Dynamic portfolio selection without risk-free assets

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Abstract. We consider the mean–variance portfolio optimization problem under the game theoretic framework and without risk-free assets. The problem is solved semi-explicitly by applying the extended Hamilton–Jacobi–Bellman equation. Although the coefficient of risk aversion in our model is a constant, the optimal amounts of money invested in each stock still depend on the current wealth in general. The optimal solution is obtained by solving a system of ordinary differential equations whose existence and uniqueness are proved and a numerical algorithm as well as its convergence speed are provided. Different from portfolio selection with risk-free assets, our value function is quadratic in the current wealth, and the equilibrium allocation is linearly sensitive to the initial wealth. Numerical results show that this model performs better than both the classical one and the variance model in a bull market.

Key words: mean–variance portfolio selection, asset allocation, time-inconsistency, equilibrium control, Hamilton–Jacobi–Bellman equation.

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1 Introduction

The portfolio selection problem deals with how to allocate the wealth among a set of assets. In his seminal work, Markowitz (1952, 1959) first proposed the mean–variance (MV) portfolio selection theory in a single period framework, which is regarded as the cornerstone in modern finance. In this pioneering theory, an investor aims at maximizing the mean return and minimizing the variance, which is regarded as the measurement of risk, at the same time. This problem has multiple solutions, which comprise the so-called efficient frontier, since it is a multi-objective optimization problem. In fact, each point on the efficient frontier is the optimal solution for the single-objective optimization problem which is to minimize the corresponding variance subject to a given level of the expected wealth. When short-selling is allowed and the covariance matrix which is composed by the volatilities of the stocks is nonnegative definite, the analytic expression of the mean–variance frontier is derived in Markowitz (1956) and Merton (1972). In the case where the covariance matrix is non-negative definite, Perold (1984) describes an algorithm for solving the MV portfolio selection problem. However, there is a criticism on how the risk is measured in the original MV framework. For the discussion on the replacement of the risk measurement, see Markowitz (1959). Besides the variance of the expected portfolio return, alternative measurements of the risk such as the semi-variance, the lower partial moment and the downside risk are proposed for constructing the optimal portfolio, see Konno and Yamazaki (1991), Markowitz et al. (1993), Zenios and Kang (1993) and Ogryczak and Ruszczyński (1999).

A main challenge for extending the original single period model to the multi-period case is the time inconsistency since the Bellman Optimality Principle is violated. In this case, the optimality of a control depends on both the current and the initial states. The concept of the term “optimality”, as well as “an optimal control law”, is therefore unclear. Technically, we cannot apply the dynamic programming directly to attack this problem.

There are three popular ways for handling a family of the time-inconsistent problems. The first one, known as the “pre-commitment” strategy in the economics literature, seeks a strategy that optimizes the objective function at the initial time. Whether it is optimal for the objective function in the future is disregarded. Here, the interpretation of “optimal” is “optimal from
the point of view of the initial time”. Richardson (1989) and Bajeux-Besnainou and Portait (1998) first develop a continuous-time version of the MV model under the pre-committed setting. Another extension to the multi-period version can be found in Li and Ng (2000). They embed the original time-inconsistent problem into a class of auxiliary stochastic linear-quadratic (LQ) control problems. Using the similar technique, Zhou and Li (2000), Lim and Zhou (2002), Lim (2004), Bielecki et al. (2005), Xia (2005) provide a solution to the continuous-time MV portfolio selection problem. With the regime switching, the MV portfolio selection and asset–liability management problems are studied by Zhou and Yin (2003), Chen et al. (2008) and Chen and Yang (2011). Dai et al. (2010) provide a pre-committed strategy when the transaction cost is taken into account.

The second approach for tackling the time inconsistency is that instead of using strategy that is fixed at the initial time, an investor keeps updating his wealth allocation in order to optimize the corresponding objective function at the current time.

The third approach is to treat the time inconsistency seriously. For this situation, a major challenge is that the dynamic programming approach cannot be applied directly since the iterated-expectations property is violated due to the variance term, which is not a linear function of the expected value of the wealth, involved in the objective function. Some of the early relevant literatures are Strotz (1955) and Pollak (1968). In Strotz (1955), the author demonstrates that if a discount function is applied to consumption plans, a certain plan which is optimal to an investor at the beginning may not be the case in the future. However, in certain cases, the strategies developed in these papers for handling the time inconsistency issue do not exist. See Peleg and Yaari (1973). In Peleg and Yaari (1973), the time-inconsistent problems are treated as noncooperative games and the optimal strategies is described using Nash equilibrium. Within this framework, there are one player at each time point and every player should find his own strategy in order to maximize his objective function. In fact, these players can be viewed as your future incarnations. From this point of view, Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) consider the Merton portfolio management problem in the context of non-constant hyperbolic discounting in deterministic and stochastic models respectively. A precise definition
of the game theoretic equilibrium concept in continuous time is provided in these two papers.

Basak and Chabakauri (2010) consider the dynamic mean–variance portfolio problem in an incomplete-market setting. They derive a recursive formulation for the mean–variance criteria and obtain the closed-form expression for its time consistent strategy via the dynamic programming approach. However, their approach can only be applied to the stochastic control problem with the MV objective function. For a more general class of the time inconsistent objective functions, Björk and Murgoci (2010) and Björk and Murgoci (2014) develop both the discrete-time and continuous-time theories within a game theoretic framework. They derive an extended Hamilton–Jacobi–Bellman (HJB) equation and provide the corresponding verification theorem. As an illustration, besides the MV utility model, the time-inconsistent control problems with non-exponential discounting and with the utility function at the terminal time depends on the current state are also solved using the developed theory. However, Björk et al. (2014) argued that the optimal control developed in Basak and Chabakauri (2010), which can be reproduced using the theory in Björk and Murgoci (2010), is not economically reasonable since it does not depend on the current wealth state. To construct a more realistic model, the authors consider the case in which the risk aversion depends on the current wealth. In particular, if the risk aversion is inversely proportional to the current wealth state, the optimal amount of money invested in the risky asset is proportional to the wealth. Under short-selling prohibition, Bensoussan et al. (2014) study the same problem with the risk aversion being inversely proportional to the current wealth in both discrete and continuous time setting and prove that the optimal control in the discrete time model converges to the one in the continuous setting.

On the other hand, numerical schemes for determining the pre-commitment strategy and the time-consistent strategy of a continuous MV asset allocation problem is proposed by Wang and Forsyth (2010) and Wang and Forsyth (2011). In their algorithms, any type of constraint can be applied to the investment behavior. Wang and Forsyth (2012) then extend the numerical techniques for determining these two policies in the mean quadratic variation problem.

Besides portfolio selection problems, there are other applications of the extend HJB equations developed in Björk and Murgoci (2010) under the mean–variance framework. An equilibrium
control for the asset-liability management problem is derived by Wei et al. (2013). In addition, the optimal time-consistent investment and reinsurance strategies using the game theoretic approach are constructed, see Li and Li (2013), Zeng et al. (2013), Li et al. (2015) and Lin and Qian (2015).

In this paper, we construct the equilibrium control for the MV asset allocation problem with multiple assets. We consider three models and in all of them, the assets an investor can trade are multiple stocks. In model 1, an additional risk-free bond with a constant interest rate is included. The objective functions considered in model 1 and model 3 are the same with the one used in the MV portfolio problem with constant risk aversion while the one in model 2 only includes the variance term. In fact, a risk-free asset can be considered as an asset with zero volatility. From this point of view, model 3 can be regarded as a generalized version of the ones in Björk and Murgoci (2010) and Basak and Chabakauri (2010). As the risk aversion goes to infinity, the equilibrium control derived in model 3 converges to the one in model 2. Furthermore, although the risk aversion considered in this paper is a constant, if the risk-free asset is not available, the optimal amount of money invested in each risky asset still depends on the current wealth, which is unexpected.

The remainder of the paper is organized as follows. In Section 2, we present the formulation of our problem as well as the game theoretic framework. In Section 3, we state the three models with different types of assets and objective functions. For each model, we state the corresponding extended HJB system. With suitable Ansatzs we can solve each system explicitly. For model 3, the existence and the uniqueness of the solution is proved. We also provide a numerical algorithm for computing the solution as well as its convergence speed. Two special cases are also presented in Section 3. In Section 4, some nature parameter combinations in model 3 is provided. Graphical illustrations of the three models are also presented for comparison. Section 5 concludes this paper. The main technical proofs of the proposition and theorems are given in appendices.
2 Problem formulation in a game theoretic framework

Assume that the state $X_t$ (typically the wealth process) at time $t$ is given by a linear stochastic differential equation:

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t$$

where $\mu, \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying suitable conditions such that the stochastic differential equation has a unique solution.

We first recall the problem formulation from Björk and Murgoci (2010). For deterministic functions $F(x, y)$ and $G(x, y)$, we consider a reward function of the form

$$J(t, x, u) = E_{t,x}\{F(X_T^u)\} + G(x, E_{t,x}(X_T^u))$$

where $(t, x)$ is the fixed initial point of time and wealth. It is pointed out in Björk and Murgoci (2010) that the optimization problem for maximizing this reward function does not satisfy the Bellman optimality principle due to the dependence on initial state and the appearance of the second term which is a nonlinear function of the expectation and thus is a time-inconsistent problem. Dynamic programming is therefore not available for solving this problem.

We can formulate the problem in the game theoretic framework established in Björk and Murgoci (2010) and construct a time-consistent optimal strategy rather than a precommitted one. Within this framework, the optimization problem is treated as a non-cooperate game and at each point of time $t$, there is a player $t$ which can be regarded as an incarnation of the investor. Then the optimal time consistent strategy $\hat{u}$ is defined as: for an arbitrary time point $t$, the optimal strategy for player $t$ is $\hat{u}(t, \cdot)$ suppose that each player $s$ where $s > t$ uses the strategy $\hat{u}(s, \cdot)$.

We now provide a formal definition of equilibrium control adopted in this paper. This definition is given by Björk and Murgoci (2010).

**Definition 1.** (Equilibrium Control Law). An admissible control law $\hat{u}$ is called equilibrium
control if for every admissible control law \( u \) valued in \( \mathbb{R} \) and \( h > 0 \),

\[
u_h(s, y) = \begin{cases} 
u, & \text{for } t \leq s < t + h, \quad y \in \mathbb{R}^n \\ \hat{u}, & \text{for } t + h \leq s \leq T, \quad y \in \mathbb{R}^n, \end{cases}
\]

such that

\[
\liminf_{h \to 0^+} \frac{J(t, x, \hat{u}) - J(t, x, u_h)}{h} \geq 0
\]

for any \((t, x) \in [0, T] \times \mathbb{R}\). The equilibrium value function \( V \) is defined as

\[
V(t, x) = J(t, x, \hat{u}).
\]

For a control law \( u \), we first define an infinitesimal operator \( A^u \):

\[
A^u = \frac{\partial}{\partial t} + \mu(x, u(t, x)) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(x, u(t, x)) \frac{\partial^2}{\partial x^2}.
\]

**Definition 2.** (Extended HJB Equation). For the Nash equilibrium problem, the extended HJB system of equations for \( J \) is

\[
\begin{align*}
\sup_{u \in \mathcal{U}} \{(A^u V)(t, x) - (A^u f)(t, x, x) + (A^u f^x)(t, x) \\
-A^u (G \circ g)(t, x) + G_y(x, g(t, x)) \cdot A^u g(t, x)\} & = 0, \quad 0 \leq t \leq T, \\
A^\hat{u} f^y(t, x) & = 0, \quad 0 \leq t \leq T, \\
A^\hat{u} g(t, x) & = 0, \quad 0 \leq t \leq T, \\
V(T, x) & = F(x, x) + G(x, x), \\
f(T, x, y) & = F(y, x), \\
g(T, x) & = x.
\end{align*}
\]

In this HJB system, \( \hat{u} \) is the optimal control law for the first equation. The notations \( G_y, f^y \)
and \( G \circ g \) are defined as

\[
G_y(x, y) = \frac{\partial}{\partial y} G(x, y),
\]

\[
f^y(t, x) = f(t, x, y),
\]

\[
(G \circ g)(t, x) = G(x, g(t, x)).
\]

For functions \( f \) and \( g \), we have the following probabilistic interpretations:

\[
f(t, x, y) = E_{t,x}\{F(y, X^u_t)\}, \tag{2.1}
\]

\[
g(t, x) = E_{t,x}(X^u_t). \tag{2.2}
\]

**Theorem 1.** (Verification Theorem). Suppose that \((V, f, g)\) is a solution of the HJB system defined in Definition 2 and the supremum in the first equation is attained at \( \hat{u}(t, x) \). Then \( \hat{u} \) is an equilibrium control law and \( V(t, x) \) is the corresponding value function. In addition, \( f \) and \( g \) allow for the probabilistic interpretations (2.1) and (2.2).

The proof of the Verification Theorem can be found in Section 3 of Björk and Murgoci (2010). This theorem states that the solution of the extended HJB system yields the optimal control and the value function of the original stochastic control problem.

### 3 Portfolio selection

We consider the dynamics for a bank account \( B \) and the prices of risky stocks \( S_i \):

\[
\begin{align*}
    dB_t &= r B_t dt \\
    dS_{it} &= \alpha_i S_{it} dt + \sum_{j=1}^d \sigma_{ij} S_{it} dW_{jt}, \quad i = 1, \ldots, n,
\end{align*}
\]

where \( r \) is the risk-free rate of a bank account \( B \), \( \alpha_i \) is the mean return of stock \( i \) and \( \sigma_{ij} \) is the volatility amount of stock \( i \) affected by risk source \( j \). Denote \( u_i \) to be the dollar amount invested in the \( i \)-th stock and \( u_t = (u_{1t}, \ldots, u_{nt}) \) to be the corresponding vector. Two scenarios
are considered in this section: the first one is to include a risk-free asset and multiple risky assets in our portfolio and the other one is to include only multiple risky assets in our portfolio. With different objective functions, we are going to derive the corresponding optimal dollar amount $\hat{u}_t$.

If the Brownian motions are correlated such that $\mathbb{E}(dW_{it}dW_{jt}) = \rho_{ij}dt$ and $\rho_{ii} = 1$, then the covariance between stock prices $S_{it}$ and $S_{jt}$ is

$$\text{Cov}(S_{it}, S_{jt}) = S_{i0}S_{j0}e^{(\alpha_i + \alpha_j)t}\left(e^{\sum_{k=1}^{d}\tilde{\sigma}_{ik}\tilde{\sigma}_{jk}t} - 1\right),$$

where $\tilde{\sigma}_{i1} = \sum_{k=1}^{d}\rho_{1k}\sigma_{ik}$ and $\tilde{\sigma}_{il} = \sqrt{1 - \rho_{il}^2}\sigma_{il}$ for $l = 2, \ldots, d$.

For simplicity, in the models we are going to discuss, we consider the case with $n = 2$ and $d = 2$, i.e.:

$$
\begin{align*}
    dS_{1t} &= \alpha_1 S_{1t}dt + \sigma_{11} S_{1t}dW_{1t} + \sigma_{12} S_{1t}dW_{2t}, \\
    dS_{2t} &= \alpha_2 S_{2t}dt + \sigma_{21} S_{2t}dW_{1t} + \sigma_{22} S_{2t}dW_{2t},
\end{align*}
$$

(3.1)

In addition, for two Brownian motions $W_{it}$ and $W_{jt}$ with correlation coefficient $\rho_{ij}$, we can make the transformation

$$
\begin{align*}
    W_{it} &= \tilde{W}_{it}, \\
    W_{jt} &= \rho_{ij} \tilde{W}_{it} + \sqrt{1 - \rho_{ij}^2} \tilde{W}_{jt},
\end{align*}
$$

where $\mathbb{E}(d\tilde{W}_{it}d\tilde{W}_{jt}) = 0$. Therefore, we can assume that the correlation coefficient between two different Brownian motions is zero, i.e., $\rho_{ij} = 0$ for $i \neq j$ and $\rho_{ii} = 1$.

### 3.1 Revisit the mean–variance optimization with risk-free asset

In this section, we revisit the model from Basak and Chabakauri (2010) in a framework of complete market and do some special analysis which will be easily compared with the next two models. In our model 1, a risk-free asset and two risky assets are included in the portfolio. The
objective function is

\[ J(t, x, u) = E_{t,x} (X_T^u) - \frac{\gamma}{2} \text{Var}_{t,x} (X_T^u) \]

where \( \gamma \in \mathbb{R} \) is the risk aversion coefficient. Denote \( \sigma_i = (\sigma_{i1}, \sigma_{i2})^T \) for \( i = 1, 2 \), \( \sigma = (\sigma_1, \sigma_2)^T \) and \( W_t = (W_{1t}, W_{2t})^T \). Let \( 1 = (1, 1)^T \). The dynamic of the investor’s wealth is

\[ dX_t^u = \{ rX_t^u + (\alpha - r1)^T u_t \} dt + u_t^T \sigma dW_t \]

where \( u_t = (u_{1t}, u_{2t})^T \) is the vector of dollar investments in the two stocks at time \( t \).

In this case, the functions \( F(x) = x - \frac{\gamma}{2} x^2 \) and \( G(x) = \frac{\gamma}{2} x^2 \). The corresponding extended HJB equation is given by

\[
\begin{align*}
V_t + \sup_{u \in \mathbb{R}^2} \left\{ r x + (\alpha - r1)^T u \right\} V_x + \frac{1}{2} \left( V_{xx} - \gamma g_x^2 \right) u^T \sigma \sigma^T u &= 0, \\
g_t + \left\{ r x + (\alpha - r1)^T u \right\} g_x + \frac{1}{2} g_{xx} u^T \sigma \sigma^T u &= 0, \\
V(T, x) &= x, \\
g(T, x) &= x.
\end{align*}
\]

Assuming the \( 2 \times 2 \) matrix \( \sigma \) is invertible, By Section 3.2 in Basak and Chabakauri (2010), we have the solutions for equation (3.2):

\[
\begin{align*}
V(t, x) &= e^{r(T-t)} x + \frac{1}{2\gamma} (\alpha - r1)^T (\sigma \sigma^T)^{-1} (\alpha - r1)(T - t), \\
g(t, x) &= e^{r(T-t)} x + \frac{1}{\gamma} (\alpha - r1)^T (\sigma \sigma^T)^{-1} (\alpha - r1)(T - t),
\end{align*}
\]

and the equilibrium control is

\[
\hat{u}(t, x) = \frac{1}{\gamma} e^{-r(T-t)} (\sigma \sigma^T)^{-1} (\alpha - r1).
\]

The numbers of risk sources and risky assets affect the equilibrium drastically, as we will see in the following analysis.
One Brownian motion case:

Suppose that there is only one random factor affecting the market. Without loss of generality, let $\sigma_{12} = \sigma_{22} = 0$ in (3.1). The wealth process can be written as

$$dX_t^u = \{X_t^u r + (\alpha_1 - r)u_1 + (\alpha_2 - r)u_2\} dt + (\sigma_{11}u_1 + \sigma_{21}u_2)dW_t$$

(3.3)

where $W_t$ is a 1-dim Brownian motion. From the first equation of (3.2) and the first order condition, we have

$$(\sigma^T \sigma)u = \frac{V_x}{\gamma g_x^2 - V_{xx}}(\alpha - r1),$$

or precisely,

$$\begin{cases} 
\sigma_{11}u_1 + \sigma_{21}u_2 = \frac{\alpha_1 - r}{\sigma_{11}} \frac{V_x}{\gamma g_x^2 - V_{xx}}, \\
\sigma_{11}u_1 + \sigma_{21}u_2 = \frac{\alpha_2 - r}{\sigma_{21}} \frac{V_x}{\gamma g_x^2 - V_{xx}}.
\end{cases}$$

(3.4)

(i) If the market prices of risk of two stocks are equal, i.e., $(\alpha_1 - r)/\sigma_{11} = (\alpha_2 - r)/\sigma_{21}$, the optimal amounts of money $\hat{u}_1$ and $\hat{u}_2$ only satisfy

$$\sigma_{11}\hat{u}_1 + \sigma_{21}\hat{u}_2 = \frac{\alpha_1 - r}{\sigma_{11}} \frac{V_x}{\gamma g_x^2 - V_{xx}},$$

which are not unique. With this relationship, we can obtain the solution for (3.2):

$$V(t,x) = e^{r(T-t)}x + \frac{1}{2\gamma} \left( \frac{\alpha_1 - r}{\sigma_1} \right)^2 (T - t),$$

$$g(t,x) = e^{r(T-t)}x + \frac{1}{\gamma} \left( \frac{\alpha_1 - r}{\sigma_1} \right)^2 (T - t).$$

The corresponding linear combination of the optimal amounts of money is thus

$$\hat{u}_1\sigma_{11} + \hat{u}_2\sigma_{21} = \frac{1}{\gamma} \frac{\alpha_1 - r}{\sigma_{11}} e^{-r(T-t)}.$$

This is consistent with financial intuition, because with the same price of market risk, it
does not matter to buy one of them more or less.

(ii) If \((\alpha_1 - r)/\sigma_{11} \neq (\alpha_2 - r)/\sigma_{21}\), since \(V_x \neq 0\), there exists no solution for (3.4).\(^1\) We have two prices of market risk. It is clear to see the case when \(\sigma_{11} = \sigma_{21}\) but \(\alpha_1 > \alpha_2\). This obviously implies arbitrage. Intuitively in this circumstance, one should buy the stock with higher price of market risk, i.e., buy stock 1 as many as he can.

**Two Brownian motions case:**

If we still have two risky assets but the uncertainties of the prices of these two assets are decided by two independent Brownian motions, i.e., \(\sigma_{12} = \sigma_{21} = 0\) in (3.1). Without loss generosity, we further assume that \(\sigma_{11} = \sigma_{22} = \sigma\), i.e., the volatilities of two stocks are the same.

(i) If \(\alpha_1 = \alpha_2 = \alpha\), then
\[
\hat{u}_1(t, x) = \hat{u}_2(t, x) = \frac{1}{\gamma} e^{-r(T-t)} \frac{\alpha - r}{\sigma^2},
\]
i.e., they are the same with the optimal one derived for the situation when only one stock is available. Note that here the two stocks are not exactly the same with each other because they are randomized by two independent Brownian motions, which demonstrates that an investor’s decision will be affected by the appearance parameters \(\alpha\) and \(\sigma\).

(ii) If \(\alpha_1 \neq \alpha_2\), then
\[
\hat{u}_i(t, x) = \frac{1}{\gamma} e^{-r(T-t)} \frac{\alpha_i - r}{\sigma^2}, \quad i = 1, 2,
\]
i.e., if the two stocks have the same volatility, the money invested on each stock is positive-related to its appreciation rate.

\(^1\)If \(V_x \equiv 0\), the first equation of the HJB system (3.2) becomes \(V_t \equiv 0\) and thus \(V(t, x)\) is a constant for all \(t \in [0, T]\) and \(x \in R\). This contradicts with the boundary condition \(V(T, x) = x\).
3.2 The variance model with only two stocks

In this model, we consider that a risk-loving investor who will not put money into bank account and that the variance as the objective function

\[ J(t, x, u) = -\frac{\gamma}{2} \text{Var}_{t,x}(X_t^u). \]

Assume there are only two stocks available, we denote the amount of money invested in stock 1 at time \( t \) to be \( u_t \). The amount of money invested in stock 2 is thus \( X_t - u_t \) and the dynamic of the value process of the portfolio is

\[
dX_t^u = \{u_t \alpha_1 + (X_t^u - u_t) \alpha_2\} dt + \{u_t \sigma_{11} + (X_t^u - u_t) \sigma_{21}\} dW_{1t} + \{u_t \sigma_{12} + (X_t^u - u_t) \sigma_{22}\} dW_{2t}.
\]

The corresponding extended HJB equation is given by

\[
V_t + \sup_{u \in U} \left( \left\{ x \alpha_2 + (\alpha_1 - \alpha_2)u \right\} V_x + \frac{1}{2} \left[ \left\{ x \sigma_{21} + (\sigma_{11} - \sigma_{21})u \right\}^2 \right] \right) = 0,
\]

\[
g_t + \left\{ x \alpha_2 + (\alpha_1 - \alpha_2)u \right\} g_x + \frac{1}{2} \left[ \left\{ x \sigma_{22} + (\sigma_{12} - \sigma_{22})u \right\}^2 \right] g_{xx} = 0,
\]

\[
V(T, x) = 0, \quad g(T, x) = x.
\]

For the optimal solution \( \hat{u} \), we make the Ansatz:

\[
\hat{u}(t, x) = k(t)x. \quad (3.7)
\]

As to why make this Ansatz, we have first tried the form \( \hat{u}(t, x) = k(t)x + c(t) \) which leads to a contradiction when putting \( g(t, x) = a(t)x, V(t, x) = A(t)x \) into the HJB system (3.6).
By substituting equation (3.7) into equation (3.5), we obtain the wealth process:

\[ dX_t^\mu = \{\alpha_2 + (\alpha_1 - \alpha_2)k(t)\}X_t^\mu dt + \{\sigma_2 + (\sigma_1 - \sigma_2)\}X_t^\mu dW_t + \{\sigma_2 + (\sigma_1 - \sigma_2)\}X_t^\mu dW_{2t}. \]

From this equation, we can obtain the expected values:

\[
E_t, x(X_T^\mu) = e^{\int_t^T \{\alpha_2 + k(s)(\alpha_1 - \alpha_2)\} ds} x, \\
E_t, x\{ (X_T^\mu)^2 \} = e^{2 \int_t^T \{\alpha_2 + k(s)(\alpha_1 - \alpha_2) + 0.5(\sigma_2 + k(s)(\sigma_1 - \sigma_2))^2 + 0.5(\sigma_2 + k(s)(\sigma_1 - \sigma_2))^2 \} ds} x^2.
\]

Therefore, the conditional variance of wealth is

\[
\text{Var}_{t,x}(X_T^\mu) = E_t, x\{ (X_T^\mu)^2 \} - \{ E_t, x(X_T^\mu) \}^2 \\
= e^{2 \int_t^T \{\alpha_2 + k(s)(\alpha_1 - \alpha_2)\} ds} \left( 1 - e^{2 \int_t^T \{\sigma_2 + k(s)(\sigma_1 - \sigma_2)^2 + 0.5(\sigma_2 + k(s)(\sigma_1 - \sigma_2))^2 \} ds} \right) x^2.
\]

The solution for equation (3.6) is given by

\[
g(t, x) = E_t, x(X_T^\mu) \\
= e^{\int_t^T \{\alpha_2 + k(s)(\alpha_1 - \alpha_2)\} ds} x, \\
V(t, x) = -\frac{\gamma}{2} \text{Var}_{t,x}(X_T^\mu) \\
= \frac{\gamma}{2} e^{2 \int_t^T \{\alpha_2 + k(s)(\alpha_1 - \alpha_2)\} ds} \left( 1 - e^{2 \int_t^T \{\sigma_2 + k(s)(\sigma_1 - \sigma_2)^2 + 0.5(\sigma_2 + k(s)(\sigma_1 - \sigma_2))^2 \} ds} \right) x^2.
\]

Denote

\[
a(t) = e^{\int_t^T \{\alpha_2 + k(s)(\alpha_1 - \alpha_2)\} ds}, \\
A(t) = \frac{\gamma}{2} e^{2 \int_t^T \{\alpha_2 + k(s)(\alpha_1 - \alpha_2)\} ds} \left( 1 - e^{2 \int_t^T \{\sigma_2 + k(s)(\sigma_1 - \sigma_2)^2 + 0.5(\sigma_2 + k(s)(\sigma_1 - \sigma_2))^2 \} ds} \right),
\]

then \( V(t, x) = A(t)x^2 \) and \( g(t, x) = a(t)x \).

**Theorem 2.** Assume the volatilities of the two stocks are not exactly the same, i.e., \((\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2 \neq 0\), from HJB equation (3.6) and the first order condition, we obtain the
expression of the optimal allocation

\[ \hat{u}(t, x) = \frac{(\alpha_2 - \alpha_1)V_x - \{\sigma_21(\sigma_11 - \sigma_21) + \sigma_22(\sigma_12 - \sigma_22)\}(V_{xx} - \gamma^2 g_x^2)x}{\{(\sigma_11 - \sigma_21)^2 + (\sigma_12 - \sigma_22)^2\}(V_{xx} - \gamma^2 g_x^2)} \]

\[ = \frac{2(\alpha_2 - \alpha_1)A(t) - \{\sigma_21(\sigma_11 - \sigma_21) + \sigma_22(\sigma_12 - \sigma_22)\}\{2A(t) - \gamma a^2(t)\}}{\{(\sigma_11 - \sigma_21)^2 + (\sigma_12 - \sigma_22)^2\}\{2A(t) - \gamma a^2(t)\}}x \]

\[ = k(t)x, \]

where \( k(\cdot) \) satisfies the following ordinary differential equation (ODE):\(^3\)

\[ k(t) = \frac{1}{(\sigma_11 - \sigma_21)^2 + (\sigma_12 - \sigma_22)^2} \left\{ (\alpha_1 - \alpha_2) \left( e^{-\int_t^T [(\sigma_21 + k(s)(\sigma_11 - \sigma_21))^2 + (\sigma_22 + k(s)(\sigma_12 - \sigma_22))^2]ds} - 1 \right) - \sigma_21(\sigma_11 - \sigma_21) - \sigma_22(\sigma_12 - \sigma_22) \right\}. \quad (3.8) \]

In the case where the first asset is a stock with \( \alpha_1 = \alpha, \sigma_11 = \sigma > 0, \sigma_12 = 0 \) and the second asset degenerates to risk-free asset with \( \alpha_2 = r, \sigma_21 = \sigma_22 = 0 \), this equation has a unique solution \( k(t) \equiv 0 \) for all \( t \in [0, T] \), i.e., we do not invest any money into the risky asset. This is reasonable because in this model, the only consideration for an investor to construct his portfolio is to minimize his risk, therefore he invests all of his wealth into the risk-free asset in order to avoid taking any risk.

**One Brownian motion case**

Suppose that there is only one random factor affecting the market. Without loss of generality, let \( \sigma_12 = \sigma_22 = 0 \). By applying the first order condition to the first equation of the HJB system (3.6), we have:

\[ \{(\sigma_11 - \sigma_21)^2 + (\sigma_12 - \sigma_22)^2\}u = (\alpha_1 - \alpha_2) \frac{V_x}{\gamma^2 g_x^2 - V_{xx}} - \{\sigma_21(\sigma_11 - \sigma_21) + \sigma_22(\sigma_12 - \sigma_22)\}x. \quad (3.9) \]

\(^3\)The proof of the uniqueness and existence of a solution to (3.8) is the same as that for the ODE of \( k_1(t) \), which is given in Appendix B.
(i) If $\sigma_{11} = \sigma_{21}$ and $\alpha_1 = \alpha_2$, (3.9) is always true no matter what real number $u$ takes as the two sides of this equation will always be 0. In fact, these two stocks are “the same”.

(ii) If $\sigma_{11} = \sigma_{21}$ but $\alpha_1 \neq \alpha_2$, since $V_x \neq 0$, no solution exists for (3.9).

Two Brownian motions case:

If uncertainties of the prices of the two assets are decided by two independent Brownian motions with the same volatility amount, i.e., $\sigma_{11} = \sigma_{22} = \sigma$ and $\sigma_{12} = \sigma_{21} = 0$, the optimal allocation is $\hat{u}(t, x) = k(t)x$ where

$$k(t) = \frac{\alpha_1 - \alpha_2}{2\sigma_{11}^2} \left[ e^{-\int_t^T \sigma_{11}^2 (2k^2(s) - 2k(s) + 1) ds} - 1 \right] + \frac{1}{2}.$$ 

At time $t \in [0, T]$, the expectation of the wealth at the end of the time period is given by

$$E_{t,x}(X^u_T) = xe^{\int_t^T (\alpha_2 + k(s)(\alpha_1 - \alpha_2)) ds},$$

while its variance is

$$\text{Var}_{t,x}(X^u_T) = x^2 e^{2\int_t^T (\alpha_2 + k(s)(\alpha_1 - \alpha_2)) ds} \left[ e^{\int_t^T \sigma_{11}^2 (2k^2(s) - 2k(s) + 1) ds} - 1 \right].$$

(i) If $\alpha_1 = \alpha_2 = \alpha$, then

$$\hat{u}_1(t, x) = \hat{u}_2(t, x) = \frac{1}{2} x,$$

i.e., the amounts of money invested on two stocks are both a half at any time $t$ and without short-selling. This is reasonable from the point of finance: there is no rank between the two random factors (Brownian motions), and the two stocks perform at the same level ($\sigma_{11} = \sigma_{22}$, $\alpha_1 = \alpha_2$). So there is no reason to put more emphasis on one stock. But things become different for the next two cases.

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4By substituting $V_x \equiv 0$ into the first equation of the HJB system (3.6), we obtain $V_t = \frac{1}{2} \gamma \sigma_{11}^2 x^2 g_x^2$ for any function $u$. Therefore, we can set $u = 0$ in the second equation of (3.6) and the corresponding solution is $g(t, x) = xe^{\alpha_2(T-t)}$. With this, we have $V_t = \frac{1}{2} \gamma \sigma_{11}^2 e^{2\alpha_2(T-t)} x^2$ which leads to a contradiction with $V_x \equiv 0$. 

16
(ii) if $\alpha_1 > \alpha_2$, then
\[ \hat{u}_1(t, x) < \frac{1}{2} x < \hat{u}_2(t, x). \]
(iii) if $\alpha_1 < \alpha_2$, then
\[ \hat{u}_1(t, x) > \frac{1}{2} x > \hat{u}_2(t, x). \]

From (ii) and (iii), we can see that under the same volatility, the amount of money invested into the stock with higher appreciation rate is less than that invested into the stock with lower appreciation rate. This leads to no contradiction under the criteria of variance because a large $\alpha$ yields a large variance.

### 3.3 Mean–variance criteria without bank account

For the third model, as in model 2, the portfolio only includes two risky assets. However, the objective function is the same with model 1, i.e.,

\[ J(t, x, u) = E_{t,x}(X_T^u) - \frac{\gamma}{2} \text{Var}_{t,x}(X_T^u). \]

Given the amount of money invested in stock 1 $u(t, x)$, the dynamic of wealth is the same with the one in model 2, i.e., equation (3.5). The corresponding extended HJB equation is given by

\[ V_t + \sup_{u \in U} \left( \{x\alpha_2 + u(\alpha_1 - \alpha_2)\} V_x + \frac{1}{2} \left\{x\sigma_{21} + u(\sigma_{11} - \sigma_{21})\right\}^2 (V_{xx} - \gamma g_x^2) \right) = 0, \]

\[ g_t + \{x\alpha_2 + u(\alpha_1 - \alpha_2)\} g_x + \frac{1}{2} \left\{x\sigma_{21} + u(\sigma_{11} - \sigma_{21})\right\}^2 g_{xx} = 0, \]

\[ V(T, x) = x, \]

\[ g(T, x) = x. \]
The only difference between (3.6) and (3.12) is that $V(T, x)$ takes different value. Suppose the optimal allocation for this problem is in the form of \(^5\)

$$\hat{u}(t, x) = k_1(t)x + k_2(t).$$  \(3.13\)

**Theorem 3.** Assume that $(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22}) \neq 0$. The optimal allocation is \(\hat{u}(t, x) = k_1(t)x + k_2(t)\) where $k_1$ and $k_2$ satisfy the ODE system:

$$k_1(t) = \frac{1}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left\{ \alpha_1 - \alpha_2 \left( e^{-\int_t^T \left[ (\sigma_{21} + k_1(s)(\sigma_{11} - \sigma_{21}))^2 + (\sigma_{22} + k_1(s)(\sigma_{12} - \sigma_{22}))^2 \right] ds} - 1 \right) ight. \right.$$ 

$$\left. \left. - \sigma_{21}(\sigma_{11} - \sigma_{21}) - \sigma_{22}(\sigma_{12} - \sigma_{22}) \right\} \right),$$  \(3.14\)

$$k_2(t) = \frac{\alpha_1 - \alpha_2}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left\{ \frac{1}{\gamma} I_1(t, T) I_2(t, T) + \int_t^T I_1(t, v) I_3(t, v) k_2(v) dv \right\},$$  \(3.15\)

where

$$I_1(t, v) = e^{-\int_t^v (\alpha_2 + k_1(s)(\alpha_1 - \alpha_2)) ds},$$

$$I_2(t, v) = e^{-\int_t^v \left[ (\sigma_{21} + k_1(s)(\sigma_{11} - \sigma_{21}))^2 + (\sigma_{22} + k_1(s)(\sigma_{12} - \sigma_{22}))^2 \right] ds},$$

$$I_3(t, v) = (\alpha_1 - \alpha_2) I_2(t, T) - [(\alpha_1 - \alpha_2) + (\sigma_{11} - \sigma_{21})(\sigma_{21} + k_1(v)(\sigma_{11} - \sigma_{21}))]
$$

$$+ (\sigma_{12} - \sigma_{22})(\sigma_{22} + k_1(v)(\sigma_{12} - \sigma_{22})) I_2(t, v).$$

**Proof.** The proof of Theorem 3 is quite tedious. We put it in Appendix A.

**Proposition 1.** The ODE system (3.14) and (3.15) admits a unique solution $(k_1(t), k_2(t))^T$ where $k_1, k_2 \in C[0, T]$. **Proof.** See Appendix B.

We now make some comments and analysis.

---

\(^5\)The procedure for solving this HJB system heavily depends on the *Ansatz* of $\hat{u}(t, x)$. We tried this general linear form and as we will see later, neither $k_1$ nor $k_2$ equals to zero in this model.
Different from the equilibrium control law \( \dot{u}(t,x) = c(t)x \) in Björk et al. (2014) where \( c(t) \) is a function of \( \gamma \), the corresponding part which associates with the current wealth state, \( k_1(t) \), in the equilibrium control law of model 3 does not depend on \( \gamma \).

Suppose that \( (\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2 \) is strictly positive,

1. If \( \alpha_1 > (\leqslant) \alpha_2 \), the values of \( k_2(t) \) and \( \dot{u}(t,x) \) decrease (increase) when \( \gamma \) increases.

2. If \( \alpha_1 = \alpha_2 \), then \( \forall t \in [0,T] \), we have

\[
k_1(t) \equiv -\frac{1}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left\{ \sigma_{21} (\sigma_{11} - \sigma_{21}) + \sigma_{22} (\sigma_{12} - \sigma_{22}) \right\}, \tag{3.16}
k_2(t) \equiv 0,
\]

i.e., \( k_1(t) \) and \( k_2(t) \) are constants with respect to \( t \). In this case, \( \dot{u}(t,x) = k_1(t)x \) is directly proportional to \( x \) with a proportionality constant that does not change with time \( t \). If we further assume \( \sigma_{12} = \sigma_{22} = 0 \), the constant \( k_1(t) \) is larger than one and thus \( \dot{u}(t,x) > x \) when \( \sigma_{11} < \sigma_{21} \). In this case, we long stock 1 and short stock 2. On the other hand, if \( \sigma_{11} > \sigma_{21} \), we have \( \dot{u}(t,x) < 0 \) and this indicates that we short stock 1 and long stock 2.

Furthermore, by differentiating \( k_1(t) \) in equation (3.14) with respect to \( t \), we have

\[
\frac{d}{dt} k_1(t) = \frac{\alpha_1 - \alpha_2}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} e^{-\int_t^T \left[ \{\sigma_{21} + k_1(s)(\sigma_{11} - \sigma_{21})\}^2 + \{\sigma_{22} + k_1(s)(\sigma_{12} - \sigma_{22})\}^2 \right] ds}
\times \left[ \{\sigma_{21} + k_1(t)(\sigma_{11} - \sigma_{21})\}^2 + \{\sigma_{22} + k_1(t)(\sigma_{12} - \sigma_{22})\}^2 \right].
\]

Therefore, if \( \alpha_1 > (\leqslant) \alpha_2 \), then \( dk_1(t)/dt > (\leqslant)0 \) and thus the value of \( k_1(t) \) increases (decreases) as \( t \) increases;

Notice that for this model, if we let \( \sigma_{12} = \sigma_{21} = \sigma_{22} = 0 \), then the uncertainty of the price of the first stock is controlled by only one Brownian motion and the second stock becomes a riskless asset. In such case, model 3 is identical to the one considered in Section
6.1 of Björk and Murgoci (2010). The optimal allocation \( \hat{u}(t, x) \), the expected value of the optimal portfolio \( E_{t,x}(X^\hat{u}_T) \) and the equilibrium value function \( V(t, x) \) obtained using equation (3.13), (A.3) and (A.4) coincide with those in Section 6.1 of Björk and Murgoci (2010).

**One Brownian motion case:**

For the case where there is only one Brownian motion, i.e., \( \sigma_{12} = \sigma_{22} = 0 \) and \( \sigma_{11} = \sigma_{21} > 0 \). Since the first equation of the HJB system (3.12) is the same with the one in (3.6). Therefore, from the first order condition, we have (3.9).

(i) If \( \alpha_1 = \alpha_2 \), we have the same conclusion with the one made in model 2.

(ii) If \( \alpha_1 \neq \alpha_2 \), since \( V_x \neq 0 \) with the similar deduction in model 2, therefore, no optimal solution exists for (3.12).

**Two Brownian motions case:**

Suppose there are two independent Brownian motions, i.e., \( \sigma_{12} = \sigma_{21} = 0 \). In addition, we assume \( \sigma_{11} = \sigma_{22} > 0 \).

(i) If \( \alpha_1 = \alpha_2 = \alpha \), then
\[
\hat{u}(t, x) = \frac{1}{2} x,
\]
i.e., the amount of money invested on each stock is the same at any time \( t \) which coincides with the same case in model 2. With this optimal allocation, the expectation and variance of the wealth \( X_T \) are (3.10) and (3.11). The reward function can be written accordingly.

(ii) If \( \alpha_1 \neq \alpha_2 \), then no conclusion can be made yet since the explicit solution of \( \hat{u}(t, x) = k_1(t)x + k_2(t) \) cannot be obtained. Numerical analysis is required for studying the behavior of \( k_1(t, x) \) and \( k_2(t, x) \). As a demonstration here, we calculate \( k_1(t) \) and \( k_2(t) \) for several combinations of \( T \) and \( \gamma \). The parameters are given by: \( \alpha_1 = 0.2, \alpha_2 = 0.12, \sigma_{11} = \sigma_{22} = 0.25, \sigma_{12} = \sigma_{21} = 0 \). The results are shown in Figure 1. From **Figure 1**, we have the following observations:
For $k_1(t)$, we have the same conclusion as the one obtained in model 2, i.e., $k_1(t) < 0.5$ if $\alpha_1 > \alpha_2$. However, since the first stock has a higher appreciation rate, when taking the expected wealth at time $T$ into consideration in our objective function, an additional positive amount of money $k_2(t)$ ($k_2(t) \equiv 0$ in model 2) is required to invest on stock 1 at any time $t \in [0, T]$. This can be viewed as a tradeoff between maximizing the expected wealth and minimizing the corresponding variance.

As $\gamma$ increases from 1 to 3, the value of $k_2(t)$ decreases drastically. This is expected as $\gamma$ increases, the solution of model 3 will converge to that of model 2. However, since $k_1(t)$ is not a function of $\gamma$, as illustrated in (3.14), $k_1(t)$ does not change with $\gamma$.

Now consider the case where $T$ increase while $\gamma$ is fixed. For $\gamma = 1$, $k_1(T) = 0.5$ and $k_2(T) = 0.64$ for both $T = 1$ and $T = 3$. For $\gamma = 3$, $k_1(T) = 0.5$ and $k_2(T) = 0.2133$ for both $T = 1$ and $T = 3$. As can be seen in Figure 1, both $k_1(t)$ and $k_2(t)$ decreases as $T$ increases from 1 to 3. When approaching the expiry date, since the investment time horizon is becoming shorter, an investor will put more and more money on the
stock with higher appreciation rate in order to have a higher expected wealth on the expiry date.

- Suppose that for all $t \in [0, T]$, $X_t \equiv 1$, i.e., the initial wealth at any time $t$ is 1. The amount of money invested in stock 1 at time $t$ is thus $\hat{u}_1(t, x) = k_1(t) + k_2(t)$. It is observed that $\hat{u}_1(t, 1)$ is always larger than 0.5. Therefore, we invest more than a half of our wealth into stock 1. Furthermore, the total amount of money invested in stock 1 is larger than 1 for all $t \in [0, T]$ when $T = 1$ and $\gamma = 1$. For $T = 3$ and $\gamma = 1$, this also happens when approaching the expiry date. In this case, one holds a long position of stock 1 and a short position of stock 2. However, as the investor becomes more and more risk averse, the amount of money invested in stock 1 reduces and is less than 1 when $\gamma = 3$. In this circumstance, an investor holds long positions of both stock 1 and stock 2.

### 3.4 A numerical algorithm for $k_1$ and $k_2$

The algorithm is an analog of Björk et al. (2014) in which a 1-dim ODE is dealt with.

**Theorem 4.** Suppose the sequence $\{k_1^{(n)}\}$ is constructed by

\[
\begin{align*}
  k_1^{(0)}(t) & = 1, \\
  k_1^{(n)}(t) & = \frac{1}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left\{ -\sigma_{21}(\sigma_{11} - \sigma_{21}) - \sigma_{22}(\sigma_{12} - \sigma_{22}) \\
  & \quad + (\alpha_1 - \alpha_2) \left( e^{-\int_{t}^{T} \left\{ (\sigma_{21} + k_1^{(n-1)})(\sigma_{11} - \sigma_{21}) \right\} ds} - 1 \right) \right\} ,
\end{align*}
\]

for $n = 1, \ldots$. Then we have

\[
|k_1^{(n)}(t) - k_1(t)| \leq \sum_{i=n}^{\infty} \frac{1}{i!} K^{i+1}(T - t)^i, \quad n = 1, 2, \ldots
\]
With the known \( k_1(t) \), we can construct another sequence \( \{ k_2^{(n)} \} \):

\[
k_2^{(0)}(t) = 1,
\]

\[
k_2^{(n)}(t) = \frac{\alpha_1 - \alpha_2}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2}\left\{ \frac{1}{\gamma} I_1(t,T)I_2(t,T) + \int_t^T I_1(t,v)I_3(t,v)k_2^{(n-1)}(v)dv \right\}.
\]

For this sequence, we have

\[
|k_2^{(n)}(t) - k_2(t)| \leq \sum_{j=n}^{\infty} \frac{1}{j!} K^{j+1}(T-t)^j, \quad n = 1, 2, \ldots
\]

**Proof.** See Appendix C

4 Numerical results

4.1 The solution of model 3

The parameters chosen are \( \alpha_1 = 0.2, \sigma_{11} = 0.3, \sigma_{12} = 0, \alpha_2 = 0.12, \sigma_{21} = 0, \sigma_{22} = 0.2 \). Figure 2 and Figure 3 are the plots of \( k_1(t) \) and \( k_2(t) \) with various \( \gamma = 1, 3, 5, 10 \) and \( T = 0.5, 1, 5, 10 \).

The dynamics of stock 1 and stock 2 are

\[
\begin{cases}
    dS_{1t} = S_{1t}(0.20dt + 0.3dW_{1t}), \\
    dS_{2t} = S_{2t}(0.12dt + 0.2dW_{2t}).
\end{cases}
\]

From Figure 2, we can see that \( k_2(t) \) decreases as the risk aversion coefficient \( \gamma \) increases. This is also reflected in the structures of the optimal allocations and reward functions for model 2 and model 3. Frankly speaking, model 3 converges to model 2 as \( \gamma \) converges to infinity.

On the other hand, Figure 3 shows that both \( k_1(t) \) and \( k_2(t) \) decrease as the terminal time \( T \) increases. However, for different time horizons, both \( k_1(t) \) and \( k_2(t) \) coincide at the date of maturity.

Figure 4 shows the effect of \( t \) and \( x \) on \( \hat{u}_1(t, x) \) and \( \hat{u}_2(t, x) = x - \hat{u}_1(t, x) \), which are the
Figure 2: The functions $k_1(t)$ and $k_2(t)$ for various choices of $\gamma$ with $T = 1$.

amounts of money invested in stock 1 and stock 2, respectively. The terminal time $T = 10$ and the risk aversion coefficient $\gamma = 3$. From Figure 2, we can see that $0 < k_1(t) < 1$ for all $t \in [0, 10]$ and thus both $\hat{u}_1(t, x)$ and $\hat{u}_2(t, x)$ increase as the value of $x$ increase. Moreover, since both $k_1(t)$ and $k_2(t)$ are increasing with time $t$, therefore for each fixed wealth state $x$, $\hat{u}_1(t, x)$ is an increasing function and $\hat{u}_2(t, x)$ is a decreasing function of $t$.

4.2 Comparison of three models

Here, the parameters are $\gamma = 3$, $T = 1$, $\alpha_1 = 0.2$, $\sigma_{11} = 0.3$, $\sigma_{12} = 0$, $\alpha_2 = 0.12$, $\sigma_{21} = 0$, $\sigma_{22} = 0.2$ and $r = 0.04$. In order to compare the investment strategies of the three models at different wealth level, we fix $t = 0$ and plot Figure 5 and Figure 6. From Figure 5, we can see that in model 2, since the objective in this model is to minimize the variance of the expected wealth at the terminal time $T$, the amount of money invested in stock 2 is larger than that invested in stock 1 for all wealth level since stock 2 has the variance which is less than that of stock 1. In model 3, the amount of money invested in stock 1 is larger than that in model 2 while the one invested in stock 2 is less than that in model 3. This is because besides minimizing
the variance, maximizing the expected wealth is also our objective in model 3. Therefore, one will invest more money into the stock with larger price of market risk.\footnote{One can calculate that the price of market price for stock 1 and stock 2 are \( (\alpha_1 - r)/\sigma_{11} = 0.5333 \) and \( (\alpha_2 - r)/\sigma_{22} = 0.4000 \).} In addition, under model 3, we can see that when with low initial wealth, one even holds short position of stock 2. Furthermore, from Figure 6, the expected wealth and the conditional variance of wealth of model 3 are both larger than the ones of model 2.

For demonstration, we simulate two paths of the price, one for stock 1 and the other for stock 2. We then calculate the amounts and proportions of money invested into these two stocks, the wealth processes, the expected wealths and the variances for the three models based on these two paths. The amounts and proportions of money invested in stock 1 and 2 can be found in Figure 7.

Observing that stock 1 has a larger volatility than that of stock 2, it is not surprising that the amount of money invested in stock 1 is the least for model 2 because it puts all efforts to minimize the variance. And obviously in model 1, the investor borrows money to make risky investment since the sum of money invested into stocks exceeds 1.
Figure 4: The effect of parameters on the amount of money invested in stock 1 and stock 2 in model 3.

Figure 8 is the simulated paths of the wealth processes, means, variances and objective functions for the three models. Model 3 has the largest expected wealth with middle variance. Model 2 has the lowest variance with also lowest expected wealth. Overall, model 3 performs out of the three models with the largest reward in a bull market.  

5 Conclusion

In this paper, we construct the optimal time-consistent portfolio selection strategy for correlated risky assets explicitly without risk-free asset under the game theoretic framework. The key idea is an application of the extended HJB system developed in Björk and Murgoci (2010). The equilibrium control is linear in wealth. If a risk-free asset is involved, the equilibrium control has zero slope (i.e., it is independent of the current wealth) and is consistent with the ones in Basak and Chabakauri (2010) and Björk and Murgoci (2010). Therefore, model 3 in Section 3 can be considered as an extension of their models. On the other hand, as risk aversion approaches infinity, the intercept term of the optimal control tends to 0. In this sense, the equilibrium control in model 3 converges to that in model 2. Theorem 1 provides the existence and uniqueness of the optimal solution. We also present an iterative scheme for the determination of the optimal solution and its convergence speed is given in Theorem 4. We conduct numerical studies for the comparisons of the amounts and proportions of money invested in the assets, the expected values of the terminal wealth, the conditional variances and the objectives functions of the three assets.

\[^7\]Here we have assumed both \( \alpha_1 \) and \( \alpha_2 \) are greater than the riskless interest rate \( r \).
models. Comparisons demonstrate that model 3 performs better than the previous two in a bull market.

All the three models choose optimal strategies according to the prices of market risk, i.e., assets with higher prices of market risk are allocated more and assets with the same level price of market risk share the same allocation.

Different from Björk et al. (2014), the optimal allocations in model 2 and model 3 depend on the initial wealth although the risk aversion coefficient $\gamma$ is a constant. The optimal reward functions are quadratic in initial wealth $x$. Usually a risk-seeking investor would put all his money into risky assets when he is optimistic in the market.
Figure 6: The expected wealths and the conditional variances of the wealth in the three models with \( t = 0 \).

Figure 7: Simulated paths of the amounts and proportions of money invested into stock 1 and 2 for the three models.
Figure 8: Simulated paths of the expected wealths, the conditional variances of wealths and the reward functions for the three models.
Appendix A

Proof of Theorem 3. The dynamic of the value process of the portfolio is given by

\[dX_t = \left\{ \alpha_2 + k_1(t)(\alpha_1 - \alpha_2) \right\} X_t + k_2(t)(\alpha_1 - \alpha_2) dt
+ \left[ \{\sigma_21 + k_1(t)(\sigma_{11} - \sigma_{21})\} X_t + k_2(t)(\sigma_{11} - \sigma_{21}) \right] dW_{1t}
+ \left[ \{\sigma_22 + k_1(t)(\sigma_{12} - \sigma_{22})\} X_t + k_2(t)(\sigma_{12} - \sigma_{22}) \right] dW_{2t},\]

\[dX_t^2 = \left\{ 2\{\alpha_2 + k_1(t)(\alpha_1 - \alpha_2)\} + \{\sigma_21 + k_1(t)(\sigma_{11} - \sigma_{21})\} \right\}^2 + \left\{ \{\sigma_22 + k_1(t)(\sigma_{12} - \sigma_{22})\} \right\}^2 X_t^2 dt
+ 2\left\{ (\alpha_1 - \alpha_2) + \{\sigma_21 + k_1(t)(\sigma_{11} - \sigma_{21})\} \right\} \left\{ (\sigma_{11} - \sigma_{21}) + \{\sigma_22 + k_1(t)(\sigma_{12} - \sigma_{22})\} \right\} k_2(t) X_t dt
+ \left\{ (\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2 \right\} k_2(t) dt
+ 2\left\{ (\sigma_21 + k_1(t)(\sigma_{11} - \sigma_{21})) X_t + k_2(t)(\sigma_{11} - \sigma_{21}) \right\} X_t dW_{1t}
+ 2\left\{ (\sigma_22 + k_1(t)(\sigma_{12} - \sigma_{22})) X_t + k_2(t)(\sigma_{12} - \sigma_{22}) \right\} X_t dW_{2t},\]

Denote \( \mu_{t,x}(T) = E_{t,x}(X_T^\nu) \) and \( q_{t,x}(T) = E_{t,x}\left\{ (X_T^\nu)^2 \right\} \). By taking expectations on both sides of these two equations, we have

\[\mu_{t,x}(T) = x + \int_t^T \left\{ \alpha_2 + k_1(s)(\alpha_1 - \alpha_2) \right\} \mu_{t,x}(s) + k_2(s)(\alpha_1 - \alpha_2) ds, \quad (A.1)\]

\[q_{t,x}(T) = x^2 + \int_t^T \left\{ 2\{\alpha_2 + k_1(s)(\alpha_1 - \alpha_2)\} + \{\sigma_21 + k_1(s)(\sigma_{11} - \sigma_{21})\} \right\}^2 + \left\{ \{\sigma_22 + k_1(s)(\sigma_{12} - \sigma_{22})\} \right\}^2 q_{t,x}(s) ds
+ \int_t^T 2\left\{ (\alpha_1 - \alpha_2) + \{\sigma_21 + k_1(s)(\sigma_{11} - \sigma_{21})\} \right\} \left\{ (\sigma_{11} - \sigma_{21}) + \{\sigma_22 + k_1(s)(\sigma_{12} - \sigma_{22})\} \right\} k_2(s) \mu_{t,x}(s) ds
+ \int_t^T \left\{ (\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2 \right\} k_2^2(s) ds. \quad (A.2)\]

Equation (A.1) is a general linear differential equation. The solution of this equation is

\[\mu_{t,x}(T) = \left\{ I_1(t,T) \right\}^{-1} \left\{ x + (\alpha_1 - \alpha_2) \int_t^T I_1(t,v) k_2(t,v) dv \right\}.\]
With this known $\mu_{t,x}(T)$, equation (A.2) is also a general linear differential equation and its solution is

\[
q_{t,x}(T) = x^2 I_1^{-2}(t,T)I_2^{-1}(t,T)
+ 2x I_1^{-2}(t,T)I_2^{-1}(t,T) \int_t^T [(\alpha_1 - \alpha_2) + \{\sigma_{21} + (\sigma_{11} - \sigma_{21})k_1(v)\}(\sigma_{11} - \sigma_{21})
+ \{\sigma_{22} + (\sigma_{12} - \sigma_{22})k_1(v)\}(\sigma_{12} - \sigma_{22})]I_1(t,v)I_2(t,v)k_2(v)dv
+ 2(\alpha_1 - \alpha_2)I_1^{-2}(t,T)I_2^{-1}(t,T) \int_t^T I_1(t,v)I_2(t,v)k_2(v)(\alpha_1 - \alpha_2)
+ \{\sigma_{21} + k_1(v)(\sigma_{11} - \sigma_{21})\}(\sigma_{11} - \sigma_{21})
+ \{\sigma_{22} + (\sigma_{12} - \sigma_{22})k_1(v)\}(\sigma_{12} - \sigma_{22})] \int_t^v k_2(w)I_1(t,w)dwdv
+ I_1^{-2}(t,T)I_2^{-1}(t,T)\{((\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2) \int_t^T I_1^2(t,v)I_2(t,v)k_2^2(v)dv.
\]

Therefore, the expectation and variance are

\[
E_{t,x}(X_T^{\hat{u}}) = \mu_{t,x}(T)
= \{I_1(t,T)\}^{-1} \left\{ x + (\alpha_1 - \alpha_2) \int_t^T I_1(t,v)k_2(v)dv \right\},
\]

\[
\text{Var}_{t,x}(X_T^{\hat{u}}) = q_{t,x}(T) - \{\mu_{t,x}(T)\}^2
= \{I_1(t,T)\}^{-2} \left\{ c_0(t)x^2 + c_1(t)x + c_2(t) \right\},
\]

where

\[
c_0(t) = \{I_2(t,T)\}^{-1} - 1,
\]

\[
c_1(t) = -2I_2^{-1}(t,T) \int_t^T I_1(t,v)I_3(t,v)k_2(v)dv,
\]

\[
c_2(t) = \{I_2(t,T)\}^{-1}\left\{ ((\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2) \int_t^T I_1^2(t,v)I_2(t,v)k_2^2(v)dv
- 2(\alpha_1 - \alpha_2) \int_t^T I_1(t,v)I_3(t,v)k_2(v)\left\{ \int_t^v I_1(t,w)k_2(w)dw \right\}dv \right\}.
\]
The functions $g(t, x)$ and $V(t, x)$ are given by

$$
g(t, x) = E_{t,x}(X^{\hat{u}_T}) = \{I_1(t, T)\}^{-1} x + (\alpha_1 - \alpha_2)\{I_1(t, T)\}^{-1} \int_t^T I_1(t, v)k_2(v)dv, \tag{A.3}
$$

$$
V(t, x) = E_{t,x}(X^{\hat{u}_T}) - \frac{\gamma}{2} \text{Var}_{t,x}(X^{\hat{u}_T}) = -\frac{\gamma}{2} \{I_1(t, T)\}^{-1} - \frac{\gamma}{2} \{I_1(t, T)\}^{-1} c_2(t) \frac{\gamma}{2} \{I_1(t, T)\}^{-2} c_2(t). \tag{A.4}
$$

Here, $g(t, \cdot)$ is a linear function while $V(t, \cdot)$ is a linear–quadratic function of the current state. By the first order condition and the HJB system (3.12), the optimal allocation $\hat{u}(t, x)$ has the expression:

$$
\hat{u}(t, x) = (\alpha_2 - \alpha_1)V_x - \{\sigma_{21}(\sigma_{11} - \sigma_{21}) + \sigma_{22}(\sigma_{12} - \sigma_{22})\}(V_{xx} - \gamma g_x^2)x.
$$

By substituting (A.3) and (A.4) into this equation and after some tedious algebra, we obtain the ODE system (3.14)–(3.15) for $k_1(t)$ and $k_2(t)$.

**Appendix B**

**Proof of Position 1.** We assume that $\alpha_1 > \alpha_2$ and the situation where $\alpha_1 < \alpha_2$ can be similarly treated. We first prove that the integral equation (3.14) admits a unique solution $k_1 \in C[0, T]$. Construct a sequence

$$
k_1^{(0)}(t) = 1,
$$

$$
k_1^{(n)}(t) = \frac{1}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left\{ -\sigma_{21}(\sigma_{11} - \sigma_{21}) - \sigma_{22}(\sigma_{12} - \sigma_{22}) \right\} + (\alpha_1 - \alpha_2) \left( e^{-\int_t^T \left\{ k_1^{(n-1)}(s)(\sigma_{11} - \sigma_{21}) \right\}^2 + \left\{ k_2^{(n-1)}(s)(\sigma_{12} - \sigma_{22}) \right\}^2 ds - 1 \right). \tag{B.1}
$$
for \(n = 1, \ldots\).

For all \(t \in [0, T]\), from

\[
0 \leq e^{-\int_t^T \left[\left\{\sigma_{21} + k_1^{(n-1)}(s)(\sigma_{11} - \sigma_{21})\right\}^2 + \left\{\sigma_{22} + k_1^{(n-1)}(s)(\sigma_{12} - \sigma_{22})\right\}^2\right] ds} \leq 1,
\]

we have

\[
\begin{align*}
\dot{k}_1^{(n)}(t) &\geq -\frac{1}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left\{\sigma_{21} \left(\sigma_{11} - \sigma_{21}\right) + \sigma_{22} \left(\sigma_{12} - \sigma_{22}\right) + (\alpha_1 - \alpha_2)\right\}, \\
\dot{k}_1^{(n)}(t) &\leq -\frac{1}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left\{\sigma_{21} \left(\sigma_{11} - \sigma_{21}\right) + \sigma_{22} \left(\sigma_{12} - \sigma_{22}\right)\right\}.
\end{align*}
\]

Therefore, the sequence \(\{k_1^{(n)}\}\) is uniformly bounded in \(C[0, T]\).

We now consider the sequence \(\{\dot{k}_1^{(n)}\}\) where \(\dot{k}_1^{(n)} = dk_1^{(n)}(t)/dt\). The derivative \(\dot{k}_1^{(n)}\) has the expression:

\[
\dot{k}_1^{(n)} = \frac{\alpha_1 - \alpha_2}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} \left[\frac{\left\{\sigma_{21} + k_1^{(n-1)}(t)(\sigma_{11} - \sigma_{21})\right\}^2 + \left\{\sigma_{22} + k_1^{(n-1)}(t)(\sigma_{12} - \sigma_{22})\right\}^2}{\left\{\sigma_{21} + k_1^{(n-1)}(t)(\sigma_{11} - \sigma_{21})\right\}^2} + \left\{\sigma_{22} + k_1^{(n-1)}(t)(\sigma_{12} - \sigma_{22})\right\}^2\right] e^{-\int_t^T \left[\left\{\sigma_{21} + k_1^{(n-1)}(s)(\sigma_{11} - \sigma_{21})\right\}^2 + \left\{\sigma_{22} + k_1^{(n-1)}(s)(\sigma_{12} - \sigma_{22})\right\}^2\right] ds}.
\]

Since we have proved that \(\{k_1^{(n)}\}\) is uniformly bounded in \(C[0, T]\), from (B.2), we can conclude that the sequence \(\{\dot{k}_1^{(n)}\}\) is also uniformly bounded in \(C[0, T]\). Denote \(|k_1^{(n)}| < M_1\) for all \(n\) and all \(t \in [0, T]\). Therefore, for any \(t_1, t_2 \in [0, T]\) and \(t_1 < t_2\), we have

\[
|k_1^{(n)}(t_1) - k_1^{(n)}(t_2)| = \left|\int_{t_1}^{t_2} \frac{d}{ds} k_1^{(n)}(t_1 + s(t_2 - t_1)) ds\right| = \left|(t_2 - t_1) \int_{0}^{1} \dot{k}_1^{(n)}(t_1 + s(t_2 - t_1)) ds\right| \leq (t_2 - t_1) \max_{0 \leq t \leq T} |\dot{k}_1^{(n)}(t)| = M_1(t_2 - t_1).
\]

Therefore, the sequence \(\{k_1^{(n)}\}\) is also equicontinuous. According to Arzela–Ascoli Theorem,
there exists a subsequence of \( \{k_1^{(n_i)}\} \), \( \{k_1^{(n_i)}\} \), and a \( k_1 \in C[0,T] \) such that \( k_1^{(n_i)} \to k_1 \) as \( i \to \infty \). Since \( \{k_1^{(n_i)}\} \) satisfies (B.1), by letting \( i \to \infty \), we can conclude that \( k_1 \) is a solution to (B.1).

For the uniqueness of the solution, suppose \( k_1 \) and \( k_2 \) are two solutions to equation (B.1). Since \( k_1 \) and \( l_1 \) are bounded in \([0,T]\), therefore the functions

\[
- \int_t^T \left[ \{\sigma_{21} + k_1(s)(\sigma_{11} - \sigma_{21})\}^2 + \{\sigma_{22} + k_1(s)(\sigma_{12} - \sigma_{22})\}^2 \right] ds
\]

and

\[
- \int_t^T \left[ \{\sigma_{21} + l_1(s)(\sigma_{11} - \sigma_{21})\}^2 + \{\sigma_{22} + l_1(s)(\sigma_{12} - \sigma_{22})\}^2 \right] ds
\]

are also bounded for all \( t \in [0,T] \). Since the function \( f(x) = e^x \) is Lipschitz on bounded set, it is easy to derive that

\[
|k_1(t) - l_1(t)| \leq M_2 \int_t^T |k_1(s) - l_1(s)| ds.
\]

This Gronwall inequality implies that \( k_1(t) = l_1(t) \) for all \( t \in [0,T] \).

We thus proved that equation (3.14) admits a unique solution \( k_1 \in C[0,T] \). For equation (3.15), denote \( \lambda = (\alpha_1 - \alpha_2)/\{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2\} \), \( \phi(t) = \gamma^{-1}l_1(t,T)I_2(t,T) \) and \( L(t,v) = I_1(t,v)I_3(t,v) \). Then we have

\[
k_2(t) = \phi(t) + \lambda \int_t^T L(t,v)k_2(v)dv. \tag{B.3}
\]

Equation (B.3) is a Volterra integral equation of the second kind. Consider the mapping \( F : C[0,T] \to C[0,T], \)

\[
Fk_2(t) = \phi(t) + \lambda \int_t^T L(t,v)k_2(v)dv.
\]
Then for all $k_2, l_2 \in C[0, T]$,

$$|Fk_2(t) - Fl_2(t)| = |\lambda| \left| \int_t^T L(t, v)\{k_2(v) - l_2(v)\}dv \right|$$

$$\leq |\lambda|M_3(T - t) \max_{t \leq s \leq T} |k_2(s) - l_2(s)|$$

where $M_3 = \max_{0 \leq t_1, t_2 \leq T} |L(t_1, t_2)|$.

Therefore,

$$|F^2k_2(t) - F^2l_2(t)| = |\lambda| \left| \int_t^T L(t, v)\{Fk_2(v) - Fl_2(v)\}dv \right|$$

$$\leq |\lambda|M_3 \int_t^T |\lambda|M_3 \max_{t \leq s \leq T} |k_2(s) - l_2(s)|(T - v)dv$$

$$= \frac{1}{2}|\lambda|^2M_3^2(T - t)^2 \max_{t \leq s \leq T} |k_2(s) - l_2(s)|.$$ 

By induction, we have

$$|F^n k_2(t) - F^n l_2(t)| \leq \frac{1}{n!} |\lambda|^n M_3^n (T - t)^n \max_{t \leq s \leq T} |k_2(s) - l_2(s)|$$

$$\leq \frac{1}{n!} |\lambda|^n M_3^n T^n \max_{t \leq s \leq T} |k_2(s) - l_2(s)|.$$ 

Since

$$\lim_{n \to \infty} \frac{(|\lambda|M_3T)^n}{n!} = 0,$$

therefore, for the given fixed values of $\lambda, M_3$ and $T$, there exists an integer $N$ such that

$$0 \leq \frac{(|\lambda|M_3T)^N}{N!} < 1.$$ 

So the mapping $F^N$ is a contraction and thus equation (B.3) has one and only one solution.
Appendix C

**Proof of Theorem 4.** The proof of the convergence speed of the iterative scheme (3.17)–(3.18) is similar to Theorem 4.9 of Björk et al. (2014). We thus omit it here.

Using the notations in Appendix B, the iteration scheme (3.19)–(3.20) can be written as

\[ k_2^{(n)}(t) = \phi(t) + \lambda \int_t^T L(t, v)k_2^{(n-1)}(v)dv. \]

Denote \( \bar{k}_2^{(n)} = k_2^{(n)} - k_2^{(n-1)}, \) for \( \forall t \in [0, T], \) we have

\[
|\bar{k}_2^{(n)}(t)| = |\lambda| \int_t^T L(t, v)(k_2^{(n-1)}(v) - k_2^{(n-2)}(v))dv \\
\leq |\lambda|M_3 \int_t^T |\bar{k}_2^{(n-1)}(v)|dv. \tag{C.1}
\]

Let \( \omega_n(t) = \int_t^T |\bar{k}_2^{(n)}(s)|ds. \) From Equation (C.1), we have

\[
\frac{d}{dt} \omega_n(t) + |\lambda|M_3\omega_{n-1}(t) \geq 0,
\]

\[
\frac{d}{dt} \omega_0(t) + |\lambda|M_3\omega_0(t) \geq 0,
\]

\[
\frac{d}{dt} \omega_1(t) + |\lambda|M_3\omega_{1}(t) \geq 0.
\]
and thus

\[ \omega_n(t) \leq |\lambda| M_3 \int_t^T \omega_{n-1}(v) dv \]

\[ \leq (|\lambda| M_3)^2 \int_t^T \int_v^T \omega_{n-1}(s) ds dv \]

\[ = (|\lambda| M_3)^2 \int_t^T \int_t^s \omega_{n-1}(s) dv ds \]

\[ = (|\lambda| M_3)^2 \int_t^T (s - t) \omega_{n-1}(s) dv \]

\[ \leq (|\lambda| M_3)^3 \int_t^T (s - t) \int_s^T \omega_{n-2}(v) dv ds \]

\[ \leq (|\lambda| M_3)^3 \int_t^T \int_t^v (s - t) \omega_{n-2}(v) dv ds \]

\[ \leq (|\lambda| M_3)^3 \int_t^T \frac{1}{2!} (v - t)^2 \omega_{n-2}(v) dv \]

\[ \leq \cdots \]

\[ \leq (|\lambda| M_3)^n \int_t^T \frac{1}{(n-1)!} (v - t)^{n-1} \omega_1(v) dv \]

\[ \leq \frac{1}{n!} (|\lambda| M_3)^n (T - t)^n \omega_1(0). \]

Therefore,

\[ |k_2^{(n)}(t) - k_2(t)| = \left| - \sum_{j=n}^{\infty} \bar{k}_2^{(j+1)} \right| \]

\[ \leq \sum_{j=n}^{\infty} |\bar{k}_2^{(j+1)}| \]

\[ \leq \sum_{j=n}^{\infty} \frac{1}{j!} \omega_1(0)(|\lambda| M_3)^{j+1}(T - t)^j \]

\[ \leq \sum_{j=n}^{\infty} \frac{1}{j!} K^{j+1}(T - t)^j \]

for \( n = 1, \ldots \). Here, \( K \) can be selected as any positive constant larger than \( \max(|\lambda| M_3, \omega_1(0)|\lambda| M_3) \).
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