The inversion formula of polylogarithms and the Riemann-Hilbert problem

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Abstract

In this article, we set up a method of reconstructing the polylogarithms $\text{Li}_k(z)$ from zeta values $\zeta(k)$ via the Riemann-Hilbert problem. This is referred to as “a recursive Riemann-Hilbert problem of additive type.” Moreover, we suggest a framework of interpreting the connection problem of the Knizhnik-Zamolodchikov equation of one variable as a Riemann-Hilbert problem.

1 Introduction

Polylogarithms $\text{Li}_k(z)$ ($k \geq 2$) satisfy the inversion formula

$$\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \text{Li}_{2,1,...,1}(1-z) = \zeta(k).$$

Applying the Riemann-Hilbert problem of additive type (alternatively, Plemelj-Birkhoff decomposition) \cite{BiMuPl} to this inversion formula, we show that $\text{Li}_k(z)$ can be reconstructed from boundary values $\zeta(k)$. We prove this by using the Riemann-Hilbert problem recursively so that we refer to this method as a recursive Riemann-Hilbert problem of additive type.

As a generalization of this method, we can reconstruct multiple polylogarithms $\text{Li}_{k_1,...,k_r}(z)$ from multiple zeta values $\zeta(k_1,\ldots,k_r)$. This is nothing but interpreting the connection relation \cite{OiU}

$$\mathcal{L}(z) = \mathcal{L}^{(1)}(z) \Phi_{\text{KZ}}$$

between the fundamental solutions of the Knizhnik-Zamolodchikov equation of one variable (KZ equation, for short)

$$\frac{dG}{dz} = \left( \frac{X_0}{z} + \frac{X_1}{1-z} \right) G$$

as a Riemann-Hilbert problem. Here $\Phi_{\text{KZ}}$ is Drinfel’d associator and $\mathcal{L}(z)$ (resp. $\mathcal{L}^{(1)}(z)$) is the fundamental solution of KZ equation normalized at $z = 0$ (resp. $z = 1$). We have completely solved this problem and a preprint is now in preparation.

2010 Mathematics Subject Classification. Primary 34M50,11G55; Secondary 30E25,11M06,32G34;
Acknowledgment  The first author is supported by Waseda University Grant for Special Research Projects No. 2011B-095. The second author is partially supported by JSPS Grant-in-Aid No. 22540035.

2 The inversion formula of polylogarithms

For positive integers \( k \), polylogarithms \( \text{Li}_k(z) \) are introduced as follows: First we set \( \text{Li}_1(z) = -\log(1-z) \). In the domain \( D = \mathbb{C} \setminus \{ z = x \mid 1 \leq x \} \), \( \text{Li}_1(z) \) has a branch such that \( \text{Li}_1(0) = 1 \) (the principal value of \( \text{Li}_1(z) \)). Starting from the principal value of \( \text{Li}_1(z) \), we introduce \( \text{Li}_k(z) \), which are holomorphic on \( D \), recursively by

\[
\text{Li}_k(z) = \int_0^z \frac{\text{Li}_{k-1}(t)}{t} \, dt \quad (k \geq 2).
\]

where the integral contour is assumed to be in \( D \). Then \( \text{Li}_k(z) \) has a Taylor expansion

\[
\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}
\]

on \( |z| < 1 \). We obtain, for \( k \geq 2 \),

\[
\lim_{z \rightarrow 1, z \in D} \text{Li}_k(z) = \zeta(k),
\]

where \( \zeta(k) \) is the Riemann zeta value \( \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \).

From (1), we have differential recursive relations:

\[
\frac{d}{dz} \text{Li}_1(z) = \frac{1}{1-z}, \quad \frac{d}{dz} \text{Li}_k(z) = \frac{\text{Li}_{k-1}(z)}{z} \quad (k \geq 2).
\]

By virtue of (1), \( \text{Li}_k(z) \) is analytically continued to a many-valued analytic function on \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \). However, in this article, we will use the notation \( \text{Li}_k(z) \) as the principal value stated previously.

We also define multiple polylogarithms \( \text{Li}_{2,1,\ldots,1}(z) \) \( (k \geq 2) \) as

\[
\text{Li}_{2,1,\ldots,1}(z) = \int_0^z \frac{(-1)^{k-1} \log^{k-1}(1-t)}{(k-1)!} \frac{t}{1-t} \, dt.
\]

By using these relations and (3), one can obtain easily the inversion formula of polylogarithms.

Proposition 1 (the inversion formula of polylogarithms). For \( k \geq 2 \), the following functional relation holds.

\[
\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \text{Li}_{2,1,\ldots,1}(1-z) = \zeta(k).
\]
Theorem 2. Put \( f_1^{(+)}(z) = \text{Li}_1(z) \). For \( k \geq 2 \), we assume that \( f_k^{(\pm)}(z) \) are holomorphic functions on \( D^{(\pm)} \) satisfying the functional relation
\[
 f_k^{(+)}(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} f_{k-j}^{(+)}(z) + f_k^{(-)}(z) = \zeta(k) \quad (z \in D^{(+)} \cap D^{(-)}),
\]
the asymptotic conditions
\[
 \frac{d}{dz} f_k^{(\pm)}(z) \to 0 \quad (z \to \infty, \ z \in D^{(\pm)}),
\]
and the normalization condition
\[
 f_k^{(+)}(0) = 0.
\]
Then we have
\[ f_k^+(z) = \text{Li}_k(z), \quad f_k^-(z) = \underbrace{\text{Li}_{2, \ldots, 1}(1 - z)}_{k-2} \quad (k \geq 2). \]

**Proof.** We prove the theorem by induction on \( k \geq 2 \). For the case \( k = 2 \), the proof can be done in the same manner as the case \( k > 2 \) from the definition of \( f_j^{(1)}(z) \). So we assume that \( f_j^{(1)}(z) = \text{Li}_j(z) \) and \( f_j^{(-)}(z) = \underbrace{\text{Li}_{2, \ldots, 1}(1 - z)}_{k-2} \) for \( 2 \leq j \leq k - 1 \). Now we show that \( f_k^+(z) = \text{Li}_k(z), f_k^-(z) = \underbrace{\text{Li}_{2, \ldots, 1}(1 - z)}_{k-2} \).

From the assumption, the equation (7) becomes
\[ f_k^+(z) + \sum_{j=1}^{k-1} (-1)^j \log^j z \text{Li}_{k-j}(z) + f_k^-(z) = \zeta(k). \] (10)

Differentiating this equation, we have
\[
0 = \frac{d}{dz} \left( f_k^+(z) + \sum_{j=1}^{k-1} (-1)^j \log^j z \text{Li}_{k-j}(z) + f_k^-(z) \right)
= \frac{d}{dz} f_k^+(z) + \sum_{j=1}^{k-2} \left( \frac{(-1)^j \log^{j-1} z}{j!} \text{Li}_{k-j}(z) + \frac{(-1)^j \log^j z \text{Li}_{k-j-1}(z)}{j!} \right)
+ \frac{1}{z} \frac{(-1)^{k-1} \log^{k-2} z}{(k-2)!} \text{Li}_1(z) + \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} \frac{1}{1-z}
+ \frac{d}{dz} f_k^-(z)
= \frac{d}{dz} f_k^+(z) - \frac{\text{Li}_{k-1}(z)}{z} + \frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} + \frac{d}{dz} f_k^-(z).
\]

Thus we obtain
\[
\frac{d}{dz} f_k^+(z) - \frac{\text{Li}_{k-1}(z)}{z} = -\frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} - \frac{d}{dz} f_k^-(z) \quad (11)
\] on \( z \in D^+ \cap D^- \). Here, the left hand side of (11) is holomorphic on \( D^+ \) and the right hand side of (11) is holomorphic on \( D^- \). Therefore both sides of (11) are entire functions. Using the asymptotic condition (8) and
\[ \frac{\text{Li}_{k-1}(z)}{z} \rightarrow 0 \quad (z \rightarrow \infty, z \in D^+), \quad \frac{\log^{k-1} z}{1-z} \rightarrow 0 \quad (z \rightarrow \infty, z \in D^-), \]
we have that both sides of (11) are 0 by virtue of Liouville’s theorem. Therefore we have
\[
f_k^+(z) = \int_2^z \frac{\text{Li}_{k-1}(z)}{z} dz = \text{Li}_k(z) + c_k^+, \quad f_k^-(z) = \int_{-\infty}^z \frac{1}{1-z} \frac{(-1)^{k-1} \log^{k-1} z}{(k-1)!} dz = \underbrace{\text{Li}_{2, \ldots, 1}(1 - z)}_{k-2} + c_k^-.
\]
where $c_k^{(+)}$, $c_k^{(-)}$ are integral constants. From the normalization condition (9), it is clear that $c_k^{(+)}$ is equal to 0. Finally, substituting $f_k^{(+)}(z)$ and $f_k^{(-)}(z)$ in (7), we obtain

$$\text{Li}_k(z) + \sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z) + \text{Li}_{2,\ldots,1}(1-z) - c_k^{(-)} = \zeta(k). \quad (12)$$

Comparing the inversion formula (10), we have $c_k^{(-)} = 0$. This concludes the proof.

The equation (10) is interpreted as the decomposition of the holomorphic function

$$\sum_{j=1}^{k-1} \frac{(-1)^j \log^j z}{j!} \text{Li}_{k-j}(z)$$

on $z \in D^{(+)} \cap D^{(-)}$ to a sum of a function $f_k^{(+)}(z)$, which is holomorphic on $D^{(+)}$, and a function $f_k^{(-)}(z)$, which is holomorphic on $D^{(-)}$. This decomposition is nothing but a Riemann-Hilbert problem of additive type. The theorem says that polylogarithms $\text{Li}_k(z)$ can be constructed from the boundary value $\zeta(k)$ by applying this Riemann-Hilbert problem recursively. In this sense, we call (7) the recursive Riemann-Hilbert problem of additive type.

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