The Mutating Contact Process: Model
Introduction and Qualitative Analysis of Phase
Transitions in its Survival

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Abstract
We introduce and study the mutating contact process, a variant of the
multitype contact process, where one type mutates at a constant rate to
the other type. We prove that on \( \mathbb{Z} \) a single mutant cannot survive while
on \( \mathbb{T}_d \) there are distinct weak survival and extinction of a single mutant
phases, yet the limiting distribution concentrates on configurations with
no mutants of the first type for any values of the parameters.

1 Introduction and Results
In this paper we wish to introduce and study a variant of the multitype contact
process introduced by Neuhauser in [6]. In that model, two populations, say
mutation strains of a virus, are in competition for space on a certain graph (the
lattice \( \mathbb{Z}^d \) or the regular tree \( \mathbb{T}_d \) most usually): each member of population
\( i \) reproduces to its nearest neighbors at rate \( \lambda_i (i = 1, 2) \) if they are unoccupied
and dies at rate 1.

The variant which we wish to study henceforward is the mutating contact
process: in it each individual of the population reproduces at rate 1 to nearest
neighbors if they are unoccupied, dies at rate \( \delta \) and mutates to a completely
novel strain at rate \( \mu \). The scope of our discussion will be limited to the question
of the survival of a single strain: under what conditions does a single strain
overcome death and mutation pressures to persist in existence on the chosen
graph. To this end it suffices to consider the model with only two types: 1. A
specific strain. 2. All other strains.

With this in mind, the model description is as follows. The system is de-
scribed by an evolving configuration \( \xi \in \{0, 1, 2\}^S \) where \( \xi (a) = 0 \) if the site
\( a \in S \) is vacant, \( \xi (a) = 1 \) if it is occupied by a mutant of the specific strain
whose survival we study and \( \xi (a) = 2 \) if \( a \) is occupied by a mutant of any other

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strain. Denote by \( n(x, \xi, i) \) the number of neighbors of \( x \) that are of type \( i \) in the configuration \( \xi \). That is,

\[
n(x, \xi, i) = \sum_{y \sim x, \xi(y) = i} 1, \quad i = 1, 2.
\]

The mutating contact process \( \xi_t \) is the Feller process on \( \{0, 1, 2\}^S \) which makes local transitions at site \( x \in S \) according to the following rules:

1. \( i \to 0 \) at rate \( \delta, i = 1, 2 \).
2. \( 0 \to i \) at rate \( n(x, \xi, i), \quad i = 1, 2 \).
3. \( 1 \to 2 \) at rate \( \mu \).

The multitype contact process introduced by Neuhauser [6] follows similar rules of evolution: It is a Feller process defined on \( \{0, 1, 2\}^S \) evolving at \( x \in S \) according to the rules:

1. \( i \to 0 \) at rate \( 1, i = 1, 2 \).
2. \( 0 \to i \) at rate \( \lambda_i n(x, \xi, i), \quad i = 1, 2 \).

The variations in our model being the introduction of the third evolution rule, modelling mutation, the requirement that all strains reproduce at the same constant rate 1, modelling selective neutrality, and the introduction of a varying death rate instead of a varying birth rate, which is more amenable to our analysis as will be seen in what follows.

The one-type contact process (obtained for instance by initializing the two-type process with only one of the types) has been studied extensively, see [5] for a review on the main results. However, there are relatively few papers on the multitype contact process, with [6, 2, 1] as representative examples. Further, a review paper by Durrett [3] summarized the results and open questions in spatial competition models.

In this paper we wish to study the phases of survival of the mutating contact process. On \( \mathbb{Z} \), the situation is straightforward - a single mutant always dies while the entire process behaves exactly as a single type contact process. The first of these assertions is proved in the following theorem. In what follows, we will denote the set of all vertices occupied by type \( i \) in the mutating contact process by \( ^i\xi_t \), that is, \( ^i\xi_t = \{ x \in S | \xi_t(x) = i \} \).

**Theorem 1.** for the mutating contact process on \( S = \mathbb{Z} \), with any finite initial configuration of 1s, and any choice of death and mutation rates, \( \delta, \mu > 0 \), no single mutant can survive:

\[
P(^i\xi_t \neq \emptyset, \forall t > 0) = 0.
\]
This result builds upon the results of Andjel et. al. [1] in which it is shown that for the multitype contact process with equal birth rates for the two strains, initialized with a finite number of 1s bounded from both sides by an infinite number of 2s, the survival of the 1s is impossible in the same manner as stated above. In the proof of Theorem 1 we will in effect reduce our problem to theirs.

The behaviour of the model on $\mathbb{T}_d$ is more involved - while there are distinct weak and strong survival phases for the one type contact process, it turns out that there is a weak survival phase for a single mutant in the mutating contact process as well, though strong survival is an impossibility, these are the results of the next theorem and the two propositions that follow it.

**Theorem 2.** For the mutating contact process on $S = \mathbb{T}_d$ or $S = \mathbb{Z}^d$ with any initial configuration, and any choice of $\delta, \mu > 0$, type 1 mutants do not survive in the limiting distribution, that is, for all $x \in S$,

$$P(\xi_t(x) = 1) \rightarrow 0 \text{ as } t \rightarrow \infty.$$  

The proof of this theorem follows lines similar to the proof of Theorem 2 in Cox and Schinazi [2], which states that for the multitype contact process on $\mathbb{T}_d$ or $\mathbb{Z}^d$ at the phase of strong survival of 2s, there can be no coexistence in the limiting distribution.

The main idea behind the proof is that for $\xi_t(x) = 1$ to occur, it must have descended from an initial 1 without mutating. As $t \rightarrow \infty$ it becomes very likely that the ancestor of $x$ has mutated along its path, and so $\xi_t(x) = 1$ becomes unlikely.

We will further show that with small enough $\delta$ and $\mu$, weak survival of 1s on $\mathbb{T}_d$ is a possibility, this we state as a proposition and its proof follows from observing an embedded supercritical Galton-Watson tree.

**Proposition 3.** for the mutating contact process on $S = \mathbb{T}_d$ with any initial finite configuration such that $\{x : \xi_0(x) = 1\} \neq \emptyset$, a single mutant can survive in the weak sense,

$$P(1^{\xi_t} \neq \emptyset, \forall t > 0) > 0,$$

if $\delta + \mu$ is sufficiently small.

The final results which is needed for drawing a qualitative phase diagram of the mutating contact on $S = \mathbb{T}_d$ is that if $\delta + \mu$ is sufficiently large then even weak survival of 1s is impossible. This is the next proposition, the proof of which follows from a coupling with a dominating one type contact process which also does not survive weakly.

**Proposition 4.** For the mutating contact process with on $\mathbb{T}_d$ with any initial finite configuration of 1s, a single mutant dies out,

$$P(1^{\xi_t} \neq \emptyset, \forall t > 0) = 0,$$

if $\delta + \mu > \delta^*$ where $\delta^*$ is the upper critical value.
Figure 1.1: Phase diagram of the mutating contact process on $\mathbb{T}_d$. We show that in the area $A$ (below line $l_1$) there is weak survival of a single mutant, and in Area $C$ (above line $l_2$) there is extinction of a single mutant. In between the two lines (area $B$) a single phase transition must occur, depicted as the dashed curve $b$. The entire process survives strongly if $\delta < \delta^*$, dies out if $\delta < \delta^*$ and survives weakly in between. Note that $l_2$ indeed intersects the $\delta$ axis at $\delta^*$ as shown, while the relation of $\delta^*$ to the weak survival phase is unknown.

We wish to comment that since the process is monotone (an instance of the model with smaller values of $\delta$ or $\mu$ can be attained by thinning the corresponding Poisson processes, see the graphical construction in the next section for the details), there can be only a single phase transition. Finally, note that the support of $\xi_t$ on $\mathbb{T}_d$ follows the evolution of the single type contact process with death rate $\delta$ and birth rate 1, as mutation events are muted through this projection. Therefore, while the survival of a single strain depends on both $\delta$ and $\mu$, the survival of the entire species depends only on the value of $\delta$ in relation to the two critical values $\delta^* < \delta^*$, which will be defined in the next section.

2 Previous Results and Model Construction

Before we continue, it will be useful to recall basic results concerning the one-type contact process. Please note that in the mutating contact process, the free parameter is the death rate $\delta$, while the birth rate is set at 1. It is usually the opposite case in previous results: the death rate is set to 1 while the birth rate is a free parameter $\lambda$. Through a rescaling of time by a factor of $1/\delta$, one may transform the first into the second with $\lambda = 1/\delta$ while maintaining equality of the law of the one-type contact process. Thus, most results transfer directly, mutatis mutandis. For convenience we will restate the results directly translated through this transformation.

We will denote the set of occupied vertices of a one type contact process on
S with death rate $\delta$ and birth rate 1 by $\zeta_t$, and if it is initialized at a single site $x$, then we will denote it $\zeta^x_t$. With this notation we will define two critical values,

$$\delta_* = \sup \{ \delta : P(\zeta^x_t \neq \emptyset \; \forall t > 0) > 0 \},$$
$$\delta^* = \sup \{ \delta : P(x \in \zeta^x_t \; i.o \; as \; t \to \infty) > 0 \}.$$  

These critical values partition the survival of the contact process into the following phases:

1. If $\delta < \delta^*$ then $P(x \in \zeta^x_t \; i.o \; as \; t \to \infty) > 0$ and we say that the process survives strongly.
2. If $\delta^* < \delta < \delta_*$ then $P(x \in \zeta^x_t \; i.o \; as \; t \to \infty) = 0$ but $P(\zeta^x_t \neq \emptyset \; \forall t > 0) > 0$ and we say that the process survives weakly.
3. If $\delta_* < \delta$ then $P(\zeta^x_t \neq \emptyset \; \forall t > 0) = 0$ and we say that the process dies out.

It has been proven that on $S = \mathbb{Z}^d$, $\delta^* = \delta_* < \infty$, so that weak survival is an impossibility and that for $S = T_d$, $0 < \delta^* < \delta_* < \infty$ and the phases are right continuous at the critical values, See chapter I.4 in Ligget [5].

The complete convergence theorem states that no matter the initial configuration $\zeta_t$ converges weakly to a mixture of the point mass on the empty configuration, $\delta_\emptyset$, and another measure called the upper invariant measure, denoted by $\bar{\nu}$. In details, denoting by $\alpha$ the probability of survival (which obviously depends on the initial configuration), that is, $\alpha = P(\zeta^x_t \neq \emptyset \; \forall t > 0)$, then the complete convergence theorem states that, $\zeta_t \Rightarrow (1 - \alpha) \delta_\emptyset + \alpha \bar{\nu}$. Confer chapter I.4 of Ligget [5] for the details.

Model Construction.

The graphical construction of the contact process is a method of realizing the contact process through the use of independent Poisson processes, introduced by Harris [4]. The construction applied to the mutating contact processes is as follows. For each vertex $x \in S$ define two Poisson processes independent of all other processes:

1. The first of these will have constant rate $\delta$ and will be appropriately named the death process at $x$. Denote its arrival times by $\{\delta^x_n : n \geq 1\}$.
2. The second will have constant rate $\mu$ and will be named the mutation process at $x$. Denote its arrival times by $\{\mu^x_n : n \geq 1\}$.

Further, for each ordered pair of neighboring vertices, $(x, y) \in S \times S$, define independently another Poisson process, with rate 1, and name it the infection process from $x$ to $y$. Denote its arrival times by $\{(x \to y)_n : n \geq 1\}$.
For some vertices \((x, y) \in S\) and times \(s \leq t\), we say that there is an open 1-path from \((x, s)\) (read - \(x\) at time \(s\)) to \((y, t)\) if one can connect the two space-time values by use of the timelines attached to any vertex and outgoing arrows without going back in time and without crossing any arrival of the death or mutation processes.

More formally, there is open 1-path from \((x, s)\) to \((y, t)\) if there exists a sequence of space-time values \(\{(x_i, t_i)\}_{i=0}^{n}\) such that:

1. The sequence starts at \((x_0, t_0) = (x, s)\) and ends with \((x_n, t_n) = (y, t)\),
2. The time values are increasing, \(t_0 < t_1 < ... < t_n\),
3. Consecutive vertices are neighbors, \(x_i \sim x_{i-1}\), \(i \geq 1\).
4. For \(0 < i < n\), the time \(t_i\) is an arrival of the infection process from \(x_{i-1}\) to \(x_i\), \(t_i = (x_{i-1} \to x_i)_k\) for some \(k\).
5. For \(0 \leq i < n\), there is no arrival of the death process at \(x_i\) during \((t_i, t_{i+1})\).
6. For \(0 \leq i < n\), there is no arrival of the mutation process at \(x_i\) during \((t_i, t_{i+1})\).

Similarly we will say that there is an open 2-path from \((x, s)\) to \((y, t)\) if an appropriate sequence obeying 1-5 of the above exists. By this definition, every 1-path is immediately also a 2-path.

Given an initial configuration, one may evolve it according to the above graphical construction by propagating initial 1s through open 1-paths and initial 2s through open 2-paths. One must take care to change 1s into 2s whenever they encounter a mutation arrival at their site, and continue to propagate the resulting 2s through open 2-paths.

The resulting configuration at time \(t\) obeys the law of the mutating contact process and therefore may be rightly denoted \(\xi_t\). We will omit the proof of this assertion, and defer to Harris [4].

**Known Results**

Two results from [1] are used in the proof of Theorem 1. We bring them here for the convenience of the reader, while rephrasing them with our notation. The first is Corollary 2.4.

**Corollary.** Suppose \(\delta < \delta_*\) and let \(A\) be an infinite subset of the positive integers, \(\mathbb{Z}_+\), then there exists a constant \(v > 0\), such that for any \(v' < v\) there exists some \(x \in A\) with an open 2-path which is to the right of the line \(\{(v't, t) : t \geq 0\}\).

Note that \(v\) is the speed of propagation of the one-type contact process, and is dependent on (and negatively correlated to) the death rate. The second result required is Theorem 3.9.
Theorem. Consider the two-type contact process \( \eta_t \) on \( \mathbb{Z} \) with initial configuration \( \eta_0 \) such that there is a finite number of 1s bounded by an infinite number of 2s on both sides. Then

\[
P (1^t \xi_t \neq \emptyset, \forall t > 0) = 0.
\]

The last result we wish to cite is Proposition 1 from [2], which will be used in the proof of Theorem 2.

Proposition. Assume \( S \) is \( \mathbb{Z}^d \) or \( T^d_\epsilon \) and \( \bar{\nu} \) is the upper invariant measure of the one-type contact process with death rate \( \delta < \delta_* \). Define

\[
\delta_L = \sup_{|A| > L} \bar{\nu} (\zeta : \zeta \cap A = \emptyset).
\]

Then \( \delta_L \to 0 \) as \( L \to \infty \).

3 Proof of Results

In this section we will prove Theorems 1 and 2, and propositions 3 and 1.

Proof of Theorem 1. We will consider the case where \( \delta < \delta_* \) such that the one-type contact process has a positive probability of survival. As well, it suffices to consider all probabilities on the event of species survival. The certainty of the extinction of a single mutant is trivial in the other cases.

Denote the leftmost site occupied by a 1 at time \( t \) by \( L_t = \min \{ x : \xi_t (x) = 1 \} \) and the rightmost site by \( R_t = \max \{ x : \xi_t (x) = 1 \} \). \( L_t \) and \( R_t \) both undergo mutation at a constant rate (without relation to the specific spatial location) so that if \( s_0 \) is an arrival time of such a mutation, then a site to the right of \( R_{s_0} \), \( \lim_{s \to s_0^-} R_s \), will be occupied by a 2 at time \( s_0 \). Since there are w.p. 1 infinitely many such arrivals, the set \( 2^t \xi_t \cap (R_t, \infty) \) is non-empty infinitely many times almost surely.

Corollary 2.4 of [1] implies that there is a positive probability, \( p_0 \), such that an infinite open path starting from \( (0, 0) \) lying on the right of the line \( \{(kt, t) : t \geq 0\} \) for some \( k > 0 \) exists. From the time and space homogeneity of the graphical construction it follows that this statement applies to any starting point \( (x, s) \) with regards to the line \( \{(x + kt, s + t) : t \geq 0\} \). Thus, combined with the previous results, almost surely there exists such a path with \( \xi_t (x) = 2 \).

By the same argument regarding \( L_t \), almost surely there exists an infinite open path lying on the left of \( \{(x - kt, s + t) : t \geq 0\} \) such that its beginning \( \xi_\ast (x) = 2 \). It follows that w.p. 1 there exists a time \( t_0 \) such that for all \( t \geq t_0 \), the set of all 1s, \( \xi_t \), is contained in an interval \( (z (t), y (t)) \) where \( \xi_t (z (t)) = \xi_t (y (t)) = 2 \).

Now, we argue the evolution of \( \xi_{t_0} \) inside the interval \( (z (t), y (t)) \) is independent the configuration outside of the interval: for the purpose of determining the value of \( \xi_t \) on \( (z (t), y (t)) \), any open path from outside the interval into it, may as well be truncated to the time-space point where it crosses \( z (t) \) or \( y (t) \).
since these endpoints are already occupied by a 2 which will propagate along the rest of the path. Thus we may assume
\[ \xi_{t_0} \left((-\infty, z(t_0)] \cup [y(t_0), \infty)\right) = 2 \]
without affecting the evolution of 1s in the model.

At this point we will couple the mutating contact process with an instance of the standard two-type contact process \( \eta_t \) with initial configuration \( \eta_{t_0} = \xi_{t_0} \) such that the following monotonicity property holds,
\[ 1_{\xi_t} \subseteq \{ x \in \mathbb{Z} : \eta_t(x) = 1 \} \text{ for all } t \geq t_0. \]
This coupling is attained by applying the same graphical construction to the processes from the time \( t_0 \), while ignoring the mutation arrivals when evolving the two-type process. It is evident that for such a coupling, starting from the same initial configuration, the monotonicity holds: if \( \xi_t(x) = 1 \) then an open path with no mutation arrivals from an initial 1 to \((x,t)\) must exist. The same open path implies \( \eta_t(x) = 1 \). An application of Theorem 3.9 of [1] on this instance of the two-type process with its initial conditions yields
\[ P \left( \{ x : \eta_t(x) = 1 \} \neq \emptyset, \forall t > 0 \right) = 0 \]
which by the monotonicity property also gives
\[ P \left( 1_{\xi_t} \neq \emptyset, \forall t > 0 \right) = 0, \]
as required.

Proof of Theorem 2. We note that \( 1_{\xi_t} \) can be coupled to a one type contact process with birth rate 1 and death rate \( \delta + \mu \) by using the same graphical construction as ours and treating mutation arrivals as death arrivals. We will denote this coupled process by \( 1_{\zeta_t} \). In this coupling it is true \( 1_{\xi_t} \subseteq 1_{\zeta_t} \) for the same initial configuration of 1s, as both propagate through the same open 1-paths with the added restriction for the first that some vertices along those open paths may already be occupied by 2s. Thus, if \( \delta + \mu > \delta^* \) then \( P (\xi_t(x) = 1) \rightarrow 0 \) follows immediately from the lack of strong survival of the coupled one-type process.

If it is true that \( \delta + \mu < \delta^* \), the proof becomes more complicated and is inspired by the proof of Theorem 2 in [2]. Before we begin the proof in this case, we may as well discuss the matter conditioned on the event of weak survival of type 1, as well as assuming that weak survival of type 1 may occur with positive probability. The result is obvious in the other cases. In other words, throughout this proof we consider all probabilities to be conditional on the event \( \{ 1_{\xi_t} \neq \emptyset, \forall t > 0 \} \). On this event, it also follows that \( |2_{\xi_s}| \neq \emptyset \) infinitely often, as type 1 mutates into type 2 at a constant rate. Since individuals die at a constant rate, the descendants of any single individual must either die out or
tend to infinity as \( t \to \infty \). This follows from the fact that with some constant positive probability, \( M \) individuals (or less) will die before propagating to other vertices in a single time unit. If at an infinite and unbounded set of times \( t \) the number of descendants is bounded above by some fixed \( M > 0 \), then as \( t \to \infty \), the (at most) \( M \) individuals will die out with probability tending to 1 as a result of geometric sampling of the aforementioned event. Thus, under the assumption that \( \{ \xi_t \neq \emptyset, \forall t > 0 \} \) occurs, it follows that for fixed \( M > 0 \)

\[
P(\left| \xi_t \right| \geq M) \to 1 \text{ as } t \to \infty,
\]

which also implies that for fixed \( L > 0 \),

\[
P(\left| \xi_t \right| \geq L) \to 1 \text{ as } t \to \infty
\]

since 1s mutate to 2s at a constant rate.

Let \( x \in S \) be some vertex. We wish to show that \( P(\xi_t(x) = 1) \to 0 \) as \( t \to \infty \). Using the previous result, we may restrict the events in question to \( \{ \xi_{t+u}(x) = 1, \left| \xi_u \right| \geq L \} \), which are subsets of \( \{ \xi_{t+u}(x) = 1 \} \) and show

\[
\lim_{u \to \infty} \limsup_{t \to \infty} P(\xi_{t+u}(x) = 1, \left| \xi_u \right| \geq L) = 0.
\]

To this end we need to express the event

\[
\{ \xi_{t+u}(x) = 1, \left| \xi_u \right| \geq L \}
\]

with the notation of the ancestor process, defined in a similar fashion to [2]. Let an ancestor configuration \( \hat{\xi} \) be a (possibly empty) sequence of pairs \(((a_1, b_1), ..., (a_n, b_n))\) for some \( n \geq 1 \), where each \( a_j \in S \) and \( b_j \in \{1, 2\} \). The vertices \( a_j \) will denote the possible ancestors in descending order of precedence, and \( b_j \) will indicate whether a certain possible ancestor is viable only if it is occupied by a 2 (2-viable) or if it is viable with no regard to its type (in which case we will say it is 1-viable).

In more details, we define the ancestor process \( \hat{\xi}_{s}^{(x,t)} \) to be the list of possible ancestors of \((x,t)\) at time \( t - s \) for some \( s \in [0,t] \). \( \hat{\xi}_{s}^{(x,t)} \) is a Markov process defined recursively as follows: At time \( t \) the only possible ancestor of \((x,t)\) is \( x \) itself, so that \( \hat{\xi}_{0}^{(x,t)} = ((x,1)) \). Now, under the assumption that \( \hat{\xi}_{n}^{(x,t)} \) has been properly defined for all \( u \leq s \):

1. If \( \hat{\xi}_{s}^{(x,t)} \) is empty, define \( \hat{\xi}_{v}^{(x,t)} \) to be empty as well for all \( v \in (s,t] \).

2. Otherwise, if \( \hat{\xi}_{s}^{(x,t)} = ((a_1, b_1), ..., (a_n, b_n)) \), then observing the graphical construction, look for the most recent past event affecting any of the \( a_j \) (an arrival of any of the Poisson processes involved). If no such event occurs till time 0 then define \( \hat{\xi}_{v}^{(x,t)} = \hat{\xi}_{s}^{(x,t)} \) for all \( v \in (s,t] \).

3. If Such an event occurs, denote its time of occurrence by \( r \) and define \( \hat{\xi}_{v}^{(x,t)} = \hat{\xi}_{s}^{(x,t)} \) for all \( v \in (s,r] \). \( \hat{\xi}_{v}^{(x,t)} \) itself is defined in accordance with the specific event that occurred.
(a) If the event is an arrow pointing from some \( a \in S \) to \( a_j \) then insert \((a,b)\) into the sequence \( \hat{\xi}^{(x,t)}(x,t) \) immediately after each appearance of \( a_j \) and set the value of the type viability \( b \) to be identical to that of the preceding appearance of \( a_j \) (i.e. if \( b_j = x \) set \( b = x \) only for the following insertion).

(b) If the event is a mutation arrival at \( a_j \), set \( b_j = 2 \) for all appearances of \( a_j \) in the sequence (if it isn’t set already).

(c) If the event is a death arrival, delete all appearances of \( a_j \) from the sequence. If all the \( a_j \) have been deleted, set \( \hat{\xi}^{(x,t)}(x,t) \) to be empty and proceed as in 1.

This algorithm defines \( \hat{\xi}^{(x,t)}(x,t) \) in such a way that \( \xi_t(x) \) takes the value of the first vertex in \( \hat{\xi}^{(x,t)}(x,t) \) which is occupied at time \( t-s \) (conditional on its 1-viability, if the vertex is occupied by a 1). The ancestor process \( \hat{\xi}^{(x,t)}(x,t) \) moves backwards in time and so can only be defined up to time \( s=t \), that is for bounded intervals. To overcome this difficulty, using the time reversibility of the underlying Poisson processes, we can switch to forward time as in [2, 6], and define a variation of the process \( \xi^{(x,t)}(x,t) \) for all \( s \geq 0 \) such that the laws of \( \hat{\xi}^{(x,t)}(x,t) \) and \( \xi^{(x,t)}(x,t) \) agree for all \( s \leq t \). Henceforward, if the ancestral process is strated at \((x,0)\), we will denote it \( \hat{\xi}_x \).

We may now return to discuss the events \( \{\xi_{t+u}(x) = 1, |^{2}\xi_u| \geq L\} \). First assume that \( \xi_u = \eta \) where \( \eta \) is some non-random configuration of 1s and 2s on \( S \) with \( |^{2}\eta| \geq L \). We will later use the law of total probability to generalize the following argument. Note that \( \xi_{t+u}(x) = 1 \) if and only if the following event, denoted \( OC\left(\xi^{(x,t+u)}_{t},\eta\right) \), occurs: the first occupied vertex in the ancestor list \( \xi^{(x,t+u)}_{t} \) is 1-viable and is actually occupied by a 1 in the configuration \( \eta \). Since \( \xi^{(x,t+u)}_{t} \) has the same law as \( \xi^{x}_{t} \) it follows that

\[
P(OC\left(\xi^{(x,t+u)}_{t},\eta\right)) = P(OC\left(\xi^{x}_{t},\eta\right)).
\]

(3.3)

Define the list of first priority 2-viable ancestors, \( A^x_t \) to be empty if the first ancestor in the list \( \xi^x_t \) is 1-viable, otherwise define it to be the set of all of the 2-viable ancestors that appear on the list before the first 1-viable ancestor.

Now, from these definitions we can deduce that that \( OC\left(\xi^x_{t},\eta\right) \) does not co-occur with \( A^x_t \cap \eta \neq \emptyset \) and that \( OC\left(\xi^x_{t},\eta\right) \subseteq \{\xi^x_t \neq \emptyset\} \). Thus,

\[
P(OC\left(\xi^x_{t},\eta\right)) \leq P(\xi^x_t \neq \emptyset, A^x_t \cap \eta = \emptyset).
\]

(3.4)

Summing over all configurations \( \eta \) and using the independence of disjoint
time-space regions we get

\begin{equation}
P\left(\xi_{t+u}(x) = 1, |\xi_u| \geq L \right)
= \int_{|\eta| \geq L} P\left(\xi_u \in d\eta \right) P\left(OC\left(\xi^{(x,t+u)}_t, \eta\right)\right)
= \int_{|\eta| \geq L} P\left(\xi_u \in d\eta \right) P\left(OC\left(\xi^{\ast}_t, \eta\right)\right)
\leq \int_{|\eta| \geq L} P\left(\xi_u \in d\eta \right) P\left(\xi^{\ast}_t \neq \emptyset, A^*_t \cap 2^|\eta| = \emptyset \right) .
\end{equation}

In what follows it will be shown that for large \( t \) and \( u \), very rarely do the events \( \{\xi^x_t \neq \emptyset\} \) and \( \{A^*_t \cap 2^|\eta| = \emptyset\} \) co-occur. For this purpose, we define a combination of events that render a location only 2-viable. Call a space-time point \((y,s)\) good if a mutation arrival occurs at \( y \) during \((s,s+1)\) and no other event affecting \( y \) occur during this time frame. Then \( p_0 = P((y,s) \text{ is good}) \) is a positive constant that does not depend on \((y,s)\). Considering a site \( x \) with non-empty ancestry at time \( s \): \( \xi^x_s = ((a_1(s), b_1(s)), \ldots, (a_n(s), b_n(s))) \), if it occurs that \((a_1(s), s)\) is good then \( a_1(s) \) remains the primary ancestor at least until time \( s+1 \) because no deaths occur and it becomes 2-viable because of the mutation: \((a_1(s+1), b_1(s+1)) = (a_1(s), 2)\). Therefore, for all \( v \geq s+1 \), The forward time ancestry list of \( x \), \( \xi^x_v \), begins with the list of ancestors of \( a_1(s+1): \xi^{(a_1(s+1),s+1)}_v \), and every ancestor from this list is 2-viable for \( x \).

We will now turn to the geometric sampling argument mentioned at the beginning of the proof. Fix some \( T > 0 \) and define \( s_k = k(T+1) \) , \( t_k = s_k + 1, k \geq 0 \). Let

\begin{equation}
R = \inf \left\{ k | \xi^x_{s_k} \neq \emptyset, (a_1(s_k), s_k) \text{ is good and } \xi^{(a_1(t_k),t_k)}_{s_{k+1}} \neq \emptyset \right\} .
\end{equation}

\( R \) is the first time that \((a_1(s_k), s_k)\) is good and that \( a_1(s_k) \), which is also \( a_1(t_k) \), has an ancestry that lasts at least \( T \) time units further than \( t_k \). To see that \( R \) is dominated by a Geometric r.v., consider from the self-duality of the contact process that \( P\left(\xi^{(a_1(t_k),t_k)}_{s_{k+1}} \neq \emptyset\right) \geq \alpha \), where \( \alpha \) is the probability of weak survival in the coupled contact process, \( \xi_t \) with initial configuration \( a_1(t_k) = 2 \). Also consider that the events \( \{(a_1(s_k), s_k) \text{ is good}\} \) and \( \{\xi^{(a_1(t_k),t_k)}_{s_{k+1}} \neq \emptyset\} \) are determined by disjoint regions of space-time and are thus independent. Deduce that

\begin{equation}
P\left( (a_1(s_k), s_k) \text{ is good and } \xi^{(a_1(t_k),t_k)}_{s_{k+1}} \neq \emptyset \right)
= P\left( (a_1(s_k), s_k) \text{ is good } \right) P\left( \xi^{(a_1(t_k),t_k)}_{s_{k+1}} \neq \emptyset \right)
\geq p_0 \alpha .
\end{equation}
Now, this inequality holds for any \( k \), so that by iteration and the Markov property,
\[
P\left( \xi_{s_k+1}^x \neq \emptyset \text{ and } R > k \right) \leq (1 - \alpha p_0)^k,
\]

consequently, for any \( k_0 > 0 \) and \( t > s_{k_0+1} \) we may partition the event
\[
\left\{ \xi_t^x \neq \emptyset, A_t^x \cap 2\eta = \emptyset \right\}
\]

by the value of \( R \):
\[
P\left( \xi_t^x \neq \emptyset, A_t^x \cap 2\eta = \emptyset \right) \leq (1 - \alpha p_0)^{k_0} + \sum_{k=0}^{k_0} P\left( R = k, \xi_t^x \neq \emptyset, A_t^x \cap 2\eta = \emptyset \right).
\]

We will further partition the events under the sum by the following, defined for all \( a \in S \),
\[
G_k(a) = \left\{ R > k - 1, \xi_{s_k}^x \neq \emptyset, a_1(s_k) = a \text{ and } (a, s_k) \text{ is good} \right\}. \tag{3.10}
\]

\( G_k(a) \) is the event that the first time that \( (a_1(s_m), s_m) \) is good is attained at \( m = k \) and it is attained at \( a_1(s_m) = a \). Note that for all \( k \leq k_0 \) and on the event \( G_k(a) \), the list of first priority 2-viable ancestors of \( x \) at time \( t \), \( A_t^x \), contains the vertices listed in \( \xi_t^{(a,s_k)} \) because \( (a, s_k) \) is good, so that its ancestors are first in the ancestry list of \( x \), and they must be 2-viable for \( x \) because of the mutation arrival that must occur. Denoting the actual vertices listed (instead of the pairs \( (a, b) \)) by \( \text{supp}\left( \xi_t^{(a,s_k)} \right) \), we incorporate this relation into the partition, and get the following inequality,
\[
P\left( R = k, \xi_t^x \neq \emptyset, A_t^x \cap 2\eta = \emptyset \right) \leq \sum_{a \in S} P\left( G_k(a) \cap \left\{ \xi_{s_k+1}^{(a,t_k)} \neq \emptyset, \text{supp}\left( \xi_t^{(a,t_k)} \right) \cap 2\eta = \emptyset \right\} \right)
\]

and apply the independence of disjoint space-time regions on each summand:
\[
= \sum_{a \in S} P\left( G_k(a) \cap \left\{ \xi_{s_k+1}^{(a,t_k)} \neq \emptyset, \text{supp}\left( \xi_t^{(a,t_k)} \right) \cap 2\eta = \emptyset \right\} \right) \tag{3.11}
\]

\[
= P\left( G_k(a) \right) P\left( \xi_{s_k+1}^{(a,t_k)} \neq \emptyset, \text{supp}\left( \xi_t^{(a,t_k)} \right) \cap 2\eta = \emptyset \right). \tag{3.12}
\]

Now, since the set \( \text{supp}\left( \xi_t^{(x,s)} \right) \) is not affected by mutation arrivals in the graphical construction (they only change the second term of the pairs in \( \xi_t^{(a,t_k)} \)), it follows the evolution of a time-reversal of the regular one-type contact process,
and therefore has the same law as \( 2 \xi_{t-\xi0} \) initialized at \( 2 \xi0 = \{ x \} \) and null elsewhere. From this we deduce the following, while noting that \( s_k + 1 - t_k = T \) and that \( t > s_{k+1} + 1 > t_k \).

\[
P(\xi^{(a, t_k)}_{s_{k+1}} \neq \emptyset, \text{ sup} \left( \xi^{(a, t_k)}_t \right) \cap 2 \eta = \emptyset) = P \left( |2 \xi_T| \geq 1, 2 \xi_{t-k} (2 \eta) \equiv 0 \right)
\]

(3.13)

This last event may occur in two distinct ways: either the entire contact process dies out between time \( T \) and time \( t - t_k \), or the process lives till time \( t - t_k \) but does not appear on the vertices in \( 2 \eta \). Denote the probability of the first by \( \rho (T) \) and observe that by the complete convergence theorem, \( \rho (T) \to 0 \) as \( T \to \infty \). We get the following partition,

\[
P \left( |2 \xi_T| \geq 1, 2 \xi_{t-k} (2 \eta) \equiv 0 \right) = \rho (T) + P \left( |2 \xi_{t-k} \geq 1, 2 \xi_{t-k} (2 \eta) \equiv 0 \right)
\]

(3.14)

Combining (3.9) to (3.14) we have,

\[
P \left( \xi^0_t \neq \emptyset, A^\infty_t \cap 2 \eta = \emptyset \right)
\]

\[
\leq (1 - \alpha p_0)^{k_0} + \sum_{k=0}^{k_0} P \left( \text{R = k, } \xi^0_t \neq \emptyset, A^\infty_t \cap 2 \eta = \emptyset \right)
\]

\[
\leq (1 - \alpha p_0)^{k_0} + \sum_{k=0}^{k_0} \sum_{a \in S} P \left( \text{G}_k (a) \cap \left\{ \xi^{(a, t_k)}_{s_{k+1}} \neq \emptyset, \text{ sup} \left( \xi^{(a, t_k)}_t \right) \cap 2 \eta = \emptyset \right\} \right)
\]

\[
= (1 - \alpha p_0)^{k_0} + \sum_{k=0}^{k_0} \sum_{a \in S} P \left( \text{G}_k (a) \right) P \left( \xi^{(a, t_k)}_{s_{k+1}} \neq \emptyset, \text{ sup} \left( \xi^{(a, t_k)}_t \right) \cap 2 \eta = \emptyset \right)
\]

\[
= (1 - \alpha p_0)^{k_0} + \sum_{k=0}^{k_0} \sum_{a \in S} P \left( \text{G}_k (a) \right) P \left( |2 \xi_T| \geq 1, 2 \xi_{t-k} (2 \eta) \equiv 0 \right)
\]

\[
\leq (1 - \alpha p_0)^{k_0} + \sum_{k=0}^{k_0} \sum_{a \in S} P \left( \text{G}_k (a) \right) \left( \rho (T) + P \left( |2 \xi_T| \geq 1, 2 \xi_{t-k} (2 \eta) \equiv 0 \right) \right)
\]

\[
\leq (1 - \alpha p_0)^{k_0} + \left( k_0 + 1 \right) \rho (T) + \sum_{k=0}^{k_0} P \left( |2 \xi_{t-k} | \geq 1, 2 \xi_{t-k} (2 \eta) \equiv 0 \right)
\]

Where the last inequality follows from the fact that the events \( \text{G}_k (a) \) are mutually exclusive, implying \( \sum_{a \in S} \text{G}_k (a) \leq 1 \).

Now, by the complete convergence theorem and Proposition 1 from [2], we have

\[
\lim_{t \to \infty} \text{sup} P \left( |2 \xi_{t-k} | \geq 1, 2 \xi_{t-k} (2 \eta) \equiv 0 \right) \leq \tilde{v} \left( \left\{ \gamma : \gamma \cap 2 \eta = \emptyset \right\} \right) \leq \delta_L,
\]

(3.16)

where \( \delta_L \) tends to zero as \( L \to \infty \)(recall that \( |2 \eta| \geq L \)).
3.15 together with 3.16 and Fatou’s lemma yield
\[
\limsup_{t \to \infty} P \left( \xi^*_t \neq \emptyset, A^*_t \cap 2 \eta = \emptyset \right) \leq (1 - \alpha p_0)^k_0 + (k_0 + 1) \left( \rho(T) + \delta_L \right). \quad (3.17)
\]

Taking \( L \to \infty \) and using Fatou’s lemma once more it follows from 3.2, 3.5 and 3.17 that
\[
\limsup_{t \to \infty} \limsup_{L \to \infty} P \left( \xi_{t+u}(x) = 1 \right) = \limsup_{L \to \infty} \limsup_{t \to \infty} P \left( \xi_t \in d\eta \right) \quad (3.18)
\]
\[
\leq \limsup_{L \to \infty} \int_{|\eta| \geq L} P \left( \xi_t \in d\eta \right) \limsup_{t \to \infty} P \left( \xi^*_t \neq \emptyset, A^*_t \cap 2 \eta = \emptyset \right)
\]
\[
\leq (1 - \alpha p_0)^k_0 + (k_0 + 1) \rho(T). \quad (3.17)
\]

Finally, taking \( u \to \infty \) then \( T \to \infty \) and finally \( k_0 \to \infty \) we deduce
\[
\lim_{u \to \infty} \limsup_{t \to \infty} P \left( \xi_{t+u}(x) = 1 \right) = 0 \quad (3.19)
\]
as required.

\textbf{Proof of Proposition 3.} Signify some vertex \( x_0 \in T_d \) as the root and assume the initial configuration is \( \xi_0(x_0) = 1 \) and zero elsewhere. With some positive probability any other finite initial configuration with
\[
\{ x, \xi_0(x) = 1 \} \neq \emptyset
\]
may evolve into the one above, and thus the proof with that specific initial configuration will be applicable to all others considered.

We will consider \( x_0 \) to be at level 0 of the tree, and any vertex \( y \) that is removed by \( n - 1 \) distinct vertices from \( x_0 \) to be at level \( n \) and we will call the immediate neighbors of \( y \) that are at level \( n + 1 \) its children. Its single neighbor at level \( n - 1 \) will be called its parent. We will denote the level of a vertex by \( l(y) \).

For each \( y \in T_d \), denote the first time it is infected with 1 by \( t(y) \),
\[
t(y) = \inf \{ t| \xi_t(y) = 1 \} ,
\]
and the time this first arrival of 1 dies or mutates by \( s(y) \),
\[
s(y) = \inf \{ s > t(y) | \xi_s(y) \neq 1 \} .
\]

Note that by the memorylessness property, \( s(y) - t(y) \) is an exponential r.v. with parameter \( \delta + \mu \), independent of \( t(y) \).
Let \( W^0_{x_0} \) be the set of all children of \( x_0 \) that \( x_0 \) infects with 1 before dying or mutating. That is,
\[
W^0_{x_0} = \{ y \in \mathbb{T}_d | l(y) = 1, t(y) < s(x_0) \}
\]. Further, let \( W^1_y \) for each \( y \in W^0_{x_0} \) be the set of all its children that it infects before dying or mutating for the first time,
\[
W^1_y = \{ z \in \mathbb{T}_d | l(y) = 2, t(z) < s(y) \}.
\]
We continue recursively with these definitions further down the levels of the tree so that for each \( z \in W^{n-1}_y \) we have the set
\[
W^n_z = \{ w \in \mathbb{T}_d | l(w) = n + 1, t(w) < s(z) \}.
\]
Note that \( |W^n_z| \) is the number of children \( z \) infects before dying or mutating for the first time. Also note that each \( |W^n_z| \) is determined by different parts of time-space and thus are all independent. Further, each child is infected at rate 1 while \( z \) dies or mutates at rate \( \delta + \mu \) and the times (beyond \( t(z) \)) these occur are independent. Thus \( |W^n_z| \) has a binomial distribution with parameters \((d-1, 1/(1 + \delta + \mu))\).

From this discussion we can conclude that the following is a Galton Watson process,
\[
S_n = \sum_{z \in W^{n-1}_z} |W^n_z|.
\]
For small enough \( \delta + \mu \), the expectation of \( |W^n_z| \) will be larger than 1, so that \( S_n \) is supercritical and thus its survival probability is positive. Since \( S_n \) counts some (albeit not all) of the vertices \( x \in \mathbb{T}_d \) such that there exists \( t > 0 \) with \( \xi_t(x) = 1 \), and since with positive probability \( S_n \to \infty \), it follows that
\[
P(\{x, \xi_t(x) = 1\} \neq \emptyset, \forall t > 0) > 0.
\]

**Proof of Proposition 4.** Recalling from the beginning of the proof of Theorem 2 that \( ^1\xi_t \) can be coupled monotonically to \( ^1\zeta_t \) which is a one-type process with birth rate 1 and death rate \( \delta + \mu \). The result follows when considering that \( \delta + \mu \geq \delta_* \) implies the a.s. extinction of \( ^1\zeta_t \) and that of \( ^1\xi_t \) follows by the monotonicity. \( \square \)

4 Notation

In this section we will provide a summary of the notation used throughout the paper for reference.
General notation.

$S$, the graph on which our processes are defined.

$\xi_t$, the mutating contact process.

$\delta$, the per capita death rate, a free parameter.

$^i\xi_t = \{x \in S|\xi_t (x) = i\}$, the set of all vertices occupied by type $i$ in the mutating contact process.

$\zeta_t$, the one type contact process with death rate $\delta$ and birth rate 1.

$^1\zeta_t$, a one-type process coupled to $^1\xi_t$ by using the same graphical construction with mutations treated as deaths.

$^2\zeta_t$, a one-type process coupled to $^2\xi_t$ by using the same graphical construction and ignoring mutations.

$\delta_*, \delta^*$, the weak, and strong survival critical values of the contact process, respectively.

$\alpha = P (\zeta_t^\infty \neq \emptyset \forall t > 0)$, the probability of survival of the contact process.

$\eta_t$, the two-type contact process (on $S = \mathbb{Z}$).

Notation specific to the proof of Theorem 2.

$\hat{\xi}_{s_t}^{(x,s)}$, the backward time ancestral process of $x$ at time $s$, $t$ time units in the past. (the list of possible ancestors of $(x, s)$ at time $s - t$, in order of primality).

$\overline{\xi}_s^t$, the forward time ancestral process, defined for all $s \geq 0$, with the same law as $\hat{\xi}_{s_t}^{(x,t)}$ for all $s \leq t$.

$\xi_t^{(x,s)}$, the forward time ancestral process of $(x,s)$ ($x$ at time $s$), evaluated at time $t$.

$A_t^x$, a list of first priority possible ancestors (which can only propagate into type 2) of $(x, t)$.

$OC (\hat{\xi}_t^{(x,s)}, \eta)$, the event that the first occupied vertex of $\eta$ in the ancestor list $\hat{\xi}_t^{(x,s)}$ is 1-viable and is actually occupied by a 1 in the configuration $\eta$.

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