Linear and nonlinear versions of Phase Retrieval

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Abstract. The problem of Phase Retrieval or Phaseless Recovery can be briefly stated as follows: for a given amplitude or measurements obtained using a redundant system of measuring vectors to restore the original signal. This problem first arose in X-ray crystallography at the beginning of the 20th century. A similar non-linear reconstruction problem arises in the digital processing of speech signals, especially in speech recognition problems. Some achievements in this direction will be shown with simple examples in spaces of small dimensions.

1. Introduction

In this paper, the simple examples will show the main two ideas of finite-dimensional reconstruction based on the amplitudes (modules) of the frame coefficients. Various aspects of this problem arise in many areas of science and technology.

In particular, in X-ray crystallography, the researcher measures the amplitudes of the Fourier transform of electron density, based on these measurements, makes conclusions about the atomic structure of the crystal [1].

In digital speech processing, in automatic speech recognition devices use cepstral coefficients, which are absolute values of linear combinations of short-term Fourier coefficients.

Phaseless recovery combines and unifies the ideas described above. Any complete set of vectors forms a frame of the space in a finite-dimensional situation. Let $\mathbb{H}$ will denote either $\mathbb{R}^M$, either $\mathbb{C}^M$. Let $F = \{f_1, f_N\}$ be the set with $N \geq M$ vectors and $\text{span} F = \mathbb{H}$. In general, the task is to find out when the vector $x \in \mathbb{H}$ can be reconstructed from the amplitudes of measurements $\{\langle x, f_n \rangle, 1 \leq n \leq N\}$, and how to make this recovery process more efficient.

Real and complex situations are very often different. For example, the property of alternative completeness gives a clear criterion for a system that restores without phases, but in a complex case gives only a necessary condition.

An analysis operator is connected with each frame:

$T : \mathbb{H} \rightarrow \mathbb{C}^N, (T x)_n = \langle x, f_n \rangle, 1 \leq n \leq N$;

and conjugate synthesis operator

$T^* : \mathbb{C}^N \rightarrow \mathbb{H}, T^* (c) = \sum_{n=1}^{N} c_n f_n$.
The main nonlinear operator in this problem is \( x \mapsto (|\langle x, f_n \rangle|)_{1 \leq n \leq N} \). The equivalence relation between \( x, y \in H \) is introduced \( x \sim y \) there is a constant \( c \) with \( |c| = 1 \) such that \( x = cy \).

Let \( \tilde{H} = H/\sim \) denote the quotient space. The nonlinear map is well defined on \( \tilde{H} \) since \(|\langle cx, f_n \rangle| = |\langle x, f_n \rangle|\) for all scalar \( c \) with \( |c| = 1 \).

We’ll consider two maps on this quotient space:

\[
\alpha : \tilde{H} \to \mathbb{R}^N, (\alpha(x))_n = |\langle x, f_n \rangle|, 1 \leq n \leq N;
\]

\[
\beta : \tilde{H} \to \mathbb{R}^N, (\beta(x))_n = |\langle x, f_n \rangle|^2, 1 \leq n \leq N.
\]

**Definition 1**. We say that the frame \( F \) restores \( x \) without phases (RWP) if the maps \( \alpha \) and \( \beta \) are injective.

2. **Nonlinear aspect in the real case**

The main results in this case, were obtained in [2] and [3].

**Theorem 1**. Assume \( F \subset \mathbb{R}^M \). The following are equivalent:

1) \( F \) is RWP-frame for \( \mathbb{R}^M \);

2) \( R(x) = \sum_{n=1}^{N} |\langle x, f_n \rangle|^2 f_n f_n^* \), \( x \in \mathbb{R}^M \) is invertible for every \( x \in \mathbb{R}^M \), \( x \neq 0 \);

3) If for all \( 1 \leq n \leq N \) \( \langle u, f_n \rangle \langle f_n, v \rangle = 0 \), then at least one of the vectors \( u = 0 \) or \( v = 0 \);

4) (Alternative completeness) For any disjoint partition of the frame set \( F = F_1 \cup F_2 \) either \( F_1 \) spans \( \mathbb{R}^M \) or \( F_2 \) spans \( \mathbb{R}^M \).

**Corollary 1**. [2] Assume \( F \subset \mathbb{R}^M \). Then

1) If \( F \) is RWP-frame, then \( N \geq 2M - 1 \);

2) If \( N = 2M - 1 \), then \( F \) is RWP-frame if and only if \( F \) is full spark.

Recall a set \( F \subset \mathbb{H} \) is called full spark if any subset of \( M \) vectors is linearly independent, and hence is complete in \( \mathbb{H} \).

The proof of the existence full spark systems for each pair of numbers \( (M, N) \) with \( N \geq M \) can be implicitly detected in the classical paper of B. Kashin [4]. The current results on full spark frames are in [5].

Now we’ll see two examples in \( \mathbb{R}^3 \). In both of them \( M = 3 \), \( N = 6 \).

**Example 1**. We consider the frame

\[
F_1 = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\]

from [6] in \( \mathbb{R}^3 \).
This matrix is the matrix of the synthesis operator. The frame operator

\[
S = T_{\mathcal{F}_1}^* T_{\mathcal{F}_1} = \begin{pmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{pmatrix}
\]

Calculations of determinants show that \( \mathcal{F}_1 \) is full spark frame, so according to the Theorem 1 the matrix

\[
R(x) = \begin{pmatrix}
|x_1|^2 + \lambda_{12}^2 + \lambda_{13}^2 & \lambda_{12}^2 & \lambda_{13}^2 \\
\lambda_{12}^2 & |x_2|^2 + \lambda_{12}^2 + \lambda_{23}^2 & \lambda_{23}^2 \\
\lambda_{13}^2 & \lambda_{23}^2 & |x_3|^2 + \lambda_{13}^2 + \lambda_{23}^2
\end{pmatrix},
\]

where \( \lambda_{12} = |x_1 + x_2|, \lambda_{13} = |x_1 + x_3|, \lambda_{23} = |x_2 + x_3| \), is invertible for \( x \in \mathbb{R}^3 \).

So frame \( \mathcal{F}_1 \) is RWP-frame.

The following result [8] allows obtaining new RWP-frames.

**Theorem 2.** If \( \mathcal{F} \) is RWP-frame, and \( V \) is the invertible operator, then the frame \( \mathcal{G} = \{g_1, ..., g_N\}, g_n = z_n V f_n \) is also RWP-frame for non zero constants \( z_1, ..., z_N \).

In such a way we obtain new RWP frame

\[
g_n = V f_n = \begin{pmatrix}
0 & 1 & -1 \\
1 & -1 & 0 \\
1 & 1 & 0
\end{pmatrix} f_n,
\]

\[
\mathcal{G}_1 = \{g_1, ..., g_6\} = \begin{pmatrix}
0 & 1 & -1 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 & 1 & -1 \\
1 & 1 & 0 & 2 & 1 & 1
\end{pmatrix}.
\]

According to the Theorem 2 the dual frame retains the RWP-property. We’ll look at the dual frame to \( \mathcal{F}_1 : \mathcal{F}_1^* f'_n = S^{-1} f_n \).

\[
S^{-1} = \frac{1}{10} \begin{pmatrix}
4 & -1 & -1 \\
-1 & 4 & -1 \\
-1 & -1 & 4
\end{pmatrix},
\]

\[
F'_1 = \frac{1}{10} \begin{pmatrix}
4 & -1 & -1 & 3 & 3 & -2 \\
-1 & 4 & -1 & 3 & -2 & 3 \\
-1 & -1 & 4 & -2 & 3 & 3
\end{pmatrix}
\]

Well-known process of constructing the appropriate Parseval frame for \( \mathcal{F}_1 \) gives the following matrix, it will be written by parts:

\[
\mathcal{F}_0^1 = \{f_0^1, ..., f_N^1\}, f_n^0 = S^{-1/2} f_n,
\]

\[
\mathcal{F}_0^1 = \frac{1}{30} \begin{pmatrix}
10\sqrt{2} + 2\sqrt{5} & -5\sqrt{2} + 2\sqrt{5} & -5\sqrt{2} + 2\sqrt{5} & ... \\
-5\sqrt{2} + 2\sqrt{5} & 10\sqrt{2} + 2\sqrt{5} & -5\sqrt{2} + 2\sqrt{5} & ... \\
-5\sqrt{2} + 2\sqrt{5} & 2\sqrt{5} & 10\sqrt{2} + 2\sqrt{5} & ... \\
-5\sqrt{2} + 2\sqrt{5} & -5\sqrt{2} + 2\sqrt{5} & 10\sqrt{2} + 2\sqrt{5} & ...
\end{pmatrix}
\]
\[
\begin{pmatrix}
... & 5\sqrt{2} + 4\sqrt{5} & 5\sqrt{2} + 4\sqrt{5} & 10\sqrt{5} - 10\sqrt{2} \\
... & 5\sqrt{2} + 4\sqrt{5} & -10\sqrt{2} + 4\sqrt{5} & 5\sqrt{2} + 4\sqrt{5} \\
... & -10\sqrt{2} + 5\sqrt{5} & 5\sqrt{2} + 4\sqrt{5} & 5\sqrt{2} + 4\sqrt{5}
\end{pmatrix}
\]

In some cases, to evaluate the quality of a frame, the mutual coherence \( \mu \) is used. For the frame \( F_1 \) this characteristic is far from optimal, which is achieved on equiangular frames and is equal to \( \sqrt{(N-M)/(M(N-1))} \).

\[
\mu(F_1) = \max_{k \neq k'} \frac{|\langle f_k, f_{k'} \rangle|}{\|f_k\| \|f_{k'}\|} = \frac{\sqrt{2}}{2} > \frac{\sqrt{5}}{5}.
\]

**Example 2.** In order to improve the mutual coherence we consider the equiangular tight frame \( F_2 \in \mathbb{R}^3 \). It is a RWP-frame.

\[
F_2 = \frac{1}{\sqrt{1 + \nu^2}} \begin{pmatrix}
0 & 0 & 1 & 1 & \nu & -\nu \\
1 & 1 & \nu & -\nu & 0 & 0 \\
\nu & -\nu & 0 & 0 & 1 & 1
\end{pmatrix}, \quad \nu = \frac{1 + \sqrt{5}}{2}.
\]

We retain the designations of the Example 1. The frame \( G_2 = \{g_1, ..., g_N\} \), \( g_k = Vf_k \) will be obtained with the help of the invertible matrix \( V \), dual frame and Parseval frame as above.

\[
V = \begin{pmatrix}
0 & 1 & -1 \\
1 & -1 & 0 \\
1 & 1 & 0
\end{pmatrix},
\]

\[
G_2 = \frac{1}{\sqrt{1 + \nu^2}} \begin{pmatrix}
-1 + \nu & -1 - \nu & -1 - \nu & -1 + \nu & 1 - \nu & 1 + \nu \\
-\nu & 1 - \nu & 1 & 1 & 1 + \nu & 1 - \nu \\
1 + \nu & 1 - \nu & \nu & -\nu & 1 & 1
\end{pmatrix},
\]

\[
S = F_2F_2^T = \frac{1}{1 + \nu^2} \begin{pmatrix}
2 + 2\nu^2 & 0 & 0 \\
0 & 2 + 2\nu^2 & 0 \\
0 & 0 & 2 + 2\nu^2
\end{pmatrix},
\]

\[
F_2' = \frac{1}{2\sqrt{1 + \nu^2}} \begin{pmatrix}
0 & 0 & 1 & 1 & \nu & -\nu \\
1 & 1 & \nu & -\nu & 0 & 0 \\
\nu & -\nu & 0 & 0 & 1 & 1
\end{pmatrix} = \frac{1}{2}F_2.
\]

\[
F_2^0 = \frac{\sqrt{2}}{4}F_2.
\]

Let’s calculate \( \mu \) for \( F_2 \).

\[
\mu(F_2) = \max_{k \neq k'} \frac{|\langle f_k, f_{k'} \rangle|}{\|f_k\| \|f_{k'}\|} = \frac{\sqrt{5}}{5} = \frac{\sqrt{N-M}}{\sqrt{M(N-1)}}.
\]
3. Linearization. Complex case

In a complex case, the situation is more complicated. In [3, 7] it was developed a procedure for linearization, which allowed us to obtain RWP-criteria in this case. Let’s go to show this approach with a simple example of the frame in $\mathbb{C}^2$.

For the beginning let’s define the map

$$j(x) = \begin{bmatrix} \text{real}(x) \\ \text{imag}(x) \end{bmatrix}.$$ 

Next to the matrix

$$J = \begin{pmatrix} 0 & -I_M \\ I_M & 0 \end{pmatrix}$$

is defined in $\mathbb{R}^{2M}$, where $I_M$ denotes the unitary $M \times M$-matrix.

Each vector $f_n$ is mapped onto the vector

$$\varphi_n = \begin{bmatrix} \text{real}(f_n) \\ \text{imag}(f_n) \end{bmatrix}$$

in $\mathbb{R}^{2M}$. After that let’s define matrices

$$\Phi_n = \varphi_n \varphi_n^T + J \varphi_n \varphi_n^T J^T, \quad F_n = f_n f_n^*.$$ 

In addition to the map $R(x)$ already discussed above, the following are entered:

$$R : \mathbb{C}^M \to \text{Sym} (\mathbb{C}^M), \quad R(x) = \sum_{n=1}^N |\langle x, f_n \rangle|^2 f_n f_n^*, \quad x \in \mathbb{C}^M;$$

$$\mathcal{R} : \mathbb{R}^{2M} \to \text{Sym} (\mathbb{R}^{2M}), \quad \mathcal{R}(\xi) = \sum_{n=1}^N \Phi_n \xi \xi^T \Phi_n, \quad \xi \in \mathbb{R}^{2M};$$

$$\mathcal{S} : \mathbb{R}^{2M} \to \text{Sym} (\mathbb{R}^{2M}), \quad \mathcal{S}(\xi) = \sum_{n: \Phi_n \xi \neq 0} \frac{1}{\langle \Phi_n \xi, \xi \rangle} \Phi_n \xi \xi^T \Phi_n, \quad \xi \in \mathbb{R}^{2M};$$

$$\mathcal{Z} : \mathbb{R}^{2M} \to \text{Sym} (\mathbb{R}^{2M \times N}), \quad \mathcal{Z}(\xi) = [\Phi_1 \xi | \ldots | \Phi_N \xi], \quad \xi \in \mathbb{R}^{2M}.$$ 

It was noted in [6] that

$$|\langle x, f_n \rangle|^2 = \text{trace}(F_n X) = \langle F_n, X \rangle_{\text{HS}},$$

where $X = xx^*$. 


So the nonlinear map $\beta$ induces the linear map on the real vector space $\text{Sym}(\mathbb{C}^M)$ of symmetric forms on $\mathbb{C}^M$:

$$A : \text{Sym}(\mathbb{C}^M) \rightarrow \mathbb{R}^N,$$

$$(A(T))_n = \langle T, F_n \rangle = \langle Tf_n, f_n \rangle, \quad 1 \leq n \leq N.$$

Similarly $\beta$ induces the linear map on $\text{Sym}(\mathbb{R}^{2M})$ of symmetric forms on $\mathbb{R}^{2M} = j(\mathbb{C}^M)$ in the following way:

$$A : \text{Sym}(\mathbb{R}^{2M}) \rightarrow \mathbb{R}^N,$$

$$(A(T))_n = \langle T, \Phi_n \rangle_{HS} = \langle Tf_\varphi, \varphi_n \rangle + \langle TJf_\varphi, J\varphi_n \rangle, \quad 1 \leq n \leq N.$$

**Theorem 3.** [3, 7]
The following assertions are equivalent:

1) $\mathcal{F}$ is RWP-frame for $\mathbb{C}^M$;
2) $\text{rank}Z(\xi) = 2M - 1$ for all $\xi \in \mathbb{R}^{2M}$, $\xi \neq 0$;
3) $\dim \ker R(\xi) = 1$ for all $\xi \in \mathbb{R}^{2M}$, $\xi \neq 0$;
4) If $\langle \Phi_n \xi, \eta \rangle = 0$, $1 \leq n \leq N$, and $\langle J\xi, \eta \rangle = 0$, then at least one of the vectors $\xi \in \mathbb{R}^{2M}$ or $\eta \in \mathbb{R}^{2M}$ is equal to 0.

**Example 3.**
Now we’ll see the frame in $\mathbb{C}^2$ with 4 vectors, i.e. $M = 2$ and $N = 4$. This frame was considered in [9].

$$\mathcal{F}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 1 & 1 & 1 \\ 0 & \sqrt{2} & \sqrt{2}e^{\frac{i3\pi}{4}} & \sqrt{2}e^{\frac{i5\pi}{4}} \end{pmatrix} =$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 1 & -\sqrt{2} \pm \frac{\sqrt{6}}{2}i & -\sqrt{2} \mp \frac{\sqrt{6}}{2}i \end{pmatrix}.$$ 

We’ll use the second assertion in Theorem 3 for verification the RWP-property.

Here are the vectors $\Phi_n$, $n = 1, \ldots, 4$.

$$\Phi_1 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Phi_2 = \frac{1}{3} \begin{pmatrix} 1/3 & \sqrt{2}/3 & 0 & 0 \\ \sqrt{2}/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/3 & \sqrt{2}/3 \\ 0 & 0 & \sqrt{2}/3 & 2/3 \end{pmatrix},$$

$$\Phi_3 = \frac{1}{3} \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 & \sqrt{6}/2 \\ -\sqrt{2}/2 & 2 & -\sqrt{6}/2 & 0 \\ 0 & -\sqrt{6}/2 & 1 & -\sqrt{2}/2 \\ \sqrt{6}/2 & 0 & -\sqrt{2}/2 & 2 \end{pmatrix}.$$
\[ \Phi_4 = \frac{1}{3} \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 & -\sqrt{6}/2 \\ -\sqrt{2}/2 & 2 & \sqrt{6}/2 & 0 \\ 0 & \sqrt{6}/2 & 1 & -\sqrt{2}/2 \\ -\sqrt{6}/2 & 0 & -\sqrt{2}/2 & 2 \end{pmatrix} . \]

For some non-zero vector \( \xi = (a_1, a_2, a_3, a_4)^T \) let's define \( Z(\xi) \), and calculate its rank.

\[ \Phi_1 \xi = \begin{pmatrix} a_1 \\ 0 \\ a_1 \\ 0 \end{pmatrix}, \quad \Phi_2 \xi = \begin{pmatrix} \frac{1}{3} a_1 + \frac{\sqrt{2}}{3} a_2 \\ \frac{\sqrt{2}}{3} a_1 + \frac{\sqrt{6}}{3} a_2 \\ \frac{1}{3} a_3 + \frac{\sqrt{2}}{3} a_4 \\ \frac{\sqrt{2}}{3} a_3 + \frac{\sqrt{6}}{3} a_4 \end{pmatrix}, \]
\[ \Phi_3 \xi = \begin{pmatrix} a_1 - \frac{\sqrt{2}}{2} a_2 + \frac{\sqrt{6}}{2} a_4 \\ -\frac{\sqrt{2}}{2} a_1 + 2 a_2 - \frac{\sqrt{6}}{2} a_3 \\ -\frac{\sqrt{6}}{2} a_2 + a_3 - \frac{\sqrt{2}}{2} a_4 \\ \frac{\sqrt{6}}{2} a_1 - \frac{\sqrt{2}}{2} a_3 + 2 a_4 \end{pmatrix}, \]
\[ \Phi_4 \xi = \begin{pmatrix} a_1 - \frac{\sqrt{2}}{2} a_2 - \frac{\sqrt{6}}{2} a_4 \\ -\frac{\sqrt{2}}{2} a_1 + 2 a_2 + \frac{\sqrt{6}}{2} a_3 \\ \frac{\sqrt{6}}{2} a_2 + a_3 - \frac{\sqrt{2}}{2} a_4 \\ -\frac{\sqrt{6}}{2} a_1 - \frac{\sqrt{2}}{2} a_3 + 2 a_4 \end{pmatrix}. \]

So we have, that \( \text{rank}(Z(\xi)) = 3 \), and \( F_3 \) is RWP-frame. But it's not equiangular, and

\[ \mu(F_3) = \max_{n \neq n'} \frac{|\langle f_n, f_{n'} \rangle|}{\|f_n\| \|f_{n'}\|} = 1 > \frac{1}{\sqrt{3}}. \]

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