Heavy-tails in economic data: fundamental assumptions, modelling and analysis

João P. da Cruz¹,² and Pedro G. Lind¹,³

¹ Center for Theoretical and Computational Physics, University of Lisbon, Av. Prof. Gama Pinto 2, 1649-003 Lisbon, Portugal
² Closer, Consultoria, Lda, Av. Eng. Duarte Pacheco Torre 2, 14C Lisboa, 1070-102 LISBOA, Portugal,
³ Departamento de Física, Faculdade de Ciências da Universidade de Lisboa, 1649-003 Lisboa, Portugal

February 2, 2012

1 Introduction: fundamental postulates

The study of heavy-tailed distributions in economic and financial systems has been widely addressed since financial time series has become a research subject. In the end of 19ᵗʰ century, Louis Jean-Baptiste Bachelier[3], who first suggested that what was latter known as Brownian Motion[18] could explain stock market price fluctuations, already mentions in his “The Theory of Speculation” that not all fluctuations can be described by Gaussian stochastic forces and in 1963 a seminal paper of Mandelbrot[33] clearly shows that a market time series exhibit a much higher frequency for big variations than the expected if one assumes a stochastic process with Gaussian noise.

Nevertheless, with the introduction of Ito’s lemma by Black, Scholes and Merton[37, 9] in European options valuation, Gaussian noise(GN) based processes become a market standard, due to the imposition of a revolution in the derivative market that multiplied several times business volumes and influenced financial market practices to the point where regulation itself is based on the same assumptions[38, 39, 7]. After the eighties, several “highly improbable” market drops were observed (e.g. the 1987 stock market drop known as “Black Monday” and on even more recent ones, already in the 21st century) that produce heavy losses that were unexplainable in a GN environment. The losses incurred in these large market drop events did not change significantly the market practices or the way regulation is done but drove some attention back to the study of heavy-tails and their underlying mechanisms. Some recent findings in these context is the scope of this chapter.

Heavy-tails of financial and economic variables distribution can be investigated through two different approaches.

One is what we call the epistemological approach, in which the behavior of the economic system is explained by a few key features of the behavior itself, e.g. the amplitude of price fluctuations or the analytical form of the return distributions[36, 10]. The ultimate goal in such an epistemological approach is to construct a function from the scratch that fits any set of empirical data just by building up parameters until the plotted function fits it.

The other approach we call the ontological approach and is based in what is commonly known as agent modeling[41] to try to find the mechanisms responsible for the emergence of the heavy-tails. Agent-based models for describing and addressing the evolution of markets has become an issue of increasing interest[21] and has been subjected to significant developments[23, 26, 42].

Still, from a point of view of market acceptance and practice adoption, the epistemological approach is the dominant one and both researchers and practitioners take Gaussian distributions as Ansatzes for the
return distributions observed in markets or, if motivated by the findings of Mandelbrot, \(\alpha\)-stable Lévy distributions or truncated Lévy distributions. And, in fact, such approach would be the best one, if economic processes were stationary.

Unfortunately, they are not and there is no guarantee that today’s fitting on such distribution will be the same as tomorrow’s. This means that we cannot disregard the underlying mechanisms in the analysis of empirical data, which appeals for the agent modeling approach.

The ontological approach has three main advantages. First, the system is able in this way to be decomposed directly into sellers and buyers, being a straightforward translation of the finance system itself. Second, one does not need to assume the system to be in equilibrium, a very important aspect regarding the fact that financial markets are not systems in equilibrium. Third, by properly incorporating the ingredients of financial agents and the trades among them one can directly investigate the impact of trades in the price, according to some prescribe scheme, investigating directly possible future scenarios of the finance system.

From the bottom to the top, looking from the agent model to the distribution that results from their aggregation, both Gaussian and heavy-tailed distributions are of interest. The common usage of either distribution types is given by the Central Limit Theorem, which states roughly that the aggregation of a growing number of random variables converges to an \(\alpha\)-stable Lévy distribution. More precisely, in its classical and stronger version, the Central Limit Theorem states that a sequence of independent and identically distributed random variables with finite expected values and variance, sum up to a Gaussian distribution when the sample size is increased. Lyapunov version of the theorem proves that the same is valid in the case where the random variables that are not identically distributed, requiring that they have moments of order greater than two and its growth rate limited by the Lyapunov condition or Lindeberg condition.

In both cases the random variables are independent from each other. If they are not, one may still group sets of such dependent variables into independent and identical distributed clusters of variables and, again, the result is a Gaussian distribution. One exception gets out from these Gaussian distributed situations: one where the size of these clusters is of the order of the total size of the system, i.e. when the second moment diverges, \(E(X^2) \to \infty\). For such kind of dependent processes instead of a Gaussian distribution, a \(\alpha\)-stable Lévy distribution with \(\alpha < 2\) will result and the typical heavy tails and power-laws are observed.

The particular case \(\alpha = 2\) in a stable Lévy distribution yields the Gaussian distribution. One feature of these \(\alpha\)-stable Lévy distribution are their heavy tails for \(\alpha < 2\), which will be addressed in the following Sections.

So, in order to a aggregation of an agent related metric have an heavy-tailed distribution a dependence mechanism is needed that bind each individual agent distribution. Several agent models in finance were already proposed. The Solomon-Levy model defines each agent as a wealth function \(\omega_i(t)\) that cannot go bellow a floor level. This floor level is given by \(\omega_i(t) \geq \omega_0 \bar{\omega}(t)\) where \(\omega(t)\) is the agent average \(\omega\) at instant \(t\), \(\omega_0\) being a proper constant. The imposition of the floor based on the mean field \(\omega(t)\) introduces the individual distribution dependence and, on average, \(|\omega_i(t) - \bar{\omega}(t)| \sim N\) and therefore the variance scales with \(N^2\).

Other authors like Cont-Bouchaud or Solomon-Weisbuch use percolation based models, which also bring up variations of system size order of magnitude and, obviously, will lead also to Lévy-type distributions.

In a nut shell, if a multi-agent model introduces a dependence mechanism that leads to infinite variance, it results in a heavy-tailed distribution. But, unlike Gaussian distribution, which can be the result of several different system configurations, the form of heavy-tailed distributions depend strongly on each detail of the system. Any small change in model parameters will produce different results, i.e. different tail exponents. This means that, if Gaussian models can be safely fitted against empirical data, one cannot fit an heavy-tailed model to empirical data unless the data is produced by a system with the same model parameters, namely the ones that form the dependence mechanism. So if a multi-agent model does not give additional information and only provides a system with infinite second moment, there is no particular surplus in the application of such a model compared with a direct fitting of an \(\alpha\)-stable Lévy
distribution. Moreover, all complexity used to build in the individual behavior of the agents\cite{32}, like dividing agents by their rationality, will not contribute to the actual solution of the problem because is equivalent to a direct fitting.

Following the above considerations, we can eliminate from our reasoning every microscopic scenario that result on either weak or strong forms of the Central Limit Theorem that will result on a Gaussian distribution. What remains? Every single microscopic scenario that generates a system size order variance. From this point forward we can choose arbitrarily our model composition, which opens a rather broad panoply of possibilities. This mathematical freedom to choose the model for the underlying mechanisms is only constraint by the requirement of bridging from the biological foundations of an economic system to the behavior of a market curve. With such bridging we are able to anchor information from the market curve to established theoretical principles of economy without additional assumptions like in\cite{15}. We will show here that it can be accomplished by using epistemological and ontological approaches at the same time.

Here, we aim to propose a minimal agent model based in the fundamental properties for observing heavy-tailed distributions. In other words, the question we are interested in is what is the minimum model for an economic system, where the emergence of heavy-tails take place?

In short, our answer is that there are three fundamental postulates for observing heavy-tails and therefore critical behavior and crisis in economic systems:

\begin{enumerate}
\item P1 First, agents tend to deal with each other and promote trading. Human beings are more efficient doing specialized labor than being self-sufficient and for that they need to exchange labor\cite{31}. The more labor exchanges they make the more specialized they can be. The usage of the expression ‘labor’ can be regarded as excessive by economists, but readers can look at it as the fundamental quantity that is common to labor, money, wage, etc. that justifies the exchange. Something must be common to all these quantities; if not, we wouldn’t exchange them. The physicists can regard such fundamental quantity as an ‘economic energy’.

\item P2 Second, each agent has some attractiveness. Human agents are different and attract differently other agents to trade. As we see bellow, this difference should reflect some imitation, where agents tend to prefer to consume (resp. produce) from (resp. for) the agents with the largest number of consumers (resp. producers). The number of producer and consumer neighbors reflects supply and demand of its labor, respectively, and combining both kind of neighbors should suffice to quantify the price of the labor exchanged. Further, the attractiveness of each agent leads to the formation of an attractive field that generates a given topology, as explained below.

\item P3 Third, it is not possible for the system to consume infinite energy and, for that reason, a limit leverage for each agent must exist. We only consume and produce a finite amount of the overall product. Thus, if an agent transposes that finite amount it loses his (consumption) trade connectivity in the economic environment in order to return to an admissible state that guarantee the finiteness of the overall product within the system. This assumption is similar to the non-linear threshold of leverage introduced by Merton\cite{37} in his approach to the valuation of corporate debt.
\end{enumerate}

In this paper we show that these three postulates P1, P2 and P3 form the minimal model that can explain the phenomena associated with critical behavior of the underlying economic systems as we will show in the next sections. In particular, heavy-tailed return distributions or power-laws emerge due to the economic organization and (complex) structure of trades among agents governed by the above three postulates.

We also show that the power-law tails are characterized by an exponent that can be measured and it is constrained by upper and lower bounds, which we deduce analytically. The knowledge of such boundaries are of great importance for risk estimates: by deriving upper and lower bounds, one avoids underestimates which enable the occurrence of crisis unexpectedly and avoids overestimates which prevents profit maximization of the trading agents.

We start in Section\cite{2} to show that postulates P1 and P2 naturally define a multi-agent system with an attracting pair potential that underlies the emergence of a scale-invariant geometry, characterized by
a heavy-tailed distribution of single agent connections with a given exponent. Next, in Section 3 we show that this geometry is constrained, namely the exponent that describes the heavy-tails for the agent interconnections is bounded by minimum and maximum values. In Section 4 we add the third postulate, P3, and translate all three postulates into a model that produces macroscopic observables in economic systems. In Section 5 we show that the model behaves critically, transiting between two phases, and its macroscopic observables reproduce statistically critical features observed in real economic systems. In particular, we focus on the fluctuations of such observables that are related to the underlying geometry of economic relations and, consequently, can be helpful to derived model risk measures. In section 6 we describe in detail a specific application, showing how the model can be applied to address the impact of financial rules in the stability of the economic system. Finally in Sec. 7 we will draw some conclusions.

2 Definition of attractive fields in economic networks

Much of advertising is about showing people that a product or service is chosen by many others and there is no record of a single case where the lack of customers is promoted as a competitive advantage. Advertisers make use of what physicists call “preferential attachment” [43, 5], while some authors in Economics call it imitation [11], and it is a representation of P1 and P2 postulates. Here we will show that such postulates in fact imply preferential attachment and that have important consequences in the topology of the economic system, by making use of many-particle physics.

Based on P1 and P2, we can regard a system of economic agents in the same way we regard a system of mechanical particles that attract each other. Since every particle influences every particle, potential wells are formed associated with each particle in the same way orbits are formed. So any new particle that enters the system will choose a well were it will be trapped due to the dominant influence of one particle relatively to every others.

How can we express this dominance that makes a particle to choose a well and not the others? According to P2, each particle has a different attractiveness. Assuming that no other mechanical influence is present besides the particle-particle attraction, then we can represent the attraction field $G_i \equiv G(M_i)$ that particle $i$ exerts over the other ones due to its attractiveness $M_i$. The nature of $M_i$ is irrelevant for our purposes, as latter will become evident.

In such a context, if a new particle enters the system, it will suffer the influence of all existing particles, each one with a relative intensity given by

$$I_i \approx \frac{G(M_i)}{\sum_j G(M_j)}.$$  \hspace{1cm}(1)

where $N$ is the total number of existing particles (agents), which we assume as $N \to \infty$. At first approximation, the absolute field intensity can be made equal to attractiveness $G(M_i) \approx M_i$ and

$$I_i = \frac{M_i}{\sum_j M_j}.$$  \hspace{1cm}(2)

Even not knowing what is the nature of $M_i$, the relative intensity can be safely assumed as measurable since it is dimensionless. Furthermore, Eq. (2) is in fact the relative distribution of attractiveness over the system. Since no other mechanical influence is present, attractiveness distribution can be translated as the relative number $k_i$ of neighboring agents trapped by the well of each agent $i$, i.e.,

$$I_i \equiv P_i = \frac{k_i}{\sum_j k_j}$$  \hspace{1cm}(3)

which is perfectly equivalent to a discretization of Eq. (2), supported as $N \to \infty$. For a better understanding, since the nature of attractiveness is not known, as it happens in economic systems, Eq. (3)
says that it is deduced by the number of agents each agent $i$ attracted, since it is an indirect measure of attractiveness. At the same time, statistically speaking, Eq. (3) represents an histogram of the relative frequency of agents attracted by other agents and, by the Law of Large Numbers [19], it converges to a probability distribution that equals $P_i$ in Eq. (3).

The expression (3) has been used as an assumption on several previous works [11, 5, 17, 22] for the representation of the natural tendency that humans have to connect to the ones that already have more connections. As we will deduced below, there is an important consequence of this conclusion [4]: if only this attractiveness is considered in the network, the probability of randomly finding a body with $k_i$ connections is

$$\Pi(k_i) \propto k_i^{-\gamma}$$

where $\gamma$ is one exponent characterizing the system topology. A power-law as the one in Eq. (4) is a rather general law for the structure of a system with different bodies out of equilibrium, and it can be observed in many economic-like systems ranging from airports [30] to the World Wide Web [12].

Thus, from this section we conclude that economic agents organize themselves in a scale invariant topology given by Eq. (4), which follows from postulates P1, P2 and the Law of Large Numbers.

3 From geometric constraints in economic networks to bounded heavy-tails

Speaking about geometrical constraints in Economics may sound odd and too abstract but the fundamental message that will be evident bellow, tells that heavy-tails in Economy are characterized by exponents whose values are bounded, due to the geometry of economical trading connection. We start by considering the Eq. (4) that was derived in the previous section based on P1 and P2 and that has a significant meaning in terms of the topology of the economic system. It means that the probability law is scale invariant or, geometrically speaking, self-similar [34]: the entire economic system, is equal to a part of itself, at least in statistical terms.

The question we want to answer in this section is can $\gamma$ in Eq. (4) assume any (positive) value? Real economic networks show typically values in the range $2 < \gamma < 3$ [1, 30], but that is taken from the empirical data of electronic databases for networks that have some kind of computational treatment. The empirical analysis of such networks does not provide a rigorous range for every other network type, in particular network underlying commercial and market systems. Next, we will show that postulates P1 and P2 imply that indeed $\gamma$ is indeed bounded in [2, 3]. To this end, we separate the cases of directed and undirected networks and use the method introduced by Song, Havlin and Makse [45].

Since the arguments for this demonstration are mainly geometric, readers should follow the figures together with the text. Figure 1 represents a ‘renormalization’ of an undirected network, i.e. a network where connections have no direction, are symmetric: connecting $i$ to $j$ is equivalent to connection $j$ to $i$.

When renormalizing such a network, one intends at each scale to rescale each object according to a common particular structure. When rescaling, connection $AB$ gives rise to two additional connections, $A_1A_2$ and $B_1B_2$, and remains as a connection of the structure. We are searching the conditions in which the distribution $P(k)$ of the number of connections one agent has is of the form of Eq. (4). To that end, we observe that only agents $A_1$ and $A_2$ can have their connectivity changed, i.e. their degree $k$ affected, and that occurs if $AB$ starts from $A_1$ or from $A_2$. So, whatever is the form of distribution $P(k)$ it is invariant under renormalization from one scale (generation) to a following one if, and only if, it changes with the number $Z$ of new agents that emerge after renormalizing, i.e. $\frac{dZ}{dp}P_{\gamma}(p,k) = 0$ where $p$ represents the scale.

Analogously, for undirected networks, the renormalization affects the connectivity of all emergent agents producing this time new $Z^2$ additional states, since $AB$ can connect any pair of them. So, in this case, $\frac{dZ}{dp}P_{\gamma}(p,k) = 0$. Thus, we can in general write

$$Z^\ell P_d(k)dk = Z_p^\ell P_d(k_p)dk_p$$

5
Figure 1: Illustration of renormalization in complex networks. Starting at connection $AB$ between two clusters of agents, one scales down finding each cluster composed by two sets of agents, $A_1$ and $A_2$ on the left and $B_1$ and $B_2$ on the right connected again by $A_1A_2$ and $B_1B_2$ respectively. Each sets of agents of this new “generation” can also be decomposed in two connected sets and so on downside.

With such construction, in directed networks, the number of probable states grows with the number $Z$ of renormalized agents, since only the nodes (agents) in one of the two sets can be chosen. Differently, for undirected networks, the number of probable states grows with the square of the number of renormalized agents $Z^2$, since all agents in all steps contribute to the number of probable states. Such difference leads to two extreme exponent values which bounded the possible exponents in heavy-tails of economic agent-models.

where $\ell = 1$ for directed networks and $\ell = 2$ for undirected networks, and where $k$ and $k_p$ are, respectively, an original and renormalized quantity of links and $Z$ and $Z_p$ are the correspondent quantity for agents.

Since, according to $P1$ and $P2$, the system converges to a topology defined by Eq. (4), then

$$P(k) \rightarrow P(k_p) \propto k_p^{-\gamma}$$

If the renormalization of the links are expressed as $k \rightarrow k_p = \alpha_p k$, renormalization will result in

$$Z^\ell = Z_p^{\alpha_p - \gamma + 1}.$$

To establish the geometrical form of the economic network, we need to find the relation between agent renormalization and link renormalization which is partially translated by Eq. (4). With that objective, we now define $l_B$ as a distance in the economic space between agents. The fractal dimension $d_B$ of the network of agents can then be calculated using a box-counting technique as

$$Z_p = Z l_B^{-d_B}$$

and the links will scale as

$$k_p = k l_B^{-d_k}$$

where $d_k$ is the fractal dimension of the network of links, which yields $\alpha_p = l_B^{-d_k}$.

Then, using Eqs. 4, 5, 8, and 9, the $\gamma$ value can be obtained as a function of the fractal dimension of both agents and links as

$$\gamma = 1 + \ell \frac{d_B}{d_k}.$$
which is the geometrical representation of $\gamma$ in a space of economic agents and economic links.

Some additional discussion here is needed. Let us suppose (wrongly!) that economic networks are unweighted, i.e., all links are equal. Then, one can prove that adding one new agent with a link to an existing agent, chosen from probability distribution as in Eq. (3) (preferential attachment), will lead to a topology with degree distribution given by Eq. (4) with $\gamma = 3$. That is coherent with our result since at the end of each link there is one agent and for each agent there is one link.

On the other hand, if we deny P2 and assume that all agents have equal attractiveness. In such a case, each node would link to every other node and we would have $Z^2 - Z$ links for each set of $Z$ nodes. Meaning that, asymptotically with $Z \to \infty$, $Z^2 - Z \sim Z^2$, yielding $d_k = 2d_B$ and $\gamma = 2$.

Since economic networks are not unweighted or agents are not created continuously, then due to P1, links are created between the existing nodes leading to a weighted network. Would P2 be not valid, links would be created randomly and, on average, it would result in an unweighted network, holding the above results. On the contrary, taking P2 as valid, we can normalize the link weights relatively to the maximum, meaning that $\gamma = 3$ happens when all weights are equal. If not, then $\gamma < 3$ because in such a case one has $d_k > d_B$.

In one sentence, this section leads us to the conclusion that postulates P1 and P2 imply a bounded geometry of the interconnected set of agents namely $2 < \gamma < 3$. This result is not only corroborated by empirical data but it is of utmost importance: independently of the complexity associated with economic processes and with the network of economic relations, the economic network is geometrically bounded. Consequently, the heavy-tailed distributions extracted when observing the network should also follow similar constraint which result in bounded exponent values, as we will show below, in Section 5.

4 Minimal agent model for the emergence of heavy-tailed return distributions

In this section we will introduce a minimal model for economical heavy-tails, which incorporates postulates P1, P2 and, P3, introduced above in order that becomes possible to analyze empirical data, to simulate hypothetical scenarios and deduce macroscopic relations.

We start by defining an economic connection as an exchange of labor between two agents, $i$ and $j$, which dissipates an amount $U_i$ in agent $i$. Agent $i$ delivers an amount of labor $W_{ij}$ to agent $j$ and gets a proportional amount of labor $E_{ij} = \alpha_{ij}W_{ij}$ (11) where $\alpha_{ij}$ is the ‘exchange rate’ of labor. Figure 2 illustrates the economic connection between one agents and some of its neighbors $j_1$, $j_2$, $j_3$ and $j_4$. Each one of the directed connections from $i$ to $j$, are seen differently by each one of the linked agents: while agent $i$ takes the connection as a production (or outgoing) connection, agent $j$ as a consumption (or incoming) connection.

Assuming that each agent can connect to several other, since every agent will have the propensity to establish new economic links to get specialized labor from other agents, the balance equation on each agent $i$ will be

$$U_i = \sum_{j \in L_{\text{out}}} k_{\text{out}}W_{ij}(1 - \alpha_{ij}) + \sum_{m \in L_{\text{in}}} k_{\text{in}}W_{mi}(\alpha_{mi} - 1)$$

where $L_{\text{out}}$ is the set of neighboring agents to which $i$ delivers labor and $L_{\text{in}}$ the set of neighbors from which $i$ gets labor. For illustrative purposes we will call henceforth $W_{ij}$, $E_{ij}$ and $U_i$ energies, though the balance equation in Eq. (12) has no straightforward parallel to the energy balance in a physical system.

Since labor in this context have arbitrary units we approximate

$$W_{ij} = \langle W \rangle_{t=0},$$

7
Figure 2: Illustration of economical connections between economic agents. Agent \( i \) transfers an amount of “labor” \( W_{ij} \) to agent \( j \) receiving in return an amount \( E_{ij} = \alpha_{ij} W_{ij} \), where \( \alpha_{ij} \) measures how well the labor is rewarded. For this trade “interaction” agent \( i \) establishes an outgoing connection with (production to) agent \( j \), while agent \( j \) establishes an incoming connection with (consumption from) agent \( i \). The balance of this interaction yields for agent \( i \) an amount of “internal energy” \( U_i = \sum_{jk} U_{ijk} = \sum_{jk} W_{ijk} - E_{ijk} \) that can be summed up over all agents connections.
and representing $U_i$ in units of $\langle W \rangle$, namely

$$U_i = k_{i,\text{out}} - k_{i,\text{in}} + \sum_{m \in L_{\text{in}}} k_{m} - \sum_{j \in L_{\text{out}}} k_{j} \alpha_{ij}. \quad (14)$$

Assuming that the exchange rate $\alpha_{ij}$ of any agent is approximately the rate average $\alpha = \langle \alpha_{ij} \rangle_{\text{tot}}$ over the system, we can make a mean-field approximation as

$$U_i = \beta (k_{i,\text{out}} - k_{i,\text{in}}) \quad (15)$$

with

$$\beta = (1 - \alpha). \quad (16)$$

This balance, translated in Eq. (15), between consumption and production must be bounded for each single agent because the only way the system dissipates physical energy if through the agents and dissipation must be finite according to the laws of thermodynamics. Thus, agents cannot leverage themselves to infinity, i.e. one agent cannot deliver more then a certain amount of labor from other agents. Similarly, it cannot consume an infinite amount of labor. This is what is reflected in postulate P3. To account for this bounded balance that takes place for each agent, we define a property $c_i$ proportional to $U_i$ which is bounded by a threshold $c_{th}$ below which the other agents break their production connections to it. We can regard this threshold the same way as a limit for default in credit risk modeling [48] and in the following we derive an heuristic expression for it.

The quantity $c_i$ should not only be proportional to $U_i$ but also inversely proportional to some fitness of the agent, namely a quantity that measures its importance in the system, i.e. a turnover. The total turnover can be defined as the total number $k_{i,\text{out}} + k_{i,\text{in}}$ of connections involving agent $i$, both incoming and outgoing [14]. Also, since we are assuming P3, from a dynamical point of view, $k_{i,\text{out}}$ or $k_{i,\text{in}}$ can represent individually a turnover since, on average, the system tends to correlate both. Thus, the threshold is defined as relative to this economic importance in a similar way as the usage of financial ratios as a measure of probability of default [17]. By other words, and due to what was shown in the previous sections, this threshold should be invariant in connectivity transformations.

For one agent $i$, considering the turnover as the total connectivity,

$$c_i = \frac{U_i}{T_i} = \frac{k_{i,\text{out}}}{k_{i,\text{in}}} - 1. \quad (17)$$

Thus, each agent sees his “relative” internal energy $c_i$ bounded, namely

$$c_i \geq c_{th} \quad (18)$$

where $c_{th}$ is taken as independent from connections.

The quantity $c_i$ in Eq. (17) takes values above $-1$ and is well-defined for any agents with at least one incoming connection ($k_{i,\text{in}} > 0$), a condition fulfilled by all agents, since economic agents cannot sustain zero consumption. The particular lower threshold value $\bar{c}_i = -1$ indicates an infinite leverage (asymptotic level), $k_{i,\text{in}} \to \infty$.

When Eq. (18) is not fulfilled, the agent loses its incoming connections and will not be able to “pay” to its neighbors that will stop “working” for him. When this occurs the following is imposed to the collapsed agent $i$ and its neighbors, indexed as $j$:

$$U_i \to U_i + \beta k_{i,\text{in}} \quad (19a)$$
$$T_i \to T_i - k_{i,\text{in}} \quad (19b)$$
$$U_j \to U_j - \beta \quad (19c)$$
$$T_j \to T_j - 1. \quad (19d)$$
Finally, one ingredient must also be properly modeled, namely the exchange rate defined above. The exchange rate of labor $\alpha_{ij}$ can be taken as a dimensionless price and is defined heuristically as follows. See also Fig. 3.

From Eq. (15) one can take $k_{in,i}$ as a measure of the supply available by agent $i$. Similarly, $k_{out,j}$ measures the demand that one of its neighbors $j$ has. When $k_{in,i}$ is large it means that many agents are working for agent $i$ and therefore each one, in particular neighbor $j$, has a shrunk importance as working agent. Thus, $\alpha_{ij}$ should reduced with increasing $k_{in,i}$. Similarly when $k_{out,j}$ is large it means that neighbor $j$ is working for many other agents, each one with a proportionally small importance, in particular the one of agent $i$. Thus, the exchange rate for agent $i$ should increase with $k_{out,j}$. Considering simultaneously both number of connections, $\alpha_{ij}$ must increase and decrease monotonically with $k_{out,j} - k_{in,i}$. Additionally, $\alpha_{ij} \geq 0$ and the first derivatives must decrease when $|k_{out} - k_{in}| \to \infty$.

Putting all such features together, $\alpha_{ij}$ must be a step function of $(k_{out,i} - k_{in,j})$ with $\alpha_{ij}$ attaining a maximal (resp. minimal) value for $k_{out} \gg k_{in}$ (resp. $k_{out} \ll k_{in}$).

So, our Ansatz for numeric simulation purposes is

$$\alpha_{ij} = \frac{\alpha_{\text{max}}}{1 + e^{\frac{k_{out,i} - k_{in,j}}{\Delta_{ij}}}}$$

which is basically a step function with average value at $\alpha_{\text{max}}/2$, similar to the hyperbolic tangent proposed by Cont-Bouchaud [13]. We call $\Delta_{ij}$ the absorbing length from agent $i$ to agent $j$. To understand the role of the absorbing length one may easily consider two extreme situations: (i) for $\Delta_{ij} \to \infty$ one has $\alpha_{ij} = 1$ always, i.e. independently of the degree of agents $i$ and $j$ the trade is purely symmetric ($W_{ij} = E_{ij}$) and (ii) for $\Delta_{ij} \to 0$ the dimensionless price $\alpha_{ij}$ is either $\alpha_{\text{max}}$ or 0, depending whether $k_{out,i} > k_{in,j}$ or $k_{out,i} < k_{in,j}$ respectively and equals one for $k_{out,i} = k_{in,j}$. Henceforth we consider $\alpha_{\text{max}} = 2$ and $\Delta_{ij} = 1$ without loss of generality.
In general, other functional forms of steps functions describing $\alpha_{ij}$ yield similar results as the ones shown below.

The “multi-agent” model described in this section is thus a direct and complete translation of the three postulates P1, P2 and P3, the latter being introduced in order that agents are not independent from each other. Moreover, their interdependence is expressed in an intuitive form, through Eqs. (19), associated to a nonlinear threshold widely use in finance and risk modeling [48] translated here as Eq. (17), completely justified by the postulates.

5 Emergence of heavy tails in empirical financial indices

Having translated the postulates into a model, we next show next that under them the economic system remains at a critical state generating the expected heavy-tailed distributions. To this end, we need to describe how agents chose their trading neighbors.

We can assume that a consumption (incoming) connection of agent $i$ from agent $j$ occurs with a probability proportional to the outgoing connections (demand) agent $j$ already has. Such scheme follows the above described imitation attachment mechanism [10], widely study in the context of complex networks where it is called a preferential attachment scheme [1]. Thus, the system of $N$ agents is initialized by addressing incoming connections between one agents and another neighbor with a probability proportional to its outgoing degree.

Every time, one new agent enters the system, e.g. after one leaves due to collapse (see above), the same preferential attachment scheme is applied. See Eq. (3).

Due to the preferential attachment scheme [1, 4], in the initial state of the system, the outgoing connections follow a $\delta$-distribution $P_{out}(k) = \delta(k - k_{out})$ and the incoming connections follow a scale-free distribution $P_{in}(k) = k^{-\gamma_{in}}$, where $\gamma_{in}$ is the exponent of the degree distribution. As the system evolves, the number of agents remains constant but at each event-time $n$ one new connection joining two agents is introduced, with both agents independently chosen according to the preferential attachment scheme mentioned above. Thus, through evolution both consumption and production networks are pushed to a degree distribution of the form [4]

$$P(k) \sim q k^{-\gamma}.$$

being $q$ the connection creation rate, as translation of P1.

To characterize the system during its evolution, we measure the total system internal energy, which accounts for all outgoing connections in the system at each time step, namely

$$U_T = \sum_{i=1}^{N} \sum_{j \in V_{out,i}} (W_{ij} - E_{ij}),$$

where $V_{out,i}$ is the set of neighbors to which agent $i$ has outgoing connections.

We call the quantity $U_T$ the overall product and its evolution reflects the development or fail of the underlying economy, similar to a financial index or GDP. Alternatively, $U_T$ can be calculated from the incoming connections.

Instead of the time series for $U_T$, one observes the corresponding logarithmic returns $dU_T/U_T$, shown in the inset of Fig. 4, because it is a relative, dimensionless, quantity. Figure 4 clearly shows that the distribution $P(dU_T/U_T)$ of the logarithmic returns is non-Gaussian.

Parallel to $U_T$ we also keep track of the number of collapses (see Eqs. (19)) occurring successively as described below. In the economical context, one such chain reaction is called a “crisis” while in the physical context one calls them “avalanches”. Henceforth, we address to these chain reactions as crisis or avalanches indistinctly and we symbolize their size by $s$. In Fig. 4b the cumulative distribution $A(s)$ of the avalanche size $s$ is plotted showing a power-law whose fit yields $A(s) \sim s^{-m}$ with an exponent $m = 2.51$ ($R^2 = 0.99$).

11
Figure 4: Return distributions and avalanches size distributions in the economy agent model for the variation of the total internal energy, \(dU_T/U_T\). (a) Probability density function (PDF), showing the asymmetry between positive and negative variations and (b) the avalanche size distribution (ASD) similar to the ones observed for empirical data (compare with Fig. 2). In the inset one sees the time series for which the PDF and the ASD are computed. Here \(N = 1000\), \(q = 1\) and \(W_{ij} = 1\) for all \(i\) and \(j\).

The size distribution of avalanches can be derived also in the scope of the mean-field approach \((\alpha_{ij} \sim \alpha)\) as follows.

An agent collapses whenever Eq. (18) does not hold, which yields the following conditions:

\[
\begin{align*}
  k_{i,\text{out}} - k_{i,\text{in}} &> u_{th}(k_{i,\text{out}} + k_{i,\text{in}}) \\
  k_{i,\text{out}} - 1 - k_{i,\text{in}} &\leq u_{th}(k_{i,\text{out}} - 1 + k_{i,\text{in}})
\end{align*}
\]

or more simply

\[
\omega k_{i,\text{in}} < k_{i,\text{out}} \leq \omega k_{i,\text{in}} + 1
\]

with \(\omega = \frac{1+u_{th}}{1-u_{th}}\). Therefore, the probability for agent \(i\) to collapse is

\[
P_{br} = P(\omega k_{i,\text{in}} < k_{i,\text{out}} \leq \omega k_{i,\text{in}} + 1).
\]

Since \(k_{i,\text{in}}\) and \(k_{i,\text{out}}\) are integers, there is only one integer value for \(k_{i,\text{out}}\) in the interval \(\omega k_{i,\text{in}} < k_{i,\text{out}} \leq \omega k_{i,\text{in}} + 1\), given approximately by \(k_{i,\text{out}} \sim \omega k_{i,\text{in}} + \frac{1}{2}\) and leading to \((\omega k_{i,\text{in}} \gg 1)\):

\[
P_{br} \approx q(\omega k_{i,\text{in}})^{-\gamma}.
\]
Whenever an agent collapses, if the number of neighboring agents expected to collapse is bigger than one, the system would be in permanent destruction. On the other hand, if that expected number is lower than one, the system would grow indefinitely. In the former case, one is lead to a scenario where agents are isolated and in the latter case one observes a situation of infinite energy consumption which is also not realistic. Therefore, the only possible state of economy is the one where the expected value of neighbor agents to collapse is precisely one, which corresponds to a critical state. In practice this critical state is characterized by a constant intermittency from one phase of economical growth (e.g. bubble) and another of economical drop (crisis). Indeed critical behavior addresses precisely phenomena with such alternate growing and dropping regimes, as economists always said they exist in our modern economies and as
everybody experiences for good or worse.

Since the expected value must be equal to one then
\[ \sum_{k_{in}=1}^{\infty} k_{in} P(k_{in}) P_{br} = 1 \] (28)

and from Eq. (21)
\[ \omega^\gamma = q^\gamma \sum_{k_{in}=1}^{\infty} (k_{in})^{-2\gamma+1} = q^\gamma \zeta(2\gamma - 1) \] (29)

where \( \zeta \) is Riemann Zeta function.

With expression (29) we show that, only by imposing first principles in Economy, \( P_1, P_2 \) and \( P_3 \) above, the economic system is trapped in a critical state defined through the relation between economic growth \( q \), leverage level \( \omega \) and the economic organization characterized by \( \gamma \). A similar derivation can be done for positive variations, instead of drops: instead of limiting the leverage to collapse, one limits the savings for a consumption chain of events that can lead to a heavy-tailed distribution with different exponents on the positive and negative side since the mechanisms are not dependent.

Since we know that the economical network is in a critical state, transiting between two phases, we can look how the microscopic mechanisms that make neighbor agents to collapse generates the heavy-tails. Such mechanisms should be the same for the emergence of bubbles, but here we focus only in the negative part of the distribution \( P(dU_T/U_T) \).

To arrive to the size distribution of avalanches further derivations are necessary. First one should notice that the collapse of a node leads to the breaking of its consumption links or, equivalently, its neighbors’ production links. Meaning that each collapse will provoke a chain reaction of size \( r \), i.e., it originates a branching process as illustrated in Fig. 5 with a probability according to Otter’s theorem \[40\], given by \[40\] [21] [2] [28]
\[ P(z = r) \propto r^{-\frac{3}{2}} \] (30)

where \( z \) is the total number of nodes involved in a single branching process. Equation (30) holds independently of the topology, as long as the branching process is critical, but the number of collapsed agents in a real economic network is difficult to recount for. What is measured when an avalanche occurs, in such a real network, is the number of links destroyed during the avalanche. This number of links accounts for a macroscopic property of the system, namely the overall product \( U_T \) which sums up all outgoing product of all agents. Therefore, we want to express \( P(z) \) in terms of the total number of destroyed links.

From the total number \( r \) of nodes included in one avalanche, since \( P(k) \sim k^{-\gamma} \), the number of nodes with \( k_j \) connections involved in the avalanche is \( n_j = r k_j^{-\gamma} \) which corresponds to \( k_j n_j = r k_j^{-\gamma+1} \) destroyed links in nodes with \( k_j \) links. Thus, the total number of links destroyed is given by
\[ K_T = r \sum_{j=k_{min}}^{k_{max}} k_j^{-\gamma+1} \] (31)

where \( k_{min} \) and \( k_{max} \) are the lowest and highest degree involved. Since the sum on the right side of (31) are the degrees, not the links, we can substitute each \( k_j \) by \( \alpha_j K_T \), where \( \alpha_j \) are leverage dependent coefficient, which leads to \( r \propto K_T^{2\gamma} \), i.e. \( P(K_T) \propto K_T^{2\gamma} \).

So, the fraction of avalanches of size – number of lost links – larger than size \( s \) is given by
\[ P(K_T \geq s) \propto \int_s^{+\infty} x^{-\frac{3}{2}} dx \propto s^{-\frac{3}{2}\gamma+1} \equiv s^{-m} \] (32)

yielding
\[ m = \frac{3}{2}\gamma - 1 \] (33)
Figure 6: (a) Time evolution of the logarithmic returns of the DJIA index (partial). The box is zoomed in (b) to emphasize the contrast between real data (solid line) throughout time $t$ and the succession of events (dashed line) where the variation changes sign. (c) Probability density functions for some important financial indices, including interest rate options (CBOE). (d) Distribution of avalanche sizes detected throughout the evolution of each financial index showing critical behavior with (e) an exponent $m$ approximately invariant and similar to the one obtained for branching processes and to our model (see text). Solid line in (e) indicates the theoretical value obtained for $\gamma = 3$ (see Eq. (32)), while dashed lines indicate the bounding values of $m$ for $2 < \gamma < 3$ (see Sec. 3).

which connects the exponent $m$ characterizing dynamical events (avalanches) with $\gamma$, the exponent characterizing the structure (degree distribution) of the underlying economical network.

Looking again to Fig. 4b, with the help of Eq. (32) and borrowing from the literature[1], the values of $\gamma$ of empirical networks, typically in the range $[2.1, 2.7]$, one concludes that the size distribution exponent should take typical values $m \in [2.15, 3.05]$ which agrees with the results from our model, as shown not
only in Fig. 4b, but also in Fig. 6 which presents data from several stock market indexes.

The time-series of the logarithmic returns shown in Fig. 6 must first be mapped in a series of events as shown in Fig. 4b. One event is defined as a (typically small) set of successive instants in the original time-series having the same derivative sign, either positive (monotonically increasing values) or negative (monotonically decreasing values). Each time the derivative changes sign a new event starts. In the continuous limit, events would correspond to the instants in the time-series with vanishing first-derivative.

The non-Gaussian distribution of the logarithmic returns (Fig. 6c) were extracted from the logarithm returns of the original series of each index, similarly to what is done in Ref. [27]. The characteristic heavy tails observed by Kiyono and co-workers are observed for short time lag (hours or smaller), whereas in Fig. 6c the daily closure values are considered. The power-law behavior of the avalanche size (Fig. 6d) is indeed similar to the simulated results in Fig. 4b.

Figure 6e shows the exponent $m$ in Eq. (33) for each one of the stock market indexes. All of them take values around the average theoretical prediction $m = 5/2$.

Further, all values lay between to bounds, $m_{min} \equiv 2 < m < m_{max} \equiv 7/2$, as we explain in Section 3.

These two limits play a major role in the description of the critical behavior. Further, since a return distribution can be bounded, it is possible to measure the risk of a wrong model choice. That kind of risk is called “model risk”.

The result in Eq. (32) relates the microscopic economic relations – trades among agents – with the form of the heavy-tailed return distributions. This relation emerges from the interplay between the distribution of the returns, one macroscopic quantity chosen as the overall product $U_T$, and the topology of the social network. The economic connection was defined as one exchange of labor, in one of its forms. No additional assumptions were taken. Since we never made any conjectures regarding the specific type of economic interaction. Equation (32) should hold for all types of trades and macroscopic observables of the economical product in one economic network, since we instantiate economic connections on the more abstract level possible.

All empirical indices are sampled daily but in different time periods. For FTSE 6498 days in London stock market were considered, starting on April 2nd 1984 and ending on December 18th 2009. For DJIA 20395 days in New York stock market were considered, starting on October 1st and ending on December 18th 2009. For DAX 4815 days in Frankfurt stock market were considered, starting on November 26th 1990 and ending on December 18th 2009. For CAC 5003 days in Paris stock market were considered, starting on March 5th 1990 and ending on December 18th 2009. For ALLORDS 6555 days in Australian stock market were considered, starting on August 3rd 1984 and ending on June 30th 2010. For HSI 5701 days were considered in Hong Kong stock market, starting on December 31st 1986 and ending on December 18th 2009. For NIKKEI 6386 days were analyzed in Tokyo stock market, starting on January 4th 1984 and ending on December 18th 2009. For CBOE IR10Y 12116 days in Chicago derivative market, starting on January 2nd 1962 and ending on June 30th 2010.

6 Consequences of heavy-tails in financial stability

In this section we apply our model to investigate the consequences of recent directives in the banking system, which are totally based in the Merton-Vasicek model for credit portfolio risk [35], and simulate it in a P1, P2 and P3 context.

Banks also follow common rules, settled under general agreement. One very important rule is to fix a minimum fraction of invested money as money belonging to the shareholders, i.e. to the bank itself, and not to the depositors. Such money is called minimum capital level and the rules to regulate it establish resilience for the bank system. This minimum capital level is a direct financial translation of postulate P3, because it imposes a minimum threshold to banks that what to stay in business.
Since a bank could, in principle, collect an arbitrary amount of money from its depositors and invest it – or loose it – each country created its own rules for the minimum amount of capital, expressed as a percentage of the total investment. In 1998, a group of central bank governors called the Basel Committee on Banking Regulation unified these rules to all banks, intending to protect the global banking system[6]. Among other, these rules impose a minimum capital level at 8% without any empirical reason[38]. In the second version of the accord[39], the Value-at-Risk paradigm[25] was adopted to calculate the risk weights.

The social pressure over the Committee to tighten the minimum capital rules became very strong after the 2008 financial turmoil, which lead to an unexpected avalanche of bank insolvencies all around the globe, questioning the ability of regulators to make effective bank system protection rules. Thus, in 2010 they issue the third version, Basel III[71], to improve bank system resilience by raising the levels of capital. At the same time, these directives were subjected to significant criticism by the academic community[16] arguing how they “even created the potential for new sources of instability”.

In this section we address this critic to nowadays bank regulation rules by showing that the bank system resilience does not necessarily improve with such raise of the capital levels. To this end we instantiate our model for generic economic networks into the banking network and enable to vary the threshold $c_{th}$.

The agents represent now banks, which provide production of credit (money) to their neighboring banks, through an integer number $k_{out,i}$ of outgoing connections, using its own capital $c_i$ and gets credit through other $k_{in,i}$ incoming connections from agents which make deposits in bank $i$. We call the first (outgoing) connections the creditor connections and the second (incoming) connections the debtor connections. See Fig. 7. The chain of collapses are now taken as chains of bankruptcies.

The bankruptcy of a bank $i$ occurs when the number of destructed creditor connections becomes such that the number of shareholder debtor connections drops below a regulatory minimum level, i.e. $c_{th} > c_i$, leading to the removal of all banks debtor connections of bank $i$, setting $k_{in} = 1$, implying an update in bank $i$ and its neighbors $j$ according to Eqs. (19).
Figure 8: Considering the overall product $U_T$ (units of $W$) of a reference minimum capital level $c_{th} = -0.71$ (solid line), there are two different processes for achieving quasi-stationary states when minimum capital level is increased up to $c_{th} = -0.69$ (a) by keeping the number of agents constant at $L=2000$ (dotted line) or (b) by maintaining the business level constant at $\Omega = 2.88$ in units of $W$ (dotted line), which leads to a decrease of $L = 1500$. (c) The average business level $\Omega$ per bank (Eq. (35)) decreases when raising the minimum capital level and keeping the number of agents constant. (d) Assuming that banks do not want to see their business level dropping, it is natural to expect that they will choose a process with constant business level to face an increase of the minimum capital level.

Once again, having an agent that bankrupts, the natural question that follows is what is the probability for that bankruptcy to trigger an avalanche of bankruptcies (financial global crisis). In the situation above of infinite leverage, when one agent is bankrupted the average number of neighbors that are induced to bankrupt is smaller than one. In the situation of zero leverage this average is larger than one. Since the financial system remains “trapped” between two phases, the average number of bankrupted agents that follow a bank bankruptcy in a chain reaction must be equal to one. And this makes all the difference in terms of banking system stability, especially when we study the magnitude of the avalanche.

Summing up the product due to the creditor connections in the network one gets the overall product $U_T$ in Eq. (22). Figure S.a and S.b shows the evolution of overall products for two different ways of raising minimum capital levels. One time step corresponds to one new trade connection. Black solid line is the reference state. In Fig. S.a minimum capital level $c_{th}$ is raised keeping the size of the system constant (green dashed line), while Fig. S.b shows what happens when the same raise of minimum capital level is accompanied by a decreasing of the system size.

Keeping constant the number of agents that trade within the system, means for each agent to maintain the same neighborhood as previously, before the raise of minimum capital level. We call the system incorporating this neighborhood, the operating neighborhood and its size is given by the number $L$ of agents. The raise of minimum capital level can however occur by maintaining other properties constant.
Figure 9: Keeping the same operating neighborhood for each agent \((L = 2000)\), different minimum capital levels yield different avalanche (crisis) size distributions. For small sizes the Central Limit Theorem holds and different distribution match with each other. Differently, large size avalanches occur at critical states of the system, and therefore the distributions deviate from each other. In the inset one shows the exponent obtained for the critical region for each scenario of minimum capital level: increasing minimum capital level decreases the probability for a large avalanche to occur, which supports the intentions of Basel III accords. However, such behavior is only guaranteed under the assumption of keeping the same operating neighborhood \(L\) for the minimum level raise, a scenario which is not natural in a trading system of economic agents (see text and Fig. 10).

Namely, the average business level per bank, defined as the moving average in time of \(U_T\) per bank:

\[
\Omega = \frac{1}{L} \frac{1}{T_S} \int_{t}^{t+T_S} U_T(x) dx
\]

where \(T_S\) is a sufficiently large period for taking time averages and the \(dt\) in the integrand is given by the time step of our simulation. Similar quantities are taken in Economics as indicators of individual average standards of living\[50\]. Roughly the time derivative of \(\Omega\) gives the overall product uniformly distributed by all agents in the network. Figures 8c and 8d shows the business level per bank corresponding to the overall product in Figs. 8a and 8b respectively. In particular, while in Fig. 8c the business level decreases significantly with the raise of \(c_{th}\), in Fig. 8d the business level is approximately constant, with the smaller size of the operating neighborhood normalizing the overall product. See horizontal lines, showing average values for business level.

Figure 8 shows that keeping constant the size \(L\) of the operating neighborhood while raising the minimum capital level induces a decrease of the business level. Differently, if the business level is kept constant, the size of the operating neighborhood shrinks. One should stress that in real financial systems, to keep the same business level, agents merge which indeed induces the network to shrink.

Next we investigate the first situation, i.e. we consider a raise of the minimum capital level with constant
Figure 10: (a) Normalized critical exponent $\tilde{m}$ as a function of the minimum capital level $c_{th}$ and the size $L$ of the operating neighborhood. For an initial financial state $F_0$ an increase of the minimum capital level to state $F_L$ means to follow an isoline of constant $L$. This leads to a larger $m$ value. A path to a different state $F_L$ having the same minimum capital level raise can be reached, following an isoline of constant business $\Omega$. In this case a smaller value for $m$ is found. As shown in plot (b) while the business $\Omega$ remains constant in the latter case, for the former case (constant $L$) it decreases. Assuming that agents will try to optimize their gains, it is more naturally to expect for them to follow isolines of constant business level which leads to the increase of probability for large crisis to occur (see text).

$L$. To this end we compute the fraction $P_c(s)$ of avalanches of size larger than $s$, yielding the cumulative size distribution of avalanches. Numerically $P_c(s)$ for a given $s$ is obtained by counting at each iteration the total number of debtor connections involved in each chain of bankruptcies, choosing those chains whose total number is larger than $s$.

Figure 9 shows the cumulative size distributions of avalanches in our model for different minimum capital levels, keeping the number of agents constant ($L = 2000$). For small avalanche sizes, the Central Limit Theorem holds\cite{36} and thus all size distributions match independently of the minimum capital level. For large enough avalanches (‘critical region’), the size distributions deviate from each other, showing a power-law tail $P_c(s) \sim s^{-m}$ with an exponent $m$ depending on the minimum capital level $c_{th}$.

As one sees in the inset of Fig. 9 the exponent increases in absolute value for larger minimum capital levels, which prevents large avalanches to occur. Though, such scenario occurs only when the size (number of agents) of the financial subsystem where the agent makes its trades is kept constant. For a scenario where the size of the operating neighborhood is adjusted to maintain the business level constant Eq. (35) the situation is different as shown next, in Fig. 10.

Figure (10a) and (10b) show the critical exponent $m$ and the business level respectively, as a function of the minimum capital level $c_{th}$ and the operating neighborhood size $L$. For easy comparison, both quantities are normalized in the unit interval of accessible values.

Roughly, the critical exponent shows a tendency to increase with both the minimum capital level and the operating neighborhood size, while with the business level the opposite occurs. Considering a reference state $F_0$ with $c_{th,0}$, $L_0$ and $\Omega_0$ there is one isoline of constant system size, $\Gamma_0^L$ and another of constant business level $\Gamma_0^\Omega$, both crossing at $F_0$.

From Fig. (10a), one sees that assuming a transition from state $F_0$ to a state with larger minimum capital level keeping the system size constant, one follows the isoline $\Gamma_0^L$, arriving to the new state $F_L$. This state has a larger value of the critical exponent $m$, which means a lower probability for large avalanches to occur, as explained above. However in such situation the new business level $\Omega_f < \Omega_0$ is smaller than the previous one, as clearly illustrated in Fig. (10b).

On the contrary, if we assume the transition to the higher minimum capital level occurring at constant
business level, i.e. along the isoline $\Gamma_0^\Omega$ one arrives to a state $F_{1\Omega}$ for which the critical exponent is approximately the same, with a larger probability for large avalanches to occur.

From economical and financial reasoning, one assumes typically that independently of external directives, under unfavorable situations, economical and financial agents try, at least, to maintain their business level. Such behavior from agents leads to a situation which contradicts the expectations in Basel accords and raises the question if such regulation will indeed prevent a larger avalanches to occur again in the future.

7 Conclusions

The emergence of heavy-tails on distributions associated with economic and financial phenomena has a direct link with the underlying mechanisms of interaction. Looking from the perspective of the distribution we can deduce the type of mechanisms and from the type of mechanisms we can explain how the heavy-tails occur.

In this chapter we showed the basic postulates describing economic interactions that underlying the emergence of heavy tails in economy: (i) all individuals tend to establish trades, (ii) different individuals have different attractiveness to get into new trades, (iii) all individual have finite leverage.

We incorporated such postulates in a minimal model to demonstrate why a economic system is a system in a critical state. Further, we showed that the geometrical structure composed by the trade connections in the entire economic network has constraints that reflect a finite range for the values of exponents describing heavy-tails. These boundary values are able to account for improved risk measures.

A specific application of our model for critical behavior among economic agents was also addressed, in the context of global banking regulation, showing that the actual tendency and principles for establishing banking rules at the level of individuals bank entities can have been driven on the opposite direction of its goals, since they ignore the complex nature of the economic network that is revealed by the heavy-tailed distributions of the variables that characterize the system.

Two notes are due here. First, in practice, capital is a consumption of wealth from agents called shareholders and could be represented by the amount $k_{in}$. The reason why we do not do it is because banking regulation separate shareholders from depositors and to map the two approaches we need to make the same segregation. Second, the difference between credit connections and debtor connections is not equal to the difference between assets and liabilities, it only represents it. That is, the break of the debtor connections on the agents that connect with the bank represents a destruction of a creditor connection, an asset. The main difference resides on the fact that we are not interested in fixed or non-performing assets. A default represents a destruction of a creditor connection in opposition to bank accounting where the loan becomes a depreciating stock.

Acknowledgments

The authors thank Nuno Araújo for helpful discussions and also PEst-OE/FIS/UI0618/2011 for partial financial support. PGL thanks Fundação para a Ciência e a Tecnologia – Ciência 2007 for financial support.

References

[1] R. Albert and A.L. Barabási. Statistical mechanics of complex networks. Rev.Modern Phys., 74:47, 2002.
[2] K.B. Athreya and P.E. Ney. Branching Processes. Courier Dover Publications, 2004.
[3] L. Bachelier. Théorie de la spéculation. Annales scientifiques de l’É.N.S., 17:21–86, 1900.
[4] A. Barabási, R. Albert, and H. Jeong. Mean-field theory for scale-free random networks. \textit{Physica A}, 272:173, 1999.

[5] A.-L. Barabási and Réka Albert. Emergence of scaling in random networks. \textit{Science}, 286:589, 1999.

[6] Basel Committee on Banking Supervision. History of the Basel Committee and its Membership, 2009.

[7] Basel Committee on Banking Supervision. \textit{Basel III: A global regulatory framework for more resilient banks and banking systems}. Bank for International Settlements Communications, CH-4002 Basel, Switzerland, 2010.

[8] P. Billingsley. \textit{Probability and Measure}. series in probability and mathematical statistic. Willey, 3 edition, 1995.

[9] F. Black and M. Scholes. The pricing of options and corporate liabilities. \textit{J. of Polit. Econ.}, 81:637, 1973.

[10] L. Borland, J.P. Bouchaud, J.F. Muzy, and G. Zumbach. The dynamics of financial markets - mandelbrot’s multifractal cascades, and beyond. \textit{Wilmott Magazine}, March 2005.

[11] J.P. Bouchaud and M. Potters. \textit{Theory of Financial Risk and Derivative Pricing From Statistical Physics to Risk Management}. Cambridge University Press, 2003.

[12] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins, and J. Wiener. Graph structure in the web. \textit{Computer Networks}, 33:309, 2000.

[13] R. Cont and J.P. Bouchaud. Herd behavior and aggregate fluctuations in financial markets. \textit{Macroeconomic dynamics}, 4(2):170, 2000.

[14] João P. da Cruz and Pedro G. Lind. \textit{Physica A}.

[15] J.P. da Cruz and P.G. Lind. The dynamics of financial stability. Submitted(available at http://arxiv.org/abs/1103.0717), 2011.

[16] J Danielsson, Paul Embrechts, Charles Goodhart, Con Keating, Felix Muennich, Olivier Renault, and Hyun Song Shin. An academic response to Basel II. 2001.

[17] S.N. Dorogovtsev, J.F.F. Mendes, and A.N. Samukhin. Structure of growing networks with preferential linking. \textit{Phys. Rev. Lett.}, 85(21):4633, 2000.

[18] A. Einstein. über die von der molekularkinetischen theorie der wärme geforderte bewegung von in ruhenden flüssigkeiten suspendierten teilchen. \textit{Annalen der Physik}, 17:549–560, 1905.

[19] P. Embrechts, C. Kluppelberg, and T. Mikosch. \textit{Modelling Extreme Events in Insurance and Finance}. Springer-Verlag, 1997.

[20] K. Falconer. \textit{Techniques in Fractal Geometry}. J. Wiley & Sons, 1997.

[21] J.D. Farmer and D. Foley. The economy needs agent-based modeling. \textit{Nature}, 460:685, 2009.

[22] M.C. González, P.G. Lind, and H.J. Herrmann. A system of mobile agents to model social networks. \textit{Phys. Rev. Lett.}, 96(088702), 2006.

[23] A.G. Haldane and R.M. May. Systemic risk in banking ecosystems. \textit{Nature}, 469(7330):351, 2011.

[24] T.E. Harris. \textit{The Theory of Branching Processes}. Springer, 1963.

[25] G. Holton. \textit{ValueatRisk: Theory and Practice}. Academic Press, 2003.

[26] N. Johnson and T. Lux. Financial systems: Ecology and economics. \textit{Nature}, 469:302, 2011.
[27] Ken Kiyono, Zbigniew Struzik, and Yoshiharu Yamamoto. Criticality and Phase Transition in Stock-Price Fluctuations. *Physical Review Letters*, 96(6):068701, 2006.

[28] K.B. Lauritsen, S. Zapperi, and H.E. Stanley. Self-organized branching processes: Avalanche models with dissipation. *Phys. Rev. E*, 54(3):2483, 1996.

[29] M. Levy, H. Levy, and S. Solomon. A microscopic model of the stock market: Cycles, booms and crashes. *Economics Letters*, 45:103, 1994.

[30] W. Li and X. Cai. Statistical analysis of airport network of china. *Phys. Rev. E.*, 69(4):1, 2004.

[31] R. Lipsey and A. Chrystal. *Economics*. OUP Oxford, 11 edition, 2007.

[32] T. Lux and M. Marchesi. Volatility clustering in financial markets: a micro-simulation of interacting agents'. *Int. J. Theor. Appl. Finance*, 3, 2000.

[33] B. Mandelbrot. *The journal of business*, 36(4):394–419, 1963.

[34] B. Mandelbrot and R. Hudson. *The (mis)Behavior of Markets - A Fractal View of Risk, Ruin, and Reward*. Basic Books, USA, 2004.

[35] R. Mantegna and H.E. Stanley. Stochastic process with ultraslow convergence to a gaussian: The truncated lévy flight. *Phys. Rev. Let.*, 73(22):2946, 1994.

[36] R. Mantegna and H.E. Stanley. Scaling behaviour in the dynamics of an economic index. *Nature*, 376(6535):46, 1995.

[37] R.C. Merton. Theory of rational option pricing. *Bell J. of Econ. Manag. Sci.*, 4, 1973.

[38] Basel Committee on Banking Supervision. International convergence of capital measurement and capital standards. (available at http://bis.org/publ/bcbs04a.htm), 1998.

[39] Basel Committee on Banking Supervision. Basel ii: International convergence of capital measurement and capital standards: a revised framework. (available at http://bis.org/publ/bcbs107.htm), 2004.

[40] R. Otter. The multiplicative process. *The Annals of Mathematical Statistics*, 20:206, 1949.

[41] E. Samanidou, E. Zschischang, D. Stauffer, and T. Lux. Agent-based models of financial markets. *Rep. Prog. Phys.*, 70(3):489, 2007.

[42] E. Samanidou, E. Zschischang, D. Stauffer, and T. Lux. Agent-based models of financial markets. *Rep. Prog. Phys.*, 70(3):409, 2007.

[43] H.A. Simon. On a class of skew distribution functions. *Biometrika*, 42(3):425, 1955.

[44] S. Solomon and G. Weisbuch. Social percolation. (available at http://xxx.lanl.gov/abs/adap-org/9909001), 1999.

[45] C. Song, S. Havlin, and H. Makse. Selfsimilarity of complex networks. *Nature*, 433:392, 2005.

[46] D. Sornette. *Why Stock Markets Crash?* Princeton University Press, 2003.

[47] L.C. Thomas, D.B. Edelman, and J.N. Crook. *Credit Scoring and Its Applications*. Society for Industrial and Applied Mathematics, 2002.

[48] O. Vasicek. Probability of loss on loan portfolio. KMV Corporation (available at http://www.kmv.com), 1987.

[49] J. Voit. *The Statistical Mechanics of Financial Markets*. Springer, 2003.

[50] Tina Wenzel. *Beyond GDP - Measuring the Wealth of Nations*. GRIN Verlag, 2009.