A SPECTRAL DECOMPOSITION OF ORBITAL INTEGRALS FOR PGL(2, F) (WITH AN APPENDIX BY S. DEBACKER)

DAVID KAZHDAN

ABSTRACT. Let \( F \) be a local non-archimedian field, \( G \) a semisimple \( F \)-group, \( dg \) a Haar measure on \( G \) and \( \mathcal{S}(G) \) be the space of locally constant complex valued functions \( f \) on \( G \) with compact support. For any regular elliptic conjugacy class \( \Omega = hG \subset G \) we denote by \( I_\Omega \) the \( G \)-invariant functional on \( \mathcal{S}(G) \) given by

\[
I_\Omega(f) = \int_G f(g^{-1}hg)dg
\]

This paper provides the spectral decomposition of functionals \( I_\Omega \) in the case \( G = \text{PGL}(2, F) \) and in the last section first steps of such an analysis for the general case.

Dedicated to A. Beilinson on the occasion of his 60th birthday.

Acknowledgments. Many thanks for J. Bernstein, S. Debacker and Y. Flicker who corrected a number of imprecisions in the original draft and S. Debacker for writing an Appendix.

I am partially supported by the ERC grant 669655-HAS.

1. Introduction

Let \( F \) be a local non-Archimedean field, \( \mathcal{O} \) be the ring of integers of \( F \), \( \mathcal{P} \subset \mathcal{O} \) the maximal ideal, \( k = \mathcal{O}/\mathcal{P} \) the residue field, \( \varpi \) a generator of \( \mathcal{P} \), \( q = |k| \), \( \text{val} : F^\times \to \mathbb{Z} \) the valuation such that \( \text{val}(\varpi) = 1 \) and \( \|a\| = q^{-\text{val}(a)} \), \( a \in F^\times \), the normalized absolute value. For any analytic \( F \)-variety \( Y \) we denote by \( \mathcal{S}(Y) \) the space of locally constant complex-valued functions on \( Y \) with compact support and by \( \mathcal{S}'(Y) \) the space of distributions on \( Y \).

Let \( G \) be a group of \( F \)-points of a reductive group over \( F \), \( Z \) be the center of \( G \), \( dz \) a Haar measure on \( Z \) and \( dg \) a Haar measure on \( G \). We denote by \( \mathcal{H}(G) \) the space of compactly supported measures on \( G \) invariant under shifts by some open subgroup. The map \( f \to fdg \) defines an isomorphisms between the spaces \( \mathcal{S}(G) \) and \( \mathcal{H}(G) \).

We denote by \( \hat{G}_{\text{cusp}} \subset \hat{G}_{2} \subset \hat{G}_{1} \subset \hat{G} \) the subsets of cuspidal, square-integrable and tempered representations. For any \( \pi \in \hat{G}_{2} \) there exists a notion of the \textit{formal degree} \( d(\pi, dg) \) of \( \pi \) which depends on a choice of a Haar measure \( dg \). We chose this measure in such a way that the formal degree of the \textit{Steinberg representation} is equal to 1 and write \( d(\pi) \) instead of \( d(\pi, dg) \). (See Section 3 for definitions in the case \( G = \text{PGL}(2, F) \)).

Given a regular elliptic conjugacy class \( \Omega \subset G \) we denote by \( I_\Omega \) the functional on the space \( \mathcal{H}(G) \) given by \( fdg \to \int_{G/Z} f(hgh^{-1})dg/dz, h \in \Omega \) where \( dg/dz \) the invariant measure on \( G/Z \) corresponding to our choice of measures \( dg \) and \( dz \).

Date: October 8, 2018.

1
Remark 1.1. The functional $I_{\Omega}$ does not depend on a choice of a Haar measure $dg$. In particular these functionals are canonically defined in the case when $G$ is semisimple.

We denote by $\hat{G}$ the set of equivalent classes of smooth irreducible complex representations. For any $\pi \in \hat{G}$ we denote by $\chi_\pi$ the character of $\pi$ which the functional on $\mathcal{H}(G)$ given by $\chi_\pi(\mu) = \text{tr}(\pi(\mu))$ where $\pi(\mu) = \int_G \pi(g) \mu$.

**Conjecture 1.1.** For any regular elliptic conjugacy class $\Omega \subset G$ there exists unique measure $\mu_{\Omega}$ on the subset $\hat{G}_t \subset \hat{G}$ of tempered representations such that

$$I_{\Omega} = \int_{\pi \in \hat{G}} \chi_\pi(\mu_{\Omega}).$$

We say that the measure $\mu_{\Omega}$ gives the spectral description of the functional $I_{\Omega}$. If $t \in G$ is a regular elliptic element we will write $I_t = I_{\Omega}$, $\Omega = tG$.

The main goal of this paper is to find the spectral description of the functionals $I_{\Omega}$ in the case $G = \text{PGL}(2, F)$. When the residual characteristic of $F$ is odd such a description (based on the knowledge of formulas for characters $\chi_\pi$) was given in [SS84].

We discuss the general of case of a general reductive group in the last Section 10, but until Section 10 we assume that $G = \text{PGL}(2, F)$.

Let $B \subset G = \text{PGL}(2, F)$ be the subgroup of upper triangular matrices. Then $B = AU$ where $A \subset B$ is the subgroup of diagonal and $U \subset B$ of unipotent matrices. We denote by $A(O)$ the maximal compact subgroup of $A$ and by $X$ be the group of characters of $A(O)$. For any $x \in X$ we define in Section 2 the notion of depth $d(x) \in \mathbb{Z}_+$ of $x$.

We denote by $U \subset G$ the subset of non-trivial unipotent elements and by $\nu$ a $G$-invariant measure on $U$.

For any $x \in X$ we denote by $\rho_x$ the representation of $G$ induced from the character $x$ of $A(O)U \subset B$ and define $\hat{G}_x \subset \hat{G}$ as the subset of irreducible representations of $G$ which appear as subquotients of $\rho_x$. Then $\hat{G}_x = \hat{G}_{i(x)}$, $i(x) = x^{-1}$ and we have a decomposition of $\hat{G}$ in the disjoint union

$$\hat{G} = \bigcup_{x \in X/i} \hat{G}_x \cup \hat{G}_{\text{cusp}}$$

where $\hat{G}_{\text{cusp}} \subset \hat{G}$ is the subset of cuspidal representations. This decomposition induces a direct sum decomposition

$$(\star) \mathcal{S}(G) = \bigoplus_{x \in X/i} \mathcal{S}(G)_x \oplus \mathcal{S}(G)_{\text{cusp}}$$

and the analogous direct sum decomposition of the space $\mathcal{S}^\vee(G)$ of distributions.

Let $\delta, \omega$ be the distributions on $G$ given by

$$\delta(f) = f(e), \omega(f) = \int_U f(u) \nu$$

We denote by $\delta_x, \omega_x \in \mathcal{S}^\vee(G)$ the components of $\delta$ and $\omega$ in the decomposition $(\star)$ and for $r \geq 0$ define

$$\delta_r = \sum_{x \in X/i | d(x) \leq r} \delta_x$$
and

$$\omega_r = \sum_{x \in X/\mu(d(x)) \leq r} \omega_x.$$  

In Section 2 we define the discriminant $d(\Omega) \in \mathbb{Z}_+$ of a regular elliptic conjugacy class $\Omega \subset G$.

**Theorem 1.2.** For any elliptic torus $T \subset G$ there exist functions $c_e(t), c_u(t)$ on $T$ such that any regular elliptic conjugacy class $\Omega = t^G \subset G, t \in T$ we have an equality

$$I_r = \sum_{\pi \in \mathcal{G}_{\text{cusp}}} d(\pi) \chi(\pi(t)) + c_e(t) \delta_{d(\Omega)} + c_u(t) \omega_{d(\Omega)}$$

of distributions.

The Plancherel formula [3.1] and Claim [3.2] provide spectral descriptions of functionals $\delta_r$ and $\omega_r$ and there the spectral descriptions of $I_r$.

2. THE STRUCTURE OF GROUPS $A$ AND $\text{PGL}(2, F)$

For any $r \in \mathbb{Z}_{\geq 0}$ we define $U_r \subset O^\times$ by

$$U_0 = O^\times; \quad U_r = 1 + P^r, \quad r > 0.$$  

We denote by $da$ the Haar measure on $F$ with $\int_O da = 1$ and by $d^\times a$ the Haar measure on $F^\times$ with $\int_{O^\times} d^\times a = 1$. Then $d^\times a = (1 - q^{-1})da/\|a\|$.

We denote by $\Theta$ the group of characters of $F^\times$, by $i$ the involution of $\Theta$ given by $\theta \mapsto \theta^{-1}$ and by $\Theta_2 \subset \Theta$ the subgroup of characters $\theta$ such that $\theta^2 = \text{Id}$. We can consider $\theta \in \Theta_2$ as a character of $F^\times/(F^\times)^2$.

We denote by $\Theta_{\text{ram}} \subset \Theta$ the subgroup of unramified characters and write $\Theta_{\text{ram}} = \Theta_2 \cap \Theta_{\text{ram}}$. It is clear that $|\Theta_{\text{ram}}| = 2$.

We denote by $X$ the group of characters of $O^\times$ and by $X_2 \subset X$ the subgroup of characters $x$ such that $x^2 = \text{Id}$. For any $x \in X$ we denote by $\Theta_x \subset \Theta$ the subset of characters $\theta$ such that $\theta|O^\times = x$. For any $x \in X_2$ we define

$$\Theta_{x} = \Theta_x \cap \Theta_2.$$  

It is clear that $|\Theta_{\text{ram}}| = 2$ and that the group $\Theta_{\text{ram}}$ acts simply transitively on $\Theta_{x}$ for all $x \in X$. So $|\Theta_{x}| = 2$ for all $x \in X$.

It is clear that the map

$$\Theta \to \mathbb{C}^\times, \quad \theta \mapsto \theta(x),$$  

defines a bijection $\Theta_x \to \mathbb{C}^\times$ for any $x \in X$. This isomorphism induces a structure of an algebraic variety on $\Theta_x$, for each $x \in X$. We denote by $\mathbb{C}[\Theta_x]$ the algebra of regular functions on $\Theta_x$ which is isomorphic to $\mathbb{C}[z, z^{-1}]$.

In the case when $x^2 = \text{Id}$ the involution $i$ acts on $\Theta_x$. We denote by $\mathbb{C}[\Theta_x/i] \subset \mathbb{C}[\Theta_x]$ the subring of invariant functions. It is clear that $\mathbb{C}[\Theta_x/i]$ is isomorphic to $\mathbb{C}[z'], z' = z + z^{-1}$.

**Definition 2.1.** For any $x \in X$ we denote by $d(x)$ the minimal integer $r \geq 0$ such that the restriction of $x^2$ to $U_r$ is trivial. Thus $x^2|U_{d(x)} = 1$ and $x^2|U_{d(x)-1} \neq 1$ where subgroups $U_r$ are defined in the beginning of this Section.
\[ \tilde{G} = \text{GL}(2, F), G' = \{ g \in \text{GL}(2, F) | \max_{i,j \in \{1,2\}} \| g_{ij} \| = 1 \} \]

\[ \tilde{p} : \tilde{G} \to G = \text{PGL}(2, F) \text{ be the natural projection, and } p \text{ the restriction of } \tilde{p} \text{ on } G'. \]
The map \( p : G' \to G \) is surjective and the group \( O^\times \) acts simply transitively on fibers of \( p \).

**Claim 2.1.** For any \( \mu \in \mathcal{H} = \mathcal{H}(G) \) there exists unique \( O^\times \)-invariant measure \( \tilde{\mu} \in \mathcal{H}(\tilde{G}) \) supported on \( G' \) such that \( p_\ast \tilde{\mu} = \mu \).

We denote by \( p^\ast : \mathcal{H}(G) \to \mathcal{H}(\tilde{G}) \) the map \( \mu \to \tilde{\mu} \). It is clear that the map \( p^\ast \) is \( G \)-equivariant.

We often describe elements \( g \in G = \text{PGL}(2, F) \) in terms of a preimage \( \tilde{g} \) in \( \text{GL}(2, F) \) under the map \( p : \text{GL}(2, F) \to G \) and matrix coefficients of \( \tilde{g}_{ij} \). For any \( g \in G \) the ratio \( \tilde{g}_{11} / \det(\tilde{g}) \) does not depend on a choice of a representative \( \tilde{g} \). We denote it by \( \frac{a_{11}}{\det(g)} \).

We denote by \( K \subset G \) the image of \( \text{GL}(2, \mathcal{O}) \) and by \( A \) the image of the group of diagonal matrices. We use the map

\[ \left( \begin{array}{cc} a_{11} & 0 \\ 0 & a_{22} \end{array} \right) \mapsto a_{11} / a_{22} \]

to identify \( A \) with \( F^\times \) and \( A(\mathcal{O}) \) with \( O^\times \), where \( A(\mathcal{O}) = A \cap K \). We denote by \( \tilde{\xi} \in \text{GL}(2, F) \) the matrix

\[ \left( \begin{array}{cc} 1 & 0 \\ 0 & \omega \end{array} \right) \mapsto a_{11} / a_{22} \]

and by \( t \in A \) the image of \( \tilde{t} \) in \( A \).

The following is well known.

**Claim 2.2.** The subsets \( K t^n K \subset G, n \geq 0, \) are disjoint, and \( G = \cup_{n \geq 0} K t^n K \).

We write \( G^{\leq m} := \cup_{0 \leq n \leq m} K t^n K \).

We define \( \Gamma_0 = K \) and for any \( d \geq 1 \) we denote by \( \Gamma_d \subset K \) the image of the subgroup \( \tilde{\Gamma}_d \) of matrices \( g \) in \( \text{GL}(2, \mathcal{O}) \) with \( g_{21} \in \mathcal{P}^d \). So \( I := \Gamma_1 \) is an Iwahori subgroup of \( G \). We denote by \( p_1 : \tilde{\Gamma}_1 \to \Gamma_1 \) the restriction of \( p \) on \( \tilde{\Gamma}_1 \).

We denote by \( U \subset G \) the image of the subgroup of matrices of the form

\[ g_a = \left( \begin{array}{cc} 1 & \gamma \\ 0 & 1 \end{array} \right), \]

write \( B = AU \), and denote by \( b \mapsto \tilde{b} \) the projection \( B \to B/U \simeq A \simeq F^\times \). For any \( \theta \in \Theta \) we denote by the same letter \( \theta \) the character of \( B \) given by \( b \mapsto \theta(\tilde{b}) \).

We denote by \( \text{det}_2 \) the map

\[ \text{det}_2 : G \to F^\times / (F^\times)^2, \quad g \mapsto \det(\tilde{g})(F^\times)^2 \]

and by \( G^0 \) the kernel of \( \text{det}_2 \). If \( \text{char}(F) \neq 2 \) then \( G^0 \) is an open subgroup of \( G \).

For any character \( x \in X \) such that \( d = d(x) > 0 \) we denote by \( \tilde{x} : \Gamma_d \to \mathbb{C}^\times \) the map

\[ g \mapsto x \left( \frac{g_{11}^2}{\det(\tilde{g})} \right). \]

If \( d(x) = 0 \), that is \( x^2 = \text{Id} \), we define a character \( \tilde{x} \) of \( \Gamma_0 = K \) by \( \tilde{x}(g) = x(\text{det}_2(g)) \).

**Claim 2.3.** For any \( x \in X \) the map \( \tilde{x} \) is a character of \( \Gamma_{d(x)} \).

**Definition 2.2.** For any regular elliptic conjugacy class \( \Omega \subset G \) we let \( d(\Omega) \) be the biggest number \( d \) such that \( \Omega \) intersects \( \Gamma_d \).
3. Basic structure of representations of \( G \)

We say that a measure \( \mu \) on \( G \) is smooth if it is \( R \)-invariant for some open subgroup \( R \subset G \). Let \( \mathcal{H} \) be the space of complex-valued compactly supported smooth measures \( \mu \) on \( G \). For any open compact subgroup \( R \subset G \) we denote by \( \text{ch}_R \subset \mathcal{H} \) the normalized Haar measure on \( R \).

Convolution, denoted by \(*\), defines an algebra structure on \( \mathcal{H} \). The algebra \( \mathcal{H} \) acts on \( \mathcal{S}(G) \) by convolution from the right, \( (f, \mu) \mapsto f * \mu \), and also from the left.

The group \( G \) acts on \( \mathcal{H} \) by conjugation. We denote by \( \mathcal{H}_G \) the space of coinvariants which is equal to the quotient \( \mathcal{H} / [\mathcal{H}, \mathcal{H}] \).

We denote by \( \mathcal{C} \) the category of smooth complex representations of \( G \) and by \( \hat{G} \) the set of equivalence classes of smooth irreducible representations of \( G \).

The group \( G \) acts on \( \mathbb{P}^1(F) \) and therefore on the spaces \( \mathcal{S}(\mathbb{P}^1(F)) \). It is clear that the subspace \( \mathbb{C} \) of constant functions invariant and we obtain the Steinberg representation \( St \) on the space \( \mathcal{S}(\mathbb{P}^1(F))/\mathbb{C} \). It is well known (see [GGP]) the representation \( St \) of \( G \) is irreducible. For any \( \theta \in \Theta_2 \) we denote by \( C_{\theta} \) the one-dimensional representation \( g \mapsto \theta(\text{det}_2(g)) \), and define \( St_{\theta} = St \otimes C_{\theta} \).

For any \((\pi, V) \in \mathcal{C}, \mu \in \mathcal{H}\), we define
\[
\pi(\mu) = \int_G \pi(g)\mu \in \text{End}(V).
\]

For irreducible representations \( \pi \) of \( G \) the operator \( \pi(\mu) \) is of finite rank for any \( \mu \in \mathcal{H} \) and we define the character \( \chi_\pi \) on \( G \), as a generalized function (a functional on \( \mathcal{H} \)) by
\[
\chi_\pi(\mu) = \text{tr} \pi(\mu), \quad \mu \in \mathcal{H}.
\]

By [JL70], there exist a locally \( L^1 \)-function on \( G \), (that we denote by \( \chi_\pi \)) such that
\[
\chi_\pi(\mu) = \int_G \chi_\pi \mu
\]

We define a map \( \kappa : \mu \mapsto \hat{\mu} \) from \( \mathcal{H} \) to functions on \( \hat{G} \) by
\[
\hat{\mu}(\pi) = \text{tr} \pi(\mu).
\]

It is clear that \( \kappa \) descends to a map from \( \mathcal{H}_G \) to functions on \( \hat{G} \).

We say that an irreducible representation \((\pi, V) \) of \( G \) is square-integrable if it is unitarizable (that is, there exists a nonzero \( G \)-invariant Hermitian form \((, ) \) on \( V \)), and for every \( v \in V \) the function \( g \mapsto (\pi(g)v, v) \) on \( G \) belongs to \( L^2(G) \). We denote by \( \hat{G}_2 \subset \hat{G} \) the subset of square-integrable representations. Let \( dg \) be a Haar measure on \( G \).

The following Claim follows from [HC70].

Claim 3.1. a) For every \((\pi, V) \in \hat{G}_2 \) there exists a number \( \deg(\pi) = \deg(\pi, dg) > 0 \), called the formal degree of \( \pi \), such that
\[
\int_G |m_v(g)|^2 \, dg = \frac{1}{\deg(\pi)}, \quad m_v(g) = (\pi(g)v, v),
\]

for any \( v \in V, (v, v) = 1 \), where \( dg \) is a Haar measure on \( G \).

b) There exists a unique choice of \( dg \) with \( \deg(St, dg) = 1 \).
c) For any irreducible square-integrable representation \((\pi, V)\) and any \(v \in V, (v, v) = 1\), the sequence of locally constant functions 
\[
(I_n(v))(g) := \int_{h \in G \leq \sigma_m} m_v(hgh^{-1}) dh
\]
on \(G\) converges as a generalized function to the character \(\chi_\pi/\deg(\pi, dg)\). In other words, for any \(\mu \in \mathcal{H}\) the sequence \(\{\int I_n(v)\mu\}\) converges to \(\hat{\mu}(\pi)/\deg(\pi)\).

For any smooth representation \((\pi, V)\) of \(G\) we denote by \(J(V)\) the normalized Jacquet functor which is a representation of \(A\) acting on the space \(V/V(U)\) where \(V(U)\) is the span of \(\{\pi(u)v - v, u \in U, v \in V\}\). We define the action of \(A\) on \(J(V)\) by \(a \mapsto ||a||^{1/2}\pi(a), a \in A\), of \(A\) on \(V_U\). Here \(||a|| = ||t_1/t_2||\) for a represented by \(\begin{pmatrix} a & 0 \\ 0 & t_2 \end{pmatrix}\).

We say that \(V\) is cuspidal if \(J(V) = \{0\}\).

We denote by \(C_{\text{cusp}}\) the subcategory of cuspidal representations and by \(\hat{G}_{\text{cusp}} \subset \hat{G}\) the subset of equivalence classes of irreducible cuspidal representations. Since matrix coefficients of cuspidal representation of \(G\) have compact support (see [JL70]) we have an inclusion \(\hat{G}_{\text{cusp}} \subset \hat{G}_2\).

4. Induced representations

For any \(\theta \in \Theta\) we denote by \((\pi_\theta, R_\theta)\) the representation of \(G\) unitarily induced from the character \(b \mapsto \theta(\bar{b})\) of \(B\). So \(R_\theta\) is the space of locally constant complex valued functions \(f\) on \(G\) such that
\[
f(gb) = \theta(\bar{b})||\bar{b}||^{1/2}f(g), \quad g \in G, \quad b \in B,
\]
and \(G\) acts on \(R_\theta\) by left shifts: \((\pi_\theta(x)f)(g) = f(x^{-1}g)\).

Since \(G = KB\), the restriction to \(K\) identifies the space \(R_\theta\) with the space \(R_x, x = \theta|O^x\), where \(R_x\) is the space of locally constant functions \(f\) on \(K\) such that
\[
f(kb) = \theta(\bar{b})f(k), \quad k \in K, \quad b \in B \cap K.
\]
It is clear that in this realization the operator \(\pi_\theta(\mu) \in \text{End}(R_x)\) is a regular function on \(\theta \in \Theta_x\) for any \(\mu \in \mathcal{H}\) and so the function
\[
\hat{\mu}_x : \theta \mapsto \hat{\mu}(\pi_\theta), \quad \theta \in \Theta_x,
\]
belongs to \(C[\Theta_x]\).

The following result is well known, see [JL70].

**Proposition 4.1.** a) For any \(\theta \in \Theta\) we have \(\text{End}_G(R_\theta) = C\).

b) A representation \(\pi_\theta\) is reducible if and only if \(\theta(a) = \theta_2(a)||a||^{1/2}\) or \(\theta(a) = \theta_2(a)||a||^{-1/2}\) where \(\theta_2 \in \Theta_2\). In the second case \(\pi_\theta\) has a one-dimensional subrepresentation \(C_{\theta_2}\), and the quotient is isomorphic to \(\text{St}_{\theta_2}\). In the first case \(\pi_\theta\) has \(\text{St}_{\theta_2}\) as a subrepresentation and the quotient is isomorphic to \(C_{\theta_2}\).

c) Let \(\theta, \theta' \in \Theta\) be such that \(\pi_\theta, \pi_{\theta'}\) are irreducible. Then the representations \(\pi_\theta, \pi_{\theta'}\) are isomorphic iff \(\theta' = \theta \pm 1\).

d) We have a disjoint union decomposition
\[
\hat{G} = \hat{G}_2 \cup (\cup_{\theta_2 \in \Theta_2} C_{\theta_2}) \cup \left(\cup_{\theta \in (\Theta - \Theta_2)/i} \pi_\theta\right).
\]
e) We have a disjoint union decomposition
\[ \hat{G}_2 = \hat{G}_{\text{cusp}} \cup (\cup \theta_2 \in \Theta_2 \text{St}_{\theta_2}). \]

f) For any \( \theta \in \Theta \) the character \( \chi_\theta := \chi_{x_\theta} \) is given by a locally \( L^1 \)-function on \( G \) supported on split elements such that
\[ \chi_\theta \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) = \theta(a/b) + \theta(b/a) \frac{\|a - b\|^2/ab}{\|a - b\|^2}. \]

**Definition 4.1.**
(1) For any \( x \in X \), we denote by \( (\tau_x, V_x) \) the representation of \( G \) by left shifts on the space of locally constant compactly supported functions \( f \) on \( G \) such that \( f(gtu) = x(t)f(g), \ g \in G, \ t \in A(O), \ u \in U. \)
(2) We denote by \( f_0 \in V_x \) the function supported on \( \Gamma_d U \), \( d = d(x) \), and such that \( f_0(\gamma u) = \widetilde{x}(\gamma), \ \gamma \in \Gamma_d, \ u \in U. \)
(3) We denote by \( p_\theta, \theta \in \Theta_x \), the map
\[ p_\theta : V_x \to R_\theta, \ \ (p_\theta(f))(g) = \sum_{n \in \mathbb{Z}} f(gt^n)q^n\theta(\varpi^n), \]
where as before \( t \) is the image in \( G \) of \( \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi \end{array} \right) \in \text{GL}(2, F). \)

We also have

**Proposition 4.2.** If \( M \subset V_x \) is a \( G \)-invariant subspace such that \( p_\theta(M) = R_\theta \) for all \( \theta \in \Theta_x \) then \( M = V_x. \)

**Proof.** As follows [Be92] it suffices to show that there is no nonzero morphism from \( V_x/M \) to an irreducible representation of \( G \). But as follows from [BZ76] all morphisms from \( V_x \) to an irreducible representation of \( G \) are factorizable through a projection \( p_\theta \) for some \( \theta \in \Theta_x. \)

**Corollary 4.3.** If \( x^2 \neq \text{Id} \) then the function \( f_0 \) generates \( V_x \) as an \( \mathcal{H} \)-module.

**Proof.** It is clear that \( f_{\theta,0} := p_\theta(f_0) \in R_\theta \) is not equal to 0. Moreover, it follows from Proposition 4.1.a),b) that it generates \( R_\theta \) as an \( \mathcal{H} \)-module. But then Proposition 4.2 implies that \( f_0 \) generates \( V_x \) as an \( \mathcal{H} \)-module.

The following result follows from Corollary 4.3. We assume that \( x^2 \neq \text{Id} \) and use the identification of the ring \( \mathbb{C}[\Theta] \) with \( \mathbb{C}[z, z^{-1}] \) as in the Introduction. Let
\[ \alpha : \mathbb{C}[\Theta_x] \simeq \mathbb{C}[z, z^{-1}] \to \text{End}_{G}(V_x) \]
be the algebra morphism defined by \( ((\alpha(z))(f))(g) = q^{-1}f(gt^{-1}), \ f \in V_x. \)
Corollary 4.4. a) For any $S \in \text{End}_G(V_x)$, and $\theta \in \Theta_x$, the map $S$ preserves the subspace $\ker(p_0) \subset V_x$ and so defines $\tilde{S}(\theta) \in \text{End}_G(R_0) = \mathbb{C}$.

b) For any $S \in \text{End}_G(V_x)$, the function $S$ on $\Theta_x$ belongs to $\mathbb{C}[\Theta_x] \simeq \mathbb{C}[z, z^{-1}]$.

c) The maps

$$\text{End}_G(V_x) \to \mathbb{C}[\Theta_x], S \mapsto \tilde{S}$$

and

$$\mathbb{C}[\Theta_x] \to \text{End}_G(V_x), s \mapsto \alpha(s)$$

are mutually inverse.

5. Structure of the representation $(\tau_x, V_x)$ when $d(x) > 0$

In this section we fix a character $x \in X$ such that $d(x) > 0$ (so $x^2 \neq Id$).

Definition 5.1. (1) We denote by $\rho_x$ the representation $\text{ind}^G_{\Gamma_d} \tilde{x}$ of $G$ on the space $\tilde{W}_x$ of locally constant functions $\phi$ on $G$ such that

$$\phi(g\gamma) = \tilde{x}(\gamma)\phi(g), \quad g \in G, \quad \gamma \in \Gamma_d,$$

and by $W_x \subset \tilde{W}_x$ the subspace of functions with compact support.

(2) Denote by $\phi_0 \in W_x$ the function supported on $\Gamma_d$ and equal to $\tilde{x}$ there, and define

$$\mu_0 := \phi_0 \text{ch}_{\Gamma_d} \in \mathcal{H}.$$

(3) Let $A : V_x \to \tilde{W}_x$, $B : W_x \to V_x$ be the $G$-morphisms defined by

$$A(f) = f \ast \mu_0, \quad B(\phi) = \phi_U, \quad \phi_U(g) = \int_U \phi(gu) du,$$

where $du$ is the Haar measure on $U$ which is normalized by $\int_{U \cap K} du = 1$.

Lemma 5.1. a) $B(\phi_0) = f_0$.

b) $A(f_0) = \phi_0$.

c) $A$ defines an isomorphism $A : V_x \to W_x$.

d) $\text{End}_G(V_x) \simeq \text{End}_G(W_x)$.

Proof. Part a) is clear. It is also clear that the restriction of $A(f_0)$ to $\Gamma_d$ is equal to $\tilde{x}$ and that supp$(A(f_0)) \subset \Gamma_d U \Gamma_d$. So to prove (b) it suffices to check that for any $u \in F, \|u\| > 1$, we have

$$\int_{\Gamma_d} f_0(gu\gamma) \tilde{x}(\gamma)^{-1} d\gamma = 0,$$

where $d\gamma$ is the normalized Haar measure on $\Gamma_d$ and $g_u = (1 \ y \ 1 \ 0)$. To see this, write $\gamma$ as $\gamma_0 \gamma_1$, $\gamma_0 = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, $\gamma_1 = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$. Note that $\tilde{x}(\gamma) = \tilde{x}(\gamma_1) = x(a/d)$. The integral equals

$$\int f_0 \begin{pmatrix} 1+uc \\ c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} dc d\gamma_1 = \int f_0 \begin{pmatrix} 1+uc \\ c \end{pmatrix} \gamma_1 \begin{pmatrix} 1 & \frac{uc}{1+uc} \\ 0 & 1 \end{pmatrix} x(a/d)^{-1} dc d\gamma_1$$

$$= \int x(1+uc)^2 dc, \quad \{c \in \mathcal{P}^d; 1+uc \in \mathcal{O}^\times\},$$
since $f_0$ is supported on $\Gamma_d U$ (so it vanishes unless $|1 + uc| = 1$) and we are integrating a nontrivial character the integral is 0.

The part c) follows from the parts a), b) since by Corollary 4.3 the function $f_0$ generates $V_x$ as an $H$-module and, as easy to check, the function $\phi_0$ generates $W_x$ as an $H$-module. The part d) follows from c).

\[ \Box \]

6. Algebras of endomorphisms

As before we fix (in this section) a character $x \in X$ such that $x^2 \neq 1$.

**Lemma 6.1.** Let $H'_x \subset H$ be the subalgebra of measures $\mu$ with

\[ l_\gamma(\mu) = r\gamma(\mu) = \tilde{x}(\gamma)\mu, \quad \gamma \in \Gamma_d, \]

where $l_\gamma$, $r_\gamma$ are left and right shifts by $\gamma$. Then

1. $\mu_0 = \phi_0 \text{ch}_{\Gamma_d}$ is the unit of $H'_x$.
2. Convolution on the right defines an isomorphism $\beta : H'_x \to \text{End}_G(W_x)$.

**Proof.**

(1) is clear.

For (2), note that the map $S \mapsto S(\phi_0) \text{ch}_{\Gamma_d}$ defines a morphism $\tilde{\beta} : \text{End}_G(W_x) \to H'_x$. One checks that the compositions $\beta \circ \tilde{\beta}$ and $\tilde{\beta} \circ \beta$ are the identity maps. So we can identify $\text{End}_G(W_x)$ with $H'_x$.

\[ \Box \]

As follows from Lemma 5.1 we identify the ring $\text{End}_G(W_x)$ with $\text{End}_G(V_x)$ and therefore (by Corollary 4.4) with the ring $\mathbb{C}[z, z^{-1}]$.

For any $n \in \mathbb{Z}$ we denote by $\phi_n \in W_x$ the function supported on $\Gamma_d t^n \Gamma_d$ with

\[ \phi_n(\gamma t^n \gamma') = \tilde{x}(\gamma' \gamma''), \quad \gamma', \gamma'' \in \Gamma_d, \]

and write $\mu_n = \phi_n \text{ch}_{\Gamma_d} \in H'_x$.

The following result follows from the commutativity of the algebra $H'_x$ and Frobenius reciprocity.

**Claim 6.2.** Let $p : (\rho_x, W_x) \to (\pi, W)$ be an irreducible quotient of $W_x$. Then

a) The action of $\text{End}_G(W_x) = H_x$ on $W_x$ preserves $\ker(p)$ and therefore induces a homomorphism $\tilde{p} : H'_x \to \text{End}_G(W) = \mathbb{C}$.

b) For any $\mu \in H'_x$ we have $\tilde{p}(\mu) = \hat{\mu}(\pi)$.

g) The set $\{\mu_n; n \in \mathbb{Z}\}$ is a basis of the space $H'_x$.

**Lemma 6.3.** a) The map $\mu \mapsto \hat{\mu}(\pi_\theta)$, $\theta \in \Theta_x$, defines an isomorphism $H'_x \to \mathbb{C}[\Theta_x]$.

b) $\hat{\mu}_n(\theta) = cz^n$, $c \neq 0$ where we identify the space $\Theta_x$ with $\mathbb{C}^\times$, $\theta \mapsto z = \theta(\overline{\omega})$.

c) The set $\{\mu_n; n \in \mathbb{Z}\}$ is a basis of the space $H'_x$.

**Proof.**

a) The first part follows immediately from Lemma 5.1 and Proposition 6.2.

b) Let

\[ R^\theta_\phi = \{ f \in R_\phi | \pi_\theta(\gamma) f = \tilde{x}(\gamma^{-1}) f, \quad \gamma \in \Gamma_d \}. \]
It follows from Lemma 5.1 and Claim 6.2 that \( \dim(R^0_\theta) = 1 \) and that this space is equal to \( \mathbb{C} \cdot f_{\theta,0} \), where \( f_{\theta,0} \) was defined to be \( p_{\theta}(f_0) \) in the proof of Corollary 4.3. Since \( f_{\theta,0}(e) = 1 \), it is sufficient to show that 
\[
((\pi_\theta(\mu_n))(f_{\theta,0}))(e) = cz^n,
\]
but this is immediate. Since the map \( \mu \to \hat{\mu} \) is not a zero map we see that \( c \neq 0 \).

The part c) follows from b).

7. Categories of representations

Fix \( x \in X \). Let \( \widehat{G}_x \subset \widehat{G} \) be the set of equivalence classes of irreducible representations of \( G \) which appear as subquotients of \( (\tau_x, V_x) \). It can be described as the set of equivalence classes of irreducible subquotients of the representations \( \{ R_\theta \} \) for \( \theta \in \Theta_x \). It follows from Proposition 4.1 that the set \( \widehat{G}_x \) depends only on the image of \( x \) in \( X/i \) and that for distinct \( x, x' \in X/i \) the sets \( \widehat{G}_x \) and \( \widehat{G}_{x'} \) are disjoint.

We denote by \( C_x \subset C \) the subcategory of representations the equivalence classes of whose irreducible subquotients belong to \( \widehat{G}_x \).

The following result is well known (see [BZ76] for parts a) and b) and [BDK86] for part c).

Proposition 7.1. a) We have a decomposition
\[
(1) C = C_{\text{cusp}} \oplus (\bigoplus_{x \in X/i} C_x).
\]
This decomposition defines the direct sum decompositions
\[
(2) S(G) = S(G)_{\text{cusp}} \oplus (\bigoplus_{x \in X/i} S(G)_x)
\]
and
\[
(3) S'(G) = S'(G)_{\text{cusp}} \oplus (\bigoplus_{x \in X/i} S'(G)_x)
\]
and
\[
(4) H_G = H_{G,\text{cusp}} \oplus (\bigoplus_{x \in X/i} H_{G,x}).
\]

b) For any \( x \in X \), \( x^2 \neq \text{Id} \), and \( \mu \in H_{G,x} \), the function \( \hat{\mu} \) is supported on \( \Theta_x \), and the map \( \kappa : \mu \to \hat{\mu} \) defines an isomorphism from \( \mathbb{H}_{G,x} \) to \( \mathbb{C}[\Theta_x] \).

c) For any \( x \in X_2 \), \( \mu \in H_{G,x} \), the function \( \hat{\mu} \) is supported on \( (\Theta_x/i) \cup (\bigcup_{\theta \in \Theta_{x_2}} \text{St}_\theta) \),

and the map \( \kappa : \mu \to \hat{\mu} \) defines an isomorphism from \( \mathbb{H}_{G,x} \) to
\[
\mathbb{C}[\Theta_x/i] \oplus (\bigoplus_{\theta \in \Theta_{x_2}} C_\theta).
\]

Lemma 7.2. a) For \( x \in X - X_2 \) we have \( \mathcal{H}'_x \subset \mathcal{H}_x \).

b) The map \( \mathcal{H}'_x \to \mathcal{H}_{G,x} \) is an isomorphism.

Proof. a) Let \( (\pi, V) \) be an irreducible representation such that \( \pi(\mu) \neq 0 \) for some \( \mu \in \widehat{H}_x \). We want to show that \( \pi \in \widehat{G}_x \).

Since \( \pi(\mu) \neq 0 \) we see that \( V^0 \neq \{0\} \), where
\[
V^0 = \{ v \in V | \pi(\gamma)v = \bar{x}(\gamma^{-1})v, \gamma \in \Gamma_d \}.
\]

Since \( V|_{\Gamma_d} = V^0 \neq \{0\} \),
By the Frobenius reciprocity we have
\[ \text{Hom}_G(W_x, V) = \text{Hom}_{\Gamma_d}(\widetilde{x}, V) = V_0. \]
Therefore \( V \) is a quotient of \( W_x \). So the lemma follows from Lemma 5.1 c) which asserts the equivalence of the representations \( V_x \) and \( W_x \) of \( G \).

b) follows now from [BDK86]. □

Corollary 7.3. Let \( \Omega \subset G \) be a regular elliptic conjugacy class and \( x \in X \) be such that \( d(x) > d(\Omega) \). Then \( I_{\Omega}(\mu) = 0 \) for any \( \mu = fdg \in \mathcal{H}_x' \) where

\[ I_{\Omega}(fdg) = \int_G f(ghg^{-1})dg, \quad h \in \Omega. \]

**Proof.** Since \( d = d(x) > 0 \) it follows from Lemma 6.3 b) that the set \( \{\mu_n = \phi_n dg; n \in \mathbb{Z}\} \) is a basis of the space \( \widetilde{\mathcal{H}}_x \) which (by Lemma 6.3 a) and Proposition 7.1 b)) is isomorphic to \( \mathcal{H}_{G,x} \). So it suffices to check that \( \int_G \phi_n(ghg^{-1})dg = 0 \) for all \( n \in \mathbb{Z} \). If \( n \neq 0 \) then all elements of \( \Gamma_d^n \Gamma_d \) are split and so the support of \( \phi_n \) is disjoint from \( \Omega \). On the other hand if \( n = 0 \) then \( \Omega \cap \Gamma_d = \emptyset \) by definition of \( d(\Omega) \). □

8. The Plancherel formula

Consider the distribution \( \delta \) on \( \mathcal{S}(G), \delta(f) := f(e) \). The part a(1) of Proposition 7.1 implies the decomposition

\[ \delta = \delta_{\text{cusp}} + \sum_{x \in X/i} \delta_x \]

where

\[ \delta_{\text{cusp}} \in \mathcal{S}^\vee(G)_{\text{cusp}}, \delta_c \in \mathcal{S}^\vee(G)_x \]

The Plancherel formula (see [AP]) describes the functionals \( \delta_x \) and \( \delta_{\text{cusp}} \). Let \( S^1 = \{z \in \mathbb{Z}||z|| = 1\} \) and \( |dz| \) the Haar measure on \( S^1 \) such that \( \int_{S^1} |dz| = 1 \). I will use notations of the section 1 and in particular the identification \( z \to x^z \in \Theta_x \) of \( \mathbb{C}^\times \) with \( \Theta_x \).

**Proposition 8.1.** a) \( \delta_{\text{cusp}} = \sum_{\pi \in \hat{G}_{\text{cusp}}} d(\pi, dg)\chi_{\pi} \).

b) If \( x^2 = Id \) then

\[ \delta_x = \sum_{\theta \in \Theta_{2,x}} \chi_{S\theta} + \int_{z \in S^1} \frac{|(z - 1)(z^{-1} - 1)|^2}{|(z/q - 1)(z^{-1}/q - 1)|^2} \chi_{x^z} |dz| \]

c) If \( x^2 \neq Id \) then

\[ \delta_x = \int_{z \in S^1} \gamma(x) \chi_{x^z} |dz| \]

where \( \gamma(x) \) are explicit constants (see [AP]).

Let \( dt \) be the Haar measure on \( A \) such that \( \int_{A(\mathbb{Q})} dt = 1 \) and \( dg/dt \) the corresponding \( G \)-invariant measure on \( G/A \). For any \( s \in F - \{0, 1\} \) we write

\[ a_s = (\delta, 0, 1) \in A \]
and denote by $\omega_s$ the functional on $\mathcal{S}(G)$ given by

$$\omega_s(f) = |s - 1| \int_{\gamma \in G/A} f(\gamma a_{\gamma} g^{-1}) d\gamma / dt$$

Let $U \subset G$ be the set of regular unipotent elements. Since $G$ acts transitively on $U$ and the stationary subgroup is unimodular (it actually is isomorphic to $U \subset G$), there exists a unique (up to a scalar) $G$-invariant measure $\nu$ on $U$.

The following claim is well known and is an easy exercise.

**Claim 8.2.** a) For any $f \in \mathcal{S}(G)$ the integral

$$\omega(f) := \int_{G/U} f(g (\frac{1}{2} 1)) g^{-1} d\gamma$$

is absolutely convergent.

b) One can choose a $G$-invariant measure $\nu$ on $U$ in such a way that $\omega(f) \equiv \lim_{s \to 1} \omega_s(f)$

We define $\omega_{s,x}$ be the components of $\omega$ in the decomposition of Proposition 7.1. The following claim follows from Proposition 4.1 and Claim 8.2.

**Lemma 8.3.** a) $\omega_{s,x} = \omega_x = 0$ for all cuspidal representations $\pi$.

b) $\omega_{s,x} = \int_{s \in S} \chi_s x(s) |dz|$ and $\omega_x = \int_{s \in S} \chi_x |dz|$ for all $x \in X$.

9. **Proof of Theorem 1.2**

The proof of Theorem 1.2 using results on orbital integrals for the group $\tilde{G} = GL(2, F)$. We denote by $\mathcal{H}$ the Hecke algebra for $G$ and for any $r \geq 0$ define subalgebras $\mathcal{H}_r \subset \mathcal{H}$ as in [Ka]. We fix a Haar measure on $\tilde{G}$ identify $\mathcal{H}$ with $\mathcal{S}(\tilde{G})$. For any $x \in X - X_2$ we denote by $\phi_x \in \mathcal{H}$ the function supported on $\Gamma_{d(x)}$ and equal to $\tilde{x}(p(\gamma))$ on $\Gamma_{d(x)}$ and write $\bar{\mu}_x = \phi_x d\gamma$.

The following result is immediate.

**Claim 9.1.** $\bar{\mu}_x \in \mathcal{H}_{d(x)}$ for any $x \in X, \mu \in \mathcal{H}_x$.

Let $\tilde{T} \subset \tilde{G}$ be a maximal elliptic torus, $\mathcal{T} \subset M_2(F)$ the Lie algebra of $\tilde{T}$. Define

$$\tilde{T}_0 = \tilde{T} \cap \mathfrak{a} M_2(\mathcal{O}), \tilde{T}_0 = \tilde{T} \cap \mathcal{K}_1.$$  

It is clear that $\tilde{T}_0$ is invariant under multiplications by $c \in U_1 \subset \mathcal{O}^\times$ and the projection $p: \tilde{G} \rightarrow G$ induces a bijection $\tilde{T}_0 / U_1 \rightarrow T_0, T_0 = p(\tilde{T}_0)$. For any $a \in \tilde{T}_0$ we have $Id_2 + a \in \tilde{T}_0$.

The following Claim follows from [Sh72] and [HC99].

**Claim 9.2.** a) For any maximal elliptic torus $\tilde{T} \subset \tilde{G}$ there exist functions $\check{c}_e, \check{c}_u$ on $\tilde{T}$ such that 

$$c_e(ca) = c_e(a), c_u(ca) = \|c|^{-1} c_u(a), c \in F^\times, a, ca \in \tilde{T}_0$$

and for any $\tilde{\mu} = \tilde{f} d\gamma \in \mathcal{H}$ there exists a neighborhood $Y_\mu$ of $0$ in $\tilde{T}$ such that $a \in Y_\mu$ we have

$$\Omega (\tilde{f}) = \check{c}_e(a) \delta(\tilde{f}) + \check{c}_u(a) \omega(\tilde{f})$$

where

$$\Omega = (Id_2 + a)^G, \delta(\tilde{f}) = \tilde{f}(e), \omega(\tilde{f}) = \int_U \tilde{f} \nu.$$
Define functions $c_e, c_U$ on $T_0$ by
\[ c_e(g) := \tilde{c}_e(g - Id_2), \quad c_U(g) := \tilde{c}_U(g - Id_2). \]

Then
\[ b) \] the functions $c_e, c_U$ are invariant under multiplication by $cId_2 \in \tilde{G}, c \in U_1$.
Therefore these functions define functions on $T_0$ which we also denote by $c_e$ and $c_U$.

As follows from Appendix the results of [Ka] are applicable for all local fields $F$. So we have the following statement.

**Claim 9.3.** The equality $(\ast)$ holds for all $a \in \omega^r \cap T$ if $\tilde{\mu} = \tilde{f}dg \in \tilde{H}_r$.

**Corollary 9.4.** For any elliptic torus $T \subset G, t \in T \cap \Gamma_1, t \neq e$ and any $\mu = fdg \in H_x, d(x) \geq 1$ we have
\[ (\ast)I_t(f) = c_e(t)\delta(f) + c_U(t)\omega(f) \]
where $I_t := I_{tG}$.

**Proof.** Apply the Corollary 9.3 to $\tilde{\mu} = p^*(\mu)$. □

**Proposition 9.5.** For any $f \in H_{cusp}$ we have
\[ I_\Omega(f) = \sum_\pi \text{tr}(\pi(f))d(\pi) \]
where $\pi$ runs through the set of equivalent classes of irreducible cuspidal representations of $G$.

**Proof.** Since $H_{cusp}$ is spanned by matrix coefficients of irreducible cuspidal representations it is sufficient to check the equality in the case when $f = m_\xi$ is a matrix coefficient of an irreducible cuspidal representation $(\pi, V), \xi \in V$ but in this case the equality follows from Proposition 3.1(c).

**Theorem 9.6.** For any elliptic torus $T \subset G$ and any regular elliptic conjugacy class $\Omega = t^G \subset G, \Omega \cap T_0 \neq \emptyset$ and any $\mu \in H_0$, we have
\[ (\ast)I_\Omega(f) = \sum_\pi \text{tr}(\pi(f))d(\pi, dg) + \sum_{x \in X, d(x) \leq d(\Omega)} (c_e(t)\delta_x(f) + c_U(t)\omega_x(f)) \]

**Proof.** As follows from Proposition 9.1 the equality is true for $f \in S(G)_{cusp}$. So as follows from Lemma 7.2 it is sufficient to check the equality in the case of $f = \phi_{n,x} \in S(G)_x, x \in X, n \in \mathbb{Z}$.

If $n \neq 0$ then $\Omega$ does not intersect the support of $\phi_{n,x}$. On the other hand it follows from Lemma 6.3(b) and the Plancherel formula (Proposition 8.1) that the right side of $\ast$ also vanishes in this case.

We see that for a proof of Theorem 9.1 it is sufficient to check the equation $\ast$ in the case $n = 0$.
Since $\delta_e(\tilde{\mu}_e) = \delta(\mu_{0,x})$ and $\omega_x(\tilde{\mu}_x) = \omega(\mu_{0,x})$ the equality $\ast$ follows from Claim 9.3 in the case when $d(x) \leq d(\Omega)$.
On the other hand iff $d(x) > d(\Omega)$, the equality follows from Corollary 7.2 □
10. The case of general groups

Let $G$ be a split semisimple $F$-group. I fix a Haar measure $dg$ on $G$ and often write $G$ instead of $G(F)$. Using the Haar measure $dg$ one identifies $f \mapsto f dg$ the space $S(G)$ of locally constant $\mathbb{C}$-valued compactly supported functions on $G$ with the space $\mathcal{H} = S(G)$ of locally constant $\mathbb{C}$-valued compactly supported measures on $G$. The convolution defines an algebra structure on $\mathcal{H}$ and we define

$$\mathcal{H}_G := \mathcal{H} / [\mathcal{H}, \mathcal{H}]$$

For $f \in S(G)$ we denote by $f_G$ the image of $f$ in $\mathcal{H}_G$. For any regular elliptic element $t \in G$ we define a functional $I_t$ on $\mathcal{H}$ by

$$I_t(fdg) = \int_G f(g t g^{-1}) dg$$

It is clear that $I_t$ does not depend on a choice a Haar measure and depends only on the conjugacy class $c = c_t$ of $t$ in $G$. We write $I_c$ instead of $I_t$.

Let $C$ be the set of regular elliptic conjugacy classes of $G$. There exists (see [K]) a measure $dc$ on $C$ such that

$$\int_G f(g) dg = \int_{c \in C} I_c(f) dc$$

for any $f \in S(G)$ supported on the subset $G_e \subset G$ of regular elliptic elements.

Let $A(G) \subset S(G)$ be the subset of functions such that $\int_{\Omega} f d\omega = 0$ for any regular non-elliptic conjugacy class $\Omega$ where $\omega$ is an invariant measure on $\Omega$. We denote by $A_G$ the image of $A(G)$ in $\mathcal{H}_G$. For any $a \in A_G$ we define a function $[a]$ on $C$ by $[a](c) = I_c([a])$. As follows from Theorem F in [K] for any $[a], [b] \in A_G$ the scalar product

$$<[a], [b]> := \int_C [a][\bar{b}] dc$$

is well defined.

Let $Z$ be the Bernstein center of $G$. As follows from Theorem B in [K] there exists a countable subset $S \subset Spec( mcZ)$ of characters and a decomposition

$$A_G = \oplus \omega, \omega \in S$$

of $A$ into a direct sum of finite dimensional subspaces such that $z \in Z$ acts by $\omega(z)$ on $A_\omega, \omega \in S$.

As follows from [K] the subspaces $A_\omega$ are mutually orthogonal and the restrictions of the form $<,>$ on the subspaces $A_\omega$ are positive definite. For any $\omega \in S$ we define a function $\phi_{\omega,c}$ on $C$ with values in complex-valued functions on $C$ by

$$\phi_{\omega,c} = \sum_i [a_i](c)[a_i]$$

where $\{a_i\}$ is an orthonormal basis of $A_\omega$. We can consider $\phi_{\omega,c}$ as a distribution on $\mathcal{H}$ where

$$\phi_{\omega,c}(f) = \int_C \phi_{\omega,c} I_c(f)$$
For a regular elliptic conjugacy class $c \subset G$ we define a functional $\alpha_c$ on $\mathcal{H}$ by

$$\alpha_c(f) := I_c(f) - \sum_{\omega \in S} <[f], \phi_\omega(c)> .$$

It is clear that for any $f \in S(G)$ almost all summands of the sum $\sum_{\omega \in S} <[f], \phi_\omega(c)>$ vanish.

If $\text{char}(F) = 0$ then it follows from [HC99] that $A(G) \subset S(G)$ can be described as the subspace of functions $f$ such that $tr(\pi(f)) = 0$ for all representations $\pi$ induced from an irreducible representations of a proper Levi subgroup of $G$. Therefore (see Theorem B in [K1]) one can express the functional $\alpha_c$ in terms of traces of representations induced from an irreducible representations of a proper Levi subgroup of $G$.

It would be interesting to find such an expression.

**Appendix A. On homogeneity for characters of $GL(n)$**

Stephen DeBacker

The following homogeneity result for $GL(n, F)$, which is a refinement of the Harish-Chandra–Howe local character expansion [HC99, LO74], is known to hold when the residue characteristic of $F$ is sufficiently large [D02, W93].

Let $g$ denote the Lie algebra of $GL(n, F)$, let $O(0)$ denote the set of nilpotent orbits in $g$, and for $O \in O$ let $\hat{\mu}_O$ denote the function which represents the Fourier transform of the nilpotent orbital integral $\mu_O$.

**Theorem A.1.** Suppose $(\pi, V)$ is an irreducible smooth representation of $GL(n, F)$ of depth $\rho(\pi)$. If $\chi_\pi$ denotes the character of $\pi$, then there exist complex constants $c_O(\pi)$, indexed by $O \in O(0)$, such that

$$\chi_\pi(1 + X) = \sum_{O \in O(0)} c_O(\pi) \hat{\mu}_O(X)$$

for all regular semisimple $X \in g_{\rho(\pi)^+}$.

In this appendix we (a) explain the notation that occurs in Theorem A.1 and its proof; (b) state a conjecture whose validity would imply Theorem A.1 for $GL(n, F)$ independent of the residue characteristic of $F$; and (c) prove this conjecture when $n = 2$.

**Notation.** Recall that $O$ denotes the ring of integers of $F$, and $\varpi$ denotes a uniformizer so that $\mathcal{P} = \varpi O$ where $\mathcal{P}$ is the prime ideal. We define $\mathcal{P}^m = \varpi^m \cdot O$ for $m \in \mathbb{Z}$. We fix an additive character $\Lambda$ of $F$ that is trivial on $\mathcal{P}$ and not trivial on $O$.

We realize $GL(n, F)$ as the group of $n \times n$ matrices with entries in $F$ having nonzero determinant. We let $A$ denote the subgroup consisting of diagonal matrices in $GL(n, F)$.

We realize $g$, the Lie algebra of $GL(n, F)$, as the algebra of $n \times n$ matrices with entries in the field $F$ with the usual bracket operation. The set of nilpotent matrices in $g$ is denoted by $N$. The group $GL(n, F)$ acts on $N$, and $O(0)$ denotes the corresponding finite set of nilpotent orbits.

For $i \in \mathbb{Z}$, we define the standard filtration lattices $\mathfrak{k}_i = \varpi^i \cdot M_n(O)$ of $g$ and the Iwahori filtration lattices $\mathfrak{b}_{i/n} = \mathfrak{b}_{i/n} = \{ Y \in g \mid Y_{jk} \in \varpi^{\lceil \frac{i-k}{n} \rceil} \cdot O \}$. Note that for all integers $i, j,$
we have $\omega^j \cdot \xi_i = \xi_{i+j}$ and $\omega^j \cdot b_{i/n} = b_{j+1}$. More concretely, for $n = 2$ we have
\[
\xi_1 = \left( \begin{array}{c} p \\ p \\ p \end{array} \right)
\]
and
\[
b_0 = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \supset b_{1/2} = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \supset b_1 = \omega \cdot b_0.
\]

Let $B$ denote the reduced Bruhat-Tits building of $\text{GL}(n, F)$ and $A \subset B$ the apartment corresponding to $A$. Let $C_0$ be an alcove in $A$. The group $\text{GL}(n, F)$ acts on $B$, and the orbit of every point in $B$ intersects the closure of $C_0$ at least once. Moy and Prasad [MP94, MP96] associated to each $x \in B$ and $r \in \mathbb{R}$ a lattice $g_{x,r}$ in $\mathfrak{g}$ and when $r \geq 0$ a compact open subgroup $G_{x,r}$ of $\text{GL}(n, F)$. For $\text{GL}(n, F)$ we have $G_{x,0} = g_{x,0}^\times$ and $G_{x,r} = 1 + g_{x,r}$ for $r > 0$. Moy and Prasad define the property that $\omega g_{x,r} = g_{x,r+1}$ and $g_{x,s} \subset g_{x,r}$ for $s > r$.

Since $g \cdot g_{x,r} = g_{x,r}$, it is enough to understand the Moy-Prasad lattices for $x$ in the closure of $C_0$. and we do this for $\text{GL}(2, F)$ in Figure 1. Here the apartment $A$ is identified with the horizontal axis and the vertical axis measures $r$. The chamber $C_0$ has end points $x_0$ and $x'$.

The plane has been divided into polygonal regions by dotted lines and each polygonal region has been labeled by a lattice. If $(x, r)$ lies in the interior of one of these polygonal regions, then $g_{x,r}$ is the lattice so labeled. If $(x, r)$ lies on a dotted line, then $g_{x,r}$ is given by the label of the polygonal region directly above the point $(x, r)$. Moy and Prasad define
\[
g_{x,r} = \bigcup_{s > r} g_{x,s}.
\]

Note that $g_{x, r+} \subset g_{x,r}$. In Figure 1 we have $g_{x,r} = g_{x,r}$ unless $(x, r)$ lies on a dotted line, in which case $g_{x,r+}$ is given by the label of the polygonal region directly below the point $(x, r)$. Similar notation is used for the Moy-Prasad subgroups.

For $r \in \mathbb{R}$ we define
\[
g_r = \bigcup_{x \in B} g_{x,r} \quad \text{and} \quad g_{r+} = \bigcup_{x \in B} g_{x,r+}.
\]
We have $g_{r+} \subset g_r$ and $g_r \neq g_{r+}$ if and only if $n \cdot r \in \mathbb{Z}$. From [AD02] we have
\[
g_{r+} = \bigcap_{x \in B} (g_{x,r+} + \mathcal{N})
\]
Note that $N \subset g_s$ for all $s \in \mathbb{R}$ and $g = \bigcup_s g_s$.

For $f \in \mathcal{S}(\mathfrak{g})$, the space of compactly supported, complex valued, locally constant functions on $\mathfrak{g}$, we define $\hat{f}$, the Fourier transform of $f$, by the formula
\[
\hat{f}(X) = \int_{\mathfrak{g}} f(Y) \cdot \Lambda(\text{tr}(X \cdot Y)) dY
\]
for $X$ in $\mathfrak{g}$. Here $dY$ is a fixed Haar measure on $\mathfrak{g}$.

If $L$ is a lattice in $\mathfrak{g}$, let $C_c(\mathfrak{g}/L)$ be the subspace of $\mathcal{S}(\mathfrak{g})$ consisting of functions that are locally constant with respect to $L$. If $L$ and $L'$ are lattices in $\mathfrak{g}$ with $L' \subset L$, then $C(L/L')$ denotes the subspace of $\mathcal{S}(\mathfrak{g})$ consisting of functions supported in $L$ and locally constant with respect to $L'$. Set
\[
D_{r+} = \sum_{x \in B} C_c(\mathfrak{g}/g_{x,r+}).
\]
From [AD02] we have $D_{r+} = \widehat{S(g_{-r})}$.

We denote by $J(g)$ the space of invariant distributions on $g$. For example, if $O \in O(0)$, then $\mu_O$, the corresponding orbital integral, lies in $J(g)$. If $\omega$ is a closed, $G$-invariant subset of $g$ (for example, $N$ or $g_{r+}$), then $J(\omega)$ denotes the subspace of $J(g)$ consisting of invariant distributions with support in $\omega$. If $\omega$ is compactly generated and $T \in J(\omega)$, then [HC99, Hu97] the distribution $\hat{T}$ defined by $\hat{T}(f) = T(\hat{f})$ for $f \in S(g)$ is represented by a locally integrable function, which is also denoted $\hat{T}$, on the set of regular semisimple elements in $g$.

A conjecture. Fix an irreducible smooth representation $(\pi, V)$ of $GL(n, F)$. The depth of $\pi$, denoted by $\rho(\pi)$, is the smallest non-negative real number for which there exists $x \in B$ so that $V$ has non-trivial fixed vectors with respect to $G_{x, \rho(\pi)+}$. Choose $r$ such that $g_r \neq g_{r+} = g_{-\rho(\pi)}$; such an $r$ must be of the form $k/n$ with $k \in \mathbb{Z}$.

For $x \in B$ and $s \leq r$, define

$$\tilde{J}_{x,s,r+} = \{ T \in J(g) \mid f \in C(g_{x,s}/g_{x,r+}), if g_{s+r} \cap \text{supp}(f) = \emptyset, then T(f) = 0 \}.$$ 

Since $N \subset g_{r+}$, every invariant distribution supported on the set of nilpotent elements belongs to $\tilde{J}_{x,s,r+}$. Set

$$\tilde{J}_r = \bigcap_{x \in B} \bigcap_{s \leq r} \tilde{J}_{x,s,r+}.$$ 

Note that $J(N) \subset \tilde{J}_{r+}$.

For $T \in J(g)$ denote by $\text{res}_{D_{r+}} T$ the restriction of $T$ to the space of functions $D_{r+}$. It is shown in [AD02, §§3.1–3.5] that Theorem A.1 follows from the following conjecture.

Conjecture A.2. We have

$$\text{res}_{D_{r+}} \tilde{J}_{r+} = \text{res}_{D_{r+}} J(N).$$

For $z \in F^\times$, $f \in S(g)$, and $T \in J(g)$, define $T_z(f) = T(fz)$ and $f_z(X) = f(zX)$ for $X \in g$. If $z$ has valuation $v$, then we have: (i) $T \in \tilde{J}_{r+}$ if and only if $T_z \in \tilde{J}_{(r+v)+}$; (ii) $f \in D_{r+}$ if and only if $f_z \in D_{(r-v)+}$; and (iii) $T' \in J(N)$ if and only if $T'_z \in J(N)$. Thus, since $T(f) = T_z(f_{z^{-1}})$, it is enough to verify Conjecture A.2 for $r \in \{ k/n \mid 0 \leq k < n \}$.

A proof for $GL(2, F)$. Thanks to the remarks at the end of the previous section, we only need to verify two statements:

(A.2) $\text{res}_{D_{0+}} \tilde{J}_{0+} = \text{res}_{D_{0+}} J(N)$

and

(A.3) $\text{res}_{D_{1/2+}} \tilde{J}_{1/2+} = \text{res}_{D_{1/2+}} J(N)$.

We will prove Statement (A.3). A proof of Statement (A.2) may be carried out in a similar fashion (see also [D04]).
Descent and recovery. Fix $T \in \tilde{J}(g_{1/2^+})$. The goal of this section is to show that $\text{res}_{D_{1/2^+}} T$ is completely determined by $\text{res}_{C(b_{1/2}/b_1)} T$, where $b_{1/2}$ and $b_1$ are Iwahori filtration lattices.

Fix $f \in D_{1/2^+}$. We write $f = \sum_i f_i$ with $f_i \in C_c(\overline{g}/g_{x_i,1/2^+})$ for some $x_i \in \mathcal{B}$. Since $T$ is linear, without loss of generality we may assume that $f \in C_c(\overline{g}/g_{x,1/2^+})$ for some $x \in \mathcal{B}$. We can write

$$f = \sum_{Z \in b \cap g_{x,1/2^+}} c_Z \cdot [Z + g_{x,1/2^+}]$$

where $[Z + g_{x,1/2^+}]$ denotes the characteristic function of the coset $Z + g_{x,1/2^+}$ and all but finitely many of the complex constants $c_Z$ are equal to zero. Again, since $T$ is linear, without loss of generality we may assume that $f = [Z + g_{x,1/2^+}]$.

Choose $s \leq 1/2$ with the property that $Z + g_{x,1/2^+} \subset g_{x,s} \setminus g_{x,s^+}$. By the definition of $\tilde{J}_r^+$ and Property [A.1] we have $T(f) = 0$ if the support of $f$ does not intersect $g_{x,s^+} \setminus \mathcal{N}$. That is, $T(f) = 0$ unless

$$(Z + g_{x,s^+}) \cap \mathcal{N} = \emptyset.$$ 

So, without loss of generality, we may assume $Z = X + Y$ with $X \in \mathcal{N} \cap (g_{x,s} \setminus g_{x,s^+})$ and $Y \in g_{x,s^+}$.

Up to conjugacy, we have two choices for $g_{x,1/2^+}$; it is either $\mathfrak{t}_1$ or $b_1$. In what follows, the reader is encouraged to consult Figure [I].

We first examine the $g_{x,1/2^+} = b_1$ case. In this case, we are looking at the coset $X + Y + g_{y,1/2^+}$ where $y$ is the barycenter of $C_0$, $X \in \mathcal{N} \cap (g_{y,s} \setminus g_{y,s^+})$, and $Z \in g_{x,s^+}$. Since $\mathcal{N} \cap (g_{y,s} \setminus g_{y,s^+})$ is empty unless $s = -m + 1/2$ for $m \in \mathbb{Z}_{\geq 0}$, we may assume $s$ has this form.

Since we are trying to show that $\text{res}_{D_{1/2^+}} T$ is completely determined by $\text{res}_{C(b_{1/2}/b_1)} T$, we may assume $m \geq 1$. Since $T$ is a $G$-invariant distribution, after conjugating by $\text{stab}_{\text{GL}(2,F)}(y) = \langle ((0 \ 1) \ 0) \rangle \times \overline{b_0^s}$ we may assume that $X$ is

$$\begin{pmatrix} 0 & 0 \\ \overline{\omega}(-m)u & 0 \end{pmatrix}$$

with $m > 0$ and $u \in \mathcal{O}^\times$. Let $A_m = A \cap G_{x_0,m}$. We write

$$T([X + Y + b_1]) = \frac{1}{q^2} \cdot \sum_{t \in A_m/A_{m+1}} T([t(X + Y)t^{-1} + b_1])$$

$$= \frac{1}{q^2} \cdot \sum_{\alpha, \beta \in \mathcal{O}/\mathcal{P}} T([X + Y + \left(\frac{0}{\omega u(\alpha - \beta)} 0\right) + b_1])$$

$$= \frac{1}{q} \cdot T([X + Y + \mathfrak{t}_1]).$$

Note that $X + Y + \mathfrak{t}_1 \subset \overline{\omega^{-m}} \mathfrak{t}_1 = g_{x_0,s^+}$. Thus, we have expressed $T$ evaluated at $f = [X + Y + b_1]$ in terms of $T$ evaluated at $f' = \frac{1}{q} \cdot [X + Y + \mathfrak{t}_1]$ where $f' \in D_{1/2^+}$ has support closer to the origin with respect to the $x_0$ filtration than $f$ had with respect to the $y$ filtration.

We now examine the $g_{x,1/2^+} = \mathfrak{t}_1$ case. We may suppose that $s = -m$ for some $m \in \mathbb{Z}_{\geq 0}$, so that $X \in \mathcal{N} \cap (g_{x_0,-m} \setminus g_{x_0,(-m)^+})$ and $Y \in g_{x_0,(-m)^+}$. Since $T$ is $G$-invariant, after
conjugating by $\xi_0^x$ we may assume that $X$ is

\[
\begin{pmatrix}
0 & \omega^{-m}v \\
0 & 0
\end{pmatrix}
\]
with $m \geq 0$ and $u \in \mathcal{O}^\times$. We then have

$$T([X + Y + \mathfrak{k}_1]) = \sum_{\delta \in \mathcal{P}/\mathcal{P}^2} T([X + Y + (0 \ 0 \ \gamma) + \mathfrak{b}_1])$$

Note that $X + Y + (0 \ 0 \ \gamma) \in \mathfrak{g}_{y,s}$. Thus, we have expressed $T$ evaluated at $f = [X + Y + \mathfrak{k}_1]$ in terms of $T$ evaluated at

$$f' = \sum_{\delta \in \mathcal{P}/\mathcal{P}^2} [X + Y + (0 \ 0 \ \gamma) + \mathfrak{b}_1]$$

where $f' \in D_{0+}$ has support closer to the origin with respect to the $y$ filtration than $f$ had with respect to the $x_0$ filtration.

To summarize, the point of descent and recovery is as follows. We begin with a simple function $f \in C((\mathfrak{g}_{x,s} \setminus \mathfrak{g}_{x,1/2^+})/\mathfrak{g}_{x,1/2^+})$ for some $x \in \mathcal{B}$. From this function, we find a point $y \in \mathcal{B}$ and a function $f' \in C(\mathfrak{g}_{y,s}/\mathfrak{g}_{y,1/2^+})$ so that $T(f) = T(f')$. After a finite number of steps, we will have shown that $T(f)$ is completely determined by $\text{res}_{C(b_{1/2}/b_1)} T$.

**Counting.** From [HC99] we know that the dimension of the complex vector space $\text{res}_{D_{0+}} J(N)$ is equal to the number of nilpotent orbits. Since $J(N) \subset J_{1/2^+}$, we have

$$2 = \dim_{\mathbb{C}} \text{res}_{D_{0+}} J(N) \leq \dim_{\mathbb{C}} \text{res}_{D_{0+}} \tilde{J}_{1/2^+}.$$ 

From our work above we have

$$\dim_{\mathbb{C}} \text{res}_{D_{0+}} \tilde{J}_{1/2^+} = \dim_{\mathbb{C}} \text{res}_{C(b_{1/2}/b_1)} \tilde{J}_{1/2^+}.$$ 

Consequently, we need only show that $\dim_{\mathbb{C}} \text{res}_{C(b_{1/2}/b_1)} \tilde{J}_{1/2^+} \leq 2$. Since $\mathfrak{g}_{1/2^+} \subset \mathfrak{g}_{x,1/2^+} + \mathcal{N}$ for any $x \in \mathcal{B}$, we have that for $T \in \tilde{J}_{1/2^+}$ the restriction of $T$ to $C(b_{1/2}/b_1)$ is completely determined by

$$T([(0 \ 1 \ \gamma) + \mathfrak{b}_1]) \text{ and } T([(\mathfrak{b}_1)]).$$

**References**

[AD92] Adler and S. DeBacker, *Some applications of Bruhat-Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group*, Mich. Math. J., 50 (2002), no. 2, pp. 263–286.

[AJ96] Aizenbud, Avraham; Avni, Nir: *Representation growth and rational singularities of the moduli space of local systems*, Invent. Math. 204 (2016), no. 1, 245–316.

[AP] A. Aubert and R. Plymen *Plancherel measure for GL(n, F) and GL(m, D): explicit formulas and Bernstein decomposition*, Journal of Number Theory 112 (2005), no. 1, 26–66.

[BDK86] J. Bernstein, P. Deligne, D. Kazhdan, *Trace Paley-Wiener theorem for reductive p-adic groups*. J. Analyse Math. 47 (1986), 180-192.

[BZ76] I. N. Bernstein, A. V. Zelevinski, *Representations of the group GL(n, F), where F is a local non-Archimedean field*. (Russian) Uspehi Mat. Nauk 31 (1976), no. 3 (189), 5-70.

[Be92] J. Bernstein, *Representations of p-adic groups*. Lectures at Harvard University, Fall 1992. Notes by Karl E. Rumelhart.

[C72] W. Casselman, *The Steinberg character as a true character*. Harmonic analysis on homogeneous spaces. Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., (1972), 413–417. Amer. Math. Soc., Providence, R.I., 1973.

[D02] *Homogeneity results for invariant distributions of a reductive p-adic group*, Ann. Sci. École Norm. Sup., 35 (2002), no. 3, pp. 391–422.
[D04] S. DeBacker, *Lectures on harmonic analysis for reductive p-adic groups*, Representations of real and p-adic groups, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 2, Singapore Univ. Press, Singapore, 2004, pp. 47–94

[GGP] W. Gelfand, I. M.; Graev, M. I.; Pyatetskii-Shapiro, I. I. *Generalized functions. Vol. 6. Representation theory and automorphic functions.*

[HC70] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*. Notes by G. van Dijk. Lecture Notes in Mathematics 162. Springer-Verlag, Berlin-New York, 1970. iv+125 pp.

[HC99] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Preface and notes by Stephen DeBacker and Paul J. Sally, Jr., University Lecture Series, 16, American Mathematical Society, Providence, RI, 1999.

[Ho74] R. Howe, *The Fourier transform and germs of characters (case of $\text{GL}_n$ over a p-adic field)*, Math. Ann. 208 (1974), pp. 305–322.

[Hu97] R. Huntsinger, *Some aspects of invariant harmonic analysis on the Lie algebra of a reductive p-adic group*, Ph.D. Thesis, The University of Chicago, 1997.

[JL70] H. Jacquet, R. P. Langlands, *Automorphic forms on GL(2)*. Lecture Notes in Mathematics 114. Springer-Verlag, Berlin-New York, 1970.

[Ka] D. Kazhdan, *On Shalika germs*. Selecta Mathematica 22 (2016), no. 4, 1821–1824.

[K] D. Kazhdan, *Cuspidal geometry of $p$-adic groups*. J. Analyse Math. 47 (1986), 136.

[Sh72] J. Shalika, *A theorem on semi-simple $\mathfrak{p}$-adic groups*. Ann. of Math. 95 (1972), 226-242.

[SS84] P. Sally, J. Shalika, *The Fourier transform of orbital integrals on $\text{SL}(2)$ over a p-adic field*. Lie group representations, II (College Park, Md., 1982/1983), 303-340, Lecture Notes in Math., 1041, Springer, Berlin, 1984.

[W93] Waldspurger, *Quelques resultats de finitude concernant les distributions invariantes sur les algebres de Lie $p$-adiques*, preprint, 1993.