A NEW TYPE OF BUBBLE SOLUTIONS FOR A SCHRÖDINGER EQUATION WITH CRITICAL GROWTH

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ABSTRACT. In this paper, we investigate the following critical elliptic equation

$$-\Delta u + V(y)u = u^{\frac{N+2}{N-2}}, \quad u > 0, \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

where $V(y)$ is a bounded non-negative function in $\mathbb{R}^N$. Assuming that $V(y) = V(|\hat{y}|, y^*)$, $y = (\hat{y}, y^*) \in \mathbb{R}^4 \times \mathbb{R}^{N-4}$ and gluing together bubbles with different concentration rates, we obtain new solutions provided that $N \geq 7$, whose concentrating points are close to the point $(r_0, y^*_0)$ which is a stable critical point of the function $r^2V(r, y^*)$ satisfying $r_0 > 0$ and $V(r_0, y^*_0) > 0$. In order to construct such new bubble solutions for the above problem, we first prove a non-degenerate result for the positive multi-bubbling solutions constructed in [21] by some local Pohozaev identities, which is of great interest independently. Moreover, we give an example which satisfies the assumptions we impose.

Key words : critical; new bubble solutions; non-degeneracy; local Pohozaev identities.

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1. Introduction and the main result

Standing waves for the following nonlinear Schrödinger equation in $\mathbb{R}^N$,

$$i \frac{\partial \psi}{\partial t} = \Delta \psi - \tilde{V}(y)\psi + |\psi|^{p-1}\psi,$$

(1.1)

are solutions of the form $\psi(t, y) = e^{i\lambda t}u(y)$, where $i$ denotes the imaginary part and $\lambda \in \mathbb{R}$, $p > 1$. Assuming that $u(y)$ is positive and vanishes at infinity, we see $\psi$ satisfies (1.1) if and only if $u$ satisfies the following nonlinear elliptic problem

$$-\Delta u + V(y)u = u^p, \quad u > 0, \quad \lim_{|y| \to \infty} u(y) = 0,$$

(1.2)

where $V(y) = \tilde{V}(y) - \lambda$. Hereafter, we assume that $V(y)$ is bounded and $V(y) \geq 0$. When $1 < p < \frac{N+2}{N-2}$ in (1.2) i.e. the subcritical exponent case, in [23] Wei and Yan showed the equation has infinitely many non-radial positive solutions when $V(y)$ is a radially positive function. There are various existence results for the subcritical case, such as [5, 6, 8].

In this paper, we will investigate the critical case i.e. $p = \frac{N+2}{N-2}$:

$$-\Delta u + V(y)u = u^{\frac{N+2}{N-2}}, \quad u > 0, \quad u \in H^1(\mathbb{R}^N),$$

(1.3)

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where $V(y) \geq 0$ and $V \not\equiv 0$. It corresponds to the following well-known Brezis-Nirenberg problem in $S^N$

$$- \Delta_{S^N} u = u^{\frac{N+2}{N-2}} + \mu u, \quad u > 0, \quad \text{on } S^N. \quad (1.4)$$

Indeed, after using the stereographic projection, problem (1.4) can be reduced to (1.3) with

$$V(y) = \frac{-4\mu - N(N-2)}{1 + |y|^2},$$

and $V(y) > 0$ if $\mu < -\frac{N(N-2)}{4}$. Problem (1.4) has been studied extensively. In [3], Brezis and Li proved if $\mu > -\frac{N(N-2)}{4}$, then the only solutions to (1.4) is the constant $u = (-\mu)^{\frac{N-2}{4}}$. When $\mu = -\frac{N(N-2)}{4}$, in [10] Druet (see also Druet and Hebey [11, 12]) proved that the set of positive solutions to (1.4) is compact provided that the energy is bounded. Furthermore, in [11, 14] it has been proved that there are more and more non-radial solutions as $\mu \to -\infty$.

In [7], Chen, Wei and Yan proved that $\mu < -\frac{N(N-2)}{4}$ and $N \geq 5$, there are infinitely many non-radial solutions to (1.4) whose energy can be made arbitrarily large. This implies that the boundedness of energy in [10, 11] is necessary. When $\mu = -\frac{N(N-2)}{2}$ and $u^{\frac{N+2}{N-2}}$ is taken place of $K(y)u^{\frac{N+2}{N-2}}$ in (1.4) where $K(y)$ being a fixed smooth function, in [24] Wei and Yan showed that it has infinitely many non-radial positive solutions. More results on the existence, multiplicity and qualitative properties of solutions for non-compact elliptic problems can also be found in [3, 9, 13, 16, 19, 20, 25] and the references therein.

It is not difficult to see that if $V \geq 0$ and $V \not\equiv 0$, then the mountain pass value for problem (1.3) is not a critical value of the corresponding functional. Hence all the arguments based on the concentration compactness arguments [17, 18] can not be used to obtain an existence result of solutions for (1.3). To our best knowledge, the first existence result for (1.3) is due to Benci and Cerami [2]. They proved that if $\|V\|_{L^{\infty}(\mathbb{R}^N)}$ is suitably small, (1.3) has a solution whose energy is in the interval $\left(\frac{1}{N}S^\frac{2}{N}, \frac{2}{N}S^\frac{2}{N}\right)$, where $S$ is the best Sobolev constant in the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. For the Brezis–Nirenberg problem in $S^N$, Benci and Cerami’s result only yields an existence result if $-4\mu - N(N-2) > 0$ is suitably small. After [2], there is no other result for (1.3) for a long time until the work by Chen, Wei and Yan [7]. In [7], it is proved that (1.3) has infinitely many nonradial solutions if $N \geq 5$, $V(y)$ is radially symmetric and $r^2V(r)$ has a local maximum point, or a local minimum point $r_0 > 0$ with $V(r_0) > 0$. Note that this condition is necessary for the existence of solutions since by the following Pohozaev identity

$$\int_{\mathbb{R}^N} \left(V(|y|) + \frac{1}{2}|y|V'(|y|)\right)u^2 = 0, \quad (1.5)$$

(1.3) has no solution if $r^2V(r)$ is always non-decreasing, or non-increasing. Recently, in [21] Peng, Wang and Yan showed problem (1.3) has infinitely many solutions by introducing some local Pohozaev type identities in the finitely dimensional reduction method, where $V(y)$ satisfies the following condition:
suppose that \( V(y) = V(|y'|, y'') = V(r, y'') \), \((y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \) and \( r^2 V(r, y'') \) has a critical point \((r_0, y_0'') \) satisfying \( r_0 > 0 \) and \( V(r_0, y_0'') > 0 \) and \( \deg(\nabla (r^2 V(r, y'')) , (r_0, y_0'')) \neq 0 \).

It is well known that the functions
\[
U_{x, \lambda}(y) = \left[ N(N - 2) \right]^{-\frac{N-2}{4}} \left( \frac{\lambda}{1 + \lambda^2 |y-x|^2} \right)^{\frac{N-2}{2}} , \lambda > 0, \ x \in \mathbb{R}^N,
\]
are the only solutions to the problem
\[
- \Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } \mathbb{R}^N . \tag{1.6}
\]

Define
\[
H_s = \left\{ u : u \in C^{1,2} (\mathbb{R}^N), u(y_1, -y_2, y'') = u(y_1, y_2, y''), \ u(r \cos \theta, r \sin \theta, y'') = u \left( r \cos \left( \theta + \frac{2 \pi j}{m} \right), r \sin \left( \theta + \frac{2 \pi j}{m} \right), y'' \right) \right\} .
\]

Let
\[
x_j = \left( \bar{r} \cos \frac{2(j-1) \pi}{m}, \bar{r} \sin \frac{2(j-1) \pi}{m}, \bar{y}'' \right), \ j = 1, 2, \ldots, m,
\]
where \( \bar{y}'' \) is a vector in \( \mathbb{R}^{N-2} \).

Let \( \delta > 0 \) be a small constant, such that \( r^2 V(r, y'') > 0 \) if \(|(r, y'') - (r_0, y_0'')| \leq 10 \delta \). Let \( \zeta(y) = \zeta(|y'|, y'') \) be a smooth function satisfying \( \zeta = 1 \) if \(|(r, y'') - (r_0, y_0'')| \leq \delta \), \( \zeta = 0 \) if \(|(r, y'') - (r_0, y_0'')| \geq 2 \delta \), and \( 0 \leq \zeta \leq 1 \). Denote
\[
Z_{x_j, \lambda}(y) = \zeta U_{x_j, \lambda}, \ Z^*_{\bar{r}, \bar{y}'', \lambda} = \sum_{j=1}^{m} U_{x_j, \lambda}, \ Z_{\bar{r}, \bar{y}'', \lambda}(y) = \sum_{j=1}^{m} Z_{x_j, \lambda}(y).
\]

Set
\[
x_j = \left( \bar{r} \cos \frac{2(j-1) \pi}{m}, \bar{r} \sin \frac{2(j-1) \pi}{m}, \bar{y}'' \right), \ j = 1, \ldots, m, \ \bar{y}'' \in \mathbb{R}^{N-2}.
\]

We recall that the result obtained in \([21]\) is as follows.

**Theorem A.** Suppose that \( V \geq 0 \) is bounded and belongs to \( C^1 \). If \( V(|y'|, y'') \) satisfies \((V')\) and \( N \geq 5 \), then there exists a positive integer \( m_0 > 0 \), such that for any integer \( m \geq m_0 \), \((1.3)\) has a solution \( u_m \) of the form
\[
u_m = Z_{\bar{r}_m, \bar{y}_m''}, \ \varphi_m = \sum_{j=1}^{m} \zeta U_{x_j, \lambda_m} + \varphi_m, \tag{1.7}
\]
where \( \varphi_m \in H_s \). Moreover, as \( m \to +\infty \), \( \lambda_m \in [L_0 m^{\frac{N-4}{2}}, L_1 m^{\frac{N-4}{2}}] \), \((\bar{r}_m, \bar{y}_m'') \to (r_0, y_0'') \), and \( \lambda_m^{-\frac{N-2}{2}} \| \varphi_m \|_{L^\infty} \to 0 \).

To construct new bubble solutions for problem \((1.3)\), we first want to apply some local Pohozaev identities to prove the multi-bubbling solutions in **Theorem A** above is non-degenerate.

In order to state our main result, we give some assumptions of the function \( V(y) : \)
(V) suppose that \( V(y) = V(|\hat{y}|, y^*) = V(r, y^*), (\hat{y}, y^*) \in \mathbb{R}^4 \times \mathbb{R}^{N-4} \) and \( r^2 V(r, y^*) \) has a critical point \( (r_0, y_0^*) \) satisfying \( r_0 > 0 \) and \( V(r_0, y_0^*) > 0 \) and \( \text{deg}(\nabla (r^2 V(r, y^*)), (r_0, y_0^*)) \neq 0 \).

\( \tilde{V} \)

\[
det(A_{i,l})_{(N-3) \times (N-3)} \neq 0, \quad i, l = 1, 2, \ldots, N - 3,
\]

where

\[
A_{i,l} = \left\{
\begin{array}{l}
\left[ \frac{\partial^2 V}{\partial \nu_i} - \left( \frac{\partial \Delta V}{2 \partial^2 V} + \frac{\nu_i}{(\nu, x_1)} \right) \left( r \frac{\partial^2 V}{\partial r^2} + \sum_{j=5}^{N} y_j \frac{\partial^2 V}{\partial \nu_j \partial \nu_j} \right) \right] (r_0, y_0^*), \text{ when } i = l = 1;

\left[ \frac{\partial^2 V}{\partial \nu_i \partial \nu_{i+3}} - \left( \frac{\partial \Delta V}{2 \partial^2 V} + \frac{\nu_{i+3}}{(\nu, x_1)} \right) \left( r \frac{\partial^2 V}{\partial r^2} + \sum_{j=5}^{N} y_j \frac{\partial^2 V}{\partial \nu_j \partial \nu_j} \right) \right] (r_0, y_0^*), \text{ when } i = 1, l = 2, 3, ..., N - 3;

\cos \frac{2\pi}{m} \left[ \frac{\partial^2 V}{\partial \nu_i \partial \nu_{i+3}} - \left( \frac{\partial \Delta V}{2 \partial^2 V} + \frac{\nu_{i+3}}{(\nu, x_1)} \right) \left( r \frac{\partial^2 V}{\partial r^2} + \sum_{j=5}^{N} y_j \frac{\partial^2 V}{\partial \nu_j \partial \nu_j} \right) \right] (r_0, y_0^*), \text{ when } i = 2, 3, ..., N - 3, l = 1;

\left[ \frac{\partial^2 V}{\partial \nu_i \partial \nu_{i+3}} - \left( \frac{\partial \Delta V}{2 \partial^2 V} + \frac{\nu_{i+3}}{(\nu, x_1)} \right) \left( r \frac{\partial^2 V}{\partial r^2} + \sum_{j=5}^{N} y_j \frac{\partial^2 V}{\partial \nu_j \partial \nu_j} \right) \right] (r_0, y_0^*), \text{ when } i, l = 2, 3, ..., N - 3,
\end{array}\right.
\]

\( \nu_i \) and \( \nu \) are the \( i \)-th unit outward normal and unit outward normal respectively on \( \Omega_1 \) defined in [2, 3].

Assume that \( \delta > 0 \) is a small constant such that \( r^2 V(r, y^*) > 0 \) if \(|(r, y^*) - (r_0, y_0^*)| \leq 10\delta \). We also define a cut-off function \( \hat{\zeta}(y) = \hat{\zeta}(|\hat{y}|, y^*) \) be a smooth function satisfying \( \hat{\zeta} = 1 \) if \(|(r, y^*) - (r_0, y_0^*)| \leq \delta \), \( \hat{\zeta} = 0 \) if \(|(r, y^*) - (r_0, y_0^*)| \geq 2\delta \), and \( 0 \leq \hat{\zeta} \leq 1 \).

**Remark 1.1.** From the proof of Theorem A in [21], if we substitute the assumption \( (V) \) for the assumption \( (V') \), then only by making some minor modifications we can also prove that the result of Theorem A is still true. For simplicity of notations, we still denote the solution as \( u_m \), and \( u_m = \sum_{j=1}^{m} \hat{\zeta}(y) U_{\tilde{x}_j, \lambda} + \varphi_m \), where

\[
\hat{x}_j = \left( \frac{\bar{r} \cos \frac{2(j-1)\pi}{m}}{m}, \frac{\bar{r} \sin \frac{2(j-1)\pi}{m}}{m}, 0, 0, \bar{y}^* \right); \quad j = 1, \ldots, m.
\]

Then, like Remark 1.1, we can also find a solution with \( n \)-bubbles, whose centers lie near the surface \((r_0, y_0^*)\) satisfying \(|\hat{y}| = |(y_1, y_2, y_3, y_4)| = r_0\). The question we want to discuss in this paper is whether these two solutions can be glued together to generate a new type of solutions. In other words, we are concerned with looking for a new solution to \((1.3)\), whose shape is, at main order

\[
u \approx \sum_{j=1}^{m} \hat{\zeta}(y) U_{\tilde{x}_j, \lambda} + \sum_{j=1}^{n} \hat{\zeta}(y) U_{p_j, \mu} := \sum_{j=1}^{m} Z_{\tilde{x}_j, \lambda} + \sum_{j=1}^{n} Z_{p_j, \mu},
\]

for \( m \) and \( n \) big integers, where we take
\[ p_j = \left(0, 0, t \cos \frac{2(j-1)\pi}{n}, t \sin \frac{2(j-1)\pi}{n}, \tilde{y}^*\right), \quad j = 1, \ldots, n, \quad \tilde{y}^* \in \mathbb{R}^{N-4}. \]

Here \( \tilde{r} \) and \( t \) are close to \( r_0 \) and \( \tilde{y}^* \rightarrow y_0^* = (y_{0,5}, y_{0,6}, \ldots, y_{0,N}) \).

The energy functional corresponding to equation (1.3) is

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*}, \quad u \in H^1(\mathbb{R}^N). \]

Therefore, generally speaking, a function of the form (1.8) is an approximate solution to (1.3) provided that \( \bar{r}, t, \tilde{y}^* \) and the parameters \( \mu \) and \( \lambda \) are such that

\[ I'(m \sum_{j=1}^{m} Z_{\hat{x}_j, \lambda} + \sum_{j=1}^{n} Z_{p_j, \mu}) \sim 0. \]

Letting that \( \lambda, \mu \rightarrow \infty, \bar{r}, t \rightarrow r_0 \) and \( \tilde{y}^* \rightarrow y_0^* \), we can easily obtain that

\[ I\left( \sum_{j=1}^{m} Z_{\hat{x}_j, \lambda} + \sum_{j=1}^{n} Z_{p_j, \mu} \right) \]

\[ = (m + n)A + m \left( \frac{B_1 V(\tilde{r}, \tilde{y}^*)}{\lambda^2} - \frac{B_2}{\lambda^{N-2} |\hat{x}_1 - \hat{x}_j|^{N-2}} + O\left( \frac{1}{\lambda^{2+\epsilon}} \right) \right) \]

\[ + n \left( \frac{C_1 V(t, \tilde{y}^*)}{\mu^2} - \frac{C_2}{\mu^{N-2} |p_1 - p_j|^{N-2}} + O\left( \frac{1}{\mu^{2+\epsilon}} \right) \right) \quad (1.9) \]

where \( A = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla U_{0,1}|^2 \) and \( B_1, B_2, C_1, C_2 \) are some positive constants and \( \epsilon > 0 \) is a small constant. Note that if \( n \gg m \), then the two terms in (1.9) are of different orders, which causes it not easy to find a critical point of \( I \). Hence it is very difficult to apply a reduction argument to construct solutions of the form (1.8).

In this paper, we use a new method which was first introduced by Guo, Musso, Peng and Yan recently in [15] where they studied the prescribed scalar curvature equation with a radial potential function. Recall that we intend to glue \( n \)-bubbles, whose centers lie on the surface \((r_0, y_0^*)\) to the \( m \)-bubbling solution \( u_m \) described in Remark 1.1. The linear operator for such a problem is

\[ Q_n \eta = -\Delta \eta + V(y) \eta - (2^* - 1) \left( u_m + \sum_{j=1}^{n} Z_{p_j, \mu} \right)^{2^* - 2} \eta. \]

Away from the points \( p_j \), the operator \( Q_n \) can be approximated by the linearized operator around \( u_m \), defined by

\[ L_m \eta = -\Delta \eta + V(y) \eta - (2^* - 1) u_m^{2^* - 2} \eta. \quad (1.10) \]
The approach we use here is to construct the solution with \( m \)-bubbles whose center is close to \((r_0, y_0^*)\) and \( n \)-bubbles whose center lies near \((r_0, y_0^*)\) as a perturbation of the solution with the \( m \)-bubbles whose center lie near \((r_0, y_0^*)\).

The main result of this paper is as follows:

**Theorem 1.2.** Suppose \( V(y) \) satisfies the assumptions \((V), (\tilde{V})\) and \( N \geq 7 \). Let \( u_m \) be a solution in Remark 1.1 and \( m > 0 \) is a large even number. Then there is an integer \( n_0 > 0 \), depending on \( m \), such that for any even number \( n \geq n_0 \), (1.3) has a solution whose main order is of the form (1.8) for some \( t_n \to r_0, \tilde{y}^* \to y_0^* \) and \( \mu_n \sim n^{\frac{N-2}{2}} \).

**Remark 1.3.** Like [21], in section 3 to deal with the slow decay of the function \( U_{p_j,\mu}(y) \) when the dimension \( N \) is not big, we introduce the cut-off function \( \hat{\zeta}(y) \).

**Remark 1.4.** We want to point out that if we assume that the small constants \( \delta \) in the definition of the cut-off function \( \hat{\zeta}(y) \) and \( \vartheta \) in (3.10) which are less or equal to \( c\mu^{-\frac{1}{2}} \), the result in Theorem 1.2 can hold for \( N \geq 5 \). It is just technical. In this case, we have the following relation

\[
\frac{1}{\mu^{\frac{1}{2}}} \leq \frac{C}{1 + \mu|y - p_j|},
\]

which can help us deal with some estimates such as (3.38).

**Remark 1.5.** In Theorem 1.2 in order to obtain the solution \( u(y) \) satisfying that it is even about \( y_h, h = 1, 2, 3, 4 \), we assume that \( m, n \) are both even integers. Otherwise, only to obtain the existence of the solution \( u(y) \), we do not need this requirement.

To prove Theorem 1.2, first it is very crucial to understand the spectral properties of the liner operator \( L_m \) and study its invertibility in some suitable space. We will mainly do this in section 2. Moreover, since the concentration points of the bump solutions include a saddle point of \( V(r, y^*) \), we can not estimate directly the derivatives of the reduced functional as usual. We will apply some local Pohozaev identities to locate the concentration points of the bump solutions as [21] [22]. However, in the process of doing the finite-dimensional reduction we need to compute more accurately, such as the estimate of \( J_4 \) in Lemma 3.2 where we follow some ideas from [14]. Finally, we would like to point out that the new solutions we construct here which are different from the solutions obtained [21].

Our paper is organized as follows. In section 2 we will prove a non-degenerate result by some local Pohozaev identities, which is very crucial in constructing a new type of bubbling solutions by applying the finitely dimensional reduction method. Applying the non-degenerate result, we construct new solutions and prove Theorem 1.2 in section 3. In section A we give some Pohozaev identities. We give an example of the potential \( V(r, y^*) \) which satisfies the assumptions \((V)\) and \((\tilde{V})\) in appendix C.

### 2. The Non-Degeneracy of the Solutions

In this section, we mainly prove the non-degeneracy of the multi-bubbling solutions obtained by Peng, Wang and Yan in [21].
Define
\[
\|u\|_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{m} \frac{1}{(1 + \lambda_m |y - x_{m,j}|^{N/2})^{N-2}} \right)^{-1} \lambda_m^{-\frac{N-2}{2}} |u(y)|
\] (2.1)
and
\[
\|f\|_{**} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{m} \frac{1}{(1 + \lambda_m |y - x_{m,j}|^{N/2})^{N-2}} \right)^{-1} \lambda_m^{-\frac{N+2}{2}} |f(y)|,
\] (2.2)
where \(x_{m,j} = (r_m \cos \frac{(j-1)\pi}{m}, r_m \sin \frac{(j-1)\pi}{m}, \bar{x}_m)\), \(\tau = \frac{N-4}{N-2}\).

Let
\[
\Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left( \frac{y''}{|y'|}, \frac{x_{m,j}'}{|x_{m,j}'|} \right) \geq \cos \frac{\pi}{m} \right\}.
\] (2.3)

Define the linear operator
\[
L_m \eta = -\Delta \eta + V \eta - (2^* - 1) U_m^{2^*-2} \eta.
\]

**Lemma 2.1.** There is a positive constant \(C\) such that
\[
|u_m(y)| \leq C \sum_{j=1}^{m} \frac{\lambda_m^{-\frac{N-2}{2}}}{1 + (\lambda_m |y - x_{m,j}|)^{N-2}} \quad \text{for all } y \in \mathbb{R}^N.
\]

**Proof.** Since the proof is just the same as Lemma 2.2 of [15], here we omit it. \(\square\)

The main result of this section is the following.

**Proposition 2.1.** Suppose \(N \geq 5\). Assume that \(V(y)\) satisfies \((V')\) and \((\tilde{V}')\). Let \(\eta \in H_s\) be a solution of \(L_m \eta = 0\). Then \(\eta = 0\).

Now we will prove Proposition 2.1 by an indirect method. Assume that there are \(m_k \to +\infty\), satisfying \(\|\eta_k\|_* = 1\) and
\[
L_{m_k} \eta_k = 0.
\] (2.4)

Denote
\[
\tilde{\eta}_k(y) = \lambda_{m_k}^{-\frac{N-2}{2}} \eta_k(\lambda_{m_k}^{-1} y + x_{m_k,1}).
\] (2.5)

**Lemma 2.2.** It holds
\[
\tilde{\eta}_k \to b_0 \psi_0 + \sum_{i=1, i \neq 2}^{N} b_i \psi_i,
\] (2.6)
uniformly in \(C^1(B_R(0))\) for any \(R > 0\), where \(b_0\) and \(b_i (i = 1, 3, \ldots, N)\) are some constants,
\[
\psi_0 = \frac{\partial U_{0,\lambda}}{\partial \lambda} \bigg|_{\lambda = 1}, \quad \psi_i = \frac{\partial U_{0,1}}{\partial y_i}, \quad i = 1, 3, \ldots, N.
\]

**Proof.** Observing that \(|\tilde{\eta}_k| \leq C\), we may suppose that \(\tilde{\eta}_k \to \eta\) in \(C_{\text{loc}}(\mathbb{R}^N)\). Then \(\eta\) satisfies
\[
-\Delta \eta = (2^* - 1) U^{2^*-2} \eta, \quad x \in \mathbb{R}^N,
\]
which implies
\[ \eta = \sum_{i=0}^{N} b_i \psi_i. \]

Since \( \eta_k \) is even in \( y_2 \), there holds \( b_2 = 0 \).

We decompose
\[ \eta_k(y) = b_{0,m} \lambda_{mk} \sum_{j=1}^{m_k} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial \lambda_{mk}} + b_{1,m} \lambda_{mk}^{-1} \sum_{j=1}^{m_k} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial \eta^{*}} \]
\[ + \sum_{i=3}^{N} b_{1,m} \lambda_{mk}^{-1} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial y_i} + \eta^*, \]

where \( \eta^*_k \) satisfies
\[ \int_{\mathbb{R}^N} Z_{x_{mk,j},\lambda_{mk}}^{2^*-2} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial \lambda_{mk}} \eta^*_k = \int_{\mathbb{R}^N} Z_{x_{mk,j},\lambda_{mk}}^{2^*-2} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial \eta^*} \eta^*_k \]
\[ = \int_{\mathbb{R}^N} Z_{x_{mk,j},\lambda_{mk}}^{2^*-2} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial y_i} \eta^*_k = 0, \quad (i = 3, \ldots, N). \]

It follows from Lemma 2.2 that \( b_{0,m}, b_1,m \) and \( b_{1,m} (i = 3, \ldots, N) \) are bounded.

**Lemma 2.3.** There holds
\[ \|\eta^*_k\| \leq C \lambda_{mk}^{-1 - \epsilon}, \]
where \( \epsilon > 0 \) is a small constant.

**Proof.** One can see easily that
\[ L_{mk} \eta^*_k \]
\[ = -\Delta \eta^*_k + V \eta^*_k - (2^* - 1) u_{mk}^{2^*-2} \eta^*_k \]
\[ = -V \left( b_{0,m} \lambda_{mk} \sum_{j=1}^{m_k} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial \lambda_{mk}} + b_{1,m} \lambda_{mk} \sum_{j=1}^{m_k} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial \eta^*} \right) \]
\[ + \sum_{i=3}^{N} b_{1,m} \lambda_{mk}^{-1} \frac{\partial Z_{x_{mk,j},\lambda_{mk}}}{\partial y_i} \]
\[ + 2 \nabla \xi \left( b_{0,m} \lambda_{mk} \sum_{j=1}^{m_k} \frac{\partial U_{x_{mk,j},\lambda_{mk}}}{\partial \lambda_{mk}} + b_{1,m} \lambda_{mk} \sum_{j=1}^{m_k} \frac{\partial U_{x_{mk,j},\lambda_{mk}}}{\partial \eta^*} \right) \]
\[ + \Delta \xi \left( b_{0,m} \lambda_{mk} \sum_{j=1}^{m_k} \frac{\partial U_{x_{mk,j},\lambda_{mk}}}{\partial \lambda_{mk}} + b_{1,m} \lambda_{mk} \sum_{j=1}^{m_k} \frac{\partial U_{x_{mk,j},\lambda_{mk}}}{\partial \eta^*} \right) \]
\[ L = L_1 + L_2 + L_3 + L_4. \]

Similar to the proof of \( J_1 \) in Lemma 2.4 in [21], we can prove
\[
\|L_1\|_{**} \leq C\lambda_m^{-1-\epsilon}. \tag{2.7}
\]
By the same argument as that of [15] in Lemma 2.4, we can estimate
\[
\|L_2\|_{**} \leq C\lambda_m^{-1-\epsilon}. \tag{2.8}
\]
Similar to \( J_2 \) of Lemma 2.4 in [21], we can check
\[
|L_3| \leq C\left(\frac{1}{\lambda_m}\right)^{1+\epsilon} \sum_{j=1}^{m_k} \frac{\lambda_m^{N+2}}{(1 + \lambda_m|y - x_{m,k,j}|)^{N+2+\tau}}.
\]
Hence, we obtain
\[
\|L_3\|_{**} \leq C\left(\frac{1}{\lambda_m}\right)^{1+\epsilon}. \tag{2.9}
\]
Also, similar to \( J_3 \) of Lemma 2.4 in [21], we can prove
\[
|L_4| \leq C\left(\frac{1}{\lambda_m}\right)^{1+\epsilon} \sum_{j=1}^{m_k} \frac{\lambda_m^{N+2}}{(1 + \lambda_m|y - x_{m,k,j}|)^{N+2+\tau}},
\]
which yields that
\[
\|L_4\|_{**} \leq C\left(\frac{1}{\lambda_m}\right)^{1+\epsilon}. \tag{2.10}
\]
It follows from (2.7) to (2.10) that
\[
\|L_{m_k} \eta^*_k\|_{**} \leq C\lambda_m^{-1-\epsilon}. \tag{2.11}
\]
Furthermore, from
\[
\int_{\mathbb{R}^N} Z_{x_{m,k,j},\lambda_m}^{2r-2} \frac{\partial Z_{x_{m,k,j},\lambda_m}}{\partial \lambda_m} \eta^*_k = \int_{\mathbb{R}^N} Z_{x_{m,k,j},\lambda_m}^{2r-2} \frac{\partial Z_{x_{m,k,j},\lambda_m}}{\partial \eta_i} \eta^*_k
\]
\[
= \int_{\mathbb{R}^N} Z_{x_{m,k,j},\lambda_m}^{2r-2} \frac{\partial Z_{x_{m,k,j},\lambda_m}}{\partial \eta_i} \eta^*_k = 0, \quad (i = 3, \ldots, N)
\]
and Lemma [21] we can prove that there exists \( \rho > 0 \) such that
\[
\|L_{m_k} \eta^*_k\|_{**} \geq \rho \|\eta^*_k\|_{*}. \tag{2.12}
\]
Combining (2.11) and (2.12), the result is true. \( \square \)

Now we give another assumption of \( V(y) \):
\[ (\tilde{V}') \]
\[ det(A_{i,l})_{(N-1)\times(N-1)} \neq 0, \quad i, l = 1, 2, \ldots, N - 1, \]
where

\[ A_{i,l} = \begin{cases} \frac{\partial^2 \nu}{\partial x^2} - \left( \frac{\partial \nu}{\partial y} + \nu \right) \left( r \frac{\partial^2 \nu}{\partial y^2} + \sum_{j=1}^{N} y_j \frac{\partial^2 \nu}{\partial y \partial y_j} \right), & \text{when } i = l = 1; \\ \cos \frac{2\pi}{m} \left[ \frac{\partial^2 \nu}{\partial y^2} + \frac{\nu_{i+1}}{(\nu_{i+1})} \right] \left( r \frac{\partial^2 \nu}{\partial y^2} + \sum_{j=1}^{N} y_j \frac{\partial^2 \nu}{\partial y \partial y_j} \right), & \text{when } i = 1, l = 2, 3, ..., N - 1; \\ \cos \frac{2\pi}{m} \left[ \frac{\partial^2 \nu}{\partial y^2} + \frac{\nu_{i+1}}{(\nu_{i+1})} \right] \left( r \frac{\partial^2 \nu}{\partial y^2} + \sum_{j=1}^{N} y_j \frac{\partial^2 \nu}{\partial y \partial y_j} \right), & \text{when } i = 2, 3, ..., N - 1, l = 1; \\ \cos \frac{2\pi}{m} \left[ \frac{\partial^2 \nu}{\partial y^2} + \frac{\nu_{i+1}}{(\nu_{i+1})} \right] \left( r \frac{\partial^2 \nu}{\partial y^2} + \sum_{j=1}^{N} y_j \frac{\partial^2 \nu}{\partial y \partial y_j} \right), & \text{when } i, l = 2, 3, ..., N - 1, \end{cases} \]

\[ (2.13) \]

\( \nu_i \) and \( \nu \) are the \( i \)-th unit outward normal and unit outward normal respectively on \( \Omega_1 \) defined in \[ (2.3) \].

**Lemma 2.4.** If \( \bar{V}' \) holds, then

\( \tilde{\eta}_k \rightarrow 0 \)

uniformly in \( C^1(B_R(0)) \) for any \( R > 0 \).

**Proof.** Step 1. Recall that

\[ \Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \frac{y'}{|y'|}, \frac{x_{m,j}'}{|x_{m,j}'|} \geq \cos \frac{\pi}{m} \right\}. \]

In order to prove \( b_{i,k} \rightarrow 0 (i = 1, 3, \cdots, N) \), we apply the identities in Lemma A.1 in the domain \( \Omega_1 \),

\[ -\int_{\partial \Omega_1} \frac{\partial u_{m_k}}{\partial \nu} \frac{\partial \eta_k}{\partial y_i} - \int_{\partial \Omega_1} \frac{\partial \eta_k}{\partial \nu} \frac{\partial u_{m_k}}{\partial y_i} + \int_{\partial \Omega_1} \left( \nabla u_{m_k}, \nabla \eta_k \right) \nu_i + \int_{\partial \Omega_1} V u_{m_k} \eta_k \nu_i \]

\[ - \int_{\partial \Omega_1} u_{m_k}^{2^* - 1} \eta_k \nu_i = \int_{\Omega_1} \frac{\partial V}{\partial y_i} u_{m_k} \eta_k, \quad i = 1, 3, \cdots, N. \]

\[ (2.14) \]

By the symmetry, we have \( \frac{\partial u_{m_k}}{\partial \nu} = 0 \) and \( \frac{\partial \eta_k}{\partial \nu} = 0 \) on \( \partial \Omega_1 \). Hence

the left hand side of \[ (2.14) \]

\[ = \nu_i \left( \int_{\partial \Omega_1} \left( \nabla u_{m_k}, \nabla \eta_k \right) + \int_{\partial \Omega_1} V u_{m_k} \eta_k - \int_{\partial \Omega_1} u_{m_k}^{2^* - 1} \eta_k \right). \]

\[ (2.15) \]

Combining \[ (2.14) \] and \[ (2.15) \], we obtain

\[ \nu_i \left( \int_{\partial \Omega_1} \left( \nabla u_{m_k}, \nabla \eta_k \right) + \int_{\partial \Omega_1} V u_{m_k} \eta_k - \int_{\partial \Omega_1} u_{m_k}^{2^* - 1} \eta_k \right) = \int_{\Omega_1} \frac{\partial V}{\partial y_i} u_{m_k} \eta_k. \]

\[ (2.16) \]
To estimate the left hand side in (2.16), we use (A.4) in \( \Omega \). Applying the symmetry, we have
\[
\int_{\Omega} u_{mk} \eta_k \langle \nabla V, y - x_{mk,1} \rangle + 2 \int_{\Omega} V \eta_k u_{mk} = - \int_{\partial \Omega} u_{mk}^{2-1} \eta_k \langle \nu, y - x_{mk,1} \rangle \\
+ \int_{\partial \Omega} \langle \nabla u_{mk}, \nabla \eta_k \rangle \langle \nu, y - x_{mk,1} \rangle + \int_{\partial \Omega} V u_{mk} \eta_k \langle \nu, y - x_{mk,1} \rangle.
\] (2.17)

On \( \partial \Omega \), it holds \( \langle \nu, y \rangle = 0 \). Then, (2.17) becomes
\[
\int_{\Omega} u_{mk} \eta_k \langle \nabla V, y - x_{mk,1} \rangle + 2 \int_{\Omega} V \eta_k u_{mk} = - \langle \nu, x_{mk,1} \rangle \left( - \int_{\partial \Omega} u_{mk}^{2-1} \eta_k + \int_{\partial \Omega} \langle \nabla u_{mk}, \nabla \eta_k \rangle + \int_{\partial \Omega} V u_{mk} \eta_k \right).
\] (2.18)

It follows from (2.10) and (2.18) that
\[
\int_{\Omega} \frac{\partial V}{\partial y_i} u_{mk} \eta_k = \frac{-\nu_i}{\nu, x_{mk,1}} \left( \int_{\Omega} u_{mk} \eta_k \langle \nabla V, y - x_{mk,1} \rangle + 2 \int_{\Omega} V \eta_k u_{mk} \right).
\] (2.19)

Since \( \nabla V(x_{mk,1}) = O(|x_{mk,1} - y_0|) \) and
\[
\int_{\Omega} u_{mk} \eta_k = \int_{(\Omega_1) \times x_{mk,1} \times x_{mk,1}} \left( \lambda_{mk}^{\frac{N}{2}} u_{mk} (\lambda_{mk}^{-1} y + x_{mk,1}) \right) \eta_k
\]
\[
= \frac{1}{\lambda_{mk}^2} \int_{\mathbb{R}^N} U \left[ b_{0,m} \psi_0 + b_{1,m} \psi_1 + \sum_{i=3}^{N} b_{i,m} \psi_i + \lambda_{mk}^{\frac{N}{2}} \eta_k (\lambda_{mk}^{-1} y + x_{mk,1}) \right] + O(\lambda_{mk}^{-3-\epsilon})
\]
\[
= O(\lambda_{mk}^{-3-\epsilon}),
\] (2.20)

where \((\Omega_1) \times x_{km,1} = \{ y, \lambda_{km}^{-1}, y + x_{km,1} \in \Omega \}\), we have
\[
\int_{\Omega} \frac{\partial V}{\partial y_i} u_{mk} \eta_k = \int_{\Omega} u_{mk} \eta_k \left( \frac{\partial V}{\partial y_i} - \frac{\partial V(x_{mk,1})}{\partial y_i} \right) + \int_{\Omega} \frac{\partial V(x_{mk,1})}{\partial y_i} u_{mk} \eta_k
\]
\[
= \int_{\Omega} u_{mk} \eta_k \left[ \langle \nabla V(x_{mk,1}), y - x_{mk,1} \rangle + \frac{1}{2} \left( \nabla^2 \frac{\partial V(x_{mk,1})}{\partial y_i} \right) (y - x_{mk,1}, y - x_{mk,1}) + O(|y - x_{mk,1}|^3) \right] + O(\lambda_{mk}^{-4-\epsilon})
\]
\[
= \frac{1}{\lambda_{mk}^2} \int_{\mathbb{R}^N} U \left( b_{0,k} \psi_0 + b_{1,k} \psi_1 + \sum_{l=3}^{N} b_{l,k} \psi_l \right) \left( \langle \nabla \frac{\partial V(x_{mk,1})}{\partial y_i}, y_{\lambda_{mk}} \rangle + \frac{1}{2} \left( \nabla^2 \frac{\partial V(x_{mk,1})}{\lambda_{mk}} \right) \frac{y_{\lambda_{mk}}}{\lambda_{mk}} \right) + O(\lambda_{mk}^{-4-\epsilon})
\]
\[
= \frac{\delta_{y_i} \delta_{y_j}}{\lambda_{mk}^3} b_{1,k} \int_{\mathbb{R}^N} U \psi_j y_j + \sum_{l=3}^{N} b_{l,k} \frac{\delta^2 V}{\delta y_i \delta y_j} \frac{y_j}{\lambda_{mk}^3} \int_{\mathbb{R}^N} U \psi_j y_j
\]
Hence, (2.21) and (2.22) give
\[ \frac{\partial \Delta V(x_{mk,1})}{\partial y} b_{0,k} \int_{\mathbb{R}^N} U \psi_0 |y|^2 + O(\lambda_{mk}^{-4-\epsilon}). \] (2.21)

Moreover, we can estimate
\[
\int_{\Omega_1} u_{mk} \eta_k \langle \nabla V, y - x_{mk,1} \rangle \\
= \int_{\Omega_1} u_{mk} \eta_k \langle \nabla V(y) - \nabla V(x_{mk,1}), y - x_{mk,1} \rangle + \int_{\Omega_1} u_{mk} \eta_k \langle \nabla V(x_{mk,1}), y - x_{mk,1} \rangle \\
= \int_{\Omega_1} u_{mk} \eta_k \langle \nabla V(y) - \nabla V(k_{mk,1}), y - x_{mk,1} \rangle + O(\lambda_{mk}^{-4-\epsilon}) \\
= \int_{\Omega_1} u_{mk} \eta_k \langle \nabla^2 V(x_{mk,1})(y - x_{mk,1}), y - x_{mk,1} \rangle + O(\lambda_{mk}^{-4-\epsilon}) \\
= \frac{1}{\lambda_{mk}^2} \int_{\mathbb{R}^N} U \left( b_{0,k} \psi_0 + b_{1,k} \eta_1 + \sum_{l=3}^N b_{l,k} \eta_l \right) \langle \nabla^2 V(x_{mk,1}) \lambda_{mk}^{-1} y, \lambda_{mk}^{-1} y \rangle + O(\lambda_{mk}^{-4-\epsilon}) \\
= \frac{b_{0,k} \Delta V(x_{mk,1})}{N \lambda_{mk}^4} \int_{\mathbb{R}^N} U \psi_0 |y|^2 + O(\lambda_{mk}^{-4-\epsilon}). \] (2.22)

Hence, (2.21) and (2.22) give
\[
b_{0,k} \frac{1}{\lambda_{mk}} \left( \frac{\partial \Delta V(x_{mk,1})}{\partial y} \right) + \frac{\nu}{\langle \nu, x_{mk,1} \rangle} \frac{\Delta V(x_{mk,1})}{N} \int_{\mathbb{R}^N} U \psi_0 |y|^2 \\
+ b_{1,k} \frac{\partial^2 V(x_{mk,1})}{\partial y \partial y_i} \int_{\mathbb{R}^N} U \psi_1 y_1 + \sum_{l=3}^N b_{l,k} \frac{\partial^2 V(x_{mk,1})}{\partial y_l \partial y_i} \int_{\mathbb{R}^N} U \psi_l y_l = O(\lambda_{mk}^{-1-\epsilon}). \] (2.23)

Step 2. Next, we apply (A.4) to get
\[
\int_{\mathbb{R}^N} u_{mk} \eta_k \langle \nabla V(y), y \rangle = 0, 
\]
which implies
\[
\int_{\Omega_1} u_{mk} \eta_k \langle \nabla V(y), y \rangle = 0. \] (2.24)

On the other hand, proceeding as in the proof of (2.20), we have
\[
\int_{\Omega_1} u_{mk} \eta_k \langle \nabla V(x_{mk,1}), y \rangle \\
= \int_{\Omega_1} u_{mk} \eta_k \langle \nabla V(x_{mk,1}), y - x_{mk,1} \rangle + \int_{\Omega_1} u_{mk} \eta_k \langle \nabla V(x_{mk,1}), x_{mk,1} \rangle \\
= O(\lambda_{mk}^{-4-\epsilon}).
\]
Therefore, from (2.23), we have

\[
\int_{\Omega_1} u_{mk} \eta \langle \nabla V(y), y \rangle = \int_{\Omega_1} u_{mk} \eta \langle \nabla V(y) - \nabla V(x_{mk,1}), y \rangle + O(\lambda_{mk}^{-4-\epsilon})
\]

\[
= \int_{\Omega_1} u_{mk} \eta \langle \nabla^2 V(x_{mk,1})(y - x_{mk,1}), y \rangle + O(\lambda_{mk}^{-4-\epsilon})
\]

\[
= \frac{1}{\lambda_{mk}^2} \int_{\mathbb{R}^N} U \left( b_{0,k} \psi_0 + b_{1,k} \psi_1 + \sum_{l=3}^N b_{l,k} \psi_l \right) \langle \nabla^2 V(x_{mk,1}) \lambda_{mk}^{-1} y, \lambda_{mk}^{-1} y + x_{mk,1} \rangle + O(\lambda_{mk}^{-4-\epsilon})
\]

\[
= \frac{b_{0,k} \Delta V(x_{mk,1})}{N \lambda_{mk}^4} \int_{\mathbb{R}^N} U \psi_0 |y|^2 + b_{1,k} \lambda_{mk}^3 \sum_{j=1}^N (x_{mk,1})_j \frac{\partial^2 V}{\partial y_j \partial y_1}(x_{mk,1}) \int_{\mathbb{R}^N} U \psi_1 y_1
\]

\[
+ \sum_{l=3}^N b_{l,k} \sum_{j=1}^N (x_{mk,1})_j \frac{\partial^2 V}{\partial y_j \partial y_l}(x_{mk,1}) \int_{\mathbb{R}^N} U \psi_1 y_l + O(\lambda_{mk}^{-4-\epsilon}),
\]

which combining with (2.24) implies that

\[
b_{0,k} \frac{\Delta V(x_{mk,1})}{N \lambda_{mk}^4} \int_{\mathbb{R}^N} U \psi_0 |y|^2 + b_{1,k} \lambda_{mk}^3 \sum_{j=1}^N (x_{mk,1})_j \frac{\partial^2 V}{\partial y_j \partial y_1}(x_{mk,1}) \int_{\mathbb{R}^N} U \psi_1 y_1
\]

\[
+ \sum_{l=3}^N b_{l,k} \sum_{j=1}^N (x_{mk,1})_j \frac{\partial^2 V}{\partial y_j \partial y_l}(x_{mk,1}) \int_{\mathbb{R}^N} U \psi_1 y_l = O(\lambda_{mk}^{-1-\epsilon}). \tag{2.25}
\]

It follows from (2.23) and (2.24) that

\[
b_{1,k} \left[ \frac{\partial^2 V}{\partial y_1 \partial y_i}(x_{mk,1}) - \frac{\partial \Delta V(x_{mk,1})}{2 \Delta V(x_{mk,1})} \sum_{j=1}^N (x_{mk,1})_j \frac{\partial^2 V}{\partial y_j \partial y_1}(x_{mk,1}) \right] \int_{\mathbb{R}^N} U \psi_1 y_1
\]

\[
+ \sum_{l=3}^N b_{l,k} \left[ \frac{\partial^2 V}{\partial y_1 \partial y_l}(x_{mk,1}) - \frac{\partial \Delta V(x_{mk,1})}{2 \Delta V(x_{mk,1})} \sum_{j=1}^N (x_{mk,1})_j \frac{\partial^2 V}{\partial y_j \partial y_l}(x_{mk,1}) \right] \int_{\mathbb{R}^N} U \psi_1 y_l
\]

\[
= O(\lambda_{mk}^{-1-\epsilon}), \ i = 1, 3, 4, ..., N,
\]
which implies that

\[
\begin{aligned}
&\left[ \frac{\partial^2 V}{\partial r^2} - \left( \frac{\partial^2 V}{\partial y_{j+i}^2} + \frac{\nu_1}{\nu_{x_1}} \right) \left( r \frac{\partial^2 V}{\partial y_{j+i}^2} + \sum_{j=3}^N y_j \frac{\partial^2 V}{\partial y_j^2} \right) \right] (r_0, y_0) b_{1,k} \\
&+ \sum_{l=2}^{N-1} \left[ \frac{\partial^2 V}{\partial y_{j+l}^2} - \left( \frac{\partial^2 V}{\partial y_{j+l}^2} + \frac{\nu_1}{\nu_{x_1}} \right) \left( r \frac{\partial^2 V}{\partial y_{j+l}^2} + \sum_{j=3}^N y_j \frac{\partial^2 V}{\partial y_j^2} \right) \right] (r_0, y_0') b_{l+1,k} = O(\lambda^{-1-\epsilon}_{m_k}) \\
&\cos 4\pi \left[ \frac{\partial^2 V}{\partial y_{j+i}^2} - \left( \frac{\partial^2 V}{\partial y_{j+i}^2} + \frac{\nu_1}{\nu_{x_1}} \right) \left( r \frac{\partial^2 V}{\partial y_{j+i}^2} + \sum_{j=3}^N y_j \frac{\partial^2 V}{\partial y_j^2} \right) \right] (r_0, y_0'') b_{1,k} \\
&+ \sum_{l=2}^{N-1} \left[ \frac{\partial^2 V}{\partial y_{j+l}^2} - \left( \frac{\partial^2 V}{\partial y_{j+l}^2} + \frac{\nu_1}{\nu_{x_1}} \right) \left( r \frac{\partial^2 V}{\partial y_{j+l}^2} + \sum_{j=3}^N y_j \frac{\partial^2 V}{\partial y_j^2} \right) \right] (r_0, y_0'') b_{l+1,k} = O(\lambda^{-1-\epsilon}_{m_k}).
\end{aligned}
\]

(2.26)

Obviously, the coefficient matrix of the system (2.26) is just the matrix \((A_{i,l})_{(N-1)\times(N-1)}\), where \(A_{i,l}, i, l = 1, 2, ..., N - 1\) are defined in (2.13).

By the assumption \((\tilde{V}'')\) and the theory of solutions for homogenous linear equations in linear Algebra, we know that the only solution of the system (2.26) is \(b_{1,k} = o(1), b_{l,k} = o(1)(l = 3, \cdots, N)\).

\[\square\]

\textbf{Proof of Proposition 2.1} First we have

\[
|\eta_k(y)| \leq C \int_{\mathbb{R}} \frac{1}{|y-z|^N} |u_{m_k}^{2,2}(z)||\eta_k(z)| dz
\]

\[
\leq C\|\eta_k\| \int \frac{1}{|y-z|^{N-2}} |u_{m_k}^{2,2}(z)| \sum_{j=1}^{m_k} \frac{\lambda_{m_k}^{N-2}}{(1 + \lambda_{m_k}|z-x_{m,k,j}|)^{\frac{N-2}{2} + \tau}}
\]

\[
\leq C\|\eta_k\| \sum_{j=1}^{m_k} \frac{\lambda_{m_k}^{N-2}}{(1 + \lambda_{m_k}|y-x_{m,k,j}|)^{\frac{N-2}{2} + \tau + \theta}},
\]

(2.27)
for some \( \theta > 0 \). Hence we have

\[
\frac{|\eta_k(y)|}{\sum_{j=1}^{m_k} \frac{\lambda_{m_k}^{\frac{N-2}{2}+\tau}}{(1 + \lambda_{m_k}|y - x_{m_k,j}|)^{\frac{N-2}{2}+\tau}}} \leq C \|\eta_k\|_* \leq \frac{\sum_{j=1}^{m_k} \frac{\lambda_{m_k}^{\frac{N-2}{2}+\tau}}{(1 + \lambda_{m_k}|y - x_{m_k,j}|)^{\frac{N-2}{2}+\tau}}}{\sum_{j=1}^{m_k} \frac{\lambda_{m_k}^{\frac{N-2}{2}}} {(1 + \lambda_{m_k}|y - x_{m_k,j}|)^{\frac{N-2}{2}}}}.
\]

Since \( \eta_k \to 0 \) in \( B_{R_{\lambda_{m_k}}^{-1}}(x_{m_k,j}) \) and \( \|\eta_k\|_* = 1 \), we know that

\[
\frac{|\eta_k(y)|}{\sum_{j=1}^{m_k} \frac{\lambda_{m_k}^{\frac{N-2}{2}+\tau}}{(1 + \lambda_{m_k}|y - x_{m_k,j}|)^{\frac{N-2}{2}+\tau}}}
\]

attains its maximum in \( \mathbb{R}^N \setminus \bigcup_{j=1}^{m_k} B_{R_{\lambda_{m_k}}^{-1}}(x_{m_k,j}) \). Therefore

\[
\|\eta_k\|_* \leq o(1) \|\eta_k\|_*.
\]

Hence \( \|\eta_k\|_* \to 0 \) as \( k \to \infty \). This contradicts with \( \|\eta_k\|_* = 1 \). \( \square \)

**Remark 2.5.** If \( V(y) \) is radial, then the assumption \( (\tilde{V}') \) is just

\[
\Delta V - \left( \Delta V + \frac{1}{2}(\Delta V') \right) r \neq 0 \quad \text{at} \quad r = r_0.
\]

We would like to point out that the local Pohozaev identities play a crucial role in the investigation of the non-degeneracy of the multi-bubbling solutions. This novel idea first comes from [15]. Also, the non-degeneracy of the solution and the uniqueness of such a solution are two very closely related problems which are both of great interest.

**Remark 2.6.** From the proof of Proposition 2.1 if we substitute the assumption \( (\tilde{V}') \) for the assumption \( (\tilde{V}) \), then only by making some minor modifications we can also prove the blowing solution \( u_m \) in Remark 1.1 is non-degenerate.

It follows from Remark 1.1 and Remark 2.6 that if the assumptions \( (V) \) and \( (\tilde{V}) \) hold, then problem (1.3) has a non-degenerate \( m \)-bubbling solution of the form \( u_m = Z_{\tilde{r}_m, \tilde{y}_m, \lambda_m} + \varphi_m \), where \( \varphi_m \in H_s \). Moreover, as \( m \to +\infty \), \( \lambda_m \in [L_0 m^{\frac{N-2}{N}}, L_1 m^{\frac{N-2}{N-2}}] \), \( (\tilde{r}_m, \tilde{y}_m) \to (r_0, y_0^*) \), and \( \lambda_m^{\frac{N-2}{2}} \|\varphi_m\|_{L^\infty} \to 0 \).

3. **Construction of a new bubble solution**

With the non-degenerate result obtained in section 2 at hand, we can construct a new multi-bubbling solution for (1.3) as in [21] [22].

Set \( n \geq m \) be a large even integer. Recall that

\[
\hat{x}_j = \left( \hat{r} \cos \frac{2(j-1)\pi}{m}, \hat{r} \sin \frac{2(j-1)\pi}{m}, 0, 0, \tilde{y}^* \right), \quad j = 1, \ldots, m, \quad \tilde{y}^* = (\tilde{y}_5, \tilde{y}_6, \ldots, \tilde{y}_N),
\]

\[
\lambda_m^{\frac{N-2}{2}} \|\varphi_m\|_{L^\infty} \to 0.
\]
and
\[ p_j = \left( 0, 0, t \cos \frac{2(j-1)\pi}{n}, t \sin \frac{2(j-1)\pi}{n}, \bar{y}^* \right), \]
where \( t \) is close to \( r_0 \) and \( \bar{y}^* \) is close to \( y_0^* = (y_{0,5}, y_{0,6}, \ldots, y_{0,N}) \in \mathbb{R}^{N-4} \).

Define
\[ \|u\|_z = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{n} \frac{1}{(1 + \mu_n|y - p_{n,j}|)^{\frac{n+2}{2}}} \right)^{-1} \left| u(y) \right| \] (3.1)
and
\[ \|f\|_z = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{n} \frac{1}{(1 + \mu_n|y - p_{n,j}|)^{\frac{n+2}{2}}} \right)^{-1} \left| f(y) \right|, \] (3.2)
where \( p_{n,j} = (t_n \cos \frac{2(j-1)\pi}{n}, t_n \sin \frac{2(j-1)\pi}{n}, x_n^*), \quad \tau = \frac{N-4}{2}. \)

Let \( u_m \) be the \( m \)-bubbling solutions in Remark 2.6, where \( m > 0 \) is a large even integer.
Since \( m \) is even, \( u_m \) is even in \( y_j, j = 1, 2, 3, 4. \)

We define
\[ X_s = \{ u : u \in H_s, u \text{ is even in } y_h, h = 1, 2, 3, 4, \]
\[ u(y_1, y_2, t \cos \theta, t \sin \theta, y^*) = u(y_1, y_2, t \cos(\theta + \frac{2\pi j}{n}), t \sin(\theta + \frac{2\pi j}{n}), y^*) \}, \]
where \( y^* = (y_5, y_6, \ldots, y_N). \)

Denote
\[ \mathcal{M}_j = \left\{ y = (y', y_3, y_4, y^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-4} : \left| \left( \frac{y_3, y_4}{|y_3, y_4|} \right), \left( \frac{p_{j3}, p_{j4}}{|p_{j3}, p_{j4}|} \right) \right| \geq \cos \frac{\pi}{n} \right\}. \]

Assume that
\[ |(t, \bar{y}^*) - (r_0, y_0^*)| \leq \vartheta, \] (3.3)
where \( \vartheta > 0 \) is a small constant.

Observe that both \( u_m \) and \( \sum_{j=1}^{n} U_{p_{j,\mu}} \) belong to \( X_s \), while \( u_m \) and \( \sum_{j=1}^{n} U_{p_{j,\mu}} \) are separated from each other. We intend to construct a solution for (1.3) of the form
\[ u = u_m + \sum_{j=1}^{n} \tilde{\zeta}(y) U_{p_{j,\mu}} + \psi := u_m + \tilde{\zeta}(y) Z^*_{t,\bar{y}^*,\mu}(y) + \psi := u_m + Z_{t,\bar{y}^*,\mu}(y) + \psi, \]
where \( \psi \in X_s \) is a small perturbed term. Recall that \( Z_{p_{j,\mu}} = \tilde{\zeta}(y) U_{p_{j,\mu}}. \)

Define the linear operator
\[ Q_n \psi = -\Delta \psi + V(y) \psi - (2^*-1) \left( u_m + \sum_{j=1}^{n} Z_{p_{j,\mu}} \right)^{2^*-2} \psi, \quad \psi \in X_s. \] (3.4)

Denote
\[ D_{j,1} = \frac{\partial Z_{p_{j,\mu}}}{\partial \mu}, \quad D_{j,2} = \frac{\partial Z_{p_{j,\mu}}}{\partial t}, \quad D_{j,k} = \frac{\partial Z_{p_{j,\mu}}}{\partial y^*_k}, k = 5, 6, \ldots, N. \]
Let \( g_n \in X_s \). Now we consider
\[
Q_n \psi_n = g_n + \sum_{i=1}^{N-2} a_{n,i} \sum_{j=1}^{n} Z_{p_{j},\mu}^{2} D_{j,i},
\]
for some constants \( a_{n,i} \), depending on \( \psi_n \).

**Lemma 3.1.** Suppose that \( \psi_n \) solves (3.5). If \( \|g_n\|_{L^2} \to 0 \), then \( \|\psi_n\|_{L^2} \to 0 \).

**Proof.** We argue by contradiction. Suppose that there exist \( n \to +\infty, \tilde{t}_n \to r_0, \tilde{y}_n^* \to y_0^* \), \( \mu_n \in [L_0 n^{\frac{N-2}{2}}, L_1 n^{\frac{N-2}{2}}] \) and \( \psi_n \) solving (3.5) for \( g = g_n, \mu = \mu_n, \tilde{t} = \tilde{t}_n, \tilde{y}^* = \tilde{y}_n^* \) with \( \|g_n\|_{L^2} \to 0 \) and \( \|\psi_n\|_{L^2} \geq c > 0 \). We may assume that \( \|\psi_n\|_{L^2} = 1 \). For simplicity, we drop the subscript \( n \).

We have
\[
|\psi(y)| \leq C \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} (u_m(z) + Z_{t,\tilde{y}^* \cdot \mu}(z))^2 \cdot |\psi(z)| \, dz
\]
\[
+ C \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} (|g| + \left| \sum_{l=1}^{N-2} a_l \sum_{j=1}^{m} D_{p_{j},\mu}^{2} D_{j,l} \right|) \, dz.
\]

Applying the following inequality that for \( \beta, \gamma \) being any two complex numbers, there holds
\[
|\beta + \gamma|^p \leq \max\{1, 2^{p-1}\}(|\beta|^p + |\gamma|^p), \quad (0 < p < +\infty)
\]
we have
\[
(\sum_{l=1}^{N-2} a_l \sum_{j=1}^{m} D_{p_{j},\mu}^{2} D_{j,l})^2 \leq C \left( \sum_{l=1}^{N-2} a_l \sum_{j=1}^{m} D_{p_{j},\mu}^{2} D_{j,l} \right)^2 \leq C \left( Z_{t,\tilde{y}^* \cdot \mu} \right)^2 \quad \text{in every } M_j,
\]
where we use the fact that \( u_m \) is bounded in \( M_j \).

As in [24], using Lemmas B.2 and B.3 from (3.8) we can prove
\[
\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} (u_m + Z_{t,\tilde{y}^* \cdot \mu})^{2} \cdot |\psi| \, dz
\]
\[
\leq \int_{\bigcup_{j=1}^{m} M_j} \frac{1}{|z-y|^{N-2}} (u_m + Z_{t,\tilde{y}^* \cdot \mu})^{2} \cdot |\psi| \, dz
\]
\[
\leq C \sum_{j=1}^{m} \int_{M_j} \frac{1}{|z-y|^{N-2}} (u_m + Z_{t,\tilde{y}^* \cdot \mu})^{2} \cdot |\psi| \, dz
\]
\[
\leq C \sum_{j=1}^{m} \int_{M_j} \frac{1}{|z-y|^{N-2}} (Z_{t,\tilde{y}^* \cdot \mu})^{2} \cdot |\psi| \, dz
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} (Z_{t,\tilde{y}^* \cdot \mu})^{2} \cdot |\psi| \, dz
\]
\[
\leq C \left\| \psi \right\|_{L^2} \sum_{j=1}^{m} \frac{\mu^{N-2}}{(1 + \mu |y - p_j|)^{N-2 + \tau + \ell}},
\]
Assume that
\[ |(t, \tilde{y}^*) - (r_0, y_0^n)| \leq \vartheta, \] (3.10)
where \( \vartheta > 0 \) is a small constant.

\[
\int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} |g(z)| dz \leq C\|g\|_{\mathcal{L}^1} \sum_{j=1}^{n} \frac{1}{(1 + \mu|y - x_j|)^{\frac{N-2}{2} + \tau}}
\] (3.11)

and
\[
\int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \sum_{j=1}^{N} D_x^{2-\gamma} Z_{j,l} dz \leq C\mu^{\frac{N-2}{2} + n_1} \sum_{j=1}^{n} \frac{1}{(1 + \mu|y - p_j|)^{\frac{N-2}{2} + \tau}},
\] (3.12)

where \( n_j = 1, j = 2, \cdots, N - 2 \), and \( n_1 = -1 \).

To estimate \( a_l, l = 1, 2, \cdots, N - 2 \), multiplying \( (3.5) \) by \( D_{1,l}(l = 1, 2, \cdots, N - 2) \) and integrating, we see that \( a_l \) satisfies
\[
\sum_{h=1}^{n} a_h \sum_{j=1}^{n_{\mu}} Z_{p_j,H_1} D_{j,h} D_{1,l} \nonumber = \left( -\Delta \psi + V(r, y^*) \psi - (2^* - 1)Z_{l,y_0^n,\mu} \psi, D_{1,l} \right) - \left( g, D_{1,l} \right).
\] (3.13)

It follows from Lemma B.1 that
\[
\left| \left( g, D_{1,l} \right) \right| \leq C\|g\|_{\mathcal{L}^{\frac{1}{2}}} \sum_{j=1}^{n} \frac{\mu^{\frac{N-2}{2} + n_1}}{(1 + \mu|y - p_j|)^{\frac{N-2}{2} + \tau}} \sum_{j=1}^{n} \frac{\mu^{\frac{N-2}{2} + n_1}}{(1 + \mu|y - p_j|)^{\frac{N-2}{2} + \tau}} \leq C\mu^{n_1} \|g\|_{\mathcal{L}^{\frac{1}{2}}}.
\] (3.14)

Similar to (2.10) in [21], we can estimate
\[
\left| \left( V(r, y^*) \psi, D_{1,l} \right) \right| = O\left( \frac{\mu^n \|\psi\|_{\mathcal{L}^{\frac{1}{2}}}}{\mu^{1+\epsilon}} \right),
\] (3.15)

where we use the fact that for any \( |(r, y^*) - (r_0, y_0^n)| \leq 2\delta, \)
\[
\frac{1}{\mu} \leq \frac{C}{1 + \mu|y - p_j|}.
\] (3.16)

On the other hand, direct calculation gives
\[
\left| \left( -\Delta \psi - (2^* - 1)(u_m + Z_{l,y_0^n,\mu})^{2-2} \psi, D_{1,l} \right) \right| = O\left( \frac{\mu^n \|\psi\|_{\mathcal{L}^{\frac{1}{2}}}}{\mu^{1+\epsilon}} \right).
\] (3.17)

Combining (3.14), (3.15), (3.17), we have
\[
\left( -\Delta \psi + V(t, y^*) \psi - (2^* - 1)(u_m + Z_{l,y_0^n,\mu})^{2-2} \psi, D_{1,l} \right) - \left( g, D_{1,l} \right) = O\left( \frac{\mu^n \|\psi\|_{\mathcal{L}^{\frac{1}{2}}}}{\mu^{1+\epsilon}} + \|g\|_{\mathcal{L}^{\frac{1}{2}}}, \right)
\] (3.18)
It is easy to check that
\[
\sum_{j=1}^{n} \langle D_{p_{j}, \mu}^{2^*-2} D_{j,h, l}, D_{1,l} \rangle = (\bar{c} + o(1)) \delta_{j,l} \mu^{2n_l} \tag{3.19}
\]
for some constant \( \bar{c} > 0 \).

Now inserting (3.18) and (3.19) into (3.13), we find
\[
a_l = \frac{1}{\mu^{n_l}} \left( o(\|\psi\|_{*}) + O(\|g\|_{*}) \right). \tag{3.20}
\]
So,
\[
\|\psi\|_{*} \leq \left( o(1) + \|g_n\|_{*} + \sum_{j=1}^{n} \frac{1}{(1 + \mu|y - p_j|)^{\frac{N-2}{2}}} \right) \tag{3.21}
\]
Since \( \|\psi\|_{*} = 1 \), we obtain from (3.21) that there is \( R > 0 \) such that
\[
\|\mu^{-\frac{N-2}{2}} \psi\|_{L^\infty(B_{R, \mu}(p_j))} \geq a > 0, \tag{3.22}
\]
for some \( j \). But \( \tilde{\psi}(y) = \mu^{-\frac{N-2}{2}} \psi(\mu(y - p_j)) \) converges uniformly in any compact set to a solution \( u \) of
\[
-\Delta u + V u - (2^* - 1) u_m^{2^*-2} u = 0, \quad \text{in} \quad \mathbb{R}^N, \tag{3.23}
\]
for some \( \Lambda \in [\Lambda_1, \Lambda_2] \). So it follows from Proposition 2.1 that \( u = 0 \). This is a contradiction to (3.22). \( \square \)

We want to construct a solution \( u \) for (1.3) with
\[
u = u_m + \sum_{j=1}^{n} \hat{\zeta} U_{p_j, \mu} + \psi,
\]
where \( \psi \in X_{s} \) is a small perturbed term, satisfying
\[
\int_{\mathbb{R}^N} Z_{p_{j}, \mu}^{2^*-2} D_{j,l} \psi = 0, \quad j = 1, \ldots, n, l = 1, 2, \ldots, N - 2.
\]
Then \( \psi \) satisfies
\[
Q_n \psi = l_n + R_n(\psi),
\]
where
\[
Q_n \psi = -\Delta \psi + V(y) \psi - (2^* - 1) \left( u_m + \sum_{j=1}^{n} Z_{p_j, \mu} \right)^{2^*-2} \psi,
\]
\[
l_n = \left( u_m + \sum_{j=1}^{n} Z_{p_j, \mu} \right)^{2^*-1} - u_m^{2^*-1} - \sum_{j=1}^{n} \hat{\zeta} U_{p_j, \mu}^{2^*-1} - V(y) \sum_{j=1}^{n} Z_{p_j, \mu} + Z_{t, \tilde{g}, \mu}^{*} \Delta \hat{\zeta} + 2 \nabla \hat{\zeta} \nabla Z_{t, \tilde{g}, \mu}.
\]
and
\[ R_n(\psi) = \left( u_m + \sum_{j=1}^{n} Z_{p_j,\mu} + \psi \right)^{2^*-1} - \left( u_m + \sum_{j=1}^{n} Z_{p_j,\mu} \right)^{2^*-1} - (2^* - 1) \left( u_m + \sum_{j=1}^{n} Z_{p_j,\mu} \right)^{2^*-2} \psi. \]

We have the following estimate for \( \| l_n \|_{\tilde{\tau}} \).

**Lemma 3.2.** There exists a small \( \epsilon > 0 \), such that
\[ \| l_n \|_{\tilde{\tau}} \leq \frac{C}{\mu^{1+\epsilon}}. \] (3.24)

**Proof.** First, we write
\[ l_n = \left[ \left( \sum_{j=1}^{n} \hat{\zeta}_{p_j,\mu} \right)^{2^*-1} - \sum_{j=1}^{n} \hat{\zeta}_{2^*-1} \right] - V(y) \sum_{j=1}^{n} \hat{\zeta}_{p_j,\mu} 
+ Z_{t,\tilde{\gamma}^*,\mu} \Delta \hat{\zeta} + 2 \nabla \hat{\zeta} \nabla Z_{t,\tilde{\gamma}^*,\mu} 
+ \left[ \left( u_m + \sum_{j=1}^{n} \hat{\zeta}_{p_j,\mu} \right)^{2^*-1} - u_m^{2^*-1} - \left( \sum_{j=1}^{n} \hat{\zeta}_{p_j,\mu} \right)^{2^*-1} \right] 
:= J_0 + J_1 + J_2 + J_3 + J_4. \] (3.25)

Just by the same argument as that of Lemma 2.5 in [21], we can estimate
\[ J_0 + J_1 + J_2 + J_3 \leq \frac{C}{\mu^{1+\epsilon}} \sum_{j=1}^{n} \mu^{\frac{N+2}{2}} \left( 1 + \mu |y - p_j| \right)^{\frac{N+2}{2}}. \] (3.26)

Now we estimate \( J_4 \). Using the assumed symmetry, we just need to estimate \( J_4 \) in \( M_1 \).

Denote \( B_1 = M_1 \cap B_{\mu^{-\frac{1}{2}}}(p_1) \). Note that, it holds \( U_{p_1,\mu} \geq c_0 \) in \( B_1 \).

When \( y \in B_1 \), applying the following formula
\[ (1 + t)^p = 1 + pt + O(t^p), \quad \text{for} \quad t \geq 0 \quad \text{and} \quad p \in (1, 2], \]
we have
\[ |J_4| \leq C U_{p_1,\mu}^{2^*-2} \left( u_m + \sum_{j=2}^{n} U_{p_j,\mu} \right) \left( \sum_{j=2}^{n} U_{p_j,\mu} \right)^{2^*-1} + \tilde{J}_4 := J_{4,1} + J_{4,2} + \tilde{J}_4, \] (3.27)

where \( \tilde{J}_4 \leq C \) in \( B_1 \).

For \( y \in M_1 \), by Lemma [B.1] we obtain
\[ J_{4,2} \leq C \mu^{\frac{N+2}{2}} \left( \sum_{j=2}^{n} \frac{1}{(1 + \mu |y - p_j|)^{N-2}} \right)^{\frac{N+2}{N-2}} \leq C \mu^{\frac{N+2}{2}} \left( \sum_{j=2}^{n} \frac{1}{(1 + \mu |y - p_1|)^{\frac{N+2}{2}}} \frac{1}{(1 + \mu |y - p_j|)^{\frac{N-2}{2}}} \right)^{\frac{N+2}{N-2}} \]
Similar to Lemma 2.5 in [21], we can prove noting that
\[
N
\]
we have
\[
y
\]
On the other hand, when \( y \in \mathbb{B}_1 \), there holds
\[
J_{4,1} \leq \left| U_{p_1,\mu}^{2^*-2} (u_m + \sum_{j=2}^n U_{p_j,\mu}) \right| \leq C U_{p_1,\mu}^{2^*-2} + C U_{p_1,\mu}^{2^*-2} \sum_{j=2}^n U_{p_j,\mu}
\]
\[
:= J_{4,1,1} + J_{4,1,2}.
\]
Similar to Lemma 2.5 in [21], we can prove
\[
|J_{4,1,2}| \leq \frac{C \mu^{N+2}}{(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in \mathbb{B}_1.
\]
Moreover, if \( N \geq 5 \) and \( y \in \mathbb{B}_1 \),
\[
(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau - 4} \leq C \mu^{\frac{1}{2} \left( \frac{N+2}{2} + \tau - 4 \right)},
\]
noting that \( \frac{N+2}{4} - \frac{\tau}{2} > 1 \), then we have
\[
J_{4,1,1} \leq \frac{\mu^{N+2}}{(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}} \left( \frac{\mu^{2 - \frac{N+2}{2}}}{(1 + \mu |y - p_1|)^{4 - \frac{N+2}{2} + \tau}} \right)
\]
Observe that

\[ \frac{M}{\mu^{N+2}} \leq C \left( 1 + \mu |y - p_1| \right)^{N+2} \]

On the other hand, noting that in \( \mathbb{M}_1 \setminus \mathbb{B}_1 \), it holds \( U_{p_1, \mu} \leq C \), therefore,

\[ |J_4| \leq C \sum_{j=1}^{n} Z_{p_j, \mu} + C \left( \sum_{j=1}^{n} U_{p_j, \mu} \right)^{2^{r-1}} = \hat{J}_{4,1} + \hat{J}_{4,2}. \]

Observe that

\[ \hat{J}_{4,1} = \sum_{j=1}^{n} \hat{\zeta}(y) U_{p_j, \mu} \leq \hat{\zeta}(y) U_{p_1, \mu} + \sum_{j=2}^{n} \hat{\zeta}(y) U_{p_j, \mu}. \]

We have

\[ \hat{\zeta} U_{p_1, \mu} \leq \frac{\hat{\zeta}(y) \mu^{N+2}}{(1 + \mu |y - p_1|)^{N-2}} \leq \frac{\hat{\zeta}(y) \mu^{N+2}}{\mu^2 (1 + \mu |y - p_1|)^{N-2}} \leq \frac{C}{\mu^{1+\epsilon} (1 + \mu |y - p_1|)^{N-1-\epsilon}} \]

since \( N - 1 - \epsilon > \frac{N+2}{2} + \tau \). Also for \( y \in \mathbb{M}_1 \setminus \mathbb{B}_1 \) we can estimate

\[ \sum_{j=2}^{n} \hat{\zeta}(y) U_{p_j, \mu} \leq C \sum_{j=2}^{n} \frac{\hat{\zeta}(y) \mu^{N+2}}{(1 + \mu |y - p_j|)^{N-2}} \leq \frac{C}{\mu^{1+\epsilon} (1 + \mu |y - p_1|)^{N-2}} \leq \frac{C}{\mu^{1+\epsilon} (1 + \mu |y - p_1|)^{N-1-\epsilon}} \]

since \( N - 1 - \tau - \epsilon \geq \frac{N+2}{2} + \tau \).
Finally, we have

\[ |\hat{J}_{4,2}| \leq CU_{p_j,\mu}^{2^*-1} + C \left( \sum_{j=2}^{n} U_{p_j,\mu} \right)^{2^*-1} = \hat{J}_{4,2,1} + \hat{J}_{4,2,2}. \]

And from (3.28), we have

\[ |\hat{J}_{4,2,2}| \leq C J_{4,2} \leq C \frac{\mu^{\frac{N+2}{2}}}{(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}} \quad y \in \mathbb{M}_1 \setminus \mathbb{B}_1. \tag{3.37} \]

For \( y \in \mathbb{M}_1 \setminus \mathbb{B}_1, |y - p_1| \geq \mu^{-\frac{1}{2}} \) and \( \mu |y - p_1| \geq \mu^{\frac{1}{2}}, \) then from \( \frac{1}{2}(\frac{N+2}{2} - \tau) = \frac{N+2}{4} - \frac{N-4}{2(N-2)} > 1, \) we have

\[
\hat{J}_{4,2,1} \leq C \frac{\mu^{\frac{N+2}{2}}}{(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}} \leq C \frac{\mu^{\frac{N+2}{2}}}{(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}} \frac{1}{(1 + \mu |y - p_1|)^{\frac{N+2}{2} - \frac{1}{2}(\frac{N+2}{2} - \tau)}}
\leq C \frac{\mu^{\frac{N+2}{2}}}{\mu^{1+\epsilon}(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in \mathbb{M}_1 \setminus \mathbb{B}_1. \tag{3.38} \]

From (3.34) to (3.38), we obtain

\[ J_4 \leq C \frac{\mu^{\frac{N+2}{2}}}{\mu^{1+\epsilon}(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in \mathbb{M}_1 \setminus \mathbb{B}_1. \tag{3.40} \]

Combining (3.33) and (3.40), applying the symmetry we have

\[ J_4 \leq C \frac{\mu^{\frac{N+2}{2}}}{\mu^{1+\epsilon}(1 + \mu |y - p_1|)^{\frac{N+2}{2} + \tau}} \sum_{j=1}^{n} \frac{\mu^{\frac{N+2}{2}}}{(1 + \mu |y - p_j|)^{\frac{N+2}{2} + \tau}}. \tag{3.41} \]

By (3.25), (3.26) and (3.41), we obtain

\[ \|l_n\|_{\tilde{\mathcal{S}}} \leq \frac{C}{\mu^{1+\epsilon}}. \]

We also need the following estimates.

**Lemma 3.3.** If \( N \geq 5, \) then there holds

\[ \|R_n(\psi)\|_{\tilde{\mathcal{S}}} \leq C \|\psi\|_{\mathcal{S}}^{\min\{2^*-1, 2\}}. \]

**Proof.** Since it can be proved by the same argument as that of Lemma 2.4 in [21], here we omit its proof. \( \square \)
We define
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(y)u^2) \, dy - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \, dy. \]

Set
\[ F(t, \tilde{y}^*, \mu) = I \left( u_m + \sum_{j=1}^n Z_{p_j, \mu} + \psi_n \right). \]

To get a solution of the form \( u_m + \sum_{j=1}^n Z_{p_j, \mu} + \psi_n \), we only need to find a critical point for \( F(t, \tilde{y}^*, \mu) \) in \( B_{\vartheta}(r_0, y_0^*) \times \left[ C_1 n^{\frac{N-2}{4}}, C_2 n^{\frac{N-2}{4}} \right] \), where \( \vartheta > 0 \) is small, \( C_1, C_2 \) are different constants.

Now we will prove Theorem 1.2.

**Proof of Theorem 1.2** By direct computation, we have
\[ F(t, \tilde{y}^*, \mu) = I \left( u_m + \sum_{j=1}^n Z_{p_j, \mu} \right) + nO \left( \frac{1}{\mu^{2+\epsilon}} \right). \] (3.42)

On the other hand, we get
\[
I \left( u_m + \sum_{j=1}^n Z_{p_j, \mu} \right) = I \left( \sum_{j=1}^n Z_{p_j, \mu} \right) + I(u_m) + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{j=1}^n u_m^{2^*-1} Z_{p_j, \mu} \nonumber \]
\[ - \frac{1}{2^*} \int_{\mathbb{R}^N} \left( \left( u_m + \sum_{j=1}^n Z_{p_j, \mu} \right)^{2^*} - u_m^{2^*} - \left( \sum_{j=1}^n Z_{p_j, \mu} \right)^{2^*} \right). \] (3.43)

It is not difficult to check
\[ \int_{\mathbb{R}^N} u_m^{2^*-1} Z_{p_j, \mu} = O \left( \frac{1}{\mu^{\frac{N-2}{2}}} \right). \]

For \( y \in \mathbb{R}^N \setminus \bigcup_{j=1}^n B_j \), we have
\[
\left| \left( u_m + \sum_{j=1}^n Z_{p_j, \mu} \right)^{2^*} - u_m^{2^*} - \left( \sum_{j=1}^n Z_{p_j, \mu} \right)^{2^*} \right| \leq C u_m^{2^*-1} \sum_{j=1}^n Z_{p_j, \mu} + \left( \sum_{j=1}^n Z_{p_j, \mu} \right)^{2^*} \leq C u_m^{2^*-1} \sum_{j=1}^n Z_{p_j, \mu}. \]

Hence we obtain
\[
\int_{\mathbb{R}^N \setminus \bigcup_{j=1}^n B_j} \left| \left( u_m + \sum_{j=1}^n Z_{p_j, \mu} \right)^{2^*} - u_m^{2^*} - \left( \sum_{j=1}^n Z_{p_j, \mu} \right)^{2^*} \right| \leq C \int_{\mathbb{R}^N} u_m^{2^*-1} \sum_{j=1}^n Z_{p_j, \mu} = O \left( \frac{n}{\mu^{\frac{N-2}{2}}} \right). \]
By symmetry, we have
\[
\int_{B_1} \left| (u_m + \sum_{j=1}^{n} Z_{p_j, \mu})^{2^*} - u_m^{2^*} - (\sum_{j=1}^{n} Z_{p_j, \mu})^{2^*} \right| \leq n \int_{B_1} \left| (u_m + \sum_{j=1}^{n} Z_{p_j, \mu})^{2^*} - u_m^{2^*} - (\sum_{j=1}^{n} Z_{p_j, \mu})^{2^*} \right|
\]

There holds
\[
\int_{B_1} u_m^{2^*} = O\left(\frac{1}{\mu^{N/2}}\right),
\]
and
\[
\int_{B_1} \left| (u_m + \sum_{j=1}^{n} Z_{p_j, \mu})^{2^*} - (\sum_{j=1}^{n} Z_{p_j, \mu})^{2^*} \right| \leq C \int_{B_1} \left( \sum_{j=1}^{n} Z_{p_j, \mu} \right)^{2^* - 1}
\]
\[
\leq C \int_{B_1} \left( U_{p_1, \mu}^{2^* - 1} + \frac{\mu^{N/2}}{(1 + \mu |y - p_1|)(2^* - 1)(N-2)(1-\tau_1)} \right) \leq \frac{C}{\mu^{N/2}},
\]
where \( \tau_1 = \frac{N-4}{N(N-2)^2} \).

Hence we have proved
\[
I(u_m + \sum_{j=1}^{n} Z_{p_j, \mu}) = I(\sum_{j=1}^{n} Z_{p_j, \mu}) + I(u_m) + O\left(\frac{n}{\mu^{N/2}}\right). \tag{3.44}
\]

Moreover, by direct computation we can obtain
\[
I(\sum_{j=1}^{n} Z_{p_j, \mu}) = n \left( A_1 + A_2 \frac{V(t, \tilde{y}^*)}{\mu^2} - \sum_{j=2}^{n} \frac{A_3}{\mu^{N-2}|p_j - p_1|^{N-2}} + O\left(\frac{1}{\mu^{2+\epsilon}}\right) \right), \tag{3.45}
\]
where \( A_1 = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} U_{0,1}^{2^*} \) and \( A_i(i = 2, 3) \) are some positive constants.

It follows from (3.42), (3.44) and (3.45) that
\[
F(t, \tilde{y}^*, \mu) = I(\sum_{j=1}^{n} Z_{p_j, \mu}) + I(u_m) + nO\left(\frac{1}{\mu^{2+\epsilon}}\right)
\]
\[
= I(u_m) + nA_1 + n\left( A_2 \frac{V(t, \tilde{y}^*)}{\mu^2} - \sum_{j=2}^{n} \frac{A_3}{\mu^{N-2}|p_j - p_1|^{N-2}} \right) + O\left(\frac{n}{\mu^{2+\epsilon}}\right), \tag{3.46}
\]
where \( A_i(i = 1, 2, 3) \) are the same as those of (3.45).
Now in order to find a critical point for $F(t, y^*, \mu)$, we only need to continue exactly as section 3 in \cite{21}. One can also see section 3 in \cite{22}. Here we omit the detailed process of its proof.

\section*{Appendix A. Some Pohozaev identities}

Set

$$-\Delta u + V(|y'|, y'')u = u^{2^*-1}, \quad (A.1)$$

and

$$-\Delta \eta + V(|y'|, y'')\eta = (2^* - 1)u^{2^*-2}\eta. \quad (A.2)$$

Suppose that $\Omega$ is a smooth domain in $\mathbb{R}^N$.

We have the following identities which are used in section 2 by proving the non-degeneracy of the multi-bubbling solutions obtained in \cite{21}.

\textbf{Lemma A.1.} There holds

$$-\int_{\partial \Omega} \frac{\partial u}{\partial \nu} \frac{\partial \eta}{\partial y_i} - \int_{\partial \Omega} \frac{\partial \eta}{\partial \nu} \frac{\partial u}{\partial y_i} + \int_{\partial \Omega} \langle \nabla u, \nabla \eta \rangle \nu_i + \int_{\partial \Omega} Vu \eta \nu_i - \int_{\partial \Omega} u^{2^*-1} \eta \nu_i$$

$$= \int_\Omega \frac{\partial V}{\partial y_i} u \eta, \quad (A.3)$$

and

$$\int_\Omega u \eta \langle \nabla V, y - x_0 \rangle + 2 \int_\Omega V \eta u = - \int_{\partial \Omega} u^{2^*-1} \eta \langle \nu, y - x_0 \rangle - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \langle \nabla \eta, y - x_0 \rangle$$

$$- \int_{\partial \Omega} \frac{\partial \eta}{\partial \nu} \langle \nabla u, y - x_0 \rangle + \int_{\partial \Omega} \langle \nabla u, \nabla \eta \rangle \langle \nu, y - x_0 \rangle + \int_{\partial \Omega} V \eta \langle \nu, y - x_0 \rangle$$

$$+ \frac{2 - N}{2} \int_{\partial \Omega} \eta \frac{\partial u}{\partial \nu} + \frac{2 - N}{2} \int_{\partial \Omega} u \frac{\partial \eta}{\partial \nu}. \quad (A.4)$$

\textbf{Proof. Proof of (A.3).} First we have

$$\int_\Omega (-\Delta u + Vu) \frac{\partial \eta}{\partial y_i} = \int_\Omega u^{2^*-1} \frac{\partial \eta}{\partial y_i},$$

and

$$\int_\Omega (-\Delta \eta + V \eta) \frac{\partial u}{\partial y_i} = \int_\Omega (2^* - 1)u^{2^*-2} \eta \frac{\partial u}{\partial y_i},$$

which implies that

$$\int_\Omega \left( -\Delta u \frac{\partial \eta}{\partial y_i} + (-\Delta \eta) \frac{\partial u}{\partial y_i} + Vu \frac{\partial \eta}{\partial y_i} + V\eta \frac{\partial u}{\partial y_i} \right) = \int_\Omega \left( u^{2^*-1} \frac{\partial \eta}{\partial y_i} + (2^* - 1)u^{2^*-2} \eta \frac{\partial u}{\partial y_i} \right). \quad (A.5)$$
It is easy to check that
\[
\int_{\Omega} \left( u^{2^*-1} \frac{\partial \eta}{\partial y_i} + (2^* - 1) u^{2^*-2} \eta \frac{\partial u}{\partial y_i} \right) = \int_{\Omega} \frac{\partial (u^{2^*-1} \eta)}{\partial y_i} = \int_{\partial \Omega} u^{2^*-1} \eta \nu_i. \tag{A.6}
\]
Moreover, similar to (2.7) in [15], we have
\[
\int_{\Omega} \left( - \Delta u \frac{\partial \eta}{\partial y_i} + (- \Delta \eta) \frac{\partial u}{\partial y_i} \right) = - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \frac{\partial \eta}{\partial y_i} - \int_{\partial \Omega} \frac{\partial \eta}{\partial \nu} \frac{\partial u}{\partial y_i} + \int_{\partial \Omega} \langle \nabla u, \nabla \eta \rangle \nu_i, \tag{A.7}
\]
and
\[
\int_{\Omega} \left( V u \frac{\partial \eta}{\partial y_i} + V \eta \frac{\partial u}{\partial y_i} \right) = \int_{\Omega} V \frac{\partial}{\partial y_i} (u \eta) = \int_{\partial \Omega} V u \eta \nu_i - \int_{\Omega} u \eta \frac{\partial V}{\partial y_i}. \tag{A.8}
\]
It follows from (A.5) to (A.8) that (A.3) holds.

**Proof of (A.4).** It is easy to check that
\[
\int_{\Omega} \left( \left( - \Delta u + V u \right) \langle \nabla \eta, y - x_0 \rangle + (- \Delta \eta + V \eta) \langle \nabla u, y - x_0 \rangle \right) = \int_{\Omega} \left( u^{2^*-1} \langle \nabla \eta, y - x_0 \rangle + (2^* - 1) u^{2^*-2} \eta \langle \nabla u, y - x_0 \rangle \right). \tag{A.9}
\]
We find that
\[
\int_{\Omega} \left( u^{2^*-1} \langle \nabla \eta, y - x_0 \rangle + (2^* - 1) u^{2^*-2} \eta \langle \nabla u, y - x_0 \rangle \right) = \int_{\Omega} \langle \nabla (u^{2^*-1} \eta), y - x_0 \rangle = \int_{\partial \Omega} u^{2^*-1} \eta \nu_i - N \int_{\Omega} u^{2^*-1} \eta. \tag{A.10}
\]
Also similar to (2.10) in [15], we have
\[
\int_{\Omega} \left( (- \Delta u \langle \nabla \eta, y - x_0 \rangle + (- \Delta \eta) \langle \nabla u, y - x_0 \rangle \right) = - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \langle \nabla \eta, y - x_0 \rangle - \int_{\partial \Omega} \frac{\partial \eta}{\partial \nu} \langle \nabla u, y - x_0 \rangle + \int_{\partial \Omega} \langle \nabla u, \nabla \eta \rangle \nu_i + (2 - N) \int_{\Omega} \langle \nabla u, \nabla \eta \rangle. \tag{A.11}
\]
On the other hand, there holds
\[
2^* \int_{\Omega} u^{2^*-1} \eta = \int_{\Omega} \left( (- \Delta u \eta + u (- \Delta \eta) + V u \eta + V \eta u) \right) = 2 \int_{\Omega} \langle \nabla u, \nabla \eta \rangle - \int_{\partial \Omega} u \frac{\partial \eta}{\partial \nu} - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} + 2 \int_{\Omega} V u \eta, \tag{A.12}
\]
which yields
\[
\int_{\Omega} \langle \nabla u, \nabla \eta \rangle = \frac{2^*}{2} \int_{\Omega} u^{2^*-1} \eta + \frac{1}{2} \int_{\partial \Omega} u \frac{\partial \eta}{\partial \nu} + \frac{1}{2} \int_{\partial \Omega} \frac{\partial u}{\partial \nu} - \int_{\Omega} V u \eta. \tag{A.13}
\]
Moreover, we obtain
\[
\int_{\Omega} (Vu(\nabla \eta, y - x_0) + V\eta(\nabla u, y - x_0)) = \int_{\Omega} V(\nabla (u\eta), y - x_0)
\]
\[
\int_{\partial \Omega} Vu\eta(\nu, y - x_0) - \int_{\Omega} u\eta(\nabla V, y - x_0) - N \int_{\Omega} Vu\eta.
\]  
(A.14)

Therefore, from (A.9) to (A.14) we know that (A.4) holds. \(\square\)

**Appendix B. Basic estimates**

For each fixed \(k\) and \(j\), \(k \neq j\), we consider the following function
\[
g_{k,j}(y) = \frac{1}{(1 + |y - x_j|)^\alpha} \frac{1}{(1 + |y - x_k|)^\beta}.
\]
(B.1)

where \(\alpha \geq 1\) and \(\beta \geq 1\) are two constants.

**Lemma B.1.** (Lemma B.1, [24]) For any constants \(0 < \delta \leq \min\{\alpha, \beta\}\), there is a constant \(C > 0\), such that
\[
g_{k,j}(y) \leq C \frac{1}{|x_k - x_j|^\delta} \left( \frac{1}{(1 + |y - x_k|)^{\alpha+\beta-\delta}} + \frac{1}{(1 + |y - x_j|)^{\alpha+\beta-\delta}} \right).
\]

**Lemma B.2.** (Lemma B.2, [24]) For any constant \(0 < \delta < N - 2\), there is a constant \(C > 0\), such that
\[
\int \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z|)^{2+\delta}} dz \leq \frac{C}{(1 + |y|)^{\delta}}.
\]

Let us recall that
\[
Z_{t,\tilde{y},\mu}(y) = \sum_{j=1}^{n} \hat{\zeta}(y)U_{p_j,\mu} = [N(N-2)]^{\frac{N-2}{4}} \sum_{j=1}^{n} \hat{\zeta} \left( \frac{\mu}{1 + \mu^2|y - p_j|^2} \right)^{\frac{N-2}{2}}.
\]

Just by the same argument as that of Lemma B.3 in [21], we can prove

**Lemma B.3.** Suppose that \(N \geq 5\). Then there is a small constant \(\iota > 0\), such that
\[
\int \frac{1}{|y - z|^{N-2}} Z_{t,\tilde{y},\mu}^{\frac{N-2}{2}}(z) \sum_{j=1}^{n} \frac{1}{(1 + \mu|z - p_j|)^{\frac{N-2}{2} + 1}} dz \leq \sum_{j=1}^{n} \frac{C}{(1 + \mu|y - p_j|)^{\frac{N-2}{2} + \iota + 1}}.
\]

**Appendix C. An example of the potential \(V(r, y^*)\)**

Here we give an example of \(V(\tilde{y}, y^*)\) which satisfies the assumptions \((V)\) and \((\tilde{V})\). We define
\[
V(r, y^*) = \begin{cases} 
    r^2 - 4r \left( \sum_{j=5}^{N} y_j \right) + \left( \sum_{j=5}^{N} y_j^2 \right) + 1, & B_\rho(r_0, y_0^*), \\
    \geq 0, & \mathbb{R}^N \setminus B_\rho(r_0, y_0^*).
\end{cases}
\]
where $\rho$ is the same as that of [21] and $(r_0, y_0^*)$ is defined below. By some direct computations, we can check that

$$f(r, y^*) := r^2 V(r, y^*) = r^4 - 4r^3 \left( \sum_{j=5}^{N} y_j \right) + r^2 \left( \sum_{j=5}^{N} y_j^2 \right) + r^2,$$

We have

$$\frac{\partial f}{\partial r} = 4r^3 - 12r^2 \left( \sum_{j=5}^{N} y_j \right) + 2r \left( \sum_{j=5}^{N} y_j^2 \right) + 2r,$$

and

$$\frac{\partial f}{\partial y_i} = -4r^3 + 2r^2 y_i, \quad i = 5, \cdots, N.$$

Suppose that $\frac{\partial f}{\partial r} = 0$, $\frac{\partial f}{\partial y_i} = 0$, we obtain

$$y_i = 2r, \quad \text{for} \quad i = 5, \cdots, N, \quad r_0 = \sqrt{\frac{1}{8N - 34}}, \quad y_{0,i} = 2\sqrt{\frac{1}{8N - 34}}(i = 5, \cdots, N).$$

Therefore, $(r_0, y_0^*)$ is a critical point of the function $f(r, y^*)$ and $V(r_0, y_0^*) = \frac{1}{2} > 0$. Also

$$\frac{\partial^2 f}{\partial r^2} = 12r^2 - 24r \left( \sum_{j=5}^{N} y_j \right) + 2 \left( \sum_{j=5}^{N} y_j^2 \right) + 2,$$

and

$$\frac{\partial^2 f}{\partial r \partial y_i} = -12r^2 + 4ry_i, \quad i = 5, \cdots, N,$$

and

$$\frac{\partial^2 f}{\partial y_i^2} = 2r^2 (i = 5, \cdots, N), \quad \frac{\partial^2 f}{\partial y_i \partial y_j} = 0(i, j = 5, \cdots, N, i \neq j).$$

By direct computation, we obtain

$$B = \begin{pmatrix}
\frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial y_5} & \cdots & \frac{\partial^2 f}{\partial r \partial y_N} \\
\frac{\partial^2 f}{\partial r \partial y_5} & \frac{\partial^2 f}{\partial y_5^2} & \cdots & \frac{\partial^2 f}{\partial y_5 \partial y_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial r \partial y_N} & \frac{\partial^2 f}{\partial y_N \partial y_5} & \cdots & \frac{\partial^2 f}{\partial y_N \partial y_N}
\end{pmatrix}_{(N-3) \times (N-3)}.$$n

Also by some tedious computation, we can obtain the eigenvalues of the matrix $B$ are as follows:

$$|\lambda I - B|_{(r_0, y_0^*)} = \left[ (\lambda - \frac{\partial^2 f}{\partial r^2})(\lambda - 2r^2) - \sum_{j=5}^{N} (-12r^2 + 4ry_j)^2 \right] (\lambda - 2r^2)^{N-5} = 0,$$
which implies that
\[
\left( \lambda - \frac{\partial^2 f}{\partial r^2} \bigg|_{(r_0, y_0^*)} \right)(\lambda - 2r_0^2) = \sum_{j=5}^{N} (-12r_0^2 + 4r_0y_{0j})^2, \quad \text{or} \quad \left( \lambda - 2r^2 \right)^{N-5} \bigg|_{(r_0, y_0^*)} = 0.
\]

By further computation, we can check that \(\min\{\lambda_1, \lambda_2\} < 0\) and \(\lambda_3 = \cdots = \lambda_{N-3} = 2r_0^2 > 0\).

Hence the assumption \((V)\) holds.

On the other hand, we recall
\[
V(r, y) = r^2 - 4r \left( \sum_{j=5}^{N} y_j \right) + \left( \sum_{j=5}^{N} y_j^2 \right) + 1, \quad \text{in} \ B_\rho(r_0, y_0^*).
\]

We obtain in \(B_\rho(r_0, y_0^*)\)
\[
\frac{\partial V}{\partial r} = 2r - 4 \sum_{j=5}^{N} y_j, \quad \frac{\partial V}{\partial y_i} = 2y_i - \frac{4y_i}{r} \left( \sum_{j=5}^{N} y_j \right), \quad i = 1, 2, 3, 4,
\]
\[
\frac{\partial V}{\partial y_k} = -4r + 2y_k, \quad k = 5, \cdots, N,
\]
\[
\frac{\partial^2 V}{\partial r^2} = 2, \quad \frac{\partial^2 V}{\partial r \partial y_i} = \frac{2y_i}{r_0^2} (i = 1, 2, 3, 4), \quad \frac{\partial^2 V}{\partial r \partial y_k} = -4(k = 5, \cdots, N), \tag{C.1}
\]
\[
\frac{\partial^2 V}{\partial y_i \partial y_i} = 2 - \frac{4r^2 - 4y_i^2}{r^3} \left( \sum_{j=5}^{N} y_j \right) (i = 1, 2, 3, 4), \quad \frac{\partial^2 V}{\partial y_k^2} = 2(k = 5, \cdots, N). \tag{C.2}
\]

and
\[
\frac{\partial^2 V}{\partial y_k \partial y_j} = 0(j \neq k, k, j = 5, \cdots, N). \tag{C.3}
\]

Then from (C.1) to (C.3), we obtain in \(B_\rho(r_0, y_0^*)\)
\[
\Delta V = \sum_{j=1}^{N} \frac{\partial^2 V}{\partial y_i \partial y_i} = \sum_{i=1}^{4} \left( 2 - \frac{4r^2 - 4y_i^2}{r^3} (y_5 + \cdots + y_N) \right) + 2(N - 4)
\]
\[
= 2N - \frac{12}{r} (y_5 + \cdots + y_N).
\]

Hence
\[
\Delta V \bigg|_{(r_0, y_0^*)} = 96 - 22N.
\]

For \(y_0^* = (y_{0,5}, y_{0,6}, \cdots, y_{0,N})\), one have
\[
\left. \frac{\partial \Delta V}{\partial y_i} \right|_{(r_0, y_0^*)} = \frac{12y_i}{r^3} (y_5 + \cdots + y_N) \bigg|_{(r_0, y_0^*)} = \frac{24y_{0,1}}{r_0^2} (N - 4),
\]
Since \( y_{0,i} = 2r_0(i = 5, \ldots, N) \), we obtain
\[
A_{i,l} = \begin{cases} 
2 - \left( \frac{24y_{0,i}(N-4)}{2r_0^2(822N)} + \frac{\nu_1}{\langle \nu, x_1 \rangle} \right)(34 - 8N)r_0, & \text{when } i = l = 1; \\
-4, & \text{when } i = 1, l = 2, 3, \ldots, N - 3; \\
\cos \frac{2\pi m}{N} (-4 - \left( \frac{-6}{r_0 (822N)} + \frac{\nu_{i+3}}{\langle \nu, x_1 \rangle} \right)(34 - 8N)r_0), & \text{when } i = 2, 3, \ldots, N - 3, l = 1; \\
2, & \text{when } i = 2, 3, \ldots, N - 3, l = 1, \ldots, N - 3; \\
0, & \text{when } i \neq l, i, l = 2, 3, \ldots, N - 3.
\end{cases}
\tag{C.4}
\]

Therefore, from (C.4) we have
\[
\det(A_{i,l})_{(N-3) \times (N-3)} = \prod_{i=2}^{N-3} A_{i,i} \left( A_{1,1} - \sum_{k=2}^{N-3} A_{k,1}A_{1,k} \right) \\
= \prod_{i=2}^{N-3} 2 \left( 2 - \left( \frac{24y_{0,i}(N-4)}{2r_0^2(822N)} + \frac{\nu_1}{\langle \nu, x_1 \rangle} \right)(34 - 8N)r_0 \right) \\
+ \sum_{k=2}^{N-3} 2 \cos \frac{2\pi m}{N} \left( -4 - \left( \frac{-6}{r_0 (822N)} + \frac{\nu_{i+3}}{\langle \nu, x_1 \rangle} \right)(34 - 8N)r_0 \right) \\
\neq 0.
\]

Hence, the assumption \( (\tilde{V}) \) also holds.

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