The Spectral Properties of a Pre-Stressed Thermoelastic Layer Coupled to a Half-Space

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Abstract. In the framework of the linearized theory of the propagation of thermoelastic waves, a dynamic coupled problem of oscillations of an inhomogeneous medium under the influence of a thermal load oscillating on its surface simulating the action of a frequency-modulated laser beam is considered. The medium is a thermoelastic prestressed layer rigidly adhered to a thermoelastic prestressed half-space. The surface of the layer is free from mechanical stress and insulated outside the area of thermal load acting. At the media boundary thermal insulation condition is considered: ideal thermal contact and. The initial stress state is assumed to be uniform and is created due to preheating and the action of mechanical forces. As an example, the distribution of the phase velocities of the body with initial heating and without it differences are obtained and shown graphically.

1. Introduction

The development of modern technologies causes a significant interest in the processes of excitation of mechanical vibrations due to exposure to laser radiation. A fairly complete review of the papers in this section is given in [1, 2]. In [3, 4], the most effective methods for analyzing various aspects of dynamic processes in thermoelastic media are presented. These are the propagation features of interface [3] waves, Lamb [4] waves. A special place in the problem of the propagation of thermoelastic waves in semi-bounded bodies is occupied by the problems with the presence of initial stresses [5]. In the article [6], following the approach described in the monograph [7]. sequential linearization of the nonlinear equations of a thermoelastic medium is carried out, and equations of motion and defining relations of the dynamics of a prestressed thermoelastic medium are constructed. The equations are constructed in tensor form, allowing generalization to curvilinear coordinates. The constructed relationships are used in [8] to construct the green functions of the thermoelastic prestressed layer and study the temperature effects that accompany the process of propagation of thermoelastic waves. Of considerable interest are questions of the contact interaction of thermoelastic bodies [9]. In [9], the contact interaction of a thermoelastic layer and a half-space was investigated. The equations of motion constructed in [6] are generalized according to the class of a prestressed thermoelastic medium with an inhomogeneous coating. Of particular interest are the problems of the propagation of Rayleigh waves in piezoelectric thermoelastic media. Due to the urgency of the problems in [10], the equations of motion and defining relations, which take into account the presence of initial stresses, were generalized to prestressed thermoelastic media in [6]. In works [12-16] the magneto-electro-elastic problems for a semi-infinite bodies are considered. In the present work the...
boundary problem of oscillation of a layered half-space under the action of a thermal load given on the surface of the medium is considered. The effect of preheating and biaxial initial strain on layered thermoelastic half-space phase velocity is investigated.

2. Formulation of the problem
The thermoelastic layer \( -1 \leq x_3 \leq 0 \) coupled with the thermoelastic half-space \( x_3 \leq -1 \) is considered. A thermal load, heat flux \( q_{40} \), acts on a layer surface in a certain region \( \Omega = \{ |x_1| \leq 1, |x_2| \leq \infty \} \). The layer surface is assumed to be free of mechanical stress and outside the load area is thermal insulated. At the boundary between the layer and the half-space conditions of thermal insulation and rigidly coupling are applied. The propagation of harmonic plane oscillations is assumed. [17-20]

3. Governing equations
For convenience an extended vector of the medium displacements \( u^{(n)} = \{u_1^{(n)}, u_2^{(n)}, u_3^{(n)}, u_4^{(n)} \} \) - components of the vector of mechanical displacements, \( u_1^{(n)} \) (temperature) is introduced. The equations of motion and thermal conductivity for a prestressed thermoelastic layered half-space in the case of plane oscillations are [6, 8-11, 18-20]:

\[
c_{1111} \frac{\partial^2 u_1^{(n)}}{\partial x_1^2} + c_{3113} \frac{\partial^2 u_3^{(n)}}{\partial x_3^2} + \omega^2 u_1^{(n)} + (c_{1133} + c_{1313}) \frac{\partial^2 u_3^{(n)}}{\partial x_1 \partial x_3} - B_{11} \frac{\partial u_2^{(n)}}{\partial x_1} = 0,
\]

\[
(c_{1333} + c_{1133}) \frac{\partial^2 u_3^{(n)}}{\partial x_1 \partial x_3} + \omega^2 u_3^{(n)} + c_{3333} \frac{\partial^2 u_3^{(n)}}{\partial x_3^2} - B_{33} \frac{\partial u_2^{(n)}}{\partial x_3} = 0,
\]

\[
K_{11} \frac{\partial^2 u_4^{(n)}}{\partial x_1^2} + K_{33} \frac{\partial^2 u_4^{(n)}}{\partial x_3^2} + i \omega \theta_4 u_4^{(n)} + i \omega \theta_4^{(n)} E \left( B_{11}^{(n)} \frac{\partial u_4^{(n)}}{\partial x_1} + B_{33}^{(n)} \frac{\partial u_4^{(n)}}{\partial x_3} \right) = 0.
\]

In Eqs. (1)-(3) the elastic and thermal material tensors taking into account the presence of initial strains and pre-heating obey the relations [6]:

\[
c_{ijkl}^{(n)} = \delta_{kl} \frac{\rho_0^{(n)}}{c_{lmm}^{(n)}} (c_{lmm}^{(n)} - 1) + c_{ijkl}^{(n)} V_k^{(n)} - \delta_{kl} \left( \theta_0^{(n)} - \theta_0 \right) B_{ij}^{(n)}, B_{ij}^{(n)} = B_{ij}^{(n)} B_{ij}^{(n)}.
\]

In equations (1) - (4) \( c_{ijkl}^{(n)}, K_{ij}^{(n)}, A_{ij}^{(n)}, B_{kl}^{(n)} = A_{ij}^{(n)} c_{ijkl}^{(n)} \) are the components of the elastic constant tensors, thermal conductivity coefficients, thermal expansion, thermoelasticity, \( \rho_0^{(n)} \) is density of a material in its natural state, \( c_{ijkl}^{(n)} \) is specific heat capacity. \( \theta_0 \) and \( \theta_0^{(n)} \), respectively, are the body temperature in the undeformed state and the temperature of the nth layer in the initial deformed state, \( V_k^{(n)} = 1 + \delta_k^{(n)} \) where \( \delta_k^{(n)} (k = 1, 2, 3) \) is the relative elongation of the fibers, \( E^{(n)} \) is the constant of thermoelastic relation, \( \omega^{(n)} \) is the normalized frequency of the half-space, \( V_k^{(n)} \) is the velocity of the longitudinal wave of the non-deformed material. The problem is indicated in dimensionless quantities as proposed by [17].

4. Boundary conditions
We explore a problem on the excitation of layered thermoelastic half-space oscillations by an oscillating heat flow acting on a layer surface in a certain region \( x_1 \in [-a, a] \). At the boundary between layer and half-space conditions of rigidly coupling and thermal insulation are considered (INS problem). The linearized boundary conditions for plane waves case are written as [20]:

\[
x_3 = 0: - K_{33} \frac{\partial u_4^{(n)}}{\partial x_3} = \left\{ \begin{array}{ll}
q_{40}(x_1), & x_1 \in [-a, a], \\
0, & x_1 \not\in [-a, a],
\end{array} \right.
\]

(5)
\[
\begin{align*}
\left\{ \begin{array}{l}
\begin{aligned}
&c_{1311}^{(i)*} \partial u_1^{(i)} / \partial x_3 + c_{1313}^{(i)*} \partial u_1^{(i)} / \partial x_1 = 0, \\
&c_{3311}^{(i)*} \partial u_1^{(i)} / \partial x_1 + c_{3333}^{(i)*} \partial u_3^{(i)} / \partial x_3 - B_3^{(i)*} u_4^{(i)} = 0,
\end{aligned}
\end{array}
\right.
\end{align*}
\]

(6)

\[
\begin{align*}
\left\{ \begin{array}{l}
\begin{aligned}
&c_{1311}^{(i)*} \partial u_1^{(i)} / \partial x_3 + c_{1313}^{(i)*} \partial u_1^{(i)} / \partial x_3 = c_{3311}^{(i)*} \partial u_1^{(0)} / \partial x_3 + c_{3333}^{(i)*} \partial u_3^{(0)} / \partial x_1, \\
&c_{3311}^{(i)*} \partial u_1^{(i)} / \partial x_1 + c_{3333}^{(i)*} \partial u_3^{(i)} / \partial x_3 - B_3^{(i)*} u_4^{(i)} = \\
&= c_{3311}^{(i)*} \partial u_1^{(0)} / \partial x_1 + c_{3333}^{(i)*} \partial u_3^{(0)} / \partial x_3 - B_3^{(i)*} u_4^{(0)},
\end{aligned}
\end{array}
\right.
\end{align*}
\]

(7)

\[
\begin{align*}
\begin{aligned}
-K_3^{(i)*} u_4^{(0)} &= 0, \\
-K_3^{(i)*} u_4^{(0)} &= 0.
\end{aligned}
\end{align*}
\]

(INS)

(8)

5. Construction of the problem solving

To build a solution of the boundary problem INS an integral Fourier conversion respectively axis \(x_1\) is used. Thus equations of motion and thermal conductivity (1)-(3) are written in the form [18]:

\[
\begin{align*}
-\alpha^2 c_{1111}^{(n)} U_1^{(n)} + c_{1131}^{(n)} U_1^{(n)} + \omega^2 U_1^{(n)} - i\alpha (c_{1113}^{(n)} + c_{1133}^{(n)}) U_3^{(n)} + i\alpha B_1^{(n)*} U_4^{(n)} &= 0, \\
-\alpha^2 c_{1311}^{(n)} U_1^{(n)} + c_{1313}^{(n)} U_1^{(n)} - \alpha^2 (c_{1313}^{(n)} + c_{1333}^{(n)}) U_3^{(n)} + \omega^2 U_3^{(n)} + c_{3333}^{(n)} U_3^{(n)} - B_3^{(n)*} U_4^{(n)} &= 0,
\end{align*}
\]

(9)

(10)

The boundary conditions in Fourier transform are as follows:

\[
\begin{align*}
\begin{aligned}
&x_3 = 0: \\
&c_{1313}^{(i)*} U_1^{(i)} - i\alpha c_{1313}^{(i)*} U_3^{(i)} = 0, \\
&c_{3333}^{(i)*} U_3^{(i)} - i\alpha c_{3313}^{(i)*} U_1^{(i)} - B_3^{(i)*} U_4^{(i)} = 0, \\
&U_1^{(i)} = U_1^{(0)}, \\
&U_3^{(i)} = U_3^{(0)}, \\
&U_4^{(i)} = Q_4,
\end{aligned}
\end{align*}
\]

(INS)

(12)

\[
\begin{align*}
\begin{aligned}
&x_3 = -h: \\
&c_{3311}^{(i)*} U_1^{(i)} - i\alpha c_{1313}^{(i)*} U_3^{(i)} - \\
&\quad - c_{3313}^{(i)*} U_1^{(0)} + i\alpha c_{1313}^{(i)} U_3^{(0)} = 0, \\
&c_{3333}^{(i)*} U_3^{(i)} - i\alpha c_{3313}^{(i)*} U_1^{(i)} - B_3^{(i)*} U_4^{(i)} - \\
&\quad - c_{3333}^{(i)*} U_3^{(0)} + i\alpha c_{1313}^{(i)} U_1^{(0)} + B_3^{(i)*} U_4^{(0)} = 0,
\end{aligned}
\end{align*}
\]

(13)

\[
\begin{align*}
\begin{aligned}
-K_3^{(i)*} U_4^{(i)} &= 0, \\
-K_3^{(i)*} U_4^{(0)} &= 0.
\end{aligned}
\end{align*}
\]

(INS)

(14)

Notation used here \(f' = df / dx_3\), \(f'' = d^2 f / dx_3^2\).

The solution of Eqs. (9)-(14) we obtained in form [20]:

\[
\begin{align*}
U_1^{(i)}(\alpha, x_3, \omega) &= -i\alpha \sum_{k=1}^{3} f_{1k}^{(i)} \left( C_k s h \sigma_k^{(i)} x_3 + C_{k+3} c h \sigma_k^{(i)} x_3 \right), \\
U_3^{(i)}(\alpha, x_3, \omega) &= \sum_{k=1}^{3} f_{3k}^{(i)} \left( C_k c h \sigma_k^{(i)} x_3 + C_{k+3} s h \sigma_k^{(i)} x_3 \right),
\end{align*}
\]

(15)

(16)

\[
\begin{align*}
U_4^{(i)}(\alpha, x_3, \omega) &= \sum_{k=1}^{3} f_{4k}^{(i)} \left( C_k s h \sigma_k^{(i)} x_3 + C_{k+3} c h \sigma_k^{(i)} x_3 \right) - h \leq x_3 \leq 0;
\end{align*}
\]

(17)
\[ U_1^{(0)}(\alpha, x_3, \omega) = -i\alpha \sum_{k=1}^{3} f_{1k}^{(0)} D_k e^{i\omega x_3} , \] (18)

\[ U_3^{(0)}(\alpha, x_3, \omega) = \sum_{k=1}^{3} f_{3k}^{(0)} D_k e^{i\omega x_3} , \] (19)

\[ U_4^{(0)}(\alpha, x_3, \omega) = \sum_{k=1}^{3} f_{4k}^{(0)} D_k e^{i\omega x_3} , \] (20)

In Eqs. (15)-(20), \( \sigma, k \) are found numerically for each value of \( \alpha \) and \( \omega \) from the characteristic equation given in [11] in accordance with the condition of wave attenuation in depth. To obtain the unknown coefficients, \( D_k \), we substitute the representation of the solution (15)-(20) into the boundary conditions (12)-(14) and construct a system of linear algebraic equations, which can be written in matrix form:

\[ \mathbf{L} \mathbf{C} = \mathbf{Q}^T , \] (21)

where \( \mathbf{Q} = (0 \ 0 \ Q_4 \ 0 \ 0 \ 0) \) is the \( q_{40} \) Fourier transform.

Taking an inverse Fourier conversion to relations (15)-(20) the solution of boundary problem (1)-(8) can be written as follows [11]:

\[ u_i^{(n)}(x_1, 0) = \frac{1}{2\pi} \int_{-1}^{1} k_{i4}^{(n)}(x_1 - \xi, 0, \omega) q_{40}(\xi)d\xi, \ i = 1, 3, 4, \ k_{i4}^{(n)}(s, 0, \omega) = \int k_{i4}^{(n)}(\alpha, 0, \omega)e^{-i\omega \alpha} d\alpha . \] (22)

### 6. The results discussion

We consider the boundary problems (1)-(8) of a cadmium sulfide (CdS) layer coupled to a magnesium oxide (MgO) half-space vibrations, which are induced by heat flux \( q_{40} \) distributed in the region of \([-1,1]\) on the surface of the medium. The initial pre-stress state created by using biaxial initial deformations and preheating conditions. Below in Table 1 materials physical characteristics are described.

Figures 1 show the effect of preheating on the phase velocities differences \( dV_f = V_f - V_f^{\theta} \) depending on the biaxial along \( x_1, x_2 \) axis initial strain of the thermoelastic layered half-space. There \( V_f^{\theta} \) and \( V_f \) are phase velocities of the first mode in the presence of prestressing and in the natural state, respectively. Hereinafter, the following designations of layer preheating are introduced: a solid line \( d\theta = 0.5 \) (150 K), a dotted line \( d\theta = 0.3 \) (100 K), a dashed line \( d\theta = 0.1 \) (30 K).

|                | 10^10 [N/m^2] | 10^6 [N/K/m^2] | [W/K/m] | [J/kg/K] | 10^5 [kg/m^3] |
|----------------|---------------|---------------|---------|----------|--------------|
| **CdS**        |               |               |         |          |              |
| \( C_{11} \)   | 9.07          | 0.709         |         |          |              |
| \( C_{33} \)   | 9.38          | 0.724         |         |          |              |
| **MgO**        |               |               |         |          |              |
| \( C_{11} \)   | 30            | 4.49          |         |          |              |
| \( C_{33} \)   | 30            | 4.49          |         |          |              |
| **Physical Characteristics** | | | | | | |
ω \, dV \quad \text{and} \quad \omega \, dV_f 
\begin{align*}
a) \quad \nu_1 = \nu_2 = 0.999 & \quad \text{b) } \nu_1 = \nu_2 = 1.001 \\
c) \quad \nu_1 = \nu_2 = 1
\end{align*}

Figure 1. The influence of preheating and initial strain on differences of phase velocity.

As we can see from figures 1 for all pre-stressing modes there are maximum of the phase velocity differences, as well as frequency ranges at which the frequency value have no affects the value considered. Moreover, with compression and simple pinching of the body, there are two frequencies at which preheating does not influence on phase velocities. In the case of stretching, one of these frequencies disappears. The initial stretching along $x_1, x_2$ axis increases the difference between the phase velocities of a heated and unheated body, and compression reduces it.

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