AN EXTENSION OF PYTHAGORAS THEOREM

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Abstract. This article proves a Pythagoras-type formula for the sides and diagonals of a polygon inscribed in a semicircle having one of the sides of the polygon as diameter.

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This paper presents a Pythagoras type theorem regarding a cyclic polygon having one of the sides as a diameter of the circumscribed circle.

Let us first observe that Pythagoras theorem [1–5] and its reciprocal can be stated differently.

Theorem 1. If a triangle is inscribed in a semicircle with radius $R$, such that its longest side is the diameter of the semicircle, then the following relation exists between its sides:

\[ a^2 = b^2 + c^2, \]  

which can also be expressed as

\[ 4R^2 = b^2 + c^2. \]

Its reciprocal can be states as follows: if the relation $a^2 = b^2 + c^2$ exists between the sides of a triangle, then that triangle can be inscribed in a semicircle such that the longest side is also the diameter of the semicircle.

We can state and prove a similar theorem for quadrilaterals. Let $ABCD$ be a quadrilateral with vertices $B$ and $C$ located on the semicircle that has the longest side $AD$ as diameter.

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Theorem 2. If a quadrilateral $ABCD$ with its longest side $AD = d$ can be inscribed in a semicircle with the diameter $AD$, then its sides satisfy the following relation:

\[ d^2 = a^2 + b^2 + c^2 + \frac{2abc}{d}. \]  

Proof. We can successively deduce the following:

\[
AD^2 = AB^2 + BD^2 \\
= AB^2 + BC^2 + CD^2 - 2BC \cdot CD \cdot \cos(C) \\
= AB^2 + BC^2 + CD^2 + 2BC \cdot CD \cdot \cos(A) \\
= AB^2 + BC^2 + CD^2 + 2BC \cdot CD \cdot \frac{AB}{AD}.
\]

\[ \blacksquare \]

Particular case. If $B$ coincides with $A$, then $a = 0$, the quadrilateral becomes triangle $ACD$, and relation (3) becomes $d^2 = b^2 + c^2$, which represents Pythagoras theorem for this right triangle.

The reciprocal of Theorem 2 does not hold true: if the sides of a quadrilateral $ABCD$ satisfy relation (3), the quadrilateral does not have to be inscribed in a semicircle having the longest side as diameter. For example, a quadrilateral with sides $AD = 4\sqrt{2}$, $AB = \sqrt{2}$, $BC = 3 + \sqrt{5}$, $CD = 3 - \sqrt{5}$ and $m(\angle A) = 90^\circ$ does satisfy relation (3), has $AD$ as its longest side, but cannot be inscribed in a semicircle with diameter $AD$.

However, the following statement is true:

Lemma. Let four segments have the lengths $a$, $b$, $c$, and $d$. If they satisfy relation (3), then there is at least one quadrilateral with these sides that can be inscribed in a semicircle having the longest side as diameter.
Proof. On a circle with diameter $AD = d = 2R$, we set a point $B$ such that $AB = a$, and a point $C$ such that $BC = b$. We can prove that $CD = c$ when the sides satisfy relation (3). We have successively:

$$
CD = AD \cos(D) = -AD \cos(B) = -AD \cdot \frac{AB^2 + BC^2 - AC^2}{2 \cdot AB \cdot BC} = -AD \cdot \frac{AB^2 + BC^2 - (AD^2 - CD^2)}{2 \cdot AB \cdot BC}
$$

$$
= -d \cdot \frac{a^2 + b^2 + c^2 - d^2}{2ab} = c.
$$

We thus found a quadrilateral with sides of lengths $a, b, c, d$.

Let us also observe that if $a, b, c$ are distinct, there are three incongruent quadrilaterals that satisfy this requirement, depending on which of the three sides, $a, b$, or $c$, is selected as the opposite side to $d$. □

Theorem 3. If a pentagon $ABCDE$ with sides $AB = a$, $BC = b$, $CD = c$, $DE = d$, $AE = 2R$ is inscribed in a circle with radius $R$, then its sides satisfy the following relation:

$$
4R^2 = a^2 + b^2 + c^2 + \frac{aby + xcd}{R},
$$

where $x = AC$ and $y = CE$.
Proof. In the quadrilateral $ABCE$ we can apply relation (3):

\[ 4R^2 = a^2 + b^2 + y^2 + \frac{aby}{R}. \]

Using the Law of Cosines we have

\[ y^2 = c^2 + d^2 - 2cd \cos(D) = c^2 + d^2 + 2cd \cos(\angle CAE) = c^2 + d^2 + 2cd \frac{x}{2R}. \]

Replacing $y^2$ from this relation in (5), we obtain relation (4).

For a hexagon, we can similarly prove the following:

**Theorem 4.** If we inscribe a hexagon in a semicircle such that the longest side is also the diameter, then the following relation is true:

\[ 4R^2 = a^2 + b^2 + c^2 + d^2 + e^2 + \frac{abz + ycx + ude}{R}. \]

We can restate the theorem in the case of a polygon with $n$ sides.

**Theorem 5.** If a polygon $A_1A_2A_3 \ldots A_{n-1}A_n$ can be inscribed in a semicircle with diameter $A_1A_n$, then the following relation is true:

\[ (A_1A_n)^2 = \sum_{k=1}^{n-1} (A_kA_{k+1})^2 + 2 \sum_{k=1}^{n-3} \frac{(A_1A_{k+1})(A_{k+1}A_{k+2})(A_{k+2}A_n)}{A_1A_n}. \]

**Observation.** The points $A_1, A_{k+1}, A_{k+2},$ and $A_n$, for $1 \leq k \leq n - 3$, involved in the sum from the right side of formula (7), are the vertices of a cyclic quadrilateral inscribed in a semicircle.
Proof. By using the mathematical induction technique, we have already demonstrated the particular cases for \( n = 3 \) and \( n = 4 \).

Let us assume that the property is true for any polygon with \( n \) sides and let us consider the case of a polygon \( A_1 A_2 A_3 \ldots A_n A_{n+1} \), for \( n \geq 4 \), inscribed in a circle with diameter \( A_1 A_{n+1} \). Following the induction hypothesis, in an \( n \)-sided polygon \( A_1 A_2 A_3 \ldots A_{n-1} A_{n+1} \) the following relation is true:

\[
(A_1 A_{n+1})^2 = \sum_{k=1}^{n-2} (A_k A_{k+1})^2 + (A_{n-1} A_{n+1})^2 + 2 \sum_{k=1}^{n-3} \frac{(A_1 A_{k+1}) (A_{k+1} A_{k+2}) (A_{k+2} A_{n+1})}{A_1 A_{n+1}}.
\]

Applying the Law of Cosines in triangle \( A_{n-1} A_n A_{n+1} \), and using the facts that:

\[
m(\angle A_{n-1} A_n A_{n+1}) + m(\angle A_{n-1} A_1 A_{n+1}) = 180^\circ
\]

and

\[
m(\angle A_1 A_{n-1} A_{n+1}) = 90^\circ,
\]

we obtain:

\[
(A_{n-1} A_{n+1})^2 = (A_{n-1} A_n)^2 + (A_n A_{n+1})^2 - 2 (A_{n-1} A_n) (A_n A_{n+1}) \cos (A_n)
\]

\[
= (A_{n-1} A_n)^2 + (A_n A_{n+1})^2 + 2 (A_{n-1} A_n) (A_n A_{n+1}) \cos (\angle A_{n-1} A_1 A_{n+1})
\]

\[
= (A_{n-1} A_n)^2 + (A_n A_{n+1})^2 + 2 (A_{n-1} A_n) (A_n A_{n+1}) \cdot \frac{A_1 A_{n-1}}{A_1 A_{n+1}}.
\]

Substituting \((A_{n-1} A_{n+1})^2\) into relation (8), we obtain:

\[
(A_1 A_{n+1})^2 = \sum_{k=1}^{n-2} (A_k A_{k+1})^2 + (A_{n-1} A_n)^2 + (A_n A_{n+1})^2 + 2 (A_{n-1} A_n) (A_n A_{n+1}) \cdot \frac{A_1 A_{n-1}}{A_1 A_{n+1}}
\]

\[
+ 2 \sum_{k=1}^{n-3} \frac{(A_1 A_{k+1}) (A_{k+1} A_{k+2}) (A_{k+2} A_{n+1})}{A_1 A_{n+1}}.
\]

The proof is now complete. \(\square\)
We can also note the validity of a reciprocal of this theorem: Let $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$ be $n$ segments with lengths $A_1A_2 = a_1, A_2A_3 = a_2, \ldots, A_{n-1}A_n = a_{n-1}, A_nA_1 = a_n$. If these segments satisfy relation (7), then there is at least one polygon with these segments as sides that can be inscribed in the semicircle with diameter $A_1A_n$. The proof can be developed in the same manner as for the case of $n = 4$.

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