A novel greedy Gauss-Seidel method for solving large linear least squares problem

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Summary
We present a novel greedy Gauss-Seidel method for solving large linear least squares problem. This method improves the greedy randomized coordinate descent (GRCD) method proposed recently by Bai and Wu [Bai ZZ, and Wu WT. On greedy randomized coordinate descent methods for solving large linear least-squares problems. Numer Linear Algebra Appl. 2019;26(4):1–15], which in turn improves the popular randomized Gauss-Seidel method. Convergence analysis of the new method is provided. Numerical experiments show that, for the same accuracy, our method outperforms the GRCD method in term of the computing time.

KEYWORDS:
greedy Gauss-Seidel method, greedy randomized coordinate descent method, randomized Gauss-Seidel method, large linear least squares problem

1 | INTRODUCTION

Linear least squares problem is a classical linear algebra problem in scientific computing, arising for instance in many parameter estimation problems. In the literature, several direct methods for solving this problem are studied. Such methods including the use of QR factorization with pivoting and the use of singular value decomposition (SVD)\(^1\)\(^2\) require high storage and are expensive when the matrix is large-scale. Hence, iterative methods are considered for solving large linear least squares problem, such as the famous Gauss-Seidel method\(^3\).

Inspired by a work of Strohmer and Vershynin\(^4\) which shows that the randomized Kaczmarz method converges linearly in expectation to the solution, Leventhal and Lewis\(^5\) obtained a similar result for the randomized Gauss-Seidel (RGS) method, which is also called the randomized coordinate descent method. This method works on the columns of the matrix \(A\) to minimize \(\|b - Ax\|_2^2\) randomly according to an appropriate probability distribution and has attracted much attention recently due to its better performance; see for example\(^6\)\(^7\)\(^8\)\(^9\)\(^10\)\(^11\)\(^12\)\(^13\)\(^14\) and references therein.

Recently, Bai and Wu\(^15\) proposed a greedy randomized coordinate descent (GRCD) method by introducing an efficient probability criterion for selecting the working columns from the matrix \(A\), which avoids a weakness of the one adopted in the RGS method. The GRCD method is faster than the RGS method in terms of the number of iterations and computing time. By the way, the idea of greed applied in\(^15\) has wide applications, see for example\(^16\)\(^17\)\(^18\)\(^19\)\(^20\)\(^21\)\(^22\)\(^23\) and references therein.

In the present paper, we develop a novel greedy Gauss-Seidel (GGS) method for solving large linear least squares problem, which adopts a quite different way to determine the working columns of the matrix \(A\) compared with the GRCD method and hence needs less computing time in each iteration; see the detailed analysis before Algorithm\(^2\) below. In theory, we prove the convergence of the GGS method. In numerical experiments, we compare the performance of the GGS and GRCD methods using the examples from\(^15\). Numerical results show that, for the same accuracy, the GGS method requires almost the same number of iterations as that of the GRCD method, however, the GGS method spends less computing time in all the cases.
2 | NOTATION AND PRELIMINARIES

For a vector $z \in \mathbb{R}^n$, $z^{(j)}$ represents its $j$th entry. For a matrix $G = (g_{ij}) \in \mathbb{R}^{m \times n}$, $G_{(ij)}$, $\|G\|_2$, and $\|G\|_F$ denote its $j$th column, spectral norm, and Frobenius norm, respectively. Moreover, if the matrix $G \in \mathbb{R}^{m \times n}$ is positive definite, then we define the energy norm of any vector $x \in \mathbb{R}^n$ as $\|x\|_G := \sqrt{x^T G x}$, where $(\cdot)^T$ denotes the transpose of a vector or a matrix. In addition, we denote the identity matrix by $I$, its $j$th column by $e_j$, the smallest positive eigenvalue of $G^T G$ by $\lambda_{\min}(G^T G)$ and the number of elements of a set $\mathcal{W}$ by $|\mathcal{W}|$.

In what follows, as done in [15], we use $x_\star = A^T b$, with $A^\dagger = (A^T A)^{-1} A^T$ being the Moore-Penrose pseudoinverse, to denote the unique least squares solution to the linear least squares problem:

$$\min_{x \in \mathbb{R}^n} \|b - Ax\|_2^2,$$

(1)

where $A \in \mathbb{R}^{m \times n}$ is of full column rank and $b \in \mathbb{R}^m$. As we know, the solution $x_\star = \arg \min_{x \in \mathbb{R}^n} \|b - Ax\|_2^2$ is the solution to the following normal equation [24] for (1):

$$A^T A x_\star = A^T b.$$

(2)

Based on the normal equation (2), Bai and Wu [24] proposed the GRCD method listed as follows, where $r_k = b - A x_k$ denotes the residual vector.

**Algorithm 1.** The GRCD method

**INPUT:** $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $\ell$, initial estimate $x_0$

**OUTPUT:** $x_{\ell}$

For $k = 0, 1, 2, \ldots, \ell - 1$ do

Compute

$$\delta_k = \frac{1}{2} \left( \frac{1}{\|A^T r_k\|_2} \max_{1 \leq j \leq n} \left\{ \frac{\|A^T r_k\|_2^2}{\|A_{(j)}\|_2} \right\} + \frac{1}{\|A\|_F} \right).$$

Determine the index set of positive integers

$$\mathcal{V}_k = \left\{ j \left| \|A^T r_k\|_2^2 \geq \delta_k \|A^T r_k\|_2 \|A_{(j)}\|_2 \right. \right\}.$$

Let $s_k = A^T r_k$ and define $\bar{s}_k$ as follows

$$\bar{s}_k^{(j)} = \begin{cases} s_k^{(j)}, & \text{if } j \in \mathcal{V}_k, \\ 0, & \text{otherwise}. \end{cases}$$

Select $j_k \in \mathcal{V}_k$ with probability $\Pr(\text{column} = j_k) = \frac{|s_k^{(j_k)|^2}{\|s_k\|_2^2}$.

Set

$$x_{k+1} = x_k + \frac{s_k^{(j_k)}}{\|A_{(j_k)}\|_2^2} e_{j_k}.$$ 

End for

From the definitions of $\delta_k$ and $\mathcal{V}_k$ in Algorithm 1 we have that if $\ell \in \mathcal{V}_k$, then

$$\frac{\|A^T r_k\|_2^2}{\|A_{(\ell)}\|_2} \geq \frac{1}{2} \left( \max_{1 \leq j \leq n} \left\{ \frac{\|A^T r_k\|_2^2}{\|A_{(j)}\|_2} \right\} + \frac{\|A^T r_k\|_2^2}{\|A\|_F} \right).$$
Note that
\[
\max_{1 \leq j \leq n} \left\{ \frac{A^T_{(j)}r_k}{\|A_{(j)}\|_2} \right\}^2 \geq \sum_{j=1}^n \frac{\|A_{(j)}\|^2_F}{\|A\|^2_F} \frac{A^T_{(j)}r_k}{\|A_{(j)}\|_2} = \frac{\|A^T r_k\|^2}{\|A\|^2_F}.
\]
Thus, we can’t conclude that if \( \ell \in \mathcal{V}_k \), then
\[
\frac{A^T_{(\ell)}r_k}{\|A_{(\ell)}\|_2} \geq \max_{1 \leq j \leq n} \left\{ \frac{A^T_{(j)}r_k}{\|A_{(j)}\|_2} \right\}^2,
\]
i.e.,
\[
\max_{1 \leq j \leq n} \left\{ \frac{A^T_{(j)}r_k}{\|A_{(j)}\|_2} \right\}^2 = \max_{1 \leq j \leq n} \left\{ \frac{A^T_{(\ell)}r_k}{\|A_{(\ell)}\|_2} \right\}^2.
\]
As a result, there may exist some \( \ell \in \mathcal{V}_k \) such that
\[
\frac{A^T_{(\ell)}r_k}{\|A_{(\ell)}\|_2} < \max_{1 \leq j \leq n} \left\{ \frac{A^T_{(j)}r_k}{\|A_{(j)}\|_2} \right\}^2.
\]
Meanwhile, from the update formula, for any \( j_k \in \mathcal{V}_k \), we have
\[
\|A x_{k+1} - A x_k \|_2 = \frac{\|A^T_{(\ell)}r_k\|^2}{\|A_{(\ell)}\|^2_2}.
\]
Thus, combining (3) and (4), we can find that we can’t make sure any column with the index from the index set \( \mathcal{V}_k \) make the distance between \( A x_{k+1} \) and \( A x_k \) be the largest when finding \( x_{k+1} \). Furthermore, to compute \( \delta_k \), we have to calculate the norm of each column of the matrix \( A \).

3.1 A NOVEL GREEDY GAUSS-SEIDEL METHOD

Considering that a column with the index from the index set \( \mathcal{V}_k \) in the GRCD method may make the distance between \( A x_{k+1} \) and \( A x_k \) be the largest and to compute \( \delta_k \) needs to calculate the norm of each column of the matrix \( A \), and inspired by some recent works on selection strategy for working index based on the maximum residual \( 20, 25, 26 \), we design a new method which includes two main steps. In the first step, we use the maximum entries of the residual vector \( s_k \) of the normal equation (2) to determine an index set \( \mathcal{R}_k \) whose specific definition is given in Algorithm 2. In the second step, we capture an index from the set \( \mathcal{R}_k \) with which we can make sure the distance between \( A x_{k+1} \) and \( A x_k \) be the largest for any possible \( x_{k+1} \). On a high level, the new method seems to change the order of the two main steps of Algorithm 1. However, comparing with the GRCD method, besides making the distance between \( A x_{k+1} \) and \( A x_k \) always be the largest when finding \( x_{k+1} \), we also do not need to calculate the norm of each column of the matrix \( A \) any longer in Algorithm 2. Moreover, we can also find that the number of elements in set \( \mathcal{R}_k \) may be less than the number of elements in set \( \mathcal{V}_k \), i.e., \( |\mathcal{R}_k| < |\mathcal{V}_k| \) because \( \mathcal{R}_k \) is determined by the maximum entries of the vector \( s_k \). Consequently, our method can reduce the computation cost at each iteration and hence behaves better in the computing time, which is confirmed by extensive numerical experiments given in Section 4.

Based on the above introduction, we propose the following algorithm, i.e., Algorithm 2.

**Algorithm 2.** The GGS method

**INPUT:** \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \ell, \) initial estimate \( x_0 \)

**OUTPUT:** \( x_\ell \)

For \( k = 0, 1, 2, \ldots, \ell - 1 \) do

Determine the index set of positive integers
\[
\mathcal{R}_k = \left\{ j_k | j_k = \max_{1 \leq j \leq n} \left| A^T_{(j)}r_k \right| \right\}.
\]
Compute
\[ j_k = \arg \max_{j \in R_k} \left\{ \frac{\| A^T_{(j)} r_k \|^2}{\| A_{(j)} \|^2} \right\}. \]

Set
\[ x_{k+1} = x_k + \frac{A^T_{(j_k)} r_k}{\| A_{(j_k)} \|^2} e_{j_k}. \]

End for

**Remark 1.** Note that if
\[ \left| A^T_{(j_k)} r_k \right| = \max_{l \neq j \in \mathcal{S}_n} \left| A^T_{(j)} r_k \right|, \]
then \( j_k \in \mathcal{R}_k \). So the index set \( \mathcal{R}_k \) in Algorithm [2] is nonempty for all iteration index \( k \).

**Remark 2.** Like Algorithm [1] we can use the values of \( \left| A^T_{(j_k)} r_k \right| \) for \( j_k \in \mathcal{R}_k \) as a probability selection criterion to devise a randomized version of Algorithm [2]. In this case, the convergence factor may be a little worse than that of Algorithm [2] because, for the latter, the index is selected based on the largest value of \( \frac{\left| A^T_{(j_k)} r_k \right|}{\| A_{(j_k)} \|^2} \) for \( j_k \in \mathcal{R}_k \), which make the distance between \( Ax_{k+1} \) and \( Ax_k \) be the largest for any possible \( x_{k+1} \).

In the following, we give the convergence theorem of the GGS method.

**Theorem 1.** The iteration sequence \( \{ x_k \}_{k=0}^{\infty} \) generated by Algorithm [2] starting from an initial guess \( x_0 \in \mathbb{R}^n \), converges linearly to the unique least squares solution \( x_* = A^T b \) and
\[
\| x_1 - x_* \|^2_{A^T A} \leq \left( 1 - \frac{1}{| \mathcal{R}_0 |} \cdot \sum_{j \in \mathcal{R}_0} \frac{1}{\| A_{(j)} \|^2} \cdot \frac{1}{n} \cdot \lambda_{\min} \left( A^T A \right) \right) \| x_0 - x_* \|^2_{A^T A},
\] (5)
and
\[
\| x_{k+1} - x_* \|^2_{A^T A} \leq \left( 1 - \frac{1}{| \mathcal{R}_k |} \cdot \sum_{j \in \mathcal{R}_k} \frac{1}{\| A_{(j)} \|^2} \cdot \frac{1}{n - 1} \cdot \lambda_{\min} \left( A^T A \right) \right) \| x_k - x_* \|^2_{A^T A}, \quad k = 1, 2, \ldots.
\] (6)

Moreover, let \( \alpha = \max \{| \mathcal{R}_k | \}, \beta = \max \{ \sum_{j \in \mathcal{R}_k} \| A_{(j)} \|^2 \}, k = 0, 1, 2, \ldots \). Then,
\[
\| x_k - x_* \|^2_{A^T A} \leq \left( 1 - \frac{\lambda_{\min} \left( A^T A \right)}{\alpha \cdot \beta \cdot (n - 1)} \right)^{k-1} \left( 1 - \frac{\lambda_{\min} \left( A^T A \right)}{| \mathcal{R}_0 |} \cdot \sum_{j \in \mathcal{R}_0} \| A_{(j)} \|^2 \cdot n \right) \| x_0 - x_* \|^2_{A^T A}, \quad k = 1, 2, \ldots.
\] (7)

**Proof.** From the update rule in Algorithm [2] we have
\[ A(x_{k+1} - x_k) = \frac{A^T_{(j_k)} r_k}{\| A_{(j_k)} \|^2} A_{(j_k)}, \]
which implies that \( A(x_{k+1} - x_k) \) is parallel to \( A_{(j_k)} \). Meanwhile,
\[
A(x_{k+1} - x_*) = A \left( x_k - x_* + \frac{A^T_{(j_k)} r_k}{\| A_{(j_k)} \|^2} e_{j_k} \right) = A \left( x_k - x_* \right) + \frac{A^T_{(j_k)} r_k}{\| A_{(j_k)} \|^2} A_{(j_k)},
\]
which together with the fact \( A^T x_\ast = A^T b \) gives

\[
A(x_{k+1} - x_\ast) = \left( I - \frac{A_{(j)}A^T_{(j)}}{\|A_{(j)}\|^2} \right) A(x_k - x_\ast).
\]

Then

\[
A^T_{(j)}A(x_{k+1} - x_\ast) = A^T_{(j)} \left( I - \frac{A_{(j)}A^T_{(j)}}{\|A_{(j)}\|^2} \right) A(x_k - x_\ast) = 0,
\]

and hence \( A(x_{k+1} - x_\ast) \) is orthogonal to \( A_{(j)} \). Thus, the vector \( A(x_{k+1} - x_k) \) is perpendicular to the vector \( A(x_{k+1} - x_\ast) \). By the Pythagorean theorem, we get

\[
\| A(x_{k+1} - x_\ast) \|^2 = \| A(x_k - x_\ast) \|^2 - \| A(x_{k+1} - x_k) \|^2,
\]

or equivalently,

\[
\| x_{k+1} - x_\ast \|^2_{A^T A} = \| x_k - x_\ast \|^2_{A^T A} - \| x_{k+1} - x_k \|^2_{A^T A}.
\]

On the other hand, from Algorithm[2] we have

\[
\left| A^T_{(j)} r_k \right| = \max_{1 \leq j \leq n} \left| A^T_{(j)} r_k \right| \quad \text{and} \quad \frac{|A^T_{(j)} r_k|^2}{\|A_{(j)}\|^2} = \max_{j \in R_k} \frac{|A^T_{(j)} r_k|^2}{\|A_{(j)}\|^2}.
\]

Then

\[
\| x_{k+1} - x_k \|^2_{A^T A} = \| A(x_{k+1} - x_k) \|^2 = \frac{|A^T_{(j)} r_k|^2}{\|A_{(j)}\|^2} \geq \sum_{j \in R_k} \frac{|A^T_{(j)} r_k|^2}{\|A_{(j)}\|^2} = \sum_{j \in R_k} \left[ \frac{1}{|R_k|} \right] \| A_{(j)} \|^2.
\]

Thus, substituting (9) into (8), we obtain

\[
\| x_{k+1} - x_\ast \|^2_{A^T A} \leq \| x_k - x_\ast \|^2_{A^T A} - \sum_{j \in R_k} \frac{1}{|R_k|} \cdot \max_{1 \leq j \leq n} \left| A^T_{(j)} r_k \right|^2.
\]

For \( k = 0 \), we have

\[
\max_{1 \leq j \leq n} \left| A^T_{(j)} r_0 \right|^2 = \max_{1 \leq j \leq n} \left| A^T_{(j)} r_0 \right|^2 \cdot \frac{\| A^T r_0 \|^2}{\sum_{j=1}^n \left| A^T_{(j)} r_0 \right|^2} \geq \frac{1}{n} \cdot \frac{\| A^T r_0 \|^2}{\| r_0 \|^2},
\]

which together with a result from[18],

\[
\| A^T x \|^2 \geq \lambda_{\min} (A^T A) \| x \|^2
\]

is valid for any vector \( x \) in the column space of \( A \), implies

\[
\max_{1 \leq j \leq n} \left| A^T_{(j)} r_0 \right|^2 \geq \frac{1}{n} \cdot \lambda_{\min} (A^T A) \cdot \| Ax_\ast - Ax_0 \|^2 \\| A^T r_0 \|^2 \geq \frac{1}{n} \cdot \lambda_{\min} (A^T A) \cdot \| x_0 - x_\ast \|^2_{A^T A}.
\]
Thus, substituting (12) into (10), we obtain
\[
\|x_1 - x_*\|_{A^T A}^2 \leq \|x_0 - x_*\|_{A^T A}^2 - \sum_{j_0 \in R_0} \frac{1}{|R_0|} \cdot \frac{1}{\lambda_{\min}(A^T A)} \cdot \frac{1}{n} \cdot \|x_0 - x_*\|_{A^T A}^2 = \left( 1 - \frac{1}{|R_0|} \cdot \sum_{j_0 \in R_0} \frac{1}{\lambda_{\min}(A^T A)} \cdot \frac{1}{n} \right) \cdot \|x_0 - x_*\|_{A^T A}^2,
\]
which is just the estimate (5).

For \( k \geq 1 \), we have
\[
\max_{1 \leq j \leq n} \left| A^T_{(j-k)} r_k \right|^2 = \max_{1 \leq j \leq n} \left| A^T_{(j-k)} r_k \right|^2 \cdot \frac{n}{\sum_{j=1}^n \left| A^T_{(j)} r_k \right|^2},
\]
Note that, according to the update formula in Algorithm 2, it is easy to obtain
\[
A^T_{(j-k)} r_k = A^T_{(j-k-1)} \left( r_{k-1} - A^T_{(j-k-1)} \frac{r_{k-1} - 1}{\|A^T_{(j-k-1)} r_{k-1}\|_2^2} A^T_{(j-k-1)} \right)
= A^T_{(j-k)} (r_{k-1} - A^T_{(j-k)} r_{k-1}) = 0. \quad (13)
\]
Then
\[
\max_{1 \leq j \leq n} \left| A^T_{(j)} r_k \right|^2 = \max_{1 \leq j \leq n} \left| A^T_{(j)} r_k \right|^2 \cdot \frac{\|A^T r_k\|_2^2}{\sum_{j \neq k} \left| A^T_{(j)} r_k \right|^2} \geq \frac{1}{n-1} \cdot \left| A^T r_k \right|^2,
\]
which together with (11) yields
\[
\max_{1 \leq j \leq n} \left| A^T_{(j)} r_k \right|^2 \geq \frac{1}{n-1} \cdot \lambda_{\min}(A^T A) \|x_* - x_k\|_2^2 = \frac{1}{n-1} \cdot \lambda_{\min}(A^T A) \|x_* - x_k\|_{A^T A}^2. \quad (14)
\]
Thus, substituting (14) into (10), we get
\[
\|x_{k+1} - x_*\|_{A^T A}^2 \leq \|x_0 - x_*\|_{A^T A}^2 - \sum_{j_0 \in R_k} \frac{1}{|R_k|} \cdot \frac{1}{\lambda_{\min}(A^T A)} \cdot \frac{1}{n} \cdot \|x_0 - x_*\|_{A^T A}^2 = \left( 1 - \frac{1}{|R_k|} \cdot \sum_{j_0 \in R_k} \frac{1}{\lambda_{\min}(A^T A)} \cdot \frac{1}{n} \right) \cdot \|x_0 - x_*\|_{A^T A}^2, \quad (15)
\]
So the estimate (6) is obtained. By induction on the iteration index \( k \), we have the estimate (7).

\[\square\]

Remark 3. Since \( 1 \leq \alpha \leq n \) and \( \min_{1 \leq j \leq n} \|A_{(j)}\|_2^2 \leq \beta \leq \|A\|_F^2 \), it holds that
\[
\left( 1 - \frac{\lambda_{\min}(A^T A)}{\min_{1 \leq j \leq n} \|A_{(j)}\|_2^2 \cdot (n-1)} \right) \leq \left( 1 - \frac{\lambda_{\min}(A^T A)}{\alpha \cdot \beta \cdot (n-1)} \right) \leq \left( 1 - \frac{\lambda_{\min}(A^T A)}{n \cdot \|A\|_F^2 \cdot (n-1)} \right).
\]
Hence, the convergence factor of the GGS method is small when the parameters \( \alpha \) and \( \beta \) are small. So, the smaller size of \( |R_k| \) is, the better convergence factor of the GGS method is when \( \beta \) is fixed. From the analysis before Algorithm 2, we know that the size of \( |R_k| \) may be smaller than that of \( |V_k| \). This is one of the reasons that our algorithm behaves better in the computing time.

Remark 4. If \( \alpha = 1 \) and \( \beta = \min_{1 \leq j \leq n} \|A_{(j)}\|_2^2 \), the right side of (6) is smaller than
\[
\left( 1 - \frac{1}{\min_{1 \leq j \leq n} \|A_{(j)}\|_2^2 \cdot (n-1)} \lambda_{\min}(A^T A) \right) \|x_k - x_*\|_{A^T A}^2.
\]
Since
\[
\min_{1 \leq j \leq n} \|A_{(j)}\|^2_2 \cdot (n - 1) \leq \|A\|^2_F - \min_{1 \leq j \leq n} \|A_{(j)}\|^2_2 < \|A\|^2_F,
\]
which implies
\[
\frac{1}{\min_{1 \leq j \leq n} \|A_{(j)}\|^2_2 \cdot (n - 1)} > \frac{1}{\|A\|^2_F - \min_{1 \leq j \leq n} \|A_{(j)}\|^2_2} + \frac{1}{\|A\|^2_F},
\]
we have
\[
\left(1 - \frac{1}{\min_{1 \leq j \leq n} \|A_{(j)}\|^2_2 \cdot (n - 1)} j_{\min}(A^T A)\right) \|x_k - x_*\|^2_{A^T A} < \left(1 - \frac{1}{2} \left(\|A\|^2_F - \min_{1 \leq j \leq n} \|A_{(j)}\|^2_2 + \frac{1}{\|A\|^2_F}\right) j_{\min}(A^T A)\right) \|x_k - x_*\|^2_{A^T A}.
\]
Note that the error estimate in expectation of the GRCD method in\textsuperscript{15} is
\[
\mathbb{E}_k \|x_{k+1} - x_*\|^2_{A^T A} \leq \left(1 - \frac{1}{2} \left(\|A\|^2_F - \min_{1 \leq j \leq n} \|A_{(j)}\|^2_2 + \frac{1}{\|A\|^2_F}\right) j_{\min}(A^T A)\right) \|x_k - x_*\|^2_{A^T A},
\]
where \(k = 1, 2, \ldots\). So the convergence factor of GGS method is slightly better for the above case.

\section{NUMERICAL EXPERIMENTS}

In this section, we report the numerical results of the GGS and GRCD methods for solving the linear least squares problem with the matrix \(A \in \mathbb{R}^{m \times n}\) from two sets. One is generated randomly by using the MATLAB function \texttt{randn}, and the other includes some sparse matrices originating in different applications from\textsuperscript{22}. To compare the GGS and GRCD methods fairly and directly, we use the examples from\textsuperscript{15}.

We compare the two methods mainly in terms of the iteration numbers (denoted as “IT”) and the computing time in seconds (denoted as “CPU”), and the IT and CPU listed in our numerical results denote the arithmetical averages of the required iteration numbers and the elapsed CPU times with respect to 50 times repeated runs of the corresponding methods. Furthermore, to give an intuitive compare of the two methods, we also present the iteration number speed-up of the GGS method against the GRCD method, which is defined as
\[
\text{IT speed-up} = \frac{\text{IT of GRCD}}{\text{IT of GGS}},
\]
and the computing time speed-up of the GGS method against the GRCD method, which is defined as
\[
\text{CPU speed-up} = \frac{\text{CPU of GRCD}}{\text{CPU of GGS}}.
\]
In addition, for the sparse matrices from\textsuperscript{22}, we define the density as follows
\[
\text{density} = \frac{\text{number of nonzero of an } m \times n \text{ matrix}}{mn},
\]
and use \(\text{cond}(A)\) to represent the Euclidean condition number of the matrix \(A\).

In our specific experiments, the solution vector \(x_*\) is generated randomly by the MATLAB function \texttt{randn}. For the consistent problem, we set the right-hand side \(b = Ax_*\). For the inconsistent problem, we set the right-hand side \(b = Ax_* + r_0\), where \(r_0\) is a nonzero vector belonging to the null space of \(A^T\), which is generated by the MATLAB function \texttt{null}. All the test problems are started from an initial zero vector \(x_0 = 0\) and terminated once the relative solution error (RES), defined by
\[
\text{RES} = \frac{\|x_k - x_*\|^2_2}{\|x_*\|^2_2},
\]
satisfies \(\text{RES} \leq 10^{-6}\) or the number of iteration steps exceeds 200,000.
| $m \times n$ | IT | CPU | IT | CPU | IT | CPU |
|-------------|-----|-----|-----|-----|-----|-----|
| $1000 \times 50$ | 126.0000 | 128.2400 | 1.0178 | 0.0138 | 0.0631 | 4.5909 |
| $1000 \times 100$ | 374.0000 | 361.5000 | 0.9666 | 0.0466 | 0.1703 | 3.6577 |
| $1000 \times 150$ | 603.0000 | 600.5000 | 0.9990 | 0.1044 | 0.3194 | 3.0590 |
| $2000 \times 50$ | 108.0000 | 106.2600 | 0.9839 | 0.0125 | 0.0525 | 4.2000 |
| $2000 \times 100$ | 246.0000 | 245.7200 | 0.9899 | 0.0466 | 0.1313 | 2.8188 |
| $2000 \times 150$ | 439.0000 | 445.6800 | 1.0152 | 0.1094 | 0.2691 | 2.4600 |
| $3000 \times 50$ | 105.0000 | 104.9600 | 0.9990 | 0.0172 | 0.0556 | 3.2844 |
| $3000 \times 100$ | 231.0000 | 236.8800 | 1.0255 | 0.0619 | 0.1444 | 2.3333 |
| $3000 \times 150$ | 409.0000 | 409.0400 | 1.0001 | 0.1400 | 0.2834 | 2.0246 |
| $4000 \times 50$ | 96.0000 | 96.7400 | 1.0390 | 0.0194 | 0.0572 | 2.9516 |
| $4000 \times 100$ | 205.0000 | 209.1200 | 1.0201 | 0.0678 | 0.1388 | 2.0451 |
| $4000 \times 150$ | 337.0000 | 334.6600 | 1.0198 | 0.1638 | 0.2662 | 1.6260 |
| $5000 \times 50$ | 96.0000 | 95.3800 | 0.9935 | 0.0259 | 0.0660 | 2.4000 |
| $5000 \times 100$ | 195.0000 | 203.0800 | 1.0414 | 0.0728 | 0.1569 | 2.1545 |
| $5000 \times 150$ | 340.0000 | 337.0200 | 0.9912 | 0.1819 | 0.2978 | 1.6375 |

For the first class of matrices, that is, the randomly generated matrices, the numerical results on IT and CPU are listed in Table 1 when the linear system is consistent, and in Table 2 when the linear system is inconsistent. From Tables 1 and 2, we see that the GGS method requires almost the same number of iterations as that of the GRCD method, but the GGS method is more efficient in term of the computing time. The computing time speed-up is at least 1.626 (see Table 1 for the $4000 \times 150$ matrix) and at most 4.7632 (see Table 2 for the $2000 \times 50$ matrix).

| $m \times n$ | IT | CPU | IT | CPU | IT | CPU |
|-------------|-----|-----|-----|-----|-----|-----|
| $1000 \times 50$ | 120.0000 | 124.8600 | 1.0405 | 0.0125 | 0.0591 | 4.7250 |
| $1000 \times 100$ | 329.0000 | 321.3800 | 0.9768 | 0.0400 | 0.1591 | 3.9766 |
| $1000 \times 150$ | 589.0000 | 579.5600 | 0.9400 | 0.0994 | 0.3009 | 3.0283 |
| $2000 \times 50$ | 113.0000 | 110.2000 | 0.9752 | 0.0119 | 0.0566 | 4.7632 |
| $2000 \times 100$ | 245.0000 | 250.0600 | 1.0207 | 0.0531 | 0.1322 | 2.4882 |
| $2000 \times 150$ | 434.0000 | 444.7200 | 1.0247 | 0.1113 | 0.2666 | 2.3961 |
| $3000 \times 50$ | 107.0000 | 105.0800 | 0.9621 | 0.0194 | 0.0553 | 2.3544 |
| $3000 \times 100$ | 235.0000 | 232.3600 | 0.9888 | 0.0609 | 0.1412 | 2.3179 |
| $3000 \times 150$ | 399.0000 | 401.4600 | 1.0062 | 0.1403 | 0.2769 | 1.9733 |
| $4000 \times 50$ | 95.0000 | 97.4800 | 1.0261 | 0.0194 | 0.0537 | 2.7342 |
| $4000 \times 100$ | 220.0000 | 216.7400 | 0.9852 | 0.0694 | 0.1444 | 2.0811 |
| $4000 \times 150$ | 348.0000 | 356.8000 | 1.0251 | 0.1525 | 0.2772 | 1.8176 |
| $5000 \times 50$ | 87.0000 | 91.9400 | 1.0568 | 0.0187 | 0.0559 | 2.9833 |
| $5000 \times 100$ | 212.0000 | 215.9600 | 1.0187 | 0.0862 | 0.1566 | 1.8152 |
| $5000 \times 150$ | 356.0000 | 339.2600 | 1.0097 | 0.1641 | 0.3050 | 1.8590 |

For the second class of matrices, that is, the sparse full column rank matrices from Table 2 when the linear system is consistent, and in Table 4 when the linear system is inconsistent. In both tables, the iteration numbers of the GGS and GRCD methods are almost the same except for the case of the matrix Trefethen_300, which is very ill-conditioned. But for all the matrices, the CPUs of the GGS method are smaller than those of the GRCD method, with the CPU speed-up being at least 1.5315 (the matrix abtahal in Table 3) and at most 11.2222 (the matrix divorce in Table 3).
TABLE 4 Numerical results for the GGS and GRCD methods when the system is inconsistent.

| name  | ablahal | Cities | divorce | WorldCities |
|-------|---------|--------|---------|-------------|
| $m \times n$ | 14596 × 2099 | 55 × 46 | 30 × 9 | 315 × 100 |
| density | 1.68% | 53.04% | 50.00% | 23.87% |
| cond (A) | 12.23 | 207.15 | 19.39 | 66.00 |
| IT | GGS | 11264 | 28449 | 552 | 3532 |
| | GRCD | 12571 | 39752 | 496.68 | 3576.2 |
| IT speed-up | | 1.1160 | 1.3973 | 0.8998 | 1.0125 |
| CPU | GGS | 6.2750 | 0.1716 | 0.0028 | 0.0550 |
| | GRCD | 11.3034 | 1.8278 | 0.0213 | 0.2050 |
| CPU speed-up | | 1.8013 | 10.6539 | 7.5556 | 3.7273 |

Therefore, in all the cases, although the GGS method requires almost the same number of iterations as that of the GRCD method except for a very special case, the former outperforms the latter in term of the computing time, which is consistent with the analysis before Algorithm 2.

References

1. Björck A. Numerical Methods for Least Squares Problems. SIAM, Philadelphia; 1996.
2. Higham NJ. Accuracy and Stability of Numerical Algorithms. SIAM, Philadelphia; 2002.
3. Saad Y. Iterative Methods for Sparse Linear Systems. SIAM, Philadelphia; 2003.
4. Strohmer T, and Vershynin R. A randomized Kaczmarz algorithm with exponential convergence. J Fourier Anal Appl. 2009;15:262–278.
5. Leventhal D, and Lewis AS. Randomized methods for linear constraints: Convergence rates and conditioning. Math Oper Res. 2010;35(3):641–654.
6. Ma A, Needell D, and Ramdas A. Convergence properties of the randomized extended Gauss–Seidel and Kaczmarz methods. SIAM J Matrix Anal Appl. 2015;36(4):1590–1604.
7. Edalatpour V, Hezari D, and Salkuyeh DK. A generalization of the Gauss–Seidel iteration method for solving absolute value equations. Appl Math Comput. 2017;293:156–167.
8. Hefny A, Needell D, and Ramdas A. Rows versus columns: Randomized Kaczmarz or Gauss–Seidel for ridge regression. SIAM J Sci Comput. 2017;39(5):S528–S542.
9. Tu S, Venkataraman S, Wilson AC, Gittens A, Jordan MI, and Recht B. Breaking locality accelerates block Gauss-Seidel. in ICML. 2017;70:3482–3491.
10. Chen L, Sun DF, and Toh KC. An efficient inexact symmetric Gauss–Seidel based majorized ADMM for high-dimensional convex composite conic programming. Math Program. 2017;161:237–270.
11. Tian ZL, Tian MY, Liu ZY, and Xu TY. The Jacobi and Gauss–Seidel–type iteration methods for the matrix equation $AXB = C$. Appl Math Comput. 2017;292:63–75.
12. Xu YY. Hybrid Jacobian and Gauss–Seidel proximal block coordinate update methods for linearly constrained convex programming. SIAM J Optimization. 2018;28(1):646–670.
13. Du K. Tight upper bounds for the convergence of the randomized extended Kaczmarz and Gauss–Seidel algorithms. Numer Linear Algebra Appl. 2019;26(3):e2233.
14. Razaviyayn M, Hong M, Rezayian N, and Luo ZQ. A linearly convergent doubly stochastic Gauss–Seidel algorithm for solving linear equations and a certain class of over–parameterized optimization problems. Math Program. 2019;176:465–496.
15. Bai ZZ, and Wu WT. On greedy randomized coordinate descent methods for solving large linear least-squares problems. Numer Linear Algebra Appl. 2019;26(4):1–15.

16. Griebel M, and Oswald P. Greedy and randomized versions of the multiplicative Schwarz method. Linear Algebra Appl. 2012;437:1596–1610.

17. Nguyen N, Needell D, and Woolf T. Linear convergence of stochastic iterative greedy algorithms with sparse constraints. IEEE Trans Inf Theory. 2017;63:6869–6895.

18. Bai ZZ, and Wu WT. On greedy randomized Kaczmarz method for solving large sparse linear systems. SIAM J Sci Comput. 2018;40(1):A592–A606.

19. Bai ZZ, and Wu WT. On relaxed greedy randomized Kaczmarz methods for solving large sparse linear systems. Appl Math Lett. 2018;83:21–26.

20. Nutini J. Greed is Good: Greedy Optimization Methods for Large–Scale Structured Problems. PhD thesis, University of British Columbia; 2018.

21. Zhang JJ. A new greedy Kaczmarz algorithm for the solution of very large linear systems. Appl Math Lett. 2019;91:207–212.

22. Du K, and Gao H. A new theoretical estimate for the convergence rate of the maximal weighted residual Kaczmarz algorithm. Numer Math Theor Meth Appl. 2019;12(2):627–639.

23. Liu Y, and Gu CQ. Variant of greedy randomized Kaczmarz for ridge regression. Appl Numer Math. 2019;143:223–246.

24. Osborne EE. On least squares solution of linear equations. J Assoc Comput Mach. 1961;8:628–636.

25. Haddock J, and Needell D. On Motzkin’s method for inconsistent linear systems. BIT Numer Math. 2019;59:387–401.

26. Rebrova E, and Needell D. Sketching for Motzkin’s iterative method for linear systems. Proc. 50th Asilomar Conf. on Signals, Systems and Computers; 2019.

27. A Davis T, and Hu YF. The university of florida sparse matrix collection. ACM Trans Math Softw. 2011;38(1):1–25.