Vortex precession in a rotating nonaxisymmetric trapped Bose-Einstein condensate

Alexander L. Fetter\textsuperscript{1,2,3} and Jong-kwan Kim\textsuperscript{1,3}
\textsuperscript{1}Geballe Laboratory for Advanced Materials, Stanford University, Stanford, CA 94305-4045
\textsuperscript{2}Department of Physics, Stanford University, Stanford, CA 94305-4060
\textsuperscript{3}Department of Applied Physics, Stanford University, Stanford, CA 94305-4090
(October 4, 2001)

Abstract

We study the precession of an off-axis straight vortex in a rotating non-axisymmetric harmonic trap in the Thomas-Fermi (TF) regime. A time-dependent variational Lagrangian analysis yields the dynamical equations of the vortex and the precessional angular velocity in two-dimensional (2D) and three-dimensional (3D) condensates.

PACS numbers: 03.75.Fi, 05.30.Jp
I. INTRODUCTION

Since the original experimental observation of Bose-Einstein condensation in dilute atomic gases \cite{1,2}, the creation and detection of vortices has attracted great interest (for a recent review article, see, for example Ref. \cite{4}). In particular, one experiment has measured the precession frequency $\omega_p$ of a vortex line in a nonrotating stationary spherical condensate \cite{5} and compared $\omega_p$ to various theoretical estimates \cite{6,7,8,9,10}. Although these theoretical approaches largely predict the same results, perhaps the most transparent and physical picture \cite{7,8} relies on a variational Lagrangian method \cite{12} that focuses directly on the position of the vortex core. For an axisymmetric condensate in rotational equilibrium at a small angular velocity $\Omega \ll \omega_\perp$, the original precession frequency $\omega_p$ is altered to $\omega_p(\Omega) = \omega_p - \Omega$ by the external rotation. The present work describes the nontrivial generalization of the variational Lagrangian method to a large rotating nonaxisymmetric condensate in the Thomas-Fermi (TF) limit. For extreme asymmetry, we may note that the precession frequency $\omega_p(\Omega)$ becomes nearly independent of $\Omega$ (for $\Omega \ll \omega_\perp$) because the irrotational flow induced by the rotating asymmetry screens the effect of the external rotation.

II. PRECESSIONAL DYNAMICS OF A STRAIGHT VORTEX

A Bose condensate is described by a macroscopic order parameter (the condensate wave function) $\Psi$, with particle density $n = |\Psi|^2$. Consider a condensate in a nonaxisymmetric harmonic trap potential $V_{tr}(r) = \frac{1}{2}M(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$. The condensate experiences both the trap potential $V_{tr}$ and the self-consistent Hartree interaction $V_H = g|\Psi|^2 = gn$ arising from the interaction with all the other particles, where $g = 4\pi \hbar^2 a/M$ is the effective interparticle interaction strength and $a > 0$ is the $s$-wave scattering length \cite{13,14}. At zero temperature, the order parameter obeys the time-dependent Gross-Pitaevskii (GP) equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2 \nabla^2}{2M} \Psi + V_{tr} \Psi + g|\Psi|^2 \Psi.$$  \hspace{1cm} (1)

When the trap rotates with angular velocity $\Omega$ about the $z$ axis, the corresponding GP equation in a co-rotating frame acquires an additional term $-\Omega L_z \Psi = i\hbar \Omega \mathbf{r} \times \nabla \Psi$ on the right-hand side, arising from the transformation to the rotating frame \cite{15}.

Instead of working directly with the time-dependent GP equation (1), it is often preferable to introduce a Lagrangian formalism based on the Lagrangian functional

$$\mathcal{L}[\Psi] = \mathcal{T}[\Psi] - \mathcal{F}[\Psi],$$  \hspace{1cm} (2)

where

$$\mathcal{T}[\Psi] = \int dV \frac{i\hbar}{2} \left( \Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right)$$  \hspace{1cm} (3)

is the time-dependent part of the Lagrangian that is analogous to the kinetic energy in classical particle mechanics. Similarly, the free energy $\mathcal{F}[\Psi]$ here plays the role of the potential energy. In a nonrotating system, $\mathcal{F}$ is simply the energy functional $\mathcal{E}$, which includes the gradient (bending) energy, the trap energy, and the self-consistent Hartree...
energy. When the system rotates, however, the free energy has an extra term $F = E - \Omega L_z$ [12], and the appropriate free-energy functional of the system is

$$F[\Psi] = \int dV \left( \frac{\hbar^2}{2M} |\nabla \Psi|^2 + V_{\text{tr}} |\Psi|^2 + \frac{g}{2} |\Psi|^4 + i\hbar \Omega \Psi^* \frac{\partial \Psi}{\partial \phi} \right).$$

(4)

Given the Lagrangian, the associated action is the time integral $S[\Psi] = \int_{t_1}^{t_2} dt [L[\Psi]]$, and it is straightforward to verify that the action is stationary with respect to small variations of $\Psi$ and $\Psi^*$ when $\Psi$ obeys the time-dependent GP equation (1). Thus the time-dependent GP equation is the Euler-Lagrange equation for this problem.

This Lagrangian formalism is exact, but it also provides the basis for a powerful approximate variational method. If the wave function $\Psi$ depends on various parameters, the integral in Eq. (4) will involve the first time derivatives of these parameters. Consequently, the resulting $L$ will serve as the effective Lagrangian that provides the corresponding dynamical equations for these parameters. This method has been used to study the quadrupole oscillations of a vortex-free condensate [12]. Here, we use it to study the dynamics of a vortex in a rotating nonaxisymmetric trap.

Specifically, we assume that a straight singly quantized vortex is displaced from the center of the trap with transverse coordinates $x_0$ and $y_0$; these variables will serve as time-dependent variational parameters that obey Lagrange’s equations [12,7]. In the TF limit, the vortex induces negligible change in the condensate density, and, in the presence of the rotation, $|\Psi|^2$ is given by [13]

$$g|\Psi(r)|^2 = \tilde{\mu} - \tilde{V}_{\text{tr}}(r) = \tilde{\mu} - \frac{1}{2}M \left( \tilde{\omega}_x^2 x^2 + \tilde{\omega}_y^2 y^2 + \tilde{\omega}_z^2 z^2 \right)$$

(5)

where $\tilde{\omega}_x$ and $\tilde{\omega}_y$ are effective oscillator frequencies that determine the shape of the condensate in the rotating asymmetric trap. Specifically, they are given in terms of a parameter $\alpha$:

$$\tilde{\omega}_x^2 = \omega_x^2 + \alpha^2 - 2\alpha \Omega, \quad \tilde{\omega}_y^2 = \omega_y^2 + \alpha^2 + 2\alpha \Omega,$$

(6)

where $\alpha$ satisfies a cubic equation

$$2\alpha^3 + \alpha(\omega_x^2 + \omega_y^2 - 4\Omega^2) + \Omega(\omega_x^2 - \omega_y^2) = 0.$$  

(7)

The parameter $\alpha$ has a simple physical interpretation [16], for it determines the irrotational flow $\mathbf{v}_0$ induced by the rotating nonaxisymmetric trap through the velocity potential $\Phi_0 = \alpha xy$, with $\mathbf{v}_0 = \nabla \Phi_0$. As noted below, $\alpha = -\Omega (\tilde{\omega}_x^2 - \tilde{\omega}_y^2) / (\tilde{\omega}_x^2 + \tilde{\omega}_y^2)$.

Equation (5) shows that the condensate density has the familiar TF parabolic form

$$|\Psi|^2 = n_0 \left( 1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2} \right),$$

(8)

if the right-hand side is positive and zero otherwise. Here $n_0 = \tilde{\mu}/g = M\tilde{\mu}/4\pi ah^2$ is the central density, $\tilde{R}_i = \sqrt{2\tilde{\mu}/M \tilde{\omega}_i^2}$ (for $i = x,y$), $\tilde{R}_z = \sqrt{2\tilde{\mu}/M \tilde{\omega}_z^2}$ are the TF radii, and

$$\tilde{\mu} = \frac{1}{2}h\tilde{\omega}_0 \left( 15Na/\tilde{d}_0 \right)^{2/5} \text{ with } \tilde{\omega}_0^3 = \tilde{\omega}_x \tilde{\omega}_y \tilde{\omega}_z \text{ and } \tilde{d}_0 = h/M\tilde{\omega}_0.$$  

For a small angular velocity
\( \Omega \ll \omega_{\perp} \) where \( \omega_{\perp}^2 \equiv \frac{1}{2}(\omega_y^2 + \omega_x^2) \), the renormalization of the oscillator frequencies and TF radii is of order \( \Omega^2 \) and thus small.

In our variational approach, we assume that the dominant vortex-induced contribution to the Lagrangian arises from the condensate’s superfluid motion and use the following trial wave function

\[
\Psi = |\Psi| e^{i(S_0 + S_1)},
\]

(9)

where \( |\Psi| \) follows from the TF density profile in Eq. (8). The first term of the phase is taken as \( S_0 = (M\alpha/\hbar)xy \), which represents the irrotational motion induced by the rotating nonaxisymmetric trap (note that \( \alpha \) vanishes if \( \omega_x = \omega_y \)). The second term is taken to be

\[
S_1 = \arctan \left( \frac{y - y_0}{x - x_0} \right),
\]

(10)

which characterizes the circulating flow around the vortex line. In contrast to the analysis of Refs. [18,7], we do not include an image vortex because the form of the TF condensate density ensures that the particle current vanishes automatically at the TF surface.

Variation of the functional (4) yields an Euler-Lagrange equation for \( S_0 \), whose solution gives [8,16,17,19,20]

\[
S_0 = -\frac{M\Omega}{\hbar} \left( \frac{\tilde{\omega}_x^2 - \tilde{\omega}_y^2}{\tilde{\omega}_x^2 + \tilde{\omega}_y^2} \right) xy = \frac{M\Omega}{\hbar} \left( \frac{\tilde{R}_x^2 - \tilde{R}_y^2}{R_x^2 + R_y^2} \right) xy.
\]

(11)

Note that the time-dependent parameters \( x_0 \) and \( y_0 \) appear only in the phase \( S_1 \), so that

\[
\partial_t \Psi = i \Psi \partial_t S_1
\]

\[
= -i \Psi \hat{r}_0 \cdot \nabla S_1
\]

\[
= -i \Psi \frac{-\dot{x}_0 (y - y_0) + \dot{y}_0 (x - x_0)}{(x - x_0)^2 + (y - y_0)^2}.
\]

(12)

**III. TWO-DIMENSIONAL CONDENSATE**

For simplicity, we first consider a two-dimensional condensate that is unbounded in the \( z \) direction (taking \( \omega_z \rightarrow 0 \)), in which case the various terms in the Lagrangian are interpreted per unit length. To evaluate the time-dependent part (3), it is convenient to introduce dimensionless coordinates scaled with the renormalized TF radii and shift the origin of coordinates to the position of the vortex. In this way, we find

\[
\mathcal{T}[\Psi] = \hbar n_0 \tilde{R}_x^2 \tilde{R}_y^2 \int d^2r \frac{-\dot{x}_0 y + \dot{y}_0 x}{\tilde{R}_x^2 x^2 + \tilde{R}_y^2 y^2} \left[ 1 - (r \cos \phi + x_0)^2 - (r \sin \phi + y_0)^2 \right]
\]

\[
= \hbar n_0 \tilde{R}_x^2 \tilde{R}_y^2 \int d\phi \frac{-\dot{x}_0 \sin \phi + \dot{y}_0 \cos \phi}{\tilde{R}_x^2 \cos^2 \phi + \tilde{R}_y^2 \sin^2 \phi} \left( -\frac{2}{3} B^3 - AB \right),
\]

(13)

where
\[ A \equiv 1 - x_0^2 - y_0^2, \]
\[ B \equiv x_0 \cos \phi + y_0 \sin \phi, \] (14)

with \( x_0 \) and \( y_0 \) also dimensionless. As anticipated in the general discussion, \( T \) indeed depends on the first time derivatives of the parameters \( x_0 \) and \( y_0 \).

Lagrange’s equations now take the familiar form
\[ D_x (T - F) = 0, \quad D_y (T - F) = 0, \] (15)

where
\[ D_x \equiv \frac{d}{dt} \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_0}, \] (16)

and similarly for \( D_y \). If this operator is applied to functions of the form appearing in Eq. (13), it is straightforward to verify that
\[ D_x \left[ (-\dot{x}_0 \sin \phi + \dot{y}_0 \cos \phi) f(x_0, y_0, \phi) \right] = -\dot{y}_0 \left( \cos \phi \frac{\partial}{\partial x_0} + \sin \phi \frac{\partial}{\partial y_0} \right) f(x_0, y_0, \phi). \] (17)

In our case,
\[ (\cos \phi \frac{\partial}{\partial x_0} + \sin \phi \frac{\partial}{\partial y_0}) A = -2B, \quad (\cos \phi \frac{\partial}{\partial x_0} + \sin \phi \frac{\partial}{\partial y_0}) B = 1. \] (18)

Applying this operator to the time-dependent term (13) simplifies the integral considerably and yields
\[ D_x T[\Psi] = \dot{y}_0 h n_0 \tilde{R}_x \tilde{R}_y \int d\phi \frac{A}{\tilde{R}_x^2 \cos^2 \phi + \tilde{R}_y^2 \sin^2 \phi} \]
\[ = \dot{y}_0 2\pi h n_0 \tilde{R}_x \tilde{R}_y (1 - x_0^2 - y_0^2). \] (19)

The first of Lagrange’s equations (15) then becomes
\[ D_x T[\Psi] = -\frac{\partial \Delta F[\Psi]}{\partial x_0}, \] (20)

where \( \Delta F \) denotes the extra free energy associated with the presence of a vortex. This equation, along with a similar one for the \( y_0 \) dependence can be rewritten as
\[ 2\pi \hbar n_0 \tilde{R}_x \tilde{R}_y \tilde{z} \times \dot{r}_0 = \nabla_0 \Delta F(r_0), \] (21)

where \( r_0 = (x_0, y_0) \) is the dimensionless coordinate vector of the displaced vortex. As is familiar in the context of two-dimensional vortices in superfluid helium [21], this dynamical equation can be interpreted as a balance between the Magnus force and the gradient of the free energy with respect to the coordinates of the vortex [6,7,10].

We can calculate the free-energy part with logarithmic accuracy [18] by substituting (4)-(11) into the energy functional (4):
\[ \Delta F(r_0, \Omega) = 2\pi \tilde{\mu} \xi^2 n_0 (1 - r_0^2) \left[ \ln \left( \frac{\tilde{R}_x}{\xi} \right) - \frac{M \Omega}{\hbar} \frac{\tilde{R}_x^2 \tilde{R}_y^2}{\tilde{R}_x^2 + \tilde{R}_y^2} (1 - r_0^2) \right], \] (22)
where \( \tilde{R}_x = 2\tilde{R}_y^2/(\tilde{R}_x^2 + \tilde{R}_y^2) \) and \( \xi^2 = \hbar^2/2M\hat{\mu} \). With these results, we can derive the equations of motion and obtain the angular frequency \( \omega_p(\Omega) \) of the vortex precession. An easy calculation yields

\[
\dot{x}_0 = -\omega_p(\Omega)y_0, \quad \dot{y}_0 = +\omega_p(\Omega)x_0,
\]

where

\[
\omega_p(\Omega) \equiv \omega_p^0 - 2\tilde{R}_y\tilde{R}_y\Omega \quad \text{and} \quad \xi^2 = \bar{\mu}/2M\tilde{R}_y^2\ln(\tilde{R}_\perp/\xi) 1/1 - r_0^2
\]

is the precession frequency in a nonrotating condensate of stretched dimensions \( \tilde{R}_x \) and \( \tilde{R}_y \). For a not-too-large angular velocity \( \Omega \ll \omega_\perp \), note that \( \omega_p^0 \simeq \omega_p(0) \). An earlier more intricate derivation of this result analyzed the dynamical motion of each element of the vortex core. If \( \omega_p(\Omega) \) is positive, the vortex precesses in the positive (counterclockwise) sense, namely with the same sense as the circulating fluid around the vortex. Such behavior is indeed seen in the recent JILA experiments. For an axisymmetric condensate with \( \tilde{R}_x = \tilde{R}_y = \tilde{R}_\perp \), the precession frequency \( \omega_p(0) \) in the absence of rotation is shifted to \( \omega_p(\Omega) = \omega_p^0 - \Omega \) by the applied rotation (as expected from the transformation to the rotating frame [22]). For an asymmetric trap, however, the induced irrotational flow in Eq. (11) acts to screen the effect of the external rotation. In the extreme limit \( \tilde{R}_y \ll \tilde{R}_x \), for example, we have \( \omega_p(\Omega) \approx \omega_p^0 - (2\tilde{R}_y/\tilde{R}_x)\Omega \), so that the precession then becomes nearly independent of \( \Omega \) for small \( \Omega \).

This behavior can be understood by examining the background fluid flow velocity in the rotating frame \( v = (\hbar/M)\nabla S_0 - \Omega \hat{z} \times r' \) in the absence of the vortex [21]; for clarity, we now use primes to denote the original dimensional coordinates with, for example, \( x' = x\tilde{R}_x \). In particular, \( v_x(x', y') = 2\Omega y' \tilde{R}_y^2/(\tilde{R}_x^2 + \tilde{R}_y^2) \). This induced velocity appears in the dimensional equation of motion as

\[
\frac{\dot{x}_0}{\tilde{R}_x} = -\omega_p^0 \frac{y_0}{\tilde{R}_y} + \frac{v_x(x'_0, y'_0)}{\tilde{R}_x} = -\omega_p^0 \frac{y'_0}{\tilde{R}_y} + \frac{2\Omega \tilde{R}_x}{\tilde{R}_x^2 + \tilde{R}_y^2} y'_0.
\]

which readily reproduces the first of Eqs. (23), and similarly for the second equation.

**IV. THREE-DIMENSIONAL CONDENSATE**

It is not difficult to generalize these results to a three-dimensional TF condensate. For example, the explicitly time-dependent part of the Lagrangian now contains an additional integral over the axial coordinate \( z \):
\[ \mathcal{T}[\Psi] = \hbar n_0 \tilde{R}_x \tilde{R}_y \tilde{R}_z \int d^3r \, \frac{-\dot{x}_0 y + \dot{y}_0 x}{R^2_x x^2 + R^2_y y^2} \left[ 1 - (r \cos \phi + x_0)^2 - (r \sin \phi + y_0)^2 - z^2 \right] \]
\[ = 2 \hbar n_0 \tilde{R}_x \tilde{R}_y \tilde{R}_z \int d^2r \int_{r=0}^{z_{\max}} dz \, \frac{-\dot{x}_0 y + \dot{y}_0 x}{R^2_x x^2 + R^2_y y^2} \left[ 1 - (r \cos \phi + x_0)^2 - (r \sin \phi + y_0)^2 - z^2 \right] \]
\[ = \frac{\hbar n_0 \tilde{R}_x \tilde{R}_y \tilde{R}_z}{6} \int d\phi \, \frac{-\dot{x}_0 \sin \phi + \dot{y}_0 \cos \phi}{R^2_x \cos^2 \phi + R^2_y \sin^2 \phi} \times \left\{ -5BA^3/2 - 3B^3A^{1/2}/2 + (A + B^2) \left[ \frac{\pi}{2} + \arctan \left( \frac{B}{-A^{1/2}} \right) \right] \right\}, \quad (27) \]

where \( z_{\max}^2 = 1 - (r \cos \phi + x_0)^2 - (r \sin \phi + y_0)^2 \). Although this angular integral is difficult to evaluate directly, only the simpler quantity \( \mathcal{D}_x \mathcal{T}[\Psi] \) is needed, and a straightforward analysis yields
\[ \mathcal{D}_x \mathcal{T}[\Psi] = y_0 \frac{8\pi \hbar n_0}{3} \tilde{R}_x \tilde{R}_y \tilde{R}_z (1 - r_0^2)^{3/2}. \quad (28) \]

In addition, the free energy is given by [8]
\[ \Delta \mathcal{F}(r_0, \Omega) = \frac{8\pi \hbar^2 n_0 \tilde{R}_z}{3 (1 - r_0^2)^{3/2}} \left[ \ln \left( \frac{\tilde{R}_z}{\xi} \right) - \frac{4 \, M \Omega}{5 \, \hbar} \frac{\tilde{R}_x^2}{R_x^2 + R_y^2} (1 - r_0^2) \right]. \quad (29) \]

The dynamical equation again reduces to (21), and the only difference is that the angular frequency \( \omega_p(\Omega) \) has a modified numerical coefficient
\[ \omega_p(\Omega) \equiv \frac{3}{2} \frac{\hbar}{M \tilde{R}_x \tilde{R}_y} \ln \left( \frac{\tilde{R}_z}{\xi} \right) \frac{1}{1 - r_0^2} - \frac{2 \tilde{R}_x \tilde{R}_y}{\tilde{R}_x^2 + \tilde{R}_y^2} \Omega; \quad (30) \]

this expression agrees with an earlier result based on the method of matched asymptotic expansions [3].

Note that our analysis for a three-dimensional condensate considers only a straight vortex line. In the present Thomas-Fermi limit of a large condensate, the axis of the vortex line must lie perpendicular to the condensate’s surface [11]. Hence the approximation of a straight vortex is directly applicable to the central region of a disk-shaped condensate. For cigar-shaped condensates, in contrast, the bending of the vortex generally plays an essential role. Indeed, for sufficiently elongated condensates, a straight vortex on the central axis can become unstable with respect to bending deformations [11].

**V. DISCUSSION AND CONCLUSIONS**

The Lagrangian functional [3] contains two parts, \( \mathcal{T}[\Psi] \) with explicit time dependence and \( \mathcal{F}[\Psi] \) involving the time-independent free energy in the rotating frame. If only the latter quantity is considered (which is the Hamiltonian in the rotating frame), the stability can be inferred directly by considering how \( \mathcal{F} \) changes for small lateral displacements of the vortex from the central position. Such methods have been used to analyze the onset of metastability in a rotating nonaxisymmetric TF condensate [4].
In contrast, the full Lagrangian allows a more complete dynamical description. In the present case, the time dependence of the vortex position $r_0$ yields equations of motion (21), which readily give the precession frequency $\omega_p(\Omega)$ in a rotating condensate. As mentioned in Sec. [1], the present description does not include an image vortex. Although such an image vortex is needed to satisfy the appropriate boundary condition for a condensate with uniform density in a rigid container [19], the present TF condensate density vanishes at the TF boundary so that the superfluid current automatically vanishes there. In addition, the image vortex has negligible effect on the energy when calculated with logarithmic accuracy. As emphasized in Refs. [4,10], the vortex is unstable (metastable) whenever the gradient of $\Delta F(r_0, \Omega)$ with respect to $r_0$ in Eq. (21) acts to move the vortex away from (back toward) the center of the trap.

The present Lagrangian approach gives a direct physical derivation of the precession frequency. Although the results describe only a single straight vortex displaced from the $z$ axis, the analysis is far simpler than the method of matched asymptotic expansions applied to the full Gross-Pitaevskii equation [8]. For a one-component condensate, it fully justifies the intuitive dynamical equation (21) suggested earlier by McGee and Holland [10]. It would be valuable to generalize the present approach to the more interesting and challenging case of two components, where additional restoring (buoyancy) forces have been proposed [10].

ACKNOWLEDGMENTS

This work has benefited from many helpful discussions with M. J. Holland and A. A. Svidzinsky. It has been supported in part by the NSF, Grant No. 99-71518.
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