The Static Maxwell System in Three Dimensional Inhomogeneous Isotropic Media, Generalized Non-Euclidean Modification of the System \((\mathbb{R})\) and Fueter Construction

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Abstract. The novel approach of modified quaternionic analysis in \(\mathbb{R}^3\) was initiated by Leutwiler and Eriksson since 1992. Further results have led to modern hyperbolic function theory. Now relevant applications in terms of electrostatic fields are presented in accordance with generalized approach of Bryukhov and Kähler, first formulated in Aveiro, Portugal, 2015. In particular, new properties of \(\alpha\)-hyperbolic harmonic fields and \(\alpha\)-axial-hyperbolic harmonic fields are described. Applications of tools of radially holomorphic functions and the Fueter potential allow us to characterize various subclasses of meridional fields in an axially symmetric inhomogeneous medium in the framework of GASPT.

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1. Notations, Introduction and Preliminaries

1.1. Notations

The real algebra of quaternions \(\mathbb{H}\) is a four dimensional skew algebra over the real field generated by real unity 1. Three imaginary unities \(i, j, \) and \(k\) satisfy to the following multiplication rules

\[i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k.\]  (1.1)

The independent quaternionic variable is written as

\[x = x_0 + ix_1 + jx_2 + kx_3,\]
its scalar and vectorial parts are defined as

\[ Sc(x) = x_0, \quad Vec(x) = ix_1 + jx_2 + kx_3 \]

respectively. In the case \( \rho = \sqrt{x_1^2 + x_2^2 + x_3^2} \neq 0 \), \( I = \frac{ix_1 + jx_2 + kx_3}{\rho} \) we obtain \( x = x_0 + I\rho \), where \( I^2 = -1 \).

The conjugation of the independent quaternionic variable is defined as the automorphism

\[ x \mapsto \overline{x} := Sc(x) - Vec(x) = x_0 - ix_1 - jx_2 - kx_3. \]

In this way we have to deal with the Euclidean norm in \( \mathbb{R}^4 \)

\[ \|x\|^2 := x\overline{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2, \]

and the identification

\[ x = x_0 + ix_1 + jx_2 + kx_3 \sim (x_0, x_1, x_2, x_3) \]

between \( \mathbb{H} \) and \( \mathbb{R}^4 \) is valid. Moreover, for each non-zero value of \( x \) an unique inverse value exists: \( x^{-1} = \overline{x}/\|x\|^2 \).

We obtain the space of reduced quaternions by imposing \( x_3 = 0 \). Hereby, the independent reduced quaternionic variable is \( x = x_0 + ix_1 + jx_2 \) and, from now on, we identified it with the vector \( (x_0, x_1, x_2) \in \mathbb{R}^3 \).

Let us denote by \( x_0 \) the longitudinal variable and denote by \( \rho = \sqrt{x_1^2 + x_2^2} \) the cylindrical radial variable in \( \mathbb{R}^3 \). In the case \( \rho \neq 0 \) the zenith angle \( \varphi \) and the azimuth angle \( \theta \) are represented by formulas:

\[ \varphi = \arccos \frac{x_0}{\rho} \quad (0 \leq \varphi \leq \pi), \quad \theta = \arcsin \frac{x_2}{\rho} \quad (0 \leq \theta \leq 2\pi). \]

As seen, the independent reduced quaternionic variable can be described by formulas:

\[ x = x_0 + \rho(i \cos \theta + j \sin \theta) = r(\cos \varphi + i \sin \varphi \cos \theta + j \sin \varphi \sin \theta). \]

The dependent reduced quaternionic variable is written as

\[ u = u_0 + iu_1 + ju_2 \sim (u_0, u_1, u_2), \]

its scalar and vectorial parts are defined as

\[ Sc(u) = u_0, \quad Vec(u) = iu_1 + ju_2. \]

The conjugation of the dependent reduced quaternionic variable is defined as the automorphism

\[ u \mapsto \overline{u} := Sc(u) - Vec(u) = u_0 - iu_1 - ju_2. \]

**Definition 1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be an open set. Each continuously differentiable mapping \( u = u_0 + iu_1 + ju_2 : \Omega \to \mathbb{R}^3 \) is called the reduced quaternion-valued \( C^1 \)-function \( u = u(x) \) in \( \Omega \).
1.2. Introduction and Preliminaries

Some important classes of three dimensional harmonic solutions of the static Maxwell system in homogeneous media (or the Riesz system, see, e.g., [9, 41]):

\[
\begin{align*}
\text{div } \vec{E} &= 0, \\
\text{curl } \vec{E} &= 0,
\end{align*}
\] (1.2)

where \( \vec{E} = (E_0, E_1, E_2) \) in \( \mathbb{R}^3 = \{ (x_0, x_1, x_2) \} \), have been constructed by Brackx, Delange, Sommen et al. in terms of the reduced quaternion-valued monogenic functions \( u = u(x) = u_0(x) + iu_1(x) + ju_2(x) \) in the framework of quaternionic analysis in \( \mathbb{R}^3 \) (see, e.g., [8, 9, 42, 58, 20, 40, 41, 64]).

General class of analytic solutions of the system (1.2) can be equivalently represented in terms of general class of analytic solutions of the system (1.3):

\[
\begin{align*}
\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} &= 0, \\
\frac{\partial u_0}{\partial x_1} &= -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} &= -\frac{\partial u_2}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} &= \frac{\partial u_2}{\partial x_1}.
\end{align*}
\] (1.3)

In this way the scalar potential \( h = h(x_0, x_1, x_2) \) in simply connected open domains \( \Lambda \subset \mathbb{R}^3 \), where \( \vec{E} = \text{grad } h \), satisfies the Laplace equation:

\[
\text{div}(\text{grad } h) = \Delta h = 0. 
\] (1.4)

Some important classes of exact solutions of the static Maxwell system in an \( x_2 \)-inhomogeneous isotropic medium \( (x_2 > 0) \):

\[
\begin{align*}
\text{div } (x_2^{-1}\vec{E}) &= 0, \\
\text{curl } \vec{E} &= 0
\end{align*}
\] (1.5)

have been constructed, in fact, by Leutwiler and Eriksson in terms of the reduced quaternionic power series with complex coefficients \( u(x) = u_0(x) + iu_1(x) + ju_2(x) \) in the framework of modified quaternionic analysis in \( \mathbb{R}^3 \) (see, e.g., [56, 57, 58, 27]), where \( (E_0, E_1, E_2) = (u_0, -u_1, -u_2) \). This area of applications in mathematical physics was briefly mentioned by Leutwiler in 2000 [58].

General class of \( C^1 \)-solutions of the system (1.5) can be equivalently represented in terms of general class of \( C^1 \)-solutions of the system (1.6):

\[
\begin{align*}
x_2\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right) + u_2 &= 0, \\
\frac{\partial u_0}{\partial x_1} &= -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} &= -\frac{\partial u_2}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} &= \frac{\partial u_2}{\partial x_1}.
\end{align*}
\] (1.6)

**Definition 1.2.** Each \( C^1 \)-solution of the system (1.5) in a simply connected open domain \( \Lambda \subset \mathbb{R}^3 \) \( (x_2 > 0) \) is called a continuously differentiable hyperbolic harmonic electrostatic field \( \vec{E} \) in \( \Lambda \).

The system (H) can be considered as a non-Euclidean hyperbolic modification of the system (R) with respect to the hyperbolic metric, defined on
the halfspace \( \{ x_2 > 0 \} \) by formula (see e.g., [1, 55, 56, 57, 58]):

\[
ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2}{x_2^2}.
\] (1.7)

The scalar potential \( h = h(x_0, x_1, x_2) \) in simply connected open domains \( \Lambda \) \((x_2 > 0)\), where \( \vec{E} = \text{grad} h \), satisfies the hyperbolic version of the Laplace equation:

\[
x_2^2 \text{div} \left( x_2^{-1} \text{grad} h \right) = x_2 \Delta h - \frac{\partial h}{\partial x_2} = 0.
\] (1.8)

Some important classes of exact solutions of the static Maxwell system in an axially symmetric inhomogeneous isotropic medium \((\rho > 0)\):

\[
\begin{align*}
\text{div} \left( \rho^{-1} \vec{E} \right) &= 0, \\
\text{curl} \vec{E} &= 0
\end{align*}
\] (1.9)

have been constructed by Bryukhov, Kähler and Aksenov in three dimensional setting, using the separation method in cylindrical coordinates (see, e.g., [11, 12, 15, 4]).

General class of \( C^1 \)-solutions of the static Maxwell system (1.9) can be equivalently represented in terms of general class of \( C^1 \)-solutions of the system (1.10):

\[
(A_3) \left\{ \begin{array}{l}
(x_1^2 + x_2^2) \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) + (x_1 u_1 + x_2 u_2) = 0, \\
\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1},
\end{array} \right.
\] (1.10)

where \( \vec{E} = (u_0, -u_1, -u_2) \).

**Definition 1.3.** Each \( C^1 \)-solution of the system (1.9) in a simply connected open domain \( \Lambda \subset \mathbb{R}^3 \) \((\rho > 0)\) is called a continuously differentiable axial-hyperbolic harmonic electrostatic field \( \vec{E} \) in \( \Lambda \).

The system \((A_3)\) can be considered as a non-Euclidean axial-hyperbolic modification of the system \((R)\) with respect to the conformal metric, defined outside the axis \( x_0 \) by formula (see e.g., [55, 12]):

\[
ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2}{\rho^2}.
\] (1.11)

The scalar potential \( h = h(x_0, x_1, x_2) \) in simply connected open domains \( \Lambda \) \((\rho > 0)\), where \( \vec{E} = \text{grad} h \), satisfies an axially symmetric Laplace-Beltrami equation:

\[
\rho^3 \text{div} \left( \rho^{-1} \text{grad} h \right) = (x_1^2 + x_2^2) \Delta h - \left( x_1 \frac{\partial h}{\partial x_1} + x_2 \frac{\partial h}{\partial x_2} \right) = 0.
\] (1.12)

One of the main stones of stumbling of applications of modified quaternionic analysis in mathematical physics is the problem of holistic interpretation of Fueter construction in \( \mathbb{R}^3 \) [35, 55, 15]:

\[
F = F(x) = u_0 + iu_1 + ju_2 = u_0(x_0, \rho) + I u_\rho(x_0, \rho),
\] (1.13)
\[ x = x_0 + I\rho, \quad I = \frac{i x_1 + j x_2}{\rho} = i \cos \theta + j \sin \theta, \quad I^2 = -1, \]  \hspace{1cm} (1.14)

where

\[ u_1 = \frac{x_1}{\rho} u_\rho = u_\rho \cos \theta, \quad u_2 = \frac{x_2}{\rho} u_\rho = u_\rho \sin \theta. \]

A special class of functions of the reduced quaternionic variable, associated with classical holomorphic \( F = F(x) = u_0 + i u_1 + j u_2 \), can be characterized as joint class of analytic solutions of the system (\( H \)) and the system (\( A_3 \)) under the special condition of Fueter construction in \( \mathbb{R}^3 \) (see, e.g., \([55, 56, 10, 12, 15]\)):

\[ u_1 x_2 = u_2 x_1. \]  \hspace{1cm} (1.15)

Elementary functions of the reduced quaternionic variable belong to the special class, in particular:

- the power function \( (\forall n \in \mathbb{Z}) \quad F(x) = x^n = r^n (\cos n\varphi + i \sin n\varphi) \);

- the exponential function \( F(x) = e^x = e^{x_0} (\cos \rho + i \sin \rho) \);

- the cosine function \( F(x) = \cos x = \frac{1}{2} (e^{-lx} + e^{lx}) \);

- the sine function \( F(x) = \sin x = \frac{1}{2} (e^{-lx} - e^{lx}) \);

- the logarithmic function (principal value) \( F(x) = \ln x = \ln r + i \varphi \).

**Definition 1.4.** Let \( \Lambda \subset \mathbb{R}^3 (\rho > 0) \) be a simply connected open domain. Let \( F = F(x) \) be a function of the reduced quaternionic variable, associated with classical holomorphic in \( \Lambda \). Each continuously differentiable mapping \( F = u_0 + i u_1 + j u_2 : \Lambda \rightarrow \mathbb{R}^3 \) under condition of non-vanishing Jacobian determinant in a point \( x^* \in \Lambda \) is called a regular Fueter mapping of the first kind in \( x^* \). Each continuously differentiable mapping \( \overline{F} = u_0 - i u_1 - j u_2 : \Lambda \rightarrow \mathbb{R}^3 \) under condition of non-vanishing Jacobian determinant in a point \( x^* \in \Lambda \) is called a regular Fueter mapping of the second kind in \( x^* \).

**2. The Static Maxwell System in Three Dimensional Inhomogeneous Isotropic Media, Principal Invariants of the \( EFG \) Tensor, the System \((GR)\), the System \((GHR)\) and \( \alpha \)-Hyperbolic Harmonic Electrostatic Fields**

General class of \( C^1 \)-solutions of the static Maxwell system in inhomogeneous isotropic media, provided by a variable \( C^1 \)-coefficient \( \phi = \phi(x_0, x_1, x_2) > 0 \), \( \phi^{-1}(x_0, x_1, x_2) > 0 \):

\[
\begin{aligned}
\text{div} (\phi \vec{E}) &= 0, \\
\text{curl} \, \vec{E} &= 0,
\end{aligned}
\]  \hspace{1cm} (2.1)

can be equivalently represented in terms of general class of \( C^1 \)-solutions of the system (2.2), first described by Bryukhov and Kähler in Aveiro, 2015:

\[
(GR) \quad \left\{ \begin{array}{l}
\phi \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) + \left( \frac{\partial \phi}{\partial x_0} u_0 - \frac{\partial \phi}{\partial x_1} u_1 - \frac{\partial \phi}{\partial x_2} u_2 \right) = 0, \\
\frac{\partial u_1}{\partial x_1} = -\frac{\partial u_0}{\partial x_0}, \quad \frac{\partial u_2}{\partial x_2} = -\frac{\partial u_0}{\partial x_0}, \quad \frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_2},
\end{array} \right.
\]  \hspace{1cm} (2.2)

where \( \vec{E} = (u_0, -u_1, -u_2) \).
The system \( GR \) can be considered as a generalized non-Euclidean modification of the system \( R \) with respect to the conformal metric
\[
ds^2 = \phi^2(dx_0^2 + dx_1^2 + dx_2^2).\]

The scalar potential \( h = h(x_0, x_1, x_2) \) in simply connected open domains \( \Lambda \subset \mathbb{R}^3 \) \((\phi^{-1} > 0)\), where \( \vec{E} = \text{grad} h \), satisfies a second order linear elliptic equation of divergence form:
\[
\text{div}(\phi \text{ grad } h) = 0. \tag{2.4}
\]

The Eq. \((2.4)\) in general three dimensional setting leads to an equation (see, e.g., \[74\]):
\[
\phi \left( \frac{\partial^2 h}{\partial x_0^2} + \frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} \right) + \frac{\partial \phi}{\partial x_0} \frac{\partial h}{\partial x_0} + \frac{\partial \phi}{\partial x_1} \frac{\partial h}{\partial x_1} + \frac{\partial \phi}{\partial x_2} \frac{\partial h}{\partial x_2} = 0. \tag{2.5}
\]

The rich variety of modern models of electrostatic fields in terms of the static Maxwell system \((2.1)\) is closely related with problems of non-Euclidean geometry (Euclidean geometry is realized in the case \( \phi = 1 \)). In this way we have to deal with the Laplace-Beltrami equation
\[
\Delta_{LB} h := \phi^{-3} \text{div}(\phi \text{ grad } h) = 0 \tag{2.6}
\]
with respect to the conformal metric \((2.3)\) (see e.g., \[24, 74, 1\]). Local isothermal coordinates allow us to demonstrate various orthogonal families of curves and geometric properties of surfaces of equal scalar potential (see e.g., \[24, 74, 23, 1, 62, 65, 6\]).

**Definition 2.1.** In terms of the static Maxwell system \((2.1)\) the Jacobian matrix \( J(\vec{E}) \) with components \( J_{lm}(\vec{E}) = \frac{\partial E_l}{\partial x_m} \) \((l, m = 0, 1, 2)\) is called the electric field gradient (EFG) tensor.

Principal invariants of the symmetric EFG tensor \( J(\vec{E}) \)
\[
\left( \begin{array}{ccc} 
\frac{\partial E_0}{\partial x_0} & \frac{\partial E_0}{\partial x_1} & \frac{\partial E_0}{\partial x_2} \\
\frac{\partial E_1}{\partial x_0} & \frac{\partial E_1}{\partial x_1} & \frac{\partial E_1}{\partial x_2} \\
\frac{\partial E_2}{\partial x_0} & \frac{\partial E_2}{\partial x_1} & \frac{\partial E_2}{\partial x_2} 
\end{array} \right) = \left( \begin{array}{ccc} 
\frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial x_1} & \frac{\partial u_0}{\partial x_2} \\
\frac{\partial u_1}{\partial x_0} & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\
\frac{\partial u_2}{\partial x_0} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} 
\end{array} \right) \tag{2.7}
\]
are characterized by formulas:
\[
\begin{cases}
I_{J(\vec{E})} = \text{tr}J(\vec{E}), \\
II_{J(\vec{E})} = J_{00}(\vec{E})J_{11}(\vec{E}) + J_{11}(\vec{E})J_{22}(\vec{E}) + J_{22}(\vec{E})J_{00}(\vec{E}) - (J_{01}(\vec{E}))^2 - (J_{12}(\vec{E}))^2 - (J_{20}(\vec{E}))^2, \\
III_{J(\vec{E})} = \text{det}J(\vec{E}),
\end{cases} \tag{2.8}
\]
where \( I_{J(\vec{E})} = \lambda_1 + \lambda_2 + \lambda_3, \ II_{J(\vec{E})} = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1, \ III_{J(\vec{E})} = \lambda_1 \lambda_2 \lambda_3, \)
and \( \lambda_1, \lambda_2, \lambda_3 \) are real eigenvalues of an orthogonal system of eigenvectors in \( \mathbb{R}^3 \).

**Definition 2.2.** Let \( \Lambda \subset \mathbb{R}^3 \) be a simply connected open domain in terms of the static Maxwell system \((2.1)\) in an inhomogeneous isotropic medium, and \( \vec{E} = (u_0, -u_1, -u_2) \). Each point \( x^* \in \Lambda, \) where continuously differentiable
mapping $\mathbf{u} = u_0 - iu_1 - ju_2 : \Lambda \to \mathbb{R}^3$ satisfies condition $\det J(\mathbf{E}) = 0$, is called a singular point of the EFG tensor $J(\mathbf{E})$. The set of singular points $x^* \in \Lambda$ is called the singular set of the EFG tensor $J(\mathbf{E})$ in $\Lambda$.

**Remark 2.3.** Geometric properties of the singular sets of continuously differentiable mappings often play crucial roles in applied problems (see, e.g., [5, 71, 37]). Meanwhile, properties of the singular sets of the EFG tensor $J(\mathbf{E})$ in an axially symmetric inhomogeneous isotropic medium under the special condition of Fueter construction in $\mathbb{R}^3$ (1.15) have not been studied in detail.

**Definition 2.4.** Let $\Lambda \subset \mathbb{R}^3 (\rho > 0)$ be a simply connected open domain. Let $F = F(x)$ be a function of the reduced quaternionic variable, associated with classical holomorphic in $\Lambda$. Each continuously differentiable mapping $F = u_0 + iu_1 + ju_2 : \Lambda \to \mathbb{R}^3$ under condition of vanishing Jacobian determinant in a point $x^* \in \Lambda$ is called a singular Fueter mapping of the first kind in $x^*$. Each continuously differentiable mapping $\bar{F} = u_0 - iu_1 - ju_2 : \Lambda \to \mathbb{R}^3$ under condition of vanishing Jacobian determinant in a point $x^* \in \Lambda$ is called a singular Fueter mapping of the second kind in $x^*$.

The characteristic equation of the EFG tensor (2.7) can be described as (see, e.g., [13]):

$$\lambda^3 - I J(\mathbf{E}) \lambda^2 + II J(\mathbf{E}) \lambda - III J(\mathbf{E}) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0. \quad (2.9)$$

The Eq. (2.9) after the substitution $\lambda = \mu + \frac{I J(\mathbf{E})}{3}$ leads to an equation:

$$\left(\mu + \frac{I J(\mathbf{E})}{3}\right)^3 - I J(\mathbf{E}) \left(\mu + \frac{I J(\mathbf{E})}{3}\right)^2 + II J(\mathbf{E}) \left(\mu + \frac{I J(\mathbf{E})}{3}\right) - III J(\mathbf{E}) = 0.$$

Let us denote by $p, q, Q$ expressions in terms of principal invariants:

$$p = II J(\mathbf{E}) \frac{1}{3}(I J(\mathbf{E}))^2, \quad q = -III J(\mathbf{E}) + \frac{1}{3}I J(\mathbf{E}) II J(\mathbf{E}) - \frac{2}{27}(I J(\mathbf{E}))^3, \quad Q = \frac{p^3}{27} + \frac{q^2}{4}.$$ 

In this way we have to deal with an incomplete cubic equation

$$\mu^3 + p \mu + q = 0. \quad (2.10)$$

The roots of the Eq. (2.10) are expressed in the form of Cardano’s solution (see, e.g., [69]):

$$\mu_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} + \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}, \quad \mu_2, 3 = -\frac{\mu_1}{2} \pm \frac{i\sqrt{3}}{2} \left(\sqrt[3]{-\frac{q}{2} + \sqrt{Q}} - \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}\right), \quad (2.11)$$

herein

$$\sqrt[3]{-\frac{q}{2} + \sqrt{Q}} \sqrt[3]{-\frac{q}{2} - \sqrt{Q}} = -\frac{p}{3}.$$
The number of real roots of the Eq. (2.10) depends on the sign of $Q$.
In the case $Q = 0$ there is one real root (2.11) and another real root (2.12) of double multiplicity (in particular, in the case $p = q = 0$).

In the case $Q < 0$ there are three real roots (2.11), (2.12).

General class of $C^1$-solutions of the static Maxwell system in $x_2$-inhomogeneous isotropic media, provided by a variable $C^1$-coefficient $\phi = \phi(x_2) > 0$, $\phi^{-1}(x_2) > 0$:

$$\begin{align*}
\text{div} (\phi(x_2) \vec{E}) &= 0, \\
\text{curl} \vec{E} &= 0,
\end{align*}$$

(2.13)
can be equivalently represented in terms of general class of $C^1$-solutions of the system (2.14):

$$\begin{align*}
(GHR) \begin{cases}
\phi(x_2) \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) - \frac{d\phi}{dx_2} u_2 &= 0, \\
\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0},
\end{cases}
\end{align*}$$

(2.14)

where in simply connected open domains $\Lambda \subset \mathbb{R}^3$ $\vec{E} = (u_0, -u_1, -u_2) = \text{grad} h$.

The system $(GHR)$ can be considered as a generalized non-Euclidean hyperbolic modification of the system $(R)$ with respect to the conformal metric

$$ds^2 = \phi^2(x_2) (dx_0^2 + dx_1^2 + dx_2^2).$$

(2.15)

The equation (2.5) in given setting can be described as:

$$\phi \left( \frac{\partial^2 h}{\partial x_0^2} + \frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} \right) + \frac{d\phi}{dx_2} \frac{\partial h}{\partial x_2} = 0.$$  

If $\phi(x_2) = x_2^{-\alpha}$ ($\alpha \in \mathbb{R}$), then

$$x_2 \Delta h - \alpha \frac{\partial h}{\partial x_2} = 0.$$  

(2.16)

The Eq. (2.16) is called the Weinstein equation or sometimes “generalized axially symmetric potential equation in three variables” (up to the sign of the parameter $\alpha$) (see, e.g., [80, 43, 74, 2, 31, 81, 6]). Some important properties of solutions of the Weinstein equation have been studied, in particular, in the framework of hyperbolic function theory (see, e.g., [28, 29, 30, 31, 59, 60]).

In the case $\alpha = 1$ solutions of the Eq. (2.16) are called hyperbolic harmonic functions with respect to the hyperbolic metric (1.7) (see, e.g., [1, 2, 58, 31]). In the general case of real $\alpha$ exact solutions of the Eq. (2.16) shall henceforth be called $\alpha$-hyperbolic harmonic functions with respect to the conformal metric, defined on the halfspace $\{x_2 > 0\}$ by formula (see, e.g., [59, 60])

$$ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2}{x_2^{2\alpha}}.$$  

(2.17)
Occasionally in the case \( \alpha = -1 \) solutions of the Eq. (2.16) are called “modified harmonic functions”, herewith elements of the class of the restrictions of the polynomial modified harmonic functions to the half-sphere \( S_+ = \{(x_0, x_1, x_2) : x_0^2 + x_1^2 + x_2^2 = 1, \ x_2 > 0 \} \) are called “modified spherical harmonics” \[59\]. An orthonormal system of modified spherical harmonics has been recently constructed by Leutwiler, using the separation method in polar coordinates and results of modified quaternionic analysis in \( \mathbb{R}^3 \) \[6, 66, 60\].

In the case of \( x_2 \)-inhomogeneous isotropic media, provided by a variable coefficient \( \phi(x_2) = x_2 - \alpha^2 \), the static Maxwell system (2.13) takes the form

\[
\begin{align*}
\text{div} \ (x_2^{-\alpha} \vec{E}) &= 0, \\
\text{curl} \ \vec{E} &= 0,
\end{align*}
\]

whereas the system \((GHR)\) becomes the system (2.19):

\[
(H_3^\alpha) \begin{cases}
x_2 \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) + \alpha u_2 = 0, \\
\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0},
\end{cases}
\]

**Definition 2.5.** Each \( C^1 \)-solution of the system (2.18) in a simply connected open domain \( \Lambda \subset \mathbb{R}^3 \) \((x_2 > 0)\) is called a continuously differentiable \( \alpha \)-hyperbolic harmonic electrostatic field \( \vec{E} \) in \( \Lambda \).

The system \((H_3^\alpha)\) can be considered as a generalized non-Euclidean \( \alpha \)-hyperbolic modification of the system \((R)\) with respect to the conformal metric (2.17).

Some important applied properties of \( \alpha \)-hyperbolic harmonic electrostatic fields in three dimensional setting can be explicitly demonstrated by means of the separation method in Cartesian coordinates (see, e.g., \[65, 6, 66, 77, 50, 59\]).

Let us first look for a class of exact solutions of the equation (2.16) under condition \( h(x_0, x_1, x_2) = g(x_0, x_2)s(x_1) \):

\[
s x_2 \left( \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial x_2^2} \right) - \alpha s \frac{\partial g}{\partial x_2} + g x_2 \frac{d^2 s}{dx_1^2} = 0.
\]

If \( g(x_0, x_2)s(x_1) \neq 0 \), the first relations of separation of variables can be introduced:

\[
\frac{1}{g} \left( \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial x_2^2} \right) - \alpha \frac{\partial g}{g x_2} \frac{\partial}{\partial x_2} = -\frac{1}{s} \frac{d^2 s}{dx_1^2} = \tilde{\lambda}^2 \quad (\tilde{\lambda} = \text{const} \in \mathbb{R}).
\]

The relations (2.20) are equivalent to the following system of equations:

\[
\begin{align*}
\frac{\partial^2 q}{\partial x_0^2} + \frac{\partial^2 q}{\partial x_2^2} - \alpha \frac{\partial q}{x_2} \frac{\partial}{\partial x_2} - \tilde{\lambda}^2 q &= 0, \\
\frac{d^2 s}{dx_1^2} + \tilde{\lambda}^2 s &= 0.
\end{align*}
\]

In particular, exact solutions of the second equation can be characterized in terms of trigonometric functions (see, e.g., \[15\]):

\[
s_{\tilde{\lambda}}(x_1) = C_{1,\tilde{\lambda}} \cos \tilde{\lambda} x_1 + C_{2,\tilde{\lambda}} \sin \tilde{\lambda} x_1.
\]

The periodicity of the solutions \( s = s_{\tilde{\lambda}}(x_1) \) implies \( \tilde{\lambda} \in \mathbb{Z} \).
Let us further look for a class of exact solutions of the first equation of the system (2.21) under condition \( g(x_0, x_2) = \Xi(x_0)\Y(x_2) \neq 0 \):

\[
\frac{1}{\Xi} \frac{d^2 \Xi}{dx_0^2} + \frac{1}{\Y} \frac{d^2 \Y}{dx_2^2} - \frac{\alpha}{\Y x_2} \frac{d\Y}{dx_2} - \lambda^2 = 0.
\]

The second relations of separation of variables can be introduced:

\[
\frac{1}{\Y} \frac{d^2 \Y}{dx_2^2} - \frac{\alpha}{\Y x_2} \frac{d\Y}{dx_2} - \lambda^2 = -\frac{1}{\Xi} \frac{d^2 \Xi}{dx_0^2} = -\beta^2 \quad (\beta = \text{const} \in \mathbb{R}). \tag{2.22}
\]

The relations (2.22) are equivalent to the following system of ordinary differential equations:

\[
\begin{cases}
  x_2 \frac{d^2 \Y}{dx_2^2} - \alpha x_2 \frac{d\Y}{dx_2} + (\beta^2 - \lambda^2)x_2^2 \Y = 0, \\
  \frac{d^2 \Xi}{dx_0^2} - \beta^2 \Xi = 0.
\end{cases} \tag{2.23}
\]

In particular, exact solutions of the second equation can be characterized in terms of hyperbolic functions (see, e.g., [44]):

\[
\Xi_\beta(x_0) = B_{1,\beta} \cosh \beta x_0 + B_{2,\beta} \sinh \beta x_0, \quad \text{where} \quad B_{1,\beta} = \Xi_\beta(0).
\]

If \( B_{1,\beta} = 1 \) and \( B_{2,\beta} = 1 \), then \( \Xi_\beta(x_0) = e^{\beta x_0} \).

Under relations (2.20) and (2.22) in the case \( \lambda^2 < \beta^2 \) exact solutions of the first equation of the system (2.23) with the separation parameters \( \lambda, \beta \) can be characterized in terms of linear independent solutions (see, e.g., [78, 70, 59]):

\[
\Y_{\lambda,\beta}(x_2) = x_2^{\alpha + 1} \left[ A_{1,\lambda,\beta} J_{\alpha + 1} \left( x_2 \sqrt{\beta^2 - \lambda^2} \right) + A_{2,\lambda,\beta} Y_{\alpha + 1} \left( x_2 \sqrt{\beta^2 - \lambda^2} \right) \right],
\]

where \( A_{1,\lambda,\beta}, A_{2,\lambda,\beta} = \text{const} \); herewith \( J_\nu(\varpi) \) and \( Y_\nu(\varpi) \) are the Bessel functions of the first and second kind of real order \( \nu = \frac{\alpha + 1}{2} \) and real argument \( \varpi = x_2 \sqrt{\beta^2 - \lambda^2} \).

Correspondingly under relations (2.20) and (2.22) in the case \( \lambda^2 > \beta^2 \) exact solutions of the first equation of the system (2.23) can be characterized in terms of linear independent solutions:

\[
\Y_{\lambda,\beta}(x_2) = x_2^{\alpha + 1} \left[ A_{1,\lambda,\beta} J_{\alpha + 1} \left( ix_2 \sqrt{\lambda^2 - \beta^2} \right) + A_{2,\lambda,\beta} Y_{\alpha + 1} \left( ix_2 \sqrt{\lambda^2 - \beta^2} \right) \right],
\]

where \( J_\nu(\varpi) \) and \( Y_\nu(\varpi) \) are the Bessel functions of the first and second kind of real order \( \nu = \frac{\alpha + 1}{2} \) and purely imaginary argument \( \varpi = ix_2 \sqrt{\lambda^2 - \beta^2} \).

This implies the following formulation.

**Theorem 2.6.** A class of exact solutions of equation (2.16) under relations (2.20) and (2.22), \( \beta \notin \mathbb{Z} \) in three dimensional setting can be represented as:

\[
h_\beta(x_0, x_1, x_2) = \sum_{\lambda=\infty}^{\infty} \left( C_{1,\lambda} \cos(\lambda x_1) + C_{2,\lambda} \sin(\lambda x_1) \right) g_{\lambda,\beta}(x_0, x_2), \tag{2.24}
\]

where

\[
g_{\lambda,\beta}(x_0, x_2) = \left( B_{1,\beta} \cosh(\beta x_0) + B_{2,\beta} \sinh(\beta x_0) \right) \Y_{\lambda,\beta}(x_2); \tag{2.25}
\]
in the case $\tilde{\lambda}^2 < \tilde{\beta}^2$

$$\Upsilon_{\lambda,\tilde{\beta}}(x_2) = x_2^{\alpha+1} \left[ A_{1,\lambda,\tilde{\beta}}J_{\alpha+1} \left( x_2 \sqrt{\tilde{\beta}^2 - \tilde{\lambda}^2} \right) + A_{2,\lambda,\tilde{\beta}}Y_{\alpha+1} \left( x_2 \sqrt{\tilde{\beta}^2 - \tilde{\lambda}^2} \right) \right]$$

(2.26)

and in the case $\tilde{\lambda}^2 > \tilde{\beta}^2$

$$\Upsilon_{\lambda,\tilde{\beta}}(x_2) = x_2^{\alpha+1} \left[ A_{1,\lambda,\tilde{\beta}}J_{\alpha+1} \left( ix_2 \sqrt{\tilde{\lambda}^2 - \tilde{\beta}^2} \right) + A_{2,\lambda,\tilde{\beta}}Y_{\alpha+1} \left( ix_2 \sqrt{\tilde{\lambda}^2 - \tilde{\beta}^2} \right) \right].$$

(2.27)

If $\tilde{\lambda}^2 = \tilde{\beta}^2$, the first equation of the system (2.23) leads to the Euler equation (see, e.g., \[51, 70\]):

$$x_2^2 \frac{d^2 \Upsilon}{dx_2^2} - \alpha x_2 \frac{d \Upsilon}{dx_2} = 0.$$  

(2.28)

Exact solutions of the Eq. (2.28) can be characterized in terms of power functions: $\Upsilon(x_2) = A_{1,\lambda}x_2^{\alpha+1} + A_{2,\lambda}; A_{1,\lambda}, A_{2,\lambda} = \text{const.}$

In two dimensional setting, where $h(x_0, x_1, x_2) = \Xi(x_0)\Upsilon(x_2)s(x_1)$, $\Xi(x_0) = \text{const.}$, the longitudinal component $E_0 = \frac{\partial h}{\partial x_0}$ of the vector $\vec{E}$ vanishes identically. In the case of transverse fields the vector $\vec{E}$ is independent of the longitudinal variable $x_0$ (see, e.g., \[46\]). Hereby, we have to deal with a subclass of transverse electrostatic fields, including the first transverse component in the form $E_1 = \frac{\partial h}{\partial x_1} = \Xi(x_0)\Upsilon(x_2)\frac{ds}{dx_1}$ and the second transverse component in the form $E_2 = \frac{\partial h}{\partial x_2} = \Xi(x_0)\frac{d\Upsilon}{dx_2}s(x_1)$. Under relations (2.20) and (2.22), in the case $\tilde{\beta} = 0$ and $\Xi(x_0) = \text{const.}$, exact solutions of the first equation of the system (2.23) with the separation parameters $\tilde{\lambda}, 0$ can be characterized in terms of the Bessel functions of the first and second kind of real order and purely imaginary argument:

$$\Upsilon_{\lambda,0}(x_2) = x_2^{\alpha+1} \left[ A_{1,\lambda,0}B_{\alpha+1} \left( ix_2 \tilde{\lambda} \right) + A_{2,\lambda,0}C_{\alpha+1} \left( ix_2 \tilde{\lambda} \right) \right];$$

$$A_{1,\lambda,0}, A_{2,\lambda,0} = \text{const.}$$

In two dimensional setting, where $h = h(x_0, x_1, x_2) = g(x_0, x_2)s(x_1)$, $s(x_1) = \text{const.}$, the first transverse component $E_1 = \frac{\partial h}{\partial x_1} = \Xi(x_0)\Upsilon(x_2)\frac{ds}{dx_1}$ vanishes identically. Under relations (2.20), $\tilde{\lambda} = 0$ and $s(x_1) = \text{const.}$, the first equation of the system (2.21) leads to the elliptic Euler-Poisson-Darboux equation in Cartesian coordinates (see, e.g., \[51, 52\]):

$$x_2 \left( \frac{\partial^2 g}{\partial x_2^2} + \frac{\partial^2 g}{\partial x_1^2} \right) - \alpha \frac{\partial g}{\partial x_2} = 0.$$  

(2.29)

New properties of exact solutions of the Eq. (2.29) have been recently studied by Konopelchenko and Ortenzi in the case $\alpha = -\frac{1}{2}$ (see, e.g., \[46\]).

In accordance with the Eq. (2.29), the system $(H^\alpha_3)$ leads to a Vekua-type system (see e.g., \[76, 73, 72, 32, 33, 34\]):

$$(V^\alpha_H) \left\{ \begin{array}{l}
\frac{\partial^2 u}{\partial x_0^2} - \alpha \frac{\partial u}{\partial x_0} + x_2 \frac{\partial u}{\partial x_2} = 0, \\
\alpha \frac{\partial u}{\partial x_2} = - \frac{\partial u}{\partial x_0},
\end{array} \right. \right.$$  

(2.30)
where
\[ u_0 = \frac{\partial g}{\partial x_0}, \quad u_2 = -\frac{\partial g}{\partial x_2}. \]  

(2.31)

3. The Static Maxwell System in Three Dimensional Axially Symmetric Inhomogeneous Isotropic Media, the System \((GAR)\) and \(\alpha\)-Axial-Hyperbolic Harmonic Electrostatic Fields

General class of \(C^1\)-solutions of the static Maxwell system in axially symmetric inhomogeneous isotropic media, provided by a variable \(C^1\)-coefficient \(\phi = \phi(\rho) > 0, \ \phi^{-1}(\rho) > 0:\)

\[
\begin{cases}
\text{div} (\phi(\rho)\vec{E}) = 0, \\
\text{curl} \vec{E} = 0
\end{cases}
\]

(3.1)

can be equivalently represented in terms of general class of \(C^1\)-solutions of the system \((3.2):\)

\[
(GAR) \begin{cases}
\phi(\rho) (\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}) - (\frac{\partial \phi(\rho)}{\partial x_1} u_1 + \frac{\partial \phi(\rho)}{\partial x_2} u_2) = 0, \\
\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0},
\end{cases}
\]

(3.2)

where in simply connected open domains \(\Lambda \subset \mathbb{R}^3 \vec{E} = (u_0, -u_1, -u_2) = \text{grad} \ h.\)

The system \((GAR)\) can be considered as a generalized non-Euclidean axial-hyperbolic modification of the system \((R)\) with respect to the conformal metric

\[
ds^2 = \phi^2(\rho)(dx_0^2 + dx_1^2 + dx_2^2). \]

(3.3)

The equation \((2.5)\) in general three dimensional axially symmetric setting can be described as:

\[
\phi \left( \frac{\partial^2 h}{\partial x_0^2} + \frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} \right) + \frac{d\phi}{d\rho} \left( \frac{\partial h}{\partial x_1} \cos \theta + \frac{\partial h}{\partial x_2} \sin \theta \right) = 0.
\]

If \(\phi(\rho) = \rho^{-\alpha} \ (\alpha \in \mathbb{R}),\) then

\[
(x_1^2 + x_2^2)\Delta h - \alpha \left( x_1 \frac{\partial h}{\partial x_1} + x_2 \frac{\partial h}{\partial x_2} \right) = 0.
\]

(3.4)

In the case of axially symmetric inhomogeneous isotropic media, provided by a variable coefficient \(\phi(\rho) = \rho^{-\alpha} \ (\rho > 0),\) the static Maxwell system \((3.1)\) takes the form

\[
\begin{cases}
\text{div} (\rho^{-\alpha}\vec{E}) = 0, \\
\text{curl} \vec{E} = 0,
\end{cases}
\]

(3.5)
The Static Maxwell System in Inhomogeneous Media

whereas the system \((GAR)\) becomes the system \((3.6)\):

\[
\begin{cases}
(A_3^α) \\
\end{cases}
\begin{align*}
\left( x_1^2 + x_2^2 \right) \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) + \alpha \left( x_1 u_1 + x_2 u_2 \right) &= 0, \\
\frac{\partial u_0}{\partial x_1} &= -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} &= -\frac{\partial u_2}{\partial x_0}, \\
\end{align*}
\tag{3.6}
\]

**Definition 3.1.** Each \(C^1\)-solution of the system \((3.5)\) in a simply connected open domain \(\Lambda \subset \mathbb{R}^3 (\rho > 0)\) is called a continuously differentiable \(\alpha\)-axial-hyperbolic harmonic electrostatic field \(\vec{E}\) in \(\Lambda\).

The system \((A_3^α)\) can be considered as a generalized non-Euclidean \(\alpha\)-axial-hyperbolic modification of the system \((R)\) with respect to the conformal metric, defined outside the axis \(x_0\) by formula

\[
ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2}{\rho^{2\alpha}}. \tag{3.7}
\]

Some important properties of \(\alpha\)-axial-hyperbolic harmonic electrostatic fields in three dimensional setting can be explicitly demonstrated by means of the separation method in cylindrical coordinates (see, e.g., [65, 66, 63, 15]).

The Eq. \((3.4)\) in cylindrical coordinates can be described as:

\[
\rho^2 \left( \frac{\partial^2 h}{\partial x_0^2} + \frac{\partial^2 h}{\partial \rho^2} \right) + (1 - \alpha) \rho \frac{\partial h}{\partial \rho} + \frac{\partial^2 h}{\partial \theta^2} = 0. \tag{3.8}
\]

Let us first look for a class of exact solutions of the Eq. \((3.8)\) under condition \(h(x_0, \theta, \rho) = g(x_0, \rho)s(\theta)\):

\[
s(\theta)\rho^2 \left( \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial \rho^2} \right) + s(\theta)(1 - \alpha) \rho \frac{\partial g}{\partial \rho} + \frac{\partial^2 s}{\partial \theta^2} = 0.
\]

If \(g(x_0, \rho)s(\theta) \neq 0\), the first relations of separation of variables can be introduced:

\[
\rho^2 \left( \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial \rho^2} \right) + (1 - \alpha) \rho \frac{\partial g}{\partial \rho} - \frac{\partial^2 s}{s \partial \theta^2} = \lambda^2 \quad (\lambda = \text{const} \in \mathbb{R}). \tag{3.9}
\]

The relations \((3.9)\) are equivalent to the following system of equations:

\[
\left\{ \begin{array}{l}
\frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial \rho^2} + (1 - \alpha) \rho \frac{\partial g}{\partial \rho} - \frac{\lambda^2}{\rho^2} g = 0, \\
\frac{\partial^2 s}{\partial \theta^2} = \lambda^2 s = 0.
\end{array} \right. \tag{3.10}
\]

In particular, exact solutions of the second equation in \((3.10)\) can be characterized in terms of trigonometric functions (see, e.g., [15]):

\[
s_\lambda(\theta) = C_{1,\lambda} \cos \lambda \theta + C_{2,\lambda} \sin \lambda \theta.
\]

The periodicity of the solutions \(s = s_\lambda(\theta)\) implies \(\lambda \in \mathbb{Z}\).

Let us further look for a class of exact solutions of the first equation of the system \((3.10)\) under condition \(g(x_0, \rho) = \Xi(x_0)\Upsilon(\rho) \neq 0\):

\[
\frac{1}{\Xi} \frac{d^2 \Xi}{dx_0^2} + \frac{1}{\Upsilon} \frac{d^2 \Upsilon}{d\rho^2} + \frac{(1 - \alpha)}{\Upsilon \rho} \frac{d \Upsilon}{d \rho} - \frac{\lambda^2}{\rho^2} = 0.
\]
The second relations of separation of variables can be introduced:

\[
\frac{1}{Y} \frac{d^2 Y}{d \rho^2} + \frac{(1 - \alpha) \rho}{Y \rho} \frac{d Y}{d \rho} - \frac{\check{\lambda}^2}{\rho^2} = - \frac{1}{\Xi} \frac{d^2 \Xi}{d x_0^2} = - \beta^2 \quad (\beta = \text{const} \in \mathbb{R}). \tag{3.11}
\]

The relations (3.11) are equivalent to the following system of ordinary differential equations:

\[
\left\{ \begin{array}{l}
\rho^2 \frac{d^2 Y}{d \rho^2} + (1 - \alpha) \rho \frac{d Y}{d \rho} + (\beta^2 \rho^2 - \check{\lambda}^2) Y = 0, \\
\frac{d^2 \Xi}{d x_0^2} - \beta^2 \Xi = 0.
\end{array} \right. \tag{3.12}
\]

In particular, exact solutions of the second equation can be characterized in terms of hyperbolic functions (see, e.g., [44]):

\[
\Xi_\beta(x_0) = B_{1,\beta} \cosh \beta x_0 + B_{2,\beta} \sinh \beta x_0, \quad \text{where} \quad B_{1,\beta} = \Xi_\beta(0).
\]

If \(B_{1,\beta} = 1\) and \(B_{2,\beta} = 1\), then \(\Xi_\beta(x_0) = e^{\beta x_0}\) (see, e.g., [15]).

Under relations (3.9) and (3.11), \(\check{\beta} \neq 0\) exact solutions of the first equation of the system (3.12) with the separation parameters \(\check{\lambda}, \check{\beta}\) can be characterized in terms of linear independent solutions (see, e.g., [78, 70]):

\[
\Upsilon_{\check{\lambda},\check{\beta}}(\rho) = \rho^{\frac{\check{\lambda}}{2}} \left[ A_{1,\check{\lambda},\check{\beta}} J_{\sqrt{\alpha^2 + 4 \lambda^2}}(\rho \check{\beta}) + A_{2,\check{\lambda},\check{\beta}} Y_{\sqrt{\alpha^2 + 4 \lambda^2}}(\rho \check{\beta}) \right],
\]

where \(A_{1,\check{\lambda},\check{\beta}}, A_{2,\check{\lambda},\check{\beta}} = \text{const}\); herewith \(J_\nu(\varpi)\) and \(Y_\nu(\varpi)\) are the Bessel functions of the first and second kind of real order \(\nu = \sqrt{\alpha^2 + 4 \lambda^2}\) and real argument \(\varpi = \rho \check{\beta}\).

This implies the following formulation.

**Theorem 3.2.** A class of exact solutions of equation (3.8) under relations (3.9) and (3.11), \(\check{\beta} \neq 0\) in three dimensional setting can be represented as:

\[
h_{\beta}(x_0, \theta, \rho) = \sum_{\check{\lambda} = -\infty}^{\infty} \left( C_{1,\check{\lambda}} \cos(\check{\lambda} \theta) + C_{2,\check{\lambda}} \sin(\check{\lambda} \theta) \right) g_{\check{\lambda},\check{\beta}}(x_0, \rho), \tag{3.13}
\]

where

\[
g_{\check{\lambda},\check{\beta}}(x_0, \rho) = \left( B_{1,\check{\lambda}} \cosh(\check{\beta} x_0) + B_{2,\check{\lambda}} \sinh(\check{\beta} x_0) \right) \Upsilon_{\check{\lambda},\check{\beta}}(\rho) \tag{3.14}
\]

and

\[
\Upsilon_{\check{\lambda},\check{\beta}}(\rho) = \rho^{\frac{\check{\lambda}}{2}} \left[ A_{1,\check{\lambda},\check{\beta}} J_{\sqrt{\alpha^2 + 4 \lambda^2}}(\rho \check{\beta}) + A_{2,\check{\lambda},\check{\beta}} Y_{\sqrt{\alpha^2 + 4 \lambda^2}}(\rho \check{\beta}) \right]. \tag{3.15}
\]

In two dimensional setting new properties of classes of transverse and meridional electrostatic fields in axially symmetric inhomogeneous media have been recently studied in the framework of applied quaternionic analysis by Khmelnytskaya, Kravchenko and Oviedo (see e.g., [45, 48, 49]). In the case of meridional fields the vector \(\vec{E}\) is independent of the azimuth angle \(\theta\).

In two dimensional setting in cylindrical coordinates, where \(h(x_0, \theta, \rho) = \Xi(x_0) \Upsilon(\rho) s(\theta), \quad \Xi(x_0) = \text{const}\), the longitudinal component \(E_0 = \frac{\partial h}{\partial x_0}\) of the vector \(\vec{E}\) vanishes identically. We have to deal with a subclass of transverse
electrostatic fields (see, e.g., [15, 45]), including the first transverse component in the form \( E_1 = \frac{\partial h}{\partial x_1} = \Xi(x_0) \frac{d\bar{\Upsilon}}{d\rho} s(\theta) \cos \theta - \Xi(x_0) \bar{\Upsilon}(\rho) \frac{d\bar{\Upsilon}}{d\rho} \cos \theta \) and the second transverse component in the form \( E_2 = \frac{\partial h}{\partial x_2} = \Xi(x_0) \frac{dx}{d\rho} s(\theta) \sin \theta + \Xi(x_0) \bar{\Upsilon}(\rho) \frac{dx}{d\rho} \cos \theta \).

Under relations (3.9) and (3.11), in the case \( \bar{\beta} = 0 \) and \( \Xi(x_0) = \text{const} \), the first equation of the system (3.12) leads to the Euler equation in the framework of transverse electrostatic fields (see, e.g., [17, 70]):

\[
\rho^2 \frac{d^2 \bar{\Upsilon}(\rho)}{d\rho^2} + (1 - \alpha) \rho \frac{d\bar{\Upsilon}(\rho)}{d\rho} - \bar{\lambda}^2 \bar{\Upsilon}(\rho) = 0. \tag{3.16}
\]

Exact solutions of the Eq. (3.16) can be characterized in terms of power functions:

\( \bar{\Upsilon}(\rho) = A_{1,\lambda} \rho^\lambda \left( \frac{1}{\sqrt{\lambda^{2} + \lambda^2}} \right) + A_{2,\lambda} \rho^{-\lambda} \left( \frac{1}{\sqrt{-\lambda^{2} + \lambda^2}} \right); \quad A_{1,\lambda}, A_{2,\lambda} = \text{const}. \)

In two dimensional setting in cylindrical coordinates, where \( h(x_0, \theta, \rho) = g(x_0, \rho) s(\theta), \: s(\theta) = \text{const}, \) we have to deal with various subclasses of meridional electrostatic fields \( \bar{E} \) in axially symmetric inhomogeneous media, here-with \( \frac{\partial h}{\partial \theta} = 0 \) (see e.g., [15, 45]). The longitudinal component takes the form \( E_0 = \frac{\partial h}{\partial x_0} = \frac{\partial g}{\partial x_0} s(\theta) \), whereas the first transverse component takes the form \( E_1 = \frac{\partial h}{\partial x_1} = \frac{\partial g}{\partial \rho} \cos \theta - \frac{\partial h}{\partial \theta} \frac{\sin \theta}{\rho} = \frac{\partial g}{\partial \rho} s(\theta) \cos \theta \), simultaneously the second transverse component takes the form \( E_2 = \frac{\partial h}{\partial x_2} = \frac{\partial h}{\partial \rho} \sin \theta + \frac{\partial h}{\partial \theta} \frac{\cos \theta}{\rho} = \frac{\partial g}{\partial \rho} s(\theta) \sin \theta \).

Under relations (3.9), in the case \( \bar{\lambda} = 0 \) and \( s(\theta) = \text{const} \), the first equation of the system (3.10) leads to the elliptic Euler-Poisson-Darboux equation in cylindrical coordinates (occasionally called ”the generalized axially symmetric potential equation” (GASPE), see e.g., [3, 19, 81]) in the framework of meridional electrostatic fields:

\[
\rho \left( \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial \rho^2} \right) + (1 - \alpha) \frac{\partial g}{\partial \rho} = 0. \tag{3.17}
\]

Approach of generalized axially symmetric potential theory (GASPT) has been originally initiated by Weinstein in cylindrical coordinates in two dimensional setting (see e.g., [70, 80, 26, 36]).

New properties of exact solutions of the Eq. (3.17) have been recently studied by Plaksa and Grishchuk in the case \((1 - \alpha) = \tilde{m} (\tilde{m} \in \mathbb{N})\) (see, e.g., [68, 38, 39]). Additionally new properties of linear differential relations between exact solutions of the Eq. (3.17) have been studied by Aksenov in the general case of real \( \alpha \) by means of the symmetry operators (see, e.g., [3]).

In accordance with the Eq. (3.17) the system (A\( \alpha \)) leads to a Vekua-type system in cylindrical coordinates \((x_0, \rho)\) (see e.g., [76, 73, 67, 34]):

\[
(V\alpha_A) \left\{ \begin{array}{l}
\rho \frac{\partial u_0}{\partial x_0} - \frac{\partial u_0}{\partial \rho} - (1 - \alpha) u_\rho = 0, \\
\frac{\partial u_0}{\partial x_0} = -\frac{\partial u_\rho}{\partial \rho},
\end{array} \right. \tag{3.18}
\]

where

\[
u_0 = \frac{\partial g}{\partial x_0}, \quad u_\rho = -\frac{\partial g}{\partial \rho}. \tag{3.19}
\]
4. GASPT, the Fueter Potential in an Axially Symmetric Inhomogeneous Medium and New Applications of Radially Holomorphic Functions in $\mathbb{R}^3$

As seen, in cylindrical coordinates in two dimensional setting under conditions $h(x_0, \theta, \rho) = g(x_0, \rho)s(\theta)$, $s(\theta) = 1$, we have to deal with meridional electrostatic fields $\vec{E}$ in axially symmetric inhomogeneous isotropic media, provided by a variable coefficient $\phi(\rho) = \rho^{-\alpha}$ ($\rho > 0$), where $E_0 = \frac{\partial g}{\partial x_0}$, $E_1 = \frac{\partial g}{\partial \rho} \cos \theta$, $E_2 = \frac{\partial g}{\partial \rho} \sin \theta$.

The Vekua-type system $(V_\alpha)$ in the case $\alpha = 1$ can be considered as a Cauchy-Riemann system in the meridional plane spanned by $(x_0, \rho)$ (see, e.g., [3, 16, 17, 18, 15]):

$$\begin{cases}
\frac{\partial u_0}{\partial x_0} - \frac{\partial u_\rho}{\partial \rho} = 0, \\
\frac{\partial u_0}{\partial \rho} = -\frac{\partial u_\rho}{\partial x_0}.
\end{cases}$$ (4.1)

In accordance with Fueter construction in $\mathbb{R}^3$ (1.13), the special class of meridional solutions of the static Maxwell system in an inhomogeneous isotropic medium, provided by a variable coefficient $\phi(\rho) = \rho^{-1}$ ($\rho > 0$), takes the form:

$$\begin{cases}
\frac{\partial E_0}{\partial x_0} + \frac{\partial E_\rho}{\partial \rho} = 0, \\
\frac{\partial E_0}{\partial \rho} = -\frac{\partial E_\rho}{\partial x_0}.
\end{cases}$$ (4.2)

where

$$E_0 + iE_1 + jE_2 = E_0 + IE_\rho = u_0 - Iu_\rho, \quad E_1 = \frac{x_1}{\rho} E_\rho = E_\rho \cos \theta, \quad E_2 = \frac{x_2}{\rho} E_\rho = E_\rho \sin \theta.$$ (4.3)

In cylindrical coordinates we obtain:

$$\text{div} \vec{E} = \frac{\partial u_0}{\partial x_0} - \frac{\partial u_\rho}{\partial \rho} - \frac{u_\rho}{\rho} = \frac{E_\rho}{\rho}.$$ (4.5)

In the framework of GASPT the elliptic Euler-Poisson-Darboux equation (3.17) in the case $\alpha = 1$ in simply connected open domains $\Lambda$ ($\rho > 0$) in terms of the scalar potential $g_0 = g_0(x_0, \rho)$

$$\frac{\partial^2 g_0}{\partial x_0^2} + \frac{\partial^2 g_0}{\partial \rho^2} = 0$$ (4.6)

indicates the existence of the stream function $g_\rho = g_\rho(x_0, \rho)$, which is defined by the system (4.7) [79, 80]:

$$\begin{cases}
\frac{\partial g_0}{\partial x_0} - \frac{\partial g_\rho}{\partial \rho} = 0, \\
\frac{\partial g_0}{\partial \rho} = -\frac{\partial g_\rho}{\partial x_0}.
\end{cases}$$ (4.7)

It follows that in the case $\alpha = 1$ the stream function satisfies the same equation:

$$\frac{\partial^2 g_\rho}{\partial x_0^2} + \frac{\partial^2 g_\rho}{\partial \rho^2} = 0.$$ (4.8)

A lot of important problems of electrostatics in homogeneous media have been solved by means of the complex potential (see, e.g., [65, 53, 21, 61]).
Physical applications of the Fueter potential under conditions of Fueter construction in $\mathbb{R}^3$ can be explicitly demonstrated in the framework of GASPT.

Each solution of the system \((4.7)\) in cylindrical coordinates ($\rho > 0$) can be expressed in terms of radially holomorphic functions [40], using radial differential operator:

$$\partial_{rad} U := U' := \frac{1}{2} \left( \frac{\partial}{\partial x_0} - I \frac{\partial}{\partial \rho} \right) U.$$ 

**Definition 4.1.** Each reduced quaternion-valued function $U = g_0 + Ig_\rho$, satisfying a Cauchy-Riemann type differential equation

$$\partial_{rad} U := \frac{1}{2} \left( \frac{\partial}{\partial x_0} + I \frac{\partial}{\partial \rho} \right) U = 0 \quad (4.9)$$

is called a radially holomorphic function in $\mathbb{R}^3$ ($\rho > 0$).

As seen, the Eq. \((4.9)\) is equivalent to the Cauchy-Riemann system in the meridional plane spanned by $(x_0, \rho)$ \((4.1)\).

The notation $\partial_{rad} U := U'$ for the radial differential operator is justified in [40] by clear statements. In particular, in terms of elementary functions of the reduced quaternionic variable classical relations are satisfied:

$$(x^n)' = nx^{n-1};$$
$$(e^x)' = e^x;$$
$$(\cos x)' = - \sin x;$$
$$(\sin x)' = \cos x;$$
$$(\ln x)' = x^{-1}.$$ 

In addition, the Eq. \((4.9)\) implies

$$U' = \frac{\partial U}{\partial x_0}. \quad (4.10)$$

**Definition 4.2.** If a radially holomorphic function $U = g_0 + Ig_\rho$ satisfies a differential equation

$$U' = F, \quad (4.11)$$

where $F = u_0 + Iu_\rho$ is a radially holomorphic function, $U$ is called a Fueter holomorphic primitive of the function $F$.

**Definition 4.3.** Each Fueter holomorphic primitive $U = g_0 + Ig_\rho$ in simply connected open domains $\Lambda$ ($\rho > 0$) in terms of the scalar potential $g_0 = g_0(x_0, \rho)$ and the stream function $g_\rho = g_\rho(x_0, \rho)$ in applied problems of mathematical physics shall henceforth be called the Fueter potential in $\Lambda$.

**Theorem 4.4 (On zero divergence).** Let $F = F(x) = u_0 + iu_1 + ju_2$ be a function of the reduced quaternionic variable, associated with classical holomorphic in $\Lambda$ ($\rho > 0$), and $\vec{E} = (u_0, -u_1, -u_2)$. In terms of the static Maxwell system \((1.9)\) the EFG tensor \((2.7)\) with condition of zero divergence in a point $x^* \in \Lambda$ is characterized by a singular Fueter mapping of the second kind, herewith $E_1(x_0^*, x_1^*, x_2^*) = E_2(x_0^*, x_1^*, x_2^*) = 0$. 


Remark 4.5. The result was first formulated as a hypothesis in 2014 [14]. The proof in cylindrical coordinates was first obtained by Bryukhov and Kähler in Aveiro, 2015. As we can see, \( \text{div}\overrightarrow{E} = \frac{E_\rho}{\rho} \), whereas \( \text{det} \, J(\overrightarrow{E}) = -\frac{E_\rho}{\rho} \left[ \left( \frac{\partial E_\rho}{\partial \rho} \right)^2 + \left( \frac{\partial E_\rho}{\partial x_0} \right)^2 \right] \).

Applications of tools of radially holomorphic functions and the Fueter potential allow us to solve new problems of meridional electrostatic fields in an axially symmetric inhomogeneous medium, provided by a variable coefficient \( \phi(\rho) = \rho^{-1} \). On the other hand, in accordance with problems of applied quaternionic analysis and geometrical optics (see, e.g., [48, 49, 45, 50]), properties of electrostatic fields in an inhomogeneous medium with dielectric permittivity \( \varepsilon = \varepsilon(\rho) = \rho^{-1} \) are relevant. On the other hand, in accordance with problems of electrical engineering and geo-electromagnetism (see, e.g., [21] [77]), properties of electrostatic fields in an inhomogeneous medium with electrical conductivity \( \sigma = \sigma(\rho) = \rho^{-1} \) are relevant.

Example 4.6. A linear superposition of two reduced quaternionic power functions with negative exponents: \( F(x) = \gamma_1 x^{-1} + \gamma_2 x^{-2} \), where \( \gamma_1, \gamma_2 \in \mathbb{R} \);

\[ F(x) = \gamma_1 \frac{1}{x} + \gamma_2 \frac{1}{x^2}. \]

Fueter holomorphic primitive of the function \( F(x) \) takes the form:

\[ U(x) = \gamma_1 \ln x - \gamma_2 x^{-1}. \]

Meridional electrostatic field \( \overrightarrow{E} = (\overrightarrow{E}^1 + \overrightarrow{E}^2) \), where \( \overrightarrow{E}^1 = (u_0^1, -u_1^1, -u_2^1) \), \( \overrightarrow{E}^2 = (u_0^2, -u_1^2, -u_2^2) \).

The \( EFG \) tensor \( J(\overrightarrow{E}^1) \) under condition \( \text{det} J(\overrightarrow{E}^1) \neq 0 \) is characterized by a regular Fueter mapping of the second kind:

\[
\gamma_1 \begin{pmatrix}
-\frac{x_0^2 + x_1^2 + x_2^2}{(x_0^2 + x_1^2 + x_2^2)^2} & -\frac{2x_0 x_1}{(x_0^2 + x_1^2 + x_2^2)^2} & -\frac{2x_0 x_2}{(x_0^2 + x_1^2 + x_2^2)^2} \\
-\frac{2x_0 x_1}{(x_0^2 + x_1^2 + x_2^2)^2} & -\frac{x_0^2 - x_1^2}{(x_0^2 + x_1^2 + x_2^2)^2} & -\frac{x_0^2 - x_2^2}{(x_0^2 + x_1^2 + x_2^2)^2} \\
-\frac{2x_0 x_2}{(x_0^2 + x_1^2 + x_2^2)^2} & -\frac{x_0^2 - x_2^2}{(x_0^2 + x_1^2 + x_2^2)^2} & -\frac{x_0^2 - x_1^2}{(x_0^2 + x_1^2 + x_2^2)^2}
\end{pmatrix}
\]

Similarly the \( EFG \) tensor \( J(\overrightarrow{E}^2) \) under condition \( \text{det} J(\overrightarrow{E}^2) \neq 0 \) is characterized by a regular Fueter mapping of the second kind:

\[
\gamma_2 \begin{pmatrix}
\frac{2x_0(-x_0^2 + 3x_1^2 + 3x_2^2)}{(x_0^2 + x_1^2 + x_2^2)^3} & -\frac{4x_0^2 x_1}{(x_0^2 + x_1^2 + x_2^2)^3} & -\frac{4x_0^2 x_2}{(x_0^2 + x_1^2 + x_2^2)^3} \\
-\frac{4x_0^2 x_1}{(x_0^2 + x_1^2 + x_2^2)^3} & \frac{2x_0(x_0^2 - 3x_1^2 + x_2^2)}{(x_0^2 + x_1^2 + x_2^2)^3} & -\frac{4x_0 x_1 x_2}{(x_0^2 + x_1^2 + x_2^2)^3} \\
-\frac{4x_0^2 x_2}{(x_0^2 + x_1^2 + x_2^2)^3} & -\frac{4x_0 x_1 x_2}{(x_0^2 + x_1^2 + x_2^2)^3} & \frac{2x_0(x_0^2 - 3x_1^2 + x_2^2)}{(x_0^2 + x_1^2 + x_2^2)^3}
\end{pmatrix}
\]

Condition of zero divergence

\[
\text{div}\overrightarrow{E} = \frac{\gamma_1}{x_0^2 + x_1^2 + x_2^2} + \frac{2\gamma_2 x_0}{(x_0^2 + x_1^2 + x_2^2)^2} = 0 \tag{4.12}
\]

leads to an equation of a sphere of a radius \( \frac{2\gamma_2}{\gamma_1} \) with center at the point \((-\frac{\gamma_2}{\gamma_1}, 0, 0)\):

\[
\gamma_1(x_0^2 + x_1^2 + x_2^2) + 2\gamma_2 x_0 = 0. \tag{4.13}
\]
The \textit{EFG} tensor \( J(\vec{E}) \) with condition of zero divergence (4.12) in a point \( x^* \in \Lambda \) is characterized by a singular Fueter mapping of the second kind, herewith \( E_1(x_0^*, x_1^*, x_2^*) = E_2(x_0^*, x_1^*, x_2^*) = 0 \).

\textbf{Example 4.7.} A linear superposition of two reduced quaternionic exponential functions \( F(x) = e^{-b_1x} - e^{-b_2x} = e^{-b_1x_0} \cos(-b_1\rho) + I \sin(-b_1\rho) - e^{-b_2x_0} \cos(-b_2\rho) + I \sin(-b_2\rho) \), where \( b_1, b_2 \in \mathbb{R} \); \( \vec{F}(x) = e^{-b_1x} + (-e^{-b_2x}) \).

Fueter holomorphic primitive of the function \( F(x) \) takes the form:
\[
U(x) = -\frac{1}{b_1}e^{-b_1x} + \frac{1}{b_2}e^{-b_2x}.
\]

Meridional electrostatic field \( \vec{E} = (\vec{E}^1 + \vec{E}^2) \), where \( \vec{E}^1 = (u_0^1, -u_1^1, -u_2^1) \), \( \vec{E}^2 = (u_0^2, -u_1^2, -u_2^2) \).

Let us denote by \( P_1, Q_1, P_2, Q_2 \) expressions in terms of real-valued exponential and trigonometric functions:
\[
P_1 = b_1 e^{-b_1x_0} \cos(b_1\rho), \quad Q_1 = e^{-b_1x_0} \sin(b_1\rho),
\]
\[
P_2 = b_2 e^{-b_2x_0} \cos(b_2\rho), \quad Q_2 = e^{-b_2x_0} \sin(b_2\rho).
\]

The \textit{EFG} tensor \( J(\vec{E}^1) \) under condition \( \det J(\vec{E}^1) \neq 0 \) is characterized by a regular Fueter mapping of the second kind:
\[
\left(\begin{array}{ccc}
-p_{1,1} & -b_{1,1}Q_1 & -b_{1,2}Q_2 \\
-p_{1,2} & x_{1,2}P_1 + x_{2,2}Q_1 & x_{1,2}P_2 - x_{1,1}Q_1 \\
-p_{1,3} & x_{1,2}P_3 - x_{1,1}Q_1 & x_{2,2}Q_1 + x_{2,2}Q_1
\end{array}\right).
\]

Similarly the \textit{EFG} tensor \( J(\vec{E}^2) \) under condition \( \det J(\vec{E}^2) \neq 0 \) is characterized by a regular Fueter mapping of the second kind:
\[
\left(\begin{array}{ccc}
-p_{2,2} & b_{2,2}Q_2 & b_{2,3}Q_2 \\
b_{2,2}P_2 & \frac{-x_{2,2}P_2}{\rho^3} - x_{2,2}Q_2 & \frac{-x_{2,2}P_2}{\rho^3} + x_{2,3}Q_2 \\
b_{2,3}Q_2 & \frac{-x_{2,2}P_2}{\rho^3} + x_{2,3}Q_2 & \frac{-x_{2,2}P_2}{\rho^3} - x_{2,3}Q_2
\end{array}\right).
\]

Condition of zero divergence
\[
\text{div} \vec{E} = e^{-b_1x_0} \sin(b_1\rho) - e^{-b_2x_0} \sin(b_2\rho) = 0 \quad (4.14)
\]
leads to the following equation:
\[
e^{(b_2-b_1)x_0} \sin(b_1\rho) - \sin(b_2\rho) = 0. \quad (4.15)
\]

In particular, under condition \( b_2 = 2b_1 \) we obtain an equation of circular cylinders of increasing radius:
\[
\sin(b_1\rho) = 0, \quad \rho = \frac{\pi m}{b_1}, \quad m = +1, +2, \ldots, \quad (4.16)
\]
and an equation with separable variables \( \rho \) and \( x_0 \):
\[
\cos(b_1\rho) = \frac{e^{b_1x_0}}{2}. \quad (4.17)
\]
The $EFG$ tensor $J(\vec{E})$ with condition of zero divergence (4.14) in a point $x^* \in \Lambda$ is characterized by a singular Fueter mapping of the second kind, herewith $E_1(x_0^*, x_1^*, x_2^*) = E_2(x_0^*, x_1^*, x_2^*) = 0$.

Fairly wide subclasses of meridional electrostatic fields $\vec{E}$ in an inhomogeneous isotropic medium, provided by a variable coefficient $\phi(\rho) = \rho^{-1}$ ($\rho > 0$), are realized in terms of the one-sided and two-sided reduced quaternionic Laplace transforms of real-valued originals $\tilde{\eta}(\tau)$.

**Definition 4.8.** A real-valued function $\tilde{\eta} = \tilde{\eta}(\tau)$ with a real argument $\tau$ is called a real-valued original, if

1. the function $\tilde{\eta}(\tau)$ satisfies the Hölder’s condition for each $\tau$, except some points $\tau = \tau^n, \tau_0^2, \ldots$ (there exists a finite quantity or zero of such points for each finite interval), where $\tilde{\eta}(\tau)$ has gaps of the first kind,
2. $\tilde{\eta}(\tau) = 0$ for $\tau < 0$,
3. there exist constants $B_{\tilde{\eta}} > 0, \alpha_{\tilde{\eta}} \geq 0$ : for all $\tau$ $|\tilde{\eta}(\tau)| < B_{\tilde{\eta}} e^{\alpha_{\tilde{\eta}} \tau}$.

The Hölder’s condition for the function $\tilde{\eta}(\tau)$ has the form:

for each $\tau$ there exist constants $A_{\tilde{\eta}} > 0, 0 < \lambda_{\tilde{\eta}} \leq 1, \delta_{\tilde{\eta}} > 0$, such that $|\tilde{\eta}(\tau + \delta) - \tilde{\eta}(\tau)| \leq A_{\tilde{\eta}} |\delta|^{\lambda_{\tilde{\eta}}}$ for each $\delta, |\delta| \leq \delta_{\tilde{\eta}}$.

**Definition 4.9.** Let us assume that $\rho > 0$. Linear integral transform of a real-valued original $\tilde{\eta}(\tau)$ with kernel in the form of the exponential function of the reduced quaternionic variable, including real-valued parameter $\tau$:

$$F(x) := L\{\tilde{\eta}(\tau); x\} = \int_{0}^{\infty} \tilde{\eta}(\tau) e^{-x\tau} d\tau$$ (4.18)

is called the one-sided reduced quaternionic Laplace transform of $\tilde{\eta}(\tau)$.

$$\overline{F}(x) = \int_{0}^{\infty} \tilde{\eta}(\tau) e^{-x\tau} d\tau = \int_{0}^{\infty} \tilde{\eta}(\tau) e^{-x_0\tau}[\cos(\rho\tau) + I \sin(\rho\tau)]d\tau.$$ (4.19)

Meridional electrostatic fields $\vec{E}$, described by the one-sided reduced quaternionic Laplace transform of $\tilde{\eta}(\tau)$, are characterized by formulas:

$$E_0 = \int_{0}^{\infty} \tilde{\eta}(\tau) e^{-x_0\tau} \cos(\rho\tau) d\tau, \quad E_\rho = \int_{0}^{\infty} \tilde{\eta}(\tau) e^{-x_0\tau} \sin(\rho\tau) d\tau.$$ (4.20)

Condition of zero divergence leads to the following equation:

$$\int_{0}^{+\infty} \tilde{\eta}(\tau) e^{-x_0\tau} \sin(\rho\tau) d\tau = 0.$$ (4.21)

The two-sided reduced quaternionic Laplace transform can be naturally realized, if $\tilde{\eta}(\tau)$ does not vanish identically for $\tau < 0$ (see, e.g. [75, 10]).

**Definition 4.10.** Let us assume that $\rho > 0$. Linear integral transform of a real-valued original $\tilde{\eta}(\tau)$ with kernel in the form of the exponential function of the reduced quaternionic variable, including real-valued parameter $\tau$:

$$F(x) := L_{-\infty}^{+\infty}\{\tilde{\eta}(\tau); x\} = \int_{-\infty}^{\infty} \tilde{\eta}(\tau) e^{-x\tau} d\tau$$ (4.21)
is called the two-sided reduced quaternionic Laplace transform of $\tilde{\eta}(\tau)$.

**Example 4.11.** The Euler’s Gamma function of the reduced quaternionic argument $\Gamma(x)$, $x_0 > 0$ [10].

$$\Gamma(-x) = \mathcal{L}^{+\infty}_{-\infty} \{e^{-x\tau}; x\} = \int_{-\infty}^{\infty} e^{-e^\tau} e^{-x_0\tau}[\cos(\rho\tau) - I \sin(\rho\tau)]d\tau,$$

$$\Gamma(-x) = \int_{-\infty}^{\infty} e^{-e^\tau} e^{-x_0\tau}[\cos(\rho\tau) + I \sin(\rho\tau)]d\tau.$$  

Meridional electrostatic field $\vec{E}$, described by the Euler’s Gamma function of the reduced quaternionic argument, is characterized by formulas:

$$E_0 = \int_{-\infty}^{\infty} e^{-e^\tau} e^{-x_0\tau} \cos(\rho\tau) d\tau, \quad E_\rho = \int_{-\infty}^{\infty} e^{-e^\tau} e^{-x_0\tau} \sin(\rho\tau) d\tau.$$  

Condition of zero divergence leads to the following equation:

$$\int_{-\infty}^{+\infty} e^{-e^\tau} e^{-x_0\tau} \sin(\rho\tau) d\tau = 0. \quad (4.22)$$  

Some theorems on properties of the classical Laplace transform can be reformulated in terms of the reduced quaternionic Laplace transform (in complex analysis see, e.g., [53 22 10 11]).

**Theorem 4.12 (On superposition).** The two-sided reduced quaternionic Laplace transform of a linear superposition of real-valued originals with reduced quaternion-valued coefficients $\gamma_1$ and $\gamma_2$, $\tilde{\eta}(\tau) = \gamma_1 \tilde{\eta}_1(\tau) + \gamma_2 \tilde{\eta}_2(\tau)$, in the framework of Fueter construction in $\mathbb{R}^3$ generates a linear superposition of the two-sided reduced quaternionic Laplace transforms:

$$\mathcal{L}^{+\infty}_{-\infty} \{\tilde{\eta}(\tau); x\} = \gamma_1 \int_{-\infty}^{\infty} \tilde{\eta}_1(\tau)e^{-x\tau} d\tau + \gamma_2 \int_{-\infty}^{\infty} \tilde{\eta}_2(\tau)e^{-x\tau} d\tau. \quad (4.23)$$

Now let us clarify roles of the reduced quaternionic Fourier cosine and sine transforms (in complex analysis see, e.g., [25 7]).

**Definition 4.13.** Let us assume that $\rho > 0$. Linear integral transform of a real-valued original $\tilde{\eta}(\tau)$ with kernel in the form of the cosine function of the reduced quaternionic variable, including real-valued parameter $\tau$:

$$\mathcal{F}\mathcal{C}\{\tilde{\eta}(\tau); x\} := \int_{0}^{\infty} \tilde{\eta}(\tau) \cos(x\tau) d\tau = \frac{1}{2} \int_{0}^{\infty} \tilde{\eta}(\tau)(e^{-ix\tau} + e^{ix\tau}) d\tau \quad (4.24)$$

is called the reduced quaternionic Fourier cosine transform of $\tilde{\eta}(\tau)$.

Let us denote by $y$ the independent reduced quaternionic variable of the form $y = Ix = -\rho + Ix_0$, then the reduced quaternionic Fourier cosine transform of $\tilde{\eta}(\tau)$ is expressed in terms of the one-sided reduced quaternionic Laplace transform:

$$\mathcal{F}\mathcal{C}\{\tilde{\eta}(\tau); x\} = \frac{1}{2} [\mathcal{L}\{\tilde{\eta}(\tau); y\} + \mathcal{L}\{\tilde{\eta}(\tau); -y\}]. \quad (4.25)$$
Meridional electrostatic fields $\vec{E}$, described by the reduced quaternionic Fourier cosine transform of $\tilde{\eta}(\tau)$, are characterized by formulas:

$$E_0 = \int_0^\infty \tilde{\eta}(\tau) \frac{e^{\rho \tau} + e^{-\rho \tau}}{2} \cos(x_0 \tau) d\tau = \int_0^\infty \tilde{\eta}(\tau) \cosh(\rho \tau) \cos(x_0 \tau) d\tau;$$

$$E_\rho = \int_0^\infty \tilde{\eta}(\tau) \frac{e^{\rho \tau} - e^{-\rho \tau}}{2} \sin(x_0 \tau) d\tau = \int_0^\infty \tilde{\eta}(\tau) \sinh(\rho \tau) \cos(x_0 \tau) d\tau.$$  

Condition of zero divergence leads to the following equation:

$$\int_0^\infty \tilde{\eta}(\tau) \sinh(\rho \tau) \sin(x_0 \tau) d\tau = 0. \quad (4.29)$$

**Definition 4.14.** Let us assume that $\rho > 0$. Linear integral transform of a real-valued original $\tilde{\eta}(\tau)$ with kernel in the form of the sine function of the reduced quaternionic variable, including real-valued parameter $\tau$:

$$\mathcal{F}_s\{\tilde{\eta}(\tau)\,;\,x\} := \int_0^\infty \tilde{\eta}(\tau) \sin(x \tau) d\tau = \frac{I}{2} \int_0^\infty \tilde{\eta}(\tau) (e^{-Ix \tau} - e^{Ix \tau}) d\tau \quad (4.27)$$

is called the reduced quaternionic Fourier sine transform of $\tilde{\eta}(\tau)$.

The corresponding relation in $\mathbb{R}^3$ is satisfied:

$$\mathcal{F}_s\{\tilde{\eta}(\tau)\,;\,x\} = \frac{I}{2} [\mathcal{L}\{\tilde{\eta}(\tau)\,;\,y\} - \mathcal{L}\{\tilde{\eta}(\tau)\,;\,-y\}]. \quad (4.28)$$

Meridional electrostatic fields $\vec{E}$, described by the reduced quaternionic Fourier sine transform of $\tilde{\eta}(\tau)$, are characterized by formulas:

$$E_0 = \int_0^\infty \tilde{\eta}(\tau) \cosh(\rho \tau) \cos(x_0 \tau) d\tau; \quad E_\rho = \int_0^\infty \tilde{\eta}(\tau) \sinh(\rho \tau) \cos(x_0 \tau) d\tau.$$ 

Condition of zero divergence leads to the following equation:

$$\int_0^\infty \tilde{\eta}(\tau) \sinh(\rho \tau) \cos(x_0 \tau) d\tau = 0. \quad (4.29)$$

**Definition 4.15.** Let us assume that $\rho > 0$. Linear integral transform of a real-valued original $\tilde{\eta}(\tau)$ with kernel in the form of the exponential function of the reduced quaternionic variable, including real-valued parameter $\tau$:

$$\mathcal{F}\{\tilde{\eta}(\tau)\,;\,x\} := \int_0^\infty \tilde{\eta}(\tau) e^{-Ix \tau} d\tau \quad (4.30)$$

is called the one-sided reduced quaternionic exponential Fourier transform of $\tilde{\eta}(\tau)$.

As seen, the one-sided reduced quaternionic exponential Fourier transform of $\tilde{\eta}(\tau)$ is expressed in terms of the reduced quaternionic Fourier cosine and sine transforms:

$$\mathcal{F}\{\tilde{\eta}(\tau)\,;\,x\} = \mathcal{L}\{\tilde{\eta}(\tau)\,;\,y\} = \mathcal{F}_c\{\tilde{\eta}(\tau)\,;\,x\} - I \mathcal{F}_s\{\tilde{\eta}(\tau)\,;\,x\}. \quad (4.31)$$
Meridional electrostatic fields $\vec{E}$, described by the one-sided reduced quaternionic exponential Fourier transform of $\tilde{\eta}(\tau)$, are characterized by formulas:

\[
E_0 = \int_0^\infty \tilde{\eta}(\tau)e^{\rho\tau} \cos(x_0\tau)d\tau; \quad E_\rho = \int_0^\infty \tilde{\eta}(\tau)e^{\rho\tau} \sin(x_0\tau)d\tau.
\]

Condition of zero divergence leads to the following equation:

\[
\int_0^{+\infty} \tilde{\eta}(\tau)e^{\rho\tau} \sin(x_0\tau)d\tau = 0.
\]

Similarly the two-sided reduced quaternionic exponential Fourier transform of real-valued originals $\tilde{\eta}(\tau)$ can be defined. If a real-valued original $\tilde{\eta} = \tilde{\eta}(\tau)$ does not vanish identically for $\tau < 0$, then

\[
\mathcal{F}_{-\infty}^{+\infty} \{\tilde{\eta}(\tau); x\} := \int_{-\infty}^{+\infty} \tilde{\eta}(\tau)e^{-I\tau x}d\tau,
\]

where

\[
\int_{-\infty}^{+\infty} \tilde{\eta}(\tau)e^{-I\tau x}d\tau = \int_0^{\infty} \tilde{\eta}(\tau)e^{-I\tau x}d\tau - \int_0^{\infty} \tilde{\eta}(\tau)e^{-(-1)\tau_1}d\tau_1,
\]

$\tau_1 = -\tau$ (in complex analysis see, e.g., [75]).

The two-sided reduced quaternionic exponential Fourier transform can be constructed for some subclasses of meridional electrostatic fields, described by the one-sided reduced quaternion-valued originals $\tilde{\eta}(\tau)$ in accordance with Theorem on superposition.

Some properties of similar versions of the quaternionic exponential Fourier transforms have been recently studied by Snopek [72].

Properties of subclasses of meridional electrostatic fields, described by the reduced quaternionic Fourier cosine transform, Fourier sine transform and exponential Fourier transform, have not been previously studied.

Meanwhile, $\vec{E} = (E_0, E_\rho \cos \theta, E_\rho \sin \theta)$ in accordance with (4.4), and the EFG tensor (2.7) can be rewritten in the form:

\[
\begin{pmatrix}
-\frac{\partial E_\rho}{\partial \rho} & \frac{\partial E_\rho}{\partial x_0} \cos \theta & \frac{\partial E_\rho}{\partial x_0} \sin \theta \\
\frac{\partial E_\rho}{\partial \rho} \cos \theta & \frac{\partial E_\rho}{\partial x_0} \cos^2 \theta + \frac{E_\rho}{\rho} \sin^2 \theta & \frac{\partial E_\rho}{\partial x_0} \sin \theta \cos \theta \\
\frac{\partial E_\rho}{\partial \rho} \sin \theta & \frac{\partial E_\rho}{\partial x_0} \sin \theta \cos \theta & \frac{\partial E_\rho}{\partial x_0} \sin^2 \theta + \frac{E_\rho}{\rho} \cos^2 \theta
\end{pmatrix}
\]

The principal invariants of the EFG tensor (2.7) are represented by formulas:

\[
I_1(\vec{E}) = \frac{E_\rho}{\rho}, \quad I_2(\vec{E}) = -\left[\left(\frac{\partial E_\rho}{\partial x_0}\right)^2 + \left(\frac{\partial E_\rho}{\partial \rho}\right)^2\right], \quad I_3(\vec{E}) = -\frac{E_\rho}{\rho} \left[\frac{\partial E_\rho}{\partial x_0}\right]^2 + \left(\frac{\partial E_\rho}{\partial \rho}\right)^2.
\]

It follows that expressions in terms of principal invariants take the form:

\[
p = -\frac{1}{3} \frac{E_\rho^3}{\rho^3} = -\left[\left(\frac{\partial E_\rho}{\partial x_0}\right)^2 + \left(\frac{\partial E_\rho}{\partial \rho}\right)^2\right], \quad q = -\frac{2}{27} \frac{E_\rho^3}{\rho^3} + \frac{2}{3} \frac{E_\rho}{\rho} \left[\frac{\partial E_\rho}{\partial x_0}\right]^2 + \left(\frac{\partial E_\rho}{\partial \rho}\right)^2,
\]

\[
Q = -\frac{1}{27} \left[\left(\frac{\partial E_\rho}{\partial x_0}\right)^2 \left(\frac{\partial E_\rho}{\partial \rho}\right)^2 \left[\frac{E_\rho^2}{\rho^2} - \left(\frac{\partial E_\rho}{\partial x_0}\right)^2 - \left(\frac{\partial E_\rho}{\partial \rho}\right)^2\right]^2 \leq 0.
\]

Hereby, the characteristic equation (2.9) of the EFG tensor $J(\vec{E})$

\[
\lambda^3 - \frac{E_\rho}{\rho} \lambda^2 - \left[\left(\frac{\partial E_\rho}{\partial x_0}\right)^2 + \left(\frac{\partial E_\rho}{\partial \rho}\right)^2\right] \lambda + \frac{E_\rho}{\rho} \left[\left(\frac{\partial E_\rho}{\partial x_0}\right)^2 + \left(\frac{\partial E_\rho}{\partial \rho}\right)^2\right] = 0
\]
can be reformulated:

\[
\left( \lambda - \frac{E_\rho}{\rho} \right) \left[ \lambda^2 - \left( \frac{\partial E_\rho}{\partial x_0} \right)^2 - \left( \frac{\partial E_\rho}{\partial \rho} \right)^2 \right] = 0.
\]

**Theorem 4.16 (On real roots in accordance with Fueter construction in \( \mathbb{R}^3 \)).**

Real roots of the characteristic equation (2.9) of the \( EFG \) tensor (2.7) in accordance with Fueter construction in \( \mathbb{R}^3 \) are represented by formulas:

\[
\lambda_1 = \frac{E_\rho}{\rho}, \quad \lambda_{2,3} = \pm \sqrt{\left( \frac{\partial E_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial E_\rho}{\partial \rho} \right)^2}. \tag{4.34}
\]

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