Theory of metal-insulator transitions in graphite under high magnetic field

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Graphite under high magnetic field exhibits consecutive metal-insulator (MI) transitions as well as re-entrant insulator-metal (IM) transition in the quasi-quantum limit at low temperature. In this paper, we identify the low-T insulating phases as excitonic insulators with spin nematic orderings. We first point out that graphite under the relevant field regime is in the charge neutrality region, where electron and hole densities compensate each other. Based on this observation, we introduce interacting electron models with electron pocket(s) and hole pocket(s) and enumerate possible umklapp scattering processes allowed under the charge neutrality. Employing effective boson theories for the electron models and renormalization group (RG) analyses for the boson theories, we show that there exist critical interaction strengths above which the umklapp processes become relevant and the system enter excitonic insulator phases with long-range order of spin superconducting phase fields (“spin nematic excitonic insulator”). We argue that, when a pair of electron and hole pockets get smaller in size, a quantum fluctuation of the spin superconducting phase becomes larger and destabilizes the excitonic insulator phases, resulting in the re-entrant IM transitions. We also show that an odd-parity excitonic pairing between the electron and hole pockets reconstruct surface chiral Fermi arc states of electron and hole into a 2-dimensional helical surface state with a gapless Dirac cone. We discuss field- and temperature-dependences of in-plane resistance by surface transports via these surface states.

I. INTRODUCTION

Graphite under high magnetic field exhibits a metal-insulator transition at low temperature \((H \geq H_{c,1} \approx 30 \text{ T})\) \cite{1,2}. The transition has been often considered as a prototype of one-dimensional Peierls density-wave instability associated with the \(2k_F\) logarithmic singularity in the Lindhard response function \cite{3,12}. A transition temperature \(T_c\) of the density wave ordering is determined by a BCS type gap equation, \(\ln T_c \propto -1/\rho(0)\). The density of states at the Fermi level \(\rho(0)\) is proportional to the magnetic field \(H\), so that \(T_c\) increases monotonically in the magnetic field \cite{3,6}. Further experiments discovered that graphite shows another metal-insulator transition \((H \geq H_0 \approx 53 \text{ T})\) \cite{13,21} as well as an insulator-metal re-entrant transition at higher magnetic field \((H = H_{c,2} \approx 75 \text{ T})\) \cite{17,21}. So far, there exist at least two distinct low-temperature insulating phases in graphite under high magnetic field: one insulating phase ranges in \(H_{c,1} < H < H_0\) and the other ranges in \(H_0 < H < H_{c,2}\). The re-entrant transition at \(H = H_{c,2}\) indicates a presence of a normal metal phase with pristine electron and hole pockets above the transition field, bringing about a skepticism against the density wave scenarios. Namely, the transition temperature of the density wave phase would increase monotonically in the field, until the electron and hole pockets that would form the Peierls density wave leave the Fermi level \cite{5,9}.

Theoretically, stabilities of the Peierls density wave phases against random single-particle backward scatters depend crucially on a commensurability condition of an electron filling \cite{25,28}. From preceding ab-initio band calculations of graphite under high magnetic field \cite{8,19}, a sequence of specific values of the field in a range of \(30 \text{ T} \leq H \leq 50 \text{ T}\) satisfy the commensurability condition. Nonetheless, experimental transition temperatures of the two insulating phases do not show any dramatic sensitivities on certain values of the field in the range. Both of the insulating phases range rather broadly in field (over 20 Tesla) \cite{17,21}.

In this paper, we explain these two low-T insulating phases in graphite under the high field as manifestation of excitonic insulators with spin nematic orderings. We first argue that graphite under high magnetic field \((H \geq 20 \text{ T})\) is in the charge neutrality region, where electron density and hole density compensate each other. Based on this observation, we begin with interacting electron models with electron pockets and hole pockets, to enumerate possible umklapp scattering processes allowed under the charge neutrality condition. Using perturbative renormalization group (RG) analyses on their effective boson theories, we show that the umklapp terms have critical interaction strength above/below which they become relevant/irrelevant on the renormalization. Above the critical interaction strength, the umklapp term locks the total displacement field as well as spin superconducting phase field. The former locking causes the insulating behavior along the field direction, while the latter results in a long-range order of spin quadrupole moment. We explain the re-entrant insulator-metal transition in graphite, through a quantum fluctuation of the spin superconducting phase field. We characterize the spin nematic excitonic insulator phases by out-of-plane (infrared optical) conductivity as well as in-plane transport property [out-of-plane current is parallel to the field]. The field and temperature

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H < H_{\text{phases (strong-coupling phase). For II" stand for two distinct spin nematic excitonic insulator Eqs. (67,68,69) for H < H_0 under high magnetic field. The phase diagram is obtained FIG. 1. (color online) Theoretical phase diagram for graphite ↓n↑tronics (state near the Fermi level comprises of two electron pockets from the RG equations, Eqs. (42,43,44) for H < H_0). “SNEI-I” and “SNEI-II” stand for two distinct spin nematic excitonic insulator phases (strong-coupling phase). For H < H_0, the electronic state near the Fermi level comprises of two electron pockets (n = 0 LL with ↑ spin and ↓ spin) and two hole pockets (n = −1 LL with ↑ spin and ↓ spin). At H = H_0, the outer two pockets (n = 0 LL with ↑ spin and n = −1 LL with ↓ spin) leave the Fermi level. For H_0 < H < H_1, the electronic state has one electron pocket (n = 0 LL with ↓) and one hole pocket (n = −1 LL with ↑). We choose H_0 = 50 T and H_1 = 120 T. For a detailed parameter set of the RG equations, see Appendix C4. Our theory may not be able to predict much about a transition between SNEI-I and SNEI-II phases (a shaded area around H = H_0); see a discussion in Sec. XB, T = 0 metal-insulator transition at H = H_{c,1} and insulator-metal transition at H = H_{c,2} are the quantum phase transition with the dynamical exponent z = 1.

A. issues to be addressed in this paper

1. direct metal-insulator transition

Under the magnetic field H (∥ z), kinetic energy part of the three-dimensional semimetal takes a form of decoupled one-dimensional quantum chains (or quantum wires). Namely, the kinetic energy within the xy plane is quenched by the Landau quantization, while the kinetic energy along the field direction remains intact, forming one-dimensional momentum-energy dispersion. As a result, the RPA density correlation function is characterized by the Lindhard response function in the one-dimension [3,4]. The function has the logarithmic singularity at $k_z = 2k_F$, where 2$k_F$ is a distance between the left and right Fermi points in the same energy band [22]. Thereby, the system has a generic instability toward the charge density wave ordering, that breaks the spatially translational symmetry along the field direction [3,12]. Meanwhile, graphite under the relevant field regime has four bands that run across the Fermi level (two electron pockets and two hole pockets; $H \leq H_0 \simeq 53$ T) or two bands (one electron pocket and one hole pocket; $H_0 \leq H$) [8,19]. When each of these four (or two) bands would undergo the Peierls density wave (DW) instabilities individually, the respective instabilities would appear at different critical temperatures or critical fields. In other words, the graphite transport experiment would observe a step-wise increase of the (out-of-plane) resistance $R_{zz}$ on lowering temperature or on increasing the magnetic field.

Nonetheless, the graphite experiment observed a direct phase transition from high-T normal metal phase to the low-T insulating phase [16-18,20,21]. Around the transition, the resistance along the field direction $R_{zz}$ continuously increases [16,21] and it increases by 100 times within windows of several Kelvin or Tesla [16,18,21]. These experimental observations clearly dictate that all the energy bands (pockets) at the Fermi level are gapped out simultaneously at the transition point. Thereby, the key question to be asked here is; what is a “talking-channel” among these four (or two) bands that enables this direct metal-insulator transition?

In this paper, we consider this channel as umklapp scattering terms and construct a mean-field theory that explains this direct metal-insulator transition. To be more specific, all the excitonic insulator phases discussed in this paper are stabilized by the umklapp terms that lock a total displacement field along the field direction, a sum of the displacement fields of the four (or two) bands. When the umklapp terms become relevant in the standard RG argument sense, the total displacement field (electric polarization) is locked, resulting in the electrically insulating behaviour along the field direction. By calculating an optical conductivity along the field direction, we explicitly demonstrate the presence of finite mobility gaps in the excitonic insulator phases.

2. re-entrant insulator-metal transition

The second issue is the re-entrant insulator-metal transition observed at the higher field region in the graphite experiment [14,21], that can hardly be explained by the conventional Peierls DW scenarios. Namely, the RPA density correlation function at finite temperature suggests that the transition temperature of the Peierls DW phase increases monotonically in the field, until the electron and/or hole pockets that would form the DW leave
the Fermi level. When they leave the Fermi level, however, the electronic state simply ends up in semiconductor phase rather than metallic phase. Contrary to this, the graphite experiments clearly observe the insulator-metal re-entrant transition in the higher-field region. The low-$T$ electric transport along the field direction above the critical field ($H > H_{c,2} \approx 75 \, \text{T}$) is as metallic as the electric transport in the high-$T$ normal metal phase [17, 18, 21]. The experiment clearly indicates a presence of pristine electron and hole pockets at the Fermi level above the critical field.

In this paper, we explain this re-entrant insulator-metal transition as a consequence of quantum spin fluctuation enhanced by raising the magnetic field. To be more specific, we first point out that the umklapp terms lock not only the total displacement field but also a spin superconducting phase field, a difference between a superconducting phase field of an electron/hole pocket with $\uparrow$ spin and hole/electron pocket with $\downarrow$ spin respectively. The higher magnetic field makes the electron and hole pockets to be smaller in size in the $k_z$ space. In the presence of the repulsive electron-electron interaction, the smaller pockets make their Luttinger parameters to be smaller than the unit. Smaller Luttinger parameters mean larger quantum fluctuation of superconducting phase field as well as the spin superconducting phase field. Thus, we can naturally argue that, in the presence of such smaller electron and hole pockets, the umklapp terms suffer from the enhanced quantum spin fluctuation, and become irrelevant in the RG argument sense. When the umklapp terms become irrelevant, the spin superconducting phase field as well as the total displacement field are unlocked, resulting in the re-entrant insulator-metal transition. Importantly, the electronic state still possesses electron and hole pockets above the critical field, though their sizes in the $k_z$ space might be small.

3. field-dependence of in-plane resistance

The third issue to be addressed in this paper is an unusual field-dependence of the electric transport in the directions transverse to the magnetic field [13, 21]. Generally, the bulk electric transport perpendicular to the field is quenched in the clean limit at low temperature ($T \ll \hbar \omega_0$; $\hbar \omega_0$ is the cyclotron frequency). Nonetheless, the system still has low-$T$ electric transport perpendicular to the field through the so-called surface chiral Fermi arc (SCFA) states [23, 24]. The associated surface resistance is inversely proportional to a length of the arc in the $k_z$ space. The length is approximately equal to the size of the respective electron (or hole) pocket in the bulk. The size of the pocket generally decreases in the field. Thereby, the surface resistance perpendicular to the field is expected to increase in the field. Contrary to this theory expectation, the in-plane resistance $R_{xx}$ in the graphite under the field ($H \parallel z$) shows an unusual field-dependence. The low-$T$ resistance $R_{xx}$ shows a broad peak around $15 < H < 30 \, \text{T}$ [2, 5, 6, 13, 21]. From $H = 30 \, \text{T}$ to $H = H_0 \approx 53 \, \text{T}$, $R_{xx}$ typically reduces by half [13, 21]. Inside the high-field-side insulating phase ($H_0 < H < H_{c,2} \approx 75 \, \text{T}$), the low-$T$ in-plane resistance $R_{xx}$ stays nearly constant in the field [17, 21]. For $H_{c,2} < H$, $R_{xx}$ starts increasing in the field again [21].

Field-(nearly) independent and metallic $R_{xx}$ in the high-field-side insulating phase can be naturally explained by a novel surface reconstruction of the surface chiral Fermi arc (SCFA) states due to the excitonic pairing in the bulk. To be more specific, we will show that an odd-parity excitonic pairing between electron and hole pockets in the bulk reconstructs the SCFA state of electron and that of hole into a $(2 + 1)$-d helical surface state with a gapless Dirac cone. $R_{xx}$ through such a Dirac-cone surface state is determined by carrier density doped in the surface region, that is typically independent from the magnetic field. Namely, unlike ‘decoupled’ SCFA states of electron and hole, the reconstructed Dirac-cone surface state barely changes its shape as a function of the magnetic field. At the zeroth order approximation, the field only changes a ‘depth’ of a band inversion between electron and hole pockets, while the shape of the Dirac-cone surface state is mainly determined by the excitonic pairing strength inside the inverted band gap. Thereby, one can naturally expect that the surface resistance due to the reconstructed Dirac-cone surface state is much less field-dependent than that of the decoupled SCFA states of electron and hole.

B. structure of the paper

The structure of the paper is as follows. In the next section with a help of appendix A, we argue that the graphite under the relevant field regime ($20 < H < 30 \, \text{T}$) is in the charge neutrality region, where electron and hole densities compensate each other. Based on this observation, we enumerate in Sec. III possible umklapp terms that are allowed under the charge neutrality condition in the four pockets model (a model with two electron pockets and two hole pockets; $H \leq H_0 \approx 53 \, \text{T}$). Employing a Hartree-Fock approximation, we construct effective field theories for possible insulating phases that can be stabilized by these umklapp terms (Sec. IV). There are three such phases; spin-nematic excitonic insulator, magnetic Mott insulator and plain excitonic insulator phases. Using renormalization group (RG) analyses, we argue typical field-dependences of the respective transition temperatures of these three phases and conclude that the spin nematic excitonic insulator (SNEI-I) phase could naturally fit in the phenomenology of the low-field-side out-of-plane insulating phase ($H_{c,1} < H < H_0$) in the graphite experiment (Sec. V). In Sec. VI, we enumerate possible umklapp terms that are allowed under the charge neutrality condition in the two-pockets model (one electron and one hole pocket; $H_0 < H$). We construct effective field theories for the possible insulating phases that can be sta-
bilized by the umklapp terms. We found two such phases: a phase with two superposed charge density waves and a spin nematic excitonic insulator (SNEI-II) phase. Using the RG analyses, we conclude that the SNEI-II phase can naturally explain the high-field-side out-of-plane insulating phase \( (H_0 < H < H_{c2}) \). In Secs. VII and VIII, we argue field-dependences of the in-plane resistance in the graphite experiment by the surface electric transports. Especially, we show in Sec. VIII that the odd-parity excitonic pairing in the two-pockets model reconstructs the surface chiral Fermi arc (SCFA) states of electron and hole into a \((2 + 1)\)-d helical surface state with a gapless Dirac cone. The surface Dirac-cone state could naturally explain field-(nearly) independent and metallic behaviour of the in-plane resistance inside the high-field-side insulating phase. After a brief summary in Sec. IX, we give a discussion with complementary viewpoint (Sec. X).

II. CHARGE NEUTRALITY REGIME IN GRAPHITE UNDER HIGH MAGNETIC FIELD

Low-temperature transport properties of graphite are dominated by four \( \pi \)-orbital bands around zone boundaries of the first Brillouin zone \([32,34]\). Graphite is a three-dimensional \( AB \) stacking of graphene layers. A unit cell has two graphene layers and it has four inequivalent carbon sites. Call them as \( A, A', B, \) and \( B' \). \( A \) and \( B \) share the same layer, and so do \( A' \) and \( B' \). \( A \) comes right above \( A' \) in the cell. The electronic band structure near the Fermi level of graphite is composed by \( \pi \) orbitals of carbon atoms that are odd under the mirror with respect to the layer, for example, \( 2p_z \) orbital \([32,34]\). \( \pi \) orbitals of \( A \) and \( A' \) carbon atoms hybridize rather strongly, forming two \( \pi \) orbital bands at the zone boundaries that have large momentum-energy dispersions along the \( c \)-axis \((4000 \, \text{K})\). Call these \( \pi \) orbitals as \( \pi_A \) and \( \pi_{A'} \) respectively. \( \pi \) orbitals of \( B \) and \( B' \) hybridize much weakly, as \( B \) and \( B' \) locate right above the centers of the hexagon in their neighboring layers. These two, which we call \( \pi_B \) and \( \pi_{B'} \) henceforth, form two degenerate bands at the zone boundaries that have a weaker energy-momentum dispersion along the \( c \)-axis \((400 \, \text{K})\).

Under the field along the \( c \)-axis, the four bands in the zone boundaries are split into Landau levels (LLs) \([8,19,34,37]\). For \( H \gtrsim 30 \, \text{T} \), the \( n = 0 \) LLs with \( \uparrow \) spin and \( \downarrow \) spin form two electron pockets around \( k_z = 0 \), and the \( n = -1 \) LLs with \( \uparrow \) spin and \( \downarrow \) spin form two hole pockets around \( k_z = \pi/c_0 \). Here \( c_0 \) is a lattice constant along the \( c \)-axis. According to the band calculation, the outer electron pocket \((n = 0 \, \text{LL with } \uparrow \text{ spin}) \) and the outer hole pocket \((n = -1 \, \text{LL with } \downarrow \text{ spin}) \) leave the Fermi level at \( H = H_0 \approx 53 \, \text{T} \).

The Hall conductivity measurements in a regime of \( 20 \, \text{T} \lesssim H \lesssim 60 \, \text{T} \) \([18,29,31]\) suggest that the number of the electron states and that of the hole states compensate each other almost completely. An estimation gives

\[ N_e - N_h : L_z/c_0 = 10^{-4} : 1 \text{, where } L_z \text{ is a linear dimension along the } c \text{-axis, } N_e \text{ and } N_h \text{ are numbers of the } k_z \text{ points within the electron pockets and hole pockets respectively } [k_z \text{ is a crystal momentum along the } c \text{-axis}] \text{ (see Appendix A for a validity of the estimation).} \]

III. FOUR POCKETS MODEL \((H < H_0)\)

Based on this observation, we consider an electron model with two electron pockets \((n = 0 \, \text{LL with } \uparrow \text{ spin and that with } \downarrow \text{ spins}) \) and two hole pockets \((n = -1 \, \text{LL with } \uparrow \text{ spin and that with } \downarrow \text{ spins}) \) with the charge neutrality condition \((N_e = N_h)\):

\[ H_{\text{kin}} = \sum_{k_z,j} \sum_{n=0,-1} \sum_{\sigma=\uparrow,\downarrow} E_{n,\sigma}(k_z)c_{n,\sigma,j}(k_z)^\dagger c_{n,\sigma,j}(k_z). \]

![FIG. 2](color online) Schematic picture of electronic states of graphite under high field \((H < H_0)\). Solid/dotted lines describe Fermi surfaces of two electron/hole pockets in both bulk and edge regions. Two electron/hole pockets in the bulk region are terminated by electron/hole-type surface chiral Fermi arc states in edge regions respectively. Namely, \(E_{0,\sigma}(k_z,y_j)/E_{1,\sigma}(k_z,y_j)\) goes higher/lower in energy, when \(y_j\) goes from the bulk region to the edge regions (see Appendix A).
The two electron pockets encompass \( k_z = 0 \) and two hole pockets are around the zone boundary \( k_z = \pm \pi/c_0 \),

\[
E_{n,\sigma}(k_z) = -2\gamma_2 \left[ \cos(k_z c_0) - \cos(k_{F,n,\sigma}c_0) \right],
\]

with \( 0 < k_{F,0,\uparrow} < k_{F,0,\downarrow} < \pi/(2c_0) < k_{F,-1,\downarrow} < k_{F,-1,\uparrow} < \pi/c_0 \) [4]. An index \( j = 1, 2, \ldots, (L_xL_y)/(2\pi^2) \) counts degenerate electron states within each LL. \( l \) denotes a magnetic length, \( l \equiv \sqrt{\hbar/c_eH} \). Since the kinetic energy part takes the exactly same momentum-energy dispersion \( \sqrt{k_z c_0} \) direction for different \( j \), we regard the system as coupled chains and call \( j \) a ‘chain index’ [38, 40]. The charge neutrality condition takes a form of

\[
k_{F,0,\uparrow} + k_{F,0,\downarrow} + k_{F,-1,\downarrow} + k_{F,-1,\uparrow} = \frac{2\pi}{c_0}.
\]

For low-temperature electric transports at those temperature much below the band width (2\( \gamma_2 \approx 400 \) K), the kinetic energy part can be linearized around the Fermi points of each pockets (\( k_z \approx \pm k_{F,n,\sigma} \));

\[
H_{\text{kin}} = \sum_j \sum_{\alpha,\tau} \tau V_{\alpha,\tau} \int dz \psi_{\alpha,\tau,j}(z) \partial_z \psi_{\alpha,\tau,j}(z) + \cdots .
\]

A chirality index \( \tau \) specifies left mover (\( \tau = -1 \)) or right mover (\( \tau = +1 \)). \( v_{F,\alpha} \) is a bare Fermi velocity of each pocket with \( \alpha \equiv (n, \sigma) \). For simplicity, we label \((n, \sigma) = (0, \uparrow), (0, \downarrow), (-1, \uparrow) \) and \((-1, \downarrow) \) as \( \alpha = 1, 2, 3 \) and 4 respectively throughout this paper, e.g., \( k_{F,\alpha} \equiv k_{F,n,\sigma} \), \( c_{\alpha,j}(k_z) \equiv c_{n,\sigma,j}(k_z) \), and \( \psi_{\alpha,\pm,j}(z) \equiv \psi_{(n,\sigma),\pm,j}(z) \). \( \psi_{\alpha,\pm,j}(z) \) is a slowly-varying Fourier transform of those \( c_{\alpha,j}(k_z) \) around \( k_z \approx \pm k_{F,\alpha} \);

\[
\psi_{\alpha,\tau,j}(z) \equiv \frac{1}{\sqrt{\mathcal{L}_z}} \sum_{|k_z - k_{F,\alpha}| < A} c_{\alpha,j}(k_z) e^{i(k_z - \tau k_{F,\alpha})z}.
\]

A short-ranged repulsive interaction is considered;

\[
H_{\text{int}} = \sum_{\alpha,\alpha',c,d} \sum_{A',B,B'} \int dr \int dr' V(r - r') \psi_{\alpha}^\dagger(r, c) \psi_{\alpha'}^\dagger(r', d) \psi_{\alpha'}(r', d) \psi_{\alpha}(r, c),
\]

where

\[
V(r) = \frac{\eta_{1,0} \eta_{2,0}}{(2\pi)^3 l_{0,x} l_{0,y}} e^{-\frac{r^2}{2l_{0,x}^2} - \frac{r^2}{2l_{0,y}^2} - \frac{r^2}{2l_{0,z}^2}},
\]

with \( r \equiv (x, y, z) \), \( \eta_{1,0} \neq 0 \), and \( \eta_{2,0} > 0 \). \( l_{0,\mu} \) denotes an interaction length along the \( \mu \)-direction. \( \psi_{\alpha}^\dagger(r, c) \) denotes an electron creation at \( \pi \)-orbital \( \pi_c(\alpha = A, A', B, B') \) of carbon atom at \( r \) with spin \( \sigma \). The creation field can be expanded in term of single-particle bases of the \( n = 0 \) and \( n = -1 \) LLs in the Landau gauge;

\[
\psi_{\alpha}(r, A) = \sum_j \sum_{\alpha,\tau} \tau V_{\alpha,\tau} \int dz \psi_{\alpha,\tau,j}(z) \partial_z \psi_{\alpha,\tau,j}(z) + \cdots .
\]

where

\[
Y_{0,\alpha}(y) = \frac{1}{\sqrt{\sqrt{\pi}l}} e^{-\frac{(y-y_j)^2}{2l^2}},
\]

\[
Y_{1,\alpha}(y) = \frac{\sqrt{2l}}{d y_j} \frac{d}{d y} Y_{0,\alpha}(y) = \frac{\sqrt{2l}}{\sqrt{\pi}l^2} e^{-\frac{(y-y_j)^2}{2l^2}},
\]

with \( y_j \equiv k_j l^2 \) and \( k_j \equiv 2\pi j/L_x \). The slowly varying field \( \psi_{(n,\sigma),\tau,j}(z) \equiv \psi_{\alpha,\tau,j}(z) \) was defined in Eq. 3 with \( \alpha \equiv (n, \sigma) \). \( \gamma_{c,\sigma}(c = A, A', B, B') \) comprises an eigenvector of a 4 by 4 SWM (Slonczewski-Weiss-McClure) Hamiltonian at \( k_z = \pm k_{F,n,\sigma} \) [33, 47]. \( \eta_{c,\sigma}(c = B, B') \) comprises the eigenvector at \( k_z = \pm k_{F,-1,\sigma} \). \( L_z \) is a linear dimension of the system size along the \( z \)-direction. A substitution of Eqs. [3, 9, 10] into Eq. 4 and expansion in \( \psi_{(n,\sigma),\tau,j}(z) \equiv \psi_{\alpha,j}(z) \) lead to scatterings between different pockets (inter-pocket scattering) and scatterings within the same pocket (intra-pocket scattering).

In this paper, we take into consideration only umklapp scattering terms that are allowed under the charge neutrality condition (Fig. 3), inter-pocket scattering terms between opposite chiralities (Fig. 4), and intra-pocket scatterings \( H_I \). This is because, in fermionic functional renormalization group analyses [12, 44, 45], these scatterings are coupled with one another at the one-loop level; they have larger chances to become relevant upon the renormalization than those scattering terms omitted.

Under the charge neutrality condition (Eq. 3), the interaction allows the following four umklapp terms and their hermitian conjugates;

\[
H_u = \sum_{j, m, n} \left( \begin{array}{c c c}
\psi_{4,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger & \psi_{2,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger & \psi_{2,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger \\
\psi_{2,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger & \psi_{4,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger & \psi_{4,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger \\
\psi_{2,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger & \psi_{2,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger & \psi_{4,+,n,\psi_{1,+,j+m-n},\psi_{1,+,j}}^\dagger \\
\end{array} \right).
\]
Due to the translational symmetry along $x$ in the Landau gauge, the scattering processes conserve a momentum $k_j \equiv 2\pi j/L_x$ that is conjugate to $x$. In eq. (11), integrals over the spatial coordinate $z$, and the scattering matrix elements that depend on $z$ and $j, m, n = 1, 2, \cdots, (L_x L_y)/(2\pi l^2)$ are omitted for clarity. For example, the first and fourth terms in Eq. (11) take the following explicit form with their Hermitian conjugates,

\begin{equation}
(1st \text{ and } 4th \text{ terms in Eq. } (11))
\begin{aligned}
&= 2 \sum_{j, m, n} \psi_{m-n, j-n} \int dz \int dz' e^{-i z z' / 2l_x} \\
&\left\{ e^{-i k_F z - i k_F z'} - i k_F z z' \right\} \\
&\left\{ \psi_{4,+n}^{\dagger} \psi_{4,+j+m-n}^{\dagger} \psi_{1,-m} \psi_{2,-j} + \psi_{2,+n}^{\dagger} \psi_{1,+j+m-n}^{\dagger} \psi_{3,-m} \psi_{4,-j} + h.c. \right\}. 
\end{aligned}
\end{equation}

The matrix element in Eq. (12) are given by a dimensionless function $f^{(12)}(x, y)$ as

\begin{equation}
V^{(12)}_{m,n} = \frac{g}{L_x} \frac{1}{2\pi l z} f^{(12)}(y_m/l, y_n/l).
\end{equation}

The function $f^{(12)}(x, y)$ can be calculated by the direct substitution of Eqs. (7, 8, 9, 10) into Eq. (6). $g$ is from Eq. (7).

In addition to $H_a$, we consider the inter-pocket scatterings between the opposite chirality ($H_b$) as well as the intra-pocket scatterings ($H_f$). They are

\begin{equation}
H_b = \sum_{j, m, n} \left( \psi_{4,+,n}^{\dagger} \psi_{4,+,j+m-n}^{\dagger} \psi_{1,+,m} \psi_{4,+,j} + \psi_{4,-n}^{\dagger} \psi_{4,-j+m-n}^{\dagger} \psi_{2,+,m} \psi_{4,-j} + \psi_{3,+,n}^{\dagger} \psi_{3,+,j+m-n}^{\dagger} \psi_{1,+,m} \psi_{3,+,j} + \psi_{3,-n}^{\dagger} \psi_{3,-j+m-n}^{\dagger} \psi_{2,+,m} \psi_{3,-j} + \psi_{2,+,n}^{\dagger} \psi_{2,+,j+m-n}^{\dagger} \psi_{1,+,m} \psi_{2,+,j} + \psi_{2,-n}^{\dagger} \psi_{2,-j+m-n}^{\dagger} \psi_{3,+,m} \psi_{2,-j} \right),
\end{equation}

and

\begin{equation}
H_f = \sum_a \sum_{j, m, n} \psi_{a,n}^{\dagger} \psi_{a,j+m-n}^{\dagger} \psi_{a,m} \psi_{a,j},
\end{equation}

with $\psi_{a,n}(z) \equiv e^{ik_{F} z} \psi_{a,+,n}(z) + e^{-ik_{F}z} \psi_{a,-n}(z) (a = 1, 2, 3, 4)$; see Eq. (11) for an actual form of $H_f$. We do not take into account the inter-pocket scatterings between the same chirality, because, at the one-loop level of the fermionic renormalization group equations (12, 14, 15), they are decoupled from $H_a$, $H_b$, and $H_f$, and do not grow up into larger values upon the renormalization.

**IV. EFFECTIVE Boson Theory**

In this section, we construct effective field theories of possible insulating phases that are stabilized by the umklapp scattering terms in Eq. (11). To this end, we first assume that the low-T insulating phases in the graphite experiment do not break the translational symmetries within the graphene plane [the graphene plane is perpendicular to the field $(z)$]. The assumption apparently does not contradict with any experimental observations in the past literatures [1, 2, 3, 9, 10]. We thus introduce as ‘mean fields’ the pairings among electron creation/annihilation operators within the same chain, and treat the inter-chain electron-electron interactions within a Hartree-Fock approximation. To be more specific, we keep only the direct process (Hartree term: $j = n$) and exchange process (Fock term: $m = n$) in Eqs. (11, 14, 15). Within such effective theory framework, the metal-insulator transitions in the graphite experiment are described by a phase transition between a phase with the mean fields being zero and a phase with the mean fields being finite. The former phase corresponds to the high-T normal metal phase and the latter corresponds to the low-T insulating phases.

To do this construction transparently, we first bosonize the slowly-varying fermion field in terms of two phase
variables defined for each pocket \( a = (n, \sigma) \) and each chain \( j = 1, \ldots, (L_xL_y)/(2\pi t^2) \) (28, 41):

\[
\begin{align*}
\psi_{a,+}(z) &= \eta_{a,j} e^{-i(\phi_{a,j}(z)-\theta_{a,j}(z))}, \\
\psi_{a,-}(z) &= \eta_{a,j} e^{-i(-\phi_{a,j}(z)-\theta_{a,j}(z))}.
\end{align*}
\] (16)

Here \( (n, \sigma) = (0, \uparrow), (0, \downarrow), (-1, \uparrow) \) and \((-1, \downarrow)\) are abbreviated as \( a = 1, 2, 3 \) and \( 4 \) respectively. \( \alpha \) is a short-range cutoff for the spatial coordinate \( z \). \( \phi_{a,j}(z), \theta_{a,j}(z) \) and \( \partial_z \theta_{a,j}(z) \) are a displacement field along the field direction \( z \), superconducting phase field, and current density field along the field respectively. They are associated with the pocket \( a \) and the \( j \)-th chain. The displacement field and superconducting phase field cannot be simultaneously definite; they are canonical conjugate to each other;

\[
[\phi_{a,j}(z), \partial_z \theta_{b,m}(z')] = i\delta_{a,b}\delta_{j,m}\delta(z-z').
\] (17)

\( \eta_{a,j} \) and \( \eta_{\bar{a},j} \) in Eq. (16) are Klein factors ensuring the anticommutation relation among fermion fields on different chains \( j \), pockets \( a \) and chiralities \( \tau = \pm \); \( \{\eta_{a,j}, \eta_{\bar{a},m}\} = \{\eta_{\bar{a},j}, \eta_{a,m}\} = \delta_{a,\bar{a}}\delta_{j,m} \), and \( \{\eta_{a,j}, \eta_{\bar{a},m}\} = 0 \). Due to the Klein factor, the interaction parts given in Eqs. (11) cannot be fully bosonized without approximation.

To obtain the effective boson theories of the insulating phases, we thus employ the Hartree-Fock approximation for the inter-chain interactions in Eqs. (11), to keep only direct process \( j = n \) and exchange process \( m = n \) in Eqs. (11). This leads to a fully-bosonized Hamiltonian

\[
H_{\text{kin}} + H_u + H_b + H_f = H_0 + \sum_{i=1}^{4} H_{u,i} + \sum_{i=1}^{4} H_{b,i} + \cdots ,
\] (18)

\[
H_0 = \sum_{m} \sum_{a} \int dz \left\{ \frac{u_a K_a}{2} |\Pi_{a,m}(z)|^2 + \frac{u_a}{2\pi F_a} |\partial_z \phi_{a,m}(z)|^2 \right\},
\] (19)

and \( \pi \Pi_{a,j}(z) \equiv \partial_z \theta_{a,j}(z) \). \( K_a \) and \( u_a \) are Luttinger parameter and Fermi velocity of a pocket with \( a = (n, \sigma) \) that are renormalized by the intra-pocket forward scatterings \( H_f \) (see appendix B for its details). As in the standard bosonization, the renormalizations are given by two parameters \( g_{2,a}(>0) \) and \( g_{4,a}(>0) \) as,

\[
\frac{u_a}{v_{F,a}} = \sqrt{(1 + \frac{g_{4,a}}{2\pi v_{F,a}})^2 - (\frac{g_{2,a}}{2\pi v_{F,a}})^2},
\] (20)

\[
K_a = \frac{2\pi v_{F,a} + g_{4,a} - g_{2,a}}{2\pi v_{F,a} + g_{4,a} + g_{2,a}}.
\] (21)

See also Appendix B for expressions of \( g_{2,a} \) and \( g_{4,a} \) in terms of \( g \) in Eq. (7). The Hartree and Fock terms in the umklapp scatterings of Eq. (11) are bosonized as

\[
H_{u,1} = \sum_{j,m} M_{j-m}^{(1)} \int dz \left\{ \sigma_{\bar{a}} z_{j} \sigma_{4z,m} \cos [Q_{+3} + Q_{+4,m}] \right. \\
+ \sigma_{\bar{a}} z_{j} \sigma_{4T,m} \cos [Q_{+1} + Q_{+4,m}] \right\},
\] (22)

\[
H_{u,2} = \sum_{j,m} M_{j-m}^{(2)} \int dz \left\{ \sigma_{\bar{a}} z_{j} \sigma_{4z,m} \cos [Q_{-3} + Q_{-4,m}] \right. \\
+ \sigma_{\bar{a}} z_{j} \sigma_{4T,m} \cos [Q_{-1} + Q_{-4,m}] \right\},
\] (23)

\[
H_{u,3} = \sum_{j,m} M_{j-m}^{(3)} \int dz \left\{ \sigma_{\bar{a}} z_{j} \sigma_{2z,m} \cos [Q_{+3} + Q_{+4,m}] \right. \\
+ \sigma_{\bar{a}} z_{j} \sigma_{2T,m} \cos [Q_{-3} + Q_{-4,m}] \right\},
\] (24)

\[
H_{u,4} = \sum_{j,m} M_{j-m}^{(4)} \int dz \left\{ \sigma_{\bar{a}} z_{j} \sigma_{2z,m} \cos [Q_{+3} + Q_{+4,m}] \right. \\
+ \sigma_{\bar{a}} z_{j} \sigma_{2T,m} \cos [Q_{-3} + Q_{-4,m}] \right\},
\] (25)

where

\[
Q_{a,b}^{\pm,j} \equiv \phi_{a,j} + \phi_{b,j} \pm (\theta_{a,j} - \theta_{b,j}),
\] (26)

with \( a, b = 1, 2, 3, 4 \). \( \sigma_{\bar{a}} z_{j} \) and \( \sigma_{\bar{a}} z_{j} \sigma_{4z,m} \) are Ising variables associated with the Klein factors within the same chain, \( \sigma_{\bar{a}} z_{j} \equiv i\eta_{a,j} \eta_{\bar{a},j} \), and \( \sigma_{\bar{a}} z_{j} \sigma_{4z,m} \equiv i\eta_{a,m} \eta_{\bar{a},m} \). The Ising variables take \( \pm 1 \).

The Fock term \( (m = n) \) of the inter-pocket scatterings,
Eq. (14), are bosonized as

\[ H_{b,13} = \sum_{j,m} H_{j-m}^{(13)} \int dz \{ \sigma_{4\ell,j} \sigma_{4\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{13} \right] + \sigma_{7\ell,j} \sigma_{7\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{13} \right] \} + \sum_{j,m} H_{j-m}^{(14)} \int dz \{ \sigma_{4\ell,j} \sigma_{4\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{14} \right] + \sigma_{7\ell,j} \sigma_{7\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{14} \right] \}, \]

\[ H_{b,2} = \sum_{j,m} H_{j-m}^{(2)} \int dz \{ \sigma_{3\ell,j} \sigma_{3\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{23} \right] + \sigma_{7\ell,j} \sigma_{7\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{23} \right] \} + \sum_{j,m} H_{j-m}^{(4)} \int dz \{ \sigma_{3\ell,j} \sigma_{3\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{43} \right] + \sigma_{7\ell,j} \sigma_{7\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{43} \right] \} + \sum_{j,m} \bar{H}_{j-m}^{(2)} \int dz \{ \sigma_{3\ell,j} \sigma_{3\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{12} \right] + \sigma_{7\ell,j} \sigma_{7\ell,m} \cos \left[ \Delta_{jm} Q_{+}^{12} \right] \}, \]

with \( \Delta_{jm} \equiv f_j - f_m \). The Hartree term \((j = n)\) of the inter-pocket scatterings in Eq. (14) could also renormalize the Luttinger parameters and Fermi velocities. For simplicity, however, we consider the renormalizations of \( K_a \) and \( u_a \) only by the intra-pocket scatterings \( H_l \) as given in Appendix B. Figs. 3 and 4 schematically show the inter-pocket scattering processes that lead to \( H_{b,2} \) and \( H_{b,4} \) respectively.

The inter-chain interactions in \( H_{b,1} \) and \( H_{b,3} \) are given by the magnetic length:

\[ M_{j-m}^{(n)} = \frac{g}{L_{2\alpha^2 \ell}} \mathcal{M}^{(n)}((y_j - y_m)/l), \]

\[ H_{j-m}^{(n)} = \frac{g}{L_{2\alpha^2 \ell}} \mathcal{H}^{(n)}((y_j - y_m)/l), \]

\[ \bar{H}_{j-m}^{(n)} = \frac{g}{L_{2\alpha^2 \ell}} \bar{H}^{(n)}((y_j - y_m)/l), \]

with \( n = 1, 2, 3, 4 \) and 13. \( \mathcal{M}(y) \) and \( \mathcal{H}(y) \) as well as \( \bar{H}(y) \) are dimensionless functions. For example, Eqs. (22, 28, 30) are obtained from the direct \((j = n)\) and exchange processes \((m = n)\) of Eq. (12) respectively with

\[ \mathcal{M}^{(1)}(x) = -e^{-x/(k_F + k_F \cos(k_F \cdot k_{F,4})^2/2\alpha^2 \ell)} f^{(12)}(0, x) \]

\[ \mathcal{M}^{(2)}(x) = e^{-x/(k_F \cdot k_{F,4})^2/2\alpha^2 \ell)} f^{(12)}(0, x) \]

For the repulsive interaction case \((g > 0)\), integrals of Eqs. (31, 32) over \( y \equiv (y_j - y_m)/l \) give negative values for any \( n = 13, 2, 4 \).

In the next section, we will use a perturbative renormalization group (RG) analyses on the effective boson model, \( H_0 + \sum_i H_{a,i} + \sum_i H_{b,i} \), where \( H_{a,i} \) and \( H_{b,i} \) are treated perturbatively (appendix C). We show that, at the one-loop level of the perturbative RG equations, \( H_{b,13} \), \( H_{b,1} \) and \( H_{b,3} \) are coupled with one another and stabilize (what we call) a plain excitonic insulator phase. Meanwhile, \( H_{a,4} \) and \( H_{b,4} \) stabilize magnetic Mott insulator phase.

V. RENORMALIZATION GROUP ANALYSES

A. spin-nematic excitonic insulator (SNEI-I) phase

We begin with the spin-nematic excitonic insulator phase stabilized by \( H_{b,2} \) and \( H_{a,2} \). The renormalization group (RG) equations for the inter-chain interaction functions in \( H_{b,2} \) and \( H_{a,2} \) take following forms at the one-loop level;

\[ \frac{dM_{j-m}^{(2)}}{dnb} = \frac{A_{23} + A_{14}}{2} H_{j-m}^{(2)} - 2C_{32} \sum_n H_{j-n}^{(2)} H_{n-m}^{(2)} - 2C_{14} \sum_n M_{j-n}^{(2)} \bar{H}_{n-m}^{(2)} \]

\[ \frac{dH_{j-m}^{(2)}}{dnb} = A_{23} H_{j-m}^{(2)} - \frac{1}{2} \sum_n \left( C_{14} M_{j-n}^{(2)} M_{n-m}^{(2)} + 4C_{23} H_{j-n}^{(2)} H_{n-m}^{(2)} \right) \]

\[ \frac{dH_{j-m}^{(2)}}{dnb} = A_{14} \bar{H}_{j-m}^{(2)} - \frac{1}{2} \sum_n \left( C_{23} M_{j-n}^{(2)} M_{n-m}^{(2)} + 4C_{14} \bar{H}_{j-n}^{(2)} \bar{H}_{n-m}^{(2)} \right) \]

\( \ln b > 0 \) is a scale change of the RG equations [see appendix C for their derivations]. The temperature \( T \) increases monotonically on renormalization; \( dT/dnb = T \).

\( A_{23}, A_{14} \) and their linear combination are the scaling dimensions of \( H_{j-m}, \bar{H}_{j-m} \) and \( M_{j-m} \) at the tree-loop level;

\[ A_{ab} = 2 - \frac{1}{2} \sum_{c=a,b} \left( K_c + K_c^{-1} \right) \coth \left( \frac{u_A}{2T} \right) < 0. \]

\( a, b = 1, 2, 3, 4 \) are the pocket indice, where \( 1 \equiv (0, \uparrow), 2 \equiv (0, \downarrow), 3 \equiv (-1, \uparrow) \) and \( 4 \equiv (-1, \downarrow) \). \( \Lambda \) is a short-range cutoff in the momentum space, \( \Lambda = \alpha^{-1} \). \( C_{ab} \) in Eqs. (35,36,37) is always finite positive definite constant for any \( a, b = 1, 2, 3, 4 \) (see Appendix C3). We assume that \( C_{ab} \) has no dependence on temperature and magnetic field. Eqs. (35,36,37) are functional RG equations.
under which inter-chain interactions change their functional forms. To gain a simpler idea of these functional RG equations, we take a sum of the inter-chain interactions over their chain indices. The sum reduces the inter-chain coupling functions into coupling constants as follows,

\[ m_{(2)} = 2\pi^2 \sum_j M_{j}^{(2)} = \frac{g}{\alpha^2} \int M^{(2)}(y) \, dy, \]  

\[ h_{(2)} = 2\pi^2 \sum_j H_{j}^{(2)} = \frac{g}{\alpha^2} \int H^{(2)}(y) \, dy < 0, \]  

\[ \vec{h}_{(2)} = 2\pi^2 \sum_j \vec{H}_{j}^{(2)} = \frac{g}{\alpha^2} \int \vec{H}^{(2)}(y) \, dy < 0. \]

As mentioned above, the inequalities in Eqs. (40,41) hold true for the repulsive interaction case. Considering the repulsive interaction case, we assume the negative bare values of \( m_{(2)} \) and \( \vec{h}_{(2)} \) in the followings.

The RG equations for the coupling constants take forms of

\[ \frac{d m_{(2)}}{d n} = \frac{A_{23} + A_{14}}{2} m_{(2)} - \frac{1}{\pi^2} m_{(2)} (C_{23} h_{(2)} + C_{14} \vec{h}_{(2)}), \]

\[ \frac{d h_{(2)}}{d n} = \frac{A_{23} h_{(2)}}{4\pi^2} - \frac{1}{4\pi^2} (C_{14} m_{(2)}^2 + 4C_{23} h_{(2)}^2), \]

\[ \frac{d \vec{h}_{(2)}}{d n} = \frac{A_{14} \vec{h}_{(2)}}{4\pi^2} - \frac{1}{4\pi^2} (C_{23} m_{(2)}^2 + 4C_{14} \vec{h}_{(2)}^2). \]

The equations dictate that the umklapp term as well as the inter-chain backward scattering are irrelevant at the tree-loop level, as \( A_{ab} \) is negative semi-definite (Eq. (38)). Smaller \( m_{(2)}, h_{(2)} \) and \( \vec{h}_{(2)} \) are always renormalized into zero (‘weak coupling phase’; normal metal phase).

\( C_{ab} \) is positive definite. Thus, the bare repulsive interaction \( g \) has a critical strength, above which \( m_{(2)}, h_{(2)} \) and \( \vec{h}_{(2)} \) help one another to grow up into larger values (‘strong coupling phase’). The critical strength decreases not only on increasing the magnetic field through a dependence of the one-loop terms on the magnetic length \( l \), but also on decreasing the temperature through a dependence of \( A_{ab} \) on the temperature. This suggests that the strong coupling phase generally appears in low temperature side and a transition temperature of the strong coupling phase increases in larger magnetic field (e.g. see a field-dependence of the transition temperature of the SNEI-I phase in Fig. [1] in a region of \( H < 40 \) T).

The transition temperature can also decrease when the Luttinger parameters \( K_a \) (\( a = 1, \ldots, 4 \)) deviate largely from the unit, \( |A_{ab}| \) has a global minimum at \( K_a = K_b = 1 \) and \( T = 0 \). When \( K_a \) deviates away from 1, \( A_{ab} \) becomes negatively larger and thus the critical strength for \( g \) increases; the transition temperature decreases. Physically speaking, \( K_a \) being greater/smaller than the unit means stronger quantum fluctuation of the displacement field/superconducting phase field of the \( a \)-th pocket [Eqs. [19,17]]. The enhanced quantum fluctuations generally destabilize the strong coupling phase.

This observation readily lets us propose a new microscopic mechanism for the re-entrant transition from the strong-coupling to weak-coupling phases; the transition induced by raising the magnetic field. The higher magnetic field generally makes the electron pocket (\( a \)) and hole pocket (\( b \)) to be smaller in size in the \( k_z \) space. This makes their bare Fermi velocities, \( v_{F,a}, v_{F,b} \), to be smaller with respect to the electron interaction energy scale. Thus, in the presence of the repulsive interaction, \( g_{2,a}, g_{2,b} > 0 \) in Eq. (21), the smaller Fermi velocities make their Luttinger parameters to be smaller than the unit, \( K_a, K_b < 1 \). Especially, for \( H < H_0 \), \( K_1 \) and \( K_4 \) are expected to be much smaller than the unit near \( H = H_0 \), where the electron pocket with \( a = 1 \) (\( n, \sigma = (0, \uparrow) \)) and the hole pocket with \( b = 4 \) (\( n, \sigma = (−1, \downarrow) \)) are about to leave the Fermi level. Thus, the transition temperature of the strong coupling phase reduces dramatically near \( H = H_0 \) through an enhancement of \( K_{a}^{-1} \) and \( K_{b}^{-1} \) in Eq. (38) (e.g. see the field-dependence of \( T_c \) of the SNEI-I phase in Fig. [1] in a region of \( 40 \) T < \( H < 50 \) T).

Physically speaking, this reduction is nothing but a consequence of the enhanced quantum fluctuation of spin superconducting phase variable.

When the bare repulsive interaction is greater than the critical value (strong coupling phase), the umklapp and inter-pocket backward scattering terms grow up into larger values;

\[ h_{(2)}, \vec{h}_{(2)} \to −\infty, \quad m_{(2)} \to ±\infty. \]

The following argument does not depend on the sign of \( m_{(2)} \), so that we set \( m_{(2)} > 0 \) henceforth. In the strong coupling regime, \( H_{a,2} \) and \( H_{b,2} \) are maximally minimized by

\[ \phi_{3,m} = \phi_{2,m} = \Phi_{−}, \quad \phi_{4,m} = \phi_{1,m} = \left\{ \begin{array}{ll} 2n\pi − \Phi_{−} & (2n + 1)\pi − \Phi_{−} \end{array} \right. \]

\[ \theta_{3,m} − \theta_{2,m} = \Theta_{−}, \quad \theta_{4,m} − \theta_{1,m} = \left\{ \begin{array}{ll} 2(n + 1)\pi − \Theta_{−} & 2n\pi − \Theta_{−} \end{array} \right. \]

with

\[ \sigma_{32,m} = \sigma_{41,m} = \sigma_{32,m} = \sigma_{31,m} = \sigma. \]

The locking of the total displacement field, \( \phi_{3,m} + \phi_{2,m} + \phi_{4,m} + \phi_{1,m} = 2n\pi \) or \( (2n + 1)\pi \), dictates that the system is electrically insulating along the field direction. Meanwhile, any electron densities, \( ρ(r, c) = \sum_{c=\uparrow,\downarrow} \psi_{j,a}^{\dagger}(r, c) \psi_{j,a}(r, c) \) with \( c = A, A', B, B' \), do not break the translational symmetry along the field direction \( z \), because

\[ \langle \psi_{j,n,\sigma}^{\dagger}(z) \psi_{j,n,\sigma}(z) \rangle = 0, \]

\[ \langle \psi_{j,0,\sigma}^{\dagger}(z) \psi_{j,0,\sigma}(z) \rangle = 0, \]

with \( n = 0, −1, \tau, \tau' = ±, τ = −τ, \sigma = \uparrow, \downarrow \). Due to the charge neutrality condition, the mean electron density
is 2 per two LLs, \( n = 0 \) and \( n = -1 \) LLs, and per the unit cell along the \( c \)-axis. Besides, the insulating phase is associated with particle-hole pairings between \( n = 0 \) LL (electron pocket) and \( n = -1 \) LL (hole pocket). Thus, we regard this phase as excitonic insulator \([18, 20, 42–45]\) instead of charge density wave phase.

An insulating property is manifested by the optical conductivity along the \( c \)-axis, \( \sigma_{zz}(\omega) \). In the strong coupling phase with large \( m_{(2)} \), we may employ a Gaussian approximation for the cosine terms in \( H_{\text{q},2} \), \( \sigma_{zz}(\omega) \) is calculated within the linear response theory as \( \sigma_{zz}(\omega) = (e^2uK)/(2\pi|\ell|)\delta(\omega - \omega^\prime) \), where \( uK \equiv \sum_{\sigma} u_{\sigma}K_{\sigma} \). \( \omega^\prime \equiv 2\pi uK\sum_{\sigma} M_{\sigma}^{(2)} \) defines a gap for collective particle-hole excitations associated with a fluctuation of the total displacement field. An inclusion of a short-ranged dielectric disorder renormalizes the gap into a smaller value \( \omega_{\text{q}} \) with a smaller spectral weight for the delta function (see Appendix D). Meanwhile, it adds a continuum spectrum in higher energy region. The continuum spectra compensate the reduced spectral weight of the delta function. The observation concludes that the excitonic insulator phase is robust against any small dielectric disorder, provided that the renormalized gap size and the spectral weight of the delta function remains finite (see appendix D).

The long-range order of the spin superconducting phases such as \( \theta_4 - \theta_2, \theta_4 - \theta_1 \) in Eq. [43] breaks the U(1) spin-rotational symmetry around the \( c \)-axis. The breaking of the continuous spin-rotational symmetry is manifested by a long-range ordering of spin quadrupole moment (‘spin-nematic excitonic insulator’). The quadrupole moment that exhibits the order is a symmetric part of a 2nd-rank spin tensor composed by two spin-1/2 moments [see appendix E]. One spin-1/2 is from the \( \pi \) orbital of \( A \) or \( A' \) carbon atom, while the other spin-1/2 is from the \( \pi \) orbital of \( B \) or \( B' \) carbon atom.

The 2nd rank spin tensor is defined as

\[
Q_{\mu\nu}^{cd}(r) = \langle S_{c,\mu}(r)S_{d,\nu}(r) + S_{c,\nu}(r)S_{d,\mu}(r) \rangle - \delta_{\mu\nu}\langle S_{c,\perp}(r) \cdot S_{d,\perp}(r) \rangle,
\]

with \( c = A, A', d = B, B' \), \( \mu, \nu = x, y \), \( S_{c,\perp} \equiv \{S_{c,x}, S_{c,y}\} \), and \( S_{c,\nu} = \{S_{c,x}, S_{c,y}\} \). The order of the spin superconducting phase [Eq. (46)] leads to a ferro type as well as density-wave type ordering of the 2nd rank spin tensor, e.g.

\[
Q_{xx}^{AB}(r) + iQ_{xy}^{AB}(r) = e^{2i\Theta}(u + u\cos(\Delta K z - 2\Phi_+)),
\]

where \( \Delta K \equiv k_{F,3} + k_{F,2} - k_{F,4} - k_{F,1} \). \( u \) is a complex-valued coefficient. Symmetry-wise speaking, the long-range order given in Eq. (46) can be also accompanied by a helical magnetic order whose magnetic moment lies in the \( xy \) plane. The helical order has two spatial pitches along the \( c \)-axis, \( 2\pi/(k_{F,3} + k_{F,2}) \) and \( 2\pi/(k_{F,1} + k_{F,4}) \). Microscopically speaking, however, an amplitude of the magnetic moment is tiny and, if any, it appears only in those spatial regions in the cell where two neighboring \( \pi \) orbitals in the same graphene layer overlap [Appendix E].

On increasing the magnetic field, the outer electron pocket with \( (n, \sigma) = (0, \uparrow) \) and hole pocket with \( (-1, \downarrow) \) leave the Fermi level at \( H = H_0 \). Ab-initio electronic band structure calculations evaluate \( H_0 \) around 53 T [8]. For \( H \rightarrow H_0 \) \( (H < H_0) \), the bare Fermi velocities of the two pockets \( v_{F,1} \) and \( v_{F,4} \) become smaller. So do the Luttinger parameters of the two pockets \( K_1 \) and \( K_4 \) [Eq. (21)]. The reduction of the Luttinger parameters makes the tree-level scaling dimension \( A_{14} \) negatively very large [Eq. (35)]. Thus, according to Eqs. (12–14), \( m_{(2)} \) and \( \tilde{\eta}_{(2)} \) are renormalized to smaller values at an early stage of the RG flow for \( H \lesssim H_0 \), irrespective of bare values of \( m_{(2)} \) and \( \tilde{\eta}_{(2)} \). Meanwhile, \( A_{23} \) as well as \( v_{F,2} \) and \( v_{F,3} \) remain rather constant around \( H = H_0 \). Thus, according to Eq. (43), \( h_{(2)} \) grows up to a larger value and eventually diverges, provided that its bare (initial) value is greater than a critical value (see below for the critical value). Larger \( h_{(2)} \) then helps \( m_{(2)} \) and \( \tilde{\eta}_{(2)} \) to grow up at a late stage of the RG flow, by way of the one-loop terms in Eqs. (12–14). The argument so far concludes that, for \( H \lesssim H_0 \), the transition temperature of the strong coupling phase is determined only by Eq. (43) with \( m_{(2)} = 0 \):

\[
\frac{dh_{(2)}}{dnb} = A_{23}h_{(2)} - \frac{C_{23}}{\pi b_{h_{(2)}}^2}.
\]

At the zero temperature, Eq. (49) gives the critical value.
When \( h_{(2)} < h_{(2),c} < 0 \), the spin nematic excitonic insulator phase always appears below a finite critical temperature \( T_c \) at \( H \leq H_0 \) (Fig. 1). The situation is consistent with the experimental phase diagram of the graphite under high field. Meanwhile, RG phase diagrams of the other insulators stabilized by \( H_{u,1} \), \( H_{u,3} \) or \( H_{u,4} \) are not consistent with the graphite experiment.

### B. Magnetic Mott Insulator and Plain Excitonic Insulator Phases

To see this, let us next consider a nature and a RG phase diagram of the magnetic Mott insulator phase stabilized by \( H_{u,4} \) and \( H_{b,4} \). By exchanging 2 and 4 in Eqs. (42, 43, 44), we can readily obtain corresponding one-loop RG equations for their coupling constants:

\[
\frac{dm_{(4)}}{dn} = \frac{A_{24} + A_{12}}{2} m_{(4)} - \frac{1}{\pi^2} m_{(4)} (C_{34} h_{(4)} + C_{12} \bar{h}_{(4)}),
\]

\[
\frac{dh_{(4)}}{dn} = A_{34} h_{(4)} - \frac{1}{4\pi^2} (C_{12} m_{(4)}^2 + 4 C_{34} h_{(4)}^2),
\]

\[
\frac{d\bar{h}_{(4)}}{dn} = A_{12} \bar{h}_{(4)} - \frac{1}{4\pi^2} (C_{34} m_{(4)}^2 + 4 C_{12} \bar{h}_{(4)}^2).
\]

Here, the coupling constants are integrals of the interchain coupling functions in \( H_{b,4} \) and \( H_{u,4} \):

\[
m_{(4)} \equiv 2\pi^2 \sum_j M_j^{(4)} = \frac{g}{\alpha^2} \int M^{(4)}(y) dy,
\]

\[
h_{(4)} \equiv 2\pi^2 \sum_j H_j^{(4)} = \frac{g}{\alpha^2} \int H^{(4)}(y) dy < 0,
\]

\[
\bar{h}_{(4)} \equiv 2\pi^2 \sum_j \bar{H}_j^{(4)} = \frac{g}{\alpha^2} \int \bar{H}^{(4)}(y) dy < 0.
\]

The inequalities hold true for bare values of \( h_{(4)} \) and \( \bar{h}_{(4)} \) in the presence of the repulsive interaction \( g \) (> 0).

The RG equations tell that the bare value of the repulsive interaction \( g \) has a critical strength above/below which \( m_{(4)} \) as well as \( h_{(4)} \) and \( \bar{h}_{(4)} \) become relevant/irrelevant on the renormalization. In the strong coupling phase with \( m_{(4)} \to \pm \infty \) and \( h_{(4)}, \bar{h}_{(4)} \to -\infty \), \( H_{b,4} \) and \( H_{u,4} \) are maximally minimized by

\[
\phi_{3,j} + \phi_{4,j} = \Phi_-, \quad \phi_{2,j} + \phi_{1,j} = \left\{ \begin{array}{ll}
2n\pi - \Phi_- & (2n + 1)\pi - \Phi_- \\
(2n + 1)\pi - \Theta_- & 2n\pi - \Theta_- 
\end{array} \right.,
\]

\[
\theta_{3,j} - \theta_{1,j} = \Theta_- , \quad \theta_{2,j} - \theta_{1,j} = \left\{ \begin{array}{ll}
(2n + 1)\pi - \Theta_- & 2n\pi - \Theta_- \\
(2n + 1)\pi - \Theta_- & 2n\pi - \Theta_- 
\end{array} \right.,
\]

with

\[
\sigma_{3\alpha,m} = \sigma_{2\alpha,m} = \sigma_{34,m} = \sigma_{\alpha,m} = \sigma,
\]

for \( m_{(4)} > 0 \). The locking of the total displacement field results in an electrically insulating behavior along the field direction, while the long-range order of the spin-superconducting phases leads to a long-range helical magnetic order, e.g.

\[
\langle S_{A,x}(r) \rangle + i \langle S_{A,y}(r) \rangle = \nu \epsilon^i \Theta^e \cos((kF,1 + kF,2)z),
\]

\[
\langle S_{B,x}(r) \rangle + i \langle S_{B,y}(r) \rangle = \nu \epsilon^i \Theta^e \cos((kF,1 + kF,2)z) + \nu' \epsilon^i \Theta^e \cos((kF,3 + kF,4)z).
\]

As for the charge degree of freedom, the insulating phase does not break the translational symmetry; \( \langle \rho(r,c) \rangle \) always respects the translational symmetry for \( c = A, B, A', B' \). The phase is stabilized by the pairings with the same LL but between the different spins, so that we call this phase as a magnetic Mott insulator.

Unlike the spin-nematic excitonic insulator, a transition temperature of the magnetic Mott insulator goes to zero at a certain critical field below \( H_0 \). For \( H \to H_0 \) (\( H < H_0 \)), where \( K_1 \) and \( K_4 \) become very small, both \( A_{34} \) and \( A_{12} \) in Eqs. (51, 52, 53) become negatively very large. Accordingly, unlike in the spin nematic excitonic insulator case in the previous section, all of the three coupling constants, \( m_{(4)}, h_{(4)} \) and \( \bar{h}_{(4)} \), are renormalized to zero for those \( H \) sufficiently close to \( H_0 \) (\( H < H_0 \)). In other words, the transition temperature of the magnetic Mott insulator always goes to zero at a certain critical field below \( H_0 \). This is also the case with the plain excitonic insulator stabilized by \( H_{u,4} \). These RG phase diagrams are not consistent with the experimental phase diagram of graphite under the high field [13, 18, 20].

Besides, the helical magnetic order in the Mott insulator is expected to be weak against magnetic disorders. Considering an anisotropy of \( g \)-factor in graphite [40], it is natural to assume that the high magnetic field allows the system to have single-particle backward scatterings between two electron pockets with \( (n, \sigma) = (0, \uparrow) \) and \( (0, \downarrow) \), and also that between two hole pockets with \( (n, \sigma) = (-1, \uparrow) \) and \( (-1, \downarrow) \). The backward scatterings do exist, especially when graphite contains those graphene layers whose normal vectors (\( c \)-axis) have non-zero angles with respect to the field direction. Such graphene layers can appear anywhere and randomly.
along the c-axis, so that the backward scatterings are generally accompanied by random U(1) phases $\lambda_{j,\pm}(z)$:

$$H'_{\text{imp}} = \sum_j \int dz A_{j,+}(z) \left\{ e^{i\lambda_{j,+}(z)} \psi_{1,+}^\dagger(z) \psi_{2,-}(z) + \text{c.c.} \right\}$$

$$+ \sum_j \int dz A_{j,-}(z) \left\{ e^{i\lambda_{j,-}(z)} \psi_{1,-}^\dagger(z) \psi_{2,+}(z) + \text{c.c.} \right\} + \cdots$$

When bosonized, these single-particle backward scatterings add random $U(1)$ phases into $\Phi_\pm \Theta_\pm$ in Eqs. \[54\]-\[55\] respectively:

$$H'_{\text{imp}} = \sum_j \int dz A_{j,+}(z) \sigma_{12,j}$$

$$\times \cos \left[ \phi_{2,j} + \phi_{1,j} - \theta_{2,j} + \theta_{1,j} + \lambda_{+,j}(z) \right]$$

$$+ \sum_j \int dz A_{j,-}(z) \sigma_{12,j}$$

$$\times \cos \left[ \phi_{2,j} + \phi_{1,j} + \theta_{2,j} - \theta_{1,j} + \lambda_{-,j}(z) \right] + \cdots$$

Since $\Phi_\pm$ and $\Theta_\pm$ comprise gapless Goldstone modes in the magnetic Mott insulator, the added random $U(1)$ phases readily kill the long-range orders of $\Phi_\pm$ and $\Theta_\pm$, however small the amplitudes $A_{j,\pm}(z)$ are \[25\]-\[28\]. Likewise, the plain excitonic insulator phase stabilized by $H_{u,1}$, $H_{u,3}$ and $H_{b,13}$ is expected to be weak against short-ranged charged disorders. The short-ranged disorder causes single-particle type backward scatterings between $(0,\uparrow)$ and $(-1,\uparrow)$ pockets and between $((0,\downarrow)$ and $(-1,\downarrow)$ pockets. From these reasonings as well as inconsistency between their RG phase diagrams and the experimental phase diagram of graphite, we conclude that the magnetic Mott insulator as well as the plain excitonic insulator can hardly explain the graphite experiment coherently.

One may expect that the spin-nematic excitonic insulator could also suffer from random single-particle backward scatterings between $(0,\uparrow)$ and $(-1,\downarrow)$ pockets or those between $(0,\downarrow)$ and $(-1,\uparrow)$ pockets. Nonetheless, these scatterings unlikely exist in the real system. Or, if any, they are much smaller than the others, because the relativistic spin-orbit interaction is needed for them, and it is extremely small in graphite \[36\]-\[40\]. Without the spin-orbit interaction, these backward scatterings need both the magnetic scatter and the short-ranged charged scatter on the same spatial point. Microscopically, however, these two types of the scatterings are of different origins and they have no correlation at all. From these reasonings as well as the generic consistency between the RG phase diagram ($H < H_0$ in Fig. 1) and the experimental phase diagram, we conclude that an insulating phase in graphite at $H < H_0$ is the spin-nematic excitonic insulator stabilized by the interval between $H_{u,2}$ and $H_{b,2}$.

VI. TWO POCKETS MODEL ($H > H_0$)

For $H > H_0$, both the electron pocket with $(n,\sigma) = (0,\uparrow)$ and hole pocket with $(n,\sigma) = (-1,\downarrow)$ leave the Fermi level \[8\]-\[19\]. The low-energy electronic system for $H > H_0$ comprises only of the electron pocket with $(n,\sigma) = (0,\downarrow)$ and the hole pocket with $(n,\sigma) = (-1,\uparrow)$. As before, we call $(n,\sigma) = (0,\downarrow)$ as $a = 2$ and $(n,\sigma) = (-1,\uparrow)$ as $a = 3$. The charge neutrality condition is given by $k_F a_0 + k_F a_{-1} = \pi/c_0$. Under the condition, the interaction allows the following umklapp term:

$$H'_u = \sum_{j,m,n} \psi_{3,+,n}^\dagger \psi_{2,+,j+m-n} \psi_{2,-,m} \psi_{3,-,j} + \text{c.c.}, \quad (57)$$

where the integrals over $z$ and scattering matrix elements are omitted. Other two-particle interaction terms that are linked with the umklapp term at the one-loop level of the fermionic RG equations are inter-pocket and intra-pocket scatterings between different chiralities \[45\]. They are

$$H'_u = \sum_{j,m,n} \psi_{3,+,n}^\dagger \psi_{2,+,j+m-n} \psi_{2,-,m} \psi_{3,-,j} + \text{c.c.}, \quad (58)$$

$H'_d = \sum_{j,m,n} \left\{ \psi_{2,+,n}^\dagger \psi_{2,+,j+m-n} \psi_{2,-,m} \psi_{2,-,j} + \psi_{3,+,n}^\dagger \psi_{3,+,j+m-n} \psi_{3,-,m} \psi_{3,-,j} + \cdots \right\}$

and

$$H'_d = \sum_{j,m,n} \left\{ \psi_{2,+,n}^\dagger \psi_{2,+,j+m-n} \psi_{2,-,m} \psi_{2,-,j} + \psi_{3,+,n}^\dagger \psi_{3,+,j+m-n} \psi_{3,-,m} \psi_{3,-,j} + \cdots \right\}$$

respectively.

To construct effective boson theories of possible insulating phases stabilized by $H'_u$, we first assume the in-plane (graphene-plane) translational symmetry of the insulating phases, consider electron pairing within the same chain, and treat the inter-chain electron-electron interactions by the Hartree-Fock approximation. Specifically, we keep only the direct process (Hartree: $j = n$) and the exchange process (Fock: $m = n$) in Eqs. \[57\]-\[58\], and bosonize them into cosine terms:

$$H'_u + H'_d + H'_u = H'_{u,1} + H'_{u,2} + H'_{d,1} + H'_{d,2} + \cdots,$$

$$H'_{u,1} = \sum_{j,m} N_{j-m}^{(1)} \int dz \sigma_{33,j} \sigma_{22,m} \cos \left[ 2\phi_{3,j} + 2\phi_{2,m} \right], \quad (60)$$

$$H'_{u,2} = \sum_{j,m} N_{j-m}^{(2)} \int dz \sigma_{23,j} \sigma_{32,m} \cos \left[ Q_{23}^{+} + Q_{23}^{-} \right], \quad (61)$$

$$H'_{d,1} = \sum_{j,m} O_{j-m}^{(1)} \int dz \sigma_{33,j} \sigma_{33,m} \cos \left[ 2\phi_{3,j} - 2\phi_{3,m} \right]$$

$$+ \sum_{j,m} O_{j-m}^{(1)} \int dz \sigma_{23,j} \sigma_{23,m} \cos \left[ 2\phi_{2,j} - 2\phi_{2,m} \right], \quad (62)$$

$$H'_{d,2} = \sum_{j,m} P_{j-m}^{(2)} \int dz \sigma_{23,j} \sigma_{32,m} \cos \left[ Q_{23}^{+} - Q_{23}^{-} \right]$$

$$+ \sum_{j,m} P_{j-m}^{(2)} \int dz \sigma_{33,j} \sigma_{33,m} \cos \left[ Q_{23}^{+} - Q_{23}^{-} \right], \quad (63)$$
Here $H_{u,1}'$ is from the Hartree process ($j = n$) of Eq. (57), while $H_{u,2}'$, $H_{d,1}'$ and $H_{d,2}'$ are from the Fock processes $(m = n)$ of Eq. (57), Eq. (59) and Eq. (58) respectively (Fig. 6). The Hartree processes of $H_{d}'$ and $H_{u}'$ renormalize the Luttinger parameters and Fermi velocities in $H_0$ [Eq. (19)]. Especially, the Hartree term of $H_{u}'$ gives rise to positive $g_{2,a}$ ($a = 2,3$) in Eqs. (20,21) in the presence of the repulsive electron interaction ($g > 0$).

As in the previous section, we carried out the perturbative RG analyses on these effective boson models, $H_0 + H_{u,1}' + H_{u,2}' + H_{d,1}' + H_{d,2}'$. At the one-loop level of the perturbative RG equations, $H_{u,1}'$ and $H_{d,1}'$ are coupled with each other, and so are $H_{u,2}'$ and $H_{d,2}'$. When the bare interaction strength $g$ is greater than critical interaction strength, respective pairs of the cosine terms grow up, to have larger amplitudes and the system enters strong coupling phases. In the following two subsections, we argue that $H_{u,1}'$ and $H_{d,1}'$ stabilize a plain superposed CDW phase, while $H_{u,2}'$ and $H_{d,2}'$ stabilize spin-nematic excitonic insulator phase.

A. a superposed CDW phase

When $H_{u,1}'$ and $H_{d,1}'$ become relevant, the cosine terms in Eqs. (60,62) are maximally minimized by a charge density wave (CDW) phase, where a displacement field of the electron pocket and that of the hole pocket exhibit long-range orders individually:

$$2\phi_{2,j} = \Phi_2,$$  
$$2\phi_{3,j} = \Phi_3,$$  
$$\Phi_2 + \Phi_3 = \left\{ \begin{array}{ll} 2n\pi & (n(1) < 0) \\ (2n + 1)\pi & (n(1) > 0) \end{array} \right.$$

with

$$g_{22,j} = g_{33,j} = \sigma.$$

Such CDW is a plain superposition of a charge density wave of the electron pocket with $\downarrow$ spin and $\pi/k_{F,0,\downarrow}$ spatial pitch and that of the hole pocket with $\uparrow$ spin and $\pi/k_{F,-1,\uparrow}$ spatial pitch. Since this strong coupling phase is not accompanied by any long-range order of spin superconducting phase fields, the transition temperature of the superposed CDW phase increases monotonically in the magnetic field in the presence of the repulsive electron-electron interaction; $g_{2,a=2}, g_{2,b=3} > 0$. Such behaviour of the transition temperature is not consistent with the graphite’s experimental phase diagram; the experiment shows the re-entrant insulator-metal transition at $H = H_{c2} \simeq 75$ T.

Besides, the long-range order of the relative displacement between the two charge density waves, $\Phi_2 - \Phi_3$, is weak against random charged impurities, unless their spatial pitches are commensurate to the underlying lattice constant $c_0$ [25,28]. Namely, the impurity potentials induce single-particle backward scatterings within the same electron pocket and/or within the same hole pocket. The impurities appear spatially randomly as a function of the coordinate $z$. Thus, the scatterings add random U(1) phases into $2\phi_{2,j}$ and $2\phi_{3,j}$, unless $\Phi_2$ and $\Phi_3$ in Eq. (64) have finite mass in the CDW phase. When $2k_{F,0,\downarrow}$ or $2k_{F,-1,\uparrow}$ is incommensurate with respect to $2\pi/c_0$, the long-range ordering of $\Phi_2 - \Phi_3$ in Eq. (64) is generally accompanied by a gapless phason excitation. Thereby, even small random charged impurities wipe out the long-range order of the relative phase between the two density waves. Meanwhile, being locked into the discrete values by the cosine potential in the umklapp term ($H_{u,1}'$), the total displacement field, $\Phi_2 + \Phi_3$, always has a finite mass in the superposed CDW phase. The locking is therefore robust against the random charged impurities, as far as their amplitudes are small.

B. spin nematic excitonic insulator (SNEI-II) phase

When $H_{u,2}'$ and $H_{d,2}'$ become relevant, the cosine terms in Eqs. (61,63) are maximally minimized by the excitonic insulator phase with broken U(1) spin rotational symmetry. To see a nature and an RG phase diagram of this strong coupling phase, let us first reduce the inter-chain coupling functions in $H_{u,2}'$ and $H_{d,2}'$ into coupling con-
constants;

\[ n(2) \equiv 2\pi l^2 \sum_j N_j^{(2)}, \]
\[ p(2) \equiv 2\pi l^2 \sum_j P_j^{(2)}, \quad \overline{p}(2) \equiv 2\pi l^2 \sum_j \overline{P}_j^{(2)}. \]

For the repulsive interaction case \((g > 0)\), bare values of these three coupling constants are negative; the cosine terms in \(H'_{u,2}\) and \(H'_{h,2}\) are all from the exchange processes. The one-loop RG equations for these coupling constants take the following forms;

\[ \frac{dn(2)}{d\ln b} = A_{23} n(2) - \frac{C_{23}}{\pi l^2} n(2) \left( p(2) + \overline{p}(2) \right), \tag{67} \]
\[ \frac{dp(2)}{d\ln b} = A_{23} p(2) - \frac{C_{23}}{\pi l^2} \left( n(2)^2 + p(2)^2 \right), \tag{68} \]
\[ \frac{d\overline{p}(2)}{d\ln b} = A_{23} \overline{p}(2) - \frac{C_{23}}{\pi l^2} \left( n(2)^2 + \overline{p}(2)^2 \right). \tag{69} \]

Negative semi-definite \(A_{23}\) and positive definite \(C_{23}\) were already defined in Eq. (68) and appendix C8 respectively. Thanks to an inversion symmetry \((Q_{ab} \rightarrow -Q_{ab}^{-1})\), the coupled equations as well as the bare values of the coupling constants are symmetric with respect to an exchange between \(p(2)\) and \(\overline{p}(2)\). This decouples the RG equations into

\[ \frac{df_{\pm}}{d\ln b} = A_{23} f_{\pm} + \frac{C_{23}}{\pi l^2} f_{\pm}^2, \tag{70} \]

where \(f_{\pm} \equiv n(2) \pm p(2) = n(2) \pm \overline{p}(2)\). At the zero temperature, \(A_{23}\) and \(C_{23}\) have no dependence of the scale change \(b\). Thereby, the equations immediately give out a RG flow diagram as in Fig. 7. The strong and weak coupling phases at \(T = 0\) are defined by

\[ \{ \begin{array}{ll} |n(2)| - p(2) > x_c & \text{(strong coupling phase),} \\ |n(2)| - p(2) < x_c & \text{(weak coupling phase),} \end{array} \tag{71} \]

with

\[ x_c \equiv -\frac{\pi l^2}{C_{23}} A_{23} > 0. \tag{72} \]

In the strong-coupling side, the cosine terms in the bosonized Hamiltonian are maximally minimized by

\[ \sigma_{\sigma,j} = \sigma_{\Delta,j} = \sigma, \tag{73} \]
\[ \theta_{2,j} - \theta_{3,j} = \Theta, \tag{74} \]
\[ 2(\phi_{2,j} + \phi_{3,j}) = \left\{ \begin{array}{ll} 2n\pi & (n(2) < 0) \\ (2n + 1)\pi & (n(2) > 0) \end{array} \right. \tag{75} \]

The locking of a sum of the two displacement fields leads to an electrically insulating property along the field direction. The optical conductivity calculated within the Gaussian approximation shows a gap behavior, \(\sigma_{zz}(\omega) = (e^2 uK)/(2\pi l^2)\delta(\omega - \omega_g)\) with \(uK = \sum_{a=2,3} u_a K_a\) and \(\omega_g^2 = 2\pi uK \sum_j N_j^{(2)}\) (see also an inset of Fig. 5). The long-range order of the spin superconducting phase \(\theta_2 - \theta_3\) in Eq. (74) breaks the global U(1) spin rotational symmetry. The strong coupling phase is accompanied by particle-hole pairing between the electron pocket with \(\downarrow\) spin and the hole pocket with \(\uparrow\) spin, so that we name the phase also as spin nematic excitonic insulator phase. Nonetheless, the phase could be symmetrically distinct from the spin nematic excitonic insulator discussed in the previous section, depending on a spatial parity of the excitonic pairing (see also Sec. VIIIA and Sec. XB). We thus distinguish these two by calling them as SNEI-I for \(H < H_0\) and SNEI-II for \(H > H_0\) respectively.

The phase boundary condition, Eqs. (71,72), explains the metal-insulator transition at a lower field regime. For simplicity, we assume that the bare values of \(n(2)\) and \(p(2) = \overline{p}(2)\) as well as \((C_{23})_{T=0}\) have no \(H\)-dependence. For a low \(H\) regime, the magnetic length \(l\) is large, so is the critical value \(x_c\) in Eq. (72). Thereby, a given bare value of \(|n(2)| - p(2)\) can be below the critical value \(x_c\) in lower \(H\) regime (weak-coupling phase; normal metal phase). On increasing \(H\), the magnetic length \(l\) decreases and so does the critical value \(x_c\). Thus, the bare value of \(|n(2)| - p(2)\) exceeds the critical value \(x_c\) at a certain critical magnetic field \((H = H_c^*)\). For \(H > H_c\), the system enters the strong coupling phase (SNEI-II phase). From a comparison with the graphite experiment, \(\downarrow\tag{18}\tag{20}\downarrow\), we assume that \(H_c^*\) is smaller than \(H_0\). In this case, the system at \(T = 0\) undergoes a phase transition from SNEI-I to SNEI-II at \(H = H_0\).

The phase boundary condition Eqs. (71,72) also explains the \(T = 0\) insulator-metal re-entrant transition at a higher field regime. When the field \(H\) increases further, both electron and hole pockets become smaller in size in the \(k_z\) space. This makes their bare Fermi veloc-
ities $v_{F,2}$ and $v_{F,3}$ as well as Luttinger parameters $K_2$ and $K_3$ smaller. The smaller Luttinger parameters can increase the critical value $x_c$ through the dependence of $A_{23}$ on $K_2^{-1}$ and $K_3^{-1}$ [Eq. (88)]. To be more precise, suppose that the electron pocket with $n = 0$ LL with $\downarrow$ spin and the hole pocket with $n = -1$ LL with $\uparrow$ spin leave the Fermi level at $H = H_1$. When $H$ gets ‘closed’ to $H_1$ from below ($H < H_1$), the increase of $-A_{23}$ can overcome the decrease of $I^2$ in Eq. (72), such that $x_c$ increases again. Namely, for $H < H_1$, $I^2$ is always bounded by $(\hbar c)/(e H_1)$ from below, while $K_2^{-1}$ and $K_3^{-1}$ as well as $-A_{23}$ have no upper bound in principle. Thus, for some magnetic field $H_{c,2}$ with $H^*_{c} < H_0 < H_{c,2} < H_1$, $x_c$ exceeds the bare value of $|n(2)| - p(2)$ again and the system falls into the weak coupling phase (normal metal phase) again. From a set of reasonable parameter values used in Fig. 1 (see appendix C4 for a set of parameters used in Fig. 1), we obtain $H_{c,2} = 82$ T and $H_1 = 120$ T.

C. critical natures of the MI and re-entrant IM transitions

The re-entrant transition point at $H = H_{c,2}$ is a zerotemperature continuous phase transition with dynamical exponent $z = 1$. Toward this quantum critical point, a correlation length along the field direction $\xi_z$ diverges as

$$\xi_z \propto |H - H_{c,2}|^{-1/\nu_2}. \quad (76)$$

A critical exponent $\nu_2$ is given only by the Luttinger parameters at the critical point ($H = H_{c,2}$),

$$\nu_2 = \frac{1}{2} \sum_{a=2,3} (K_a + K_a^{-1}) - 2. \quad (77)$$

Since $z = 1$, the correlation length is inversely proportional to the gap $\omega_g$ in the optical conductivity along the field direction, $\sigma_{zz}(\omega)$:

$$\omega_g \propto (H_{c,2} - H)^{z/\nu_2} = (H_{c,2} - H)^{1/\nu_2}, \quad (78)$$

for $H < H_{c,2}$. By measuring how the gap vanishes toward $H = H_{c,2}$ as a function of the field, one can determine the values of the Luttinger parameters at the quantum critical point. By seeing how much the Luttinger parameters thus determined deviate from 1, one could also test a validity of our theory of the re-entrant insulator-metal transition.

The low-$H$ metal-insulator transition between the normal metal and SNEI-I phases is also a quantum critical point. Toward this point, $H = H_{c,1}$, the gap $\omega_g$ in the SNEI-I phase also vanishes,

$$\omega_g \propto (H - H_{c,1})^{1/\nu_1}, \quad (79)$$

for $H_{c,1} < H$. The critical exponent $\nu_1$ is given by the Luttinger parameters at $H = H_{c,1}$:

$$\nu_1 = \frac{1}{2} \sum_a (K_a + K_a^{-1}) - 2, \quad (80)$$

where the summation in the pocket index $a$ is taken over

$$\begin{aligned}
\{ a = 1, 4 & \quad (|A_{14}|C_{23}h(2) \gg |A_{23}|C_{14}h(2)), \\
\{ a = 2, 3 & \quad (|A_{14}|C_{23}h(2) \ll |A_{23}|C_{14}h(2)).
\end{aligned} \quad (81)$$

Meanwhile, the gap in $\sigma_{zz}(\omega)$ reaches finite constant values at $H = H_0 = 0$, when the phase transition from SNEI-I phase to SNEI-II phase is of the first order. This is the case when the spatial parities of the excitonic pairings in the two phases are different from each other (see also Sec. VIII A and Sec. XB).

VII. IN-PLANE RESISTANCE IN THE FOUR POCKETS MODEL ($H < H_0$)

Generally, in-plane current operators in the clean limit have finite matrix elements only between neighboring Landau levels. When the temperature is much lower than the cyclotron frequency $\hbar \omega_0$, the in-plane resistance increases on increasing magnetic field $H$. Contrary to this expectation, the low-temperature in-plane resistance in graphite under high magnetic field shows an unexpected $H$-dependence\cite{14, 15, 17, 18}. It shows a broad peak around $15 T \sim 30$ T, and then decreases slowly on further increasing $H$. From $H = 30$ T to $H = H_0$ $\approx 53$ T, the resistance reduces by half or more. Besides, when the system enters the low-field-side out-of-plane insulating phase ($H_{c,1} < H < H_0$), the in-plane resistance shows an additional steep increase by 15% to 30%\cite{14, 15, 18}. Unlike the out-of-plane resistivity, the additional increase amount becomes smaller for lower temperature.

A. $H$-dependence of $R_{xx}$ at $H < H_0$

The $H$-dependence of the in-plane resistance in $30 T < H < H_0 \approx 53$ T can be explained by charge transports along the surface chiral Fermi arc (SCFA) states. To see this, notice first that the electron/hole pockets in the bulk are terminated with SCFA states of the electron/hole type around the boundary regions of the system (see Fig. 2 and appendix A). A SCFA state of the electron/hole type is a bundle of $N_a$-number of chiral edge modes of electron/hole type respectively, where $N_a$ is a number of $k_z$ points within the electron/hole pocket ($a = 1, 2, 3, 4$). Here ‘$a$’ denotes the pocket index: $1 \equiv (0, \uparrow)$, $2 \equiv (0, \downarrow)$, $3 \equiv (-1, \uparrow)$, and $4 \equiv (-1, \downarrow)$. The chiral edge mode enables unidirectional electric current flow along the boundary in a $xy$ plane. The chiral directions of the electric current flows of the electron-type and hole-type edge modes are opposite to each other.

In the presence of short-ranged charged impurities, the current flow along the electron-type edge mode with $\sigma$ spin can be scattered into the hole-type edge mode with the same $\sigma$ spin. In this respect, the SCFA state with $(-1, \sigma)$ (hole-type) and that with $(0, \sigma)$ (electron-type) cancel each other by the intra-surface backward scatterings due to the charged impurities. In the absence
of any backward scatterings between \((0, \sigma)\) and \((-1, \bar{\sigma})\) \((\sigma, \bar{\sigma}) = (\uparrow, \downarrow), (\downarrow, \uparrow); \) see the last paragraph in Sec. V for the reasonings of the absence), both \((N_2 - N_4)\)-number of anticlockwise (electron-type) chiral edge modes with \(\downarrow\) spin and \((N_3 - N_1)\)-number of clockwise (hole-type) chiral edge modes with \(\uparrow\) spin individually contribute to the two-terminal conductance within the \(xy\) plane;

\[
G_n = \frac{e^2}{\hbar} \left( N_2 - N_4 + N_3 - N_1 \right) = \frac{2e^2}{\hbar} \left( N_3 - N_1 \right).
\]

From the first line to the second line, we used the charge-neutrality condition; \(N_1 + N_2 = N_3 + N_4\). Importantly, the in-plane conductance given by Eq. (82) usually increases on increasing \(H\) for \(H < H_0\). This is because a variation of \(N_1\) with respect to \(H\) is larger than that of \(N_3\): \(dN_1/dH < dN_3/dH < 0\). For \(N_3 = (L_z/(2c_0))(1 - H/H_1)\), and \(N_1 = (L_z/(2c_0))(1 - H/H_0)\), the \(H\)-dependence of the resistance due to the surface charge transport is given by

\[
R_s = \frac{\hbar}{e^2 L_z H (H_1 - H_0)}.
\]

The resistance is on the order of 1 \(\Omega\) at \(H = 30\) \(T\) \([L_z = 50\) \(\mu m, c_0 = 0.67\) \(nm, H_0 = 50\) \(T\) and \(H_1 = 120\) \(T\).] The value is on the same order of the experimental value \([2 \Omega \sim 3 \Omega]\) [17].

B. \(T\)-dependence of \(R_{xx}\) at \(H < H_0\)

The \(T\)-dependence of the in-plane resistance inside the low-field-side insulating phase \((H_{c,1} < H < H_0)\) can be explained by a coupling between the SCFA states and gapless Goldstone modes associated with the spin-nematic order in the bulk. The spin-nematic excitonic insulator (SNEI-I) phase breaks two global \(U(1)\) symmetries. They are the \(U(1)\) spin-rotational symmetry around the field direction and a translational symmetry associated with a spatial polarization of the spin \((\uparrow \text{ or } \downarrow)\) and pseudospin \((n = 0 \text{ LL or } n = -1 \text{ LL})\) densities.

Such SNEI-I phase has two low-energy gapless excitations. They are space-time fluctuations of the following two phase variables [Eqs. (45, 46)]:

\[
f_j(z) \equiv (\theta_{3,j}(z) - \theta_{2,j}(z)) - (\theta_{4,j}(z) - \theta_{1,j}(z)) - 2\Theta_-, \quad (84)
\]

\[
g_j(z) \equiv (\phi_{3,j}(z) + \phi_{2,j}(z)) - (\phi_{4,j}(z) + \phi_{1,j}(z)) - 2\Phi_-. \quad (85)
\]

When they vary slowly in \(z/c_0\) and \(y_j/l \equiv 2\pi l j/L_{zx}\), their energy dispersions become linear in the momenta;

\[
H_{sw} = \frac{1}{2L_z N} \sum_k \left( B_1 k^2 + C_1 k_z^2 \right) f^\dagger(k) f(k) + \frac{1}{2L_z N} \sum_k \left( B_2 k^2 + C_2 k_z^2 \right) g^\dagger(k) g(k), \quad (86)
\]

with positive \(B_t\) and \(C_t\) \((t = 1, 2)\), and \(k \equiv (k_z, k)\). \(k_z\) and \(k\) are conjugate to \(z\) and \(y_j \equiv 2\pi l j/L_{zx}\) respectively,

\[
f_j(z) \equiv \frac{1}{L_z N} \sum_k e^{ik_z z + iky_j} f(k),
\]

\[
g_j(z) \equiv \frac{1}{L_z N} \sum_k e^{ik_z z + iky_j} g(k).
\]

The gapless modes couple with the SCFA states through a simple density-density interaction, e.g.,

\[
H' = \frac{1}{L_x} \sum_{a, \tau, b} \sum_n \sum_m \int dz A_{a,\tau,b}^* \left( y_n, y_m \right) \times \rho_{\tau,n}(z) \left( \psi^\dagger_{b,m}(z) \psi_{b,m}(z) \right), \quad (87)
\]

with bulk density operator \(\rho_{a,\tau,n}(z) \equiv \psi^\dagger_{a,\tau,n}(z) \psi_{a,\tau,n}(z)\) \(\left(|y_n| \leq L_y/2\right)\). \(a, b = 1, 2, 3, 4\) denote the pocket indices, and \(\tau = \pm\) is the chirality index. By definition, the summations over the chain indices \(n\) and \(m\) in Eq. (87) are restricted within the bulk region and edge region respectively; \(L_y/2 \leq |y_n|\).

When bosonized, the density operator in the bulk region is given by a linear combination of the phase variables, \(2\pi \rho_{a,\tau,n}(z) \equiv \partial_z \phi_{a,n}(z) - \tau \partial_\tau \theta_{a,n}(z)\). Such phase variables generally contain the two low-energy gapless excitations with the linear dispersions. Thus, the situation becomes precisely analogous to the electron-phonon interaction in metal [47, 48]. The coupling gives the SCFA electrons with finite transport life times [18]. When the temperature is on the order of a band width of the gapless Goldstone modes (but smaller than the transition temperature of the SNEI-I phase), the transport life time of the SCFA states is linear in temperature \(T\); so is the resistivity due to the surface charge transport. This can explain the \(T\)-dependence of the in-plane resistance in the insulating phases in graphite [14, 15, 18].

VIII. IN-PLANE RESISTANCE IN THE TWO POCKET MODELS \((H > H_0)\)

The in-plane resistance of graphite under the high magnetic field stays almost constant in the field inside the high-field-side out-of-plane insulating phase \((H < H_{c,2})\) [17, 21]. Above the re-entrant insulator-metal (IM) transition field \((H_{c,2} < H)\), the resistance shows the normal behaviour; \(R_{xx}\) increases in the field [21].

In the following, we will argue that the SNEI-II phase in \(H > H_0\) can be either topological [19, 51] or topologically trivial, depending on the spatial parity of the excitonic pairing between electron pocket \((n = 0, \downarrow)\) and hole pocket \((n = -1, \uparrow)\). When the excitonic pairing field is an odd function in the momentum \(k_z\), the SNEI-II phase becomes topological and thereby the SCFA state of electron type \((n = 0, \downarrow)\) and the SCFA state of hole type \((n = -1, \uparrow)\) are reconstructed into a helical surface state with a gapless Dirac cone. The electric transport through
Eq. (88) takes a number defined for the bulk 1-dimensional Hamiltonian the former/latter case, the following topological winding number:

\[ H \equiv \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dq_z}{q_z} \left( \hat{N} \times \partial_{q_z} \hat{N} \right)_3, \]

with \( \hat{N} \equiv (N_1(q_z), N_2(q_z), 0) \).

The non-zero bulk winding number reconstructs the SCFA state of the electron type and that of the hole type into a 2-d surface state with a gapless Dirac cone at side surfaces. The side surface is subtended by \( z \parallel H \) (either or). To be concrete, impose the periodic boundary conditions along \( z \) and \( x \), put a confining potential along \( y \) direction. The mass term \( M \) in Eq. (88) enters a normal 1-dimensional semiconductor regime, \( M > 2\gamma_2 \); the winding number takes zero in the vacuum. In the bulk region, \( |y| < L_y/2 \), the gapped mean-field Hamiltonian with the negative \( n_{(2)} \) is in the band-inverted regime, \( M < 2\gamma_2 \); the winding number takes \( \pm1 \). Such two topologically distinct 1-dimensional gapped systems are inevitably separated by a 1-d gapless Dirac Hamiltonian, that should come somewhere around \( |y| = L_y/2 \). In other words, the side surface has a 2-d helical surface state that forms a gapless Dirac cone as a function of \( k_z \) and \( y \equiv k_z l^2 \) (Fig. 9(A)).

The reconstructed surface state has the helical velocities not only along the \( z \)-direction but also along the \( x \)-direction. To see this, notice that the velocity along the \( x \)-direction is given by a derivative of the single-particle energy with respect to the spatial coordinate \( y \): \( v_x \equiv \partial E_{EI}(q_z, \sigma) / \partial y \). Such velocity changes its sign around \( y = L_y/2 \), where \( E_{EI}(q_z) \) forms the 1-d gapless Dirac dispersion: \( v_x < 0 \) for \( y < L_y/2 \) and \( v_x > 0 \) for \( y > L_y/2 \) (See also Figs. 8(B,C)).

Quantitatively, the Dirac cone is highly anisotropic in its velocity within the side surface. Namely, the velocity along the \( x \) direction is determined by a work function in the edge region: \( v_x = \Omega(\partial M / \partial y) \). Conventionally, the work function varies in energy on the order of eV within a length scale of \( A \): \( \partial M / \partial y = \Omega(eV/A) \). Thus, the velocity along \( x \) direction is much faster than that along \( z \) direction, the latter of which is given by an energy scale of the band width (2\( \gamma_2 \)) or the excitonic pairing (\( \Delta_{EI} \)).

The 2-d helical surface state in the side surface is continuously connected to a 2-d critical wavefunction sitting on a top (bottom) surface. The top (bottom) surface is subtended by \( x \) and \( y \) coordinates (Fig. 9(B)).
Graphite under high magnetic field exhibits mysterious metal-insulator (MI) transitions as well as insulator-metal (IM) re-entrant transitions. We discuss these enigmatic electronic phase transitions in terms of perturbative RG analyses of effective boson theories. We argue that the two insulating phases in graphite under high field are excitonic insulators with spin nematic orderings. Similar conclusions were suggested by experimental works both for $H < H_0$ [20] and $H > H_0$ [18]. This paper enumerates possible umklapp terms allowed under the charge neutrality condition, clarifies natures of insulating states stabilized by each of them, and argues that excitonic insulators with long-range orderings of spin superconducting phases can give a possible explanation to the graphite experiments.

IX. SUMMARY

Based on this, we propose a new mechanism for the re-entrant IM transition. When a pair of electron and hole pockets get smaller in size, strong quantum fluctuation of the spin superconducting phase destabilizes the spin-nematic excitonic insulator, causing the re-entrant IM transition. The strength of the quantum fluctuation is quantified by the Luttinger parameters of the electron and hole pockets. We relate the Luttinger parameters with the critical exponent of the $T = 0$ re-entrant IM transition point. We show that the exponent can be experimentally determined from the infrared optical spectroscopy. By determining the Luttinger parameters at the transition point, experimentalists can test a validity of our theory for the re-entrant IM transition.

We attribute an ‘unexpected’ field- and temperature-dependences of the in-plane electric transport in graphite under the high field as surface charge transports through surface chiral Fermi arc (SCFA) states and reconstructed Dirac-cone surface state. We first argue that a metallic temperature dependence of the in-plane transport observed in the low-field-side insulating phases is due to bulk-edge couplings between the SCFA states and gapless Goldstone modes associated with the spin nematic orderings. Being gapless excitations, the Goldstone modes in the spin-nematic excitonic insulator phases could be experimentally detected through ultrasound measurements [58]. We also argue that the odd-parity excitonic pairing in the bulk reconstructs SCFA states of electron and hole into a $(2 + 1)$-d helical surface state with a gapless Dirac cone. Based on this finding, we discuss the field-(nearly) independent and metallic behaviour of the in-plane transport inside the high-field-side insulating phase [13] [15] [17] [18].
FIG. 11. (color online) Schematic picture of a possible RG phase diagram in the presence of a relevant perturbation (denoted by ‘$X$’) around the decoupled Luttinger liquid (LL) fixed point (denoted by ‘FP0’). In the presence of such relevant perturbation, the LL fixed point is unstable; the normal metal phase is characterized by a new stable fixed point (denoted by ‘FP3’). A horizontal axis (‘$Y$’) denotes the umklapp and inter-pocket scattering terms that drive the system into the excitonic insulator (EI) phases. ‘FP2’ represents a stable fixed point characterizing the EI phases. The critical properties of the metal-insulator (MI) and re-entrant insulator-metal (IM) transitions are characterized by a new saddle fixed point (denoted by ‘FP4’) instead of by the FP1. In this schematic picture, we assume that the fixed point for the EI phases (‘FP2’) is locally stable against the small perturbation $X$.

X. DISCUSSION

A. natures of the ‘normal’ metal phase and criticalities of metal-insulator transitions

Our theory regards the ‘normal’ metallic phase in the graphite experiment as decoupled Luttinger liquid (LL) phase, where we assume that inter-chain electron-electron interactions only renormalize the Luttinger parameters and Fermi velocities as in Eqs. (20,21,24,27). Nonetheless, it could be possible that a fixed point of the decoupled LL phase (a gaussian theory given by Eq. (19); schematically denoted by ‘FP0’ in Fig. 11) is unstable against a certain perturbation associated with the inter-chain interactions (denoted by ‘$X$’ in Fig. 11) and, as a result, the ‘normal’ metal phase is characterized by a new stable fixed point (schematically denoted by ‘FP3’ in Fig. 11). The stable fixed point could be the Fermi-liquid fixed point [50,64] or the sliding Luttinger-liquid fixed point [65,66]. One of the experimental evidences that could support our theory’s assumption of the decoupled Luttinger liquid is a $T$-linear behaviour (or at least non-Fermi-liquid behaviour) in the out-of-plane resistivity in the high-$T$ ‘normal’ metal phase. To our best knowledge, however, no comprehensive experimental studies have been carried out so far for the temperature-dependence of the resistivity in the ‘normal’ metal phase in the graphite under the high magnetic field [17,20].

When the metal phase is characterized by a new free theory instead of the free theory of the decoupled LL phase (a gaussian theory given by Eq. (19)), critical properties of the metal-insulator (MI) and re-entrant insulator-metal (IM) transitions are characterized by a new saddle-point fixed point (schematically denoted by ‘FP4’ in Fig. 11) rather than by the FP1 that leads to the argument in Sec. VIB. Meanwhile, having a finite charge gap, a fixed point of the excitonic insulator (EIs) is expected to be locally stable against the small perturbation. Thereby, the primary features of the two EI phases discussed in the paper will not change dramatically even in the presence of such perturbations. These features include the finite mobility gaps in $\sigma_{zz}(\omega)$ in the two SNEI phases, an overall structure of the $H$-$T$ phase diagram as well as topological Dirac-cone surface state in the SNEI-II phase and in-plane electric transport due to the surface state.

B. excitonic BCS-BEC crossover and nature of a transition between SNEI-I and SNEI-II phases

Our theory does not include an effect of an excitonic condensation, as emphasized in Ref. [20]. When $H$ approaches $H_0$ from above ($H > H_0$), electron-hole bound states formed by electron in $(n,\sigma) = (0,\uparrow)$ LL and hole in $(n,\sigma) = (-1,\downarrow)$ LL could undergo the Bose-Einstein condensation. Such condensation further assists electron-hole BCS pairings between $(n,\sigma) = (0,\downarrow)$ and $(n,\sigma) = (-1,\uparrow)$ LLs, through the umklapp term $H_{n,2}$. This leads to a phase with electrically insulating behaviour along the field direction; the phase is essentially same as the SNEI-I phase discussed in the paper. When the exciton BEC effect is included into our theory, the phase boundary between SNEI-I and SNEI-II phases (say $H = H_{c,3}$) will presumably go above $H_0$ ($H < H_{c,3}$).

For $H \geq H_{c,3}$, the long-range phase coherences defined by $\theta_{1,j} - \theta_{1,j} = n\pi - \Theta_-$ and $\phi_{1,j} + \phi_{1,j} = (m+1)\pi - \Phi_-$ in Eqs. (45,46) fade away, while the other long-range phase coherences defined by $\theta_{3,j} - \theta_{2,j} = \Theta_-$ and $\phi_{3,j} + \phi_{2,j} = \Phi_-$ may survive, leading to a phase similar to the spin nematic excitonic insulator phase discussed in Sec. VIB. From this viewpoint, the SNEI-II phase could be regarded as a ‘partial ordered phase’ derived from the SNEI-I phase. Nonetheless, it can be entirely possible that these two SNEI phases are symmetrically distinct from each other, depending on the spatial parities of the excitonic pairings in the two phases, whose importance was emphasized in Sec. VIII A. Qualitative natures of the phase transition between these two excitonic insulator phases need further theoretical studies.
Appendix A: charge neutrality condition

Transverse conductivity \( \sigma_{xy} \) gives a precise information of electron carrier density \( n_e \) and hole carrier density \( n_h \) in any given metal and semimetal under high magnetic field through the following formula;

\[
\sigma_{xy} H = ec(n_e - n_h). \tag{A1}
\]
e (0) and \( c \) are the electron charge, and the speed of light respectively. In the main text, we use the formula and evaluate the total number of \( k_z \) points in the electron/hole pockets \( N_e/N_h \) in graphite under the field. With the formula, the previous Hall conductivity measurement in the regime of \( 20 \text{T} \lesssim H \lesssim 55 \text{T} \) \cite{18} gives \( N_e - N_h \): \( L_z/c_0 = 10^{-4} \times 1 \). Using the Kubo formula of the Hall conductivity, Akiba discussed a validity of the formula in the quasi-quantum limit in the graphite \cite{18}. In the following, we employ the Buttiker’s theory of the Hall conductivity \cite{67, 68}, to demonstrate a validity of the formula in a generic three-dimensional metal and semimetal under high field.

Use the Landau gauge and assume that a given three-dimensional system is translational symmetric along \( x \) and \( z \) directions. Electrons are confined along \( y \) direction within \( |y| < L_y/2 \) by a confining potential. A single-particle Hamiltonian comprises of two parts;

\[
\hat{H}_T \equiv \hat{H}_0(k_z; \hat{k}_z) + \hat{V}(k_z; \hat{k}_z, \hat{y}), \tag{A2}
\]
with \( \hat{k}_z \equiv (-i\partial_y) \pm i(-k_x + \mu V/H) \). \( \hat{H}_0 \) is a bulk Hamiltonian that depends on the coordinate \( y \) through \( \hat{k}_x \) \( \hat{k}_z \). \( \hat{V} \) describes an effect of the confining potential; \( \hat{V} \equiv 0 \) when \( |y| \ll L_y/2 \). \( \hat{V} \) depends on \( y \) explicitly. \( \hat{H}_T \) in Eq. (A2) is already Fourier-transformed with respect to \( x \) and \( z \). they are functions of the conjugate momenta \( k_x \) and \( k_z \). In a system with multiple energy bands, \( \hat{H}_T \) takes a matrix form. For the spinless graphite case, \( \hat{H}_T \) is a four by four matrix; the four bases are from the \( \pi \) orbitals in \( A, A', B \) and \( B' \) carbon atoms within the unit cell. Using the \( k \cdot p \) expansion, Slonczewski, Weiss and McClure derived \( \hat{H}_0 \) around the zone boundary of the first Brillouin zone of the graphite.

In the following, we only assume that \( \hat{H}_0(k_z; \hat{k}_z) \) as well as \( \hat{V}(k_z; \hat{k}_z, \hat{y}) \) are given by finite order polynomials in \( \kappa_{\pm} \) and \( y \). Under this assumption, the explicit \( y \)-dependence of \( \hat{V} \) can be rewritten into the \( y_c \)-dependence by use of \( y \equiv (-i)(l^2/2)(\kappa_+ - \kappa_-) + y_c \) and \( y_c \equiv k_x l^2 \);

\[
\hat{H}_T(k_z; \hat{k}_z, \hat{y}) = \hat{H}_T'(k_z, y_c; \hat{k}_z) . \tag{A3}
\]
Eigenstates of such \( \hat{H}_T \) are localized in the \( y \) coordinate at \( y = y_c \). Eigenvalues depend on \( k_z \), \( y_c \) and the Landau index \( n \);

\[
\hat{H}_T \phi_n(k_z, y_c; y - y_c) = E_n(k_z, y_c) \phi_n(k_z, y_c; y - y_c). \tag{A4}
\]
A single-particle velocity operator along \( x \) is given by a \( k_x \) derivative of \( \hat{H}_T \). With \( k_x l^2 \equiv y_c \), an expectation value of the velocity with respect to the eigenstate is given by a \( y_c \)-derivative of the eigenvalue. Besides, the eigenstate is uniformly extended along \( x \). Thus, an electric current carried by the eigenstate is given by

\[
J_{x,n,k_z,y_c} = \left(-e\right)^2 \frac{\partial E_n(k_z, y_c)}{\hbar L_x} \frac{\partial y_c}{\partial y_c} . \tag{A5}
\]
The total current density from the \( n \)-the Landau level is a sum of \( J_{x,n,k_z,y_c} \) over all the filled \( k_z \) and \( k_x \equiv y_c/l^2 \) points;

\[
\begin{align*}
J_x &= \frac{1}{L_z L_y} \sum_{k_z} \sum_{k_x} J_{x,n,k_z,y_c, f_T(E_n(k_z, y_c))} \\
&= \left(-e\right) \frac{\hbar L_y}{L_z} \int_{-\pi}^{\pi} \frac{dk_z}{2\pi} \int_{-\infty}^{+\infty} \frac{dy_c}{2\pi} \frac{\partial E_n}{\partial y_c} f_T(E_n(k_z, y_c)).
\end{align*} \tag{A6}
\]
\( f_T(E) \) is a Fermi distribution function. At the zero temperature, this reduces to a step function,

\[
\begin{align*}
f_{T=0}(E_n) &= \begin{cases} 
\theta(\mu_+ - E_n(k_z, y_c)) & (y_c \approx L_y/2), \\
\theta(\mu_- - E_n(k_z, y_c)) & (y_c \approx -L_y/2). 
\end{cases}
\end{align*} \tag{A7}
\]
\( \mu_+ \) are Fermi levels around \( y = \pm L_y/2 \) respectively. In the presence of a Hall voltage \( V_H \) in the \( +y \) direction, \( \mu_+ - \mu_- = -eV_H \).

In graphite under the high field, the two electron/hole pockets in the bulk region (\( n = 0/n = -1 \) LLs with \( \uparrow \) and \( \downarrow \) spins) end up with two electron/hole surface chiral Fermi arc (SCFA) states in the boundary region. Namely, \( E_n=0/-1,\sigma(k_z, y_c) \) increases/decreases in energy, when \( y_c \) goes from the bulk region to the boundary region (Fig. 2);

\[
\begin{align*}
&\exists \ \{ E_{n=0,\sigma}(k_z, y_c) \uparrow (|y_c|/\gamma), \\
&\exists \ \{ E_{n=-1,\sigma}(k_z, y_c) \downarrow (|y_c|/\gamma) \}.
\end{align*} \tag{A8}
\]
Accordingly, the current density induced by the finite Hall voltage comprises of two parts that cancel each other:

\[
\begin{align*}
J_x &= -e \frac{\mu_+ - \mu_-}{\hbar L_y} \left( \int_{-k_1}^{k_1} \frac{dk_z}{2\pi} + \int_{-k_2}^{k_2} \frac{dk_z}{2\pi} \right) \\
& \quad + e \frac{\mu_- - \mu_+}{\hbar L_y} \left( \int_{k_3}^{k_3} \frac{dk_z}{2\pi} + \int_{k_4}^{k_4} \frac{dk_z}{2\pi} \right). \tag{A9}
\end{align*}
\]
The first part is from the two electron surface states that subtend chiral arcs from \( k_z = -k_1 \) to \( k_z = k_1 \) and from \( k_z = -k_2 \) to \( k_z = k_2 \) respectively. The other part is from the two hole surface states that subtend chiral arcs from \( k_z = k_3 \) to \( 2\pi/c_0 - k_3 \) and from \( k_z = k_4 \) to \( 2\pi/c_0 - k_4 \) respectively (Fig. 2). To have Eq. (A9), we assume that the hole pocket energies are same in the vacuum,

\[
E_{n=-1,\sigma}(k_z, y_c = -\infty) = E_{n=-1,\sigma}(k_z, y_c = +\infty). \tag{A10}
\]
Eq. [A9] gives the Hall conductivity as,
\[ \sigma_{xy} = \frac{e^2}{H} \frac{1}{L_z} \left( N_1 + N_2 - N_3 - N_4 \right) \]
\[ = \frac{e^2}{H} (n_e - n_h), \]  
(A11)
with \((N_1 + N_2)/L_z = 2\pi l^2 n_e\) and \((N_3 + N_1)/L_z = 2\pi l^2 n_h\). From the previous Hall conductivity measurement [31], we typically have
\[ n_e - n_h = 5 \times 10^{15} \text{[cm}^{-3}], \]
\[ c_0 = 6.7 \times 10^{-10} \text{[m]}, \quad l = 45 \times 10^{-10} \text{[m]}, \]
for \(H = 30 \text{ T}\) and
\[ n_e - n_h = -10 \times 10^{15} \text{[cm}^{-3}], \]
\[ c_0 = 6.7 \times 10^{-10} \text{[m]}, \quad l = 40 \times 10^{-10} \text{[m]}, \]
for \(H = 55 \text{ T}\). This gives out a ratio between \(N_e - N_h\) and \(L_z/c_0\) as
\[ N_e - N_h : L_z/c_0 = \pm 3 \times 10^{-4} : 1 \]  
(A12)
for \(30 \text{ T} < H < 55 \text{ T}\). From this very small number, we conclude that graphite under this field regime safely satisfies the charge neutrality condition.

### Appendix B: renormalization of Luttinger parameters and Fermi velocities

In the main text, we use the Hartree-Fock approximation for the four pockets model or two pockets model, to introduce effective boson Hamiltonians, such as Eqs. [18][19][20][21] with Eqs. [22][23][24][25][26][27][28][29][30][31] or with Eqs. [32][33][34][35][36][37][38][39][40]. Thereby, the bare kinetic energy part takes a quadratic form in the phase variables, Eq. (19), whose coefficients (Luttinger parameters and Fermi velocities) are further renormalized by intra-pocket forward scattering terms. In the following, we summarize how the intra-pocket forward scattering terms renormalize the Luttinger parameters and Fermi velocities.

The electron interaction within the same pockets is given by
\[ H_f = \sum_{j,m,n,a=1,2,3,4} \int dz \int dz' e^{-i(x-x')^2/2a} V^{(1),a}_{n-m,n-j} \psi^\dagger_{a,n}(z) \psi_{a,j+m-n}(z') \psi_{a,m}(z') \psi_{a,j}(z), \]  
(B1)
with \(\psi_{a,n}(z) \equiv e^{ik_{F,a}z}\psi_{a,+n}(z) + e^{-ik_{F,a}z}\psi_{a,-n}(z)\). The matrix element \(V^{(1),a}_{n,m} (a = 1, 2, 3, 4)\) is obtained by the substitutions of Eqs. (18,19,20) into Eq. (19). In the limit of short interaction length \((l_{0,z} \ll l)\), the matrix element takes a form of
\[ V^{(1),a}_{n,m} = \frac{g}{L_x} \int_{l_{0,z}} f^{(1),a}(y_n/l, y_m/l). \]  
(B2)
Dimensionless functions \(f^{(1),a}(x,y)\) decay quickly for \(|x|, |y| > 1\). With the Hartree-Fock approximation, \(H_f\) is bosonized into the followings,
\[ H_f = \sum_{a \sum_j \int dz' \sqrt{2\pi l_{0,z}} \left( V^{(1),a}_{j-m,0} - V^{(1),a}_{j-m} \right) \times \left( \rho_{a,j+1} \rho_{a,j-1} + \rho_{a,j-1} \rho_{a,j+1} \right) \]
\[ + \sum_{a \sum_j \int dz' \sqrt{2\pi l_{0,z}} \left( V^{(1),a}_{j-m,0} - V^{(1),a}_{j-m} e^{-2(k_{F,a} l_{0,z})^2} \right) \times \left( \rho_{a,j+1} \rho_{a,j-1} + \rho_{a,j-1} \rho_{a,j+1} \right) \]
\[ + 2 \sum_{a \sum_j \int dz' \sqrt{2\pi l_{0,z}} \left( V^{(1),a}_{j-m,0} e^{-2(k_{F,a} l_{0,z})^2} - V^{(1),a}_{j-m} \right) \times \eta_{a,j} \eta_{a,-j} \eta_{a,+m} \eta_{a,-m} \cos \left[ 2(\phi_{a,j}(z) - \phi_{a,m}(z)) \right] \]
\[ + \cdots, \]  
(B3)
where \(\rho_{a,j}(z)\) stands for an electron density in the right (+) or left (−) branch in the \(a\)-th pocket \((a = 1, 2, 3, 4)\) of the \(j\)-th chain \((j = 1, 2, 3, \cdots, z_{j}^{2}/l_{0,2}^2)\).

\(\rho_{a,j}(z) \equiv \psi_{a,j}(z) \psi_{a,j}(z) = -\frac{1}{2\pi} \left( \partial_z \phi_{a,j} + \partial_y \theta_{a,j} \right).\)

The third term in Eq. (B3) represents a rigidity between two displacement fields in different chains in the same pocket. When the corresponding inter-chain interaction is negative definite, this could result in charge density wave orders with broken translational symmetry along the field direction. An interplay between this inter-chain rigidity term and one of the umklapp term is discussed for the two-pocket model case (see Sec. V).

The first two terms in Eq. (B3) lead to the renormalizations of the Luttinger parameters and Fermi velocities. To quantify them, we employ a gradient expansion with respect to the chain index,
\[ \rho_{a,j}(z) = \rho_{a,j}(y_m - y_j) \partial_{y_j} \rho_{a,j} + \frac{1}{2} \left( y_m - y_j \right)^2 \partial_{y_j}^2 \rho_{a,j} + \cdots, \]  
(B4)
to keep only the leading order. This leads to
\[ H_f = \sum_a \sum_j \int dz \left\{ \frac{g_2 a + g_4 a}{(2\pi)^2} \left( \partial_z \phi_{a,j} \right)^2 + \frac{-g_2 a + g_4 a}{(2\pi)^2} \left( \partial_y \theta_{a,j} \right)^2 \right\} + \cdots, \]  
(B5)
with
\[ g_{2,a} = 2\sqrt{2\pi l_{0,z}} \sum_m \left( V^{(1),a}_{m,0} - V^{(1),a}_{0,m} e^{-2(k_{F,a} l_{0,z})^2} \right) \]
\[ = \frac{2}{\pi l_{0,z}} \int dx \left\{ f^{(1),a}(x,0) - f^{(1),a}(0,x) e^{-2(k_{F,a} l_{0,z})^2} \right\}, \]  
(B6)
\[ g_{4,a} = 2\sqrt{2\pi l_{0,z}} \sum_m \left( V^{(1),a}_{m,0} - V^{(1),a}_{0,m} \right) \]
\[ = \frac{2}{\pi l_{0,z}} \int dx \left\{ f^{(1),a}(x,0) - f^{(1),a}(0,x) \right\}. \]  
(B7)
When combined with the bare kinetic energy part:

\[ H_{\text{kin}} = \sum_{a,j} \frac{v_{F,a}}{2\pi} \int dz \left\{ \left( \partial_z \phi_{a,j} \right)^2 + \left( \partial_r \theta_{a,j} \right)^2 \right\}, \]  

(E8)

Eq. (B5) gives out Eq. (19) with Eqs. (20,21).

**Appendix C: derivation of renormalization group (RG) equations**

In the main text, we employ one-loop RG equations, Eqs. [35,36,37], and clarify possible insulating phases as well as natures of \( T = 0 \) metal-insulator and insulator-metal transition points in graphite under the high field.

We solve the RG equations numerically to obtain a finite temperature phase diagram as in Fig. 1. The RG equations are derived perturbatively by use of the standard momentum-shell renormalization method [11]. In the following, we briefly summarize how to derive the one-loop RG equations for \( H_{n,2} \) and \( H_{b,2} \), Eqs. [35,36,37].

We begin with a partition function of the effective field theory:

\[ Z = \sum_{\sigma} \int D\phi D\theta e^{-S_0[\phi,\theta] - S_1[\phi,\theta]}. \]  

(C1)

An action \( S \) comprises of a gaussian part \( S_0 \) and non-gaussian part \( S_1 \):

\[ S_0 = \int^\beta_0 d\tau \sum_{a,j} \frac{1}{2\pi} \left\{ -2i\partial_{\tau} \theta_{a,j}(r) \partial_{\tau} \phi_{a,j}(r) + u_a K_a \left[ \partial_z \phi_{a,j}(r) \right]^2 + \frac{u_a}{K_a} \left[ \partial_r \phi_{a,j}(r) \right]^2 \right\}, \]  

\[ S_1 = \int^\beta_0 d\tau \left\{ H_{u,2} + H_{b,2} + \cdots \right\}. \]  

(C2)

Here \( a \) is the pocket index (\( a = 1, 2, 3, 4 \)). The summation over Ising variables \( \sigma \) represent traces over two-dimensional Hilbert spaces subtended by two Klein factors associated with the bosonization. With \( r = (z, \tau) \), \( q = (k_z, i\omega_n) \) and Matsubara frequency \( \omega_n = 2n\pi/\beta \), the Fourier transforms of \( \phi_{j,a}(z, \tau) \) and \( \theta_{j,a}(z, \tau) \) are given by

\[ \phi_{j,a}(r) = \frac{1}{\beta L_z} \sum_{\omega_n} \sum_{|k_z|<\Lambda} e^{ik_z z - i\omega_n \tau} \phi_{j,a}(q). \]  

(C4)

\( \Lambda \) is a cutoff in the momentum space. We decompose the field operators into a slow mode and a fast mode in the momentum space,

\[ \phi_{j,a}(r) = \phi^{<}_{j,a}(r) + \phi^{>}_{j,a}(r), \]

\[ \phi^{<}_{j,a}(r) = \frac{1}{\beta L_z} \sum_{\omega_n} \sum_{|k_z|<\Lambda'} e^{iF_{\phi}^a(k_z)} \phi_{j,a}(q), \]

\[ \phi^{>}_{j,a}(r) = \frac{1}{\beta L_z} \sum_{\omega_n} \sum_{\Lambda'<|k_z|<\Lambda} e^{iF_{\phi}^a(k_z)} \phi_{j,a}(q), \]

with \( \Lambda' = \Delta b^{-1} \). \( b > 1 \) denotes a scale change.

First integrate out the fast mode \( \phi^{>} \) and \( \theta^{>} \) in the partition function and rescale spatial and temporal length scales as

\[ z_{\text{new}} = z_{\text{old}} b^{-1}, \tau_{\text{new}} = \tau_{\text{old}} b^{-1}, \beta_{\text{new}} = \beta_{\text{old}} b^{-1}. \]  

(C5)

This gives a partition function for the slow mode. The partition function takes essentially the same form as Eqs. [C2,C3], while the interchain interactions in Eqs. [C3] are renormalized. The renormalization is calculated with respect to an infinitesimally small scale change \( \ln b \ll 1 \). This gives the RG equations for the interactions as in Eqs. [35,36,37].

We derive the partition function for the slow mode perturbatively in the non-gaussian part \( S_1 \). We do so up to the second order in \( S_1 \):

\[ Z = Z_0^{<} \int D\phi^{<} D\theta^{<} e^{-S_0^{<} e^{-(S_{U})}} \]  

(C6)

where

\[ \langle S_{U} \rangle = \left< S_1 \right> - \frac{1}{2} \left< \left< S_1 \right> \right> \]  

(C7)

and

\[ \langle \cdots \rangle = \frac{1}{Z_0^{<}} \int D\phi^{<} D\theta^{<} \cdots e^{-S_0^{<}}, \]

\[ S_0^{<} = \frac{1}{2} \beta L_z \sum_{a,j} \sum_{\omega_n} \sum_{|k_z|<\Lambda'} \cdots, \]

\[ S_0^{>} = \frac{1}{2} \beta L_z \sum_{a,j} \sum_{\omega_n} \sum_{|k_z|<\Lambda'} \cdots, \]

with \( Z_0^{>} = \left< 1 \right> \). “...” in the right-hand sides of \( S_0^{</>} \) is a Fourier transform of the integrand in Eq. [C2]. The first term in Eq. [C7] gives a tree-level renormalization to the interchain interactions, while the second term gives a one-loop level renormalization.

### 1. tree-level renormalization

\( \langle S_1 \rangle \) in Eq. [C7] gives the tree-level renormalization to the interchain interactions:

\[ \langle \int^\beta_0 H_{u,2} d\tau \rangle = \frac{1}{2} \int d^2r \sum_{j,m} \sum_{c} \sum_{\pm} M^{(2)}_{j-m} \eta^c \eta^c e^{iM^{(2)}_{j-m}(r)e^{-(M^{(2)}_{j-m}(r)^2)}} \]  

(C8)

\[ \langle \int^\beta_0 H_{b,2} d\tau \rangle = \frac{1}{2} \int d^2r \sum_{j-m} \sum_{c} \sum_{\pm} \left\{ \begin{array}{c} H^{(2)}_{j-m} \eta^c \eta^c e^{iH^{(2)}_{j-m}(r)e^{-(H^{(2)}_{j-m}(r)^2)}} \\ + H^{(2)}_{j-m} \eta^c \eta^c e^{iH^{(2)}_{j-m}(r)e^{-(H^{(2)}_{j-m}(r)^2)}} \end{array} \right\} \]  

(C9)
where
\[ M_j^\eta (r) = Q^{23}_j^\eta (r) + Q^{14}_j^\eta (r), \]
\[ H_j^\eta (r) = Q^{23}_j^\eta (r) - Q^{23}_m^\eta (r), \]
\[ \mathcal{H}_j^\eta (r) = Q^{14}_j^\eta (r) - Q^{14}_m^\eta (r), \]
and
\[ \sigma_j^+ = \sigma_{2j}, \quad \sigma_j^- = \sigma_{2j}, \]
\[ \tau_j^+ = \sigma_{4j}, \quad \tau_j^- = \sigma_{4j}. \]
As the leading order in the infinitesimally small \( b \), we obtain,
\[ \langle M_{j,m}^{\eta,\eta} (r)^2 \rangle = \sum_{a=1,2,3,4} \frac{1}{2} (K_a + 1/K_a) \cosh \frac{\beta u_a A}{2} \ln b, \]
\[ \langle H_{j,m}^{\eta,\eta} (r)^2 \rangle = 2 \sum_{a=2,3} \frac{1}{2} (K_a + 1/K_a) \cosh \frac{\beta u_a A}{2} \ln b, \]
\[ \langle \mathcal{H}_{j,m}^{\eta,\eta} (r)^2 \rangle = 2 \sum_{a=1,4} \frac{1}{2} (K_a + 1/K_a) \cosh \frac{\beta u_a A}{2} \ln b. \]
This leads to the tree-level RG equation as
\[ \frac{dM_{j,m}^{(2)}}{d\ln b} = \left[ 2 - \frac{1}{2} \sum_{a=1,2,3,4} (K_a + 1/K_a) \cosh \frac{\beta u_a A}{2T} \right] M_{j,m}^{(2)}, \]
\[ \frac{dH_{j,m}^{(2)}}{d\ln b} = \left[ 2 - \frac{1}{2} \sum_{a=2,3} (K_a + 1/K_a) \cosh \frac{\beta u_a A}{2T} \right] H_{j,m}^{(2)}, \]
\[ \frac{d\mathcal{H}_{j,m}^{(2)}}{d\ln b} = \left[ 2 - \frac{1}{2} \sum_{a=1,4} (K_a + 1/K_a) \cosh \frac{\beta u_a A}{2T} \right] \mathcal{H}_{j,m}^{(2)}. \]

2. one-loop level renormalization

\[ \langle S_t S_j \rangle_{>,c} = \langle S_t^2 \rangle_{>,-} - \langle S_j \rangle^2_{>-} \]
where \( \langle AB \rangle_{>,c} = \langle AB \rangle - \langle A \rangle \langle B \rangle \). When \( i \neq m, n \) and \( j \neq m, n \) in Eq. \( \text{[C12]} \), the right hand side vanishes identically. The terms with \( i = m \) and \( j \neq m \) or those with \( i = n \) and \( j = m \) are negligibly smaller than the others in the larger \( L_x \) limit. We thus consider only those terms in Eq. \( \text{[C12]} \) with \( i = m, n \) and \( j \neq m, n \) and/or those terms with \( i \neq m, n \) and \( j = m, n \).

The one-loop renormalization in Eq. \( \text{[C12]} \) generates \( S_M, S_H \) and \( S_\mathcal{H} \) as well as other types of cosine terms. Nonetheless, tree-level scaling dimensions of all the other cosine terms thus generated are negatively much larger than those of \( S_M, S_H \) and \( S_\mathcal{H} \). Namely, they are much more irrelevant than \( S_M, S_H \) and \( S_\mathcal{H} \) at the tree-level renormalization group flow. Thus, we only keep those terms in Eq. \( \text{[C12]} \) that generate \( S_M, S_H \) and \( S_\mathcal{H} \). \( S_M^2 \) with \( \epsilon = -\epsilon', \eta = \eta' \) and \( i = m \) (or \( j = n \)) generates \( S_M^2 \) (or \( S_H^2 \)) respectively. \( S_M^2 \) with \( \epsilon = -\epsilon', \eta = \eta' \), and \( i = m \) or \( j = n \) or with \( \epsilon = \epsilon', \eta = \eta' \), and \( i = m \) (or \( j = n \)) generates \( S_M, S_H \). \( S_M^2 \mathcal{H} \) does not generate any of \( S_M, S_H \) and \( S_\mathcal{H} \). In the following, we only demonstrate how \( S_M^2 \) generates \( S_\mathcal{H} \).

With \( \epsilon = -\epsilon', \eta = \eta' \) and \( i = m \), Eq. \( \text{[C12]} \) with \( I = J = M \) reduces to
\[ \langle S_M^2 \rangle_{>,c} = \frac{1}{4} \int d^2 r \int d^2 r' \sum_{j,m} \sum_{i,m} \sum_{\eta,\eta'} \tau_j^{\eta,\eta} M_j^{(2)} M_{j,m}^{(2)} \]
\[ e^{i(M_j^{\eta,\eta}(r) - M_j^{\eta,\eta}(r'))} (e^{iM_j^{\eta,\eta}(r)}) (e^{-iM_j^{\eta,\eta}(r')}) \]
\[ = \frac{1}{2} \int d^2 r \int d^2 r' \sum_{j,m} \sum_{i,m} \sum_{\eta,\eta'} \tau_j^{\eta,\eta} M_j^{(2)} M_{j,m}^{(2)} \]
\[ \cos [M_j^{\eta,\eta}(r) - M_{j,m}^{\eta,\eta}(r')] \langle M_j^{\eta,\eta}(r) M_{j,m}^{\eta,\eta}(r') \rangle_{>,c} \]
\[ = \text{[C13]} \]
where
\[ \cos [M_j^{\eta,\eta}(r) - M_{j,m}^{\eta,\eta}(r')] = \]
\[ \cos [Q_{j,m}^{14,14}(r) - Q_{j,m}^{14,14}(r')] \cos [Q_{j,m}^{23,23}(r) - Q_{j,m}^{23,23}(r')] \]
\[ - \sin [Q_{j,m}^{14,14}(r) - Q_{j,m}^{14,14}(r')] \sin [Q_{j,m}^{23,23}(r) - Q_{j,m}^{23,23}(r')]. \]
\[ \text{[C14]} \]
The largest part of the contribution comes from \( r = r' \). In this case, the second term in Eq. \( \text{[C14]} \) vanishes (see the next subsection for a justification of this approximation). For the first term with \( j \neq n \), we replace
\[ \cos [Q_{j,m}^{23,23}(r) - Q_{j,m}^{23,23}(r')] \]
by its normal ordering with use of a formula \( \cos \Phi = \exp[-(\Phi^2)/2 \text{H11,69}] \).
Within the normal order, we employ a Taylor expansion with respect to small \( r - r' \). At the leading order expansion, Eq. \( \text{[C14]} \) becomes
\[ \cos [Q_{j,m}^{14,14}(r) - Q_{j,m}^{14,14}(r')] e^{-\frac{1}{2} (Q_{j,m}^{14,14}(r) - Q_{j,m}^{14,14}(r'))^2} \]
\[ \text{[C15]} \]
Thereby, we have
\[
\langle S_M^2 \rangle_{\epsilon,c} = \int d^4 r \sum_{j \neq n} \sum_{\epsilon_\eta} \pi^{\eta_\eta} \cos [F_{jn}(r)] \times C_{23} \sum_{i} M^{(2)}_{i-n} M^{(2)}_{i-n} \ln b, \quad (C16)
\]
where
\[
C_{cd} \ln b = \frac{1}{2} \int dr \frac{1}{2}((Q^{cd}_{\eta,i} (r) - Q^{cd}_{\eta,i} (r'))^2) \langle Q^{cd}_{\eta,i} (r) Q^{cd}_{\eta,i} (r') \rangle_c,
\]
\[
(C17)
\]
with \(c, d = 1, 2, 3, 4\). Note that the integrand in Eq. (C17) is short-ranged in \(r - r'\) and \(C_{cd}\) is a positive definite real-valued quantity (see the next subsection). Eq. (C16) in combination with Eqs. (C7) - (C10) dictates that the one-loop renormalization
\[
\frac{dH^{(2)}_{j-n}}{db} = \cdots - C_{23} \sum_{i} M^{(2)}_{i-n} M^{(2)}_{i-n} + \cdots . \quad (C18)
\]
Since \(M^{(2)}_{i-n} = M^{(2)}_{i-n}\), this is nothing but the first term of the one-loop renormalization in Eq. (37). Similarly, one can show all the other terms of the one-loop renormalizations in Eqs. (35), (36), (37). A factor “4” in the second term of the one-loop renormalization in Eq. (37) is due to the four distinct contributions to \(S_M\) from \(S_{j}^2\); (i) \(\epsilon = -\epsilon', \eta = \eta', i = m, (ii) \epsilon = -\epsilon', \eta = \eta', j = n, (iii) \epsilon = \epsilon', \eta = \eta', i = n, (iv) \epsilon = \epsilon', \eta = \eta', j = m\) in Eq. (C12). Likewise, \(2S_M S_H\) has two distinct contributions to \(S_M\), giving rise to the second (second) term of the one-loop renormalization in Eq. (37); (i) \(\epsilon = \epsilon', \eta = \eta', i = n, (j = n), (ii) \epsilon = -\epsilon', \eta = \eta', i = m, (j = m)\) in Eq. (C12). This completes the derivation of Eqs. (35), (36), (37).

3. evaluation of \(C_{cd}\)

\(C_{ab}\) is defined in Eq. (C17). Let us first calculate the integrand in Eq. (C17):
\[
\langle Q^{ab}_{\eta,i} (r) Q^{ab}_{\eta,i} (r') \rangle_c = \frac{1}{(\beta L_z)^2} \sum_{A' < |k| < A} \sum_{\omega_n} \epsilon_i^{2q(r-r')} \langle Q^{ab}_{\eta,i} (q) Q^{ab}_{\eta,i} (q) \rangle_c,
\]
\[
\langle (Q^{ab}_{\eta,i} (r) - Q^{ab}_{\eta,i} (r'))^2 \rangle_c = \frac{1}{(\beta L_z)^2} \sum_{|k| < A} \sum_{\omega_n} 2(1 - \epsilon_i^{2q(r-r')}) \langle Q^{ab}_{\eta,i} (q) Q^{ab}_{\eta,i} (q) \rangle_c,
\]
where
\[
\langle Q^{ab}_{\eta,i} (q) Q^{ab}_{\eta,i} (q) \rangle_c = \sum_{c=a,b} \left\{ (\phi^{*}_{\epsilon,c,i}(q) \phi_{\epsilon,c,i}(q))_{c} + (\theta^{*}_{\epsilon,c,i}(q) \theta_{\epsilon,c,i}(q))_{c} \right\}.
\]
\[
(C19)
\]
with \((-1)^a = 1 \text{ and } (-1)^b = -1\). We used Fourier transform in Eq. (C4). The gaussian integrals over the fast/slow modes lead to
\[
\langle \phi^{*}_{\epsilon,c,i}(q) \phi_{\epsilon,c,i}(q) \rangle_c = \frac{\beta L_z \pi u_c K_c}{u_c^2 k_z^2 + \omega_n^2},
\]
\[
\langle \theta^{*}_{\epsilon,c,i}(q) \theta_{\epsilon,c,i}(q) \rangle_c = \frac{\beta L_z \pi u_c K_c^{-1}}{u_c^2 k_z^2 + \omega_n^2},
\]
\[
\langle \phi^{*}_{\epsilon,c,i}(q) \phi_{\epsilon,c,i}(q) \rangle_c = -\frac{\beta L_z \pi \omega_n}{k_z (u_c^2 k_z^2 + \omega_n^2)}.
\]

Accordingly, we have
\[
\langle Q^{ab}_{\eta,i} (r) Q^{ab}_{\eta,i} (r') \rangle_c = \frac{1}{2} \sum_{c=a,b} (K_c + K_c^{-1}) M_c (r - r') + \sum_{c=a,b} \eta(-1)^c F_{2,c}(r - r'),
\]
\[
\langle (Q^{ab}_{\eta,i} (r) - Q^{ab}_{\eta,i} (r'))^2 \rangle_c = \frac{1}{2} \sum_{c=a,b} (K_c + K_c^{-1}) F_{1,c}(r - r') + \sum_{c=a,b} \eta(-1)^c F_{2,c}(r - r'),
\]
\[
(C20)
\]
with
\[
M_c (r) = \int_{A' < |k| < A} dk \frac{1}{\beta} \sum_{\omega_n} \frac{u_c e^{iqr}}{\omega_n^2 + u_c^2 k_z^2}
\]
\[
= \cos(\Delta z) e^{-\omega_c A |r|} \ln b,
\]
\[
F_{2,c}(r) = -\int_{A' < |k| < A} dk \frac{1}{\beta} \sum_{\omega_n} \frac{i \omega_n}{k_z} e^{iqr} e^{-\omega_c A |r|} \ln b,
\]
\[
F_{1,c}(r) = \int_{|k| < A'} dk \frac{1}{\beta} \sum_{\omega_n} \frac{2(1 - \cos(qr)) u_c}{\omega_n^2 + u_c^2 k_z^2}
\]
\[
= \log \left[ (x^2 + y_c^2) / a^2 \right],
\]
\[
F_{2,c}(r) = \int_{|k| < A'} dk \frac{1}{\beta} \sum_{\omega_n} \frac{2 \cdot e^{iqr}}{k_z} \omega_n^2 + u_c^2 k_z^2
\]
\[
= 2 \text{Arg} \left[ y_c - ix \right] \equiv 2 \theta_c (r),
\]
and \(y_c = u_c + a \text{sgn}(\tau)\). In the right hand side, \(M_c (r), F_{2,c}(r), F_{1,c}(r)\) and \(F_{2,c}(r)\) are evaluated at the zero temperature. Substituting these into Eq. (C17), we obtain
$C_{ab}$ at $T = 0$ as,

$$C_{ab, T=0} = \sum_{c=a,b} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dz \left( \frac{\alpha^2}{z^2 + y_a^2} \right) \lambda_a \left( \frac{\alpha^2}{z^2 + y_b^2} \right)^{\lambda_b}$$

\[ \approx \sum_{c=a,b} \lambda_c \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dz \left( \frac{\alpha^2}{z^2 + y_a^2} \right)^{\lambda_a} \left( \frac{\alpha^2}{z^2 + y_b^2} \right)^{\lambda_b} \cos(\Lambda z), \quad (C21) \]

with $y_c^2 \equiv (u_c \tau + \alpha^2)^2$, $\lambda_c \equiv \frac{1}{\Lambda} \left. \left( K_a + K_a^{-1} \right) \right| \Delta_{ab}(r) \equiv \theta_a(r) - \theta_b(r)$. The integrand in the first line is short-ranged in $r$, justifying a posteriori the approximations made in Eqs. (C1-15). Based on the same spirit, we approximate $\Delta_{ab}(r)$ by zero, to obtain the second line.

$C_{ab}$ is positive definite. One can show this by carrying out the $z$-integral formally,

$$C_{ab, T=0} = \sum_{c=a,b} \lambda_c \int_{-\infty}^{\infty} dt G(\tau) e^{-u_c \Lambda |\tau|}, \quad (C22)$$

and

$$G(\tau) \equiv \int_{-\infty}^{\infty} d\xi F_a(\xi; \tau) F_b(\Lambda - \xi; \tau) d\xi,$$  \[ \quad \text{(C23)} \]

$$F_a(\xi; \tau) \equiv \int_{-\infty}^{\infty} dz e^{i \xi z} \left( \frac{\alpha^2}{z^2 + y_a^2} \right)^{\lambda_a},$$

$$\approx 2\sqrt{\pi} \alpha^2 \lambda_a \left( \frac{\Lambda}{2|y_a|} \right)^{\lambda_a - \frac{1}{2}} K_{\lambda_a - \frac{1}{2}} \left( \frac{|y_a||\xi|}{\Lambda} \right), \quad (C24)$$

with the Bessel function $K_\nu(x)$ and the Gamma function $\Gamma(x)$. Since $\lambda_a > 1/2$, $F_a(\xi; \tau)$ is positive definite and so is $G(\tau)$. With Eq. (C22) this assures the positive definiteness of $C_{ab, T=0}$.

$C_{ab, T=0}$ in Eq. (C21) depends on the Luttinger parameters $K_a$ and $K_b$. Nonetheless, the dependence is much weaker than that of $A_{ab}$ in Eq. (38). One can see this, by evaluating an upper bound of $C_{ab, T=0}$.

$$C_{ab, T=0} < \sum_{c=a,b} \lambda_c \int_{-\infty}^{\infty} dz \left( \frac{\alpha^2}{z^2 + \alpha^2} \right)^{\lambda_a + \lambda_b} \int d\tau e^{-u_c \Lambda |\tau|}$$

$$\sum_{c=a,b} \alpha \lambda_c \Gamma(\frac{1}{2}) \Gamma(\lambda_a + \lambda_b + \frac{1}{2}) \equiv C_u.$$  \[ \text{(C25)} \]

$\Lambda_E$ denotes a finite high-energy cutoff in the energy scale, $\Lambda_E = \Lambda \times \max_{a,b}(u_a)$. When the Luttinger parameters get much smaller/larger than 1, $\lambda_a + \lambda_b \to \pm \infty$, the upper bound of $C_{ab, T=0}$ as well as $|A_{ab, T=0}|$ diverge;

$$C_u \to \frac{\alpha}{\Lambda_E} \Gamma\left( \frac{1}{2} \right) (\lambda_a + \lambda_b)^{\frac{1}{2}},$$

$$|A_{ab, T=0}| \to 2(\lambda_a + \lambda_b).$$  \[ \text{(C25)} \]

Meanwhile, $C_{ab, T=0}/|A_{ab, T=0}|$ goes to the zero in the limit of $\lambda_a + \lambda_b \to \pm \infty$. For simplicity, we assume that $A_{ab, T=0}$ does not depend on the magnetic field $H$ in the main text. A typical value of $C_{ab, T=0}$ is evaluated in a simple case with $K_a = K_b = 1$ and $u_a = u_b = u$,

$$C_{ab, T=0;K_a=1,u_a=u_b=u} = \int d\tau e^{-u \Lambda |\tau|} \int dz \left( \frac{\alpha^2}{z^2 + (u |\tau| + \alpha^2) \cos(\Lambda z)} \right)$$

$$\approx e^{-u \Lambda} \int d\tau e^{-u \Lambda |\tau|} \int dz \left( \frac{\alpha^2}{z^2 + (u |\tau| + \alpha^2) \cos(\Lambda z)} \right)$$

$$= e^{-u \Lambda} \frac{2 \alpha^2}{u} \int_0^\infty dx e^{-\alpha^2 x} = e^{-u \Lambda} \frac{2 \alpha^2}{u} E_1(2\alpha). \quad (C26)$$

$E_1(x)$ is the exponential integral. $\alpha$ is a lattice constant along the $z$-direction while $\Lambda$ is a high energy cutoff in the momentum space; $\Lambda \alpha = O(1)$.

4. parameters used in Fig. 1

To obtain theoretical phase diagram at finite temperature as in Fig. 1, we solved numerically the RG equations Eqs. (12, 23, 24) for $H < H_0$ and Eqs. (17, 18, 19) for $H_0 < H < H_1$. Thereby, a set of parameters in the RG equations are chosen in the following way.

$A_{ab}$ has an engineering dimension of [length]/[energy]. From Eq. (C26), we set

$$C_{ab} = \frac{2 \alpha}{\Lambda_E}, \quad (C27)$$

for any $a, b = 1, 2, 3, 4$. $\alpha$ is the lattice constant of the graphite along the c-axis, $\alpha = \alpha_0 = 6.7\text{Å}$. $\Lambda_E$ is a high energy cutoff in the energy scale. We set this to be a band width of the four pockets, $\Lambda_E = 40 \mathrm{[meV]}$.

According to Eqs. (39, 40, 41), $m_{(2)}, h_{(2)}, \bar{h}_{(2)}, n_{(2)}, p_{(2)}$, and $\bar{p}_{(2)}$ have the same engineering dimension as $\bar{g} \equiv g/\alpha^2$, where $g$ represents an interaction strength as in Eq. (7). For initial values of $m_{(2)}, \ldots, \bar{p}_{(2)}$ in the RG flow, we set

$$\left\{ \begin{array}{l} (m_{(2)}, h_{(2)}, \bar{h}_{(2)}) = \bar{g} (3, -1.25, -1.25), \\ (n_{(2)}, p_{(2)}, \bar{p}_{(2)}) = \bar{g} (1.1, -1.25, -1.25). \end{array} \right. \quad (C28)$$

A value of $\bar{g}$ is set in the following way. We consider that the interaction is from the Coulomb interaction and therefore its typical interaction energy scale is given by

$$E_{int} = \frac{e^2}{cl}.$$  \[ \text{(C29)} \]

The magnetic length $l$ depends on the magnetic field and the relative permittivity $\varepsilon$ is set to 13 for graphite. We regard that the Coulomb interaction ranges over the magnetic length in the $xy$ plane, and ranges over the Thomas-Fermi screening length along the $z$ direction $\lambda_{TF}$. We thus compare $E_{int}$ with $g/(\bar{g}^2 \lambda_{TF})$ [see Eq. (7)]. This leads to

$$\bar{g} = \frac{g}{\alpha^2} = \frac{e^2}{cl \alpha^2 \lambda_{TF}}.$$  \[ \text{(C30)} \]
The screening length along the c-axis is set to $\lambda_{TF} = c_0/\sqrt{6}$.

$A_{ab}$ in the RG equations is given by Eq. (38). $u_c A$ in Eq. (38) ($c = 1, 2, 3, 4$) is set to the high-energy cutoff in the energy scale, $\Lambda_c = 40$ [meV]. For the Luttinger parameters $K_a$ in Eq. (38), we use Eq. (21). The intra-pocket forward scattering strengths in Eq. (21) are set as,

\[
\begin{align*}
g_{4,a=1} &= g_{4,a=4} = \hat{g}, \\
g_{4,a=2} &= g_{4,a=3} = \hat{g}, \\
g_{2,a=1} &= g_{2,a=4} = \hat{g}/1.6, \\
g_{2,a=2} &= g_{2,a=3} = \hat{g}/1.1,
\end{align*}
\]

where $\hat{g}$ is given in Eq. (C30). The bare Fermi velocity in Eq. (21) $v_{F,a}$ is a $k_z$ derivative of the energy dispersion of the four pockets given in Eq. (2):

\[
v_{F,a} = \frac{\partial E_n,\sigma(k_z)}{\partial k_z} \bigg|_{k_z=k_{F,n,\sigma}} = -2\gamma_2 c_0 \sin(2\pi n_\sigma)
\]

with $a = (n, \sigma); 1 = (0, \uparrow), 2 = (0, \downarrow), 3 = (-1, \uparrow),$ and $4 = (-1, \downarrow).$ We set $2\gamma_2 = 40$ [meV], and

\[
\begin{align*}
\xi_{0,\uparrow} &= \frac{1}{2} + \frac{H}{200[T]}, \\
\xi_{0,\downarrow} &= \frac{1}{2} + \frac{H}{200[T]}, \\
\xi_{-1,\uparrow} &= \frac{1}{2} + \frac{H}{200[T]}, \\
\xi_{-1,\downarrow} &= \frac{1}{2} + \frac{H}{200[T]},
\end{align*}
\]

Eq. (C32) realizes $H_0 = 50$ [T] and $H_1 = 120$ [T].

**Appendix D: calculation of optical conductivity $\sigma_{zz}(\omega)$**

In the main text, we describe how the longitudinal optical conductivity along the field direction behaves in the SNEI phases as well as the metal-insulator transition points at $H = H_{c,1}$ and $H = H_{c,2}$. According to the linear response theory, the conductivity is given by a retarded correlation function between an electron polarization operator $P_z$ and current operator $J_z$. In the bosonization language, the former is a sum of the displacement fields over the pocket index $(a)$ and the chain index $(j)$,

\[
\hat{P}_z = -\frac{e}{\pi} \sum_j \sum_a \int dz \phi_{a,j}(z).
\]

The latter is a sum of the current density fields,

\[
\hat{J}_z = \frac{e}{\pi} \sum_j \sum_a u_a K_a \int dz \partial_z \theta_{a,j}(z).
\]

The correlation function is calculated with respect to a mean field action for the SNEI phases. For the mean field action, we employ a Gaussian approximation for $H_{u,2}$ and $H'_{u,2}$, to replace their cosine terms by proper quadratic terms,

\[
\begin{align*}
H_{u,2} &\simeq \sum_{j,m} M_{j-m}^{(2)} \int dz \{ (\phi_{2,j} + \phi_{3,j} + \phi_{1,m} + \phi_{4,m})^2 \\
&+ (\theta_{2,j} - \theta_{3,j} - \theta_{1,m} - \theta_{4,m})^2 \} \\
H'_{u,2} &\simeq \frac{1}{2} \sum_{j,m} N_{j-m}^{(2)} \int dz \{ (\phi_{2,j} + \phi_{3,j} + \phi_{2,m} + \phi_{3,m})^2 \\
&+ (\theta_{2,j} - \theta_{3,j} - \theta_{2,m} + \theta_{3,m})^2 \}.
\end{align*}
\]

This in combination with $H_0$ in Eq. (19), gives a gaussian ('mean-field') action that takes a form of

\[
S_{MF} = \frac{1}{2\beta L_z N} \sum_K \left( \phi_K^\dagger \theta_K^\dagger \right) [M_{0,K}] \left( \phi_K^\dagger \theta_K^\dagger \right),
\]

with $K \equiv (k_z, k, i\omega_n)$. The Fourier transform is taken with respect to the spatial coordinate $z$, imaginary time $\tau$ and the chain index $j$ ($y_j \equiv 2\pi i j/L_x$);

\[
\phi_{a,j}(z,\tau) \equiv \frac{1}{\beta L_z N} \sum_K e^{ik_z z + i k y_j - i\omega_n \tau} \phi_{a,K}.
\]

In the following, we briefly summarize how to calculate the retarded correlation function with respect to $S_{MF}$ in the SNEI-I phase with/without disorder.

For the model with two electron pockets and two hole pockets, the gaussian action is described by a $8 \times 8$ matrix,

\[
[M_{0,K}] = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix},
\]

A 4 by 4 matrix $A_K$ is for the displacement fields of the four pockets $\phi_a$ ($a = 1, 2, 3, 4$), and 4 by 4 matrix $D_K$ is for the superconducting phas fields of the four pockets $\theta_a$ ($a = 1, 2, 3, 4$). They are given by
\[ A_K = \begin{bmatrix} \frac{u_1}{\pi K^2} k_z^2 + 2M(0) & 2M(0) & 2M*(k) & 2M*(k) \\ 2M(0) & \frac{u_1}{\pi K^2} k_z^2 + 2M(0) & 2M*(k) & 2M*(k) \\ 2M(k) & 2M(k) & \frac{u_2}{\pi K^2 } k_z^2 + 2M(0) & 2M(0) \\ \end{bmatrix}, \]

\[ D_K = \begin{bmatrix} \frac{u_1}{\pi K^2} k_z^2 + 2M(0) & -2M(0) & 2M*(k) & -2M*(k) \\ -2M(0) & \frac{u_1}{\pi K^2} k_z^2 + 2M(0) & -2M*(k) & 2M*(k) \\ 2M(k) & -2M(k) & \frac{u_2}{\pi K^2 } k_z^2 + 2M(0) & -2M(0) \\ -2M(k) & 2M(k) & -2M(0) & \frac{u_2}{\pi K^2 } k_z^2 + 2M(0) \\ \end{bmatrix}, \]

where \( M(k) = \sum_j M_j^{(2)} e^{iky_j} \). The other 4 by 4 matrices \( B_K \) and \( C_K \) connect the four \( \phi \) fields and the four \( \theta \) fields,

\[ B_K = C_K = \frac{ik_z \omega_n}{\pi} 1_{4 \times 4}. \]  

\( 1_{4 \times 4} \) stands for the 4 by 4 unit matrix.

For later convenience, we introduce a new basis with respect to the pocket index:

\[ \Phi \equiv \begin{bmatrix} \Phi_1 \\ \Phi_{1I} \\ \Phi_{1II} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_4 \\ \phi_2 \\ \phi_3 \end{bmatrix} = T \tilde{\Phi}, \]

\[ \tilde{\Theta} \equiv T \tilde{\Theta}. \]

In the right hand side, we omitted the subscript \( K \) for the \( \phi, \theta, \Phi \) and \( \Theta \) fields. With the new basis, the gaussian action is given by

\[ S_{MF} = \frac{1}{2\beta L_z N} \sum_K \left[ \tilde{\Phi}_K \tilde{\Theta}_K^T \right] \left[ M_{c,K} \right] \left[ \tilde{\Phi}_K \tilde{\Theta}_K \right], \]

and

\[ \left[ M_{c,K} \right] = \begin{bmatrix} T A_K T & B_K \\ C_K & T D_K T \end{bmatrix}. \]

We consider that the total displacement field \( \Phi_+ \) couples with a disorder potential through;

\[ \hat{H}_{imp} = \sum_j \int dz \epsilon_j(z) \Phi_+^2(z). \]

Physically, such disorder potential \( \epsilon_j(z) \) is nothing but a local fluctuation of the dielectric constant. We take a quenched average over the local fluctuation as

\[ \cdots = \frac{\int dz \epsilon_j(z) \cdots e^{-\frac{1}{2} \sum_j \int dz \epsilon_j^2(z)}}{\int dz \epsilon_j(z) e^{-\frac{1}{2} \sum_j \int dz \epsilon_j^2(z)}}. \]

gy stands for a disorder strength associated with spatially (but not temporally) fluctuating dielectric constant.

We first calculate an imaginary-time time-ordered correlation function between \( \tilde{P}_z \) and \( \tilde{J}_z \), and then take an analytic continuation, \( i\omega \rightarrow \omega + i\eta \). This gives the retarded correlation function. The real part of the retarded correlation function is nothing but the optical conductivity \( \sigma_{zz}(\omega) \);

\[ \sigma_{zz}(\omega) = \text{Re} \left\{ \frac{\bar{\sigma}_{zz}(i\omega_n)}{i\omega_n = \omega + i\eta} \right\} \]

\[ \sigma_{zz}(i\omega_n) = \tilde{e}^2 U^{-1} T \overline{Q_{zz}^c(i\omega_n)} T \tilde{e}^+, \]

with \( \tilde{e}^+ = (1, 1, 1, 1)^T \). \( U^{-1} \) and \( Q_{zz}^c(i\omega_n) (\overline{Q_{zz}^c(i\omega_n)}) \) is the quenched average of \( Q_{zz}^c(i\omega_n) \) as well as \( T \) are 4 by 4 matrices,

\[ U^{-1} = \begin{bmatrix} u_1 K_1 & u_4 K_4 \\ u_2 K_2 & u_3 K_3 \end{bmatrix}. \]

\( Q_{zz}^c(i\omega_n) \) is a Fourier transform of the imaginary-time time-ordered correlation function between four \( \Phi \) fields and four \( \Theta \) fields,

\[ Q_{zz}^c(i\omega_n) = \int_0^\beta d\tau Q_{zz}^c(\tau, \tau) e^{i\omega_n \tau}, \]

\[ [Q_{zz}^c(\tau)]_{\alpha \beta} = \frac{\tilde{e}^2}{\pi^2 V} \sum_{j,m} \int dz \int dz' [R_{jm}^c(\tau, z, 0, z')]_{\alpha \beta}, \]

\[ [R_{jm}^c(\tau, z, 0, z')]_{\alpha \beta} = \frac{\int d\tilde{\Phi}_d \tilde{\Theta}_e^{S_{MF}} \partial_{\alpha \beta} \tilde{\Theta}_{a,j}(z, \tau) \tilde{\Phi}_{\beta,m}(z, 0)}{\int d\tilde{\Phi}_d \tilde{\Theta}_e^{S_{MF}}}, \]

with \( \alpha, \beta = +, I, II, III \) and the chain index \( j, m = 1, \ldots, S/(2\pi l^2) \).

With use of a Born approximation [28], we can take the quenched average of \( Q_{zz}^c(i\omega_n) \),

\[ Q_{zz}^c(-i\omega_n) = \frac{2\tilde{e}^2}{\pi^2 V} \sum_m \int dz'' \sum_k e^{-ikz''-ikym-ikz} \]

\[ \left[ 1_{4 \times 4} - \left[ M_{c,K}^{-1} \right]_{\Phi \Phi} \left[ P(i\omega_n) \right]^{-1} \left[ M_{c,K}^{-1} \right]_{\Phi \Theta} \right], \]

where \( \left[ M_{c,K}^{-1} \right]_{\Phi \Phi} \) and \( \left[ M_{c,K}^{-1} \right]_{\Phi \Theta} \) are 4 by 4 matrices. \( \left[ M_{c,K}^{-1} \right]_{\Phi \Phi} \) and \( \left[ M_{c,K}^{-1} \right]_{\Phi \Theta} \) are 4 by 4 blocks of
\[ [M_{c,k}]^{-1}_{\Phi} \Phi \equiv T (A - BD^{-1}C)^{-1} T, \quad (D17) \]
\[ [M_{c,k}]^{-1}_{\Phi+} \Phi \equiv T (A - BD^{-1}C)^{-1} BD^{-1} T. \quad (D18) \]

4 by 4 matrices \( A, B, C, D \) and \( T \) in the right hand sides are given by Eqs. \([D6, D7, D8, D9]\). \( P(i\omega_n) \) is a 4 by 4 diagonal matrix that represents an effect of the disorder,

\[ P(i\omega_n) = \begin{bmatrix} g_m(i\omega_n) & 0 & 0 & 0 \\ 0 & g_m(i\omega_n) & 0 & 0 \\ 0 & 0 & g_m(i\omega_n) & 0 \\ 0 & 0 & 0 & g_m(i\omega_n) \end{bmatrix}. \quad (D19) \]

\( m(i\omega_n) \) is a sum of the \((\Phi_+, \Phi_+)-\)component of the inverse of the 8 by 8 matrix \([M_{c,k}] \) over \( k \equiv (k_z, k) \):

\[ m(i\omega_n) = \frac{2}{L^D N} \sum_k [M_{c,k}]^{-1}_{\Phi+}. \quad (D20) \]

Note that \( m(i\omega_n) \) is an even function of \( \omega_n \) (see below).

One may rewrite Eq. \([D16]\) into

\[
\frac{Q_{zz}^c(-i\omega_n)}{\pi^2} = \frac{2\pi^2}{\pi^2 V} \sum_m \int d\pi' z'' \sum_k e^{-ik_z z''} e^{-ik_m (-\omega_n)} T_{[k_z^2] [\pi^2 \{(DA - DTPT) + \omega_n^2 1_{4 \times 4}\}]_{k=0}}^{-1} T = -\frac{2\pi^2}{\pi^2 V} T_{[k_z^2, [\pi^2 \{(DA - DTPT) + \omega_n^2 1_{4 \times 4}\}]_{k=0}]}^{-1} T.
\]

From the first to the second line, we took the sum over the chain index \( l \) and the integral over \( z'' \);\n
\[ \frac{1}{V} \sum_m \int d\pi' e^{-ik_z z''} e^{-ik_m (-\omega_n)} = \frac{1}{2\pi^2 l^2} \delta_{k_0,0}. \quad (D22) \]

Substituting Eq. \([D21]\) into Eq. \([D13]\), we obtain the imaginary-time optical conductivity as

\[
\frac{\sigma_{zz}(i\omega_n)}{\pi^2 l^2} = \frac{2\pi^2}{\pi^2 l^2} T \pi^2 \left[[\pi^2 \{(DA - DTPT) + \omega_n^2 1_{4 \times 4}\}]_{k=0} \right]^{-1} U^{-1} \tilde{c}_+. \quad (D23) \]

The \( k = 0 \) limit in the integrand is well-defined.

To see this, use Taylor expansions of \( A \) and \( D \) in small \( k \);

\[
A_K = 2M(0)A_0 + k_z^2 A_1 + O(k), \quad D_K = 2M(0)D_0 + k_z^2 D_1 + O(k),
\]

with

\[
A_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \quad (D24)
\]

and

\[
A_1 = \frac{1}{\pi} \begin{bmatrix} u_1 & u_1 & u_1 & u_1 \\ u_1 & u_1 & u_1 & u_1 \\ u_1 & u_1 & u_1 & u_1 \\ u_1 & u_1 & u_1 & u_1 \end{bmatrix}, \quad D_1 = \frac{1}{\pi} \begin{bmatrix} u_2 & u_2 & u_2 & u_2 \\ u_2 & u_2 & u_2 & u_2 \\ u_2 & u_2 & u_2 & u_2 \\ u_2 & u_2 & u_2 & u_2 \end{bmatrix}.
\]

Since \( D_0 T P = 0 \) and \( D_0 A_0 = 0 \), the integrand in the \( k = 0 \) limit takes a finite value;

\[
\lim_{k_z \to 0} \lim_{k \to 0} \frac{1}{k_z^2} \left(D_K A_K - D K T P T\right) = 2M(0) \left(D_0 A_1 + D_1 A_0 \right) - D_1 T P T = 2M(0)D_0 A_1 + \frac{1}{\pi} \left(2M(0) - \frac{g_m(i\omega_n)}{4}\right) U^{-1} A_0.
\]

From the second to the last line, we used \( T P T = \frac{g_m(i\omega_n)}{4} A_0 \) and \( \pi D_1 = U^{-1} \).

The imaginary-time optical conductivity is further calculated from Eq. \([D23]\) as,

\[
\frac{\sigma_{zz}(i\omega_n)}{\pi^2 l^2} = \frac{2\pi^2}{\pi^2 l^2} \frac{e^2 \omega_n}{2} \tilde{e}_+ T \pi^2 \left[[\pi^2 \{(DA - DTPT) + \omega_n^2 1_{4 \times 4}\}]_{k=0} \right]^{-1} U^{-1} \tilde{e}_+. \quad (D25)
\]

with \( uK \equiv \sum_{a=1,2,3,4} u_a K_a \). From the first to the second line, we used \( A_0 D_0 = 0, \tilde{e}_+ T D_0 = 0 \) and \( A_0 = \tilde{e}_+ T \). From the second to the last line, we used \( e^2 \tilde{U}^{-1} \tilde{e}_+ = uK \). In the clean limit (\( g_{y} = 0 \)), this gives \( \sigma_{zz}(\omega) = (e^2 uK)/(2\pi^2 l^2) \delta(\omega - \omega_y) \) with \( \omega_y \equiv 2\pi uK \sum_j M_j \) after the analytic continuation.

The effect of the disorder average is included in \( m(i\omega_n) \). To see this effect in \( \sigma_{zz}(\omega) \), let us take \( u_1 = u_4, K_1 = K_4, u_2 = u_3 \) and \( K_2 = K_3 \) for simplicity. With use of \( M(k) = 0 \) for \( k > 1/l [28] \), we obtain the following expression for \( m(i\omega_n) \),

\[
m(i\omega_n) = \frac{\pi}{2} \left(\frac{K_1}{\sqrt{\omega_n^2 + \omega_1^2}} + \frac{K_2}{\sqrt{\omega_n^2 + \omega_2^2}}\right), \quad (D25)
\]

with \( \omega_1^2 \equiv 4\pi M(0)u_1 K_1 < 4\pi M(0)u_2 K_2 \equiv \omega_2^2 \). After the analytic continuation, we finally obtain the optical
conductivity as follows,

\[ \sigma_{zz}(\omega) = \begin{cases} \frac{e^2 u K}{\pi^2} \frac{\omega_1}{g(\omega)} \delta(\omega - \omega_s) & 0 < \omega < \omega_1, \\ \frac{e^2 u K}{\pi^2} \frac{\omega_1}{\omega_1^2 + \omega_2^2} & \omega_1 < \omega < \omega_2, \\ \frac{e^2 u K}{\pi^2} \frac{\omega_2}{\omega_2^2 + \omega_2^2} & \omega_1 < \omega < \omega_1, \end{cases} \]

(D26)

where

\[ g(\omega) = -\omega^2 - \omega_y^2 - \frac{g_y \pi^2 u K}{4 \left( \frac{K_1}{\omega_1^2 - \omega^2} + \frac{K_2}{\omega_1^2 - \omega^2} \right)}, \]

and

\[ a_1(\omega) = -\omega^2 - \omega_y^2 - \frac{g_y \pi^2 u K}{4 \left( \frac{K_1}{\omega_1^2 - \omega^2} + \frac{K_2}{\omega_1^2 - \omega^2} \right)}, \]

\[ b_1(\omega) = \frac{g_y \pi^2 u K}{4 \left( \frac{K_1}{\omega_1^2 - \omega^2} + \frac{K_2}{\omega_1^2 - \omega^2} \right)}, \]

and

\[ a_2(\omega) = -\omega^2 - \omega_y^2, \]

\[ b_2(\omega) = \frac{g_y \pi^2 u K}{4 \left( \frac{K_1}{\omega_1^2 - \omega^2} + \frac{K_2}{\omega_1^2 - \omega^2} \right)} \]

Note that \( \omega = \omega_n(\omega_y) \) in Eq. (D26) is one and only one solution of \( g(\omega) = 0 \) within \( 0 < \omega < \omega_1 \). The renormalized gap \( \omega_n \) becomes progressively smaller, when the disorder strength increases. There exists a critical value of the disorder,

\[ g_y,c = \frac{1}{\pi^2 u K} \frac{8\omega_n^2 \omega_1 \omega_2}{K_1 \omega_2 + K_2 \omega_1}. \]  

(D27)

When \( g_y \) approaches the critical value, the renormalized gap \( \omega_n \) reduces to zero continuously. At \( g_y = g_y,c \), the system undergoes a quantum phase transition from the SNEI-I phase \( (g_y < g_y,c) \) to a disorder-driven phase \( (g_y > g_y,c) \). To obtain Fig. 15, we use the same parameter sets as in the appendix C. We set \( u_1 = u_4 \) and \( u_2 = u_3 \) by Eq. (20). We set \( g_y \) to be smaller than \( g_y,c \).

**Appendix E: magnetism and spin nematicity in SNEI phases**

SNEI phases introduced in the main text are characterized by particle-hole pairings between \( n = 0 \) LL with \( \uparrow (\downarrow) \) spin and \( n = -1 \) LL with \( \downarrow (\uparrow) \) spins. The phases break the \( U(1) \) spin rotational symmetry around the field direction. Nonetheless, neither \( A \)-carbon site \( \pi \)-orbital electron spin nor \( B \)-carbon site electron spins exhibit magnetic order in the SNEI phases:

\[ \langle S_{A,+}(r) \rangle = \langle \psi_{1+}^\dagger(r, A) \psi_{1+}(r, A) \rangle = 0, \]

\[ \langle S_{B,+}(r) \rangle = \langle \psi_{1+}^\dagger(r, B) \psi_{1+}(r, B) \rangle = 0, \]

\[ \frac{1}{L_x} \sum_j \left( Y_{1,j}(y) Y_{0,j}(y) \right) \times \]

\[ \times \sum_{r=\pm} \left( \eta_B^r \eta_B^r e^{-i\pi(\phi_{F,1}+\phi_{F,3})z} \psi_{1,r,j}^\dagger \psi_{4,-r,j}^\dagger \right) + \eta_B^r \eta_B^r e^{-i\pi(\phi_{F,2}+\phi_{F,3})z} \psi_{3,r,j}^\dagger \psi_{2,-r,j}^\dagger \right) \]

\[ = 0, \]  

(E1)

because

\[ \lim_{L_x \to \infty} \frac{1}{L_x} \sum_j Y_{1,j}(y) Y_{0,j}(y) \]

\[ = \frac{1}{2\pi^2} \int dy Y_{1,j}(y) Y_{0,j}(y) = 0. \]

Magnetism of the SNEI-I phase is most explicitly manifested by a long-range order of a symmetric part of a 2nd rank spin tensor composed of spin-\( \frac{1}{2} \) moment of \( A \)-carbon-site \( \pi \)-orbital electron and that of \( B \)-carbon-site. Such 2nd rank spin tensor has two components,

\[ Q_{AB}^{++}(r) \equiv \langle S_{A,+}(r) S_{B,-}(r) \rangle, \]

\[ Q_{AB}^{++}(r) \equiv \langle S_{A,+}(r) S_{B,+}(r) \rangle. \]

In the SNEI-I phase, \( Q_{AB}^{++}(r) \) vanishes identically, while \( Q_{AB}^{++}(r) \) exhibits both a ferro-type and a density-wave-type order:

\[ Q_{AB}^{++}(r) = \langle \psi_{1+}^\dagger(r, A) \psi_{1+}(r, A) \psi_{1+}^\dagger(r, B) \psi_{1+}(r, B) \rangle \]

\[ = \frac{1}{L_x} \left( \sum_j Y_{0,j}(y) \right) \frac{1}{L_x} \left( \sum_m Y_{0,m}(y) \right) \]

\[ \times \left\{ \gamma_A^r \gamma_A^r \eta_B^r \eta_B^r e^{-2i\theta_{-}} + \gamma_A^r \gamma_A^r \eta_B^r \eta_B^r e^{-2i\theta_{-}} + \gamma_A^r \gamma_A^r \eta_B^r \eta_B^r e^{-i\Delta K e^{-i2\Phi_{-}} 2i\theta_{-}} + \gamma_A^r \gamma_A^r \eta_B^r \eta_B^r e^{-i\Delta K e^{-i2\Phi_{-}} 2i\theta_{-}} \right\}, \]

with \( \Delta K \equiv k_{F,2} + k_{F,3} - k_{F,1} - k_{F,4} \). Here we used Eqs. 45 46 17 and

\[ \langle \psi_{1+}^\dagger(r, \downarrow) \psi_{1+}(r, \downarrow) \rangle = \delta_{jm} l \sigma_{31,m} e^{i(\phi_{1} + \phi_{4}) + i(\theta_{1} - \theta_{4})}, \]

\[ \langle \psi_{1+}^\dagger(r, \uparrow) \psi_{1+}(r, \uparrow) \rangle = \delta_{jm} l \sigma_{31,m} e^{-i(\phi_{1} + \phi_{4}) - i(\theta_{1} - \theta_{4})}, \]

\[ \langle \psi_{2+}^\dagger(r, \downarrow) \psi_{3+}(r, \downarrow) \rangle = \delta_{jm} l \sigma_{32,m} e^{-i(\phi_{2} + \phi_{3}) - i(\theta_{1} - \theta_{2})}, \]

\[ \langle \psi_{2+}^\dagger(r, \uparrow) \psi_{3+}(r, \uparrow) \rangle = \delta_{jm} l \sigma_{32,m} e^{i(\phi_{2} + \phi_{3}) - i(\theta_{1} - \theta_{2})}, \]

The spatial inversion symmetry generally allows

\[ \gamma_A^r \gamma_A^r \eta_B^r \eta_B^r = u \]

(E2)
This gives

$$Q_{ab}^{+\pm}(r) = \frac{e^{-2i\Theta_\pm}}{(2\pi l^2)^2} \left( u + u \cos \left( \Delta K z - 2\Phi_\pm \right) \right).$$  \hspace{1cm} (E3)

Note also that the SNEI phases could be accompanied by a long-range ordering of small magnetic moments within the xy plane. Nonetheless, the moment does exist only in those spatial regions in the unit cell where two π-orbitals of A-carbon site and B-carbon site overlap. This statement is suggested by Eq. (E1) and finite expectation values of the following two quantities in the SNEI phases;

$$\langle \psi^\dagger \langle r, A \rangle \psi \langle r, B \rangle \rangle = \frac{\sqrt{2iw}}{\pi l^2} e^{-i\Theta_\pm} \cos \left( (k_{F,1} + k_{F,4})z + \Phi_\pm \right),$$

$$\langle \psi^\dagger \langle r, A \rangle \psi \langle r, B \rangle \rangle = \frac{\sqrt{2iw}}{\pi l^2} e^{i\Theta_\pm} \cos \left( (k_{F,2} + k_{F,3})z - \Phi_\pm \right),$$

with

$$v \equiv \gamma_{A,\uparrow} \eta_{B,\downarrow} \langle \sigma_{r,1} \rangle \neq 0,$$

$$w \equiv \gamma_{A,\downarrow} \eta_{B,\uparrow} \langle \sigma_{r,4} \rangle \neq 0.$$

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