Monstrous Moonshine: The first twenty-five years

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Abstract

Twenty-five years ago, Conway and Norton published in this journal† their remarkable paper ‘Monstrous Moonshine’, proposing a completely unexpected relationship between finite simple groups and modular functions. This paper reviews the progress made in broadening and understanding that relationship.

1. Introduction

It has been approximately twenty-five years since John McKay remarked that

\[ 196\,884 = 196\,883 + 1. \] (1.1)

That time has seen the discovery of important structures, the establishment of another deep connection between number theory and algebra, and a reinforcement of a new era of cooperation between pure mathematics and mathematical physics. It is a beautiful and accessible example of how mathematics can be driven by strictly conceptual concerns, and of how the particular and the general can feed off each other. Now, six years after Borcherds’ Fields Medal, the original flurry of activity is over; the new period should be one of consolidation and generalisation and should witness the gradual movement of this still rather esoteric corner of mathematics toward the mainstream.

The central question McKay’s equation (1.1) raises, is: What does the \( j \)-function (the left side) have to do with the Monster finite group (the right side)? Many would argue that we still don’t have our finger on the essence of the matter. But what is clear is that we understand far more about this central question today than we did in 1978. Today we say that there is a vertex operator algebra, called the Moonshine module \( V^{♮} \), which interpolates between the left and right sides of (1.1): its automorphism group is the Monster and its graded dimension is the \( j \)-function \((-744)\).

This paper tries to summarise this work of the past twenty-five years in about as many pages. The original article [24] is still very readable and contains a wealth of information not found in other sources. Other reviews are [21], [87], [12], [39], [90], [48], [15], [78], [102], [16], [46] and the introductory chapter in [44], and each has its own emphasis. Our own bias here has been to breadth at the expense of depth, which probably limits this review to be a mere annotated sampling of representative literature.

2. Background

In §2.1 we describe the finite simple groups and in particular the Monster. In §2.2 we focus on the modular groups and functions which arise in Monstrous Moonshine.

2.1. The Monster. By definition, a simple group is one whose only normal subgroups are the trivial ones: \( \{1\} \) and the group itself. The importance of the finite simple groups

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lies in their role as building blocks, in the sense that any finite group \( G \) can be constructed from \( \{1\} \) by extending successively by a (unique up to order) sequence of finite simple groups. For example the symmetric group \( S_4 \) arises in this way from the cyclic groups \( C_2, C_2, C_3, C_2 \).

A formidable accomplishment last century was determining the explicit list of all finite simple groups. See e.g. [49] for more details and references. These groups are:

(i) the cyclic groups \( C_p, \) \( p \) prime;
(ii) the alternating groups \( A_n, \) \( n \geq 5 \);
(iii) 16 infinite families of groups of Lie type;
(iv) 26 sporadic groups.

An example of a family of finite simple group of Lie type is \( \text{PSL}_n(\mathbb{F}_q) \), i.e. the group of \( n \times n \) matrices with determinant 1, and entries from the finite field \( \mathbb{F}_q \), quotiented out by its centre (the scalar matrices \( aI \), where \( a^n = 1 \)).

The mysteriousness of the sporadics is due to their falling outside those infinite families. They range in size from the Mathieu group \( M_{11} \), with order 7920 and discovered in 1861, to the Monster \( \mathbb{M} \), with order

\[
|\mathbb{M}| = 2^{45} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53} .
\]  

(2.1)

The existence of \( \mathbb{M} \) was conjectured in 1973 by Fischer and Griess, and finally constructed in 1980 by Griess [50]. Most sporadics arise in \( \mathbb{M} \) (e.g. as quotients of subgroups). We’ll encounter many of these sporadics in the coming pages, but most of our attention will be directed at \( \mathbb{M} \).

Griess showed in fact that \( \mathbb{M} \) was the automorphism group of a 196883-dimensional commutative nonassociative algebra, now called the Griess algebra, but the construction was somewhat artificial. We now understand [44] the Griess algebra as the first nontrivial tier of an infinite-dimensional graded algebra, the Moonshine module \( V^\natural \), which lies at the heart of Monstrous Moonshine. We’ll discuss \( V^\natural \) in §4.2; we will find that it has a very rich algebraic structure, is conjectured to obey a strong uniqueness property, and has automorphism group \( \mathbb{M} \).

The Monster \( \mathbb{M} \) has a remarkably simple presentation. As with any noncyclic finite simple group, it is generated by its involutions (i.e. elements of order 2) and so will be a homomorphic image of a Coxeter group. Let \( \mathcal{G}_{pqr} \), \( p \geq q \geq r \geq 2 \), be the graph consisting of three strands of lengths \( p+1, q+1, r+1 \), sharing a common endpoint. Label the \( p+q+r+1 \) nodes as in Figure 1. Given any graph \( \mathcal{G}_{pqr} \), define \( Y_{pqr} \) to be the group consisting of a generator for each node, obeying the usual Coxeter group relations (i.e. all generators are involutions, and the product \( gg' \) of two generators has order 3 or 2, depending on whether or not the two nodes are adjacent), together with one more relation:

\[
(ab_1b_2ac_1c_2ad_1d_2)^{10} = 1 .
\]  

(2.2)

The groups \( Y_{pqr} \), for \( p \leq 5 \), have now all been identified (see e.g. [61]). Conway conjectured and, building on work by Ivanov [60], Norton proved [98] that \( Y_{555} \cong Y_{444} \) is the ‘Bimonster’, the wreathed-square \( \mathbb{M} \wr C_2 \) of the Monster (so has order \( 2|\mathbb{M}|^2 \)). A closely related presentation of the Bimonster has 26 involutions as generators and has relations
given by the incidence graph of the projective plane of order 3; the Monster itself arises from 21 involutions and the affine plane of order 3 \[25\]. Likewise, \(Y_{553} \cong Y_{443} \cong \mathbb{M} \times C_2\). Other sporadics arise in e.g. \(Y_{533} \cong Y_{433}\) (the Baby Monster \(\mathbb{B}\)), \(Y_{552} \cong Y_{442}\) (the Fischer group \(Fi_{24}'\)), and \(Y_{532} \cong Y_{432}\) (the Fischer group \(Fi_{23}\)). The Coxeter groups of \(G_{555}, G_{553}, G_{533}, G_{552},\) and \(G_{532}\), are all infinite groups of hyperbolic reflections in e.g. \(\mathbb{R}^{17,1}\), and contain copies of groups such as the affine \(E_8\) Weyl group, so the geometry here should be quite pretty. What role, if any, these remarkable presentations have in Moonshine hasn’t been established yet. As a first step though, \([97]\) identifies in \(\text{Aut}(V^2)\) the 21 involutions generating \(\mathbb{M}\).

\[\begin{align*}
&\text{Figure 1. The graph } G_{555} \text{ presenting the Bimonster} \\
\end{align*}\]

The Monster has 194 conjugacy classes, and so that number of irreducible representations. Its character table (and much other useful information) is given in the Atlas \([22]\), where we also find analogous data for the other simple groups of ‘small’ order. For example, we find that \(\mathbb{M}\) has exactly 2, 3, 4 conjugacy classes of elements of order 2, 3, 4, respectively — these classes are named 2A, 2B, 3A, etc. We also find that the dimensions of the smallest irreducible representations of \(\mathbb{M}\) are 1, 196883, 21296876, and 842609326. This is the same 196883 as on the right side of (1.1), and as the dimension of the Griess algebra.

2.2. The \(j\)-function. The group \(\text{SL}_2(\mathbb{R})\), consisting of \(2 \times 2\) matrices of determinant 1 with real entries, acts on the upper half-plane \(\mathbb{H} := \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}\) by fractional linear transformations
\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot \tau = \frac{a\tau + b}{c\tau + d}.
\]
(2.3)

Of course this is really an action of \(\text{PSL}_2(\mathbb{R}) := \text{SL}_2(\mathbb{R})/\{\pm I\}\) on \(\mathbb{H}\), but it is more convenient to work with \(\text{SL}_2(\mathbb{R})\). \(\mathbb{H}\) is the hyperbolic plane, one of the three possible geometries in two dimensions (the others are the sphere and the Euclidean plane), and \(\text{PSL}_2(\mathbb{R})\) is its group of orientation-preserving isometries.

Let \(G\) be a discrete subgroup of \(\text{SL}_2(\mathbb{R})\). Then the space \(G \backslash \mathbb{H}\) has a natural structure of an orientable surface, and inherits a complex structure from \(\mathbb{H}\) (so can be regarded as a complex curve). By the genus of the group \(G\), we mean the genus of the resulting real surface \(G \backslash \mathbb{H}\). For example, the choice \(G = \text{SL}_2(\mathbb{Z})\) yields the sphere with one puncture, so \(\text{SL}_2(\mathbb{Z})\) has genus 0. Moreover, any curve \(\Sigma\) with genus \(g\) and \(n\) punctures, for \(3g+n > 3\), is equivalent as a complex curve to the space \(G \backslash \mathbb{H}\), for some subgroup \(G\) of \(\text{SL}_2(\mathbb{R})\) isomorphic to the fundamental group \(\pi_1(\Sigma)\).
The most important choice for $G$ is $\text{SL}_2(\mathbb{Z})$, thanks to its interpretation as the modular group of the torus. Most groups $G$ of interest are commensurable with $\text{SL}_2(\mathbb{Z})$, i.e. $G \cap \text{SL}_2(\mathbb{Z})$ has finite index in both $G$ and $\text{SL}_2(\mathbb{Z})$. Examples of these are the congruence subgroups

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} , \quad (2.4a)$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid N \text{ divides } c \right\} . \quad (2.4b)$$

For example $\Gamma_0(N)$ has genus 0 for $N = 2, 13, 25$, while $N = 50$ has genus 2 and $N = 24$ has genus 3. The following definition includes all groups arising in Monstrous Moonshine.

**Definition 1.** Call a discrete subgroup $G$ of $\text{SL}_2(\mathbb{R})$ a moonshine-type modular group, if it contains some $\Gamma_0(N)$, and also obeys the condition that

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in G \text{ iff } t \in \mathbb{Z} .$$

Such a modular group is necessarily commensurable with $\text{SL}_2(\mathbb{Z})$. Note that for such a $G$, any meromorphic function $f : G \backslash \mathbb{H} \to \mathbb{C}$ will have a Fourier expansion of the form $f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n$, where $q = e^{2\pi i \tau}$.

**Definition 2.** Let $G$ be any subgroup of $\text{SL}_2(\mathbb{R})$ commensurable with $\text{SL}_2(\mathbb{Z})$. By a modular function $f$ for $G$ we mean any meromorphic function $f : \mathbb{H} \to \mathbb{C}$, such that

$$f(\frac{a\tau + b}{c\tau + d}) = f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

and such that, for any $A \in \text{SL}_2(\mathbb{Z})$, the function $f(A.\tau)$ has Fourier expansion of the form

$$\sum_{n=-\infty}^{\infty} b_n q^n/N$$

for some $N$ and $b_n$ (both depending on $A$), and where $b_n = 0$ for all but finitely many negative $n$.

This definition simply states that $f$ is a meromorphic function on the compact surface $\Sigma_G := G \backslash \overline{\mathbb{H}}$, where $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$. The $G$-orbits of $\mathbb{Q} \cup \{i\infty\}$ are called *cusps*; their role is to fill in the punctures of $G \backslash \mathbb{H}$, compactifying the surface, as there are much fewer meromorphic functions on compact surfaces than on noncompact ones (compare the Riemann sphere to the complex plane!).

We are especially interested in genus 0 groups $G$ of moonshine-type. Their modular functions are particularly easy to characterise: there will be a unique modular function $J_G$ for $G$, with $q$-expansion of the form

$$J_G(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n ; \quad (2.6)$$

the modular functions for $G$ are precisely the rational functions $f(\tau) = \frac{\text{poly}(J_G(\tau))}{\text{poly}(J_G(\tau))}$ in $J_G$. This function $J_G$ is called the (normalised) *Hauptmodul* for the genus 0 group $G$. For example, the modular group $\text{SL}_2(\mathbb{Z})$ has Hauptmodul

$$J_{\text{SL}_2(\mathbb{Z})}(\tau) = J(\tau) = q^{-1} + 196884 q + 21493760 q^2 + 8642909970 q^3 + \cdots . \quad (2.7)$$
This 196884 is the same as that on the left-side of (1.1). Historically, in place of this Hauptmodul was the equivalent

\[ j(\tau) = \frac{(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8)^3}{8q(\tau)^{24}} = J(\tau) + 744. \]

As we know, there are other genus 0 modular groups. For example the Hauptmoduls for \( \Gamma_0(2) \), \( \Gamma_0(13) \), and \( \Gamma_0(25) \), are respectively

\[ J_2(\tau) = q^{-1} + 276q - 2048q^2 + 11202q^3 - 49152q^4 + 184024q^5 + \cdots, \]
\[ J_{13}(\tau) = q^{-1} - q + 2q^2 + q^3 + 2q^4 - 2q^5 - 2q^7 - 2q^8 + q^9 + \cdots, \]
\[ J_{25}(\tau) = q^{-1} - q + q^4 + q^6 - q^{11} - q^{14} + q^{21} + q^{24} - q^{26} + \cdots. \]

Thompson [115] proved there are only finitely many modular groups of moonshine-type in each genus. Cummins [28] has found all of these of genus 0 and 1. In particular there are precisely 6486 genus 0 moonshine-type groups. Exactly 616 of these have Hauptmoduls with rational (in fact integral) coefficients, the remainder have cyclotomic integer coefficients. There are some natural equivalences (e.g. a Galois action) which collapse this number to 371, 310 of which have integral Hauptmoduls.

In genus > 0, two functions are needed to generate the function field. A complication facing the development of a higher-genus Moonshine is that, unlike the situation in genus 0 considered here, there is no canonical choice for these generators.

See e.g. [92] for a very readable account of some of the circle of ideas meandering through this subsection. Modular functions are discussed in e.g. [75].

### 3. The Monstrous Moonshine conjectures

The number on the left of (1.1) is the first nontrivial coefficient of the \( j \)-function, and the numbers on the right are the dimensions of the smallest irreducible representations of the Fischer–Griess Monster \( \mathbb{M} \). On the one side we have a modular function; on the other, a sporadic finite simple group. Moonshine is the explanation and generalisation of this unlikely connection.

But first, why can’t (1.1) merely be a coincidence? This is soon dispelled by comparing the next few coefficients of \( J \) with the dimensions of irreducible representations of \( \mathbb{M} \):

\[ 21493760 = 21296876 + 196883 + 1, \quad (3.1a) \]
\[ 864299970 = 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1. \quad (3.1b) \]

#### 3.1. The fundamental conjecture of Conway and Norton

The central structure in the attempt to understand equations (1.1) and (3.1) is an infinite-dimensional graded module for the Monster:

\[ V = V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots. \quad (3.2a) \]

If we let \( \rho_d \) denote the \( d \)-dimensional irreducible representation of \( \mathbb{M} \), then the first few subspaces will be \( V_0 = \rho_1, V_1 = \{0\}, V_2 = \rho_1 \oplus \rho_{196883}, \) and \( V_3 = \rho_1 \oplus \rho_{196883} \oplus \rho_{21296876} \). This module is to have graded dimension

\[ \dim_V(\tau) = \sum_{n=0} q^n \dim(V_n) = 1 + 196884q^2 + 21493760q^3 + \cdots = qJ(\tau). \quad (3.2b) \]
Of course, (3.2b) alone certainly doesn’t uniquely determine $V$, but assume for now this $V$ has been found. Thompson [114] suggested studying in addition the graded traces

$$T_g(\tau) := q^{-1} \sum_{n=0}^{\infty} \text{ch}_{V_n}(g) q^n$$

(3.2c)

for all $g \in \mathbb{M}$, where the $\text{ch}_{V_n}$ are characters. As taking $g = 1$ recovers $J$, (3.2c) is a natural twist of (3.2b). The functions $T_g$ are now called the McKay–Thompson series.

Conway and Norton conjectured [24] that for each element $g$ of the Monster $\mathbb{M}$, $T_g$ is the Hauptmodul

$$J_{G_g}(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n(g) q^n$$

(3.3)

for a genus 0 subgroup $G_g$ of $\text{SL}_2(\mathbb{R})$. So for each $n$ the coefficient $g \mapsto a_n(g)$ defines a character $\text{ch}_{V_n}(g)$ of $\mathbb{M}$. They explicitly identify each of the groups $G_g$; these groups each contain $\Gamma_0(N)$ as a normal subgroup, for some $N$ dividing $\text{o}(g) \gcd(24, \text{o}(g))$ ($\text{o}(g)$ is the order of $g$), and the quotient group $G_g/\Gamma_0(N)$ has exponent 2 (or 1).

Since $T_g = T_{gh^{-1}}$ by definition, there are at most 194 distinct McKay–Thompson series. All coefficients $a_n(g)$ are integers (as are in fact most entries of the character table of $\mathbb{M}$). This implies that $T_g = T_h$ whenever the cyclic subgroups $\langle g \rangle$ and $\langle h \rangle$ are equal. In fact, the total number of distinct McKay–Thompson series $T_g$ arising in Monstrous Moonshine turns out to be only 171. The first 50 coefficients $a_n(g)$ of each $T_g$ are given in [91]. Together with the recursions given in §3.3 below, this allows one to effectively compute arbitrarily many coefficients $a_n(g)$ of the Hauptmoduls. It is also this which uniquely defines $V$, up to equivalence, as a graded $\mathbb{M}$-module.

For example, there are two different conjugacy classes of order 2 elements. One of these gives the Hauptmodul $J_2$ in (2.8a), while the other corresponds to (3.4) below. Similarly, (2.8b) corresponds to an order 13 element, but $J_{25}$ in (2.8c) doesn’t equal any $T_g$. Recall that there are exactly 616 Hauptmoduls of moonshine-type with integer coefficients, so most of these don’t arise as $T_g$. Recently [23], a fairly simple characterisation has been found of the groups arising as $G_g$ in Monstrous Moonshine. Their proof of this characterisation is by exhaustion.

Conway coined this conjecture Monstrous Moonshine. The word ‘moonshine’ here is English slang for ‘insubstantial or unreal’, ‘idle talk or speculation’, ‘an illusive shadow’. It is meant to give the impression that matters here are dimly lit, and that [24] is ‘distilling information illegally’ from the character table of $\mathbb{M}$.

Monstrous Moonshine began, unofficially, in 1975 when Andrew Ogg remarked that the list of primes $p$ for which the group

$$\Gamma_0(p)+ := \langle \Gamma_0(p), \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \rangle$$

(3.4)

has genus 0, is precisely equal to the list of primes $p$ dividing the order of $\mathbb{M}$. Indeed, in the tables of [24] we find that, for each prime $p$ dividing $|\mathbb{M}|$, an element $g$ of $\mathbb{M}$ of order $p$ is assigned the group $G_g = \Gamma_0(p)+$. 

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3.2. Lie theory and Moonshine. McKay not only noticed (1.1), but also observed that
\[
j(\tau)^{\frac{2}{3}} = q^{\frac{1}{3}} (1 + 248q + 4124q^2 + 34752q^3 + \cdots). \tag{3.5}
\]
The point is that 248 is the dimension of the defining representation of the \(E_8\) Lie group, while 4124 = 3875 + 248 + 1 and 34 752 = 30 380 + 3875 + 2 \cdot 248 + 1. Incidentally, \(j^{\frac{2}{3}}\) is a generating modular function for the genus-0 group \(\Gamma(3)\). Thus Moonshine is related somehow to Lie theory.

McKay later found independent relationships with Lie theory \([89], [15], [47]\), reminiscent of his famous A-D-E correspondence with finite subgroups of \(SU_2(\mathbb{C})\). As mentioned earlier, \(M\) has two conjugacy classes of involutions. Let \(K\) be the smaller one, called ‘2A’ in \([22]\) (the alternative, class ‘2B’, has almost 100 million times more elements). The product of any two elements of \(K\) will lie in one of nine conjugacy classes: namely, 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A, corresponding respectively to elements of orders 1, 2, 2, 3, 3, 4, 4, 5, 6. It is surprising that, for such a complicated group as \(M\), that list stops at only 6 — we call \(M\) a \textit{6-transposition group} for this reason (more on this in §5.2). The punchline: McKay noticed that those nine numbers are precisely the labels of the affine \(E_8\) diagram (see Figure 2). Thus we can attach a conjugacy class of \(M\) to each vertex of the \(\widehat{E}_8\) diagram. An interpretation of the \textit{edges} in the \(\widehat{E}_8\) diagram, in terms of \(M\), is unfortunately not known.

![Figure 2. The affine \(E_8\), \(F_4\), and \(G_2\) diagrams with labels](image)

We can’t get the affine \(E_7\) labels in a similar way, but McKay noticed that an order two folding of affine \(E_7\) gives the affine \(F_4\) diagram, and we can obtain its labels using the Baby Monster \(\mathbb{B}\) (the second largest sporadic). In particular, let \(K\) now be the smallest conjugacy class of involutions in \(\mathbb{B}\) (also labelled ‘2A’ in \([22]\)); the conjugacy classes in \(KK\) have orders 1, 2, 2, 3, 4 (\(\mathbb{B}\) is a 4-transposition group), and these are the labels of \(\widehat{F}_4\). Of course we’d prefer \(\widehat{E}_7\) to \(\widehat{F}_4\), but perhaps that two-folding has something to do with the fact that an order-\textit{two} central extension of \(\mathbb{B}\) is the centraliser of an element \(g \in M\) of order \textit{two}.

Now, the \textit{triple}-folding of affine \(E_6\) is affine \(G_2\). The Monster has three conjugacy classes of order \textit{three}. The smallest of these (‘3A’) has a centraliser which is a \textit{triple} cover of the Fischer group \(Fi'_{24}\). Taking the smallest conjugacy class of involutions in \(Fi'_{24}\), and multiplying it by itself, gives conjugacy classes with orders 1, 2, 3 (hence \(Fi'_{24}\) is a 3-transposition group) — and those not surprisingly are the labels of \(G_2\)!
Although we now understand (3.5) (see §4.1) and have proven the fundamental Conway–Norton conjecture (see §§4.2–4.4), McKay’s $E_8, F_4, G_2$ observations still have no explanation. In [47] these patterns are extended, by relating various simple groups to the $E_8$ diagram with deleted nodes.

3.3. Replicable functions. There are several other less important conjectures. One which played an important role in ultimately proving the main conjecture involves the replication formulae. Conway–Norton want to think of the Hauptmoduls $T_g$ as being intimately connected with $M$; if so, then the group structure of $M$ should somehow directly relate different $T_g$. Considering the power map $g \mapsto g^n$ leads to the following.

It was well-known classically that $j(\tau)$ has the property that $j(p\tau) + j(\frac{\tau}{p}) + j(\frac{\tau+1}{p}) + \cdots + j(\frac{\tau+p-1}{p})$ is a polynomial in $j(\tau)$, for any prime $p$ (proof: it’s a modular function for $SL_2(\mathbb{Z})$, and hence equals a rational function of $j(\tau)$; since its only poles will be at the cusps, the denominator polynomial must be trivial). Hence the same will hold for $J$. More generally, we get

$$\sum_{ad=n, 0 \leq b < d} J(\frac{a\tau + b}{d}) = Q_n(J(\tau)),$$  

where $Q_n$ is the unique polynomial for which $Q_n(J(\tau)) - q^{-n}$ has a $q$-expansion with only strictly positive powers of $q$. For example, $Q_2(x) = x^2 - 2a_1$ and $Q_3(x) = x^3 - 3a_1x - 3a_2$, where we write $J(\tau) = \sum_n a_n q^n$. The left side of (3.6a) is really a Hecke operator applied to $J$. These equations (3.6a) can be rewritten into recursions such as $a_4 = a_3 + (a_1^2 - a_1)/2$, or can be collected together into the remarkable expression (originally due to Zagier)

$$p^{-1} \prod_{m > 0, n \in \mathbb{Z}} (1 - p^m q^n)^{a_{mn}} = J(z) - J(\tau),$$  

where $p = e^{2\pi iz}$.

Conway and Norton conjectured [24] that these formulas have an analogue for any McKay–Thompson series $T_g$. In particular, (3.6a) becomes

$$\sum_{ad=n, 0 \leq b < d} T_g(a\tau + b) = Q_{n,g}(T_g(\tau)),$$  

where $Q_{n,g}$ plays the same role for $T_g$ that $Q_n$ played for $J$. These are called the replication formulae. Again, these yield recursions like $a_4(g) = a_3(g) + (a_1(g)^2 - a_1(g^2))/2$, or can be collected together into the expression

$$p^{-1} \exp[- \sum_{k > 0} \sum_{m > 0, n \in \mathbb{Z}} a_{mn}(g^k) p^m q^n k] = T_g(z) - T_g(\tau).$$  

This looks a lot more complicated than (3.6b), but you can glimpse the Taylor expansion of $\ln(1 - p^m q^n)$ there and in fact for $g = 1$ (3.7b) reduces to (3.6b).
Axiomatising (3.7a) leads to Norton’s notion of replicable function [96], [1]. Write
\[ f^{(1)}(\tau) = q^{-1} + \sum_{k=1}^{\infty} b_k^{(1)} q^k, \]
and replacing each \( T_g \) in (3.7a) with \( f^{(n)} \), use (3.7a) to recursively define each \( f^{(n)} \). If each \( f^{(n)} \) has a \( q \)-expansion of the form
\[ f^{(n)}(\tau) = q^{-1} + \sum_{k=1}^{\infty} b_k^{(n)} q^k \]
— i.e. no fractional powers of \( q \) arise — then we call \( f = f^{(1)} \) replicable. Equation (3.7a) says the McKay–Thompson series are replicable, and [30] proved that the Hauptmodul of any genus 0 modular group of moonshine-type is replicable, provided its coefficients are rational. Conversely, Norton conjectured that any replicable function with rational coefficients is either such a Hauptmodul, or one of the ‘modular fictions’ \( f(\tau) = q^{-1}, f(\tau) = q^{-1} \pm q \). This conjecture seems difficult and is still open. Incidentally, if the coefficients \( b_k^{(1)} \) are irrational, then the definition (3.7a) of replicability should be modified to include Galois automorphisms (see §8 of [29]). Replication in positive genus is discussed in [109].

Replication (3.7a) concerns the power map \( g \mapsto g^n \) in \( \mathbb{M} \). Can Moonshine see more of the group structure of \( \mathbb{M} \)? We explored one step in this direction in §3.2, where McKay modeled products of conjugacy classes using Dynkin diagrams. A different idea is given in §5.1. It would be very desirable to find other direct connections between the group operation in \( \mathbb{M} \) and e.g. the McKay–Thompson series.

3.4. The Leech lattice and Moonshine. The Leech lattice \( \Lambda = \Lambda_{24} \) is a 24-dimensional even self-dual lattice [26] which is to lattices much as the \( \mathbb{M} \)-module \( V \) of (3.2a) turns out to be for vertex operator algebras (see §4.2 below). \( \Lambda \) has no vectors of odd norm, no norm-2 vectors, and precisely 196560 norm-4 vectors — a number remarkably close to the monstrous 196883. In fact its theta series \( \Theta_\Lambda(\tau) = \sum_{v \in \Lambda} q^{v \cdot v/2} \), when divided by \( \eta(\tau)^{24} \), equals \( J(\tau) + 24 \). Is this another example of Moonshine?

Indeed it is. However we have:

**Theorem 3.** Let \( L \subset \mathbb{R}^n \) be any \( n \)-dimensional positive-definite lattice whose norms \( \mathbf{v} \cdot \mathbf{v} \) are all rational. Let \( \mathbf{t} \in \mathbb{R}^n \) be any vector with finite order in \( L \): i.e. \( m \mathbf{t} \in L \) for some nonzero \( m \in \mathbb{Z} \). Then the theta series
\[ \Theta_{L+\mathbf{t}}(\tau) := \sum_{\mathbf{v} \in L} e^{\pi i \mathbf{v} \cdot (\mathbf{v} + \mathbf{t})^2}, \]
divided by \( \eta(\tau)^n \), is a modular function for some \( \Gamma(N) \).

See e.g. Theorem 20 of [100] for a proof of this classical result. If \( L \) is in fact an even lattice (i.e. all norms \( \mathbf{v} \cdot \mathbf{v} \) lie in \( 2\mathbb{Z} \)), we can say more. Let \( L^* := \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot L \subset \mathbb{Z} \} \) be the dual lattice. It contains \( L \) with finite index: write \( \mathbf{t}_i + L, i = 1, \ldots, M \), for the finitely many cosets in \( L^* / L \). Define a column vector \( \tilde{\chi}_L(\tau) \) with \( i \)th component \( \Theta_{\mathbf{t}_i + L}(\tau) / \eta(\tau)^n \). Then \( \tilde{\chi}_L \) forms a vector-valued modular function for \( \text{SL}_2(\mathbb{Z}) \): for any \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \),
\[ \tilde{\chi}_L(a\tau + b) \begin{pmatrix} c \tau + d \end{pmatrix} = \rho(A) \tilde{\chi}_L(\tau) \tag{3.8} \]
for some \( M \)-dimensional unitary matrix representation \( \rho \) of \( \text{SL}_2(\mathbb{Z}) \). In particular, for the Leech lattice \( L = \Lambda, M = 1 \) and we can quickly identify \( \Theta_\Lambda(\tau) \) in terms of \( J(\tau) \). Although
the 196560 \approx 196884 coincidence is thus trivial to explain, it will turn out to be a very instructive example of Moonshine.

The lattices are related to groups through their automorphism groups, which will always be finite for positive-definite lattices. The automorphism group \( Co_0 := \text{Aut}(\Lambda) \) of the Leech lattice has order about \( 8 \times 10^{18} \), and is the direct product of \( C_2 \) with Conway’s sporadic group \( Co_1 \). Several other sporadics are also involved in \( \text{Aut}(\Lambda) \). To each automorphism \( \alpha \in \text{Aut}(\Lambda) \), let \( \theta_\alpha \) denote the theta series of the sublattice of \( \Lambda \) fixed by \( \alpha \). [24] also associate to each automorphism \( \alpha \) a certain function \( \eta_\alpha(\tau) \) of the form \( \prod_i \eta(a_i \tau)/\prod_j \eta(b_j \tau) \) built out of the Dedekind eta. Both \( \theta_\alpha \) and \( \eta_\alpha \) are constant on each conjugacy class in \( \text{Aut}(\Lambda) \), of which there are 202. [24] remarks that the ratio \( \theta_\alpha/\eta_\alpha \) always seems to equal some McKay–Thompson series \( T_{g(\alpha)} \). See also [87].

It turns out that this observation isn’t quite correct [74]. For each \( \alpha \in \text{Aut}(\Lambda) \), the subgroup of \( \text{SL}_2(\mathbb{R}) \) which fixes \( \theta_\alpha/\eta_\alpha \) is indeed genus 0, but for exactly 15 conjugacy classes in \( \text{Aut}(\Lambda) \), \( \theta_\alpha/\eta_\alpha \) is not the Hauptmodul.

Similarly, one can ask this for the \( E_8 \) lattice, whose automorphism group is the Weyl group (of order \( \approx 7 \times 10^8 \)) of the \( E_8 \) Lie group. The automorphisms \( \alpha \) of the \( E_8 \) lattice which yield a Hauptmodul were classified in [19].

4. Proof of the Monstrous Moonshine conjectures

At first glance, the significance of the Moonshine conjectures seems very unlikely: they constitute after all a finite set of very specialised coincidences. The whole point though is to try to understand why such seemingly incomparable objects as the Monster and the Hauptmoduls can be so related, and to try to extend and apply this understanding to other contexts. Establishing the truth (or falsity) of the conjectures was merely meant as an aid to uncovering the why of Moonshine. Indeed, in achieving this understanding, important new algebraic structures were formulated. We will sketch this theory below.

The main Conway–Norton conjecture was attacked almost immediately. Thompson showed [113] (see also [103]) that if \( g \mapsto a_n(g) \) is a character for all sufficiently small \( n \) (apparently \( n \leq 1300 \) is sufficient), then it will be for all \( n \). He also showed that if certain congruence conditions hold for a certain number of \( a_n(g) \) (all with \( n \leq 100 \)), then all \( g \mapsto a_n(g) \) will be virtual characters (i.e. differences of true characters of \( \mathbb{M} \)). Atkin, Fong, and Smith (see [110] for details) used that to prove on a computer that indeed all \( a_n(g) \) were virtual characters (they didn’t quite reach \( n = 1300 \) though). But their work doesn’t say anything more about the underlying (possibly virtual) representation \( V \), other than its existence. Their work plays no role in the following.

We want to show that the McKay–Thompson series \( T_g(\tau) \) of (3.2c) equals the Hauptmodul \( J_G(\tau) \) in (3.3). First, we need to construct the infinite-dimensional module \( V \) of \( \mathbb{M} \). We discuss this, and the underlying theory of vertex operator algebras, in §4.2. Borcherds’ strategy [11] was to bring in Lie theory, by associating to the module \( V \) a ‘Monster Lie algebra’. This algebra, and the underlying theory of generalised Kac–Moody algebras, is described in §4.3. In the final subsection we go from the Monster Lie algebra to the replication formulae, and conclude the proof. We begin this section though by explaining the much simpler connection of \( E_8 \) with \( j^{\frac{1}{3}} \).

4.1. \( E_8 \) and \( j^{\frac{1}{3}} \). An explanation for the relation between \( E_8 \) and \( j^{\frac{1}{3}} \) was found
almost immediately, by Kac and Lepowsky [65], [76]: $j^\frac{\lambda}{2}$ is the (normalised) character of a representation of the affine Kac–Moody algebra $E_8^{(1)}$. Given a finite-dimensional simple Lie algebra $\mathfrak{g}$, the affine algebra $\mathfrak{g}^{(1)}$ is the infinite-dimensional Lie algebra consisting of all Laurent polynomials $\sum_{n=-\infty}^{\infty} a_n t^n$ where $a_n \in \mathfrak{g}$ and $t$ is an indeterminate, together with a central term and derivation $D = -L_0$ (see [66], [70]). Highest weight representations are defined in the usual way. Thanks largely to the fact that the affine Weyl group is a semi-direct product of the additive group $\mathbb{Z}^r$ ($r = \text{rank}(\mathfrak{g})$) with the finite Weyl group, the characters of these representations (especially the ‘integrable’ highest weight ones, which are the direct analogue of the finite-dimensional representations of $\mathfrak{g}$) transform nicely with respect to $SL_2(\mathbb{Z})$. See e.g. Chapter 13 of [70] for details. This is probably the single biggest reason Kac–Moody algebras are so well-known.

**Theorem 4.** [69] Let $\mathfrak{g}$ be any finite-dimensional Lie algebra, and $\mathfrak{g}^{(1)}$ denote the corresponding affine algebra. Let $P^k_\pm$ denote the finitely many ‘level $k$ integrable highest weight modules’ $L_\lambda$ of $\mathfrak{g}^{(1)}$. The $\mathfrak{g}^{(1)}$-module $L_\lambda$ has a natural $\mathbb{Z}$-grading $L_\lambda = \bigoplus_{n=0}^{\infty} (L_\lambda)_n$ into finite-dimensional $\mathfrak{g}$-modules $(L_\lambda)_n$. Let $\chi_\lambda(\tau) = q^{h_\lambda - c/24} \sum_{n=0}^{\infty} \dim((L_\lambda)_n) q^n$ be the corresponding normalised character (for some appropriate choice of $h_\lambda - c/24 \in \mathbb{Q}$). Then each $\chi_\lambda$ is a holomorphic function in $\mathbb{H}$, and the vector $\vec{\chi}_k(\tau)$ with entry $\chi_n(\tau)$ for each $n \in P^k_\pm$ defines a vector-valued modular function for $SL_2(\mathbb{Z})$, as in (3.8), for some finite-dimensional unitary representation $\rho$ of $SL_2(\mathbb{Z})$.

In fact each character $\chi_\lambda$ will be a rational function in lattice theta series, and so will be a modular function for some $\Gamma(N)$. It turns out that there is only one level 1 integrable highest-weight representation of $E_8^{(1)}$, and its character equals $j^\frac{\lambda}{2}$. The modularity of $j^\frac{\lambda}{2}$ is thus predicted by Kac–Moody theory, and the fact that the coefficients are dimensions of $E_8$ representations is automatic. We will see a simultaneous generalisation of Theorems 3 and 4 next section.

We’ve already encountered the mysterious normalisation $q^{-\frac{1}{24}}$ of the McKay–Thompson series, and the $q^{-\frac{\lambda}{2}}$ of the $E_8^{(1)}$ character, and more generally the $q^{h_\lambda - c/24}$ of $\chi_\lambda$. Many explanations have been provided for this pervasive factor. For example, te [2] it is topological in origin, and related to the Atiyah–Singer Index Theorem; a geometric interpretation using determinant line bundles is due to Segal [107]. In quantum physics it’s called the conformal anomaly (a breakdown of manifest conformal symmetry when the classical system is quantised), and is introduced in regularisation as a vacuum energy. Probably the simplest instance of it is the prefactor in the familiar definition $\eta(\tau) = q^{\frac{\tau}{24}} \prod_{n=1}^{\infty} (1-q^n)$ for the Dedekind eta: reading through classical proofs for its modularity we find that $\frac{1}{24}$ here arises through the combination $\zeta(2)/(2\pi)^2$; $\eta$ appears also in physics in the partition function of the bosonic string, and that same $\frac{1}{24}$ arises there via regularisation as $-\zeta(-1)/2$. The equivalence of these two expressions for $\frac{1}{24}$ comes from the functional equation of the Riemann zeta. This same zeta value appears famously in the central term of the Virasoro algebra (4.3), and Bloch [8] found other zeta values appearing in other algebras of differential operators, many of which have now been interpreted and generalised (starting with [77]) within the vertex operator algebra framework.

Although a direct explanation for Monstrous Moonshine using affine algebras has never been found (and certainly isn’t expected), the theory of Kac–Moody algebras influenced
every stage of the ultimate proof. For this reason we'll briefly sketch their theory. A simple finite-dimensional Lie algebra \( g \) is built out of the 3-dimensional algebra \( \text{sl}_2 \), in a simple way; the Dynkin diagram of \( g \) encodes the exact presentation. In the identical way, Kac–Moody algebras are also built out of copies of \( \text{sl}_2 \)— the only difference is that the finite-dimensionality constraint (a positive-definiteness condition on the Cartan matrix) is lifted. Their structure is completely analogous to that of the simple Lie algebras: e.g. it has a grading by roots into finite-dimensional spaces; it has a triangular decomposition (making Verma modules possible); it has an invariant symmetric bilinear form. See e.g. [66], [70] for details. The affine algebras are the class of Kac–Moody algebras especially analogous to the finite-dimensional ones.

4.2. The Moonshine module \( V^\natural \). A vital component of the Monstrous Moonshine conjectures came a few years after [24]. In a deep work, Frenkel–Lepowsky–Meurman [43], [44] constructed a graded infinite-dimensional representation \( V^\natural \) of \( \mathbb{M} \) and conjectured (correctly) that it is the representation \( V \) in (3.2a). \( V^\natural \) has a very rich algebraic structure: it is in fact a vertex operator algebra!

A vertex operator algebra [9], [44], [67], [39], [79] is an infinite-dimensional vector space \( V \) with infinitely many heavily constrained bilinear products \( u \ast_n v \). The name means ‘algebra of (generalised) vertex operators’; vertex operators are formal differential operators which originally appeared in physics as quantum fields describing the creation and propagation of physical strings (see §6 below), and were constructed later but independently by Lie theorists (starting with Lepowsky and Wilson) to realise affine Kac–Moody algebras as algebras of differential operators. Because there were vertex operator constructions associated to lattices, affine algebra modules, and string theory, and all of these have connections to modular functions, it was natural to use vertex operators to try to construct the \( \mathbb{M} \)-module \( V \) of (3.2a).

The definition of vertex operator algebra (VOA) is too complicated to give in detail here. A VOA is a graded infinite-dimensional vector space \( V = \bigoplus_{n=0}^{\infty} V_n \), where each \( V_n \) is finite-dimensional. To simplify the discussion, we will limit ourselves in this paper to VOAs with one-dimensional \( V_0 \), which is typical of the examples relevant to Moonshine (and conformal field theory). To each vector \( v \in V \) we assign a vertex operator \( Y(v, z) \), which is a formal power series \( Y(v, z) = \sum_{m \in \mathbb{Z}} v(m) z^{-m-1} \), with coefficients \( v(m) \in \text{End}(V) \). The vertex operator is just the generating function for the products: \( u \ast_n v = u(n)(v) \). These products respect the grading — in particular,

\[
V_k \ast_n V_\ell \subseteq V_{k+\ell-n-1} .
\]

A key axiom, which collects together all the identities obeyed by the products \( u \ast_n v \), can be written as

\[
(z - w)^M [Y(u, z), Y(v, w)] = 0 \quad \forall u, v \in V ,
\]

for some integer \( M \) (depending on \( u, v \)), where the bracket in (4.2a) means the commutator \( Y(u, z) Y(v, w) - Y(v, w) Y(u, z) \). This strange-looking formula really says that each such commutator is a linear combination of Dirac deltas and their derivatives, all centred at \( z = w \) (see e.g. Corollary 2.2 in [67]). Equation (4.2a) implies more down-to-earth identities,
such as
\[(u_\ell v)_n = \sum_{i \geq 0} (-1)^i \binom{\ell}{i} (u_{\ell-i} \circ v_{n+i} - (-1)^{\ell} v_{\ell+n-i} \circ u_i) . \quad (4.2b)\]

There are two distinguished elements in \( \mathcal{V} \): the identity \( 1 \in \mathcal{V}_0 \) (so \( \mathcal{V}_0 = \mathbb{C}1 \)) and the conformal vector \( \omega \in \mathcal{V}_2 \). The identity obeys \( Y(1, z) = id \), i.e. \( 1_{(n)} v = \delta_{n,-1} v \). More interesting is the conformal vector: writing \( L_n = \omega_{(n+1)} \), the operators \( L_n \) are required to form a representation on \( \mathcal{V} \) of the Virasoro algebra:
\[
[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} c id_{\mathcal{V}} ,
\]
for some number \( c \in \mathbb{R} \) (an important numerical invariant of \( \mathcal{V} \)) called the rank or central charge of the VOA. In addition we require \( L_0 u = nu \) whenever \( u \in \mathcal{V}_n \), and \( L_{-1} \) acts on \( \mathcal{V} \) as a derivation.

The appearance here of the Virasoro algebra is fundamental. It is the unique nontrivial central extension of the (polynomial) vector fields on \( S^1 \), which in turn is the Lie algebra associated to the group \( \text{Diff}(S^1) \) of diffeomorphisms of the circle.

The notion of a VOA may seem very arbitrary, but as we’ll mention in §6 it is the ‘chiral algebra’ of a conformal field theory. The simplest (and least interesting) special case of a VOA occurs when \( M = 0 \) in (4.2a) — i.e. when all vertex operators commute. Then it is not hard to see that, for each choice of \( z \neq 0 \), \( \mathcal{V} \) would be a commutative associative algebra with unit, whose product is given by \( u *_z v := Y(u, z)v \). A more honest way to motivate VOAs has been suggested by Huang: binary trees can be used to keep track of the brackets in nested products, e.g. \( a((bc)d) \), and e.g. Lie algebras can be easily formulated using this language [58]; in a monumental work [59], Huang ‘two-dimensionalised’ this Lie algebra formulation by replacing binary trees with spheres with tubes, and showed that the result is equivalent to a VOA.

A relation between VOAs and Lie algebras also exists at a more elementary level. Placing \( \ell = n = 0 \) in (4.2b), and evaluating it on the right by \( w \in \mathcal{V} \), gives
\[(u_0 v)_0 w = u_0(v_0 w) - v_0(u_0 w) . \quad (4.4)\]
Writing \([xy]\) for \( x_0 y \), this becomes the Lie algebra Jacobi identity, at least when \([xy]=-[yx]\). There are different ways to obtain from this a true Lie algebra. The simplest is that \( \mathcal{V}_1 \) will be a Lie algebra for that bracket; moreover, the number \( \langle u, v \rangle \) defined by \( u_1 v = \langle u, v \rangle 1 \) is an invariant bilinear form for this Lie algebra (by (4.2b) with \( \ell = 0, n = 1 \)).

Equation (4.4) tells us each \( \mathcal{V}_n \) is a \( \mathcal{V}_1 \)-module, and in fact \( e^u \) will be an automorphism of \( \mathcal{V} \) for any \( u \in \mathcal{V}_1 \). In the most common examples, \( \mathcal{V}_1 \) will be reductive (i.e. a direct sum of simple and abelian Lie algebras).

Modules \( M \) of \( \mathcal{V} \) can be defined in the obvious way [42], [79] — e.g. for each \( u \in \mathcal{V} \), each \( u_{(n)} \) will be in \( \text{End}(M) \). All (irreducible) modules \( M \) come with a \( \mathbb{Z} \)-grading \( M = \oplus_{n=0}^\infty M_n \), where \( u_k M_n \subseteq M_{n+k-1} \) for all \( u \in \mathcal{V}_k \), and \( L_0 x = (n + h) x \) for any \( x \in M_n \), where \( h \) is some number (the conformal weight) depending only on \( M \). The (normalised) character of \( M \) is
\[
\chi_M(\tau) = q^{-c/24} \text{Tr}_M q^{L_0} = q^{h-c/24} \sum_{n=0}^\infty \dim(M_n) q^n .
\]
It takes a little effort to construct even the simplest examples of VOAs. The best-behaved ones are called rational VOAs [124] (borrowing on terminology from physics) and have only finitely many irreducible modules. A rational VOA is associated to any even positive-definite lattice $L$, and their modules are in one-to-one correspondence with the cosets $L^*/L$. Another important example: to any affine non-twisted Kac–Moody algebra and choice of positive integer $k$ (the 'level'), the highest weight module $L_{k\Lambda_0}$ has a natural VOA structure, and its modules are precisely the affine algebra modules $L_\lambda$ for each highest weight $\lambda \in P^+_1$.

One of the deepest results in the theory of VOAs is due to Zhu:

**Theorem 5.** [124] Let $\mathcal{V}$ be a rational VOA. Its characters $\chi_M(\tau)$ are holomorphic in $\mathbb{H}$, and the subspaces $M_n$ carry representations for $\text{Aut}(\mathcal{V})$. Write $\tilde{\chi}_\mathcal{V}(\tau)$ for the vector whose components are the characters $\chi_M(\tau)$ of irreducible modules $M$. Then $\tilde{\chi}_\mathcal{V}$ is a vector-valued modular function for $\text{SL}_2(\mathbb{Z})$.

It is believed that the characters $\chi_M(\tau)$ themselves will be modular functions for some $\Gamma(N)$; significant progress towards this was made in [5] (see also [71]). The proof of Zhu’s Theorem is much more difficult than that of Theorem 4, which it generalises.

The automorphism group $\text{Aut}(\mathcal{V})$ is by definition required to fix $\omega$, which is why it respects the grading of $\mathcal{V}$. $\text{Aut}(\mathcal{V})$ is how group theory impinges on VOA theory. Since the automorphism group $\text{Aut}(\mathcal{V})$ of a VOA contains $e^{v_1}$ as a (normal) subgroup, $\text{Aut}(\mathcal{V})$ can be finite only when $\mathcal{V}_1 = 0$. Zhu’s Theorem tells us that Moonshine (without the genus-0 aspect) will hold between the group $\text{Aut}(\mathcal{V})$ and the functions $\chi_M(\tau)$, for any rational VOA.

The most famous example of a VOA is the Moonshine module $V^\sharp$ of [44]. It is the orbifold of the Leech lattice VOA $\mathcal{V}_\Lambda$ by the $\pm 1$-symmetry of $\Lambda$, which means it’s the direct sum of two parts: an invariant part $V^\sharp_+$ and a twisted part $V^\sharp_-$ (more on this in §5.1). The orbifold serves two purposes: it removes the constant term ‘24’ from the graded dimension $J + 24$ (hence the subspace $(\mathcal{V}_\Lambda)_1$ of $\mathcal{V}_\Lambda$) and it enhances the symmetry from the discrete part of $\text{Aut}(\mathcal{V}_\Lambda)$, which is an extension of $Co_0$ by $(C_2)^{24}$, to all of $\mathbb{M}$.

A major claim of [44] was that $V^\sharp$ is a ‘natural’ structure (hence their notation). Even so, this bipartite structure to $V^\sharp$ complicates its study. We have $V^\sharp_0 = C\mathbf{1}$, as usual, but the Lie algebra $\mathcal{V}_1 = \{0\}$ is trivial. For any such VOA, the space $\mathcal{V}_2$ will be a commutative nonassociative algebra with product $u \times v := u_1v$ and identity $\frac{1}{2}\omega$. For the Moonshine VOA $V^\sharp$, this can be shown (with effort!) to be the 196883-dimensional Griess algebra extended by an identity element. From this, we find the automorphism group of $V^\sharp$ to be the Monster $\mathbb{M}$. The only irreducible module for $V^\sharp$ is itself — such a VOA is called holomorphic. Together with Zhu’s Theorem, this implies that its character, namely $J(\tau)$, must be a modular function for $\text{SL}_2(\mathbb{Z})$ (strictly speaking, we only get invariance up to a 1-dimensional character of $\text{SL}_2(\mathbb{Z})$, but it is easy to show that character must be identically 1). We’ll see in §5.1 how to obtain the other McKay–Thompson series from $V^\sharp$.

Conjecturally, there are 71 holomorphic VOAs with rank $c = 24$ [106]. Much as the Leech lattice is the unique even self-dual positive-definite lattice of dimension 24 containing no norm-2 vectors [26], the Moonshine module $V^\sharp$ is conjecturally [44] the unique holomorphic VOA with $c = 24$ and with trivial $\mathcal{V}_1$. Thus, just as the Leech lattice is the unique lattice with theta series $\Theta_\Lambda$, so (conjecturally) is the Moonshine module the unique
holomorphic VOA with (normalised) graded dimension $J$. Proving this is one of the most important (and difficult) challenges in the subject.

4.3. The Monster Lie algebra $\mathfrak{m}$. To show that all of the McKay–Thompson series $T_g$ are indeed Hauptmoduls, Borcherds needed identities satisfied by their $q$-expansions. He obtained these through a Lie algebra he associated to $V^\natural$. Before discussing it, let’s briefly describe Borcherds’ generalisation of Kac–Moody algebras [10].

A Borcherds–Kac–Moody algebra differs from a Kac–Moody algebra in that it is built up from Heisenberg algebras as well as $\text{sl}_2$, and these subalgebras intertwine in more complicated ways. Nevertheless much of the theory for finite-dimensional simple Lie algebras continues to find an analogue in this much more general setting (e.g. root-space decomposition, Weyl group, character formula,…). This unexpected fact is the point of Borcherds–Kac–Moody algebras. For reasons of space we avoid giving here the fairly simple definition, but for this and much more see the review articles [53], [63], [102].

Their basic structure theorem is that of Kac–Moody algebras. In particular, there is a grading by roots into finite-dimensional spaces (except that the 0-graded piece, corresponding to the Cartan subalgebra, may be infinite-dimensional). They also have a triangularisable decomposition and an invariant symmetric bilinear form. Indeed, these structural properties characterise Borcherds–Kac–Moody algebras. In this sense Borcherds–Kac–Moody algebras are the ultimate generalisation of simple Lie algebras, in that any further generalisation would lose some basic structural ingredient.

In short, Borcherds’ algebras strongly resemble the Kac–Moody ones and constitute a natural and nontrivial generalization. The main differences are that they can be generated by copies of the 3-dimensional Heisenberg algebra as well as $\text{sl}_2$, and that there can be imaginary simple roots. Borcherds introduced these algebras and developed their theory in order to understand the Monster Lie algebra $\mathfrak{m}$.

We want to construct $\mathfrak{m}$ from the Moonshine module $V^\natural = V^\natural_0 \oplus V^\natural_1 \oplus \cdots$. For later convenience, relabel its subspaces $V^i := V^\natural_{i+1}$. Of course the obvious choice $V^\natural_1 = V^0$ is 0-dimensional, so we must modify $V^\natural_1$ first. Let $\mathfrak{H}_{1,1}$ denote the even self-dual indefinite lattice consisting of all pairs $(m, n) \in \mathbb{Z}^2$ with inner product $(m, n) \cdot (m', n') = mn' + nm'$. Because it is indefinite, the usual construction of a VOA from a lattice will fail here to produce a true VOA, but most properties will be obtained. Call this near-VOA, $\mathcal{V}_{1,1}$.

The Monster Lie algebra $\mathfrak{m}$ is a Lie algebra associated to the near-VOA $V^\natural \otimes \mathcal{V}_{1,1}$ — see [11] for the details. $\mathfrak{m}$ inherits a $\mathfrak{H}_{1,1}$-grading from $\mathcal{V}_{1,1}$, and this is its root space decomposition: the $(m, n)$ root space is isomorphic (as a vector space) to $V^{mn}$, if $(m, n) \neq (0, 0)$; the $(0, 0)$ piece is isomorphic to $\mathbb{R}^2$. Structurally, the Monster Lie algebra has a decomposition $\mathfrak{m} = u^+ \oplus \text{gl}_2 \oplus u^-$ into a sum of Lie subalgebras, where $u^\pm$ are free Lie algebras (see e.g. [64]). It inherits the action of $\mathbb{M}$ from $V^\natural$.

This construction of $\mathfrak{m}$ may seem indirect; an alternate approach, anticipated in [11] and [12], uses Moonshine cohomology [81] — a functor, inspired by BRST cohomology in conformal field theory, assigning to certain $c = 2$ near-VOAs some Lie algebra carrying an action of $\mathbb{M}$. To $\mathcal{V}_{1,1}$ this functor associates $\mathfrak{m}$.

4.4. Denominator identities and modular equations. It was discovered early on that the Hauptmoduls all obey the replication formulae, and that anything obeying those for-
mulae will be determined by their first few coefficients. The idea then is to show that the McKay–Thompson series \( T_\varnothing \) of (3.2c) also are replicable. Borcherds did this using Lie algebra denominator identities [11].

Finite-dimensional simple Lie algebras \( g \) possess a very useful formula for their characters, due to Weyl: the (formal) character \( \chi_\lambda \) of a module \( L_\lambda \) equals

\[
\chi_\lambda := \sum_\mu \dim(L_\lambda(\mu)) e^\mu = e^{-\rho} \sum_{w \in W} \epsilon(w) \frac{e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})} ,
\]

(4.5)

where \( W \) is the Weyl group, \( \Delta_+ \) the positive roots, \( \epsilon(w) = \det(w) \) is a sign, and where \( \oplus_\mu L_\lambda(\mu) \) is the weight-space decomposition of \( L_\lambda \). As the weights \( \mu \) by definition lie in the dual \( h^* \) of the Cartan subalgebra of \( g \), the character \( \chi_\lambda \) can be regarded as a complex-valued function on the space \( h \cong \mathbb{C}^r \) (\( r = \text{rank}(g) \)).

Consider the trivial representation: i.e. \( x \mapsto 0 \) for all \( x \in g \). Its character \( \chi_0 \) will be identically 1. Thus the character formula (4.5) tells us that a certain alternating sum over a Weyl group, equals a certain product over positive roots. These formulas, called denominator identities, are nontrivial even in this finite-dimensional case.

In a famous paper [83], Macdonald generalised the denominator identity for (4.5), to infinite sum/product identities, corresponding to the extended Dynkin diagrams. The simplest one was known classically as the Jacobi triple product identity:

\[
\sum_{n=-\infty}^{\infty} (-1)^n x^{n^2} y^n = \prod_{m=1}^{\infty} (1 - x^{2m})(1 - x^{2m-1}y)(1 - x^{2m-1}y^{-1}) .
\]

(4.6)

Macdonald’s identities were later reinterpreted, by Kac and Moody, as denominator identities for the affine algebras. For example, we now know (4.6) to be the denominator identity for the algebra \( A_1^{(1)} \).

In particular, the same formula (4.5) holds for Kac–Moody algebras, except that the sum and product are now infinite, the positive roots now come with multiplicities, and the characters are usually normalised by a prefactor \( q^{h_\lambda - c/24} \). The variable \( \tau \) in Theorem 4 is one of the coordinates in the Cartan subalgebra \( \mathbb{C}^{r+2} \) of the affine algebra (see e.g. equation (13.2.4) of [66]). In that theorem we dropped the remaining variable dependence of the \( \chi_\lambda \) for readability, although those additional coordinates serve the important role of guaranteeing linear independence of the characters, and of giving us an action of \( \text{SL}_2(\mathbb{Z}) \) rather than merely \( \text{PSL}_2(\mathbb{Z}) \).

Because a Borcherds–Kac–Moody algebra \( g \) is triangularisable, highest weight \( g \)-modules can be defined in the usual way from Verma modules. The character formula becomes

\[
\chi_\lambda = e^{-\rho} \sum_{w \in W} \epsilon(w) \frac{w(\lambda+\rho) S_\lambda}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult} \alpha}} ,
\]

(4.7)

where \( S_\lambda \) is a correction factor due to imaginary simple roots.

The corresponding denominator identity of the Monster Lie algebra \( m \) can be computed, and is given in (3.6b). Its Weyl group is \( C_2 \) and sends the \( (m,n) \)-root space to
\((n, m)\); the \((m, n)\) root has multiplicity given by coefficient \(a_{mn}\) of \(J\); for each \(n > 0\) we have an imaginary simple root \((1, n)\) with multiplicity \(a_n\). Because of a cohomological interpretation of all denominator identities, (3.6b) can be ‘twisted’ by each \(g \in \mathbb{M}\), and this gives (3.7b). These formulas are equivalent to the replication formula (3.7a) conjectured in §3.3.

Identities equivalent to (3.7b) were obtained by more elementary means — i.e. methods requiring less of the theory of Borcherds–Kac–Moody algebras — in [64] and [68], permitting a simplification of Borcherds’ proof at this stage.

Now, it turns out that if we verify for each conjugacy class \(K_g\) of \(\mathbb{M}\) that the first, second, third, fourth and sixth coefficients of the McKay–Thompson series \(T_g\) and the corresponding Hauptmodul \(J_{G_g}\) agree, then indeed \(T_g = J_{G_g}\). That is precisely what Borcherds then did: he compared finitely many coefficients, and as they all equalled what they should, this concluded the proof [11] of Monstrous Moonshine!

However, this case-by-case verification occurred at the critical point where the McKay–Thompson series were being compared directly to the Hauptmoduls, and so provides little insight into why the \(T_g\) are genus 0. Fortunately a more conceptual explanation of their equality has since been found.

A function \(f\) obeying the replication formulae (3.7a) will also obey modular equations — i.e. a 2-variable polynomial identity satisfied by \(f(x)\) and \(f(nx)\). The simplest examples come from the exponential and cosine functions: note that for any \(n > 0\), \(\exp(nx) = (\exp(x))^n\) and \(\cos(nx) = T_n(\cos(x))\) where \(T_n\) is a Tchebychev polynomial. It was known classically that \(j\) (hence \(J\)) satisfied a modular equation for any \(n\): e.g. put \(X = J(\tau)\) and \(Y = J(2\tau)\), then

\[
(X^2 - Y)(Y^2 - X) = 393768 (X^2 + Y^2) + 42987520 XY + 40491318744 (X + Y) - 120981708338256 .
\]

The only functions \(f(\tau) = q^{-1} + a_1 q + \cdots\) which obey modular equations for all \(n\), are \(J(\tau)\) and the ‘modular fictions’ \(q^{-1}\) and \(q^{-1} \pm q\) (which are essentially exp, cos, and sin) [72]. More generally, we have:

**Theorem 6.** [29] A function \(B(\tau) = q^{-1} + \sum_{n=1}^{\infty} b_n q^n\) which obeys a modular equation for all \(n \equiv 1 \pmod{N}\), will either be of the form \(B(\tau) = q^{-1} + b_1 q\), or will be a Hauptmodul for a modular group of moonshine-type.

The converse is also true [29]. The denominator identity argument tells us each \(T_g\) obeys a modular equation for each \(n \equiv 1 \pmod{N}\) modulo the order of \(g\), so Theorem 6 then concludes the proof of Monstrous Moonshine.

The computer searches in [20] suggest that the hypothesis of Theorem 6 may be considerably weakened, perhaps all the way down to the existence of modular equations for any two distinct primes.

5. Further developments

5.1. Orbifolds. About a third of the McKay–Thompson series \(T_g\) will have some negative coefficients. In §5.4 we’ll see Borcherds interpret them as dimensions of superspaces
(which come with signs). In an important announcement [97], on par with [24], Norton proposed that, although $T_g(-1/\tau)$ will not usually be another McKay–Thompson series, it will always have nonnegative integer $q$-coefficients, and these can be interpreted as ordinary dimensions. In the process, he extended the $g \mapsto T_g$ assignment to commuting pairs $(g,h) \in \mathbb{M} \times \mathbb{M}$.

In particular, to each such pair we have a function $N(g,h;\tau)$, which we will call a Norton series, such that

$$N(g^ah^c, g^bh^d; \tau) = \alpha N(g,h; \frac{a\tau + b}{c\tau + d}) \quad \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}),$$

for some root of unity $\alpha$ (of order dividing 24, and depending on $g,h,a,b,c,d$). The Norton series $N(g,h;\tau)$ is either constant, or generates the modular functions for a genus-0 subgroup of $\text{SL}_2(\mathbb{Z})$ containing some $\Gamma(N)$ (but otherwise not necessarily of moonshine-type). Constant $N(g,h;\tau)$ arise when all elements of the form $g^ah^b$ (for $\gcd(a,b) = 1$) are ‘non-Fricke’ (an element $g \in \mathbb{M}$ is called Fricke if the group $G_g$ contains an element sending 0 to $\infty$ — the identity 1 is Fricke, as are 120 of the 171 $G_g$). Each $N(g,h;\tau)$ has a $q^{1/24}$-expansion for that $N$; the coefficients of this expansion are characters evaluated at $h$ of some central extension of the centralizer $C_M(g)$. Simultaneous conjugation of $g,h$ leaves the Norton series unchanged: $N(aga^{-1}, aha^{-1};\tau) = N(g,h;\tau)$.

For example, when $(g,h) \cong C_2 \times C_2$ and $g,h,gh$ are all in class 2A, then $N(g,h;\tau) = \sqrt{J(\tau)} - 984$. The McKay–Thompson series are recovered by the $g = 1$ specialisation: $N(1,h;\tau) = T_h(\tau)$. This action (5.1) of $\text{SL}_2(\mathbb{Z})$ is related to its natural action on the fundamental group $\mathbb{Z}^2$ of the torus, as we’ll see in §6, as well as a natural action of the braid group, as we’ll see next subsection. Norton arrived at his conjecture empirically, by studying the data of Queen (see §5.3).

The basic tool we have for approaching Moonshine conjectures is the theory of VOAs, so we need to understand Norton’s suggestion from that point of view. For reasons of space, we’ll limit this discussion to $V^x$, but it generalises. Given any automorphism $g \in \text{Aut}(V^x)$, we can define $g$-twisted modules in the obvious way [36]. Then for each $g \in \mathbb{M}$, there is a unique $g$-twisted module, call it $V^x(g)$, for $V^x$ — this statement generalises the holomorphicity of $V^x$ mentioned in §4.2. More generally, given any automorphism $h \in \text{Aut}(V^x)$ commuting with $g$, $h$ will yield an automorphism of $V^x(g)$, so we can perform Thompson’s twist (3.2c) and write

$$q^{-c/24} \text{Tr}_{V^x(g)} h q^{L_0} =: \mathcal{Z}(g,h;\tau).$$

These $\mathcal{Z}(g,h)$’s can be thought of as the building blocks of the graded dimensions of various eigenspaces in $V^x(g)$: e.g. if $h$ has order $m$, then the subspace of $V^x(g)$ fixed by automorphism $h$ will have graded dimension $m^{-1} \sum_{i=1}^m \mathcal{Z}(g,h^i)$. In the case of the Monster considered here, we have $\mathcal{Z}(g,h) = N(g,h)$.

The important paper [36] proves that, whenever the subgroup $(g,h)$ generated by $g$ and $h$ is cyclic, then $N(g,h)$ will be a Hauptmodul satisfying (5.1). One way this will happen of course is whenever the orders of $g$ and $h$ are coprime. Extending [36] to all commuting pairs $g,h$ is one of the most pressing tasks in Moonshine.
This orbifold construction is the same as was used to construct \( V^\natural \) from \( \mathcal{V}_\Lambda \): \( V^\natural \) is the sum of the ‘\( i \)'-invariant subspace \( V_\natural^+ \) of \( \mathcal{V}_\Lambda \) with the ‘\( i \)'-invariant subspace \( V_\natural^- \) of the unique ‘\(-1\)'-twisted module for \( \mathcal{V}_\Lambda \), where \( i \in \text{Aut}(\Lambda) \) is some involution. The graded dimensions of \( V_\natural^\pm \) are \( 2^{-1}(\mathbb{Z}(\pm 1, 1) + \mathbb{Z}(\pm 1, i)) \), respectively, and these sum to \( J \).

The orbifold construction is also involved in an interesting reformulation of the Hauptmodul property, due to Tuite [116]. Assume the uniqueness conjecture: \( V^\natural \) is the only VOA with graded dimension \( J \). He argues from this that, for each \( g \in \mathbb{M} \), \( T_g \) will be a Hauptmodul if the only orbifolds of \( V^\natural \) are \( \mathcal{V}_\Lambda \) and \( V^\natural \) itself. In e.g. [62], this analysis is extended to some of Norton’s \( N(g, h) \)'s, where the subgroup \( \langle g, h \rangle \) is not cyclic (thus going beyond [36]), although again assuming the uniqueness conjecture.

5.2. Why the Monster? That \( \mathbb{M} \) is associated with modular functions can be explained by it being the automorphism group of the Moonshine VOA \( V^\natural \). But what is so special about this group \( \mathbb{M} \) that these modular functions \( T_g \) and \( N(g, h) \) should be Hauptmoduls? This is still open. One approach is due to Norton, and was first (rather cryptically) stated in [97]: the Monster is probably the largest (in a sense) group with the 6-transposition property. Recall from §3.2 that a \( k \)-transposition group \( G \) is one generated by a conjugacy class \( K \) of involutions, where the product \( gh \) of any two elements of \( K \) has order \( \leq k \). For example, taking \( K \) to be the transpositions in the symmetric group \( G = S_n \), we find that \( S_n \) is 3-transposition.

A transitive action of \( \Gamma := \text{PSL}_2(\mathbb{Z}) \) on a finite set \( X \) with one distinguished point \( x_0 \in X \), is equivalent to specifying a finite index subgroup \( \Gamma_0 \) of \( \Gamma \). In particular, \( \Gamma_0 \) is the stabiliser \( \{ g \in \Gamma \mid g.x_0 = x_0 \} \) of \( x_0 \). \( X \) can be identified with the cosets \( \Gamma_0 \setminus \Gamma \), and \( x_0 \) with the coset \( \Gamma_0 \). (If we avoid specifying \( x_0 \), then \( \Gamma_0 \) will be identified only up to conjugation.)

To such an action, we can associate an interesting triangulation of the closed surface \( \Gamma_0 \setminus \mathbb{H} \), called a (modular) quilt. The definition, originally due to Norton and further developed by Parker, Conway, and Hsu, is somewhat involved and will be avoided here (but see especially Chapter 3 of [57]). It is so-named because there is a polygonal ‘patch’ covering every cusp of \( \Gamma_0 \setminus \mathbb{H} \), and the closed surface is formed by sewing together the patches along their edges (‘seams’). There are a total of \( 2n \) triangles and \( n \) seams in the triangulation, where \( n \) is the index \( \| \Gamma_0 \setminus \Gamma \| = \| X \| \). The boundary of each patch has an even number of edges, namely the double of the corresponding cusp width. The familiar formula

\[
\gamma = \frac{n}{12} - \frac{n_2}{4} - \frac{n_3}{3} - \frac{n_\infty}{2} + 1
\]

for the genus \( \gamma \) of \( \Gamma_0 \setminus \mathbb{H} \) in terms of the index \( n \) and the numbers \( n_i \) of \( \Gamma_0 \)-orbits of fixed-points of order \( i \), can be interpreted in terms of the data of the quilt (see (6.2.3) of [57]), and we find in particular that if every patch of the quilt has at most 6 sides, then the genus will be 0 or 1, and genus 1 only exceptionally.

In particular, we’re interested in one class of these \( \Gamma \)-actions (actually an \( \text{SL}_2(\mathbb{Z}) \)-action, but this doesn’t matter). Recall that the braid group \( B_3 \) has presentation

\[
\langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle , \tag{5.3a}
\]

and centre \( Z = \langle (\sigma_1 \sigma_2 \sigma_1)^2 \rangle \) [7]. It is related to the modular group by

\[
B_3/Z \cong \text{PSL}_2(\mathbb{Z}) , \quad B_3/\langle (\sigma_1 \sigma_2 \sigma_1)^4 \rangle \cong \text{SL}_2(\mathbb{Z}) . \tag{5.3b}
\]
Fix a finite group $G$ (we’re most interested in the choice $G = M$). We can define a right action of $B_3$ on triples $(g_1, g_2, g_3) \in G^3$ by

$$(g_1, g_2, g_3)\sigma_1 = (g_1 g_2 g_1^{-1}, g_1, g_3), \quad (g_1, g_2, g_3)\sigma_2 = (g_1, g_2 g_3 g_2^{-1}, g_2). \quad (5.4a)$$

We will be interested in this action on the subset of $G^3$ where all $g_i \in G$ are involutions. The action (5.4a) is equivalent to a reduced version, where we replace $(g_1, g_2, g_3)$ with $(g_1 g_2, g_2 g_3) \in G^2$. Then (5.4a) becomes

$$(g, h)\sigma_1 = (g, gh), \quad (g, h)\sigma_2 = (gh^{-1}, h). \quad (5.4b)$$

These $B_3$ actions come from specialisations of the Burau and reduced Burau representations [7], respectively, and generalise to actions of $B_n$ on $G^n$ and $G^{n-1}$. We can get an action of $\text{SL}_2(\mathbb{Z})$ from the $B_3$ action (5.4b) in two ways: either

(i) by restricting to commuting pairs $g, h$; or
(ii) by identifying each pair $(g, h)$ with all its conjugates $(aga^{-1}, aha^{-1})$.

Norton’s $\text{SL}_2(\mathbb{Z})$ action of §5.1 arises from the $B_3$ action (5.4b), when we perform both (i) and (ii).

The quilt picture was designed for this $\text{SL}_2(\mathbb{Z})$ action. The point of this construction is that the number of sides in each patch is determined by the orders of the corresponding elements $g, h$. If $G$ is say a 6-transposition group (such as the Monster), and we take the involutions $g_i$ from 2A, then each patch will have $\leq 6$ sides, and the corresponding genus will be 0 (usually) or 1 (exceptionally if at all). In this way we can relate the Monster with a genus-0 property.

Based on the actions (5.4), Norton anticipates some analogue of Moonshine valid for noncommuting pairs. CFT considerations (‘higher genus orbifolds’) alluded to in §6 suggest that more natural should be e.g. quadruples $(g, g', h, h') \in M^4$ obeying $ghg^{-1}h^{-1} = h'g'h'^{-1}g'^{-1}$.

An interesting question is, how much does Monstrous Moonshine determine the Monster? How much of $M$’s structure can be deduced from e.g. McKay’s $\hat{E}_8$ Dynkin diagram observation, and/or the (complete) replicability of the $T_g$, and/or Norton’s conjectures in §5.1, and/or Modular Moonshine in §5.4 below? A small start toward this is taken in [99], where some control on the subgroups of $M$ isomorphic to $C_p \times C_p$ ($p$ prime) was obtained, using only the properties of the series $N(g, h)$. For related work, see Chapter 8 of [57].

5.3. Other finite groups. It is natural to ask about Moonshine for other groups. Indeed, the Hauptmodul for $\Gamma_0(2)+$ looks like

$$q^{-1} + 4372q + 96256q^2 + 1240002q^3 + \cdots \quad (5.5a)$$

and we find the relations

$$4372 = 4371 + 1, \quad 96256 = 96255 + 1, \quad 1240002 = 1139374 + 4371 + 2 \cdot 1, \quad (5.5b)$$

where 1, 4371, 96255, and 1139374 are all dimensions of irreducible representations of the Baby Monster $\mathbb{B}$. Thus we find Moonshine for $\mathbb{B}$! We will return to this example shortly.
Of course any subgroup of $\mathbb{M}$ automatically inherits Moonshine by restriction, but obviously this isn’t interesting. Most constructions of the Leech lattice start with Mathieu’s sporadic $M_{24}$ (see e.g. Chapters 10 and 11 of [26]), and most constructions of the Monster involve the Leech lattice. Thus we are led to the following natural hierarchy of (most) sporadics:

1. $M_{24}$ (from which we can get $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$); which leads to
2. $Co_0 = C_2 \times Co_1$ (from which we get $HJ$, $HS$, $McL$, $Suz$, $Co_3$, $Co_2$); which leads to
3. $\mathbb{M}$ (from which we get $He$, $Fi_{22}$, $Fi_{23}$, $Fi_{24}$, $HN$, $Th$, $\mathbb{B}$).

It can thus be argued that we could approach problems in Monstrous Moonshine, by first addressing in order $M_{24}$ and $Co_1$, which should be much simpler. Indeed, the full VOA orbifold theory — i.e. the complete analogue of §5.1 — for $M_{24}$ has been established in [38] (the relevant series $Z(g,h)$ had already been constructed in [88]).

Largely by trial and error, Queen [101] established Moonshine for the following groups (all essentially centralisers of elements of $\mathbb{M}$): $Co_0$, $Th$, $3.2.Suz$, $2.HJ$, $HN$, $2.A_7$, $He$, $M_{12}$ (by e.g. ‘$2.HJ$’ we mean $C_2$ is normal and $HJ$ is the quotient $2.HJ/C_2$). In particular, to each element $g$ of these groups, there corresponds a series $Q_g(\tau) = q^{-1} + \sum_{n=0}^{\infty} a_n(g)q^n$, which is a Hauptmodul for some modular group of moonshine-type, and where each $g \mapsto a_n(g)$ is a virtual character. For $Th$, $HN$, $He$ and $M_{12}$ it is a proper character. Other differences with Monstrous Moonshine are that there can be a preferred nonzero value for the constant term $a_0$, and that although $\Gamma_0(N)$ will be a subgroup of the fixing group, it won’t necessarily be normal. We will return to these results next section, where we will see that many seem to come out of the Moonshine for $\mathbb{M}$. About half of Queen’s Hauptmoduls $Q_g$ for $Co_0$ do not arise as a McKay–Thompson series for $\mathbb{M}$. Norton’s conjectures in §5.1 are a reinterpretation and extension of Queen’s work.

Queen never reached $\mathbb{B}$ because of its size. However, the Moonshine (5.5) for $\mathbb{B}$ falls into her and Norton’s scheme because (5.5a) is the McKay–Thompson series associated to class $2A$ of $\mathbb{M}$, and the centraliser of an element in $2A$ is a double cover of $\mathbb{B}$.

There can’t be a VOA $V = \bigoplus_n V_n$ with graded dimension (5.5a) and automorphisms in $\mathbb{B}$, because e.g. the $\mathbb{B}$-module $V_3$ doesn’t contain $V_2$ as a submodule. However, Höhn deepened the analogy between $\mathbb{M}$ and $\mathbb{B}$ by constructing a vertex operator superalgebra $V\mathbb{B}^2$ of rank $c = 23.5$, called the shorter Moonshine module, closely related to $V^2$ (see e.g. [56]). Its automorphism group is $C_2 \times \mathbb{B}$. Just as $\mathbb{M}$ is the automorphism group of the Griess algebra $V_2^2$, so is $\mathbb{B}$ the automorphism group of the algebra $(V\mathbb{B}^2)_2$. Just as $V^2$ is associated to the Leech lattice $\Lambda$, so is $V\mathbb{B}^2$ associated to the shorter Leech lattice $O_{23}$, the unique 23-dimensional positive-definite self-dual lattice with no vectors of length 2 or 1 (see e.g. Chapter 6 of [26]). The automorphism group of $O_{23}$ is $C_2 \times Co_2$.

There has been no interesting Moonshine rumoured for the remaining six sporadics (the pariahs $J_1$, $J_3$, $Ru$, $ON$, $Ly$, $J_4$). There will be some sort of Moonshine for any group which is an automorphism group of a vertex operator algebra (so this means any finite group [37]!). Many finite groups of Lie type should arise as automorphism groups of VOAs associated to affine algebras except defined over finite fields. But apparently all known examples of genus-0 Moonshine are limited to the groups involved with $\mathbb{M}$.

### 5.4. Modular Moonshine

Consider an element $g \in \mathbb{M}$. We expect from [101], [97], [36] that there is a Moonshine for the centraliser $C_\mathbb{M}(g)$ of $g$ in $\mathbb{M}$, governed by the $g$-
twisted module $V^\natural(g)$. Unfortunately, $V^\natural(g)$ is not usually itself a VOA, so the analogy with $\mathbb{M}$ is not perfect. Ryba found it interesting that, for $g \in \mathbb{M}$ of prime order $p$, Norton’s series $N(g, h)$ is a McKay–Thompson series (and has all the associated nice properties) whenever $h$ is $p$-regular (i.e. $h$ has order coprime to $p$). This special behaviour of $p$-regular elements suggested to him to look at modular representations.

The basics of modular representations and Brauer characters are discussed in sufficient detail in Chapter 2 of [31]. A modular representation $\rho$ of a group $G$ is a representation defined over a field of positive characteristic $p$ dividing the order $|G|$ of $G$. Such representations possess many special (i.e. unpleasant) features. For one thing, they are no longer completely reducible (so the role of irreducible modules as direct summands will be replaced with their role as composition factors). For another, the usual notion of character (the trace of representation matrices) loses its usefulness and is replaced by the more subtle Brauer character $\beta(\rho)$: a complex-valued class function on $M$ which is only well-defined on the $p$-regular elements of $G$.

**Theorem 7.** [105], [17], [13] Let $g \in \mathbb{M}$ be any element of prime order $p$, for any $p$ dividing $|\mathbb{M}|$. Then there is a vertex operator superalgebra $^g\mathcal{V} = \bigoplus_{n \in \mathbb{Z}}^g\mathcal{V}_n$ defined over the finite field $\mathbb{F}_p$ and acted on by the centraliser $C_{\mathbb{M}}(g)$. If $h \in C_{\mathbb{M}}(g)$ is $p$-regular, then the graded Brauer character

$$R(g, h; \tau) := q^{-1} \sum_{n \in \mathbb{Z}} \beta(^g\mathcal{V}_n)(h) q^n$$

equals the McKay–Thompson series $T_{gh}(\tau)$. Moreover, for $g$ belonging to any conjugacy class in $\mathbb{M}$ except 2B, 3B, 5B, 7B, or 13B, this is in fact an ordinary VOA (i.e. the ‘odd’ part vanishes), while in the remaining cases the graded Brauer characters of both the odd and even parts can separately be expressed using McKay–Thompson series.

By a vertex operator superalgebra, we mean there is a $\mathbb{Z}_2$-grading into even and odd subspaces, and for $u, v$ both odd the commutator in (4.2a) is replaced by an anticommutator. In the proof, the superspaces arise as cohomology groups, which naturally form an alternating sum. The centralisers $C_{\mathbb{M}}(g)$ in the Theorem are quite nice: e.g. for $g$ in classes 2A, 2B, 3A, 3B, 3C, 5A, 5B, 7A, 11A, respectively, these involve the sporadic groups $\mathbb{B}$, $Co_1$, $Fi'_{24}$, $Suz$, $Th$, $HN$, $HJ$, $He$, and $M_{12}$. The proof for $p = 2$ is not complete at the present time. The conjectures in [105] concerning modular analogues of the Griess algebra for several sporadics follow from Theorem 7.

Can these modular $^g\mathcal{V}$s be interpreted as a reduction mod $p$ of (super)algebras in characteristic 0? Also, what about elements $g$ of composite order?

**Conjecture 8.** [13] Choose any $g \in \mathbb{M}$ and let $n$ denote its order. Then there is a $\frac{1}{n}\mathbb{Z}$-graded superspace $^g\hat{\mathcal{V}} = \bigoplus_{i \in \frac{1}{n}\mathbb{Z}}^g\hat{\mathcal{V}}_i$ over the ring of cyclotomic integers $\mathbb{Z}[e^{2\pi i/n}]$. It is often (but probably not always) a vertex operator superalgebra — in particular $1\hat{\mathcal{V}}$ is an integral form of the Moonshine module $V^\natural$. Each $^g\hat{\mathcal{V}}$ carries a representation of a central extension of $C_{\mathbb{M}}(g)$ by $C_n$. Define the graded trace

$$B(g, h; \tau) = q^{-1} \sum_{i \in \frac{1}{n}\mathbb{Z}} \text{ch}_{^g\hat{\mathcal{V}}_i}(h) q^i.$$
If \( g, h \in M \) commute and have coprime orders, then \( B(g, h; \tau) = T_{gh}(\tau) \). If all \( q \)-coefficients of \( T_g \) are nonnegative, then the ‘odd’ part of \( ^g\hat{V} \) vanishes, and \( ^g\hat{V} \) is the \( g \)-twisted module \( V^g(g) \) of [36]. If \( g \) has prime order \( p \), then the reduction mod \( p \) of \( ^g\hat{V} \) is the modular vertex operator superalgebra \( ^g\hat{V} \) of Theorem 7.

When we say \( ^1\hat{V} \) is an integral form for \( V^1 \), we mean that \( ^1\hat{V} \) has the same structure as a VOA, with everything defined over \( \mathbb{Z} \), and tensoring it with \( \mathbb{C} \) recovers \( V^2 \). This remarkable conjecture, which tries to explain Theorem 7, is completely open.

5.5. The geometry of Moonshine. Algebra is the mathematics of structure, and so of course it has a profound relationship with every area of mathematics. Therefore the trick for finding possible fingerprints of Moonshine in say geometry is to look there for modular functions. And that search quickly leads to the elliptic genus.

For details see e.g. [55], [108], [112]. All manifolds here are compact, oriented and differentiable. In Thom’s cobordism ring \( \Omega \), elements are equivalence classes of cobordant manifolds, addition is connected sum, and multiplication is Cartesian product. The universal elliptic genus \( \phi(M) \) is a ring homomorphism from \( \mathbb{Q} \otimes \Omega \) to the ring of power series in \( q \), which sends \( n \)-dimensional manifolds with spin connections to a weight \( n/2 \) modular form of \( \Gamma_0(2) \) with integer coefficients. Several variations and generalisations have been introduced, e.g. the Witten genus assigns spin manifolds with vanishing first Pontryagin class a weight \( n/2 \) modular form of \( \text{SL}_2(\mathbb{Z}) \) with integer coefficients.

Several deep relationships between elliptic genera and the general material reviewed elsewhere in this paper, have been uncovered. For instance, the important rigidity property of the Witten genus with respect to any compact Lie group action on the manifold, is a consequence of the modularity of the characters of affine algebras (our Theorem 4) [81]. The elliptic genus of a manifold \( M \) has been interpreted as the graded dimension of a vertex operator superalgebra constructed from \( M \) [111]. Seemingly related to this, [18] recovered the elliptic genus of a Calabi–Yau manifold \( X \) from the sheaf of vertex algebras in the chiral de Rham complex [85] attached to \( X \). Unexpectedly, the elliptic genus of even-dimensional projective spaces \( \hat{P}^{2n} \) has nonnegative coefficients and in fact equals the graded dimension of a certain vertex algebra [86]; this suggests interesting representation-theoretic questions in the spirit of Monstrous Moonshine. In physics, elliptic genera arise as partition functions of \( N = 2 \) superconformal field theories [120]. Mason’s constructions [88] associated to Moonshine for the Mathieu group \( M_{24} \) have been interpreted as providing a geometric model (‘elliptic system’) for elliptic cohomology \( \text{Ell}^* (BM_{24}) \) of the classifying space of \( M_{24} \) [112], [39]. The Witten genus (normalised by \( \eta^8 \)) of the Milnor–Kervaire manifold \( M^8_0 \), an 8-dimensional manifold built from the \( E_8 \) diagram, equals \( j^{1/2} [55] \) (recall (3.5)).

Hirzebruch’s ‘prize question’ (p.86 of [55]) asks for the construction of a 24-dimensional manifold \( M \) with Witten genus \( J \) (after being normalised by \( \eta^{24} \)). We would like \( \mathbb{M} \) to act on \( M \) by diffeomorphisms, and the twisted Witten genera to be the McKay–Thompson series \( T_g \). It would also be nice to associate Norton’s series \( N(g, h) \) to this Moonshine manifold. Constructing such a manifold is perhaps the remaining Holy Grail of Monstrous Moonshine.

Hirzebruch’s question was partially answered by Mahowald and Hopkins [84], who
constructed a manifold with Witten genus $J$, but couldn’t show it would support an effective action of $M$. Related work is [3], who constructed several actions of $M$ on e.g. 24-dimensional manifolds (but none of which could have genus $J$), and [73], who showed the graded dimensions of the subspaces $V^2_{\pm}$ of the Moonshine module are twisted $\hat{A}$-genera of Milnor–Kervaire’s manifold $M^8_0$ (the $A$-genus is the specialisation of elliptic genus to the cusp $i\infty$).

There has been a second conjectured relationship between geometry and Monstrous Moonshine. Mirror symmetry says that most Calabi–Yau manifolds come in closely related pairs. Consider a 1-parameter family $X_z$ of Calabi–Yau manifolds, with mirror $X^*$ given by the resolution of an orbifold $X/G$ for $G$ finite and abelian. Then the Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X^*)$ will be equal, and more precisely the moduli space of (complexified) Kähler structures on $X$ will be locally isometric to the moduli space of complex structures on $X^*$. The ‘mirror map’ $z(q)$, which can be defined using the Picard–Fuchs equation [95], gives a canonical map between those moduli spaces. For example, $x_1^4 + x_2^4 + x_3^4 + x_4^4 + z^{-1/4}x_1x_2x_3x_4 = 0$ is such a family of K3 surfaces, where $G = C_4 \times C_4$. Its mirror map is given by

$$z(q) = q - 104q^2 + 6444q^3 - 311744q^4 + 13018830q^5 - 493025760q^6 + \cdots . \quad (5.6)$$

Lian–Yau [80] noticed that the reciprocal $1/z(q)$ of the mirror map in (5.6) equals the McKay–Thompson series $T_g(\tau) + 104$ for $g$ in class 2A of $M$. After looking at several other examples with similar conclusions, they proposed their Mirror-Moonshine Conjecture: The reciprocal $1/z$ of the mirror map of a 1-parameter family of K3 surfaces with an orbifold mirror, will be a McKay–Thompson series (up to an additive constant).

A counterexample (and more examples) are given in §7 of [118]. In particular, although there are relations between mirror symmetry and modular functions (see e.g. [51] and [54]), there doesn’t seem to be any special relation with the Monster. Doran [40] ‘de-mystifies the Mirror-Moonshine phenomenon’ by finding necessary and sufficient conditions for $1/z$ to be a modular function for a modular group commensurable with $SL_2(\mathbb{Z})$.

6. The physics of Moonshine

The physical side (perturbative string theory, or equivalently conformal field theory) of Moonshine was noticed early on, and has profoundly influenced the development of Moonshine and VOAs. This is a very rich subject, which we can only superficially touch on. The book [32], with its extensive bibliography, provides an introduction but will be difficult reading for many mathematicians (as will this section!) The treatment in [45] is more accessible and shows how naturally VOAs arise from the physics. This effectiveness of physical interpretations isn’t magic — it merely tells us that many of our finite-dimensional objects are seen much more clearly when studied through infinite-dimensional structures (often by being ‘looped’). Of course Moonshine, which teaches us to study the finite group $M$ via its infinite-dimensional module $V^2$, fits perfectly into this picture.

A conformal field theory (CFT) is a quantum field theory on 2-dimensional space-time, whose symmetries include the conformal transformations. In string theory the basic objects are finite curves (‘strings’) rather than points (‘particles’), and the CFT lives on the surface
traced by the strings as they evolve (colliding and separating) through time. Each CFT is associated with a pair \( \mathcal{V}_L, \mathcal{V}_R \) of mutually commuting VOAs, called its chiral algebras [6]. For example, strings living on a compact Lie group manifold (the so-called Wess–Zumino–Witten model) will have chiral algebras given by affine algebra VOAs. The space \( \mathcal{H} \) of states for the CFT carries a representation of \( \mathcal{V}_L \otimes \mathcal{V}_R \), and many authors have (somewhat optimistically) concluded that the study of CFTs reduces to that of VOA representation theory. Rational VOAs correspond to the important class of rational CFTs, where \( \mathcal{H} \) decomposes into a finite sum \( \oplus M_L \otimes M_R \) of irreducible modules. The Virasoro algebra (4.3) arises naturally in CFT through infinitesimal conformal transformations. The vertex operator \( Y(\phi, z) \), for the space-time parameter \( z = e^{t+ix} \), is the quantum field which creates from the vacuum \( |0\rangle \in \mathcal{H} \) the state \( |\phi\rangle \in \mathcal{H} \) at time \( t = -\infty \): \( |\phi\rangle = \lim_{z \to 0} Y(\phi, z) |0\rangle \). In particular, Borcherds’ definition [9] of VOAs can be interpreted as an axiomatisation of the notion of chiral algebra in CFT, and for this reason alone is important.

In CFT, the Hauptmodul property of Moonshine is hard to interpret, and a less direct formulation like that in [116] is needed. However, both the statement and proof of Theorem 5 are natural from the CFT framework (see [45]) — e.g. the modularity of the series \( T_g \) and \( N(g, h) \) are automatic in CFT. This modularity arises in CFT through the equivalence of the Hamiltonian formulation, which describes concretely the graded spaces we take traces on (and hence the coefficients of our \( q \)-expansions), and the Feynman path formalism, which interprets these graded traces as sections over moduli spaces (and hence makes modularity manifest). Beautiful reviews are sketched in [119], [120].

Because \( V^2 \) is so mathematically special, it may be expected that it corresponds to interesting physics. Certainly it has been the subject of some speculation. There will be a \( c = 24 \) rational CFT whose chiral algebra \( \mathcal{V}_L \) and state space \( \mathcal{H} \) are both \( V^2 \), while \( \mathcal{V}_R \) is trivial (this is possible because \( V^2 \) is holomorphic). This CFT is nicely described in [34]; see also [35]. The Monster is the symmetry of that CFT, but the Bimonster \( \mathbb{M} \wr C_2 \) will be the symmetry of a rational CFT with \( \mathcal{H} = V^2 \otimes \overline{V^2} \). The paper [27] finds a family of D-branes for the latter theory which are in one-to-one correspondence with the elements of \( \mathbb{M} \), and their ‘overlaps’ \( \langle g \parallel q^2(\overline{L_0}+L_0-\pi \pi)\parallel h \rangle \) equal the McKay–Thompson series \( T_{g^{-1}h} \). However, we still lack any explanation as to why a CFT involving \( V^2 \) should yield interesting physics.

Almost every facet of Moonshine finds a natural formulation in CFT, where it often was discovered first. For example, the ‘No-Ghost’ Theorem of Brower–Goddard–Thorn was used to great effect in [11] to understand the structure of the Monster Lie algebra \( \mathfrak{m} \). On a finite-dimensional manifold \( M \), the index of the Dirac operator \( D \) in the heat kernel interpretation is a path integral in supersymmetric quantum mechanics, i.e an integral over the free loop space \( \mathcal{L}M = \{ \gamma : S^1 \to M \} \); the string theory version of this is that the index of the Dirac operator on \( \mathcal{L}M \) should be an integral over \( \mathcal{L}(\mathcal{L}M) \), i.e. over smooth maps of tori into \( M \), and this is just the elliptic genus, and explains why it should be modular. The orbifold construction of [36] comes straight from CFT (although [43]’s construction of \( V^2 \) predates CFT orbifolds by a year and in fact influenced their development in physics). That said, the translation process from physics to mathematics of course is never easy — Borcherds’ definition [9] is a prime example!

But from this standpoint, what is most exciting is what hasn’t yet been fully exploited.
String theory tells us that CFT can live on any surface $\Sigma$. The VOAs, including the geometric VOAs of [59], capture CFT in genus 0. The graded dimensions and traces considered above concern CFT quantities (‘conformal blocks’) at genus 1: $\tau \mapsto e^{2\pi i \tau}$ maps $\mathbb{H}$ onto a cylinder, and the trace identifies the two ends. But there are analogues of all this at higher genus [123] (though the formulas can rapidly become awkward). For example, the graded dimension of e.g. the $V^2$ CFT in genus 2 is computed in [117], and involves e.g. Siegel theta functions. The orbifold theory in §5.1 is genus 1: each ‘sector’ $(g, h)$ corresponds to a homomorphism from the fundamental group $\mathbb{Z}^2$ of the torus into the orbifold group $G$ (e.g. $G = \mathbb{M}$) — $g$ and $h$ are the targets of the two generators of $\mathbb{Z}^2$ and hence must commute. More generally, the sectors will correspond to each homomorphism $\varphi : \pi_1(\Sigma) \to G$, and to each we will get a higher genus trace $Z(\varphi)$, which will be a function on the Teichmüller space $T_g$ (generalising the upper half-plane $\mathbb{H}$ for genus 1). The action of $\text{SL}_2(\mathbb{Z})$ on the $N(g, h)$ generalises to the action of the mapping class group on $\pi_1(\Sigma)$ and $T_g$. See e.g. [4] for some thoughts in this direction.

7. Conclusion

There are different basic aspects to Monstrous Moonshine: (i) why modularity enters at all; (ii) why in particular we have genus 0; and (iii) what does it have to do with the Monster. We understand (i) best. There will be a Moonshine-like relation between any (subgroup of the) automorphism group of any rational VOA, and the characters $\chi_M$, and the same can be expected to hold of the orbifold characters $Z$ in §5.1.

To prove the genus 0 property of the $T_g$, we needed recursions obtained one way or another from the Monster Lie algebra $\mathfrak{m}$, and from these we apply Theorem 6. These recursions are very special, but so presumably is the genus 0 property. The suggestion of [20] though is that we may be able to considerably simplify this part of the argument.

Every group known to have rich Moonshine properties is contained in the Monster. Our understanding of this seemingly central role of $\mathbb{M}$ is the poorest of those three aspects.

It should be clear from this review, of the central role VOAs play in our current understanding of Moonshine. The excellent review [39] makes this point even more forcefully. It can be (and has been) questioned though whether the full and difficult machinery of VOAs is really needed to understand this, i.e. whether we really have isolated the key conjunction of properties needed for Moonshine to arise. CFT has been an invaluable guide thus far, but perhaps we are a little too steeped in its lore.

Moonshine (in its more general sense) is a relation between algebra and number theory, and its impact on algebra has been dramatic (e.g. VOAs, $V^2$, Borcherds–Kac–Moody algebras). Its impact on number theory has been far less so. This may merely be a temporary accident due to the backgrounds of most researchers (including the mathematical physicists) working to date in the area. But the most exciting prospects for the future of Moonshine (in this writer’s opinion) are in the direction of number theory. Hints of this future can be found in e.g. [121], [41], [14], [33], [52], [94].

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