On the diffeomorphism groups of elliptic surfaces

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Abstract

In this paper we determine for relatively minimal elliptic surfaces with positive Euler number the image of the natural representation of the group of orientation preserving self-diffeomorphisms on $H$, the second homology group reduced modulo torsion. To this end we construct as many embedded spheres of square -2 such that an isometry not induced from any combination of reflections at such spheres or from 'complex conjugation' can be shown not to be induced from some diffeomorphism at all. This is done with the help of Seiberg-Witten-invariants.

INTRODUCTION

It is known that in dimension four corresponding notions of topological and smooth manifold theory differ much more than in any other dimension. This fact is best illustrated by the complete classification of simply connected topological manifolds as given by Freedman [F] versus non-existence results for smooth structures on some topological manifolds and multitudes of them on others which had been obtained by Donaldson [D] and others, e.g. in [F/M1].

A further example is provided by the respective groups of self-equivalences, homeomorphisms and diffeomorphisms, which we assume to be orientation preserving without further mentioning. They are studied via their representations on the second homology and general results have been obtained on simply connected manifolds: The representations descend to faithful representations of isotopy classes of homeomorphisms [Q] and pseudo-isotopy classes of diffeomorphisms respectively [K]. The group of homeomorphisms in this case maps onto the orthogonal group [Q], but the image of the diffeomorphism group defies computation except for special manifolds [F/M1], [E/O], and after stabilization [W2].

It is the purpose of this paper to enlarge the set of special manifolds by proving the following result, of which a special case answers a question raised by Friedman and Morgan in their recent preprint [F/M2]:

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Main Theorem\hspace{1em} Let $X$ be a minimal elliptic surface with positive Euler number. \linebreak[1]\hspace{1em} $\text{Diff}(X)$ its group of diffeomorphisms and $\mathbb{H}_2$ its second homology reduced modulo torsion. Then there exists an isometry $\sigma_*$ such that the image of the natural representation of $\text{Diff}$ in the orthogonal group $O$ of $\mathbb{H}_2$ with respect to the intersection form is

- $O_k \cdot \{\pm \text{id}\}$ the subgroup generated by $\pm \text{id}$ and the stabilizer group of the homology class of the canonical divisor in the case of non-rational surfaces with geometric genus zero,
- $O'_k \cdot \{\sigma_*, \text{id}\}$ the subgroup generated by $\sigma_*$ and the stabilizer group of the canonical class in the subgroup of elements with real spinor norm one in the case of surfaces with positive geometric genus,
- $O$ in the case of rational surfaces.

We will address the existence and non-existence claims of the theorem separately. So in the first part we prove the existence of ‘many’ smoothly embedded 2-spheres of (intersection-)square $-2$ and exploit the interplay between these spheres, the diffeomorphisms they give rise to, and algebraic results on groups of orthogonal transformations. This is just the sort of argument Friedman and Morgan set up when proving the existence result for regular elliptic surfaces $X_d$ without multiple fibres and of positive Euler number $e(X_d) = 12d > 0$. They also got corollaries for regular elliptic surfaces $X_{d,m_1,\ldots,m_n}$ with fibres of multiplicities $m_1,\ldots,m_n$. Based on their first result we will be able to generalize it to arbitrary elliptic surfaces $X_{d,q}$ without multiple fibres, of positive Euler number $12d$ and irregularity $q$ possibly non-vanishing. These surfaces fibre over curves which are smooth closed 2-manifolds $\Sigma_q$ of genus $q$. We will then proceed to the case of elliptic surfaces $X_{d,q,m_1,\ldots,m_n}$ obtained from the former ones by logarithmic transformations, i.e. by inserting multiple fibres. In the regular case we improve the result of Friedman and Morgan by a small margin.

Our result just fails to cover all minimal elliptic surfaces since we restrict ourselves to the case of positive Euler number. The remaining surfaces will require essentially different methods because even in the topological category only few homology classes are represented by spheres.

The main tool in the second part are Seiberg Witten invariants which have already proved invaluable in the realms of smooth four manifolds. We start setting up the terminology in line with the definitions in [B]. This provides the means to link the results from [B] to the applications we have in mind. Our non-existence claims are then proved using the invariance of the homology class of a canonical divisor and exploiting the subtleties of the homology orientation very much in the spirit of [E/O]. Taking into account the existence of a diffeomorphism induced from complex conjugation, cf. [F/M], is the final bit to accomplish the proof of the theorem.

In the course of the argument we will freely use Poincaré duality to identify homology classes of divisors with the Chern classes of the line bundles they determine. A fair knowledge of the geometry of elliptic surfaces as can be found in [F/M] or
[BPV] is assumed tacitly.
Embedded spheres in elliptic surfaces

**Theorem 1** ([F/M]) Every class in $H_2(X_1 - nf)$ of square -2 is represented by a 2-sphere smoothly embedded in $X_1 - nf$. In fact there is a set of $9 + 2n$ such spheres

$$\{\alpha_1, \ldots, \alpha_{2n+1}, \beta_1, \ldots, \beta_8\}$$

whose classes form a basis for $H_2(X_1 - nf)$ such that

- the $\{\beta_j\}$ intersect according to the Dynkin diagram of $-E_8$,
- the intersection pattern of the $\{\alpha_i\}$ is given by a complete graph on $2n + 1$ vertices whose edges are weighted by the intersection number -2,
- every $\alpha_i$ intersects $\beta_8$ algebraically once,
- $\{\ell + \alpha_i\}$ represents any given basis for the radical of the intersection pairing on $H_2(X_1 - nf)$; $\ell$ denoting the long vector of $-E_8$.

Moreover a set $\{\beta'_j\}$ with the first property can be supplemented to such a basis.

*Proof:* Except for the addendum the claim is that of thm.5.1 in [F/M] generalized along the lines of thm.5.11, loc.cit..

Given any basis as in the theorem we get another for every isometry of $(H_2,q_H)$. Such an isometry we get by mapping $\{\beta_j\}$ of the theorem to the $\{\beta'_j\}$ and extending by an isomorphism of the radical. The images of the $\{\alpha_i\}$ then are represented by spheres according to the first assertion of the theorem and are easily seen to supplement the $\{\beta'_j\}$ to a basis. \hfill $\Box$

**Lemma 1** An elliptic surface $X_{d,q}$ which is properly irregular, i.e. $q \geq 1$, and has positive Euler number $12d$ decomposes as

$$X_{d,q} = X_1 \# f \ldots \# f X_1 \#_{2f} B$$

with $B := \Sigma_{q-1} \times f$. This is equivalent to

$$X_{d,q} - f = (X_1 \# f) \cup_{f \times S^1} \ldots \cup_{f \times S^1} (X_1 \# f) \cup_{f \times S^1} (X_1 - 3f) \cup_{2(f \times S^1)} (B - 2f).$$

*Proof:* There is a well known decomposition, cf. [F/M] pp. 159, 190, 195;
with $B' = \Sigma_q \times f$ which can be considered as lifted from a decomposition of the base

$$\Sigma = S^2 \# pt \ldots \# pt S^2 \# pt \Sigma_q.$$ 

Since $\Sigma_q$ is $\Sigma_{q-1} \#_{2pts} S^2$ and we may transfer the new $S^2$ summand to the neighbouring one, we get:

$$\Sigma = S^2 \# pt \ldots \# pt S^2 \# pt (S^2 \# pt S^2) \#_{2pts} \Sigma_{q-1},$$

which induces the claimed decomposition on the total space. Cut out a regular fibre and reformulate the fibre connected sum in terms of the union along submanifolds to get the second assertion. \qed

**Lemma 2** There is a basis of $H_2(B, 2f)$ geometrically represented by

- a section to the fibration map,
- smoothly embedded cylinders with vanishing selfintersection and the distinct boundary components mapping to the distinct fibres.

**Proof:** $(B, 2f) = (f \times (\Sigma_{q-1}, 2pts))$. Choose injective maps $\psi_i : S^1 \hookrightarrow f$ and $\phi_j : (I, \{0, 1\}) \hookrightarrow (\Sigma_{q-1}, 2pts)$ representing bases of $H_1(f)$, $H_1(\Sigma_{q-1}, 2pts)$ respectively. The product maps

$$\Psi_{i,j} : (S^1 \times I, S^1 \times \{0, 1\}) \hookrightarrow (B, 2f)$$

then represent classes in $H_2(B, 2f)$. These are obviously embedded cylinders of the kind we want and a look at the exact homology sequence of the pair $(B, 2f)$ should convince us that together with a section they form a basis for $H_2(B, 2f)$ \qed

**Lemma 3** There are four smoothly embedded vanishing cells of square -1 disjoint off their boundaries

$$(D^2, S^1) \hookrightarrow (X_1 - f, 2f)$$

representing relative classes in $H_2(X_1 - f, 2f)$ such that their boundaries are arbitrarily given simple closed curves generating $H_1(2f)$.

**Proof:** Without loss of generality we may assume that $X_1 - f$ contains six cusp neighbourhoods disjoint from $2f$ and each other. Given a curve then we construct a cell as follows:
Choose a regular fibre in the boundary of the cusp neighbourhood. This fibre and 
the given curve map to points in the base which we connect by a simple arc such that 
the fibration over it is trivial. So we may transfer the homology class of the curve 
to the other end. Pick an associated vanishing cell. Its boundary can be isotoped 
over the arc to the given curve. Glue the cylinder thus obtained to the cell to get 
the desired embedding. Different cells won't intersect as long as we choose different 
cusp neighbourhoods for different curves and disjoint arcs - which obviously is a 
condition easily met.

\[\text{Theorem 2} \quad \text{In the proper irregular case, i.e. } q \geq 1, \text{ there is a generating set for } 
H_2(X_{1,q} - f) \text{ of square } -2 \text{ classes which are represented by } 2\text{-spheres} \]
\[
\{\alpha_1, \ldots, \alpha_7, \beta_1, \ldots, \beta_8, \gamma_1, \ldots, \gamma_{2q-1}, \delta_1, \ldots, \delta_{2q-1}, \varepsilon_1, \varepsilon_2, \varepsilon_3\},
\]
smoothly embedded in \(X_{1,q} - f\) and such that

- the \(\{\alpha_i\}\) intersect according to a complete graph on 7 vertices: \(\alpha_i \cdot \alpha_j = -2\),
- the \(\{\beta_j\}\) intersect according to the Dynkin diagram of \(-E_8\),
- every \(\alpha_i\) intersects \(\beta_8\) algebraically once,
- the intersection graph spanned by \(\{\beta_8, \gamma_1, \delta_1, \varepsilon_1, \varepsilon_2, \varepsilon_3\}\) is:

Proof: Decompose \(X_{1,q} - f\) according to lemma 1:
\[
X_{1,q} - f = (X_1 - 3f) \cup_{2(f \times S^1)} (B - 2f) = (X_1 - f) \# 2f B.
\]
This is a fibre connected sum along two disjoint regular fibres. Choosing injective 
maps \(\phi_{1,2} : S^1 \hookrightarrow f\) generating the first homology of one of these we get a second pair 
of such maps to the other fibre by the product structure on \(B\). Together they give 
rise to embedded cylinders in \((B, 2f)\) and embedded cells in \((X_1 - f, 2f)\) according 
to the preceding lemmas.
If we glue these in the obvious way we get smoothly embedded 2-spheres in \(X_{1,q} - f\) with square \(-2\). Call them \(\gamma_k, \delta_l\) such that the index refers to an indexing of a basis for \(H_1(\Sigma_{q-1}, 2\text{pts.})\), cf. lemma 2, and the \(\gamma\)'s are those constructed from \(\phi_1\), the \(\delta\)'s those from \(\phi_2\).

Let’s now have a closer look at the homology sequence of the pair \((X_{1,q} - f, X_1 - 3f)\):

\[
H_2(X_1 - 3f) \xrightarrow{\partial} H_2(X_{1,q} - f) \xrightarrow{\partial} H_2(X_{1,q} - f, X_1 - 3f) \rightarrow H_1(X_1 - 3f).
\]

Here replace the relative group by \(H_2(B, 2f)\) by means of a suitable excision lemma, cf. [La, §3.0]. Since the section in \(H_2(B, 2f)\) maps neither to zero nor to a torsion class in \(H_1(X_1 - 3f)\) the image of the relative map is spanned by the cylinders of lemma 2. Of course our new spheres \(\gamma_k, \delta_l\) are lifts of those. Thus any generating set for \(H_2(X_1 - 3f)\) will be supplemented by these spheres to a generating set for \(H_2(X_{1,q} - f)\).

Pick such a set \(\{\alpha_i', \beta_j'\}\) as given by theorem [La] such that the basis of the radical \(\{\ell' + \alpha_i'\}\) is dual to \(\gamma_k, \delta_l\), i.e.:

\[
\langle \ell' + \alpha_i', \gamma_k \rangle = \delta_{i1}^\text{Kronecker}, \quad \langle \ell' + \alpha_i', \delta_l \rangle = \delta_{i2}^\text{Kronecker}
\]

Note that we may speak unambiguously of intersection with \(\gamma_k\), for it is determined by the relative class it represents in \(H_2(X_1 - f, 2f)\) which is independent of the index, ditto for \(\delta_l\).

Now we adjust this set for our purposes. First set:

\[
\beta_i := \beta_i' \quad i = 1, \ldots, 7; \quad \beta_8 := \beta_8' - \langle \beta_8' - \ell', \gamma_k \rangle \alpha_1' - \langle \beta_8' - \ell', \delta_1 \rangle \alpha_2'.
\]

It is easily checked that:

- the \(\{\beta_j\}\) span an intersection graph \(-E_8\),
- \(\beta_8 \cdot \gamma_k = 0 = \beta_8 \cdot \delta_1\).

With theorem [La] we may supplement the new \(\beta_j\) by a set \(\{\alpha_i\}\) subject to a duality property as above relative to the new \(\ell\).

Finally define the \(\varepsilon_1, \varepsilon_2, \varepsilon_3\) to be:

\[
\varepsilon_1 = \langle \gamma_k, \ell \rangle (\ell + \alpha_1) + \langle \delta_1, \ell \rangle (\ell + \alpha_2) + \alpha_1 \\
\varepsilon_2 = \langle \gamma_k, \ell \rangle (\ell + \alpha_1) + \langle \delta_1, \ell \rangle (\ell + \alpha_2) + \alpha_2 \\
\varepsilon_3 = \langle \gamma_k, \ell \rangle (\ell + \alpha_1) + \langle \delta_1, \ell \rangle (\ell + \alpha_2) + \alpha_3
\]

And now we are finished. The intersection pattern of the set chosen is seen to meet the requirements of the theorem. Moreover the set has already been constructed as a set of spheres except for \(\varepsilon_1, \varepsilon_2, \varepsilon_3\), but as they can be considered as elements in \(H_2(X_1 - 3f)\) there are embedded spheres representing them by theorem [La]. \(\square\)
Corollary 1 In the proper irregular case there is a generating set for \( H_2(X_{d,q} - f) \) of square \(-2\) classes which are represented by 2-spheres

\[
\{ \alpha^1_{1,\ldots,7}, \beta^1_{1,\ldots,7}, \gamma_k, \delta_{1,2,3}, \alpha^n_{l,\ldots,2}, \beta^n_{l,\ldots,2}, \zeta_{i,\ldots,2}^n | 1 \leq i \leq 5, 1 \leq k \leq 2q-1, 1 \leq j \leq 8, 2 \leq n \leq d \}
\]

smoothly embedded in \( X_{d,q} - f \) and such that:

- the \( \{ \alpha^1_{1,\ldots,7}, \beta^1_{1,\ldots,7}, \gamma_k, \delta_{1,2,3} \} \) intersect as in theorem 2,
- the \( \{ \alpha^n_{l,\ldots,2}, \beta^n_{l,\ldots,2} \} \) intersect as in theorem 7,
- for every sphere \( \zeta^1_{i,2} \) there are two spheres among the \( \alpha \)'s of the \( n \)th and the \( (n-1) \)th summand respectively which intersect \( \zeta \) algebraically once.

Proof: Start with a decomposition of \( X_{d,q} - f \) as given by lemma 1:

\[ X_{d,q} - f = X_{1,q} \# f X_1 \ldots \# f (X_1 - f). \]

By a Mayer-Vietoris argument a generating set is given as soon as there are such sets for each summand and spheres which after restriction generate the first homology of all those fibres on which fibre connected sum has been performed. Sets of the first kind we get from theorem 1 and theorem 2 so our set will automatically be furnished with the first two properties.

For the construction of the spheres of the second kind recall that two fibres and their neighbourhoods in the summands had to be identified in order to perform fibre connected sum. Choose generating curves for the first homology of these fibres compatible with the identification. Then inside the connected sum we can glue the vanishing cells in each summand which we obtain as in lemma 4. The outcome is obviously a 2-sphere embedded in the connected sum of the adjacent summands of the fibre we started with.

By construction in each of this summands a sphere \( \zeta \) can be considered as a relative homology class, thus there is a class \( \ell + \alpha \) dual to it, cf. the proof of the preceding theorem. Hence

\[ \langle \zeta, \langle \zeta, \ell \rangle (\ell + \alpha) + \alpha \rangle = 1, \]

and we are left with showing that we may even assume the class \( \langle \zeta, \ell \rangle (\ell + \alpha) + \alpha \) to be represented by one of the \( \alpha_i \)'s. But this is indeed the case as its square is \(-2\) and the sum with \( \ell \) is an element of the radical for the summand under consideration, thus a isometry as in the proof of theorem 2 will do the job.

\[ \Box \]

Theorem 3 If \( X_{d,q} \) is an elliptic surface without multiple fibres and Euler number \( 12d > 0 \) and \( f \) is a generic regular fibre, then:

- every class in \( H_2(X_{d,q} - f) \) of square \(-2\) is represented by a 2-sphere smoothly embedded in \( X_{d,q} - f \).
• in the rational case, i.e. $X_{d,q} = X_1$, reflections in classes of square -2 generate the group of all isometries of $H_2(X_{d,q} - f)$ which restrict to the identity on the radical, in all other cases they generate a subgroup of index at most 2,

• every such reflection is realized by a diffeomorphism which is the identity on a neighbourhood of $f$.

The proof mimics that given for thm.6.2 in [F/M]. Its geometric ingredient is the fact that a 2-sphere of square -2 gives rise to a diffeomorphism which is the identity outside a neighbourhood of the sphere and which acts on homology as reflection in the class of the sphere, cf. [F/M], p.358f. The algebraic counterpart is a theorem due to Ebeling in the general case and the slightly enhanced lemma 5.9 of [F/M] in the rational case:

**Theorem 4** ([P]) Let $(L, \langle, \rangle)$ be an even lattice. Let $\Delta \subset L$ be a set of vectors of square -2, and let $\Gamma_\Delta$ be the subgroup of the isometry group of $L$ generated by reflections in $\lambda \in \Delta$. Suppose that $\Delta$ satisfies the conditions i), ii), and iii) below:

i) $\Delta$ spans $L$,

ii) $\Delta$ is contained in a single $\Gamma_\Delta$-orbit,

iii) in $\Delta$ there are six elements $\lambda_1, \ldots, \lambda_6$ which intersect as given by the following diagram:

```
1   -2
\lambda_1 --- \lambda_3
    /\     /\      /\    /\  \
   /  \    /  \    /  \  /  \  \
  /    \  /    \  /    \ /    \  \
\lambda_2      \lambda_4           \lambda_5
               \     \    \     \    \  \
               /\   /\   /\   /\   /\  \\
              /  \ /  \ /  \ /  \ /  \  \\
             /    \ /    \ /    \ /    \  \\
           \lambda_4 \lambda_1 \lambda_6
```

Then $\Gamma_\Delta$ is the subgroup of the isometry group of $(L, \langle, \rangle)$ consisting of automorphisms of real spinor norm one which are the identity on $\operatorname{Hom}(L, \mathbb{Z})/\operatorname{im} L$, and $\Gamma_\Delta \cdot \Delta$ is the set of all vectors of square -2 in $L$. \qed

**Lemma 4** Let $(L, \langle, \rangle)$ be a lattice which decomposes as a direct sum of the radical and a unimodular lattice $E$. Let $\Delta \subset L$ be a set of vectors of square -2. Suppose that:
i) $\Delta$ spans $L$,

ii) $\Gamma_{\Delta \cap E} \cdot (\Delta \cap E)$ is a single $\Gamma_{\Delta \cap E}$-orbit and is the set of all vectors of square $-2$ in $E$,

iii) $\Gamma_{\Delta \cap E}$ contains all isometries of $(E, \langle \cdot, \cdot \rangle_E)$.

Then $\Delta$ is contained in a single $\Gamma_\Delta$-orbit which consists of all vectors of square $-2$ in $L$ and $\Gamma_\Delta$ is the group of all isometries of $(L, \langle \cdot, \cdot \rangle)$ which are the identity on the radical.

Proof of theorem 2: As the theorem has been proved in [F/M] in the regular case we may first consider irregular surfaces; so we may take $\Delta$ to be the set of spheres given in the corollary to theorem 2. Thereby the hypotheses i) and iii) of theorem 2 are automatically satisfied. To check hypothesis ii) we make use of the following criterion: two elements of $\Delta$ are in the same orbit of $\Gamma_\Delta$ if they intersect algebraically once up to sign. Since being conjugate is a transitive relation the verification of hypothesis ii) boils down to checking whether the intersection graph is connected by edges which carry intersection numbers $\pm 1$. This in fact is ensured by the various intersection properties of the preceding results.

In the rational case we take $\Delta$ to be the set of spheres given in thm.1. Then the first hypothesis of the lemma is satisfied, and so are the others by well known properties of the $E_8$ lattice.

Now let’s exploit the conclusions: In both cases $\Gamma_\Delta$ is shown to be the group of all reflections in classes of square $-2$. Its generators are given by spheres which are disjoint from the fibre $f$. By the geometrical ingredient mentioned above the generators are realized by diffeomorphisms which are the identity on a neighbourhood of $f$, and so is the entire group! Moreover because there is only one orbit for classes of square $-2$, an orbit of an embedded sphere under the diffeomorphisms just constructed, will supply embedded spheres representing all classes of square $-2$.

Finally note that in the general case the homology lattice modulo its radical is unimodular. Hence we may identify the group of isometries which induce the identity on $\text{Hom}(L, \mathbb{Z})/\text{im} L$ with the group of isometries which are the identity on the radical. Thus of the latter $\Gamma_\Delta$ is a subgroup of index at most two, since the real spinor norm is a group homomorphism to $\mathbb{Z}_2$. In the rational case the stronger conclusion is already given by the lemma. 

Corollary 2 If $X_{d,q}$ is an elliptic surface without multiple fibres and Euler number $12d > 0$ and $2f$ are two disjoint regular fibres, then:

- every class in $H_2(X_{d,q} - 2f)$ of square $-2$ is represented by a 2-sphere smoothly embedded in $X_{d,q} - 2f$,

- two such spheres are conjugate under the group of reflections on classes of square $-2$. 

}\end{proof}
Lemma 5 An elliptic surface $X_{d,q;m_1,...,m_n}$ with positive Euler number has a decomposition

$X_{d,q;m_1,...,m_n} - f = (X_{d,q} - 2f) \cup_{f \times S^1} (S_{m_1,...,m_n} - f) = (X_{d,q} - f)\# f S_{m_1,...,m_n}$,

with $S_{m_1,...,m_n}$ a Seifert fibration of tori over $S^2$ such that its multiple fibres have multiplicities $m_i$.

This decomposition induces an exact homology sequence:

$$H_i(X_{d,q} - 2f) \rightarrow H_i(X_{d,q;m_1,...,m_n} - f) \rightarrow H_i(S_{m_1,...,m_n}, f) \rightarrow H_{i-1}(X_{d,q} - 2f).$$

Proof: A decomposition of the base curve of the fibration into a disc to which all multiple fibres map and the remaining part to which all the other singular fibres map lifts to the decomposition of the total space as in the claim above. The exact sequence is then derived from the exact homology sequence of the pair $(X_{d,q;m_1,...,m_n}, X_{d,q} - 2f)$ and the isomorphism, cf. [La, §3.0]:

$$\Psi : H_i(X_{d,q;m_1,...,m_n} - f, X_{d,q} - 2f) \xrightarrow{\sim} H_i(S_{m_1,...,m_n}, f).$$

Lemma 6 There is an exact sequence

$$H_2(X_{d,q} - 2f) \rightarrow H_2(X_{d,q;m_1,...,m_n} - f) \rightarrow \mathbb{Z}_m \rightarrow 0,$$

with $\overline{H}$ the quotient group of $H$ modulo its subgroup of torsion elements.

In this sequence the multiple fibres taken as geometric preimages under the fibration map, i.e. without their multiplicities, represent classes which map to a set of generators for the quotient group $\mathbb{Z}_m$ and $m$ is given as the least common multiple of the multiplicities.

To make the following argument clearer at least in a typographical respect we introduce $\text{Tors}_i$ as a shorthand notation for the torsion subgroup of $H_i$, as well as:

$$X := X_{d,q}, \quad S_* := S_{m_1,...,m_n}, \quad X_* := X_{d,q;m_1,...,m_n}.$$

Proof of lemma: Take the following part of the sequence from the lemma above and compute the rank for each group:

$$H_2(S_*, f) \xrightarrow{\partial \circ \Psi^{-1}} H_1(X - 2f) \rightarrow H_1(X_* - f) \rightarrow H_1(S_*, f).$$

Thus the rank of $\ker \partial \circ \Psi^{-1}$ vanishes. Moreover $H_1(X - 2f)$ is torsionfree, so in fact $\ker \partial \circ \Psi^{-1} = \text{Tors}_2(S_*, f)$ and we can write down an exact sequence:

$$H_2(X - 2f) \rightarrow H_2(X_* - f) \rightarrow \text{Tors}_2(S_*, f) \rightarrow 0.$$
Since $H_2(X - 2f)$ is torsionfree, the torsion subgroup of $H_2(X_* - f)$ is mapped isomorphically onto its image. Dividing both out yields:

$$H_2(X - 2f) \xrightarrow{j} \tilde{H}_2(X_* - f) \to \text{Tors}_2(S_*, f)/\text{Tors}_2(X_* - f) \to 0.$$  

This must be shown to be the sequence of the claim. To this end consider the following commutative diagram of exact rows and columns:

\[
\begin{array}{cccc}
H_2(f) & & & \\
H_3(S^2 \times f, f) & \to & H_2(X - 2f) & \to & H_2(X - f) \\
\downarrow & & \downarrow j & & \downarrow \\
H_2(S_* - f) & \to & H_2(X_* - f) & \to & H_2(X_* - f, f) \\
\end{array}
\]

The left hand column is part of the homology sequence of the pair $(S_*, S_* - f)$ modified by the excision lemma from [La]. With duality and computations of fundamental groups we obtain ranks for the following part of this sequence:

\[
\begin{array}{cccc}
H_3(S_* - f) & \to & H_3(S_*) & \to & H_3(S^2 \times f, f) & \to & H_2(S_* - f) \\
\end{array}
\]

thus the last map which is the map of the diagram maps to $\text{Tors}_2(S_* - f)$. Keeping this in mind a diagram chase shows that a class in the image of $j$ and commensurable with $[f]$ in $H_2(X_* - f)$ must be an integer multiple of $[f]$. On the other hand a multiple fibre without its multiplicity represents the class $\frac{1}{m}[f]$ hence $[f]$ is at least $m$-divisible in $H_2(X_* - f)$ with $m = \text{lcm}(m_i)$. To state it differently the multiple fibres without their multiplicities represent classes which generate a subgroup $\mathbb{Z}_m$ in $\text{Tors}_2(S_*, f)/\text{Tors}_2(X_* - f)$. Computation using duality and fundamental groups

\[
\begin{array}{cccc}
\text{Tors}_2(S_*, f) & \cong & \text{Tors}_1(S_* - f) & \cong & \bigoplus \mathbb{Z}_{m_i} \\
\text{Tors}_2(X_* - f) & \cong & \text{Tors}_1(X_* - f) & \cong & \bigoplus \mathbb{Z}_{m_i}/\mathbb{Z}_m \\
\end{array}
\]

proves this inclusion to be an isomorphism since the cardinalities are shown to coincide, and so we are done. \(\Box\)

**Theorem 5** Let $mf_m$ be a multiple fibre of multiplicity $m$ in the elliptic surface $X_{d,q;m_1,...,m_n}$, so $m = m_i$ for some $1 \leq i \leq n$. Then in the complement of a generic regular fibre there exist three smoothly embedded spheres $\sigma_{1,2,3}$ of square -2, such that:

- $\sigma_3$ is disjoint from all multiple fibres, $\sigma_{1,2}$ from all multiple fibres except $mf_m$,
- in $H_2(X_{d,q;m_1,...,m_n})$ the difference of the first two spheres represents $[f_m]$,
- $\sigma_1 \cdot \sigma_3 = \sigma_2 \cdot \sigma_3 = \pm 1$, 

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the class of \( \sigma_3 \) is in the image of the map

\[
H_2(X_{d,q} - 2f) \to H_2(X_{d,q;m_1,\ldots,m_n - f}).
\]

Proof: A fibred neighbourhood of a multiple fibre is diffeomorphic to \( S^1 \times S^3_m \), the product of the loop with a Seifert fibration of loops over the disk with a central fibre of multiplicity \( m > 1 \). As sometimes earlier we may assume that there exist three cusp neighbourhoods.

A path

\[
w : (I, \partial I) \to (S^3_m, \partial S^3_m)
\]

which meets the central fibre in exactly one point gives rise to a cylinder of square -2 in the fibred neighbourhood simply by taking the cartesian product with the loop. By methods exploited already in lemma \( 3 \) we may glue two vanishing cells from different cusp neighbourhoods to this cylinder to get the embedded sphere \( \sigma_2 \). Apply the same construction to a new path. Up to the central fibre it coincides with the old one, then pursues the fibre once before taking up the old path again up to its end. Just homotop it off itself and the construction yields another embedded sphere \( \sigma_1 \). The sphere \( \sigma_3 \) is obtained by gluing one of the vanishing cells just used to another one in the third cusp neighbourhood. These spheres can now be seen to satisfy the claims of the theorem:

- the neighbourhoods we used can be chosen disjoint from all the other multiple fibres and a generic regular one; out of these neighbourhoods the constructed spheres map to simple arcs on the base and these again can be chosen in an appropriate way,
- the difference of the classes represented by the first two spheres is represented by the cartesian product of the central fibre in the Seifert fibration with the loop,
- the intersection with the third sphere is by construction up to sign just the selfintersection of the vanishing cell in the first cusp neighbourhood,
- because of the first assertion the third sphere is liftable.

\[ \square \]

**Theorem 6** If \( X_{d,q;m_1,\ldots,m_n} \) is an elliptic surface with non-vanishing Euler number and \( f \) a generic regular fibre, then:

- every class in \( H_2(X_{d,q;m_1,\ldots,m_n - f}) \) of square -2 is represented by a 2-sphere smoothly embedded in \( X_{d,q;m_1,\ldots,m_n - f} \),
• in the case of surfaces with \( p_g = 0 \), reflections in classes of square \(-2\) generate the group of all isometries of \( \overline{H}_2(X_{d,q:m_1,...,m_n} - f) \) which restrict to the identity on the radical, in all other cases they generate the subgroup of index at most 2 which consists of all element of real spinor norm one,

• every such reflection is realized by a diffeomorphism which is the identity on a neighbourhood of \( f \).

Proof: The claim is proved already for elliptic surfaces without multiple fibres. Moreover we are left to check the hypotheses for Ebeling’s theorem and lemma 4 in the appropriate cases since the rest of the proof of theorem 3 goes through unchanged. But they are a consequence of the exact sequence of lemma 6

\[
H_2(X_{d,q} - 2f) \to \overline{H}_2(X_{d,q:m_1,...,m_n} - f) \to \mathbb{Z}_m \to 0,
\]

and the assertion proved there that the span of classes represented by the multiple fibres maps surjective onto the quotient:

• the image of \( H_2(X_{d,q} - 2f) \) has a generating set represented by spheres according to the corollary to theorem 2, enlarge this set by three spheres as in theorem 5 for each multiple fibre, then multiple fibres are given as differences of spheres, thus the total lattice is generated,

• the intersection numbers in theorem 5 are such that the new spheres are conjugate to spheres representing elements in \( H_2(X_{d,q} - 2f) \), in this lattice all spheres of square \(-2\) are conjugate by the corollary to theorem 5,

• since in \( \overline{H}_2(X_{d,q:m_1,...,m_n} - f) \) there is a sublattice isomorphic to \( H_2(X_{d,q} - f) \), the special intersection lattice of the last hypothesis in Ebeling’s result can be found again in case that \( p_g > 0 \),

• by the same argument the lattice \( \overline{H}_2(X_{d,q:m_1,...,m_n} - f) \) decomposes as a direct sum of the radical and an \(-E_8\) lattice in the case that \( p_g = 0 \).  

\[\square\]

Corollary 3 Let \( X \) be an elliptic surface with non-vanishing Euler number, then on homology the diffeomorphisms just constructed generate the stabilizer group of the class of a general fibre in case that \( p_g(X) = 0 \), they generate the elements of real spinor norm one only in case that \( p_g(X) > 0 \).

Proof: This is immediate from the fact that the isometries of the homology lattice \( H_2(X - f) \) stabilizing its radical map surjectively onto the isometries of the lattice \( H_2(X) \) stabilizing the class of a general fibre. This argument is given in greater detail in the proof of thm.6.5 in [F/M].  

\[\square\]
Theorem 7 Let $X$ be an Enriques surface or a K3 surface, then:

- every class in $\overline{H}_2(X)$ of square $-2$ is represented by a 2-sphere smoothly embedded in $X$,
- the reflections in classes of square $-2$ generate the group of all isometries of $\overline{H}_2(X)$ with real spinor norm one,
- every such reflection is realized by a diffeomorphism, so diffeomorphisms of $X$ generate the group of isometries of real spinor norm one.

This extension to the previous results is possible since there are special K3 surfaces and Enriques surfaces which by their complex structure have sections and two-sections respectively which provide embedded spheres of square $-2$ complementing the previous sets to a basis of $\overline{H}_2$. This fact has already been exploited in the K3 case in the proof of thm.6.6 in [F/M]. Therefore and since the proofs are conceptionally analogous we restrict ourselves to the case of Enriques surfaces. Again we make use of an algebraic result:

Lemma 7 ([F]) Let $L$ be a lattice and $\Delta \subset L$ be a set of vectors of square $-2$ such that:

1) $\Delta$ is a basis for $L$,
2) the elements of $\Delta$ intersect according to the Dynkin diagram of $-E_{10}$.

Then $\Gamma_\Delta$ is the subgroup of the isometry group of $L$ consisting of automorphisms of real spinor norm one and $\Delta$ is contained in a single $\Gamma_\Delta$-orbit which consists of all vectors of square $-2$ in $L$.

Proof of theorem: As already stated there is a sphere representing a class $\sigma$ of square $-2$ such that $\overline{H}_2(X)$ is spanned by $\sigma$ and classes from $\overline{H}_2(X-f)$. In the image of $\overline{H}_2(X-f)$ choose a set of classes $\{\beta'_i\}_{1 \leq i \leq 8}$ intersecting according to the Dynkin diagram of $-E_8$ and let $f$ be the unique class in the radical which pairs to one with $\sigma$. Then set

$$
\beta_i = \beta'_i - \langle \beta'_i, \sigma \rangle f \quad 1 \leq i \leq 8 \\
\beta_9 = f - \ell \\
\beta_{10} = \sigma,
$$

where $\ell$ is the long vector of the $-E_8$ spanned by $\{\beta_i\}_{1 \leq i \leq 8}$, and it is to check these $\beta_i$ intersect according to the Dynkin diagram of $-E_{10}$. Moreover they all are represented by embedded spheres since the $\{\beta_i\}_{1 \leq i \leq 9}$ are given by classes from $\overline{H}_2(X-f)$ which are all represented by spheres according to thm.7.

We finally apply the lemma and conclude as in the proof of thm.6. Since generators of $\Gamma_\Delta$ are induced from diffeomorphisms, so is the whole group of reflections which proves part of the third claim. There are classes which are represented by embedded spheres and hence they are all, for they form a single orbit under $\Gamma_\Delta$, proving the first claim. The rest is immediate from the lemma. \(\square\)
Applications of Seiberg-Witten invariants

We introduce the following terminology for Seiberg-Witten theory which is for convenience (almost) that of our main reference, cf. [B, pp. 11,12,14].

**SC-structure**

An equivalence class of a complex vector bundle $W$, which is fibrewise irreducible as module of the Clifford bundle; the set of SC-structures corresponds bijectively to the set of Spin$^c$-structures.

**SW-multiplicity map**

The Seiberg-Witten map $n_X$ which maps a SC-structure $W$ to the Euler class of a moduli space of solutions to the monopole equations on $W$.

**SW-multiplicity map with integer values**

The map $n_X^\sigma$ which assigns to a SC-structure $W$ the class $n_X(W)$ evaluated with respect to an orientation determined by a choice of a homology orientation $\sigma$, i.e. an orientation for $H^1(X, \mathbb{R}) \oplus H^{2, +}(X, \mathbb{R})$.

**SW-structure**

A SC-structure with non-trivial SW-multiplicity.

**Basic class**

The Chern class of a SW-structure.

---

Theorem 8 ([B]) If $X$ is a minimal Kähler surface of non-negative Kodaira dimension then the homology class $k$ of its canonical divisor is invariant up to sign under oriented diffeomorphism.

Theorem 9 ([B]) Let $X$ be a Kähler surface with positive geometric genus then $k$ and $-k$ are basic classes. Moreover there is a unique SW-structure with determinant equal to $-k$ modulo torsion.

Theorem 10 ([B],[W]) The SW-multiplicities with integer values of Kähler surfaces with $b^+ \geq 3$ have the following properties:

i) Invariance under diffeomorphisms $\varphi : X \to Y$:

$$n_Y^\varphi(W) = n_X^{\varphi^*}(\varphi^*W).$$

ii) Antisymmetry under change of homology orientation:

$$n_X^{\sigma}(W) = -n_X^{\varphi}(W).$$

Lemma 8 Let $\varphi$ be a diffeomorphism of a Kähler surface $X$ changing the homology orientation, $c$ a class fixed by $\varphi$ up to torsion. If $W$ is a SW-structure with determinant equal to $c$ modulo torsion, then so is $\varphi^*W$. Moreover the two structures are different.
Proof: Obviously $W$ and $\varphi^*W$ have the same determinant up to torsion. By the preceding theorem and the assumption on $\varphi$ we have:

$$n^\varphi_X(\varphi^*W) = -n^\varphi_X(\varphi^*W) = -n^{\varphi^*\sigma}_X(\varphi^*W) = -n^\sigma_X(W).$$

Hence the values of $n^\sigma_X$ are non-trivial both, and different. ⧫

Lemma 9 The homology lattice $(\bar{H}_2, \langle, \rangle)$ of an elliptic surface with positive Euler number and positive geometric genus contains a hyperbolic direct summand orthogonal to the homology class $k$ of its canonical divisor.

Proof: This is an obvious corollary to the classification of indefinite even unimodular forms in the $K3$ case where $k$ is trivial. Otherwise choose a class which pairs to one with a primitive class commensurable with $k$. Together they span a unimodular sublattice, thus its orthogonal complement is an orthogonal direct summand of $\bar{H}$. The assumption on the geometric genus implies that the intersection form restricted to this summand is indefinite. Of course it is unimodular, and it is even, because $k$ is a characteristic element for the intersection form. Therefore as in the trivial case we may appeal to the classification of indefinite even unimodular forms to get a hyperbolic summand as claimed. ⧫

Lemma 10 In $O_k$ there is an isometry $\iota$ of the homology lattice $\bar{H}$ of an elliptic surface as above which is not induced from any diffeomorphism of the surface.

Proof: Let $\iota$ be inversion at zero on a hyperbolic summand as given in lemma 9 and the identity on the orthogonal complement, then $\iota$ stabilizes the canonical class but changes the homology orientation. Given a diffeomorphism which changes the homology orientation, then by lemma 9 a class which is fixed up to torsion does not have a unique SW-structure. But as we know from thm. 8 there is a unique SW-structure with determinant $-k$ modulo torsion, so we may reverse the conclusion above to disprove the existence of a diffeomorphism inducing $\iota$. ⧫

Lemma 11 ([F/M1]) The diffeomorphism group of an elliptic surface with positive Euler number contains a diffeomorphism $\sigma$ with $\sigma(k) = -k$.

The idea of the construction is to embed the surface in projective space. Complex conjugation then maps it to another surface, the fibre class to the negative of the fibre class on the second surface.
On the other hand the two surfaces are deformation equivalent, a family containing both gives rise to at least one diffeomorphism between them preserving the canonical class. Since the canonical class is a multiple of the fibre class, the composite map does the job.

**Theorem 11** Let $X$ be a minimal elliptic surface with positive Euler number, $k$ the homology class of its canonical divisor and

$$
\psi : \text{Diff}(X) \longrightarrow O(\overline{\mathcal{H}}_2(X))
$$

the natural homomorphism, then the image of $\psi$ is given by

$$
im \psi = \begin{cases} 
O'_k \cdot \{\sigma, \text{id}\} & \text{in case } p_g > 0, \\
O' & \text{if } X \text{ is a K3 surface}, \\
O_k \cdot \{\pm \text{id}\} & \text{in case } p_g = 0, X \text{ non rational}, \\
O & \text{in the special case of Enriques surfaces}, \\
O & \text{in case } X \text{ rational}.
\end{cases}
$$

**Proof:** The result is well known in the rational case, cf. [Wa], and has been previously established for K3 surfaces by Donaldson [D]. Our proof for K3 surfaces is based on the fact $O' \subset \text{im } \psi \subset O$, which is obvious by thm [7] and on the existence of $\iota \in O, \iota \notin \text{im } \psi$ as given by lemma [10]. Then the result follows from $[O:O'] = 2$ (notice that consequently $\sigma \in O'$).

For Enriques surfaces we have $O' \subset \text{im } \psi \subset O$ as well, but this time lemma [11] does not apply due to the fact that $p_g = 0$, i.e. the canonical divisor - though numerically trivial - is not effective. Instead we use $\sigma \in \text{im } \psi$ of lemma [11] which can be shown not to be contained in $O'$.

In the case of positive geometric genus and $k$ non-trivial we have $O'_k \subset \text{im } \psi$ by the corollary to thm [8], since $k$ is a non-trivial multiple of the fibre class. On the other hand we just got $\text{im } \psi \subset O_k \cdot \{\pm \text{id}\}$ which contains $O'_k$ as subgroup of index at most four. Moreover the preceding lemmas provide us with elements $\sigma, \iota \in O_k \cdot \{\pm \text{id}\}$ such that $\sigma \in \text{im } \psi, \sigma \notin O'_k$ and $\iota \in O_k, \iota \notin \text{im } \psi$. Hence $O_k \cdot \{\pm \text{id}\}$ contains $\text{im } \psi = O'_k \cdot \{\sigma, \text{id}\}$ as subgroup of index two.

The argument is similar in the case of vanishing geometric genus and $X$ non-rational, i.e. with a nef canonical divisor, so $k$ is a non-trivial positive multiple: We have $O_k \subset \text{im } \psi \subset O_k \cdot \{\pm \text{id}\}$ by the appropriate results, the corollary to thm [8] and thm [8]. Since $\sigma \in \text{lemma [11]}$ is in $\text{im } \psi$ but not in $O_k$ the second relation is in fact an equality of the groups. \(\square\)
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