The first two coefficients of the Bergman function expansions for Cartan-Hartogs domains

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Abstract Let $\phi$ be a globally defined real Kähler potential on a domain $\Omega \subset \mathbb{C}^d$, and $g_F$ be a Kähler metric on the Hartogs domain $M = \{(z, w) \in \Omega \times \mathbb{C}^d : \|w\|^2 < e^{-\phi(z)}\}$ associated with the Kähler potential $\Phi_F(z, w) = \phi(z) + F(\phi(z) + \ln \|w\|^2)$. Firstly, we obtain explicit formulas of the coefficients $a_j$ ($j = 1, 2$) of the Bergman function expansion for the Hartogs domain $(M, g_F)$ in a momentum profile $\phi$. Secondly, using explicit expressions of $a_j$ ($j = 1, 2$), we obtain necessary and sufficient conditions for the coefficients $a_j$ ($j = 1, 2$) to be constants. Finally, we obtain all the invariant complete Kähler metrics on Cartan-Hartogs domains such that their the coefficients $a_j$ ($j = 1, 2$) of the Bergman function expansions are constants.

Key words: Kähler metrics · Coefficients of the Bergman function expansion · Hartogs domains · Cartan-Hartogs domains

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1 Introduction

Let $M$ be a domain in $\mathbb{C}^n$, $\phi$ be a Kähler potential on $M$, and $g$ be a Kähler metric on $M$ associated with the Kähler form $\omega = \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \phi$. Set

$$H_\alpha = \left\{ f \in \text{Hol}(M) : \int_M |f|^2 e^{-\alpha \phi(z)} \omega^n < +\infty \right\},$$

where $\text{Hol}(M)$ denotes the space of holomorphic functions on $M$. For $\alpha > 0$, let $K_\alpha(z, \overline{z})$ be the Bergman kernel (namely, the reproducing kernel) of the Hilbert space $H_\alpha$ if $H_\alpha \neq \{0\}$, the Bergman function on $M$ associated with the metric $g$ defined by

$$\varepsilon(\alpha; z) = e^{-\alpha \phi(z)} K_\alpha(z, \overline{z}), \quad z \in M.$$ (1.1)

The metric $g$ is called balanced when $\varepsilon(1; z)$ is constant. Balanced metric plays an important role in the quantization of a Kähler manifold, see Berezin [2], Cahen-Gutt-Rawnsley [5], Engliš [10], Marinescu [28], Ma [27] and Luč [26].

Under given some conditions for $(M, \omega)$, $\varepsilon(\alpha; z)$ admits an asymptotic expansion as $\alpha \to +\infty$

$$\varepsilon(\alpha; z) \sim \sum_{j=0}^{\infty} a_j(z) \alpha^{n-j},$$ (1.2)

where the expansion coefficients $a_0, a_1, a_2$ in Lu [25] and Engliš [12] are given by

$$\begin{cases}
  a_0 = 1, \\
  a_1 = \frac{1}{2} k_g, \\
  a_2 = \frac{1}{3} \Delta k_g + \frac{1}{24} |R_g|^2 - \frac{1}{6} |\text{Ric}_g|^2 + \frac{1}{8} k_g^2.
\end{cases}$$ (1.3)

Here $k_g$, $\Delta g$, $R_g$ and $\text{Ric}_g$ denote the scalar curvature, the Laplace, the curvature tensor and the Ricci curvature associated with the metric $g$, respectively. If for all sufficiently large positive numbers $\alpha$,
\[ \epsilon(\alpha; z) \text{ are constants in } z \text{ on } M, \text{ Loi in } [21] \text{ has proved that there exists an asymptotic expansion on } (M, \omega) \text{ for } \epsilon(\alpha; z) \text{ as } (1.2) \text{ and } (1.3), \text{ and all coefficients } a_j \text{ are constants. For graph theoretic formulas of coefficients } a_j, \text{ see Xu } [33]. \]

For the general reference of the Bergman function expansions, refer to Berezin [2], Catlin [7], Zelditch [35], Engliš [11], Dai-Liu-Ma [8], Ma-Marinescu [28, 29, 30], Berman-Berndtsson-Sjöstrand [3], Hsiao [17] and Hsiao-Marinescu [18].

In [9] Donaldson used the first coefficient \( a_1 \) in the expansion of the Bergman function to give conditions for the existence and uniqueness of constant scalar curvature Kähler metrics (cscK metrics). This work inspired many papers on the subject since then.

The main purpose of this paper is to study cscK metrics on Hartogs domains such that the coefficients \( a_2 \) of the Bergman function expansion are constants. For the study of Kähler metrics with the constant coefficients \( a_2 \), see Loi [21], Loi-Zuddas [24], Zedda [34], Feng-Tu [15], Loi-Zedda [22] and Feng [14].

In [4] and [16], for Fock-Bargmann-Hartogs domains and Cartan-Hartogs domains, we studied balanced metrics associated with Kähler forms

\[ \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \left( \nu \phi(z) - \ln(e^{-\phi(z)} - \|w\|^2) \right), \nu \geq 0. \]

From Lemma 4.1 below, we know that the Kähler forms of \( G \)-invariant Kähler metrics on Cartan-Hartogs domains \( \Omega(\mu, d_0) \) can be written as

\[ \omega = \frac{\nu \sqrt{-1}}{2\pi} \partial\bar{\partial} \left( \phi(z) + F(\phi(z)) \|w\|^2 \right), \nu > 0. \]

So in this paper, we will study Kähler metrics associated with Kähler forms

\[ \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \left( \phi(z) + F(\phi(z)) + \ln \|w\|^2 \right) \]

on Hartogs domains

\[ M = \{(z, w) \in \Omega \times \mathbb{C}^d : \|w\|^2 < e^{-\phi(z)} \}. \]

Now we give main results of this paper, namely the following theorems.

**Theorem 1.1.** Let \( g_\phi \) be a complete Kähler metric on the domain \( \Omega \) associated with the Kähler form \( \omega_\phi = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \phi \), where \( \phi \) is a globally defined Kähler potential on a domain \( \Omega \subset \mathbb{C}^d \), defined a Hartogs domain

\[ M = \{(z, w) \in \Omega \times \mathbb{C}^d : \|w\|^2 < e^{-\phi(z)} \}. \]

Set \( g_F \) is a complete Kähler metric on the domain \( M \) associated with the Kähler form \( \omega_F = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \Phi_F \), here

\[ \Phi_F(z, w) = \phi(z) + F(\phi(z)) + \ln \|w\|^2, \quad F(-\infty) = 0. \]

Let \( k_\phi \), \( \Delta_\phi \), \( \text{Ric}_g \) and \( \text{Ric}_g \) be the scalar curvature, the Laplace, the Ricci curvature and the curvature tensor on the domain \( \Omega \) with respect to the metric \( g_\phi \), respectively. Put

\[ a_1 = \frac{1}{2} k_\phi, \quad a_2 = \frac{1}{3} \Delta_\phi k_\phi + \frac{1}{24} \| \text{Ric}_g \|^2 - \frac{1}{6} \| \text{Ric}_g \|^2 + \frac{1}{8} k_\phi^2 \]

and

\[ a_1 = \frac{1}{2} k_F, \quad a_2 = \frac{1}{3} \Delta_\phi k_F + \frac{1}{24} \| \text{Ric}_g \|^2 - \frac{1}{6} \| \text{Ric}_g \|^2 + \frac{1}{8} k_F^2. \]

Then both \( a_1 \) and \( a_2 \) are constants on \( M \) if and only if

(i) \( F(t) = -\frac{1}{2} \ln(1 - e^t) \)
and $a_2 = 0$ for $d = 1$, where $A = \frac{d_0 - a_1}{d_0 + 1}$ is a positive constant.

(ii) $F(t) = -\ln \left(1 - e^t\right)$, $a_1 = -\frac{d(d+1)}{2}$ and $a_2 = \frac{(d-1)d(d+1)(3d+2)}{24}$ for $d > 1$.

Let $\Omega$ be a bounded symmetric domain in $\mathbb{C}^d$, that is, $\Omega$ is bounded and there exists a holomorphic automorphism $\gamma_0$ for every $z_0 \in \Omega$ such that $\gamma_0$ is involutive (i.e. $\gamma_0^2 = id$) and $z_0$ is an isolated fixed point of $\gamma_0$. If $\Omega$ can not be expressed as the product of two bounded symmetric domains, then $\Omega$ is called an irreducible bounded symmetric domain. If we put the Bergman metric on $\Omega$, then $\Omega$ is a Hermitian symmetric space. By selecting proper coordinate systems, irreducible bounded symmetric domains can be realized as circular convex domains. The following assume that irreducible bounded symmetric domains are circle convex domains.

Every Hermitian symmetric space of noncompact type can be realized as a bounded symmetric domain in some $\mathbb{C}^d$ by the Harish-Chandra embedding theorem. In 1935, E. Cartan has showed that there exist only six types of irreducible bounded symmetric domains. They are four types of classical bounded symmetric domains ($\Omega_I(m,n), \Omega_{II}(n), \Omega_{III}(n), \Omega_{IV}(n)$) and two exceptional domains ($\Omega_{V}(16), \Omega_{V}(27)$). So irreducible bounded symmetric domains are called Cartan domains.

For an irreducible bounded symmetric domain $\Omega$, we denote by $r, a, b, d, p$ and $N$, the rank, the characteristic multiplicities, the dimension, the genus, and the generic norm of $\Omega$, respectively; thus

$$p = (r-1)a + b + 2, \quad d = \frac{r(r-1)}{2}a + br + r.$$ 

For convenience, we list the characteristic multiplicities, the rank, and the generic norm $N$ for the classical domain $\Omega$ according to its type as the following table.

| $\Omega$ | $(a, b, r)$ | $N$ |
|----------|-------------|-----|
| $\Omega_I(m,n) = \{ z \in \mathbb{M}_{m,n} : I - z\overline{z} > 0 \}$ (1 $\leq m \leq n$) | $(2, n-m, n)$ | $\det(I - z\overline{z})$ |
| $\Omega_{II}(2n) = \{ z \in \mathbb{M}_{2n,2n} : z^t = -z, I - z\overline{z} > 0 \}$ (n $\geq 3$) | $(4, 0, n)$ | $\sqrt{\det(I - z\overline{z})}$ |
| $\Omega_{II}(2n+1) = \{ z \in \mathbb{M}_{2n+1,2n+1} : z^t = -z, I - z\overline{z} > 0 \}$ (n $\geq 2$) | $(4, 2, n)$ | $\sqrt{\det(I - z\overline{z})}$ |
| $\Omega_{III}(n) = \{ z \in \mathbb{M}_{n,n} : z^t = z, I - z\overline{z} > 0 \}$ (n $\geq 2$) | $(1, 0, n)$ | $\det(I - z\overline{z})$ |
| $\Omega_{IV}(n) = \{ z \in \mathbb{C}^n : 1 - 2z\overline{z} + |z\overline{z}|^2 > 0, |z\overline{z}|^2 < 1 \}$ (n $\geq 5$) | $(n - 2, 0, 2)$ | $1 - 2z\overline{z} + |z\overline{z}|^2$ |

The above, $\mathbb{M}_{m,n}$ denotes the set of all $m \times n$ matrices $z = (z_{ij})$ with complex entries, $\overline{z}$ is the complex conjugate of the matrix $z$, $z^t$ is the transpose of the matrix $z$, $I$ denotes the identity matrix, and $z > 0$ indicates that the square matrix $z$ is positive definite. For the reference of the irreducible bounded symmetric domains, refer to Hua [19] and Faraut-Kaneyuki-Korányi-Lu-Roos [13].

For the Cartan domain $\Omega$ in $\mathbb{C}^d$, a positive real number $\mu$ and a positive integer number $d_0$, let

$$\phi(z) := -\mu \ln N(z, \overline{z}),$$

the Cartan-Hartogs domain $\Omega(\mu, d_0)$ is defined by

$$\Omega(\mu, d_0) := \left\{ (z, w) \in \Omega \times \mathbb{C}^{d_0} \subset \mathbb{C}^d \times \mathbb{C}^{d_0} : ||w||^2 < e^{-\phi(z)} \right\},$$

where $N$ is the generic norm of $\Omega$, and $|| \cdot ||$ the standard Hermitian norm in $\mathbb{C}^{d_0}$. Below we assume $\frac{\partial^2 \phi}{\partial z \partial \overline{z}}(0) = \mu I_d$.

From Lemma 3.1 of Ahn-Byun-Park [1], the Cartan-Hartogs domain $\Omega(\mu, d_0)$ is homogeneous if and only if $\Omega$ is the unit ball in $\mathbb{C}^d$ and $\mu = 1$. For the general reference of Cartan-Hartogs domains, see Ahn-Byun-Park [1], Feng-Tu [15], Loi-Zedda [23], Wang-Yin-Zhang-Roos [31], Zedda [34] and references therein.
Theorem 1.2. For a given positive integer $d_0$ and a positive real number $\mu$, let

$$\Omega(\mu, d_0) := \left\{ (z, w) \in \Omega \times \mathbb{B}^{d_0} : \|w\|^2 < N(z, \overline{z})^\mu \right\}$$

be the Cartan-Hartogs domain, where $N(z, \overline{z})$ is the generic norm of an irreducible bounded symmetric domain $\Omega$ in $\mathbb{C}^d$.

Let $G$ be the group of holomorphic automorphism mappings generated by (1.5) on the Cartan-Hartogs domain $\Omega(\mu, d_0)$, $g$ be a $G$-invariant complete Kähler metric on the domain $\Omega(\mu, d_0)$ associated with the Kähler form $\omega_g = \frac{1}{2\pi} \partial \overline{\partial} \Phi$, that is

$$\partial \overline{\partial} (\Phi \circ \Upsilon) = \partial \overline{\partial} \Phi, \ \forall \ \Upsilon \in G.$$

Then the first two coefficients $a_j (j = 1, 2)$ of the Bergman function expansion for $(\Omega(\mu, d_0), \omega_g)$ are constants if and only if

(i) $\Omega = \mathbb{B}^d := \{ z \in \mathbb{C}^d : \|z\|^2 < 1 \}$, and

$$\omega_g = -\frac{\nu \sqrt{-1}}{2\pi} \partial \overline{\partial} \ln(1 - \|z\|^2 - \|w\|^2), \ \nu > 0$$

for $d > 1$.

(ii) $\Omega = \mathbb{B} := \{ z \in \mathbb{C} : |z| < 1 \}$ and

$$\omega_g = -\frac{\nu \sqrt{-1}}{2\pi} \partial \overline{\partial} \left\{ \mu \ln(1 - |z|^2) + \frac{(d_0 + 1)\mu}{d_0\mu + 1} \ln \left( 1 - \frac{\|w\|^2}{(1 - |z|^2)^\mu} \right) \right\}, \ \nu > 0$$

for $d = 1$.

Remark 1.1. (i) From [23], for $\nu > d$, Kähler metrics associated with

$$\omega = -\frac{\nu \sqrt{-1}}{2\pi} \partial \overline{\partial} \ln(1 - \|z\|^2)$$

are balanced on unite balls $\mathbb{B}^d$.

(ii) Using methods of [4], we can prove that for $\nu > \frac{1}{\mu} + d_0$, Kähler metrics associated with

$$\omega = -\frac{\nu \sqrt{-1}}{2\pi} \partial \overline{\partial} \left\{ \mu \ln(1 - |z|^2) + \frac{(d_0 + 1)\mu}{d_0\mu + 1} \ln \left( 1 - \frac{\|w\|^2}{(1 - |z|^2)^\mu} \right) \right\}$$

are balanced on Cartan-Hartogs domains

$$\left\{ (z, w) \in \mathbb{B} \times \mathbb{C}^{d_0} : \|w\|^2 < (1 - |z|^2)^\mu \right\}.$$
To prove Theorem 1.1, let
\[ x = F'(t), \quad \varphi(x) = F''(t), \]
then the scalar curvature of \( g_F \) is given by a linear second-order differential expression in \( \varphi(x) \). Consequently, the \( \varphi(x) \) is an explicit quadratic function of \( x \) when both \( a_1 \) and \( a_2 \) are constants. This method referred to as the momentum construction (see [6], [20]), the function \( \varphi(x) \) is called the momentum profile of \( \omega_F \).

The paper is organized as follows. In Section 2, by calculating the scalar curvature, the squared norm of the Ricci curvature tensor, the squared norm of the curvature tensor, and the Laplace of the scalar curvature, we obtain explicit expressions of the coefficients \( a_j \) \((j = 1, 2)\) of the Bergman function expansion for \((M, g_F)\) in the momentum profile \( \varphi(x) \). In Section 3, using the expressions of the coefficients \( a_j \) \((j = 1, 2)\) of the Bergman function expansion for \((M, g_F)\), we obtain an explicit expression of the function \( F \) when \( a_1 \) and \( a_2 \) on Hartogs domain \((M, g_F)\) are constants, thus we obtain necessary and sufficient conditions for the coefficients \( a_j \) \((j = 1, 2)\) to be constants. In Section 4, we first give the general expressions of holomorphic invariant Kähler metrics on Cartan-Hartogs domain \( \Omega(\mu, d_0) \), and then give all invariant complete Kähler metrics such that their the coefficients \( a_j \) \((j = 1, 2)\) of the Bergman function expansions are constants.

2 The first two coefficients of the Bergman function expansions for Hartogs domains

The following we first compute the scalar curvature, the squared norm of the Ricci curvature tensor, and the Laplace of the scalar curvature for the metric \( g_F \) on \( M \). Secondly, we obtain an expression of the squared norm of the curvature tensor. Finally we get expressions of the coefficients \( a_j \) \((j = 1, 2)\) of the Bergman function expansion on \((M, g_F)\). As applications, we obtain the necessary and sufficient conditions for the coefficients \( a_j \) \((j = 1, 2)\) to be constants when \( \varphi(x) = x(1 + x) \) or \( \varphi(x) = Ax^2 + x \) with \( d = 1 \).

To prove the Theorem 2.3, we need the following Lemma 2.1 and Lemma 2.2.

Lemma 2.1. Let \( \phi \) be a globally defined real Kähler potential on a domain \( \Omega \), \( F \) be a real function on \([-\infty, 0)\) and
\[ \Phi_F(z, w) = \phi(z) + F(t), \]
where \( z \in \mathbb{C}^d \), \( w \in \mathbb{C}^{d_0} \), and
\[ t := \phi(z) + \ln r^2, \quad r^2 := \|w\|^2. \]

Set
\[ Z = (z, w), \quad T = (T_{ij}) := \frac{\partial^2 \Phi_F}{\partial z^i \partial \bar{z}^j} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \phi}{\partial z^1 \partial \bar{z}^1} & \frac{\partial^2 \phi}{\partial z^1 \partial \bar{w}^1} \\ \frac{\partial^2 \phi}{\partial z^1 \partial \bar{w}^2} & \frac{\partial^2 \phi}{\partial z^1 \partial \bar{w}^3} \end{pmatrix}, \]
and
\[ T^{-1} = (T^j_i) := \begin{pmatrix} (T^{-1})_1 & (T^{-1})_2 \\ (T^{-1})_3 & (T^{-1})_4 \end{pmatrix}. \]

Then
\[ T_1 = (1 + F')^d \frac{\partial^2 \phi}{\partial z^1 \partial \bar{z}^1} + F'' \frac{\partial \phi}{\partial z^1} \frac{\partial \phi}{\partial \bar{z}^1}, \quad T_2 = \frac{F''}{r^2} \frac{\partial \phi}{\partial z^1} w, \quad (2.1) \]
\[ T_3 = \frac{F''}{r^2} \frac{\partial \phi}{\partial \bar{z}^1} \bar{w}, \quad T_4 = \frac{F'}{r^2} I_{d_0} + \frac{F''}{r^4} \bar{w}^4 w, \quad (2.2) \]
\[ \det T = \frac{1}{r^{2d_0}} (F')^{d_0-1} F''(1 + F')^d \det \left( \frac{\partial^2 \phi}{\partial z^1 \partial \bar{z}^1} \right), \quad (2.3) \]
\[ (T^{-1})_1 = \frac{1}{1 + F'} \left( \frac{\partial^2 \phi}{\partial z^1 \partial \bar{z}^1} \right)^{-1}, \quad (2.4) \]
\[(T^{-1})_2 = -\frac{1}{1 + F^t} \left( \frac{\partial^2 \phi}{\partial z^t \partial z} \right)^{-1} \frac{\partial \phi}{\partial z^t} w, \tag{2.5} \]
\[(T^{-1})_3 = -\frac{1}{1 + F^t} \bar{w}^t \frac{\partial \phi}{\partial \bar{z}} \left( \frac{\partial^2 \phi}{\partial \bar{z}^t \partial \bar{z}} \right)^{-1}, \tag{2.6} \]
\[(T^{-1})_4 = \frac{r^2}{F^t I_{d_0}} + \left( \frac{1}{F^n} - \frac{1}{F^t} \right) \bar{w}^t w + \frac{1}{1 + F^t} \bar{w}^t \frac{\partial \phi}{\partial \bar{z}} \left( \frac{\partial^2 \phi}{\partial \bar{z}^t \partial \bar{z}} \right)^{-1} \frac{\partial \phi}{\partial \bar{z}^t} w. \tag{2.7} \]

Where \( Z^t \) and \( \bar{Z} \) denote the transpose and the conjugate of the row vector \( Z = (z, w) \), respectively, \( I_{d_0} \) denotes the identity matrix of order \( d_0 \), and symbols

\[
\frac{\partial}{\partial z^t} := \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d} \right)^t, \quad \frac{\partial}{\partial \bar{z}} := \left( \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_d} \right), \quad \frac{\partial^2}{\partial \bar{z}^t \partial \bar{z}} := \left( \frac{\partial^2}{\partial \bar{z}_1 \partial \bar{z}_1} \right). \]

Proof. By direct computation, it follows that (2.1) and (2.2).

Using (2.1) and (2.2), we have

\[
T_4^{-1} = \frac{r^2}{F^t I_{d_0}} + \left( \frac{1}{F^n} - \frac{1}{F^t} \right) \bar{w}^t w, \]
\[
T_2 T_4^{-1} T_3 = F'^n \frac{\partial \phi}{\partial z^t} \frac{\partial \phi}{\partial \bar{z}},
\]
and
\[
T_1 - T_2 T_4^{-1} T_3 = (1 + F^t) \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}}.
\]

So we get

\[
\det T = \det(T_1 - T_2 T_4^{-1} T_3) \det T_4 = \frac{1}{r^{2d_0}} (F'^d)^{d_0 - 1} F'^d (1 + F^d) \det \left( \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}} \right),
\]
\[
(T_1 - T_2 T_4^{-1} T_3)^{-1} = \frac{1}{1 + F^t} \left( \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}} \right)^{-1},
\]
\[
-(T_1 - T_2 T_4^{-1} T_3)^{-1} T_2 T_4^{-1} = -\frac{1}{1 + F^t} \left( \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}} \right)^{-1} \frac{\partial \phi}{\partial z^t} w,
\]
\[
-T_4^{-1} T_3 (T_1 - T_2 T_4^{-1} T_3)^{-1} = -\frac{1}{1 + F^t} \bar{w}^t \frac{\partial \phi}{\partial \bar{z}} \left( \frac{\partial^2 \phi}{\partial \bar{z}^t \partial \bar{z}} \right)^{-1},
\]
and

\[
T_4^{-1} + T_4^{-1} T_3 (T_1 - T_2 T_4^{-1} T_3)^{-1} T_2 T_4^{-1} = \frac{r^2}{F^t I_{d_0}} + \left( \frac{1}{F^n} - \frac{1}{F^t} \right) \bar{w}^t w + \frac{1}{1 + F^t} \bar{w}^t \frac{\partial \phi}{\partial \bar{z}} \left( \frac{\partial^2 \phi}{\partial \bar{z}^t \partial \bar{z}} \right)^{-1} \frac{\partial \phi}{\partial \bar{z}^t} w.
\]

Since

\[
T^{-1} = \begin{pmatrix}
(T_1 - T_2 T_4^{-1} T_3)^{-1} & -(T_1 - T_2 T_4^{-1} T_3)^{-1} T_2 T_4^{-1} \\
-T_4^{-1} T_3 (T_1 - T_2 T_4^{-1} T_3)^{-1} & T_4^{-1} + T_4^{-1} T_3 (T_1 - T_2 T_4^{-1} T_3)^{-1} T_2 T_4^{-1}
\end{pmatrix},
\]
we have (2.4), (2.5), (2.6) and (2.7). \(\square\)
Remark 2.1. Under assumptions of Lemma 2.1, from

\[
\begin{pmatrix}
I_d & -T_2T_4^{-1} \\
0 & I_{d_0}
\end{pmatrix}
\begin{pmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{pmatrix}
\begin{pmatrix}
I_d & 0 \\
-T_4^{-1}T_3 & I_0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 + F' \frac{\partial^2 \phi}{\partial z \partial \bar{z}} \\
F' r I_{d_0} + F'' - F' \bar{w}^t w
\end{pmatrix}
\]

we obtain that \(\Phi_F\) is a Kähler potential on

\[M^* = \{(z, w) \in \Omega \times \mathbb{C}^{d_0} : 0 < \|w\|^2 < e^{-\phi(z)}\}\]

if and only if

\[
(1 + F') \frac{\partial^2 \phi}{\partial z \partial \bar{z}} > 0 \quad \text{(2.8)}
\]

and

\[
\frac{F'}{r^2} I_{d_0} + \frac{F'' - F'}{r^4} \bar{w}^t w > 0. \quad \text{(2.9)}
\]

For the case of \(d_0 = 1\), (2.8) and (2.9) are equivalent to

\[
1 + F'(t) > 0, \quad \frac{F''(t)}{e^t} > 0, \quad t \in (-\infty, 0),
\]

respectively.

For the case of \(d_0 > 1\), by the eigenvalues of the matrix \(\frac{F'}{r^2} I_{d_0} + \frac{F'' - F'}{r^4} \bar{w}^t w\) are

\[
\frac{F'(t)}{r^2}, \ldots, \frac{F'(t)}{r^2}, \frac{F''(t)}{r^2},
\]

so (2.8) and (2.9) are equivalent to

\[
\frac{F'(t)}{e^t} > 0, \quad \frac{F''(t)}{e^t} > 0, \quad t \in (-\infty, 0),
\]

respectively.

Lemma 2.2. Let \(\phi\) be a globally defined real Kähler potential on a domain \(\Omega\),

\[
\Phi_F(z, w) = \phi(z) + F(\phi(z) + \ln \|w\|^2)
\]

and

\[M = \{(z, w) \in \Omega \times \mathbb{C}^{d_0} : \|w\|^2 < e^{-\phi(z)}\}.\]

For given \((z_0, w_0) \in M\), let

\[
\tilde{\phi}(u) := \phi(u + z_0) - \phi(z_0) - \frac{\partial \phi}{\partial z}(z_0) u^t - \frac{\partial \phi}{\partial \bar{z}}(z_0) \bar{u},
\]

\[
\tilde{\Omega} := \{u \in \mathbb{C}^d : u + z_0 \in \Omega\},
\]

\[
\tilde{M} := \{(u, v) \in \tilde{\Omega} \times \mathbb{C}^{d_0} : \|v\|^2 < e^{-\tilde{\phi}(u)}\}
\]

and

\[
\tilde{\Phi}_F(u, v) := \tilde{\phi}(u) + F(\tilde{\phi}(u) + \ln \|v\|^2).
\]
Define the holomorphic mapping $\Upsilon$

$$\Upsilon : M \to \tilde{M},
\begin{align*}
(z, w) &\mapsto (u, v) = \left( z - z_0, e^{\frac{1}{2}\phi(z_0)} + \frac{1}{2}\frac{\partial \phi}{\partial z}(z_0)(z - z_0)t \right).
\end{align*}$$

Then

$$\phi(z) + \ln \|w\|^2 = \tilde{\phi}(u) + \ln \|v\|^2$$
and

$$\partial\bar{\partial}\Phi_F = \partial\bar{\partial}(\tilde{\Phi}_F \circ \Upsilon).$$

**Proof.** The proof is trivial, we omit it.

By Lemma 2.2, the scalar curvature, the Laplace, the squared norm of the curvature tensor and the squared norm of the Ricci curvature at $(z_0, w_0)$ associated with the Kähler potential $\Phi_F$ on the domain $M$ are equal to the scalar curvature, the Laplace, the squared norm of the curvature tensor and the squared norm of the Ricci curvature at $(0, e^{\frac{1}{2}\phi(z_0)}w_0)$ associated with the Kähler potential $\tilde{\Phi}_F$ on the domain $\tilde{M}$, respectively. For convenience, the following we assume that $0 \in \Omega$ and

$$\phi(0) = 0, \quad \frac{\partial \phi}{\partial z^2}(0) = 0, \quad \frac{\partial \phi}{\partial \bar{z}}(0) = 0.$$

**Theorem 2.3.** Assume that $\phi$ is a globally defined Kähler potential on a domain $\Omega \subset \mathbb{C}^d$. Let $g_\phi$ be a Kähler metric on the domain $\Omega$ associated with the Kähler form $\omega_\phi = \sqrt{-1}\pi \partial\bar{\partial}\phi$, and $g_F$ be a Kähler metric on the domain $M$ associated with the Kähler form $\omega_F = \sqrt{-1}\pi \partial\bar{\partial}\Phi_F$, where

$$\Phi_F(z, w) = \phi(z) + F(t)$$

with $t = \phi(z) + \ln \|w\|^2$ is a Kähler potential on a Hartogs domain

$$M = \left\{ (z, w) \in \Omega \times \mathbb{C}^{d_0} : \|w\|^2 < e^{-\phi(z)} \right\}.$$

Set

$$G = (d_0 - 1) \ln F' + \ln F'' + d \ln(1 + F'), \quad (2.12)$$

$$\psi_1 = -d \frac{G'}{1 + F'} + (d_0 - 1) \frac{d_0 - G'}{F'} - \frac{G''}{F''}, \quad (2.13)$$

and

$$\psi_2 = d \left( \frac{G'}{1 + F'} \right)^2 + (d_0 - 1) \left( \frac{G' - d_0}{F'} \right)^2 + \left( \frac{G''}{F''} \right)^2. \quad (2.14)$$

Then

$$k_{g_F} = \frac{1}{1 + F'} k_{g_\phi} + \psi_1, \quad (2.15)$$

$$|\text{Ric}_{g_F}|^2 = \frac{1}{(1 + F')^2} |\text{Ric}_{g_\phi}|^2 - \frac{2G'}{(1 + F')^2} k_{g_\phi} + \psi_2, \quad (2.16)$$

$$\triangle_{g_F} k_{g_F} = \frac{1}{(1 + F')^2} (\triangle_{g_\phi} k_{g_\phi}) + \frac{1}{F'} \frac{\partial^2 k_{g_F}}{\partial t^2} + \left( \frac{d}{1 + F'} + \frac{d_0 - 1}{F'} \right) \frac{\partial k_{g_F}}{\partial t}, \quad (2.17)$$

where $k_{g_\phi}$, $\Delta_{g_\phi}$ and $\text{Ric}_{g_\phi}$ denote the scalar curvature, the Laplace and the Ricci curvature on the domain $\Omega$ with respect to the metric $g_\phi$, respectively.
Then, we obtain (2.18) and

\[ \phi(z) = \|z\|^2 + \sum_{i,j,k,l=1}^d c_{ijkl} z_i \bar{z}_j z_k \bar{z}_l + O(\|z\|^5). \]

By Lemma 2.2, to compute \( k_{gf}(z,w) \), \( |\text{Ric}_{gf}|^2(z,w) \), \( \triangle_{gf} k_{gf}(z,w) \), we only need to calculate \( k_{gf}(0,w) \), \( |\text{Ric}_{gf}|^2(0,w) \) and \( \triangle_{gf} k_{gf}(0,w) \).

Using Lemma 2.1, we get

\[
T(0,w) = \begin{pmatrix}
(1+F')I_d & \frac{F'}{r}I_d \\
0 & \frac{F''}{r^2} I_d + \frac{F''-F'}{r^2} \bar{w}^i w^i \\
\end{pmatrix},
\]

(2.19)

and

\[
(0, w) = \begin{pmatrix}
(1+F')I_d & \frac{F'}{r}I_d \\
0 & \frac{F''}{r^2} I_d + \frac{F''-F'}{r^2} \bar{w}^i w^i \\
\end{pmatrix},
\]

(2.20)

\[
\ln \det T = G - d_0 \ln \|w\|^2 + \ln \left( \det \left( \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} \right) \right).
\]

(2.21)

Let

\[
\text{Ric}_{gf} = \frac{\partial^2 \ln \det T}{\partial \bar{Z}^i \partial Z^i}(0, w).
\]

From (2.18) and (2.21), it follows that

\[
\text{Ric}_{gf}(0,w) := \frac{\partial^2 \ln \det T}{\partial \bar{Z}^i \partial Z^i}(0, w)
\]

\[
= - \begin{pmatrix}
G'I_d - \text{Ric}_{gf}(0) & 0 \\
0 & \frac{G''-d_0}{F'} I_d + \frac{G''-F'}{F'} \bar{w}^i w^i \\
\end{pmatrix},
\]

which implies that

\[
(T^{-1} \text{Ric}_{gf})(0,w)
\]

\[
= - \begin{pmatrix}
G'I_d - \frac{1}{1+F'} \text{Ric}_{gf}(0) & 0 \\
0 & \frac{G''-d_0}{F'} I_d + \frac{d_0 F''+F'G''-F''G'}{F'} \bar{w}^i w^i \\
\end{pmatrix}.
\]

So

\[
k_{gf}(0,w) = \text{Tr}(T^{-1} \text{Ric}_{gf})(0,w) = \frac{1}{1+F'} k_{gf} + \psi_1(t)
\]

and

\[
|\text{Ric}_{gf}|^2(0,w) = \text{Tr}(T^{-1} \text{Ric}_{gf} T^{-1} \text{Ric}_{gf})(0,w)
\]

\[
= \frac{1}{(1+F')^2} k_{gf}^2(0) - 2 \frac{G'}{(1+F')^2} k_{gf}(0)
\]

\[
+ d \left( \frac{G'}{1+F'} \right)^2 + (d_0 - 1) \left( \frac{G''-d_0}{F'} \right)^2 + \left( \frac{G''}{F'} \right)^2,
\]

where

\[
k_{gf}(0) = \text{Tr}(\text{Ric}_{gf})(0), \quad |\text{Ric}_{gf}|^2(0) = \text{Tr}(\text{Ric}_{gf} \text{Ric}_{gf})(0).
\]

Then, we obtain (2.15) and (2.16).
Applying (2.15) and (2.18), we obtain

$$\frac{\partial^2 k_{gf}}{\partial Z^i \partial Z^j}(0, w)$$

$$= \left( -\frac{\partial k_{gf}}{\partial t} I_d + \frac{1}{1+F'} \frac{\partial^2 k_{gf}}{\partial Z^2}(0) - \frac{F''}{r^2(1+F')^2} \frac{\partial^2 k_{gf}}{\partial t^2}(0) \right)$$

$$- \frac{1}{r^2} \frac{\partial k_{gf}}{\partial t} I_d + \frac{1}{r^2} \left( \frac{\partial^2 k_{gf}}{\partial t^2} - \frac{\partial k_{gf}}{\partial t} \right) \bar{w}^i w_j,$$

here $Z = (z, w)$. Therefore

$$\frac{\partial^2 k_{gf}}{\partial Z^i \partial Z^j}(0, w)$$

$$= \text{Tr} \left( T^{-1} \frac{\partial^2 k_{gf}}{\partial Z^i \partial Z^j} \right) (0, w)$$

$$= \frac{d}{1+F'} \frac{\partial k_{gf}}{\partial t} + \frac{1}{(1+F')^2} \frac{\partial^2 k_{gf}}{\partial t^2} + \frac{d_0 - 1}{F'} \frac{\partial k_{gf}}{\partial t} + \frac{1}{F''} \frac{\partial^2 k_{gf}}{\partial t^2},$$

thus we get (2.17).

Let

$$x = F'(t), \ \varphi(x) = F''(t).$$

In Lemma 2.4 below, we obtain expressions in $\varphi(x)$ for the scalar curvature $k_{gf}$, the squared norm $|\text{Ric}_{gf}|^2$ of the Ricci curvature tensor, and the Laplace $\Delta_{gf} k_{gf}$ of the scalar curvature.

**Lemma 2.4.** Under assumptions of Theorem 2.3, let

$$x = F'(t), \ \varphi(x) = F''(t),$$

$$\sigma = \frac{((1+x)^d x^{d_0-1} \varphi)'}{(1+x)^{d x^{d_0-1}}},$$

and

$$\chi = \frac{d_0 (d_0 - 1)}{x} \frac{((1+x)^d x^{d_0-1} \varphi)''}{(1+x)^{d x^{d_0-1}}}.$$ (2.23)

Then

$$k_{gf} = \frac{1}{1+x} k_{g \phi} + \frac{d_0 (d_0 - 1)}{x} \frac{((1+x)^d x^{d_0-1} \varphi)''}{(1+x)^{d x^{d_0-1}}},$$ (2.24)

$$|\text{Ric}_{gf}|^2 = \frac{1}{(1+x)^2} |\text{Ric}_{g \phi}|^2 - 2 \frac{\sigma}{(1+x)^2} k_{g \phi} + (\sigma')^2$$

$$+ d \left( \frac{\sigma}{1+x} \right)^2 + (d_0 - 1) \left( \frac{\sigma - d_0}{x} \right)^2,$$ (2.25)

and

$$\Delta_{gf} k_{gf}$$

$$= \frac{1}{(1+x)^2} (\Delta_{g \phi} k_{g \phi}) - \frac{((\varphi(1+x)^{d-2} x^{d_0-1})')}{(1+x)^{d x^{d_0-1}}} k_{g \phi} + \frac{((\varphi(1+x)^{d} x^{d_0-1})')}{(1+x)^{d x^{d_0-1}}},$$ (2.26)

where $k_{g \phi}$, $\Delta_{g \phi}$ and $\text{Ric}_{g \phi}$ denote the scalar curvature, the Laplace and the Ricci curvature on the domain $\Omega$ with respect to the metric $g \phi$, respectively.
Proof. Using 

\[ x = F'(t), \ \varphi(x) = F''(t), \]

we give 

\[ F'''(t) = \varphi'(x) \frac{dx}{dt} = \varphi'(x) F''(t) = \varphi'(x) \varphi(x). \]

Then 

\[ G' = (d_0 - 1) \frac{F''}{F} + \frac{F'''}{F} + \frac{d}{1 + F'} = \frac{((1 + x)^d x^{d_0 - 1} \varphi)'}{(1 + x)^d x^{d_0 - 1}}. \]

Let 

\[ \sigma = \frac{((1 + x)^d x^{d_0 - 1} \varphi)'}{(1 + x)^d x^{d_0 - 1}}, \]

we have 

\[ G'(t) = \sigma(x) \] (2.27)

and 

\[ G''(t) = \sigma'(x) \frac{dx}{dt} = \sigma'(x) \varphi(x). \] (2.28)

By (2.27) and (2.28), we obtain 

\[ \psi_1 = -d \frac{G'}{1 + F'} + (d_0 - 1) \frac{d_0 - G'}{F'} - \frac{G''}{F'} \]

\[ = \frac{d_0 (d_0 - 1)}{x} - \frac{((1 + x)^d x^{d_0 - 1} \sigma)'}{(1 + x)^d x^{d_0 - 1}}, \]

which implies (2.24).

From (2.27) and (2.28), we also give 

\[ \psi_2 = d \left( \frac{G'}{1 + F'} \right)^2 + (d_0 - 1) \left( \frac{G' - d_0}{F'} \right)^2 + \left( \frac{G''}{F'} \right)^2 \]

\[ = \frac{d}{(1 + x)^2} \sigma^2 + (d_0 - 1) \left( \frac{\sigma - d_0}{x} \right)^2 + (\sigma')^2, \]

thus 

\[ |\text{Ric}_{g_F}|^2 \]

\[ = \frac{1}{(1 + F')^2} |\text{Ric}_{g_0}|^2 - \frac{2G'}{(1 + F')^2} k_{g_0} + \psi_2 \]

\[ = \frac{1}{(1 + x)^2} |\text{Ric}_{g_0}|^2 - 2 \frac{\sigma}{(1 + x)^2} k_{g_0} + (\sigma')^2 + d \frac{\sigma^2}{(1 + x)^2} + (d_0 - 1) \left( \frac{\sigma - d_0}{x} \right)^2. \]

Since 

\[ \frac{\partial k_{g_F}}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{1 + x} k_{g_0} + \chi \right) \frac{dx}{dt} = -\frac{\varphi}{(1 + x)^2} k_{g_0} + \varphi \chi' \]

and 

\[ \frac{\partial^2 k_{g_F}}{\partial t^2} = \left( - \left( \frac{\varphi}{(1 + x)^2} \right)' k_{g_0} + (\varphi \chi')' \right) \varphi, \]
then
\[
\begin{align*}
\triangle_{g_F} k_{g_F} &= \frac{1}{(1 + F')^2}(\triangle g_o k_{g_o}) + \frac{1}{F''} \frac{\partial^2 k_{g_F}}{\partial t^2} + \left(\frac{d}{1 + F'} + \frac{d_0 - 1}{F'}\right) \frac{\partial k_{g_F}}{\partial t} \\
&= \frac{1}{(1 + x)^2}(\triangle g_o k_{g_o}) - \left(\frac{dx + (d_0 - 1)(1 + x)}{x(1 + x)^3}\varphi + \left(\frac{\varphi}{(1 + x)^2}\right)'\right) k_{g_o} \\
&\quad + (\varphi'')' + \frac{dx + (d_0 - 1)(1 + x)}{x(1 + x)} x' \varphi'.
\end{align*}
\]

In order to obtain the coefficient \(a_2\) of the Bergman function expansion for \((M, g_F)\), we give a key Lemma 2.5 which gives an explicit expression of the squared norm \(|R_{g_F}|^2\) of the curvature tensor of the metric \(g_F\).

**Lemma 2.5.** Under the situation of Theorem 2.3, let
\[
t = \phi(z) + \ln \|w\|^2, \quad x = F'(t), \quad \varphi(x) = F''(t).
\]
Then
\[
|R_{g_F}|^2 = \frac{1}{(1 + x)^2}|R_{g_o}|^2 - \frac{4\varphi}{(1 + x)^3} k_{g_o} + 2d(d + 1) \frac{\varphi^2}{(1 + x)^4} + 4d \left(\frac{\varphi}{1 + x}\right)'^2 \\
+ (\varphi'')^2 + (d_0 - 1) \left\{4d \left(\frac{\varphi}{x(1 + x)}\right)^2 + 4 \left(\frac{\varphi'}{x}\right)^2 + 2d_0 \left(\frac{\varphi - x}{x^2}\right)^2\right\},
\]
where \(k_{g_o}\) and \(R_{g_o}\) denote the scalar curvature and the curvature tensor on the domain \(\Omega\) with respect to the metric \(g_o\), respectively.

**Proof.** Let \(Z = (z, w)\), \(T = \frac{\partial^2 k_{g_F}}{\partial Z \partial \bar{Z}}\),
\[
R_{g_F} \equiv (R_{ij}) := -\partial \bar{T} + (\partial T) T^{-1} \wedge (\bar{T})
\]
and
\[
R_{ij} = \sum_{k,l=1}^n R_{ijkl} dZ_k \wedge d\bar{Z}_l.
\]

Since the metric \(g_F\) is invariant under transformations
\[
(z, w) \in M \mapsto (z, wU) \in M, \quad U \in \mathcal{U}(d_0),
\]
where \(\mathcal{U}(d_0)\) indicates the unitary group of order \(d_0\), we only need to compute the squared norm of the curvature tensor of the metric \(g_F\) at \((z, w) = (0, w_1, 0, \cdots, 0)\).

Let \(\phi\) be given locally by
\[
\phi(z) = \|z\|^2 + \sum_{i,j,k,l=1}^d c_{ijkl} z_i \bar{z}_j z_k \bar{z}_l + O(\|z\|^5).
\]
Then
\[
\partial \phi(0) = 0, \quad \bar{\partial} \phi(0) = 0, \quad \partial \bar{\partial} \phi(0) = \sum_{k=1}^d dz_k \wedge d\bar{z}_k, \quad (\partial \frac{\partial \phi}{\partial z})(0) = 0
\]
and 
\[(\partial \frac{\partial \phi}{\partial z})(0) = dz, (\partial \frac{\partial \phi}{\partial z^t})(0) = (dz)^t, (\partial \frac{\partial \phi}{\partial z})(0) = 0, \partial \bar{\partial} (\partial \frac{\partial \phi}{\partial z^t})(0) = 0, \partial \bar{\partial} (\partial \frac{\partial \phi}{\partial z})(0) = 0.\]

The following we set \(z = 0, e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^d, w = w_1 e_1, du = (dw_2, dw_3, \ldots, dw_d), dw = (dw_1, du),\) we get
\[
\partial t = t,w_1 dw_1, \quad \bar{\partial} t = t,w_1 dw_1, \quad \partial \bar{\partial} t = dz \wedge (dz)^t + \frac{1}{r^2} du \wedge (du)^t,
\]
where \(r^2 = |w_1|^2.\)

By using Lemma 2.1 and
\[
x = F'(t), \quad \varphi(x) = F''(t), \quad F'''(t) = \varphi(x) \varphi'(x), \quad F^{(4)}(t) = \varphi^2(x) \varphi''(x) + \varphi(x) (\varphi'(x))^2,
\]
we have
\[
\partial T = \begin{array}{c|c|c|c}
1 & d & d_0 \\
\hline
0 & (\partial T)_1 & (\partial T)_2 \\
\hline
0 & (\partial T)_3 & (\partial T)_4 \\
\end{array}
\]
(2.32)
\[
(\partial T)_4 = \begin{array}{c}
1 \\
\hline
\end{array} \\
\begin{array}{c}
d_0-1 \\
\hline
\end{array} \\
\begin{array}{c|c|c|c}
\begin{array}{c}
F'' - F' \\
\hline
0 \\
\end{array} & \begin{array}{c}
\frac{e'' - e'}{r^4} dw_1 du \\
\hline
0 \\
\end{array} & \begin{array}{c}
dw_1 I_{d_0-1} \\
\hline
\end{array}
\end{array}
\]
(2.33)
\[
\bar{T} = \begin{array}{c|c|c|c}
1 & d & d_0 \\
\hline
0 & (\bar{T})_1 & (\bar{T})_2 \\
\hline
0 & (\bar{T})_3 & (\bar{T})_4 \\
\end{array}
\]
(2.34)
\[
(\bar{T})_4 = \begin{array}{c}
1 \\
\hline
\end{array} \\
\begin{array}{c}
d_0-1 \\
\hline
\end{array} \\
\begin{array}{c|c|c|c}
\begin{array}{c}
F'' - F' \\
\hline
0 \\
\end{array} & \begin{array}{c}
\frac{e'' - e'}{r^4} (dz)^t e_1 \\
\hline
0 \\
\end{array} & \begin{array}{c}
dw_1 I_{d_0-1} \\
\hline
\end{array}
\end{array}
\]
(2.35)
and

\[ \partial \overline{\partial} T = \frac{d}{d_0} \left( \frac{d}{(\partial \overline{\partial} T)_3} \frac{d_0}{(\partial \overline{\partial} T)_4} \right), \]  
\[ \text{(2.36)} \]

where

\[ (\partial \overline{\partial} T)_1 = \left\{ \frac{F''}{r^2} dz \wedge (d\bar{z})^t + \frac{F'''}{r^2} dw_1 \wedge d\overline{w_1} + \frac{F'''}{r^2} du \wedge (d\bar{u})^t \right\} I_d + (1 + F') \frac{d \partial}{\partial z} \frac{\partial^2 \phi}{\partial 
abla \bar{z}}(0) - F''(d\bar{z})^t \wedge dz \]
\[ = \left\{ \varphi dz \wedge (d\bar{z})^t + \frac{\varphi'}{r^2}dw_1 \wedge d\overline{w_1} + \frac{\varphi}{r^2} du \wedge (d\bar{u})^t \right\} I_d + (1 + x) \frac{d \partial}{\partial z} \frac{\partial^2 \phi}{\partial 
abla \bar{z}}(0) - \varphi(d\bar{z})^t \wedge dz, \]  
\[ \text{(2.37)} \]

\[ (\partial \overline{\partial} T)_2 = - \left\{ \frac{F''}{r^2} dw_1 \wedge (d\bar{z})^t e_1 + \frac{F''}{r^2} (d\bar{z})^t \wedge dw \right\} = - \left\{ \frac{\varphi(1 - \varphi')}{r^2}dw_1 \wedge (d\bar{z})^t e_1 + \frac{\varphi}{r^2} (d\bar{z})^t \wedge dw \right\}, \]  
\[ \text{(2.38)} \]

\[ (\partial \overline{\partial} T)_3 = - \left\{ \frac{F'' - F'''}{r^2} e_1^t dz \wedge d\overline{w_1} + \frac{F''}{r^2} (d\bar{w})^t \wedge dz \right\} = - \left\{ \frac{\varphi(1 - \varphi')}{r^2} e_1^t dz \wedge d\overline{w_1} + \frac{\varphi}{r^2} (d\bar{w})^t \wedge dz \right\} \]  
\[ \text{(2.39)} \]

and

\[ (\partial \overline{\partial} T)_4 = \frac{1}{d_0 - 1} \left( \frac{1}{(\partial \overline{\partial} T)_{41}} \frac{d_0 - 1}{(\partial \overline{\partial} T)_{43}} \right), \]  
\[ \text{(2.40)} \]

with

\[ (\partial \overline{\partial} T)_{41} = \frac{F''}{r^2} dz \wedge (d\bar{z})^t + \frac{F^{(4)} - 2F'' + F'}{r^4} dw_1 \wedge d\overline{w_1} + \frac{F''' - 2F'' + F'}{r^4} du \wedge (d\bar{u})^t \]

\[ = \frac{\varphi'}{r^2} dz \wedge (d\bar{z})^t + \frac{\varphi' \varphi + \varphi(1 - \varphi)^2}{r^4} dw_1 \wedge d\overline{w_1} + \frac{\varphi' - 2\varphi + x}{r^4} du \wedge (d\bar{u})^t, \]

\[ (\partial \overline{\partial} T)_{42} = \frac{F'' - F'''}{r^4} du \wedge d\overline{w_1} = \frac{\varphi' - 2\varphi + x}{r^4} du \wedge d\overline{w_1}, \]

\[ (\partial \overline{\partial} T)_{43} = \frac{F'' - F'''}{r^4} dw_1 \wedge (d\bar{w})^t = \frac{\varphi' - 2\varphi + x}{r^4} dw_1 \wedge (d\bar{w})^t \]

and

\[ (\partial \overline{\partial} T)_{44} = \left\{ \frac{F''}{r^2} dz \wedge (d\bar{z})^t + \frac{F'' - 2F'' + F'}{r^4} dw_1 \wedge d\overline{w_1} + \frac{F' - F'}{r^4} du \wedge (d\bar{u})^t \right\} I_{d_0 - 1} \]

\[ - \frac{F''}{r^4} (d\bar{w})^t \wedge du \]

\[ = \left\{ \frac{\varphi'}{r^2} dz \wedge (d\bar{z})^t + \frac{\varphi' - 2\varphi + x}{r^4} dw_1 \wedge d\overline{w_1} + \frac{\varphi - x}{r^4} du \wedge (d\bar{u})^t \right\} I_{d_0 - 1} \]

\[ - \frac{\varphi - x}{r^4} (d\bar{w})^t \wedge du. \]
According to (2.20), (2.32), (2.33), (2.34) and (2.35), we get

\[
((\partial T)^{-1} \wedge (\partial T)) = \frac{d}{d_0} \left( \frac{((\partial T)^{-1} \wedge (\partial T))_1}{((\partial T)^{-1} \wedge (\partial T))_1} \right)
\]

(2.41)

here

\[
((\partial T)^{-1} \wedge (\partial T))_1 = \frac{\varphi^2}{1 + x} dz \wedge (d\bar{z})^t + \varphi \frac{(1-x)^2}{r^4} du \wedge (d\bar{u})^t - \frac{(\varphi - x)^2}{x^4 r^4} dw \wedge (d\bar{w})^t
\]

\[
((\partial T)^{-1} \wedge (\partial T))_2 = \frac{\varphi^2}{1 + x} dz \wedge (d\bar{z})^t + \varphi \frac{(1-x)^2}{r^4} du \wedge (d\bar{u})^t - \frac{(\varphi - x)^2}{x^4 r^4} dw \wedge (d\bar{w})^t
\]

and

\[
((\partial T)^{-1} \wedge (\partial T))_3 = \frac{\varphi^2}{1 + x} dz \wedge (d\bar{z})^t
\]

Now (2.36) and (2.41) give

\[
R_{gf} = \frac{d}{d_0} \left( \frac{(R_{gf})_1}{(R_{gf})_1} \frac{(R_{gf})_2}{(R_{gf})_2} \frac{(R_{gf})_3}{(R_{gf})_3} \frac{(R_{gf})_4}{(R_{gf})_4} \right)
\]

(2.42)

where

\[
(R_{gf})_1 = \left\{ -\varphi dz \wedge (d\bar{z})^t - \frac{1}{r^2} \varphi \frac{(1-x)^2}{1+x} du \wedge (d\bar{u})^t - \frac{(\varphi - x)^2}{x^4 r^4} dw \wedge (d\bar{w})^t \right\} I_d
\]

\[
-(1 + x) \partial \bar{\partial} (\partial^2 \phi) (0) + \varphi (d\bar{z})^t \wedge dz
\]

(2.43)

\[
(R_{gf})_2 = -\frac{(1+x)}{r^2} \left( \frac{\varphi}{1+x} \right)' dw_1 \wedge (d\bar{z})^t e_1 + \frac{\varphi}{r^2} (d\bar{z})^t \wedge dw
\]

\[
= -\left( \frac{(1+x)}{r^2} \left( \frac{\varphi}{1+x} \right)' dw_1 \wedge (d\bar{z})^t + \frac{\varphi}{r^2} du \wedge (d\bar{u})^t \right)
\]

\[
\vdots
\]

\[
= -\left( \frac{(1+x)}{r^2} \left( \frac{\varphi}{1+x} \right)' dw_1 \wedge (d\bar{z})^t + \frac{\varphi}{r^2} du \wedge (d\bar{u})^t \right)
\]

(2.44)

\[
(R_{gf})_3 = -\frac{(1+x)}{r^2} \left( \frac{\varphi}{1+x} \right)' dz_1 \wedge (d\bar{w})^t + \frac{\varphi}{r^2} (d\bar{w})^t \wedge dz
\]

\[
= -\left( \frac{(1+x)}{r^2} \left( \frac{\varphi}{1+x} \right)' dz_1 \wedge (d\bar{w})^t + \frac{\varphi}{r^2} (d\bar{w})^t \wedge dz \right)
\]

(2.45)
(R_{gF})_4 = \frac{1}{d_0-1} \left( \frac{(R_{gF})_{41}}{(R_{gF})_{43}} \left( \frac{(R_{gF})_{42}}{(R_{gF})_{44}} \right) \right),

(2.46)

Here

\begin{align*}
(R_{gF})_{41} &= -\frac{(1+x)\varphi}{r^2} \left( \frac{\varphi}{1+x} \right)' dz \wedge (dz)^t - \frac{\varphi^2\varphi''}{r^4} dw_1 \wedge dw_1 \\
&\quad - \frac{x \varphi}{r^4} \left( \frac{\varphi}{x} \right)' du \wedge (d\bar{u})^t,
\end{align*}

\begin{align*}
(R_{gF})_{42} &= -\frac{x \varphi}{r^4} \left( \frac{\varphi}{x} \right)' du \wedge d\bar{w}^t,
\end{align*}

\begin{align*}
(R_{gF})_{43} &= -\frac{x \varphi}{r^4} \left( \frac{\varphi}{x} \right)' dw_1 \wedge (d\bar{u})^t
\end{align*}

and

\begin{align*}
(R_{gF})_{44} &= \left\{ -\frac{\varphi}{r^2} dz \wedge (dz)^t - \frac{x \varphi}{r^4} \left( \frac{\varphi}{x} \right)' du \wedge d\bar{w}^t - \frac{\varphi - x}{r^4} du \wedge (d\bar{u})^t \right\} I_{d_0-1} \\
&\quad + \frac{\varphi - x}{r^4} (d\bar{u})^t \wedge du.
\end{align*}

Set

\[ \phi_{ijkl} := \frac{\partial^4 \phi}{\partial z_i \partial z_j \partial z_k \partial z_l}, \]

then

\[ R_{g\phi}(0) = -\partial \bar{\partial} (\frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j})(0) = -\left( \sum_{k,l=1}^{d} \phi_{ijkl}(0) dz_k \wedge d\bar{z}_l \right), \]

(2.47)

Let

\[ R_{ij} = (R_{ijkl})_{1 \leq k,l \leq d+d_0}, \quad A_{ij} = \text{Tr} \left( T^{-1} R_{ij} T^{-1} \right), \]

Using

\[ R_{ijkl} = R_{kjl}, \quad R_{ijkl} = R_{ilkj}, \quad R_{ijkl} = R_{klij}, \quad R_{ijkl} = R_{ijkl}, \]

we get

\[ A_{ij} = A_{ji}. \]

By

\[ T^{-1} = \begin{pmatrix}
\frac{1}{1+x} I_d & 0 & 0 \\
0 & \frac{\varphi}{2} & 0 \\
0 & 0 & \frac{\varphi^2}{2} I_{d_0-1}
\end{pmatrix}, \]

at (0, w), we have

\[ |R_{g\phi}|^2 = \sum_{t_1,t_2,t_3,t_4=1}^{d+d_0} \sum_{i,j,k,l=1}^{d+d_0} T^t_{t_1} T^t_{t_2} T^t_{t_3} T^t_{t_4} R_{t_1 t_2 t_3 t_4} R_{ijkl}, \]

\[ = \sum_{i,j=1}^{d+d_0} (T^{ij} T^{\bar{ij}}) \sum_{k,l=1}^{d+d_0} (T^{kk} T^{\bar{k}\bar{k}}) |R_{ijkl}|^2 \]

\[ = \sum_{i,j=1}^{d+d_0} (T^{ij} T^{\bar{ij}}) A_{ij}, \]
thus
\[
|R_{ij}|^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{1}{(1+x)^2} A_{ij} + \sum_{i=1}^{d} \frac{r^2}{(1+x)\varphi} A_{d+1} + \sum_{i=1}^{d} \sum_{j=d+2}^{d+d_0} \frac{r^2}{(1+x)} A_{ij}
\]
\[+ \frac{r^4}{x^2} A_{d+d+1} + 2 \sum_{j=d+2}^{d+d_0} \frac{r^4}{x^2} A_{d+1} + \sum_{i=d+2}^{d+d_0} \sum_{j=d+2}^{d+d_0} \frac{r^4}{x^2} A_{ij}.\]  

(2.48)

Let \( E_{ij} \) be the \( d \times d \) matrix whose \((i, j)\) entry is 1, other entries are 0; \( E_{ij}^\phi \) be the \((d + d_0) \times (d + d_0)\) matrix whose \((i, j)\) entry is 1, other entries are 0. Set \( R_{ij}^\phi = (\phi_{ijl}(0))_{1 \leq k,l \leq d}. \)

From (2.42), (2.43), (2.44), (2.45) and (2.46), it follows that

(i) For \( 1 \leq i = j \leq d, \)
\[
R_{ii} = \begin{pmatrix}
-\varphi I_d - (1+x)R_{ii}^\phi & 0 \\
0 & -\frac{(1+x)\varphi}{r^2} \\
0 & 0 \\
-\frac{x}{r} I_{d_0-1}
\end{pmatrix},
\]
so
\[
A_{ii} = \frac{1}{(1+x)^2} \text{Tr} \left\{ \left( \varphi I_d + (1+x)R_{ii}^\phi + \varphi E_{ii} \right) \left( \varphi I_d + (1+x)(R_{ii}^\phi)^t + \varphi E_{ii} \right) \right\}
\]
\[+ \frac{r^4}{x^2} \left( \frac{\varphi}{1+x} \right)^2 + (d_0 - 1) \frac{r^4}{x^2} \frac{\varphi^2}{r^4}
\]
\[= \frac{(d + 3)\varphi^2}{(1+x)^2} + \frac{2\varphi}{1+x} \sum_{k=1}^{d} \phi_{iikk}(0) + \sum_{k,l=1}^{d} |\phi_{ijkl}(0)|^2 + \frac{2\varphi}{1+x} \phi_{i\bar{i}\bar{i}\bar{i}}(0)
\]
\[+ \left( (1+x) \left( \frac{\varphi}{1+x} \right)^2 \right) + (d_0 - 1) \frac{\varphi^2}{x^2}.\]

(ii) For \( 1 \leq i \neq j \leq d, \)
\[
R_{ij} = \begin{pmatrix}
-(1+x)R_{ij}^\phi - \varphi E_{ij} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix},
\]
then
\[
A_{ij} = \frac{1}{(1+x)^2} \text{Tr} \left\{ \left( 1+x \right) R_{ij}^\phi + \varphi E_{ij} \right\} \left( (1+x)(R_{ij}^\phi)^t + \varphi E_{ij} \right) \right\}
\]
\[= \frac{\varphi^2}{(1+x)^2} + \frac{2\varphi}{1+x} \phi_{ij\bar{i}}(0) + \sum_{k,l=1}^{d} |\phi_{ijkl}(0)|^2.\]

(iii) For \( 1 \leq i \leq d, j = d + 1 \)
\[
R_{i(d+1)} = -\frac{\varphi}{r^2} \left( \frac{\varphi}{1+x} \right)^t E_{d+1},
\]
thus
\[
A_{i(d+1)} = \frac{r^2}{(1+x)\varphi} \left( \frac{\varphi}{r^2} \left( \frac{\varphi}{1+x} \right)^t \right)^2 = \frac{(1+x)\varphi}{r^2} \left( \left( \frac{\varphi}{1+x} \right)^t \right)^2.
\]

(iv) For \( 1 \leq i \leq d, d + 2 \leq j \leq d + d_0, \)
\[
R_{ij} = -\frac{\varphi}{r^2} E_{ij},
\]
we get

\[ A_{ij} = \frac{r^2}{x(1+x)} \left( \frac{\varphi}{r^2} \right)^2 = \frac{\varphi^2}{x(1+x)} r^2. \]

(v) For \( i = j = d + 1 \), since

\[
\mathbf{R}_{d+1}^T = \begin{pmatrix}
-\frac{(1+x)\varphi}{r^2} \left( \frac{\varphi}{1+x} \right)' I_d & 0 & 0 \\
0 & -\frac{\varphi^2}{x^2} & 0 \\
0 & 0 & -\frac{x}{r^2} \left( \frac{\varphi}{x} \right)' I_{d-1}
\end{pmatrix},
\]

therefore

\[
A_{d+1} = d \frac{1}{(1+x)^2} \left( \frac{1+x}{r^2} \left( \frac{\varphi}{1+x} \right)' \right)^2 + \frac{r^4}{\varphi^2} \left( \frac{\varphi}{r^4} \right)^2 + (d_0 - 1) \frac{r^4}{x^2} \left( \frac{x \varphi}{r^4} \right)^2 \left( \frac{\varphi}{x} \right)'^2
\]

\[
= \frac{\varphi^2}{r^4} \left\{ d \left( \frac{\varphi}{1+x} \right)'^2 + \left( \frac{\varphi'}{x} \right)^2 + (d_0 - 1) \left( \frac{\varphi}{x} \right)'^2 \right\}.
\]

(vi) For \( i = d + 1, d + 2 \leq j \leq d + d_0 \), using

\[
\mathbf{R}_{d+1} = -\frac{x \varphi}{r^4} \left( \frac{\varphi}{x} \right)' \mathbf{E}_{d+1},
\]

we have

\[
A_{d+1} = \frac{r^4}{x^2} \left( \frac{x \varphi}{r^4} \right)^2 \left( \frac{\varphi}{x} \right)'^2 = \frac{x \varphi}{r^4} \left( \frac{\varphi}{x} \right)'^2.
\]

(vii) For \( d + 2 \leq i = j \leq d + d_0 \), by

\[
\mathbf{R}_{ij} = \begin{pmatrix}
-\frac{\varphi}{r^2} I_d & 0 & 0 \\
0 & -\frac{x \varphi}{r^2} \left( \frac{\varphi}{x} \right)' & 0 \\
0 & 0 & -\frac{x}{r^2} \left( \frac{\varphi}{x} \right)' I_{d-1}
\end{pmatrix},
\]

we obtain

\[
A_{ij} = d \frac{1}{(1+x)^2} \left( \frac{\varphi^2}{r^2} \right) + \frac{r^4}{\varphi^2} \left( \frac{x \varphi}{r^4} \right)^2 \left( \frac{\varphi}{x} \right)'^2 + (d_0 + 2) \frac{r^4}{x^2} \left( \frac{x \varphi}{r^4} \right)^2 \left( \frac{\varphi}{x} \right)'^2
\]

\[
= \frac{1}{r^4} \left\{ d \left( \frac{\varphi}{1+x} \right)^2 + \left( \frac{x \varphi}{x} \right)'^2 + (d_0 + 2) \left( \frac{\varphi}{x} \right)'^2 \right\}.
\]

(viii) For \( d + 2 \leq i \neq j \leq d + d_0 \), from

\[
\mathbf{R}_{ij} = -\frac{\varphi}{r^4} \left( \frac{\varphi}{x} \right)' \mathbf{E}_{ij},
\]

then

\[
A_{ij} = \frac{r^4}{x^2} \left( \frac{\varphi}{x} \right)'^2 = \frac{1}{r^4} \left( \frac{\varphi}{x} \right)'^2.
\]

Combining the above (i) - (viii), from (2.48), we have

\[
|R_{\varphi x}|^2 = \frac{4 \varphi}{(1+x)^3} \sum_{i=1}^{d} \sum_{k=1}^{d} \phi_{i k k}(0) + 2d(d + 1) \frac{\varphi^2}{(1+x)^4}
\]

\[
+ 4d \left( \frac{\varphi}{1+x} \right)^2 + \left( \frac{\varphi'}{x} \right)^2 + 4d(d_0 - 1) \left( \frac{\varphi}{x(1+x)} \right)^2
\]

\[
+ 4(d_0 - 1) \left( \frac{x \varphi}{x^2} \right)^2 + 2d_0(d_0 - 1) \left( \frac{\varphi}{x^2} \right)^2.
\]
Since
\[ k_\phi(0) = -\sum_{i,k=1}^d \phi_{ik}(0), \quad |R_{g_\phi}|^2(0) = \sum_{i,j,k,l=1}^d |\phi_{ijkl}(0)|^2, \]
we get
\[ |R_{g_F}|^2 = \frac{1}{(1+x)^2}|R_{g_\phi}(0)|^2 - \frac{4\varphi}{(1+x)^2} k_\phi(0) + 2d(d+1)\frac{\varphi^2}{(1+x)^2} + 4d \left( \frac{\varphi}{1+x} \right)^2 \]
\[ + (\varphi'')^2 + (d_0-1) \left\{ 4d \left( \frac{\varphi}{x(1+x)} \right)^2 + 4 \left( \frac{\varphi'}{x} \right)^2 + 2d_0 \left( \frac{\varphi-x}{x^2} \right)^2 \right\}, \]
which completes the proof of (2.29). □

Applying Lemma 2.4 and Lemma 2.5, we obtain explicit expressions of the coefficients \( a_j \) \((j = 1, 2)\) of the Bergman function expansion for \((M, g_F)\).

**Theorem 2.6.** Assume that \( \phi \) is a globally defined Kähler potential on a domain \( \Omega \subset \mathbb{C}^d \). Let \( g_\phi \) be a Kähler metric on the domain \( \Omega \) associated with the Kähler form \( \omega_\phi = \sqrt{\pi} \partial \bar{\partial} \phi \), and \( g_F \) be a Kähler metric on the domain \( M \) associated with the Kähler form \( \omega_F = \sqrt{\pi} \partial \bar{\partial} \Phi_F \), where
\[ \Phi_F(z, w) = \phi(z) + F(\phi(z) + \ln \|w\|^2) \]
is a Kähler potential on a Hartogs domain
\[ M = \left\{ (z, w) \in \Omega \times \mathbb{C}^{d_0} : \|w\|^2 < e^{-\phi(z)} \right\}. \]

Set
\[ t = \phi(z) + \ln \|w\|^2, \quad x = F'(t), \quad \varphi(x) = F''(t), \]
\[ \sigma(x) = \frac{(1+x)^d x^{d-1} \varphi(x)}{(1+x)^d x^{d-1}}, \]
and
\[ \chi(x) = \frac{d_0(d_0-1)}{x} - \frac{(1+x)^d x^{d-1} \varphi(x)}{(1+x)^d x^{d-1}}. \]

Let \( k_{g_\phi}, \Delta_{g_\phi}, \text{Ric}_{g_\phi} \) and \( R_{g_\phi} \) be the scalar curvature, the Laplace, the Ricci curvature and the curvature tensor on the domain \( \Omega \) with respect to the metric \( g_\phi \), respectively. Put
\[ a_1 = \frac{1}{2} k_{g_\phi}, \quad a_2 = \frac{1}{3} \Delta_{g_\phi} k_{g_\phi} + \frac{1}{24} |R_{g_\phi}|^2 - \frac{1}{6} |\text{Ric}_{g_\phi}|^2 + \frac{1}{8} k_{g_\phi}^2, \]
and
\[ a_1 = \frac{1}{2} k_{g_F}, \quad a_2 = \frac{1}{3} \Delta_{g_F} k_{g_F} + \frac{1}{24} |R_{g_F}|^2 - \frac{1}{6} |\text{Ric}_{g_F}|^2 + \frac{1}{8} k_{g_F}^2. \]
Then
\[ a_1 = \frac{1}{1+x} a_1 + \frac{d_0(d_0-1)}{2x} - \frac{(1+x)^d x^{d-1} \varphi''}{2(1+x)^d x^{d-1}} \]
(2.49)
and

\[
\begin{align*}
\mathbf{a}_2 &= \frac{1}{(1 + x)^2} a_2 + \left\{ \frac{1}{2(1 + x)} \chi + \frac{\varphi}{(1 + x)^3} \right\} a_1 \\
&\quad + \frac{1}{24} \left\{ 8(\varphi \chi')' + 3\chi^2 - 4(\sigma')^2 - \frac{4d}{(1 + x)^2} \sigma^2 + (\varphi'')^2 \\
&\quad + 4d \left( \left( \frac{\varphi}{1 + x} \right)' \right)^2 + \frac{d + d_0 - 1}{x(1 + x)} \varphi \chi' + \frac{2d(d + 1)}{(1 + x)^4} \varphi^2 \right\} \\
&\quad + \frac{d_0 - 1}{6} \left\{ \frac{d\varphi^2}{x^2(1 + x)^2} + \left( \frac{\varphi}{x} \right)'^2 + \frac{d_0 (\varphi - x)^2}{2x^4} - \frac{(\sigma - d_0)^2}{x^2} \right\}.
\end{align*}
\] (2.50)

Theorem 2.6 implies the following results.

**Theorem 2.7.** Assume that

\[ \varphi(x) = x(1 + x), \]

in Theorem 2.6, then

(I)

\[
\begin{align*}
\mathbf{a}_1 &= \left( a_1 + \frac{d(d + 1)}{2} \right) \frac{1}{1 + x} - \frac{n(n + 1)}{2},
\end{align*}
\] (2.51)

and

\[
\begin{align*}
\mathbf{a}_2 &= \left\{ a_2 + \frac{(d - 1)(d + 2)}{2} \left( a_1 + \frac{d(d + 1)}{2} \right) - \frac{(d - 1)d(d + 1)(3d + 2)}{24} \right\} \\
&\quad \times \frac{1}{(1 + x)^2} - \frac{(n - 1)(n + 2)}{2} \left( a_1 + \frac{d(d + 1)}{2} \right) \frac{1}{1 + x} \\
&\quad + \frac{(n - 1)n(n + 1)(3n + 2)}{24},
\end{align*}
\] (2.52)

where \( n = d + d_0 \).

(II) \( \mathbf{a}_1 \) is a constant \( \iff \) \( a_1 = -\frac{d(d + 1)}{2} \iff \mathbf{a}_1 = -\frac{n(n + 1)}{2} \).

(III) \( \mathbf{a}_2 \) is a constant \( \iff \) \( a_2 = \frac{(n - 1)n(n + 1)(3n + 2)}{24} \iff a_1 = -\frac{d(d + 1)}{2} \) and \( a_2 = \frac{(d - 1)d(d + 1)(3d + 2)}{24} \).

**Theorem 2.8.** Assume that \( d = 1 \) and

\[ \varphi(x) = Ax^2 + x \]

in Theorem 2.6, then

(I)

\[
\mathbf{a}_1 = \frac{a_1 - d_0 + A(d_0 + 1)}{1 + x} - \frac{n(n + 1)}{2} A.
\]

If \( a_1 = d_0 - (d_0 + 1)A \), then

\[
\mathbf{a}_2 = \frac{a_2}{(1 + x)^2} + \frac{(n - 1)n(n + 1)(3n + 2)}{24} A^2.
\]

Here \( n = 1 + d_0 \).

(II) \( \mathbf{a}_1 \) is a constant \( \iff \) \( a_1 = d_0 - (d_0 + 1)A \iff a_1 = -\frac{n(n + 1)}{2} A. \)

(III) Both \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) are constants \( \iff \) \( a_1 = d_0 - (d_0 + 1)A \) and \( a_2 = 0 \iff a_1 = -\frac{n(n + 1)}{2} A \) and \( a_2 = \frac{(n - 1)n(n + 1)(3n + 2)}{24} A^2. \)
3 The Kähler metrics with constant coefficients $a_j \ (j = 1, 2)$

In this section, we first give an explicit expression of the function $\varphi(x)$ for the coefficients $a_j \ (j = 1, 2)$ to be constants, then we get an explicit expression of $F(t)$ by solving the differential equation. Finally we complete the proof of Theorem 1.1.

**Theorem 3.1.** Assume that $\phi$ is a globally defined Kähler potential on a domain $\Omega \subset \mathbb{C}^d$. Let $g_\phi$ be a Kähler metric on the domain $\Omega$ associated with the Kähler form $\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$.

Let $g_F$ be a Kähler metric on the Hartogs domain

$$M = \left\{ (z,w) \in \Omega \times \mathbb{C}^{d_0} : \|w\|^2 < e^{-\phi(z)} \right\}$$

associated with the Kähler form

$$\omega_F = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Phi_F,$$

where $\| \cdot \|$ is the standard Hermite norm in $\mathbb{C}^{d_0}$ and

$$\Phi_F(z,w) = \phi(z) + F(t), \ t = \phi(z) + \ln \|w\|^2,$$

where $F(t)$ satisfies the following conditions:

(i) \[ \lim_{t \to -\infty} F(t) = 0. \] \hspace{1cm} (3.1)

(ii) For the case of $d_0 > 1$,

$$0 < \lim_{t \to -\infty} \frac{F'(t)}{e^t} < +\infty, \ 0 < \lim_{t \to -\infty} \frac{F''(t)}{e^t} < +\infty, \ F'(t) > 0, \ F''(t) > 0, \ t \in (-\infty, 0). \] \hspace{1cm} (3.2)

(iii) For the case of $d_0 = 1$,

$$0 < \lim_{t \to -\infty} (1 + F'(t)) < +\infty, \ 0 < \lim_{t \to -\infty} \frac{F''(t)}{e^t} < +\infty, \ 1 + F'(t) > 0, \ F''(t) > 0, \ t \in (-\infty, 0). \] \hspace{1cm} (3.3)

Let $k_{g_\phi}$, $\Delta_{g_\phi}$, $\text{Ric}_{g_\phi}$ and $R_{g_\phi}$ be the scalar curvature, the Laplace, the Ricci curvature and the curvature tensor on the domain $\Omega$ with respect to the metric $g_\phi$, respectively. Put

$$a_1 = \frac{1}{2} k_{g_\phi}, \ a_2 = \frac{1}{3} \Delta_{g_\phi} k_{g_\phi} + \frac{1}{24} |R_{g_\phi}|^2 - \frac{1}{6} |\text{Ric}_{g_\phi}|^2 + \frac{1}{8} k_{g_\phi}^2$$

and

$$a_1 = \frac{1}{2} k_{g_F}, \ a_2 = \frac{1}{3} \Delta_{g_F} k_{g_F} + \frac{1}{24} |R_{g_F}|^2 - \frac{1}{6} |\text{Ric}_{g_F}|^2 + \frac{1}{8} k_{g_F}^2.$$

Set

$$x = F'(t), \ \varphi(x) = F''(t),$$

then we have the following results.

(I) For $d_0 = 1$, let

$$a_1 = -\frac{1}{2} (d+1)B, \ a_1 = -\frac{1}{2} (d+1)(d+2)A.$$  

$a_1$ is a constant if and only if $a_1$ is a constant and

$$\varphi(x) = A(1+x)^2 - B(1+x) + \frac{C_1}{1+x} + \frac{C_2}{(1+x)^2}, \hspace{1cm} (3.4)$$

where $C_1$ and $C_2$ are constants.
(II) For \(d_0 > 1\) and \(d = 1\), \(a_1\) is a constant if and only if \(a_1\) is a constant and

\[
\varphi(x) = \left(\frac{d_0 - a_1}{d_0 + 1} + \frac{d_0}{2} C\right) x^2 + (C + 1)x - C + \frac{C}{1 + x},
\]

here \(C\) is a constant.

(III) For \(d_0 = 1\), let \(a_1 = -\frac{1}{d}(d + 1)B\), \(a_1 = -\frac{1}{2}(d + 1)(d + 2)A\).

If \(a_1\) and \(a_2\) are constants, then \(a_1\) and \(a_2\) are constants,

\[
a_2 = \frac{1}{24}(d - 1)d(d + 1)(3d + 2)B^2, \quad a_2 = \frac{1}{24}d(d + 1)(d + 2)(3d + 5)A^2
\]

and

\[
\varphi(x) = A(1 + x)^2 - B(1 + x)
\]

for \(d > 1\),

\[
\varphi(x) = A(1 + x)^2 - B(1 + x) + C
\]

for \(d = 1\), here \(A, B, C\) are constants.

(IV) For \(d_0 > 1\) and \(d = 1\), let \(A = \frac{d_0 - a_1}{d_0 + 1}\). If \(a_1\) and \(a_2\) are constants, then \(a_1\) is a constant,

\[
a_2 = 0, \quad a_1 = -A\frac{(d_0 + 1)(d_0 + 2)}{2}, \quad a_2 = A^2\frac{d_0(d_0 + 1)(d_0 + 2)(3d_0 + 5)}{24}
\]

and

\[
\varphi(x) = Ax^2 + x.
\]

(V) For \(d_0 > 1\) and \(d > 1\), let \(n = d + d_0\). If \(a_1\) and \(a_2\) are constants, then

\[
\varphi(x) = x(x + 1),
\]

\[
a_1 = -\frac{d(d + 1)}{2}, \quad a_2 = \frac{(d - 1)d(d + 1)(3d + 2)}{24}
\]

and

\[
a_1 = -\frac{n(n + 1)}{2}, \quad a_2 = \frac{(n - 1)n(n + 1)(3n + 2)}{24}.
\]

Proof. (I) Using (2.49), we get (3.4).

(II) For \(d_0 > 1\) and \(d = 1\), if \(a_1\) is a constant, from (2.49), we obtain \(a_1\) is a constant, and \(\varphi(x)\) can be written as

\[
\varphi(x) = \sum_{j=0}^{2} A_j(1 + x)^j + \sum_{j=1}^{d_0 - 1} \frac{B_j}{x^j} + \frac{C}{1 + x}.
\]

Since

\[
\lim_{t \to -\infty} F'(t) = \lim_{t \to -\infty} F''(t) = 0,
\]

we have \(\varphi(0) = 0\), thus

\[
B_j = 0 \quad (1 \leq j \leq d_0 - 1), \quad \sum_{j=0}^{2} A_j + C = 0.
\]
By (2.49), we get
\[
\frac{((1 + x)^d x^{d_0 - 1} \varphi(x))'' - d_0(d_0 - 1)}{x} - \frac{2a_1}{1 + x} + 2a_1 = (1 + d_0)(2 + d_0)A_2 + 2a_1 + d_0(d_0 - 1) \frac{2A_2 + A_1 - C - 1}{x} + 2A_1 - 2(d_0 - 1)A_0 - 2a_1 + (d_0 - 1)(d_0 - 2)C
\]
\[
= 0,
\]

thus
\[
a_1 = -\frac{1}{2}(1 + d_0)(2 + d_0)A_2, \quad 2A_2 + A_1 - C - 1 = 0
\]

and
\[
2A_1 - 2(d_0 - 1)A_0 - 2a_1 + (d_0 - 1)(d_0 - 2)C = 0.
\]

In view of
\[
A_0 + A_1 + A_2 + C = 0,
\]
we obtain
\[
A_2 = \frac{1}{2}d_0C + \frac{d_0 - a_1}{d_0 + 1},
\]
then
\[
\varphi(x) = \sum_{j=0}^{2} A_j(1 + x)^j + \frac{C}{1 + x}
\]
\[
= A_2x^2 + (A_1 + 2A_2)x + (A_0 + A_1 + A_2) + \frac{C}{1 + x}
\]
\[
= \left(\frac{d_0 - a_1}{d_0 + 1} + \frac{d_0}{2}C\right)x^2 + (C + 1)x - C + \frac{C}{1 + x}.
\]

(III) By (2.49) and (2.50), if \(a_1\) and \(a_2\) are constants, then \(a_1\) and \(a_2\) are constants, thus \(|R_{g_0}|^2 - 4|\text{Ric}_{g_0}|^2\) and \(|R_{g_F}|^2 - 4|\text{Ric}_{g_F}|^2\) also are constants.

Substituting (3.4) into (2.29) and (2.25), we have
\[
\frac{|R_{g_F}|^2 - 4|\text{Ric}_{g_F}|^2}{(1 + x)^2} = \left(\frac{\sigma}{1 + x}\right)^2 - \left(\frac{4\varphi}{1 + x}\right)^2 + 4d \left(\frac{\varphi'}{1 + x}\right)^2 + \left(\varphi''\right)^2 - 4 \left(\sigma'\right)^2 - \frac{4d(1 + x)^2}{(1 + x)}
\]
\[
= \frac{A_{2d+4}}{(1 + x)^{2d+4}} + \frac{A_{2d+3}}{(1 + x)^{2d+3}} + \frac{A_{2d+2}}{(1 + x)^{2d+2}} + \frac{A_2}{(1 + x)^2} + A_0,
\]
where
\[
A_{2d+4} = d(d + 1)(d + 2)(d + 3)C_2^2, \quad A_{2d+3} = 2d(d + 1)^2(d + 2)C_1, \quad A_{2d+2} = (d - 1)d(d + 1)(d + 2)C_1^2, \quad A_2 = |R_{g_0}|^2 - 4|\text{Ric}_{g_0}|^2 + \frac{4d + 2}{d(d + 1)}k_{g_0}^2
\]
and
\[
A_0 = -2(d + 1)(d + 2)(2d + 3)A^2.
\]
From \(A_{2d+4} = 0\), we have \(C_2 = 0\), so \(A_{2d+3} = 0\).
For $d > 1$, by $A_{2d+2} = 0$, it follows that $C_1 = 0$.
For $d = 1$, for any $C_1$, $A_{2d+2} = 0$.
Substituting $\varphi(x) = A(1 + x)^2 - B(1 + x)$ and $\varphi(x) = A(1 + x)^2 - B(1 + x) + C$ into (2.50), we obtain
\[
a_2 = \frac{a_2 - \frac{1}{24}(d - 1)d(d + 1)(3d + 2)B^2}{(1 + x)^2} + \frac{d(d + 1)(d + 2)(3d + 5)A^2}{24}
\]
for $d > 1$, and
\[
a_2 = 2A^2 + \frac{a_2}{(x + 1)^2}
\]
for $d = 1$, respectively. This gives (3.6).
(IV) Using (3.5) and (2.50), we get
\[
a_2 = \frac{C^2}{(1 + x)^6} + \sum_{j=0}^{5} \frac{F_j}{(1 + x)^j},
\]
thus
\[
C = 0, \quad \varphi(x) = \frac{d_0 - a_1}{d_0 + 1}x^2 + x.
\]
Then put them into (2.49) and (2.50), we have
\[
a_1 = \frac{(a_1 - d_0)(d_0 + 2)}{2}
\]
and
\[
a_2 = \frac{a_2}{(1 + x)^2} + \frac{d_0(d_0 + 2)(3d_0 + 5)(a_1 - d_0)^2}{24(d_0 + 1)}.
\]
Let
\[
A = \frac{d_0 - a_1}{d_0 + 1},
\]
then
\[
a_2 = 0, \quad a_1 = -A\frac{(d_0 + 1)(d_0 + 2)}{2}, \quad a_2 = A^2\frac{d_0(d_0 + 1)(d_0 + 2)(3d_0 + 5)}{24}.
\]
(V) For $d > 1$ and $d_0 > 1$, according to (2.49), we obtain
\[
(1 + x)^d x^{d_0-1} \varphi'(x) = 2a_1(1 + x)^{d-1} x^{d_0-1} + d_0(d_0 - 1)(1 + x)^d x^{d_0-2} - 2a_1(1 + x)^d x^{d_0-1},
\]
using integration by parts, we obtain
\[
(1 + x)^d x^{d_0-1} \varphi(x) = \sum_{j=0}^{d_0-1} c_{j1} x^j (1 + x)^{d+d_0-j} + \sum_{j=0}^{d_0-1} c_{j2} x^j (1 + x)^{d+d_0+1-j} + (-1)^{d_0-1}(C_1 x + C_2),
\]
thus
\[
\varphi(x) = \sum_{j=0}^{d_0-1} \frac{d_1}{x^j} (1 + x)^{j+1} + \sum_{j=0}^{d_0-1} \frac{d_2}{x^j} (1 + x)^{j+2} + (-1)^{d_0-1} \frac{C_1 x + C_2}{(1 + x)^{d_0-1}},
\]
which can be written as
\[
\varphi(x) = \sum_{j=0}^{2} A_j (x + 1)^j + \sum_{j=1}^{d_0-1} \frac{B_j(a_1, a_1, d, d_0)}{x^j} + (-1)^{d_0-1} \frac{C_1 x + C_2}{x^{d_0-1}(1 + x)^d}.
\]
Owing to $\varphi(0) = 0$ and
\[
(-1)^{d_0-1} \frac{C_1 x + C_2}{x^{d_0-1}(1 + x)^d} = \sum_{j=1}^{d_0-1} \frac{D_j}{x^j} + \frac{C_1 - C_2}{(1 + x)^d} \frac{(d_0 - 1)C_2 - (d_0 - 2)C_1}{(1 + x)^{d-1}} + \sum_{j=1}^{d-2} \frac{E_j}{(1 + x)^j},
\]
we have
\[
\varphi(x) = \sum_{j=0}^{2} A_j (x + 1)^j + \frac{C_1 - C_2}{(1 + x)^d} \frac{(d_0 - 1)C_2 - (d_0 - 2)C_1}{(1 + x)^{d-1}} + \sum_{j=1}^{d-2} \frac{E_j}{(1 + x)^j}.
\]

From (2.25) and (2.29), we have
\[
|R_{g_F}|^2 - 4|Ric_{g_F}|^2 = \frac{d(d + 1)(d + 2)(d + 3)(C_1 - C_2)^2}{(1 + x)^{2d+4}} + \sum_{j=1}^{2d+3} \frac{F_j}{(1 + x)^j} + \sum_{j=0}^{4} \frac{G_j}{x^j},
\]
which implies that
\[
C_1 = C_2,
\]
so
\[
|R_{g_F}|^2 - 4|Ric_{g_F}|^2 = \frac{(d - 1)d(d + 1)(d + 2)C_1^2}{(1 + x)^{2d+2}} + \sum_{j=1}^{2d+1} \frac{F_j}{(1 + x)^j} + \sum_{j=0}^{4} \frac{G_j}{x^j},
\]
therefore
\[
C_1 = C_2 = 0,
\]
namely
\[
\varphi(x) = \sum_{j=0}^{2} A_j (1 + x)^j.
\]

By (3.9) and (2.49), we obtain
\[
a_1 = \frac{d(d - 1)}{2(1 + x)^2} A_0 + \frac{H_1}{x} + \frac{H_2}{x^2} + \frac{H_3}{1 + x} + H_0,
\]
since $a_1$ is a constant, then
\[
A_0 = 0,
\]
which combines with $\varphi(0) = 0$, we get $A_1 + A_2 = 0$, thus $\varphi(x) = A_2 x (1 + x)$.

Substituting
\[
\varphi(x) = A_2 x (1 + x)
\]
into (2.49), we have
\[
a_1 = -\frac{(d + d_0)(d + d_0 + 1)}{2} A_2 + \frac{d(d + 1)A_2 + 2a_1}{2(x + 1)} - \frac{(d_0 - 1)d_0(A_2 - 1)}{2x},
\]
notice that $d_0 > 1$, thus
\[
A_2 = 1, \ a_1 = -\frac{d(d + 1)}{2}, \ a_1 = -\frac{(d + d_0)(d + d_0 + 1)}{2}.
\]

Finally, from (2.52), we obtain
\[
a_2 = \frac{(d - 1)(d + 1)(3d + 2)}{24}, \ a_2 = \frac{(n - 1)(n + 1)(3n + 2)}{24},
\]
where $n = d + d_0$. □
Theorem 3.2. Under assumptions of Theorem 3.1, we have the following results.

(i) For \( d = 1 \), if \( a_1 \) and \( a_2 \) are constants, then
\[
\varphi(x) = Ax^2 + x
\]
and
\[
F(t) = \begin{cases} \frac{-t}{2} \ln(1 - ce^{2t}), & cA > 0, c \leq 1, \\ \mu A, & c > 0, A = 0. \end{cases}
\]

(ii) For \( d > 1 \), if \( a_1 \) and \( a_2 \) are constants, then
\[
\varphi(x) = x(x + 1)
\]
and
\[
F(t) = -\ln(1 - ce^t), \quad 0 < c \leq 1.
\]

Proof. (I) Let \( x = F'(t), \varphi(x) = F''(t) \), according to Theorem 3.1, we can assume that
\[
\varphi(x) = Ax^2 + Dx + E.
\]
Using \( \frac{dx}{dt} = \varphi(x) \), \( x = F'(t) \) and \( F''(t) > 0 \), \( \varphi(x) \) and \( F(t) \) can be expressed as follows:
\[
\varphi(x) = \begin{cases} A(x - \lambda)^2, & A > 0, \\ A((x - \lambda)^2 + \mu^2), & A > 0, \mu > 0, \\ A((x - \lambda)^2 - \mu^2), & \mu A > 0, \\ 2\lambda, & \lambda > 0, \\ \lambda(x - \mu), & \lambda \neq 0, \end{cases}
\]
\[
F(t) = \begin{cases} \frac{-t}{4} \ln |t + c| + \lambda t + c_1, & A > 0, \\ \frac{-t}{4} \ln |\cos(\mu At + c)| + \lambda t + c_1, & A > 0, \mu > 0, \\ \frac{-t}{4} \ln |1 - ce^{2\mu At}| + (\lambda + \mu)t + c_1, & \mu A > 0, cA > 0, \\ \lambda t^2 + ct + c_1, & \lambda > 0, \\ ce^{\lambda t} + \mu t + c_1, & \lambda \neq 0, c > 0. \end{cases}
\]

(II) For the case of \( d = d_0 = 1 \), by \( F(-\infty) = 0 \) and (3.15), \( \varphi(x) \) and \( F(t) \) can be expressed as
\[
\varphi(x) = \begin{cases} A(x + 2\mu), & \mu A > 0, \\ \lambda x, & \lambda > 0, \end{cases}
\]
\[
F(t) = \begin{cases} -\frac{1}{\mu} \ln |1 - ce^{2\mu At}|, & \mu A > 0, c > 0, c \leq 1, \\ ce^{\lambda t}, & c > 0, \lambda > 0, \end{cases}
\]
respectively. In view of
\[
0 < \lim_{t \to -\infty} \frac{F''(t)}{c^t} < +\infty,
\]
we get \( 2\mu A = 1 \) for \( F(t) = -\frac{1}{\mu} \ln |1 - ce^{2\mu At}|, \) and \( \lambda = 1 \) for \( F(t) = ce^{\lambda t} \). Thus \( \varphi(x) \) and \( F(t) \) can be written as (3.10) and (3.11), respectively.

(III) For the case of \( d = 1 \) and \( d_0 > 1 \), applying \( \varphi(x) = Ax^2 + x, F(-\infty) = 0 \) and (3.15), we obtain (3.10) and (3.11).

(IV) For the case of \( d > 1 \) and \( d_0 = 1 \), from \( \varphi(x) = A(1 + x)^2 - B(1 + x) \) and \( F(-\infty) = 0 \), by (3.15), we have
\[
\varphi(x) = Ax(x + 2\mu), \mu A > 0
\]
and
\[ F(t) = -\frac{1}{A} \ln |1 - ce^{2\mu A t}|, \mu A > 0, cA > 0, c \leq 1. \]

Since
\[ 0 < \lim_{t \to -\infty} \frac{F''(t)}{e^t} < +\infty, \]
then \(2\mu A = 1\). Thus \(\varphi(x) = Ax^2 + x\), using \(\varphi(-1) = 0\), we get (3.12) and (3.13).

(V) For the case of \(d > 1\) and \(d_0 > 1\), from \(\varphi(x) = x^2 + x\) and \(F(-\infty) = 0\), by (3.15), we have (3.13).

**Proof of Theorem 1.1.** According to Theorem 3.2, Theorem 2.7 and Theorem 2.8, we obtain that both \(a_1\) and \(a_2\) are constants on \((M, g_F)\) if and only if

(i) \[
F(t) = \begin{cases} 
-\frac{1}{A} \ln(1 - ce^t), & cA > 0, c \leq 1, \\
\frac{1}{ce^t}, & A = 0, c > 0 
\end{cases}
\] (3.16)

and \(a_2 = 0\) for \(d = 1\), where \(A = \frac{d_0 - a_1}{d_0 + 1}\).

(ii) \[
F(t) = -\ln(1 - ce^t), \quad 0 < c \leq 1,
\] (3.17)

\[a_1 = -\frac{d(d+1)}{2}\] and \[a_2 = \frac{(d-1)d(d+1)(3d+2)}{24}\] for \(d > 1\).

Notice that we do not consider the completeness of the metrics \(g_\phi\) and \(g_F\) in the formulas (3.16) and (3.17).

Let
\[ f(u) = \frac{1}{2} F(2u), \; \tau = f'(u), \; \varphi(\tau) = f''(u), \]

then
\[ \varphi(\tau) = 2\tau(1 + A\tau), \; \tau \in I = \left[0, \frac{1}{A} \frac{c}{1 - c}\right] \]

for
\[ F(t) = -\frac{1}{A} \ln(1 - ce^t), \; cA > 0, \; c \leq 1, \; t \in [-\infty, 0), \]

and
\[ \varphi(\tau) = 2\tau, \; \tau \in I = [0, c) \]

for
\[ F(t) = ce^t, \; c > 0, \; t \in [-\infty, 0). \]

By Proposition 2.3 of [20], \(g_F\) is a complete Kähler metric on the domain \(M\) for \(d_0 = 1\) if and only if
\[ F(t) = \begin{cases} 
-\frac{1}{A} \ln(1 - e^t), & d = 1, A > 0, \\
-\ln(1 - e^t), & d > 1. 
\end{cases} \]

Using mathematical induction, we obtain that \(g_F\) is a complete Kähler metric on the domain \(M\) if and only if
\[ F(t) = \begin{cases} 
-\frac{1}{A} \ln(1 - e^t), & d = 1, A > 0, \\
-\ln(1 - e^t), & d > 1. 
\end{cases} \]

So we complete the proof of Theorem 1.1. \(\square\)
4 Proof of Theorem 1.2

In order to prove Theorem 1.2, we first give the Kähler forms of $G$-invariant Kähler metrics on the Cartan-Hartogs domain $\Omega(\mu, d_0)$.

**Lemma 4.1.** Let $G$ be the set of mappings generated by (1.5), $\Phi$ be a Kähler potential on the Cartan-Hartogs domain $\Omega(\mu, d_0)$. If for all $\Upsilon \in G$,

$$\partial \bar{\partial} (\Phi \circ \Upsilon) = \partial \bar{\partial} \Phi,$$

then there are a unique real number $\nu > 0$ and a unique real function $F$ with $F(0) = 0$ such that

$$\partial \bar{\partial} (\nu \phi + F(\rho)) = \partial \bar{\partial} \Phi,$$

where

$$\phi(z) = -\mu \ln N(z, \overline{z}), \quad \rho = e^{\phi(z)} \|w\|^2.$$

**Proof.** 

**Step 1.** We prove that there exist a number $\nu \in \mathbb{C}$ and a function $F$ satisfying

$$\partial \bar{\partial} (\nu \phi + F(\rho)) = \partial \bar{\partial} \Phi.$$

Let $Z = (z, w)$ and

$$w = \frac{w}{N(z, \overline{z})} = e^{\frac{1}{2} \phi(z)} w, \quad w_0 = \frac{w_0}{N(z_0, \overline{z}_0)} = e^{\frac{1}{2} \phi(z_0)} w_0.$$

By

$$\partial \bar{\partial} (\Phi \circ \Upsilon) = \partial \bar{\partial} \Phi$$

and

$$\frac{\partial^2 (\Phi \circ \Upsilon)}{\partial Z^t \partial \overline{Z}}(Z) = \frac{\partial \Upsilon}{\partial Z^t}(Z) \frac{\partial^2 \Phi}{\partial Z^t \partial \overline{Z}}(\Upsilon(Z)) \left( \frac{\partial \Upsilon}{\partial Z^t} \right)^t(Z),$$

we get

$$\frac{\partial^2 \Phi}{\partial Z^t \partial \overline{Z}}(Z) = \frac{\partial \Upsilon}{\partial Z^t}(Z) \frac{\partial^2 \Phi}{\partial Z^t \partial \overline{Z}}(\Upsilon(Z)) \left( \frac{\partial \Upsilon}{\partial Z^t} \right)^t(Z).$$

Let

$$\left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) := \left( \begin{array}{cc} \frac{\partial^2 \Phi}{\partial z \partial \overline{z}} & \frac{\partial^2 \Phi}{\partial w \partial \overline{z}} \\ \frac{\partial^2 \Phi}{\partial w \partial z} & \frac{\partial^2 \Phi}{\partial w^2} \end{array} \right),$$

using

$$\frac{\partial \Upsilon}{\partial Z^t}(Z) = \left( \begin{array}{c} \frac{\partial \gamma}{\partial z}(z) \frac{\partial \psi}{\partial z}(U) \\ 0 \end{array} \right),$$

we obtain

$$A_{11}(Z) = \frac{\partial \gamma}{\partial z^t} A_{11}(\Upsilon(Z)) \left( \frac{\partial \gamma}{\partial z^t} \right)^t(Z) + \frac{\partial \psi}{\partial z^t} w U A_{21}(\Upsilon(Z)) \left( \frac{\partial \gamma}{\partial z^t} \right)^t(Z),$$

$$+ \frac{\partial \gamma}{\partial z^t} A_{12}(\Upsilon(Z)) \left( \frac{\partial \psi}{\partial z^t} w U A_{21}(\Upsilon(Z)) \left( \frac{\partial \gamma}{\partial z^t} \right)^t(Z),$$

$$A_{12}(Z) = \psi(z) \frac{\partial \gamma}{\partial z^t} A_{12}(\Upsilon(Z)) \left( \frac{\partial \gamma}{\partial z^t} \right)^t(Z) + \psi(z) \frac{\partial \psi}{\partial z^t} w U A_{22}(\Upsilon(Z)) \left( \frac{\partial \psi}{\partial z^t} w U \right)^t(Z),$$

$$A_{21}(Z) = \psi(z) U A_{21}(\Upsilon(Z)) \left( \frac{\partial \gamma}{\partial z^t} \right)^t(Z) + \psi(z) U A_{22}(\Upsilon(Z)) \left( \frac{\partial \psi}{\partial z^t} \right)^t(Z).$$
and
\[ A_{22}(Z) = |\psi(z)|^2 U A_{22}(T(Z)) \overline{U^T}. \] (4.4)
For \( Z = (z_0, w_0) \), since
\[ \gamma(z_0) = 0, T(z_0, w_0) = (0, w_0 U), \psi(z_0) = e^{\frac{i}{2} \phi(z_0)}, \frac{\partial \psi}{\partial z^t}(z_0) = e^{\frac{i}{2} \phi(z_0)} \frac{\partial \phi}{\partial z^t}(z_0), \]
we have
\[ A_{11}(z_0, w_0) = \frac{\partial \gamma}{\partial z^t}(z_0) A_{11}(0, w_0 U) \left( \frac{\partial \gamma}{\partial z^t}(z_0) \right)^t + \frac{\partial \psi}{\partial z^t}(z_0) w_0 U A_{21}(0, w_0 U) \left( \frac{\partial \gamma}{\partial z^t}(z_0) \right)^t \]
\[ + e^{\phi(z_0)} \frac{\partial \phi}{\partial z^t}(z_0) w_0 U A_{22}(0, w_0 U) \overline{U^T} w_0 \frac{\partial \phi}{\partial z^t}(z_0), \] (4.5)
\[ A_{12}(z_0, w_0) = e^{\frac{i}{2} \phi(z_0)} \frac{\partial \gamma}{\partial z^t}(z_0) A_{12}(0, w_0 U) \overline{U^T} + e^{\phi(z_0)} \frac{\partial \phi}{\partial z^t}(z_0) w_0 U A_{22}(0, w_0 U) \overline{U^T}, \] (4.6)
\[ A_{21}(z_0, w_0) = e^{\frac{i}{2} \phi(z_0)} U A_{21}(0, w_0 U) \left( \frac{\partial \gamma}{\partial z^t}(z_0) \right)^t + e^{\phi(z_0)} U A_{22}(0, w_0 U) \overline{U^T} w_0 \frac{\partial \phi}{\partial z^t}(z_0) \] (4.7)
and
\[ A_{22}(z_0, w_0) = e^{\phi(z_0)} U A_{22}(0, w_0 U) \overline{U^T}. \] (4.8)
Let \( z_0 = 0 \) in (4.8), it follows that
\[ A_{22}(0, w_0) = U A_{22}(0, w_0 U) \overline{U^T}. \]
According to Lemma 4.2 below, there is a function \( F \) such that
\[ A_{22}(0, w) = F'(\|w\|^2) I_{d_0} + F''(\|w\|^2) \overline{w^T} w. \]
Thus
\[ A_{22}(z, w) = e^{\phi(z)} \left( F'(\rho) I_{d_0} + e^{\phi(z)} F''(\rho) \overline{w^T} w \right). \] (4.9)
Set \( z_0 = 0 \) and \( U = I_{d_0} \) in (4.6), by \( \phi(0) = 0 \) and \( \frac{\partial \phi}{\partial z^t}(0) = 0 \), we have
\[ A_{12}(0, w_0) = \frac{\partial \gamma}{\partial z^t}(0) A_{12}(0, w_0) \]
for all \( \gamma \in \text{Aut}(\Omega) \) with \( \gamma(0) = 0 \). So \( A_{12}(0, w_0) = 0 \). Combining (4.6) and (4.9), we get
\[ A_{12}(z, w) = e^{\phi(z)} \left( F'(\rho) + \rho F''(\rho) \right) \frac{\partial \phi}{\partial z^t}(z) w. \] (4.10)
For the same argument, we also have
\[ A_{21}(z, w) = e^\phi(F'(\rho) + \rho F''(\rho)) w \frac{\partial \phi}{\partial z^t}. \] (4.11)
Substituting \( A_{12}(0, w) = 0, A_{21}(0, w) = 0 \) and (4.9) into (4.5), then
\[ A_{11}(z_0, w_0) = \frac{\partial \gamma}{\partial z^t}(z_0) A_{11}(0, w_0 U) \left( \frac{\partial \gamma}{\partial z^t}(z_0) \right)^t \]
\[ + \rho(F'(\rho) + \rho F''(\rho)) \frac{\partial \phi}{\partial z^t}(z_0) \frac{\partial \phi}{\partial z^t}(z_0), \] (4.12)
here \(\gamma(z_0) = 0\) and \(\rho = e^{\phi(z_0)}\|w_0\|^2\).

Let \(z_0 = 0\), (4.12) implies that

\[
A_{11}(0, w) = \frac{\partial \gamma}{\partial z^t} A_{11}(0, wU) \left( \frac{\partial \gamma}{\partial z^t} \right)^t
\]

for all \(U \in \mathcal{U}(d_0)\), \(\gamma \in \text{Aut}(\Omega)\) with \(\gamma(0) = 0\). Therefore, by Schur’s lemma, there exists a function \(P\) such that

\[
A_{11}(0, w) = \mu P(\|w\|^2) I_d.
\]

Since

\[
\partial \bar{\partial}(\phi \circ \gamma) = \partial \bar{\partial} \phi
\]

for all \(\gamma \in \text{Aut}(\Omega)\), using \(\partial^2 \phi / \partial z \partial \bar{z}(0) = \mu I_d\), it follows that

\[
\frac{\partial^2 \phi}{\partial z \partial \bar{z}}(z_0) = \mu \left( \frac{\partial \gamma}{\partial z}(z_0) \right)^t
\]

for all \(\gamma \in \text{Aut}(\Omega)\) with \(\gamma(z_0) = 0\). Hence, by (4.12), we obtain

\[
A_{11}(z, w) = P(\rho) \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}} + \rho(F'(\rho) + \rho F''(\rho)) \frac{\partial \phi}{\partial z^t}(z_0) \frac{\partial \phi}{\partial \bar{z}}(z_0).
\]  \tag{4.13}

Now, we set \(\Psi = \Phi - F(\rho)\), then

\[
A_{11} = \frac{\partial^2 \Psi}{\partial z^t \partial \bar{z}} + \rho(F'(\rho) + \rho F''(\rho)) \frac{\partial \phi}{\partial z^t}(z_0) \frac{\partial \phi}{\partial \bar{z}} - \rho F'(\rho) \frac{\partial \phi}{\partial \bar{w}} - \rho F''(\rho) \frac{\partial \phi}{\partial w}.
\]

\[
A_{12} = \frac{\partial^2 \Psi}{\partial z^t \partial w} + \rho F'(\rho) \frac{\partial \phi}{\partial \bar{w}} + \rho F''(\rho) \frac{\partial \phi}{\partial w},
\]

\[
A_{21} = \frac{\partial^2 \Psi}{\partial z^t \partial \bar{w}} + e\phi(F'(\rho) + \rho F''(\rho)) \frac{\partial \phi}{\partial \bar{w}} - \rho F'(\rho) \frac{\partial \phi}{\partial \bar{w}},
\]

and

\[
A_{22} = \frac{\partial^2 \Psi}{\partial z^t \partial \bar{z}} + e\phi(F'(\rho) + \rho F''(\rho)) \frac{\partial \phi}{\partial \bar{w}}.
\]

These combine with (4.13), (4.10), (4.11) and (4.9), we get

\[
\frac{\partial^2 \Psi}{\partial z^t \partial \bar{z}} = (P(\rho) - \rho F'(\rho)) \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}}
\]  \tag{4.14}

and

\[
\frac{\partial^2 \Psi}{\partial z^t \partial \bar{w}} = 0, \quad \frac{\partial^2 \Psi}{\partial w^t \partial \bar{z}} = 0, \quad \frac{\partial^2 \Psi}{\partial w^t \partial \bar{w}} = 0.
\]  \tag{4.15}

From (4.15), the function \(\Psi\) must be the following form

\[
\Psi = \Psi_1(z, \bar{z}) + \Psi_2(z, w) + \Psi_3(\bar{z}, \bar{w}),
\]

by (4.14), we have

\[
\frac{\partial^2 \Psi_1}{\partial z^t \partial \bar{z}} = (P(\rho) - \rho F'(\rho)) \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}}.
\]

thinks to \(\rho(z, w) = e^{\phi(z)}\|w\|^2\), so \(P(\rho) - \rho F'(\rho)\) is a constant. Finally, let \(\nu = P(0)\), then

\[
\partial \bar{\partial}(\nu \phi + F(\rho)) = \partial \bar{\partial} \Phi.
\]
Step 2. We prove that the function $F$ with $F(0) = 0$ and the number $\nu$ are unique.
We only show that if
\[
\partial \theta (\lambda \phi + Q(\rho)) \equiv 0, \quad Q(0) = 0,
\]
then $\lambda = 0$ and $Q \equiv 0$.

Since
\[
\frac{\partial^2 (\lambda \phi + Q(\rho))}{\partial z^t \partial \bar{z}} = (\lambda + \rho Q'(\rho)) \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}} + \rho (Q'(\rho) + \rho Q''(\rho)) \frac{\partial \phi}{\partial z^t \partial \bar{z}} = 0,
\]
let $\rho = 0$, which gives
\[
\lambda \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}} = 0,
\]
so $\lambda = 0$.

By $\partial \tilde{\theta}(Q(\rho)) \equiv 0$, we obtain
\[
\frac{\partial^2 Q}{\partial w^t \partial \bar{w}} = e^\phi \left( Q'(\rho) I_{d_0} + e^\phi Q''(\rho) \bar{w}^t w \right) = 0,
\]
thus $Q'(\rho) = 0$ for $d_0 > 1$ and $Q'(\rho) + \rho Q''(\rho) = 0$ for $d_0 = 1$. So $Q \equiv 0$ for $d_0 > 1$ and $Q'(\rho) = c$ for $d_0 = 1$. For $d_0 = 1$, let $\rho = 0$, we have $c = 0$, hence $Q \equiv 0$.

Step 3. We prove that $F$ with $F(0) = 0$ is a real function and $\nu > 0$.

Using
\[
\frac{\partial^2 (\nu \phi + F(\rho))}{\partial z^t \partial \bar{z}} = (\nu + \rho F'(\rho)) \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}} + \rho (F'(\rho) + \rho F''(\rho)) \frac{\partial \phi}{\partial z^t \partial \bar{z}}
\]
is positive definite, we get $\nu \frac{\partial^2 \phi}{\partial z^t \partial \bar{z}}$ is positive definite, then $\nu > 0$.

Since
\[
\frac{\partial^2 (\nu \phi + F(\rho))}{\partial w^t \partial \bar{w}} = e^\phi \left( F'(\rho) I_{d_0} + e^\phi F''(\rho) \bar{w}^t w \right)
\]
is positive definite, then
\[
(F - \bar{F})' I_{d_0} + e^\phi (F - \bar{F})'' \bar{w}^t w = 0,
\]
which implies $F = \bar{F}$, namely $F$ is a real function. \qed

**Lemma 4.2.** Let $\Psi$ be the continuous differentiable function of order 2 on $\mathbb{B}^m$. If for all $U \in \mathcal{U}(m)$,
\[
\partial \tilde{\theta}(\Psi \circ U) = \partial \tilde{\theta} \Psi,
\]

namely
\[
\frac{\partial^2 \Psi}{\partial w^t \partial \bar{w}}(w, \bar{w}) = U \frac{\partial^2 \Psi}{\partial w^t \partial \bar{w}}(wU, \bar{w}U)\overline{U}^t,
\]
then there is a function $F$ such that
\[
\partial \tilde{\theta}(F(\|w\|^2)) = \partial \tilde{\theta} \Psi.
\]

**Proof.** We only prove for the case $m = 1, 2$, the proof of the case $m > 2$ is the same as the proof of the case $m = 2$, we omit its the proof.

(i) For $m = 1$, we have
\[
\frac{\partial^2 \Psi}{\partial w^t \partial \bar{w}}(w, \bar{w}) = \frac{\partial^2 \Psi}{\partial w^t \partial \bar{w}}(e^{\sqrt{-1} \theta} w, e^{-\sqrt{-1} \theta} \bar{w}), \quad \forall \ \theta \in \mathbb{R},
\]
then there exists a function $f$ that satisfies
\[
\frac{\partial^2 \Psi}{\partial w^t \partial \bar{w}}(w, \bar{w}) = f(\|w\|^2).
\]
Let
\[ Q(x) = \begin{cases} \frac{1}{2} \int_0^x f(t) \, dt, & x \in (0, 1), \\ f(0), & x = 0 \end{cases} \]
and
\[ F(x) = \int_0^x Q(t) \, dt, \quad x \in [0, 1). \]

It is easy to see that \( F \) satisfies
\[ \partial \bar{\partial} (F(\|w\|^2)) = \partial \bar{\partial} \Psi. \]

(ii) For \( m = 2 \), let
\[ f(w, \bar{w}) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} (w, \bar{w}) = \frac{\partial^2 \Psi}{\partial w^i \partial \bar{w}^j}(w), \]
thus
\[ f(wU, \bar{w}U) = \overline{U^T f(w, \bar{w}) U}, \quad \forall \ U \in \mathcal{U}(2). \]

The following we will prove that there exist functions \( r \) and \( s \) satisfying
\[ f(w, \bar{w}) = r(\|w\|^2) I_2 + s(\|w\|^2) w. \quad (4.16) \]

Let \( U = e^{tX} \in \mathcal{U}(2), \ X = (X_{ij})_{1 \leq i, j \leq 2} \) and \( t \in \mathbb{R} \). Differentiating the left and right-hand sides at \( t = 0 \) below,
\[ f(we^{tX}, \bar{w}e^{tX}) = e^{-tX} f(w, \bar{w}) e^{tX}, \]
we find
\[ \sum_{ij=1}^2 (w_i X_{ij} \frac{\partial f}{\partial w_j} - w_j X_{ij} \frac{\partial f}{\partial w_i}) = fX - Xf. \quad (4.17) \]

Setting
\[ X = \sqrt{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \sqrt{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ \sqrt{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sqrt{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
in (4.17), we have
\[ w_1 \frac{\partial f}{\partial w_1} - w_1 \frac{\partial f}{\partial w_1} = \begin{pmatrix} 0 & -f_{12} \\ f_{21} & 0 \end{pmatrix}, \quad (4.18) \]
\[ w_2 \frac{\partial f}{\partial w_2} - w_2 \frac{\partial f}{\partial w_2} = \begin{pmatrix} 0 & f_{12} \\ -f_{21} & 0 \end{pmatrix}, \quad (4.19) \]
\[ w_1 \frac{\partial f}{\partial w_2} + w_2 \frac{\partial f}{\partial w_1} - w_1 \frac{\partial f}{\partial w_2} - w_2 \frac{\partial f}{\partial w_1} = \begin{pmatrix} f_{12} - f_{21} & f_{11} - f_{22} \\ f_{22} - f_{11} & f_{21} - f_{12} \end{pmatrix}, \quad (4.20) \]
\[ -w_1 \frac{\partial f}{\partial w_2} + w_2 \frac{\partial f}{\partial w_1} - w_1 \frac{\partial f}{\partial w_2} + w_2 \frac{\partial f}{\partial w_1} = \begin{pmatrix} f_{12} + f_{21} & -f_{11} - f_{22} \\ -f_{11} + f_{22} & f_{12} - f_{21} \end{pmatrix}, \quad (4.21) \]
respectively.

Using (4.20) and (4.21), we get
\[ w_2 \frac{\partial f}{\partial w_1} - w_1 \frac{\partial f}{\partial w_2} = \begin{pmatrix} f_{12} & 0 \\ f_{22} - f_{11} & -f_{12} \end{pmatrix}, \quad (4.22) \]
and
\[ w_1 \frac{\partial f}{\partial w_2} - w_2 \frac{\partial f}{\partial w_1} = \begin{pmatrix} -f_{21} & f_{11} - f_{22} \\ 0 & f_{21} \end{pmatrix}. \quad (4.23) \]
It is well known that the solutions of the equations
\[
x \frac{\partial g_1}{\partial x} - y \frac{\partial g_1}{\partial y} = 0, \quad x \frac{\partial g_2}{\partial x} - y \frac{\partial g_2}{\partial y} = g_2
\]
are
\[
g_1 = h_1(xy), \quad g_2 = x h_2(xy),
\]
respectively. Then from (4.18) and (4.19), \(f\) may be written as
\[
f(w, \bar{w}) = \left( \frac{h_{11}(|w_1|^2, |w_2|^2)}{w_2 w_1 h_{21}(|w_1|^2, |w_2|^2)} \right) w_1 w_2 h_{12}(|w_1|^2, |w_2|^2).
\]
(4.24)
Let \(u = w_1 \bar{w}_1, \ v = w_2 \bar{w}_2\). Substituting (4.24) into (4.22) and (4.23), we get
\[
\begin{pmatrix}
\frac{w_1 w_2}{w_1} \left( \frac{\partial h_{11}}{\partial u} - \frac{\partial h_{11}}{\partial v} \right) \\
\left( |w_2|^2 - |w_1|^2 \right) h_{21} + |w_1 w_2|^2 \left( \frac{\partial h_{21}}{\partial u} - \frac{\partial h_{21}}{\partial v} \right)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\frac{w_1 w_2}{w_1} h_{12} & 0 \\
0 & h_{22} - h_{11} - \frac{w_1 w_2}{w_1} h_{12}
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
\frac{w_1 w_2}{w_1} \left( \frac{\partial h_{11}}{\partial u} - \frac{\partial h_{11}}{\partial v} \right) \\
\left( |w_2|^2 - |w_1|^2 \right) h_{12} + |w_1 w_2|^2 \left( \frac{\partial h_{12}}{\partial u} - \frac{\partial h_{12}}{\partial v} \right)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\frac{w_1 w_2}{w_1} h_{21} & h_{11} - h_{22} \\
0 & \frac{w_1 w_2}{w_2} h_{21}
\end{pmatrix}
\]
That is
\[
\frac{\partial h_{11}}{\partial u} - \frac{\partial h_{11}}{\partial v} = h_{12},
\]
\[
\frac{\partial h_{12}}{\partial u} - \frac{\partial h_{12}}{\partial v} = 0,
\]
\[
(v - u) h_{21} + uv \left( \frac{\partial h_{21}}{\partial u} - \frac{\partial h_{21}}{\partial v} \right) = h_{22} - h_{11},
\]
\[
\frac{\partial h_{22}}{\partial u} - \frac{\partial h_{22}}{\partial v} = -h_{12},
\]
and
\[
\frac{\partial h_{11}}{\partial u} - \frac{\partial h_{11}}{\partial v} = h_{21},
\]
\[
\frac{\partial h_{21}}{\partial u} - \frac{\partial h_{21}}{\partial v} = 0,
\]
\[
(v - u) h_{12} + uv \left( \frac{\partial h_{12}}{\partial u} - \frac{\partial h_{12}}{\partial v} \right) = h_{22} - h_{11},
\]
\[
\frac{\partial h_{22}}{\partial u} - \frac{\partial h_{22}}{\partial v} = -h_{21}.
\]
Combining (4.25), (4.26), (4.29) and (4.30), we obtain that there exists a function \(s\) such that
\[
h_{12}(u, v) = h_{21}(u, v) = s(u + v).
\]
(4.33)
Comparing (4.28), (4.29), we have
\[ \frac{\partial (h_{11} + h_{22})}{\partial u} = \frac{\partial (h_{11} + h_{22})}{\partial v} = 0, \]
so there is a function \( p \) such that
\[ h_{11} + h_{22} = p(u + v). \]
(4.34)
Substituting (4.33) into (4.27), it gives that
\[ h_{11} - h_{22} = (u - v)s(u + v). \]
(4.35)
By (4.34) and (4.35), we have
\[ h_{11} = r(u + v) + us(u + v), \quad h_{22} = r(u + v) + vs(u + v), \]
(4.36)
here
\[ r(x) = \frac{1}{2}p(x) - \frac{1}{2}xs(x). \]
Substituting (4.33) and (4.36) into (4.24), we obtain (4.16).
Now let
\[ q(x) = \int_{0}^{x} (x - t)s(t)dt \quad (0 \leq x < 1), \quad \Psi_1(w, \bar{w}) = \Psi(w, \bar{w}) - q(\|w\|^2). \]
Thus (4.16) gives that
\[ \frac{\partial^2 \Psi_1}{\partial w_1 \partial \bar{w}_1}(w, \bar{w}) = \frac{\partial^2 \Psi_1}{\partial w_2 \partial \bar{w}_2}(w, \bar{w}) = r(\|w\|^2) - q'(\|w\|^2) \]
(4.37)
and
\[ \frac{\partial^2 \Psi_1}{\partial w_1 \partial \bar{w}_2} = \frac{\partial^2 \Psi_1}{\partial w_2 \partial \bar{w}_1} = 0. \]
(4.38)
From (4.38) and (4.37), it follows that
\[ \Psi_1(w, \bar{w}) = H_1(w_1, \bar{w}_1) + H_2(w_2, \bar{w}_2) + H_3(w_1, w_2) + H_4(w_1, \bar{w}_2) \]
and
\[ \frac{\partial^2 H_1}{\partial w_1 \partial \bar{w}_1}(w_1, \bar{w}_1) = r(\|w\|^2) - q'(\|w\|^2), \]
This indicates that \( r(\|w\|^2) - q'(\|w\|^2) \) is a constant.
Let \( c = r(\|w\|^2) - q'(\|w\|^2) \) and \( F(x) = cx + g(x) \), we have
\[ \partial \bar{\partial}(F(\|w\|^2)) = \partial \bar{\partial} \Psi. \]
Now we give a proof of Theorem 1.2.

Proof of Theorem 1.2. From Lemma 4.1, the Kähler potential \( \Phi \) of \( \mathcal{G} \)-invariant the Kähler metric \( g \) can be selected as
\[ \Phi = \nu \phi + \nu F(\rho), \quad \rho = e^{\phi(x)}\|w\|^2. \]
Let \( g_F \) be a Kähler metric on the domain \( \Omega(\mu, d_0) \) associated with the Kähler form
\[ \omega_{g_F} = \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} (\phi + F(\rho)), \]
namely $g = \nu g_F$, thus $a_1(g) = \frac{1}{\nu} a_1(g_F)$ and $a_2(g) = \frac{1}{\nu} a_2(g_F)$, here $a_j(g)$ are the coefficients of the Bergman function expansion with respect to the metric $g$.

It is easy to see that completeness of the metric $g$ is the same as with the metric $g_F$

We denote the rank $r$, the characteristic multiplicities $a, b$, the dimension $d$, the genus $p$, and the generic norm $N(z, \bar{w})$ for the Cartan domain $\Omega$. Then the Bergman functions for the Cartan domain $\Omega$ with respect to the potential $\phi(z) = -\mu \ln N(z, \bar{z})$ are

$$
\epsilon(\alpha; z) = \frac{1}{\mu^d} \prod_{j=1}^{r} \left( \mu \alpha - p + 1 + (j - 1) \frac{a}{2} \right)_{1+b+(r-j)a}^{1+}(4.39)
$$

for all $\alpha > \frac{p-1}{\mu}$, where $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$. In particular

$$
\epsilon(\alpha; z) = \prod_{j=1}^{d} \left( \alpha - \frac{j}{\mu} \right)
$$

for $\Omega = \mathbb{B}^d$ and $\alpha > \frac{d}{\mu}$.

Using (4.39) and Lemma 3.3 of [15], we obtain that the first two coefficients of the Bergman function expansion for the Cartan domain $\Omega$ with respect to the potential $\phi(z) = -\mu \ln N(z, \bar{z})$ are

$$
a_1 = -\frac{dp}{2\mu} \quad (4.40)
$$

and

$$
a_2 = \frac{1}{2\mu^2} \left\{ \frac{d^2p^2}{4} - \frac{r(p-1)p(2p-1)}{6} + \frac{r(r-1)a(3p^2-3p+1)}{12} \right.
$$

$$
- \left( \frac{r-1}{24} \right) \frac{r(2r-1)2(p-1)}{24} + \frac{r^2(2r-1)^2a^2}{48} \right\}. \quad (4.41)
$$

Notice that $\Omega = \mathbb{B}$, $p = 2, r = 1$ and $a = 2$ for $d = 1$, so by (4.40) and (4.41) we get

$$
a_1 = -\frac{1}{\mu}, \quad a_2 = 0.
$$

From the proof of Theorem 1.3 of [15], we have that

$$
a_1 = -\frac{d(d+1)}{2}, \quad a_2 = \frac{(d-1)d(d+1)(3d+2)}{24}
$$

if and only if $\Omega = \mathbb{B}^d$ and $\mu = 1$.

According to Theorem 1.1, both the coefficients $a_1(g_F)$ and $a_2(g_F)$ are constants iff for $d > 1$,

$$
\Omega = \mathbb{B}^d, \mu = 1, \omega_{g_F} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \ln (1 - ||z||^2 - ||w||^2);
$$

for $d = 1$,

$$
\Omega = \mathbb{B}, \omega_{g_F} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \left\{ \mu \ln (1 - |z|^2) + \frac{(d_0 + 1)\mu}{d_0\mu + 1} \ln \left( 1 - \frac{||w||^2}{(1 - |z|^2)^{\mu}} \right) \right\}.
$$

This completes the proof. $\blacksquare$

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