This document is merely a review article on quantum gravity. It is organized as follows: in the section 1, it is argued why one should construct a quantum theory of gravity. The importance of the singularity theorems of general relativity is discussed, and the famous Penrose singularity theorem is proven. In section 2.1, the covariant quantization approach of gravity is reviewed and the Feynman rules of quantum gravity are derived. The problem of divergences that occur at two loop order is mentioned in section 2.2. In section 2.3 some comments are made on the non-perturbative evaluation of the quantum gravitational path integral in the framework of Euclidean quantum gravity. In section 3, an article by Hawking is reviewed that shows the gravitational path integral at one loop level to be dominated by contributions from some kind of virtual gravitational instantons. In section 4, the canonical, non-perturbative quantization approach that is based on the Wheeler deWitt equation is reviewed. After deriving the Wheeler deWitt equation in section 4.1, arguments from deWitt are described in section 4.2 which show occurrence of infinities at small distances within the framework of canonical quantum gravity. In section 4.3, the loop quantum gravity approach is reviewed shortly and its problems are mentioned. In section 5, arguments from Hawking are reviewed which show the gravitational path integral to be an approximate solution of the Wheeler deWitt equation. In section 6, the black hole entropy is derived in various ways. Section 6.1 uses the gravitational path integral for this calculation and section 6.2 reviews how the black hole entropy can be derived from canonical quantum gravity. In section 7.1, arguments from Dvali and Gomez who claim that gravity can be quantized in a way which would be in some sense “self-complete” are critically assessed. In section 7.2 a model from Dvali and Gomez for the description of quantum mechanical black holes is critically assessed and compared with the standard quantization methods of gravity.

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1 Why one should construct a theory of quantum gravity?

A classical particle with rest mass $m$ becomes a black hole if its entire mass is confined within its Schwarzschild radius $r_s = \frac{2GM}{c^2}$ with $G$ as the gravitational constant, and $c$ as the speed of light in vacuum. Quantum effects usually begin at the Compton wavelength $\lambda = \frac{h}{mc}$, where $h$ is Planck’s constant and $m$ is the particle’s rest mass. So, for the surrounding of a black hole, quantum effects would become important if the black hole has a mass of $m = \sqrt{\frac{hc}{2G}}$. This is an energy range of around $4.31 \cdot 10^{18} \text{GeV}/c^2$, which is far beyond the energy of current particle accelerators. Therefore, one might question whether it is reasonable, to do research on quantum gravity. In the following, we will set $\hbar = c = G = 1$ if not explicitly stated otherwise.

Well known classical solutions of general relativity, are e.g the Schwarzschild solution, which describes static spherical black holes of mass $M$

$$
\begin{align*}
    ds^2 &= - \left( 1 - \frac{2M}{r} \right) dt + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega,
\end{align*}
$$

($d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric of a unit two sphere in spherical coordinates). Physically more realistic than the Schwarzschild solution is the Kerr solution which describes rotating black
\[ ds^2 = -\left( 1 - \frac{2Mr}{\rho} \right) dt^2 - \frac{2Mar \sin^2 \theta}{\rho^2} (dt d\phi + d\phi dt) \]
\[ + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left( r^2 + a^2 \right)^2 - a^2 \Delta \sin^2 \theta d\phi, \]

\((\Delta(r) = r^2 - 2Mr + a^2 \text{ and } \rho(r, \theta) = r^2 + acos^2 \theta, \text{ with } a \text{ as some constant})\).

And there is the important Friedmann Robertson Walker metric that describes the evolution of a spatially homogeneous and isotropic universe

\[ ds^2 = -dt^2 + R(t) d\sigma, \]

(where \( t \) is the timelike coordinate and \( d\sigma^2 = \gamma_{ij}(u) du^i du^j \) is the line element of a maximally symmetric three manifold \( \Sigma \) with \( u^1, u^2, u^3 \) as coordinates and \( \gamma_{ij} \) as a symmetric three dimensional metric).

Unfortunately, in all these solutions, singularities are present where the curvature becomes infinite. More precisely, with the Riemannian tensor \( R_\lambda^{\mu \nu \beta} \) defined by

\[
R_\lambda^{\mu \nu \beta} = \partial_\mu \Gamma_\nu^\rho_\sigma - \partial_\nu \Gamma_\mu^\rho_\sigma + \Gamma_{\mu \lambda}^\rho \Gamma^{\lambda \nu \sigma} - \Gamma_{\nu \lambda}^\rho \Gamma^{\lambda \mu \sigma},
\]

where

\[
\Gamma_{\mu \nu}^\sigma = \frac{1}{2} g^{\sigma \rho} \left( \partial_\mu g_{\nu \rho} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu} \right),
\]
is the Christoffel connection on the spacetime with metric tensor \( g_{\mu \nu} \), the solutions above have singularities in the sense that at some point scalars like \( R = R_\mu^\nu R_\nu^\mu, R_\mu^\nu R_\nu^\rho R_\rho^\mu, R_\mu^\rho R^\rho_\nu R_\mu^\nu, \text{ or } R_\mu^\nu R_\nu^\sigma R_\sigma^\lambda R_\lambda^\mu \) become infinite, where \( R_{\mu \nu} = g^{\alpha \beta} R_{\alpha \mu \nu \beta} \) with \( R_{\alpha \mu \nu \beta} = g_{\alpha \lambda} R_{\mu \nu \beta}^{\lambda} \).

In the early years physicists were skeptical that these solutions given above should have physical significance, due to the singularities they contain. However, beginning in 1965, Penrose and Hawking have shown a variety of singularity theorems. These theorems imply that under reasonable conditions, the manifolds described by classical general relativity must contain singularities. In the following section, we review the derivation of the Penrose singularity theorem. Then, we argue that the singularity theorems make the development of a theory of quantum gravity necessary, since the singularities inside black holes would render the physics of infalling matter inconsistent.

### 1.1 The occurrence of singularities in gravity

In this section, we give a short review of the first of the singularity theorem that were published by Penrose [11]. The information found in this section is merely a short summary of the excellent descriptions in [6, 3, 1, 12], with most of the proofs changed only slightly. The focus of the text in this section is on topology, which has seemingly beautiful applications on relativity. In order to keep this section short, some important proofs from differential geometry have been omitted. When necessary, the reader is pointed out to appropriate references for these geometric results.

We begin by noting some basic definitions and theorems from point set topology.
1.1.1 Basic definitions and theorems of point set topology

Definition 1.1. Basic definitions of point set topology

- A topological space \((X, \mathcal{T})\) is a set \(X \neq \emptyset\) with a collection of subsets \(\mathcal{T} \subset \mathcal{P}(X) := \{Y \subset X\}\), where

\[
\begin{aligned}
X, \emptyset \in \mathcal{T}, \\
\forall U_\lambda \in \mathcal{T} \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}, \text{ with } \lambda \in \Lambda \text{ and } \Lambda \text{ as arbitrary index set,} \\
\forall U_1, \ldots, U_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^n U_i \in \mathcal{T} \text{ with } n \in \mathbb{N}.
\end{aligned}
\]  

(4)

- The sets \(V \in \mathcal{T}\) are called open sets.

- Let \(x \in X\) and \(A \subset X\). Then \(A\) is called a neighborhood of \(x\) if \(\exists V \in \mathcal{T} : x \in V \subset A\) and \(x\) is then called an inner point of the neighborhood \(A\).

- A subset \(C \subset X\) is called closed, if the complement \(C^c \equiv X \setminus C \equiv \{x \in X : x \notin C\}\) is open, i.e. \(C^c \in \mathcal{T}\).

- A collection \(\{U_\lambda\}\) of sets \(U_\lambda \in \mathcal{T}\), where \(\lambda \in \Lambda\), and \(\Lambda\) is an arbitrary index set, is called an open cover of a set \(A \subset X\), if

\[
A \subset (\bigcup_{\lambda \in \Lambda} U_\lambda)
\]  

(5)

- A subcover \(\{U_\alpha\}\) of \(A\) is a subset \(\{U_\alpha\} \subset \{U_\lambda\}\), where \(\alpha \in \Theta, \Theta \subset \Lambda\), which fulfills

\[
A \subset (\bigcup_{\alpha \in \Theta} U_\alpha)
\]  

(6)

- The set \(A\) is called compact if every open sub cover of \(A\) has a finite subcover.

- Let \(\{U_\lambda\}\) be an open cover of \(X\). An open cover \(\{V_\beta\}\) with \(\beta \in \Psi\), where \(\Psi\) is an arbitrary index set, is called a refinement of \(\{U_\lambda\}\) if

\[
\forall \beta \exists \lambda : V_\beta \subset U_\lambda.
\]  

(7)

- an open cover \(\{U_\beta\}\) is locally finite if \(\forall x \in X\) there exists an open neighborhood \(U(x)\) such that the set

\[
\{\beta \in \Psi : U_\beta \cap U(x) \neq \emptyset\}
\]  

(8)

is finite.

- A topological space \((X, \mathcal{T})\) is called paracompact if every open cover \(\{O_\lambda\}\) of \(X\) has a locally finite refinement \(\{V_\beta\}\).

- \((X, \mathcal{T})\) is Hausdorff if for all \(x, y \in X, x \neq y\) there exist neighborhoods \(U\) of \(x\) and \(V\) of \(y\) such that \(U \cap V = \emptyset\). In the following, we will consider only Hausdorff spaces.

- \((X, \mathcal{T})\) is regular, if for each pair of a closed set \(A\) and a point \(p \notin A\), there exists neighborhoods \(U, V\), where \(A \subset U\) and \(p \in V\) and \(U \cap V = \emptyset\).
• Let \((X, \mathcal{T})\) be a topological space which is Hausdorff. A sequence of points \(\{x_n\} \in X\) is said to converge to a point \(x\) if given any open neighborhood \(O\) of \(x\),

\[ \exists n : x_n \in O \forall n \in \mathbb{N}. \quad (9) \]

• The point \(x\) is then the limit point of \(\{x_n\}\).

• A subset \(Y \subset X\) is called dense in \(X\) if for every point \(x \in X\), \(x\) is either in \(Y\) or is a limit point of \(Y\).

• \((X, \mathcal{T})\) is separable if \(X\) contains a countable dense subset.

• Let \(Y \subset X\) and \(Y'\) be the set of all limit points of \(Y\). Then, the closure of \(Y\) is \(\overline{Y} = Y \cup Y'\).

• A point \(y \in X\) is an accumulation point of \(\{x_n\}\) if every open neighborhood of \(y\) contains infinitely many points of \(\{x_n\}\).

• A topological space \((X, \mathcal{T})\) is first countable if for each point \(x \in X\)

\[ \exists U_1, \ldots, U_n \in \mathcal{T}, n \in \mathbb{N} \quad (10) \]

such that for any open neighborhood \(V\) of \(x\), there \(\exists i : U_i \subset V\).

• \((X, \mathcal{T})\) is second countable if there exists a countable collection \(\{U_i\}_{i=1}^{\infty}\) of open subsets of \(\mathcal{T}\) such that every open set \(A \in \mathcal{T}\) can be expressed as

\[ A = \bigcup_{i=1}^{\infty} U_i \quad (11) \]

• Let \((M, g)\) be a metric space with metric \(g\). A metric topology is the collection of sets that can be realized as union of open balls

\[ B(x_0, \epsilon) \equiv \{x \in X | g(x_0, x) < \epsilon\} \quad (12) \]

where \(x_0 \in X\) and \(\epsilon > 0\),

• A topological space \((X, \mathcal{T})\) is called metrizable, if there is a metric \(g : (X, X) \to [0, \infty)\) on \(X\) such that the metric topology of \((X, d)\) equals \(\mathcal{T}\).

• A topological space \((X, \mathcal{T})\) is called connected, if the only subsets that are both open and closed are the sets \(X\) and \(\emptyset\).

• Let \((X, \mathcal{T})\) be a topological space. A function \(h : V \subset X \to U \subset \mathbb{R}^4\) is called 4 dimensional chart if \(h\) is a homeomorphism of the open set \(V \subset X\) to an open set \(U \subset \mathbb{R}^4\).

• A family of cards \(A = \{h_\alpha : X_\alpha \to U_\alpha\}_{\alpha \in \Lambda}\), where \(\Lambda\) is an arbitrary index set and \(\cup_\alpha X_\alpha = X\), is called an atlas of \(X\).

• An atlas \(A\) is called differentiable if \(\forall \alpha, \beta \in \Lambda \times \Lambda : h_\beta \circ (h_\alpha | X_\alpha \cap X_\beta)^{-1}\) is a diffeomorphism.

• Two differentiable atlases \(A\) and \(B\) are equivalent, if \(A \cup B\) is a differentiable atlas.
• A 4-dimensional differentiable manifold is a set $X$ of a topological space which is connected, Hausdorff and second countable, together with an equivalence class of differentiable atlases that are homeomorphisms to $\mathbb{R}^4$.

One has the following famous theorems in point set topology, which we will use often:

**Theorem 1.2.** Let $(X, T)$ be any topological space which is metrizable. Then, if $(X, T)$ is separable, $(X, T)$ is second countable.

*Proof.* See [4] for a proof of this rather weak statement. This follows as part of Urysohn’s lemma but actually, the Urysohn lemma is much more powerful. It shows that a topological space is separable and metrizable if and only if it is regular, Hausdorff and second-countable. We will use theorem when we show that $C(p, q)$ is compact. We need the result only in the weak form above. □

**Theorem 1.3.** (Bolzano-Weierstrass) Let $(X, T)$ be a topological space and $A \subset X$. If $A$ is compact, then every sequence of points $\{x_n\} \in A$ has an accumulation point in $A$. Conversely, if $(X, T)$ is second countable, and every sequence has an accumulation point in $A$ then $A$ is compact.

*Proof.* Stated in [1] on p. 426, proven in almost every analysis script one can get. □

**Theorem 1.4.** (Heine-Borel) A subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ is compact if and only if it is bounded and closed.

*Proof.* Stated in [1] on p. 425, proven in almost every analysis script one can get. □

One can use the Heine-Borel theorem for proving:

**Theorem 1.5.** (Extreme value theorem) A continuous function $f : X \to \mathbb{R}$, from a compact set of a topological space is bounded and attains its maximum and minimum values.

*Proof.* Stated in [1] on p. 425. Proven in almost every analysis script one can get. □

**Theorem 1.6.** Let $(X, T)$ be a topological space and $A_i \subset X$, $i \in \Lambda$, with $\Lambda$ as an arbitrary index set. If $A_i$ are compact, then $\bigcup A_i$ is is compact.

*Proof.* Let $U$ be an open cover of $\bigcup A_i$. Since $U$ is an open cover of each $A_i$, there exists a finite subcover $U_i \subseteq U$ for each $A_i$. Then $\bigcup U_i \subseteq U$ is a finite subcover of $\bigcup A_i$. □

**Theorem 1.7.** Let $M$ be a paracompact differentiable manifold. Then, $M$ is metrizable with a positive definite Riemannian metric.

*Proof.* This is shown in most texts on Riemannian geometry. For example in [8] on p. 25 as corollary 1.4.4. Note that the author of [8] already defines the term manifolds such that they must be paracompact. □
1.1.2 The Raychaudhuri’s equation and conjugate points on geodesics

Most of the topology definitions and lemmas will be used later. For now, we need some definitions for the spacetime manifold, and the curves in that manifold:

Definition 1.8. Basic definitions from Lorentzian geometry:

- A pair \((M, g)\) is a Lorentzian manifold if \(M\) is a paracompact, 4 dimensional differentiable manifold and \(g\) is a symmetric non-degenerate 2-tensor field on \(M\), called metric, which has the signature \((- , + , + , +)\).

- Let \((M, g)\) be a Lorentzian manifold and \(p \in M\). A vector \(v \in T_p M\) is said to be timelike, if \(g_{\mu\nu} v^\mu v^\nu < 0\), spacelike, if \(g_{\mu\nu} v^\mu v^\nu > 0\) and null if \(g_{\mu\nu} v^\mu v^\nu = 0\). The length of \(v\) is
  \[
  |v| = \sqrt{|g_{\mu\nu} v^\mu v^\nu|},
  \]
  (13)

- Two timelike tangent vectors \(x, y \in T_p M\) define the same time orientation, if
  \[
  g_{\mu\nu} x^\mu y^\nu > 0.
  \]
  (14)

- A Lorentzian manifold is stably causal if and only if there exists a global time function \(t : M \rightarrow \mathbb{R}\), where \(\nabla^a t(p)\) is a timelike vectorfield \(\forall p \in M\).

- A timelike vector \(v \in T_p M\) is future pointing if and only if \(g_{\mu\nu} \nabla^\mu t(p) v^\nu > 0\) and past pointing if \(g_{\mu\nu} \nabla^\mu t(p) v^\nu < 0\).

- A Lorentzian manifold is time ordered if a continuous designation of future pointing and past pointing for timelike vectors can be made over the entire manifold.

- A differentiable curve \(\gamma : I \subset \mathbb{R} \to M\) is said to be timelike, spacelike or null, if its tangent vector \(\dot{\gamma} = \frac{d\gamma(\zeta)}{d\zeta}\) is timelike, spacelike, or null for all \(\lambda \in I\). For a timelike curve, the tangent vector can not be null. A differential curve is a causal curve if \(\dot{\gamma}\) is either timelike or null.

- If a curve \(\gamma\) is a differentiable spacelike curve connecting the points \(p, q \in M\), it has a the length function
  \[
  l(\gamma) = \int_a^b (g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu)^{1/2} dt
  \]
  (15)
  where \(\gamma(a) = p, \gamma(b) = q\). If instead \(\gamma\) is a differentiable timelike curve, the length is defined by the so called proper time
  \[
  \tau(\gamma) = \int_a^b (-g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu)^{1/2} dt
  \]
  (16)

- A curve \(\gamma\) is called a geodesic if and only if it fulfills the geodesic equation
  \[
  \frac{d^2 \gamma^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{d\gamma^\mu}{d\lambda} \frac{d\gamma^\nu}{d\lambda} = 0
  \]
  (17)
There is an important theorem on the uniqueness of geodesics around small neighborhoods that we will use often:

**Theorem 1.9.** Let \((M,g)\) be a time ordered Lorentzian manifold. Then, each point \(p_0 \in M\) has an open neighborhood \(V \subset M\) called geodesically convex, such that the spacetime \((V,g)\) obtained by restricting \(g\) to \(V\) satisfies: if \(p,q \in V\) then there exists a unique geodesic joining \(p\) to \(q\), with the geodesic confined entirely to \(V\).

**Proof.** Shown in [5] as proposition 31 on p. 72.

We now have to make some definitions that link the curves in a spacetime manifold to the causality of events that happen in that spacetime.

**Definition 1.10.** Causality

- A differentiable curve \(\gamma\) is a future (past) directed causal curve, if \(\dot{\gamma}\) is non-spacelike and future (past) directed.

- A continuous curve \(\lambda\) is future directed, if for each \(p \in \lambda\) there exists a geodesically convex neighborhood \(U\) such that if \(\lambda(\zeta_1), \lambda(\zeta_2) \in U\) with \(\zeta_1 < \zeta_2\), then there exists a future directed differentiable curve connecting \(\lambda(\zeta_1)\) to \(\lambda(\zeta_2)\).

- The chronological future of a set \(p \subset M\) is the set \(I^+(p)\) of all points in \(M\) that can be connected from \(p\) by a future directed timelike curve.

- The causal future of a set \(p \subset M\) is the set \(J^+(p)\) of all points in \(M\) that can be connected from \(p\) by a future directed causal curve.

- The chronological future of a set \(p \subset M\) relative to \(U \subset M\) is the set \(I^+(p,U) \equiv U \cap I^+(p)\).

- The causal future of a set \(p \subset M\) relative to \(U \subset M\) is the union \(J^+(p,U) \equiv (p \cap U) \cup (U \cap J^+(p))\).

In the following, we often use an important theorem on the lengths of timelike geodesics in geodesically convex neighborhoods:

**Theorem 1.11.** Let \((M,g)\) be a time oriented spacetime and \(p_0 \in M\). Then there exists a geodesically convex open neighborhood \(V \subset M\) of \(p_0\) such that the spacetime \((V,g)\) obtained by restricting \(g\) to \(V\) satisfies the following property: If \(q \in I^+(p)\), \(c\) is the unique timelike geodesic connecting \(p\) to \(q\) and \(\gamma\) is any piecewise smooth timelike curve connecting \(p\) to \(q\) then \(\tau(\gamma) \leq \tau(c)\), with equality if and only if \(\gamma\) is the unique geodesic.

**Proof.** Shown in [5] on p. 133-134 as Lemma 14, note also the refinement, on p. 134, which shows that Lemma 14 holds also for piecewise smooth curves.

On a manifold, one can have entire families of geodesics. The following provides some useful definitions for them.

**Definition 1.12.** Definitions for geodesic congruences
Let $\gamma_s(t), t, s \in \mathbb{R}$ denote a smooth one parameter family of geodesics on a manifold $(g, M)$. The collection of these geodesics defines a smooth two-dimensional surface. The coordinates of this surface may be chosen to be $s$ and $t$, provided that the geodesics do not cross. Such a family of geodesics is called geodesic congruence.

Denote the entire surface generated by the geodesic congruence as the set of coordinates $x^\mu(t, s) \in M$. Then, the tangent vectors to the geodesic congruence are defined by $T^\mu = \frac{\partial x^\mu}{\partial t}$, and the geodesic deviation vectors are $S^\mu = \frac{\partial x^\mu}{\partial s}$.

The vectorfield $v^\mu = T^\nu \nabla_\nu S^\mu$ defines the relative velocity of the geodesic congruence and $a^\mu = T^\lambda \nabla_\lambda v^\mu = T^\lambda \nabla_\lambda (T^\nu \nabla_\nu S^\mu)$ is the relative acceleration.

For a tensor $T_{\mu_1, \ldots, \mu_l}$ we define the symmetric part $T_{(\mu_1, \ldots, \mu_l)} = \frac{1}{l!} \sum_{\sigma} T_{\mu_{\sigma(1)}, \ldots, \mu_{\sigma(l)}}$ where the sum goes over all permutations $\sigma$ of $1, \ldots, l$. The antisymmetric part $T_{[\mu_1, \ldots, \mu_l]} = \frac{1}{l!} \sum_{\sigma} \text{sign}(\sigma) T_{\mu_{\sigma(1)}, \ldots, \mu_{\sigma(l)}}$ where $\text{sign}(\sigma)$ is 1 for even and $-1$ for odd permutations.

Let $T^\mu$ be the tangent vector field to a timelike geodesic congruence parametrized by proper time $\tau$. We define the projection tensor $P_{\mu\nu} = g_{\mu\nu} + T_\mu T_\nu$.

Any vector in the tangent space of a point $p \in M$ can be projected by $P_{\mu\nu}$ into the subspace of vectors normal to $T^\mu$. We now can define a tensorfield $B_{\mu\nu} = \nabla_\nu T_\mu$ where we call $\theta = B^{\mu\nu} P_{\mu\nu}$ the expansion.

The shear is defined by the symmetric tensorfield $\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3} \theta P_{\mu\nu}$.

The twist is an antisymmetric tensorfield defined by $\omega_{\mu\nu} = B_{[\mu\nu]}$.

Let $(M, g)$ be a manifold on which a Christoffel connection is defined and let $\gamma$ be a geodesic with tangent $T$. A solution $S$ of the geodesic deviation equation $T^\lambda \nabla_\lambda (T^\nu \nabla_\nu S^\mu) = R^\mu_{\nu\rho\delta} T^\nu T^\rho S^\sigma$, is called Jacobi field.

We can use these definitions to derive equations that describe whether the geodesics in a given congruence are converging to each other. The following equations are described in most relativity books, e.g. [3][1]. They will be used during the proofs of the singularity theorem.
Lemma 1.13. Let the tangent vector of a geodesic congruence be parametrized by proper time $t = \tau$. Then, the tangent field fulfills $T_\mu T^\mu = -1$ and $T^\lambda \nabla_\lambda T^\mu = 0$. The geodesic acceleration vector fulfills the geodesic deviation equation:

$$a^\mu = R^\mu_{\nu \rho \sigma} T^\nu T^\rho T^\sigma$$ \hfill (23)

Proof. Because of

$$d\tau^2 = -g_{\mu \nu} dx^\mu dx^\nu$$ \hfill (24)

and $T^\mu = \frac{\partial x^\mu}{\partial \tau}$, we have

$$T^\mu T^\mu = -1$$ \hfill (25)

The expression $T^\lambda \nabla_\lambda T^\mu = 0$ follows from the geodesic equation for $x^\mu$. For the relative acceleration vector, we have:

$$a^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu)$$
$$= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$
$$= (T^\rho \nabla_\rho S^\sigma) (\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$
$$= (S^\rho \nabla_\rho T^\sigma) (\nabla_\sigma T^\mu) + T^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) - (S^\sigma \nabla_\sigma T^\rho) \nabla_\rho T^\mu + R^\mu_{\nu \rho \sigma} T^\nu T^\rho T^\sigma$$ \hfill (26)

where in the second line it was used that

$$T^\rho \nabla_\rho X^b = X^a \nabla_a X^b,$$ \hfill (27)

the third line uses Leibniz’s rule, the fourth line replaces a double covariant derivative by the derivative in the opposite order plus the Riemann tensor. Finally, in the fifth line, $T^\rho \nabla_\rho T^\mu = 0$ is used together with the Leibniz’s rule for canceling two identical terms.

Furthermore, one can establish the following theorem:

Theorem 1.14. Given the definitions 1.12 then

$$B^\mu_{\nu \mu} T^\nu = B^\mu_{\nu \nu} T^\nu = \sigma^\mu_{\nu \nu} T^\nu = \sigma^\mu_{\nu \nu} T^\nu = \omega^\mu_{\nu \nu} T^\nu = \omega^\mu_{\nu \nu} T^\nu = 0$$ \hfill (28)

and Raychaudhuri’s equation

$$\frac{d \theta}{d \tau} = -\frac{1}{3} g^{\mu \nu} \sigma^\mu_{\nu \nu} + \omega^\mu_{\nu \nu} \omega^{\mu \nu} - R^\mu_{\nu \mu} T^\nu T^\nu$$ \hfill (29)

holds.

Proof. Because of

$$\nabla_\nu (T^\mu T^\mu) = \nabla_\nu (-1) = 0,$$ \hfill (30)

we get, using

$$T^\mu B^\mu_{\nu \mu} = T^\nu \nabla_\nu T^\mu$$ \hfill (31)

the result $T^\mu B^\mu_{\nu \nu} = 0$. With

$$T^\nu B^\mu_{\nu \nu} = T^\nu \nabla_\nu T^\mu$$ \hfill (32)

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and the geodesic equation $T^\nu \nabla_\nu T_\mu = 0$, we arrive at $T^\nu B_{\mu\nu} = 0$. Since we have

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3} \theta P_{\mu\nu} \tag{33}$$

and

$$\theta = B^{\mu\nu} P_{\mu\nu}, \tag{34}$$

it follows from

$$B_{\mu\nu} T^\mu = B_{\mu\nu} T^\nu = 0 \tag{35}$$

that

$$\sigma_{\mu\nu} T^\mu = \sigma_{\mu\nu} T^\nu = 0 \tag{36}$$

and similarly $\omega_{\mu\nu} T^\nu = \omega_{\mu\nu} T^\mu = 0$. Furthermore

$$T^\rho \nabla_\rho B_{\mu\nu} = T^\rho \nabla_\rho \nabla_\mu T_\nu = T^\rho \nabla_\rho T_\mu + T^\rho R^\lambda_{\mu\nu\rho} T_\lambda \tag{37}$$

$$= \nabla_\nu (T^\rho \nabla_\rho T_\mu) - (\nabla_\nu T^\rho)(\nabla_\rho T_\mu) - R_{\lambda\mu\nu\sigma} T^\sigma T^\lambda$$

$$= -B^\sigma_{\nu\rho} B_{\mu\sigma} - R_{\lambda\mu\nu\sigma} T^\sigma T^\lambda \tag{38}$$

taking the trace of this equation yields:

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - R^{cd} T_c T_d \tag{39}$$

**Theorem 1.15.** Given the definitions \[11\] then $\sigma_{\mu\nu} \sigma^{\mu\nu} \geq 0$, $\omega_{\mu\nu} \omega^{\mu\nu} \geq 0$. Furthermore $\omega_{\mu\nu} = 0$ if and only if $T^\nu$ is orthogonal to a family of hypersurfaces.

**Proof.** $\sigma_{\mu\nu}$ is a spatial tensorfield that is $\sigma_{\mu\nu} T^\nu = 0$, where $T^\nu$ is timelike. Therefore, we have $\sigma_{\mu\nu} \sigma^{\mu\nu} \geq 0$ and similarly $\omega_{\mu\nu} \omega^{\mu\nu} \geq 0$. By Frobenius’ theorem, see \[1\] p. 434-436, a vector field $T^\nu$ is hypersurface orthogonal, if and only if $T_{[\mu} \nabla_\nu T_{\lambda]} = 0$. With $B_{\mu\nu} = \nabla_\nu T_\mu$, $\omega_{\mu\nu} = B_{[\mu\nu]}$ and $T^\nu \omega_{\mu\nu} = T^\nu \omega_{\mu\nu} = 0$, it follows that $\omega_{\mu\nu} = 0$ if and only if $T^\nu$ is orthogonal to a family of hypersurfaces.

Having defined the equations for families of geodesics, we now give some further definitions related to the causality structure of the Lorentzian manifold under study.

**Definition 1.16.** Basic definitions for curvature and causality structures

- Let $(M, g)$ be a stably causal manifold and $\lambda$ a differentiable future directed causal curve. We say $p$ is a future endpoint of $\lambda$ if for every neighborhood $U$ of $p$ there exists a $t_0$ such that $\lambda(t) \in U$ for $t > t_0$. A past endpoint is defined similarly, with $\lambda(t) \in U$ for $t < t_0$.
- The curve $\lambda$ is said to be future (past) inextendible if it has no future (past) endpoint.
- A subset $\Sigma \subset M$ is called achronal, if there do not exist $p, q \in \Sigma$ such that $q \in I^+(p)$ or:

$$I^+(\Sigma) \cap \Sigma = \emptyset. \tag{40}$$
The past (future) domain of dependence of the achronal set \( \Sigma \subset M \) is the set \( D^{-(+)}(\Sigma) \) of all points \( p \in M \) such that any future (past) inextendible curve starting at \( p \) intersects \( \Sigma \).

The domain of dependence is the set
\[
D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)
\] (41)

A spacetime \((M, g)\) is strongly causal if \( \forall p \in M \) and every open neighborhood \( O \) of \( p \) there is a neighborhood \( V \subset O \) of \( p \) such that no causal curve intersects \( V \) more than once.

An achronal subset \( \Sigma \subset M \) is called a Cauchy surface, if \( D(\Sigma) = M \).

A Lorentzian manifold is globally hyperbolic, if it is stably causal and possesses a Cauchy surface \( \Sigma \).

Let \((M, g)\) be a globally hyperbolic manifold with a Cauchy surface \( \Sigma \). A point \( p \) on a geodesic \( \gamma \) of the geodesic congruence orthogonal to \( \Sigma \) is called a conjugate point to \( \Sigma \) along \( \gamma \) if there exists a Jacobi-field \( S^\mu \) which is non-vanishing except on \( p \).

We define the extrinsic curvature \( K_{ab} \) of \( \Sigma \) as
\[
K_{ab} = B_{ba} \quad \text{(42)}
\]
and its trace as
\[
K = P^{ab}K_{ab}. \quad \text{(43)}
\]

A standard proof from differential geometry lectures shows that a timelike geodesic of maximal length that starts on a Cauchy surface must be orthogonal to it:

**Theorem 1.17.** Let \((M, g)\) be a globally hyperbolic manifold with a Cauchy surface \( \Sigma \) and let \( \gamma \) be a timelike geodesic starting at \( p \in \Sigma \) to a point \( q \in I^+(\Sigma) \). Then, \( \gamma \) only maximizes length if and only if it is orthogonal to \( \Sigma \).

**Proof.** Proof is given in [5] on p. 280. \( \square \)

The following proof is taken from Wald’s book [1]. It relates the conditions for stable and strong causality:

**Theorem 1.18.** Let \((M, g)\) be a spacetime which is stably causal. Then, it is also strongly causal.

**Proof.** Let \( t \) be the global time function on \( M \). Since \( g_{\mu\nu} \gamma^\nu \nabla^\mu t(p) > 0 \) for all \( p \in \gamma \) where \( \gamma \) is a future directed timelike curve, \( t \) strictly increases along every future directed timelike curve. Given any \( p \in M \) and any open neighborhood \( O \) of \( p \), we can choose an open neighborhood \( V \subset O \) shaped such that the limiting value of \( t \) along every future directed causal curve leaving \( V \) is greater than the limiting value of \( t \) on every future directed causal curve entering \( V \). Since \( t \) increases along every future directed causal curve, no causal curve can enter \( V \) twice. \( \square \)

**Theorem 1.19.** Let \((M, g)\) be a globally hyperbolic spacetime satisfying \( R_{\mu\nu}\zeta^\mu \zeta^\nu \geq 0 \) for all timelike \( \zeta \), Let \( \Sigma \) be a Cauchy surface with \( K = \theta < 0 \). Then within proper time \( \tau \leq 3/|K| \) there exists a conjugate point \( q \) along the timelike geodesic \( \gamma \) orthogonal to \( \Sigma \), assuming that \( \gamma \) extends that far.
The proof is adapted from Wald’s book[1], where one can find a proof for the situation where a point is conjugate to another point on a geodesic. Only minor changes were necessary to adapt this proof for the situation of a geodesic orthogonal to a hypersurface.

**Proof.** Let γ be a timelike geodesic orthogonal to Σ with tangent $T^\mu = \frac{dx^\mu}{d\tau}$. We consider the congruence of all timelike geodesics passing through Σ that are orthogonal to Σ. Since the deviation vector $S^\mu = \frac{dx^\mu}{ds}$ of this geodesic congruence is orthogonal to γ, we have $S^\mu T^\mu = 0$. Therefore, we can introduce an orthonormal basis $e^\mu_a(\tau)$, $a = 1,\ldots,3$ which is orthogonal to γ, i.e. where $e^\mu_b e^b_\mu = \delta_ab$, and $e^\mu_a T^\mu = 0$ and is parallel transported along γ, that is

$$\frac{d}{d\tau} \nabla_\nu e^\mu_a := \frac{D}{d\tau} e^\mu_a = 0. \tag{44}$$

holds. Inserting this with $S^\mu = (e^\mu)_a S^a$ in the geodesic deviation equation

$$T^\lambda \nabla_\lambda (T^\nu \nabla_\nu S^\mu) = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma, \tag{45}$$

the term on the left hand side simplifies to

$$\frac{dS^\mu}{d\tau^2} = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma. \tag{46}$$

Since this is an ordinary linear differential equation, $S^\mu$ must depend linearly on the initial $S^\mu(0)$ and $\frac{dS^\mu(0)}{d\tau}$. Since we are searching for conjugate points, the congruence $S^\mu(0)$ should not be vanishing on Σ but all geodesics of the congruence should be hypersurface-orthogonal to Σ. Therefore, $\frac{d}{d\tau} S^\mu(0) = 0$, and noting that $S^\mu = (e^\mu)_a S^a$ with $a = 1,\ldots,3$, we can make an ansatz,

$$S^\sigma(\tau) = A^\sigma_a S^a(0). \tag{47}$$

Plugging this in the coordinate form of the geodesic deviation equation gives

$$\frac{dA^\mu_a}{d\tau^2} = R^\mu_{\nu\rho\sigma} T^\nu T^\rho A^\sigma_a \tag{48}$$

A point p that is conjugate to Σ develops by definition if and only if $S^\mu(\tau) = 0$. With our ansatz, this condition becomes $det(A^\mu_a) = 0$. Between Σ and p, the Jacobi-field should be non vanishing. Therefore $det(A^\mu_a) \neq 0$ for this region and an inverse of $A^\mu_a$ exists. We have

$$\frac{dS^\nu}{d\tau} = T^\mu \nabla_\mu S^\nu$$

$$= T^\mu \nabla_\mu ((e^\nu)_a S^a)$$

$$= (e^\nu)_a T^\mu \nabla_\mu S^a$$

$$= (e^\nu)_a B^\rho_a S^\rho$$

$$= B^\nu_a S^a$$

$$= B^\nu_a A^\sigma_b S^b(0). \tag{49}$$

with $b = 1,\ldots,3$. From our ansatz, we get

$$\frac{dS^\nu}{d\tau} = \frac{dA^\mu_a}{d\tau} S^a(0)$$

$$= B^\nu_a A^\sigma_b S^b(0) \tag{50}.$$
or in matrix notation \( \frac{dA}{d\tau} = BA \) that is

\[
B = \frac{dA}{d\tau} A^{-1}
\]

Since \( \theta = tr(B) \) and for any non-singular matrix \( A \):

\[
tr \left( \frac{dA}{d\tau} A^{-1} \right) = \frac{1}{det(A)} \frac{d}{d\tau} (det(A))
\]

it follows that

\[
\theta = \frac{1}{det(A)} \frac{d}{d\tau} (det(A)).
\]

Since \( A \) satisfies the ordinary linear differential equation \( \frac{d}{d\tau} \) \( det(A) \) can not become infinite. Hence if and only if \( \theta = -\infty \) at a point \( q \) on \( \gamma \), then \( A \to 0 \) and \( det(A) \to 0 \) which implies that \( q \) will be a conjugate point.

Our congruence is hypersurface orthogonal on \( \Sigma \) by construction. Therefore, \( \omega_{bc} = 0 \) on \( \Sigma \). The antisymmetric part of eq. (37) is

\[
T^c \nabla_c \omega_{ab} = -\frac{2}{3} \theta \omega_{ab} - 2 \sigma_{[b} \omega_{a]c}
\]

see [1] p. 218, or [3] p. 461. Therefore, if \( \omega \) is orthogonal to a family of hypersurfaces at at one time, it remains so, along the geodesic \( \gamma \) which implies \( \omega_{ab} = 0 \) for all the time. Since \( \sigma_{ab} \sigma^{ab} \geq 0 \) and by proposition

\[
R_{ab} T^a T^b \geq 0,
\]

we have from Raychaudhuri’s equation (37):

\[
\frac{d\theta}{d\tau} + \frac{1}{3} \theta^2 \leq 0
\]

and hence

\[
\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3} \tau,
\]

where \( \theta_0 \)is the initial value of \( \theta \) on \( \Sigma \). As a result, if \( \theta_0 \leq 0 \) on \( \Sigma \), the expansion will converge to \( \theta \to -\infty \) at \( \tau \leq 3/|\theta_0| \), and there will be a conjugate point \( p \) on \( \gamma \).

The intrinsic curvature \( K_{ab} \) is \( K_{ab} = B_{ba} = B_{ab} \). Since the antisymmetric part \( B_{[ab]} = \omega_{ab} = 0 \) only the symmetric part

\[
\sigma_{ab} + \frac{1}{3} \theta P_{ab} = B_{(ab)}
\]

is non-vanishing. With

\[
\theta = P^{ab} B_{ab},
\]

we get

\[
K = P^{ab} K_{ab} = \theta.
\]

Therefore, if \( K < 0 \) on \( \Sigma \) and

\[
R_{ab} T^a T^b \geq 0,
\]

there will be a conjugate point to \( \Sigma \) on \( \gamma \) within proper time \( \tau \leq 3/|\theta_0| \).
Theorem 1.20. Let \((M, g)\) be a globally hyperbolic Lorentzian manifold with a Cauchy surface \(\Sigma\), and \(\gamma\) a timelike geodesic orthogonal to \(\Sigma\), and \(p\) a point on \(\gamma\). If there exists a conjugate point between \(\Sigma\) and \(p\) then \(\gamma\) does not maximize length among the timelike curves connecting to \(\Sigma\) to \(p\).

Proof. A heuristic argument would be the following: Let \(q\) be a conjugate point along \(\gamma\) between \(\Sigma\) and \(p\). Then, there exists another geodesic \(\Tilde{\gamma}\) orthogonal to \(\Sigma\) with the same approximate length. This geodesic must intersect \(\gamma\) at \(q\). Let \(V\) be a geodesically convex neighborhood of \(q\) and \(r \in V\) a point along \(\gamma\) between \(q\) and \(p\). Then, we get a piecewise smooth timelike curve \(\tilde{\gamma}\) obtained by following: 1) \(\gamma\) between \(\Sigma\) and \(r\) then 2) the unique geodesic between \(r\) and \(s\) and, finally, 3) the geodesic \(\gamma\) between \(s\) and \(p\). This curve \(\tilde{\gamma}\) connects \(\Sigma\) to \(p\) and has strictly bigger length than \(\gamma\) by the twin paradox \([1, 1]\). Moreover, \(\tilde{\gamma}\) can be smoothed while retaining bigger length. A rigorous version of this statement is provided by Penrose in \([6]\) on p. 65 as theorem 7.27. Penrose uses in his didactically excellent introduction a so called synchronous coordinate system that simplifies his calculation.

1.1.3 The topological space \(C(p, q)\) of continuous curves from \(p\) to \(q\) in \(M\)

Now we define the topological space of the set of continuous curves \(C(p, q)\) from points \(p\) to \(q \in M\). We will use this in the proof of the singularity theorem.

Definition 1.21. The topological space \(C(p, q)\) of continuous curves from \(p\) to \(q\) in \(M\)

- Let \((M, g)\) be a strongly causal spacetime. The set \(C(p, q)\) denotes the set of continuous future directed causal curves from \(p\) to \(q \in I^+(p)\).
- Similarly, The set \(\tilde{C}(p, q)\) denotes the set of smooth future directed causal curves from \(p\) to \(q\).
- A trip from \(p\) to \(q\) where \(q \in I^+(p)\) is a piecewise future directed timelike geodesic with past endpoint \(p\) and future endpoint \(q\). The set of trips from \(p\) to \(q\) is denoted by \(\kappa(p, q)\).
- It is obviously a subset of \(C(p, q)\)
- Following Geroch’s article \([7]\), we will define a topology \(\mathcal{T}\) on \(C(p, q)\). First we define sets
  \[
  O(U) \equiv \{\lambda \in C(p, q) | \lambda \subset U\} 
  \]
  and a set \(O\) is called open if it can be expressed as \(O = \cup O(U) \in \mathcal{T}\), where \(U \subset M\) is an open set.
- Let \(\{\lambda_n\} \subset C(p, q)\) be a sequence of curves with \(n \in \Lambda \subseteq \mathbb{N}\) and \(\Lambda\)as some index set. A point \(x \in M\) is a convergence point of \(\{\lambda_n\}\) if, given any open neighborhood \(U\) of \(x\), there exist an \(N \in \mathbb{N}: \lambda_n \cap U \neq \emptyset, \forall n > N\)
- A curve \(\lambda \in C(p, q)\) is a convergence curve of \(\{\lambda_n\}\) if each point \(x \in \lambda\) is a convergence point of \(\{\lambda_n\}\).
- A point \(x \in M\) is a limit point of \(\{\lambda_n\}\) if for every \(U\) which is a neighborhood of \(x\) the set
  \[
  \{n \in \mathbb{N} : \lambda_n \cap U \neq \emptyset\}
  \]
  is infinite.
• A curve $\lambda \in C(\Sigma, q)$ is a limit curve of $\{\lambda_n\}$ if there exists a subsequence $\{\lambda_{n'}\} \subset \{\lambda_n\}_{n=1}^{\infty}$ with $n' \in \Lambda' \subset \mathbb{N}$ and $\Lambda'$ as some index set, for which $\lambda$ is a convergence curve.

• Let $(M, g)$ be a globally hyperbolic spacetime with a Cauchy surface $\Sigma$ and $q \in I^+(\Sigma)$. Then, $C(\Sigma, q)$ is the set

$$C(\Sigma, q) = \cup_{p \in \Sigma} C(p, q)$$

(65)

• Similarly, the set $\tilde{C}(p, q)$ is the set

$$\tilde{C}(\Sigma, q) = \cup_{p \in \Sigma} \tilde{C}(p, q)$$

(66)

Theorem 1.22. Let $(M, g)$ be strongly causal spacetime. Then, $(C(p, q), T)$ defines a topological space.

Proof. For all $O(U_1), O(U_2) \in T$, we have

$$O(U_1) \cap O(U_2) = O(U_1 \cap U_2) \in T$$

(67)

and $\forall O(U_i) \in T$ we have

$$(\cup_i O(U_i)) \in T$$

(68)

Therefore, $(C(p, q), T)$ describes a topological space.

Theorem 1.23. Let $(M, g)$ be strongly causal spacetime. Then, $(C(p, q), T)$ is Hausdorff.

Proof. $(M, g)$ is Hausdorff. This means that for each of the two curves $\lambda$ and $\lambda'$, with $\lambda \neq \lambda'$, we can find two open neighborhoods $X, Y$ where $\lambda \subset X$ and $\lambda' \subset Y$ with $X \cap Y = \emptyset$. Since $(M, g)$ is strongly causal, we can find two neighborhoods $U_\lambda, U_{\lambda'}$ where $\lambda \subset U_\lambda \subset X$ and $\lambda' \subset U_{\lambda'} \subset Y$ where $\lambda$ intersects $U_\lambda$ only twice, and similarly $\lambda'$ intersects $U_{\lambda'}$ only twice. Since $U_\lambda \cap U_{\lambda'} = \emptyset$, it follows that, $U_\lambda \cap \lambda' = \emptyset$ and $U_2 \cap \lambda = \emptyset$ and we have

$$O(U_\lambda) =: \{\lambda \in C(p, q) | \lambda \subset U_\lambda\}$$

(69)

and

$$O(U_{\lambda'}) =: \{\lambda' \in C(p, q) | \lambda' \subset U_{\lambda'}\}$$

(70)

such that

$$O(U_\lambda) \cap O(U_{\lambda'}) = \emptyset$$

(71)

which implies that $(C(p, q), T)$ is Hausdorff.

Theorem 1.24. Let $(M, g)$ be a strongly causal spacetime, Then $C(p, q)$ is separable.

In his book [1], Wald writes on p. 206: “See Geroch [7] for a sketch of the proof of this result”. Unfortunately, Geroch just writes the following on this in a footnote at p. 445:

To construct a countable dense set in $C(p, q)$, choose a countable dense set $p_i$ including the points $p$ and $q$, in $M$, [That one can find such a countable dense set was shown by Geroch in [81]]. Consider curves $\gamma \in C(p, q)$ which consist of geodesic segments, each of which lies in a normal neighborhood and joins two $p_i$. 

16
By our basic definitions, let $X, \mathcal{T}$ be a topological space. A subset $Y \subset X$ is called dense in $X$ if for every point $x \in X$, $x$ is either in $Y$ or is a limit point of $Y$. The “proof” of Geroch above merely states that one can construct a dense set of points in $M$. But by simply connecting these points to construct a set of curves, it does not become clear that $C(p, q)$ contains a subset of curves such an arbitrary curve $\gamma \in C(p, q)$ is either in $\kappa(p, q)$ or a limit curve of $\kappa(p, q)$.

Fortunately, in his article [6], Penrose provides on p. 50 a picture, which seems to indicate a proof idea. The proof below is an own attempt to realize the proof idea that is indicated by Penrose in that picture.

**Proof.** Let the curve $\lambda \in C(p, q)$ and $U_{\lambda 1} \subset M$ be an open set where $\lambda \subset U_{\lambda 1}$. Since $M$ is a paracompact, by theorem 1.67 one can always define assign a positive definite Riemannian metric to $M$ and define an open ball $B(p, a)$ as in eq[12] around each point in $p \in M$ with radius $a$ with respect to the Riemannian metric.

Without loss of generality, we can define $O_{1,0} = B(p_{0,0}, r_{1,0}) \subset U_{\lambda 1}$ to be an open neighborhood of $p_{0,0} = p$ with radius $r_{1,0}$ that is geodesically convex. Since $M, g$ is strongly causal, one can then define a closed neighborhood $\overline{X}_{1,0} \subset O_{1,0}$ with such that $\overline{X}_{1,0}$ is intersected by any curve only twice.

We now denote the intersection point of $\overline{X}_{1,0}$ and $\lambda$ as $p_{1,0} \in I^+(p_{0,0})$. For $p_{1,0}$ we can then find a geodesically convex open neighborhood $O_{2,0} = B(p_{1,0}, r_{2,0})$ with radius $r_{2,0}$. We can find a closed neighborhood $\overline{X}_{2,0} \subset O_{2,0}$ that is intersected by any curve only twice. We then go to the intersection point $p_{2,0} \in I^+(p_{1,0})$ of $\overline{X}_{2,0}$ and $\lambda$. We can continue inductively, constructing $n \in \mathbb{N}$ balls with radii $r_{i,0}$ and $n$ intersection points $p_{i,0}$ until we reach a neighborhood $X_{n,0}$ that contains $q = p_{n+1,0}$. So we get a countable series of points in geodesically convex neighborhoods.

We connect the pairs of points $p_{i,0}, p_{i+1,0}$ with the unique timelike geodesic given by theorem 1.11 for $i = 0, \ldots, n$ and get a trip $\lambda_1$ from $p$ to $q$ which is in an open set $O_{\lambda_1} = \bigcup_i O_{i,0} \subset U_{\lambda_1}$.

Now we begin the sequence again, but this time, we let the open balls $O_{i,1}$ have radii of $r_{i,1} = r_{i,0}/2$. By connecting the points $p_{i,1}, p_{i+1,1}$ for all $i = 0, \ldots, n$, we get another trip $\lambda_2$ from $p$ to $q$ which is in a neighborhood $O_{\lambda_2} = \bigcup_i O_{i,2} \subset O_{\lambda_1}$. We continue inductively, with $O_{i,j} = B(p_{i,j}, r_{i,0}/2^j)$ and we get a countable series of trips $\lambda_j$ in open neighborhoods

$$O_{\lambda_1} = \bigcup_i O_{i,j+1} \subset O_{\lambda_j} = \bigcup_i O_{i,j}$$

(72)

where $O_{\lambda_1} \subset U_{\lambda_1}$. Since the radii of the balls $O_{i,j}$ around the curve converge to zero with increasing $j$, there exists for any open set $U_{\lambda_j} \subset M$ where $\lambda \subset U_{\lambda_j}$ an $N \in \mathbb{N}$ such that the trip $\lambda_j \subset U$ for all $j > N$. Therefore, if $\lambda \in C(p, q)$ is not itself a trip, it will be the limit curve of a countable sequence of trips. Thereby, $C(p, q)$ contains a countable dense subset, which implies that $C(p, q)$ is separable. $\Box$

**Theorem 1.25.** Let $(M, g)$ be a strongly causal spacetime. Then, $(C(p, q), \mathcal{T})$ is second countable

**Proof.** The proof idea is taken from Geroch’s article. Since the set of trips $\kappa(p, q)$ is dense in $C(p, q)$, $C(p, q)$ must be separable. A metric on $C(p, q)$ is obtained using the maximal distance between curves with respect to a positive definite metric that we can always assign to $M$ by theorem 1.7. As $C(p, q)$ is a separable metrizable space, by theorem 1.23 it is then second countable. $\Box$
The proof is taken from Geroch’s article. I have added some comments and changes to avoid some potential problems. In the proof of Geroch, $t$ lies in the interval $[0, 1]$. However, it is not guaranteed that the curve will have a parametrization which, if $t$ lies in this interval, leads $\lambda$ to move out of the geodesically convex neighborhood. Then, the existence of a curve $\gamma(t)$ in this interval could not be guaranteed.

**Theorem 1.26.** (Geroch) Let $(M, g)$ be stably causal and let $\lambda$ be a past inextendible curve starting at point $p \in M$. Then, there exists a past inextendible timelike curve $\gamma$ such that $\gamma(t) \in I^+(\lambda(t))$.

**Proof.** Since $M$ is paracompact, one can, by theorem 1.7, assign a positive definite Riemannian metric to $M$. Without loss of generality, we assume the curve parameter to be in $[0, \infty)$ and $\lambda(0) = p$. Let $d(p, q)$ denote the infimum of the length of all curves connecting $p$ with $q$ measured with respect the positive definite Riemannian metric that we can always assign to $M$. We may choose a point $q \in I^+(p)$, where $p, q \in U_1$ with $U_1$ as a geodesically convex neighborhood of $p$ and

$$d(p, q) < C,$$

where $C$ is some positive constant.

Starting at $q$ we can, using theorem 1.11, construct a timelike curve $\gamma(t)$ consisting of segments that connect two points in geodesically convex neighborhoods. Then, one can find an $\epsilon_1 > 0$ such that this curve can be defined for all $t \in [0, \epsilon_1]$ where $\epsilon_1 > 0$ and

$$\gamma(t) \in I^+(\lambda(t)) \text{ where } \gamma(t) \in U_1 \text{ and } d(\gamma(t), \lambda(t)) < \frac{C}{1+\epsilon},$$

(73)

Now we can find an $\epsilon_2 > \epsilon_1$ such that

$$\gamma(\epsilon_1) \in I^+(\lambda(\epsilon_2)), \gamma(t) \in U_2$$

(74)

and $U_2$ is a convex neighborhood with $\gamma(\epsilon_1) \in U_2$ and $\lambda(\epsilon_2) \in U_2$. With theorem 1.11, we may extend $\gamma$ in $U_2$ such that it is timelike and $\forall t \in [\epsilon_1, \epsilon_2]$ the curve $\gamma$ is subject to eq. (73) with $U_1$ replaced by $U_2$. By induction, we continue to extend $\gamma$ to a curve, defined for $t \in [0, \infty)$. Since

$$d(\gamma(\infty), \lambda(\infty)) = 0,$$

(75)

(76)

The resulting curve will be past inextendible because any endpoint of $\gamma$ would also be an endpoint of $\lambda$.

**Theorem 1.27.** (Hawking-Ellis) Let $(M, g)$ be a time ordered manifold and let $\{\lambda_n\}$ be a sequence of past inextendible causal curves which have a limit point $p$. Then there exists a past inextendible causal curve $\lambda$ passing through $p$ which is a limit curve of $\{\lambda_n\}$.

The proof is taken from the book of Hawking and Ellis. I have added some minor adaptations that are necessary for our situation. In his book Hawking considers a sequence of future inextendible curves.

**Proof.** Since $M$ is a paracompact, by theorem 1.7, one can always assign a positive definite Riemannian metric to $M$ and define an open ball $B(p, a)$ as in eq. 12 around each point in $p \in M$ with radius $a$ with respect to the Riemannian metric. Let $U_1$ be a geodesically convex
neighborhood as in theorem 1.11 around $p$. Let $b > 0$ be such that $B(p, b)$ is defined on $M$ and let $\{\lambda_{1,0,n}\}$ be a subsequence of $\lambda_n \cap U_1$ which converges to $p$ since $B(p, b)$ is compact. By theorem 1.3, it will contain limit points of $\{\lambda_{1,0,n}\}$. Any such limit point $y$ must either lie in $J^-(p, U_1)$ or $J^+(p, U_1)$ since otherwise, there would be a neighborhood $V_1$ of $y$ and $V_2$ of $p$ between which there would be no spacelike curve in $U_1$. Let

$$x_{11} \in J^-(p, U_1) \cap \overline{B}(p, b)$$

be one of these limit points, and let $\{\lambda_{1,1,n}\}$ be a subsequence of $\{\lambda_{1,0,n}\}$ which converges to $x_{11}$. Then $x_{11}$ will be a point of our limit curve $\lambda$.

Define

$$x_{ij} \in J^-(p, U_1) \cap \overline{B}(p, i^{-1} jb)$$

as a limit point of the subsequence $\{\lambda_{i-1,i-1,n}\}$ for $j = 0$ and of $\{\lambda_{i,j-1,n}\}$ for $i \geq j \geq 1$ and define $\{\lambda_{i,j,0}\}$ as subsequence of the above subsequence which converges to $x_{ij}$. Any two of the $x_{ij}$ will have non spacelike separation. Therefore, the closure of the union of all $x_{ij}$ for $j \geq i$ will give a non spacelike curve $\lambda$ from $p = x_{i0}$ to $x_{11} = x_{ii}$.

The subsequence

$$\{\lambda'_{n}\} = \{\lambda_{m,m,n}\}$$

of $\{\lambda_n\}$ intersects each of the balls

$$B(x_{mj}, m^{-1}b)$$

for $0 \leq j \leq m$ and therefore $\lambda$ will be a limit curve of $\{\lambda_n\}$ from $p$ to $x_{11}$. Now we let $U_2$ be a convex neighborhood about $x_{11}$ and repeat this construction, where we begin this time with the sequence $\{\lambda'_{n}\}$.

**Theorem 1.28.** Let $(M, g)$ be a globally hyperbolic Lorentzian manifold with a Cauchy surface $\Sigma$ and $\lambda$ be a past inextendible curve. Then $\lambda$ intersects $I^-(\Sigma)$.

**Proof.** Suppose $\lambda$ did not intersect $I^-(\Sigma)$. By theorem 1.18, $(M, g)$ is stably causal and then by theorem 1.22 we could find a past inextendible curve $\gamma$, such that

$$\forall t : \gamma(t) \in I^+(\lambda(t)),$$

where

$$I^+(\lambda) \subset I^+(\Sigma \cup I^+(\Sigma)) = I^+(\Sigma)$$

that is $\gamma(t) \subset I^+(\Sigma)$. Even if we extend $\gamma$ indefinitely into the future, it can not intersect $I^-(\Sigma)$ because otherwise the achronality of $\Sigma$ would be violated. Since by definition all past inextendible curves must intersect $\Sigma$, no such $\gamma$ can exist and $\lambda$ must enter $I^-(\Sigma)$.

**Theorem 1.29.** Let $(M, g)$ be a globally hyperbolic spacetime $\Sigma$ a Cauchy surface and $q \in I^+(\Sigma)$, then $C(\Sigma, q)$, is compact.

This proof is adapted from Wald [11] on p. 206, who took it from Hawking and Ellis[2]. I have merely added some additional explanations.
Proof. By theorem 1.18, $(M, g)$ is strongly causal. By theorem 1.25, $C(p, q)$ is the second countable for all $p, q \in M$. We begin by showing that $C(p, q)$ with $p \in \Sigma, q \in D^+(p)$ is compact. By theorem 1.23, one only has to show that every sequence of curves $\{\lambda_n\}_{n=1}^\infty \in C(p, q)$ has a limit curve $\lambda \in C(p, q)$. Then $C(p, q)$ is compact. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of future directed causal curves from $p \in \Sigma$ to $q \in I^+(p)$. Hence, $\{\lambda_n\}$ is a sequence of causal curves, all passing through $q$ by construction. Therefore, $q$ is a limit point of this sequence.

Now, we temporarily remove $p$. The spacetime $(M, g)$ is Hausdorff. Hence, we can choose an open neighborhood $V_i$ of a point $x_i \neq p$ with $p \notin V_i$. We can do this for all points $x \in M\setminus p$. The union $\cup V_i$ is then open and $\cup V_i = M/p$. With $M\setminus p$ open and $g$, $(M\setminus p, g)$ is still a Lorentzian manifold. Furthermore, $(M\setminus p, g)$ is still strongly causal since we can define a global time function $\tilde{t} = M\setminus p \to \mathbb{R}$ with $\tilde{t} = t x \in M\setminus p$ where $t$ is the time function of $(M, g)$. Strong causality of $(M\setminus p, g)$ implies stable causality by theorem 1.18. Hence, the removal of $p$ does not change the definitions of past or future directions of the curves. However, the resulting curves $\{\lambda_n\} \in M\setminus p$ do not have a past endpoint $p \in M$ anymore. Since all curves $\{\lambda_n\}$ now end before $\Sigma$, the removal of $p$ from $M$ implies that $\Sigma$ is not a Cauchy surface anymore.

Since $\{\lambda_n\}$ are past inextendible in $M\setminus p$ and $(M\setminus p, g)$ is stably causal, we can apply theorem 1.27 in with some modification and get a limit curve $\lambda \in M\setminus p \{\lambda_n\}$. One has, however, to be cautious with the compact balls $B(q, b_i)$ that are used with theorem 1.24. The radiuses $b_i$ of these balls have to be such that the removed point $p$ is never contained in the interior of any one of these balls or intersected by their closure. That is, the balls must lie entirely within $M\setminus p$. This can be easily achieved by choosing the $B(q, b)$ sufficiently small. The limit curve $\lambda$ obtained from theorem 1.27 is passing through $q$ and comes infinitely close to $p$.

By theorem 1.28 there can not be a past inextendible curve in a globally hyperbolic spacetime that does not intersect $I^- (\Sigma)$, since none of the curves $\lambda_n$ enter $I^- (\Sigma)$, the spacetime $(M\setminus p, g)$ cannot be a globally hyperbolic spacetime. If we add $p$ again to $M$, $(M, g)$ becomes globally hyperbolic by definition. All $\{\lambda_n\}_{n=1}^\infty$ are future directed causal curves beginning in $\Sigma$. Hence, they do not enter $I^- (\Sigma)$. With $p \in \Sigma$ restored, either $\lambda$ will remain past inextendible, or it will end in $p \in \Sigma$. Suppose, $\lambda$ remains past-inextendible. Then, it must enter $\Sigma$ since in a globally hyperbolic spacetime, all past inextendible curves enter $\Sigma$. Suppose $\lambda$ also enters $I^- (\Sigma)$. $(M, g)$ is Hausdorff, therefore, we can find for any point $u \in \lambda \cap I^- (\Sigma)$ a neighborhood that does not lie in $p \in \Sigma$. Hence, if $\lambda$ were past inextendible, $\lambda$ could not be a limit curve of $\{\lambda_n\}_{n=1}^\infty \in C(p, q)$, where $p \in \Sigma$. Therefore, $\lambda$ must end in $p$. As a result, $\lambda \in C(p, q)$, where $p \in \Sigma, q \in I^+(p)$. So, every sequence $\{\lambda_n\}_{n=1}^\infty \in C(p, q)$ has an accumulation curve $\lambda \in C(p, q)$ and by theorem 1.3 $C(p, q)$ then must be compact. By definition, $C(\Sigma, q)$ is the union of compact sets. Therefore, by theorem 1.29, $C(\Sigma, q)$ is also compact. 

**Definition 1.30.** Finally, we need some definitions for continuous, non differentiable curves

- We call the length functional $\tau$ of a curve in $\tilde{C}(p, q)$ upper semicontinuous in $\tilde{C}(p, q)$ if for each $\lambda \in \tilde{C}(p, q)$ given $\epsilon > 0$ there exists an open neighborhood $O \subset \tilde{C}(p, q)$ of $\lambda$ such that for all $\lambda' \in O$, we have
  \[ \tau(\lambda') \geq \tau(\lambda) + \epsilon \]  
  where $\tau$ is the length functional. (83)

- For a function $\gamma \in C(p, q)$ and an open neighborhood $O \subset C(p, q)$ of $\gamma$ we define
  \[ T(O) = \sup \{ \tau(\lambda)| \lambda \in O, \lambda \in \tilde{C}(p, q) \} \]  
  (84)
and
\[ \tau(\gamma) = \inf \left\{ T(O) | O \text{ is an open neighborhood of } \mu \right\} \] (85)

**Theorem 1.31.** (Hawking-Ellis) Let \((M, g)\) be a strongly causal spacetime and \(p, q \in M, q \in I^+(p)\) then, \(\tau(\lambda)\) with \(\lambda \in \hat{C}(p, q)\) is upper semicontinuous.

**Proof.** The proof is taken from Wald[1], who took it from the book of Hawking and Ellis[2]. I only have added some additional explanations. Let \(\lambda \in \hat{C}(p, q)\) be parametrized by proper time
\[ t = \int_{t=0}^{t(p)} \sqrt{-T^\mu T_\mu} dt \] (86)
with a tangent \(T\). Within a geodesically convex neighborhood of each point \(r \in \lambda\), the spacelike geodesics orthogonal to \(T\) form a three dimensional spacelike hypersurface. Within a small open neighborhood \(U\) of \(\lambda\), a unique hypersurface will pass through each point of \(U\). On \(U\) we can define a function \(F(p) = t\) which corresponds to the proper time value of \(\lambda\) at the hypersurface in which \(p\) lies. From eq. (86), we get \(-\nabla^\mu F = T^\mu\) on \(\lambda\). Since \(T^\mu\) is timelike, so is \(\nabla^\mu F\) on \(\lambda\). Furthermore, the tangent vector is normalized, and so
\[ T^\mu T_\mu = \nabla^\mu F \nabla_\mu F = -1 \] (87)
on \(\lambda\).

Let \(\gamma \in \hat{C}(p, q)\) with \(\gamma \in U\). We parametrize \(\gamma\) by \(F\) and let it have a tangent \(l^\mu\). Then \(l^\mu \nabla_\mu F = 1\). We can compose \(l^\mu\) in a spacelike and timelike part
\[ l^\mu = \alpha \nabla^\mu F + m^\mu \] (88)
where \(m^\mu \nabla_\mu F = 0\) in order to fulfill \(l^\mu \nabla_\mu F = 1\), and thus \(m^\mu\) is spacelike. Then
\[ l^\mu \nabla_\mu F = 1 = \alpha \nabla^\mu F \nabla_\mu F \] (89)
and therefore
\[ l^\mu = \frac{\nabla^\mu F}{\nabla_\mu F \nabla_\mu F} + m^\mu \] (90)
and
\[ l_\mu l^\mu = \frac{1}{\nabla^\mu F \nabla_\mu F} + m_\mu m^\mu \] (91)
since \(m_\mu m^\mu\) is spacelike it is \(\ge 0\) therefore
\[ \sqrt{-l_\mu l^\mu} \le \sqrt{-\frac{1}{\nabla^\mu F \nabla_\mu F}} \] (92)
The function \(\nabla^\mu F\) is continuous with
\[ \nabla^\mu F \nabla_\mu F = -1 \] (93)
on \(\lambda\). Therefore, given an \(\epsilon > 0\), we can choose a neighborhood \(U' \subset U\) of \(\lambda\) on which
\[ \sqrt{-\frac{1}{\nabla^\mu F \nabla_\mu F}} \le 1 + \frac{\epsilon}{\tau(\lambda)} \] (94)
Hence, for any $\gamma \in \tilde{C}(p, q)$ contained in $U'$, after we switch the parametrization to the arc length where $\frac{dt}{d\tau} = 1$:

$$\tau(\gamma) = \int_0^{\ell(p)} \sqrt{-l^a l_a} dF$$

$$= \int_0^{\ell(p)} \sqrt{-l^a l_a} \frac{d\tau}{dt} \, dt = \int_0^{\tau(\lambda)} \sqrt{-l^a l_a} \, d\tau$$

$$\leq \left(1 + \frac{\epsilon}{\tau(\lambda)}\right) \tau(\lambda) = \tau(\lambda) + \epsilon \quad (95)$$

As a result $\tau(\gamma) \leq \tau(\lambda) + \epsilon$ for all $\lambda \in \tilde{C}(p, q)$ and $\gamma \in \tilde{C}(p, q)$ where $\gamma \in U$, with $U$ as a neighborhood of $\lambda$. This argument holds for all curves in $\tilde{C}(p, q)$. Therefore, it also holds for curves in $\tilde{C}(\Sigma, q)$.

**Theorem 1.32.** Let $(M, g)$ be a global hyperbolic spacetime with Cauchy surface $\Sigma$ and a point $p \in D^+(\Sigma)$. Then among the timelike curves connecting $p$ to $\Sigma$ there exists a cube $\gamma$ with maximal length. This is a timelike geodesic orthogonal to $\Sigma$.

The first paragraph of this proof is from Wald[1], the rest is an own attempt.

**Proof.** The length functional $\tau(\gamma)$ of a curve $\gamma \in C(\Sigma, q)$ is a continuous function. Because of theorem [1.29] $C(\Sigma, q)$ is compact and due to theorem [1.5] the length functional then becomes maximal for a curve in $C(\Sigma, q)$. We have to show that this maximum is attained for a timelike geodesic. We begin by proving that the timelike geodesic maximizing length compared to any other continuous curve in a geodesically convex neighborhood. Given a geodesically convex neighborhood $O$, and two points $p, q$ in $O$, by theorem [1.11] the unique geodesic $\gamma$ connecting $p, q$ has greater than or equal length than any piecewise smooth curve $\mu$ connecting $p, q$. Therefore, by upper semicontinuity, we must have

$$\tau(\mu) \leq \tau(\gamma) \quad (96)$$

for any continuous curve $\mu$. If equality held with $\mu \neq \gamma$ define a point $u \in \mu, u \notin \gamma$. Let $\gamma_1$ be the geodesic segment connecting $p, u$ and $\gamma_2$ be the geodesic segment connecting $u, q$. Since by theorem [1.11] $\gamma_1$ maximizes length from $p$ to $u$ and $\gamma_2$ maximizes length from $u$ to $q$, we have

$$\tau(\gamma_1) + \tau(\gamma_2) \geq \tau(\mu) = \tau(\gamma) \quad (97)$$

which contradicts theorem [1.11] that implies $\gamma$ having greater length than any other piecewise smooth curve connecting $p, q$. Therefore we have $\tau(\mu) < \tau(\gamma)$ on $O$.

Let $\gamma$ be an arbitrary curve in $C(p, q)$. With the construction in theorem [1.24] we can cover $\gamma$ with a countable set of points $\{p_{ij}\}$ in geodesically convex open neighborhoods $O_{i,j}$, where $\{p_{i,j}\} \subset \{p_{i,j+1}\}$. The pairs of points $p_{i,j}, p_{i+1,j}$ can be piecewise connected by unique timelike geodesics $\lambda_{i,j}$ of theorem [1.11]. The piecewise curve $\lambda_j \cup \lambda_{ij}$ is then a trip in $\kappa(p, q) \subset C(p, q)$ with a length

$$\tau(\lambda_j) = \sum_i \tau(\lambda_{ij}) \quad (98)$$
The length fiction $\tau(\gamma)$ of the original curve $\gamma$ can also be described by a sum of lengths of its own segments $\gamma_{ij}$ that connect the points $p_{i,j}, p_{i+1,j} \in O_{ij}$:

$$\tau(\gamma) = \sum_i \tau(\gamma_{ij})$$

(99)

By the first paragraph, we have

$$\tau(\lambda_{ij}) \geq \tau(\gamma_{ij}) \forall i,j$$

(100)

and therefore

$$\tau(\lambda_j) = \sum_i \tau(\lambda_{ij}) \geq \sum_i \tau(\gamma_{ij}) = \tau(\gamma) \forall j,$$

(101)

with equality in eqs (100) and (101) if and only if $\gamma$ is a timelike geodesic. In this case, the segments $\lambda_{ij}$ would all be lying on $\gamma_{ij}$ for all $i,j$, resulting in the equality of the length functions $\tau(\lambda_j)$ and $\tau(\gamma)$.

Since $\{p_{i,j}\} \subset \{p_{i,j+1}\}$ are in convex neighborhoods, repeated application of theorem 1.11 leads for a continuous curve $\gamma$ to the following expression for the approximating trips:

$$\tau(\lambda_j) \geq \tau(\lambda_{j+1})$$

(102)

Again, in eq. (102), we have equality if and only if $\gamma$ is a timelike geodesic. In the latter case, the segments $\lambda_{ij}$ and $\lambda_{j+1}$ would be just a description of the same curve $\gamma$ and thereby have equal length.

For a continuous curve $\gamma$ eqs. (102) and (101) imply that the length

$$\tau(\lambda) = \lim_{j \to \infty} \sum_i \tau(\lambda_{ij})$$

(103)

of the limit curve

$$\lambda = \lim_{j \to \infty} \lambda_j$$

(104)

is approximating the length of $\gamma$ from above, i.e.

$$\tau(\lambda) \to \tau(\gamma),$$

(105)

where $\tau(\lambda) = \tau(\gamma)$ if and only if $\gamma$ is a timelike geodesic.

Assume $\gamma$ were, on some segment $\gamma_{ij}$ connecting $p_{i,j}$ and $p_{i+1,j}$, not a timelike geodesic. Then, one could get a longer curve by replacing the segment $\gamma_{ij}$ by the corresponding segment $\lambda_{ij}$ of our construction. Hence $\tau$ would not reach its maximum on $\gamma$.

Now assume $\gamma$ would be a timelike geodesic. Then, we have equality in eqs (102) and (101), which yields

$$\tau(\lambda) = \lim_{j \to \infty} \sum_i \tau(\lambda_{ij}) = \sum_i \tau(\lambda_{i+1}) = \tau(\lambda) = \tau(\gamma)$$

(106)

for the entire curve. Hence, $\tau(\gamma)$ would have reached its maximum, since on each segment $\lambda_{ij}$, the length $\tau(\lambda_{ij})$ is at its maximum.

Finally, a length maximizing geodesic must be orthogonal to $\Sigma$, since by theorem 1.17 a length maximizing geodesic starting on $\Sigma$ must be orthogonal to $\Sigma$ or would otherwise be possible to increase its length. One can choose some geodesically convex open neighborhood $U$,
where \( U \cap \Sigma \neq \emptyset \) and \( U \cap \gamma \neq \emptyset \). Then one can connect a point \( u \in \gamma \) where \( u \in U \) to a point \( p \in \Sigma \) with a geodesic segment orthogonal to \( \Sigma \). The curve obtained by following \( \gamma \) from \( q \) to \( u \) and the segment connecting \( u \) to \( \Sigma \) with a geodesic segment orthogonal to \( \Sigma \), would by theorem \ref{1.17} have bigger length than \( \gamma \).

### 1.1.4 The Penrose and Penrose-Hawking singularity theorems of General Relativity

**Proposition 1.33.** Let \((M, g)\) be a global hyperbolic spacetime with Cauchy surface where \( R_{\mu\nu} \zeta^\mu \zeta^\nu \geq 0 \) for all timelike \( \zeta \). Suppose there exists a Cauchy surface for which the trace of the extrinsic curvature \( K < 0 \). Then no future directed timelike curve can be extended beyond proper time \( \tau_0 = \frac{3}{|K|} \).

**Proof.** Suppose there exists a future directed timelike curve \( \lambda \) from \( \Sigma \) that can be defined for some proper time greater than \( \tau_0 \). Then, one can define the curve up to a point \( p = \lambda(\tau_0 + \epsilon) \) with \( \epsilon > 0 \). According to theorem \ref{1.32} there would exist a timelike geodesic \( \gamma \) with maximal length that connects \( \Sigma \) to \( p \), with \( \gamma \) being orthogonal to \( \Sigma \). Because \( \tau(\lambda(p)) = \tau_0 + \epsilon \), we would have \( \tau(\gamma(p)) \geq \tau_0 + \epsilon \). Theorem \ref{1.19} implies that \( \gamma \) would develop conjugate points at \( \tau_0 \). Finally, theorem \ref{1.20} implies that then \( \gamma \) then fails to maximize length. Hence, no timelike curve can be extended beyond \( \tau_0 \).

The most problematic assumption in this theorem is that the spacetime should be globally hyperbolic. However, Hawking and Penrose were able to eliminate this condition in \cite{12}.

**Definition 1.34.** To state the Penrose-Hawking singularity theorem, one needs the following definitions:

- An edge of a closed achronal set \( \Sigma \) is the set of all points \( p \in \Sigma \) such that every open neighborhood \( O \) of \( p \) contains a point \( q \in I^+(p) \).

- A spacetime \((M, g)\) satisfies the timelike generic condition if each timelike geodesic with tangent \( \zeta^\mu \) possesses at least one point where \( R_{\mu\nu} \zeta^\mu \zeta^\nu \neq 0 \).

- \((M, g)\) satisfies the null generic condition if every null geodesic with tangent \( k^\mu \) possesses at least one point where either \( R_{\mu\nu} k^\mu k^\nu \neq 0 \) or

\[
 k_{[\mu} R_{\nu]\rho] \sigma [\lambda k_{\tau]} k^\rho k^\sigma \neq 0
\]

- A compact two dimensional smooth spacelike submanifold \( \Omega \) where the expansion \( \theta \) along all ingoing and outgoing future directed null geodesics orthogonal to \( \Omega \) is negative, is called a trapped surface.

Then, the Penrose-Hawking singularity theorem \cite{12} reads:

**Proposition 1.35.** Let \((M, g)\) be a spacetime where 1) \( R_{\mu\nu} \zeta^\mu \zeta^\nu \geq 0 \) for all timelike and null \( \zeta \) 2) The timelike and null generic conditions are satisfied, 3) no closed timelike curves exist, 4) one of the following a) \((M, g)\) possesses a compact achronal set without edge, b) \((M, g)\) possesses a trapped null surface, c) there exists a point \( p \in M \) such that the expansion of future or past directed null geodesics emanating from \( p \) becomes negative along each geodesic in this congruence. Then \((M, g)\) must contain at least one incomplete timelike or null geodesic.
The generic conditions can be assumed to hold in any reasonable model of the universe. Similarly, closed timelike curves must reasonably be excluded for a physical spacetime. Contracting Einstein’s equation with matter,
\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \]  
we get
\[ R = -8\pi T, \]  
or
\[ R_{\mu\nu} = 8\pi (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \]
we get from \( R_{\mu\nu} \zeta^{\mu} \zeta^{\nu} \geq 0 \)
\[ 8\pi \left( T_{\mu\nu} \zeta^{\mu} \zeta^{\nu} - \frac{1}{2} T \zeta^{\nu} \zeta^{\nu} \right) \geq 0 \]
which is called strong energy condition. In general, the energy momentum tensor \( T_{\mu\nu} \) is not diagonalizable. However, it seems that in most cases an energy momentum tensor for a perfect fluid
\[ T_{\mu\nu} = (\rho + p) \zeta^{\mu} \zeta^{\nu} + pg_{\mu\nu} \]
is a reasonable assumption, where \( \zeta \) is the fluid’s four velocity, \( \rho \) the energy density and \( p \) the pressure, the latter being assumed to be equal in every direction. Then, \( T_{\mu\nu} \zeta^{\mu} \zeta^{\nu} = \rho \) and \( T_{\mu\nu} g^{\mu\nu} \zeta^{\nu} \zeta^{\nu} = -(-\rho + p + 4p) = \rho - 3p \). Therefore, the strong energy condition becomes:
\[ (\rho - \frac{1}{2} \rho + \frac{1}{2} 3p) \geq 0 \]  
or
\[ \rho + 3p \geq 0 \]
If we neglect shear and rotation, the Raychaudhuri equation is then equivalent to
\[ \frac{d\theta}{d\tau} = -8\pi (\rho + 3p) \leq 0. \]
So, the strong energy condition is responsible for gravity being attractive. This is a condition that must have been violated during the inflation era of the universe, however, it seems to hold for the universe that we observe today.
A trapped surface is formed in the interior of the solutions for black holes. The latter can be indirectly observed by their orbiting stars [10]. The trajectory of those stars reveal a super massive black hole with large Schwarzschild radius in the center of most galaxies. By the singularity theorems above, it seems inevitable that by classical general relativity, a singularity must form. One can show, see e.g. [11] p. 862 that all matter inside a Schwarzschild black hole falls inevitably towards the singularity, and the length scales of the matter coming close to the singularity are getting proportional to
\[ V \propto \tau_{\text{singularity}} - \tau \]  
25
where \( \tau \) is the proper time since the objects have crossed the event horizon and \( \tau_{\text{singularity}} \) is the proper time where the matter has arrived at the singularity. Since no quantum mechanical object can be pressed in lengths shorter than its Compton wavelength, one has to seek for an alternative theory that describes the behavior of matter and the gravitational field close to a singularity. Because the curvature in the vicinity of a singularity is extremely large and even becomes infinite at the singularity, a first attempt would be to use quantum field theory in curved spacetimes. In this framework, one unfortunately finds that close to the singularity, one is not able to define ordinary particle propagators, see [30]. In order to find a resolution of this problem, one has to define a consistent theory of quantum gravity. This theory should replace classical general relativity on the Planck scale and yield expressions that allow a consistent description of matter inside of black holes.

2 Covariant quantization of general relativity

For computing scattering amplitudes, the covariant quantization based on the perturbative evaluation of the path integral provides an excellent tool. In the first subsection, we will derive the Feynman rules for gravitational amplitudes. In the second subsection we will review the problems associated with loop calculations.

2.1 The Feynman rules of gravitation

This section is merely a review of the excellent introductions [21, 22, 18, 19, 27]. With the Lagrangian density \( \mathcal{L} = \sqrt{-g} R + \xi \mathcal{L} \), the action of the gravitational field

\[
S = \int dt L = \int \sqrt{-g} d^4 x
\]

is invariant under infinitesimal gauge transformations of the form

\[
g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = g_{\mu\nu} + g_{\rho\nu} \partial_\mu \eta^\rho + g_{\mu\alpha} \partial_\nu \eta^\alpha + \eta^\alpha \partial_\alpha g_{\mu\nu}
\]

where \( \eta^\mu \) are the components of an arbitrary infinitesimal vector field. In general relativity, one can show a local flatness theorem. This theorem says that around a given point \( p \) in the spacetime manifold described by Einstein’s equation, there is a local neighborhood where \( g_{\mu\nu} = \eta_{\mu\nu} \) with \( \eta_{\mu\nu} \) as the Minkowski metric. As we are developing a local field theory, we now choose such a neighborhood and decompose the metric as follows:

\[
g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

The symmetric tensor \( g_{\mu\nu} \) has the function of a classical background metric and the symmetric tensor \( h_{\mu\nu} \) is supposed to be described by some quantum field theory. It is invariant under gauge transformations

\[
h_{\mu\nu} \rightarrow h^\alpha_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \eta_\nu + \nabla_\nu \eta_\mu
\]

Although a decomposition of the metric as in eq. 117 can always be made as a kind of gauge, one should note that this composition of the metric is in fact highly problematic. The idea that \( h_{\mu\nu} \) is subject to quantum processes implies that a finite change of \( h_{\mu\nu} \) through some quantum process could, according to Faddeev and Popov (see [22] on p. 780), for example,
change the signature of the metric $\mathcal{g}_{\mu\nu}$, if $h_{\mu\nu}$ is not sufficiently small. A change of the signature of the metric may not be problematic, since in the path integral over metrics, one would have to sum over all possible metrics, including ones with different signature. However, Wald notes in [1] on p. 384 that the perturbation theory we will obtain below satisfies, “for all orders, the causality relations with respect to the background metric $g_{\mu\nu}$ and not with respect to $\mathcal{g}_{\mu\nu}$.” Wald writes that the entire summed series may still obey the correct causality relation with respect to $g_{\mu\nu}$, if it were to converge. Later, we will see, however that it does not converge. This would effectively imply that the theory makes only sense for small $h_{\mu\nu}$.

Here, however, we will not be bothered by such problems, and postpone the descriptions of quantum mechanical black holes. Instead, we derive the Feynman rules for perturbative quantum gravity. We can derive them from the path integrals like

$$Z = \int \mathcal{D}g_{\mu\nu} e^{iS}.$$ (119)

where $\mathcal{D}g_{\mu\nu}$ is understood as the usual summation over all possible metrics. Using the expansion of the metric in eq. (117), we can expand the action around the background $\eta_{\mu\nu}$ as

$$S = \int d^4x \left( \sqrt{-g} R + \mathcal{L} + \mathcal{L} \right)$$ (120)

where $g$ and $R$ are constructed from the background metric $g_{\mu\nu} = \eta_{\mu\nu}$ and $\mathcal{L}$ is linear in $h_{\mu\nu}$ and $\mathcal{L}$ is quadratic in $h_{\mu\nu}$. The path integral of eq (119) will then become an expression like

$$Z = e^{i \int d^4x \sqrt{-g} R \int \mathcal{D}h_{\mu\nu} e^{i \int d^4x (L + L + \ldots)}$$ (121)

where $\int d^4x \sqrt{-g} R = 0$ if $g_{\mu\nu} = \eta_{\mu\nu}$.

The part of the action quadratic in $h_{\mu\nu}$ can be used to derive the propagator. In the following, we will adopt the notation of ‘t Hooft and Veltman for the metric tensor $g^{\alpha\beta} h_{\alpha\beta} = h^0_0$. Since the metric is a symmetric tensor, we do not need to worry on which index comes first. Following Veltman and ‘t Hooft, we begin the calculation by setting

$$\overline{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} = g_{\mu\nu} (\delta^\alpha_\nu + h^\alpha_\nu)$$ (122)

where $g_{\mu\nu}$ is at first arbitrary, and indices are raised and lowered by $g_{\mu\nu}$. ‘t Hooft and Veltman get

$$\overline{g}^{\mu\nu} = g^{\alpha\nu} (\delta^\alpha_\mu - h^\alpha_\mu + h^\alpha_\beta h^\beta_\nu) = g^{\mu\nu} - h^{\mu\nu} + h^{\alpha\beta} h^{\mu\nu}$$ (123)

and using

$$\sqrt{-\overline{g}} = \sqrt{-\text{det}(\overline{g})} = e^{\frac{1}{2} \text{tr}(\ln(-\text{det}(\overline{g})))} = \sqrt{-\text{det}(g)} e^{\frac{1}{2} \text{tr}(\ln(\delta^\alpha_\nu + h^\alpha_\nu))}$$

they arrive at

$$\sqrt{-\overline{g}} = \sqrt{-g}(1 + h^\alpha_\mu - h^\alpha_\mu h^\alpha_\nu + h^\alpha_\mu h^\alpha_\nu + \frac{1}{8} (h^\alpha_\mu)^2).$$ (124)

With eq. (3), ‘t Hooft and Veltman decompose the Christoffel connection as

$$\Gamma^\alpha_{\mu\nu} = +\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\nu}$$ (125)
with
\[ \Gamma^\alpha_{\mu
u} = \frac{1}{2} \left( \nabla_\mu h^\alpha_\nu + \nabla_\nu h^\alpha_\mu - \nabla^\alpha h_{\mu
u} \right) \] (126)
containing terms linear in \( h_{\mu
u} \) and
\[ \Gamma^\alpha_{\mu
u} = -\frac{1}{2} h^{\alpha\gamma}(\nabla_\gamma h_{\mu
u} + \nabla_\nu h_{\mu\gamma} - \nabla_\gamma h_{\mu\nu}) \] (127)
containing terms quadratic in \( h_{\mu\nu} \). Following \([21]\), we can, using eq. (2), then decompose the Riemann tensor in separate parts \( R^\mu_{\nu\alpha\beta}, R^\mu_{\nu\alpha\beta} \), each depending on \( \Gamma^\alpha_{\mu
u}, \Gamma^\alpha_{\mu\nu} \), or \( \Gamma^\alpha_{\mu\nu} \):
\[ T_{\nu\alpha\beta} = R^\mu_{\nu\alpha\beta} + R^\mu_{\nu\alpha\beta} + R^\mu_{\nu\alpha\beta} \] (128)
\[ R^\mu_{\nu\alpha\beta} = \frac{1}{2}\nabla_\alpha \nabla_\nu h^\mu_\beta - \nabla^\mu \nabla_\nu h_\beta - \nabla_\beta \nabla_\nu h^\mu_\alpha + \nabla^\beta \nabla_\nu h_\alpha + \frac{1}{2} R^\mu_{\gamma\alpha\beta} h^\gamma_\nu + \frac{1}{2} \frac{1}{2} R^\gamma_{\nu\alpha\beta} h^\gamma_\mu \] (129)
\[ R^\mu_{\nu\alpha\beta} = \partial_\nu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\nu\gamma} \Gamma^\gamma_{\alpha\beta} - \Gamma^\mu_{\alpha\gamma} \Gamma^\gamma_{\nu\beta} \] (130)
one then can construct \( \bar{R}_{\nu\alpha} = \bar{R}_{\nu\alpha} \) and \( \bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} \).

Veltman and 't Hooft find for the Lagrangian density up to second order the expression:
\[ \mathcal{L} = \sqrt{-g} R \]
\[ = \sqrt{-g} R + \mathcal{L} + \mathcal{L} \] (131)
where
\[ \mathcal{L} = \sqrt{-g}(h^\alpha_\alpha R^\alpha_\beta - \frac{1}{2} h^\alpha_\alpha R) \] (132)
is the part linear in \( h_{\mu\nu} \) and
\[ \mathcal{L} = \sqrt{-g} \left( -R \left( \frac{1}{8} h^\alpha_\alpha \right) + \frac{1}{4} h^\alpha_\beta h^\beta_\alpha - \frac{1}{4} h^\alpha_\beta R^\beta_\alpha + \frac{1}{2} h^\alpha_\beta h^\beta_\mu R^\mu_\alpha - \frac{1}{4} \nabla_\mu h^\alpha_\beta \nabla^\mu h^\alpha_\beta \right. \]
\[ + \frac{1}{2} \nabla_\mu h^\alpha_\beta \nabla^\mu h^\alpha_\beta - \frac{1}{2} \nabla_\beta h^\alpha_\alpha \nabla^\mu h^\mu_\beta + \frac{1}{2} \nabla^\alpha h^\beta_\beta \nabla^\mu h^\mu_\alpha \] (133)
is part of the Lagrangian that is quadratic in \( h_{\mu\nu} \) (with total derivatives being omitted).

We now choose the background \( g_{\mu\nu} = \eta_{\mu\nu} \). The scalar \( R \) and the tensor \( R^\alpha_\beta \) contain sums of derivatives of the metric, which must vanish for \( g_{\mu\nu} = \eta_{\mu\nu} \). Therefore \( R \) and \( R^\alpha_\beta \) must vanish too. Furthermore \( -g = 1 \) and in the second line, the covariant derivatives become partial ones. The result is:
\[ \mathcal{L} = -\frac{1}{4} \partial_\nu h^\beta_\alpha \partial^\mu h^\alpha_\beta + \frac{1}{4} \partial_\mu h^\alpha_\beta \partial^\mu h^\beta_\alpha - \frac{1}{2} \partial_\beta h^\alpha_\alpha \partial^\mu h^\beta_\mu + \frac{1}{2} \partial^\nu h^\beta_\mu \partial_\nu h^\alpha_\beta - \frac{1}{2} \partial_\nu h^\alpha_\beta \partial^\nu h^\alpha_\beta \]
\[ + \frac{1}{2} \partial_\mu h^\alpha_\beta \partial^\mu h^\beta_\alpha - \frac{1}{2} \partial^\mu h^\alpha_\beta \partial_\mu h^\beta_\alpha + \frac{1}{4} \partial^\nu h^\mu_\mu \partial^\nu h^\mu_\mu \] (134)
where partial integration was used for the last term in the sum.

Historically, this was first obtained by Pauli and Fierz \([33]\). Feynman noted in \([24]\) that this expression does not have a generalized inverse, and one must add a gauge breaking term, in
order to define a propagator. This was similar to the electromagnetic field. However, Feynman’s
group also found that when computing loop amplitudes, the theory with this propagator still
failed to be unitary. For one loop amplitudes, Feynman was able to find a method around this
problem. In a loop “fictitious particles” or “ghosts” have to be inserted in order to compensate
overcounting. This happens in the path integral because of the summation over fields that are
physically equivalent by a gauge transformation. In 1967, deWitt [25] was able to generalize this
procedure to all orders of perturbation theory. Finally, Fadeev and Popov [26] found a simple
way to describe this procedure, which we will outline here. This is merely a summary of [18],
p. 51.

The path integral from which the Feynman diagrams of the S-matrix are constructed is
formally an expression

\[ Z = \int D h_{\mu\nu} e^{iS} \]  

with \( S = \int d^4 x \sqrt{-g} R \). We described the relevant gauge transformations \( \eta_\mu \) in eq. (116). Now we
choose a gauge constraint to fix a gauge \( \eta_\mu \). This corresponds to four conditions \( G_\alpha(h_{\mu\nu}) = 0 \).
Then, we define a functional

\[ \Delta^{-1}_G(h_{\mu\nu}) = \int D \eta_\mu \prod_\alpha \delta(G_\alpha(h^\eta_{\mu\nu})) \]  

where \( h^\eta_{\mu\nu} \) is the gauge transformed metric and \( \eta_\mu \) is the gauge transformation from eq. (116). For compact
gauge groups, the integration measure \( D \eta_\mu \) is left-invariant. Therefore, we have
for the gravitational field

\[ D(\eta'_\mu \eta_\mu) = D \eta_\mu, \]  

and it follows that

\[ \Delta^{-1}_G(h^\eta_{\mu\nu}) = \int D \eta_\mu \prod_\alpha \delta(G_\alpha(h^\eta'_{\mu\nu})) \]
\[ = \int D \eta'_\mu \eta_\mu \prod_\alpha \delta(G_\alpha(h^\eta'_{\mu\nu})) \]
\[ = \Delta^{-1}_G(h_{\mu\nu}) \]  

hence \( \Delta^{-1}_G(h^\eta_{\mu\nu}) \) is gauge invariant. This can be inserted into the path integral as

\[ Z = \int D h^\eta_{\mu\nu} \Delta_G(h^\eta_{\mu\nu}) \int D \eta_\mu \prod_\alpha \delta(G_\alpha(h^\eta_{\mu\nu})) e^{iS} \]
\[ = \int D h_{\mu\nu} \Delta_G(h_{\mu\nu}) \int D \eta_\mu \prod_\alpha \delta(G_\alpha(h_{\mu\nu})) e^{iS} \]
\[ = \int D \eta_\mu \int D h_{\mu\nu} \prod_\alpha \delta(G_\alpha(h_{\mu\nu})) \Delta_G(h_{\mu\nu}) e^{iS} \]  

where we have used the gauge invariance of \( S, G_\alpha(h_{\mu\nu}) \) and \( D h^\eta_{\mu\nu} \). The factor \( \int D \eta_\mu \) only
contributes a multiplicative constant, since the rest of the integrand does not depend on \( \eta \).
Therefore, \( \int D \eta_\mu \) may be dropped.
Because of the deltafunction in the integrand, we can expand $G_\alpha(h^\eta_{\mu\nu})$ in eq. (136) around $\eta_{\mu} = 0$, since for other gauges, the contribution of the integrand is zero. We get

$$G_\alpha(h^\eta_{\mu\nu}) = G_\alpha(h_{\mu\nu}) + (A\eta)_\alpha$$

(140)

where

$$G_\alpha(h_{\mu\nu}) = 0$$

(141)

from the gauge condition, and

$$A_{\alpha\beta} = \frac{\delta G_\alpha(h^\eta_{\mu\nu})}{\delta \eta^\beta}$$

(142)

is a matrix of the derivatives of $G_\alpha(h^\eta_{\mu\nu})$ with respect to $\eta^\beta$.

For n-dimensional vectors $\vec{a}$ and some vectorfield $\vec{g}(\vec{a})$ we have the identity

$$1 = \text{det} \left( \frac{\partial g_i}{\partial a_j} \right) \prod_i \int da_i \delta^n(\vec{g}(\vec{a}))$$

(143)

From eq. (136), we have

$$1 = \Delta_G(h_{\mu\nu}) \int D\eta_{\mu} \prod_\alpha \delta(G_\alpha(h^\eta_{\mu\nu}))$$

(144)

which is the continuous analogue of eq. (143). Thereby,

$$\Delta_G(h_{\mu\nu}) = \text{det}(A_{\alpha\beta}).$$

(145)

This determinant is conventially expressed in terms of an integral over anticommuting grassman variables $(\bar{\eta}^\alpha)^*$ and $\bar{\eta}^\beta$ which are called ghost fields:

$$\text{det}(A) = \int \prod_\alpha D(\bar{\eta}^\alpha)^* D\bar{\eta}^\alpha e^{-i \int d^4x (\bar{\eta}^\alpha)^* A_{\alpha\beta} \bar{\eta}^\beta}$$

(146)

inserting this into the path integral, we get

$$Z = \int Dh_{\mu\nu} \prod_\alpha \delta(G_\alpha(h_{\mu\nu})) \Delta_G(h_{\mu\nu}) e^{iS}$$

$$= \int D\eta_{\mu} \int \prod_\alpha D(\bar{\eta}^\alpha)^* D\bar{\eta}^\alpha$$

$$\delta(G_\alpha(h_{\mu\nu})) e^{iS + i \int d^4x (\bar{\eta}^\alpha)^* A_{\alpha\beta} \bar{\eta}^\beta}$$

(147)

We can also choose $G_\alpha(h_{\mu\nu}) = c_\alpha$ as gauge condition. This leads to an integral

$$Z = \int D\eta_{\mu} \int \prod_\alpha D(\bar{\eta}^\alpha)^* D\bar{\eta}^\alpha$$

$$\delta(G_\alpha(h_{\mu\nu}) - c_\alpha) e^{iS + i \int d^4x (\bar{\eta}^\alpha)^* A_{\alpha\beta} \bar{\eta}^\beta}$$

(148)

Since $c_\alpha$ does not depend on $h_{\mu\nu}$, $\Delta_G(h_{\mu\nu})$ and also $Z$ are not changed by $c_\alpha$. Since $Z$ is independent of $c_\alpha$, we may integrate over $c_\alpha$. Inserting the normalization factor

$$\int Dc_\alpha e^{-\frac{i}{\hbar} \int d^4x c_\alpha}$$
we get

\[ Z = \int Dc_\alpha e^{-\frac{i}{\hbar} \int d^4x c_\alpha} \int D\eta_{\alpha\beta} \int \prod_{\alpha} D(\tilde{\eta}^\alpha)^* D\tilde{\eta}^\alpha \delta(G_\alpha(h_{\mu\nu}) - c_\alpha) e^{i S + i \int d^4x (\tilde{\eta}^\alpha)^* A_{\alpha\beta} \tilde{\eta}^\beta} \]

\[ = \int D\eta_{\mu\nu} \int \prod_{\alpha} D(\tilde{\eta}^\alpha)^* D\tilde{\eta}^\alpha e^{i S^* - \frac{i}{\hbar} \int d^4x G_\alpha G^\alpha + i \int d^4x \tilde{\eta}^\alpha)^* A_{\alpha\beta} \tilde{\eta}^\beta}. \quad (149) \]

The expression

\[ \frac{i}{4\zeta} \int d^4x G_\alpha G^\alpha \]

is called gauge fixing term.

We can derive the propagator of the gravitational field, if we take the part of \( S = \int d^4x \sqrt{-g} R \) which is quadratic in \( h_{\mu\nu} \) together with some suitable gauge fixing. Choosing, for example the DeDonder gauge, which is equal to

\[ G_\mu = \partial_\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h_{\nu\nu} \]

and subtracting \( \frac{1}{2} G_\mu^2 \) from eq (133), we obtain, using in the 6th line that \( h_{\mu\nu} \) must be symmetric:

\[ \mathcal{E} - \frac{1}{2} C_{\mu}^2 = -\frac{1}{4} \partial_\nu h_{\alpha\beta} \partial^\nu h^{\alpha\beta} + \frac{1}{4} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \frac{1}{2} \partial^\nu h_{\alpha\beta} \partial^\mu h_{\beta\mu} + \frac{1}{2} \partial^\nu h_{\mu\nu} \partial_\beta h^\mu_\beta \]

\[ = \frac{1}{4} \partial_\nu h_{\alpha\beta} \partial^\nu h^{\alpha\beta} + \frac{1}{4} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \frac{1}{2} \partial^\nu h_{\alpha\beta} \partial^\mu h_{\beta\mu} + \frac{1}{2} \partial^\nu h_{\mu\nu} \partial_\beta h^\mu_\beta \]

\[ = 0 \]

\[ V_{\alpha\beta\mu} = \frac{1}{2} \eta_{\alpha\mu} \eta_{\beta\nu} - \frac{1}{4} \eta_{\alpha\beta} \eta_{\mu\nu} \quad (151) \]

Inversion of the matrix \( V_{\alpha\beta\mu\nu} \) yields \( \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta} \) and leads to the graviton propagator

\[ \mu\nu \kappa \lambda \beta \alpha = D_{\mu\alpha\beta}(k) = \frac{1}{k^2 - \kappa} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}) \]

Note that similar to the situation in electrodynamics, the form of the graviton propagator depends crucially on the gauge.
In their article [27], 't Hooft and Veltman also derive the propagator for the Prentki gauge which we will not consider here. The ghost Lagrangian is obtained by subjecting $C_{\mu}$ to the gauge transformation of eq. (118). We get,

$$G_{\mu} = \frac{\partial}{\partial \nu} (h_{\mu \nu} + \nabla_{\mu} \eta_{\nu} + \nabla_{\nu} \eta_{\mu}) - \frac{1}{2} \partial_{\mu} (h_{\nu \nu} + \nabla_{\nu} \eta_{\nu} + \nabla_{\nu} \eta_{\nu})$$

$$= \frac{\partial}{\partial \nu} h_{\mu \nu} - \frac{1}{2} \partial_{\mu} h_{\nu \nu} + \partial_{\nu} (\nabla_{\mu} \eta_{\nu} + \nabla_{\nu} \eta_{\mu}) - \partial_{\mu} \nabla_{\nu} \eta_{\nu}$$

$$= \frac{\partial}{\partial \nu} h_{\mu \nu} - \frac{1}{2} \partial_{\mu} h_{\nu \nu} + \partial_{\nu} \partial_{\nu} \eta_{\mu} - \partial_{\nu} \Gamma_{\mu \nu}^{\lambda} \eta_{\lambda} + \partial_{\nu} \partial_{\nu} \eta_{\lambda} - \partial_{\nu} \Gamma_{\nu \mu}^{\lambda} \eta_{\lambda} - \partial_{\nu} \Gamma_{\nu \lambda}^{\mu} \eta_{\lambda} - \partial_{\nu} \Gamma_{\nu \lambda}^{\nu} \eta_{\lambda}$$

$$= \frac{\partial}{\partial \nu} h_{\mu \nu} - \frac{1}{2} \partial_{\mu} h_{\nu \nu} + \partial_{\nu} \partial_{\nu} \eta_{\mu} - \partial_{\nu} \Gamma_{\mu \nu}^{\lambda} \eta_{\lambda} - \partial_{\nu} \Gamma_{\nu \mu}^{\lambda} \eta_{\lambda} + \partial_{\nu} \Gamma_{\nu \lambda}^{\nu} \eta_{\lambda}$$

(152)

We note that in the expansion of the Christoffel symbols, eq. (125), we have $\Gamma_{\mu \nu}^{\alpha} = 0$ with $g_{\mu \nu} = \eta_{\mu \nu}$, and the part $\Gamma_{\mu \nu}^{\alpha}$ depends linearly on $h_{\mu \nu}$. Taking the derivative of $G_{\mu}$ with respect to $\eta_{\mu \nu}$, we observe that the part of $G_{\mu}$ which does not include any order of $h_{\mu \nu}$ is $\partial_{\nu}^{2}$. Therefore, the lowest order ghost Lagrangian is

$$\mathcal{L}_{\text{ghost}} = -\partial_{\nu} \tilde{\eta}_{\alpha} \partial_{\nu} \eta_{\alpha} + \mathcal{O}(h_{\mu \nu}) = -\partial_{\nu} \tilde{\eta}_{\alpha} \eta_{\mu \nu} \partial_{\mu} \eta_{\alpha} + \mathcal{O}(h_{\mu \nu})$$

(153)

In diagrammatic language, this corresponds to a ghost particle with a propagator given by, see [19]:

$$\mathcal{D}_{\mu \nu}(k) = \frac{\eta_{\mu \nu}}{k^{2}}$$

The graviton couples to itself. The three graviton vertex was first given in the third article on quantum gravity by deWitt [28]. It is a quite complicated expression. Nevertheless, deWitt notes that it could be “straightforwardly” computed from the cubic part of the action. I have however, not tried to derive it and I merely state this result here. In his article, deWitt also describes the four graviton vertex coming from the quartic part of the action, which I do not present here, simply because of the length of that formula. We have
where have chosen the notation of [19]. Furthermore, the graviton couples to the ghosts with the vertex \( V(k^1, k^2, k^3)_{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3} \) that is given by

\[
V(k^1, k^2, k^3)_{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3} = - \left( k_{(\alpha_1}^2 k_{\beta_1)}^3 \left( 2\eta_{\alpha_2(\alpha_3} \eta_{\beta_3)} \eta_{\beta_2} - \eta_{\alpha_2 \beta_2} \eta_{\alpha_3 \beta_3} \right) + k_{(\alpha_2}^1 k_{\beta_2)}^3 \left( 2\eta_{\alpha_1(\alpha_3} \eta_{\beta_3)} \eta_{\beta_1} - \eta_{\alpha_1 \beta_1} \eta_{\alpha_3 \beta_3} \right) + k_{(\alpha_3}^1 k_{\beta_3)}^3 \left( 2\eta_{\alpha_1(\alpha_2} \eta_{\beta_2)} \eta_{\beta_1} - \eta_{\alpha_1 \beta_1} \eta_{\alpha_2 \beta_2} \right) + 2k_{(\alpha_1}^1 (\alpha_3) \eta_{\beta_1)} \eta_{\alpha_2 \beta_2} + 2q_{(\alpha_3}^1 \eta_{\beta_3)\eta_{\alpha_2}^{\gamma} (\alpha_2 \eta_{\beta_2)} (\alpha_1 \eta_{\beta_1}) + 2k_{(\alpha_1}^1 \eta_{\beta_1)} (\alpha_3) \eta_{\beta_3) (\alpha_2 k_{(\beta_2)}^3) + k_{\alpha_2}^3 (\eta_{\alpha_1(\alpha_2} \eta_{\beta_2)} \eta_{\alpha_3 \beta_3} + \eta_{\alpha_1(\alpha_3} \eta_{\beta_3)} \eta_{\alpha_2 \beta_2} - 2\eta_{\alpha_1(\alpha_2} \eta_{\beta_2) (\alpha_3 \eta_{\beta_3)} \eta_{\beta_1}) + k_{\alpha_2}^3 (\eta_{\alpha_2(\alpha_3} \eta_{\beta_3)} \eta_{\alpha_2 \beta_2} + \eta_{\alpha_2(\alpha_2} \eta_{\beta_2)} \eta_{\alpha_3 \beta_3} \eta_{\alpha_1 \beta_1} - 2\eta_{\alpha_2(\alpha_2} \eta_{\beta_2) (\alpha_3 \eta_{\beta_3)} \eta_{\beta_2}) + k_{\alpha_2}^3 (\eta_{\alpha_3(\alpha_1} \eta_{\beta_1)} \eta_{\alpha_2 \beta_2} + \eta_{\alpha_3(\alpha_2} \eta_{\beta_2)} \eta_{\alpha_3 \beta_3} \eta_{\alpha_1 \beta_1} - 2\eta_{\alpha_3(\alpha_1} \eta_{\beta_1)} (\alpha_2 \eta_{\beta_2)} (\alpha_3) \eta_{\beta_3) (\alpha_2 k_{(\beta_2)}^3) \right)
\]

(154)

It should be derivable from eq. (152) with eq. (126) and is, see [19, eq. (70), or [22]:

\[
V(k^1, k^2, k^3)_{\alpha \beta \gamma \mu} = -\eta_{\gamma(\alpha}^1 k_{\beta)}^3 \eta_{\mu}^2 + \eta_{\gamma \mu} k_{(\alpha}^1 k_{\beta)}^3
\]

(155)

Gravity does not only couple to itself but also to ordinary matter. For example, we can add to the Lagrangian density \( \mathcal{L} = \sqrt{-g} R \) a massive scalar field with a Lagrangian density

\[
\mathcal{L}_m = \sqrt{-g} \left( \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)
\]

(156)

and a well known propagator:
\[ k = \frac{1}{i} \frac{1}{(2\pi)^4} k^* + m^2 - i\epsilon \]

Using \( \nabla_\mu \phi = \partial_\mu \phi, \nabla^\nu \phi = g^{\mu\nu} \partial_\nu \phi \) and eqs. (23)–(24) with \( g_{\mu\nu} = \eta_{\mu\nu} \), we get, neglecting terms of quadratic or higher order

\[
\mathcal{L}_m = \sqrt{-g} \left( -\frac{1}{2} \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) \\
\approx (1 + \frac{1}{2} h_\alpha^\alpha) \left( -\frac{1}{2} \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi + \frac{1}{2} \partial_\mu \phi h^{\mu\nu} \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) \\
\approx -\frac{1}{2} \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi + \frac{1}{2} \partial_\mu \phi h^{\mu\nu} \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \\
\approx -\frac{1}{2} \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \\
\approx -\frac{1}{2} \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{4} h_\alpha^\alpha (\partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi + m^2 \phi^2) + \frac{1}{2} \partial_\mu \phi h^{\mu\nu} \partial_\nu \phi \\
\approx \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} h_\alpha^\alpha (\partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi + m^2 \phi^2) + \frac{1}{2} \partial_\mu \phi h^{\mu\nu} \partial_\nu \phi \\
\approx \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \\
\approx \frac{1}{2} \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \\
-\frac{1}{4} h_\alpha^\alpha \left( \frac{1}{2} \eta_{\mu\nu} (\partial_\mu \phi \partial^\nu \phi + m^2 \phi^2) - \partial_\mu \phi \partial_\nu \phi \right) (157)
\]

The vertex function between the scalar field and the graviton is then in momentum space:

\[
V(k_1, k_2, k_3)_{\mu\nu} = (2\pi^4) \left( \frac{1}{2} \eta_{\mu\nu} (k_1^2 k_2^2) - \frac{1}{2} \eta_{\mu\nu} m^2 - k_\mu^1 k_\nu^2 \right) (158)
\]

Having the Feynman rules defined, one can use them to describe scattering processes. In his article, deWitt reports the result of the cross section for gravitational scattering of scalar particles in the center of mass frame. deWitt gets, with \( v = p/E \)

\[
\frac{d\sigma}{d\Omega} \propto \left( \frac{(1 + 3v^2)(1 - v^2) + 4v^2(1 + v^2)c^2(\theta/2)}{v^2\sin^2(\theta/2)} + \frac{(1 + 3v^2)(1 - v^2) + 4v^2(1 + v^2)\sin^2(\theta/2)}{v^2\cos^2(\theta/2)} + (3 - v^2)(1 + v^2) + 2v^2\sin^2\theta \right)^2 (159)
\]

From the vertices of eqs. (158) and (154) it also follows that the theory is non-renormalizable. By counting momentum powers one observes that the superficial degree of divergence is

\[
D = -2I + 2V + 4L (160)
\]

34
where $I$ is the number of internal lines, $V$ is the number of vertices, and $L$ the number of loops. Because the graviton propagators are $\propto 1/k^2$, each graviton propagator lowers the degree of divergence by 2. On the other hand, the vertices have terms $\propto k^\mu k^\nu$ which implies that each vertex adds 2 to the degree of divergence. However, the number of loops is

$$L = 1 + I - V$$  \hspace{1cm} (161)

or

$$2(-L + 1) = -2I + 2V$$  \hspace{1cm} (162)

that is

$$D = 2 - 2L + 4L = 2 + 2L = 2(L + 1)$$  \hspace{1cm} (163)

which is independent of the number of external lines and increases with every loop. This estimation should, however, not be taken as a definite result. It may be that because of some symmetry principle, or an invariant, divergences cancel and the theory is still finite. In the next sub section, we will see that in fact this happens at the first loop level for pure gravity. Unfortunately, the divergences appear on the first loop level if matter fields are coupled to gravity. This stimulated the hope that with the addition of the right matter fields, e.g. within a supersymmetric theory, one could get a finite theory of quantum gravity for all orders. Unfortunately, the divergences appear on the two loop level even for pure gravitation.

### 2.2 Loops and divergencies in the framework of covariant quantization of gravity

The investigation of loops in the covariant quantization of gravity began with early computations by Feynman [24] and deWitt [25], but the complete understanding of perturbative quantum gravity at one loop level was only achieved by the articles of ‘t Hooft and Veltman. [27, 21]. By computing the counterterm for all divergent one loop graviton amplitudes, they found that pure gravity is actually one loop finite. In the following, we will sketch their derivation.

In his famous article [25] deWitt introduced the so called background field method. The introduction of the background field method given here follows the excellent articles [31, 32, 33]. A usual generating functional of a non gauge field theory with quantum field $\phi$ is

$$Z(J) = \int D\phi e^{\int d^4x L(\phi) + J\phi}$$  \hspace{1cm} (164)

where $J$ is called source and $J\phi = \int d^4x J\phi$. The Green’s functions are defined by

$$1 \left. \frac{\delta^n Z(J)}{\delta^n J(x_1) \ldots J(x_n)} \right|_{J=0} = \langle 0 | T(\phi(x_1), \ldots \phi(x_n)) | 0 \rangle$$

For a non-gauge theory, it is found that the two point Green’s functions, i.e. the propagators, are simply the inverse of the part of $\mathcal{L}$ that is quadratic in $\phi$. For a gauge theory, the gauge invariance has to be broken by the addition of a gauge fixing term and a ghost Lagrangian, before one can define a propagator as a generalized inverse of the quadratic part of $\mathcal{L}$, as we saw in the sub section above. In general, the Green’s functions obtained from $Z$ contain connected graphs like
as well as disconnected graphs like

Disconnected graphs do not contribute to the s-matrix. The so called energy-functional

$$W(J) = -i \ln(Z(J))$$  \hspace{1cm} (165)

generates only connected graphs. A diagram is called one particle irreducible (1PI), if it can be split into two disjoined pieces by cutting a single internal line. It is simpler to compute the entire collection of Feynman diagramms by calculating first 1PI diagrams and then connecting these graphs together. The 1PI graphs are generated by the so called effective action, it is the Legendre transform

$$\Gamma(\hat{\phi}) = W(J) - J\hat{\phi}$$  \hspace{1cm} (166)

where

$$\hat{\phi} = \frac{\delta W(J)}{\delta J}$$  \hspace{1cm} (167)

In the background field method, instead of using eq. (164) one uses

$$\tilde{Z}(J,\phi_B) = \int D\phi e^{i \int d^4xL(\phi+\phi_B)+J\phi}$$  \hspace{1cm} (168)

where we have split the original field into $\bar{\phi} = \phi_B + \phi$. Only the part $\phi$, which is called the quantum part of the field, appears in the measure of the path integral and is coupled to the source. The part $\phi_B$ is called the background field, for which the classical equations of motion

$$\frac{\delta L(\bar{\phi})}{\delta \bar{\phi}}|_{\bar{\phi} = \phi_B} = 0$$

are assumed to hold. One defines the so called Background energy functional

$$\tilde{W}(J,\phi_B) = -i \ln(\tilde{Z}(J,\phi_B))$$

and with

$$\tilde{\phi} = \frac{\delta \tilde{W}(J,\phi_B)}{\delta J}$$  \hspace{1cm} (169)

we define the background effective action

$$\tilde{\Gamma}(\tilde{\phi},\phi_B) = \tilde{W}(J,\phi_B) - J\tilde{\phi}$$  \hspace{1cm} (170)

By the variable shift $\phi \to \phi - \phi_B$ in eq (168), we observe that

$$\tilde{Z}(J,\phi_B) = Z(J)e^{-J\phi_B}$$  \hspace{1cm} (171)

and

$$\tilde{W}(J,\phi_B) = W(J) - J\phi_B$$  \hspace{1cm} (172)

Using eq. (169), we get

$$\tilde{\phi} = \frac{\delta \tilde{W}(J,\phi_B)}{\delta J} = \frac{\delta W(J)}{\delta J} - \phi_B = \hat{\phi} - \phi_B$$  \hspace{1cm} (173)
With help of eqs. (172) and (173), we get from eq. (170)

\[ \tilde{\Gamma}(\tilde{\phi}, \phi_B) = W(J) - J\phi_B - J\tilde{\phi} = \Gamma(\tilde{\phi}) = \Gamma(\tilde{\phi} + \phi_B) \] (174)

eq. (174) implies that the background effective action \( \tilde{\Gamma}(\tilde{\phi}, \phi_B) \) is equal to the ordinary effective action in the presence of a background field \( \phi_B \). Therefore \( \tilde{\Gamma}(\tilde{\phi}, \phi_B) \) generates the same 1PI Diagrams than the conventional approach. Hence, the background field method therefore produces the same S-Matrix as usual field theory.

As in eq. (122), we define the metric as

\[ g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \] (175)

with the arbitrary background field \( g_{\mu\nu} \) and the quantum field \( h_{\mu\nu} \). Using the expression for the path integral, eq. (149), with this metric and a covariant source term \( J^{\mu\nu} \), we can write the generating functional for the gravitational field as

\[ Z(J^{\mu\nu}) = \int Dh_{\mu\nu} \int \prod_{\alpha} D(\bar{\eta}^{\alpha}) D\eta^{\alpha} e^{iS - \frac{1}{4}\int G_{\alpha\beta}A_{\alpha\beta} + \int d^4x(\bar{\eta}^{\alpha})^* A_{\alpha\beta} \eta^{\alpha} + J^{\mu\nu} h_{\mu\nu}} \] (176)

where \( \tilde{R} \) and \( \tilde{T} \) are constructed from the metric in eq. (122). In his famous second article [28] on quantum gravity, deWitt has shown this action functional is independent of the gauge fixing term, provided that the gauge fixing term is background invariant and that the classical equations of motion hold for the background field. It is for this reason, why the background method can be applied to gauge theories.

We can now expand the Lagrangian as in eq. (131). The part of the Lagrangian that is linear in \( h_{\mu\nu} \) can be written as

\[ L_B = \sqrt{-g}(-\frac{1}{2}h_\alpha R + h_\alpha R_\alpha) = h_{\mu\nu}\sqrt{-g}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) \] (177)

In \( L_B \), \( R \) and \( R^{\mu\nu} \) are constructed with the background field \( g_{\mu\nu} \) for which the classical equations of motion are assumed to hold. For \( g_{\mu\nu} \) these are the Einstein equations in vacuum

\[ R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 0 \] (178)

and therefore eq. (177) implies that \( L_B \) must vanish. The Lagrangian in eq. (131) is invariant with respect to the following gauge transformation

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + (g_{\alpha\nu} + h_{\alpha\nu})\nabla_\mu \eta^\alpha + (g_{\mu\alpha} + h_{\mu\alpha})\nabla_\nu \eta^\alpha + \eta^\alpha \nabla_\alpha h_{\mu\nu} \] (179)

This is a gauge invariance of the quantum field \( h_{\mu\nu} \). Of course the Lagrangian is also a gauge invariant with respect to a gauge transformation of the background field, but for now this invariance will not interest us further, and we use a gauge fixing term that only breaks the gauge invariance with respect to \( h_{\mu\nu} \). ’t Hooft and Veltman have chosen

\[ C_\mu = \sqrt{-g}(\nabla_\mu h_{\nu} - \frac{1}{2}\nabla_\nu h_{\mu})e^{\nu\alpha} \] (180)
where $\mu^{\alpha \lambda \nu} = g^{\mu \nu}$. Subjecting this gauge fixing term to the gauge transformation of eq. (179), ’t Hooft and Veltman find a ghost Lagrangian:

$$L_g = \sqrt{-g} \eta_{\mu}^{\alpha} (g_{\mu \nu} \Box - R_{\mu \nu}) \eta^{\nu}$$ (181)

where terms containing $h_{\mu \nu}$ have been omitted. For the full ghost Lagrangian, see Goroff and Sagnotti [38] on p 215.

As the gauge invariance with respect to the background field is not broken, the entire generating functional in the background field method is gauge invariant with respect to $g_{\mu \nu}$. Therefore, the counterterms must be solely composed of expressions that are invariant with respect to gauge transformations of the $g_{\mu \nu}$. The counter Lagrangian $\Delta L$ must be dimensionless. From the integration $\int dx^4$ at the one loop level, we deduce that $\Delta L$ must have dimension $d^4 x^{-4}$ in order for $\int d^4 x \Delta L$ to be dimensionless. This is achieved by invariant scalars involving 4 derivatives. Hence the counterterm must be a linear combination of invariant quadratic functions of the curvature. It can therefore only be composed of terms like

$$\Delta L = \alpha R^2_{\mu \alpha \beta} + \beta R^2_{\mu \nu} + \gamma R^2$$ (182)

the term $R^2_{\mu \alpha \beta}$ needs not to be considered since one can show that

$$R^2_{\mu \alpha \beta} - 4R^2_{\mu \nu} + R^2$$ (183)

is a total derivative in 4 dimensions, see Appendix B of [27] for a proof. Therefore, we end up with a counter Lagrangian

$$\Delta L = \beta R^2_{\mu \nu} + \gamma R^2$$ (184)

In [27, 21], ’t Hooft and Veltman succeeded in computing the coefficients $\beta, \gamma$. First, they showed that to a Lagrangian

$$\mathcal{L} = \sqrt{-g} (-\partial_\mu \phi^* g^{\mu \nu} \partial_\nu \phi + 2 \phi^*_\mu N^\mu \partial_\mu \phi + \phi^* M \phi)$$ (185)

where $N^\mu$ and $M$ are c-number functions of spacetime, there corresponds a counterterm

$$\mathcal{L} = \frac{\sqrt{-g}}{2\epsilon} \text{tr} \left( \frac{1}{12} Y_{\mu \nu} Y_{\mu \nu} + \frac{1}{2} \left( M - N^\mu N_\mu - \nabla_\mu N^\mu - \frac{1}{6} R \right)^2 + \frac{1}{60} (R_{\mu \nu} R^{\mu \nu} - \frac{1}{3} R^2) \right)$$ (186)

where

$$Y_{\mu \nu} = \nabla_\mu N_\nu - \nabla_\nu N_\mu + [N_\mu, N_\nu]$$ (188)

Then, ’t Hooft and Veltman consider the Lagrangian of gravity coupled to a massless scalar field

$$\mathcal{L} = \sqrt{-g} (\overline{\phi} - \frac{1}{2} \partial_\mu \overline{\phi} g^{\mu \nu} \partial_\nu \overline{\phi})$$ (189)

where $\overline{\phi} = \phi_B + \phi$ with $\phi_B$ as a background field. After expanding this Lagrangian to second order in the quantum fields $h_{\mu \nu}$ and $\phi_B$ and finding the appropriate gauge fixing terms, ’t Hooft
and Veltmann bring their Lagrangian into a similar form as eq. (185). After some arithmetic
manipulation, they get a counterterm

\[ \Delta L = \sqrt{-g} \left( \frac{9}{720} R^2 + \frac{43}{120} R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{2} (\partial_{\mu} \phi_B g^{\mu\nu} \phi_B)^2 \right) \]

(190)

this counterterm still contains contributions from closed loops of \( \phi \) particles. The contribution
of these closed loops can be obtained from eq (186) by setting \( M = N = 0 \) and is

\[ \frac{1}{2} \sqrt{-g} \left( \frac{1}{144} R^2 + \frac{1}{120} (R_{\mu\nu} R^{\mu\nu}) - \frac{1}{3} R^2 \right) \]

(191)

Subtracting this contribution and setting \( \phi_B = 0 \) gives the counterterm for pure gravitation

\[ :L = \sqrt{-g} \left( \frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu} R^{\mu\nu} \right) \]

(192)

The equations of motion for the background field can be inferred from setting \( L \) to zero, similar
as in eq. (177). From these equations, one gets

\[ \nabla_{\mu} \nabla_{\nu} \phi_B = 0, \]

(193)

\[ R_{\mu\nu} = -\frac{1}{2} \nabla_{\mu} \phi_B \nabla_{\nu} \phi_B, \]

(194)

and

\[ R = -\frac{1}{2} \nabla_{\mu} \phi_B \nabla^{\mu} \phi_B. \]

(195)

For pure gravitation, we have \( \phi_B = 0 \) and therefore \( R = R_{\mu\nu} = 0 \) and the counterterm vanishes. 
On the other hand, inserting eqs (193), (194) and (195) into eq. (190), gives

\[ \Delta L = \frac{203 \sqrt{-g}}{80 \epsilon} R^2. \]

(196)

In case of pure gravity, the counterterm vanishes on shell if the equations of motion are fulfilled. 
It therefore can be written as

\[ \Delta L = \frac{1}{\epsilon} F(g_{\mu\nu}) \frac{\delta L}{\delta g_{\mu\nu}} \]

(197)

where \( F \) is a covariant combination of \( g_{\mu\nu} \) such that \( \Delta L \) consists of invariant quadratic functions
of curvature. This actually corresponds to a to a simple field redefinition

\[ L(g_{\mu\nu} + \frac{1}{\epsilon} F(g_{\mu\nu})) = L(g_{\mu\nu}) + \frac{1}{\epsilon} F(g_{\mu\nu}) \frac{\delta L}{\delta g_{\mu\nu}} \]

(198)

Such a redefinition of the quantum fields can not have any observable effect, as it does not lead
to different Feynman diagrams. Therefore, the vanishing of \( \Delta L \) in case of pure gravity implies
that there are no divergences at the one loop level. With the inclusion of matter, the scalar
$R^2$ appears in the counterterm that does not vanish. Moreover, a term like $R^2$ is absent in
the original Lagrangian and therefore, the theory of gravity coupled to a massless scalar has
nonrenormalizable divergences at the one loop level.

One may ask whether the massless scalar is just the wrong matter field. However, using
similar techniques than 't Hooft and Veltman, Deser, Tsao, and Nieuwenhuizen found non
renormalizable divergences at one loop level if the Lagrangian of gravitation was coupled to
Yang Mills [34], or Maxwell Fields[35]. Finally, Deser and Nieuwenhuizen also computed non
renormalizable divergences in one loop amplitudes if the gravitational field is coupled to a
fermion field [36]. The latter is more difficult to compute since the Lagrangian of gravitation
can only be coupled to Dirac fermions if one describes the gravitational field by using tetrads
tetrads.

All this leads to the assumption that perhaps an additional symmetry is needed that cancels
the divergences. One candidate for such a symmetry is supersymmetry, where for each fermion
there exists a supersymmetric partner boson. In 1986, however, Goroff and Sagnotti [37, 38],
using the help of computer algebra, calculated counterterms for the second loop level. They
found nonrenormalizable divergences even in pure gravity.

At two loop order, one has an additional integration and the counterterm therefore must be
composed out of three Riemann tensors or out of two Riemann tensors and two derivatives. This
leaves the combinations $\nabla^\mu R^\mu R$, $R^3$, $\nabla^\mu R_{\alpha\beta} \nabla^\mu R^{\alpha\beta}$, $RR_{\alpha\beta} R^{\alpha\beta}$, $R_{\alpha\gamma} R_{\beta\delta} R^{\alpha\beta\gamma\delta}$, $R_{\alpha\gamma} R_{\beta\delta} R^{\gamma\epsilon\delta\gamma\epsilon}$, $RR_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$, $RR_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$, $RR_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$, the first six terms vanish
on mass shell, due to eq.(194) and (195). This leaves the last two expressions- According to
Goroff and Sagnotti, they are linearly dependent, and so we are left with a two loop counternet
in form of

$$\Delta L = \frac{\alpha}{\epsilon} \sqrt{-g} R_{\gamma \delta} R_{\epsilon \zeta} R_{\alpha \beta}^{\gamma \delta \epsilon \zeta}.$$  (199)

Goroff and Sagnotti found that the one shell diagrams contributing to this invariant are the
following two loop vertex corrections:
In the computation, the one loop counterterms where treated as a field redefinition, in order to prevent problems with terms mixed of background and quantum field that arise in loop computations with the background field method. Furthermore, a combined notation for gravitons and ghosts was used that made it possible to implement the calculation more easily on a computer (the solid circles in the diagrams actually represent these combined internal ghost graviton lines, while the whiggled external lines are the gravitons). The calculation took three days on a VAX 11/780 gave the result

$$\Delta \mathcal{L} = \frac{209}{2880 (4\pi)^4} e \int d^4 x \sqrt{-g} R_{\gamma\delta} R^{\gamma\delta} R^{\epsilon\zeta} R_{\epsilon\zeta}$$

(200)

Unfortunately, the descriptions in the article of Goroff and Sagnotti are not very detailed. More details of a different approach were given by deVen [39], who used a different technique to reproduce the result of Goroff and Sagnotti, However, he also had to resort to a computer implementation.

2.3 Considerations on the non perturbative evaluation of the gravitational path integral

One can also try to evaluate the path integral without using a perturbation expansion of the action. In the following we make some comments on attempts to evaluate the path integral non-perturbatively by the methods of Euclidean quantum gravity. Usually, the action of the
gravitational field is taken to be \( S = \int d^4x \sqrt{-g} R \). However, in this action a boundary term is omitted that will be crucial for the following. If we vary the action, we get

\[
\delta S = \int d^4x \left( \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g} \right)
\]

\[
= \int d^4x \left( \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} - \frac{1}{2} R \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \right)
\]

\[
= \int d^4x \left( \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} \right)
\]

(201)

From the last term on the right hand side, Einstein’s equations can be derived. The first term yields with

\[
\delta R_{\mu\nu} = \nabla_\lambda \delta \Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta \Gamma^\lambda_{\lambda\mu}
\]

(202)

the result

\[
\int d^4x \sqrt{-g} \nabla_\sigma \left( g^{\mu\nu} \delta \Gamma^\sigma_{\mu\nu} - g^{\mu\nu} \delta \Gamma^\lambda_{\lambda\mu} \right) = \int d^4x \sqrt{-g} \nabla_\sigma \left( g_{\mu\nu} \nabla^\sigma \delta g^{\mu\nu} - \nabla_\lambda \delta g^{\sigma\lambda} \right)
\]

(203)

By Stokes theorem, the volume integral over the divergence can be converted into a surface integral around the boundary \( \partial M \) of the spacetime manifold. Further evaluation of the integrand, see [1], shows that it is exactly equal to \(-2 \sqrt{-\gamma} K\) where \( K \) is the trace of the extrinsic curvature and \( \gamma \) is the three metric on the boundary. Accordingly, we have to add this term to the action for the gravitational field. Our new action now reads

\[
S = \int d^4x \sqrt{-g} R + 2 \int_{\partial M} d^3x \sqrt{-\gamma} K - C
\]

(204)

where \( C \) is a constant that is independent of \( g \). In asymptotically flat space a natural choice for \( C \) is such that \( S = 0 \) for the Minkowski metric \( \eta_{\mu\nu} \), or

\[
C = -2 \int_{\partial M} d^3x \sqrt{-\gamma} K^0
\]

(205)

with \( K^0 \) as the extrinsic curvature at the boundary embedded in flat space. Thereby, the action becomes

\[
S = \int d^4x \sqrt{-g} R + 2 \int_{\partial M} d^3x \sqrt{-\gamma} K - 2 \int_{\partial M} d^3x \sqrt{-\gamma} K^0
\]

(206)

For a quantum theory with a scalar field \( \phi \), the path integral

\[
Z = \int \mathcal{D}x(t) e^{iS(\phi)}
\]

(207)

oscillates and does not converge. However, the integral can be made to converge by a Wick rotation. One replaces \( t \) by \(-i\tau\), which introduces a factor of \(-i\) in the volume integral of the action. The path integral then becomes

\[
Z_{eu} = \int \mathcal{D}\varphi(t) e^{-I(\varphi)}
\]

(208)
where

\[ I(\varphi) = -iS(\varphi) \]

is the Euclidean action. Since \( I(\varphi) \in \mathbb{R} \) and \( I(\varphi) \geq 0 \) for fields that are real on the Euclidean space \((\tau, x^1, x^2, x^3)\), the path integral converges, which makes its evaluation possible.

Unfortunately, this is not the case with the gravitational field. After a Wick rotation, the volume element \( d^4x \sqrt{-g} \) becomes \(-id^4x \sqrt{g}\) and Euclidean action for the gravitational field is:

\[ I = -\int d^4x \sqrt{g} R - 2 \int_{\partial M} d^3x \sqrt{\gamma} (K - K^0) \]

The minus sign comes from the fact that the direction of rotation into the complex plane must be the same as that for the matter fields that might be included into the action.

According to Gibbons, Hawking and Perry, [68], a conformal transformation \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \), with \( \Omega \) as a positive conformal factor that is equal to one on \( \partial M \) changes this action into

\[ I(\Omega^2, g) = -\int d^4x \sqrt{\tilde{g}} \left( \Omega^2 R + 6g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \right) - 2 \int_{\partial M} d^3x \sqrt{\gamma} \Omega^2 (K - K^0) \]

Because of the covariant derivatives of the conformal factor, \( I(\Omega, g) \) can be arbitrarily negative in case a rapidly varying conformal factor is chosen.

The Euclidean path integral can be formulated with a quantity

\[ Y(g) = \int D\Omega e^{-I(\Omega^2, g)} \]

as

\[ Z_{eu} = \int Dg Y(g) \]

where one might as well include the necessary ghost and gauge fixing terms.

The path integral in this formulation contains a summation over all possible conformal factors \( \Omega \). Some of these factors lead to a negative action, which result in a divergent \( Y(g) \). Thereby, even when evaluated without the appeal to a perturbation series in terms of Feynman diagrams, the path integral of gravitation is highly divergent. However, Schoen and Yau [84] were able to prove that at least for asymptotically Euclidean spacetimes with \( R = 0 \), the so called positive action conjecture holds, which states that \( I \geq 0 \), with \( I = 0 \) if \( g \) is flat.

3 Hawking’s articles on the topology of spacetime at Planck-scale

It was Stephen Hawking who gave a more rigorous outline of the topology of spacetime that follows from the quantum gravitational path integral [48]. For understanding this work, we will need some basic definitions of algebraic topology.

3.1 Some definitions from algebraic topology

For what follows, a detailed description of the mathematics involved would lead too far. We therefore redirect the interested reader to the literature. For example the expository article of
Eguchi, Gilkey and Hanson [49] will provide a nice entry. Also the book of Nakahara [50] will cover most of the relevant mathematics, and many of the mathematical definitions and proofs below can be found in similar, and sometimes much more detailed form in Nakahara’s book. Also, the books from R. C. Kirby [77] as well as A. Scorpan on four manifolds [57] were used. Sometimes, the book of Bredon [51] was of help. We confine us here to give a few: definitions on topology.

**Definition 3.1.** Topology definitions

- A topological space $X$ is said to be disconnected if $X = T_1 \cup T_2$, where $T_i \subset X, T_i \neq \emptyset$ are open sets with $T_1 \cap T_2 = \emptyset$. If $X$ is not disconnected, $X$ is called connected.

- We say $T \subset X$ is a connected component of $X$ if $T$ is a maximal connected subset of $X$ (the connected subsets are ordered by inclusion).

- A connected manifold is a connected topological space that is also a manifold.

- Let $M$ be a connected manifold covered by charts $\{U_j\}$. We say that $M$ is orientable if for any overlapping charts $U_i, U_j$ there exists a local coordinate system $\{x^\mu\}$ for $U_i$ and $\{y^\nu\}$ for $U_j$ such that $\det(\partial x^\mu / \partial y^\nu) > 0$

- An $n$-manifold with boundary is a topological space $X$ in which each point $x \in X$ has a neighborhood $U_x$ homeomorphic to either $\mathbb{R}^n$ or to $H^n := \{ (x_1, \ldots x_n) \in \mathbb{R}^n | x_n \geq 0 \}$.

- The set of all points $x \in X$ of an $n$-dimensional manifold with boundary that only have neighborhoods homeomorphic to $H^n$ is called the boundary of $X$ and denoted by $\partial X$. If $\partial X = \emptyset$, then the manifold is called closed.

- A continuous function $f : [0,1] \rightarrow X$ with $f(0) = x \in X$ and $f(1) = y \in X$ is called path.

- If there is a path joining any two points $x,y \in X$ the topological space $X$ is called path connected.

- A homotopy between two continuous functions $f,g : X \rightarrow Y$ is a continuous function $H : X \times [0,1] \rightarrow Y$ where, $\forall x \in X : H(x,0) = f(x)$ and $H(x,1) = g(x)$. Continuous functions $f,g : X \rightarrow Y$ are said homotopic, or $f \sim g$, if and only if there exists a homotopy $H$ between them. This describes an equivalence class of homotopic functions, see, e.g. Nakahara [50], p. 124 for a proof.

- If $f$ and $g$ are continuous maps from $X$ to $Y$ and $K$ is a subset of $X$, then we say that $f$ and $g$ are homotopic relative to $K$ if there exists a homotopy $H : X \times [0,1] \rightarrow Y$ between $f$ and $g$ such that $\forall k \in K, \forall t \in [0,1] : H(k,t) = f(k) = g(k)$.

- A topological space $X$ is simply connected if and only if it is path-connected, and whenever two continuous maps $u : [0,1] \rightarrow X$ and $w : [0,1] \rightarrow X$ with the same starting and ending points $u(0) = w(0)$ and $u(1) = w(1)$, are homotopic relative to $[0,1]$.

- Two topological spaces $X$ and $Y$ are homotopy equivalent if $\exists$ continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f$ is homotopic to the identity map of $X$ and $f \circ g$ homotopic to the identity map of $Y$. 

44
• Let $e_0, e_1, \ldots, e_r$ linearly independent vectors in $\mathbb{R}^m$ where $m \geq r$. Then an $r$-simplex is defined by

$$\sigma_r \equiv (e_0, e_1, \ldots, e_r) \equiv \left\{ x \in \mathbb{R}^m | x = \sum_{i=0}^{r} \lambda_i e_i \land \sum_{i=0}^{r} \lambda_i = 1 \land 0 \leq \lambda_i \leq 1 \right\}$$

(214)

with the $\lambda_i$ called barycentric coordinates.

• Given an $r$ simplex $\sigma_r \equiv (e_0, e_1, \ldots, e_r)$ we define the $r-1$ simplex

$$(e_0, e_1, \hat{e}_i, \ldots, e_r) \equiv (e_0, e_1, e_{i-1}, e_{i+1} \ldots e_r)$$

(215)

which we call the $i$-th $r-1$ face of $\sigma_r$.

• Let $M$ be an $m$ dimensional manifold and $\sigma_r$ an $r$-simplex. We define a smooth map $f : \sigma_r \rightarrow M$ where $f(\sigma_r) = \emptyset$ if $\sigma_r = \emptyset$. We denote the image of $\sigma_r$ by $f(\sigma_r) = s_r$, and call $s_r$ a singular $r$ simplex in $M$.

• If $\{s_{r,i}\}$ is a set of singular $r$-simplexes in $M$, we define a singular $r$-chain in $M$ by a formal sum

$$c_r = \sum_i a_i s_{r,i}$$

(216)

which goes over all non-empty $s_{r,i}$ and where $a_i \in \mathbb{R}$. These singular $r$-chains form a group $C_r(M)$ with multiplication

$$c_r * c_r' = \sum_i (a_i + a_i') s_{r,i}$$

(217)

unit element

$$e = \sum_i 0 s_{r,i}$$

(218)

and inverse

$$c_r^{-1} = -c_r = \sum_i (-a_i) s_{r,i}$$

(219)

• Given two groups $(A, *)$ and $(B, \cdot)$, a group homomorphism is a map $h : A \rightarrow B$ where $\forall u, v \in A : h(u * v) = h(u) \cdot h(v)$.

• If $h$ is a group homomorphism, we define $im(h) = \{x | x \in h(A) \subset B\}$ and $ker(h) = \{x | x \in A : h(x) = e\}$ where $e$ is the identity element of $B$.

• The boundary operator on a singular $r$-chain is a map $\partial_r : C_r(M) \rightarrow C_{r-1}(M)$ which is defined by

$$\partial_r c_r \equiv \sum_i a_i \partial_r s_{r,i}$$

(220)

where

$$\partial_r s_{r,i} = f(\partial_r \sigma_{r,i})$$

(221)

The boundary of a 0 simplex is defined to be zero:

$$\partial_0(e_0) \equiv 0$$

(222)
and we define the boundary of a general r-simplex as

$$\partial_r \sigma_r \equiv \sum_{i=0}^{r} (-1)^i (e_0, e_1, \hat{e}_i \ldots e_r)$$ (223)

Now we have the following theorems:

**Lemma 3.2.** \(\partial_r\) is a group homomorphism

**Proof.** We have

$$\partial_r (c_r * c'_r) = \partial_r \sum_i (a_i + a'_i) s_{r,i} = \sum_i (a_i + a'_i) \partial_r s_{r,i} = c_{r-1} * c_{r-1}$$ (224)

**Lemma 3.3.** If \(f : G_1 \rightarrow G_2\) is a homomorphism, with \(e_i\) as unit element of \(G_i\) and \(x_i^{-1}\) as inverse of \(x_i \in G_i\), then \(\ker(f)\) is a subgroup of \(G_1\) and \(\text{im}(f)\) is a subgroup of \(G_2\)

**Proof.** If \(x, y \in \ker(f) \Rightarrow x * y \in \ker(f)\) (225)

since

$$f(x * y) = f(x) * f(y) = e_2 * e_2 = e_2$$ (226)

similarly, we have \(e_1 \in \ker(f)\), because

$$e_2 * f(a) = f(a) = f(e_1 * a) = f(e_1) * f(a) \Rightarrow e_2 = f(e_1).$$ (227)

If \(x \in \ker(f)\) then we have \(x^{-1} \in \ker(f)\) since

$$f(e_1) = e_2 = f(x^{-1} x) = f(x^{-1}) * f(x) = f(x^{-1}) * e_2.$$ (228)

With \(f(x_1), f(x_2) \in \text{im}(f)\) we have

$$f(x_1) * f(x_2) = f(x_1 * x_2) \in \text{im}(f)$$ (229)

and \(e_2 \in \text{im}(f)\) because \(f(e_1) = e_2\). Furthermore, \(f(x) \in \text{im}(f)\), therefore, \((f(x))^{-1} \in \text{im}(f)\) since

$$e_2 = f(e_1) = f(x * x^{-1}) = f(x) * f(x^{-1}) \Rightarrow (f(x))^{-1} = f(x^{-1}) \in \text{im}(f).$$ (230)

We can use this result in the following definition:

**Definition 3.4.** Definition

- Let \(c \in C_r(M)\). If \(\partial_r c = 0\), then \(c\) is called r-cycle. The set of r-cycles \(Z_r(M) = \ker\partial_r\) is called r-cycle group. If \(\exists d \in C_{r+1}(M) : c = \partial_{r+1} d\) then, \(c\) is called an r-boundary. The set of r-boundaries \(B_r(M) = \text{im}\partial_{r+1}\) is called the r-boundary group.

And we can show an important theorem:
Theorem 3.5. \( B_r(M) \subset Z_r(M) \) and they are both subgroups of \( C_r(M) \)

Proof. We first prove that \( \partial_r(\partial_{r+1}\sigma_{r+1}) = \emptyset \) for an \( r \)-simplex \( \sigma_{r+1} \). This can be done by simply repeated application of the boundary operator:

\[
\partial_r(\partial_{r+1}\sigma_{r+1}) = \sum_{i=0}^{r+1} (-1)^i \partial_r(e_0, e_1, \hat{e}_i \ldots e_{r+1}) = 0
\]

Using our definitions, we get

\[
\partial_r(\partial_{r+1}s_{r+1,i}) = f(\partial_r(\partial_{r+1}\sigma_{r+1,i})) = f(\emptyset) = \emptyset
\]

But

\[
\partial_r(\partial_{r+1}c_{r+1}) = \sum a_i \partial_r(\partial_{r+1}s_{r+1,i})
\]

where the sum goes over all non empty \( s_{r+1,i} \). Therefore, \( \partial_r(\partial_{r+1}c_{r+1}) = 0 \).

If \( c \in B_r(M) \), then \( c = \partial_{r+1}d \) for a \( d \in C_{r+1}(M) \) and

\[
\partial_r c = \partial_r(\partial_{r+1}d) = 0
\]

implies that \( c \in Z_r(M) \) so \( B_r(M) \subset Z_r(M) \). Since the boundary operator is a homomorphism, and because the image and kernel of a homomorphism is a subgroup of the image and inverse image, both \( Z_r(M) \) and \( B_r(M) \) are subgroups of \( C_r \).

These results can now be used to define the singular cohomology group.

Definition 3.6. Singular cohomology group

- Two \( r \)-cycles \( z, z' \in Z_r(M) \) are homologous if \( z - z' \in B_r(M) \) and we say \( z \sim z' \). The homologous \( r \)-cycles define an equivalence class, \([z]\) were \([z] = [z']\).

- We define \( H_r(M) \) as the set of equivalence classes of \( r \)-cycles

\[
H_r(M) = \{ [z] | z \in Z_r(M) \}
\]

and call it a singular homology group.
• We define \( \text{dim}(H_r(M)) = b_r \) as the \( r \)-th Betti number of \( M \).

• The Euler characteristic, or Euler number of an \( n \)-dimensional manifold \( M \) is

\[
\chi = \sum_{i=0}^{n} (-1)^i b_i
\]

where \( b_i \) is the \( i \)-th Betti number.

Heuristically, the \( r \)-th Betti number of a manifold \( M \) is the number of independent closed \( r \)-surfaces that are not boundaries of some \( r + 1 \) surface. This means, for example \( b_0 \) is the number of connected components of \( M \), \( b_1 \) is the number of one-dimensional or "circular" holes in \( M \) and \( b_2 \) is the number of two-dimensional "voids" or "cavities" in \( M \).

We now need some definitions on \( r \)-forms:

**Definition 3.7. \( r \)-forms**

• A differential form \( \omega_r \) of order \( r \) or an \( r \)-form is a totally antisymmetric tensor of type \((0, r)\) at \( p \in M \), with \( M \) as a differentiable manifold. As a \((0, r)\) tensor, \( \omega_r \) is a map \( \omega_r : (V_1, \ldots, V_r) \rightarrow \mathbb{R} \) where \( (V_1, \ldots, V_r) \in T_p(M) \).

• A wedge product of \( r \) one forms is an \( r \)-form, given by the totally antisymmetric tensor product

\[
dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} = \sum_\sigma \text{sign}(\sigma(r)) dx^{\mu_{\sigma(1)}} \otimes dx^{\mu_{\sigma(2)}} \otimes \ldots \otimes dx^{\mu_{\sigma(r)}}
\]

where the sum goes over all permutations \( \sigma \) of \( 1, 2, \ldots, r \) and \( \text{sign}(\sigma(r)) = +1 \) for an even permutation and \( \text{sign}(\sigma(r)) = -1 \) for an odd permutation.

• We denote the vector space of all \( r \) forms at \( p \in M \) by \( \Omega^r_p(M) \) and expand an element \( \omega \in \Omega^r_p(M) \) as

\[
\omega = \frac{1}{r!} \omega_{\mu_1 \mu_2 \ldots \mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r}
\]

where \( \omega_{\mu_1 \mu_2 \ldots \mu_r} \) are totally antisymmetric.

• For a \( p \) form \( \omega \) and an \( r \) form \( \eta \) acting, we define the wedge product acting on vectors \( V_1, \ldots, V_{p+r} \) as follows:

\[
\omega \wedge \eta(V_1, \ldots, V_{p+r}) = \frac{1}{p!} \sum_\sigma \text{sign}(\sigma) \omega(V_{\sigma(1)}, \ldots, V_{\sigma(p)}) \eta(V_{\sigma(p+1)}, \ldots, V_{\sigma(p+r)})
\]

• We now assign an \( r \) form on every point in \( M \) and denote the space of all \( r \) forms on \( M \) as \( \Omega^r(M) \).

• The exterior derivative of an \( r \)-form is a map \( d_r : \Omega^r(M) \rightarrow \Omega^{r+1}(M) \), where

\[
d_r \omega = \frac{1}{r!} \frac{\partial}{\partial x^{\nu}} \omega_{\mu_1 \ldots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r}
\]
Theorem 3.8. We have \( d_{r+1}(d_r \omega_r) = 0 \)

Proof. According to our definitions, 

\[
d_{r+1}(d_r \omega_r) = \frac{1}{(r!)^2} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \omega_{\mu_1,\ldots,\mu_r} dx^\lambda \wedge dx^\nu, \ldots, dx^\mu_r \tag{241}
\]

which is zero because \( \frac{\partial^2}{\partial x^\mu \partial x^\nu} \omega_{\mu_1,\ldots,\mu_r} \) is symmetric with respect to \( \lambda, \nu \) while \( dx^\lambda \wedge dx^\nu \) is anti-symmetric.

Similar as in the case with the singular homology, we can use this result able to state the following theorem:

Theorem 3.9. \( \text{im}(d_r) \subseteq \text{ker}(d_{r+1}) \)

Proof. If \( \omega_r \in \Omega^r(M) \) then \( d_r \omega_r \in \text{im}(d_r) \) and \( d_{r+1}(d_r \omega_r) = 0 \), hence \( d_r \omega \in \ker(d_{r+1}) \) and therefore \( \text{im}(d_r) \subseteq \ker(d_{r+1}) \)

Definition 3.10. DeRahm cohomology group

- An r-form \( \omega_r \in \text{im}(d_{r-1}) \) is called exact, an r-form \( \omega_r \in \ker(d_{r+1}) \) is called closed, i.e. if an r-form is exact there exists an r-1 form \( \psi_{r-1} : \omega_r = d_{r-1} \psi_{r-1} \) and if an r-form is closed, we have \( d_r \omega_r = 0 \)

- The set of closed r-forms is called the r-th cocycle group \( Z^r(M) = \ker(d_{r+1}) \) and the set of exact r-forms is called r-th coboundary group \( B^r(M) = \text{im}(d_{r-1}) \). We say the r-th DeRahm cohomology group \( H^r(M) \) is the set of equivalence classes \( [\omega] \in H^r(M) \), where

\[
\{ \omega' \in Z^r(M) \ | \ \omega' = \omega + d\psi, \psi \in \Omega^{r-1} \} \tag{242}
\]

- for \( [c] \in H_r(M) \) and \( [\omega] \in H^r(M) \), we define

\[
\Lambda_r : H_r(M) \times H^r(M) \rightarrow \mathbb{R}, \Lambda_r([c], [\omega]) = \int_c \omega \tag{243}
\]

- A bilinear form \( f(x,y) \rightarrow \mathbb{R}, x \in V, y \in V^* \) where \( V, V^* \) are some finite dimensional vector spaces, defines a pair of linear maps

\[
f_1 : V \rightarrow V^* : x \mapsto (y \mapsto f(x,y)) \tag{244}
\]

and

\[
f_2 : V^* \rightarrow V, y \mapsto (x \mapsto f(x,y)) \tag{245}
\]

If \( V \) is finite dimensional and \( f_1 \) or \( f_2 \) is an isomorphism, then both \( f_1 \) and \( f_2 \) are isomorphisms and the bilinearform \( f(x,y) \) is called non-degenerate. Clearly, \( f_1 \) can be an isomorphism, if and only if

\[
f(x,y) = 0 \Rightarrow x = 0 \tag{246}
\]

and

\[
f(x,y) = 0 \Rightarrow y = 0 \tag{247}
\]

If \( V \) is finite dimensional, then \( \text{rank}(f_1) = \text{rank}(f_2) \) and if additionally \( \dim(V) = \text{rank}(f_1) \) then \( f_1 \) and \( f_2 \) are isomorphisms and the bilinearform is non-degenerate.
A bilinear form \( f(x, y) \rightarrow \mathbb{R}, x, y \in V \) is symmetric if \( f(x, y) = f(y, x) \) and skew-symmetric if \( f(x, y) = -f(y, x) \).

**Theorem 3.11.** \( \Lambda_r([c], [\omega]) \) is a bilinear form and it does not depend on the representation of \( \omega \) and \( c \)

**Proof.** \( \Lambda_r([c + c'], [\omega]) = \int_c \omega + \int_{c'} \omega \) and

\[
\Lambda_r([c + c'], [\omega + \omega']) = \int_c (\omega + \omega') = \int_c \omega + \int_c \omega' \tag{248}
\]

Furthermore, we have with Stokes theorem \( \int_c d\omega = \int_{\partial c} \omega \) that

\[
\Lambda_r([c + \partial c'], [\omega]) = \int_c \omega + \int_{\partial c'} \omega = \int_c \omega + \int_{c'} d\omega \tag{249}
\]

From \( [\omega] \in H^r(M) \Rightarrow \omega \in Z^r(M) \) and thereby \( \omega \) is closed, and \( d\omega = 0 \), thus \( \int_{c'} d\omega = 0 \) and

\[
\Lambda_r([c + \partial c'], [\omega]) = \int_c \omega. \tag{250}
\]

Similarly

\[
\Lambda_r([c], [\omega + d\psi]) = \int_c \omega + \int_c d\psi \tag{251}
\]

but we have \( d^2 = 0 \), thereby

\[
\Lambda_r([c], [\omega + d\psi]) = \int_c \omega \tag{252}
\]

In 1931, DeRahm proved the following theorem:

**Theorem 3.12.** DeRahm: Let \( M \) be a differentiable manifold. Then \( H_r(M) \) and \( H^r(M) \) are finite dimensional and the map \( \Lambda_r : H_r(M) \times H^r(M) \rightarrow \mathbb{R} \) is bilinear and non degenerate

**Proof.** Shown for example in Bredon \[51\] on p. 286ff. \( \Box \)

Because \( \Lambda(\cdot, [\omega]) \) and \( \Lambda([c], \cdot) \) are isomorphisms, \( H_r(M) \) and \( H^r(M) \) are dual to another, and we have for the Betti numbers

\[
dim(H_r(M)) = \text{rank}(\Lambda([c], \cdot)) = \text{rank}(\Lambda(\cdot, [\omega])) = \dim(H^r(M)) = b_r \tag{253}
\]

Now we need some more definitions on one forms

**Definition 3.13.** Definitions on one forms

- With \( M \) as a differentiable closed m-dimensional manifold, \( r \leq m, [\omega] \in H^r(M) \) and \( [\eta] \in H^{m-r} \) we define a map

\[
\sigma_{r,m} : H^r(M) \times H^{m-r} \rightarrow \mathbb{R}, \sigma_{r,m}([\omega], [\eta]) = \int_M \omega \wedge \eta \tag{254}
\]
• If \( m = 4 \), we call \( \sigma_{2,2} \) the intersection form of the manifold. For non differentiable manifolds, see the more general definitions in [57] on p. 115.

• We say that the parity of the intersection form \( \sigma_{2,2} \) is even if \( \sigma_{2,2}(\omega, \omega) \) is even for any \( \omega \), where \( [\omega] \in H^r(M) \), and \( \sigma_{2,2}(\omega, \omega) \) is odd otherwise.

**Theorem 3.14.** Poincaré Duality: Let \( M \) be a closed manifold. There is an isomorphism between \( H^r(M) \) and \( H^{m-r}(M) \) and for the Betti numbers, we have \( b_r = b_{m-r} \).

**Proof.** (The proof below is valid only for differentiable manifolds, since our definition of the intersection form was only for these manifolds. For non-differentiable manifolds, see [51] p. 352): The form \( \sigma_{r,m}([\omega], [\eta]) \) is bilinear and because of the properties of the wedge product, it is also non-degenerate. Therefore, there is an isomorphism between \( H^r(M) \) and \( H^{m-r}(M) \) and this implies

\[
b_r = \dim(H^r(M)) = \dim(H^{m-r}(M)) = b_{m-r}
\] (255)

We still need to define the Hodge star and the Laplacian of an \( r \)-form

**Definition 3.15.** Hodge star and Laplacian

• A Hodge star \( * \) is a linear map \( *: \Omega^r(M) \to \Omega^{m-r}(M) \) which transforms an \( r \)-form

\[
\omega = \frac{1}{r!} \omega_{\mu_1 \mu_2 \ldots \mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r}
\]

on a differentiable manifold with metric \( g \) into

\[
* \omega_r = \frac{\sqrt{|g|}}{r!(m-r)!} \omega_{\mu_1 \mu_2 \ldots \mu_r} \epsilon^{\mu_1 \mu_2 \ldots \mu_r \nu_{r+1} \ldots \nu_m} : dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_m}
\]

(256)

where \( \epsilon \) is the totally antisymmetric tensor.

• Let \( d : \Omega^{r-1}(M) \to \Omega^r(M) \) be the exterior derivative for \( r-1 \) forms on a Lorentzian manifold \( M \) with dimension \( m \). We define the adjoint \( d^\dagger : \Omega^r(M) \to \Omega^{r-1}(M) \) of \( d \) as

\[
d^\dagger = (-1)^{mr+m} * d *
\]

(258)

and if \( M \) is Riemannian, we have

\[
d^\dagger = (-1)^{mr+m+1} * d *
\]

(259)

• The Laplacian \( \Delta : \Omega^r(M) \to \Omega^r(M) \) is then defined by

\[
\Delta = dd^\dagger + d^\dagger d
\]

(260)

• An \( r \) form \( \omega_r \) is called harmonic, if \( \Delta \omega_r = 0 \) and the space of harmonic \( r \) forms is denoted by \( \text{Har}^r(M) \)
• The inner product of two $r$-forms $\omega$ and $\eta$ is defined by

\[
(\omega, \eta) = \int \omega \wedge * \eta = \frac{1}{r!} \int \omega_{\mu_1, \ldots, \mu_r} \eta^{\mu_1, \ldots, \mu_r} \sqrt{|g|} dx^1 \ldots dx^m
\]  

(261)

which implies that $(\omega, \eta) = (\eta, \omega)$ and $(\omega, \omega) \geq 0$

**Theorem 3.16.** The Hodge star satisfies

\[
* * \omega_r = (-1)^r (m-r)
\]  

(262)

if $M$ is an $m$ dimensional Riemannian manifold and

\[
* * \omega_r = (-1)^{1+\tau(m-r)} \omega_r
\]  

(263)

if $M$ is Lorentzian.

**Proof.** Shown in Nakahara [50], p. 291

We have an important theorem by Hodge:

**Theorem 3.17.** Hodge: Let $M$ be a compact orientable differentiable manifold. Then, there is an isomorphism between $H^r(M)$ and $\text{Harm}^r(M)$

**Proof.** See Nakahara, p. 296 [50]

As a result of this theorem we have

\[
b_r = \text{dim}(H^r(M)) = \text{dim}(\text{Harm}^r(M))
\]  

(264)

for a compact manifold, and if the manifold is 4 dimensional, Poincare duality implies that $b_p = b_{4-p}$ or $b_0 = b_4$ and $b_3 = b_1$. For a compact manifold, a harmonic 0 form, i.e. a scalar function $\omega$ where $d\omega = 0$, is a constant. Thus, for compact 4 manifolds, $\text{dim}(H^0(M)) = 1 = b_0 = b_4$. If $M$ is simply connected, the number of its one dimensional holes is zero, so for simply connected 4 dimensional manifolds $0 = b_1 = b_3$. A more formal proof of this goes by showing that $H_1$ is trivial if $M$ is simply connected, see for example Nakahara [50], p. 241.

We can now define the signature of a manifold:

**Definition 3.18.** Signature of a manifold

• Let $M$ be a closed manifold of dimension $m = 2l$, the form $\sigma_{l,l} : H^l(M) \times H^l \rightarrow \mathbb{R}$ is non degenerate and therefore it has the rank $b_l = \text{dim}(H^l(M))$. Furthermore, $\sigma_{l,l}$ is symmetric, if $l$ is even and $\sigma_{l,l}$ is skew-symmetric if $l$ is odd. If $l$ is even, $\sigma$ has $b_+$ positive and $b_-$ negative real eigenvalues where $b_+ + b_- = b_l$. The signature of $M$ is defined as

\[
\tau(M) \equiv b_+ - b_-
\]  

(265)

if $l$ is odd, then we define $\tau(M) \equiv 0$.

• We say that the form $\sigma_{l,l}$ is definite if $b_l = |\tau|$ and indefinite otherwise.

For a closed simply connected 4 dimensional manifold, this means that $\sigma_{22}$is definite if $|\tau| = \chi - 2$ and indefinite otherwise.
Theorem 3.19. Let $M$ be a closed orientable $4r$ dimensional differentiable Riemannian manifold. We have

$$\text{Harm}^{2r}(M) = \text{Harm}^{2r}_+(M) \otimes \text{Harm}^{2r}_-(M)$$

(266)

where $\text{Harm}^{2r}_+(M)$ is the space of harmonic one forms with negative (positive) eigenvalues of $\ast$. Furthermore,

$$\tau(M) = \dim(\text{Harm}^{2r}_+(M)) - \dim(\text{Harm}^{2r}_-(M))$$

(267)

Proof. For a $2r$ form on a $4r$ dimensional Riemannian manifold, we have $\ast^2 \omega_{2r} = 1 \omega_{2r}$, hence the eigenvalues of $\ast$ are $\pm 1$. We can therefore separate the space of harmonic one forms $\text{Harm}^{2r}(M)$ into two a disjoined subspaces $\text{Harm}^{2r}_\pm(M)$ of harmonic one forms with negative (positive) eigenvalues of $\ast$. So we have

$$\text{Harm}^{2r}(M) = \text{Harm}^{2r}_+(M) \otimes \text{Harm}^{2r}_-(M)$$

(268)

Since $M$ is closed and orientable, we can apply Hodge’s theorem and deRham’s theorem:

$$b_r = \dim(H_r(M)) = \dim(H^r(M)) = \dim(\text{Harm}^r(M))$$

(269)

$$= \dim(\text{Harm}^r_+(M)) + \dim(\text{Harm}^r_-(M))$$

we define $\omega^+_{2r} \in \text{Harm}^{2r}_+(M)$ and $\omega^-_{2r} \in \text{Harm}^{2r}_-(M)$ and we have using the properties of the direct product between one forms:

$$\sigma_{2r,2r}(\omega^+_{2r}, \omega^+_{2r}) = \int_M \omega^+_{2r} \wedge \omega^+_{2r} = \int_M \omega^+_{2r} \wedge \ast \omega^+_{2r} = (\omega^+_{2r}, \omega^+_{2r}) \geq 0$$

(270)

and

$$\sigma_{2r,2r}(\omega^-_{2r}, \omega^-_{2r}) = \int_M \omega^-_{2r} \wedge \omega^-_{2r} = -\int_M \omega^-_{2r} \wedge \ast \omega^-_{2r} = -(\omega^-_{2r}, \omega^-_{2r}) \leq 0$$

(271)

as well as

$$\sigma_{2r,2r}(\omega^+_{2r}, \omega^-_{2r}) = -\int_M \omega^+_{2r} \wedge \ast \omega^-_{2r}$$

$$= -\int_M \omega^-_{2r} \wedge \ast \omega^+_{2r}$$

$$= -\sigma_{2r,2r}(\omega^-_{2r}, \omega^+_{2r})$$

$$= 0$$

(272)

Therefore $\sigma_{2r,2r}$ is block diagonal and we have $b_r = b_+ + b_-$ with $b_\pm = \dim(\text{Harm}^{2r}_\pm(M))$ and

$$\tau(M) = \dim(\text{Harm}^{2r}_+(M)) - \dim(\text{Harm}^{2r}_-)$$

(273)

We have the following general theorems on the topology of 4 manifolds:

Theorem 3.20. Markov[52]: The problem of homotopy equivalence of two $n$-dimensional manifolds $M, N$ is undecidable for $n > 3$. The problem of homeomorphy of two $n$-dimensional manifolds $M, N$ is undecidable for $n > 3$. 53
However, Milnor has been able to show, using earlier work of Whitehead [54, 55], that

**Theorem 3.21.** Two simply connected, closed, orientable 4-manifolds are homotopy equivalent if and only if their intersection forms are isomorphic.

A theorem of Freedman [56] from 1981 shows that the classification of oriented simply connected 4-manifolds up to homeomorphism can be reduced to the classification of intersection forms:

**Theorem 3.22.** Freedman [56]: Given an even (odd) intersection form, there exists up to homeomorphism exactly one (two) simply connected, closed 4 dimensional manifolds represented by that intersection form. If the form is odd, the two 4 dimensional manifolds \( M_1 \) and \( M_2 \) are distinguished by their Kirby Siebenmann invariant \( \kappa(M_i) \) which is equal to 0 or 1, and vanishes if and only if the product manifold \( M_i \times \mathbb{R} \) is differentiable.

One then can invoke Serre’s classification theorem for indefinite intersection forms [58]:

**Theorem 3.23.** Serre: Let \( Q \) and \( Q' \) be two symmetric bilinear unimodular forms. If both \( Q \) and \( Q' \) are indefinite, then they are isomorphic if and only if they have the same rank, signature and parity. [58]

For definite intersection forms, Donaldson [59] showed

**Theorem 3.24.** Donaldson [59]: If a definite form is the intersection form of a smooth simply connected closed 4 manifold \( M \) then it is \( \pm \otimes b_r \). (1).

As a consequence of all these theorems, the topology of smooth simply connected closed oriented 4 manifolds is entirely characterized up to homeomorphism by \( \chi \) and \( \tau \) and the parity of the intersection form.

### 3.2 Hawking’s method to estimate the topologies that contribute dominantly in the gravitational path integral

Even if we are only able to evaluate the gravitational path integral up to one loop order, one can actually use it to derive some estimates how the spacetime at Planck scale looks like. The gravitational action is, unfortunately, not scale invariant. In the following, we will review an argument first made by Wheeler in [69] and then used by Hawking in [48] which shows that fluctuations of the metric at short length scales, even if they can change the entire topology of the spacetime, do not have a large action and thus are not damped in the path integral.

Regge calculus [70] is a method for approximately computing a manifold. With this method, spacetime is decomposed into a 4 simplicial complex, where each 4 simplex is taken to be flat and to be determined by its edge lengths. The angles between the 2 simplices that connect different 4 simplices are such that they could not be connected together in 4 dimensional flat space. (In the following, we will use the action with the correct factor \( \tilde{I} = \frac{1}{16\pi} I \), that we discarded for simplicity in section 2). It was found by Regge [70] that the Euclidean gravitational action is equal to

\[
- \frac{1}{16\pi} \int d^4x \sqrt{g} R = - \frac{1}{8\pi} \sum A_i \delta_i
\]

where \( A_i \) is the area of the i-th 2 simplex and

\[
\delta_i = 2\pi - \sum_k \theta_k
\]

54
is the deficit angle, where $\sum_k \theta_k$ is sum of the angles between the 3 simplices which are connected by the $i$-th 2 simplex. A simplicial complex that is stationary under small variations of the edge length is an approximation to a smooth solution of the Einstein Field equation. In the path integral, one integrates over all metrics, and a manifold described by Regge calculus can then be regarded as a certain metric in the summation of the path integral without any approximation.

If the edge lengths are chosen such that some 4 simplices collapse to simplices of lower dimension, the action is still well defined and finite. For example, let $a, b, c$ be the edges of a triangle, which is a 2 simplex. Then we must have $a + b > c$ and if $a + b = c$ the 2 simplex collapses to a one simplex (which is an edge). We may choose an approximation of the spacetime metric with very small simplicial complexes. We can make some of the simplices collapse to lower dimensions, and then blow up some of the simplices by an arbitrary small amount. Thereby, we can change the topology of the manifold. For example the Euler number or signature may be changed this way, but the action will remain finite by eq. (274). Moreover, if the simplices we made our modification with have a vanishingly small edge length, the action will, by Regge’s formula (274), only change by an infinitesimally small amount.

Without any cutoffs being employed, the gravitational amplitude is a path integral over all possible metrics, including ones with topology changing quantum fluctuations that can be described approximately with simplices of very small edge lengths. Because the fluctuations at small length scales have a small Euclidean action, the formula of the Euclidean path integral

$$ Z = \int \mathcal{D}g_{\mu\nu} e^{-I} $$

then implies the contributions from these quantum fluctuations are not highly damped. Therefore, they should lead to comparatively large contributions to the amplitude.

In the following, we want to give a review of Hawking’s ideas which describe a method for evaluating the topologies that give the dominant contribution in the gravitational path integral. Hawking considers a path integral

$$ Z(\Lambda) = \int \mathcal{D}g_{\mu\nu} e^{-I} $$

that goes over all closed Euclidean metrics with some 4 volume $V$ and an Euclideanized action

$$ I = -\frac{1}{16\pi} \int d^4x (\sqrt{g} R - \sqrt{g} 2\Lambda) $$

where $\Lambda$ is a Lagrange multiplicator. Hawking writes that the finite volume should be considered merely as a normalization device, similar to periodic boundary conditions in field theory. The factor $\Lambda$ is similar to the cosmological constant but its difference is that the Lagrange multiplicator can get very large while the cosmological constant is small.

This partition function can be represented by

$$ Z(\Lambda) = tr(e^{-AV/(8\pi)}) $$

where the trace goes over all “states” of the gravitational field. These “states” are however, not to be confused with quantum mechanical “graviton” states, but they are simply the solutions over which the path integral is integrated. One can regard the sum above as the Laplace transform.
of a function \( N(V) \) where \( N(V)dV \) is the number of states in the gravitational field between \( V \) and \( V + dV \)

\[
Z(\Lambda) = \int_0^\infty dVN(V)e^{-\frac{4\Lambda V}{8\pi}}
\]

(280)

and from the inverse Laplace transform, we get

\[
N(V) = \frac{1}{16\pi^2} \int_{-i\infty}^{i\infty} d\Lambda Z(\Lambda)e^{\frac{4\Lambda V}{8\pi}}
\]

(281)

where the contour of integration is taken to the right of any singularities of \( Z(\Lambda) \) in order to ensure that \( N(V) = 0 \) for \( V \leq 0 \).

One would expect the dominant contributions to the path integral to be close to solutions of Einstein’s field equation

\[
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0
\]

(282)

Contracting with \( g_{\mu\nu} \) yields

\[
R = 4\Lambda
\]

(283)

or

\[
R_{\mu\nu} = \Lambda g_{\mu\nu}
\]

(284)

Inserting \( R = 4\Lambda \) into the action of Eq. (278), and integrating we get for a spacetime that fulfills the Einstein equations

\[
I = -\frac{\Lambda V}{8\pi}
\]

(285)

From \( R = 4\Lambda \) one observes that \( \Lambda \) must have the same dimensions than the curvature scalar so we can set

\[
\Lambda = -8\pi c \sqrt{V}
\]

(286)

where the \( 8\pi \) is for convenience and \( c \) is a constant that will turn out to depend on the topology. From this, we have

\[
V = \frac{8^2\pi^2c^2}{\Lambda^2}
\]

(287)

setting this into the action, and integrating, we get for a spacetime that satisfies Einstein’s equation:

\[
I = -\frac{\Lambda V}{8\pi} = -\frac{8\pi c^2}{\Lambda}
\]

(288)

One can show a so called Atiyah Singer index theorem, see [62]. From a version of it follows the Chern-Gauss-Bonnet theorem [60]. The latter states that the Euler characteristic of a 4 dimensional manifold \( M \) is given by:

\[
\chi(M) = \frac{1}{32\pi^2} \int_M \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta} \wedge R^{\gamma\delta} - \frac{1}{32\pi^2} \int_{\partial M} \epsilon_{\alpha\beta\gamma\delta} (2K^{\alpha\beta} \wedge R^{\gamma\delta} - \frac{4}{3} K^{\alpha\beta} \wedge K^{\gamma} \wedge K^{\delta})
\]

(289)

In the following we assume the manifold to be closed. Then the boundary terms vanish. The integral over \( M \) can be expressed as (see [61]):

\[
\chi(M) = \frac{1}{32\pi^2} \int_M d^4\sqrt{g} \left( C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{2}{3} R^2 \right)
\]

(290)
where
\[ C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - (g_{\mu[\alpha} R_{\beta]\nu} - g_{\nu[\alpha} R_{\beta]\mu}) + \frac{1}{3} R g_{\mu[\alpha} g_{\beta]\nu} \] (291)
is the Weyl tensor.

Inserting eqs. (284) and (283) into the expression for \( \chi(M) \) yields
\[ \chi(M) = \frac{1}{32\pi^2} \int d^4 x \sqrt{g} \left( C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \frac{8}{3} \Lambda^2 \right) \] (292)

Note the typo in Hawking’s original paper [48], where he writes:
\[ \chi(M) = \frac{1}{32\pi^2} \int M d^4 x \sqrt{g} \left( C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \frac{8}{3} \Lambda^2 \right) \] (293)

Similarly, from another version of the Atiyah Singer index theorem [62], it follows that the signature is equal to
\[ \tau(M) = \frac{1}{48\pi^2} \int_M R_{\alpha}^\alpha \wedge R_{\beta}^\beta - \frac{1}{48\pi^2} \int_{\partial M} K_{\alpha}^\alpha \wedge R_{\beta}^\beta - \eta(0) \] (294)

where \( \eta(s) \) is the so called eta function of a certain differential operator on \( \partial M \). Assuming that \( M \) is closed, one can derive (see [61]):
\[ \tau(M) = \frac{1}{48\pi^2} \int_M d^4 x \sqrt{g} C_{\mu\nu\alpha\beta} \ast C^{\mu\nu\alpha\beta} \] (295)

with \( \ast C^{\mu\nu\alpha\beta} \) as the Hodge dual to \( C_{\mu\nu\alpha\beta} \). One has, see [61]
\[ C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} \geq |C_{\mu\nu\alpha\beta} \ast C^{\mu\nu\alpha\beta}| \] (296)

and thereby
\[
2\chi - 3|\tau| = \frac{2}{32\pi^2} \int_M d^4 x \sqrt{g} \left( C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \frac{8}{3} \Lambda^2 \right) \\
- \frac{3}{48\pi^2} \int_M d^4 x \sqrt{g} |C_{\mu\nu\alpha\beta} \ast C^{\mu\nu\alpha\beta}| \\
\geq \frac{2}{32\pi^2} \int_M d^4 x \sqrt{g} \left( C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \frac{8}{3} \Lambda^2 \right) \\
- \frac{3}{48\pi^2} \int_M d^4 x \sqrt{g} |C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}| \\
= \frac{2}{32\pi^2} \int_M d^4 x \sqrt{g} \left( \frac{8}{3} \Lambda^2 \right) \\
= \frac{5}{3} \pi^2 V \left( \frac{8}{3} \Lambda^2 \right) \\
= \frac{32}{3} \pi^2 \varepsilon^2 \] (297)

where in the last line, we have used eq. (280)
The following paragraph lists the signature $\tau$ and Euler number $\chi$ for some solutions of Einstein’s field equations. The $S^4$ space with a metric, see [44] p. 35

$$ds^2 = (1 - \frac{2}{3} \Lambda r^2) dt + (1 - \frac{2}{3} \Lambda r^2)^{-1} dr^2 + r^2 d\Omega^2$$

(298) has $\chi = 2$ and $\tau = 0$, the $CP^2$ space with its metric, see [63]

$$ds^2 = \frac{\rho^2}{\rho^2 + x^2} \left( \delta_{\mu\nu} - \frac{x_{\mu} x_{\nu} + \eta_{\mu\sigma} n_{\nu}^\lambda x^\sigma x^\lambda}{\rho^2 + x^2} \right) dx^\mu dx^\nu$$

(299) where $\rho$ is some length scale and

$$\eta_{\mu\sigma} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(300) has $\chi = 3$ and $\tau = -1$, see [64]. The space $S^2 \times S^2$ has a metric, see [44] p. 244

$$ds^2 = (1 - \Lambda r^2) dt + \frac{dr^2}{1 - \Lambda r^2} + \frac{1}{\Lambda} d\Omega^2$$

(301) and $\chi = 4, \tau = 0$. Kummer’s quartic surface $K^3$, see [73] for a physical description of this metric, has $\chi = 24$ and $\tau = 16$ and the Schwarzschild solution has $\chi = 2$ and $\tau = 0$.

According to Hawking [44], p. 59, the Euclidean action of eq. (278)

$$I = -\frac{1}{16\pi} \int (R - 2\Lambda) \sqrt{g} dx^4$$

(302) attains a minimum value of

$$I_{min} = -\frac{3\pi}{\Lambda}$$

(303) on $S^4$. By using eqs. (280), (287), and (288), we get for a spacetime that satisfies Einstein’s equation

$$I_{min} = -\frac{3\pi}{\Lambda} = -\frac{8\pi c^2}{\Lambda} = c\sqrt{V} = c\sqrt{\frac{8\pi^2 c^2}{\Lambda^2}}$$

(304) therefore, the constant $c$ has a lower bound of

$$c \geq -\sqrt{\frac{3}{8}}$$

(305) From equation (292), one sees that for large $\chi$ either $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ must be large, and / or $\Lambda^2$ must be large. By eq. (280), a large $\Lambda^2$ implies a large $c^2$. Since by eq. (305), $c$ is bounded from below, a large $\Lambda^2$ implies a positive $c$.

If $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ is large in eq. (292), we get a converging effect on geodesics. We can not build our spacetime from a microscopic model, where all geodesics will develop conjugate points after short distances, since this would conflict with the existence of macroscopic geodesics, whose existence is confirmed by measurements. In order to prevent the Weyl curvature from

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converging, a considerably large negative $\Lambda$ constant has to be put in, and due to eq. (286) this would imply a large constant $c$ because $c$ is bounded from below by eq. (305).

From the inequality of eq. (297)

$$2\chi - 3|\tau| \geq \frac{32}{3}c^2$$  \hspace{1cm} (306)

one would then expect a considerably small $|\tau|$ for large $\chi$. Since $|\tau|$ can be considered to be small for large $\chi$ we can assume that then

$$c \approx d\sqrt{\chi}$$  \hspace{1cm} (307)

where

$$d = \frac{\sqrt{3}}{4}$$  \hspace{1cm} (308)

To compute the $\chi$ for the manifolds that give the dominant contribution in the path integral, Hawking employs the so called zeta function renormalization of the gravitational path integral. The following short introduction is a review of [75] and [44] p. 45. For a scalar field with an Euclidean action

$$S = \frac{1}{2} \int d^4x \sqrt{g} (\phi^{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi + V\phi^2)$$  \hspace{1cm} (309)

zeta function renormalization of a path integral like $Z = \int \mathcal{D}\phi e^{-S}$ goes as follows: After integration by parts, the action becomes

$$S = \frac{1}{2} \int d^4x (\phi \nabla_{\nu} (\sqrt{g} g^{\mu\nu} \nabla_{\mu} \phi) + \sqrt{g} V\phi^2)$$

$$= \frac{1}{2} \int d^4x (\phi F\phi)$$  \hspace{1cm} (310)

with

$$F = -\Box + V$$  \hspace{1cm} (311)

The quantity $F$ is a self adjoint operator which has an eigenvalue problem

$$F\phi_n = \lambda_n \phi_n$$  \hspace{1cm} (312)

Because $F$ is self adjoint, the eigenfunctions form an orthonormal base

$$\int d^4x \sqrt{g} \phi_n \phi_m = \delta_{nm}$$  \hspace{1cm} (313)

and can be expanded as

$$\phi = \sum_{n=0}^{\infty} c_n \phi_n$$  \hspace{1cm} (314)

where

$$c_n = \int d^4x \sqrt{g} \phi_n$$  \hspace{1cm} (315)
Putting the expansion for $\phi$ into the action yields

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{g} \sum_{m,n} c_m c_n \lambda_m \phi_m \phi_n = \frac{1}{2} \sum_n c_n^2 \lambda_n$$  \hspace{1cm} (316)

Once an orthonormal base of eigenfunctions is chosen, the coefficients $c_n$ characterize the space of the functions $\phi_n$ over which the path integral is performed. Given that $\mathcal{D}\phi$ must be covariant and that $c_n$ are coordinate independent, one makes the guess

$$\mathcal{D}\phi = \prod_n f(c_n) dc_n$$  \hspace{1cm} (317)

The simplest choice for $f$ would be a constant and comparison with the measure of the path integral for flat space suggests

$$\mathcal{D}\phi = \prod_{n=0}^{\infty} \frac{c_n}{\sqrt{2\pi}}$$  \hspace{1cm} (318)

Then, the Euclidean path integral $Z = \int \mathcal{D}\phi e^{-\mathcal{S}}$ becomes

$$Z = \int \prod_{n=0}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2} \lambda_n c_n^2} = \left( \prod_{n=0}^{\infty} \lambda_n \right)^{-1/2}$$  \hspace{1cm} (319)

By using the characteristic polynomial, one can show for a finite dimensional matrix $A$ that the product of its eigenvalues $\lambda_n$ is equal to its determinant $\det(A)$. In our case, we have a differential operator which corresponds to an infinitely dimensional matrix. For this, the product of the eigenvalues is infinite, and one must find some way to regularize it. For this reason, we define a generalized zeta function

$$\zeta(s) = \sum_{n=0}^{\infty} \left( \frac{1}{\lambda_n} \right)^s$$  \hspace{1cm} (320)

It will converge for $\text{Re}(s) > 2$ and can be analytically extended to a meromorphic function of $s$ with poles only at $s = 1$ and $s = 2$. The gradient of $\zeta(s)$ is formally equal to

$$\zeta'(s) = \frac{d}{ds} \sum_{n=0}^{\infty} e^{-s \ln(\lambda_n)} = -\sum_{n=0}^{\infty} e^{-s \ln(\lambda_n)} \ln(\lambda_n)$$  \hspace{1cm} (321)

and we get

$$\zeta'(0) = -\sum_{n=0}^{\infty} \ln(\lambda_n) = -\ln \left( \prod_{n=0}^{\infty} \lambda_n \right)$$  \hspace{1cm} (322)

or

$$\ln(Z) = -\frac{1}{2} \ln \left( \prod_{n=0}^{\infty} \lambda_n \right) = \frac{1}{2} \zeta'(0)$$  \hspace{1cm} (323)

note that this computations are formal, because when applying mathematical rigor, the sums $\sum_n \ln(\lambda_n)$ can not be handled as if they were finite, as we did above. By analytic continuation of the zeta function, this method should remove the problematic divergences of the path integral, at least when the theory considered is renormalizable.
For computation of the zeta function, one employs the notion of the heat kernel of an operator \( F(x, y, \tau) \). The heat kernel is a solution of the generalized heat equation

\[
\frac{d}{d\tau} K(x, y, \tau) + F K(x, y, \tau) = 0
\]  

(324)

where \( x, y \) are points in spacetime, \( \tau \) is an additional parameter, and \( F \) is an operator acting on the last argument of \( K(x, y, \tau) \). The heat kernel can be expressed as

\[
K(x, y, \tau) = e^{-F}\tag{325}
\]

or in terms of eigenvalues of \( F \)

\[
K(x, y, \tau) = \sum_{n=0}^{\infty} e^{-\lambda_n \tau} \varphi_n(x)\varphi_n(y)\tag{326}
\]

since

\[
\frac{d}{d\tau} \sum_{n=0}^{\infty} e^{-\lambda_n \tau} \varphi_n(x)\varphi_n(y) = \sum_{n=0}^{\infty} (-\lambda_n)e^{-\lambda_n \tau} \varphi_n(x)\varphi_n(y) = -FK(x, y, \tau)\tag{327}
\]

One defines the “trace” of the heat kernel as

\[
tr(K(\tau)) = \int d^4x \sqrt{g} F(x, x, \tau) = \int d^4x \sqrt{g} \sum_{n=0}^{\infty} e^{-\lambda_n \tau} \varphi_n(x)\varphi_n(x) = \sum_{n=0}^{\infty} e^{-\lambda_n \tau}\tag{328}
\]

where we have used in the last line that

\[
\int d^4x \sqrt{g} \varphi_n(x)\varphi_n(x) = \delta_{nn} = 1\tag{329}
\]

The generalized zeta function is related to the trace of the heat kernel by a Mellin transformation

\[
\zeta(s) = \sum_{n=0}^{\infty} \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} tr(K(\tau)). \tag{330}
\]

One can determine \( K(x, y, \tau) \) by solving the heat equation (324), then compute \( tr(K(\tau)) \) and from this one gets by eq. (330) the generalized zeta function. For an operator \( \Box + \xi R \) on a four dimensional compact manifold, deWitt [76] computed an expansion

\[
tr(K) = \sum B_n \tau^{n-2}\tag{331}
\]

where

\[
B_n = \int d^4x b_n \sqrt{g}, \tag{332}
\]

and

\[
b_0 = (4\pi)^{-2}, \tag{333}
\]

\[
b_1 = (4\pi)^{-2}(1 - \frac{1}{6} - \xi)R, \tag{334}
\]
and
\[ b_2 = \frac{1}{2880\pi^2} (R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - R^{\mu\nu} R_{\alpha\beta} + 30(1 - 6\xi)^2 R^2 + (6 - 30\xi)\Box R) \] (335)

In their remarkable article [68], Gibbons, Hawking and Perry used this technique to threat the gravitational path integral itself. One may express the metric with a classical background as \( g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \) and then expand the Euclidean action perturbatively as in section 2.1:
\[ I(g_{\mu\nu}) = I(g_{\mu\nu}) + I(h_{\mu\nu}) + I(h_{\mu\nu}) + \text{higher order terms} \] (336)

where \( I(h_{\mu\nu}) \) is linear and \( I(h_{\mu\nu}) \) is quadratic in the quantum field. As in Section 2.2, we have 
\[ I(h_{\mu\nu}) = 0 \] by the equations of motion, and so, omitting ghost and gauge fixing terms, the Euclidean path integral in the background field method is given by
\[ Z_{eu} = e^{-I(g_{\mu\nu})} \int \mathcal{D}h_{\mu\nu} e^{-I(h_{\mu\nu})} \] (337)

Then one must consider the addition of gauge fixing and ghost terms in order to make the one loop path integral unitary. Gibbons, Hawking and Perry find that one can express the terms for \( I(h_{\mu\nu}) \) by determinants of certain operators \( F, G, C \) that have positive eigenvalues. Gibbons, Hawking and Perry find for the amplitude
\[ \ln(Z) = -I(g_{\mu\nu}) - \frac{1}{2} \ln \left( \det \left( \frac{1}{2} \pi^{-1} \mu^{-2} (-F + G) \right) \right) + \ln \left( \det \left( \frac{1}{2} \pi^{-1} \mu^{-2} C \right) \right) \] (338)

where \( \mu \) is some normalization factor, \( C \) is the operator for the ghosts and \( -F + G \) is an operator for the Euclidean action and the gauge fixing.

For an operator whose eigenvalues are \( k^{-1}\lambda_n \), one has
\[ \zeta(s) = k^s \zeta(s) \] (339)

or
\[ \zeta'(0) = \ln(k)\zeta(0) + \zeta'(0) \] (340)

Therefore, the one loop gravitational amplitude becomes with the zeta functions expressing the eigenvalues of \( F, G, \) and \( C \):
\[ \ln(Z) = -I(g_{\mu\nu}) + \frac{1}{2}\zeta_F'(0) + \frac{1}{2}\zeta_G'(0) - \zeta_C'(0) + \frac{1}{2} \ln(2\pi\mu^2)(\zeta_F(0) + \zeta_G(0) - 2\zeta_C(0)) \] (341)

From the expansion of the heat kernel, Gibbons, Perry, and Hawking then find the astonishing result that
\[ \zeta_F(0) + \zeta_G(0) - 2\zeta_C(0) = \int d^4x\sqrt{g} \left( \frac{53}{720\pi^2} C_{abcd} C^{abcd} + \frac{763}{540\pi^2} \Lambda^2 \right) \]
\[ = \frac{106}{45} \chi + \frac{1168}{15} \zeta^2 \]
\[ \equiv \gamma \] (342)

where eqs. (292) and (307) have been used.
If we introduce a scale factor into the metric so that
\[ \tilde{g}_{ab} = k g_{ab} \]  
we have to consider that the background action transforms under a change
\[ \tilde{g}_{ab} = k g_{ab} \]  
into
\[ I(\tilde{g}_{ab}) = k I(g_{ab}), \]  
and that the eigenvalues of the operators \( F, G, C \) will get multiplied by \( k^{-1} \). So we get a new amplitude of the form
\[ \ln(Z) = \ln(Z) + (1 - k) I(g_{\mu\nu}) + \frac{1}{2} \gamma \ln(k) \]  
Compared to a solution of Einstein’s equation with
\[ R_{\mu\nu} = g_{\mu\nu} \]  
eq (284)
\[ R_{\mu\nu} = \Lambda g_{\mu\nu} \]  
eq (348)
formally looks like a rescaling with a conformal factor \( k = \Lambda \) and from eq. (346), we can conclude that the one loop amplitude behaves as
\[ Z \propto (\frac{\Lambda}{\Lambda_0})^{-\gamma} \]  
where \( \Lambda_0 \) is some normalization factor related to \( \mu \). In eq. (280) we had an amplitude
\[ Z(\Lambda) = \int \mathcal{D}g_{\mu\nu} e^{-I} \]  
and we saw in eq. (283) that for a spacetime which fulfills Einstein’s field equations, we have
\[ I = -\frac{1}{16\pi} \int (R - 2\Lambda) \sqrt{g} dx^4 = -\frac{\Lambda V}{8\pi} = -\frac{8\pi c^2}{\Lambda} \]  
Using eq. (349), and eq. (288), Hawking approximates the action with a quantum and background part as
\[ Z(\Lambda) \propto (\frac{\Lambda}{\Lambda_0})^{-\gamma} e^{\frac{a\chi^2}{\Lambda}} = (\frac{\Lambda}{\Lambda_0})^{-\gamma} e^{\frac{a\chi^2}{\Lambda}} \]  
By eq. (288), the action is minimized for large \( c \) and from eq. (297), it follows that large \( c \) implies \( |\tau| \) is small. Vanishing \( |\tau| \) implies \( \frac{dV}{d\chi} \geq c^2 \) by eq. (297) and we can write in eq (312):
\[ \gamma \approx a\chi \]  
where \( a > 106/45 \). Thereby the amplitude gets the simple form
\[ Z(\Lambda) \propto (\frac{\Lambda}{\Lambda_0})^{-a\chi} e^{\frac{a\chi^2}{\Lambda}} \]  
63
In eq. (281), we considered a formula for the number of states between the volume elements \( V \) and \( V + dV \)

\[
N(V) = \frac{1}{16\pi^2} \int_{i\infty}^{i\infty} d\Lambda Z(\Lambda) e^{\Lambda V} \tag{355}
\]

In order to estimate the topologies that give the most dominant contribution in the path integral, a saddle point approximation with the amplitude of eq. (354) is employed and yields

\[
N(V) \approx \left( \frac{\Lambda}{\Lambda_0} \right)^{-a\chi} e^{\frac{64\pi^2\chi^2 + \Lambda^2 V}{2\Lambda}} \tag{356}
\]

Setting \( \frac{dN}{d\Lambda} = 0 \) gives

\[
e^\Lambda \frac{64\pi^2\chi^2 + \Lambda^2 V}{\Lambda^2} \left( \frac{\Lambda}{\Lambda_0} \right)^{-a\chi} \left( -64\pi\chi d^2 - 8\pi\Lambda a\chi + \Lambda^2 V \right) = 0 \tag{357}
\]

and solving this for \( \Lambda \) results in

\[
\Lambda_s = \frac{4 \left( \pi a\chi \pm \sqrt{(a\pi\chi)^2 + 4\pi^2 V \chi d^2} \right)}{V} \tag{358}
\]

where, because the contour integral in eq. (281) should pass to the right of the singularity at \( \Lambda_0 \), one has to take the positive sign of the square root. We can get the dominant topologies in the path integral from

\[
\frac{dN(V)}{d\chi} \bigg|_{\Lambda = \Lambda_s} = 0 \tag{359}
\]

which yields after simplification

\[
e^\Lambda \frac{64\pi^2\chi^2 + \Lambda^2 V}{\Lambda^2} \left( \frac{\Lambda}{\Lambda_0} \right)^{-a\chi} \left( a\Lambda_s \ln \left( \frac{\Lambda_s}{\Lambda_0} \right) - 8\pi d^2 \right) = 0 \tag{360}
\]

or

\[
a\ln \left( \frac{\Lambda_s}{\Lambda_0} \right) \Lambda_s - 8\pi d^2 = 0 \tag{361}
\]

If \( \Lambda_0 \geq 1 \), this will be satisfied by

\[
\Lambda_s \approx \Lambda_0 \tag{362}
\]

and if \( \Lambda_0 < 1 \), we get

\[
\Lambda_s \approx \Lambda_0^{a/(8\pi d^2)} \tag{363}
\]

Setting \( \Lambda_s = \Lambda_0 \) into eq. (358), one arrives, after solving for \( \chi \), at

\[
\frac{\Lambda_0^2}{8\pi (8\pi d^2 + \Lambda_0 a)} V = \chi \tag{364}
\]

or

\[
\chi \propto hV
\]

where \( h = \frac{\Lambda_0^2}{8\pi (8\pi d^2 + \Lambda_0 a)} \) is some constant depending on the cutoff \( \Lambda_0 \) that is likely to be set somewhere near the length scales of the Planck scale. A similar expression follows with \( \Lambda_s \approx \Lambda_0^{a/(8\pi d^2)} \). Finally, Hawking concludes:
This supports the picture of spacetime foam because it says that the dominant contribution to the number of states comes from metrics with one gravitational instanton per unit Planck volume $h^{-1}$.

4 Canonical quantization of general relativity

4.1 The Wheeler deWitt equation

In section 3, we have seen that the gravitational path integral is dominated by metrics which describe virtual gravitational instantons. One therefore could believe that the divergence of the two loop amplitude is due to a general failure of perturbation theory in general relativity. For example, Hawking writes in [44]:

"Attempts to quantize gravity ignoring the topological possibilities and simply drawing Feynman diagrams around flat space have not been very successful. It seems to me that the fault lies not with the pure gravity or supergravity theories themselves but with the uncritical application of perturbation theory to them. In classical relativity we have found that perturbation theory has only limited range of validity. One can not describe a black hole as a perturbation around flat space. Yet this is what writing down a string of Feynman diagrams amounts to."

Therefore, we will describe the standard non-perturbative quantization framework for gravity in this section. The canonical quantization of gravity provides an excellent tool especially for the quantization of closed spacetimes, e.g. the closed Friedmann Robertson Walker universe.

We consider the quantization of systems where the Lagrangian $L(q, \dot{q})$ with $q$ as canonical coordinate, is singular. By this, it is meant that that the canonical momentum

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}}$$

(365)

can not be solved for $\dot{q}$. Gravity is such a theory and the Hamiltonian description of those theories were investigated at first by Dirac [13], who presented a method for converting a gauge field theory with singular Lagrangian into a form with a Hamiltonian. In 1967, deWitt used this model to derive a quantum mechanical equation for relativistic spacetimes [14]. In this section, we will shortly review some aspects of this work.

We begin with the assumption that spacetime is globally hyperbolic. Then, one can find a time function $t$ such that each surface $t = \text{const}$ is a Cauchy surface $\Sigma$ and there exists a corresponding time flow vector field $t^\mu \nabla_\mu t = 1$. The metric tensor $g_{\mu\nu}$ induces a spatial metric $\gamma_{ij}$ on $\Sigma$ and one can decompose the four metric $g_{\mu\nu}$ as follows:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

(366)

This metric corresponds to a line element

$$ds^2 = -N^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

(367)

$$= -N^2 dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$
The function $N$ is called lapse function and it is defined by

$$N = \frac{1}{n^\mu \nabla_\mu t}$$

(368)

where $n^\mu$ is the unit normal to $\Sigma_t$, is called lapse function. The three dimensional vector $\beta_k$ is the component of $t^\mu$ tangential to $\Sigma_t$ and is called shift vector. If one imagines spacetime as being foliated by a family of hypersurfaces with $t = const$, $N dt$ is then the proper time lapse between the upper and lower hypersurfaces, and the shift vector gives the correspondence between two points in the hypersurfaces. The point $(x^i + dx^i, \beta^i dt)$ in the lower hypersurface corresponds to $(x^i + dx^i, t + dt)$ in the upper hypersurface. The spatial indices are raised and lowered using the 3 metric $\gamma_{ij}$ and its inverse, with $\gamma_{ik} \gamma^{kj} = \delta^i_j$, $\beta^i = \gamma^{ij} \beta_j = \text{det}(\gamma_{ij})$. The Lagrangian of gravity has the form

$$L = \frac{1}{16\pi} \int d^3x \sqrt{-g} R$$

(369)

In the following sections 3 and 4, we neglect the factor $1/16\pi$ for simplicity. This Lagrangian can, up to a total derivative be expressed as, see [1] p. 464:

$$L = \int d^3x \sqrt{-g} R = \int d^3x N \sqrt{\gamma} \left( K_{ij} K^{ij} - K^2 + (^3 R) \right)$$

(370)

where

$$K_{ij} = \frac{1}{2} N^{-1} (D_j \beta_i + D_i \beta_j - \partial_t \gamma_{ij})$$

(371)

is the extrinsic curvature of the hypersurface $x^0 = const$ where $D_i$ denotes covariant derivation with respect to the i-th direction based on the three metric $\gamma_{ij}$ and $(^3 R)$ is the curvature scalar with respect to $\gamma_{ij}$.

One can define conjugate momenta for $N, \beta_i, \gamma_{ij}$

$$\pi = \frac{\delta L}{\delta \partial_t N} = 0,$$

(372)

and

$$\pi^i = \frac{\delta L}{\delta \partial_i \beta_i} = 0,$$

(373)

and with help of

$$\frac{\delta K_{ij}}{\delta \partial_t \gamma_{kl}} = -\frac{\delta_{ik} \delta_{jl}}{2N},$$

(374)

and

$$K = \gamma^{ij} K_{ij}$$

(375)
we can derive

\[
\pi^{kl} = \frac{\delta L}{\delta \partial_t \gamma_{kl}} = N \sqrt{\gamma} \left( 2K_{ij} \frac{\delta K_{ij}}{\delta \partial_t \gamma_{kl}} - 2K \frac{\delta K}{\delta \partial_t \gamma_{kl}} \right) = N \sqrt{\gamma} \left( \frac{-2K_{kl} + 2K_{kl}^i K}{2N} \right) = -\sqrt{\gamma} (K^{kl} - \gamma^{kl} K) = \sqrt{\gamma} \left( \gamma^{kl} K - K^{kl} \right),
\]

(376)

The Hamiltonian is then

\[
H = \int d^3x (\pi \partial_t N + \pi^i \partial_t \gamma_i + \pi^{ij} \partial_t \gamma_{ij}) - L
\]

\[
= \int d^3x (\pi^i \partial_t \gamma_i) - L
\]

\[
= \int d^3x \left( \sqrt{\gamma} (\gamma^{ij} K - K^{ij}) \partial_t \gamma_{ij} - L \right)
\]

\[
= \int d^3x \left( -2N \sqrt{\gamma} (\gamma^{ij} K - K^{ij}) \frac{1}{2N} (D_j \beta_i + D_i \beta_j - \partial_t \gamma_{ij}) + \sqrt{\gamma} (K^{ij} K - K^{ij}) (D_j \beta_i + D_i \beta_j) \right) - L
\]

\[
= \int d^3x \left( -2N \sqrt{\gamma} (\gamma^{ij} K - K^{ij}) \frac{1}{2N} (D_j \beta_i + D_i \beta_j - \partial_t \gamma_{ij}) + \pi^{ij} (D_j \beta_i + D_i \beta_j) + N \sqrt{\gamma} \left( K^2 - K_{ij} K^{ij} - (3) R \right) \right)
\]

\[
= \int d^3x \left( -2N \sqrt{\gamma} (\gamma^{ij} K - K^{ij}) K_{ij} + N \sqrt{\gamma} \left( K^2 - K_{ij} K^{ij} - (3) R \right) + \pi^{ij} (D_j \beta_i + D_i \beta_j) \right)
\]

\[
= \int d^3x \left( -2N \sqrt{\gamma} \gamma^{ij} K K_{ij} + 2N \sqrt{\gamma} K^{ij} K_{ij} + N \sqrt{\gamma} \left( K^2 - K_{ij} K^{ij} - (3) R \right) + 2\pi^{ij} D_j \beta_i \right)
\]

\[
= \int d^3x \left( -2N \sqrt{\gamma} K^2 + 2N \sqrt{\gamma} K^{ij} K_{ij} + N \sqrt{\gamma} \left( K^2 - K_{ij} K^{ij} - (3) R \right) + 2\pi^{ij} D_j \beta_i \right)
\]

\[
= \int d^3x \left( N \sqrt{\gamma} (K_{ij} K^{ij} - K^2 - (3) R) - 2\beta_i D_j \left( \gamma^{-1/2} \pi^{ij} \right) + 2D_i (\gamma^{-1/2} \beta_j \pi^{ij}) \right)
\]

\[
= \int d^3x \left( N \sqrt{\gamma} (K_{ij} K^{ij} - K^2 - (3) R) - 2\beta_i D_j \left( \gamma^{-1/2} \pi^{ij} \right) + 2D_i (\gamma^{-1/2} \beta_j \pi^{ij}) \right)
\]

(377)

The term \(2D_i (\gamma^{-1/2} \beta_j \pi^{ij})\) only contributes a boundary term to \(H\), which, for finite spacetimes can be neglected after the integration. It therefore will be dropped and we arrive at

\[
H = \int d^3x \left( N \sqrt{\gamma} (K_{ij} K^{ij} - K^2 - (3) R) - \beta_i 2D_j \left( \gamma^{-1/2} \pi^{ij} \right) \right) \int d^3x (N \mathcal{H}_\phi + \beta_i \chi^i)
\]

(378)
with
\[ \chi^i = 2D_j \left( \gamma^{-1/2} \pi^{ij} \right) \] (379)
and
\[ \mathcal{H}_G = \sqrt{\gamma} (K_{ij} K^{ij} - K^2 - R) \] (380)
When writing this, one should emphasize that the Lagrangian \( L \) and the Hamiltonian \( H \) were derived both by omitting boundary terms. These terms do not contribute anything to \( H \) in case of finite worlds. For asymptotically flat worlds, one must add, see [14], or [1], p. 469, a contribution:
\[ E_\infty = \int_\Sigma N \sqrt{\gamma} (\gamma_{ik,j} - \gamma_{ij,k}) \] (381)
to the Hamiltonian.
Since \( \pi = 0 \) we have \( \partial_t \pi = 0 \). Therefore, the Poisson bracket yields
\[ \{ \pi, H \} = \partial_t \pi = 0 = \frac{\partial H}{\partial \mathcal{N}} = \mathcal{H}_g \] (382)
Similarly, we have \( \pi^i = 0 \), or \( \partial_t \pi^i = 0 \), which implies
\[ \{ \pi^i, H \} = \partial_t \pi^i = 0 = \frac{\partial H}{\partial \beta^i} = \chi^i = 0 \] (383)
Eq. (383) is associated with spatial diffeomorphism invariance and therefore called diffeomorphism constraint.
Using
\[ \pi^{kl} \gamma_{kl} = \sqrt{\gamma} (\delta^i_j K - K^{kl} \gamma_{kl}) = \sqrt{\gamma} (\delta^i_j K - K) = \sqrt{\gamma} 2K \] (384)
yields
\[ K = \frac{1}{2} \gamma^{-1/2} \pi^{kl} \gamma_{kl}, \] (385)
and we can simplify \( \mathcal{H}_G \) even further:
\[
\mathcal{H}_G = \sqrt{\gamma} K_{ij} K^{ij} - \sqrt{\gamma} K^2 - \sqrt{\gamma} (3) R
\]
\[ = \sqrt{\gamma} (K_{ij} - \gamma_{ij} K) K_{ij} + \pi^{ij} K \gamma_{ij} - \pi^{ij} K \gamma_{ij} - \sqrt{\gamma} (3) R
\]
\[ = -\pi^{ij} (K_{ij} - \gamma_{ij}) - \pi^{ij} K \gamma_{ij} - \sqrt{\gamma} (3) R
\]
\[ = \gamma^{-1/2} \pi^{ij} \pi_{ij} - \pi^{ij} K \gamma_{ij} - \sqrt{\gamma} (3) R
\]
\[ = \gamma^{-1/2} \pi^{ij} \pi_{ij} - \frac{1}{2} \sqrt{\gamma} (3) R
\]
\[ = \mathcal{G}_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{\gamma} (3) R \] (386)
where
\[ \mathcal{G}_{ijkl} = \frac{1}{2} \gamma^{-1/2} (\gamma_{ik} \gamma_{jl} + \gamma_{il} \gamma_{jk} - \gamma_{ij} \gamma_{kl}) \] (387)
Since \( \gamma_{ij} \) and \( \pi^{ij} \) are canonical coordinates. Therefore, we have the following Poisson bracket:
\[ \{ \gamma_{ij}(x), \pi^{kl}(x') \} = \delta_{(i}^k \delta_{j)}^l \delta(x, x') \] (388)
In the quantum theory, this becomes:

\[ [\hat{\gamma}_{ij}(x), \hat{\pi}^{kl}(x')] = i\delta^k_i \delta^l_j \delta(x, x') \]  

(389)

where \( \hat{\gamma}_{ij} \) and \( \hat{\pi}^{ij} \) are now operators acting on a state functional \( \Psi \) that depends on the three metric \( \gamma_{ij} \) for which we will use the symbolic notation \( \Psi(\gamma) \). The relation (389) is fulfilled if

\[ \hat{\gamma}_{ij} \Psi(\gamma) = \gamma_{ij} \Psi(\gamma) \]  

(390)

and

\[ \hat{\pi}^{ij} \Psi(\gamma) = \frac{1}{i \delta\gamma_{ij}} \Psi(\gamma) \]  

(391)

The Hamiltonian constraint (382) then becomes:

\[ \left( G_{ijkl} \frac{\delta}{\delta\gamma_{ij}} \frac{\delta}{\delta\gamma_{kl}} + \sqrt{\gamma^{(3)}} R \right) \Psi(\gamma) = 0 \]  

(392)

This is the Wheeler-deWitt equation, which describes a Schroedinger like equation for the universe. And the diffeomorphism constraint becomes

\[ 2D_j \left( \gamma^{-1/2} \frac{1}{i \delta\gamma_{ij}} \Psi(\gamma) \right) = 0. \]  

(393)

The Wheeler deWitt equation (392) can be approximately solved with a semiclassical WKB-like ansatz

\[ \Psi = C(\gamma)e^{iS(\gamma)} \]  

(394)

where it is assumed that

\[ \left| \frac{\delta C(\gamma)}{\delta\gamma_{ij}} \right| << \left| C(\gamma) \frac{\delta S(\gamma)}{\delta\gamma_{ij}} \right| \]  

(395)

deWitt [14] gets from the Hamiltonian constraint of eq. (392) an equation for the phase

\[ G_{ijkl} \frac{\delta S(\gamma)}{\delta\gamma_{ij}} \frac{\delta S(\gamma)}{\delta\gamma_{kl}} = \sqrt{\gamma^{(3)}} R \]  

(396)

and one for the amplitude

\[ \frac{\delta}{\delta\gamma_{ij}} \left( C^2(\gamma) \frac{\delta S(\gamma)}{\delta\gamma_{ij}} \right) = 0 \]  

(397)

Additionally, deWitt found from the diffeomorphism

\[ D_j \left( \frac{\delta S(\gamma)}{\delta\gamma_{ij}} \right) = 0 \]  

(398)

and

\[ D_j \left( \frac{\delta C(\gamma)}{\delta\gamma_{ij}} \right) = 0 \]  

(399)

From a time integration of equation (371),

\[ - \frac{\partial\gamma_{ij}}{\partial x^b} = 2NK_{ij} - D_j \beta_i - D_i \beta_j \]  

(400)
one can determine the four geometry of spacetime. By setting
\[ \pi^{ij} = \frac{\delta S(\gamma)}{\delta \gamma_{ij}} \] (401)
and noting that
\[ \pi^{ij} = -\sqrt{\gamma}(\Gamma^{ij} - \gamma^{ij} K) \] (402)
deWitt gets the equation
\[ \frac{\partial \gamma_{ij}}{\partial x^0} = 2NG_{ijkl} \frac{\delta S(\gamma)}{\delta \gamma_{kl}} - D_j \beta_i - D_i \beta_j \] (403)
Differentiating this equation by \( x^0 \), deWitt derived a set of differential equations that together with eqs. (396), (398) and (399) turned out to be equivalent to the classical Einstein equations.

Hence, the ansatz of eq. (394) is indeed similar to a WKB approximation.

The Hamiltonian generates time translations but it vanishes in general relativity. To get a dynamical theory, deWitt proposed to use wave packets in form of superpositions like
\[ \Psi = Ce^{-iS} + Ce^{iS} \] (404)
In his article, deWitt used such an ansatz to describe the properties of the quantized Friedmann Robertson Walker universe [14].

4.2 Problems of canonical quantum gravity, occurrence of inconsistencies and infinities at low distance physics.

Unfortunately, until now, no exact solutions of the Wheeler deWitt equation have been found. In his article [14], deWitt also noted that the quantization leads to a severe inconsistency even without considering the problems with the definition of a scalar product. Contracting all indices in eq. (388) and setting \( x' = x \) yields
\[ \left[ \hat{\gamma}_{ij}(x), \hat{\pi}^{ij}(x) \right] = 6i\delta(x, x) \] (405)
And therefore, we have
\[ \left[ 6i\hbar \delta(x, x), i \int \chi_{k'} \delta \zeta^{k'} d^3x' \right] = 0. \] (406)
where denotes \( \delta \zeta^{k} \) an infinitesimal displacement. On the other hand, deWitt computed the commutators (the notation \( ,k \) means partial differentiation with respect to \( x^k \) )
\[ \left[ \hat{\gamma}_{ij}, i \int \chi_{k'} \delta \zeta^{k'} d^3x' \right] = -\hat{\gamma}_{ij,k} \delta \zeta^k - \hat{\gamma}_{k,j} \delta \zeta^k + \hat{\gamma}_{ik} \delta \zeta^k \] (407)
and
\[ \left[ \hat{\pi}^{ij}, i \int \chi_{k'} \delta \zeta^{k'} d^3x' \right] = -\left( \hat{\pi}^{ij} \delta \zeta^k \right) + \hat{\pi}^{kj} \delta \zeta^k + \hat{\pi}^{ik} \delta \zeta^k \] (408)
Using
\[ [A - B, C] = [A, C] - [B, C] \] (409)
and

\[ [AB, C] = A[B, C] + [A, C]B \] (410)

we can evaluate the commutator in eq. (406) as

\[
\left[ \tilde{\gamma}_{ij}, \hat{\pi}^{ij} \right] = \left( \hat{\gamma}_{ij} \hat{\pi}^{ij} - \hat{\pi}^{ij} \hat{\gamma}_{ij} \right), i \int \chi_k \delta \zeta^k d^3 x'
\]

\[
= \left[ \hat{\gamma}_{ij} \hat{\pi}^{ij}, i \int \chi_k \delta \zeta^k d^3 x' \right] - \left[ \hat{\pi}^{ij} \hat{\gamma}_{ij}, i \int \chi_k \delta \zeta^k d^3 x' \right]
\]

\[
= \tilde{\gamma}_{ij} \left[ \hat{\pi}^{ij}, i \int \chi_k \delta \zeta^k d^3 x' \right] + \left[ \tilde{\gamma}_{ij}, i \int \chi_k \delta \zeta^k d^3 x' \right] \hat{\pi}^{ij}
\]

\[
- \hat{\pi}^{ij} \left[ \tilde{\gamma}_{ij}, i \int \chi_k \delta \zeta^k d^3 x' \right] - \left[ \hat{\pi}^{ij}, i \int \chi_k \delta \zeta^k d^3 x' \right] \tilde{\gamma}_{ij}
\]

\[
= \tilde{\gamma}_{ij} \left( -\hat{\pi}^{ij} \delta \zeta^k \right) \delta \zeta^k_{,k} + \hat{\gamma}_{ij} \hat{\pi}^{ij} \delta \zeta^i_{,k} + \hat{\gamma}_{ij} \hat{\pi}^{ij} \delta \zeta^j_{,k}
\]

\[
+ (\hat{\pi}^{ij} \delta \zeta^k), k \tilde{\gamma}_{ij} - \hat{\pi}^{ij} \delta \zeta^i_{,k} \tilde{\gamma}_{ij} - \hat{\pi}^{ij} \delta \zeta^j_{,k} \tilde{\gamma}_{ij}
\]

\[
= \left( \hat{\pi}^{ij} \delta \zeta^k \right) \delta \zeta^k_{,k} + \left( \hat{\gamma}_{ij} \hat{\pi}^{ij} - \hat{\gamma}_{ij} \hat{\pi}^{ij} \right) \delta \zeta^i_{,k} + \left( \hat{\gamma}_{ij} \hat{\pi}^{ij} - \hat{\gamma}_{ij} \hat{\pi}^{ij} \right) \delta \zeta^j_{,k}
\]

\[
+ (\hat{\pi}^{ij} \delta \zeta^k), k \tilde{\gamma}_{ij} - \hat{\pi}^{ij} \delta \zeta^i_{,k} \tilde{\gamma}_{ij} - \hat{\pi}^{ij} \delta \zeta^j_{,k} \tilde{\gamma}_{ij}
\]

simplifying further, we get

\[
= -6i \delta (x, x) \delta \zeta^k \] (411)

All in all, we have:

\[
\left[ \tilde{\gamma}_{ij} (x), \hat{\pi}^{ij} (x) \right] = 6i \delta (x, x) \int \chi_k \delta \zeta^k d^3 x' \] (413)

\[
= 0
\]

\[
= -6i \delta (x, x) \delta \zeta^k \]

which is a contradiction. The delta function is a distribution that, according to deWitt, “may, without inconsistency, be taught as a limit of a sequence of successively narrower twin peaked
functions, all of which are smooth, have unit integral and vanish at point \( x' = x \) in the valley between the peaks

\[
\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \left( f_{\epsilon}(x - \sqrt{\epsilon}) + f_{\epsilon}(x + \sqrt{\epsilon}) - \frac{2f_{\epsilon}(x)}{1 + \epsilon} \right)
\]

where

\[
f_{\epsilon}(x) = \frac{\epsilon}{x^2 + \epsilon^2}
\]

In his article [14], deWitt notes on p. 1121: “In an infinite world, passage to \( \epsilon \to 0 \) would correspond to the usual cutoff going to infinity in momentum space”. On p. 1120, deWitt writes that the then appearing inconsistency from eq. (413) “bears on problems of interpreting divergences”. Apparently, the canonical version of quantum gravity becomes inconsistent at high energies. The article by Goroff and Sagnotti [37, 38] that demonstrated inconsistencies of covariant quantum gravity at high momentum was published in the year 1985. It seems that deWitt arrived at a similar conclusion 18 years earlier.

### 4.3 A short review of Loop quantum gravity and its problems

One attempt to regularize the Hamiltonian is the so-called loop quantum gravity. In the following, we give a short review of these methods. We can, however, not give all details on the rather involved techniques. We confine us here to simply stating the main results of this regularization procedure. The text below in this section can be viewed as a summary of [17] and sect 4.3 and 6 of [18], also [15] was of some help. The interested reader is referred to the excellent introductions [17] [15] [10] [18].

First, one writes the Hamiltonian in so-called Ashtekar variables. Using an orthonormal basis \( e^i_a(x) \) where \( i, a \in 1, 2, 3 \) one can define

\[
E^i_a(x) = |\text{det}(e^i_a)| e^i_a(x)
\]

and a connection

\[
A^i_a(x) = \Gamma^i_a(x) + \gamma K^i_a(x),
\]

where

\[
\Gamma^i_a(x) = -\frac{1}{2} \omega_{ajk} \epsilon^{ijk},
\]

is the Levi-Civita Connection with \( \epsilon^{ijk} \) as anti symmetric tensor and the spin connection

\[
\omega^i_a = \Gamma^i_{kj} e^k_a
\]

and

\[
K^i_a = K_{ab}(x)e^b(x)
\]

is the extrinsic curvature on the three dimensional spacetime manifold. The parameter \( \gamma \) in Eq. (416) is called Barbero Immirizi parameter. \( E^i_a \) and \( A^i_a \) are canonically conjugate variables and the Hamiltonian constraint of eq. (382) expressed in Ashtekar variables, becomes after some calculation, see [17]:

\[
\mathcal{H} = \frac{E^m_a E^n_b}{\sqrt{|\text{det}(E^i_d)|}} \left( \epsilon^{abc} F_{mnc} - \frac{1}{2} (1 + \gamma^2) K^a_{[m} K^b_{n]} \right)
\]
where
\[
F_{mnc} = \partial_m A_{nc} - \partial_n A_{mc} + \epsilon_{cvw} A_{mv} A_{nw}
\]
\[
= -\frac{1}{2} \epsilon_{cvw} R_{mnvw} + \gamma(D_m K_{nc} - D_n K_{mc}) + \gamma^2 \epsilon_{cvw} K^u m K^v n
\]
is the field strength of \( A \). This could be simplified by Thiemann, see [20, 17] to an expression for the Hamiltonian operator
\[
H_G = \hat{d} x N \epsilon_{mnp} \text{tr} \left( F_{mn} \{ A_p, V \} - 1 + \gamma^2 \{ A_m, K \} \{ A_n, K \} \{ A_p, V \} \right)
\]
\[
= H_1 + H_2
\]
where
\[
H_1 = \int d^3 x N(x) \epsilon^{mnp} tr \left( F_{mn} \{ A_p, V \} \right)
\]
and \( N(x) \) is the lapse function, \( A_p = \tau_i A^i_p \), and \( \tau_i = i \sigma_i / 2 \) with \( \sigma_i \) being Pauli matrices,
\[
V = \int \sqrt{1 \frac{1}{3!} \epsilon_{abc} \epsilon^{mnp} E^a m E^b n E^c p}
\]
is the volume, and
\[
K = \int d^3 x K^i_a E^a_i
\]
is the trace of the extrinsic curvature.

Loop quantum gravity considers quantum states on a Hilbert space of so-called spin networks. The Hamiltonian gets converted to an operator on such spin network states, which allows a proper regularization. We call a function \( \lambda : [0,1] \rightarrow \Sigma, s \mapsto \{ \lambda^a(s) \} \) a link or edge. Then, \( h_\lambda[A](s) \in SU(2) \) is called a holonomy along \( \lambda \) corresponding to \( A_a = A^i_a \tau_i \) if it fulfills
\[
h_\lambda[A](0) = 1
\]
and
\[
\frac{d}{ds} h_\lambda[A](s) - A_a(\lambda(s)) \frac{d\lambda^a(s)}{ds} h_\lambda[A](s)
\]
The formal solution of this differential equation is
\[
U[A, \lambda] = 1 + \int_0^1 ds A(\lambda(s)) + \int_0^1 ds \int_0^1 dt A(\lambda(t)) A(\lambda(s)) + \ldots
\]
The holonomies transform in matrix valued \( SU(2) \) representations \( \rho_{j_\lambda} \) for arbitrary spins \( j_\lambda = \frac{1}{2}, 1, \frac{3}{2} \). We will, following Nicolai [17], denote them by \( (\rho_{j_\lambda}, h_\lambda[A])_{\alpha\beta} \) where \( \alpha, \beta \) are indices of the \( SU(2) \) representation.

A spin network is a graph \( \Gamma \) of finitely many vertices \( v_i \) that are connected finitely many edges \( \lambda_i \), where each edge is associated with a holonomy. With a function \( \psi_n : [SU(2)]^n \rightarrow \mathbb{C} \), we can define a cylindrical function
\[
\Psi_{\Gamma \psi}[A] = \psi \left( (\rho_{j_{\lambda_1}, h_{\lambda_1}[A])_{\alpha_1\beta_1}, (\rho_{j_{\lambda_2}, h_{\lambda_2}[A])_{\alpha_2\beta_2}, \ldots, (\rho_{j_{\lambda_n}, h_{\lambda_n}[A])_{\alpha_n\beta_n} \right).
\]

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When going to Ashtekar variables, one can derive from eq. (416) a so called Gauss constraint:
\[ \partial_mE^m_a + \epsilon_{abc}A^b_aE^c_m = 0 \]
This constraint is satisfied for SU(2) invariant cylindrical functions. These are constructed from holonomies whose spins \( j \) obey the Clebsch-Gordan rules for pairs of spins of the holonomies that are connected at each vertex. The SU(2) invariant cylindrical function is then defined by contracting the SU(2) indices of the holonomies at each vertex with Clebsch-Gordan coefficients.

For example, for a graph with three edges such that one vertex connects all of these edges, the SU(2) gauge invariant spin state is
\[ \Psi_{\Gamma,(J,C)}[A] = (\rho_{j_1}, h_{\lambda_1}[A])_{\alpha_1\beta_1} (\rho_{j_2}, h_{\lambda_2}[A])_{\alpha_2\beta_2} (\rho_{j_3}, h_{\lambda_3}[A])_{\alpha_3\beta_3} C^{j_1, j_2, j_3}_{\beta_1 \beta_2 \beta_3} \]
where \( C^{j_1, j_2, j_3}_{\beta_1 \beta_2 \beta_3} \) are the Clebsch-Gordan coefficients of the representations connected to the edges \( \lambda_1, \ldots, \lambda_3 \). For graphs where a vertex connects more than three links, different choices for the state functions are possible, see [17].

Given a graph \( \Gamma \) and two gauge invariant cylindrical functions on this graph, one can then define a norm
\[ \langle \Psi_{\Gamma,(J,C)}[A] | \Psi_{\Gamma',(J',C')}[A'] \rangle = \int \Pi_{\lambda_n \in \Gamma} dh_{\lambda_n} \Psi_{\Gamma,(J,C)}[A] \Psi_{\Gamma'(J',C')}[A'] \]
where \( dh_{\lambda_n} \) is the SU(2) Haar measure. The scalar product for states with different graphs is defined to be zero. From this norm, one can construct a Hilbert space \( \mathcal{H}_G \) of gauge invariant states, if one considers only those states which have finite norm.

The diffeomorphism constraint of canonical quantum gravity is then implemented with a subspace of \( \mathcal{H}_G \). Following Nicolai, we define for a state \( |\Psi_{\Gamma,(J,C)}[A]\rangle \in \mathcal{H}_G \) the sum
\[ \eta(\Psi_{\Gamma,(J,C)}[A]) = \sum_{\phi \in Diff \setminus \Gamma} \Psi_{\Gamma,(J,C)}[A \circ \phi] \]
where \( Diff \setminus \Gamma \) denotes the set of diffeomorphisms on the three dimensional spacetime manifold that do not leave \( \Gamma \) invariant. The sum
\[ \eta(\Psi_{\Gamma',(J,C')}[A]) = \sum_{\phi \in Diff \setminus \Gamma'} \langle \Psi_{\Gamma',(J,C')}[A \circ \phi] | \Psi_{\Gamma,(J,C)}[A] \rangle \]
consists of only finite terms, since the scalar product will be zero if the graph \( \Gamma' \circ \phi \) is different from the graph \( \Gamma \). If the graphs do not differ, then \( \phi \) can only change the orientation or order of the vertices, but since there are finite vertices in each graph, and the scalar product is finite for \( |\Psi_{\Gamma,(J,C)}[A]\rangle \in \mathcal{H}_G \), the sum must be finite. Since \( \phi \) is in the set of diffeomorphisms there are only contributions if \( \Gamma \) and \( \Gamma' \) are diffeomorphic. By
\[ \langle \eta(\Psi_{\Gamma',(J,C')}[A]) | \eta(\Psi_{\Gamma,(J,C)}[A]) \rangle \equiv \langle \eta(\Psi_{\Gamma,(J,C)}[A]) | \Psi_{\Gamma,(J,C)}[A] \rangle \]
we then get a finite norm for the diffeomorphism invariant states from which we can construct \( \mathcal{H}_{Diff} \subset \mathcal{H}_G \).

With the area element
\[ dF_a = \epsilon_{mnp}E^m_a dx^n dx^p \]
the flux-vector

\[ F^a_S(E) = \int_S dF^a \]

for a surface \( S \) in the three dimensional manifold is the conjugate variable to a holonomy \( h_\lambda[A] \) and one can compute the Poisson bracket as

\[ \{ (\rho_{j_1} h_\lambda[A])_{\alpha\beta}, F^a_S(E) \} = \iota(\lambda,S) \gamma \left( \rho_{j_1} h_{\lambda_1}[A] \right) \tau^a \left( \rho_{j_2} h_{\lambda_2}[A] \right)_{\alpha\beta} \]

(436)

where \( \lambda_1 \) and \( \lambda_2 \) are the parts of the curve that lie on different sides of the surface \( S \) and the intersection number \( \iota(\lambda,S) \) is defined by:

\[ \iota(\lambda,S) = \int_\lambda dx^m \int_S dy^n dy^p \epsilon_{mnp} \delta^3(x,y) \in \{ \pm 1, 0 \} \]

(437)

In the quantum theory, the Poisson brackets become commutators

\[ \left[ (\rho_{j_1} h_\lambda[A])_{\alpha\beta}, \tilde{F}^a_S(E) \right] = i\hbar^2 \iota(\lambda,S) \gamma \left( \rho_{j_1} h_{\lambda_1}[A] \right) \tau^a \left( \rho_{j_2} h_{\lambda_2}[A] \right)_{\alpha\beta} \]

(438)

where \( \hbar = 1.62 \times 10^{-33} \text{ cm} \) is the Planck length.

If an holonomy operator \( (\rho_{j_1} h_\lambda[A]) \) acts on a spin network state, the holonomy just adds some edge to the graph that is

\[ (\rho_{j_1} h_\lambda[A])_{\alpha\beta} \Psi_{\Gamma,\{J,C\}}[A] = (\rho_{j_1} h_\lambda[A])_{\alpha\beta} \Psi_{\Gamma,\{J,C\}}[A] \]

(439)

The action of a flux on a spin network state is zero if the surface \( S \) does not intersect the Graph of \( |\Psi_{\Gamma,\{J,C\}}[A]\rangle \). If \( S \) intersects the graph on an edge, a spin matrix \( \tau_\alpha \) is inserted and the spin network changes as:

\[ \tilde{F}^a_S(E) \left( \ldots, (\rho_{j_1} h_\lambda[A])_{\alpha\beta}, \ldots \right) = 8\pi i\hbar^2 \iota(\lambda,S) \gamma \left( \ldots, (\rho_{j_1} h_{\lambda_1}[A])_{\alpha\beta} (\tau_\iota)_{\alpha\beta} (\rho_{j_2} h_{\lambda_2}[A])_{\alpha\beta}, \ldots \right) \]

(440)

where \( \lambda = \lambda_1 \cup \lambda_2 \) and \( (\tau_\iota)_{\alpha\beta} \) is in the \( SU(2) \) generator representation corresponding to \( j \). If \( S \) intersects \( \lambda \) at a vertex, the result depends on the position and orientation of the surface with respect to the edge, as well as on the choice for the Clebsch Gordan coefficients, see [17]. Unfortunately, this makes the computations in LQG highly dependent on the vertex triangulation that is chosen for \( \Sigma \).

Using the formula \( A(S) = \int_S \sqrt{dF^a dF^a} \) for the area of a surface \( S \), we can define an area operator by replacing the integral by a sum made of \( N \) infinitesimal surfaces \( S_i \) with \( S = \bigcup_i S_i \).

\[ \hat{A}(S) = \lim_{N \to \infty} \sum_{i=1}^N \sqrt{\tilde{F}^a_{S_i}(E) \tilde{F}^a_{S_i}(E)} \]

(441)

Using a coordinate system \( \Gamma \), where each of the small areas are intersected by only one edge, we find by eq. (441) and

\[ (\tau_\iota)_{\alpha\beta} (\tau_\iota)_{\alpha\beta} = -j_\lambda(j_\lambda + 1) \]

(442)
that

\[ \hat{A}(S)\Psi_{\Gamma,(J,C)} = 8\pi^2 \gamma \sum_{p=1}^{N(\Gamma)} \sqrt{j_p (j_p + 1)} \Psi_{\Gamma,(J,C)}, \]  

(433)

where \( N(\Gamma) \) is the number of links in \( \Gamma \). In eq. (433), no functions are present that could become infinite. This together with fact that the area has a discrete spectrum in the spin network states implies that in loop quantum gravity, one can not run into problems with infinities as with the Wheeler deWitt equation in section 2.2. However, the result for the operators depend highly on the coordinate system of the graph that \( \Psi_{\Gamma,(J,C)} \) is associated with, e.g it depends on the number of links \( N(\Gamma) \) in the graph, and the position of the nodes as well as the spin representations.

Like the area operator, one can define the volume operator of a three dimensional volume \( \Omega \) as a finite Riemann sum of \( N \) small volume elements \( \Omega_k \) with \( \Omega = \bigcup^N_k \Omega_k \): \n
\[ V(\Omega) = \lim_{N \to \infty} \frac{1}{\sqrt{\text{Vol}(\Omega)}} \sum_{i=1}^{N} \sqrt{\frac{1}{3} \varepsilon_{abc} \hat{F}_{S_1^a}^3(E) \hat{F}_{S_2^a}^3(E) \hat{F}_{S_3^a}^3(E)} \]  

(444)

where \( S_i^a : S_1^a \cup S_2^a \cup S_3^a = \Omega \) are three non-coincident surfaces. When computing this operator, one must, however, choose the coordinate system appropriately, since otherwise the operator would either diverge, or vanish, see \[17 \] p. 29.

Now we fix a point \( x \) on the three manifold and a tangent vector \( u \) at \( x \) and consider a path \( \lambda_{\epsilon,x,u} \) of length \( \epsilon \) starting at \( x \) tangent to \( u \). Then the holonomy for this configuration can be expanded as:

\[ h_{\lambda_{\epsilon,x,u}}[A] = 1 + \epsilon u^a A_a(x) + O(\epsilon^2) \]  

(445)

see \[15 \] \[16 \] \[20 \]. Similarly, we fix two tangent vectors \( u, v \) at \( x \) and consider the triangular loop \( x, u, v \) denoted by \( \lambda_{\epsilon,x,u,v} \) with one vertex at \( x \), and two edges each of length \( \epsilon \) and tangent to \( u, v \), then

\[ h_{\lambda_{\epsilon,x,u,v}}[F] = 1 + \frac{1}{2} \epsilon^2 u^a v^b F_{ab}(x) + O(\epsilon^3) \]  

(446)

We can now define tetrahedra of edge length \( \epsilon \) with tangents \( u_1, u_2, u_3 \) at each point \( x \) whose triple product \( u_1 (u_2 \times u_3) = 1 \). Thiemann noted that in \[20 \] that the \( H_1 \) part of the Hamiltonian in eq. (422) can then be written as

\[ H_1 = \int d^3x \mathcal{N} \mathcal{H}_G = \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int d^3x \mathcal{N}(x) \epsilon^{ijk} \text{tr} \left( h_{\lambda_{\epsilon,x,u}} h_{\lambda_{\epsilon,x,v}} h_{\lambda_{\epsilon,x,k}} \left( h_{\lambda_{\epsilon,x,u}}^{-1}, V \right) \right) \]  

(447)

the integral can be replaced by a Riemannian sum of small three dimensional regions \( \Omega_m \) of volume \( \epsilon^3 \) with \( x_m \) being an arbitrary point in \( \Omega_m \)

\[ H_1 = \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \sum_m \epsilon^3 \mathcal{N}(x_m) \epsilon^{ijk} \text{tr} \left( h_{\lambda_{\epsilon,x_m,u}} h_{\lambda_{\epsilon,x_m,v}} h_{\lambda_{\epsilon,x_m,k}} \left( h_{\lambda_{\epsilon,x_m,u}}^{-1}, V(\Omega_m) \right) \right) \]  

(448)

Finally, \( \epsilon \) drops out, and we can replace the Poisson bracket in the above sum by commutators.

We now have to choose the points \( x_m \), the tangents \( u_1, u_2, u_3 \) and the paths \( \lambda_{\epsilon,x,u} \) and \( \lambda_{\epsilon,x,v} \) such that the operator \( H_1 \) is well defined, gauge invariant and nontrivial. For \( H_1 \), Thiemann found that such a choice can be made, see \[13 \] p 279 or Thiemann’s article \[20 \]. So we end up
with a well defined first part of the Hamilton operator that can act on a spin network state. Unfortunately, the second part, $H_2$, of the Hamiltonian is omitted in the loop quantum gravity literature, due to computational complexity. If one chooses the Barbero Immirizi parameter to be $\gamma = \pm i$ the second part of the Hamiltonian drops out. However, Nicolai notes on p. 6 of [82] that such a choice would lead to the phase space of general relativity to be complexified. Then a reality constraint would have to be imposed in order to recover the original phase space. Nicolai writes that quantizing this reality constraint would lead to additional difficulties.

Even if we assume that it is possible to give a meaningful interpretation of the second part of the Hamiltonian or to solve the reality constraint, loop quantum gravity requires a specific choice of the spin network graph, i.e. a special coordinate system of points has to be chosen. Some people might think that this procedure is at odds with the principles of general relativity. Furthermore, the results of this theory seem to depend on the spin representations that were chosen for the holonomies. Since these representations are arbitrary, there is a quantization ambiguity incorporated in loop quantum gravity.

Finally, the Wheeler deWitt equation [392] admits an approximate solution with superpositions of WKB states. We noted in section 4.1 that by using these approximate solutions, deWitt was able to show a connection of the Wheeler deWitt theory to Einstein’s equations of classical general relativity. Unfortunately, it is not clear at all, how to derive the WKB states of the Wheeler deWitt theory from some semi-classical limit of the spin network states that one has in loop quantum gravity. Worse, no one has even succeeded to derive Einstein’s classical equations from the loop quantum gravity. Therefore, the author of this note is highly skeptical of the loop quantization approach. However, it is the opinion of the author that the certain techniques of LQG to describe a spacetime as a discrete graph may turn out to be of some use, when one is dealing with nonperturbative phenomena in quantum gravity.

5 The path integral of gravitation as solution of the Wheeler-deWitt equation

When the articles of deWitt appeared, the relation between the perturbative and the canonical versions of quantum gravity were unknown. This changed with the work of Hartle and Hawking [41]. For a quantum theory with a scalar field, the path integral

$$Z = \psi(x, t) = N \int D\phi(t)e^{iS(\phi(t))}$$

(449)

where N is some normalization factor, fulfills a Schroedinger equation

$$i\frac{\partial}{\partial t}\psi(x, t) = H\psi$$

(450)

see, e.g [40]. In analogy to this, the path integral for gravity should satisfy the Wheeler deWitt equation. Hartle and Hawking [41] showed that this is the case for

$$Z = \int Dg_{\mu\nu}e^{iS}$$

(451)

if one does not include ghost and gauge fixing terms. Later, Barvinsky showed in [42] that the inclusion of ghosts and gauge fixing leads to factor ordering ambiguities as they were present in section 4.2.
The path integral of eq. (451) can be used to define expectation values of the form
\[
\langle F(g) \rangle = \int \mathcal{D}g_{\mu\nu} F(g) e^{iS} = Z^{-1} F Z
\] (452)

The wave function of the Wheeler deWitt equation depends only on the three geometry \(\gamma_{ij}\). The metric in eq. (366) has a line element:
\[
\text{d}s^2 = -N^2 \text{d}t^2 + \gamma_{ij}(\text{d}x^i + \beta^i \text{d}t)(\text{d}x^j + \beta^j \text{d}t)
\] (453)

where \(N\) is the lapse function that gives the proper time lapse between the upper and lower hypersurfaces. When we vary \(Z\) with respect to \(N\) we therefore put it forward or backward in time, but as the wave function of general relativity should be independent on time, we have
\[
0 = \langle \delta F / \delta N \rangle = Z^{-1} \left( \delta Z / \delta N \right)
\] (454)

From eq. (377), we have
\[
H = \int d^3 x \left( \pi \partial_t N + \pi^i \partial_i \beta_i + \pi^{ij} \partial_j \gamma_{ij} - L \right)
\] (455)
or
\[
S = \int dt L = \int d^4 x \left( \pi \partial_t N + \pi^i \partial_i \beta_i + \pi^{ij} \partial_j \gamma_{ij} - \int dt H \right)
\] (456)

with \(\pi = \pi^i = 0\), and \(\gamma_{ij}\) being independent of \(N\), and
\[
H = \int d^3 x \left( N \mathcal{H}_G + \beta_i \chi^i \right)
\] (457)

As a result, we are led to the Hamiltonian constraint
\[
\frac{i \delta S}{\delta N} Z = -i \mathcal{H}_G Z = 0
\] (458)

where
\[
\mathcal{H}_G = G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{\gamma^{(3)}} R
\] (459)

Similarly, computation of \(\langle \delta F / \delta \beta^i \rangle = 0\) immediately leads to the diffeomorphism constraints of eq. (383).

If we compute
\[
Z^{-1} \left( -i \delta / \delta \gamma_{ij} \right) Z = Z^{-1} \int \mathcal{D}g_{\mu\nu} \delta S / \delta \gamma_{ij} e^{iS}
\] (460)

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we get, using
\[ \frac{\delta S}{\delta \gamma_{ij}} = \sqrt{\gamma} (\gamma^{ij} K - K^{ij}) = \pi^{ij} \] (461)
the expression
\[ Z^{-1} \left( -i \frac{\delta}{\delta \gamma_{ij}} \right) Z = Z^{-1} \int \mathcal{D}g_{\mu\nu} \pi^{ij} e^{iS} \] (462)
which yields
\[ \hat{\pi}^{ij} = -i \frac{\delta}{\delta \gamma_{ij}} \] (463)
and is equal to the momentum operator from eq. (461). Putting eq. (463) into eq. (459),
we arrive at the Wheeler deWitt equation. Unfortunately, this is true only approximately.
Barvinsky considered the path integral with gauge-fixing and ghost contributions [42]. He found
that these terms lead to additional delta functions \( \delta(0) \) in the constraint equations. Similar terms
arose in our discussion of the factor ordering problem above. Apparently, the Wheeler deWitt
equation can only derived from the path integral if we set \( \delta(0) = 0 \) and thereby ignore the factor
ordering problem.

6 Two ways of the description of black holes in quantum gravity

6.1 Calculation of the black hole entropy from the path integral

This section reviews the computation of the black hole entropy from Euclidean quantum gravity
by Gibbons and Hawking [43]. One may express the metric with a classical background as
\[ g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \] and then expand the Euclidean action perturbatively as in section 2.1:
\[ I(g_{\mu\nu}) = I(g_{\mu\nu}) + I(h_{\mu\nu}) + I(h_{\mu\nu}) + \text{higher order terms} \] (464)
where \( I(h_{\mu\nu}) \) is linear and \( I(h_{\mu\nu}) \) is quadratic in the quantum field. As in Section 2.2, we have
\( I(h_{\mu\nu}) = 0 \) by the equations of motion, and so, omitting ghost and gauge fixing terms, the
Euclidean path integral in the background field method is given by
\[ Z_{eu} = e^{-I(g_{\mu\nu})} \int \mathcal{D}h_{\mu\nu} e^{-I(h_{\mu\nu})} \] (465)
The path integral up to the action quadratic in the metric perturbation can be evaluated with
the same techniques of dimensional regularization from section 2.2. In their book on Euclidean
quantum gravity [44], Gibbons and Hawking use a slightly different technique, namely zeta
function regularization, see section 3.

The background field method can handle the possibility that the background \( g_{\mu\nu} \) is a classical
black hole. We now assume that \( g_{\mu\nu} \) is given by the Schwarzschild metric of eq. (1). Going over
to Kruskal coordinates, the metric becomes
\[ ds^2 = \frac{32M^3}{r} e^{\frac{-r}{2M}} (-dz + dy) + r^2 d\Omega^2 \] (466)
with
\[- z^2 + y^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}\] (467)

and
\[(y + z) = e^{\frac{r}{2M}} \quad \text{(468)}\]

The singularity lies at \(- z^2 + y^2 = -1\). Setting \(\zeta = iz\), the metric becomes positive definite:
\[ds^2 = \frac{32M^3}{r} e^{\frac{r}{2M}} (d\zeta + dy) + r^2 d\Omega^2\] (469)

with
\[\zeta^2 + y^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}\] (470)

the coordinate \(r\) will be real and greater than or equal to \(2M\) as long as \(y\) and \(\zeta\) are real. eq. (468) shows that setting \(t = -i\tau\) implies \(\tau\) has a period of \(8\pi M\).

The Euclidean path integral has a connection to the canonical partition function. We have for field configurations of a scalar field \(\varphi_1\) at \(t_1\) and \(\varphi_2\) at \(t_2\) with a Hamiltonian \(H\)
\[\langle \varphi_2, t_2 | \varphi_1, t_1 \rangle = \int \mathcal{D}\varphi e^{iS} = \langle \varphi_2, t_2 | e^{-iH(t_2-t_1)} | \varphi_1, t_1 \rangle\] (471)

where the path integral is over all field configurations that take the value \(\varphi_1\) at \(t_1\) and \(\varphi_2\) at \(t_2\), and on the right hand side, the Schroedinger picture for the amplitude was invoked. Setting \(t_2 - t_1 = -i\beta\) and \(\varphi_1 = \varphi_2\), a summation over all \(\varphi_1\) yields the canonical partition sum
\[tr(e^{-\beta H}) = \int \mathcal{D}\varphi e^{-i} = Z_{eu}\] (472)

with the path integral now taken over all fields with period in \(\beta\) in imaginary time. Since the Euclidean section of the Schwarzschild metric is periodic in \(\beta = 8\pi M\) (473)

one should be able to compute the canonical partition sum from the Euclidean path integral. The Euclidean section of the Schwarzschild metric has \(R = 0\). Therefore, the non-zero part comes from the boundary part of the action in eq. (210):
\[I = -\int d^4x \sqrt{g} R - 2 \int_{\partial M} d^3x \sqrt{\gamma} (K - K^0)\] (474)

which is an integral over the intrinsic curvature. Evaluation of this integral yields, see [43]:
\[I = 4\pi M^2 = \frac{\beta^2}{16\pi}\] (475)

(Note that to obtain the correct result, one has to use the action with the correct factor \(\tilde{I} = \frac{1}{4\pi} I\), which we discarded for simplicity in section 2).

From statistical mechanics, we have
\[\langle E \rangle = -\frac{\partial}{\partial \beta} \ln(Z_{eu})\] (476)
inserting the contribution of the background

\[ Z_{eu} \approx e^{-\frac{\beta}{8\pi}} \]  

(477)

we get

\[ \langle E \rangle = M = \frac{\beta}{8\pi} \]

The entropy of the canonical ensemble is defined by

\[ S = \beta \langle E \rangle + \ln(Z_{eu}) = \frac{\beta^2}{8\pi} - \frac{\beta^2}{16\pi} = \frac{\beta^2}{16\pi} \]

(478)

which yields

\[ S = 4\pi M^2 = \frac{1}{4} A \]

(479)

where \( A \) is the area of the event horizon.

Gibbons and Hawking mention in their article [43] that one can use this technique also for different spacetimes and write: “Because \( R \) and \( K \) are holomorphic functions on the complexified spacetime except at singularities, the action integral is really a contour integral and will have the same value on any section of the complexified spacetime which is homologous to the Euclidean section even though the metric on this section may be complex. This allows to extend the procedure to other spacetimes which do not necessarily have a real Euclidean section.” Gibbons and Hawking then discuss the Kerr solution and mention that one can use this technique also in the presence of matter fields, where a black hole is surrounded by a perfect fluid rotating at some angular velocity. Finally, they also consider a star of rotating matter without an event horizon and find that without an event horizon, the gravitational field apparently does not contribute to entropy.

6.2 Calculation of the black hole entropy from the Wheeler deWitt equation

One can derive the entropy of a black hole not only from the gravitational path integral, but also from the canonical formalism involving the Hamiltonian and the Wheeler deWitt equation. The following section is merely a summary of the excellent explanations [47, 46]. The line element of a spacetime with spherical symmetry is, see the calculation in [46]:

\[ ds^2 = -N^2(r,t)dt^2 + \Lambda^2(r,t)(dr + \beta^r dt)^2 + R^2(r,t)dt^2 \]

(480)

where \((r,t)\) is a parametrization of the spacetime, \( N \) is the lapse and \( \beta^r \) the shift function. The Hamiltonian constraint of such a spacetime can be derived as,

\[ \mathcal{H}_G = \frac{\Lambda P_A}{2R^2} - \frac{P_A P_R}{R} + V_G = 0 \]

(481)

see Kuchar [45], and Louko and Whiting [46], where

\[ V_G = \frac{RR'}{A} - \frac{RR'A'}{A^2} + \frac{R'^2}{2A} - \frac{\Lambda}{2} \]

(482)
The diffeomorphism constraint becomes

\[ \mathcal{H}_r = P_R R' - \Lambda P_\Lambda' = 0 \quad (483) \]

with

\[ P_\Lambda = -N^{-1} R(\dot{R} - R' N') \quad (484) \]

and

\[ P_R = -N^{-1}(\Lambda(\dot{R} - R' N') + R(\dot{\Lambda} - (\Lambda N')') \quad (485) \]

as canonical momenta that one obtains from varying the action with respect to \( \dot{R} \) and \( \dot{\Lambda} \). Using the canonical momenta from eqs. (484-485) and the Hamiltonian constraint of eq. (481), the Wheeler deWitt equation is

\[ \left( \frac{-\Lambda \delta^2}{2R^2 \delta \Lambda^2} + \frac{1}{R} \frac{\delta^2}{\delta \Lambda \delta R} + V_G \right) \psi(\Lambda, R) = 0 \quad (486) \]

A semi-classical solution of this equation is given by the WKB ansatz

\[ \psi(\Lambda, R) = C(\Lambda, R)e^{iS_0(\Lambda, R)} \quad (487) \]

with

\[ \left| \frac{\delta C(\Lambda, R)}{\delta \Lambda} \right| < \left| C(\Lambda, R) \frac{\delta S_0(\Lambda, R)}{\delta \Lambda} \right| \quad (488) \]

and

\[ \left| \frac{\delta C(\Lambda, R)}{\delta R} \right| < \left| C(\Lambda, R) \frac{\delta S_0(\Lambda, R)}{\delta R} \right| \quad (489) \]

which implies for the Hamiltonian constraint

\[ \left( \frac{-\Lambda \delta^2}{2R^2 \delta \Lambda^2} + \frac{1}{R} \frac{\delta^2}{\delta \Lambda \delta R} + V_G \right)^2 + \frac{1}{R} \frac{\delta^2}{\delta \Lambda \delta R} + V_G = 0. \quad (490) \]

We have for the exterior of the Schwarzschild metric an action of the form

\[ S = \int dt L = \int dt \int_0^\infty dr \left( \dot{P_\Lambda} + \dot{P_R} - \mathcal{H}_G - \beta^\nu \mathcal{H}_r \right) \quad (491) \]

See [47, 49]. The action in eq. (491) does not take boundary terms into account. In order to describe a spacetime with an event horizon and a flat curvature at infinity, the action must therefore be supplied with appropriate boundary terms. These then lead to additional constraints. The additional constraints then lead to modifications in the Hamiltonian and the functional \( S_0 \).

According to Louko and Whiting [46], the following boundary conditions are appropriate for a non-degenerate event horizon, which we locate at the parameter value \( r = 0 \):

\[ R(t, r) = R_0(t) + R_2(t) r^2 + \mathcal{O}(r^4), \]

where \( R_0 \) is defined by \( \left( 1 - \frac{2M}{R} \right) \big|_{r=0} \),

\[ N(t, r) = N_1(t) r + \mathcal{O}(r^3), \quad (493) \]
\[ \beta'(t, r) = \beta'_1(t)r + \mathcal{O}(r^3), \quad (494) \]
\[ \Lambda(t, r) = \Lambda_0(t) + \mathcal{O}(r^2) \quad (495) \]

Kiefer and Brotz write in [47] that the variation of the action \( S \) from eq. (491) leads with the above boundary conditions to the following boundary term at \( r = 0 \):

\[ \delta S \bigg|_{r=0} = -\frac{\partial}{\partial r} \left( \Lambda \frac{\partial H_G}{\partial \Lambda'} \right) \bigg|_{r=0} = -\frac{N_1 R_0}{\Lambda_0} \delta R_0 \quad (496) \]

This term must be subtracted from the action if \( N_1 \neq 0 \). Similarly, it was found by Regge and Teitelboim in 1974 [87] that in case of asymptotically flat spacetimes, another boundary term must be subtracted. This was also mentioned earlier by deWitt in [14]. With all these boundary terms included, the action now becomes:

\[ S_{\text{total}} = \int dt L = \int dt \int_0^\infty dr \left( P_\Lambda \dot{\Lambda} + P_R \dot{R} - N H_G - \beta^r H_r \right) + \frac{1}{2} \int dt \frac{N_1 R_0^2}{\Lambda} - \int dt N_+ M \quad (497) \]

with \( M \) as the so-called ADM mass and \( N_+ \) as the lapse function at infinity. A variation of this action would lead to an unwanted term

\[ \int dt \frac{R_0^2}{2} \delta \left( \frac{N_1}{\Lambda} \right) \quad (498) \]

This term is zero if we assume that

\[ \left( \frac{N_1}{\Lambda} \right) \equiv N_0(t) \quad (499) \]

is fixed at \( r = 0 \), i.e. it can not be varied. The need of fixing \( N_1 \) and \( N_+ \) can be removed if we define suitable parametrizations \( N_0(t) \equiv \dot{\tau}(t) \) and \( N_+ \equiv \dot{\tau}_+(t) \) and consider \( \tau \) and \( \tau_+ \) as additional variables. We then get an action:

\[ S_{\text{total}} = \int dt \int_0^\infty dr \left( P_\Lambda \dot{\Lambda} + P_R \dot{R} - N H_G - \beta^r H_r \right) + \int dt \frac{R_0^2}{2} \dot{\tau} - \int dt M \dot{\tau}_+ \quad (500) \]

In the canonical framework, the functions \( \tau \) and \( \tau_+ \) represent additional degrees of freedom that must be supplied with corresponding canonical momenta \( \pi_0 \) and \( \pi_+ \). These momenta can only brought consistently into the action as additional variables, if we impose additional constraints

\[ C_0 = \pi_0 - \frac{R_0^2}{2} = 0 \quad (501) \]

and

\[ C_+ = \pi_+ + M = 0 \quad (502) \]
With $N_0$ and $N_+$ now acting as Lagrange multipliers, the action becomes

$$S_{\text{total}} = \int dt L = \int dt \int_0^\infty dr \left( P\dot{\Lambda} + P_R \dot{R} - N\mathcal{H}_G - \beta^r \mathcal{H}_r \right)$$

$$+ \int dt \pi_0 \dot{\tau}_0 + \pi_+ \dot{\tau}_+ - N_0 C_0 - N_+ C_+$$

(503)

In the quantum theory, the additional momenta become $\pi = -i \frac{\delta}{\delta \tau_0}$ and $\pi_+ = -i \frac{\delta}{\delta \tau_+}$ and the WKB ansatz

$$\psi = C(\Lambda, R)e^{iS_0(\Lambda, R, \tau_0, \tau_+)}$$

with the additional degrees of freedom $\tau_+$ and $\tau_0$ then leads to the quantum mechanical constraints

$$\frac{\partial_0 S_0}{\partial \tau_0} - \frac{R_0^2}{2} = 0$$

(504)

$$\frac{\partial_0 S_0}{\partial \tau_+} + M = 0$$

(505)

This changes the solution $S_0$ into

$$S_0(\Lambda, R, \tau_+, \tau_0) \rightarrow S_0 + \frac{R_0^2}{2} \tau_0 - M \tau_+$$

(506)

Setting $r - 2M = \zeta(r)$ in the Schwarzschild metric, we get, up to order $\frac{1}{2M}$, see [86] on p. 4:

$$ds^2 = -\frac{\zeta}{2M} dt^2 + \frac{2M}{\zeta} (d\zeta)^2 + (2M)^2 d\Omega^2$$

(507)

Defining

$$\rho^2 = 8M\zeta$$

(508)

yields

$$d\zeta^2 \frac{2M}{\zeta} = d\rho^2$$

(509)

and the line element becomes

$$ds^2 = -\frac{\rho^2}{16M^2} dt^2 + d\rho^2 + (2M)^2 d\Omega^2$$

(510)

This corresponds to our previous line element with $\beta^r = 0$, $\Lambda = 1$, $N(t, \rho) = N_1(t)\rho$, and $N_1 = \frac{1}{1t}$. Setting $t = -it_\epsilon$, the euclideanized version of this line element is

$$ds^2 = N_1^2 \rho^2 dt_\epsilon^2 + d\rho^2 + (2M)^2 d\Omega^2$$

(511)

The Euclideanized Schwarzschild metric has a time coordinate that is periodic with $8\pi M$. Thereby, using $N_1 = \frac{1}{1t}$, the variable $\tau_0$ becomes

$$\tau_0 = \int_0^{8\pi M} dt N_1 = 8\pi M N_1 = 2\pi$$

(512)
and we get
\[ S_0 + R_0^2 \pi - M \tau_+ = S_0 + \frac{1}{4} A - M \tau_+ \] (513)

Similarly, the parameter \( \beta = \tau_+ = \int dt N_+ \) is interpreted by Louko and Whiting in [46] as the inverse of the normalized temperature at infinity. The euclideanized WKB solution of the Wheeler deWitt equation is then
\[ \psi(\Lambda, R) = \psi_0(\Lambda, R) e^{-\beta M + \frac{A}{4}} \] (514)

We have seen in section 5 that the gravitational path integral is a solution of the Wheeler deWitt equation. Similarly, \( \psi_0(\Lambda, R) = e^{-S_0(\Lambda, R)} \) can now be considered as the quantum contribution to the path integral and \( e^{-\beta M + \frac{A}{4}} \) is considered as the background contribution to the amplitude. By comparison with
\[ \tilde{S} - \beta \langle E \rangle = \ln(Z_{eu}) \] (515)

where \( \tilde{S} \) is the entropy and in the Euclidean path integral \( Z_{eu} \), only background contributions are considered, we can indentify \( \tilde{S} = \frac{1}{4} A \) as black hole entropy.

7 A comment on recent papers by Dvali and Gomez.

7.1 A comment on articles by Dvali and Gomez regarding a proposed “self-completeness” of quantum gravity

In the following, we will critically assess various statements and claims that Dvali and Gomez make in [78]. The main subject of this article seems to circle around the question whether one could design an experiment to resolve so called trans-Planckian states. Dvali and Gomez first consider a gravitational amplitude on p. 5
\[ T^{\mu\nu} \Delta_{\mu\nu\alpha\beta} = \frac{T^{\mu\nu} \tau_{\mu\nu} - \frac{1}{2} T^{\mu\nu} \tau_{\mu\nu}}{p^2} \] (516)

with two energy momentum sources \( T^{\mu\nu} \) and \( \tau_{\mu\nu} \). They then consider a modification of this amplitude of the form
\[ T^{\mu\nu} \Delta_{\mu\nu\alpha\beta} = \frac{T^{\mu\nu} \tau_{\mu\nu} - \frac{1}{2} T^{\mu\nu} \tau_{\mu\nu}}{p^2} + \frac{a T^{\mu\nu} \tau_{\mu\nu} - b \frac{1}{3} T^{\mu\nu} \tau_{\mu\nu}}{p^2 + (1/L)^2} \] (517)

where the parameters a and b are fixed according to the spin of the new particle and \( (1/L) \) symbolizes the energy of the new particle. The latter is assumed to have a mass of \( 1/L \) which is higher than the Planck mass. Dvali and Gomez then write that such a modification of a propagator would be impossible to detect since a scattering experiment at that energies would create a black hole of a Schwarzschild radius larger than the impact parameter of the scattering experiment. For this, Dvali and Gomez use a version of the so called generalized uncertainty principle in quantum gravity. In the following paragraph, we will mention constants like \( G, c \) and the Planck’s constant \( h \) explicitly. Multiplying the equations for the Schwarzschild radius
\[ r_s = \frac{2Gm}{c^2} \] (518)

85
and the reduced Compton wavelength

$$\lambda_c = \frac{\lambda_c}{2\pi} = \frac{\hbar}{mc}$$  \hspace{1cm} (519)$$

gives the so called Planck length $l_p$

$$r_s\lambda_c = \frac{2G\hbar}{c^3} = 2l_p^2 \geq l_p^2$$  \hspace{1cm} (520)$$

Since the Planck mass $m_p$ is defined by $r_s = \lambda_c$ any attempt to generate a black hole of mass greater than $m_p = \sqrt{\hbar c / 2\pi}$ will give a black hole larger than its Compton wavelength. This black hole should be regarded as a classical black hole for large $r$ or $m > m_p$. Then, Dvali and Gomez give an argument that should approximately be something like: “putting a quantum state within a box of width $L$ with infinitely high walls would require an energy of $E \propto \frac{1}{L^2}$.” Actually, they write that $E = \frac{1}{L}$ on p. 4, but this seems to be a typo. Converting this kinetic energy into a mass with $E = mc^2$ yields $m = E/c^2 \propto \frac{1}{L^2}$ or $\lambda_c \propto L^2\hbar c$ and setting this into the formula for the definition of the Planck length yields

$$r_s \propto \frac{2l_p^2}{L^2\hbar c}$$  \hspace{1cm} (521)$$

Based on these approximations, whose validity is by no means guaranteed even approximately at “trans-Planckian energies”, Dvali and Gomez conclude that any attempt to resolve distances smaller than the Planck length $l_p$ should create an even larger black hole, which could then be regarded as classical. Thereby, the modifications in the graviton propagator in eq. (517) would not show up in an experiment, since any attempt to resolve the added trans-Planckian degrees of freedom would lead to the creation of a black hole with a Schwarzschild radius larger than the Planck length. This very same argument is then repeated in several different physical situations. Finally, Dvali and Gomez note on p. 5 in eq. (8) that the path integral of classical black holes is exponentially suppressed, as we saw in section 6. From all this, Dvali and Gomez concludes on p. 3 that ”Deep – UV – gravity = Deep – IR – gravity” and they state on p. 2

“The existing common knowledge about Einstein gravity is that it becomes inapplicable at deep UV and that it must be completed by a more powerful theory that will restore consistency at sub-Planckian distances. We wish to question the above statement and suggest that pure Einstein gravity is self complete in deep-UV. In other words, we argue that for restoring consistency no new propagating degrees of freedom are necessary at energies $>> M_p$.”

In the article of Dvali and Gomez, there are several aspects which in the view of the author of this text are somewhat problematic. For example, on p. 6, Dvali and Gomez write

“However, for $L^{-1} >> M_{\text{Planck}}$, the same vertex describes evaporation of a classical BH of mass $1/L$ into a single particle-anti-particle pair”.

But in their entire article, they do not seem to specify the mathematical details of a quantum field theory that would describe an “evaporation of a black hole” as a “vertex”. On p. 32, they write:
In other words, a String theory with order one String coupling is built into Einstein gravity. To put it differently, by writing down Einstein's action, we are committing ourselves to a String theory.

As we saw in section 2, the amplitude of Einstein gravity is divergent at two loops. This is different in String theory, whose two loop finiteness has been shown by D'Hoker and Phong. Furthermore, String theory reduces in the classical limit to a version of Supergravity and not Einstein gravity. In the opinion of the author of this text, this renders the claims of Dvali and Gomez regarding String theory highly problematic.

However, most obvious problem inherent in the assumptions of Dvali and Gomez is that the formula of the Schwarzschild radius, their eqs. (5) and (7), which they invoke on p. 4, 5, 12, 15, 16, 23, 24, 25, and 29 of their article in various different physical situations, are something that comes entirely from purely classical physics. Without a proper description of a quantum black hole, it is not clear what the analogue of the Schwarzschild radius will look like, when a quantum state of high energy is put into a box of “trans-Planckian width” as Dvali and Gomez call it.

In fact, the equations for the black hole argument of Dvali and Gomez do not seem to follow at all from the quantum field theoretical amplitudes themselves. As we saw in section 3, up to one loop order of the gravitational amplitude, only some estimates can be made about the topologies that give the dominant contributions to the path integral at Planck scale. The analysis of section 3 implies that the euler characteristic $\chi$ of the metrics which are dominant contributions fulfills 

$$\chi \propto hV$$

where $V$ is the volume and $h$ is some constant. According to Hawking, this result means that we will find one gravitational instanton per unit volume $h^{-1}$ at Planck scale. Thereby, the obvious disconnectedness of the black hole argument from Dvali and Gomez with the quantum field theoretical amplitude of quantum gravity can, at least for amplitudes up to one loop order, be somewhat removed with Hawking’s work on spacetime foam from 1978. However, the amplitude that follows from quantum gravity seems to always imply a summation over several metrics, and not only over ones that describe classical black hole metrics. Hence there remain important differences between the proposal from Dvali and Gomez and the gravitational path integral even at one loop order.

As we saw in section 2.2, an evaluation of the gravitational path integral in two loop order leads to divergencies. One could suspect that these divergences arise because of non-local topological features of the spacetime that we have to expect at planck scale. The one loop analysis in section 3 reveals a spacetime filled with some sort of gravitational instantons. Since a gravitational instanton is a non-local entity, further analysis of the path integral would likely need non-perturbative methods. The traditional way of computing a perturbation series with the gravitational path integral in terms of Feynman diagrams may then no longer be a valid procedure at all. To quote Hawking:

"Attempts to quantize gravity ignoring the topological possibilities and simply drawing Feynman diagrams around flat space have not been very successful. It seems to me that the fault lies not with the pure gravity or supergravity theories themselves but with the uncritical application of perturbation theory to them. In classical relativity we have found that perturbation theory has only limited range of..."
validity. One can not describe a black hole as a perturbation around flat space. Yet this is what writing down a string of Feynman diagrams amounts to."

That one likely has to do with non-perturbative physics at the Planck scale and below is acknowledged by Dvali and Gomez when they write on p. 16:

"However, for $m >> m_p$ the new degree of freedom is no longer a perturbative state.

Unfortunately, a non-perturbative evaluation of the path integral turned out to yield a divergent amplitude again. In section 2.3, we arrived at the conclusion that the path integral of Euclidean quantum gravity can be described by decomposing the metric into $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with $\Omega$ as a conformal factor. We can now write the path integral as

$$Z_{eu} = \int \mathcal{D}gY(g)$$

where

$$Y(g) = \int \mathcal{D}\Omega e^{-I(\Omega^2, g)}$$

and with an action

$$I(\Omega^2, g) = -\int d^4x \sqrt{\gamma} \left( \Omega^2 R + 6g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \right) - 2 \int_{\partial M} d^3x \sqrt{\gamma} \Omega^2 (K - K^0)$$

Because of the derivatives of $\Omega$, the action $I(\Omega^2, g)$ can be arbitrarily negative if a rapidly varying conformal factor is chosen. However, eq. (523) says that one has to do an integration over all possible conformal factors, including ones that lead to a divergent path integral.

The non-perturbative evaluation of the gravitational path integral does not contain any mechanism that would exclude the summation over metrics $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with a rapidly varying conformal factor. For the thought experiment that is envisaged by Dvali and Gomez where a collider probes distances at the Planck scale, the quantum mechanical formulas above would therefore imply a divergent amplitude.

Another way of doing non-perturbative quantum gravity would be the Wheeler deWitt equation. Unfortunately, this theory is only consistent for low energy states within a WKB approximation. As we observed in section 4.2, at high energies, severe factor ordering problems appear and the theory becomes inconsistent, yielding equations like

$$0 = -6i \left( \delta(x, x) \delta \xi^k \right)_k$$

where $\xi$ is an infinitesimal displacement and $, k$ denotes a partial derivative with respect to $x^k$.

The author of this text has the opinion that one could perhaps speak of some kind of “self completeness”, if there were, for example, a mechanism in the Wheeler deWitt equation, which would show that high energy states would automatically lead to the creation of classical solutions for black holes. Within the scope of a semi classical WKB approximation, the Wheeler deWitt theory is able to handle large non-local quantized metrics, like the quantized Friedmann Robertson Walker universe, see [14], or a quantized Schwarzschild metric, see section 6.2. Furthermore, from the WKB ansatz, deWitt was able to show a connection between the classical Einstein equations and the Wheeler deWitt theory, see section 4.1 and [14]. Therefore, it would
be conceivable if this theory would predict the emergence of classical black holes at high energies.

Unfortunately, such a mechanism seems not to be there in the Wheeler deWitt theory, but one gets severe factor ordering inconsistencies at high energies instead, as we have demonstrated in section 4.2.

The artificial invocation of the classical Schwarzschild radius at trans-Planckian energies in the article [78] by Dvali and Gomez does not follow from the equations of non-perturbative quantum gravity. The author of this text therefore has the opinion that the proposals of Dvali and Gomez on the alleged “self-completeness” of gravity are not justified. Instead the author of this note advocates the view that quantum general relativity must be replaced by some other theory, like loop quantum gravity, or String theory that both support the definition of a complete Hilbert space of states without giving rise to severe inconsistencies at high energies.

But there is an additional problem in the proposal of Dvali and Gomez. Any theory that proposes an amplitude which is dominated by virtual gravitational instantons must take into account that the trajectories of ordinary particles flying in this spacetime might be altered by these instanton metrics. Hawking, Page and Pope figured that this might even lead to predictions that can be experimentally tested [65, 66]. They devised an approximation scheme to evaluate the path integral of gravity and matter fields non-perturbatively.

A typical amplitude for a scalar particle propagating from a initial field mode \( u(x') \) to a final mode \( v(x') \) would be

\[
- \int \Sigma^{\mu}(x') \nabla^{\mu} G(x', y') \nabla^{\nu} v(x') d\Sigma^{\mu}(x') d\Sigma^{\nu}(y')
\]  

(525)

where \( \Sigma^{\mu}(x') \) and \( \Sigma^{\nu}(y') \) are the Cauchy data for the initial and final states and \( G(x', y') \) is the Green's function of the metric. The s-matrix is computed from initial and final states emerging from infinity, where they are supposed to be non-interacting, and must be governed by flat space equations of motion. Therefore, such amplitudes make only sense in asymptotically Euclidean metrics and we must try to convert metrics that are not asymptotically Euclidean into an asymptotically Euclidean form with an appropriate conformal factor.

Taking the idea of the non-perturbative evaluation of the path integral seriously, one can not confine oneself to simply consider an amplitude composed of black hole metrics, but one has to sum the path integral over all possible metrics. Unfortunately, the Green’s functions for most of these metrics are not known and we can not compute the expression of a scattering amplitude of a particle within such an arbitrary metric.

Hawking, Page and Pope note that by using topological sums of a certain number of copies of \( CP^2 \) and \( CP^2 \) (the bar means opposite orientation), one can construct a simply connected closed manifold of arbitrary signature \( \tau \) and Euler characteristic \( \chi \), with an odd and definite intersection form. Similarly, by using certain numbers of copies of \( S^2 \times S^2 \) and \( K^3 \) if \( \tau > 0 \) or \( \bar{K^3} \) if \( \tau < 0 \), one can construct a simply connected closed manifold with even and indefinite intersection form, arbitrary signature and Euler number, see [77] p. 26. By Freedman’s theorem, the topology of the simply connected spacetimes from these construction would then, up to homeomorphy, be equivalent to an arbitrary simply connected spacetime with the same Euler number and signature. Note, however that this equivalence just holds for the topology, and not the metric.

Hawking, Page and Pope then propose the view that one should restrict the path integral to simply connected spacetimes. With the argument above, the topology of this simply connected spacetime can be built out of building blocks like \( CP^2 \), \( \bar{CP^2}, S^2 \times S^2 \), \( K^3, \bar{K^3} \). Hawking et
al then proceed to calculate the scattering amplitudes of particles that are moving in these building blocks. They weight these amplitudes with \( e^{-I} \) where \( I \) is the Euclidean action of the metric, and include some weighting factor that depends on the conformal transformation which was employed to make the manifolds asymptotically Euclidean. Then, Hawking, Page and Pope average over all parameter values (like scale parameters, or certain orientations) that these metrics have.

For example, we noted in section 3.2 that \( CP^2 \) has the following metric

\[
d s' = \frac{\rho'^2}{\rho'^2 + x'^2} \left( \delta_{\mu\nu} - \frac{x'^\mu x'^\nu + n'^{\mu}_{\sigma} n'^{\nu}_{\lambda} x'^{\sigma} x'^{\lambda}}{\rho'^2 + x'^2} \right) \, dx'^\mu \, dx'^\nu
\]

(526)

with a scale parameter \( \rho \) and

\[
\eta'^{\mu}_{\sigma} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

(527)

(see [65, 66]). From the metric of \( CP^2 \), an asymptotically Euclidean metric can be obtained by an appropriate conformal transformation \( g = \Omega^2 g' \) which sets the origin of \( CP^2 \) to infinity. Thereby we get a space of the same topology as \( CP^2 \) (up to homeomorphy), but with the appropriate infinity structures. The green functions of \( CP^2 \) have the form, see [65, 66]:

\[
G(x', y') = \frac{1}{4\pi^2 \rho(1 - L)}
\]

(528)

where

\[
L = \frac{(\rho' + x'y' - i n^l_{\mu\nu} x'^\mu y'^\nu)(\rho'^2 + x'^2)(\rho'^2 + y'^2)}{(\rho'^2 + x'^2)(\rho'^2 + y'^2)}
\]

(529)

Note that this Green’s function has additional singularities compared to the Green’s functions in flat space. For spaces like \( S^2 \times S^2 \) the Green’s functions are not known, so Hawking, Page and Pope made an approximation to \( S^2 \times S^2 \) by a metric that describes a conformally flat manifold with conical singularities.

After weighting the individual amplitudes and integrating over the metric parameters, Hawking et al conclude that the amplitudes are of order

\[
A \propto \left( \frac{k_1 k_2}{m_p} \right)^s
\]

(530)

where \( k_1 \) and \( k_2 \) are the momenta of the in and out states, \( m_p \) is the Planck mass and \( s \) is the spin. For a scalar particle, like the Higgs field, we have \( s = 0 \) and therefore the amplitudes would be of order one.

Hawking writes that this would suggest that the Higgs particle is of composite nature. Unfortunately, in 2012, the Higgs particle has been found at the Large Hadron Collider in Genf, and further analysis of the data provided evidence for the Higgs field to be indeed a scalar particle [80]. This puts the approximations of Hawking et al severely into question. Additionally, Warner [67] has analyzed scattering amplitudes of Spin 1 fields with Hawking’s model, and he also found large amplitudes that are in disagreement with observation.
From this one can conclude that either the approximation of the gravitational path integral by restricting it to \( CP^2, \overline{CP}^2, S^2 \times S^2, K^3, \overline{K}^3 \) metrics is wrong, or that the idea of a gravitational scattering amplitude dominated at Planck scale by virtual gravitational instantons is physically incorrect.

Unfortunately, any model that proposes a scattering amplitude being dominated by non-perturbative virtual gravitational instantons will have to confront the problem that the Green’s functions for these instantons will, in general, look very different from flat space. Thereby some changes in the particle trajectories should be expected in all these models.

Hawking et al tried to approximate the Euclidean path integral by restricting the summation to \( CP^2, \overline{CP}^2, S^2 \times S^2, K^3, \overline{K}^3 \) metrics because one does not know the Green’s function of all the metrics that the path integral has to be summed over. The fact that their approximation turned out to be wrong also suggests that entirely new methods are needed to describe the spacetime at Planck scale.

### 7.2 A comment on articles by Dvali and Gomez on black holes

In Section 3, we have shown that the gravitational one loop amplitude should be dominated by virtual gravitational instantons at Planck scale. So perhaps a way to get further understanding of gravity would be to create reasonable quantum mechanical models for spacetimes with high Euler numbers and signature. Recently, Dvali and Gomez have proposed something like this. In his article “black holes as critical point of quantum phase transition,” they write on p. 2

“Black holes represent Bose-Einstein-Condensates of gravitons at the critical point of a quantum phase transition”.

Dvali and Gomez proceed by writing on p. 10

“We now wish to establish the connection between the black hole quantum portrait and critical phenomena in ordinary BEC. We shall consider a simple prototype that captures the key phenomenon. Let \( \psi(x) \) be a field operator describing the order parameter of a Bose gas. The simplest Hamiltonian that takes into account the self interaction of the order parameter can be written in the form

\[
H = -\hbar L_0 \int d^3 x \psi(x) \nabla^2 \psi(x) - g \int d^3 x \psi(x) \psi(x) \psi(x) \psi(x)
\]

Dvali and Gomez then write

“We shall put the system in a box of Size R and periodic boundary conditions

\[
\psi(0) = \psi(2\pi R)
\]

Performing a plane wave expansion

\[
\psi = \sum_k \frac{a_k}{\sqrt{V}} e^{i \frac{\vec{k} \cdot \vec{x}}{\hbar}}
\]

we can rewrite the Hamiltonian as

\[
\mathcal{H} = \sum_k k^2 a_k^\dagger a_k - \frac{1}{4} \alpha \sum_k a_k^\dagger a_{k+\vec{p}} a_{k-\vec{p}} a_k a_{k'}
\]
In contrast to these statements by Dvali and Gomez, we have derived the classical Hamiltonian of general relativity in section 4 as

\[ H = \int d^3x \left( N \sqrt{\gamma} (K_{ij} K^{ij} - K^2 - 3R) - \beta_i 2D_j \left( \gamma^{-1/2} \pi_{ij} \right) \right) \]

(531)

In the quantum theory, this gave rise to a Hamiltonian constraint that turned out to be

\[ \left( G_{ijkl} \frac{\delta}{\delta \gamma_{ij}} \frac{\delta}{\delta \gamma_{kl}} + \sqrt{\gamma} (3) R \right) \Psi(\gamma_{ij}) = 0 \]  

(532)

and a diffeomorphism constraint

\[ 2iD_j \left( \frac{\delta}{\delta \gamma_{ij}} \Psi(\gamma_{ij}) \right) = 0 \]

For comparison with the Hamiltonian proposed by Dvali and Gomez, we now try to put the Hamiltonian constraint of quantum gravity into a form with creation and annihilation operators. To the knowledge of the author of this note, there seems to be just one article [72] that has investigated this question. Usually, one defines, in analogy to the Klein-Gordon equation, the following norm for the wave-functionals of the Wheeler deWitt equation:

\[ \langle \Psi_1 | \Psi_2 \rangle = \int d^3x \, d\Sigma_{ij}(x) \Psi^*_1 \left( G_{ijkl} \frac{\delta}{\delta \gamma_{ij}} \frac{\delta}{\delta \gamma_{kl}} \Psi_n(\gamma_{ij}) + \frac{\gamma_{ij}}{i \beta_i} \Psi^*_n(\gamma_{ij}) \right) \]

(533)

where \( d\Sigma \) is the a surface element of the \( 6 \times \infty^3 \) dimensional space spanned up by the pseudo-metric \( G_{ijkl} \).

McGuigan writes [72] that in order to convert the Hamiltonian into a form with creation and annihilation operators, one would have to find a complete set of solutions \( \Psi_n \) which are orthonormal with respect to the above norm of eq. (533). Then one could make an ansatz

\[ \Psi(\gamma_{ij}) = \sum_k \left( a_k^{ij} \Psi_n(\gamma_{ij}) + a_k^{ij*} \Psi_n^*(\gamma_{ij}) \right) \]

(534)

where \( a_k^{ij} \) and \( a_k^{ij*} \) are the creation and annihilation operators for gravitons with \( k \) momentum. However, McGuigan states

"The presence of the metric \( G_{ijkl} \) as well as the term \( \sqrt{\gamma} (3) R \) which are not quadratic in the \( \gamma_{ij} \) or its derivatives will prevent us from finding such solutions here".

Please note that the situation here is entirely different with that from zeta function renormalization from section 3. There, we had an operator where an orthonormal base could be found. This was the case because we made a perturbation expansion that we previously have cut at second order, and then we were able to find an orthonormal base for these second order terms. In contrast to a perturbation series at one loop, the Hamiltonian incorporates the full non-perturbative information of the theory. Unfortunately, for non perturbative quantum gravity
which incorporates the full nonlinearity of the theory, an orthonormal base of quantum states cannot be found.

One should also note that similar problems occur for a Hamiltonian that describes spherical black holes. In the Wheeler deWitt equation for spherical black holes from section 6.2

\[
\left( -\Lambda \delta^2 \frac{2}{2R^2 \delta R^2} + \frac{1}{R} \frac{\delta^2 \delta R^2}{\Lambda} - \frac{RR' \delta R}{\Lambda} + \frac{R'^2}{\Lambda^2} \right) \psi(\Lambda, R) = 0
\]

there occur terms like \(1/R^2\) or \(1/\Lambda^2\) which would prevent finding a complete orthonormal set of solutions.

McGuigan then goes on noting that the situation would not be that way in linearized gravity, where he investigates a topology \(S^1 \times S^1 \times S^1\). Expanding the metric as

\[
\gamma_{ij}(x) = \hat{d}^3k (2\pi)^3 \gamma_{ij}(k) e^{kx}
\]

and defining “zero modes” \(\gamma_{ij0} = \gamma_{ij}(k = 0)\), McGuigan finds a Hamiltonian

\[
H = H_{0G} + H_{osc}
\]

where

\[
H_{0G} = \frac{1}{l^2} c \left( -\pi a^4 + \frac{1}{2\pi} \gamma_{ij0} \pi_{ij0} - \frac{a}{2} (1 - V(\gamma_{ij0})) \right)
\]

is a “zero mode” part. In \(H_{0G}\), we have \(c = g_{00}\), the constant \(a^3\) denotes the volume of the system, \(l\) is the planck length and \(\pi_{ij0}\) are the canonical momenta associated to the zero modes \(\gamma_{ij0}\). The function \(V\) is, according to McGuigan, a placeholder for “the complicated dependence” on the zero modes that comes from the \(3^3 R\) part of the Hamiltonian.

Furthermore we have

\[
H_{osc} = a \int \frac{d^3k}{(2\pi)^3} |k| \left( a_{ij}^{ij} a_{ij}^{ij} + \frac{1}{2} \right)
\]

which is the part of the Hamiltonian that describes gravitons with annihilation and creation operators.

When writing this, one should note that one can quantize linearized gravity much more easily by methods that were first developed by Gupta in 1928 \[85\]. Gupta started from a metric

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

where \(|h_{\mu\nu}| << 1\) and the linearized Einstein equations for weak fields:

\[
8\pi T_{\mu\nu} = \frac{1}{2} \left( \partial_{\rho} \partial_{\nu} h_{\mu}^\rho + \partial_{\nu} \partial_{\rho} h_{\mu}^\rho - \partial_{\mu} \partial_{\nu} h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_{\rho} \partial_{\lambda} h^{\rho\lambda} + \eta_{\mu\nu} \Box h \right)
\]

For the linearized metric \(h_{\mu\nu}\) Gupta then inserted an expansion in terms of creation and annihilation operators for the gravitons. The creation and annihilation operators fulfilled the usual commutator relations for bosons. Historically, it was this investigation by Gupta that showed gravitons to be spin 2 particles.
The author of this note wants to emphasize that McGuigan concludes a description with creation and annihilation operators can not be given for the Hamiltonian of the full non-linearized theory of gravitation. This is in direct contrast to the Hamiltonian proposed by Dvali and Gomez. But unfortunately, there are an additional problematic aspects:

We noted in section 4.2 that deWitt has found severe factor ordering inconsistencies that always should occur for high energy states with the Wheeler deWitt theory. This problem essentially forbids the construction of a full quantum theory for the Wheeler deWitt equation, which necessarily would have to incorporate states of high energy. In section 4.1, we therefore were only able to solve the Wheeler deWitt equation approximately by a WKB ansatz

$$\Psi = Ce^{iS_0}$$

However, state of the form $a^+|\psi\rangle$ that Dvali and Gomez want to construct in their articles is in general not a semi classical WKB state. Instead, expressions like $a^+|\psi\rangle$ suggest that one would not work in a semiclassical limit, but then the Wheeler deWitt theory becomes inconsistent.

In Loop quantum gravity that we have reviewed in section 4.3, one writes the Hamiltonian in Ashtekar variables, and then one defines a state space with holonomies that are connected to points on a suitable triangulization of the spacetime manifold. One does this in Loop quantum gravity just in order to get a quantum theory of gravity that is free of the factor ordering inconsistencies that the usual Wheeler deWitt theory is plagued with. If one could consistently describe a black hole in form of a condensate consisting of ordinary gravitons, one would simply not need theories like loop quantum gravity or String theory, where one introduces an entire set of additional assumptions just in order to be able to define a consistent quantum theory.

The Hamiltonian proposed by Dvali

$$H = -\hbar L_0 \int d^3x \Psi(x) \nabla^2 \Psi(x) - g \int d^3x \Psi(x) a^+ \Psi(x)$$

seems to be devoid of all the problems given above that one faces when one considers the usual Hamiltonian of non perturbative quantum gravity, eq. (532), which also seems to look completely different than Dvali’s proposal. Furthermore, from the usual Hamiltonian of eq. (532), a connection to the classical Einstein equations could be derived with the WKB ansatz, as we saw in section 4.1. In contrast, a rigorous derivation of Einstein’s equation, or a derivation of the Schwarzschild metric is absent in the corresponding article of Dvali and Gomez. All this suggests that the model of Dvali does not correspond to the usual quantized Hamiltonian of general relativity and it does not seem to describe linearized gravity either.

Finally, I want to mention an additional problem of the proposal by Dvali and Gomez. They claim that they are able to derive the black hole entropy from their Bose Einstein condensate model. Dvali and Gomez write on p. 14 of [71] after a step by step calculation where their Hamiltonian of eq. (542) was used as a starting point:

Thus, we have reproduced the black hole evaporation law from the depletion of the cold Bose-Einstein condensate at criticality.

However, during the standard calculation of the black hole entropy from the path integral in section 7.1, it became apparent that the black hole entropy always emerges from the proper
inclusion of boundary terms at the event horizon. The entropy is computed from the background term
\[ e^{-I(g_{\mu\nu})} \]
in the Euclidean path integral
\[ Z_{eu} = e^{-I(g_{\mu\nu})} \int \mathcal{D}h_{\mu\nu} e^{-I(h_{\mu\nu})} \]  \hspace{1cm} (543)
with the Euclidean action
\[ I = -\int d^4x \sqrt{g} R - 2 \int \partial_{\partial M} d^3x \sqrt{\gamma} (K - K^0), \]  \hspace{1cm} (544)
Since the Euclidean section of the Schwarzschild solution has \( R = 0 \), only the boundary term
\[ 2 \int_{\partial M} d^3x \sqrt{\gamma} (K - K^0) \]  \hspace{1cm} (545)
at the event horizon gives a non-vanishing result and contributes to the gravitational entropy.

The standard Hamiltonian of quantum gravity was derived in section 4 by omitting such boundary terms. Therefore, the black hole entropy could only be computed with the canonical formalism in section 6.2 after we included the necessary boundary terms into the action. Using a description with canonical variables, the action with the boundary terms included was given by eq. (500):

\[ S_{\text{total}} = \int dt \int_0^\infty dr \left( P_\Lambda \dot{\Lambda} + P_R \dot{R} - N \mathcal{H}_G - \beta^r \mathcal{H}_\tau \right) + \int dt \frac{R_0^2}{2} \dot{\tau} - \int dt M \dot{\tau} \]  \hspace{1cm} (546)
The boundary terms are given by \( \int dt \frac{R_0^2}{2} \dot{\tau} - \int dt M \dot{\tau} \) and they contain the new degrees of freedom \( \tau \) and \( \tau^+ \). The latter have to be quantized with additional constraints and corresponding canonical momenta \( \pi_0 \) and \( \pi^+ \). The classical version of the constraints turned out to be, see eq. (501-502):

\[ C_0 = \pi_0 - \frac{R_0^2}{2} = 0 \]  \hspace{1cm} (547)
and
\[ C_+ = \pi^+ + M = 0 \]  \hspace{1cm} (548)
Upon quantization, the canonical momenta were converted into operators \( \pi = -i \frac{\delta}{\delta \tau_0} \) and \( \pi^+ = -i \frac{\delta}{\delta \tau^+} \) that act on a state functional
\[ \psi = C(\Lambda, R) e^{iS_0(\Lambda, R, \tau_0, \tau^+)} \]
of the Wheeler deWitt equation for the black hole. The quantum mechanical constraints for the boundary terms become, see eq. (504-505):

\[ \frac{\partial_0 S_0}{\partial \tau_0} - \frac{R_0^2}{2} = 0 \]  \hspace{1cm} (549)
\[
\frac{\partial S_0}{\partial \tau_+} + M = 0
\] (550)

The inclusion of the boundary terms changes the wave functional as

\[
\psi = C(\Lambda, R)e^{S_0(\Lambda, R, \tau_+, \tau_0)} \rightarrow \psi = C(\Lambda, R)e^{S_0 + \frac{R^2}{2}\tau_0 - M\tau_+}
\] (551)

In section 5 we have shown that the amplitude from the path integral corresponds to a solution of the Wheeler–deWitt equation. Hence, the black hole entropy could finally be derived from the changed wave functional in section 6.2 after the values of \(\tau_0\) and \(\tau_+\) were computed using the properties of the Euclideanized section of the Schwarzschild solution.

The fact that Hawking radiation emerges from the proper quantization of boundary terms at the event horizon can also be seen by the calculation of Gibbons and Hawking [13] who showed that a spherical star without an event horizon has no gravitational entropy at all.

In the Hamiltonian of eq. (542) from Dvali and Gomez, boundary terms that give raise to additional degrees of freedom at the event horizon, which then change the wavefunctional of the black hole, do not seem to be present. For this season, I question whether one can, as Dvali and Gomez claim in [71], correctly derive the black hole entropy from a Hamiltonian like the one they propose. It seems to me that their Hamiltonian would, at best, resemble something like the quantum Hamiltonian in eq. (542), which alone does not suffice to derive the black hole entropy, since the boundary terms that give rise to a change of the wave functional are missing. It is for all these reasons, why I find the proposals by Dvali and Gomez on black holes to be highly problematic.

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