Cash Accumulation Strategy based on Optimal Replication of Random Claims with Ordinary Integrals

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This paper presents a numerical model to solve the problem of cash accumulation strategies for products with an unknown future price, like assets. Stock prices are modeled by a discretized Wiener Process, and by the means of ordinary integrals this Wiener Process will be exactly matched at a preset terminal time. Three applications of the model are presented: accumulating cash for a single asset, for set of different assets, and for a proportion of the excess achieved by a certain asset. Furthermore, an analysis of the efficiency of the model as function of different parameters is performed.

1. Introduction

Cash accumulation strategies deal with the question on how much money should be deposited into an account to be able to buy something at a specific price some time in the future. When the future price of the desired buy is known, cash accumulation strategies are quite straightforward. However, it becomes more complex when the future price is unknown and due to unpredictable changes. For example, looking at financial markets, stock prices seem to experience some kind of unpredictable behaviour. There a multiple factors influencing the stock, such as market sentiment, speculations, and demand and supply.

Therefore, the challenge is to generate a cash accumulation strategy that will accumulate the exact amount of cash at a terminal time in the future needed to buy a specific stock, taking into account the fact that the stock price at this terminal time cannot be known any time before the terminal time. A common approach to this problem is to assume that the changes

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in stock prices are basically small random daily changes. Using this assumption stock prices can be modelled as a random walk, that is generated by, for instance, a Wiener Process, or a Brownian Motion. According to the Martingale Representation Theorem, stochastic integrals can replicate these random variables from a Wiener Process (Dokuchaev, 2013a). This way, a solution can be found such that the final value of the random walk can be matched exactly. Backward stochastic differential equations (BSDE) can be used to find this solution. (Brunick & Shreve, 2013). However, it is hard to find an explicit and unique solution for a BSDE, and if a solution can be found, it can take much computational effort (Huijskens, 2013). Therefore, it is desired to find a simpler solution to this problem. A solution using ordinary integrals is presented by Dokuchaev (2013a). Using ordinary integrals an exact solution can be obtained, but this solution is not unique. Therefore, an optimized solution is presented.

The aim of this report is to present a MATLAB model that can implement the equations presented by Dokuchaev (2013a; 2013b), using the technique of optimal replication of random claims with ordinary integrals, applied to the problem of cash accumulation strategy.

The structure of this report is as follows. Section 2 will describe the method used for this problem, presenting the equations for the general problem and how these equations will be incorporated in the model for this application. Section 3 will present numerical experiments performed with the model by showing 3 example cash accumulation policy problems, and 2 experiments analyzing the efficiency of the model. Subsequently, section 4 will present discussion points that arise from analyzing the model.

2. Method

This section will describe the method for generating the cash accumulation strategy model. First the equations for the general problem are presented in subsection 2.1. Afterwards, in subsection 2.2 it is shown how these equations can be adapted for the purposes of a cash accumulation strategy problem and how these equations can be incorporated in the model. Subsequently, in subsection 2.3 and subsection 2.4 2 extensions to the model are presented: respectively adding multidimensionality, and the accumulation of a proportion of an excess that is achieved by an asset.

2.1. General Problem Statement

Suppose $f$ is a random vector defined by Equation 1 (Dokuchaev, 2013a). The random vector is defined as the sum of its expected value $\mathbb{E}f$ and the integral of random changes generated
by a $d$-dimensional Wiener Process $w(t)$. The coefficient $k_f(t)$ can influence these random changes, and it may depend on the history of the Wiener Process up until the current time $t$ (Dokuchaev, 2013a).

\[ f = Ef + \int_0^T k_f(t)dw(t) \]  

(1)

The aim this problem is to find a solution function $x(t)$ that, starting from a certain point $x(0) = a$ with $a \in \mathbb{R}^n$, has the exact same value as the random vector at the terminal time $T$, so $x(T) = f$ (Dokuchaev, 2013a). Generally speaking, $x(t)$ can be described by the differential equation as can be seen in Equation 2, subject to the just before mentioned boundary conditions. In this equation, $A \in \mathbb{R}^{n \times n}$ is a given matrix, and $b \in \mathbb{R}^{n \times n}$ is a non degenerate matrix (Dokuchaev, 2013a).

\[ \frac{dx}{dt} = Ax(t) + bu(t), \quad t \in (0, T) \]  

(2)

As the solution for $x(t)$ is not unique, an optimal solution should be found. For this optimal solution in an integral norm, the value for $E \int_0^T u(t)^\top \Gamma(t)u(t)dt$ should be minimized (Dokuchaev, 2013a). The parameter $\Gamma(t) = g(t)G(t)$ is related to the penalty for fast growing $u(t)$ as $t \to T$ (Dokuchaev, 2013a). For $g(t)$, the restrictions shown in Equation 3 and Equation 4 apply over the entire time interval $(0, T]$, and $G(t) > 0$ has to be a symmetric positively defined matrix (Dokuchaev, 2013a).

\[ 0 < g(t) \leq c(T - t)^\alpha \]  

(3)

\[ g(t)^{-1} \leq c(1 + (T - t))^{-\alpha} \]  

(4)

Defining the following for parameters, the optimal solution for $u(t)$ can be expressed using Equation 5 (Dokuchaev, 2013a). In combination with Equation 2 and the specified boundary conditions, $x(t)$ can be found (Dokuchaev, 2013a).

\[ \hat{k}_\mu(t) = R(t)^{-1}k_f(t), \quad R(s) \triangleq \int_s^T Q(t)dt, \quad Q(t) = e^{A(T-t)}b\Gamma(t)^{-1}b^\top e^{A^\top(T-t)} \]

\[ \hat{\mu} = R(0)^{-1}(Ef - e^{AT}a) + M(t), \quad \text{with} \quad M(t) = \int_0^t \hat{k}_\mu(t)dw(s) \]

\[ \hat{u}(t) = \Gamma(t)^{-1}b^\top e^{A^\top(T-t)}\hat{\mu}. \]  

(5)
2.2. Application to Cash Accumulation Policy Model

This general optimal solution can now be applied to create the desired cash accumulation strategy model. Imagine an investor wanting to buy an asset with stock price $S(t)$ at a specified terminal time $T$. The stock price $S(t)$ is subject to random continuous changes $dS(t)$ that can be described by Equation 6, in which $a(t)$ can be described as a drift of the stock price, and $\sigma(t)dw(t)$ is the randomly changing part induced by the continuous Wiener Process $w(t)$ with $w(0) = 0$, as could comparably be seen in Equation 1 (Dokuchaev, 2013a).

$$dS(t) = S(t)[a(t)dt + \sigma(t)dw(t)]$$

However, for modelling purposes, a discrete Wiener Process will be applied with time step $\Delta t$, such that over the time interval $(0, T]$ $N = \frac{T}{\Delta t}$ times steps will be present. In this discretization, the change in the random walk $dW$ is normally distributed, with mean 0 and standard deviation $\sqrt{\Delta t}$ (Highman, 2013). The stock price can now be expressed using Equation 7.

$$S(t_k) = S(0) + S(t_{k-1}) \sum_{k=1}^{N} (a(t_{k-1})\Delta t + \sigma(t_{k-1})dw(t_k))$$

Consider the first and easiest case, in which there are no interest rates to consider. In this case, $A = 0$, $b = 1$, $n = 1$, and $f = S(T)$. The function $u(t)$ describes the density of cash deposits and withdrawals, in units of dollars per unit time step. Therefore, the amount of cash deposited or withdrawn within one time step can be represented by $u(t)\Delta t$ (Dokuchaev, 2013b). The value for $a$ represents the starting amount of cash in the account, which may take both positive and negative values. A negative value for $a$ would indicate a debt at starting time 0. With these assumptions, the function $R(s)$ simplifies to Equation 8 and the optimal solution for $u(t)$ is found using Equation 9 (Dokuchaev, 2013b). Integrating $u(t)$ over the entire interval ensures that the integral matches the terminal value $f = S(T)$ exactly, as according to Equation 10 (Dokuchaev, 2013b).

$$R(s) \triangleq \int_{s}^{T} \Gamma(t)^{-1} dt.$$

$$u(t) = \Gamma(t)^{-1} \left[ R(0)^{-1}(S(0) - a) + \int_{0}^{t} R(s)^{-1}dS(t) \right]$$

$$\int_{0}^{T} u(t) dt = f$$

In the next case, there are interest rates to consider, and for both loans and savings
the interest rate is \( r \geq 0 \). For this case, \( A = r \). Now, the functions for \( Q(t) \) and \( R(s) \) can be evaluated using Equation 11 and Equation 12 (Dokuchaev, 2013b). Subsequently, the cash flow density \( u(t) \) can be evaluated using Equation 13 (Dokuchaev, 2013b). To find the final amount of cash in the account at the terminal time \( T \), the cash flows deposited at time \( t \) should be accumulated using the accumulation factor \( e^{r(T-t)} \). In accordance with Equation 14, the integral of the accumulated cash flows over the entire interval will match the terminal value \( f = S(T) \) exactly (Dokuchaev, 2013b).

\[
Q(t) = e^{2r(T-t)} \Gamma(t)^{-1}
\]  

(11)

\[
R(s) = \int_s^T Q(t) dt
\]  

(12)

\[
u(t) = \Gamma(t)^{-1} \left[ R(0)^{-1}(S(0) - e^{rT}a) + \int_0^t R(s)^{-1}dS(t) \right]
\]  

(13)

\[
\int_0^T e^{r(T-t)} u(t) = f
\]  

(14)

This last case is the case that will be used for the model, as for \( A = r = 0 \), all equations will be similar to the first case. First, for each time step \( M(t_k) \) will be calculated using Equation 15. Subsequently, \( \mu(t_k) \) will be calculated using Equation 16. Afterwards, the cash flow density \( u(t_k) \) will evaluated using Equation 17. Finally, the amount of accumulated cash in the account \( x(t) \) can be evaluated using Equation 18. In this equation, for each time step the cash that is deposited or withdrawn, is accumulated to the end of the time step using the accumulation factor \( e^{r\Delta t} \).

\[
M(t_k) = M(t_{k-1}) + R(t_k)^{-1}dW(t_k)
\]  

(15)

\[
\mu(t_k) = R(0)^{-1}(S(0) - e^{rT}a) + M(t_k)
\]  

(16)

\[
u(t_k) = \Gamma(t)^{-1} e^{r(T-t_k)} \mu(t_k)
\]  

(17)

\[
x(t_k) = (x(t_{k-1}) + u(t_k)\Delta t) e^{r\Delta t}
\]  

(18)

Plotting both \( S(t) \) and \( x(t) \) will give a visual representation of the development of the stock price \( S(t) \) and the amount of cash \( x(t) \) over time. Using the above equations, at terminal time \( T \), \( S(T) = x(T) \).
2.3. Extension of Model: Accumulation Cash for Set of Different Assets

The current model is suitable if an investor wants to buy one or a multiple of the same asset. Now, the model will be extended for the case that this investor wants to buy a package consisting of different assets at the terminal time.

Instead of all equations being scalar equations, for this situation, they will have to be vector equations. In Equation 19, \( S(t_k) \) now represents the vector in \( \mathbb{R}^m \), where \( m \) stands for the number of different assets, containing all individual stock prices at time step \( t_k \). Similarly to the case in which only one stock was bought, every different stock will experience changes induced by an independent Wiener Process. Therefore, each entry of \( dW(t_k) \) contains a random number from a normal distribution with mean 0 and standard deviation \( \sqrt{\Delta t} \). This will ensure that each stock price changes independently, and that the changes in one stock price do not influence any other stock price.

\[
S(t_k) = S(0) + S(t_{k-1}) \sum_{k=1}^{N} (a(t_{k-1})\Delta t + \sigma(t_{k-1})dW(t_k))
\] (19)

To solve this problem of buying multiple different assets, the model will now initiate an individual cash accumulation process for each different asset. Note that all deposits and withdrawals for each individual process are from the same bank account. Therefore, the interest rate \( r \) will be the same for each process.

Equation 20 and Equation 21 are the vector equivalent versions of Equation 15 and Equation 16, with all vectors in \( \mathbb{R}^m \). The vector \( a \) represents the starting amount of cash that is available for each individual cash accumulation process. If a company has for instance $80 as a starting amount of cash, and wishes to buy 4 different assets, it will be assumed that $20 is available as a starting amount for each individual process.

\[
M(t_k) = M(t_{k-1}) + R(t_k)^{-1}dW(t_k)
\] (20)

\[
\mu(t_k) = R(0)^{-1}(S(0) - e^{(rT)}a) + M(t_k)
\] (21)

The vector function \( u(t_k) \) now represents the cash flow density for each individual process, and is given by Equation 22. Subsequently, the amount of cash that is accumulated in each process \( x(t_k) \), can be calculated using Equation 23. These equations ensure that at the terminal time, each individual process exactly matches its corresponding stock price at the
terminal time.

\[ u(t_k) = \Gamma(t)^{-1} e^{r(T-t_k)} \mu(t_k) \]  

(22)

\[ x(t_k) = (x(t_{k-1}) + u(t_k) \Delta t) e^{r \Delta t} \]  

(23)

However, since all cash for each process will be accumulated in the same bank account, it is more interesting to know what the total amount of cash is that has to be deposited or withdrawn at each time step. This can be represented by \( \text{sum}(u(t_k)) \Delta t \). The total amount of cash that is available in the account at each time step, can correspondingly be represented by \( \text{sum}(x(t_k)) \). The total cash in the account at the terminal time will now exactly match the sum of all different stock prices at the terminal time.

### 2.4. Extension of Model: Accumulation of Proportion of Equity Excess

Another extension of the model can be obtained when looking at the problem when the accumulated amount of cash has to be a certain proportion of the positive difference between the stock price and a certain strike price that is set at the start of the process. If there’s no difference, or the difference turns out to be negative, the function will take value 0. Therefore, the function \( f \) will be given by Equation 24 (Dokuchaev, 2013b). In this, equation \( c \) represents the proportion, and \( K \) represents the preset strike price that will be used to calculate the achieved excess. An alternative expression for \( f \), that is used in the model, is Equation 25 (Dokuchaev, 2013b).

\[ f = c \max(S(T) - K, 0) \]  

(24)

\[ f = H(S(0), 0) + \int_0^T \frac{\partial H}{\partial x}(S(t), t) dS(t). \]  

(25)

In this formulation, \( H(S(t), t) \) can be calculated using the Black-Scholes formula for a call option (Dokuchaev, 2013b). Using this formula, an estimate can be calculated for an European call option. The expression for the Black-Scholes equation can be seen in Equation 26 (Hull, 2012). In this equation, \( N(\cdot) \) stands for the cumulative normal distribution. The variables \( d_1 \) and \( d_2 \) can be calculated using Equation 27 and Equation 28 respectively (Hull, 2012). Note that \( \sigma \) in these equations should be constant, so \( \sigma(t) = \sigma \) compared to Equation 6 (Dokuchaev, 2013; Hull, 2012).

\[ H(S(t), t) = N(d_1) S(t) - N(d_2) Ke^{-r(T-t)} \]  

(26)
\[ d_1 = \frac{1}{\sigma \sqrt{T - t}} \left[ \ln \left( \frac{S(t)}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \right] \] (27)

\[ d_2 = d_1 - \sigma \sqrt{T - t} \] (28)

Comparing Equation 26 with Equation 1, it can be seen that the coefficient \( k_f \) from Equation 1 can be calculated using Equation 29 again assuming that \( \sigma \) is constant (Dokuchaev, 2013b). Subsequently, \( \frac{\partial H}{\partial x} \) can be calculated using Equation 30 (Haug, 2007). Whereas \( k_f(t) \) could be any random constant in Equation 1, it has to be calculated for each independent time step for Equation 25. Therefore, \( k_f(t) \) can now be considered a function of time, that depends on the history of the random claim itself. The cash flow density for this problem can now be calculated using Equation 31 (Dokuchaev, 2013b).

\[ k_f = \frac{\partial H}{\partial x}(S(t), t) \sigma dS(t) \] (29)

\[ \frac{\partial H}{\partial x}(S(t), t) = N(d_1) \] (30)

\[ u(t) = c \Gamma(t)^{-1} \left[ R(0)^{-1} H(S(0), 0) + \int_0^t R(s)^{-1} \frac{\partial H}{\partial x}(S(t), t) \sigma dS(t) \right] \] (31)

For the implementation of this application in the model, several equations have to be adapted slightly. Whereas in the in the previous example the stock prices could directly be used for the replication, now the function \( f \) will have to be calculated first. Therefore, for each time step, the value for \( f(t_k) \) will be calculated using Equation 32. Subsequently, \( M(t_k) \) and \( \mu(t_k) \) will be calculated according to Equation 33 and Equation 34 respectively. The cash flow density \( u(t_k) \) can now be represented by Equation 35.

\[ f(t_k) = f(t_{k-1}) + N(d_1(t_k)) \sigma dS(t_k) \] (32)

\[ M(t_k) = M(t_{k-1}) + R(t)^{-1} N(d_1(t_k)) \sigma dS(t_k) \] (33)

\[ \mu(t_k) = R(0)^{-1} H(S(0), 0) + M(t_k) \] (34)

\[ u(t_k) = c \Gamma(t)^1 e^{A(T-t)} \mu(t_k) \] (35)
3. Numerical Experiments

This section will present the main results obtained from the implementation from the last section into the MATLAB model. Different experiments will be presented in order to show how the model behaves. In subsection 3.1, 3 numerical examples will be presented, and in subsection 3.2, 2 experiments regarding the time performance of the model will be presented.

3.1. Application of Model to Example Problems

3.1.1. Accumulating Cash to Buy Asset after 1 Year

For this first example, a company wanting to accumulate enough cash in order to be able to buy an asset after exactly 1 year is considered. The price of this asset at time 0 $S(0)$ is $150. The company currently has $50 available in the account. Given an interest rate $r$ of 12% for both loans and savings, the company wishes to automatically make a withdrawal or deposit once a day such that after 1 year, the company has exactly enough money in order to buy this asset.

For this rendering, the following parameters have entered into the program. The drift of the stock price $a(t)$ is assumed to be 0, and the coefficient $\sigma(t)$, that influences the random changes of the stock price is assumed to be 0.5. Note that these two parameters can have any value in $\mathbb{R}$, or may be any function of time in order to match the stock price at the terminal time. Furthermore, $A = r = 0.12, b = 1, \Gamma(t) = 1$. Since the company wants to update their account once a day for 1 year, the number of time steps $N$ is 365.

The rendering of the model for this example with the current parameters can be seen in Figure 1. Both the development of the stock price and the amount of cash in the account can be seen. Looking at the plot, it can be noted that the changes in the stock price might be relatively big, but the changes in the amount of accumulated cash remain quite smoothly. Therefore, the amount of cash deposited or withdrawn is relatively small at each update. At the terminal time of 1 year, it can be seen that the amount of cash matches the stock price as desired.

If the company would have liked to buy a multiple of this asset, the model will have to be adapted slightly. If the starting stock price $S(0)$ as well as the observed changes $dS$ at each time step are multiplied by the desired number of assets, running the model will ensure that cash is accumulated to buy this number of assets at the terminal time of 1 year.
3.1.2. Accumulating Cash to Buy Set of 2 Independent Stocks after 1 Year

For the second example, now consider the same company wanting to buy a set of two different, independently behaving assets after exactly 1 year. At time 0, the first asset has a stock price $S(0)$ of $200, whereas the second asset has a stock price $S(0)$ at time 0 of $400. Currently, the company has a starting amount of cash of $40 and wishes to update their account once every day, given that the interest rate $r$ is now 30%.

The model will now be used to simultaneously accumulate cash to exactly buy both assets after one year. Similarly to the first example, $a(t) = 0$ and $\sigma(t) = 0.5$. The other parameters are $A = r = 0.3$, $b = 1$, and $\Gamma(t) = 1$. Updating the account once a day for 1 year leads to the number of time steps $N$ being equal to 365.

The model will work as follows. Each time step, both stock prices will change independently. Subsequently, the model will try to match both stock prices independently by withdrawing or depositing the right amount of cash. As the starting amount of cash is $40, it will be assumed that the starting value for matching each independent stock price is $20. So, by depositing the independent amounts of cash at each time step in the same account, this will ensure that there will be enough cash in the account after one year in order to be able to exactly buy both assets at the terminal time of 1 year.
In Figure 2, two renderings for this example are presented. In Figure 2a, the behaviour of the stock prices of both assets can be found. As can be seen, the stock prices behave completely independently, and the changes of one stock price do not influence the changes the other stock price. Subsequently, in Figure 2b, the sum of the two independent stock prices is presented as well as the total amount of cash that is accumulated in the account. As can be seen, now the total amount of accumulated cash at the terminal time of 1 year, matches the sum of the two stock prices.

Similarly to the first example, the model should be changed slightly if the company wants to buy a multiple of each of the assets. As in the model $S(0)$ and $dS$ are now 2x1 vectors, the inner product of these vectors and the 2x1 vector containing the desired number of assets should be taken. Subsequently, running the program will ensure that the accumulated amount of cash in order to exactly buy the desired amount of each asset.

### 3.1.3. Accumulating 50 % of the Excess Achieved by Equity over 2 Years

For the last case example, the accumulated cash has to be 50% of the excess achieved by a certain equity. The starting price $S(0)$ is $75 and the strike price $K$ is $30. Cash has to be accumulated in 2 years time, updating the account once every 2 days. The interest rate $r$ for this example is 3%.
For this example, \( a(t) = 0, \sigma(t) = 0.3 \). The other parameters are \( A = r = 0.03, b = 1, \) and \( \Gamma(t) = 1 \). The proportion factor \( c \) is 0.5. Updating the account once every 2 days for 2 years, leads to \( N = 365 \).

![Figure 3: Rendering for Cash Accumulation of 50% of the Excess](image)

Figure 3 shows the rendering of the model for this example. Instead the stock price being plotted as in the other examples, now the proportion of the current excess is plotted. As expected, the accumulated cash matches this excess at the terminal time of 2 years.

### 3.2. Model Performance Analysis

This subsection will describe the influence of choosing different parameters on the overall performance of the model. Both the effect of changing the time step size and the number of dimensions are considered.

#### 3.2.1. Impact of Discretization Rate

First the impact of changing the time step size is considered. Changing the time step size changes the way the stock price behaves. As the changes in the random walks \( dW \) are normally distributed with mean 0, and standard deviation \( \sqrt{\Delta t} \), for smaller time steps \( \Delta t \) the change at each time step is also expected to be smaller. However, as the time steps are smaller, this means that over the entire time interval, more time steps will have to be evaluated. Therefore, the absolute changes that will be experienced are comparable to the
case with larger time steps. This can also be seen in Figure 4 where two the model is run with two different time step sizes.

![Figure 4](image-url)  
Figure 4: Renderings for different values of $\Delta t$

To calculate how different time step sizes influence the overall efficiency of the program, the model is ran and timed at a different number of time steps $N$. At each number of time steps, the model is ran and timed 5 times. The average time for these five runs is used a measure for the time performance at that specific number of time steps. The results can be seen in Figure 5. The relationship between the number of time step appears to be linear. When $N = 10,000$, the time it takes to run the model has increased by 550%, compared to $N = 5$.

### 3.2.2. Impact of Dimensionality

Next, the effect of changing the number of different stocks $m$ on the efficiency of the program is analyzed. Similarly to the different time step sizes, for different numbers the model is run and timed 5 times. Except for the number of different stocks, all other parameters are the same to allow for fair comparison.

The average of these 5 runs is used as a measure of the model performance for that specific number of different stocks. As can be seen in Figure 6, the relationship between the number of different stocks and the time performance of the model seems to be linear. At $m = 10,000$, the time it takes for the model to run, is approximately 35% higher compared...
to $m = 5$. This means that if the appropriate starting conditions are defined, a set of 10,000 different independently behaving stock prices can be matched at the terminal time, and that it takes only 35% more effort time wise to calculate.
4. Discussion

This section will present the main discussion points that arise when analyzing the model. In subsection 4.1, a modeling error due to discretization will be discussed, and subsection 4.2 discusses an assumption regarding interest rates.

4.1. Error at Last Time Step due to Discretization

When analyzing the plots generated by the model, it should be noted that the very last time step is not plotted. This can clearly be seen in Figure 7. The reason for this can be found in Equation 12, which is being used to calculate \( R(s) \). At the terminal time \( T \), \( R(T) \) will have to be 0. Subsequently, Equation 13 makes use of \( R(s)^{-1} \). With \( R(T) = 0 \), this automatically means that \( R(T)^{-1} \rightarrow \infty \). Therefore, also \( u(T) \rightarrow \infty \). As this result can’t be right, this very last time step will not be considered and will be excluded from the plot.

![Figure 7: Rendering of model at \( \Delta t = 0.05 \)](image)

This result seems contradictory to the method, which is designed to ensure that at the terminal time \( x(T) = S(T) \). However, this turns out only to be true when the equations are evaluated continuously in time. The singularities that arise at the terminal time are defined, such that the desired solution can be obtained. When making use of the discretized model, these singularities are not defined, and therefore, the observed error is obtained.

The impact of this error, however, is not significant. Whereas for relatively large time steps, such as in Figure 7, this error can be in the order of several dollars, for small time steps such as the ones used in Section 3, the order of error might be several cents.
4.2. Assumption on Constant Interest Rate

The next discussion point will deal with the assumption in the model that the interest rate \( r \) is constant over the entire time interval in which the cash is accumulated. However, in reality this is not the case.

Banks can change their interest rates on a daily basis. Even though the daily changes in the interest rates may be relatively small, this could still mean that at the terminal time, which may be in the course of years, the interest is significantly different compared to the time at which the cash accumulation started.

Therefore, in future research, a way has to be found to incorporate this changing interest rate into the model. A suggestion to model the interest rates, is to model it with its own Wiener Process, like the stock prices are modeled. However, it should be verified if the equations still produce the right outcome. If not, further research must be conducted such that the equations are found to produce the right income with changing interest rates.

5. Conclusions

The aim of this report was to create a model to solve the problem of cash accumulation strategies for purposes of buying things with an unknown future price, like assets, at a pre-defined terminal time. The model is based on an optimal replication of random claims with ordinary integrals. For the model the stock prices were modeled after a discretized Wiener Process. Subsequently, all other provided equations for this problem were discretized in order to be simulated in the model.

The model is able to solve three different problems regarding buying assets. The first is to accumulate cash to buy one or a multiple of the same asset. The second is accumulating cash to buy a set consisting of different, independently behaving assets. Finally, the model can accumulate a proportion of the excess achieved by a certain asset.

One drawback of the model consists of the fact that due to discretization the last time step cannot be modeled correctly. However, this is not of great significance when a large number of time steps is used, and it only leads to minor inaccuracies in the end result. Further research could be conducted in correcting the assumption that interest rates are constant over the entire interval.
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