Abstract

Coisotropic algebras are used to formalize coisotropic reduction in Poisson geometry as well as in deformation quantization and find applications in various other fields as well. In this paper we prove a Serre-Swan Theorem relating the regular projective modules over the coisotropic algebra built out of a manifold $M$, a submanifold $C$ and an integrable smooth distribution $D \subseteq TC$ with vector bundles over this geometric situation and show an equivalence of categories for the case of a simple distribution.

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1 Introduction

In Poisson geometry, coisotropic reduction is one of the common reduction schemes to construct out of a Poisson manifold \((M, \pi)\) with a coisotropic submanifold \(\iota: C \to M\) a new Poisson manifold: being coisotropic means that the Hamiltonian vector fields \(X_f \in \Gamma^\infty(TM)\) are tangent to \(C\) for all functions \(f \in \mathcal{C}^\infty(M)\) with \(\iota^* f = 0\). Thus they define a smooth, but in general singular, distribution which turns out to be integrable nevertheless. The quotient of \(C\) with respect to the orbit relation \(\sim\) induced by this distribution then becomes a Poisson manifold \(M_{\text{red}} = C/\sim\) with Poisson structure \(\pi_{\text{red}}\) whenever it is actually a manifold at all. This construction generalizes the Marsden-Weinstein reduction at momentum level zero, where the coisotropic submanifold is the level set of the momentum map for value zero.

Being omnipresent in Poisson geometry it raises immediately the question whether and how one can pass to a quantum analogue. In fact, this question was first posed by Dirac \[9\] when discussing the quantization of first class constraints which, in modern language, are essentially coisotropic submanifolds. The physical necessity of understanding such constraint systems comes from quantum field theoretical models whenever gauge degrees of freedom are involved.

Within deformation quantization \[1\] many general results on the quantization of coisotropic submanifolds have been obtained, most notably by Bordemann \[2\] for the global aspects in the symplectic case and by Cattaneo and Felder \[3, 4\] for the general Poisson case. The principal idea is to find a star product \(\ast\) on \((M, \pi)\) in such a way that the classical vanishing ideal \(J_C\) of the constraint surface \(C\) becomes or is deformed into a left ideal inside \(\mathcal{C}^\infty(M)[\hbar]\) with respect to \(\ast\). Moreover, one wants the classical Lie normalizer of \(J_C\) with respect to the Poisson bracket to be deformed into the Lie normalizer with respect to the star product commutator. Then the quotient algebra of this normalizer by the (therein two-sided associative) ideal \(J_C\) should provide a quantization of the reduced phase space \((M_{\text{red}}, \pi_{\text{red}})\). In simple enough geometric situations this program is shown to be successful in the above references.

Slightly more general and beyond the original Poisson geometric motivation is the situation of a manifold \(M\) with a submanifold \(C\) being equipped with a possibly singular but integrable smooth distribution \(D \subseteq TC\). This will be our geometric framework in this paper. In particular, in this case one can still consider the orbit space \(M_{\text{red}} = C/D\) whenever it turns out to be a manifold again.

In \[7, 8\] a more algebraic and conceptual approach was proposed: the notion of coisotropic (triples of) algebras is designed to provide an algebraic formulation for the above reduction scheme. One considers a total algebra \(\mathcal{A}_{\text{tot}}\) together with another algebra \(\mathcal{A}_N\) and an algebra homomorphism \(\iota: \mathcal{A}_N \to \mathcal{A}_{\text{tot}}\). Finally, we require a two-sided ideal \(\mathcal{A}_0 \subseteq \mathcal{A}_N\). Then such triples \(\mathcal{A} = (\mathcal{A}_{\text{tot}}, \mathcal{A}_N, \mathcal{A}_0)\) of algebras are suitable to encode both the classical situation as well as the quantized version. They always admit a reduction given by the quotient \(\mathcal{A}_{\text{red}} = \mathcal{A}_N/\mathcal{A}_0\). In the previous works a good category of bimodules over such triples of algebras was studied in detail allowing for Morita theory. In addition, their deformation theory was discussed, including a first investigation of the relevant Hochschild cohomologies.

In our geometric setting, we can associate a coisotropic algebra \(\mathcal{C}^\infty(M, C, D)\) by taking \(\mathcal{C}^\infty(M)\) as tot-component, the functions on \(M\) whose restrictions to \(C\) are constant in direction of \(D\) as N-component and the vanishing ideal of \(C\) as 0-component. If the quotient \(C/D\) is a manifold, the reduced algebra \(\mathcal{C}^\infty(M, C, D)_{\text{red}}\) becomes isomorphic to \(\mathcal{C}^\infty(M_{\text{red}})\).

While \[7, 8\] focus on general algebraic features, the aim of this paper is to give a first contact to the underlying geometry for a particular class of coisotropic algebras by formulating a Serre-Swan Theorem in this context.

Recall that the classical Serre-Swan Theorem for smooth manifolds identifies vector bundles over manifolds with finitely generated projective modules over the algebra of smooth functions. Thus, we would like to understand vector bundles over manifolds that are equipped with a distribution on a submanifold by comparing them to certain projective modules over the corresponding coisotropic
algebra \( \mathcal{C}\infty(M, C, D) \).

To achieve such a theorem we first have to specify what \textit{projective} module over a coisotropic algebra should really mean: this is complicated by the fact that the module category of a coisotropic algebra is \textit{not} an abelian category. Hence the usual standard definition has to be interpreted in the correct way. We base our definition on the notion of free modules and a splitting property with respect to \textit{regular} epimorphisms instead of all epimorphisms. This seems to be the most reasonable choice for a category \( \text{Proj}(s\mathcal{A}) \) of projective modules over a coisotropic algebra \( s\mathcal{A} \).

On the geometric side, we investigate vector bundles \( E_{\text{tot}} \to M \) with a specified subbundle \( E_N \to M \) and a subbundle \( E_0 \subseteq \iota^* E_N \to C \) inside the pull-back bundle on the submanifold \( C \). Moreover, we need a holonomy-free \( D \)-connection for the vector bundle \( \iota^* E_N \) preserving the subbundle \( E_0 \). With an obvious notion of morphisms such \((E_{\text{tot}}, E_N, E_0, \nabla)\) form a category \( \text{Vect}_3(M, C, D) \).

Taking sections of such \((E_{\text{tot}}, E_N, E_0, \nabla)\) then yields a module over the corresponding coisotropic algebra \( \mathcal{C}\infty(M, C, D) \). However, without additional assumptions it will be hard to control whether this module is indeed projective.

Conversely, if we have a finitely generated regular projective module over \( \mathcal{C}\infty(M, C, D) \), then we can construct \((E_{\text{tot}}, E_N, E_0, \nabla)\) whose sections reproduce the module we started with: here we do not yet need any additional assumptions about \( D \).

The final Serre-Swan Theorem we state assumes that the quotient \( M_{\text{red}} = C/D \) is a manifold, i.e. \( D \) is a simple distribution. Then the functor of taking sections yields the equivalence of categories

\[
\Gamma^\infty: \text{Vect}_3(M, C, D) \to \text{Proj}(\mathcal{C}\infty(M, C, D)).
\]  

(1.1)

We finally show that reduction of modules matches the geometric reduction of vector bundles in a functorial way.

The Serre-Swan Theorem raises several questions some of which will be pursued in future works:

- Having a meaningful definition of projective modules one can start investigating the algebraic \( K \)-theory of coisotropic algebras, both in the commutative case but also in general. The Serre-Swan Theorem then provides a passage to geometric \( K \)-theory adapted to submanifolds equipped with an integrable distribution. In this context it would be interesting to understand the deformation quantization of projective modules, see also [4].

- It would be interesting to understand in which scenarios of non-simple distributions the equivalence (1.1) still holds and when it ultimately breaks down.

- Coisotropic algebras arise in other contexts in differential geometry as well: one particularly interesting situation would be a submanifold \( C \subseteq M \) in an ambient manifold with a group action of a discrete group \( G \) on \( C \). Then the \( N \)-component would be smooth functions on \( M \) whose restrictions to \( C \) are \( G \)-invariant and the \( 0 \)-component is still the vanishing ideal. In this situation, one can also try to find an analogue of the above Serre-Swan Theorem.

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2 Coisotropic Structures

In order to formulate a coisotropic version of the Serre-Swan Theorem we first need to find a well-behaved notion of projective coisotropic module. To this end we will first recall the definitions and some properties of coisotropic modules and algebras, before studying coisotropic index sets and the free coisotropic modules generated by such sets. This will finally allow us to find a reasonable definition of projective coisotropic modules and prove many characterizations analogous to the classical case, such as a dual basis lemma and the equivalence to direct summands of free modules. In the following, \( k \) denotes a commutative unital ring.
2.1 Coisotropic Algebras and their Modules

In this preliminary section we introduce coisotropic algebras and their modules following [7,8]. For this we will first need to consider coisotropic \( k \)-modules as the fundamental algebraic structure underlying coisotropic algebras and their modules.

**Definition 2.1 (Coisotropic \( k \)-modules)**

i.) A triple \( E = (E_{\text{tot}}, E_N, E_0) \) of \( k \)-modules together with a module homomorphism \( \iota_E : E_N \rightarrow E_{\text{tot}} \) is called a coisotropic \( k \)-module if \( E_0 \subseteq E_N \) is a sub-module.

ii.) A morphism \( \Phi : E \rightarrow F \) between coisotropic \( k \)-modules is a pair \( (\Phi_{\text{tot}}, \Phi_N) \) of module homomorphisms \( \Phi_{\text{tot}} : E_{\text{tot}} \rightarrow F_{\text{tot}} \) and \( \Phi_N : E_N \rightarrow F_N \) such that \( \Phi_{\text{tot}} \circ \iota_E = \iota_F \circ \Phi_N \) and \( \Phi_N(E_0) \subseteq F_0 \).

iii.) The category of coisotropic \( k \)-modules is denoted by \( C_3\text{Mod} \).

Note that we suppress the ring \( k \) in our notation.

We now collect some categorical properties of coisotropic modules. All of the following can be proved by directly checking the categorical properties, see e.g. [18]. For this let \( E, F \) be coisotropic modules and let \( \Phi, \Psi : E \rightarrow F \) be morphisms of coisotropic modules.

a) The morphism \( \Phi \) is a monomorphism iff \( \Phi_{\text{tot}} \) and \( \Phi_N \) are injective module homomorphisms.

b) The morphism \( \Phi \) is an epimorphism iff \( \Phi_{\text{tot}} \) and \( \Phi_N \) are surjective module homomorphisms.

c) The morphism \( \Phi \) is a regular monomorphism iff it is a monomorphism with \( \Phi_N^{-1}(F_0) = E_0 \).

d) The morphism \( \Phi \) is a regular epimorphism iff it is an epimorphism with \( \Phi_N(E_0) = F_0 \).

Note that monomorphisms (epimorphisms) in \( C_3\text{Mod} \) do in general not agree with regular monomorphisms (epimorphisms), showing that \( C_3\text{Mod} \) is not an abelian category, unlike the usual categories of modules.

e) The kernel of \( \Phi \) is given by the coisotropic module

\[
\ker(\Phi) = (\ker(\Phi_{\text{tot}}), \ker(\Phi_N), \ker(\Phi_N) \cap E_0)
\]

and \( \iota_{\ker} : \ker(\Phi_N) \rightarrow \ker(\Phi_{\text{tot}}) \) the morphism induced by \( \iota_E \).

f) The image of \( \Phi \) is given by the coisotropic module

\[
\im(\Phi) = (\im(\Phi_{\text{tot}}), \im(\Phi_N), \im(\Phi_N|_{E_0}))
\]

and \( \iota_{\im} : \im(\Phi_N) \rightarrow \im(\Phi_{\text{tot}}) \) the morphism induced by \( \iota_F \).

g) The morphisms \( \text{Hom}(E, F) \) of coisotropic modules over \( k \) form a \( k \)-module which can be enlarged to a coisotropic \( k \)-module by defining

\[
C_3\text{Hom}(E, F)_{\text{tot}} := \text{Hom}(E_{\text{tot}}, F_{\text{tot}})
\]

\[
C_3\text{Hom}(E, F)_N := \text{Hom}(E, F)
\]

\[
C_3\text{Hom}(E, F)_0 := \{ (\Phi_{\text{tot}}, \Phi_N) \in \text{Hom}(E, F) \mid \Phi_N(E_0) \subseteq F_0 \},
\]

where \( \iota : \text{Hom}(E, F) \rightarrow \text{Hom}(E_{\text{tot}}, F_{\text{tot}}) \) is the projection onto the first component.

Using the image every morphism of coisotropic modules can be factorized into a regular epimorphism and a monomorphism.

h) The coproduct of \( E \) and \( F \) is given by

\[
E \oplus F = (E_{\text{tot}} + F_{\text{tot}}, E_N + F_N, E_0 + F_0)
\]

with \( \iota_{\oplus} = \iota_E + \iota_F \). It is also called the *direct sum* of \( E \) and \( F \).
It should be clear that also infinite direct sums can be defined this way.

The definition of coisotropic modules allows us to reinterpret several (geometric) reduction procedures in a completely algebraic fashion, as stated in the following straightforward proposition.

**Proposition 2.2 (Reduction)** Mapping a coisotropic module \( \mathcal{E} \) to the quotient \( \mathcal{E}_{\text{red}} = \mathcal{E}_N/\mathcal{E}_0 \) and morphisms of coisotropic modules to the induced morphisms yields a functor

\[
\text{red}: \mathcal{C}_3\text{Mod} \to \text{Mod},
\]

to the category \( \text{Mod} \) of \( \mathbb{k} \)-modules.

The coisotropic modules that arise in geometric examples will be equipped with an additional multiplicative structure. This is captured by the following definition.

**Definition 2.3 (Coisotropic algebra)**

i.) A coisotropic algebra is a triple \( \mathcal{A} = (\mathcal{A}_\text{tot}, \mathcal{A}_N, \mathcal{A}_0) \) consisting of unital associative \( \mathbb{k} \)-algebras \( \mathcal{A}_\text{tot} \), \( \mathcal{A}_N \) and a two-sided ideal \( \mathcal{A}_0 \subseteq \mathcal{A}_N \) together with a unital algebra homomorphism \( \iota: \mathcal{A}_N \to \mathcal{A}_\text{tot} \).

ii.) A morphism of coisotropic algebras \( \mathcal{A} \) and \( \mathcal{B} \) is given by a pair of unital algebra homomorphisms \( \Phi_\text{tot}: \mathcal{A}_\text{tot} \to \mathcal{B}_\text{tot} \) and \( \Phi_N: \mathcal{A}_N \to \mathcal{B}_N \) such that \( \iota_B \circ \Phi_N = \Phi_\text{tot} \circ \iota_A \) and \( \Phi_N(\mathcal{A}_0) \subseteq \mathcal{B}_0 \).

iii.) The category of coisotropic algebras is denoted by \( \mathcal{C}_3\text{Alg} \).

Note that \( \mathcal{C}_3\text{Alg} \) forms indeed a category. A coisotropic algebra \( \mathcal{A} \) is called commutative if \( \mathcal{A}_\text{tot} \) and \( \mathcal{A}_N \) are commutative algebras. The reduction functor of Proposition 2.2 clearly induces a functor \( \text{red}: \mathcal{C}_3\text{Alg} \to \text{Alg} \) with the reduced algebra given by \( \mathcal{A}_{\text{red}} = \mathcal{A}_N/\mathcal{A}_0 \).

**Example 2.4** From differential geometry we obtain the following two basic examples:

i.) Let \( \iota: C \to M \) be a submanifold of a manifold \( M \) and let \( D \subseteq TC \) be an integrable distribution on \( C \). We denote the functions on \( C \) constant along the leaves of \( D \) by \( \mathcal{E}_D^\infty(C) \). Then

\[
\mathcal{E}_D^\infty(M, C, D) := (\mathcal{E}_D^\infty(M), \mathcal{E}_D^\infty(M), j_C),
\]

with

\[
\mathcal{E}_D^\infty(M) := \{ f \in \mathcal{E}_D^\infty(M) \mid \iota^* f \in \mathcal{E}_D^\infty(C) \}
\]

and the vanishing ideal

\[
j_C := \{ f \in \mathcal{E}_D^\infty(M) \mid \iota^* f = 0 \}
\]

is a commutative coisotropic algebra. As soon as the leaf space \( C/D \) carries a canonical manifold structure we have \( \mathcal{E}_D^\infty(M, C, D)_{\text{red}} = \mathcal{E}_D^\infty(C/D) \). Note the slight abuse of notation: \( \mathcal{E}_D^\infty(C) \) denotes the algebra of functions on \( C \) which are constant along the distribution on \( C \), whereas \( \mathcal{E}_D^\infty(M) \) denotes the algebra of functions on \( M \) which are constant along the distribution only on the submanifold \( C \).

ii.) Let \( (M, \pi) \) be a Poisson manifold together with a coisotropic submanifold \( \iota: C \to M \). Then \( \mathcal{A} = (\mathcal{E}_D^\infty(M), \mathcal{E}_D^\infty(M), j_C) \) is a commutative coisotropic algebra and \( \mathcal{A}_{\text{red}} \simeq \mathcal{E}_D^\infty(M)/j_C \) is even a Poisson algebra.

Having these examples in mind we want to understand vector bundles that are compatible with the submanifold and the distribution on it. On the algebraic side this will be captured by the following notion of module over a coisotropic algebra.

**Definition 2.5 (Right module over coisotropic algebra)** Let \( \mathcal{A} \in \mathcal{C}_3\text{Alg} \) be a coisotropic algebra.
i.) A triple $\mathcal{E} = (\mathcal{E}_\text{tot}, \mathcal{E}_N, \mathcal{E}_0)$ consisting of a right $A_{\text{tot}}$-module $\mathcal{E}_\text{tot}$ and right $A_N$-modules $\mathcal{E}_N$ and $\mathcal{E}_0$ together with a module morphism $\iota_0: \mathcal{E}_0 \to \mathcal{E}_\text{tot}$ along the morphism $\iota_\text{cl}: A_N \to A_{\text{tot}}$ is called a coisotropic right $\mathcal{A}$-module if $\mathcal{E}_0 \subseteq \mathcal{E}_N$ is a sub-module such that

$$\mathcal{E}_N \cdot \mathcal{A}_0 \subseteq \mathcal{E}_0.$$  \hfill (2.9)

ii.) A morphism $\Phi: \mathcal{E} \to \mathcal{F}$ between coisotropic right $\mathcal{A}$-modules is a pair $(\Phi_\text{tot}, \Phi_N)$ consisting of an $A_{\text{tot}}$-module morphism $\Phi_\text{tot}: \mathcal{E}_\text{tot} \to \mathcal{F}_\text{tot}$ and an $A_N$-module morphism $\Phi: \mathcal{E}_N \to \mathcal{F}_N$ such that $\Phi_\text{tot} \circ \iota_0 = \iota_\mathcal{F} \circ \Phi_N$ and $\Phi_\text{tot}(\mathcal{E}_0) \subseteq \mathcal{F}_0$.

iii.) The category of coisotropic right $\mathcal{A}$-modules is denoted by $\mathcal{C}_3\text{Mod}(\mathcal{A})$.

There is an obvious notion of left modules and bimodules over coisotropic algebras, see [7], but in the following we will only need right modules. Therefore we will shorten our notation and just say module instead of right module from now on. If we consider the coisotropic algebra $\mathbb{k} = (k, k, 0)$, then modules over the coisotropic algebra $\mathbb{k}$ agree with coisotropic $\mathbb{k}$-modules as introduced in [Definition 2.1] i.e. $\mathcal{C}_3\text{Mod}(\mathbb{k}) = \mathcal{C}_3\text{Mod}$.

**Example 2.6** Let $C \subseteq M$ be a submanifold and let $D \subseteq TC$ be an integrable distribution on $C$. Let, moreover, $E \to M$ be a vector bundle over $M$ and $\nabla$ a covariant derivative on $E$. Then setting $\mathcal{E}_\text{tot} = \Gamma^\infty(E)$,

$$\mathcal{E}_N = \{ s \in \Gamma^\infty(E) \mid \nabla_X s|_C = 0 \text{ for all } X \in \Gamma^\infty(TM) \text{ with } X|_C \in D \} \quad (2.10)$$

and $\mathcal{E}_0 = \{ s \in \Gamma^\infty(E) \mid s|_C = 0 \}$ defines a coisotropic $\mathcal{A}$-module $\mathcal{E}$ for $\mathcal{A} = (\mathcal{C}_\text{tot}(M), \mathcal{C}_{\mathcal{B}}(M), \mathcal{J}_C)$ as in [Example 2.3][i.] Note that the construction of $\mathcal{E}_N$ strongly depends on the choice of the covariant derivative.

Clearly, the quotient $\mathcal{E}_N/\mathcal{E}_0$ is an $\mathcal{A}_\text{red}$-module for any $\mathcal{A}$-module $\mathcal{E}$. Thus, we can easily extend the reduction of [Proposition 2.2][i.] to the case of modules over coisotropic algebras by constructing the functor $\red: \mathcal{C}_3\text{Mod}(\mathcal{A}) \to \mathcal{C}_3\text{Mod}(\mathcal{A}_\text{red})$ using $\mathcal{E}_\text{red} = \mathcal{E}_N/\mathcal{E}_0$.

### 2.2 Coisotropic Index Sets

Forgetting all algebraic structure of a given coisotropic module or coisotropic algebra yields a triple of sets $(M_\text{tot}, M_N, M_0)$ such that $M_0 \subseteq M_N$ and a map $\iota_M: M_N \to M_\text{tot}$.

**Definition 2.7 (Coisotropic index set)** i.) A triple $(M_\text{tot}, M_N, M_0)$ of sets with $M_0 \subseteq M_N$ together with a map $\iota_M: M_N \to M_\text{tot}$ is called a coisotropic index set.

ii.) A morphism of coisotropic index sets $M = (M_\text{tot}, M_N, M_0)$ and $N = (N_\text{tot}, N_N, N_0)$ is a pair $(f_\text{tot}, f_N)$ with $f_\text{tot}: M_\text{tot} \to N_\text{tot}$ and $f_N: M_N \to N_N$ such that $f_N(M_0) \subseteq N_0$ and $f_\text{tot} \circ \iota_M = \iota_N \circ f_N$.

iii.) The category of coisotropic index sets is denoted by $\text{Set}_3$.

**Remark 2.8** Instead of keeping the underlying set of the 0-component of a given coisotropic module we could also use the equivalence relation on the N-component induced by the 0-component. This leads to the notion of coisotropic sets as introduced in [8]. These two concepts do not agree, but for our purpose coisotropic index sets will be more useful.

The category $\text{Set}_3$ inherits a lot of structure from the category $\text{Set}$ of sets. In particular, $\text{Set}_3$ has all finite limits and colimits. But it does not resemble $\text{Set}$ completely, as the next result shows.

**Proposition 2.9 (Mono- and epimorphisms in $\text{Set}_3$)** Let $f: M \to N$ be a morphism of coisotropic index sets.
i.) The morphism $f$ is a monomorphism iff $f_{\text{tot}}$ and $f_N$ are injective.

ii.) The morphism $f$ is an epimorphism iff $f_{\text{tot}}$ and $f_N$ are surjective.

iii.) The morphism $f$ is a regular monomorphism iff $f_{\text{tot}}$ and $f_N$ are injective and $f_N^{-1}(N_0) = M_0$.

iv.) The morphism $f$ is a regular epimorphism iff $f_{\text{tot}}$ and $f_N$ are surjective and $f_N(M_0) = N_0$.

Remark 2.10 The relationship between the categories $\text{Set}$ and $\text{Set}_3$ can be made more precise: In contrast to $\text{Set}$ the category $\text{Set}_3$ is not a topos, but only a quasi-topos, since it does only admit a weak subobject classifier. See e.g. [20] and [14] for more on quasi-topoi.

Recall, that in $\text{Set}$ the axiom of choice can be rephrased by saying that every epimorphism splits, i.e. for every surjective map $f: M \to N$ there exists a map $g: N \to M$ such that $f \circ g = \text{id}_M$. The next example shows that the equivalent statement need not be true in $\text{Set}_3$. In particular, not every epimorphism splits.

Example 2.11 Let $M = (\{0,1\}, \{0,1\}, \emptyset)$ with $\iota_M$ the identity, and $N = (\{0\}, \{0,1\}, \emptyset)$ with $\iota_N = 0$. Then $\Phi = (0, \text{id}_{\{0,1\}}): M \to N$ is a regular epimorphism. Suppose there exists a split $\Psi: N \to M$, then $\Psi_N = \text{id}_{\{0,1\}}$. But $\Psi_{\text{tot}}$ would need to fulfill both $\Psi_{\text{tot}}(0) = \Psi_{\text{tot}}(\iota_N(0)) = \iota_M(\Psi_N(0)) = 0$ and $\Psi_{\text{tot}}(0) = \Psi_{\text{tot}}(\iota_N(1)) = \iota_M(\Psi_N(1)) = 1$. Therefore $(\Psi_{\text{tot}}, \Psi_N)$ can not be chosen to be a morphism of triples of sets.

The next result characterizes completely the coisotropic index sets for which any regular epimorphism into them splits.

Proposition 2.12 Let $P \in \text{Set}_3$ be a coisotropic index set. Then the following statements are equivalent:

i.) Every regular epimorphism $M \to P$ splits.

ii.) For every regular epimorphism $\Phi: M \to N$ and every morphism $\Psi: P \to N$ there exists a morphism $\chi: P \to M$ such that $\Phi \circ \chi = \Psi$.

iii.) The map $\iota_P: P_N \to P_{\text{tot}}$ is injective.

Proof: For $[\text{i.}] \implies [\text{ii.}]$ suppose every regular epimorphism into $P$ splits. Let $\Phi: M \to N$ and $\Psi: P \to N$ be given, with $\Phi$ a regular epimorphism. We can construct the pullback $P_{\Phi \times \Phi} M$ of coisotropic index sets given by

$$(P_{\Phi \times \Phi} M)_{\text{tot}} = \{(x, y) \in P_{\text{tot}} \times M_{\text{tot}} \mid \Psi_{\text{tot}}(x) = \Phi_{\text{tot}}(y)\}$$

$$(P_{\Phi \times \Phi} M)_N = \{(x, y) \in P_N \times M_N \mid \Psi_N(x) = \Phi_N(y)\}$$

$$(P_{\Phi \times \Phi} M)_0 = \{(x, y) \in P_0 \times M_0 \mid \Psi_N(x) = \Phi_N(y)\}$$

with the morphism $\iota_P \times \iota_M$. Then it is easy to see that $\text{pr}_P: P_{\Phi \times \Phi} M \to P$ is a regular epimorphism. Thus by assumption there exists a split $\tilde{\chi}: P \to P_{\Phi \times \Phi} M$, and thus $\chi = \text{pr}_M \circ \tilde{\chi}$ fulfills $\Phi \circ \chi = \Psi$ as wanted. Next, we consider $[\text{ii.}] \implies [\text{iii.}]$ Construct a new coisotropic index set $M$ by defining $M_{\text{tot}} = P_N \sqcup (P_{\text{tot}} \setminus \text{im}(\iota_P))$, $M_N = P_N$, $M_0 = P_0$ and $\iota_M$ by the obvious inclusion. Then $\iota_M$ is injective and $\Phi: M \to P$ defined by $\Phi_N = \text{id}_{P_N}$ and

$$\Phi_{\text{tot}}(p) = \begin{cases} \iota_P(p) & \text{if } p \in P_N \\ p & \text{if } p \in P_{\text{tot}} \setminus \text{im}(\iota_P) \end{cases}$$

is a regular epimorphism. Hence there exists a morphism $\chi: P \to M$ such that $\Phi \circ \chi = \text{id}_P$. But then $\chi_N$ is injective and because of $\chi_{\text{tot}} \circ \iota_P = \iota_M \circ \chi_N$ this implies the injectivity of $\iota_P$. Finally, we show $[\text{iii.}] \implies [\text{i.}]$ Let $\Phi: M \to P$ be a regular epimorphism in $\text{Set}_3$, i.e. $\Phi_{\text{tot}}$ and $\Phi_N$ are
surjective and $\Phi_N(M_0) = P_0$. Thus $\Phi_0 : M_0 \to P_0$ is a surjective map of sets. Hence there exists a map $\Psi_0 : P_0 \to M_0$ such that $\Phi_0 \circ \Psi_0 = \id_{P_0}$. This map can now clearly be extended to $P_N$, giving $\Psi_N : P_N \to M_N$ with $\Phi_N \circ \Psi_N = \id_{P_N}$ preserving the 0-component.

\[
\begin{array}{c}
M_{\text{tot}} \xrightarrow{\Phi_{\text{tot}}} P_{\text{tot}} \\
\downarrow{\iota_M} \quad \downarrow{\iota_P} \\
M_N \xrightarrow{\Phi_N} P_N
\end{array}
\]

Now if we restrict the codomain of $\iota_P$ to $\im(\iota_P)$ then $\iota_P : P_N \to \im(\iota_P)$ is surjective, hence there exists a split $\tau : \im(\iota_P) \to P_N$. This allows us to define $\Psi_{\text{tot}} = \iota_M \circ \Psi_N \circ \tau$. This is a split of $\Phi_{\text{tot}} \mid_{\im(\iota_P)}$ since

$\Phi_{\text{tot}} \circ \Psi_{\text{tot}} = \Phi_{\text{tot}} \circ \iota_M \circ \Psi_N \circ \tau = \iota_P \circ \Phi_N \circ \Psi_N \circ \tau = \id_{\im(\iota_P)}$.

Then extending $\Psi_{\text{tot}}$ to the whole of $P_{\text{tot}}$ yields a section $\Psi_{\text{tot}} : P_{\text{tot}} \to M_{\text{tot}}$ of $\Phi_{\text{tot}}$. To show that this is now a morphism of coisotropic index sets we need the injectivity of $\iota_P$. We have

$\Psi_{\text{tot}} \circ \iota_P = \iota_M \circ \Psi_N \circ \tau \circ \iota_P = \iota_M \circ \Psi_N$

since $\tau \circ \iota_P = \id_{P_N}$.

We will denote the category of coisotropic index sets $P$ with injective $\iota_P$ by $\Set_{3}^{\text{inj}}$. Proposition 2.12 shows that $\Set_{3}^{\text{inj}}$ is exactly the subcategory of regular objects in $\Set_{3}$.

### 2.3 Free Coisotropic Modules

After having established a reasonable notion of sets underlying coisotropic modules we now want to construct free coisotropic modules with basis given by a coisotropic index set.

**Lemma 2.13** Let $M \in \Set_3$ and $A \in C_3\text{Alg}$ be given. Then $A(M)$ defined by

\[
\begin{align*}
& (A(M))_{\text{tot}} := A(M_{\text{tot}}) \quad (2.11) \\
& (A(M))_N := A(M_N) \quad (2.12) \\
& (A(M))_0 := A(M_0) + A(M_N) \quad (2.13)
\end{align*}
\]

with $\iota_{A(M)} : A(M_N) \to A(M_{\text{tot}})$ given by

\[
\iota_{A(M)} \left( \sum_{m \in M_N} b^N_m a_m \right) := \sum_{m \in M_{\text{tot}}} b^\text{tot}_m \left( \sum_{n \in \iota^{-1}_{\iota_M}(m)} \iota_{A}(a_n) \right) \quad (2.14)
\]

is a coisotropic $A$-module. Here $b^\text{tot}_m$ and $b^N_m$ denote the standard bases of $A(M_{\text{tot}})$ and $A(M_N)$, respectively.

The following result shows that these coisotropic modules fulfill the usual universal property for free modules.
Proposition 2.14 Let $M \in \text{Set}_3$ and $\mathcal{A} \in \text{C}_3\text{Alg}$ be given. The coisotropic $\mathcal{A}$-module $\mathcal{A}^{(M)}$ together with the morphism $i: M \to \mathcal{A}^{(M)}$ given by the usual embedding of the standard basis fulfills the following universal property: For every $E \in \text{C}_3\text{Mod}(\mathcal{A})$ and every morphism $\phi: M \to E$ of coisotropic index sets there exists a unique morphism $\Phi: \mathcal{A}^{(M)} \to E$ of coisotropic $\mathcal{A}$-modules such that the diagram

$$
\begin{aligned}
\mathcal{A}^{(M)} & \xrightarrow{\Phi} E \\
i & \downarrow \phi \\
M & \xrightarrow{i} \mathcal{A}^{(M)}
\end{aligned}
$$

(2.15)

commutes.

PROOF: Note that $(\mathcal{A}^{(M)})_\text{tot}$ and $(\mathcal{A}^{(M)})_\text{N}$ are free. Therefore, they fulfill the corresponding universal property separately. Moreover, we clearly have

$$\Phi_N(\mathcal{A}_0^{(M)}) = \Phi_N(\mathcal{A}_N^{(M)} \cdot \mathcal{A}_0) = \Phi_N(\mathcal{A}_N^{(M)}) \cdot \mathcal{A}_0 \subseteq E_N \cdot \mathcal{A}_0 \subseteq E_0$$

and $\Phi_N(\mathcal{A}_N^{(M)}) = \Phi_N(i_N(M_0)) = \phi_N(M_0) \subseteq E_0$. \qed

Definition 2.15 (Set$_3$-free coisotropic $\mathcal{A}$-module) A coisotropic $\mathcal{A}$-module of the form $\mathcal{A}^{(M)}$ for some coisotropic index set $M \in \text{Set}_3$ is called Set$_3$-free with basis $M$. It is called finitely generated if $M$ consists of finite sets.

Coisotropic modules of the form $\mathcal{A}^{(M)}$ with $M$ a coisotropic index set with injective $\iota_M: M_\text{N} \to M_\text{tot}$ will be called Set$_3^{\text{inj}}$-free.

Remark 2.16 It is not hard to show that mapping coisotropic index sets to free coisotropic modules defines a functor from Set$_3$ to $\text{C}_3\text{Mod}(\mathcal{A})$ which is left adjoint to the forgetful functor from $\text{C}_3\text{Mod}(\mathcal{A})$ to Set$_3$, hence justifying the name free coisotropic module.

Example 2.17 \begin{enumerate}[i.)]
\item Every coisotropic algebra $\mathcal{A}$ regarded as a module over itself is free with basis $M = (\{\text{pt}\}, \{\text{pt}\}, \emptyset)$.
\item Analogously to (2.3) we can associate to every coisotropic right $\mathcal{A}$-module $E$ a dual $E^*$ with
\begin{align}
E^*_\text{tot} &= \text{Hom}_{\mathcal{A}_\text{tot}}(E_\text{tot}, \mathcal{A}_\text{tot}), \\
E^*_N &= \text{Hom}_\mathcal{A}(E, \mathcal{A}_N), \\
E^*_0 &= \{ (\alpha_\text{tot}, \alpha_N) \in \text{Hom}_{\mathcal{A}}(E, \mathcal{A}_N) \mid \alpha_N(E_N) \subseteq E_0 \}.
\end{align}
\end{enumerate}

It has the structure of a coisotropic left $\mathcal{A}$-module. Let now $\mathcal{A}^{(M)}$ be a finitely generated Set$_3^{\text{inj}}$-free coisotropic module with $M_0 \subseteq M_\text{N} \subseteq M_\text{tot}$. Then its dual is given by

$$
\begin{aligned}
(\mathcal{A}^{(M)})^*_\text{tot} &\simeq (\mathcal{A}^{(M)}_\text{tot})^*, \\
(\mathcal{A}^{(M)})^*_N &\simeq (\mathcal{A}^{(M)}_0)^* \oplus (\mathcal{A}^{(M)}_N \setminus M_0)^* \oplus (\mathcal{A}^{(M)}_\text{tot} \setminus M_N)^*, \\
(\mathcal{A}^{(M)})^*_0 &\simeq (\mathcal{A}^{(M)}_0)^* \oplus (\mathcal{A}^{(M)}_\text{tot} \setminus M_N).
\end{aligned}
$$

(2.17)

As the N-component might fail to be a free $\mathcal{A}_N$-module, this is in general not a free coisotropic $\mathcal{A}$-module. Note, however, that its reduction is given by the free module $(\mathcal{A}^{(M)}_\text{red})^{(M_N \setminus M_0)}$ and that there exists a direct sum decomposition

$$
(\mathcal{A}^{(M)})^* = \mathcal{F} \oplus \mathcal{G}
$$

(2.18)

into coisotropic left $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{G}$ where $\mathcal{F}$ is free and $\mathcal{G}_\text{red} = \{0\}$. 

9
2.4 Projective Coisotropic Modules

In this section we want to give characterizations of regular projective coisotropic modules that more closely resemble the usual characterizations of projective modules via projections, dual bases, or as direct summands of free modules. For this let us start with a categorical definition of projective objects using a lifting property based on regular epimorphisms instead of general epimorphisms:

**Definition 2.18 (Regular projective module)** A coisotropic $\mathcal{A}$-module $\mathcal{P} \in C_3\text{Mod}(\mathcal{A})$ is called regular projective if for every morphism $\Psi: \mathcal{P} \to \mathcal{F}$ and every regular epimorphism $\Phi: \mathcal{E} \to \mathcal{F}$ there exists a morphism $\chi: \mathcal{P} \to \mathcal{E}$ such that $\Phi \circ \chi = \Psi$. Diagrammatically:

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\chi} & \mathcal{E} \\
\downarrow{\Phi} & & \downarrow{\Psi} \\
\mathcal{F} & & \\
\end{array}
\]

We start with an important class of regular projective coisotropic modules: the $\text{Set}_{inv}^3$-free coisotropic modules are regular projective.

**Lemma 2.19** Every $\text{Set}_{inv}^3$-free coisotropic module in $C_3\text{Mod}(\mathcal{A})$ is regular projective.

**Proof:** Let $\mathcal{A}(M)$ be a free coisotropic module with $M \in \text{Set}_{inv}^3$. Suppose the following morphisms are given:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\Phi} & \mathcal{F} \\
\mathcal{P} \xrightarrow{\Psi} \\
\end{array}
\]

with $\Phi$ a regular epimorphism. Since $\Phi$ and $\Psi$ induce morphisms $\phi: E \to F$ and $\psi: M \to F$ of coisotropic index sets we know by [Proposition 2.12] that there exists $\xi: M \to E$ such that $\phi \circ \xi = \psi$. Then by the freeness of $\mathcal{A}(M)$ there exists $\Xi: \mathcal{A}(M) \to E$ such that $\Phi \circ \Xi$ restricted to $M$ is just $\psi$. Hence $\Phi \circ \Xi = \Psi$. $\Box$

The category $C_3\text{Mod}(\mathcal{A})$ has enough $\text{Set}_{inv}^3$-projectives in the following sense:

**Proposition 2.20** For every coisotropic module $\mathcal{E} \in C_3\text{Mod}(\mathcal{A})$ there exists $M \in \text{Set}_{inv}^3$ and a regular epimorphism $\Phi: \mathcal{A}(M) \to \mathcal{E}$.

**Proof:** As constructed in the proof of [Proposition 2.12] there exists $M \in \text{Set}_{inv}^3$ and a regular epimorphism $\phi: M \to \mathcal{E}$ of coisotropic index sets. Then by the universal property of $\mathcal{A}(M)$ there exists $\Phi: \mathcal{A}(M) \to \mathcal{E}$ such that $\Phi \circ \iota = \phi$. Then $\Phi$ is a regular epimorphism since so is $\phi$. $\Box$

Another important notion in the characterization of regular projective coisotropic modules is that of a split exact sequence. A sequence of morphisms

\[
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0
\]

is a called short exact if $f$ is a monomorphism, $\text{im}(f) = \ker(g)$, and $g$ is a regular epimorphism. It is called split exact if in addition there exists $h: C \to B$ such that $g \circ h = \text{id}_C$.

**Proposition 2.21** Let $\mathcal{P} \in C_3\text{Mod}(\mathcal{A})$ be a regular projective coisotropic module. Then every short exact sequence of the form

\[
0 \longrightarrow \mathcal{E} \xrightarrow{\Phi} \mathcal{F} \xrightarrow{\Psi} \mathcal{P} \longrightarrow 0
\]

is a split exact sequence.
Proof: Since $\mathcal{P}$ is regular projective and $\Psi$ is a regular epimorphism the sequence splits by the universal property of $\mathcal{P}$. □

For us split exact sequences are important because of the so-called splitting lemma. The splitting lemma is a basic result in homological algebra for modules over rings. Despite $C_3\text{Mod}(\mathcal{A})$ not being an abelian category, the splitting lemma nevertheless holds for coisotropic modules.

**Proposition 2.22 (Splitting lemma in $C_3\text{Mod}(\mathcal{A})$)** A short exact sequence

$$0 \longrightarrow E \overset{\Phi}{\longrightarrow} F \overset{\Psi}{\longrightarrow} G \longrightarrow 0$$

in $C_3\text{Mod}(\mathcal{A})$ splits if and only if it is isomorphic as a sequence to

$$0 \longrightarrow E \overset{i_E}{\longrightarrow} E \oplus G \overset{p_G}{\longrightarrow} G \longrightarrow 0$$

with the canonical inclusion $i_E$ and projection $p_G$.

Proof: Suppose there exists $\xi : F \longrightarrow E$ such that $\xi \circ \Phi = \text{id}_E$. Then we know that $F_{\text{tot}} \simeq E_{\text{tot}} \oplus \Theta_{\text{tot}}$ and $F_N \simeq E_N \oplus \Theta_N$ by the splitting lemma in the respective categories of modules. We denote these isomorphisms by $\theta_{\text{tot}}$ and $\theta_N$, respectively. To show that these form a morphism of coisotropic triples consider that $\theta = i_E \circ \xi + i_G \circ \Psi$ is a composition of morphisms of coisotropic triples. Thus $\theta$ itself is a morphism of coisotropic triples. Moreover, $\theta_N(\mathcal{F}_N) = \xi_N(\mathcal{F}_N) + \Psi_N(\mathcal{F}_N) = \mathcal{E}_0 + \mathcal{G}_0$ holds since $\xi$ and $\Psi$ are regular epimorphisms. Hence, $\theta$ is an isomorphism of coisotropic modules. Conversely, suppose $\theta : \mathcal{F} \longrightarrow E \oplus G$ is an isomorphism such that $\theta \circ \Phi = i_E$ and $p_G \circ \theta = \Psi$. Then $\theta^{-1} \circ i_G$ is clearly a splitting for (2.22). □

The following result shows that regular projective coisotropic modules can be described as direct summands of $\text{Set}^{\text{inj}}_3$-free modules. The proof is completely analogous to the usual case, see e.g. [13, Prop. 3.10].

**Theorem 2.23 (Regular projective modules)** Let $\mathcal{P} \in C_3\text{Mod}(\mathcal{A})$ be given. The following statements are equivalent:

i.) The module $\mathcal{P}$ is regular projective.

ii.) Every short exact sequence $0 \rightarrow E \rightarrow \mathcal{F} \rightarrow \mathcal{P} \rightarrow 0$ splits.

iii.) The module $\mathcal{P}$ is a direct summand of a $\text{Set}^{\text{inj}}_3$-free module, i.e. there exists $M \in \text{Set}^{\text{inj}}_3$ and $E \in C_3\text{Mod}(\mathcal{A})$ such that $\mathcal{A}(M) \simeq \mathcal{P} \oplus E$.

iv.) There exists $M \in \text{Set}^{\text{inj}}_3$ and $e = (e_{\text{tot}}, e_N) \in \text{End}_{\mathcal{A}}(\mathcal{A}(M))$ such that $e^2 = e$ and $\mathcal{P} \simeq e\mathcal{A}(M) = \text{im}(e)$.

Proof: We first show the equivalence of [i.), ii.) and iii.)]. The implication [i.) $\implies$ ii.) is given by [Proposition 2.21]. Assume [ii.) and iii.)]. By [Proposition 2.20] there exists a short exact sequence $0 \rightarrow E \rightarrow \mathcal{A}(M) \rightarrow \mathcal{P} \rightarrow 0$ with $M \in \text{Set}^{\text{inj}}_3$. This sequence splits by assumption, and therefore by the splitting lemma we have $\mathcal{A}(M) \simeq E \oplus \mathcal{P}$. Now assuming [iii.)] we have a split exact sequence $0 \rightarrow E \rightarrow \mathcal{A}(M) \rightarrow \mathcal{P} \rightarrow 0$ with $M \in \text{Set}^{\text{inj}}_3$. Let $\Psi : \mathcal{P} \rightarrow \mathcal{F}$ and $\Phi : \mathcal{G} \rightarrow \mathcal{F}$ be given with $\Phi$ regular epimorphism. We get the following diagram:

$$
\begin{array}{c}
0 \longrightarrow E \overset{i}{\longrightarrow} \mathcal{A}(M) \overset{\pi}{\longrightarrow} \mathcal{P} \longrightarrow 0 \\
\mathcal{G} \overset{\Phi}{\longrightarrow} \mathcal{F}
\end{array}
$$
Since $\mathcal{A}^{(M)}$ is regular projective there exists a morphism $\eta: \mathcal{A}^{(M)} \to \mathcal{P}$ such that $\Phi \circ \eta = \Psi \circ \pi$. Then $\eta \circ (\eta': \mathcal{P} \to \mathcal{G})$ yields the desired morphism making $\mathcal{P}$ projective. Statements $[iii.]$ and $[iv.]$ are equivalent: If $\mathcal{A}^{(M)} \simeq \mathcal{P} \oplus \mathcal{E}$, then choose for $e \in \text{End}_{\mathcal{A}}(\mathcal{A}^{(M)})$ the projection on $\mathcal{P}$. If $\mathcal{P} \simeq e\mathcal{A}^{(M)}$, then $\mathcal{E} \simeq \ker(e)$ provides the correct direct summand.

The equivalence of $[i.]$ and $[ii.]$ can also be phrased as $\mathcal{P}$ is regular projective if and only if every regular epimorphism $\mathcal{F} \rightarrow \mathcal{P}$ splits, compare Proposition 2.12. A coisotropic index set $M$ such that $\mathcal{A}^{(M)} \simeq \mathcal{P} \oplus \mathcal{E}$ is called generating set of the regular projective module $\mathcal{P}$. In addition to the above characterizations of regular projective modules we can also use a coisotropic version of a dual basis.

**Proposition 2.24 (Dual basis)** Let $\mathcal{P} \in \text{C3Mod}(\mathcal{A})$. Then $\mathcal{P}$ is regular projective with generating set $M \in \text{Set}^{\text{inj}}$ if and only if there exist families $(e_n)_{n \in M_{\text{tot}}} \subseteq \mathcal{P}_{\text{tot}}$ and $(f_m)_{m \in M_N} \subseteq \mathcal{P}_N$ of elements and families $(e^n)_{n \in M_{\text{tot}}} \subseteq (\mathcal{P}_{\text{tot}})^*$ = $\text{Hom}_{\mathcal{A}}(\mathcal{P}_{\text{tot}}, \mathcal{A}_{\text{tot}})$ and $(f^m)_{m \in M_N} \subseteq (\mathcal{P}_N)^*$ = $\text{Hom}_{\mathcal{A}_N}(\mathcal{P}_N, \mathcal{A}_N)$ of functionals such that

$$
x_{\text{tot}} = \sum_{n \in M_{\text{tot}}} e_n e^n(x_{\text{tot}}) \quad \text{and} \quad x_N = \sum_{m \in M_N} f_m f^m(x_N)
$$

(2.24)

for all $x_{\text{tot}} \in \mathcal{P}_{\text{tot}}, x_N \in \mathcal{P}_N$ where for fixed $x_{\text{tot}}, x_N$ only finitely many of the $e^n(x_{\text{tot}}), f^m(x_N)$ differ from 0. Moreover, the following properties need to be satisfied:

i.) One has $e_{\text{tot}}(m) = \iota_{\mathcal{P}}(f_m)$ for $m \in M_N$.

ii.) One has $f_m \in \mathcal{P}_0$ for $m \in M_0$.

iii.) One has $e^{\mathcal{P}}(m) \circ \iota_{\mathcal{P}} = \iota_{\mathcal{A}} \circ f^m$ for $m \in M_N$.

iv.) One has $e^n \circ \iota_{\mathcal{P}} = 0$ for $n \in M_{\text{tot}} \setminus e_{\text{tot}}(M_N)$.

v.) One has $f^m(x) \in \mathcal{A}_0$ for $x \in \mathcal{P}_0, m \in M_N \setminus M_0$.

**Proof:** Let $\mathcal{P} \simeq e\mathcal{A}^{(M)}$ be regular projective with idempotent $e \in \text{End}_{\mathcal{A}}(\mathcal{A}^{(M)})$ and generating set $M \in \text{Set}^{\text{inj}}$. Denote by $b_n \in \mathcal{A}^{(M_{\text{tot}})}_{\text{tot}}$ and $c_m \in \mathcal{A}^{(M_N)}_N$ the standard bases and by $b^n$ and $e^n$ the canonical coordinate functionals. Defining $e_n = e_{\text{tot}}(b_n)$ and $f_m = e_N(c_m)$ for $n \in M_{\text{tot}}$ and $m \in M_N$ as well as $e^n = b^n_{\mathcal{A}^{(M)}}$ and $f^m = c^m_{\mathcal{A}^{(M)}}$ gives usual dual bases for $\mathcal{A}^{(M_{\text{tot}})}_{\text{tot}}$ and $\mathcal{A}^{(M_N)}_N$. Thus we get (2.24). Since $e$ is a morphism of coisotropic modules we have $[i.]$ and $[ii.\text{]}$ by the definition of $e_n$ and $f_m$. For $x \in \mathcal{A}^{(M_N)}_N$ it holds that

$$b^n(\iota_{\mathcal{A}^{(M)}}(x)) = b^n\left(\iota_{\mathcal{A}^{(M)}}\left(\sum_{m \in M_N} c_m x_m\right)\right) = b^n\left(\sum_{m \in M_N} b_{\text{tot}}(m)\iota_{\mathcal{A}}(x_m)\right),$$

and therefore we get $b^{\mathcal{P}}(m) \circ \iota_{\mathcal{A}^{(M)}} = \iota_{\mathcal{A}} \circ e^n$ by setting $n = e_{\text{tot}}(m)$ and $b^n \circ \iota_{\mathcal{A}^{(M)}} = 0$ for $n \in M_{\text{tot}} \setminus e_{\text{tot}}(M_N)$. Since $e^n$ and $f^m$ are defined as restrictions of $b^n$ and $c^m$ to $\mathcal{A}^{(M)}$ we get $[iii.]$ and $[iv.]$. If now $x \in \mathcal{P}_0 \simeq e_N(\mathcal{A}^{(M)})_0 \subseteq \mathcal{A}^{(M_N)}_N + \mathcal{A}^{(M_0)}_0$, then $e^n(x) \in \mathcal{A}_0$ for $m \in M_N \setminus M_0$ and therefore $[v.]$ holds. Let now such a dual basis in the above sense be given. The map $M \to \mathcal{P}$ of coisotropic index sets defined by $n \mapsto e_n$ and $m \mapsto f_m$ is a morphism of coisotropic index sets because of $[i.]$ and $[ii.\text{]}$. By the universal property of free coisotropic modules we thus get an induced morphism $q: \mathcal{A}^{(M)} \to \mathcal{P}$. We define $i: \mathcal{P} \to \mathcal{A}^{(M)}$ by

$$i_{\mathcal{P}}: x_{\mathcal{P}} \mapsto \sum_{n \in M_{\text{tot}}} b_n e^n(x_{\mathcal{P}}) \quad \text{and} \quad i_N: x_N \mapsto \sum_{m \in M_N} c_m f^m(x_N).$$

The maps $i_{\mathcal{P}}$ and $i_N$ are clearly module morphisms as the $e^n$ and $f^m$ are. To see that these form a morphism of coisotropic modules observe that

$$(i_{\mathcal{P}} \circ \iota_{\mathcal{P}})(x_{\mathcal{P}}) = \sum_{n \in M_{\text{tot}}} b_n e^n(\iota_{\mathcal{P}}(x_{\mathcal{P}})) = \sum_{m \in M_N} b_{\text{tot}}(m)\iota_{\mathcal{A}}(f^m(x_N)))$$
Theorem 2.23 shows that for a regular projective module \( E \) over a coisotropic algebra \( \mathcal{A} \) the components \( E_{\text{tot}} \) and \( E_N \) are projective \( \mathcal{A}_{\text{tot}-} \)- and \( \mathcal{A}_N \)-modules, respectively. Thus, in order to describe the regular projective modules over \( \mathcal{C}_{(M,C,D)} = (\mathcal{C}_{\infty}(M),\mathcal{C}_{\omega}(M),\mathcal{J}_{\omega}(M),f_C) \) as in Example 2.24 \([17]\) it makes sense to first focus on the tot- and N-components separately.

It is well-known that projective modules over the smooth functions on a manifold are essentially sections of a vector bundle over said manifold, see e.g. \([19]\) Thm. 11.32 and Remark thereafter:
Remark 3.1 (Serre-Swan Theorem) Let \( E \to N \) be a vector bundle over a smooth manifold.

i.) The sections \( \Gamma^\infty(E) \) are a finitely generated projective module over \( \mathcal{C}^\infty(N) \).

ii.) If \( N \) is connected, then taking sections yields an equivalence of categories

\[
\Gamma^\infty : \text{Vect}(N) \to \text{Proj}(\mathcal{C}^\infty(N)),
\]

where \( \text{Vect}(N) \) is the category of vector bundles over \( N \) together with vector bundle morphisms over the identity and \( \text{Proj}(\mathcal{C}^\infty(N)) \) are the finitely generated projective modules over \( \mathcal{C}^\infty(N) \) together with module morphisms.

This already describes the \( \text{tot} \)-components of the regular projective modules of interest. To investigate the \( \text{N} \)-components, we first restrict ourselves to the submanifold \( C \subseteq M \): Given an integrable distribution \( D \subseteq TC \) on a smooth manifold \( C \) we want to understand the projective modules over \( \mathcal{C}^\infty_D(C) \), the functions constant along the leaves of \( D \).

### 3.1 \( D \)-Connections and Quotients of Vector Bundles

If \( D \subseteq TC \) is simple, i.e. the leaf space \( C/D \) carries the structure of a smooth manifold such that the projection \( \pi : C \to C/D \) is a surjective submersion, then

\[
\pi^* : \mathcal{C}^\infty(C/D) \to \mathcal{C}^\infty_D(C)
\]

is an isomorphism of algebras. Therefore, instead of understanding modules over \( \mathcal{C}^\infty_D(C) \) we can as well consider modules over \( \mathcal{C}^\infty(C/D) \).

It turns out that—similarly to \( \mathcal{C}^\infty_D(C) \) being the functions constant along the leaves—the projective \( \mathcal{C}^\infty_D(C) \)-modules are sections on vector bundles over \( C \) which are parallel with respect to some covariant derivative. However, only the part of the covariant derivative in direction of \( D \) plays a role.

This leads to the following definition:

**Definition 3.2 (\( D \)-Connection)** Let \( D \subseteq TC \) be a smooth (possibly singular) distribution and let \( E \to C \) be a vector bundle. A map \( \nabla : D \times \Gamma^\infty(E) \to E \) is called \( D \)-connection on \( E \) if it is the restriction of a covariant derivative \( \nabla : TC \times \Gamma^\infty(E) \to E \).

Similar concepts, sometimes called partial connections, are used in [3, Ch. 6], [10], [15, Def. 2.2]. A generalization of this notion is used in the context of Lie algebroids, see e.g. [11]. Later, requiring the \( D \)-connections to be restrictions of covariant derivatives on \( TC \) will ensure some smoothness also in direction transverse to the leaves. Indeed, in the case of a locally constant rank the following statement is easy to prove using projections onto the subbundle \( D \) (on connected components of \( C \)):

**Lemma 3.3** Let \( D \subseteq TC \) be a smooth distribution with locally constant rank. Let \( \nabla : D \times \Gamma^\infty(E) \to E \) be a \( \mathbb{R} \)-bilinear map such that:

i.) We have \( \nabla_{v_p}s \in E_p \) for \( v_p \in D \).

ii.) There is the following Leibniz rule in the second argument, i.e.

\[
\nabla_{v_p}(fs) = f(p)\nabla_{v_p}s + v_p(f)s(p)
\]

for \( v_p \in D \), \( f \in \mathcal{C}^\infty(C) \), and \( s \in \Gamma^\infty(E) \).

iii.) The induced map

\[
\nabla : \Gamma^\infty(D) \times \Gamma^\infty(E) \to \Gamma^\infty(E) \quad \text{with} \quad \nabla_Xs|_p := \nabla_{X(p)}s
\]

indeed maps into \( \Gamma^\infty(E) \).
Then $\nabla$ is a $D$-connection.

In the singular setting there are maps fulfilling the requirements of Lemma 3.3 which are not restrictions of a covariant derivative on $TC$:

**Example 3.4** Consider the (integrable) smooth singular distribution $D \subseteq T\mathbb{R}$ defined by

$$D_p := \begin{cases} T_p\mathbb{R} & \text{if } p > 0 \\ \{0_p\} & \text{if } p \leq 0. \end{cases}$$ (3.5)

We define $\nabla: D \times \Gamma^\infty(T\mathbb{R}) \to T\mathbb{R}$ by

$$\nabla_{v\partial_x}|_p (g\partial_x) := v \cdot g'(p) \cdot \partial_x|_p + v \cdot \frac{1}{p} \cdot g(p) \cdot \partial_x|_p$$ (3.6)

for $p > 0$, $v \in \mathbb{R}$, and $g \in \mathcal{C}^\infty(\mathbb{R})$. For $p \leq 0$ the only element of $D_p$ is $0_p$ and we set

$$\nabla_{0_p}(g\partial_x) := 0.$$ (3.7)

This fulfills the requirements of Lemma 3.3 Indeed, it maps smooth sections to smooth sections because for $f \in \mathcal{C}^\infty(\mathbb{R})$ we have $f\partial_x \in \Gamma^\infty(D)$ iff $f(p) = 0$ for all $p \leq 0$. But clearly $\nabla$ can not be the restriction of a covariant derivative $\nabla$ in direction $T\mathbb{R}$ because we would have

$$\hat{\nabla}_{\partial_x}|_p (\partial_x) = \frac{1}{p} \partial_x|_p$$ (3.8)

for all $p > 0$.

In order to show projectivity of parallel sections on a vector bundle over $C$ we want to identify them with sections over a suitable vector bundle over $C/D$. Denote by $P_{\gamma,p\rightarrow q}$ the parallel transport along a curve $\gamma$ from $p$ to $q$ inside a fixed leaf with respect to a given $D$-connection. We need to assume that the parallel transport $P$ is path-independent and call such $D$-connections holonomy-free. Then we define the following map on the fibered product $E \times_\pi C = \{(v_p, q) \subseteq E \times C | \pi(p) = \pi(q)\}$:

**Definition 3.5** ($\nabla$-Transport map) Let $D \subseteq TC$ be simple with projection $\pi: C \to C/D$ and $\nabla$ a holonomy-free $D$-connection on a vector bundle $E \to C$. Then we call

$$T^\nabla: E \times_\pi C \to E, \ (v_p, q) \mapsto P_{\gamma,p\rightarrow q}(v_p),$$ (3.9)

where $\gamma$ is any smooth curve inside the leaf $\pi^{-1}(\{\pi(p)\}) = \pi^{-1}(\{\pi(q)\})$ connecting $p$ and $q$ the $\nabla$-transport map.

Note that the parallel transport along leaves does not depend on the extension of $\nabla$ to $TC$. Some properties of this map are summarized in the next statement:

**Lemma 3.6** Let $\nabla$ be a holonomy-free $D$-connection with transport map $T^\nabla$.

i.) For $(v_p, q) \in E \times_\pi C$ one has $T^\nabla(v_p, q) \in E_q$.

ii.) The transport map $T^\nabla$ is linear in the first argument, i.e. one has

$$T^\nabla(\lambda v_p + \mu w_p, q) = \lambda T^\nabla(v_p, q) + \mu T^\nabla(w_p, q) \text{ for } (v_p, q), (w_p, q) \in E \times_\pi C.$$

iii.) For $v_p \in E$ one has $T^\nabla(v_p, p) = v_p$.

iv.) For $(v_p, q) \in E \times_\pi C$ and $r \in \pi^{-1}(\pi(\{p\})) = \pi^{-1}(\pi(\{q\}))$ one has $T^\nabla(T^\nabla(v_p, q), r) = T^\nabla(v_p, r)$.

v.) The transport map $T^\nabla$ is smooth.
Proof: The algebraic properties are clear. For smoothness note that by standard arguments from the theory of ordinary differential equations, for any \( p \in C \) there exists an open neighborhood \( U \subseteq C \) such that \( T^\nabla \) is smooth on \( E|_{U \times \pi} \). Using the property \([HE]\) we can iterate this statement to see that for any \( p,q \in C \) there exist open neighborhoods of \( U_p \) of \( p \) and \( U_q \) of \( q \) such that \( T^\nabla \) is smooth on \( E|_{U_p \times \pi} \).

In \([17]\) Def. 2.1.1 this is called a linear action of \( R(\pi) \) on \( E \) where \( R(\pi) \subseteq C \times C \) is the equivalence relation induced by \( \pi \). The \( \nabla \)-transport map induces an equivalence relation on the vector bundle:

**Definition 3.7 (\( \nabla \)-Equivalence)** Let \( D \subseteq TC \) be simple with projection \( \pi: C \to C/D \) and let \( \nabla \) be a holonomy-free \( D \)-connection. Then for \( v_p,w_q \in E \) we set
\[
v_p \sim^\nabla w_q \iff \pi(p) = \pi(q) \quad \text{and} \quad T^\nabla(v_p,q) = w_q.
\]

Its set of equivalence classes \( E/\sim^\nabla \) offers a good candidate for a projection onto \( C/D \) which could turn it into a vector bundle over \( C/D \): We simply map an equivalence class \([v_p]\) to \( \pi(p) \). Indeed, we can equip \( E/\sim^\nabla \) with the structure of a vector bundle over \( C/D \):

**Proposition 3.8** Let \( D \subseteq TC \) be simple with projection \( \pi: C \to C/D \) and let \( \nabla \) be a holonomy-free \( D \)-connection on a vector bundle \( E \to C \). Then there exists a unique vector bundle structure on
\[
pr_C: E/\sim^\nabla \to C/D, \quad [v_p] \mapsto \pi(p)
\]
such that the quotient map
\[
\pi_C: E \to E/\sim^\nabla, \quad v_p \mapsto [v_p]
\]
is both a submersion and a vector bundle morphism over \( \pi \). Moreover, there exists an isomorphism
\[
\Xi: E \to \pi^2(E/\sim^\nabla), \quad v_p \mapsto (p,[v_p])
\]
of vector bundles over the identity \( \text{id}_C \) and this isomorphism fulfills
\[
P_{\gamma,p \leadsto q}(v_p) = \Xi^{-1}(q,[v_p])
\]
for \( v_p \in E \) and \( \pi(p) = \pi(q) \).

Proof: Using Lemma 3.6 \([12]\) Prop. 4.1] gives the result.

A similar quotient is considered in the context of infinitesimal ideal systems in \([21]\) Thm. 3.7 and in \([15]\) Ch. 6]: If we turn a given vector bundle \( E \to C \) into a Lie algebroid with trivial anchor and trivial Lie bracket and use \((D, C \times \{0\}, \nabla)\) as infinitesimal ideal system, then the constructions of \([15]\) Cor. 6.3 and Proposition 3.8 coincide.

**Remark 3.9** Other examples of linear actions of \( R(\pi) \) on \( E \), i.e., maps \( T \) with the properties of \([\text{Lemma 3.6}]\) can be found in the context of group actions: Let \( \pi: C \to C/G \) be a left principal fiber bundle with structure group \( G \) and principal action \( \ast^C \). Moreover, let \( E \to C \) be a \( G \)-equivariant vector bundle with left action \( \ast^E \). Then using the ratio map \( r: C \times_C G \to G \) defined by \( r(p,q)\ast^C p = q \) we can set
\[
T(v_p,q) := r(p,q)\ast^E v_p.
\]
With \( v_p \sim_T w_q \iff T(v_p,q) = w_q \) we obtain \( E/\sim_T = E/G \to C/G \) as quotient vector bundle.

Ultimately, we are interested in sections which are parallel in the direction of a distribution:
Definition 3.10 (\(\nabla\)-Parallel sections) Let \(D \subset TC\) be a distribution and let \(\nabla\) be a \(D\)-connection on a vector bundle \(E \to C\). Then the \(\nabla\)-parallel sections of \(E\) are

\[
\Gamma_D^\nabla(E,\nabla) := \{ s \in \Gamma^\infty(E) \mid \nabla_X s = 0 \ \forall X \in \Gamma^\infty(D) \}.
\]  

(3.16)

As to be expected this condition can be integrated to a condition involving the parallel transport, cf. [13, Prop. 6.5]:

Lemma 3.11 Let \(D \subset TC\) be an integrable distribution and let \(\nabla\) be a \(D\)-connection on a vector bundle \(E \to C\). Then

\[
\Gamma_D^\nabla(E,\nabla) = \{ s \in \Gamma^\infty(E) \mid \mathcal{P}_{\gamma,a,b}(\nabla(s(\gamma(a)))) = s(\gamma(b)) \text{ for all smooth curves } \gamma \text{ inside a leaf of } D \}.
\]  

(3.17)

With (3.13) at hand, we see that the \(\nabla\)-parallel sections are nothing else but the sections over the quotient vector bundle:

Lemma 3.12 Let \(D \subset TC\) be simple with projection \(\pi: C \to C/D\) and let \(\nabla\) be a holonomy-free \(D\)-connection on a vector bundle \(E \to C\). Then with \(\Xi\) as in (3.13)

\[
\Gamma^\infty(E/\sim\nabla) \to \Gamma_D^\nabla(E,\nabla), \quad r \mapsto (C \ni p \mapsto \Xi^{-1}(p, r(\pi(p))) \in E)
\]  

(3.18)

is an isomorphism of modules over the isomorphism \(\pi^*: \mathcal{C}^\infty(C/D) \to \mathcal{C}^\infty(C)\) of algebras. Its inverse is given by

\[
\Gamma_D^\nabla(E,\nabla) \to \Gamma^\infty(E/\sim\nabla), \quad s \mapsto (C/D \ni q \mapsto [s(\pi^{-1}(q))] \in E/\sim\nabla),
\]  

(3.19)

where \(\pi^{-1}(q)\) is an arbitrary element in \(\pi^{-1}(\{q\})\).

3.2 The Module of Invariant Sections

Using [Remark 3.1] and applying [Lemma 3.12] to both the sections and the dual sections we finally obtain a dual basis for invariant sections:

Proposition 3.13 Let \(D \subset TC\) be simple with projection \(\pi: C \to C/D\) and let \(\nabla\) be a holonomy-free \(D\)-connection on a vector bundle \(E \to C\). Then there exists a finite index set \(I\) and \(\nabla\)-parallel sections \(e_i \in \Gamma_D^\nabla(E,\nabla)\) and \(e^i \in \Gamma_D^\nabla(E^*,\nabla^*)\) for \(i \in I\) such that

\[
v_p = \sum_{i \in I} e_i(p)e^i(v_p)
\]  

(3.20)

for all \(v_p \in E\).

PROOF: Let \(I\) be a finite index set and let \(f_i \in \Gamma^\infty(E/\sim\nabla)\) and \(f^i \in \Gamma^\infty((E/\sim\nabla)^*)\) for \(i \in I\) such that

\[
u_p = \sum_{i \in I} f_i(x)f^i(u_x)
\]  

for all \(u_x \in E/\sim\nabla\) at \(x \in C/D\). [Lemma 3.12] gives \(e_i \in \Gamma_D^\nabla(E,\nabla)\) with \([e_i(p)] = f_i(\pi(p))\) for all \(p \in C\). Setting \(e^i(v_p) := f^i([v_p])\) gives \(e^i \in \Gamma_D^\nabla(E^*,\nabla^*)\) with

\[
\left[ \sum_{i \in I} e_i(p)e^i(v_p) \right] = \sum_{i \in I} [e_i(p)]f^i([v_p]) = [v_p]
\]

for all \(v_p \in E\). But then also

\[
\sum_{i \in I} e_i(p)e^i(v_p) = v_p
\]

because \(\pi\nabla: E_p \to (E/\sim\nabla)_{\pi(p)}\) is bijective. \(\square\)
For singular $D$ we are not guaranteed that taking parallel sections gives us a finitely generated projective $\mathcal{E}_D^\infty(C)$-module. Nevertheless, in one dimension we have the following positive result:

**Example 3.14** Let $C = \mathbb{R}$ and $E \to \mathbb{R}$ be a vector bundle. For dimensional reasons the curvature of any covariant derivative on $E$ vanishes and therefore the parallel transport is always path-independent. In particular, if $D \subseteq TC$ is some integrable distribution and $\nabla$ is a $D$-connection on $E$, then it has an extension to a holonomy-free $TC$-connection $\nabla$. Therefore by Proposition 3.13 there exists a finite dual basis $e_i \in \Gamma_\infty(T_C(E, \nabla))$ and $e^i \in \Gamma_\infty(T_C(E^*, \nabla^*))$ for $i \in I$ with

$$v_p = \sum_{i \in I} e_i(p)e^i(v_p) \quad (3.21)$$

for all $v_p \in E$. In particular $e_i \in \Gamma_\infty(E, \nabla)$ and $e^i \in \text{Hom}_{\mathcal{E}_D^\infty}(\Gamma_\infty(E, \nabla), \mathcal{E}_D^\infty(C))$ yield a dual basis of $\Gamma_\infty(E, \nabla)$. Therefore $\Gamma_\infty(E, \nabla)$ is a finitely generated projective $\mathcal{E}_D^\infty(C)$-module.

Our condition that a $D$-connection arises from a covariant derivative defined on all tangent vectors becomes crucial in the singular case. Without this assumption, the module $\Gamma_\infty(E, \nabla)$ might fail to be finitely generated and projective:

**Example 3.15** We reconsider the situation from Example 3.4 i.e.

$$D_p := \begin{cases} T_p\mathbb{R} & \text{if } p > 0 \\ \{0_p\} & \text{if } p \leq 0, \end{cases} \quad (3.22)$$

and the $\nabla$ which was not the restriction of a covariant derivative. Then the parallel sections are given by

$$A := \{ f \in \mathcal{C}^\infty(\mathbb{R}) \mid f(x) = 0 \text{ for } x > 0 \}. \quad (3.23)$$

This $\mathcal{E}_D^\infty(C)$-module $A$ is neither finitely generated nor projective. In [10] Prop. 5.3 it is shown that $A$ is not a finitely generated $\mathcal{C}^\infty(\mathbb{R})$-module. Thus it is not a finitely generated $\mathcal{E}_D^\infty(C)$-module either. To see that it is not projective, assume that there exists a dual basis $e_i \in A$, $e^i \in \text{Hom}_{\mathcal{E}_D^\infty}(A, \mathcal{E}_D^\infty(C))$ for some (possibly infinite) index set $I$ such that for fixed $s \in A$ only finitely many $e^i(s) \neq 0$ and

$$s = \sum_{i \in I} e_i e^i(s). \quad (3.24)$$

If $r \in A$ and $r(x) = 0$ for some $x < 0$, then we can choose some bump function $\zeta \in A \subseteq \mathcal{E}_D^\infty(C)$ with $\zeta(x) = 1$. It follows that

$$e^i(r)|_x = \zeta(x)e^i(r)|_x = e^i(\zeta r)|_x = r(x)e^i(\zeta)|_x = 0. \quad (3.25)$$

Now for $s(x) := \chi(-x)$ with $\chi$ as above we have $s \in A$ and therefore only finitely many of the $e^i(s) \neq 0$. We denote the indices by $j_1, \ldots, j_n$. If $s \in A$ is arbitrary, then for fixed $x < 0$

$$s' := s - \frac{s(x)}{s(x)} \zeta s \in A \quad (3.26)$$

fulfills $s'(x) = 0$ and therefore $e^i(s')|_x = 0$ for all $i \in I$. This shows that because $\frac{s(x)}{s(x)} \zeta \in A \subseteq \mathcal{E}_D^\infty(C)$ we have

$$e^i(s)|_x = \frac{s(x)}{s(x)} \zeta e^i(s)|_x = \frac{s(x)}{s(x)} e^i(s)|_x. \quad (3.27)$$
In particular, \( e^i(s) \big|_x = 0 \) for all \( i \not\in \{j_1, \ldots, j_n\} \), all \( x < 0 \) and all \( s \in A \). As \( e_i|_y = 0 \) for all \( y \geq 0 \) we have \( e_i e^i(s) = 0 \) for all \( i \not\in \{j_1, \ldots, j_n\} \) and all \( s \in A \) which implies

\[
s = \sum_{k=1}^{n} e_j e^j(s). \quad (3.28)
\]

In particular, \( A \) would be finitely generated in contradiction to what we saw before. Therefore \( A \) can not be projective.

The situation in the one-dimensional case was very special as we could extend any \( D \)-connection to a covariant derivative with globally path-independent parallel transport. The next example considers a situation in which this is not the case:

**Example 3.16** Let \( C = \mathbb{R}^2 \), let \( E \to C \) be a vector bundle and let \( D \subseteq TC \) be the integrable distribution defined by

\[
D_p := \begin{cases} T_p \mathbb{R}^2 & \text{if } p_x > 0 \\ \{0_p\} & \text{if } p_x \leq 0. \end{cases} \quad (3.29)
\]

Let \( \nabla \) be a holonomy-free \( D \)-connection. For \( p, q \in C \) we define a smooth path connecting \( p \) and \( q \) by

\[
\gamma_{p,q}(t) := (1 - t)p + tq. \quad (3.30)
\]

Clearly \( \gamma_{p,q} \) depends smoothly on \( p \) and \( q \) and we obtain a smooth map

\[
T^\prime: E \times C \to E, \quad (v_p, q) := P^\nabla_{\gamma_{p,q},0\to1}(v_p). \quad (3.31)
\]

Fixing some point \( p_0 \) in the non-trivial leaf \( \mathbb{R}^+ \times \mathbb{R} \) of \( D \), for example \( p_0 = (1,0) \), and a basis \( b_1, \ldots, b_n \) of \( E_{p_0} \) with dual basis \( b^1, \ldots, b^n \) we can define sections by setting

\[
e_i(p) := T^\prime(b_i, p) \quad \text{and} \quad e^i(p) := b^i(T^\prime(v_p, p_0)) \quad (3.32)
\]

for \( p \in C \). Then

\[
\sum_{i=1}^{n} e_i(p) e^i(v_p) = \sum_{i=1}^{n} T^\prime \left( b^i(T^\prime(v_p, p_0)) b_i, p \right) = T^\prime(T^\prime(v_p, p_0), p) = v_p \quad (3.33)
\]

as \( \gamma_{p_0,p} \) is the reverse curve to \( \gamma_{p_0,p} \). Moreover, the \( e_i, e^i \) really are a dual basis of \( \Gamma^\infty(E, \nabla) \) because \( T^\prime \) restricted to \( \mathbb{R}^+ \times \mathbb{R} \) is just \( T^\nabla \).

## 4 Projective Coisotropic \( \mathcal{C}^\infty(M, C, D) \)-Modules

Now that we have described the finitely generated projective modules over \( \mathcal{C}^\infty_D(C) \) we again look at the whole coisotropic algebra \( \mathcal{C}^\infty(M, C, D) \) and want to find a description of its regular projective coisotropic modules. There are two main steps we have to take: First of all, we need to extend from our submanifold \( C \subseteq M \) to \( M \) in order to pass from \( \mathcal{C}^\infty_D(C) \)-modules to \( \mathcal{C}^\infty_D(M) \)-modules. As a second step we need to incorporate the relations between the tot-, the \( N \) and the 0-components which become evident in **Proposition 2.24**.
4.1 Regular Projective Coisotropic $\mathcal{E}(M, C, D)$-Modules

As we have seen in the previous section we need to look at sections of vector bundles but also need the additional data of a covariant derivative. In the following we will use $i^*$ to denote the restriction of vector bundles on $M$ as well as their sections to the submanifold $C$.

**Proposition 4.1 (Triples of invariant sections)** Let $\iota: C \to M$ be a submanifold and $D \subseteq TC$ an integrable distribution. Let $E_{\text{tot}} \to M$ be a vector bundle, $E_N \subseteq E_{\text{tot}}$ and $E_0 \subseteq \iota^* E_N \to C$ vector subbundles, and $\nabla$ a $D$-connection on $\iota^* E_N$. Then for $E := (E_{\text{tot}}, E_N, E_0, \nabla)$ the $\mathcal{E}(M, C, D)$-module $\Gamma^\infty(E)$ defined by

$$
\Gamma^\infty(E)_{\text{tot}} := \Gamma^\infty(E_{\text{tot}})
$$

$$
\Gamma^\infty(E)_{N} := \{ s \in \Gamma^\infty(E_N) \mid \nabla_X i^* s = 0 \ \forall X \in \Gamma^\infty(D) \}
$$

$$
\Gamma^\infty(E)_0 := \{ s \in \Gamma^\infty(E_N) \mid i^* s \in \Gamma^\infty(E_0) \text{ and } \nabla_X i^* s = 0 \ \forall X \in \Gamma^\infty(D) \}
$$

(4.1)

together with $\iota^* (E)$ being the inclusion of sections $\Gamma^\infty(E_N) \to \Gamma^\infty(E_{\text{tot}})$ is called the triple of invariant sections subject to $E$.

**Remark 4.2** Triples of the form as in Proposition 4.1 are not the only possible way to define $\mathcal{E}(M, C, D)$-modules using sections of vector bundles. For example, it would suffice to specify $E_N$ on the submanifold $C$ and $\nabla$ for sections of $E_N/E_0$: More precisely, given a vector bundle $E_{\text{tot}} \to M$ and subbundles $E_0 \subseteq E_N \subseteq i^* E_{\text{tot}} \to C$ we could consider the coisotropic module $\mathcal{E}$ defined by

$$
\mathcal{E}_{\text{tot}} := \Gamma^\infty(E_{\text{tot}})
$$

$$
\mathcal{E}_N := \{ s \in \Gamma^\infty(E_N) \mid i^* s \in \Gamma^\infty(E_N) \text{ and } \nabla_X i^* s = 0 \ \forall X \in \Gamma^\infty(D) \}
$$

$$
\mathcal{E}_0 := \{ s \in \Gamma^\infty(E_N) \mid i^* s \in \Gamma^\infty(E_0) \}.
$$

(4.2)

However, triples of this type will not fit into the description of regular projective $\mathcal{E}(M, C, D)$-modules obtained in Theorem 4.12. Roughly speaking, the reason for this is the constancy of the rank of $e_N$ for an idempotent $e^2 = e$ on a free coisotropic $\mathcal{E}(M, C, D)$-module with $\im e = \mathcal{E}$.

Under certain additional assumptions we can prove that triples of the form as in Proposition 4.1 are regular projective:

**Proposition 4.3** Let $\iota: C \to M$ be a closed submanifold and let $D \subseteq TC$ be simple with projection $\pi: C \to C/D$. Moreover, let $E := (E_{\text{tot}}, E_N, E_0, \nabla)$ be as in Proposition 4.1 with the additional assumptions that

i.) $E_0$ is closed under $\nabla$, i.e. $\nabla_X i^* s \in \Gamma^\infty(E_0)$ for all $X \in \Gamma^\infty(D), s \in \Gamma^\infty(E_0)$,

ii.) $\nabla$ is holonomy-free.

Then $\Gamma^\infty(E)$ is a finitely generated regular projective coisotropic $\mathcal{E}(M, C, D)$-module.

Before we prove projectivity, we will need to extend sections of $i^* E_N$ to sections on $E_N$. This can be done by locally writing a vector bundle $E$ over $M$ as the pull-back vector bundle of the restriction $i^* E$ to $C$.

**Lemma 4.4** Let $\iota: C \to M$ be a submanifold and $E \to M$ a vector bundle. Then on an open neighborhood $U \supseteq \iota(C)$ there exists a surjective submersion $t: U \to C$ with $t \circ \iota = \text{id}_C$ and a vector bundle isomorphism

$$
\Theta: E|_U \to i^*(i^* E)
$$

over the identity with $t^* \circ \Theta \circ i^* = \text{id}_{i^* E}$ for the projections $i^*: E^* \to E$ and $i^*: i^*(i^* E) \to i^* E$.

(4.3)
Proof: We choose a vector bundle \( p_{F}: F \to C \), an open neighborhood \( U \subseteq M \) of \( \iota(C) \subseteq M \) and a diffeomorphism \( \tau: F \to U \) with \( \tau \circ \iota_{F} = \iota \) for the zero section \( \iota_{F}: C \to F \). This can be done using a tubular neighborhood. We can define \( t := p_{F} \circ \tau^{-1} \). Now we choose an arbitrary covariant derivative \( \nabla \) on \( E|_{U} \to U \). For \( p \in U \) we define

\[
\gamma_{p}: I \to U, \quad \gamma_{p}(s) := \tau((1 - s)\tau^{-1}(p)).
\]

We then have \( \gamma_{p}(0) = p \) and \( \gamma_{p}(1) = t(p) \). We obtain a family of smooth curves which depends smoothly on \( p \in U \). Setting

\[
\Theta(v_{p}) := (p, P_{\gamma_{p},0 \to 1}(v_{p})),
\]

we obtain a vector bundle morphism \( \Theta: E|_{U} \to t^{*}(t^{*}E) \). Its inverse is given by

\[
(p, v_{t(p)}) \mapsto P_{\gamma_{p},1 \to 0}(v_{t(p)}).
\]

□

Moreover, we will need a decomposition of \( \iota^{*}E_{N} \) into \( E_{0} \) and a complementary subbundle \( E_{\perp} \) which is also closed under \( \nabla \):

**Lemma 4.5** Let \( C \) be a smooth manifold, \( D \subseteq TC \) simple with projection \( \pi: C \to C/D \) and \( \nabla \) a holonomy-free \( D \)-connection on a vector bundle \( E \to C \). If \( F \subseteq E \) is a vector subbundle such that

\[
\nabla_{X}s \in \Gamma^{\infty}(F)
\]

for all \( s \in \Gamma^{\infty}(F), X \in \Gamma^{\infty}(D) \), then there exists a complementary vector subbundle \( F_{\perp} \) such that

\[
\nabla_{X}r \in \Gamma^{\infty}(F_{\perp})
\]

for all \( r \in \Gamma^{\infty}(F_{\perp}), X \in \Gamma^{\infty}(D) \).

Proof: By assumption the restriction \( \nabla^{F} \) of \( \nabla \) to \( F \) defines a \( D \)-connection on \( F \). Its parallel transport coincides with the parallel transport of elements of \( F \) with respect to \( \nabla \). In particular, the parallel transport with respect to \( \nabla^{F} \) is path-independent along the leaves of \( D \). Therefore, we can take the vector bundle quotient of \( F \) with respect to \( \nabla^{F} \) and obtain a vector subbundle \( F/\sim_{\nabla^{F}} \subseteq E/\sim_{\nabla} \): Indeed, a suitable injective vector bundle morphism over the identity is given by

\[
F/\sim_{\nabla^{F}} \ni [v_{p}] \mapsto [v_{p}] \in E/\sim_{\nabla}.
\]

The fact that we have a vector subbundle \( F/\sim_{\nabla^{F}} \subseteq E/\sim_{\nabla} \) implies that there exists a complementary subbundle \( Q \subseteq E/\sim_{\nabla} \), i.e.

\[
(F/\sim_{\nabla^{F}}) \oplus Q = E/\sim_{\nabla}.
\]

Pulling \( Q \) back gives a subbundle \( \pi^{*}Q \subseteq \pi^{*}(E/\sim_{\nabla}) \). Using the isomorphism

\[
\Phi: E \to \pi^{*}(E/\sim_{\nabla}), \quad v_{p} \mapsto (p, [v_{p}])
\]

we obtain a subbundle \( F_{\perp} := \Phi^{-1}(\pi^{*}Q) \subseteq E \). This is clearly complementary to \( F \) and also closed under \( \nabla \) as \( \Phi \) is compatible with covariant derivatives. □

A decomposition like that now allows us to construct a dual basis to show regular projectivity of the triple of invariant sections:
This way, we obtain a dual basis of $\tilde{\Gamma}$ yielding sections for $\Pi$. We choose a finite dual basis and a finite dual basis for $\Pi_0$. We extend the open neighborhood as in Lemma 4.4. We define $\tilde{\Gamma}$, which leads to $E$ which we still denote by the same symbol) map parallel sections in $\Gamma_\infty(i^\sharp E_N)$ to functions in $\mathcal{C}_\infty(D)$. Moreover, for $v_p \in i^\sharp E_N$ we have

$$\sum_{i \in I_0} f_i(p)f_i(v_p) + \sum_{i \in I_\tilde{\Gamma}} g_i(p)g_i(v_p) = \sum_{i \in I_0} f_i(p)f_i(\Pi_0v_p) + \sum_{i \in I_\tilde{\Gamma}} g_i(p)g_i(\Pi_\tilde{\Gamma}v_p) = \Pi_0v_p + \Pi_\tilde{\Gamma}v_p = v_p.$$  

Now we extend the $f_i, f_i^*, g_i, g_i^*$ to a dual basis of $\{ s \in \Gamma_\infty(E_N|_U) \mid \nabla_X i^\sharp s = 0 \ \forall X \in \Gamma_\infty(D) \}$ for an open neighborhood $U \subseteq M \subseteq C$. To do so, let $\Theta : E_N|_U \to i^\sharp(i^\sharp E_N)$ be a vector bundle isomorphism as in Lemma 4.4. We define

$$\tilde{f}_i(p) := \Theta^{-1}(p, f_i(t(p)))$$

for $p \in U$ which leads to $i^\sharp \tilde{f}_i = f_i$ and

$$\tilde{f}_i(p)(v_p) := f_i(t(p))(\Theta(v_p))$$

for $v_p \in E_N|_U$ which leads to $i^\sharp \tilde{f}_i = f_i$. Then we have

$$\sum_{i \in I_0} \tilde{f}_i(p)\tilde{f}_i(v_p) + \sum_{i \in I_\tilde{\Gamma}} \tilde{g}_i(p)\tilde{g}_i(v_p) = \sum_{i \in I_0} \Theta^{-1}(p, f_i(t(p)))(\tilde{f}_i(v_p)) + \sum_{i \in I_\tilde{\Gamma}} \Theta^{-1}(p, g_i(t(p)))(\tilde{g}_i(v_p))$$

$$= \Theta^{-1}(p, f_i(t(p)))(\tilde{f}_i(v_p)) + \sum_{i \in I_\tilde{\Gamma}} g_i(t(p))\tilde{g}_i(v_p)$$

$$= \Theta^{-1}(p, i^\sharp(\Theta(v_p)))$$

$$= v_p$$

for $v_p \in E|_U$. Again, we denote the extensions by $f_i, f_i^*, g_i, g_i^*$. Next, we choose a dual basis $(h_i, h_i^*)_{i \in J}$ of $\Gamma_\infty(E_N|_M\setminus C)$. Such a dual basis exists by Remark 3.1. Now we choose a quadratic partition of unity $\chi_1, \chi_2 \in \mathcal{C}_\infty(M, [0, 1])$ with $\chi_1^2 + \chi_2^2 = 1$ and supp $\chi_1 \subseteq U$ and supp $\chi_2 \subseteq M \setminus C$. Existence is guaranteed by the closedness of $C$. Multiplying the $f_i, f_i^*, g_i, g_i^*$ with $\chi_1$ and $h_i, h_i^*$ with $\chi_2$ yields sections $\tilde{f}_i, f_i^*, \tilde{g}_i, g_i^*$ defined on the whole of $M$. Moreover, the $h_i, h_i^*$ vanish on $C$ and the $\tilde{f}_i, f_i^*, \tilde{g}_i, g_i^*$ vanish on $M \setminus U$. This also shows that the $\tilde{h}_i, \tilde{h}_i^*$ coincide with $h_i, h_i^*$ on $M \setminus U$ and that the $\tilde{f}_i, f_i^*, \tilde{g}_i, g_i^*$ coincide with the $f_i, f_i^*, g_i, g_i^*$ on $C$. All those $\tilde{f}_i, f_i^*, \tilde{g}_i, g_i^*$ together yield a dual basis of $\{ s \in \Gamma_\infty(E_N) \mid \nabla_X i^\sharp s = 0 \ \forall X \in \Gamma_\infty(D) \}$ as for $v_p \in E_N$ with $p \in M \setminus U$ we have

$$v_p = \sum_{i \in J} h_i(p)h_i^*(v_p) = \sum_{i \in J} \tilde{h}_i(p)\tilde{h}_i^*(v_p) = \sum_{i \in I_0} \tilde{f}_i(p)\tilde{f}_i^*(v_p) + \sum_{i \in I_\tilde{\Gamma}} \tilde{g}_i(p)\tilde{g}_i^*(v_p) + \sum_{i \in J} \tilde{h}_i(p)\tilde{h}_i^*(v_p),$$

if $p \in C$ we have

$$v_p = \sum_{i \in I_0} f_i(p)f_i^*(v_p) + \sum_{i \in I_\tilde{\Gamma}} g_i(p)g_i^*(v_p) = \sum_{i \in I_0} \tilde{f}_i(p)\tilde{f}_i^*(v_p) + \sum_{i \in I_\tilde{\Gamma}} \tilde{g}_i(p)\tilde{g}_i^*(v_p) + \sum_{i \in J} \tilde{h}_i(p)\tilde{h}_i^*(v_p),$$

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and if $p \in U \setminus C$ we have

$$v_p = \chi_1^2(p)v_p + \chi_2^2(p)v_p$$

$$= \chi_1^2(p)\left(\sum_{i \in I_0} f_i(p)f^i(v_p) + \sum_{i \in I_0^\bot} g_i(p)g^i(v_p)\right) + \chi_2^2(p)\sum_{i \in J} h_i(p)h^i(v_p)$$

$$= \sum_{i \in I_0} \tilde{f}_i(p)\tilde{f}^i(v_p) + \sum_{i \in I_0^\bot} \tilde{g}_i(p)\tilde{g}^i(v_p) + \sum_{i \in J} \tilde{h}_i(p)\tilde{h}^i(v_p).$$

For simplicity, we again denote the resulting $\tilde{f}_i, \tilde{f}^i, \tilde{g}_i, \tilde{g}^i, \tilde{h}_i, \tilde{h}^i$ by $f_i, f^i, g_i, g^i, h_i, h^i$.

It remains to extend it to a dual basis of $\Gamma^\infty(E_{\text{tot}})$. For this, we choose a subbundle $E_N^\bot$ complementary to $E_N$, i.e. $E_{\text{tot}} = E_N \oplus E_N^\bot$. We denote the projection onto $E_N$ by $\Pi_N$ and the projection onto $E_N^\bot$ by $\Pi_N^\bot$. We choose a dual basis $(e_k, e^k)_{k \in K}$ of $\Gamma^\infty(E^\bot_N)$. We can embed the $e_k$ into $\Gamma^\infty(E_{\text{tot}})$ and turn the elements $e^k$ into elements of $\Gamma^\infty(E_{\text{tot}})$ by precomposing with $\Pi_N^\bot$. For $i \in I_0 \sqcup I_0^\bot \sqcup J$ and $d \in \{f, g, h\}$ we set

$$e_i := d_i \quad \text{and} \quad e^i := d^i \circ \Pi_N.$$

This gives us elements $e_i \in \Gamma^\infty(E_{\text{tot}})$ and $e^i \in \Gamma^\infty(E^\bot_{\text{tot}})$.

Now we define the generator $N \in \text{Set}_{3}^{\text{inj}}$ by $N_{i_0} = I_0 \sqcup I_0^\bot \sqcup J \sqcup K$, $N_{i_N} = I_0 \sqcup I_0^\bot \sqcup J$ and $N_0 = I_0$. As $\iota_N: N_0 \rightarrow N_{i_0}$ we take the obvious inclusion. So, our candidate for a $\text{Set}_{3}^{\text{inj}}$-dual basis of $\Gamma^\infty(E)$ consists of the $f, g, h$ in the $N$-component, in the $0$-component we have the $f_i, f^i$ and in the $\text{tot}$-component we have the $e_i, e^i$ and the $e_k, e^k$. In the following, we check that this indeed fulfills the required properties: First of all, for $v_p \in E_N$ we have already seen that

$$\sum_{i \in I_0 \sqcup I_0^\bot \sqcup J} (d_i(p)d^i(v_p)) = v_p.$$

Moreover, on $C$ we have $\nabla_X d_i = 0$ and $\nabla_X d^i = 0$. For $i \in I_0$ we have $i^*f_i \in \Gamma^\infty_{\text{tot}}(E_0)$ by definition of $f_i$. This shows that for $i \in I_0$ we have $d_i \in \Gamma^\infty_{\text{tot}}(E_0)$.

For $i \in J$ the $h_i, h^i$ vanish on $C$ and therefore the condition $h^i(s) \in \mathcal{F}_C$ for $s \in \Gamma^\infty(E_0)$ is trivially fulfilled. For $i \in I_0^\bot$ it is fulfilled for the $g^i$ because $g^i(s)|_{\mathcal{F}} = g^i(\Pi_0 \circ s)|_{\mathcal{F}} = g^i(0)|_{\mathcal{F}} = 0$ for $s \in \Gamma^\infty(E_0)$. This shows that for $i \in I_0^\bot \sqcup J$ we have $d^i(s) \in \mathcal{F}_C$ for $s \in \Gamma^\infty(E_0)$.

The $e_i, e^i$ for $i \in N_{i_0}$ are a dual basis of $\Gamma^\infty(E_{\text{tot}})$ because

$$\sum_{i \in I_0 \sqcup I_0^\bot \sqcup J} e_i(p)e^i(v_p) + \sum_{i \in K} e_i(p)e^i(v_p) = \sum_{i \in I_0 \sqcup I_0^\bot \sqcup J} d_i(p)(\Pi_N(v_p)) + \Pi_N^\bot(v_p)$$

$$= \Pi_N(v_p) + \Pi_N^\bot(v_p)$$

$$= v_p$$

for all $v_p \in E_{\text{tot}}$. For $i \in N_N$ we have $e_{\iota_N(i)} = \iota_{\Gamma^\infty(E)}(d_i)$. For $i \in N_{i_0} \setminus \iota_N(N_N)$, so $i \in K$, we have $e^i \circ \iota_{\Gamma^\infty(E)} = 0$ as $\Pi_N^\bot|_{E_N} = 0$. For $i \in I_0 \sqcup I_0^\bot \sqcup J$ we have $e^{\iota_{\Gamma^\infty(E)}} = d^i$ as $\Pi_N|_{E_N} = \text{id}_{E_N}$.

This shows that we have finally found a $\text{Set}_{3}^{\text{inj}}$-dual basis of $\Gamma^\infty(E)$. Therefore, $\Gamma^\infty(E)$ is a projective coisotropic $\mathcal{C}^\infty(M, C, D)$-module with finite generator $N \in \text{Set}_{3}^{\text{inj}}$. \qed

Up to now, triples of invariant sections over triples of vector bundles as in Proposition 4.3 are merely an example of regular projective coisotropic $\mathcal{C}^\infty(M, C, D)$-modules. In the following, we want to see that essentially all regular projective coisotropic $\mathcal{C}^\infty(M, C, D)$-modules are of this form. As a first step, we formulate the following theorem, which does not need to assume any regularity of the integrable distribution $D$:

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Theorem 4.6 Let $\mathcal{P}$ be a finitely generated regular projective coisotropic module over the coisotropic algebra $\mathcal{C}\mathcal{E}_{\infty}(M,C,D)$. Assume that $M$ is connected and $C/D$ is connected with respect to the final topology induced by the projection $C \to C/D$. Then there exists a vector bundle $E_{\text{tot}} \to M$, vector subbundles $E_N \subseteq E_{\text{tot}}$ and $E_0 \subseteq \iota^* E_N$ and a holonomy-free $D$-connection $\nabla$ on $\iota^* E_N$ such that $E_0$ is closed under $\nabla$ and the coisotropic module $\mathcal{P}$ is isomorphic to $\Gamma^\infty(E_{\text{tot}}, E_N, E_0, \nabla)$.

Moreover, we can choose $(E_{\text{tot}}, E_N, E_0, \nabla)$ in such a way that there exists a decomposition $\iota^* E_N = E_0 \oplus E_N^\perp$ and $\nabla = \nabla^0 \oplus \nabla^\perp$ where $\nabla^0$ and $\nabla^\perp$ are holonomy-free $D$-connections on the subbundles $E_0$ and $E_N^\perp$, respectively.

Proof: Let $\mathcal{P} \cong \im e$ for some $e = e^2 \colon \mathcal{A}^{(N)} \to \mathcal{A}^{(N)}$ with $N \in \mathcal{S}_3^{\text{inj}}$ and $\mathcal{A} = \mathcal{C}\mathcal{E}_{\infty}(M,C,D)$. Then using the identification of module homomorphisms of free modules with matrices we find $\hat{e}_{\text{tot}} \in M_{N_{\text{tot}}}(\mathcal{C}\mathcal{E}_{\infty}(M))$ and $\hat{e}_N \in M_{N_N}(\mathcal{C}\mathcal{E}_{\infty}^D(M))$ with

$$\begin{pmatrix} e_{\text{tot}}(f_1) \\
\vdots \\
 f_{N_{\text{tot}}} \end{pmatrix} = \sum_{j \in N_{\text{tot}}} (\hat{e}_{\text{tot}})_{ij} f_j$$

and similarly for the $N$-component. We define the vector bundle morphisms

$$\phi_{\text{tot}} : M \times \mathbb{R}^{N_{\text{tot}}} \to M \times \mathbb{R}^{N_{\text{tot}}}, \quad (p, v) \mapsto (p, \hat{e}_{\text{tot}}(p)v)$$

$$\phi_N : M \times \mathbb{R}^{N_N} \to M \times \mathbb{R}^{N_N}, \quad (p, v) \mapsto (p, \hat{e}(p)v).$$

Because $\hat{e}_{\text{tot}}(p)^2 = \hat{e}_{\text{tot}}(p)$ we have $\text{tr}(\hat{e}_{\text{tot}}(p)) = \dim \im \hat{e}_{\text{tot}}(p) = \dim \ker(1 - \hat{e}_{\text{tot}}(p)) \in \mathbb{N}_0$. As $M$ is assumed to be connected, the rank of $\phi_{\text{tot}}$ (and similarly of $\phi_N$) is constant and their images define vector subbundles $E_{\text{tot}} := \im \phi_{\text{tot}} \subseteq M \times \mathbb{R}^{N_{\text{tot}}}$ and $E_N := \im \phi_N \subseteq M \times \mathbb{R}^{N_N}$. The map

$$\psi : E_{\text{tot}} \to E_{\text{tot}}, \quad (p, v) \mapsto \left(p, \sum_{i \in N_N} c_{\iota_N(i)} v^i\right)$$

with the standard basis $(c_j)$ of $\mathbb{R}^{N_{\text{tot}}}$ is an injective vector bundle morphism over the identity as $e_{\text{tot}} \circ \iota_{\mathcal{A}^{(N)}} = \iota_{\mathcal{A}^{(N)}} \circ e_N$ and $\iota_N$ is assumed to be injective. Therefore, we can consider $E_N \subseteq E_{\text{tot}}$ as a subbundle.

Now we define a subbundle $E_0$ of $\iota^* E_N$ using the restriction of $\hat{e}_N$ to the subspace $V_0 \subseteq \mathbb{R}^{N_N}$ spanned by the standard basis $b_i$ for $i \in N_0$. Indeed, we have

$$\hat{e}_N(p)|_{V_0} : V_0 \to V_0$$

for $p \in C$ as $e_N(\mathcal{A}_0^{(N)}) \subseteq \mathcal{A}_0^{(N)}$ and therefore $(\hat{e}_N(p)(v))^t = 0$ for $i \in N_N \setminus N_0$. Now we can proceed as before and define a vector bundle morphism

$$\phi_0 : C \times V_0 \to C \times V_0, \quad (p, v) \mapsto (p, \hat{e}_N(p)(v))$$

over the identity. As the entries of $(\hat{e}_N)_{ij}|_C$ are in $\mathcal{C}\mathcal{E}_{\infty}^D(C)$ we have a well-defined and continuous map

$$C/D \ni [p] \mapsto \hat{e}_N(p)|_{V_0} \in M_{N_0}(\mathbb{R}).$$

Using $\hat{e}_N(p)^2 = \hat{e}_N(p)$ and the fact that $C/D$ is assumed to be connected with respect to the final topology we see analogously that $\phi_0$ has constant rank. This way we obtain a subbundle $E_0 := \im \phi_0 \subseteq \iota^* E_N$. Choosing some complementary subspace $V_0^\perp$ to $V_0 \subseteq \mathbb{R}^{N_N}$ we obtain another subbundle $E_0^\perp := \iota^* E_N \cap (C \times V_0^\perp)$ of $\iota^* E_N$. Indeed, $E_0^\perp$ is of constant rank because $E_0^\perp \oplus E_0 = \iota^* E_N$ and it is the intersection of two vector bundles over the manifold $C$. 

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Having defined the subbundles, we now look at the covariant derivatives: On \( \iota^\ast E_N \) we can define a covariant derivative

\[
\nabla_X s := \sum_{i,j \in N_N} b_i(\hat{e}_N(p))_{ij} L_X(s^i),
\]

where \( X \in \Gamma^\infty(TC) \), \( s \in \Gamma^\infty(\iota^\ast E_N) \) and \( L_X \) denotes the Lie derivative of functions. Because of \((\hat{e}_N)_{ij}|_C \in \mathcal{C}^\infty_D(C) \) and \( s^i = \sum_{j \in N_N} (\hat{e}_N(p))_{ij} s^j \) for \( s \in \Gamma^\infty(\iota^\ast E_N) \) we have

\[
\nabla_X s = \sum_{i \in N_N} b_i L_X(s^i)
\]

for \( X \in \Gamma^\infty(D) \) which makes it clear that the parallel transport is path-independent along paths inside the leaves of \( D \). It remains to show that the subbundles \( E_0 \) and \( E_0^\perp \) are closed under \( \nabla|_D \). In fact, for any subspace \( W \subseteq R^{N_N} \) it holds that \( \Gamma^\infty(\iota^\ast E_N \cap (C \times W)) \) is closed under differentiating in the direction of \( D \): To see this, let \( \Pi^\perp: R^{N_N} \to W^\perp \) be some linear projection onto a complementary subspace \( W^\perp \) of \( W \). Then for \( s \in \Gamma^\infty(\iota^\ast E_N \cap (C \times W)) \) and \( v_p \in D \) we have

\[
\Pi^\perp(\nabla_v s) = \sum_{i \in N_N} \Pi_{ij}^\perp (b_i v_p(s^i)) = \sum_{i \in N_N} b_i \Pi_{ij}^\perp v_p(s^i) = \sum_{i,j \in N_N} b_j v_p(\Pi_{ij}^\perp s^i) = \sum_{j \in N_N} b_j v_p((\Pi^\perp(s))^j) = 0
\]

for the constant matrix coefficients \( \Pi_{ij}^\perp \) of \( \Pi^\perp \). Therefore, \( \nabla_v s \in \iota^\ast E_N \cap (C \times W) \). This shows that \( E_0 \) and \( E_0^\perp \) are closed under \( \nabla \).

It remains to show that for \( \mathfrak{E} := (E_{tot}, E_N, E_0, \nabla) \) the sections \( \Gamma^\infty(\mathfrak{E}) \) are isomorphic to the module \( \mathfrak{D} \) we started with. For this, we show that \( \Gamma^\infty(\mathfrak{E}) \cong \text{im} \circ \iota^\ast E_N \) by \( \Xi_{tot}: \text{im}(\epsilon_{tot}) \ni \left( \begin{array}{c} \hat{s}^1 \\ \vdots \\ \hat{s}^{N_{tot}} \end{array} \right) \mapsto (p \mapsto (p, \hat{s}(p))) \in \Gamma^\infty(\mathfrak{E})_{tot} \)

and

\[
\Xi_N: \text{im}(\epsilon_N) \ni \left( \begin{array}{c} \hat{r}^1 \\ \vdots \\ \hat{r}^{N_N} \end{array} \right) \mapsto (p \mapsto (p, \hat{r}(p))) \in \Gamma^\infty(\mathfrak{E})_N.
\]

Here the components fulfill \( \hat{s}^i \in \mathcal{C}^\infty(M) \) and \( \hat{r}^i \in \mathcal{C}^\infty_D(M) \). The map \( \Xi_{tot} \) indeed maps into \( \Gamma^\infty(\mathfrak{E}_{tot}) \) because

\[
\hat{s} \in \text{im}(\epsilon_{tot}) \iff \epsilon_{tot}(\hat{s}) = \hat{s} \iff \hat{e}_{tot}(p) = \hat{s}(p) \quad \forall p \in M,
\]

so \((p, \hat{s}(p)) \in (E_{tot}, p)\) for all \( p \in M \). Moreover, the second component of a section \( s \in \Gamma^\infty(\mathfrak{E}_{tot}) \) is just an element \( \hat{s} \) of \( \mathcal{C}^\infty(M)^{\text{Ntot}} \) which fulfills \( \epsilon_{tot}(p) = \hat{s}(p) \) for all \( p \in M \). Therefore \( \Xi_{tot} \) is surjective. Injectivity is clear. Next, \( \Xi_N \) maps into \( \Gamma^\infty(\mathfrak{E})_N \) because it maps into \( \Gamma^\infty(\mathfrak{E})_N \) for the same reasons as above (the components now being in \( \mathcal{C}^\infty_D(M) \subseteq \mathcal{C}^\infty(M) \)) and because

\[
(\nabla_{v_p}(t^\ast r))^i = \sum_{j \in N_N} v_p(\hat{r}^j \circ i) = 0
\]

for \( r := \Xi_N(\hat{r}) \) and \( v_p \in D \) as \( \hat{r}^j \circ i \in \mathcal{C}^\infty_D(C) \). For surjectivity of \( \Xi_N \) note that \( \nabla_{v_p} s^i = v_p(s^i) \) for \( v_p \in D, s \in \Gamma^\infty(\iota^\ast E_N) \). Then it is clear that the second component of an element in \( \Gamma^\infty(\mathfrak{E})_N \) is just an element \( \hat{r} \in \mathcal{C}^\infty_D(M)^{\text{NN}} \) which fulfills \( \hat{e}_N(p) = \hat{r}(p) \) for all \( p \in M \). Therefore, \( \Xi_N \) is surjective. Clearly, \( \Xi_N \) is injective. We check the condition \( \Xi_{tot} \circ \iota_{\text{im}(\epsilon)} = \iota_{\Gamma^\infty(\mathfrak{E})} \circ \Xi_N \): Let \( \hat{r} \in \text{im}(\epsilon_N) \). Then

\[
\Xi_{tot}(\iota_{\text{im}(\epsilon)}(\hat{r})) = \Xi_{tot}(\sum_{i \in N_N} e_N(i) \hat{r}^i) = \left( p \mapsto (p, \left( \sum_{i \in N_N} e_N(i) \hat{r}^i(p) \right)) \right)
\]

and

\[
\iota_{\Gamma^\infty(\mathfrak{E})}(\Xi_N(\hat{r})) = \iota_{\Gamma^\infty(\mathfrak{E})}(p \mapsto (p, \hat{r}(p)))
\]

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Therefore \( \Xi \) is compatible with the \( i \)'s. We now have to look at the 0-component: If \( \hat{r} \in \text{im}(e)_0 \), then by definition \( \hat{r}^i \) is zero on \( C \) for \( i \in N \setminus N_0 \). In fact for \( \hat{r} \in \text{im}(e) \) we have \( \hat{r} \in \text{im}(e)_0 \) iff \( \hat{r}^i|^C \equiv 0 \) for all \( i \in N \setminus N_0 \). If \( \hat{r}^i|^C \equiv 0 \) for all \( i \in N \setminus N_0 \), then \( \hat{r} = e_N(\hat{r}) \in \text{im}(e|_{\mathcal{A}^0(N)}) \) = \( \text{im}(e)_0 \).

Conversely if \( \hat{r} \in \text{im}(e|_{\mathcal{A}^0(N)}) = \text{im}(e)_0 \), then \( \hat{r} \in \mathcal{A}^0(N) \) as \( e_N \) respects the 0-components. This shows \( \hat{r}^i|^C \equiv 0 \) for all \( i \in N \setminus N_0 \). With this equivalence we see that \( (p, \hat{r}(p)) \in F \) for \( p \in C \) and \( \hat{r} \in \text{im}(e)_0 \). Therefore \( \Xi_N(\text{im}(e)_0) \subseteq \Gamma^\infty(C_0) \). Conversely, for \( r = (p \mapsto (p, \hat{r}(p))) \in \Gamma^\infty(C_0) \) we first of all have \( \hat{r} \in \text{im}(e)_N \) but also \( \hat{r}^i|^C \equiv 0 \) for \( i \in N \setminus N_0 \). This shows that \( \hat{r} \in \text{im}(e)_0 \) and therefore \( \Xi_N|_{\text{im}(e)_0} : \text{im}(e)_0 \to \Gamma^\infty(C_0) \) is surjective. Therefore \( \Xi \) is a morphism of coisotropic \( \mathcal{A} \)-modules which has bijective \( \Xi_{\text{tot}} \) and \( \Xi_N \) and \( \Xi_N \) is surjective on the 0-components. This is equivalent to being an isomorphism of coisotropic modules.

\[ \text{Remark 4.7} \] Of course the existence of such a decomposition in \text{Theorem 4.6} always implies that \( E_0 \) is closed under \( \nabla \) and that \( \nabla \) is a holonomy-free \( D \)-connection. However, without assumptions on the structure of \( C/D \) the converse is not clear, see \text{Lemma 4.5}.

In the singular case, we are not guaranteed that taking triples of invariant sections gives us a finitely generated projective \( \mathcal{A}\infty(M,C,D) \)-module. Nevertheless, if for simplicity we consider \( M = C, \) \( E_{\text{tot}} = E_N = E \) and \( E_0 = C \times \{0\} \) (or \( E=N \)), \text{Example 3.14} and \text{Example 3.16} give us some positive examples in a singular setting.

In the regular case, \text{Theorem 4.6} lets us formulate an equivalence of categories as in \text{Remark 3.1}.

\text{To do so, we define the following category:}

\text{Definition 4.8 (The category} \text{Vect}_3(M,C,D) \text{)} \text{Let} \ i : C \to M \text{ be a submanifold of a manifold} \ M \text{ and} \ D \subseteq TC \text{ an integrable distribution. Then the category} \text{Vect}_3(M,C,D) \text{ has}

\begin{enumerate}
\item[i)] \text{as objects quadruplets} \ (E_{\text{tot}}, E_N, E_0, \nabla) \text{ consisting of a vector bundle} \ E_{\text{tot}} \to M, \text{ a vector subbundle} \ E_N \subseteq E_{\text{tot}} \text{ and a vector subbundle} \ E_0 \subseteq i^*E_N \text{ together with a holonomy-free} \ D \text{-connection on} i^*E_N \text{ such that} \ E_0 \text{ is closed under} \nabla.
\item[ii)] \text{as morphisms vector bundle morphisms} \ \Phi : E_{\text{tot}} \to F_{\text{tot}} \text{ over the identity such that}
\end{enumerate}

\[ \Phi(E_N) \subseteq F_N, \quad i^*\Phi(E_0) \subseteq F_0 \] (4.6)

\text{and}

\[ i^*\Phi \circ (\nabla^F_X s) = \nabla^E_X ((i^*\Phi_N) \circ s) \] (4.7)

\text{for all} \ s \in \Gamma^\infty(i^*E_N) \text{ and all} \ X \in \Gamma^\infty(D).

\text{Example 4.9} \text{Consider the special case} \ M = C \text{ and} \ D = 0, \text{ corresponding to a classical manifold. Then every vector bundle} \ E \text{ over} \ M \text{ can be considered as an object in} \text{Vect}_3(M,M,0) \text{ with identical} \text{tot-} \text{and} \ N \text{-component and vanishing 0-component. The} \ D \text{-connection} \nabla \text{ is trivial since} \ D = 0.

\text{Note that—similarly to} \text{Lemma 3.11} \text{—the infinitesimal condition of preserving the covariant derivative can be integrated to a condition preserving the parallel transport:}

\text{Lemma 4.10} \text{Let} \ D \subseteq TC \text{ be an integrable distribution. Let} \ \Phi : E \to F \text{ be a vector bundle morphism over the identity} \text{ id}_C. \text{ Let} \nabla^E \text{ and} \nabla^F \text{ be} \ D \text{-connections on} \ E \text{ (resp.} \ F). \text{ Then the following is equivalent:}
i.) The vector bundle morphism $\Phi$ fulfills
\[
\Phi \circ (\nabla^E_X s) = \nabla^E_X (\Phi \circ s)
\]
for sections $X \in \Gamma^\infty(D)$ and $s \in \Gamma^\infty(E)$.

ii.) For any smooth curve $\gamma: I \to C$ inside a leaf of $D$ one has for $a, b \in I$ that
\[
\Phi \circ \Gamma^E_{\gamma,a \to b} = \Gamma^E_{\gamma,a \to b} \circ \Phi.
\]

In the category $\text{Vect}_3(M, C, D)$ we can turn taking invariant sections into a functor:

**Definition 4.11 (Taking sections in $\text{Vect}_3(M, C, D)$)** Let $\iota: C \to M$ be a submanifold of a manifold $M$ and $D \subseteq TC$ an integrable distribution. Then we define the functor
\[
\Gamma^\infty: \text{Vect}_3(M, C, D) \to C_3\text{Mod}(\mathcal{E}^\infty(M, C, D))
\]
by

i.) mapping objects $E := (E_{\text{tot}}, E_N, E_0, \nabla^E)$ to the triple $\Gamma^\infty(E)$ of invariant sections.

ii.) mapping morphisms $\Phi: E \to F$ between $E := (E_{\text{tot}}, E_N, E_0, \nabla^E)$ and $F := (F_{\text{tot}}, F_N, F_0, \nabla^F)$ to
\[
\Gamma^\infty(\Phi)_{\text{tot}}: \Gamma^\infty(E)_{\text{tot}} \ni s \mapsto \Phi \circ s \in \Gamma^\infty(F)_{\text{tot}} \quad \text{and}
\Gamma^\infty(\Phi)_N: \Gamma^\infty(E)_N \ni r \mapsto \Phi|_{E_N} \circ r \in \Gamma^\infty(F)_N.
\]

It is easily checked that this indeed gives a functor into $C_3\text{Mod}(\mathcal{E}^\infty(M, C, D))$. By **Proposition 4.3** we know that if $C$ is closed and $D \subseteq TC$ is simple, then
\[
\Gamma^\infty: \text{Vect}_3(M, C, D) \to \text{Proj}(\mathcal{E}^\infty(M, C, D)),
\]
where on the right hand side we have the full subcategory of finitely generated regular projective coisotropic $\mathcal{E}^\infty(M, C, D)$-modules. Moreover, we have seen in **Theorem 4.6** that if $M$ and $C/D$ are connected, then
\[
\Gamma^\infty: \text{Vect}_3(M, C, D) \to \text{Proj}(\mathcal{E}^\infty(M, C, D))
\]
is essentially surjective. It remains to show fullness and faithfulness in order to prove the following result:

**Theorem 4.12 (Coisotropic Serre-Swan Theorem)** Let $\iota: C \to M$ be a closed submanifold of a connected manifold $M$ and let $D \subseteq TC$ be simple with connected leaf space $C/D$. Then the functor
\[
\Gamma^\infty: \text{Vect}_3(M, C, D) \to \text{Proj}(\mathcal{E}^\infty(M, C, D))
\]
yields an equivalence of categories.

**Proof:** Let $E, F \in \text{Vect}_3(M, C, D)$. We claim that
\[
\text{Hom}_{\text{Vect}_3}(E, F) \ni \Phi \mapsto (\Gamma^\infty(\Phi)_{\text{tot}}, \Gamma^\infty(\Phi)_N) \in \text{Hom}_{C_3\text{Mod}(\mathcal{E}^\infty(M, C, D))}(\Gamma^\infty(E), \Gamma^\infty(F))
\]
is a bijection. Injectivity is clear because if $\Gamma^\infty(\Phi)_{\text{tot}} = \Gamma^\infty(\Phi')_{\text{tot}}$, then $\Phi \circ s = \Phi' \circ s$ for all $s \in \Gamma^\infty(E_{\text{tot}})$.

For surjectivity let a morphism $(\chi_{\text{tot}}, \chi_N)$ of the coisotropic $\mathcal{E}^\infty(M, C, D)$-modules $\Gamma^\infty(E)$ and $\Gamma^\infty(F)$ be given. We know that as $\chi_{\text{tot}} \in \text{Hom}_{\mathcal{E}^\infty(M)}(\Gamma^\infty(E_{\text{tot}}), \Gamma^\infty(F_{\text{tot}})) \simeq \text{Hom}_{\text{Vect}(M)}(E_{\text{tot}}, F_{\text{tot}})$ there exists a (unique) vector bundle morphism $\Phi$ over $id_M$ with $\Phi \circ s = \chi_{\text{tot}}(s)$ for all $s \in \Gamma^\infty(E_{\text{tot}})$. For $v_p \in E_N$ the section $s$ defined by
\[
s(q) := \sum_{i \in N_N} d_i(q) d^i(v_p)
\]
for all $q \in C$. This section fulfills $\Gamma^\infty(\Phi)|_{E_N} \circ s = \chi_{\text{tot}}|_{E_N} \circ s$. Therefore, $\Phi$ is given by
\[
\Phi|_{E_N} \circ s = \chi_{\text{tot}}|_{E_N} \circ s
\]
for all $s \in \Gamma^\infty(E_{\text{tot}})$.
with a dual basis as in Proposition 4.3 is an element of $\Gamma^\infty(E_N)$ and fulfills $s(p) = v_p$. Therefore

$$\Phi(v_p) = \chi_{\text{tot}}(t_1(s))|_p = \nu_2(\chi_N(s))|_p \in F_N$$

as $\chi_N(s) \in \Gamma^\infty(E_N)$. Similarly, for $v_p \in E_0$ the section defined by

$$s(q) \coloneqq \sum_{i \in I_0} f_i(q) f^i(v_p)$$

with the $f_i, f^i$ as in Proposition 4.3 is an element of $\Gamma^\infty(E_0)$ and fulfills $s(p) = v_p$. Therefore

$$\Phi(v_p) = \chi_{\text{tot}}(t_1(s))|_p = \nu_2(\chi_N(s))|_p \in F_0$$

as $\chi_N(s) \in \Gamma^\infty(E_0)$. This shows that

$$\Phi(E_N) \subseteq F_N \quad \text{and} \quad \nu^2 \Phi(E_0) \subseteq F_0.$$  

We need to check that $\nu^2 \Phi_N$ preserves the covariant derivatives. For this let $v_p \in \nu^2 E_N$. Then as already used above there exists a section $s \in \Gamma^\infty(E_N)$ with $s(p) = v_p$ and for $\gamma$ being an arbitrary curve in a leaf of $D$ connecting $p, q \in C$ we have

$$P^E_{\gamma,p\rightarrow q}(\Phi(v_p)) = P^E_{\gamma,p\rightarrow q}(\chi_N(s))|_q = \chi_N(s)|_q = \Phi(s(q)) = \Phi(P^E_{\gamma,p\rightarrow q}(s(p))) = \Phi(P^E_{\gamma,p\rightarrow q}(v_p)),$$

because of $\chi_N(s)|_C \in \Gamma^\infty(E_F, \nabla^F)$ and Lemma 3.11. By Lemma 4.10 this shows that $\nu^2 \Phi_N$ preserves the covariant derivatives. Therefore, $\Phi$ is a preimage of $(\chi_{\text{tot}}, \chi_N)$ in $\text{Hom}_{\text{Vect}}(M,C,D)(E,F)$. 

Remark 4.13 The classical Serre-Swan Theorem can be understood as a special case of Theorem 4.12 as follows: By Example 4.9 the category $\text{Vect}(M)$ of vector bundles over a manifold $M$ can be embedded into the category $\text{Vect}_3(M,M,0)$. Similarly, the category $\text{Proj}(\mathcal{C}^\infty(M))$ of finitely generated projective modules over $\mathcal{C}^\infty(M)$ can be embedded into the category $\text{Proj}(\mathcal{C}^\infty(M,M,0))$, see Example 2.27. Then it is easy to see that the functor of taking sections preserves these subcategories, reproducing the classical Serre-Swan Theorem.

4.2 Reduction of Regular Projective Coisotropic $\mathcal{C}^\infty(M,C,D)$-Modules

As a final step, we want to look at the reduction functor $\text{red} : \mathcal{C}_3\text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{A}_\text{red})$: Knowing that—under some regularity assumptions—regular projective coisotropic $\mathcal{C}^\infty(M,C,D)$-modules are essentially triples of vector bundles together with a holonomy-free $D$-connection, we can of course ask whether the quotient from Proposition 3.8 is in some sense compatible with reduction of coisotropic modules, i.e. taking the quotient of the $N$- and the 0-component. As we will see in the following, this is indeed the case. However, we need yet another quotient of vector bundles in order to cope with the sections taking values in $E_0$. For a vector bundle $E \rightarrow C$ and a subbundle $F \subseteq E$ we can consider the fiberwise quotient vector bundle $\frac{E}{F} \rightarrow C$. It has the property that the projection $\Pi : E \rightarrow \frac{E}{F}$ is both a submersion and a vector bundle morphism. Moreover, if $\nabla$ is a $D$-connection on $E$ such that $F$ is closed under $\nabla$, then there exists a unique $D$-connection $\tilde{\nabla}$ on $\frac{E}{F}$ which fulfills

$$\tilde{\nabla}_X(\Pi \circ s) = \Pi \circ (\nabla_X s)$$

for all $X \in \Gamma^\infty(D), s \in \Gamma^\infty(E)$. Note also that for any complementary vector subbundle $F^\perp$ to $F$ the map

$$\psi : F^\perp \rightarrow E_{\frac{E}{F}}, \quad v_p \mapsto [v_p]$$

is an isomorphism of vector bundles. Using this, it is straightforward to prove the following lemma:
Lemma 4.14 Let \( \iota: C \to M \) be a closed submanifold of a manifold \( M \) and \( D \subseteq TC \) an integrable distribution. Let \( E_N \to M \) be a vector bundle and \( E_0 \subseteq \iota^* E_N \) a subbundle. Assume that \( \nabla \) is a \( D \)-connection on \( \iota^* E_N \), such that \( E_0 \) is closed under \( \nabla \) and there exists a complementary subbundle \( E_0^\perp \) which is also closed under \( \nabla \). Then for

\[
\Gamma_D^\infty(E_N, \nabla) \coloneqq \{ s \in \Gamma^\infty(E_N) \mid \nabla_X s = 0 \quad \forall X \in \Gamma^\infty(D) \} \tag{4.17}
\]

and

\[
\Gamma_D^\infty(E_N, \nabla)_0 \coloneqq \{ s \in \Gamma_D^\infty(E_N, \nabla) \mid \iota^* s \in \Gamma^\infty(E_0) \} \tag{4.18}
\]

the map

\[
\Gamma_D^\infty(E_N, \nabla)/\Gamma_D^\infty(E_N, \nabla)_0 \ni [s] \mapsto \Pi \circ \iota^* s \in \Gamma_D^\infty(\frac{\iota^* E_N}{E_0}, \nabla) \tag{4.19}
\]

with \( \Pi: \iota^* E_N \to \frac{\iota^* E_N}{E_0} \) and \( \nabla \) as in (4.15) defines an isomorphism of \( \mathcal{E}_D^\infty(C) \)-modules.

In the case of a simple \( D \subseteq TC \) we can thus prove that reduction is compatible with taking sections:

**Theorem 4.15 (Reduction of triples of invariant sections)** Let \( \iota: C \to M \) be a closed submanifold and let \( D \subseteq TC \) be simple. Moreover, let \( E \in \text{Vect}_3(M, C, D) \). Then

\[
(\Gamma^\infty(E))_{\text{red}} \simeq \Gamma^\infty(\frac{\iota^* E_N}{E_0}/\sim_\nabla) \tag{4.20}
\]

with \( \nabla \) as in (4.15). More precisely, the diagram

\[
\begin{array}{ccc}
\text{Vect}_3(M, C, D) & \xrightarrow{\text{red}} & \text{Vect}(C/D) \\
\downarrow^{\Gamma^\infty} & & \downarrow^{\Gamma^\infty} \\
\text{Proj}(\mathcal{E}_D^\infty(M, C, D)) & \xrightarrow{\text{red}} & \text{Proj}(\mathcal{E}_D^\infty(C/D))
\end{array}
\tag{4.21}
\]

commutes up to a natural isomorphism, where \( \text{red}: \text{Vect}_3(M, C, D) \to \text{Vect}(C/D) \) is the functor mapping objects \((E_{tot}, E_N, E_0, \nabla)\) to \( \frac{\iota^* E_N}{E_0}/\sim_\nabla \) and morphisms to their obvious induced counterparts.

**Proof:** By **Lemma 4.13** there exists a complementary subbundle \( E_0^\perp \), which is closed under \( \nabla \). Thus by **Lemma 4.14** we have

\[
(\Gamma^\infty(E))_{\text{red}} \simeq \Gamma_D^\infty(\frac{\iota^* E_N}{E_0}, \nabla).
\]

Using the fact that \( \Pi: \iota^* E_N \to \frac{\iota^* E_N}{E_0} \) is compatible with the covariant derivatives we see that by **Lemma 4.10** \( \nabla \) is a holonomy-free \( D \)-connection. Thus **Lemma 3.12** yields the first result. To see that the diagram of functors commutes up to natural isomorphism first note that red: \( \text{Vect}_3(M, C, D) \to \text{Vect}(C/D) \) maps morphisms \( \Phi: E \to F \) in \( \text{Vect}_3(M, C, D) \) to vector bundle morphisms \( \phi \) over the identity \( \text{id}_{C/D} \), explicitly given by

\[
\phi([v_p]_{E_0})_{\nabla_F} = [\Phi([v_p])]_{E_0} \nabla_F.
\]

The map \( \phi \) is well-defined by the assumptions on \( \Phi \) and a vector bundle morphism because the projections \( \iota^* E_N \to \frac{\iota^* E_N}{E_0}, \frac{\iota^* E_N}{E_0}/\sim_{\nabla_E}, \iota^* F_N \to \frac{\iota^* F_N}{E_0} \) and \( \frac{\iota^* F_N}{E_0}/\sim_{\nabla_F} \) are submersions and vector bundle morphisms. Then, unwinding the definitions of the functors, one sees that the diagram commutes up to the natural isomorphisms \( \eta \) given by

\[
\eta_E: (\Gamma^\infty(E))_{\text{red}} \to \Gamma^\infty(\frac{\iota^* E_N}{E_0}/\sim_\nabla),
\]

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where $\pi^{-1}(q) \subseteq C$ is an arbitrary element of $\pi^{-1}(\{q\})$: This $\eta_\pi$ is just the composition of (4.19) and (3.19).

A result like this was to be expected: In Proposition 2.26 we already saw that the reduction of a regular projective coisotropic $\mathcal{E}^\infty(M,C,D)$-module yields a projective $\mathcal{E}_D^\infty(C) \simeq \mathcal{E}^\infty(C/D)$-module.

Remark 3.1 tells us that those are essentially given by sections of vector bundles over $C/D$.

Example 4.16 (Bott connection) Let $D \subseteq TC$ be an integrable distribution of constant rank. The Bott connection with

$$\nabla: \Gamma^\infty(D) \times \Gamma^\infty\left(\frac{TC}{D}\right) \to \Gamma^\infty\left(\frac{TC}{D}\right), \quad \nabla_X(\Pi \circ Y) = \Pi([X,Y])$$ (4.22)

for $X \in \Gamma^\infty(D)$, $Y \in \Gamma^\infty(TC)$ and the projection $\Pi: TC \to \frac{TC}{D}$ is flat in direction $D$, cf. [3 Lemma 6.3]. It is not hard to check that if $\pi: C \to C/D$ is a submersion,

$$\frac{TC}{D} \ni \Pi(v_p) \mapsto (p, T\pi(v_p)) \in \pi^\perp(T(C/D))$$ (4.23)

is a vector bundle isomorphism over the identity and

$$\frac{TC}{D} / \sim_\nabla \simeq T(C/D).$$ (4.24)

To interpret this in the context of coisotropic triples, first we need to come up with a $D$-connection $\tilde{\nabla}$ on $TC$ which induces $\nabla$ on $\frac{TC}{D}$ via (4.15). This can be done by choosing an arbitrary complement $D^\perp \subseteq TC$ to $D$ and using the isomorphism $\psi: D^\perp \ni v_p \mapsto [v_p] \in \frac{TC}{D}$ of vector bundles to carry $\nabla$ over to $D^\perp$, yielding some $\tilde{\nabla}^D_{D^\perp}$. Now we need to choose an arbitrary $D$-connection $\tilde{\nabla}^D_D$ on the vector bundle $D$. Their direct sum $\tilde{\nabla} := \tilde{\nabla}^D_D \oplus \tilde{\nabla}^D_{D^\perp}$ then is a $D$-connection on $TC$ inducing $\nabla$ on $\frac{TC}{D}$ via (4.15). Hence, Lemma 4.14 implies

$$\Gamma_D^\infty(TC, \tilde{\nabla}) / \Gamma_D^\infty(D, \tilde{\nabla}) \simeq \Gamma_D^\infty\left(\frac{TC}{D}, \nabla\right) \simeq \Gamma^\infty\left(\left(\frac{TC}{D}, \nabla\right) / \sim_\nabla\right) \simeq \Gamma^\infty(T(C/D)).$$ (4.25)

To incorporate this construction into the setting of coisotropic $\mathcal{E}^\infty(M,C,D)$-modules, we might need to restrict to a tubular neighborhood of $C$: Lemma 4.14 allows to find a neighborhood $U \subseteq M$ of a closed submanifold $\iota: C \to M$ such that $TM|_U = TU \simeq t^\sharp(\iota^\sharp(TM))$ for a submersion $t: U \to C$ with $t \circ \iota = \text{id}_C$. Therefore,

$$\widetilde{TC} := t^\sharp(TC) \subseteq t^\sharp(\iota^\sharp(TM)) \simeq TU$$ (4.26)

defines a subbundle of $TU$ which restricts to $TC$ on $C$. We arrive at the coisotropic $\mathcal{E}^\infty(U,C,D)$-module $\Gamma^\infty(TU, \widetilde{TC}, D, \tilde{\nabla})$ with

$$\Gamma^\infty(TU, \widetilde{TC}, D, \tilde{\nabla})_{\text{red}} \simeq \Gamma^\infty(T(C/D))$$ (4.27)

by (4.20). If in addition $\tilde{\nabla}$ is a holonomy-free $D$-connection also on $D$, (it is always holonomy-free on $D^\perp$), then Proposition 4.3 tells us that $\Gamma^\infty(TU, \widetilde{TC}, D, \tilde{\nabla})$ is a finitely generated regular projective coisotropic $\mathcal{E}^\infty(U,C,D)$-module.

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