ON THE CRAW–ISHII CONJECTURE

SEUNG-JO JUNG

ABSTRACT. In [2], Craw and Ishii proved that for a finite abelian group $G \subset \text{SL}_3(\mathbb{C})$ every (projective) relative minimal model of $\mathbb{C}^3/G$ is isomorphic to the fine moduli space $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations for some GIT parameter $\theta$. In this article, we conjecture that the same is true for a finite group $G \subset \text{GL}_3(\mathbb{C})$ if a relative minimal model $Y$ of $X = \mathbb{C}^3/G$ is smooth. We prove this for some abelian groups.

CONTENTS

1. Introduction 1
   1.1. Overview of the article 3
2. $G$-constellations and $G$-bricks 4
   2.1. Moduli spaces of $G$-constellations 4
   2.2. Cyclic quotients and toric lattices 5
   2.3. $G$-bricks and the birational component $Y_\theta$ 5
3. Star subdivisions and moduli descriptions 7
   3.1. Star subdivisions and round down functions 7
   3.2. Star subdivisions and bricksets 12
   3.3. Star subdivisions and stability parameters 13
4. Main theorem 14
   4.1. Toric minimal model program 14
   4.2. The Craw–Ishii conjecture 15
   4.3. The first case: $r = abc + a + b + 1$ 16
   4.4. The second case: $r = abc + a - 2b + 1$ 22
   4.5. Discussions 27
Appendix A. $\frac{1}{39}(1, 5, 11)$ type 28
References 29

1. Introduction

Let $G$ be a finite subgroup in $\text{GL}_3(\mathbb{C})$. A $G$-cluster $Z$ is a $G$-invariant subscheme of $\mathbb{C}^n$ with $\text{H}^0(\mathcal{O}_Z)$ isomorphic to $\mathbb{C}[G]$ the regular representation of $G$. For a finite group $G$ in $\text{SL}_2(\mathbb{C})$, Ito and Nakamura [8] showed that the minimal resolution of $\mathbb{C}^2/G$ is isomorphic to the $G$-Hilbert scheme $G$-Hilb $\mathbb{C}^2$ that is the fine moduli space of $G$-clusters.

Date: 21st January 2016.
In [1], Bridgeland, King and Reid proved that for a finite group $G$ in $\text{SL}_3(\mathbb{C})$ the $G$-Hilbert scheme $\text{G-Hilb} \mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$.

In [2], Craw and Ishii introduced a generalised notion of $G$-clusters. A $G$-constellation $\mathcal{F}$ is a $G$-equivariant sheaf on $\mathbb{C}^n$ with $H^0(\mathcal{F})$ isomorphic to $\mathbb{C}[G]$. Define the GIT stability parameter space

$$\Theta = \{ \theta \in \text{Hom}_G(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \},$$

where $R(G)$ denotes the representation space of $G$. For $\theta \in \Theta$, we say that $G$-constellation $\mathcal{F}$ is $\theta$-(semi)stable if $\theta(\mathcal{G}) > 0$ ($\theta(\mathcal{G}) \geq 0$) for every nonzero proper subsheaf $\mathcal{G}$ of $\mathcal{F}$. A stability parameter $\theta$ is called generic if every $\theta$-semistable $G$-constellation is $\theta$-stable.

Furthermore, Craw and Ishii constructed the moduli space $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations using GIT [2]. They conjectured that for a finite subgroup $G \subset \text{SL}_3(\mathbb{C})$, every projective crepant resolution of $\mathbb{C}^3/G$ is isomorphic to $\mathcal{M}_\theta$ for some $\theta \in \Theta$ and proved this for $G$ being abelian. Note that if $G \subset \text{SL}_3(\mathbb{C})$, then $\mathbb{C}^3/G$ has Gorenstein canonical singularities. Being motivated by this, this article makes the following conjecture and proves the conjecture for some cases.

**Conjecture 1.1 (Craw–Ishii conjecture).** Let $G$ be a finite subgroup in $\text{GL}_3(\mathbb{C})$. Suppose $X = \mathbb{C}^3/G$ has a smooth relative minimal model. Then every relative minimal model of $X$ is isomorphic to (an irreducible component of) $\mathcal{M}_\theta$ for a suitable GIT parameter $\theta$.

On the other hand, $\mathcal{M}_\theta$ need not be irreducible in general [3]. However, if $G$ is abelian, Craw, Maclagan and Thomas [4] showed that $\mathcal{M}_\theta$ has a unique irreducible component $Y_\theta$ containing the torus $(\mathbb{C}^*)^n/G$ for generic $\theta$. Furthermore, they proved that $Y_\theta$ can be obtained by variation of GIT from $\mathbb{C}^n/G$. The component $Y_\theta$ is called the birational component of $\mathcal{M}_\theta$.

**Theorem 1.2 (Main Theorem).** Let $G \subset \text{GL}_3(\mathbb{C})$ be the finite group of type $\frac{r}{t}(1,a,b)$ with $b$ coprime to $a$ satisfying one of the following:

(i) $r = abc + a + b + 1$ for some positive integer $c$;

(ii) $r = abc + a - 2b + 1$ and $b = ak + 1$ for some positive integers $c, k$ with $c \geq 2$ and $a \geq 3$.

Then every relative minimal model $Y \rightarrow X := \mathbb{C}^3/G$ is isomorphic to the birational component $Y_\theta$ of the moduli space $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.

Moreover the main theorem implies that the relative minimal model can be obtained by variation of GIT from $\mathbb{C}^3/G$ for the cases.

**Corollary 1.3.** In the situation as in Theorem 1.2, every relative minimal model $Y \rightarrow X = \mathbb{C}^3/G$ is obtained by variation of GIT quotient.
1.1. Overview of the article. Let $G$ be a finite group in $\text{GL}_3(\mathbb{C})$. We say that $\varphi : Y \to X := \mathbb{C}^3/G$ is a relative minimal model if:

(i) $Y$ has only $\mathbb{Q}$-factorial terminal singularities;
(ii) $K_Y$ is $\varphi$-nef;
(iii) $\varphi$ is projective.

For example, for the case where $G \subset \text{SL}_3(\mathbb{C})$, a projective crepant resolution $Y \to \mathbb{C}^3/G$ is a relative minimal model.

For the group $G$ of type $\frac{1}{r}(\alpha_1, \ldots, \alpha_n)$, i.e.

\[ G = \langle \text{diag}(e^{\alpha_1}, \ldots, e^{\alpha_n}) | e^r = 1 \rangle \subset \text{GL}_n(\mathbb{C}), \]

by toric geometry, the quotient variety $X = \mathbb{C}^n/G$ is given by the toric cone

\[ \sigma_{+} := \text{Cone}(e_1, \ldots, e_n) \]

with the lattice

\[ L = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \ldots, \alpha_n). \]

Fix a primitive interior lattice point $v = \frac{1}{r}(a_1, \ldots, a_n) \in L \cap \sigma_{+}$. The star subdivision of $\sigma_{+}$ at $v$ is the minimal fan containing the following $n$-dimensional cones $\sigma_k$ for $k = 1, \ldots, n$:

\[ \sigma_k := \text{Cone}(e_1, \ldots, \hat{e}_k, v, \ldots, e_n). \]

Then the corresponding toric variety $X_v$ admits the induced projective toric morphism $\nu : X_v \to X = \mathbb{C}^n/G$. If $v$ generates $L/\mathbb{Z}^n$, then the affine open set $U_k$ of $X_v$ corresponding to $\sigma_k$ has a quotient singularity $\mathbb{C}^n/G_k$ for some abelian group $G_k$, e.g. $G_1$ is the group of type $\frac{1}{r}(a_1, a_2, \ldots, a_n)$. Note that the order of $G_k$ is smaller than the order of $G$. Thus we can use induction on the order of groups.

The groups in the main theorem satisfy:

(i) every relative minimal model $Y \to X$ is smooth;
(ii) every relative minimal model $Y \to X$ has a projective morphism $Y \to X_v$ for $v = \frac{1}{r}(1, a, b)$.

For the proof, the notion of $G$-bricks and round down functions is essential, which was recently developed in [9, 10].

A $G$-brick $\Gamma$ is a certain $\mathbb{C}$-basis of $H^0(\mathcal{F})$ for a torus invariant $G$-constellation $\mathcal{F}$ on the birational component (see Definition 2.7). We say that $\Gamma$ is $\theta$-stable if the corresponding $G$-constellation $\mathcal{F}$ is $\theta$-stable. Using suitable $G$-bricks, we are able to describe an affine local chart of the birational component $Y_\theta$ (see Theorem 2.13).

The round down functions for the star subdivisions at $v$ are maps between monomial lattices compatible with the star subdivision. Using the round down functions, we produce a set $\mathcal{G}$ of $G$-bricks from $G_k$-bricks.

Since the set $\mathcal{G}$ is a $G$-brickset (see Definition 2.15), it suffices to find a GIT parameter $\theta$ such that every $G$-brick $\Gamma \in \mathcal{G}$ is $\theta$-stable.
After finding a parameter $\theta$, we conclude that $Y$ is isomorphic to $Y_\theta$ for some $\theta$.

**Acknowledgement.** I am deeply grateful to Miles Reid for his valuable advice and encouragement. I would like to thank Alastair Craw, Akira Ishii, Yukari Ito, Yujiro Kawamata for kind explanations. I am grateful to Sara Muhvić for sharing her examples. I would like to thank the University of Warwick for its hospitality, where this work was done in June 2015.

2. **$G$-constellations and $G$-bricks**

2.1. **Moduli spaces of $G$-constellations.** In this section, we briefly review moduli spaces of $\theta$-stable $G$-constellations (see e.g. [2, 4, 11]).

Consider a finite diagonal group $G$ in $\text{GL}_n(\mathbb{C})$.

**Definition 2.1.** A $G$-equivariant coherent sheaf $F$ on $\mathbb{C}^n$ is called a $G$-constellation if $H^0(F)$ is isomorphic to the regular representation $\mathbb{C}[G]$ of $G$ as a $\mathbb{C}[G]$-module.

**Remark 2.2.** For a free $G$-orbit $Z$ in $\mathbb{C}^3$, $O_Z$ is a $G$-constellation.

Define the GIT stability parameter space

$$\Theta = \left\{ \theta \in \text{Hom}_Z(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \right\}$$

where $R(G) := \bigoplus_{\rho \in \text{Irr} G} \mathbb{Z} \cdot \rho$ is the representation space of $G$.

**Definition 2.3.** For a stability parameter $\theta \in \Theta$, we say that:

(i) a $G$-constellation $F$ is $\theta$-semistable if $\theta(G) \geq 0$ for every subsheaf $G \subsetneq F$;

(ii) a $G$-constellation $F$ is $\theta$-stable if $\theta(G) > 0$ for every subsheaf $0 \neq G \subsetneq F$;

(iii) $\theta$ is generic if every $\theta$-semistable object is $\theta$-stable.

By King [11], it is known that if $\theta$ is generic, then there exists a quasiprojective scheme $M_\theta$ which is a fine moduli space of $\theta$-stable $G$-constellations.

Moreover, in [7], Ito–Nakajima showed that $M_\theta$ is canonically isomorphic to $G$-$\text{Hilb} \mathbb{C}^n$ for $\theta \in \Theta_+$ where

$$\Theta_+ := \left\{ \theta \in \Theta \mid \theta(\rho) > 0 \text{ for } \rho \neq \rho_0 \right\}.$$ 

In particular, $M_\theta$ can be obtained by variation of GIT from $G$-$\text{Hilb} \mathbb{C}^n$.

Assume that $\theta$ is generic. Let $\mathcal{M}_\theta$ denote the fine moduli space of $\theta$-stable $G$-constellations. Craw, Maclagan and Thomas showed that the moduli space $\mathcal{M}_\theta$ need not be irreducible [5]. Furthermore, they proved that $\mathcal{M}_\theta$ has a distinguished component $Y_\theta$ which is birational to $\mathbb{C}^n/G$ if $G$ is abelian [3].
Theorem 2.5 (Craw–Maclagan–Thomas [4]). Assume that $G$ be a finite abelian group in $\text{GL}_n(\mathbb{C})$. For a generic parameter $\theta \in \Theta$, the moduli space $\mathcal{M}_\theta$ has a unique irreducible component $Y_\theta$ that contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover:

(i) $Y_\theta$ is a not-necessarily-normal toric variety which is birational to the quotient variety $\mathbb{C}^n/G$;

(ii) there is a projective morphism $Y_\theta \to \mathbb{C}^n/G$ obtained by variation of GIT quotient.

Definition 2.6. The unique irreducible component $Y_\theta$ in Theorem 2.5 is called the birational component of $\mathcal{M}_\theta$.

2.2. Cyclic quotients and toric lattices. Consider the group $G$ of type $1^r(\alpha_1, \ldots, \alpha_n)$, i.e.$G = \langle \text{diag}(\epsilon^{\alpha_1}, \ldots, \epsilon^{\alpha_n}) \mid \epsilon^r = 1 \rangle \subset \text{GL}_n(\mathbb{C})$.

As $G$ is abelian, the set of irreducible representations of $G$ can be identified with the character group $G^\vee := \text{Hom}(G, \mathbb{C}^\times)$ of $G$.

For the group $G$ of type $1^r(\alpha_1, \ldots, \alpha_n)$, define the lattice $L = \mathbb{Z}^n + \mathbb{Z} \cdot 1^r(\alpha_1, \ldots, \alpha_n)$.

Set $L = \mathbb{Z}^n \subset L$. Consider the two dual lattices $M = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$, $\overline{M} = \text{Hom}_\mathbb{Z}(\overline{L}, \mathbb{Z})$. Note that we can consider the two dual lattices $\overline{M}$ and $M$ as Laurent monomials and $G$-invariant Laurent monomials, respectively.

The embedding of $G$ into the torus $(\mathbb{C}^\times)^n \subset \text{GL}_n(\mathbb{C})$ induces a surjective homomorphism $\text{wt} : \overline{M} \to G^\vee$ with kernel $M$. Note that there are two isomorphisms of abelian groups $L/\mathbb{Z}^n \to G$ and $\overline{M}/M \to G^\vee$.

Let $\overline{M}_{\geq 0}$ denote genuine monomials in $\overline{M}$, i.e.$\overline{M}_{\geq 0} = \{x_1^{m_1} \cdots x_n^{m_n} \in \overline{M} \mid m_i \geq 0 \text{ for all } i\}$.

For a set $A \subset \mathbb{C}[x_1^+, \ldots, x_n^+]$, $\langle A \rangle$ denotes the $\mathbb{C}[x_1, \ldots, x_n]$-submodule of $\mathbb{C}[x_1^+, \ldots, x_n^+]$ generated by $A$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}^n$ and $\sigma_+$ the cone generated by $e_1, \ldots, e_n$. By toric geometry, the corresponding affine toric variety $U_{\sigma_+} = \text{Spec} \mathbb{C}[\sigma_+^\vee \cap M]$ is the quotient variety $X = \mathbb{C}^3/G$.

2.3. $G$-bricks and the birational component $Y_\theta$. In this section, we review the notion of $G$-bricks introduced in [9,10]. Using $G$-bricks, we can describe an affine local chart of the birational component $Y_\theta$.

Definition 2.7. A $G$-prebrick $\Gamma$ is a subset of Laurent monomials in $\mathbb{C}[x_1^+, \ldots, x_n^+]$ satisfying:

(i) the monomial 1 is in $\Gamma$;
(ii) for each weight \( \rho \in G^\vee \), there exists a unique Laurent monomial \( m_\rho \in \Gamma \) of weight \( \rho \), i.e. \( \text{wt}: \Gamma \to G^\vee \) is bijective;
(iii) if \( \mathbf{p}' \cdot \mathbf{p} \cdot m_\rho \in \Gamma \) for \( \mathbf{m}_\rho \in \Gamma \) and \( \mathbf{p}, \mathbf{p}' \in M_{\geq 0} \), then \( \mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma \);
(iv) the set \( \Gamma \) is connected in the sense that for any element \( \mathbf{m}_\rho \), there is a (fractional) path in \( \Gamma \) from \( \mathbf{m}_\rho \) to 1 whose steps consist of multiplying or dividing by one of \( x_i \).

For a Laurent monomial \( \mathbf{m} \in M' \), let \( \text{wt}_\Gamma (\mathbf{m}) \) denote the unique element \( \mathbf{m}_\rho \in \Gamma \) of the same weight as \( \mathbf{m} \).

For a \( G \)-prebrick \( \Gamma = \{ \mathbf{m}_\rho \} \), we define \( S(\Gamma) \) to be the subsemigroup of \( M \) generated by \( \mathbf{p} \cdot \mathbf{m}_\rho \) for all \( \mathbf{p} \in M_{\geq 0}, \mathbf{m}_\rho \in \Gamma \). We define a cone \( \sigma(\Gamma) \) in \( L_\mathbb{R} = \mathbb{R}^n \) as follows:

\[
\sigma(\Gamma) = S(\Gamma)^\vee = \left\{ \mathbf{u} \in L_\mathbb{R} \mid \left( \mathbf{u}, \frac{\mathbf{p} \cdot \mathbf{m}_\rho}{\text{wt}_\Gamma (\mathbf{p} \cdot \mathbf{m}_\rho)} \right) \geq 0, \ \forall \mathbf{m}_\rho \in \Gamma, \ \mathbf{p} \in M_{\geq 0} \right\}.
\]

As is proved in [10], the semigroup \( S(\Gamma) \) is finitely generated as a semigroup. Thus the semigroup \( S(\Gamma) \) defines an affine toric variety.

Define two affine toric varieties:
\[
U(\Gamma) := \text{Spec} \mathbb{C}[S(\Gamma)],
\]
\[
U''(\Gamma) := \text{Spec} \mathbb{C}[\sigma(\Gamma)^\vee \cap M].
\]

**Definition 2.8.** For a \( G \)-prebrick \( \Gamma \),

\[
B(\Gamma) := \left\{ x_i \cdot \mathbf{m}_\rho \mid \mathbf{m}_\rho \in \Gamma, \right\} \setminus \Gamma
\]

is called the **Border bases** of \( \Gamma \).

Let \( \Gamma \) be a \( G \)-prebrick. Define

\[
C(\Gamma) := (\Gamma) / \langle B(\Gamma) \rangle.
\]

The module \( C(\Gamma) \) is a torus invariant \( G \)-constellation. A submodule \( \mathcal{G} \) of \( C(\Gamma) \) is determined by a subset \( A \subset \Gamma \), which forms a \( \mathbb{C} \)-basis of \( \mathcal{G} \).

**Lemma 2.9.** Let \( A \) be a subset of \( \Gamma \). The following are equivalent.

(i) The set \( A \) forms a \( \mathbb{C} \)-basis of a submodule of \( C(\Gamma) \).
(ii) If \( \mathbf{m}_\rho \in A \), then \( x_i \cdot \mathbf{m}_\rho \in \Gamma \) implies \( x_i \cdot \mathbf{m}_\rho \in A \) for all \( i \).

**Definition 2.10.** Let \( \Gamma \) be a \( G \)-prebrick.

(i) A \( G \)-prebrick \( \Gamma \) is called a **\( G \)-brick** if the affine toric variety \( U(\Gamma) \) contains a torus fixed point.
(ii) A \( G \)-prebrick \( \Gamma \) is called \( \theta \)-**stable** if the torus invariant \( G \)-constellation \( C(\Gamma) \) is \( \theta \)-stable.

Note that from toric geometry, \( U(\Gamma) \) has a torus fixed point if and only if \( S(\Gamma) \cap (S(\Gamma))^{-1} = \{1\} \), i.e. the cone \( \sigma(\Gamma) \) is an \( n \)-dimensional cone.
Proposition 2.11 ([9]). For generic \( \theta \), let \( \Gamma \) be a \( \theta \)-stable \( G \)-brick and \( Y_\theta \) the birational component of \( M_\theta \). There exists an open immersion
\[
U(\Gamma) = \text{Spec } \mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta.
\]

Remark 2.12. The \( G \)-brick \( \Gamma \) forms a \( \mathbb{C} \)-basis of \( G \)-constellations parametrised by \( U(\Gamma) \). ♦

Theorem 2.13 ([9]). Let \( G \subset \text{GL}_n(\mathbb{C}) \) be a finite diagonal group and \( \theta \) a generic GIT parameter for \( G \)-constellations. Assume that \( S \) is the set of all \( \theta \)-stable \( G \)-bricks.

(i) The birational component \( Y_\theta \) of \( M_\theta \) is isomorphic to the not-necessarily-normal toric variety \( \bigcup_{\Gamma \in S} U(\Gamma) \).

(ii) The normalisation of \( Y_\theta \) is isomorphic to the normal toric variety whose toric fan consists of the \( n \)-dimensional cones \( \sigma(\Gamma) \) for \( \Gamma \in S \) and their faces.

Remark 2.14. For given \( G \) and \( \theta \), it is difficult to find all \( \theta \)-stable \( G \)-bricks in general. ♦

Definition 2.15. For a finite diagonal group \( G \subset \text{GL}_n(\mathbb{C}) \), assume that \( Y \) is a normal toric variety admitting a proper birational morphism \( Y \to X := \mathbb{C}^n/G \). Let \( \Sigma_{\text{max}} \) denote the set of the \( n \)-dimensional cones in the fan of \( Y \). A set \( \mathcal{S} \) of \( G \)-bricks is called a \( G \)-brickset for \( Y \) if \( \mathcal{S} \) satisfies:

(i) there is a bijective map \( \Sigma_{\text{max}} \to \mathcal{S} \) sending \( \sigma \) to \( \Gamma_\sigma \);

(ii) \( S(\Gamma_\sigma) = \sigma^\vee \cap M \).

Proposition 2.16. Suppose that \( Y \to X := \mathbb{C}^n/G \) is a proper birational morphism. Let \( \mathcal{S} \) be a \( G \)-brickset for \( Y \). Then \( Y \) is isomorphic to the toric variety \( \bigcup_{\Gamma \in \mathcal{S}} U(\Gamma) \). Moreover, if there exists \( \theta \in \Theta \) such that every \( \Gamma \) in \( \mathcal{S} \) is \( \theta \)-stable, then \( Y \) is isomorphic to \( Y_\theta \).

Proof. By definition, it is clear that \( Y \) is isomorphic to the toric variety \( \bigcup_{\Gamma \in \mathcal{S}} U(\Gamma) \). Assume that there exists \( \theta \in \Theta \) such that every \( \Gamma \) in \( \mathcal{S} \) is \( \theta \)-stable. From Proposition 2.11, we can conclude that there exists an open immersion \( \iota: Y \hookrightarrow Y_\theta \). Furthermore, since \( Y \to X \) is proper and \( Y_\theta \to X \) is projective, \( \iota \) is a closed embedding between \( n \)-dimensional toric varieties. Thus \( \iota \) is an isomorphism. □

3. Star subdivisions and moduli descriptions

3.1. Star subdivisions and round down functions. Fix a primitive lattice point \( v = \frac{1}{2}(a_1, \ldots, a_n) \in L \cap \sigma_+ \). The star subdivision (or barycentric subdivision) \( \Sigma \) of \( \sigma_+ \) at \( v \) is the minimal fan containing all cones \( \text{Cone}(\tau, v) \) where \( \tau \) varies over all faces of \( \sigma_+ \) with \( v \not\in \tau \). Let \( X := U_{\sigma_+} \) be the affine toric variety corresponding to \( \sigma_+ \) and \( X_\Sigma \) the toric variety corresponding to the fan \( \Sigma \). Then the star subdivision
induces a projective toric morphism $\nu: X_{\Sigma} \to X = \mathbb{C}^n/G$ with the ramification formula

$$rK_{X_{\Sigma}} - \nu^*rK_X \equiv_{\text{num}} \left( \sum_i a_i - r \right) E_v,$$

where $E_v$ is the torus invariant prime divisor corresponding to the 1-dimensional cone Cone$(v)$.

The fan $\Sigma$ consists of the $n$-dimensional cone $\sigma_k$ and its faces for $k = 1, \ldots, n$:

$$\sigma_k := \text{Cone}(e_1, \ldots, \hat{e}_k, v, \ldots, e_n).$$

Assume that $v$ generates $L/\mathbb{Z}^n$. Fix $k \in \{1, \ldots, n\}$. Let $L_k$ be the sublattice of $L$ generated by $e_1, \ldots, \hat{e}_k, v, \ldots, e_n$. Let us consider the dual lattice $M_k := \text{Hom}_\mathbb{Z}(L_k, \mathbb{Z})$ with the corresponding dual basis $\{\xi_1, \ldots, \xi_n\}$

$$\xi_j = \begin{cases} x_jx_k^{-a_j} & \text{if } j \neq k, \\ x_k^{a_k} & \text{if } j = k. \end{cases}$$

Note that $M_k$ contains the lattice $M$ and that the lattice inclusion $L_k \hookrightarrow L$ induces a toric morphism

$$\varphi: \text{Spec} \mathbb{C}[\sigma_k^\vee \cap M_k] \to U_k := \text{Spec} \mathbb{C}[\sigma_k^\vee \cap M].$$

With eigencordinates $\{\xi_1, \ldots, \xi_n\}$, the toric affine variety $U_k$ has a quotient singularity of type

$$\frac{1}{a_k}(a_1, \ldots, -r, \ldots, a_n).$$
Example 3.2. Consider the group $G$ of type $\frac{1}{20}(1,3,4)$. Figure 3.2 shows the star subdivision at $v = \frac{1}{20}(1,3,4)$.

The cone $\sigma_2$ corresponds to the quotient singularity of type $\frac{1}{3}(1,1,1)$ with eigencordinates $xy^{\frac{1}{3}}, y^{\frac{20}{3}}, y^{-\frac{4}{3}}z$. There exists a unique lattice point $v_7 = \frac{1}{20}(7,1,8)$ on the plane containing $e_1, v, e_3$. On the other hand, the cone $\sigma_3$ on the right side of $v$ has a singularity of type $\frac{1}{4}(1,3,0)$. Note that there are three other lattice points $v_5, v_{10}, v_{15}$ on the plane containing $e_1, e_2, v$. Lastly, the affine toric variety corresponding to the cone $\sigma_1 = \text{Cone}(e_2, e_3, v)$ is smooth as $v, e_2, e_3$ form a $\mathbb{Z}$-basis of $L$.

Definition 3.3 (Round down functions). With the notation above, for $k \in \{1, \ldots, n\}$, the $k$-th round down function $\phi_k : M \to M_k$ of the star subdivision at $\frac{1}{r}(a_1, \ldots, a_n)$ is defined by

$$\phi_k(x_1^{m_1} \cdots x_n^{m_n}) = \xi_1^{m_1} \cdots \xi_k^{\left\lfloor \frac{1}{r} \sum a_i m_i \right\rfloor} \cdots \xi_n^{m_n},$$

where $\left\lfloor \right\rfloor$ is the floor function.

Observe that since $v \in L$, $\frac{1}{r} \sum a_i m_i$ is an integer if and only if the monomial $x_1^{m_1} \cdots x_n^{m_n}$ is $G$-invariant.

For the star subdivision at $\frac{1}{r}(a_1, \ldots, a_n)$, let $G_k$ denote the abelian group $L/L_k$ for $k \in \{1, \ldots, n\}$.

Lemma 3.4. For each $k$, let $\phi_k$ be the round down function of the star subdivision at $\frac{1}{r}(a_1, \ldots, a_n)$ and $G_k$ the abelian group $L/L_k$. For a

---

1As is stated in [9, 10], Davis, Logvinenko, and Reid introduced a similar construction in a more general setting.
$G$-invariant monomial $p \in M$ and a monomial $m \in \overline{M}$,
\[
\phi_k(m \cdot p) = \phi_k(m) \cdot p.
\]
In particular, the weights of $\phi_k(m \cdot p)$ and $\phi_k(m)$ are the same with respect to the $G_k$-action. Thus $\phi_k$ induces a well-defined surjective map
\[
\phi_k: G^\vee_k \to G^\vee_k, \quad \rho \mapsto \phi_k(\rho),
\]
where $\phi_k(\rho)$ is the weight of $\phi_k(m)$ for a monomial $m \in \overline{M}$ of weight $\rho$.

**Remark 3.5.** In the lemma above, $\phi_k: G^\vee \to G^\vee_k$ can be described as follows. Let $p_i$ be the irreducible representation of $G$ whose weight is $i$. Then the weight of the representation $\phi_k(p_i)$ is $j$ where $j$ is the residue of $i$ modulo $a_k$, i.e. $j = i \mod a_k$. \hfill \Box

**Lemma 3.6.** Let $m \in \overline{M}$ be a monomial of weight $j$. The weight $j$ satisfies $0 \leq j < r - a_k$ if and only if
\[
\phi_k(x_k \cdot m) = \phi_k(m).
\]
Proof. Assume that $m = x_1^{m_1} \cdots x_n^{m_n}$ is a monomial of weight $j$ with $0 \leq j < r - a_k$, i.e.
\[
0 \leq \frac{1}{r} \sum a_im_i - \frac{1}{r} \sum a_im_i < \frac{r - a_k}{r}.
\]
This is equivalent to the condition that $\phi_k(x_k \cdot m) = \phi_k(m)$. \hfill \Box

**Lemma 3.7.** If $\phi_k(m) = \phi_k(m')$ for some $k$, then $m = p \cdot m'$ or $m' = p \cdot m$ for some $p \in \overline{M}_{\geq 0}$.

Proof. Let us suppose that $\phi_k(m) = \phi_k(m')$ for $m = x_1^{m_1} \cdots x_n^{m_n}$ and $m' = x_1^{m'_1} \cdots x_n^{m'_n}$ with $m_k \geq m'_k$. From the definition of the round down function $\phi_k$, we have $m_i = m'_i$ for all $i \neq k$. Thus $m = p \cdot m'$ with $p = x_k^{m_k - m'_k} \in \overline{M}_{\geq 0}$. \hfill \Box

**Definition 3.8.** The star subdivision of $\sigma_+$ at $v = \frac{1}{r}(a_1, \ldots, a_n)$ is said to be **good** if:

(i) $v$ generates $L/\mathbb{Z}^n$;
(ii) for every $i \neq j$, $a_i + a_j \leq r$.

**Lemma 3.9.** Let $\phi_k$ be the $k$-th round down function of the good star subdivision at $\frac{1}{r}(a_1, \ldots, a_n)$. For any monomial $n$ in the lattice $M_k$ and any degree one monomial $\xi_j$ in $M_k$, there exist $x_i$ and a monomial $m \in \overline{M}$ such that
\[
\phi_k(x_i \cdot m) = \xi_j \cdot n \quad \text{with} \quad \phi_k(m) = n.
\]

Proof. Fix $k$. Suppose that $n$ is a monomial in $M_k$ and that $\xi_j$ is a degree one monomial in $M_k$.

First consider the case where $j = k$, i.e. $\xi_j = \xi_k$. As the round down function $\phi_k$ is surjective, there exists $m \in \overline{M}$ such
that \( \phi_k(m) = n \). By Lemma 3.4 after multiplying \( x_k \) enough, we may assume that \( \phi_k(x_k \cdot m) \neq \phi_k(m) \). This means that
\[
\frac{1}{r} \sum_i a_i m_i + \frac{a_k}{r} \geq \left[ \frac{1}{r} \sum_i a_i m_i \right] + 1.
\]
Thus we have \( \phi_k(x_k \cdot m) = \xi_k \cdot n \).

For the case where \( j \neq k \), consider \( m = x_1^{m_1} \cdots x_n^{m_n} \in \overline{M} \) such that \( \phi_k(m) = n \) with \( \phi_k(x_k^{-1} \cdot m) \neq \phi_k(m) \), i.e.
\[
\frac{1}{r} \sum_i a_i m_i - \frac{a_k}{r} < \left[ \frac{1}{r} \sum_i a_i m_i \right].
\]
Since the star subdivision is good, we have \( a_k + a_j \leq r \). This implies that \( \phi_k(x_j \cdot m) = \xi_j \cdot n \).

**Proposition 3.10.** Let \( \phi_k \) be the \( k \)-th round down function of the good star subdivision at \( \frac{1}{r}(a_1, \ldots, a_n) \). For a \( G_k \)-brick \( \Gamma' \), define
\[
\Gamma := \{ m \in \overline{M} \mid \phi_k(m) \in \Gamma' \}.
\]

(i) The set \( \Gamma \) is a \( G \)-brick with \( S(\Gamma) = S(\Gamma') \).
(ii) For \( m \in \overline{M} \), we have \( \text{wt}_{\Gamma'}(\phi_k(m)) = \phi_k(\text{wt}_{\Gamma'}(m)) \).

**Proof.** First we show (ii) assuming that \( \Gamma \) is a \( G \)-prebrick. It follows that \( \phi_k(m) \) is of the same weight as \( \phi_k(\text{wt}_{\Gamma'}(m)) \) from Lemma 3.4. Since \( \phi_k(\text{wt}_{\Gamma'}(m)) \in \Gamma' \), the assertion is proved.

To prove (i), note that \( 1 \in \Gamma \) as \( \phi_k(1) = 1 \in \Gamma' \). Second we show that there exists a unique monomial of weight \( \rho \) in \( \Gamma \) for each \( \rho \in G' \). Fix \( \rho \in G' \). Since the star subdivision is good, we have a monomial \( m \in \overline{M} \) such that the weight of \( m \) is \( \rho \). Note that \( \text{wt}_{\Gamma'}(\phi_k(m)) \in \Gamma' \) and that \( \frac{\text{wt}_{\Gamma'}(\phi_k(m))}{\phi_k(m)} \) is in the lattice \( M \). From Lemma 3.4
\[
\phi_k : m \cdot \left( \frac{\text{wt}_{\Gamma'}(\phi_k(m))}{\phi_k(m)} \right) \mapsto \text{wt}_{\Gamma'}(\phi_k(m))
\]
so \( m \cdot \left( \frac{\text{wt}_{\Gamma'}(\phi_k(m))}{\phi_k(m)} \right) \) is an element of weight \( \rho \) in \( \Gamma \). From Lemma 3.4, the uniqueness is followed. Lemma 3.9 implies that \( \Gamma \) is connected as \( \Gamma' \) is connected.

To show (iii) in Definition 2.7, suppose that \( p' \cdot p \cdot m_\rho \in \Gamma \) for \( m_\rho \in \Gamma \) and \( p, p' \in \overline{M}_{\geq 0} \). Note that
\[
\phi_k(p' \cdot p \cdot m_\rho) = \frac{\phi_k(p' \cdot p \cdot m_\rho)}{\phi_k(p \cdot m_\rho)} \cdot \frac{\phi_k(p \cdot m_\rho)}{\phi_k(m_\rho)} \cdot \phi_k(m_\rho) \in \Gamma'.
\]
Since \( \Gamma' \) is a \( G_k \)-brick, \( \phi_k(p \cdot m_\rho) \in \Gamma' \). Thus \( p \cdot m_\rho \) is in \( \Gamma \). Therefore \( \Gamma \) is a \( G \)-prebrick.
To show that $S(\Gamma) = S(\Gamma')$, note that for $p \in \overline{M}_{\geq 0}$ and $m_\rho \in \Gamma$, $$\frac{p \cdot m_\rho}{\text{wt}_\Gamma(p \cdot m_\rho)} = \frac{\phi_k(p \cdot m_\rho)}{\phi_k(\text{wt}_\Gamma(p \cdot m_\rho))} = \frac{n \cdot \phi_k(m_\rho)}{\text{wt}_{\Gamma'}(n \cdot \phi_k(m_\rho))} \in S(\Gamma')$$ where $n = \frac{\phi_k(p \cdot m_\rho)}{\phi_k(m_\rho)}$. Since $S(\Gamma)$ is generated by $p \cdot m_\rho \in \Gamma$, we proved that $S(\Gamma) \subset S(\Gamma')$.

For the opposite inclusion, suppose that $n \in \Gamma'$. Let $\{\xi_j\}$ be the eigencoordinates with respect to the $G_k$-action. Lemma 3.9 shows that for every $\xi_j$ there exist $x_i, m_\rho \in \Gamma$ such that $\phi_k(x_i \cdot m_\rho) = \xi_j \cdot n$ with $\phi_k(m_\rho) = n$. Then $$\frac{\xi_j \cdot n}{\text{wt}_{\Gamma'}(\xi_j \cdot n)} = \frac{\phi_k(x_i \cdot m_\rho)}{\phi_k(\text{wt}_\Gamma(x_i \cdot m_\rho))} = \frac{\phi_k(x_i \cdot m_\rho)}{\text{wt}_{\Gamma'}(\phi_k(x_i \cdot m_\rho))} = \frac{x_i \cdot m_\rho}{\text{wt}_\Gamma(x_i \cdot m_\rho)}.$$ This completes the proof. □

**Definition 3.11.** The $G$-brick $\Gamma$ in Proposition 3.10 is called the natural inverse of $\Gamma'$ and denoted by $\phi_k^*(\Gamma')$.

### 3.2. Star subdivisions and bricksets.

Let $G$ be a finite diagonal group in $\text{GL}_n(\mathbb{C})$. Let $X$ denote the quotient variety $\mathbb{C}^n/G$ and $X_v$ the toric variety given by the good star subdivision at $v = \frac{1}{r}(a_1, \ldots, a_n)$. Recall that the toric fan of $X_v$ contains the $n$-dimensional cone $$\sigma_k := \text{Cone}(e_1, \ldots, \hat{e}_k, v, \ldots, e_n).$$

Note that $X_v$ is covered by the affine toric open sets $$U_k = \text{Spec} \mathbb{C}[\sigma_k^\vee \cap M] \cong \mathbb{C}^n/G_k,$$ where $G_k = L/L_k$.

Assume that $Y$ is a normal toric variety admitting a proper birational morphism $Y \to X$. Let $\Sigma$ denote the toric fan of $Y$. Assume further that there exists a dominant toric morphism $\overline{\varphi}: Y \to X_v$ fitting into the commutative diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{\overline{\varphi}} & X_v \\
\downarrow & & \downarrow \\
X. & & 
\end{array}$$

As is standard in toric geometry (see eg. Section 3.3 in [3]), for each cone $\sigma \in \Sigma$, there exists a cone $\sigma_k$ such that $\sigma \subset \sigma_k$. Therefore for each $k$, $\varphi$ induces the following toric morphism $$\overline{\varphi}_k: Y_k \to U_k,$$ where $Y_k$ is the toric variety whose fan consists of the cones $\sigma \in \Sigma$ satisfying $\sigma \subset \sigma_k$. Note that since $X_v \to X$ is projective, if the morphism $Y \to X$ is projective, then so is $\varphi_k$. 
Theorem 3.12. With the assumption above, further assume that each $Y_k \to U_k$ has a $G_k$-brickset $\mathcal{S}_k$. Define

$$\mathcal{S} := \bigcup_k \{ \phi_k^*(\Gamma') \mid \Gamma' \in \mathcal{S}_k \}. $$

Then $\mathcal{S}$ is a $G$-brickset for the morphism $Y \to X$.

Proof. Let $\Sigma_{\text{max}}$ be the set of the $n$-dimensional cones in the fan $\Sigma$ of $Y$. From Proposition 3.10, it follows that every object in the set $\mathcal{S}$ is a $G$-brick. It suffices to prove that the set $\mathcal{S}$ satisfies:

(i) there exists a bijection $\Sigma_{\text{max}} \to \mathcal{S}$ sending $\sigma$ to $\Gamma_{\sigma}$;
(ii) $S(\Gamma_{\sigma}) = \sigma^\vee \cap M$.

Let $\sigma$ be an arbitrary $n$-dimensional cone in $\Sigma$. Then there exists a unique cone $\sigma_k$ such that $\sigma \subset \sigma_k$. By the assumption, there is a unique $G_k$-brick $\Gamma' \in \mathcal{S}_k$ such that $S(\Gamma') = \sigma^\vee \cap M$. Define

$$\Gamma_{\sigma} = \phi_k^*(\Gamma') := \{ m \in M \mid \phi_k(m) \in \Gamma' \}. $$

By Proposition 3.10 we have

$$S(\Gamma_{\sigma}) = S(\Gamma') = \sigma^\vee \cap M,$$

and the proof is completed. \square

Please note that by Proposition 2.16 if there is $\theta \in \Theta$ satisfying that every $\Gamma$ in $\mathcal{S}$ is $\theta$-stable, then $Y$ is isomorphic to $Y_{\theta}$.

3.3. Star subdivisions and stability parameters. In this section, we discuss the existence of a stability parameter $\theta$ such that every $G$-brick in the brickset described in Theorem 3.12 is $\theta$-stable.

Consider the good star subdivision at $v = \frac{1}{r}(a_1, \ldots, a_n)$. For each $k$, let $\Theta^{(k)}$ be the GIT parameter space of $G_k$-constellations. Remember that by Lemma 3.4 we have the well-defined surjective map

$$\phi_k : G^\vee \to G_k^\vee$$

induced by the round down function $\phi_k$. Note that the linear map

$$(\phi_k)_* : \Theta \to \Theta^{(k)}$$

defined by

$$(3.13) \quad [(\phi_k)_*(\theta)](\chi) = \sum_{\phi_k(\rho) = \chi} \theta(\rho) \quad \text{for} \quad \chi \in G_k^\vee$$

is well-defined.

Let $\rho_i$ denote the irreducible representation of $G$ whose weight is $i$ and $\chi_j$ the irreducible representation of $G_k$ whose weight is $j$. First
note that Θ, which is a \( \mathbb{Q} \)-vector space of \((r - 1)\)-dimension, has a \( \mathbb{Q} \)-basis \( \{ \theta_i \in \text{Hom}_\mathbb{Z}(R(G), \mathbb{Q}) \mid 1 \leq i < r \} \) where

\[
\theta_i(\rho_l) =\begin{cases} 
1 & \text{if } \rho_l = \rho_i, \\
-1 & \text{if } \rho_l \text{ is trivial}, \\
0 & \text{otherwise},
\end{cases}
\]

for \( \rho_l \in G^\vee \). By the definition of the round down functions (see Lemma 3.4 or Remark 3.5), we have

\[
[(\phi_k)_*(\theta_i)](\chi_j) =\begin{cases} 
1 & \text{if } j \equiv i \pmod{a_k}, \\
-1 & \text{if } \chi_j \text{ is trivial}, \\
0 & \text{otherwise},
\end{cases}
\]

for \( \chi_j \in G^\vee_k \). In particular, \( (\phi_k)_*(\theta_i) \equiv (\phi_k)_*(\theta_{i'}) \) if \( i \equiv i' \pmod{a_k} \).

Remark 3.15. As is discussed above, \( (\phi_k)_* : \Theta \to \Theta^{(k)} \) is surjective. Indeed, \( (\phi_k)_*(\theta_i) \) for \( 1 \leq i < a \) form a \( \mathbb{Q} \)-basis of \( \Theta^{(k)} \).

From now on, we only consider 3-dimensional cases. Consider the good star subdivision at \( v = \frac{1}{r}(1, a, b) \) with \( a < b \).

Lemma 3.16. Consider the good star subdivision at \( v = \frac{1}{r}(1, a, b) \) with \( a < b \). Assume that \( a \) and \( b \) are coprime. Given \( \theta^{(k)} \in \Theta^{(k)} \) for \( k = 2, 3 \), there exists \( \theta_P \in \Theta \) such that

\[
(\phi_k)_*(\theta) \equiv \theta^{(k)}
\]

for all \( k \).

Proof. Consider the linear map

\[
\phi_* = ((\phi_2)_*, (\phi_3)_*) : \Theta \to \Theta^{(2)} \oplus \Theta^{(3)}.
\]

We need to prove that \( \phi_* \) is surjective. It suffices to show that

\[
\{ \phi_*(\theta_i) \mid 1 \leq i \leq a + b - 2 \}
\]

is linearly independent as the dimension of \( \Theta^{(2)} \oplus \Theta^{(3)} \) is \( a + b - 2 \). Using the fact that \( a \) and \( b \) are coprime, the assertion follows from a direct calculation.

4. Main theorem

4.1. Toric minimal model program. In this section, we recall the birational geometry of toric varieties (see [13]). Reid [13] introduced a combinatorial criterion for a toric variety to have terminal singularities and canonical singularities.

Theorem 4.1 (Reid [13]). Let \( X \) be the toric variety corresponding to a fan \( \Sigma \) with a lattice \( L \) and the dual lattice \( M \). Then \( X \) has only terminal singularities (resp. canonical singularities) if and only if any cone \( \sigma \in \Sigma \) satisfies the conditions (i) and (ii) (resp. (i) and (iii)).
(i) There exists an element $m \in M_\mathbb{Q}$ such that $\langle u, m \rangle = 1$ for any primitive vector $u$ of $\sigma$;
(ii) There are no other lattice points in the set $\{ u \in \sigma \mid \langle u, m \rangle \leq 1 \}$ except vertices;
(iii) There are no other lattice points in the set $\{ u \in \sigma \mid \langle u, m \rangle < 1 \}$ except the origin.

Theorem 4.2 (Reid [13]). Let $X$ be a quasiprojective toric variety and $V \rightarrow X$ a projective birational toric morphism with $V$ smooth. Then there exists the following diagram

\[
\begin{array}{ccc}
V & \longrightarrow & \varphi \rightarrow X_{\text{can}} \\
\downarrow \varphi & & \downarrow \nu \\
X, & & \\
\end{array}
\]

where

(i) $X_{\text{can}}$ has canonical singularities, $\nu : X_{\text{can}} \rightarrow X$ is a projective birational morphism, and $K_{X_{\text{can}}}$ is $\nu$-ample;
(ii) $Y$ has $\mathbb{Q}$-factorial terminal singularities, $\varphi : Y \rightarrow X$ is a projective birational morphism, and $K_Y$ is $\varphi$-nef, i.e. $\varphi$ is crepant.

Definition 4.3. In Theorem 4.2, we say that:

(i) the variety $X_{\text{can}}$ is a relative canonical model of $X$;
(ii) the variety $Y$ is a relative minimal model of $X$.

Convention 4.4. In this article, relative minimal models of $X$ are always projective over $X$.

4.2. The Craw–Ishii conjecture. For a finite abelian subgroup $G$ of $\text{SL}_3(\mathbb{C})$, Craw and Ishii proved that every projective crepant resolution of $\mathbb{C}^3/G$ is isomorphic to $M_\theta$ for a suitable parameter $\theta$.

Theorem 4.5 (Craw–Ishii [2]). For a finite abelian subgroup $G$ of $\text{SL}_3(\mathbb{C})$, let $Y$ be a relative minimal model of $\mathbb{C}^3/G$. Then $Y$ is isomorphic to $M_\theta$ for a suitable $\theta$.

They conjectured that the same holds without the abelian assumption. We further conjecture that the same is true for all finite group $G \subset \text{GL}_3(\mathbb{C})$ if $Y$ is a smooth relative minimal model.

Conjecture 4.6 (Craw–Ishii conjecture). For a finite subgroup $G$ of $\text{GL}_3(\mathbb{C})$, let $Y$ be a relative minimal model of $\mathbb{C}^3/G$. If $Y$ is smooth, then $Y$ is isomorphic to (the birational component $Y_\theta$ of) $M_\theta$ for a suitable $\theta$.

To prove the theorem above, Craw and Ishii showed that a flop of $M_\theta$ is isomorphic to $M_{\theta'}$ for some parameter $\theta'$ as two crepant resolutions are connected by a sequence of flops. This completes the proof because
we already knew that $G$-Hilb $\mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$ by Bridgeland–King–Reid [11] for $G \subset \text{SL}_3(\mathbb{C})$.

Note that for $G \not\subset \text{SL}_3(\mathbb{C})$ we do not have a moduli description of any relative minimal model of $\mathbb{C}^3/G$ yet.

**Remark 4.7.** From Theorem 4.5 we have a simple corollary as follows. For $G$ a finite abelian subgroup in $\text{SL}_3(\mathbb{C})$ and $Y$ a projective crepant resolution of $X = \mathbb{C}^3/G$, there exist:

(i) a $G$-brickset $\mathcal{S}$ for $Y \to X$;
(ii) a stability parameter $\theta$ such that $\Gamma \in \mathcal{S}$ is $\theta$-stable.

We use this to prove the Craw–Ishii conjecture for some cases. ♦

4.3. The first case: $r = abc + a + b + 1$. In this section, as the first example, we prove the Craw–Ishii conjecture for the group $G$ of type $\frac{1}{r}(1, a, b)$ with $r = abc + a + b + 1$ where $a, b, c$ are positive integers with $b$ coprime to $a$. Consider the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, b),$$

and the cone $\sigma_+ = \text{Cone}(e_1, e_2, e_3)$. The lattice point $v := \frac{1}{r}(1, a, b)$ is in the interior of the simplex $\Delta$ where

$$\Delta := \{ u \in \sigma_+ \mid \langle u, x_1x_2x_3 \rangle \leq 1 \},$$

with considering the monomial $x_1x_2x_3$ as an element in $M_Q$. Thus the quotient singularity $X := \mathbb{C}^3/G$ defined by the cone $\sigma_+$ is not a canonical singularity.

Consider the star subdivision of $\sigma_+$ at $v$. Let $X_v$ denote the toric variety corresponding to the star subdivision. From Section 3.1 we have:

(i) the cone $\sigma_1 = \text{Cone}(v, e_2, e_3)$ defines a smooth open set $U_1$;
(ii) the cone $\sigma_2 = \text{Cone}(e_1, v, e_3)$ defines the affine toric variety $U_2 = \mathbb{C}^3/G_2$ with $G_2$ of type $\frac{1}{a}(1, -r, b)$;
(iii) the cone $\sigma_3 = \text{Cone}(e_1, e_2, v)$ defines the affine toric variety $U_3 = \mathbb{C}^3/G_3$ with $G_3$ of type $\frac{1}{b}(1, a, -r)$.

Note that $\sigma_2$ and $\sigma_3$ define Gorenstein 3-fold abelian quotient singularities. Hence the star subdivision at $v$ has only canonical singularities. Since a star subdivision induces a projective toric morphism, from the ramification formula (3.1), it follows that the star subdivision of $\sigma_+$ at $v$ defines the relative canonical model $X_{\text{can}}$ of $X$, i.e. $X_v$ is the relative canonical model of $X$. 


Suppose that $\varphi : Y \to X$ is a relative minimal model. Then there exists a projective crepant morphism $\overline{\varphi} : Y \to X_v$ fitting into the following commutative diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & X_v \\
\downarrow & & \downarrow \\
X & & X
\end{array}
$$

As is discussed in Section 3.2, we have the following three induced projective crepant morphisms:

(i) $\overline{\varphi}_1 : Y_1 \to U_1 = \mathbb{C}^3$;
(ii) $\overline{\varphi}_2 : Y_2 \to U_2 = \mathbb{C}^3/G_2$;
(iii) $\overline{\varphi}_3 : Y_3 \to U_3 = \mathbb{C}^3/G_3$.

Here $Y_k$ denotes the toric variety given by the cones $\sigma$ such that $\sigma \subset \sigma_k$.

Note that $\overline{\varphi}_1$ is an isomorphism. From the Craw–Ishii Theorem [2], it follows that there exists a generic GIT parameter $\theta^{(k)}$ such that $Y_k$ is the moduli space of $\theta^{(k)}$-stable $G_k$-constellations. Thus we have:

(i) a $G_k$-brickset $\mathcal{G}_k$ for $Y_k \to U_k$;
(ii) a stability parameter $\theta^{(k)}$ such that $\Gamma \in \mathcal{G}_k$ is $\theta^{(k)}$-stable.

By Theorem 3.12, there exists a $G$-brickset $\mathcal{G}$ for $Y \to X$. Note that to the cone $\sigma_1$, we assign the $G$-brick

$$
\Gamma := \phi_1^{-1}(1) = \{1, x_1, x_1^2, \ldots, x_1^{r-1}\},
$$

which satisfies $S(\Gamma) = \sigma_1^g \cap M$.

Now we show that there exists a parameter $\theta$ such that every $\Gamma \in \mathcal{G}$ is $\theta$-stable. First note that since $a$ and $b$ are coprime, by Lemma 3.16 there exists $\theta \in \Theta$ satisfies (3.17) for given $\theta^{(2)}$ and $\theta^{(3)}$.

Define the GIT parameter $\psi \in \Theta$ by

$$
\psi(\rho) = \begin{cases} 
-1 & \text{if } 0 \leq \text{wt}(\rho) < b, \\
-1 & \text{if } \text{wt}(\rho) = a + b, \\
1 & \text{if } r - b - 1 \leq \text{wt}(\rho) < r, \\
0 & \text{otherwise.}
\end{cases}
$$

Remark 4.9. The parameter $\psi$ in (4.8) has the following properties for each $k$.

(i) For any $\chi \in G_k^g$, we have $[(\phi_k)_*(\psi)](\chi) \equiv 0$, i.e.

$$
\sum_{i=j \mod a_k} \psi(\rho_i) = 0
$$

for each $0 \leq j < a_k$.

(ii) For any $i$ with $0 \leq i < a_k$, we have $\psi(\rho_i) < 0$.

(iii) For a monomial $m$ of weight $i$ with $a_k \leq i < r$, define

$$
A := A(m) := \{x_k^l \cdot m \mid \phi_k(x_k^l \cdot m) = \phi_k(m) \text{ for some } l \geq 0\}.
$$
Then we have $\psi(A) > 0$.
These are the key properties we use for the existence of a suitable parameter.

\[\theta := \theta_P + m\psi,\]

where $\theta_P$ satisfying (3.17). If a $G$-brick $\Gamma$ is in $\mathcal{S}$ described above, then $\Gamma$ is $\theta$-stable.

**Proof.** Let $\Gamma$ be a $G$-brick in $\mathcal{S}$ and $\sigma$ the corresponding cone.

If the cone $\sigma$ is contained in $\sigma_1$, then the corresponding $G$-graph is $\Gamma = \{1, x_1, x_2, \ldots, x_{r-2}, x_{r-1}\}$. Note that any nonzero proper submodule $\mathcal{G}$ of $C(\Gamma)$ is given by $A = \{x_j^1, x_j^2, \ldots, x_j^{r-2}, x_j^{r-1}\}$ for some $1 \leq j \leq r - 1$ by Lemma 2.9. Thus $\psi(\mathcal{G}) > 0$ by definition.

From this, it follows that $\Gamma$ is $\theta$-stable for sufficiently large $m$.

For the other cases, assume that $\Gamma$ is the $G$-brick corresponding to a cone $\sigma \subset \sigma_k$. Let $\Gamma'$ be the $G_k$-brick corresponding to $\Gamma$ and $\mathcal{G}$ a nonzero proper submodule of $C(\Gamma)$ with $\mathbb{C}$-basis $A \subset \Gamma$. Recall that

$$\Gamma = \{m \in M | \phi_k(m) \in \Gamma'\},$$

as in Proposition 3.10. We have the following two cases:

(i) $A = \phi_k^{-1}(\phi_k(A)) := \{m \in M | \phi_k(m) \in \phi_k(A)\};$

(ii) $A \subsetneq \phi_k^{-1}(\phi_k(A))$.

In case (i), $\psi(\mathcal{G}) = 0$ by definition. Moreover, we can see that the set $\phi_k(A)$ defines a nonzero proper submodule $\mathcal{G}'$ of $C(\Gamma')$ as follows. Let $\xi_1, \xi_2, \xi_3$ be the eigencoordinates with respect to the $G_k$-action on $\mathbb{C}^3$. Suppose that $\xi_j \cdot \phi_k(m_\rho) \in \Gamma'$ for some $m_\rho \in A$. Lemma 3.9 implies that there exist $m_\rho'$ and $x_i$ such that

$$\phi_k(x_i \cdot m_\rho') = \xi_j \cdot \phi_k(m_\rho) \quad \text{with} \quad \phi_k(m_\rho') = \phi_k(m_\rho).$$

As $A$ is a $\mathbb{C}$-basis of $\mathcal{G}$, Lemma 2.9 implies that $x_i \cdot m_\rho' \in A$. Thus $\xi_j \cdot \phi_k(m_\rho)$ is in $\phi_k(A)$. This shows that $\phi_k(A)$ is a $\mathbb{C}$-basis of a nonzero proper submodule $\mathcal{G}'$ of $C(\Gamma')$. Since

$$(\phi_k)_*(\theta) \equiv \theta^{(k)},$$

we have $\theta(\mathcal{G}) = \theta^{(k)}(\mathcal{G}') > 0$ as $\mathcal{G}'$ is a submodule of the $\theta^{(k)}$-stable constellation $C(\Gamma')$.

Consider case (ii). Observe that

$$\sum_{\phi_k(\rho') \in \phi_k(A)} \psi(\rho') = 0$$
by the definition of $\psi$. Lemma 3.6 implies that if $m$ in $\phi_k^{-1}(\phi_k(A)) \setminus A$, then $0 \leq \text{wt}(\rho) < r - b$. Moreover we have

$$\sum_{\rho' \in \phi_k^{-1}(\phi_k(A)) \setminus A} \psi(\rho') < 0.$$  

Thus $\psi(G) > 0$. Therefore $\theta(G) > 0$ for sufficiently large $m$.

Since there exist a finite number of $G$-bricks in $\mathcal{G}$, we are done. □

Remark 4.12. The parameter $\theta$ in Proposition 4.10 does not need to be generic. However, since the condition for $\theta$ is an open condition, there exists a generic parameter in a small neighbourhood of $\theta$. ◊

As we have proved the existence of a suitable generic parameter $\theta$, we have the following theorem.

Theorem 4.13. For positive integers $a, b, c$ with $b$ coprime to $a$, let $G$ be the group of type $\frac{1}{2}(1, a, b)$ with $r = abc + a + b + 1$. Assume that $Y \to X := \mathbb{C}^3/G$ is any relative minimal model of $X$. Then $Y$ is isomorphic to the birational component $Y_\theta$ of the moduli space $M_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.

In some small cases (e.g., $\frac{1}{20}(1, 3, 4)$), the irreducible component $Y_\theta$ is actually a connected component of $M_\theta$. However, we do not have a big example such that $M_\theta$ itself is irreducible.

Question 4.14. In the situation as above, is $M_\theta$ in the theorem above irreducible?

Example 4.15. Let $G$ be the group of type $\frac{1}{20}(1, 3, 4)$ as in Example 3.2. Consider the star subdivision at $v = \frac{1}{20}(1, 3, 4)$. Then the star subdivision gives the relative canonical model of $X = \mathbb{C}^3/G$.

Let $\phi: Y \to X$ be a relative minimal model whose fan is shown in Figure 4.1.

There exist the two induced projective crepant resolutions:

(i) $\phi_2: Y_2 \to U_2 = \mathbb{C}^3/G_2$;

(ii) $\phi_3: Y_3 \to U_3 = \mathbb{C}^3/G_3$.

Here $G_2$ is of type $\frac{1}{2}(1, 1, 1)$ and $G_3$ is of type $\frac{1}{4}(1, 3, 0)$. Note that $Y_2$ and $Y_3$ are $G_2$-Hilb $\mathbb{C}^3$ and $G_3$-Hilb $\mathbb{C}^3$, respectively.

We illustrate how to calculate $G$-bricks associated to the following cones:

$$\sigma_1 := \text{Cone} \left( (1, 0, 0), \frac{1}{20}(1, 3, 4), \frac{1}{20}(7, 1, 8) \right),$$

$$\sigma_2 := \text{Cone} \left( (1, 0, 0), \frac{1}{20}(1, 3, 4), \frac{1}{20}(15, 5, 0) \right).$$

Note that the cone $\sigma_1$ is in $\text{Cone}(e_1, v_1, e_3)$. Moreover, observe that the left fan corresponds to $G_2$-Hilb $\mathbb{C}^3$ with $G_2$ of type $\frac{1}{2}(1, 1, 1)$. Consider the cone $\sigma_1'$ in the fan of $G_2$-Hilb $\mathbb{C}^3$ corresponding to $\sigma_1$. Let
\( \xi, \eta, \zeta \) denote the eigencoordinates for \( G_2 \). The corresponding \( G_2 \)-brick is

\[
\Gamma_1' = \{1, \zeta, \zeta^2\}.
\]

The \( G \)-brick \( \Gamma_1 \) corresponding to \( \sigma_1 \) is

\[
\Gamma_1 = \{x_1^{m_1}y_1^{m_2}z_1^{m_3} \in \overline{M} | \phi_2(x_1^{m_1}y_1^{m_2}z_1^{m_3}) \in \Gamma_1' \}
\]

where the left round down function \( \phi_2 \) is defined by

\[
\phi_2(x_1^{m_1}y_1^{m_2}z_1^{m_3}) = \xi^{m_1} \eta^{\left\lfloor \frac{1}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3}m_3 \right\rfloor} \zeta^{m_3}.
\]

Thus

\[
\Gamma_1 = \begin{pmatrix}
 y^{-2}z^2 & y^{-1}z^2 & z^2 & y_2z^2 & y_3z^2 \\
y^{-1}z & yz & y^2z & y^3z & y^4z & y^5z & y^6
\end{pmatrix}
\]

Observe that the cone \( \sigma_2 \) is in \( \text{Cone}(e_1, e_2, v_1) \). The right fan is the fan of \( G_3 \)-Hilb \( \mathbb{C}^3 \), where \( G_3 \) is of type \( \frac{1}{3}(1, 3, 0) \). Let \( \alpha, \beta, \gamma \) be the eigencoordinates. For the cone \( \sigma_2' \) corresponding to \( \sigma_2 \), observe that the corresponding \( G_3 \)-brick is

\[
\Gamma_2' = \{1, \beta, \beta^2, \beta^3\}.
\]

The \( G \)-brick \( \Gamma_2 \) corresponding to \( \sigma_2 \) is

\[
\Gamma_2 = \{x_1^{m_1}y_1^{m_2}z_1^{m_3} \in \overline{M} | \phi_3(x_1^{m_1}y_1^{m_2}z_1^{m_3}) \in \Gamma_2' \}
\]
where the right round down function $\phi_3$ is

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \alpha^{m_1} \beta^{m_2} \gamma^{m_3} \lfloor \frac{1}{\gamma} m_1 + \frac{1}{\gamma} m_2 + \frac{1}{\gamma} m_3 \rfloor.$$

Thus

$$\Gamma_2 = \begin{cases} y^3 z^{-2} & y^3 z^{-1} & y^3 z & y^3 z^2 \\ y^2 z^{-1} & y^2 z & y^2 z^2 & y^2 z^3 \\ y & yz & yz^2 & yz^3 \\ 1 & z & z^2 & z^3 \end{cases}.$$

Note that $S(\Gamma_1) = \sigma_Y^1 \cap M$ and $S(\Gamma_2) = \sigma_Y^2 \cap M$.

Now we turn to stability parameters. Since $Y_k$ is $G_k$-Hilb for each $k = 2, 3$, from (2.4) we can take

$$\theta^{(2)} = (-2, 1, 1), \quad \theta^{(3)} = (-3, 1, 1, 1).$$

Then the condition (3.17) of $\theta_P$ for given $\theta^{(2)}$ and $\theta^{(3)}$ is

$$\begin{cases} -2 = \sum_{t=0}^6 \theta_P(\rho_d), \\ 1 = \sum_{t=0}^6 \theta_P(\rho_{d+1}), \\ 1 = \sum_{t=0}^5 \theta_P(\rho_{d+2}), \\ -3 = \sum_{t=0}^4 \theta_P(\rho_d), \\ 1 = \sum_{t=0}^4 \theta_P(\rho_{d+1}), \\ 1 = \sum_{t=0}^4 \theta_P(\rho_{d+2}), \\ 1 = \sum_{t=0}^3 \theta_P(\rho_{d+3}). \end{cases}$$

Take

$$\theta_P = (-3, 0, 0, 0, 0, 1, 1, 1, 0, \ldots, 0)$$

as a solution of the equations above. For $\psi$ in (4.8), define $\theta = \theta_P + m\psi$:}

$$\theta(\rho_i) = (\theta_P + m\psi)(\rho_i) = \begin{cases} -3 - m & \text{if } i = 0, \\ -m & \text{if } 1 \leq i \leq 3, \\ 0 & \text{if } i = 4, \\ 1 & \text{if } i = 5 \text{ or } 6, \\ 1 - m & \text{if } i = 7, \\ m & \text{if } 15 \leq i \leq 19, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the $G$-brick $\Gamma_2$ above:

$$\Gamma_2 = \begin{cases} y^3 z^{-2} & y^3 z^{-1} & y^3 z & y^3 z^2 \\ y^2 z^{-1} & y^2 z & y^2 z^2 & y^2 z^3 \\ y & yz & yz^2 & yz^3 \\ 1 & z & z^2 & z^3 \end{cases}.$$
As examples, consider the two submodules $G$, $H$ generated by $A$ and $B$, respectively, where
\[
A = \left\{ y^3z^{-2}, y^3z^{-1}, y^3, y^3z, y^3z^2, y^2z^2 \right\},
\]
\[
B = \left\{ y^3, y^3z, y^3z^2, y^2, y^2z, y^2z^2, y, yz, yz^2, yz^3, yz^4, 1, z, z^2, z^3, z^4 \right\}.
\]
First consider the submodule $G$. Note that $\psi(G) = 0$. By definition, note that $\phi_3(A) = \{\beta^2, \beta^3\}$ forms a basis of a submodule $G'$ of $C(\Gamma_2')$ with $\theta(G) = \theta^{(3)}(G')$. Thus
\[
\theta(G) = \theta^{(3)}(G') = 2 > 0.
\]
For the submodule $H'$, note that $\phi_3^{-1}(\phi_3(B))$ contains $y^2z^{-1}, y^3z^{-1}$ and $y^3z^{-2}$. Observe that $\psi(H) > 0$. Thus $\theta(H)$ is positive for large enough $m$. More precisely,
\[
\theta(H) = -3 + 1 + m + m = 2m - 1
\]
is positive if $m > \frac{1}{2}$.
\[\Box\]

4.4. The second case: $r = abc + a - 2b + 1$. Consider the group of type $\frac{1}{r}(1, a, b)$. Assume that the star subdivision at $v = \frac{1}{r}(1, a, b)$ gives:
(i) $\sigma_2 := \text{Cone}(e_1, v, e_3)$ is of type $\frac{1}{a}(1, 1, 1)$ for $a \geq 4$;
(ii) $\sigma_3 := \text{Cone}(e_1, e_2, v)$ is a Gorenstein quotient singularity.

This means that:
(i) $-r \equiv 1 \mod a$;
(ii) $1 - r + b \equiv 3 \mod a$;
(iii) $1 - r + a \equiv 0 \mod b$.

In the rest of this section, we consider the case where
\[
r = abc - 2b + a + 1 \quad \text{with} \quad b = ak + 1, a \geq 4
\]
for some positive integers $c, k$. Consider the lattice
\[
L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, b),
\]
Let $v$ and $w$ denote the lattice points
\[
v := \frac{1}{r}(1, a, b) \quad \text{and} \quad w := \frac{1}{r}\left(\frac{r + 1}{a}, 1, \frac{r + b}{a}\right).
\]
Let $X_v$ denote the toric variety corresponding to the star subdivision at $v$. In this case, $X_v$ is not the relative canonical model of $X = \mathbb{C}^3/G$ because the quotient of type $\frac{1}{a}(1, 1, 1)$ is not canonical for $a \geq 4$. The relative canonical model depends on $c$. We have the two cases:
(a) $c \geq 2$;
Case (a): $c \geq 2$. Consider the case where $c \geq 2$. In this case, the relative canonical model is given by the fan consisting of the following five cones and their faces:

\[
\begin{align*}
\sigma_1 &= \text{Cone}(v, e_2, e_3), \\
\sigma_3 &= \text{Cone}(e_1, e_2, v), \\
\sigma_4 &= \text{Cone}(w, v, e_3), \\
\sigma_6 &= \text{Cone}(e_1, v, w), \\
\sigma_7 &= \text{Cone}(e_1, w, e_3).
\end{align*}
\]

Indeed, the cone $\sigma_2$ defines a Gorenstein quotient singularity and the others define smooth affine toric open sets. We can check directly $K_{X_{\text{can}}}$ is ample over $X$.

Since there exists a projective morphism $X_{\text{can}} \to X_v$, for every relative minimal model $\varphi: Y \to X$, we have a projective morphism $\overline{\varphi}: Y \to X_v$ fitting into:

\[
\begin{array}{ccc}
Y & \longrightarrow & X_{\text{can}} \\
\downarrow \varphi & & \downarrow \\
X_v & \longrightarrow & X
\end{array}
\]

The morphism $\overline{\varphi}$ induces two projective morphisms:

(i) $\overline{\varphi}_2: Y_2 \to U_2 = \mathbb{C}^3/G_2$;
(ii) $\overline{\varphi}_3: Y_3 \to U_3 = \mathbb{C}^3/G_3$. 

Figure 4.2. Canonical model for $c \geq 2$
As is seen above, $G_2$ is of type $\frac{1}{a}(1,1,1)$ and the induced morphism $\varphi_2$ is given by

$$G_2\text{-Hilb } \mathbb{C}^3 \to \mathbb{C}^3/G_2$$

where $G_2$ is of type $\frac{1}{a}(1,1,1)$. Thus it follows that there exist a brickset $\mathcal{S}_2$ for $Y_2 \to U_2$ and $\theta^{(2)}$ for the brickset $\mathcal{S}_2$. On the other hand, since $U_3$ is a Gorenstein quotient singularity, by the Craw–Ishii Theorem [2], there exists a brickset $\mathcal{S}_3$ for $Y_3 \to U_3$ and $\theta^{(3)}$ for the brickset $\mathcal{S}_3$.

From Theorem 3.12, there is a $G$-brickset for $Y \to X$. Now it suffices to find a GIT parameter $\theta$ such that every $\Gamma \in \mathcal{S}$ is $\theta$-stable. Define the GIT parameter $\psi \in \Theta$ by

$$\psi(\rho) = \begin{cases} 
-1 & \text{if } 0 \leq \text{wt}(\rho) < b, \\
-1 & \text{if } \text{wt}(\rho) = 2ab - 5b + 3, \\
1 & \text{if } \text{wt}(\rho) = r - a - b + 2, \\
1 & \text{if } r - b \leq \text{wt}(\rho) < r, \\
0 & \text{otherwise.}
\end{cases}$$

Note that $\psi$ above has the same properties in Remark 4.9. Thus the same proof works for the existence of $\theta$ as in Proposition 4.10. Therefore the following theorem follows.

**Theorem 4.16.** Consider positive integers $a, k, c$ with $c \geq 2$, $a \geq 4$ and $b = ak+1$. Let $G$ be the group of type $\frac{1}{a}(1,a,b)$ with $r = abc + a - 2b + 1$. Let $Y \to X := \mathbb{C}^3/G$ be a relative minimal model of $X$. Then $Y$ is isomorphic to the birational component $Y_\theta$ of the moduli space $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.

Case (b): $c = 1$. For the case where $c = 1$, the fan of the relative canonical model consists of the following four cones and their faces:

$$\sigma_1 = \text{Cone}(v, e_2, e_3), \quad \sigma_5 = \text{Cone}(e_1, e_2, v, w),$$

$$\sigma_4 = \text{Cone}(w, v, e_3), \quad \sigma_7 = \text{Cone}(e_1, w, e_3).$$

Indeed, the cone $\sigma_5$ defines a toric Gorenstein singularity and hence it is canonical. Note that since the cone $\sigma_5$ is not simplicial, the corresponding affine toric variety is not a quotient type. In particular, the relative canonical model does not need to be obtained by a sequence of star subdivisions because a star subdivision of a simplicial fan is simplicial.

Note that since $X_{can}$ is Gorenstein, every relative minimal model is smooth. However, some relative minimal of $X$ does not have a morphism to $X_\theta$ (See Example 4.18). For a relative minimal model $Y$ admitting a morphism to $X_\theta$ and for $a \geq 6$, we can prove in the same
Figure 4.3. Canonical model for $c = 1$

way as in Case (a) with the following $\psi$:

$$
\psi(\rho) = \begin{cases} 
-1 & \text{if } 0 \leq \text{wt}(\rho) < b, \\
-1 & \text{if } \text{wt}(\rho) = ab - 5b + 3, \\
1 & \text{if } \text{wt}(\rho) = r - a - b + 2, \\
1 & \text{if } r - b \leq \text{wt}(\rho) < r, \\
0 & \text{otherwise.}
\end{cases}
$$

Proposition 4.17. For positive integers $a,k$, let $G$ be the group of type $\frac{1}{r}(1,a,b)$ with $r = ab + a - 2b + 1$ and $b = ak + 1$. Furthermore assume that $a \geq 6$. Let $X_v$ denote the toric variety given by the star subdivision of $\sigma_+$ at $v = \frac{1}{r}(1,a,b)$. Let $Y \to X := \mathbb{C}^3/G$ be a relative minimal model admitting a morphism $Y \to X_v$. Then $Y$ is isomorphic to the birational component $Y_\theta$ of the moduli space $\mathcal{M}_\theta$ of $\theta$-stable $G$-constellations for a suitable parameter $\theta$.

Example 4.18. Consider the group $G$ of type $\frac{1}{39}(1,5,11)$. Then the star subdivision at $v = \frac{1}{39}(1,5,11)$ gives:

(i) $\sigma_2 := \text{Cone}(e_1,v,e_3)$ corresponds to the quotient singularity of type $\frac{1}{8}(1,1,1)$;

(ii) $\sigma_3 := \text{Cone}(e_1,e_2,v)$ corresponds to the quotient singularity of type $\frac{1}{11}(1,5,5)$.

As is discussed above, the relative canonical model $X_{\text{can}}$ of $X = \mathbb{C}^3/G$ is Gorenstein, but not $\mathbb{Q}$-factorial.

Let $v_i$ denote the lattice point $\frac{1}{r}(i,5i,11i)$ where $\bar{\cdot}$ denotes the residue modulo $r$. In particular, $v_1 = v$ and $v_8 = w$. Note that there exists a plane $\Pi$ containing $e_1$, $e_2$, $v_1$ and $v_8$. Observe that the lattice
points \( v_4, v_{11}, v_{18}, v_{25}, \) and \( v_{32} \) lie on the plane \( \Pi \). Thus subdividing the cone \( \sigma_5 \) into smooth cones only using these points defines a crepant resolution of the toric singularity given by \( \sigma_5 \) where

\[
\sigma_5 = \text{Cone}(e_1, e_2, v_1, v_8).
\]

In Figure 4.4, the variety \( Y \) is a relative minimal model of \( X \) admitting a morphism to \( X_v \). Actually one can prove that \( Y \) is isomorphic to \( Y_\theta \) for some \( \theta \). On the other hand, the variety \( Z \) is a relative minimal model having no morphism to \( X_v \). At this moment, we do not know that whether \( Z \) is isomorphic to \( Y_\theta \) for some \( \theta \). \( \diamond \)
4.5. Discussions.

4.5.1. Smoothness of minimal models. From Theorem 4.2, it follows that if the relative canonical model \( X_{\text{can}} \) is Gorenstein, then any relative minimal model is Gorenstein. Since a toric Gorenstein 3-fold terminal singularity is smooth, every relative minimal model is smooth if the relative canonical model \( X_{\text{can}} \) is Gorenstein for \( G \) being abelian. However, we do not know any sufficient condition for the group \( G \) of type \( \frac{1}{r}(1, a, b) \) having the Gorenstein relative canonical model \( X_{\text{can}} \) of \( \mathbb{C}^3/G \).

**Question 4.19.** Let \( G \) be the group of type \( \frac{1}{r}(1, a, b) \) with \( a+b+1 < r \). Let \( X \) be the quotient \( \mathbb{C}^3/G \) and \( X_{\text{can}} \) the relative canonical model of \( X \). When is \( X_{\text{can}} \) Gorenstein? If so, when can we obtain \( X_{\text{can}} \) by a sequence of star subdivisions?

4.5.2. Other stability parameters. The theorems above said that for a relative minimal model \( Y \) there exists some parameter \( \theta \) such that \( Y_{\theta} \) is isomorphic to \( Y \). We can ask whether \( Y_{\theta} \) is a relative minimal model for all generic \( \theta \) or not.

Sara Muhvić calculated the following:

(i) for the type of \( \frac{1}{10}(1, 2, 3) \), \( G\)-Hilb \( \mathbb{C}^3 \) is smooth but not a relative minimal model;

(ii) for the type of \( \frac{1}{24}(1, 3, 5) \), \( G\)-Hilb \( \mathbb{C}^3 \) is not even smooth.

Thus it seems that there exist few chambers in \( \Theta \) giving a relative minimal model of \( \mathbb{C}^3/G \).

**Question 4.20.** Let \( G \) be the group of type in Section 4.3 or Section 4.4. For which \( \theta \), is \( Y_{\theta} \) a relative minimal model?

4.5.3. Existence of stability parameters. Let \( Y \to X = \mathbb{C}^3/G \) be a relative minimal model admitting a morphism to \( X_v \), where \( X_v \) is the toric variety given by the star subdivision of \( \sigma_+ \) at \( v \). The main theorem was proved by showing the three statements:

(i) there exists a \( G \)-brickset \( \mathcal{S} \) for \( Y \to X \) using round down functions;

(ii) the linear map \( \Theta \to \Theta^{(1)} \oplus \Theta^{(2)} \oplus \Theta^{(3)} \) is surjective in (3.17);

(iii) there exists a stability parameter \( \psi \) satisfying (4.9).

To prove (i), we only used the existence of a \( G_k \)-brickset for \( G_k \), whose order is smaller than that of \( G \). When we proved (ii), we only use the assumption that \( a \) and \( b \) are coprime. However, showing the existence of \( \psi \) in (iii) was done on a case by case basis in Section 4.3 and Section 4.4. It would be interesting if we have a systematic way to produce such a parameter \( \psi \).

**Question 4.21.** Is there a systematic method to find a stability parameter \( \psi \) satisfying the properties in Remark 4.9 for a star subdivision?
Appendix A. \(\frac{1}{39}(1,5,11)\) Type

Let \(G\) be the group of type \(\frac{1}{39}(1,5,11)\) as in Example 4.18. Consider the relative minimal model \(Z\) in Figure 4.4. In this section, although we cannot see that \(Z\) is isomorphic to the birational component \(Y_\theta\) of \(\mathcal{M}_\theta\), we show that there exists a \(G\)-brickset for \(Z \to X = \mathbb{C}^3/G\).

Although there is no morphism \(Z \to X_u\), there exists a morphism \(\varphi: Z \to X_u\) where \(X_u\) is the toric variety given by the star subdivision at \(u = v_4 = \frac{1}{39}(4,20,5)\). The star subdivision of \(\sigma_+\) at \(u\) produces the three cones:

\[\sigma_1 = \text{Cone}(u,e_2,e_3), \quad \sigma_2 = \text{Cone}(e_1,e_2,u), \quad \sigma_3 = \text{Cone}(e_1,u,e_3).\]

The morphism \(\varphi\) induces the following three morphisms:

(i) \(\varphi_1: Z_1 \to \mathbb{C}^3/G_1\), where \(G_1\) is of type \(\frac{1}{4}(1,0,1)\);
(ii) \(\varphi_2: Z_2 \to \mathbb{C}^3/G_2\), where \(G_2\) is of type \(\frac{1}{20}(4,1,5)\);
(iii) \(\varphi_3: Z_3 \to \mathbb{C}^3/G_3\), where \(G_3\) is of type \(\frac{1}{5}(4,0,1)\).

As is shown in Figure A.1, note that for \(k = 1,3\), the morphism \(\varphi_{2k}: Z_k \to \mathbb{C}^3/G_k\) is given by

\[G_k\text{-Hilb}\mathbb{C}^3 \to \mathbb{C}^3/G_k.\]

Thus, to show the existence of a \(G\)-brickset \(\mathcal{G}\) for \(Z \to X\), it only remains to show there exists a \(G_2\)-brickset for \(\varphi_2: Z_2 \to \mathbb{C}^3/G_2\) by Theorem 3.12. Considering the star subdivision of \(\text{Cone}(e_1,v_4,e_3)\) at \(v_8\), one can see that there exists a stability parameter \(\theta^{(2)}\) such that \(Z_2\) is isomorphic to the birational component of the moduli space of \(\theta^{(2)}\)-stable \(G_2\)-constellations in a similar way to the case in the main
theorem. Therefore we can conclude that there exists a $G$-brickset\(^2\) $\mathcal{S}$ for $Z \to X$.

Finally, we discuss why we cannot see the existence of $\theta$. First, a parameter $\psi$ satisfying (4.9) can be found, e.g. $\psi$ can be defined to be

$$
\psi(\rho_i) = \begin{cases} 
-1 & \text{if } 0 \leq i \leq 18, \\
1 & \text{if } i = 19, \\
1 & \text{if } 20 \leq i \leq 38.
\end{cases}
$$

On the other hand, the linear map

$$
\phi_* = ((\phi_1)_*, (\phi_2)_*, (\phi_3)_*) : \Theta \to \Theta^{(1)} \oplus \Theta^{(2)} \oplus \Theta^{(3)}
$$

is not surjective. Therefore, we cannot tell if there exists a solution for (3.17). However, this does not mean that there are no parameters $\theta$ for the $G$-brickset $\mathcal{S}$. Using a computer, we might be able to find a parameter $\theta$ such that every $\Gamma \in \mathcal{S}$ is $\theta$-stable.

REFERENCES

[1] T. Bridgeland, A. King, M. Reid, The McKay correspondence as an equivalence of derived categories, J. Amer. Math. Soc. 14 (2001), no. 3, 535–554.
[2] A. Craw, A. Ishii, Flops of $G$-Hilb and equivalences of derived categories by variation of GIT quotient, Duke Math. J. 124 (2004), no. 2, 259–307.
[3] D. Cox, J. Little, H. Schenck, Toric Varieties, Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011.
[4] A. Craw, D. Maclagan, R. R. Thomas, Moduli of McKay quiver representations I: The coherent component, Proc. Lond. Math. Soc. (3) 95 (2007), no. 1, 179–198.
[5] A. Craw, D. Maclagan, R. R. Thomas, Moduli of McKay quiver representations II: Gröbner basis techniques, J. Algebra 316 (2007), no. 2, 514–535.
[6] S. Davis, T. Logvinenko, M. Reid, How to calculate $A$-Hilb $\mathbb{C}^n$ for $1/(a, b, 1, \ldots, 1)$, preprint.
[7] Y. Ito, H. Nakajima, McKay correspondence and Hilbert schemes in dimension three, Topology 39 (2000), no. 6, 1155–1191.
[8] Y. Ito, I. Nakamura, Hilbert schemes and simple singularities, New trends in algebraic geometry (Warwick, 1996), 151–233, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.
[9] S.-J. Jung, McKay Quivers and Terminal Quotient Singularities in Dimension 3, PhD thesis, University of Warwick, 2014.
[10] S.-J. Jung, Terminal quotient singularities in dimension three via variation of GIT, in preprint, arxiv:1502.03579.
[11] A. King, Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford Ser.(2) 45 (1994), no. 180, 515–530.
[12] I. Nakamura, Hilbert schemes of abelian group orbits, J. Algebraic. Geom. 10 (2001), no.4, 757–779.
[13] M. Reid, Decomposition of toric morphisms, Arithmetic and geometry, Vol. II, 395–418, Progr. Math., 36, Birkhäuser, Boston, MA, 1983.

\(^2\)You can find the $G$-brickset $\mathcal{S}$ on my website: [http://newton.kias.re.kr/~seungjo/CI1.html](http://newton.kias.re.kr/~seungjo/CI1.html)
[14] M. Reid, *Young person’s guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

**Korea Institute for Advanced Study**, 85 Hoegiro, Dongdaemun-gu, Seoul, 130-722, Republic of Korea

E-mail address: seungjo@kias.re.kr