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Arithmetical proofs of strong normalization results for symmetric $\lambda$-calculi

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Abstract. We give arithmetical proofs of the strong normalization of two symmetric $\lambda$-calculi corresponding to classical logic. The first one is the $\lambda_{\mu\tilde{\mu}}$-calculus introduced by Curien & Herbelin. It is derived via the Curry-Howard correspondence from Gentzen’s classical sequent calculus LK in order to have a symmetry on one side between “program” and “context” and on other side between “call-by-name” and “call-by-value”. The second one is the symmetric $\lambda\mu$-calculus. It is the $\lambda\mu$-calculus introduced by Parigot in which the reduction rule $\mu'$, which is the symmetric of $\mu$, is added. These results were already known but the previous proofs use candidates of reducibility where the interpretation of a type is defined as the fix point of some increasing operator and thus, are highly non arithmetical.

Keywords: $\lambda$-calculus, symmetric calculi, classical logic, strong normalization.

1. Introduction

Since it has been understood that the Curry-Howard correspondence relating proofs and programs can be extended to classical logic (Felleisen [13], Griffin [15]), various systems have been introduced: the $\lambda_c$-calculus (Krivine [17]), the $\lambda_{exn}$-calculus (de Groote [6]), the $\lambda_{\mu\text{Sym}}$-calculus (Barbanera & Berardi [1]), the $\lambda_{\Delta}$-calculus (Rehof & Sorensen [29]), the $\lambda_{\mu\tilde{\mu}}$-calculus (Curien & Herbelin [4]), the dual calculus (Wadler [31]), ... Only a few of them have computation rules that correspond to the symmetry of classical logic.

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We consider here the $\lambda\mu$-$\tilde{\mu}$-calculus and the symmetric $\lambda\mu$-calculus and we give arithmetical proofs of the strong normalization of the simply typed calculi. Though essentially the same proof can be done for the $\lambda^{sym}$-calculus, we do not consider here this calculus since it is somehow different from the previous ones: its main connector is not the arrow but the connectors or and and and the symmetry of the calculus comes from the de Morgan laws. This proof will appear in Battyanyi’s PhD thesis [2] who will also consider the dual calculus. Note that Dougherty & all [12] have shown the strong normalization of this calculus by the reducibility method using the technique of the fixed point construction.

The first proof of strong normalization for a symmetric calculus is the one by Barbanera & Berardi for the $\lambda^{Sym}$-calculus. It uses candidates of reducibility but, unlike the usual construction (for example for Girard’s system $F$), the definition of the interpretation of a type needs a rather complex fix-point operation. Yamagata [32] has used the same technic to prove the strong normalization of the symmetric $\lambda\mu$-calculus where the types are those of system $F$ and Parigot, again using the same ideas, has extended Barbanera & Berardi’s result to a logic with second order quantification. Polonovsky, using the same technic, has proved in [27] the strong normalization of the $\overline{\lambda}\mu\tilde{\mu}$-reduction. These proofs are highly non arithmetical.

The two proofs that we give are essentially the same but the proof for the $\overline{\lambda}\mu\tilde{\mu}$-calculus is much simpler since some difficult problems that appear in the $\lambda\mu$-calculus do not appear in the $\overline{\lambda}\mu\tilde{\mu}$-calculus. In the $\overline{\lambda}\mu\tilde{\mu}$-calculus, a $\mu$ or a $\lambda$ cannot be created at the root of a term by a reduction but this is not the case for the symmetric $\lambda\mu$-calculus. This is mainly due to the fact that, in the former, there is a right-hand side and a left-hand side whereas, in the latter, this distinction is impossible since a term on the right of an application can go on the left of an application after some reductions.

The idea of the proofs given here comes from the one given by the first author for the simply typed $\lambda$-calculus: assuming that a typed term has an infinite reduction, we can define, by looking at some particular steps of this reduction, an infinite sequence of strictly decreasing types. This proof can be found either in [7] (where it appears among many other things) or as a simple unpublished note on the web page of the first author (www.lama.univ-savoie.fr/~david).

We also show the strong normalization of the $\mu\tilde{\mu}$-reduction (resp. the $\mu\mu'$-reduction) for the un-typed calculi. The first result was already known and it can be found in [27]. The proof is done (by using candidates of reducibility and a fix point operator) for a typed calculus but, in fact, since the type system is such that every term is typable, the result is valid for every term. It was known that, for the un-typed $\lambda\mu$-calculus, the $\mu$-reduction is strongly normalizing (see [28]) but the strong normalization of the $\mu\mu'$-reduction was an open problem raised long ago by Parigot. Studying this reduction by itself is interesting since a $\mu$ (or $\mu'$)-reduction can be seen as a way “to put the arguments of the $\mu$ where they are used” and it is useful to know that this is terminating.

This paper is an extension of [11]. In particular, section 4 essentially appears there. It is organized as follows. Section 2 gives the syntax of the terms of the $\overline{\lambda}\mu\tilde{\mu}$-calculus and the symmetric $\lambda\mu$-calculus and their reduction rules. Section 3 is devoted to the proof of the normalization results for the $\overline{\lambda}\mu\tilde{\mu}$-calculus and section 4 for the symmetric $\lambda\mu$-calculus. We conclude in section 5 with some remarks and future work.
2. The calculi

2.1. The $\lambda\mu\bar{\mu}$-calculus

2.1.1. The un-typed calculus

There are three kinds of terms, defined by the following grammar, and there are two kinds of variables. In the literature, different authors use different terminology. Here, we will call them either $c$-terms, or $l$-terms or $r$-terms. Similarly, the variables will be called either $l$-variables (and denoted as $x, y, ...$) or $r$-variables (and denoted as $\alpha, \beta, ...$).

In the rest of the paper, by term we will mean any of these three kind of terms.

$$\begin{align*}
c & ::= \langle t_l, t_r \rangle \\
t_l & ::= x \mid \lambda x t_l \mid \mu \alpha c \mid t_r.t_l \\
t_r & ::= \alpha \mid \lambda \alpha t_r \mid \mu x c \mid t_l.t_r
\end{align*}$$

Remark 2.1. $t_l$ (resp. $t_r$) stands of course for the left (resp. right) part of a $c$-term. At first look, it may be strange that, in the typing rules below, left terms appear in the right part of a sequent and vice-versa. This is just a matter of convention and an other choice could have been done. Except the change of name (done to make easier the analogy between the proofs for $\lambda\mu\bar{\mu}$-calculus and the symmetric $\lambda\mu$-calculus) we have respected the notations of the literature on this calculus.

2.1.2. The typed calculus

The logical part of this calculus is the (classical) sequent calculus which is, intrinsically, symmetric. The types are built from atomic formulas with the connectors $\to$ and $\neg$ where the intuitive meaning of $A \to B$ is “$A$ and not $B$”. The typing system is a sequent calculus based on judgments of the following form:

$$c : (\Gamma \vdash \Delta) \quad \Gamma \vdash t_l : A, \Delta \quad \Gamma, t_r : A \vdash \Delta$$

where $\Gamma$ (resp. $\Delta$) is a $l$-context (resp. a $r$-context), i.e. a set of declarations of the form $x : A$ (resp. $\alpha : A$) where $x$ (resp. $\alpha$) is a $l$-variable (resp. a $r$-variable) and $A$ is a type.

$$\begin{align*}
\Gamma, x : A \vdash x : A, \Delta \\
\Gamma \vdash \lambda x t_l : A \to B, \Delta \\
\Gamma \vdash t_l : A, \Delta \quad \Gamma, t_r : B \vdash \Delta \\
\Gamma \vdash t_l.t_r : A \to B, \Delta
\end{align*}$$

$$\begin{align*}
\Gamma \vdash \lambda \alpha t_r : A \to B, \Delta \\
\Gamma, t_l : A \vdash \alpha : B, \Delta \\
\Gamma, \lambda \alpha t_r : A \to B \vdash \Delta
\end{align*}$$

$$\begin{align*}
\Gamma, t_l : A, \Delta \quad \Gamma, t_r : A \vdash \Delta \\
\langle t_l, t_r \rangle : (\Gamma \vdash \Delta)
\end{align*}$$
2.1.3. The reduction rules

The cut-elimination procedure (on the logical side) corresponds to the reduction rules (on the terms) given below.

- \((\lambda x \, t_1, t'_1, t_r) \triangleright \lambda t'_1, \mu x (t_1, t_r)\)
- \((t'_r, t_1, x_\alpha t_r) \triangleright \mu (t_1, t_r), t'_r\)
- \((\mu \alpha \, c, t_r) \triangleright \mu c[\alpha := t_r]\)
- \((t_1, \mu x \, c) \triangleright \mu c[x := t_1]\)
- \((\mu \alpha \, (t_1, \alpha) \triangleright s_l, t_l) \quad \text{if} \quad \alpha \notin Fv(t_l)\)
- \((\mu x \, (x_\alpha t_r) \triangleright s_r, t_r) \quad \text{if} \quad x \notin Fv(t_r)\)

**Remark 2.2.** It is easy to show that the \(\mu \tilde{\mu}\)-reduction is not confluent. For example \(\langle \mu \alpha \, (x_\beta), \mu y (x_\alpha) \rangle\) reduces both to \(\langle x_\beta \rangle\) and to \(\langle x_\alpha \rangle\).

**Definition 2.1.**

- We denote by \(\triangleright_l\) the reduction by one of the logical rules i.e. \(\triangleright_l \lambda, \triangleright_l \mu, \triangleright_l \tilde{\mu}\).
- We denote by \(\triangleright_s\) the reduction by one of the simplification rules i.e. \(\triangleright_s s_l, \triangleright_s s_r\).

2.2. The symmetric \(\lambda \mu\)-calculus

2.2.1. The un-typed calculus

The set (denoted as \(T\)) of \(\lambda \mu\)-terms or simply terms is defined by the following grammar where \(x, y, \ldots\) are \(\lambda\)-variables and \(\alpha, \beta, \ldots\) are \(\mu\)-variables:

\[
T ::= x \mid \lambda x T \mid (T \, T) \mid (\alpha \, T) \mid (\alpha \, T)
\]

Note that we adopt here a more liberal syntax (also called de Groote’s calculus) than in the original calculus since we do not ask that a \(\mu \alpha\) is immediately followed by a \((\beta \, M)\) (denoted \([\beta] M\) in Parigot’s notation).

2.2.2. The typed calculus

The logical part of this calculus is natural deduction. The types are those of the simply typed \(\lambda \mu\)-calculus i.e. are built from atomic formulas and the constant symbol \(\bot\) with the connector \(\rightarrow\). As usual \(\neg A\) is an abbreviation for \(A \rightarrow \bot\).

The typing rules are given below where \(\Gamma\) is a context, i.e. a set of declarations of the form \(x : A\) and \(\alpha : \neg A\) where \(x\) is a \(\lambda\) (or intuitionistic) variable, \(\alpha\) is a \(\mu\) (or classical) variable and \(A\) is a formula.
\[ \Gamma, x : A \vdash M : B \quad \Gamma \vdash \lambda x M : A \rightarrow B \quad \Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A \quad \Gamma \vdash (M \ N) : B \]

\[ \Gamma, \alpha : \neg A \vdash M : \bot \quad \Gamma \vdash \mu \alpha M : A \quad \Gamma, \alpha : \neg A \vdash M : \bot \quad \Gamma, \alpha : \neg A \vdash (\alpha \ M) : \bot \]

Note that, here, we also have changed Parigot’s notation but these typing rules are those of his classical natural deduction. Instead of writing

\[ M : (A_{x_1}^{x_1}, \ldots, A_{x_n}^{x_n} \vdash B, C_{\alpha_1}^{\alpha_1}, \ldots, C_{\alpha_m}^{\alpha_m}) \]

we have written

\[ x_1 : A_{x_1}, \ldots, x_n : A_{x_n}, \alpha_1 : \neg C_{\alpha_1}, \ldots, \alpha_m : \neg C_{\alpha_m} \vdash M : B \]

### 2.2.3. The reduction rules

The cut-elimination procedure (on the logical side) corresponds to the reduction rules (on the terms) given below. Natural deduction is not, intrinsically, symmetric but Parigot has introduced the so called Free deduction [22] which is completely symmetric. The \( \lambda \mu \)-calculus comes from there. To get a confluent calculus he had, in his terminology, to fix the inputs on the left. To keep the symmetry, it is enough to add a new reduction rule (called the \( \mu \)'-reduction) which is the symmetric rule of the \( \mu \)-reduction and also corresponds to the elimination of a cut.

- \( (\lambda x M \ N) \nrightarrow_{\beta} M[x := N] \)
- \( (\mu \alpha M \ N) \nrightarrow_{\mu} \mu \alpha M[\alpha = r \ N] \)
- \( (N \ \mu \alpha M) \nrightarrow_{\mu'} \mu \alpha M[\alpha = l \ N] \)
- \( (\alpha \ \mu \beta M) \nrightarrow_{\rho} M[\beta := \alpha] \)
- \( \mu \alpha (\alpha \ M) \nrightarrow_{\theta} M \) if \( \alpha \) is not free in \( M \).

where \( M[\alpha = r \ N] \) (resp. \( M[\alpha = l \ N] \)) is obtained by replacing each sub-term of \( M \) of the form \( \langle \alpha \ U \rangle \) by \( \langle \alpha \ (U \ N) \rangle \) (resp. \( \langle \alpha \ (N \ U) \rangle \)). This substitution is called a \( \mu \)-substitution (resp. a \( \mu \)'-substitution).

**Remark 2.3.**

1. It is shown in [23] that the \( \beta \mu \)-reduction is confluent but neither \( \mu \mu \)' nor \( \beta \mu \)' is. For example \( (\mu \alpha x \ \mu \beta y) \) reduces both to \( \mu \alpha x \) and to \( \mu \beta y \). Similarly \( (\lambda x x \ \mu \beta y) \) reduces both to \( x \) and to \( \mu \beta y \).

2. Unlike for a \( \beta \)-substitution where, in \( M[x := N] \), the variable \( x \) has disappeared it is important to note that, in a \( \mu \) or \( \mu \)'-substitution, the variable \( \alpha \) has not disappeared. Moreover its type has changed. If the type of \( N \) is \( A \) and, in \( M \), the type of \( \alpha \) is \( \neg (A \rightarrow B) \) it becomes \( \neg B \) in \( M[\alpha = r \ N] \). If the type of \( N \) is \( A \rightarrow B \) and, in \( M \), the type of \( \alpha \) is \( \neg B \) it becomes \( \neg B \) in \( M[\alpha = l \ N] \).
3. In section 4, we will not consider the rules $\theta$ and $\rho$. The rule $\theta$ causes no problem since it is strongly normalizing and it is easy to see that this rule can be postponed. However, unlike for the $\lambda\mu\tilde{\mu}$-calculus where all the simplification rules can be postponed, this is not true for the rule $\rho$ and, actually, Battyanyi has shown in [2] that $\mu\mu\rho$ is not strongly normalizing. However he has shown that $\mu\mu\rho$ (in the untyped case) and $\beta\mu\mu\rho$ (in the typed case) are weakly normalizing.

2.3. Some notations

The following notations will be used for both calculi. It will also be important to note that, in section 3 and 4, we will use the same notations (for example $\Sigma_l$, $\Sigma_r$) for objects concerning respectively the $\lambda\mu\tilde{\mu}$-calculus and the symmetric $\lambda\mu$-calculus. This is done intentionally to show the analogy between the proofs.

**Definition 2.2.** Let $u, v$ be terms.

1. $cxyt(u)$ is the number of symbols occurring in $u$.
2. We denote by $u \leq v$ (resp. $u < v$) the fact that $u$ is a sub-term (resp. a strict sub-term) of $v$.
3. A proper term is a term that is not a variable.
4. If $\sigma$ is a substitution and $u$ is a term, we denote by
   - $\sigma + [x := u]$ the substitution $\sigma'$ such that for $y \neq x$, $\sigma'(y) = \sigma(y)$ and $\sigma'(x) = u$
   - $\sigma[x := u]$ the substitution $\sigma'$ such that $\sigma'(y) = \sigma(y)[x := u]$.

**Definition 2.3.** Let $A$ be a type. We denote by $lg(A)$ the number of symbols in $A$.

In the next sections we will study various reductions. The following notions will correspond to these reductions.

**Definition 2.4.** Let $\triangleright$ be a notion of reduction.

1. The transitive (resp. reflexive and transitive) closure of $\triangleright$ is denoted by $\triangleright^+$ (resp. $\triangleright^*$). The length (i.e. the number of steps) of the reduction $t \triangleright^* t'$ is denoted by $lg(t \triangleright^* t')$.
2. If $t$ is in $SN$ i.e. $t$ has no infinite reduction, $\eta(t)$ will denote the length of the longest reduction starting from $t$ and $\eta_c(t)$ will denote $(\eta(t), cxyt(t))$.
3. We denote by $u \prec v$ the fact that $u \leq w$ for some $w$ such that $v \triangleright^* w$ and either $v \triangleright^+ w$ or $u < w$. We denote by $\preceq$ the reflexive closure of $\prec$.

**Remark 2.4.** - It is easy to check that the relation $\preceq$ is transitive, that $u \preceq v$ iff $u \leq w$ for some $w$ such that $v \triangleright^* w$. We can also prove (but we will not use it) that the relation $\preceq$ is an order on the set $SN$.
- If $v \in SN$ and $u \prec v$, then $u \in SN$ and $\eta_c(u) < \eta_c(v)$.
- In the proofs done by induction on some $k$-uplet of integers, the order we consider is the lexicographic order.
3. Normalization for the $\overline{\lambda \mu \tilde{\mu}}$-calculus

The following lemma will be useful.

**Lemma 3.1.** Let $t$ be a $l$-term (resp. a $r$-term). If $t \in SN$, then $\langle t, \alpha \rangle \in SN$ (resp. $\langle x, t \rangle \in SN$).

**Proof.** By induction on $\eta(t)$. Since $\langle t, \alpha \rangle \not\in SN$, $\langle t, \alpha \rangle \not\ll u$ for some $u$ such that $u \not\in SN$. If $u = \langle t', \alpha \rangle$ where $t \ll t'$ we conclude by the induction hypothesis since $\eta(t') < \eta(t)$. If $t = \mu \beta c$ and $u = c[\beta := \alpha] \not\in SN$, then $c \not\in SN$ and $t \not\in SN$. Contradiction. □

3.1. $\not\ll_s$ can be postponed

**Definition 3.1.** Let $\not\ll_{\mu_0}, \not\ll_{\tilde{\mu}_0}$ be defined as follows:

- $\langle \mu \alpha c, t_1 \rangle \not\ll_{\mu_0} c[\alpha := t_1]$ if $\alpha$ occurs at most once in $c$
- $\langle t_1, \mu x c \rangle \not\ll_{\tilde{\mu}_0} c[x := t_1]$ if $x$ occurs at most once in $c$

2. Let $\not\ll_{l_0} = \not\ll_{\mu_0} \cup \not\ll_{\tilde{\mu}_0}$.

**Lemma 3.2.** If $u \llervt v \ll_{l_0} w$, then there is $t$ such that $u \llervt t \llervt w$ or $u \llervl t \llervl w$.

**Proof.** By induction on $u$. □

**Lemma 3.3.** If $u \llervt v \llervl w$, then either $u \llervl w$ or, for some $t$, $u \llervl t \llervl w$ or $u \llervl t \llervl w$.

**Proof.** By induction on $u$. □

**Lemma 3.4.** If $u \llervt v \llervl w$ then, for some $t$, $u \llervl t \llervt w$ and $lg(u \llervt v \llervl w) \leq lg(u \llervl t \llervt w)$.

**Proof.** By induction on $lg(u \llervt v \llervl w)$. Use lemma 3.3. □

**Lemma 3.5.** If $u \llervt v \llervl w$ then, for some $t$, $u \llervl t \llervt w$.

**Proof.** By induction on $lg(u \llervt v \llervl w)$. Use lemmas 3.2 and 3.4. □

**Corollary 3.1.** $\llervt$ can be postponed.

**Proof.** By lemma 3.5. □

**Lemma 3.6.** The $s$-reduction is strongly normalizing.

**Proof.** If $u \llervt v$, then $ctxy(u) > ctxy(v)$. □

**Theorem 3.1.** 1. If $t$ is strongly normalizing for the $l$-reduction, then it is also strongly normalizing for the $ls$-reduction.

2. If $t$ is strongly normalizing for the $\mu \tilde{\mu}$-reduction, then it is also strongly normalizing for the $\mu \tilde{\mu} s$-reduction.

**Proof.** Use lemmas 3.6 and 3.1. It is easy to check that the lemma 3.1 remains true if we consider only the reduction rules $\mu$ and $\tilde{\mu}$. □
3.2. The $\mu\overline{\nu}$-reduction is strongly normalizing

In this section we consider only the $\mu\overline{\nu}$-reduction and we restrict the set of terms to the following grammar.

\[
\begin{align*}
c & ::= \langle t_l, t_r \rangle \\
t_l & ::= x \mid \mu\alpha c \\
t_r & ::= \alpha \mid \mu x c
\end{align*}
\]

It is easy to check that, to prove the strong normalization of the full calculus with the $\mu\overline{\nu}$-reduction, it is enough to prove the strong normalization of this restricted calculus.

Remember that we are, here, in the un-typed calculus and thus our proof does not use types but the strong normalization of this calculus actually follows from the result of the next section: it is easy to check that, in this restricted calculus, every term is typable by any type, in the context where the free variables are given this type. We have kept this section since the main ideas of the proof of the general case already appear here and this is done in a simpler situation.

The main point of the proof is the following. It is easy to show that if $t \in SN$ then the real cause of non SN is, in some sense, $[x := t_l]$.

Having this result, we show, essentially by induction on $\eta c(t_l) + \eta c(t_r)$, that if $t_l, t_r \in SN$ then $\langle t_l, t_r \rangle \in SN$. The point is that there is, in fact, no deep interactions between $t_l$ and $t_r$ i.e. in a reduct of $\langle t_l, t_r \rangle$ we always know what is coming from $t_l$ and what is coming from $t_r$. The final result comes then from a trivial induction on the terms.

**Definition 3.2.** We denote by $\Sigma_l$ (resp. $\Sigma_r$) the set of simultaneous substitutions of the form $[x_1 := t_1, \ldots, x_n := t_n]$ (resp. $[\alpha_1 := t_1, \ldots, \alpha_n := t_n]$) where $t_1, \ldots, t_n$ are proper l-terms ($r$-terms).

- For $s \in \{l, r\}$, if $\sigma = [\xi_1 := t_1, \ldots, \xi_n := t_n] \in \Sigma_s$, we denote by $dom(\sigma)$ (resp. $dom(\sigma)$) the set $\{\xi_1, \ldots, \xi_n\}$ (resp. $\{t_1, \ldots, t_n\}$).

**Lemma 3.7.** Assume $t_l, t_r \in SN$ and $\langle t_l, t_r \rangle \notin SN$. Then either $t_l = \mu\alpha c$ and $c[\alpha := t_r] \notin SN$ or $t_r = \mu x c$ and $c[x := t_l] \notin SN$.

**Proof.** By induction on $\eta(t_l) + \eta(t_r)$. Since $\langle t_l, t_r \rangle \notin SN$, $\langle t_l, t_r \rangle \triangleright t$ for some $t$ such that $t \notin SN$. If $t = \langle t'_l, t_r \rangle$ where $t_l \triangleright t'_l$, we conclude by the induction hypothesis since $\eta(t'_l) + \eta(t_r) < \eta(t_l) + \eta(t_r)$. If $t = \langle t_l, t'_r \rangle$ where $t_r \triangleright t'_r$, the proof is similar. If $t_l = \mu\alpha c$ and $t = c[\alpha := t_r] \notin SN$ or $t_r = \mu x c$ and $t = c[x := t_l] \notin SN$, the result is trivial. \hfill $\Box$
Lemma 3.8. 1. Let \( t \) be a term, \( t_1 \) a \( t \)-term and \( \tau \in \Sigma_t \). Assume \( t_1 \in SN \), \( x \) is free in \( t \) but not free in \( Im(\tau) \). If \( t[\tau] \in SN \) but \( t[\tau][x := t_1] \notin SN \), there is \( \langle x, t_r \rangle < t \) and \( \tau' \in \Sigma_t \) such that \( t_r[\tau'] \in SN \) and \( \langle t_1, t_r[\tau'] \rangle \notin SN \).

2. Let \( t \) be a term, \( t_r \) a \( r \)-term and \( \sigma \in \Sigma_r \). Assume \( t_r \in SN \), \( \alpha \) is free in \( t \) but not free in \( Im(\sigma) \). If \( t[\sigma] \in SN \) but \( t[\sigma][\alpha := t_r] \notin SN \), there is \( \langle t_1, \alpha \rangle < t \) and \( \sigma' \in \Sigma_r \) such that \( t_1[\sigma'] \in SN \) and \( \langle t_1[\sigma'], t_r \rangle \notin SN \).

**Proof** We prove the case (1) (the case (2) is similar). Note that \( t_1 \) is proper since \( t[\tau] \in SN \), \( t[\tau][x := t_1] \notin SN \) and \( x \) is not free in \( Im(\tau) \). Let \( Im(\tau) = \{ t_1, \ldots, t_k \} \). Let \( U = \{ u / u \) is proper and \( u \leq t \} \) and \( V = \{ v / v \) is proper and \( v \leq t_i \) for some \( i \} \).

Define inductively the sets \( \Sigma_1' \) and \( \Sigma_1'' \) of substitutions by the following rules:

\[
\rho \in \Sigma_1' \text{ iff } \rho = \emptyset \text{ or } \rho = \rho' + [y := v[\delta]] \text{ for some l-term } v \in V, \delta \in \Sigma_r' \text{ and } \rho' \in \Sigma_1' \text{ and } \delta \in \Sigma_r', \text{ iff } \delta = \emptyset \text{ or } \delta = \delta' + [\beta := u[\rho']] \text{ for some r-term } u \in U, \rho' \in \Sigma_1' \text{ and } \delta' \in \Sigma_r'.
\]

Denote by \( C \) the conclusion of the lemma, i.e. there is \( \langle x, t_r \rangle < t \) and \( \tau' \in \Sigma_t \) such that \( t_r[\tau'] \in SN \) and \( \langle t_1, t_r[\tau'] \rangle \notin SN \). We prove something more general.

1. If \( u \in U, \rho \in \Sigma_1', u[\rho] \in SN \) and \( u[\rho][x := t_1] \notin SN \), then \( C \) holds.
2. If \( v \in V, \delta \in \Sigma_1'', v[\delta] \in SN \) and \( v[\delta][x := t_1] \notin SN \), then \( C \) holds.

The term \( t \) is proper since \( t[\tau] \in SN \). Then conclusion \( C \) follows from (1) with \( t \) and \( \tau \).

The properties (1) and (2) are proved by a simultaneous induction on \( \eta c(u[\rho]) \) (for the first case) and \( \eta c(v[\delta]) \) (for the second case). We only consider (1), the case (2) is proved in a similar way.

- If \( u \) begins with a \( \mu \). The result follows from the induction hypothesis.
- If \( u = \langle u_1, u_r \rangle \).
  - If \( u_r[\rho][x := t_1] \notin SN \): then \( u_r \) is proper and the result follows from the induction hypothesis.
  - If \( u_1[\rho][x := t_1] \notin SN \) and \( u_1 \) is proper: the result follows from the induction hypothesis.
  - If \( u_1[\rho][x := t_1] \notin SN \) and \( u_1 = y \in dom(\rho) \). Let \( \rho(y) = \mu \beta d[\delta] \). Then \( \mu \beta d[\delta][x := t_1] \notin SN \) and the result follows from the induction hypothesis with \( \mu \beta d \) and \( \delta \) (case (2)) since \( \eta c(\mu \beta d[\delta]) < \eta c(u[\rho]) \).
  - Otherwise, by lemma 3.7, there are two cases to consider. Note that \( u_r \) cannot be a variable because, otherwise, \( u[\rho][x := t_1] = \langle u_1[\rho][x := t_1], u_r \rangle \) and thus, by lemma 3.7, \( u[\rho][x := t_1] \) would be in \( SN \).
    1. \( u_1[\rho][x := t_1] = \mu \alpha c \) and \( c[\alpha := u_r[\rho][x := t_1]] \notin SN \).
    - If \( u_1 = \mu \alpha d \), then \( d[\alpha := u_r[\rho][x := t_1]] \notin SN \) and the result follows from the induction hypothesis with \( d[\alpha := u_r[\rho]] \) and \( \rho \) since \( \eta(d[\alpha := u_r[\rho]] < \eta(u[\rho])) \).
    - If \( u_1 = \mu \alpha c \) and \( \rho(y) = \mu \beta d[\delta], then d[\delta'][x := t_1] \notin SN \) where \( \delta' = \delta + [\beta := u_r[\rho]] \) and the result follows from the induction hypothesis with \( d \) and \( \delta' \) (case (2)).
Proof

By induction on the term. It is enough to show that, if \( t \in SN \), then \( \langle t, u_t \rangle \) and \( v_r = \rho[x := t] \) satisfy the desired conclusion.

(2) \( u_r[\rho[x := t]] = \mu y c \) and \( c[\alpha := u_r[\rho[x := t]]] \notin SN \). Then \( u_r = \mu y d \) and \( d[y := u_r][\rho[x := t]] \notin SN \). The result follows from the induction hypothesis with \( d[y := u_r] \) and \( \rho \) since \( \eta(d[y := u_r][\rho]) < \eta(u_r) \).

\( \Box \)

**Theorem 3.2.** The \( \mu \mu \)-reduction is strongly normalizing.

**Proof** By induction on the term. It is enough to show that, if \( t_i, t_r \in SN \), then \( \langle t_i, t_r \rangle \in SN \). We prove something more general: let \( \tau \) (resp. \( \sigma \)) be in \( \Sigma_r \) (resp. \( \Sigma_l \)) and assume \( t_i[\sigma], t_r[\tau] \in SN \). Then \( \langle t_i[\sigma], t_r[\tau] \rangle \in SN \). Assume it is not the case and choose some elements such that \( t_i[\sigma], t_r[\tau] \in SN \), \( t_i[\sigma], t_r[\tau] \notin SN \) and \( (\eta(t_i) + \eta(t_r) + cxy(t_i) + cxy(t_r)) \) is minimal. By lemma 3.7, either \( t_i[\sigma] = \mu \alpha c \) and \( c[\alpha := t_r[\tau]] \notin SN \) or \( t_r[\tau] = \mu x c \) and \( c[x := t_i[\sigma]] \notin SN \). Look at the second case (the first one is similar). We have \( t_r = \mu x d \) and \( d[\tau] = c \), then \( d[\tau][x := t_i[\sigma]] \notin SN \).

By lemma 3.8, let \( u_r < d \) and \( r' \in \Sigma_l \) be such that \( u_r[r'] \in SN \), \( t_i[\sigma], u_r[r'] \notin SN \). This contradicts the minimality of the chosen elements since \( \eta(u_r) < \eta(c) \).

\( \Box \)

3.3. The typed \( \lambda \mu \mu \)-calculus is strongly normalizing

In this section, we consider the typed calculus with the \( l \)-reduction. By theorem 3.2, this is enough to prove the strong normalization of the full calculus. To simplify notations, we do not write explicitly the type information but, when needed, we denote by \( type(t) \) the type of the term \( t \).

The proof is essentially the same as the one of theorem 3.2. It relies on lemma 3.10 for which type considerations are needed: in its proof, some cases cannot be proved “by themselves” and we need an argument using the types. For this reason, its proof is done using the additional fact that we already know that, if \( t_i, t_r \in SN \) and the type of \( t_r \) is small, then \( t[x := t_r] \) also is in \( SN \). Since the proof of lemma 3.10 is done by induction on the type, when we will use lemma 3.10, the additional hypothesis will be available.

**Lemma 3.9.** Assume \( t_i, t_r \in SN \) and \( \langle t_i, t_r \rangle \notin SN \). Then either \( \langle t_i \rangle + \langle t_r \rangle \notin SN \) or \( \langle t_r \rangle \notin SN \) or \( \langle t_i \rangle \notin SN \) or \( \langle t_r \rangle \notin SN \) or \( \langle t = \lambda x u_r, t_i = u_r[u], \eta(u_r) \in SN \) or \( \langle t_r = \lambda x u_r, t_l = u_l[u] \rangle \notin SN \) or \( \langle u_r \rangle \notin SN \) or \( \langle \mu \alpha \langle u_r, u_i \rangle, u' \rangle \notin SN \).

**Proof** By induction on \( \eta(t_i) + \eta(t_r) \).

\( \Box \)

**Definition 3.3.** Let \( A \) be a type. We denote \( \Sigma_{A,l} \) (resp. \( \Sigma_{A,r} \)) the set of substitutions of the form \( [x_1 := t_1, ..., x_n := t_n] \) where \( t_1, ..., t_n \) are proper \( l \)-terms (resp. \( r \)-terms) and the type of the \( x_i \) (resp. \( \alpha_i \)) is \( A \).

**Lemma 3.10.** Let \( n \) be an integer and \( A \) be a type such that \( lg(A) = n \). Assume \( H \) holds where \( H \) is: for every \( u, v \in SN \) such that \( lg(type(v)) < n, u[x := v] \in SN \).

1. Let \( t \) be a term, \( l \) a \( l \)-term and \( \tau \in \Sigma_{A,l} \). Assume \( l \in SN \) and has type \( A \), \( x \) is free in \( t \) but not free in \( lm(\tau) \). If \( l[\tau] \notin SN \) but \( l[\tau][x := l] \notin SN \), there is \( \langle x, l \rangle < t \) and \( \tau' \in \Sigma_{A,l} \) such that \( l[\tau'] \in SN \) and \( \langle l, \tau'[\tau] \rangle \notin SN \).
2. Let \( t \) be a term, \( t_r \) a \( r \)-term and \( \sigma \in \Sigma_{A,r} \). Assume \( t_r \in SN \) and has type \( A \), \( \alpha \) is free in \( t \) but not free in \( Im(\tau) \). If \( t[\sigma] \in SN \) but \( t[\sigma'][\alpha := t_r] \notin SN \), there is \( \langle t_1, \alpha \rangle \prec t \) and \( \sigma' \in \Sigma_{A,r} \) such that \( t_1[\sigma'] \in SN \) and \( \langle t_1[\sigma'], t_r \rangle \notin SN \).

**Proof** We only prove the case (1), the other one is similar. Note that \( t_1 \) is proper since \( t[\tau] \in SN \) and \( t[\tau][x := t_1] \notin SN \). Let \( Im(\tau) = \{ t_1, \ldots, t_k \} \). Let \( U = \{ u / u \) is proper and \( u \leq t \} \) and \( V = \{ v / v \) is proper and \( u \leq t_i \) for some \( i \} \). Define inductively the sets \( \Sigma'_{A,l} \) and \( \Sigma'_{A,r} \) of substitutions by the following rules:

\[
\rho \in \Sigma'_{A,l} \text{ iff } \rho = \emptyset \text{ or } \rho = \rho' + [y := v[\delta]] \text{ for some } l\text{-term } v \in \mathcal{V}, \delta \in \Sigma'_{A,r}, \rho' \in \Sigma'_{A,l} \text{ and } y \text{ has type } A.
\]

\[
\delta \in \Sigma'_{A,r} \text{ iff } \delta = \emptyset \text{ or } \delta = \delta' + [\beta := u[\rho]] \text{ for some } r\text{-term } u \in \mathcal{U}, \rho \in \Sigma'_{A,l}, \delta' \in \Sigma'_{A,r} \text{ and } \beta \text{ has type } A.
\]

Denote by \( C \) the conclusion of the lemma, i.e. there is \( \langle t, t_r \rangle \prec t \) and \( \tau' \in \Sigma'_{A,l} \) such that \( t_r[\tau'] \in SN \) and \( \langle t_1, t_r[\tau''] \rangle \notin SN \). We prove something more general.

(1) If \( u \in \mathcal{U}, \rho \in \Sigma'_{A,l}, u[\rho] \in SN \) and \( u[\rho][x := t_1] \notin SN \), then \( C \) holds.

(2) If \( v \in \mathcal{V}, \delta \in \Sigma'_{A,r}, v[\delta] \in SN \) and \( v[\delta][x := t_1] \notin SN \), then \( C \) holds.

Note that, since \( t[\tau][x := t_1] \notin SN \), \( t \) is proper and thus, \( C \) follows from (1) with \( t \) and \( \tau \). The properties (1) and (2) are proved by a simultaneous induction on \( \eta(c(u[\rho])) \) (for the first case) and \( \eta(c(v[\delta])) \) (for the second case). We only consider (1) since (2) is similar.

The proof is as in lemma 5.8. We only consider the additional cases: \( u = \langle u_1, u_r \rangle \), \( u_1[\rho][x := t_1] \in SN \), \( u_r[\rho][x := t_1] \in SN \), \( u_r \) is proper and one of the two following cases occurs.

- \( u_1[\rho][x := t_1] = \lambda x v_1, u_r[\rho][x := t_1] = v'_1, v_r \) and \( \langle v'_1, \mu(x)v_1, v_r \rangle \notin SN \). Then, \( u_r = u'_1, w_r, v_r = u'_1[\rho][x := t_1] \) and \( v_r = w_r[\rho][x := t_1] \). There are three cases to consider.

  - \( u_1 = \lambda x w_1 \) and \( u_1[\rho][x := t_1] = v_1 \), then the result follows from the induction hypothesis with \( \langle u'_1, \mu(x)w_1, w_r \rangle \) and \( \rho \) since \( \eta(\langle u'_1, \mu(x)w_1, w_r \rangle[\rho]) < \eta(u[\rho]) \).

  - \( u_1 = y \in dom(\rho) \). Let \( \rho(y) = \lambda z w_1[\delta] \), then \( a = \langle u'_1[\rho], \mu x(w_1[\delta], w_r[\rho]) \rangle[\rho] \not\in SN \). But,

    - \( b = u'_1[\rho][x := t_1], c = w_1[\delta][x := t_1], d = w_r[\rho][x := t_1] \in SN \),
    - \( l(\text{type}(b)) < n, l(\text{type}(c)) < n \),
    - \( a = \langle x_2, \mu(x_1, d) \rangle[x_1 := c][x_2 := b] \)

and this contradicts the hypothesis \((H)\).

  - \( u_1 = x \), then \( \langle x, u_r \rangle \) and \( \tau' = \tau[x := t_1] \) satisfy the desired conclusion.

- \( u_1[\rho][x := t_1] = u'_1, v_1, u_r[\rho][x := t_1] = \lambda x v_r \) and \( \langle \mu(x)v_1, v_r, \tau' \rangle \notin SN \). The proof is similar. \( \Box \)
Lemma 3.11. If \( t, t_1, t_r \in SN \), then \( t[x := t_1], t[\alpha := t_r] \in SN \).

Proof We prove something a bit more general: let \( A \) be a type and \( t \) a term.

1. Let \( t_1, ..., t_k \) be \( t \)-terms and \( \tau_1, ..., \tau_k \) be substitutions in \( \Sigma_{A,t} \). If, for each \( i \), \( t_i \) has type \( A \) and \( t_i[\tau_i] \in SN \), then \( t[x_1 := t_1[\tau_1], ..., x_k := t_k[\tau_k]] \in SN \).

2. Let \( t_1, ..., t_k \) be \( r \)-terms and \( \tau_1, ..., \tau_k \) be substitutions in \( \Sigma_{A,r} \). If, for each \( i \), \( t_i \) has type \( A \) and \( t_i[\tau_i] \in SN \), then \( t[\alpha_1 := t_1[\tau_1], ..., \alpha_k := t_k[\tau_k]] \in SN \).

We only consider (1) since (2) is similar. This is proved by induction on \((lg(A), \eta(t), cxtv(t), \Sigma \eta(t), \Sigma cxtv(t))\) where, in \( \Sigma \eta(t) \) and \( \Sigma cxtv(t) \), we count each occurrence of the substituted variable. For example if \( k = 1 \) and \( x_1 \) has \( n \) occurrences, \( \Sigma \eta(t_1) = n \eta(t_1) \).

The only no trivial case is \( t = \langle u_1, u_r \rangle \). Let \( \sigma = [x_1 := t_1[\tau_1], ..., x_k := t_k[\tau_k]] \). By the induction hypothesis, \( u_1[\sigma], u_r[\sigma] \in SN \). By lemma 3.9, there are four cases to consider.

- If \( u_1 = \mu \alpha \) and \( c[\alpha := u_r[\sigma]] \notin SN \).
  
  - If \( u_1 = \mu \alpha d \) and \( d[\sigma] = c \). Then \( d[\alpha := u_r][\sigma] \notin SN \) and, since \( \eta(d[\alpha := u_r][\sigma]) < \eta(t) \), this contradicts the induction hypothesis.
  
  - If \( u_1 = x_i, t_i = \mu \alpha d \) and \( d[\tau_i][\alpha := u_r[\sigma]] \notin SN \). By lemma 3.10, there is \( v_i \in \Sigma_{A,t} \) such that \( v_i[\tau_i] \in SN \) and \( \langle v_i[\tau_i], u_r[\sigma] \rangle \notin SN \). Let \( t' = \langle y, u_r \rangle \) where \( y \) is a fresh variable and \( \sigma' = \sigma + [y := v_i[\tau_i]] \). Then \( \langle v_i[\tau_i], u_r[\sigma] \rangle = t'[\sigma'] \) and, since \( \eta(v_i), cxtv(t_i) < \eta(t_i), cxtv(t_i) \) we get a contradiction from the induction hypothesis.

- If \( u_r[\sigma] = \lambda x w, u_r[\sigma] = v'_r v_r \) and \( \langle v'_r, \mu x (v_1, v_r) \rangle \notin SN \), then \( u_r = w'_r, w_r, v'_r[\sigma] = v'_r \) and \( w_r[\sigma] = v_r \).
  
  - If \( u_1 = \lambda x w_1 \) and \( w_1[\sigma] = v_1 \). Then \( \langle w'_1, \mu x (w_1, w_r) \rangle [\sigma] \notin SN \) and this contradicts the induction hypothesis, since \( \eta(w'_1, \mu x (w_1, w_r)) < \eta(t) \).
  
  - If \( u_1 = x_i, t_i = \lambda x w_1 \) and \( \langle w'_r[\sigma], \mu x (w_1[\tau_i], w_r[\sigma]) \rangle \notin SN \). Then \( \langle w_1[\tau_i], w_r[\sigma] \rangle = \langle y, u_r[\sigma] \rangle [y := w_1[\tau_i]] \) where \( y \) is a fresh variable and thus \( (w_1[\tau_i], w_r[\sigma]) \in SN \), since \( lg(type(w_1[\tau_i])) < lg(A) \).
  
    Since \( \langle w'_1[\sigma], \mu x (w_1[\tau_i], w_r[\sigma]) \rangle = \langle z, \mu x (w_1[\tau_i], w_r[\sigma]) \rangle [z := w'_1[\sigma]] \) where \( z \) is a fresh variable and \( lg(type(w'_1[\sigma])) < lg(A) \), this contradicts the induction hypothesis.

- If \( u_r[\sigma] = \lambda y v_r, u_1[\sigma] = v'_r v_1 \) and \( \langle \mu y v_1, v_r \rangle, v'_r \notin SN \). This is proved in the same way.

\( \square \)

Theorem 3.3. Every typed term is in \( SN \).

Proof By induction on the term. It is enough to show that if \( t_l, t_r \in SN \), then \( \langle t_l, t_r \rangle \in SN \). Since \( \langle t_l, t_r \rangle = \langle x, \alpha \rangle [x := t_l] [\alpha := t_r] \) where \( x, \alpha \) are fresh variables, the result follows from lemma 3.11.
4. Normalization for the symmetric $\lambda\mu$-calculus

4.1. The $\mu\mu'$-reduction is strongly normalizing

In this section we consider the $\mu\mu'$-reduction, i.e. $M \triangleright M'$ means $M'$ is obtained from $M$ by one step of the $\mu\mu'$-reduction. The proof of theorem 4.1 is essentially the same as the one of theorem 3.2. We first show (cf. lemma 4.2) that a $\mu$ or $\mu'$-substitution cannot create a $\mu$ and then we show (cf. lemma 4.3) that, if $M[\sigma] \in SN$ but $M[\sigma][\alpha = \tau P] \not\in SN$, then the real cause of non $SN$ is, in some sense, $[\alpha = \tau P]$. The main point is again that, in a reduction of $(M N) \in SN$, there is, in fact, no deep interactions between $M$ and $N$ i.e. in a reduct of $(M N)$ we always know what is coming from $M$ and what is coming from $N$.

Definition 4.1. The set of simultaneous substitutions of the form $[\alpha_1 =_{s_1} P_1 \ldots, \alpha_n =_{s_n} P_n]$ where $s_i \in \{l, r\}$ will be denoted by $\Sigma$.

- For $s \in \{l, r\}$, the set of simultaneous substitutions of the form $[\alpha_1 =_{s} P_1 \ldots \alpha_n =_{s} P_n]$ will be denoted by $\Sigma_s$.
- If $\sigma = [\alpha_1 =_{s_1} P_1 \ldots, \alpha_n =_{s_n} P_n]$, we denote by $dom(\sigma)$ (resp. $Im(\sigma)$) the set $\{\alpha_1, \ldots, \alpha_n\}$ (resp. $\{P_1, \ldots, P_n\}$).
- Let $\sigma \in \Sigma$. We say that $\sigma \in SN$ iff for every $N \in Im(\sigma)$, $N \in SN$.
- If $\overline{P}$ is a sequence $P_1, \ldots, P_n$ of terms, $(M \overline{P})$ will denote $(M P_1 \ldots P_n)$.

Lemma 4.1. If $(M N) \triangleright^* \mu\alpha P$, then either $M \triangleright^* \mu\alpha M_1$ and $M_1[\alpha = \tau N] \triangleright^* P$ or $N \triangleright^* \mu\alpha N_1$ and $N_1[\alpha =_{\tau} M] \triangleright^* P$.

Proof By induction on the length of the reduction $(M N) \triangleright^* \mu\alpha P$. □

Lemma 4.2. Let $M$ be a term and $\sigma \in \Sigma$. If $M[\sigma] \triangleright^* \mu\alpha P$, then $M \triangleright^* \mu\alpha Q$ for some $Q$ such that $Q[\sigma] \triangleright^* P$.

Proof By induction on $M$. $M$ cannot be of the form $(\beta M')$ or $\lambda x M'$. If $M$ begins with a $\mu$, the result is trivial. Otherwise $M = (M_1 M_2)$ and, by lemma 4.1, either $M_1[\sigma] \triangleright^* \mu\alpha R$ and $R[\alpha = \tau M_2[\sigma]] \triangleright^* P$ or $M_2[\sigma] \triangleright^* \mu\alpha R$ and $R[\alpha =_{\tau} M_1[\sigma]] \triangleright^* P$. Look at the first case (the other one is similar). By the induction hypothesis $M_1 \triangleright^* \mu\alpha Q$ for some $Q$ such that $Q[\sigma] \triangleright^* R$ and thus $M \triangleright^* \mu\alpha Q[\alpha = \tau M_2]$. Since $Q[\alpha = \tau M_2][\sigma] = Q[\sigma][\alpha = \tau M_2[\sigma]] \triangleright^* R[\alpha = \tau M_2[\sigma]] \triangleright^* P$ we are done. □

Lemma 4.3. Assume $M, N \in SN$ and $(M N) \not\in SN$. Then either $M \triangleright^* \mu\alpha M_1$ and $M_1[\alpha = \tau N] \not\in SN$ or $N \triangleright^* \mu\beta N_1$ and $N_1[\beta =_{\tau} M] \not\in SN$.

Proof By induction on $\eta(M) + \eta(N)$. Since $(M N) \not\in SN$, $(M N) \triangleright P$ for some $P$ such that $P \not\in SN$. If $P = (M' N)$ where $M \triangleright M'$ we conclude by the induction hypothesis since $\eta(M') + \eta(N) < \eta(M) + \eta(N)$. If $P = (M N')$ where $N \triangleright N'$ the proof is similar. If $M = \mu\alpha M_1$ and $P = \mu\alpha M_1[\alpha = \tau N]$ or $N = \mu\beta N_1$ and $P = \mu\beta N_1[\beta =_{\tau} M]$ the result is trivial. □
Lemma 4.4. Let $M$ be a term and $\sigma \in \Sigma_s$. Assume $\delta$ is free in $M$ but not free in $\text{Im}(\sigma)$. If $M[\sigma] \in SN$ but $M'[\sigma][\delta = s_P] \notin SN$, there is $M' < M$ and $\sigma'$ such that $M'[\sigma'] \in SN$ and, if $s = r$, $(M'[\sigma'] P) \notin SN$ and, if $s = l$, $(P M'[\sigma']) \notin SN$.

Proof. Assume $s = r$ (the other case is similar). Let $\text{Im}(\sigma) = \{N_1, \ldots, N_k\}$. Assume $M, \delta, \sigma, P$ satisfy the hypothesis. Let $U = \{U / U \leq M\}$ and $V = \{V / V \leq N_i \text{ for some } i\}$. Define inductively the sets $\Sigma_m$ and $\Sigma_n$ of substitutions by the following rules:

\[ \rho \in \Sigma_m \text{ iff } \rho = \emptyset \text{ or } \rho = \rho' + [\beta = r V[\tau]] \text{ for some } V \in V, \tau \in \Sigma_n \text{ and } \rho' \in \Sigma_m. \]

\[ \tau \in \Sigma_n \text{ iff } \tau = \emptyset \text{ or } \tau = \tau' + [\alpha = l U[\rho]] \text{ for some } U \in U, \rho \in \Sigma_m \text{ and } \tau' \in \Sigma_n. \]

Denote by $C$ the conclusion of the lemma, i.e. there is $M' < M$ and $\sigma'$ such that $M'[\sigma'] \in SN$, and $(M'[\sigma'] P) \notin SN$.

We prove something more general.

1. Let $U \in U$ and $\rho \in \Sigma_m$. Assume $U[\rho] \in SN$ and $U[\rho][\delta = r P] \notin SN$. Then, $C$ holds.

2. Let $V \in V$ and $\tau \in \Sigma_n$. Assume $V[\tau] \in SN$ and $V[\tau][\delta = r P] \notin SN$. Then, $C$ holds.

The conclusion $C$ follows from (1) with $M$ and $\sigma$. The properties (1) and (2) are proved by a simultaneous induction on $\eta_c(U[\rho])$ (for the first case) and $\eta_c(V[\tau])$ (for the second case).

Look first at (1)

- if $U = \lambda x U'$ or $U = \mu \alpha U'$: the result follows from the induction hypothesis with $U'$ and $\rho$.

- if $U = (U_1 U_2)$: if $U_1[\rho][\delta = r P] \notin SN$ for $i = 1$ or $i = 2$, the result follows from the induction hypothesis with $U_i$ and $\rho$. Otherwise, by lemma 4.2 and 4.3, say $U_1 \triangleright^* \mu \alpha U'$ and, letting $U' = U_1[\alpha = l, U_2], U'[\rho][\delta = r P] \notin SN$ and the result follows from the induction hypothesis with $U'$ and $\rho$.

- if $U = (\delta U_1)$: if $U_1[\rho][\delta = r P] \in SN$, then $M = U_1$ and $\sigma' = \rho[\delta = r P]$ satisfy the desired conclusion. Otherwise, the result follows from the induction hypothesis with $U_1$ and $\rho$.

- if $U = (\alpha U_1)$: if $\alpha \notin \text{dom}(\rho)$ or $U_1[\rho][\delta = r P] \notin SN$, the result follows from the induction hypothesis with $U_1$ and $\rho$. Otherwise, let $\rho(\alpha) = V[\tau]$. If $V[\tau][\delta = r P] \notin SN$, the result follows from the induction hypothesis with $V$ and $\tau$ (with (2)). Otherwise, by lemmas 4.2 and 4.3 there are two cases to consider.

- $U_1 \triangleright^* \mu \alpha_1 U_2$ and $U_2[\rho'][\delta = r P] \notin SN$ where $\rho' = \rho + [\alpha_1 = r V[\tau]]$. The result follows from the induction hypothesis with $U_2$ and $\rho'$.

- $V \triangleright^* \mu \beta V_1$ and $V_1[\tau'][\delta = r P] \notin SN$ where $\tau' = \tau + [\beta = l U_1[\rho]]$. The result follows from the induction hypothesis with $V_1$ and $\tau'$ (with (2)).

The case (2) is proved in the same way. Note that, since $\delta$ is not free in the $N_i$, the case $b = (\delta V_1)$ does not appear. \qed

Theorem 4.1. Every term is in $SN$.

Proof. By induction on the term. It is enough to show that, if $M, N \in SN$, then $(M N) \in SN$. We prove something more general: let $\sigma$ (resp. $\tau$) be in $\Sigma_r$ (resp. $\Sigma_l$) and assume $M[\sigma], N[\tau] \in SN$. Then $(M[\sigma] N[\tau]) \in SN$. Assume it is not the case and choose some elements such that $M[\sigma], N[\tau] \in SN$, $(M[\sigma] N[\tau]) \notin SN$ and
(η(M) + η(N), cxty(M) + cxty(N)) is minimal. By lemma 4.3, either M[σ] ▷⁺ μδM₁ and M₁[δ =r, N[τ]] ∉ SN or N[τ] ▷⁺ μβN₁ and N₁[δ =i, M[σ]] ∉ SN. Look at the first case (the other one is similar). By lemma 4.2, contradicts the minimality of the chosen elements since looking at the first case (the other one is similar). By lemma 4.2, contradicts the minimality of the chosen elements since looking at the first case (the other one is similar).

4.2. The simply typed symmetric λμ-calculus is strongly normalizing

In this section, we consider the simply typed calculus with the βμμ'-reduction i.e. M ▷ M' means M' is obtained from M by one step of the βμμ'-reduction. The strong normalization of the βμμ'-reduction is proved essentially as in theorem 5.3.

There is, however, a new difficulty: a β-substitution may create a μ, i.e. the fact that M[x := N] ▷⁺ μαP does not imply that M ▷⁺ μαQ. Moreover the μ may come from a complicated interaction between M and N and, in particular, the alternation between M and N can be lost. Let e.g. M = (M₁(x (λy₁λy₂μαM₁) M₂ M₃)) and N = λz(z N₁). Then M[x := N] ▷⁺ (M₁ (μαM₁' M₃)) ▷⁺ μαM₁'[α =r, M₃][α =i, M₁]. To deal with this situation, we need to consider some new kind of μμ'-substitutions (see definition 4.2). Lemma 4.10 gives the different ways in which a μ may appear. The difficult case in the proof (when a μ is created and the control between M and N is lost) will be solved by using a typing argument.

To simplify the notations, we do not write explicitly the type information but, when needed, we denote by type(M) the type of the term M.

Lemma 4.5. 1. If (M N) ▷⁺ λxP, then M ▷⁺ λyM₁ and M₁[y := N] ▷⁺ λxP.

2. If (M N) ▷⁺ μαP, then either (M ▷⁺ λyM₁ and M₁[y := N] ▷⁺ μαP) or (M ▷⁺ μαM₁ and M₁[α =r, N] ▷⁺ P) or (N ▷⁺ μαN₁ and N₁[α =i, M] ▷⁺ P).

Proof (1) is trivial. (2) is as in lemma 4.3.

Lemma 4.6. Let M ∈ SN and σ = [x₁ := N₁, ..., xₖ := Nₖ]. Assume M[σ] ▷⁺ λyP. Then, either M ▷⁺ λyP₁ and P₁[σ] ▷⁺ P or M ▷⁺ (x₁ Q) and (N₁ Q[σ]) ▷⁺ λyP.

Proof By induction on ηc(M). The only non immediate case is M = (R S). By lemma 4.5, there is a term R₁ such that R[σ] ▷⁺ λzR₁ and R₁[z := S[σ]] ▷⁺ λyP. By the induction hypothesis (since ηc(R) < ηc(M)), we have two cases to consider.

1. R ▷⁺ λzR₂ and R₂[σ] ▷⁺ R₁, then R₂[z := S][σ] ▷⁺ λyP. By the induction hypothesis (since η(R₂[z := S]) < η(M)),
   - either R₂[z := S] ▷⁺ λyP₁ and P₁[σ] ▷⁺ P; but then M ▷⁺ λyP₂ and we are done.
   - or R₂[z := S] ▷⁺ (x₁ Q) and (N₁ Q[σ]) ▷⁺ λyP, then M ▷⁺ (x₁ Q) and again we are done.

2. R ▷⁺ (x₁ Q) and (N₁ Q[σ]) ▷⁺ λzR₁. Then M ▷⁺ (x₁ Q S) and the result is trivial.

Proof (1) is trivial. (2) is as in lemma 4.3.
Definition 4.2. • An address is a finite list of symbols in \{l, r\}. The empty list is denoted by [] and, if \(a\) is an address and \(s \in \{l, r\}\), \([s :: a]\) denotes the list obtained by putting \(s\) at the beginning of \(a\).

• Let \(a\) be an address and \(M\) be a term. The sub-term of \(M\) at the address \(a\) (denoted as \(M_a\)) is defined recursively as follows: if \(M = (P \ Q)\) and \(a = [r :: b]\) (resp. \(a = [l :: b]\)) then \(M_a = Q_b\) (resp. \(P_b\)) and undefined otherwise.

• Let \(M\) be a term and \(a\) be an address such that \(M_a\) is defined. Then \(M(a = N)\) is the term \(M\) where the sub-term of \(M_a\) has been replaced by \(N\).

• Let \(M, N\) be some terms and \(a\) be an address such that \(M_a\) is defined. Then \(N[\alpha = a M]\) is the term \(N\) in which each sub-term of the form \((\alpha U)\) is replaced by \((\alpha M(a = U))\).

Remark 4.1. Let \(N = \lambda x(\alpha \lambda y(\mu \beta(\alpha y)))\), \(M = (M_1 \ (M_2 \ M_3))\) and \(a = [r :: l]\). Then \(N[\alpha = a M] = \lambda x(\alpha \ (M_1 \ (\lambda y(\mu \beta(\alpha (M_1 \ (y \ M_3)))) \ M_3))).\)

• Let \(M = (P \ ((R \ (x \ T)) \ Q))\) and \(a = [r :: l :: r :: l]\). Then \(N[\alpha = a M] = N[\alpha = r \ T][\alpha = l \ R][\alpha = r \ Q][\alpha = r \ P].\)

• Note that the sub-terms of a term having an address in the sense given above are those for which the path to the root consists only on applications (taking either the left or right son).

• Note that \([\alpha = r] M\) is not the same as \([\alpha = l] M\) but \([\alpha = l] M\) is the same as \([\alpha = r] (M \ N)\) where \(N\) does not matter. More generally, the term \(N[\alpha = a M]\) does not depend of \(M_a\).

• Note that \(M(a = N)\) can be written as \(M'[x_a := N]\) where \(M'\) is the term \(M\) in which \(M_a\) has been replaced by the fresh variable \(x_a\) and thus (this will be used in the proof of lemma 4.12) if \(M_a\) is a variable \(x\), \((\alpha U)[\alpha = a M] = (\alpha M_1[y := U[\alpha = a M]])\) where \(M_1\) is the term \(M\) in which the particular occurrence of \(x\) at the address \(a\) has been replaced by the fresh name \(y\) and the other occurrences of \(x\) remain unchanged.

Lemma 4.7. Let \(M\) be a term and \(\sigma = [\alpha_1 = a_1, N_1, ..., \alpha_n = a_n, N_n]\).

1. If \(M[\sigma] \triangleright^+ \lambda x P\), then \(M \triangleright^+ \lambda x Q\) and \(Q[\sigma] \triangleright^+ P\).

2. If \(M[\sigma] \triangleright^+ \mu \alpha P\), then \(M \triangleright^+ \mu \alpha Q\) and \(Q[\sigma] \triangleright^+ P\).

Proof By induction on \(M\). Use lemma 4.3. \(\square\)

Lemma 4.8. Assume \(M, N \in SN\) and \((M N) \not\in SN\). Then, either \((M \triangleright^+ \lambda y P\) and \(P[y := N] \not\in SN)\) or \((M \triangleright^+ \mu \alpha P\) and \(P[\alpha = r N] \not\in SN)\) or \((N \triangleright^+ \mu \alpha P\) and \(P[\alpha = l M] \not\in SN)\).

Proof By induction on \(\eta(M) + \eta(N)\). \(\square\)

Lemma 4.9. If \(\Gamma \vdash M : A\) and \(M \triangleright^+ N\) then \(\Gamma \vdash N : A\).

Proof Straightforward. \(\square\)
Lemma 4.10. Let \( n \) be an integer, \( M \in SN \), \( \sigma = [x_1 := N_1, \ldots, x_k := N_k] \) where \( lg(type(N_i)) = n \) for each \( i \). Assume \( M[\sigma] \triangleright^* \mu \alpha P \). Then,

1. either \( M \triangleright^* \mu \alpha P_1 \) and \( P_1[\sigma] \triangleright^* P \)
2. or \( M \triangleright^* Q \) and, for some \( i \), \( N_i \triangleright^* \mu \alpha N_i' \) and \( N_i'[\alpha = a Q[\sigma]] \triangleright^* P \) for some address \( a \) in \( Q \) such that \( Q_a = x_i \).
3. or \( M \triangleright^* Q, Q_a[\sigma] \triangleright^* \mu \alpha N' \) and \( N'[\alpha = a Q[\sigma]] \triangleright^* P \) for some address \( a \) in \( Q \) such that \( lg(type(Q_a)) < n \).

Proof. By induction on \( \eta \sigma(M) \). The only non immediate case is \( M = (RS) \). Since \( M[\sigma] \triangleright^* \mu \alpha P \), the application \( (R[\sigma] S[\sigma]) \) must be reduced. Thus there are three cases to consider.

- It is reduced by a \( \mu' \)-reduction, i.e. there is a term \( S_1 \) such that \( S[\sigma] \triangleright^* \mu \alpha S_1 \) and \( S_1[\alpha = R[\sigma]] \triangleright^* P \). By the induction hypothesis:
  - either \( Sb \triangleright^* \mu \alpha Q \) and \( Q[\sigma] \triangleright^* S_1 \), then \( M \triangleright^* \mu \alpha Q[\alpha = \eta R] \) and \( Q[\alpha = \eta R][\sigma] \triangleright^* P \).
  - or \( S \triangleright^* Q \) and, for some \( i \), \( N_i \triangleright^* \mu \alpha N_i' \), \( Q_a = x_i \) for some address \( a \) in \( Q \) and \( N_i'[\alpha = a Q[\sigma]] \triangleright^* S_1 \). Then \( M \triangleright^* (RQ) = Q' \) and letting \( b = [r :: a] \) we have \( N_i'[\alpha = b Q'[\sigma]] \triangleright^* P \).
  - or \( S \triangleright^* Q, Q_a[\sigma] \triangleright^* \mu \alpha N' \) for some address \( a \) in \( Q \) such that \( lg(type(Q_a)) < n \) and \( N'[\alpha = a Q[\sigma]] \triangleright^* S_1 \). Then \( M \triangleright^* (RQ) = Q' \) and letting \( b = [r :: a] \) we have \( N'[\alpha = b Q'[\sigma]] \triangleright^* P \) and \( lg(type(Q'_a)) < n \).

- It is reduced by a \( \mu \)-reduction. This case is similar to the previous one.

- It is reduced by a \( \beta \)-reduction, i.e. there is a term \( U \) such that \( R[\sigma] \triangleright^* \lambda y U \) and \( U[y := S[\sigma]] \triangleright^* \mu \alpha P \). By lemma 4.7, there are two cases to consider.
  - either \( R \triangleright^* \lambda y R_1 \) and \( R_1[\sigma][y := S[\sigma]] = R_1[y := S][\sigma] \triangleright^* \mu \alpha P \). The result follows from the induction hypothesis since \( \eta(R_1[y := S]) < \eta(M) \).
  - or \( R \triangleright^* (x_i R_1) \). Then \( Q = (x_i R_1 S) \) and \( a = [] \) satisfy the desired conclusion since then \( lg(type(M)) < n \).

Definition 4.3. Let \( A \) be a type. We denote by \( \Sigma_A \) the set of substitutions of the form \( [\alpha_1 = a_1, M_1, \ldots, \alpha_n = a_n, M_n] \) where the type of the \( \alpha_i \) is \( \neg A \).

Remark 4.2. Remember that the type of \( \alpha \) is not the same in \( N \) and in \( N[\alpha = a M] \). The previous definition may thus be considered as ambiguous. When we consider the term \( N[\sigma] \) where \( \sigma \in \Sigma_A \), we assume that \( N \) (and not \( N[\sigma] \)) is typed in the context where the \( \alpha_i \) have type \( A \). Also note that considering \( N[\alpha = a M] \) implies that the type of \( M_a \) is \( A \).

Lemma 4.11. Let \( n \) be an integer and \( A \) be a type such that \( lg(A) = n \). Let \( N, P \) be terms and \( \tau \in \Sigma_A \). Assume that,

- for every \( M, N \in SN \) such that \( lg(type(N)) < n \), \( M[x := N] \in SN \).
- \( N[\tau] \in SN \) but \( N[\tau][\delta = a P] \notin SN \).
\( \delta \) is free and has type \(-A\) in \( N \) but \( \delta \) is not free in \( Im(\tau) \).

Then, there is \( N' \prec N \) and \( \tau' \in \Sigma_A \) such that \( N'[\tau'] \in SN \) and \( P(\alpha = N'[\tau']) \notin SN \).

**Proof**

The proof looks like the one of lemma 4.4. Denote by (H) the first assumption i.e. for every \( M, N \in SN \) such that \( lg(type(N)) < n, M[x := N] \in SN \).

Let \( \tau = [\alpha_1 = a_1 M_1, \ldots, \alpha_n = a_n M_n], U = \{ U / U \lea N \} \) and \( V = \{ V / V \lea M_i \text{ for some } i \} \). Define inductively the sets \( \Sigma_m \) and \( \Sigma_n \) of substitutions by the following rules:

- Assume next the interaction between \( V \) and \( \Sigma_m \) such that \( \Sigma_m[\tau'] \in SN \) and \( P(\alpha = N'[\tau']) \notin SN \).
- Assume first the interaction between \( \Sigma_n \) and \( \Sigma_m \) such that \( \Sigma_n[\tau'] \in SN \) and \( P(\alpha = N'[\tau']) \notin SN \).

Denote by \( C \) the conclusion of the lemma. We prove something more general.

(1) Let \( U \in U \) and \( \rho \in \Sigma_n \). Assume \( U[\rho] \in SN \) and \( U[\rho][\delta = a P] \notin SN \). Then, \( C \) holds.

(2) Let \( V \in V \) and \( \sigma \in \Sigma_m \). Assume \( V[\sigma] \in SN \) and \( V[\sigma][\delta = a P] \notin SN \). Then, \( C \) holds.

Note that the definitions of the sets \( \Sigma_n \) and \( \Sigma_m \) are not the same as the ones of lemma 4.4. We gather here in \( \Sigma_n \) all the \( \mu\mu'\)-substitutions getting thus the new substitutions of definition 4.3 and we put in \( \Sigma_m \), only the \( \lambda\)-substitutions.

The conclusion \( C \) follows from (1) with \( N \) and \( \tau \). The properties (1) and (2) are proved by a simultaneous induction on \( \eta c(U[\rho]) \) (for the first case) and \( \eta c(V[\tau]) \) (for the second case).

The proof is by case analysis as in lemma 4.4. We only consider the new case for \( V[\sigma] \), i.e. when \( V = (V_1 V_2) \) and \( V_1[\sigma][\delta = a P] \in SN \). The other ones are done essentially in the same way as in lemma 4.4.

- Assume first the interaction between \( V_1 \) and \( V_2 \) is a \( \beta\)-reduction. If \( V_1 \triangleright^* \lambda x V_1' \), the result follows from the induction hypothesis with \( V_1'[x := V_2][\sigma] \). Otherwise, by lemma 4.4, \( V_1 \triangleright^* (x \bar{W}) \). Let \( \sigma(x) = U[\rho] \). Then \( (U[\rho][\bar{W}[\sigma]] \triangleright^* \lambda y Q \) and \( Q[y := V_2[\sigma]][\delta = a P] \notin SN \). But, since the type of \( y \) is \( A \), the type of \( y \) is less than \( A \) and since \( Q[\delta = a P] \) and \( V_2[\sigma][\delta = a P] \) are in \( SN \) this contradicts (H).

- Assume next the interaction between \( V_1 \) and \( V_2 \) is a \( \mu \) or \( \mu'\)-reduction. We consider only the case \( \mu \) (the other one is similar). If \( V_1 \triangleright^* \mu \alpha V_1' \), the result follows from the induction hypothesis with \( V_1'[\alpha = \bar{r}, V_2][\sigma] \). Otherwise, by lemma 4.10, there are two cases to consider.

  - \( V_1 \triangleright^* Q, \ Q_c = x \) for some address \( c \) in \( Q \) and \( x \in dom(\sigma), \sigma(x) = U[\rho], U[\rho] \triangleright^* \mu \alpha U_1 \) and \( U_1[\alpha = c Q[\sigma]][\alpha = \bar{r}, V_2[\sigma]][\delta = a P] \notin SN \). By lemma 4.7, we have \( U \triangleright^* \mu \alpha U_2 \) and \( U_2[\rho] \triangleright^* U_1 \), then \( U_2[\rho][\alpha = c Q[\sigma]][\alpha = \bar{r}, V_2[\sigma]][\delta = a P] \notin SN \). Let \( V' = (Q V_2) \) and \( b = l :: c \). The result follows then from the induction hypothesis with \( U_2[\rho'] \) where \( \rho' = \rho + [\alpha = b V'[\sigma]] \).

  - \( V_1 \triangleright Q, Q_c[\sigma][\delta = a P][\rho] \triangleright^* \mu \alpha \bar{R} \) for some address \( c \) in \( Q \) such that \( lg(type(Q_c)) < n, R[\alpha = c Q[\sigma]][\delta = a P][\alpha = \bar{r}, V_2[\sigma]][\delta = a P] \notin SN \). Let \( V' = (Q V_2) \) where \( Q' \) is the same as \( Q \) but \( Q_c \) has been replaced by a fresh variable \( y \) and \( b = l :: c \). Then \( R[\alpha = b V'[\sigma]][\delta = a P] \notin SN \). Let \( R' \) be such that \( R' \prec R, R'[\alpha = b V'[\sigma]][\delta = a P] \notin SN \) and \( \eta c(R') \) is minimal. It is easy to check that \( R' = (\alpha R'') \), \( R''[\alpha = b V'[\sigma]][\delta = a P] \in SN \) and \( V'[\sigma'][\delta = a P] \notin SN \) where \( \sigma' = \sigma + \)
If \( y := R''[\alpha \mapsto_b V''[\sigma]] \). If \( V''[\sigma][\delta \mapsto_a P] \not\in SN \), we get the result by the induction hypothesis since \( \eta(c(V''[\sigma])) < \eta(c(V[\sigma])) \). Otherwise, this contradicts the assumption (H) since \( V''[\sigma][\delta \mapsto_a P], R''[\alpha \mapsto_b V''[\sigma][\delta \mapsto_a P]] \in SN, V''[\sigma][\delta \mapsto_a P] [y := R''[\alpha \mapsto_b V''[\sigma][\delta \mapsto_a P]]] \not\in SN \) and the type of \( y \) is less than \( n \).

**Lemma 4.12.** If \( M, N \in SN \), then \( M[x := N] \in SN \).

**Proof.** We prove something a bit more general: let \( M, N, x \) be terms and \( \tau_1, \ldots, \tau_k \) be substitutions in \( \Sigma_A \). Assume that, for each \( i \), \( N_i \) has type \( A \) and \( N_i[\tau_i] \in SN \). Then \( M[x_1 := N_1[\tau_1], \ldots, x_k := N_k[\tau_k]] \in SN \). This is proved by induction on \( \{lg(A), \eta(M), cxy(M), \Sigma \eta(N_i), \Sigma cxy(N_i)\} \) where, in \( \Sigma \eta(N_i) \) and \( \Sigma cxy(N_i) \), we count each occurrence of the substituted variable. For example if \( k = 1 \) and \( x_1 \) has \( n \) occurrences, \( \Sigma \eta(N_i) = n \eta(N_i) \).

If \( M \) is \( \lambda y M_1 \) or \( (\alpha M_1) \) or \( \mu \alpha M_1 \) or a variable, the result is trivial. Assume then that \( M = (M_1, M_2) \). Let \( \sigma = [x_1 := N_1[\tau_1], \ldots, x_k := N_k[\tau_k]] \). By the induction hypothesis, \( M_1[\sigma], M_2[\sigma] \in SN \). By lemma 4.13 there are 3 cases to consider.

1. \( M_1[\sigma] \triangleright^* \lambda y P \) and \( P[y := M_2[\sigma]] \not\in SN \). By lemma 4.10 there are two cases to consider.

   - \( M_1 \triangleright^* \lambda y Q \) and \( Q[\sigma] \triangleright^* P \). Then \( Q[y := M_2][\sigma] = Q[\sigma][y := M_2[\sigma]] \triangleright^* P[y := M_2[\sigma]] \) and, since \( \eta(Q[y := M_2]) < \eta(M) \), this contradicts the induction hypothesis.

   - \( M_1 \triangleright^* (x_i \overline{Q}) \) and \( (N_i \overline{Q[\sigma]}) \triangleright^* \lambda y P \). Then, since the type of \( N_i \) is \( A \), \( lg(type(y)) < lg(A) \). But \( M_2[\sigma] \in SN \) and \( P[y := M_2[\sigma]] \not\in SN \). This contradicts the induction hypothesis.

2. \( M_1[\sigma] \triangleright^* \mu \alpha P \) and \( P[\alpha \mapsto_r M_2[\sigma]] \not\in SN \). By lemma 4.10 there are three cases to consider.

   - \( M_1 \triangleright^* \mu \alpha Q \) and \( Q[\sigma] \triangleright^* P \). Then, \( Q[\alpha \mapsto_r M_2][\sigma] = Q[\sigma][\alpha \mapsto_r M_2[\sigma]] \triangleright^* P[\alpha \mapsto_r M_2[\sigma]] \) and, since \( \eta(Q[\alpha \mapsto_r M_2]) < \eta(M) \), this contradicts the induction hypothesis.

   - \( M_1 \triangleright^* Q, N_i[\tau_i] \triangleright^* \mu \alpha L' \) and \( Q_a = x_i \) for some address \( a \) in \( Q \) such that \( L'[\alpha \mapsto_a Q[\sigma]] \triangleright^* P \) and thus \( L'[\alpha \mapsto_a M'[\sigma]] \not\in SN \) where \( b = (l :: a) \) and \( M' = (Q, M_2) \).

   By lemma 4.12, \( N_i \triangleright^* \mu \alpha L \) and \( L[\tau_i] \triangleright^* L' \). Thus, \( L[\tau_i][\alpha \mapsto_b M'[\sigma]] \not\in SN \). By lemma 4.11, there is \( L_1 < L \) and \( \tau' \) such that \( L_1[\tau'] \in SN \) and \( M'[\sigma][\mu \alpha L'] \not\in SN \). Let \( M'' \) be \( M' \) where the variable \( x_i \) at the address \( b \) has been replaced by the fresh variable \( y \) and let \( \sigma_1 = \sigma + [y := L_1[\tau']] \). Then \( M''[\sigma_1] = M'[\sigma][\mu \alpha L'] \not\in SN \).

   If \( M_1 \triangleright^* Q \) we get a contradiction from the induction hypothesis since \( \eta(M'') < \eta(M) \). Otherwise, \( M'' \) is the same as \( M \) up to the change of name of a variable and \( \sigma_1 \) differs from \( \sigma \) only at the address \( b \). At this address, \( x_i \) was substituted in \( \sigma \) by \( N_i[\tau_i] \) and in \( \sigma_1 \) by \( L_1[\tau'] \) but \( \eta(c(L_1)) < \eta(c(N_i)) \) and thus we get a contradiction from the induction hypothesis.
– $M \triangleright^* Q, Q_a[\sigma] \triangleright^* \mu \alpha L$ for some address $a$ in $Q$ such that $\text{lg}(\text{type}(Q_a)) < \text{lg}(A)$ and $L[\alpha =_b M'[\sigma]] \not\in SN$ where $b = [l :: a]$ and $M' = (Q M_2)$.

By lemma 4.11, there is an $L'$ and $\tau'$ such that $L'[\tau'] \in SN$ and $M'[\sigma][b = L'[\tau']] \not\in SN$. Let $M''$ be $M'$ where the variable $x_i$ at the address $b$ has been replaced by the fresh variable $y$. Then $M''[\sigma][y := L'[\tau']] = M'[\sigma][b = L'[\tau']] \not\in SN$.

But $\eta(M'') \leq \eta(M)$ and $\text{ctxy}(M'') < \text{ctxy}(M)$ since, because of its type, $Q_a$ cannot be a variable and thus, by the induction hypothesis, $M''[\sigma] \in SN$. Since $M''[\sigma][y := L'[\tau']] \not\in SN$ and $\text{lg}(\text{type}(L')) < \text{lg}(A)$, this contradicts the induction hypothesis.

• $M_2[\sigma] \triangleright^* \mu \alpha P$ and $P[\alpha =_l M_1[\sigma]] \not\in SN$. This case is similar to the previous one. □

**Theorem 4.2.** Every typed term is in $SN$.

**Proof** By induction on the term. It is enough to show that if $M, N \in SN$, then $(M N) \in SN$. Since $(M N) = (x y)[x := M][y := N]$ where $x, y$ are fresh variables, the result follows by applying theorem 4.12 twice and the induction hypothesis. □

5. Remarks and future work

5.1. Why the usual candidates do not work?

In [26], the proof of the strong normalization of the $\lambda\mu$-calculus is done by using the usual (i.e. defined without a fix-point operation) candidates of reducibility. This proof could be easily extended to the symmetric $\lambda\mu$-calculus if we knew the following properties for the un-typed calculus:

1. If $N$ and $(M[x := N] \overrightarrow{P})$ are in $SN$, then so is $(\lambda x M N \overrightarrow{P})$.
2. If $N$ and $(M[\alpha =_\tau N] \overrightarrow{P})$ are in $SN$, then so is $(\mu \alpha M N \overrightarrow{P})$.
3. If $\overrightarrow{P}$ are in $SN$, then so is $(x \overrightarrow{P})$.

These properties are easy to show for the $\beta\mu$-reduction but they were not known for the $\beta\mu\mu'$-reduction.

The third property is true but the properties (1) and (2) are false. The proof of (3) and the counter-examples for (1) and (2) can be found in [10].

5.2. Future work

We believe that our technique, will allow to give explicit bounds for the length of the reductions of a typed term. This is a goal we will try to manage.
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