ORSAY LECTURES ON CONFINEMENT(III)

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Light quark confinement$^{1,2}$

We have described the confinement of heavy quarks in an analogy with the theory of the supercharged nucleus [1,2]. Let us now suppose again that $\alpha$ is behaving like

\[
\begin{align*}
\alpha & \quad \alpha_c \\
\frac{1}{q} & \quad \frac{1}{\lambda}
\end{align*}
\]

Fig. 1.

Making this assumption, we are neglecting gluon-gluon interactions and the existence of gluons as real particles. Our aim is to see, what can arise from the discussion of light quarks only. We introduce $\lambda$ corresponding to $\alpha_c$ and consider quark masses $m_0 \ll \lambda$. The interactions of light quarks (for which $m_0 \ll \lambda$) will be discussed in a rather simplified way. We will take into account all possible interactions

where the gluon propagator (considered as an effective photon), corresponding to the dotted line is

\[
D_{\mu\nu} = \frac{\alpha}{q^2} \delta_{\mu\nu}.
\] (1)

Further, we look for a model which enables us to see, what happens to the fermions if there is an interaction between them, as indicated above. The question is, how the bound states or the Green function behave in such a case.

Let us consider the energy of two quarks, $u$ and $\overline{d}$, for example. Without interaction there will be positive energy states with $E > 2m$ and negative energy

$^1$This is the third lecture on quark confinement given by V.N.Gribov in 1992 in Orsay. An extensive discussion of the consequences of all this for the structure of the Green function can be found in [5,6] - in the two last papers concluding his 20 years long study of the problem of quark confinement in QCD.

$^2$The text was prepared for publication by Yu. Dokshitzer, B. Metsch and J. Nyiri on the basis of a tape recording and notes taken during the lecture.
states with $E < -2m$:

\[ E(ud) \]

\[
\begin{array}{c}
2m \\
0 \\
-2m
\end{array}
\]

Introducing the interaction, for small $\alpha$ we will find that there are some bound states near $2m$ and $-2m$.

\[ E(ud) \]

\[
\begin{array}{c}
2m \\
0 \\
-2m
\end{array}
\]

So far, we consider the usual Dirac vacuum: the negative energy states are occupied, and the positive ones are empty. Increasing the coupling, i.e. increasing $\alpha$, we could expect that the magnitude of the energy is decreasing and the levels corresponding to the bound states will come closer and closer to zero.

\[ E(ud) \]

\[
\begin{array}{c}
2m \\
0 \\
-2m
\end{array}
\]

With a further increase of the coupling up to a critical value, one possibility for the levels will be just to approach the zero line and never cross. There is, however, also a possibility of crossing. We will see that the first case corresponds to normal spontaneous symmetry breaking. But, if the levels cross, and especially in the most clear case, when they pass the lines $2m$ and $-2m$, respectively, everything
will change and we arrive at very different phenomena:

Indeed, now the original vacuum which corresponds to the case when level 2 is empty and level 1 is occupied, is absolutely unstable. We have to fill the new negative energy state and leave the positive energy level empty. But by filling this new state, we get an excitation, a meson-type state with a mass $\mu$. For free quarks this would mean that the quark with negative energy decays into a negative energy meson (filling the negative energy levels) and creates a positive energy quark. As a result, the Dirac picture in which all negative energy levels are filled up and all positive energy levels are empty, is destroyed. But if so, a positive energy quark also decays into a positive energy meson and a quark with negative energy. This means that both decays

$$q^- \rightarrow \mu^- + q^+,$$
$$q^+ \rightarrow \mu^+ + q^-$$

are possible, and both $q^-$ and $q^+$ are unstable.

The question is now, how to deal with the bound state problem. Of course, we could just start to calculate the bound states, considering the interactions without corrections to the Green function. However, one has to take into account that the fermion-fermion interaction changes the effective mass of the quarks and this in its turn will change the bound states considerably,

\[\text{Diagram of fermion-fermion interaction.}\]

which makes the problem more complicated. We thus will have to consider bound states and the Green functions on equal footing.

Up to now, the only approach which deals with this problem and is self-consistent is the Nambu-Jona-Lasinio model [3]. It considers the fermion Green
function corrections due to a four-fermion interaction:

In spite of the strong dependence on the cut-off, the model preserves all sym-
metries in the Green function and in two-particle interactions. Let us present
the result of Nambu and Jona-Lasinio in a way somewhat different from what
is given in [3]. We express it as the dependence of the renormalized mass $m$
on
the bare mass $m_0$. They found that if the effective coupling (it depends on the
definition in their case) $\frac{g^2}{2\pi}$ is less than unity, the curve will be just the usual one:

If, however, the coupling $\frac{g^2}{2\pi}$ is larger than unity, the dependence will be like $\frac{g^2}{2\pi} > 1$:

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Fig. 2.

---

$^3$This result is not always quoted, but it is present in their paper.
According to the interpretation of Nambu and Jona-Lasinio, the upper part of the curve, which at \( m_0 = 0 \) reaches a finite point, corresponds to the spontaneous symmetry breaking. But there are three solutions at \( m_0 = 0 \) and at sufficiently small \( m_0 \) values. What Nambu and Jona-Lasinio claim is that the lower part of the curve is unstable, and there is a real vacuum. I agree with this, if \( m_0 > m_{\text{crit}} \). We can ask: what is the source of instability of this curve? The general argumentation is the following. Talking about spontaneous symmetry breaking, \( m \) is like a magnetic field in a ferromagnetic; we just choose a definite direction. But \( m_0 \) is like an external field, and the system is like a compass. If the external field and the induced field are pointing at the same direction, the situation is stable. If they point to opposite directions, the compass will change.

I am, however, not sure that the instability of the almost perturbative solution which contains no condensate at all can be explained in such a way. The explanation may be right for the part \( b \) of the curve in Fig.2 which corresponds to a big spontaneous magnetic moment and the opposite direction of \( m_0 \). It does not work for the part \( a \) close to perturbation theory which has no spontaneous magnetic moment. And, looking more carefully at the curve, we see that the part \( b \) corresponds to pseudoscalar states inside the Dirac sea, while on the piece \( a \) both the pseudoscalar and scalar states are inside, both levels passed. Recognizing this, one can conclude that indeed, the mentioned state is unstable, but for a trivial reason: the corresponding level is inside the Dirac sea and it is not filled up. The problem is, what happens if we fill up this level. It remains an open question, what can be considered as a ground state under these conditions. And it is a problem how to get these results in a more self-consistent way, not depending on the cut-off so strongly.

The Nambu - Jona-Lasinio model can be reproduced in our picture. For this purpose, just as a theoretical exercise, let us use \( \alpha \) not going to unity, i.e. draw instead of the curve in Fig.1. In this case there will be a second scale \( \lambda_2 \), and outside this scale we will have just point-like interaction. This reminds of the Nambu - Jona-Lasinio picture, which, apparently, can be reached somehow in our approach. The problem is, how to write constructively the corresponding
equation, and whether this can be done at all. Of course, this constructive part can be only approximate. But if we recognize that it can be written, then we will be able to develop a theory in which we put the main ingredient of our discussion as an input into our solution and try to find the real construction. The main difference will appear in the analytic properties of the Green function. The Green function of a fermion for such a case would be quite different in its analytic structure compared to the usual one.

I am afraid I will not have the time to come to this point today, but I would like to explain just the physics.

How to write the equation? What happens in the real case and how to deal with it? Let us start with with the Green function. What do we know, what are we supposed to know about this Green function as a function of $q^2$?

![Diagram](image)

Beyond a certain $\lambda$ in the region where the coupling is small, it is asymptotically free; here the Green function has to satisfy the renorm group equation. But if as a result of the interaction a mass is acquired, this mass would be somewhere at smaller $q^2$; here the equation becomes essentially very complicated and we are not able to extract a reasonable structure.

The idea to write an equation which is correct in both regions, near the threshold and at large $q^2$, and to match these two solutions, comes from the following consideration. Suppose that $\alpha_{\text{crit}}/\pi$ is small:

$$\frac{\alpha_{\text{crit}}}{\pi} \approx 1 - \sqrt{\frac{2}{3}} = 0.2.$$ 

Now, however, we may ask: how could new masses, new solutions etc. appear at all at such a small $\alpha$. Obviously, this $\alpha$ has to be multiplied by something large. What happens, for example, at large $q$? We know, that there is always a logarithm of $q^2/\lambda^2$ and the real parameter becomes

$$\alpha_0 \ln \frac{q^2}{\lambda^2}$$

which is, in spite of the smallness of $\alpha$, big enough to change the Green function essentially. But near the threshold there is also a logarithm:

$$\ln \frac{q^2 - m^2}{\lambda^2}$$

with some scale $\tilde{\lambda} = \lambda$ or $m$.
which is always present. In other words, in this region there could be also a quantity which changes seriously in spite of the relative smallness of \( \alpha \). Hence, we want to write the equation which is correct near the threshold, taking into account correctly the singularity of a supposed mass, and after that compare this with the renorm group equation; we shall see whether it is possible to write an equation which is correct in both regions, and if yes, we will try to solve it. In order to get the singularity correctly, we take the second derivative of \( G^{-1} \) with respect to the momenta.

\[
G^{-1} = G_0^{-1} + \int \frac{1}{(q-q')^2} + \int \frac{G(q')}{q} \tag{2}
\]

The contribution of the first term is trivial, the second derivative of \( q + m \) gives zero. Taking the second derivative of the first graph, it can be easily seen that

\[
\partial^2 \frac{1}{(q-q')^2 - i\epsilon} = -4\pi i\delta^{(4)}(q - q'). \tag{3}
\]

This gives for the first diagram \( \gamma_\mu G(q)\gamma_\mu \alpha \) – just by direct calculation. In other words, we make it local. From this diagram we take the contribution where \( k \equiv (q - q') \) – the momentum of the photon – essentially equals zero.

Let us now look at the second diagram. We have here the choice of taking the second derivative at one of the photon lines, or to differentiate once at one line and once at the second line. Having in mind that all the integrations would have a structure which need some logarithmic enhancement, it would mean that the most important regions of integration in this integral would be those where \( k_1 \ll k_2 \) and \( k_2 \ll k_1 \). We take the derivative at \( k_1 \) and then set it to be zero, but for \( k_2 \) the integration will give the same as before. If this integration gives us two logarithms, we kill one and recover it after the integration of our differential equation; but we still have the first one. But if we differentiate once one line and once the other, we will always sit on the region \( k_1 \sim k_2 \), because they have to be of the same order. And, in this case, there is no logarithm at all; after the integration, we will recover one, but one order will be lost. Clearly, a possible approach is to try not to choose different diagrams, but to use the small \( k \) region of integration. Ordering the integration inside the diagram in such a way that one momentum is much smaller than the others, and differentiating this diagram, we will find a relatively simple answer. Indeed, suppose that we have any diagrams with any loops. If we differentiate some lines twice (it can be any line) and neglect
all first derivatives, we get an amplitude of the following structure:

\[ k = 0 \quad k = 0 \]

\[ q \quad q \]

This is just the Compton scattering of a zero momentum photon \( k = 0 \), and for this quantity the most singular contribution is obviously

\[ \Gamma_\mu(0, q)G(q)\Gamma_\mu(0, q) \]

which corresponds to the diagram

\[ \Gamma_\mu \quad G(q) \quad \Gamma_\mu \]

But the vertex \( \Gamma \) is at zero momentum and hence \( \Gamma_\mu(0, q) = \partial_\mu G^{-1}(q) \). In this approximation we can write a very simple equation:

\[ \partial^2 G^{-1}(q) = \frac{\alpha(q)}{\pi} \partial_\mu G^{-1}(q)G(q)\partial_\mu G^{-1}(q) \] \hspace{1cm} (4)

which differs essentially from any Bethe-Salpeter type equation. Indeed, using a Bethe-Salpeter type equation, we do not change the vertex part and end up with rather bad properties. Equation (4) is scale invariant, it is \( \gamma_5 \)-invariant, it has many nice symmetry properties and, what is most important, it has a correct behaviour near the threshold.

The gauge is fixed, because we used

\[ D_{\mu\nu} = \frac{\delta_{\mu\nu}}{q^2}. \]

It is an important question, what we would get in different gauges. In Feynman gauge we are very lucky: we find an expression which does not depend explicitly on the expression for the Green function. Using a different gauge, we would find the infrared behaviour of this diagram to be more complicated, and we would not be able to extract universally the region of small \( q \). We would have integrals over \( q \) which are also possible to use, but with the necessity to think about the behaviour near the threshold.

We, however, have chosen this gauge; we did not destroy the general features and used the current conservation which just corresponds to \( \Gamma_\mu = \partial_\mu G^{-1} \). Accepting this, we can now ask, what is the relation to the renorm group equation.
Suppose that we would like to write the renorm group equation in the same spirit. Let us take again the second derivatives. In this case we would be definitely correct, because we know that it is a logarithmic approximation.

In our logarithmic approximation we will do exactly the same with the only difference that $\alpha$ would be $\alpha(q^2)$. In the renorm group equation $\alpha$ is a function of $q^2$. But, of course, $\alpha$ in general depends on two momenta: $k^2$ and $q^2$. And in the renorm group equation at large momenta, in the ultraviolet region, $\alpha$ depends on the variable which is the largest. Since we are close to $k = 0$, this means that here we will have $\alpha(q^2)$, and we will recover the renorm group equation at large $q^2$. If we solve this equation with a slowly varying $\alpha$, we will be correct in both the threshold region and the ultraviolet region.

We also have to formulate an equation for the bound states under the same assumption. Looking for bound states, we consider scalar and pseudoscalar vertices. This vertex

\[
\begin{align*}
q + \frac{p}{2} & \quad \varphi(q, p) \quad p \\
q - \frac{p}{2} &
\end{align*}
\]

has to be equal to

\[
+ + - + \cdots
\]

Here $\varphi(q, p)$ depends on $p$, the total momentum of a pair, and $q$, the quark momenta being given by $q + p/2$ and $q - p/2$. With the same procedure as in obtaining the equation for the Green function we find for the vertex (see [4] for some details):

\[
\partial^2 \varphi(q, p) = \frac{\alpha}{\pi} \left[ A_\mu(q) \partial_\mu \varphi(q, p) + \partial_\mu \varphi \tilde{A}_\mu(q) - A_\mu(q) \varphi(q, p) \tilde{A}_\mu(q) \right], \quad (5)
\]

where $A_\mu = \partial_\mu G^{-1} G$, $\tilde{A}_\mu = G \partial_\mu G^{-1}$. It means that we have two equations in this approximation. We used this approximation just to be constructive and to study what will result if we make this approximation. In principle, solving both equations we will get everything what is necessary: we know $G$ and $A_\mu$, we have a linear equation for bound states, we can see what is the type of the energy etc. The equation for the bound states has very nice features. It is beautiful from the point of view of the Goldstone theorem in the following sense.

Suppose I have some symmetry in my equation, e.g. $\gamma_5$-invariance. Since there is no $\gamma_5$ in equation (5), it is $\gamma_5$-invariant. But of course the boundary condition
for \( G^{-1} \) at \( q \to \infty \) is just \( G_0^{-1} = (\hat{q} - m_0) \), and thus destroys the symmetry. But suppose that \( m_0 = 0 \). In this case there would be symmetry here, which means that the Green function will not be unique, since it can be

\[
G^{-1} + \delta G^{-1},
\]

where \( G^{-1} \) is some solution and \( \delta G^{-1} \propto \gamma_5 G^{-1} \). This means that the variation \( \delta G^{-1} \) also is important. What would be the equation for the variation? If we calculate the variation on both sides of equation (4) we obtain

\[
\partial_\mu (\delta G^{-1}) = \frac{\alpha}{\pi} \left( \partial_\mu (\delta G^{-1}) G \partial_\mu G^{-1} + \partial_\mu G^{-1} G \partial_\mu (\delta G^{-1}) - \partial_\mu G^{-1} G (\delta G^{-1}) G \partial_\mu G^{-1} \right),
\]

so we find that \( \phi = \delta G^{-1} \) fulfils equation (5) at \( p = 0 \). It means that if some symmetry is broken, i.e. if there are multiple solutions of the equation for the Green function, we always will have some solution of the equation for the vertex at \( p = 0 \), which is the Goldstone.

It is clear, that in the present model we can discuss many questions, use a running coupling \( \alpha \) as in Fig. 1 and reproduce the NJL-features without any essentials depending on a cutoff. Before discussing this point further, we will first look for the solution of (4) and discuss the result.

Above, we introduced \( A_\mu = (\partial_\mu G^{-1}) G \), which is a very useful quantity. Since \( G^{-1} = a \hat{q} + b \) is essentially a \( 2 \times 2 \) matrix, \( A_\mu \) is a \( U(2) \) gauge potential:

\[
\partial^2 G^{-1} = \partial_\mu ((\partial_\mu G^{-1}) G G^{-1}) = (\partial_\mu A_\mu) G^{-1} + A_\mu (\partial_\mu G^{-1}) = \frac{\alpha}{\pi} A_\mu \partial_\mu G^{-1}
\]

where in the last step we used Eq. (4). Multiplying from the right by \( G \) we thus find

\[
\partial_\mu A_\mu + A_\mu A_\mu = \frac{\alpha}{\pi} A_\mu A_\mu.
\]

This means that

\[
\partial_\mu A_\mu = -\beta A_\mu A_\mu;
\]

and \( A_\mu \) is a pure \( U(2) \)-gauge potential with a condition \( \beta = 1 - \frac{\alpha}{\pi} \). Of course, this is just a useful trick. Important is to write down the real equation for the Green function. The most natural thing is to express \( G^{-1} \) in the form

\[
G^{-1} = \rho e^{\hat{n} \hat{n}},
\]

where \( \hat{n} \) is a \( 2 \times 2 \)-matrix

\[
\hat{n} = \frac{\hat{q}}{q}.
\]

It is just easier to use this form for our purpose: we can find an equation for \( \rho \) and an equation for \( \phi \). Both are functions of \( q^2 \): \( \rho(q^2) \), \( \varphi(q^2) \). There is, however, no scale in the equation; it contains only a derivative of \( q \). If we introduce

\[
\xi = \ln q,
\]
we will find an equation in which $\xi$ can be considered as "time", and which is an oscillator equation. In fact there are two oscillators, one for $\rho$, the other for $\varphi$, and they will satisfy non-linear equations. For $\varphi$ we find

$$\ddot{\varphi} + 2 \left( 1 + \beta \frac{\dot{\rho}}{\rho} \right) \dot{\varphi} - 3 \sinh \varphi = 0.$$  \hspace{1cm} (8)

This is just an oscillator with damping; a similar equation can be written for $\rho$. Important is that that there has to be "energy" conservation in this equation. Indeed, we said that $\xi$ plays the role of time; it, however, did not enter the equation explicitly. Thus there has to be a conservation law which, as it is easy to show, leads to

$$\left( 1 + \beta \frac{\dot{\rho}}{\rho} \right)^2 = 1 + \beta^2 \left( \frac{\dot{\varphi}^2}{4} - 3 \sinh^2 \frac{\varphi}{2} \right).$$  \hspace{1cm} (9)

We thus can eliminate $\rho$ altogether, and find the equation for $\varphi$

$$\ddot{\varphi} + 2 \sqrt{1 - \beta^2 \left( 3 \sinh^2 \frac{\varphi}{2} - \frac{\dot{\varphi}^2}{4} \right)} \dot{\varphi} - 3 \sinh \varphi = 0,$$

which is an oscillator with damping. Having this in mind is sufficient to understand the structure of the solution. Indeed, what is this $\varphi$? We have

$$G^{-1} = \rho \cosh \frac{\varphi}{2} + \frac{\dot{\varphi}}{q} \rho \sinh \frac{\varphi}{2}.$$  \hspace{1cm} (10)

The perturbative solution is $\varphi$ close to $i\pi$. In this case the first term is zero, the other is proportional to $\dot{q}/q$ - this corresponds to the massless solution. Since $m_0$ is small, we have to have solutions like this at $q \to \infty$.

Now we have to find the solution everywhere. Let us first investigate the equation without damping; we get

$$\ddot{\varphi} - 3 \sinh \varphi = 0.$$

If we go to the Euclidean space, $\varphi = i\psi$, the potential becomes a periodical potential:

$$\ddot{\psi} - 3 \sin \psi = 0.$$
We have to look for a possible solution for this structure with damping. What does this mean? For the oscillator with damping any solution at $\xi \to \infty$ has to be in a minimum, because the energy is decreasing. But if $\xi$ is going to $-\infty$, the energy is growing. What could be in this case a normal, reasonable solution? It is almost clear that the only possibility is to put at $\xi \to -\infty$ the ”particles” in this oscillator at the maximum, and start to move them slowly; eventually, they will appear inside the well.

There is a most important question, namely: what is the critical coupling in this case? What do we know about an oscillator with damping? If the damping is large enough, all the trajectories will go monotonically to the minimum. If the damping is sufficiently small, the solution will start to oscillate. In order to see when this transition happens, we have to look for the equation just near the minimum $\psi = \pi$. With $\phi \equiv \pi - \psi$ we have for small $\phi$

$$\ddot{\phi} + 2\sqrt{1 + 3\beta^2}\dot{\phi} + 3\phi = 0,$$

with fundamental solutions

$$\phi_{1,2} = e^{\nu_{1,2}\xi} \text{ where } \nu_{1,2} = -\sqrt{1 + 3\beta^2} \pm \sqrt{3\beta^2 - 2}.$$

So we have monotonic behaviour for $3\beta^2 - 2 > 0$. On the other hand if $\beta^2 < \frac{2}{3}$, i.e.

$$\frac{\alpha_{\text{crit}}}{\pi} = 1 - \sqrt{\frac{2}{3}} < \frac{\alpha}{\pi} < 1 + \sqrt{\frac{2}{3}},$$

we will have oscillations before reaching the minimum. The critical angle $\psi_c$, which separates the regions where the solution is monotonic and where it oscillates can be shown (see e.g. (4)) to be given by

$$\sin^2 \frac{\psi}{2} = \left(\frac{2}{3} - \beta^2\right) \sqrt{\frac{1 + 3\beta^2}{1 - \beta^2}} \frac{1}{1 + \sqrt{(1 + 3\beta^2)(1 - \beta^2)}}.$$
Up to now we have considered a constant coupling $\alpha$. We know that for $q > \lambda$ the Green function is determined by perturbation theory, which has to match the solutions in the region of smaller $q$. If $\beta^2 > \frac{2}{3}$ for all $q^2$, the solution which goes as $\psi \approx \frac{q}{m_c}$ for $q \to 0$ matches the solution $\frac{i}{2}(\psi - \pi) \approx \frac{m_0}{q} + \frac{q^2}{\lambda}$ for $q \to \infty$ monotonically. This determines $m_0$ as a function of $m_c$ in a unique way. Let $\lambda$ be the value of $q$ where $\beta^2(\lambda^2) = \frac{2}{3}$. If, however, $\beta^2 < \frac{2}{3}$ below $q = \lambda$, the solutions can oscillate and $m_0(m_{c_i}) = 0$ for some $m_{c_i}$ as indicated in Fig.6.

This then is a solution corresponding to broken chiral symmetry.

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