Young Measures Generated by Ideal Incompressible Fluid Flows

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Abstract

In their seminal paper [11] R. DiPerna and A. Majda introduced the notion of measure-valued solution for the incompressible Euler equations in order to capture complex phenomena present in limits of approximate solutions, such as persistence of oscillation and development of concentrations. Furthermore, they gave several explicit examples exhibiting such phenomena. In this paper we show that any measure-valued solution can be generated by a sequence of exact weak solutions. In particular this gives rise to a very large, arguably too large, set of weak solutions of the incompressible Euler equations.

1 Introduction

The incompressible Euler equations

\[
\begin{align*}
\partial_t v + \text{div}(v \otimes v) + \nabla p &= 0, \\
\text{div} v &= 0
\end{align*}
\]

describe the motion of an inviscid fluid with constant density in \(d\) dimensions, \(d \geq 2\). If we are given an initial velocity field \(v_0 \in L^2(\mathbb{R}^d)\) with \(\text{div} v_0 = 0\) weakly and a positive time \(0 < T \leq \infty\), then the weak formulation of these equations reads

\[
\int_0^T \int_{\mathbb{R}^d} \left( v \cdot \partial_t \phi + v \otimes v : \nabla \phi \right) dx dt + \int_{\mathbb{R}^d} v_0(x) \phi(x, 0) dx = 0; \tag{1}
\]

that is, we say that \(v \in L^2_{\text{loc}}(\mathbb{R}^d \times [0, T]; \mathbb{R}^d)\) is a weak solution if it is weakly divergence-free and satisfies (1) for every \(\phi \in C^\infty_c(\mathbb{R}^d \times [0, T]; \mathbb{R}^d)\) with \(\text{div} \phi = 0\).

Whereas classical solutions, if they exist, are unique in the class of dissipative solutions (see [19] pp. 153-158) and moreover conserve energy, it has been known since the seminal work of Scheffer and Shnirelman that weak solutions are not unique and need not conserve energy. In [21] V. Scheffer constructed a weak solution in two dimensions with compact support in space and time, thus disproving uniqueness even for zero initial data (see also [22] for a different proof). A. Shnirelman in [23] later showed that there exist weak solutions with decreasing energy. In [8] and [9] these results were put in a unified framework based on convex integration and Baire category techniques. In particular in [9]
the authors show that various admissibility criteria, like energy conservation or energy dissipation, are neither sufficient to restore uniqueness nor can they provide for any regularity higher than $L_t^\infty L_x^2$.

Several weaker concepts of solutions for Euler have arisen in the literature, for example Brenier’s generalised flows [4, 5], Lions’ dissipative solutions [19], and DiPerna-Majda’s measure-valued solutions [11]. The latter can be briefly described as follows: Given a sequence of velocity fields $v_n(x,t)$, it is known from classical Young measure theory (see e.g. [2, 15, 20, 25, 26]) that there exists a subsequence (not relabeled) and a parametrised probability measure $\nu_{x,t}$ on $\mathbb{R}^d$ such that for all bounded test functions $f$, $f(v_n(x,t)) \rightharpoonup \hat{f}_{x,t}$ weakly* in $L^\infty$. One can interpret the measure $\nu_{x,t}$ as the probability distribution of the velocity field at the point $x$ at time $t$ when the sequence $(v_n)$ exhibits faster and faster oscillations as $n \to \infty$. Since we only have an $L^2$ bound on $(v_n)$, concentrations could occur for non-bounded $f$, in particular for the energy density $f(v) = \frac{1}{2} |v|^2$. DiPerna and Majda addressed this issue in [11], providing a framework in which both oscillations and concentrations can be described. To this end they introduced a generalised Young measure and defined a measure-valued solution for Euler to be a generalised Young measure that satisfies the Euler equations in an average sense (see Section 2.3 below). By considering sequences of Leray solutions for the Navier-Stokes equations with viscosities tending to zero, they show global existence of measure-valued solutions for arbitrary initial data. In the context of the calculus of variations Alibert and Bouchitté later introduced a modified version of these generalised Young measures [1], which we will work with.

Our main result is the following (the relevant definitions can be found in the next section):

**Theorem 1.** A Young measure $(\nu, \lambda, \nu^\infty)$ on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d \times [0, T]$ is a measure-valued solution of the Euler equations with bounded energy if and only if there exists a sequence $(v_n)_{n \in \mathbb{N}}$ of weak solutions to the Euler equations bounded in $C([0, T]; L^2_w(\mathbb{R}^d; \mathbb{R}^d))$ which generate the Young measure $(\nu, \lambda, \nu^\infty)$ in the sense that

$$f(v_n) dx dt \rightharpoonup \left( \int_{\mathbb{R}^d} f dv \right) dx dt + \left( \int_{S^{d-1}} f^\infty dv^\infty \right) \lambda$$

in the sense of measures for every $f : \mathbb{R}^d \to \mathbb{R}$ that possesses an $L^2$-recession function $f^\infty$.

The proof relies on the techniques developed in [8,9], in particular on the notion of subsolution. As a byproduct of our analysis, we establish a link between Euler subsolutions and measure-valued solutions in Section 2.4.

Theorem 1 shows that in a sense measure-valued solutions and weak solutions are essentially the same for the incompressible Euler equations in dimension.
In other words, we see that in the absence of any regularity, that is, on the level of $L^2$ or $L^\infty$ solutions, the notion of weak solution is too weak to yield any information on the correlations of the velocities at different space-time points. Indeed, measure-valued solutions merely describe the one-point statistics $\nu_{x,t}$ of the velocity field in a weakly convergent sequence.

This is in contrast with weak and measure-valued solutions in other contexts, such as hyperbolic conservation laws in one space dimension, where the two defining aspects of a measure-valued solution – the microscopic nonlinearity and the macroscopic conservation laws – are strong enough to lead to compensated compactness, see e.g. [10]. In such situations the equations are usually complemented by a suitable entropy condition. As is well known, for the incompressible Euler equations a possible entropy condition is related to the kinetic energy $\frac{1}{2}\int |v|^2 \, dx$. Indeed, imposing the admissibility condition, that the energy should be bounded by the initial energy for all times, leads to the weak-strong uniqueness for the Euler equations: any weak solution with this initial data that satisfies the weak energy inequality is a dissipative solution in the sense of P.-L. Lions and therefore coincides with the smooth solution as long as the latter exists. For the admissibility condition in the context of hyperbolic conservation laws see [7,10]. The weak-strong uniqueness for admissible measure-valued solutions for the Euler equations (see Section 2.3) was proved in [6]. Here we prove:

**Theorem 2.** Suppose that $(\nu, \lambda, \nu^\infty)$ is an admissible measure-valued solution with initial data $v_0 \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ ($\text{div} \, v_0 = 0$). Then the generating sequence $(v_n)$ as in Theorem 1 may be chosen such that in addition

$$\|v_n(t=0) - v_0\|_{L^2(\mathbb{R}^d)} < \frac{1}{n}$$

and

$$\sup_{t \in [0,T]} \frac{1}{2} \int_{\mathbb{R}^d} |v_n(x,t)|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_n(x,0)|^2 \, dx.$$

The following existence result can be easily deduced from the proof of this theorem and the existence of admissible measure-valued solutions for arbitrary $L^2$-initial data (cf. e.g. [6]):

**Corollary 3.** There exists an $L^2$-dense subset $\mathcal{E}$ of the set of solenoidal $L^2$-vectorfields on $\mathbb{R}^d$ such that for every initial data in $\mathcal{E}$, there exist infinitely many admissible weak solutions of Euler.

This is shown at the end of this work. Whether one can improve on the set $\mathcal{E}$ of such “wild” initial data, and obtain an existence result for admissible weak solutions for a larger set of initial data, seems to be a very delicate issue. In particular, such an initial data needs to be highly irregular, for otherwise we would contradict the weak-strong uniqueness and classical local existence theorems (see e.g. Section 2.3 in [9]). Without the admissibility condition, existence of weak solutions has been shown in [24] for all initial data.
Finally, it should be mentioned that generalised Young measures are of importance not only in fluid mechanics, where they emerged, but have also been recognised a useful tool in the calculus of variations. In particular, the question has been of some interest how Young measures that arise from certain constrained sequences can be characterised: The prototypic result is the theorem of Kinderlehrer and Pedregal [16] which states that a (classical) Young measure is generated by a sequence of gradients if and only if it satisfies a certain Jensen-type inequality. The result has been generalised to so-called $A$-free sequences [13] and to generalised Young measures [12, 14, 17]. Theorem 1 also gives a characterisation of Young measures that are generated by a constrained sequence (namely a sequence of Euler solutions), but it differs from the previously known results in two important respects: First, our problem does not fit into the $A$-free framework since the constant rank condition is not satisfied; and second, our sequence not only satisfies a linear system of PDE’s, but in addition a nonlinear pointwise constraint. More concretely, not only do we generate the Young measure with an $A$-free sequence, but with a sequence of exact solutions of the Euler equations.

The rest of this paper is organised as follows: In Section 2 we recall the notion and key properties of generalised Young measures and admissible measure-valued solutions. Section 3 is devoted to the proof of Theorems 1 and 2. It is split into several independent parts: First we apply the results of [9] to reduce the problem to finding appropriate subsolutions in Section 3.1, and we then use some more or less standard Young measure techniques in Section 3.2 to reduce to discrete homogeneous oscillation Young measures. In Section 3.3 we present an explicit construction of a generating sequence for discrete oscillation Young measures. Finally, in Section 3.4 we complete the proofs of Theorem 2 and Corollary 3 using an argument from [9].

2 Preliminaries

2.1 Basic Notation

Given a locally compact separable metric space $X$, we denote by $C_c(X)$ the space of continuous functions with compact support and $C_0(X)$ the Banach space obtained from the completion of $C_c(X)$ with respect to the supremum norm. Using the Riesz representation theorem the space of finite Radon measures, denoted $\mathcal{M}(X)$, can be identified with the dual space of $C_0(X)$. We denote by $\mathcal{M}^+(X)$ and $\mathcal{M}^1(X)$ the subspaces of positive finite measures and probability measures, respectively.

For an open or closed subset $U \subseteq \mathbb{R}^m$, $\mu \in \mathcal{M}^+(U)$ and an open or closed subset $V \subseteq \mathbb{R}^l$, we denote by $L^\infty_w(U, \mu; \mathcal{M}^1(V))$ the space of $\mu$-weakly*-measurable maps from $U$ into $\mathcal{M}^1(V)$. That such a map $\nu$ is $\mu$-weakly*-measurable means
that for each bounded Borel function $f : V \to \mathbb{R}$, the map

$$ x \mapsto (\nu_x, f) := \int_V f(z) d\nu_x(z) $$

is $\mu$-measurable. In case $\mu$ is the Lebesgue measure we omit the specification of the measure.

We will denote by $L^2_2$ the space $L^2(\mathbb{R}^d)$, by $L^\infty_2 L^2_2$ the space $L^\infty([0, T]; L^2_2)$, and by $C L^2_2$ the space $C([0, T]; L^2_2(\mathbb{R}^d))$ of functions that are weakly continuous in time and $L^2$ in space; more precisely, it is the space of maps $v : [0, T] \to L^2(\mathbb{R}^d)$ such that the map

$$ t \mapsto \int_{\mathbb{R}^d} v(x, t) \phi(x) dx $$

is continuous for each test function $\phi \in L^2(\mathbb{R}^d)$.

We shall write $A : B$ for the scalar product of two matrices in $\mathbb{R}^{d \times d}$, that is, $A : B = \sum_{i,j} A_{ij} B_{ij}$, and $v \otimes w$ for the tensor product of two vectors in $\mathbb{R}^d$, which is defined as a $(d \times d)$-matrix with entries $(v \otimes w)_{ij} = v_i w_j$. Moreover we define for $v \in \mathbb{R}^d$

$$ v \circ v := v \otimes v - \frac{1}{d} |v|^2 I_d, $$

where $I_d$ is the $d \times d$ identity matrix. Note that $v \circ v$ is symmetric and has zero trace. The space of symmetric $(d \times d)$-matrices is denoted by $S^d$ and the space of traceless symmetric $(d \times d)$-matrices by $S^d_0$. If $\phi : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is a differentiable matrix-valued function, then $\text{div} \phi$ is a vector field defined by $(\text{div} \phi)_i = \sum_j \partial_{x_j} \phi_{ij}$.

If $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are maps from some sets $X$, $Y$ into, say, $\mathbb{R}$, then $f \otimes g$ is a map $X \times Y \to \mathbb{R}$ defined by $f \otimes g(x, y) = f(x) g(y)$, whereas for two measures $\mu$ and $\nu$ living on two measurable spaces $X$ and $Y$ respectively, $\mu \otimes \nu$ is a measure on $X \times Y$ defined by $(\mu \otimes \nu)(A \times B) = \mu(A) \nu(B)$ for measurable subsets $A \subseteq X$, $B \subseteq Y$.

Finally, $S^{d-1} \subset \mathbb{R}^d$ is the $(d-1)$-dimensional unit sphere.

## 2.2 Generalised Young Measures

In this section we recall the notion of generalised Young measure as introduced in [11], [1]. For a more detailed and exhaustive discussion of (generalised) Young measures, see e.g. [2, 6, 15, 17, 20].

Let $\Omega \subseteq \mathbb{R}^m$ be an open or closed set, $p \in [1, \infty)$, and $(w_n)_{n \in \mathbb{N}}$ a sequence of maps $\Omega \to \mathbb{R}^l$ bounded in $L^p(\Omega)$. We want to study the limit behaviour of sequences of the form $(f(w_n(y)))_{n \in \mathbb{N}}$ and, more generally, $(f(y, w_n(y)))_{n \in \mathbb{N}}$ for a certain class of test functions $f$. Given $f \in C(\Omega \times \mathbb{R}^l)$, its $L^p$-recession function $f^\infty$ is defined as

$$ f^\infty(y, z) = \lim_{y' \to y \atop z' \to z \atop s \to \infty} \frac{f(y', s z')}{s^p}, $$
provided the limit exists. Observe that in this case $f^\infty$ is $p$-homogeneous, i.e. $f^\infty(y, \alpha z) = \alpha^p f^\infty(y, z)$ for all $\alpha \geq 0$, $y, z \in \mathbb{R}^l$. In this paper we consider test functions in the class

$$\mathcal{F}_p := \{ f \in C(\mathbb{R}^l) : f^\infty \text{ exists and is continuous on } S^{l-1} \},$$

and more generally

$$\mathcal{F}_p(\Omega) := \{ f \in C(\Omega \times \mathbb{R}^l) : f^\infty \text{ exists and is continuous on } \Omega \times S^{l-1} \}.$$

Examples of functions in $\mathcal{F}_p(\Omega)$ are given by continuous functions satisfying $|f(y, z)| \leq C(1 + |z|^q)$ with $0 \leq q < p$, in which case $f^\infty = 0$, or by continuous functions which are $p$-homogeneous in $z$, in which case $f^\infty = f$. Of course, functions in $\mathcal{F}_p(\Omega)$ always satisfy a bound $|f(y, z)| \leq C(1 + |z|^q)$ (where $C$ may depend on $y$, however).

A generalised Young measure on $\mathbb{R}^l$ with parameters in $\Omega$ is defined as a triple $(\nu, \lambda, \nu^\infty)$ such that

$$\nu \in L_w^\infty(\Omega; \mathcal{M}^1(\mathbb{R}^l)),$$

$$\lambda \in \mathcal{M}^+(\mathbb{R}^l),$$

and

$$\nu^\infty \in L_w^\infty(\mathbb{R}^l; \mathcal{M}^1(S^{l-1})).$$

Observe that $\nu$ is only defined Lebesgue-a.e. on $\Omega$ and $\nu^\infty$ is defined only $\lambda$-a.e. on $\mathbb{R}^l$. Classical Young measures are simply those where $\lambda = 0$ (in which case $\nu^\infty$ is immaterial). In this case we simply write $\nu$ instead of a triple.

We are now able to state the following important result of Alibert and Bouchitté, which is a refinement of the construction in [11] (for proofs, see [1], [17], [6]):

Theorem 4. (Fundamental Theorem for Generalised Young Measures.)

For $p \in [1, \infty)$ let $(w_n)_{n \in \mathbb{N}}$ be a sequence of maps $\Omega \to \mathbb{R}^l$ bounded in $L^p(\Omega)$. Then there exists a subsequence (not relabeled) and a generalised Young measure $(\nu, \lambda, \nu^\infty)$ such that, for every $f \in \mathcal{F}_p(\Omega)$,

$$f(y, w_n(y)) dy \overset{\ast}{\rightharpoonup} \langle \nu_y, f(y, \cdot) \rangle dy + \langle \nu_y^\infty, f^\infty(y, \cdot) \rangle \lambda$$

in the sense of measures, where $\langle \nu_y, f(y, \cdot) \rangle = \int_{\mathbb{R}^l} f(y, z) d\nu_y(z)$ and $\langle \nu_y^\infty, f^\infty(y, \cdot) \rangle = \int_{S^{l-1}} f^\infty(y, z) d\nu^\infty_y(z)$.

Moreover, we then have that $\int_\Omega \langle \nu_y, | \cdot |^p \rangle dy < \infty$.

In the situation of the theorem, we say that the subsequence $(w_n)$ generates the Young measure $(\nu, \lambda, \nu^\infty)$ in $L^p(\Omega)$, and write

$$w_n \overset{Y_p}{\rightharpoonup} (\nu, \lambda, \nu^\infty). \quad (2)$$

With the notation

$$\langle \nu, \lambda, \nu^\infty; f \rangle := \int_\Omega \langle \nu, f \rangle dy + \int_\Omega \langle \nu^\infty, f^\infty \rangle d\lambda,$$
we can write this as

\[
\int_{\Omega} f(y, w_n(y)) \, dy \to \langle \nu, \lambda, \nu^\infty; f \rangle \quad \text{for all } f \in \mathcal{F}_p(\Omega).
\]

In the same manner we define convergence of generalised Young measures: we say that \((\nu^k, \lambda^k, \nu^\infty,k) \xrightarrow{Y_p} (\nu, \lambda, \nu^\infty)\) if

\[
\langle \nu^k, \lambda^k, \nu^\infty,k; f \rangle \to \langle \nu, \lambda, \nu^\infty; f \rangle \quad \text{for all } f \in \mathcal{F}_p(\Omega). 
\]

Indeed, (2) is a special case of (3), since the function \(w_n\) can be identified with the classical Young measure \(\delta_{w_n}\).

The following proposition collects some well-known properties of generalised Young measures. The proofs for the case \(p = 1\) can be found for instance in [17], but can easily be modified for general \(p \in [1, \infty)\).

**Proposition 5.**

a) There exists a countable set of functions \(f_k = \phi_k \otimes h_k, k \in \mathbb{N}, \) with \(\phi_k \in C_c(\Omega), h_k \in \mathcal{F}_p\) such that

\[
\langle \nu, \lambda, \nu^\infty; f_k \rangle = \langle \tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty; f_k \rangle \quad \forall k \implies (\nu, \lambda, \nu^\infty) = (\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty).
\]

b) If \(w_n \xrightarrow{Y_p} (\nu, \lambda, \nu^\infty), \tilde{w}_n \xrightarrow{Y_p} (\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)\) and \(w_n - \tilde{w}_n \to 0\) locally in measure, then \(\nu = \tilde{\nu}\).

c) If \(w_n \xrightarrow{Y_p} (\nu, \lambda, \nu^\infty)\) and \(w_n - \tilde{w}_n \to 0\) in \(L^p_{loc}\), then \(\tilde{w}_n \xrightarrow{Y_p} (\nu, \lambda, \nu^\infty)\).

d) \(w_n \to w\) strongly in \(L^p_{loc}\) if and only if \(w_n \xrightarrow{Y_p} \delta_w\).

e) Suppose \(w_n \xrightarrow{Y_p} (\nu, \lambda, \nu^\infty)\) and let \(w \in L^p(\Omega)\). Then \(w_n + w \xrightarrow{Y_p} (T_w \nu, \lambda, \nu^\infty)\), where \(T_w \nu\) is the oscillation measure defined by

\[
\langle (T_w \nu)_y, f \rangle := \int_{\mathbb{R}} f(z + w(y)) \, d\nu_y(z) \quad \text{for } f \in C_0(\mathbb{R}^d), \text{ a.e. } y \in \Omega
\]

In general the Young measure records the defect from strong convergence, this is signified by d). In our case the defect can come from oscillation, recorded by the oscillation measure \(\nu\) or from concentration, recorded by the concentration measure \(\lambda\) and the concentration-angle measure \(\nu^\infty\).

In e) the Young measure \((T_w \nu, \lambda, \nu^\infty)\) is said to be the shift of the Young measure \((\nu, \lambda, \nu^\infty)\). This operation is useful in separating the microscopic oscillatory or concentration behaviour from the macroscopic coarse-grained state.

Part a) of the proposition implies that it suffices to test the convergence with functions \(f \in \mathcal{F}_p\), i.e. those which are independent of \(y \in \Omega\). A further consequence of part a) is that the convergence notion in (3) is metrizable on bounded sets. This immediately leads to the following diagonal-sequence extraction principle, which we prefer to state explicitly as a proposition:
Proposition 6. Suppose that for each \( k \in \mathbb{N} \),
\[
\left( \nu^{k,n}, \lambda^{k,n}, \nu^{\infty,k,n} \right) \xrightarrow{Y_p} \left( \nu^k, \lambda^k, \nu^{\infty} \right)
\]
as \( n \to \infty \)
and moreover
\[
\left( \nu^k, \lambda^k, \nu^{\infty} \right) \xrightarrow{Y_p} \left( \nu, \lambda, \nu^{\infty} \right)
\]
as \( k \to \infty \).

Then there exists a sequence \( n(k) \to \infty \) with \( k \to \infty \) such that
\[
\left( \nu^{k,n(k)}, \lambda^{k,n(k)}, \nu^{\infty,k,n(k)} \right) \xrightarrow{Y_p} \left( \nu, \lambda, \nu^{\infty} \right)
\]
as \( k \to \infty \).

2.3 Measure-Valued Solutions of the Euler Equations

A measure-valued solution to the Euler equations is a generalised Young measure on \( \mathbb{R}^d \) with parameters in \( \mathbb{R}^d \times [0,T] \) which satisfies the Euler equations in an average sense. This means that
\[
\int_0^T \int_{\mathbb{R}^d} \partial_t \mathbf{\phi} \cdot (\nu, \xi) + \nabla \mathbf{\phi} : (\nu, \xi \otimes \xi) \, dx \, dt + \int_{\mathbb{R}^d \times (0,T)} \nabla \mathbf{\phi} : (\nu^{\infty}, \theta \otimes \theta) \, d\lambda = 0 \quad (4)
\]
for all \( \phi \in C^\infty_c \left( \mathbb{R}^d \times (0,T) ; \mathbb{R}^d \right) \) with \( \text{div} \, \phi = 0 \), and
\[
\int_{\mathbb{R}^d} \nabla \psi \cdot (\nu_{x,t}, \xi) \, dx = 0 \quad (5)
\]
for all \( \psi \in C^\infty_c (\mathbb{R}^d) \) and for almost every \( t \). Here, the quantity
\[
\bar{v}(x,t) := \langle \nu_{x,t}, \xi \rangle
\]
is called the barycentre of \( \nu_{x,t} \) and signifies the coarse-grained, or macroscopic, flow. As usual, we have written \( \langle \nu, \xi \otimes \xi \rangle = \int \xi \otimes \xi \nu(d\xi) \) etc.

In light of the energy bound for weak solutions of the Navier-Stokes equations, it is natural to restrict attention to measure-valued solutions to the Euler equations which inherit this bound.

Proposition 7 (\cite{6}). Let \( (v_n(x,t)) \) be a sequence of functions \( \mathbb{R}^d \times [0,T] \to \mathbb{R}^d \) which is bounded in \( L^\infty \left( [0,T] ; L^2 (\mathbb{R}^d) \right) \) and generates a Young measure \( (\nu, \lambda, \nu^{\infty}) \) in \( L^2 \left( \mathbb{R}^d \times [0,T] \right) \). Then
\[
\text{esssup}_t \left( \int_{\mathbb{R}^d} (v_{x,t}, \cdot)^2 \, dx \right) < \infty, \quad (7)
\]
and the concentration measure \( \lambda \) admits a disintegration of the form
\[
d\lambda(x,t) = \lambda_t(dx) \otimes dt, \quad (8)
\]
where \( t \mapsto \lambda_t \) is a bounded (w.r.t. the total variation norm) measurable map from \( [0,T] \) into \( \mathcal{M}^+ \left( \mathbb{R}^d \right) \).
In particular, in this case Jensen’s inequality implies that \( \bar{v}(x, t) \in L^\infty_t L^2_x \). A well-known consequence of (4) is then that \( \bar{v} \) can be re-defined on a set of times of measure zero so that it belongs to the space \( CL^2_w \) (see Appendix A of [9]), and therefore, the initial average \( \bar{v}(\cdot, 0) \) is a well-defined \( L^2 \) function that is assumed in the sense that \( \bar{v}(\cdot, t) \rightharpoonup \bar{v}(\cdot, 0) \) weakly in \( L^2 \) as \( t \to 0 \). Thus, we may write equation (4) as

\[
\int_0^T \int_{\mathbb{R}^d} \partial_t \phi \cdot \langle \nu, \xi \rangle dx dt + \int_0^T \int_{\mathbb{R}^d} \nabla \phi : \langle \nu, \xi \rangle dx dt = -\int_{\mathbb{R}^d} \phi(x, 0) \bar{v}(x, 0) dx
\]

for all \( \phi \in C_\infty([0, T); \mathbb{R}^d) \) with \( \text{div} \phi = 0 \).

Finally, the energy of the measure-valued solution can be defined for almost every time \( t \)

\[
E(t) := \frac{1}{2} \int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2 \rangle dx + \frac{1}{2} \lambda_t(\mathbb{R}^d).
\]

**Definition 8. (Measure-Valued Solutions.)**

a) A Young measure \( (\nu, \lambda, \nu^\infty) \) on \( \mathbb{R}^d \) with parameters in \( \mathbb{R}^d \times [0, T] \) is called a measure-valued solution of the Euler equations with barycentre \( \bar{v} := \langle \nu, \xi \rangle \) if it satisfies (4)-(5).

b) A Young measure \( (\nu, \lambda, \nu^\infty) \) on \( \mathbb{R}^d \) with parameters in \( \mathbb{R}^d \times [0, T] \) is called an admissible measure-valued solution of the Euler equations with initial data \( v_0 \in L^2(\mathbb{R}^d) \) if it satisfies (7)-(8), the equations (9) and (5) hold, and moreover

\[
E(t) \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 dx \text{ for a.e. } t > 0.
\]

**Proposition 9.** Let \( (\nu, \lambda, \nu^\infty) \) be an admissible measure-valued solution of the Euler equations and \( \bar{v} \) its barycentre as in (4). Then

\[
\bar{v}(\cdot, t) \to \bar{v}(\cdot, 0) = v_0
\]

strongly in \( L^2(\mathbb{R}^d) \) as \( t \to 0 \).

**Proof.** We have already seen that \( \bar{v} \in CL^2_w \), and therefore

\[
\liminf_{t \to 0} \| \bar{v}(t) \|_{L^2} \geq \| \bar{v}(0) \|_{L^2}.
\]

On the other hand,

\[
\int |\bar{v}(t)|^2 dx = \int |\nu_{x,t}, \xi|^2 dx \leq \int |\nu_{x,t}, \xi|^2 dx + \lambda_t(\mathbb{R}^d)
\]

\[
= 2E(t) \leq \int |\bar{v}(0)|^2 dx,
\]

9
where we used the weak energy inequality in Definition 8. Combining both inequalities yields $\| \bar{v}(t) \|_{L^2} \to \| \bar{v}(0) \|_{L^2}$ as $t \to 0$, and since weak convergence together with convergence of the norms implies strong convergence, we are done.

2.4 Subsolutions

We recall from [8] that the Euler equations can be written in a way that separates them into a linear differential constraint and a nonlinear constitutive relation.

Lemma 10. Let $v \in L^\infty ([0, T]; L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d))$, $u \in L^\infty ([0, T]; L^1_{\text{loc}}(\mathbb{R}^d; S^d_0))$ and $q$ be a distribution such that

$$\partial_t v + \text{div} u + \nabla q = 0$$
$$\text{div} v = 0. \tag{11}$$

If it also holds that

$$u = v \circ v \tag{12}$$

for almost every $(x, t) \in \mathbb{R}^d \times [0, T]$, then $v$ and $p := q - \frac{1}{d}|v|^2$ are a weak solution to the Euler equations. Conversely, if $(v, p)$ is a weak solution of Euler, then $(v, u, q)$ with $u := v \circ v$ and $q := p + \frac{1}{d}|v|^2$ solve (11) and (12).

A pair $(v(x, t), u(x, t))$ for which there exists a pressure $q(x, t)$ such that (11) is satisfied is called a subsolution for the Euler equations. Thus, a subsolution is a solution precisely if a certain pointwise nonlinear equation, namely (12), holds.

2.4.1 Measure-Valued Subsolutions

The concept of subsolution easily leads to the corresponding measure-valued notion. For the concentration-angle measure we define the set

$$S^{d-1} = \left\{ (v, u) \in \mathbb{R}^d \times S^d_0 : \frac{1}{d}|v|^2 + |u|_\infty = 1 \right\},$$

where $|u|_\infty$ denotes the operator norm of the matrix $u$. Notice that $(v, v \circ v) \in S^{d-1}$ whenever $v \in S^{d-1}$. Motivated by Lemma 10 consider a sequence

$$w_n = (v_n, u_n) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \times S^d_0$$

bounded in $L^\infty ([0, T]; L^2(\mathbb{R}^d) \times L^1(\mathbb{R}^d))$. Analogously to Section 2.2 we define the space of test-functions

$$\mathcal{F}_{2,1} := \left\{ f \in C(\mathbb{R}^d \times S^d_0) : f^\infty(v, u) := \lim_{\substack{s', s \to \infty \\ \ \ s' \to s \\ \ s \to \infty}} \frac{f(sv', sv^2 u')}{s^2} \epsilon C(S^{d-1}) \text{ exists} \right\},$$

and similarly also the $(x, t)$-dependent version $\mathcal{F}_{2,1}(\mathbb{R}^d \times [0, T])$. We have the obvious analogue of the Fundamental Theorem for Young measures:
Theorem 11. Suppose $w_n = (v_n, u_n)$ is a sequence bounded in $L^\infty([0, T]; L^2 \times L^1(\mathbb{R}^d))$, and $f \in \mathcal{F}_{2,1}(\mathbb{R}^d \times [0, T])$. Then there exists a subsequence (not relabeled) and a Young measure $(\nu, \lambda, \nu^\infty)$, with $\nu \in L^\infty(\mathbb{R}^d \times [0, T]; \mathcal{M}(\mathbb{R}^d \times \mathcal{S}^d))$, $\lambda \in \mathcal{M}^*(\mathbb{R}^d \times [0, T])$, $\nu^\infty \in L^\infty(\mathbb{R}^d \times [0, T], \lambda; \mathcal{M}(\mathcal{S}^{d-1}))$, such that

$$f(x, t; w_n(x, t))dxdt \sim (\nu_{x,t}, f(x, t; \cdot))dxdt + (\nu_{x,t}^\infty, f^\infty(x, t; \cdot))\lambda$$

in the sense of measures for all $f \in \mathcal{F}_{2,1}(\mathbb{R}^d \times [0, T])$.

In this case we write

$$w_n \xrightarrow{\mathcal{F}_{2,1}} (\nu, \lambda, \nu^\infty).$$

Proof. Consider the homeomorphism $\mathcal{S}^d_0 \to \mathcal{S}^d_0$, $u \mapsto |u|_\infty u$, with inverse $u \mapsto \frac{u}{\sqrt{|u|_\infty}}$. Given $f \in \mathcal{F}_{2,1}$ define

$$g(v, u) := f \left( \sqrt{|v|} |u|_\infty u \right).$$

It is easy to see that $g \in \mathcal{F}_2$. Indeed, let $(v, u)$ such that $|v|^2 + |u|^2 = 1$. Then

$$\lim_{(v', u') \to (v, u)} \frac{g(s v', s u')}{s^2} = \lim_{(v', u') \to (v, u)} \frac{f(\sqrt{sv'}, s^2 u'|u|_\infty)}{s^2} = f^\infty(\sqrt{v}, u|u|_\infty).$$

Applying Theorem 10 to $(\frac{\sqrt{|v|}}{\sqrt{|u|}})$ in $L^2$ yields a generalised Young measure $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ such that

$$f(v, u)dxdt = g \left( \frac{\sqrt{|v|}}{\sqrt{|u|}} \frac{u}{|u|_\infty} \right) dxdt \sim (\tilde{\nu}, \tilde{\lambda})$$

where $\tilde{\lambda} = \lambda$ and

$$\int_{\mathbb{R}^d \times \mathcal{S}^d_0} f(\xi, \zeta) d\nu(\xi, \zeta) = \int_{\mathbb{R}^d \times \mathcal{S}^d_0} f(\sqrt{|\xi|} |\zeta|_\infty) d\tilde{\nu}(\xi, \zeta),$$

$$\int_{\mathbb{S}^{d-1}} f^\infty(\xi, \zeta) d\nu^\infty(\xi, \zeta) = \int_{\{|\xi|^2 + |\zeta|^2 = 1\}} f^\infty(\sqrt{|\xi|} |\zeta|_\infty) d\tilde{\nu}^\infty(\xi, \zeta).$$

Let $(\nu, \lambda, \nu^\infty)$ be a Young measure on $\mathbb{R}^d \times \mathcal{S}^d_0$ with parameters in $\mathbb{R}^d \times [0, T]$. Its barycentre $\bar{w} = (\bar{v}, \bar{u})$ is defined by

$$\bar{v}(x, t) := (\nu_{x,t}, \pi_1) \quad \text{(13)}$$

$$\bar{u}(x, t) := (\nu_{x,t}, \pi_2) dxdt + (\nu_{x,t}^\infty, \pi_2) \lambda \quad \text{(14)}$$

for a.e. $x, t$, where $\pi_1$ and $\pi_2$ are the canonical projections from $\mathbb{R}^d \times \mathcal{S}^d_0$ onto $\mathbb{R}^d$ and $\mathcal{S}^d_0$, respectively. Note that $\bar{u}(x, t)$ is only a measure. Such a Young measure is called a measure-valued subsolution if $(\bar{v}, \bar{u})$ is a subsolution in the sense of distributions, i.e. if it satisfies (11) for some distribution $q$. 


2.4.2 Energy and Admissibility

Definition 12. For \((v, u) \in \mathbb{R}^d \times S_0^d\) we define the generalised energy by

\[
e(v, u) := \frac{d}{2} \lambda_{\text{max}}(v \otimes v - u),
\]

where \(\lambda_{\text{max}}\) denotes the largest eigenvalue.

Lemma 13 (Lemma 3.2 in [9]).

a) \(e : \mathbb{R}^d \times S_0^d \to \mathbb{R}\) is convex.

b) For every \((v, u) \in \mathbb{R}^d \times S_0^d\), \(\frac{1}{2} |v|^2 \leq e(v, u)\), with equality if and only if \(u = v \circ v\).

c) For every \((v, u) \in \mathbb{R}^d \times S_0^d\), \(|u|_\infty \leq 2\frac{d-1}{d} e(v, u), |u|_\infty\) being the operator norm of the matrix \(u\).

In particular \(e(v, u) \geq 0\). Observe moreover that \(e \in \mathcal{F}_{2,1}\) with \(e^\infty = e\). Then the energy of a Young measure on \(\mathbb{R}^d \times S_0^d\) is defined by

\[
E(t) = \int_{\mathbb{R}^d} (\nu_{x,t}, e) dx + \int_{\mathbb{R}^d} (\nu_{x,t}^\infty, e) \lambda t(dx). \quad (15)
\]

If for a measure-valued subsolution \(E(t) \leq \frac{1}{2} \int |\bar{v}(x,0)|^2 dx\) for a.e. \(t \geq 0\), we call it an admissible measure-valued subsolution.

2.4.3 Lifting

Finally, we “lift” measure-valued solutions to the space of measure-valued subsolutions, i.e. Young measures from \(\mathbb{R}^d\) to \(\mathbb{R}^d \times S_0^d\). Let \(Q : \mathbb{R}^d \to \mathbb{R}^d \times S_0^d\) be defined by

\[
Q(\xi) = (\xi, \xi \circ \xi).
\]

It is easy to see that

\[
f \in \mathcal{F}_{2,1} \Rightarrow f \circ Q \in \mathcal{F}_{2,1} \quad \text{with} \quad (f \circ Q)^\infty = f^\infty \circ Q.
\]

Given now a Young measure \((\nu, \lambda, \nu^\infty)\) on \(\mathbb{R}^d\), we define a Young measure \((\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)\) on \(\mathbb{R}^d \times S_0^d\) by

\[
(\tilde{\nu}_{x,t}, f) = (\nu_{x,t}, f \circ Q) \quad \text{for} \quad f \in C_0(\mathbb{R}^d \times S_0^d) \text{ for a.e. } (x,t),
\]

\[
(\tilde{\nu}_{x,t}^\infty, g) = (\nu_{x,t}^\infty, g \circ Q) \quad \text{for} \quad g \in C(S^{d-1}) \text{ for } \lambda\text{-a.e. } (x,t).
\]

Then we have

Proposition 14. Let \((\nu, \lambda, \nu^\infty)\) be a measure-valued solution with bounded energy and \((\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)\) be defined as above. Suppose \((v_n, u_n)\) is bounded in \(L^\infty_t(L^2_x \times L^1_x)\) and \((v_n, u_n) \overset{Y_{2,1}}{\to} (\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)\). Then

a) the barycentre \(\bar{v}, \bar{u}\) of \((\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)\) forms an Euler subsolution, i.e. satisfies \((\ref{eq:Euler})\) for some distribution \(\varrho\);
b) if \( \bar{E}(t) \) denotes the energy of the Young measure \((\bar{\nu}, \lambda, \bar{\nu}^\infty)\) in the sense of Equation (15) and \( E(t) \) the energy of \((\nu, \lambda, \nu)\) in the sense of (10), then \( \bar{E}(t) = E(t) \) for a.e. \( t \);

c) \( v_n \to (\nu, \lambda, \nu^\infty) \);

d) \( |u_n - v_n \circ v_n| \to 0 \) in \( L^1_{loc}(\mathbb{R}^d \times [0, T]) \).

**Proof.** a) follows straightforwardly by the definition of \((\bar{\nu}, \lambda, \bar{\nu}^\infty)\) and the fact that \((\nu, \lambda, \nu^\infty)\) is a solution to (4)-(5).

b) By definition of \((\bar{\nu}, \lambda, \bar{\nu}^\infty)\), and applying Lemma 13,
\[
\bar{E}(t) = \int (\bar{\nu}_{x,t}, c) dx + \int (\bar{\nu}_{x,t}, e) \lambda_i(dx) = \int (\nu, c(\xi, \xi \circ \xi)) dx + \int (\nu^\infty, c(\xi, \xi \circ \xi)) \lambda_i(dx)
\]
\[
= \frac{1}{2} \int (\nu, |\xi|^2) dx + \frac{1}{2} \int (\nu^\infty, |\xi|^2) \lambda_i(dx)
\]
\[
= \frac{1}{2} \int (\nu, |\xi|^2) dx + \frac{1}{2} \lambda_i(\mathbb{R}^d) = E(t),
\]
where we used that \( \nu^\infty \) is supported on \( S^{d-1} \).

c) Let \( f \in F_2 \) and define \( g := f \circ \pi_1 \). Then \( g \in F_{2,1} \) with \( g^\infty(\xi, \zeta) = f^\infty(\xi) \).

Therefore
\[
f(v_n) dx dt = g(v_n, u_n) dx dt
\]
\[
\overset{\star}{=} (\bar{\nu}, g) dx dt + (\bar{\nu}^\infty, g^\infty) \lambda
\]
\[
= (\nu, f) dx dt + (\nu^\infty, f^\infty) \lambda
\]
by definition of \((\bar{\nu}, \lambda, \bar{\nu}^\infty)\) and since \( g \circ Q = f \).

d) Note that the function \( f(\xi, \zeta) = |\zeta - \xi \circ \xi| \) belongs to \( F_{2,1} \) with \( f^\infty = f \).
We can thus apply Theorem 11 with \( f \) to obtain
\[
|u_n - v_n \circ v_n| dx dt \overset{\star}{=} (\bar{\nu}, f) dx dt + (\bar{\nu}^\infty, f^\infty) \lambda dt = 0
\]
because the set \( \{(\xi, \xi \circ \xi) : \xi \in \mathbb{R}^d\} \) contains the supports of \( \bar{\nu} \) and \( \bar{\nu}^\infty \), respectively, and on this set, \( f \) and \( f^\infty \) vanish. \( \square \)

3 **Proof of Theorems 1 and 2**

First of all observe that whenever a sequence of weak Euler solutions bounded in \( L^\infty_t L^2_x \) generates a generalised Young measure, then this will be a measure-valued solution with bounded energy in the sense of Definition 8 a). If the generating sequence consists of admissible weak solutions with initial data \( v_0 \), then the measure-valued solution will be admissible as in Definition 8 b). This follows directly from the Fundamental Theorem of Young measures (see also 11, 4) as well as the discussion in Section 2.3.
Before we begin to prove the converse, we state a weaker version of Theorem 2 that we can prove along with Theorem 1. In Section 3.4 we then conclude from this weaker statement the full assertion of Theorem 2.

Proposition 15. Let \((\nu, \lambda, \nu^{\infty})\) be an admissible measure-valued solution with initial data \(v_0 \in L^2(\mathbb{R}^d)\). Then the generating sequence \((v_n)\) as in Theorem 1 may be chosen such that in addition

\[
\|v_n(t = 0) - v_0\|_{L^2(\mathbb{R}^d)} < \frac{1}{n}
\]

and

\[
\sup_{t \in [0, T]} \frac{1}{2} \int_{\mathbb{R}^d} |v_n(x, t)|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 \, dx + \frac{1}{n}.
\]

We prove this Proposition in three steps: In Section 3.1 we use a result of [9] to show that it suffices to generate measure-valued subsolutions by sequences of subsolutions. Section 3.2 adapts various well-known Young measure techniques to our framework to show that it suffices to construct generating sequences for discrete homogeneous oscillation measures. This is rather general and does not use any specific properties of the Euler equations. Finally, in Section 3.3 we show how to generate discrete homogeneous Young measures from subsolutions, where the plane wave analysis of the system (11) is exploited to give an explicit construction of the generating sequence.

3.1 From Subsolutions to Exact Solutions

The goal of this section is to prove

Proposition 16. a) We can generate \((\nu, \lambda, \nu^{\infty})\) as required in Theorem 1 provided we can generate the lifted Young measure (see Subsection 2.4) \((\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^{\infty})\) in the sense of Theorem 11 by a sequence \((v_n, u_n)\) bounded in \(L^\infty_t (L^2_x \times L^1_x)\) with the properties

- \((v_n, u_n)\) are smooth in \(\mathbb{R}^d \times [0, T]\);
- \((v_n, u_n)\) is a subsolution.

b) If \((\nu, \lambda, \nu^{\infty})\) is admissible with initial data \(v_0\) and energy \(E\), then we can generate it as required in Proposition 15 if the sequence \((v_n, u_n)\) additionally satisfies

- \(\limsup_n \sup_t \int e(v_n, u_n) \, dx \leq \text{esssup}_t E(t)\);
- \(v_n(\cdot, 0) \to v_0\) strongly in \(L^2\).

Proof. Suppose now \((v_n, u_n)\) generates the Young measure \((\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^{\infty})\) as in part a) of the proposition. We choose for each \(n\) a function \(\tilde{e}_n \in C\left(\mathbb{R}^d \times (0, T); \mathbb{R}\right) \cap C\left([0, T]; L^1(\mathbb{R}^d; \mathbb{R})\right)\) such that \(\tilde{e}_n > e_n := e(v_n, u_n)\) on \(\mathbb{R}^d \times [0, T]\) and

\[
\sup_t \int_{\mathbb{R}^d} (\tilde{e}_n - e_n) \, dx + \int_0^T \int_{\mathbb{R}^d} (\tilde{e}_n - e_n) \, dx \, dt < \frac{1}{n}.
\]  

(16)
For fixed $n$, let $q_n$ be a pressure field such that $(v_n, u_n, q_n)$ satisfies (11). Consider now vectorfields $v \in C([0, T]; L^2_w)$ that are smooth on $\mathbb{R}^d \times (0, T)$ and for which there exists a smooth matrix field $u \in C^\infty(\mathbb{R}^d \times (0, T))$; $S_0^d$ such that

$$
\begin{align*}
\partial_t v + \text{div} u + \nabla q_n &= 0 \\
\text{div} v &= 0,
\end{align*}
$$

(17)

and

$$
\begin{align*}
v(\cdot, 0) &= v_n(\cdot, 0), \\
v(\cdot, T) &= v_n(\cdot, T),
\end{align*}
$$

(18)

(19)

for a.e. $x \in \mathbb{R}^d$ and all $t \in (0, T)$. We then define the function space $X_n^0$ by

$$
X_n^0 = \{ v \in C^\infty(\mathbb{R}^d \times (0, T)) \cap C([0, T]; L^2_w) : v \text{ satisfies } (17), (18), (19), (20) \}
$$

and we denote the closure of $X_n^0$ under the $C([0, T]; L^2_w)$-topology by $X^n$. Note that $X_n^0$ is non-empty, since $(v_n, u_n) \in X_n^0$. The following result is Proposition 3.3. in [9] (the density of the set of solutions in $X^n$ is not explicitly stated in the Proposition, but is an immediate consequence of its proof):

**Theorem 17.** The set of solutions $v \in X^n$ to the Euler equations with energy density

$$
\frac{1}{2}|v(x, t)|^2 = \bar{e}_n(x, t)
$$

for every $t \in (0, T)$ and a.e. $x$ and pressure

$$
p = q_n - \frac{1}{d}|v|^2
$$

is dense in $X^n$ (w.r.t. the $C([0, T]; L^2_w)$-topology). In particular, there are infinitely many such solutions.

Therefore, for $n \in \mathbb{N}$, we can find a sequence $(v_n^k)_{k \in \mathbb{N}} \subset CL^2_w$ of weak solutions with $v_n^k \rightharpoonup v_n$ as $k \to \infty$ in the $CL^2_w$-topology, i.e.

$$
\sup_{t \in (0, T)} \int (v_n^k - v_n) \cdot \phi \, dx \to 0 \quad \forall \phi \in L^2(\mathbb{R}^d).
$$

Since $v_n \in C^\infty(\mathbb{R}^d \times [0, T])$, we can then choose $k = k(n)$ so large that

$$
\sup_{t \in (0, T)} \left| \int v_n \cdot (v_n^k - v_n) \, dx \right| < \frac{1}{n}.
$$

Next, since $\frac{1}{2}|v_n^k|^2 = \bar{e}_n$ for a.e. $(x, t)$ by Theorem [17] we have

$$
\left| \int \int |v_n^k|^2 - |v_n|^2 \, dx \, dt \right| \leq 2 \int \int (\bar{e}_n - e_n) \, dx \, dt + 2 \int \int (e_n - \frac{1}{2}|v_n|^2) \, dx \, dt,
$$

15
where, by choice of \( e_n \), the first expression is less than \( \frac{1}{n} \). Concerning the last term we have

\[
\left| \int \int (e_n - \frac{1}{2} |v_n^2|) \, dx \, dt \right| = \left| \int \int \frac{1}{2} \lambda_{\max}(v_n \otimes v_n - u_n) - \frac{1}{2} |v_n|^2 \, dx \, dt \right|
\]

\[
= \left| \int \int \frac{1}{2} \lambda_{\max}(v_n \circ v_n - u_n + \frac{1}{d} |v_n|^2 I_d) - \frac{1}{2} |v_n|^2 \, dx \, dt \right|
\]

\[
= \left| \int \int \frac{1}{2} \lambda_{\max}(v_n \circ v_n - u_n) \, dx \, dt \right|
\]

\[
\leq C \int \int |v_n \circ v_n - u_n| \, dx \, dt \to 0
\]

as \( n \to \infty \), by Proposition 14. We also used in this calculation that for a matrix \( A \), \( \lambda_{\max}(A + \alpha I_d) = \lambda_{\max}(A) + \alpha \).

Since

\[
\int \int |v_n^k - v_n|^2 \, dx \, dt = \int \int |v_n^k|^2 - |v_n|^2 \, dx \, dt - 2 \int \int v_n \cdot (v_n^k - v_n) \, dx \, dt,
\]

we deduce that there exists a subsequence \( v_n^{k(n)} \) of Euler solutions such that \( v_n - v_n^{k(n)} \to 0 \) in \( L^2_{loc}(\mathbb{R}^d \times [0,T]) \). Hence by Propositions 13 and 15, this yields that the sequence \( (v_n^{k(n)}) \) generates the Young measure \( (\nu, \lambda, \nu^\infty) \) in \( L^2 \). This proves part a) of Proposition 16.

For part b), recall that in fact \( \frac{1}{2} |v_n^{k(n)}|^2 = \bar{e}_n \) for a.e. \( x \in \mathbb{R}^d \) for all \( t \in (0,T) \). Therefore by Proposition 16 and the assumption about the energy in part b) of Proposition 14 we have

\[
\limsup_n \sup_t \frac{1}{2} \int |v_n^{k(n)}|^2 \, dx \leq \text{esssup}_t E(t) \leq \frac{1}{2} \int |v_0|^2 \, dx.
\]

(21)

Since \( v_n^{k(n)}(\cdot,0) = v_n(\cdot,0) \), we also get

\[
\|v_n^{k(n)}(\cdot,0) - v_0\|_{L^2} = \|v_n(\cdot,0) - v_0\|_{L^2} \to 0
\]

as \( n \to \infty \), which, together with (21), completes the proof of the proposition. \( \Box \)

### 3.2 Approximation of Generalised Young Measures

In this section we reduce the problem of generating an arbitrary measure-valued solution \( (\nu, \lambda, \nu^\infty) \) to generating discrete homogeneous (i.e. independent of \( x \) and \( t \)) oscillation measures, i.e. where \( \lambda = 0 \) and

\[
\nu_{x,t} = \sum_{i=1}^{N} \mu_i \delta(v_{i}, u_i)
\]

(22)

for all \( x, t \), with \( \mu_i > 0 \), \( \sum_{i=1}^{N} \mu_i = 1 \), and \( (v_{i}, u_i) \in \mathbb{R}^d \times S^d_0 \). This reduction is achieved by approximating measure-valued solutions by a coupling of discrete
oscillation measures and smooth subsolutions. The latter represents the macro-
scopic flow whereas the former amounts to the microscopic oscillations encoded
by the Young measure. The general techniques for such an approximation are
well known, see for instance [17].

Given \( k \in \mathbb{N} \) let \( Q^k \) be the collection of open cubes \( Q^k \subset \mathbb{R}^d \times (0,T) \) of
sidelength \( \frac{1}{k} \), whose vertices are neighbouring points on the lattice \( \frac{1}{k} \mathbb{Z}^{d+1} \). Our
precise statement is the following:

**Theorem 18.** Let \( (\nu, \lambda, \nu^{\infty}) \) be a measure-valued subsolution with bounded en-
ergy \( E(t) \). Then there exists a sequence of

- **discrete oscillation measures** \( \nu^{k,\alpha} \) on \( \mathbb{R}^d \times S^d_0 \) with zero barycentres which
  are piecewise constant with respect to \( Q^k \),
- **smooth subsolutions** \( (\bar{v}^k, \bar{u}^k) \) bounded in \( L_1^\infty(L^2_x \times L^1_x) \),

such that

\[
\mathcal{T}(\bar{v}^k, \bar{u}^k) \nu^{k} \overset{Y_{2,1}}{\to} (\nu, \lambda, \nu^{\infty})
\]

and

\[
\int_{\mathbb{R}^d} \langle \mathcal{T}(\bar{v}^k, \bar{u}^k) \nu^{k}, e \rangle dx \leq \text{esssup}_t E(t) + \frac{1}{k} \quad \forall t \in [0,T].
\]

Furthermore, if \( (\nu, \lambda, \nu^{\infty}) \) is an admissible measure valued subsolution with ini-
tial data \( v_0 \), then in addition

\[
\| \bar{v}^k(t=0) - v_0 \|_{L^2(\mathbb{R}^d)} < \frac{1}{k}.
\]

**Proof.** Using Proposition 6 we can reduce the proof of the theorem to a series
of approximation steps.

First of all we show that a homogeneous Young measure can be approximated
by discrete oscillation measures. More precisely, let \( (\nu, \lambda, \nu^{\infty}) \) be homogeneous
in the sense that \( \nu \) and \( \nu^{\infty} \) are independent of \( x,t \) and \( \lambda \) is a constant multiple of
Lebesgue measure \( L^{d+1} \). The approximability is then equivalent to the following

**Claim 1.** Let

\[
\nu \in \mathcal{M}^1(\mathbb{R}^d \times S^d_0), \quad \alpha \in [0, \infty), \quad \nu^{\infty} \in \mathcal{M}^1(S^{d-1})
\]
such that \( (\nu, e) < \infty \), where \( e \) is the generalised energy from Definition 12 and
assume that

\[
\{\nu, \pi_1\} = 0, \quad \langle \nu, \pi_2 \rangle + \alpha \langle \nu^{\infty}, \pi_2 \rangle = 0.
\]

We claim that there exists a sequence \( \nu^k \in \mathcal{M}^1(\mathbb{R}^d \times S^d_0) \) of \textit{discrete} probability
measures of the form \( \nu^k \) with zero barycentre, such that

\[
\nu^k(t=0) \to \nu(t), \quad \nu^k \to \nu^{\infty} + \alpha \nu^{\infty} \quad \text{for all } f \in \mathcal{F}_{2,1}.
\]

**Step 1. From classical to generalised measures.**
Let us assume that $\nu, \nu^\infty$ are discrete probability measures, i.e.

$$\nu = \sum_{i=1}^{N} \mu_i \delta_{(v_i, u_i)}, \quad \nu^\infty = \sum_{i=1}^{M} \tau_i \delta_{(v_i^\infty, u_i^\infty)}$$

with $(v_i, u_i) \in \mathbb{R}^d \times S_0^d$, $(v_i^\infty, u_i^\infty) \in S^{d-1}$, such that

$$\sum_{i=1}^{N} \mu_i v_i = 0, \quad \sum_{i=1}^{N} \mu_i u_i + \alpha \sum_{j=1}^{M} \tau_j u_j^\infty = 0.$$ 

Define a sequence $(\nu^m)$ of probability measures by

$$\nu^m = \left(1 - \frac{1}{m}\right) \sum_{i=1}^{N} \mu_i \delta_{(v_i, u_i)} + \frac{1}{m} \sum_{j=1}^{M} \tau_j \delta_{(\sqrt{am} v_j^\infty, am u_j^\infty)},$$

A direct calculation shows that the barycentre $(\bar{v}^m, \bar{u}^m)$ of $\nu^m$ satisfies $(\bar{v}^m, \bar{u}^m) \to 0$ as $m \to \infty$. Moreover, for any $f \in \mathcal{F}_{2,1}$

$$\langle \nu^m, f \rangle = \left(1 - \frac{1}{m}\right) \sum_{i=1}^{N} \mu_i f(v_i, u_i) + \frac{1}{m} \sum_{j=1}^{M} \tau_j f(\sqrt{am} v_j^\infty, am u_j^\infty)$$

$$\xrightarrow{m \to \infty} \sum_{i=1}^{N} \mu_i f(v_i, u_i) + \alpha \sum_{j=1}^{M} \tau_j f^\infty(v_j^\infty, u_j^\infty) = \langle \nu, f \rangle + \alpha \langle \nu^\infty, f^\infty \rangle.$$ 

Therefore also the shifted measure $\mathcal{T}_{(\bar{v}^m, \bar{u}^m)} \nu^m$ satisfies

$$\langle \mathcal{T}_{(\bar{v}^m, \bar{u}^m)} \nu^m, f \rangle \to \langle \nu, f \rangle + \alpha \langle \nu^\infty, f^\infty \rangle \quad \text{for all } f \in \mathcal{F}_{2,1}. \quad (28)$$

**Step 2. From discrete to general measures with compact support**

More generally, assume that $\nu, \nu^\infty$ are probability measures with compact support. By standard measure theory (see e.g. [3], §30), we find sequences of discrete measures $\nu^k$ with uniformly compact support and $\nu^{k,\infty}$ such that

$$\nu^k \Rightarrow^* \nu \text{ in } \mathcal{M}(\mathbb{R}^d \times S_0^d), \quad \nu^{k,\infty} \Rightarrow^* \nu^\infty \text{ in } \mathcal{M}(S).$$

It follows easily that

$$\langle \nu^k, f \rangle + \alpha \langle \nu^{k,\infty}, f^\infty \rangle \to \langle \nu, f \rangle + \alpha \langle \nu^\infty, f^\infty \rangle \quad \text{for all } f \in \mathcal{F}_{2,1}. \quad (29)$$

**Step 3. From compact support to finite energy**

First of all note that the assumption $\langle \nu, e \rangle < \infty$ implies $\langle \nu, f \rangle < \infty$ for any $f \in \mathcal{F}_{2,1}$. Indeed, any $f \in \mathcal{F}_{2,1}$ satisfies a bound of the form $|f(\xi, \zeta)| \leq C(|\xi|^2 + |\zeta|)$, therefore by Lemma [18] we have $|f| \leq C' e$. Using an idea from [18], we may then approximate $\nu$ by compactly supported measures in the following way:
For $\rho \in \mathbb{N}$, let $r^\rho : \mathbb{R}^d \times S^d_0 \to \mathbb{R}$ be a smooth function which is 1 on $B_\rho$, zero on $(\mathbb{R}^d \times S^d_0) \setminus B_{\rho + 1}$ and $0 \leq r \leq 1$ everywhere. Define also a number $s^\rho$ by

$$s^\rho = \langle \nu, 1 - r^\rho \rangle,$$

which measures how much mass $\nu$ carries outside of $B_\rho(0)$. We then define $\nu^\rho := r^\rho \nu + s^\rho \delta_0$, which is a probability measure with support in $B_{\rho + 1}$.

In order to keep the condition (26), let $\bar{v}^\rho = \langle \nu^\rho, \pi_1 \rangle$ and $\bar{u}^\rho = \langle \nu^\rho, \pi_2 \rangle$ and consider the shifted measure $T_{(\bar{v}, \bar{u})} \nu^\rho$. Using (30) we see that $(\bar{v}^\rho, \bar{u}^\rho) \to 0$ as $\rho \to \infty$, hence

$$\langle T_{(\bar{v}, \bar{u})} \nu^\rho, f \rangle \to \langle \nu, f \rangle$$

for all $f \in F_{2,1}$. (31)

Claim 1 then follows by choosing a diagonal sequence in the three approximations (28), (29), (31).

Next, we show how to discretize a measure-valued subsolution so that Claim 1 can be applied to each homogeneous part separately.

**Claim 2.** Let $(\nu, \lambda, \nu^\infty)$ be a measure-valued subsolution with bounded energy and barycentre $(\bar{v}, \bar{u})$. Then there exists a sequence of smooth subsolutions $(\bar{v}^k, \bar{u}^k)$ and a sequence of generalised Young measures $(\nu^k, \lambda^k, \nu^k, \nu^\infty)$ with zero barycentre which are piecewise constant with respect to $Q^k$, such that

$$\langle T_{(\bar{v}^\rho, \bar{u}^\rho)} \nu^\rho, f \rangle \to \langle \nu, f \rangle$$

for all $f \in F_{2,1}$. (31)

Moreover, if $(\nu, \lambda, \nu^\infty)$ is an admissible measure-valued subsolution with initial data $v_0$, then in addition

$$\| \bar{v}^k(t = 0) - v_0 \|_{L^2(\mathbb{R}^d)} < \frac{1}{k}.$$ (34)

**Step 4. Regularizing.**
Let \( \psi : \mathbb{R}^d \to \mathbb{R} \) be a standard mollification kernel, that is, smooth and non-negative, supported on \( B_1(0) \), and \( \int \psi dx = 1 \). Let furthermore \( \chi : \mathbb{R} \to \mathbb{R} \) be another mollification kernel with the same properties as \( \psi \), but whose support is required to be contained in \((-1,0)\). Define now \( \psi_\epsilon(x) = \frac{1}{\epsilon^d} \psi \left( \frac{x}{\epsilon} \right) \) and \( \chi_\epsilon(t) = \frac{1}{\epsilon} \chi \left( \frac{t}{\epsilon} \right) \), so that the mass is still 1 and the supports are in \( B_\epsilon(0) \) and \((-\epsilon,0)\) respectively. Set \( \phi_\epsilon(x,t) = \psi_\epsilon(x) \chi_\epsilon(t) \). We can now define for every \( t \in [0,T-\varepsilon] \) and \( x \in \mathbb{R}^d \) another Young measure \( (\nu^\varepsilon, \lambda^\varepsilon, \nu^{\varepsilon,\infty}) \) by
\[
\langle \nu^\varepsilon, f \rangle \equiv \langle \nu, f \rangle \ast \phi_\epsilon \quad \text{for all } f \in C_0(\mathbb{R}^d \times S^d_0),
\]
\[
\lambda^\varepsilon = \lambda \ast \phi_\epsilon,
\]
\[
\langle \nu^{\varepsilon,\infty}, g \rangle = \left( \frac{\langle \nu^\infty, g \rangle \lambda^\varepsilon + \phi_\epsilon}{\lambda^\varepsilon} \right) \quad \text{for all } g \in \mathcal{C}(S).
\]

Observe that \( \|\nu^\infty, g\| \leq \sup|g| \), so that with the above definition \( \langle \nu^{\varepsilon,\infty}, g \rangle \in L^\infty(\mathbb{R}^d \times [0,T-\varepsilon]) \), whereas \( \langle \nu^\varepsilon, f \rangle, \lambda^\varepsilon \in C^\infty(\mathbb{R}^d \times [0,T-\varepsilon]) \). Also, the mollified Young measure is only defined for \( t \in [0,T-\varepsilon] \). However, a simple rescaling of time \( t \mapsto \frac{t}{\varepsilon} \) can then restore the original domain \( t \in [0,T] \). Therefore, we may as well assume that \( (\nu^\varepsilon, \lambda^\varepsilon, \nu^{\varepsilon,\infty}) \) is defined for \( (x,t) \in \mathbb{R}^d \times [0,T] \). Moreover, we have
\[
(\nu^\varepsilon, \lambda^\varepsilon, \nu^{\varepsilon,\infty}) \overset{Y_2.1}{\longrightarrow} (\nu, \lambda, \nu^\infty) \quad \text{as } \varepsilon \to 0,
\]
(35)
since \( \mu \ast \phi_\epsilon \overset{*}{\rightharpoonup} \mu \) in \( \mathcal{M} \) as \( \varepsilon \to 0 \) for any \( \mu \). Moreover, letting
\[
E_\varepsilon(t) = \int_{\mathbb{R}^d} \langle \nu^\varepsilon, e \rangle \, dx + \int_{\mathbb{R}^d} \langle \nu^{\varepsilon,\infty}, e \rangle \lambda^\varepsilon \, (dx) \quad \text{for } t \in [0,T],
\]
we easily see that
\[
E_\varepsilon(t) = E \ast \chi_\epsilon(t) \quad \text{for all } t \in [0,T].
\]
In particular for every \( t \in [0,T] \) we have
\[
E_\varepsilon(t) = \int E(t-s) \chi_\epsilon(s) \, ds \leq \text{esssup}_t E(t) \int \chi = \text{esssup}_t E(t).
\]
(36)
For the barycentre \( (\bar{v}_\varepsilon, \bar{u}_\varepsilon) \) of this measure we have
\[
\bar{v}_\varepsilon = \bar{v} \ast \phi_\epsilon, \quad \bar{u}_\varepsilon = \bar{u} \ast \phi_\epsilon,
\]
(37)
so the barycentre is smooth and, by linearity, is a subsolution.

Finally, assume that \( (\nu, \lambda, \nu^\infty) \) is an admissible measure-valued subsolution with initial data \( v_0 \). We claim that in this case
\[
\bar{v}_\varepsilon(0) \to \bar{v}(0) \quad \text{in } L^2(\mathbb{R}^d),
\]
(38)
where we write \( \bar{v}(t) := x \mapsto \bar{v}(x,t) \). Indeed, we have
\[
\bar{v}_\varepsilon(x,0) = \int_0^x \int_{\mathbb{R}^d} \bar{v}(x-y,s) \psi_\varepsilon(y) \, dy \chi_\varepsilon(-s) \, ds
\]
\[
= \int_0^x [\bar{v}(s) \ast \psi_\varepsilon] \chi_\varepsilon(-s) \, ds,
\]
20
To this end define \( R_l \) Proposition 7 in [17]. For

we use the well-known technique of averaging, see also Lemma 4.22 in [20] and

Concerning the energy, we claim that

for all \((\epsilon, x, t)\) and any \(g\) and any \(\nu \rightarrow 0\) as \(l \rightarrow \infty\) by Proposition 9. This proves our claim [38].

**Step 5. Averaging.**

Next, fix \( \epsilon > 0 \) and consider the shifted regular Young measure

with barycentre zero, together with the “macroscopic” state

We use the well-known technique of averaging, see also Lemma 4.22 in [20] and Proposition 7 in [17]. For \( l \in \mathbb{N} \) let \( Q^l = \{ Q_i^l \} \) be the collection of open cubes in \( \mathbb{R}^d \times (0, T) \) of sidelength \( \frac{1}{l} \) with vertices on the lattice \( \frac{1}{l} \mathbb{Z}^d \). We define \((\tilde{\nu}^l, \tilde{\lambda}^l, \tilde{\nu}^{\infty, l})\) by

for all \((x, t) \in Q_i^l\) for every \(i\), where \( \int_{Q_i^l} g d\mu := \frac{1}{\mu(Q_i^l)} \int_{Q_i^l} g d\mu \) for any measure \(\mu\) and any \(g \in L^1(Q_i^l; \mu)\).

Then \((\tilde{\nu}^l, \tilde{\lambda}^l, \tilde{\nu}^{\infty, l})\) is homogeneous on each \( Q_i^l \), and also has zero barycentre for a.e. \((x, t)\). Moreover, it follows from Proposition 8 in [17] that

Concerning the energy, we claim that

To this end define

\[
(\tilde{\nu}^l, \tilde{\lambda}^l, \tilde{\nu}^{\infty, l}) \xrightarrow{Y_2, 1} (\hat{\nu}, \hat{\lambda}, \hat{\nu}^{\infty}) \quad \text{as } l \to \infty. \tag{39}
\]
and for a.e. $t \in (0, T)$:

$$F_0(t) = \int_{\mathbb{R}^d} \langle T(\hat{v}, \hat{\nu}) \hat{v}, e \rangle \, dx, \quad F_1(t) = \int_{\mathbb{R}^d} \langle T(\hat{v'}, \hat{\nu'}) \hat{v'}, e \rangle \, dx,$$

$$F_2(t) = \int_{\mathbb{R}^d} \langle T(\hat{v'}, \hat{\nu'}) \hat{v'}, e \rangle \, dx, \quad F_3(t) = \int_{\mathbb{R}^d} \langle T(\hat{\nu}, \hat{\nu}) \hat{\nu}, e \rangle \, dx.$$

By the definition of $\hat{v}'$ and since $(\hat{v'}, \hat{u}')$ is constant on $Q^l_1$, we have $\langle T(\hat{v'}, \hat{\nu'}) \hat{v}', f \rangle = \int_Q \langle T(\hat{v'}, \hat{\nu'}) \hat{v'}, f \rangle \, dx \, dt$. Hence

$$\sup_t F_2(t) \leq \sup_t F_1(t).$$

On the other hand $e$ satisfies the pointwise estimate

$$|e(\xi_1, \zeta_1) - e(\xi_2, \zeta_2)| \leq C(|\xi_1|, |\xi_2| + |\xi_1 - \xi_2|^2 + |\zeta_1 - \zeta_2|), \quad (41)$$

from which we obtain for a.e. $t$

$$|F_0(t) - F_1(t)| \leq C \left( \int_{\mathbb{R}^d} \langle \hat{v}, |\xi|^2 \rangle + |\hat{v}|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} |\hat{v}' - \hat{u}'|^2 \, dx \right)^{1/2} + \int_{\mathbb{R}^d} |\hat{v}' - \hat{v}|^2 + |\hat{u}' - \hat{u}| \, dx.$$ 

Since $(\hat{v}, \hat{u}) \in L^\infty_{v, u}(L^2_v \times L^2_u) \cap C^\infty(\mathbb{R}^d \times (0, T))$, the right hand side converges to zero as $l \to \infty$, uniformly in $t$. Similarly $\sup_t |F_2(t) - F_3(t)|$ can be made arbitrarily small. Combining these estimates yields

$$\sup_t F_3(t) \leq \sup_t F_0(t) + o(1) \quad \text{as } l \to \infty$$

as claimed in $\text{(40)}$.

Combining the approximations $\text{34-37}$ and $\text{39}$ leads to the approximation as claimed in $\text{32}$. The bound on the energy $\text{33}$ follows from $\text{36}$ and $\text{40}$, and $\text{34}$ follows from $\text{35}$. This proves Claim 2.

**Step 6. Conclusion of the argument.**

Given a measure-valued subsolution $(\nu, \lambda, \nu^\infty)$, by Claim 2 there exists a sequence of generalised Young measures $(\nu^k, \lambda^k, \nu^{\infty, k})$ which are piecewise constant with respect to $Q^k$, and there exist smooth subsolutions $(\hat{v}^k, \hat{u}^k)$ such that

$$(\langle T(\hat{v}^k, \hat{u}^k) \nu^k, \lambda^k, \nu^{\infty, k} \rangle) \xrightarrow{Y_{2,1}} (\nu, \lambda, \nu^\infty) \quad \text{as } k \to \infty$$

and $\text{33}$ holds. Then, using Claim 1 we can approximate the homogeneous Young measure on each cube $Q^l_k \in Q^k$ separately by a discrete oscillation measure as in $\text{27}$. In this way we obtain a sequence of piecewise constant discrete oscillation measures $\nu^{k,l}$ such that

$$\nu^{k,l} \xrightarrow{Y_{2,1}} (\nu^k, \lambda^k, \nu^{k,\infty}) \quad \text{as } l \to \infty.$$
In particular we have for any fixed \( k \in \mathbb{N} \) and \( j = 0, 1, \ldots \)

\[
\int_{\mathbb{T}} \int_{\mathbb{T}} \langle T(\bar{v}^k, \bar{u}^k) \nu^{k,l}, e \rangle dx dt \to \int_{\mathbb{T}} \int_{\mathbb{T}} \langle T(\bar{v}^k, \bar{u}^k) \nu^k, e \rangle dx dt + \int_{\mathbb{T}} \langle \nu^{k,\infty}, e \rangle \lambda_t(dx) dt
\]

as \( l \to \infty \), and since \( \nu^{k,l} \) and \( (\nu^k, \lambda^k, \nu^{k,\infty}) \) are \( t \)-independent in the time interval \( (\frac{j}{k}, \frac{j+1}{k}) \) and \( (\bar{v}^k, \bar{u}^k) \) is smooth, we conclude (analogously to (40) and after passing to a subsequence) that for a.e. \( t \in (0,T) \)

\[
\int_{\mathbb{T}} \langle T(\bar{v}^k, \bar{u}^k) \nu^{k,l}, e \rangle dx \leq \text{esssup}_t \int_{\mathbb{T}} \langle T(\bar{v}^k, \bar{u}^k) \nu^k, e \rangle dx + \int_{\mathbb{T}} \langle \nu^{k,\infty}, e \rangle \lambda_t(dx) + \frac{1}{k}
\]

for all \( k \) and \( l = l(k) \) large enough. Together with (44) this implies (24). If the measure-valued solution is admissible, we have (34), from which (25) follows. This concludes the proof.

### 3.3 Discrete Homogeneous Young Measures

Let \( Q = (0,1)^{d+1} \). In light of Theorem 18 it remains to show the following:

**Proposition 19.** Let

\[
\nu = \sum_{i=1}^{N} \mu_i \delta_{w_i}
\]

be a probability measure on \( \mathbb{R}^d \times S^d_0 \) with zero barycentre. Then there exists a sequence \( w^k = (v^k, u^k) \in C^\infty_c(Q; \mathbb{R}^d \times S^d_0) \) of smooth subsolutions such that

\[
\|w^k\|_{L^\infty(Q)} \leq C(\nu, e)
\]

for some fixed constant \( C \), for any \( f \in C(\mathbb{R}^d \times S^d_0) \)

\[
f(w^k) \rightharpoonup (\nu, f) \quad \text{in} \quad L^\infty(\mathbb{R}^d \times (0,T))
\]

and moreover if \( f \) is convex, then

\[
\limsup_{k \to \infty} \sup_{t \in [0,1]} \int_{[0,1]^d} f(w^k(x,t))dx \leq \langle \nu, f \rangle.
\]

Concerning the above proposition we remark that (43) is the classical Young measure convergence for bounded sequences. In particular it follows that

\[
\int_Q f(w^k(x,t))dx dt \to \langle \nu, f \rangle.
\]

The crucial point about estimate (45) is that it is uniform in \( t \).

In order to construct generating sequences consisting of subsolutions, we use the localized plane-wave construction developed in [5].
**Proposition 20.** Let \( \bar{w} = (\bar{v}, \bar{u}) \in \mathbb{R}^d \times S_d^0 \) with \( \bar{v} \neq 0 \).

1. There exists \( \eta \in S_d \setminus \{e_{d+1}\} \) such that
   \[
   w(x,t) := \bar{w} h((x,t) \cdot \eta)
   \]
   is a subsolution for any profile \( h \in C^\infty(\mathbb{R}) \).

2. There exists a second order homogeneous linear operator \( L_{\bar{w}} \) such that
   \[
   w := L_{\bar{w}}[\varphi]
   \]
   is a solution of \( 11 \) for any \( \varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}) \).

3. Moreover, if \( \varphi(x,t) = H((x,t) \cdot \eta) \) for \( H \in C^\infty(\mathbb{R}) \), then
   \[
   L_{\bar{w}}[\varphi](x,t) = \bar{w} H''((x,t) \cdot \eta)
   \]

**Proof.** Recall that we defined subsolutions as pairs \((v,u)\) such that there exists a distribution \( q \) so that \( 11 \) holds.

Therefore statement 1. of the proposition follows immediately from the plane-wave analysis of the Euler equations performed in Section 2 of [8] and Remark 1 therein. Statements 2. and 3. follow directly from Lemma 3.2 and Lemma 3.3 in [8] using the identification

\[
(v, u, q) \mapsto U = \begin{pmatrix} u + qI_d & v \\ v & 0 \end{pmatrix}.
\]

\(\square\)

For the proof of Proposition 19 we start with the following sharpened variant of Proposition 2.2 from [8]:

**Proposition 21.** Let \( \mu_i \geq 0 \) and \( w_i = (v_i, u_i) \in \mathbb{R}^d \times S_d^0 \) for \( i = 1, 2 \) with \( \mu_1 + \mu_2 = 1 \) and \( \mu_1 w_1 + \mu_2 w_2 = 0 \). Assume also that \( v_1 \neq v_2 \). Moreover, let \( f \in C(\mathbb{R}^d \times S_d^0) \) be convex. For any \( \epsilon > 0 \) there exists

\[
w \in C^\infty_c(Q; \mathbb{R}^d \times S_d^0)
\]

such that

(i) \( w = (v,u) \) is a subsolution;

(ii) \( \|w\|_{L^\infty(Q)} \leq \max_i |w_i| + \epsilon \);

(iii) There exist open subsets \( A_1, A_2 \subset Q \) such that for \( i = 1, 2 \)

\[
w(x,t) = w_i \text{ for } (x,t) \in A_i, \quad \text{and} \quad |\mathcal{L}^{d+1}(A_i) - \mu_i| < \epsilon;
\]

24
(iv) For the convex function \( f \) we have
\[
\int_{[0,1]^d} f(w(x,t)) \, dx \leq \mu_1 f(w_1) + \mu_2 f(w_2) + \epsilon \quad \text{for all } t \in [0,1].
\]

Furthermore, concerning property (iii) we even have for \( i = 1, 2 \)
\[
|L^d \{ x \in [0,1]^d : (x,t) \in A_i \} - \mu_i | < \epsilon \quad \text{for all } t \in (\epsilon, 1 - \epsilon).
\]

Proof. Fix \( \delta > 0 \) small. Let \( h : \mathbb{R} \to \mathbb{R} \) be the 1-periodic extension of
\[
h(s) = \begin{cases} 
\mu_1 & \text{if } s \in [0,\mu_2), \\
-\mu_2 & \text{if } s \in [\mu_2, 1) 
\end{cases}
\]
and let \( h_\delta = h \ast \zeta_\delta \), where \( \zeta \in C_\infty([-1,1]) \) is a standard mollifying kernel. Since \( h_\delta \) is 1-periodic with mean zero, there exists \( H_\delta \in L^\infty(\mathbb{R}) \cap C_\infty(\mathbb{R}) \) such that \( H''_\delta = h_\delta \).

Let \( \bar{w} = w_2 - w_1 \) and consider the wave direction \( \eta \in S^d \setminus \{ e_{d+1} \} \) as well as the operator \( L_{\bar{w}} \) obtained from Proposition 20. For any \( k \in \mathbb{N} \) set
\[
\varphi^k(x,t) := \frac{1}{k^2} H_\delta(k(x,t) \cdot \eta).
\]

Next, let \( \phi \in C_\infty(Q) \) be a cutoff function, i.e. such that
\[
0 \leq \phi \leq 1 \quad \text{in } Q \quad \text{and } \phi = 1 \quad \text{in } [\delta,1-\delta]^{d+1},
\]
and set \( w^k := L_{\bar{w}}[\phi \varphi^k] \). Then
\[
w^k \in C_\infty(Q; \mathbb{R}^d \times S^d) \quad \text{and } (\phi^k, u^k) = w^k \text{ is a subsolution.}
\]

Furthermore, let
\[
W^k(x,t) := \bar{w} \, h(k(x,t) \cdot \eta).
\]

Since \( L_{\bar{w}} \) is a homogeneous second order differential operator, \( L_{\bar{w}}[\phi \varphi^k] - \phi L_{\bar{w}}[\varphi^k] \) can be written as a sum of products of first order derivatives of \( \phi \) with first order derivatives of \( \varphi^k \) and of second derivatives of \( \phi \) with \( \varphi^k \). Therefore
\[
\|w^k - \phi L_{\bar{w}}[\varphi^k]\|_{L^\infty(Q)} \leq \frac{C(\delta)}{k}.
\]

Also, by Proposition 20 we have \( L_{\bar{w}}[\varphi^k](x,t) = \bar{w} \, h_\delta(k(x,t) \cdot \eta) \), whence we conclude using the definition of \( h \) and the form of \( L_{\bar{w}}[\varphi^k] \) that
\[
\|w^k\|_{L^\infty(Q)} \leq \max_i |w_i| + \frac{C(\delta)}{k},
\]

Moreover, since \( \phi = 1 \) on \([\delta,1-\delta]^{d+1}\), we actually have
\[
w^k(x,t) = \bar{w} \, h_\delta(k(x,t) \cdot \eta) \quad \text{in } (\delta,1-\delta)^{d+1}.
\]
Since \( h = h_S \) on \((\delta, \mu_2 - \delta)\) as well as on \((\mu_2 + \delta, 1 - \delta)\), it follows that \( w^k = w_i \) on \( A^k_i \), defined by

\[
A^k_1 = \{(x, t) \in (\delta, 1 - \delta)^d : k(x \cdot \eta) \in (\delta, \mu_2 - \delta) + \mathbb{Z}\}
\]
\[
A^k_2 = \{(x, t) \in (\delta, 1 - \delta)^d : k(x \cdot \eta) \in (\mu_2 + \delta, 1 - \delta) + \mathbb{Z}\}
\]

and furthermore

\[
\left| \mathcal{L}^d \{ x \in [0, 1]^d : w^k(x, t) = w_i \} - \mu_i \right| \leq C \delta \quad \text{for } i = 1, 2
\]

for any \( t \in (\delta, 1 - \delta) \), where the constant is independent of \( t \) and \( \delta \).

Next, let us write \( \eta = (\eta', \eta_{d+1}) \) for \( \eta' \in \mathbb{R}^d \) and observe that by Proposition 20 and the assumption \( v_1 \neq v_2 \) we have \( \eta' \neq 0 \). Fix \( t \in [0, 1] \) and let

\[
a = \frac{\eta_{d+1}}{\| \eta \|^2}, \quad x' = x + at
\]

so that \((x, t) \cdot \eta = x' \cdot \eta'\). We can then write for any \( f \in C(\mathbb{R}^d \times S_0^d) \)

\[
\int_{[0, 1]^d} f(\phi(x, t)W^k(x, t)) \, dx = \int_{\mathbb{R}^d} f(\phi(x, t)W^k(x, t)) \, dx
\]
\[
= \int_{\mathbb{R}^d} f(\bar{\omega} \phi(x' - at, t)h(kx' \cdot \eta')) \, dx'
\]
\[
= \int_{\mathbb{R}^d} f(\bar{\omega} \phi_t(x')h(kx' \cdot \eta')) \, dx',
\]

where we have written \( \phi_t(x') = \phi(x' - at, t) \). Now, standard Young measure theory implies that for any fixed \( t \)

\[
\int_{\mathbb{R}^d} f(\bar{\omega} \phi_t(x')h(kx' \cdot \eta')) \, dx' \xrightarrow{k \to \infty} \int_{\mathbb{R}^d} \left[ \mu_1 f(w_1 \phi_t(x')) + \mu_2 f(w_2 \phi_t(x')) \right] \, dx'.
\]

Moreover, since the family \( \{ \phi_t \}_{t \in [0, 1]} \) is equicontinuous with \( \| \phi_t - \phi_{t'} \|_{L^\infty(\mathbb{R}^d)} \leq C|t - t'| \) for some fixed constant \( C \), the convergence above in fact holds uniformly in \( t \in [0, 1] \). Furthermore, if \( f \) is convex, then, since \( |\phi_t(x')| \leq 1 \) for all \( x' \),

\[
\int_{\mathbb{R}^d} \left[ \mu_1 f(w_1 \phi_t(x')) + \mu_2 f(w_2 \phi_t(x')) \right] \, dx' \leq \int_{\mathbb{R}^d} \phi_t(x') \left[ \mu_1 f(w_1) + \mu_2 f(w_2) \right] \, dx'
\]
\[
= \int_{[0, 1]^d} \phi(x, t) \left[ \mu_1 f(w_1) + \mu_2 f(w_2) \right] \, dx
\]
\[
\leq \mu_1 f(w_1) + \mu_2 f(w_2).
\]

Since any continuous convex function is locally Lipschitz, the \( L^\infty \) bound together with the uniform \( L^\infty \)-boundedness of \( \phi W^k \) and the fact that \( w^k = \phi W^k \) on \( A^k_1 \cup A^k_2 \) implies that, by choosing first \( \delta > 0 \) sufficiently small and then \( k \) sufficiently large, we can ensure that

\[
\left| \int_{[0, 1]^d} f(w^k(x, t)) \, dx - \int_{[0, 1]^d} f(\phi(x, t)W^k(x, t)) \, dx \right| \leq \epsilon/2 \quad \text{for all } t \in [0, 1].
\]
Consequently, for \( k \) sufficiently large we obtain
\[
\int_{[0,1]^d} f(w^k(x,t)) \, dx \leq \mu_1 f(w_1) + \mu_2 f(w_2) + \epsilon.
\]
This concludes the proof.

**Proof of Proposition 19.** Let
\[
\sum_{i=1}^{N} \mu_i \delta_{w_i}
\]
with \( w_i = (v_i, u_i) \in \mathbb{R}^d \times S_0^d \) be a probability measure with barycentre zero, and such that
\[
\text{span}(v_1, \ldots, v_N) \text{ has maximal rank (48)}
\]
such that the constraint \( \sum_i \mu_i v_i = 0 \). We prove by induction on \( N \) that for any \( \epsilon > 0 \) and a given convex function \( f \in C(\mathbb{R}^d \times S_0^d) \) there exists a smooth subsolution \( w = (v, u) \in C^\infty(Q; \mathbb{R}^d \times S_0^d) \) such that
\[
\begin{align*}
(i) & \quad w = (v, u) \text{ is a subsolution;} \\
(ii) & \quad \|w\|_{L^\infty(Q)} \leq \max_{i} |w_i| + \epsilon; \\
(iii) & \quad \text{There exist open subsets } A_i \subset Q \text{ for } i = 1, \ldots, N \text{ such that} \\
& \quad w(x,t) = w_i \text{ for } (x,t) \in A_i, \quad \text{and} \quad |\mathcal{L}^{d+1}(A_i) - \mu_i| < \epsilon; \\
(iv) & \quad \int_{[0,1]^d} f(w(x,t)) \, dx \leq \sum_{i=1}^{N} \mu_i f(w_i) + \epsilon \quad \text{for all } t \in [0,1].
\end{align*}
\]
Proposition 19 then follows easily. Indeed, the non-degeneracy assumption (48) can be achieved by perturbing \( v_i \) slightly, and then (ii),(iii),(iv) implies (43), (44) and (45), respectively. Observe in particular that we can find a fixed sequence \( w^k \) as required in Proposition 19 so that (45) holds for all convex \( f \); indeed, this can be obtained by a diagonal argument and the observation that it suffices to show (45) for a countable set of convex functions.

The case \( N = 1 \) is trivial, since then \( \nu = \delta_0 \) and we can simply take \( w \equiv 0 \).

**Induction step.** Let \( \nu = \sum_{i=1}^{N+1} \mu_i \delta_{w_i} \) for \( N \geq 2 \). Using condition (48) and by a reordering of the vectors \( v_1 \ldots v_{N+1} \) if necessary, we may assume without loss of generality that
\[
v_{N+1} \neq 0 \text{ and } (v_1, \ldots, v_N) \text{ satisfies (48)}.
\]
Define the probability measures
\[ \nu_1 := \mu_{N+1} \delta_{w_{N+1}} + (1 - \mu_{N+1}) \delta_{\bar{w}} \]
\[ \nu_2 := \sum_{i=1}^{N} \frac{\mu_i}{1 - \mu_{N+1}} \delta_{w_i}, \]
where
\[ \bar{w} := \sum_{i=1}^{N} \frac{\mu_i}{1 - \mu_{N+1}} w_i. \]

Observe that both \( \nu_1 \) and \( \nu_2 \) have zero barycentre. Moreover, by a direct calculation we check that \( \nu_{N+1} \neq \bar{w} \). Therefore \( \nu_1 \) satisfies the assumptions of Proposition 21 and \( \nu_2 \) satisfies the induction hypothesis. Therefore, given \( \epsilon > 0 \) we obtain two subsolutions
\[ W_1, W_2 \in C_c^\infty(Q; \mathbb{R}^d \times S_0^d) \]
satisfying properties (i)-(iv) with respect to the measures \( \nu_1, \nu_2 \), and moreover \( W_1 \) satisfies the time-slice estimate (46).

Let \( A, B \subset Q \) be the open sets from property (iii) for \( W_1 \), with \( B \) corresponding to the value \( \bar{w} \), i.e. such that
\[ W_1(x,t) = \bar{w} \text{ for } (x,t) \in B, \quad \text{and} \quad |\mathcal{L}^{d+1}(B) - (1 - \mu_{N+1})| < \epsilon, \quad (49) \]
and let
\[ B_\epsilon = \{(x,t) \in B : \epsilon < t < 1 - \epsilon\}. \]

Fix a finite family of disjoint cubes
\[ \tilde{B} = \bigcup_{j=1}^{M} ((x_j, t_j) + \alpha_j Q), \quad (x_j, t_j) \in Q, \quad \alpha_j > 0, \]
such that
\[ \tilde{B} \subset B_\epsilon, \quad |\mathcal{L}^{d+1}(B_\epsilon \setminus \tilde{B})| < \epsilon, \quad (50) \]
and set
\[ w(x,t) = W_1(x,t) + \sum_{j=1}^{M} W_2\left(\frac{x - x_j}{\alpha_j}, \frac{t - t_j}{\alpha_j}\right). \]

We claim that \( w \) satisfies (i)-(iv) for the measure \( \nu \).

To start with, it is easy to see by linearity that \( w \in C_c^\infty(Q; \mathbb{R}^d \times S_0^d) \) and (i),(ii) hold. Concerning (iii), let \( A_{N+1} = A \) and \( A_i \subset Q \) for \( i = 1, \ldots, N \) be defined by
\[ A_i = \bigcup_{j=1}^{M} ((x_j, t_j) + \alpha_j \tilde{A}_i), \]
where \( \tilde{A}_i \) are the open sets from property (iii) for \( W_2 \). Then
\[ w(x,t) = w_i \text{ for } (x,t) \in A_i \text{ for } i = 1, \ldots, N+1, \]
and moreover, for \( i = 1, \ldots, N \)

\[
\mathcal{L}^{d+1}(A_i) = \sum_{j=1}^{M} \alpha_j^{d+1} \mathcal{L}^{d+1}(\tilde{A}_i).
\]

On the other hand \( \sum_j \alpha_j^{d+1} = \mathcal{L}^{d+1}(\tilde{B}) \), hence from [30] and the definition of \( B_\varepsilon \) we deduce \( |\mathcal{L}^{d+1}(B) - \sum_j \alpha_j^{d+1}| < 5\varepsilon \). Combined with [19] this yields \( |1 - \mu_{N+1} - \sum_j \alpha_j^{d+1}| < 6\varepsilon \), therefore we obtain

\[
|\mathcal{L}^{d+1}(A_i) - \mu_i| \leq \sum_j \alpha_j^{d+1} |\mathcal{L}^{d+1}(\tilde{A}_i) - \frac{\mu_i}{1-\mu_{N+1}}| + \frac{\mu_i}{1-\mu_{N+1}} |1 - \mu_{N+1} - \sum_j \alpha_j^{d+1}| \\
\leq C\varepsilon
\]

for some fixed constant \( C \). Thus, by replacing \( \varepsilon \) by \( \varepsilon/C \) in the above we can ensure property (iii) for \( w \).

Next, recall the convex function \( f \in C(\mathbb{R}^d \times S^n) \). Concerning property (iv) for \( w \), assume first that \( t \in [0, \varepsilon] \cup [1-\varepsilon, 1] \). Note that in this case \( w(x, t) = W_1(x, t) \) for all \( x \), hence using property (iv) for \( W_1 \) and the convexity of \( f \) we obtain

\[
\int_{[0,1]^d} f(w(x, t)) \, dx = \int_{[0,1]^d} f(W_1(x, t)) \, dx \\
\leq \mu_{N+1} f(w_{N+1}) + (1 - \mu_{N+1}) f(\bar{w}) + \varepsilon \\
\leq \mu_{N+1} f(w_{N+1}) + \sum_{i=1}^{N} \mu_i f(w_i) + \varepsilon.
\]

Next, let \( t \in (\varepsilon, 1-\varepsilon) \), and recall that by Proposition [21] the function \( W_1 \) satisfies additionally the time-slice estimate [40]. Define the sets

\[
A_t = \{ x : (x, t) \in A \}, \quad B_t = \{ x : (x, t) \in B \}, \quad \tilde{B}_t = \{ x : (x, t) \in \tilde{B} \}.
\]

By estimate [40] we have

\[
|\mathcal{L}^d(A_t) - \mu_{N+1}| < \varepsilon, \quad |\mathcal{L}^d(B_t) - (1 - \mu_{N+1})| < \varepsilon,
\]

hence \( \mathcal{L}^d([0,1]^d \setminus (A_t \cup B_t)) < 2\varepsilon \). We can therefore estimate

\[
\int_{[0,1]^d} f(w(x, t)) \, dx \\
\leq \int_{A_t} f(w(x, t)) \, dx + \int_{B_t} f(w(x, t)) \, dx + \int_{\tilde{B}_t} f(w(x, t)) \, dx + O(\varepsilon) \\
= \mathcal{L}^d(A_t) f(w_{N+1}) + \mathcal{L}^d(B_t \setminus \tilde{B}_t) f(\bar{w}) + \int_{\tilde{B}_t} f(w(x, t)) \, dx + O(\varepsilon).
\]
For the third integral we calculate
\[
\int_{\tilde{B}_t} f(w(x,t))dx = \sum_j \int_{x_j+\alpha_j[0,1]^d} f\left(\tilde{w} + W_2\left(\frac{x-x_j}{\alpha_j}, \frac{t-t_j}{\alpha_j}\right)\right)dx \\
= \sum_j \alpha_j^d \int_{[0,1]^d} f\left(\tilde{w} + W_2(x, \frac{t}{\alpha_j})\right)dx \\
\leq L^d(\tilde{B}_t) \sum_{i=1}^N \frac{\mu_i}{1 - \mu_{N+1}} f(w_i) + \epsilon,
\]
where \(\sum_j\) denotes summation over those \(j\) for which \(t \in t_j + (0, \alpha_j)\), and in the last inequality we have used property (iv) for \(W\). Furthermore, by convexity of \(f\) we also have \(f(\tilde{w}) \leq \sum_{i=1}^N \frac{\mu_i}{1 - \mu_{N+1}} f(w_i)\). It follows that
\[
\int_{[0,1]^d} f(w(x,t))dx \leq \sum_{i=1}^{N+1} \mu_i f(w_i) + O(\epsilon),
\]
implying property (iv) for \(w\) as required.

\(\square\)

### 3.4 Proof of Proposition 15 and Theorem 2

#### 3.4.1 Proof of Proposition 15

Let \((\nu, \lambda, \nu^\infty)\) be an admissible measure-valued solution with initial data \(v_0\). By Proposition 16 it suffices to generate the lifted admissible measure-valued subsolution by a suitable sequence of subsolutions. By abuse of notation, let us still denote the lifted Young measure by \((\nu, \lambda, \nu^\infty)\), and its energy by \(E(t)\).

By Theorem 18 we obtain a sequence of smooth subsolutions \(\tilde{w}^k = (\tilde{v}^k, \tilde{u}^k)\) uniformly bounded in \(L^\infty_t(L^2_x \times L^1_t)\) and a sequence of discrete oscillation measures \(\nu^k\) that are piecewise constant with respect to \(Q_k\), such that (23), (24) and (25) hold.

Let us fix \(Q \in Q_k\) for the moment. By Proposition 19 there exists a sequence \(\tilde{w}^{k,n} = (\tilde{v}^{k,n}, \tilde{u}^{k,n}) \in C^\infty_c(Q; \mathbb{R}^d \times \mathcal{S}^d_0)\) of subsolutions such that
\[
\|\tilde{w}^{k,n}\|_{L^\infty(Q)} \leq \langle \nu^k|Q, c\rangle
\]
and generating \(\nu^k\) on \(Q\). Next, for \(l \geq 1\), let \(\tilde{w}^{k,l} = (\tilde{v}^{k,l}, \tilde{u}^{k,l})\) be defined as the average of \((\tilde{v}^k, \tilde{u}^k)\) over cubes of size \(2^{-l-1}\), i.e.
\[
(\tilde{v}^{k,l}, \tilde{u}^{k,l})(x,t) := \int_{Q_i^l} (\tilde{v}^k, \tilde{u}^k) \quad \text{for } (x,t) \in Q_i^l \text{ for all } Q_i^l \in Q_{2^l}^k.
\]

Since \((\tilde{v}^k, \tilde{u}^k) \in L^\infty_t(L^2_x \times L^1_t) \cap C^\infty(\mathbb{R}^d \times [0,T])\), we have
\[
\sup_l \int_{\mathbb{R}^d} |\tilde{v}^{k,l} - \tilde{v}^k|^2 dx + \sup_l \int_{\mathbb{R}^d} |\tilde{u}^{k,l} - \tilde{u}^k|^2 dx \to 0 \quad \text{as } l \to \infty.
\]
Consequently, from (41) we deduce
\[
\sup_{t} \left| \int_{Q_t} e(\bar{w}^k(x,t) + \bar{w}^{k,n}(x,t)) \, dx - \int_{Q_t} e(\tilde{w}^{k,l}(x,t) + \tilde{w}^{k,n}(x,t)) \, dx \right| \to 0
\]
as \( l \to \infty \), where \( Q_t = \{ x : (x,t) \in Q \} \), and the convergence is uniform in \( k, n \), owing to the \( L^\infty \) bound on \( \bar{w}^{k,n} \) as well as the energy bound (24) on \( \nu^k \). On the other hand, since \( e \in C(\mathbb{R}^d \times S^d_t) \) is convex, from (45) we have
\[
\limsup_{n \to \infty} \sup_{t} \int_{Q_t} e(\bar{w}^k,l + \bar{w}^{k,n}(x,t)) \, dx \leq \langle T_{\bar{w}^k,l}, \nu^k|_Q, e \rangle.
\]
Thus, given any \( \delta > 0 \) we can first choose \( l \) and then \( n \) sufficiently large, so that
\[
\sup_{t} \int_{Q_t} e(\bar{w}^k + \bar{w}^{k,n}(x,t)) \, dx \leq \langle T_{\bar{w}^k,l}, \nu^k|_Q, e \rangle + \delta.
\]
Summing over all cubes in \( Q_k \) and taking a suitable diagonal subsequence, we obtain \( (\tilde{v}^k, \tilde{u}^k) \) such that \( (\bar{v}^k + \tilde{v}^k, \bar{u}^k + \tilde{u}^k) \) is uniformly bounded in \( L^\infty_t(L^2_x \times L^1_x) \) (this follows from (24) and (43)), consists of subsolutions, and
\[
(\tilde{v}^k, \tilde{u}^k) \overset{Y_{2,1}}{\to} (\nu, \lambda, \nu^\infty).
\]
Furthermore,
\[
\sup_{t} \int_{\mathbb{R}^d} e(\bar{v}^k + \tilde{v}^k, \bar{u}^k + \tilde{u}^k) \, dx \leq \text{esssup}_t E(t) + \frac{1}{k}
\]
and, by (26),
\[
\int_{\mathbb{R}^d} |\bar{v}^k(x,0) + \tilde{v}^k(x,0) - v_0(x)|^2 \, dx = \int_{\mathbb{R}^d} |\bar{v}^k(x,0) - v_0(x)|^2 \, dx \to 0
\]
as \( k \to \infty \). Therefore the sequence \( (\bar{v}^k + \tilde{v}^k, \bar{u}^k + \tilde{u}^k) \) satisfies the conditions in Proposition 16b). Proposition 15 follows, since \( (\nu, \lambda, \nu^\infty) \) is admissible and therefore \( \text{esssup}_t E(t) = \frac{1}{2} \int_{\mathbb{R}^d} |v_0|^2 \, dx \).

3.4.2 Proof of Theorem 2

As above, let \( (\nu, \lambda, \nu^\infty) \) be an admissible measure-valued solution, and, by abuse of notation, let us denote by \( (\nu, \lambda, \nu^\infty) \) also the corresponding lifted Young measure on \( \mathbb{R}^d \times S^d_t \) as in Section 2.4.3. In the proof of Proposition 15 above, we showed that for every \( k \in \mathbb{N} \) there exists a subsolution \( (v^k, u^k) \in C^\infty(\mathbb{R}^d \times [0,T]) \) such that
\[
(v^k, u^k) \overset{Y_{2,1}}{\to} (\nu, \lambda, \nu^\infty) \quad \text{as} \quad k \to \infty
\]
(51)
\[
\frac{1}{2} \int |v^k(x,0)|^2 \, dx - \frac{1}{2} \int |v_0(x)|^2 \, dx < \frac{1}{k}
\]
(52)
\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^d} e(v^k, u^k) \, dx < \frac{1}{2} \int |v_0|^2 \, dx + \frac{1}{k}
\]
(53)
In this final step, we want to deduce from this the full statement of Theorem
We will do so by using the argument from [9] for the construction of wild initial data. A slight modification of the proof of Proposition 5.1 in [9] yields the following statement:

**Proposition 22.** Let \((v, u) \in L^\infty_v (L^2_v \times L^1_v) \cap C^\infty (\mathbb{R}^d \times [0, T])\) be a subsolution, and let
\[
\bar{e} \in C(\mathbb{R}^d \times [0, T]) \cap C([0, T]; L^1(\mathbb{R}^d))
\]
such that \(e(v(x, t), u(x, t)) < \bar{e}(x, t)\) for all \((x, t) \in \mathbb{R}^d \times [0, T]\). Then there exists a sequence of subsolutions \((v^n, u^n) \in C^\infty (\mathbb{R}^d \times [0, T])\) such that
\[
v^n \to v \quad \text{in } C([0, T]; L^2_v(\mathbb{R}^d)),
\]
\[
e(v^n(x, t), u^n(x, t)) < \bar{e}(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times [0, T]
\]
and
\[
\frac{1}{2} |v^n(x, 0)|^2 - \bar{e}(x, 0) < \epsilon \quad \text{for } a.e. x \in \mathbb{R}^d.
\] (54)

Indeed, the proof follows along the lines of the proof of Proposition 5.1 in [9], by iterating the Claim from there with the functional \(\int_0^t \frac{1}{2} |v(x, 0)|^2 - \bar{e}(x, 0) \, dx\) instead of \(\int_0^t \frac{1}{2} |v(x, 0)|^2 - 1 \, dx\), and only considering the functions for times \(t \geq 0\).

With this assertion at hand, we proceed as in the proof of Proposition 16. Choose \(\bar{e}_k \in C(\mathbb{R}^d \times [0, T]) \cap C([0, T]; L^1(\mathbb{R}^d))\) so that

- \(e(v^k(x, t), u^k(x, t)) < \bar{e}_k(x, t)\) for all \((x, t) \in \mathbb{R}^d \times [0, T]\);
- \(\sup_{\mathbb{R}^d} \int_{\mathbb{R}^d} \bar{e}_k - e(v^k, u^k) \, dx < \frac{1}{k}\);
- \(\int_{\mathbb{R}^d} \bar{e}_k(x, t) \, dx \leq \int_{\mathbb{R}^d} \bar{e}_k(x, 0) \, dx\) for all \(t \in [0, T]\).

Such a choice is possible because of (52) - (53).

Then, apply Proposition 22 to each \((v^k, u^k)\) with \(\bar{e}_k\). We obtain sequences \((v^{k,n}, u^{k,n})\), and as in the proof of Proposition 16 we can extract a diagonal subsequence \(n = n(k)\) such that
\[
v^{k,n(k)} - v^k \to 0 \quad \text{in } L^2_{loc}(\mathbb{R}^d \times (0, T)).
\]
Moreover, we have that \(v^{k,n(k)}(t = 0) \to v_0\) strongly in \(L^2\) if \(n(k)\) is chosen sufficiently large: Indeed, on the one hand, we know that \(v^{k,n(k)}(t = 0) \to v^k(t = 0)\) weakly (cf. Proposition 22). On the other hand, by (54), the choice of \(\bar{e}_k\), (53), and (52), we have
\[
\left\| v^{n,k}(t = 0) \right\|_2^2 - \left\| v^k(t = 0) \right\|_2^2 < \frac{8}{k}.
\]
These two facts imply that \(v^{k,n(k)}(t = 0) \to v^k(t = 0)\) strongly as \(n \to \infty\), and hence, if \(n(k)\) is suitably chosen, (52) yields \(v^{k,n(k)}(t = 0) \to v_0\) strongly in \(L^2\) as \(k \to \infty\).
Finally, with each \( v^{k,n}(k) \) we argue as in the proof of Proposition 16b) to find exact Euler solutions close to \( v^{k,n}(k) \) in \( C([0, T]; L^2_w(\mathbb{R}^d)) \) and hence, due to the choice of \( \bar{e}_k \), in \( L^2_{loc}(\mathbb{R}^d \times (0, T)) \). Therefore these Euler solutions generate the same Young measure. Furthermore, their initial values \( v^{k,n}(k)(t = 0) \) are close to \( v_0 \), and because \( v^{k,n}(k) \) satisfies (54), they are admissible. This proves Theorem 2.

3.4.3 Proof of Corollary 3

As observed by DiPerna and Majda in [11], given any initial data \( v_0 \in L^2(\mathbb{R}^d) \), a sequence of Leray solutions with viscosity \( \rightarrow 0 \) generates a measure-valued solution of Euler. Moreover, such a measure-valued solution has initial data \( v_0 \), and it is easy to see from the energy bound that it will be admissible. Hence, as in the previous Subsection 3.4.2, we find a sequence of subsolutions \( (v^{k,n}(k), u^{k,n}(k)) \) such that (passing to a subsequence if necessary)

\[
\|v^{k,n}(k)(t = 0) - v_0\|_2 < \frac{1}{k},
\]

\[
\frac{1}{2} |v^{k,n}(k)(x, 0)|^2 = \bar{e}_k(x, 0) \quad \text{for a.e. } x \in \mathbb{R}^d,
\]

and

\[
e(v^{k,n}(k)(x, t), u^{k,n}(k)(x, t)) < \bar{e}_k(x, t) \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and all } t > 0.
\]

Hence, by virtue of Theorem 17 (which is Proposition 3.3. in [9]), for each such \( (v^{k,n}(k), u^{k,n}(k)) \) there exist infinitely many solutions for Euler with initial data \( v^{k,n}(k)(t = 0) \) and energy density \( \bar{e}_k \) for all times \( t \geq 0 \). By choice of \( \bar{e}_k \) (see Subsection 3.4.2), these solutions are admissible. This shows that in every \( L^2 \)-neighbourhood of \( v_0 \) there exists initial data admitting infinitely many admissible solutions of Euler and thus the corollary is proved.

Finally we remark that if we choose \( \bar{e}_k \) such that \( \int \bar{e}_k \, dx \) is constant in time, then the Euler solutions obtained in this way will conserve energy. This shows the existence of energy-conserving weak solutions for a dense subset of initial data.

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