Criteria for transience and recurrence of regime-switching diffusion processes∗

Jinghai Shao†

School of Mathematical Sciences, Beijing Normal University, 100875, Beijing, China

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Abstract

We provide some on-off type criteria for recurrence and transience of regime-switching diffusion processes using the theory of M-matrix and the Perron-Frobenius theorem. State-independent and state-dependent regime-switching diffusion processes in a finite space and a countable space are both studied. We put forward a finite partition method to deal with switching process in a countable space. As an application, we improve the known criteria for recurrence of linear regime-switching diffusion processes, and provide an on-off type criterion for a kind of nonlinear regime-switching diffusion processes.

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1 Introduction

Regime-switching diffusion processes have received much attention lately, and they can provide more realistic formulation for many applications such as biology, mathematical finance, etc. See

∗Supported in part by NSFC (No.11301030), 985-project and Beijing Higher Education Young Elite Teacher Project.
†Email: shaojh@bnu.edu.cn
and references therein for more details on their application. The regime-switching diffusion process (for short, \textbf{RSDP}) studied in this work can be viewed as a number of diffusion processes modulated by a random switching device or as a diffusion process which lives in a random environment. More precisely, \textbf{RSDP} is a two-component process \((X_t, \Lambda_t)\), where \((X_t)\) describes the continuous dynamics, and \((\Lambda_t)\) describes the random switching device. \((X_t)\) satisfies the stochastic differential equation (for short, SDE)

\[
dX_t = b(X_t, \Lambda_t)dt + \sigma(X_t, \Lambda_t)dB_t, \quad X_0 = x \in \mathbb{R}^d,
\]

where \((B_t)\) is a Brownian motion in \(\mathbb{R}^d, d \geq 1\), \(\sigma\) is \(d \times d\)-matrix, and \(b\) is a vector in \(\mathbb{R}^d\). While \((\Lambda_t)\) is a continuous time Markov chain on the state space \(\mathcal{M} = \{1, 2, \ldots, N\}, 2 \leq N \leq \infty\), satisfying

\[
P(\Lambda_{t+\delta} = l | \Lambda_t = k, X_t = x) = \begin{cases} 
q_{kl}(x)\delta + o(\delta), & \text{if } k \neq l, \\
1 + q_{kk}(x)\delta + o(\delta), & \text{if } k = l,
\end{cases}
\]

for \(\delta > 0\). The \(Q\)-matrix \(Q_x = (q_{kl}(x))\) is irreducible and conservative for each \(x \in \mathbb{R}^d\). If the \(Q\)-matrix \((q_{kl}(x))\) does not depend on \(x\), then \((X_t, \Lambda_t)\) is called a state-independent \textbf{RSDP}; otherwise, it is called a state-dependent one. When \(N\) is finite, namely, \((\Lambda_t)\) is a Markov chain on a finite state space, we call \((X_t, \Lambda_t)\) a \textbf{RSDP} in a finite state space. When \(N\) is infinite, we call \((X_t, \Lambda_t)\) a \textbf{RSDP} in an infinite state space. Next, we collect some conditions used later.

\textbf{(H)} There exists constant \(\bar{K} > 0\) such that

\begin{itemize}
  \item[(i)] \(x \mapsto q_{ij}(x)\) is a bounded continuous function for each pair of \(i, j \in \mathcal{M}\).
  \item[(ii)] \(|b(x, i)| + \|\sigma(x, i)\| \leq \bar{K}(1 + |x|), \quad x \in \mathbb{R}^d, \quad i \in \mathcal{M}\).
  \item[(iii)] \(|b(x, i) - b(y, i)| + \|\sigma(x, i) - \sigma(y, i)\| \leq \bar{K}|x - y|, \quad x, y \in \mathbb{R}^d, \quad i \in \mathcal{M}\).
  \item[(iv)] For each \(i \in \mathcal{M}\), \(a(x, i) = \sigma(x, i)\sigma(x, i)^*\) is uniformly positive definite.
\end{itemize}

Here and in the sequel, \(\sigma^*\) stands for the transpose of matrix \(\sigma\), and \(\|\sigma\|\) denotes the operator norm. Hypothesis (Hi),(Hii) and (Hiii) guarantee the existence of a unique nonexplosive solution of (1.1) and (1.2) (cf. [16 Theorem 2.1]). Hypothesis (Hiv) is used to ensure that \((X_t, \Lambda_t)\) possesses strong Feller property (cf. [15, 17]), which will be used in the study of exponential ergodicity.

Corresponding to the process \((X_t, \Lambda_t)\), there is a family of diffusion processes defined by

\[
dX^{(i)}_t = b(X^{(i)}_t, i)dt + \sigma(X^{(i)}_t, i)dB_t,
\]
for each $i \in \mathcal{M}$. These processes $(X_t^{(i)}) (i \in \mathcal{M})$ are the diffusion processes associated with $(X_t, \Lambda_t)$ in each fixed environment. The recurrent behavior of $(X_t, \Lambda_t)$ is intensively connected with its recurrent behavior in each fixed environment. But this connection is rather complicated as having been noted by [11]. In [11], some examples in $[0, \infty)$ with reflecting boundary at 0 and $\mathcal{M} = \{1, 2\}$ were constructed. They showed that even when $(X_t^{(1)})$ and $(X_t^{(2)})$ are both positive recurrent (transient), $(X_t, \Lambda_t)$ could be transient (positive recurrent, respectively) by choosing suitable transition rate $(q_{ij})$ between two states. In view of this complicity, it is a challenging work to determine the recurrent property of a regime-switching diffusion process. There are lots of work having been dedicated to this task. See, for instance, [3, 11, 10, 16, 4] and references therein. Except giving the examples we mentioned above, [11] also studied the reversible state-independent RSDP. In [10], the author provided a theoretically complete characterization of recurrence and transience for a class of state-independent RSDP, which we will state more precisely later. In [4], some on-off type criteria were established to justify the exponential ergodicity of state-independent and state-dependent RSDP in a finite state space. The convergence in total variation norm and in Wasserstein distance were both studied in [4]. The cost function used to define the Wasserstein distance in [4] is bounded. All the previously mentioned work considered only the RSDP in a finite state space. Although the general criteria by the Lyapunov functions for Markov processes still work for RSDP, it is well known that finding a suitable Lyapunov function is a difficult task for RSDP due to the coexistence of generators for diffusion process and jump process. So it is better to provide some easily verifiable criteria in terms of the coefficients of diffusion process $(X_t)$ and the $Q$-matrix of $(\Lambda_t)$. In this direction, [17] has provided some criteria for a class of state-dependent RSDP $(X_t, \Lambda_t)$ in a finite state space. Precisely, the continuous component $(X_t)$ considered in [17] behaves like a linear one and $Q$-matrix $(q_{ij}(x))$ behaves like a state-independent $Q$-matrix $(\hat{q}_{ij})$ in a neighborhood of $\infty$. In addition, the recurrent property for geometric Brownian motion in a two-state random environment was studied in [12].

In [13], we studied the ergodicity for RSDP in Wasserstein distance. Both state-independent and state-dependent RSDP in finite and infinite state spaces are studied in [13]. The cost function used in [13] is not necessarily bounded. We put forward some new criteria for ergodicity based on the theory of M-matrix and Perron-Frobenius theorem. Our present work is devoted to studying the recurrent property of RSDP in total variation norm. Compared with [4, 13], the on-off type criteria given there own only one hand “on”, that is, if the condition holds,
then the process is ergodic. Examples can show these criteria are sharp. In the present work, we shall show that these on-off type criteria can own two hands, both “on” and “off”, that is, if the condition holds, then the process is recurrent, and if not, then the process is transient.

Another contribution of this work is that we put forward a finite partition method to study the transience and recurrence for state-dependent RSDP in an infinite state space. Up to our knowledge, there is few result for the recurrent property of RSDP in an infinite state space. In this work, based on the criteria given by M-matrix theory, we put forward a finite partition approach study the recurrent properties of RSDP in an infinite state space. Its basic idea is to transform the state-dependent RSDP in an infinite state space into a new state-independent RSDP in a finite state space (see Theorem 2.6 for details).

As an application of our criteria, we develop the study in [10] and [17]. In [10], the authors considered the state-independent RSDP \((X_t, \Lambda_t)\) in \(\mathbb{R}^d \times \mathcal{M}\) with \(d \geq 2\) and \(\mathcal{M}\) a finite set. For each \(i \in \mathcal{M}\), the associated diffusion \((X_t^{(i)})\) has the infinitesimal generator \(L^{(i)} = \frac{1}{2} \Delta + V\), where

\[
V(x, i) = |x|^\delta \hat{b}(x/|x|, i) \cdot \nabla, \quad \delta \in [-1, 1).
\]  

Let \(S^{d-1}\) denote the \(d-1\)-dimension sphere, and \(\mu\) be the invariant probability measure for \((\Lambda_t)\). In [10], they studied the process under the condition that \(\hat{b}(\phi, i) \neq 0, \hat{b}(\phi, i) \in C^1(S^{d-1})\) for each \(i \in \mathcal{M}\), and

\[
\sum_{i \in \mathcal{M}} \hat{b}(\phi, i) \mu_i = 0 \quad \text{for each } \phi \in S^{d-1}.
\]  

Condition (1.5) allows them to transform the problem into studying the recurrent behavior of the generator

\[
\hat{L} = r^\gamma \left[ c_1(\phi) \frac{\partial^2}{\partial r^2} + c_2(\phi) \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} D_{S^{d-1}} + \frac{1}{r^2} L_{S^{d-1}} \right],
\]

where \(\gamma = 0\), if \(-1 \leq \delta \leq 0\), and \(\gamma = 2\delta\), if \(0 < \delta < 1\), \(c_1(\phi) \geq 0\), \(D_{S^{d-1}}\) is a first-order operator on \(S^{d-1}\) and \(L_{S^{d-1}}\) is a (possible degenerate) diffusion generator on \(S^{d-1}\). By posing some further conditions on \(c_1(\phi)\) and \(c_2(\phi)\), they got a quantity \(\rho\) expressed in terms of \(c_1(\phi), c_2(\phi)\) and the density of invariant probability measure of the process corresponding to \(\hat{L}\). They showed that \((X_t, \Lambda_t)\) is recurrent or transient according to whether \(\rho \leq 0\) or \(\rho > 0\). Theoretically, this result is complete although calculating \(\rho\) is a difficult task, which has been pointed out in [10].

In this work, roughly speaking, we consider the processes corresponding to \(\sum_{i \in \mathcal{M}} \mu_i \hat{b}_i(\phi, i) \neq 0\). In Section 3, we consider the case \(\delta = 1\) and in Section 4, after developing the criteria given in Section 2, we consider the case \(\delta \in [-1, 1)\). Some easily verifiable criteria are provided.
The usefulness and sharpness of the criteria established in this work can be seen from the following example. Let \((\Lambda_t)\) be a continuous time Markov chain on \(\{1, 2, \ldots, N\}\), \(N < \infty\), equipped with an irreducible conservative \(Q\)-matrix \((q_{ij})\). Let \(\mu\) be the invariant probability measure of \((\Lambda_t)\). Let \((X_t)\) be a random diffusion on \([0, \infty)\) with reflecting boundary at 0 satisfying
\[
dX_t = b_{\Lambda_t} X_t^\delta dt + dB_t, \quad \delta \in [-1, 1].
\]
In the case \(\delta \in [-1, 1)\), if \(\sum_{i=1}^N \mu_i b_i \leq 0\), then \((X_t, \Lambda_t)\) is recurrent; if \(\sum_{i=1}^N \mu_i b_i > 0\), then \((X_t, \Lambda_t)\) is transient. In the case \(\delta = 1\), if \(\sum_{i=1}^N \mu_i b_i < 0\), then \((X_t, \Lambda_t)\) is exponentially ergodic; if \(\sum_{i=1}^N \mu_i b_i > 0\), then \((X_t, \Lambda_t)\) is transient.

This work is organized as follows. We shall provide some new criteria on transience, recurrence and exponential ergodicity for RSDP in Section 2. In Section 2, we first study the state-independent RSDP in a finite state space, then study the state-dependent RSDP in an infinite state space. In Section 3, we consider the recurrent property of Ornstein-Uhlenbeck process and linear diffusion in random environments. In Section 4, we provide another kind of criteria then apply these criteria to study the recurrent properties of nonlinear regime-switching diffusion processes.

2 Criteria for recurrence and transience: I

Let \((X_t, \Lambda_t)\) be defined by (1.1) and (1.2). We first consider the situation that \(Q\)-matrix of \((\Lambda_t)\) is independent of \(x\) and \(N < \infty\). For a diffusion process in \(\mathbb{R}^d\) with generator \(L = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}\), we write \(L \sim (a(x), b(x))\) for simplicity, where \(a(x) = (a_{ij}(x)), \ b(x) = (b_i(x))\). For each \(i \in \mathcal{M}\), \((X_t^{(i)})\) is defined by (1.3), its generator \(L^{(i)} \sim (a^{(i)}(x), b^{(i)}(x))\), where \(a^{(i)}(x) = \sigma(x, i) \sigma(x, i)^*\), \(b^{(i)}(x) = b(x, i)\).

For the vector \(\beta = (\beta_1, \ldots, \beta_N)^*\), we use \(\text{diag}(\beta) = \text{diag}(\beta_1, \ldots, \beta_N)\) to denote the diagonal matrix generated by vector \(\beta\) as usual. Before stating our results, we introduce some notation and basic properties on M-matrix. We refer the reader to [2] for more discussion on this topic.

Let \(B\) be a matrix or vector. By \(B \geq 0\) we mean that all elements of \(B\) are non-negative. By \(B > 0\) we mean that \(B \geq 0\) and at least one element of \(B\) is positive. By \(B \gg 0\), we mean that all elements of \(B\) are positive. \(B \ll 0\) means that \(-B \gg 0\).
Definition 2.1 (M-matrix) A square matrix $A = (a_{ij})_{n \times n}$ is called an M-Matrix if $A$ can be expressed in the form $A = sI - B$ with some $B \geq 0$ and $s \geq \text{Ria}(B)$, where $I$ is the $n \times n$ identity matrix, and $\text{Ria}(B)$ the spectral radius of $B$. When $s > \text{Ria}(B)$, $A$ is called a nonsingular M-matrix.

We cite some conditions equivalent to that $A$ is a nonsingular M-matrix as follows, and refer to [2] for more discussion on this topic.

Proposition 2.2 The following statements are equivalent.

1. $A$ is a nonsingular $n \times n$ M-matrix.

2. All of the principal minors of $A$ are positive; that is,
\[
\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \ldots, n.
\]

3. Every real eigenvalue of $A$ is positive.

4. $A$ is semipositive; that is, there exists $x \gg 0$ in $\mathbb{R}^n$ such that $Ax \gg 0$.

5. There exists $x \gg 0$ with $Ax > 0$ and $\sum_{j=1}^i a_{ij}x_j > 0$, $i = 1, \ldots, n$.

Next result is our first main result for the recurrent property of state-independent regime-switching diffusion processes in a finite state space. We will use often the following condition for a function $V \in C^2(\mathbb{R}^d)$.

(A1) There exist constants $r_0 > 0$ and $\beta_i \in \mathbb{R}$, $i \in \mathcal{M}$ such that
\[
V(x) > 0, \quad L^{(i)}V(x) \leq \beta_i V(x), \quad |x| > r_0.
\]  

Here the constant $\beta_i$ could be negative or positive.

Theorem 2.3 Assume that (Hi), (Hi), (Hi) hold and $N < \infty$. Assume there exists a function $V \in C^2(\mathbb{R}^d)$ such that condition (A1) is satisfied and the matrix $-(Q + \text{diag}(\beta))$ is a nonsingular M-matrix. Then $(X_t, \Lambda_t)$ is positive recurrent if $\lim_{|x| \to \infty} V(x) = \infty$, and is transient if $\lim_{|x| \to \infty} V(x) = 0$. Assume further that (Hiv) holds. Then $(X_t, \Lambda_t)$ is exponentially ergodic if $\lim_{|x| \to \infty} V(x) = \infty$.  

6
\textbf{Proof.} Denote by $\mathcal{A}$ the generator of $(X_t, \Lambda_t)$. Due to \cite{13},

$$\mathcal{A} f(x, i) = L^{(i)} f(\cdot, i)(x) + Q f(x, \cdot)(i),$$

where $Qg(i) = \sum_{j \neq i} q_{ij}(g_j - g_i)$ for $g \in \mathcal{B}(\mathcal{M})$. As $-(Q + \text{diag}(\beta))$ is a nonsingular M-matrix, by Proposition \cite{22} there exists a vector $\xi = (\xi_1, \ldots, \xi_N)^* \gg 0$ such that

$$\lambda = (\lambda_1, \ldots, \lambda_N)^* = -(Q + \text{diag}(\beta)) \xi \gg 0.$$  

Take $f(x, i) = V(x)\xi_i, \quad x \in \mathbb{R}^d, i \in \mathcal{M}$, then, for $|x| > r_0$, $i \in \mathcal{M}$,

$$\mathcal{A} f(x, i) = Q \xi(i) V(x) + \xi_i L^{(i)} V(x) \leq (Q \xi(i) + \beta_i \xi_i) V(x) = -\lambda_i V(x) \leq -\min_{1 \leq i \leq N} \left( \frac{\lambda_i}{\xi_i} \right) f(x, i).$$  

As $N < \infty$, $\min_{1 \leq i \leq N}(\lambda_i/\xi_i) > 0$, and we have $\mathcal{A} f(x, i) \leq 0$. If $\lim_{|x| \to \infty} V(x) = \infty$, furthermore, we can get $\mathcal{A} f(x, i) \leq -\varepsilon$ for some $\varepsilon > 0$ by choosing $r_0 > 0$ large enough. Therefore, according to the Foster-Lyapunov drift conditions (cf. \cite{10} Section 2 or \cite{16} Chapter 3), we obtain that $(X_t, \Lambda_t)$ is positive recurrent if $\lim_{|x| \to \infty} V(x) = \infty$, and $(X_t, \Lambda_t)$ is transient if $\lim_{|x| \to \infty} V(x) = 0$. According to \cite{15} Theorem 5.1, when (Hiv) holds and $\lim_{|x| \to \infty} V(x) = \infty$, \cite{22} yields that $(X_t, \Lambda_t)$ is exponentially ergodic.  

\textbf{Theorem 2.4} Assume that (Hi), (Hi1), (Hi1ii) hold and $N < \infty$. Suppose that there exists a function $V \in C^2(\mathbb{R}^d)$ such that condition (A1) holds and

$$\sum_{i \in \mathcal{M}} \mu_i \beta_i < 0, \quad (2.3)$$

where $\mu = (\mu_i)_{i \in \mathcal{M}}$ is the invariant probability measure of $(\Lambda_t)$. Then $(X_t, \Lambda_t)$ is transient if $\lim_{|x| \to \infty} V(x) = 0$, and is positive recurrent if $\lim_{|x| \to \infty} V(x) = \infty$. Assume further (Hiv) holds, if $\lim_{|x| \to \infty} V(x) = \infty$, then $(X_t, \Lambda_t)$ is exponentially ergodic.

\textbf{Proof.} Let $Q_p = Q + p \text{diag}(\beta)$, and

$$\eta_p = -\max_{\gamma \in \text{spec}(Q_p)} \text{Re} \gamma, \quad \text{where spec}(Q_p) \text{ denotes the spectrum of } Q_p.$$  

Let $Q_{(p,t)} = e^{tQ_p}$, then the spectral radius $\text{Ria}(Q_{(p,t)})$ of $Q_{(p,t)}$ equals to $e^{-\eta_p t}$. Since all coefficients of $Q_{(p,t)}$ are positive, Perron-Frobenius theorem (see \cite{2} Chapter 2) yields $-\eta_p$ is a simple
eigenvalue of $Q_p$. Moreover, note that the eigenvector of $Q_{(p,t)}$ corresponding to $e^{-\eta_p t}$ is also an eigenvector of $Q_p$ corresponding to $-\eta_p$. Then Perron-Frobenius theorem ensures that there exists an eigenvector $\xi \gg 0$ of $Q_p$ associated with the eigenvalue $-\eta_p$. Then applying Proposition 4.2 of [1] (by replacing $A_p$ there with $Q_p$), if $\sum_{i=1}^{N} \mu_i \beta_i < 0$, then there exists some $p_0 > 0$ such that $\eta_p > 0$ for any $0 < p < p_0$. Fix a $p$ with $0 < p < \min\{1, p_0\}$ and an eigenvector $\xi \gg 0$, then we obtain

$$Q_p \xi = (Q + p \text{diag}(\beta))\xi = -\eta_p \xi \ll 0.$$  

Put $f(x,i) = V(x)^p \xi_i$, $x \in \mathbb{R}^d$, $i \in \mathcal{M}$. For $|x| > r_0$, $i \in \mathcal{M}$,

$$\mathcal{A} f(x,i) = Q \xi(i) V(x)^p + \xi_i L^{(i)} V(x)^p \leq (Q \xi(i) + p \beta_i \xi_i) V(x)^p = -\eta_p \xi_i V(x)^p = -\eta_p f(x,i).$$

Then analogous to the argument of Theorem 2.3 we can conclude the proof.

\[ \textbf{Remark 2.5} \] We give a heuristic explanation of the condition (2.3) in previous theorem. As $\mu$ is the invariant probability measure of $(\Lambda_t)$, $\mu_i$ represents in some sense the time ratio spent by $(\Lambda_t)$ in the state $i$. $\beta_i$ represents the recurrent behavior of $(X_t^{(i)})$. Therefore, the quantity $\sum_{i \in \mathcal{M}} \mu_i \beta_i$ averages the recurrent behavior of $(X_t^{(i)})$ with respect to $\mu$, which determine the recurrent behavior of $(X_t, \Lambda_t)$ according to previous theorem.

Now we go to consider the state-dependent RSDP with $(\Lambda_t)$ being a Markov chain in a countable space $\mathcal{M}$. Namely, the $Q$-matrix $(q_{ij}(x))$ is dependent on $x$ and $N = \infty$. Let $V \in C^2(\mathbb{R}^d)$ such that (A1) holds and $M = \sup_{i \in \mathcal{M}} \beta_i < \infty$. As the M-matrix theory is about matrices with finite size, we shall put forward a finite partition method to transform the RSDP in an infinite state space into a new RSDP in a finite state space. Let

$$\Gamma = \{-\infty = k_0 < k_1 < \ldots < k_{m-1} < k_m = M\}$$

be a finite partition of $(-\infty, M]$. Corresponding to $\Gamma$, there exists a finite partition $F = \{F_1, \ldots, F_m\}$ of $\mathcal{M}$ defined by

$$F_i = \{j \in \mathcal{M}; \beta_j \in (k_{i-1}, k_i]\}, \quad i = 1, 2, \ldots, m.$$
We assume each $F_i$ is nonempty, otherwise, we can delete some points in the partition $\Gamma$. Set

$$
\beta_i^F = \sup_{j \in F_i} \beta_j, \quad q_{ii}^F = -\sum_{k \neq i} q_{ik}^F,
$$

(2.4)

$$
q_{ik}^F = \begin{cases} 
\sup_{x \in \mathbb{R}^d} \sup_{r \in F_k} \sum_{j \in F_i} q_{rj}(x), & \text{for } k < i, \\
\inf_{x \in \mathbb{R}^d} \inf_{r \in F_k} \sum_{j \in F_i} q_{rj}(x), & \text{for } k > i.
\end{cases}
$$

(2.5)

Then

$$
\beta_j \leq \beta_i^F, \quad \forall j \in F_i, \text{ and } \beta_{i-1}^F < \beta_i^F, \quad i = 2, \ldots, m.
$$

After doing these preparation, we can get the following result.

**Theorem 2.6** Assume (Hi), (Hii), (Hiii) hold and $N = \infty$. Let $V \in C^2(\mathbb{R}^d)$ such that (A1) is satisfied and $M = \sup_{i \in \mathcal{M}} \beta_i < \infty$. Define the partition $\Gamma$ and the corresponding vector $(\beta_i^F)$, finite matrix $Q^F$ as above. Suppose that the $m \times m$ matrix $-(\text{diag}(\beta_1^F, \ldots, \beta_m^F) + Q^F)H_m$ is a nonsingular $M$-matrix, where

$$
H_m = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}_{m \times m}.
$$

(2.6)

Then $(X_t, \Lambda_t)$ is recurrent if $\lim_{|x| \to \infty} V(x) = \infty$, and is transient if $\lim_{|x| \to \infty} V(x) = 0$.

**Proof.** As $-(Q^F + \text{diag}(\beta_1^F, \ldots, \beta_m^F))H_m$ is a nonsingular $M$-matrix, by Proposition 2.2, there exists a vector $\eta^F = (\eta_1^F, \ldots, \eta_m^F)^* \gg 0$ such that

$$
\lambda^F = (\lambda_1^F, \ldots, \lambda_m^F)^* = -(Q^F + \text{diag}(\beta_1^F, \ldots, \beta_m^F))H_m\eta^F \gg 0.
$$

Hence, $\bar{\lambda} := \max_{1 \leq i \leq m} \lambda_i^F > 0$. Set $\xi^F = H_m\eta^F$. Then

$$
\xi_i^F = \eta_m^F + \cdots + \eta_i^F, \quad i = 1, \ldots, m,
$$

which implies that $\xi_{i+1}^F < \xi_i^F$ for $i = 1, \ldots, m - 1$, and $\xi^F \gg 0$. For each $j \in \mathcal{M}$, we define $\xi_j = \xi_i^F$ if $j \in F_i$, which is reasonable as $(F_i)$ is a finite partition of $\mathcal{M}$. Via this method, we get a vector $\xi = (\xi_1, \xi_2, \ldots)^*$ from $\xi^F$. 

9
Let $\mathcal{F} : \mathcal{M} \to \{1,2,\ldots,m\}$ be a map defined by $\mathcal{F}(j) = k$ if $j \in F_k$. Let $Q_x g(i) = \sum_{j \neq i} q_{ij}(x)(g_j - g_i)$ for $g \in \mathcal{B}(\mathcal{M})$. Set $f(x, r) = V(x)\xi_r$, $x \in \mathbb{R}^d$, $r \in \mathcal{M}$. By the definition of $(\beta_t^F)$ and $Q^F$, we have, for $r \in F_i$,

$$Q_x \xi(r) = \sum_{j \neq r} q_{rj}(x)(\xi_j - \xi_i) = \sum_{j \neq F_i} q_{rj}(x)(\xi_j - \xi_i)$$

Moreover,

$$\mathcal{A} f(x, r) = Q_x \xi(r)V(x) + \xi_r L^r V(x)$$

$$\leq (Q^F \xi^F(\mathcal{F}(r)) + \beta_{F(r)}^F \xi^F(r))V(x)$$

$$= -\lambda_{\mathcal{F}(r)} V(x) \leq 0.$$

Let

$$\tau = \inf \{t > 0; (X_t, \Lambda_t) \in \{x \in \mathbb{R}^d; |x| \leq r_0\} \times \{1,2,\ldots,m_0\}\},$$

where $m_0 < \infty$ is a fixed number. Applying Itô’s formula to $(X_t, \Lambda_t)$ with $X_0 = x$, $\Lambda_0 = l$, and $|x| > r_0$, $l > m_0$ (cf. [14]),

$$\mathbb{E} f(X_{t\wedge \tau}, \Lambda_{t\wedge \tau}) = f(x, l) + \mathbb{E} \int_0^{t\wedge \tau} \mathcal{A} f(x, \Lambda_s)ds \leq f(x, l) = V(x)\xi_l.$$

(1) When $\lim_{|x| \to \infty} V(x) = 0$, if $\mathbb{P}(\tau < \infty) = 1$, then passing $t \to \infty$ in (2.8), we get

$$\inf_{\{y; |y| \leq r_0\}} V(y) \leq \mathbb{E} V(X_\tau) \leq \max_{i,k} \left( \frac{\xi_k^F}{\xi_i^F} \right) V(x),$$

as $|X_\tau| \leq r_0$. We get $\inf_{\{y; |y| \leq r_0\}} V(y) > 0$ by the compactness of set $\{y; |y| \leq r_0\}$ and positiveness of function $V$. So letting $|x|$ tend to $\infty$ in previous inequality, the right hand goes to 0, but the left hand is strictly bigger than a positive constant, which is a contradiction. Therefore, $\mathbb{P}(\tau < \infty) > 0$, and the process $(X_t, \Lambda_t)$ is transient.

(2) Consider the case $\lim_{|x| \to \infty} V(x) = \infty$. Introduce another stopping time

$$\tau_K = \inf\{t > 0; |X_t| \geq K\}.$$
As the process \((X_t, \Lambda_t)\) is nonexplosive, \(\tau_K\) increases to \(\infty\) almost surely as \(K \to \infty\). Itô’s formula also yields that
\[
\mathbb{E}[V(X_{t\wedge\tau_K\wedge\tau})\xi_{\Lambda_{t\wedge\tau_K\wedge\tau}}] \leq V(x)\xi_t.
\]
Letting \(t \to \infty\), Fatou’s lemma implies that
\[
\mathbb{E}[V(X_{\tau_K\wedge\tau})] \leq \max_{i,k} \left( \frac{\xi_i}{\xi_k} \right) V(x),
\]
and hence,
\[
\mathbb{P}(\tau \geq \tau_K) \leq \max_{i,k} \left( \frac{\xi_i}{\xi_k} \right) \frac{V(x)}{\inf_{|y|\leq K} V(y)}.
\]
Since \(\lim_{|x| \to \infty} V(x) = \infty\), letting \(K \to \infty\) in the previous inequality, we obtain that \(\mathbb{P}(\tau = \infty) \leq 0\). We have completed the proof till now.

As an application of Theorem 2.6, we construct an example of state-independent RSDP in an infinite state space.

**Example 2.1** Let \((\Lambda_t)\) be a birth-death process on \(\mathcal{M} = \{1, 2, \ldots\}\) with \(b_i \equiv b > 0\) and \(a_i \equiv a > 0\). Let \(X_t\) be a random diffusion process on \([0, \infty)\) with reflecting boundary at 0 and satisfies
\[
dX_t = \beta_{\Lambda_t} X_t dt + \sqrt{2} dB_t,
\]
where \(\beta_i = \kappa - i^{-1}\) for \(i \geq 1\). First, set \(V(x) = x\). Let us take the finite partition \(F = \{F_1, F_2\}\) to be \(F_1 = \{1\}\) and \(F_2 = \{2, 3, \ldots\}\). It is easy to see that \(q_{12}^F = b_1\) and \(q_{21}^F = a_2\). Then
\[
L^{(i)} V(x) = \beta_i V(x), \quad x > 1, \quad i \geq 1.
\]
So \(\beta_1^F = \kappa - 1, \beta_2^F = \kappa,\) and
\[
-(Q^F + \text{diag}(\beta_1^F, \beta_2^F)) H_2 = \begin{pmatrix} b_1 - \beta_1^F & -\beta_1^F \\ -a_2 & -\beta_2^F \end{pmatrix}.
\]
By Proposition 2.2, previous matrix is a nonsingular M-matrix if and only if
\[
\kappa < b + 1, \quad \text{and} \quad \kappa^2 - (b + a + 1)\kappa + a > 0. \quad (2.9)
\]
Therefore, according to Theorem 2.6, if (2.9) holds, the process \((X_t, \Lambda_t)\) is recurrent.
Second, set $V(x) = x^{-1}$. For $m \geq 2$, we take $F_1 = \{1, 2, \ldots, m-1\}$ and $F_2 = \{m, m+1, \ldots\}$, so $F = \{F_1, F_2\}$ is a finite partition of $\mathcal{M}$. Then

$$L^{(i)}V(x) = (-\beta_i + 2x^{-2})x^{-1} \leq (-\beta_i + 2r_0^{-2})x^{-1}, \quad \text{for } x > r_0.$$ 

Therefore, in this case, $\beta_1^F = -\kappa + 1 + 2r_0^2$ and $\beta_2^F = -\kappa + \frac{1}{m} + 2r_0^2$. By the arbitrariness of $r_0 > 0$ and $m \geq 2$, we get that the matrix $-(Q^F + \text{diag}(\beta_1^F, \beta_2^F))H_2$ is a nonsingular M-matrix if

$$\kappa + b - 1 > 0 \quad \text{and} \quad \kappa^2 + (b + a - 1)\kappa - a > 0. \quad (2.10)$$

Therefore, if (2.10) holds, then $(X_t, \Lambda_t)$ is transient.

By this example, we also want to show that when $(\Lambda_t)$ is a Markov chain on a countable set, the process $(\Lambda_t)$ and $(X_t, \Lambda_t)$ may have very different recurrent property. More precisely, if we take $b = 2$ and $a = 1$, then $(\Lambda_t)$ is transient, but for $\kappa < 2 - \sqrt{3}$, (2.9) holds and hence $(X_t, \Lambda_t)$ is recurrent. If we take $b = 1$ and $a = 2$, then $(\Lambda_t)$ is exponentially ergodic, but for $\kappa > \sqrt{3} - 1$, (2.10) holds and hence $(X_t, \Lambda_t)$ is transient.

3 Recurrent property of Ornstein-Uhlenbeck process and linear diffusion in random environments

In this section, we first consider the Ornstein-Uhlenbeck type process in random environment, that is, the process $(X_t, \Lambda_t)$ satisfies:

$$dX_t = b_{\Lambda_t}X_t dt + \sigma_{\Lambda_t}dB_t, \quad X_0 = x \in \mathbb{R}, \quad (3.1)$$

where $(B_t)$ is a Brownian motion in $\mathbb{R}$, and $(\Lambda_t)$ is a continuous Markov chain on the space $\mathcal{M} = \{1, \ldots, N\}$ with $N < \infty$. $(\Lambda_t)$, $(B_t)$ are mutually independent. The $Q$-matrix $(q_{ij})$ of $(\Lambda_t)$ is independent of $(X_t)$, and is irreducible and conservative. We assume that $d \times d$ matrix $\sigma_i$ is positive definite for every $i \in \mathcal{M}$. Let $\mu = (\mu_i)$ be the invariant probability measure of $(q_{ij})$. In [3], the authors showed that when $\sum_{i \in \mathcal{M}} \mu_i b_i < 0$, the process $(X_t, \Lambda_t)$ is ergodic in weak topology, that is, the distribution of $(X_t, \Lambda_t)$ converges weakly to a probability measure $\nu$. In [6, 1], the tail behavior of $\nu$ was studied. Using the criteria in Section 2, we can get the following result.
Proposition 3.1 Let \((X_t, \Lambda_t)\) be defined as above. If \(\sum_{i \in M} \mu_i b_i < 0\), then \((X_t, \Lambda_t)\) is exponentially ergodic. If \(\sum_{i \in M} \mu_i b_i > 0\), then \((X_t, \Lambda_t)\) is transient.

Proof. (1) By (3.1), we get the generator \(L^{(i)}\) of \((X_t^{(i)})\) is given by
\[
L^{(i)} = \frac{1}{2} \sum_{k,l=1}^d a_{kl}^{(i)} \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{k=1}^d b_k \frac{\partial}{\partial x_k},
\]
where \(a^{(i)} = \sigma_i \sigma_i^*\). Take \(V(x) = |x|\), then for each \(i \in M\),
\[
L^{(i)} V(x) = b_i |x|, \quad \text{for } |x| > 1.
\]
As \(\lim_{|x| \to \infty} |x| = \infty\), by Theorem 2.4 we get \((X_t, \Lambda_t)\) is exponentially ergodic if \(\sum_{i \in M} \mu_i b_i < 0\).

(2) Now we take \(V(x) = |x|^{-\gamma}\) with \(\gamma > 0\). We have
\[
L^{(i)} V(x) = \frac{\gamma(\gamma + 2)}{2} \sum_{k,l} a_{kl}^{(i)} |x|^{-\gamma-4} x_k x_l - \frac{\gamma}{2} \left( \sum_{k} a_{kk}^{(i)} \right) |x|^{-\gamma - 2} - \gamma b_i |x|^{-\gamma}
\]
\[
= |x|^{-\gamma} \left( -\gamma b_i + \frac{\gamma(\gamma + 2)}{2} |x|^{-4} \sum_{k,l} a_{kl}^{(i)} x_k x_l - \frac{\gamma}{2} |x|^{-2} \sum_{k} a_{kk}^{(i)} \right),
\]
for \(|x| > r_0 > 0\). When \(r_0\) is sufficiently large, it is easy to see that
\[
\frac{\gamma(\gamma + 2)}{2} |x|^{-4} \sum_{k,l} a_{kl}^{(i)} x_k x_l - \frac{\gamma}{2} |x|^{-2} \sum_{k} a_{kk}^{(i)} \leq \frac{1}{r_0}, \quad \forall |x| > r_0.
\]
Therefore, we get
\[
L^{(i)} V(x) \leq (-\gamma b_i + \frac{1}{r_0}) V(x), \quad |x| > r_0.
\] (3.2)
By Theorem 2.4 as \(\lim_{|x| \to \infty} |x|^{-\gamma} = 0\), if
\[
\sum_{i=1}^N \mu_i (-\gamma b_i + \frac{1}{r_0}) = -\gamma \sum_{i=1}^N \mu_i b_i + \frac{1}{r_0} \leq 0,
\]
then \((X_t, \Lambda_t)\) is transient. When \(\sum_{i=1}^N \mu_i b_i > 0\), we can always find a constant \(r_0 > 0\) sufficiently large such that \(-\gamma \sum_{i=1}^N \mu_i b_i + \frac{1}{r_0} < 0\). Hence, when \(\sum_{i=1}^N \mu_i b_i > 0\), \((X_t, \Lambda_t)\) is transient. \(\blacksquare\)
Therefore, for \( i \) in component form, \( \sigma(i) \) is mutually independent. The Brownian motion in \( \mathbb{R}^d \) is a continuous time Markov chain on a finite state space \( \mathcal{M} = \{1, \ldots, N\} \). We first consider the state-independent switching, and in this case \( (\Lambda_t, (B_t)) \) are mutually independent. The \( Q \)-matrix \((q_{ij})\) of \((\Lambda_t)\) is irreducible and conservative. Rewrite

in component form,

\[
dX_j(t) = \sum_{k=1}^d b_{jk}(\Lambda_t)X_k(t)dt + \sum_{l=1}^d \sum_{k=1}^d (\sigma_l(\Lambda_t))_{jk}X_k(t)dB_l(t), \quad j = 1, \ldots, d.
\]

Therefore, for \( i \in \mathcal{M} \), the generator \( L^{(i)} \) is given by

\[
L^{(i)} = \frac{1}{2} \sum_{j,m=1}^d a_{jm}(i) \frac{\partial^2}{\partial x_j \partial x_m} + \sum_{j=1}^d \left( \sum_{k=1}^d b_{jk}(i)x_k \right) \frac{\partial}{\partial x_j},
\]

where \( a_{jm}(i) = \sum_{l=1}^d (\sigma_l(i)x)_j(\sigma_l(i)x)_m \) or in matrix form \( (a_{jm}(i)) = \sum_{l=1}^d (\sigma_l(i)x)(\sigma_l(i)x)^* \). For every \( p \in \mathbb{R} \), \( \frac{\partial|x|^p}{\partial x_j} = p|x|^{p-2}x_j \) and \( \frac{\partial^2|x|^p}{\partial x_j \partial x_m} = p|x|^{p-2}\delta_{jm} + p(p-2)|x|^{p-4}x_jx_m \), \( j, m = 1, \ldots, d \).

By direct calculation,

\[
L^{(i)}|x|^p = \frac{p}{2}|x|^{p-2} \sum_{l=1}^d x^*\sigma_l(i)^*\sigma_l(i)x + \frac{p(p-2)}{2}|x|^{p-4} \sum_{l=1}^d (x^*\sigma_l(i)x)(x^*\sigma_l(i)^*x) + p|x|^{p-2}x^*b(i)x.
\]

Set

\[
\lambda^{(i)}_{\text{max}}(A + b) = \max_{\phi \in S^{d-1}} \sum_{l=1}^d \phi^*(\frac{\sigma_l(i)^*\sigma_l(i)}{2} + b(i))\phi,
\]

\[
\lambda^{(i)}_{\text{min}}(A + b) = \min_{\phi \in S^{d-1}} \sum_{l=1}^d \phi^*(\frac{\sigma_l(i)^*\sigma_l(i)}{2} + b(i))\phi,
\]

and

\[
\vartheta^{(i)}_{\text{max}} = \max_{\phi \in S^{d-1}} \sum_{l=1}^d (\phi^*\sigma_l(i)\phi)(\phi^*\sigma_l(i)^*\phi), \quad \vartheta^{(i)}_{\text{min}} = \min_{\phi \in S^{d-1}} \sum_{l=1}^d (\phi^*\sigma_l(i)\phi)(\phi^*\sigma_l(i)^*\phi).
\]

Then for \( 0 < p < 1 \),

\[
L^{(i)}|x|^p \leq p\left(\lambda^{(i)}_{\text{max}}(A + b) + \frac{p-2}{2}\vartheta^{(i)}_{\text{min}}\right)|x|^p.
\]

Now we consider the linear diffusion in random environments. Let \((X_t, \Lambda_t)\) satisfy

\[
dX_t = b(\Lambda_t)X_tdt + \sigma(\Lambda_t)X_tdB_t, \quad X_0 = x \in \mathbb{R}^d,
\]

where \( b(i) \in \mathbb{R}^d \times \mathbb{R}^d \), \( \sigma(i) = (\sigma_1(i), \ldots, \sigma_d(i)) \) and each \( \sigma_j(i) \in \mathbb{R}^d \times \mathbb{R}^d \), \( i \in \mathcal{M} \), \((B_t)\) is a Brownian motion in \( \mathbb{R}^d \), \((\Lambda_t)\) is a continuous time Markov chain on a finite state space \( \mathcal{M} = \{1, \ldots, N\} \). We first consider the state-independent switching, and in this case \((\Lambda_t, (B_t))\) are mutually independent. The \( Q \)-matrix \((q_{ij})\) of \((\Lambda_t)\) is irreducible and conservative. Rewrite

in component form,
For $p < 0$, \[ L^{(i)}|x|^p \leq p \left( \lambda^{(i)}_{\min}(A + b) + \frac{p-2}{2} \vartheta^{(i)}_{\max} \right) |x|^p. \] Consequently, according to Theorem 2.4 and letting $p \to 0^+$ or $p \to 0^-$, we obtain the following result.

**Theorem 3.2** If \( \sum_{i \in \mathcal{M}} \mu_i \left( \lambda^{(i)}_{\max}(A + b) - \vartheta^{(i)}_{\min} \right) < 0 \), then \((X_t, \Lambda_t)\) defined by (3.3) is positive recurrent. If \( \sum_{i \in \mathcal{M}} \mu_i \left( \lambda^{(i)}_{\min}(A + b) - \vartheta^{(i)}_{\max} \right) > 0 \), then \((X_t, \Lambda_t)\) is transient.

**Remark 3.3** It is easy to see that Theorem 3.2 improves the result of [17, Theorem 5.1], where they showed that if \( \sum_{i \in \mathcal{M}} \mu_i \lambda^{(i)}_{\max}(b_i + b(i)^* + \sum_{j=1}^{d} \sigma_i^2(i)\sigma_j^2(i)^*) < 0 \), then \((X_t, \Lambda_t)\) is positive recurrent.

**Corollary 3.4** When $d = 1$, i.e. \((X_t)\) satisfies
\[
dX_t = b_i \Lambda_t dt + \sigma_i \Lambda_t dB_t, \quad X_0 = x > 0,
\]
where $b_i, \sigma_i$ are constants for $i \in \mathcal{M}$. Then \((X_t, \Lambda_t)\) is positive recurrent if \( \sum_{i \in \mathcal{M}} \mu_i (b_i - \frac{1}{2} \sigma_i^2) < 0 \) and is transient if \( \sum_{i \in \mathcal{M}} \mu_i (b_i - \frac{1}{2} \sigma_i^2) > 0 \).

**Proof.** It is easy to check that in 1-dimensional case
\[
\lambda^{(i)}_{\max} = \lambda^{(i)}_{\min} = b_i + \sigma_i^2/2, \quad \text{and} \quad \vartheta^{(i)}_{\max} = \vartheta^{(i)}_{\min} = \sigma_i^2.
\]
Therefore, we conclude the proof by applying Theorem 3.2.

Next, we shall consider the state-dependent regime-switching diffusion \((X_t, \Lambda_t)\) with \((X_t)\) satisfying (3.3). Assume the $Q$-matrix \((q_{ij}(x))\) is irreducible and conservative for each $x \in \mathbb{R}^d$ and (Hi) holds. Take $V(x) = |x|^p$ for $p \in (0, 1)$ or $p \in (-1, 0)$. By (3.7) and (3.8), we introduce the following notation. Let \( \tilde{\beta}_i = \lambda^{(i)}_{\max}(A + b) - \vartheta^{(i)}_{\min} \) and \( \hat{\beta}_i = \lambda^{(i)}_{\min}(A + b) - \vartheta^{(i)}_{\max} \) for each $i \in \mathcal{M}$. Reorder \((\tilde{\beta}_i)\) and \((\hat{\beta}_i)\) to get
\[
\tilde{\beta}_{i_1} \leq \tilde{\beta}_{i_2} \leq \ldots \leq \tilde{\beta}_{i_N},
\]
and
\[
\hat{\beta}_{j_1} \leq \hat{\beta}_{j_2} \leq \ldots \leq \hat{\beta}_{j_N}.
\]
Define two new conservative $Q$-matrices $\tilde{Q}$ and $\hat{Q}$ by
\[
\tilde{q}_{kl} = \begin{cases} 
\inf_x q_{ik,i}(x), & \text{if } l > k, \\
\sup_x q_{ik,i}(x), & \text{if } l < k, 
\end{cases} 
\hat{q}_{kl} = \begin{cases} 
\inf_x q_{kj,j}(x), & \text{if } l > k, \\
\sup_x q_{kj,j}(x), & \text{if } l < k. 
\end{cases}
\tag{3.9}
\]

Applying Theorem 2.6, we obtain the following result.

**Proposition 3.5** If $-(\text{diag}(\tilde{\beta}_i, \ldots, \tilde{\beta}_N) + \tilde{Q})H_N$ is a nonsingular $M$-matrix, then $(X_t, \Lambda_t)$ is recurrent. If $-(\text{diag}(\hat{\beta}_j, \ldots, \hat{\beta}_{j_N}) + \hat{Q})H_N$ is a nonsingular $M$-matrix, then $(X_t, \Lambda_t)$ is transient.

## 4 Criteria for transience and recurrence: II

According to Foster-Lyapunov drift condition for diffusion processes, if there exists a $V \in C^2(\mathbb{R}^d)$ satisfying (A1) with $\beta_i \leq 0$ and $\lim_{|x| \to \infty} V(x) = \infty$, then the diffusion process $(X^{(i)}_t)$ is exponentially ergodic. When there is no diffusion process $(X^{(i)}_t)$, $i \in \mathcal{M}$, being exponentially ergodic, we can not find suitable function $V \in C^2(\mathbb{R}^d)$ satisfying (A1), so the criteria introduced in Section 2 are useless for this kind of RSDP. For example, the diffusion process corresponding to $L^{(i)} = \frac{1}{2} \Delta + |x|^\delta \hat{b}(x/|x|, i) \cdot \nabla$ with $\delta \in [0, 1)$ is not exponentially ergodic. Therefore, to deal with this kind of processes, we need to extend our method introduced in Section 2. Let $(X_t, \Lambda_t)$ be defined by (1.1) and (1.2) and $(X^{(i)}_t)$ be the corresponding diffusion process in the fixed environment $i \in \mathcal{M}$ with the generator $L^{(i)}$. Instead of finding a function $V$ satisfying condition (A1), we need to find two functions $h, g \in C^2(\mathbb{R}^d)$ satisfying the following condition:

(A2) There exists some constant $r_0 > 0$ such that for each $i \in \mathcal{M}$,
\[
h(x), \ g(x) > 0, \quad L^{(i)} h(x) \leq \beta_i g(x), \quad |x| > r_0, \\
\lim_{|x| \to \infty} \frac{g(x)}{h(x)} = 0, \quad \lim_{|x| \to \infty} \frac{L^{(i)} g(x)}{g(x)} = 0.
\]

**Theorem 4.1** Let $(X_t, \Lambda_t)$ be state-independent RSDP defined by (1.1) and (1.2) with $N < \infty$. Assume that (Hi), (Hii), (Hiii) hold. Let $\mu$ be the invariant probability measure of the process $(\Lambda_t)$. Suppose that there exist two functions $h, g \in C^2(\mathbb{R}^d)$ such that (A2) holds and
\[
\sum_{i=1}^{N} \mu_i \beta_i < 0.
\]
Then $(X_t, \Lambda_t)$ is recurrent if $\lim_{|x| \to \infty} h(x) = 0$, is transient if $\lim_{|x| \to \infty} h(x) = \infty$. 

16
Proof. As $\sum_{i=1}^{N} \mu_{i} \beta_{i} < 0$, by the Fredholm alternative we obtain that there exist a constant $\kappa > 0$ and a vector $\xi$ such that

\[ Q \xi(i) = -\kappa - \beta_{i}, \quad i \in \mathcal{M}. \]

Set $f(x, i) = h(x) + \xi_{i}g(x)$. We obtain

\[ \mathcal{A} f(x, i) = L^{(i)} h(x) + \xi_{i}L^{(i)} g(x) + Q \xi(i)g(x) \]

\[ \leq \left( \beta_{i} + \xi_{i} \frac{L^{(i)} g(x)}{g(x)} \right) g(x). \quad (4.1) \]

By (4.1) and condition (A2), we get

\[ \mathcal{A} f(x, i) \leq \left( -\kappa + \xi_{i} \frac{L^{(i)} g(x)}{g(x)} \right) g(x) \leq 0, \quad \text{as} \ |x| \to \infty. \quad (4.2) \]

As $N < \infty$, $\xi$ is bounded. Since $\lim_{|x| \to \infty} \frac{g(x)}{h(x)} = 0$ and $f(x, i) = (1 + \xi_{i} \frac{g(x)}{h(x)})h(x)$ for $|x| > r_{0}$, it is easy to see that there exists $r_{1} > 0$ such that $f(x, i) > 0$ for $|x| > r_{1}$. In addition, if $\lim_{|x| \to \infty} h(x) = \infty$, then $\lim_{|x| \to \infty} f(x, i) = \infty$; if $\lim_{|x| \to \infty} h(x) = 0$, then $\lim_{|x| \to \infty} f(x, i) = 0$. By the method of Lyapunov function, inequality (4.2) yields that $(X_{t}, \Lambda_{t})$ is recurrent if $\lim_{|x| \to \infty} h(x) = \infty$ and is transient if $\lim_{|x| \to \infty} h(x) = 0$.

Now we consider the following state-independent RSDP in a finite state space,

\[ dX_{t} = |X_{t}|^\delta \hat{b}(X_{t}/|X_{t}|, \Lambda_{t})dt + \sigma(X_{t}, \Lambda_{t})dB_{t}, \quad X_{0} = x \in \mathbb{R}^{d}, d \geq 1, \quad (4.3) \]

where $\delta \in [-1, 1)$, $\hat{b}(\cdot, \cdot) : S^{d-1} \times \mathcal{M} \to \mathbb{R}^{d}$, $\sigma(\cdot, \cdot) : \mathbb{R}^{d} \times \mathcal{M} \to \mathbb{R}^{d \times d}$, and $(B_{t})$ is a $d$-dimensional Brownian motion. Let $(\Lambda_{t})$ be a continuous time Markov chain on $\mathcal{M}$ with irreducible conservative $Q$-matrix $(q_{ij})$, which is also independent of $(B_{t})$. Let $\mu$ be the invariant probability measure of $(\Lambda_{t})$. Set $a^{(i)}(x) = \sigma(x, i)\sigma(x, i)^{\ast}$. Suppose conditions (Hi), (Hii), (Hiii) are satisfied. In [10], the authors considered the recurrent property of $(X_{t}, \Lambda_{t})$ under the condition

\[ \sum_{i \in \mathcal{M}} \mu_{i} \hat{b}(\phi, i) = 0, \quad \forall \phi \in S^{d-1}. \]

In this section, we shall study the case $\sum_{i \in \mathcal{M}} \mu_{i} \hat{b}(\phi, i) \neq 0$.

**Theorem 4.2** Assume that $\|a^{(i)}(\cdot)\|$ is bounded on $\mathbb{R}^{d}$ for every $i \in \mathcal{M}$. Let

\[ \beta_{i} = \begin{cases} \limsup_{|x| \to \infty} \frac{\hat{b}_{i}(\phi, i)|x|^{\delta}}{|x|}, & \text{if } \delta \in (-1, 1), \\ \limsup_{|x| \to \infty} \left( \frac{1}{2} \sum_{k=1}^{d} a_{kk}^{(i)}(x) - \frac{\sum_{k=1}^{d} a_{k1}^{(i)}(x)k}{2|x|^{2}} + \frac{\sum_{k=1}^{d} b_{k}(\phi, i)x_{k}}{|x|^{2}} \right), & \text{if } \delta = -1, \end{cases} \quad (4.4) \]

17
and
\[
\hat{\beta}_i = \begin{cases} 
\liminf_{|x| \to \infty} \frac{d}{k=1} \hat{b}_k\left(\frac{x}{|x|}, i\right) \frac{\xi_k}{|x|}, & \text{if } \delta \in (-1, 1), \\
\liminf_{|x| \to \infty} \left(\frac{1}{2} \sum_{k=1}^{d} a_{kk}(x) - \frac{\sum_{k,l=1}^{d} a_{kl}(x) x_k x_l}{2 |x|^2} + \sum_{k=1}^{d} \hat{b}_k\left(\frac{x}{|x|}, i\right) x_k\right), & \text{if } \delta = -1.
\end{cases}
\]

(4.5)

If \( \sum_{i \in M} \mu_i \beta_i < 0 \), then \((X_t, \Lambda_t)\) is recurrent. If \( \sum_{i \in M} \mu_i \beta_i > 0 \), then \((X_t, \Lambda_t)\) is transient.

**Proof.** (1) Set \( h(x) = |x|^\gamma, \gamma > 0 \), and \( g(x) = |x|^\gamma+\delta-1 \). Then it holds that
\[
\lim_{|x| \to \infty} \frac{g(x)}{h(x)} = 0, \quad \lim_{|x| \to \infty} \frac{L^{(i)}g(x)}{g(x)} = 0.
\]

By direct calculation we get
\[
L^{(i)}h(x) = \left(\gamma - 1 \right) \frac{\sum_{k,l=1}^{d} a_{kl}(x) x_k x_l}{2 |x|^\delta+3} + \frac{\sum_{k=1}^{d} a_{kk}(x)}{2 |x|^\delta+1} + \frac{\sum_{k=1}^{d} \hat{b}_k\left(\frac{x}{|x|}, i\right) x_k}{|x|} \right) \gamma g(x).
\]

(4.6)

When \( \delta \in (-1, 1) \),
\[
L^{(i)}h(x) = \left[O\left(|x|^{-\delta-1}\right) + \frac{\sum_{k=1}^{d} \hat{b}_k\left(\frac{x}{|x|}, i\right) x_k}{|x|}\right] \gamma g(x),
\]

which implies that if \( \sum_{i \in M} \mu_i \beta_i < 0 \), then there exists \( r_0 > 0 \) such that
\[
\sum_{i \in M} \mu_i \left[O\left(|x|^{-\delta-1}\right) + \frac{\sum_{k=1}^{d} \hat{b}_k\left(\frac{x}{|x|}, i\right) x_k}{|x|}\right] < 0, \quad \text{for } |x| > r_0.
\]

Applying Theorem 4.1 as \( \gamma > 0 \), we obtain that \((X_t, \Lambda_t)\) is recurrent if \( \sum_{i \in M} \mu_i \beta_i < 0 \).

When \( \delta = -1 \), it holds
\[
\limsup_{|x| \to \infty} \lim_{\delta \to 0} \left(\gamma - 1 \right) \frac{\sum_{k,l=1}^{d} a_{kl}(x) x_k x_l}{2 |x|^\delta+3} + \frac{\sum_{k=1}^{d} a_{kk}(x)}{2 |x|^\delta+1} + \frac{\sum_{k=1}^{d} \hat{b}_k\left(\frac{x}{|x|}, i\right) x_k}{|x|} = \beta_i.
\]

Therefore, if \( \sum_{i \in M} \mu_i \beta_i < 0 \), by choosing \( \gamma > 0 \) sufficiently small and \( r_0 > 0 \) sufficiently large, we can use Theorem 4.1 to show that \((X_t, \Lambda_t)\) is recurrent.

(2) Now we set \( h(x) = |x|^{-\gamma} \) and \( g(x) = |x|^{-\gamma+\delta-1} \) for \( \gamma > 0 \). Then it still holds
\[
\lim_{|x| \to \infty} \frac{g(x)}{h(x)} = 0, \quad \lim_{|x| \to \infty} \frac{L^{(i)}g(x)}{g(x)} = 0.
\]
and
\[ L^{(i)} h(x) = \left[ - (\gamma + 1) \frac{\sum_{k,l=1}^{d} a_{kl}^{(i)}(x)x_kx_l}{2|x|^{\delta+3}} + \sum_{k=1}^{d} a_{kk}^{(i)}(x) + \frac{\sum_{k=1}^{d} \hat{b}_k(x_k)g(x)}{|x|} \right] (-\gamma)g(x). \] (4.7)

Note that it is \(-\gamma < 0\) before \(g(x)\) in above equality. Similar to the argument in step (1), we can conclude the proof.

When the dimension \(d\) is equal to 1, we can obtain more explicit and complete criteria as presented as follows.

**Corollary 4.3** Let \((X_t, \Lambda_t)\) be a regime-switching diffusion on \([0, \infty)\) with reflecting boundary at 0. \((X_t)\) satisfies
\[ dX_t = b_{\Lambda_t}X_t^\delta dt + \sigma_{\Lambda_t}dB_t, \quad \delta \in [-1, 1), \]
where \(b_i, \sigma_i\) are constants for \(i\) in a finite set \(\mathcal{M}\). \((\Lambda_t)\) is a continuous time Markov chain on \(\mathcal{M}\) independent of \((B_t)\). Then \((X_t, \Lambda_t)\) is recurrent if and only if \(\sum_{i \in \mathcal{M}} \mu_i b_i \leq 0\).

**Proof.** By taking \(h(x)\) and \(g(x)\) as in the Theorem 4.2, it is easy to check that \(\beta_i = \tilde{\beta}_i = b_i\). So according to Theorem 4.2, \((X_t, \Lambda_t)\) is recurrent if \(\sum_{i \in \mathcal{M}} \mu_i b_i < 0\) and transient if \(\sum_{i \in \mathcal{M}} \mu_i b_i > 0\). Therefore, we only need to consider the case \(\sum_{i \in \mathcal{M}} \mu_i b_i = 0\). To deal with this situation, we have to consider it separately according to the range of \(\delta\). Note that it holds \(\sum_{i \in \mathcal{M}} \mu_i b_i (Q^{-1}b)(i) < 0\) as \(\sum_{i \in \mathcal{M}} \mu_i b_i = 0\) (cf. [10]).

Case 1: \(\delta \in (0, 1)\). For \(p > 0\), set
\[ f(x, i) = x^p - p(Q^{-1}b)(i)x^{p-1+\delta} + c_i x^{p-2+2\delta}, \]
where the vector \((c_i)\) would be determined later. By noting that \(\delta \in (0, 1)\), we obtain
\[ \mathcal{A} f(x, i) = \left[ - p(p-1+\delta)b_i(Qb)(i) + Qc(i) \right] x^{p-2+2\delta} + o(x^{p-2+2\delta}). \]

Take \(p \in (0, 1-\delta)\), then \(\sum_{i \in \mathcal{M}} p(p-1+\delta)\mu_i b_i (Q^{-1}b)(i) > 0\). By the Fredholm alternative, there exist a constant \(\beta > 0\) and a vector \((c_i)\) such that \(Qc(i) = p(p-1+\delta)b_i(Q^{-1}b)(i) - \beta\). Choosing these \(p\) and \((c_i)\), we have \(\mathcal{A} f(x, i) = -\beta x^{p-2+2\delta} + o(x^{p-2+2\delta})\). As \(\lim_{|x| \to \infty} f(x, i) = \infty\) for each \(i \in \mathcal{M}\), we obtain that \((X_t, \Lambda_t)\) is recurrent when \(\sum_{i \in \mathcal{M}} \mu_i b_i = 0\) and \(\delta \in (0, 1)\).
Case 2: $\delta \in [-1, 0)$. In this situation, we take $f(x, i) = x^p - p(Q^{-1}b)(i)x^{p-\delta}$. Then

$$
\mathcal{A}f(x, i) = \frac{1}{2}\sigma_i^2 p(p-1)x^{p-2} - p(Q^{-1}b)(i)\left[\frac{1}{2}(p-1+\delta)(p-2+\delta)\sigma_i^2 x^{p-3+\delta} + b_i(p-1+\delta)x^{p-2+2\delta}\right] \\
= \frac{1}{2}\sigma_i^2 p(p-1)x^{p-2} + o(x^{p-2}).
$$

By setting $p \in (0, 1)$, we have $\lim_{x \to \infty} f(x, i) = \infty$ and $\mathcal{A}f(x, i) \leq 0$. Hence, $(X_t, \Lambda_t)$ is recurrent.

Case 3: $\delta = 0$. We take $f(x, i) = x^p - p(Q^{-1}b)(i)x^{p-1} + c_i x^{p-2}$. Then

$$
\mathcal{A}f(x, i) = \left[\frac{1}{2}\sigma_i^2 p(p-1) - p(p-1)b_i(Q^{-1}b)(i) + Qc(i)\right]x^{p-2} + o(x^{p-2}).
$$

Putting $p \in (0, 1)$, as $p(p-1)\sum_{i \in \mathcal{M}} \mu_i\left(\sigma_i^2 - b_i(Q^{-1}b)(i)\right) < 0$, there exist a vector $(c_i)$ and a positive constant $\beta$ such that

$$
Qc(i) + \frac{1}{2}\sigma_i^2 p(p-1) - p(p-1)b_i(Q^{-1}b)(i) = -\beta < 0.
$$

Therefore, we get $\mathcal{A}f(x, i) \leq 0$ and $\lim_{x \to \infty} f(x, i) = \infty$, which implies that $(X_t, \Lambda_t)$ is recurrent. We complete the proof.

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