Diabolical Entropy

Neil Dobbs¹, Nicolae Mihalache²

¹ School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 14, Ireland. E-mail: neil.dobbs@ucd.ie
² Université Paris Est - Créteil, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France. E-mail: nicolae.mihalache@u-pec.fr

Received: 15 September 2017 / Accepted: 26 September 2018
Published online: 6 February 2019 – © Springer-Verlag GmbH Germany, part of Springer Nature 2019

In memory of Tan Lei.

Abstract: Milnor and Thurston’s famous paper proved monotonicity of the topological entropy for the real quadratic family. Guckenheimer showed that it is Hölder continuous. We obtain a precise formula for the Hölder exponent at almost every quadratic parameter. Furthermore, the entropy at most parameters is proven to be in a set of Hausdorff dimension smaller than one, while most values of the entropy arise from a set of parameters of dimension smaller than one.

1. Introduction

This paper studies the regularity of the topological entropy of maps from the quadratic (or logistic) family \( f_a(x) = x^2 + a, a \in [-2, \frac{1}{4}] \). The fundamentals concerning topological entropy for piecewise-monotone maps of the interval were developed by Misiurewicz and Szlenk [38]. They showed that the entropy \( h_{\text{top}}(g) \) of a map \( g \) is the exponential growth rate of the number of monotonic laps of iterates of \( g \) and that, for smooth unimodal maps, the entropy varies continuously. Let us write \( h(a) \) for \( h_{\text{top}}(f_a) \). Milnor and Thurston [35]¹ proved that \( h \) is a monotone function, a result recently generalised to the multimodal setting by Bruin and van Strien [12]. The function \( h \) has range \([0, \log 2]\), yet is locally constant on an open dense subset of \([-2, \frac{1}{4}]\) and therefore quite irregular. On the other hand, Guckenheimer [24] proved that \( h \) is a Hölder continuous map (for \( a \) away from a neighbourhood of the null-set of \( h \)). Our most striking result, Theorem 1, provides an exact formula for the Hölder exponent of \( h \) at most parameters, given in terms of the value of \( h \) itself and the Lyapunov exponent of the critical value.

¹Douady, Hubbard, and Sullivan had proven that the number of periodic orbits of some fixed period is monotonically decreasing, which implies the monotonicity of entropy. This result was unpublished, a later version was published by Douady [19].

N.D. was supported by the ERC Bridges grant while at the University of Geneva.
N.M. was supported by the ERC AG COMPAS grant, CNRS semester and ANR LAMBDA.
A long-standing problem in one-dimensional dynamics is whether the measure of maximal entropy can be absolutely continuous [36, Theorem 9.6], [9], [16], [10, pp 1758-9] or, in Misiurewicz’ words [36], “when is the measure-theoretical entropy equal to the topological entropy”? We resolve this in Theorem 2 using a recent result of Inou. Together with advanced thermodynamic formalism developed in [18] and extended in Sect. 5, we obtain some uniformity estimates for the limit in Theorem 1, in turn giving us dimension estimates for images and preimages of large sets under $h$, see Theorem 4. We extend Guckenheimer’s Hölder continuity result to the entire quadratic family in Theorem 8.

1.1. Precise Hölder exponent almost everywhere. In the presence of an attracting periodic orbit, a map $f_a$ is hyperbolic, the non-wandering set is structurally stable and the entropy $h$ is locally constant. We denote the set of hyperbolic parameters in $[-2, \frac{1}{4}]$ by $\mathcal{H}$. Hyperbolic parameters form an open, dense set [23,33], so $h$ is locally constant on an open dense set. The maximal set $\mathcal{F}$ on which $h$ is locally constant (or flat) strictly contains $\mathcal{H}$. Let us define $\mathcal{V}$ as the set of parameters $a$ at which $h$ is not locally constant at $a$ on either side of $a$. By monotonicity and continuity of $h$,

$$\mathcal{V} = \left\{ a \in \left[-2, \frac{1}{4}\right] : \{a\} = h^{-1}(h(a)) \right\}.$$

Non-renormalisable parameters with positive entropy are contained in $\mathcal{V} \cup \{-2\}$. As a by-product of Jakobson’s theorem [28], $\mathcal{V}$ has positive measure.

The lower Lyapunov exponent $\lambda_l(a)$ of $f_a(a)$ is defined by

$$\lambda_l(a) := \liminf_{n \to \infty} \frac{\log |(f^a)'(a)|}{n};$$

the upper Lyapunov exponent $\lambda_u(a)$ is defined with a $\limsup$ instead.

When $\lambda(a) = \lambda_l(a)$, we call the common value $\lambda(a)$ the (pointwise) Lyapunov exponent of the critical value.

Tsujii’s weak regularity condition [53]

$$\lim_{\delta \to 0^+} \liminf_{n \to \infty} \frac{1}{n} \sum_{|f_a^j(0)| \leq \delta} \log |f_a^j(0)| = 0 \tag{WR}$$

will be rather important in this work. It says that the critical orbit may recur, but not too close too soon and not too often. Let

$$\mathcal{W} := \left\{ a \in \left[-2, \frac{1}{4}\right] : \lambda_u(a) > 0 \text{ and } f_a \text{ verifies (WR)} \right\}. \tag{1}$$

In particular, for all $a \in \mathcal{V}$, $\lambda(a)$ exists. We shall deduce (in Proposition 4.2, based on work of Tsujii, Avila and Moreira and Lyubich [1,2,34,53]) that $\mathcal{W} \subseteq \mathcal{H}^c$ has full measure in $\mathcal{H}^c$ and thus in $\mathcal{V}$. Our first theorem uncovers the local behaviour of $h$ almost everywhere.

**Theorem 1.** For every $a \in \mathcal{V} \cap \mathcal{W}$,

$$\lim_{t \to 0} \frac{\log |h(a + t) - h(a)|}{\log |t|} = \frac{h(a)}{\lambda(a)}. \tag{2}$$
Consequently, the derivative satisfies \( h'(a) = 0 \), for \( a \in \mathcal{V} \cap \mathcal{W} \), if \( h(a)/\lambda(a) > 1 \); if \( h(a)/\lambda(a) < 1 \), \(|h'(a)| = \infty\). Corollary 3 and Theorems 4 and 5 below give more precise information on the structure of \( h \). In particular, \( h' \) exists almost everywhere and is null.

We say that a real map \( g : I \to \mathbb{R} \) is \((C, \beta)\)-Hölder continuous at \( x \in I \) if \( C, \beta > 0 \) and if, for all \( y \in I \),
\[
|g(y) - g(x)| \leq C|y - x|^\beta.
\]
(3)

If the constants are not specified, we will say the map is \( \beta \)-Hölder continuous or Hölder continuous at \( x \). We omit “at \( x \)” if (3) holds at every \( x \in I \). Let
\[
\text{Höl}(g, x) := \sup\{\beta > 0 : g \text{ is } \beta\text{-Hölder continuous at } x\}.
\]

A weaker formulation of Theorem 1 would be that
\[
\text{Höl}(h, a) = \frac{h(a)}{\lambda(a)}
\]
for every \( a \in \mathcal{V} \cap \mathcal{W} \).

While \( h \) is monotone, \( \lambda(a) \) can vary wildly with \( a \). As \( \lambda(a) \) in general does not exist, one cannot do much better than Theorem 1 (which we show for all parameters \( a \in \mathcal{F}^c \) subject to relative full-measure hypotheses: existence of \( \lambda(a) \) and Tsujii’s weak regularity condition). From the proof, one can extrapolate that, if \( \lambda_n(a) = \frac{1}{n} \log |Df^n_a(a)| \) oscillates slowly but with large range as \( n \) grows, the limit in (2) does not exist.

Collet–Eckmann [14] parameters are those for which \( \lambda(a) > 0 \). Benedicks and Carleson [4] proved that they form a positive measure set. Each parameter in \( \mathcal{W} \) is Collet–Eckmann. The set \([-2, 1/4] \setminus (\mathcal{V} \cup \mathcal{F}) \) is necessarily countable; to one side of each parameter, the entropy is locally constant; each parameter in the set is either parabolic or preperiodic. For preperiodic parameters, (2) still holds on the non-locally-constant side, see Theorem 6. For parabolic parameters, we obtain infinite flatness, see Theorem 7.

Isola and Politi [27] performed numerical experiments and some analysis on the regularity of \( h \). They suggested that its local Hölder exponent at some parameter \( a \), as a function of a number \( \tau(a) \) related to the kneading determinant, is its value \( h(a) \). Their hypothesis was recently confirmed [52]. Bruin [9] showed that \( h \) is not absolutely continuous, see also [41]. Besides this, Theorem 1 is the first non-experimental result concerning the regularity of \( a \mapsto h(a) \) since the works of Guckenheimer [24] and Milnor and Thurston [35].

1.2. Uniform estimates and dimension. The orbit of the critical value determines to a large extent the ergodic properties of the map. Conversely, the critical value is often typical with respect to a measure so, knowing the ergodic properties, one can sometimes determine properties of the post-critical orbit. These will permit us to deduce dimension estimates in Theorem 4.

To avoid confusion with the pointwise Lyapunov exponent of the critical value, given a map \( f \), we shall denote the Lyapunov exponent of an \( f \)-invariant probability measure \( \mu \) by
\[
\chi(\mu) = \int \log |f'| \, d\mu.
\]

There are two rather special types of such measures,
\( \mu_{\text{max}} \): the measure of maximal entropy;

\( \mu_{\text{acip}} \): an absolutely-continuous invariant probability (acip).

The Feigenbaum (–Coullet–Tresser) parameter \( a_F \) is the leftmost parameter \( a \) satisfying \( h(a) = 0 \). For a quadratic map \( f_a, a \in [-2, a_F) \), the measure of maximal entropy always exists, is unique and its metric entropy \( h(\mu_{\text{max}}) \) equals \( h(a) \) [45] (this function overloading of \( h \), so it can take as an argument either a parameter or an invariant measure, ought not cause confusion). There may or may not be an acip, but if there is, by [6,31] it is unique and

\[ \chi(\mu_{\text{acip}}) = h(\mu_{\text{acip}}) > 0. \]

We shall use \( \mu_{\text{max}}^a, \mu_{\text{acip}}^a \) to indicate dependence on \( f_a \).

Let us define

\[ \hat{X} := \{ a \in \mathcal{W} : \lambda(a) = \chi(\mu_{\text{acip}}^a) \}, \] (4)

\[ X := \{ a \in \mathcal{V} \cap \mathcal{W} : \lambda(a) = \chi(\mu_{\text{acip}}^a) \}, \] (5)

\[ Y := \{ a \in \mathcal{V} \cap \mathcal{W} : \lambda(a) = \chi(\mu_{\text{max}}^a) \}. \] (6)

\( \hat{X} \) will have full measure in \( \mathcal{H}^c \) and \( X = \hat{X} \cap \mathcal{V} \) will have full measure in \( \mathcal{F}^c \) (Proposition 4.2). It follows from [9,47] (see Proposition 4.3) that \( h(Y) \) has full measure in \([0, \log 2)\).

**Definition 1.1.** We say that a continuous map \( g : I \to I \) defined on a compact interval \( I \) is unimodal if \( g \) has exactly one turning point \( c \) with \( c \in I \setminus \partial I \). We say \( g \) is a smooth unimodal map if, moreover, \( g \) is continuously differentiable and \( c \) is the unique (critical) point satisfying \( g'(c) = 0 \). The map and the critical point are non-degenerate if \( g''(c) \neq 0 \).

**Definition 1.2.** A map \( g : I \to I \) is \( S \)-unimodal if it is a \( C^2 \) smooth unimodal map with critical point \( c, |g'|^{-1/2} \) is convex on each component of \( I \setminus \{c\}, g(\partial I) \subset \partial I \) and \( |g'| > 1 \) on \( \partial I \).

The convexity condition is equivalent [40], for \( C^3 \) maps, to having non-positive Schwarzian derivative, while strict convexity corresponds to negative Schwarzian derivative. Quadratic maps have negative Schwarzian derivative.

Except for some special cases [16], if \( g \) is \( S \)-unimodal with an acip \( \mu_{\text{acip}} \) of positive entropy, then \( h(\mu_{\text{acip}}) < h(\mu_{\text{max}}) = h_{\text{top}}(g) \). In the context of the quadratic family, we obtain the following result of independent interest (whose likelihood was suggested by Bruin [9], see also [11]), improving on the work of the first author [16] and of Misiurewicz [36, Theorem 9.6]. It depends on a recent result of Inou [26] on extensions of conjugacies of polynomial-like maps and is proven in Sect. 5.1.

**Theorem 2.** If \( f_a : x \mapsto x^2 + a, \mu_{\text{max}}^a = \mu_{\text{acip}}^a \) if and only if \( a = -2 \).

It was known already to von Neumann and Ulam [55] that \( \mu_{\text{acip}}^{-2} \) is the pullback of Lebesgue measure by the smooth conjugacy to the tent map \( x \mapsto 1 - 2|x| \). It follows that \( h(\mu_{\text{acip}}^{-2}) = \log 2 \) and \( \mu_{\text{acip}}^{-2} = \mu_{\text{max}}^{-2} \). In Sect. 5.1 we show that absolute continuity of \( \mu_{\text{max}}^a \) implies \( a = -2 \).

For \( a \neq -2, h(\mu_{\text{acip}}^a) < h(a) \) (when \( \mu_{\text{acip}}^a \) exists). For \( a = -2 \), one can verify that \( \lambda(a) = \log 4 = 2 \chi(\mu_{\text{max}}^a) = 2 \chi(\mu_{\text{acip}}^a) \). Thus \( -2 \notin \hat{X} \cup Y \). On \( \hat{X} \), we deduce

\[ \lambda(a) = \chi(\mu_{\text{acip}}^a) = h(\mu_{\text{acip}}^a) < h(\mu_{\text{max}}^a) = h(a). \] (7)
For $a \in \mathcal{H}$, $\lambda(a) < h(a)$. On $\mathcal{F}$, $h' = 0$ by definition. Applying Theorem 1 on $X$, a full measure subset of $\mathcal{F}^c$, we obtain the following.

**Corollary 3.** For almost every $a \in [-2, \frac{1}{4}]$, $\lambda(a) < h(a)$ and $h'(a) = 0$.

Thus $h$ is absolutely singular. A *devil’s staircase* function can be defined as a non-constant, continuous, monotone function which has derivative 0 almost everywhere and is locally constant on an open dense set. In particular, $h$ is a devil’s staircase function. We shall improve on this, obtaining uniform dimension estimates away from $\{-2, a_F\}$.

Recent tools from [18] allow us to prove uniformity in (7) and to prove continuous dependence of $\chi(\mu_{\text{max}}^a)$ on $a$ (Theorem 5.8). We obtain the following, where $\dim_H(A)$ denotes the Hausdorff dimension of a set $A$.

**Theorem 4.** For every $\varepsilon > 0$,

$$\dim_H(h(X \cap [-2 + \varepsilon, a_F - \varepsilon])) < 1,$$

$$\dim_H(Y \cap [-2 + \varepsilon, a_F - \varepsilon]) < 1.$$  

This is much stronger than just absolute singularity of $h$. Neglecting a neighbourhood of two points, full measure (note that $\dim_H(h(\mathcal{F})) = 0$) gets mapped to dimension strictly less than one and dimension strictly less than one gets mapped to full measure. We shall also show that removing a neighbourhood of $a = -2$ is necessary, for otherwise the above dimensions tend to 1, see Theorem 5.13. Straightforward numerical estimates indicate that the uniformity estimates extend to a neighbourhood of $a_F$; however, the method known to the authors requires estimating multipliers in a renormalisation limit, whose rigorous proof would be inappropriately lengthy.

It is possible to find positive measure subsets of $X$ on which $h(a)/\lambda(a)$ is arbitrarily large. This leads to:

**Theorem 5.** Given $\varepsilon > 0$, there is a positive measure subset of $\mathcal{F}^c$ whose image under $h$ has dimension at most $\varepsilon$.

### 1.3. Renormalisation.

**Definition 1.3.** We say that a unimodal map $g : I \to I$ is renormalisable of period $n \geq 2$ if there is an open interval $J \subset I$ containing its turning point such that its images $g^i(J)$ with $i = 0, 1, \ldots, n - 1$ are disjoint and the restriction of $g^n$ to $J$ is unimodal. The interval $J$ is called a *restrictive interval*. A period-two renormalisation is called a *Feigenbaum renormalisation*.

The flat set $\mathcal{F} \subset [-2, \frac{1}{4}]$ is the union of the interval $(a_F, \frac{1}{4}]$ and the interior of all non-Feigenbaum renormalisation windows [19].

**Definition 1.4.** A unimodal map $g$ is said to be *pre-Chebyshev* if: a) $g$ is exactly $m$ times renormalisable, for some $m \geq 0$, and each renormalisation is of period two; b) if $J$ is the restrictive interval for the $m^{\text{th}}$ renormalisation, $g_{|J}^{2m} : J \to J$ is smoothly conjugate on $J$ to $x \mapsto 1 - 2|x|$ on $(-1, 1)$.  

1.4. Unimodal families. It is reasonable to ask for local results for more general uni-modal families than the quadratic family. One must pass from the small scale in parameter space to the large scale in phase space to obtain such results. Tsujii [53] provided the tools to do this and we work within his setting. More general results are possible, but would require for example developing Tsujii-Benedicks-Carleson type results for uni-modal maps with degenerate critical points or, for example, using Sands’ techniques [48] to deal with a lack of transversality.

Well-rooted family. Let $I$ denote the compact interval $[-1, 1]$. Consider $G : I \times [0, 1] \rightarrow I$ of class $C^2$ with $g_t = G(\cdot, t)$, so $G$ defines a one-parameter family of interval maps $g_t : I \rightarrow I$. We denote by $\partial_1 G$, $\partial_2 G$ the partial derivatives with respect to the first and second variables. Suppose each $g_t$ is a smooth unimodal map with critical point situated at 0 and $g_t(\partial I) \subset \partial I$. For $g_0$ we impose that

- $g_0$ is a non-degenerate S-unimodal map;
- the Lyapunov exponent $\lambda_0$ of the critical value of $g_0$ exists and $\lambda_0 > 0$;
- Tsujii’s weak regularity condition holds:

$$\lim_{\delta \to 0^+} \liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log |g_0'(g_j(0))| = 0;$$

which means all denominators are non-zero, the series converges and its value is non-zero.

Such a $g_0$ satisfies the backward Collet–Eckmann condition and all periodic points of $g_0$ are hyperbolic repelling [40, Theorem A]. For such $g_0$, the topological entropy is positive. As $\lambda_0 > 0$, convergence in (9) at $t = 0$ is automatic.

Definition 1.5. We call $G$, as above, a well-rooted unimodal family.

Let us simplify the notation by $h(t) := h_{\text{top}}(g_t)$. We have the local, general form of Theorem 1.

Theorem 6. Let $G$ be a well-rooted family and assume that $h$ is monotone and that $h$ is not locally constant at $t = 0$. Then

$$\lim_{t \to 0^+} \frac{\log |h(t) - h(0)|}{\log |t|} = \frac{h(0)}{\lambda_0}.$$ 

It is unclear whether the monotonicity hypothesis is necessary. Without it, one can obtain partial results. Monotonicity is an interesting, subtle question, see the discussions and results in [8,12].

Corollary 1.6. Theorem 1 holds.
Proof. Set \( r_a = (1 + \sqrt{1 - 4a})/2 \). A quadratic map \( f_a \) maps the interval \( I_a = [-r_a, r_a] \) into itself, while points outside \( I_a \) escape to infinity, and \( f_a(\pm r_a) = r_a \). Quadratic maps are non-degenerate and have negative Schwarzian derivative, properties preserved by affine conjugacies. The definition of \( W \) gives existence of \( \lambda(a) > 0 \) and that Tsuji’s weak regularity condition holds, properties preserved by smooth conjugacy. Levin [32] proved that transversality holds in the quadratic family if

\[
\sum_{k \geq 0} \frac{1}{|f_a^k'(a)|} < \infty.
\]

The above sum is finite for all \( a \in W \) since \( \lambda(a) > 0 \), so transversality does indeed hold. Transversality is preserved by smooth families of diffeomorphic conjugacies.

Thus \( g_t(x) := r_a^{-1} f_a(r_a x) \) defines a well-rooted unimodal family to which we can apply Theorem 6. Since entropy is preserved by topological conjugacy and \( a \mapsto r_a \) is smooth (for \( a \in [-2, 1/4] \)), Theorem 1 holds. \( \square \)

When \( g_0 \) is preperiodic, one could refine the estimates to obtain actual Hölder continuity of \( h \) at 0 with exponent \( \text{Höl}(h, 0) \). The argument follows the proof of Theorem 6, noting that for every time \( n \), one can pass with bounded distortion from small scale in parameter space to large scale in phase space (see for example [48]).

At the boundary of a flat interval for \( h \), there is either a pre-Chebyshev or a (primitive) parabolic parameter. A pre-Chebyshev renormalises to a full-branched map, thus it is preperiodic and fully described by the previous remark. At parabolic parameters \( h : t \mapsto h_{\text{top}}(g_t) \) is very flat:

**Theorem 7.** Let \( G \) be a \( C^2 \) family of \( C^2 \) unimodal maps \( \{g_t\}_{t \in [0, 1]} \) so that \( g_0 \) has a parabolic periodic orbit which attracts, but is disjoint from, its critical orbit. Then

\[
\text{Höl}(h, 0) = \infty.
\]

For S-unimodal maps, the existence of a parabolic periodic point implies that the critical point is contained in the immediate basin of attraction. In Sect. 3.2, we shall present more precise estimates for the flatness of entropy at parabolic parameters.

1.5. Hölder estimates. Guckenheimer [24] assumes that the entropy is bounded away from zero to obtain uniform Hölder regularity for unimodal families. We extend this result to the full quadratic family.

**Theorem 8 ([24]).** For the quadratic family, the function \( h \) is Hölder continuous.

The exponent \( \beta \) of Hölder continuity which is obtained will depend on a bound from renormalisation theory. Numerical and heuristic estimates lead us to formulate the following, recalling \( a_F = \inf \{a : h(a) = 0\} \).

**Conjecture 1.** There exists \( \beta > \frac{1}{2} \) such that, on any compact subinterval of \((-2, a_F)\), \( h \) is \( \beta \)-Hölder continuous.

Finding the optimal \( \beta \) appears difficult. As estimated in [24], \( \text{Höl}(h, -2) = 1/2 \), while straightforward computation gives

\[
\text{Höl}(h, a_F) = \frac{\log 2}{\log \delta^*} = 0.449 \ldots,
\]

where \( \delta^* = 4.669 \ldots \) is the first Feigenbaum constant. In particular, the exponent is worse at the boundary of \([-2, a_F]\).
1.6. Further remarks. Recent results [3,13] consider the regularity of the entropy in families of dynamical systems with holes. That is, for a fixed map on the circle or the interval, they remove an interval and all its preimages from the phase space, thus modelling physical systems which leak mass. By varying the size and the position of the initial gap, they show that the entropy is Hölder continuous. In both cases, the regularity constant involves the value of the entropy itself.

Dudko and Schleicher [20] and Tiozzo [51] have developed the theory of core entropy of complex quadratic polynomials, as introduced by Thurston. They proved continuous dependence of core entropy on the parameter. Their work does not apply to regularity of $h$.

1.7. Structure. Unimodal maps are canonically semi-conjugate to tent maps with the same entropy. Facts concerning tent maps, together with some further preliminaries, are presented in the following section.

The proof of Theorem 6 will depend on studying the movement of critical orbits as one varies the parameter, both for unimodal families of smooth maps and the family of tent maps. The speed depends on the Lyapunov exponents of the image of the respective turning points. For the tent map, the Lyapunov exponent is just the entropy, and the parameter change is comparable to the entropy change. Combining these elements forms the backbone of the proof, presented in Sect. 3.1. Theorem 1 follows from Theorem 6, as already noted. In Sect. 3.2, we provide refined estimates near parabolic parameters which imply Theorem 7.

In Sect. 4, we show that certain sets of parameters have large measure.

Theorem 2 is proven in Sect. 5.1. More generally, in Sect. 5, we study the relation between $h(a)$ and $\lambda(a)$. For $a \in X \cup Y$, $h(a) \neq \lambda(a)$. We use Theorem 2 and thermodynamic formalism for families of unimodal maps to obtain uniformity in $h(a) \neq \lambda(a)$ on $(X \cup Y) \cap [-2+\varepsilon, a_F - \varepsilon]$. Using Theorem 1 and a classical dimension lemma, we obtain Theorems 4 and 5.

In the final section we prove Theorem 8, giving uniform Hölder continuity for the entire quadratic family.

2. Preliminaries

2.1. Tent maps, kneading itineraries and conjugacies. Let us introduce the family of (symmetric) tent maps

$$T_b : x \mapsto 1 - b|x|$$

for $b \in (1, 2]$ and $x \in \mathbb{R}$. The orientation-preserving fixed point is $\frac{-1}{b-1}$. $T_b$ restricted to the interval

$$\hat{I}_b := \left[ \frac{-1}{b-1}, \frac{1}{b-1} \right]$$

is unimodal. If $b > \sqrt{2}$ then $T_b$, restricted to $\hat{I}_b$, is not renormalisable, while if $b \in (1, \sqrt{2}]$, $T_b$ is (Feigenbaum) renormalisable of period two. The tent map $T_b$ has slope $\pm b$ and topological entropy $\log b$. We call a tent map periodic if the orbit of the turning point 0 is periodic. The period is necessarily at least 3.
Given a unimodal map \( g \), smooth on the complement of its turning point \( c \), let

\[
K(g) := \varepsilon_1 \varepsilon_2 \ldots, \quad \text{where } \varepsilon_i = \text{sgn}(g'(g^i(c))) \in \{0, \pm1\}.
\]

We call \( K(g) \) the \textit{kneading itinerary} of \( g \).

If the critical point is not periodic, for \( i \geq 0 \), let

\[
d_i := \prod_{j=1}^{i} \varepsilon_j,
\]

thus

\[
d_0 = 1, \quad d_1 = \text{sgn}(g'(g(c))) \quad \text{and} \quad d_i = \text{sgn}((g^i)'(g(c))).
\]

If the critical point is periodic of minimal period \( n \), then we set \( d_n = 1 \), see Lemma 4.5 in [35]. We use the above definition for \( d_0, \ldots, d_{n-1} \) and set the coefficients \( d_k \) with \( k > n \) so that the sequence \((d_k)_k\) is periodic with period \( n \).

In both cases, for all \( k \in \mathbb{N} \), \( d_k \in \{-1, 1\} \). The kneading determinant \( D_g \) is defined by

\[
D_g(t) := 1 + d_1 t + d_2 t^2 + \cdots.
\]

By Milnor and Thurston [35],

\[
s := \inf \{ t > 0 : D_g(t) = 0 \} \in (0, 1)
\]

if and only if

\[
h_{\text{top}}(g) > 0,
\]

and in this case

\[
h_{\text{top}}(g) = -\log s.
\]

In particular, the kneading itinerary determines the entropy.

Following the notation of [7], for \( n \geq 0 \), let

\[
\varphi_n(b) := T^n_b(1). \quad (10)
\]

Observe that

\[
\varphi_i(b) = 1 - b|\varphi_{i-1}(b)| = 1 - b \text{sgn}(\varphi_{i-1}(b)) \varphi_{i-1}(b).
\]

Note that \( \varphi_{i-1}(b) \in \hat{I}_b \) so, differentiating the above formula and dividing across by \( \varphi_{i-1}'(b) \), if \( |\varphi_{i-1}'(b)| \) is sufficiently large then

\[
\frac{b+1}{2} < \left| \frac{\varphi_i'(b)}{\varphi_{i-1}'(b)} \right| < b + 1. \quad (11)
\]

**Lemma 2.1.** Given \( b_* > 1 \), there are constants \( C, N \geq 1 \) such that, if \( b \geq b_* \) and \( n \geq N \), the derivative \( \varphi_n'(b) \) exists unless

\[
\varphi_j(b) = 0
\]

for some \( j \) with \( 1 \leq j < n \). If \( \varphi_n' \) exists on an interval \( V \subset [b_*, 2] \) then

\[
C^{-1} b_0^n \leq |\varphi_n'(b)| \leq C b_0^n \quad (12)
\]

for all \( b, b_0 \in V \).
Proof. Brucks and Misiurewicz [7] proved this lemma for \( b_\ast = \sqrt{2} \). We shall consider the case \( b \in (1, \sqrt{2}] \). On every interval on which \( \varphi_j(b) \neq 0 \) for all \( 1 \leq j < n \), \( \varphi_n \) is a polynomial of degree \( n \), thus continuously differentiable.

For \( m \geq 0 \) and \( b_m = 2^{-m} \), one has \( \varphi_j(b_m) \neq 0 \) for any \( j \geq 1 \). The derivative is continuous at each \( b_m \) and they form (for bounded \( m \)) a discrete set of parameters, so we can omit them from our considerations.

Let \( m \geq 1 \) be such that \( b_{2m} = \sqrt{2}, \sqrt{2} \). Note that \( T_b \) is \( m \)-times renormalisable [7, 30]. Let \( A_b \) denote the affine rescaling which maps the closure of the restrictive interval for \( T_{b_{2m}} \) to the interval \( \hat{I}_{b_{2m}} = \hat{I}_{b_{2m}} \). Since \( A_b(0) = 0 \), \( A_b \) is just multiplication by some number \( ab \) bounded away from 0. The boundaries of the restrictive interval and of \( \hat{I}_{b_{2m}} \) are preperiodic points with smooth continuations, so \( b \mapsto ab \) is smooth as long as \( T_b \) remains \( m \)-times renormalisable. Differentiating

\[
\varphi_{2^m k - 1}(b) = a_b^{-1} \varphi_{k - 1}(b_{2^m})
\]

one obtains

\[
\varphi'_{2^m k - 1}(b) = \frac{2^m b_{2^m - 1}}{a_b} \varphi'_{k - 1}(\hat{b}) - \varphi_{k - 1}(\hat{b}) \frac{a'_b}{a_b^2}.
\]

By [7], if \( \varphi_j(\hat{b}) \neq 0 \) for \( j = 1, \ldots, k - 2 \), then \( \varphi'_{k - 1}(\hat{b}) \) exists and is comparable with \( \hat{b}^{k - 1} \), so this term dominates for large \( k \), \( a_b \) and \( a'_b \) being bounded functions. Bound (12) thus holds for \( n = 2^m k - 1 \), for all large \( k \), for \( b = b_0 \). Applying (11) allows one to interpolate, giving (12) for all large \( n \), for \( b = b_0 \).

Let \( V \subset [b_\ast, 2] \) be an interval on which \( \varphi'_{n} \) exists. On \( V \), \( |\varphi'_{n}| \geq C^{-1} b_\ast^{n} \). For \( b, b_0 \in V \), we deduce

\[
|b - b_0| \leq C b_\ast^{-n}.
\]

Therefore,

\[
b^n/b_0^n = \left(1 + \frac{b - b_0}{b_0}\right)^n \leq C_1.
\]

Redefining \( C \), (12) is proven. \( \square \)

For a quadratic map \( f_a : x \mapsto x^2 + a, a \in [-2, 1/4] \), the fixed points solve \( x^2 + a = x \), so \( f_a \) restricted to the interval

\[
I_a := \left[ \frac{-1 - \sqrt{1 - 4a}}{2}, \frac{1 + \sqrt{1 - 4a}}{2} \right]
\]

is unimodal.

Given any unimodal map \( g \), there is a canonical semi-conjugacy \( \phi \),

\[
\phi \circ T_b \circ \phi = T_{b_{2m}}
\]
between $g$ and $T_b$, where $b = \exp(h_{\text{top}}(g))$. See [15, Chapter II] as a general reference for this and the following statements. S-unimodal maps do not have wandering intervals, and any non-repelling periodic orbit (or non-degenerate interval of periodic points) has the critical point in its immediate basin. If $T_b$ is not periodic and $g$ is S-unimodal, then $\phi$ is a conjugacy. If $T_b$ is periodic, the interior of the preimage of the turning point by the conjugacy, i.e. of $\phi^{-1}(0)$, is a restrictive interval for $g$. Since the period of $T_b$ is greater than two, $g$ is renormalisable of some period other than two.

If $G$ is a well-rooted family, the entropy $h_{\text{top}}(g_t)$ is a continuous function of $t$ [37, Theorem 2]. Thus $h_{\text{top}}([g_{t_0}, g_{t_1}])$ contains $[h_{\text{top}}(g_{t_0}), h_{\text{top}}(g_{t_1})]$. If $\log b \in (h_{\text{top}}(g_{t_0}), h_{\text{top}}(g_{t_1}))$, there exists $t \in (t_0, t_1)$ for which $g_t$ and $T_b$ have the same kneading itinerary, that is,

$$K(g_t) = K(T_b),$$

which implies that $g_t$ and $T_b$ are semi-conjugate and have the same topological entropy $\log b$. If $T_b$ is periodic and $g_t$ satisfies (13), the critical orbit of $g_t$ is periodic and, therefore, super-attracting.

If $g$ is S-unimodal with $h_{\text{top}}(g) \leq \frac{\log 2}{2m}$, then (since the same holds for tent maps as noted during the proof of Lemma 2.1) $g$ is $m$ times Feigenbaum renormalisable (that is, with period two) to a unimodal map $\hat{g}$ with

$$h_{\text{top}}(\hat{g}) = 2^m h_{\text{top}}(g).$$

### 2.2. Notations and conventions.

Given two non-negative expressions $A(\cdot)$ and $B(\cdot)$, we say that $A$ dominates $B$ and write

$$A \gtrsim B,$$

if there exists a constant $C > 0$ such that

$$A \geq CB.$$  

We say that $A$ and $B$ are comparable and write

$$A \simeq B$$

if $A \gtrsim B$ and $B \gtrsim A$. As entropy is a logarithmic expression, we will need an additive equivalent. We say

$$A \simeq_+ B$$

if $e^A \simeq_+ e^B$.

We write $(x, y)$ for the open interval bounded by $x, y$ irrespective of which of $x, y$ is greater, and similarly for half-open or closed intervals.

### 3. Proofs of Theorems 6 and 7

We denote the distortion [15, Sect. I.2] of a smooth map $\psi$ on an interval $J$ by

$$\text{Dist}(\psi, J) := \sup_{x, y \in J} \log \frac{\psi'(x)}{\psi'(y)} \in \mathbb{R}^+.$$
Let \( G : I \times [0, 1] \rightarrow I \), \( G : (x, t) \mapsto g_t(x) \) be a \( C^2 \) family of \( C^2 \) unimodal maps with each \( g_t \) having its critical point at 0. Define, for \( n \geq 0 \),

\[ \xi_n(t) := g^n_t(g_t(0)). \]

These functions describe the displacement of the critical orbit as one varies the parameter. It is straightforward to check that

\[
\frac{\xi_n'(t)}{(g^n_t)'(g_t(0))} = \sum_{j=0}^{n} \frac{\partial^2 G(g_j^t(0), t)}{(g_j^t)'(g_t(0))}. 
\] (15)

For the remainder of the section, set \( h(t) := h_{\text{top}}(g_t) \).

### 3.1. Proof of Theorem 6

Let \( G : I \times [0, 1] \rightarrow I \), \( G : (x, t) \mapsto g_t(x) \) be a well-rooted unimodal family as per Definition 1.5. The unique critical point of each \( g_t \) is at 0. Let us denote \( g := g_0 \). The Lyapunov exponent of its critical value \( v := g(0) \) exists and is denoted \( \lambda_0 \). Let \( b_0 := \exp(h(0)) \).

To prove Theorem 6, the first step is to find an increasing sequence of times \((k_n)_n \) and decreasing parameter intervals \( \omega_n \) for which \( \xi_{k_n} \) maps \( \omega_n \) to the large scale in phase space with bounded distortion.

Next, we introduce safe elements for tent maps. These elements and their preimages under the (semi-)conjugacy are topologically-defined points. Some of them are contained in \( \xi_{k_n}(\omega_n) \). One can catch these topologically-defined points, that is, there is a parameter \( t_n \in \omega_n \) which gets mapped by \( \xi_{k_n} \) onto such a point.

The logarithmic size of \( t_n \) is roughly \(-k_n \lambda_0\). Doing the same thing for tent maps, for the corresponding tent parameter \( b_n = \exp(h(t_n)) \), the logarithmic size of \( |b_n - b_0| \) is roughly \(-k_n \log b_0 \), and \( |b_n - b_0| \) is comparable with \( |\log b_n - \log b_0| \), the difference in the entropies. One obtains an estimate for the entropy at each \( t_n \), which gives the result.

The details constitute the remainder of this subsection. As in [53], we use \( \omega \) and \( \omega_n \) to denote small parameter intervals; the reader should be aware that their images (for example \( \xi_n(\omega_n), p(\omega) \) in what follows) will also be intervals.

### Lemma 3.1 ([53])

Let \( \delta, \varepsilon > 0 \). There exist \( r_0, m_0 > 0 \), a strictly increasing sequence \((k_n)_{n \geq 0} \) of positive integers and a decreasing sequence of intervals \( \omega_n = [0, s_n) \) such that, for all \( n \geq 0 \),

\[
\frac{k_{n+1}}{k_n} \leq 1 + \delta, \\
|\xi_{k_n}(\omega_n)| > r_0, \\
\text{Dist}(\xi_j, \omega_n) \leq 1 \quad \text{for} \quad m_0 \leq j \leq k_n, \\
\text{(16)}
\]

and, for all \( t \in \omega_n \),

\[
\lambda_0 - \varepsilon \leq \frac{1}{k_n} \log |\xi_{k_n}'(t)| \leq \lambda_0 + \varepsilon. \\
\text{(17)}
\]
Proof. This was essentially shown by Tsujii [53] and relies on the properties, including transversality, of $g_0$. Since $\lambda_0$ exists, for any $\kappa > 0$, for all large $m$,

$$\lambda_0 - \kappa \leq \frac{1}{m} \log |(g^m)'(v)| \leq \lambda_0 + \kappa. \quad (18)$$

This implies

$$\liminf_{n \to \infty} \frac{1}{n} \log |g'(g^n(v))| = 0,$$

which is condition (W) in [53]. Since $a \in W$, we can apply [53, Lemma 4.5] [our $g_t$ is Tsujii's $f_t$]. Let $\zeta > 0$ be small. Then there exists $\rho > 0$ and a number $a^+(m)$ [the extra variables in Tsujii's $a^+(\cdot, \cdot; \cdot)$ are constant, for our purposes] such that for each sufficiently large $L$, there exists $m = m(L)$ with

$$L \leq \log |(g^m)'(v)| \leq L(1 + \zeta) \quad (19)$$

and

$$|(g^m)'(v)| \cdot a^+(m) > \rho. \quad (20)$$

From (18) and (19) with $\kappa, \zeta$ sufficiently small, we obtain a strictly increasing sequence $(\kappa_n)$ with $\kappa_{n+1}/\kappa_n \leq 1 + \delta$ for which (20) holds for each $m = m_n$.

Now we apply [53, Lemma 5.2]. Inequality (5.1) there is verified since $g$ is Collet–Eckmann. We obtain, for some constant $l_1 > 1$, for each $n$,

$$l_1^{-1} < \frac{|\xi_{k_n}'(0)|}{|(g^{k_n})'(v)|} < l_1$$

and

$$\text{Dist}(\xi_j, \omega_n) \leq 1,$$

for $m_0 \leq j \leq k_n$ (for some fixed $m_0$), on an interval $\omega_n$ of length at least $l_1^{-1} \cdot a^+(k_n)$.

Thus

$$|\xi_{k_n}(\omega_n)| > l_1^{-1} \cdot a^+(k_n) \cdot l_1^{-1} |(g^{k_n})'(v)|.$$

The result follows, taking $r_0 := l_1^{-2} \rho$. □

Lemma 3.2. In the setting of Lemma 3.1, for $\varepsilon, \delta > 0$ small enough, for $n$ large enough and all $m < k_n$, the image $\xi_m(\omega_n)$ does not contain the critical point 0.

Proof. Since the $C^1$-norm of $G$ is bounded, by $C > 0$ say, and

$$\xi_{m+1}'(t) = g'_t(\xi_m(t))\xi_m'(t) + \partial_2 G(\xi_m(t), t),$$

$|\xi_m'|$ cannot grow too fast, in particular $|\xi_{m+1}'(t)| \leq C|\xi_m'(t)| + C$ so, for $j \geq 1$,

$$|\xi_{m+j}'(t)| \leq (2C)^j \max(|\xi_m'(t)|, 1). \quad (21)$$

Moreover, if $\xi_m(t) = 0$, then

$$|\xi_{m+1}'(t)| = |\partial_2 G(0, t)| \leq C.$$
Take $\varepsilon \in (0, \lambda_0/4)$ and $\delta > 0$ small enough that $\delta \log(2C) < \lambda_0/4$. For large $n$, $k_{n+1} \leq k_n + \delta k_n$ and
\[
\frac{3}{4} \lambda_0 \leq \lambda_0 - \varepsilon \leq \frac{1}{k_{n+1}} \log |\xi'_{k_{n+1}}(0)|. \tag{22}
\]

For $m$ between $k_n$ and $k_{n+1}$, $|\xi'_m(0)| > 1$, and (21) implies that
\[
\log |\xi'_{k_{n+1}}(0)| \leq k_n \delta \log(2C) + \log |\xi'_m(0)| \leq k_n \frac{\lambda_0}{4} + \log |\xi'_m(0)|.
\]
Combining this with (22), we deduce that
\[
\frac{\lambda_0}{2} \leq \frac{1}{k_{n+1}} \log |\xi'_m(0)| \leq \frac{1}{m} \log |\xi'_m(0)|.
\]
From this and bounded distortion (16), there exists $M \geq 1$ for which, for $n \geq 1$ and $M \leq m \leq k_n$,
\[
|\xi'_m(t)| \geq e^{\frac{m\lambda_0}{2} - 1} > C
\]
for all $t \in \omega_n$. Thus $0 \notin \xi_{m-1}(\omega_n)$ for such $m, n$. Now we deal with $m \leq M$. As $|\omega_n| \to 0$ and the critical orbit of $g_0$ is not periodic, there exists $N$ such that if $m \leq M$ and $n \geq N$, $0 \notin \xi_m(\omega_n)$. \qed

**Safe elements.** Given a tent map $T$ and an integer $N \geq 1$, we call an element $x$ of $T^{-N}(0)$ safe if $x \notin \text{Orb}(0)$. Considering the minimal $n$ with $T^n(x) = 0$, safe elements admit continuations under small perturbations of $T$. Focusing on our base tent parameter $b_0$, we denote by
\[
S_N := \bigcup_{j=1}^{N} T_{b_0}^{-j}(0) \setminus \text{Orb}(0) \tag{23}
\]
the collection of safe elements of $T_{b_0}$ for times up to $N$. Denote by $S = S_\infty$ the collection of all safe elements for $T_{b_0}$. This set is dense in $\hat{I}_{b_0}$. Let $\phi$ be the semi-conjugacy of $g$ to $T_{b_0}$
\[
\phi \circ g = T_{b_0} \circ \phi. \tag{24}
\]
For each safe element $x$ with corresponding minimal $n$, there may be an interval mapped by $\phi$ to $x$. However, there is a unique point of $g^{-n}(0)$ mapped by $\phi$ to $x$. If $\phi$ is a conjugacy, then $\phi$ is bijective on $\phi^{-1}(S)$ and the points in $\phi^{-1}(S)$ admit continuations (as points of $g_0^{-n}(0)$ as one perturbs $g$).

**Lemma 3.3.** Suppose $h$ is not locally constant at $t = 0$. Given $r_0 > 0$, there exist $\varepsilon_0, N, N_1 > 0$ for which the following holds. If $0 < \varepsilon < \varepsilon_0$ and $n \geq N_1$ are such that $\xi_n$ is diffeomorphic on $[0, \varepsilon]$ and
\[
|\xi_n([0, \varepsilon])| \geq r_0,
\]
then
\[
\# (S_N \cap \phi(\xi_n([0, \varepsilon]))) \geq 3.
\]
Proof. If \( \phi \) is a conjugacy, then \( T \) is not periodic, all preimages of 0 are safe and \( \phi^{-1}(S) \) is dense in \( I \). In this case, the lemma is trivial.

Henceforth, suppose \( \phi \) is not a conjugacy, which implies \( g \) is renormalisable of some type other than Feigenbaum. Since the entropy is not locally constant, there are \( t \) arbitrarily close to 0 for which \( g_t \) is not renormalisable of the same type as \( g \). Since \( G \) is well-rooted, \( g \) is not parabolic. It follows that the final renormalisation of \( g \) is conjugate to the Chebyshev map and any previous renormalisation is of Feigenbaum type.

In particular, high iterates of 0 periodically lie on the boundary of the smallest restrictive interval \( (p, q) \), say at \( p \). Set \( p_0 = p, q_0 = q, p_k = g^k(p) = g^k(q) \) and \( q_k = g^k(g^m(0)) \), for \( k = 1, \ldots, m - 1 \), where \( m \geq 3 \) is minimal with \( g^m([p, q]) \subset [p, q] \). For each \( k \), the restriction

\[
g^m : [p_k, q_k] \to [p_k, q_k]
\]

is unimodal. Each interval \([p_k, q_k]\) is the preimage of a point under \( \phi \). Therefore the points \( p_0, \ldots, p_{m-1} \) are all in the accumulation set of \( \phi^{-1}(S) \). Remark that

\[
\phi^{-1}(S) \subset I \setminus \bigcup_{k=0}^{m-1} [p_k, q_k].
\]

Using continuations of points in \( \phi^{-1}(S) \) for \( \epsilon > 0 \) arbitrarily small, we only need to show that the interval \( \xi_n([0, \epsilon]) \) is disjoint from \( \cup_{k=0}^{m-1}(p_k, q_k) \). We may assume that \( r_0 \) is smaller than the distance between any two intervals \((p_k, q_k)\). Those intervals are not adjacent, as \( g \) is renormalisable of some type other than Feigenbaum.

The points \( p_k, q_k \) have continuations \( p_k(t), q_k(t) \) with

\[
g_t^m(p_k(t)) = g_t^m(q_k(t)) = p_k(t).
\]

In this preperiodic transversal setting, \( |\xi'_n(0)| \to \infty \) as \( n \to \infty \), as it is comparable to \( |(g^n)'(v)| \), by Eq. (15). Hence

\[
|p'_k(0)|, |q'_k(0)| < |\xi'_n(0)|
\]

for all \( k \) and all large \( n \).

Let \( n \) be large and suppose \( \xi_n(0) = p_k \). Relative to \( p_k(t) \) at \( t = 0 \), \( \xi_n(t) \) can go in two directions. In one direction, \( \xi_n(t) \) goes inside the interval \((p_k(t), q_k(t))\). If this happens,

\[
g_t^m([p_k(t), q_k(t)]) \subset [p_k(t), q_k(t)]
\]

for all sufficiently small \( t \geq 0 \), and \( g_t \) has a restrictive interval \((p_0(t), q_0(t))\). This we exclude, for otherwise the entropy would be locally constant at \( t = 0 \), a contradiction.

Consequently, \( \xi_n(t) \) starts off going in the other direction and, for all sufficiently small positive \( \epsilon', \xi_n([0, \epsilon']) \) is an interval bordering but disjoint from \((p_k, q_k)\). Thus, if \( \xi_n \) is diffeomorphic on \([0, \epsilon]\),

\[
\xi_n([0, \epsilon]) \cap (p_k(0), q_k(0)) = \emptyset,
\]

which completes the proof. \( \Box \)

Let us recall that we work with a well-rooted unimodal family \( G \) and that \( S_N \) and \( \phi \) are defined by (23) and (24).
Proposition 3.4. Given $K_0 > 0$ and $N \geq 1$, there exist $K \geq 1$ and $\varepsilon_0, \gamma > 0$ such that the following holds. If $\omega = [0, \varepsilon)$ for some $\varepsilon \in (0, \varepsilon_0)$, if $n \geq 1$ is large, if $\text{Dist}(\xi_n, \omega) \leq K_0$, if $0 \notin \cup_{j=1}^{n-1} \xi_j(\omega)$ and if $\phi(\xi_n(\omega))$ contains (at least) three safe elements of $S_N$, then there exists $t \in \omega$ with

$$
\gamma |\omega| < t < |\omega|
$$

and

$$
K^{-1} t^\frac{h(0)}{\lambda_n} \leq |h(t) - h(0)| \leq K t^\frac{h(0)}{\lambda_n},
$$

where

$$
\lambda_n := \frac{1}{n} \log |\xi_n'(0)|.
$$

Proof. Let $\theta > 0$ denote half the minimal distance between any pair of elements of $S_N$ or of $\phi^{-1}(S_N)$. Let $q$ be a non-extremal safe element of $S_N \cap \phi(\xi_n(\omega))$ and $p \in g^{-1}(0) \cap \xi_n(\omega)$ with $\phi(p) = q$, for some minimal $j \leq N$. Then, using $d(\cdot, \cdot)$ for distance,

$$
d(q, \phi(\xi_n(\partial \omega))), d(p, \xi_n(\partial \omega)) \geq 2\theta.
$$

Let $p(t), q(t)$ denote the continuations of $p, q$ (it makes sense to write $q(t)$, since to each $g_t$ there corresponds a unique $T_{\exp(h(t))}$). As $S_N$ is finite, we can choose $\varepsilon_0 > |\omega|$ such that $p(t)$ and $q(t)$ are always well-defined on $\omega$ and such that

$$
d(q(\omega), \phi(\xi_n(\partial \omega))), d(p(\omega), \xi_n(\partial \omega)) > \theta.
$$

Hence, there exists $t'$ with $\xi_n(t') = p(t')$. For this $t'$,

$$
\theta < |\xi_n(0) - \xi_n(t')| < |\xi_n(\omega)| \leq 4.
$$

As $t' \in \omega$, by bounded distortion of $\xi_n$ on $\omega$, we obtain

$$
\frac{\theta}{4eK_0} < \frac{|\xi_n([0, t'))|}{eK_0|\xi_n(\omega)|} < \frac{t'}{|\omega|}\tag{26}
$$

and

$$
\frac{\theta}{eK_0|\xi'_n(0)|} < t' < \frac{4eK_0}{|\xi'_n(0)|}\tag{27}
$$

From (26), we can choose $\gamma := \frac{\theta}{4eK_0}$.

Let $b' := \exp(h(t'))$. Now the corresponding semi-conjugacy maps $p(t')$ to

$$
q(t') = T^n_{b'}(1),
$$

and $b_0 \neq b'$ since

$$
\theta < |q(t') - T^n_{b_0}(1)| < |\hat{I}_{b_0}|.
$$

We introduced in (10) the notation $\varphi_k(b) = T^n_k(1)$, the equivalent of $\xi_n$ for tent maps. From Eq. (13) and the hypothesis $0 \notin \cup_{j=1}^{n-1} \xi_j(\omega)$, we deduce $\varphi_k(b) \neq 0$ for all $k = 1, \ldots, n-1$ and all $b \in (b_0, b')$. By Lemma 2.1, $\varphi_n$ is a diffeomorphism on $(b_0, b')$ and

$$
|\varphi_n'(b)| \simeq b_0^n
$$
for all $b \in (b_0, b')$. Meanwhile $\theta < |\varphi_n(b_0) - \varphi_n(b')| \leq |\hat{I}_{b_0}|$. Hence,

$$|b_0 - b'| \simeq b_0^{-n}.$$ 

As $|h(0) - h(t')| = |\log b_0 - \log b'| \simeq |b_0 - b'|$ and

$$|\xi_n'(0)| \frac{h_0}{\lambda_n} = \exp(nh(0)) = b_0^n,$$

we deduce, using (27) for the final step, that

$$|h(0) - h(t')| \simeq b_0^{-n} \simeq |\xi_n'(0)| \frac{h_0}{\lambda_n} \simeq t \frac{h_0}{\lambda_n},$$

noting $e^{K_0}$, $\theta$, $h(0)$ are constants and $\lambda_n$ converges to $\lambda_0 > 0$ as $n \to \infty$. $\square$

**Proof of Theorem 6.** Given a well-rooted unimodal family $G$, we must show that, if $h$ is monotone and not locally constant at $t = 0$,

$$\lim_{t \to 0^+} \frac{\log |h(t) - h(0)|}{\log t} = \frac{h(0)}{\lambda_0}.$$ 

By monotonicity, it suffices to find sequences of parameters $t_n \nearrow 0$ and of numbers $v_n \to h(0)/\lambda_0$ as $n \to \infty$, for which

$$|h(t_n) - h(0)| \simeq t_n^{v_n}$$

and with

$$\frac{\log t_{n+1}}{\log t_n} \to 1 \quad \text{as} \quad n \to \infty.$$

Let $\delta, \varepsilon > 0$ and let $r_0, m_0, k_n, \omega_n$ be given by Lemma 3.1, which applies exactly in the case of a well-rooted family. From Lemma 3.1, we have $|\xi_{kn}(\omega_n)| > r_0$,

$$1 < \frac{k_{n+1}}{k_n} \leq 1 + \delta, \quad \text{Dist}(\xi_j, \omega_n) \leq 1 \quad \text{for} \quad j = m_0, \ldots, k_n, \quad \text{and, for} \quad t \in \omega_n,$$

$$\lambda_0 - \varepsilon \leq \frac{1}{k_n} \log |\xi_{kn}'(t)| \leq \lambda_0 + \varepsilon.$$

By Lemma 3.3, there is some fixed $N$ for which, for all large $n$, $\phi(\xi_{kn}(\omega_n))$ contains at least three safe elements of $S_N$. By Lemma 3.2, we can apply Proposition 3.4 to obtain $K \geq 1$, $\gamma > 0$ and a sequence of parameters $t_n$ for which the following holds:

$$\gamma |\omega_n| \leq t_n \leq |\omega_n|$$

and

$$K^{-1}t_n^{\frac{h_0}{\lambda_0}} \leq |h(t_n) - h(0)| \leq K t_n^{\frac{h_0}{\lambda_0}}. \quad (28)$$

Let us finally check that the $t_n$ do not decrease too fast. By bounded distortion (and assuming $n$ is large),

$$1 - \varepsilon \leq \frac{\log |\xi_{kn}'(0)|}{-\log t_n} \leq 1 + \varepsilon.$$ 

We deduce

$$\frac{\log t_{n+1}}{\log t_n} \leq (1 + \delta) \frac{(\lambda_0 + \varepsilon)(1 + \varepsilon)}{(\lambda_0 - \varepsilon)(1 - \varepsilon)} \quad (29).$$

Since $\varepsilon, \delta > 0$ are arbitrary, this completes the proof. $\square$
3.2. Parabolic parameters. **Quadratic maps.** We call a map $f_a$ parabolic if it has a periodic point $p$ (of period $k$, say) with multiplier 1 or $-1$.

In the case that the multiplier is $-1$, then $f_a^k$ is orientation-reversing, locally. By Singer’s theorem [15, Theorem II.6.1] (using negative Schwarzian derivative), $p$ is attracting on at least one side, so in this case it is attracting on both sides. Arbitrarily small neighbourhoods of $p$ are mapped diffeomorphically and compactly inside themselves by $f_a^{2k}$. As one perturbs $f_a$, this topological situation persists and there is an (at least one-sided) attracting periodic orbit of period $k$ or $2k$. Again by Singer’s theorem, the critical point lies in the basin of attraction. The kneading itinerary of the critical point remains the same and the entropy is locally constant.

If the multiplier is 1, the entropy may vary with $a$, in which case the parabolic orbit disappears in a *saddle-node bifurcation*, whence the entropy is decreasing and $a$ is increasing. In this subsection, we prove that the entropy function is infinitely flat at parabolic parameters for general unimodal families.

**Tent maps.** Multiplication by $b$ is the same as iteration of the function whose graph is given by $y = bx$. After $n$ iterates starting from $x$, one is at a distance $b^nx$ from 0. Translating this behaviour, we understand iterates of a piecewise-linear map restricted to one branch.

Let $T_{b_0}$ be a preperiodic tent map and let $\alpha, k, N$ satisfy

$$\alpha := T_{b_0}^{N+k}(0) = T_{b_0}^N(0),$$

with $k \geq 1$, $N \geq 0$ minimal with this property. Note $\alpha$ is periodic with period $k$. Near $\alpha$, if $N = 0$ the graph of $T_{b_0}^k$ is on one side only of the diagonal; if $N > 0$ the graph of $T_{b_0}^k$ crosses the diagonal at $\alpha$. For nearby parameters $b$, if $N > 0$ the graph still crosses the diagonal and, in a small neighbourhood of $\alpha$, there is only one branch. If $N = 0$, we need the following.

**Lemma 3.5.** Let $T_b$, $b \in (1, 2]$ be a tent map. If $T_b^k(0) = 0$ for some $k \geq 1$ then $b^k > 2\sqrt{2}$.

**Proof.** If $b \in (2^{2m-1}, 2^{2m}]$, then $T_b$ is $m$-times renormalisable of period 2. It follows that $k = 2^m p$ for some $p \geq 3$ and $b^k > 2^{h/2}$. \(\square\)

**Lemma 3.6.** There exist $C_1, l \geq 1$ and a neighbourhood $U_\alpha$ of $\alpha$ for which, for all $b$ close to $b_0$, the following holds. If $n \geq 1$ and

$$T_b^{N+jk}(0) \in U_\alpha$$

for $j = 1, \ldots, n$, then

$$| \log b_0 - \log b | \leq C_1 b_0^{-nk/l}.$$

**Proof.** There are three cases. First if $N > 0$ and $U_\alpha$ is small, then iterates in $U_\alpha$ all lie in the same branch of $T_b^k$. If $\alpha_b$ denotes the continuation of the periodic point $\alpha$, then

$$|T_b^{N+2k}(0) - T_b^N(0)| \geq |T_b^{N+2k}(0) - \alpha_b|.$$

Hence

$$b^{(n-2k)} |T_b^{N+2k}(0) - T_b^N(0)| < |U_\alpha|.$$
Next if $N = 0$ and the graph of $T^b_k$ does not touch the diagonal in a small neighbourhood, then for $U_\alpha$ small enough, $T^{jk}_b(0)$ for $j = 1, \ldots, n$ are monotone and lie in the same branch. We obtain

$$b^{(n-1)k}|T^b_k(0) - 0| < |U_\alpha|.$$  

If $N = 0$ and the graph crosses the diagonal, $\alpha$ splits into an orientation-preserving fixed point (for $T^b_k$) and an orientation-reversing fixed point. Since the slope of $T^b_k$, for $b$ near $b_0$, is at least $2\sqrt{2}$, the sequence $0, \alpha_b, T^b_k(0)$ is monotone and

$$|T^{2k}_b(0) - \alpha_b| \geq |T^k_b(0) - 0|.$$  

Furthermore, iterates $T^{jk}_b(0)$, $j = 2, \ldots, n$ are all in the same branch (monotonically receding from 0). Thus

$$b^{(n-2)k}|T^k_b(0) - 0| < |U_\alpha|.$$  

Moving to the next stage of the proof, $T_b$ is piecewise degree 1 in $b$ and $x$ so

$$T^{N+Mk}_b(0) - T^N_b(0),$$

for $M \in \{1, 2\}$, is (piecewise) a polynomial of degree $N + Mk$ in $b$. Let $1 \leq l \leq N + Mk$ be the order of the zero $b_0$. The entropy changes with $b$, so the polynomial is not constant 0 and $l$ is finite. [One can perhaps show that $l = 1$, but this is not needed.] Then

$$|T^{N+Mk}_b(0) - T^N_b(0)| \simeq |b_0 - b|^l.$$  

Consequently $|b_0 - b|^l < C_0/b^{nk}$, whence

$$|\log b_0 - \log b| \leq C_1b_0^{-nk/l}.$$  

$C^2$ unimodal families. The following standard lemma will provide a lower bound for the time spent by the critical orbit in an almost-parabolic funnel. We use $\rho_t$ to study the local behaviour around a parabolic fixed point which is at least one-sided attracting. We may assume it is attracting to the right.

**Lemma 3.7.** Let $\rho_t : (-2\delta, 2\delta) \to \rho_t((-2\delta, 2\delta)) \subset \mathbb{R}$, $t \in [0, 2\delta^2)$, be a $C^2$ family of $C^2$ diffeomorphisms with $0 < \delta < 1$. Let $\rho := \rho_0$ and assume that $\rho(0) = 0$, $\rho'(0) = 1$ and that, for all $x \in (0, \delta)$,

$$\rho(x) < x.$$  

There exist arbitrarily small $\varepsilon_0, t_0 > 0$ and some $C > 1$ such that for all $t \in [0, t_0]$ and all $x \in (\sqrt{t}, \varepsilon_0)$,

$$\rho^n_t(x) \in (0, \varepsilon_0) \quad \text{for all } n \leq \frac{1}{C\sqrt{t}}.$$  

$\square$
Proof. For \( \varepsilon_0 \) small enough and \( x \in (0, \varepsilon_0] \), the hypotheses entail that \( \rho'(x) > 4/5 \) and \( 3x/4 \leq \rho(x) < x \). By continuity of the family, there exists \( C_0 > 1 \) and \( t_0 > 0 \) such that \( C_0 t_0 \leq \sqrt{t_0}/6 \) and

\[
x/2 - C_0 t < \rho_t(x) < \varepsilon_0 \tag{32}
\]

for \( t \in (0, t_0] \) and \( x \in [0, \varepsilon_0] \). Consequently, if \( x \in [0, \varepsilon_0] \) and if \( n \) is minimal such that \( \rho^{n}_t(x) \notin [0, \varepsilon_0] \), then \( \rho^n_t(x) < 0 \). By monotonicity, for a given \( x \), either \( \rho^{j+1}_t(x) \geq \rho^j_t(x) \) for all \( j \geq 0 \), in which case \( n = +\infty \) and the lemma holds, or

\[
\rho^{j+1}_t(x) < \rho^j_t(x)
\]

for \( j < n \). It remains to treat this second case.

Given \( x \geq \sqrt{t} \), there may be a first iterate \( y \) of \( x \) with \( y \in [0, \sqrt{t}) \). Then by (32),

\[
\sqrt{t}/3 \leq \sqrt{t}/2 - C_0 t < y < \sqrt{t}.
\]

We shall estimate the time needed for this iterate \( y \) to leave.

By hypothesis and Taylor expansion at \((0, 0)\), there exist \( C_1 > 0 \) and \( \alpha \in \mathbb{R} \) such that for all \( t \in [0, \delta^2) \), \( x \in (-\delta, \delta) \),

\[
|\rho_t(x) - x - \alpha t| \leq C_1 (|x| + |t|)^2. \tag{33}
\]

For \( 0 \leq x \leq \sqrt{t} \), setting \( C = |\alpha| + 4C_1 \),

\[
x - \rho_t(x) \leq Ct.
\]

Hence,

\[
y - \rho^j_t(y) \leq jC t \leq y
\]

for \( j \leq 1/(3C \sqrt{t}) \leq y/(Ct) \). In particular, for this range of \( j \), \( \rho^j_t(y) \geq 0 \). This completes the proof. \( \square \)

Remark that in the above proof, \( \rho_t \) may still have a fixed point for some \( t > 0 \). By (32), the rightmost one in \([0, \varepsilon_0]\) must be attracting on its right. For \( t \) small, it must also be close to 0.

We are now in position to prove the following result, which has Theorem 7 as an immediate corollary.

**Proposition 3.8.** Let \( G \) be a \( C^2 \) family of \( C^2 \) unimodal maps, \( G(\cdot, t) = g_t(\cdot) \), so that \( g_0 \) has a parabolic periodic orbit \( p \) which attracts, but is disjoint from, the critical orbit. Then for \( t > 0 \) sufficiently small

\[
\frac{\log |h(t) - h(0)|}{\log t} \geq -\frac{1}{\sqrt{t} \log t}.
\]

**Proof.** We may suppose that the entropy is not locally constant at \( t = 0 \), with the convention \( \log 0 = -\infty \). Furthermore, we suppose that the critical point of each \( g_t \) is 0. The kneading itinerary \( K(g_0) \) of \( g_0 \) is preperiodic (periodic if one assumes that \( g_0 \) is \( S \)-unimodal), so

\[
K(g_0) \neq K(f_{af}).
\]
the kneading itinerary of the Feigenbaum map. Therefore, if \( h(0) = 0 \) then \( h = 0 \) on some neighbourhood \([0, t_0)\). Henceforth assume \( h(0) > 0 \).

Let \( b_0 := \exp(h(0)) > 1 \), so \( g_0 \) is semi-conjugate to the (necessarily preperiodic) tent map \( T_{b_0} \). Let \( N \geq 0, k \geq 1 \) be minimal with

\[
\alpha := T_{b_0}^{N+k}(0) = T_{b_0}^N(0).
\]

Let \( \phi_t \) denote the semi-conjugacy between \( g_t \) and \( T_{\exp(h(t))} \). Set \( V = \phi_0^{-1}(\alpha) \) and let \( p \in V \) denote the parabolic periodic point whose orbit attracts 0. Note that \( V \) contains the immediate basin of attraction of \( p \) and \( g_0^{N+Mk}(0) \) is sufficiently close to \( p \) that one can apply Lemma 3.7. We obtain, for small enough \( t \),

\[
|\log b - \log b_0| \leq C_1 b_0^{-k \lfloor Ct^{-1/2} \rfloor / l}.
\]

Taking log,

\[
\log |h(t) - h(0)| \leq -C_2 t^{-1/2} + \log C_1.
\]

### 4. Parameter Space

In this section we collect some facts concerning large-measure sets of parameters in the quadratic family \( f_a : x \mapsto x^2 + a, a \in [-2, \frac{1}{4}] \).

A parameter \( a \) is called Collet–Eckmann if \( \lambda(a) > 0 \). Let

\[
E := \{ a \in [-2, a_F] : \lambda(a) > 0 \text{ and } \lim \inf_{n \to \infty} \frac{1}{n} \log |f_a^n(f_a^n(a))| = 0 \},
\]

be the set of Collet–Eckmann parameters whose critical orbit is *slowly recurrent* (or non-recurrent).

Set \( \xi_n(a) := f_a^n(a) \). Computing gives

\[
\xi'_n(a) = 1 + f_a'(f_a^{k-1}(a)) + \cdots + (f_a^{k-1})'(f_a(a)) + (f_a^k)'(a).
\]

Consider (compare (15))

\[
Q_n(a) := \frac{\xi'_n(a)}{(f_a^n)'(a)} = \sum_{k=0}^n \frac{1}{(f_a^k)'(a)}.
\]

If the limit of \( Q_n(a) \) as \( n \to \infty \) exists and is non-zero, we say that \( f_a \) satisfies the *transversality condition*. Recall that Levin [32] proved transversality holds if

\[
\sum_{k \geq 0} \frac{1}{|(f_a^k)'(a)|} < \infty.
\]
For Collet–Eckmann parameters the sum is finite, so transversality does hold at each $a \in E$. Nowicki [39] proved that the Collet–Eckmann condition implies uniform exponential growth at preimages of the critical point. By Singer’s theorem [49], since $\lambda(a) > 0$, all periodic orbits are hyperbolic repelling.

Therefore, all the hypotheses required to apply [53, Theorem 1] for parameters in $E$ are verified. We obtain:

**Fact 4.1** ([53, Theorem 1(I)]). _Almost every $a \in E$ satisfies Tsujii’s weak regularity condition (WR)._  

**Proposition 4.2** ([1,53]). _The set_  

$$\mathcal{W} := \{a \in \mathcal{H}_c : \lambda(a) > 0 \text{ and } f_a \text{ verifies (WR)}\}$$  

_has full measure in $\mathcal{H}_c$ (and thus in $\mathcal{F}_c$). Moreover, for almost every $a \in \mathcal{W}$,  

$$\lambda(a) = \chi(\mu^a_{\text{acip}}).$$

**Proof.** Avila and Moreira [1,2] (building on Lyubich’s [34]) showed that $E$ has full measure in $\mathcal{H}_c$, that $\lambda(a)$ exists and that $\lambda(a) = \chi(\mu^a_{\text{acip}}) > 0$ for almost every $a \in E$. Fact 4.1 implies (WR) holds almost everywhere in $E$. $\square$

Let $Y$ be given by (6). Recall that $a \in Y$ if $f_a$ satisfies (WR) and  

$$\lambda(a) = \chi(\mu^a_{\text{max}}) > 0.$$  

**Proposition 4.3** ([9,47]). _The set $h(Y)$ has full measure in $[0, \log 2]$. _

**Proof.** For almost every value of the entropy, the weak regularity condition was shown in the doctoral thesis of Sands [47]. This can be extracted as follows: his Theorem 54 states that, for almost every value of the entropy, the kneading invariant of the corresponding tent map is “slowly recurrent”.\(^2\) Slow recurrence is defined on page 57 in terms of a function $\mathcal{R}$, itself defined in section 2.3. $\mathcal{R}(j)$ is large if $|f^j(0)|$ is small. Combined with Lemma 39, one obtains weak regularity [see also his footnote on page 57] for the quadratic map.

For almost every $w \in [0, \log 2]$, there is a unique $a_w \in [-2, \frac{1}{3}]$ with $h(a_w) = w$. By [9, Corollary 1], for almost every $w \in [0, \log 2]$, for the map $f_{a_w}$, the critical point 0 is typical with respect to the measure of maximal entropy. With Birkhoff’s theorem, we nearly obtain $\lambda(a_w) = \chi(\mu^a_{\text{max}})$; the only missing ingredient (since $\log |f'_{a_w}|$ is unbounded at 0) is the weak regularity condition, which we have shown. $\square$

We say a parameter $a$ is Misiurewicz if all periodic orbits of $f_a$ are hyperbolic repelling and 0 is non-recurrent. If $a$ is non-renormalisable, the entropy is at least $\log \frac{2}{\lambda^2}$ and is not locally constant at $a$.

**Fact 4.4** ([18, Theorem 1.30]). _Given any $\varepsilon > 0$ and any non-renormalisable Misiurewicz parameter $a_0$, there is a non-renormalisable Misiurewicz parameter $a$ arbitrarily close to $a_0$ such that $\mu^a_{\text{acip}}$ exists and $h(\mu^a_{\text{acip}}) \in (0, \varepsilon)$. _

**Proposition 4.5.** Given $\varepsilon > 0$, there exists a positive measure set $A$ of non-renormalisable Collet–Eckmann parameters with $0 < \lambda(a) = \chi(\mu^a_{\text{acip}}) < \varepsilon$ for all $a \in A$.

\(^2\) Sands’ definition of slow recurrence differs from the standard one (which we use) and more resembles Tsujii’s weak regularity condition.
Proof. By Fact 4.4, there exists a non-renormalisable Misiurewicz parameter $a_1$ with $h(a_1) \geq \frac{\log 2}{2}$ and $h(\mu_{\text{acip}}^{a_1}) \in (0, \varepsilon/2)$. By [54, Main Theorem], there is a positive measure set $A'$ of parameters, with $a_1$ as a density point, such that, for any sequence $a_p \to a_1$, $\mu_{\text{acip}}^{a_p}$ exists (and is necessarily unique) and converges to $\mu_{\text{acip}}^{a_1}$, while by [54, Inequality (2.1)],

$$\limsup_{a_p \to a_1} h(\mu_{\text{acip}}^{a_p}) \leq h(\mu_{\text{acip}}^{a_1}).$$

Entropy equals Lyapunov exponent for acips. For $r$ small enough and $A_r = A' \cap B(a_1, r)$, we deduce

$$0 < \chi(\mu_{\text{acip}}^{a}) < \varepsilon$$

for all $a \in A_r$. By Proposition 4.2, for almost every $a \in A_r$, $\lambda(a) = \chi(\mu_{\text{acip}}^{a})$.

From Tsujii's (and Benedicks-Carleson's) construction, for parameters in $A'$, there are arbitrarily small neighbourhoods of 0 mapped to some fixed large scale. On the other hand, $f_{a_1}$ is non-renormalisable, so for nearby renormalisable parameters, the restrictive intervals are very small and do not get mapped to the fixed large scale. Therefore, they are not in $A'$. Hence, taking $r$ small enough, all parameters in the set $A_r$ are non-renormalisable. Set $A = A_r$ to complete the proof. \(\square\)

5. Typical Values

For large sets of parameters, one can compare the Lyapunov exponent of an invariant measure with the Lyapunov exponent along the critical orbit. This allows one to estimate $h(a)/\lambda(a)$ on large sets and constitutes the goal of this section.

5.1. Visible measures of maximal entropy. The most interesting dynamical measures for a unimodal map are the measure of maximal entropy $\mu_{\text{max}}$ and the absolutely continuous invariant probability measure $\mu_{\text{acip}}$ (supposing the latter exists). To prove Theorem 2, we must show that if $\mu_{\text{max}}^a = \mu_{\text{acip}}^a$ for a quadratic map $f_a : x \mapsto x^2 + a$, then $a = -2$.

Recall Definition 1.4 of pre-Chebyshev maps.

Proposition 5.1. The only pre-Chebyshev quadratic map $f_a$ is

$$f_{-2} : x \mapsto x^2 - 2.$$
$J \subset U$ is equivalent to Lebesgue measure, itself equivalent to the measure of maximal entropy.

There is extra rigidity in analytic setting which comes from a dichotomy of Zdunik and Popovici and Volberg concerning harmonic measure \[42,56\]: for a polynomial-like map, either the measure of maximal entropy and the harmonic measure are mutually singular or the map is conformally equivalent to a polynomial (on some neighbourhoods of the maps’ Julia sets). In particular, $f^n : U \to V$ is conformally equivalent on $U$ to $z \mapsto z^2 - 2$ on a corresponding neighbourhood of $[-2, 2]$.

With Inou’s [26, Theorem 1], we go a step further. Since $f$ and $g : z \mapsto z^2 - 2$ are entire and have bounded degree and $f^n : U \to V$ and $g$ are conformally equivalent, $f^n$ and $g$ have the same degree on $\mathbb{C}$. Therefore $n = 1$ and $f = g$. \qed

**Proof of Theorem 2.** Combining the above proposition with the following fact completes the proof of Theorem 2.

**Fact 5.2** ([16, Theorem 2]). Given an $S$-unimodal map $g$ with positive entropy, the measure of maximal entropy is absolutely continuous if and only if $g$ is pre-Chebyshev.

### 5.2. Comparing topological entropy and Lyapunov exponents.

In this section, we consider the quadratic family $f_a, a \in [-2, \frac{1}{4}]$, and the entropy function $h : a \mapsto h_{\text{top}}(f_a)$.

Given a piecewise-smooth map $g$, we define $\mathcal{M}(g)$ as the set of ergodic, $g$-invariant, probability measures with non-negative Lyapunov exponent. We have the pressure function

$$P_g(t) := \sup_{\mu \in \mathcal{M}(g)} \{ h(\mu) - t \chi(\mu) \}.$$ 

For $g = f_a$, we write $P_a$ for $P_{f_a}$. As a supremum of lines (of slope $-\chi(\mu)$), the pressure is Lipschitz, convex and decreasing. A measure realising the supremum for $t$ is an equilibrium measure for the parameter $t$.

**Fact 5.3.** Let $g$ be a $C^2$ unimodal map with positive entropy and suppose $\mu \in \mathcal{M}(g)$. The following statements are true.

1. $h(\mu) \leq \chi(\mu)$;
2. if $\chi(\mu) > 0$ then $h(\mu) = \chi(\mu) \iff \mu$ is an acip;
3. if $\chi(\mu) > 0$ then $\dim_H(\mu) = \frac{h(\mu)}{\chi(\mu)}$;
4. $P_g(1) \leq 0$.

**Proof.** 1. is Ruelle’s Inequality [46] (note that $\chi(\mu) \geq 0$ by definition of $\mathcal{M}(f)$); 2. is Pesin’s formula [31]; 3. is the Dynamical Volume Lemma [25] and [17]; 4. follows from the definition of pressure and 1. \qed

As a consequence of Theorem 2 and Fact 5.3 we obtain the following inequalities.

**Corollary 5.4.** If $a \in (2, a_F)$ and $\mu_{\text{acip}}^a$ exists, then

$$\chi(\mu_{\text{acip}}^a) = h(\mu_{\text{acip}}^a) < h(a) = h(\mu_{\text{max}}^a) < \chi(\mu_{\text{max}}^a).$$
In Sect. 1.2 we set

\[ X := \{ a \in V \cap W : \lambda(a) = \chi(\mu_{\text{acip}}) \}, \]
\[ Y := \{ a \in V \cap W : \lambda(a) = \chi(\mu_{\text{max}}) \} \]

and noted that \(-2 \notin X \cup Y\).

By the previous corollary, for \(a \in X\), \(h(a) > \lambda(a)\), while for \(a \in Y\), \(h(a) < \lambda(a)\). In the sequel, we gather the necessary tools to obtain uniformity in these inequalities. Convergence of maps will be with respect to the sup-norm.

**Proposition 5.5.** Let \((f_k)_{k \geq 1}\) be a convergent sequence of maps with limit \(f_0\), and suppose each \(f_k, k \geq 0\), is an S-unimodal map with positive entropy and with derivative bounded by some \(M > 0\). For each \(t > -h_{\text{top}}(f_0)/2 \log M\) such that \(\inf_{k \geq 0} P_{f_k}(t) > 0\)

\[ P_{f_0}(t) = \lim_{k \to \infty} P_{f_k}(t). \]

**Proof.** To apply a result in [18], we will need a uniform lower bound on the entropy of the equilibrium states for \(f, f_k\). Let \(f := f_k\) for some \(k \geq 0\).

For \(t \leq 0\), \(P_f(t) \geq h_{\text{top}}(f)\). For any measure \(\mu\), \(\chi(\mu) \leq \log M\). Simple geometry implies that for \(t \in \left[ -h_{\text{top}}(f)/2 \log M, 0 \right]\), any equilibrium measure \(\mu_t^f\) for \(f, t\) must have entropy \(h(\mu_t^f) \geq h_{\text{top}}(f)/2\). Such equilibrium measures exist by upper-semicontinuity of metric entropy and Lyapunov exponents (for a fixed map). For smooth unimodal maps, \(h_{\text{top}}\) depends continuously on the map [37]. Thus for sufficiently large \(k\),

\[ h(\mu_t^{f_k}) \geq h_{\text{top}}(f_0)/3. \]

For \(t > 0\), \(P_f(t) \geq \varepsilon > 0\) implies \(h(\mu_t^f) \geq \varepsilon\), if the equilibrium measure exists. It exists by [18, Corollary 1.20].

According to [18, Remark 1.14], for smooth unimodal maps, the **decreasing critical relations** [18, Definition 1.13] hypothesis is unnecessary and we may apply [18, Lemma 13.1] to obtain convergence of the pressure, as required. \(\square\)

**Remark 5.6.** According to [18], if one extends the domain of a unimodal map to a slightly larger interval with boundary mapped outside the interval, the only possible critical relation occurs if the critical point is periodic. A sequence of maps will have decreasing critical relations if the limit map has no extra critical relations. In the unimodal context, if the critical point of the limit map is not periodic, then the sequence has decreasing critical relations. A small perturbation deals with the super-attracting case.

**Proposition 5.7.** Given an S-unimodal map \(f\) with positive entropy, the map \(t \mapsto P_f(t)\) is real-analytic in a neighbourhood of 0.

**Proof.** We shall use the decomposition of Jonker and Rand [30, Theorem 1] (for unimodal maps) of the non-wandering set \(\Omega(f) = \Omega_0 \cup \Omega_1 \cup \cdots\) into strata. If \(\phi\) denotes the semi-conjugacy of \(f\) with the tent map \(T\) having the same entropy and the orientation-preserving fixed point of \(T\) is denoted by \(\beta\), then \(\Omega_0 = \phi^{-1}(\beta)\) and

\[ \Omega_1 = \phi^{-1}(\Omega(T) \setminus (\{ \beta \} \cup \bigcup_{n \geq 0} T^{-n}(0))). \]
If $T$ is not periodic and the slope is not $\pm 2$, one can equivalently write $\Omega_1 = \phi^{-1}(\Omega(T))$. The topological entropy is carried by $\Omega_1$: if $h_i$ denotes the topological entropy of $f$ restricted to $\Omega_i$, then $h_0 = 0$, while

$$h_{\text{top}}(f) = h_1 > h_j \geq h_{j+1}$$

for $j \geq 2$. For tent maps, the decomposition stops at $\Omega_1$. If it stops at $\Omega_1$ for $f$ too, let $W := \emptyset$. If it does not stop at $\Omega_1$ for $f$, then there is a maximal open interval $W \ni 0$ collapsed by the semi-conjugacy to the corresponding tent map, and $\Omega_j$ is contained in the (periodic) orbit of $W$ for $j \geq 2$. Therefore, if we denote by $g$ the restriction of $f$ to the complement of $W$, the non-wandering set of $g$ is just $\Omega_0 \cup \Omega_1$. Now $\Omega_0$ is just the orientation-preserving fixed point(s), while $\Omega_1$ consists of a finite (possibly 0) number of isolated periodic orbits and a transitive set $\Omega_1'$. From the entropy estimates $h_{\text{top}}(f) > \sup_{j \geq 2} h_j \geq h_{j+1}$, it follows that the pressures of $g$ and $f$ coincide on a neighbourhood of zero.

Apply [18, Theorem 1.28] for $g$, noting that a transitive point in $\Omega_1'$ is a transitive point in $J(g)$ [18, Definition 1.25], to obtain analyticity of the pressure function for $g$ on a neighbourhood of zero. An alternative is, provided $g$ has no parabolic points, to apply [44, Theorem A] for $g$ and $\Omega_1'$. ☐

**Theorem 5.8.** Let $s \mapsto g_s$ be a continuous one-parameter family of $S$-unimodal maps with positive entropy. Then

$$s \mapsto h_{\text{top}}(g_s), \quad s \mapsto \mu_{\text{max}}^{g_s}, \quad s \mapsto \chi(\mu_{\text{max}}^{g_s})$$

are continuous.

**Proof.** The first claim was shown by Misiurewicz [37]. The second claim was shown by Raith [45]. Let us show the third. The derivative of $P_{g_s}$ at 0 exists, since $P_{g_s}$ is real-analytic on a neighbourhood of 0. The value of the derivative is $-\chi(\mu_{\text{max}}^{g_s})$. Since the pressures converge (Proposition 5.5) on a neighbourhood of zero and the pressure functions are convex, the derivatives converge. Consequently, the Lyapunov exponents converge. ☐

**Lemma 5.9.** Let $(g_k)_{k \geq 1}$ be a sequence of maps converging to a map $g_0$, and suppose each $g_k$, $k \geq 0$, is an $S$-unimodal map with positive entropy and derivative bounded by some $M > 0$. Suppose each $g_k$, $k \geq 1$, has an acip $\mu_{\text{acip}}^k$ and

$$h_{\text{top}}(g_k)/\chi(\mu_{\text{acip}}^k) \to 1$$

as $k \to \infty$. Then the measure of maximal entropy for $g_0$ is absolutely continuous.

**Proof.** We have $h(\mu_{\text{acip}}^k) = \chi(\mu_{\text{acip}}^k)$ for all $k \geq 0$. Since $h_{\text{top}}(g_0) > 0$ and $h_{\text{top}}$ is continuous, we deduce

$$\lim_{k \to \infty} h(\mu_{\text{acip}}^k) = \lim_{k \to \infty} \chi(\mu_{\text{acip}}^k) = h_{\text{top}}(g_0).$$

Consequently,

$$\lim \inf_{k \to \infty} P_{g_k}(t) \geq h_{\text{top}}(g_0) - th_{\text{top}}(g_0).$$

Thus, by Proposition 5.5, $P_{g_0}(t) \geq h_{\text{top}}(g_0) - th_{\text{top}}(g_0)$ for $t \in [0, 1)$. Meanwhile, $P_{g_0}(1) \leq 0$, so $P_{g_0}(1) = 0$. Since $P_{g_0}$ is convex and its graph passes through $(0, h_{\text{top}}(g_0))$
and \((1, 0)\), its graph over the interval \([0, 1]\) is in fact the straight line joining these points with slope \(-h_{\text{top}}(g_0)\). The graph of the line \(h_{\text{top}}(g_0) - t\chi(\mu_{\text{max}}^{g_0})\) is tangent to \(P_{g_0}\) at \(t = 0\). Since \(P_{g_0}\) is analytic on a neighbourhood of 0, we deduce that \(\chi(\mu_{\text{max}}^{g_0}) = h_{\text{top}}(g_0) = h(\mu_{\text{max}}^{g_0}),\) which implies \(\mu_{\text{max}}^{g_0}\) is absolutely continuous. □

**Proposition 5.10.** Given \(\varepsilon > 0\), there exists \(\delta > 0\) for which

- for all \(a \in [-2 + \varepsilon, a_F - \varepsilon]\), if \(\mu_{\text{acip}}^a\) exists then \(h(a)/\chi(\mu_{\text{acip}}^a) > 1 + \delta;\)
- for all \(a \in [-2 + \varepsilon, a_F - \varepsilon]\), \(\chi(\mu_{\text{max}}^a)/ h(a) > 1 + \delta.\)

**Proof.** The first statement is an immediate corollary of Lemma 5.9 and Theorem 2. The second follows from Theorem 5.8. □

**Proposition 5.11.** For all \(\varepsilon > 0\) and \(\delta > 0\)

\[
\text{Leb}(\{a \in X \cap (-2, -2 + \varepsilon) : 1 < h(a)/\lambda(a) < 1 + \delta\}) > 0,
\]

\[
\text{Leb}(h \{(a \in Y \cap (-2, -2 + \varepsilon) : 1 > h(a)/\lambda(a) > 1 - \delta\}) > 0.
\]

**Proof.** Freitas [22] showed that \(-2\) is a one-sided Lebesgue density point of a positive measure set of (Benedicks-Carleson) parameters on which \(a \mapsto h(\mu_{\text{acip}}^a)\) and thus \(a \mapsto \chi(\mu_{\text{acip}}^a)\) vary continuously, which implies the first statement.

The second statement follows from Theorem 5.8 and Proposition 4.3, noting that \(h\) is not locally constant at \(a = -2.\) □

### 5.3. Dimension estimates

The following lemma is a variant of [5, Lemma 5.1.3].

**Lemma 5.12.** Let \(u : E \to \mathbb{R}\) be a real map defined on a set \(E \subseteq \mathbb{R}\). For every \(\alpha > 0\) let the \(\alpha\)-flat set of \(u\) be

\[
F(u, \alpha) := \left\{ x \in E : \liminf_{y \to x} \frac{\log |u(y) - u(x)|}{\log |y - x|} \geq \alpha \right\}.
\]

If \(A \subseteq F(u, \alpha)\) then

\[
\dim_H(u(A)) \leq \frac{\dim_H(A)}{\alpha}.
\]

**Proof.** Let \(d > \dim_H(A)\) and let \(0 < \beta < \alpha.\) We set

\[
A_N := \{ x \in A : |u(y) - u(x)| \leq N|y - x|^\beta \text{ for all } y \in E \cap B(x, N^{-1})\}.
\]

Then \(A = \bigcup_N A_N.\) Given \(\varepsilon > 0,\) let \(\{U_j\}\) be a covering of \(A_N\) of diameter \(< N^{-1}\) such that \(\sum_j |U_j|^d \leq \varepsilon.\) The covering \(\{U(U_j)\}\) of \(u(A_N)\) satisfies

\[
\sum_j |u(U_j)|^{d/\beta} \leq N^{d/\beta} \sum_j |U_j|^d \leq N^{d/\beta} \varepsilon.
\]

Consequently, \(\dim_H(u(A_N)) \leq d/\beta.\) Since \(d > \dim_H(A)\) and \(\beta < \alpha\) are arbitrary, this completes the proof. □
As $X, Y \subset V$, $h$ is bijective on $X \cup Y$. In the remaining lines of this section, $h$ will stand for its restriction to $X \cup Y$. Applying Theorem 1, for each $a \in X \cup Y \subset V \cap W$, $a \in F(h, h(a)/\lambda(a))$ and $h(a) \in F(h^{-1}, \lambda(a)/h(a))$.

**Proof of Theorem 4.** Lemma 5.12 and Proposition 5.10 now imply
\[
\dim_H(h(X \cap [-2 + \varepsilon, a_F - \varepsilon])), \dim_H(Y \cap [-2 + \varepsilon, a_F - \varepsilon]) < 1
\]
for each $\varepsilon > 0$, as required. \qed

**Proof of Theorem 5.** Given $\varepsilon' > 0$, let $A$ be given by Proposition 4.5. For $a \in A$,
\[
\chi(\mu_a^{\text{acip}}) = \lambda(a) \leq \varepsilon',
\]
while $h(a) \geq \log 2^2$. By Theorem 1 and Lemma 5.12,
\[
\dim_H(h(A)) \leq \varepsilon' \frac{2}{\log 2} \dim_H(A).
\]
Noting that $\dim_H(A) = 1$ and that $\varepsilon'$ can be taken arbitrarily small, there exist positive measure sets $A$ with $\dim_H(h(A)) < \varepsilon$, as required. \qed

**Theorem 5.13.** For every $\varepsilon > 0$,
\[
\dim_H(h(X \cap (-2, -2 + \varepsilon))) = \dim_H(Y \cap (-2, -2 + \varepsilon)) = 1.
\]

**Proof.** This follows from Proposition 5.11 and Theorem 1. \qed

### 6. Uniform Hölder Regularity

In this section we shall prove Theorem 8, namely that the entropy function is uniformly Hölder continuous for the quadratic family.

Recall that, if $h(a) < \log 2^2$, then $f_a$ is $m$ times Feigenbaum renormalisable and the $m$th renormalisation is a unimodal map $g$ with entropy
\[
h_{\text{top}}(g) = 2^m h(a).
\]

By the theory of renormalisation [50, Theorem 1], there is a universal bound (independent of $m \geq 0$) $\Gamma \geq 4$:
\[
||g'||_\infty \leq \Gamma. \quad (37)
\]

Let $a_m$ denote the quadratic parameter with $h(a_m) = \frac{\log 2}{2m}$. Note that $a_0 = -2$, corresponding to the Chebyshev map. For each $m \geq 0$, $f_{a_m}$ is $m$ times Feigenbaum renormalisable and the $m$th renormalisation is conjugate (on the restrictive interval) to Chebyshev. Again from renormalisation theory [21],
\[
\lim_{m \to \infty} \left| \frac{a_m - a_{m-1}}{a_{m+1} - a_m} \right| = \delta_* \approx 4.67.
\]

It follows that, given $\alpha_0 < \frac{\log 2}{\log \delta_*}$, for some $C_0 > 1$, $h$ restricted to the set $\{a_m : m \geq 0\}$ is $(C_0, \alpha_0)$-Hölder. To prove Theorem 8, it therefore suffices to prove the existence of $C_1, \alpha_1$ for which, on each interval $[a_m, a_{m+1}]$, $h$ is $(C_1, \alpha_1)$-Hölder. The following lemma is primarily due to Guckenheimer, but the proof uses Brucks and Misiurewicz’ (Benedicks-Carleson-type) estimates for tent maps instead of studying kneading determinants.
Lemma 6.1. ([24, Lemma 3]) There is a constant \( C > 0 \) such that, for every \( n \) and \( a, a' \in [a_m, a_{m+1}] \), if
\[
|h(a) - h(a')| > C2^{-m}2^{-n/2},
\] (38) then there is a periodic tent map \( T_b \) with period at most \( 2^m n \) for which \( h^{-1}(\log \hat{b}) \subset [a, a'] \).

Proof. Recall that by definition (10), \( \varphi_n(b) = T_{b}^{n}(1) \), the \( n \)th iterate of the critical value for the tent map \( T_b \). By [7], for some constant \( \rho > 0 \) and all \( b \geq \sqrt{2} \),
\[
|\varphi_n'(b)| \geq \rho b^n
\]
wherever \( \varphi_n \) is differentiable, and \( \varphi_n \) is differentiable at \( b \) unless \( \varphi_j(b) = 0 \) for some \( j \leq n - 1 \). If \( \varphi_j(b) = 0 \), then 0 is periodic of period \( j + 1 \). Let \( b \geq \sqrt{2} \). Now
\[
\varphi_n(b) \in [-1/(\sqrt{2} - 1), 1/(\sqrt{2} - 1)],
\]
so if we look at the maximal parameter interval \( (b, b') (b' > b) \) on which \( \varphi_n \) is differentiable,
\[
|b' - b| < \frac{2}{\sqrt{2} - 1} b^{-n} \rho^{-1}.
\]
Noting \( b, b' \geq \sqrt{2} \), there is a constant \( C \) for which
\[
|\log b' - \log b| < C2^{-n/2}.
\]
Consequently, if one has \( \sqrt{2} \leq b < b' \) and \( \log b' - \log b > C2^{-n/2} \), there is a periodic tent map \( T_{b^*} \) with period \( j \leq n \) and slope \( b^* \) lying strictly between \( b \) and \( b' \). Then \( T_{b^*} \) has entropy \( 2^{-m} \log b^* \) and is periodic with period \( 2^m j \). One immediately obtains the corresponding statement for renormalisable tent maps: If one has
\[
2^{-m-1} \log 2 \leq \log b < \log b' \leq 2^{-m} \log 2
\]
and \( \log b' - \log b > C2^{-m}2^{-n/2} \), there is a periodic tent map with period at most \( 2^m n \) and slope in \( (b, b') \).

Now with \( m, a, a', n \) satisfying (38), let \( b, b' \) be the corresponding tent maps and \( \hat{b} \in (b, b') \) a periodic tent map parameter with period at most \( 2^m n \). Then \( h^{-1}(\log \hat{b}) \subset [a, a'] \), by monotonicity of entropy, completing the proof. □

We will use the following simple but fruitful observation by Przytycki. If some critical value comes back too soon too close to the critical point 0, then the map has an attracting cycle:

Lemma 6.2 ([43]). Let \( C, \gamma \geq 1 \) and let \( g \) be a \( C^1 \) map with derivative satisfying
\[
|g'(x)| \leq \min(C|x|, \gamma).
\]
If \( g \) does not have an attracting periodic orbit of period \( n + 1 \), then
\[
|g^{n+1}(0)| > 2^{-2} C^{-1} \gamma^{-n}.
\]

Proof. Consider \( r = 2^{-1} C^{-1} \gamma^{-n} \). Then \( |(g^{n+1})'| \leq 2^{-1} \) on \( B(0, r) \). If \( g^{n+1} \) does not have an attracting fixed point, \( g^{n+1}(B(0, r)) \not\subset B(0, r) \) so \( |g^{n+1}(0)| > r/2 \), as required. □
Lemma 6.3. Suppose \( h(a_0) < 2^{-m} \log 2 \), so \( f_{a_0} \) is \( m \) times Feigenbaum renormalisable. Then

\[
|\xi'_{2m_{n-1}}(a_0)| < \Gamma^{n+3m}.
\]

Proof. Let \( g_k \) denote the \( k \)th (Feigenbaum) renormalisation of \( f_a \), omitting the dependence on \( a \). Denote by \( J_k \) the corresponding restrictive interval containing 0. Thus

\[
g_k := f_{2^k a} |_{J_k}.
\]

Denote by \( g_a \) the \( m \)th renormalisation of \( f_{a_0} \) for \( a \) in a neighbourhood of \( a_0 \). It is necessary to compute bounds for the derivative of the \( n \)th critical value of \( g_a \) with respect to \( a \) in a neighborhood of \( a_0 \). Observe that whenever \( 2^k | n \),

\[
(f_{2^k a}^n)'(f_a^n(0)) = |g_k'(f_a^n(0))| \leq \Gamma.
\]

Decomposing the orbit of \( f_a^j(a) \) according to visits to 

\[
J_1, \ldots, J_{m-1}, J_m, J_{m+1}, \ldots, J_{k+1}, J_k,
\]

we obtain

\[
\frac{\log |(f_a^{\hat{n}})'(f_a^j(a))|}{\log \Gamma} \leq \left\lfloor \frac{\hat{n}}{2^m} \right\rfloor + 2m,
\]

where \( \lfloor y \rfloor \) is the integer part of \( y \in \mathbb{R} \). If we plug these estimates into (35), for any \( n \geq 1 \) we obtain

\[
|\partial_a g_a^{n-1}(a)| = |\xi'_{2m_{n-1}}(a)| \leq 2^m \Gamma^{2m} \sum_{i=0}^{n-1} \Gamma^i < \Gamma^{n+3m}.
\]

(39)

This is the desired bound. \( \square \)

Lemma 6.4. There exists \( \rho > 0 \) such that, if

\[
\log \hat{b} \in (2^{-m-1} \log 2, 2^{-m} \log 2)
\]

and \( T_\hat{b} \) is periodic with period \( 2^m n \), then the length of the interval \( h^{-1}(\hat{b}) \) is at least \( \exp(-(n + m)\rho) \).

Proof. Each element of \( h^{-1}(\hat{b}) \) is a quadratic parameter which is \( m \) times Feigenbaum renormalisable and then once renormalisable of period \( n \). The left endpoint \( a_C \) has a final renormalisation which is conjugate to Chebyshev; \( h^{-1}(\hat{b}) \) contains a super-attracting parameter \( a' \) of period \( 2^m n \), so \( \xi_{2m_{n-1}}(a') = 0 \). Let \( k := 2^m n - 1 \). We will estimate \( \xi_k(a_C) \), and \( \xi_k' \) on \( (a_C, a') \), giving a lower bound on \( a' - a_C \).

If we set \( J_0 := I_{a_C} \) (defined on page 12) and denote by

\[
J_1 \supset J_2 \supset \cdots \supset J_m
\]

the first \( m \) restrictive intervals for \( f_{a_C} \), it follows from [50, Theorem 1] that for some universal \( \kappa > 0 \),

\[
|J_{k+1}|/|J_k| \geq e^{-\kappa}.
\]

N. Dobbs, N. Mihalache
In particular, the restrictive interval $J_m$ for the $m$th renormalisation $g = f_a^{2^m}C$ has length $\delta \geq \exp(-m\kappa)$. Again by [50, Theorem 1],

$$|g'(x)| < Cx/\delta \leq C,$$

for some universal constant $C$. Since $g$ renormalises into Chebyshev, it does not have an attracting orbit. Applying Lemma 6.2,

$$|g^n(0)| > 2^{-2}\delta C^{-1}C^{-n+1} > \exp(-m\kappa - n\rho_1)$$

for some universal $\rho_1 > 0$. Reformulating,

$$|\xi_k(aC)| > \exp(-m\kappa - n\rho_1).$$

Meanwhile, from Lemma 6.3, $|\xi_k'| < \Gamma^{n+3m}$. Thus

$$|a' - aC| > \exp(-m(\kappa + 3\log \Gamma) - n(\rho_1 + \log \Gamma)).$$

This completes the proof of Theorem 8.

Acknowledgements. The authors thank Magnus Aspenberg, Viviane Baladi, Michael Benedicks, Davoud Cheraghi, Jean-Pierre Eckmann, Jacek Graczyk, Hans Koch and Masato Tsujii for helpful comments and conversations and the referees for insightful reports.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

1. Avila, A., Moreira, C.G.: Statistical properties of unimodal maps: physical measures, periodic orbits and pathological laminations. Publ. Math. Inst. Hautes Études Sci. (101), 1–67 (2005)

2. Avila, A., Moreira, C.G.: Statistical properties of unimodal maps: the quadratic family. Ann. Math. (2) 161(2), 831–881 (2005)

3. Bandtlow, O.F., Rugh, H.H.: Entropy continuity for interval maps with holes. Ergodic Theory and Dynamical Systems, 1–26 (2017)

4. Benedicks, M., Carleson, L.: The dynamics of the Hénon map. Ann. Math. (2) 133(1), 73–169 (1991)

5. Bishop, C.J., Peres, Y.: Fractals in probability and analysis, volume 162 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2017)

6. Blokh, A.M., Lyubich, M.Y.: Measurable dynamics of $S$-unimodal maps of the interval. Ann. Sci. École Norm. Sup. (4) 24(5), 545–573 (1991)

7. Brucks, K., Misriurewicz, M.: The trajectory of the turning point is dense for almost all tent maps. Ergod. Theory Dyn. Syst. 16(6), 1173–1183 (1996)

8. Bruin, H.: Non-monotonicity of entropy of interval maps. Phys. Lett. A, 202(5-6), 359–362 (1995)

9. Bruin, H.: For almost every tent map, the turning point is typical. Fund. Math. 155(3), 215–235 (1998)

10. Bruin, H., Holland, M., Nicol, M.: Livšic regularity for Markov systems. Ergod. Theory Dyn. Syst. 25(6), 1739–1765 (2005)

11. Bruin, H., Holland, M., Nicol, M.: Livšic regularity for Markov systems. Ergod. Theory Dyn. Syst. 25(6), 1739–1765 (2005)

12. Bruin, H., Strien, S.van: Monotonicity of entropy for real multimodal maps. J. Am. Math. Soc. 28(1), 1–61 (2015)

13. Carminati, C., Tiozzo, G.: The local Hölder exponent for the dimension of invariant subsets of the circle. Ergod. Theory Dyn. Syst. 37(6), 1825–1840 (2017)

14. Collet, P., Eckmann, J.-P.: On the abundance of aperiodic behaviour for maps on the interval. Commun. Math. Phys. 73(2), 115–160 (1980)

15. de Melo, W., van Strien, S.: One-dimensional dynamics, volume 25 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer, Berlin (1993)

16. Dobbs, N.: Visible measures of maximal entropy in dimension one. Bull. Lond. Math. Soc. 39(3), 366–376 (2007)

17. Dobbs, N.: On cusps and flat tops. Ann. Inst. Fourier (Grenoble), 64(2), 571–605 (2014)

18. Dobbs, N., Todd M.: Free energy jumps up. Preprint arXiv:1512.09245, (2015)

19. Douady, A.: Topological entropy of unimodal maps: monotonicity for quadratic polynomials. In: Real and complex dynamical systems (Hillerød, 1993), volume 464 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., pp. 65–87. Kluwer Academic Publisher, Dordrecht, (1995)

20. Dudko, D., Schleicher, D.: Core entropy of quadratic polynomials. Preprint arXiv:1412.8760, (2014)

21. Eckmann, J.-P., Wittwer, P.: A complete proof of the Feigenbaum conjectures. J. Stat. Phys. 46(3-4), 455–475 (1987)

22. Freitas, J.M.: Continuity of SRB measure and entropy for Benedicks–Carleson quadratic maps. Nonlinearity, 18(2), 831–854 (2005)

23. Graczyk, J., Świątek, Grzegorz: Generic hyperbolicity in the logistic family. Ann. Math. (2) 146(1), 1–52 (1997)

24. Guckenheimer, J.: The growth of topological entropy for one-dimensional maps. In: Global Theory of Dynamical Systems (Proceedings of Internatinal Conference, Northwestern University, Evanston, Ill., 1979), volume 819 of Lecture Notes in Math., pp. 216–223. Springer, Berlin (1980)

25. Hofbauer, F., Raith, P.: The Hausdorff dimension of an ergodic invariant measure for a piecewise monotonic map of the interval. Can. Math. Bull. 35(1), 84–98 (1992)

26. Inou, H.: Extending local analytic conjugacies. Trans. Am. Math. Soc. 363(1), 331–343 (2011)

27. Isola, S., Politi, A.: Universal encoding for unimodal maps. J. Stat. Phys. 61(1-2), 263–291 (1990)

28. Jakobson, M.V.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Commun. Math. Phys. 81(1), 39–88 (1981)

29. Jiang, Y.: Renormalization and geometry in one-dimensional and complex dynamics, volume 10 of Advanced Series in Nonlinear Dynamics. World Scientific Publishing Co., Inc., River Edge, NJ, (1996)

30. Jonker, L., Rand, D.: Bifurcations in one dimension. II. A versal model for bifurcations. Invent. Math. 63(1), 1–15 (1981)

31. Ledrappier, F.: Some properties of absolutely continuous invariant measures on an interval. Ergod. Theory Dyn. Syst. 1(1), 77–93 (1981)

32. Levin, G.: On an analytic approach to the Fatou conjecture. Fund. Math. 171(2), 177–196 (2002)

33. Lyubich, M.: Dynamics of quadric polynomials. I, II. Acta Math. 178(2):185–247, 247–297 (1997)

34. Lyubich, M.: Almost every real quadratic map is either regular or stochastic. Ann. Math. (2) 156(1), 1–78 (2002)
35. Milnor, J., Thurston W.: On iterated maps of the interval. In: Dynamical systems (College Park, MD, 1986–87), volume 1342 of Lecture Notes in Math., pp. 465–563. Springer, Berlin, (1988)

36. Misiurewicz, M.: Absolutely continuous measures for certain maps of an interval. Inst. Hautes Études Sci. Publ. Math. (53):17–51 (1981)

37. Misiurewicz, M.: Jumps of entropy in one dimension. Fund. Math. 132(3), 215–226 (1989)

38. Misiurewicz, M., Szlenk W.: Entropy of piecewise monotone mappings. In: Dynamical systems, Vol. II—Warsaw, pp. 299–310. Astérisque, No. 50. Soc. Math. France, Paris, (1977)

39. Nowicki, T.: Some dynamical properties of \( S \)-unimodal maps. Fund. Math. 142(1), 45–57 (1993)

40. Nowicki, T., Sands, D.: Non-uniform hyperbolicity and universal bounds for \( S \)-unimodal maps. Invent. Math. 132(3), 633–680 (1998)

41. Perera, M., Perrier A.: Sur l’absolue continuité de l’entropie dans la famille quadratique. Master’s thesis, (2012)

42. Popovici, I., Volberg, A.: Rigidity of harmonic measure. Fund. Math. 150(3), 237–244 (1996)

43. Przytycki, F.: Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps. Fund. Math. 144(3), 259–278 (1994)

44. Przytycki, F., Rivera-Letelier, J.: Geometric pressure for multimodal maps of the interval. Accepted, Memoirs of the AMS. Preprint arXiv:1405.2443, (2014)

45. Raith, P.: Continuity of the measure of maximal entropy for unimodal maps on the interval. Qual. Theory Dyn. Syst. 4(1), 67–76 (2003)

46. Ruelle, D.: An inequality for the entropy of differentiable maps. Bol. Soc. Brasil. Mat. 9(1), 83–87 (1978)

47. Sands, D.: Topological conditions for positive Lyapunov exponent in unimodal maps. Ph.D. thesis, University of Cambridge, Cambridge (1993)

48. Sands, D.: Misiurewicz maps are rare. Commun. Math. Phys. 197(1), 109–129 (1998)

49. Singer, D.: Stable orbits and bifurcation of maps of the interval. SIAM J. Appl. Math. 35(2), 260–267 (1978)

50. Sullivan, D.: Bounds, quadratic differentials, and renormalization conjectures. In: American Mathematical Society centennial publications, vol. II (Providence, RI, 1988), pp. 417–466. Amer. Math. Soc., Providence, RI, (1992)

51. Tiozzo, G.: Continuity of core entropy of quadratic polynomials. Invent. Math. 203(3), 891–921 (2016)

52. Tiozzo, G.: The local Hölder exponent for the entropy of real unimodal maps. arXiv preprint arXiv:1707.01575, 2017

53. Tsujii, M.: Positive Lyapunov exponents in families of one-dimensional dynamical systems. Invent. Math. 111(1), 113–137 (1993)

54. Tsujii, M.: On continuity of Bowen-Ruelle-Sinai measures in families of one-dimensional maps. Commun. Math. Phys. 177(1), 1–11 (1996)

55. Ulam, S.M., von Neumann, J.: On combination of stochastic and deterministic processes. Bull. A.M.S. 53(11), 1120 (1947)

56. Zdunik, A.: Harmonic measure on the Julia set for polynomial-like maps. Invent. Math. 128(2), 303–327 (1997)

Communicated by C. Liverani