Asymptotics of the Charlier polynomials via difference equation methods

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Abstract

We derive uniform and non-uniform asymptotics of the Charlier polynomials by using difference equation methods alone. The Charlier polynomials are special in that they do not fit into the framework of the turning point theory, despite the fact that they are crucial in the Askey scheme. In this paper, asymptotic approximations are obtained respectively in the outside region, an intermediate region, and near the turning points. In particular, we obtain uniform asymptotic approximation at a pair of coalescing turning points with the aid of a local transformation. We also give a uniform approximation at the origin by applying the method of dominant balance and several matching techniques.

Keywords: Asymptotic approximation; difference equation; Charlier polynomials; Airy function; matching.

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1 Introduction and statement of results

The Charlier polynomials $C_n^{(a)}(x)$ are discrete orthogonal polynomials such that
\begin{equation}
\sum_{k=0}^{\infty} C_n^{(a)}(k)C_m^{(a)}(k) \frac{a^k}{k!} = e^a a^n n! \delta_{mn}, \quad m, n = 0, 1, \cdots,
\end{equation}
with parameter $a > 0$. An explicit expression for the polynomials is
\begin{equation}
C_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k! (-a)^{n-k};
\end{equation}
cf. Szegő [19] pp.34-35. Here the notation $C_n^{(a)}(x)$ refers to the monic polynomial, as in Bo and Wong [4]. Other notations are also used to stand for the Charlier polynomials. For example, $C_n(x,a) = (-a)^{-n}C_n^{(a)}(x)$; cf. [17] Ch.18. The family of Charlier polynomials occupies a crucial position in the Askey scheme, as illustrated in [17] Fig.18.21.1.

In 1994, Bo and Wong [4] considered the uniform asymptotic expansions for $C_n^{(a)}(n\beta)$ as $n \to \infty$. The uniformity is for $\beta$ in compact subsets of the real interval $(0, +\infty)$. Integral methods are used to obtain the asymptotic formulas, starting with the generating function
\begin{equation}
e^{-aw}(1+w)^x = \sum_{n=0}^{\infty} C_n^{(a)}(x) \frac{w^n}{n!}.
\end{equation}
The uniform interval of \([4]\) covers the seven regions in a 1998 paper \([12]\) of Goh. In these regions, Plancherel-Rotach asymptotics are obtained for the Charlier polynomials, using the integral methods as well.

It is readily seen that (1.2) follows from (1.3); cf. \([19]\). Also, Darboux’s method and the steepest descent method can be applied to derive non-uniform asymptotic approximations. Earlier in 1985, an asymptotic formula for \(C_n^{(a)}(x)\) when \(x < 0\) has been obtained by Maejima and Van Assche \([16]\), using probabilistic arguments.

In \([4]\), Bo and Wong commented that in regard to the asymptotics of the Charlier polynomials, not much is known in the literature. Since then, a quarter century sees new observations made and quite a number of novel tools developed. For instance, in 2001, Dunster \([10]\) made use of the connection with Laguerre polynomials, namely, \(C_n^{(a)}(x) = n! L_n^{(x-n)}(a)\), considered a differential equation with respect to the parameter \(a\), and obtained asymptotic formulas, uniformly in \(x\) or \(a\).

The Riemann-Hilbert approach is a powerful new tool in asymptotic analysis; see Deift \([9]\). For discrete orthogonal polynomials, pioneering work has been done by Baik et al. \([2]\), published in 2007. The method of Baik et al. is applied by Ou and Wong \([18]\), with modifications, to obtain uniform asymptotic approximations for \(C_n^{(a)}(z)\) with non-rescaled complex variable \(z\). A significant feature of \([18]\) is the global uniformity. Asymptotic expansions are derived in three regions that cover the whole complex plane. Yet formulas with explicitly leading coefficients are not written down.

Thus, up to now, quite a lot of facts have been known about the Charlier polynomials, including generating functions, differential equations, recurrence relations, and the orthogonal measure. Uniform and non-uniform asymptotics are derived. Still, the global uniform asymptotic approximations need to be clarified, and the turning point asymptotics of the polynomials are of great interest. The polynomials can, in a sense, serve as a touchstone for new tools and techniques developed.

The main focus of the present investigation will be on difference equation methods.

In a review \([1]\) of Chihara’s book \([6]\), Richard Askey commented on the reason for the renewal of interest in orthogonal polynomials in the 70-80s, that General orthogonal polynomials are primarily interesting because of their 3-term recurrence relation.

The three-term recurrence formula for Charlier polynomials is
\[
xC_n^{(a)}(x) = C_{n+1}^{(a)} + (n + a)C_n^{(a)}(x) + anC_{n-1}^{(a)}(x), \quad n = 0, 1, \ldots ,
\]
with fixed \(a > 0\), and initial data \(C_{-1}^{(a)} = 0\) and \(C_0^{(a)} = 1\); cf. \([4, 6]\).

1.1 Non-oscillatory regions and a neighborhood of the origin

A non-oscillatory region considered here is described as
\[
x = ny, \quad \text{dist}(y, [0, 1]) > r
\]
for an arbitrary positive constant \(r\). The natural re-scaling \(x = ny\) aims to normalize the limiting oscillatory interval to \(y \in [0, 1]\); see Kuijlaars and Van Assche \([15]\) Sec.4.5. There are several ways to derive the asymptotics in these unbounded regions via difference equation. In what follows we take arguments from Wang and Wong \([21]\), and in sprit similar to Van Assche and Geronimo \([20]\).
Denote the coefficients in (1.4) as $a_n = n + a$ and $b_n = an$, and introduce

$$C_n^{(a)}(x) = \prod_{k=1}^{n} w_k(x).$$

We see that $w_1(x) = x - a$, and

$$w_k(x) = x - a_{k-1} - \frac{b_{k-1}}{w_{k-1}(x)}, \quad k = 2, 3, \ldots.$$  \(1.6\)

It is verified in Lemma 1 that

$$w_k = (x - k) \left\{ 1 + \frac{(1 - a)x - k}{(x - k)^2} + O\left(\frac{1}{n^2}\right) \right\},$$

in which the error term is uniform in $k = 1, 2, \ldots, n$, and in dist \((y, [0, 1]) > r\) for arbitrary positive $r$; see Section 2.1 for full details of the proof.

Now substituting (1.7) into (1.5), and making use of the trapezoidal rule, we obtain the asymptotic formula

$$C_n^{(a)}(ny) = n^\frac{y}{y-1} \exp\left(-\frac{a}{y-1}\right) \exp\left\{ n \left[ y \log \frac{y}{y-1} - 1 \right] \right\} (y-1)^n \left[ 1 + O\left(\frac{1}{n}\right) \right],$$

holding uniformly for large $n$ and $y$ keeping a constant distance from \([0, 1]\). The logarithms and square roots take principal branches.

It is worth mentioning that one may derive non-uniform asymptotic approximations for $C_n^{(a)}(x)$ by applying Darboux’s method, starting from (1.3). Other integral methods can be used as well. However, we stick to our theme, using difference equation methods alone.

Section 2.2 will be devoted to a uniform asymptotic analysis of $C_n^{(a)}(x)$ in a neighborhood of $x = 0$. To this end, we apply the method of dominant balance to (1.4), to obtain a subdominant asymptotic solution $C_s(n,x)$, and a dominant asymptotic solution $C_d(n,x)$. The latter is then extended uniformly to a domain $|y| \leq r < 1$, with $x = ny$. The uniform asymptotic approximation for $C_n^{(a)}(x)$ in the domain of uniformity is then determined by matching the outer asymptotics (1.13), as stated in Lemma 2.

A combination of formulas (1.8) with (2.8) yields the following uniform asymptotic approximation:

**Theorem 1.** For arbitrary positive constant $\delta$, it holds

$$C_n^{(a)}(ny) = (-1)^n e^{\frac{a}{y}} \frac{\Gamma(n-ny)}{\Gamma(-ny)} + \varepsilon_1,$$

where

$$|\varepsilon_1| \leq \frac{M_1}{n} \left| e^{\frac{a}{y}} \frac{\Gamma(n-ny)}{\Gamma(-ny)} \right|, \quad n \to \infty,$$

holding uniformly for $|y - 1| \geq \delta$, with $M_1$ being a constant.

Indeed, taking $\delta$ in Lemma 2 to be $1 - \delta'$ with sufficiently small $\delta'$, we see that (1.9) holds for $y \in D_1 : |y| < 1 - \delta'$. On the other hand, from (1.8), putting in use Stirling’s formula, we have (1.9) for $y \in D_2 : |y - [0, 1]| > \delta''$. With appropriately small $\delta'$ and $\delta''$, we have $D_1 \cup D_2$...
Figure 1: Left: Domains of validity. $D_1$ is the shaded disc $|y| < 1 - \delta'$, $D_2$ denotes the outer region $|y-0, 1| > \delta''$ in gray, then $D_1 \cup D_2$ covers $|y-1| \geq \delta$, with small positive $\delta$ and appropriately chosen $\delta'$ and $\delta''$. Right: The intermediate region, bounded by $|y-1| = \delta$ with $y = 1 + \frac{t}{\sqrt{a}}$, and lying outside of the neighborhoods of the turning points $t = \pm 2\sqrt{a}$. The gray parts are where the intermediate asymptotics and the turning point asymptotics match.

covers $|y-1| \geq \delta$; see Figure 1 for an illustration. For $y$ in a neighborhood of the positive real axis, one may have to use the identity $(-1)^n \frac{\Gamma(n-x)}{\Gamma(-x)} = \frac{\Gamma(x+1)}{\Gamma(1+x-n)}$.

It is asked in Bo and Wong [4] whether there exists a uniform asymptotic expansion for $C_n^{(a)}(ny)$ in the interval $-\infty < \beta \leq \delta$, where the constant $\delta \in (0, 1)$. Theorem 1 provides an answer.

1.2 An intermediate region

Now we have explicitly derived an asymptotic formula of $C_n^{(a)}(ny)$ for $|y-1| \geq \delta$. The domain of uniformity contains a neighborhood of infinity, and a neighborhood of $y = 0$, that is, an end-point of the equilibrium measure. What left is an $O(1)$ neighborhood of $y = 1$, namely, $|y-1| < \delta$. We will see that $y = 1$ is of significance since a pair of turning points coalesce there.

The last two decades see dramatic changes in difference equation methods in this respect. For example, Wang and Wong [23, 24] established a turning point theory for second order linear difference equations, the theory is further completed by several authors, including Cao and Li [5]; see also the review article [25]. In the mentioned works, special functions, such as the Airy functions and Bessel functions, are employed to describe the asymptotic behavior at the turning point. However, there is a connection problem to be solved, namely, one has to further determine which solution behaves as this asymptotic solution. In many cases, solving such a connection problem turns out to be a hard question, in particularly using the difference equation methods alone. An attempt to overcome the difficulty has been made by Geronimo [11]. Some of the material here also appears in Huang [13].

In the present paper, we demonstrate, using the Charlier polynomials as an example, how to solve the connection problem. We use three types of asymptotics, respectively in non-oscillatory region, intermediate region (with constants to be determined), and at the turning point (in the forms of a linear combination), all obtained via difference equation methods. Matching adjacent regions outside in, we determine asymptotics in the inner regions, step by step.
Substituting \( C_n^{(a)}(x) = (2a) \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} P_n(x) \) into (1.14) gives the symmetric canonical form
\[
P_{n+1}(x) - (A_n x + B_n) P_n(x) + P_{n-1}(x) = 0, \tag{1.10}
\]
where, as \( n \to \infty, \)
\[
A_n = \frac{1}{\sqrt{2a}} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{5}{2} + 1 \right)} \sim \frac{1}{\sqrt{n}} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \quad B_n = -\frac{n + a}{\sqrt{2a}} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{5}{2} + 1 \right)} \sim \sqrt{n} \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}, \tag{1.11}
\]
with \( \alpha_0 = \frac{1}{\sqrt{a}}, \alpha_1 = -\frac{1}{4\sqrt{a}}, \beta_0 = -\frac{1}{\sqrt{a}}, \beta_1 = \frac{1}{\sqrt{a}} - \sqrt{a}, \) and \( \beta_s = -\alpha_s - \alpha_{s-1} \) for \( s = 1, 2, \ldots. \)

Now we introduce a new local variable \( t \) at \( y \sim 1, \)
\[
x = n \left( 1 + \frac{t}{\sqrt{n}} \right). \tag{1.12}
\]
In what follows, we consider the case when \( t \) is taken away from the real interval \([-\sqrt{n}, 2\sqrt{n}],\) to which all zeros belong. An idea in [14] applies, with modifications, so that in such an outer region, (1.10) possesses a pair of non-vanishing asymptotic solutions of the form
\[
P_n(x) \sim \exp \left( \sqrt{n} \phi_{-1}(t) + \phi_0(t) + \sum_{k=1}^{\infty} \frac{\phi_k(t)}{n^{k/2}} \right), \tag{1.13}
\]
where \( \phi_k(t), \) to be determined, are functions independent of \( n. \) Here, unlike [14], instead of work out an infinite series, we focus on the leading terms \( \phi_{-1}(t) \) and \( \phi_0(t). \)

From (1.12), one writes
\[
x = n \left( 1 + \frac{t}{\sqrt{n}} \right) = (n + 1) \left( 1 + \frac{t}{\sqrt{n+1}} \right) = (n - 1) \left( 1 + \frac{t}{\sqrt{n-1}} \right). \tag{1.14}
\]

Substituting (1.13) into (1.10), and equalizing the constant terms on both sides, we have the first order differential equation
\[
e^{-\phi_{-1}(t)} + e^{\phi_{-1}(t)} = \frac{t}{\sqrt{a}} \tag{1.15}
\]
for \( \phi_{-1}, \) which implies \( e^{\phi_{-1}(t)} = \frac{t+\sqrt{t^2-4a}}{2\sqrt{a}} \). We pick the minus sign first, and obtain a solution to (1.15),
\[
\phi_{-1}(t) = t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} + C_{-1},
\]
such that \( e^{\phi_{-1}(t)} = \frac{t-\sqrt{t^2-4a}}{2\sqrt{a}}, \) where \( C_{-1} \) is a constant independent of \( n. \) Substituting the expression into (1.10) and further equalizing the coefficients of \( 1/\sqrt{n} \) gives
\[
\phi_0(t) = \frac{a}{\sqrt{t^2 - 4a}} + \frac{1}{2} \sqrt{t^2 - 4a} - \frac{t}{2(t^2 - 4a)} + \frac{1}{2} C_{-1}.
\]
Thus we have
\[
\phi_0(t) = -\frac{1}{4} \log(t^2 - 4a) + \frac{1}{4} t \sqrt{t^2 - 4a} + \frac{1}{2} C_{-1} t + C_0,
\]
where \( C_0 \) is a constant. With such \( \phi_{-1}(t) \) and \( \phi_0(t) \), the asymptotic solution (1.13) now reads
\[
P_n(x) \sim C \exp \left( \sqrt{n} \left[ t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} \right] - \frac{1}{4} \log(t^2 - 4a) + \frac{t}{4} \sqrt{t^2 - 4a} \right),
\] (1.16)
where \( x = n(1 + \frac{t}{\sqrt{n}}) \), and \( C = C(n, t) = e^{\sqrt{n}C_{-1} + \frac{1}{2}C_{-1}t + C_0} \).

Similarly, if we take the alternative choice \( e^{\phi'_{-1}(t)} = \frac{t + \sqrt{t^2 - 4a}}{2\sqrt{a}} \), we have the other formal (asymptotic) solution
\[
\tilde{P}_n(x) \sim \tilde{C} \exp \left( -\sqrt{n} \left[ t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} \right] - \frac{1}{4} \log(t^2 - 4a) - \frac{t}{4} \sqrt{t^2 - 4a} \right),
\] (1.17)
where \( \tilde{C} = \tilde{C}(n, t) = e^{-\sqrt{n}\tilde{C}_{-1} - \frac{1}{2}\tilde{C}_{-1}t - \tilde{C}_0} \), with \( \tilde{C}_{-1} \) and \( \tilde{C}_0 \) being constants.

Now we match the asymptotic approximations (1.8) and
\[
C_n^{(a)}(x) \sim (2a)^{\frac{n}{2}} \frac{\Gamma((n + 1)/2)}{\Gamma(1/2)} \left( A(x)P_n(x) + B(x)\tilde{P}_n(x) \right) \quad \text{as} \quad n \to \infty,
\] (1.18)
with \( x = ny = n(1 + \frac{t}{\sqrt{n}}) \), at the transition area described as
\[
n^{1/6} \ll |t| \ll n^{1/4}.
\]

Here \( A(x) \) and \( B(x) \), to be determined, depend only on \( x \).

First, we see that for \( t \in (2\sqrt{a}, +\infty) \), \( \tilde{P}_n(x) \) is dominant and \( P_n(x) \) is recessive. The approximation in (1.18) is also recessive, hence the coefficient \( B(x) \) vanishes. The remaining coefficient in (1.18) and the constants \( C_{-1} \) and \( C_0 \) can be determined by the matching process to give \( A(x) = \left( \frac{\Gamma((x+1)/a)}{\Gamma(1/2)} \right)^{1/2} \), \( C_{-1} = 0 \) and \( C_0 = -\frac{3}{4} \log 2 - \frac{1}{4} \log \pi + \frac{a}{2} \). As a result, we have

**Theorem 2.** For \( x = ny = n(1 + \frac{t}{\sqrt{n}}) \), with \( |y - 1| < \delta \) and \( |t - (\pm \infty, 2\sqrt{a})| > r, \delta \) and \( r \) being positive constants, it holds
\[
C_n^{(a)}(x) \sim \frac{C}{\sqrt{w(x)}} \exp \left( \sqrt{n} \left[ t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} \right] - \frac{1}{4} \log(t^2 - 4a) + \frac{t}{4} \sqrt{t^2 - 4a} \right)
\] (1.19)
as \( n \to \infty \), where \( w(x) = \frac{n^a}{\Gamma(x+1)} \) and \( C = (2a)^{\frac{n}{2}} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} 2^{-\frac{3}{4}} \pi^{-\frac{1}{4}} e^{\frac{a}{4}} \).

**Remark 1.** The assumptions (1.11) are not the same as in Huang-Cao-Wang [13] in that here \( B_n \sim \sqrt{n} \sum \frac{\beta_s}{n^s} \), instead of \( B_n \sim \sum \frac{\beta_s}{n^s} \) in [14]. Nevertheless, after the change of variable (1.12), which is nonlinear in \( n \), we can still apply the method in [13] to extract the leading terms. In this intermediate case the method seems to be superior to the one illustrated in the previous subsection.

It is worth pointing out that altering the assumption has significant impact to the turning point analysis that follows: The Charlier case seems to go beyond the framework of the turning point theory; see Wong [25]. The general assumption in [25] is the difference equation (1.10) with
\[
A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s},
\]
just as in \cite{7}. The types of asymptotic solutions are described by the characteristic equation 
\[ \lambda^2 - (\alpha y + \beta_0)\lambda + 1 = 0, \quad x = n^\beta y, \] 
which has two roots that coincide when \( \alpha y + \beta_0 = \pm 2 \).

Such a \( y \) is a turning point; see \cite{7}. However, in the Charlier case, the \( P_n \) term in (1.10) is
dominant, and the characteristic equation degenerates to \( \alpha y + \beta_0 = 0 \) with \( y = n^{-1} x \), giving rise
to a single critical point at \( y = 1 \). Yet in local variable \( t \), introduced in (1.12), the characteristic
equation takes the form \( \lambda^2 - \frac{t}{\sqrt{a}} \lambda + 1 = 0 \), which locates two turning points at \( t = \pm 2\sqrt{a} \).

1.3 At turning points

In the terminology of Baik et al. \cite{2}, there is a saturated-band-void configuration in a shrinking
neighborhood of \( y = 1 \). Here saturated region is defined as an open subinterval of maximal
length in which the equilibrium measure realizes the lower constraint. A void by definition
is an open subinterval of maximal length in which the equilibrium measure realizes the upper
constraint, namely, 0. The band lies in between.

As mentioned in Remark 1 for the Charlier difference equation (1.10), the index equation
degenerates. With the local re-scaling (1.12), we are capable of determining the band in variable
\( t \) as \( t \in (-2\sqrt{a}, 2\sqrt{a}) \). We proceed to derive the transition asymptotics of \( C_n^{(a)}(x) \) in the band,
or, more challenging, at the turning points \( t = \pm 2\sqrt{a} \), where \( x = n(1 + \frac{x}{\sqrt{n}}) \).

Bo and Wong \cite{4} used Bessel functions to give the approximation of \( C_n^{(a)}(ny) \) for \( 0 < \varepsilon \leq y \leq M < \infty \), restricted to the real axis. In \cite{18}, Ou and Wong addressed global asymptotic
formulas using Riemann-Hilbert approach. The uniform expansions they derived involve a quite
complicated combination of the Airy functions. Tedium calculation is needed if one has to draw
local leading asymptotics from their formulas.

In what follows, we apply the theory of Wang and Wong (see \cite{25}) to obtain the asymptotic
form of solutions at turning points. The most attention will be paid to connect the asymptotic
solution with the Charlier polynomials, by matching the approximation here with those in earlier
subsections.

Still, we start from the standard form difference equation (1.10), relaxing the initial conditions.
As mentioned in Remark 1 the fundamental assumptions on \( A_n \) and \( B_n \) are not fulfilled,
as compared with \cite{20,25}. However, by introducing the local transformation \( x = n(1 + \frac{x}{\sqrt{n}}) \),
the method of Wang and Wong keeps applicable.

As a first step, we treat the turning point \( t = t_0 = 2\sqrt{a} \). The idea in \cite{23} is to seek an
asymptotic solution to (1.10), of the form

\[ Q_n(x) = \sum_{s=0}^{\infty} \chi_s(\xi) \delta^s; \]
as suggested by Costin and Costin \cite{7}, where \( \delta = n^{-\alpha}, \xi = n^\sigma \eta(t) \), with \( \eta(t_0) = 0, \eta(t) \) being a
one-to-one mapping in a neighborhood of \( t = t_0 \). Substituting the expressions in the difference
equation and matching the leading terms, we determine \( \sigma = \frac{1}{3} \) and \( \alpha = \frac{1}{2} \). Ignoring lower order
terms in \( n \), we have

\[ \chi_0(\xi_+) - (A_n x + B_n) \chi_0(\xi) + \chi_0(\xi_-) = 0, \]

where \( \xi_+ = (n+1)^{1/3} \eta(t_+) \) and \( \xi_- = (n-1)^{1/3} \eta(t_-) \) vary around \( \xi, t_\pm \) being given in (1.14). Repeatedly using Taylor expansions, constantly ignoring lower order terms, and taking advantage
of the symmetry form, we further obtain the differential equation

\[ \chi''_0(\xi) = c^3 \xi \chi_0(\xi), \]
where the constant \( c = a^{-1/6}/a'(t_0) \). Similarly, other \( \chi_k \) solve inhomogeneous equations of the same type. This indicates the involvement of Airy functions. As in \([23]\), we may write the asymptotic solution in the more accurate form

\[
Q_n(x) = \chi \left( n^{1/3} \eta + n^{-1/6} \Phi \right) \sum_{k=0}^{\infty} \frac{A_k(\eta)}{n^{k/2}} + n^{-1/6} \chi' \left( n^{1/3} \eta + n^{-1/6} \Phi \right) \sum_{k=1}^{\infty} \frac{B_k(\eta)}{n^{k/2}}, \tag{1.20}
\]

and proceed to determine \( \eta(t), \Phi(t) \) and \( A_0(\eta) \). Here we have used the fact that \( \eta(t) \) defines a conformal mapping at \( t = t_0 \), therefore, the non-oscillating coefficients \( A_k \) and \( B_k \), as functions of \( t \), can be regarded as functions of \( \eta \). Substituting \( \chi \) in (1.20) into (1.10), with details given in Section 4.1, we do have \( \eta(t) \), a conformal mapping at \( t = 2\sqrt{a} \), being positive for \( t > 2\sqrt{a} \), such that

\[
\frac{2}{3} (\eta(t))^{3/2} = t \log \frac{t + \sqrt{t^2 - 4a}}{2\sqrt{a}} - \sqrt{t^2 - 4a}, \quad t \in \mathbb{C} \setminus (-\infty, 2\sqrt{a}). \tag{1.21}
\]

We can further determine

\[
\sqrt{\eta(t)} \Phi(t) = -\frac{t \sqrt{t^2 - 4a}}{4}, \tag{1.22}
\]

where \( \sqrt{\eta(t)} \) and \( \sqrt{t^2 - 4a} \) are positive for \( t > 2\sqrt{a} \). Also, \( A_0(\eta) \) is solved up to a constant factor independent of both \( n \) and \( t \). We pick

\[
A_0(\eta) = \left( \frac{t^2 - 4a}{4a} \right)^{-1/4}. \tag{1.23}
\]

Choosing the Airy function \( \chi \) in (1.20) to be \( \text{Ai} \) and \( \text{Bi} \), and denoting the asymptotic solution respectively as \( Q_n(x) \) and \( \tilde{Q}_n(x) \), we have

\[
C_n^{(a)}(x) \sim (2a)^{\frac{2}{3}} \frac{\Gamma(n + 1/2)}{\Gamma(1/2)} \left( K_1(x)Q_n(x) + K_2(x)\tilde{Q}_n(x) \right);
\]

cf. \([4.20]\). The coefficients \( K_1(x) \) and \( K_2(x) \) are determined by matching the approximation with the intermediate asymptotic behavior \([1.19]\), as conducted in Section 4.1.

**Theorem 3.** For \( x = n \left( 1 + \frac{i}{n} \right) \), in a neighborhood of the turning point \( t = 2\sqrt{a} \), as \( n \to \infty \), it holds

\[
C_n^{(a)}(x) \sim C_{K,n}x^{\frac{1}{12}} \left( \frac{a}{\Gamma(x + 1)} \right)^{-\frac{1}{2}} \left( \frac{t^2 - 4a}{4a} \right)^{-1/4} \text{Ai} \left( n^{1/3} \eta(t) + n^{-1/6} \Phi(t) \right), \tag{1.24}
\]

where \( \eta(t) \) and \( \Phi(t) \) are given in (1.21) and (1.22), and \( C_{K,n} = (2a)^{\frac{2}{3}} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} \left( \frac{\pi}{2a} \right)^{1/4} e^{\frac{a}{2}} \).

Now we turn to the other turning point \( t = -2\sqrt{a} \). In this case, we substitute

\[
C_n^{(a)}(x) = (-1)^n(2a)^{\frac{2}{3}} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} Q_n(x)
\]

into (1.4), to give the canonical difference equation

\[
Q_{n+1}(x) + (A_nx + B_n)Q_n(x) + Q_{n-1}(x) = 0. \tag{1.25}
\]
We still seek asymptotic solutions to \( (1.25) \) of the form \( (1.20) \), with \( \tilde{\eta}, \Phi, A_k \) and \( B_k \) taking the places of \( \eta, \Phi, A_k \) and \( B_k \), respectively. Solving differential equations yields

\[
\frac{2}{3} \tilde{\eta}^{3/2} = t \log \frac{-t + \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a},
\]

(1.26)
such that \( \tilde{\eta}(t) \) is a conformal mapping at \( t = -2\sqrt{a} \) with \( \tilde{\eta}(-2\sqrt{a}) = 0 \), \( \tilde{\eta}(t) > 0 \) for \( t < -2\sqrt{a} \), and \( \tilde{\eta}(t) \sim \left( \frac{3}{2} \right)^{2/3} a^{-1/6}(-(t + 2\sqrt{a})) \) for \( t \sim -2\sqrt{a} \). Here, as before, \( \sqrt{t^2 - 4a} \) is analytic in \( \mathbb{C} \setminus [-2\sqrt{a}, 2\sqrt{a}] \), and behaves like \( t \) as \( t \to \infty \). The logarithm takes principal branch.

Also, one obtains

\[
\sqrt{\tilde{\eta}} \tilde{\Phi} = \frac{1}{4} t \sqrt{t^2 - 4a}.
\]

(1.27)
where \( \tilde{\Phi}(t) \) is analytic in a neighborhood of \( t = -2\sqrt{a} \), and \( \sqrt{\tilde{\eta}(t)} \) take positive values for \( t < -2\sqrt{a} \). The leading coefficient \( A_0(\tilde{\eta}) \) can be determined up to a factor independent of \( n \) and \( t \). We pick

\[
A_0(\tilde{\eta}) = \left( \frac{t^2 - 4a}{4a\tilde{\eta}} \right)^{-1/4} = \exp \left( \frac{\pi i}{2} - \frac{1}{4} \log(t^2 - 4a) + \frac{1}{4} \log(4a\tilde{\eta}) \right),
\]

(1.28)
so that \( A_0(\tilde{\eta}) \) is analytic in a neighborhood of \( t = -2\sqrt{a} \), being real positive for \( t < -2\sqrt{a} \). Here in the neighborhood, \( t \) is an analytic function of \( \tilde{\eta} \).

Now we determine the asymptotic approximation in a upper half neighborhood of \( t = -2\sqrt{a} \), again by a matching process. Careful choice of \( \chi \) has to be made regarding the Stokes line \( \arg \tilde{\eta} = -\frac{2\pi}{3} \), that is, \( \arg\{-t + 2\sqrt{a}\} = -\pi + \arg(t + 2\sqrt{a}) \sim -\frac{2\pi}{3} \) for \( t \) close to \( -2\sqrt{a} \). Hence the solution to \( (1.25) \), corresponding to the Charlier polynomials, has the asymptotic approximation \( K_1(x) \tilde{Q}_n(x) + K_2(x) \hat{Q}_n(x) \); cf. \( (4.23) \), where

\[
\tilde{Q}_n(x) \sim A_0(\tilde{\eta}) \text{Ai} \left( \omega n^{\frac{1}{3}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right), \quad \hat{Q}_n(x) \sim A_0(\tilde{\eta}) \text{Ai} \left( \omega^2 n^{\frac{1}{3}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right),
\]

with \( \omega = e^{2\pi i/3} \), and with \( K_1(x) \) and \( K_2(x) \) to be determined by matching \( (4.23) \) with the intermediate asymptotic formula \( (1.19) \), as carried out in Section 4.2.

**Theorem 4.** For \( x = \left( 1 + \frac{i}{\sqrt{n}} \right) \), in an upper half neighborhood of the turning point \( t = -2\sqrt{a} \), as \( n \to \infty \), it holds

\[
C_n(a) \sim C_{K,n} \frac{x^{\frac{n}{2}} A_0(\tilde{\eta})}{\sqrt{w(x)}} e^{-(x\pi + \frac{\pi}{2})i} \text{Ai} \left( \omega n^{\frac{1}{3}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right),
\]

(1.29)
where \( w(x) = \frac{\Gamma(x+i\pi)}{\Gamma(x+1)} \), \( \tilde{\eta}(t) \) and \( \tilde{\Phi}(t) \) are given in \( (1.26) \) and \( (1.27) \), \( A_0(\tilde{\eta}) \) takes the branch as in \( (1.28) \), and \( C_{K,n} = (-1)^n (2a)^{n/2} \frac{\Gamma((n+1)/2)(\pi/2a)^{1/4}}{\Gamma(1/2)} e^{\pi/2} \). In a real interval containing \( t = -2\sqrt{a} \),

\[
C_n(a) \sim C_{K,n} \frac{x^{\frac{n}{2}} A_0(\tilde{\eta})}{\sqrt{w(x)}} \left[ \cos(x\pi) \text{Ai} \left( n^{\frac{1}{3}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right) - \sin(x\pi) \text{Bi} \left( n^{\frac{1}{3}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right) \right].
\]

(1.30)
While the asymptotic approximation in the lower half neighborhood is obtained by taking complex conjugate simultaneously with respect to the function and the variable.
It is worth mentioning that $e^{\mp \pi i/3} \text{Ai} \left( se^{\pm 2\pi i/3} \right) = \frac{1}{2} \left( \text{Ai}(s) + iB(s) \right)$; cf. [17, (9.2.11)].

Later in Section 4.2, a coherence check is made. Approximations for $t < -2\sqrt{a}$ and $t > 2\sqrt{a}$ are derived and are shown in consistence with (2.4) and (3.2), obtained respectively from the non-oscillatory region asymptotics, and intermediate asymptotics.

**Remark 2.** The asymptotic forms (1.24) and (1.29) differ from [25] in the appearance of a shift $n^{-1/6} \Phi(t)$ in the variable, yet they do agree with [23]. For sure one can re-expand the Airy function by using (4.4) and (4.9) to get rid of $\Phi$. However, by this way the leading term can be extracted appropriately; see the absence of $B_0$ in (1.20).

In applying the turning point theory via difference equation, it is common to expand the asymptotic series in descending powers of some $\nu = n + \tau_0$, where $\tau_0$ is a constant shift. The purpose is to ensure the existence of a formal solution. Such a shift of the large degree seems unnecessary in our derivation. A possible reason might lie in the nonlinearity in $n$ of the local transformation $x = n + \sqrt{nt}$.

**1.4 Discussion and arrangement of the rest of the paper**

The motivation of the present investigation is twofold. First, from a difference equation point of view, the Charlier polynomials are special, as indicated in the basic assumptions (1.11), in that they can not fit into any of the known cases; cf. Wong [25], see also Remark 1. Therefore, it is desirable to derive uniform and non-uniform asymptotics of the polynomials via difference equations. As described in the present section, we have put in use various methods, all of a difference equation nature, to deal with asymptotic approximations respectively in the outside region, an intermediate region, and near the turning points. The overlapping domains of validity actually cover the whole complex $y$-plane. Here $y = x/n$ is the re-scaling that normalizes the support of the equilibrium measure to $y \in [0, 1]$. In particular, we obtain uniform asymptotic expansions at a pair of coalescing turning points in the $y$-plane.

Another motivation is that the Charlier polynomials may serve as a model in the study of the Heun class equations. For example, when one considers the connection problems between fundamental solutions for the confluent Heun’s equation (CHE), and the doubly-confluent Heun’s equation (DHE), a central piece seems to be a three term recurrence relation satisfied by the coefficients of a certain Frobenius solution, essentially similar to (1.4), in that they share the same equilibrium support $y \in [0, 1]$, after a re-scaling $y = x/n^2$. Here $x$ is the accessory parameter in CHE or DHE. Most likely, the techniques used here, such as those in Section 2.2, could play a role. Eigenvalue problems and root polynomials for CHE and DHE also seem to be relevant.

It is worth mentioning that in a paper of Dai-Ismail-Wang [8], non-oscillatory asymptotic approximations are derived, and matched with asymptotic behavior resulted from the turning point techniques of Wang and Wong (see [25]). The present investigation repeatedly makes use of matching processes, just as [8] did, yet the treatment of coalescing turning points (in Section 4) and the uniform approximation at the end-point $y = 0$ (in Section 2.2) seem to be novel, and applicable elsewhere. It is also worth noticing that in Section 4.1-4.2, the matching processes are closely related to the Stokes lines of the asymptotic solutions; see also Remark 3.

The rest of the paper is arranged as follows. In Section 2.1, we derive a uniform asymptotic approximation for $C_n^{(a)}(ny)$ as $\text{dist}(y, [0, 1]) > r$ and $n \to \infty$, $r$ being a generic positive constant. In Section 2.2, we provide uniform asymptotics for $C_n^{(a)}(ny)$ as $|y| < r$ and $n \to \infty$, where $r \in (0, 1)$. A combination of the results covers the $y$-domain $|y - 1| > \delta$, and thus proves Theorem 1. Then, Section 3 is devoted to the asymptotic approximation of $C_n^{(a)}(x)$ in the
intermediate region described as $|t \pm 2\sqrt{a}| > r$ and $|y-1| < \delta$, where $x = ny = n \left(1 + \frac{t}{\sqrt{n}}\right)$, and again $r$ and $\delta$ are generic positive constants. The result turns out to be Theorem 2. In the last section, Section 4, we determine two uniform asymptotic approximations at the turning points $t = 2\sqrt{a}$ and $t = -2\sqrt{a}$, respectively, and we prove Theorems 3 and 4 in this section.

2 Non-oscillatory regions, the origin, and proof of Theorem 1

2.1 Non-oscillatory asymptotics

We are in a position to prove (1.8). To this aim, we need to show the validity of (1.7) beforehand. It is appropriate to write (1.7) as

$$w_k(x) := (x-k) \left\{ 1 + \frac{1-a}{x-k} - \frac{ak}{(x-k)^2} + \varepsilon_k \right\} := (x-k) \{1 + \delta_k\}. \tag{2.1}$$

We estimate the error terms as follows.

Lemma 1. Assume that $x = ny$ with $\text{dist}(y,[0,1]) > r$ for a certain positive constant $r$. Then, there exist positive constants $M_0, M_1$ and $N$, such that

$$|\delta_k| \leq \frac{M_0}{n} \tag{2.2}$$

and

$$|\varepsilon_k| \leq \frac{M_1}{n^2} \tag{2.3}$$

for $k = 1, 2, \ldots, n$ and $n > N$.

Proof: We prove the lemma by induction in $k$. Comparing $w_1(x) = x-a$ with (2.1), we see that

$$|\delta_1| \leq \frac{|1-a|}{n} \quad \text{and} \quad |\varepsilon_1| \leq \frac{a}{n^2},$$

as long as $\text{dist}(y,[0,1]) > r$.

Assume the validity of (2.2) for $k-1$, we proceed to show that (2.3) holds for index $k$. Indeed, substituting (2.1) into (1.6) with $a_k = k+a$ and $b_k = ak$, we have

$$(x-k)\varepsilon_k = \left(\frac{ak}{x-k} - \frac{a(k-1)}{x-k+1}\right) + \frac{a(k-1)}{x-k+1} \left( 1 - \frac{1}{1+\delta_{k-1}} \right).$$

Straightforward estimation then gives

$$|\varepsilon_k| \leq \frac{a}{r^3} \left( \frac{1}{n^2} + \frac{3a}{r^2} + \frac{a}{r^3} \right) \frac{1}{n},$$

for $k \leq n$ and $n > N = 2M_0$. Accordingly, from (2.1) we have

$$|\delta_k| \leq \left( \frac{|1-a|}{r} + \frac{3a}{r^2} + \frac{a}{r^3} \right) \frac{1}{n}.$$
for \( k \leq n \) and \( n > N = 2M_0 \). Hence, assigning
\[
M_0 = \frac{|1-a|}{r} + \frac{3a}{r^2} + \frac{a}{r^3}, \quad M_1 = \frac{a(1+r+2rM_0)}{r^3} \quad \text{and} \quad N = 2M_0,
\]
we complete the proof of the lemma.

Now substituting (1.7) into (1.5), we obtain the asymptotic approximation (1.8) for \( y \) keeping a distance \( r \) from \([0,1]\). Here use has been made of the trapezoidal rule to give
\[
\sum_{k=1}^{n} \log \left( y - \frac{k}{n} \right) = n \int_{0}^{1} \log(y-t) dt + \frac{1}{2} \log \frac{y-1}{y} + O \left( \frac{1}{n} \right),
\]
and the right-hand side terms explicitly give
\[
n [y \log y - (y-1) \log (y-1) - 1] + \frac{1}{2} \log \frac{y-1}{y}.
\]
We also pick up the later terms in (1.7) by approximating
\[
\sum_{k=1}^{n} \log \left\{ 1 + \frac{(1-a)x-k}{(x-k)^2} \right\} \sim \sum_{k=1}^{n} \frac{(1-a)x-k}{(x-k)^2},
\]
which in turn can be approximated by
\[
\int_{0}^{1} \frac{(1-a)y-t}{(y-t)^2} dt = \log \frac{y}{y-1} - \frac{a}{y-1},
\]
each time with an error \( O(1/n) \).

It is worth mentioning that formal derivation from (1.8) might give a formula for fixed \( y \in (0,1) \), namely,
\[
C_n^{(a)}(ny) \sim 2n^n \sqrt{\frac{y}{1-y}} \exp \left( -\frac{a}{y-1} \right) \exp \left\{ n \left[ y \log \frac{y}{1-y} - 1 \right] \right\} (y-1)^n \cos \left( n y \pi + \frac{\pi}{2} \right). \quad (2.4)
\]
As mentioned in Szegő [19, p. 395], forming the real part of the approximation of a polynomial \( P_n(x) \), the asymptotic formula for \( \frac{1}{2} P_n(x) \) arises. A similar idea has been applied to, e.g., [14]. It is readily verified that (2.4) agrees with Theorem 1 for \( y \in (0,1) \).

2.2 Asymptotics for \( C_n^{(a)}(x) \) as \( |x| < n \)

More precisely, we derive a uniform asymptotic formula for \( C_n^{(a)}(x) \) as \( \frac{|x|}{n} \leq r < 1 \) and \( n \to \infty \), \( r \) being an arbitrary constant. To do so, We apply the method of dominant balance to the three-term recurrence formula (1.4); cf. Bender and Orszag [3, Ch.5] for the method. Indeed, by matching the \( C_n^{(a)} \) term with \( C_{n-1}^{(a)} \) we obtain an asymptotic solution of (1.4),
\[
C_s(n) = C_s(n,x) \sim (-a)^n \frac{\Gamma(n+1-a)}{\Gamma(n+1-a-x)}.
\]

While a comparison of the \( C_{n+1}^{(a)} \) and \( C_n^{(a)} \) terms gives rise to another asymptotic solution
\[
C_d(n) = C_d(n,x) \sim (-1)^n \Gamma(n-x).
\]
These are the only possible balances. It is readily seen that $C_d(n)$ is dominant over $C_s(n)$ as $n \to \infty$. Hence we may focus on $C_d(n)$.

Refinements are available. The purpose now is to derive an approximation holding uniformly in the domain $|y| \leq r$, with $x = ny$ and $r < 1$, of the form

$$C_d(n, x) \sim (-1)^n \Gamma(n - x) \sum_{k=0}^{\infty} \frac{\varphi_k(y)}{n^k}, \quad (2.5)$$

where $\varphi_k(y)$ are functions independent of $n$. We proceed to determine the leading coefficient $\varphi_0(y)$. To this end, we write $x = ny = (n \pm 1)y$, and we see that the shifts $y_+ - y = (\pm \frac{1}{n} + \frac{1}{n^2} + \cdots) y$. Substituting the formal solution (2.5) into (1.4) yields

$$(n - x) \left[ \varphi_0(y) + \frac{\varphi_1(y)}{n + 1} \right] - (n - x + a) \left[ \varphi_0(y) + \frac{\varphi_1(y)}{n} \right] + \frac{an}{n - x - 1} \varphi_0(y) - \varphi_0(y) = O \left( \frac{1}{n} \right),$$

in which the leading $O(n)$ terms cancel automatically. Using Taylor expansions, and picking up the constant terms, we have

$$-a\varphi_0 - (1 - y)\varphi'_0 + \frac{a}{1 - y} \varphi_0 = 0.$$

Solving this first order differential equation gives

$$\varphi_0(y) = \exp \left( \frac{ny}{1 - y} \right),$$

up to a constant factor. Therefore, it follows that

$$C_d(n, x) = (-1)^n \Gamma(n - x) \exp \left( \frac{ax}{n - x} \right) \left\{ 1 + O \left( \frac{1}{n} \right) \right\}, \quad (2.6)$$

where the $O(1/n)$ term is uniform for $\frac{|x|}{n} \leq r$ and $n \to \infty$. Later terms can be determined recursively.

The uniformity of (2.5) and (2.6) can be justified by using arguments similar to [26]. To begin with, we write $C_d(n, x) = (-1)^n \Gamma(n - x)e^{\frac{ax}{n - x}} \left\{ 1 + \epsilon_n \right\}$, and substitute it into (1.4). As a result we obtain the non-homogeneous equation

$$\left( n - x - \frac{ax}{n - x} \right) \epsilon_{n+1}(x) - (n - x + a)\epsilon_{n}(x) + \left( a + \frac{ax}{n - x} \right) \epsilon_{n-1}(x) = O \left( \frac{1}{n} \right),$$

where again the right-hand-side term, may involving $\epsilon_n$ this time, is uniform in $n$ and $x$ such that $\frac{|x|}{n} \leq r < 1$. Following Wong and Li [26], by applying the method of variation of parameters [3, p.49] and the method of successive approximation, we see that there is a solution of the form $\epsilon_n = O(1/n)$ in the domain of uniformity.

Now that $C_d(n, x)$ is the dominant solution, we have

$$C_n^{(a)}(x) \sim K_d(x) C_d(n, x) \quad (2.7)$$

for $x$ in the mentioned domain of uniformity, where the coefficient $K_d(x)$ is to be determined by matching (1.8). For $|x| \sim \nu n$, we rewrite (1.8) as $C_n^{(a)}(x) \sim (-1)^n e^{\frac{ax}{n - x}} \Gamma(n - x)^{1/(n - x)}$ by applying Stirling’s formula, and compare both sides of (2.7). We see that $K_d(x) = e^a / \Gamma(-x)$, being analytic in complex $x$. Therefore, we have
Lemma 2. There is a large-$n$ asymptotic approximation

$$C_n^{(a)}(x) = (-1)^{n} e^{\frac{ax}{n-x}} \frac{\Gamma(n-x)}{\Gamma(-x)} + \varepsilon_1,$$

(2.8)

holding uniformly in $|y| = |x|/n < \delta$ for arbitrary constant $\delta \in (0,1)$, where

$$|\varepsilon_1| \leq \frac{M_1}{n} \left| e^{\frac{an}{n-x}} \frac{\Gamma(n-x)}{\Gamma(-x)} \right|,$$

with $M_1$ being a constant.

3 Intermediate asymptotics and proof of Theorem 2

The difference equation (1.10) has asymptotic solutions of the form (1.13) in the intermediate region, namely, in $|y - 1| < \delta$ and $|t \pm 2\sqrt{a}| > \delta$, where $x = ny = n \left( 1 + \frac{t}{\sqrt{n}} \right)$, $\delta$ and $\bar{\delta}$ are constants. Now we take the leading terms. More precisely, we determine $\phi_{-1}$ and $\phi_0$, so that $P_n(x) \sim e^{\sqrt{n} \phi_{-1}(t) + \phi_0(t)}$. The derivation is similar to [14], with modifications.

Temporary, we assume that $|t - (-\infty, 2\sqrt{a})| > \delta$. From (1.12) and (1.14), for the same $x$ when $n$ varies, we write

$$x = n \left( 1 + \frac{t}{\sqrt{n}} \right) = (n+1) \left( 1 + \frac{t_+}{n+1} \right) = (n-1) \left( 1 + \frac{t_-}{n-1} \right),$$

and see that, for large $n$,

$$t_+ = -\frac{1}{\sqrt{n+1}} + \sqrt{\frac{n}{n+1}} t = -\frac{1}{\sqrt{n}} + \frac{t}{2n} + O\left(n^{-3/2}\right) + O\left(\frac{t}{n^2}\right),$$

and

$$t_- = \frac{1}{\sqrt{n-1}} + \sqrt{\frac{n}{n-1}} t = \frac{1}{\sqrt{n}} + \frac{t}{2n} + O\left(n^{-3/2}\right) + O\left(\frac{t}{n^2}\right).$$

Substituting (1.13) into (1.10) gives

$$e^{\sqrt{n+1} \phi_{-1}(t_+) + \phi_0(t_+)} - e^{\sqrt{n} \phi_{-1}(t) - \phi_0(t)} + O(1/n) + e^{\sqrt{n} \phi_{-1}(t_+) + \phi_0(t_+)} - e^{\sqrt{n} \phi_{-1}(t) + \phi_0(t)} + O(1/n)$$

$$= \alpha_0 t + \frac{\alpha_1 + \beta_1}{\sqrt{n}} + O(1/n).$$

Expanding $\phi_{-1}$ and $\phi_0$ at $t$, we have

$$e^{-\phi_{-1}(t)} + \frac{1}{2\sqrt{n}} \phi_{-1}'(t) t + \phi_{-1}''(t) + O(1/n) + e^{-\phi_{-1}(t)} + \frac{1}{2\sqrt{n}} \phi_0''(t) + O(1/n)$$

$$= \frac{t}{\sqrt{n}} - \frac{\sqrt{a}}{\sqrt{n}} + O\left(\frac{1}{n}\right).$$

Equalizing the coefficients of 1 and $1/\sqrt{n}$, respectively, on both sides, we have the first order differential equations (1.15), namely, $e^{-\phi_{-1}'(t)} + e^{\phi_{-1}'(t)} = \frac{t}{\sqrt{n}}$, for $\phi_{-1}$, and

$$\left(e^{\phi_{-1}'(t)} - e^{-\phi_{-1}'(t)}\right) \phi_0' = -\sqrt{a} - e^{-\phi_{-1}'} \left[ \frac{1}{2} \phi_{-1}' - \frac{t}{2} \phi_{-1}'' + \frac{1}{2} \phi_{-1}''\right] - e^{\phi_{-1}'} \left[ -\frac{1}{2} \phi_{-1}' + \frac{t}{2} \phi_{-1}'' + \frac{1}{2} \phi_{-1}''\right],$$

(3.1)
for \( \phi_0 \).

From (1.15) one sees that \( e^{\phi_0(t)} = \frac{t + \sqrt{t^2 - 4a}}{2\sqrt{a}} \), each corresponds to an asymptotic solution of (1.10). For example, taking the minus sign and integrating, we have

\[
\phi_1(t) = t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} + C_{-1},
\]
such that \( e^{\phi_1(t)} = \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} \), where \( C_{-1} \) is a constant independent of \( n \). Substituting the expression into (3.1), we obtain

\[
\phi'_0(t) = \frac{a}{\sqrt{t^2 - 4a}} + \frac{t}{2} \sqrt{t^2 - 4a} - \frac{t}{2(t^2 - 4a)} + \frac{1}{2} C_{-1}.
\]

Then the asymptotic solution (1.16) is readily derived.

Now we match the asymptotic approximations (1.8) and (1.18) with \( x = ny \) and \( y = 1 + t/\sqrt{n} \), at the transition area described as

\[
n^{1/6} \ll |t| \ll n^{1/4}.
\]

As mentioned in Section 1.2, we see from (1.8) or (1.9) that \( C_n^{(a)}(n + \sqrt{n}t) \) is recessive for \( t \in (2\sqrt{a}, +\infty) \). The asymptotic solution \( \tilde{P}_n(x) \) in (1.17) is dominant as compared with \( P_n(x) \). Hence the coefficient \( B(x) \) vanishes. Thus \( C_n^{(a)}(x) \sim (2a)^{n/2} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} A(x) P_n(x) \).

Substituting in \( y = 1 + \frac{t}{\sqrt{n}} \) with \( |t| \ll n^{1/4} \), we can write (1.8) as

\[
C_n^{(a)}(x) \sim \left( \frac{n}{e} \right)^{n} \left( \frac{1}{\sqrt{2\pi n}} \right) x^{n/2 - 1/4} \exp \left( -\frac{a\sqrt{n}}{t} - \sqrt{n} t \log t + \frac{t}{2} \sqrt{n} t \log n + \sqrt{n} t + \frac{t^2}{2} - \frac{t^3}{6\sqrt{n}} \right).
\]

On the other hand, a combination of (1.16) with (1.18) as \( |t| \gg n^{1/6} \) gives

\[
C_n^{(a)}(x) \sim A(x) C_K(n, t) \exp \left( \sqrt{n} \left[ t \log \sqrt{n} - t \log t + t - \frac{a}{2} \right] - \frac{1}{2} \log t + \frac{t^2}{4} - \frac{a}{2} \right),
\]

where \( x = n(1 + \frac{t}{\sqrt{n}}) \), and \( C_K(n, t) = (2a)^{n/2} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} e^{\sqrt{n}C_{-1} + \frac{1}{2} C_{-1}t + C_0} \). A comparison of these formulas, with the aid of Stirling’s formula, gives

\[
A(x) \sim 2^{1/2} e^{-C_{-1}\sqrt{n} + \frac{1}{2} C_{-1}t - C_0 + \frac{1}{2} x^{1/2} \log x - \frac{1}{4} a^{1/2}}
\]

for large \( n \). Here use has been made of the fact that \( \frac{1}{2} x \log x - \frac{1}{2} \frac{1}{2} n \log n - \frac{n}{2} + \frac{1}{2} \sqrt{n} \log n + \frac{t^2}{4} \) and \( x^{1/4} \sim n^{1/4} \). We also see the involvement of \( \phi_1(t) \); see (1.13). The function \( A(x) \) is independent of \( n \), so we have \( C_{-1} = 0 \), and we may take

\[
A(x) = (w(x))^{-1/2} = \left( \frac{\Gamma(x + 1)}{a^x} \right)^{1/2} \quad \text{and} \quad C_0 = -\frac{3}{4} \log 2 - \frac{1}{4} \log \pi + \frac{a}{2},
\]

where \( w(x) = \frac{a^x}{\Gamma(x+1)} \) is the discrete weight shown in (1.1). Thus completes the proof of (1.19) in Theorem 2.

The same idea leading to (2.4) applies. From (1.19) we have

\[
C_n^{(a)}(x) \sim \frac{2C}{w(x) \left( (4a - t^2)^{1/4} \right)} \cos \left( 2\sqrt{a} \theta \right) \sin \theta - \theta \cos \theta + a \sin \theta \cos \theta - \frac{\pi}{4},
\]

where \( x = n(1 + \frac{t}{\sqrt{n}}) \) with \( t = 2\sqrt{a} \cos \theta \in (-2\sqrt{a}, 2\sqrt{a}) \), \( \theta \in (0, \pi) \), and as in (1.19), \( w(x) = \frac{a^x}{\Gamma(x+1)} \) and \( C = (2a)^{n/2} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} 2^{1/4} \pi^{-1/2} e^{1/2} \).
4 Airy-type approximations

Adapting the change of variable \( x = n(1 + \frac{t}{\sqrt{n^3}}) \), we consider the turning points \( t = t_0 = \pm 2\sqrt{a} \). The aim is to determine the asymptotic solution (1.20), or, the leading term

\[
Q_n(x) \sim A_0(\eta) \chi \left( n^{1/3} \eta(t) + n^{-1/6} \Phi(t) \right),
\]

namely, to determine \( \eta(t) \), \( \Phi(t) \) and \( A_0(\eta) \). Here \( \chi \) solves the Airy equation \( \chi''(\tau) = \tau \chi(\tau) \). The derivation is very similar to Wang and Wong [22]. But, since the basic assumption (1.11), and the form in (1.20), differ from those in [22], we need to describe the key steps briefly.

We begin with \( Q_{n+1}(x) \), preserving \( x \) while \( n \) varies. We write \( x = n + \sqrt{n} t = (n \pm 1) + \sqrt{n \pm 1} t_\pm \) as in (1.14). For large \( n \), we see that the shifts

\[
t_+ - t \sim -\frac{1}{\sqrt{n}} - \frac{t}{2n} \quad \text{and} \quad t_- - t \sim \frac{1}{\sqrt{n}} + \frac{t}{2n}.
\]

Also we turn to the variable of \( \chi \). To this aim, we denote

\[
(n \pm 1)^{1/3} \eta(t_\pm) + (n \pm 1)^{-1/6} \Phi(t_\pm) := n^{1/3} \eta(t) + n^{-1/6} \Phi(t) + n^{-1/6} u_\pm. \tag{4.1}
\]

It is readily seen that both \( u_\pm \) admit asymptotic expansion of the form \( \sum_{k=0}^{\infty} \frac{u_{k,\pm}}{n^{k/2}} \), with the leading terms given as

\[
u_+ \sim \sum_{k=0}^{\infty} \frac{u_{+,k}}{n^{k/2}}, \quad u_- \sim \sum_{k=0}^{\infty} \frac{u_{-,k}}{n^{k/2}}
\]

and

\[
u_+ \sim \sum_{k=0}^{\infty} \frac{u_{+,k}}{n^{k/2}} = -\eta'(t) + \left( \frac{\eta(t)}{3} - \frac{t \eta'(t)}{2} + \frac{\eta''(t)}{2} - \Phi(t) \right) \frac{1}{\sqrt{n}} + \ldots \tag{4.2}
\]

and

\[
u_- \sim \sum_{k=0}^{\infty} \frac{u_{-,k}}{n^{k/2}} = \eta'(t) + \left( -\frac{\eta(t)}{3} + \frac{t \eta'(t)}{2} + \frac{\eta''(t)}{2} + \Phi(t) \right) \frac{1}{\sqrt{n}} + \ldots \tag{4.3}
\]

Regarding \( \zeta \) and \( \mu \) as free variables, we take a closer look at \( \chi(n^{1/3} \zeta + n^{-1/6} \mu) \).

Lemma 3. Assume that \( \chi \) solves the Airy equation. Then the equality holds

\[
\chi \left( n^{1/3} \zeta + n^{-1/6} \mu \right) = \chi \left( n^{1/3} \zeta \right) X(n; \zeta, \mu) + n^{-1/6} \chi' \left( n^{1/3} \zeta \right) Y(n; \zeta, \mu),
\]

where the coefficients

\[
X(n; \zeta, \mu) = \sum_{k=0}^{\infty} \frac{X_k(\zeta, \mu)}{n^{k/2}} \quad \text{and} \quad Y(n; \zeta, \mu) = \sum_{k=0}^{\infty} \frac{Y_k(\zeta, \mu)}{n^{k/2}},
\]

with

\[
X_0(\zeta, \mu) = \frac{1}{2} \left( e^{\sqrt{\zeta} \mu} + e^{-\sqrt{\zeta} \mu} \right), \quad Y_0(\zeta, \mu) = \frac{1}{2\sqrt{\zeta}} \left( e^{\sqrt{\zeta} \mu} - e^{-\sqrt{\zeta} \mu} \right), \tag{4.6}
\]

\[
X_k(\zeta, \mu) = \frac{1}{2\sqrt{\zeta}} \int_0^\mu s X_{k-1}(\zeta, s) \left( e^{\sqrt{\zeta}(s-\mu)} - e^{\sqrt{\zeta}(s+\mu)} \right) ds, \quad k = 1, 2, \ldots \tag{4.7}
\]

and

\[
Y_k(\zeta, \mu) = \frac{1}{2\sqrt{\zeta}} \int_0^\mu s Y_{k-1}(\zeta, s) \left( e^{\sqrt{\zeta}(s-\mu)} - e^{\sqrt{\zeta}(s+\mu)} \right) ds, \quad k = 1, 2, \ldots, \tag{4.8}
\]

in which \( \sqrt{\zeta} \) takes the principal branch.
These differential equations, with initial data, have formal (asymptotic) solutions (4.5), where

\[ \zeta \]

For convenience, we may write for short

\[ t \]

4.1 Turning point

Solving these equations gives (4.6), (4.7) and (4.8). This completes the proof of the Lemma.

Furthermore, differentiating (4.4) twice with respect to \( \mu \), we obtain the equations

\[ \frac{\partial}{\partial \mu} X(n; \zeta, 0) = 0, \quad \frac{\partial}{\partial \mu} Y(n; \zeta, 0) = 1. \]

Hence we also have

\[ \frac{\partial^2}{\partial \mu^2} X(n; \zeta, \mu) = \left( \zeta + \frac{\mu}{n^{1/2}} \right) X(n; \zeta, \mu), \quad X(n; \zeta, 0) = 1, \quad \frac{\partial}{\partial \mu} X(n; \zeta, 0) = 0, \]

and

\[ \frac{\partial^2}{\partial \mu^2} Y(n; \zeta, \mu) = \left( \zeta + \frac{\mu}{n^{1/2}} \right) Y(n; \zeta, \mu), \quad Y(n; \zeta, 0) = 0, \quad \frac{\partial}{\partial \mu} Y(n; \zeta, 0) = 1. \]

These differential equations, with initial data, have formal (asymptotic) solutions (4.5), where \( X_k \) and \( Y_k \) are determined, iteratively, by equations

\[
\begin{aligned}
\frac{\partial^2 X_0(\zeta, \mu)}{\partial \mu^2} &= \zeta X_0(\zeta, \mu), \quad X_0(\zeta, 0) = 1, \quad \frac{\partial X_0(\zeta, 0)}{\partial \mu} = 0, \\
\frac{\partial^2 X_k(\zeta, \mu)}{\partial \mu^2} &= \zeta X_k(\zeta, \mu) + \mu X_{k-1}(\zeta, \mu), \quad X_k(\zeta, 0) = 0 \quad \frac{\partial X_k(\zeta, 0)}{\partial \mu} = 0, \quad k = 1, 2, \ldots,
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{\partial^2 Y_0(\zeta, \mu)}{\partial \mu^2} &= \zeta Y_0(\zeta, \mu), \quad Y_0(\zeta, 0) = 0, \quad \frac{\partial Y_0(\zeta, 0)}{\partial \mu} = 1, \\
\frac{\partial^2 Y_k(\zeta, \mu)}{\partial \mu^2} &= \zeta Y_k(\zeta, \mu) + \mu Y_{k-1}(\zeta, \mu), \quad Y_k(\zeta, 0) = 0 \quad \frac{\partial Y_k(\zeta, 0)}{\partial \mu} = 0, \quad k = 1, 2, \ldots.
\end{aligned}
\]

Solving these equations gives (4.6), (4.7) and (4.8). This completes the proof of the Lemma.

4.1 Turning point \( t = 2\sqrt{a} \) and proof of Theorem 3

For convenience, we may write for short \( \zeta(t) = \zeta(n, t) = \eta(t) + \Phi(t)/\sqrt{n} \) and \( \mu = u_\pm \).

To evaluate \( Q_{n+1}(x) \), we need to work out \( \chi \left( (n \pm 1)^{1/3} \eta(t_\pm) + (n \pm 1)^{-1/6} \Phi(t_\pm) \right) \), that is, \( \chi \left( n^{1/3} \zeta + n^{-1/6} u_\pm \right) \); cf. (4.1). Applying Lemma 3, we have

\[
\chi \left( n^{1/3} \zeta + n^{-1/6} u_\pm \right) \sim \chi \left( n^{1/3} \zeta \right) \sum_{k=0}^\infty \frac{X_k(\zeta, u_\pm)}{n^{k/2}} + n^{-1/6} \chi' \left( n^{1/3} \zeta \right) \sum_{k=0}^\infty \frac{Y_k(\zeta, u_\pm)}{n^{k/2}}. \tag{4.10}
\]
The situation here slightly differs from Lemma 3 in the dependence on \( n \) of \( \zeta \) and \( u_\pm \) in lower order terms; see (4.2)-(4.3). Similarly, in view of (4.9), we obtain

\[
\chi' \left( n^{1/3} \zeta + n^{-1/6} u_\pm \right) \sim n^{1/6} \chi \left( n^{1/3} \zeta \right) \sum_{k=0}^{\infty} \frac{\partial}{\partial \zeta} X_k(\zeta, u_\pm) \frac{n^{k/2}}{n^{k/2}} + \chi' \left( n^{1/3} \zeta \right) \sum_{k=0}^{\infty} \frac{\partial}{\partial \mu} Y_k(\zeta, u_\pm) \frac{n^{k/2}}{n^{k/2}}, \tag{4.11}
\]

where \( X_k(\zeta, \mu) \) and \( Y_k(\zeta, \mu) \) are given in (4.6)-(4.8).

Combining (4.10)-(4.11) with (1.20) yields

\[
Q_{n+1}(x) \sim A_\pm(n, \eta) \chi \left( n^{1/3} \zeta \right) + B_\pm(n, \eta) n^{-1/6} \chi' \left( n^{1/3} \zeta \right), \tag{4.12}
\]

where the coefficients demonstrate a complicated dependence on \( n \) and \( t \) (equivalently, on \( \eta \)), as

\[
A_\pm(n, \eta) = \sum_{k=0}^{\infty} \frac{X_k(\zeta, u_\pm)}{n^{k/2}} \sum_{k=0}^{\infty} \frac{A_k(u_\pm)}{(n + 1)^{k/2}} + \left( \frac{n}{n + 1} \right) \frac{1}{6} \sum_{k=0}^{\infty} \frac{\partial}{\partial \zeta} X_k(\zeta, u_\pm) \frac{n^{k/2}}{n^{k/2}} \sum_{k=1}^{\infty} \frac{B_k(u_\pm)}{(n + 1)^{k/2}},
\]

and

\[
B_\pm(n, \eta) = \sum_{k=0}^{\infty} \frac{Y_k(\zeta, u_\pm)}{n^{k/2}} \sum_{k=0}^{\infty} \frac{A_k(u_\pm)}{(n + 1)^{k/2}} + \left( \frac{n}{n + 1} \right) \frac{1}{6} \sum_{k=0}^{\infty} \frac{\partial}{\partial \zeta} Y_k(\zeta, u_\pm) \frac{n^{k/2}}{n^{k/2}} \sum_{k=1}^{\infty} \frac{B_k(u_\pm)}{(n + 1)^{k/2}},
\]

where \( u_\pm = \eta(t_\pm) \), \( t_\pm \) are the shifted variables; see (1.14). Finally, substituting (1.20) and (4.12) into the difference equation (1.10), equalizing the coefficients of \( \chi \) and \( \chi' \), we have

\[
A_+(n, \eta) + A_-(n, \eta) \sim (A_n x + B_n) \sum_{k=0}^{\infty} \frac{A_k(\eta)}{n^{k/2}} \tag{4.13}
\]

and

\[
B_+(n, \eta) + B_-(n, \eta) \sim (A_n x + B_n) \sum_{k=1}^{\infty} \frac{B_k(\eta)}{n^{k/2}}. \tag{4.14}
\]

Now we are in a position to determine the mapping \( \eta(t) \). Indeed, picking up the \( O(1) \) terms from (4.13) we have the first order equation

\[
e^{-\eta'(t)\sqrt{\eta(t)}} + e^{\eta'(t)\sqrt{\eta(t)}} = \frac{t}{\sqrt{a}}. \tag{4.15}
\]

Here use has been used of the fact that \( A_n x + B_n \sim \frac{t}{\sqrt{a}} - \frac{\sqrt{a}}{\sqrt{n}}, \) non-vanishing \( A_0(\eta) \sim A_0(\eta) \equiv \frac{A_0(\eta)\eta'}{\sqrt{\eta}}, \)

\[
X_0(\zeta, u_\pm) = \frac{\partial}{\partial \mu} Y_0(\zeta, u_\pm) \sim \frac{1}{2} \left( e^{-\eta'\sqrt{\eta}} + e^{\eta'\sqrt{\eta}} \right) + \frac{\sqrt{\eta}}{2\sqrt{n}} \left( -\frac{\Phi_{\eta'}^2}{2\eta} \pm u_{\pm, 1} \right) \left( e^{-\eta'\sqrt{\eta}} - e^{\eta'\sqrt{\eta}} \right)
\]

and

\[
Y_0(\zeta, u_\pm) \sim \pm e^{\eta'\sqrt{\eta}} \left( e^{-\eta'\sqrt{\eta}} - e^{\eta'\sqrt{\eta}} \right) \left( 1 - \frac{\Phi_{\eta'}}{2\eta} \right) + \frac{1}{2\sqrt{n}} \left( u_{\pm, 1} \pm \Phi_{\eta'} \right) \left( e^{\eta'\sqrt{\eta}} + e^{-\eta'\sqrt{\eta}} \right),
\]
with $u_{±,1}$ being given in (4.2)–(4.3). Each of the approximations has an error $O(1/n)$; cf. (4.2), (4.3), (4.6) and (1.14). While the leading $O(1)$ terms on both sides of (4.11) vanish.

Near the turning point $t_0$, $η(t)$ is uniquely determined from (4.15) by the initial condition $η(t_0) = 0$, and the assumption that $η(t)$ is monotone increasing for real $t > t_0 = 2\sqrt{a}$. Accordingly, we have

$$η'(t) \sqrt{η(t)} = \log \frac{t + \sqrt{t^2 - 4a}}{2\sqrt{a}}, \quad t \in \mathbb{C} \setminus (-\infty, 2\sqrt{a}], \quad (4.16)$$

where the logarithm takes principal branch, and $\sqrt{t^2 - 4a}$ is analytic in $\mathbb{C} \setminus [-2\sqrt{a}, 2\sqrt{a}]$ and is positive for $t > 2\sqrt{a}$. As a result,

$$\frac{2}{3} (η(t))^{3/2} = t \log \frac{t + \sqrt{t^2 - 4a}}{2\sqrt{a}} - \sqrt{t^2 - 4a}, \quad t \in \mathbb{C} \setminus (-\infty, 2\sqrt{a}). \quad (4.17)$$

It is readily verified that $\frac{2}{3} (η(t))^{3/2}$ takes purely imaginary values as $t$ approaches $(-2\sqrt{a}, 2\sqrt{a})$ from above and below, and that if we choose the branch of $η(t)$ in (4.17), to be positive on $t > 2\sqrt{a}$, then $η(t)$ is analytic and univalent in $\mathbb{C} \setminus (-\infty, -2\sqrt{a}]$.

Having had $η(t)$, we proceed to determine $Φ(t)$ and $A_0(η)$. To this aim, we need to work out more details. For example, from (4.6)–(4.8), for free variables $ζ$ and $μ$, we have

$$X_1(ζ, μ) = \frac{1}{8\sqrt{ζ}} \left( μ^2 + \frac{1}{ζ} \right) \left( e^{\sqrt{ζ}μ} - e^{-\sqrt{ζ}μ} \right) - \frac{μ}{8ζ} \left( e^{\sqrt{ζ}μ} + e^{-\sqrt{ζ}μ} \right)$$

and

$$Y_1(ζ, μ) = \frac{μ^2}{8ζ} \left( e^{\sqrt{ζ}μ} + e^{-\sqrt{ζ}μ} \right) - \frac{μ}{8ζ^{3/2}} \left( e^{\sqrt{ζ}μ} - e^{-\sqrt{ζ}μ} \right).$$

Resuming that $ζ = η(t) + \frac{Φ(t)}{\sqrt{η}}$, we have, with errors $O(1/\sqrt{η})$,

$$X_1(ζ, u_{±}) \sim X_1(η, ±η') = ±X_1(η, η'), \quad Y_1(ζ, u_{±}) \sim Y_1(η, ±η') = Y_1(η, η'),$$

and

$$\frac{∂}{∂μ} X_0(ζ, u_{±}) \sim ±\sqrt{η} \left( e^{-η'\sqrt{η}} - e^{η'\sqrt{η}} \right), \quad A_1(η_{±}) \sim A_1(η), \quad B_1(η_{±}) \sim B_1(η).$$

Making use of all these, bring together the $O(1/\sqrt{η})$ terms in (4.13) and (4.14), we have

$$\sqrt{η} \left( \frac{Φη'}{2η} + Φ' - \frac{η}{3} + \frac{tη}{2} \right) = -\frac{a}{\sqrt{t^2 - 4a}} \quad (4.18)$$

and

$$\frac{η'}{\sqrt{η}} \frac{\sqrt{t^2 - 4a}}{\sqrt{a}} \frac{dA_0(η)}{dη} + \left( \frac{η''}{2η} \right) \frac{t}{2\sqrt{a}} - \frac{η' \sqrt{t^2 - 4a}}{4η^{3/2} \sqrt{a}} A_0(η) = 0. \quad (4.19)$$

In view of (4.16), (4.17), and the fact that $η(t_0) = 0$, we solve the differential equation (4.18) to give $\sqrt{η(t)}Φ(t) = -\frac{t\sqrt{t^2 - 4a}}{4}$, which is (1.22), where $\sqrt{η(t)}$ and $\sqrt{t^2 - 4a}$ are positive for $t > t_0 = 2\sqrt{a}$. 

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One can solve $A_0(\eta)$ up to a constant factor. Indeed, we can write (4.19) as

$$\frac{dA_0}{A_0} = \left( -\frac{t}{2(\eta^2 - 4\alpha)} + \frac{1}{4\eta} \right) d\eta,$$

where $\frac{1}{\eta} = \frac{d\eta}{dt}$, and $t$ is regarded as a function of $\eta$. Thus $A_0(\eta)$ is determined up to a constant factor independent of both $n$ and $\eta$. From the above equation we readily pick one solution $A_0(\eta) = \left( \frac{e^{2-4\alpha}}{4\eta} \right)^{-1/4}$, which is (1.23).

Now we choose $\chi$ to be $\text{Ai}$ and $\text{Bi}$, and have the following asymptotic formula

$$C_n^{(a)}(x) \sim (2a)^{\frac{3}{4}} \frac{\Gamma((n + 1)/2)}{\Gamma(1/2)} \left( K_1(x)Q_n(x) + K_2(x)\tilde{Q}_n(x) \right),$$

(4.20)

where the asymptotic solutions

$$Q_n(x) \sim A_0(\eta)\text{Ai} \left( n^{1/3} \eta(t) + n^{-1/6} \Phi(t) \right), \quad \tilde{Q}_n(x) \sim A_0(\eta)\text{Bi} \left( n^{1/3} \eta(t) + n^{-1/6} \Phi(t) \right).$$

To determine the coefficients $K_1(x)$ and $K_2(x)$, again we apply a matching process: we match the approximation (4.20) with the intermediate asymptotic formula (1.19). Since $\eta(t)$ is monotone increasing for $t > 2\sqrt{\alpha}$, and the solution in (1.19) is exponentially small for $t > 2\sqrt{\alpha}$, we must have $K_2(x) \equiv 0$, that is, the exponentially large part in (4.20) vanishes. To find $K_1(x)$, we apply $\text{Ai}(s) \sim \frac{s^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}s^{3/2}}$, $s \to \infty$; cf. [17] (9.7.5). In view of (1.22) and (1.23), we deduce from (4.20) that

$$C_n^{(a)}(x) \sim (2a)^{\frac{3}{4}} \frac{\Gamma((n + 1)/2)}{\Gamma(1/2)} \frac{K_1(x)(4a)^{\frac{3}{4}} n^{-\frac{1}{12}} (t^2 - 4a)^{-\frac{1}{2}}}{2\sqrt{\pi}} e^{-\frac{2}{3}\sqrt{n\eta^3} + \frac{\sqrt{n^2 - 4a}}{4}},$$

where $\frac{2}{3}\eta^{3/2}$ is explicitly given in (4.17). The approximation will agree with (1.19) if

$$\frac{K_1(x)(4a)^{\frac{3}{4}} n^{-\frac{1}{12}}}{2\sqrt{\pi}} \sim 2^{-\frac{2}{3}} \pi^{-\frac{1}{4}} e^{\frac{3}{2}} \sqrt{w(x)}.$$

Hence we choose

$$K_1(x) = \left( \frac{\pi}{2a} \right)^{1/4} e^{\frac{3}{2}} x^{\frac{1}{12}} w(x)^{-\frac{1}{2}}.$$

Here use has been made of the fact that $x^{\frac{1}{12}} \sim n^{\frac{1}{12}}$ for $n$ large and $|t| \ll \sqrt{n}$. Substituting it and $K_2(x) \equiv 0$ into (4.20), we obtain the uniform asymptotic approximation (1.24) in a neighborhood of the turning point $t = t_0 = 2\sqrt{\alpha}$.

**Remark 3.** In (4.20), $\tilde{Q}_n(x)$ is dominant in $|\arg \eta| < \frac{\pi}{3}$, and is subdominant in $\frac{\pi}{3} < |\arg \eta| < \pi$, as compared with $Q_n(x)$. Here $\arg \eta \sim \arg(t - 2\sqrt{\alpha})$ for small $t < 2\sqrt{\alpha}$. In the matching process we see that the intermediate asymptotics matches the subdominant term involving $Q_n(x)$ for $|\arg \eta| < \frac{\pi}{3}$, thus $K_2(x)$ must be asymptotically zero, and $K_1(x)$ determined accordingly. Beyond this sector, $K_1(x)$ and $K_2(x)$ are preserved, for $\tilde{Q}_n(x)$ is subdominant and can not be observed. The ray $t > 2\sqrt{\alpha}$ is a Stokes line. The sector is illustrated in Figure 7.
As a coherence check, we may specify \( t = 2\sqrt{a}\cos \theta \in (-2\sqrt{a}, 2\sqrt{a}), \theta \in (0, \pi) \). Recall that \( \text{Ai}(-x) \sim \frac{1}{\sqrt{\pi x^{1/4}}} \cos \left( \frac{2}{3}x^{3/2} - \frac{\pi}{4} \right) \) as \( x \to \infty \) with \(| \text{arg } x | < \frac{2\pi}{3}\), cf. [17, (9.7.9)]. For fixed \( \theta \in (0, \pi) \), we can rewrite (1.24) as

\[
C^{(a)}_n(x) \sim \frac{2C}{\sqrt{w(x)}(4a-t^2)^{1/4}} \cos \left( \frac{2}{3}\sqrt{\eta}(-\eta)^{3/2} + (-\eta)^{3/2} \frac{\Phi}{\eta} - \frac{\pi}{4} \right),
\]

where \( C \) is the same as in (3.2), and we have used \( x^{1/2} \sim n^{1/2} \) in deriving the approximation. From (4.17) and (1.22), it is easily seen that

\[
\frac{2}{3}(-\eta)^{3/2} = 2\sqrt{a}(\sin \theta - \theta \cos \theta) \text{ and } (-\eta)^{3/2} \frac{\Phi}{\eta} = \frac{a}{2} \sin(2\theta).
\]

Hence the approximation is exactly (3.2), derived from intermediate asymptotics.

### 4.2 Turning point \( t = -2\sqrt{a} \) and proof of Theorem 4

In this case we employ a slightly different canonical form. Substituting

\[
C^{(a)}_n(x) = (-1)^n(2a)^{n/2} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} \mathcal{Q}_n(x)
\]

into (1.4) gives the following difference equation

\[
\mathcal{Q}_{n+1}(x) + (A_n x + B_n) \mathcal{Q}_n(x) + \mathcal{Q}_{n-1}(x) = 0,
\]

(4.21)

where the coefficients \( A_n \) and \( B_n \) are the same as in (1.11). Once again, we assume that we have an asymptotic solution to (4.21) of the form (1.20), and proceed to determine the functions \( \tilde{\eta}(t) \), \( \tilde{\Phi}(t) \) and \( A_0(\tilde{\eta}) \) in this case. Here we use the notations \( \tilde{\eta}(t) \) and \( \tilde{\Phi}(t) \), in stead of \( \eta(t) \) and \( \Phi(t) \) in the previous subsection.

All the derivation leading to (4.13)-(4.14) holds, one need only to replace the factor \( A_n x + B_n \) with \( -(A_n x + B_n) \sim -\frac{t}{\sqrt{a}} + \frac{\sqrt{a}}{n} \) on the righthand sides. The equation (4.15) now reads

\[
e^{-\tilde{\eta}(t)\sqrt{\tilde{\eta}(t)}} + e^{\tilde{\eta}(t)\sqrt{\tilde{\eta}(t)}} = -\frac{t}{\sqrt{a}},
\]

where we further require that \( \tilde{\eta}(-2\sqrt{a}) = 0 \), and \( \tilde{\eta}(t) \) is monotone decreasing for \( t < -2\sqrt{a} \). Hence we have

\[
\frac{2}{3}\tilde{\eta}^{3/2} = t \log \frac{-t + \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a},
\]

(4.22)

such that \( \tilde{\eta}(t) > 0 \) for \( t < -2\sqrt{a} \), \( \tilde{\eta}(t) \sim \left( \frac{3}{2} \right)^{2/3} a^{-1/6}(-t + 2\sqrt{a}) \) for \( t \sim -2\sqrt{a} \), and \( \tilde{\eta}'\sqrt{\tilde{\eta}} = \log \frac{-t + \sqrt{t^2 - 4a}}{2\sqrt{a}} \).

Now compare the \( O(1/\sqrt{n}) \) terms in the modified version (4.13)-(4.14), instead of (4.18) and (4.19), we have

\[
\sqrt{\tilde{\eta}} \left( \frac{\tilde{\Phi}\tilde{\eta}'}{2\tilde{\eta}} + \tilde{\Phi}' - \frac{\tilde{\eta}}{3} + \frac{t\tilde{\eta}'}{2} \right) = \frac{a}{\sqrt{t^2 - 4a}}
\]
Comparing it with (1.19), and recalling that $x$ at $t$ $\eta$ Indeed, the above matching process holds for arg $\tilde{\eta}$ twice real part of (1.29). The asymptotic approximation in the lower half neighborhood is (1.29) in the upper half $t<$ $\eta$ valid since $Q$ asymptotics matches the subdominant solution $\eta$. Hence $\eta$ $\omega$ $t<$ $\eta$ $\omega$ $t$ $\eta$ $\omega$ $t$ at $t = -2\sqrt{a}$, such that $A_0(\tilde{\eta}) \sim a^{1/8} \left( \frac{-(t + 2\sqrt{a})}{\tilde{\eta}} \right)^{-1/4}$ as $t \to -2\sqrt{a}$, and is real positive for $t < -2\sqrt{a}$, where arg$\{-(t + 2\sqrt{a})\} \in (-\pi, \pi)$.

Denote by $Q_n(x)$ and $\tilde{Q}_n(x)$ the asymptotic solutions to (4.21), such that $Q_n(x) \sim A_0(\tilde{\eta}) \text{Ai} \left( \frac{\omega}{\sqrt{n}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right)$, $\tilde{Q}_n(x) \sim A_0(\tilde{\eta}) \text{Ai} \left( \frac{\omega}{\sqrt{n}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right)$, where $\omega = e^{2\pi i/3}$, functions $\tilde{\eta}(t), \tilde{\Phi}(t)$ and $A_0(\tilde{\eta})$ are given in the present subsection. Accordingly we can write

$$C_n(a) \sim (-1)^n (2a)^{\frac{n}{2}} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} \left( K_1(x) Q_n(x) + K_2(x) \tilde{Q}_n(x) \right),$$

(4.23)

with $K_1(x)$ and $K_2(x)$ to be determined by matching (4.23) with the intermediate asymptotic formula (1.19). As in Remark 3, we pay attention to the dominant and subdominant solutions. First we consider the case when arg $\tilde{\eta} \sim -\frac{2\pi}{3}$, or, approximately, arg$(t + 2\sqrt{a}) = \pi + \text{arg}(-t + 2\sqrt{a}) \sim \frac{\pi}{3}$. Actually, the curve arg $\tilde{\eta}(t) = -\frac{2\pi}{3}$ is a Stokes line; cf. Figure 1. In this case, $\tilde{Q}_n(x)$ is dominant as compared with $Q_n(x)$. The dominant $\tilde{Q}_n(x)$ does not match (1.19), hence we set $K_2(x) = 0$. To determine $K_1(x)$, we expand (4.23) to give

$$C_n(a) \approx (2a)^{\frac{n}{2}} \frac{\Gamma((n+1)/2)}{\Gamma(1/2)} \text{Ai} \left( x^{\frac{\pi}{2}} \sqrt{\pi} \right) e^{-\frac{\pi}{2} \log(x^{\frac{1}{2}} + \sqrt{x}) - \frac{1}{2} \log(x^{\frac{1}{2}} - \sqrt{x}) + \frac{1}{4} x^{\frac{1}{2}} + \frac{1}{4} \sqrt{x}}. $$

Comparing it with (1.19), and recalling that $x^{\frac{1}{2}} \sim n^{\frac{1}{2}}$, we obtain

$$K_1(x) = \left( \frac{\pi}{2a} \right)^{1/4} e^{a/2} x \pi/w(x)^{-1/2} e^{-(x + \frac{\pi}{3})i}.$$ 

Indeed, the above matching process holds for arg $\tilde{\eta} \in (-\pi, -\frac{\pi}{3})$, in which the intermediate asymptotics matches the subdominant solution $Q_n(x)$. For arg $\tilde{\eta} \in (-\frac{\pi}{3}, 0)$, the result is still valid since $Q_n(x)$ is the dominant solution. Therefore, for $t + 2\sqrt{a} = O(1)$, Im $t > 0$, we have (1.29) in the upper half $t$-neighborhood, where the special function employed is an Airy function $e^{-\pi/4} \text{Ai} \left( \frac{\pi(n+1)/2}{2\sqrt{\pi}} \right) = \frac{1}{2} \left( \text{Ai}(s) - i\text{Bi}(s) \right)$; see [17] (9.2.11). The formula (1.30) follows from taking twice real part of (1.29). The asymptotic approximation in the lower half neighborhood is
obtained by symmetry, involving the Airy function $e^{\pi i/3} \text{Ai}(s e^{-2\pi i/3}) = \frac{1}{2} (\text{Ai}(s) + i\text{Bi}(s))$. This proves Theorem 4.

As an application of Theorem 4, we check the oscillating in an interval around $t = -2\sqrt{a}$. When $t \sim -2\sqrt{a}$, the zeros of the Charlier polynomials will be represented in terms of the zeros of Airy function. Here, we would rather do a coherence check to show that how the density of zeros changes from (2.4) to (3.2). To this aim, we require $t$ to keep away from $-2\sqrt{a}$.

For $t < -2\sqrt{a}$, $s = \tilde{\eta}(t) + \tilde{\Phi}(t)/\sqrt{n}$ is positive, and $\text{Bi}(n^{1/3}s)$ is dominant over $\text{Ai}(n^{1/3}s)$. Hence from (1.30) we deduce

$$C_n(a)^{(a)}(x) \sim C_K n^{1/2} A_0(\tilde{\eta}) \frac{\sqrt{\pi}}{\sqrt{w(x)}} \text{Bi} \left( n^{\frac{3}{2}} \left( \tilde{\eta} + \frac{\tilde{\Phi}}{\sqrt{n}} \right) \right) \cos \left( x\pi + \frac{\pi}{2} \right).$$

Here use has been made of the fact that $\text{Bi}(s) \sim \frac{1}{\sqrt{\pi} s^{1/2}} e^{3/4 s^{3/4}}$ for large positive $s$; see [17 (9.7.7)]. It is readily seen that the above formula agrees with (2.4) as $1 \ll -(t + 2\sqrt{a}) \ll \sqrt{n}$.

On the other side of $t = -2\sqrt{a}$, namely, $t > -2\sqrt{a}$, $s = \tilde{\eta}(t) + \tilde{\Phi}(t)/\sqrt{n}$ is negative. Parameterizing $t = 2\sqrt{a} \cos \theta$ with $\theta < \pi$, and using the asymptotic formula [17 (9.7.5)] again for $\text{Ai}$, from (1.29) one deduce

$$C_n(a)^{(a)}(x) \sim (-1)^n C_K n^{1/2} A_0(\tilde{\eta}) \frac{\sqrt{\pi}}{\sqrt{w(x)}} (\tilde{\eta})^{1/4} \cos \left( 2\sqrt{a n} (\sin \theta - \theta \cos \theta) + a \sin \theta \cos \theta - \frac{\pi}{4} \right)$$

for $t > -2\sqrt{a}$, as $n \to \infty$; see the discussion right after (2.4). Here we have used

$$\frac{2}{3} (\tilde{\eta})^{3/2} = 2\sqrt{a} [ (\pi - \theta) \cos \theta + \sin \theta ] , \quad (\tilde{\eta})^{3/2} \frac{\tilde{\Phi}}{\tilde{\eta}} = a \cos \theta \sin \theta \quad \text{and} \quad A_0(\tilde{\eta}) = \left( \frac{-\tilde{\eta}}{\sin^2 \theta} \right)^{1/4}.$$

The above approximation for $C_n(a)^{(a)}(x)$ is exactly (3.2).

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