The variational bicomplex on graded manifolds and its cohomology

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Abstract: Lagrangian formalism on graded manifolds is phrased in terms of the Grassmann-graded variational bicomplex, generalizing the familiar variational bicomplex for even Lagrangian systems on fiber bundles.

Lagrangian systems of odd and affine even fields on a smooth manifold $X$ ($\dim X = n$) can be described in algebraic terms of the Grassmann-graded variational bicomplex [2, 4, 8], generalizing the variational bicomplex for even Lagrangian systems on fiber bundles [1, 7, 14]. Here, this bicomplex is stated in a general setting, when a fiber bundle $Y \to X$ of even fields need not be affine. For this purpose, we consider graded manifolds whose body is a fiber bundle $Y \to X$ and its jet manifolds $J^r Y$, but not $X$. We show that the relevant cohomology of the Grassmann-graded variational bicomplex on these graded manifolds reduces to that of the variational bicomplex on a fiber bundle $Y \to X$.

Remark 1. Smooth manifolds throughout are assumed to be real, finite-dimensional, Hausdorff, second-countable (consequently, paracompact) and connected. By a Grassmann algebra over a ring $\mathcal{K}$ is meant a $\mathbb{Z}_2$-graded exterior algebra of some $\mathcal{K}$-module. We restrict our consideration to graded manifolds $(Z, \mathfrak{A})$ with structure sheaves $\mathfrak{A}$ of Grassmann algebras of finite rank [3, 9]. The symbols $|.|$ and $[.]$ stand for the form degree and Grassmann parity, respectively. We denote by $\Lambda, \Sigma, \Xi, \Omega$ the symmetric multi-indices, e.g., $\Lambda = (\lambda_1 ... \lambda_k), \lambda + \Lambda = (\lambda \lambda_1 ... \lambda_k)$. Summation over a multi-index $\Lambda = (\lambda_1 ... \lambda_k)$ throughout means separate summation over each its index $\lambda_i$.

Let $J^r Y$, $r \in \mathbb{N}$, be finite order jet manifolds of sections of $Y \to X$, where $r = 0$ conventionally stands for $Y$. They make up the inverse system

$$X \leftarrow \pi Y \leftarrow J^1 Y \leftarrow \cdots J^{r-1} Y \leftarrow J^r Y \leftarrow \cdots,$$

(1)

where $\pi_{r-1}$ are affine bundles and, hence, open maps. Its projective limit $(J^\infty Y, \pi^\infty_r : J^\infty Y \to J^r Y)$ is a paracompact Fréchet manifold, called the infinite order jet manifold. Moreover, $Y$ is the strong deformation retract of $J^\infty Y$. A bundle atlas $\{(U, x^\lambda, y^\mu)\}$ of $Y \to X$ yields the coordinate atlas

$$\{((\pi^\infty_0)^{-1}(U); x^\lambda, y^\mu)\}, \quad y^\mu_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x^\lambda} d^\mu y^\mu_\Lambda, \quad 0 \leq |\Lambda|,$$

(2)

of $J^\infty Y$, where

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} y^i_{\lambda+\Lambda} \partial^\Lambda_i, \quad d_\Lambda = d_{\lambda_1} \circ \cdots \circ d_{\lambda_k},$$

(3)

are total derivatives. Let us fix an atlas of $Y$ containing a finite number of charts [10].
The inverse system (1) yields the direct system

\[
\mathcal{O}^* x \xrightarrow{\pi^*} \mathcal{O}^* y \xrightarrow{\pi^*_1} \mathcal{O}^*_1 y \rightarrow \cdots \mathcal{O}^*_{r-1} y \xrightarrow{\pi^*_r} \mathcal{O}^*_r y \rightarrow \cdots ,
\]

of graded differential algebras (henceforth GDAs) \(\mathcal{O}^*_y\) of exterior forms on jet manifolds \(J^r Y\) with respect to the pull-back monomorphisms \(\pi^*_r\). Its direct limit is the GDA \(\mathcal{O}^*_\infty Y\) of all exterior forms on finite order jet manifolds modulo the pull-back identification. One can think of elements of \(\mathcal{O}^*_\infty Y\) as being exterior forms on the infinite order jet manifold \(J^\infty Y\) as follows. Let \(\mathcal{O}^*_r\) be the sheaf of germs of exterior forms on \(J^r Y\) and \(\mathfrak{D}^*_r\) the canonical presheaf of local sections of \(\mathfrak{D}^*_r\), seen as a particular topological bundle over \(Y\) (we follow the terminology of [12]). Since \(\pi^*_r\) are open maps, there is the direct system of presheaves

\[
\mathfrak{D}^*_X \xrightarrow{\pi^*} \mathfrak{D}^*_0 \xrightarrow{\pi^*_1} \mathfrak{D}^*_1 \xrightarrow{\pi^*_r} \mathfrak{D}^*_r \rightarrow \cdots .
\]

Its direct limit \(\mathfrak{D}^*_\infty\) is a presheaf of GDAs on \(J^\infty Y\). Let \(\Sigma^*\) be the sheaf of GDAs of germs of \(\mathfrak{D}^*_\infty\) on \(J^\infty Y\). The structure module \(\Gamma(\Sigma^*)\) of global sections of \(\Sigma^*\) is a GDA such that, given an element \(\phi \in \Gamma(\Sigma^*)\) and a point \(z \in J^\infty Y\), there exist an open neighbourhood \(U\) of \(z\) and an exterior form \(\phi^{(k)}\) on some finite order jet manifold \(J^k Y\) so that \(\phi|_U = \pi^*_k \phi^{(k)}|_U\). Therefore, there is the GDA monomorphism

\[
\mathcal{O}^*_\infty Y \rightarrow \Gamma(\Sigma^*).
\]

It should be emphasized that the paracompact space \(J^\infty Y\) admits a partition of unity by elements of the ring \(\Gamma(\Sigma^*)\).

Due to the monomorphism (5), one can restrict \(\mathcal{O}^*_\infty Y\) to the coordinate chart (2) where horizontal forms \(\{dx^\lambda\}\) and contact one-forms \(\{\theta^\lambda = dy^\lambda - y^\lambda d\lambda\}\) make up a local basis for the \(\mathcal{O}^0 Y\)-algebra \(\mathcal{O}^*_\infty Y\). Though \(J^\infty Y\) is not a smooth manifold, elements of \(\mathcal{O}^*_\infty Y\) are exterior forms on finite order jet manifolds and, therefore, their coordinate transformations are smooth. There is the canonical decomposition \(\mathcal{O}^*_\infty Y = \oplus \mathcal{O}^{k,m}_\infty Y\) of \(\mathcal{O}^*_\infty Y\) into \(\mathcal{O}^0 Y\)-modules \(\mathcal{O}^{k,m}_\infty Y\) of \(k\)-contact and \(m\)-horizontal forms together with the corresponding projectors

\[
h_k : \mathcal{O}^*_\infty Y \rightarrow \mathcal{O}^{k,*}_\infty Y, \quad h^m : \mathcal{O}^*_\infty Y \rightarrow \mathcal{O}^{*,m}_\infty Y.
\]

Accordingly, the exterior differential on \(\mathcal{O}^*_\infty Y\) is split into the sum \(d = d_H + d_V\) of the nilpotent total and vertical differentials, where

\[
d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H \circ h_0 = h_0 \circ d, \quad d_H(\phi) = dx^\lambda \wedge d\lambda(\phi).
\]

One also introduces the \(\mathbb{R}\)-module projector

\[
\varrho : \mathcal{O}^{k,n}_\infty Y \rightarrow E_k \subset \mathcal{O}^{k,n}_\infty Y, \quad k = 1, \ldots ,
\]

such that \(\varrho \circ d_H = 0\) and the nilpotent variational operator \(\delta = \varrho \circ d\) on \(\mathcal{O}^{k,n}_\infty Y\). Then the GDA \(\mathcal{O}^*_\infty Y\) is split into the above mentioned variational bicomplex. This contains the variational subcomplex

\[
0 \rightarrow \mathbb{R} \rightarrow \mathcal{O}^0_\infty Y \xrightarrow{d_H} \mathcal{O}^{0,1}_\infty Y \cdots \xrightarrow{d_H} \mathcal{O}^{0,n}_\infty Y \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \rightarrow \cdots ,
\]

\(2\).
whose elements \( L \in \mathcal{O}_{\infty}^0 Y \) and \( \delta L \in E_1 \) are respectively finite order Lagrangians and their Euler–Lagrange operators on a fiber bundle \( Y \rightarrow X \).

Turn now to Lagrangian systems both of even and odd fields. Though there are different approaches to treat odd fields on a smooth manifold \( X \), the following variant of the Serre–Swan theorem motivates us to describe them in terms of graded manifolds whose body is \( X \).

**Theorem 1.** Let \( Z \) be a smooth manifold. A Grassmann algebra \( \mathcal{A} \) over the ring \( C^\infty(Z) \) of smooth real functions on \( Z \) is isomorphic to the Grassmann algebra of graded functions on a graded manifold with a body \( Z \) iff it is the exterior algebra of some projective \( C^\infty(Z) \)-module of finite rank.

**Proof.** The proof follows at once from Batchelor’s theorem [3] and the Serre–Swan theorem generalized to an arbitrary smooth manifold [9, 13]. By virtue of the first one, any graded manifold \( (Z, \mathcal{A}) \) with a body \( Z \) is isomorphic to the one \( (Z, \mathcal{A}_Q) \) with the structure sheaf \( \mathcal{A}_Q \) of germs of sections of the exterior bundle product

\[
\bigwedge Q^* = \mathbb{R} \oplus \mathbb{Q} \oplus \mathbb{Q} \wedge \mathbb{Q} \oplus \cdots,
\]

where \( Q^* \) is the dual of some vector bundle \( Q \rightarrow Z \). We agree to call \((Z, \mathcal{A}_Q)\) the simple graded manifold modelled over the structure vector bundle \( Q \rightarrow X \). Its structure ring \( \mathcal{A}_Q \) of graded functions (sections of \( \mathcal{A}_Q \)) consists of sections of the exterior bundle \( \bigwedge Q^* \rightarrow Z \). The Serre–Swan theorem states that a \( C^\infty(Z) \)-module is isomorphic to the module of sections of a smooth vector bundle over \( Z \) iff it is a projective module of finite rank. \( \square \)

In field models, Batchelor’s isomorphism is usually fixed from the beginning. Therefore, we restrict our consideration to simple graded manifolds \((Z, \mathcal{A}_Q)\). One associates to \((Z, \mathcal{A}_Q)\) the following bigraded differential algebra (henceforth BGDA) \( S^*[Q; Z] \) [3, 9]. Let us consider the sheaf \( \mathcal{D}_A Q \) of graded derivations of \( \mathcal{A}_Q \). One can show that its sections over an open subset \( U \subset Z \) exhaust all \( \mathbb{Z}_2 \)-graded derivations of the \( \mathbb{Z}_2 \)-graded \( \mathbb{R} \)-ring \( \mathcal{A}_U \) of graded functions on \( U \) [3]. Global sections of \( \mathcal{D}_A Q \) make up the real Lie superalgebra of \( \mathbb{Z}_2 \)-graded derivations of the \( \mathbb{R} \)-ring \( \mathcal{A}_Q \). Then one can construct the Chevalley–Eilenberg complex of \( \mathcal{D}_A Q \) with coefficients in \( \mathcal{A}_Q \) [6]. Its subcomplex \( S^*[Q; Z] \) of \( \mathcal{A}_Q \)-linear morphism is the \( \mathbb{Z}_2 \)-graded Chevalley–Eilenberg differential calculus

\[
0 \rightarrow \mathbb{R} \rightarrow \mathcal{A}_Q \rightarrow S^1[Q; Z] \rightarrow S^2[Q; Z] \rightarrow \cdots
\]

over a \( \mathbb{Z}_2 \)-graded commutative \( \mathbb{R} \)-ring \( \mathcal{A}_Q \) [9]. The Chevalley–Eilenberg coboundary operator \( d \) and the graded exterior product \( \wedge \) make \( S^*[Q; Z] \) into a BGDA whose elements obey the relations

\[
\phi \wedge \phi' = (-1)^{\lvert \phi \rvert \lvert \phi' \rvert + [\phi][\phi']} \phi' \wedge \phi,
\]

\[
d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{\lvert \phi \rvert} \phi \wedge d\phi'.
\]

Given the GDA \( \mathcal{O}^* Z \) of exterior forms on \( Z \), there are the canonical monomorphism \( \mathcal{O}^* Z \rightarrow S^*[Q; Z] \) and body epimorphism \( S^*[Q; Z] \rightarrow \mathcal{O}^* Z \).
Lemma 2. The BGDA $S^*[Q; Z]$ is a minimal differential calculus over $A_Q$, i.e., it is generated by elements $df$, $f \in A_Q$.

Proof. One can show that elements of $\mathcal{A}Q$ are represented by sections of some vector bundle over $Z$, i.e., $\mathcal{A}Q$ is a projective $C^\infty(Z)$- and $A_Q$-module of finite rank, and so is its $A_Q$-dual $S^1[Q; Z]$ [8, 9]. Hence, $\mathcal{A}Q$ is the $A_Q$-dual of $S^1[Q; Z]$ and, consequently, $S^1[Q; Z]$ is generated by elements $df$, $f \in A_Q$ [9]. □

This fact is essential for our consideration because of the following [9].

Lemma 3. Given a ring $R$, let $K, K'$ be $R$-rings and $A, A'$ the Grassmann algebras over $K$ and $K'$, respectively. Then any homomorphism $\rho : A \to A'$ yields the homomorphism of the minimal Chevalley–Eilenberg differential calculus over a $\mathbb{Z}_2$-graded $R$-ring $A$ to that over $A'$ given by the map $da \mapsto d(\rho(a))$, $a \in A$. This map provides a monomorphism if $\rho$ is a monomorphism of $R$-algebras.

One can think of elements of the BGDA $S^*[Q; Z]$ as being graded exterior forms on $Z$ as follows. Given an open subset $U \subset Z$, let $A_U$ be the Grassmann algebra of sections of the sheaf $A_Q$ over $U$, and let $S^*[Q; U]$ be the corresponding Chevalley–Eilenberg differential calculus over $A_U$. Given an open set $U' \subset U$, the restriction morphisms $A_U \to A_{U'}$ yield the restriction morphism of the BGDAs $S^*[Q; U] \to S^*[Q; U']$. Thus, we obtain the presheaf $\{U, S^*[Q; U]\}$ of BGDAs on a manifold $Z$ and the presheaf $S^*[Q; Z]$ of BGDAs of germs of this presheaf. Since $\{U, A_U\}$ is the canonical presheaf of the sheaf $A_Q$, the canonical presheaf of $S^*[Q; Z]$ is $\{U, S^*[Q; U]\}$. In particular, $S^*[Q; Z]$ is the BGDA of global sections of the sheaf $S^*[Q; Z]$, and there is the restriction morphism $S^*[Q; Z] \to S^*[Q; U]$ for any open $U \subset Z$. Due to this morphism, elements of $S^*[Q; Z]$ can be written in the following local form.

Given bundle coordinates $(z^A, q^a)$ on $Q$ and the corresponding fiber basis $\{c^a\}$ for $Q^* \to X$, the tuple $(z^A, c^a)$ is called a local basis for the graded manifold $(Z, A_Q)$ [3]. With respect to this basis, graded functions read

$$f = \sum_{k=0} \frac{1}{k!} f_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k}, \quad (11)$$

where $f_{a_1 \cdots a_k}$ are smooth real functions on $Z$, and we omit the symbol of the exterior product of elements $c^a$. Due to the canonical splitting $VQ = Q \times Q$, the fiber basis $\{\partial_a\}$ for vertical tangent bundle $VQ \to Q$ of $Q \to Z$ is the dual of $\{c^a\}$. Then graded derivations take the local form $u = u^A \partial_A + u^a \partial_a$, where $u^A, u^a$ are local graded functions. They act on graded functions (11) by the rule

$$u(f_{a_1 b} c^a \cdots c^b) = u^A \partial_A (f_{a_1 b} c^a \cdots c^b) + u^d f_{a_1 \cdots b} \partial_d (c^a \cdots c^b). \quad (12)$$

Relative to the dual local bases $\{dz^A\}$ for $T^*Z$ and $\{dc^b\}$ for $Q^*$, graded one-forms read $\phi = \phi_A dz^A + \phi_a dc^a$. The duality morphism is given by the interior product

$$u|\phi = u^A \phi_A + (-1)^{|\phi_a|} u^a \phi_a, \quad u \in \mathcal{A}Q, \quad \phi \in S^1[Q; Z].$$

The Chevalley–Eilenberg coboundary operator $d$, called the graded exterior differential, reads

$$d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \partial_a \phi,$$
where the derivations $\partial_A$ and $\partial_a$ act on coefficients of graded exterior forms by the formula (12), and they are graded commutative with the graded exterior forms $dz^A$ and $dc^a$.

Involving even fields which need not be affine, we come to graded manifolds whose body is a fiber bundle $Y \to X$. We define jets of odd fields as simple graded manifolds modelled over jet bundles over $X$ [4, 8]. This definition differs from that of jets of a graded commutative ring [9] and jets of a graded fiber bundle [11], but reproduces the heuristic notion of jets of odd ghosts in the Lagrangian BRST theory [2, 5].

Given a vector bundle $F \to X$, let us consider the simple graded manifold $(J^rY, \mathcal{A}_{Fr})$ whose body is $J^rY$ and structure vector bundle is the pull-back

$$F_r = J^rY \times_X J^rF$$

onto $J^rY$ of the jet bundle $J^rF \to X$. Given the simple graded manifold $(J^{r+1}Y, \mathcal{A}_{Fr+1})$, there is an epimorphism of graded manifolds

$$(J^{r+1}Y, \mathcal{A}_{Fr+1}) \to (J^rY, \mathcal{A}_{Fr})$$

seen as local-ringed spaces. It consists of the surjection $\pi_r^{r+1}$ and the sheaf monomorphism $\pi_r^{r+1*} \mathcal{A}_{Fr} \to \mathcal{A}_{Fr+1}$, where $\pi_r^{r+1*} \mathcal{A}_{Fr}$ is the pull-back onto $J^{r+1}Y$ of the topological fiber bundle $\mathcal{A}_{Fr} \to J^rY$. This sheaf monomorphism induces the monomorphism of the canonical presheaves

$$\mathcal{A}_{Fr} \to \mathcal{A}_{Fr+1},$$

which associates to each open subset $U \subset J^{r+1}Y$ the ring of sections of $\mathcal{A}_{Fr}$ over $\pi_r^{r+1}(U)$. Accordingly, there is the monomorphism of $\mathbb{Z}_2$-graded rings $\mathcal{A}_{Fr} \to \mathcal{A}_{Fr+1}$. By virtue of Lemmas 2 and 3, this monomorphism yields the monomorphism of BGDA

$$S^*[F_r; J^rY] \to S^*[F_{r+1}; J^{r+1}Y].$$

As a consequence, we have the direct system of BGDA

$$S^*[Y \times_X F; Y] \longrightarrow S^*[F_1; J^1Y] \longrightarrow \cdots S^*[F_r; J^rY] \longrightarrow \cdots,$$

whose direct limit $S^*_\infty[F; Y]$ is a BGDA of all graded differential forms $\phi \in S^*[F_r; J^rY]$ on jet manifolds $J^rY$ modulo monomorphisms (14). Its elements obey the relations (9) – (10).

The monomorphisms $O^*_r Y \to S^*[F_r; J^rY]$ provide a monomorphism of the direct system (4) to the direct system (15) and, consequently, the monomorphism

$$O^*_\infty Y \to S^*_\infty[F; Y]$$

of their direct limits. In particular, $S^*_\infty[F; Y]$ is an $O^0_\infty Y$-algebra. Accordingly, the body epimorphisms $S^*[F_r; J^rY] \to O^*_r Y$ yield the epimorphism of $O^0_\infty Y$-modules

$$S^*_\infty[F; Y] \to O^*_\infty Y.$$

If $Y \to X$ is an affine bundle, we recover the BGDA introduced in [4, 8] by restricting the ring $O^0_\infty Y$ to its subring $P^0_\infty Y$ of polynomial functions, but now one should regard elements of $S^*_\infty[F; Y]$ as graded exterior forms on the infinite order jet manifold $J^\infty Y$, but not $X$. 

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Indeed, let \( \mathfrak{S}^*[F_\times; J^r Y] \) be the sheaf of BGDAs on \( J^r Y \) and \( \mathfrak{S}^*[F_\times; J Y] \) its canonical presheaf whose elements are the Chevalley–Eilenberg differential calculus over elements of the presheaf \( \mathfrak{A}_{F_\times} \). Then the presheaf monomorphisms (13) yield the direct system of presheaves

\[
\mathfrak{S}^*[Y \times F; Y] \longrightarrow \mathfrak{S}^*[F_\times; J^1 Y] \longrightarrow \cdots \mathfrak{S}^*[F_\times; J^r Y] \longrightarrow \cdots,
\]

whose direct limit \( \mathfrak{S}^\infty_{F_\times}[F; Y] \) is a presheaf of BGDAs on the infinite order jet manifold \( J^\infty Y \). Let \( \mathfrak{S}^\infty_{F_\times}[F; Y] \) be the sheaf of BGDAs of germs of the presheaf \( \mathfrak{S}^\infty_{F_\times}[F; Y] \). The structure module \( \Gamma(\mathfrak{S}^\infty_{F_\times}[F; Y]) \) of sections of \( \mathfrak{S}^\infty_{F_\times}[F; Y] \) is a BGDCA such that, given an element \( \phi \in \Gamma(\mathfrak{S}^\infty_{F_\times}[F; Y]) \) and a point \( z \in J^\infty Y \), there exist an open neigbourhood \( U \) of \( z \) and a graded exterior form \( \phi^{(k)} \) on some finite order jet manifold \( J^k Y \) so that \( \phi|_U = \pi^\infty_k \phi^{(k)}|_U \). In particular, there is the monomorphism \( \mathfrak{S}^\infty_{F_\times}[F; Y] \rightarrow \Gamma(\mathfrak{S}^\infty_{F_\times}[F; Y]) \).

Due to this monomorphism, one can restrict \( \mathfrak{S}^\infty_{F_\times}[F; Y] \) to the coordinate chart (2) and say that \( \mathcal{O}_N^\infty Y \) as an \( \mathcal{O}_N^\infty Y \)-algebra is locally generated by the elements

\[
(1, c^a_\lambda, dx^\lambda, \theta^\lambda) = dc^a_\lambda - c^a_{\lambda+\Lambda}dx^\lambda, \theta^\lambda = dy^\lambda - y^i_{\lambda+\Lambda}dx^\lambda), \quad 0 \leq |\Lambda|,
\]

where \( c^a_\lambda, \theta^\lambda \) are odd and \( dx^\lambda, \theta^i_\lambda \) are even. We agree to call \((y^i, c^a)\) the local basis for \( \mathfrak{S}^\infty_{F_\times}[F; Y] \). Let the collective symbol \( s^A \) stand for its elements. Accordingly, the notation \( s^A_\lambda = ds^A_\lambda - s^A_{\lambda+\Lambda}dx^\lambda \) is introduced. For the sake of simplicity, we further denote \([A] = [s^A]\).

Similarly to \( \mathcal{O}^\infty_{F_\times} Y \), the BGDCA \( \mathfrak{S}^\infty_{F_\times}[F; Y] \) is decomposed into \( \mathfrak{S}^\infty_{F_\times}[F; Y] \)-modules \( \mathfrak{S}^k_{F_\times}[F; Y] \) of \( k \)-contact and \( r \)-horizontal graded forms. Accordingly, the graded exterior differential \( d \) on \( \mathfrak{S}^\infty_{F_\times}[F; Y] \) falls into the sum \( d = d_H + d_V \) of the total and vertical differentials, where

\[
d_H(\phi) = dx^\lambda \wedge d(\lambda(\phi)), \quad d(\lambda) = d(\lambda) + \sum \left( \begin{array}{c} 1, c^a_\lambda, dx^\lambda, \theta^\lambda \\ 0 \leq |\Lambda| \end{array} \right) s^A_{\Lambda+\Lambda}d(\lambda(\phi)),
\]

Given the graded projection endomorphism

\[
\varrho = \sum_{k>0} \frac{1}{k} \circ h_k \circ h^n, \quad \varrho(\phi) = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta^\Lambda \wedge [d(\lambda(\phi))], \quad \phi \in \mathfrak{S}^{0,n}_F Y,
\]

and the graded variational operator \( \delta = \varrho \circ d \), the BGDCA \( \mathfrak{S}^\infty_{F_\times}[F; Y] \) is split into the Grassmann-graded variational bicomplex analogous to the above mentioned variational bicomplex of \( \mathcal{O}^\infty_{F_\times} Y \). We restrict our consideration to its short variational subcomplex

\[
0 \longrightarrow \mathfrak{S}^0_{F_\times}[F; Y] \xrightarrow{d_H} \mathfrak{S}^{0,1}_{F_\times}[F; Y] \cdots \xrightarrow{d_H} \mathfrak{S}^{0,n}_{F_\times}[F; Y] \xrightarrow{\delta} \mathbf{E}_1, \quad \mathbf{E}_1 = \varrho(\mathfrak{S}^{1,n}_{F_\times}[F; Y]), \quad (19)
\]

and subcomplex of one-contact graded forms

\[
0 \longrightarrow \mathfrak{S}^{1,0}_{F_\times}[F; Y] \xrightarrow{d_H} \mathfrak{S}^{1,1}_{F_\times}[F; Y] \cdots \xrightarrow{d_H} \mathfrak{S}^{1,n}_{F_\times}[F; Y] \xrightarrow{\varrho} \mathbf{E}_4 \rightarrow 0. \quad (20)
\]

One can think of its elements

\[
L = L \omega \in \mathfrak{S}^{0,0}_{F_\times}[F; Y], \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad (21)
\]

\[
\delta L = \theta^A \wedge \mathcal{E}^A \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta^\Lambda \wedge d(\lambda(\varphi)L) \omega \in \mathbf{E}_1 \quad (22)
\]
as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

Our goal now is cohomology of the subcomplexes (19) – (20) of the BGDA $S^*_\infty[F; Y]$ and its de Rham complex

$$0 \rightarrow \mathbb{R} \rightarrow S^0_\infty[F; Y] \xrightarrow{d} S^1_\infty[F; Y] \cdots \xrightarrow{d} S^k_\infty[F; Y] \rightarrow \cdots. \quad (23)$$

Theorem 4. **There is an isomorphism**

$$H^*(S^*_\infty[F; Y]) = H^*(Y) \quad (24)$$

of cohomology $H^*(S^*_\infty[F; Y])$ of the de Rham complex (23) to the de Rham cohomology $H^*(Y)$ of $Y$.

**Proof.** The complex (23) is the direct limit of the de Rham complexes of the BGDA $S^*[J^r Y \times J^r F; J^r Y]$, $r \in \mathbb{N}$. Therefore, the direct limit of cohomology groups of these complexes is cohomology of the de Rham complex (23). Cohomology of the de Rham complex of $S^*[J^r Y \times J^r F; J^r Y]$ equals the de Rham cohomology of $J^r Y$ [3, 9] and, consequently, that of $Y$, which is the strong deformation retract of any $J^r Y$. Hence, the isomorphism (24) holds. $\square$

One can say something more. The isomorphism (24) is induced by the cochain monomorphisms

$$\mathcal{O}^*Y \rightarrow S^*[Y \times F; Y] \rightarrow S^*_\infty[F; Y].$$

Therefore, any closed graded exterior form $\phi \in S^*_\infty[F; Y]$ is split into the sum $\phi = d\sigma + \varphi$ of an exact graded exterior form and a closed exterior form $\varphi$ on $Y$.

Turn now to the complexes (19) – (20). We have proved that, in the case of an affine bundle $Y \rightarrow X$, cohomology of the short variational complex (19) equals the de Rham cohomology of $X$, while the complex (20) is exact [8]. Let us generalize this result to the case of an arbitrary fiber bundle $Y \rightarrow X$.

**Lemma 5.** If $Y = \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$, the complexes (19) – (20) at all the terms, except $\mathbb{R}$, are exact.

**Proof.** This is the case of an affine bundle $Y$, and the above mentioned exactness has been proved when the ring $\mathcal{O}^0_\infty Y$ is restricted to the subring $\mathcal{D}^0_\infty Y$ of polynomial functions (see [8], Lemmas 4.2 – 4.3). The proof of these lemmas is straightforwardly extended to $\mathcal{O}^0_\infty Y$ if the homotopy operator (4.5) in [8], Lemma 4.2 is replaced with that (4.8) in [8], Remark 4.1. $\square$

**Theorem 6.** Cohomology of the complex (19) equals the de Rham cohomology $H^*(Y)$ of $Y$. The complex (20) is exact.

**Proof.** The proof follows that of [8], Theorem 2.1. We first prove Theorem 6 for the above mentioned BGDA $\Gamma(\mathfrak{F}^*_\infty[F; Y])$. Similarly to $S^*_\infty[F; Y]$, the sheaf $\mathfrak{F}^*_\infty[F; Y]$ and the BGDA
where $\mathfrak{c}_1 = \varrho(\mathfrak{q}_{1,0}^m[F; Y])$. By virtue of Lemma 5, the complexes (25) – (26) at all the terms, except $\mathbb{R}$, are exact. The terms $\mathfrak{q}_{0,0}^m[F; Y]$ of the complexes (25) – (26) are sheaves of $\Gamma(\mathfrak{q}_{0,0}^0)$-modules. Since $J^1Y$ admits a partition of unity just by elements of $\Gamma(\mathfrak{q}_{0,0}^0)$, these sheaves are fine and, consequently, acyclic. By virtue of the abstract de Rham theorem (see [8], Theorem 8.4, generalizing [12], Theorem 2.12.1), cohomology of the complex (27) equals the cohomology of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$ and, consequently, the de Rham cohomology of $Y$, which is the strong deformation retract of $J^\infty Y$. Similarly, the complex (28) is proved to be exact. It remains to prove that cohomology of the complexes (19) – (20) equals that of the complexes (27) – (28). The proof of this fact straightforwardly follows the proof of [8], Theorem 2.1, and it is a slight modification of the proof of [8], Theorem 4.1, where graded exterior forms on the infinite order jet manifold $J^\infty Y$ of an affine bundle are treated as those on $X$. $\square$

**Proposition 7.** Every $d_H$-closed graded form $\phi \in S_{0,m}^{0,n}[F; Y]$ falls into the sum

$$\phi = h_0 \psi + d_H \xi, \quad \xi \in S_{0,m}^{0,n-1}[F; Y],$$

where $\psi$ is a closed $m$-form on $Y$. Every $\delta$-closed graded Lagrangian $L \in S_{0,n}^0[F; Y]$ is the sum

$$\phi = h_0 \psi + d_H \xi, \quad \xi \in S_{0,n}^{0,n-1}[F; Y],$$

where $\psi$ is a closed $n$-form on $Y$.

**Proof.** The complex (19) possesses the same cohomology as the similar part of the variational complex (6) of the GDA $\mathcal{O}_{1,0}^* Y$. The monomorphism (16) and the body epimorphism (17) yield the corresponding cochain morphisms of the complexes (6) and (31). Therefore, cohomology of the complex (6) is the image of cohomology of $\mathcal{O}_{1,0}^* Y$. $\square$

The global exactness of the complex (20) at the term $S_{0,n}^1[F; Y]$ results in the following [8].

**Proposition 8.** Given a graded Lagrangian $L = L\omega$, there is the decomposition

$$dL = \delta L - d_H \Xi, \quad \Xi \in S_{1,n}^{1,n-1}[F; Y],$$

$$\Xi = \sum_{s=0}^A \theta^A_{\nu_s \ldots \nu_1} \Lambda^{\nu_s \ldots \nu_1} \omega_\lambda, \quad F_A^{\nu_1 \ldots \nu_n} = \partial_A^{\nu_1 \ldots \nu_n} L - d_L F_A^{\lambda k \ldots n} + h_A^{\nu_k \ldots n},$$
where local graded functions $h$ obey the relations $h^\nu_a = 0$, $h^{(\nu_1 \nu_2 \cdots \nu_k)}_{a_1 a_2 \cdots a_k} = 0$. Locally, one can always choose $\Xi$ (33) where all functions $h$ vanish.

The decomposition (32) leads to the first variational formula for graded Lagrangians as follows [4, 8]. Let $\vartheta \in \mathcal{D}S_\infty^0 [F; Y]$ be a graded derivation of the $\mathbb{R}$-ring $S_\infty^0 [F; Y]$. The interior product $\vartheta | \phi$ and the Lie derivative $L_\vartheta \phi$, $\phi \in S_\infty^0 [F; Y]$, are defined by the formulae

\[
\vartheta | \phi = \vartheta^\lambda \phi_\lambda + (-1)^{[\vartheta, \phi]} \vartheta^A \phi_A, \quad \phi \in S_\infty^1 [F; Y],
\]

\[
\vartheta | (\phi \wedge \sigma) = (\vartheta | \phi) \wedge \sigma + (-1)^{[\vartheta, \phi]} \phi \wedge (\vartheta | \sigma), \quad \phi, \sigma \in S_\infty^* [F; Y],
\]

\[
L_\vartheta \phi = \vartheta | d\phi + d(\vartheta | \phi), \quad L_\vartheta (\phi \wedge \sigma) = L_\vartheta (\phi) \wedge \sigma + (-1)^{[\vartheta, \phi]} \phi \wedge L_\vartheta (\sigma).
\]

A graded derivation $\vartheta$ is said to be contact if the Lie derivative $L_\vartheta$ preserves the ideal of contact graded forms of the BGDA $S_\infty^* [F; Y]$. With respect to the local basis $\{ s^A \}$ for the BGDA $S_\infty^* [F; Y]$, any contact graded derivation takes the form

\[
\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda dx_\lambda + \sum_{\vartheta < |A|} \vartheta^A \partial_A,
\]

where the tuple of graded derivations $\{ \partial_A, \vartheta^A \}$ is defined as the dual of the tuple $\{ dx_\lambda, ds^A \}$ of generating elements of the $S_\infty^0 [F; Y]$-algebra $S_\infty^* [F; Y]$, and $\vartheta^A, \vartheta^0$ are local graded functions [8]. One can justify that any vertical contact graded derivation $\vartheta$ (34) satisfies the relations

\[
\vartheta | d_H \phi = -d_H (\vartheta | \phi), \quad L_\vartheta (d_H \phi) = d_H (L_\vartheta \phi), \quad \phi \in S_\infty^* [F; Y].
\]

Then it follows from the splitting (32) that the Lie derivative $L_\vartheta L$ of a Lagrangian $L$ along a contact graded derivation $\vartheta$ (34) fulfills the first variational formula

\[
L_\vartheta L = \vartheta_V | \delta L + d_H (h_0 (\vartheta | \Xi_L)) + d_V (\vartheta_H | \omega) L,
\]

where $\Xi_L = \Xi + L$ is a Lepagean equivalent of $L$ given by the coordinate expression (33).

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