TWO-DIRECTION MULTIWAVELET MOMENTS

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ABSTRACT. Two-direction multiscaling functions $\phi$ and two-direction multiwavelets $\psi$ associated with $\phi$ are a more general and more flexible setting than one-direction multiscaling functions and multiwavelets. In this paper, we derive two methods for computing continuous moments of orthogonal two-direction multiscaling functions $\phi$ and orthogonal two-direction multiwavelets $\psi$ associated with $\phi$. The first method is by doubling and the second method is by separation. Two examples for both methods are given.

1. INTRODUCTION

Two-direction multiscaling functions $\phi$ and two-direction multiwavelets $\psi$ associated with $\phi$, which are a more general setting than the one-direction multiscaling functions and multiwavelets, are investigated in [4, 5, 8, 9, 10, 11, 12, 13]. The two-direction setting is more flexible than the one-direction setting.

In multiwavelet theory, computation of continuous moments for two-direction multiwavelets associated with two-direction multiscaling functions is important, since vanishing continuous moments for two-direction multiwavelets provide the approximation order for two-direction multiscaling functions. The discrete or continuous moments for the orthogonal two-direction multiscaling functions and multiwavelets are necessary in theory such as for establishing Condition E and approximation order, and many applications such as solutions of differential equations, signal processing, and image processing, especially for prefiltering or balancing two-direction multiscaling functions. It is well-known that continuous moments of one-direction orthogonal multiscaling functions and multiwavelets can be computed if their recurrence coefficients are given, for example, see [1, 2, 3, 6].

The main objective of this paper is to derive two different methods for computing continuous moments of the orthogonal two-direction multiscaling functions $\phi$ and multiwavelets $\psi$ associated with $\phi$. To derive the method for computing continuous moments by doubling, we investigate the following: for $s = 1, 2, \ldots, d - 1$, where $d$ is a dilation factor

- investigate recursion formulas for continuous moments of doubled $\Phi(x) = [\phi(x), \phi(-x)]^T$ and $\Psi^{(s)}(x) = [\psi^{(s)}(x), \psi^{(s)}(-x)]^T$;
- investigate normalization for the zeroth continuous moment of $\Phi$;
- investigate the continuous moments of $\Phi$ and $\Psi^{(s)}$;
- choose the continuous moments of $\phi$ and $\psi^{(s)}$ from the continuous moments of $\Phi$ and $\Psi^{(s)}$, respectively.

To derive the method for computing continuous moments by separation, we investigate the following: for $s = 1, 2, \ldots, d - 1$, where $d$ is a dilation factor

- investigate recursion formulas for the continuous moments of $\phi$ and $\psi^{(s)}$ by separation;
- investigate normalization for the zeroth continuous moment of $\phi$;

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• investigate the continuous moments of $\phi$ and $\psi^{(s)}$.

We provide two algorithms, one by doubling and the other by separation, for computing continuous moments of the orthogonal two-direction multiscaling functions $\phi$ and multiwavelets $\psi$ associated with $\phi$.

This paper is organized as follows. Orthogonal two-direction multiscaling functions $\phi$ and multiwavelets $\psi$ are introduced in section 2. The main results for computing continuous moments of the orthogonal two-direction multiscaling functions $\phi$ and multiwavelets $\psi$ are stated in sections 3 and 4. In section 3, one method for computing moments by doubling is introduced. In section 4, the other method for computing moments by separation is introduced. Finally, two examples for illustrating the general theory in sections 2, 3, and 4 are given in section 5.

2. TWO-DIRECTION MULTISCALING FUNCTIONS AND MULTIWAVELETS

In this section we review orthogonal two-direction multiscaling functions and multiwavelets associated with the orthogonal two-direction multiscaling functions (see \cite{9,10,11,12,13}).

A two-direction multiscaling function of multiplicity $r$ and dilation factor $d$ is a vector of $r$ real or complex-valued functions

$$\phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_r(x)]^T, \quad x \in \mathbb{R},$$

which satisfies a recursion relation

$$\phi(x) = \sqrt{d} \sum_{k \in \mathbb{Z}} \left[ P_k^+ \phi(dx - k) + P_k^- \phi(k - dx) \right]$$  \hspace{1cm} (2.1)

and generates a multiresolution approximation of $L^2(\mathbb{R})$. The $P_k^+, P_k^-$, called positive- and negative-direction recursion coefficients for $\phi$, respectively, are $r \times r$ matrices.

Two-direction multiwavelets $\psi^{(s)}$, $s = 1, 2, \ldots, d - 1$, associated with $\phi$ satisfy

$$\psi^{(s)}(x) = \sqrt{d} \sum_{k \in \mathbb{Z}} \left[ Q_k^{(s)+} \phi(dx - k) + Q_k^{(s)-} \phi(k - dx) \right].$$  \hspace{1cm} (2.2)

The $Q_k^{(s)+}, Q_k^{(s)-}$ are called positive- and negative-direction recursion coefficients for $\psi^{(s)}$, $s = 1, 2, \ldots, d - 1$, respectively.

Two-direction multiscaling function $\phi$ and multiwavelets $\psi^{(s)}$ associated with $\phi$ are orthogonal if for all $j, k \in \mathbb{Z}$, and $s, t = 1, 2, \ldots, d - 1$,

$$\langle \phi(x-j), \phi(x-k) \rangle = \delta_{jk} I_r,$$

$$\langle \phi(x-j), \phi(k-x) \rangle = O,$$

$$\langle \psi^{(s)}(x-j), \psi^{(t)}(x-k) \rangle = \delta_{st} \delta_{jk} I_r,$$

$$\langle \psi^{(s)}(x-j), \psi^{(t)}(k-x) \rangle = O,$$

$$\langle \phi(x-j), \psi^{(s)}(x-k) \rangle = O,$$

$$\langle \phi(x-j), \psi^{(s)}(k-x) \rangle = O,$$

where $I_r$ is the $r \times r$ identity matrix and $O$ is the $r \times r$ zero matrix. Here the inner product is defined by

$$\langle \phi, \psi^{(s)} \rangle = \int \phi(x)\psi^{(s)*}(x) \, dx,$$

where $*$ denotes the complex conjugate transpose. This inner product is an $r \times r$ matrix.

By taking the Fourier transform on both sides of (2.1), we have

$$\hat{\phi}(\xi) = P^+(e^{-i\xi/d}) \hat{\phi}(\xi/d) + P^-(e^{-i\xi/d}) \hat{\phi}(\xi/d),$$  \hspace{1cm} (2.3)
where
\[ P^+(z) = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} P^+_k z^k \quad \text{and} \quad P^-(z) = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} P^-_k z^k \quad (2.4) \]
are called positive- and negative-direction mask symbols, respectively.

We rewrite the two-direction recursion relation (2.1) as
\[ \phi(-x) = \sqrt{d} \sum_{k \in \mathbb{Z}} \left[ P^+_k \phi(-dx - k) + P^+_k \phi(k + dx) \right] \quad (2.5) \]

By taking the Fourier transform on both sides of (2.5), we have
\[ \hat{\phi}(\xi) = P^+(e^{-i\xi/d}) \hat{\phi}(\xi/d) + P^-(e^{-i\xi/d}) \hat{\phi}(\xi/d). \quad (2.6) \]

From (2.3) and (2.6), we have
\[ \begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\phi}(\xi/d) \end{bmatrix} = \begin{bmatrix} P^+(e^{-i\xi/d}) & P^-(e^{-i\xi/d}) \\ P^- (e^{-i\xi/d}) & P^+(e^{-i\xi/d}) \end{bmatrix} \begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\phi}(\xi/d) \end{bmatrix} \quad (2.7) \]

We see that (2.1) has a solution if and only if (2.7) has a solution. That is, (2.1) is equivalent to (2.7).

Let
\[ \Phi(x) = \begin{bmatrix} \phi(x) \\ \phi(-x) \end{bmatrix} = \sqrt{d} \sum_{k \in \mathbb{Z}} \begin{bmatrix} P^+_k & P^-_k \\ P^-_k & P^+_k \end{bmatrix} \Phi(dx - k) \quad (2.8) \]
be the matrix refinable equation of \( \Phi \). Equation (2.8) is called the deduced \( d \)-scale matrix refinement equation of the two-direction refinement equation (2.1). Then (2.7) is the \( d \)-scale matrix refinement equation in the frequency domain of \( \Phi \). Its refinement mask is
\[ P_\Phi(z) = \begin{bmatrix} P^+(z) \\ P^-(z) \\ P^+(z) \end{bmatrix}. \quad (2.9) \]

If \( P^+_k \) and \( P^-_k \), \( k \in \mathbb{Z} \), are real, then
\[ P_\Phi(z) = \begin{bmatrix} P^+(z) \\ P^-(z) \\ P^+(z) \end{bmatrix} = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} \begin{bmatrix} P^+_k & P^-_k \\ P^-_k & P^+_k \end{bmatrix} z^k. \quad (2.10) \]

In this paper we only consider real recursion coefficients \( P^+_k \), \( P^-_k \), \( Q^{(s)+}_k \), and \( Q^{(s)-}_k \) in \( \mathbb{R}^{r \times r} \) for \( s = 1, 2, \ldots, d - 1 \) and \( k \in \mathbb{Z} \).

**Condition E.** A matrix \( A \) satisfies Condition E if it has a simple eigenvalue of 1, and all other eigenvalues are smaller than 1 in absolute value.

Condition E for \( \phi \) defined in (2.1) means that the matrix
\[ \begin{bmatrix} P^+(1) & P^-(1) \\ P^-(1) & P^+(1) \end{bmatrix} = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} \begin{bmatrix} P^+_k & P^-_k \\ P^-_k & P^+_k \end{bmatrix} \quad (2.11) \]
satisfies Condition E (see [9] Theorem 3) for \( d = 2 \).

For the scalar case \( r = 1 \), since the eigenvalues of the matrix in (5.2) are \( P^+(1) + P^-(1) \) and \( P^+(1) - P^-(1) \), Condition E for \( \phi \) is equivalent to
\[ P^+(1) + P^-(1) = 1, \quad |P^+(1) - P^-(1)| < 1, \]
or
\[ P^+(1) - P^-(1) = 1, \quad |P^+(1) + P^-(1)| < 1, \]
that is,
\[
\frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} [P_k^+ + P_k^-] = 1, \\
\frac{1}{\sqrt{d}} \left| \sum_{k \in \mathbb{Z}} [P_k^+ - P_k^-] \right| < 1,
\]
or
\[
\frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} [P_k^+ - P_k^-] = 1, \\
\frac{1}{\sqrt{d}} \left| \sum_{k \in \mathbb{Z}} [P_k^+ + P_k^-] \right| < 1
\]
(see [11, Theorem 2]).

It is well-known that if \( \phi \) is a compactly supported \( L^2 \)-stable solution of (2.1), then Condition E for \( \phi \) is satisfied (see [9, Theorem 3] for \( d = 2 \)).

Condition E for \( \phi \) is the condition for the stability of the multiresolution approximation produced by \( \phi \) or for the existence of \( \phi \) in (2.1) (see [7, Theorem 3]).

**Approximation Order.** The two-direction multiscaling function approximation to a function \( f \) at resolution \( d^{-j} \) is given by the series
\[
P_j f = \sum_{k \in \mathbb{Z}} [(f, \phi_{jk}^+) \phi_{jk}^+ + (f, \phi_{jk}^-) \phi_{jk}^-],
\]
(2.12)
where
\[
\phi_{jk}^+(x) = d^{-j/2} \phi(d^j x - k) \quad \text{and} \quad \phi_{jk}^-(x) = d^{-j/2} \phi(k - d^j x).
\]
The two-direction multiscaling function \( \phi \) provides approximation order \( p \) if
\[
\|f - P_n f\| = O(d^{-np}),
\]
whenever \( f \) has \( p \) continuous derivatives.

The two-direction multiscaling function \( \phi \) has accuracy \( p \) if all polynomials of degree up to \( p - 1 \) can be expressed locally as linear combinations of integer shifts of \( \phi(x) \) and \( \phi(-x) \). That is, there exist row vectors \((c_{j,k}^+)^*\) and \((c_{j,k}^-)^*\), \( j = 0, \ldots, p - 1, \ k \in \mathbb{Z} \), so that
\[
x^j = \sum_{k \in \mathbb{Z}} [(c_{j,k}^+)^* \phi(x - k) + (c_{j,k}^-)^* \phi(k - x)].
\]
(2.13)
It is well-known that \( \phi \) provides approximation order \( p \) if and only if \( \phi \) has accuracy \( p \) (see [3, 7]).

A high approximation order is desirable in applications. A minimum approximation order of 1 is a required condition in many theorems.

Throughout this paper we assume that the two-direction multiscaling function \( \phi \) is orthogonal, has compact support, is continuous (which implies approximation order at least 1), and satisfies Condition E.

3. Moments of Two-Direction Multiwavelets by Doubling

It is well-known that the discrete and continuous moments of one-direction orthogonal (or biorthogonal) multiscaling functions and multiwavelets associated with multiscaling functions can be computed if the recurrence coefficients of multiscaling functions and multiwavelets are given, for example, see [11, 2, 3, 6].

The discrete or continuous moments for the orthogonal two-direction multiscaling functions and multiwavelets are necessary in theory such as for establishing Condition E, approximation order, etc.

In this section we investigate how to compute the integral \( m_0 \), the zeroth continuous moment, and higher moments of the orthogonal two-direction multiscaling functions \( \phi \) in terms of the recurrence coefficients \( P_k^+, P_k^- \) of \( \phi \).
Similarly, we investigate how to compute the continuous moments of orthogonal two-direction multiwavelets \( \psi^{(s)} \) associated with the orthogonal two-direction multiscaling function \( \phi \) in terms of the recurrence coefficients \( Q_k^{(s)+}, Q_k^{(s)-} \) of \( \psi^{(s)} \) for \( s = 1, 2, \ldots, d - 1 \).

The main idea of the method by doubling in this section is to consider the two-direction multiscaling function \( \Phi(x) \) and \( \phi(-x) \) together, and the two-direction multiwavelets \( \psi^{(s)}(x) \) and \( \psi^{(s)}(-x) \) together. That is, we consider refinable functions \( \Phi(x) = [\phi(x), \phi(-x)]^T \) and \( \Psi^{(s)}(x) = [\psi^{(s)}(x), \psi^{(s)}(-x)]^T \). Then for \( \Phi \) and \( \Psi^{(s)} \), we develop a theory for computing continuous moments. Finally, we choose parts of the continuous moments of \( \Phi \) and \( \Psi^{(s)} \) for the continuous moments of \( \phi \) and \( \psi^{(s)} \), respectively for \( s = 1, 2, \ldots, d - 1 \).

The \( j \)th discrete moment \( M_j \) of the two-direction multiscaling function \( \phi \) is an \( r \times r \) matrix defined by

\[
M_j = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^j [P_k^+ + P_k^-], \quad j = 0, 1, 2, \ldots \tag{3.1}
\]

The \( j \)th discrete moment \( M_j^\pm \) of the one-direction multiscaling function \( \Phi \) is a \( 2r \times 2r \) matrix defined by

\[
M_j^\pm = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^j \begin{bmatrix} P_k^+ & P_k^- \\ P_{-k}^- & P_{-k}^+ \end{bmatrix}, \quad j = 0, 1, 2, \ldots \tag{3.2}
\]

In particular,

\[
M_0^\pm = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} \begin{bmatrix} P_k^+ & P_k^- \\ P_{-k}^- & P_{-k}^+ \end{bmatrix} = P_\Phi(1)
\]

and \( M_0^\pm \) is used in defining Condition E for \( \phi \) in (5.2).

The \( j \)th continuous moment \( m_j \) of \( \phi \) is a vector of size \( r \) defined by

\[
m_j = \int_{-\infty}^{\infty} x^j \phi(x) \, dx, \quad j = 0, 1, 2, \ldots \tag{3.3}
\]

The \( j \)th continuous moment \( m_j^\pm \) of \( \Phi \) is a vector of size \( 2r \) defined by

\[
m_j^\pm = \int_{-\infty}^{\infty} x^j \Phi(x) \, dx, \quad j = 0, 1, 2, \ldots \tag{3.4}
\]

In multiwavelet theory, computation of continuous moments for two-direction multiwavelets is important, since vanishing continuous moments for two-direction multiwavelets provide the approximation order for two-direction multiscaling functions.

Let \( \psi^{(s)} \), \( s = 1, 2, \ldots, d - 1 \), be orthogonal two-direction multiwavelets associated with the orthogonal two-direction multiscaling function \( \phi \). Let

\[
\Psi^{(s)}(x) = [\psi^{(s)}(x), \psi^{(s)}(-x)]^T \tag{3.5}
\]

for \( s = 1, 2, \ldots, d - 1 \).

The \( j \)th discrete moments \( N_j^{(s)} \) of \( \Psi^{(s)} \), \( s = 1, 2, \ldots, d - 1 \), are \( 2r \times 2r \) matrices defined by

\[
N_j^{(s)} = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^j \begin{bmatrix} Q_k^{(s)+} & Q_k^{(s)-} \\ Q_{-k}^{(s)-} & Q_{-k}^{(s)+} \end{bmatrix}, \quad j = 0, 1, 2, \ldots \tag{3.6}
\]

The \( j \)th continuous moments \( n_j^{(s)} \) of \( \psi^{(s)} \), \( s = 1, 2, \ldots, d - 1 \), are vectors of size \( r \) defined by

\[
n_j^{(s)} = \int_{-\infty}^{\infty} x^j \psi^{(s)}(x) \, dx, \quad j = 0, 1, 2, \ldots \tag{3.7}
\]
We give some further explanatory remarks.

Remark 3.2. After simplification that the continuous moments $n_j^{(s)\pm}$ as in (3.5) and (2.1), let the two-direction multiscaling functions $\phi$ and multiwavelets $\psi^{(s)}$ be defined as in (2.11) and (2.2), respectively, and the refinable functions $\Phi$ and $\Psi^{(s)}$ defined as in (2.8) and (3.5), respectively for $s = 1, 2, \ldots, d - 1$. Then the $j$th continuous moments $m_j^{\pm}$ of $\Phi$ and $n_j^{(s)\pm}$ of $\Psi^{(s)}$ satisfy the following theorem.

**Theorem 3.1.** Let the two-direction multiscaling functions $\phi$ and multiwavelets $\psi^{(s)}$ be defined as in (2.11) and (2.2), respectively, and the refinable functions $\Phi$ and $\Psi^{(s)}$ defined as in (2.8) and (3.5), respectively for $s = 1, 2, \ldots, d - 1$. Then the $j$th continuous moments $m_j^{\pm}$ of $\Phi$ and $n_j^{(s)\pm}$ of $\Psi^{(s)}$ associated with $\Phi$ satisfy the following:

\[
m_j^{\pm} = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} M_{j-\ell}^{\pm} m_{\ell}^{\pm}, \quad j = 0, 1, 2, \ldots,
\]

\[
n_j^{(s)\pm} = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} N_{j-\ell}^{(s)\pm} m_{\ell}^{\pm}, \quad j = 0, 1, 2, \ldots,
\]

where $\binom{j}{\ell} = \frac{j!}{\ell!(j-\ell)!}$ stands for the binomial coefficient.

**Proof.** By substituting the recursion formula (2.8) into the integral in (3.4), we find after simplification that the continuous moments $m_j^{\pm}$ satisfy (3.9).

By substituting the recursion formula (2.2) and $\psi^{(s)}(-x)$ into the integral in (3.8), we find after simplification that the continuous moments $n_j^{(s)\pm}$ satisfy (3.10).

**Remark 3.2.** We give some further explanatory remarks.

For $s = 1, 2, \ldots, d - 1$ and $j = 0, 1, 2, \ldots$,

1. By substitution, we obtain other formulas for $m_j^{\pm}$ and $n_j^{(s)\pm}$:

\[
m_j^{\pm} = \left[ \frac{1}{(-1)^j} \right] m_j, \quad n_j^{(s)\pm} = \left[ \frac{1}{(-1)^j} \right] n_j^{(s)}.
\]

2. (3.11) is useful for double checking the results of $m_j^{\pm}$ and $n_j^{(s)\pm}$.

3. Theorem 3.1 provides one way to compute the continuous moments $m_j^{\pm}$ of $\Phi$ and $n_j^{(s)\pm}$ of $\Psi^{(s)}$. We will pursue this in the following.

The normalizing condition for one-direction multiscaling functions $\phi$ is given in [3, 6]. In the following Lemma we derive the normalizing condition for the two-direction multiscaling functions $\phi$.

**Theorem 3.3.** (Normalization for $m_0$) Let $\phi$ be a two-direction orthogonal multiscaling function. Then the normalizing condition for $m_0$ of $\phi$ is

\[
m_0^* m_0 = \frac{1}{2}.
\]

**Proof.** By expanding the constant 1 in an orthogonal two-direction multiscaling function series, we have

\[
1 = \sum_{k \in \mathbb{Z}} \left[ \langle 1, \phi(x - k) \rangle \phi(x - k) + \langle 1, \phi(k - x) \rangle \phi(k - x) \right]
\]

\[
= m_0^* \sum_{k \in \mathbb{Z}} [\phi(x - k) + \phi(k - x)].
\]
By integrating (3.13) on [0, 1], we have
\[ 1 = \int_{0}^{1} 1 \, dx = m_{0}^{*} \sum_{k \in \mathbb{Z}} \int_{0}^{1} [\phi(x - k) + \phi(k - x)] \, dx = m_{0}^{*} \int_{-\infty}^{\infty} [\phi(x) + \phi(-x)] \, dx = 2 m_{0}^{*} m_{0}. \]
Hence, we have equation (3.12). □

For the scalar case \( r = 1 \), that is, for orthogonal two-direction scaling functions \( \phi \), the normalization is
\[ m_{0} = \frac{\sqrt{2}}{2}. \] (3.14)

**Theorem 3.4.** (Normalization for \( m_{0}^{\pm} \)) Let \( \phi \) be a two-direction orthogonal multiscaling function. Then the normalizing condition for \( m_{0}^{\pm} \) of \( \Phi \) is
\[ (m_{0}^{\pm})^{*} m_{0}^{\pm} = 1. \] (3.15)

**Proof.** By (3.11) and theorem 3.3, we have
\[ (m_{0}^{\pm})^{*} m_{0}^{\pm} = m_{0}^{*} [1 \ 1] m_{0} = 2 m_{0}^{*} m_{0} = 1. \] □

From equation (2.8), \( \Phi \) can be considered as a one-direction multiscaling function. The result in theorem 3.4 coincides with the normalizing condition for the one-direction multiscaling functions which is known as \( (m_{0})^{*} m_{0} = 1 \) (see [3, 6]).

Now we summarize the above results in the following theorem. The following theorem is the main result of this section.

**Theorem 3.5.** Assume that a two-direction multiscaling function \( \phi \) is orthogonal, has compact support, is continuous (which implies approximation order at least 1), and satisfies Condition E. Let \( \Phi \) be defined as in (2.8). Then the continuous moments \( m_{j}^{\pm} \) of \( \Phi \) and \( n_{j}^{(s)\pm} \) of \( \Psi^{(s)} \) can be computed recursively as:
for \( j = 0 \),
\[ m_{0}^{\pm} = M_{0}^{\pm} m_{0}^{\pm}, \quad \text{with normalization} \quad (m_{0}^{\pm})^{*} m_{0}^{\pm} = 1; \]
for \( j = 1, 2, 3, \ldots \),
\[ m_{j}^{\pm} = (d^{j} I_{2r} - M_{0}^{\pm})^{-1} \sum_{\ell=0}^{j-1} \binom{j}{\ell} M_{j-\ell}^{\pm} m_{\ell}^{\pm}; \] (3.16)
for \( j = 0, 1, 2, \ldots \),
\[ n_{j}^{(s)\pm} = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} N_{j-\ell}^{(s)\pm} m_{\ell}^{\pm}. \] (3.17)

where \( I_{2r} \) is the \( 2r \times 2r \) identity matrix.

Once \( m_{0}^{\pm} \) has been chosen, all other continuous moments are uniquely defined and can be computed recursively from (3.16) and (3.17).

Finally, the \( j \)th continuous moment \( m_{j} \) of \( \phi \) and \( n_{j}^{(s)} \) of \( \psi^{(s)} \) are the upper half of \( m_{j}^{\pm} \) and \( n_{j}^{(s)\pm} \), respectively for \( s = 1, 2, \ldots, d - 1 \).

**Proof.** By setting \( j = 0 \) in equation (3.9), we have
\[ m_{0}^{\pm} = M_{0}^{\pm} m_{0}^{\pm}. \]
By theorem 3.4 we normalize \( m_{0}^{\pm} \) so that \( (m_{0}^{\pm})^{*} m_{0}^{\pm} = 1. \)
From equation (3.9), we collect \( m_j^\pm \) terms on the left-hand side and multiply by \( d^j \) on both sides. Then the coefficient matrix

\[
(d^j I_{2r} - M_0^\pm)
\]

of \( m_j^\pm \) is nonsingular by Condition E for \( \phi \). Hence, \( m_j^\pm \) can be determined recursively as

\[
m_j^\pm = (d^j I_{2r} - M_0^\pm)^{-1} \sum_{\ell=0}^{j-1} \binom{j}{\ell} M_{j-\ell}^\pm m_\ell^\pm, \quad j = 1, 2, 3, \ldots
\]

(3.17) is derived in (3.10).

For the scalar case \( r = 1 \), that is, for orthogonal two-direction scaling function \( \phi \), the zeroth continuous moment of \( \Phi \) is

\[
m_0^\pm = \sqrt{\frac{2}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

by remark 3.2 and theorem 3.4. Hence, \( m_0 = \sqrt{2}/2 \), which coincides with (3.14).

Now we provide an algorithm for computing continuous moments \( m_j \) and \( n_j^{(s)} \) of \( \phi \) and \( \psi^{(s)} \), respectively, for \( s = 1, 2, \ldots, d-1 \) and \( j = 0, 1, 2, \ldots \) by doubling.

**Algorithm 3.6.** Let an orthogonal two-direction multiscaling function \( \phi \) be given. Let \( \psi^{(s)} \), \( s = 1, 2, \ldots, d-1 \), be orthogonal multiwavelets associated with \( \phi \). Then an algorithm for computing continuous moments \( m_j \) of \( \phi \) and \( n_j^{(s)} \) of \( \psi^{(s)} \) by doubling is the following:

1. **step 1:** Compute the \( j \)th discrete moment \( M_j^\pm \) of \( \Phi \) for \( j = 0, 1, 2, \ldots \);
2. **step 2:** Compute the \( j \)th discrete moment \( N_j^{(s)\pm} \) of \( \Psi^{(s)} \) for \( s = 1, 2, \ldots, d-1 \) and \( j = 0, 1, 2, \ldots \);
3. **step 3:** Compute the zeroth continuous moment \( m_0^\pm \) of \( \Phi \) as the corresponding eigenvector to the eigenvalue 1 of \( M_0^\pm \);
4. **step 4:** Normalize \( m_0^\pm \) so that \( (m_0^\pm)^* m_0^\pm = 1 \);
5. **step 5:** Compute the \( j \)th continuous moment \( m_j^\pm \) recursively for \( j = 1, 2, \ldots \);
6. **step 6:** Take the upper half of \( m_j^\pm \) as the \( j \)th continuous moments \( m_j \) of \( \phi \) for \( j = 0, 1, 2, \ldots \);
7. **step 7:** Compute the \( j \)th continuous moment \( n_j^{(s)\pm} \) recursively for \( s = 1, 2, \ldots, d-1 \) and \( j = 0, 1, 2, \ldots \);
8. **step 8:** Take the upper half of \( n_j^{(s)\pm} \) as the \( j \)th continuous moment \( n_j^{(s)} \) of \( \psi^{(s)} \) for \( s = 1, 2, \ldots, d-1 \) and \( j = 0, 1, 2, \ldots \).

Step 2 can be placed anywhere between step 1 and step 7.

The \( j \)th discrete moments \( M_j \) of \( \phi \) can be computed directly from (3.1).

4. **Moments of two-direction multiwavelets by separation**

In this section we provide another simple and efficient method for computing the discrete and continuous moments of the two-direction multiscaling function and two-direction multiwavelet associated with the two-direction multiscaling function.

The main idea of the method in this section is to separate the two-direction multiscaling function \( \phi \) and two-direction multiwavelet \( \psi \) into two parts, positive and negative, and then use the direct method on each part.
Two-Direction Multiwavelet Moments

We recall (3.1) that the \( j \)th discrete moment \( M_j \) of the two-direction multiscaling function \( \phi \) is an \( r \times r \) matrix defined by

\[
M_j = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^j [P_k^+ + P_k^-], \quad j = 0, 1, 2, \ldots. \tag{4.1}
\]

Now we separate the \( j \)th discrete moment \( M_j \) into positive and negative parts. We define the \( j \)th positive and negative discrete moments, \( M_j^+ \) and \( M_j^- \) respectively, of the discrete moment \( M_j \) of \( \phi \) by

\[
M_j^+ = \frac{1}{\sqrt{d}} \sum_{k} k^j P_k^+, \quad j = 0, 1, 2, \ldots, \tag{4.2}
\]

\[
M_j^- = \frac{1}{\sqrt{d}} \sum_{k} k^j P_k^-, \quad j = 0, 1, 2, \ldots, \tag{4.3}
\]

so that \( M_j = M_j^+ + M_j^- \).

By applying (2.1) in (3.3), the \( j \)th continuous moment \( m_j \) of two-direction multiscaling function \( \phi \) can be expressed as

\[
m_j = \sqrt{d} \sum_{k} \left[ P_k^+ \int_{-\infty}^{\infty} x^j \phi(dx - k) \, dx + P_k^- \int_{-\infty}^{\infty} x^j \phi(k - dx) \, dx \right], \quad j = 0, 1, 2, \ldots. \tag{4.4}
\]

Now we separate the \( j \)th continuous moment \( m_j \) into positive and negative parts. We define the \( j \)th positive and negative continuous moments, \( m_j^+ \) and \( m_j^- \) respectively, of the continuous moment \( m_j \) of \( \phi \) by

\[
m_j^+ = \sqrt{d} \sum_{k} P_k^+ \int_{-\infty}^{\infty} x^j \phi(dx - k) \, dx, \quad j = 0, 1, 2, \ldots, \tag{4.5}
\]

\[
m_j^- = \sqrt{d} \sum_{k} P_k^- \int_{-\infty}^{\infty} x^j \phi(k - dx) \, dx, \quad j = 0, 1, 2, \ldots, \tag{4.6}
\]

so that \( m_j = m_j^+ + m_j^- \).

Let us define the \( j \)th discrete moment of the two-direction multiwavelet \( \psi^{(s)} \), \( s = 1, 2, \ldots, d - 1 \) by an \( r \times r \) matrix

\[
N_j^{(s)} = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^j [Q_k^{(s)+} + Q_k^{(s)-}], \quad j = 0, 1, 2, \ldots. \tag{4.7}
\]

Now we separate the \( j \)th discrete moment \( N_j \) into positive and negative parts. We define the \( j \)th positive and negative discrete moments, \( N_j^{(s)+} \) and \( N_j^{(s)-} \) respectively, of the discrete moment \( N_j^{(s)} \) of \( \psi^{(s)} \), \( s = 1, 2, \ldots, d - 1 \), by

\[
N_j^{(s)+} = \frac{1}{\sqrt{d}} \sum_{k} k^j Q_k^{(s)+}, \quad j = 0, 1, 2, \ldots, \tag{4.8}
\]

\[
N_j^{(s)-} = \frac{1}{\sqrt{d}} \sum_{k} k^j Q_k^{(s)-}, \quad j = 0, 1, 2, \ldots, \tag{4.9}
\]

so that \( N_j^{(s)} = N_j^{(s)+} + N_j^{(s)-} \).

We recall (3.7), the \( j \)th continuous moment \( n_j^{(s)} \) of the two-direction multiwavelet \( \psi^{(s)} \), \( s = 1, 2, \ldots, d - 1 \), is a vector of size \( r \) defined by

\[
n_j^{(s)} = \int_{-\infty}^{\infty} x^j \psi^{(s)}(x) \, dx, \quad j = 0, 1, 2, \ldots. \tag{4.10}
\]
Theorem 4.1. Let $\phi$ be a two-direction multiscaling function defined as (2.1) and assume that the zeroth discrete moment $M_0$ satisfies Condition E. Let $\phi$ be a two-direction multiscaling function defined as (2.1) and $\psi$ be two-direction multiwavelet associated with $\phi$. Then the $j$th continuous moments $m_j$ of $\phi$ and $n_j^{(s)}$ of $\psi^{(s)}$, $s = 1, 2, \ldots, d - 1$, associated with $\phi$ satisfy the following:

$$m_j = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} \left[ M_{j-\ell}^+ + (-1)^\ell M_{j-\ell}^- \right] m_\ell, \quad j = 0, 1, 2, \ldots,$$

$$n_j^{(s)} = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} \left[ N_{j-\ell}^{(s)+} + (-1)^\ell N_{j-\ell}^{(s)-} \right] m_\ell, \quad j = 0, 1, 2, \ldots,$$

where $\binom{j}{\ell} = \frac{j!}{\ell!(j-\ell)!}$ stands for the binomial coefficient.

Proof. After substituting the recursion formula (2.1) into the integral in (3.4), we separate the $j$th continuous moment $m_j$ into two parts, positive and negative continuous moments $m_j^+$ and $m_j^-$. By changing variables for $dx - k$ in $m_j^+$, we have

$$m_j^+ = \sqrt{d} \sum_{k \in \mathbb{Z}} P_k^+ \int_{-\infty}^{\infty} x^j \phi(dx - k) \, dx$$

$$= d^{-j} \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} P_k^+ \int_{-\infty}^{\infty} (x + k)^j \phi(x) \, dx.$$

By using the binomial expansion, we have

$$m_j^+ = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^{j-\ell} P_k^+ \int_{-\infty}^{\infty} x^\ell \phi(x) \, dx$$

$$= d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} M_{j-\ell}^+ m_\ell.$$

By changing variables for $k - dx$ in $m_j^-$, we have

$$m_j^- = \sqrt{d} \sum_{k \in \mathbb{Z}} P_k^- \int_{-\infty}^{\infty} x^j \phi(k - dx) \, dx$$

$$= d^{-j} \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} P_k^- \int_{-\infty}^{\infty} (-x + k)^j \phi(x)(-1) \, dx$$

$$= d^{-j} \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} P_k^- \int_{-\infty}^{\infty} (-x + k)^j \phi(x) \, dx.$$

By using the binomial expansion, we have

$$m_j^- = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} k^{j-\ell} P_k^- \int_{-\infty}^{\infty} x^\ell \phi(x) \, dx$$

$$= d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell M_{j-\ell}^- m_\ell.$$
Hence,
\[ m_j = m_j^+ + m_j^- = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} \left[ M_{j-\ell}^+ + (-1)^{\ell} M_{j-\ell}^- \right] m_\ell, \quad j = 0, 1, 2, \ldots. \]

For \( n_j^{(s)} \), it is similar to the proof of \( m_j \).

We have proved the normalization
\[ m_0^* m_0 = \frac{1}{2} \]
in theorem 3.3. Hence, it is not necessary to derive normalization here again.

Now we summarize the above results in the following theorem. The following theorem is the main result of this section.

**Theorem 4.2.** Assume that a two-direction multiscaling function \( \phi \) is orthogonal, has compact support, is continuous (which implies approximation order at least 1), and satisfies Condition E. Then the continuous moments \( m_j \) of \( \phi \) and \( n_j^{(s)} \) of \( \psi^{(s)} \) can be computed recursively as:

- For \( j = 0 \),
  \[ m_0 = M_0 m_0, \quad \text{with normalization} \quad m_0^* m_0 = \frac{1}{2}; \]
- for \( j = 1, 2, 3, \ldots \),
  \[ m_j = (d^j I_r - \left[ M_0^+ + (-1)^{j} M_0^- \right])^{-1} \sum_{\ell=0}^{j-1} \binom{j}{\ell} M_{j-\ell} m_\ell; \quad (4.12) \]
  \[ n_j^{(s)} = d^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} \left[ N_j^{(s)+} + (-1)^{\ell} N_j^{(s)-} \right] m_\ell, \quad (4.13) \]

where \( I_r \) is the \( r \times r \) identity matrix.

Once \( m_0 \) has been chosen, all other continuous moments are uniquely defined and can be computed recursively from (4.12) and (4.13).

**Proof.** By setting \( j = 0 \) in equation (4.10), we have
\[ m_0 = M_0 m_0. \]

By theorem 3.3, we normalize \( m_0 \) so that \( m_0^* m_0 = 1/2 \).

From equation (4.10), we collect \( m_j \) terms on the left-hand side and multiply by \( d^j \) on both sides. Then the coefficient matrix
\[ (d^j I_r - \left[ M_0^+ + (-1)^{j} M_0^- \right]) \]
of \( m_j \) is nonsingular by Condition E for \( \phi \). Hence, \( m_j \) can be determined recursively as
\[ m_j = (d^j I_r - \left[ M_0^+ + (-1)^{j} M_0^- \right])^{-1} \sum_{\ell=0}^{j-1} \binom{j}{\ell} M_{j-\ell} m_\ell, \quad j = 1, 2, 3, \ldots \]

(4.13) is derived in (4.11).
For future reference, we list some in detail.

\[
\begin{align*}
m_0 &= M_0 m_0 \quad \text{with normalization} \quad m_0^* m_0 = \frac{1}{2}, \\
m_1 &= (d I_r - M_0^+ + M_0^-)^{-1} M_1 m_0, \\
m_2 &= (d^2 I_r - M_0)^{-1} [M_2 m_0 + 2(M_1^+ - M_1^-) m_1], \\
m_3 &= (d^3 I_r - M_0^+ + M_0^-)^{-1} [M_3 m_0 + 3(M_2^+ - M_2^-) m_1 + 3M_1 m_2].
\end{align*}
\]

For \( s = 1, 2, \ldots, d - 1, \)

\[
\begin{align*}
n_0^{(s)} &= N_0^{(s)} m_0, \\
n_1^{(s)} &= d^{-1} [N_1^{(s)} m_0 + (N_0^{(s)+} - N_0^{(s)-}) m_1], \\
n_2^{(s)} &= d^{-2} [N_2^{(s)} m_0 + 2(N_1^{(s)+} - N_1^{(s)-}) m_1 + N_0 m_2], \\
n_3^{(s)} &= d^{-3} [N_3^{(s)} m_0 + 3(N_2^{(s)+} - N_2^{(s)-}) m_1 + 3N_1^{(s)} m_2 + (N_0^{(s)+} - N_0^{(s)-}) m_3].
\end{align*}
\]

Now we provide an algorithm for computing continuous moments \( m_j \) and \( n_j^{(s)} \) of \( \phi \) and \( \psi^{(s)} \), respectively, for \( j = 0, 1, 2, \ldots \) by separation.

**Algorithm 4.3.** Let an orthogonal two-direction multiscaling function \( \phi \) be given. Let \( \psi^{(s)}, \ s = 1, 2, \ldots, d - 1 \), be orthogonal multiwavelets associated with \( \phi \). Then an algorithm for computing continuous moments \( m_j \) of \( \phi \) and \( n_j^{(s)} \) of \( \psi^{(s)} \) by separation is the following:

- step 1: Compute the \( j \)th positive and negative discrete moments, \( M_j^+ \) and \( M_j^- \) respectively, and the \( j \)th discrete moment \( M_j \) of \( \phi \) for \( j = 0, 1, 2, \ldots \);
- step 2: Compute the \( j \)th positive and negative discrete moments, \( N_j^{(s)+} \) and \( N_j^{(s)-} \) respectively, and the \( j \)th discrete moment \( N_j^{(s)} \) of \( \psi^{(s)} \) for \( s = 1, 2, \ldots, d - 1 \) and \( j = 0, 1, 2, \ldots \);
- step 3: Compute the zeroth continuous moment \( m_0 \) of \( \phi \) as the corresponding eigenvector to the eigenvalue 1 of \( M_0 \);
- step 4: Normalize \( m_0 \) so that \( m_0^* m_0 = 1/2 \);
- step 5: Compute the \( j \)th continuous moment \( m_j \) of \( \phi \) for \( j = 1, 2, \ldots \);
- step 6: Compute the \( j \)th continuous moments \( n_j^{(s)} \) of \( \psi^{(s)} \) for \( s = 1, 2, \ldots, d - 1 \) and \( j = 0, 1, 2, \ldots \).

Step 2 can be placed anywhere between step 1 and step 6.

**Remark 4.4.** We give some further explanatory remarks.

1. Two methods for computing continuous moments for \( \phi \) and \( \psi^{(s)} \) for \( s = 1, 2, \ldots, d - 1 \) by doubling and by separation generate the same results. This will be demonstrated in Example 5.1.
2. The discrete and continuous moments for \( \phi \) and \( \psi^{(s)} \) for \( s = 1, 2, \ldots, d - 1 \) can be computed by both doubling and separation.
3. The method for computing moments by separation in section 4 is simpler, faster, and more efficient than that by doubling in section 3. So, it is recommended to use the method for computing continuous moments by separation in practice.

5. **Examples**

In this section, we provide two examples for illustrating the general theory in sections 2, 3 and 4.
Example 5.1. In this example we take two-direction scaling function $\phi$ and two-direction wavelet $\psi$ associated with $\phi$ given in [13]. We note that multiplicity $r = 1$ and dilation factor $d = 2$.

The nonzero recursion coefficients for $\phi$ are

$$P_1^+ = \frac{1}{\sqrt{2}} \frac{3}{4}, \quad P_2^+ = \frac{1}{\sqrt{2}} \frac{2 - \sqrt{7}}{4}, \quad P_2^- = \frac{1}{\sqrt{2}} \frac{2 + \sqrt{7}}{4}, \quad P_3^- = \frac{1}{\sqrt{2}} \frac{1}{4}.$$ 

The nonzero recursion coefficients for $\psi$ are

$$Q_{-2}^+ = \frac{1}{\sqrt{2}} \frac{3}{4}, \quad Q_{-3}^+ = -\frac{1}{\sqrt{2}} \frac{2 - \sqrt{7}}{4}, \quad Q_{-1}^- = -\frac{1}{\sqrt{2}} \frac{2 + \sqrt{7}}{4}, \quad Q_{-2}^- = \frac{1}{\sqrt{2}} \frac{1}{4}.$$ 

These differ from Yang and Xie [13] by a factor of $1/\sqrt{2}$, due to differences in notation.

$\phi$ is supported on $[0,2]$ and $\psi$ associated with $\phi$ is supported on $[-2,0]$. $\phi$ is orthogonal and $\psi$ is also orthogonal.

The matrix

$$\begin{bmatrix} P^+(1) & P^-(1) \\ P^-(1) & P^+(1) \end{bmatrix} = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} \begin{bmatrix} P_k^+ & P_k^- \\ P_k^- & P_k^+ \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 5 - \sqrt{7} & 3 + \sqrt{7} \\ 3 + \sqrt{7} & 5 - \sqrt{7} \end{bmatrix}$$

has two eigenvalues 1 and $(1 - \sqrt{7})/4 \approx -0.4114$. Hence, Condition E for $\phi$ is satisfied.

Moments of two-direction multiwavelets by doubling.

The discrete moments $M_j^\pm$ of $\Phi(x) = [\phi(x), \phi(-x)]^T$ for $j = 0, 1, 2$ are

$$M_0^\pm = \frac{1}{8} \begin{bmatrix} 5 - \sqrt{7} & 3 + \sqrt{7} \\ 3 + \sqrt{7} & 5 - \sqrt{7} \end{bmatrix}, \quad M_1^\pm = \frac{1}{8} \begin{bmatrix} 7 - 2\sqrt{7} & 7 + 2\sqrt{7} \\ -7 - 2\sqrt{7} & -7 + 2\sqrt{7} \end{bmatrix}, \quad M_2^\pm = \frac{1}{8} \begin{bmatrix} 11 - 4\sqrt{7} & 17 + 4\sqrt{7} \\ 17 + 4\sqrt{7} & 11 - 4\sqrt{7} \end{bmatrix}.$$ 

The eigenvalues of $M_0^\pm$ are 1 and $(1 - \sqrt{7})/4$, and the corresponding eigenvector to the eigenvalue 1 is $[1,1]^T$. Normalization factor is $\sqrt{2}/2$. Normalized $m_0^\pm$ is $\sqrt{2}/2 [1 \ 1]^T$. We find that $2^j I_2 - M_j^\pm$ is invertible for $j = 1, 2, 3, \ldots$. Hence, we can find the $j$th continuous moments $m_j^\pm$ of $\Phi$ recursively, for $j = 1, 2, 3, \ldots$.

The continuous moments $m_j^\pm$ of $\Phi$ for $j = 0, 1, 2$ are

$$m_0^\pm = \sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad m_1^\pm = \frac{7\sqrt{2} - \sqrt{14}}{12} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad m_2^\pm = \frac{28\sqrt{2} - 7\sqrt{14}}{36} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

Hence, the $j$th continuous moments $m_j$ of $\phi$ for $j = 0, 1, 2$ are

$$m_0 = \frac{\sqrt{2}}{2}, \quad m_1 = \frac{7\sqrt{2} - \sqrt{14}}{12}, \quad m_2 = \frac{28\sqrt{2} - 7\sqrt{14}}{36}.$$ 

The discrete moments $N_j^\pm$ of $\Psi(x) = [\psi(x), \psi(-x)]^T$ for $j = 0, 1, 2$ are

$$N_0^\pm = \frac{1}{8} \begin{bmatrix} 1 + \sqrt{7} & -1 - \sqrt{7} \\ -1 - \sqrt{7} & 1 + \sqrt{7} \end{bmatrix}, \quad N_1^\pm = \frac{\sqrt{7}}{8} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}, \quad N_2^\pm = \frac{1}{8} \begin{bmatrix} -6 + 9\sqrt{7} & 2 - \sqrt{7} \\ 2 - \sqrt{7} & -6 + 9\sqrt{7} \end{bmatrix}.$$ 

The continuous moments $n_j^\pm$ of $\Psi$ for $j = 0, 1, 2$ are

$$n_0^\pm = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad n_1^\pm = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad n_2^\pm = \frac{4\sqrt{2} - \sqrt{14}}{48} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Hence, the $j$th continuous moments $n_j$ of $\psi$ for $j = 0, 1, 2$ are

$$n_0 = 0, \quad n_1 = 0, \quad n_2 = \frac{4\sqrt{2} - \sqrt{14}}{48}.$$  

**Moments of two-direction multiwavelets by separation.** 
Positive and negative discrete moments, $M_j^+$ and $M_j^−$ respectively, of $\phi$ for $j = 0, 1, 2$ are

$$M_0^+ = \frac{5 - \sqrt{7}}{8}, \quad M_1^+ = \frac{7 - 2\sqrt{7}}{8}, \quad M_2^+ = \frac{11 - 4\sqrt{7}}{8},$$

$$M_0^- = \frac{3 + \sqrt{7}}{8}, \quad M_1^- = \frac{7 + 2\sqrt{7}}{8}, \quad M_2^- = \frac{17 + 4\sqrt{7}}{8}.$$  

Hence, the discrete moments $M_j$ of $\phi$ for $j = 0, 1, 2$ are

$$M_0 = 1, \quad M_1 = \frac{7}{4}, \quad M_2 = \frac{7}{2}.$$  

Without computing the positive and negative continuous moments, $m_j^+$ and $m_j^−$ respectively, of $\phi$, we can compute the $j$th continuous moments $m_j$ of $\phi$ for $j = 0, 1, 2, \ldots$ by equation (4.12).

The fact that $M_0 = 1$ implies that the eigenvalue of $M_0$ is 1 and the corresponding eigenvector to the eigenvalue 1 is 1. Normalization factor is $\sqrt{2}/2$. Hence, $m_0 = \sqrt{2}/2$. We find that

$$2^j - [M_0^+ + (-1)^j M_0^-] = \begin{cases} 2^j - \frac{1 - \sqrt{7}}{4}, & \text{for odd } j, \\ 2^j - 1, & \text{for even } j. \end{cases}$$

$2^j - [M_0^+ + (-1)^j M_0^-]$ is not 0 and therefore invertible for $j = 1, 2, 3, \ldots$. Hence, we can find the $j$th continuous moments $m_j$ of $\phi$ recursively, for $j = 1, 2, 3, \ldots$.

The $j$th continuous moments $m_j$ of $\phi$ for $j = 0, 1, 2$ are

$$m_0 = \frac{\sqrt{2}}{2}, \quad m_1 = \frac{7\sqrt{2} - \sqrt{14}}{12}, \quad m_2 = \frac{28\sqrt{2} - 7\sqrt{14}}{36}.$$  

The positive and negative discrete moments, $N_j^+$ and $N_j^−$ respectively, of $\psi$ for $j = 0, 1, 2$ are

$$N_0^+ = \frac{1 + \sqrt{7}}{8}, \quad N_1^+ = \frac{-3\sqrt{7}}{8}, \quad N_2^+ = \frac{-6 + 9\sqrt{7}}{8},$$

$$N_0^- = \frac{-1 + \sqrt{7}}{8}, \quad N_1^- = \frac{\sqrt{7}}{8}, \quad N_2^- = \frac{2 - \sqrt{7}}{8}.$$  

Hence, the discrete moments $N_j$ of $\psi$ for $j = 0, 1, 2$ are

$$N_0 = 0, \quad N_1 = -\frac{\sqrt{7}}{4}, \quad N_2 = -\frac{1}{2} + \sqrt{7}.$$  

The $j$th continuous moments $n_j$ of $\psi$ for $j = 0, 1, 2$ are

$$n_0 = 0, \quad n_1 = 0, \quad n_2 = \frac{4\sqrt{2} - \sqrt{14}}{48}.$$  

Hence, two methods by doubling and by separation generate the same results for the continuous moments $m_j$ and $n_j$ of $\phi$ and $\psi$, respectively for $j = 0, 1, 2, \ldots$.

Since $n_0 = 0 = n_1$ and $n_2 \neq 0$, $\psi$ has 2 vanishing moments and $\phi$ provides approximation order 2.  

$\square$
Example 5.2. In this example we take two-direction multiscaling function $\phi$ and two-direction multiwavelet $\psi$ associated with $\phi$ given in [8]. We note that multiplicity $r = 2$ and dilation factor $d = 2$.

The nonzero recursion coefficients for $\phi$ are

$$
P_{0}^+ = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ -2\sqrt{3} + \sqrt{21} & 0 \end{bmatrix}, \quad P_{0}^- = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 3\sqrt{3} & 0 \end{bmatrix},
$$

$$
P_{1}^+ = \frac{1}{8\sqrt{2}} \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}, \quad P_{1}^- = \frac{1}{8\sqrt{2}} \begin{bmatrix} 4 - 2\sqrt{7} & 0 \\ 0 & 2 + \sqrt{7} \end{bmatrix},
$$

and

$$
P_{2}^+ = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ \sqrt{3} & 0 \end{bmatrix}, \quad P_{2}^- = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ -2\sqrt{3} - \sqrt{21} & 0 \end{bmatrix},
$$

$$
P_{3}^+ = \frac{1}{8\sqrt{2}} \begin{bmatrix} 4 + 2\sqrt{7} & 0 \\ 0 & 2 + \sqrt{7} \end{bmatrix}, \quad P_{3}^- = \frac{1}{8\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.
$$

The nonzero recursion coefficients for $\psi$ are

$$
Q_{0}^+ = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & -4 + 2\sqrt{7} \\ -2 + \sqrt{7} & 0 \end{bmatrix}, \quad Q_{0}^- = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 6 \\ 3 & 0 \end{bmatrix},
$$

$$
Q_{1}^+ = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & -3\sqrt{3} \end{bmatrix}, \quad Q_{1}^- = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & 2\sqrt{3} + \sqrt{21} \end{bmatrix},
$$

and

$$
Q_{2}^+ = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & 2\sqrt{3} - \sqrt{21} \end{bmatrix}, \quad Q_{2}^- = \frac{1}{8\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}.
$$

These differ from Wang, Zhou, and Wang [8] by a factor of $1/\sqrt{2}$, due to differences in notation.

$\phi_1$ is supported on $[0, 2]$ and support of $\phi_2$ is contained in $[-3, 3]$. $\psi_1$ is supported on $[-2, 0]$ and support of $\psi_2$ is contained in $[-3, 3]$.

$\phi$ is orthogonal and $\psi$ is also orthogonal.

The matrix

$$
\begin{bmatrix}
P^+(1) & P^-(1)

P^-(1) & P^+(1)
\end{bmatrix} = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}} \begin{bmatrix} P_k^+ & P_k^-

P_k^- & P_k^+ \end{bmatrix}
$$

has four eigenvalues $1, 1/2, (3-3\sqrt{7} + \sqrt{8-2\sqrt{7}})/16 \approx -0.2057$ and $(3-3\sqrt{7} - \sqrt{8-2\sqrt{7}})/16 \approx -0.4114$. Hence, Condition E for $\phi$ is satisfied.

In this example, we only compute moments by separation for simplicity.

**Moments of two-direction multiwavelets by separation.**

The positive and negative zeroth discrete moments, $M_0^+$ and $M_0^-$, of $\phi$ are

$$
M_0^+ = \frac{1}{16} \begin{bmatrix} 10 - 2\sqrt{7} \\ \sqrt{3} + \sqrt{21} \end{bmatrix}, \quad M_0^- = \frac{1}{16} \begin{bmatrix} 2\sqrt{7} + 6 \\ -\sqrt{3} - \sqrt{21} \end{bmatrix}.
$$

The discrete moments, $M_j$ of $\phi$ for $j = 0, 1, 2$ are

$$
M_0 = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_1 = \frac{1}{8} \begin{bmatrix} 14 \\ -2\sqrt{21} \end{bmatrix}, \quad M_2 = \frac{1}{4} \begin{bmatrix} 14 \\ -\sqrt{3} + 2\sqrt{21} \end{bmatrix}.
$$
The eigenvalues of $M_0$ are 1 and $1/2$, and the corresponding eigenvector to the eigenvalue 1 is $[1, 0]^T$. Normalization factor is $\sqrt{2}/2$. Hence,

$$m_0 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We find that $2^j I_2 - \left[ M_0^+ + (-1)^j M_0^- \right]$ is invertible for $j = 1, 2, 3, \ldots$. Hence, we can find the $j$th continuous moments $m_j$ of $\phi$ recursively, for $j = 1, 2, 3, \ldots$. The $j$th continuous moments $m_j$ of $\phi$ for $j = 0, 1, 2$ are

$$m_0 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad m_1 = \frac{7\sqrt{2} - \sqrt{14}}{12} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad m_2 = \frac{4\sqrt{2} - \sqrt{14}}{252} \begin{bmatrix} 49 \\ 3\sqrt{3} \end{bmatrix}.$$

The discrete moments $N_j$ of $\psi$ for $j = 0, 1, 2$ are

$$N_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 \\ -\sqrt{\frac{3}{2}} \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 \\ -2\sqrt{7} \end{bmatrix}.$$

The $j$th continuous moments $n_j$ of $\psi$ for $j = 0, 1, 2$ are

$$n_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad n_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad n_2 = \frac{1}{168} \begin{bmatrix} 0 \\ 4\sqrt{2} - \sqrt{14} \end{bmatrix}.$$

Hence, $\psi$ has 2 vanishing moments and $\phi$ provides approximation order 2.

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