Abstract. We show that every Abelian group satisfying a mild cardinal inequality admits a pseudocompact group topology from which all countable subgroups inherit the maximal totally bounded topology (we say that such a topology satisfies property ♯).

Every pseudocompact group $G$ with cardinality $|G| \leq 2^{2^{c}}$ satisfies this inequality and therefore admits a pseudocompact group topology with property ♯. Under the Generalized Continuum Hypothesis (GCH) this criterion can be combined with an analysis of the algebraic structure of pseudocompact groups to prove that every pseudocompact Abelian group admits a pseudocompact group topology with property ♯.

We also observe that pseudocompact groups with property ♯ contain no infinite compact subsets and are examples of Pontryagin reflexive precompact groups that are not compact.

1. Introduction

A topological space $X$ is pseudocompact if every real-valued continuous function on $X$ is bounded. Pseudocompactness is greatly enhanced by the addition of algebraic structure. This fact was discovered in 1966 by Comfort and Ross [8] who proved that pseudocompact topological groups are totally bounded or, what is the same, that they always appear as subgroups of compact groups. They went even further and precisely identified pseudocompact groups among subgroups of topological groups: a subgroup of a compact group is pseudocompact if and only if it is $G_{δ}$-dense (i.e., meets every nonempty $G_{δ}$-subset).

A powerful tool to study totally bounded topologies is Pontryagin duality. This is because a totally bounded group topology is always induced

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by a group of characters [7] and Pontryagin duality is based on relating a
topological group with its group of continuous characters. We recall here
that a character of a group $G$ is nothing but a homomorphism of $G$ into the
multiplicative group $\mathbb{T}$ of complex numbers of modulus one.

If $G$ is an Abelian topological group, the topology of uniform convergence
on compact subsets of $G$ makes the group of continuous characters of $G$, denoted $G^\wedge$, into a topological group. Evaluations then define a homomor-
phism $\alpha_G: G \to G^{\wedge\wedge}$ between $G$ and the group of all continuous characters
on the dual group, the so-called bidual group $G^{\wedge\wedge}$. When $\alpha_G$ is a topologi-
cal isomorphism we say that $G$ is Pontryagin reflexive. It will be necessary
for the development of this paper to keep in mind that character groups of
discrete groups are compact groups. Even if it is not relevant for our pur-
poses we cannot resist here to add that character groups of compact groups
are again discrete, and that the Pontryagin van-Kampen theorem proves
that all locally compact Abelian groups (discrete and compact ones are thus
comprised) are reflexive.

In the present paper Pontryagin duality will appear both as a tool for
constructing pseudocompact group topologies and as an objective itself. To
be precise, this paper is motivated by the following two questions

**Question 1.1** ([3]). *Is every Pontryagin reflexive totally bounded Abelian
group a compact group?*

**Question 1.2** ([11], Question 25 of [12]). *Does every pseudocompact group
admit a pseudocompact group topology with no infinite compact subsets?*

In this paper we obtain a negative answer to Question 1.1 and a positive
answer, valid under GCH, to Question 1.2. The focus of the paper will be on
Question 1.2 with the analysis of Question 1.1 and its relation with Question
1.2 deferred to Section 6.

Our approach to Question 1.2 consists in combining techniques that can
be traced back at least to [22] with the ideas of [17]. Our construction
actually produces pseudocompact groups with all countable subgroups $h$-
embedded. This is stronger (see Section 2) that finding pseudocompact
group topologies with no infinite compact subsets. With the aid of results
from [20] this construction will yield a wide range of negative answers to
Question 1.1. As pointed to us by M. G. Tkachenko, Question 1.1 has been
answered independently in [1].
On notation and terminology. All groups considered in this paper will be Abelian. So, the specification *Abelian group* to be found at some points will respond only to a matter of emphasis. To further avoid the cumbersome use of the word ”Abelian”, free Abelian groups will simply be termed as *free groups*.

The symbol $\mathbb{P}$ will denote the set of all prime numbers. *Faute de mieux*, we will use the unusual symbol $\mathbb{P}^+$ to denote the set of all prime powers, i.e., an integer $k \in \mathbb{P}^+$ if and only if $k = p^n$ for some $p \in \mathbb{P}$ and some positive integer $n$.

For a set $X$ and a cardinal number $\alpha$, $[X]^\alpha$ stands for the collection of all subsets of $X$ with cardinality $\alpha$.

Following Tkachenko [22], we say that a subgroup $H$ of a topological group $G$ is $h$-embedded if every homomorphism of $H$ to the unit circle $\mathbb{T}$ can be extended to a *continuous* homomorphism of $G$ to $\mathbb{T}$. If $G$ is totally bounded and $H$ is $h$-embedded in $G$, then the topology of $H$ must equal the maximal totally bounded topology of $H$ (or, using van Douwen’s terminology, $H = H^\sharp$).

The cardinal function $m(\alpha)$ will be often used. The cardinal $m(\alpha)$ is defined for every infinite cardinal $\alpha$ as the least cardinal number of a $G_\delta$-dense subset of a compact group $K_\alpha$ of weight $\alpha$. It is proved in [6] that this definition does not depend on the choice $K_\alpha$ and therefore makes sense. The same reference contains proofs of the following basic essential features of $m(\alpha)$:

$$\log(\alpha) \leq m(\alpha) \leq \log(\alpha)^\omega \quad \text{and} \quad \text{cf}(m(\alpha)) > \omega, \quad \text{for every } \alpha \geq \omega.$$  

It is well known that every compact group has cardinality $2^\kappa$ for some cardinal $\kappa$. The question on which cardinals can appear as the cardinal of a pseudocompact group is not so readily answered. We will say that a cardinal $\kappa$ is *admissible* provided there is a pseudocompact group of cardinal $\kappa$. The first obstructions to admissibility were found by van Douwen [14], the main one being that the cardinality $|G|$ of a pseudocompact group cannot be a strong limit cardinal of countable cofinality, see [10, Chapter 3] for more information on admissible cardinals.

Our task will mainly consist in constructing pseudocompact group topologies on a given group $G$. As indicated in the introduction every pseudocompact group topology is totally bounded and a totally bounded group topology $T$ on a group $G$ is always induced by a group of characters $H \subset$
Hom(G, T), [7, 8]. To stress this latter fact we will usually refer to T as T_H.

We have also introduced above the symbol G^\wedge to denote the group of all continuous characters of a topological Abelian group equipped with the compact-open topology. We will use in this context the subscript d to indicate that G carries the discrete topology. Thus (G_d)^\wedge equals the set Hom(G, T) of all homomorphisms into T. Being a closed subgroup T^G, (G_d)^\wedge is always a compact group.

Several purely algebraic notions from the theory of infinite Abelian groups will be necessary, as for instance the notion of basic subgroup and the related one of pure subgroup, we refer to [16] for the meaning and significance of these properties. As usual, the symbol t(G) stands for the torsion subgroup of the group G.

2. The dual property to pseudocompactness

The following theorem is at the heart of the relationship between questions 1.2 and 1.1.

**Theorem 2.1** ([20]). Let (G, T_H), H ⊂ Hom(G, T), be a totally bounded group. (G, T_H) is pseudocompact if and only if every countable subgroup of (H, T_G) is h-embedded in G^\wedge_d.

**Definition 2.2.** We say that a topological group G has property ∗ if every countable subgroup of G is h-embedded in G.

Thus property ∗ is, in the terminology of [20], the dual property of pseudocompactness.

The relation between property ∗ and Question 1.2 is clear from the following Lemma. Although a combination of Propositions 3.4 and 4.4 of [20] would provide an indirect proof, we offer a direct proof for the reader’s convenience.

**Lemma 2.3.** Let (G, T_H) denote a totally bounded group with property ∗. Then (G, T_H) has no infinite compact subsets.

**Proof.** We first see that all countable subgroups of G are T_H-closed. Suppose otherwise that x ∈ cl_{(G,T_H)} N \ N with N a countable subgroup of G. The subgroup \( \tilde{N} = \langle N \cup \{x\} \rangle \) is also countable and, by hypothesis, inherits its maximal totally bounded group topology from (G, T_H). Since subgroups are
necessarily closed in that topology it follows that \( N \) is closed in \( \tilde{N} \), which goes against \( x \in \tilde{N} \setminus N \).

Now suppose \( K \) is an infinite compact subset of \( G \) and let \( S \subset K \) be a countable subset of \( K \). Denote by \( (G, \mathcal{T}_H) \) the completion of \( G \) and by \( b(G_d) \) the Bohr compactification of \( G \) as a discrete group. There is then a continuous map \( \bar{\imath}: b(G_d) \to (G, \mathcal{T}_H) \) that extends the identity map \( \imath: G \to (G, \mathcal{T}_H) \). Then \( \bar{\imath}(\text{cl}_{b(G_d)} S) = \text{cl}_{(G, \mathcal{T}_H)} \bar{\imath}(S) \subset K \), therefore \( \text{cl}_{b(G_d)} S \subset \langle S \rangle \).

Since \( \langle S \rangle \) is a countable subgroup of \( G \), we have by the preceding paragraph that \( \text{cl}_{b(G_d)} S \subset \langle S \rangle \) and, hence, \( \text{cl}_{b(G_d)} S = \text{cl}_{G^2} S \subset \langle S \rangle \). By a well known theorem of van Douwen [15] (see also [18] and [2, Theorem 9.9.51] for different proofs and [19] for extensions of that result) \( |\text{cl}_{b(G_d)} S| = 2^c \) and therefore it is impossible that \( \text{cl}_{b(G_d)} S \subset \langle S \rangle \).

□

We establish next some easily deduced permanence properties.

**Proposition 2.4.** The class of groups having property \( \sharp \) is closed for finite products.

**Proof.** Let \( G_1 \) and \( G_2 \) be two topological Abelian groups with property \( \sharp \) and let \( N \) be a countable subgroup of \( G_1 \times G_2 \). Let \( h \) be a homomorphism from \( N \) to \( T \). By considering an arbitrary extension of \( h \) to \( G_1 \times G_2 \) we may assume that \( h \) is actually defined on \( G_1 \times G_2 \). Since both \( \pi_1(N) \) and \( \pi_2(N) \) are countable there will be continuous homomorphisms \( h_i: G_i \to T, i = 1, 2 \), with \( h_1(x) = h(x, 0) \) and \( h_2(y) = h(0, y) \) for all \( x \in \pi_1(N) \) and \( y \in \pi_2(N) \).

The homomorphism \( \bar{h}: G_1 \times G_2 \to T \) given by \( \bar{h}(x, y) = h_1(x) \cdot h_2(y) \) is then a continuous extension of \( h \).

□

**Lemma 2.5.** Let \( \pi: K \to L \) be a continuous surjection between two compact Abelian groups \( K \) and \( L \) and suppose that \( N \) is a subgroup of \( L \) that, as subspace of \( L \), carries the maximal totally bounded topology. If \( M \) is a subgroup of \( K \) such that \( \pi|_M \) is a group isomorphism between \( M \) and \( N \), then \( M \) also inherits from \( K \) the maximal totally bounded topology.

**Proof.** Denote by \( \mathcal{T}_K \) and \( \mathcal{T}_L \) the topologies that \( M \) inherit from \( K \) and \( L \) respectively (the latter obtained through \( \pi|_M \)). Since \( \pi \) is continuous, the topology \( \mathcal{T}_K \) is finer than \( \mathcal{T}_L \), but \( \mathcal{T}_K \) is the maximal totally bounded topology, therefore \( \mathcal{T}_K = \mathcal{T}_L \).

□

3. Property \( \sharp \) on torsion-free and bounded groups

We will make a heavy use of products of groups in the sequel. If \( \sigma \) is a cardinal number, \( K^\sigma \) stands for such products. We use calligraphical letters,
to denote sets of coordinates, that is, subsets of \( \sigma \). If \( D \subset \sigma \), we will denote by \( \pi_K \) the projection from \( K^\alpha \) to \( K^D \), if no confusion is possible we will simply use \( \pi_D \).

**Lemma 3.1.** Let \( G \) be a metrizable group and let \( \sigma \geq \alpha \) be cardinal numbers with \( m(\sigma) \leq \alpha \), and \( \alpha^\omega \leq \sigma \).

For every \( \theta \in [\alpha]^\omega \) and \( \eta < \alpha \), it is possible to find two collections of sets of coordinates \( \mathcal{S}_\theta, \mathcal{N}_\eta \subset \sigma \) and an independent \( G_\delta \)-dense subset \( D = \{d_\eta; \eta < m(\sigma)\} \) of \( G^\alpha \) with \( |D| = m(\sigma) \) satisfying the following properties:

1. \( |\mathcal{S}_\theta| = \sigma \).
2. \( \mathcal{S}_\theta \cap \mathcal{S}_{\theta'} = \emptyset \).
3. \( \mathcal{S}_\theta \setminus \bigcup_{\eta \in \theta} \mathcal{N}_\eta = \sigma \) for every countable family \( \theta \in [\alpha]^\omega \).
4. Every subset \( \{g_\eta; \eta < \alpha\} \) of \( G^\alpha \) with \( \pi_{\mathcal{N}_\eta}(g_\eta) = \pi_{\mathcal{N}_\eta}(d_\eta) \), for all \( \eta < \alpha \) is \( G_\delta \)-dense.

**Proof.** Let \( A_\beta = \{a_\gamma; \gamma < \alpha\} \) be a set with \( |A_\beta| = \sigma \) and consider the disjoint union \( A = \bigcup_{\beta < \epsilon} A_\beta \). We identify \( G^\alpha \) with \( G^A \). Since \( \alpha^\omega \leq \sigma \), we can as well decompose each \( A_\beta \) as a disjoint union \( A_\beta = \bigcup_{\theta \in [\alpha]^\omega} A_{\theta, \beta} \) of sets of cardinality \( |A_{\theta, \beta}| = \sigma \).

For each \( N \in [\alpha]^\omega \), let next \( F_N = \{f_{(N, \eta)}; \eta < \alpha\} \) be an independent \( G_\delta \)-dense subset of the product \( G^{\bigcup_{\gamma \in N} A_\gamma} \). Assume that each \( f_{(N, \eta)} \) actually belongs to \( G^A \) by putting \( \pi_{A_\gamma}(f_{(N, \eta)}) = 0 \) if \( \gamma \notin N \).

We now order \( \alpha = [c]^\omega \times \alpha \) lexicographically and define the sets \( \mathcal{N}_{\mathcal{N}} \), \( \tilde{\eta} \in [c]^\omega \times \alpha \) and \( \mathcal{S}_\theta, \theta \in [c]^\omega \times \alpha \). For \( \tilde{\eta} = (N, \eta) \in [c]^\omega \times \alpha \) define \( \mathcal{N}_{(N, \eta)} = \cup_{\gamma \in N} A_\gamma \) and given \( \theta = \{(N_k, \eta_k)\}; k < \omega, (N_k, \eta_k) \in [c]^\omega \times \alpha \) we define \( \mathcal{S}_\theta = A_{\theta, \beta} \) where \( \beta_0 \) is such that \( \beta \in N_k \) for some \( k \), implies \( \beta < \beta_0 \) (recall that \( \epsilon \) has uncountable cofinality). By construction of the sets \( A_{\theta, \beta} \), we have \( \mathcal{S}_\theta \cap \mathcal{S}_{\theta'} = \emptyset \), when \( \theta \neq \theta' \). Condition (3) obviously holds, since \( \mathcal{S}_\theta \) and \( \bigcup_{\eta \in \theta} \mathcal{N}_{\mathcal{N}} \) are even disjoint.

Define finally \( D = \{f_{\tilde{\eta}}; \tilde{\eta} \in [c]^\omega \times \alpha\} = \bigcup_{N \in [c]^\omega} F_N \).

Suppose \( \tilde{D} = \{g_{\tilde{\eta}}; \tilde{\eta} \in [c]^\omega \times \alpha\} \) is such that \( \pi_{\mathcal{N}_\tilde{\eta}}(g_{\tilde{\eta}}) = \pi_{\mathcal{N}_\tilde{\eta}}(f_{\tilde{\eta}}) \), for all \( \tilde{\eta} \in [c]^\omega \times \alpha \).

To check that \( \tilde{D} \) is indeed \( G_\delta \)-dense we choose a \( G_\delta \)-subset \( U \) of \( G^A \). There will be then \( N = \{\alpha_n; n < \omega\} \in [c]^\omega \) and a \( G_\delta \)-set \( V \subset G^{\bigcup_{\epsilon = \alpha_n} A_{\epsilon}} \) such that \( \{\tilde{x}; \tilde{x} \in G^A: \pi_{\cup_{\epsilon = \alpha_n}}(\tilde{x}) \in V \) for each \( n < \omega \} \subset U \). Since \( F_N \) is \( G_\delta \)-dense in \( G^{\bigcup_{\gamma \in N} A_\gamma} = G^{\bigcup_{\epsilon = \alpha_n} A_{\epsilon}} \), there will be an element \( f_{(N, \eta)} \in F_N \) with \( \pi_{\cup_{\epsilon = \alpha_n}}(f_{(N, \eta)}) \in V \) for every \( \alpha_n \in N \).
As \( g(N, \eta) \) and \( f(N, \eta) \) have the same \( \cup_{\gamma \in N} A_\gamma \)-coordinates, we conclude that \( g(N, \eta) \in U \cap \tilde{D} \). □

If \( \chi \) is a homomorphism between two groups \( G_1 \) and \( G_2 \) and \( \sigma \) is a cardinal number, we denote by \( \chi^\sigma \) the product homomorphism \( \chi^\sigma : G_1^\sigma \to G_2^\sigma \) defined by \( \chi^\sigma((g_\eta)_{\eta < \sigma}) = (\chi(g_\eta))_{\eta < \sigma} \). It is easily verified that, for any \( D \subseteq \sigma \), the projections \( \pi_D^{G_i} : G_i^\sigma \to G_i^D \), \( i = 1, 2 \) satisfies

\[
\pi_D^{G_2} \circ \chi^\sigma = \chi^D \circ \pi_D^{G_1}.
\]

**Corollary 3.2.** Let \( \chi : G_1 \to G_2 \) be a surjective homomorphism between two metrizable groups \( G_1 \) and \( G_2 \). If \( \sigma \) and \( \alpha \) are cardinal numbers with \( m(\sigma) \leq \alpha \) and \( \alpha^\omega \leq \sigma \), then it is possible to find an independent \( G_\delta \)-dense subset \( D \) of \( G_1^\sigma \) satisfying the properties of Proposition 3.1 such that in addition \( \chi^\sigma(D) \) is an independent subset of \( G_2^\sigma \).

**Proof.** It suffices to repeat the proof of Lemma 3.1 taking care to choose the sets \( F_N \) in such a way that \( \chi^{\cup_{\gamma \in N} A_\gamma}(F_N) \) is also independent. □

**Proposition 3.3.** Let \( \chi : G \to T \) be a surjective character of a compact metrizable group \( G \). If \( \sigma \) and \( \alpha \) are cardinal numbers with \( m(\sigma) \leq \alpha \), and \( \alpha^\omega \leq \sigma \), then the topological group \( G^\sigma \) contains an independent \( G_\delta \)-dense subset \( F \) of cardinality \( \alpha \) such that \( F \) and \( \chi^\sigma(F) \) are isomorphic and generate subgroups with property \( \# \).

**Proof.** We begin with a \( G_\delta \)-dense subset of \( G^\sigma \), \( D = \{ d_\eta : \eta < \alpha \} \), with the properties of Lemma 3.1 and Corollary 3.2. We have thus for each \( \eta < \alpha \) and each \( \theta \in [\alpha]^\omega \) sets \( N_\eta \) and \( S_\theta \) with the properties (1) through (4) of that Lemma.

Next, for every \( \theta \in [\alpha]^\omega \), we choose and fix a set of coordinates \( D_\theta \subseteq \sigma \) of cardinality \( |D_\theta| = \sigma \) in such a way that

\[
D_\theta \subseteq S_\theta \setminus \bigcup_{\eta \in \theta} N_\eta
\]

(recall that by Lemma 3.1, \( |S_\theta \setminus \bigcup_{\eta \in \theta} N_\eta| = \sigma \))

Given each \( \theta \in [\alpha]^\omega \), we consider the free subgroup \( \langle \chi^\sigma(d_\eta) : \eta \in \theta \rangle \) and equip it with its maximal totally bounded topology. Denoting the resulting topological group as \( \langle \chi^\sigma(d_\eta) : \eta \in \theta \rangle^\sharp \), and taking into account that it has weight \( c \), we can find an embedding

\[
j_\theta : \langle \chi^\sigma(d_\eta) : \eta \in \theta \rangle^\sharp \to T^{D_\theta}.
\] (3.1)
For each \( \theta \in [\alpha]^\omega \) and each \( \eta \in \theta \), let \( g_{\eta, \theta} \) denote an element of \( G^{D_\theta} \) with \( \chi^{D_\theta}(g_{\eta, \theta}) = j_\theta(\chi(\delta_\eta)) \). Observe that the set \( \{g_{\eta, \theta} : \eta \in \theta \} \) is independent.

We finally define the elements \( f_\eta \), \( \eta < \alpha \), by the rules:

\[
\pi^{D_\theta}_G(f_\eta) = g_{\eta, \theta}, \text{ if } \theta \in [\alpha]^\omega \text{ is such that } \eta \in \theta, \quad \text{ and }
\pi^{D_\theta}_G(f_\eta) = \pi^{D_\theta}_G(d_\eta) \text{ if } \gamma \notin D_\theta \text{ for any } \theta \in [\alpha]^\omega \text{ with } \eta \in \theta.
\]

Let us see that \( F = \{ f_\eta : \eta < \alpha \} \) satisfies the desired properties:

1. \( F \) and \( \chi^\sigma(F) \) are independent. Suppose that \( \sum_{k=1}^m n_k f_{\eta_k} = 0 \) with \( n_k \in \mathbb{Z} \). Choose then \( \theta \in [\alpha]^\omega \) with \( \eta_1, \ldots, \eta_m, \in \theta \). Since \( \pi^{D_\theta}_G(f_{\eta_k}) = g_{\eta_k, \theta} \) and the set \( \{g_{\eta, \theta} : \eta \in \theta \} \) is independent, the independence of \( F \) follows. Since \( \pi^{D_\theta}(\chi^\sigma(f_\eta)) = \chi^{D_\theta}(g_{\eta, \theta}) \), \( \chi^\sigma(F) \) is also independent. It is easy to see, now, that \( F \) and \( \chi^\sigma(F) \) are isomorphic.

2. The subgroup \( \langle \chi^\sigma(F) \rangle \) has property \( \sharp \). Let \( N \) be a countable subgroup of \( \langle \chi^\sigma(F) \rangle \). Let \( \theta \in [\alpha]^\omega \) be such that \( N \subseteq \langle \chi^\sigma(f_\eta) : \eta \in \theta \rangle \) and define \( N_\theta := \langle f_\eta : \eta \in \theta \rangle \).

Observe finally that \( \pi^\tau_T(N) = \chi^{D_\theta}(\pi^{D_\theta}_G(N_\theta)) \). This last subgroup is just \( j_\theta(\langle \chi(\delta_\eta) : \eta \in \theta \rangle) \) and the latter carries by construction its maximal totally bounded topology, since the restriction of \( \pi^{D_\theta}_G : T^\sigma \to T^{D_\theta} \) to \( N \) is a group isomorphism onto \( \pi^\tau_T(N) = \chi^{D_\theta}(\pi^{D_\theta}_G(N_\theta)) \), Lemma 2.5 applies.

3. \( \langle F \rangle \) has property \( \sharp \). Take \( \pi = \chi^\sigma \), \( K = G^\sigma \) and \( L = T^\sigma \). Bearing in mind that the restriction to \( \langle F \rangle \) is an isomorphism because \( F \) and \( \chi^\sigma(F) \) are independent sets, Lemma 2.5 applies again.

4. \( F \) is a \( G_\delta \)-dense subset of \( G^\sigma \). Observe that, for every \( \eta < \alpha \), \( f_\eta \) coincides with \( d_\eta \) on the set of coordinates \( N_\eta \), for \( D_\theta \subseteq S_\theta \setminus \bigcup_{\eta \in \theta} N_\eta \).

Since \( D \) has the properties of Lemma 3.1, we conclude that \( F \) is \( G_\delta \)-dense.

\[ \square \]

**Proposition 3.4.** Let \( \sigma \) and \( \alpha \) be cardinal numbers with \( m(\sigma) \leq \alpha \), and \( \alpha^\omega \leq \sigma \). The topological group \( \mathbb{Z}(p)^\sigma \) contains an independent \( G_\delta \)-dense subset \( H \) with property \( \sharp \).

**Proof.** Proceed exactly as in Proposition 3.3 and construct an embedding into \( \mathbb{Z}(p)^\sigma \). To obtain the \( \sharp \)-property we identify countable subgroups with Bohr groups of the form \( \oplus_\alpha \mathbb{Z}(p) \). \( \square \)
4. The algebraic structure of pseudocompact Abelian groups

We obtain here some results on the algebraic structure of pseudocompact that will be useful in the next section. The first of them is inspired (and shares a part of its proof) from the first part of the proof of Lemma 3.2 of [17]. We sketch here the proof for the reader’s convenience. We thank Dikran Dikranjan for pointing a misguiding sentence in a previous version of this proof.

**Lemma 4.1.** Every Abelian group admits a decomposition

\[ G = \left( \bigoplus_{p^k \in \mathcal{P}_0} \mathbb{Z}(p^k) \right) \bigoplus H \]

where \( \mathcal{P}_0 \) is a finite subset of \( \mathcal{P}^1 \) and \( H \) is a subgroup of \( G \) with

\[ |nH| = |H| \text{ for all } n \in \mathbb{N}. \]

**Proof.** Decompose \( t(G) = \bigoplus_p G_p \) as a direct sum of \( p \)-groups \( G_p \) and let \( B_p \) denote a basic subgroup of \( G_p \) for each \( p \). This in particular means that \( B_p \) is a direct sum of cyclic \( p \)-groups,

\[ B_p = \bigoplus_{n < \omega} B_{p,n} \text{ with } B_{p,n} \cong \bigoplus_{\beta_n} \mathbb{Z}(p^n) \]

and that \( G_p/B_p \) is divisible. Define \( \mathcal{D} = \{ |B_{p,n}| : p^n \in \mathcal{P}^1 \} \). If \( \mathcal{D} \) has no maximum or \( \beta_0 = \max \mathcal{D} \) is attained at an infinite number of \( |B_{p,n}| \)'s we stop here. If, otherwise, \( \beta_0 = \max \mathcal{D} = |B_{p_1,n_1}| = \ldots = |B_{p_r,n_r}| \) and \( |B_{p_i,n_j}| < \beta_0 \) for all the remaining \( p^n_j \in \mathcal{P}^1 \) we repeat the process with the set \( \mathcal{D} \setminus |B_{p_1,n_1}| \). After a finite number of steps we obtain in this manner a finite collection of cardinals \( F \subset \mathcal{D} \) such that either:

1. **Case 1:** the supremum \( \beta := \sup (\mathcal{D} \setminus F) \) is not attained, or
2. **Case 2:** the supremum \( \beta := \sup (\mathcal{D} \setminus F) \) is attained infinitely often, i.e., there is an infinite subset \( I \subset \mathcal{P}^1 \) with \( |B_{p,n}| = \beta \) for all \( p^n \in I \).

Define \( \mathcal{P}_0 = \{ p^n \in \mathcal{P}^1 : |B_{p,n}| \in F \} \) (observe that \( \mathcal{P}_0 \) is necessarily finite), and set \( \gamma(p^n_k) = |B_{p_{\alpha},n_k}| \) if \( p^n_k \in \mathcal{P}_0 \). Since the subgroups \( B_{p_k,n_k} \) are bounded pure subgroups there will be [16, Theorem 27.5] a subgroup \( H \) of \( G \) such that

\[ G = \left( \bigoplus_{p^n_k \in \mathcal{P}_0} B_{p_k,n_k} \right) \bigoplus H, \]
For each prime $p$, consider a $p$-basic subgroup $B_{p,H} = \oplus_n B_{p,n,H}$ of $H_p$, the $p$-part of $t(H)$, it is immediately checked that either $B_{p,H}$ itself (if $p \not\in P_0^f$) or $B_{p,H} \bigoplus \left( \bigoplus_{p^k_i \in P_0^f} \bigoplus \gamma(p^k_i) B_{p^k_i,n^k_i} \right)$ (if $p \in P_0^f$) is also $p$-basic in $G$.

Since different basic subgroups are necessarily isomorphic [16, Theorem 35.], we have that $B_{p,H}$ or $B_{p,H} \bigoplus \left( \bigoplus_{p^k_i \in P_0^f} \bigoplus \gamma(p^k_i) B_{p^k_i,n^k_i} \right)$ is isomorphic to $B_p$. We have therefore that, for each $p$, either $\sup |B_{p,n,H}|$ is not attained (case 1 above) or attained at infinitely many $p^k_i$’s (case 2).

Let now $n$ be any natural number. Then $|nB_{p,n,H}| = |B_{p,n,H}|$ unless $p^k_i$ divides $n$. Since this will only happen for finitely many $p^k_i$’s, we conclude, in both cases 1 and 2 that $|nB_{p,H}| = |B_{p,H}|$.

Using that $B_{p,H}$ is pure in $H_p$ and that $H_p/B_{p,H}$ is divisible we have that,

$$|nH_p| = \frac{|nH_p|}{|nB_{p,H}|} + |nB_{p,H}| = n \left( \frac{H_p}{B_{p,H}} \right) + |B_{p,H}| = \frac{H_p}{B_{p,H}} + |B_{p,H}| = |H_p|.$$ 

Since $|H| = \sum_p H_p + r_0(H)|$ for every infinite group $H$ and $r_0(nH) = r_0(H)$ we have finally that $|H| = |nH|$, for every $n \in \mathbb{Z}$. □

The terminology introduced in the next definition is motivated in the present context by Theorem 4.3 below.

**Definition 4.2.** If $G$ is an Abelian group, the set $P_0^f$ of Lemma 4.1 can be partitioned as $P_0^f = P_1^f \cup P_2^f$ with $p^k_i \in P_1^f$ if and only if $m(\gamma(p^k_i)) > 2^{r_0(G)}$.

The cardinal numbers $\gamma(p^k_i)$ with $p^k_i \in P_1^f$ will be called the dominant ranks of $G$.

**Theorem 4.3.** Let $G$ be an Abelian group. If $G$ admits a pseudocompact group topology, then $G$ can be decomposed as

$$G = \left( \bigoplus_{p^k \in P_1^f} \bigoplus \mathbb{Z}(p^k) \right) \oplus G_0$$

where $\gamma(p^k_i) \in P_1^f$ are the dominant ranks of $G$ and $m(|G_0|) \leq 2^{r_0(G)}$. 
Proof. Decompose $G$ as in Lemma 4.1:

$$
\left( \bigoplus_{p^k \in P^0} \mathbb{Z}(p^k) \right) \bigoplus H
$$

with $P^0$ a finite subset of $P^1$ and

$$|nH| = |H| \text{ for all } n \in \mathbb{N}.$$ 

Split $P^0 = P^1 \cup P^2$ as in Definition 4.2. Define $P_f = P^1$ and

$$G_0 = \bigoplus_{p^k \in P^2} \mathbb{Z}(p^k) \bigoplus H.$$ 

It remains to check that $m(|G_0|) \leq 2^{\alpha_0(G)}$. This is obvious if $|G_0| = \gamma(p^k_i)$ (as $p^k_i \in P^2$), we can assume therefore that $|G_0| = |H|$.

By [10, Theorem 3.19] there is some $n$ such that $|nG| \leq 2^{\alpha_0(G)}$. Therefore $|H| = |nH| \leq 2^{\alpha_0(G)}$ and, thus,

$$m(|H|) \leq \left( \log 2^{\alpha_0(G)} \right)^\omega \leq \left( 2^{\alpha_0(G)} \right)^\omega = 2^{\alpha_0(G)}.$$ 

We end this section with a mention to a recent result of Dikranjan and Shakmatov that will be an essential tool to extend the scope of our results.

**Theorem 4.4** (Dikranjan and Shakmatov [13], see also Corollary 1.19 of [9]). If $G$ is a pseudocompact group, then $\alpha_0(G)$ is an admissible cardinal.

5. **Pseudocompact groups with property $\sharp$**

The results of the previous sections will be used here to obtain sufficient conditions for the existence of pseudocompact group topologies with property $\sharp$.

**Lemma 5.1.** Let $\pi : G_1 \to G_2$ be a quotient homomorphism between two Abelian topological groups $G_1$ and $G_2$ and let $L$ be another topological group. Assume that the following conditions hold:

1. $G_1$ contains a free $G_\delta$-dense subgroup $H_1$ such that $H_1$ and $\pi(H_1)$ are isomorphic and have property $\sharp$.
2. $G_1$ contains another free subgroup $H_2$ such that $H_1 \cap H_2 = \{0\}$, $H_1 + H_2$ and $\pi(H_1 + H_2)$ are isomorphic and have property $\sharp$.
3. $m(|L|) \leq |H_2|$.
Under these conditions the product $G_1 \times L$ contains a $G_δ$-dense subgroup $\tilde{H}$ such that both $\tilde{H}$ and $\pi \left( p_1(\tilde{H}) \right)$ have property $\mathfrak{z}$, where $p_1 : G_1 \times L \to G_1$ denotes the first projection.

Proof. We first enumerate the elements of $H_1$ and $H_2$ as $H_1 = \{ f_β : \kappa < \beta \}$ and $H_2 = \{ g_η : \eta < \alpha \}$. Since $m(|L|) \leq \alpha = |H_2|$, we can also enumerate a $G_δ$-dense subgroup $D$ of $L$ (allowing repetitions if necessary) as $D = \{ d_η : \eta < \alpha \}$. We now define the subgroup $\tilde{H}$ of $G_1 \times L$ as

$$\tilde{H} = \{ (f_κ + g_η, d_η) : \eta < \alpha, \kappa < \beta \}.$$ 

It is easy to check that $\tilde{H}$ is a $G_δ$-dense subgroup of $G_1 \times L$ with $\tilde{H} \cap \{ 0 \} \times L = \{ 0, 0 \}$.

Since the homomorphism $p_1$ is continuous and establishes a group isomorphism between $\tilde{H}$ and $H_1 + H_2$, Lemma 2.5 shows that $\tilde{H}$ has property $\mathfrak{z}$. The same argument applies to the group $\pi \left( p_1(\tilde{H}) \right) = \pi(H_1 + H_2)$. ∎

Definition 5.2. Let $\alpha \geq \omega$ be a cardinal. We say that $\alpha$ satisfies property $(\ast)$ if:

there is a cardinal $κ$ with $κ^\omega \leq \alpha \leq 2^κ$.  \(\ast\)

Every cardinal $α$ with $α^\omega = α$ satisfies property $(\ast)$. This condition is equivalent to the condition $m(α)^\omega \leq α$.

To apply Lemma 5.1 we need the following result:

Theorem 5.3 (Theorem 4.5 of [4]). Let $G = (G, T_1)$ be a pseudocompact Abelian group with $w(G) = \alpha > \omega$, and set $σ = \min\{ r_0(N) : N \text{ is a closed } G_δ\text{-subgroup of } G \}$. If $α^\omega \leq σ$ and if $λ \geq ω$ satisfies $m(λ) \leq σ$, then $G$ admits a pseudocompact group topology $T_2$ such that $w(G, T_2) = α + λ$ and $T_1 \setminus T_2$ is pseudocompact. Moreover, every closed $G_δ$-subgroup of $(G, T_1)$ is $G_δ$-dense $(G, T_2)$.

Corollary 5.4. Let $σ, α$ and $λ$ be cardinals with $α^\omega \leq σ$ and $m(λ) \leq σ$. If $H$ is a free, dense subgroup of $T^σ$ with property $\mathfrak{z}$ and cardinality $α$, then $T^σ$ contains another subgroup $H_2$ with $H \cap H_2 = \{ 0 \}$, $|H_2| = λ + α$ and such that $H + H_2$ has property $\mathfrak{z}$.

Proof. Let $F(σ)$ denote the free Abelian group of rank $σ$. We apply Theorem 5.3 to the pseudocompact group $(F(σ), T_H)$ defined by $H$. We obtain thus a pseudocompact topology $T_{H_2}$ on $F(σ)$ induced by a subgroup $H_2$ of $T^σ$ of cardinality $|H_2| = α + λ$ such that $T_H \setminus T_{H_2} = T_{H + H_2}$ is pseudocompact. By Theorem 2.1 the subgroup $H + H_2$ has property $\mathfrak{z}$ and, since closed $G_δ$-subgroups of $T_H$ are $G_δ$-dense in $T_{H_2}$, we also have that $H \cap H_2 = \{ 0 \}$. ∎
Theorem 5.5. Let $G$ be a pseudocompact group with dominant ranks $\gamma(p_1^{n_1}), \ldots, \gamma(p_k^{n_k})$ and suppose that $\gamma(p_i^{n_i})$, $1 \leq i \leq k$, satisfy property (*). If $r_0(G)$ also satisfies property (*) for some $\kappa$ with $m(|G_0|) \leq 2^\kappa$, then $G$ admits a pseudocompact topology with property $\sharp$.

Proof. Decompose, following Theorem 4.3, $G$ as a direct sum

$$G = \left( \bigoplus_{\gamma(p_1^{n_1})} \mathbb{Z}(p_1^{n_1}) \bigoplus \cdots \bigoplus_{\gamma(p_k^{n_k})} \mathbb{Z}(p_k^{n_k}) \right) \bigoplus G_0$$

with $m(|G_0|) \leq 2^{r_0(G)}$.

Let $F$ denote a free Abelian group of cardinality $r_0(G)$ contained in $G_0$ and denote by $D(F)$ and $D(t(G_0))$ divisible hulls of $F$ and $t(G_0)$, respectively. There is then a chain of group embeddings (here we use [16, Lemmas 16.2 and 24.3])

$$F \xrightarrow{j_1} G_0 \xrightarrow{j_2} D(F) \oplus D(t(G_0)) \quad \text{(5.1)}$$

Denote by $\chi$ the quotient homomorphism obtained as the dual map of the canonical embedding $\mathbb{Z} \to \mathbb{Q}$. Observe that identifying $F$ with $\oplus_{r_0(G)} \mathbb{Z}$ and $D(F)$ with $\oplus_{r_0(G)} \mathbb{Q}$, the dual map of $j_2 \circ j_1$ is exactly $\chi^{r_0(G)}$.

Taking $\sigma = r_0(G)$, $G = \mathbb{Q}_d^\sigma$ and $\alpha = \kappa^\omega$, we can apply Proposition 3.3 to get a $G_\delta$-dense subgroup $H_1$ of $(D(F)_d)^\wedge = \left( \mathbb{Q}_d \right)^{r_0(G)}$ with $|H_1| = \kappa^\omega$ and such that $H_1$ and $\chi^{r_0(G)}(H_1)$ are isomorphic and have property $\sharp$ (notice that $\kappa^\omega$ and $r_0(G)$ satisfy the hypothesis of that Proposition).

We now apply Corollary 5.4 to $\chi^{r_0(G)}(H_1)$ to obtain another free subgroup $H_2'$ of $\mathbb{T}^{r_0(G)}$ with $\chi^{r_0(G)}(H_1) \cap H_2' = \{0\}$, $|H_2'| = 2^\kappa$ and such that $\chi^{r_0(G)}(H_1) + H_2'$ has property $\sharp$. By lifting (through $\chi^{r_0(G)}$) the free generators of $H_2'$ to $(D(F)_d)^\wedge$, we obtain a free subgroup $H_2$ of $(D(F)_d)^\wedge$ such that $H_1 \cap H_2 = \{0\}$ and $|H_2| = 2^\kappa$. Clearly $H_1 + H_2$ is isomorphic to $\chi^{r_0(G)}(H_1) + H_2'$ and therefore $H_1 + H_2$ has property $\sharp$ by Lemma 2.5.

We finally apply Lemma 5.1. The role of $G_1 \times L_1$ is played by $(D(F)_d)^\wedge \times (D(t(G_0))_d)^\wedge$; $G_2$ is here identified with $\mathbb{T}^{r_0(G)}$ and $\pi$ is $\chi^{r_0(G)}$. Lemma 5.1 then provides a $G_\delta$-dense subgroup $\tilde{H}$ of $\left( D(F)_d \right)^\wedge \times (D(t(G_0))_d)^\wedge$ such that both $\tilde{H}$ and $\chi^{r_0(G)}(p_1(\tilde{H}))$ have property $\sharp$. This subgroup generates a pseudocompact topology $T_{\tilde{H}}$ on $D(F) \oplus D(t(G_0))$ with property $\sharp$ that makes $F$ pseudocompact (the induced topology on $F$ is just the topology inherited
from \( \chi^{r_0(G)}(p_1(\widetilde{H})) \). Since \( G_0 \) sits between \( F \) and \( D(F) \oplus D(t(G_0)) \), it follows that the restriction of \( T_{\widetilde{H}} \) to \( G_0 \) is pseudocompact and has property \( \sharp \).

By Proposition 3.4 the bounded group \( \bigoplus_{\alpha(p_{n_1}^1)} Z(p_{n_1}^1) \cdots \bigoplus_{\alpha(p_{n_k}^k)} Z(p_{n_k}^k) \) also admits a pseudocompact group topology with property \( \sharp \) and the theorem follows.

\( \square \)

**Corollary 5.6.** Let \( G \) denote a pseudocompact group with dominant ranks \( \gamma(p_{n_1}^{n_1}), \ldots, \gamma(p_{n_k}^{n_k}) \). If the dominant ranks satisfy property (*) and \( r_0(G)^\omega = r_0(G) \), then \( G \) admits a pseudocompact topology with property \( \sharp \).

**Proof.** To see that the hypothesis of Theorem 5.5 are satisfied, it suffices to observe that \( r_0(G)^\omega = r_0(G) \) implies that \( r_0(G) \) satisfies (*) with \( \kappa = r_0(G) \), and the condition \( m(|G_0|) \leq 2^\kappa \) is guaranteed by Lemma 4.3.

Dikranjan and Shakmatov [11] prove under a set-theoretic axiom called \( \nabla \kappa \) (that implies \( c = \omega_1 \) and \( 2^c = \kappa \) with \( \kappa \) being any cardinal \( \kappa \geq \omega_2 \)) that every pseudocompact group of cardinality at most \( 2^c \) has a pseudocompact group topology with no infinite compact subsets. It follows from Theorem 5.5 that the result is true in ZFC, even for larger cardinalities.

**Corollary 5.7.** Let \( G \) be a pseudocompact group of cardinality \( |G| \leq 2^{2^c} \). Then \( G \) admits a pseudocompact topology with property \( \sharp \) (and thus a pseudocompact topology with no infinite compact subsets).

**Proof.** Since a pseudocompact group with \( r_0(G) < c \) is a bounded group it will suffice to check that every cardinal \( \alpha \) with \( \alpha \leq 2^{2^c} \) satisfies property (*), Theorem 5.5 will then be applied. We consider the following two cases:

- **Case 1:** \( c \leq \alpha \leq 2^c \). In this case we put \( \kappa = c \).
- **Case 2:** \( \alpha > 2^c \). Choose \( \kappa = 2^c \) for this case.

Observe that in both cases \( m(|G|) \leq 2^\kappa \) and hence that all hypothesis of Theorem 5.5 are fulfilled.

By van Douwen’s theorem [14], a strong limit admissible cardinal must have uncountable cofinality. Therefore GCH implies that every admissible cardinal must have property (*), [10, Lemma 3.4]. Combining this fact with Theorem 4.4 it turns out that, under GCH, every pseudocompact group admits a pseudocompact group topology with property \( \sharp \).

**Theorem 5.8** (GCH). Every pseudocompact group admits a pseudocompact group topology with property \( \sharp \).
Proof. Let $\gamma(p_i^{n_i}) \geq \cdots \geq \gamma(p_k^{n_k})$ be the dominant ranks of $G$. Then $|G| = \gamma(p_1^{n_1})$ and $\gamma(p_1^{n_1})$ is admissible. Since we can assume that $n_i < n_j$ when $j > i$ and $p_i = p_j$, $p_1 G$ will be a pseudocompact group of cardinality $|p_1 G| = \gamma(p_2^{n_2})$. Proceeding in the same way we obtain that the dominant ranks are admissible cardinals. Theorem 4.4 shows on the other hand that the cardinal $r_0(G)$ is also admissible for every pseudocompact group $G$.

We have therefore that $r_0(G)$ and all dominant ranks of $G$ are admissible cardinals. Since GCH implies that $\alpha^\omega = \alpha$ for all admissible cardinals, see [10, Lemma 3.4], Corollary 5.6 finishes the proof. □

6. Property ♯ and the duality of totally bounded Abelian groups

Pontryagin duality was designed to work in locally compact groups and usually works better for complete groups. This behaviour raised the question (actually our first motivating Question 1.1) as to whether all totally bounded reflexive group should be compact, [3]. We see next that this is not the case.

**Theorem 6.1.** If a pseudocompact group contains no infinite compact subsets, then it is Pontryagin reflexive.

Proof. Let $G = (G, T_H)$ be a pseudocompact group with no infinite compact subsets. The group of continuous characters of $G$ is then precisely $H$ and since $G$ has no infinite compact subsets, the topology of this dual group will equal the topology of pointwise convergence on $G$, therefore $G^\wedge = (H, T_G)$ (see in this connection [21]). By Theorem 2.1, $(H, T_G)$ must be again a totally bounded group with property ♯ and hence with no infinite compact subsets, the same argument as above then shows that $G^\wedge \wedge = (H, T_G)^\wedge = (G, T_H)$ and therefore that $G$ is reflexive. □

This last theorem combined with Lemma 2.3 and the results of Section 5 provides a wide range of examples that answer negatively Question 1.1. This question has also been answered independently in [1] where another collection of examples has been obtained.

**Corollary 6.2** (GCH). *Every infinite pseudocompact group $G$ supports a noncompact, pseudocompact group topology $T_H$ such that $(G, T_H)$ is reflexive.*

**Corollary 6.3.** *Every infinite pseudocompact group $G$ with $|G| \leq 2^{2^i}$ supports a noncompact, pseudocompact group topology $T_H$ such that $(G, T_H)$ is reflexive.*
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References

[1] S. Ardanza-Trevijano, M.J. Chasco, X. Domínguez, and M. G. Tkachenko. Precompact noncompact reflexive abelian groups. Preprint, 2008.
[2] A. Arhangel’skii and M. G. Tkachenko. Topological groups and related structures, volume 1 of Atlantis Studies in Mathematics. Atlantis Press, Paris, 2008.
[3] M. J. Chasco and E. Martín-Peinador. An approach to duality on abelian precompact groups. Journal of Group Theory, 11:5:635–643, 2008.
[4] W. W. Comfort and Jorge Galindo. Pseudocompact topological group refinements of maximal weight. Proc. Amer. Math. Soc., 131(4):1311–1320 (electronic), 2003.
[5] W. W. Comfort and Dieter Remus. Imposing pseudocompact group topologies on abelian groups. Fund. Math., 142(3):221–240, 1993.
[6] W. W. Comfort and Lewis C. Robertson. Cardinality constraints for pseudocompact and for totally dense subgroups of compact topological groups. Pacific J. Math., 119(2):265–285, 1985.
[7] W. W. Comfort and K. A. Ross. Topologies induced by groups of characters. Fund. Math., 55:283–291, 1964.
[8] W. W. Comfort and Kenneth A. Ross. Pseudocompactness and uniform continuity in topological groups. Pacific J. Math., 16:483–496, 1966.
[9] Dikran Dikranjan and Anna Giordano Bruno. w-divisible groups. Topology Appl., 155(4):252–272, 2008.
[10] D. Dikranjan and D. Shakhmatov. Algebraic structure of pseudocompact groups. Mem. Amer. Math. Soc., 133(633):x+83, 1998.
[11] D. Dikranjan and D. Shakhmatov. Forcing hereditarily separable compact-like group topologies on abelian groups. Topology Appl., 151(1-3):2–54, 2005.
[12] D. Dikranjan and D. Shakhmatov. Selected topics from the structure theory of topological groups. In Elliott Pearl, editor, Open problems in topology, pages 389–406. Elsevier, 2007.
[13] Dikran Dikranjan and Dmitri Shakhmatov. Algebraic structure of pseudocompact abelian groups. Preprint, 2008.
[14] E. K. van Douwen. The weight of a pseudocompact (homogeneous) space whose cardinality has countable cofinality. Proc. Amer. Math. Soc., 80(4):678–682, 1980.
[15] E. K. van Douwen. The maximal totally bounded group topology on $G$ and the biggest minimal $G$-space, for abelian groups $G$. Topology Appl., 34(1):69–91, 1990.
[16] L. Fuchs. Infinite abelian groups. Vol. I. Academic Press, New York, 1970.
[17] J. Galindo and S. Garcia-Ferreira. Compact groups containing dense pseudocompact subgroups without non-trivial convergent sequences. Topology Appl., 154(2):476–490, 2007.
[18] J. Galindo and S. Hernández. On a theorem of van Douwen. Extracta Math., 13(1):115–123, 1998.
[19] J. Galindo and S. Hernández. The concept of boundedness and the Bohr compactification of a MAP abelian group. *Fund. Math.*, 159(3):195–218, 1999.

[20] S. Hernández and S. Macario. Dual properties in totally bounded abelian groups. *Arch. Math. (Basel)*, 80(3):271–283, 2003.

[21] S. U. Raczkowski and F. Javier Trigos-Arrieta. Duality of totally bounded abelian groups. *Bol. Soc. Mat. Mexicana (3)*, 7(1):1–12, 2001.

[22] M. G. Tkačenko. Compactness type properties in topological groups. *Czechoslovak Math. J.*, 38(113)(2):324–341, 1988.

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