Learning with risks based on M-location

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Abstract

In this work, we study a new class of risks defined in terms of the location and deviation of the loss distribution, generalizing far beyond classical mean-variance risk functions. The class is easily implemented as a wrapper around any smooth loss, it admits finite-sample stationarity guarantees for stochastic gradient methods, it is straightforward to interpret and adjust, with close links to M-estimators of the loss location, and has a salient effect on the test loss distribution.

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1 Introduction

In machine learning, the important yet ambiguous notion of “good off-sample generalization” (or “small test error”) is typically formalized in terms of minimizing the expected value of a random loss $E_\mu L(h)$, where $h$ is a candidate decision rule and $L(h)$ is a random variable on an underlying probability space $(\Omega, \mathcal{F}, \mu)$. This setup based on average off-sample performance has been famously called the “general setting of the learning problem” by Vapnik [48], and is central to the decision-theoretic formulation of learning in the influential work of Haussler [17]. This is by no means a purely theoretical concern; when average performance dictates the ultimate objective of learning, the data-driven feedback used for training in practice will naturally be designed to prioritize average performance [8, 20, 24, 43]. Take the default optimizers in popular software libraries such as PyTorch or TensorFlow; virtually without exception, these methods amount to efficient implementations of empirical risk minimization. While the minimal expected loss formulation is clearly an intuitive choice, the tacit emphasis on average performance represents an important and non-trivial value judgment, which may or may not be appropriate for any given real-world learning task.

To make this value judgment an explicit part of the machine learning workflow, in this work we consider a generalized class of risk functions, designed to give the user much greater flexibility in terms of how they choose to evaluate performance, while still allowing for theoretical performance guarantees. One core statistical concept is that of the M-location of the loss distribution under a candidate $h$, defined by

$$M(h) := \arg \min_{\theta \in \mathbb{R}} E_\mu \rho \left( \frac{L(h) - \theta}{\sigma} \right).$$

Here $\rho : \mathbb{R} \rightarrow [0, \infty)$ is a function controlling how we measure deviations, and $\sigma > 0$ is a scaling parameter. Since the loss distribution $\mu$ is unknown, clearly $M(h)$ is an ideal, unobservable quantity. If we replace $\mu$ with the empirical distribution induced by a sample $(L_1, \ldots, L_n)$, then for certain special cases of $\rho$ we get an M-estimator of the location of $L(h)$, a classical notion dating back to Huber [19], which justifies our naming. Ignoring integrability concerns for the moment, note that in the special case of $\rho(u) = u^2$, we get the classical risk $M(h) = E_\mu L(h)$, and in the case of $\rho(u) = |u|$, we get $M(h) = \inf\{u : \mu\{L(h) \leq u\} \geq 0.5\}$, namely the median of the loss distribution. This rich spectrum of evaluation metrics makes the notion of casting learning problems in terms of minimization of M-locations (via their corresponding M-estimators) very conceptually appealing. However, while the statistical properties of the minima of M-estimators in special cases are understood [10], the optimization involved is both costly and difficult, making the task of designing and studying M(·)-minimizing learning algorithms highly intractable. With these issues in mind, we study a closely related alternative which retains the conceptual appeal of raw M-locations, but is more computationally congenial.

Our approach With $\sigma$ and $\rho$ as before, our generalized risks will be defined implicitly by

$$\min_{\theta \in \mathbb{R}} \left[ \theta + \eta E_\mu \rho \left( \frac{L(h) - \theta}{\sigma} \right) \right],$$

where $\eta > 0$ is a weighting parameter that controls the balance of priority between location and deviation. A more formal definition will be given in section 2 (see equations (3)–(5)), including concrete forms for $\rho(\cdot)$ that are conducive to both fast computation and meaningful learning guarantees. In addition, we will see (cf. Proposition 3) that after minimization, this objective
can be written as
\[ [M(h) - c_M] + \eta E_{\mu} \rho \left( \frac{L(h) - [M(h) - c_M]}{\sigma} \right), \]
where the shift term \( c_M > 0 \) can be simply characterized by the equality
\[ E_{\mu} \rho' \left( \frac{L(h) - [M(h) - c_M]}{\sigma} \right) = \frac{\sigma}{\eta}, \]
noting that \( c_M \to 0_+ \) as \( \eta \to \infty \). By utilizing smoothness properties of loss functions typical to machine learning problems (e.g., squared error, cross-entropy, etc.), even though the generalized risks need not be convex, they can be shown to satisfy weak notions of convexity, which still admit finite-sample guarantees of near-stationary for stochastic gradient-based learning algorithms (details in section 3). This approach has the additional benefit that implementation only requires a single wrapper around any given loss which can be set prior to training, making for easy integration with frameworks such as PyTorch and TensorFlow, while incurring negligible computational overhead.

Our contributions The key contribution here is a new concrete class of risk functions, defined and analyzed in section 2. These risks are statistically easy to interpret, their empirical counterparts are simple to implement in practice, and as we prove in section 3, their design allows for standard stochastic gradient-based algorithms to be given competitive finite-sample excess risk guarantees. We also verify empirically (section 4) that the proposed feedback generation scheme has a demonstrable effect on the test distribution of the loss, which as a side-effect can be easily leveraged to outperform traditional ERM implementations, a result which is in line with early insights from Breiman [9] and Reyzin and Schapire [34] regarding the impact of the loss distribution on generalization. More broadly, the choice of which risk to use plays a central role in pursuit of increased transparency in machine learning, and our results represent a first step towards formalizing this process.

Relation to existing literature With respect to alternative notions of “risk” in machine learning, perhaps the most salient example is conditional value-at-risk (CVaR) [33, 18], namely the expected loss conditioned on it exceeding a quantile at a pre-specified probability level. CVaR allows one to encode a sensitivity to extremely large losses, and admits convexity when the underlying loss is convex, though the conditioning often leaves the effective sample size very small. Other notions such as cumulative prospect theory (CPT) scores have also been considered [7, 26, 23], but the technical difficulties involved with computation and analysis arguably outweigh the conceptual benefits of learning using such scores. These proposals can all be interpreted as “location” parameters of the underlying loss distribution; our risks take the form of a sum of a location and a deviation term, where the location is a shifted M-location, as described above. The basic notion of combining location and deviation information in evaluation is a familiar concept; the mean-variance objective \( E_{\mu} L(\cdot) + \text{var}_{\mu} L(\cdot) \) dates back to classical work by Markowitz [28]; our proposed class includes this as a special case, but generalizes far beyond it. Mean-variance and other risk function classes are studied by Ruszczyński and Shapiro [40, 41], who give minimizers a useful dual characterization, though our proposed class is not captured by their work (see also Remark 4). We note also that the recent (and independent) work of Lee et al. [25] considers a form which is similar to (2) in the context of empirical risk minimizers; the critical technical difference is that their formulation is restricted to \( \rho \) which is monotonic, an assumption which enforces convexity. The special case of mean-variance is also
treated in depth in more recent work by Duchi and Namkoong [14], who consider stochastic learning algorithms for doing empirical risk minimization with variance-based regularization. Finally, we note that our technical analysis in section 3 makes crucial use of weak convexity properties of function compositions, an area of active research in recent years [15, 13, 12]. Since our proposed objective can be naturally cast as a composition taking us from parameter space to a Banach space and finally to \( \mathbb{R} \), leveraging the insights of these previous works, we extend the existing machinery to handle learning over Banach spaces, and give finite-sample guarantees for arbitrary Hilbert spaces. More details are provided in section 3, plus the appendix.

**Notation and terminology** To give the reader an approximate idea of the technical level of this paper, we assume some familiarity with probability spaces, the notion of sub-gradients and the sub-differential of convex functions, as well as special classes of vector spaces like Banach and Hilbert spaces, although the main text is written with a wide audience in mind. Strictly speaking, we will also deal with sub-differentials of non-convex functions, but these technical concepts are relegated to the appendix, where all formal proofs are given. In the main text, to improve readability, we write \( \partial f(x) \) to denote the sub-differential of \( f \) at \( x \), regardless of the convexity of \( f \). When we refer to a function being \( \lambda \)-smooth, this refers to its gradient being \( \lambda \)-Lipschitz continuous, and weak smoothness just requires such continuity on directional derivatives; all these concepts are given a detailed introduction in the appendix. Throughout this paper, we use \( E[\cdot] \) for taking expectation, and \( P \) as a general-purpose probability function.

For indexing, we will write \( [k] := \{1, \ldots, k\} \). Distance of a vector \( v \) from a set \( A \) will be denoted by \( \text{dist}(v; A) := \inf\{\|v - v'\| : v' \in A\} \).

### 2 A concrete class of risk functions

The risks described by (2) are fairly intuitive as-is, but a bit more structure is needed to ensure they are well-defined and useful in practice. To make things more concrete, let us fix \( \rho \) as

\[
\rho(u) := u \tan(u) - \frac{\log(1 + u^2)}{2}, \quad u \in \mathbb{R}.  \tag{3}
\]

This function is handy in that it behaves approximately quadratically around zero, and it is both \( \pi/2 \)-Lipschitz and strictly convex on the real line.\(^1\) Fixing this particular choice of \( \rho \) and letting \( Z \) be a random variable (any \( \mathcal{F} \)-measurable function), we interpolate between mean- and median-centric quantities via the following class of functions, indexed by \( \sigma \in [0, \infty) \):

\[
r_\sigma(Z, \theta) := \theta + \eta E_\mu \rho_\sigma(Z - \theta), \quad \text{where } \rho_\sigma(u) := \begin{cases} |u|, & \text{if } \sigma = 0 \\ \rho(u/\sigma), & \text{if } 0 < \sigma < \infty \\ u^2, & \text{if } \sigma = \infty. \end{cases} \tag{4}
\]

With this class of ancillary functions in hand, it is natural to define

\[
R_\sigma(Z) := \inf_{\theta \in \mathbb{R}} r_\sigma(Z, \theta) \tag{5}
\]

to construct a class of risk functions. In the context of learning, we will use this risk function to derive a generalized risk, namely the composite function \( h \mapsto R_\sigma(L(h)) \). As a special case, clearly this includes risks of the form (2) given earlier. Visualizations of these functions are given in the supplementary appendix. Minimizing \( R_\sigma(L(h)) \) in \( h \) is our formal learning problem of interest.

\(^1\)To see this, note that \( \rho'(u) = \tan(u) \) and \( \rho'(\mathbb{R}) \subset (-\pi/2, \pi/2) \), and \( \rho''(u) = 1/(1 + u^2) > 0 \).
Before considering learning algorithms, we briefly cover the basic properties of the functions $r_{\sigma}$ and $R_{\sigma}$. Without restricting ourselves to the specialized context of “losses,” note that if $Z$ is any square-$\mu$-integrable random variable, this immediately implies that $|r_{\sigma}(Z, \theta)| < \infty$ for all $\theta \in \mathbb{R}$, and thus $R_{\sigma}(Z) < \infty$. Furthermore, the following result shows that it is straightforward to set the weight $\eta$ to ensure $R_{\sigma}(Z) > -\infty$ also holds, and a minimum exists.

**Proposition 1** (Well-defined risks). Assuming that $E_{\mu} Z^2 < \infty$, set $\eta$ based on $\sigma \in [0, \infty]$ as follows: if $\sigma = 0$, take $\eta > 1$; if $0 < \sigma < \infty$, take $\eta > 2\sigma/\pi$; if $\sigma = \infty$, take any $\eta > 0$. Under these settings, the function $\theta \mapsto r_{\sigma}(Z, \theta)$ is bounded below and takes its minimum on $\mathbb{R}$. Thus, for each square-$\mu$-integrable $Z$, there always exists a (non-random) $\theta_{Z} \in \mathbb{R}$ such that

$$R_{\sigma}(Z) = \theta_{Z} + \eta E_{\mu} \rho_{\sigma}(Z - \theta_{Z}). \quad (6)$$

Furthermore, when $\sigma > 0$, this minimum $\theta_{Z}$ is unique.

**Remark 2** (Mean-median interpolation). In order to ensure that risk modulation via $\sigma \in [0, \infty]$ smoothly transitions from a median-centric ($\sigma = 0$ case) to a mean-centric ($\sigma = \infty$ case) location, the parameter $\eta$ plays a key role. Noting that for any $u \in \mathbb{R}$, for $\rho$ defined by (3) we have $2\sigma^2 \rho(u/\sigma) \to u^2$ as $\sigma \to \infty$, and thus for large values of $\sigma > 0$ it is natural to set $\eta = 2\sigma^2$. On the other end of the spectrum, since $\log(1 + (u/\sigma)^2) \to 0_+$ whenever $\sigma \to 0_+$, it is thus natural to set $\eta = \sigma/\pi \to 2\sigma/\pi$ when $\sigma > 0$ is small. Strictly speaking, in light of the conditions in Proposition 1, to ensure $R_{\sigma}$ is finite one should take $\eta > 2\sigma/\pi$.

What can we say about our risk functions $R_{\sigma}$ in terms of more traditional statistical risk properties? The form of $R_{\sigma}$ given in (6) has a simple interpretation as a weighted sum of “location” and “deviation” terms. In the statistical risk literature, the seminal work of Artzner et al. [1] gives an axiomatic characterization of location-based risk functions that can be considered coherent, while Rockafellar et al. [36] characterize functions which capture the intuitive notion of “deviation,” and establish a lucid relationship between coherent risks and their deviation class. The following result describes key properties of the proposed risk functions, in particular highlighting the fact that while our location terms are monotonic, our risk functions are non-traditional in that they are non-monotonic.

**Proposition 3** (Non-monotonic risk functions). Let $\mathcal{B}$ be a Banach space of square-$\mu$-integrable functions. For any $\sigma \in [0, \infty]$, let $\eta > 0$ be set as in Proposition 1. Then, the functions $r_{\sigma} : \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ and $R_{\sigma} : \mathcal{B} \to \mathbb{R}$ satisfy the following properties:

- Both $r_{\sigma}$ and $R_{\sigma}$ are continuous, convex, and sub-differentiable.
- The location in (6) is monotonic (i.e., $Z_1 \leq Z_2$ implies $\theta_{Z_1} \leq \theta_{Z_2}$) and translation-equivariant (i.e., $\theta_{Z+a} = \theta_{Z} + a$ for any $a \in \mathbb{R}$), for any $0 < \sigma \leq \infty$.
- The deviation in (6) is non-negative and translation-invariant, namely for any $a \in \mathbb{R}$, we have $E_{\mu} \rho_{\sigma}(Z + a - \theta_{Z+a}) = E_{\mu} \rho_{\sigma}(Z - \theta_{Z})$, for any $0 < \sigma \leq \infty$.
- The risk $R_{\sigma}$ is not monotonic, i.e., $\mu\{Z_1 \leq Z_2\} = 1$ need not imply $R_{\sigma}(Z_1) \leq R_{\sigma}(Z_2)$.

In particular, the risk $h \mapsto R_{\sigma}(L(h))$ need not be convex, even if $L(\cdot)$ is.

**Remark 4.** Since our risk function $R_{\sigma}$ is not monotonic, standard results in the literature on optimizing generalized risks do not apply here. We remark that our proposed risk class does not appear among the comprehensive list of examples given in the works of Ruszczyński and Shapiro [40, 41], aside from the special case of $\sigma = \infty$ with $\eta = 1$. While the continuity and sub-differentiability of any risk function which is convex and monotonic is well-known for a large class of Banach spaces [40, Sec. 3], in Proposition 3 we obtain such properties without monotonicity by using square-$\mu$-integrability combined with properties of our function class $\rho_{\sigma}$.
We call this approach “naive” since it is exactly what we would do if we knew
Algorithm 1

\[ \alpha \]

\[
\text{Given some initial value } f(0), \theta(0) \in H \times \mathbb{R}, \text{ step sizes } (\alpha_t), \text{ and max iterations } n. \\
\text{for } t \in \{0, \ldots, n-1\} \text{ do} \\
\quad \text{Sample } G_t \text{ via (9).} \\
\quad \text{Update } (h_t, \theta_t) \mapsto (h_{t+1}, \theta_{t+1}) \text{ via (8).} \\
\text{end for} \\
\text{Sample } T \in \{0, \ldots, n-1\} \text{ with probabilities } P\{T = t\} = \alpha_t / (\sum_{k=0}^{n-1} \alpha_k), t \in [n-1]. \\
\text{return: } (\bar{h}_{[n]}, \bar{\theta}_{[n]}) := (h_T, \theta_T).
\]

Since our principal interest is the case where \( Z = L(h) \), the key takeaways from this section are that while the proposed risk \( h \mapsto R_\sigma(L(h)) \) is well-defined and easy to estimate given a random sample \( L_1(h), \ldots, L_n(h) \), the learning task is non-trivial since \( R_\sigma(L(\cdot)) \) is not differentiable (and thus non-smooth) when \( \sigma = 0 \), and for any \( \sigma \in [0, \infty] \) need not be convex, even when the underlying loss is both smooth and convex. Fortunately, smoothness properties of the losses typically used in machine learning can be leveraged to overcome these technical barriers, opening a path towards learning guarantees for practical algorithms. This is the topic of the next section.

3 Learning algorithm analysis

Thus far, we have only been concerned with \emph{ideal} quantities \( R_\sigma \) and \( r_\sigma \) used to define the ultimate formal goal of learning. In practice, the learner will only have access to noisy, incomplete information. In this work, we focus on iterative algorithms based on stochastic gradients, largely motivated by their practical utility and ubiquity in modern machine learning applications. For the rest of the paper, we overload our risk definitions to enhance readability, writing \( r_\sigma(h, \theta) := r_\sigma(L(h), \theta) \) and \( R_\sigma(h) := R_\sigma(L(h)) \). First note that we can break down the underlying joint objective as \( r_\sigma(h, \theta) = \mathbb{E}_\mu(f_2 \circ F_1)(h, \theta) \), where we have defined

\[ F_1(h, \theta) := (L(h), \theta), \quad f_2(u, \theta) := \theta + \eta \rho_\sigma(u - \theta). \tag{7} \]

From the point of view of the probability space \( (\Omega, \mathcal{F}, \mu) \), the function \( F_1 \) is random, whereas \( f_2 \) is deterministic; our use of upper- and lower-case letters is just meant to emphasize this. Given some initial value \( (h_0, \theta_0) \in H \times \mathbb{R} \), one naively hopes to construct an efficient stochastic gradient algorithm using the update

\[ (h_{t+1}, \theta_{t+1}) = \Pi_C [ (h_t, \theta_t) - \alpha_t G_t ] , \tag{8} \]

where \( \alpha_t > 0 \) is a step-size parameter, \( \Pi_C [\cdot] \) denotes projection to some set \( C \subset H \times \mathbb{R} \), and the stochastic feedback \( G_t \) is just a composition of sub-gradients, namely

\[ G_t \in \partial f_2(L(h_t), \theta_t) \circ \partial F_1(h_t, \theta_t). \tag{9} \]

We call this approach “naive” since it is exactly what we would do if we knew \emph{a priori} that the underlying objective was convex and/or smooth.\footnote{The convex and smooth case is the classical setting [29]; see Shalev-Shwartz and Ben-David [42] for a modern textbook introduction. When convex but non-smooth, see Shamir and Zhang [44]. When smooth but non-convex, see Ghadimi and Lan [16].} The precise learning algorithm studied here is summarized in Algorithm 1. Fortunately, as we describe below, this naive procedure actually enjoys lucid non-asymptotic guarantees, on par with the smooth case.
How to measure algorithm performance? Before stating any formal results, we briefly discuss the means by which we evaluate learning algorithm performance. Since the sequence \( (R_\sigma(h_t)) \) cannot be controlled in general, a more tractable problem is that of finding a stationary point of \( r_\sigma \), namely any \((h^*, \theta^*)\) such that \( 0 \in \partial r_\sigma(h^*, \theta^*) \). However, it is not practical to analyze \( \text{dist}(0; \partial r_\sigma(h_t, \theta_t)) \) directly, due to a lack of continuity. Instead, we consider a smoothed version of \( r_\sigma \):

\[
\tilde{r}_{\sigma,\beta}(h, \theta) := \inf_{h', \theta'} \left[ r_\sigma(h', \theta') + \frac{1}{2\beta} \| (h, \theta) - (h', \theta') \|^2 \right].
\]  

(10)

This is none other than the Moreau envelope of \( r_\sigma \), with weighting parameter \( \beta > 0 \). A familiar concept from convex analysis on Hilbert spaces [5, Ch. 12 and 24], the Moreau envelope of non-smooth functions satisfying weak convexity properties has recently been shown to be a very useful metric for evaluating stochastic optimizers [12, 13]. Our basic performance guarantees will first be stated in terms of the gradient of the smoothed function \( \tilde{r}_{\sigma,\beta} \). We will then relate this to the joint risk \( r_\sigma \) and subsequently the risk \( R_\sigma \).

Guarantees based on joint risk minimization

Within the context of the stochastic updates characterized by (8)–(9), we consider the case in which \( H \) is any Hilbert space. All Hilbert spaces are reflexive Banach spaces, and the stochastic sub-gradient \( G_t \in (H \times \mathbb{R})^* \) (the dual of \( H \times \mathbb{R} \)) can be uniquely identified with an element of \( H \times \mathbb{R} \), for which we use the same notation \( G_t \). Denoting the partial sequence \( G_{[t]} := (G_0, \ldots, G_t) \), we formalize our assumptions as follows:

A1. For all \( h \in H \), the random loss \( L(h) \) is square-\( \mu \)-integrable, locally Lipschitz, and weakly \( \lambda \)-smooth, with a gradient satisfying \( E_\mu |L(h; \cdot)|^2 < \infty \).

A2. \( H \) is a Hilbert space, and \( C \subset H \times \mathbb{R} \) is a closed convex set.

A3. The feedback (9) satisfies \( E(G_{[t]} | G_{[t-1]}) = E_\mu G_t \) for all \( t > 0 \).

A4. For some \( 0 < \kappa < \infty \), the second moments are bounded as \( E_\mu \|G_t\|^2 \leq \kappa^2 \) for all \( t \).

The following is a performance guarantee for Algorithm 1 in terms of the smoothed joint risk.

**Theorem 5** (Nearly-stationary point of smoothed objective). If \( 0 < \sigma < \infty \), set smoothing parameter \( \gamma = (1 + \eta \pi / (2\sigma)) \max\{1, \lambda \} \). Otherwise, if \( \sigma = 0 \), set \( \gamma = (1 + \eta) \max\{1, \lambda \} \).

Under these \( \sigma \)-dependent settings and assumptions A1–A4, let \( (\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \) denote the output of Algorithm 1, with \( r^*_C := \inf \{ r_\sigma(h, \theta) : (h, \theta) \in C \} \) denoting the minimum over the feasible set and \( \Delta_0 := \tilde{r}_{\sigma,\beta}(h_0, \theta_0) - r^*_C \) denoting the initialization error. Then, for any choice of \( n > 1 \), \( \eta > 0 \), and \( \beta < 1 / \gamma \), we have that

\[
E \| \tilde{r}'_{\sigma,\beta}(\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \|^2 \leq \left( \frac{1}{1 - \beta \gamma} \right) \frac{\Delta_0 + \gamma \kappa^2 \sum_{t=0}^{n-1} \alpha_t^2 / 2}{\sum_{t=0}^{n-1} \alpha_t},
\]

where expectation is taken over all the feedback \( G_{[n-1]} \).

**Remark 6** (Sample complexity). Let us briefly describe a direct take-away from Theorem 5. If \( \Delta_0 \), \( \gamma \), and \( \kappa \) are known (upper bounds will of course suffice), then constructing step sizes as \( \alpha_t^2 \geq \Delta_0 / (n^2 \gamma^2) \), if we set \( \beta = 1 / (2 \gamma) \), it follows immediately that

\[
E \| \tilde{r}'_{\sigma,\beta}(\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \|^2 \leq \left( \frac{2 \gamma^2 \Delta_0}{n} \right)^{1/2}.
\]

\(^3\)The expectation on the left-hand side is with respect to the joint distribution of \( G_{[t]} \) conditioned on \( G_{[t-1]} \).
Fixing some desired precision level of \( \sqrt{\mathbb{E} \left[ \| r_{\sigma,\beta} (\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \| \right]^2} \leq \varepsilon \), the sample complexity is \( \mathcal{O}(\varepsilon^{-4}) \). This matches guarantees available in the smooth (but non-convex) case [16], and suggests that the “naive” strategy implemented by Algorithm 1 in fact comes with a clear theoretical justification.

**Implications in terms of the original objective** The results described in Theorem 5 and Remark 6 are with respect to a smoothed version of the joint risk function \( r_{\sigma} \). Linking these facts to insights in terms of the original proposed risk \( R_{\sigma} \) can be done as follows. Assuming we take \( n \geq 2 \gamma \kappa^2 \Delta_0 / \varepsilon^4 \) to achieve the \( \varepsilon \)-precision discussed in Remark 6, the immediate conclusion is that the algorithm output is \( (\varepsilon/2\gamma) \)-close to a \( \varepsilon \)-nearly stationary point of \( r_{\sigma} \). More precisely, we have that there exists an ideal point \( (h_{[n]}^*, \theta_{[n]}^*) \) such that

\[
\mathbb{E} \left[ \text{dist} (0; \partial r_{\sigma} (h_{[n]}^*, \theta_{[n]}^*)) \right] \leq \varepsilon, \quad \text{and} \quad \mathbb{E} \left[ \| (\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) - (h_{[n]}^*, \theta_{[n]}^*) \| \right] = \frac{\varepsilon}{2\gamma}.
\]

The above fact follows from basic properties of the Moreau envelope (cf. Appendix B.4). These non-asymptotic guarantees of being close to a “good” point extend to the function values of the risk \( R_{\sigma} \) since we are close to a candidate \( h_{[n]}^* \) whose risk value can be no worse than

\[
\mathbb{E} \left[ R_{\sigma} (h_{[n]}^*) \right] \leq \mathbb{E} \left[ r_{\sigma} (h_{[n]}^*, \theta_{[n]}^*) \right] \leq \mathbb{E} \left[ r_{\sigma} (\tilde{h}_{[n]}, \tilde{\theta}_{[n]}) \right].
\]

We remark that these learning guarantees hold for a class of risks that are in general non-convex and need not even be differentiable, let alone satisfy smoothness requirements.

**Key points in the proof of Theorem 5** Here we briefly highlight the key sub-results involved in proving Theorem 5; please see Appendix C.2 for all the details. The key structure that we require is a smooth loss, reflected in assumption A1. This along with the Lipschitz property of our function \( \rho_{\sigma} \) for all \( 0 \leq \sigma < \infty \) allows us to prove that the underlying objective \( r_{\sigma} \) is weakly convex, where \( \mathcal{H} \) can be any Banach space (Proposition 12); this generalizes a result of Drusvyatskiy and Paquette [13, Lem. 4.2] from Euclidean space to any Banach space. This alone is not enough to obtain the desired result. Note that the assumption A3 is very weak, and trivially satisfied in most traditional machine learning settings (e.g., where losses are based on a sequence of iid data points). The question of whether the feedback is unbiased or not, i.e., whether \( \mathbb{E}_\mu G_t \) is in the sub-differential of \( r_{\sigma} \) at step \( t \) or not, is something that needs to be formally verified. In Proposition 14 we show that as long as the gradient has a finite expectation, this indeed holds for the feedback generated by (9), when \( \mathcal{H} \) is any Banach space. With the two key properties of a weakly convex objective and unbiased random feedback in hand, we can leverage the techniques used in Davis and Drusvyatskiy [12, Thm. 3.1] for proximal stochastic gradient methods applied to weakly convex functions on \( \mathbb{R}^d \), extending their core argument to the case of any Hilbert space. Combining this technique with the proof of weak convexity and unbiasedness lets us obtain Theorem 5.

# 4 Empirical analysis

In this section we introduce representative results for a series of experiments designed to investigate the quantitative and qualitative repercussions of modulating the underlying risk function class.\(^4\)

\(^4\)Repository with software and demos: https://github.com/feedbackward/mrisk
Figure 1: A simple toy example using $L(h) = h L_{\text{wide}} + (1 - h) L_{\text{thin}}$. Trajectories shown are the sequence $(h_t)$ generated by running (8) on $\mathbb{R}^2$, with $h_0 = 0.5$ and $\theta_0 = 0.5$, averaged over all trials. Densities of $L_{\text{wide}}$ (red) and $L_{\text{thin}}$ (blue) are also plotted, with additional details in the main text.

Sanity check in one dimension As a natural starting point, we use a toy example to ensure that Algorithm 1 takes us where we expect to go for a particular risk setting. Consider a loss on $\mathbb{R}$ with the form $L(h) = h L_{\text{wide}} + (1 - h) L_{\text{thin}}$, where $L_{\text{wide}}$ and $L_{\text{thin}}$ are random variables independent of $h$ and each other. As a simple example, we use a folded Normal distribution for both, namely $L = |\text{Normal}(a, b^2)|$, where $a_{\text{wide}} = 0$, $a_{\text{thin}} = 2$, $b_{\text{wide}} = 1.0$, and $b_{\text{thin}} = 0.1$. For simplicity, we fix $\alpha_t = 0.001$ throughout, and each step uses a mini-batch of size 8. Regarding the risk settings, we look in particular at the case of $\sigma = \infty$, where we modify the setting of $\eta = 2^k$ over $k = 0, 1, \ldots, 7$. Results averaged over 100 trials are given in Figure 1. By modifying $\eta$, we can control whether the learning algorithm “prefers” candidates whose losses have a high degree of dispersion centered around a good location, or those whose losses are well-concentrated near a weaker location.

Impact of risk choice on linear regression Next we consider how the key choice of $\sigma$ (and thus the underlying risk $R_\sigma$) plays a role on the behavior of Algorithm 1. As another simple, yet more traditional example, consider linear regression in one dimension, where $Y = w_0^* + w_1^* X + \epsilon$, where $X$ and $\epsilon$ are independent zero-mean random variables, and $(w_0^*, w_1^*) \in \mathbb{R}^2$ are unknown to the learner. Using squared error $L(h) = (Y - h(X))^2$, we run Algorithm 1 again with mini-batches of size 8 and $\alpha_t = 0.001$ fixed throughout, over a range of $\sigma \in [0, \infty]$ settings, for the same number of iterations as in the previous experiment. The initial value $(h_0, \theta_0)$ is initialized at zero plus uniform noise on $[-0.05, 0.05]$. We also consider multiple noise distributions; as a concrete example, letting $N = \text{Normal}(0, (0.8)^2)$, we consider both $\epsilon = N$ (Normal case) and $\epsilon = e^N - \mathbb{E} e^N$ (log-Normal case). In Figure 2, we plot the learned regression lines (averaged over 100 trials) for each choice of $\sigma$ and each noise setting. By modulating the target risk function, we can effectively choose between a self-imposed bias (smaller slope, lower intercept here), and a sensitivity to outlying values.

Tests using real-world data Finally, we consider an application to some well-known benchmark datasets for classification. At a high level, we run Algorithm 1 for multi-class logistic regression for 10 independent trials, where in each trial we randomly shuffle and re-split each full dataset (88% training, 12% testing), and randomly re-initialize the model weights identically to the previous paragraph, again with mini-batches of size 8, and step sizes fixed to $\alpha_t = 0.01 / \sqrt{d}$, where $d$ is the number of free parameters. Additional background on the datasets is given in appendix E. The key question of interest is how the test loss distribution changes as we modify the learner’s feedback to optimize a range of risks $R_\sigma$. In Figure 3, we see a stark difference.
Figure 2: Learned regression lines (solid; colors denote $\sigma \in [0, \infty]$) are plotted along with the true model $(w_0^*, w_1^*) = (1.0, 1.0)$ (dashed; black). Histograms are of independent samples of $w_0^* + \epsilon$. The left plots are the Normal case, and right plots are the log-Normal case.

Figure 3: Top row: average test error (zero-one loss) as a function of epochs, for four datasets and five $\sigma$ levels, plus traditional ERM (denoted “off”). Bottom row: histograms of the test error (logistic loss) incurred after the final epoch for one trial under the `emnist_balanced` dataset.

between doing traditional empirical risk minimization (ERM, denoted “off”) and using $R_\sigma$-based feedback, particularly for moderately large values of $\sigma$. The logistic losses are concentrated much more tightly (visible in the bottom row histograms), and this also leads to a better classification error (visible in the top row plots), an interesting trend that we observed across many distinct datasets.
Figure 4: Left: graphs of $\rho$ defined in (3) (solid line), $\rho'$ (dashed line), and $\rho''$ (dot-dash line). Middle–right: these are graphs of $\theta \mapsto \eta\rho_\sigma(1.0 - \theta)$, where the minimizer is $\theta_{\text{min}} = 1.0$, the colors correspond to different $\sigma$ values, and $\eta$ is set following Remark 2. That is, for $0 < \sigma < 1.0$, set $\eta = \sigma / \text{atan}(\infty)$. For $\sigma = 0$, set $\eta = 1.05$. For $1.0 \leq \sigma < \infty$, set $\eta = 2\sigma^2$. For $\sigma = \infty$, set $\eta = 1.0$.

A Overview of appendix contents

Our appendix is comprised of several sections, ordered as follows:

B Background and setup

C Detailed proofs

D Helper results

E Empirical supplement

As with the main paper, we handle theoretical topics before diving into empirical topics. Section B gives a very detailed background including numerous formal definitions, supporting lemmas, and discussion on results used later in the detailed proofs (section C) for the main paper’s results. Additional numerical test results are at the very end of section E.

To provide additional visual intuition for the reader, we include at the start of this appendix several figures related to $\rho$ defined in (3), $\rho_\sigma$ defined in (4), and the resulting risk functions. In Figure 4 we plot $\rho$ and its derivatives, plus $\rho_\sigma$ for a wide variety of $\sigma \in [0, \infty]$ values. Additional details are given in the figure caption. In Figure 5, we show how specific choices of standard loss functions lead two different forms of the function composition $h \mapsto L(h) \mapsto \eta\rho_\sigma(L(h) - \theta)$.

B Background and setup

B.1 Preliminaries

General notation (probability) Underlying all our analysis is a probability space $(\Omega, \mathcal{F}, \mu)$.$^5$

All random variables, unless otherwise specified, will be assumed to be $\mathcal{F}$-measurable functions with domain $\Omega$. Integration using $\mu$ will be denoted by $E_\mu Z := \int_{\Omega} Z(\omega) \mu(d\omega)$, and $P$ will be used as a generic probability function, typically representing $\mu$ itself, or the product measure.

$^5$For basic measure-theoretical facts supporting our main arguments, we use Ash and Doléans-Dade [2] as a well-established and accessible reference. We will cite the exact results that pertain to our arguments in the main text as they become necessary.
Figure 5: Graphs of $h \mapsto \eta \rho_\sigma(L(h) - \theta)$, with $h \in \mathbb{R}$, over different choices of $\sigma$ and $L$, with $\theta = 3.0$ fixed, and $\eta$ set as in Figure 4. Colors correspond to $\sigma$, and each plot corresponds to a choice of $L(\cdot)$. Absolute: $L(h) = |h - h^*|$. Squared: $L(h) = (h - h^*)^2$. Hinge: $L(h) = \max\{0, 1 - hh^*\}$. Cross-entropy: $L(h) = \log(1 + \exp(-hh^*))$. In all cases, we have fixed $h^* = \pi$.

induced by a sample of random variables on $(\Omega, \mathcal{F}, \mu)$. We use $\mathcal{L}_2 := L_2(\Omega, \mathcal{F}, \mu)$ to denote the set of all square-$\mu$-integrable functions.\(^6\)

General notation (normed spaces)  Let $\mathcal{V}$ denote an arbitrary vector space. When we call $\mathcal{V}$ a normed (linear) space, we are referring to $(\mathcal{V}, \| \cdot \|)$, where $\| \cdot \| : \mathcal{V} \to \mathbb{R}$ denotes the relevant norm. For any normed space $\mathcal{V}$, we shall denote by $\mathcal{V}^*$ the usual dual space of $\mathcal{V}$, namely all continuous linear functionals defined on $\mathcal{V}$. The space $\mathcal{V}^*$ is equipped with the norm $\|v^*\| := \sup\{v^*(u) : \forall u \in \mathcal{V}, \|u\| \leq 1\}$. We shall use the notation $\langle \cdot, \cdot \rangle$ to represent the “coupling” function between $\mathcal{V}$ and $\mathcal{V}^*$, that is for any $u \in \mathcal{V}$ and $v^* \in \mathcal{V}^*$, we will write $\langle u, v^* \rangle := v^*(u)$. For any sequence $(x_n)$ of elements $x_1, x_2, \ldots \in \mathcal{V}$, we denote convergence of $(x_n)$ to some element $x'$ by $x_n \to x'$. When we take limits and do not specify a particular sequence, for example writing $x \to x'$, then this refers to any sequence (of elements from $\mathcal{V}$) that converges to $x'$. In the special case of real-valued sequences (where $\mathcal{V} \subset \mathbb{R}$), if we write $x_n \to x'_+$ (respectively $x_n \to x'_-$), this refers to all sequences from above (resp. below), i.e., any convergent sequence such that $x_n \geq x'$ (resp. $x_n \leq x'$) for all $n$. We denote the open ball of radius $r > 0$ centered at $x_0 \in \mathcal{V}$ by $B(x_0; r) := \{x \in \mathcal{V} : \|x_0 - x\| < r\}$. We denote the extended real line by $\mathbb{R}$. On normed space $\mathcal{V}$, we denote the interior of a set $U \subset \mathcal{V}$ by $\text{int} U$ (all $x \in U$ such that $B(x; \delta) \subset U$ for some $\delta$).

\(^6\)Strictly speaking, this is the set of all equivalence classes of square-$\mu$-integrable functions, where $f \in \mathcal{L}_2(\Omega, \mathcal{F}, \mu)$ represents all functions that are equal $\mu$-almost everywhere.
General terminology On any normed linear space $\mathcal{V}$, a set $A \subset \mathcal{V}$ is said to be **compact** if for any sequence of elements in $A$, there exists a sub-sequence which converges on $A$. We denote the **effective domain** of an extended real-valued function $f$ by $\text{dom } f := \{ x : f(x) < \infty \}$. We call a convex function $f : \mathcal{V} \to \mathbb{R}$ proper if $f > -\infty$ and $\text{dom } f \neq \emptyset$. We say that $f$ is coercive if $\|x\| \to \infty$ implies $f(x) \to \infty$.\(^7\) For a function $f : \mathcal{X} \to \mathcal{Y}$, with $\mathcal{X}$ and $\mathcal{Y}$ being normed spaces, we say $f$ is (locally) **Lipschitz** at $x_0 \in \mathcal{X}$ if there exists $\delta > 0$ and $\lambda > 0$ such that $x_1, x_2 \in B(x_0; \delta)$ implies $\| f(x_1) - f(x_2) \| \leq \lambda \| x_1 - x_2 \|$. We say $f$ is $\lambda$-Lipschitz on $\mathcal{X}$ if this property holds with a common coefficient $\lambda$ for all $x_0 \in \mathcal{X}$.

**Semi-continuous functions** We say that a function $f$ is **lower semi-continuous**\(^8\) (LSC) at a point $x$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\| x - x' \| < \delta$ implies $f(x') > f(x) - \varepsilon$. If $-g$ is LSC, then we say $g$ is **upper** semi-continuous (USC). The property that $f$ is LSC at a point $x$ is equivalent\(^9\) to the property that for any sequence $x_n \to x$, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

(12)

Ordinary continuity is equivalent to being both USC and LSC, but the added generality of these weaker notions of continuity is often useful.

**Differentiability** We start by introducing some common notions of directional differentiability at a high level of generality.\(^10\) Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces, $U \subset \mathcal{X}$ an open set, and $f : \mathcal{X} \to \mathcal{Y}$ a function of interest. The **radial derivative** of $f$ at $x \in U$ in direction $u$ is defined

$$f'_r(x; u) := \lim_{\alpha \to 0^+} \frac{f(x + \alpha u) - f(x)}{\alpha}.$$  

(13)

A slight modification to this gives us the (Hadamard) **directional derivative** of $f$ at $x \in U$ in direction $u$:

$$f'(x; u) := \lim_{(\alpha, u') \to (0, u)} \frac{f(x + \alpha u') - f(x)}{\alpha}$$  

(14)

When $f'_r(x; u)$ exists for all directions $u$, we say that $f$ is **radially differentiable** at $x$. Identically, when $f'(x; u)$ exists for all directions $u$, we say that $f$ is **directionally differentiable** at $x$. When the map $u \mapsto f'_r(x; u)$ is continuous and linear, we say that $f$ is **Gateaux differentiable** at $x$. When the map $u \mapsto f'(x; u)$ is continuous and linear, we say $f$ is **Hadamard differentiable** at $x$. If $f$ is Hadamard differentiable, then it is Gateaux differentiable. The converse does not hold in general, but if $f$ is Lipschitz on a neighborhood of $x \in U$, then radial differentiability and directional differentiability (at $x$) are equivalent.\(^11\)

When we simply say that a function $f : \mathcal{X} \to \mathcal{Y}$ is **differentiable** at $x \in U$, we mean that there exists a function $f'(x)(\cdot) : \mathcal{X} \to \mathcal{Y}$ that is linear, continuous, and which satisfies

$$\lim_{\| u \| \to 0} \frac{\| f(x + u) - f(u) - f'(x)(u) \|}{\| u \|} = 0.$$  

(15)

---

\(^7\)For example, the function $f(x) = x^2$ is coercive, but $f(x) = \exp(x)$ is not.

\(^8\)Nice references on semi-continuity: Ash and Doléans-Dade [2, Appendix 2], Luenberger [27, Ch. 2], Barbu and Precupanu [3, Sec. 2.1], Penot [31, Ch. 1].

\(^9\)Ash and Doléans-Dade [2, Thm. A2.2], Penot [31, Lem. 1.18].

\(^10\)We follow basic notation and terminology of the authoritative text by Penot [31].

\(^11\)Penot [31, Prop. 2.25].
This property is often referred to as Fréchet differentiability. When $f$ is differentiable at $x$, the map $f'(x)$ is uniquely determined.\(^{12}\) In the special case where $\mathcal{V} \subseteq \mathbb{R}$, the linear functional represented by $f'(x) \in \mathcal{X}^*$ is called the gradient of $f$ at $x$. Differentiability is also closely related to directional differentiability; if $f$ is Gateaux differentiable on $U$ and the map $x \mapsto f'(x; \cdot)$ is continuous at $x$, then $f$ is differentiable at $x$.\(^{13}\)

**Sub-differentials** Let $\mathcal{V}$ be any normed linear space. If $f : \mathcal{V} \to \mathbb{R}$ is a (proper) convex function, the sub-differential of $f$ at $x \in \text{dom } f$ is defined

$$\partial f(x) := \{ v^* \in \mathcal{V}^* : f(u) - f(x) \geq \langle u - x, v^* \rangle, u \in \mathcal{V} \}$$

(16)

The second characterization of $\partial f(x)$, given using the radial derivative \(^\text{13}\), is useful and intuitive.\(^{14}\) Some authors refer to this as the Moreau-Rockafellar sub-differential to emphasize the context of convex analysis.

More generally, however, if $f$ is not convex, then the strong global property used to define the MR sub-differential is so restrictive that most interesting functions are left out. A more general notion is that of the Fréchet sub-differential.\(^{15}\) Denoted $\partial_F f(x)$, the Fréchet sub-differential of $f$ at $x$ is the set of all bounded linear functionals $v^* \in \mathcal{V}^*$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\| x - u \| \leq \delta \implies f(u) - f(x) \geq \langle u - x, v^* \rangle - \varepsilon \| x - u \|.$$  

(17)

This local requirement is much weaker than the condition characterizing the MR-sub-differential, and clearly we have $\partial f(x) \subseteq \partial_F f(x)$. When $f$ is assumed to be locally Lipschitz, another class of sub-differentials is often useful. Define the Clarke directional derivative of $f$ at $x$ in the direction $u$ by

$$f'_C(x; u) := \limsup_{(\alpha, x') \to (0_+, x)} \frac{f(x' + \alpha u) - f(x')}{\alpha},$$

(18)

The corresponding Clarke sub-differential is defined as

$$\partial_C f(x) := \{ v^* \in \mathcal{V}^* : f'_C(x; u) \geq \langle u, v^* \rangle, u \in \mathcal{V} \}.$$  

(19)

In the special case where $f$ is convex, all the sub-differentials coincide, i.e., $\partial f(x) = \partial_F f(x) = \partial_C f(x)$.\(^{16}\) We say that a function $f$ is sub-differentiable at $x$ if its sub-differential (in any sense) at $x$ is non-empty. Finally, a remark on notation when using set-valued functions like $x \mapsto \partial_C f(x)$. When we write something like “we have $\langle u, \partial_C f(x) \rangle \geq g(u)$,” it is the same as writing “we have $\langle u, v^* \rangle \geq g(u)$ for all $v^* \in \partial_C f(x)$.” This kind of notation will be used frequently.

**B.2 Generalized convexity**

Let $\mathcal{X}$ be a normed linear space. Take an open set $U \subseteq \mathcal{X}$ and fix some point $x_0 \in U$. For a function $f : \mathcal{X} \to \mathbb{R}$ and parameter $\gamma \in \mathbb{R}$, say that there exists $\delta > 0$ such that for all

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\(^{12}\)See for example Luenberger [27, Ch. 7] or Penot [31, Ch. 2].

\(^{13}\)Penot [31, Prop. 2.51].

\(^{14}\)See Penot [31, Thm. 3.22] for this fact.

\(^{15}\)We follow Penot [31, Sec. 4.1] for terms and notation here.

\(^{16}\)This follows from Penot [31, Prop. 5.34]. See also Penot [31, Sec. 4.1.1, Exercise 1].
When \( x, x' \in B(x_0; \delta) \) and \( \alpha \in (0, 1) \), we have

\[
f(\alpha x + (1 - \alpha)x') \leq \alpha f(x) + (1 - \alpha)f(x') + \frac{\gamma\alpha(1 - \alpha)}{2} \|x - x'\|^2.
\]  

(20)

When \( \gamma \geq 0 \), we say \( f \) is \( \gamma \)-\textit{weakly convex} at \( x_0 \). When \( \gamma \leq 0 \), we say \( f \) is \( (-\gamma) \)-\textit{strongly convex} at \( x_0 \). When (20) holds for all \( x_0 \in U \), we say that \( f \) is \( \gamma \)-weakly/strongly convex on \( U \). The special case of \( \gamma = 0 \) is the traditional definition of convexity on \( U \).\(^{17}\)

The ability to construct a quadratic lower-bounding function for \( f \) is closely related to notions of weak/strong convexity. Consider the following condition: given \( \gamma \in \mathbb{R} \), there exists \( \delta > 0 \) such that for all \( x, x' \in B(x_0; \delta) \) we have

\[
f(x') \geq f(x) + \langle x' - x, \partial_C f(x) \rangle - \frac{\gamma }{2} \|x - x'\|^2.
\]  

(21)

Here \( \partial_C f \) denotes the Clarke sub-differential of \( f \), defined by (19). Let us assume henceforth that \( \mathcal{X} \) is Banach, \( f \) is locally Lipschitz, and \( \partial_C f(x) \) is non-empty for all \( x \in U \).\(^{19}\) As such, for Banach spaces and locally Lipschitz functions, we have that the conditions (20), (21), and (22) are all equivalent for the general case of \( \gamma \in \mathbb{R} \).

Let us consider one more closely related property on the same open set \( U \subset \mathcal{X} \):

\[
x \mapsto f(x) + \frac{\gamma }{2} \|x\|^2 \text{ is convex on } U.
\]  

(23)

In the special case where \( \mathcal{X} \) is a real Hilbert space and the norm \( \| \cdot \| \) is induced by the inner product as \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \), then for any \( x, x' \in U \) and \( \alpha \in \mathbb{R} \), the equality

\[
\alpha \|x\|^2 + (1 - \alpha)\|x'\|^2 = \|\alpha x + (1 - \alpha)x'\|^2 + \alpha(1 - \alpha)\|x - x'\|^2
\]  

(24)

is easily checked to be valid.\(^{21}\) In this case, the equivalence (20) \( \iff \) (23) follows from direct verification using (24).\(^{22}\)

The facts above are summarized in the following result:

**Proposition 7** (Characterization of generalized convexity). Consider a function \( f : \mathcal{X} \rightarrow \mathbb{R} \) on normed linear space \( \mathcal{X} \). When \( \mathcal{X} \) is Banach and \( f \) is locally Lipschitz, then with respect to open set \( U \subset \mathcal{X} \) we have the following equivalence:

\[
(20) \iff (21) \iff (22).
\]

When \( \mathcal{X} \) is Hilbert, this equivalence extends to (23).

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\(^{17}\)See for example Nesterov [30, Ch. 3].

\(^{18}\)Penot [31, Prop. 5.3].

\(^{19}\)See for example Daniilidis and Malick [11, Thm. 3.1]; in particular their proof of (i) \( \Rightarrow \) (iii). Their result is stated for \( \mathcal{X} = \mathbb{R}^d \) and locally Lipschitz \( f \), but the proof easily generalizes to Banach spaces. See also the remarks following their proof about how the local Lipschitz condition can be removed.

\(^{20}\)Just apply the argument for (ii) \( \Rightarrow \) (i) employed by Daniilidis and Malick [11, Thm. 3.1], and strengthen their argument by using a more general form of Lebourg’s mean value theorem [31, Thm. 5.12].

\(^{21}\)Bauschke and Combettes [5, Cor. 2.15].

\(^{22}\)See also Davis and Drusvyatskiy [12, Lem. 2.1] for a similar result when \( \mathcal{X} = \mathbb{R}^d \) and \( f \) is LSC.
B.3 Function composition on normed spaces

Next we consider the properties of compositions involving functions which are smooth and/or convex. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be normed linear spaces. Let \( g : \mathcal{X} \to \mathcal{Y} \) and \( h : \mathcal{Y} \to \mathbb{R} \) be the maps used in our composition, and denote by \( f := h \circ g \) the composition, i.e., \( f(x) = h(g(x)) \) for each \( x \in \mathcal{X} \). Our goal will be to present sufficient conditions for the composition \( f \) to be weakly convex on an open set \( U \subset \mathcal{X} \), in the sense of (20). If we assume simply that \( h \) is convex, fixing any point \( x_0 \in U \) such that \( h \) is sub-differentiable at \( g(x_0) \), it follows that for any choice of \( x \in \mathcal{X} \) we have

\[
f(x) = h(g(x)) \geq h(g(x_0)) + \langle g(x) - g(x_0), \partial h(g(x_0)) \rangle.
\]

Let us further assume that \( h \) is \( \lambda_0 \)-Lipschitz, and \( g \) is \textit{smooth} in the sense that it is differentiable on \( U \) and the map \( x \mapsto g'(x) \) is \( \lambda_1 \)-Lipschitz. For readability, denote the derivative \( g'(x_0) : \mathcal{X} \to \mathcal{Y} \) by \( g'_0(\cdot) := g'(x_0)(\cdot) \). Taking any choice of \( v_0 \in \partial h(g(x_0)) \), we can write

\[
\langle g(x) - g(x_0), v_0 \rangle = \langle g'_0(x-x_0), v_0 \rangle + \langle g(x) - g(x_0) - g'_0(x-x_0), v_0 \rangle \\
\geq \langle g'_0(x-x_0), v_0 \rangle - \|g(x) - g(x_0) - g'_0(x-x_0)\| \|v_0\| \\
\geq \langle g'_0(x-x_0), v_0 \rangle - \frac{\lambda_1}{2} \|x-x_0\|^2 \|v_0\| \\
\geq \langle g'_0(x-x_0), v_0 \rangle - \frac{\lambda_0 \lambda_1}{2} \|x-x_0\|^2
\]

(26)

The first inequality follows from the definition of the norm for linear functionals and the fact that \( \partial h(g(x_0)) \subset \mathcal{Y}^* \). The second inequality follows from a Taylor approximation for Banach spaces (Proposition 16), using the smoothness of \( g \). The final equality follows from the fact that for convex functions, the Lipschitz coefficient implies a bound on all sub-gradients, see (47). To deal with the remaining term, note that we can write

\[
\langle g'_0(x-x_0), \partial h(g(x_0)) \rangle = \langle x-x_0, (g'_0)^*(\partial h(g(x_0))) \rangle = \langle x-x_0, \partial h(g(x_0)) \circ g'_0 \rangle.
\]

(27)

To explain the notation here, we use \( (\cdot)^* \) to denote the adjoint, namely \( A^*(y^*) := y^* \circ A \), induced by any continuous linear map \( A : \mathcal{X} \to \mathcal{Y} \), defined for each \( y^* \in \mathcal{Y}^* \). The special case we have considered here is where \( Au = g'_0(u) \), noting that differentiability means that the map \( u \mapsto g'_0(u) \) is continuous and linear. Recalling the desired form of (21), we need to establish a connection with \( \partial \mathcal{C} f(x_0) \). If we further assume that \( g \) is locally Lipschitz, then we have

\[
\partial \mathcal{C} f(x_0) \subset \partial \mathcal{C} h(g(x_0)) \circ g'_0 = \partial h(g(x_0)) \circ g'_0,
\]

(28)

where the equality follows from the coincidence of sub-differentials in the convex case, and the key inclusion follows from direct application of a generalized chain rule. Taking these facts together yields the following result.

**Proposition 8** (Weak convexity for composite functions). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces. Let \( g : \mathcal{X} \to \mathcal{Y} \) be locally Lipschitz and \( \lambda_1 \)-smooth on an open set \( U \subset \mathcal{X} \). Let \( h : \mathcal{Y} \to \mathbb{R} \) be convex and \( \lambda_0 \)-Lipschitz on \( g(U) \subset \mathcal{Y} \). Furthermore, let \( g(U) \subset \text{dom} \ h \). Then, the composite function \( f := h \circ g \) is \( \gamma \)-weakly convex on \( U \), for \( \gamma = \lambda_0 \lambda_1 \).

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23Written explicitly, for any \( u \in \mathcal{X} \), we have \( \langle u, A^*(y^*) \rangle = A^*(y^*)(u) = y^*(Au) = \langle Au, y^* \rangle \). If \( A : \mathcal{X} \to \mathcal{Y} \) is linear and continuous, this implies that the adjoint \( A^* \) is a continuous linear map from \( \mathcal{Y}^* \) to \( \mathcal{X}^* \). For more general background: Luenberger [27, Ch. 6], Penot [31, Ch. 1].

24See for example Penot [31, Thm. 5.13(b)]. This inclusion holds as long as \( g \) is strictly differentiable [31, Defn. 2.54], a property implied by the smoothness we have assumed.
Proof. The desired result just requires us to piece together the key facts we have outlined in the main text. Local Lipschitz properties for $g$ and $h$ imply that $f$ is locally Lipschitz, and thus $\partial_c f(x_0) \neq \emptyset$ for all $x_0 \in U$. Using (28), we have that $\partial h(g(x_0)) \neq \emptyset$ as well. With this in mind, linking up (25)–(27), under the assumptions stated, for each $x, x_0 \in U$ we have

$$f(x) \geq f(x_0) + \langle x - x_0, \partial h(g(x_0)) \circ g'_0 \rangle - \frac{\lambda_0 \lambda_1}{2} \|x - x_0\|^2. \quad (29)$$

Using the inclusion (28) with (29), we have (21) for $\gamma = \lambda_0 \lambda_1$, and the desired result holds since (21) implies (20). \hfill \Box

Remark 9. We note that Proposition 8 extends a result of Drusvyatskiy and Paquette [13, Lem. 4.2] from the case where $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional Euclidean spaces, to the general Banach space setting considered here. For the classical case of Euclidean spaces, exact chain rules are well-known [38, Ch. 10.B].

B.4 Proximal maps of weakly convex functions

For normed linear space $\mathcal{X}$ and function $f : \mathcal{X} \rightarrow \mathbb{R}$, the Moreau envelope $\text{env}_\beta f$ and proximal mapping (or proximity operator) $\text{prox}_\beta f$ are respectively defined for each $x \in \mathcal{X}$ as follows:

$$\text{env}_\beta f(x) := \inf \left\{ f(x') + \frac{1}{2\beta} \|x - x'\|^2 : x' \in \mathcal{X} \right\} \quad (30)$$

$$\text{prox}_\beta f(x) := \arg \min_{x' \in \mathcal{X}} \left[ f(x') + \frac{1}{2\beta} \|x - x'\|^2 \right]. \quad (31)$$

Here $\beta > 0$ is a parameter. In the case where $f$ is convex, the basic properties of the proximal map and envelope are well-understood, particularly when $\mathcal{X}$ is a Hilbert space.\(^{25}\) These insights extend readily to the setting of weak convexity. Under the assumption that $\mathcal{X}$ is Hilbert, let $f$ be $\gamma$-weakly convex on $\mathcal{X}$. Trivially we can write

$$f(x') + \frac{1}{2\beta} \|x - x'\|^2 = \left( f(x') + \frac{\gamma}{2} \|x - x'\|^2 \right) + \frac{1}{2} \left( \frac{1}{\beta} - \gamma \right) \|x - x'\|^2.$$ 

If we write $f_{\gamma,x}(u) := f(u) + (\gamma/2)\|x - u\|^2$ and $\beta_\gamma := (\beta^{-1} - \gamma)^{-1}$ for readability, then as long as $\beta_\gamma > 0$ we have for all $x \in \mathcal{X}$ that $\text{env}_\beta f(x) = \text{env}_{\beta_\gamma} f_{\gamma,x}(x)$ and $\text{prox}_\beta f(x) = \text{prox}_{\beta_\gamma} f_{\gamma,x}(x)$. By leveraging Proposition 7 under the Hilbert space assumption, we have that for any $x \in \mathcal{X}$, the function $f_{\gamma,x}(\cdot)$ is convex. This means that as long as $\beta_\gamma > 0$, namely whenever $\gamma < \beta^{-1}$, all the standard results available for the case of convex functions can be brought to bear on the problem.\(^{26}\) Of particular importance to us is the fact that when $f$ is LSC and $\gamma$-weakly convex, the Moreau envelope is differentiable, with gradient

$$(\text{env}_\beta f)'(x) = \frac{1}{\beta} \left( x - \text{prox}_\beta f(x) \right), \quad (32)$$

well-defined for all $\beta < \gamma^{-1}$ and $x \in \mathcal{X}$.\(^ {27}\) We will be interested in finding stationary points of $f$, namely those $x \in \mathcal{X}$ such that $0 \in \partial_c f(x)$. From the basic properties of the envelope and

\(^{25}\)See for example Bauschke and Combettes [5, Ch. 12 and 24]. For Banach spaces, modified notions of “proximity” measured using Bregman divergences have also been developed [4, 46]. See also Jourani et al. [21] for more analysis of the Moreau envelope in more general spaces.

\(^{26}\)For example, see Bauschke and Combettes [5, Sec. 12.4].

\(^{27}\)One can just apply standard arguments such as given by Bauschke and Combettes [5, Prop. 12.30], while utilizing the weak convexity property described. See also Drusvyatskiy and Paquette [13, Lem. 4.3], Davis and Drusvyatskiy [12, Lem. 2.2], and Poliquin and Rockafellar [32, Thm. 4.4].
proximal mapping, for γ-weakly convex \( f \) we have
\[
\text{dist}(0; \partial C f(\text{prox}_\beta f(x))) \leq \| (\text{env}_\beta f)'(x) \|. \tag{33}
\]
That is, for any point \( x \in \mathcal{X} \), the point \( \text{prox}_\beta f(x) \in \mathcal{X} \) is approximately stationary. The degree of precision is controlled by the gradient of \( \text{env}_\beta f \) evaluated at \( x \). In addition, it follows immediately from (32) that
\[
\| x - \text{prox}_\beta f(x) \| = \beta \| (\text{env}_\beta f)'(x) \|. \tag{34}
\]
Since one trivially also has \( f(\text{prox}_\beta f(x)) \leq f(x) \), the norm of the gradient of \( \text{env}_\beta f \) evaluated at \( x \) also tells us how far we are from a point (namely \( \text{prox}_\beta f(x) \in \mathcal{X} \)) which is no worse than \( x \) in terms of function value. These basic facts directly motivate the use of the Moreau envelope norm to quantify algorithm performance.\(^{28}\)

C  Detailed proofs

C.1  Proofs for section 2

Lemma 10  (Lower semi-continuity). Let \( \mathcal{Z} \) be a linear space of \( \mathcal{F} \)-measurable random variables, and let \( \rho : \mathbb{R} \rightarrow [0, \infty) \) be any non-negative LSC function that is Borel-measurable. Then we have that the functional \( (Z, \theta) \mapsto \mathbb{E}_\mu \rho(Z - \theta) \) is also LSC.

\begin{proof}
\text{Proof of Lemma 10.} \text{Let } (Z_k) \text{ and } (\theta_k) \text{ respectively be convergent sequences on } Z \text{ and } \mathbb{R}. \text{ As we take } k \rightarrow \infty, \text{ say } Z_k \rightarrow Z_* \text{ pointwise, for some } Z_* \in Z, \text{ and } \theta_k \rightarrow \theta_* \in \mathbb{R}. \text{ Since by assumption } \rho \text{ is LSC on } \mathbb{R}, \text{ using (12) we have (again, pointwise) that}
\[
\rho(Z_* - \theta_*) \leq \liminf_{k \rightarrow \infty} \rho(Z_k - \theta_k).
\]
Writing \( \rho_k := \rho(Z_k - \theta_k) \) for each \( k \geq 1 \) and \( \rho_* = \rho(Z_* - \theta_*) \), it follows that
\[
\mathbb{E}_\mu \rho_* \leq \mathbb{E}_\mu \left( \liminf_{k \rightarrow \infty} \rho_k \right) \leq \liminf_{k \rightarrow \infty} (\mathbb{E}_\mu \rho_k). \tag{35}
\]
The former inequality follows from monotonicity of the integral, and the latter inequality follows from an application of Fatou’s inequality, which is valid since \( \rho_k \geq 0 \).\(^{29}\) Taking both ends of (35) together, since the choice of sequences \( (Z_k) \) and \( (\theta_k) \) were arbitrary, it follows again from the equivalence (12) that the functional \( (Z, \theta) \mapsto \mathbb{E}_\mu \rho((Z - \theta)/\sigma) \) is LSC on \( Z \times \mathbb{R} \). \end{proof}

Lemma 11  (Basic integration properties). Let \( \mathbb{E}_\mu Z^2 < \infty \) hold, and take any \( \theta \in \mathbb{R} \). Then the following properties of integrals based on \( \rho_\sigma \) defined in (5) hold:

- \( \text{For all } \sigma \in [0, \infty], \text{ we have } 0 \leq \mathbb{E}_\mu \rho_\sigma(Z - \theta) < \infty. \)
- \( \text{For } 0 < \sigma \leq \infty, \rho_\sigma(\cdot) \text{ is differentiable, and we have } \mathbb{E}_\mu |\rho_\sigma'(Z - \theta)| < \infty. \)
- \( \text{For } 0 < \sigma \leq \infty, \rho_\sigma'(\cdot) \text{ is differentiable, and we have } 0 \leq \mathbb{E}_\mu \rho_\sigma''(Z - \theta) < \infty. \)

\(^{28}\)This is highlighted in works such as Drusvyatskiy and Paquette [13] and Davis and Drusvyatskiy [12].

\(^{29}\)Ash and Doléans-Dade [2, Lem. 1.6.8].
Furthermore, the Leibniz integration property holds for both derivatives, that is

\[
\frac{d}{d\theta} E_\mu \rho_\sigma(Z - \theta) = -\frac{E_\mu \rho'_\sigma(Z - \theta)}{\sigma}, \quad \frac{d}{d\theta} E_\mu \rho'_{\infty}(Z - \theta) = -\frac{E_\mu \rho''_{\infty}(Z - \theta)}{\sigma}\tag{36}
\]

for any \( 0 < \sigma < \infty \), and for the special case of \( \sigma = \infty \), we have

\[
\frac{d}{d\theta} E_\mu \rho_\infty(Z - \theta) = -2(E_\mu Z - \theta), \quad \frac{d}{d\theta} E_\mu \rho'_{\infty}(Z - \theta) = -2. \tag{37}
\]

These equalities hold for any \( \theta \in \mathbb{R} \).

Proof of Lemma 11. Non-negativity of \( \rho_\sigma \) implies \( 0 \leq E_\mu \rho_\sigma(Z - \theta) \) for all \( \sigma \). Regarding finiteness, starting with \( \sigma = 0 \) we have \( E_\mu \rho_0(Z - \theta) = E_\mu |Z - \theta| < \infty \), which follows from Hölder’s inequality and \( \mu \)-integrability of \( Z^2 \). For \( 0 < \sigma < \infty \), first note that \( \text{atan}'(u) \leq 1 \) for all \( u \in \mathbb{R} \), and thus since \( \text{atan}(0) = 0 \), we have \( |\text{atan}(u)| \leq |u| \) and \( u \text{atan}(u) \leq u^2 \) for all \( u \). In particular, this means \( \rho_\sigma(Z - \theta) \leq (Z - \theta)^2/\sigma^2 \), and thus square-\( \mu \)-integrability of \( Z \) implies \( E_\mu \rho_\sigma(Z - \theta) < \infty \). The \( \sigma = \infty \) case follows identically.

Moving to \( \rho'_\sigma(\cdot) \), for \( 0 < \sigma < \infty \) we have that \( |\rho'_\sigma(u)| < \pi/2 \) for all \( u \in \mathbb{R} \), and thus \( E_\mu |\rho'_\sigma(Z - \theta)| < \infty \). The exact same argument holds for \( \rho''_\sigma(\cdot) \), since \( 0 < \rho''_\sigma(u) \leq 1 \) for all \( u \in \mathbb{R} \). The \( \sigma = \infty \) case follows analogously.

For the Leibniz property, let \( (a_k) \) be any real sequence such that \( a_k \to 0 \). Using the fact that \( \rho'_\sigma \) is bounded, the dominated convergence theorem lets us deduce the following:

\[
\lim_{k \to \infty} \frac{E_\mu \rho_\sigma(Z - (\theta + a_k)) - E_\mu \rho_\sigma(Z - \theta)}{a_k} = \lim_{k \to \infty} E_\mu \left[ \frac{\rho_\sigma(Z - (\theta + a_k)) - \rho_\sigma(Z - \theta)}{a_k} \right] = E_\mu \left[ \lim_{k \to \infty} \frac{\rho_\sigma(Z - (\theta + a_k)) - \rho_\sigma(Z - \theta)}{a_k} \right] = -\frac{E_\mu \rho'_\sigma(Z - \theta)}{\sigma}.
\]

We note that the first equality just uses \( \mu \)-integrability and linearity of the Lebesgue integral, the second equality uses boundedness and integrability of the derivative, plus dominated convergence (e.g., Ash and Doléans-Dade [2, Thm. 1.6.9]). The last equality is just the chain rule applied to the differentiable function \( \rho_\sigma(\cdot) \). Since the sequence \( (a_k) \) was arbitrary, we conclude that the first equality of (36) holds. The second equality of (36), as well as both equalities in (37) hold via an identical argument. \( \square \)

Proof of Proposition 1. First, note that the convexity of \( \rho_\sigma(\cdot) \) implies that \( \theta \mapsto r_\sigma(Z, \theta) \) is convex. We start by showing that \( r_\sigma \) is also coercive, namely that \( |\theta| \to \infty \) implies \( r_\sigma(Z, \theta) \to \infty \). For all cases \( \sigma \in [0, \infty] \), the non-negativity of \( \rho_\sigma \) and \( \eta \) trivially implies that \( \theta \to \infty \) implies \( r_\sigma(Z, \theta) \to \infty \), and thus we need only consider the negative direction, where \( \theta \to -\infty \).

For the case of \( \sigma = 0 \), note that

\[
\theta + \eta E_\mu |Z - \theta| \geq \theta - \eta|\theta| + \eta E_\mu |Z|.
\]

Clearly, with \( \eta > 1 \) the right-hand side grows without bound as \( \theta \to -\infty \). For the case of \( 0 < \sigma = \infty \), writing \( Z_\theta := (Z - \theta)/\sigma \) for readability, the joint risk can be written conveniently as

\[
r_\sigma(Z, \theta) = \theta \left( 1 - \frac{\eta}{\sigma} E_\mu \text{atan}(Z_\theta) \right) + \frac{\eta}{\sigma} E_\mu Z \text{atan}(Z_\theta) - \frac{\eta}{2} E_\mu \log(1 + Z^2_\theta). \tag{38}
\]

Since \( \text{atan}(\cdot) \) is monotonic (increasing) on \( \mathbb{R} \), bounded as \( |\text{atan}(\cdot)| < \pi/2 \), and \( \text{atan}(u) \to \pi/2 \) as \( u \to \infty \), we have that \( E_\mu \text{atan}(Z_\theta) \to \pi/2 \) as \( \theta \to -\infty \), by monotone convergence. Thus, \(30\) Ash and Doléans-Dade [2, Thm. 1.6.7].
taking \( \eta > 2\sigma / \pi \) ensures that eventually as \( \theta \to \infty \), the first term on the right-hand side of (38) will become positive. Since this term grows linearly, it dominates the other unbounded term (which is logarithmic), and thus we have shown that \( r_\sigma \) is coercive whenever \( \sigma \geq 0 \). Convexity and coercivity together imply that \( \theta \mapsto r_\sigma(Z, \theta) \) takes its minimum on \( \mathbb{R} \); see Bertsekas [6, Sec. B.3.2] or Barbu and Precupanu [3, Thm. 2.11] for standard references.

The case of \( \sigma = \infty \) is easy, since by direct inspection we can write
\[
\begin{align*}
    r_\infty(Z, \theta) &= \theta(1 - 2 E_\mu, Z) + \eta \theta^2 + \eta E_\mu, Z^2.
\end{align*}
\]
Since the sum of a strongly convex function and an affine function is strongly convex, we have that \( \theta \mapsto r_\infty(Z, \theta) \) has a unique minimum on \( \mathbb{R} \).

It only remains to prove the uniqueness of \( \theta_Z \) in the proposition statement for the case of \( 0 < \sigma < \infty \). The most direct way of doing this is to use the Leibniz property (36) proved in our helper Lemma 11, which in particular tells us that
\[
\frac{d^2}{d\theta^2} r_\sigma(Z, \theta) = \frac{\eta}{\sigma} E_\mu \rho'' \left( \frac{Z - \theta}{\sigma} \right) > 0,
\]
where positivity follows from the fact that \( \rho''(\cdot) = 1/(1 + (\cdot)^2) > 0 \). This implies strict convexity, and thus that the minimizer \( \theta_Z \) is unique.

**Proof of Proposition 3.** We take the points in the statement of the proposition in order, one at a time. To be seen, the (joint) convexity of \( r_\sigma \) follows from direct inspection, using the convexity of \( \rho_\sigma \) for any \( \sigma \in [0, \infty] \). With this fact in mind, note that the convexity of \( R_\sigma \) can be checked easily as follows. For any \( Z_1, Z_2 \in \mathbb{Z} \) and \( \theta_1, \theta_2 \in \mathbb{R} \), the definition and convexity of \( r_\sigma \) immediately implies that for any \( \alpha \in (0, 1) \) we have
\[
\begin{align*}
    R_\sigma(\alpha Z_1 + (1 - \alpha) Z_2) &\leq \alpha r_\sigma(\alpha Z_1 + (1 - \alpha) Z_2, \alpha \theta_1 + (1 - \alpha) \theta_2) \\
    &\leq \alpha r_\sigma(Z_1, \theta_1) + (1 - \alpha) r_\sigma(Z_2, \theta_2).
\end{align*}
\]
Using the notation (and statement) of Proposition 1, we can set \( \theta_1 = \theta_{Z_1} \in \mathbb{R} \) and \( \theta_2 = \theta_{Z_2} \in \mathbb{R} \), and plugging this in to the above inequalities, we obtain
\[
R_\sigma(\alpha Z_1 + (1 - \alpha) Z_2) \leq \alpha R_\sigma(Z_1) + (1 - \alpha) R_\sigma(Z_2),
\]
and thus both \( r_\sigma \) and \( R_\sigma \) are convex for any \( \sigma \in [0, \infty] \). Note that this does not require the minima \( \theta_{Z_1} \) and \( \theta_{Z_2} \) to be unique, and thus holds for the \( \sigma = 0 \) case without issue. From Lemma 11, we also have that \( |r_\sigma(\cdot, \cdot)| < \infty \) and \( |R_\sigma(\cdot)| < \infty \), so both functions are proper convex. As for continuity, note that from Lemma 10 and the continuity of \( \rho_\sigma(\cdot) \) for all \( \sigma \in [0, \infty] \), we can immediately infer that \( r_\sigma \) is LSC. It is well-known that on Banach spaces, any proper convex LSC function is continuous and sub-differentiable on the interior of the effective domain.\(^{31}\) Since our integrability assumptions imply \( \text{dom} r_\sigma = \mathbb{Z} \times \mathbb{R} \), the continuity and sub-differentiability of \( r_\sigma \) is thus proved. To handle \( R_\sigma \), take any sequence \( (Z_k) \) converging to an arbitrarily chosen point \( Z_* \in \mathbb{Z} \). Let \( (\theta_k) \) be any sequence converging to \( \theta_{Z_*} \in \mathbb{R} \). Then by definition of \( R_\sigma \) and continuity of \( r_\sigma \), we have
\[
\limsup_{k \to \infty} R_\sigma(Z_k) \leq \limsup_{k \to \infty} r_\sigma(Z_k, \theta_k) = \lim_{k \to \infty} r_\sigma(Z_k, \theta_k) = R_\infty(Z_*, \theta_{Z_*}) = R_\sigma(Z_*).
\]
The two ends of the inequality (39) imply that \( R_\sigma \) is USC, via (12) and the relation of USC to LSC functions. On the effective domain of any convex USC function, the function is in fact
\[^{31}\text{Actually, via Barbu and Precupanu [3, Prop. 2.16], this holds for every point in the algebraic interior of its effective domain; the fact stated follows as the algebraic interior contains the interior.}\]
continuous.\textsuperscript{32} Thus, we have that $R_\sigma$ is continuous. Furthermore, the sub-differentiability of $R_\sigma$ follows in the exact same fashion as for $r_\sigma$.

Next, for the monotonicity of the location term $Z \mapsto \theta_Z$ in (6), with $0 < \sigma \leq \infty$, recall that we can utilize the Leibniz properties (36)–(37) from the integration Lemma 11. To start, we know that for any $Z$, the corresponding $\theta_Z$ must satisfy the following first-order optimality condition:

$$
\begin{align*}
\text{If } \sigma < \infty: & \quad E_\mu \rho'_\sigma(Z - \theta_Z) = \frac{\sigma}{\eta}, & \quad \text{If } \sigma = \infty: & \quad \theta_Z = E_\mu Z - \frac{1}{2\eta}. \\
\end{align*}
$$

(40)

The desired monotonicity property is obvious for the $\sigma = \infty$ case using (40). As for the case of $0 < \sigma < \infty$, it is evident from the second equality of (36) that the function $\theta \mapsto E_\mu \rho_\sigma(Z - \theta)$ is monotonically decreasing on $\mathbb{R}$. Thus, if we assume $Z_1 \leq Z_2$ almost surely but $\theta_{Z_1} > \theta_{Z_2}$, the first order optimality combined with monotonicity implies

$$
\frac{\sigma}{\eta} = E_\mu \rho_\sigma(Z_1 - \theta_{Z_1}) < E_\mu \rho_\sigma(Z_1 - \theta_{Z_2}) \leq E_\mu \rho_\sigma(Z_2 - \theta_{Z_2}) = \frac{\sigma}{\eta},
$$

which is a contradiction. Thus, $\theta_{Z_1} \leq \theta_{Z_2}$ as desired for the $0 < \sigma < \infty$ case as well.

The translation-equivariance property of $Z \mapsto \theta_Z$ follows from direct inspection using the condition (40), that is for $0 < \sigma < \infty$, we trivially have

$$
\frac{\sigma}{\eta} = E_\mu \rho_\sigma(Z - \theta_Z) = E_\mu \rho_\sigma(Z + a - (\theta_Z + a)),
$$

and thus $r_\sigma(Z + a, \theta_Z + a) = R_\sigma(Z + a)$. Since Proposition 1 guarantees that the minimizer of $\theta \mapsto R_\sigma(Z, \theta)$ is unique, we can safely write $\theta_{Z + a} = \theta_Z + a$. The proof for the $\sigma = \infty$ case is analogous.

Finally, to prove that $R_\sigma$ is not in general monotonic, we give a concrete example of $Z_1$ and $Z_2$ such that $Z_1 \leq Z_2$ almost surely, but $R_\sigma(Z_1) > R_\sigma(Z_2)$. For simplicity, consider the case of $\sigma = \infty$, where $\rho_\infty(\cdot) = (\cdot)^2$, and for any $\eta > 0$ direct inspection shows that

$$
R_\infty(Z) = E_\mu Z + \eta \var_\mu Z - \frac{1}{4\eta}.
$$

(41)

That is, the special case of $\sigma = \infty$ is equivalent to the mean-variance risk function of classical portfolio theory, dating back to Markowitz\textsuperscript{[28]}. The random variables $Z_1$ and $Z_2$ are constructed as follows. Let $c_1$ and $c_2$ be the respective centers, $w_1$ and $w_2$ the respective widths, and $v_1 < w_2$ and $v_2 < w_2^2$ the respective scaling factors of $Z_1$ and $Z_2$, which are characterized as

$$
P\{Z_j = c_j - w_j\} = \mu \{Z_j = c_j + w_j\} = \frac{v_j}{2w_j^2}, \quad P\{Z_j = c_j\} = 1 - P\{Z_j \neq c_j\}
$$

for each $j \in \{1, 2\}$. Note that our assumptions imply $0 < \mu \{Z_j = c_j\} < 1$, and direct inspection shows that $E Z_j = c_j$ and $\var Z_j = v_j$, again for each $j \in \{1, 2\}$. As a simple concrete example, note that setting $c_2 = c_1 + w_1 + w_2$ guarantees $Z_1 \leq Z_2$ with probability 1. From the equality (41) given above, the difference in risks can be written as

$$
R_\infty(Z_1) - R_\infty(Z_2) = E Z_1 - E Z_2 + \eta (\var Z_1 - \var Z_2) = -(v_1 + w_2) + \eta(v_1 - v_2).
$$

Thus, $R_\infty(Z_1) > R_\infty(Z_2)$ holds whenever $v_1 - v_2 > w_1 + w_2$. For concreteness, say for some $\varepsilon > 0$, we fix the variance factors to $v_1 = w_1^2 - \varepsilon$ and $v_2 = w_2^2 - \varepsilon$ respectively. Then the condition simplifies to $w_1^2 > w_1 + w_2 + w_2^2$. As an example, setting $w_1 = 2$ and $w_2 = 1/2$, the condition holds, implying $R_\infty(Z_1) > R_\infty(Z_2)$, despite the fact that $Z_1 \leq Z_2$. This gives us a simple but intuitive example where monotonicity of $R_\sigma$ does not hold, and concludes the proof.\hfill \Box

\textsuperscript{32}Penot\textsuperscript{[31, Prop. 3.2].}
C.2 Proofs for section 3

Recall that our basic probabilistic setup for the learning problem has an underlying probability space \((\Omega, \mathcal{F}, \mu)\), a hypothesis class \(\mathcal{H}\), and a random loss \(L(h)\) indexed by \(\mathcal{H}\). That is, we consider any \(\mathcal{F}\)-measurable function \(L(\cdot; \cdot) : \Omega \rightarrow \mathbb{R}\) as a loss. When a particular realization \(\omega \in \Omega\) is important, we will write \(L(h; \omega)\), but otherwise, for readability we will typically write \(L(h) := L(h; \cdot)\). Our basic integrability assumption, carried over from section 2, is that of square-\(\mu\)-integrability, which in the context of losses is written explicitly as

\[
\mathbb{E}_\mu |L(h)|^2 = \int_\Omega |L(h; \omega)|^2 \mu(d\omega) < \infty
\]

for all \(h \in \mathcal{H}\). This requirement is made in assumption A1 in the main text. It follows that \(\{L(h) : h \in \mathcal{H}\} \subset L^2(\Omega, \mathcal{F}, \mu)\). Thus the map \(h \mapsto L(h)\) takes us from \(\mathcal{H}\) to \(L^2(\Omega, \mathcal{F}, \mu)\).

Loss-specific terminology To ensure our use of formal terms is clear, we apply the definitions of section B.1 to losses here. We shall typically suppress the dependence on \(\omega \in \Omega\) in directional derivatives and gradients, writing \(L'_\lambda(h; g) = L'_\lambda(h; g, \cdot)\), \(L'(h; g) = L'(h; g, \cdot)\), and \(L(h) := L(h; \cdot)\). Let \(H \subset \mathcal{H}\) be an open set. We say that \(L\) is radially differentiable at \(h \in H\) if the radial derivative \(L'_\lambda(h; g)\) exists for all directions \(g \in \mathcal{H}\), \(\mu\)-almost surely. We say that \(L\) is directionally differentiable at \(h \in H\) if the directional derivative \(L'(h; g)\) exists for all directions \(g \in \mathcal{H}\), \(\mu\)-almost surely. On this “good” event of probability 1, if the map \(g \mapsto L'_\lambda(h; g)\) is linear and continuous, we say \(L\) is Gateaux differentiable at \(h\), and if the map \(g \mapsto L'(h; g)\) is linear and continuous, we say \(L\) is Hadamard differentiable at \(h\). When we say that \(L\) is (Fréchet) differentiable at \(h \in H\), we mean that there exists a function \(L'(h)(\cdot) : H \rightarrow \mathbb{R}_+\) that is linear, continuous, and which satisfies (15) \(\mu\)-almost surely.\(^{33}\) We say that \(L\) is \(\lambda\)-Lipschitz at \(h \in H\) if there exists a \(\delta > 0\) such that \(\|h - h'\| < \delta \implies \|L(h) - L(h')\| \leq \lambda\|h - h'\|\). With the running assumption about second moments, this amounts to requiring

\[
\|h - h'\| < \delta \implies \mathbb{E}_\mu |L(h) - L(h')|^2 \leq \lambda^2\|h - h'\|^2.
\]  

(42)

We say that \(L\) is weakly \(\lambda\)-smooth at \(h \in H\) if \(L\) is Gateaux differentiable and the map \(h \mapsto L'_\lambda(h; \cdot)\) is \(\lambda\)-Lipschitz \(\mu\)-almost surely at \(h\). That is, if for small enough \(\delta > 0\) we have

\[
\|h - h'\| < \delta \implies \|L'_\lambda(h; \cdot) - L'_\lambda(h'; \cdot)\| \leq \lambda\|h - h'\|.
\]  

(43)

Note that the norm used here is the operator norm applied to the linear map \(L'_\lambda(h; \cdot) - L'_\lambda(h'; \cdot)\).

C.2.1 Weak convexity of joint composition function

The joint risk function \(r_\sigma(L(h), \theta)\) can be written as a simple composition \((h, \theta) \mapsto (L(h), \theta) \mapsto \theta + \eta \mathbb{E}_\mu \rho_\sigma(L(h) - \theta)\). For any \(0 \leq \sigma < \infty\) and any smooth loss, using the preliminary results established section B.3, it is straightforward to show the weak convexity of this composite function.

**Proposition 12.** Let the hypothesis class \(\mathcal{H}\) be Banach. Let the loss \(L\) be locally Lipschitz and weakly \(\lambda'\)-smooth on \(\mathcal{H}\). Then, for any \(0 \leq \sigma < \infty\), defining a \(\sigma\)-dependent factor \(\lambda_\sigma\) as

\[
\lambda_\sigma := \begin{cases} 
1, & \text{if } \sigma = 0 \\
\pi/(2\sigma), & \text{if } 0 < \sigma < \infty
\end{cases}
\]

we have that \((h, \theta) \mapsto r_\sigma(L(h), \theta)\) is \(\gamma\)-weakly convex with \(\gamma = (1 + \eta \lambda_\sigma) \max\{1, \lambda'\}\).

\(^{33}\)In our particular setting with losses here, the norm used in the numerator of (15) will be the \(L_2\) norm.
Remark 13. The result in the preceding Proposition 12 is rather useful, and it does not require the loss to be convex. When the loss is convex, the analysis becomes somewhat simpler and stronger arguments are naturally possible; composite risks under convex losses and convex, monotonic risk functions is the setting considered by Ruszczyński and Shapiro [40, Sec. 3.2], for example.

We have established conditions under which the intermediate joint objective $r_\sigma(L(h), \theta)$ is weakly convex, and characterized this weak convexity with respect to properties of the underlying risk function and data distribution. Since the data distribution $\mu$ is unknown, we can never actually compute $r_\sigma(L(h), \theta)$. Any learning algorithm will only have access to feedback of a stochastic nature which provides incomplete, noisy information. Our next task is to establish conditions under which the feedback available to the learner is “good enough” to ensure reasonable performance guarantees.
C.2.2 Unbiased stochastic feedback

In considering stochastic feedback, recall that \( r_\sigma(L(h), \theta) = E_\mu(f_2 \circ F_1)(h, \theta) \), with \( F_1 \) and \( f_2 \) given by (7) in the main text. For each \( h \in \mathcal{H} \) and \( \theta \in \mathbb{R} \), the value \( F_1(h, \theta) \) returned by \( F_1 \) is a random vector. We shall assume that for any \( h \in \mathcal{H} \), the learner can obtain independent random samples of the loss \( L(h) \) and the associated gradient \( L'(h) \). Since \( F'_1(h, \theta) = (L'(h), 1) \) by which we mean \( F'_1(h, \theta)(g, r) = (L'(h)(g), r) \) for all \( g \in \mathcal{H} \) and \( r \in \mathbb{R} \), clearly the learner can also independently sample from \( F'_1(h, \theta) \) and \( F'_1(h, \theta) \). Sub-differentiability is already guaranteed by Proposition 3, and since \( \rho_\sigma \) and \( \eta \) are by design known to the learner, they can readily acquire an element from \( \partial f_2(u, \theta) \). Thus if \( (L(h), \theta) \in \text{int}(\text{dom} \ f_2) = \mathbb{R}^2 \) \( \mu \)-almost surely, it follows that the learner can sample from \( \partial f_2(L(h), \theta) \circ F'_1(h, \theta) \). This is the stochastic feedback available to the learner, and when we ask that it be “good enough,” this means we require it to be an unbiased estimator of the (Clarke) sub-differential of \( r_\sigma \). The following result gives mild conditions under which this is achieved.

**Proposition 14.** Under the conditions of Proposition 12, for any \( h \in \mathcal{H} \) and \( \theta \in \mathbb{R} \), as long as \( E_\mu|L'(h; \cdot)| < \infty \), the stochastic sub-differential is an unbiased estimator in that

\[
E_\mu [\partial f_2(L(h), \theta) \circ F'_1(h, \theta)] \subset \partial_C r_\sigma(L(h), \theta),
\]

and this holds for any choice of 0 \( \leq \sigma < \infty \).

**Proof.** Using the weak smoothness of \( L \), with probability 1, the map \( (h, \theta) \mapsto F'_1(h, \theta) \) is continuous, plus \( F_1 \) is locally Lipschitz and \( \partial_C F_1(h, \theta) = \{ F'_1(h, \theta) \} \).\(^{38}\) Furthermore, since

\[
F'_1(h, \theta)(g, r) = (L'(h)(g), r),
\]

the linearity of \( L'(h)(\cdot) \) implies that \( F'_1(h, \theta)(\mathcal{H} \times \mathbb{R}) = \mathbb{R}^2 \) \( \mu \)-almost surely. Since \( F_1 \) and \( f_2 \) are (locally) Lipschitz, the facts we have just laid out imply a strong chain rule.\(^{39}\) That is, it holds \( \mu \)-almost surely that

\[
\partial_C(f_2 \circ F_1)(h, \theta) = \partial_C f_2(L(h), \theta) \circ F'_1(h, \theta) = \partial f_2(L(h), \theta) \circ F'_1(h, \theta),
\]

where the second equality follows from the convexity of \( f_2 \).

Next, the Lipschitz property of \( \rho_\sigma \) implies that \( f_2 \) is \( \lambda \)-Lipschitz for some \( \lambda > 0 \); the actual value is not important for this proof. Using the Lipschitz continuity of \( f_2 \) and the fact that \( \partial_C F_1(h, \theta) = \{ F'_1(h, \theta) \} \) implies \( \partial_C L(h) = \{ L'(h) \} \), we have

\[
|(f_2 \circ F_1)'(h, \theta)(g, r)| \leq \limsup_{(a, h') \to (0_+, h)} \frac{|(f_2 \circ F_1)(h' + \alpha g, \theta' + \alpha r) - (f_2 \circ F_1)(h', \theta')|}{\alpha}
\]

\[
\leq \lambda \left( \limsup_{(a, h') \to (0_+, h)} \frac{|L(h' + \alpha g) - L(h')|}{\alpha} + |r| \right)
\]

\[
= \lambda (|L'(h)(g)| + |r|).
\]

Thus, using \( r_\sigma(L(h), \theta) = E_\mu(f_2 \circ F_1)(h, \theta) \) and \( E_\mu|L'(h)(g)| < \infty \) for all \( h, g \in \mathcal{H} \), we have

\[
E_\mu (f_2 \circ F_1)'(h, \theta; \cdot) = (r_\sigma)'(L(h), \theta; \cdot)
\]

by a direct application of dominated convergence.\(^{40}\)

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\(^{38}\)Penot [31, Prop. 5.6].

\(^{39}\)Penot [31, Prop. 5.13].

\(^{40}\)Ash and Doléans-Dade [2, Thm. 1.6.9].
To conclude, taking \( G(h, \theta) \in \partial f_2(L(h), \theta) \circ F_1(h, \theta) \), by (44) it follows that we have \( G(h, \theta) \in \partial C(f_2 \circ F_1)(h, \theta) \), and thus by definition of the Clarke sub-differential, monotonicity of the integral, and finally (45), we obtain

\[
E_\mu G(h, \theta)(\cdot) \leq E_\mu (f_2 \circ F_1)_C(h, \theta; \cdot) = (r_\sigma)_C(L(h), \theta; \cdot). \tag{46}
\]

Linearity of \( E_\mu G(h, \theta)(\cdot) \) follows from the linearity of both \( G(h, \theta)(\cdot) \) and the integral. Finally, applying (46) we have

\[
\sup_{\|u,r\|=1} E_\mu G(h, \theta)(g, r) \leq \sup_{\|g,r\|=1} (r_\sigma)_C(L(h), \theta; u, r) < \infty,
\]

where finiteness holds because \( r_\sigma \) is locally Lipschitz.\(^41\) Thus \( E_\mu G(h, \theta) \in (H \times \mathbb{R})^* \), and with (46) we have \( E_\mu G(h, \theta) \in \partial C r_\sigma(L(h), \theta) \) as desired. \(\square\)

**Remark 15.** The validity of interchanging the operations of (sub-)differentiation and expectation is a topic of fundamental importance in stochastic optimization and statistical learning theory. A useful, modern reference on this topic is included in Ruszczyński and Shapiro [39, Ch. 2]. A classical reference is Rockafellar and Wets [37]; see also Rockafellar [35] for a look at measurability of convex integrands. The interchangeability problem appears in various places in the literature over the years, see for example Shapiro [45] as well as Kall and Mayer [22, Ch. 3, Rmk. 2.2]. See also Ruszczyński and Shapiro [40, Eqn. (3.9)], who refer to generalized versions of a classic result due to Strassen [47].

### C.2.3 Proximity to a nearly-stationary point

**Proof of Theorem 5.** With all the results established thus far, this proof has just two simple parts. First, we need to show that the objective function of interest is weakly convex, and that we have access to unbiased estimates of the sub-differential; this is done here. This is done using the critical preparatory results in Propositions 12 and 14. Once this has been established, the remaining part just has us applying recent results from the literature for non-asymptotic control of the envelope gradient norm.

To begin, the assumptions of Proposition 12 are satisfied by A1, which ensures that \( r_\sigma \) is \( \gamma \)-weakly convex for \( \gamma = (1 + \eta \lambda_r) \max\{1, \lambda\} \). Furthermore, the \( \mu \)-integrability assumption on \( L'(h) \) lets us use Proposition 14 to ensure that feedback drawn from (9) is such that \( E_\mu G_t \in \partial C r_\sigma(h_t, \theta_t) \) for all \( t \). Furthermore, using A3 implies that \( E[G_t | G_{k-1}] \in \partial C r_\sigma(h_t, \theta_t) \) for all \( t \), since Algorithm 1 uses \( G_t \) sampled via (9).

The desired result follows from an application of Davis and Drusvyatksiyi [12, Thm. 3.1], where their objective function \( f \) corresponds to our \( r_\sigma \).\(^42\) While their proof is given for the case of \( H = \mathbb{R}^d \), using assumption A2, if we leverage our characterization of weak convexity (Proposition 7), and replace their Lemma 2.2 with our (32), it is straightforward to see that their insights extend to arbitrary Hilbert spaces using the usual norm induced by the inner product. Thus with the moment bound A4 in hand, the generalized result can be applied to Algorithm 1, for objective function \( r_\sigma \), which has just been proved to be \( \gamma \)-weakly convex. The desired result follows immediately. \(\square\)

### D Helper results

In this section, we provide some standard results that are leveraged in the main paper.

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\(^41\)Penot [31, Prop. 5.2(b)].

\(^42\)It also relies on the observation that a proximal stochastic sub-gradient update using the indicator function of \( C \) as a regularizer is equivalent to the projected sub-gradient update we do here.
D.1 Useful results based on Lipschitz properties

Let $\mathcal{X}$ be a normed linear space, and let $f : \mathcal{X} \to \mathbb{R}$ be convex and $\lambda$-Lipschitz. If $f$ is sub-differentiable at a point $x$, then using the definition of the sub-differential, we have that

$$|\langle x' - x, \partial f(x) \rangle| \leq |f(x') - f(x)| \leq \lambda \|x' - x\|.$$ 

It immediately follows that

$$\|\partial f(x)\| \leq \lambda. \tag{47}$$

That is, all sub-gradients of $f$ at $x$ have norm no greater than the Lipschitz coefficient $\lambda$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be differentiable on $U \subset \mathcal{X}$, an open set. Further, assume that the derivative is $\lambda$-Lipschitz on $U$, that is, for each $x, x' \in U$, we have $\|f'(x) - f'(x')\| \leq \lambda \|x - x'\|$. First-order Taylor approximations have direct analogues in this general setting, as the following result shows.\(^{43}\)

**Proposition 16.** Let $f : \mathcal{X} \to \mathcal{Y}$ be differentiable on an open set $U \subset \mathcal{X}$, with $\mathcal{X}$ and $\mathcal{Y}$ assumed to be Banach. If $f'(\cdot)$ is $\lambda$-Lipschitz on $U$, then for any $x, u \in U$ such that $x + u \in U$, we have

$$\|f(x + u) - f(x) - f'(x)(u)\| \leq \frac{\lambda}{2} \|u\|^2. \tag{48}$$

D.2 Radial derivatives of convex functions

Say a function $f : \mathcal{V} \to \mathbb{R}$ is convex. Take any $u, v \in \text{dom } f$, and any scalar $c \geq 0$ such that $v + c(v-u) \in \text{dom } f$. Then writing $u' := v + c(v-u)$, note that

$$v = \frac{1}{1+c} (u' + cu) = (1-\beta)u' + \beta u,$$

where $\beta := c/(1+c) \in [0,1)$. By convexity we have $f((1-\beta)u' + \beta u) \leq (1-\beta)f(u') + \beta f(u)$. Filling in definitions and rearranging we have

$$f(v + c(v-u)) - f(v) \geq c(f(v) - f(u)). \tag{48}$$

Note that this can be done for any pair of $u, v$ and scalar $c$ that keeps the relevant points on the domain. Clearly this property is necessary for convexity, but it is in fact also sufficient.\(^{44}\)

For any function $f : \mathcal{V} \to \mathbb{R}$ and open set $U \subset \mathcal{X}$, fix a point $x \in U$. We denote the difference quotient of $f$ at $x$, incremented in the direction $u$, modulated by scalar $\alpha \neq 0$ as

$$q(\alpha) := q(\alpha; f, x, u) := \frac{f(x + \alpha u) - f(x)}{\alpha}. \tag{49}$$

Consider the map $g(t) := f(x + tu) - f(x)$, with all elements but $t \geq 0$ fixed. When $f$ is convex, direct inspection immediately shows that $t \mapsto g(t)$ is convex. For any $0 \leq t_1 < t_2$, take some $t' \in (t_1, t_2)$. Clearly, there exists a $\beta \in (0,1)$ such that $t' = \beta t_1 + (1-\beta)t_2$. Then, we have

$$\frac{g(t') - g(t_1)}{t' - t_1} = \frac{g(\beta t_1 + (1-\beta) t_2) - g(t_1)}{(1-\beta)(t_2 - t_1)} \leq \frac{\beta g(t_1) + (1-\beta)g(t_2) - g(t_1)}{(1-\beta)(t_2 - t_1)} = \frac{(t_2 - g(t_1))}{t_2 - t_1},$$

where the inequality follows from convexity of $g$. If we use this inequality in the special case of $t_1 = 0$, alongside the basic relation $q(\alpha) = (g(\alpha) - g(0))/\alpha$, it immediately follows that $\alpha \mapsto q(\alpha)$.

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\(^{43}\)See for example Luenberger [27, Sec. 7.3, Prop. 2–3] and Nesterov [30, Ch. 1].

\(^{44}\)For example, see Nesterov [30, Thm. 3.1.1].
is monotonic (non-decreasing) on the positive reals. Furthermore, the set \( \{q(\alpha) : \alpha > 0\} \) is bounded below. To see this, take some \( \gamma > 0 \) small enough that \( x - \gamma u \in \text{dom } f \), and note that by direct application of convexity and the basic property (48), it follows that

\[
\left( \frac{\alpha}{\gamma} \right) (f(x) - f(x - \gamma u)) \leq f \left( x + \frac{\alpha}{\gamma} (x - (x - \gamma u)) \right) - f(x) = f(x + \alpha u) - f(x).
\]

That is, dividing both sides by \( \alpha \), we have

\[
\left( \frac{1}{\gamma} \right) (f(x) - f(x - \gamma u)) \leq \frac{f(x + \alpha u) - f(x)}{\alpha} = q(\alpha; f, u).
\]

Since the choice of \( \gamma > 0 \) depends only on \( x \) and \( u \), and is free of \( \alpha \), it follows that the set \( \{q(\alpha) : \alpha > 0\} \) is bounded below, as desired. Using this boundedness alongside the monotonicity of \( \alpha \mapsto q(\alpha) \), we have that the infimum is finite. Thus, recalling the definition (13) of the radial derivative of \( f \) at \( x \) in the direction \( u \), since we have

\[
f'_r(x; u) = \lim_{\alpha \to 0^+} \frac{f(x + \alpha u) - f(x)}{\alpha} = \inf \{q(\alpha; f, x, u) : \alpha > 0\},
\]

it follows immediately that the radial derivative always exists (i.e., \( f'_r(x; u) \in \mathbb{R} \)). Note also that using convexity, direct inspection shows that for all \( u \) we have

\[
f(u) - f(x) \geq f'_r(x; u - x).
\]

Furthermore, it is easily verified that whenever \( x \in \text{dom } f \), the map \( u \mapsto f'_r(x; u) \) is sub-additive and positively homogeneous, i.e., a sub-linear functional.\(^{45}\) The basic facts of interest here are summarized in the following proposition.

**Proposition 17** (Difference quotients for convex functions). Let \( V \) be a vector space. If function \( f : V \to \mathbb{R} \) is proper and convex, then it is radially differentiable on \( \text{int(} \text{dom } f \text{)} \).

**Proof.** The desired result follows immediately from previous discussion leading up to (51), and the fact that if \( x \) is an interior point of the effective domain of \( f \), it follows that for any \( u \in V \), we can find a \( \gamma > 0 \) small enough that \( x - \gamma u \in \text{dom } f \), which means we can apply the lower bound of (50) to the difference quotients \( q(\alpha; f, x, u) \) indexed by \( \alpha > 0 \).

\[\square\]

### D.3 Loss example

**Example 18.** While stated with a somewhat high degree of abstraction, let us give a concrete example to emphasize that the assumptions of Proposition 12 are readily satisfied under natural and important learning settings. Consider the regression problem, where we observe random pairs \( (X, Y) \sim \mu \), assuming that \( X \) is a finite-dimensional real-valued random vector, and \( Y \) is a real-valued random variable, related to the inputs by the relation \( Y = h^*(X) + \epsilon \), where \( \epsilon \) is a zero-mean random noise term. For simplicity, let \( h^* \) be a continuous linear map, and let \( \mathcal{H} \) be the set of all continuous linear maps on the space that \( X \) is distributed over. Finally, let the loss by the squared error, such that

\[
L(h) = (h(X) - Y)^2 = (\langle X, h - h^* \rangle - \epsilon)^2
\]

\[
L'(h)(u) = 2(\langle X, h - h^* \rangle - \epsilon)(u, X).
\]

\(^{45}\)This means that the Hahn-Banach theorem can be applied to construct a linear functional \( g \) bounded above as \( g(u) \leq f'_r(x; u) \), for all \( u \). See for example Luenberger [27, Sec. 5.4] or Ash and Doléans-Dade [2, Thm. 3.4.2]. This \( g \) is not necessarily a sub-gradient of \( f \) at \( x \), since it need not be continuous in general; such functions are sometimes called *algebraic sub-gradients* [40, Sec. 3].
Since we make almost no assumptions on the nature of the underlying noise distribution, clearly both the losses and the “gradients” can be unbounded and heavy-tailed. Fix any $h_0 \in \mathcal{H}$, and note that for any $h \in \mathcal{H}$, we have
\[
L(h) - L(h_0) = \langle X, h - h^* \rangle^2 - \langle X, h_0 - h^* \rangle^2 - 2\epsilon \langle X, h - h_0 \rangle \\
= \langle X, h - h_0 \rangle \langle X, (h - h^*) + (h_0 - h^*) \rangle - 2\epsilon \langle X, h - h_0 \rangle.
\]
Absolute values can be bounded above as
\[
|L(h) - L(h_0)| \leq \|X\|^2 \|h - h_0\| (\|h - h^*\| + \|h_0 - h^*\|) + 2|\epsilon|\|X\||\|h - h_0\|.
\]
It follows immediately that as long as $E_\mu \|X\|_4^4 < \infty$, we have that the local Lipschitz property (42) of the loss is satisfied, for arbitrary choice of $h_0$. As for the weak smoothness requirement on the loss, note that
\[
\|L'(h) - L'(h_0)\| = \sup_{\|u\|=1} \langle u, L'(h) - L'(h_0) \rangle = \sup_{\|u\|=1} 2\langle (X, h - h_0) \rangle (u, X) \leq 2\|X\|^2 \|h - h_0\|.
\]
Thus, if the random inputs $X$ are $\mu$-almost surely bounded, the desired smoothness condition (43) holds. Note that this does not preclude heavy-tailed losses and gradients since no additional assumptions have been made on the noise term.

E Empirical supplement

Due to limited space, we could only include key details and a few representative results in section 4 of the main text. Here we fill in those additional details. To begin, all our numerical experiments have been implemented entirely in Python (v. 3.8) using the following additional open-source software: Jupyter notebook (for interactive demos),\(^{46}\) matplotlib (v. 3.4.1, for all visuals),\(^{47}\) PyTables (v. 3.6.1, for dataset handling),\(^{48}\) NumPy (v. 1.20.0, for almost all computations),\(^{49}\) and SciPy (v. 1.6.2, for random variable statistics and special functions). In the following paragraphs, we provide information about the benchmark datasets used, as well as several figures including additional experimental results.

Dataset description The real-world benchmark datasets used in our classification tests are as follows: adult,\(^{50}\) australian,\(^{51}\) cifar10,\(^{52}\) cod_rna,\(^{53}\) covtype,\(^{54}\) emnist_balanced,\(^{55}\) fashion_mnist,\(^{56}\) and mnist.\(^{57}\) See Table 1 for a summary. Further background on all datasets is available at the URLs provided in the footnotes. Dataset size reflects the size after removal of instances with missing values, where applicable. For all datasets with categorical features, the “input features” given in Table 1 represents the number of features after doing a one-hot encoding of all such features. The “model dimension” is just the product of the number of input

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\(^{46}\)https://jupyter.org/  
\(^{47}\)https://matplotlib.org/  
\(^{48}\)https://www.pytables.org/  
\(^{49}\)https://numpy.org/  
\(^{50}\)https://archive.ics.uci.edu/ml/datasets/Adult  
\(^{51}\)https://archive.ics.uci.edu/ml/datasets/statlog+(australian+credit+approval)  
\(^{52}\)https://www.cs.toronto.edu/~kriz/cifar.html  
\(^{53}\)https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html  
\(^{54}\)https://archive.ics.uci.edu/ml/datasets/covtype  
\(^{55}\)https://www.nist.gov/itl/products-and-services/emnist-dataset  
\(^{56}\)https://github.com/zalandoresearch/fashion-mnist  
\(^{57}\)http://yann.lecun.com/exdb/mnist/
| Dataset      | Size   | Input features | Number of classes | Model dimension |
|--------------|--------|----------------|-------------------|-----------------|
| adult        | 45,222 | 105            | 2                 | 210             |
| australian   | 690    | 43             | 2                 | 86              |
| cifar10      | 60,000 | 3,072          | 10                | 30,720          |
| cod_rna      | 331,152| 8              | 2                 | 16              |
| covtype      | 581,012| 54             | 7                 | 378             |
| emnist_balanced | 131,600 | 784          | 47                | 36,848          |
| fashion_mnist | 70,000 | 784            | 10                | 7,840           |
| mnist        | 70,000 | 784            | 10                | 7,840           |

Table 1: A summary of the benchmark datasets used for performance evaluation.

In Figures 6–8, we give additional results that complement Figure 3 in the main text. The trends in terms of the histograms of test loss distributions are essentially uniform across this wide variety of datasets. We also see that a sharply-concentrated logistic (test) loss tends to correlate with better classification error (average zero-one error), with cifar10 being the only exception to this trend. As another point not raised in the main text, intuitively we would hope that Algorithm 1 performs well in terms of the risk $R_\sigma$ corresponding to its particular $\sigma$ setting; we have found this to be true across the benchmark datasets studied here. See Figure 9 for an example from the adult dataset. Moving from top to bottom, the order of colors shows a rather clear reversal. Very similar trends can be observed on the other datasets as well. In estimating $R_\sigma$ on the test set, we use an empirical mean estimate of $r_\sigma$, and then minimize with respect to $\theta$ using the minimize_scalar function of the SciPy (v. 1.6.2) optimize module.

Additional results
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Figure 7: Additional test error histograms.
Figure 8: Additional test error histograms.
Figure 9: Empirical mean estimates of $R_\sigma$ for a variety of $\sigma \in [0, \infty]$ settings. The colored curves correspond to different $\sigma$ settings in running Algorithm 1, just as in previous plots, whereas the distinct plots correspond to the different $\sigma$ used in evaluation.