ABSTRACT. In this paper, we establish sufficient conditions for a singular integral $T$ to be bounded from certain Hardy spaces $H^p_L$ to Lebesgue spaces $L^p$, $0 < p \leq 1$, and for the commutator of $T$ and a BMO function to be weak-type bounded on Hardy space $H^1_L$. We then show that our sufficient conditions are applicable to the following cases: (i) $T$ is the Riesz transform or a square function associated with the Laplace-Beltrami operator on a doubling Riemannian manifold, (ii) $T$ is the Riesz transform associated with the magnetic Schrödinger operator on an Euclidean space, and (iii) $T = g(L)$ is a singular integral operator defined from the holomorphic functional calculus of an operator $L$ or the spectral multiplier of a non-negative self-adjoint operator $L$.

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1. Introduction and statement of main results

The Calderón-Zygmund theory of singular integral operators has been a central part of Harmonic analysis and has had extensive applications to estimates on regularity of solutions to partial differential equations. There are a number of recent works which study singular integral operators with non-smooth kernels that are beyond the standard Calderón-Zygmund theory. See, for example [20], [14], [11] and [5]. This paper is a study in that direction, aiming to study boundedness of certain singular integral operators and their commutators with BMO functions. Our results are applicable to large classes of differential and integral operators which include the Riesz transforms on manifolds, the holomorphic functional calculi and spectral multipliers of non-negative self adjoint operators such as the Laplace-Beltrami operators on manifolds and magnetic Schrödinger operators on Euclidean spaces.

Let us first explain our framework. Assume that \((X,d,\mu)\) is a metric measure space endowed with a distance \(d\) and a nonnegative Borel doubling measure \(\mu\) on \(X\). Recall that a measure is doubling provided that there exists a constant \(C > 0\) such that for all \(x \in X\) and for all \(r > 0\),

\[
V(x, 2r) \leq CV(x, r) < \infty,
\]

where \(B(x, r) = \{y \in X : d(x, y) < r\}\) and \(V(x, r) = \mu(B(x, r))\). In particular, \(X\) is a space of homogeneous type. A more general definition and further studies of these spaces can be found in [17]. Note that the doubling property implies the following strong homogeneity property,

\[
V(x, \lambda r) \leq c\lambda^n V(x, r)
\]

for some \(c, n > 0\) uniformly for all \(\lambda \geq 1\) and \(x \in X\). The smallest value of the parameter \(n\) in the right hand side of (2) is a measure of the dimension of the space. There also exist \(c\) and \(N, 0 \leq N \leq n\), so that

\[
V(y, r) \leq c \left(1 + \frac{d(x, y)}{r}\right)^N V(x, r)
\]

uniformly for all \(x, y \in X\) and \(r > 0\). Indeed, property (3) with \(N = n\) is a direct consequence of the triangle inequality of the metric \(d\) and the strong homogeneity property.

To simplify notation, we will often just use \(B\) for \(B(x_B, r_B)\). Also given \(\lambda > 0\), we will write \(\lambda B\) for the \(\lambda\)-dilated ball, which is the ball with the same center as \(B\) and with radius \(r_{\lambda B} = \lambda r_B\). For each ball \(B \subset X\) we set

\[
S_0(B) = B \text{ and } S_j(B) = 2^j B \setminus 2^{j-1} B \text{ for } j \in \mathbb{N}.
\]

In this paper we will assume that there exists an operator \(L\) defined on \(L^2(X)\). We consider the following conditions:

\textbf{(H1)} \(L\) is a non-negative self-adjoint operator on \(L^2(X)\);

\textbf{(H2)} The operator \(L\) generates an analytic semigroup \(\{e^{-tL}\}_{t>0}\) which satisfies the Davies-Gaffney estimate. That is, there exist constants \(C, c > 0\) such that for any open subsets \(U_1, U_2 \subset X\),
\[(4) \quad |\langle e^{-tL}f_1, f_2 \rangle| \leq C \exp \left(-\frac{\text{dist}(U_1, U_2)^2}{ct}\right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}, \quad \forall t > 0,\]

for every \(f_i \in L^2(X)\) with \(\text{supp} f_i \subset U_i, \ i = 1, 2\), where \(\text{dist}(U_1, U_2) := \inf_{x \in U_1, y \in U_2} d(x, y)\).

**Remark 1.1.** It is easy to check that the Gaussian bound \((H3)\) implies condition \((H2)\).

We list a number of examples:

(i) It is well known that the Laplace operator \(\Delta\) on the Euclidean space \(\mathbb{R}^n\) satisfies \((H1)\) and \((H3)\). So do the second order non-negative self-adjoint divergence form operators with real bounded measurable coefficients on \(\mathbb{R}^n\). Second order divergence form operators with complex bounded measurable coefficients on \(\mathbb{R}^n\) would satisfy \((H2)\), and satisfy \((H3)\) for low dimensions \(n\) but might not satisfy \((H3)\) for higher dimensions \(n\). See for example [18].

(ii) Schrödinger operators or magnetic Schrödinger operators with real potentials satisfy \((H1)\) and \((H3)\), see [39].

(iii) Laplace-Beltrami operators on all complete Riemannian manifolds satisfy \((H1)\) and \((H2)\) but do not satisfy \((H3)\) in general. [18].

(iv) Laplace type operators acting on vector bundles satisfy \((H1)\) and \((H2)\), see [38].

Our aim in this paper is to obtain boundedness of certain singular integral operators with non-smooth kernels and boundedness of their commutators via estimates on related function spaces. Recently, the theory of Hardy spaces associated with operators was studied by many authors, see for examples [4], [6], [7], [26], [31], [30], [32], [19] and [41]. We denote by \(H^p_L(X), 0 < p \leq 1\), the Hardy spaces associated to the operator \(L\). For the precise definition, we refer the reader to Section 2.

Assume that \(T\) is a bounded operator on \(L^2(X)\). There are a number of known sufficient conditions on \(T\) or its associated kernel \(k(x, y)\) so that \(L^2\) boundedness of \(T\) can be extended to other spaces such as Lebesgue space \(L^p, p \neq 2\), Hardy spaces, and BMO spaces. See, for example [31] and [19] for boundedness of holomorphic functional calculi of certain generators of analytic semigroups on Hardy spaces. It is also a natural question to consider boundedness of the commutator of a BMO function \(b\) and \(T\) which is given by

\([b, T]f(x) := T((b(x) - b)f)(x)\)
for all functions \( f \) with compact supports. See, for example [40] Chapter 7 and [25].

In this paper, we establish a sufficient condition on an \( L^2 \) bounded operator \( T \) so that it implies both the following:

(i) \( T \) is bounded from the Hardy spaces \( H^p_2(X) \) to \( L^p(X), 0 < p \leq 1 \); and
(ii) the commutator \( [b, T] \) is bounded from \( H^1_2(X) \) to \( L^{1, \infty}(X) \) under the extra assumption that \( T \) is of weak type \((1, 1)\).

While the boundedness of singular integral operators whose kernels are not smooth enough to belong to the standard class of Calderón-Zygmund operators and the boundedness of the commutators of BMO functions and these operators was studied extensively, see for example [20], [11], [3], [25], [5] and their references, the boundedness of the commutators of BMO functions and these operators at the end-point spaces is much less well known. Our main results in this paper include a sufficient condition so that weak type \((1, 1)\) estimate of \( T \) implies certain weak type boundedness of its commutators \([b, T]\) and the condition is general enough to be applicable to a wide range of operators in Sections 4, 5 and 6. The main result is as follows.

**Theorem 1.2.** Assume that \( L \) is an operator which satisfies (H1) and (H2). Let 0 < \( p \leq 1 \). Let a denote a \((p, 2, m)\)-atom in the Hardy space \( H^p_2(X) \) associate to the operator \( L \). (See Section 2 for the precise definition). Assume that \( T \) is a bounded operator on \( L^2(X) \) so that \( Ta \) satisfies the estimate

\[
\left( \int_{S_j(B)} |Ta|^2 d\mu \right)^{\frac{1}{2}} \leq C 2^{-2jm} V(B)^{\frac{1}{2} - \frac{1}{p}}
\]

for any \((p, 2, m)\)-atom \( a \) supported in the ball \( B \) and all \( j \geq 2 \). Then we have:

(i) \( T \) is bounded from \( H^p_2(X) \) to \( L^p(X) \); and
(ii) in addition, if \( T \) is of weak type \((1, 1)\) then the commutator \([b, T]\), where \( b \) is a BMO function, maps continuously from \( H^1_2(X) \) to \( L^{1, \infty}(X) \).

**Remark 1.3.** (a) The main result of Theorem 1.2 is the boundedness of the commutator in (ii). The result in (i) on boundedness of \( T \) on Hardy spaces was proved in many situations and can be considered as folklore.

(b) There is no regularity condition on the kernel of \( T \), so in general \( T \) is not a standard Calderón-Zygmund singular integral operator (whose kernel is required to be Hölder continuous or at least to satisfy the Hörmander condition).

(c) In Sections 4, 5 and 6, we apply Theorem 1.2 to prove the boundedness of various singular integral operators and their commutators which do not belong to the class of Calderón-Zygmund operators.

(d) It follows from (i) and interpolation (see [31] Theorem 9.3) that \( T \) is bounded from \( H^p_2(X) \) to \( L^p(X) \) for 0 < \( p \leq 2 \).
The paper is organized as follows. In Section 2, we recall the definition of \( H^p_L(X) \), the Hardy space associated to the operator \( L \), and some characterizations of \( H^p_L(X) \). The proof of Theorem 1.2 is given in Section 3. In Section 4, we consider the Riesz transform \( T \) on a doubling manifold and use Theorem 1.2 to obtain some endpoint estimates of commutators of Riesz transforms on manifolds. In Section 5, we study the boundedness of Riesz transforms of magnetic Schrödinger operators and their commutators. We will show that the Riesz transforms of magnetic Schrödinger operators are bounded from \( H^p_A(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) for all \( 0 < p \leq 1 \) and the commutators of Riesz transforms and BMO functions are bounded from \( H^1_A(\mathbb{R}^n) \) to \( L^1_{\infty}(\mathbb{R}^n) \). In Section 6, we show boundedness of holomorphic functional calculus and spectral multipliers of an operator which generates a holomorphic semigroup with suitable kernel bounds.

2. Hardy spaces associated to operators

The theory of Hardy spaces associated to non-negative self-adjoint operators satisfying the Davies-Gaffney estimate was developed recently by Hofmann et al. [31]. Here, we use the definitions and characterizations of Hardy spaces \( H^p_L(X) \) in [31] and [19].

2.1. Hardy spaces \( H^p_{L,Sh}(X) \) for \( p \geq 1 \). Let \( L \) be an operator which satisfies (H1) and (H2). Set

\[
H^2(X) := \overline{R(L)} = \{ Lu \in L^2(X) : u \in D(L) \}
\]

where \( D(L) \) is the domain of \( L \).

It is known that \( L^2(X) = \overline{R(L)} \oplus \mathcal{N}(L) \), where \( R(L) \) and \( \mathcal{N}(L) \) stand for the range and the kernel of \( L \), and the sum is orthogonal.

Consider the following quadratic operators associated to \( L \)

\[
S_{h,K} f(x) = \left( \int_{0}^{\infty} \int_{d(x,y)<t} |(t^2 L)^K e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \quad x \in X
\]

where \( K \) is a positive integer and \( f \in L^2(X) \). We shall write \( S_h \) in place of \( S_{h,1} \). For each integer \( K \geq 1 \) and \( 1 \leq p < \infty \), we now define

\[
D_{K,p} = \left\{ f \in H^2(X) : S_{h,K} f \in L^p(X) \right\}, \quad 1 \leq p < \infty.
\]

**Definition 2.1.** Let \( L \) satisfy (H1) and (H2).

(i) For each \( 1 \leq p \leq 2 \), the Hardy space \( H^p_{L,Sh}(X) \) associated to \( L \) is the completion of the space \( D_{1,p} \) in the norm

\[
\|f\|_{H^p_{L,Sh}(X)} = \|S_h f\|_{L^p(X)}.
\]

(ii) For each \( 2 < p < \infty \), the Hardy space \( H^p_L(X) \) associated to \( L \) is the completion of the space \( D_{K_0,p} \) in the norm

\[
\|f\|_{H^p_L(X)} = \|S_{h,K_0} f\|_{L^p(X)}, \quad K_0 = \left\lfloor \frac{n}{2} \right\rfloor + 1.
\]
It can be verified that $H^2_{L,S_h}(X) = H^2(X) \subset L^2(X)$ and the dual space of $H^p_{L,S_h}(X)$ is $H^{p'}_{L,S_h}(X)$ where $1/p + 1/p' = 1$ (see Proposition 9.4 of [31]). If $L$ satisfies (H1) and (H3), then it was proved in [3] that $H^p_{L,S_h}(X)$ and $L^p(X)$ coincide for all $p \in (1, \infty)$.

2.2. The atomic Hardy spaces $H^p_{L,at,m}(X)$ for $p \leq 1$. In what follows, assume that

$$m \in \mathbb{N} \quad \text{and} \quad m > \frac{n(2-p)}{4p},$$

where the parameter $n$ is a constant in [2]. Let us denote by $\mathcal{D}(T)$ the domain of an operator $T$.

The notion of a $(p, 2, m)$-atom, $0 < p \leq 1$, associated to operators on spaces $(X, d, \mu)$, is defined as follows.

**Definition 2.2.** A function $a \in L^2(X)$ is said to be a $(p, 2, m)$-atom associated to an operator $L$ if there exist a function $b \in \mathcal{D}(L^m)$ and a ball $B$ such that

1. $a = L^mb$;
2. $\text{supp} L^kb \subset B$, $k = 0, 1, \ldots, m$;
3. $\|(r_B^2 L)^k b\|_{L^2(X)} \leq r_B^{2m} V(B)^{\frac{1}{2} - \frac{1}{p}}$, $k = 0, 1, \ldots, m$.

Obviously, in the case $\mu(X) < \infty$ the constant function $[\mu(X)]^{-\frac{1}{p}}$ is also considered to be an atom.

**Definition 2.3.** Given $0 < p \leq 1$ and $m > \frac{n(2-p)}{4p}$, we say that $f = \sum \lambda_j a_j$ is an atomic $(p, 2, m)$-representation if $\{\lambda_j\}_{j=0}^\infty \in l^p$, each $a_j$ is a $(p, 2, m)$-atom, and the sum converges in $L^2(X)$. Set

$$\mathbb{H}^p_{L,at,m}(X) = \{ f : f \text{ has an atomic } (p, 2, m)\text{-representation} \},$$

with the norm given by

$$||f||_{\mathbb{H}^p_{L,at,m}(X)} = \inf\{ \left( \sum |\lambda_j|^p \right)^{1/2} : f = \sum \lambda_j a_j \text{ is an atomic } (p, 2, m)\text{-representation} \}.$$

The space $H^p_{L,at,m}(X)$ is then defined as the completion of $\mathbb{H}^p_{L,at,m}(X)$ with respect to the quasi-metric $d$ defined by $d(h, g) = ||h - g||_{\mathbb{H}^p_{L,at,m}(X)}$ for all $h, g \in \mathbb{H}^p_{L,at,m}(X)$.

In this case the mapping $h \rightarrow ||h||_{H^p_{L,at,m}(X)}$, $0 < p < 1$ is not a norm and $d(h, g) = ||h - g||_{H^p_{L,at,m}(X)}$ is a quasi-metric. For $p = 1$, the mapping $h \rightarrow ||h||_{H^1_{L,at,m}(X)}$ is a norm. A straightforward argument shows that $H^p_{L,at,m}(X)$ is complete. In particular, $H^1_{L,at,m}(X)$ is a Banach space. A basic result concerning these spaces is the following proposition.

**Proposition 2.4.** If an operator $L$ satisfies conditions (H1) and (H2), then for every $0 < p \leq 1$ and for all integers $m \in \mathbb{N}$ with $m > \frac{n(2-p)}{4p}$, the spaces $H^p_{L,at,m}(X)$ coincide and their norms are equivalent.
For the proof, we refer to Theorem 5.1 of [31] for $p = 1$, and Section 3 of [19] for $p < 1$.

The notion of a $(p, 2, m, \epsilon)$-molecule associated to an operator $L$ will be described as follows.

**Definition 2.5.** Let $0 < p \leq 1$, $\epsilon > 0$ and $m \in \mathbb{N}$. We say that a function $\alpha \in L^2$ is called a $(p, 2, m, \epsilon)$-molecule associated to $L$ if there exist a function $b \in D(L^m)$ and a ball $B$ such that

(i) $\alpha = L^m b$;
(ii) For every $k = 0, 1, \ldots, m$ and $j = 0, 1, \ldots$, the following estimate holds

$$\|(r_B^2 L)^k b\|_{L^2(S_j(B))} \leq C B^{2 - \frac{j}{2} - \frac{1}{p}}.$$

**Proposition 2.6.** Suppose $0 < p \leq 1$, $m > \frac{n(2 - p)}{4p}$ and $\epsilon > 0$. If $\alpha$ is a $(p, 2, m, \epsilon)$-molecule associated to $L$, then $\alpha \in H^p_{L, at, m}(X)$. Moreover, $\|\alpha\|_{H^p_{L, at, m}(X)}$ is independent of $\alpha$.

For the proof, we refer to [31] for $p = 1$, and [19] for $p < 1$.

### 2.3. A characterization of Hardy spaces associated to operators in terms of square functions.

In Section 2.1, we had the definitions of the Hardy spaces $H^p_{L, S_h}(X)$ for $p \geq 1$. Now consider the case $0 < p < 1$. The space $H^p_{L, S_h}(X)$ is defined as the completion of

$$\{f \in H^2(X) : |S_h f|_{L^p(X)} < \infty\}$$

under the norm defined by the $L^p$ norm of the square function; i.e.,

$$\|f\|_{H^p_{L, S_h}(X)} = |S_h f|_{L^p(X)}, \ 0 < p < 1.$$

Then the “square function” and “atomic” $H^p$ spaces are equivalent, if the parameter $m > \frac{n(2 - p)}{4p}$. In fact, we have the following result.

**Proposition 2.7.** Suppose $0 < p \leq 1$ and $m > \frac{n(2 - p)}{4p}$. Then we have $H^p_{L, at, m}(X) = H^p_{L, S_h}(X)$ and their norms are equivalent.

*Proof:* For the proof, see [19].

Consequently, as in the next definition, one may write $H^p_{L, at}(X)$ in place of $H^p_{L, at, m}(X)$ when $m > \frac{n(2 - p)}{4p}$. Precisely, we have the following definition.

**Definition 2.8.** The Hardy space $H^p_L(X), 0 < p \leq 1$, is the space

$$H^p_L(X) := H^p_{L, S_h}(X) := H^p_{L, at}(X) := H^p_{L, at, m}(X), \ m > \frac{n(2 - p)}{4p}.$$
3. BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS AND THEIR COMMUTATORS

To prove that an operator $T$ is bounded on the Hardy space $H^p_L(X)$ which possesses an atomic decomposition, it is not enough in general to prove that $Ta$ is uniformly bounded for all atomic functions $a$. However, if the operator $T$ satisfies extra condition such as being $L^2(X)$ bounded (or even the weaker condition of weak type $(2, 2)$), then the uniform boundedness of $Ta$ does imply the boundedness of $T$ on $H^p_L(X)$. More precisely, we have the following result.

**Proposition 3.1.** Suppose that $T$ is a linear (resp. nonnegative sublinear) operator which maps $L^2(X)$ continuously into $L^{2, \infty}(X)$. If there exists, for $0 < p \leq 1$, a constant $C$ such that

$$||Ta||_{L^p} \leq C$$

for all $(p, 2, m)$-atoms $a \in H^p_L(X)$, then $T$ extends to a bounded linear (resp. sublinear) operator from $H^p_L(X)$ to $L^p(X)$.

**Proof.** The proof of this proposition is standard. For completeness and convenience of reader, we give a proof here.

Suppose that $f \in H^p_L(X) \cap H^2(X)$ so that we may write $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $L^2$ sense where $a_j$ are $(p, 2, m)$-atoms and $\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \approx ||f||_{H^p_L(X)}$. It suffices to show that $|Tf| \leq \sum_{j=1}^{\infty} |\lambda_j||Ta_j|$.

If $T$ is a linear operator, then from the fact that the sum $\sum_{j=1}^{\infty} \lambda a_j$ converges in $L^2(X)$ and $T$ is of weak type $(2, 2)$, we conclude that $Tf = \sum_{j=1}^{\infty} \lambda_j Ta_j$.

If $T$ is a nonnegative sublinear operator and $T$ is of weak type $(2, 2)$, one has

$$\mu\{ x \in X : |Tf - T\left(\sum_{j=1}^{N} \lambda_j a_j\right)| > t \} \leq \frac{C}{t^2} \left\| \sum_{j=N+1}^{\infty} \lambda_j a_j \right\|_{L^2(X)}.$$

Hence

$$\lim_{N \to \infty} \mu\{ x \in X : |Tf - T\left(\sum_{j=1}^{N} \lambda_j a_j\right)| > t \} \leq C \lim_{N \to \infty} \frac{1}{t^2} \left\| \sum_{j=N+1}^{\infty} \lambda_j a_j \right\|_{L^2(X)} = 0.$$

This implies that $T\left(\sum_{j=1}^{N} \lambda_j a_j\right) \to Tf$ a.e. as $N \to \infty$. Since $T$ is a nonnegative sublinear operator, we have

$$Tf - \sum_{j=1}^{\infty} T(\lambda_j a_j) = Tf - T\left(\sum_{j=1}^{N} \lambda_j a_j\right) + T\left(\sum_{j=1}^{N} \lambda_j a_j\right) - \sum_{j=1}^{\infty} T(\lambda_j a_j) \leq Tf - T\left(\sum_{j=1}^{N} \lambda_j a_j\right) \to 0, \text{ as } N \to \infty.$$
Thus, $Tf \leq \sum_{j=1}^{\infty} T(\lambda_j a_j)$. The proof is complete. \hfill \Box

Proof of Theorem 1.2: (i) Proposition 3.1, it suffices to show that for any $(p, 2, m)$-atom $a$, for $m > \frac{n(2-p)}{4p}$, associated to the ball $B$, we have $||Ta||_{L^p} \leq C$. Indeed, we have

$$\int_X |Ta(x)|^p d\mu(x) = \sum_{j=0}^{\infty} \int_{S_j(B)} |Ta(x)|^p d\mu(x)$$

$$= \sum_{j=0}^{\infty} K_j.$$

By Jensen’s and Hölder’s inequalities and (6), one has, for each $j$,

$$K_j \leq V(S_j(B))^{1-\frac{p}{2}} ||Ta||_{L^2(S_j(B))}^p$$

$$\leq CV(2^j B)^{1-\frac{p}{2}} 2^{-2jmp} V(B)^{\frac{p}{2}-1}$$

$$\leq C(2^j)^{n-\frac{np}{2}} - 2mp.$$

This together with $m > \frac{n(2-p)}{4p}$ gives

$$\int_X |Ta(x)|^p d\mu(x) \leq C \sum_{j=0}^{\infty} 2^{j(n-\frac{np}{2}) - 2mp} \leq C.$$

The proof of (i) is complete.

(ii) Suppose that $f \in H_1^1(X) \cap H^2(X)$ so that we may write $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $L^2$ sense where $a_j$ are $(1, 2, m)$-atoms associated to balls $B_j$ and $\sum_{j=1}^{\infty} |\lambda_j| \approx ||f||_{H_1^1}$. It suffices to show that there exists a constant $c > 0$ such that

$$\mu\{x \in X : [b, T](\sum_{j=1}^{\infty} \lambda_j a_j)(x) > \lambda\} \leq \frac{c}{\lambda} ||f||_{H_1^1} ||b||_{BMO}.$$

Using the commutator technique as in (37), we can assume that $b \in L^\infty$. Setting $b_B = \frac{1}{V(B)} \int_B bd\mu$, we have

$$|[b, T]f(x)| = |T((b(x) - b)f)(x)|$$

$$\leq |T((b(x) - b)(\sum_{j \geq 0} \lambda_j a_j))(x)|$$

$$\leq |T[\sum_{j \geq 0} \lambda_j (b(x) - b_{B_j})a_j](x)| + |T[\sum_{j \geq 0} \lambda_j (b - b_{B_j})a_j](x)|$$

where in the last inequality we use that fact that both series $\sum_{j \geq 0} \lambda_j (b(x) - b_{B_j})a_j$ and $\sum_{j \geq 0} \lambda_j (b - b_{B_j})a_j$ converge in $L^1(X)$. 

At this stage, by the similar argument as in Proposition 3.1, we can write
\[
\left| T \left( \sum_{j \geq 0} \lambda_j (b(x) - b_{B_j}) a_j \right) (x) \right| \leq \sum_j |\lambda_j| \left| [b(x) - b_{B_j}] T a_j (x) \right|.
\]
This gives
\[
\left| [b, T] f(x) \right| = \left| T((b(x) - b)f)(x) \right| \\
\leq \sum_j |\lambda_j| \left| [b(x) - b_{B_j}] T a_j (x) \right| + \left| T\left( \sum_j \lambda_j [b_{B_j} - b] a_j \right)(x) \right|.
\]
Therefore,
\[
\lambda \mu \{ x \in X : |[b, T] f(x)| > \lambda \} \leq \lambda \mu \{ x \in X : \sum_j |\lambda_j| \left| [b(x) - b_{B_j}] T a_j (x) \right| > \lambda / 2 \} \\
+ \lambda \mu \{ x \in X : \left| T\left( \sum_j \lambda_j [b_{B_j} - b] a_j \right)(x) \right| > \lambda / 2 \} \\
:= E_1 + E_2.
\]
Let us estimate \( E_2 \) first. Since \( T \) is of weak type \((1, 1)\), one has, by Hölder’s inequality
\[
E_2 \leq C \sum_{j \geq 0} |\lambda_j| \int_X |[b_{B_j} - b(x)] a_j(x)| d\mu(x) \\
\leq C \sum_{j \geq 0} |\lambda_j| \|b_{B_j} - b\|_{L^2(B_j)} \|a_j\|_{L^2(B_j)} \\
\leq C \sum_{j \geq 0} |\lambda_j| \|b\|_{BMO} V(B_j)^{1/2} V(B_j)^{-1/2} \leq C \|f\|_{H^1}\|b\|_{BMO}.
\]
We now estimate \( E_1 \). Obviously,
\[
E_1 \leq C \sum_{j \geq 0} |\lambda_j| \int_X |[b(x) - b_{B_j}] T a_j(x)| d\mu(x) \\
= C \sum_{j \geq 0} |\lambda_j| \sum_{k=0}^{\infty} \int_{S_k(B_j)} |[b(x) - b_{B_j}] T a_j(x)| d\mu(x) \\
\leq C \sum_{j \geq 0} |\lambda_j| \sum_{k=0}^{\infty} \int_{S_k(B_j)} |[b(x) - b_{2kB_j}] T a_j(x)| d\mu(x) \\
+ C \sum_{j \geq 0} |\lambda_j| \sum_{k=0}^{\infty} \int_{S_k(B_j)} |[b_{B_j} - b_{2kB_j}] T a_j(x)| d\mu(x).
\]
By Hölder’s inequality, (9) and the fact that \(|b_{B_j} - b_{2^kB_j}| \leq ck\|b\|_{BMO}\), we have,

\[
\int_{S_k(B_j)} \left| [b(x) - b_{2^kB_j}] T a_j(x) \right| d\mu(x) \leq C \left( \|b(x) - b_{2^kB_j}\|_{L^2(S_k(B_j))} \|T a_j\|_{L^2(S_k(B_j))} \right)
\]

\[
\leq CV(2^k B_j)^{1/2} \|b\|_{BMO} 2^{-2mk} V(B_j)^{-1/2}
\]

\[
\leq C 2^{k(\frac{n}{2} - 2m)} \|b\|_{BMO}.
\]

and

\[
\int_{S_k(B_j)} \left| [b_{2^kB_j} - b_{B_j}] T a_j \right| d\mu(x) \leq C j \|b\|_{BMO} \int_{S_k(B_j)} \left| T a_j \right| d\mu(x)
\]

\[
\leq C k V(2^k B_j)^{1/2} \|b\|_{BMO} 2^{-2mk} V(B_j)^{-1/2}
\]

\[
\leq C k 2^{k(\frac{n}{2} - 2m)} \|b\|_{BMO}.
\]

The estimates (9), (10) together with \(m > \frac{n}{4}\) imply that

\[
E_1 \leq C \sum_{j \geq 0} |\lambda_j| \|b\|_{BMO} \approx \|f\|_{H^1(X)} \|b\|_{BMO}.
\]

It remains to extend \([b, T]\) to the whole space \(H^1(X)\). For each \(f \in H^1(X)\), there exists a sequence \((f_n)\) in \(H^1(X) \cap H^2(X)\) so that \(f_n \to f\) in \(H^1(X)\). We have, for all \(k,n \in \mathbb{N}\),

\[
\left\| [b, T] f_n - [b, T] f_k \right\|_{L^1,\infty(X)} \leq \left\| [b, T] (f_n - f_k) \right\|_{L^1,\infty(X)} \leq C \|b\|_{BMO} \|f_n - f_k\|_{H^1(X)}.
\]

Since \(L^1,\infty(X)\) is complete, we can define \([b, T] f = \lim_{n \to \infty} [b, T] f_n\) in \(L^1,\infty(X)\). It can be verified that

\[
\left\| [b, T] f \right\|_{L^1,\infty(X)} \leq C \|b\|_{BMO} \|f\|_{H^1(X)}.
\]

The proof of (ii) is complete.

\[\square\]

**Remark 3.2.** (a) In our approach, to obtain (9), we split

\[
T a = T (I - e^{-r_B^2L})^m a + T (I - e^{-r_B^2L})^n a.
\]

Observe that

\[
I - (I - e^{-r_B^2L})^m = \sum_{k=1}^{m} c_k e^{-kr_B^2L},
\]

where \(c_k = (-1)^{k+1} \frac{m!}{(m-k)!k!}\). Therefore,

\[
T [I - (I - e^{-r_B^2L})^m] a = \sum_{k=1}^{m} a_k T \left( r_B^2 L e^{-\frac{k}{m}r_B^2L} \right)^m (r_B^{-2m} b).
\]
where \( a = L^m b \) and \( a_k = c_k \left( \frac{\lambda}{m} \right)^m \).

Therefore, it is not difficult to see that condition (6) holds if the following two estimates are satisfied:

\[
\left( \int_{S_j(B)} |T(I - e^{-r^2 B L})^m f|^2 d\mu \right)^{1/2} \leq C 2^{-2jm} \left( \int_B |f|^2 d\mu \right)^{1/2}
\]

and

\[
\left( \int_{S_j(B)} |T(r^{2m} L^m e^{-kr^2 B L}) g|^2 d\mu \right)^{1/2} \leq C 2^{-2jm} \left( \int_B |g|^2 d\mu \right)^{1/2}
\]

for all integers \( j \geq 2 \), \( k = 1, \cdots, m \) and for all \( f \) and \( g \) with their supports contained in the ball \( B \).

(b) Theorem 1.2 still holds if the value \( 2^{-2jm} \) in (6), (11) and (12) is replaced by \( 2^{-2j\delta} \) for some \( \delta > \frac{n(2-p)}{4p} \).

(c) The estimates in (11) and (12) do not hold without the terms \( (I - e^{-r^2 B L})^m \) and \( r^{2m} L^m e^{-kr^2 B L} \) in the left hand sides, respectively. The effect of these terms is to make the kernels of \( T(I - e^{-r^2 B L})^m \) and \( T(r^{2m} L^m e^{-kr^2 B L}) \) decay faster than the kernel of \( T \) when \( x \) is away from \( y \).

4. Commutators of BMO functions and the Riesz transforms or square functions on doubling manifolds

Let \( X \) be a complete non-compact connected Riemannian manifold, \( \mu \) the Riemannian measure, \( \nabla \) the Riemannian gradient. Denote by \( |\cdot| \) the length in the tangent space, and by \( ||\cdot||_p \) the norm in \( L^p(X,\mu), 1 \leq p \leq \infty \). For simplicity we will write \( L^p(X) \) instead of \( L^p(X,\mu) \). Let \( \Delta \) be the Laplace-Beltrami operator. Denote by \( B(x,r) \) the open ball of radius \( r > 0 \) and center \( x \in X \), and by \( V(x,r) \) its measure \( \mu(B(x,r)) \). Throughout this section, assume that \( X \) satisfies the doubling property (1). It is well-known that the Laplace-Beltrami operator \( \Delta \) satisfies conditions (H1) and (H2). So let us denote the Hardy space associated to \( \Delta \) by \( H^1_\Delta(X) \).

Let us consider \( T = \nabla \Delta^{-1/2} \), the Riesz transform on \( X \), and take \( b \in \text{BMO}(X) \) (the space of functions of bounded mean oscillations on \( X \)). We define the commutator

\[
[b, T] g = bT g - T(bg),
\]

where \( g, b \) are scalar valued and \( [b, T] g \) is valued in the tangent space. In [5], it was proved that for any \( b \in \text{BMO}(X) \), under the doubling condition and Gaussian upper bound for the heat kernel, the commutator \([b, T] \) is bounded on \( L^p(X) \) with appropriate weights, for \( 1 < p < 2 \). The case of end-point value \( p = 1 \) was not considered in [5].
Our following theorem gives the endpoint estimate for the commutator $[b,T]$ when $p = 1$.

**Theorem 4.1.** Assume that $X$ satisfies the doubling property (1) and $b$ is a function in $\text{BMO}(X)$. Then, the Riesz transform $T = \nabla \Delta^{-1/2}$ is bounded from $H^p_\Delta(X)$ to $L^p(X)$, for all $0 < p \leq 1$. Moreover, if the Riesz transform $T = \nabla \Delta^{-1/2}$ is of weak type $(1,1)$ then the commutator $[b,T]$ maps $H^1_\Delta(X)$ continuously into $L^{1,\infty}(X)$.

**Proof.** By a similar argument to the proof of Lemma 2.2 in [29], it can be verified that for every $m \in \mathbb{N}$, all closed sets $E,F$ in $X$ with $d(E,F) > 0$ and every $f \in L^2(X)$ supported in $E$, one has

\begin{equation}
\|\nabla \Delta^{-1/2}(I - e^{-t\Delta})^m f\|_{L^2(F)} \leq C \left(\frac{t}{d(E,F)^2}\right)^m \|f\|_{L^2(E)}, \ \forall t > 0,
\end{equation}

and

\begin{equation}
\|\nabla \Delta^{-1/2}(t \Delta e^{-t\Delta})^m f\|_{L^2(F)} \leq C \left(\frac{t}{d(E,F)^2}\right)^m \|f\|_{L^2(E)}, \ \forall t > 0.
\end{equation}

Obviously, (13) and (14) imply (11) and (12), respectively. Hence our results follow from Theorem 1.2. This completes our proof. □

Note that in (ii) of Theorem 1.2 we need the Riesz transform $T = \nabla \Delta^{-1/2}$ to be of weak type $(1,1)$ and this condition on the Riesz transform can be obtained from the assumptions of doubling condition and the Gaussian bound $(H3)$. For reader’s convenience, we recall the following result in [13].

**Proposition 4.2 ([13]).** Assume that $X$ satisfies the doubling property (1) and the kernels $p_t(x,y)$ of $e^{-t\Delta}$ have Gaussian upper bounds $(H3)$. Then the Riesz transform $T = \nabla \Delta^{-1/2}$ is bounded on $L^p(X)$, $1 < p \leq 2$ and of weak type $(1,1)$.

From Theorems 4.1 and Proposition 4.2 we obtain the following result.

**Corollary 4.3.** Assume that $X$ satisfies the doubling property (1), the kernels $p_t(x,y)$ of $e^{-t\Delta}$ have Gaussian upper bounds $(H3)$ and $b \in \text{BMO}(X)$. Then the commutator $[b,T]$ maps $H^1_\Delta(X)$ continuously into $L^{1,\infty}(X)$.

In [31], it was shown that under condition $(H3)$, $H^1_\Delta(X) = H^1_\Delta(X)$ and $H^p_{\Delta,m}(X) = L^p(X)$ for all $p > 1$ and $m \geq 1$, see also [6]. It implies that if $T$ is a bounded operator on $L^p(X), p > 1$ and $T$ maps $H^1_\Delta(X)$ continuously into $L^{1,\infty}(X)$ then by interpolation (Theorem 9.7 in [31]), $T$ maps $H^p_{\Delta,m}(X)$ into $L^q(X)$ whenever $1 < q < p$, and hence $T$ extends to a bounded operator on $L^q(X)$ for all $1 < q < p$. 


Definition 4.4. We say that \( X \) satisfies an \( L^2 \) Poincaré inequality on balls if there exists \( C > 0 \) such that for any ball \( B \subset X \) and any function \( f \in C_\infty(2B) \),

\[
\int_B |f(x) - f_B|^2 d\mu \leq C r_B^2 \int_{2B} |\nabla f(x)|^2 d\mu,
\]

where \( f_B \) denotes the mean-value of \( f \) on the ball \( B \) and \( r_B \) the radius of \( B \).

Under the conditions of doubling property and \( L^2 \) Poincaré inequality, it is known that the Hardy space \( H^1_\Delta(X) \) coincides with the standard Hardy space. Indeed, we have the following result.

Proposition 4.5 ([6]). Assume that \( X \) satisfies the doubling property (1) and Poincaré inequality (15) then the Hardy space \( H^1_\Delta(X) \) and the Coifman-Weiss Hardy space \( H^1_{CW}(X) \) coincide.

This Proposition and Corollary 4.3 give the following Corollary.

Corollary 4.6. Assume that \( X \) satisfies the doubling property (1) and Poincaré inequality (15) and let \( b \in \text{BMO}(X) \). Then the commutator \([b,T]\) maps \( H^1_{CW}(X) \) continuously into \( L^{1,\infty}(X) \).

We now show that similar results hold when we replace the Riesz transforms of the Laplace-Beltrami operator by square functions of the Laplace-Beltrami operator. Consider the following four versions of the square functions

\[
G f(x) := \left( \int_0^\infty t |\nabla e^{-t\Delta} f(x)|^2 dt \right)^{1/2}, \quad H f(x) := \left( \int_0^\infty |\nabla e^{-t\Delta} f(x)|^2 dt \right)^{1/2},
\]

\[
g f(x) := \left( \int_0^\infty t |\Delta e^{-t\Delta} f(x)|^2 dt \right)^{1/2}, \quad h f(x) := \left( \int_0^\infty t |\Delta e^{-t\Delta} f(x)|^2 dt \right)^{1/2}.
\]

By similar arguments used in Lemma 2.2 of [29], it can be verified that \( G, H, g \) and \( h \) satisfy (13) and (14) and hence they satisfy (11) and (12). For reader’s convenience, we sketch the proof for \( H \) only. The remainders are treated similarly.

The first ingredient is that the Davies-Gaffney estimate (4) is valid in a general complete, connected Riemannian manifold (see for example [3]): There exist two constants \( C \geq 0 \) and \( c > 0 \) such that, for every \( t \geq 0 \), every closed subsets \( E \) and \( F \) of \( X \), and every function \( f \) supported in \( E \), one has

\[
||e^{-t\Delta} f||_{L^2(F)} + ||t\Delta e^{-t\Delta} f||_{L^2(F)} + ||\sqrt{t} |\nabla e^{-t\Delta} f||_{L^2(F)} \leq C \exp\left\{ - \frac{d^2(E,F)}{t} \right\} ||f||_{L^2(E)}.
\]

Secondly, we recall the following result in [29].

Lemma 4.7. Assume that the two families of operators \( \{S_t\}_{t>0} \) and \( \{T_t\}_{t>0} \) satisfy the Davies-Gaffney estimate (4). Then there exist two constants \( C \geq 0 \)
and \( c > 0 \) such that, for every \( t > 0 \), every closed subsets \( E \) and \( F \) of \( X \), and every function \( f \) supported in \( E \), one has
\[
\|S_{t}f\|_{L^2(F)} \leq C \exp \left\{ -\frac{d(E, F)^2}{c \max\{s, t\}} \right\} \|f\|_{L^2(E)}.
\]

We now show that \( \mathcal{H} \) satisfies (13) and (14). Let us prove condition (13) first. We have
\[
\|\mathcal{H}(I - e^{-t\Delta})^m f\|_{L^2(F)} := \left\| \left( \int_0^\infty |\nabla e^{-s\Delta} (I - e^{-t\Delta})^m f|^2 \frac{ds}{s} \right)^{1/2} \right\|_{L^2(F)}
\]
\[
\leq C \left( \int_0^\infty \left\| \nabla e^{-s\Delta} (I - e^{-t\Delta})^m f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2}
\]
\[
\leq C \left( \int_0^t \left\| \nabla e^{-s\Delta} (I - e^{-t\Delta})^m f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2}
\]
\[
+ C \left( \int_t^\infty \left\| \nabla e^{-s\Delta} (I - e^{-t\Delta})^m f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2} = I_1 + I_2.
\]

To estimate the term \( I_1 \), we note that
\[
(I - e^{-t\Delta})^m = I + \sum_{k=1}^m c_k e^{-tk\Delta}.
\]

Hence,
\[
I_1 \leq C \left( \int_0^t \left\| \nabla e^{-s\Delta} (I - e^{-t\Delta})^m f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2}
\]
\[
+ C \sup_{1 \leq k \leq m} \left( \int_0^t \left\| \nabla e^{-s\Delta} e^{-tk\Delta} f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2}
\]
\[
\leq C \left( \int_0^t \left\| \nabla e^{-s\Delta} f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2}
\]
\[
+ C \sup_{1 \leq k \leq m} \left( \int_0^t \left\| \nabla e^{-s\Delta} e^{-tk\Delta f} \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2}.
\]

Using Lemma 4.7 and (16), one has
\[
I_1 \leq C \left( \int_0^t \exp \left\{ -\frac{d(E, F)^2}{c t} \right\} \frac{ds}{s} \right)^{1/2} \|f\|_{L^2(E)}
\]
\[
+ C \left( \exp \left\{ -\frac{d(E, F)^2}{c t} \right\} \int_0^t \frac{ds}{t} \right)^{1/2} \|f\|_{L^2(E)}.
\]

It is not difficult to see that the expression above is bounded by \( C \left( \frac{d(E, F)}{d(E, F)} \right)^m \|f\|_{L^2(E)} \) as desired.

We now estimate the second term \( I_2 \). We have
\[
(17) \quad I_2 \leq C \left( \int_t^\infty \left\| \nabla e^{-s\Delta} (e^{-s\Delta} - e^{-(s+t)\Delta})^m f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{1/2}.
\]
It was observed in [29] that
\[ \left\| \frac{s}{t}(e^{-s\Delta} - e^{-(s+t)\Delta})g \right\|_{L^2(F)}^2 \leq C \exp \left\{ -\frac{d(E, F)^2}{cs} \right\} \|g\|_{L^2(E)}. \]

Multiplying and dividing (17) by \((\frac{s}{t})^{2m}\) and using Lemma 4.7 for \(\sqrt{s}\nabla e^{-s\Delta}\) and \(m\) copies of \(\frac{s}{t}(e^{-s\Delta} - e^{-(s+t)\Delta})\), we get that
\[ I_2 \leq C \left( \int t^\infty \exp \left\{ -\frac{d(E, F)^2}{cs} \right\} \left( \frac{t}{s} \right)^{2m} ds \right)^{1/2} \|f\|_{L^2(E)}. \]

Next, making a change of variables \(r := \frac{d(E, F)^2}{cs}\), we can control the RHS of the above expression by \(C \left( \frac{d(E, F)^2}{cs} \right)^m \|f\|_{L^2(E)}\) as desired.

This finishes the proof of (13) for \(H\). The proof for (14) can be done essentially the same way, hence it is omitted here. This completes our proof.

Let us recall that under the Gaussian condition \((H3)\), \(G, H, g\) and \(h\) are of weak type \((1, 1)\), see [14]. Hence the following result follows from Theorem 1.2.

**Theorem 4.8.** (i) Assume that \(X\) satisfies the doubling property \((1)\) and \(b\) is a function in \(BMO(X)\). Then \(G, H, g\) and \(h\) are bounded from \(H^p_\Delta(X)\) to \(L^p(X)\) for any \(0 < p \leq 1\).

(ii) If the Gaussian condition \((H3)\) is satisfied, then the commutators of a \(BMO\) function \(b\) and each of \(G, H, g\) and \(h\) are bounded from \(H^1_\Delta(X)\) to \(L^{1, \infty}(X)\).

5. **Commutators of BMO functions and Riesz transforms associated with magnetic Schrödinger operators**

The approach in Section 4.1 can be used to obtain the boundedness of Riesz transforms of Schrödinger operators and their commutators but is not applicable in the case of magnetic Schrödinger operators. Indeed, a different approach is needed for magnetic Schrödinger operators.

Consider magnetic Schrödinger operators in general setting as in [24]. Let the real vector potential \(\vec{a} = (a_1, \cdots, a_n)\) satisfy
\[ a_k \in L^2_{\text{loc}}(\mathbb{R}^n), \quad \forall k = 1, \cdots, n, \]
and an electric potential \(V\) with
\[ 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n). \]

Let \(L_k = \partial/\partial x_k - ia_k\). We define the form \(Q\) by
\[ Q(f, g) = \sum_{k=1}^n \int_{\mathbb{R}^n} L_k f \overline{L_k g} dx + \int_{\mathbb{R}^n} V f \overline{g} dx. \]
with domain $\mathcal{D}(Q) = \mathcal{Q} \times \mathcal{Q}$ where 
$$\mathcal{Q} = \{ f \in L^2(\mathbb{R}^n), L_k f \in L^2(\mathbb{R}^n) \text{ for } k = 1, \cdots, n \text{ and } \sqrt{V} f \in L^2(\mathbb{R}^n) \}.$$ 

It is well known that this symmetric form is closed and this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$ (the space of $C^\infty$ functions with compact supports). See, for example [39].

Let us denote by $A$ the self-adjoint operator associated with $Q$. The domain of $A$ is given by 
$$\mathcal{D}(A) = \left\{ f \in \mathcal{D}(Q), \exists g \in L^2(\mathbb{R}^n) \text{ such that } Q(f, \varphi) = \int_{\mathbb{R}^n} g \varphi dx, \forall \varphi \in \mathcal{D}(Q) \right\},$$

and $A$ is given by the expression 
$$Af = \sum_{k=1}^{n} L_k^2 L_k f + V f. \tag{20}$$

Formally, we write $A = -\left(\nabla - i\vec{a}\right) \cdot \left(\nabla - i\vec{a}\right) + V$. For $k = 1, \cdots, n$, the operators $L_k A^{-1/2}$ are called the Riesz transforms associated with $A$. It is easy to check that 
$$\|L_k f\|_{L^2(\mathbb{R}^n)} \leq \|A^{1/2} f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{D}(Q) = \mathcal{D}(A^{1/2}) \tag{21}$$

for any $k = 1, \cdots, n$, and hence the operators $L_k A^{-1/2}$ are bounded on $L^2(\mathbb{R}^n)$. Note that this is also true for $V^{1/2} A^{-1/2}$. Moreover, it was recently proved in Theorem 1.1 of [24] that for each $k = 1, \cdots, n$, the Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$, i.e., there exists a constant $C_p > 0$ such that
$$\|V^{1/2} A^{-1/2} f\|_{L^p(\mathbb{R}^n)} + \sum_{k=1}^{n} \left\|L_k A^{-1/2} f\right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \tag{22}$$

for $1 < p \leq 2$.

The $L^p$-boundedness of Riesz transforms for the range $p > 2$ can be obtained if one imposes certain additional regularity conditions on the potential $V$, see for example [2].

**Remark 5.1.** (i) In [24], the boundedness of the Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ was proved for $L^p(\mathbb{R}^n)$ spaces with $1 < p < 2$;

(ii) In [25], $L^p$ boundedness of commutators of a BMO function and the Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ was proved for the range $1 < p < 2$;

(iii) Recently, [8] extended the results in [24] and [25] to weighted weak type $L^{1,\infty}$ estimates and weighted $L^p$ estimates with an appropriate range of $p$ (depending on the weight).

It is a natural open question to consider the endpoint estimates for the commutators of a BMO function and the operators $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$, and the boundedness of the Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ on the range $0 < p \leq 1$. Our aim in this section is to establish the end point estimates
for the commutators of the Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ and a BMO function $b$ when $p = 1$ and the estimates for Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ for $0 < p \leq 1$. Our main result of this section is the following theorem.

**Theorem 5.2.** (i) The Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are bounded from $H^p_A(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $0 < p \leq 1$.

(ii) Let $b \in \text{BMO}(\mathbb{R}^n)$. Then the commutators $[b, V^{1/2} A^{-1/2}]$ and $[b, L_k A^{-1/2}]$ map $H^1_A(\mathbb{R}^n)$ continuously into $L^{1, \infty}(\mathbb{R}^n)$.

Recently, we had learnt that in [33] the authors also obtained the results in (i) of Theorem 5.2 by using the different approach.

### 5.1. Some kernel estimates on heat semigroups.

Let $A = -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$ be the magnetic Schrödinger operator in (24). By the well known diamagnetic inequality (see, Theorem 2.3 of [39] and [15] for instance) we have the pointwise inequality

$$|e^{-tA} f(x)| \leq e^{t\Delta} (|f|)(x) \quad \forall t \geq 0, \quad f \in L^2(\mathbb{R}^n).$$

This inequality implies in particular that the semigroup $e^{-tA}$ maps $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and that the kernel $p_t(x, y)$ of $e^{-tA}$ satisfies

$$|p_t(x, y)| \leq (4\pi t)^{-\frac{n}{2}} \exp \left( -\frac{|x - y|^2}{4t} \right)$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^n$.

Note that $A$ satisfies conditions (H1) and (H2). So, for $0 < p \leq 1$, we denote by $H^p_A(\mathbb{R}^n)$ the Hardy space associated to the operator $A$. Note that Gaussian upper bounds carry over from heat kernels to their time derivatives of its kernels. That is, for each $k \in \mathbb{N}$, there exist two positive constants $c_k$ and $C_k$ such that the time derivatives of $p_t$ satisfy

$$|\partial_t^k p_t(x, y)| \leq C_k t^{-(n+2k)/2} \exp \left( -\frac{|x - y|^2}{c_k t} \right)$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^n$. For the proof of (24), see, for example, [12], [18] and [36, Theorem 6.17].

We let $\tilde{p}_t^k(x, y) = t^k (d^k / dt^k) p_t(x, y)$. In the sequel, we always use the notation $L_k \tilde{p}_t^k(x, y)$ to mean $L_k \tilde{p}_t^k(\cdot, y)(x)$.

### 5.2. The proof of boundedness of commutators.

To prove the main result of this section, we need the following lemma which gives a weighted estimate for $L_k \tilde{p}_t^k(x, y)$.

**Lemma 5.3.** Let $A = -(\nabla - i\vec{a})(\nabla - i\vec{a}) + V$ be the magnetic Schrödinger operator in (24). For each $k$ and $\gamma > 0$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} |V^{1/2}(x)| \tilde{p}_t^k(x, y)|^2 e^{\gamma |x - y|^2} \, dx + \sum_{k=1}^n \int_{\mathbb{R}^n} |L_k \tilde{p}_t^k(x, y)|^2 e^{\gamma |x - y|^2} \, dx \leq \frac{C}{t^{n/2}}$$
for all $t$ and $y \in \mathbb{R}^n$.

**Proof.** In [24], the authors proved this for $k = 1$ and in the case when $k = 0$, the proof can be found in [8]. We now adapt these estimates to prove (25). Let $\psi$ be a $C^\infty$ function with compact support on $\mathbb{R}^n$ such that $0 \leq \psi \leq 1$. Consider

$$I_t(\psi) = \sum_{k=1}^{n} \int_{\mathbb{R}^n} |L_k \tilde{p}_t^k(x, y)|^2 e^{\gamma |x-y|^2} \psi(x) dx.$$ 

Using Lemma 2.5 in [39], we have

$$I_t(\psi) = \sum_{k=1}^{n} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \left( e^{-i\lambda_k \tilde{p}_t^k(x, y)} \frac{\partial}{\partial x_k} \left( e^{-i\lambda_k \tilde{p}_t^k(x, y)} e^{\gamma |x-y|^2} \psi(x) dx \right) \right.$$ 

$$= \sum_{k=1}^{n} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \left( e^{-i\lambda_k \tilde{p}_t^k(x, y)} \frac{\partial}{\partial x_k} \left( e^{-i\lambda_k \tilde{p}_t^k(x, y)} e^{\gamma |x-y|^2} \psi(x) dx \right) \right.$$ 

$$- \sum_{k=1}^{n} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \left( e^{-i\lambda_k \tilde{p}_t^k(x, y)} \frac{\partial}{\partial x_k} \left( e^{-i\lambda_k \tilde{p}_t^k(x, y)} e^{\gamma |x-y|^2} \psi(x) dx \right) \right.$$ 

(26) $$= II_1 - II_2,$$

where $\lambda_1, \ldots, \lambda_n$ are functions in $L^2_{\text{loc}}$ satisfying

$$L_k = e^{i\lambda_k} \frac{\partial}{\partial x_k} e^{-i\lambda_k}, \quad k = 1, \ldots, n.$$ 

From the fact that $\psi$ has compact support, we have

$$\tilde{p}_t^k(\cdot, y)e^{\gamma |x-y|^2} \psi(\cdot) \in \mathcal{D}(Q) \subset \mathcal{D}(L_k).$$

We can then write the first term $II_1$ as

$$II_1 = \sum_{k=1}^{n} \int_{\mathbb{R}^n} L_k \tilde{p}_t^k(x, y) L_k(\tilde{p}_t^k(\cdot, y)e^{\gamma |x-y|^2} \psi(x) dx.$$ 

Since $0 \leq V$ and $0 \leq \psi$, we obtain

$$II_1 \leq Q(\tilde{p}_t^k(\cdot, y), \tilde{p}_t^k(\cdot, y)e^{\gamma |x-y|^2} \psi)$$

$$= \int_{\mathbb{R}^n} \mathcal{A}_{\tilde{p}_t^k}(x, y) \tilde{p}_t^k(x, y) e^{\gamma |x-y|^2} \psi(x) dx.$$ 

On the other hand, we have $\mathcal{A}_{\tilde{p}_t^k}(x, y) = t^k \frac{d^{k+1}}{dt^{k+1}} p_t(x, y)$. We then apply [24] to obtain

$$II_1 \leq \int_{\mathbb{R}^n} \frac{1}{t^{n/2+1}} e^{-c_2 |x-y|^2} \frac{1}{t^{n/2}} e^{-c_1 |x-y|^2} e^{\gamma |x-y|^2} \psi(x) dx.$$ 

Hence for any $\gamma < c_1$ there exists a constant $c > 0$ independent of $\psi$ such that

(27) $$II_1 \leq \frac{c}{t^{n/2+1}}.$$
since $0 \leq \psi \leq 1$.

Next, we rewrite the term $II_2$ as follows:

$$II_2 = \sum_{k=1}^{n} \int_{\mathbb{R}^n} e^{i\lambda_k} \frac{\partial}{\partial x_k} \left( e^{-i\lambda_k \tilde{p}_t^k(x,y)} \tilde{p}_t(x,y) e^{\gamma \frac{|x-y|^2}{t}} \right) dx.$$

This gives

$$II_2 = \sum_{k=1}^{n} \frac{c}{\sqrt{t}} \int_{\mathbb{R}^n} |L_k \tilde{p}_t^k(x,y)||p_t(x,y)| e^{2\gamma \frac{|x-y|^2}{t}} \psi(x) dx$$

$$+ \sum_{k=1}^{n} \int_{\mathbb{R}^n} |L_k \tilde{p}_t^k(x,y)||\tilde{p}_t^k(x,y)| e^{\gamma \frac{|x-y|^2}{t}} \left| \frac{\partial}{\partial x_k} \psi(x) \right| dx$$

$$= J_1(\psi) + J_2(\psi).$$

Then by (24) and Cauchy-Schwarz inequality,

$$J_1(\psi) \leq \frac{c}{\sqrt{t}} \sum_{k=1}^{n} \left( \int_{\mathbb{R}^n} t^{-n} e^{\beta \frac{|x-y|^2}{t}} dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |L_k \tilde{p}_t(x,y)|^2 e^{\gamma \frac{|x-y|^2}{t}} \psi(x) dx \right)^{\frac{1}{2}}$$

$$\leq \frac{c}{\sqrt{t^{n/2+1}}} I_t(\psi),$$

provided $\beta < \frac{2c}{3}$.

Using this estimate and (27), we have

$$I_t(\psi) \leq c \left( \frac{1}{t^{n/2+1}} + J_2(\psi) \right),$$

where $c$ is a constant independent of $\psi$.

Now apply (28) with $\psi_j(x) = \psi(x/j)$, where $\psi$ is a function such that $\psi(x) = 1$ for all $x$ with $|x| \leq 1$. It is not difficult to show that $\lim_{j \to \infty} J_2(\psi_j) = 0$. Then apply Fatou’s lemma to (28) with $\psi_j$ we have

$$\int_{\mathbb{R}^n} |L_k \tilde{p}_t^k(x,y)|^2 e^{\gamma \frac{|x-y|^2}{t}} dx \leq \frac{c}{t^{n/2+1}}.$$

For the estimate of $\int_{\mathbb{R}^n} |V^{1/2}(x) \tilde{p}_t^k(x,y)|^2 e^{\gamma \frac{|x-y|^2}{t}} dx$, we note that

$$\int_{\mathbb{R}^n} |V^{1/2}(x) \tilde{p}_t^k(x,y)|^2 e^{\gamma \frac{|x-y|^2}{t}} \psi(x) dx = Q(\tilde{p}_t^k(\cdot,y), \tilde{p}_t^k(\cdot,y)) e^{\gamma \frac{|x-y|^2}{t}} \psi(x) - II_1.$$

From the estimates of both terms, we obtain that

$$\int_{\mathbb{R}^n} |V^{1/2}(x) \tilde{p}_t^k(x,y)|^2 e^{\gamma \frac{|x-y|^2}{t}} \psi(x) dx \leq \frac{c}{t^{n/2+1}}.$$

At this stage, repeating the above argument, one has

$$\int_{\mathbb{R}^n} |V^{1/2}(x) \tilde{p}_t^k(x,y)|^2 e^{\gamma \frac{|x-y|^2}{t}} dx \leq \frac{c}{t^{n/2+1}}.$$
This finishes our proof. □

The following proposition gives an estimate for the operators $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ which will be useful for the proof of Theorem 5.2.

**Proposition 5.4.** For all $m \geq 1$ there exists $C > 0$ such that for all $j \geq 2$, all balls $B$ and all $f \in L^1(\mathbb{R}^n)$ with support in $B$

\begin{equation}
(\int_{S_j(B)} |V^{1/2} A^{-1/2} (I - e^{-r_B^2 A})^m f(x)|^2 \, dx)^{\frac{1}{2}} \leq C \, 2^{-2j(m-1)} \left( \int_B |f|^2 \, dx \right)^{1/2}
\end{equation}

and for all $k = 1, \ldots, n,$

\begin{equation}
(\int_{S_j(B)} |L_k A^{-1/2} (I - e^{-r_B^2 A})^m f(x)|^2 \, dx)^{\frac{1}{2}} \leq C \, 2^{-2j(m-1)} \left( \int_B |f|^2 \, dx \right)^{1/2}.
\end{equation}

**Proof.** We will prove only (30). The inequality (29) can be treated by a similar argument.

We adapt an argument used in [3] (see also [8]) to our situation. Fix a ball $B$ with radius $r_B$ and $f \in L^1(\mathbb{R}^n)$ supported in $B$. Observe that

$$A^{-1/2} \equiv \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t A} \frac{dt}{\sqrt{t}}.$$

We have

$$L_k A^{-1/2} (I - e^{-r_B^2 A})^m f = c \int_0^\infty L_k e^{-t A} (I - e^{-r_B^2 A})^m \frac{dt}{\sqrt{t}} = c \int_0^\infty g_r (t) L_k e^{-t A} dt,$$

where $g_r : \mathbb{R}^+ \to \mathbb{R}$ is a function such that

$$\int_0^\infty |g_r (t)| e^{-c_4 r_B^2 t} \frac{dt}{\sqrt{t}} \leq C m^{4-jm}.$$

See [ACDH, p.932]. Hence the composite operator $L_k A^{-1/2} (I - e^{-r_B^2 A})^m$ has an associated kernel $\mathcal{K}_{s,k}(y,z)$ defined by

$$\mathcal{K}_{s,k}(y,z) = c \int_0^\infty g_r (t) L_k p_s(y,z) dt.$$

By invoking Lemma 5.3,

$$\left( \int_{\mathbb{R}^n} |L_k p_t(x,y)|^2 e^{\gamma \frac{|x-y|^2}{t}} \, dx \right)^{\frac{1}{2}} \leq \frac{C}{t^{n+2}}.$$
for all $t > 0$ and $y \in \mathbb{R}^n$ and some $\gamma > 0$. This implies that for all $j > 0$, $y \in B$ and all $t > 0$,

$$
\left( \int_{S_j(B)} |L_k p_s(x, y)|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{t}} e^{-c \frac{r_B^2}{t}} \frac{|2^j B|^{1/2}}{|t^{\frac{1}{2}}|^{1/2}} \frac{1}{|2^j B|^{1/2}}
$$

for some $\alpha < c$. Note that in the last inequality we use the fact that $s^b e^{-cs} < Ce^{-\alpha s}$ for all $s > 0$ and $\alpha < c$.

Using Minkowski’s inequality we obtain that the LHS of (30) is dominated by

$$
\int_0^\infty |g_r(t)| \int |f(y)| \left( \int_{S_j(B)} |L_k p_s(x, y)|^2 \, dx \right)^{1/2} \, dy \, dt
$$

$$
\leq C \frac{1}{|B|^{1/2}} \int_0^\infty |g_r(t)| e^{-c \frac{r_B^2}{t}} \frac{dt}{\sqrt{t}} \left( \int_B |f(y)| \, dy \right)
$$

$$
\leq C 4^{-m} \frac{1}{|B|^{1/2}} \int_B |f(y)| \, dy
$$

$$
\leq C 4^{-m} \left( \int_B |f(x)|^2 \, dx \right)^{1/2}.
$$

The proof is complete. \(\square\)

**Proposition 5.5.** For all $m \geq 1$ and $t \geq 0$ there exists $C > 0$ such that for all $j \geq 2$, all balls $B$, $\frac{1}{m} r_B^2 \leq t \leq r_B^2$ and all $f$ with support in $B$

$$
\left( \int_{S_j(B)} |V^{1/2} A^{-1/2} (tAe^{-tA})^m f(x)|^2 \, dx \right)^{\frac{1}{2}} \leq C 2^{-2jm} \left( \int_B |f(x)|^2 \, dx \right)^{1/2}
$$

and for all $k = 1, \ldots, n$,

$$
\left( \int_{S_j(B)} |L_k A^{-1/2} (tAe^{-tA})^m f(x)|^2 \, dx \right)^{\frac{1}{2}} \leq C 2^{-2jm} \left( \int_B |f(x)|^2 \, dx \right)^{1/2}.
$$

**Proof.** Firstly, observe that

$$
L_k A^{-1/2} f = \frac{1}{2 \sqrt{\pi}} \int_0^\infty L_k e^{-sA} f \frac{ds}{\sqrt{s}} = \frac{m}{2 \sqrt{\pi}} \int_0^\infty L_k e^{-msA} f \frac{ds}{\sqrt{s}}.
$$
So, we obtain
\[
\left( \int_{S_j(B)} |L_k A^{-1/2}(tA e^{-tA})^m f(x)|^2 \, dx \right)^{\frac{1}{2}}
= C \left( \int_{S_j(B)} \left| \int_0^\infty L_k e^{-msA}(tA e^{-tA})^m f(y) \frac{ds}{\sqrt{s}} \right|^2 \, dx \right)^{\frac{1}{2}}
= C \left( \int_{S_j(B)} \int_0^\infty \int_B \left( \frac{t}{t + s} \right)^m L_k \tilde{P}^m_{m(t+s)}(x,y) f(y) \frac{ds}{\sqrt{s}} \, dx \right)^{\frac{1}{2}}
\]
By Minkowski’s inequality and Lemma \ref{lem:5.3},
\[
\left( \int_{S_j(B)} |L_k A^{-1/2}(tA e^{-tA})^m f(x)|^2 \, dx \right)^{\frac{1}{2}}
\leq C \int_0^\infty \left( \frac{t}{t + s} \right)^m \int_B f(y) \left( \int_{S_j(B)} \left| L_k \tilde{P}^m_{m(t+s)}(x,y) \right|^2 \, dx \right)^{\frac{1}{2}} \frac{ds}{\sqrt{s}}
\leq C \int_0^\infty \left( \frac{t}{t + s} \right)^m \exp \left\{ - c \frac{d^2(S_j(B), B)}{t + s} \right\} \frac{ds}{\sqrt{s(t + s)^{\frac{n+2}{4}}} \int_B f(y)dy}
\leq C \int_0^\infty \left( \frac{t}{t + s} \right)^m \exp \left\{ - c \frac{4^j r_B^2}{t + s} \right\} \frac{ds}{\sqrt{s(t + s)^{\frac{n+2}{4}}} r_B^{n/2} \|f\|_{L^2(B)}}
\leq C \|f\|_{L^2(B)} \left( \int_t^l \ldots + \int_l^\infty \ldots \right) = C \|f\|_{L^2(B)} (I + II).
\]
Let us estimate I first. We have
\[
\int_0^l \left( \frac{t}{t + s} \right)^m \exp \left\{ - c \frac{4^j r_B^2}{t + s} \right\} \left( \frac{r_B^2}{t + s} \right)^{\frac{n+2}{4}} \frac{ds}{r_B \sqrt{s}}
\leq C \int_0^l \left( \frac{t}{t + s} \right)^m \left( \frac{t + s}{4^j r_B^2} \right)^m \left( \frac{r_B^2}{t} \right)^{\frac{n+2}{4}} \frac{ds}{r_B \sqrt{s}}
\leq C \int_0^l \left( \frac{t}{4^j r_B^2} \right)^m \frac{ds}{r_B \sqrt{s}}
\leq C 4^{-jm}.
\]
We now estimate the second term II. We have
\[
\int_l^\infty \left( \frac{t}{t + s} \right)^m \exp \left\{ - c \frac{4^j r_B^2}{t + s} \right\} \left( \frac{r_B^2}{t + s} \right)^{\frac{n+2}{4}} \frac{ds}{r_B \sqrt{s}}
\leq C \int_l^\infty \left( \frac{t}{t + s} \right)^m \left( \frac{t + s}{4^j r_B^2} \right)^{m-1} \left( \frac{r_B^2}{t} \right)^{\frac{n+2}{4}} \frac{ds}{r_B \sqrt{s}}
\leq C 4^{-j(m-1)} \int_l^\infty \left( \frac{t}{r_B^2} \right)^m r_B ds \frac{ds}{s \sqrt{s}}
\leq C 4^{-j(m-1)}.
\]
This completes our proof. □
We are now ready to give the proof of Theorem 5.2.

**Proof of Theorem 5.2.** Firstly, it can be proved that the Riesz transforms $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are of weak type $(1, 1)$, see [DOY, p. 273]. So, combining Propositions 5.4 and 5.5 together with Theorem 1.2 and Remark 3.2 (b), Theorem 5.2 is proved.

\[\square\]

6. Holomorphic functional calculi and spectral multipliers

6.1. Preliminaries on holomorphic functional calculus. We now give some preliminary definitions of holomorphic functional calculi as introduced by A. McIntosh [35].

Let $0 \leq \omega < \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$

$$S_\omega = \{ z \in \mathbb{C} : |\arg z| \leq \omega \}$$

and denote the interior of $S_\omega$ by $S_0^\omega$. We employ the following subspaces of the space $H(S_0^\nu)$ of all holomorphic functions on $S_0^\nu$:

$$H_\infty(S_0^\nu) = \{ g \in H(S_0^\nu) : ||g||_\infty < \infty \},$$

where $||g||_\infty = \sup \{|g(z)| : z \in S_0^\nu\}$, and

$$\Psi(S_0^\nu) = \{ \psi \in H(S_0^\nu) : \exists s > 0, |\psi(z)| \leq c|z|^s(1 + |z|^{2s+1})^{-1} \}.$$

Let $0 \leq \omega < \pi$. A closed operator $L$ in $L^2(X)$ is said to be of type $\omega$ if $\sigma(L) \subset S_\omega$, and for each $\nu > \omega$ there exists a constant $c_\nu$ such that

$$||(L - \lambda I)^{-1}|| \leq c_\nu|\lambda|^{-1}, \lambda \notin S_\nu.$$

If $L$ is of type $\omega$ and $\psi \in \Psi(S_0^\nu)$, we define $\psi(L) \in \mathcal{L}(L^2, L^2)$ by

$$\psi(L) = \frac{1}{2\pi i} \int_{\Gamma} (L - \lambda I)^{-1}\psi(\lambda)d\lambda,$$

where $\Gamma$ is the contour $\{ \xi = re^{\pm i\xi} : r > 0 \}$ parametrized clockwise around $S_\omega$, and $\omega < \xi < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(L^2, L^2)$, and it is straightforward to show, using Cauchy’s theorem, that the definition is independent of the choice of $\xi \in (\omega, \nu)$. If, in addition, $L$ is one-one and has dense range and if $\phi \in H_\infty(S_0^\nu)$, then $\phi(L)$ can be defined by

$$\phi(L) = [\psi(L)]^{-1}(\phi\psi)(L),$$

where $\psi(z) = z(1 + z)^{-2}$.

It can be shown that $\phi(L)$ is a well-defined linear operator in $L^2$. We say that $L$ has a bounded $H_\infty$ calculus in $L^2$ if there exists $c_{\nu,2} > 0$ such that $\phi(L) \in \mathcal{L}(L^2, L^2)$, and for $\phi \in H_\infty(S_0^\nu)$,

$$||\phi(L)|| \leq c_{\nu,2}||\phi||_\infty.$$
In [35], it was proved that $L$ has a bounded $H_\infty$-calculus in $L^2$ if and only if for any non-zero function $\psi \in \Psi(S^0_\nu)$, $L$ satisfies the square function estimate and its reverse
\begin{equation}
(33) \quad c_1 \|f\|_2 \leq \left( \int_0^\infty \|\psi_t(L)f\|_2^2 \, dt \right)^{1/2} \leq c_2 \|f\|_2
\end{equation}
for some $0 < c_1 \leq c_2 < \infty$, where $\psi_t(x) = \psi(tx)$. Note that different choices of $\nu > \omega$ and $\psi \in \Psi(S^0_\nu)$ lead to equivalent quadratic norms of $f$. As noted in [35], positive self-adjoint operators satisfy the quadratic estimate (33). So do normal operators with spectra in a sector, and maximal accretive operators. For definitions of these classes of operators, we refer the reader to [34]. For detailed studies of operators which have bounded holomorphic functional calculi, see for example [11, 35, 28, 21].

6.2. Application to holomorphic functional calculus. We first show that the holomorphic functional calculi $g(L)$ satisfy (11) and (12).

**Proposition 6.1.** Assume that $L$ satisfies (H1) and (H2). Let $0 < \nu < \pi$. Then for any $g \in H_\infty(S^0_\nu)$, $m \in \mathbb{N}$, all closed sets $E, F$ in $X$ with $d(E,F) > 0$ and any $f \in L^2(X)$ supported in $E$, one has
\begin{equation}
(34) \quad \|g(L)(I - e^{-tL})^m f\|_{L^2(F)} \leq C \max \left\{ \left( \frac{t}{d(E,F)^2} \right)^{m-1}, \left( \frac{t}{d(E,F)^2} \right)^{m+1} \right\} \|f\|_{L^2(E)} \|g\|_\infty, \forall t > 0,
\end{equation}
and
\begin{equation}
(35) \quad \|g(L)(tLe^{-tL})^m f\|_{L^2(F)} \leq C \max \left\{ \left( \frac{t}{d(E,F)^2} \right)^{m-1}, \left( \frac{t}{d(E,F)^2} \right)^{m+1} \right\} \|f\|_{L^2(E)} \|g\|_\infty, \forall t > 0.
\end{equation}

Before giving the proof of Proposition 6.1, we state the following lemma.

**Lemma 6.2.** Assume that $L$ satisfies conditions (H1) and (H2). Then for any $z = re^{i\theta}$ with $\theta \in (-\pi/2, \pi/2)$, all closed sets $E, F$ in $X$ with $d(E,F) > 0$ and any $f \in L^2(X)$ supported in $E$, one has
\[ \left\| e^{-zL} f \right\|_{L^2(F)} \leq C \exp \left\{ - \frac{d(E,F)^2}{|z|} \cos \theta \right\} \|f\|_{L^2(E)}. \]

The proof is similar to the proof of Proposition 3.1 of [31] and hence we omit details here.

**Proof of Proposition 6.1:** We prove the estimate (34) first. We define
\[ g_{s,t}(z) = z^s (1 + z)^{-2s} g(z) (1 - e^{-tz})^m. \]
Then $g_{s,t} \in \Psi(S^0_\nu)$. Moreover, $\lim_{s \to 0} g_{s,t}(z) = g(z)(1 - e^{-tz})^m$ uniformly in any compact set in $S^0_\nu$. Therefore, by the Convergence Theorem in [35],
\begin{equation}
(36) \quad \lim_{s \to 0} g_{s,t}(L)f = g(L)(1 - e^{-tL})^m f
\end{equation}
in $L^2(X)$ norm for every $f \in L^2(X)$.

Choose $0 < \theta < \mu < \nu$ and $\mu < \frac{\pi}{2}$. On one hand, we have

$$g_{s,t}(L) = \frac{1}{2\pi i} \int_{\gamma} (L - \lambda I)^{-1} \frac{\lambda^s}{(1 + \lambda)^2s} g(\lambda) (1 - e^{-t\lambda})^m d\lambda,$$

here $\gamma = \gamma_+ + \gamma_-$, where $\gamma_+(t) = te^{i\theta}$ if $0 \leq t < \infty$ and $\gamma_-(t) = -te^{-i\theta}$ if $0 \leq t < \infty$ with $0 < \mu < \nu$. On the other hand, we have

$$(L - \lambda I)^{-1} = \int_{\Gamma} e^{\lambda z} e^{-zL} dz,$$

here $\Gamma = \Gamma_+ + \Gamma_-$, where $\Gamma_+(t) = te^{i\beta}$ if $0 \leq t < \infty$ and $\gamma_-(t) = -te^{-i\beta}$ if $0 \leq t < \infty$ with $\beta = \frac{\pi - \theta}{2}$. Therefore

$$\|g_{s,t}(L)f\|_{L^2(F)} \leq \left\| \int_{\Gamma_+} e^{-zL} f \int_{\gamma_+} \frac{\lambda^s}{(1 + \lambda)^2s} g(\lambda) e^{\lambda z} (1 - e^{-t\lambda})^m d\lambda dz \right\|_{L^2(F)} + \left\| \int_{\Gamma_+} e^{-zL} f \int_{\gamma_-} \frac{\lambda^s}{(1 + \lambda)^2s} g(\lambda) e^{\lambda z} (1 - e^{-t\lambda})^m d\lambda dz \right\|_{L^2(F)} = I_1 + I_2.$$

Let us estimate $I_1$ first. Observe that, by Lemma 6.2,

$$I_1 \leq C \int_{\Gamma_+} \left\| e^{-zL} f \right\|_{L^2(F)} \int_{\gamma_+} \frac{\lambda^s}{(1 + \lambda)^2s} g(\lambda) e^{\lambda z} (1 - e^{-t\lambda})^m d\lambda dz$$

$$\leq C \|f\|_{L^2(E)} \int_0^\infty \exp \left\{ - c \frac{d(E,F)^2}{|z|} \cos \beta \right\} \int_0^\infty \|g(\lambda)\|_{\infty} |e^{\lambda z} (1 - e^{-t\lambda})^m| |d\lambda| |dz|. $$

Since $\mu < \pi/2$, we obtain $|(1 - e^{-t\lambda})^m| \leq c(t|\lambda|)^m$. Hence,

$$I_1 \leq C \|f\|_{L^2(E)} \|g(\lambda)\|_{\infty} \int_0^\infty \exp \left\{ - c \frac{d(E,F)^2}{|z|} \right\} \int_0^\infty e^{-c \lambda |z|} (t|\lambda|)^m |d\lambda| |dz|$$

$$\leq C \|f\|_{L^2(E)} \|g(\lambda)\|_{\infty} \int_0^\infty \exp \left\{ - c \frac{d(E,F)^2}{|z|} \right\} \frac{t^m}{|z|^m} |dz|$$

$$\leq C \|f\|_{L^2(E)} \|g(\lambda)\|_{\infty} \left( \int_0^t \ldots + \int_t^\infty \ldots \right) = C \|f\|_{L^2(E)} \|g(\lambda)\|_{\infty} (I_{11} + I_{12}).$$

Concerning the term $I_{11}$, we have

$$I_{11} = \int_0^t \exp \left\{ - c \frac{d(E,F)^2}{|z|} \right\} \frac{t^m}{|z|^m} |dz|$$

$$\leq \int_0^t \left( \frac{|z|}{d(E,F)^2} \right)^{m+1} \frac{t^m}{|z|^m} |dz| = \left( \frac{t}{d(E,F)^2} \right)^{m+1}.$$

Concerning the term $I_{11}$, we have

$$I_{12} = \int_t^\infty \exp \left\{ - c \frac{d(E,F)^2}{|z|} \right\} \frac{t^m}{|z|^m} |dz|$$

$$\leq \int_t^\infty \left( \frac{|z|}{d(E,F)^2} \right)^{m-1} \frac{t^m}{|z|^m} |dz| = \left( \frac{t}{d(E,F)^2} \right)^{m-1}. $$
The term \( I_2 \) can be treated by the same way. These estimates together with (36) give (34).

We proceed to prove (35). Repeating the arguments above, we obtain

\[
\|g(L)(tLe^{-tL})^m f\|_{L^2(F)} \leq \left\| \int_{\Gamma_+} e^{-zL} f \int_{\gamma_+} g(\lambda)e^{\lambda z}(t\lambda e^{-t\lambda})^m d\lambda dz \right\|_{L^2(F)} + \left\| \int_{\Gamma_-} e^{-zL} f \int_{\gamma_-} g(\lambda)e^{\lambda z}(t\lambda e^{-t\lambda})^m d\lambda dz \right\|_{L^2(F)} = I_{11} + I_{12}.
\]

We need only estimate \( I_{11} \). The estimate for \( I_{12} \) is proved similarly. One has,

\[
I_{11} \leq C \int_{\Gamma_+} \left\| e^{-zL} f \right\|_{L^2(E)} \int_{\gamma_+} g(\lambda)e^{\lambda z}(t\lambda e^{-t\lambda})^m d\lambda dz \\
\leq C \| f \|_{L^2(E)} \int_0^\infty \exp \left\{ -c \frac{d(E, F)^2}{|z|} \cos \beta \right\} \int_0^\infty \| g(\lambda) \|_\infty e^{\lambda |z|} \lambda |d\lambda| |z| \\
\leq C \| f \|_{L^2(E)} \int_0^\infty \exp \left\{ -c \frac{d(E, F)^2}{|z|} \right\} \int_0^\infty \| g(\lambda) \|_\infty e^{-|\lambda| (c_1|z| + c_2 t)} (t|\lambda|)^m d\lambda |d| |z| \\
\leq C \| f \|_{L^2(E)} \| g(\lambda) \|_\infty \left( \int_0^t \cdots + \int_t^\infty \cdots \right) = C \| f \|_{L^2(E)} \| g(\lambda) \|_\infty (I_{111} + I_{112})
\]

For the term \( I_{111} \),

\[
I_{111} = \int_0^t \exp \left\{ -c \frac{d(E, F)^2}{|z|} \right\} \left( c_1 |z| + c_2 t \right)^m |d| |z| \\
\leq C \int_0^t \left( \frac{|z|}{d(E, F)^2} \right)^m \left( c_1 |z| + c_2 t \right)^m |d| |z| = C \left( \frac{t}{d(E, F)^2} \right)^m.
\]

The remaining term \( I_{112} \) is dominated by

\[
C \int_0^\infty \left( \frac{|z|}{d(E, F)^2} \right)^{m-1} \left( c_1 |z| + c_2 t \right)^m |d| |z| = C \left( \frac{t}{d(E, F)^2} \right)^{m-1}.
\]

This completes our proof. \( \square \)

Note that from Proposition 6.1 \( g(L) \) satisfies (11) and (12) with \( m \) being replaced by \( m - 1 \). Moreover, if the Gaussian upper bound condition (H3) is satisfied then \( g(L) \) is of weak type \((1,1)\), see [20]. Hence we obtain the following result.

**Theorem 6.3.** Assume that \( L \) satisfies conditions (H1) and (H2). Let \( g \in H_\infty(S^0) \). Then \( g(L) \) is bounded from \( H_p^1(X) \) to \( L^p(X) \) for all \( 0 < p \leq 1 \). Moreover, if the Gaussian upper bound condition (H3) is satisfied, then the commutator of a BMO function \( b \) and \( g(L) \) is bounded from \( H_1^1(X) \) to \( L^{1,\infty}(X) \).
Remark 6.4. We can obtain the following estimate which is sharper than (37):
\begin{equation}
\|g(L)(tLe^{-tL})^m f\|_{L^2(F)} \leq C \left( \frac{t}{d(E, F)^2} \right)^m \|f\|_{L^2(E)} \|g\|_{\infty}, \forall t > 0.
\end{equation}
See for example [10]. We remark that the estimate (37) implies the boundedness for the holomorphic functional calculus $g(L)$ from $H^p_L(X)$ to $H^p_L(X)$ and hence $g(L)$ is bounded from $H^p_L(X)$ to $L^p(X)$, see [10, 19]. In this section, we obtain the $H^p_L - L^p$ boundedness of $g(L)$ and our main result is the endpoint estimate of the commutator $[b, g(L)]$ where $b$ is a BMO function.

6.3. Application to spectral multipliers. Assume that $L$ satisfies conditions (H1). Let 
\begin{equation*}
F(L) = \int_0^\infty F(\lambda) dE_L(\lambda)
\end{equation*}
be the spectral multiplier $F(L)$ defined on $L^2$ by using the spectral resolution of $L$.

Our main result on spectral multipliers is the following.

Proposition 6.5. Assume that $L$ satisfies conditions (H1) and (H2). Let $F$ be a bounded function defined on $(0, \infty)$ such that for some real number $\alpha > \frac{n(2-p)}{2p} + \frac{1}{2}$ and any non-zero function $\eta \in C_c^\infty(\frac{1}{2}, 2)$ there exists a constant $C_\eta$ such that
\begin{equation}
\sup_{t > 0} \|\eta(\cdot)F(t\cdot)\|_{W^{2,\alpha}(\mathbb{R})} \leq C_\eta
\end{equation}
where $\|F\|_{W^{2,\alpha}(\mathbb{R})} = \|(I - d^2/dx^2)^{\alpha/2} F\|_{L^2}$. Then the multiplier operator satisfies the following estimate
\begin{equation}
\left( \int_{S_j(B)} |F(\sqrt{L})a|^2 d\mu \right)^{\frac{1}{2}} \leq C 2^{-j\delta} V(B)^{\frac{1}{2} - \frac{1}{4}}
\end{equation}
for some $\delta > \frac{n(2-p)}{4p}$, for any $(p, 2, m)$-atom $a$ supported in $B$ and sufficiently large $m$.

Before giving the proof we state the following result in [22].

Lemma 6.6. Let $\gamma > 1/2$ and $\beta > 0$. Then there exists a constant $C > 0$ such that for every function $F \in W^{2,\gamma+\beta/2}$ and every function $g \in L^2(X)$ supported in the ball $B$, we have
\begin{equation*}
\int_{d(x, x_B) > 2r_B} |F(2^j \sqrt{L})g(x)|^2 \left( \frac{d(x, x_B)}{r_B} \right)^\beta d\mu(x) \leq C(r_B 2^j)^{-\beta} \|F\|_{W^{2,\gamma+\beta/2}}^2 \|g\|_{L^2}^2
\end{equation*}
for $j \in \mathbb{Z}$.

Proof of Proposition 6.5.

Obviously, since $F(\sqrt{L})$ is bounded on $L^2(X)$, (39) holds for $j = 0, 1, 2$. For $j > 2$, we will exploit some ideas in [22] to our situation.

Fix $\epsilon > 0$ and $\gamma > 1/2$ such that $\gamma + \epsilon + \frac{n(2-p)}{2p} = \alpha$. Set $\beta = \frac{n(2-p)}{p} + 2 \epsilon$. Then
\( \gamma + \beta/2 = \alpha \). Let \( a = L^m b \) be a \((p, m, m)\)-atom supported in \( B \) with \( m > \beta/4 \) and \( \ell_0 = -\log_2 r_B \).

Fix a function \( \phi \in C^\infty_c \left( \frac{1}{2}, 2 \right) \) such that
\[
\sum_{\ell \in \mathbb{Z}} \phi(2^{-\ell} \lambda) = 1 \text{ for } \lambda > 0.
\]

Then, one has
\[
F(\sqrt{L})a = \sum_{\ell \geq \ell_0} \phi(2^{-\ell} \sqrt{L}) F(\sqrt{L})a + \sum_{\ell < \ell_0} \phi(2^{-\ell} \sqrt{L}) L^m F(\sqrt{L})b.
\]

Recall that by definition of \((p, m, m)\) atoms,
\[
\|a\|_{L^2} \leq V(B)^{\frac{1}{2} - \frac{1}{p}} \text{ and } \|b\|_{L^2} \leq r_B^2 V(B)^{\frac{1}{2} - \frac{1}{p}}.
\]

Set
\[
F_\ell(\lambda) = \begin{cases} F(2^\ell \lambda) & \text{if } \ell \geq \ell_0 \\ 2^{m \ell} F(2^\ell \lambda) \lambda^m \phi(\lambda) & \text{if } \ell < \ell_0. \end{cases}
\]

and extend \( F_\ell \) to the even function. Obviously,
\[
\|F_\ell\|_{W^{2,\alpha}} \leq \begin{cases} C & \text{if } \ell \geq \ell_0 \\ C2^{m \ell} & \text{if } \ell < \ell_0. \end{cases}
\]

Applying Lemma 6.6, we obtain
\[
\left( \int_{S_j(B)} |F(\sqrt{L})a(x)| (\frac{d(x, x_B)}{r_B})^{\beta} \, d\mu(x) \right)^{1/2}
\leq C \sum_{\ell \geq \ell_0} (r_B 2^\ell)^{-\beta/2} \|F_\ell\|_{W^{2,\alpha}} \|a\|_{L^2} + C \sum_{\ell < \ell_0} (r_B 2^\ell)^{-\beta/2} \|F_\ell\|_{W^{2,\alpha}} \|b\|_{L^2}
\leq C \left( \sum_{\ell \geq \ell_0} (r_B 2^\ell)^{-\beta/2} + \sum_{\ell < \ell_0} (r_B 2^\ell)^{2m - \beta/2} \right) V(B)^{\frac{1}{2} - \frac{1}{p}}
\leq CV(B)^{\frac{1}{2} - \frac{1}{p}}.
\]

This implies that
\[
\left( \int_{S_j(B)} |F(\sqrt{L})a|^2 \, d\mu \right)^{1/2} \leq C2^{-j\beta/2} V(B)^{\frac{1}{2} - \frac{1}{p}}.
\]

The proof is complete.

From Proposition 6.5 and Theorem 1.2 we obtain the following result.

**Theorem 6.7.** (i) Assume that \( L \) satisfies conditions (H1) and (H2). Let \( F \) be a bounded function defined on \((0, \infty)\) such that for some real number \( \alpha > \frac{n(\frac{2}{p} - 1)}{2p} + \frac{n}{2p} \) and any non-zero function \( \eta \in C^\infty_c \left( \frac{1}{2}, 2 \right) \) satisfies the condition (38). Then the multiplier operator \( F(L) \) is bounded from \( H^p_\ell(X) \) to \( L^p(X) \) for \( 0 < p < 1 \).
(ii) Under the same assumptions as (i), the operators \( F(L) \) is bounded from \( H^p_1(X) \) to \( H^p_2(X) \) for all \( 0 < p \leq 1 \).

(iii) Assume that \( L \) satisfies (H1) and (H3). Let \( F \) be a bounded function defined on \((0, \infty)\) such that for some real number \( \alpha > \frac{n}{2} + \frac{1}{2} \) and any non-zero function \( \eta \in C_c^\infty(\frac{1}{2}, 2) \) there exists a constant \( C_\eta \) such that

\[
\sup_{t > 0} \| \eta(\cdot)F(t \cdot) \|_{W^{2,\infty}(\mathbb{R})} \leq C_\eta.
\]  

Then the commutator of \( F(L) \) and a BMO function \( b \) is bounded from \( H^1_1(X) \) to \( L^{1,\infty}(X) \).

**Proof.** (i) is a direct consequence of Proposition 6.5 and Theorem 1.2.

To prove (iii), we note that the Gaussian upper bound condition (H3) together with (40) implies that \( F(L) \) is of weak type \((1, 1)\), see [23]. Therefore, by Proposition 6.5 and Theorem 1.2, the commutator of \( F(L) \) and a BMO function \( b \) is bounded from \( H^1_1(X) \) to \( L^{1,\infty}(X) \).

Concerning (ii), we can prove (ii) by first showing that \( F(L) \) maps a \((1, 2, m)\) atom to \( H^1_1(X) \). It then follows that \( F(L) \) is bounded on \( H^p_1(X) \) for all \( 0 < p \leq 1 \). We sketch the proof here for reader’s convenience.

Due to Proposition 2.7 and the fact that the condition (38) is invariant under the change of variable \( \lambda \mapsto \lambda^* \), it suffices to show that there exists \( \epsilon > 0 \) such that for any \((p, 2, 2m)\)-atom \( a \) associated to the ball \( B \), the function

\[
\tilde{a} = F(\sqrt{L})a
\]

is a multiple of \((p, 2, m, \epsilon)\)-molecule.

Fix \( \epsilon > 0 \) and \( \gamma > 1/2 \) such that \( \gamma + \epsilon + \frac{n(2-p)}{2p} = \alpha \). Set \( \beta = \frac{n(2-p)}{2p} + 2\epsilon \). Then \( \gamma + \beta/2 = \alpha \). Let \( j_0 = -\log_2 r_B \), \( a = L^{2m}b \) with \( m > \beta/4 \) and \( \tilde{b} = F(\sqrt{L})L^mb \). Then \( \tilde{a} = L^m \tilde{b} \). We need to verify that

\[
\| (r_B^2L)^{\ell/2} \tilde{b} \|_{L^2(S_k(B))} \leq C 2^{-k \epsilon} r_B^{2m} V(2^k B)^{1/2-1/p}
\]

for all \( 0 \leq \ell \leq m \) and \( k = 0, 1, \ldots \).

It is easy to check that (11) holds for \( k = 0, 1, 2 \). To check (11) for \( k \geq 3 \), we fix a function \( \phi \in C_c^\infty(\frac{1}{2}, 2) \) such that

\[
\sum_{j \in \mathbb{Z}} \phi(2^{-j} \lambda) = 1 \text{ for } \lambda > 0.
\]

Then, for \( 0 \leq \ell \leq m \), one has

\[
(r_B^2L)^{\ell/2} \tilde{b} = r_B^{2\ell} \sum_{j \geq j_0} \phi(2^{-j} \sqrt{L}) F(\sqrt{L}) L^{\ell+m} b
\]

\[
+ r_B^{2\ell} \sum_{j < j_0} \phi(2^{-j} \sqrt{L}) L^m F(\sqrt{L}) L^\ell b
\]

\[
= r_B^{2\ell} \sum_{j \geq j_0} \phi(2^{-j} \sqrt{L}) F(\sqrt{L}) b_1 + r_B^{2\ell} \sum_{j < j_0} \phi(2^{-j} \sqrt{L}) L^m F(\sqrt{L}) b_2.
\]

\[
\sum_{j \geq j_0} \phi(2^{-j} \sqrt{L}) F(\sqrt{L}) b_1 = 2^{-\epsilon} r_B^{2m} V^2(2^{k+1} B) \leq C 2^{-k \epsilon} r_B^{2m} V(2^k B)^{1/2-1/p}
\]

for all \( k = 0, 1, 2, \ldots \).
It is easy to see that
\[ \|b_1\|_{L^2} \leq r_B^{2m-2\ell} V(B)^{\frac{1}{2} - \frac{1}{\ell}} \text{ and } \|b_2\|_{L^2} \leq r_B^{4m-2\ell} V(B)^{\frac{1}{2} - \frac{1}{\ell}}. \]
Set
\[ F_j(\lambda) = \begin{cases} F(2^j \lambda) \phi(\lambda) & , j \geq j_0 \\ 2^{2mj} F(2^j \lambda) \lambda^{2m} \phi(\lambda) & , j < j_0. \end{cases} \]
and extend \( F_j \) to the even function. Obviously,
\[ \|F_j\|_{W^{2,\alpha}} \leq \begin{cases} C & , j \geq j_0 \\ C2^{2mj} & , j < j_0. \end{cases} \]
Applying Lemma 6.6, we obtain
\[
(\int_{d(x,B)}>2r_B} \left| (r_B^2 L)\tilde{b}(x) \right|^2 \left( \frac{d(x,x_B)}{r_B} \right)^\beta d\mu(x) \right)^{1/2} \\
\leq C r_B^{2\ell} \sum_{j \geq j_0} (r_B 2^j)^{-\beta/2} \|F_j\|_{W^{2,\alpha}} \|b_1\|_{L^2} + C r_B^{2\ell} \sum_{j < j_0} (r_B 2^j)^{-\beta/2} \|F_j\|_{W^{2,\alpha}} \|b_2\|_{L^2} \\
\leq C r_B^{2m} V(B)^{\frac{1}{2} - \frac{1}{\ell}}.
\]
So, for \( k \geq 3 \), one has
\[
\left\| (r_B^2 L)\tilde{b}\right\|_{L^2(S_k(B))} \leq C2^{-j(\beta/2-n(\frac{1}{2}-\frac{1}{\ell}))} r_B^{2m} V(2^k B)^{\frac{1}{2} - \frac{1}{\ell}} \\
\leq C2^{-j} r_B^{2m} V(2^k B)^{\frac{1}{2} - \frac{1}{\ell}}.
\]
This implies that \( \tilde{a} = F(\sqrt{\ell}) a \) is a multiple of a \((p, m, \epsilon, 2, \ell)\)-molecule and the multiple constant is independent of \( a \). Our proof is complete. \( \Box \)

**Remark 6.8.** i) When \( p = 1 \), it was shown in [22] that \( F(L) \) maps a \((1, 2, m)\) atom into \( H^1_L(X) \), but the boundedness of \( F(L) \) on the Hardy spaces was only obtained under the extra assumption that the measure of any ball \( B(x, r) \) has a lower bound \( cr^\kappa \) for some \( \kappa > 0 \), see [22]. In this article, using the fact that the convergence in the atomic decomposition in Hardy spaces \( H^1_L(X) \) is in the sense of \( L^2(X) \), we can obtain the boundedness of \( F(L) \) on \( H^1_L(X) \) without the assumption of the measure of a ball having the lower bound \( c r^\kappa \).

ii) The condition \( \ref{4.4} \) in (iii) of Theorem 6.7 can be replaced by the following condition
\[ \sup_{t > 0} \|\eta(\cdot) F(t)\|_{W^{\infty,\alpha}(R)} \leq C_\eta \]
with \( \alpha > n/2 \). In this situation, we can use the following estimates in [23, Lemma 4.3] instead of Lemma 6.6 to obtain Proposition 6.8. Since the proof is quite similar to that of Proposition 6.5, we omit details here.

**Lemma 6.9.** Suppose that \( L \) satisfies (H1) and (H3), \( R > 0 \) and \( \alpha > 0 \). Then for any \( \epsilon > 0 \), there exists a constant \( C = C(\alpha, \epsilon) \) such that
\[
\int_X \left| K_{F(\sqrt{\ell})}(x,y) \right|^2 \left( 1 + Rd(x,y) \right)^\alpha d\mu(x) \leq \frac{C}{V(y,R^{-1})} \|\delta_R F\|_{W^{\infty,\alpha+\epsilon}}^2.
\]
for all Borel functions $F$ such that $\text{supp} F \subseteq [R/4, R]$, where $K_{F(\sqrt{\mathcal{T}})}(x, y)$ is the associated kernel to $F(\sqrt{\mathcal{T}})$.

iii) The approach in this paper can be applied to consider the boundedness of the commutators of generalized fractional integrals on Hardy spaces associated to operators. This will appear in the forthcoming paper [9].

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