BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR POSITIVE OPERATORS BETWEEN CLASSICAL BANACH SPACES

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Dedicated to the memory of Victor Lomonosov

ABSTRACT. We prove that the class of positive operators from $L_\infty(\mu)$ to $L_1(\nu)$ has the Bishop-Phelps-Bollobás property for any positive measures $\mu$ and $\nu$. The same result also holds for the pair $(c_0, \ell_1)$. We also provide an example showing that not every pair of Banach lattices satisfies the Bishop-Phelps-Bollobás property for positive operators.

1. Introduction

In 1961 Bishop and Phelps proved that for any Banach space the set of (bounded and linear) functionals attaining their norms is norm dense in the topological dual space $[15]$. In 1970 Bollobás gave some quantified version of that result $[16]$. In order to state such result we recall the following notation. By $B_X$, $S_X$ and $X^*$ we denote the closed unit ball, the unit sphere and the topological dual of a Banach space $X$, respectively. If $X$ and $Y$ are both real or both complex Banach spaces, $L(X, Y)$ denotes the space of (bounded linear) operators from $X$ to $Y$, endowed with its usual operator norm.

Bishop-Phelps-Bollobás theorem (see $[17$, Theorem 16.1$]$ or $[19$, Corollary 2.4$]$). Let $X$ be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{2}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

After a period in which a lot of attention has been devoted to extending Bishop-Phelps theorem to operators and interesting results have been obtained about that topic (see $[2]$), in 2008 it was posed the problem of extending Bishop-Phelps-Bollobás theorem for operators.

In order to state some of these extensions it will be convenient to recall the following notion.

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Definition 1.1 ([5, Definition 1.1]). Let $X$ and $Y$ be either real or complex Banach spaces. The pair $(X, Y)$ is said to have the Bishop-Phelps-Bollobás property for operators (BPBp) if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in S_{L(X,Y)}$, if $x_0 \in S_X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $T \in S_{L(X,Y)}$ satisfying the following conditions

$$\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$ 

If $X$ and $Y$ are Banach spaces, it is known that the pair $(X, Y)$ has the Bishop-Phelps-Bollobás property in the following cases:

- $X$ and $Y$ are finite-dimensional spaces [5, Proposition 2.4].
- $X$ is any Banach space and $Y$ has the property $\beta$ (of Lindenstrauss) [5, Theorem 2.2]. The spaces $c_0$ and $\ell_\infty$ have property $\beta$.
- $X$ is uniformly convex and $Y$ is any Banach space ([9, Theorem 2.2] or [23, Theorem 3.1]).
- $X = \ell_1$ and $Y$ has the approximate hyperplane series property [5, Theorem 4.1]. For instance, finite-dimensional spaces, uniformly convex spaces, $C(K)$ and $L_1(\mu)$ have the approximate hyperplane series property.
- $X = L_1(\mu)$ and $Y$ has the Radon-Nikodým property and the approximate hyperplane series property, whenever $\mu$ is any $\sigma$-finite measure [20, Theorem 2.2] (see also [7, Theorem 2.3]).
- $X = L_1(\mu)$ and $Y = L_1(\nu)$, for any positive measures $\mu$ and $\nu$ [21, Theorem 3.1].
- $X = L_1(\mu)$ and $Y = L_\infty(\nu)$, for any positive measure $\mu$ and any localizable positive measure $\nu$ [21, Theorem 4.1] (see also [14]).
- $X = C(K)$ and $Y = C(S)$ in the real case, where $K$ and $S$ are compact Hausdorff topological spaces [6, Theorem 2.5].
- $X = C(K)$ and $Y$ is a uniformly convex Banach space, in the real case [24, Theorem 2.2] (see also [22, Corollary 2.6] and [25, Theorem 5]).
- $X = C_0(L)$, for any locally compact Hausdorff topological space $L$ and $Y$ is a $C$-uniformly convex space, in the complex case [3, Theorem 2.4]. As a consequence, the pair $(C_0(L), L_p(\mu))$ has the BPBp for any positive measure $\mu$ and $1 \leq p < +\infty$.
- $X = \ell_n^\infty$ and $Y = L_1(\mu)$ for any positive integer $n$ and any positive measure $\mu$ [10, Corollary 4.5] (see also [10, Theorem 3.3], [11, Theorem 3.3] and [8, Theorem 2.9]).
- $X$ is an Asplund space and $A \subset C(K)$ is a uniform algebra [18, Theorem 3.6] (see also [13, Corollary 2.5]).
The paper [4] contains a survey with most of the results known about the Bishop-Phelps-Bollobás property for operators.

In this short note we introduce a version of Bishop-Phelps-Bollobás property for positive operators between Banach lattices (see Definition 2.2). The only difference between this property and the previous one is that we assume that the operators appearing in Definition 1.1 are positive. In Section 2 we prove that the pair $(L_\infty(\mu), L_1(\nu))$ has the Bishop-Phelps-Bollobás property for positive operators for any positive measures $\mu$ and $\nu$. The parallel result for $(c_0, L_1(\mu))$ is shown in section 3, for any positive measure $\mu$. As a consequence, the subset of positive operators from $c_0$ to $\ell_1$ satisfies the Bishop-Phelps-Bollobás property. We remark that it is not known whether the pairs $(L_\infty(\mu), L_1(\nu))$ and $(c_0, \ell_1)$ satisfy the Bishop-Phelps-Bollobás property for operators in the real case. In both cases the set of norm attaining operators is dense in the space of operators (see [27, Theorem B] for the first case). For the second pair, a necessary condition on the range space in order to have the Bishop-Phelps-Bollobás for operators is known (see [10, Theorem 3.3]). We also provide an example showing that not every pair of Banach lattices satisfies the Bishop-Phelps-Bollobás property for positive operators.

2. Bishop-Phelps-Bollobás property for positive operators for the pair $(L_\infty, L_1)$

We begin by recalling some notions and introducing the appropriate notion of Bishop-Phelps-Bollobás property for positive operators. The concepts in the first definition are standard and can be found, for instance, in [1].

**Definition 2.1.** An ordered vector space is a real vector space $X$ equipped with a vector space order, that is, an order relation $\leq$ that is compatible with the algebraic structure of $X$. An ordered vector space is called a Riesz space if every pair of vectors has a least upper bound and a greatest lower bound. A norm $\|\|$ on a Riesz space $X$ is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A normed Riesz space is a Riesz space equipped with a lattice norm. A normed Riesz space whose norm is complete is called a Banach lattice.

An operator $T : X \to Y$ between two ordered vector spaces is called positive if $x \geq 0$ implies $Tx \geq 0$.

We remark that throughout this paper by operator we mean a linear mapping. Recall that every positive operator from a Banach lattice to a normed Riesz space is continuous [12, Theorem 4.3].
Definition 2.2. Let $X$ and $Y$ be Banach lattices. The pair $(X,Y)$ is said to have the Bishop-Phelps-Bollobás property for positive operators if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in S_{L(X,Y)}$, such that $S \geq 0$, if $x_0 \in S_X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and a positive operator $T \in S_{L(X,Y)}$ satisfying the following conditions

$$
\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.
$$

Let $(\Omega, \mu)$ be a measure space. We denote by $L_\infty(\mu)$ the space of real valued measurable essentially bounded functions on $\Omega$ and by $1$ the constant function equal to $1$ on $\Omega$. Since an element $f$ in $B_{L_\infty(\mu)}$ satisfies that $|f| \leq 1$ a.e., it is clear that a positive operator from $L_\infty(\mu)$ to any other Banach lattice satisfies the next result.

**Lemma 2.3.** Let $\mu$ and $\nu$ be positive measures and $T$ a positive operator from $L_\infty(\mu)$ to $L_1(\nu)$. Then $\|T\| = \|T(1)\|_1$.

It is trivially satisfied that $\|f + g\|_1 = \|f - g\|_1$ for any positive integrable functions $f$ and $g$ with disjoint supports. Next result shows that in case that two functions satisfy the previous assumption and the quantities $\|f + g\|_1$ and $\|f - g\|_1$ are close, then both functions can be approximated by positive functions with disjoint supports.

**Lemma 2.4.** Let $(\Omega, \mu)$ be a measure space, $0 < \varepsilon < \frac{1}{5}$ and $f_1, f_2 \in L_1(\mu)$ be positive functions such that

$$
\|f_1 + f_2\|_1 \leq 1 \quad \text{and} \quad 1 - \varepsilon^2 \leq \|f_1 - f_2\|_1.
$$

Then there are two positive functions $g_1$ and $g_2$ with disjoint supports in $L_1(\mu)$ and also satisfying that

$$
\|g_1 + g_2\|_1 = 1 \quad \text{and} \quad \|g_i - f_i\|_1 < 7\varepsilon \quad \text{for} \quad i = 1, 2.
$$

**Proof.** We define the set $W$ given by

$$
W = \{t \in \Omega : |f_1(t) - f_2(t)| \leq (1 - \varepsilon)(f_1(t) + f_2(t))\}.
$$
Clearly $W$ is a measurable subset of $\Omega$. We have that

\[
1 - \varepsilon^2 \leq \|f_1 - f_2\|_1 \\
= \int_\Omega |f_1 - f_2| \, d\mu \\
= \int_W |f_1 - f_2| \, d\mu + \int_{\Omega \setminus W} |f_1 - f_2| \, d\mu \\
\leq (1 - \varepsilon) \int_W (f_1 + f_2) \, d\mu + \int_{\Omega \setminus W} (f_1 + f_2) \, d\mu \\
\leq 1 - \varepsilon \int_W (f_1 + f_2) \, d\mu.
\]

As a consequence

\[
(2.1) \quad \int_W (f_1 + f_2) \, d\mu \leq \varepsilon.
\]

Now we define the sets given by

\[
G_1 = \Omega \setminus W \cap \{ t \in \Omega : f_1(t) > f_2(t) \} \quad \text{and} \quad G_2 = \Omega \setminus W \cap \{ t \in \Omega : f_2(t) > f_1(t) \}.
\]

Clearly $G_1$ and $G_2$ are measurable subsets and it is satisfied that

\[
(f_1 - f_2)\chi_{G_1} = |f_1 - f_2|\chi_{G_1} > (1 - \varepsilon)(f_1 + f_2)\chi_{G_1}.
\]

So $f_2\chi_{G_1} \leq (2 - \varepsilon)f_2\chi_{G_1} \leq \varepsilon f_1\chi_{G_1}$ and

\[
(2.2) \quad \int_{G_1} f_2 \, d\mu \leq \int_{G_1} \varepsilon f_1 \, d\mu \leq \varepsilon.
\]

By using the same argument with the function $f_1$ we obtain that

\[
(2.3) \quad \int_{G_2} f_1 \, d\mu \leq \varepsilon.
\]

Since the subsets $W$, $G_1$ and $G_2$ are a partition of $\Omega$, in view of (2.1) and (2.3) we deduce that

\[
\|f_1 - f_1\chi_{G_1}\|_1 = \|f_1\chi_W \cup G_2\|_1 \\
= \|f_1\chi_W\|_1 + \|f_1\chi_{G_2}\|_1 \\
= \int_W f_1 \, d\mu + \int_{G_2} f_1 \, d\mu \\
\leq 2\varepsilon.
\]
Since $f_1$ and $f_2$ satisfy the same conditions we also have that
\begin{equation}
\|f_2 - f_2 \chi_{G_2}\|_1 \leq 2\varepsilon. \tag{2.5}
\end{equation}
By using that $f_1$ and $f_2$ are positive functions we deduce that
\begin{align}
\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1 &= \|f_1 + f_2\|_1 - \|f_1 - f_1 \chi_{G_1}\|_1 - \|f_2 - f_2 \chi_{G_2}\|_1 \\
\geq & \|f_1 - f_2\|_1 - 4\varepsilon \quad \text{(by (2.4) and (2.5))} \tag{2.6} \\
\geq & 1 - \varepsilon^2 - 4\varepsilon \\
> & 1 - 5\varepsilon > 0.
\end{align}

Now we define the functions $g_1$ and $g_2$ by
\[ g_i = \frac{f_i \chi_{G_i}}{\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1}, \quad i = 1, 2. \]

It is clear that $g_i \in L_1(\mu)$ for $i = 1, 2$ and they are positive functions with disjoint supports. It is also clear that $\|g_1 + g_2\|_1 = 1$.

Since $f_1$ and $f_2$ are positive functions we have that $\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1 \leq \|f_1 + f_2\|_1 \leq 1$, so for $i = 1, 2$ we have that
\begin{align}
\|f_i \chi_{G_i}\|_1 \frac{1}{\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1} - 1 &= \|f_i \chi_{G_i}\|_1 \left( \frac{1}{\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1} - 1 \right) \\
&= \|f_i \chi_{G_i}\|_1 \frac{1 - \|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1}{\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1} \tag{2.7} \\
&\leq 1 - \|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1.
\end{align}

For $i = 1, 2$ we estimate the distance from $g_i$ to $f_i$ as follows
\begin{align*}
\|g_i - f_i\|_1 &= \left\| \frac{f_i \chi_{G_i}}{\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1} - f_i \right\|_1 \\
&\leq \left\| \frac{f_i \chi_{G_i}}{\|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1} - f_i \chi_{G_i} \right\|_1 + \left\| f_i \chi_{G_i} - f_i \right\|_1 \\
&\leq 1 - \|f_1 \chi_{G_1} + f_2 \chi_{G_2}\|_1 + 2\varepsilon \quad \text{(by (2.7), (2.4) and (2.5))} \\
&< 7\varepsilon \quad \text{(by (2.6)).}
\end{align*}

\[ \square \]

**Theorem 2.5.** For any positive measures $\mu$ and $\nu$, the pair $(L_\infty(\mu), L_1(\nu))$ has the Bishop-Phelps-Bollobás property for positive operators.
Moreover, in Definition 2.2, if the function $f_0$ where the operator $S$ is close to attain its norm is positive, then the function $f_1$ where $T$ attains its norm is also positive.

Proof. Assume that $(\Omega_1, \mu)$ is a measure space. Let $0 < \varepsilon < 1$, $f_0 \in S_{L_\infty(\mu)}$, $S \in S_{L(L_\infty(\mu),L_1(\nu))}$ and assume that $S$ is a positive operator satisfying that

$$\|S(f_0)\|_1 > 1 - \eta^2,$$

where $\eta = \left(\frac{\varepsilon}{58}\right)^2$. We define the sets $A$, $B$ and $C$ given by

$A = \{t \in \Omega_1 : -1 \leq f_0(t) < -1 + \eta\}$, $B = \{t \in \Omega_1 : 1 - \eta < f_0(t) \leq 1\}$

and

$C = \{t \in \Omega_1 : |f_0(t)| \leq 1 - \eta\}$.

By using that $S$ is a positive operator we obtain that

$$1 - \eta^2 < \|S(f_0)\|_1 = \|S(f_0 \chi_A + f_0 \chi_B + f_0 \chi_C)\|_1 \leq \|S(f_0 \chi_A)\|_1 + \|S(f_0 \chi_B)\|_1 + \|S(f_0 \chi_C)\|_1 \leq \|S(\chi_A)\|_1 + \|S(\chi_B)\|_1 + (1 - \eta)\|S(\chi_C)\|_1 \leq 1 - \eta\|S(\chi_C)\|_1.$$  

Hence $\|S(\chi_C)\|_1 \leq \eta$. By using again that $S$ is positive we deduce that

$$\|S(f \chi_C)\|_1 \leq \|S(\chi_C)\|_1 \leq \eta, \quad \forall f \in B_{L_\infty(\mu)}.$$  

On the other hand it is trivially satisfied that

$$\|f_0 \chi_A + \chi_A\|_\infty \leq \eta \quad \text{and} \quad \|f_0 \chi_B - \chi_B\|_\infty \leq \eta,$$

so

$$\|S(f_0 \chi_A + \chi_A)\|_1 \leq \eta \quad \text{and} \quad \|S(f_0 \chi_B - \chi_B)\|_1 \leq \eta.  

By using the assumption we obtain that

$$1 - \eta^2 < \|S(f_0)\|_1 \leq \|S(f_0 \chi_A + \chi_A)\|_1 + \|S(f_0 \chi_B - \chi_B)\|_1 + \|S(f_0 \chi_C)\|_1 \leq \|S(\chi_B - \chi_A)\|_1 + 3\eta \quad \text{(by (2.9) and (2.8)).}$$

As a consequence

$$\|S(\chi_B - \chi_A)\|_1 \geq 1 - 4\eta.$$
Since $S(\chi_A)$ and $S(\chi_B)$ are positive functions and $\|S(\chi_A) + S(\chi_B)\|_1 \leq 1$ we can apply Lemma 2.4 and so there are two positive functions $g_1$ and $g_2$ in $L_1(\nu)$ satisfying the following conditions

$$\|g_1 - S(\chi_A)\|_1 < 14\sqrt{\eta} = \frac{7\varepsilon}{29}, \quad \|g_2 - S(\chi_B)\|_1 < \frac{7\varepsilon}{29},$$

$$\text{supp } g_1 \cap \text{supp } g_2 = \emptyset \quad \text{and} \quad \|g_1 + g_2\|_1 = 1.$$

Assume that $\nu$ is a measure on $\Omega_2$. We obtain that

$$\|S(\chi_A)\chi_{\Omega_2 \setminus \text{supp } g_1}\|_1 = \|(g_1 - S(\chi_A))\chi_{\Omega_2 \setminus \text{supp } g_1}\|_1 \leq \|g_1 - S(\chi_A)\|_1 < \frac{7\varepsilon}{29} \tag{2.11}$$

and also

$$\|S(\chi_B)\chi_{\Omega_2 \setminus \text{supp } g_2}\|_1 < \frac{7\varepsilon}{29} \tag{2.12}.$$

Now we define the operator $V : L_\infty(\mu) \to L_1(\nu)$ as follows

$$V(f) = S(f\chi_A)\chi_{\text{supp } g_1} + S(f\chi_B)\chi_{\text{supp } g_2} \quad (f \in L_\infty(\mu)).$$

Clearly $V$ is well defined and it is a positive operator since $S \geq 0$. So

$$\|V\| = \|V(1)\|_1 = \|S(\chi_A)\chi_{\text{supp } g_1} + S(\chi_B)\chi_{\text{supp } g_2}\|_1 \leq \|S\| = 1.$$

Now we estimate the norm of $V - S$. If $f \in B_{L_\infty(\mu)}$ then we have that

$$\|(V - S)(f)\|_1 = \|S(f\chi_A)\chi_{\text{supp } g_1} + S(f\chi_B)\chi_{\text{supp } g_2} - S(f)\|_1$$

$$= \|S(f\chi_A)\chi_{\text{supp } g_1} + S(f\chi_B)\chi_{\text{supp } g_2} - S(f\chi_A) - S(f\chi_B) - S(f\chi_C)\|_1$$

$$\leq \|S(f\chi_A)\chi_{\Omega_2 \setminus \text{supp } g_1}\|_1 + \|S(f\chi_B)\chi_{\Omega_2 \setminus \text{supp } g_2}\|_1 + \|S(f\chi_C)\|_1$$

$$\leq \|S(\chi_A)\chi_{\Omega_2 \setminus \text{supp } g_1}\|_1 + \|S(\chi_B)\chi_{\Omega_2 \setminus \text{supp } g_2}\|_1 + \|S(f\chi_C)\|_1$$

$$< \frac{14\varepsilon}{29} + \eta < \frac{\varepsilon}{2} \quad (\text{by } (2.11), (2.12) \text{ and } (2.8)).$$

We proved that $\|V - S\| < \frac{\varepsilon}{2}$ and so $\|V\| \geq 1 - \frac{\varepsilon}{2} > 0$. Since $f_0 \in S_{L_\infty(\mu)}$ the function $f_1$ given by $f_1 = \chi_B - \chi_A + f_0\chi_C \in S_{L_\infty(\mu)}$ and satisfies

$$\|f_1 - f_0\|_\infty \leq \eta < \varepsilon.$$
Since $g_1$ and $g_2$ have disjoint supports we also have that
\[
\|V(f_1)\|_1 = \|S(-\chi_A)\chi_{\supp g_1} + S(\chi_B)\chi_{\supp g_2}\|_1 \\
= \|S(\chi_A)\chi_{\supp g_1} + S(\chi_B)\chi_{\supp g_2}\|_1 \\
= \|V(1)\|_1 = \|V\|.
\]

If we take $T = \frac{V}{\|V\|}$, the operator $T \in S_{L(L_\infty(\mu),L_1(\nu))}$, is a positive operator, attains its norm at $f_1$ and satisfies that
\[
\|T - S\| \leq \|T - V\| + \|V - S\| = \left|1 - \|V\|\right| + \|V - S\| \leq 2\|V - S\| < \varepsilon.
\]

We proved that the pair $(L_\infty(\mu), L_1(\nu))$ has the Bishop-Phelps-Bollobás property for positive operators. In case that $f_0 \geq 0$ the function $f_1$ also satisfies the same condition.

\[\square\]

3. A positive result for the Bishop-Phelps-Bollobás property for positive operators for $(c_0, L_1)$.

**Theorem 3.1.** For any positive measure $\mu$, the pair $(c_0, L_1(\mu))$ has the Bishop-Phelps-Bollobás property for positive operators.

Moreover, in Definition 2.2, if the element $x_0$ is positive, then the element $u_0$ where $T$ attains its norm is also positive.

**Proof.** The proof of this result is similar to the proof of Theorem 2.5. In any case we include it for the sake of completeness. Throughout this proof we denote by $\|\|$ the usual norm of $c_0$.

Assume that $\Omega$ is the set such that $(\Omega, \mu)$ is the measure space considered for $L_1(\mu)$. Let $0 < \varepsilon < 1$, $x_0 \in S_{c_0}$, $S \in S_{L(c_0,L_1(\mu))}$ and assume that $S$ is a positive operator satisfying that
\[
\|S(x_0)\|_1 > 1 - \eta^2,
\]
where $\eta = \left(\frac{\varepsilon}{2^8}\right)^2$. We define the sets $A$, $B$ and $C$ given by
\[
A = \{k \in \mathbb{N} : -1 \leq x_0(k) < -1 + \eta\}, \quad B = \{k \in \mathbb{N} : 1 - \eta < x_0(k) \leq 1\}
\]
and
\[
C = \{k \in \mathbb{N} : |x_0(k)| \leq 1 - \eta\}.
\]
Since $x_0 \in S_{c_0}$ the sets $A$ and $B$ are finite and $\{A, B, C\}$ is a partition of $\mathbb{N}$. 
For each positive integer \( n \) we denote by \( C_n = C \cap \{ k \in \mathbb{N} : k \leq n \} \), which is a finite subset of \( \mathbb{N} \). By using that \( S \) is a positive operator in \( \mathcal{L}(c_0, \mathcal{L}_1(\mu)) \) we obtain that

\[
1 - \eta^2 < \| S(x_0) \|_1 \\
= \| S(x_0 \chi_A + x_0 \chi_B + x_0 \chi_C) \|_1 \\
\leq \| S(x_0 \chi_A) \|_1 + \| S(x_0 \chi_B) \|_1 + \| S(x_0 \chi_C) \|_1 \\
\leq \| S(\chi_A) \|_1 + \| S(\chi_B) \|_1 + (1 - \eta) \lim_n \{ \| S(\chi_{C_n}) \|_1 \} \\
\leq 1 - \eta \lim_n \{ \| S(\chi_{C_n}) \|_1 \}.
\]

Hence \( \lim_n \{ \| S(\chi_{C_n}) \|_1 \} \leq \eta \). Since \( S \) is positive we get that

\[
(3.1) \quad \| S(x \chi_C) \|_1 \leq \lim_n \{ \| S(\chi_{C_n}) \|_1 \} \leq \eta, \quad \forall x \in B_{c_0}.
\]

On the other hand it is trivially satisfied that

\[
\| x_0 \chi_A + \chi_A \| \leq \eta \quad \text{and} \quad \| x_0 \chi_B - \chi_B \| \leq \eta,
\]

and so

\[
(3.2) \quad \| S(x_0 \chi_A + \chi_A) \|_1 \leq \eta \quad \text{and} \quad \| S(x_0 \chi_B - \chi_B) \|_1 \leq \eta.
\]

In view of the assumption, since \( \{ A, B, C \} \) is a partition of \( \mathbb{N} \) we obtain that

\[
1 - \eta^2 < \| S(x_0) \|_1 \\
\leq \| S(x_0 \chi_A + x_0 \chi_B) \|_1 + \| S(x_0 \chi_C) \|_1 \\
\leq \| S(x_0 \chi_A + \chi_A) \|_1 + \| S(\chi_B - \chi_A) \|_1 + \| S(x_0 \chi_B - \chi_B) \|_1 + \| S(x_0 \chi_C) \|_1 \\
\leq \| S(\chi_B - \chi_A) \|_1 + 3\eta \quad \text{(by (3.2) and (3.1))}.
\]

Hence

\[
(3.3) \quad \| S(\chi_B - \chi_A) \|_1 \geq 1 - 4\eta.
\]

Now we can apply Lemma 2.4 to the positive functions \( S(\chi_A) \) and \( S(\chi_B) \) since \( \| S(\chi_A) + S(\chi_B) \|_1 \leq \| S \| = 1 \). So there exist two positive functions \( g_1 \) and \( g_2 \) in \( L_1(\mu) \) satisfying the following conditions

\[
\| g_1 - S(\chi_A) \|_1 < \frac{7\varepsilon}{29}, \quad \| g_2 - S(\chi_B) \|_1 < \frac{7\varepsilon}{29},
\]

\[
\text{supp} \ g_1 \cap \text{supp} \ g_2 = \emptyset \quad \text{and} \quad \| g_1 + g_2 \|_1 = 1.
\]
As a consequence, we have that

\[(3.4) \quad \|S(\chi_A)\chi_{\Omega \setminus \text{supp } g_1}\|_1 = \|(g_1 - S(\chi_A))\chi_{\Omega \setminus \text{supp } g_1}\|_1 \leq \|g_1 - S(\chi_A)\|_1 < \frac{7\varepsilon}{29}\]

and also

\[(3.5) \quad \|S(\chi_B)\chi_{\Omega \setminus \text{supp } g_2}\|_1 < \frac{7\varepsilon}{29}.

We define the operator \(U : c_0 \rightarrow L_1(\mu)\) by

\[U(x) = S(x\chi_A)\chi_{\text{supp } g_1} + S(x\chi_B)\chi_{\text{supp } g_2} \quad (x \in c_0).\]

The operator \(U\) is linear, bounded and positive. Since \(U(x) = U(x\chi_{A \cup B})\) for any element \(x \in c_0\) and \(A \cup B\) is finite we obtain that

\[\|U\| = \|U(x_{A \cup B})\|_1 = \|S(\chi_A)\chi_{\text{supp } g_1} + S(\chi_B)\chi_{\text{supp } g_2}\|_1 \leq \|S\| = 1.\]

Now we estimate the distance between \(U\) and \(S\). For an element \(x \in B_{c_0}\) it is satisfied

\[
\|(U - S)(x)\|_1 = \|S(x\chi_A)\chi_{\text{supp } g_1} + S(x\chi_B)\chi_{\text{supp } g_2} - S(x)\|_1 \\
= \|S(x\chi_A)\chi_{\text{supp } g_1} + S(x\chi_B)\chi_{\text{supp } g_2} - S(x\chi_A) - S(x\chi_B) - S(x\chi_C)\|_1 \\
\leq \|S(x\chi_A)\chi_{\Omega \setminus \text{supp } g_1}\|_1 + \|S(x\chi_B)\chi_{\Omega \setminus \text{supp } g_2}\|_1 + \|S(x\chi_C)\|_1 \\
\leq \|S(\chi_A)\chi_{\Omega \setminus \text{supp } g_1}\|_1 + \|S(\chi_B)\chi_{\Omega \setminus \text{supp } g_2}\|_1 + \|S(\chi_C)\|_1 \\
< \frac{14\varepsilon}{29} + \eta < \frac{\varepsilon}{2} \quad \text{(by (3.4), (3.5) and (3.1))}.
\]

We proved that \(\|U - S\| < \frac{\varepsilon}{2}\) and so \(\|U\| \geq 1 - \frac{\varepsilon}{2} > 0\). Since \(x_0 \in S_{c_0}\) the element \(u_0\) given by \(u_0 = \chi_B - \chi_A + x_0\chi_C \in S_{c_0}\) and satisfies

\[\|u_0 - x_0\| \leq \eta < \varepsilon.\]

Since \(g_1\) and \(g_2\) have disjoint supports we also have that

\[\|U(u_0)\|_1 = \|S(-\chi_A)\chi_{\text{supp } g_1} + S(\chi_B)\chi_{\text{supp } g_2}\|_1 \\
= \|S(\chi_A)\chi_{\text{supp } g_1} + S(\chi_B)\chi_{\text{supp } g_2}\|_1 \\
= \|U(\chi_{A \cup B})\|_1 = \|U\|.
\]

If we take \(T = \frac{U}{\|U\|}\), the operator \(T \in S_{L(c_0, L_1(\mu))}\), is a positive operator, attains its norm at \(u_0\) and satisfies that

\[\|T - S\| \leq \|T - U\| + \|U - S\| = \|1 - \|U\|\| + \|U - S\| \leq 2\|U - S\| < \varepsilon.\]
We proved that the pair \((c_0, L_1(\mu))\) has the Bishop-Phelps-Bollobás property for positive operators. Notice that in case that \(x_0\) is positive, the element \(u_0\) is also positive. □

Lastly we provide an example showing that the property that we considered is not trivial.

**Example 3.2.** Let \(Y = c_0\) as a set, endowed with the norm given by

\[
\|\|x\|\| = \|x\| + \left\| \left\{ \frac{x_n}{2^n} \right\} \right\|_2 \quad (x \in c_0),
\]

where \(\|\|\) is the usual norm of \(c_0\). Then the pair \((c_0, Y)\) does not satisfy the Bishop-Phelps-Bollobás property for positive operators.

**Proof.** It is clear that \(\|\|\|\) is a norm equivalent to the usual norm of \(c_0\) and it is a lattice norm on \(Y\). Also the space \(Y\) is strictly convex. So the formal identity from \(c_0\) to \(Y\) cannot be approximated by norm attaining operators by \([26, \text{Proposition 4}]\). Since the formal identity is a positive operator we are done. □

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