SEMI-HYPERBOLIC PATCHES OF SOLUTIONS TO THE
TWO-DIMENSIONAL COMPRESSIBLE
MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. We construct semi-hyperbolic patches of solutions, in which one
family out of two families of wave characteristics start on sonic curves and end
on transonic shock waves, to the two-dimensional (2D) compressible magne-
tohydrodynamic (MHD) equations. This type of flow patches appear frequently
in transonic flow problems. In order to use the method of characteristic decom-
position to construct such a flow patch, we also derive a group of characteristic
decompositions for 2D self-similar MHD equations.

1. Introduction. The compressible magnetohydrodynamic (MHD) equations has
the form

\[
\begin{align*}
\rho_t + \text{div}(\rho \vec{U}) &= 0, \\
(\rho \vec{U})_t + \text{div}(\rho \vec{U} \otimes \vec{U} + pI) - \mu (\text{rot} \vec{H}) \times \vec{H} &= 0, \\
(\rho E + \frac{1}{2} \mu |\vec{H}|^2)_t + \text{div}(\rho \vec{U} E + \vec{U} p) - \text{div}(\mu (\vec{U} \times \vec{H}) \times \vec{H}) &= 0, \\
\vec{H}_t - \text{rot}(\vec{U} \times \vec{H}) &= 0, \\
\text{div} \vec{H} &= 0,
\end{align*}
\]

where \( \rho \) is the fluid density, \( p \) is the pressure, \( \vec{U} \) is the velocity of the fluid, \( E = e + \frac{1}{2} |\vec{U}|^2 \) is the specific total energy, \( e = e(\rho, S) \) is the specific internal energy, \( S \) is
the specific entropy, \( \vec{H} \) is the magnetic field, and the constant \( \mu \) is the magnetic
permeability; see [1, 19].

We assume that the magnetic field \( \vec{H} = (0, 0, H) \), the velocity \( \vec{U} = (u, v, 0) \),
and the variable \((u, v, p, \rho, H)\) is independent of the space variable \( z \). Then the MHD
equations (1) can be simplified as

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p + \frac{\mu}{2}H^2)_x + (\rho uv)_y &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p + \frac{\mu}{2}H^2)_y &= 0, \\
(\rho E + \frac{\mu}{2}H^2)_t + (\rho u E + \mu u H^2)_x + (\rho v E + \mu v H^2)_y &= 0, \\
H_t + (\tilde{H} u)_x + (H v)_y &= 0.
\end{align*}
\]

(2)

From the first and the fifth equations of (2), we have

\[
\left(\frac{H}{\rho}\right)_t + u \left(\frac{H}{\rho}\right)_x + v \left(\frac{H}{\rho}\right)_y = 0
\]

(3)

which implies that \(H/\rho\) is a constant along each stream line. The relation (3) is essentially an expression of the frozen-in law.

The Rankine-Hugoniot conditions for (2) are

\[
\begin{align*}
n_t[\rho] - n_x[\rho u] - n_y[\rho v] &= 0, \\
n_t[\rho u] - n_x[\rho u^2 + p + \frac{\mu}{2}H^2] - n_y[\rho uv] &= 0, \\
n_t[\rho v] - n_x[\rho uv] - n_y[\rho v^2 + p + \frac{\mu}{2}H^2] &= 0, \\
n_t[\rho E + \frac{\mu}{2}H^2] - n_x[\rho u E + \mu u H^2] - n_y[\rho v E + \mu v H^2] &= 0, \\
n_t[H] - n_x[H u] - n_y[H v] &= 0.
\end{align*}
\]

(4)

where \([\cdot]\) denotes the jump of the variable \((\cdot)\) across the discontinuity surface and \((n_t, n_x, n_y)\) is a normal to the discontinuity surface. From the first and the fifth equations of (4) we have

\[
[H/\rho] = 0
\]

(5)

across the shock waves.

According to (3) and (5), if \(H/\rho\) is a constant at the initial time then system (2) can be written as

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p + \kappa\rho^2)_x + (\rho uv)_y &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p + \kappa\rho^2)_y &= 0,
\end{align*}
\]

(6)

where \(\kappa\) is a constant. We consider a fluid with the equation of state

\[
p = \rho^\gamma \quad (1 < \gamma < 3).
\]

(7)

2D Riemann problems, which refer to Cauchy problems with special initial data that are constant along each ray from the origin, are interesting and important problems. 2D Riemann problems usually allows us to consider the so-called self-similar solutions. The self-similar solutions are the solutions which depend only on the self-similar variables \((\xi, \eta) = (\xi, \eta); \) see \[2, 32, 34\]. Then by self-similar transformation system (6) can be changed into the form

\[
\begin{align*}
(\rho U)_\xi + (\rho U)_\eta + 2\rho &= 0, \\
(\rho U^2 + p + \kappa\rho^2)_\xi + (\rho U V)_\eta + 3\rho U &= 0, \\
(\rho UV)_\xi + (\rho V^2 + p + \kappa\rho^2)_\eta + 3\rho V &= 0.
\end{align*}
\]

(8)

where \((U, V) = (u - \xi, v - \eta)\) is called the pseudo-flow velocity.
The eigenvalues of system (8) are determined by

\[(V - U\Lambda) \left[ (w^2 - U^2)\Lambda^2 + 2UV\Lambda + (w^2 - V^2) \right] = 0 \quad (9)\]

which yields

\[\Lambda = \Lambda_{\pm} = \frac{UV \pm \sqrt{w^2(U^2 + V^2 - w^2)}}{U^2 - w^2} \quad \text{and} \quad \Lambda = \Lambda_0 = \frac{V}{U}. \quad (10)\]

Here, \(w = \sqrt{c^2 + b^2}\) is the magneto-acoustic speed, \(b = \sqrt{2\kappa \rho}\) is the Alfvén speed, and \(c = \sqrt{\gamma \rho^{\gamma-1}}\) is the sound speed. Thus, system (8) is hyperbolic if and only if \(U^2 + V^2 > w^2\) (pseudosupersonic) and elliptic-hyperbolic if and only if \(U^2 + V^2 < w^2\) (pseudosubsonic).

The wave characteristics \(C_{\pm}\) are defined as the integral curves of \(C_{\pm}: \frac{d\eta}{d\xi} = \Lambda_{\pm}\). The direction of the wave characteristics is defined as the tangent direction that forms an acute angle \(\delta\) with the pseudoflow velocity \((U, V)\). By simple computation, we see that the \(C_+\) characteristic direction forms with the pseudoflow direction the angle \(\delta\) from \(C_+\) to \((U, V)\) in the clockwise direction, and the \(C_-\) characteristic direction forms with the pseudoflow direction the angle \(\delta\) from \(C_-\) to \((U, V)\) in the counterclockwise direction. By computation, we have

\[w^2 = q^2 \sin^2 \delta, \quad (11)\]

in which \(q^2 = U^2 + V^2\). The \(C_+\) (\(C_-\)) characteristic angle is defined as the angle between the \(C_+\) (\(C_-\)) characteristic direction and the positive \(\xi\)-axis.

We denote by \(\alpha\) and \(\beta\) the \(C_+\) and \(C_-\) characteristic angle, respectively, where \(0 < \alpha - \beta < \pi\). Let \(\sigma\) be the angle between the pseudoflow velocity and the positive \(\xi\)-axis. Then we have

\[\alpha = \sigma + \delta, \quad \beta = \sigma - \delta, \quad \sigma = \frac{\alpha + \beta}{2}, \quad \delta = \frac{\alpha - \beta}{2}. \quad (12)\]

Therefore, the relations between \((U, V, w)\) and \((\sigma, \delta, w)\) are

\[u - \xi = w \frac{\cos \sigma}{\sin \delta}, \quad v - \eta = w \frac{\sin \sigma}{\sin \delta}, \quad \tan \sigma = \frac{V}{U}. \quad (13)\]
In transonic flow there are two types of sonic curves, one is called the Keldysh type and the other is call the Tricomi type. In pseudosupersonic flow region adjacent to a Keldysh type sonic curve, all the wave characteristics vanish tangentially into the sonic curve; see Figure 1(I). Keldysh type sonic curves can be happened in solutions to some types of 2D Riemann problems; see for example [3, 7, 21, 33]. In pseudosupersonic flow region adjacent to a Tricomi type sonic curve, one family out of two families of wave characteristics start on the sonic curve; see Figure 1(II). Tricomi type sonic curves can be happened in the transonic flow over an airfoil and Guderley-Mach reflection; see [6, 8, 12, 13, 22, 25, 26, 27, 29]. Zheng et al. [14, 23, 34] call the pseudosupersonic flow patch adjacent to a Tricomi type sonic curve the semi-hyperbolic patch. Semi-hyperbolic patches also appear frequently in 2D steady transonic flow problems, see Courant and Friedrichs [6].

In this paper, we are concerned with semi-hyperbolic patches of solutions to the 2D self-similar compressible MHD equations (8). We consider the following problem.

**Problem.** Let ̃A ̃B be a C_+ characteristic curve with the direction of A points to B, and let ̃D ̃A be a C_- characteristic curve with the direction of D points to A. The data on ̃A ̃B and ̃D ̃A are given such that points B and D are pseudosonic. Construct a pseudo-supersonic solution of system (8) with the data on ̃A ̃B and ̃D ̃A in a region Ω bounded by ̃A ̃B, ̃D ̃A, and a sonic curve connecting D and B; see Figure 2.

This problem is a Goursat-type boundary value problem, since ̃A ̃B and ̃D ̃A are characteristics. The main result is stated as Theorem 3.8 where we obtain the existence of classical pseudosupersonic solution up to a sonic boundary ̃B ̃D to Problem 1. This generalizes the result for the 2D compressible Euler equations for polytropic gases of Li and Zheng [14] to the MHD equations. In order to use the method of characteristic decomposition introduced by Li et al. [16, 17] in investigating interactions of rarefaction waves of 2D compressible Euler equations, we derive a group of characteristic decompositions for system (8).

For semi-hyperbolic patches of solutions to some other type of hyperbolic conservation laws, the readers can see [9, 10, 11]. Song et al. [24, 28] also studied the
regularity of semi-hyperbolic patches near sonic lines. There are also some other ways to construct such a flow patch. Recently, Zhang and Zheng [30, 31] construct semi-hyperbolic flow patches of solutions to 2D steady Euler equations and 2D pressure gradient equations by solving Cauchy problems with given data on the sonic curve.

The rest of paper is organized as follows. Section 2 is concerned with characteristic equations for (8). We derive several characteristic decompositions for the magneto-acoustic speed \( w \) and wave characteristic angles \( \alpha \) and \( \beta \). Section 3 is devoted to solve Problem 1.

2. Primary systems and characteristic decompositions.

2.1. Irrotational flow. For smooth flow, system (8) can be written as

\[
\begin{cases}
(pU)_\xi + (pV)_\eta + 2\rho = 0, \\
UU_\xi + UV_\eta + \left( \frac{p}{\rho} + \kappa \rho^2 \right)_\xi + U = 0, \\
UV_\xi + VV_\eta + \left( \frac{p}{\rho} + \kappa \rho^2 \right)_\eta + V = 0.
\end{cases}
\] (14)

If flow is also irrotational, i.e., \( u_\eta = v_\xi \), then there exists a potential function \( \varphi \) such that \( \varphi_\xi = U \) and \( \varphi_\eta = V \). Thus, by the last two equations of (14) we have the pseudo-Bernoulli law

\[
\frac{1}{2}(U^2 + V^2) + \frac{c^2}{\gamma - 1} + b^2 + \varphi = \text{Const.} \tag{15}
\]

Hence, for irrotational flow (14) can be written as

\[
\begin{cases}
(u^2 - U^2) u_\xi - UV(u_\eta + v_\xi) + (w^2 - V^2) v_\eta = 0, \\
v_\xi - u_\eta = 0
\end{cases}
\] (16)

supplemented by the pseudo-Bernoulli’s law (15). Multiplying (16) by \( \Lambda_\pm \), we get

\[
\begin{cases}
\partial_+ u + \Lambda_- \partial_+ v = 0, \\
\partial_- u + \Lambda_+ \partial_- v = 0.
\end{cases}
\] (17)

where \( \partial_\pm = \partial_\xi + \Lambda_\pm \partial_\eta \).

2.2. Characteristic equations in term of \( \alpha, \beta \) and \( w \). From (13) we have

\[
\bar{\partial}_\pm u = \cos(\sigma \pm \delta) + \frac{\cos \sigma}{\sin \delta} \bar{\partial}_\pm w - \frac{w \cos \beta \bar{\partial}_\pm \alpha - w \cos \alpha \bar{\partial}_\pm \beta}{2 \sin^2 \delta},
\] (18)

\[
\bar{\partial}_\pm v = \sin(\sigma \pm \delta) + \frac{\sin \sigma}{\sin \delta} \bar{\partial}_\pm w - \frac{w \sin \beta \bar{\partial}_\pm \alpha - w \sin \alpha \bar{\partial}_\pm \beta}{2 \sin^2 \delta},
\] (19)

where

\[
\bar{\partial}_+ = \cos \alpha \partial_\xi + \sin \alpha \partial_\eta, \quad \bar{\partial}_- = \cos \beta \partial_\xi + \sin \beta \partial_\eta.
\] (20)

Inserting (18) and (19) into (17), we have

\[
\bar{\partial}_+ w = -\frac{\cos(2\delta)}{\cot \delta} + \frac{w}{\sin(2\delta)} [\bar{\partial}_+ \alpha - \cos(2\delta) \bar{\partial}_+ \beta],
\] (21)

\[
\bar{\partial}_- w = -\frac{\cos(2\delta)}{\cot \delta} + \frac{w}{\sin(2\delta)} [\cos(2\delta) \bar{\partial}_- \alpha - \bar{\partial}_- \beta].
\] (22)
Differentiating the pseudo-Bernoulli law (15) and using (18) and (19), we get
\[ \bar{\partial}_\pm w = \frac{(\gamma - 1)c^2 + b^2}{2w^2\sin^2\delta + (\gamma - 1)c^2 + b^2} \left[ -\sin(2\delta) + \frac{w}{\tan \delta} (\bar{\partial}_+ \alpha - \bar{\partial}_+ \beta) \right]. \] (23)
Inserting this into (21) and (22), we have
\[ w\bar{\partial}_+ \alpha = \Psi \cos^2 \delta [2\sin^2 \delta + w\bar{\partial}_+ \beta] \] (24)
and
\[ w\bar{\partial}_- \beta = \Psi \cos^2 \delta [-2\sin^2 \delta + w\bar{\partial}_- \alpha], \] (25)
respectively, where
\[ \Psi = M - \tan^2 \delta, \quad M = \frac{(3 - \gamma)c^2 + b^2}{(\gamma + 1)c^2 + 3b^2}. \] (26)
By computation we have
\[ \frac{dM}{d\rho} = -\frac{4b^2c^2(\gamma - 2)^2}{\rho[(\gamma + 1)c^2 + 3b^2]^2} \leq 0, \]
and consequently
\[ M \leq 1. \] (27)
Inserting (24) into (23), we have
\[ w\bar{\partial}_+ \alpha = -\frac{\sin(2\delta)}{2\chi^2} \Psi \bar{\partial}_+ w \] (28)
and
\[ w\bar{\partial}_+ \beta = -\frac{\tan \delta}{\chi^2} \bar{\partial}_+ w - 2\sin^2 \delta, \] (29)
where
\[ \chi^2 = \frac{(\gamma - 1)c^2 + b^2}{(\gamma + 1)c^2 + 3b^2}. \] (30)
Similarly, inserting (25) into (23), we have
\[ w\bar{\partial}_- \alpha = \frac{\tan \delta}{\chi^2} \bar{\partial}_- w + 2\sin^2 \delta \] (31)
and
\[ w\bar{\partial}_- \beta = \frac{\sin(2\delta)}{2\chi^2} \Psi \bar{\partial}_- w. \] (32)
By (28)–(32), we have
\[ w\bar{\partial}_+ \sigma = -\frac{(\gamma \rho^{\gamma - 2} + 2\kappa) \sin(2\delta)}{\gamma(\gamma - 1)\rho^{\gamma - 2} + 2\kappa} \bar{\partial}_+ w - \sin^2 \delta. \] (33)
\[ w\bar{\partial}_- \sigma = \frac{(\gamma \rho^{\gamma - 2} + 2\kappa) \sin(2\delta)}{\gamma(\gamma - 1)\rho^{\gamma - 2} + 2\kappa} \bar{\partial}_- w + \sin^2 \delta. \] (34)
By computation, we also have
\[ \bar{\partial}_+ u = \frac{2(\gamma \rho^{\gamma - 2} + 2\kappa) \sin \beta}{\gamma(\gamma - 1)\rho^{\gamma - 2} + 2\kappa} \bar{\partial}_+ w, \quad \bar{\partial}_- u = -\frac{2(\gamma \rho^{\gamma - 2} + 2\kappa) \sin \alpha}{\gamma(\gamma - 1)\rho^{\gamma - 2} + 2\kappa} \bar{\partial}_- w. \] (35)
and
\[ \bar{\partial}_+ v = -\frac{2(\gamma \rho^{\gamma - 2} + 2\kappa) \cos \beta}{\gamma(\gamma - 1)\rho^{\gamma - 2} + 2\kappa} \bar{\partial}_+ w, \quad \bar{\partial}_- v = \frac{2(\gamma \rho^{\gamma - 2} + 2\kappa) \cos \alpha}{\gamma(\gamma - 1)\rho^{\gamma - 2} + 2\kappa} \bar{\partial}_- w. \] (36)
2.3. Characteristic decompositions. In this part, we shall derive several characteristic decompositions for the variables $w$, $\alpha$, and $\beta$.

**Lemma 2.1.** For the variable $w$, we have the following characteristic decompositions

\[
\begin{align*}
    w\tilde{\partial}_+\tilde{\partial}_-w &= \tilde{\partial}_-w \left\{ \sin(2\delta) + \frac{\tilde{\partial}_-w}{2\chi^2\cos^2\delta} + \left( 1 + \frac{\Psi\cos(2\delta)}{2\chi^2} + \frac{2(\gamma - 2)b^2\varepsilon^2}{((\gamma - 1)c^2 + b^2)^2} \right) \tilde{\partial}_+w \right\}, \\
    w\tilde{\partial}_-\tilde{\partial}_+w &= \tilde{\partial}_+w \left\{ \sin(2\delta) + \frac{\tilde{\partial}_+w}{2\chi^2\cos^2\delta} + \left( 1 + \frac{\Psi\cos(2\delta)}{2\chi^2} + \frac{2(\gamma - 2)b^2\varepsilon^2}{((\gamma - 1)c^2 + b^2)^2} \right) \tilde{\partial}_-w \right\}.
\end{align*}
\]  

(37) 

(38)

**Proof.** This lemma was proved in [4] by a method similar to that of [5, 15]; we omit the proof here.

**Corollary 1.** The second order equations of $w$ in homogeneous form:

\[
\begin{align*}
    w\tilde{\partial}_+ \left( \frac{\tilde{\partial}_-w}{\sin^2\delta} \right) &= \tilde{\partial}_+w + \tilde{\partial}_-w \left( \frac{\tilde{\partial}_-w}{\sin^2\delta} \right) - \frac{2(\gamma - 1)c^2 + b^2 + 8w^2\sin^2\delta}{((\gamma - 1)c^2 + b^2)^2} \tilde{\partial}_+w \left( \frac{\tilde{\partial}_-w}{\sin^2\delta} \right), \\
    w\tilde{\partial}_- \left( \frac{\tilde{\partial}_+w}{\sin^2\delta} \right) &= \tilde{\partial}_-w + \tilde{\partial}_+w \left( \frac{\tilde{\partial}_+w}{\sin^2\delta} \right) - \frac{2(\gamma - 1)c^2 + b^2 + 8w^2\sin^2\delta}{((\gamma - 1)c^2 + b^2)^2} \tilde{\partial}_-w \left( \frac{\tilde{\partial}_+w}{\sin^2\delta} \right).
\end{align*}
\]  

(39) 

(40)

**Proof.** This corollary can be obtained by performing a direct calculation and simplification for (37) and (38).

**Lemma 2.2.** For the characteristic angles $\alpha$ and $\beta$, we have the following characteristic decompositions

\[
\begin{align*}
    w\tilde{\partial}_+\tilde{\partial}_-\alpha + X_1\tilde{\partial}_-\alpha &= \left\{ \tan\delta \left( 1 - 4\sin^2\delta \right) + \frac{4\tan\delta(\gamma - 2)b^2c^2}{\Psi((\gamma + 1)c^2 + 3b^2)^2} \right\} \tilde{\partial}_+\alpha, \\
    w\tilde{\partial}_-\tilde{\partial}_+\beta + X_2\tilde{\partial}_+\beta &= \left\{ \tan\delta \left( 1 - 4\sin^2\delta \right) + \frac{4\tan\delta(\gamma - 2)b^2c^2}{\Psi((\gamma + 1)c^2 + 3b^2)^2} \right\} \tilde{\partial}_-\beta,
\end{align*}
\]  

(41) 

(42)

where

\[
\begin{align*}
    X_1 &= \sin^2\delta \left[ 2\tan\delta - \Psi\sin(2\delta) \right] + \frac{2\sin(2\delta)(\gamma - 2)b^2c^2}{((\gamma + 1)c^2 + 3b^2)^2} - \frac{w}{\sin(2\delta)}\tilde{\partial}_-\alpha \\
    &+ \left\{ \frac{1}{\sin(2\delta)} - \frac{\Psi\sin(2\delta)}{2} + \frac{2(\gamma - 2)b^2c^2}{\tan\delta((\gamma + 1)c^2 + 3b^2)^2} \right\} w\tilde{\partial}_+\beta
\end{align*}
\]  

(43)

and

\[
\begin{align*}
    X_2 &= \sin^2\delta \left[ 2\tan\delta - \Psi\sin(2\delta) \right] + \frac{2\sin(2\delta)(\gamma - 2)b^2c^2}{((\gamma + 1)c^2 + 3b^2)^2} + \frac{w}{\sin(2\delta)}\tilde{\partial}_+\beta \\
    &- \left\{ \frac{1}{\sin(2\delta)} - \frac{\Psi\sin(2\delta)}{2} + \frac{2(\gamma - 2)b^2c^2}{\tan\delta((\gamma + 1)c^2 + 3b^2)^2} \right\} w\tilde{\partial}_-\alpha.
\end{align*}
\]  

(44)
Proof. In view of (28), (31) and (32), we get

\[ \partial_- w = \frac{\chi^2 w}{\tan \delta} \partial_- \alpha - \chi^2 \sin(2\delta), \quad \partial_- w = \frac{2\chi^2 w}{\Psi \sin(2\delta)} \partial_- \beta, \quad \partial_+ w = -\frac{2\chi^2 w}{\Psi \sin(2\delta)} \partial_+ \alpha. \]

(45)

Combining these with (37) we obtain

\[
w \partial_+ \left[ \frac{\chi^2 w}{\tan \delta} \partial_- \alpha - \chi^2 \sin(2\delta) \right] = \left[ \frac{\chi^2 w}{\tan \delta} \partial_- \alpha - \chi^2 \sin(2\delta) \right] \left\{ \sin(2\delta) + \frac{w \partial_- \beta}{\Psi \cos^2 \delta \sin(2\delta)} \right. \\
- \left[ 1 + \frac{\Psi \cos(2\delta)}{2 \chi^2} + \frac{2(\gamma - 2)b^2 c^2}{(\gamma - 1)c^2 + b^2} \right] \frac{2\chi^2 w}{\Psi \sin(2\delta)} \partial_+ \alpha \right\}.
\]

(46)

Due to

\[
\partial_+ \left( \frac{\chi^2 w}{\tan \delta} \partial_- \alpha - \chi^2 \sin(2\delta) \right) = \frac{w \sin(2\delta)}{\tan \delta} \frac{d \chi^2}{d \rho} \partial_+ \rho - \frac{\chi^2}{2 \sin^2 \delta} (w \partial_+ \alpha - w \partial_+ \beta) + \frac{\chi^2}{\tan \delta} \partial_+ w
\]

\[
- \frac{2(\gamma - 2)b^2 c^2}{(\gamma + 1)c^2 + 3b^2} \frac{2w^2}{(\gamma - 1)c^2 + b^2} \frac{2\chi^2 w}{\Psi \tan \delta \sin(2\delta)} \partial_+ \alpha
\]

\[
+ \frac{\chi^2 w}{\Psi \sin(2\delta) \sin^2 \delta} \left[ \tan \delta - \frac{\Psi \sin(2\delta)}{2} \right] \partial_+ \alpha - \frac{2\chi^2 w \partial_+ \alpha}{\Psi \tan \delta \sin(2\delta)} - \chi^2
\]

and

\[
w \partial_+ (\chi^2 \sin(2\delta)) = w \sin(2\delta) \frac{d \chi^2}{d \rho} \partial_+ \rho + \chi^2 \cos(2\delta) (w \partial_+ \alpha - w \partial_+ \beta)
\]

\[
= \frac{2(\gamma - 2)b^2 c^2}{(\gamma + 1)c^2 + 3b^2} \frac{2w^2}{(\gamma - 1)c^2 + b^2} \frac{2\chi^2 w}{\Psi \tan \delta \sin(2\delta)} \partial_+ \alpha
\]

\[
- \frac{2\chi^2 \cos(2\delta) w}{\Psi \sin(2\delta)} \left[ \tan \delta - \frac{\Psi \sin(2\delta)}{2} \right] \partial_+ \alpha + \chi^2 \sin^2(2\delta),
\]

we have

\[
w \partial_+ \left[ \frac{\chi^2 w}{\tan \delta} \partial_- \alpha - \chi^2 \sin(2\delta) \right] = \frac{\chi^2 w}{\tan \delta} w \partial_+ \partial_- \alpha + w \partial_- \partial_+ \alpha \partial_+ \left( \frac{\chi^2 w}{\tan \delta} \right) - w \partial_+ (\chi^2 \sin(2\delta))
\]

\[
= \frac{\chi^2 w^2 \partial_+ \partial_- \alpha \tan \delta}{\tan \delta} - \frac{2(\gamma - 2)b^2 c^2}{(\gamma + 1)c^2 + 3b^2} \frac{2w^2}{(\gamma - 1)c^2 + b^2} \frac{2\chi^2 w^2 \partial_+ \alpha \partial_- \alpha}{\Psi \tan \delta \sin(2\delta)} + \frac{\chi^2 w^2 \partial_+ \alpha \partial_- \alpha}{\Psi \sin^2 \delta \sin(2\delta)}
\] \times \left[ \tan \delta - \frac{\Psi \sin(2\delta)}{2} \right] - \frac{2\chi^2 w^2 \partial_+ \alpha \partial_- \alpha}{\Psi \tan \delta \sin(2\delta)} - \chi^2 w \partial_- \alpha + \frac{2(\gamma - 2)b^2 c^2}{(\gamma + 1)c^2 + 3b^2} \frac{2\chi^2 w}{\Psi}
\] \times \left[ \frac{2w^2 \partial_+ \alpha}{(\gamma - 1)c^2 + b^2} + \frac{\chi^2 w}{\Psi \sin(2\delta)} \left[ \tan \delta - \frac{\Psi \sin(2\delta)}{2} \right] \partial_+ \alpha - \chi^2 \sin^2(2\delta) + 2\chi^2 \sin^2 \delta.\right.\]
By a direct calculation, we get similarly. Combining the above relations, we get (41). The decomposition (42) can be proved

Moreover,

3. Semi-hyperbolic flow patch. In this part we shall construct a semi-hyperbolic flow patch to the two-dimensional self-similar MHD equations (8).
3.1. Goursat problem. We first give a detailed description of Problem 1. Assume that \( AB \) can be described by \( \xi = \xi_+(s) \), \( \eta = \eta_+(s) \) \((0 \leq s \leq s+)\), and \( DA \) can be described by \( \xi = \xi_-(s) \), \( \eta = \eta_-(s) \) \((s_- \leq s \leq 0)\).

Let \((\alpha_+, \beta_+, w_+)(s)\) be defined on \([0, s_+]\) such that the following properties are satisfied:

- \((a1)\): \((\alpha_+ - \beta_+)(s_+) = \pi;\)
- \((a2)\): \(\delta_0 := \frac{\sqrt{\pi} - \beta_+}{2} > \frac{\pi}{4};\)
- \((a3)\): \(\xi_+ \sin \alpha_+ - \eta_+ \cos \alpha_+ = 0, \xi_+ \cos \alpha_+ + \eta_+ \sin \alpha_+ > 0, \) and \(w_+ > 0\) on \([0, s_+];\)
- \((a4)\): \(w_+ \alpha_+' = -\frac{\sin \delta_0}{\xi_+(\rho_+)} \Psi(\delta_+, \rho_+) w'_+ \) on \([0, s_+];\)
- \((a5)\): \(w_+ \beta_+'' = \frac{\tan \delta_0}{\xi_+(\rho_+)} w_+'' - 2 \sqrt{(\xi'_+)^2 + (\eta'_+)^2} \sin \delta_+ \) on \([0, s_+],\)

where \(\delta_+ = \frac{\alpha_+ - \beta_+}{2}\) and \(\rho_+ = \rho(w_+).\)

Let \((\alpha_-, \beta_-, w_-)(s)\) be defined on \([s_-, 0]\) such that the following properties are satisfied:

- \((b1)\): \((\alpha_- - \beta_-)(s_-) = \pi;\)
- \((b2)\): \((\alpha_+, \beta_+, w_+)(0) = (\alpha_-, \beta_-, w_-)(0);\)
- \((b3)\): \(\xi_- \sin \beta_- - \eta_- \cos \beta_- = 0, \xi_- \cos \beta_- + \eta_- \sin \beta_- > 0, \) and \(w_- < 0\) on \([s_-, 0];\)
- \((b4)\): \(w_- \alpha_-'' = \frac{\sin \delta_-}{\xi_-(\rho_-)} \Psi(\delta_-, \rho_-) w_-' \) on \([s_-, 0];\)
- \((b5)\): \(w_- \beta_-'' = \frac{\tan \delta_-}{\xi_-(\rho_-)} w_-'' + 2 \sqrt{(\xi'_-)^2 + (\eta'_-)^2} \sin \delta_- \) on \([s_-, 0],\)

where \(\delta_- = \frac{\alpha_- - \beta_-}{2}\) and \(\rho_- = \rho(w_-).\)

Consider system (8) with the boundary data

\[
\begin{cases}
(\alpha, \beta, w)(\xi_+(s), \eta_+(s)) = (\alpha_+, \beta_+, w_+)(s) & \text{on } AB; \\
(\alpha, \beta, w)(\xi_-(s), \eta_-(s)) = (\alpha_-, \beta_-, w_-)(s) & \text{on } DA.
\end{cases}
\] (47)

From (a1)–(a5) and (b1)–(b5) we know that problem (8), (47) is a Goursat problem with \(AB\) and \(DA\) as the characteristic boundaries. Moreover, from (a3) we know that the direction of the characteristic \(AB\) is from \(A\) to \(B\), from (b3) we know that the direction of the characteristic \(DA\) is from \(D\) to \(A\), and from (a1) and (b1) we know that \(B\) and \(D\) are sonic points; see Figure 2.

3.2. Local solution.

**Lemma 3.1.** (Local solution) When \(\varepsilon > 0\) is sufficiently small the Goursat problem (8), (47) admits a unique \(C^1\) solution on a domain \(\Omega_\varepsilon\) closed by \(AB, DA\), and a level curve \(\delta(\xi, \eta) = \delta_0 + \varepsilon\). Moreover, this solution satisfies

\[
\delta > \frac{\pi}{4}, \quad \delta_+ w_+ > 0, \quad \delta_- w_- < 0, \quad \delta_+ \alpha_+ > 0, \quad \delta_- \alpha_- < 0, \quad \delta_+ \beta_+ < 0, \quad \text{and } \delta_- \beta_- > 0 \text{ in } \Omega_\varepsilon.
\] (48)

**Proof.** From (a4)–(a5) and (b4)–(b5) we know that the compatibility conditions are satisfied at the point \(A\). Therefore, the Goursat problem admits a unique local \(C^1\) solution by the method of characteristics (cf. Chapter 2 of Li and Yu [20]).

From (a3) and (b3) we have

\[
\delta_+ w_+|_{AB} = \frac{w'_+}{\sqrt{(\xi'_+)^2 + (\eta'_+)^2}} > 0 \quad \text{and} \quad \delta_- w_-|_{DA} = \frac{w'_-}{\sqrt{(\xi'_-)^2 + (\eta'_-)^2}} < 0,
\]
respectively. Thus, using the characteristic decompositions (37) and (38) we have that the solution satisfies

\[ \overline{\partial}_+ w > 0 \quad \text{and} \quad \overline{\partial}_- w < 0 \quad \text{in } \Omega_\varepsilon. \]  

(49)

From (a2), (b3), (b4), and (b5) we have

\[ \delta |\tilde{D}A| > \frac{\pi}{4}. \]  

(50)

Let \( F \) be an arbitrary point in \( \Omega_\varepsilon \). If \( \delta = \frac{\pi}{4} \) at \( F \) then by (24), (25), and (49) we have \( \overline{\partial}_+ \delta(F) > 0 \) which leads to a contradiction, since \( \delta |\tilde{D}A| > \frac{\pi}{2} \). Then we have

\[ \delta > \frac{\pi}{4} \quad \text{in } \Omega_\varepsilon. \]  

(51)

From (27), (28), (29), (32), (49), and (51) we immediately have

\[ \overline{\partial}_+ \alpha > 0, \quad \overline{\partial}_- \beta < 0, \quad \text{and} \quad \overline{\partial}_- \beta > 0 \quad \text{in } \Omega_\varepsilon. \]  

(52)

From (b3) we have

\[ \overline{\partial}_- \alpha |\tilde{D}A| = \alpha' - \sqrt{(\xi')^2 + (\eta')^2} \leq 0. \]  

(53)

Then by (41), (51), and (52) we have

\[ \overline{\partial}_+ \alpha \leq 0 \quad \text{in } \Omega_\varepsilon. \]  

(54)

We then complete the proof of the lemma.

\[ \square \]

3.3. Global solution.

**Lemma 3.2.** Suppose that the Goursat problem admits a \( C^1 \) solution on \( \Omega_\varepsilon \) where \( 0 < \varepsilon < \frac{\pi}{2} - \delta_0 \). Then we have

\[ \alpha(\beta) \in \left\{ (\alpha, \beta) \mid \alpha > \alpha(A), \beta < \beta(A), \frac{\alpha - \beta}{2} < \delta_0 + \varepsilon \right\} \quad \text{in } \Omega_\varepsilon. \]  

(55)

**Proof.** This lemma can be obtained by (48).

**Lemma 3.3.** Suppose that the Goursat problem admits a \( C^1 \) solution on \( \Omega_\varepsilon \) where \( 0 < \varepsilon < \frac{\pi}{2} - \delta_0 \). Then we have

\[ w_+(0) \leq w < w_-(s_-)e^{\pi} \quad \text{in } \Omega_\varepsilon. \]  

(56)

**Proof.** From (28) and (29), we have

\[ w\overline{\partial}_+ (\alpha - \beta) = 2\sin^2 \delta + \frac{1}{\chi^2(\rho)} (\tan^2 \delta \cos \delta \sin \delta \cos \delta) \overline{\partial}_+ w. \]  

(57)

Then by integrating (58) along \( C_+ \) characteristics we can get Lemma 3.3.

**Lemma 3.4** \( (C^0 \text{ norm estimate}) \). Assume that the Goursat problem admits a \( C^1 \) solution on \( \Omega_\varepsilon \) where \( 0 < \varepsilon < \frac{\pi}{2} - \delta_0 \). Then, there exists a \( \mathcal{H}_0 \) independent of \( \varepsilon \) such that

\[ \|(u, v, w)\|_{0; \Omega_\varepsilon} < \mathcal{H}_0. \]  

(59)
Proof. From (56) and (57) we have
\[ \bar{\partial}_+ \delta > \frac{\sin^2 \delta_0}{w_+(0)} \quad \text{in} \quad \Omega_\varepsilon, \]
which implies that the domain \( \Omega_\varepsilon \) is uniformly bounded with respect to \( \varepsilon \). Therefore, by (13) and Lemmas 3.2 and 3.3 we can get the uniform \( C^0 \) norm estimate of the variable \((u,v)\).

\[ \text{Lemma 3.5. Assume that the Goursat problem admits a } C^1 \text{ solution on } \Omega_\varepsilon \text{ where } 0 < \varepsilon < \frac{\pi}{2} - \delta_0. \text{ Then, we have} \]
\[ \left( \frac{\bar{\partial}_+ w}{\sin^2 \delta}, \frac{\bar{\partial}_- w}{\sin^2 \delta} \right) \in (0, \varrho_\varepsilon) \times [-\varrho_\varepsilon, 0) \quad \text{in} \quad \Omega_\varepsilon, \]
where
\[ \varrho_\varepsilon := \max \left\{ \sup_{AB} \frac{\bar{\partial}_+ w}{\sin^2 \delta}, \sup_{D_x A} \frac{-\bar{\partial}_- w}{\sin^2 \delta} \right\} \]
and \( B_\varepsilon \) and \( D_\varepsilon \) are points on \( \overline{AB} \) and \( \overline{DA} \), respectively, such that \( \delta(B_\varepsilon) = \delta(D_\varepsilon) = \delta_0 + \varepsilon \).

Proof. By Lemma 3.1, we have that \( \bar{\partial}_+ w > 0 \) and \( \bar{\partial}_- w < 0 \) in \( \Omega_\varepsilon \).

In order to prove this lemma, we only need to prove that for any \( \mu > 0 \),
\[ \left( \frac{\bar{\partial}_+ w}{\sin^2 \delta}, \frac{\bar{\partial}_- w}{\sin^2 \delta} \right) \in (0, \varrho_\varepsilon + \mu) \times (-\varrho_\varepsilon - \mu, 0) \quad \text{in} \quad \Omega_\varepsilon. \]
Suppose that this proposition is not valid, then there must exists a “first” point \( F \) in \( \Omega_\varepsilon \), such that
\[ \left( \frac{\bar{\partial}_+ w}{\sin^2 \delta}(F), \frac{\bar{\partial}_- w}{\sin^2 \delta}(F) \right) \in \bigcup_{i=1}^2 \gamma_i \text{ and } \left( \frac{\bar{\partial}_+ w(\xi,\eta)}{\sin^2 \delta(\xi,\eta)}, \frac{\bar{\partial}_- w(\xi,\eta)}{\sin^2 \delta(\xi,\eta)} \right) \in (0, \varrho_\varepsilon + \mu) \times (-\varrho_\varepsilon - \mu, 0) \quad \text{for any } (\xi, \eta) \in \Omega_\varepsilon \setminus \{G\}, \]
where \( \Omega_\varepsilon \) is the same as that in Lemma 3.1, \( \gamma_1 = (0, \varrho_\varepsilon + \mu) \times \{-\varrho_\varepsilon - \mu, 0\} \), and \( \gamma_2 = \{\varrho_\varepsilon + \mu\} \times (-\varrho_\varepsilon - \mu, 0) \). If \( \left( \frac{\bar{\partial}_+ c(\xi,\eta)}{\sin^2 \delta(\xi,\eta)}, \frac{\bar{\partial}_- w(\xi,\eta)}{\sin^2 \delta(\xi,\eta)} \right) \in \gamma_1 \), then by the first equation of (39) we have that
\[ \bar{\partial}_+(\frac{\bar{\partial}_- w}{\sin^2 \delta}) > 0 \quad \text{at the point } F, \quad \text{which leads to a contradiction.} \]
If \( \left( \frac{\bar{\partial}_+ w(F)}{\sin^2 \delta(F)}, \frac{\bar{\partial}_- w(F)}{\sin^2 \delta(F)} \right) \in \gamma_2 \) then by (40) we get
\[ -c \bar{\partial}_-(\frac{\bar{\partial}_+ w}{\sin^2 \delta}) < 0 \quad \text{at the point } F, \quad \text{which leads to a contradiction.} \]
Therefore, by the method of continuity we can get this lemma.

\[ \text{Lemma 3.6. Assume that the Goursat problem admits a } C^1 \text{ solution on } \Omega_\varepsilon \text{ where } 0 < \varepsilon < \frac{\pi}{2} - \delta_0. \text{ Then, there exists a } \mathcal{H}_1 \text{ such that} \]
\[ \|(Du, Dv, Dw)\|_{0, \Omega_\varepsilon} < \mathcal{H}_1. \]

Proof. This lemma can be obtained by (35), (36), Lemma 3.5, \( \sin \beta \bar{\partial}_+ - \sin \alpha \bar{\partial}_- = -\sin 2\delta \bar{\partial}_k \), and \( \cos \beta \bar{\partial}_+ - \cos \alpha \bar{\partial}_- = \sin 2\delta \bar{\partial}_k \).
where \((\alpha_{PQ}, \beta_{PQ}, w_{PQ})\) and \((\alpha_{RP}, \beta_{RP}, w_{RP})\) are the value of the solution on \(\overline{PQ}\) and \(\overline{RP}\), respectively. We then have the following theorem.

**Lemma 3.7** (Curved quadrilateral building block). When the arc lengths of \(\overline{PQ}\) and \(\overline{RP}\) are less than \(\nu_0\) where

\[
\nu_0 = \frac{w_+(0) \sin \left( \frac{\pi}{4} - \delta_0 - \varepsilon \right)}{6 \varrho_\varepsilon \tan^2 \left( \frac{\pi}{4} + \frac{\delta_0 + \varepsilon}{2} \right)} \cdot \left( \frac{\pi - 2(\delta_0 + \varepsilon)}{4} \right) \cdot \min \left\{ \frac{\gamma - 1}{\gamma + 1}, \frac{1}{3} \right\},
\]

the Goursat problem \((8), (63)\) admits a global \(C^1\) solution on a curved quadrilateral domain bounded by \(\overline{PQ}, \overline{RP}, \overline{RT},\) and \(\overline{QT}\), where \(\overline{RT}\) is the \(C^+\) characteristic passing through \(R\), \(\overline{QT}\) is the \(C^-\) characteristic passing through \(Q\), and this solution satisfies \(\delta_0 < \delta < \frac{\pi}{4} + \delta_0 + \varepsilon\).

**Proof.** From Lemma 3.5 we have \(\sup_{\overline{RP}} \partial^{+} \leq \varrho_\varepsilon\) and \(\sup_{\overline{PQ}} \partial^{-} \leq \varrho_\varepsilon\). Then by using a method similar to that of Lemma 3.5 we have that the solution of Goursat problem \((8), (63)\) satisfies \((48)\) and

\[
(\partial^+ w, \partial^- w) \in (0, \varrho_\varepsilon] \times [-\varrho_\varepsilon, 0). \tag{64}
\]

Thus, by \((28)\) and \((32)\) we have

\[
0 < \partial^+ \alpha \leq \frac{\varrho_\varepsilon \tan^2 \delta}{2 \chi^2 w_+(0)}, \quad 0 < \partial^- \beta \leq \frac{\varrho_\varepsilon \tan^2 \delta}{2 \chi^2 w_+(0)}. \tag{65}
\]

In what follows we shall show that when arc lengths of \(\overline{PQ}\) and \(\overline{RP}\) are less than \(\nu_0\), this solution satisfies \(\delta < \frac{\pi}{4} + \frac{\delta_0 + \varepsilon}{2}\). We shall prove this by contradiction. Suppose that there exists a point \(I\) such that \(\delta(I) = \frac{\pi}{4} + \frac{\delta_0 + \varepsilon}{2}\). The \(C^+\) characteristic curve passing through \(I\) intersects with \(\overline{RP}\) at a point \(I_1\), the \(C^-\) characteristic curve passing through \(I_1\) intersects with \(\overline{PQ}\) at a point \(I_2\); see Figure 3(left). Through the point \(I_1\) draw a straight line with the slope \(\tan \alpha(I)\), through the point \(I_2\) draw a straight line with the slope \(\tan \beta(I)\), the two straight lines intersect at a point \(H\).

By estimate \((65)\) we have \(\overline{HI}_1\) and \(\overline{HI}_2\) lie in the triangle \(\triangle_{H_1, I_2}\) and the arc lengths of \(\overline{HI}_1\) and \(\overline{HI}_2\) are less than the sum of the arc lengths of \(\overline{H_1I}_1\) and \(\overline{H_2I}_2\) and \(\overline{I_1I}_2\).
Thus, the arc lengths of $\tilde{I}_1$ and $\tilde{I}_2$ are less than $\frac{6\nu_0}{\sin(\frac{\pi}{2}-\delta_0-\varepsilon)}$. Then, by (48) we have
\[
\alpha(I) < \alpha(I_1) + \frac{\varrho \tan^2(\frac{\pi}{2} + \frac{\delta_0 + \varepsilon}{2})}{2\chi^2 w_+(0)} \cdot \frac{6\nu_0}{\sin(\frac{\pi}{2} - \delta_0 - \varepsilon)} < \alpha(P) + \frac{\pi - 2(\delta_0 + \varepsilon)}{4}
\]
and
\[
\beta(I) > \beta(I_2) - \frac{\varrho \tan^2(\frac{\pi}{2} + \frac{\delta_0 + \varepsilon}{2})}{2\chi^2 w_+(0)} \cdot \frac{6\nu_0}{\sin(\frac{\pi}{2} - \delta_0 - \varepsilon)} > \beta(P) - \frac{\pi - 2(\delta_0 + \varepsilon)}{4}.
\]
Thus, we have $\delta(I) < \frac{\pi}{2} + \frac{\delta_0 + \varepsilon}{2}$, which leads to a contradiction.

By a method similar to that of Lemmas 3.4 and 3.6 we can get the a priori uniform $C^1$ norm estimate of the solution to the Goursat problem (8), (63). Hence, by the theory of global classical solutions for quasilinear hyperbolic equations (cf. Li [18]) we can get this lemma.

**Theorem 3.8.** Under assumptions (a1)–(a5), (b1)–(b5), the Goursat problem (8), (47) admits a unique global $C^1$ solution on a domain $\Omega$ closed by $AB$, $DA$, and $BD$. Here, $BD$ is a sonic curve connecting $B$ and $D$.

**Proof.** Let $Q_0 = B_\varepsilon$, $Q_1$, $Q_2$, $\cdots$, $Q_n = D_\varepsilon$ be $n+1$ different points on the level curve $\delta = \delta_0 + \varepsilon$ in turn. The $C_+$ characteristic curve through $Q_i$ intersects with the $C_-$ characteristic curve through $Q_{i+1}$ at a point $P_i$, where $i = 0, 1, \cdots, n - 1$; see Figure 3(right). Since $\bar{\partial}_+ \delta > 0$ and $\bar{\partial}_- \delta < 0$, the level curve $\delta = \delta_0 + \varepsilon$ is a non-characteristic curve. Hence, $P_i \neq Q_i$ and $P_i \neq Q_{i+1}$ for any $i = 0, 1, \cdots, n - 1$. From (55) we know that when $Q_i$ and $Q_{i+1}$ are sufficiently close the arc lengths of $\overline{P_iQ_i}$ and $\overline{P_iQ_{i+1}}$ are less then $\nu_0$. Thus, by Lemma 3.7 we know that the Goursat problem for system (8) with $\overline{P_iQ_i}$ and $\overline{P_iQ_{i+1}}$ as characteristic boundaries admits a unique $C^1$ solution on a curved quadrilateral domain bounded by $\overline{P_iQ_i}$, $\overline{P_iQ_{i+1}}$, $\overline{R_{i+1}Q_i}$, and $\overline{R_{i+1}Q_{i+1}}$, where $R_{i+1}Q_i$ is the $C_-$ characteristic curve passing through $Q_i$, $R_{i+1}Q_{i+1}$ is the $C_+$ characteristic curve passing through $Q_{i+1}$.

Let $R_0 = B$, then for each $i = 0, 1, \cdots, n$, there exists $\delta_i$, $\delta_0 + \varepsilon < \delta_i < \frac{\pi}{2}$, such that the Goursat problem for system (8) with $\overline{Q_iR_i}$ and $\overline{Q_iR_{i+1}}$ as the characteristic boundaries admits a unique $C^1$ solution on a domain closed by $\overline{Q_iR_{i+1}}$, $\overline{Q_iR_i}$, and a level curve $\delta(\xi, \eta) = \delta_i$. Let $\delta_e = \min \{\delta_0, \delta_1, \cdots, \delta_n, \delta(R_1), \delta(R_2), \cdots, \delta(R_n)\}$. It is easy to see by $\partial_+ \delta > 0$ that $\delta_e > \delta_0 + \varepsilon$. Then, the local solution is extended from $\Omega_e$ to $\Omega_{\delta_e - \delta_i}$. Repeating the above processes, we can construct a solution on $\Omega$. We then have this theorem.

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