A CHARACTERIZATION OF THE WEIGHTED WEAK TYPE COIFMAN-FEFFERMAN AND FEFFERMAN-STEIN INEQUALITIES

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Abstract. We introduce a variant of the $C_p$ condition (denoted by $SC_p$), and show that it characterizes weighted weak type versions of the classical Coifman-Fefferman and Fefferman-Stein inequalities.

1. Introduction

This paper concerns two long-standing open problems of characterizing the weights $w$ that satisfy the Coifman-Fefferman [5] inequality

\[ \|Tf\|_{L^p(w)} \leq C\|Mf\|_{L^p(w)} \]

and the Fefferman-Stein [9] inequality

\[ \|f\|_{L^p(w)} \leq C\|f^\#\|_{L^p(w)}. \]

Here $T$ is a Calderón-Zygmund operator, and $M$ and $f^\#$ are the maximal and the sharp maximal operators, respectively.

Originally (1.1) was established in [5] for weights satisfying the $A_\infty$ condition (in fact, a good-$\lambda$ inequality relating $T$ and $M$ had been already obtained in the weighted setting with $w \in A_\infty$ in an earlier work of Coifman [4] and (1.1) is implicit there). Inequality (1.2) was established in [9] in the unweighted setting but the method in [9] is easily extended to $w \in A_\infty$.

Recall that one of the equivalent definitions of $A_\infty$ says that this is the class of weights satisfying the reverse Hölder inequality, namely $w \in A_\infty$ if there exist $C > 0$ and $r > 1$ such that for every cube $Q$,

\[ \left( \frac{1}{|Q|} \int_Q w^r \right)^{1/r} \leq C \frac{1}{|Q|} \int_Q w. \]
In [24], Muckenhoupt showed that the $A_\infty$ condition is not necessary for (1.1); he also established that (1.1) for the Hilbert transform implies the so-called $C_p$ condition which he conjectured to be sufficient for (1.1). Observe that this conjecture is still open.

In the $n$-dimensional case the $C_p$ condition can be formulated as follows: $w \in C_p$ if there exist $C > 0$ and $r > 1$ such that for every cube $Q$,

$$\left( \frac{1}{|Q|} \int_Q w^r \right)^{1/r} \leq C \frac{1}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w.$$

It is easy to see that for every $q > p > 0$,

$$A_\infty \subset C_q \subset C_p.$$

In [25], Sawyer extended Muchenhoupt’s result by showing that (1.1) for each of the Riesz transforms $R_j, j = 1, \ldots, n$ implies the $C_p$ condition; also Sawyer gave a partial answer to Muchenhoupt’s conjecture proving that the $C_{p+\varepsilon}$ condition for some $\varepsilon > 0$ is sufficient for (1.1).

In [29], Yabuta obtained an analogue of Sawyer’s result for (1.2). Namely, he showed that the $C_p$ condition is necessary for (1.2) and the $C_{p+\varepsilon}$ condition for some $\varepsilon > 0$ is sufficient for (1.2). Thus a natural analogue of Muckenhoupt’s conjecture for (1.2) is that the $C_p$ condition is necessary and sufficient for (1.2).

In [18], it was shown that there is a condition $\tilde{C}_p$ such that $C_{p+\varepsilon} \subset \tilde{C}_p \subset C_p$ for every $\varepsilon > 0$ and $\tilde{C}_p$ is sufficient for (1.2).

By the above results of Sawyer and Yabuta, the $C_p$ conjectures for both (1.1) and (1.2) would be easily solved if the following self-improving property $C_p \Rightarrow C_{p+\varepsilon}$ was true. However, it was shown by Kahanpää and Mejilbro [14] in the one-dimensional case that there exist $C_p$ weights that do not belong to $C_{p+\varepsilon}$ for every $\varepsilon > 0$. The Kahanpää- Mejilbro construction has been recently extended to higher dimensions in the work by Canto, Li, Roncal and Tapiola [2].

We also mention recent works [1, 3] where different aspects of the $C_p$ theory have been investigated. In particular, it was shown in [3] that for $p > 0$ the $C_{\max (1,p)+\varepsilon}$ condition is sufficient for

\begin{equation}
\|A_S f\|_{L^p(w)} \leq C \|M f\|_{L^p(w)}, \tag{1.3}
\end{equation}

which provides a different approach to (1.1). Here $A_S$ is the sparse operator defined by

$$A_S f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q,$$

and $S$ is a sparse family.
Observe that although both (1.1) and (1.2) are known to hold for all \( p > 0 \), the above mentioned results in [24, 25, 29, 18] are obtained in the case \( p > 1 \).

The proofs of the necessity of the \( C_p \) condition for (1.1) (with the Riesz transforms) in [25] and for (1.2) in [29] actually show that the \( C_p, p > 1 \), condition is also necessary for the weak type estimates

\[
\|Tf\|_{L^p,\infty(w)} \leq C \|Mf\|_{L^p(w)}
\]

and

\[
\|f\|_{L^p,\infty(w)} \leq C \|f\#\|_{L^p(w)}.
\]

However, even for these, weaker versions of (1.1) and (1.2) the sufficiency of the \( C_p \) condition is an open question.

In this paper we characterize (1.5) and a variant of (1.4) by means of a condition which seems to be stronger than the \( C_p \) condition.

**Definition 1.1.** Let \( p > 0 \). We say that a weight \( w \) satisfies the \( SC_p \) (strong \( C_p \)) condition if there exist \( C > 0 \) and \( r > 1 \) such that for every family of pairwise disjoint cubes \( \{Q_j\} \),

\[
\sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} w^r \right)^{1/r} |Q_j| \leq C \int_{\mathbb{R}^n} (M\chi_{\bigcup_j Q_j})^p w.
\]

A number of equivalent definitions of the \( SC_p \) condition is given in Section 3. In the trivial case when the family \( \{Q_j\} \) consists of one cube only, we obtain the \( C_p \) condition. So, obviously, \( SC_p \subset C_p \). On the other hand, if \( p > 1 \), then \( C_{p+r} \subset SC_p \) (see Section 6 for further discussion on the relationship between \( SC_p \) and \( C_p \)).

Given a Calderón-Zygmund operator \( T \), define its maximal truncation \( T^* \) by

\[
T^*f(x) = \sup_{\varepsilon > 0} |T(f\chi_{\{|y-x|>\varepsilon\}})(x)|.
\]

**Theorem 1.2.** If \( p > 0 \) and \( w \in SC_p \), then for every Calderón-Zygmund operator \( T \) with Dini-continuous kernel,

\[
\|T^*f\|_{L^p,\infty(w)} \leq C \|Mf\|_{L^p(w)}.
\]

Conversely, if \( p > 1 \) and (1.6) holds for each of the maximal truncated Riesz transforms \( R^*_j, j = 1, \ldots, n \), then \( w \in SC_p \).

Observe that since \( |Tf| \leq |T^*f| + c|f| \) (see [27, p. 36]), Theorem 1.2 implies that the \( SC_p \) condition is also sufficient for (1.4). However, in our proof of the necessity part of Theorem 1.2, the assumption that (1.6) holds for the maximal Riesz transforms (and for \( p > 1 \)) is crucial.
It is still not clear to us how to deduce the necessity of the \( SC_p \) condition even in the one-dimensional case assuming (1.4) for the Hilbert transform (not maximally truncated).

It turns out that the necessity of the \( SC_p \) condition for the weak Fefferman-Stein inequality (1.5) is quite easy for every \( p > 0 \), and the following theorem holds.

**Theorem 1.3.** Let \( p > 0 \). The inequality (1.5) holds if and only if \( w \) satisfies the \( SC_p \) condition.

The sufficiency parts of both Theorems 1.2 and 1.3 are corollaries of the corresponding weak type analogue of (1.3).

**Theorem 1.4.** Let \( \mathcal{D} \) be a dyadic lattice and let \( \mathcal{S} \subset \mathcal{D} \) be an \( \eta \)-sparse family. Let \( p > 0 \) and assume that \( w \) satisfies the \( SC_p \) condition. Then

\[
\| A_S f \|_{L^{p,\infty}(w)} \leq C \| M f \|_{L^p(w)},
\]

where \( C > 0 \) does not depend on \( f \).

The proof of this theorem is based essentially on the technique developed by Domingo-Salazar, Lacey and Rey [8] in order to prove a weighted weak type (1,1) estimate for \( A_S \) with an arbitrary weight.

Since (1.5) is derived from (1.7) and, by Theorem 1.3, the \( SC_p \) condition is necessary for (1.5), we obtain that the \( SC_p \) condition is also necessary for (1.7), in general.

The paper is organized as follows. Section 2 contains necessary definitions and preliminary facts. In Section 3, we obtain several characterizations of the \( SC_p \) condition. Section 4 is devoted to proving Theorem 1.4. In Section 5 we prove Theorems 1.2 and 1.3. Section 6 contains some concluding remarks and open questions.

### 2. Preliminaries

In this section we provide necessary definitions and facts that will be used in the rest of the paper.

#### 2.1. Dyadic lattices, sparse families, and Calderón-Zygmund operators.

Given a cube \( Q_0 \subset \mathbb{R}^n \), let \( \mathcal{D}(Q_0) \) denote the set of all dyadic cubes with respect to \( Q_0 \), that is, the cubes obtained by repeated subdivision of \( Q_0 \) and each of its descendants into \( 2^n \) congruent subcubes.

The following definition was given in [21].

**Definition 2.1.** A dyadic lattice \( \mathcal{D} \) in \( \mathbb{R}^n \) is any collection of cubes such that

(i) if \( Q \in \mathcal{D} \), then each child of \( Q \) is in \( \mathcal{D} \) as well;
(ii) every 2 cubes $Q', Q'' \in \mathcal{D}$ have a common ancestor, i.e., there exists $Q \in \mathcal{D}$ such that $Q', Q'' \in \mathcal{D}(Q)$;
(iii) for every compact set $K \subset \mathbb{R}^n$, there exists a cube $Q \in \mathcal{D}$ containing $K$.

Let $\mathcal{D}$ be a dyadic lattice. We say that a family $S \subset \mathcal{D}$ is $\eta$-sparse, $0 < \eta < 1$, if for every cube $Q \in S$,
\[
\left| \bigcup_{Q' \subseteq Q, Q' \in S} Q' \right| \leq (1 - \eta)|Q|.
\]
In particular, if $S \subset \mathcal{D}$ is $\eta$-sparse, then defining for every $Q \in S$,
\[
E_Q = Q \setminus \bigcup_{Q' \subseteq Q, Q' \in S} Q',
\]
we obtain that $|E_Q| \geq \eta|Q|$ and the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint.

We say that $T$ is a Calderón-Zygmund operator with Dini-continuous kernel if $T$ is a linear operator of weak type $(1, 1)$ such that
\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy \quad \text{for all } x \notin \text{supp } f
\]
with kernel $K$ satisfying the smoothness condition
\[
|K(x, y) - K(x', y)| \leq \omega\left(\frac{|x - x'|}{|x - y|}\right) \frac{1}{|x - y|^n}
\]
for $|x - x'| < |x - y|/2$, where $\int_0^1 \omega(t) \frac{dt}{t} < \infty$.

We will use the following result. Its different versions and proofs can be found in [7, 13, 16, 20, 21, 23].

**Theorem 2.2.** Let $T$ be a Calderón-Zygmund operator with Dini-continuous kernel. Then for every compactly supported $f \in L^1(\mathbb{R}^n)$, there exist $3^n$ dyadic lattices $\mathcal{D}_j$ and $\eta_n$-sparse families $S_j \subset \mathcal{D}_j$ such that for a.e. $x \in \mathbb{R}^n$,
\[
T^*f(x) \leq C_{n,T} \sum_{j=1}^{3^n} A_{S_j}f(x).
\]

### 2.2. Maximal operators and $\lambda$-oscillations.

For a locally integrable function $f$, define the Hardy-Littlewood maximal function $Mf$ and the Fefferman-Stein sharp maximal function $f^#$ by
\[
Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| \quad \text{and} \quad f^#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - f_Q|,
\]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing the point $x$, and $f_Q = \frac{1}{|Q|} \int_Q f$. 
The non-increasing rearrangement of a measurable function \( f \) on \( \mathbb{R}^n \) is defined by
\[
f^*(t) = \inf \left\{ \alpha > 0 : \left| \{ x \in \mathbb{R}^n : |f(x)| > \alpha \} \right| \leq t \right\} \quad (0 < t < \infty).
\]

Given a measurable function \( f \), a cube \( Q \) and \( 0 < \lambda < 1 \), the \( \lambda \)-oscillation of \( f \) over \( Q \) is defined by
\[
\omega_\lambda(f;Q) = \inf_c \left( (f - c)\chi_Q \right)^*(\lambda|Q|).
\]

The local sharp maximal function \( M_\#_\lambda f \) is defined by
\[
M_\#_\lambda f(x) = \sup_{Q \ni x} \omega_\lambda(f;Q) \quad (0 < \lambda < 1).
\]

It is well known (see [11, 17]) that the sharp function \( f^\# \) can be viewed as the maximal operator acting on \( M_\#_\lambda f \), namely, for all \( x \in \mathbb{R}^n \),
\[
(2.1) \quad c_1 M M_\#_\lambda f(x) \leq f^\#(x) \leq c_2 M M_\#_\lambda f(x) \quad (0 < \lambda \leq 1/2),
\]
where \( c_1 \) depends on \( \lambda \) and \( n \) and \( c_2 \) depends only on \( n \).

In [21], the notion of the \( \lambda \)-oscillation is defined a bit differently:
\[
\tilde{\omega}_\lambda(f;Q) = \inf \left\{ \omega(f;E) : E \subset Q, |E| \geq (1 - \lambda)|Q| \right\},
\]
where
\[
\omega(f;E) = \sup_E f - \inf_E f.
\]

It is easy to see that
\[
(2.2) \quad \tilde{\omega}_\lambda(f;Q) \leq 2 \omega_\lambda(f;Q) \quad (0 < \lambda < 1).
\]

Indeed, observe that for every constant \( c \),
\[
\omega(f;E) = \omega(f - c;E) \leq 2 \sup_E |f - c|.
\]

Let
\[
E = \{ x \in Q : |f(x) - c| \leq ((f - c)\chi_Q)^*(\lambda|Q|) \}.
\]

Then \( |E| \geq (1 - \lambda)|Q| \), and therefore,
\[
\tilde{\omega}_\lambda(f;Q) \leq 2 ((f - c)\chi_Q)^*(\lambda|Q|),
\]
which implies (2.2).

Let \( S_0(\mathbb{R}^n) \) be the space of measurable functions \( f \) on \( \mathbb{R}^n \) such that for any \( \alpha > 0 \),
\[
|\{ x \in \mathbb{R}^n : |f(x)| > \alpha \}| < \infty.
\]

In [21], the following result was proved (for a local version of this result see [12, 19]).
Theorem 2.3. Let $f \in S_0(\mathbb{R}^n)$. For every dyadic lattice $\mathcal{D}$, there exists a $\frac{1}{6}$-sparse family $S \subset \mathcal{D}$ (depending on $f$) such that

$$|f| \leq \sum_{Q \in S} \tilde{\omega}_{2^{-n}}(f;Q) \chi_Q$$

almost everywhere.

2.3. Hölder’s inequality for $L \log L$. Given a Young function $\Phi$ and a cube $Q$, define the normalized Orlicz average $\|f\|_{\Phi,Q}$ by

$$\|f\|_{\Phi,Q} = \inf \left\{ \alpha > 0 : \frac{1}{|Q|} \int_Q \Phi(|f(y)|/\alpha) dy \leq 1 \right\}.$$

Denote $\|f\|_{L \log L,Q}$ if $\Phi(t) = t \log(e+t)$ and $\|f\|_{\text{exp}L,Q}$ if $\Phi(t) = e^t - 1$. Then the following generalized Hölder’s inequality holds (see, e.g., [28, p. 166]):

$$\frac{1}{|Q|} \int_Q |fg| dx \lesssim \|f\|_{L \log L,Q} \|g\|_{\text{exp}L,Q}.$$

A simple computation shows that if $E \subset Q$, then

$$\|\chi_E\|_{\Phi,Q} = \frac{1}{\Phi^{-1}(|Q|/|E|)}.$$

This along with Hölder’s inequality implies

$$\int_E |f| \lesssim \frac{|Q|}{\log(1 + |Q|/|E|)} \|f\|_{L \log L,Q} \quad (E \subset Q).$$

2.4. A reverse $L \log L$ estimate for the Riesz transforms. A well known result of Stein [26] says that

$$\|f\|_{L \log L,Q} |Q| \lesssim \int_Q M(f \chi_Q) dx.$$

In the same work [26], Stein mentioned (without a proof) that for the standard Riesz transforms defined by

$$R_j f(x) = \lim_{\varepsilon \to 0} c_n \int_{|y| > \varepsilon} f(x - y) \frac{y_j}{|y|^{n+1}} dy \quad (j = 1, \ldots, n)$$

the following analogue of (2.4) holds: if $f \geq 0$ on a cube $\alpha Q$, $\alpha > 1$, and $R_j f \in L^1(\alpha Q)$ for every $j = 1, \ldots, n$, then $f \in L \log L(Q)$. We will need a quantitative version of this result, similar to (2.4). Probably the proof of the following statement is well known but we could not find it in the literature, and therefore it is given below.

Lemma 2.4. For every cube $Q$ and a non-negative function $f$ on $Q$,

$$\|f\|_{L \log L,Q} |Q| \lesssim \sum_{j=1}^n \int_{3Q} |R_j(f \chi_Q)| dx + \int_Q f dx.$$
Proof. Define the Poisson maximal function
\[ M(f, P)(x) = \sup_{t > 0} |P_t \ast f(x)|, \]
where \( P \) is the Poisson kernel. By an equivalent characterization of the Hardy space \( H^1 \) (see [10, p. 141]),
\[ \| M(f, P) \|_{L^1} \lesssim \sum_{j=1}^{n} \| R_j f \|_{L^1} + \| f \|_{L^1}. \]
(2.5)

Since \( M(f \chi_Q) \sim M(f \chi_Q, P) \), by (2.4) we obtain
\[ \| f \|_{L^1 \log L, Q} \lesssim \int_Q M(f \chi_Q, P) dx \]
\[ \lesssim \int_Q M((f - f_Q) \chi_Q, P) dx + \int_Q f dx. \]
(2.6)

Further, by (2.5),
\[ \int_Q M((f - f_Q) \chi_Q, P) dx \lesssim \sum_{j=1}^{n} \| R_j ((f - f_Q) \chi_Q) \|_{L^1} + \int_Q f dx. \]
(2.7)

By the standard estimate for singular integrals (see, e.g., [10, p. 231]),
\[ \int_{\mathbb{R}^n \setminus 3Q} |R_j ((f - f_Q) \chi_Q)| dx \lesssim \| f - f_Q \|_{L^1(Q)} \lesssim \int_Q f dx. \]
Therefore, using also that \( \| R_j (\chi_Q) \|_{L^1(3Q)} \lesssim |Q| \), we obtain
\[ \sum_{j=1}^{n} \| R_j ((f - f_Q) \chi_Q) \|_{L^1} \lesssim \sum_{j=1}^{n} \| R_j ((f - f_Q) \chi_Q) \|_{L^1(3Q)} + \int_Q f dx \]
\[ \lesssim \sum_{j=1}^{n} \| R_j (f \chi_Q) \|_{L^1(3Q)} + \int_Q f dx, \]
which, combined with (2.6) and (2.7), completes the proof. \qed

3. Characterizations of the \( SC_p \) condition

In this section we obtain several equivalent definitions of the \( SC_p \) condition. An important role will be played by the following simple lemma.

Lemma 3.1. Let \( f \in L^1_{loc} (\mathbb{R}^n) \). For all \( \alpha > 0 \) and for all \( x \in \mathbb{R}^n \),
\[ M_{\chi_{\{Mf>\alpha\}}}(x) \leq \frac{q^n}{\alpha} Mf(x). \]
Proof. By homogeneity, it suffices to prove that for every cube $Q$ containing $x$,
\begin{equation}
\frac{|Q \cap \{Mf > 1\}|}{|Q|} \leq 9^n Mf(x).
\end{equation}

Let $y \in Q$, and let $Q'$ be an arbitrary cube containing $y$. Then either $Q' \subset 3Q$ or $Q \subset 3Q'$. Therefore,
\begin{equation}
Mf(y) = \sup_{Q' \ni y} \frac{1}{|Q'|} \int_{Q'} |f| \leq \max \left( M(f\chi_{3Q})(y), 3^n \inf_Q Mf \right).
\end{equation}

If $3^n \inf_Q Mf > 1$, then (3.1) holds trivially. If $3^n \inf_Q Mf \leq 1$, then by (3.2) and by the weak type $(1,1)$ of $M$,
\begin{equation*}
|\{y \in Q : Mf(y) > 1\}| \leq |\{y \in Q : M(f\chi_{3Q})(y) > 1\}|
\leq 3^n \int_{3Q} |f|.
\end{equation*}

Therefore, in this case we again obtain (3.1). \qed

Definition 3.2. Let $R \geq 1$. We say that a family of cubes $\{Q_j\}$ is $R$-separated if the cubes $RQ_j$ are pairwise disjoint.

In the following theorem, a set $E$ is assumed to be bounded measurable set of positive measure.

Theorem 3.3. Let $p > 0$ and let $w$ be a weight. The following conditions are equivalent.
(i) $w \in SC_p$.
(ii) For every $R \geq 1$, there exists $C > 0$ such that for every $R$-separated family of cubes $\{Q_j\}$,
\begin{equation*}
\sum_j \|w\|_{L^\infty L,Q_j} |Q_j| \leq C \int_{\mathbb{R}^n} (M\chi_{\bigcup_j Q_j})^p w.
\end{equation*}
(iii) There exists a continuous function $\varphi$ on $(0,1)$ with $\lim_{\lambda \to 0} \varphi(\lambda) = 0$ such that for every set $E$ and for all $0 < \lambda < 1$,
\begin{equation*}
w(E) \leq \varphi(\lambda) \int_{\mathbb{R}^n} (M\chi_{\{M\chi_E > \lambda\}})^p w.
\end{equation*}
(iv) There exist $0 < C, \lambda_0 < 1$ such that for every set $E$,
\begin{equation*}
\int_{\mathbb{R}^n} (M\chi_E)^p w \leq C \int_{\mathbb{R}^n} (M\chi_{\{M\chi_E > \lambda_0\}})^p w.
\end{equation*}
(v) There exist $C, \delta > 0$ such that for every set $E$ and for all $0 < \lambda < 1$,
\begin{equation*}
w(E) \leq C\lambda^\delta \int_{\mathbb{R}^n} (M\chi_{\{M\chi_E > \lambda\}})^p w.
\end{equation*}
Proof. The implication (i) ⇒ (ii) is trivial since
\[ \left\| w \right\|_{L^{\log L,Q}} \leq C_r \left( \frac{1}{|Q|} \int_Q w^r \right)^{1/r} \]
for every \( r > 1 \).

Turn to the implication (ii) ⇒ (iii). Let \( E \) be a bounded set of positive measure, and let \( 0 < \lambda < 1 \). Denote
\[ \Omega = \{ x : M\chi_E > \lambda \} . \]

Let \( R \geq 1 \) as in condition (ii). By the Whitney covering lemma (as stated in [25]), there is a covering \( \Omega = \bigcup_j Q_j \) such that
\[ C_1 Q_j \cap \Omega^c \neq \emptyset \quad \text{and} \quad \sum_j \chi_{RQ_j}(x) \leq C_2, \]
where \( C_1 \) and \( C_2 \) depend only on \( R \) and \( n \).

The first condition in (3.3) implies that
\[ |Q_j \cap E| \leq C_n^2 \lambda |Q_j|. \]
In turn, the second condition in (3.3) implies that the family \( F = \{ Q_j \} \) can be written as the union of \( N \) \( R \)-separated families \( F_i \), where \( N \) depends only on \( C_2 \) and \( n \) (see [15, p. 69] for the proof of this fact).

Applying condition (ii) along with (2.3) and (3.4), we obtain
\[ \sum_{Q_j \in F_i} w(E \cap Q_j) \lesssim \frac{1}{\log(C/\lambda)} \sum_{Q_j \in F_i} \left\| w \right\|_{L^{\log L,Q}} |Q_j| \]
\[ \lesssim \frac{1}{\log(C/\lambda)} \int_{\mathbb{R}^n} (M\chi_{\{M\chi_E > \lambda\}})^p w. \]
Therefore,
\[ w(E) = \sum_{i=1}^N \sum_{Q_j \in F_i} w(E \cap Q_j) \lesssim \frac{1}{\log(C/\lambda)} \int_{\mathbb{R}^n} (M\chi_{\{M\chi_E > \lambda\}})^p w, \]
which proves (iii).

Let us show now that (iii) ⇒ (iv). Let \( 0 < \tau < 1 \). By the Calderón-Zygmund decomposition, if \( \frac{|Q \cap E|}{|Q|} \leq \tau \), then
\[ |Q \cap E| \leq 2^n \tau |Q \cap \{ M\chi_E > \tau \}|. \]
Therefore,
\[ M\chi_E(x) \leq \tau \Rightarrow M\chi_E(x) \leq 2^n \tau M\chi_{\{M\chi_E > \tau\}}. \]
From this,
\[ \int_{\mathbb{R}^n} (M\chi_E)^p w \leq (2^n \tau)^p \int_{\mathbb{R}^n} (M\chi_{\{M\chi_E > \tau\}})^p w + w(\{ M\chi_E > \tau \}). \]
Next, condition (iii) combined with Lemma 3.1 implies
\[ w(\{ M_{\chi E} > \tau \}) \leq \varphi(\lambda) \int_{\mathbb{R}^n} (M_{\chi \{ M_{\chi E} > \lambda \tau / 9^n \}^k})^p w. \]
Hence, taking \( \tau = \tau' = 2^{-n-2} \) and \( \lambda = \lambda' \) such that \( \varphi(\lambda') \leq 1/4 \), we obtain condition (iv) with \( C = 1/2 \) and \( \lambda_0 = \lambda' \tau' / 9^n \).

Turn to the proof of (iv) \( \Rightarrow \) (v). Iterating (iv) along with Lemma 3.1 yields
\[ \int_{\mathbb{R}^n} (M_{\chi E})^p w \leq C_k \int_{\mathbb{R}^n} (M_{\chi \{ M_{\chi E} > \lambda_0 / 9^n \}^k})^p w \]
for all \( k \in \mathbb{N} \). From this,
\[ w(E) \leq (9^n / \lambda_0)^{\delta \lambda} \int_{\mathbb{R}^n} (M_{\chi \{ M_{\chi E} > \lambda \}^k})^p w \]
for all \( 0 < \lambda < 1 \), where \( \delta = \frac{\log C}{\log (\lambda_0 / 9^n)} \).

It remains to show that (v) \( \Rightarrow \) \( SC_p \). Let \( \{ Q_j \} \) be a family of pairwise disjoint cubes. Take
\[ E_j \subset \{ x \in Q_j : w(x) \geq (w_{\chi Q_j})^* (\lambda |Q_j|) \} \]
with \( |E_j| = \lambda |Q_j| \). Denote \( E = \bigcup_j E_j \). Let us prove that
\[ w(E) \leq (9^n \lambda)^{1/3} \lambda \int_{\mathbb{R}^n} (M_{\chi \cup_j Q_j})^p w. \]
If \( 1/3^n \leq \lambda < 1 \), then (3.5) is trivial since \( w(E) \leq w(\cup_j Q_j) \). Suppose that \( 0 < \lambda < 1/3^n \). Then we claim that
\[ \{ M_{\chi E} > 3^n \lambda \} \subset \cup_j 3Q_j. \]
Indeed, let \( \frac{|Q \cap E|}{|Q|} > 3^n \lambda \). Denote by \( F \) a subfamily of those \( Q_j \) having non-empty intersection with \( Q \). If \( Q_j \subset 3Q \) for all \( Q_j \in F \), then
\[ |Q \cap E| \leq \sum_{E_j \subset Q_j \in F} |E_j| = \lambda \sum_{Q_j \in F} |Q_j| \leq 3^n \lambda |Q|, \]
which is a contradiction. Therefore, there exists \( Q_j \in F \) such that \( Q \subset 3Q_j \), which proves (3.6).

Applying condition (v) along with (3.6) yields
\[ w(E) \leq C (3^n \lambda)^{1/3} \lambda \int_{\mathbb{R}^n} (M_{\chi \{ M_{\chi E} > 3^n \lambda \}^k})^p w \leq C' \int_{\mathbb{R}^n} (M_{\chi \cup_j 3Q_j})^p w. \]
Since \( \cup_j 3Q_j \subset \{ M_{\chi \cup_j Q_j} \geq 1/3^n \} \), Lemma 3.1 along with the previous estimate completes the proof of (3.5).
It follows from (3.5) that
\[
\sum_j |Q_j|(w\chi_{Q_j})^*(\lambda|Q_j|) \leq \frac{1}{\lambda} w(E)
\]
\[
\lesssim (\lambda^{k-1}\chi_{(0,1/3^k)}(\lambda) + \lambda^{-1}\chi_{[1/3^k,1]}(\lambda)) \int_{\mathbb{R}^n} (M\chi_{Q_j})^p w.
\]
From this, rewriting the standard estimate
\[
\|w\|_{L^r(Q_j)} \leq \|w\|_{L^r(Q_j)}
\]
as
\[
\left(\frac{1}{|Q_j|} \int_{Q_j} w^r \right)^{1/r} \leq \int_0^1 (w\chi_{Q_j})^*(\lambda|Q_j|) \frac{d\lambda}{\lambda^{1-1/r}}
\]
and taking \( r > 1 \) such that \( 1 - 1/r < \delta \), we obtain the SC\(_p\) condition.

4. **Proof of Theorem 1.4**

As we have mentioned in the Introduction, the proof of Theorem 1.4 is an adaptation of the method from [8].

**Proof of Theorem 1.4.** Denote
\[
E = \{x : A_Sf(x) > 2, Mf(x) \leq 1/4\}.
\]
Then, by homogeneity and by Chebyshev’s inequality, it suffices to show that
\[
(4.1) \quad w(E) \lesssim \int_{\mathbb{R}^n} (Mf)^p w.
\]

By the standard limiting argument, one can assume that the family \( S \) is finite. Then \( w(E) < \infty \). For \( k \in \mathbb{N} \) denote
\[
F_k = \{Q \in S : 4^{-k-1} < |f|_Q \leq 4^{-k}\}.
\]
Then, for \( x \in E \),
\[
A_Sf(x) \leq \sum_{k=1}^{\infty} \frac{1}{4^k} \sum_{Q \in F_k} \chi_Q.
\]
Therefore, by Chebyshev’s inequality,
\[
(4.2) \quad w(E) \leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{4^k} \sum_{Q \in F_k} w(E \cap Q).
\]

Write \( F_k = \bigcup_{\nu=0}^{N} F_{k,\nu} \), where \( F_{k,0} \) is the family of the maximal cubes in \( F_k \) and \( F_{k,\nu+1} \) is the family of the maximal cubes in \( F_k \setminus \bigcup_{\nu=0}^{N} F_{k,\nu} \).

Denote \( E_Q = Q \setminus \bigcup_{Q' \in F_{k,\nu+1}} Q' \) for each \( Q \in F_{k,\nu} \). Then the sets \( E_Q \) are pairwise disjoint for \( Q \in F_k \).
For $\nu \geq 0$ and $Q \in F_{k,\nu}$ denote

$$A_k(Q) = \bigcup_{Q' \in F_{k,\nu+2^k}, Q' \subseteq Q} Q'$$

(if $F_{k,\nu+2^k} = \emptyset$, then set $A_k(Q) = \emptyset$). Observe that

$$Q \setminus A_k(Q) = 2^{k-1} \bigcup_{l=0}^{2^k-1} \bigcup_{Q' \in F_{k,\nu+l}, Q' \subseteq Q} E_{Q'}.$$

Using that the sets $E_Q$ are disjoint, we obtain

$$\sum_{Q \in F_k} w(E \cap (Q \setminus A_k(Q))) \leq \sum_{\nu=0}^{N} \sum_{Q \in F_{k,\nu}} \sum_{l=0}^{2^k-1} \sum_{Q' \in F_{k,\nu+l}, Q' \subseteq Q} w(E \cap E_{Q'}) \leq 2^k \sum_{Q \in F_k} w(E \cap E_Q) \leq 2^k w(E).$$

From this and from (4.2),

$$w(E) \leq \frac{1}{2} w(E) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{4^k} \sum_{Q \in F_k} w(A_k(Q)),$$

and hence

(4.3) $$w(E) \leq \sum_{k=1}^{\infty} \frac{1}{4^k} \sum_{Q \in F_k} w(A_k(Q)).$$

By the $\eta$-sparseness, $|A_k(Q)| \leq (1 - \eta)^{2^k} |Q|$. Take $r > 1$ as in the $SC_p$ condition. By Hölder’s inequality,

$$w(A_k(Q)) \leq (1 - \eta)^{2^k/r'} \left( \frac{1}{|Q|} \int_Q w^{r'} \right)^{1/r'} |Q|.$$

This along with (4.3) implies

(4.4) $$w(E) \leq \sum_{k=1}^{\infty} \frac{(1 - \eta)^{2^k/r'}}{4^k} \sum_{Q \in F_k} \left( \frac{1}{|Q|} \int_Q w^{r'} \right)^{1/r'} |Q|.$$

Let $Q_j$ be the maximal cubes of $F_k$. Then setting $F_k(Q_j) = \{ Q \in F_k : Q \subseteq Q_j \}$, we can write $F_k = \bigcup Q_j F_k(Q_j)$. Therefore,

$$\sum_{Q \in F_k} \left( \frac{1}{|Q|} \int_Q w^{r'} \right)^{1/r} |Q| = \sum_j \sum_{Q \in F_k(Q_j)} \left( \frac{1}{|Q|} \int_Q w^{r'} \right)^{1/r} |Q|.$$
By the sparseness and the well known fact that for $0 < \delta < 1$,
\[
\int_Q (M(f \chi_Q))^{\delta} \leq C_{\delta, n} \left( \frac{1}{|Q|} \int_Q |f| \right)^{\delta} |Q|,
\]
we obtain
\[
\sum_{Q \in F_k(Q_j)} \left( \frac{1}{|Q|} \int_Q w^{r} \right)^{1/r} |Q| \leq \frac{1}{\eta} \sum_{Q \in F_k(Q_j)} \left( \frac{1}{|Q|} \int_Q w^{r} \right)^{1/r} |E_Q|
\]
\[
\leq \frac{1}{\eta} \int_{Q_j} M(w^r \chi_{Q_j})^{1/r} \leq C \left( \frac{1}{|Q_j|} \int_{Q_j} w^{r} \right)^{1/r} |Q_j|.
\]
Combining this with (4.4) and applying the $SC_p$ condition along with Lemma 3.1, we obtain
\[
w(E) \lesssim \sum_{k=1}^{\infty} (1 - \eta)^{2k/r'} \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} w^{r} \right)^{1/r} |Q_j|
\]
\[
\lesssim \sum_{k=1}^{\infty} (1 - \eta)^{2k/r'} \int_{\mathbb{R}^n} (M \chi_{\cup_j Q_j})^{p} w
\]
\[
\lesssim \sum_{k=1}^{\infty} (1 - \eta)^{2k/r'} \int_{\mathbb{R}^n} (M \chi_{\{Mf > 4^{-k-1}\}})^{p} w
\]
\[
\lesssim \sum_{k=1}^{\infty} (1 - \eta)^{2k/r'} 4^{(p-1)k} \int_{\mathbb{R}^n} (Mf)^{p} w \lesssim \int_{\mathbb{R}^n} (Mf)^{p} w.
\]
This proves (4.1), and therefore, the theorem is proved. \hfill \Box

5. PROOF OF THEOREMS 1.2 AND 1.3

In the necessity part of Theorem 1.2 we will use the notion of the grand maximal truncated operator $M_T$ defined in [20] by
\[
M_Tf(x) = \sup_{Q \ni x} \|T(f \chi_{\mathbb{R}^n \setminus 3Q})\|_{L^\infty(Q)}.
\]
It was shown in [20] that for any Calderón-Zygmund operator with Dini-continuous kernel,
\[
M_Tf(x) \lesssim T^* f(x) + Mf(x).
\]
Therefore, the assumption
\[
\|R_k f\|_{L^{p, \infty}(w)} \lesssim \|Mf\|_{L^p(w)} \quad (k = 1, \ldots, n)
\]
implies
\[
\|M_{R_k} f\|_{L^{p, \infty}(w)} \lesssim \|Mf\|_{L^p(w)} \quad (k = 1, \ldots, n).
\]
Also, (5.1) trivially implies that

\[ \| R_k f \|_{L^{p, \infty}(w)} \lesssim \| M f \|_{L^p(w)} \quad (k = 1, \ldots, n). \]

**Proof of Theorem 1.2.** The sufficiency of the condition \( w \in SC_p \) is an immediate combination of Theorems 1.4 and 2.2.

Let us turn to the necessity of \( w \in SC_p \). We will show that (5.2) along with (5.3) implies condition (ii) of Theorem 3.3 with \( R = 3 \).

Take any sequence of cubes \( \{ Q_j \} \), which is 3-separated, and let us show that

\[ \sum_j \| w \|_{L^\log L, Q_j} |Q_j| \lesssim \int_{\mathbb{R}^n} (M \chi_{\cup_j Q_j})^p w. \]

By Lemma 2.4,

\[ \| w \|_{L^\log L, Q_j} |Q_j| \lesssim \sum_{k=1}^n \int_{3Q_j} |R_k(w \chi_{Q_j})| dx + \int_{Q_j} w dx. \]

Therefore, in order to prove (5.4), it suffices to show that for every \( k = 1, \ldots, n \),

\[ \sum_j \int_{3Q_j} |R_k(w \chi_{Q_j})| dx \lesssim \int_{\mathbb{R}^n} (M \chi_{\cup_j Q_j})^p w. \]

Denote

\[ \psi_j = \text{sign} R_k(w \chi_{Q_j}) \chi_{3Q_j} \quad \text{and} \quad \psi = \sum_j \psi_j. \]

Then

\[ \sum_j \int_{3Q_j} |R_k(w \chi_{Q_j})| dx = - \sum_j \int_{Q_j} R_k(\psi_j) w dx \]

\[ = \int_{\cup_j Q_j} R_k(-\psi) w dx + \sum_j \int_{Q_j} R_k(\psi - \psi_j) w dx. \]

Since

\[ \sum_j \int_{Q_j} R_k(\psi - \psi_j) w dx \leq \sum_j \int_{Q_j} |R_k(\psi \chi_{\mathbb{R}^n \setminus 3Q_j})| w dx \]

\[ \leq \int_{\cup_j Q_j} M R_k(\psi) w dx, \]

we obtain

\[ \sum_j \int_{3Q_j} |R_k(w \chi_{Q_j})| dx \leq \int_{\cup_j Q_j} (|R_k(\psi)| + M R_k(\psi)) w dx. \]
Applying (5.2) and (5.3) yields
\[
\int_{\cup_j Q_j} \left( |R_k(\psi)| + M_{R_k}(\psi) \right) wdx \\
\leq \|\chi_{\cup_j Q_j}\|_{L_p,1(w)} \|R_k(\psi)| + M_{R_k}(\psi)\|_{L_p,\infty(w)} \\
\lesssim w(\cup_j Q_j)^{1/p'} \|M\chi_{\cup_j,3Q_j}\|_{L_p(w)}.
\]

We have already seen in the proof of the implication (v) \(\Rightarrow\) (i) of Theorem 3.3 that
\[
\|M\chi_{\cup_j,3Q_j}\|_{L_p(w)} \lesssim \|\chi_{\cup_j,3Q_j}\|_{L_p(w)}.
\]
Therefore,
\[
\int_{\cup_j Q_j} \left( |R_k(\psi)| + M_{R_k}(\psi) \right) wdx \lesssim \int_{\mathbb{R}^n} (M\chi_{\cup_j Q_j})^p w,
\]
which, along with (5.6), proves (5.5), and therefore, the theorem is proved.

**Proof of Theorem 1.3.** Suppose that \(w \in SC_p\). Let \(f \in S_0(\mathbb{R}^n)\). Fix a dyadic lattice \(D\). By Theorem 2.3 combined with (2.2), there exists a \(1/6\)-sparse family \(S \subset D\) such that for a.e. \(x \in \mathbb{R}^n\),
\[
|f| \leq 2 \sum_{Q \in S} \omega_{2^{-n-2}}(f; Q) \chi_Q.
\]
Since \(\omega_{2^{-n-2}}(f; Q) \leq (M^\#_f)_Q\), we obtain that
\[
|f| \leq 2 A_S(M^\#_{2^{-n-2}f}).
\]
Therefore, by Theorem 1.4 combined with the left-hand side of (2.1),
\[
\|f\|_{L_p,\infty(w)} \leq 2 \|M^\#_{2^{-n-2}f}\|_{L_p,\infty(w)} \\
\lesssim \|MM^\#_{2^{-n-2}f}\|_{L_p(w)} \lesssim \|f^\#\|_{L_p(w)},
\]
proving (1.5).

Assume now that (1.5) holds. Let \(E\) be a bounded set of positive measure, and let \(0 < \lambda < 1\). Set in (1.5) \(f = \log^+(\frac{1}{\lambda}M\chi_E)\).

It is well known (see [6]) that \(f \in BMO\) and \(\|f\|_{BMO} \leq C_n\). Hence,
\[
(5.7) \quad \|M^\#_{1/2}f\|_{L^\infty} \leq 2 \|f\|_{BMO} \leq 2C_n.
\]
Also, \(\text{supp}(f) \subset \{x : M\chi_E \geq \lambda\}\). Since
\[
\text{supp}(M^\#_{1/2}f) \subset \{M\chi_{\text{supp}(f)} \geq 1/2\},
\]
by Lemma 3.1 we obtain
\[
\text{supp}(M^\#_{1/2}f) \subset \{M\chi_E \geq \lambda/2 \cdot 9^n\}.
\]
Therefore, by the right-hand side of (2.1) along with (5.7),
\[ \| f^\# \|_{L^p(w)} \lesssim \| MM_{1/2}^\# f \|_{L^p(w)} \lesssim \| M \chi_{\{M \chi \geq \lambda/2 \cdot 9^n\}} \|_{L^p(w)}. \]

On the other hand, since \( f \geq \log(1/\lambda) \chi_E \), we obtain that
\[ \log(1/\lambda) w(E)^{1/p} \leq \| f \|_{L^{p,\infty}(w)}, \]
which along with the previous estimate and (1.5) implies
\[ w(E) \leq \frac{C}{(\log(1/\lambda))^p} \int_{\mathbb{R}^n} (M \chi_{\{M \chi \geq \lambda/2 \cdot 9^n\}})^p w. \]

Thus, \( w \) satisfies condition (iii) of Theorem 3.3, which proves that \( w \in SC_p \). \( \square \)

6. Concluding remarks

Remark 6.1. The main results of this paper raise several interesting questions. The most natural one is the following.

Question 6.2. What is the relationship between the \( SC_p \) and \( C_p \) conditions? In particular, is it true that \( SC_p = C_p \)?

If \( SC_p \neq C_p \), then we would obtain that Muckenhoupt’s conjecture for (1.1) as well as its counterpart for (1.2) are not true. Probably in this case, the \( SC_p \) condition would be the natural candidate for a necessary and sufficient condition for (1.1) and (1.2).

Since the \( C_{p+\varepsilon} \) condition for \( p > 1 \) implies (1.2) (by Yabuta’s result [29]), by Theorem 1.3 we obtain that
\[ C_{p+\varepsilon} \Rightarrow SC_p \quad (p > 1). \]

Therefore, thinking about a possible counterexample to the implication \( C_p \Rightarrow SC_p \), a weight \( w \) should be from the class \( C_p \setminus \cup_{q>p} C_q \).

Remark 6.3. Observe that for the Hardy-Littlewood maximal operator \( M \), there is an argument (see [22, Cor. 1.3] ) showing that
\[ M : L^p(w) \rightarrow L^{p,\infty}(w) \Rightarrow M : L^p(w) \rightarrow L^p(w) \quad (p > 1) \]
(without the use of the implication \( M : L^p(w) \rightarrow L^{p,\infty}(w) \Rightarrow w \in A_p \)). Related to this, one can ask the following.

Question 6.4. Is it possible to deduce that the weak \( L^p(w) \) Coifman-Fefferman inequality (1.4) implies the strong Coifman-Fefferman inequality (1.1) (without appealing to any structural properties of \( w \))?
The same question can be asked about the Fefferman-Stein inequality. If the answer to Question 6.4 is positive, then the $SC_p$ condition would be necessary and sufficient for (1.1) (at least with the maximal truncated Calderón-Zygmund operator $T^*$).

**Remark 6.5.** As we have seen, the necessity part of the proof of Theorem 1.2 is based essentially on the notion of the grand maximal truncated operator $M_T$, and this explains why Theorem 1.2 is formulated for $T^*$ instead of $T$. Thus, the following question appears naturally.

**Question 6.6.** Is it possible to deduce that the $SC_p$ condition is necessary for (1.4) at least in the one-dimensional case for the Hilbert transform?

Since the $SC_p$ condition is necessary, in general, for the corresponding weak type estimate for the sparse operator, it would be very surprising if the answer to Question 6.6 is negative.

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