Efficiency Statistics and Bounds of Time-Reversal Symmetry Breaking Systems

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Universal properties of efficiency statistics for stochastic time-reversal symmetry breaking machines are studied in the Gaussian limit. Effects of thermodynamic and quantum mechanical bounds on efficiency fluctuations are revealed. Unlike the time-reversal symmetry limit, where the least probable efficiency \( \eta^\star \) always equals the Carnot efficiency \( \eta_C \), here under the condition of maximum macroscopic efficiency we obtain the universal form \( \eta^\star = r\eta_C \), with \( r \) as the asymmetry parameter of the Onsager matrix. When operating under other conditions, \( \eta^\star \) was found to depend on transport coefficients and affinities, allowing versatile tailoring of efficiency statistics. We further examine the width of efficiency distribution \( \sigma_\eta \) around its macroscopic value and reveal rich characteristics. Specifically, the tight-coupling limit becomes unfavorable when time-reversal symmetry is broken, characterized by an infinitely broad distribution of efficiency.

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Introduction.— The operation of macroscopic machines is dictated by the theory of linear irreversible thermodynamics [1]. The efficiency of heat engines, that is the ratio between extracted work and heat absorbed from the hot source, is limited by the second law of thermodynamics to provide the Carnot bound. While a conventional heat engine approaches this limit when operating reversibly, delivering zero output power, it was recently demonstrated that the Carnot efficiency may be achieved while delivering finite power, if the Onsager matrix is made asymmetric by breaking time reversal symmetry [2]. Recent studies had explored energy conversion in diverse situations, in cold atomic ensembles [3], optomechanical systems [4], superconducting materials [5, 6], photo-electric devices [7], and spin-caloritronics [8].

Understanding the operation principles of mesoscopic energy conversion devices, possibly relying on quantum coherent effects, is pivotal for addressing fundamental and practical challenges in non-equilibrium thermodynamics and energy management systems [2,3,10,12,14]. Since small devices can suffer strong fluctuations, their performance is incompletely communicated by averaged thermodynamical quantities. Indeed, in small systems the second law of thermodynamics is replaced by the fluctuation theorem [2,10,12,14] \( P_\eta(S,B) \propto e^{-S} \) at a long time \( t \) (we set \( e = h = k_B = 1 \) throughout this paper). Here \( S \) is the entropy production associated with a process of probability \( P_\eta \). A nontrivial consequence of the fluctuation theorem is that it allows a negative entropy production, hence efficiencies exceeding the thermodynamic bound.

The concept of “stochastic efficiency” has been recently coined and examined by Verley et al. [2,13] and others [1,14,22]. A remarkable finding has been that for time-reversal symmetric (TRS) devices the Carnot efficiency becomes the least probable one in the long-time limit. A geometric interpretation of stochastic fluctuations in the Gaussian approximation revealed generic features of the large deviation function (LDF) of efficiency \( G(\eta) = -\lim_{t \to \infty} t^{-1} \ln[P_\eta(\eta)] \) and consolidated the observation of the “unlikely Carnot efficiency”, a result which has been put into an experimental examination very recently [22].

In this Letter, we study the impact of time-reversal symmetry breaking on efficiency statistics. It was pointed out that in this case the least likely efficiency is no longer the Carnot efficiency [2,19]. However, the characteristics of efficiency fluctuations in time-reversal breaking (TRB) systems are unknown: What are the general properties of the LDF, specifically, the least likely efficiency? Does it depend on both linear-response coefficients and thermodynamic affinities? How do the laws of thermodynamics restrict efficiency statistics in TRB systems? To answer these questions, we derive the LDF and the least probable efficiency \( \eta^\star \) for general TRB thermodynamic systems within the Gaussian (close-to-equilibrium) approximation under (i) the condition of maximum macroscopic efficiency; (ii) for devices working at the maximum macroscopic power. Two of our main results are that beyond the TRS limit, under the condition of maximum macroscopic efficiency, \( \eta^\star = r\eta_C \), where \( r \neq 1 \) characterizes time-reversal symmetry breaking, and that the “tight-coupling limit” becomes generally unfavourable in TRB systems. Our findings are exemplified within a simple model of a thermoelectric device and backed up with a full counting statistics analysis. Our discussion offers viable relevance for ongoing studies on efficiency statistics of mesoscopic energy transducers such as biological light harvesting molecules [7], quantum optomechanical systems [4], and superconduct-
ing junctions \[3, 6\].

Efficiency statistics in the Gaussian approximation.— For devices working in the linear-response regime efficiency statistics can be well described within the Gaussian approximation \[1, 2, 3, 11, 12, 13, 10, 21, 24\]. For a TRB device with the macroscopic response \( T_i = M_{ij} A_j \), the statistics at long time \( t \) is described by the probability \[1, 24\]
\[
P(t\hat{I}) = \frac{\sqrt{\text{det}(M^{-1})_{\text{sym}}}}{4\pi} \exp\left(-\frac{1}{4} \delta\hat{I}^T \cdot M^{-1} \cdot \delta\hat{I} \right)
\]
where \( \text{det}(M^{-1})_{\text{sym}} \) is the determinant of the symmetric part of the inverse of the response matrix \( M \). Here, \( T_i \) and \( A_j \) \((i, j = 1, 2)\) are the (macroscopic) currents and affinities for the energy input \( ("1") \) and output \( ("2") \) degrees of freedom of the device; macroscopic and average quantities are denoted with a bar over the symbols throughout this paper, \( \delta\hat{I} = \hat{I} - \bar{\hat{I}} \) is the current fluctuation and \( \hat{T} \) denotes transpose. Following the treatment of Ref. \[1\], direct but lengthy calculations (see Supplementary Material) yield the following (rescaled) LDF of efficiency fluctuations for TRB systems,
\[
J(\eta) = \frac{\eta \eta^*}{S_{\text{tot}}} J(\eta_C) (\eta + \alpha^2 + \alpha q r + \alpha q \eta)^2
\]
\[
= \frac{J(\eta_C)}{1 + \alpha^2 + \alpha q r + \alpha q \eta},
\]
where \( S_{\text{tot}} = \sum_i T_i A_i \) is the average total entropy production rate and
\[
J(\eta_C) = \frac{4 - q^2 (1 + r)^2}{16 (1 - q^2 r)}
\]
is the LDF at Carnot efficiency. The dimensionless quantities here are defined as
\[
q = \frac{M_{21}}{\sqrt{M_{22}M_{11}}}, \quad r = \frac{M_{12}}{M_{21}}, \quad \alpha = \frac{A_1 \sqrt{M_{11}}}{A_2 \sqrt{M_{22}}},
\]
We term \( q \) the degree of coupling \[3, 23\], \( r \) is the TRB parameter \[2, 3, 23\], and \( \alpha \) is the affinity parameter. In addition, efficiency is defined as \[3, 2, 23\]
\[
\eta = -\frac{1}{\alpha} \frac{\partial}{\partial \lambda} \ln Z_\lambda, \quad \eta^* = J(\eta_C) \eta C = \frac{W}{Q} = \text{standard definition of efficiency,}
\]
where \( \eta \) is the Carnot efficiency; in our scheme efficiency reduces to \( \eta C \equiv 1 \).

The second law of thermodynamics requires that \[2, 3, 13\]
\[ S_{\text{tot}} \geq 0 \] and hence \( M_{11}, M_{22} \geq 0 \) and \( M_{11} M_{22} \geq (M_{21} + M_{12})^2 / 4 \), or equivalently,
\[
-2 \leq q (1 + r) \leq 2.
\]
Equality is attained only in the “tight-coupling” limit \[2, 23\] where the macroscopic efficiency reaches its upper bound (note that the determinant of the Onsager matrix may not vanish in this limit for \( r \neq 1 \)). We shall examine here how thermodynamic bounds restrict efficiency statistics.

In the TRS limit, \( r = 1 \), Eq. \[1\] goes back to the results obtained in Refs. \[1, 2, 10, 21\]. For more general situations we find that \( 0 \leq J(\eta) \leq 1/4 \) is guaranteed by the thermodynamic bound \[1\] [see Supplementary Material]. Moreover, \( J(\eta) \) has only one minimum and one maximum. While the minimum \( J(\eta) = 0 \) is reached at the macroscopic efficiency \( \eta = \eta / \eta C \), the maximum value \( J(\eta^*) = 1/4 \) is realized at
\[
\eta^* = 1 + \frac{q(r - 1)(1 + \alpha q r + \alpha q^2)}{q - q r - 2 \alpha + q^2 (1 + r) \alpha}.
\]
In the TRS limit, the least likely efficiency is always the Carnot efficiency, \( \eta^* = \eta C \equiv 1 \) \[1, 2, 10, 21\]. For TRB systems, in contrast, \( \eta^* \) depends on the TRB parameter \( r \), the degree of coupling \( q \), and the affinity parameter \( \alpha \), see Fig. \[4\].

We now discuss some general properties of the LDF. First, the LDF at Carnot (reversible) efficiency \( J(\eta_C) \) is independent of affinities but it is solely determined by the response coefficients. It is also invariant under time-reversal symmetry, \( r \rightarrow 1/r \) and \( q \rightarrow qr \) \[2, 3, 13\]. The thermodynamic bound \[41\] dictates that \( 4(1 - q^2 r)^2 \geq 4 - q^2 (1 + r)^2 \geq 0 \), hence \( 0 \leq J(\eta_C) \leq 1/4 \). \( J(\eta_C) \) can be suppressed by breaking time-reversal symmetry, partic-
ularly in approaching the tight-coupling limit. Second, at \( \alpha = 0 \), \( J(\eta) = J(\eta_C) \) for all \( \eta \). Thus the LDF becomes a constant, independent of \( \eta \). Third, \( \eta^* \) diverges at \( \alpha_c = \frac{q(1-r)}{2-q^2-r^2} \). For \( |r| > 1 \), \( \alpha_c \) leads to a positive macroscopic efficiency and output power, relevant for device operation. Besides, \( \eta^* \rightarrow \infty \) for all \( \alpha \) and \( q \). It has been shown that this limit is achievable for a triple quantum-dots (QD) thermoelectric device as \( M_{21} \rightarrow 0 \) and \( M_{12} \neq 0 \) at the same time [6].

Another key quantity for efficiency fluctuations is the width of efficiency distribution around its macroscopic value. Expanding \( J(\eta) \) around its minimum \( \eta \), one gets \( J(\eta) \approx \frac{1}{2\sigma_\eta}(\eta - \bar{\eta})^2 + O((\eta - \bar{\eta})^3) \), where the dispersion of efficiency fluctuations (for \( \alpha \neq 0 \)) is given by

\[ \sigma_\eta = \frac{2\sqrt{2}|\alpha|(1 - q^2r)(1 + \alpha^2 + \alpha qr + \alpha^2 qr)}{(1 + \alpha qr)^2\sqrt{4 - q^2(1 + r)^2}}. \]  

(6)

Efficiency fluctuations at maximum macroscopic efficiency.— The maximum macroscopic efficiency

\[ \eta\equiv \eta_{\text{max}} = r\sqrt{ZT + \frac{1}{1 - 1}} = \sqrt{\frac{ZT + \frac{1}{1 - 1}}{1 + \frac{1}{1 - 1}}} \]  

(7)

is reached when \( \alpha = -qr/(1 + \sqrt{1 - q^2r}) \) [5], where \( ZT = q^2r/(1 - q^2r) \) is the figure of merit for energy conversion [2, 5, 8, 13]. The thermodynamic upper bound of the macroscopic efficiency, reached at the tight-coupling limit, \( |q(1 + r)| \rightarrow 2 \), is [2, 5, 8, 13]

\[ \eta_{\text{bound}} = \min\{r^2, 1\}. \]  

(8)

One of the key results of this work is that under the maximum macroscopic efficiency condition, the least probable efficiency \[ \eta^* \] reduces to \( (\eta_C \equiv 1) \)

\[ \eta^* = r. \]  

(9)

It recovers recent findings that Carnot efficiency is the least probable one in a stochastic device in the TRS limit. The maximum macroscopic efficiency, the upper bound of the macroscopic efficiency, and the least probable efficiency are plotted in Fig. 2(b) as a function of \( r \) for \( q = 0.5 \). We find that generally

\[ \eta^* \geq \eta_{\text{bound}} \geq \eta_{\text{max}}, \forall r \geq 0, \]  

(10a)

\[ \eta^* < \eta_{\text{max}} \leq \eta_{\text{bound}}, \forall r < 0. \]  

(10b)

The least probable efficiency \( \eta^* \) coincides with \( \eta_{\text{bound}} \) only in the TRS limit, \( r = 1 \), or for the TRB case with \( r = 0 \).

We calculate the width of distribution \( \sigma_\eta \) and plot it in Fig. 2(b); the white region in the figure is forbidden by the thermodynamic bound [4]. The width of distribution \( \sigma_\eta \) is small when \( q \) or \( r \) are small, corresponding to the “weak-coupling limit” where the macroscopic efficiency is small. Physically, this can be understood as that two nearly uncoupled currents are very unlikely to have collective fluctuations which result in a considerably large (positive or negative) efficiency. The width of efficiency distribution is generally large when \( |qr| \sim 1 \), particularly when \( r \) is negative [see Fig. 2(b)].

Since the work of Mahan and Sofo [28], significant efforts were dedicated to realize the “tight-coupling” limit of \( T_2 = cT_1 \), with \( c \) as a constant independent of affinities, leading to the Carnot efficiency [25]. These works were confined to TRS systems. We show next that in TRB systems the tight-coupling limit [29] becomes unfavourable as it exhibits infinite efficiency fluctuations. Approaching the tight-coupling limit \( |q(1 + r)| \rightarrow 2 \) [which we cannot depict in Fig. 2(b)], we find that \( \sigma_\eta \rightarrow \infty \) for \( 0 < |r| < 1 \) and \( r = -1 \), while \( \sigma_\eta \rightarrow 0 \) for other regimes. In the TRS limit, \( r = 1 \), our results agree with Ref. [4].

The underlying physics for this behavior can be understood through Eq. (11). Upon approaching the tight-coupling limit, the denominator of \( \eta_{\text{bound}} \) vanishes. The numerator is proportional to the total entropy production rate \( S_{\text{tot}} = M_{22}A_2^2(1 + aq + aqr + a^2) \). In the tight-coupling limit, for \( |r| \geq 1 \) (unless \( r = -1 \)), the maximum macroscopic efficiency is reached when the total entropy production rate is zero. However, for \( 0 < |r| < 1 \), the maximum macroscopic efficiency is attained at finite (average) total entropy production rate [2, 5, 8], thus the width of efficiency distribution diverges. For \( r = -1 \) the tight-coupling is reached in the limit \( q \rightarrow \infty \), and again \( \sigma_\eta \) diverges. The latter case is relevant to recent studies on “chiral thermoelectrics” (e.g., Nernst engines) [3, 4].

We could also reach these conclusions by directly considering Eqs. [11] and [12]. In the tight-coupling limit \( J(\eta_C) = 0 \) (except for \( r = 1 \), the TRS limit), \( J(\eta) \) thus vanishes (except for \( \eta = \eta^* \)) whenever the average entropy production rate is nonzero, hence the dispersion \( \sigma_\eta \) diverges.

FIG. 2. Efficiency statistics under the maximum macroscopic efficiency condition. (a) Maximum macroscopic efficiency \( \eta_{\text{max}} \) (blue curve), thermodynamic upper bound of the macroscopic efficiency \( \eta_{\text{bound}} \) (red curve), and the least probable efficiency \( \eta^* \) (green curve) for \( q = 0.5 \). The blue dot represents the TRS point, \( r = 1 \). (b) Width of efficiency distribution \( \sigma_\eta \). The dashed line represents the TRS limit, \( r = 1 \). The white region is forbidden by the thermodynamic bound [4].
However, the limit with zero entropy production rate $\overline{S}_{\text{tot}}$ also needs careful examination. When $\overline{S}_{\text{tot}}$ vanishes, the LDF $\mathcal{G}(\eta) = \overline{S}_{\text{tot}} / J(\eta)$ still vanishes as the rescaled LDF $J(\eta)$ is bounded between 0 and 1/4. Therefore, even for vanishing average entropy production rate, the distribution of efficiency is infinitely broad, and the macroscopic description of efficiency fails.

Efficiency fluctuations at maximum macroscopic output power.— We now turn our attention to the maximum macroscopic output power condition$^{[23]}$, which is arrived at $\alpha = -qr/2$. Under this condition the macroscopic efficiency is $\eta = 2 \xi 8 \ 13$.

$$\eta(W_{\text{max}}) = \frac{rZT}{2(ZT + 2)} = \frac{q^2r^2}{4 - 2q^2r}, \tag{11}$$

with the thermodynamic upper bound $\eta_{\text{bound}}(W_{\text{max}}) = \frac{r^2}{1 + r^2}$.

$$\eta_{\text{bound}}(W_{\text{max}}) = \frac{r^2}{1 + r^2}, \tag{12}$$

reached in the tight-coupling limit. The least likely efficiency is found to be

$$\eta^* = r \left( \frac{4 - 3q^2r - q^2r^2}{4 - 2q^2r - 2q^2r^2} \right). \tag{13}$$

Calculations in Fig. 3 indicate that there are lines of singularities for $\eta^*$, varying with the degree of coupling $q$ and the degree of TRB $r$. These singularity lines are reached at $r = (1 \pm \sqrt{1 + 8/q^2})/2$, and they arrive at the thermodynamic bound in the $r$ parameter space at $r = 1$ and $q = \pm 1$. Therefore for $|q| > 1$, there is only one singular point for $\eta^*$ as a function of $r$, while in other cases there can be two singular points. This rich behavior demonstrates that we can largely tune $\eta^*$ and the LDF $J(\eta)$ in TRB systems.

The width of efficiency distribution under maximum macroscopic output power condition has similar features as those obtained under the maximum macroscopic efficiency condition [see Supplementary Material for details]. Nevertheless, a significant difference takes place in the tight-coupling limit: here $\sigma_{\eta}$ tends to infinity for all $r$ except at $r = 1$ (i.e., the TRS limit) where $\sigma_{\eta}$ approaches zero, since here devices are operated with a finite average entropy production rate $\overline{S}_{\text{tot}}$. In the Supplementary Material we explore $J(\eta)$ under different situations and manifest its rich features in various limits. In addition, it was pointed out in Refs. $[3, 8]$ that there exists a stronger “quantum bound” on transport coefficients in multi-terminal Landauer-Büttiker conductors due to the unitarity of the scattering matrix. The effect of the quantum bound on efficiency statistics is discussed in details in the Supplementary Material.

TRB thermoelectric transport in three-terminal systems.— To describe a thermoelectric device we consider the affinities $A_1 = \frac{\mu_d - \mu_c}{T_R}$ and $A_2 = \frac{1}{T_R} - \frac{1}{T_L}$, where $T_{L,R}$ and $\mu_{L,R}$ denote the temperatures and chemical potentials in the left ($L$) and right ($R$) electronic reservoirs. In order to receive $M_{12} \neq M_{21}$, we introduce a third (probe) $P$ terminal, responsible for introducing inelastic effects, by employing the constraints that the average thermal and electrical currents flowing out of the probe terminal $P$ are zero $^{[32]}$. We particularly consider a triple-QDs system described by the Hamiltonian $H = H_{qd} + H_{\text{lead}} + H_{\text{run}}$ where $H_{qd} = \sum_{i=1,2,3} E_i d_i^\dagger d_i + (\mu_d^0d_i^\dagger d_i + \text{H.c.})$, $H_{\text{lead}} = \sum_{i=1,2,3} \sum_k \varepsilon_k c_i^\dagger c_k$, and $H_{\text{run}} = \sum_{c,b} \sum_k V_{ab} c_i^\dagger c_k + \text{H.c.}$. Each quantum dot is coupled to a different reservoir thus we employ the indices $1/2/3$ to identify the leads $L/R/P$, respectively.

This noninteracting model has been analyzed thoroughly in Refs. $[4, 6]$ using the Landauer-Büttiker approach, to obtain the elements of the Onsager matrix. We recall here that the transmission function is obtained from the retarded Green’s function of the QDs and the damping rate $\Gamma = 2\pi \sum_k |V_k|^2 \delta(\omega - \varepsilon_k)$, assumed to be a constant (independent of energy) for all three QDs. We use $\phi = \pi/2$, $\Gamma = 0.5$ and $t = -0.2$, given in units of $k_BT = 1$. Substituting the Onsager elements into Eq. (1), we reach the least probable efficiency $\eta^*$ and the width of distribution $\sigma_{\eta}$ under the maximum macroscopic output power condition, see Fig. 3. We find that around $E_1 + E_2 = 0$, $|\eta^*|$ becomes very large (positive or negative). The underlying reasoning was discussed in Ref. $[6]$. When $E_1 + E_2$ is close to zero, the transport coefficient $|M_{21}|$ can become very small while $|M_{12}|$ is still finite, yielding a very large $|r|$. This in turn pushes $|\eta^*|$ to very large values. In addition, at this special region the width of efficiency distribution $\sigma_{\eta}$ tends to zero, although
FIG. 4. Triple-QDs thermoelectric systems at the maximum macroscopic output power condition: (a) Least probable efficiency $\eta^*$ and (b) width of efficiency distribution $\sigma_{\eta}$. The QD connected to the probe terminal is set at $E_3 = 2$, $\phi = \pi/2$, $\Gamma = 0.5$, and $t = -0.2$. The white region in (a) depicts very large or very small (negative) $\eta^*$ values which are not properly incorporated into the figure.

the average (macroscopic) efficiency is still finite. These calculations, based on the Gaussian approximation, agree with a careful full-counting statistics analysis with vertex corrections [2], carried out in the linear response regime [see Supplementary Material].

Conclusions and discussions.— We explored the operation of time-reversal breaking energy conversion devices by deriving the LDF of efficiency within the Gaussian approximation and studying its properties: the least likely efficiency, the most probable efficiency, and the width of efficiency distribution. We further showed that the long-pursued tight-coupling limit for energy conversion can become unfavourable for applications in TRB systems. Our analysis can be instrumental for understanding energy transducers based on optical [3], atomic [3], superconducting [3, 4], spintronic [3, 4], and thermoelectric nanosystems [3, 4]. Future directions include the exploration of quantum effects in TRB systems [3, 4], and going beyond the Gaussian-linear response approximation.

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[29] For TRB systems, the Onsager matrix can be decomposed as, $\hat{M} = \hat{M}^{sym} + \hat{M}^{asy}$, where $\hat{M}^{sym}$ is the symmetric (dissipative) part and $\hat{M}^{asy}$ the anti-symmetric (dissipationless) part. The currents can be decomposed similarly, $\hat{I}_i = \hat{T}_i^{sym} + \hat{T}_i^{asy}$, where $\hat{T}_i^{sym} = \sum_j M_{ij}^{sym} A_j$ and $\hat{T}_i^{asy} = \sum_j M_{ij}^{asy} A_j$. The tight-coupling limit of TRB systems gives $\hat{T}_2^{asy} = c\hat{T}_1^{sym}$ instead of $\hat{T}_2 = c\hat{T}_1$.
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SUPPLEMENTARY MATERIALS

LDF IN TRB SYSTEMS AND ITS PROPERTIES

Derivation of the LDF for TRB systems

The LDF of efficiency fluctuations was derived in Ref. [1], specific to TRS systems. Here we extend this work and obtain the LDF for TRB systems. We begin by introducing the probability distribution function (PDF) of the stochastic currents \( I_i \) (\( i = 1, 2 \))

\[
P_t(\mathbf{I}) = \frac{t \sqrt{\det((\mathbf{M}^{-1})_{sym})}}{4\pi} \exp(-\frac{t}{4} \delta \mathbf{I}^T \cdot \mathbf{M}^{-1} \cdot \delta \mathbf{I}).
\]

Replacing the stochastic currents with stochastic entropy production rates \( S_i = I_i A_i \) (\( i = 1, 2 \)) at given affinities \( A_1 \) and \( A_2 \), one finds the PDF of the entropy production \([1, 2]\)

\[
P_t(S_1, S_2) = \frac{t \sqrt{\det((\hat{C}^{-1})_{sym})}}{4\pi S_{tot}} \exp\left[ -\frac{t}{4S_{tot}} \delta \mathbf{S}^T \cdot \hat{C}^{-1} \cdot \delta \mathbf{S} \right],
\]

where \( \delta \mathbf{S} = \mathbf{S} - \mathbf{\bar{S}} \), \( \mathbf{\bar{S}} = (S_1, S_2)^T \) is the stochastic entropy production, \( \mathbf{\bar{S}} = (\mathbf{\bar{S}}_1, \mathbf{\bar{S}}_2) \) is the averaged (macroscopic) entropy production rate, and \( \det((\hat{C}^{-1})_{sym}) \) is the determinant of the symmetric part of the inverse of matrix \( \hat{C} \). The macroscopic total entropy production rate is \( S_{tot} = \mathbf{\bar{S}}_1 + \mathbf{\bar{S}}_2 \). Here, \( C_{ij} = A_i M_{ij} A_j/S_{tot} \) (\( i, j = 1, 2 \)). The PDF of stochastic efficiency is

\[
P_t(\eta) = \int dS_1 dS_2 \delta(\eta + \frac{S_1}{S_2}) P_t(S_1, S_2) = \int dS_2 |S_2| P_t(-\eta S_2, S_2).
\]

A direct calculation yields an expression

\[
P_t(-\eta S_2, S_2) = \frac{t \sqrt{\det((\hat{C}^{-1})_{sym})}}{4\pi S_{tot}} \exp\left[ -\frac{t}{4S_{tot}} (a(\eta)S_2^2 + b(\eta)S_2 + c) \right],
\]

with the coefficients

\[
a(\eta) = (C_{11} + \eta(C_{12} + C_{21}) + \eta^2 C_{22})/\det(\hat{C}), \quad c = \mathbf{\bar{S}}_2^2,
\]

\[
b(\eta) = \{\mathbf{\bar{S}}_1(C_{12} + C_{21}) - 2\mathbf{\bar{S}}_2 C_{11} + \eta[2C_{22}\mathbf{\bar{S}}_1 - (C_{12} + C_{21})\mathbf{\bar{S}}_2]\}/[2\det(\hat{C})].
\]

The coefficients \( C_{ij} \) sum up to unity \( \sum_{ij} C_{ij} = 1 \). \( \det(\hat{C}) = C_{11}C_{22} - C_{12}C_{21} \) is the determinant of \( \hat{C} \). Note that \( C_{12} \neq C_{21} \) because of time-reversal symmetry breaking. The full probability distribution of the stochastic efficiency is now found to be

\[
P_t(\eta) = \frac{\sqrt{\det((\hat{C}^{-1})_{sym})}}{\pi a(\eta)} \exp[-t\mathbf{\bar{S}}^2/4] \left[ 1 + \sqrt{\pi\mathbf{\bar{S}}_{tot}} h(\eta) \exp[t\mathbf{\bar{S}}_{tot}h^2(\eta)] \text{erf}(t\mathbf{\bar{S}}_{tot}h(\eta)) \right],
\]

with \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \) being the error function and

\[
h(\eta) = \frac{-b(\eta)}{2\mathbf{\bar{S}}_{tot}\sqrt{a(\eta)}}.
\]

The large deviation function of stochastic efficiency is obtained from

\[
J(\eta) = -\lim_{\eta \to \infty} \frac{\ln[P_t(\eta)]}{t\mathbf{\bar{S}}_{tot}} = \frac{1}{4} - \eta^2 h(\eta).
\]

By substituting the following parametrization,

\[
q \equiv \frac{M_{21}}{\sqrt{M_{22}M_{11}}}, \quad r \equiv \frac{M_{12}}{M_{21}}, \quad \alpha \equiv \frac{A_1\sqrt{M_{11}}}{A_2\sqrt{M_{22}}},
\]

(22)
we find that
\[
\eta = \frac{(1 + \alpha q + \alpha q + \alpha^2)q^2 + \alpha q(1 + r)\eta + \eta^2}{\alpha^2(1 - q^2 r)}, \quad (23a)
\]
\[
b(\eta) = M_{22} A^2_{2}(1 + \alpha q + \alpha q + \alpha^2)(\alpha^2q(r - 1) + 2(\alpha - 1)) + q(r - 1)\eta, \quad (23b)
\]
\[
\overline{S}_{\text{tot}} = M_{22} A^2_{2}(1 + \alpha q + \alpha q + \alpha^2), \quad (23c)
\]
\[
J(\eta) = \frac{[4 - q^2(1 + r)^2]}{16(1 - q^2 r)(1 + \alpha^2 + \alpha q + \alpha q r)(\alpha^2 + \alpha q \eta + \alpha q r \eta + \eta^2). \quad (23d)
\]

**Thermodynamic bounds on the LDF**

We prove here that the inequalities $0 \leq J(\eta) \leq 1/4$ are guaranteed by the thermodynamic bound

\[
|q(1 + r)| \leq 2. \quad (24)
\]

First, $4(1 - q^2 r) \geq 4 - q^2(1 + r)^2 \geq 0$, which guarantees the positive semi-definiteness of the prefactors of the numerator and denominator in Eq. (23d). Second, $1 + \alpha^2 + \alpha q + \alpha q r = \left(\alpha + \frac{\alpha^2(1 + r)}{2}\right)^2 + (1 - \frac{2^2(1 + r)^2}{4}) \geq 0$. This is consistent with the fact that this term originates from the average total entropy production rate. The last term in the denominator is also not less than zero, since $\alpha^2 + \alpha q \eta + \alpha q r + \eta^2 = \left(\alpha + \frac{\alpha^2(1 + r)}{2}\right)^2 + \eta^2(1 - \frac{2^2(1 + r)^2}{4}) \geq 0$. Therefore $J(\eta)$ is guaranteed to be greater than zero. Using exactly the same arguments, one can show that

\[
\frac{1}{4} - J(\eta) = \frac{\alpha^2 q(r - 1) + \alpha q^2(1 + r)(r - \eta) + 2(\eta - 1) + q(r - 1)\eta}{16(1 - q^2 r)(1 + \alpha^2 + \alpha q + \alpha q r)(\alpha^2 + \alpha q \eta + \alpha q r \eta + \eta^2)} \geq 0. \quad (25)
\]

Therefore, $0 \leq J(\eta) \leq 1/4$. From the above we find that $J(\eta) = 1/4$ is reached only at

\[
\eta^* = 1 + \frac{q(r - 1)(1 + \alpha q + \alpha q r + \alpha^2)}{q - qr - 2\alpha + q^2(1 + r)\alpha}. \quad (26)
\]

We also find that there is only one minimum of $J(\eta)$ which is the macroscopic efficiency $\eta$, and only one maximum of $J(\eta)$ which is precisely the least probable efficiency $\eta^*$ given above. This is confirmed by solving the extremum equation $\partial_{\eta} J(\eta) = 0$ where we find only two solutions: one is $\eta$, the other is $\eta^*$. This property determines the basic-generic shape of the LDF curve.

**$J(\eta)$ under time-reversal operation**

In the main text we defined the following parameters

\[
r = \frac{M_{12}}{M_{21}}, \quad q = \frac{M_{21}}{\sqrt{M_{22} M_{11}}} \quad (27)
\]

Under time-reversal operation, $\phi \rightarrow -\phi$, the above parameters transform as follows

\[
r(-\phi) = \frac{1}{r}, \quad q(-\phi) = qr, \quad (28)
\]

where we have used Onsager’s reciprocity relation $M_{12}(\phi) = M_{21}(-\phi)$. Denoting the LDF of efficiency for the reversed magnetic field ($\phi \rightarrow -\phi$) by $\tilde{J}(\eta)$, we obtain that

\[
J(\eta) - \tilde{J}(\eta) = \frac{J(\eta_c) - \tilde{J}(\eta_c)}{1 + \alpha^2 + \alpha q r + q} \quad (29)
\]

We find that for TRB systems Carnot efficiency ($\eta_c = 1$) appears as a special point where the distributions become invariant under time-reversal operation, i.e., $J(\eta_c) = \tilde{J}(\eta_c)$. For TRS systems ($r = 1$) this equality trivially holds for all values of efficiency.
$J(\eta)$ at various limits

In this section we illustrate the rich behavior of $J(\eta)$ at various limits: (i) weak coupling limit $r \to 0$ or $q \to 0$; (ii) tight-coupling limit with maximum macroscopic efficiency for $0 < |r| < 1$, $|r| > 1$, and $r = \pm 1$; (iii) tight coupling limit with maximum macroscopic output power for $|r| \neq 0, 1$, and for $r = \pm 1$. Results are plotted in Fig. 5.

Fig. 5(a) shows that in the weak-coupling limit ($r \to 0$), the LDF experiences a sharp transition from the minimum $\eta \to 0$ to the maximum $\eta^* \to 0$. Therefore, $J(\eta)$ behaves like a derivative of the Dirac delta function. In contrast, when $q \to 0$ but $r$ is finite, Fig. 5(b) shows that $J(\eta)$ develops an infinitely narrow dip near $\eta = 0$, i.e., it behaves like the Dirac delta function itself. Note that in the weak coupling regime with $q \to 0$ and/or $r \to 0$, the affinity parameter $\alpha \to 0$. As a result, the behavior of $J(\eta)$ is very similar either under the maximum macroscopic efficiency condition or the maximum output power condition as both the efficiency and the output power go to 0.

In the tight-coupling limit, the behavior is quite different under those two conditions. We first examine the maximum macroscopic efficiency condition. Fig. 5(c) shows that for $0 < |r| < 1$, the width of efficiency distribution $\sigma_\eta$ tends to infinity, while the maximum value of $J(\eta)$ at $\eta^* = r$ develops a very sharp peak. In contrast, for $|r| > 1$, as shown in Fig. 5(d), the width of efficiency distribution approaches zero, while the width at the least probable efficiency becomes infinite. The $r = 1$ situation, see Fig. 5(e), demonstrates a sharp transition from the minimum value to the maximum point, resembling the behavior of the derivative of the Dirac delta function. For $r = -1$, the tight coupling limit, $|q(1 + r)| \to 2$ is pushed to $q \to \infty$. Therefore, in this situation for any finite $q$ the behavior of $J(\eta)$ shows a regular behavior. Nevertheless, as shown in Fig. 5(f), the distribution of $\eta$ is quite broad for $r = -1$. This case is relevant to recent studies on “chiral thermoelectrics” (e.g., Nernst engines) where, however, a much stronger bound on $q$ was obtained [3, 4].

Under the maximum macroscopic output power condition, the behavior of $J(\eta)$ for $r \to 0$, $q \to 0$ cases is identical to that observed in Fig. 5(a) and (b). The tight-coupling limit with $0 < |r| < 1$ also shows features similar to Fig. 5(c). However, when $r = 1$ in the tight-coupling limit, see Fig. 5(g), the width around the macroscopic efficiency ($\eta^* = 0.5$) approaches zero while the width around the maximum value of $J(\eta)$, at $\eta^* = 1$, becomes infinite. Fig. 5(h) focuses on the $|r| > 1$ case for which the width of efficiency distribution tends to infinity, while the maximum of $J(\eta)$ becomes a sharp peak. Fig 5(i) shows that at $r = -1$, $J(\eta)$ becomes an extremely broad distribution.

**Width of efficiency distribution under general conditions**

The width of efficiency distribution at arbitrary $\alpha$ is found to be

$$\sigma_\eta = \frac{2\sqrt{2}\alpha(1 - q^2r)(1 + \alpha^2 + \alpha q + \alpha qr)}{(1 + \alpha q)^2 \sqrt{4 - q^2(1 + r)^2}} \quad (30)$$

In Fig. 6 we display $\sigma_\eta$ as a function of the affinity parameter $\alpha$ and the TRB parameter $r$ for $q = 0.5$. We find that the width of efficiency distribution grows with $|\alpha|$; at $\alpha \sim 0$ it is very small. Besides, it tends to very large values in approaching the tight-coupling limit.

**Width of efficiency distribution under the maximum macroscopic power condition**

Fig. 5(b) in the main text illustrates the behavior of the width of efficiency distribution $\sigma_\eta$ under the condition of maximum macroscopic efficiency. Here we complement this result and display in Fig. 7(a) the behavior of $\sigma_\eta$ under the condition of maximum macroscopic power. We see in this figure that the features of $\sigma_\eta$ are qualitatively similar in both cases. In particular, at the small $q^2 r$ limit, $\sigma_\eta$ for both cases become quantitatively similar, which is consistent with the understanding that the maximum macroscopic efficiency and maximum macroscopic power conditions are close at this limit. Nonetheless, we find that $\sigma_\eta$ diverges in the tight-coupling limit for all $r$ except at $r = 0, \pm 1$. In Fig. 7(b) we examine a particular case with $q = 0.5$. The tight-coupling limit is then reached for $r = -5$ or $r = 3$, where $\sigma_\eta$ is shown to diverge.

**QUANTUM MECHANICAL BOUND ON EFFICIENCY FLUCTUATIONS**

We examine here the effect of the quantum mechanical bound, which was introduced in Ref. 2, on efficiency fluctuations. Within our parameters the bound comes up as $1 + q^2r - q^2 - q^2r^2 \geq 0$, affecting the width of efficiency
FIG. 5. Illustrations of the rich behavior of $J(\eta)$ at various limits. (a)-(b) Weak-coupling limit. (a) The $r \to 0$ limit with $r = 0.001$ and $q = 0.5$. (b) The $q \to 0$ limit with finite $r$. We employ $q = 0.001$ and $r = 0.5$. (c)-(f) Several examples for achieving the tight-coupling limit $|q(1 + r)| \to 2$ with maximum macroscopic efficiency. (c) $r = 0.5, q = 1.3332$ (d) $r = 1, q = 0.9999, (e) r = 5, q = 0.3332$, (f) $r = -1, q = 10$. (g)-(i) Examples for the tight-coupling limit with maximum macroscopic output power. (g) $r = 1, q = 0.9999$, (h) $r = 5, q = 0.3332$, (i) $r = -1, q = 10$. Note that the scales of the horizontal axes in these figures are different.

FIG. 6. Width of efficiency distribution $\sigma_\eta$ as a function of the affinity parameter $\alpha$ and the TRB parameter $r$ for $q = 0.5$. The boundaries, $r = -5$ and $r = 3$, satisfy the tight-coupling limit. White regions appear either because $\sigma_\eta$ is too small (around $\alpha = 0$), or because it is too large (around $r = -5$ or 3).

distribution fluctuation $\sigma_\eta$. The bound was obtained by considering a three-terminal Landauer-Büttiker model, and it is a direct result of the unitarity of the scattering matrix. In Fig. 5 we plot (a) $\sigma_\eta$ under the maximum macroscopic efficiency condition, (b) the least probable efficiency $\eta^*$, and (c) $\sigma_\eta$ under the maximum macroscopic power condition. The quantum bound regularizes the divergent behavior of $\sigma_\eta$ under both conditions, as well as the divergency of $\eta^*$ for the maximum power condition. This is because the above bound prohibits us from meeting the tight-coupling limit, unless $r = 1$, see $J(\eta)$ in Fig. 5(d). However, note that the least probable efficiency can still diverge at the
following affinity parameter, even within the quantum mechanical bound,
\[ \alpha_c = \frac{q(1-r)}{2 - q^2 - q^2 r}. \] (31)

At this value we receive positive average efficiencies and average output power when \( r > 1 \) or \( r < -1 \). Inspecting Fig. 8 further, we find that the width of efficiency distribution under both maximum macroscopic efficiency and maximum macroscopic power conditions can become considerably large. In addition, the least probable efficiency \( \eta^* \) can deviate significantly from the Carnot efficiency. In fact, within the maximum macroscopic efficiency condition, \( \eta^* \) is allowed to diverge when \( r \to \infty \) [6]. We note that the above bound goes to the thermodynamic bound for \( N \)-terminal Landauer-Büttiker conductors, when \( N \to \infty \) [3]. However, the quantum bound may not necessarily hold when genuine many body interactions and inelastic processes (e.g., electron-electron scattering and electron-phonon scattering) are taken into account.

**COMPARISON BETWEEN THE GAUSSIAN APPROXIMATION AND A FULL COUNTING STATISTICS ANALYSIS FOR A TRIPLE-QDS THERMOELECTRIC MODEL**

In this section we present a comparison between the LDFs as obtained under the Gaussian approximation and a full counting statistics analysis using a three-terminal thermoelectric model. A counting statistics method including a probe terminal has been recently developed by Utsumi et al. (though the discussion there is focused on heat currents fluctuations) [7]. In this approach the LDF of efficiency is calculated through the cumulant-generating function (CGF) defining counting fields for the particle number \( \xi_j \) and energy \( \lambda_j \) in each reservoir \( j = L, R, P \). The CGF of the three-terminal system is given by [7]

\[ \mathcal{F}_{3t} = \int \frac{d\omega}{2\pi} \ln \det \left[ \mathbf{1} - \mathbf{f}(\omega) \mathbf{K}(\omega, \phi) \right]. \] (32)
Here
\[ \hat{f} = \text{diag}(f_L, f_R, f_P), \quad \hat{K} = 1 - e^{i\hat{S}}(\omega, \phi)e^{-i\hat{S}}(\omega, \phi), \quad \hat{\theta} = \text{diag}(\omega \lambda_L + \xi_L, \omega \lambda_R + \xi_R, \omega \lambda_P + \xi_P), \]
(33)
with “diag” denoting a diagonal matrix, \( f_j = [\exp(\frac{\mu_j}{T_j}) + 1]^{-1} \) (\( j = L, R, P \)) is the Fermi distribution for \( j \)-th reservoir, and \( \hat{S} \) is the S-matrix of the triple-quantum-dot (QD) system.

By integrating out the short-time dynamics it was shown in Ref. [7] that the effective CGF for particle and energy transport between the L and R reservoirs (when the particle and energy currents flowing out of the probe terminal are zero) is given by
\[ F_{2t}(\lambda_L, \lambda_R, \xi_L, \xi_R, T_L, \mu_L, T_R, \mu_R) = F_{3t}(\lambda_L, \lambda_R, \lambda_p^*, \xi_L, \xi_R, \xi_p^*, T_L, \mu_L, T_R, \mu_R, T_p^*, \mu_p^*). \]
(34)
Here \( \lambda_p^*, \xi_p^*, T_p^*, \) and \( \mu_p^* \) are determined by the saddle-point equations
\[ \frac{\partial F_{3t}}{\partial \lambda_p} = \frac{\partial F_{3t}}{\partial \xi_p} = \frac{\partial F_{3t}}{\partial T_p} = \frac{\partial F_{3t}}{\partial \mu_p} = 0, \]
(35)
which maximize the probability of processes with zero energy and particle currents flowing out of the probe terminal.

In the linear-response regime, the effective two-terminal CGF can be approximated by a second-order expansion in \( \lambda_j \) and \( \xi_j \) as well as the affinities (\( \mu_j - \mu \))/\( T \) and \( 1/T - 1/T_j \) for \( j = L, R \). Due to particle and energy conservation, the counting fields for the right reservoir can be regarded as redundant, hence we can set \( \lambda_R = \xi_R = 0 \). Furthermore we set \( T_R = T \) and \( \mu_R = \mu \). The approximate two-terminal CGF is now given by
\[ F_{2t}(\bar{a}) = \frac{1}{2} \bar{a}^T \cdot \bar{R} \bar{a}, \]
where
\[ \bar{a}^T = (\lambda_L, \xi_L, A_1, A_2), \quad \bar{R}_{\beta \beta'} = \frac{\partial^2 F_{3t}}{\partial a_\beta \partial a_{\beta'}} \bigg|_{\lambda_L = \xi_L = A_1 = A_2 = 0}, \]
(37)
with \( A_1 = (\mu_L - \mu_R)/T_R \) and \( A_2 = 1/T_R - 1/T_L \).

The matrix \( \bar{R} \) must be calculated from the second derivative tensor of the three-terminal CGF at equilibrium with vertex corrections as shown in Ref. [7],
\[ \frac{\partial^2 F_{3t}}{\partial a_\beta \partial a_{\beta'}} \bigg|_{\lambda_L = \xi_L = A_1 = A_2 = 0} = \left( \begin{array}{ccc} \frac{\partial^2 F_{3t}}{\partial a_\beta \partial a_{\beta'}} & -\frac{\partial^2 F_{3t}}{\partial a_\beta \partial b_{\gamma'}} & \frac{\partial^2 F_{3t}}{\partial a_\beta \partial b_{\gamma'}} \\ \frac{\partial^2 F_{3t}}{\partial a_\beta \partial b_{\gamma'}} & \frac{\partial^2 F_{3t}}{\partial b_{\gamma} \partial b_{\gamma'}} & \frac{\partial^2 F_{3t}}{\partial b_{\gamma} \partial b_{\gamma'}} \\ \frac{\partial^2 F_{3t}}{\partial a_\beta \partial b_{\gamma'}} & \frac{\partial^2 F_{3t}}{\partial b_{\gamma} \partial b_{\gamma'}} & \frac{\partial^2 F_{3t}}{\partial b_{\gamma} \partial b_{\gamma'}} \end{array} \right) \bigg|_{\lambda_L = \xi_L = \lambda_R = \xi_R = A_1 = A_2 = A_3 = A_4 = 0}, \]
(38)
where \( \bar{b} = (\lambda_P, \xi_P, A_3, A_4) \) with \( A_3 = (\mu_P - \mu_R)/T_R, A_4 = 1/T_R - 1/T_P, \) and \( U = \hat{U}^{-1} \) with
\[ B_{\gamma \gamma'} = \frac{\partial^2 F_{3t}}{\partial b_\gamma \partial b_{\gamma'}} \bigg|_{\lambda_L = \xi_L = \lambda_R = \xi_R = A_1 = A_2 = A_3 = A_4 = 0}. \]
(39)
We now change variables
\[ i\lambda_L \rightarrow \eta \zeta (T_L - T_R)/T^2, \quad i\xi_L \rightarrow \zeta (\mu_L - \mu_R)/T, \]
(40)
and obtain the LDF of efficiency fluctuations as [2]
\[ \mathcal{G}(\eta) = -\min_\zeta F_{2t}(-i\eta \zeta, -i\zeta (\mu_L - \mu_R)/T, A_1, A_2) \]
(41)
for any given \( A_1 \) and \( A_2 \).

In Fig. [a] we plot the LDF \( J(\eta) \) as obtained from the Gaussian approximation [Eq. (23b)] and that from the full counting statistics method. The two functions perfectly match. In Fig. [b] we plot the least probable efficiency \( \eta^* \) calculated from the Gaussian approximation and the full counting statistics method. Again, the results from the two methods agree well with each other. These results confirms the validity of our analysis in the main text based on the Gaussian approximation, a consequence of the fluctuation theorem.
FIG. 9. Comparison between $J(\eta)$ within the Gaussian approximation (labeled by “G”), and from a full counting statistics analysis (labeled by “FCS”) for the triple-QDs thermoelectric model. (a) $J(\eta)$ for $\phi = \pi/2$. (b) $\eta'$ versus $\phi$. Parameters are $E_1 = 1$, $E_2 = 0$, $E_3 = 1$, $\Gamma = 1$, $t = 0.4$, $T_l = 1.125$, $T_R = 1$, $\mu_L = -0.01$, and $\mu_R = 0$.

CHARACTERIZATION OF THERMOELECTRIC TRANSPORT IN A TRIPLE-QDS SYSTEM

We calculate linear transport coefficients for a three-terminal thermoelectric model when the average thermal and electrical currents flowing out of the probe terminal $P$ are zero. These conditions lead to $\dot{M} = \dot{M}_{LL} - \dot{M}_{LP} \dot{M}_{PP}^{-1} \dot{M}_{PL}$. Here $\dot{M}$ is the transport matrix for the total three-terminal system, i.e., $\dot{P} = \dot{M} \dot{A}$ where $\dot{P} = (\dot{P}_L, \dot{P}_P)$ and $\dot{A} = (\dot{A}_L, \dot{A}_P)$ with $\dot{P}_L = (I_{L,e}, I_{L,h})$ and $\dot{A}_\gamma = (A_{\gamma,e}, A_{\gamma,h})$ are the currents and affinities for terminals $\gamma = L, P$. $I_e$ is the charge current, $I_h$ is the heat current, and e.g., $A_{Le} = (\mu_L - \mu_R)/T_R$, $A_{Lh} = 1/T_R - 1/T_L$. Linear transport coefficients are calculated from the expression

$$\dot{M}_{\gamma\gamma'} = \int \frac{d\omega}{2}\left[\delta_{\gamma\gamma'} - |S_{\gamma\gamma'}(\omega, \phi)|^2\right] \left(\frac{1}{\omega + i\omega^2}\right)f_0(\omega)[1 - f_0(\omega)],$$

(42)

where $S_{\gamma\gamma'}(\omega, \phi) (\gamma, \gamma' = L, P)$ is the scattering matrix between terminals $\gamma$ and $\gamma'$, $\phi = 2\pi\Phi/\Phi_0$. $\Phi$ and $\Phi_0$ are the magnetic flux in our triple-quantum-dots (QDs) system and the flux quantum, respectively. The Fermi distribution $f_0(\omega) = [\exp(\frac{\omega}{\Theta}) + 1]^{-1}$ corresponds to an equilibrium state with the chemical potential set at $\mu = 0$. Onsager reciprocal symmetry originates from the symmetry $S_{\gamma\gamma'}(\omega, \phi) = S_{\gamma'\gamma}(\omega, -\phi)$.

The transmission function is obtained from the relation $\hat{S}(\omega, \phi) = -1 + i\hat{\Gamma}(\omega)$. Here $\hat{I}$ is a $3 \times 3$ identity matrix and $\hat{\Gamma}(\omega) = [\omega + i\Gamma]1 - \hat{H}_{qd}^{-1}$ is the retarded Green’s function of the QDs. The hybridization energy $\Gamma = 2\pi\sum_k |V_k|^2\delta(\omega - \varepsilon_k)$ is assumed to be a constant (independent of energy) for all three QDs.

We calculate the transport coefficients and then determine the TRB parameter $r$ and the degree of coupling $q$. In Fig. 10 we plot these parameters against $E_1$ and $E_2$. We further confirm the “quantum bound” on linear transport coefficients discovered by Brandner et al. for three-terminal TRB conductors, which using our parametrization casts into the form as $1 + q^2r - q^2 - q^2r^2 > 0$. In Fig. 11(c) we plot it and show that it is always greater than zero in our parameter region. To illustrate the divergence of $r$, we plot it in Fig. 11(d) as a functions of the magnetic flux $\phi$. We find that $r$ diverges when $M_{21}$ goes to zero with a finite $M_{12}$. This happen when $\phi \simeq -0.4\pi$ or $\phi \simeq 1.6\pi$.

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FIG. 10. Time-reversal symmetry breaking in a three-terminal triple QDs model for thermoelectrics. (a) TRB parameter $r$ and (b) degree of coupling $q$ as functions of the QDs energies $E_1$ and $E_2$. (c) The function $1 + q^2 r - q^2 - q^2 r^2$ for the same values of energies as in panel (a), demonstrating that the quantum bound is satisfied. In (a)-(c) we used the following parameters: the QD connected to the probe terminal has energy $E_3 = 2$, $\phi = \pi/2$, $\Gamma = 0.5$, and $t = -0.2$ (energy unit is $k_B T = 1$). (d) The example illustrates the appearance of $r \to \infty$ at a certain flux ($\phi$) value, when the parameters are chosen as $E_1 = 0.2$, $E_2 = -0.2$, $E_3 = 1$, $\Gamma = 0.5$, and $t = -1$. 