LIOUVILLE-TYPE RESULTS FOR STATIONARY MAPS OF A CLASS OF FUNCTIONAL RELATED TO PULLBACK METRICS

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Abstract. We study a generalized functional related to the pullback metrics (3). We derive the first variation formula which yield stationary maps. We introduce the stress-energy tensor which is naturally linked to conservation law and yield monotonicity formula via the coarea formula and comparison theorem in Riemannian geometry. A version of this monotonicity inequalities enables us to derive some Liouville type results. Also we investigate the constant Dirichlet boundary value problems and the generalized Chern type results for tension field equation with respect to this functional.

1. Introduction

Let \( u : (M^m, g) \to (N^n, h) \) be a smooth map between Riemannian manifolds \( (M^m, g) \) and \( (N^n, h) \). Recently, Kawai and Nakauchi [7] introduce a functional related to the pullback metric \( u^*h \) :

\[
\Phi(u) = \frac{1}{4} \int_M \|u^*h\|^2 dv_g
\]

where \( u^*h \) is the symetric 2-tensor ( pullback metric ) defined by

\[(u^*h)(X,Y) = \langle du(X), du(Y) \rangle_h\]

for any vector fields \( X, Y \) on \( M \) and \( \|u^*h\| \) its norm

\[
\|u^*h\| = \left( \sum_{i,j=1}^{m} \langle \langle du(e_i), du(e_j) \rangle_h \rangle_h \right)^2
\]

with respect to a local orthonormal frame \( (e_1, \ldots, e_m) \) on \( (M, g) \). The map \( u \) is stationary for \( \Phi \) if it is a critical point of \( \Phi(u) \) with respect to any compactly supported variation of \( u \) and \( u \) is stationary stable if the second variation for the functional \( \Phi(u) \) is non-negative. When \( M \) and \( N \) are compact without boundary, the same authors show the non-existence of non-constant stable stationary map for \( \Phi \) if \( M \) (respectively \( N \)) is a standard sphere \( S^m \) (respectively \( S^n \)). Also they show that a stationary map of \( \Phi \) is a constant map provided that \( M \) is compact without boundary and \( N \) is non-compact supporting a strictly \( C^2 \) convex function.

On the other hand, following Baird and Eells [2], M.Ara [1] introduced the \( F \)-harmonic maps, generalizing harmonic maps, and the stress \( F \)-energy tensor. Let \( F : [0, +\infty[ \to [0, +\infty[ \) be a \( C^2 \) function such that \( F(0) = 0 \) and \( F'(t) > 0 \) on

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A smooth map $u : M \rightarrow N$ is said to be an $F$-harmonic map if it is a critical point of the following $F$-energy functional $E_F$ given by

$$E_F(u) = \int_M F\left(\frac{\|du\|^2}{2}\right) \, dv_g,$$  \hspace{1cm} (1.2)

with respect to any compactly supported variation of $u$, where $\|du\|$ is the Hilbert-Schmid norm of the differential of $u$:

$$\|du\|^2 = \text{trace}_g u^* h = \sum_{i=1}^m <du(e_i), du(e_j)>_h.$$  

The stress $F$-energy tensor $S_F$ associated with $E_F$-energy is given at any map $u$ by

$$S_{F,u}(X,Y) = F\left(\frac{\|du\|^2}{2}\right) <X,Y>_g - F'\left(\frac{\|du\|^2}{2}\right) <du(X), du(Y)>_h$$

for any vector fields $X, Y$ on $M$. Via the stress-energy tensor $S_F$ of $E_F$, monotonicity formula, Liouville-type results and the constant Dirichlet boundary-value problem were investigated recently by Dong and Wei generalizing and refining the works of several authors (see [5] and references therein).

In this paper, we generalize and unify the concept of critical point of the functional $\Phi$. For this, we define the functional $\Phi_F$ by

$$\Phi_F(u) = \int_M F\left(\frac{\|u^* h\|^2}{4}\right) \, dv_g,$$  \hspace{1cm} (1.3)

which is $\Phi$ if $F(t) = t$. We derive the first variation formula of $\Phi_F$ and we introduce the stress-energy tensor $S_{\Phi_F}$ associated to $\Phi_F$ which is naturally linked to conservation law. The tensor $S_{\Phi_F}$ yield monotonicity formula via coarea formula and comparison theorems in Riemannian geometry. These monotonicity inequalities enable us to derive a large classes of Liouville-type results for stationary maps for the functional $\Phi_F$. As another consequence, we obtain the unique constant solutions of constant Dirichlet boundary-value problems on starlike domains for smooth maps satisfying a conservation law. We also obtain generalized Chern type results for tension field equation with respect to the functional $\Phi_F$. We mention that our results are extenstions of results of Nakauchi-Takenaka where they gave the first variation formula, the second variation formula, the monotonicity formula and the Bochner type formula [10].

The contents of this paper is as follows:

1- Introduction
2- Functionals related to pullback metrics and conservation law
3- Monotonicity formula and Liouville-type results
4- Constant Dirichlet boundary-value problems
5- Generalized Chern type results
2. Functional related to pullback metrics and conservation law

Let $F : [0, +\infty] \to [0, +\infty]$ be a $C^2$-function such that $F(0) = 0$ and $F' > 0$ on $]0, +\infty[$. Let $u : M \to N$ be a smooth map from an $m$-dimensional Riemannian manifold $(M, g)$ to a Riemannian manifold $(N, h)$. We call $u$ a stationary map for the functional

$$\Phi_F(u) = \int_M F\left(\frac{\|u^*h\|^2}{4}\right) \, dV_g$$

if

$$\frac{d}{dt} \Phi_F(u_t)_{|t=0} = 0$$

for any compactly supported variation $u_t : M \to N(-\epsilon < t < \epsilon)$ with $u_0 = u$.

Let $\nabla$ and $\nabla^N$ always denote the Levi-civita connection of $M$ and $N$ respectively. Let $\tilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\tilde{\nabla}_X W = \nabla_{du(x)} W$, where $X$ is a tangent vector of $M$ and $W$ is a section of $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ on $M$. We define an $u^{-1}TN$-valued 1-form $\sigma_{F,u}$ on $M$ by

$$\sigma_{F,u}(\cdot) = F'\left(\frac{\|u^*h\|^2}{4}\right) \sum_{j=1}^m h(du(\cdot), du(e_j)) du(e_j). \tag{2.1}$$

When $F(t) = t$, we have $\sigma_{F,u}(\cdot) = \sigma_u(\cdot) = \sum_{j=1}^m h(du(\cdot), du(e_j)) du(e_j)$, as defined in [7], which give

$$\|u^*h\|^2 = \sum_{i=1}^m \langle du(e_i), \sigma_u(e_i) \rangle_h$$

We define the tension field $\tau_{\Phi_F}(u)$ of $u$ with respect the functional $\Phi_F$ by

$$\tau_{\Phi_F}(u) := \text{div}_g \sigma_{F,u} \tag{2.2}$$

where $\text{div}_g \sigma_{F,u}$ denotes the divergence of $\sigma_{F,u}$:

$$\tau_{\Phi_F}(u) = \sum_{i=1}^m \left\{ \tilde{\nabla}_{e_i} \left( F'\left(\frac{\|u^*h\|^2}{4}\right) \sigma_u(e_i) \right) \right\} - F'\left(\frac{\|u^*h\|^2}{4}\right) \sigma_u(\nabla_{e_i} e_i)$$

$$= F'\left(\frac{\|u^*h\|^2}{4}\right) \text{div}_g \sigma_u + \sigma_u \left\{ \text{grad} \left( F'\left(\frac{\|u^*h\|^2}{4}\right) \right) \right\}$$

**Lemma 2.1 (First variation formula).** Let $u : (M, g) \to (N, h)$ be a $C^2$ map. Then

$$\frac{d}{dt} \Phi_F(u_t)_{|t=0} = - \int_M \langle \tau_{\Phi_F}(u), du(X) \rangle_h \, dV_g,$$

where $X$ (respectively $u_t$) is any smooth vector field with compact support on $M$ (respectively any $C^2$ deformation of $u$).

**Proof.** Let $X$ be a vector field on $M$ with compact support. Let $U : [-\epsilon, \epsilon] \times M \to N$ be any smooth deformation of $u$ such that

$$U(0, x) = u(x)$$

$$\frac{dU}{dt}(0, x) = du(X)$$
Put $u_t(x) = U(t, x)$ and $Y_t = du_t(\frac{\partial}{\partial t}, X)$ the variation field. Then

$$\frac{\partial}{\partial t} F \left( \frac{\|u_t^* h\|^2}{4} \right) = \frac{\partial}{\partial t} F \left( \frac{\sum_{i,j=1}^{m} \langle du_t(e_i), du_t(e_j) \rangle_h}{4} \right)$$

$$= F' \left( \frac{\|u_t^* h\|^2}{4} \right) \frac{1}{4} \frac{\partial}{\partial t} \left( \sum_{i,j=1}^{m} \langle du_t(e_i), du_t(e_j) \rangle_h \langle du_t(e_i), du_t(e_j) \rangle_h \right)$$

$$= F' \left( \frac{\|u_t^* h\|^2}{4} \right) \sum_{i,j=1}^{m} \left\langle \nabla_{\frac{\partial}{\partial t}} (du_t(e_i)), du_t(e_j) \right\rangle_h \langle du_t(e_i), du_t(e_j) \rangle_h$$

$$= \sum_{i=1}^{m} \langle \sigma_{F,u_t}(e_i), \tilde{\nabla}_{e_i} Y_t \rangle_h$$

Since $Y_t$ has compact support on $M$, using an integration by parts we obtain

$$\frac{d}{dt} \Phi_F(u_t) \bigg|_{t=0} = - \int_M \left( \sum_{i=1}^{m} \tilde{\nabla} e_i (\sigma_{F,u}(e_i)), Y_0 \right)_h \, dv_g$$

$$= - \int_M \langle \text{div}_g (\sigma_{F,u}), du(X) \rangle_h \, dv_g$$

which is the first variation formula for $\Phi_F$. □

The first variation formula allows us to define the notion of stationary maps for the functional $\Phi_F$.

**Definition 2.2** (Stationary map). A smooth map $u$ is called stationary for the functional $\Phi_F$ if it is a solution of the Euler-Lagrange equation

$$\text{div}_g \sigma_{F,u} = 0,$$

equivalently

$$F' \left( \frac{\|u^* h\|^2}{4} \right) \text{div}_g \sigma_u + \sigma_u \left( \text{grad} \left( F' \left( \frac{\|u^* h\|^2}{4} \right) \right) \right) = 0$$

**Example.** 1- Every totally geodesic map $u$, i.e $\nabla du = 0$, is stationary for $\Phi_F$. 2- If $N = \mathbb{R}$ then $\|u^* h\|^2 = \|du\|^4$. Hence $u$ is stationary for $\Phi_F$ if and only if $u$ is $G$-harmonique where $G(t) = F(t^2)$.

Following Baird [3], for a smooth map $u : (M, g) \rightarrow (N, h)$ we associate a symmetric 2-tensor $S_{\Phi_F,u}$ to the functional $\Phi_F$, called the stress-energy tensor

$$S_{\Phi_F,u}(X, Y) = F \left( \frac{\|u^* h\|^2}{4} \right) \langle X, Y \rangle_g - F' \left( \frac{\|u^* h\|^2}{4} \right) \langle \sigma_u(X), du(Y) \rangle_h \quad (2.3)$$

where $X, Y$ are vector fields on $M$.

**Proposition 2.3.** Let $u : (M, g) \rightarrow (N, h)$ a smooth map and let $S_{\Phi_F,u}$ be the associated stress-energy tensor, then for all $x \in M$ and for each vector $X \in T_x M$,

$$(\text{div} S_{\Phi_F,u})(X) = - \langle \text{div}_g \sigma_{F,u}, du(X) \rangle$$
where
\[ \sigma_{F,u}(X) = F' \left( \frac{\|u^*h\|^2}{4} \right) \sum_{i=1}^{m} \langle du(X), du(e_i) \rangle \, du(e_i). \]

**Proof.** For any vector field \( X \) on \( M \) we have
\[
\frac{1}{4} \nabla_X \|u^*h\|^2 = \sum_{i,j=1}^{m} \langle (\nabla_X du)(e_i), du(e_j) \rangle \langle du(e_i), du(e_j) \rangle = \sum_{i=1}^{m} \langle (\nabla_X du)(e_i), \sigma_u(e_i) \rangle
\]
We compute
\[
(div S_{\Phi,F,u})(X) = \sum_{k=1}^{m} \left\{ \nabla_{e_k} \left( S_{\Phi,F,u}(e_k, X) \right) - S_{\Phi,F,u}(e_k, \nabla_{e_k} X) - S_{\Phi,F,u}(\nabla_{e_k} e_k, X) \right\}
\]
\[
= \sum_{k=1}^{m} \left\{ \left( \nabla_{e_k} F \left( \frac{\|u^*h\|^2}{4} \right) \right) < e_k, X > + F \left( \frac{\|u^*h\|^2}{4} \right) < e_k, \nabla_{e_k} X > -
\nabla_{e_k} \left( < \sigma_{F,u}(e_k), du(X) > \right) - F \left( \frac{\|u^*h\|^2}{4} \right) < e_k, \nabla_{e_k} X > -
\langle \sigma_{F,u}(e_k), du(\nabla_{e_k} X) > \right\}
\]
\[
= \nabla_X F \left( \frac{\|u^*h\|^2}{4} \right) - < div_g \sigma_{F,u}, du(X) > - \sum_{k=1}^{m} \langle \sigma_{F,u}(e_k), \nabla_{e_k} (du(X)) - du(\nabla_{e_k} X) \rangle
\]
\[
= - < div_g \sigma_{F,u}, du(X) > + \sum_{i=1}^{m} \langle (\nabla_X du)(e_i), \sigma_{F,u}(e_i) \rangle \sum_{k=1}^{m} < (\nabla_{e_k} du)(X), \sigma_{F,u}(e_k) >
\]
Since \( (\nabla_{e_k} du)(X) = (\nabla_X du)(e_k) \) we obtain
\[
(div S_{\Phi,F,u})(X) = - < div_g \sigma_{F,u}, du(X) >
\]
\[
\square
\]

**Definition 2.4.** A map \( u : M \rightarrow N \) is said to satisfy an \( \Phi_F \)-conservation law if \( S_{\Phi,F,u} \) is divergence free, i.e. the \((0,1)\)-type tensor field \( div S_{\Phi,F,u} \) vanishes identically \( (div S_{\Phi,F,u} = 0) \).

In particular if \( u \) is a stationary map for \( \Phi_F \) then \( S_{\Phi,F,u} \) is divergence free. In general the conserve will not be true i.e \( div S_{\Phi,F,u} = 0 \), then we cannot conclude that \( u \) is a stationary map for the functional \( \Phi_F \). However, in the spacial case when \( u \) is a submersive mapping, we do have the equivalence.

**Corollary 2.5.** If \( u : M \rightarrow N \) is a submersion almost everywhere, then \( u \) is a stationary map for \( \Phi_F \) if and only if \( div S_{\Phi,F,u} = 0 \).
3. Monotonicity formula and Liouville-type results

In this section, we will establish the monotonicity formula for the functional $\Phi_F$ on complete Riemannian manifolds with a pole. We recall a pole is a point $x_0 \in M$ such that the exponential map $\exp_{x_0} : T_{x_0}M \to M$ is a global diffeomorphism. For a map $u : (M, g) \to (N, h)$ satisfying an $\Phi_F$-conservation law, the stress-energy 2-tensor $S_{\Phi_F, u}$ yield monotonicity formula via corea formula and comparison theorems in Riemannian geometry. These monotonicity inequalities enable us to derive a large classes of Liouville-type results for stationary maps for the functional $\Phi_F$. We mention that Nakauchi and Takenaka gave the monotonicity formula for the functional $\Phi_F$ and our result is a generalization of their result to the $F$-functional $\Phi_F$ [10].

For a vector field $X$ on $M$, we denote by $\theta_X$ its dual 1-form, i.e

$$\theta_X(Y) = \langle X, Y \rangle_g$$

Let $\nabla \theta_X$ the 2-tensor

$$(\nabla \theta_X)(Y, Z) = (\nabla_Y \theta_X)(Z) = Y (\theta_X(Z)) - \theta_X(\nabla_Y Z) = \langle \nabla_Y X, Z \rangle_g$$

Following Baird [3], the contraction of stress-energy tensor $S_{\Phi_F, u}$ with $X$ is given by

$$\text{div} \left( i_X S_{\Phi_F, u} \right) = (\text{div} S_{\Phi_F, u})(X) + \langle S_{\Phi_F, u}, \nabla \theta_X \rangle$$

where $(i_X S_{\Phi_F, u})(Y) := S_{\Phi_F, u}(X, Y)$ and

$$\langle S_{\Phi_F, u}, \nabla \theta_X \rangle = \sum_{i,j=1}^m S_{\Phi_F, u}(e_i, e_j) \langle \nabla_{e_i} X, e_j \rangle_g.$$ 

Let $D$ be any bounded domain of $M$ with $C^1$-boundary. We integrate the formula (7), by Stokes’s theorem, we obtain the basis of monotonicity formula

$$\int_{\partial D} S_{\Phi_F, u}(X, \nu) ds_g = \int_D \left\{ (\text{div} S_{\Phi_F, u})(X) + \langle S_{\Phi_F, u}, \nabla \theta_X \rangle \right\} dv_g$$

where $\nu$ is unit outward normal vector field along $\partial D$ with $(m - 1)$-dimensional volume element $ds_g$. In particular, if $u$ satisfies an $\Phi_F$-conservation law, for almost $R_2 > R_1 \geq 0$ we have

$$\int_{\partial B(R_2)} S_{\Phi_F, u}(X, \nu) ds_g - \int_{\partial B(R_1)} S_{\Phi_F, u}(X, \nu) ds_g = \int_{B(R_2) \setminus B(R_1)} \langle S_{\Phi_F, u}, \nabla \theta_X \rangle dv_g$$

Following Kassi[8], we introduce the following

**Definition 3.1.** The upper (lower) $F$-degree $d_F$ of the function $F$ is defined to be

$$d_F = \sup_{t \geq 0} \frac{t F'(t)}{F(t)} \quad (l_F = \inf_{t \geq 0} \frac{t F'(t)}{F(t)})$$

We assume that $d_F$ is finite. The main result in this section is the following theorem which give monotonicity formula for the functional $\Phi_F$. 

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Theorem 3.2. Let \((M, g)\) be a complete Riemannian manifold with a pole \(x_0\). Assume that there exist two positive functions \(h_1(r)\) and \(h_2(r)\) such that
\[
h_1(r) [g - dr \otimes dr] \leq \text{Hess}(r) \leq h_2(r) [g - dr \otimes dr]
\] (3.4)
on \(M \setminus \{x_0\}\) where \(r = d_g(x, x_0)\). Let \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) be a \(C^2\)-function such that \(\phi(0) = 0\), \(\phi'(t) > 0\) on \([0, +\infty[\) and
\[
\phi'(r)h_2(r) \geq \phi''(r)
on M \setminus \{x_0\}.
\]
If \(u : (M, g) \to (N, h)\) satisfies an \(\Phi_F\)-conservation law, then for \(0 < R_1 \leq R_2\) we have
\[
\frac{1}{e^{G(R_1)}} \int_{B(x_0, R_1)} F\left(\frac{\|u^*h\|^2}{4}\right) dv_g \leq \frac{1}{e^{G(R_2)}} \int_{B(x_0, R_2)} F\left(\frac{\|u^*h\|^2}{4}\right) dv_g
\] (3.5)
where \(G(R)\) is a primitive of the lower bound of
\[
R \to \left(\phi'(R)\right)^{-1} \inf_{B(x_0, R)} \left(\phi''(r) + ((m - 1)h_1(r) - 4d_F h_2(r))\phi'(r)\right).
\]
In particular, if
\[
\int_{B(x_0, R)} F\left(\frac{\|u^*h\|^2}{4}\right) dv_g = o(e^{G(R)})
\]
then \(u\) is a constant map.

Proof. Denoted \(D = B_R(x_0)\) the geodesic ball of radius \(R\) centred at \(x_0\). Taking \(X = \phi'(r)\frac{\partial}{\partial r} \in T_{x_0}M\) where \(\frac{\partial}{\partial r}\) denoted unit radial vector field. Choosing a local orthonormal frame field \(\{e_1, \cdots, e_{m-1}, e_m = \frac{\partial}{\partial r}\}\) on \(M\). Since \(u\) satisfies an \(\Phi_F\)-conservation law, applying formula (8) to \(D = B(x_0, R)\) and \(X = \phi'(r)\frac{\partial}{\partial r}\) we have
\[
\int_{B_R(x_0)} < S_{\Phi_F, u}, \nabla \theta_X > dv_g = \int_{\partial B_R(x_0)} S_{\Phi_F, u}(X, \nu) ds_g
\]
\[
= \int_{\partial B_R(x_0)} F\left(\frac{\|u^*h\|^2}{4}\right) g(X, \nu) ds_g - \int_{\partial B_R(x_0)} F'\left(\frac{\|u^*h\|^2}{4}\right) h(\sigma_u(X), du(\nu)) ds_g
\]
\[
= \phi'(R) \left\{ \int_{\partial B_R(x_0)} F\left(\frac{\|u^*h\|^2}{4}\right) ds_g - \int_{\partial B_R(x_0)} F'\left(\frac{\|u^*h\|^2}{4}\right) h(\sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})) ds_g \right\}
\]
\[
= \phi'(R) \left\{ \int_{\partial B_R(x_0)} F\left(\frac{\|u^*h\|^2}{4}\right) ds_g - \int_{\partial B_R(x_0)} F'\left(\frac{\|u^*h\|^2}{4}\right) \sum_{i=1}^{m} h(du(e_i), du(\frac{\partial}{\partial r}))^2 ds_g \right\}
\]
\[
\leq \phi'(R) \int_{\partial B_R(x_0)} F\left(\frac{\|u^*h\|^2}{4}\right) ds_g
\]
Now, we will compute the item \(< S_{\Phi_F, u}, \nabla \theta_X >\) on the left hand side. For this purpose, using local orthonormal frame field \(\{e_1, \cdots, e_{m-1}, \frac{\partial}{\partial r}\}\), it is easy to see that
\[
\nabla \frac{\partial}{\partial r} = \phi'' \frac{\partial}{\partial r}, \quad \nabla e_i X = \phi' \sum_{k=1}^{m} \text{Hess}(r)(e_i, e_k)e_k, \quad 1 \leq i \leq m - 1,
\]
\[ \text{div} X = \phi'' + \phi' \sum_{k=1}^{m-1} \text{Hess}(r)(e_k, e_k) \]

where \( \text{Hess}(.) \) denoted the Hessian operator, i.e.

\[ \text{Hess}(r)(e_i, e_j) = \nabla_{e_i} \nabla_{e_j} r - (\nabla_{e_i} e_j) r. \]

So

\[ h(\sigma_u(e_\alpha), du(e_\beta))g(\nabla_{e_\alpha} X, e_\beta) = \phi' \text{Hess}(r)(e_i, e_j) h(\sigma_u(e_i), du(e_j)) + h(\sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})) \]

Then

\[ < S_{\Phi_{F,u}} , \nabla \theta_X > = F \left( \frac{\| u^* h \|^2}{4} \right) \text{div} X - F' \left( \frac{\| u^* h \|^2}{4} \right) \sum_{\alpha, \beta = 1}^{m} h(\sigma_u(e_\alpha), du(e_\beta))g(\nabla_{e_\alpha} X, e_\beta) \]

\[ = \frac{\| u^* h \|^2}{4} \left( \phi'' + \phi' \sum_{k=1}^{m-1} \text{Hess}(r)(e_k, e_k) \right) \]

\[ - F' \left( \frac{\| u^* h \|^2}{4} \right) \left\{ \phi' \sum_{i,j=1}^{m-1} h(\sigma_u(e_i), du(e_j)) + \phi'' h(\sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})) \right\} \]

\[ \geq \frac{\| u^* h \|^2}{4} \left( \phi'' + (m-1)h_1 \phi' \right) - \]

\[ \frac{\| u^* h \|^2}{4} \left\{ \phi' h_2 \sum_{i=1}^{m-1} h(\sigma_u(e_i), du(e_i)) + \phi'' h(\sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})) \right\} \]

\[ = \frac{\| u^* h \|^2}{4} \left( \phi'' + (m-1)h_1 \phi' \right) - \]

\[ \phi' h_2 F' \left( \frac{\| u^* h \|^2}{4} \right) \| u^* h \|^2 + (\phi' h_2 - \phi'') \sum_{i=1}^{m} (h(du(\frac{\partial}{\partial r}), du(e_i)))^2 \]

\[ \geq \left( \phi'' + ((m-1)h_1 - 4dF h_2)\phi'(r) \right) F \left( \frac{\| u^* h \|^2}{4} \right) \]

Hence

\[ < S_{\Phi_{F,u}} , \nabla \theta_X > \geq \left( \phi''(r) + ((m-1)h_1(r) - 4dF h_2(r))\phi'(r) \right) F \left( \frac{\| u^* h \|^2}{4} \right) \]

and

\[ \int_{\partial B_R(x_0)} F \left( \frac{\| u^* h \|^2}{4} \right) d\sigma \geq H(R) \int_{B_R(x_0)} F \left( \frac{\| u^* h \|^2}{4} \right) d\nu \]

where \( H(R) \) is a lower bound of

\[ R \rightarrow \left( \phi'(R) \right)^{-1} \inf_{B(x_0, R)} \left( \phi''(r) + ((m-1)h_1(r) - 4dF h_2(r))\phi'(r) \right) \]

The coarea formula implies that

\[ \frac{d}{dR} \int_{B_R(x_0)} F \left( \frac{\| u^* h \|^2}{4} \right) d\nu = \int_{\partial B_R(x_0)} F \left( \frac{\| u^* h \|^2}{4} \right) d\sigma \]
Hence
\[
\frac{d}{dR} \int_{B_R(x_0)} F\left( \frac{\|u^*h\|^2}{4} \right) dv_g \geq H(R)
\]
(3.6)
for almost every \( R > 0 \). By integration (12) over \([R_1, R_2]\), we have
\[
\log \int_{B_{R_1}(x_0)} F\left( \frac{\|u^*h\|^2}{4} \right) dv_g - \log \int_{B_{R_2}(x_0)} F\left( \frac{\|u^*h\|^2}{4} \right) dv_g \geq G(R_2) - G(R_1)
\]
This proves monotonicity inequality. The constancy of \( u \) follows by letting \( R_2 \) to infinity in (11).

The rest of this section is devoted to derive some Liouville-type results under some explicit curvatures conditions on \((M,g)\) with a pole \( x_0 \). The radial curvature \( K_r \) (respectively radial Ricci curvature \( \text{Ric}_r \) of \( M \) is the restriction of the sectional curvature function (respectively the Ricci curvature function) to all planes which contain the unit vector \( \nabla r \) in \( T_xM \) tangent to the unique geodesic joining \( x_0 \) to \( x \) and pointing away from \( x_0 \). The tensor \( g - dr \otimes dr \) is trivial on the radial direction \( \nabla r \) and equal to \( g \) on the orthogonal complement \([\nabla r]^\perp\). We regard how \( K_r \) varies as long as we have Hessian comparison estimates with bounds satisfying (10) with \( \phi(t) = \frac{t^2}{2} \). We collect some Liouville-type results for maps satisfying an \( \Phi_F \)-conservation law in the following two theorems.

**Theorem 3.3.** Let \( u : (M,g) \to (N,h) \) be a \( C^2 \) map satisfying an \( \Phi_F \)-conservation law. The \( u \) is constant provided one of the following conditions is satisfied:

(i) \(-\alpha^2 \leq K_r \leq -\beta^2 \) with \( \alpha > 0, \beta > 0 \) and \((m-1)\beta - 4d_F\alpha \geq 0 \) and
\[
\int_{B_R(x_0)} F\left( \frac{\|u^*h\|^2}{4} \right) dv_g = o\left(R^{(m-4d_F\alpha)}\right)
\]
(ii) \( K_r = 0 \) with \( m - 4d_F > 0 \) and
\[
\int_{B_R(x_0)} F\left( \frac{\|u^*h\|^2}{4} \right) dv_g = o\left(R^{(m-4d_F)}\right)
\]
(iii) \(-\frac{A}{(1+2\epsilon)(1+\epsilon)} \leq K_r \leq \frac{B}{(1+\epsilon)^2} \) with \( \epsilon > 0, A \geq 0, 0 \leq B < 2\epsilon \) and \((m-1)\epsilon - 4d_Fe^\frac{A}{\epsilon} > 0 \) and
\[
\int_{B_R(x_0)} F\left( \frac{\|u^*h\|^2}{4} \right) dv_g = o\left(R^{(m-1)\epsilon - 4d_Fe^\frac{A}{\epsilon} + 1)}\right)
\]
(iv) \(-\alpha^2 \leq K_r \leq 0, \text{Ric}_r \leq -\beta^2 \) where \( \alpha > 0, \beta > 0, \beta - 4d_F\alpha \geq 0 \) and
\[
\int_{B_R(x_0)} F\left( \frac{\|u^*h\|^2}{4} \right) dv_g = o\left(R^{2(1-\frac{4d_F\alpha}{\beta})}\right)
\]

**Theorem 3.4.** Let \( u : (M,g) \to (N,h) \) be a \( C^2 \) map satisfying an \( \Phi_F \)-conservation law. Suppose that \( d_F \leq \frac{1}{4} \) and the radial curvature \( K_r \) of \( M \) satisfy:
\(-Ar^{2q} \leq K_r \leq -Br^{2q}\) with \(A \geq B > 0, q > 0\) and \((m-1)B_0 - 4d_F \sqrt{A} \coth \sqrt{A} \geq 0\) where \(B_0 = \min \left\{ 1, -\frac{q+1}{2} + \sqrt{B + \frac{(q+1)^2}{4}} \right\}\). Then for \(1 < R_1 \leq R_2\) we have
\[
\frac{1}{R_1^4} \int_{B_{R_1}(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) dv_g \leq \frac{1}{R_2^4} \int_{B_{R_2}(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) dv_g
\]
where \(\lambda = 1 + (m-1)B_0 - 4d_F \sqrt{A} \coth \sqrt{A}\). In particular if
\[
\int_{B_{R}(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) dv_g = o \left( R^\lambda \right)
\]
then \(u\) is constant on \(M \setminus B_1(x_0)\).

For the proof, we will need the following Hessian comparison theorems of Greene-Wu [6].

**Lemma 3.5.** Let \((M, g)\) be a complete Riemannian manifold with a pole \(x_0\). Let \(K_r\) and \(\text{Ric}_r\) are the respectively the radial sectional and Ricci curvatures of \(M\). Then

(a) If \(-\alpha^2 \leq K_r \leq -\beta^2\) with \(\alpha > 0, \beta > 0\) then

\[
\beta \coth(\beta r) [g - dr \otimes dr] \leq \text{Hess}(r) \leq \alpha \coth(\alpha r) [g - dr \otimes dr]
\]

(b) If \(K_r = 0\), then

\[
\frac{1}{r} [g - dr \otimes dr] = \text{Hess}(r)
\]

(c) If \(-A(1+r^2)^{-1-\epsilon} \leq K_r \leq -B(1+r^2)^{-1-\epsilon}\) with \(\epsilon > 0, A \geq 0\) and \(0 \leq B < 2\epsilon\), then

\[
\frac{1-B}{2\epsilon} [g - dr \otimes dr] \leq \text{Hess}(r) \leq \frac{A}{r} [g - dr \otimes dr]
\]

(d) If \(-\alpha^2 \leq K_r \leq 0\) and \(\text{Ric}_r \leq -\beta^2\) where \(\alpha > 0, \beta > 0\) then

\[
\Delta r \geq \beta \coth(\beta r), \quad \text{Hess}(r) \leq \alpha \coth(\alpha r) [g - dr \otimes dr]
\]

(e) If \(-Ar^{2q} \leq K_r \leq -Br^{2q}\) with \(A \geq B > 0\) and \(q > 0\), then

\[
B_0 r^q [g - dr \otimes dr] \leq \text{Hess}(r) \leq (\sqrt{A} \coth \sqrt{A}) r^q [g - dr \otimes dr]
\]

for \(r \geq 1\), where \(B_0 = \min \left\{ 1, -\frac{q+1}{2} + \sqrt{B + \frac{(q+1)^2}{4}} \right\}\).

3.1. **Proof of Theorem 3.3.**

Proof. In order to use theorem 3.2, we fix \(\phi(t) = \frac{t^2}{2}\).

Case (i). Since \(-\alpha^2 \leq K_r \leq -\beta^2\) with \(\alpha > 0, \beta > 0\), by comparison theorem

\[
\beta \coth(\beta r) [g - dr \otimes dr] \leq \text{Hess}(r) \leq \alpha \coth(\alpha r) [g - dr \otimes dr]
\]
Since \((m-1)\beta - 4d_F\alpha \geq 0\), by theorem
\[
H(R) = (\phi'(R))^{-1} \inf_{B(x_0, R)} \left( \phi''(r) + ((m-1)h_1(r) - 4d_Fh_2(r))\phi'(r) \right)
\]
\[
= \frac{1}{R} \inf_{B(x_0, R)} (1 + ((m-1)\beta\coth(\beta r) - 4d_F\alpha\coth(\alpha r))r)
\]
\[
= \frac{1}{R} \inf_{t\in[0,R]} (1 + (m-1)\beta t\coth(\beta t) - 4d_F\alpha t\coth(\alpha t))
\]
\[
= \frac{1 + (m-1)\beta - 4d_F\alpha}{R}
\]
Thus \(G(R) = \log R^{1 + (m-1)\beta - 4d_F\alpha}\) which implies the monotonicity inequality.

\textit{Case (ii).} Since \(K_r = 0\) by comparison theorem
\[
\frac{1}{r} [g - dr \otimes dr] = \text{Hess}(r)
\]
In this case
\[
H(R) = (\phi'(R))^{-1} \inf_{B(x_0, R)} \left( \phi''(r) + ((m-1)h_1(r) - 4d_Fh_2(r))\phi'(r) \right)
\]
\[
= \frac{m - 4d_F}{R}
\]
and \(G(R) = \log R^{(m-4d_F)}\) which implies the monotonicity inequality.

\textit{Case (iii).} Since \(-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq -\frac{B}{(1+r^2)^{1+\epsilon}}\) with \(\epsilon > 0\), \(A \geq 0\), \(0 \leq B < 2\epsilon\) by comparison theorem
\[
\frac{1 - \frac{B}{2\epsilon}}{r} [g - dr \otimes dr] \leq \text{Hess}(r) \leq \frac{\epsilon^4}{e_r} [g - dr \otimes dr]
\]
In this case
\[
H(R) = (\phi'(R))^{-1} \inf_{B(x_0, R)} \left( \phi''(r) + ((m-1)h_1(r) - 4d_Fh_2(r))\phi'(r) \right)
\]
\[
= \frac{1 + (m-1)(1-B) - 4d_Fe_r}{R}
\]
and \(G(R) = \log R^{1 + (m-1)(1-B) - 4d_Fe_r}\) which implies the monotonicity inequality.

\textit{Case (iv).} Since \(-\alpha^2 \leq K_r \leq 0\), \(\text{Ric}_r \leq -\beta^2\) where \(\alpha \geq \beta > 0\), by comparison theorem
\[
\Delta r \geq \beta \coth(\beta r), \quad \text{and} \quad \text{Hess}(r) \leq \alpha \coth(\alpha r)[g - dr \otimes dr]
\]
If \(\beta - 4d_F\alpha \geq 0\), following the proof of theorem
\[
\frac{1}{\phi'(R)} (\Delta \phi(r) - 4d_Fh_2(r)\phi'(r)) = \frac{1}{R} (1 + r\Delta r - 4d_F\alpha r \coth(\alpha r))
\]
\[
\geq \frac{1 + \beta r \coth(\beta r) - 4d_F\alpha r \coth(\alpha r)}{R}
\]
\[
\geq H(R) = \frac{2(1 - 2d_F\alpha)}{R}
\]
and \(G(R) = \log R^{2(1 - 2d_F\alpha)}\) which implies the monotonicity inequality.
3.2. Proof of Theorem 3.4.

Proof. By monotonicity formula (9)
\[
\int_{\partial B(x_0, R)} S_{\Phi, u}(X, \nu) ds_g - \int_{\partial B(x_0, 1)} S_{\Phi, u}(X, \nu) ds_g = \int_{B(x_0, R) \setminus B(x_0, 1)} \langle S_{\Phi, u}, \nabla \theta_X \rangle dv_g
\]
Since \(-Ar^{2q} \leq K_r \leq -Br^{2q}\) with \(A \geq B > 0, q > 0\), by comparison theorem
\[
B_0r^q [g - dr \otimes dr] \preceq \text{Hess}(r) \preceq (\sqrt{A} \coth \sqrt{A})r^q [g - dr \otimes dr]
\]
for \(r \geq 1\), where \(B_0 = \min \left\{ 1, -\frac{q+1}{2} + \sqrt{B + \frac{(q+1)^2}{4}} \right\} \).
Take \(X = r \frac{\partial}{\partial r}\), following the proof of theorem, we have
\[
\langle S_{\Phi, u}, \nabla \theta_X \rangle \geq (1 + (m - 1)B_0r^{q+1} - 4d_F \sqrt{A} \coth \sqrt{A})r^{q+1} F \left( \frac{\|u^*h\|^2}{4} \right)
\]
\[
\geq (1 + (\lambda - 1)r^{q+1}) F \left( \frac{\|u^*h\|^2}{4} \right)
\]
and
\[
S_{\Phi, u}(X, \frac{\partial}{\partial r}) = F \left( \frac{\|u^*h\|^2}{4} \right) - F' \left( \frac{\|u^*h\|^2}{4} \right) < \sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}) > \quad \text{on} \quad \partial B_1(x_0)
\]
\[
S_{\Phi, u}(X, \frac{\partial}{\partial r}) = RF \left( \frac{\|u^*h\|^2}{4} \right) - RF' \left( \frac{\|u^*h\|^2}{4} \right) < \sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}) > \quad \text{on} \quad \partial B_R(x_0)
\]
Hence
\[
R \int_{\partial B_R(x_0)} \left\{ F \left( \frac{\|u^*h\|^2}{4} \right) - F' \left( \frac{\|u^*h\|^2}{4} \right) < \sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}) > \right\} ds_g =
\]
\[
\int_{\partial B_1(x_0)} \left\{ F \left( \frac{\|u^*h\|^2}{4} \right) - F' \left( \frac{\|u^*h\|^2}{4} \right) < \sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}) \right\} ds_g 
\]
\[
\geq \lambda \int_{B_R(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) dv_g
\]
Since \(\|u^*h\|^2 \geq \sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}) > 0\) and \(tF'(t) \leq d_F F(t)\)
\[
\int_{\partial B_1(x_0)} \left\{ F \left( \frac{\|u^*h\|^2}{4} \right) - F' \left( \frac{\|u^*h\|^2}{4} \right) < \sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}) \right\} ds_g \geq (1 - 4d_F) \int_{\partial B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) ds_g \geq 0
\]
Hence for \(R > 1\)
\[
R \int_{\partial B_R(x_0)} \left\{ F \left( \frac{\|u^*h\|^2}{4} \right) - F' \left( \frac{\|u^*h\|^2}{4} \right) < \sigma_u(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}) > \right\} ds_g \geq \lambda \int_{B_R(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) dv_g
\]
Coarea formula then implies
\[
\frac{d}{dR} \int_{B_R(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) dv_g \geq \frac{\lambda}{R}
\]
for almost all \( R \geq 1 \). Integrating over \([R_1, R_2]\), we get

\[
\log \left( \int_{B_{R_2}(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) \, dv_g \right) - \log \left( \int_{B_{R_1}(x_0) \setminus B_1(x_0)} F \left( \frac{\|u^*h\|^2}{4} \right) \, dv_g \right) \\
\geq \lambda \log R_2 - \lambda \log R_1
\]

which implies monotonicity inequality. \( \square \)

4. Constant Dirichlet boundary-value problems

In this section we deal with constant Dirichlet boundary-value problems for maps satisfying an \( \Phi_F \)-conservation law. As in [5], we introduce starlike domains with \( C^1 \)-boundaries which generalize \( C^1 \)-convex domains.

**Definition 4.1.** A bounded domain \( D \subset (M, g) \) with \( C^1 \)-boundary \( \partial D \) is called starlike if there exist an interior point \( x_0 \in D \) such that

\[
< \frac{\partial}{\partial r_{x_0}}, \nu >_g \bigg|_{\partial D} \geq 0
\]

where \( \nu \) is the unit normal to \( \partial D \) and \( \frac{\partial}{\partial r_{x_0}} \) is the unit vector field such that for any \( x \in D \setminus \{x_0\} \cup \partial D \), \( \frac{\partial}{\partial r_{x_0}}(x) \) is the unit vector tangent to the unique geodesic joining \( x_0 \) to \( x \) and pointing away from \( x_0 \).

**Theorem 4.2.** Let \( u : (M, g) \to (N, h) \) be a \( C^2 \) map and \( D \subset M \) be a starlike domain. Assume that \( l_F \geq \frac{1}{2} \) and \( u\big|_{\partial D} \) is constant. If \( u \) satisfies an \( \Phi_F \)-conservation law, then \( u \) is constant on \( D \) provided one of the following conditions is satisfied:

(i) \(-\alpha^2 \leq K_r \leq -\beta^2 \) with \( \alpha > 0, \beta > 0 \) and \( (m - 1)\beta - 4d_F\alpha \geq 0 \),

(ii) \( K_r = 0 \) with \( m - 4d_F > 0 \),

(iii) \(-\frac{\beta}{1 + r^2} \leq K_r \leq -\frac{\beta}{1 + r^2} \) with \( \epsilon > 0, A \geq 0, 0 \leq B < 2\epsilon \) and \( 1 + (m - 1)(1 - \frac{B}{2\epsilon}) - 4d_Fe^{A \epsilon} > 0 \),

(iv) \(-\alpha^2 \leq K_r \leq 0, \ Ric_r \leq -\beta^2 \) where \( \alpha > 0, \beta > 0, \beta - 4d_F\alpha \geq 0 \).

**Proof.** Let the vector field \( X = r_{x_0}\nabla r_{x_0} \). Under the radial curvatures conditions, following the proof of theorem 2.2, localized on \( \overline{D} \), we get

\[
\langle S_{\Phi_F}, \nabla \theta_X \rangle \geq K F \left( \frac{\|u^*h\|^2}{4} \right) \quad \text{on} \quad D
\]

where

\[
K = 1 + (m - 1)\beta - 4d_F\alpha \quad \text{case (i)}
\]

\[
= m - 4d_F \quad \text{case (ii)}
\]

\[
= 1 + (m - 1)(1 - \frac{B}{2\epsilon}) - 4d_Fe^{A \epsilon} \quad \text{case (iii)}
\]

\[
= 2(1 - \frac{2d_F\alpha}{\beta}) \quad \text{case (vi)}
\]

Let \( x \in \partial D \) and choose a local orthonormal frame field \( \{e_1, \ldots, e_{m-1}, \nu\} \) on \( T_xM \) such that \( \{e_1, \ldots, e_{m-1}\} \) is an orthonormal frame field on \( T_x\partial D \). Since \( u|_{\partial D} \) is
constant, we get $du(e_i) = 0$, $i = 1, \cdots m - 1$ and $du(\frac{\partial}{\partial x_0}) = \langle \frac{\partial}{\partial x_0}, \nu \rangle > g du(\nu)$. Hence at $x$

$$S_{\Phi_f, u}(X, \nu) = r_{x_0} S_{\Phi_f, u}(\frac{\partial}{\partial x_0}, \nu)$$

$$= r_{x_0} F \left( \frac{\| u^* h \|^2}{4} \right) < \frac{\partial}{\partial x_0}, \nu \rangle > g - F' \left( \frac{\| u^* h \|^2}{4} \right) < \sigma_u(\frac{\partial}{\partial x_0}), du(\nu) > _g$$

$$= r_{x_0} < \frac{\partial}{\partial x_0}, \nu \rangle > g \left( F \left( \frac{\| u^* h \|^2}{4} \right) - F' \left( \frac{\| u^* h \|^2}{4} \right) \| u^* h \|^2 \right)$$

$$\leq r_{x_0} < \frac{\partial}{\partial x_0}, \nu \rangle > g (1 - 4F) F \left( \frac{\| u^* h \|^2}{4} \right) \leq 0$$

Hence $S_{\Phi_f, u}(X, \nu) \leq 0$ on $\partial D$. Since $u$ satisfies an $\Phi_f$-conservation law, by monotonicity formula

$$0 \leq K \int_D F \left( \frac{\| u^* h \|^2}{4} \right) dv_g \leq \int_D \langle S_{\Phi_f}, \nabla \theta_X \rangle dv_g = \int_{\partial D} S_{\Phi_f, u}(X, \nu) ds_g \leq 0$$

which implies that $u|_D$ is constant.

\[ \square \]

5. Generalized Chern type results

In this section, we deal with the following mean $F$-curvature type equation for $TN$-valued sections over on $(M, g)$:

$$u : (M, g) \rightarrow (N, h) \quad \text{with} \quad \text{div}_g \sigma_{F, u} = s$$

where $s : M \rightarrow u^{-1}TN$ is a $C^1$ section,

$$\sigma_{F, u} = F' \left( \frac{\| u^* h \|^2}{4} \right) \sigma_u$$

where $\sigma_u$ is the $\mathbb{R}$-valued 1-form on $M$ defined by

$$\sigma_u = \sum_{i=1}^{m} \langle du(\cdot), du(e_i) \rangle > _h du(e_i)$$

We observe that

$$\| \sigma_{F, u} \| \leq F' \left( \frac{\| u^* h \|^2}{4} \right) \| u^* h \|^2 \leq 4d_F F' \left( \frac{\| u^* h \|^2}{4} \right)$$

where $d_F$ is the $F$-degree. We recall that $(M, g)$ is said to have the doubling property if there exist a constant $D(M) > 0$ such that $\forall R > 0$, $\forall x \in M$,

$$\text{Vol}_g(B(x, 2R)) \leq D(M) \text{Vol}_g(B(x, R)).$$

**Theorem 5.1.** Let $(M, g)$ be a complete non-compact Riemannian manifold with doubling property. Let $u : (M, g) \rightarrow (N, h)$ be a $C^2$ map such that

$$\text{div}_g \sigma_{F, u} = s$$

off a bounded set $K \subset M$ where $s$ is a parallel $C^1$ section of $u^{-1}TN$ over $M$ i.e $\nabla s = 0$. Let $\phi : ]0, +\infty[ \rightarrow ]0, +\infty[ \quad \text{be an increasing function}$. Assume that

$$\sup_{x \in B(R)} \| s(x) \| \phi(r(x)) = o(R)$$
If
\[
\limsup_{R \to \infty} \frac{1}{\text{Vol}_g(B(R))} \int_{B(R)} \|\sigma_{F,u}\| \, dv_g < \infty
\]
then \( \inf_{x \in M \setminus K} \|s(x)\| = 0 \).

Proof. For any section \( t \) of \( u^{-1}TN \) over \( M \), let \( Z \) be the vector field on \( M \) defined by
\[
< Z, X >_g = < t, \sigma_{F,u}(X) >_h
\]
for all vector fields \( X \) on \( M \). We choose a local orthonormal frame field \( \{e_i\}_{i=1}^m \) on \( M \) and we compute
\[
\text{div}_g Z = \sum_{i=1}^m < \nabla_{e_i}^M Z, e_i >_g
\]
\[
= \sum_{i=1}^m e_i < Z, e_i >_g + \sum_{i=1}^m \left\{ < \tilde{\nabla}_{e_i} t, i, e_i, \sigma_{F,u} > + < t, \tilde{\nabla}_{e_i} \sigma_{F,u}(e_i) > \right\}
\]
\[
= < \sum_{i=1}^m \theta_{e_i} \wedge \tilde{\nabla}_{e_i} t, \sigma_{F,u} > + < t, \sum_{i=1}^m \tilde{\nabla}_{e_i} \sigma_{F,u}(e_i) >
\]
\[
= < \tilde{\nabla} t, \sigma_{F,u} > + < t, \text{div}_g \sigma_{F,u} >
\]
where \( \tilde{\nabla} \) is the induced connection on \( u^{-1}TN \) from \( \nabla^M \) and \( \nabla^N \) and \( \theta_{e_i} \) the dual 1-form of \( e_i \). Then for any bounded open set \( D \subset M \setminus K \) with smooth boundary \( \partial D \), we have
\[
\int_{\partial D} \langle t, \sigma_{F,u}(\nu) \rangle \, ds_g = \int_D \langle \tilde{\nabla} t, \sigma_{F,u} \rangle \, dv_g + \int_D < t, \text{div}_g \sigma_{F,u} > \, dv_g \quad (5.1)
\]
where \( \nu \) denotes the unit outward normal vector field on \( \partial D \). The formula (5.1) with \( t = \psi s \), where \( \psi \in C^2(M, \mathbb{R}^+) \), gives
\[
\int_D \psi \|s(x)\|^2 \, dv_g = - \int_D < d\psi \otimes s, \sigma_{F,u} > \, dv_g - \int_D \psi < \tilde{\nabla} s, \sigma_{F,u} > \, dv_g
\]
\[
+ \int_{\partial D} \psi < s, \sigma_{F,u}(\nu) > \, ds_g
\]
Since \( K \subset M \) is compact, choose a sufficiently large \( R_0 < R \) such that \( K \subset B(x_0, R_0) \). Let \( 0 \leq \psi \leq 1 \) be the cut-off function i.e \( \psi = 1 \) on \( \overline{B(x_0, R)} \), \( \psi = 0 \) off \( B(x_0, 2R) \), and \( |\nabla \psi| \leq \frac{C}{R} \). The formula (14) with \( D = B(x_0, 2R) \setminus B(x_0, R_0) \) implies
\[
\inf_{x \in M \setminus K} \|s(x)\|^2 \left( 1 - \frac{V(R)}{V(R_0)} \right) \leq \frac{C}{RV(R)} \int_{B(x_0, 2R) \setminus B(x_0, R_0)} \|s\| \|\sigma_{F,u}\| \, dv_g +
\]
\[
\frac{1}{V(R)} \int_{B(x_0, 2R) \setminus B(x_0, R_0)} \|\tilde{\nabla} s\| \|\sigma_{F,u}\| \, dv_g +
\]
\[
\frac{1}{V(R)} \int_{\partial B(x_0, R_0)} \|s\| \|\sigma_{F,u}(\nu)\| \, ds_g
\]
where \( V(R) = \text{Vol}_g(B(x_0, R)) \). Since \( \sup_{x \in B(R)} \| \nabla s(x) \| = o \left( \frac{1}{\phi(R)} \right) \), for each \( \eta > 0 \)
\[
\sup_{x \in B(R)} \| \nabla s(x) \| \leq \frac{\eta}{\phi(R)} \quad R \geq R_1
\]

By mean value inequality
\[
\| s(x) - s(x_0) \| \leq \frac{R}{\phi(R)}
\]
in \( B(x_0, R) \) for \( R \geq R_1 \). By doubling property, the inequality (15) become

\[
\inf_{x \in M \setminus K} \| s(x) \|^2 \leq C D(M) \left( \eta + \| s(x_0) \| \frac{\phi(2R)}{R} \right) \frac{1}{V(2R)} \int_{B(2R)} \frac{\| \sigma_{F,u} \|}{\phi(r(x))} dv_g + \frac{D(M)}{\eta V(2R)} \int_{\partial B(2R)} \frac{\| \sigma_{F,u} \|}{\phi(r(x))} dv_g + \frac{1}{V(R)} \int_{\partial B(x_0, R_0)} \| s \| \| \sigma_{F,u}(\nu) \| ds_g
\]

where \( D(M) \) is the constant in doubling property. Since \( (M, g) \) is complete, the doubling property implies that \( (M, g) \) has infinite volume \([9]\). Letting \( R \) to infinity, we get
\[
\inf_{x \in M \setminus K} \| s(x) \|^2 \leq \eta D(M)(C + 1) \limsup_{R \to \infty} \frac{1}{V(2R)} \int_{B(2R)} \frac{\| \sigma_{F,u} \|}{\phi(r(x))} dv_g
\]
and \( \inf_{x \in M \setminus K} \| s(x) \| = 0 \) since \( \eta \) is arbitrary. \( \square \)

The following is an analogous of Chern’s result \([4]\):

**Corollary 5.2.** Let \( (M, g) \) be a complete non-compact Riemannian manifold with doubling property and \( (N, h) \) be a Riemannian manifold. Let \( u : (M, g) \to (N, h) \) be a \( C^2 \) map such that \( \text{div}_g \sigma_{F,u} = s \) where \( s \) is a constant section of \( u^{-1}TN \). Let \( \phi : ]0, +\infty[ \to ]0, +\infty[ \) be an increasing function such that \( \phi(R) = o(R) \). If
\[
\limsup_{R \to \infty} \frac{1}{V(R)} \int_{B(R)} \frac{\| \sigma_{F,u} \|}{\phi(r(x))} dv_g < \infty
\]
then \( u \) is a stationary map for the functional \( \Phi_F \).

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