An Elliptic Type Gradient Estimate For the Schrödinger Equation

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Abstract

In this paper, the author discusses the elliptic type gradient estimate for the solution of the time-dependent Schrödinger equations on noncompact manifolds. As its application, the dimension-free Harnack inequality and the Liouville type theorem for the Schrödinger equation are proved.

Keywords: Gradient estimate, Schrödinger equation, Liouville type theorem.

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1. Introduction

In [3], Cheng and Yau proved the classical gradient estimate for harmonic functions. Later Li and Yau [10] obtained the following parabolic type gradient estimate for the heat equation $u_t = \Delta u$ and where the subscript $t$ denotes the partial differentiation with respect to the $t$-variable, $\Delta$ is the Laplacian operator on $M$.

Theorem 1.1. (Li – Yau) Let $M$ be an $n$-dimensional complete manifold with Ricci curvature bounded below by $-K$, $K \geq 0$. Suppose that $u$ is any positive solution to the heat equation $u_t = \Delta u$ in $B(x_0, 2R) \times [t_0 - 2T, t_0]$, then for $\forall \alpha > 1$,

$$\frac{\left| \nabla u \right|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{C}{R^2} + \frac{na^2}{2T} + \frac{na^2}{\sqrt{2(\alpha - 1)}}K$$

in $B(x_0, R) \times [t_0 - T, t_0]$. Here $\nabla$ denotes the gradient operator on $M$, and the positive constant $C$ depends only on dimension $n$. 

Recently X.D.Li \cite{Li} studied the heat equation \( u_t = \Delta u + \nabla \phi \cdot \nabla u, \phi \in C^2(M) \), on a manifold with Bakry-Emery’s Ricci curvature \( \widetilde{\text{Ric}} := \text{Ric} - \nabla^2 \phi - \frac{1}{m-n} \nabla \phi \otimes \nabla \phi \), where the constant \( m > n \), \( \text{Ric} \) denotes the Ricci curvature of \( M \) and \( m = n \) if and only if \( \phi = 0 \). If the curvature condition in Theorem 1.1 is replaced by the following curvature condition:

\[
\widetilde{\text{Ric}} \geq -K, \tag{1.2}
\]

then Li obtained a similar gradient estimate for the heat equation \( u_t = \Delta u + \nabla \phi \cdot \nabla u \):

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{C}{R^2} + \frac{m\alpha^2}{2T} + \frac{m\alpha^2}{\sqrt{2}(\alpha - 1)} K.
\]

The Bakry-Emery Ricci curvature has some interesting properties. In \cite{Bakry-Emery}, the author established some connections between the Bakry-Emery Ricci curvature and Ricci flow. In \cite{Li}, he also proved some Liouville type theorems for the above heat equation \( u_t = \Delta u + \nabla \phi \cdot \nabla u \) on a Manifold with the Bakry-Emery Ricci curvature.

As was shown in \cite{Lott-Villani}, using (1.1) we can derive the following Harnack inequality:

\[
u(x_1, t_1) \leq u(x_2, t_2) \left( \frac{t_2}{t_1} \right)^{\frac{m}{2}} \exp \left( \frac{\alpha \rho^2(x_1, x_2)}{4(t_2 - t_1)} + \frac{n\alpha K}{\sqrt{2(\alpha - 1)}} (t_2 - t_1) \right)
\]

where \( \forall x_1, x_2 \in M \), \( \rho(x_1, x_2) \) denotes the geodesic distance between \( x_1 \) and \( x_2 \), and \( 0 < t_1 < t_2 < +\infty \). We notice that from this kind of Harnack inequality we can only compare the solutions at different times. In \cite{Hamilton}, Hamilton got the following elliptic type gradient estimate on compact manifolds. From this gradient estimate one can compare the solution of two different points at the same time.

**Theorem 1.2. (Hamilton)** Let \( M \) be a compact manifold without boundary and with Ricci curvature bounded below by \( -K \), \( K \geq 0 \). Suppose that \( u \) is any positive solution to the heat equation \( u_t = \Delta u \) with \( u \leq C \) for all \( (x,t) \in M \times (0, +\infty) \). Then

\[
\frac{|\nabla u|^2}{u^2} \leq \left( \frac{1}{t} + 2K \right) (\ln \frac{C}{u}). \tag{1.3}
\]

Recently Souplet and Zhang \cite{Souplet} generalized the elliptic type gradient estimate to noncompact manifolds.

**Theorem 1.3. (Souplet – Zhang)** Let \( M \) be an \( n \)-dimensional complete noncompact manifold with Ricci curvature bounded below by \( -K \), \( K \geq 0 \). Suppose that \( u \) is any positive solution to the heat equation \( u_t = \Delta u \) in \( Q_{2R,2T} \equiv B(x_0, 2R) \times [t_0 - 2T, t_0] \), and \( u \leq C \) in \( Q_{2R,2T} \). Then

\[
\frac{|\nabla u|}{u} \leq C_1 \left( \frac{1}{R} + \frac{1}{T^\frac{n}{2}} + \sqrt{K} \right) (1 + \ln \frac{C}{u}) \tag{1.4}
\]

in \( Q_{R,T} \). Here the positive constant \( C_1 \) depends only on dimension \( n \), moreover \( C_1 \) increases to infinity as \( n \) goes to infinity.
We know that (1.4) is a local gradient estimate. Letting $R \to +\infty$, we can obtain the global gradient estimate:

$$\frac{|\nabla u|}{u} \leq C_1 \left( \frac{1}{T^2} + \sqrt{K} \right) (1 + \ln \frac{C}{u})$$

(1.5)

We notice that the constant $C_1$ in (1.5) depends on the dimension $n$, moreover $C_1$ increases to infinity as $n$ goes to infinity, so this gradient estimate cannot be applied to infinite dimensional manifolds. However Hamilton’s gradient estimate (1.3) does not depend on dimension $n$. In this paper, we want to prove the following dimension-free elliptic type gradient estimate for a more general equation—the Schrödinger equation with potential $h(x,t)$:

$$u_t = \triangle u + \nabla \phi \cdot \nabla u + hu.$$  

(1.6)

**Theorem 1.4.** Let $M$ be an $n$-dimensional complete noncompact manifold satisfying the curvature condition (1.2). Suppose that the potential $h$ is a negative function defined on $M \times (0, +\infty)$ which is $C^1$ in the $x$-variable, and that $u$ is any positive solution to the Schrödinger equation (1.6) with $u \leq C$ for all $(x, t) \in M \times (0, +\infty)$. Then

$$\frac{|\nabla u|}{u} \leq \left( \frac{1}{T^2} + \sqrt{2K} + |\nabla \sqrt{-h}| \right) (1 + \ln \frac{C}{u}).$$

(1.7)

As proved by Li and Yau in [10], we can derive the dimension-free Harnack inequality for the equation (1.8). We must point out that the dimension-free Harnack inequality for the heat equation $u_t = \triangle u + \nabla \phi \cdot \nabla u$ was first observed by Wang [18]. This kind of inequality can be widely applied in geometric analysis and probability, more details can be seen in Wang’s survey article [19].

**Corollary 1.1.** Let $M$ be an $n$-dimensional complete noncompact manifold satisfying the curvature condition (1.2). Suppose that the potential $h$ is a negative function defined on $M \times (0, +\infty)$ which is $C^1$ in the $x$-variable, and $|\nabla \sqrt{-h}| \leq C_2$ for a positive constant $C_2$, and that $u$ is any positive solution to the Schrödinger equation (1.6) with $u \leq 1$ for all $(x, t) \in M \times (0, +\infty)$. Then for $\forall x_1, x_2 \in M$,

$$u(x_2, t) \leq u(x_1, t) e^{(1-\beta)}$$

where $\beta = \exp(-\frac{\rho(x_1, x_2)}{t^2} - (\sqrt{2K} + \sqrt{C_2})\rho) \rho(t)$ and $\rho(x_1, x_2)$ denotes the geodesic distance between $x_1$ and $x_2$.

When potential $h(x,t) = -\lambda$, $\lambda$ is a positive constant, (1.6) becomes

$$u_t = \triangle u + \nabla \phi \cdot \nabla u - \lambda u.$$  

(1.8)

Then we can easily obtain the following dimension-free elliptic type gradient estimate for the equation (1.8), which improves Souplet and Zhang’s gradient estimate (1.5).
Corollary 1.2. Let $M$ be an $n$-dimensional complete noncompact manifold satisfying the curvature condition (1.2). Suppose that $u$ is any positive solution to the equation (1.8) with $u \leq C$ for any $(x,t) \in M \times (0, +\infty)$. Then

$$\frac{|\nabla u|}{u} \leq \left( \frac{1}{t^{\frac{n}{2}}} + \sqrt{2K} \right) \left( 1 + \ln \frac{C}{u} \right).$$  \quad (1.9)$$

We notice that when the solution of (1.8) does not depend on time, the equation (1.8) becomes

$$\Delta u(x) + \nabla \phi(x) \cdot \nabla u(x) - \lambda u(x) = 0. \quad (1.10)$$

From (1.9), letting $t \to \infty$, we have the gradient estimate for the equation (1.10):

$$\frac{|\nabla u|}{u} \leq \sqrt{2K} \left( 1 + \ln \frac{C}{u} \right). \quad (1.11)$$

From Schoen and Yau’s nice book [16], it is known that the gradient estimate of the positive solution for the equation $\Delta u - \lambda u = 0$ is that $\frac{|\nabla u|}{u} \leq C(n, K, \lambda)$ where the positive constant $C(n, K, \lambda)$ depends on $n$, $K$, $\lambda$. However the gradient estimate (1.11) doesn’t depend on $\lambda$, so this result allows us to deduce the following Liouville type theorem:

Corollary 1.3. Let $M$ be an $n$-dimensional complete noncompact manifold with nonnegative Bakry-Emery’s Ricci curvature, i.e. $K = 0$ in (1.2). Then there does not exist any positive and bounded solution to Schrödinger equation (1.10).

Remark: The Liouville type theorem for the Schrödinger equation was investigated by many authors. In [8], [10], [11] and [12], the authors considered the Schrödinger equation with nonnegative potential and obtained a similar Liouville type theorem. However in our case the potential is negative. In [4], [5] and [7], the authors discussed the Liouville type theorem for the Schrödinger equation on parabolic manifolds. However in our case, the manifold need not be parabolic.

2. Proof of Main Theorem and Corollaries

In this section, we will prove Theorem 1.4 and its corollaries.

Proof: First we define the operators $L = \Delta + \nabla \phi$ and $\Box = L - \partial_t$. Since $u$ is the solution of the equation $\Box u = -hu$, we notice that the solution $u$ is invariant under scaling, so we can assume $u \leq 1$. Setting $f = \ln u$, $w = |\nabla \ln (1 - f)|^2$, then $f \leq 0$ and also satisfies the following equation

$$\Box f = u^{-1} \Box u - |\nabla f|^2 = -h - |\nabla f|^2. \quad (2.1)$$
By (2.1), we see that
\[ \Box \ln(1 - f) = -\frac{\Box f}{1 - f} - |\nabla \ln(1 - f)|^2 = \frac{h}{1 - f} - fw. \] (2.2)

For convenience of computation, we introduce the two curvature operators defined by Bakry-Emery \[1\]. We define the curvature operator \( \Gamma \) by
\[ \Gamma(f, g) = \frac{1}{2} \{ L(fg) - fLg - gLf \}, \ \forall f, g \in C^2(M) \]

It is obvious that
\[ \Gamma(f, f) = |\nabla f|^2 \] (2.3)

And the Bakry-Emery curvature operator \( \Gamma_2 \) is defined to be a bilinear map:
\[ \Gamma_2(f, g) = \frac{1}{2} \{ L\Gamma(f, g) - \Gamma(Lf, g) - \Gamma(f, Lg) \}, \ \forall f, g \in C^2(M) \]

In particular, we have that
\[ \Gamma_2(f, f) = \frac{1}{2} \{ L\Gamma(f, f) - 2\Gamma(Lf, f) \} \]

Through direct calculation, we find that the Bakry-Emery curvature operator \( \Gamma_2 \) also satisfies
\[ \Gamma_2(f, f) = \frac{1}{2} \{ \Box \Gamma(f, f) - 2\Gamma(\Box f, f) \}, \ \forall f \in C^2(M). \] (2.4)

In the sequel, for convenience we sometimes denote \( \Gamma(f, f) \) and \( \Gamma_2(f, f) \) by \( \Gamma(f) \) and \( \Gamma_2(f) \) respectively.

By Bochner’s formula, under the curvature condition (1.2), we have that
\[ \Gamma_2(f) \geq -K\Gamma f, \ \forall f \in C^2(M). \] (2.5)

This inequality was proved in \[1\] (see also \[9\]).

By (2.2) – (2.5), we know that
\[ \Box w = \Box \Gamma(\ln(1 - f)) \]
\[ = 2\Gamma_2(\ln(1 - f)) + 2\Gamma(\Box \ln(1 - f), \ln(1 - f)) \]
\[ = 2\Gamma_2(\ln(1 - f)) + 2\Gamma(\frac{h}{1 - f}, \ln(1 - f)) - 2\Gamma(fw, \ln(1 - f)) \]
\[ \geq -2Kw - \frac{2h}{1 - f}w - \frac{2|\nabla h|}{1 - f}w^{\frac{3}{2}} + 2(1 - f)w^2 + 2f|\nabla w|w^{\frac{1}{2}} \]
\[ \geq -2Kw - \frac{|\nabla \sqrt{h}|^2}{1 - f} + 2(1 - f)w^2 + 2f|\nabla w|w^\frac{1}{2} \]
\[ \geq -2Kw - |\nabla \sqrt{h}|^2 + 2(1 - f)w^2 + 2f|\nabla w|w^\frac{1}{2} \]

where we use the fact that \( h < 0 \), \( \frac{1}{1 - f} < 1 \) in the above last two inequalities. Hence we obtain

\[ \Box w \geq -2Kw - |\nabla \sqrt{h}|^2 + 2(1 - f)w^2 + 2f|\nabla w|w^\frac{1}{2}. \quad (2.6) \]

Now we choose a cut-off function \( \eta \) to be a \( C^2 \) function on \([0, +\infty)\) satisfying

\[ \eta(t) = 1, \quad \text{for} \quad 0 \leq t \leq 1; \quad \eta(t) = 0, \quad \text{for} \quad t \geq 2; \]
and

\[ 0 \leq \eta(t) \leq 1, \quad -C\eta(t)^\frac{1}{2} \leq \eta'(t) \leq 0, \quad \eta''(t) \geq -C \quad \text{for} \quad \forall t \geq 0. \]

Let \( \rho(p, x) \) be the geodesic distance between \( p \) and \( x \), and define \( \psi(x) = \eta(\frac{\rho(p, x)}{R}) \). Then we have that

\[ \frac{|\nabla \psi|^2}{\psi} = \frac{\eta'^2|\nabla \rho|^2}{R^2 \eta} = \frac{\eta'^2}{R^2 \eta} \leq C^2 \quad (2.7) \]

and

\[ L\psi(x) = \eta''|\nabla \rho|^2 \frac{R^2}{R^2} + \eta' L\rho \frac{R}{R} \geq -mC \frac{R}{R^2} - \frac{mK}{R}, \quad x \notin \text{cut}(p), \quad (2.8) \]

where we use a general Laplacian comparison theorem: \( L\rho \leq \frac{m-1}{\rho} + (m - 1)K \) under the curvature condition (1.2), which was proved by Qian \[13\].

Let \( \varphi = t\psi \). Suppose that \( \varphi w \) attains its maximum at the point \((x_0, t_0) \in B(p, 2R) \times [0, T]\). According a well known argument of Calabi\[2\], we can assume that \( x_0 \) is not in the cut locus of \( p \). Then at \((x_0, t_0)\) we have

\[ \nabla (\varphi w) = 0, \quad \Delta (\varphi w) \leq 0, \quad \partial \frac{\partial}{\partial t} (\varphi w) \geq 0 \]

Thus we derive that at \((x_0, t_0)\)

\[ \Box (\varphi w) \leq 0, \quad \nabla w = -\frac{\nabla \varphi}{\varphi} w. \quad (2.9) \]

Hence (2.9) implies that at \((x_0, t_0)\)

\[ \varphi \Box w + \Box \varphi w + 2\Gamma(\varphi, w) \leq 0 \]

Combining the above inequality with (2.6) and (2.9), we have at \((x_0, t_0)\)

\[ \varphi \{-2Kw - |\nabla \sqrt{h}|^2 + 2(1 - f)w^2 + 2f \frac{|\nabla \varphi|}{\varphi} w^\frac{1}{2} \} + \Box \varphi w - \frac{2|\nabla \varphi|^2}{\varphi} w \leq 0 \]
By the Young inequality: \(2ab \leq a^2 + b^2\) and \(f < 0\), we see at \((x_0, t_0)\)

\[-2K\varphi w - |\nabla \sqrt{-h}|^2 \varphi + (1 - f)\varphi w^2 - \frac{f^2|\nabla \varphi|^2}{(1 - f)}w + \Box \varphi w - \frac{2|\nabla \varphi|^2}{\varphi}w \leq 0\]

Multiplying the both sides of the above inequality by \(\varphi\), we notice that \(0 \leq \psi \leq 1\), \(f^2(1 - f) \leq 1\) and \(\frac{1}{1 - f} \leq 1\), then from (2.7) and (2.8) we know at \((x_0, t_0)\)

\[(\varphi w)^2 - T(2K + \frac{3C^2}{R^2} + \frac{mC}{R} + \frac{mK}{R} + \frac{1}{T})(\varphi w) - |\nabla \sqrt{-h}|^2 T^2 \leq 0\]

Since \(\psi = 1\) on \(B(p, R)\), then we have

\[w(x, T) \leq 2K + \frac{3C^2}{R^2} + \frac{mC}{R} + \frac{mK}{R} + \frac{1}{T} + |\nabla \sqrt{-h}|\]

for \(\forall x \in B(p, R)\). Since \(T\) is arbitrary, let \(R \to +\infty\), then we obtain

\[\frac{|\nabla u|}{u} \leq \left(1 + \frac{1}{t^2} + \sqrt{2K} + \sqrt{|\nabla \sqrt{-h}|^2}\right)(1 - \ln u)\]

The scaling invariance of \(u\) implies (1.7). This completes the proof of Theorem 1.4.

Now we give the proof of Corollary 1.5:

**Proof:** Let \(\gamma(s)\) be a minimal geodesic joining \(x_1\) and \(x_2\), \(\gamma : [0, 1] \to M\), \(\gamma(0) = x_2\), \(\gamma(1) = x_1\). Then by (1.9)

\[\ln \frac{1 - f(x_1, t)}{1 - f(x_2, t)} = \int_0^1 \frac{d\ln(1 - f(\gamma(s), t))}{ds} ds\]

\[\leq \int_0^1 |\dot{\gamma}| \cdot \frac{|\nabla u|}{u(1 - \ln u)} ds\]

\[\leq \frac{\rho}{t^2} + (\sqrt{2K} + \sqrt{C_2})\rho\]

Then

\[\frac{1 - f(x_1, t)}{1 - f(x_2, t)} \leq \exp\{\frac{\rho}{t^2} + (\sqrt{2K} + \sqrt{C_2})\rho\}\]  \hspace{1cm} (2.10)

Let \(\beta = \exp\left(-\frac{\rho}{t^2} - (\sqrt{2K} + \sqrt{C_2})\rho\right)\), (2.10) then implies that

\[\frac{1 - \ln u(x_1, t)}{1 - \ln u(x_2, t)} \leq \frac{1}{\beta}\]

Through some easy computation, we obtain

\[u(x_2, t) \leq u(x_1, t)^\beta e^{1-\beta}\]
This completes the proof of Corollary 1.5.

Since the potential in (1.8) is a constant, the term $|\nabla \sqrt{-h}|^{\frac{1}{2}}$ in (1.7) is equal to zero. So Corollary 1.6 follows by (1.7). Let $t \to \infty$ and $K = 0$ in (1.9), then $|\nabla \log u| = 0$. Thus $u$ must be a constant. From (1.10), we know $u \equiv 0$. So there does not exist any positive and bounded solution of the equation (1.10). This completes the Corollary 1.7.

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