}(x)} |\nabla u|^2 \delta^{-\epsilon} dy \right)^{1/2} \leq C \left( \int_{\Psi_{16r}(x)} |\nabla u|^p dy \right)^{1/p} \left( \int_{\Psi_{16r}(x)} \delta^{-\epsilon p/(p-2)} dy \right)^{1/2 - 1/p} \leq C r^{\frac{n(1 - \frac{1}{p}) - \epsilon}{2}} \left( \int_{\Psi_{16r}(x)} |\nabla u|^p dy \right)^{1/p}.$$

The second inequality depends on our assumption on $\Lambda$ and holds when $\epsilon p/(p-2) < 2$ or $p > 2/(1 - \epsilon/2)$. Now we may use the three previous displayed equations and Lemma 3.19 to obtain

$$\left( \frac{1}{r^{n-1}} \int_{\Delta_{8r}(x)} |\nabla u|^q d\sigma \right)^{1/q} \leq C \left[ \left( \frac{1}{r^n} \int_{\Delta_{32r}(x)} |\nabla u|^2 dy \right)^{1/2} + r^{1-n} \right].$$

In this last step, we have used the normalization of $a$, $\|a\|_{L^\infty} \leq 1/\sigma(\Delta_{r}(x))$ to estimate the term involving the Neumann data from Lemma 3.19. Finally, we may use the Lemma 4.10 and the normalization of the atom to obtain that $(r^{-n} \int_{\Omega} |\nabla u|^2 dy)^{1/2} \leq C r^{1-n}$ which gives the estimate (4.12).

In case b), we use Lemma 4.11 to conclude that $\Delta_{16r}(x) \subset N$. Next, we use Hölder’s inequality, that $a$ is supported in $\Delta_{r}(x)$ and Lemma 4.4 to obtain

$$(\frac{1}{r^{n-1}} \int_{\Delta_{8r}(x)} |\nabla u|^q d\sigma)^{1/q} \leq C \left[ \left( \frac{1}{r^n} \int_{\Delta_{r}(x) \cap N} |a|^2 d\sigma \right)^{1/2} + \left( \frac{1}{r^n} \int_{\Psi_{16r}(x)} |\nabla u|^2 dy \right)^{1/2} \right].$$
Using the normalization of the atom $a$ and Lemma 4.10, the right-hand side of (4.12) may be estimated by $\sigma(\Delta_{R}(x))^{-1}$ and we obtain (4.2) in this case.

Now we turn our attention to the proof of the estimate (4.3). Our first step is to observe that the solution $u$ satisfies the estimate

$$|u(y)| \leq \frac{C_{r}}{|x-y|^{n-2\beta}}, \quad |y-x| > 2r. \quad (4.13)$$

To establish (4.13), we begin with the representation formula in part 3) of Lemma 4.9 and claim that we may find $\bar{x}$ in $\Delta_{r}(x)$ so that

$$u(y) = -\int_{\Delta_{r}(x) \cap N} a(z)(G(y,z) - G(y,\bar{x})) \, d\sigma.$$

If $\Delta_{r}(x) \subset N$, then we may let $\bar{x} = x$ and use that $a$ has mean value zero to obtain the above representation. If $\Delta_{r}(x) \cap D \neq \emptyset$, then we choose $\bar{x} \in D \cap \Delta_{r}(x)$ and use that $G(y,\cdot)$ vanishes on $D$. Now the estimate (4.13) follows easily from the normalization of the atom and the estimates for the Green function in part 4) of Lemma 4.9.

We will consider three cases in the proof of (4.3): a) $2^{k}r < r_{0}$ and $\delta(x) \leq 2 \cdot 2^{k}r \sqrt{1 + M^{2}}$, b) $2^{k}r < r_{0}$ and $\delta(x) > 2 \cdot 2^{k}r \sqrt{1 + M^{2}}$, c) $2^{k}r \geq r_{0}$. The details are similar to the proof of (4.2), thus we will be brief.

We begin with case a) and use Hölder’s inequality with exponents $2/q$ and $2/(2 - q)$ to obtain

$$\left( \int_{\Sigma_{k}} |\nabla u|^{q} \, d\sigma \right)^{1/q} \leq C \left( \int_{\Sigma_{k}} |\nabla u|^{2} \delta^{1-\epsilon} \, d\sigma \right)^{1/2} \left( 2^{k}r \right)^{(n-1)(\frac{1}{q} - \frac{1}{2}) + \frac{\epsilon}{2}}.$$

As in the proof of the estimate (4.2), this requires that $1 < q < 1/(1 - \epsilon/2)$. From Lemma 4.6 we have

$$\left( \int_{\Sigma_{k}} |\nabla u|^{2} \delta^{1-\epsilon} \, d\sigma \right)^{1/2} \leq C \left( \sum_{j=k-1}^{k+1} \int_{S_{j}} |\nabla u|^{2} \delta^{-\epsilon} \, dy \right)^{1/2}.$$

This estimate requires $k \geq 2$ so that $\Sigma_{k-1} \cap \Delta_{r}(x) = \emptyset$. Then Hölder and the reverse Hölder estimate in Lemma 3.19 gives

$$\left( \int_{S_{k}} |\nabla u|^{2} \delta^{-\epsilon} \, dy \right)^{1/2} \leq C \left( \int_{S_{k}} |\nabla u|^{p} \, dy \right)^{1/2} \leq C \left( 2^{k}r \right)^{-n/2} \left( \sum_{j=k-1}^{k+1} \int_{S_{j}} |\nabla u|^{2} \, dy \right)^{1/2}.$$

Here we need $k \geq 2$ so that $\Delta_{r}(x) \cap \Sigma_{k-1} = \emptyset$ and the term in involving the Neumann data in Lemma 3.19 vanishes. Finally, from Caccioppoli and our estimate (4.13) for $u$, we obtain that

$$\left( \int_{S_{k}} |\nabla u|^{2} \, dy \right)^{1/2} \leq \frac{C}{2^{k}r} \left( \sum_{j=k-1}^{k+1} \int_{S_{j}} |u|^{2} \, dy \right)^{1/2} \leq C 2^{-k\beta} (2^{k}r)^{1-n/2}.$$
Again, we need $k \geq 2$ so that the data for the mixed problem is zero when we apply Caccioppoli’s inequality. Combining the four previous estimates gives

\[
\left( \int_{\Sigma_k} |\nabla u|^q \, d\sigma \right)^{1/q} \leq C 2^{-k\beta} \sigma(\Sigma_k)^{-1/q'}
\]

for $k \geq 4$. We need $k \geq 4$ in order to fatten up the set $\Sigma_k$ three times: once to apply Lemma 4.6, once to apply Lemma 4.19 and once to apply Caccioppoli’s inequality.

Now we consider case b). Since $\delta(x) \geq 2(2^k r)\sqrt{1 + M^2}$, we have $\Delta_{2^k r}(x) \subset N$ by Lemma 4.11. Hence, we may use Lemma 4.4, Caccioppoli’s inequality and (4.13) to obtain (4.3).

Finally, we consider case c) where $2^k r > r_0$. We recall that we have a covering of $\partial\Omega$ by coordinate cylinders. In each coordinate cylinder, we may use Lemma 4.4, Lemma 4.5 or Lemma 4.6 and the techniques given above to obtain

\[
\left( \int_{Z_{r_0} \cap \partial\Omega} |\nabla u|^q \, d\sigma \right)^{1/q} \leq C r_0^{(1-n)/q'}.
\]

Adding these estimates gives (4.3) with a constant that depends on the global character of the domain.

We now show that the non-tangential maximal function of our weak solutions lies in $L^1$ when the Neumann data is an atom.

**Theorem 4.14** Let $\Omega$, $N$ and $D$ be a standard domain for the mixed problem. If $f_N$ is in $H^1(N)$, then there exists $u$ a solution of the $L^1$-mixed problem (1.1) with Neumann data $f_N$ and zero Dirichlet data and this solution satisfies

\[
\| (\nabla u)^* \|_{L^1(\partial\Omega)} \leq C \| f_N \|_{H^1(N)}.
\]

The constant $C$ in this estimate depends on the global character of $\Omega$, $N$ and $D$.

**Proof.** We begin by considering the case when $f_N$ is an atom and we let $u$ be the weak solution of the mixed problem with Neumann data an atom $a$ and zero Dirichlet data. The result for data in $H^1(N)$ follows easily from the result for an atom.

We establish a representation for the gradient of $u$ in terms of the boundary values of $u$. Let $x \in \Omega$ and $j$ be an index ranging from 1 to $n$. We claim

\[
\frac{\partial u}{\partial x_j}(x) = \int_{\partial\Omega} \sum_{i=1}^n \frac{\partial \Xi}{\partial y_i}(x - \cdot)(\nu_i \frac{\partial u}{\partial y_j} - \frac{\partial u}{\partial y_i} \nu_j) + \frac{\partial \Xi}{\partial y_j}(x - \cdot) \frac{\partial u}{\partial \nu} \, d\sigma.
\]

(4.15)
If $u$ is smooth up to the boundary, the proof of (4.15) is a straightforward application of the divergence theorem. However, it takes a bit more work to establish this result when we only have that $u$ is a weak solution.

Thus, we suppose that $\eta$ is a smooth function which is zero in a neighborhood of $\Lambda$ and supported in a coordinate cylinder. Using the coordinate system for our coordinate cylinder, we set $u_\tau(y) = u(y + \tau e_n)$ where $e_n$ is the unit vector the $x_n$ direction and $\tau > 0$. Applying the divergence theorem gives

$$
\int_{\partial \Omega} \eta \frac{\partial \Xi}{\partial \nu}(x - \cdot) \frac{\partial u_\tau}{\partial y_j} - \nabla \Xi(x - \cdot) \cdot \nabla u_\tau \nu_j + \frac{\partial \Xi}{\partial y_j}(x - \cdot) \frac{\partial u_\tau}{\partial \nu} \, d\sigma
= \eta(x) \frac{\partial u_\tau}{\partial x_j}(x) + \int_{\Omega} \nabla \eta \cdot \nabla \Xi(x - \cdot) \frac{\partial u_\tau}{\partial y_j} \\
- \nabla_y \Xi(x - \cdot) \cdot \nabla u_\tau \frac{\partial \eta}{\partial y_j} \\
+ \frac{\partial \Xi}{\partial y_j}(x - \cdot) \nabla u_\tau \cdot \nabla \eta \, dy.
$$

(4.16)

Thanks to the truncated maximal function estimate in Lemma 4.6, we may let $\tau$ tend to zero from above and conclude that the same identity holds with $u_\tau$ replaced by $u$. Next, we suppose that $\eta$ is of the form $\eta \phi_\varepsilon$ where $\phi_\varepsilon = 0$ on $\{x : \delta(x) < \varepsilon\}$, $\phi_\varepsilon = 1$ on $\{x : \delta(x) > 2\varepsilon\}$ and we have the estimate $|\nabla \phi_\varepsilon(x)| \leq C/\varepsilon$. Since we assume the boundary between $D$ and $N$ is a Lipschitz surface, we have the following estimate for $\varepsilon$ sufficiently small

$$
|\{x : \delta(x) \leq 2\varepsilon\}| \leq C\varepsilon^2.
$$

(4.17)

Using our estimate for $\nabla \phi_\varepsilon$ and the inequality (4.17), we have

$$
| \int_{\Omega} \eta \nabla \phi_\varepsilon \cdot \nabla \Xi(x - \cdot) \frac{\partial u}{\partial y_j} \, dy | \leq C \left( \int_{\{y : \delta(y) < 2\varepsilon\}} |\nabla u|^2 \, dy \right)^{1/2}
$$

and the last term tends to zero with $\varepsilon$ since the gradient of a weak solution lies in $L^2(\Omega)$. Using this and similar estimates for the other terms in (4.16), gives

$$
\lim_{\varepsilon \to 0^+} \int_{\Omega} \nabla(\phi_\varepsilon \eta) \cdot \nabla \Xi(x - \cdot) \frac{\partial u}{\partial y_j} - \nabla_y \Xi(x - \cdot) \cdot \nabla u \frac{\partial (\phi_\varepsilon \eta)}{\partial y_j} \\
+ \frac{\partial \Xi}{\partial y_j}(x - \cdot) \nabla u \cdot \nabla (\phi_\varepsilon \eta) \, dy 
= \int_{\Omega} \nabla \eta \cdot \nabla_y \Xi(x - \cdot) \frac{\partial u}{\partial y_j} \\
- \nabla_y \Xi(x - \cdot) \cdot \nabla u \frac{\partial \eta}{\partial y_j} \\
+ \frac{\partial \Xi}{\partial y_j}(x - \cdot) \nabla u \cdot \nabla \eta \, dy.
$$

Thus we obtain the identity (4.16) with $u_\tau$ replaced by $u$ and without the support restriction on $\eta$. Finally, we choose a partition of unity which consists of functions that are either supported in a coordinate cylinder, or whose support
does not intersect the boundary of $\Omega$. Summing as $\eta$ runs over this partition gives us the representation formula (4.15) for $u$. As we have $\nabla u \in L^q(\partial \Omega)$ for some $q > 1$, it follows from the theorem of Coifman, McIntosh and Meyer [7] that $(\nabla u)^*$ lies in $L^q(\partial \Omega)$. However, a bit more work is needed to obtain the correct $L^1$ estimate for $(\nabla u)^*$.

We claim

$$\int_{\partial \Omega} \frac{\partial u}{\partial \nu} \, d\sigma = 0$$
$$\int_{\partial \Omega} \nu_j \frac{\partial u}{\partial y_i} - \nu_i \frac{\partial u}{\partial y_j} \, d\sigma = 0.$$ 

Since $(\nabla u)^*$ lies in $L^q(\partial \Omega)$, the proof of these two identities is a standard application of the divergence theorem. Using these results and the estimates for $\nabla u$ in Theorem 4.1, we can show that $\partial u/\partial \nu$ and $\nu_j \partial u/\partial y_i - \nu_i \partial u/\partial y_j$ are molecules on the boundary (see [8]) and hence it follows from the representation formula (4.15) that $(\nabla u)^*$ lies in $L^1(\partial \Omega)$ and satisfies the estimate

$$\|(\nabla u)^*\|_{L^1(\partial \Omega)} \leq C.$$ 

Finally, the existence of non-tangential limits at the boundary follows from the estimate for the non-tangential maximal function. Once we know the limits exist it is easy to see that the boundary data for the $L^1$-mixed problem must agree with the boundary data for the weak formulation.

5 Uniqueness of solutions

In this section we establish uniqueness of solutions to the $L^1$-mixed problem (1.1). We use the existence result established in section 4 and argue by duality that if $u$ is a solution of the mixed problem with zero Dirichlet and Neumann data, then $u$ is also a solution of the regularity problem with zero data and hence is zero.

**Theorem 5.1** Let $\Omega$, $N$ and $D$ be a standard domain for the mixed problem. Suppose that $u$ solves the $L^1$-mixed problem (1.1) with data $f_N = 0$ and $f_D = 0$. If $(\nabla u)^* \in L^1(\partial \Omega)$, then $u = 0$.

Given a Lipschitz domain $\Omega$, we may construct a sequence of smooth approximating domains. A careful exposition of this construction may be found in the dissertation of Verchota ([39, Appendix A], [40, Theorem 1.12]). We will need this approximation scheme and a few extensions. Given a Lipschitz domain $\Omega$, Verchota constructs a family of smooth domains $\{\Omega_k\}$ with $\bar{\Omega}_k \subset \Omega$. In addition, he finds bi-Lipschitz homeomorphisms $\Lambda_k : \partial \Omega \to \partial \Omega_k$ which are constructed as follows.

We choose a smooth vector field $V$ so that for some $\tau = \tau(M) > 0$, $V \cdot \nu \leq -\tau$ a.e. on $\partial \Omega$ and define a flow $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by $\frac{d}{dt}f(x, t) = V(f(x, t))$,
The Jacobian determinant of $DS_t$ is bounded away from 0 and $\rho$. Assume that both are defined in a neighborhood of $\partial \Omega$. As $\nabla \pi$ is an open set and the map $(x, t) \rightarrow f(x, t)$ from $\partial \Omega \times (-\rho, \rho) \rightarrow \mathcal{O}$ is bi-Lipschitz. Since the vector field $V$ is smooth, we have

$$DF(x, t) = I_n + O(t)$$

where $I_n$ is the $n \times n$ identity matrix and $DF$ denotes the derivative of a map $F$. In addition, we have a Lipschitz function $t_k(x)$ defined on $\partial \Omega$ so that $\Lambda_k(x) = f(x, t_k(x))$ is a bi-Lipschitz homeomorphism, $\Lambda_k : \partial \Omega \rightarrow \partial \Omega_k$. We may find a collection of coordinate cylinders $\{Z_i\}$ so that each $Z_i$ serves as a coordinate cylinder for $\partial \Omega$ and for each of the approximating domains $\partial \Omega_k$. If we fix a coordinate cylinder $Z$, we have functions $\phi$ and $\phi_k$ so that $\partial \Omega \cap Z = \{(x', \phi(x')) : x' \in \mathbb{R}^{n-1}\} \cap Z$ and $\partial \Omega_k \cap Z = \{(x', \phi_k(x')) : x' \in \mathbb{R}^{n-1}\} \cap Z$. The functions $\phi_k$ are $C^\infty$ and $\|\nabla' \phi_k\|_{L^\infty(\mathbb{R}^{n-1})}$ is bounded in $k$, $\lim_{k \to \infty} \nabla' \phi_k(x') = \nabla' \phi(x')$ a.e. and $\phi_k$ converges to $\phi$ uniformly. Here we are using $\nabla'$ to denote the gradient on $\mathbb{R}^{n-1}$.

We let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the projection $\pi(x', x_n) = x'$ and define $S_k(x') = \pi(\Lambda_k(x', \phi(x')))$. According to Verchota, the map $S_k$ is bi-Lipschitz and has a Jacobian which is bounded away from 0 and $\infty$. We let $T_k$ denote $S_k^{-1}$ and assume that both are defined in a neighborhood of $\pi(Z)$. We claim that

$$\lim_{k \to \infty} DT_k(S_k(x')) = I_{n-1}, \quad \text{a.e. in } \pi(Z),$$

and the sequence $\|DT_k\|_{L^\infty(\pi(Z))}$ is bounded in $k$.

To establish (5.4), it suffices to show that $DS_k$ converges to $I_{n-1}$ and that the Jacobian determinant of $DS_k$ is bounded away from zero and infinity. The bound on the Jacobian is part of Verchota’s construction (see [39, p. 119]). As a first step, we compute the derivatives of $t_k(x', \phi(x'))$. We first observe that

$$\frac{\partial}{\partial x_i} f((x', \phi(x')), t_k(x', \phi(x'))) = \frac{\partial f}{\partial x_i}((x', \phi(x')), t_k(x', \phi(x'))) + \frac{\partial \phi}{\partial x_i}(x') \frac{\partial f}{\partial x_n}((x', \phi(x')), t_k(x', \phi(x'))) + V(f((x', \phi(x')), t_k(x', \phi(x')))) \frac{\partial}{\partial x_i} t_k(x', \phi(x')).$$

Since $f((x', \phi(x')), t_k(x', \phi(x')))$ lies in $\partial \Omega_k$, the derivative is tangent to $\partial \Omega_k$ and we have

$$\frac{\partial}{\partial x_i} f((x', \phi(x')), t_k(x', \phi(x'))) \cdot \nu_k(y) = 0, \quad \text{a.e. in } \pi(Z),$$

where $y = (S_k(x'), \phi_k(S_k(x'))) \in \partial \Omega_{k'}$. Solving equation (5.5) for $\frac{\partial}{\partial x_i} t_k$ gives

$$\frac{\partial}{\partial x_i} t_k(x', \phi(x')) = -(V(y) \cdot \nu_k(y))^{-1} \left(\frac{\partial f}{\partial x_i}((x', \phi(x')), t_k(x', \phi(x'))) + \frac{\partial \phi}{\partial x_i}(x') \frac{\partial f}{\partial x_n}((x', \phi(x')), t_k(x', \phi(x'))) \cdot \nu_k(y)\right).$$
Since \( \lim_{k \to \infty} t_k(x', \phi(x')) = 0 \) uniformly for \( x' \in \pi(Z) \), (5.3) holds, and \( \nu_k(y) \) converges pointwise a.e. and boundedly to \( \nu(x) \), we obtain that
\[
\lim_{k \to \infty} \frac{\partial}{\partial x_i} t_k(x', \phi(x')) = 0, \quad \text{a.e. in } \pi(Z). \tag{5.6}
\]
Given (5.3), (5.6), and recalling that \( S_k(x') = \pi(f((x', \phi(x'))), t_k(x', \phi(x'))) \), (5.4) follows.

**Lemma 5.7** Let \( \Omega, N \) and \( D \) be a standard domain for the mixed problem. If \( u \) is in \( W^{1,1}(\partial \Omega_k) \) and \( w \) is the weak solution of the mixed problem with Neumann data an atom for \( N \) and zero Dirichlet data, then we have
\[
\int_{\partial \Omega_k} u \frac{\partial w}{\partial \nu} d\sigma \leq C_w \|u\|_{W^{1,1}(\partial \Omega_k)}.
\]

**Proof.** This may be proven using generalized Riesz transforms as in [40, Section 5]. Also, see more recent treatments by Sykes and Brown [37, section 3] and Kilty and Shen [23, section 7]. Verchota’s argument uses square function estimates to show that the generalized Riesz transforms are bounded operators on \( L^p(\partial \Omega) \). In the proof of this Lemma, we need that the Riesz transforms of \( w \) are bounded functions. From the estimate for the Green function in Lemma 4.9 and the representation of \( w = -\langle G, a \rangle_{\partial \Omega} \), we conclude that \( w \) is Hölder continuous. The Hölder continuity, and hence boundedness, of the Riesz transforms of \( w \) follow from the following characterization of Hölder continuous harmonic functions. A harmonic function \( u \) in a Lipschitz domain \( \Omega \) is Hölder continuous of exponent \( \alpha \), \( 0 < \alpha < 1 \), if and only if \( \sup_{x \in \Omega} \text{dist}(x, \partial \Omega)^{1-\alpha} |\nabla u(x)| \) is finite.

We will need the following technical lemma on approximation of functions with \( (\nabla u)^* \) in \( L^1(\partial \Omega) \). The proof relies on the approximation scheme of Verchota outlined above. In our application, we are interested in studying functions in Sobolev spaces on the family of approximating domains. Working with derivatives makes the argument fairly intricate.

**Lemma 5.8** Let \( \Omega, N \) and \( D \) be a standard domain for the mixed problem. If \( u \) satisfies \( (\nabla u)^* \in L^1(\partial \Omega) \) and \( \nabla u \) has non-tangential limits a.e. on \( \partial \Omega \), then we may find a sequence of Lipschitz functions \( U_j \) so that
\[
\lim_{k \to \infty} \|u - U_j\|_{W^{1,1}(\partial \Omega_k)} \leq C/j.
\]

If the non-tangential limits of \( u \) are zero a.e. on \( D \), then we may arrange that \( U_j|_{\partial \Omega} \) is zero on \( D \).

The constant \( C \) may depend on \( \Omega \) and \( u \).

**Proof.** To prove the Lemma, it suffices to consider a function \( u \) which is zero outside one of the coordinate cylinders \( Z \) as given in Verchota’s approximation scheme. We have \( u(x', \phi(x')) \in W^{1,1}(\mathbb{R}^{n-1}) \), where we have set this function
bounded, we have that \( \lim_{k \to \infty} u_j \) to a neighborhood of \( \partial \Omega \) by

\[
U_j(f(x,t)) = \eta(f(x,t))u_j(x), \quad x \in \partial \Omega
\]

where \( \eta \) is a smooth cutoff function which is one on a neighborhood of \( \partial \Omega \) and supported in the set \( \mathcal{O} \) defined in (5.2). If we have that \( u \) is zero on \( D \), then standard approximation results for Sobolev spaces allow us to choose \( u_j \) to be zero in a neighborhood of \( D \). This relies on our assumption on \( \Lambda \).

We consider

\[
\int_{\pi(Z)} |\nabla' u(x', \phi_k(x')) - \nabla' U_j(x', \phi_k(x'))| dx'
\]

\[
\leq \int_{\pi(Z)} |\nabla' u(x', \phi_k(x')) - \nabla' u(x', \phi(x'))| dx'
\]

\[
+ \int_{\pi(Z)} |\nabla' u(x', \phi(x')) - \nabla' u_j(x', \phi(x'))| dx'
\]

\[
+ \int_{\pi(Z)} |\nabla' u_j(x', \phi(x')) - \nabla' U_j(x', \phi_k(x'))| dx'
\]

\[
= A_k + B + C_k.
\]

We have that \( \lim_{k \to \infty} A_k = 0 \) since we assume that \( (\nabla u)^* \in L^1(\partial \Omega) \), \( \nabla u \) has non-tangential limits a.e., and \( \nabla' \phi_k \) converges pointwise a.e. and boundedly to \( \nabla \phi \). By our choice of \( u_j \), we have \( B \leq C/j \). Finally, our construction of \( U_j \) and our definition of \( T_k \) (before (5.4)) imply that \( U_j(x', \phi_k(x')) = u_j(T_k(x'), \phi(T_k(x'))) \) and hence we have

\[
C_k \leq \int_{\pi(Z)} |(I_{n-1} - DT_k(x'))\nabla' u_j(x', \phi(x'))| dx'
\]

\[
+ \int_{\pi(Z)} |DT_k(x')(\nabla' u_j(x', \phi(x'))) - \nabla' u_j(T_k(x'), \phi(T_k(x')))| dx'
\]

\[
= C_{k,1} + C_{k,2}.
\]

We have that \( \lim_{k \to \infty} C_{k,1} = 0 \) since \( \nabla' u_j \) is bounded and (5.4) holds. Since \( T_k(x') \) converges uniformly to \( x' \), \( DT_k \) is bounded and the Jacobian of \( S_k \) is bounded, we have that \( \lim_{k \to \infty} C_{k,2} = 0 \).

**Proof of Theorem 5.7.** We let \( u \) be a solution of the \( L^1 \)-mixed problem, (1.1), with \( f_N = 0 \) and \( f_D = 0 \) and we wish to show that \( u \) is zero. We fix \( a \) an atom for \( N \) and let \( w \) be a solution of the mixed problem with Neumann data \( a \) and zero Dirichlet data. Our goal is to show that

\[
\int_N au \, d\sigma = 0. \quad (5.9)
\]

This implies that \( u \) is zero on \( \partial \Omega \) and then Dahlberg and Kenig’s result for uniqueness of solutions of the regularity problem [10] implies that \( u = 0 \) in \( \Omega \).
We turn to the proof of (5.9). Applying Green’s second identity in one of the approximating domains \( \Omega_k \) gives us
\[
\int_{\partial \Omega_k} w \frac{\partial u}{\partial \nu} \, d\sigma = \int_{\partial \Omega_k} u \frac{\partial w}{\partial \nu} \, d\sigma, \quad k = 1, 2, \ldots \tag{5.10}
\]
We have \( \nabla u \) is in \( L^1(\partial \Omega) \) while \( w \) Hölder continuous and hence bounded. Recalling that \( w \) is zero on \( D \) and \( \partial u / \partial \nu \) is zero on \( N \), we may use the dominated convergence theorem to obtain
\[
\lim_{k \to \infty} \int_{\partial \Omega_k} w \frac{\partial u}{\partial \nu} \, d\sigma = 0. \tag{5.11}
\]
Thus, our claim will follow if we can show that
\[
\lim_{k \to \infty} \int_{\partial \Omega_k} u \frac{\partial w}{\partial \nu} \, d\sigma = \int_{\partial \Omega} u a \, d\sigma. \tag{5.12}
\]
Note that the existence of the limit in (5.12) follows from (5.10) and (5.11). We let \( U_j \) be the sequence of functions from Lemma 5.8 and consider
\[
\left| \int_{\partial \Omega} u a \, d\sigma - \lim_{k \to \infty} \int_{\partial \Omega_k} u \frac{\partial w}{\partial \nu} \, d\sigma \right| \tag{5.13}
\]
\[
\leq \left| \int_{\partial \Omega} u a \, d\sigma - \lim_{k \to \infty} \int_{\partial \Omega_k} U_j \frac{\partial w}{\partial \nu} \, d\sigma \right| + \limsup_{k \to \infty} \left| \int_{\partial \Omega_k} (u - U_j) \frac{\partial w}{\partial \nu} \, d\sigma \right|. 
\]
Because we have that \( \nabla w \) is in \( L^1(\partial \Omega) \) and \( U_j \) is bounded, we may take the limit of the first term on the right of (5.13) and obtain
\[
\left| \int_{\partial \Omega} u a \, d\sigma - \lim_{k \to \infty} \int_{\partial \Omega_k} U_j \frac{\partial w}{\partial \nu} \, d\sigma \right| = \left| \int_N (u - U_j) a \, d\sigma \right| \leq C/j.
\]
Here we use that \( U_j |_D = 0 \). According to Lemmata 5.7 and 5.8 the second term on the right of (5.13) is bounded by \( C_w/j \). As \( j \) is arbitrary, we obtain (5.12) and hence the Theorem.

6 A Reverse Hölder inequality at the boundary

In this section we establish an estimate in \( L^p(\partial \Omega) \) for the gradient of a solution to the mixed problem. This is the key estimate that is used in section 7 to establish \( L^p \)-estimates for the mixed problem.

Lemma 6.1 Let \( \Omega, N, D \) be a standard domain for the mixed problem. Let \( u \) be a weak solution of the mixed problem with Neumann data \( f_N \) in \( L^\infty(N) \)
and zero Dirichlet data. Let \( p_0 > 2 \) be as in Lemma 3.19 and fix \( q \) satisfying \( 1 < q < p_0/2 \). For \( x \in \partial \Omega \) and \( r \) with \( 0 < r < r_0 \) we have

\[
\left( \int_{\Delta_r(x)} (\nabla u)^{*q} \, d\sigma \right)^{1/q} \leq C \left[ \int_{\Psi_2r(x)} |\nabla u| \, dy + \|f_N\|_{L^\infty(\Delta_{2r}(x) \cap N)} \right].
\]

The constant \( c = 1/16 \) and \( C \) depends on \( M, n \) and \( q \).

**Proof.** We fix \( x \in \partial \Omega \) and \( r \) with \( 0 < r < r_0 \). We claim that we have

\[
\left( \int_{\Delta_r(x)} |\nabla u|^q \, d\sigma \right)^{1/q} \leq C \left( \int_{\Psi_{16r}(x)} |\nabla u| \, dy + \|f_N\|_{L^\infty(\Delta_{16r}(x) \cap N)} \right). \tag{6.2}
\]

We will consider two cases: a) \( \delta(x) \leq 8r\sqrt{1+M^2} \), b) \( \delta(x) > 8r\sqrt{1+M^2} \). We give the proof in case a). Since we assume \( 1 < q < p_0/2 \), we may choose \( \epsilon \) satisfying \( 2 - \frac{2}{q} < \epsilon < 2 - \frac{4}{p_0} \). We apply Hölder’s inequality with exponents \( \frac{2}{q} \) and \( \frac{2}{(2 - q)} \) to obtain

\[
\left( \int_{\Delta_r(x)} |\nabla u|^q \, d\sigma \right)^{\frac{1}{q}} \leq \left( \int_{\Delta_r(x)} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma \right)^{\frac{1}{2}} \left( \int_{\Delta_r(x)} \delta^{(\epsilon - 1)q/(2 - q)} \, d\sigma \right)^{\frac{1}{2} - \frac{1}{q}} \leq Cr^{(n-1)(\frac{1}{q} - \frac{1}{2}) + \frac{1}{2}} \left( \int_{\Delta_r(x)} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma \right)^{1/2}
\]

where we use that \( q(\epsilon - 1)/(2 - q) > -1 \) or \( 2 - \frac{2}{q} < \epsilon \) which implies that the integral of \( \delta^{(\epsilon - 1)q/(2 - q)} \) is finite. Next, we use Lemma 4.6 and our hypothesis that \( \delta(x) \leq 8r\sqrt{1+M^2} \) to obtain

\[
\left( \int_{\Delta_r(x)} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma \right)^{1/2} \leq C \left[ \left( \int_{\Psi_{8r}(x)} |\nabla u|^2 \delta^{-\epsilon} \, dy \right)^{1/2} + \left( \int_{\Delta_{8r}(x) \cap N} |f_N|^2 \delta^{1-\epsilon} \, d\sigma \right)^{1/2} \right] \leq C \left[ \left( \int_{\Psi_{8r}(x)} |\nabla u|^2 \delta^{-\epsilon} \, dy \right)^{1/2} + r \frac{n-\epsilon}{2} \|f_N\|_{L^\infty(\Delta_{8r}(x) \cap N)} \right].
\]

To estimate \( \left( \int_{\Psi_{8r}(x)} |\nabla u|^2 \delta^{-\epsilon} \, dy \right)^{1/2} \), we choose \( p > 2 \), use Hölder’s inequality

\[27\]
with exponents p/2 and p/(p - 2), and Lemma 3.19 to find
\[
\left( \int_{\Psi_{4r}(x)} |\nabla u|^2 \delta^{-\epsilon} \, dy \right)^{1/2} \\
\leq \left( \int_{\Psi_{4r}(x)} \delta^{-\epsilon p/(p-2)} \, dy \right)^{1/2} \left( \int_{\Psi_{4r}(x)} |\nabla u|^p \, dy \right)^{1/p} \\
\leq Cr^{-\frac{\theta}{2} + \frac{\theta}{2}} \left[ \int_{\Psi_{16r}(x)} |\nabla u| \, dy + \left( \int_{\Delta_{16r}(x) \cap N} \|f_N\|_{L^\infty(\Delta_{16r}(x) \cap N)} \right)^{\frac{2}{p}} \right].
\]
Combining the two previous displayed inequalities gives the estimate
\[
\left( \int_{\Delta_{4r}(x)} |\nabla u|^q \, d\sigma \right)^{1/q} \leq Cr^{(n-1)/q} \left( \int_{\Psi_{16r}(x)} |\nabla u| \, dy + \|f_N\|_{L^\infty(\Delta_{16r}(x) \cap N)} \right),
\]
which gives the claim (6.2).

Now we consider the proof of (6.2) in case b). Here, we use \( \delta(x) > 8r \sqrt{1 + M^2} \) and Lemma 4.11 to conclude that \( \Delta_{8r}(x) \subset N \) or that \( \Delta_{8r}(x) \subset D \). Then we may use Lemma 4.4 or Lemma 4.5 to conclude that
\[
\int_{\Delta_{4r}(x)} |\nabla u|^2 \, d\sigma \leq C \left( \int_{\Delta_{8r}(x) \cap N} |f_N|^2 \, d\sigma + \frac{1}{r} \int_{\Psi_{8r}(x)} |\nabla u|^2 \, dy \right).
\]
Next, Lemma 3.19 gives
\[
\left( \int_{\Psi_{8r}(x)} |\nabla u|^2 \, dy \right)^{1/2} \leq C \left( \int_{\Psi_{16r}(x)} |\nabla u| \, dy + \|f\|_{L^\infty(\Delta_{16r}(x) \cap N)} \right).
\]
Using the two previous estimates and Hölder’s inequality, we obtain the claim (6.2) in case b).

To obtain the estimate for the non-tangential maximal function, we choose a cutoff function \( \eta \) which is one on \( B_{3r}(x) \) and supported in \( B_{4r}(x) \). By repeating the arguments in the proof of Theorem 4.14, we may show that for \( z \in \Omega \) and \( j = 1, \ldots, n \), we have the following representation for the derivatives of \( u \):
\[
(\eta \frac{\partial u}{\partial z_j})(z) = \int_{\partial \Omega} \eta(\frac{\partial \Xi}{\partial \nu}(z - \cdot) \frac{\partial u}{\partial y_j} - \nu_j \nabla y \cdot \nabla u) \, d\sigma \\
- \int_{\Omega} \nabla \eta \cdot \nabla y \Xi(z - \cdot) \frac{\partial u}{\partial y_j} - \frac{\partial \eta}{\partial y_j} \nabla y \Xi(z - \cdot) \cdot \nabla u \\
+ \nabla \eta \cdot \nabla u \frac{\partial \Xi}{\partial y_j}(z - \cdot) \, dy.
\]
From this representation and the theorem of Coifman, McIntosh and Meyer [7], we obtain
\[
\left( \int_{\Delta_{r}(x)} (\nabla u)^*_r \, d\sigma \right)^{1/q} \leq C \left[ \int_{\Psi_{4r}(x)} |\nabla u| \, dy + \left( \int_{\Delta_{4r}(x)} |\nabla u|^2 \, d\sigma \right)^{1/q} \right]^{1/q}.
\]
From this estimate, the claim (6.2) and a covering argument, we obtain the Theorem.

7 Estimates for solutions with data from $L_p$, $p > 1$

In this section, we use the following variant of an argument developed by Shen [35] to establish $L_p$-estimates for elliptic problems in Lipschitz domains. Shen’s argument is based on earlier work of Caffarelli and Peral [6].

As the argument depends on a Calderón-Zygmund decomposition into dyadic cubes, it will be stated using surface cubes rather than the surface balls $\Delta_r(x)$ used elsewhere in this paper.

Let $Q_0$ be a cube in the boundary and let $F$ be defined on $4Q_0$. Let the exponents $p$ and $q$ satisfy $1 < p < q$. Assume that for each $Q \subset Q_0$, we may find two functions $F_Q$ and $R_Q$ defined in $2Q$ such that

$$|F| \leq |F_Q| + |R_Q|,$$

$$\frac{1}{2Q} \int |F_Q| \, d\sigma \leq C \left( \frac{1}{4Q} \int |f|^p \, d\sigma \right)^{1/p},$$

$$\left( \frac{1}{2Q} \int |R_Q|^q \, d\sigma \right)^{1/q} \leq C \left[ \frac{1}{2Q} \int |F| \, d\sigma + \left( \frac{1}{4Q} \int |f|^p \, d\sigma \right)^{1/p} \right].$$

Under these assumptions, for $r$ in the interval $(p, q)$, we have

$$\left( \frac{1}{Q_0} \int |F|^r \, d\sigma \right)^{1/r} \leq C \left[ \frac{1}{4Q_0} \int |F| \, d\sigma + \left( \frac{1}{4Q_0} \int |f|^p \, d\sigma \right)^{1/r} \right].$$

The constant in this estimate will depend on the Lipschitz constant of the domain, the $L_p$ indices involved and the constants in the estimates in the conditions (7.2)–(7.3). The argument to obtain this conclusion is more or less the same as in Shen [35, Theorem 3.2]. The main differences arise because the last term in (7.3) require us to substitute the maximal function $M(|f|^p)^{1/p}$ for $M(f)$. We omit a detailed proof. Our hypotheses differ from Shen’s in that Shen has $p = 1$ in (7.2) and (7.3) while we have $p > 1$. We need to change Shen’s formulation because we begin with results in Hardy spaces, rather than $L_p$-spaces.

In our application, we will let $4Q_0$ be a cube with sidelength comparable to $r_0$. We let $u$ be a solution of the mixed problem with Neumann data $f$ in $L_p(N)$ and Dirichlet data zero. We define $f$ to be zero in $D$. Since $L_p(N)$ is contained in the Hardy space $H^1(N)$, we may use Theorem 4.1 to obtain a solution of the mixed problem with Neumann data $f$ on $N$ and zero Dirichlet data on $D$. Let $F = (\nabla u)^*$ and given a cube $Q \subset Q_0$ and with diameter $r$, define $F_Q$ and $R_Q$ as follows. We let $\bar{f}_{4Q} = 0$ if $4Q \cap D \neq \emptyset$ and $\bar{f}_{4Q} = \frac{1}{4Q} \int f \, d\sigma$.
if $4Q \subset N$. Set $g = \chi_{4Q}(f - \bar{f}_{4Q})$ and $h = f - g$. As both $g$ and $h$ are elements of the Hardy space $H^1(N)$, we may use Theorem 4.14 to find solutions of the $L^1$-mixed problem with Neumann data $g$ or $h$. We let $v$ be the solution with Neumann data $g$ and $w$ be the solution with Neumann data $h$. According to the uniqueness result Theorem 5.1 we have $u = v + w$. We let $R_Q = (\nabla v)^*$ and $F_Q = (\nabla v)^*$ so that (7.1) holds. We turn our attention to establishing (7.2) and (7.3).

To establish (7.2), observe that the $H^1$-norm of $g$ satisfies the bound

$$
\|g\|_{H^1(N)} \leq C \|f\|_{L^p(4Q)} \sigma(Q)^{1/p'}.
$$

With this, the estimate (7.2) follows from Theorem 4.1. Now we turn to the estimate (7.3) for $F_Q = (\nabla w)^*$. We note that the Neumann data $h$ is constant on $4Q \cap N$. We define a maximal operator by taking the supremum over that part of the cone that is far from the boundary,

$$(\nabla w)^+_+(x) = \sup_{y \in \Gamma(x) \setminus B_A(x)} |\nabla w(y)|$$

where $A$ is to be chosen.

A simple geometric argument gives that

$$(\nabla w)^+_+(x) \leq C \int_{4Q} (\nabla w)^* d\sigma, \quad x \in 2Q. \quad (7.4)$$

The estimate for $(\nabla w)^+_+_{A^r}$ uses the local estimate for the mixed problem in Lemma 6.1 to conclude that

$$
\left( \int_{2Q} (\nabla w)^+_+_{A^r} d\sigma \right)^{1/q} \leq C \left[ \|h\|_{L^\infty(4Q)} + \int_{T(3Q)} |\nabla w| d\sigma \right] \leq C \left[ \int_{4Q} |f| d\sigma + \int_{4Q} (\nabla w)^* d\sigma \right]. \quad (7.5)
$$

provided that the constant $A$ in the definition of $(\nabla w)^+_+$ is chosen sufficiently small. Recall that $T(Q)$ was defined at the beginning of the proof of Lemma 4.6. From the estimates (7.4) and (7.5), we conclude that

$$
\left( \int_{2Q} (R_Q)^q d\sigma \right)^{1/q} \leq C \left[ \int_{4Q} |f| d\sigma + \int_{4Q} (\nabla w)^* d\sigma \right]^{1/p}. \quad (7.6)
$$

We have $(\nabla w)^* \leq (\nabla v)^* + (\nabla u)^*$ and hence we may estimate the term involving $(\nabla w)^*$ by

$$
\int_{4Q} (\nabla w)^* d\sigma \leq \int_{4Q} (\nabla u)^* d\sigma + \int_{4Q} (\nabla v)^* d\sigma \leq \int_{4Q} (\nabla u)^* d\sigma + C \left( \int_{4Q} |f|^p d\sigma \right)^{1/p}
$$

where we have used Theorem 4.14 to estimate the term involving $(\nabla v)^*$. Combining this with (7.6) gives (7.3).

Applying the technique of Shen outlined above gives the $L^p$-estimate and thus we obtain the following theorem.
**Theorem 7.7** Let $\Omega$, $N$ and $D$ be a standard domain for the mixed problem and let $p$ satisfy $1 < p < p_0/2$ where $p_0$ is from Lemma 3.19.

Given data $f_N$ in $L^p(N)$, we may solve the $L^p$-mixed problem with Neumann data $f_N$ and Dirichlet data 0 and this solution satisfies the estimate

$$\| (\nabla u)^* \|_{L^p(\partial \Omega)} \leq C \| f_N \|_{L^p(\partial \Omega)}.$$

The constant $C$ depends on the global character of the domain and the index $p$.

**8 Further questions**

This work adds to our understanding of the mixed problem in Lipschitz domains. However, there are several avenues which are not yet explored.

1. Can we study the inhomogeneous mixed problem and obtain results similar to those of Fabes, Mendez and M. Mitrea [12] and I. Mitrea and M. Mitrea [31]?

2. Is there an extension to $p < 1$ as the work of Brown [4]?

3. Can we study the mixed problem for more general decompositions of the boundary, $\partial \Omega = D \cup N$? To what extent is the condition that the boundary between $D$ and $N$ be a Lipschitz graph needed?

4. Can we extend these techniques to elliptic systems and higher order elliptic equations?

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A Correction to the Proof of Theorem 7.7.

In an earlier work [4, Equation (7.4)] we make a technical claim about non-tangential maximal functions which is an essential step in our study of the $L^p$-mixed problem. The example below shows that the statement (7.4) is incorrect. A correct substitute for this claim may be found in (A.3) below. This note provides a proof of the main theorem of [4] which avoids the problematic claim. We will follow the notation, equation references, and results from our earlier work. We thank L. Croyle for pointing out this error. Her dissertation [2] includes a different approach to correcting this error and the paper [1] includes a version of the argument presented here. The correction outlined here should also be applied to several subsequent papers including [3, 5] that make use of the method from [4].

We begin with an example which shows that the claim (7.4) may fail.

**Example.** Consider the domain $\Omega$ in $\mathbb{R}^2$ which lies above the graph of the Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x_1) = \max(-2|x_1|, 2|x_1|-4)$. For this domain, we may find $y \in \Gamma(0)$ with $|y|$ large, and $\epsilon = (1 + \alpha) \text{dist}(y, \partial \Omega) - |y|$ small. This requires that $\alpha$ be sufficiently large. We claim that we have
\[
\{x : y \in \Gamma(x), x \in \Delta_{1/2}(0)\} \subset \Delta_{C\epsilon}(0).
\]
To see this, note that if $x \in \Delta_{1/2}(0)$, then we have $|x - y| \geq |y| + c|x|$. This follows because the line segments that form the boundary near zero are not tangent to the boundary of the ball $\partial B_{|y|}(y)$. Thus $y \in \Gamma(x)$ will hold only if $|x|$ is at most a multiple of $\epsilon$. If we let $u(z) = \chi_{B_{|y|}(y)}(z)$, then we have $u^*(0) = 1$, but $\int_{\Delta_{1/2}(0)} u^* d\sigma \leq C\epsilon$. Thus (7.4) fails.

To avoid making use of (7.4) in Ott and Brown, we apply Shen’s argument outlined in (7.1-3) to $M((\nabla u^*)^{1/2})^2$ rather than to $\nabla u^*$. Here, $M$ is the Hardy-Littlewood maximal function which we define by
\[
M(f)(x) = \sup_{s > 0} \int_{\Delta_s(x)} |f| \, d\sigma.
\]
We will need several auxiliary maximal functions which we define here. For these definitions we need a parameter $r$ which will give a division between small and large scales. In our applications, $r$ will be comparable to the sidelength of the cube $Q$ that appears in Shen’s argument outlined in (7.1-3). We let
\[
M_0(f)(x) = \sup_{0 < s < r} \int_{\Delta_s(x)} |f| \, d\sigma, \quad M_\infty(f)(x) = \sup_{s \geq r} \int_{\Delta_s(x)} |f| \, d\sigma.
\]
We also define truncated non-tangential maximal functions by
\[
u^\gamma(x) = \sup_{y \in \Gamma(x), |x-y| < cr} |u(y)|, \quad \nu^\delta(x) = \sup_{y \in \Gamma(x), |x-y| > cr} |u(y)|.
\]
We define $F = M((\nabla u^*)^{1/2})^2$ and fix a cube $Q$. Let $v$ and $w$ be defined as in section 7 of [4]. We set $F_Q = M((\nabla u^*)^{1/2})^2$ and $R_Q = M((\nabla w^*)^{1/2})^2$. By uniqueness
for the $L^1$-mixed problem (Theorem 5.1), we have that $u = v + w$ and thus it follows that we have (7.1). By Theorem 4.17, we have the estimate

$$
\int_{\partial \Omega} \nabla v^* \, d\sigma \leq C \|g\|_{H^1(N)} \leq \sigma(4Q)^{1-1/p} \left( \int_{4Q} |f|^p \, d\sigma \right)^{1/p}.
$$

From this and the Hardy-Littlewood maximal theorem, we obtain

$$\int_{2Q} M((\nabla v^*)^2) \, d\sigma \leq C \left( \int_{4Q} |f|^p \, d\sigma \right)^{1/p} \quad (A.1)$$

which is (7.2).

Before estimating $R_\Omega$, we give two technical lemmata.

**Lemma A.2** Suppose that $x, y$ are in $\partial \Omega$ and $|x - y| < Ar$, then we have

$$M_{\infty}(f)(x) \leq C_A M_{\infty}(f)(y).$$

**Proof.** By the triangle inequality, we have $\Delta_s(x) \subset \Delta_{s+Ar}(y)$. Thus it follows that

$$\int_{\Delta_s(x)} |f| \, d\sigma \leq \sigma(\Delta_{s+Ar}(y)) / \sigma(\Delta_s(x)) \int_{\Delta_{s+Ar}(y)} |f| \, d\sigma.$$

If we require that $s \geq r$, then we have a constant so that $\sigma(\Delta_{s+Ar}(y)) / \sigma(\Delta_s(x)) \leq C_A$ and the Lemma follows.

**Lemma A.3** For $p > 0$, we have

$$u^\Delta(x) \leq C M_{\infty}((u^*)^p)^{1/p}(x).$$

The constant depends on the value of $p$ and the constant $c$ entering into the definition of $u^\Delta$.

**Proof.** Fix $x \in \partial \Omega$ and suppose that $y \in \Gamma(x)$. Fix $\hat{y}$ so that $|y - \hat{y}| = d(y) = \text{dist}(y, \partial \Omega)$ and observe that if $|z - \hat{y}| < ad(y)$, we have $y \in \Gamma(z)$. This implies $|u(y)| \leq u^*(z)$ for $z \in \Delta_{ad(y)}(\hat{y})$. By the triangle inequality $|x - \hat{y}| \leq |x - y| + |y - \hat{y}| \leq (2 + \alpha)d(y)$. Hence we have that $\Delta_{ad(y)}(\hat{y}) \subset \Delta_{(2+2\alpha)d(y)}(x)$. It follows that

$$|u(y)| \leq \frac{\sigma(\Delta_{(2+2\alpha)d(y)}(x))}{\sigma(\Delta_{ad(y)}(\hat{y}))} \int_{\Delta_{(2+2\alpha)d(y)}(x)} u^* \, d\sigma.$$

If we assume that $|x - y| > cr$, then we have $d(y) > cr/(1 + \alpha)$ and obtain

$$u^\Delta(x) \leq C M_{\infty}(u^*)(x)$$

which is our result with $p = 1$. To obtain the Lemma for other values of $p$, apply the case with $p = 1$ to $|u|^p$. 

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To estimate $R_Q$, we will fix $x \in 2Q$ and $r > c \text{diam}(Q)$ so that $\Delta_{4r}(x) \subset 4Q$. Our goal is to show that
\[
\left( \int_{\Delta_r(x)} M((\nabla w^*)^\frac{1}{2})^{2q} \ d\sigma \right)^{1/q} \leq C \left[ \int_{\Delta_{4r}(x)} M((\nabla u^*)^\frac{1}{2})^2 \ d\sigma + \left( \int_{4Q} |f|^p \ d\sigma \right)^{1/p} \right].
\] (A.4)

Covering $2Q$ by a finite collection of balls we may obtain (7.3) from (A.4).

To estimate $R_Q = M((\nabla w^*)^\frac{1}{2})^2 \leq C(M_\infty((\nabla w^*)^\frac{1}{2})^2 + M_0((\nabla w^\gamma)^\frac{1}{2})^2 + M_0((\nabla w^\triangle)^\frac{1}{2})^2)$ (A.5)
and will proceed to estimate the three terms on the right of (A.5).

From Lemma A.2 we have
\[
\sup_{y \in \Delta_r(x)} M_\infty((\nabla w^*)^\frac{1}{2})^2(y) \leq C \inf_{y \in \Delta_r(x)} M_\infty((\nabla w^*)^\frac{1}{2})^2(y) \leq C \int_{\Delta_r(x)} M_\infty((\nabla w^*)^\frac{1}{2})^2 \ d\sigma.
\] (A.6)

Next we observe that Lemma 6.1 and the Hardy-Littlewood maximal theorem gives that
\[
\left( \int_{\Delta_r(x)} M_0((\nabla w^\gamma)^\frac{1}{2})^{2q} \ d\sigma \right)^{1/q} \leq C \left( \int_{\Delta_{4r}(x)} \nabla w^* \ d\sigma + |f|_{4Q} | \right) .
\] (A.7)

Finally, to estimate $M_0((\nabla w^\triangle)^\frac{1}{2})^2$, we may use Lemma A.2 and Lemma A.3 to see that
\[
\nabla w^\triangle(y) \leq CM_\infty((\nabla w^*)^\frac{1}{2})^2(z), \quad y \in \Delta_2r(x), \ z \in \Delta_{4r}(x).
\]

For $y \in \Delta_r(x)$, the maximal function $M_0(h)(y)$ depends only on the values of $h$ in $\Delta_2r(x)$, thus we obtain
\[
\sup_{y \in \Delta_r(x)} M_0((\nabla w^\triangle)^\frac{1}{2})^2(y) \leq C \int_{\Delta_{4r}(x)} M((\nabla w^*)^\frac{1}{2})^2 \ d\sigma.
\] (A.8)

From (A.5) (A.8) we conclude that
\[
\left( \int_{\Delta_r(x)} M((\nabla w^*)^\frac{1}{2})^{2q} \ d\sigma \right)^{1/q} \leq C \left[ \int_{\Delta_{4r}(x)} M((\nabla u^*)^\frac{1}{2})^2 \ d\sigma + \left( \int_{2Q} |f|^p \ d\sigma \right)^{1/p} \right].
\]

To complete the proof of (A.4), we write $w = u - v$ and then use (A.1) to obtain
\[
\left( \int_{\Delta_r(x)} M((\nabla w^*)^\frac{1}{2})^2 \ d\sigma \right) \leq C \left( \int_{\Delta_{4r}(x)} M((\nabla u^*)^\frac{1}{2})^2 \ d\sigma + \int_{\Delta_{4r}(x)} M((\nabla v^*)^\frac{1}{2})^2 \ d\sigma \right)
\]
\[
\leq C \left( \int_{\Delta_{4r}(x)} M((\nabla u^*)^\frac{1}{2})^2 \ d\sigma + \left( \int_{2Q} |f|^p \ d\sigma \right)^{1/p} \right).
\] (A.9)

Our claim (A.4) follows.
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