A non-singular universe with vacuum energy

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Abstract

We construct a model for the universe based on the existence of quantum fields at finite temperature in the background of Robertson-Walker spacetime in presence of a non-zero cosmological constant. We discuss the vacuum regime in the light of the results obtained through previous studies of the back-reaction of massless quantum fields in the static Einstein universe, and we argue that an adiabatic vacuum state and thermal equilibrium is achieved throughout this regime. Results shows that such a model can explain many features of the early universe as well as the present universe. The model is free from the basic problems of the standard Friedmann cosmology, and is non-singular but involves a continuous creation of energy at a rate proportional to the size of the universe, which is lower than that suggested by the steady-state cosmology.
I. INTRODUCTION

The Friedmann models for the universe which were deduced from the Einstein field equations with a vanishing cosmological constant and vanishing covariant derivative of the energy-momentum tensor, described an expanding universe that starts from a singularity at \( t = 0 \). This develops into a universe that is either: (a) ever expanding with acceleration if the curvature is negative (\( k = -1 \)), or (b) ever expanding with an ultimate constant speed if the geometry of the spatial section is flat (\( k = 0 \)), or (c) expanding until reaching a maximum radius, then collapsing under its own gravitational field to an ultimate singularity, a model which is described as a closed spacetime with positive curvature (\( k = 1 \)). These models found their applications in two main observational achievements of cosmology during the last century; the Hubble discovery of the redshift-distance relationship, and the discovery of the Cosmic Microwave Background radiation (CMB).

The existence of a homogenous and isotropic CMB was a prediction of the Gamow and collaborators’s big bang theory, which has assumed that the universe started as a homogeneous and isotropic distribution of particles in thermal equilibrium at very high temperatures. As the universe expanded and cooled down, massive particles decoupled from thermal equilib-
rium. Accordingly radiation was released as electrons combined with hydrogen and helium nuclei to form atoms. These radiations are believed to have left a relic which is the detected CMB. This scenario was further developed and refined and was called the Standard Big Bang model (SBB).

The SBB utilized statistical physics, particle physics and the standard general relativity in which particles were not considered in their field theoretic description, but rather were dealt with through their general phenomenological description. On the other hand the SBB model did not start with the universe at $t = 0$ since physics cease to work in such a limit. For these reasons some basic questions remained unanswered; the existence of the initial singularity posed a challenge for the physicists to define the initial conditions for the universe. Indeed it was only assumed that the universe originated from vacuum without giving any explanation for such a birth. Quantum effects were not investigated within the standard model and therefore, some important contribution is missing from the SBB model.

Recent analysis of the main features of the CMB suggested that the universe is highly homogeneous and isotropic on very large scale and that, if it is to be described geometrically by the Friedmann solutions, then it has to be nearly flat (i.e. $k = 0$) \[1\]. This result was confirmed by subsequent observations from WMAP by Spergel et al. \[2\] and also later by Dunkley et al. \[3\]. This conclusion was based on the fact that direct observational data suggests that the total matter and radiation density in the universe is about the same as the critical density needed to flatten the universe. On the other hand, it well known that the SBB model suffers from some basic shortcomings, these are: the horizon, the flatness, the magnetic monopoles and the formation of large cosmic structures problems. To resolve these problems Guth \[4\] was the first to suggest that the universe may have experienced a state of inflation during very early times of its development, such that the flat geometry of the universe was attained at a very early stage. This theory was further developed into a main trend in cosmology where a diversity of inflationary models were suggested (for a comprehensive review see \[5\]). A model of inflation typically amounts to choosing a form for the potential, perhaps supplemented with a mechanism for bringing inflation to an end, and perhaps may involve more than one scalar field. In an ideal world the potential would be predicted from fundamental particle physics, but unfortunately there are many proposals for possible forms. It has become customary to assume that the potential can be freely chosen, and then one seek to constrain it with observations.
However, some authors [6] argue that explanations provided by inflation for the homogeneity, isotropy, and flatness of our universe are not satisfactory, and that a proper explanation of these features will require a much deeper understanding of the initial state of our universe. On the other hand, and although inflationary models are spectacularly successful in providing an explanation of the deviations from homogeneity, these authors point out that the fundamental mechanism responsible for providing deviations from homogeneity, namely, the evolutionary behavior of quantum modes with wavelength larger than the Hubble radius, will operate whether or not inflation itself occurs. However, if inflation did not occur, one must directly confront the issue of the initial state of modes whose wavelength was larger than the Hubble radius at the time at which they were born. Under some simple hypotheses concerning the "birth time" and initial state of these modes (but without any fine tuning), it is shown that non-inflationary fluid models, in the extremely early universe, would result in the same density perturbation spectrum and amplitude as in inflationary models, although there would be no "slow roll" enhancement of the scalar modes [6]. Other authors believe that inflationary theories are incomplete since it does not deal with basic puzzles such as the initial singularity, nor with dark energy indicated by the recent observations [7].

Ozer and Taha [8] devised a model universe free from the basic problems of the SBB model by assuming a universe predominantly kept at the state of critical density and, accordingly the scale factor of their model is given by

\[ R^2(t) = R^2_0 + t^2. \]  

(1)

On the other hand Chen-Wu [9] adopted the prescription that the cosmological constant varies like \(1/a^2(t)\). Their study resulted in a model that has the same form for the variation of the scale factor as in the SBB model, therefore would have no problem integrating the Gamow explanation of natural abundance, but it would again lead to a singular universe this time with continuous particle creation at a rate comparable to that suggested by the steady state theory. The Chen-Wu suggestion was shown to explain the cosmological constant problem through a phenomenological approach. The good feature of the Ozer and Taha model is the fluent removal of the SBB problems without the need for an inflation stage. However it remains that the assumption of having a universe starting up with a density exactly equal to the critical density will surely need justification.

It is quite possible that the scale factor do not follow a monotonic behavior during the
whole history of the universe, rather it is quite expectable that it follows different schemes during the different stages of the development of the universe. It is certain now that the universe has passed through many phase-transition states, that had different variations of the scale factor. If so, then one can say that the Friedmann models will stand as a simplified picture for the development of the universe and that it is good for describing the basic theme only. The generally accepted scenario now stipulates that there are at least three regimes: the vacuum-dominated regime which can be described by the control of the cosmological constant, the radiation-dominated regime during which massive particles were in an ultra-relativistic state in thermal equilibrium with radiation, and the matter-dominated regime which developed when particles attained non-relativistic states and settled to form a dust-like fluid.

In this paper we construct a model for the universe with the background geometry of Robertson-Walker metric endowed with quantum fields at finite temperature. We assume the presence of a non-zero cosmological constant and seek the solution of the Einstein field equation. Therefore, the present model does not satisfy the standard Friedmann paradigm, and in order to construct the model we utilize the results of previous studies and calculations of the vacuum energy density in the static Einstein universe. The universe is shown to have a violent start from a non-singular Plank sized patch developing through the interaction of vacuum energy and curvature into a very hot spot and then transiting smoothly into a thermal universe that coasts into the present one. The model is free from the standard problems of the SBB based on the standard closed Friedmann model including the initial singularity which get smoothed-out by the quantum-vacuum effects.

II. THE STATIC EINSTEIN UNIVERSE

The investigation of the back-reaction effect of massless quantum fields at finite temperatures in the background of the static Einstein universe resulted in a relationship between the radius of the Einstein universe and its temperature showing some interesting features [11]. First, all solutions were shown to possess two regimes, the vacuum (Casimir) regime, through which the temperature rises from zero sharply reaching a maximum value of order of $10^{32}K$ within a very small change of radii, and a Planck regime, through which the temperature decays exponentially to zero following a Planckian behavior (see Fig. 1). Second, it was
shown that the universe has no singular state; no static Einstein universe can be singular, rather the ground state is at zero temperature with a non-zero radius of order of Planck length. Third, a background (Tolman) temperature was calculated for states of Einstein universe with large radii, and it was found that the background temperature equals the observed value of 2.73 K at a radius nearly two order of magnitudes as compared to the present Hubble length.

It is well known the static Einstein universe is a solution of the field equations with a cosmological constant, that can always be adjusted to balance the gravitational attraction of matter contained within the spatial section of the universe. The study of the variation of the value of the cosmological constant with temperature for successive states of the Einstein universe resulted in showing that the cosmological constant has large and nearly constant value during the Casimir regime, decaying according to the inverse square law ($\Lambda \sim 1/a^2$) during the Planck regime (see Fig. 2). The exceptional case was with the minimally coupled massless scalar field, where the cosmological constant was shown to have an infinite value at start, decaying exponentially to small values already within the Casimir regime [12].

III. A TIME-DEPENDENT COSMOLOGICAL CONSTANT

The above mentioned results are quite motivating to consider a more realistic case of a time-dependent model in which the cosmological constant is changing with the scale factor according to the inverse square law.

In order to include any and all sources of energy that would contribute to the total energy of the universe, let us write the Friedmann equations in the following form

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} (\rho_{m,r})_{\text{eff}} + \frac{\lambda_{\text{eff}}}{3},$$

where $(\rho_{m,r})_{\text{eff}}$ is the energy density for matter and radiation, and $\lambda_{\text{eff}}$ represents any and all contributions coming from the cosmological constant or any other source of energy-momentum density including the vacuum energy density. In other form the Friedmann equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} (T^0_0)_{\text{total}} + \frac{\lambda}{3},$$
where \((T^0_0)_{\text{total}} = \rho_{m,r} + \rho_v\), with \(\rho_v = \langle 0|T^0_0|0\rangle_{\text{tot}}\) is the total vacuum energy density. Eq. (3) can be re-written as

\[
6 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} - \frac{8}{3} \pi \left( 2\rho_{m,r} - \frac{3}{8\pi} \left( \frac{\dot{a}}{a} \right)^2 \right) - \frac{2\lambda}{3} \right] = 32\pi \langle 0|T^0_0|0\rangle_{\text{tot}} - \frac{6k}{a^2}, \tag{4}
\]

Now if

\[
\left[ 2\rho_{m,r} - \frac{3}{8\pi} \left( \frac{\dot{a}}{a} \right)^2 \right] = (\rho_{m,r})_{\text{eff}} = \rho_c \tag{5}
\]

and

\[
2\lambda = \lambda_{\text{eff}}, \tag{6}
\]

then we obtain

\[
6 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} - \frac{8}{3} \pi (\rho_{m,r})_{\text{eff}} - \frac{\lambda_{\text{eff}}}{3} \right] = 32\pi \langle T^0_0 \rangle_{\text{tot}} - \frac{6k}{a^2}. \tag{7}
\]

Comparing the left hand side of (7) with (2) we get

\[
\frac{6k}{a^2} = 32\pi \langle 0|T^0_0|0\rangle_{\text{tot}}. \tag{8}
\]

for \(k = 1\) this will give the same result as that obtained in the case of closed static universe \([11]\).

From the basic Einstein field equation and for the Robertson-Walker metric we can deduce that

\[
T^\mu_{\ 0 ; \mu} = \dot{\rho} + 3 \left( \frac{\dot{a}}{a} \right) (\rho + p). \tag{9}
\]

Applying the covariant derivative to the Einstein field equation we obtain

\[
\dot{\rho} + 3 \left( \frac{\dot{a}}{a} \right) (\rho + p) = -\frac{1}{8\pi} \dot{\lambda}. \tag{10}
\]

This equation will stand as a replacement for the equation of state used by the standard Friedmann models where the right-hand side is taken to vanish on the assumption that \(\lambda\) is constant.

The second law of thermodynamics requires that
\[
dE + pdV = TdS,
\]
which means that
\[
\frac{dE}{dt} + p \frac{dV}{dt} = T \frac{dS}{dt},
\]
where \( E = m_0c^2 = (\rho V), \) and \( V = 2\pi^2 a^3 \) is the volume of the closed universe. Therefore
\[
\frac{dE}{dt} = 6\pi^2 a^2 \rho \frac{da}{dt} + 2\pi^2 a^3 \frac{d\rho}{dt},
\]
so that for Eq. (11) we obtain
\[
V \left[ \dot{\rho} + 3 \left( \frac{\dot{a}}{a} \right) (\rho + p) \right] = T \frac{dS}{dt},
\]
where the dot denotes differentiation with respect to time. Substituting (13) in (10) we get
\[
TdS = -\frac{V}{8\pi} d\lambda.
\]
Therefore, a variable \( \lambda \) may solve the entropy problem without the introduction of specific fields or irreversible processes. The idea that \( \lambda \) may be variable has previously been suggested by several authors \[8], \[12\] and \[17\].

For the homogeneous isotropic model with Robertson-Walker metric with \( \rho = \rho(t), \) \( p = p(t) \) and \( \lambda = \lambda(t), \) Eq. \[2\], \[5\] and \[6\] yield
\[
\left( \frac{\dot{a}}{a} \right)^2 = \alpha^{-1} (\rho_{m,r} (t) + \Lambda(t)) - \frac{k}{2a^2},
\]
where \( \alpha = (3/8\pi) \) and \( \Lambda = (\lambda/8\pi). \)

Thus \( dS/dt \geq 0 \) requires that \( d\Lambda/dt \leq 0 \) for all \( t \geq 0. \) Then From Eq. (15) we may interpret \( \Lambda(t) \) as a vacuum or cosmological energy density.

Now from (11) and (14) we get
\[
\frac{d(\rho V)}{dt} + p \frac{dV}{dt} + \frac{V}{8\pi} \frac{d\lambda}{dt} = 0,
\]
which means that
\[
\frac{d (\rho a^3)}{dt} + p \frac{da^3}{dt} + a^3 d\Lambda = 0. \tag{16}
\]

Taking \( p = \frac{1}{3} \rho \) for radiation filled universe we get

\[
\frac{d\rho}{da} + \frac{d\Lambda}{da} + \frac{4\rho}{a} = 0. \tag{17}
\]

To solve this equation we consider the general form

\[
\frac{dy}{dx} + P(x) y = Q(x), \tag{18}
\]

where the solution is given by

\[
y = \exp[-I(x)] \int Q(x) \exp[I(x)] dx + b \exp[-I(x)], \tag{19}
\]

where

\[
I(x) = \int P(x) dx, \tag{20}
\]

and \( b \) is a constant. Here we have \( P(a) = \frac{4}{a} \) so that

\[
I(a) = \int_{a_0}^{a} P(a) da = \ln \left( \frac{a}{a_0} \right)^4 \tag{21}
\]

Also

\[
Q(a) = -\frac{d\Lambda}{da}. \tag{22}
\]

consequently we obtain the solution

\[
\rho = \rho_0 \left( \frac{a_0}{a} \right)^4 - \left( \frac{1}{a} \right)^4 \int_{a_0}^{a} (a')^4 \frac{d\Lambda}{da'} da'. \tag{23}
\]

We shall take \( \rho_0 \) and \( a_0 \) to be the values of the energy density and scale factor at \( t = 0 \).

If the total energy of the universe, including the vacuum energy, is to be constant then we have to take \( dE = 0 \), which implies that

\[
TdS + a^3 d\Lambda = 0. \tag{24}
\]

One observes that the conditions \( \dot{\Lambda} \leq 0 \) and \( \dot{a} \geq 0 \) imply that the integral in (23) is negative so that the decrease of \( \Lambda \) as \( a \) increases generates a positive contribution to \( \rho \). Thus
the cosmological energy density $\Lambda$ is continually depleted and transformed into radiation energy density in accordance with (23). This consolidates the conjecture of Altaie and Setare [12].

Following [8] we take the initial time $t = 0$ to be the moment when $a = 0$. Then if $\rho_0 = 0$, Eq. (23) is still valid and the empty curved space-time metric which is governed by

$$R^\mu_\nu - \frac{1}{2} g^\mu_\nu R = -8\pi [T^\mu_\nu]^{(\text{vac})}, \quad t = 0, \quad \text{(25)}$$

soon will generate a non-empty universe, i.e., one with non-vanishing radiation density $\rho$. One might thus take Eq. (23) to imply that radiation is being created out of the space-time curvature all times. The physical picture one has is therefore the one in which the universe is continuously created by the unfolding of space-time curvature. This is different from the continuous creation in the Bondi-Gold-Hoyle [18] model which was suggested in the context of steady-state cosmology. The steady-state model is also based on classical general relativity with modified $T^\mu_\nu$. The modification is by the addition of a covariant term which was chosen so that $\rho$, $p$ and $H$ remain constant throughout the de Sitter expansion of the universe. Continuous creation and the absence of an initial singularity are features of steady-state cosmology that are shared by the present model.

In the present formulation the cosmological energy density $\Lambda$ is related to entropy by

$$TdS + a^3 d\Lambda = 0. \quad \text{(26)}$$

This equation may in fact be interpreted as an expression of the constancy of the total entropy of the cosmos, substance and spacetime, i.e.,

$$dS + dS_c = 0, \quad \text{(27)}$$

where

$$dS_c = \frac{a^3}{T} d\Lambda, \quad \text{(28)}$$

is the change in the entropy of the curved spacetime. On the long run ($t \to \infty$) this change flattens the spacetime. Under these conditions one would intuitively expect a decrease in the entropy $S_c$ since the number of degrees of freedom one might associate with a state of high curvature should be larger than those associated with an almost flat space.
We note that in a universe of pure radiation the choice of \( \Lambda \) completely determines the model, since \( \Lambda \) determines \( \rho \) by Eq. (23), leaving Eq. (15) to be solved for \( a(t) \). However when, in addition to radiation matter is present, extra assumptions are needed to uniquely determine the model. In this case \( \rho = \rho_m + \rho_r \), where \( \rho_m \) is the rest-mass energy density and \( \rho_r \) is the energy density of radiation and relativistic matter (to be referred simply as radiation in what follows). Then Eq. (16) reads

\[
d (\rho_m a^3) + d (\rho_r a^3) + p da^3 = -a^3 d\Lambda. \tag{29}
\]

A plausible assumption that may readily be made is that the processes responsible for changes in rest-mass energy density are, except for the matter creation period, much slower than those responsible for creation of radiation, (see [8]); i.e.,

\[
\left| \frac{d}{dt} (\rho_m a^3) \right| \ll \left| \frac{d}{dt} (\rho_r a^3) \right|. \tag{30}
\]

We also make the assumption that the pressure of the universe, under these conditions, is caused by its radiation

\[
p = p_r = \frac{1}{3} \rho_r. \tag{31}
\]

Eq. (16) then yield

\[
d (\rho_r a^3) + \frac{1}{3} \rho_r da^3 = -a^3 d\Lambda, \tag{32}
\]

and

\[
d (\rho_m a^3) \approx 0, \tag{33}
\]

implying that Eq. (16) may be assumed to be approximately valid for the radiation component in both the radiation-dominated as well as the matter-dominated eras. This may alternatively be interpreted in the sense that a substance is created, by the unfolding of curved spacetime, as massless radiation. Yet another interpretation could be that the change in entropy of non-relativistic matter is much less than that of radiation, i.e., \( \Delta S_m \ll \Delta S_r \), since \(-a^3 d\Lambda\) is a measure of the total change in the entropy.
IV. THE MODEL

We now deduce a particular function for $\Lambda$, thereby defining a specific cosmological model in the classical class. Our starting point is the observation that the present value of the energy density of the universe is close to its critical value $\rho_c$. Theoretically the investigations of the back-reaction effects of quantum fields at finite temperatures indicates that the density of the universe was always fixed at the critical value

$$\rho_c (t) = \frac{3H^2}{8\pi} = \frac{3}{8\pi} \left( \frac{\dot{a}}{a} \right)^2.$$  \hspace{1cm} (34)

From Eq. (15) we notice that the condition $\rho = \rho_c$ requires that

$$\left( \frac{\dot{a}}{a} \right)^2 = \alpha^{-1} (\rho_c + \Lambda (t)) - \frac{k}{2a^2}$$

$$= \alpha^{-1} \left( \alpha \left( \frac{\dot{a}}{a} \right)^2 + \Lambda (t) \right) - \frac{k}{2a^2}. \hspace{1cm} (35)$$

So that $\Lambda(t)$ comes to be

$$\Lambda (t) = \frac{\alpha k}{2a^2}. \hspace{1cm} (36)$$

The conditions $\dot{\Lambda} \leq 0$ and $\dot{a} \geq 0$ then immediately give $k \geq 0$ so that $k = 1$. This is a significant deviation from the standard model where $\rho = \rho_c$ implies $k = 0$. This is an important feature of the present model, therefore, we can confidently conclude that

$$\Lambda (t) = \frac{\alpha}{2a^2}. \hspace{1cm} (37)$$

The model now is completely specified in its dynamical structure. The rest of the work will mostly run in similar fashion to that of Ozer and Taha with a difference by a factor of 2 in some equations. However, as for the initial start of the universe we here do not need to follow the assumption of Ozer and Taha but would rather resort to take the results of previous works which defined for us the initial radius of the universe as a result of self-consistency condition applied on the Einstein field equations for a given quantum field source. The provisions of an equation of state determines the physical content. In the following sections we present the results of the calculations with some outlines.
A. The very early universe

In our model the initial moment $t = 0$ has been chosen to coincide with the state of the universe when its energy content is specified by the presence of Casimir energy resulting from the high initial curvature. Self-consistency of the Einstein field equation requires the universe to have a non-zero radius at $T = 0$. This was already calculated in a previous work (see [15]). There are, initially, no such excitations since $\dot{a}(0) = 0$ and the total energy is locked up in potential form in spacetime curvature. With $\rho_0 = 0$, Eqs. (23) and (37) yield

$$\rho = \frac{\alpha}{2a^2} \left( 1 - \frac{a_0^2}{a^2} \right), \quad (38)$$

Note that the condition $\rho_0 = 0$ requires $a_0 \neq 0$. This implies that all functions $a, \rho, S, T$ are finite at $t = 0$ (as well as for all finite $t$ as will soon become clear) so that the initial singularity of the standard model does not exist.

Eq. (38) with $\rho = \rho_c = \alpha\left(\frac{a}{a}\right)^2$, may now be solved for $a(t)$ giving

$$\frac{ada}{[a^2 - a_0^2]^{1/2}} = \left(\frac{1}{2}\right)^{1/2} dt. \quad (39)$$

This can be easily integrated to give

$$a(t) = \left[a_0^2 + \frac{t^2}{2}\right]^{1/2}. \quad (40)$$

It is clear that the universe starts accelerating during the Casimir regime and then soon get to an ultimate speed when it reaches the velocity of light. In this model we have no inflation but a direct parametric dependence of the radius on time. Note that in this solution of the field equations $a \to \infty$ as $t \to \infty$ i.e., the model is continuously expanding although $k = 1$. This is due to the variable cosmological energy density $\Lambda$ in Eq. (35) which renders the characterization of the asymptotic behavior being $\Lambda$—dependent. Different choices of $\Lambda$, for fixed $k$, can give different types of asymptotic behavior. The intrinsic geometry, for fixed $t$, is governed by the parameter $k$ and is independent of $\Lambda$.

The time-dependence of all functions in the model is completely determined by Eqs. (38) and (40). For the radiation energy density we have
\[ \rho(t) = \frac{\alpha}{2} \left[ \frac{(t/\sqrt{2})^2}{\left(a_0^2 + (t/\sqrt{2})^2\right)^2} \right] \] (41)

The radiation temperature \( T \) is assumed to be related to \( \rho \) by

\[ \rho(T) = \frac{1}{30} \pi^2 N(T) T^4 \] (42)

The effective number of spin degree of freedom \( N(T) \) at temperature \( T \) is given by \( N(T) = N_b(T) + \frac{2}{3} N_f(T) \), where \( N_b(T) \) and \( N_f(T) \) refer to bosons and fermions respectively.

We take \( N(T) \) to be constant throughout the pure radiation era. From Eqs. (41) and (42) we obtain

\[ T(t) = \left( \frac{15\alpha}{\pi^2 N} \right)^{1/4} \left[ \frac{t^2/2}{\left(a_0^2 + t^2/2\right)^2} \right]^{1/4} \] (43)

This is qualitatively the same result that was obtained by Ozer and Taha. In terms of the radius \( a \) the dependence of \( T \) is given by

\[ T(a) = \left( \frac{15\alpha}{\pi^2 N} \right)^{1/4} \left[ \frac{a^2 - a_0^2}{a^4} \right]^{1/4} . \] (44)

Thus, according to this model the universe have a cold start since \( T = 0 \) at \( t = 0 \). For small \( a_0 \), this need, however, not to be different from a hot universe since temperature increases rapidly within a time-scale of order \( a_0 \). Fig. 3 shows the qualitative time development of the temperature of the universe \( T \) according to Eq. (44). Also, from Eqs. (37) and (40) we get Fig. 4, which illustrates a qualitative relationship between \( T \) and \( \Lambda \).

The maximum temperature \( T_{\text{max}} \) is obtained at \( t = \sqrt{2}a_0 \) and is given by

\[ T_{\text{max}} = \left( \frac{1}{2} \right)^{1/4} \left( \frac{15\alpha}{2\pi^2 Na_0^2} \right)^{1/4} , \] (45)

For \( t \gg a_0 \), Eqns. (41) and (43) gives

\[ \rho = \frac{\alpha}{t^2} \] (46)

and

\[ T = \left( \frac{30\alpha}{\pi^2 N} \right)^{1/4} \left[ \frac{1}{t} \right]^{1/2} . \] (47)
These equations are to be compared to those of the standard model, namely

\[ \rho_{SM} = \frac{\alpha}{(2t)^2}, \]  
\[(48)\]

and

\[ T_{SM} = \left( \frac{30\alpha}{\pi^2 N} \right)^{1/4} \left[ \frac{1}{2t} \right]^{1/2}. \]
\[(49)\]

Thus for \( t \geq a_0 \) the values of the energy density and temperature attained at a time \( t \) in the standard model are attained at time \( 2t \) in the present model. For \( t \approx a_0 \), Eqs. (41) and (43) coincide with Eqs. (48) and (49) of the standard model. One may, therefore, conclude that although this model is clearly different from the standard model in several aspects, such as having cold initiation, non-adiabaticity and regularity at \( t = 0 \), it possesses for \( T \geq T_{\text{max}} \) essentially the same thermal history as the standard model. This is somewhat surprising since the dependence of \( a \) on \( t \), in the present model, is completely different from that of the standard model: \( a_{SM} \sim t^{1/2} \). On the other hand we should, however, expect some substantial deviation in the time-dependence as compared to the standard model if the cosmological problems of the standard model are to be avoided.

If we have to look at the variation of the cosmological constant with the temperature of the universe, then we will obtain the temperature dependence depicted in Fig. 4. The figure shows a dependence which is similar to that we obtained for the Einstein universe for conformally coupled scalar field.

In particular, the dependence of the cosmic scale factor \( a \) on \( t \) determines the causal structure of the model. The horizon distance \( d_H (t) \) at time \( t \) is the proper distance travelled by light emitted at \( t = 0 \)

\[ d_H (t) = a(t) \int_0^t \frac{dt}{a(t)} \]  
\[(50)\]

For the universe around us to be causally-connected to us at cosmic time \( t \) it is necessary that \( d_H (t) \geq d_{\text{proper}} (t) \) where \( d_{\text{proper}} (t) \) is the proper distance, at time \( t \) between our galaxy and another galaxy most distant from us assuming that our galaxy is at \( r = 0 \).

\[ d_{\text{proper}} (t) = a(t) \int_0^{r_{\text{max}}} \frac{dr}{\sqrt{1 - kr^2}} = a(t) \begin{cases} r_{\text{max}}, & k = 0 \\ \sinh^{-1} r_{\text{max}}, & k = -1 \\ \sin^{-1} r_{\text{max}}, & k = 1 \end{cases} \]  
\[(51)\]
For \( k = 0 \) and \( k = -1 \) the universe is spatially infinite so that \( r_{\text{max}} = \infty \). This implies that for \( k = 0 \) and \( k = -1 \) global causal connection, i.e. causal connection for the whole space, is never established at any finite time \( t \). The region of the universe which is causally connected at time \( t \) is limited in coordinate space to \( 0 \leq r \leq r_H(t) \), where

\[
\int_0^{r_H(t)} \frac{dr}{\sqrt{1 - kr^2}} = \int_0^t \frac{dt'}{a(t')}. \tag{52}
\]

The horizon (or causality) problem for \( k = 0 \) and \( k = -1 \) in the standard model may be formulated only for the currently observed universe, i.e. in non-global terms.

For \( k = 1 \) the universe is spatially finite and \( r_{\text{max}} = 1 \). It is then possible to determine the time \( t = t_{\text{caus}} \) when the whole universe is causally connected. This is given by

\[
\int_0^{t_{\text{caus}}} \frac{dt'}{a(t)} = \int_0^1 \frac{dr}{\sqrt{1 - kr^2}} = \frac{\pi}{2}. \tag{53}
\]

For the standard model with \( k = 1 \), one finds that by the end of the radiation-dominated era, \( t_r = 10^{12} \) s, only a small part of the whole space is causally connected. To see this

\[
\int_0^{t_r} \frac{dt}{a(t)} = 2 \left[ \frac{t_r}{A^2} \right]^{1/2}. \tag{54}
\]

Using \( a = At^{1/2} \) in the radiation-dominated era of the standard model where \( A = (2\pi^2 N / 15\alpha)^{1/4} a_p T_P \). For \( N = 100, a_p = 10^{10} \) years and \( T_P = 2.7 \)K , the right-hand side of (54) is 0.014 so that Eq. (52) with \( k = 1 \) gives \( r_H(t_r) = \sin 0.014 \approx 0.014 \ll 1 \). Global causal connection, i.e. up to \( r = 1 \), is realized during the matter-dominated era as may be verified using (53). Thus most of the currently observable universe was not in causal contact at the end of the radiation-dominated era. One is then unable to explain the observed isotropy of the background black-body radiation.

In the present model Eqs. (40) and (53) determine the time \( t_{\text{caus}} \) when global causality is established. One finds

\[
\int_0^{t_{\text{caus}}} \frac{dt'}{\left( a_0^2 + \left( t/\sqrt{2} \right)^2 \right)^{1/2}} = \frac{\pi}{2}, \tag{55}
\]

Now let

\[
\frac{t}{\sqrt{2}} = a_0 \sinh w \quad \Rightarrow \quad dt = \sqrt{2}a_0 \left( \cosh w \right) dw,
\]
then Eq. (55) becomes

$$\int_{w_{\text{caus}}}^{\infty} \sqrt{2a_0 \cosh w} \; dw = \int_0^{\infty} \frac{2a_0 \cosh w}{\left(a_0^2 + (a_0 \sinh w)^2\right)^{1/2}} \; dw = \frac{\pi}{2},$$

which then would yield

$$t_{\text{caus}} = \sqrt{2}a_0 \sinh \left(\frac{\pi}{2\sqrt{2}}\right) = 1.9a_0.$$  \hfill (57)

Note that for the integral in (53) to converge it is necessary to have $a_0 \neq 0$. Eq. (57) indicates that global causal connection in the present model has been established at a very early time. Thus the present model does not possess a horizon problem.

We observe that the “cold era” in this model is restricted to an interval when the causally connected part of the universe covers a tiny fraction of the whole space. When global causality is established at $t_{\text{caus}} = 1.9a_0$ the maximum temperature is surpassed and the whole universe attains the temperature $T_{\text{caus}}$. From Eq. (45), we have:

$$T_{\text{max}} = \left(\frac{15\alpha}{4\pi N}\right)^{1/4} \left(\frac{1}{a_0^2}\right)^{1/4},$$

also with the help of Eqs. (45) and (57), we get:

$$T_{\text{caus}} = 0.6916 \left(\frac{15\alpha}{\pi^2 N}\right)^{1/4} \left(\frac{1}{a_0^2}\right)^{1/4}.$$ \hfill (59)

this shows that

$$T_{\text{caus}} = 0.978T_{\text{max}}$$  \hfill (60)

**B. Radiation and matter**

The very early pure radiation era soon gives way to a period of matter generation. Throughout this period, $a_1 \leq a \leq a_2$ say, Eq. (29) is valid. For $a \geq a_2$ Eqs. (32) and (33) hold, so that the total rest-mass energy

$$E_m = \rho_m a^3,$$ \hfill (61)
remains approximately constant, i.e.,

$$\rho_m = \frac{E_m}{a^3}. \tag{62}$$

Note that although the proper volume is $2\pi^2a^3$ we shall, in accordance with general convention, take it to be simply $a^3$. This is of no consequence since the measurable quantity is the energy density.

The solution to Eq. (32) for $\rho_r$ with $\Lambda$ given by Eq. (37) is

$$\rho = \rho_0 \exp \left[ -\ln \left( \frac{a}{a_p} \right)^4 \right] + \exp \left[ -\ln \left( \frac{a}{a_p} \right)^4 \right] \int_{a_p}^{a} \left( \frac{\alpha}{a_p} + \frac{E_m}{a} \right) \exp \left[ \ln \left( \frac{a'}{a_p} \right)^4 \right]. \tag{63}$$

Now, if $\rho = \rho_0$ when $a = a_0$, then $\rho$ becomes

$$\rho = \left( \frac{a_p}{a} \right)^4 \int_{a_p}^{a} \left( \frac{\alpha}{a^3} + \frac{E_m}{a^4} \right) \left( \frac{a'}{a_p} \right)^4 \, da'$$

$$= \frac{\alpha}{2a^2} \left[ 1 + \left( -1 - \frac{2E_m}{\alpha a_p} \right) \left( \frac{a_p^2}{a^2} \right) \right] + \frac{E_m}{a^3}$$

$$= \frac{\alpha}{2a^2} \left[ 1 + \omega \left( \frac{a_p^2}{a^2} \right) \right] + \frac{E_m}{a^3}. \tag{64}$$

so that

$$\rho - \frac{E_m}{a^3} = \frac{\alpha}{2a^2} \left[ 1 + \omega \left( \frac{a_p^2}{a^2} \right) \right]. \tag{65}$$

Accordingly

$$\rho_r = \frac{\alpha}{2a^2} \left[ 1 + \omega \left( \frac{a_p^2}{a^2} \right) \right], \quad a \geq a_2, \tag{66}$$

where $\omega$ is a dimensionless constant and $a_p$ is the present value of the scale factor. Note that although Eq. (32) for $\rho_r$ is the same as Eq. (16), its solution (66) is not the analytic continuation of $\rho$ in Eq. (38). The reason is that the system is subject to different equations of state in the two regions: $p = \frac{1}{3}\rho$ for $a_0 \leq a \leq a_1$ and $p = \frac{1}{3} \left( \rho - \frac{E_m}{a^3} \right)$ for $a \geq a_2$. These two regions do not, therefore, belong to the same phase. The region $a_1 \leq a \leq a_2$ during which rest-mass is created may, therefore, be thought of as a region of phase transition. This will be further discussed in the next section.

When $\omega$ is expressed in terms of present values of $\rho_r$ and $a$ one obtains
\[ \omega + 1 = 2\alpha^{-1}a_p^2\rho_p^p = \rho_r^p/\rho_p^p. \] (67)

The total radiation energy \( E_r = \rho_r a^3 \) is then

\[ E_r \approx \frac{\alpha}{2}a + \frac{\alpha\omega}{2}a^2, \quad a \geq a_2. \] (68)

For matter dominance to occur there must exist a value of \( a, a = a_{eq} \) say, such that

\[ E_r(a_{eq}) = E_m^p, \] (69)

where for all \( a \geq a_2, \)

\[ E_r(a) \geq E_m^p \quad \text{when} \quad a \leq a_{eq}. \] (70)

This requires that for \( a_2 \leq a \leq a_{eq} \) and at least in the neighborhood of \( a_{eq}, a \) must be small enough for \( E_r \) to be decreasing at \( a = a_{eq} \) so that

\[ \omega \geq \left( \frac{a_{eq}}{a_p} \right)^2. \] (71)

The value of \( a_{eq} \) is given by

\[ a_{eq} = \frac{\rho_m^p a_p^3 \pm \left[ (\rho_m^p a_p^3)^2 - 4\alpha \left( 2\omega\alpha a_p^2 \right) \right]^{1/2}}{2\alpha}. \] (72)

then from Eq. (67)

\[ a_p = \frac{\alpha^{1/2} (1 + \omega)^{1/2}}{\sqrt{2} (\rho_r^p)^{1/2}} \] (73)

\[ a_{eq} = \frac{\rho_m^p \alpha^{3/2} (1 + \omega)^{3/2} / \sqrt{8} (\rho_r^p)^{3/2}}{2\alpha} \pm \frac{[(\rho_m^p)^2 \alpha^3 (1 + \omega)^3 / 8 (\rho_r^p)^3 - 8\alpha^2 \omega \alpha (1 + \omega)/2\rho_r^p]^{1/2}}{2\alpha} \]

\[ = \frac{1}{\sqrt{8}} \frac{\rho_m^p \alpha^{1/2} (1 + \omega)^{3/2}}{2 (\rho_r^p)^{3/2}} \left\{ 1 \pm \left[ 1 - 8 \frac{4\omega}{(1 + \omega)^2} \left( \frac{\rho_r^p}{\rho_m^p} \right)^2 \right]^{1/2} \right\}, \] (74)
\[ a_{eq}^1 = \frac{1}{\sqrt{8}} \rho_m^p \alpha^{1/2} (1 + \omega)^{3/2} \left\{ 1 - \left[ 1 - \frac{32 \omega}{(1 + \omega)^2} \left( \frac{\rho_p^p}{\rho_m^p} \right)^2 \right]^{1/2} \right\}. \quad (75) \]

The temperature \( T_{eq} \) at \( a = a_{eq} \) is given by

\[
\rho_p^p = \frac{1}{30} \pi^2 N (T_P) T_P^4
\]

\[
E_r^p = \rho_p^p a_p^3 = \frac{1}{30} \pi^2 N (T_P) T_P^4 a_p^3
\]

\[
E_r (a_{eq}) = \frac{1}{30} \pi^2 N (T_{eq}) T_{eq}^4 a_{eq}^3
\]

Remembering that \( E_r (a_{eq}) = E_m^p = \rho_m^p a_p^3 \), therefore Eq. (78) becomes

\[
E_m^p = \rho_m^p a_p^3 = \frac{1}{30} \pi^2 N (T_{eq}) T_{eq}^4 a_{eq}^3
\]

and

\[
\frac{\rho_m^p}{\rho_p^p} = \frac{T_{eq}^4 a_{eq}^3}{T_P^4 a_p^3}
\]

so that

\[
T_{eq}^4 = \left( \frac{\rho_m^p}{\rho_p^p} \right) \left( \frac{a_p}{a_{eq}} \right)^3 T_P^4
\]

with

\[
a_p = \frac{\alpha^{1/2} (1 + \omega)^{1/2}}{\sqrt{2} (\rho_p^p)^{1/2}}.
\]

Now from Eq. (74)

\[
\frac{a_p}{a_{eq}} = \left[ \frac{\alpha (1 + \omega)}{2 \rho_p^p} \right]^{1/2} \left[ \rho_m^p \alpha^{1/2} (1 + \omega)^{3/2} \left\{ 1 \pm \left[ 1 - \frac{(8) \times 4 \omega}{(1 + \omega)^2} \left( \frac{\rho_p^p}{\rho_m^p} \right)^2 \right]^{1/2} \right\} \right] (83)
\]

\[
\frac{a_p}{a_{eq}} = \frac{1}{1 + \omega} \left( \frac{\rho_p^p}{\rho_m^p} \right) \left\{ 1 \pm \left[ 1 - \frac{4 \omega}{(1 + \omega)^2} \left( \frac{\rho_p^p}{\rho_m^p} \right)^2 \right]^{1/2} \right\}.
\]

substituting this into (81), we have
\[ T_{eq} = \left( \frac{\rho_m^p}{\rho_r^p} \right)^{1/4} \left( \frac{\alpha_p}{\alpha_{eq}} \right)^{3/4} T_P. \]  

so that

\[ T_{eq} = T_P \left( \frac{\rho_p^b}{\rho_m^p} \right)^{1/2} (1 + \omega)^{-3/4} \left\{ 1 \pm \left[ 1 - \frac{32 \omega}{(1 + \omega)^2} \left( \frac{\rho_p^b}{\rho_m^p} \right)^2 \right]^{1/2} \right\}^{-3/4}. \]  

The radiation energy density \( \rho_p^b \) today is given by

\[ \rho_p^b = \frac{1}{30} \pi^2 N (T_P)^4, \]  

where as shown in the Appendix A, \( N (T_P) = \frac{43}{11} \) in the present model (for neutrino filled universe). The observational value of \( T_P = 2.7^\circ \) \( K \) yields

\[ \rho_p^b = 3.8 \times 10^{-51} \text{ (GeV)}^4. \]  

The total energy density today is

\[ \rho = \alpha H_P^2 \approx 4 \times 10^{-47} \text{ (GeV)}^4, \]  

where we have used \( H_P = 72 \text{ km s}^{-1} \text{ Mpc}^{-1} \). This shows that the universe is matter-dominated at present with \( \rho_m^p \approx \rho_p^b \). This feature is not permanent in the model under consideration, since a second era of radiation-dominance starts when

\[ a = a_{eq}^{1/4} = \frac{1}{\sqrt{8}} \frac{\rho_m^p \alpha^{1/2} (1 + \omega)^{3/2}}{2 (\rho_{eq}^p)^{3/2}} \left\{ 1 + \left[ 1 - \frac{32 \omega}{(1 + \omega)^2} \left( \frac{\rho_p^b}{\rho_m^p} \right)^2 \right]^{1/2} \right\}. \]  

With the ratio \( \rho_p^b / \rho_m^p \approx 1 \times 10^{-4} \) one can approximate the expressions for \( a_{eq}, T_{eq} \) and \( a_p \), obtaining

\[ a_{eq} \approx \sqrt{8} \frac{\omega}{(1 + \omega)^{1/2}} \frac{\alpha^{1/2} (\rho_p^b)^{1/2}}{\rho_m^p} \approx \frac{\omega}{(1 + \omega)^{1/2}} \times \sqrt{8} \frac{1}{2}, \]  

and

\[ T_{eq} \approx \frac{1}{\sqrt{8}} \left( \frac{1 + \omega}{\omega} \right)^{3/4} \left( \frac{\rho_m^p}{\rho_p^b} \right) T_P \approx \frac{2}{\sqrt{8}} \left( \frac{1 + \omega}{\omega} \right), \]  

and
\[ a_p \approx (1 + \omega)^{1/2} \frac{\alpha^{1/2}}{\sqrt{2} (\rho_r)^{1/2}} \approx (1 + \omega)^{1/2} \times \frac{0.7}{\sqrt{2}} \times 10^{44} \text{ (GeV)}^{-1}. \]  

(92)

From these we deduce that the \(1/a^4\) term in \(\rho_r\) is dominant at \(a = a_{eq}\), since

\[ \frac{\omega a_p^2}{a_{eq}^2} \approx \frac{(1 + \omega)^2}{\omega} \times \frac{1}{4} \times 3 \times 10^7, \]

(93)

for all \(\omega\). This implies that one has \(1/a^4\) dominance throughout \(a_1 \leq a \leq a_{eq}\) except perhaps for the matter generation periods. Dominance of \(\rho_r\) by this term extends well beyond \(a_{eq}\) to values of \(a\) such that \(a \approx 10^{-1} \omega^{1/2} a_p \gg a_{eq}\). However, to decide which of the two terms in \(\rho_r\), if any, is dominant at present would depend upon the value of the parameter \(\omega\).

One notes that with \(1/a^4\) dominance of \(\rho_r\) one approximately has \(aT\) constant as in the standard model. However this does not imply that the entropy is constant under these conditions, since the variation of \(aT\) over the whole range amounts to considerable generation of entropy. In the present model there does not exist an entropy problem in any case since the entropy is initially zero. In the pure radiation era one has

\[ \frac{dS}{da} = \frac{\alpha}{T}, \]

(94)

with

\[ \rho = \frac{1}{30} \pi^2 N T^4 = \frac{\alpha}{2a^2} (a^2 - a_0^2), \]

(95)

and

\[ T = \left( \frac{15\alpha}{\pi^2 N} \right)^{1/4} \left( \frac{1}{a} \right) (a^2 - a_0^2)^{1/4}, \]

(96)

we obtain

\[ \frac{dS}{da} = \left( \frac{\pi^2 N}{15\alpha} \right)^{1/4} \frac{\alpha a}{(a^2 - a_0^2)^{1/4}}, \]

(97)

so that

\[ S = \left( \frac{1}{8} \right)^{3/4} \left[ \frac{4}{3} \left( \frac{\pi^2 N \alpha^3}{30} \right)^{1/4} (a^2 - a_0^2)^{3/4} \right], \]

(98)

where for \(a \geq a_2\), i.e. after matter generation, the entropy will be
\[ S = \left( \frac{1}{8} \right)^{3/4} \left[ \frac{4}{3} \left( \frac{\pi^2 N \alpha^3}{30} \right)^{1/4} \left( a^2 + \omega a^2_p \right)^{3/4} \right] + \text{constant}. \quad (99) \]

This gives for the total entropy generated during the era \( a \geq a_2 \) the expression

\[ S(a_p) - S(a_2) \approx \left( \frac{1}{8} \right)^{3/4} \left[ \frac{4}{3} \left( \frac{\pi^2 N \alpha^3}{30} \right)^{1/4} \left( (1 + \omega)^{3/4} - (\omega)^{3/4} \right)^{3/4} \right], \]

which means that

\[ S(a_p) - S(a_2) \approx \left( \frac{1}{8} \right)^{3/4} \times 2 (1 + \omega)^{3/4} \left( (1 + \omega)^{3/4} - (\omega)^{3/4} \right) \times 10^{93} \quad (100) \]

It is thus seen that a lot of entropy is produced since the end of the matter generation era. We remark that although the parameter \( \omega \) occurs in all expressions for \( a_{eq} \), \( T_{eq} \), \( a_p \) and \( S(a_p) \), it is possible to eliminate it and obtain relations between these quantities. One may also obtain bounds on these quantities that hold for all \( \omega \), such as for example,

\[ a_p \geq 5 \times 10^{43} \text{ (GeV)}^{-1}. \quad (101) \]

This bound leads to an upper bound on the present value of the cosmological constant,

\[ \lambda_p \leq 6 \times 10^{-88} \text{ (GeV)}^2, \quad (102) \]

which is well within the upper limit of \( 10^{-82} \text{ (GeV)}^2 \) placed on \( \lambda_p \) from cosmological observation [19].

We now consider the time variation of \( a \) for \( a \geq a_2 \). The field equation (15)

\[ \left( \frac{\dot{a}}{a} \right)^2 = \alpha^{-1} [\rho^r + \rho^m] + \alpha^{-1} \Lambda(t) - \frac{1}{2a^2}, \]

or

\[ \left( \frac{\dot{a}}{a} \right)^2 = \alpha^{-1} E_m^p a^3 + \frac{\omega}{2a^2} \left( \frac{a^2_p}{a^2} \right) + \frac{1}{2a^2}, \quad (103) \]

may be written in the form

\[ \frac{da}{dt} = \left( \frac{a_p}{a} \right) \left[ a_p \left( \frac{\rho_p^r}{\alpha} \right)^{1/2} \right] \left[ \frac{1}{2(1 + \omega)} \left( \frac{a}{a_p} \right)^2 + \left( \frac{a}{a_p} \right) \left( \frac{\rho_p^m}{\rho_p^m} \right) + \frac{\omega}{2(1 + \omega)} \right]. \]

Let \( (a_p/a) = x \) and \( dx = d(a_p/a) \), therefore
\[
\int_0^t dt' = \left( \frac{\alpha}{\rho_p^l} \right)^{1/2} \int_0^{a_p/a} \frac{x dx}{\left[ 2^{-1} (1 + \omega)^{-1} x^2 + (\rho_m^p/\rho_p^p) x + \omega/\left[ 2 (1 + \omega) \right] \right]^{1/2}}
\]

\[
t = \left( \frac{\alpha}{\rho_p^l} \right)^{1/2} \int_0^{a_p/a} \frac{x dx}{\left[ \frac{1}{2(1+\omega)} x^2 + (\rho_m^p/\rho_p^p) x + \frac{\omega}{2(1+\omega)} \right]^{1/2}}
\]

(104)

under the approximate boundary condition \( t = t_2 \approx 0 \), when \( a = a_2 \approx 0 \). For the present age \( t_p \) of the matter-dominated universe one obtains, for all \( \omega \),

\[
\left( \frac{\alpha}{\rho_p^l} \right)^{1/2} \int_0^1 \frac{x dx}{\left[ (\rho_m^p/\rho_p^p) x + 1/2 \right]^{1/2}} \leq t_p (\omega) \leq \left( \frac{\alpha}{\rho_p^l} \right)^{1/2} \int_0^1 \frac{x^{1/2} dx}{\left[ (\rho_m^p/\rho_p^p) + (1/2) x \right]^{1/2}},
\]

(105)

Another parameter which is almost \( \omega \)-independent is the present value \( q_p \) of the deceleration parameter defined by \( q = -a\ddot{a}/a^2 \). One obtains for this value

\[
q = -\frac{a\ddot{a}}{a^2} = -\left( \frac{\ddot{a}}{a} \right) \left( \frac{a}{a} \right)^2.
\]

(106)

Substituting for \( \left( \frac{\ddot{a}}{a} \right) \) and \( \left( \frac{a}{a} \right)^2 \) from the above we obtain

\[
q_p = \frac{1}{2} + \left( \frac{\omega}{(1 + \omega)} - \frac{1}{2} \right) \left( \frac{\rho_p^p}{\rho_m^p} \right) + \ldots,
\]

(107)

so that \( q_p \approx \frac{1}{2} \) in this model. This is nearly the same value as in the standard model with \( k = 0 \).

We now consider the period of generation of rest-mass and in particular show that in this model the pressure must have been negative during part of this period.

C. Period of matter generation

Consider Eq. (16) with \( \Lambda \) given by (37)

\[
dE + pda^3 = \alpha da,
\]

where we have written \( E \) for the total energy \( \rho a^3 \). This equation is valid for all \( a \). Integrating it between \( a = a_0 \) and \( a = a_2 \), the end of the matter generation period, one obtains:
3 \int_{a_0}^{a_2} p a^2 da = \alpha (a_2 - a_0) - E_2, \quad (108)

where $E_2 = E(a_2)$ and we used the initial condition $E(a_0) = 0$. Now,

$$E_2 = E_m(a_p) + E_r(a_2), \quad (109)$$

since we assume that the rest-mass contribution $E_m$ to the total energy remains constant for all $a \geq a_2$. Using Eqs. (61), (67) and (68), Eq. (108) becomes

$$E_2 = E_m(a_p) + E_r(a_2) = \rho^p_m a^3 + \rho r(a_2) a^3 = \frac{\alpha a_2}{2} \left[ \left( \rho^p_m \rho r \right) \left( \frac{a_p}{a_2} \right) + 1 \right] + \frac{\alpha \omega a^2_r}{2} a_2. \quad (110)$$

Substituting Eq. (110) into (108) we obtain

$$3 \int_{a_0}^{a_2} p a^2 da = \alpha (a_2 - a_0) - E_2 = -\frac{\alpha a_2}{2} \left[ (1 + \omega) \left( \frac{\rho^p_m}{\rho r} \right) \left( \frac{a_p}{a_2} \right) - 1 \right] - \frac{\alpha \omega a^2_r}{2} a_2 - \alpha a_0. \quad (111)$$

The right-hand side of (111) is clearly negative since $\rho^p_m \geq \rho^p_r$, thus the integral on the left-hand side of this equation must be negative. But

$$\int_{a_0}^{a_2} p a^2 da \geq 0 \quad (112)$$

indicating that the pressure must have been negative during part of the region of matter generation.

As mentioned before, the region of matter generation is a phase transition period between region of known and different equations of state. It now appears that the equation of state associated with the creation of rest-mass is characterized by negative pressure. Ideas on phase transitions in the early universe from unified gauge field theories indicate that phase transitions are expected to have occurred at least twice; at $T \sim 10^{15}$ GeV (GUT phase transition) and at $T \sim 10^2$ GeV (electro-weak phase transition). If the maximum temperature is of order $10^{19}$ GeV, this gives $a_1 \lesssim 10^{-10}$ (GeV)$^{-1}$ so that $t_1 \leq 10^{-34}$ s.

We now observe that the period of matter generation separates the pure radiation regime in which $\ddot{a} \geq 0$ and matter-and-radiation regime in which $\ddot{a} \leq 0$ as may be seen as
\[
\frac{\ddot{a}}{a} = -\frac{4\pi}{3} \left( \rho_{\text{effective}} + 3p \right) + \frac{\lambda_{\text{effective}}}{3}.
\]

This can be written as

\[
\frac{\ddot{a}}{a} = \alpha^{-1} \left[ \frac{2\alpha}{2a^2} - \frac{1}{2} (\rho + 3p) \right], \tag{113}
\]

which means that

\[
a\ddot{a} = 1 - \frac{1}{2\alpha} (\rho + 3p) a^2, \tag{114}
\]

and that, for \(a \geq a_2\)

\[
\frac{1}{2\alpha} (\rho + 3p) a^2 = \frac{1}{2} + \frac{\omega}{2} \frac{a_0^2}{a^2} + \frac{\rho_m}{\alpha} a^2 \geq 1. \tag{115}
\]

It thus appears that the creation of rest-mass results in the reversal of the sign of \(\ddot{a}\). Thus one might say that the presence of rest-mass causes the deceleration of the expansion of the universe. The rate of generation of energy is given by the equation

\[
\frac{dE}{da} = \alpha - 3pa^2. \tag{116}
\]

This shows that the maximum rate of energy generation occurs within the negative pressure interval after the pressure has attained its maximum negative value.

V. DISCUSSION

Motivated by previous studies of the back-reaction effect of quantum fields in the Einstein static model for the universe we have tried to construct a working model for a dynamic universe. The background geometry is assumed to be that of the closed Robertson-Walker model. We assumed the existence of a non-zero cosmological constant for start and have solved the Einstein field equations accordingly. Quantum field are shown to produce a non-zero vacuum energy density which will be a source for a critical density dynamic universe that will continue expanding without limit. The non-zero value of the vacuum expectation value of the energy-momentum tensor of the quantum field causes the universe to have a non-singular start, though the radius is of the size of Planck length. The critical density universe was originally considered in Friedmann cosmology to indicate a spatially flat universe, but
since we have introduced a non-zero cosmological constant here then it is legitimate not to consider the universe to be exactly flat but to be nearly flat. This might explain the recent analysis of the CMB measurements which showed that the universe is nearly flat rather than being exactly flat. The subsequent development of the universe is controlled by the back-reaction effects. Our model is free from all the standard big bang model problems and is non-singular. However, it is found that there should be a continuous creation of matter and energy in this universe at a rate lower than that proposed by the steady-state theory. The rate of matter/energy creation is proportional to the radius of the universe and consequently the overall density of universe drops as $1/a^2$. This creation of matter/energy can be explained by the fact of the conversion of the energy contained in the cosmological constant into particles and radiation. We did not attempt to suggest a mechanism for the creation of such energy but we have determined the necessary relations to the cosmological constant.

**Figure Caption:**

Fig. 1: The temperature-radius relationship deduced from the back-reaction effect of the vacuum energy plotted for different matter fields: the conformally coupled scalar field (1), the neutrino field (2), the photon field (3) and the minimally coupled field (4) (see ref. [10]).

Fig. 2: Depicts the contributions of the conformally coupled scalar field (1), the photon field (2), the neutrino field (3) and the minimally coupled scalar field (4) to the cosmological constant in an Einstein universe at finite temperatures (see ref. [10]).

Fig. 3: The temperature-radius relationship according to this model. Note that the x-axis does not start from zero but from 0.34.

Fig. 4: The temperature dependence of the cosmological constant calculated in accordance with the present model.

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Fig. 2

Casimir Regime

Planck Regime

\[ \Lambda \]

\[ T \]
Fig. 3
