On singular Artin monoids

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Abstract

In this paper we study some combinatorial aspects of the singular Artin monoids. Firstly, we show that a singular Artin monoid $SA$ can be presented as a semidirect product of a graph monoid with its associated Artin group $A$. Such a decomposition implies that a singular Artin monoid embeds in a group. Secondly, we give a solution to the word problem for the FC type singular Artin monoids. Afterwards, we show that FC type singular Artin monoids have the FRZ property. Briefly speaking, this property says that the centralizer in $SA$ of any non-zero power of a standard singular generator $\tau_s$ coincides with the centralizer of any non-zero power of the corresponding non-singular generator $\sigma_s$. Finally, we prove Birman’s conjecture, namely, that the desingularization map $\eta : SA \to \mathbb{Z}[A]$ is injective, for right-angled singular Artin monoids.

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1 Introduction

Let $S$ be a finite set. Recall that a Coxeter matrix over $S$ is a matrix $M = (m_{st})_{s,t \in S}$ indexed by the elements of $S$ and such that $m_{ss} = 1$ for all $s \in S$, and $m_{st} = m_{ts} \in \{2, 3, 4, \ldots, +\infty\}$ for all $s, t \in S$, $s \neq t$. A Coxeter matrix $M = (m_{st})$ is usually represented by its Coxeter graph, $\Gamma$, which is defined as follows. $S$ is the set of vertices of $\Gamma$, two vertices $s, t$ are joined by an edge if $m_{st} \geq 3$, and this edge is labelled by $m_{st}$ if $m_{st} \geq 4$. If $a, b$ are two letters and $m \in \mathbb{Z}_{\geq 2}$, then we denote by $w(m : a, b)$ the word $\ldots bab$ of length $m$. We take an abstract set $S = \{\sigma_s : s \in S\}$ in one-to-one correspondence with $S$, and we define the Artin group associated to $\Gamma$ to be the group $A = A_{\Gamma}$ presented by

$$A = A_{\Gamma} = \langle S \mid w(m_{st} : \sigma_s, \sigma_t) = w(m_{st} : \sigma_t, \sigma_s) \text{ for } s, t \in S, s \neq t \text{ and } m_{st} < +\infty \rangle.$$

The Coxeter group associated to $\Gamma$ is the quotient $W = W_{\Gamma}$ of $A$ by the relations $\sigma_s^2 = 1$, $s \in S$.

Take $X \subset S$ and put $S_X = \{\sigma_s : s \in X\}$. We denote by $A_X$ the subgroup of $A$ generated by $S_X$, and by $W_X$ the subgroup of $W$ generated by $S_X$. Let $\Gamma_X$ be the full
subgraph of $\Gamma$ generated by $X$. Then $A_X$ is the Artin group associated to $\Gamma_X$ (see [32] and [34]), and $W_X$ is the Coxeter group associated to $\Gamma_X$ (see [10]). The subgroup $A_X$ is called standard parabolic subgroup of $A$, and $W_X$ is called standard parabolic subgroup of $W$.

We say that $\Gamma$ (or $A$) is of spherical type if $W$ is finite, that $\Gamma$ (or $A$) is right-angled if $m_{st} \in \{2, +\infty\}$ for all $s, t \in S, s \neq t$, and that $\Gamma$ (or $A$) is of type FC if, for all $X \subset S$, either $W_X$ is finite, or there exist $s, t \in X, s \neq t$, such that $m_{st} = +\infty$. Note that the spherical type Artin groups as well as the right-angled Artin groups are both FC type Artin groups. The number $n = |S|$ is called the rank of $A$.

The first (non-abelian) example of Artin group which has appeared in the literature is certainly the braid group $B_n$ introduced by Artin [3] in 1925. One of the most important works in the subject is a paper by Garside [23] where the word problem and the conjugacy problem for $B_n$ are solved. Garside’s ideas have been extended to all spherical type Artin groups by Brieskorn, Saito [12], and Deligne [17] in 1972. These two papers, [12] and [17], are the foundation of the theory of Artin groups, and, more specifically, of the spherical type Artin groups. Right-angled Artin groups are also known as graph groups or as free partially commutative groups. They have been widely studied, and their applications extend to various domains like parallel computation, random walks, and cohomology of groups. We mention, for example, the paper [8] where a group which is finitely presented is constructed as a subgroup of some right-angled Artin group. Artin groups of type FC have been introduced by Charney and Davis [15] in 1995 in their study of the $K(\pi, 1)$-problem for complements of infinite hyperplane arrangements associated to reflection groups.

In the same way as the braid group $B_n$ has been extended to the singular braid monoid $SB_n$ (see [3] and [9]), the Artin groups can be extended to the singular Artin monoids as follows. Take a new abstract set $T = \{\tau_s; s \in S\}$ in one-to-one correspondence with $S$, and define the singular Artin monoid associated to $\Gamma$ to be the monoid $SA = S\Gamma$, presented as a monoid by the generating set $S \cup S^{-1} \cup T$, where $S^{-1} = \{\sigma_s^{-1}; s \in S\}$, and by the relations

\[
\begin{align*}
\sigma_s \sigma_s^{-1} &= \sigma_s^{-1} \sigma_s = 1, & \text{for } s \in S, \\
\sigma_s \tau_s &= \tau_s \sigma_s, & \text{for } s \in S, \\
w(m_{st} : \sigma_s, \sigma_t) &= w(m_{st} : \sigma_t, \sigma_s), & \text{for } s, t \in S, s \neq t, \text{ and } m_{st} < +\infty, \\
w(m_{st} - 1 : \sigma_s, \sigma_t) \tau_s &= \tau_{st} w(m_{st} - 1 : \sigma_s, \sigma_t), & \text{for } s, t \in S, s \neq t, \text{ and } m_{st} < +\infty, \\
\tau_s \tau_t &= \tau_t \tau_s, & \text{for } s, t \in S, s \neq t, \text{ and } m_{st} = 2,
\end{align*}
\]

where $s \land t = s$ if $m_{st}$ is even, and $s \land t = t$ if $m_{st}$ is odd. Observe that we have an epimorphism $\theta : SA \to A$ which sends $\sigma_s^{+1}$ to $\sigma_s^{+1}$ and $\tau_s$ to $\sigma_s$ for all $s \in S$, and that this epimorphism has a section $\iota : A \to SA$ which sends $\sigma_s^{+1}$ to $\sigma_s^{±1}$ for all $s \in S$. In particular, $A$ embeds in $SA$.

The combinatorial study of the singular Artin monoids (of spherical type) has been initiated by Corran [16] with techniques inspired from [12].

The purpose of the present paper is to study different combinatorial aspects of the monoid $SA$. 

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Firstly, in Section 2, we prove that $SA$ can be decomposed as a semidirect product of a so-called graph monoid with the Artin group $A$. A consequence of this decomposition shall be that $SA$ embeds in a group. Note that this last result has been previously proved by Basset [7] and Keyman [30] with completely different proofs.

Sections 3 and 4 concern only singular Artin monoids of type FC. We solve the word problem for $SA$ in Section 3. In [21], Fenn, Rolfsen, and Zhu proved that the centralizer in the singular braid monoid of a standard singular generator $\tau_j$ is equal to the centralizer of any non-zero power of $\tau_j$, and that this centralizer coincides with the centralizer of any non-zero power of $\sigma_j$. This property, which we like to call FRZ property, is of importance in the study of singular braids, and, in particular, in the proof of Birman’s conjecture for braid groups (see [36]). In Section 4, we extend the FRZ property to all singular Artin monoids of type FC.

Define the desingularization map as the multiplicative homomorphism $\eta : SA \to \mathbb{Z}[A]$ which sends $\sigma_s^{\pm 1}$ to $\sigma_s^{\pm 1}$ and $\tau_s$ to $\sigma_s - \sigma_s^{-1}$ for all $s \in S$. One of the main questions in the subject, known as Birman’s conjecture, is to determine whether the desingularization map is injective. This is known to be true for braid groups [36] and for rank 2 Artin groups [20]. We prove Birman’s conjecture for right-angled Artin groups in Section 5.

The last section is dedicated to some questions for which we do not have any significant result but that deserve to be mentioned.

\section{Semidirect product structure}

Our purpose in this section is to determine a decomposition of $SA$ as a semidirect product of a so-called graph monoid with the Artin group $A$. A first consequence of this decomposition is that a singular Artin monoid embeds in a group. This decomposition shall be also used in Sections 3 and 4 to solve the word problem and to prove the “FRZ property” for singular Artin monoids of type FC.

Let $G$ be a (standard) graph, let $V$ be its set of vertices, and let $E = E(G)$ be its set of edges. Define the graph monoid of $G$ to be the monoid $\mathcal{M}(G)$ presented as a monoid by

$$\mathcal{M}(G) = \langle V \mid uv = vu \text{ if } \{u, v\} \in E \rangle^+.$$ 

The first (standard) graph that we shall consider is the graph $\hat{\Omega}$ defined by the following data.

- $\hat{\Upsilon} = \{\alpha \tau_s \alpha^{-1}; \alpha \in A \text{ and } s \in S\}$ is the set of vertices of $\hat{\Omega}$;
- $\{\hat{u}, \hat{v}\}$ is an edge of $\hat{\Omega}$ if $\hat{u} \hat{v} = \hat{v} \hat{u}$ in $SA$.

**Proposition 2.1.** We have $SA = \mathcal{M}(\hat{\Omega}) \rtimes A$.

**Proof.** We have a homomorphism $f : \mathcal{M}(\hat{\Omega}) \rtimes A \to SA$ defined by $f(\alpha) = \alpha \in SA$ for all $\alpha \in A$, and $f(\hat{u}) = \hat{u} \in SA$ for all $\hat{u} \in \hat{\Upsilon}$. Conversely, one can easily verify using the presentation of $SA$ that there is a homomorphism $g : SA \to \mathcal{M}(\hat{\Omega}) \rtimes A$ which sends $\sigma_s^{\pm 1}$ to $\sigma_s^{\pm 1} \in A$, and $\tau_s$ to $\tau_s \in \hat{\Upsilon}$, for all $s \in S$. Obviously, $f \circ g = \text{Id}$ and $g \circ f = \text{Id}$. □
Corollary 2.2. SA embeds in a group.

Proof. Let $\mathcal{G}(\hat{\Omega})$ be the group presented by $\mathcal{G}(\hat{\Omega}) = \langle \hat{\Upsilon} \mid \hat{u}\hat{v} = \hat{v}\hat{u} \text{ if } \{\hat{u}, \hat{v}\} \in E(\hat{\Omega}) \rangle$. Then $\mathcal{M}(\hat{\Omega})$ embeds in $\mathcal{G}(\hat{\Omega})$ (see [19] and [18]), thus $SA = \mathcal{M}(\hat{\Omega}) \rtimes A$ embeds in $\mathcal{G}(\hat{\Omega}) \rtimes A$. □

3 The word problem

Note that a solution to the word problem for $SA$ will also give a solution to the word problem for the Artin group $A$. So, a reasonable approach would be to study the word problem for those singular Artin monoids whose associated Artin groups have known solutions to the word problem. In the case of Artin groups of type FC, a solution has been found by Alcobelli [2]. Another observation is that a given element of a graph monoid $M(G)$ has finitely many representatives and these representatives can be easily listed. Other solutions to the word problem for $M(G)$ can be found in [13], [41], and [19].

Now, assume that $A$ is of type FC and consider the decomposition $SA = M(\hat{\Omega}) \rtimes A$ of the previous section. By the above observations, in order to solve the word problem for $SA$, it suffices to find an algorithm which decides whether two elements $\alpha\tau_s\alpha^{-1}$ and $\beta\tau_t\beta^{-1}$ of $\hat{\Upsilon}$ are equal, and, if not, whether they commute or not. Such an algorithm can be easily derived from Proposition 3.1 below together with Alcobelli’s solution to the word problem for $A$.

Define the graph $\Omega$ as follows.
- $\hat{\Upsilon} = \{\alpha\sigma_s\alpha^{-1}; \alpha \in A \text{ and } s \in S\}$ is the set of vertices of $\Omega$;
- $\{u, v\}$ is an edge of $\Omega$ if $uv = vu$ in $A$.

Proposition 3.1. Assume $\Gamma$ to be of type FC. Then there exists an isomorphism $\varphi : \hat{\Omega} \rightarrow \Omega$ which sends $\alpha\tau_s\alpha^{-1}$ to $\alpha\sigma_s\alpha^{-1}$ for all $\alpha \in A$ and all $s \in S$.

The remainder of the section is dedicated to the proof of Proposition 3.1.

Define the Artin monoid associated to $\Gamma$ to be the monoid $A^+ = A^+_\Gamma$ presented as a monoid by

$$A^+ = \langle S \mid w(m_{st} : \sigma_s, \sigma_t) = w(m_{st} : \sigma_t, \sigma_s) \text{ for } s, t \in S, s \neq t, \text{ and } m_{st} < +\infty \rangle^+.$$  

By [35], the natural homomorphism $A^+ \rightarrow A$ which sends $\sigma_s$ to $\sigma_s$ for all $s \in S$ is injective. We can define the length function $\lg : A^+ \rightarrow \mathbb{N}$ which associates to each element of $A^+$ the length of any of its representatives with respect to the generating set $S$. Since the defining relations of $A^+$ are homogeneous, this function is well-defined and is a homomorphism of monoids. For the same reason, we can define a partial order $\leq_R$ on $A^+$ by setting $a \leq_R b$ if there exists $c \in A^+$ such that $ca = b$. Now, the following proposition is a mixture of several well-known facts on spherical type Artin groups.
Proposition 3.2. Assume \( \Gamma \) to be of spherical type.

1. (Brieskorn-Saito \[12\], Deligne \[17\]). \((A^+, \leq_R)\) is a lattice. The lattice operations of \((A^+, \leq_R)\) are denoted by \(\wedge_R\) and \(\vee_R\).

2. (Brieskorn-Saito \[12\], Deligne \[17\]). Let \(s, t \in S, s \neq t\). Then \(\sigma_s \vee_R \sigma_t = w(m_{st} : \sigma_s, \sigma_t) = w(m_{st} : \sigma_t, \sigma_s)\).

3. (Brieskorn-Saito \[12\], Deligne \[17\]). Let \(\Delta = \vee_R \{\sigma_s ; s \in S\}\). Then there exists a permutation \(\mu : S \to S\) such that \(\mu^2 = \text{Id}\) and \(\Delta \sigma_s \Delta^{-1} = \sigma_{\mu(s)}\) for all \(s \in S\).

4. (Brieskorn-Saito \[12\], Deligne \[17\]). Each \(\alpha \in A\) can be written as \(\alpha = a\Delta^k\) with \(a \in A^+\) and \(k \in \mathbb{Z}\).

5. (Charney \[14\]). Each \(\alpha \in A\) can be uniquely written as \(\alpha = ab^{-1}\) with \(a, b \in A^+\) and \(a \wedge_R b = 1\). Such an expression \(\alpha = ab^{-1}\) is called the Charney form of \(\alpha\).

6. (Charney \[14\]). Let \(\alpha \in A\) and \(u, v \in A^+\) such that \(\alpha = uv^{-1}\). Then \(u = ac\) and \(v = bc\), where \(c = u \wedge_R v\) and \(ab^{-1}\) is the Charney form of \(\alpha\).

Let \(s, t \in S\) and \(\omega \in A\). We say that \(\omega\) is an elementary positive \((t, s)\)-ribbon if either

- \(s = t\), and \(\omega = \sigma_s\); or
- \(s = t\), and there exists \(r \in S\) such that \(m_{sr}\) is even and \(\omega = w(m_{sr} - 1 : \sigma_s, \sigma_r)\); or
- \(s \neq t\), \(m_{st}\) is odd, and \(\omega = w(m_{st} - 1 : \sigma_s, \sigma_t)\).

Define an elementary \((t, s)\)-ribbon to be either an elementary positive \((t, s)\)-ribbon, or the inverse of an elementary positive \((s, t)\)-ribbon. Note that, if \(\omega\) is an elementary \((t, s)\)-ribbon, then \(\omega \sigma_s = \sigma_t \omega\) and \(\omega \tau_s = \tau_t \omega\). We say that \(\omega\) is a \((t, s)\)-ribbon if there exist a sequence \(s_0 = s, s_1, \ldots, s_p = t\) in \(S\), and a sequence \(\omega_1, \ldots, \omega_p\) in \(A\), such that \(\omega_i\) is an elementary \((s_i, s_{i-1})\)-ribbon for all \(i = 1, \ldots, p\), and \(\omega = \omega_p \ldots \omega_2 \omega_1\). Clearly, if \(\omega\) is a \((t, s)\)-ribbon, then \(\omega \sigma_s = \sigma_t \omega\) and \(\omega \tau_s = \tau_t \omega\).

The key point in the proof of Proposition 3.1 is the following result which can be found in \[25\] (see also \[24\]).

**Proposition 3.3 (Godelle \[25\]).** Assume \(\Gamma\) to be of type FC. Let \(s \in S, X \subset S\), and \(\alpha \in A\). Then the followings are equivalent.

1. \(\alpha \sigma_s \alpha^{-1} \in A_X\).
2. There exists \(k \in \mathbb{Z} \setminus \{0\}\) such that \(\alpha \sigma_s^k \alpha^{-1} \in A_X\).
3. There exists \(t \in X, \omega \in A\), and \(\beta \in A_X\), such that \(\omega\) is a \((t, s)\)-ribbon and \(\alpha = \beta \omega\).

**Corollary 3.4 (Godelle \[25\]).** Assume \(\Gamma\) to be of type FC. Let \(s, t \in S\) and \(\alpha \in A\). Then the followings are equivalent.

1. \(\alpha \sigma_s \alpha^{-1} = \sigma_t\).
2. There exists \(k \in \mathbb{Z} \setminus \{0\}\) such that \(\alpha \sigma_s^k \alpha^{-1} = \sigma_t^k\).
3. \(\alpha\) is a \((t, s)\)-ribbon.

**Lemma 3.5.** Assume \(\Gamma\) to be of type FC. Then there exists a bijection \(\varphi : \hat{\Gamma} \to \Gamma\) which sends \(\alpha \tau_s \alpha^{-1}\) to \(\alpha \sigma_s \alpha^{-1}\) for all \(\alpha \in A\) and all \(s \in S\).
Proof. We take $\alpha, \beta \in A$ and $s, t \in S$, and we turn to prove the following equivalence.

$$\alpha t_s a^{-1} = \beta t_i \beta^{-1} \iff \alpha \sigma_s a^{-1} = \beta \sigma_t \beta^{-1}.$$ 

Assume first that $\alpha t_s a^{-1} = \beta t_i \beta^{-1}$. Recall the epimorphism $\theta : SA \to A$ which sends $\sigma_{s}^{\pm 1}$ to $s$ and $\tau_s$ to $\sigma_s$ for all $s \in S$. Then

$$\alpha \sigma_s a^{-1} = \theta(\alpha t_s a^{-1}) = \theta(\beta t_i \beta^{-1}) = \beta \sigma_t \beta^{-1}.$$ 

Now, assume that $\alpha \sigma_s a^{-1} = \beta \sigma_t \beta^{-1}$. By Corollary 3.4, $\beta^{-1} \alpha$ is a $(t, s)$-ribbon, thus $\beta^{-1} \alpha t_s = \tau_t \beta^{-1} \alpha$, and therefore $\alpha \tau_s a^{-1} = \beta t_i \beta^{-1}$.

$\blacksquare$

Lemma 3.6. Assume $\Gamma$ to be of spherical type. Then the bijection $\varphi : \hat{\Omega} \to \Omega$ extends to an isomorphism $\varphi : \hat{\Omega} \to \Omega$.

Proof. We take $\alpha, \beta \in A$ and $s, t \in S$, and we turn to prove the following equivalence.

$$(\alpha t_s a^{-1})(\beta t_i \beta^{-1}) = (\beta t_i \beta^{-1})(\alpha t_s a^{-1}) \iff (\alpha \sigma_s a^{-1})(\beta \sigma_t \beta^{-1}) = (\beta \sigma_t \beta^{-1})(\alpha \sigma_s a^{-1}).$$

First, assume that $(\alpha t_s a^{-1})(\beta t_i \beta^{-1}) = (\beta t_i \beta^{-1})(\alpha t_s a^{-1})$. Then

$$(\alpha \sigma_s a^{-1})(\beta \sigma_t \beta^{-1}) = \theta((\alpha t_s a^{-1})(\beta t_i \beta^{-1})) = \theta((\beta t_i \beta^{-1})(\alpha t_s a^{-1})) = (\beta \sigma_t \beta^{-1})(\alpha \sigma_s a^{-1}).$$

Now, we assume that $(\alpha \sigma_s a^{-1})(\beta \sigma_t \beta^{-1}) = (\beta \sigma_t \beta^{-1})(\alpha \sigma_s a^{-1})$, and we prove that $(\alpha \tau_s a^{-1})(\beta \tau_t \beta^{-1}) = (\beta \tau_t \beta^{-1})(\alpha \tau_s a^{-1})$. Our proof is divided into 3 steps.

Step 1: Assume $\beta = 1$, $\alpha = a \in A^+$, and $a \sigma_s \wedge_R a = 1$. So, $a \sigma_s a^{-1}$ is a Charney form. Since $\sigma_t a \sigma_s a^{-1} \sigma_t^{-1} = a \sigma_s a^{-1}$, by Proposition 3.2, there exists $c \in A^+$ such that $\sigma_t a = ac$ and $\sigma_t a \sigma_s = \sigma_s c$. The element $c$ is clearly of length 1, namely, $c = \sigma_r$, for some $r \in S$, and, by Corollary 3.4, the equality $\sigma_t a = a \sigma_r$ implies that $a$ is a $(t, r)$-ribbon. Moreover, we have $a \sigma_r \sigma_s = \sigma_t a \sigma_s = a \sigma_s \sigma_r$, thus $\sigma_r \sigma_s = \sigma_s \sigma_r$, therefore $m_{sr} = 2$. So,

$$t_i a \tau_s a^{-1} = a \tau_s a^{-1} = a \tau_s a^{-1} \tau_t.$$ 

Step 2: Assume $\beta = 1$ and $\alpha = a \in A^+$. We argue by induction on the length of $a$. The case $a \sigma_s \wedge_R a = 1$ is treated in Step 1, thus we can suppose that $a \sigma_s \wedge_R a \neq 1$. In particular, there exists some $r \in S$ such that $\sigma_r \leq_R a \sigma_s \wedge_R a$. Suppose $r = s$. Then $a$ can be written as $a = a_1 \sigma_s$ with $a_1 \in A^+$, and, moreover, $\sigma_t a_1 \sigma_s a_1^{-1} = a_1 \sigma_s a_1^{-1} \sigma_t$. By the inductive hypothesis, it follows that

$$\tau_t a \tau_s a^{-1} = \tau_t a \tau_s a^{-1} = a_1 \tau_s a_1^{-1} \tau_t = a \tau_s a^{-1} \tau_t.$$ 

Suppose $r \neq s$. We have $\sigma_s, \sigma_r \leq_R a \sigma_s$, thus, by Proposition 3.2, $\sigma_s \vee_R \sigma_r = w(m_{sr} : \sigma_r, \sigma_s) \leq_R a \sigma_s$, therefore $\omega = w(m_{sr} - 1 : \sigma_s, \sigma_r) \leq_R a$. Write $u = s$ if $m_{sr}$ is even, and $u = r$ if $m_{sr}$ is odd. Then $\omega$ is an (elementary) $(u, s)$-ribbon and $a$ can be written as
\( a = a_1 \omega \) with \( a_1 \in A^+ \). Moreover, \( \sigma_t a_1 \sigma_a a_1^{-1} = \sigma_t a \sigma_a a_1^{-1} a_1 = a_1 \sigma_a a_1^{-1} \). By the inductive hypothesis, it follows that
\[
\tau_t a \tau_s a_1^{-1} = \tau_t a_1 \tau_s a_1^{-1} = a_1 \tau_s a_1^{-1} \tau_t = a \tau_s a_1^{-1} \tau_t.
\]

**Step 3:** General case. By Proposition 3.2, there exist \( a \in A^+ \) and \( k \in \mathbb{Z} \) such that \( \beta^{-1} \alpha = a \Delta^k \). Recall the permutation \( \mu : S \to S \) such that \( \Delta \sigma_t \Delta^{-1} = \sigma_{\mu(t)} \) for all \( r \in S \). Let \( r = \mu^k(s) \). Then \( \Delta^k \) is a \((r, s)\)-ribbon (by Corollary 3.4), and, moreover, \( \sigma_t a \sigma_a a_1^{-1} = \sigma_t (\beta^{-1} \alpha \sigma_a a_1^{-1} \beta) = (\beta^{-1} \alpha \sigma_a a_1^{-1} \beta) \sigma_t = a \sigma_a a_1^{-1} \sigma_t \). By Step 2, it follows that
\[
\tau_t (\beta^{-1} \alpha \sigma_a a_1^{-1} \beta) = \tau_t a \sigma_a a_1^{-1} = a \tau_s a_1^{-1} \tau_t = (\beta^{-1} \alpha \tau_s a_1^{-1} \beta) \tau_s,
\]
hence \( (\beta \tau_t \beta^{-1}) (\alpha \sigma_a a_1^{-1}) = (\alpha \sigma_a a_1^{-1}) (\beta \tau_t \beta^{-1}) \). \( \square \)

In order to extend this result to all Artin groups of type FC (namely, in order to prove Proposition 3.1), we need one more preliminary result (on amalgamated products).

**Proposition 3.7 (Serre [39]).** Let \( G = G_1 \ast_H G_2 \) be the amalgamated product of two groups \( G_1 \) and \( G_2 \) over \( H \). Let \( C_1 \) and \( C_2 \) be transversals of \( G_1/H \) and \( G_2/H \), respectively, which contain 1. For all \( g \in G \) there exists a unique sequence \((g_1, \ldots, g_l, h)\) such that

- \( g = g_1 g_2 \cdots g_l h; \)
- \( h \in H, \) and either \( g_i \in C_1 \setminus \{1\} \) or \( g_i \in C_2 \setminus \{1\} \) for all \( i = 1, \ldots, l; \)
- \( g_i \in C_1 \Rightarrow g_{i+1} \in C_2, \) and \( g_i \in C_2 \Rightarrow g_{i+1} \in C_1, \) for all \( i = 1, \ldots, l - 1. \) \( \blacksquare \)

The above sequence \((g_1, \ldots, g_l, h)\) is called the **amalgam normal form** of \( g \) (relative to the amalgamated product \( G_1 \ast_H G_2 \)). The number \( l \) is called the **amalgam norm** of \( g \) (relative to the amalgamated product \( G_1 \ast_H G_2 \)) and is denoted by \( l = |g|_s \).

**Proof of Proposition 3.1.** We take \( \alpha, \beta \in A \) and \( s, t \in S \), and we turn to prove the following equivalence.
\[
(\alpha \tau_s a_1^{-1} \beta \tau_t \beta^{-1}) = (\beta \tau_t \beta^{-1}) (\alpha \tau_s a_1^{-1}) \iff (\alpha \sigma_a a_1^{-1} \beta \sigma_t \beta^{-1}) = (\beta \sigma_t \beta^{-1}) (\alpha \sigma_a a_1^{-1}).
\]

The implication \( \Rightarrow \) can be proved exactly in the same manner as the implication \( \Rightarrow \) in the proof of Lemma 3.6. So, we assume that \( (\alpha \sigma_a a_1^{-1} \beta \sigma_t \beta^{-1}) = (\beta \sigma_t \beta^{-1}) (\alpha \sigma_a a_1^{-1}) \), and we prove that \( (\alpha \sigma_a a_1^{-1} \beta \sigma_t \beta^{-1}) = (\beta \sigma_t \beta^{-1}) (\alpha \sigma_a a_1^{-1}) \). We argue by induction on the rank of \( A \) (i.e. on the number of vertices of \( \Gamma \)). Up to changing \( \alpha \) by \( \beta^{-1} \alpha \), we can also assume that \( \beta = 1 \).

If \( m_{x, y} < +\infty \) for all \( x, y \in S, x \neq y \), then \( A \) is of spherical type and, therefore, the equality \( (\alpha \tau_s a_1^{-1} \beta \tau_t \beta^{-1}) = (\beta \tau_t \beta^{-1}) (\alpha \tau_s a_1^{-1}) \) follows from Lemma 3.6. So, we can assume that there exist \( x, y \in S, x \neq y \), such that \( m_{x, y} = +\infty \). Let \( X = S \setminus \{x\} \) and \( Y = S \setminus \{y\} \).

Then \( A = A_X \ast_{A_X \cap Y} A_Y \). We choose transversals \( C_X \) and \( C_Y \) of \( A_X/A_X \cap Y \) and \( A_Y/A_X \cap Y \), respectively, which contain 1, we consider the amalgam normal form \((\alpha_1, \ldots, \alpha_l, \gamma)\) of \( \alpha \), and we argue by induction on \( l = |\alpha|_s \).
Assume \( l = 0 \). So, \( \alpha = \gamma \in A_{X \cap Y} \). The fact that the amalgam normal forms of \( \sigma_t(\alpha \sigma_s \alpha^{-1}) \) and \( (\alpha \sigma_s \alpha^{-1})\sigma_t \) are equal implies that either \( \sigma_s, \sigma_t \in A_X \) (namely \( s, t \in X \)), or \( \sigma_s, \sigma_t \in A_Y \) (namely, \( s, t \in Y \)). Say \( s, t \in X \). We have \( s, t \in X \) and \( \alpha \in A_X \), thus, by the inductive hypothesis, \( \tau_t(\alpha \tau_s \alpha^{-1}) = (\alpha \tau_s \alpha^{-1})\tau_t \).

Now, we assume that \( l = |\alpha|_s \geq 1 \). Without loss of generality, we can assume that \( \alpha_t \in C_X \). Then \( \alpha_t \in C_Z \), where \( Z = Y \) if \( l \) is even, and \( Z = X \) if \( l \) is odd. We consider 4 different cases.

**Case 1:** \( \alpha_t \gamma \sigma_s \gamma^{-1} \alpha_t^{-1} \in A_{X \cap Y} \). By Proposition 3.3, \( \alpha_t \gamma \) can be written as \( \alpha_t \gamma = \gamma_t \omega \) where \( \omega \) is a \((r, s)\)-ribbon for some \( r \in X \cap Y \), and \( \gamma_t \in A_{X \cap Y} \). Let \( \alpha' = \alpha_1 \alpha_2 \ldots \alpha_{l-1} \gamma_1 = \alpha \omega^{-1} \). We have \( (\alpha' \sigma_t \alpha'^{-1})\sigma_t = (\alpha \sigma_s \alpha^{-1})\sigma_t = \sigma_t(\alpha \sigma_s \alpha^{-1}) \), thus, by the inductive hypothesis (on the amalgam norm of \( \alpha \)), we have \( (\alpha \tau_s \alpha^{-1})\tau_t = (\alpha' \tau_t \alpha'^{-1})\tau_t = \tau_t(\alpha' \tau_t \alpha'^{-1}) = \tau_t(\alpha \tau_s \alpha^{-1}) \).

**Case 2:** \( l = 1, s \in X \), and \( \alpha \sigma_s \alpha^{-1} \in A_X \setminus A_{X \cap Y} \). The fact that the amalgam normal forms of \( \sigma_t(\alpha \sigma_s \alpha^{-1}) \) and \( (\alpha \sigma_s \alpha^{-1})\sigma_t \) are equal implies that \( \sigma_t \in A_X \), namely, that \( t \in X \). We have \( s, t \in X \) and \( \alpha \in A_X \), thus, by the inductive hypothesis (on the rank of \( A \)), we have \( \tau_t(\alpha \tau_s \alpha^{-1}) = (\alpha \tau_s \alpha^{-1})\tau_t \).

**Case 3:** \( l \geq 2, s \in X \), and \( \alpha_t \gamma \sigma_s \gamma^{-1} \alpha_t^{-1} \in A_X \setminus A_{X \cap Y} \). Then the amalgam normal form of \( \alpha \sigma_s \alpha^{-1} \) has the form \( (\alpha_1, \ldots, \alpha_{l-1}, \beta'_1, \beta'_2, \ldots, \beta'_l, \gamma_1) \), and \( \alpha_1, \beta'_l \in C_Z \). The fact that the amalgam normal forms of \( \sigma_t(\alpha \sigma_s \alpha^{-1}) \) and \( (\alpha \sigma_s \alpha^{-1})\sigma_t \) are equal implies that \( \sigma_t \in A_Z \), namely, that \( t \in Z \). Then the amalgam normal form of \( (\alpha \sigma_s \alpha^{-1})\sigma_t \) has either the form \( (\alpha_1, \ldots, \alpha_{l-1}, \beta'_1, \beta'_2, \ldots, \beta'_l, \gamma_2) \) if \( \beta'_t \gamma_t \sigma_t \in A_{X \cap Y} \), or the form \( (\alpha_1, \ldots, \alpha_{l-1}, \beta'_1, \beta'_2, \ldots, \beta'_l, \gamma_2) \) if \( \beta'_t \gamma_t \sigma_t \notin A_{X \cap Y} \). Now, \( \sigma_t(\alpha \sigma_s \alpha^{-1}) \) has also this amalgam normal form, thus \( \alpha_1 A_{X \cap Y} = \sigma_t \alpha_1 A_{X \cap Y} \), namely, \( \alpha_1^{-1} \sigma_t \alpha_1 \in A_{X \cap Y} \). By Proposition 3.3, we deduce that \( \alpha_1 \) can be written as \( \alpha_1 = \omega \delta \), where \( \omega \) is a \((t, r)\)-ribbon for some \( r \in X \cap Y \), and \( \delta \in A_{X \cap Y} \). Let \( \alpha' = \delta \alpha_2 \ldots \alpha_t \gamma = \omega^{-1} \alpha \). We have \( \sigma_r(\alpha' \sigma_s \alpha'^{-1}) = (\alpha' \sigma_s \alpha'^{-1})\sigma_r \), thus, by the inductive hypothesis (on the amalgam norm of \( \alpha \)), we have \( \tau_r(\alpha' \tau_r \alpha'^{-1}) = (\alpha' \tau_r \alpha'^{-1})\tau_r \), therefore \( \tau_t(\alpha \tau_s \alpha^{-1}) = (\alpha \tau_s \alpha^{-1})\tau_t \).

**Case 4:** \( l \geq 1 \) and \( s = x \notin X \). Then the amalgam normal form of \( \alpha \sigma_s \alpha^{-1} \) has the form \( (\alpha_1, \ldots, \alpha_{l-1}, \alpha_t, \beta'_0, \beta'_1, \ldots, \beta'_l, \gamma_1) \). Applying the same argument as in Case 3, we conclude that \( \tau_t(\alpha \tau_s \alpha^{-1}) = (\alpha \tau_s \alpha^{-1})\tau_t \).

\[ \square \]

**4 The FRZ property**

The aim of this section is to prove the following.

**Proposition 4.1.** Assume \( \Gamma \) to be of type FC. Let \( \alpha \in SA \) and \( s, t \in S \). Then the followings are equivalent.

1. \( \alpha \sigma_s = \sigma_t \alpha \).
2. There exists \( k \in \mathbb{Z} \setminus \{0\} \) such that \( \alpha \sigma_s^k = \sigma_t^k \alpha \).
(3) $\alpha \tau_s = \tau_t \alpha$.

(4) There exists $k \in \mathbb{N} \setminus \{0\}$ such that $\alpha \tau_s^k = \tau_t^k \alpha$.

The following lemma is a preliminary result to the proof of Proposition 4.1.

**Lemma 4.2.** Let $G$ be a graph, let $V$ be its set of vertices, and let $E$ be its set of edges. Let $u_1, \ldots, u_l, v, w \in V$ and $p \in \mathbb{N} \setminus \{0\}$ such that $vp_{u_1}u_2 \ldots u_l = u_1u_2 \ldots u_lwp$ (in $\mathcal{M}(G)$). Then $v = w$, and $u_i v = vu_i$ for all $i = 1, \ldots, l$.

**Proof.** Observe that, if $H$ is a full subgraph of $G$, then there is an epimorphism $f_H : \mathcal{M}(H) \to \mathcal{M}(G)$ which sends the vertices of $H$ to themselves and sends the other vertices of $G$ to 1. First, applying this observation to the graph $H = \{v\}$ with one vertex, $v$, and no edge, we deduce that $v = w$. Now, we argue by induction on $l$. Suppose that $u_1 \neq v$ and that $u_l$ and $v$ are not joined by an edge. Let $H$ be the full subgraph of $G$ generated by $\{u_1, v\}$. Then $\mathcal{M}(H)$ is the free monoid freely generated by $\{u_1, v\}$, $f_M(vp_{u_1} \ldots u_l)$ is a word which starts with $v$, and the word $f_M(u_1 \ldots u_lwp)$ starts with $u_1$: a contradiction. So, either $u_1 = v$ or $\{u_1, v\} \in E$, that is, $u_1v = vu_1$. It follows also that $vp_{u_2} \ldots u_l = u_2 \ldots u_lwp$, and, by the inductive hypothesis, we conclude that $u_i v = vu_i$ for all $i = 2, \ldots, l$. \qed

**Proof of Proposition 4.1.** The implications (1)$\Rightarrow$(2) and (3)$\Rightarrow$(4) are obvious, thus it remains to prove (2)$\Rightarrow$(3) and (4)$\Rightarrow$(1).

**Proof of (2)$\Rightarrow$(3).** We assume that $\alpha \sigma_s^k = \sigma_t^k \alpha$ for some $k \in \mathbb{Z} \setminus \{0\}$. Recall the epimorphism $\theta : SA \to A$ which sends $\sigma_s^{\pm 1}$ to $\sigma_t^{\pm 1}$ and $\tau_s$ to $\sigma_s$ for all $s \in S$. Let $\alpha_0 = \theta(\alpha) \in A$. Then $\alpha_0 \sigma_s^k = \sigma_t^k \alpha_0$, thus, by Corollary 3.4, $\alpha_0$ is a $(t, s)$-ribbon, therefore $\alpha_0 \tau_s = \tau_s \alpha_0$. Let $\beta = \alpha_0^{-1} \alpha$. Then $\beta \sigma_s^k = \sigma_t^k \beta$. Recall the decomposition $SA = \mathcal{M}(\hat{\Omega}) \rtimes A$, and write $\beta = u_1 u_2 \ldots u_l \beta_0$ where $u_1, \ldots, u_l \in \hat{\Omega}$ and $\beta_0 \in A$. We have $\sigma_s^k \beta \sigma_s^{-k} = u'_1 u'_2 \ldots u'_l \beta'_0$, where $u'_i = \sigma_s^k u_i \sigma_s^{-k} \in \hat{\Omega}$ for all $i = 1, \ldots, l$, and $\beta'_0 = \sigma_s^k \beta_0 \sigma_s^{-k} \in A$. The equality $\sigma_s^k \beta \sigma_s^{-k} = \beta$ implies that $\beta_0 = \beta_0$, and that there exists a permutation $\chi$ in $\text{Sym}_l$ such that $u'_i = u_i \chi(i)$ for all $i = 1, \ldots, l$. Firstly, the equality $\beta_0' = \sigma_s^k \beta_0 \sigma_s^{-k} = \beta_0$ implies by Corollary 3.4 that $\beta_0$ is a $(s, s)$-ribbon, thus $\beta_0 \tau_s = \tau_s \beta_0$. Now, let $p$ be the order of $\chi$. Then $\sigma_s^k u_i \sigma_s^{-k} = u_i$ for all $i = 1, \ldots, l$. Take $i \in \{1, \ldots, l\}$ and write $u_i = \gamma_i \tau_{r_i} \gamma_i^{-1}$ where $\gamma_i \in A$ and $r_i \in S$. The element $\sigma_s^k$ commutes with $\gamma_i \tau_{r_i} \gamma_i^{-1} = u_i$, thus $\sigma_s^k \alpha$ commutes with $\gamma_i \sigma_{r_i} \gamma_i^{-1} = \theta(\gamma_i \tau_{r_i} \gamma_i^{-1})$, therefore $\sigma_s$ commutes with $\gamma_i \sigma_{r_i} \gamma_i^{-1}$ (by Corollary 3.4), and hence, by Proposition 3.1, $\tau_s$ commutes with $\gamma_i \tau_{r_i} \gamma_i^{-1} = u_i$. This shows that $\tau_s$ commutes with $\beta = u_1 \ldots u_l \beta_0$, and we conclude that $\alpha \tau_s = \alpha_0 \beta \tau_s = \alpha_0 \tau_s \beta = \tau_s \alpha_0 \beta = \tau_s \alpha$. \qed

**Proof of (4)$\Rightarrow$(1).** We assume that $\alpha \tau_s^k = \tau_t^k \alpha$ for some $k \in \mathbb{N} \setminus \{0\}$. We write $\alpha = u_1 u_2 \ldots u_l \alpha_0$, where $u_1, \ldots, u_l \in \hat{\Omega}$ and $\alpha_0 \in A$. Then the equality $\alpha \tau_s^k = \tau_t^k \alpha$ implies that we have $u_1 u_2 \ldots u_l (\alpha_0 \tau_s \alpha_0^{-1})^k = \tau_t^k u_1 u_2 \ldots u_l$ in $\mathcal{M}(\hat{\Omega})$. By Lemma 4.2, it follows that $\alpha_0 \tau_s \alpha_0^{-1} = \tau_t$, and $\tau_t u_i = u_i \tau_t$ for all $i = 1, \ldots, l$. The equality $\alpha_0 \tau_s \alpha_0^{-1} = \tau_t$ implies that $\alpha_0 \sigma_s \alpha_0^{-1} = \theta(\alpha_0 \tau_s) = \theta(\tau_t) = \sigma_t$. Now, take $i \in \{1, \ldots, l\}$ and write $u_i = \gamma_i \tau_{r_i} \gamma_i^{-1}$ where $\gamma_i \in A$ and $r_i \in S$. The element $\tau_t$ commutes with $u_i = \gamma_i \tau_{r_i} \gamma_i^{-1}$,
thus $\sigma_i = \theta(\tau_i)$ commutes with $\gamma_i \sigma_{r_i} \gamma_i^{-1} = \theta(u_i)$, that is, $\sigma_i \gamma_i \sigma_{r_i} \gamma_i^{-1} \sigma_i^{-1} = \gamma_i \sigma_{r_i} \gamma_i^{-1}$, hence, by Lemma 3.5, $\sigma_i u_i \sigma_i^{-1} = \sigma_i \gamma_i \tau_i \gamma_i^{-1} \sigma_i^{-1} = \gamma_i \tau_i \gamma_i^{-1} = u_i$. This shows that $\alpha \sigma_s = u_1 \ldots u_l \alpha_0 \sigma_s = u_1 \ldots u_l \sigma_i \alpha_0 = \sigma_i u_1 \ldots u_l \alpha_0 = \sigma_i \alpha$.

5 Birman’s conjecture

The purpose of this section is to prove Birman’s conjecture for right-angled Artin groups. As these groups are the same as graph groups, for convenience, we shall use the terminology of graph groups and graph products of groups.

Let $G$ be a graph, let $V$ be its set of vertices, and let $E$ be its set of edges. Take an (abstract) set $S = \{u; u \in V\}$ in one-to-one correspondence with $V$, and define the graph group associated to $G$ to be the group presented by

$$G(G) = \langle S \mid \sigma_u \sigma_v = \sigma_v \sigma_u \text{ for } \{u, v\} \in E \rangle.$$

Take another abstract set $T = \{\tau_u; u \in V\}$ in one-to-one correspondence with $V$, and define the singular graph monoid associated to $G$ to be the monoid $SG(G)$ presented as a monoid by the generating set $S \cup S^{-1} \cup T$ and by the relations

$$\sigma_u \sigma_u^{-1} = \sigma_u^{-1} \sigma_u = 1, \quad \sigma_u \tau_u = \tau_u \sigma_u, \quad \text{for } u \in V,$$

$$\sigma_u \sigma_v = \sigma_v \sigma_u, \quad \sigma_u \tau_v = \tau_v \sigma_u, \quad \tau_u \tau_v = \tau_v \tau_u, \quad \text{for } \{u, v\} \in E.$$  

The desingularization map is defined in this context as the multiplicative homomorphism $\eta : SG(G) \to \mathbb{Z}[G(G)]$ which sends $\sigma_u^\pm 1$ to $\sigma_u^\pm 1$ and $\tau_u$ to $\sigma_u - \sigma_u^{-1}$ for all $u \in V$.

**Theorem 5.1.** The desingularization map $\eta : SG(G) \to \mathbb{Z}[G(G)]$ is injective.

We begin with some definitions and results on graph products of groups.

Suppose given a group (or a monoid) $K_u$ for each $u \in V$. Then the graph product of the family $\{K_u\}_{u \in V}$ along the graph $G$ is the quotient $G(\{K_u\}_{u \in V}, G)$ of the free product $\ast_{u \in V} K_u$ by the relations

$$g_u g_v = g_v g_u \quad \text{for } \{u, v\} \in E, \ g_u \in K_u, \ g_v \in K_v.$$ 

Note that, if $K_u = \mathbb{N}$ for all $u \in V$, then $G(\{K_u\}, G) = M(G)$, if $K_u = \mathbb{Z}$ for all $u \in V$, then $G(\{K_u\}, G) = G(G)$, and if $K_u = \mathbb{Z} \times \mathbb{N}$ for all $u \in V$, then $G(\{K_u\}, G) = SG(G)$.

Let $G(\{K_u\}, G)$ be a graph product of groups (or monoids). Let $g \in G(\{K_u\}, G)$. An expression for $g$ is a sequence $W = (g_1, g_2, \ldots, g_l)$ such that

- there exists $u_i \in V$ such that $g_i \in K_{u_i} \setminus \{1\}$ for all $i = 1, \ldots, l$,

- $g = g_1 g_2 \ldots g_l$.

The support of $W$ is the sequence $\text{Supp}(W) = (u_1, u_2, \ldots, u_l)$, and the length of $W$ is $|W| = l$. The minimal length for an expression for $g$ is called the syllable length of $g$ and is denoted by $|g|_G$. An expression $W$ of $g$ is called reduced if its length is equal to the length of $g$. 

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Let \( g \in G(\{K_u\}, G) \), let \( W = (g_1, \ldots, g_l) \) be an expression for \( g \), and let \((u_1, u_2, \ldots, u_l)\) be the support of \( W \). Suppose there exists \( i \in \{1, \ldots, l-1\} \) such that \( u_i = u_{i+1} \), and put

\[
W' = \begin{cases} 
(\ldots, g_{i-1}, g_{i+1}, \ldots) & \text{if } g_i g_{i+1} = 1, \\
(\ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots) & \text{if } g_i g_{i+1} \neq 1.
\end{cases}
\]

We say that \( W' \) is obtained from \( W \) via an elementary \( M \)-operation of type I. This operation shortens the length of an expression by 1 or 2. Suppose there exists \( i \in \{1, \ldots, l-1\} \) such that \( \{u_i, u_{i+1}\} \in E \), and put

\[
W'' = (\ldots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \ldots).
\]

We say that \( W'' \) is obtained from \( W \) via an elementary \( M \)-operation of type II. This operation leaves the length of an expression unchanged. We say that an expression \( W \) is \( M \)-reduced if its length cannot be reduced applying a sequence of elementary \( M \)-operations.

The following proposition is essentially proved in [27] (see also [28] and [29]).

**Proposition 5.2 (Green [27]).** (1) Let \( g \in G(\{K_u\}, G) \), and let \( W_1 \) and \( W_2 \) be two \( M \)-reduced expressions for \( g \). Then \( W_1 \) and \( W_2 \) are related by a sequence of elementary \( M \)-operations of type II.

(2) Let \( g \in G(\{K_u\}, G) \), and let \( W \) be an expression for \( g \). Then \( W \) is \( M \)-reduced if and only if \( W \) is reduced. \( \Box \)

Now, assume that the set \( V \) of vertices is endowed with a total order, \( \leq \). Let \( g \in G(\{K_u\}, G) \), and let \( W \) be a reduced expression for \( g \). We say that \( W \) is a normal form for \( g \) if \( \text{Supp}(W) \) is the smallest support of a reduced expression for \( g \) with respect to the lexicographic order.

**Corollary 5.3.** (1) Each element of \( G(\{K_u\}, G) \) has a unique normal form.

(2) To be a normal form depends only on its support, namely, if \( W \) is the normal form for some \( g \in G(\{K_u\}, G) \), if \( W' \) is some expression for some \( g' \in G(\{K_u\}, G) \), and if \( \text{Supp}(W') = \text{Supp}(W) \), then \( W' \) is the normal form for \( g' \). \( \Box \)

We return to the study of the graph group \( G(G) \). Consider the homomorphism \( \text{deg} : G(G) \to \mathbb{Z} \) which sends \( \sigma_u \) to 1 for all \( u \in V \). For \( n \in \mathbb{Z} \), we put \( G_n(G) = \{ g \in G(G); \text{deg}(g) \geq n \} \), and we denote by \( \mathcal{A}_n \) the free \( \mathbb{Z} \)-module freely generated by \( G_n(G) \). Let \( \mathcal{A} = \mathbb{Z}[G(G)] \). Then \( \{\mathcal{A}_n\}_{n \in \mathbb{Z}} \) is a filtration of \( \mathcal{A} \) compatible with the multiplication, that is,

- \( \mathcal{A}_n \subset \mathcal{A}_m \) if \( n \geq m \),
- \( \mathcal{A}_p \mathcal{A}_q \subset \mathcal{A}_{p+q} \) for \( p, q \in \mathbb{Z} \),
- \( 1 \in \mathcal{A}_0 \).

Moreover, this filtration is separate, namely,

- \( \cap_{n \in \mathbb{Z}} \mathcal{A}_n = \{0\} \).
We denote by $\tilde{A}$ the completion of $A$. For $n \in \mathbb{Z}$, we put $G^{(n)}(G) = \{ g \in G(G); \deg(g) = n \}$, and we denote by $A^{(n)}$ the free $\mathbb{Z}$-module freely generated by $G^{(n)}(G)$. Then each element of $\tilde{A}$ can be uniquely written as a formal series $\sum_{n=k}^{+\infty} P_n$, where $k \in \mathbb{Z}$ (which may be negative), and $P_n \in A^{(n)}$ for all $n \geq k$.

Take a free abelian group $K_u \simeq \mathbb{Z} \times \mathbb{Z}$ of rank 2 generated by $\{ \sigma_u, \tau_u \}$ for all $u \in V$, and write $\tilde{G} = G(\{ K_u \}, G)$. Let $U(\tilde{A})$ be the group of unities of $\tilde{A}$. We have a homomorphism $\tilde{\eta} : \tilde{G} \to U(\tilde{A})$ defined by

$$\tilde{\eta}(\sigma_u) = \sigma_u, \quad \tilde{\eta}(\tau_u) = \sigma_u - \sigma_u^{-1}, \quad \text{for } u \in V.$$ 

Note that

$$\tilde{\eta}(\tau_u^{-1}) = -\sum_{n=0}^{+\infty} \sigma_u^{2n+1}, \quad \text{for } u \in V.$$ 

Note also that $SG(G)$ is a submonoid of $\tilde{G}$, and $A = \mathbb{Z}[G(G)]$ is a subalgebra of $\tilde{A}$, and that the restriction of $\tilde{\eta}$ to $SG(G)$ is the desingularization map $\eta : SG(G) \to \mathbb{Z}[G(G)]$. So, Theorem 5.1 is a consequence of the following.

**Proposition 5.4.** The homomorphism $\tilde{\eta} : \tilde{G} \to U(\tilde{A})$ is injective.

**Proof.** For $u \in V$ and $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, $(p, q) \neq (0, 0)$, we write

$$(\sigma_u - \sigma_u^{-1})^p \sigma_u^q = \sum_{n=k}^{+\infty} c_n p q \sigma_u^n.$$ 

Note that the numbers $c_n p q$ do not depend on $u$, but only on $n$, $p$, and $q$. Note also that there always exists some $a \geq k$ (which may be negative) such that $a \neq 0$ and $C_a p q \neq 0$.

Let $\tilde{g} \in \tilde{G}$, $\tilde{g} \neq 1$. Let $(\tau_{u_1}^{p_1} \sigma_{u_1}^{q_1}, \tau_{u_2}^{p_2} \sigma_{u_2}^{q_2}, \ldots, \tau_{u_l}^{p_l} \sigma_{u_l}^{q_l})$ be the normal form for $\tilde{g}$. We have

$$\tilde{\eta}(\tilde{g}) = \sum_{n_1 \geq k_{1}, \ldots, n_l \geq k_{l}} c_{n_1 p_1 q_1} c_{n_2 p_2 q_2} \cdots c_{n_l p_l q_l} \sigma_{u_1}^{n_1} \sigma_{u_2}^{n_2} \cdots \sigma_{u_l}^{n_l}.$$ 

By the above observations, we can find $a_1, \ldots, a_l \in \mathbb{Z} \setminus \{0\}$ such that $c_{a_1 p_1 q_1} \neq 0$ for all $i = 1, \ldots, l$. We turn to show that $\sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l} \neq \sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l}$ for any $l$-tuple $(n_1, \ldots, n_l)$ in $\mathbb{Z}^l$ different from $(a_1, \ldots, a_l)$. This implies that $\sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l} \neq 1$, and that the coefficient of $\sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l}$ in $\tilde{\eta}(\tilde{g})$ is $c_{a_1 p_1 q_1} c_{a_2 p_2 q_2} \cdots c_{a_l p_l q_l} \neq 0$, thus that $\tilde{\eta}(\tilde{g}) \neq 1$.

The sequence $(\tau_{u_1}^{p_1} \sigma_{u_1}^{q_1}, \tau_{u_2}^{p_2} \sigma_{u_2}^{q_2}, \ldots, \tau_{u_l}^{p_l} \sigma_{u_l}^{q_l})$ is a normal form, thus, by Corollary 5.3, the sequence $(\sigma_{u_1}^{a_1}, \sigma_{u_2}^{a_2}, \ldots, \sigma_{u_l}^{a_l})$ is the normal form for $\sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l}$. Assume $n_i \neq 0$ for all $i = 1, \ldots, l$. Again, by Corollary 5.3, the sequence $(\sigma_{u_1}^{n_1}, \sigma_{u_2}^{n_2}, \ldots, \sigma_{u_l}^{n_l})$ is the normal form for $\sigma_{u_1}^{n_1} \sigma_{u_2}^{n_2} \cdots \sigma_{u_l}^{n_l}$, thus $\sigma_{u_1}^{n_1} \sigma_{u_2}^{n_2} \cdots \sigma_{u_l}^{n_l} \neq \sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l}$ if $(n_1, n_2, \ldots, n_l) \neq (a_1, a_2, \ldots, a_l)$. Now, assume that there exists $i \in \{1, \ldots, l\}$ such that $n_i = 0$. Then

$$|\sigma_{u_1}^{n_1} \sigma_{u_2}^{n_2} \cdots \sigma_{u_l}^{n_l}|_G < l = |(\sigma_{u_1}^{a_1}, \sigma_{u_2}^{a_2}, \ldots, \sigma_{u_l}^{a_l})| = |\sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l}|_G,$$

hence $\sigma_{u_1}^{n_1} \sigma_{u_2}^{n_2} \cdots \sigma_{u_l}^{n_l} \neq \sigma_{u_1}^{a_1} \sigma_{u_2}^{a_2} \cdots \sigma_{u_l}^{a_l}$. 

\qed
6 Other questions

6.1 Topological interpretation

A geometric braid on \( n \) strings is a \( n \)-tuple \( \beta = (b_1, \ldots, b_n) \) of smooth disjoint paths in \( \mathbb{C} \times [0, 1] \) such that

- the projection of \( b_k(t) \) on the second component is \( t \) for all \( k \in \{1, \ldots, n\} \) and all \( t \in [0, 1] \),
- \( b_k(0) = (k, 0) \) and \( b_k(1) = (\xi(k), 1) \), where \( \xi \) is some permutation of \( \{1, \ldots, n\} \), for all \( k \in \{1, \ldots, n\} \).

The braid group on \( n \) strings, denoted by \( \mathcal{B}_n \), is the group of isotopy classes of braids. Define a singular braid on \( n \) strings as a \( n \)-tuple \( \beta = (b_1, \ldots, b_n) \) of smooth paths in \( \mathbb{C} \times [0, 1] \) such that

- the projection of \( b_k(t) \) on the second component is \( t \) for all \( k \in \{1, \ldots, n\} \) and all \( t \in [0, 1] \),
- \( b_k(0) = (k, 0) \) and \( b_k(1) = (\xi(k), 1) \), where \( \xi \) is some permutation of \( \{1, \ldots, n\} \), for all \( k \in \{1, \ldots, n\} \),
- the strings intersect transversely in finitely many double points.

The singular braid monoid, denoted by \( \mathcal{SB}_n \), is the monoid of isotopy classes of singular braids. By [3] (see also [4]), the braid group \( \mathcal{B}_n \) is isomorphic to the Artin group \( \mathcal{A}_\Gamma \) associated to the Coxeter graph \( \Gamma = A_{n-1} \), and, by [9], \( \mathcal{SB}_n \) is the singular Artin monoid associated to \( \Gamma = A_{n-1} \).

Some other Artin groups can be viewed as “geometric braid groups”. For instance, the Artin group associated to the Coxeter graph \( \Gamma = B_n \) is the braid group on \( n \) strings of the annulus, and the Artin group associated to the graph \( \Gamma = D_n \) is an index 2 subgroup of the braid group on \( n \) strings of the plane endowed with a singular point of degree 2 (see [1]). One can easily verify using the techniques of [26] that the singular braid monoid of the annulus coincides with the singular Artin monoid associated to \( \Gamma = B_n \). However, we did not succeed to find any embedding of the singular Artin monoid associated to \( \Gamma = D_n \) into the singular braid monoid of the plane endowed with a singular point of degree 2, and we suspect that such an embedding does not exist.

In other respects, the spherical type Artin groups can be interpreted as fundamental groups of regular orbit spaces (see [11]), and the groups of type \( A_n \) and \( D_n \) as geometric monodromy groups of simple singularities (see [37]). In both cases, we do not know whether these topological interpretations can be extended to the singular Artin monoids, even for the singular Artin monoids of type \( A_n \) (namely, the singular braid monoids).

6.2 Vassilev invariants

An invariant on \( \mathcal{A} \) is a set-map \( v : \mathcal{A} \to H \), where \( H \) is some abelian group, or, equivalently, a homomorphism \( v : \mathbb{Z}[\mathcal{A}] \to H \) of \( \mathbb{Z} \)-modules. Consider the homomorphism \( \text{ord} : SA \to \mathbb{N} \) which sends \( \sigma_s^{\pm 1} \) to 0 and \( \tau_s \) to 1 for all \( s \in S \). For \( d \in \mathbb{N} \), we put
$S_dA = \{ \alpha \in SA; \text{ord}(\alpha) = d \}$. So, $S_dA$ is the set of elements of $SA$ that have exactly $d$ “singularities”. Recall the desingularization map $\eta : SA \to \mathbb{Z}[A]$ which sends $\sigma_s^{\pm1}$ to $\sigma_s^{\pm1}$ and $\tau_s$ to $\sigma_s^{-1} - \sigma_s$ for all $s \in S$. Then define a Vassiliev invariant of type $d$ as an invariant $v : \mathbb{Z}[A] \to H$ which vanishes on $\eta(S_{d+1}A)$.

The following questions have been solved for braid groups (see [6], [31], and [33]), and remain open for the other Artin groups.

**Question 6.1.** Do Vassiliev invariants separate the elements of $A$? In other words, given two elements $\alpha, \beta \in A$, $\alpha \neq \beta$, does there exist a Vassiliev invariant $v : \mathbb{Z}[A] \to H$ such that $v(\alpha) \neq v(\beta)$?

By a universal Vassiliev invariant we mean an invariant $Z : \mathbb{Z}[A] \to A$ such that, if $v : \mathbb{Z}[A] \to H$ is some Vassiliev invariant, then there exists a homomorphism $u : A \to H$ of $\mathbb{Z}$-modules such that $v = u \circ Z$.

**Question 6.2.** Describe a universal Vassiliev invariant for $A$ in terms of generators and relations.

### 6.3 Conjugacy problem

Let $\mathcal{M}$ be a monoid, and let $\mathcal{G}$ be the group of unities of $\mathcal{M}$. We say that two elements $\alpha, \beta \in \mathcal{M}$ are conjugate if there exists some $\gamma \in \mathcal{G}$ such that $\gamma \alpha \gamma^{-1} = \beta$. A solution to the conjugacy problem in $\mathcal{M}$ is an algorithm which decides whether two given elements $\alpha, \beta \in \mathcal{M}$ are conjugate or not.

As for the word problem, a solution to the conjugacy problem for $SA$ will give a solution to the conjugacy problem for $A$ (which, by the way, is the group of unities of $SA$). So, a reasonable approach is to study the conjugacy problem for those singular Artin monoids whose associated Artin groups have known solutions to the conjugacy problem. For instance, a solution to the conjugacy problem for Artin groups of spherical type can be found in [12] (see also [38] and [22]).

A solution to the conjugacy problem for singular braid monoids is given in [40]. We suspect that this algorithm can be extended to all spherical type singular Artin monoids. On the other hand, we do not know whether the techniques introduced in the present paper, more specifically, Propositions 2.1 and 3.1, can be also used to solve the conjugacy problem in $SA$.

On other respects, let $A = \mathcal{G}(\hat{\Omega}) \rtimes A$ be the smallest group which contains $SA = \mathcal{M}(\hat{\Omega}) \rtimes A$. As far as we know, no solution to the conjugacy problem for $A$ is known, even in the case where $SA$ is the singular braid monoid.

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