Ideals of general linear Lie algebras of infinite-dimensional vector spaces

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Abstract

Let $V$ be an infinite-dimensional vector space over a field of characteristic not equal to 2. We classify ideals of the Lie algebra $\mathfrak{gl}(V)$ of all linear transformations of the space $V$.

Introduction

Let $F$ be a field of characteristic not equal to 2. Any associative algebra $A$ over the field $F$ gives rise to the Lie algebra $A^{(\cdot)} = (A, [a, b] = ab - ba)$. Let $V$ be an infinite-dimensional vector space over $F$. In this paper, we consider the algebras $\text{End}_F(V)$ of all linear transformations $V \to V$ and $\mathfrak{gl}(V) = \text{End}_F(V)^{(\cdot)}$ the general Lie algebra of $V$.

For a cardinal $\alpha \leq \dim_F V$ denote by $I_\alpha$ the ideal of all linear transformations $\varphi : V \to V$ such that $\dim_\mathbb{F} \varphi(V)$ is less than $\alpha$. In particular, for the countable

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cardinal $\aleph_0$ the ideal $I_{\aleph_0}$ consists of all linear transformations of finite ranges. Let $\text{Id}_V$ denote the identity transformation.

N. Jacobson \[9\] classified ideals of the associative ring $\text{End}_F(V)$ of linear transformations of $V$. He proved that:

\textit{all ideals of the ring $\text{End}_F(V)$ of linear transformations of an infinite-dimensional vector space $V$ are: $(0)$, $I_\alpha$, where $\aleph_0 \leq \alpha \leq \dim_F V$, and $\text{End}_F(V)$.}

The purpose of this paper is the classification of ideals of the algebra $\mathfrak{gl}(V)$.

**Theorem 1.** Let $V$ be an infinite-dimensional vector space over a field $F$ of characteristic not equal to $2$. Every ideal of $\mathfrak{gl}(V)$ belongs to one of the following families:

1. $\{0\}, F \cdot \text{Id}_V, \mathfrak{gl}(V)$;
2. $I_\alpha$ or $F \cdot \text{Id}_V + I_\alpha$, where $\aleph_0 \leq \alpha \leq \dim_F V$;
3. any subspace $U$, 
   \[ [I_{\aleph_0}, \mathfrak{gl}(V)] \subseteq U \subseteq F \cdot \text{Id}_V + I_{\aleph_0}. \]

Moreover, the co-dimension of $[I_{\aleph_0}, \mathfrak{gl}(V)]$ in $F \cdot \text{Id}_V + I_{\aleph_0}$ is 2. Therefore, if the field $F$ is infinite then the family (3) is an infinite family of ideals.

I. Penkov and V. Serganova \[14\] classified ideals of countable-dimensional Mackey Lie algebras and I. Penkov and A. Petukhov \[13\] studied ideals of the enveloping algebra of an infinite-dimensional Lie algebra viewed as the union of a chain of embeddings of simple finite-dimensional Lie algebras. The paper \[11\] contains the description of ideals of the algebra $\mathfrak{gl}(V)$ for a countable-dimensional vector space $V$.

In \[5\], it was shown that the Lie algebra $\mathfrak{gl}(V)$ of an infinite-dimensional vector space $V$ is perfect, i.e. $\mathfrak{gl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$.

A. Rosenberg \[15\] (see also \[12\]) studied normal subgroups of the group $GL(V)$ of invertible linear transformations on $V$.

For some results on the structure of other associative and Lie algebras of infinite matrices see \[2, 3, 4, 6, 7, 8\].

### 1 Ideas of the algebra $\mathfrak{gl}(V)$

Let $U$ be a subset of an associative algebra $A$. Denote by $\text{id}_A(U)$ the ideal of the algebra $A$, which is generated $U$.

**Lemma 1** (Alahmadi-Alsulami, \[11\]). Let $A$ be an associative algebra over a field $F$ of characteristic not equal to $2$ and let $U$ be an ideal of the Lie algebra $A(-)$. Then

\[ [\text{id}_A([U, U]), A] \subseteq U. \]
An associative algebra $A$ is called a prime algebra if for any nonzero ideals $I, J \in A$ the product $I \cdot J$ is also nonzero.

The following lemma is due to I. Herstein [10], Lemma 2 ([10]).

**Lemma 3.** The factor-algebra $\text{End}_F(V)/I_\alpha$ is prime for any $\alpha$, $\aleph_0 \leq \alpha \leq \dim_F V$.

**Proof.** Suppose that the algebra $\text{End}_F(V)/I_\alpha$ has two proper ideals whose product is $(0)$. N. Jacobson [9] proved that an arbitrary proper ideal of the algebra $\text{End}_F(V)$ looks as $I_\alpha$, $\aleph_0 \leq \alpha \leq \dim_F V$. Hence, by Jacobson’s Theorem these ideals are $I_\beta/I_\alpha$ and $I_\gamma/I_\alpha$, where $\alpha < \beta, \gamma \leq \dim_F V$. If $\beta \leq \gamma$ then $I_\beta \subseteq I_\gamma$ and

$$I_\beta^2 \subseteq I_\alpha.$$ 

There exists a subspace $W \subset V$, $\dim_F W = \alpha$. Let $\rho : V \to W$ be a projection of $V$ onto $W$. Then $\rho \in I_\beta \setminus I_\alpha$, $\rho^2 = \rho$, a contradiction. \hfill \square

**Proposition 1.** The center $C$ of the algebra $\text{End}_F(V)/I_\alpha$ is

$$(\mathbb{F} \cdot \text{Id}_V + I_\alpha)/I_\alpha,$$

where $\alpha$, $\aleph_0 \leq \alpha \leq \dim_F V$.

We need several lemmas to prove this proposition. Let $z \in \text{End}_F(V)$ be an element not lying in $I_\alpha$ and such that

$$[z, \text{End}_F(V)] \subseteq I_\alpha.$$

**Lemma 4.** $\dim_\mathbb{F} \ker z < \alpha$.

**Proof.** The subspace $I = I_\alpha + \text{End}_F(V)z$ of $\text{End}_F(V)$ is an ideal. The ideal $I$ contains $z$ and, therefore, is strictly larger then $I_\alpha$. By Jacobson’s Theorem, all ideals of $\text{End}_F(V)$ are of the types $(0)$, $I_\beta$, where $\aleph_0 \leq \beta \leq \dim_\mathbb{F} V$, and $\text{End}_F(V)$.

**Case 1.** $I_\alpha + \text{End}_F(V)z = I_\beta$, $\alpha < \beta \leq \dim_\mathbb{F} V$.

Suppose that $\dim_\mathbb{F} \ker z \geq \alpha$. Then there exists a subspace $V' \subseteq \ker z$ such that $\dim_\mathbb{F} V' = \alpha$.

Let $p$ be a projection $p : V \to V'$, that is $p(V) \subseteq V'$ and $p(v) = v$ for every element $v \in V'$. Then $p \in I_\beta$. Hence, there exist elements $\varphi \in I_\alpha$ and $a \in \text{End}_F(V)$ such that $p = \varphi + az$. For an arbitrary element $v \in V'$ we have

$$v = p(v) = \varphi(v) + a(z(v)).$$
Since $V' \subseteq \ker z$, it follows that $z(v) = 0$. Hence, $\varphi(v) = v$ for all elements of $V'$. Therefore, $\dim_F \varphi(V) \geq \alpha$, which contradicts the inclusion $\varphi \in \mathcal{I}_\alpha$.

**Case 2.** $I_\alpha + \operatorname{End}_F(V)z = \operatorname{End}_F(V)$.

Recall that $\operatorname{Id}_V$ is the identity transformation on $V$. Again there exist elements $\varphi \in \mathcal{I}_\alpha$ and $a \in \operatorname{End}_F(V)$ such that

$$\varphi(v) + az = \operatorname{Id}_V.$$

If $v \in \ker z$ then

$$v = \varphi(v) + a(z(v)) = \varphi(v),$$

hence $\ker z \subseteq \varphi(V)$. Since $\varphi \in \mathcal{I}_\alpha$, it implies that $\dim_F \ker z < \alpha$.

**Lemma 5.** If $W$ is a subspace of $V$ and

$$W \cap z(W) = (0)$$

then $\dim_F W < \alpha$.

**Proof.** There exists a linear transformation $\varphi \in \operatorname{End}_F(V)$ such that $\varphi(W) = (0)$ and $\varphi(v) = v$ for an arbitrary element $v \in z(W)$.

By the assumption on $z$, the image of the linear transformation $z\varphi - \varphi z$ has dimension less than $\alpha$.

For an arbitrary element $w \in W$ we have $z(\varphi(w)) = 0$, $\varphi(z(w)) = z(w)$. Therefore, $z(w) = (\varphi z - z\varphi)(w)$. The subspace $z(W)$ lies in the image of $\varphi z - z\varphi$, hence $\dim_F z(W) < \alpha$. If $\dim_F z(W) < \alpha$ and $\dim_F \ker z < \alpha$ then $\dim_F W < \alpha$.

**Proof of Proposition**

Let $W$ be a maximal subspace of $V$ such that $W \cap z(W) = (0)$ (it exists by Zorn’s Lemma). Then $\dim_F W < \alpha$.

Choose an element $v \in V \setminus (W \oplus z(W))$. Then by maximality of $W$, we have

$$(W + Fv) \cap z(W + Fv) \neq (0).$$

Choose an element $w \in W$ and a scalar $\xi \in F$ such that

$$0 \neq z(w + \xi v) \in W + Fv.$$

**Case 1.** $\xi = 0$. Then $0 \neq z(w) \in W + Fv$, which is impossible since $z(W) \cap W = (0)$ and $v \notin W \oplus z(W)$.

**Case 2.** If $\xi \neq 0$ then $z(w) + \xi z(v) = w' + \eta v$, where $w' \in W$, $\eta \in F$. So,

$$z(v) = \frac{\eta}{\xi} v \mod (W \oplus z(W)).$$

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We proved that for an arbitrary element \( v \in V \setminus (W \oplus z(W)) \) there exists a scalar \( t_v \in \mathbb{F} \) such that
\[
z(v) = t_v v \mod (W \oplus z(W)).
\]

Let \( v', v'' \in V \) be linearly independent modulo \( W \oplus z(W) \). Then
\[
z(v') = t_{v'} v' \mod (W \oplus z(W)), \tag{1}
\]
\[
z(v'') = t_{v''} v'' \mod (W \oplus z(W)), \tag{2}
\]
\[
z(v' + v'') = t_{v' + v''}(v' + v'') \mod (W \oplus z(W)). \tag{3}
\]
Subtracting the equalities (1) and (2) from the equality (3) we get
\[
(t_{v' + v''} - t_{v'}) v' + (t_{v' + v''} - t_{v''}) v'' \in (W \oplus z(W)).
\]
In view of the linear independence of \( v', v'' \) modulo \( W \oplus z(W) \) we get
\[
t_{v'} = t_{v''} = t_{v' + v''}.
\]

Hence there exists a scalar \( t \in \mathbb{F} \) such that
\[
z(v) = tv \mod (W \oplus z(W))
\]
for an arbitrary \( v \in V \). The image of the linear transformation \( z - t \cdot \text{Id}_V \) lies in \( W \oplus z(W) \). Hence
\[
\dim_{\mathbb{F}}(z - t \cdot \text{Id}_V) < \alpha \quad \text{and} \quad z - t \cdot \text{Id}_V \in I_\alpha.
\]
This completes the proof of the proposition. \( \square \)

**Lemma 6.**

1. Let \( \aleph_0 < \alpha \leq \dim_{\mathbb{F}} V \). Then \( I_\alpha = [I_\alpha, I_\alpha] \).
2. \( [I_{\aleph_0}, \mathfrak{g}(V)] = [I_{\aleph_0}, I_{\aleph_0}] \) has co-dimension 1 in \( I_{\aleph_0} \).

**Proof.** Let \( \varphi \in I_\alpha, \dim_{\mathbb{F}} \varphi(V) < \alpha \). If \( \aleph_0 < \alpha \) then there exists a subspace \( V' \subset V \) such that
\[
\varphi(V) \subseteq V' \quad \text{and} \quad \aleph_0 \leq \dim_{\mathbb{F}} V' < \alpha.
\]
Choose a subspace \( V'' \subset V \) such that \( V = V' \oplus V'' \) is a direct sum. The linear transformation \( \varphi \) can be decomposed as \( \varphi = \varphi_1 + \varphi_2 \), where
\[
\varphi_1 : V' \to V', \quad \varphi_1(V'') = (0), \quad \text{and} \quad \varphi_2 : V'' \to V', \quad \varphi_2(V'') = (0).
\]
Let $p$ be a projection from $V$ to $V'$, i.e.

$$p|_{V'} = \text{Id}_{V'}, \quad p(V'') = (0).$$

We notice that the images of $\varphi_1, \varphi_2, p$ lie in $V'$. Therefore, $\varphi_1, \varphi_2, p \in I_\alpha$.

By [5], we have

$$\varphi_1|_{V'} \in [\mathfrak{gl}(V'), \mathfrak{gl}(V')].$$

Hence $\varphi_1 \in [I_\alpha, I_\alpha]$. Furthermore,

$$\varphi_2 = [p, \varphi_2] \in [I_\alpha, I_\alpha].$$

We proved that $\varphi \in [I_\alpha, I_\alpha]$. This completes the proof of part (1) of the lemma.

Now, consider the ideal $I_{\aleph_0}$. A linear transformation $\varphi$ lies in $I_{\aleph_0}$ if and only if the subspace $V' = \varphi(V)$ is finite-dimensional.

There exists a finite-dimensional subspace $V'' \subset V$ such that $V' \subset V''$ and $\varphi(V'') = V'$. Let $\text{tr}(\varphi)$ be the trace of the restriction

$$\varphi|_{V''} \in \mathfrak{gl}(V').$$

It is easy to see that

(i) $\text{tr}(\varphi)$ does not depend on a choice of the subspace $V''$, 

(ii) $\text{tr} : I_{\aleph_0} \to \mathbb{F}$ is a linear functional, 

(iii) $\text{tr}(\varphi) = 0$ if and only if $\varphi \in [I_{\aleph_0}, I_{\aleph_0}]$.

This implies that $[I_{\aleph_0}, I_{\aleph_0}]$ has co-dimension 1 in $I_{\aleph_0}$.

It remains to show that

$$[I_{\aleph_0}, \mathfrak{gl}(V)] = [I_{\aleph_0}, I_{\aleph_0}]$$

or equivalently

$$\text{tr}([I_{\aleph_0}, \mathfrak{gl}(V)]) = (0).$$

Choose $\varphi \in I_{\aleph_0}$ and $\psi \in \mathfrak{gl}(V)$. Since $\dim \varphi(V) < \infty$ the subspace $\ker \varphi$ has a finite co-dimension in $V$. Hence, there exists a finite-dimensional subspace $V_1 \subset V$ such that $\varphi(V) \subset V_1$, $\varphi(V_1) = \varphi(V)$ and $V = V_1 + \ker \varphi$. Let $\dim V_1 = n$. Choose a subspace $V_2 \subset \ker \varphi$ such that

$$\ker \varphi = (V_1 \cap \ker \varphi) \oplus V_2$$
is a direct sum. Then $V = V_1 \oplus V_2$ is a direct sum. Let $B_1$, $B_2$ be bases of the subspaces $V_1$, $V_2$, respectively. In the basis $B = B_1 \cup B_2$ of the vector space $V$ the linear transformations $\varphi$, $\psi$ have (infinite) matrices

\[
\begin{pmatrix}
  a & 0 \\
  0 & 0
\end{pmatrix},
\begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\]

respectively, where the blocks $a$, $b_{11}$ are $n \times n$ matrices. We have

\[
\begin{pmatrix}
  a & 0 \\
  0 & 0
\end{pmatrix},
\begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix} = \begin{pmatrix}
  [a, b_{11}] & ab_{12} \\
  -b_{21}a & 0
\end{pmatrix}.
\]

The trace of this infinite matrix is equal to the trace of the $n \times n$ matrix $[a, b_{11}]$, i.e. is equal to 0. This completes the proof of the lemma.

**Proof of Theorem**. Let $U$ be a proper ideal of the Lie algebra $\mathfrak{gl}(V)$. Let $A = \text{End}_F(V)$. Consider the ideal $\text{id}_A([U, U])$. By Jacobson’s Theorem, $\text{id}_A([U, U])$ equals $(0)$, $I_\alpha$ for some $\alpha$, $\aleph_0 \leq \alpha \leq \dim_F V$, or $A$. We will consider each of these cases separately.

**Case 1.** Suppose that $\text{id}_A([U, U]) = A$. Then by Lemma we have $[\text{id}_A([U, U]), A] = [A, A] \subseteq U$. By [5], we have $U = \mathfrak{gl}(V)$, which contradicts the assumption that $U$ is proper.

**Case 2.** Suppose that $\text{id}_A([U, U]) = (0)$. Then $[U, U] = (0)$. It is easy to see that the algebra $A$ is prime. Indeed, let $\varphi, \psi \in A$ be nonzero linear transformations $\varphi(v) \neq 0$, $\psi(w) \neq 0$, where $v, w \in V$. There exists a linear transformation $\chi : V \to V$ such that $\chi(\varphi(v)) = w$. Then

$$
\psi \chi \varphi(v) = \psi(w) \neq 0,
$$

hence $\psi A \varphi \neq (0)$. By Lemma $U$ lies in the center of the algebra $A$. Hence $U = F \cdot \text{Id}_V$.

**Case 3.** Now, let $\text{id}_A([U, U]) = I_\alpha$, $\aleph_0 \leq \alpha \leq \dim_F V$. The ideal $(U + I_\alpha)/I_\alpha$ of the Lie algebra $(A/I_\alpha)/(-)$ is abelian, i.e.

$$
[(U + I_\alpha)/I_\alpha, (U + I_\alpha)/I_\alpha] = (0).
$$

Lemma implies that the algebra $A/I_\alpha$ is prime. By Lemma the ideal $(U + I_\alpha)/I_\alpha$ lies in the center $C$ of the algebra $A/I_\alpha$. By Proposition the center $C$ is $(F \cdot \text{Id}_V + I_\alpha)/I_\alpha$. Hence

$$
U \subseteq F \cdot \text{Id}_V + I_\alpha.
$$
On the other hand, by Lemma 1, 
\[ [I_\alpha, A] \subseteq U \subseteq \mathbb{F} \cdot \text{Id}_V + I_\alpha. \]

If \( \aleph_0 < \alpha \) then, by Lemma 6 (1),
\[ I_\alpha \subseteq U \subseteq \mathbb{F} \cdot \text{Id}_V + I_\alpha, \]
which implies \( U = I_\alpha \) or \( U = \mathbb{F} \cdot \text{Id}_V + I_\alpha \).

Let now \( \alpha = \aleph_0 \). Then, by Lemma 6 (2), the co-dimension of \([I_{\aleph_0}, A] = [I_{\aleph_0}, I_{\aleph_0}]\)
in \( \mathbb{F} \cdot \text{Id}_V + I_{\aleph_0} \) is equal to 2. From proved above follows that for an arbitrary subspace \( U \),
\[ [I_{\aleph_0}, A] \subseteq U \subseteq \mathbb{F} \cdot \text{Id}_V + I_{\aleph_0}, \]
we have
\[ [U, A] \subseteq [\mathbb{F} \cdot \text{Id}_V + I_{\aleph_0}, A] = [I_{\aleph_0}, A] \subseteq U. \]
Hence, \( U \) is an ideal of the Lie algebra \( \mathfrak{gl}(V) \).

Notice that in the case of a countable-dimensional vector space \( V \) our description of ideals coincides with the description of ideals in [11].

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