On the Gradient Flow Formulation of the Lohe Matrix Model with High-Order Polynomial Couplings

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Abstract
We present a first-order aggregation model for a homogeneous Lohe matrix ensemble with higher order couplings via a gradient flow approach. For homogeneous free flow with the same Hamiltonian, it is well known that the Lohe matrix model with cubic couplings can be recast as a gradient system with a potential which is a squared Frobenius norm of the averaged state. In this paper, we further derive a generalized Lohe matrix model with higher-order couplings via gradient flow approach for a polynomial potential. For the proposed model, we also provide a sufficient framework in terms of coupling strengths and initial data leading to the emergent dynamics of a homogeneous ensemble.

Keywords Complete aggregation · Emergence · Lohe matrix model · Practical aggregation · Tensors

Mathematics Subject Classification 70G60 · 34D06 · 70F10

1 Introduction

Synchronization of oscillatory complex systems often appears naturally in nature, e.g., synchronous heart beating [29] and synchronous firing of fireflies [1,4,30,31,36], etc. Then, one of natural questions lies in the design of mathematical model which exhibits collective synchronous behaviors. In this direction, Winfree [35] and Kuramoto [22] proposed analytically manageable simple mathematical models in a half century ago, and they provided sufficient frameworks leading to the emergent dynamics of weakly coupled oscillators. Recently, the authors introduced a generalized synchronization model on the space of tensors, namely

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“the Lohe tensor model” which encompasses all the previous Lohe type aggregation models such as the Kuramoto model \([3,8,9,12–14,18,19,22,23,32,33]\), the Lohe sphere model \([7,28]\) and the Lohe matrix model \([11,17,21,24–27]\). However, all interaction mechanism in aforementioned models are given by cubic couplings (see Sect. 2 for details).

In this paper, we are mainly interested in the generalization of the Lohe matrix model with high-order couplings. To fix the idea, we begin with the original Lohe matrix model with cubic couplings. Let \(U_j = U_j(t)\) be a \(d \times d\) complex state matrix of the \(j\)th particle, and the dynamics of the state matrix is given by the first-order matrix-valued continuous dynamical system:

\[
i \dot{U}_j U_j^\dagger = H_j + \frac{i \kappa}{2N} \sum_{k=1}^{N} \left( U_k U_j^\dagger - U_j U_k^\dagger \right), \quad t > 0, \quad j \in \mathcal{N} := \{1, \ldots, N\}, \quad (1.1)
\]

where \(U_j^\dagger\) is the hermitian conjugation of \(U_j\). Then, it is easy to see that the quadratic quantity \(U_j U_j^\dagger\) is conserved along (1.1) so that without loss of generality, we may assume

\[U_j U_j^\dagger = I_d, \quad j \in \mathcal{N}.
\]

Then, system (1.1) can be rewritten as follows:

\[
\dot{U}_j = -i H_j U_j + \kappa \sum_{k=1}^{N} \left( U_k U_j^\dagger U_j - U_j U_k^\dagger U_j \right), \quad j \in \mathcal{N}, \quad (1.2)
\]

or equivalently

\[
\dot{U}_j = -i H_j U_j + \frac{\kappa}{2N} \sum_{k=1}^{N} \left( U_k - U_j U_k^\dagger U_j \right), \quad j \in \mathcal{N}. \quad (1.3)
\]

The above system was first introduced in a series of works \([24,25]\) as one of possible non-abelian generalizations of the Kuramoto model. At first glance, it is not clear a priori whether only cubic couplings should appear in the R.H.S. of (1.2). The similar issues were already addressed for Kuramoto synchronization, e.g., the Kuramoto–Daido model in \([1,10]\) replaces the sinusoidal coupling by a general \(2\pi\)-periodic function and the work \([5]\) introduced a higher-order coupling of states. With the same line of spirit, we are interested in the following simple question for the Lohe matrix model:

What will be the general form of higher-order couplings which generalizes the cubic coupling in system (1.2) and can exhibit collective behaviors such as complete aggregation and practical aggregation?

We refer to Sect. 2.3 for details. Note that the Lohe matrix model was not introduced via a hamiltonian formalism or variational approach. Hence it is not clear why cubic couplings were involved in (1.3). As briefly discussed in \([16]\), any odd order of couplings will be possible for the Lohe tensor model. In a recent paper by the first author and his collaborators, the Lohe matrix model with the same hamiltonian \(H_j = H\) can recast as a gradient flow as well. Moreover, thanks to solution splitting property for the Lohe matrix model, if we substitute \(\tilde{U}_j = e^{i H t} U_j\) into (1.3), then we have

\[
\dot{\tilde{U}}_j = \frac{\kappa}{2N} \sum_{k=1}^{N} \left( \tilde{U}_k - \tilde{U}_j \tilde{U}_k^\dagger \tilde{U}_j \right), \quad j \in \mathcal{N}.
\]
That is why we can set \( H = 0 \) without loss of generality.

Consider the equivalent form of the Lohe matrix model:

\[ \dot{U}_j = \frac{\kappa}{2N} \sum_{k=1}^{N} \left( U_k U_j^\dagger U_j - U_j U_k^\dagger U_k \right), \quad j \in \mathcal{N}. \]  

(1.4)

A gradient flow formulation of (1.4) was first investigated in [21]. For a homogeneous Lohe matrix ensemble \( \{U_j\} \), we set

\[ U_c := \frac{1}{N} \sum_{j=1}^{N} U_j, \quad \mathcal{V}_1(U) := -\frac{\kappa N}{2} \| U_c \|_F^2, \]

where \( \| \cdot \|_F \) is a Frobenius norm defined as follows: for given \( d \times d \) size matrix \( A \),

\[ [A]_{\alpha\beta} : (\alpha, \beta)\text{-component of } A, \quad \|A\|_F := \sqrt{\sum_{\alpha,\beta=1}^{d} |[A]_{\alpha\beta}|^2}. \]

See the alternative notation part at the end of this section. Then, system (1.3) can recast as a gradient flow (Proposition 2.3):

\[ \dot{U}_j = -\frac{\partial \mathcal{V}_1(U)}{\partial U_j} \bigg|_{T_{U_j} U(d)}, \quad j \in \mathcal{N}, \]

(1.6)

where the derivative over complex-valued matrix is defined as follows:

\[ \left[ \frac{\partial \mathcal{V}(U)}{\partial U} \right]_{\alpha\beta} = \frac{\partial \mathcal{V}(U)}{\partial \text{Re}([U]_{\alpha\beta})} + i \frac{\partial \mathcal{V}(U)}{\partial \text{Im}([U]_{\alpha\beta})}. \]

Main question is how to define a potential suitably so that the resulting gradient flow exhibits an emergent aggregation dynamics.

Next, we briefly discuss our main results. First, we introduce a potential as follows. For \( m \geq 1 \), we set

\[ \mathcal{V}_m(U) := -\frac{\kappa N}{2m} \text{tr}(U_c (U_c^\dagger)^m) = -\frac{\kappa N}{2m} \text{tr}(U_c U_c^\dagger \cdots U_c U_c^\dagger). \]

Note that it follows from the property of trace: \( \text{tr}[AB] = \text{tr}[BA] \) that the potential \( \mathcal{V}_m(U) \) can be rewritten as

\[ \text{tr}((U_c U_c^\dagger)^m) = \text{tr}((U_c^\dagger U_c)^m). \]

(1.5)

So actually \( U_c^\dagger U_c \) and \( U_c U_c^\dagger \) have same effect in the trace function. Then the corresponding gradient flow

\[ \dot{U}_j = -\frac{\partial \mathcal{V}_m}{\partial U_j} \bigg|_{T_{U_j} U(d)}, \quad j \in \mathcal{N}. \]  

(1.6)
can be expressed as
\[
i\dot{U}_j U_j^\dagger = H_j + \frac{i\kappa}{2N^{2m-1}} \sum_{k_1,\ldots,k_{2m-1}=1}^N \left( U_{k_1} U_{k_2}^\dagger \cdots U_{k_{2m-2}}^\dagger U_{k_{2m-1}} U_j - U_j U_{k_{2m-1}}^\dagger U_{k_{2m-2}} \cdots U_{k_2} U_{k_1}^\dagger \right).
\] (1.7)

Then, it is easy to see that \( U_j U_j^\dagger \) is a conserved quantity for (1.7). Thus for \( U_j U_j^\dagger = I_d \), system (1.7) can be further rewritten as
\[
\dot{U}_j = -iH_j U_j + \frac{\kappa}{2N^{2m-1}} \sum_{k_1,\ldots,k_{2m-1}=1}^N \left( U_{k_1} U_{k_2}^\dagger \cdots U_{k_{2m-2}}^\dagger U_{k_{2m-1}} U_j - U_j U_{k_{2m-1}}^\dagger U_{k_{2m-2}} \cdots U_{k_2} U_{k_1}^\dagger \right).
\] (1.8)

Note that the order in the coupling term in the R.H.S. of (1.8) is \( 2m + 1 \). Second, we provide a general case:
\[
i\dot{U}_j U_j^\dagger = H_j + \sum_{n=1}^m \frac{i\kappa_n}{2} \left( U_{c_1} U_{c_2}^\dagger \cdots U_{c_{2n-2}}^\dagger U_{c_{2n-1}} U_j - U_j U_{c_{2n-1}}^\dagger U_{c_{2n-2}} \cdots U_{c_2} U_{c_1}^\dagger \right), \quad j \in \mathcal{N}.
\] (1.9)

It is easy to see that the R.H.S. of (1.9) is skew-hermitian so that system (1.9) conserves the quantity \( U_j U_j^\dagger \). For an ensemble, we set
\[
\mathcal{V}_{poly} := -N \text{tr}(f(U_{c_1} U_{c_1}^\dagger)), \quad f(A) := \frac{\kappa_1}{2} A + \frac{\kappa_2}{4} A^2 + \cdots + \frac{\kappa_m}{2m} A^m.
\] (1.10)

Then, one has emergent dynamics (see Theorem 4.2):
\[
\lim_{t \to \infty} \| U_j(t) - U_j^\infty \|_F = 0, \quad \lim_{t \to \infty} \frac{d}{dt} \mathcal{V}_{poly}(U) = 0, \quad \lim_{t \to \infty} \| \dot{U}_j \|_F = 0, \quad j \in \mathcal{N}.
\]

The rest of this paper is organized as follows. In Sect. 2, we briefly discuss the Lohe matrix model and its basic properties, and we also present a gradient flow formulation, and the relation between system (1.9) with \( d = 1 \) and the Kuramoto–Daido model. We also briefly review basic a priori estimates to be used crucially for a later use. In Sect. 3, we consider a monomial potential function and as a gradient flow approach, we derive a generalized Lohe matrix model with higher-order couplings, and study several emergent estimates and provide several sufficient frameworks leading to the emergent dynamics. In Sect. 4, we consider a general case with a polynomial potential and derive a generalized Lohe matrix model with higher-order couplings. Using Barbalat’s lemma, we derive an emergent dynamics of the proposed model. In Sect. 5, we derive a Gronwall type differential inequality for a ensemble diameter. This yields an exponential decay estimate of ensemble diameter. Finally, Sect. 6 is devoted to a brief summary of our main results and some unresolved issues for a future work.

**Notation** Let \( \mathbb{U}(d) \) be a unitary group manifold consisting of unitary \( d \times d \) matrix \( U U^\dagger = I_d \) and for two unitary matrices \( A, B \in \mathbb{U}(d) \), we introduce a Frobenius inner product \( \text{tr}(A^\dagger B) \).
product $\langle \cdot, \cdot \rangle_F$ and its induced norm $\| \cdot \|_F$:

$$
\langle A, B \rangle_F := \text{tr}(A^\dagger B), \quad \| A \|_F := \sqrt{\langle A, A \rangle_F}.
$$

## 2 Preliminaries

In this section, we review the Lohe matrix model \cite{24,25} on the unitary group $U(d)$ and review the basic properties of the Lohe matrix model such as conservation laws and gradient flow formulation.

### 2.1 The Lohe Matrix Model

Let $U_j$ be a $d \times d$ unitary matrix whose dynamics is governed by the first-order continuous-time dynamical system:

$$
\begin{aligned}
\dot{U}_j &= H_j + \frac{i\kappa}{2N} \sum_{k=1}^{N} \left( U_k U_j^\dagger - U_j U_k^\dagger \right), \quad t > 0, \\
U_j(0) &= U_j^0, \quad j \in \mathcal{N},
\end{aligned}
$$

where $\kappa$ is a nonnegative coupling strength, $U_j^\dagger$ denotes the hermitian conjugate of the matrix $U_j$, and $H_j$ is the Hermitian matrix with the property $H_j^\dagger = H_j$. This property results in the following relation:

$$
\langle U_j, -iH_j U_j \rangle_F + \langle -iH_j U_j, U_j \rangle_F = 0, \quad j \in \mathcal{N},
$$

where $\langle \cdot, \cdot \rangle_F$ is the Frobenius inner product on $U(d)$:

$$
\langle A, B \rangle_F := \text{tr}(A^\dagger B), \quad A, B \in U(d).
$$

Note that for zero coupling strength $\kappa = 0$, system (2.1) reduces to the usual Schrödinger equation. More precisely, consider the time-evolution operator $U = U(\cdot)$ of the quantum state:

$$
|\psi(t)\rangle = U(t)|\psi(0)\rangle.
$$

Then, by the Schrodinger equation, one has

$$
i \frac{d}{dt} U(t)|\psi(0)\rangle = HU(t)|\psi(0)\rangle,
$$

where $H$ is a Hamiltonian operator. Since this identity holds for all $|\psi(0)\rangle$, we have

$$
i \dot{U} U^\dagger = H.
$$

This is exactly the free-flow part of system (2.1). The coupling terms of (2.1) follows from the natural gradient of the given potential $\mathcal{V}_{poly}$ given as (1.10).

Next, we will see that the quadratic quantity $U_j^\dagger U_j$ is a conserved quantity (see Proposition 2.1). Since $U_j U_j^\dagger = I_d$, system (2.1) can be rewritten as

$$
\dot{U}_j = -i \quad H_j U_j + \frac{\kappa}{2N} \sum_{k=1}^{N} \left( U_k U_j^\dagger - U_j U_k^\dagger \right), \quad j \in \mathcal{N}.
$$


Moreover, system (2.2) can be further simplified as a mean-field form using the average quantity \( U_c := \frac{1}{N} \sum_{k=1}^{N} U_k \) to rewrite system (2.2) as
\[
\dot{U}_j = -iH_j U_j + \frac{\kappa}{2} (U_c - U_j U_c^\dagger U_j).
\]

Next, we list several key properties of (2.1) as follows.

**Proposition 2.1** [24,25] Let \( \{U_j\} \) be a global smooth solution to (2.1) with the initial data \( \{U_0^j\} \). Then, the following assertions hold.

1. (Conservation of amplitude): The quadratic quantity \( U_j U_j^\dagger \) is conserved along the Lohe matrix flow (2.1):
\[
U_j(t) U_j^\dagger(t) = U_0^j U_0^j, \quad t > 0.
\]

2. (Unitary invariance): Let \( \tilde{U}_j \) be a transformed state by the relation:
\[
\tilde{U}_j := U_j L, \quad L \in \mathbb{U}(d).
\]
Then, the transformed state \( \tilde{U}_j \) satisfies
\[
i \dot{\tilde{U}}_j \tilde{U}_j^\dagger = H_j + \frac{i \kappa}{2N} \sum_{k=1}^{N} (\tilde{U}_k \tilde{U}_j^\dagger - \tilde{U}_j \tilde{U}_k^\dagger), \quad t > 0.
\]

As in the Kuramoto model, system (2.1) admits a “solution splitting property” for the identical Hamiltonian case:
\[
H_j = H, \quad j \in \mathbb{N}.
\]

In this case, system (2.1) becomes
\[
\dot{U}_j = -iH U_j + \frac{\kappa}{2N} \sum_{k=1}^{N} (U_k - U_j U_k^\dagger U_j), \quad j \in \mathbb{N}.
\] (2.3)

Let \( \mathcal{R}(t) \) and \( \mathcal{L}(t) \) be the two solution operators corresponding to the following two subsystems, respectively:
\[
\dot{V}_j = -iHV_j, \quad \dot{W}_j = \frac{\kappa}{2N} \sum_{k=1}^{N} (W_k - W_j W_k^\dagger W_j).
\]

Next, we introduce solutions operators associated with the above two systems:
\[
\mathcal{R}(t) \mathcal{V}^0 := (e^{-iHt} V_1^0, \ldots, e^{-iHt} V_N^0), \quad \mathcal{W}(t) \mathcal{L}^0 := (W_1(t), \ldots, W_N(t)), \quad t \geq 0.
\]

**Proposition 2.2** [15] Let \( \mathcal{S}(t) \) be a solution operator to (2.3). Then, one has
\[
\mathcal{S}(t) = \mathcal{R}(t) \circ \mathcal{L}(t), \quad t \geq 0.
\]

It follows from Proposition 2.2 that it suffices to assume \( H = 0 \) for a homogeneous ensemble in what follows.
2.2 A Gradient Flow Formulation

Consider the Lohe matrix model with $H \equiv 0$:

$$\dot{U}_j = \frac{\kappa}{2N} \sum_{k=1}^{N} \left( U_k - U_j U_k^\dagger U_j \right), \quad j \in \mathcal{N}. \quad (2.4)$$

In [20], the authors introduced an order parameter $R$ and the corresponding potential $V_1$ for (2.4) with $H_i = 0$:

$$R^2 := \frac{1}{N^2} \sum_{i,j=1}^{N} \text{tr} \left( U_i^\dagger U_j \right) = \text{tr}(U_c^+ U_c) \quad \text{and} \quad V_1 := -\frac{\kappa N}{2} R^2 = -\frac{\kappa N}{2} \|U_c\|_F^2, \quad (2.5)$$

Then, it is easy to see that $R^2$ is analytic and

$$R = \|U_c\|_F \leq \frac{1}{N} \sum_{j=1}^{N} \|U_j\|_F = \sqrt{d}. \quad (2.6)$$

Note that the potential $V_1$ is an analytic function of states $U_j$’s, and the Riemannian metric on $\mathbb{U}(d)$ is induced by the natural inclusion $\mathbb{U}(d) \hookrightarrow M_{d,d}(\mathbb{C})$.

**Proposition 2.3** [20] The Lohe matrix model (2.4) with $H = 0$ is a gradient flow with an analytical potential $V_1$ in (2.5):

$$\dot{U}_j = -\left. \frac{\partial V_1}{\partial U_j} \right|_{U_c} , \quad t > 0, \quad j \in \mathcal{N}.$$ 

**Proof** Although a detailed proof can be found in [20], we briefly sketch its proof here for self-containedness.

- **Step A** (Expression of the potential in terms of components of $U_i$): Note that the function $\mathcal{V}$ has an obvious polynomial extension to all of $M_{d,d}(\mathbb{C})^N = \mathbb{C}^{2d^2N}$ viewed as a real analytic manifold. Since each variable $U_i$ is in $M_{d,d}(\mathbb{C}) = \mathbb{C}^{2d^2}$, the partial derivatives of a matrix can be calculated by the partial derivatives of each real and imaginary component of $U_i$ on $\mathbb{R}^{2d^2}$. Let $u^{kl}_i = a^{kl}_i + ib^{kl}_i$ be the $(k, l)$-element of matrix $U_i$, where $a^{kl}_i$ and $b^{kl}_i$ are real numbers. First, we use

$$\text{tr} \left( U_i U_j^\dagger \right) = \sum_{k,l=1}^{d} u^{kl}_i u^{kl}_j = \sum_{k,l=1}^{d} \left[ (a^{kl}_i a^{kl}_j + b^{kl}_i b^{kl}_j) + i(a^{kl}_i b^{kl}_j - a^{kl}_j b^{kl}_i) \right]$$

to see

$$V_1 = -\frac{\kappa N}{2} \|U_c\|_F^2 = -\frac{\kappa}{2N} \sum_{i,j=1}^{N} \text{tr}(U_i U_j^\dagger) = -\frac{\kappa}{2N} \sum_{i,j=1}^{N} \sum_{k,l=1}^{d} \left[ a^{kl}_i a^{kl}_j + b^{kl}_i b^{kl}_j \right],$$

where we cancel the imaginary term by symmetry of indices $i, j$.

- **Step B**: We derive

$$\frac{\partial V_1}{\partial U_i} = -\frac{\kappa}{N} \sum_{j=1}^{N} U_j = -\kappa U_c.$$
By direct calculation with (2.5), one has
\[
\frac{\partial V_1}{\partial a_i} = -\frac{\kappa}{N} \sum_{j=1}^N a_{ij}, \quad \frac{\partial V_1}{\partial b_i} = -\frac{\kappa}{N} \sum_{j=1}^N b_{ij},
\]
and we revert back to the coordinates of \( M_{d,d}(\mathbb{C})^N = \mathbb{R}^{2d^2 N} \) to obtain
\[
\frac{\partial V_1}{\partial U_i} = \sum_{k,l=1}^d \left( \frac{\partial V_1}{\partial a_{ik}} + i \frac{\partial V_1}{\partial b_{ik}} \right) E_{kl} = -\frac{\kappa}{N} \sum_{j=1}^N \sum_{k,l=1}^d u_j E_{kl} = -\frac{\kappa}{N} \sum_{j=1}^N U_j,
\]
where \( E_{kl} \) denotes the \( d \times d \) matrix whose \((k, l)\)-coordinate is 1 and the other coordinates are 0.

- Step C: We set
\[
u(d) = T_{U_d} \mathbb{U}(d) = \{ X \in M_{d,d}(\mathbb{C}) \mid X + X^\dagger = 0 \},
\]
and define an orthogonal projection:
\[
\pi : T_{U_d} M_{d,d}(\mathbb{C}) \to \nu(d) \quad \text{by} \quad A \mapsto \frac{1}{2} (A - A^\dagger).
\]
Since \( T_{U_d} \mathbb{U}(d) \) is the right translate \( \nu(d) U_i \) of \( \nu(d) \), we can see that the orthogonal projection \( \pi_{U_i} : T_{U_i} M_{d,d}(\mathbb{C}) \to T_{U_i} \nu(d) \) is given by \( A U_i \mapsto \pi(A) U_i = \frac{1}{2} (A - A^\dagger) U_i \) for an element \( A U_i \in T_{U_i} M_{d,d}(\mathbb{C}) \). Hence we may calculate
\[
\frac{\partial V_1}{\partial U_i} \bigg|_{T_{U_i} \nu(d)} = \pi_{U_i} \left( \frac{\partial V_1}{\partial U_i} \bigg|_{T_{U_i} M_{d,d}(\mathbb{C})} \right) = \pi \left( \frac{\partial V_1}{\partial U_i} \bigg|_{T_{U_i} M_{d,d}(\mathbb{C})} \right) U_i^\dagger U_i
\]
\[
= \pi \left( -\frac{\kappa}{N} \sum_{j=1}^N U_j U_i^\dagger \right) U_i = -\frac{\kappa}{2N} \sum_{j=1}^N (U_j U_i^\dagger - U_i U_j^\dagger) U_i
\]
\[
= -\frac{\kappa}{2} (U_c - U_i U_c^\dagger U_i).
\]

As a direct application of Proposition 2.3, we have the convergence of the flow \( e^{-iHt} U_i \) as \( t \to \infty \).

**Corollary 2.1** Let \( U_i = U_i(t) \) be a global solution to the Cauchy problem (2.4). Then, the flow \( U_i \) converges for any initial configuration \( \{ U_i^0 \} \).

**Proof** We use a gradient flow formulation in Proposition 2.3 and a standard argument in [20] to derive the convergence of the flow. \( \square \)

**Lemma 2.1** (Babalat’s lemma [2]) Suppose that a real-valued function \( f : [0, \infty) \to \mathbb{R} \) is continuously differentiable, and \( \lim_{t \to \infty} f(t) = \alpha \in \mathbb{R} \). If \( f' \) is uniformly continuous, then
\[
\lim_{t \to \infty} f'(t) = 0.
\]

**Proposition 2.4** Let \( \{ U_j \} \) be a global solution to the Cauchy problem (2.3). Then, the potential \( V_1 \) in (2.5) satisfies
\[
\frac{dV_1}{dt} = -\sum_{i=1}^N \| \dot{U}_i \|_F^2, \quad \exists \lim_{t \to \infty} V_1(U(t)) \quad \text{and} \quad \sup_{0 \leq t < \infty} \left| \frac{d^2}{dt^2} V_1(U(t)) \right| < \infty.
\]

\( \square \) Springer
Proof Note that $U_j$ and $U_c$ satisfy
\[
\dot{U}_j = \kappa (U_c - U_j U_c^\dagger U_j), \quad \dot{U}_c = \frac{\kappa}{N} \sum_{j=1}^{N} (U_c - U_j U_c^\dagger U_j), \quad j \in \mathbb{N}. \tag{2.7}
\]

- (Estimate of the first and second estimates): We use the above relations (2.7) to get
\[
\frac{d R^2}{dt} = \frac{1}{\kappa N} \sum_{j=1}^{N} \|\dot{U}_j\|_F^2 \geq 0. \tag{2.8}
\]
This yields the first estimate:
\[
\frac{d V_1}{dt} := - \sum_{j=1}^{N} \|\dot{U}_j\|_F^2 \leq 0.
\]
Since $V_1$ is non-decreasing and bounded below by $-\frac{1}{2} \kappa N d$ (see (2.6)),
\[
\exists \lim_{t \to \infty} V_1(U(\cdot)).
\]

- (Estimate of the third estimate): In the sequel, we will derive
\[
\|\dot{U}_i\|_F \leq \frac{\kappa}{2} \sqrt{d} (1 + d), \quad \|\dot{U}_c\|_F \leq \frac{\kappa}{2} \sqrt{d} (1 + d), \quad \left| \frac{d}{dt} \frac{\|\dot{U}_j\|_F^2}{2} \right| \leq \frac{\kappa^3}{2} d (1 + d)(1 + 2d).
\]
For the first estimate, we use (2.6) and (2.7) to get
\[
\|\dot{U}_j\|_F \leq \frac{\kappa}{2} \left(\|U_c\|_F + \|U_j U_c^\dagger U_j\|_F\right) \leq \frac{\kappa}{2} \left(\|U_c\|_F + \|U_j\|_F \cdot \|U_c^\dagger\|_F \cdot \|U_j\|_F\right) \leq \frac{\kappa}{2} \sqrt{d} (1 + d).
\]
This also implies
\[
\|\dot{U}_c\|_F = \left\| \frac{1}{N} \sum_{j=1}^{N} \dot{U}_j \right\|_F \leq \frac{1}{N} \sum_{j=1}^{N} \|\dot{U}_j\|_F \leq \frac{\kappa}{2} \sqrt{d} (1 + d).
\]
For the third estimate, we use (2.7) to obtain
\[
\frac{d}{dt} \frac{\|\dot{U}_j\|_F^2}{2} = \frac{\kappa^2}{2} \text{Re} \text{ tr} (U_c U_c^\dagger - U_j U_c^\dagger U_j U_c^\dagger) = \frac{\kappa^2}{2} \text{Re} \text{ tr} (U_c U_c^\dagger + U_c U_c^\dagger - U_j U_c^\dagger U_j U_c^\dagger - U_j U_c^\dagger U_j U_c^\dagger - U_j U_c^\dagger U_j U_c^\dagger)
\]
\[
= \frac{\kappa^2}{2} \text{Re} \text{ tr} (U_c U_c^\dagger - U_j U_c^\dagger U_j U_c^\dagger - U_j U_c^\dagger U_j U_c^\dagger).
\]
This yields
\[
\left| \frac{d}{dt} \frac{\|\dot{U}_j\|_F^2}{2} \right| \leq \frac{\kappa^2}{2} (\|U_c\|_F \|U_c^\dagger\|_F + \|U_j U_c^\dagger\|_F \|U_j U_c^\dagger\|_F + \|U_j U_c^\dagger\|_F \|U_j U_c^\dagger\|_F)
\]
\[
\leq \frac{\kappa^3}{2} d (1 + d)(1 + 2d).
\]
Finally, (2.8) yields
\[
\frac{d^2 R^2}{dt^2} = \frac{2}{\kappa N} \sum_{j=1}^{N} \frac{d}{dt} \|\dot{U}_j\|_F^2 \leq \kappa^2 d (1 + d)(1 + 2d).
This yields

\[
\sup_{0 \leq t < \infty} \left| \frac{d^2 V_1}{dt^2} \right| \leq N \kappa^3 d(1 + d)(1 + 2d).
\]

Finally, we combine Lemma 2.1 and Proposition 2.4 to get a desired result.

**Corollary 2.2** Let \( U_j = U_j(t) \) be a global solution to the Cauchy problem (2.3). Then, one has

\[
\lim_{t \to \infty} \dot{U}_j(t) = 0, \quad \text{for all } j \in \mathbb{N}.
\]

### 2.3 Relation with the Kuramoto–Daido Model

In this subsection, we briefly discuss a relationship between the Kuramoto–Daido model and the generalized Lohe model (1.9) with higher-order couplings.

Consider a phase ensemble for the Kuramoto–Daido oscillators with a size \( N \), and let \( \theta_j = \theta_j(t) \) be the phase of the \( j \)th Kuramoto–Daido oscillator. Then, the temporal dynamics of \( \theta_j \) is governed by the following system of the first-order ODEs:

\[
\dot{\theta}_j = v_j + \frac{\kappa}{N} \sum_{k=1}^{N} h(\theta_k - \theta_j), \quad j \in \mathbb{N},
\]

(2.9)

where \( v_j \) and \( \kappa \) are the natural frequency of the \( j \)th oscillator and nonnegative coupling strength. The coupling function \( h \) is assumed to be \( 2\pi \)-periodic and odd:

\[
h(\theta + 2\pi) = h(\theta), \quad h(-\theta) = -h(\theta), \quad \theta \in \mathbb{R}.
\]

(2.10)

Note that the choice \( h(\theta) = \sin \theta \) satisfies the above conditions (2.10) and system (2.9) reduces to the Kuramoto model:

\[
\dot{\theta}_j = v_j + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j), \quad j \in \mathbb{N}.
\]

(2.11)

In what follows, we consider the reduction of system (1.9) for \( d = 1 \). In this case, \( \mathbb{U}(1) = \mathbb{S}^1 \) and system (1.9) is reduced to a system on \( \mathbb{S}^1 \). More precisely, we set

\[
U_j = e^{-i\theta_j}, \quad i = 1, \ldots, N, \quad U_c := \frac{1}{N} \sum_{j=1}^{N} e^{-i\theta_j} = Re^{-i\varphi}, \quad H_j = v_j \in \mathbb{R},
\]

(2.12)

where \( R \) and \( \varphi \) correspond to amplitude and negative phase of the centroid of the ensemble \( \{e^{-i\theta_j}\} \), respectively. Now, we substitute the above ansatz (2.12) into (1.9) to obtain

\[
\dot{\theta}_j = v_j + \sum_{n=1}^{m} \kappa_n R^{2n-1} \sin(\varphi - \theta_j).
\]

(2.13)
On the other hand, note that

\[ R \sin(\varphi - \theta_j) = \frac{1}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j), \]

where \( f \) is the \( m \)th order polynomial in (1.10). Then, we use (2.14) to rewrite (2.13) as

\[
\dot{\theta}_j = \nu_j - \frac{2 f'(R^2)}{2} \sum_{k=1}^{N} \sin(\theta_k - \theta_j),
\]

(2.15)

Again, we use the last identity in (2.14) to see that system (2.15) can be written as a gradient flow:

\[
\dot{\theta}_j = -\frac{\partial}{\partial \theta_j} \left[ -\sum_{i=1}^{N} v_i \theta_i + \frac{1}{2} \frac{f'(R^2)}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j) \right], \quad \forall j \in \mathcal{N}.
\]

(2.16)

This is a consistent result to (1.5)–(1.6), and it is easy to see that if \( f(x) = \frac{\kappa x^2}{2} \), system (2.15) is exactly the Kuramoto model (2.11). We further estimate the term \( f(R^2) \) in (2.16) as follows.

\[
f(R^2) = \sum_{n=1}^{m} \frac{\kappa_n}{2n} R^{2n} = \sum_{n=1}^{m} \frac{\kappa_n}{2n} \left( \frac{1}{N} \sum_{i,j=1}^{N} \cos(\theta_i - \theta_j) \right)^n \]

\[
= \sum_{n=1}^{m} \frac{\kappa_n}{2n} \left( \frac{1}{N} \sum_{i_1,j_1=1}^{N} \cos(\theta_{i_1} - \theta_{j_1}) \right) \cdots \left( \frac{1}{N} \sum_{i_n,j_n=1}^{N} \cos(\theta_{i_n} - \theta_{j_n}) \right)^{n\text{-times}}
\]

(2.17)

Note that the R.H.S. of (2.17) involve with multiple interactions up to \( m \)-pairs. To see the apparent difference between system (1.9) with \( d = 1 \) and the Kuramoto–Daido model, we return to the gradient flow formulation of the Kuramoto–Daido model (2.9)–(2.10). For this, we set \( H \) to be the anti-derivative of the coupling function \( h \) in (2.10):

\[
H(\theta) := \int_{0}^{\theta} h(\xi) d\xi, \quad \theta \in \mathbb{R}.
\]

Then, it is easy to see that \( H \) is an even function

\[
H(-\theta) = H(\theta), \quad \theta \in \mathbb{R},
\]

and the Kuramoto–Daido model (2.9) can be rewritten as a gradient flow:

\[
\dot{\theta}_j = -\frac{\partial}{\partial \theta_j} \left[ -\sum_{i=1}^{N} v_i \theta_i + \frac{1}{2N^2} \sum_{i,k=1}^{N} H(\theta_i - \theta_k) \right], \quad \forall j \in \mathcal{N}.
\]

(2.18)
Note that unlike the potential in (2.16) and (2.17), the potential of system (2.18) is defined by the binary interaction between two particles. This is one of crucial difference between systems (2.9) and (2.15).

2.4 Previous Results

In this subsection, we brief review two closely related results from [11,20] that deal with Lohe type aggregation model on matrix Lie groups. The original Lohe matrix model was introduced as an aggregation model on the unitary group. After Lohe’s work [24,25], it was further extended to the Lohe group (whose definition is defined below) in [20]. Next, we provide a concept of the Lohe group in the following definition.

Definition 2.1 [20] Let $G$ be a subgroup of the linear group $GL(d, k)$ with $k = \mathbb{R}$ or $k = \mathbb{C}$. Then, we say $G$ is a Lohe group if the following two conditions hold.

1. $G$ is a matrix Lie group, which is a closed subgroup of $GL(d, k)$: $G \leq GL(d, k)$.

2. $G$ satisfies

$X - X^{-1} \in \mathfrak{g}$ for all $X \in G$.

Here, $\mathfrak{g}$ is the Lie algebra associated to $G$, i.e., the tangent space $T_I G$ of $G$ at the identity matrix $I$.

Remark 2.1 The following matrix groups are Lohe groups:

$GL(k, d)$, $O_d(k)$, $O_{p,q}(k)$, $U_d$, $SP_n(k)$, $SO_d(k)$, $SO_{p,q}(k)$, $SP(n)$, $SL(2, k)$, $SU(2)$.

Let $G$ and $\mathfrak{g}$ be a Lohe group and its associated Lie algebra, respectively. Then, a generalized Lohe matrix model [20] on $G$ reads as follows.

$$\left\{ \begin{array}{c}
\dot{X}_j X_j^{-1} = \Omega_j + \frac{\kappa}{2N} \sum_{k=1}^{N} \left( X_k X_k^{-1} - X_j X_j^{-1} \right),
\quad t > 0,

X_j(0) = X_j^0, \quad j \in \mathcal{N},
\end{array} \right.$$  

where $\Omega_i \in \mathfrak{g}$.

For an ensemble $\{X_j\}$, we set

$$\mathcal{D}(X) := \max_{1 \leq i,j \leq N} \|X_i - X_j\|_F.$$

Theorem 2.1 [20] Suppose that $H_i$, the coupling strength and the initial data $\{X^0\}$ satisfy

$H_i = 0, \quad i = 1, \ldots, N, \quad \kappa > 0 \text{ and } \mathcal{D}(X^0) < 1.$

Then, there exists a smooth global solution $\{X_j\}$ such that

$$\frac{\mathcal{D}(X^0)}{(1 + \mathcal{D}(X^0))e^{\kappa t} - \mathcal{D}(X^0)} \leq \mathcal{D}(X(t)) \leq \frac{\mathcal{D}(X^0)}{(1 - \mathcal{D}(X^0))e^{\kappa t} + \mathcal{D}(X^0)}, \quad t \geq 0.$$
In [11], Deville further introduced the generalized model (2.19) by introducing a polynomial coupling, namely “quantum Kuramoto model” on the Lohe group $G$ associated with Lie algebra $g$ introduced in Definition 2.1. Let $\Gamma$ be an undirected weighted graph with $N$ vertices with edge weights $\gamma_{ij} \geq 0$ with $\gamma_{ij} = \gamma_{ji}$. To be more specific, let $f$ be a real analytic function. Then, the quantum Kuramoto model reads as
\[
\dot{X}_j X_j^{-1} = \Omega_j + \frac{1}{2} \sum_{k=1}^{N} \gamma_{jk} \left( f(X_k X_j^{-1}) - f(X_j X_k^{-1}) \right),
\] (2.20)
where $\Omega_j \in g$.

Note that for $f(x) = x$ and $\gamma_{ij} = \frac{2}{N}$, the quantum Kuramoto model (2.20) becomes the generalized Lohe matrix model (2.19). In the aforementioned work, Deville studied sync and near sync solutions and investigated the stability of these solutions and twist solutions.

Other than a linear function $f$, system (2.20) cannot be rewritten as a mean-field form. For example, $f(x) = x^2$ and $\gamma_{jk} = \frac{2}{N}$, system (2.20) becomes
\[
\dot{X}_j X_j^{-1} = \Omega_j + \frac{1}{N} \sum_{k=1}^{N} \left( (X_k X_j^{-1})^2 - (X_j X_k^{-1})^2 \right).
\]
This is clearly different from our proposed model (3.9):
\[
\dot{U}_j U_j^\dagger = -iH_j + \frac{\kappa}{N^3} \sum_{k_1,k_2,k_3=1}^{N} \left( U_{k_1} U_{k_2}^\dagger U_{k_3}^\dagger U_j^\dagger - U_j U_{k_2}^\dagger U_{k_3}^\dagger U_{k_1} \right).
\]

2.5 Elementary Estimates

In this subsection, we present several elementary estimates to be used for later sections.

Let $U_i$ and $U_j$ be the unitary matrices. Hence a matrix product between two matrices $AB$ is well defined. Then we have following identities:
\[
\|U_i - U_j\|_F^2 = 2d - \text{tr}(U_i U_j^* - U_j U_i^*), \quad \|U_c\|_F^2 = d - \frac{1}{2N^2} \sum_{i,j} \|U_i - U_j\|_F^2.
\]

**Lemma 2.2** Let $A$ and $B$ be the matrices with proper size. Then we have the following inequality:
\[
\|AB\|_F \leq \|A\|_{\text{op}} \cdot \|B\|_F,
\]
where $\|\cdot\|_{\text{op}}$ is an operator norm.

**Proof** Let
\[
B = \begin{bmatrix} b_1 : b_2 : \cdots : b_n \end{bmatrix},
\]
where $b_\alpha$ is a vector. From the direct calculation, we have
\[
\|AB\|_F^2 = \sum_{\alpha,\beta} |[AB]_{\alpha\beta}|^2 = \sum_{\alpha} \|Ab_\alpha\|^2 \leq \sum_{\alpha} \left( \|A\|_{\text{op}} \cdot \|b_\alpha\| \right)^2
\]
\[
= \|A\|_{\text{op}}^2 \cdot \sum_{\alpha} \|b_\alpha\|^2 = \|A\|_{\text{op}}^2 \cdot \|B\|_F^2.
\]
Remark 2.2 If $U$ is a unitary matrix, then
\[ \|U\|_{\text{op}} = 1. \]

Lemma 2.3 Let $\{U_i\}_{i=1}^N$ be an ensemble of unitary matrices in $\mathbb{U}(d)$. Then we have
\[ \|U_c\|_{\text{op}} \leq 1. \]
Equality holds if and only if there exists $v \in \mathbb{C}^d$ with $\|v\| \neq 0$ such that
\[ U_1v = U_2v = \cdots = U_Nv. \]

Proof By direct calculation, we have
\[ \|U_c v\|_F \leq \frac{1}{N} \sum_{k=1}^{N} \|U_k v\|_F \leq \frac{1}{N} \sum_{k=1}^{N} \|U_k\|_{\text{op}} \|v\| = \|v\|. \]
So we have
\[ \|U_c\|_{\text{op}} \leq 1. \]
We can also easily show the equality condition.

Lemma 2.4 Let $A$ and $B$ be $d \times d$ matrices. Then, one has
\[ |\text{tr}(A)| \leq \sqrt{d} \|A\|_F \quad \text{and} \quad \|AB\|_F \leq \|A\|_F \cdot \|B\|_F. \]

Proof (i) By definition of a trace, one has
\[ |\text{tr}(A)|^2 = \left| \sum_{\alpha} [A]_{\alpha\alpha} \right|^2 \leq \sum_{\alpha} |[A]_{\alpha\alpha}|^2 \cdot \sum_{\alpha} 1^2 \leq d \|A\|_F^2. \]
(ii) By direct calculation, one has
\[ \|AB\|_F^2 = \sum_{\alpha, \beta} |[AB]_{\alpha\beta}|^2 = \sum_{\alpha, \beta} \left| \sum_{\gamma} [A]_{\alpha\gamma} [B]_{\gamma\beta} \right|^2 \leq \sum_{\alpha, \beta} \left( \sum_{\gamma} |[A]_{\alpha\gamma}|^2 \right) \cdot \left( \sum_{\gamma} |[B]_{\gamma\beta}|^2 \right) = \|A\|_F^2 \cdot \|B\|_F^2. \]
Thus, we have
\[ \|AB\|_F \leq \|A\|_F \cdot \|B\|_F. \]

3 The Lohe Matrix Model with a Monomial Interaction

In this section, we present a generalized Lohe matrix model with a monomial higher-order coupling via a gradient flow approach as a preliminary step for a general polynomial coupling. In the sequel, we derive a generalized Lohe matrix model with higher-order couplings: for $t > 0$,
\[
\begin{align*}
\begin{cases}
\text{i} \dot{U}_j & = H + \frac{i \kappa}{2} \left( U_c U_c^\dagger U_c \cdots U_c^\dagger U_c U_c^\dagger - U_j U_c^\dagger U_c \cdots U_c^\dagger U_c \right), \\
U_j(0) & = U_j^0,
\end{cases}
\end{align*}
\tag{3.1}
\]
where $H$ is a Hermitian matrix with $H^\dagger = H$.

Note that system (3.1) can be rewritten as summation form instead of a mean-field form using a mean-field quantity $U_c$:

$$\dot{U}^\dagger_j U_j = H + \frac{i\kappa}{2N^{2m-1}} \sum_{k_1, \ldots, k_{2m-1}=1}^N \left( U_{k_1}^\dagger U_{k_2}^\dagger \cdots U_{k_{2m-2}}^\dagger U_{k_{2m-1}}^\dagger U_j^\dagger - U_j U_{k_{2m-1}}^\dagger U_{k_{2m-2}} \cdots U_{k_2}^\dagger U_{k_1}^\dagger \right).$$

Since the R.H.S. of (3.1) is self-adjoint, and this yields the conservation of quadratic quantities $U_j U_j^\dagger$.

**Lemma 3.1** Let $\{U_i\}$ be a global smooth solution of system (3.1). Then one has

$$\frac{d}{dt} (U_j U_j^\dagger) = 0, \quad t > 0, \quad j \in N.$$

**Proof** We set

$$C_j(U_1, \ldots, U_N) := \frac{i\kappa}{2N^{2m-1}} \sum_{k_1, \ldots, k_{2m-1}=1}^N \left( U_{k_1}^\dagger U_{k_2}^\dagger \cdots U_{k_{2m-2}}^\dagger U_{k_{2m-1}}^\dagger U_j^\dagger - U_j U_{k_{2m-1}}^\dagger U_{k_{2m-2}} \cdots U_{k_2}^\dagger U_{k_1}^\dagger \right).$$

Then, it is easy to see that

$$C_j(U_1, \ldots, U_N) = C_j(U_1, \ldots, U_N). \quad (3.2)$$

Now, we return to system (3.1):

$$\dot{U}_j U_j^\dagger = -i \left( H + C_j(U_1, \ldots, U_N) \right). \quad (3.3)$$

We take a hermitian conjugate of (3.3) and use (3.2) to get

$$U_j \dot{U}_j^\dagger = i \left( H + C_j(U_1, \ldots, U_N) \right). \quad (3.4)$$

Finally, we add (3.3) and (3.4) to get

$$\frac{d}{dt} (U_j U_j^\dagger) = 0.$$

From now on, throughout the paper, we assume

$$U_j U_j^\dagger = U_j^\dagger U_j = I_d, \quad j \in N.$$

and consider emergent dynamics of the following Cauchy problem:

$$\begin{cases}
\dot{U}_j = -iH U_j + \frac{\kappa}{2} \left( U_{c} U_{c}^\dagger U_{c} \cdots U_{c}^\dagger U_{c} - U_j U_{c}^\dagger U_{c} \cdots U_{c} U_{c}^\dagger U_{c} \right), \quad t > 0, \\
U_j(0) = U_j^0, \quad j \in N.
\end{cases} \quad (3.5)$$

Now, we consider the corresponding nonlinear subsystem:

$$\begin{cases}
\dot{L}_j = \frac{\kappa}{2} \left( L_{c} L_{c}^\dagger L_{c} \cdots L_{c}^\dagger L_{c} - L_j L_{c}^\dagger L_{c} \cdots L_{c} L_{c}^\dagger L_{c} \right), \quad t > 0, \\
L_j(0) = U_j^0, \quad j \in N.
\end{cases} \quad (3.6)$$
Proposition 3.1 (Solution splitting property) Let \( \{U_j\} \) and \( \{L_j\} \) be two solutions to systems (3.5) and (3.6), respectively. Then one has

\[
U_j(t) = e^{-iHt} \circ L_j(t), \quad j \in \mathbb{N}.
\]  

(3.7)

Proof We substitute (3.7) into (3.6) and use the relations

\[
\dot{U}_j = -iHe^{-iHt}L_j + e^{-iHt}\dot{L}_j, \quad U_cU_c^\dagger = L_cL_c^\dagger, \quad U_jU_j^\dagger = L_jL_j^\dagger
\]

to see that \( L_j \) satisfies system (3.6).

From now on, we assume \( H \equiv 0 \). In what follows, we will derive system (3.1) using a gradient flow formulation with a monomial potential \( V_m(U) \): for \( m \geq 1 \), we set

\[
V_m(U) := -\kappa N^2 \text{tr}((U_cU_c^\dagger)^m) = -\kappa N^2 \frac{\text{tr}(U_cU_c^\dagger \cdots U_cU_c^\dagger)}{2m}.
\]  

(3.8)

Note that \( V_m \) is analytic and bounded:

\[
|V_m(U)| \leq \frac{\kappa}{2m} \frac{\text{tr}(U_cU_c^\dagger \cdots U_cU_c^\dagger)}{2m} \leq \frac{\kappa}{2m} \|U_c\|_F^{2m} \leq \frac{\kappa}{2m} d^m.
\]

(3.9)

In next two subsections, we consider two cases separately:

Either \( m = 2 \) or \( m \geq 3 \).

First, we get some intuition with the case \( m = 2 \), and then we treat the case with \( m \geq 3 \).

3.1 Case with \( m = 2 \)

Consider the Cauchy problem to the following system:

\[
\dot{U}_j = \frac{\kappa}{2} (U_cU_c^\dagger U_c - U_jU_j^\dagger U_cU_c^\dagger U_j), \quad t > 0,
\]

\[
U_j \bigg|_{t=0} = U_j^0 \in \mathbb{U}(d), \quad j \in \mathbb{N}.
\]  

(3.9)

3.1.1 A Gradient Flow Formulation

For an ensemble \( \{U_j\}_{j=1}^N \), consider the potential

\[
V_2(U) := -\frac{N\kappa}{4} \text{tr}((U_cU_c^\dagger)^2).
\]

(3.10)

Lemma 3.2 Let \( \{U_j\} \) be a solution of (3.9) and \( V_2 = V_2(U) \) be a potential defined by (3.10). Then, one has

\[
\frac{\partial V_2}{\partial U_i} = -\kappa U_cU_c^\dagger U_i, \quad i \in \mathbb{N}.
\]

Proof (i) We set

\[
[U_i]_{\alpha\beta} := a_i^{\alpha\beta} + ib_i^{\alpha\beta}, \quad a_i^{\alpha\beta}, b_i^{\alpha\beta} \in \mathbb{R}.
\]
Then we have

\[
\nu_2 = -\frac{\kappa}{4N^3} \sum_{i,j,k,l} [U_i]\omega\beta[U_j^\dagger]_\beta\gamma[U_k]_\gamma\delta[U_l^\dagger]_\delta\alpha \\
= -\frac{\kappa}{4N^3} \sum_{i,j,k,l} (a_i^{\alpha\beta} + ib_i^{\alpha\beta})(a_j^{\gamma\beta} - ib_j^{\gamma\beta})(a_k^{\gamma\delta} + ib_k^{\gamma\delta})(a_l^{\delta\alpha} - ib_l^{\delta\alpha}),
\]  

(3.11)

where we used Einstein summation rule. Now, we use the symmetry and (3.11) to get

\[
\nu_2 = -\frac{\kappa}{4N^3} \sum_{i,j,k,l} \left( a_i^{\alpha\beta} a_j^{\gamma\beta} a_k^{\gamma\delta} b_l^{\delta\alpha} + b_i^{\alpha\beta} b_j^{\gamma\beta} b_k^{\gamma\delta} a_l^{\delta\alpha} - 2a_i^{\alpha\beta} b_j^{\gamma\beta} a_k^{\gamma\delta} b_l^{\delta\alpha} \\
+ 2a_i^{\alpha\beta} a_j^{\gamma\beta} b_k^{\gamma\delta} b_l^{\delta\alpha} + 2b_i^{\alpha\beta} a_j^{\gamma\beta} a_k^{\gamma\delta} b_l^{\delta\alpha} \right).
\]

(3.12)

This yields

\[
\frac{\partial \nu_2}{\partial a_i^{\mu\nu}} = -\frac{\kappa}{4N^3} \sum_{j,k,l} \left( 4a^{\mu\nu} a_j a_k a_l^{\mu\delta} - 4b_j^{\mu\nu} a_k a_l^{\mu\delta} + 4a_j^{\mu\nu} b_k a_l^{\mu\delta} + 4b_j^{\mu\nu} a_k b_l^{\mu\delta} \right),
\]

\[
\frac{\partial \nu_2}{\partial a_i^{\mu\nu}} = -\frac{\kappa}{N^3} \sum_{j,k,l} \left( a^{\mu\nu} a_j a_k a_l^{\mu\delta} - b_j^{\mu\nu} a_k a_l^{\mu\delta} + b_j^{\mu\nu} a_k b_l^{\mu\delta} + a_j^{\mu\nu} b_k a_l^{\mu\delta} + a_j^{\mu\nu} b_k b_l^{\mu\delta} \right),
\]

\[
\frac{\partial \nu_2}{\partial b_i^{\mu\nu}} = -\frac{\kappa}{N^3} \sum_{j,k,l} \left( b^{\mu\nu} a_j a_k a_l^{\mu\delta} - a_j^{\mu\nu} a_k a_l^{\mu\delta} + a_j^{\mu\nu} a_k b_l^{\mu\delta} + b_j^{\mu\nu} a_k a_l^{\mu\delta} + b_j^{\mu\nu} a_k b_l^{\mu\delta} \right).
\]

Finally we can calculate

\[
\frac{\partial \nu_2}{\partial U_i} = \left( \frac{\partial \nu_2}{\partial a_i^{\mu\nu}} + \frac{\partial \nu_2}{\partial b_i^{\mu\nu}} \right) E^{\mu\nu},
\]

where \( E^{\mu\nu} \) denotes the \( d \times d \) matrix whose \((\mu, \nu)\)-coordinate is 1 and the other coordinates are 0. By direct calculation, one has

\[
\frac{\partial \nu_2}{\partial a_i^{\mu\nu}} + \frac{\partial \nu_2}{\partial b_i^{\mu\nu}} = -\frac{\kappa}{N^3} \sum_{j,k,l} \left( (a^{\mu\alpha} a_j a_k a_l^{\alpha\beta} - b_j^{\mu\alpha} a_k a_l^{\alpha\beta} + b_j^{\mu\alpha} a_k b_l^{\alpha\beta} + a_j^{\mu\alpha} b_k a_l^{\alpha\beta} + a_j^{\mu\alpha} b_k b_l^{\alpha\beta}) \\
+ i(b_j^{\mu\alpha} b_k a_l^{\mu\beta} - a_j^{\mu\alpha} b_k a_l^{\mu\beta} + a_j^{\mu\alpha} b_k b_l^{\mu\beta} + b_j^{\mu\alpha} b_k a_l^{\mu\beta} + b_j^{\mu\alpha} b_k b_l^{\mu\beta}) \right) \\
= -\frac{\kappa}{N^3} \sum_{j,k,l} [U_j]_\mu\alpha [U_k^\dagger]_\alpha\beta [U_l]_\beta\nu.
\]

This implies

\[
\frac{\partial \nu_2}{\partial U_i} = \left( \frac{\partial \nu_2}{\partial a_i^{\mu\nu}} + \frac{\partial \nu_2}{\partial b_i^{\mu\nu}} \right) E^{\mu\nu} = -\frac{\kappa}{N^3} \sum_{j,k,l} U_j U_k^\dagger U_l = -\kappa U_c U_c^\dagger U_c.
\]

\( \square \)

**Proposition 3.2** System (3.9) can be rewritten as a gradient flow with the potential \( \nu_2 \):

\[
\dot{U}_i = -\frac{\partial \nu_2}{\partial U_i} \bigg|_{U_i \in U(d)}, \quad i \in \mathcal{N}.
\]
Proof We use Lemma 2.1 to see
\[
\frac{\partial V_2}{\partial U_i} \bigg|_{T_{U_i} U(d)} = \pi U_i \left( \frac{\partial V_2}{\partial U_i} \bigg|_{T_{U_i} M_{d,d}(C)} \right) = \pi \left( \frac{\partial V_2}{\partial U_i} \bigg|_{T_{U_i} M_{d,d}(C)} U_i^\dagger \right) U_i
\]
\[
= \pi \left( -\frac{\kappa}{N^3} \sum_{j,k,l=1}^N U_j U_k^\dagger U_l^\dagger U_i \right) U_i
\]
\[
= -\frac{\kappa}{2N^3} \sum_{j,k,l=1}^N (U_j U_k^\dagger U_l^\dagger U_i - U_l U_j^\dagger U_k^\dagger U_i) U_i
\]
\[
= -\frac{\kappa}{2} (U_c U_c^\dagger U_c - U_i U_i^\dagger U_i U_i).
\]
\[\square\]

As a corollary of a gradient flow formulation of (3.9), we have the convergence of the flow.

Corollary 3.1 Suppose that coupling strength and the initial data \( \{U_j^0\} \) satisfy
\[
\kappa > 0, \quad U_j^t U_j^0 = I_d, \quad j \in \mathcal{N},
\]
and let \( \{U_j\} \) be a global solution of system (3.9). Then, there exists an equilibrium \( (U_1^\infty, \cdots, U_N^\infty) \) such that
\[
\lim_{t \to \infty} \|U_j(t) - U_j^\infty\|_F = 0, \quad j \in \mathcal{N}.
\]

Proof Since system (3.9) is a gradient flow with the analytical potential \( V_2 \), the flow \( U_j \) converges toward an equilibrium (see Theorem 5.2 in [21]). \[\square\]

3.1.2 Temporal Evolution of Potential

Next, we study temporal evolution of the potential \( V_1 \) and \( V_2 \) in the following lemma.

Lemma 3.3 Let \( \{U_j\} \) be a global solution of system (3.9) with the initial data \( \{U_j^0\} \):
\[
U_j^0 U_j^0 = I_d, \quad j \in \mathcal{N}.
\]

Then, one has
\[
(i) \quad \frac{d}{dt} V_1(U) = -\frac{\kappa^2}{4} \sum_{j=1}^N \|U_c U_c^\dagger U_j - U_j U_c^\dagger U_c\|_F^2 - \frac{\kappa^2}{8} \sum_{j=1}^N \|U_c U_c^\dagger U_j - U_j U_c^\dagger U_c\|_F^2,
\]
\[
(ii) \quad \frac{d}{dt} V_2(U) = -\frac{\kappa^2}{4} \sum_{j=1}^N \|U_c U_c^\dagger U_c U_j - U_j U_c^\dagger U_c U_c\|_F^2.
\]
Proof \( (i) \) We use (3.9) to get
\[
\frac{d}{dt} V_1(U) = -\frac{\kappa N}{2} \frac{d}{dt} \tr(U_c U_c^\dagger) = -\frac{\kappa N}{2} \tr(\dot{U}_c U_c^\dagger + U_c \dot{U}_c^\dagger) = -\frac{\kappa}{2} \sum_{j=1}^{N} \tr(\dot{U}_j U_j^\dagger + U_c \dot{U}_j^\dagger)
\]
\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \left( \tr(U_c U_c^\dagger U_c U_c^\dagger - U_j U_j^\dagger U_c U_j^\dagger U_j U_j^\dagger) \right) + (c.c.)
\]
\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \tr(2U_c U_c^\dagger U_c U_c^\dagger - U_j U_j^\dagger U_c U_j^\dagger U_j U_j^\dagger - U_c U_j U_c U_j^\dagger U_j U_j^\dagger)
\]
\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \tr((U_c U_c^\dagger U_j - U_c U_j^\dagger U_c)(U_j U_c U_j^\dagger - U_c U_j^\dagger U_c)
\]
\[
+ U_c U_j^\dagger U_c U_j^\dagger - U_c U_j^\dagger U_c U_j U_j^\dagger)
\]
\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \|U_c U_c^\dagger U_j - U_c U_j^\dagger U_c U_j^\dagger\|_F^2 - \frac{\kappa^2}{8} \sum_{j=1}^{N} \|U_c U_j^\dagger U_j - U_j U_j^\dagger U_c\|_F^2.
\]

(ii) Similarly, one has
\[
\frac{d}{dt} V_2(U) = -\frac{\kappa N}{4} \frac{d}{dt} \tr(U_c U_c^\dagger U_c U_c^\dagger) = -\frac{\kappa N}{2} \left( \tr(\dot{U}_c U_c^\dagger U_c U_c^\dagger) + \tr(U_c \dot{U}_c^\dagger U_c U_c^\dagger) \right)
\]
\[
= -\frac{\kappa}{2} \sum_{j=1}^{N} \left( \tr(\dot{U}_j U_j^\dagger U_c U_c^\dagger) + \tr(U_c \dot{U}_c^\dagger U_c U_c^\dagger) \right)
\]
\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \tr((U_c U_c^\dagger U_c - U_c U_c^\dagger U_c U_c^\dagger U_j U_j^\dagger U_c U_j^\dagger) + (c.c.)
\]
\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \|U_c U_c^\dagger U_c U_j^\dagger - U_c U_j^\dagger U_c U_j^\dagger\|_F^2.
\]

As a corollary, one has the following result.

Corollary 3.2. Let \( \{U_j\} \) be a global solution of system (3.9) with the initial data \( \{U_j^0\} \):
\[
\kappa > 0, \quad (U_j^0)^\dagger U_j^0 = I_d, \quad j \in \mathcal{N}.
\]
Then, one has
\[
\lim_{t \to \infty} \|U_c U_c^\dagger U_c U_j^\dagger - U_j U_j^\dagger U_c U_c^\dagger\|_F^2 = 0, \quad j \in \mathcal{N}.
\]

Proof \( (i) \) Since \( V_2(U) \) is bounded below and non-increasing along the flow (3.9), \( V_2(U(\cdot)) \) converges as \( t \to \infty \).

\( (ii) \) It follows from Lemma 3.3 that
\[
\frac{d}{dt} V_2(U) = -\frac{\kappa^2}{4} \sum_{j=1}^{N} \|U_c U_j^\dagger U_c U_j^\dagger - U_j U_j^\dagger U_c U_c^\dagger\|_F^2.
\]
In order to apply Babalat’s lemma (Lemma 2.1) for the derivation of the desired estimate, it suffices to show that

\[
\sup_{0 \leq t < \infty} \left| \frac{d^2}{dt^2} \mathcal{V}_2(U) \right| < \infty.
\]

We differentiate (3.13) with respect to \( t \) and obtain

\[
\frac{d^2}{dt^2} \mathcal{V}_2(U) = -\frac{\kappa^2}{4} \sum_{j=1}^N \frac{d}{dt} \| U_c U_c^\dagger U_c^\dagger U_j - U_j U_c^\dagger U_c^\dagger \|^2_F.
\]

From the direct calculation, we have

\[
\frac{d}{dt} \| U_c U_c^\dagger U_c^\dagger U_j - U_j U_c^\dagger U_c^\dagger \|^2_F = \frac{d}{dt} \text{tr}\left[(U_{k_1} U_{k_2} U_{k_3} U_j - U_j U_{k_1} U_{k_2} U_{k_3}) (U_{k_4} U_{k_5} U_{k_6} U_j - U_j U_{k_4} U_{k_5} U_{k_6})^\dagger\right].
\]

From the boundedness of \( \| U_j \|_F, \| U_{k_a} \|_F, \| U_j \|_{\text{op}}, \| U_{k_a} \|_{\text{op}}, \| \dot{U}_j \|_F, \| \dot{U}_{k_a} \|_F, \| \dot{U}_j \|_{\text{op}}, \| \dot{U}_{k_a} \|_{\text{op}} \) and Lemmas 2.2, 2.3, 2.4 we can obtain the boundedness of

\[
\frac{d}{dt} \text{tr}\left[(U_{k_1} U_{k_2} U_{k_3} U_j - U_j U_{k_1} U_{k_2} U_{k_3}) (U_{k_4} U_{k_5} U_{k_6} U_j - U_j U_{k_4} U_{k_5} U_{k_6})^\dagger\right].
\]

Hence

\[
\frac{d}{dt} \| U_c U_c^\dagger U_c^\dagger U_j - U_j U_c^\dagger U_c^\dagger \|^2_F \text{ is uniformly bounded.}
\]

Therefore, one has

\[
\frac{d^2}{dt^2} \mathcal{V}_2(U) = -\frac{\kappa^2}{4} \sum_{j=1}^N \frac{d}{dt} \| U_c U_c^\dagger U_c^\dagger U_j - U_j U_c^\dagger U_c^\dagger \|^2_F.
\]

is uniformly bounded over time. So we can apply Babalat’s lemma to obtain

\[
\lim_{t \to \infty} \frac{d}{dt} \mathcal{V}_2(U) = 0.
\]

This implies

\[
\lim_{t \to \infty} \| U_c U_c^\dagger U_c^\dagger U_j - U_j U_c^\dagger U_c^\dagger \|^2_F = 0, \quad j \in \mathcal{N}.
\]
### 3.2 Case with $m \geq 3$

Consider the Cauchy problem for (3.1) in a mean-field form:

\[
\begin{aligned}
\dot{U}_j &= \frac{\kappa}{2} \left( U_c U_c^\dagger \cdots U_c^\dagger U_c - U_j U_c U_c^\dagger \cdots U_c^\dagger U_j \right), \\
U_j \bigg|_{t=0} &= U_j^0 \in \mathbb{U}(d), \quad j \in \mathcal{N}.
\end{aligned}
\]  

(3.14)

Similar to Lemma 3.1 and Proposition 3.2, one has a gradient flow formulation to (3.14).

**Proposition 3.3** System (3.14) can be rewritten as a gradient flow with the potential $V_m$:

\[
\dot{U}_j = -\frac{\partial V_m}{\partial U_j} \bigg|_{U_j U(d)}, \quad j \in \mathcal{N}.
\]

**Proof** The proof is basically the same as in the proof of Proposition 3.2. Hence we omit its details. \(\square\)

As a corollary of a gradient flow formulation of (3.14), we have the convergence of the flow.

**Corollary 3.3** Suppose that coupling strength and the initial data $\{U_j^0\}$ satisfy

\[
\kappa > 0, \quad U_j^0 U_j^0 = I_d, \quad j \in \mathcal{N},
\]

and let $\{U_j\}$ be a global solution of system (3.14). Then, there exists an equilibrium $(U_1^\infty, \cdots, U_N^\infty)$ such that

\[
\lim_{t \to \infty} \|U_j(t) - U_j^\infty\|_F = 0, \quad j \in \mathcal{N}.
\]

**Proof** Since system (3.14) is a gradient flow with the analytical potential $V_m(U)$, the flow $U_j(\cdot)$ converges toward an equilibrium (see Theorem 5.2 in [21]). \(\square\)

Now we want to find the derivative of functional $V_m(U)$ along the dynamics (3.9).

**Lemma 3.4** Let $\{U_j\}$ be a global solution of system (3.14) with the initial data satisfying

\[
U_j^{0\dagger} U_j^0 = I_d, \quad j \in \mathcal{N}.
\]

Then we have

\[
\frac{d}{dt} V_m(U) = -\frac{\kappa^2}{4} \sum_{j=1}^N \| U_j U_c U_c^\dagger \cdots U_c^\dagger U_j^0 - U_c U_c^\dagger U_c \cdots U_c^\dagger U_j^0 \|_F^2.
\]
Proof By direct calculations, one has

\[
\frac{d}{dt} V_m(U) = -\frac{\kappa N}{2m} \text{tr}((U_c U_c^\dagger)^m) = -\frac{\kappa N}{2} \left( \text{tr}(\dot{U}_c U_c^\dagger \cdots U_c U_c^\dagger) + (c.c.) \right) 
\]

\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \text{tr}((U_c U_c^\dagger \cdots U_c U_c^\dagger - U_j U_c^\dagger U_c^\dagger \cdots U_c U_c^\dagger U_j) U_c^\dagger \cdots U_c U_c^\dagger) 
\]

\[
+ (c.c.) 
\]

\[
= -\frac{\kappa^2}{4} \sum_{j=1}^{N} \|U_j U_c^\dagger U_c^\dagger \cdots U_c U_c^\dagger - U_c U_c^\dagger U_c^\dagger \cdots U_c U_c^\dagger U_j\|_F^2. 
\]

\[
\square 
\]

Proposition 3.4 Let \(\{U_j\}\) be a global smooth solution of system (3.14) with the initial data \(\{U_j^0\}\):

\[
U_j^0 U_j^0 = I_d, \quad j \in \mathcal{N}. 
\]

Then, for \(i = 1, \ldots, N\),

\[
\lim_{t \to \infty} \|U_i U_c^\dagger U_c^\dagger \cdots U_c U_c^\dagger - U_c U_c^\dagger U_c^\dagger \cdots U_c U_c^\dagger U_i\|_F^2 = 0, \quad \lim_{t \to \infty} \|\dot{U}_j\|_F = 0.
\]

Proof (i) The first assertion follows from the gradient flow formulation (Proposition 3.2) and \(U_j \in \mathcal{U}(d)\).

(ii) We use the boundedness of \(V_2\) (see (3.8)) and Lemma 3.4 to see

\[
\exists \lim_{t \to \infty} V_m(U). 
\]

Note that

\[
\frac{d}{dt} V_m(U) = -\frac{\kappa^2}{4} \sum_{i=1}^{N} \|U_i U_c^\dagger U_c^\dagger \cdots U_c U_c^\dagger - U_c U_c^\dagger U_c^\dagger \cdots U_c U_c^\dagger U_i\|_F^2. 
\]

We claim:

\[
\sup_{0 \leq t < \infty} \left| \frac{d^2}{dt^2} V_m(U) \right| < \infty. 
\]

By (3.15) and (3.16), we can apply Babalat’s lemma to get the desired estimate:

\[
\lim_{t \to \infty} \frac{d}{dt} V_m(U) = 0.
\]

For the proof of claim (3.16), it is sufficient to prove the uniform boundedness of

\[
\frac{d^2}{dt^2} V_m(U).
\]
This proof is very similar to the proof of Corollary 3.2, so we will omit. From this result, we have

$$\lim_{t \to \infty} \frac{d}{dt} V_m(U) = -\lim_{t \to \infty} \sum_{i=1}^{N} \left\| \frac{U_i U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger - U_c U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger U_i^\dagger}{2m-1} \right\|_F^2 = 0.$$ 

So we have

$$\lim_{t \to \infty} \frac{\kappa^2}{4} \left\| \frac{U_i U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger - U_c U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger U_i^\dagger}{2m-1} \right\|_F^2 = 0,$$

for all $i \in \mathcal{N}$.

(iii) From the relation:

$$\dot{U}_j = \frac{\kappa}{2} \left( \frac{U_c U_c^\dagger \cdots U_c^\dagger U_c^\dagger - U_c U_c^\dagger \cdots U_c^\dagger U_c^\dagger U_j^\dagger}{2m-1} \right) U_j,$$

we can transform above limit as follows: for $j \in \mathcal{N}$,

$$\| \dot{U}_j \|_F^2 = \frac{\kappa^2}{4} \left\| \frac{U_j U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger - U_c U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger U_j^\dagger}{2m-1} \right\|_F^2 \to 0 \text{ as } t \to \infty.$$

\(\square\)

In next lemma, we consider the special case with $m = 2^k$. This special case contains the term $\|U_c U_c^\dagger - U_c U_c^\dagger\|_F^2$ in the derivative of order parameter $R^2$. The equilibrium set of this case should be contained in the original Lohe matrix model. This result is more powerful than the original Lohe matrix model.

**Lemma 3.5** Let \{U_j\} be a global solution of the system (3.14) with $m = 2^k$. Then we have

$$\frac{d R^2}{dt} = \frac{\kappa}{2N} \sum_{i=1}^{N} \left( \| \frac{(U_c U_i^\dagger - U_i U_c^\dagger) U_c U_c^\dagger \cdots U_c^\dagger}{2^k-1} \|_F^2 \right. 
+ \sum_{p=1}^{k} \frac{1}{2^p} \left\| \frac{U_c U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_i^\dagger - U_c U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger U_i^\dagger U_c^\dagger \cdots U_c^\dagger U_i^\dagger}{2^{2k-2p}} \right\|_F^2, \\
\left. \right\} \text{where } R^2 = \text{tr}(U_c U_c^\dagger).$$

**Proof** Note that

$$\frac{d}{dt} \text{tr}(U_c U_c^\dagger) = \text{tr}(\dot{U}_c U_c^\dagger) + (c.c.)$$

$$= \frac{\kappa}{2N} \sum_{i=1}^{N} \left( \text{tr} \left( \frac{(U_c U_c^\dagger \cdots U_c^\dagger U_c^\dagger - U_i U_c^\dagger U_c^\dagger \cdots U_c^\dagger U_c^\dagger U_i^\dagger U_c^\dagger \cdots U_c^\dagger U_i^\dagger)}{2^{k+1}-1} \right) + (c.c.) \right).$$
where \((c.c.)\) stands for the complex conjugate of the previous term. The terms inside parenthesis become

\[
\begin{align*}
\text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger U_c \cdots U_c^\dagger U_c \cdots U_c^\dagger U_i U_i^\dagger) + (c.c.)
&= \text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger (2U_c U_c^\dagger - U_i U_i^\dagger) + U_c U_c^\dagger) \\
&= \text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger (U_c U_i^\dagger - U_i U_i^\dagger) + U_c U_c^\dagger) \\
&= \|U_c U_i^\dagger - U_i U_i^\dagger\|_F^2 \\
&+ \text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger - U_i U_i^\dagger U_i \cdots U_i U_i^\dagger).
\end{align*}
\]

(3.17)

Now we define

\[
A_p := \text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger - U_i U_i^\dagger U_i \cdots U_i U_i^\dagger).
\]

Next, we derive a recursive relation between \(A_p\) and \(A_{p+1}\) when \(1 \leq p < k\):

\[
\begin{align*}
2A_p &= 2\text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger - U_c U_c^\dagger U_c \cdots U_c^\dagger U_i U_i^\dagger - U_i U_i^\dagger U_i \cdots U_i U_i^\dagger) \\
&= \|U_c U_c^\dagger U_c \cdots U_c^\dagger - U_i U_i^\dagger U_i \cdots U_i U_i^\dagger\|_F^2 \\
&+ \text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger - U_i U_i^\dagger U_i \cdots U_i U_i^\dagger) \\
&= \|U_c U_c^\dagger U_c \cdots U_c^\dagger - U_i U_i^\dagger U_i \cdots U_i U_i^\dagger\|_F^2 + A_{p+1}.
\end{align*}
\]

(3.18)

On the other hand, \(A_k\) can be estimated as follows.

\[
A_k = \text{tr}(U_c U_c^\dagger U_c \cdots U_c^\dagger - U_c U_c^\dagger U_c \cdots U_c^\dagger U_i U_i^\dagger U_i \cdots U_i U_i^\dagger) \\
= \frac{1}{2} \|U_c U_c^\dagger \cdots U_c^\dagger U_i - U_i U_i^\dagger U_i \cdots U_i U_i^\dagger\|_F^2.
\]
If we combine (3.18) and (3.19), $A_1$ can be calculated inductively.

\[
A_1 = \frac{1}{21} A_2 + \frac{1}{21} \left\| \frac{U_c U_c^\dagger \cdots U_c^\dagger - U_i U_i^\dagger U_c \cdots U_c^\dagger U_i^\dagger}{2^k} \right\|_F^2 + \frac{1}{21} \left\| \frac{U_i U_i^\dagger U_c \cdots U_c^\dagger U_i^\dagger}{2^k} \right\|_F^2
\]

\[
= \cdots
\]

\[
= \frac{1}{2^{k-1}} A_k + \sum_{p=1}^{k-1} \frac{1}{2p} \left\| \frac{U_c U_c^\dagger \cdots U_c^\dagger - U_i U_i^\dagger U_c \cdots U_i^\dagger}{2^k} \right\|_F^2 + \frac{1}{2^k} A_2
\]

Finally, we combine (3.17) and (3.20) to get

\[
\left\| \left( U_c U_c^\dagger - U_i U_i^\dagger \right) \frac{U_c U_c^\dagger \cdots U_c^\dagger - U_i U_i^\dagger U_c \cdots U_i^\dagger}{2^k} \right\|_F^2 + \text{tr} \left( U_i U_i^\dagger U_c \cdots U_i^\dagger \right)
\]

\[
= \left\| \left( U_c U_c^\dagger - U_i U_i^\dagger \right) \frac{U_c U_c^\dagger \cdots U_c^\dagger}{2^k-1} \right\|_F^2 + A_2
\]

\[
= \left\| \left( U_c U_c^\dagger - U_i U_i^\dagger \right) \frac{U_i U_i^\dagger \cdots U_i^\dagger}{2^k-1} \right\|_F^2 + \sum_{p=1}^{k} \frac{1}{2^p} \left\| \frac{U_c U_c^\dagger \cdots U_c^\dagger}{2^k} - U_i U_i^\dagger U_c \cdots U_i^\dagger}{2^k-2p} \right\|_F^2
\]

From this, we have

\[
\frac{d}{dt} \text{tr} \left( U_i U_i^\dagger \right) = \frac{k}{2N} \sum_{i=1}^{N} \left( \left\| \left( U_c U_c^\dagger - U_i U_i^\dagger \right) \frac{U_c U_c^\dagger \cdots U_c^\dagger}{2^k-1} \right\|_F^2 + \sum_{p=1}^{k} \frac{1}{2^p} \left\| \frac{U_c U_c^\dagger \cdots U_c^\dagger}{2^k} - U_i U_i^\dagger U_c \cdots U_i^\dagger}{2^k-2p} \right\|_F^2 \right).
\]

\[\square\]

**Proposition 3.5** Let $\{U_j\}$ be a global solution of system (3.14) with $m = 2^k$. Then, one has

(i) \[\lim_{t \to \infty} \left\| \left( U_c U_c^\dagger - U_i U_i^\dagger \right) \frac{U_c U_c^\dagger \cdots U_c^\dagger}{2^k-1} \right\|_F = 0.\]

(ii) \[\lim_{t \to \infty} \left\| \frac{U_c U_c^\dagger \cdots U_c^\dagger}{2^k} - U_i U_i^\dagger U_c \cdots U_i^\dagger}{2^k-2p} \right\|_F = 0,\]

for all $p = 1, 2, \cdots, k$, $i \in \mathcal{N}$.
Proof It follows from Lemma 3.5 that $R$ is non-decreasing and bounded. Hence, $R$ tends to $R^\infty$ as $t \to \infty$. On the other hand, we use the uniform boundedness of $\dot{U}_i$ and

$$\frac{dR^2}{dt} = \frac{k}{2N} \sum_{i=1}^{N} \left( \| (U_{c_i} U_i^\dagger - U_i U_{c_i}^\dagger) U_{c_i} U_i^\dagger \cdots U_{c_i}^\dagger \|^2_F \right)^{\frac{1}{2k-1}} + \sum_{p=1}^{k} \frac{1}{2^p} \| \left( U_{c_i} U_i^\dagger \cdots U_{c_i}^\dagger - U_{c_i} U_i^\dagger \cdots U_{c_i}^\dagger U_j U_i^\dagger \cdots U_{c_i}^\dagger U_{c_i}^\dagger \right)^{\frac{1}{2k-2^p}} U_{c_i} U_i^\dagger \cdots U_{c_i}^\dagger U_{c_i}^\dagger \|_F^2 \right)$$

to show

$$\sup_{0 \leq t < \infty} \left| \frac{d^2}{dt^2} R^2 \right| < \infty.$$ 

By Babalat’s lemma, one has

$$\lim_{t \to \infty} \frac{dR^2}{dt} = 0.$$ 

This implies the desired estimates. \hfill \square

4 A Gradient Flow Formulation with a Polynomial Potential

In this section, we continue the study on the Lohe matrix model with higher-order couplings. In previous section, we considered the monomial potential function so that only one pair of coupling terms is involved in the coupling. In the sequel, we consider a polynomial potential function.

Consider the Lohe matrix model in a mean-field form:

$$i\dot{U}_j = \sum_{n=1}^{m} \frac{\kappa_n}{2} \left( U_{c_i} U_i^\dagger \cdots U_{c_i}^\dagger U_i^\dagger \cdots U_{c_i}^\dagger \right)^{\frac{1}{2n-1}},$$

$$U_j(0) = U_j^0 \in \mathbb{U}(d), \quad j \in \mathcal{N}. \tag{4.1}$$

First, we study a conservation law.

Lemma 4.1 Let $\{U_j\}$ be a global solution of system (4.1). Then one has

$$\frac{d}{dt} (U_j^\dagger U_j) = 0, \quad t > 0, \quad j \in \mathcal{N}.$$ 

Proof The proof is the same as that of Lemma 3.1. Hence we omit its proof. \hfill \square

For $U_j^\dagger U_j = U_j U_j^\dagger = I_d$, system (4.1) becomes

$$\dot{U}_j = \sum_{n=1}^{m} \frac{\kappa_n}{2} \left( U_{c_i} U_i^\dagger \cdots U_{c_i}^\dagger U_i^\dagger \cdots U_{c_i}^\dagger \right)^{\frac{1}{2n-1}}, \tag{4.2}$$

From now on, we assume

$$U_j^\dagger U_j = U_j U_j^\dagger = I_d, \quad j \in \mathcal{N}.$$
Next, we study the gradient flow formulation of (4.1). For this, we consider a polynomial potential:
\[ V_{\text{poly}} := -N \text{tr}(f(U_c U_c^\dagger)), \quad f(A) := \frac{\kappa_1}{2} A + \frac{\kappa_2}{4} A^2 + \cdots + \frac{\kappa_m}{2m} A^m. \] (4.3)

Then, \( V_{\text{poly}} \) is an analytic function and since
\[ \| \text{tr} \left[ \left( U_c U_c^\dagger \right)^n \right] \|_F \leq \left( \| U_c \|_{\text{op}}^{n-1} \cdot \| U_c \|_F \right)^2 \leq d. \]
it is easy to see
\[ |V_{\text{poly}}| \leq \frac{N}{2} (\kappa_1 + \kappa_2 + \cdots + \kappa_m) d. \]

**Proposition 4.1** System (4.2) can be rewritten as a gradient flow with potential \( V_{\text{poly}} \):
\[ \dot{U}_j = -\frac{\partial V_{\text{poly}}}{\partial U_j} \bigg|_{U_j(U(d))}, \quad j \in \mathcal{N}. \]

**Proof** The proof is basically the same as in the proof of Proposition 3.2. Hence we omit its details. \( \square \)

**Lemma 4.2** Let \( \{U_j\} \) be a global solution of system (4.1) with the initial data satisfying
\[ U_j^{0\dagger} U_j^0 = I_d, \quad j \in \mathcal{N}. \]
Then, one has
\[ \frac{d}{dt} V_{\text{poly}} = -\sum_{i=1}^N \sum_{n=1}^m \frac{\kappa_n}{2} \left( \| (U_c U_c^\dagger)^{n-1} U_c U_c^\dagger U_i U_i^\dagger (U_c U_c^\dagger)^{n-1} \|_F \right)^2. \]

**Proof** We use (4.2) to see
\[ \frac{d}{dt} \text{tr}(f(U_c U_c^\dagger)) = \sum_{n=1}^m \frac{\kappa_n}{2n} \frac{d}{dt} \text{tr}((U_c U_c^\dagger)^n) \]
\[ = \sum_{n=1}^m \frac{\kappa_n}{2} \left( \text{tr}(\dot{U}_c U_c^\dagger (U_c U_c^\dagger)^{n-1}) + (c.c.) \right). \] (4.4)
The first term in the R.H.S. of (4.6) can be estimated as follows.
\[ \text{tr}(\dot{U}_c U_c^\dagger (U_c U_c^\dagger)^{n-1}) \]
\[ = \frac{1}{N} \sum_{i=1}^N \sum_{l=1}^m \frac{\kappa_l}{2} \text{tr}((U_c U_c^\dagger U_c \cdots U_c^\dagger U_c \cdots U_c^\dagger U_i) U_c^\dagger U_c^\dagger (U_c U_c^\dagger)^{n-1}) \]
\[ = \frac{1}{N} \sum_{i=1}^N \sum_{l=1}^m \frac{\kappa_l}{2} \text{tr}((U_c U_c^\dagger)^{n+l-1} - U_i U_c^\dagger U_c^\dagger U_c U_c^\dagger U_i U_c^\dagger (U_c U_c^\dagger)^{n-1}). \] (4.5)
We combine (4.4) and (4.5) to obtain
\[
\frac{d}{dt} \text{tr}(f(U_c U_c^\dagger))
= \sum_{n=1}^m \frac{\kappa_n}{2} \left( \text{tr}(U_c U_c^\dagger(U_c U_c^\dagger)^{n-1}) \right) + (c.c.)
= \frac{1}{N} \sum_{i=1}^N \sum_{l,n=1}^m \frac{\kappa_l \kappa_n}{4} \text{tr}((U_c U_c^\dagger)^{n+l-1} - U_i U_c^\dagger(U_c U_c^\dagger)^{l-1} U_i U_c^\dagger(U_c U_c^\dagger)^{n-1}) + (c.c.)
= \frac{1}{N} \sum_{i=1}^N \sum_{l,n=1}^m \frac{\kappa_l \kappa_n}{4} \left[ ((U_c U_c^\dagger)^{n-1} U_i U_c^\dagger(U_c U_c^\dagger)^{l-1} - U_i U_c^\dagger(U_c U_c^\dagger)^{l-1}) \right]
= \frac{1}{N} \sum_{i=1}^N \sum_{l,n=1}^m \frac{\kappa_l \kappa_n}{4} \left( (U_c U_c^\dagger)^{n-1} U_i U_c^\dagger - U_i U_c^\dagger(U_c U_c^\dagger)^{n-1} \right)^2_F.
\]

Therefore, we have following equality:
\[
\frac{d}{dt} V_{\text{poly}} = -N \frac{d}{dt} \text{tr}(f(U_c U_c^\dagger)) = -\sum_{i=1}^N \left\| \sum_{n=1}^m \frac{\kappa_n}{2} \left( (U_c U_c^\dagger)^{n-1} U_i U_c^\dagger - U_i U_c^\dagger(U_c U_c^\dagger)^{n-1} \right) \right\|^2_F.
\]

\[\square\]

**Remark 4.1** Since
\[
f'(A) = \frac{1}{2} \left( \kappa_1 I + \kappa_2 A + \cdots + \kappa_m A^{m-1} \right),
\]
we can express above result as follows:
\[
\sum_{i=1}^N \left\| \sum_{n=1}^m \frac{\kappa_n}{2} \left( (U_c U_c^\dagger)^{n-1} U_i U_c^\dagger - U_i U_c^\dagger(U_c U_c^\dagger)^{n-1} \right) \right\|^2_F = \sum_{i=1}^N \left\| f'(U_c U_c^\dagger) U_i U_c^\dagger - U_i U_c^\dagger f'(U_c U_c^\dagger) \right\|^2_F.
\]

**Theorem 4.1** Let \( \{U_j\} \) be a global solution of system (4.1) with the initial data \( \{U_j^0\} \):
\[
U_j^0 U_j^0 = I_d, \quad j = 1, \cdots, N.
\]
Then, there exists an equilibrium \((U_1^\infty, \cdots, U_N^\infty)\) such that
\[
\lim_{t \to \infty} \|U_j(t) - U_j^\infty\|_F = 0, \quad \lim_{t \to \infty} \frac{d}{dt} V_{\text{poly}}(U) = 0, \quad \lim_{t \to \infty} \|\dot{U}_j\|_F = 0, \quad j \in \mathcal{N}.
\]
Proof (i) By Lemma 4.1 and the assumption on the initial data, we have
\[ U_j^\dagger U_j = U_j U_j^\dagger = I_d, \quad j \in \mathbb{N}. \]
Under this circumstance, dynamics of (4.1) is equivalent to (4.2). Moreover, it follows from Proposition 4.1 and analyticity of the potential function that the ensemble \((U_1, \cdots, U_N)\) tends to an equilibrium \((U_1^\infty, \cdots, U_N^\infty)\) as \(t \to \infty\).

(ii) We can use similar argument to prove the boundedness of
\[ \sup_{0 \leq t < \infty} \left| \frac{d}{dt} V_{\text{poly}} \right|. \]
Then we can apply the Barbalat’s lemma to obtain
\[ \lim_{t \to \infty} \frac{d}{dt} V_{\text{poly}} = 0. \]

(iii) Above result yields
\[ \lim_{t \to \infty} \| \dot{U}_i \|_F^2 = \left\| \sum_{n=1}^m \frac{\kappa_n}{2} \left( (U_c U_c^\dagger)^{n-1} U_c U_i^\dagger - U_i U_c^\dagger (U_c U_c^\dagger)^{n-1} \right) \right\|_F^2 \to 0 \text{ as } t \to \infty. \]

Remark 4.2 Our governing system (4.1) can be derived from the square matrix Riccati equation in [27]:
\[ \dot{Z}_j = \Gamma_1 + i \Omega_1 Z_j + i Z_j \Omega_2 - Z_j \Gamma_2 Z_j. \] (4.7)
More precisely, recall system (4.1) for heterogeneous Hamiltonians:
\[ \dot{U}_j = -i H_j U_j + \sum_{n=1}^m \frac{\kappa_n}{2} \left( (U_c U_c^\dagger)^{n-1} U_c U_i^\dagger - U_i U_c^\dagger (U_c U_c^\dagger)^{n-1} \right) \] (4.8)
Now we take
\[ Z_j = U_j, \quad \Omega_1 = -H, \quad \Omega_2 = 0, \quad \Gamma_1 = \Gamma_2 = \sum_{n=1}^m \frac{\kappa_n}{2} \left( U_c U_c^\dagger \cdots U_c U_c^\dagger \right) \] (4.9)
and substitute (4.9) into (4.7) to get (4.8). Hence, $d \times d$ matrix cross ratios $C_{ijkl}$ introduced in [27]:

$$C_{ijkl} = (U_i - U_k)(U_i - U_l)^{-1}(U_j - U_l)(U_j - U_k)^{-1}, \quad \forall 1 \leq i, j, k, l \leq N,$$

is also a constant of motion for system (4.8). We also refer to [34] for the constant of motion to the Kuramoto type model.

Next, we consider the following special polynomial type function $f$ satisfying the following property:

$$\kappa_j \neq 0 \iff j = 2^n \quad \text{for some } n \in \mathbb{Z}_{\geq 0}.$$

i.e., $f(A)$ takes the following form:

$$f(A) = \frac{\kappa_{2^0}}{2^1} A^2 + \frac{\kappa_{2^1}}{2^2} A^{2^2} + \cdots + \frac{\kappa_{2^{l-1}}}{2^l} A^{2^{l-1}}.$$

As aforementioned, coupling term with $j = 2^n$ has a good structure of the derivative of the order parameter $R^2$. Thus, we focus on this case in the sequel. Consider the system:

$$\begin{cases}
\dot{U}_j = \sum_{k=0}^{l-1} \frac{\kappa_{2^k}}{2} (U_c U_i U_c \cdots U_c U_j - U_j U_i U_c \cdots U_c U_j) \frac{2^{k+1} - 1}{2^{k+1} - 1}, \\
U_j(0) = U_j^0 \in \mathbb{U}(d), \quad j \in \mathcal{N}.
\end{cases} \quad (4.10)$$

We have following dynamics of order parameter.

**Lemma 4.3** Let $\{U_i\}$ be a global solution of system (4.10). Then we have

$$\frac{d R^2}{dt} = \sum_{k=0}^{l-1} \sum_{i=1}^{N} \frac{\kappa_{2^k}}{2N} \left( \|U_c U_i U_c \cdots U_c - U_i U_c \cdots U_c\|_F^2 \right).$$

**Proof** By direct calculations, one has

$$\frac{d}{dt} \|U_c\|_F^2 = \text{tr}(\dot{U}_i U_i^\dagger) + (c.c.)$$

$$= \sum_{k=0}^{l-1} \frac{\kappa_{2^k}}{2} \text{tr} \left( (U_c U_i U_c \cdots U_c U_i - U_i U_c \cdots U_c U_i) U_i U_c \cdots U_c \right)$$

$$= \sum_{k=0}^{l-1} \sum_{i=1}^{N} \frac{\kappa_{2^k}}{2N} \left( \|U_c U_i U_c \cdots U_c - U_i U_c \cdots U_c\|_F^2 \right) + \sum_{p=1}^{k} \frac{1}{2^p} \left( \|U_c U_i U_c \cdots U_c - U_i U_c \cdots U_c\|_F^2 \right).$$

$\square$
Theorem 4.2 Let \( \{U_j\} \) be a global solution of system (4.10) with the initial data \( \{U^0_j\} \):

\[ U^0_j U_j^0 = I_d, \quad j \in \mathcal{N}. \]

Then, the following assertions hold.

1. For all \( i \in \mathcal{N} \) and for all \( k = 0, 1, \ldots, l - 1 \) which satisfies \( \kappa_{2k} \neq 0 \),
   \[
   \lim_{t \to \infty} \left\| (U_c U_i^\dagger - U_i U_c^\dagger) U_c U_c^\dagger \cdots U_c \right\|_F = 0.
   \]

2. For all \( i \in \mathcal{N} \), for all \( p = 1, 2, \ldots, k \) and for all \( k = 0, 1, \ldots, l - 1 \) which satisfies \( \kappa_{2k} \neq 0 \),
   \[
   \lim_{t \to \infty} \left\| \frac{U_c U_c^\dagger U_c \cdots U_c^\dagger - U_c U_c^\dagger U_c \cdots U_c^\dagger U_i U_i^\dagger U_i \cdots U_i^\dagger}{2^k} \right\|_F = 0.
   \]

Proof Since \( R^2 \) is bounded and non-increasing, \( R^2 \) converges as \( t \to \infty \). Next, we will show

\[
\lim_{t \to \infty} \frac{dR^2}{dt} = 0.
\]

For this, it suffices to check

\[
\sup_{0 \leq t < \infty} \left| \frac{d^2 R^2}{dt^2} \right| < \infty.
\]

Once the above estimate is verified, then Babalat’s lemma yields the desired estimates. However the proof of the boundedness of second derivative of \( R^2 \) is very similar to the proof of Corollary (3.2). Then we have

\[
\lim_{t \to \infty} \frac{dR^2}{dt} = \lim_{t \to \infty} \sum_{k=0}^{l-1} \sum_{i=1}^{N} \frac{\kappa_{2k}}{2N} \left( \left\| (U_c U_i^\dagger - U_i U_c^\dagger) U_c U_c^\dagger \cdots U_c \right\|_F^2 \right) + \sum_{p=1}^{k} \frac{1}{2^p} \left( \left\| \frac{U_c U_c^\dagger U_c \cdots U_c^\dagger - U_c U_c^\dagger U_c \cdots U_c^\dagger U_i U_i^\dagger U_i \cdots U_i^\dagger}{2^k} \right\|_F^2 \right) = 0.
\]

From this equality, for all \( i \) and \( k \) which satisfies \( \kappa_{2k} \), we have

\[
\lim_{t \to \infty} \left( \left\| (U_c U_i^\dagger - U_i U_c^\dagger) U_c U_c^\dagger \cdots U_c \right\|_F^2 \right) + \sum_{p=1}^{k} \frac{1}{2^p} \left( \left\| \frac{U_c U_c^\dagger U_c \cdots U_c^\dagger - U_c U_c^\dagger U_c \cdots U_c^\dagger U_i U_i^\dagger U_i \cdots U_i^\dagger}{2^k} \right\|_F^2 \right) = 0.
\]
Since each term is non-negative, each term must converge to zero, we have
\[
\|(U_c U_i^\dagger - U_i U_c^\dagger) U_c U_i^\dagger \cdots U_c\|_F \to 0 \quad \text{as} \quad t \to \infty
\]
and
\[
\| U_c U_i^\dagger U_c \cdots U_i^\dagger - U_i U_c^\dagger U_c \cdots U_i^\dagger \|_F \to 0 \quad \text{as} \quad t \to \infty
\]
for all \( i \in \mathcal{N}, \ p = 1, 2, \ldots, k, \) and \( k = 0, 1, \ldots, l - 1 \) which satisfies \( \kappa_{2^k} \neq 0. \)

\[5\]
Emergent Dynamics of Lohe Ensemble

In this section, we study a relaxation estimate toward the aggregated state for system (4.1). In previous section, we show that the state configuration tends to an equilibrium for any initial data without any explicit decay estimate. The main reason for this is that we employed a gradient flow approach and Babalat’s lemma which does not tell us any constructive decay estimate. For an explicit decay estimate, we employ a diameter functional and derive a Riccati type differential inequality for the state diameter. This yields an explicit decay estimate for some restricted class of initial data and system parameters.

### 5.1 Ensemble Diameter

For a state configuration \( \{U_j\} \), we set
\[
\mathcal{D}(U) := \max_{i,j} \|U_i - U_j\|_F.
\]

**Lemma 5.1** Let \( \{U_j\} \) be a global solution to system (4.1). Then \( \mathcal{D}(U) \) satisfies
\[
-\kappa_+ D(U)^2 - \kappa_1 D(U)^4 \leq \frac{d}{dt} D(U)^2 \leq -\kappa_- D(U)^2 + \kappa_1 D(U)^4,
\]
where \( \kappa_+ \) and \( \kappa_- \) are given by the following relations:
\[
\kappa_- := 2\kappa_1 - \sqrt{d} \sum_{n=2}^{m} \kappa_n \quad \text{and} \quad \kappa_+ := 2\kappa_1 + \sqrt{d} \sum_{n=2}^{m} \kappa_n.
\]

**Proof** Let \((i,j)\) be a pair of indices. By direct estimate, one has
\[
\frac{d}{dt} \|U_i - U_j\|_F^2
\]
\[
= \frac{d}{dt} \text{tr}(2I - U_i U_j^\dagger - U_j U_i^\dagger) = -\text{tr}(\dot{U}_i U_j^\dagger + \dot{U}_j U_i^\dagger) - (c.c.)
\]
\[
= -\sum_{n=1}^{m} \frac{\kappa_n}{2} \text{tr}(U_c U_i^\dagger U_c \cdots U_c U_i^\dagger U_j U_c \cdots U_c U_i^\dagger U_j U_c \cdots U_c U_i^\dagger U_j U_c \cdots U_c U_i^\dagger U_j)
\]
\[
- \sum_{n=1}^{m} \frac{\kappa_n}{2} \text{tr}(U_c U_i^\dagger U_c \cdots U_c U_i^\dagger U_j U_c \cdots U_c U_i^\dagger U_j U_c \cdots U_c U_i^\dagger U_j U_c \cdots U_c U_i^\dagger U_j - (c.c.)
\]
Finally, we combine (5.1), (5.2), (5.3) and (5.4) to get

\[
= - \sum_{n=1}^{m} \frac{k_n}{2} \text{tr} \left( \frac{(U_i U_j U_i \cdots U_i U_j U_i) (U_i^T + U_j^T) - (U_i U_j U_i \cdots U_i U_j U_i) (U_i U_j^T U_i - U_j U_i^T U_j)}{2^n - 1} \right)
\]

\[(c.c.)\]

\[
= - \sum_{n=1}^{m} \frac{k_n}{2} \text{tr} \left( \frac{(U_i U_j U_i \cdots U_i U_j U_i) (U_i^T + U_j^T - U_i^T U_j^T - U_j^T U_i^T)}{2^n - 1} \right) \quad (c.c.)
\]

\[
= - \sum_{n=1}^{m} \frac{k_n}{2} \text{tr} \left( (U_i U_j U_i \cdots U_i U_j U_i)^{n-1} (U_i U_j^T U_i - U_j U_i^T U_j) \right)
\]

\[
= - U_i U_j^T U_i U_j + U_j U_i^T U_i U_j - U_i U_j^T U_i U_j
\]

It follows from the Lemma 2.2 that

\[
\left| \text{tr} \left( (U_i U_j U_i \cdots U_i U_j U_i)^{n-1} (U_i U_j^T U_i - U_j U_i^T U_j) \right) \right| \leq \sqrt{d} \left\| (U_i U_j U_i \cdots U_i U_j U_i)^{n-1} (U_i U_j^T U_i - U_j U_i^T U_j) \right\|_F.
\]

(5.1)

On the other hand, Lemmas 2.3 and 2.4 imply

\[
\sqrt{d} \left\| (U_i U_j U_i \cdots U_i U_j U_i)^{n-1} (U_i U_j^T U_i - U_j U_i^T U_j) \right\|_F \leq \sqrt{d} \left\| U_i U_j^T U_i - U_j U_i^T U_j \right\|_F \leq \sqrt{d} \left\| U_i U_j^T U_i - U_j U_i^T U_j \right\|_F.
\]

(5.2)

Note that

\[
U_i U_j^T U_i - U_j U_i^T U_j = U_i (U_i - U_j)^T (U_i - U_j) (U_i - U_j)^T.
\]

(5.3)

Then, one has

\[
\left\| U_i U_j^T U_i - U_j U_i^T U_j \right\|_F \leq \left\| U_i (U_i - U_j)^T (U_i - U_j) (U_i - U_j)^T \right\|_F.
\]

(5.4)

From Lemma 2.4, we have

\[
\left\| (U_i - U_j)^T (U_i - U_j) \right\|_F \leq \left\| U_i - U_j \right\|_F^2.
\]

Finally, we combine (5.1), (5.2), (5.3) and (5.4) to get

\[
\left| \text{tr} \left( (U_i U_j U_i \cdots U_i U_j U_i)^{n-1} (U_i U_j^T U_i - U_j U_i^T U_j) \right) \right| \leq 2 \sqrt{d} \left\| U_i - U_j \right\|_F^2.
\]
It follows from [17] that we have following estimate with \( n = 1 \). If we set
\[
\mathcal{I}_n := \frac{k_n}{2} \text{tr} \left( (U_c U_c^\dagger)^{n-1} (U_c U_i^\dagger + U_i U_c^\dagger) - U_c U_i^\dagger U_i U_i^\dagger - U_i U_i^\dagger U_j U_j^\dagger \right),
\]
then we have
\[
-2\kappa_1 D(U)^2 - \kappa_1 D(U)^4 \leq -\mathcal{I}_1 \leq -2\kappa_1 D(U)^2 + \kappa_1 D(U)^4.
\]
For \( n > 1 \), we have
\[
|\mathcal{I}_n| \leq \kappa_n \sqrt{d} D(U)^2.
\]
From the equality:
\[
-\sum_{n=0}^{\infty} \mathcal{I}_n = \frac{d}{dt} D(U)^2,
\]
we have following estimate:
\[
- \left( 2\kappa_1 + \sqrt{d} \sum_{n=2}^{m} \kappa_n \right) D(U)^2 - \kappa_1 D(U)^4 \leq \frac{d}{dt} D(U)^2 \leq - \left( 2\kappa_1 - \sqrt{d} \sum_{n=2}^{m} \kappa_n \right) D(U)^2 + \kappa_1 D(U)^4.
\]
Now we set
\[
\kappa_- = 2\kappa_1 - \sqrt{d} \sum_{n=2}^{m} \kappa_n, \quad \kappa_+ = 2\kappa_1 + \sqrt{d} \sum_{n=2}^{m} \kappa_n.
\]
Then we can express above estimate as follows
\[
-\kappa_+ D(U)^2 - \kappa_1 D(U)^4 \leq \frac{d}{dt} D(U)^2 \leq -\kappa_- D(U)^2 + \kappa_1 D(U)^4.
\]
and assume that \( \kappa_- > 0 \). This implies \( \kappa_1 \) must be positive and \( \kappa_n \) with \( n > 1 \) can be negative.

5.2 Relaxation Estimate

In this subsection, we derive decay estimates for \( D(U) \). For this, we first present estimates on the Riccati type differential inequalities.

**Lemma 5.2** Suppose that a differential inequality \( X \) satisfies a differential inequality:
\[
-\kappa_+ X - \kappa_1 X^2 \leq \frac{d}{dt} X \leq -\kappa_- X + \kappa_1 X^2, \quad t > 0, \quad 0 \leq X(0) < \frac{\kappa_-}{\kappa_1}.
\]
Then we have
\[
\frac{\kappa_+}{\kappa_1} X(0) e^{\kappa_+ t} (X(0) + \kappa_+/\kappa_1) \leq X(t) \leq \frac{\kappa_-}{\kappa_1} X(0)/e^{\kappa_- t} (\kappa_-/\kappa_1 - X(0)) + X(0)
\]
Proof By direct estimates, one has
\[-\kappa_1 X \left( X + \frac{\kappa_+}{\kappa_1} \right) \leq \frac{d}{dt} X \leq \kappa_1 X \left( \frac{\kappa_-}{\kappa_1} - X \right).\]
For the lower bound estimate, we use the L.H.S. of the above differential inequality to get
\[-\kappa_+ \leq \frac{\dot{X}}{X} - \frac{\dot{X}}{X + \frac{\kappa_+}{\kappa_1}}.\]
This yields
\[
\frac{\kappa_+ X(0)/\kappa_1}{e^{\kappa_+ t} (X(0) + \kappa_+ / \kappa_1) - X(0)} \leq X(t).
\]
Similarly, one has
\[
X(t) \leq \frac{\kappa_- X(0)/\kappa_1}{e^{\kappa_- t} (\kappa_- / \kappa_1 - X(0)) + X(0)}.
\]
\[\square\]
Finally, Lemmas 5.1 and 5.2 imply the exponential decay estimate of relative states.

Theorem 5.1 Suppose that coupling strengths and initial data satisfy
\[
\kappa_- = 2 \kappa_1 - \sqrt{d \sum_{n=2}^{m} \kappa_n} > 0 \quad \text{and} \quad \max_{1 \leq i, j \leq N} \| U_i^0 - U_j^0 \|_F^2 < \frac{\kappa_-}{\kappa_1},
\]
and let \( \{ U_i \} \) be a global solution of system (4.1). Then we have
\[
O(e^{-\kappa_- t}) \leq \| U_i(t) - U_j(t) \|_F^2 \leq O(e^{-\kappa_+ t}), \quad i, j \in \mathcal{N}.
\]

5.3 Extension to a Heterogeneous Ensemble

For a heterogeneous ensemble, we can extend the generalized Lohe matrix model (4.1) by adding \( H_j \) to the R.H.S. of (4.1):
\[
i \dot{U}_j U_j^\dagger = H_j + \sum_{n=1}^{m} \frac{i \kappa_n}{2} (U_c U_c^\dagger U_c \cdots U_c^\dagger U_j U_j^\dagger - U_j U_j^\dagger U_c U_c \cdots U_c^\dagger U_j^\dagger), \quad t > 0,
\]
\[
U_j(0) = U_j^0 \in \mathbb{U}(d), \quad j \in \mathcal{N},\]
where \( H_j \) is a Hermitian matrix with \( H_j^\dagger = H_j \). In this case, it is easy to see that
\[
\frac{d}{dt} U_j^\dagger U_j = 0, \quad j \in \mathcal{N}.
\]
Hence, we have
\[
U_j^\dagger U_j = I_d, \quad j \in \mathcal{N}.
\]
In this case, system (5.5) becomes
\[
\begin{cases}
\dot{U}_j = -i H_j + \sum_{n=1}^{m} \frac{\kappa_n}{2} (U_c U_c^\dagger U_c \cdots U_c^\dagger U_j U_j^\dagger - U_j U_j^\dagger U_c U_c \cdots U_c^\dagger U_j^\dagger), \\
U_j(0) = U_j^0 \in \mathbb{U}(d), \quad j \in \mathcal{N}.
\end{cases}
\]
For an ensemble \( \{H_j\} \), we set
\[
D(H) := \max_{i,j} \|H_i - H_j\|_F.
\]

We use the same argument as in [17], one has following estimate:
\[
-2D(H)D(U) - \kappa_+ D(U)^2 - \kappa_1 D(U)^4 \leq \frac{d}{dt} D(U)^2
\]
\[
\leq 2D(H)D(U) - \kappa_- D(U)^2 + \kappa_1 D(U)^4.
\]

This yields
\[
- D(H) - \frac{\kappa_+}{2} D(U) - \frac{\kappa_1}{2} D(U)^3 \leq \frac{d}{dt} D(U) \leq D(H) - \frac{\kappa_-}{2} D(U) + \frac{\kappa_1}{2} D(U)^3.
\] (5.6)

**Theorem 5.2** Suppose that system parameters and initial data satisfy
\[
\kappa_- = 2\kappa_1 - \sqrt{d} \sum_{n=2}^{m} \kappa_n > 0, \quad D(H) < \frac{1}{3} \sqrt{\frac{\kappa_-^3}{3\kappa_1}},
\]
\[
U_j^{0}\dagger U_j^0 = I_d, \quad j = 1, \cdots, N \quad \text{and} \quad D(U^0) < \rho = \frac{2D(H)}{\kappa_1}.
\]

Then, for a global solution \( \{U_i\} \) to (5.5), we have the following practical aggregation:
\[
\lim_{\kappa_1 \to 0} \limsup_{t \to \infty} D(U) = 0.
\]

**Proof** For the decay estimate of (5.6), we set
\[
f(x) := D(H) - \frac{\kappa_-}{2} x + \frac{\kappa_1}{2} x^3.
\]

Then we have
\[
\frac{d}{dt} D(U) \leq f(D(U)).
\]

Now we want to analyze the graph of the \( f(x) \) defined on \( x \geq 0 \). Let \( \zeta \) be the positive solution of the \( f'(x) \). Since
\[
f'(x) = -\frac{\kappa_-}{2} + \frac{3\kappa_1}{2} x^2,
\]
there is only one positive solution and only one negative solution. Then the global minimum of \( f(x) \) with the range \( x \geq 0 \) is at \( x = \zeta \) with
\[
\zeta = \sqrt{\frac{\kappa_-}{3\kappa_1}}.
\]

So the global minimum is
\[
f(x) \leq f(\zeta) = D(H) - \frac{1}{3} \sqrt{\frac{\kappa_-^3}{3\kappa_1}}.
\]

From the assumption
\[
D(H) < \frac{1}{3} \sqrt{\frac{\kappa_-^3}{3\kappa_1}},
\]

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we have two distinct positive solutions $\eta_1$ and $\eta_2$ of $f(x) = 0$ with $\eta_1 < \eta_2$. Then, we know

$$f(x) > 0 \text{ at } x < \eta_1, \quad x > \eta_2; \quad f(x) < 0 \text{ at } \eta_1 < x < \eta_2.$$ 

If the initial data satisfies

$$D(U^0) < \eta_2,$$

then

$$\lim_{t \to \infty} \sup D(U) \leq \eta_1.$$ 

Now we want to find the estimate for $\eta_1$. Since

$$f''(x) = 3\kappa_1 x \geq 0 \quad \forall x \geq 0$$

If we draw the tangent line $l$ at $(0, D(H))$ on the graph of $y = f(x)$, then $l$ intersects with $x$-axis at $(\rho, 0)$ with

$$0 < \eta_1 < \rho.$$ 

Since $\rho = \frac{2D(H)}{\kappa_1}$, we have

$$\lim_{t \to \infty} \sup D(U) \leq \frac{2D(H)}{\kappa_1}.$$ 

Finally, we have the practical synchronization:

$$\lim_{\kappa_1 \to 0} \lim_{t \to \infty} D(U) = 0.$$ 

Remark 5.1 From the structure of gradient flow introduced in Proposition 3.3, we could find the sum of squares (SOS) forms of the temporal derivative of $f''(R^2)$ and the square of order parameter $R^2$ introduced in Lemmas 4.2 and 4.3. From these SOS forms, we can apply Barbalat’s lemma to get the complete (practical) aggregations.

6 Conclusion

In this paper, we have proposed a generalized Lohe matrix model with a higher-order polynomial coupling via the gradient flow approach. In [20], the first author and his collaborator have shown that the Lohe matrix model can cast as a gradient flow with a quadratic potential on the unitary group. In the original Lohe’s works [24,25], the quadratic coupling is not justified a priori. Hence it is not clear why Lohe employed a cubic interaction for the evolution of the state. In authors’ earlier work on the Lohe tensor model which is a high-dimensional generalization of the Lohe matrix model, couplings can be allowed to include odd high-order ones. To incorporate higher-order couplings, we use a gradient flow approach to derive a generalized Lohe matrix model with higher-order couplings by employing a higher-order potential and gradient flow approach altogether. For the proposed model, we presented a sufficient framework for the emergent dynamics in terms of system parameters and initial data. Our gradient flow approach is restricted to a homogeneous ensemble. In [5,6], authors introduced the reduction of the Kuramoto model on $(\mathbb{S}^1)^N$ to the hyperbolic space $\mathbb{H}^2$. Since our governing model with $H_j = H$ for all $j \in \mathcal{N}$ can be considered as the special case of
Lohe’s Riccati type matrix model, we might be able to find the reduction of our model on $(\mathbb{U}(d))^N$ to a system defined with gradient on lower-dimensional manifold. This will be left for a future work.

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