Correlations and Binding in 4D Dynamical Triangulation

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We study correlations on the euclidean spacetimes generated in Monte Carlo simulations of the model. In the elongated phase, curvature correlations appear to fall off like a fractional power. Near the transition to the crumpled phase this power is consistent with 4. We also present improved data of our computations of the binding energy of test particles.

1. Introduction

The canonical partition function of the dynamical triangulation model is defined as a sum over simplicial manifolds consisting of equilateral four-simplices glued together according to triangulations $\mathcal{T}$,

$$Z = \sum_{\mathcal{T}(N_4)} \exp(\kappa_2 N_2).$$

(1)

Here $N_2$ is the number of triangles in the simplicial manifold consisting of $N_4$ four-simplices and the topology is chosen to be that of $S^4$. (For more details see \[1\].) We recall that the system has two phases, a crumpled phase at low $\kappa_2$ and an elongated phase at high $\kappa_2$.

The model is supposed to represent the quantum gravitational path integral over euclidean spacetimes weighted with the Regge-Einstein action, with volume $V \propto N_4$ and bare Newton constant $G_0 \propto \kappa_2^{-1}$. As such it should be able to reproduce semiclassical Einstein gravity as an effective theory. This can presumably be investigated using coordinate invariant correlation functions. Another test is to see if scalar test particles form bound states, with appropriate binding energies under nonrelativistic conditions.

2. Correlations

Let $n_x$ denote the number of four-simplices winding around a triangle $x$ (often called a ‘hinge’). We measured correlations \[5\]

$$C_O = \langle O O \rangle(r) - [\langle O \rangle(r)]^2,$$

(2)

where the brackets denote the average over simplicial configurations,

$$\langle O \rangle(r) = \frac{\sum_{xy} O_x O_y \delta_{d_{xy}-r}}{\sum_{xy} \delta_{d_{xy}-r}},$$

(3)

at that time about the spacetimes described by the model. For a semiclassical interpretation in terms of an effective action it is obviously important if there is in some sense a background spacetime which resembles the classical $S^4$. To address this question we studied \[3\] the number of simplices at geodesic distance $r$, $N'(r)$. We found encouraging scaling of this function in a region around the transition between the crumpled and elongated phase. (For a different scaling analysis see \[4\].) At the transition we observed that $N'(r)$ behaves approximately like a four-sphere, $\propto \sin^3[(\pi/2)r/r_m]$, for distances $r$ outside a short distance ‘planckian regime’ up to intermediate distances of order $r_m$. For larger distances fluctuations distort the shape of $N'(r)$. This suggests that we focus our attention on this intermediate distance regime for a possible semiclassical interpretation of the data.
with \( O_x = n_x^{-1} \), \( n_x \), and similar for \( \langle O 1 \rangle \). Here \( d_{xy} \) is a ‘triangle geodesic distance’ between \( x \) and \( y \). The ‘disconnected part’ subtracted in (2) has the form of a correlation of the observable \( O_x \) with the distance to a point \( y, d_{xy} \), which is itself a function of the geometry. This at first sight unfamiliar form is needed for the correlation to vanish at larger distances.

Other expressions for similar correlations are conceivable. For example, including a volume element \( \propto n_x \) with a summation over \( x \) may appear more natural from the Regge calculus point of view. We shall argue below that this leads to the same scaling results. Taking the average \( \langle \rangle \) separately in numerator and denominator is another option, which should not make significant difference for not too large distances, because of the self-averaging effect of the summations in (3).

In Regge calculus \( n_x^{-1} \) is related to the scalar curvature \( R_x \), and \( n_x \) to a local volume element at \( x \); we write therefore \( C_R \) (\( C_V \)) for \( O_x = n_x^{-1} \) (\( O_x = n_x \)). However, generally, in a scaling region, a lattice operator is expected to reduce to a combination of various continuum operators, weighted with powers of the lattice distance according to their dimensions. Assuming that the theory can be described by a continuum metric tensor \( g_{\mu\nu} \) with corresponding curvature \( R \), the dominant continuum observable corresponding to both our lattice correlation functions \( C_R \) and \( C_V \) would be given by

\[
\frac{\langle \int dx dy \sqrt{g(x)} \sqrt{g(y)} \delta(d(x, y) - r) R(x) R(y) \rangle}{\langle \int dx dy \sqrt{g(x)} \sqrt{g(y)} \delta(d(x, y) - r) \rangle} - \left[ \frac{\langle \int dx dy \sqrt{g(x)} \sqrt{g(y)} \delta(d(x, y) - r) R(x) \rangle}{\langle \int dx dy \sqrt{g(x)} \sqrt{g(y)} \delta(d(x, y) - r) \rangle} \right]^2.
\]

Here \( d(x, y) \) is the geodesic distance between the points \( x \) and \( y \) for a given metric \( g_{\mu\nu} \), and the average over geometries is supposed to be calculable in semiclassical fashion. Of course, we do not know the effective action specifying this average. It could be a combination of \( \int dx \sqrt{g} R \) and higher order terms like \( R^2 \).

Fig. 1 shows the two terms in eq. (2) separately for \( C_R(r) \) for \( N_4 = 16000 \) and \( \kappa_2 = 0.80 \) (crumpled phase), 1.22 (transition) and 1.50 (elongated phase). The two contributions are very similar and fig. 3 shows their difference, \( C_R(r) \), which drops to zero before finite size effects take over.

Fig. 2 shows \( C_R(r) \) for a system of \( N_4 = 32000 \) simplices and \( \kappa_2 = 1.255 \), which is at the transition between crumpled and elongated phase. For this case, with the employed definition of geodesic distance \( \overline{xy} \), the ‘planckian regime’ mentioned in the Introduction is \( r \lesssim 7 \), while \( r_m \approx 11 \). So the approximate \( S^4 \) background geometry is to be found in the region \( 7 \lesssim r \lesssim 11 \).
We see an approximate power behavior $C_V \approx a r^b$ for $r \gtrsim 7$. A fit in the range $8 \leq r \leq 18$ gives $a = -0.5(2)$, $b = -4.0(2)$. The plot for $C_V$ looks similar and leads to a compatible power $b = -4.30(12)$, with $a = -5.7(1.6)$. The equality of the powers supports the argument given above that $C_R$ and $C_V$ are dominated by the same operators in the scaling region.

The power law behavior of the data may be expected for a flat spacetime background, with $C_{V,R} \propto G r^{-4}$ or $\propto G^2 r^{-8}$ for one- or two-graviton exchange. The approximate $S^4$ interpretation of the spacetimes near the transition implies finite size corrections to the power law. An $S^4$ background curvature $\mathbf{R}$ would also allow a behavior $G^2 \mathbf{R} r^{-4}$, by dimensional reasoning.

We measured $C_{V,R}$ also in the elongated phase at $\kappa_2 = 1.50$, where the system is known to behave like a branched polymer with effective dimension 2. As fig. 3 shows, a power law looks better in this phase and a fit in the region $3 \leq r \leq 15$ leads to $b = -2.56(3)$ for $C_R$ and $b = -2.57(2)$ for $C_V$. Perhaps this can be interpreted in an effective conformal mode theory [8]. In the crumpled phase the correlation data are difficult to analyse because the distances are rather short in this phase.

We draw the optimistic conclusion from the power behavior that the correlation functions indicate the presence of massless excitations.

3. Binding

The behaviour of test particles in the spacetimes generated with the dynamical triangulation model is another interesting subject. Test particles have no backreaction on the geometry and a calculation of their attraction due to fluctuations is analogous to the quenched or valence approximation in lattice QCD. Let $G_{xy}$ be a scalar field propagator on a given simplicial manifold, $(-\Box_{xy} + m^2_0 \delta_{xy}) G_{yz} = \delta_{xz}$, where now $x,y$ denote the points on the dual lattice and $\Box$ is the corresponding lattice laplacian. We measured

\begin{align*}
G(r) &= \langle \frac{\sum_y \delta_{d_{xy}-r} G_{xy}}{\sum_y \delta_{d_{xy}-r}} \rangle, \\
G^{(2)}(r) &= \langle \left[ \frac{\sum_y \delta_{d_{xy}-r} G_{xy}}{\sum_y \delta_{d_{xy}-r}} \right]^2 \rangle,
\end{align*}

where $d_{xy}$ is the (dual lattice) geodesic distance between four-simplices. On a flat background one would expect the behavior $G(r) \propto r^\alpha \exp(-mr)$, $G^{(2)}(r) \propto r^\beta \exp(-Mr)$ at large distances, with $m$ the (renormalized) mass of the scalar particle and $M$ the mass of a two-particle state (in the quenched approximation). This is a bound state if $M < 2m$, with binding energy $E_b = 2m - M > 0$. 

![Figure 3. Power fit to $C_R(d)$ near the transition.](image)

![Figure 4. $C_R(d)$ in the elongated phase.](image)
Fig. 5 shows $G(r)^2$ and $G^{(2)}(r)$ for $N_4 = 32000$ at the transition, $\kappa_2 = 1.255$, for several values of the bare mass, $m_0 = 1, 0.316, 0.1, 0.0316$. The approximate $S^4$ behavior mentioned in the Introduction is now found in the region $10 \lesssim r \lesssim 21$, with the distance definition employed here. We see roughly exponential behavior suggesting that the $S^4$ character of the background geometry can in a first approximation be neglected, similar to the correlation functions above. The corresponding renormalized masses are $m \approx 1.19, 0.57, 0.27, 0.14$. The ratio $m/m_0$ appears to increase substantially as $m_0$ gets smaller.

If the model describes semiclassical gravity we may expect hydrogen atom like behavior in a nonrelativistic situation, with small fine structure constant $\alpha \rightarrow Gm^5$:

$$E_b = \frac{1}{4} G^2 m^5. \tag{5}$$

In this way we may be able to define the renormalized Newton constant $G$. To get a rudimentary feeling for the corrections to (5) the Hamiltonian $H = 2\sqrt{m^2 + p^2} - Gm^2/r$ may be useful. Replacing $p \rightarrow 1/r$ and minimizing the energy leads to $E_b = 2m - 2m\sqrt{1 - G^2 m^4/4}$, which suggests that $Gm^2 = 2$ has to be considered 'large'.

Fig. 6 shows the effective binding energy

$$E_{b\text{eff}} = \frac{1}{r} \ln \frac{G(r)^2}{G^{(2)}(r)}. \tag{6}$$

Clearly, there is no indication of nonrelativistic $m^5$ behavior. To judge the situation we have to make sure that $Gm^2$ is not too large (assuming $G$ exists in the model), or $m$ too small for neglecting finite size $S^4$ corrections, or both. It is also desirable to get a semiclassical understanding of the quenched approximation. Finally, we may have to take into consideration that the Euclidean effective action at intermediate distances (neglecting higher order terms like $R^2$) is unbounded.

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