Linear functions preserving Green’s relations over fields

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1. A. Guterman, M. Johnson, M. Kambites. Linear isomorphisms preserving Green’s relations for matrices over anti-negative semifields, Linear Algebra Appl. 545 (2018) 1-14.

2. A. Guterman, M. Johnson, M. Kambites, A. Maksaev. Linear functions preserving Green’s relations over fields, Linear Algebra Appl. 611 (2021), 310-333.
Let $M$ be a monoid, $a, b \in M$. We say that:

**Definition**

(i) $a R b$ if $aM = bM$;
(ii) $a L b$ if $Ma = Mb$;
(iii) $a J b$ if $MaM = MbM$;
(iv) $a H b$ if $a R b$ and $a L b$;
(v) $a D b$ if $\exists c \in M: a R c$ and $c L b$.

**Simple properties:**

- $R, L, J, H, D$ are equivalence relations on $M$
- $H = R \cap L$
- $D = R \circ L = L \circ R$
- $H \subseteq R, L \subseteq D \subseteq J$
Let $K$ be a field. Consider the monoid $\mathcal{M} = (M_n(K), \cdot)$ of square matrices of order $n$.

**Theorem**

For $A, B \in M_n(K)$, it holds that:

(i) $A \mathcal{L} B \iff \text{Row}_K(A) = \text{Row}_K(B) \iff \ker A = \ker B$;

(ii) $A \mathcal{R} B \iff \text{Col}_K(A) = \text{Col}_K(B) \iff \text{Im} A = \text{Im} B$;

(iii) $A \mathcal{J} B \iff \text{rk} A = \text{rk} B$;

(iv) $A \mathcal{H} B \iff \text{Row}_K(A) = \text{Row}_K(B)$ and $\text{Col}_K(A) = \text{Col}_K(B)$;

(v) $A \mathcal{D} B \iff A \mathcal{J} B$, i.e., $\mathcal{D} = \mathcal{J}$. 
A linear map $T: M_n(K) \to M_n(K)$ preserves a relation $\mathcal{X}$ (on $M_n(K)$) if $A \mathcal{X} B \Rightarrow T(A) \mathcal{X} T(B)$ for all $A, B \in M_n(K)$.

Which of the linear maps preserve Green’s relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D}$?
Linear maps

Definition

A linear map \( T : M_n(K) \to M_n(K) \) preserves a relation \( \mathcal{X} \) (on \( M_n(K) \)) if \( A \mathcal{X} B \Rightarrow T(A) \mathcal{X} T(B) \) for all \( A, B \in M_n(K) \).

Which of the linear maps preserve Green’s relations \( \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D} \)?

- \( T \equiv 0 \)
- \( T(A) = PAQ \), where \( P, Q \in GL_n(K) \). Preserves \( \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D} \)
- \( T(A) = A^T \). Preserves \( \mathcal{H}, \mathcal{J}, \mathcal{D} \)
- \( T(A) = AX \), where \( X \in M_n(K) \). Preserves \( \mathcal{L} \)
  \((\text{Ker } A = \text{Ker } B \Rightarrow \text{Ker } AX = \text{Ker } BX)\)
- \( T(A) =XA \), where \( X \in M_n(K) \). Preserves \( \mathcal{R} \)
- Compositions of the above transformations
Let $S$ be a semifield: $\mathbb{R}_+, \mathbb{B}$ (boolean semiring), $\langle \mathbb{R} \cup \{-\infty\}, \max, + \rangle$ (tropical semifield), ... A semifield is either anti-negative (any element except 0 does not have an additive inverse) or a field.
Results for semifields

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\( \mathbb{R}_+, \mathbb{B} \) (boolean semiring), \( \langle \mathbb{R} \cup \{-\infty\}, \max, + \rangle \) (tropical semifield), \ldots
A semifield is either anti-negative (any element except 0 does not have an additive inverse) or a field.

Theorem (Guterman, Johnson, Kambites, 2018)

Let \( S \) be a semifield which is not a field and \( T: M_n(S) \to M_n(S) \) a bijective \( S \)-linear map. The following are equivalent:

(i) \( T \) preserves \( L \);
(ii) \( T \) preserves \( R \);
(iii) There exist invertible (monomial) matrices \( P, Q \in M_n(S) \) such that \( T(A) = PAQ \) for all \( A \in M_n(S) \).

Theorem (Guterman, Johnson, Kambites, 2018)

Let \( S \) be a semifield which is not a field and \( T: M_n(S) \to M_n(S) \) a bijective \( S \)-linear map. The following are equivalent:

(i) \( T \) preserves \( J \);
(ii) \( T \) preserves \( D \);
(iii) \( T \) preserves \( H \);
(iv) There exist invertible (monomial) matrices \( P, Q \in M_n(S) \) such that \( T(A) = PAQ \) for all \( A \in M_n(S) \) or \( T(A) = PATQ \) for all \( A \in M_n(S) \).
Relations $\mathcal{J} = \mathcal{D}$, and also $\mathcal{L}, \mathcal{R}$

**Theorem (Guterman, Johnson, Kambites, M., 2021)**

Let $K$ be a field and $T: M_n(K) \rightarrow M_n(K)$ a linear map. The following are equivalent:

(i) $T$ preserves $\mathcal{J}$;

(ii) Either $T \equiv 0$, or there exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PATQ$ for all $A \in M_n(K)$.

**Theorem (Guterman, Johnson, Kambites, M., 2021)**

Let $K$ be a field and $T: M_n(K) \rightarrow M_n(K)$ a bijective linear map. The following are equivalent:

(i) $T$ preserves $\mathcal{L}$;

(ii) $T$ preserves $\mathcal{R}$;

(iii) There exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$. 
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(ii) $T$ preserves $\mathcal{R}$;

(iii) There exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$.

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $n \geq 1$ and $K$ be a field in which every polynomial of degree exactly $n$ has a root. The linear maps preserving $\mathcal{L}$ on $M_n(K)$ (resp. $\mathcal{R}$) are precisely those of the form $A \mapsto PAX$ (resp. $A \mapsto XAP$), where $P \in GL_n(K)$ and $X \in M_n(K)$. 
Proof tools

Theorem (Marcus and Moyls 1959; Minc 1977)

Let $K$ be an algebraically closed field and $T : M_n(K) \to M_n(K)$ a linear map preserving the set of rank-1 matrices. Then $\exists P, Q \in GL_n(K)$:

- $T(A) = PAQ$ for all $A \in M_n(K)$ or
- $T(A) = PATQ$ for all $A \in M_n(K)$.

Theorem (Lautemann 1981)

Let $K$ be an arbitrary field and $T : M_n(K) \to M_n(K)$ a bijective linear map preserving the set of rank-1 matrices. Then $\exists P, Q \in GL_n(K)$:

- $T(A) = PAQ$ for all $A \in M_n(K)$ or
- $T(A) = PATQ$ for all $A \in M_n(K)$. 
Examples

**Statement**

For $K = \mathbb{Q}$ and every $n \geq 2$, there exist non-bijective linear $\mathcal{L}$-preservers that do not fit the conditions of the above theorems.

**Example (Botta, 1978)**

Let $f(x) \in K[x]$ be an irreducible polynomial of degree $n \geq 2$ (if exists). Let $C$ be any matrix in $M_n(K)$ whose minimal polynomial is $f(x)$. Then

$$\det(\lambda_1 I + \lambda_2 C + \lambda_3 C^2 + \ldots + \lambda_n C^{n-1}) = 0 \iff \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0.$$  

Hence the linear transformation $T$ defined as $T(A) = \sum_{i=1}^{n} a_{i1} C^{i-1}$, where $A = (a_{ij}) \in M_n(K)$, preserves $\mathcal{L}$ and $\mathcal{H}$, and is not of the form mentioned in the above theorems.
Examples

Statement

If $K = \mathbb{R}$ and $n$ is even, then there exist non-bijective linear $L$-preservers that do not fit the conditions of the above theorems.

Example (based on a construction of Petrović, 2009)

Let $n$ be even, $n = 2m$. $C_{2k-1} = E_{2k-1,1} + E_{2k,2}$ and $C_{2k} = E_{2k,1} - E_{2k-1,2}$, $k = 1, \ldots, \frac{n}{2}$.

$T : M_n(K) \to M_n(K)$ is given by

$$T(A) = \sum_{i=1}^{n} a_{i1} C_i = \begin{pmatrix}
    a_{1,1} & -a_{2,1} & 0 & \cdots & 0 \\
    a_{2,1} & a_{1,1} & 0 & \cdots & 0 \\
    a_{3,1} & -a_{4,1} & 0 & \cdots & 0 \\
    a_{4,1} & a_{3,1} & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{2m-1,1} & -a_{2m,1} & 0 & \cdots & 0 \\
    a_{2m,1} & a_{2m-1,1} & 0 & \cdots & 0
\end{pmatrix}$$
Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $K$ be a field and $T : M_n(K) \to M_n(K)$ a bijective linear map. The following are equivalent:

(i) $T$ preserves $\mathcal{H}$;
(ii) $T$ preserves $\mathcal{J}$;
(iii) There exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PATQ$ for all $A \in M_n(K)$.

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $n \geq 1$ and $K$ be a field in which every polynomial of degree exactly $n$ has a root. Then for a linear map $T$ preserving the $\mathcal{H}$-relation on $M_n(K)$, it holds that either $T \equiv 0$, or there exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PATQ$ for all $A \in M_n(K)$. 
Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $K$ be a field and $T: M_n(K) \to M_n(K)$ a linear map. Then $T$ preserves $\mathcal{H} \iff T = 0$ or $T$ preserves the set of non-singular matrices.

Theorem (de Seguins Pazzis, 2010)

Let $n \geq 2$, $K$ be any field, and $T: M_n(K) \to M_n(K)$ be a linear non-singularity preserver. Then:

(i) either $T$ is bijective and then there exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PATQ$ for all $A \in M_n(K)$;

(ii) or there exist an $n$-dimensional subspace $V \subset M_n(K)$ contained in $GL_n(K) \cup \{0_{n \times n}\}$, an isomorphism $\alpha: K^n \to V$, and a non-zero $x \in K^n$ such that

$$T(M) = \alpha(Mx^T) \quad \forall M \in M_n(K) \quad \text{or} \quad T(M) = \alpha(M^Tx^T) \quad \forall M \in M_n(K)$$
Examples over $\mathbb{R}$

There exist non-bijective maps preserving $\mathcal{H}$ (and also $\mathcal{L}$).

**Example**

Let $T : M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ be defined as follows:

$$T \begin{pmatrix} a & \ast \\ b & \ast \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

This is the matrix representation of complex numbers.
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Example

Let $T: M_4(\mathbb{R}) \rightarrow M_4(\mathbb{R})$ be defined as follows:

$$T \begin{pmatrix} a & \ast & \ast & \ast \\ b & \ast & \ast & \ast \\ c & \ast & \ast & \ast \\ d & \ast & \ast & \ast \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} = aP_1 + bP_2 + cP_3 + dP_4,$$

where $P_1 = I_4$,  

$$P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

This is the matrix representation of quaternions.
Theorem (Kervaire, Milnor–Bott, 1958)

Any finite dimensional division algebra over the real numbers has dimension 1, 2, 4 or 8.

Corollary

If $n \notin \{2, 4, 8\}$, then every linear $H$-preserver on $M_n(\mathbb{R})$ is either zero or bijective.
Thanks for attention!