Revisiting the Schrödinger Probability Current

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We revisit the definition of the probability current for the Schrödinger equation. First, we prove that the Dirac probability currents of stationary wave functions of the hydrogen atom and of the isotrop harmonic oscillator are not nil and correspond to a circular rotation of the probability. Then, we recall how it is necessary to add to classical Pauli and Schrödinger currents, an additional spin-dependent current, the Gordan current.

Consequently, we get a circular probability current in the Schrödinger approximation for the hydrogen atom and the isotrop harmonic oscillator.

I. INTRODUCTION

In this letter we revisit the definition of the probability current for the Schrödinger equation. We discuss the probability current of the hydrogen atom and the isotrop harmonic oscillator for which the classical results can be expressed in the form: "the probability current of the wave eigenfunctions of Schrödinger with m = 0 is nil", cf. for example [1]. Indeed, as the function \( \psi_{nl0} = R_{nl}(r)Y_{l0}(\theta, \phi) \) is real, the classical definition of the Schrödinger probability current leads to

\[
J_1 = \frac{i\hbar}{2m} (\psi \nabla \psi^* - \psi^* \nabla \psi) = 0. \tag{1}
\]

First, we prove that for the Dirac equation, the probability current of stationary wave functions of the hydrogen atom and of the isotrop harmonic oscillator are not nil and correspond to a circular rotation of the probability.

Then, we recall how a good approximation of the Dirac probability current is obtained in the nonrelativist case. It is necessary to add in equation (1) the Gordon current [2] which reads

\[
J_2 = \frac{\hbar}{2m} \text{rot}(\psi^* \sigma \psi)
\]

in the Pauli approximation [2], [4] and

\[
J_2 = \frac{1}{m} \nabla \rho \times \mathbf{s} \tag{2}
\]

in the Schrödinger approximation [3], [5], where \( \mathbf{s} \) corresponds at a constant spin vector. Consequently, we get a circular probability current in the Schrödinger approximation for the hydrogen atom and the isotrop harmonic oscillator.

II. THE DIRAC PROBABILITY CURRENT OF THE HYDROGEN ATOM

In the presence of electromagnetic couplings, the Dirac equation reads

\[
-i\gamma^\mu \frac{\partial \psi}{\partial x_\mu} + \frac{e}{\hbar} \gamma^\mu A_\mu \psi + \frac{mc}{\hbar} \psi = 0 \tag{3}
\]

where the \( \gamma^\mu \) (\( \mu = 0, 1, 2, 3 \)) are the Dirac matrices.

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The probability density $\rho = \psi^\dagger \psi$ and the three components of the probability current $\mathbf{J}$:

$$J^i = c\psi^\dagger \gamma^0 \gamma^i \psi$$ (4)

satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$ (5)

In the case of an electron in the hydrogen atom with a spherically symmetric potential $V = -\frac{e^2}{4\pi \varepsilon_0 r}$ and a potential vector $A = 0$, the Dirac eigenvalues equation is then

$$\gamma_0 \left(-\frac{E}{\hbar c} + \frac{eV}{\hbar c}\right) \psi(r) - i\mathbf{\Gamma} \nabla \psi(r) + \frac{mc}{\hbar} \psi(r) = 0.$$ (6)

The eigenfunctions $\psi_{n,l,j,m}(r)$ depend on the quantum numbers $n, l, j$ and $m$ with $n$ integer ($n \geq 1$), $l$ integer ($0 \leq l \leq n - 1$), $j = \frac{l}{2}$ if $l = 0$ and $j = l + m$ if $l \geq 1$, $m = \pm \frac{1}{2}$. Then $\psi_{n,l,j,m}(r)$ is given (4) by

$$\psi_{n,l,j,m}(r) = \left(\frac{f(r)}{ig(r)}\right) \Omega_{j\nu m}(\mathbf{r})$$

with $l' = j + \frac{1}{2}$ if $l = j - \frac{1}{2}$ and $l' = j - \frac{1}{2}$ if $l = j + \frac{1}{2}$, and where $f(r)$ and $g(r)$ are real functions of $r$ and where

$$\Omega_{\pm \frac{1}{2}, m} = \left(\pm \frac{\sqrt{\frac{l+\frac{1}{2}+m}{l+\frac{1}{2}}}}{\sqrt{l+\frac{1}{2}}} Y_{l,m+\frac{1}{2}} \right).$$ (7)

We obtain four classes of wave eigenfunctions: $m = \frac{1}{2}$ or $-\frac{1}{2}$, $l = 0$ or $l \geq 1$. For $m = \frac{1}{2}$

$$\psi_{n,l,j,m=\frac{1}{2}}(r, \theta, \varphi) = \left(\begin{array}{c} f(r) \\ b(r) \exp(i\varphi) \\ c(r) \\ d(r) \exp(i\varphi) \end{array}\right)$$

(6)

with for $l = 0$: $a(\theta) = Y_{00}$, $b(\theta) = 0$, $c(\theta) = -Y_{00} \cos \theta$, $d(\theta) = -Y_{00} \sin \theta$ and, for $l > 0$: $a(\theta) = \sqrt{\frac{l+1}{2l+1}} Y_{l,0}$, $b(\theta) = \sqrt{\frac{l+1}{2l+1}} Y_{l,1} \exp(-i\varphi)$, $c(\theta) = \sqrt{\frac{l+1}{2l+1}} Y_{l+1,0}$ and $d(\theta) = \sqrt{\frac{l+1}{2l+1}} Y_{l+1,1} \exp(-i\varphi)$, and for $m = -\frac{1}{2}$

$$\psi_{n,l,j,m=-\frac{1}{2}}(r, \theta, \varphi) = \left(\begin{array}{c} f(r) \\ -b(r) \exp(-i\varphi) \\ a(r) \\ -c(r) \end{array}\right)$$

(7)

with for $l = 0$: $a(\theta) = 0$, $b(\theta) = Y_{00}$, $c(\theta) = -Y_{00} \sin \theta$, $d(\theta) = Y_{00} \cos \theta$ and, for $l > 0$: $a(\theta) = -\sqrt{\frac{l+1}{2l+1}} Y_{l,-1} \exp(i\varphi)$, $b(\theta) = \sqrt{\frac{l+1}{2l+1}} Y_{l,0}$, $c(\theta) = -\sqrt{\frac{l+1}{2l+1}} Y_{l-1,-1} \exp(i\varphi)$ and $d(\theta) = \sqrt{\frac{l+1}{2l+1}} Y_{l-1,0}$.

For a wave function $\psi = \psi_1 \chi_1 + \psi_2 \chi_2$ and the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

the probability density is equal to: $\rho = \psi_1^\dagger \chi_1 + \psi_2^\dagger \chi_2$, and the probability current is given by: $J^i = c(\psi_1^\dagger \sigma^i \chi_2 + \psi_2^\dagger \sigma^i \chi_1)$.

For the functions (6) and (7), we find: $\rho = f^2(r)[a^2(\theta) + b^2(\theta) + g^2(r)[c^2(\theta) + d^2(\theta)], J^1 = -2\epsilon f(r)g(r)ad - bc \cos \varphi, J^2 = -2\epsilon f(r)g(r)ad - bc \sin \varphi, J^3 = 0$, and as probability current velocity:

$$\mathbf{v} = \mathbf{J} \rho = \frac{2\epsilon f(r)g(r)[ad - bc]}{f^2(r)[a^2 + b^2] + g^2(r)[c^2 + d^2]} \mathbf{u}_\varphi$$

(8)

which corresponds to circular probability currents.
We have the same result for all the Dirac equations with a spherically symmetric potential; in particular for the isotropic harmonic oscillator.

The Dirac eigenfunctions are not stationary. All the electron wave eigenfunctions of the hydrogen atom correspond to circular probability currents. It is not the case for the Schrödinger equation with the current given by (1), since we have then stationary solutions for the eigenfunctions \( \psi_{nl0} \). In the fundamental state \( 1s_{\frac{1}{2}} \) with the energy \( E_{1s_{\frac{1}{2}}} = mc^2 \sqrt{1 - \alpha^2} \), we have: \( f(r) = a_0(r_0) \sqrt{1 - \alpha^2 - 1} e^{-2\alpha \rho_0} \), \( g(r) = -a_0 \sqrt{1 - \alpha^2 - 1} (\frac{r}{r_0})^{2(1 - \alpha^2 - 1)} e^{-2\alpha \rho_0} \), where \( r_0 = \frac{\hbar^2}{mc\alpha} = 0.53 \text{ Å} \) is the Bohr radius. From (3), we deduce \( \rho(r, \theta, \phi) = \rho(r) = 2a_0^2 (\frac{r}{r_0})^{2(1 - \alpha^2 - 1)} e^{-2\alpha \rho_0} \), and

\[
\mathbf{v} = \alpha c \sin \theta \mathbf{u}_\phi.
\] (9)

From (7) and (8), we deduce \( \mathbf{v} = -\alpha c \sin \theta \mathbf{u}_\phi \). There is a big difference between (9) and the current nil given by (4) with the Schrödinger equation. The lesson of this example is that, although Schrödinger equation is a good approximation of the Dirac equation, the Schrödinger probability current (1) is not the good approximation of the Dirac probability current (4). This conclusion appears clearly, as we recall now [1], [2], [3], in the approximations to transform first the bispinor of the Dirac equation to the spinor of the Pauli equation, then the Pauli spinor to the wave function of the Schrödinger equation.

### III. THE PAULI AND SCHÖDINGER PROBABILITY CURRENTS

After the change of variable \( \psi = \psi' e^{-\frac{i}{\hbar} mc \omega t} \), the Dirac equation (3) for the bispinor \( \psi = (\chi_1, \chi_2) \) can be written

\[
[i\hbar \frac{\partial}{\partial t} - eV] \chi_1 = c\sigma (-i\hbar \nabla - \frac{e}{c} \mathbf{A}) \chi_2
\] (10)

\[
[i\hbar \frac{\partial}{\partial t} - eV + 2mc^2] \chi_2 = c\sigma (-i\hbar \nabla - \frac{e}{c} \mathbf{A}) \chi_1.
\] (11)

Small electron velocities allow us to neglect the two first terms of the left part of equation (10) and to eliminate \( \chi_2 \) in (10). So we obtain the Pauli equation for the spinor \( \chi_1 \)

\[
i\hbar \frac{\partial \chi_1}{\partial t} = \left[ \frac{1}{2m} (-i\hbar \nabla - \frac{e}{c} \mathbf{A})^2 + eV - \frac{\hbar}{2mc} \sigma \mathbf{H} \right] \chi_1
\]

where \( \mathbf{H} = \nabla \times \mathbf{A} \) is the magnetic field.

As \( \chi_2 \ll \chi_1 \), the Dirac density \( \rho = \psi^* \psi = \chi_1^* \chi_1 + \chi_2^* \chi_2 \) is equal in first approximation to the density \( \rho = \chi_1^* \chi_1 \) of the Pauli equation. Then the Dirac current density can be written

\[
\mathbf{J} = \frac{i}{2m}(\chi_1^* \nabla \chi_1 - \chi_1 \nabla \chi_1^*) - \frac{e}{mc} \mathbf{A} \chi_1 \chi_1^* + \frac{\hbar}{2m} \nabla \times (\chi_1^* \sigma \chi_1)
\] (12)

Thus, to obtain in Pauli a good approximation of the Dirac probability current, it is necessary to add the Gordan current (2)

\[
\mathbf{J}_2 = \frac{\hbar}{2m} \nabla \times (\chi_1^* \sigma \chi_1)
\]

to the classical Pauli current

\[
\mathbf{J}_1 = \frac{i}{2m}(\chi_1^* \nabla \chi_1 - \chi_1 \nabla \chi_1^*) - \frac{e}{mc} \mathbf{A} \chi_1 \chi_1^*.
\]

Then, the current \( \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \) verifies, as the current \( \mathbf{J}_1 \), the continuity equation [5].

To obtain the Schrödinger equation from Pauli equation, consider the case where we have no magnetic field [5] and where the system is in a spin eigenstate

\[
\chi_1(r, t) = \varphi(r, t) \chi
\]

with a constant spinor \( \chi \) such as \( \chi^* \chi = 1 \). Then the function \( \varphi(r, t) \) verifies the Schrödinger equation

\[
\hbar \frac{d\varphi}{dt} = -\frac{\hbar^2}{2m} \Delta \varphi + eV \varphi.
\] (13)
However, in this case, the current (12) does not reduce to classical Schrödinger probability current. Indeed, writing $\varphi(r, t) = \sqrt{\rho} e^{i S \hbar}$, we have

$$J = \frac{\rho}{m} \nabla S + \frac{1}{m} \nabla \rho \times s$$

with

$$s = \frac{\hbar}{2} (\chi^* \sigma \chi).$$

Indeed, to obtain a good approximation of the Dirac probability current for the Schrödinger equation, it is necessary to add the spin-dependent current

$$J_2 = \frac{1}{m} \nabla \rho \times s$$

to the classical Schrödinger current

$$J_1 = \frac{\rho}{m} \nabla S.$$

Therefore the probability current depends on a constant spin vector $s$. The add of a constant spin vector to the Schrödinger equation is not new, and has been used before by Landau [7].

This current $J_2$ gives also a simple explanation of the value 2 of the gyromagnetic ratio of the Dirac electron; Indeed, this contribution to the particle orbital angular momentum is

$$L_2 = r \times m J_2 = r \times \nabla \rho \times s$$

and this mean angular momentum is then

$$< L_2 >= \int L_2 d^3r = \int (r \cdot s) \nabla \rho d^3r - \int (r \cdot \nabla \rho) s d^3r = -s + 3s = 2s.$$

We can verify the precedent theoritical approach with the computation of the current (13) of the hydrogen atom wave eigenfunctions in the approximation of Schrödinger.

For the wave functions $\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) P_{lm}(\theta) \exp(i m \phi) \exp(-i E_n t \hbar)$, the classical Schrödinger current is

$$J_1 = \frac{\rho}{m_e} \frac{m \hbar}{r \sin \theta} \mathbf{u}_\phi$$

with $\rho = R_{nl}^2(r) P_{lm}^2(\theta)$. The current is nil for the wave functions with $m = 0$, in particular for the states $ns$. If $m \geq 0$, we take as spin vector $s = \frac{\hbar}{2} \mathbf{k}$ and $s = -\frac{\hbar}{2} \mathbf{k}$ if $m < 0$. Then, this vector can be written

$$s = sgn(m) (\cos \theta \mathbf{u}_r - \sin \theta \mathbf{u}_\phi)$$

and the additional spin-dependent current (Gordon current) $J_2$ is then equal to

$$J_2 = -sgn(m) \frac{\hbar}{m_e r \sin \theta} (R_{nl} R_{nl}' P_{lm} P_{lm}' \sin \theta + \frac{P_{nl}^2}{r} P_{lm} P_{lm}' \cos \theta) \mathbf{u}_\phi.$$

Then, the probability current velocity is

$$v = \frac{J}{\rho} = \frac{\hbar}{m_e} [-sgn(m) (\frac{R_{nl}'}{R_{nl}}) \sin \theta + \frac{P_{nl}'}{P_{lm}} \cos \theta + \frac{m}{r \sin \theta}] \mathbf{u}_\phi$$

which corresponds to circular probability currents exactly as for the Dirac equation (11). In the case of the fundamental state $1s$, we find again exactly the velocity (11) of the Dirac probability current

$$v_{1s} = \frac{\alpha c}{2} \sin \theta \mathbf{u}_\phi.$$

For the states $2s$, $2p_0$, $2p_1$ and $2p_{-1}$, we obtain for the current velocity

$$v_{2s} = \frac{\alpha c}{2} \frac{1 + \frac{1}{2 r_0}}{1 - \frac{1}{2 r_0}} \sin \theta \mathbf{u}_\phi,$$

$$v_{2p_0} = v_{2p_1} = \frac{\alpha c}{2} \sin \theta \mathbf{u}_\phi \quad \text{and} \quad v_{2p_{-1}} = -\frac{\alpha c}{2} \sin \theta \mathbf{u}_\phi.$$
IV. CONCLUSION

As we have shown, although the Schrödinger equation is a good approximation of the Dirac equation, the Schrödinger probability current \( \mathbf{J}_2 \) is not the good approximation of the Dirac probability current \( \mathbf{J}_1 \). The main conclusion is that it is necessary to add to the classical Schrödinger current the spin-dependent current

\[
\mathbf{J}_2 = \frac{1}{m} \nabla \rho \times \mathbf{s}
\]

which corresponds to a constant spin vector \( \mathbf{s} \).

In the case of the isotrop harmonic oscillator with \( V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2 \), the fundamental wave function \( 1s \)

\[
\psi_{0,0,0}(\mathbf{r}) = \left( \frac{m\omega}{\pi \hbar} \right)^{3/4} \exp(-\frac{m\omega}{2\hbar} r^2) \exp(-\frac{3i\omega t}{2})
\]

gives a Schrödinger classical current \( \mathbf{J}_1 \) nil. If we consider an oscillator with a constant spin \( \mathbf{s} = \frac{\hbar}{2} \mathbf{k} \), then the equation (13) gives a circular probability current \( \mathbf{J} \) with the velocity

\[
\mathbf{v}_1 = \frac{\mathbf{J}}{\rho} = -\omega r \sin \theta \mathbf{u}_\phi.
\] (14)

It is the same velocity as in the classical case. For the other eigenfunctions of the harmonic oscillator, we find also a non nil probability current, but non circular as in (13).

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