On Local Rank, Joinings and Asymptotic Properties of Measure-Preserving Actions

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Abstract
The note contains a collection of facts and observations around locally rank one actions as well as constructions connected with some results by T. Downarowicz, A.Katok, J.King, F.Parreau, A.A. Prikhodko, E.Roy, J.Serafin, J-P.Thouvenot et al.

1 Definitions. Results.

In this note we are speaking about connections between rank properties, self-joinings and certain asymptotic properties of measure-preserving actions. Using the term “action” instead of “transformation” we hint at the fact that most of our results can be naturally generalized to $\mathbb{Z}^n$- and $\mathbb{R}^n$-actions. Thus, let us recall necessary definitions for $\mathbb{Z}$-actions only.

Local rank. Let $T$ be an automorphism of a measure space $(X, \mu)$, where $\mu(X) = 1$. We denote by $\beta(T)$ its local rank. It is defined as maximal number $\beta$ such that there exists a sequence of finite partitions of the form

$$\xi_j = \{B_j, TB_j, T^2B_j, \ldots, T^{h_j-1}B_j, C_{j1}, \ldots, C_{jm_j} \ldots\}$$

for which $\mu(U_j) \to \beta$ and any measurable set can be approximated by $\xi_j$-measurable ones as $j \to \infty$. The sequence of sets

$$U_j = \bigsqcup_{0 \leq k < h_j} T^kB_j$$

is called a sequence of $\beta$-towers. One writes $\xi_j \to \varepsilon$ and call the sequence of tower partitions $\{B_j, TB_j, T^2B_j, \ldots, T^{h_j-1}B_j\}$ approximating.

Joinings. A self-joining (of order 2) is defined to be a $T \times T$-invariant measure $\nu$ on $X \times X$ with the marginals equal to $\mu$:

$$\nu(A \times X) = \nu(X \times A) = \mu(A).$$
A joining $\nu$ is called ergodic if the dynamical system $(T \times T, X \times X, \nu)$ is ergodic. The measures $\Delta^i = (Id \times T^i)\Delta$ defined by the formula

$$\Delta^i(A \times B) = \mu(A \cap T^i B)$$

are referred to as off-diagonals measures (for $i \neq 0$). If $T$ is ergodic, then $\Delta^i$ are ergodic self-joinings. We say that $T$ has minimal self-joinings of order 2 (and we write $T \in MSJ(2)$) if $T$ has no ergodic joinings except $\mu \otimes \mu = \mu \times \mu$ and $\Delta^i$.

**Partial mixing.** An automorphism $T$ is called partially mixing, if for some $\alpha \in (0, 1]$ and for all measurable sets $A, B$

$$\lim \inf_i \mu(A \cap T^i B) \geq \alpha \mu(A) \mu(B).$$

The maximal value of $\alpha$ satisfying this property for a given $T$ is denoted as $\alpha(T)$, and $T$ is called mixing if $\alpha(T) = 1$.

**Mildly mixing.** We say that an automorphism $T$ is mildly mixing if for any set $A$ of positive measure

$$\lim \sup_i \mu(A \cap T^i A) < \mu(A).$$

Actually, $T$ is mildly mixing if it has no rigid factors (except trivial one). Let us recall that a factor is a restriction of our action onto some invariant $\sigma$-algebra.

**Partially rigidity.** An automorphism $T$ is said to be partially rigid if for all measurable $A, B$ for some $\rho \in (0, 1]$

$$\lim \sup_i \mu(A \cap T^i A) \geq \rho \mu(A).$$

The maximal $\rho$ is denoted by $\rho(T)$ ( $T$ is called rigid as $\rho(T) = 1$).

**RESULTS.** In this note we present the following results.

1. Calculations of local rank. For any $b \in (0, 1)$ there exists a partially mixing action $\Phi$ of local rank $\beta(\Phi) = b$;

**Remark.** Let us recall A.Katok’s result: There exists a weakly mixing $T$ such that the local rank $\beta(T \otimes T) \geq \frac{1}{4}$. This property of $T$ is proved to be generic [4]. It can be shown that $\beta(T \otimes T) \leq \frac{1}{4}$ and $\text{Rank}(T \otimes T) = \infty$ (see [10]). A similar approach to getting a positive local rank works well for a Cartesian product of two different actions. In paper [5] J.King and J-P.Thouvenot state as a remark the following assertion: Given $a > 0$ there are automorphisms $S_1, S_2$ such that $T = S_1 \times S_2$ is partially mixing, and $\beta(T) > 1 - a$.  

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We propose a mildly mixing transformation close to rank one and far from MSJ-class. For any \( \varepsilon > 0 \) in the class \( \{ T : \beta(T) > 1 - \varepsilon \} \) there is a mildly mixing transformation with uncountable centralizer and uncountable system of factors.

We provide an infinite system of factors for a partially mixing action with the positive local rank via a modified Katok’s construction. For any \( b \in (0, \frac{1}{4}) \) we find a partially mixing action \( \Phi \otimes \Phi \) of local rank \( b \).

(3) A generalization of King-Thouvenot’s theorem to \( \mathbb{Z}^n \)-actions with \( n > 1 \). Given a \( \mathbb{Z}^n \)-action \( \Phi \) satisfying property \( \alpha(\Phi) + \beta(\Phi) > 1 \), we show that \( \Phi \) has the finite-joining-rank property. Its Markov centralizer is generated by combinations of action elements with a fixed finite collection of Markov operators.

For rank one partially mixing \( \mathbb{Z}^2 \)-actions nontrivial self-joinings are discovered by T. Downarowicz and J. Serafin [1].

(4) In contrast to the finite-joining-rank situation and in connection with Proposition 18 of [7], where the authors proved that a non-rigid Poisson suspension is not of rank one, we consider actions of a positive local rank assuming the existence of a sequence of special joinings tending to \( \Delta \). In this case (close to one discussed in the proof of the mentioned result of [7]) we prove the partial rigidity of such actions. This observations, combined with the proof by F. Parreau and E. Roy, gives: non-rigid Poisson suspensions are of zero local rank.

Remark. It’s an open question: do there exist actions \( T \) and \( S \) with the same spectral but different rank properties. If a Poisson suspension \( T_s \) has a positive local rank, then we get a contrast with Gaussian automorphism \( G_T \). They are of the same spectrum identical to \( \exp(\sigma_T) \), but \( G_T \) has zero local rank (T. de la Rue).

(5) Banach’s problem and Rokhlin’s multiple mixing problem are connected within the class of automorphisms having positive local rank. So, there is an additional interest to study of multiple mixing properties of locally rank one systems. In [8] A.A. Prikhodko introduced new examples of iceberg transformations with a positive local rank. Jointly with A.A. Prikhodko we state a positive result: mixing iceberg transformations are mixing of all orders.

The author thanks Alexander Prikhodko for helpful comments.

2 A positive local rank for tensor products of partially mixing systems.

Calculating local rank. We explain how to build a pair of partially mixing automorphisms \( S, \tilde{S} \) such that for \( T = S \times \tilde{S} \) the local rank \( \beta(T) > 1 - a \). Let us
consider two rank one constructions for $S$ and $\tilde{S}$ with height sequences $h_j$ and $\tilde{h}_j = h_j + 1$. Let the corresponding cut sequences $r_j$ and $\tilde{r}_j$ satisfy the condition $r_j, \tilde{r}_j \gg (h_j)^3$. As in the case of spacer sequences assume that our rank one construction have constant (flat) parts and mixing stochastic Ornstein’s parts. This provides

$$\beta(T) = (1 - \varepsilon)(1 - \tilde{\varepsilon}) > 1 - a, \quad \alpha(T) = \varepsilon\tilde{\varepsilon}.$$ 

Indeed, we have:

1. $\alpha(T) = \alpha(S)\alpha(\tilde{S}) = \varepsilon\tilde{\varepsilon}$.
2. $\beta(T) \geq (1 - \varepsilon)(1 - \tilde{\varepsilon})$ (we have a sequence of approximating $\beta$-towers of heights $h_j(h_j + 1)$ and of measures tending to $(1 - \varepsilon)(1 - \tilde{\varepsilon})$).
3. $\beta(T) \leq (1 - \varepsilon)(1 - \tilde{\varepsilon})$.

To prove this we use the following property of a transformation $T$ with $\beta(T) > 0$ (see [10]). If $T$ commutes with $R$, then for any $\delta > 0$ there is an $m > 0$ and a sequence $n(i)$ such that

$$T^{n(i)} \to (\beta(T) - \delta)R^m + \ldots$$

Our notation “…” always means “something positive”. In the above expansion we have a sum of weighted Markov operators containing some “non-$R^m$-part”

$$\varepsilon(1 - \tilde{\varepsilon})\Theta \otimes \tilde{P} + (1 - \varepsilon)\tilde{\varepsilon}P \otimes \Theta + \varepsilon\tilde{\varepsilon}\Theta \otimes \Theta,$$

where $P, \tilde{P}$ are some Markov operators and $\Theta$ is the orthoprojection onto the space of constants in $L_2(X, \mu)$. Setting $R = S \otimes I$, for arbitrary $\delta > 0$ we get

$$\beta(T) - \delta \leq (1 - \varepsilon)(1 - \tilde{\varepsilon}).$$

**Modified Katok’s examples.** For any given $a, b > 0$ with $a + 2b = 1$ there exists an automorphism $T$ such that

$$\alpha(T) = a^2, \quad \beta(T) = b^2,$$

and $T$ possesses an infinite structure of factors.

For this we combine Katok’s construction (giving positive local rank of $S \otimes S$) with Ornstein’s mixing transformation (providing partially mixing). For instance, consider a rank one construction $S$ with the spacer sequence

$$s_j(1), s_j(2), \ldots, s_j(q_j), 0, 0, \ldots, 0, 1, 1, \ldots, 1$$

having $q_j$ zeros and $q_j$ ones with $q_j \gg h_j$ corresponding to the Katok’s construction part and stochastic $s_j(i)$ (Ornstein’s part). Then for almost all rank one constructions of such kind the generated transformation $S$ has the property
\[ \alpha(S \otimes S) = \beta(S \otimes S) = 1/9. \] Now we observe a family of factors \((I \otimes T^n)\mathcal{F}\), where \(\mathcal{F}\) is the symmetric factor-algebra of \(S \otimes S\). We note also that for these constructions one has \(\beta(S \odot S) = 2\beta(S \otimes S)\).

**Mildly mixing, local rank and an abundance of joinings.** In connection with the question due to J.-P. Thouvenot on MSJ property of mildly mixing rank one transformations \((T\text{ is of Rank 1 iff } \beta(T) = 1)\) we remark that a little “rank freedom” could imply a certain abundance of joinings.

**Examples.** Given \(\varepsilon > 0\) there is a mildly mixing automorphisms \(T\) having \(\beta(T) > 1 - \varepsilon\), and possessing both an uncountable centralizer and an uncountable structure of factors.

Let us apply King-Thouvenot’s idea for infinite products. We consider partially mixing automorphisms \(S_i\) such that

\[
\beta(\bigotimes_{i=1}^{\infty} S_i) = \prod_{i=1}^{\infty} (1 - \alpha(S_i)) > 1 - \varepsilon,
\]

\[
\alpha(\bigotimes_{i}^{N} S_i) = \prod_{i}^{N} (\alpha(S_i)).
\]

Since all the finite products \(\otimes_{i}^{N} S_i\) are partially mixing, the infinite product has to be mildly mixing as well. Indeed, if there exists a rigid function \(f\), then all its projections to the finite product are rigid, hence, the projections are constant functions, so \(f\) is constant.

### 3 Joinings of partially mixing actions with positive local rank.

The *off-diagonals* measures \(\Delta^z = (Id \times T^z)\Delta\) are defined by the formula

\[
\Delta^z(A \times B) = \mu(A \cap T^z B).
\]

If an action is ergodic, then \(\Delta^z\) are ergodic self-joinings. \(T\) is called action with *minimal self-joinings of order 2* whenever \(T\) has no ergodic joinings except \(\mu \times \mu\) and \(\Delta^z\).

Let us represent a joining \(\nu\) in the form

\[
\nu(A \times B) = \int_A \nu_x(B) d\mu(x),
\]

where \(\nu_x\) are conditional measures. The relatively independent product \(\nu \otimes_X \nu'\) is defined by the equation

\[
\nu \otimes_X \nu'(A \times B) = \int_X \nu_x(A) \nu'_x(B) d\mu(x).
\]
A reader familiar with Markov operators in $L_2(\mu)$ can see that $\nu \otimes X \nu'$ corresponds to the operator $P^* P$, where $(P\chi_A, \chi_B) = \nu(A \times B)$ (here the Markov operator $P$ corresponds to the polymorphism $\nu$).

**Theorem 3.** If $\alpha(\Phi) + \beta(\Phi) > 1$, then $\mathbb{Z}^n$-action $\Phi$ has finitely many classes of equivalent ergodic joinings ($\nu \sim \nu'$ iff $\nu = (I \otimes T^z) \nu'$), and all ergodic non-trivial self-joinings are the graphs of finite-value maps.

**Remark.** For $n = 1$ King and Thouvenot proved minimal self-joinings [5]. Non-trivial joinings could appear for $n > 1$ [1].

Proof. For simplicity we consider the case when $\Phi$ is a $\mathbb{Z}$-action generated by a single transformation $T$. The proof in the general case is similar (and we assume that rank-towers are rectangle). Let $\nu \not= \mu \otimes^2$ be an ergodic self-joining. Denote

$$\eta := \nu \otimes X \nu.$$

If

$$\eta = a \Delta + ..., \quad a > 0,$$

then $\nu$ is situated on a graph of a finite-value map. We note that the case

$$\eta = a \Delta^z + ..., \quad a > 0, \quad z \not= 0,$$

is impossible. It means that $\nu$ and $(I \otimes T^z) \nu$ are not disjoint, so $\nu = (I \otimes T^z) \nu$. This implies $\nu = \mu \otimes \mu$, since $T^z$ is ergodic.

Our aim is to prove (1). Suppose $\eta$ and $\Delta$ to be disjoint, and expand $\eta$ in sum

$$\eta = c_{max} \mu \otimes^2 + (1 - c_{max}) \tilde{\eta},$$

where the number $c_{max}$ is maximal. We see that

$$\mu \otimes^2 \quad and \quad \tilde{\eta} \quad are \quad disjoint. \quad (2)$$

If $c_{max} = 1$, then $\eta = \mu \times \mu = \nu$. So let $c_{max} < 1$.

Given $\varepsilon > 0$, $0 \leq k \leq \varepsilon h_j$, we define the sets $C^k_j$ (called columns):

$$C^k_j = \bigcup_{i=0}^{h_j-k} T^i T^k E_j \times T^i B_j.$$

For negative $k$ ($-\varepsilon h_j \leq k \leq 0$), we put

$$C^k_j = \bigcup_{i=0}^{h_j+k} T^i B_j \times T^i T^{-k} B_j.$$
Next, we define
\[ D_\varepsilon^j = \bigsqcup_{k=-[\varepsilon h_j]}^{[\varepsilon h_j]} C^k_j, \quad U_\varepsilon^j = \bigsqcup_{k=0}^{[\varepsilon h_j]} T^k B_j, \]
and for \( \varepsilon > 0 \) we have
\[ \eta(D_\varepsilon^j) > \varepsilon^2 \beta(T)^2. \]
To see this we use the facts that
\[ U_\varepsilon^j \times U_\varepsilon^j \subset D_\varepsilon^j, \quad \eta(U_\varepsilon^j \times U_\varepsilon^j) \geq \mu(U_\varepsilon^j)^2, \quad \mu(U_\varepsilon^j) \approx \beta(T)\varepsilon. \]
Given \( \varepsilon > 0 \) we obtain
\[ \tilde{\eta}(D_\varepsilon^j) \geq d(\varepsilon) > 0. \]
For a subsequence \( j' \) denoted again by \( j \) one has
\[ \tilde{\eta}(\cdot \mid D_\varepsilon^j) \to \eta' \ll \tilde{\eta}. \]

Inside the domains \( D_\varepsilon^j \) our joining \( \tilde{\eta} \) is approximated by parts of off-diagonals \( \Delta_j^z = \Delta^z(\mid C_j^z) \) (see [11]):
\[ \lim_j \tilde{\eta}(\mid D_\varepsilon^j) = \lim_j \sum_{|z| < \varepsilon h_j} a_j^z \Delta_j^z, \]
where \( a_j^z = \tilde{\eta}(C_j^z \mid D_j^z) \). Note that \( \sum_{|z| < M} a_j^z \) tends to zero for fixed \( M \) and \( j \to \infty \) (we suppose now that \( \tilde{\eta} \perp \Delta^z \)). If \( z \) is large, then
\[ \Delta^z \approx \alpha(\Phi)\mu^{\otimes 2} + \ldots. \]
\[ \Delta_j^z \approx [\alpha(\Phi) - \varepsilon \beta(\Phi) - (1 - \beta(\Phi))]\mu^{\otimes 2} + \ldots \]
whenever \( |z| < \varepsilon h_j \). So \( \tilde{\eta} \) contains a part
\[ \lim_j \sum_z a_j^z \Delta_j^z = c\mu^{\otimes 2} + \ldots, \]
where \( c \geq \alpha(\Phi) + \beta(\Phi) - 1 - \varepsilon \beta(\Phi) > 0. \)

Hence, \( \mu^{\otimes 2} \) and \( \tilde{\eta} \) are not disjoint, but this contradicts (2)! So our assumption on the disjointness of \( \eta \) and \( \Delta \) is false. We have proved \( \eta \gg \Delta \) (in fact \( \eta = \frac{1}{N} \Delta + \ldots \)), so \( \nu \) is a graph of a finite-value map.

A composition of two finite-value maps is a finite-value map. This implies that a relative product \( \nu \otimes_X \nu' \) of two non-trivial joinings is supported on a graph, hence, \( \nu \otimes_X \nu' \) and \( \mu^{\otimes 2} \) are disjoint.
If we have a collection \( \nu_1, \ldots, \nu_k \neq \mu\otimes^2 \) of ergodic self-joinings, and \( k > \lfloor (\alpha + \beta - 1)^{-1} \rfloor + 1 \), then there exist different \( m, n \) \( (1 \leq m, n \leq k) \) such that
\[
\limsup_j \nu_m \otimes_X \nu_n (U_j^\varepsilon \times U_j^\varepsilon) > 0.
\]
The mentioned approximation shows that if \( \nu_m \otimes_X \nu_n \) has no \( \Delta^z \) as a component, then it must contain as a component \( \mu\otimes^2 \), but this is impossible. Thus,
\[
\nu_m \otimes_X \nu_n = a \Delta^z + \ldots .
\]
This means that \( \nu_m \) and \( \nu_n \) are equivalent.

4 Local Rank and Partial Rigidity

Remark. Consider a locally rank one ergodic \( \mathbb{Z}^n \)-action \( T \). If there is a joining \( \nu \neq \Delta^z \) \( (P \neq I) \) for \( T \) with \( \nu \otimes_X \nu \gg \Delta \) (respectively, \( P^*P = cI + \ldots \)), then \( T \) is partially rigid.

Indeed, if \( T^{z(i)} \rightarrow aP + \ldots, P \neq T^z \), then there is \( w(j) \rightarrow \infty \) such that
\[
T^{w(j)} \rightarrow a^2 P^*P + \ldots = c'I + \ldots, \quad c' > 0.
\]
So our action is partially rigid.

In a recent preprint [7] F. Parreau and E. Roy proposed an interesting approach to the study of rigidity and rank of Poisson suspensions using a certain natural sequence of joinings \( \nu_i \rightarrow \Delta \). Their work stimulated the writing of the following theorem.

**Theorem 4.** Let \( T \) be a locally rank one ergodic action. If it has a sequence of self-joinings \( \nu_i \) with \( \nu_i \otimes_X \nu_i \perp \Delta \) and \( \nu_i \rightarrow \Delta \) then \( T \) is partially rigid.

Let us consider a self-joining \( \nu \) with a joining \( \eta \) corresponding to the operator \( P^*P \), where Markov operator \( P \) corresponds to \( \nu \). Denote \( \eta = \nu \otimes_X \nu \) and consider \( U_j^\varepsilon \) and \( D_j^\varepsilon \) (as well as \( \Delta_j^z \)) introduced in the proof of Theorem 3. Since for \( \varepsilon > 0 \)
\[
\liminf_j \eta(U_j^\varepsilon \times U_j^\varepsilon) = \liminf_j (P^*P \chi_{U_j^\varepsilon}, \chi_{U_j^\varepsilon}) \geq c \geq (\varepsilon \beta(T))^2,
\]
we get
\[
\eta(D_j^\varepsilon) \rightarrow d > 0, \quad \eta(\mid D_j^\varepsilon) \rightarrow \eta', \quad \eta = d\eta' + \ldots .
\]
Applying again approximation arguments we get
\[
\sum_z a_j^z \Delta_j^z \rightarrow \eta'.
\]
Now let us turn to our sequence $\nu_i \to \Delta$. We have $P_i \to \text{w} I$, where $P_i$ corresponds to the joining $\nu_i$. Since $I$ is unitary we see that $P_i \to_s I$ and $P_i^* P_i \to_s I$ (we use the symbol $\to_s$ for strong convergence). Thus, $\eta_i \to \Delta$, and the corresponding sequence $\eta'_i \to \Delta$ as well ($\Delta$ is ergodic self-joining!). We get

$$\sum z a_{j(i)}^z \Delta_j^z \to \Delta.$$  

Choice lemma [11] and the possibility to have $a_{j(i)}^0 \to 0$ (as a consequence of $\nu_i \otimes_X \nu_i \perp \Delta$) together provide the partial rigidity of the action: in fact, there is a sequence $z_i \to \infty$ such that

$$\Delta_{j(i)}^{z_i} \to \Delta, \quad \Delta^{z_i} \to \beta(T)(1 - \varepsilon)\Delta + \ldots, \quad T^{z_i} \to \beta(T)(1 - \varepsilon)I + \ldots.$$ 

Our action $T$ is partially rigid.

5 Multiple Mixing of Iceberg Transformations

We know that the condition $\beta(\Phi) > 2^{-n}$ for mixing $\mathbb{Z}^n$-action $\Phi$ implies multiple mixing. If a mixing $T$ of positive local rank hasn’t multiple mixing, then due to B. Host et al. we can say that an automorphism is found with absolutely continuous spectrum of finite multiplicity. Thus, within the class of mixing transformations of a positive local rank a connection is observed between the spectral problem on the existence of transformations with absolutely continuous spectra of finite multiplicity and Rokhlin’s multiple mixing problem. In [8] new examples of transformations with positive local rank are introduced. Although, instead of new spectral phenomena we got the following result of positive kind.

**Theorem 5** (jointly with A.A.Prikhodko). *Mixing iceberg transformations are mixing of all orders.*

In fact, this theorem follows from certain properties of the construction [8] (there is some analogy with finite rank actions), and the theorem on multiple mixing of systems with $D$-property [9].

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