Hall Algebras Associated to Complexes of Fixed Size

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Abstract Let $A$ be a finitary hereditary abelian category with enough projectives. We study the Hall algebra of complexes of fixed size over projectives. Explicitly, we first give a relation between Hall algebras of complexes of fixed size and cyclic complexes. Second, we characterize the Hall algebra of complexes of fixed size by generators and relations, and relate it to the derived Hall algebra of $A$. Finally, we give the integration map on the Hall algebra of 2-term complexes over projectives.

Keywords Bridgeland Hall algebra, derived Hall algebra, complex of fixed size

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1 Introduction

The Hall algebra of a finite dimensional algebra $A$ over a finite field was introduced by Ringel [19] in 1990. Ringel [18, 19] proved that if $A$ is representation-finite and hereditary, the Ringel–Hall algebra of $A$ is isomorphic to the positive part of the corresponding quantized enveloping algebra. In order to give an intrinsic realization of the full quantized enveloping algebra via Hall algebra approach, one has managed to define the Hall algebra of a triangulated category satisfying some homological finiteness conditions (cf. [21, 22]). However, the root category of a finite dimensional algebra does not satisfy the homological finiteness conditions. In other words, the Hall algebra of a root category has not been defined.

In 2013, for each hereditary algebra $A$, Bridgeland [4] introduced an algebra, called the Bridgeland Hall algebra of $A$, which is the Hall algebra of 2-cyclic complexes over projective $A$-modules with some localization and reduction. He proved that the quantized enveloping algebra associated to $A$ is embedded into the Bridgeland Hall algebra of $A$. This provides a beautiful realization of the entire quantized enveloping algebra by Hall algebras.

Inspired by Bridgeland’s work, for each hereditary algebra $A$ and any nonnegative integer $m \neq 1$, Chen and Deng [8] applied Bridgeland’s construction to $m$-cyclic complexes over projective $A$-modules, and introduced the Bridgeland Hall algebra of $m$-cyclic complexes of $A$, whose algebra structure was characterized in [23].

Cluster algebras were introduced by Fomin and Zelevinsky in [10] and later the quantum cluster algebras were introduced by Berenstein and Zelevinsky in [3]. In [9], the author and his coauthors consider the Hall algebra of 2-term complexes over projective $kQ$-modules of a

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finite acyclic quiver $Q$, and realize the corresponding quantum cluster algebra with principal coefficients as a sub-quotient of the thus defined algebra, which is of some localization and twist.

In this paper, let $\mathcal{A}$ be a finitary abelian category with enough projectives and $m$ be a positive integer. The purpose of this paper is to generalize the construction of Hall algebra of 2-term complexes over projectives to $m$-term complexes, and give a characterization on the algebra structure of the thus defined algebra. In Section 2 we recall some homological properties of $m$-cyclic complexes and $m$-term complexes over projectives. We establish a relation between Hall algebras of $m$-cyclic complexes and $m$-term complexes in Section 3. From Section 4 on, we assume that $\mathcal{A}$ is hereditary. We characterize indecomposable objects in the categories of $m$-cyclic complexes and $m$-term complexes over projectives in Section 4. In order to study the Hall algebra of $m$-term complexes over projectives, in Section 5, we first give a characterization on the Hall algebra of bounded complexes over projectives, and relate it to the derived Hall algebra of $\mathcal{A}$. Section 6 is devoted to characterizing the Hall algebra of $m$-term complexes over projectives by generators and relations. Finally, we give the integration map on the Hall algebra of 2-term complexes over projectives, and propose some further research directions in Section 7.

Let us fix some notations used throughout the paper. Let $k = \mathbb{F}_q$ be always a finite field with $q$ elements. Let $\mathcal{A}$ be an (essentially small) finitary abelian $k$-category with enough projectives, and $\mathcal{P} \subset \mathcal{A}$ be the full subcategory consisting of projective objects. Given an exact category $\mathcal{E}$, we denote by $\mathcal{C}^b(\mathcal{E})$, $\mathcal{K}^b(\mathcal{E})$ and $\mathcal{D}^b(\mathcal{E})$ the category of bounded complexes over $\mathcal{E}$, the bounded homotopy category, and the bounded derived category, respectively. The Grothendieck group of $\mathcal{E}$ and the set of isomorphism classes $[X]$ of objects in $\mathcal{E}$ are denoted by $K(\mathcal{E})$ and $\text{Iso}(\mathcal{E})$, respectively. For any object $M \in \mathcal{E}$ we denote by $\hat{M}$ the image of $M$ in $K(\mathcal{E})$. For a finite set $S$, we denote by $|S|$ its cardinality. For an object $M$ in an additive category, we denote by $\text{Aut} (M)$ the automorphism group of $M$, and set $a_M := |\text{Aut} (M)|$.

2 Cyclic Complexes and Complexes of Fixed Size

In this section, we summarize some necessary homological properties of cyclic complexes and complexes of fixed size. We focus our attention on the complexes over projectives.

2.1 Cyclic Complexes

For each positive integer $m$, write $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} = \{0, 1, \ldots, m - 1\}$. By definition, an $m$-cyclic complex $M_0 = (M_i, d_i)_{i \in \mathbb{Z}_m}$ over $\mathcal{A}$ consists of objects $M_i$ in $\mathcal{A}$ and morphisms $d_i : M_i \to M_{i+1}$ with $i \in \mathbb{Z}_m$ satisfying $d_{i+1}d_i = 0$ for all $i \in \mathbb{Z}_m$. A morphism $f$ between two $m$-cyclic complexes $M_0 = (M_i, d_i)_{i \in \mathbb{Z}_m}$ and $N_0 = (N_i, c_i)_{i \in \mathbb{Z}_m}$ is given by a family of morphisms $f_i : M_i \to N_i$ with $i \in \mathbb{Z}_m$ satisfying $f_{i+1}d_i = c_if_i$ for all $i \in \mathbb{Z}_m$. The category of $m$-cyclic complexes over $\mathcal{A}$ is denoted by $C_m(\mathcal{A})$. For the simplicity of notation, we write $C_0(\mathcal{A})$ for the category $\mathcal{C}^b(\mathcal{A})$ of bounded complexes over $\mathcal{A}$, and set $\mathbb{Z}_0 := \mathbb{Z}$. The bounded complexes are called 0-cyclic complexes. For each integer $t$, we have a shift functor

$$[t] : C_m(\mathcal{A}) \to C_m(\mathcal{A}), \quad M_0 \mapsto M_0[t],$$

where $M_0[t] = (X_i, f_i)$ is defined by

$$X_i = M_{i+t}, \quad f_i = (-1)^td_{i+t}, \quad i \in \mathbb{Z}_m.$$
For $m \geq 0$, let $C_m(\mathcal{P})$ be the full subcategory of $C_m(\mathcal{A})$, which is consisting of $m$-cyclic complexes over $\mathcal{P}$. In the sense of component-wise exactness, $C_m(\mathcal{A})$ is an abelian category, and $C_m(\mathcal{P})$ is closed under extensions. For any morphism $f : Q \to P$ of projectives, if $m \neq 1$ one defines $C_f = (M_i, d_i) \in C_m(\mathcal{P})$ by

$$M_i = \begin{cases} Q & i = m - 1; \\ P & i = 0; \\ 0 & \text{otherwise,} \end{cases} \quad d_i = \begin{cases} f & i = m - 1; \\ 0 & \text{otherwise.} \end{cases}$$

If $m = 1$, one defines

$$C_f = \left( P \oplus Q, \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \right) \in C_1(\mathcal{P}).$$

So each projective object $P$ determines an object $K_P := C_{id_P}$ in $C_m(\mathcal{P})$. By [8, Lemma 2.3], all indecomposable projective (injective) objects in $C_m(\mathcal{P})$ are of the form $K_P[r]$ for some indecomposable object $P \in \mathcal{P}$ and $r \in \mathbb{Z}_m$. That is, $C_m(\mathcal{P})$ is a Frobenius exact category.

### 2.2 Complexes of Fixed Size

For each positive integer $m$, we consider the category $C^m(\mathcal{A})$. It is the full subcategory of $C^b(\mathcal{A})$ whose objects are the complexes $M_* = (M_i, d_i)$ with $M_i = 0$ if $i \notin \{1, 2, \ldots, m\}$, that is,

$$M_* = M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{m-1}} M_{m-1} \xrightarrow{d_{m-1}} M_m$$

with $d_{i+1}d_i = 0$ for $1 \leq i < m - 1$. Each object in $C^m(\mathcal{A})$ is called an $m$-term complex over $\mathcal{A}$. Let $C^m(\mathcal{P})$ be the full subcategory of $C^m(\mathcal{A})$, which is consisting of $m$-term complexes over $\mathcal{P}$. Clearly, $C^m(\mathcal{A})$ is an abelian category in the sense of component-wise exactness, and $C^m(\mathcal{P})$ is closed under extensions. It is noted that $C^1(\mathcal{P}) = \mathcal{P}$. For $m \geq 2$, the Auslander–Reiten theory and some homological properties of $C^m(\mathcal{P})$ are studied in [2, 6, 7]. In fact, it is proved in [2] that $C^m(\mathcal{P})$ is an exact category with enough projectives and injectives, and its global dimension is $m - 1$. Following [2], given an object $P \in \mathcal{P}$, we consider the following objects in $C^m(\mathcal{P})$:

- $J_P = (M_i, d_i)$ with $M_i = 0$ if $i \notin \{m - 1, m\}$, $M_{m-1} = M_m = P$ and $d_{m-1} = id_P$;
- $S_P = (M_i, d_i)$ with $M_i = 0$ if $i \neq 1$, and $M_1 = P$;
- $T_P = (M_i, d_i)$ with $M_i = 0$ if $i \neq m$, and $M_m = P$.

By [2, Corollary 3.9], all indecomposable projective objects in $C^m(\mathcal{P})$ are of the form $T_P$ for some indecomposable $P \in \mathcal{P}$ or $J_P[r]$ for some indecomposable $P \in \mathcal{P}$ and some $r \in \{0, 1, \ldots, m - 2\}$; and all indecomposable injective objects in $C^m(\mathcal{P})$ are of the form $S_P$ for some indecomposable $P \in \mathcal{P}$ or $J_P[r]$ for some indecomposable $P \in \mathcal{P}$ and some $r \in \{0, 1, \ldots, m - 2\}$. Thus, for $m \geq 2$, $C^m(\mathcal{P})$ is not Frobenius. For $m \geq 2$ and any morphism $f : Q \to P$ of projectives, define $T_f = (M_i, d_i) \in C^m(\mathcal{P})$ by

$$M_i = \begin{cases} Q & i = m - 1; \\ P & i = m; \\ 0 & \text{otherwise,} \end{cases} \quad d_i = \begin{cases} f & i = m - 1; \\ 0 & \text{otherwise.} \end{cases}$$

So for each projective object $P$, we have that $T_{id_P} = J_P$. 
3 Hall Algebras of $m$-cyclic Complexes and $m$-term Complexes

In this section, letting $m \geq 1$, we study the relation between Hall algebras of $m$-cyclic complexes and $m$-term complexes.

Given objects $L, M, N \in \mathcal{A}$, let $\text{Ext}^1_{\mathcal{A}}(M, N)_L \subset \text{Ext}^1_{\mathcal{A}}(M, N)$ be the subset consisting of those equivalence classes of short exact sequences with middle term $L$.

**Definition 3.1** The Hall algebra $\mathcal{H}(A)$ of $\mathcal{A}$ is the vector space over $\mathbb{C}$ with basis elements $[M] \in \text{Iso}(\mathcal{A})$, and with the multiplication defined by

$$[M] \diamond [N] = \sum_{[L] \in \text{Iso}(\mathcal{A})} \frac{\text{Ext}^1_{\mathcal{A}}(M, N)_L}{|\text{Hom}_{\mathcal{A}}(M, N)|}[L].$$

**Remark 3.2** Given objects $L, M, N \in \mathcal{A}$, set

$$g_{MN}^L := \{\{N' \subset L \mid N' \cong N, L/N' \cong M\}\}.$$ 

It is well known that

$$g_{MN}^L = \frac{|\text{Ext}^1_{\mathcal{A}}(M, N)_L|}{|\text{Hom}_{\mathcal{A}}(M, N)|} \cdot \frac{a_L}{a_M a_N},$$

where

$$W_{MN}^L = \{(\varphi, \psi) \mid \begin{array}{ccc} 0 & \rightarrow & N \xrightarrow{\varphi} L \xrightarrow{\psi} M \rightarrow 0 \end{array} \text{ is exact in } \mathcal{A}.\}.$$ 

By the Riedtmann–Peng formula [15, 17],

$$g_{MN}^L = \frac{|\text{Ext}^1_{\mathcal{A}}(M, N)_L|}{|\text{Hom}_{\mathcal{A}}(M, N)|} \cdot \frac{a_L}{a_M a_N}. \quad (3.1)$$

Thus,

$$[M] \diamond [N] = \sum_{[L] \in \text{Iso}(\mathcal{A})} \frac{|W_{MN}^L|}{a_L}[L]. \quad (3.2)$$

In fact, in terms of alternative generators $[[M]] = \frac{[M]}{a_M}$, the product takes the form

$$[[M]] \diamond [[N]] = \sum_{[L] \in \text{Iso}(\mathcal{A})} g_{MN}^L[[L]],$$

which is the definition used, for example, in [19, 20].

Let $\mathcal{H}(C_m(\mathcal{A}))$ (resp., $\mathcal{H}(C^m(\mathcal{A}))$) be the Hall algebra of the abelian category $C_m(\mathcal{A})$ (resp., $C^m(\mathcal{A})$) as defined in Definition 3.1. Let $\mathcal{H}(C_m(\mathcal{P}))$ (resp., $\mathcal{H}(C^m(\mathcal{P}))$) be the subspace of $\mathcal{H}(C_m(\mathcal{A}))$ (resp., $\mathcal{H}(C^m(\mathcal{A}))$) spanned by the isomorphism classes of objects in $C_m(\mathcal{P})$ (resp., $C^m(\mathcal{P})$). Since $C_m(\mathcal{P})$ (resp., $C^m(\mathcal{P})$) is closed under extensions, $\mathcal{H}(C_m(\mathcal{P}))$ (resp., $\mathcal{H}(C^m(\mathcal{P}))$) is a subalgebra of the Hall algebra $\mathcal{H}(C_m(\mathcal{A}))$ (resp., $\mathcal{H}(C^m(\mathcal{A}))$).

Define $\mathcal{I}$ to be the subspace of $\mathcal{H}(C_m(\mathcal{P}))$ spanned by elements $[M_0]$, where $M_0 = (M_i, d_i) \in C_m(\mathcal{P})$ with $d_0 \neq 0$. Denote by $\mathcal{I}'$ the complement space of $\mathcal{I}$ in $\mathcal{H}(C_m(\mathcal{P}))$, which is spanned by elements $[M_0]$ satisfying that $M_0 = (M_i, d_i) \in C_m(\mathcal{P})$ with $d_0 = 0$. Namely, as vector spaces, we have the decomposition $\mathcal{H}(C_m(\mathcal{P})) = \mathcal{I} \oplus \mathcal{I}'$.

**Lemma 3.3** $\mathcal{I}$ is an ideal of $\mathcal{H}(C_m(\mathcal{P}))$. 

There exists an isomorphism of algebras $\chi$ that

$$0 \rightarrow N_0 \xrightarrow{f} L_0 \xrightarrow{g} M_0 \rightarrow 0,$$

where $L_0 = (L, b) \in C_1(\mathcal{P})$. Then we have that $fc = bf$. Assume that $b = 0$, thus $fc = 0$. We get that $c = 0$, since $f$ is injective. This is a contradiction, and thus we conclude that $b \neq 0$.

That is, $[M_0] \circ [N_0] \in I$. Similarly, we prove that $[N_0] \circ [M_0] \in I$. That is, $I$ is an ideal of $\mathcal{H}(C_1(\mathcal{P}))$.

If $m > 1$, let $N_0 = (N, c_i) \in C_m(\mathcal{P})$ with $c_0 \neq 0$, i.e., $[N_0] \in I$. For any $M_0 = (M, d_i) \in C_m(\mathcal{P})$, consider the short exact sequence

$$0 \rightarrow N_0 \xrightarrow{f} L_0 \xrightarrow{g} M_0 \rightarrow 0,$$

where $L_0 = (L, b_i) \in C_m(\mathcal{P})$. Then we have that $f_1 c_0 = b_0 f_0$. Assume that $b_0 = 0$, thus $f_1 c_0 = 0$. We get that $c_0 = 0$, since $f_1$ is injective. This is a contradiction, and thus we conclude that $b_0 \neq 0$. That is, $[M_0] \circ [N_0] \in I$. Similarly, we prove that $[N_0] \circ [M_0] \in I$. Hence, $I$ is an ideal of $\mathcal{H}(C_m(\mathcal{P}))$.

Given $M_0 = (M_i, d_i) \in C_m(\mathcal{P})$, we fix an object $M_\bullet \in C_m(\mathcal{P})$ defined by

$$M_\bullet = M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{m-1}} M_m,$$

where $M_m := M_0$. In particular, for $M_0 = (M_0, d_0) \in C_1(\mathcal{P})$, $M_\bullet := M_0 \in C^1(\mathcal{P})$.

**Theorem 3.4** There exists an isomorphism of algebras

$$\rho : \mathcal{H}(C_m(\mathcal{P}))/I \xrightarrow{\cong} \mathcal{H}(C^m(\mathcal{P})).$$

**Proof** Let $\chi : \mathcal{H}(C_m(\mathcal{P})) \rightarrow \mathcal{H}(C^m(\mathcal{P}))$ be the linear map defined on basis elements by

$$\chi([M_0]) = \begin{cases} [M_\bullet] & \text{if } d_0 = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $M_0 = (M_i, d_i) \in C_m(\mathcal{P})$. For any objects $M_0 = (M_i, d_i), N_0 = (N_i, c_i) \in C_m(\mathcal{P})$, if $M_0 \cong N_0$, then $d_0 = 0$ if and only if $c_0 = 0$; assume that $d_0 = 0$, in this case $M_\bullet \cong N_\bullet$. Hence, $\chi$ is well defined.

Given objects $M_0 = (M_i, d_i), N_0 = (N_i, c_i) \in C_m(\mathcal{P})$, on one hand, if $d_0 \neq 0$ or $c_0 \neq 0$, i.e., $\chi([M_0]) = 0$ or $\chi([N_0]) = 0$, then $\chi([M_0]) \circ \chi([N_0]) = 0$. By Lemma 3.3 we have that $[M_0] \circ [N_0] \in I$, thus $\chi([M_0] \circ [N_0]) = 0$.

On the other hand, if $d_0 = 0$ and $c_0 = 0$, then by (3.2), we obtain that

$$[M_0] \circ [N_0] = \sum_{L_0 : L_0 = (L_i, b_i) \text{ with } b_0 = 0 \atop a_{L_0}L_0} \frac{W_{L_0^{M_0, N_0}}}{a_{L_0}} [L_0] + x,$$

where $x \in I$. Since $\chi(x) = 0$, we obtain that

$$\chi([M_0] \circ [N_0]) = \sum_{L_0 : L_0 = (L_i, b_i) \text{ with } b_0 = 0 \atop a_{L_0}L_0} \frac{W_{L_0^{M_0, N_0}}}{a_{L_0}} [L_\bullet].$$
In this case, it is clear that there exists a bijection between $W_{M_0, N_0}^{L_0}$ and $W_{M_*, N_*}^{L_*}$, and $a_{L_0} = a_{L_*}$. Hence,

$$
\sum_{[L_0]: L_0 = (L_0, h)} \frac{[W_{M_0, N_0}^{L_0}]}{a_{L_0}} [L_0] = \sum_{[L_*]} \frac{[W_{M_*, N_*}^{L_*}]}{a_{L_*}} [L_*] = \chi([N_0]) \circ \chi([M_0]).
$$

Therefore, $\chi$ is a homomorphism of algebras.

For any $M_0 = (M_0, d_i) \in C^m(\mathcal{P})$, take $M_0 = (M_0, d_i) \in C^m(\mathcal{P})$ with $M_0 := M_m$ and $d_0 = 0$, then $\chi([M_0]) = [M_*]$, and thus $\chi$ is surjective.

Since $\chi(I) = 0$, $\chi$ induces a surjective homomorphism of algebras

$$
\rho : \mathcal{H}(C_m(\mathcal{P}))/I \longrightarrow \mathcal{H}(C^m(\mathcal{P})).
$$

The injectivity of $\rho$ follows from the fact that $\rho$ sends the basis $\{[M_0] + I \mid M_0 = (M_0, d_i) \in C_m(\mathcal{P}) \text{ with } d_0 = 0 \}$ of $\mathcal{H}(C_m(\mathcal{P}))/I$ to a basis of $\mathcal{H}(C^m(\mathcal{P}))$. \hfill \Box

4 Indecomposable Objects in $C_m(\mathcal{P})$ and $C^m(\mathcal{P})$

From now onwards, we always assume that $\mathcal{A}$ is hereditary until the end of the entire paper. For each object $M \in \mathcal{A}$, according to [4, Section 4.1], it has a minimal projective resolution:

$$
0 \longrightarrow \Omega_M \overset{\delta_M}{\longrightarrow} P_M \longrightarrow M \longrightarrow 0.
$$

(4.1)

Moreover, we have the following well-known lemma (cf. [4, Lemma 4.1]).

**Lemma 4.1** Given $M \in \mathcal{A}$, each projective resolution of $M$ is isomorphic to a resolution of the form

$$
0 \longrightarrow \Omega_M \oplus R \overset{\delta_M \oplus 1}{\longrightarrow} P_M \oplus R \longrightarrow M \longrightarrow 0,
$$

for some $R \in \mathcal{P}$ and some minimal projective resolution

$$
0 \longrightarrow \Omega_M \overset{\delta_M}{\longrightarrow} P_M \longrightarrow M \longrightarrow 0.
$$

We define objects $C_M := C_{M, [M]} \in C_m(\mathcal{P})$ for $m \geq 0$ and $T_M := T_{[M]} \in C^m(\mathcal{P})$ for $m \geq 2$. By Lemma 4.1, we know that any two minimal projective resolutions of $M$ are isomorphic, so $C_M$ and $T_M$ are well defined up to isomorphism.

Now, let us give characterizations of indecomposable objects in $C_m(\mathcal{P})$ and $C^m(\mathcal{P})$.

**Lemma 4.2** ([8, Lemma 2.3]) For $m \geq 0$, the objects $C_M[r]$ and $K_P[r]$, where $r \in \mathbb{Z}_m$, $M \in \mathcal{A}$ is indecomposable and $P \in \mathcal{P}$ is indecomposable, provide a complete set of indecomposable objects in $C_m(\mathcal{P})$. Moreover, all $K_P[r]$ with $r \in \mathbb{Z}_m$ form a complete set of indecomposable projective-injective objects.

**Lemma 4.3** For $m \geq 2$, the objects $S_P$, $T_M[r]$ and $J_P[r]$, where $0 \leq r < m - 1$, $M \in \mathcal{A}$ is indecomposable and $P \in \mathcal{P}$ is indecomposable, provide a complete set of indecomposable objects in $C^m(\mathcal{P})$. Moreover, all $J_P[r]$ with $0 \leq r < m - 1$ and $T_P$ form a complete set of indecomposable projective objects; all $J_P[r]$ with $0 \leq r < m - 1$ and $S_P$ form a complete set of indecomposable injective objects.

1) The notations $P_M$ and $\Omega_M$ will be used throughout the paper.
Proof. We only prove the first statement, since the others have been given in the previous section. Take an arbitrary object in $C^m(\mathcal{P})$

$$M_\bullet = M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{m-1}} M_{m-1} \xrightarrow{d_{m-1}} M_m.$$  

For each $1 \leq i < m$, we have the short exact sequence

$$0 \rightarrow \text{Ker } d_i \xrightarrow{\lambda_i} M_i \xrightarrow{d_i} \text{Im } d_i \rightarrow 0.$$  

By the hereditary assumption, all the objects appearing in these sequences are projective. Thus these sequences split. That is, we can assume that $M_i = \text{Ker } d_i \oplus \text{Im } d_i$ for each $1 \leq i < m$. Writing $M_\bullet$ as follows:

$$M_\bullet = \text{Ker } d_1 \oplus \text{Im } d_1 \xrightarrow{(0 \ 0 \ d_1^\prime)} \text{Im } d_2 \oplus \text{Ker } d_2 \xrightarrow{(d_2 \ 0 \ 0)} M_3 \xrightarrow{d_3} \cdots \xrightarrow{d_{m-1}} M_{m-1} \xrightarrow{d_{m-1}} M_m,$$

where and elsewhere $d_i^\prime = d_i|_{\text{Im } d_i}$ for $1 \leq i < m$, we obtain that $M_\bullet$ is the direct sum of the following two objects

$$M_1^1 = \text{Ker } d_1 \xrightarrow{0} \text{Im } d_2 \xrightarrow{d_2^\prime} M_3 \xrightarrow{d_3} \cdots \xrightarrow{d_{m-1}} M_{m-1} \xrightarrow{d_{m-1}} M_m$$

and

$$N_1^1 = \text{Im } d_1 \xrightarrow{d_1^\prime} \text{Ker } d_2 \xrightarrow{0} \cdots \xrightarrow{0} 0 \rightarrow 0.$$  

Similarly, $M_1^2$ is the direct sum of the following two objects

$$M_1^2 = \text{Ker } d_1 \xrightarrow{0} \text{Im } d_2 \xrightarrow{d_2^\prime} M_3 \xrightarrow{d_3} \cdots \xrightarrow{d_{m-1}} M_{m-1} \xrightarrow{d_{m-1}} M_m$$

and

$$N_1^2 = 0 \xrightarrow{d_1^\prime} \text{Ker } d_2 \xrightarrow{0} \cdots \xrightarrow{0} 0 \rightarrow 0.$$  

Repeating this process, we get that $M_\bullet$ has a direct sum decomposition

$$M_\bullet = S_P \oplus N_1^1 \oplus \cdots \oplus N_{m-2}^1 \oplus N_{m-1}^m$$

where $P = \text{Ker } d_1$; for $1 \leq i \leq m - 2$, $N_i^i = (X_j, f_j)$ with $X_j = 0$ if $j \not\in \{i, i + 1\}$, $X_i = \text{Im } d_i$, $X_{i+1} = \text{Ker } d_{i+1}$ and $f_i = d_i^\prime$; and $N_{m-1}^m = (X_j, f_j)$ with $X_j = 0$ if $j \not\in \{m - 1, m\}$, $X_{m-1} = \text{Im } d_{m-1}$, $X_m = M_m$ and $f_{m-1} = d_{m-1}^\prime$.

Noting for each $1 \leq i \leq m - 1$ the differential $f_i$ in $N_i^i$ is injective, we have the short exact sequence

$$0 \rightarrow X_i \xrightarrow{f_i} X_{i+1} \rightarrow Y_i \rightarrow 0,$$

where $Y_i := \text{Coker } f_i$. Then, by Lemma 4.1, $N_i^i \cong T_{Y_i}[m - i - 1] \oplus J_{R_i}[m - i - 1]$ for some $R_i \in \mathcal{P}$. Set $r = m - i - 1$, since $1 \leq i \leq m - 1$ we have that $0 \leq r < m - 1$. Therefore, we complete the proof.  

In what follows, we will prove the (high) extensions in $C^m(\mathcal{P})$ coincide with those in
Lemma 4.4. For any $M_\bullet \in C^m(\mathcal{P})$, we have the following exact sequence

$$\begin{array}{c}
\Omega_{M_\bullet} : \\
{\xymatrix}{0 \ar[r] & M_1 \ar[d]^{(\frac{1}{0})} & \cdots & M_{m-2} \ar[d]^{(-\frac{1}{0})} & M_{m-1} \ar[d]^{(-\frac{1}{0})} & M_m \ar[d]^{(-\frac{1}{0})} & 0} \\
{\xymatrix}{P_{M_\bullet} :}{M_1 \ar[r]^{(\frac{1}{0})} & M_1 \oplus M_2 \ar[r]^{(d_1 1)} & \cdots & M_{m-2} \oplus M_{m-1} \ar[r]^{(d_{m-2} 1)} & M_{m-1} \oplus M_m} \\
{\xymatrix}{M_\bullet :}{M_1 \ar[r]^{d_1} & M_2 \ar[r]^{d_2} & \cdots & M_{m-1} \ar[r]^{d_{m-1}} & M_m.}
\end{array}$$

It is noted that $P_{M_\bullet}$ is projective in $C^m(\mathcal{P})$. So we obtain that for any positive integer $k$ and $N_\bullet \in C^m(\mathcal{P})$, we have that

$$\text{Ext}_{C^m(\mathcal{P})}^{k+1}(M_\bullet, N_\bullet) \cong \text{Ext}_{C^m(\mathcal{P})}^k(\Omega_{M_\bullet}, N_\bullet). \quad (4.3)$$

Lemma 4.4. For any $M_\bullet, N_\bullet \in C^m(\mathcal{P})$ and any positive integer $k$, we have that

$$\text{Ext}_{C^m(\mathcal{P})}^k(M_\bullet, N_\bullet) \cong \text{Ext}_{C^b(\mathcal{P})}^k(M_\bullet, N_\bullet).$$

Proof. We proceed the proof by induction on $k$. If $k = 1$, clearly,

$$\text{Ext}_{C^m(\mathcal{P})}^1(M_\bullet, N_\bullet) \cong \text{Ext}_{C^b(\mathcal{P})}^1(M_\bullet, N_\bullet).$$

For $k \geq 2$, without loss of generality, we assume that $M_\bullet$ is indecomposable. By Lemma 4.3, $M_\bullet$ is of the forms $S_P$, $T_M[r]$ or $J_P[r]$, for some $0 \leq r < m - 1$, indecomposable $M \in \mathcal{A}$ and indecomposable $P \in \mathcal{P}$.

If $M_\bullet = J_P[r]$ for some $0 \leq r < m - 1$, then $\text{Ext}_{C^m(\mathcal{P})}^k(M_\bullet, N_\bullet) = 0 = \text{Ext}_{C^b(\mathcal{P})}^k(M_\bullet, N_\bullet)$, since $J_P[r]$ is projective in both $C^m(\mathcal{P})$ and $C^b(\mathcal{P})$.

If $M_\bullet = T_M$ for some indecomposable $M \in \mathcal{A}$. By (4.2), we have a short exact sequence

$$0 \longrightarrow T_{\Omega_M} \longrightarrow J_{\Omega_M} \oplus T_P \longrightarrow T_M \longrightarrow 0.$$}

Noting $T_{\Omega_M}, J_{\Omega_M}$ and $T_P$ are projective in $C^m(\mathcal{P})$, we obtain that $\text{Ext}_{C^m(\mathcal{P})}^k(M_\bullet, N_\bullet) = 0$ for any $k \geq 2$. In this case, we claim that $\text{Ext}_{C^b(\mathcal{P})}^k(M_\bullet, N_\bullet) = 0$ for any $k \geq 2$. Indeed, without loss of generality, we assume that $N_\bullet$ is indecomposable. If $N_\bullet = J_P[r]$ for some $0 \leq r < m - 1$, clearly, $\text{Ext}_{C^b(\mathcal{P})}^k(M_\bullet, N_\bullet) = 0$, since each $J_P[r]$ is also projective in $C^b(\mathcal{P})$; if $N_\bullet = S_P$, $\text{Ext}_{C^b(\mathcal{P})}^k(M_\bullet, N_\bullet) \cong \text{Hom}_{K^b(\mathcal{P})}(T_M, S_P[k]) \cong \text{Hom}_{D^b(\mathcal{A})}(M, P[m - 1 + k]) = 0$; if $N_\bullet = T_N[r]$ for some $N \in \mathcal{A}$ and $0 \leq r < m - 1$, $\text{Ext}_{C^b(\mathcal{P})}^k(M_\bullet, N_\bullet) \cong \text{Hom}_{K^b(\mathcal{P})}(T_M, T_N[r + k]) \cong \text{Hom}_{D^b(\mathcal{A})}(M, N[r + k]) = 0$.

If $M_\bullet$ takes the other forms of indecomposable objects in $C^m(\mathcal{P})$, i.e., $M_\bullet$ is of the form $S_P$ or $T_M[r]$ for some $0 < r < m - 1$, in this case, the $m$-th component $M_m$ of $M_\bullet$ is zero. Thus, the complex $P_{M_\bullet}$ is also projective in $C^b(\mathcal{P})$. Hence, we obtain that for $k \geq 2$

$$\text{Ext}_{C^m(\mathcal{P})}^k(M_\bullet, N_\bullet) \cong \text{Ext}_{C^m(\mathcal{P})}^{k-1}(\Omega_{M_\bullet}, N_\bullet) \quad (\text{since } P_{M_\bullet} \text{ is projective in } C^m(\mathcal{P}))$$

$$\cong \text{Ext}_{C^b(\mathcal{P})}^{k-1}(\Omega_{M_\bullet}, N_\bullet) \quad (\text{by induction})$$

$$\cong \text{Ext}_{C^b(\mathcal{P})}^k(M_\bullet, N_\bullet) \quad (\text{since } P_{M_\bullet} \text{ is projective in } C^b(\mathcal{P})). \quad \Box$$
5 Hall Algebras of Bounded Complexes and Derived Hall Algebras

In order to study the Hall algebra of \( C^m(\mathcal{P}) \), by reformulating [23, Theorem 3.2] we first give a characterization of the Hall algebra of \( C^b(\mathcal{P}) \), and then relate it to the derived Hall algebra of \( A \). These are similar to the results given in [14, Section 5.2].

Let \( H(C^b(A)) \) be the Hall algebra of the abelian category \( C^b(A) \) as defined in Definition 3.1. Let \( \mathcal{H}(C^b(\mathcal{P})) \) be the subspace of \( \mathcal{H}(C^b(A)) \) spanned by the isomorphism classes of objects in \( C^b(\mathcal{P}) \). Since \( C^b(\mathcal{P}) \) is closed under extensions, \( \mathcal{H}(C^b(\mathcal{P})) \) is a subalgebra of the Hall algebra \( \mathcal{H}(C^b(A)) \).

For objects \( M, N \in A \), define

\[
\langle M, N \rangle := \dim_k \text{Hom}_A(M, N) - \dim_k \text{Ext}_A^1(M, N).
\]  

(5.1)

Since \( A \) is hereditary, it descends to give a bilinear form

\[
\langle \cdot, \cdot \rangle : K(A) \times K(A) \longrightarrow \mathbb{Z},
\]

known as the Euler form.

Define the Hall algebra \( \mathcal{H}_{tw}(A) \) to be the same vector space as \( \mathcal{H}(A) \), but with the twisted multiplication defined by

\[
[M] * [N] = q^{\langle M, N \rangle} [M] \circ [N].
\]

Given objects \( M_\bullet, N_\bullet \in C^b(\mathcal{P}) \), there exists a positive integer \( m \) such that \( M_\bullet, N_\bullet \in C^m(\mathcal{P}) \). Since the global dimension of \( C^m(\mathcal{P}) \) is \( m - 1 \), by Lemma 4.4, we know that \( \text{Ext}^i_{C^b(\mathcal{P})}(M_\bullet, N_\bullet) = 0 \) for any \( i \geq m \), that is, \( C^b(\mathcal{P}) \) is locally homological finite. Thus, the Euler form of \( C^b(\mathcal{P}) \)

\[
\langle \cdot, \cdot \rangle : K(C^b(\mathcal{P})) \times K(C^b(\mathcal{P})) \longrightarrow \mathbb{Z}
\]

determined by

\[
\langle M_\bullet, N_\bullet \rangle = \sum_{i \geq 0} (-1)^i \dim_k \text{Ext}^i_{C^b(\mathcal{P})}(M_\bullet, N_\bullet)
\]  

(5.2)

is also well defined.

**Lemma 5.1** For any \( M, N \in A \) and \( r, l \in \mathbb{Z} \), we have that

1. \( \text{Ext}^i_{C^b(\mathcal{P})}(C_M[r], C_N[r]) = 0 \) for any \( i \geq 2 \);
2. \( \text{Ext}^1_{C^b(\mathcal{P})}(C_M[r], C_N[r]) \cong \text{Ext}^1_A(M, N) \);
3. \( \text{Ext}^i_{C^b(\mathcal{P})}(C_M[r], C_N[r+1]) = 0 \) for any \( i \geq 1 \);
4. \( \langle C_M[r], C_N[l] \rangle = (-1)^{r-l}(M, N) \), if \( r - l \geq 1 \);
5. \( \langle C_M[r], C_N[l] \rangle = 0 \), if \( l - r > 1 \).

**Proof** For any \( i \geq 1 \), by [12, Lemma 3.1], we have that

\[
\text{Ext}^i_{C^b(\mathcal{P})}(C_M[r], C_N[l]) \cong \text{Hom}_{K^b(\mathcal{P})}(C_M[r], C_N[l + i])
\]

\[
\cong \text{Hom}_{D^b(A)}(M, N[l + r])
\]

(5.3)

\[
\begin{cases} 
\text{Hom}_A(M, N), & l - r + i = 0; \\
\text{Ext}^1_A(M, N), & l - r + i = 1; \\
0, & \text{otherwise.}
\end{cases}
\]
Thus, (1)–(3) can be easily proved. Noting that $ \text{Hom}_{\mathcal{P}}(C_M[r], C_N[l]) = 0 $ if $ r - l \geq 1 $, we prove (4). Noting that $ \text{Hom}_{\mathcal{P}}(C_M[r], C_N[l]) = 0 $ if $ l - r > 1 $, we prove (5).

Define the Hall algebra $ \mathcal{H}_{tw}(C^b(\mathcal{P})) $ to be the same vector space as $ \mathcal{H}(C^b(\mathcal{P})) $, but with the twisted multiplication

$$ [M_\bullet] * [N_\bullet] = q^{(M_\bullet \cdot N_\bullet)} [M_\bullet] \circ [N_\bullet]. $$

(5.4)

For any $ P \in \mathcal{P} $ and $ r \in \mathbb{Z} $, since $ K_P[r] $ is projective-injective in $ C^b(\mathcal{P}) $, we have that

$$ \langle K_P[r], M_\bullet \rangle = \dim_k \text{Hom}_{\mathcal{P}}(K_P[r], M_\bullet), \quad \langle M_\bullet, K_P[r] \rangle = \dim_k \text{Hom}_{\mathcal{P}}(M_\bullet, K_P[r]), $$

and thus obtain that

$$ [K_P[r]] * [M_\bullet] = [M_\bullet \oplus K_P[r]] = [M_\bullet] * [K_P[r]] $$

(5.5)

for any $ M_\bullet \in C^b(\mathcal{P}) $.

Define the Hall algebra $ \mathcal{MH}(A) $ to be the localization of $ \mathcal{H}_{tw}(C^b(\mathcal{P})) $ with respect to elements $ [K_P[r]] $ for all $ P \in \mathcal{P} $ and $ r \in \mathbb{Z} $. For each $ r \in \mathbb{Z} $ and $ \alpha \in K(A) $, by writing $ \alpha = \tilde{P} - \tilde{Q} $ for some $ P, Q \in \mathcal{P} $, we define

$$ K_{\alpha, r} = [K_P[r]] * [K_Q[r]]^{-1}. $$

For any $ \alpha, \beta \in K(A) $, it is easy to see that

$$ K_{\alpha, r} * K_{\beta, r} = K_{\alpha + \beta, r}. $$

(5.6)

Moreover, for any $ M_\bullet \in C^b(\mathcal{P}) $ we have that

$$ K_{\alpha, r} * [M_\bullet] = [M_\bullet] * K_{\alpha, r}. $$

(5.7)

Given an object $ M \in A $, for each $ r \in \mathbb{Z} $, we define

$$ E_{M, r} := K_{-\Omega_{M}, r} * [C_M[r]] \in \mathcal{MH}(A). $$

(5.8)

Let us reformulate [8, Proposition 4.4 (2)] in the following

**Proposition 5.2** For each $ r \in \mathbb{Z} $, there exists an embedding of algebras

$$ \psi_r : \mathcal{H}_{tw}(A) \xrightarrow{\cong} \mathcal{MH}(A), \quad [M] \mapsto E_{M, r}. $$

**Remark 5.3** Compared with the modified Ringel–Hall algebra studied in [14], the element $ E_{M, r} $ corresponding to $ [M] $ needs to be defined by appending the element $ K_{-\Omega_{M}, r} $ to $ [C_M[r]] $. Actually, in the modified Ringel–Hall algebra, the embedding above is immediate.

By Lemma 4.2, applying the arguments similar to those in the proof of [8, Proposition 4.4 (3)], we obtain the following

**Proposition 5.4** $ \mathcal{MH}(A) $ has a basis consisting of elements

$$ K_{\alpha, r} * K_{\alpha_{i+1}, r+1} \cdots * K_{\alpha, l} * E_{M, r} * E_{M_{r+1}, r+1} \cdots * E_{M_{l}, l}, $$

where $ r, l \in \mathbb{Z}, r \leq l, \alpha_i \in K(A) $ and $ M_i \in A $ for $ r \leq i \leq l $.

Given objects $ M, N, X, Y \in A $, we denote by $ W^Y_{MN} $ the set

$$ \{(f, g, h) \mid 0 \xrightarrow{g} X \xrightarrow{f} M \xrightarrow{h} N \xrightarrow{0} Y \quad \text{is exact in} \ A \} $$
and set
\[ \gamma_{MN}^{XY} := \frac{|W_{MN}^{XY}|}{a_M a_N}. \]

Applying the arguments similar to those in the proof of [23, Theorem 3.2] together with Lemma 5.1, we obtain the following

**Proposition 5.5** The Hall algebra \( \mathcal{MH}(A) \) is generated by the elements in
\[ \{E_{M,r}, K_{\alpha,r} \mid M \in A, \alpha \in K(A), r \in \mathbb{Z}\} \]
with the defining relations
\[
E_{M,r} \cdot E_{N,r} = \sum_{[L]} q^{(M,N)} |\text{Ext}_A^1(M,N)_L| E_{L,r}; \tag{5.9}
\]
\[
E_{M,r+1} \cdot E_{N,r} = \sum_{[X],[Y]} q^{(M,N)} \gamma_{XY}^{MN} a_M a_N E_{Y,r} \cdot E_{X,r+1} \cdot K_{M-X,r}; \tag{5.10}
\]
\[
E_{M,r} \cdot E_{N,l} = q^{(-1)^{-1}(M,N)} E_{N,l} \cdot E_{M,r}, \quad r-l > 1; \tag{5.11}
\]
\[
K_{\alpha,r} \cdot E_{M,l} = E_{M,l} \cdot K_{\alpha,r}; \tag{5.12}
\]
\[
K_{\alpha,r} \cdot K_{\beta,r} = K_{\alpha+\beta,r}, \quad K_{\alpha,r} \cdot K_{\beta,l} = K_{\beta,l} \cdot K_{\alpha,r}; \tag{5.13}
\]
where \( M, N \in A, \alpha, \beta \in K(A) \) and \( r, l \in \mathbb{Z} \).

**Remark 5.6** By Proposition 5.5 and [14, Proposition 5.3], we obtain that the Hall algebra \( \mathcal{MH}(A) \) is isomorphic to the modified Ringel–Hall algebra \( \mathcal{MH}_{tw}(A) \) defined in [14]. Explicitly, there exists an isomorphism \( \varphi : \mathcal{MH}(A) \rightarrow \mathcal{MH}_{tw}(A) \) defined on generators by \( E_{M,r} \mapsto U_{M,-r} \) and \( K_{\alpha,r} \mapsto K_{\alpha,-r} \).

For simplicity, we recall the twisted derived Hall algebra \( \mathcal{DH}(A) \) of \( A \) in the form of generators and relations in the following

**Proposition 5.7** ([21]) \( \mathcal{DH}(A) \) is an associative and unital \( \mathbb{C} \)-algebra generated by the elements in \( \{Z_{M}^{[r]} \mid M \in \text{Iso}(A), r \in \mathbb{Z}\} \) and the following relations
\[
Z_{M}^{[r]} \cdot Z_{N}^{[r]} = \sum_{[L]} q^{(M,N)} |\text{Ext}_A^1(M,N)_L| Z_{L}^{[r]}; \tag{5.14}
\]
\[
Z_{M}^{[r+1]} \cdot Z_{N}^{[r]} = \sum_{[X],[Y]} q^{(M,N)} \gamma_{XY}^{MN} a_M a_N Z_{Y}^{[r]} \cdot Z_{X}^{[r+1]}; \tag{5.15}
\]
\[
Z_{M}^{[r]} \cdot Z_{N}^{[l]} = q^{(-1)^{-1}(M,N)} Z_{N}^{[l]} \cdot Z_{M}^{[r]}, \quad r-l > 1. \tag{5.16}
\]

Now we reformulate [14, Theorem 5.5] in the following

**Theorem 5.8** There is an embedding of algebras
\[
\Psi : \mathcal{DH}(A) \hookrightarrow \mathcal{MH}(A)
\]
defined on generators (with \( n > 0 \)) by
\[
Z_{M}^{[0]} \mapsto E_{M,0}, \quad Z_{M}^{[n]} \mapsto E_{M,n} \cdot \prod_{i=1}^{n} K_{(-1)^i M,n-i}
\]
and
\[
Z^{-\eta}_M \rightarrow E_{M,-\eta} \prod_{i=1}^{\eta} K_{(-1)^i M_{-1}i} + (n-\eta+1)
\]

Let \( T^\infty(A) \) be the subalgebra of \( MH(A) \) generated by elements \( K_{\alpha,r} \) with \( \alpha \in K(A) \) and \( r \in \mathbb{Z} \). By [14, Corollary 5.7], there is an isomorphism of algebras
\[
\hat{\Psi} : DH(A) \otimes C T^\infty(A) \rightarrow MH(A), \quad x \otimes t \mapsto \Psi(x) \ast t.
\]
(5.17)
By [14, Corollary 5.7], we know that \( MH(A) \) is invariant under derived equivalences. The inverse of \( \hat{\Psi} \) is the homomorphism
\[
\hat{\Psi}^{-1} : MH(A) \rightarrow DH(A) \otimes C T^\infty(A)
\]
(5.18)
defined on generators (with \( \eta > 0 \)) by
\[
K_{\alpha,n} \mapsto K_{\alpha,n}, \quad E_{M,0} \mapsto Z^0_M,
K_{\alpha,n} \mapsto Z^n_M \prod_{i=0}^{n-1} K_{(-1)^{n-1-i} M_{-1}i} \quad \text{and} \quad E_{M,-n} \mapsto Z^{-\eta}_M \prod_{i=1}^{n} K_{(-1)^{\eta-i} M_{-1}i},
\]
where we have written elements \( x \otimes t \in DH(A) \otimes C T^\infty(A) \) as \( x \ast t \).

In fact, these results above are essentially the same as those given by Gorsky in [12, Theorem 4.2]. However, the explicit map between the Hall algebra of \( C^b(P) \) and the derived Hall algebra is not given there. Besides, the twist therein is only used in the Hall algebra of \( C^b(P) \), not in the derived Hall algebra, which not only involves \( C^b(P) \), but also \( K^b(P) \).

### 6 Hall Algebras of \( m \)-term Complexes

Since the global dimension of \( C^m(P) \) is \( m-1 \), the Euler form of \( C^m(P) \)
\[
\langle \cdot, \cdot \rangle : K(C^m(P)) \times K(C^m(P)) \rightarrow \mathbb{Z}
\]
determined by
\[
\langle M_\bullet, N_\bullet \rangle = \sum_{i=0}^{m-1} (-1)^i \dim_k \text{Ext}_C^i (M_\bullet, N_\bullet)
\]
(6.1)
is well defined.

Define the Hall algebra \( H_{tw}(C^m(P)) \) to be the same vector space as \( H(C^m(P)) \), but with the twisted multiplication
\[
[M_\bullet] \ast [N_\bullet] = q^{\langle M_\bullet, N_\bullet \rangle} [M_\bullet] \circ [N_\bullet].
\]
(6.2)
For any projective-injective object \( J_\bullet \in C^m(P) \), we have that
\[
\langle J_\bullet, M_\bullet \rangle = \dim_k \text{Hom}_{C^m(P)} (J_\bullet, M_\bullet), \quad \langle M_\bullet, J_\bullet \rangle = \dim_k \text{Hom}_{C^m(P)} (M_\bullet, J_\bullet),
\]
and thus obtain that
\[
[J_\bullet] \ast [M_\bullet] = [M_\bullet \oplus J_\bullet] = [M_\bullet] \ast [J_\bullet]
\]
(6.3)
for any \( M_\bullet \in C^m(P) \). Define the Hall algebra \( MH_m(A) \) to be the localization of \( H_{tw}(C^m(P)) \) with respect to elements \([J_\bullet] \) corresponding to projective-injective objects \( J_\bullet \) in \( C^m(P) \).

We should be reminded that \( MH_1(A) \) is isomorphic to the group algebra \( \mathbb{C}[K(A)] \) of the Grothendieck group \( K(A) \). Indeed, since \( C^1(P) = \mathcal{P} \), we get that \( H_{tw}(C^1(P)) \) is spanned
by elements \([P]\) with \(P \in \mathcal{P}\), which are subject to relations \([P] * [Q] = [P \oplus Q]\). Then we obtain that \(\mathcal{MH}_1(\mathcal{A})\) is spanned by elements \([P] * [Q]^{-1}\) with \(P, Q \in \mathcal{P}\). It is well known that the group algebra \(\mathbb{C}[K(\mathcal{A})]\) is generated by elements \(K_\alpha\) with \(\alpha \in K(\mathcal{A})\), which are subject to relations \(K_\alpha * K_{\beta} = K_{\alpha + \beta}\). Then the correspondence \(\kappa : \mathbb{C}[K(\mathcal{A})] \to \mathcal{MH}_1(\mathcal{A})\), \(K_\alpha \mapsto [P] * [Q]^{-1}\), by writing \(\alpha = \hat{P} - \hat{Q}\) for some \(P, Q \in \mathcal{P}\), is an isomorphism of algebras.

From now on, we assume that \(m \geq 2\). As before, for each \(0 \leq r < m - 1\) and \(\alpha \in K(\mathcal{A})\), by writing \(\alpha = \hat{P} - \hat{Q}\) for some \(P, Q \in \mathcal{P}\), we define
\[
J_{\alpha,r} = [J_P[r]] * [J_Q[r]]^{-1}.
\]
For any \(\alpha, \beta \in K(\mathcal{A})\), it is easy to see that
\[
J_{\alpha,r} * J_{\beta,r} = J_{\alpha+\beta,r}.
\]
Moreover, for any \(M_\bullet \in C^m(\mathcal{P})\) we have that
\[
J_{\alpha,r} * [M_\bullet] = [M_\bullet] * J_{\alpha,r}.
\]

Given an object \(M \in \mathcal{A}\), for each \(0 \leq r < m - 1\), we define
\[
X_{M,r} := J_{-Q_M,r} * [T_M[r]] \in \mathcal{MH}_m(\mathcal{A}).
\]
For each object \(P \in \mathcal{P}\), we define
\[
X_{P,m-1} := [S_P] \in \mathcal{MH}_m(\mathcal{A}).
\]

First of all, we give a basis in \(\mathcal{MH}_m(\mathcal{A})\) as follows:

**Proposition 6.1** \(\mathcal{MH}_m(\mathcal{A})\) has a basis consisting of elements
\[
J_{\alpha_0,0} * J_{\alpha_1,1} * \cdots * J_{\alpha_{m-2},m-2} * X_{M_0,0} * \cdots * X_{M_{m-2},m-2} * X_{P,m-1},
\]
where \(\alpha_r \in K(\mathcal{A})\) and \(M_r \in \mathcal{A}\) for \(0 \leq r < m - 1\), and \(P \in \mathcal{P}\).

**Proof** It is similar to (5.3) that for any objects \(M, N \in \mathcal{A}\),
\[
\text{Ext}_{C^m(\mathcal{P})}^1(T_M[r], T_N[l]) = 0, \quad 0 \leq r < l < m - 1.
\]
Thus, for \(M_0, M_1, \ldots, M_{m-2} \in \mathcal{A}\) and \(P \in \mathcal{P}\), noting that \(S_P\) is injective in \(C^m(\mathcal{P})\), we have that
\[
[T_{M_0}] * [T_{M_1}[1]] * \cdots * [T_{M_{m-2}}[m-2]] * [S_P] = q^a [T_{M_0} \oplus T_{M_1}[1] \oplus \cdots \oplus T_{M_{m-2}}[m-2] \oplus S_P]
\]
for some \(a \in \mathbb{Z}\). By Lemma 4.3, we can easily complete the proof. \(\square\)

By identifying the objects \(T_M[r], J_P[r]\) and \(S_P\) in \(C^m(\mathcal{P})\) with the objects \(C_M[r]\), \(K_P[r]\) and \(C_P[m-1]\) in \(C^b(\mathcal{P})\), respectively, we embed \(C^m(\mathcal{P})\) into \(C^b(\mathcal{P})\), i.e., we have an embedding functor \(\epsilon : \mathcal{C}^m(\mathcal{P}) \to \mathcal{C}^b(\mathcal{P})\). Clearly, \(\epsilon\) is a fully faithful exact functor. Moreover, for any objects \(M_\bullet, N_\bullet \in C^m(\mathcal{P})\), by Lemma 4.4, we have that \(\text{Ext}_{C^m(\mathcal{P})}^1(M_\bullet, N_\bullet) \cong \text{Ext}_{C^b(\mathcal{P})}^1(\epsilon(M_\bullet), \epsilon(N_\bullet))\) for all \(i \geq 1\). That is, \(\epsilon\) is an extremely faithful exact functor. By functorial properties of Hall algebras (cf., [20]), there exists an embedding of algebras \(\lambda : \mathcal{H}_m(C^m(\mathcal{P})) \to \mathcal{H}_m(C^b(\mathcal{P}))\). Thus, we have the following

**Proposition 6.2** There exists an embedding of algebras \(\lambda : \mathcal{MH}_m(\mathcal{A}) \to \mathcal{MH}(\mathcal{A})\).
Proof Let $\pi_m : \mathcal{H}_t(C^m(\mathcal{P})) \rightarrow \mathcal{M}H_m(A)$ and $\pi : \mathcal{H}_t(C^b(\mathcal{P})) \rightarrow \mathcal{M}H(A)$ be the natural homomorphisms of algebras. Since $\pi \circ \lambda'$ maps all elements $[J_P[r]]$ in $\mathcal{H}_t(C^m(\mathcal{P}))$ to the invertible elements $[K_P[r]]$ in $\mathcal{M}H(A)$, by universal properties of localizations, we obtain that there is a unique homomorphism of algebras

$$\lambda : \mathcal{M}H_m(A) \longrightarrow \mathcal{M}H(A)$$

such that $\pi \circ \lambda' = \lambda \circ \pi_m$. Explicitly, $\lambda(J_{\alpha,r}) = K_{\alpha,r}$, $\lambda(X_{M,r}) = E_{M,r}$ and $\lambda(X_{P,m-1}) = E_{P,m-1}$ for all $\alpha \in K(A)$, $M \in A$, $0 \leq r < m - 1$ and $P \in \mathcal{P}$. Thus, the injectivity of $\lambda$ follows from the fact that $\lambda$ sends the basis of $\mathcal{M}H_m(A)$ in Proposition 6.1 to a linearly independent set in $\mathcal{M}H(A)$.

Combining Proposition 6.2 with Proposition 5.2, we obtain the following

**Proposition 6.3** For each $0 \leq r < m - 1$, there exists an embedding of algebras

$$\varphi_r : \mathcal{H}_t(A) \longrightarrow \mathcal{M}H_m(A), \quad [M] \longrightarrow X_{M,r}.$$

Proof Since the image of $\psi_r$ is contained in the image of $\lambda$, taking $\varphi_r = \lambda^{-1} \circ \psi_r$ gives the desired homomorphism.

Combining Propositions 5.5, 6.1 and 6.2, we have the following

**Proposition 6.4** The Hall algebra $\mathcal{M}H_m(A)$ is generated by the elements in

$$\{J_{\alpha,r}, X_{M,r}, X_{P,m-1} | \alpha \in K(A), M \in A, P \in \mathcal{P}, 0 \leq r < m - 1\}$$

with the defining relations

\[
\begin{align*}
X_{M,r} \ast X_{N,r} &= \sum_{[L]} q^{(M,N)} \frac{[\text{Ext}^1_A(M,N)_L]}{[\text{Hom}_A(M,N)]} X_{L,r}; \quad (6.8) \\
X_{P,m-1} \ast X_{Q,m-1} &= X_{P \oplus Q,m-1}; \quad (6.9) \\
X_{M,r+1} \ast X_{N,r} &= \sum_{[X,Y]} q^{-1}(M,N)_{\gamma_{MN}} a_{MN} a_{XY} X_{Y,r} \ast X_{X,r+1} \ast J_{M-X,r}, \quad 0 \leq r < m - 2; \quad (6.10) \\
X_{P,m-1} \ast X_{M,m-2} &= \sum_{[R],[B]} q^{-(P,M)}_{(R,M)} a_{PM} a_{RM} a_{PB} X_{B,m-2} \ast X_{R,m-1} \ast J_{P-R,m-2}; \quad (6.11) \\
X_{M,r} \ast X_{N,l} &= q^{-(1)^{r-1}(M,N)}_{(1)^{r-1}(M,N)} X_{N,l} \ast X_{M,r}, \quad r - l > 1; \quad (6.12) \\
X_{P,m-1} \ast X_{M,r} &= q^{-(1)^{m-r-1}(P,M)}_{(1)^{m-r-1}(P,M)} X_{M,r} \ast X_{P,m-1}, \quad 0 \leq r < m - 2; \quad (6.13) \\
J_{\alpha,r} \ast X_{M,l} &= X_{M,l} \ast J_{\alpha,r}, \quad J_{\alpha,r} \ast X_{P,m-1} = X_{P,m-1} \ast J_{\alpha,r}; \quad (6.14) \\
J_{\alpha,r} \ast J_{\beta,r} &= J_{\alpha+\beta,r}, \quad J_{\alpha,r} \ast J_{\beta,l} = J_{\beta,l} \ast J_{\alpha,r}; \quad (6.15)
\end{align*}
\]

where $M, N \in A$, $P, Q \in \mathcal{P}$ and $\alpha, \beta \in K(A)$.

Let $T_{m-1}(A)$ be the subalgebra of $T_{\infty}(A)$ generated by elements $K_{\alpha,r}$ with $\alpha \in K(A)$ and $0 \leq r < m - 1$.

**Corollary 6.5** There is an embedding of algebras

$$\iota : \mathcal{M}H_m(A) \longrightarrow \mathcal{D}H(A) \otimes _C T_{m-1}(A).$$

Proof Taking $\iota = \tilde{\psi}^{-1} \circ \lambda$ gives the desired homomorphism.
Remark 6.6 $\mathcal{MH}_2(A)$ is used in [9] to realize quantum cluster algebra with principal coefficients as its subquotient.

7 Integration Maps Associated to Hall Algebras of $C^2(\mathcal{P})$

In this section, let $\mathcal{E}$ be a finitary exact category of global dimension at most one. By [13], Definition 3.1 also applies to the exact category $\mathcal{E}$. We also have the Riedtmann–Peng formula (3.1) in the Hall algebra $\mathcal{H}(\mathcal{E})$ of $\mathcal{E}$.

Let $T_q(\mathcal{E})$ be the $\mathbb{Z}[q,q^{-1}]$-algebra with a basis $\{X^\alpha | \alpha \in K(\mathcal{E})\}$ and the multiplication given by

$$X^\alpha * X^\beta = q^{-\langle \alpha, \beta \rangle} X^{\alpha + \beta},$$

where $\langle -, - \rangle$ is the Euler form on $K(\mathcal{E})$. According to [16], we recall the integration map on the Hall algebra $\mathcal{H}(\mathcal{E})$ as follows:

**Proposition 7.1** The integration map

$$\int: \mathcal{H}(\mathcal{E}) \longrightarrow T_q(\mathcal{E}), \quad [M] \mapsto \hat{X}^M$$

is a homomorphism of algebras.

**Proof** For reader’s convenience, we give the proof here. Given objects $M, N \in \mathcal{E}$,

$$\int [M] \circ [N] = \sum_{[L]} \frac{\text{Ext}^1_\mathcal{E}(M,N)_L}{\text{Hom}_\mathcal{E}(M,N)} X^L = \sum_{[L]} \frac{\text{Ext}^1_\mathcal{E}(M,N)_L}{\text{Hom}_\mathcal{E}(M,N)} \hat{X}^{M+\hat{N}} = q^{-\langle M,N \rangle} \hat{X}^M \hat{N} = \hat{X}^M \hat{X}^N = \int [M] \ast \int [N]. \quad \square$$

Since the global dimension of $C^2(\mathcal{P})$ is equal to one, we can apply the integration map in Proposition 7.1 to the Hall algebra of $C^2(\mathcal{P})$. Before doing this, we first give a characterization on the Grothendieck group $K(C^2(\mathcal{P}))$ of $C^2(\mathcal{P})$. By [1, Proposition 3.2], for each object $M_\bullet = (M_1 \xrightarrow{d_1} M_2) \in C^2(\mathcal{P})$, we have the following injective resolution

$$0 \longrightarrow M_\bullet \longrightarrow S_{M_1} \oplus J_{M_2} \longrightarrow S_{M_2} \longrightarrow 0.$$  \hspace{1cm} (7.1)

Let $P_1, P_2, \ldots, P_n$ be all indecomposable projective objects in $\mathcal{A}$ up to isomorphisms. Fixing a minimal injective resolution of $M_\bullet$,

$$0 \longrightarrow M_\bullet \longrightarrow \bigoplus_{i=1}^n a_i S_{P_i} \oplus \bigoplus_{i=1}^n c_i J_{P_i} \longrightarrow \bigoplus_{i=1}^n b_i S_{P_i} \longrightarrow 0,$$

we define the dimension vector $\text{dim}$ on objects in $C^2(\mathcal{P})$ by setting

$$\text{dim} M_\bullet = (b_1 - a_1, \ldots, b_n - a_n, c_1, \ldots, c_n).$$

By the dual of Lemma 4.1, we obtain that $\text{dim} M_\bullet$ does not depend on the minimality of injective resolutions. Thus, by the dual of Horseshoe Lemma (cf., [5, Theorem 12.8]), we get
the additivity of \( \text{dim} \). That is, for any short exact sequence

\[
0 \to M_\bullet \to L_\bullet \to N_\bullet \to 0
\]

in \( C^2(\mathcal{P}) \), we have that \( \text{dim} L_\bullet = \text{dim} M_\bullet + \text{dim} N_\bullet \).

**Lemma 7.2** The Grothendieck group \( K(C^2(\mathcal{P})) \) is a free abelian group having as a basis the set

\[
\{ \hat{J}_{P_i}, \hat{S}_{P_i} | 1 \leq i \leq n \}
\]

and there exists a unique group isomorphism \( f : K(C^2(\mathcal{P})) \to \mathbb{Z}^{2n} \) such that \( f(M_\bullet) = \text{dim} M_\bullet \) for each object \( M_\bullet \) in \( C^2(\mathcal{P}) \).

**Proof** For any object \( M_\bullet \in C^2(\mathcal{P}) \), taking an injective resolution of \( M_\bullet \) as (7.1), we obtain that in \( K(C^2(\mathcal{P})) \)

\[
\hat{M}_\bullet = \sum_{i=1}^{n} ((a_i - b_i)\hat{S}_{P_i} + c_i\hat{J}_{P_i}).
\]

This shows that \( \{ \hat{J}_{P_i}, \hat{S}_{P_i} | 1 \leq i \leq n \} \) generates the group \( K(C^2(\mathcal{P})) \).

For any objects \( M_\bullet, N_\bullet \in C^2(\mathcal{P}) \), it is clear that \( M_\bullet \cong N_\bullet \) implies \( \text{dim} M_\bullet = \text{dim} N_\bullet \), since their minimal injective resolutions are isomorphic. Thus, the additivity of \( \text{dim} \) implies the existence of a unique group homomorphism \( f : K(C^2(\mathcal{P})) \to \mathbb{Z}^{2n} \) such that \( f(M_\bullet) = \text{dim} M_\bullet \) for each object \( M_\bullet \) in \( C^2(\mathcal{P}) \). Since the image of the generating set \( \{ \hat{J}_{P_i}, \hat{S}_{P_i} | 1 \leq i \leq n \} \) under the homomorphism \( f \) is the canonical basis of the free group \( \mathbb{Z}^{2n} \), this set is \( \mathbb{Z} \)-linearly independent in \( K(C^2(\mathcal{P})) \). It follows that \( K(C^2(\mathcal{P})) \) is free and that \( f \) is an isomorphism. \( \Box \)

Let \( \Lambda \) be the bilinear form on \( \mathbb{Z}^{2n} \) obtained from the Euler form of \( C^2(\mathcal{P}) \) by the isomorphism \( f \) in Lemma 7.2. Define the *quantum torus* associated to the pair \( \{ \mathbb{Z}^{2n}, \Lambda \} \) to be the \( \mathbb{Z}[q, q^{-1}] \)-algebra \( T \) with a basis \( \{ X^e | e \in \mathbb{Z}^{2n} \} \) and the multiplication given by

\[
X^e \ast X^f = q^{-\Lambda(e,f)} X^{e+f}.
\]

**Corollary 7.3** The integration map

\[
\int : \mathcal{H}(C^2(\mathcal{P})) \to T, \quad [M_\bullet] \mapsto X^{\text{dim} M_\bullet}
\]

is a homomorphism of algebras.

**Remark 7.4 (Further directions)**

1. Since the global dimension of \( C^2(\mathcal{P}) \) is equal to one, we want to know whether the Hall algebra \( \mathcal{MH}_2(\mathcal{A}) \) has a bialgebra structure, or say, whether Green’s formula on the Hall numbers of a hereditary abelian category holds in \( C^2(\mathcal{P}) \).

2. Let \( Q \) be an acyclic quiver of \( n \) vertices, and take \( \mathcal{A} \) to be the category of finite dimensional \( kQ \)-modules. Whether can we use the integration map in Corollary 7.3 to relate the Hall algebra \( \mathcal{MH}_2(\mathcal{A}) \) with quantum cluster algebras (with principal coefficients). In other words, whether can we relate the integration map in Corollary 7.3 with the work in [9]. Recently, this question has been addressed in [11].

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