Conservation laws of isentropic flow perturbations and the separation of acoustic waves

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Abstract: The paper deals with the linearisation of the isentropic Navier-Stokes equation around a new pathline-averaged base flow. As a consequence of the considered base flow, the perturbation equations satisfy a conservation law. It is demonstrated that this flow perturbations can be split into acoustic and vorticity modes, with the acoustic modes being independent of the vorticity modes.

Moreover, we conclude that the present acoustic perturbation

- is propagated by the convective wave equation;
- fulfills Lighthill’s acoustic analogy;
- satisfies an inhomogeneous convective wave equation with a known sound source.

In contrast to other authors, no assumptions on a slowly varying or irrotational flow were necessary. Using a symmetry argument for the conservation laws, an energy conservation result and a generalisation of the sound intensity is provided.

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1 Introduction

Nowadays, several manufacturers have to pay increasingly attention to the noise of high-unsteady flows. On the one hand, there are many applications where the background noise disturbs the desired acoustic signal. On the other hand, the acoustic comfort has become an important selection criteria for consumers. Therefore, it is surprising that the following significant question is open until now:

Is it possible to derive a closed system of equations for the acoustic quantities in the case of high-unsteady flows?

Hence, the general mechanisms of sound generation and propagation are also unknown. The aim of the paper is to isolate equations for the acoustic fluctuations in the case of high-unsteady flows. As usual, the following steps will be considered:

1. splitting the variables of the compressible Navier-Stokes equations \( \rho, u, p \) into their base flow (\( \bar{} \)) and their fluctuations (\( \cdot \));

2. linearising around the base flow;


3. neglecting the dissipation mechanisms for the fluctuations;
4. separating the acoustic fluctuations from the non-acoustic perturbations.

The definition of an acoustic perturbation is widely accepted being that part of fluctuation which is radiating with a velocity that depends on the speed of sound, whereas non-acoustic perturbations are convected by the hydrodynamic flow. Motivated by the large disparity between the energy of the hydrodynamic and acoustic variables, assumptions like

\[
\frac{|\rho'|}{|\rho|} = \mathcal{O}(\epsilon), \quad \frac{||u'||}{||u||} = \mathcal{O}(\epsilon), \quad \frac{|p'|}{|p|} = \mathcal{O}(\epsilon), \quad 0 < \epsilon \ll 1,
\]

ensures a small linearisation error. Because of the conservation form of the Navier-Stokes equations one may expect a resulting linear conservation law after step three. However, conservation laws for the fluctuation components are only known in special cases, e.g. in the case of uniform mean flows w.r.t. time averaging. Moreover, an exact separation of acoustic waves from other compressibility effects is also known exclusively in the mentioned case.

As a consequence, there is no commonly accepted set of acoustic equations available. In order to address this situation several non-radiating base flow have been suggested in the literature, e.g. incompressible flows by the viscous/acoustic splitting method in [HP94], [SS99], [SM05] and [SM06] and time averaged flows by the linearised Euler equations (LEE) and modifications in [BBJ02] and [ES03]. Since the different length scales in acoustic and hydrodynamic waves of low Mach number flows, the latter is in general responsible for grid-dependent and unstable solutions of the perturbation equations, respectively (cf. [ES03] and [SM06]).

Therefore, the major challenge in the design of perturbation equations is to ensure stability properties. We mention the main stability arguments:

- The underlying differential operator yields a conservation of the perturbed energy (cf. [Mö99], [ES03]). This a priori energy result is necessary for the well-posedness in the sense of Hadamard.

- Motivated by the special case of uniform mean flow, where the splitting theorems [Gol76, p. 220] and [ES03, Appendix A] claim that vorticity fluctuations are non-acoustic perturbations, stability could be improved by considering the condition (cf. [ES03], [SM06])

\[
\partial_t \left( \nabla \times u' \right) = 0.
\]

Obviously, the following question is of special interest:

Are vorticity fluctuations and their derivatives always non-acoustic perturbations for more general flows?

A second approach to determine the acoustic field is the use of the so called acoustic analogies. These methods are naturally exact but incomplete theories, i.e. they consist of an exact equation for a pressure fluctuation variable, which includes the unknowns of the compressible Navier-Stokes equation on the right hand side. Hence, in order to use the exact equation as an acoustic analogy the right-hand side has to be interpreted as a known source of sound. In other words, the mechanisms responsible for sound propagation are replaced by equivalent sources. Consequently, it is impossible to separate (and therefore predict) the sound propagation mechanisms from the physical sources of sound. Thus, for example, the famous acoustic analogy of Lighthill (cf. [Lig52]) is not be able to determine refraction and convection effects. Obviously, due to the required known source terms the acoustic analogies consider sound as a by-product of fluid dynamics.
The key motivation of the paper was the following question:

Can we define a base flow such that the equations for the perturbations can be formulated as a conservation law and what are the consequences for the separation of the acoustic waves?

It turns out that it is convenient to work with a base flow which is constant along the pathlines of the isentropic Navier-Stokes equations:

$$\partial_t \bar{p} + u \cdot \nabla \bar{p} = 0, \quad \partial_t \bar{u} + u \cdot \nabla \bar{u} = 0.$$  

Namely, sound waves are interpreted as vibrations around the fluid particle path. Because of $\nabla \bar{u} = 0$ in the case of constant base flow, the associated momentum equation implies $\nabla \bar{p} = 0$. Obviously, $\partial_t \bar{u} = 0$ and $\partial_t \bar{p} = 0$ and the proposed splitting method seems to be a natural generalisation of a time averaged splitting method in the case of uniform mean flow.

One of the major result of the paper is the extension of the splitting Theorem [Gol76, p. 220], which claimed that the acoustic pressure is only related to the rotational-free velocity. To the best of the author’s knowledge, the splitting Theorem under consideration represents the most general available result on the decomposition of compressible pressure fluctuations. As a consequence, the acoustic pressure fluctuation may be interpreted as the solution of Lighthill’s analogy and of a convective wave equation with a true sound source.

The paper is organised as follows. Section 2 introduces the compressible Navier-Stokes equations and the associated equations of state. After a short review of conservation laws and perturbation equations in the presence of uniform mean flow, Section 4 concerns the pathline-averaged base flow. The following Section is devoted to a rigorous investigation of the propagation of sound and proves some of the key results: a conservation law for the first order fluctuations of a high-unsteady flow and the Splitting Theorem, that is a closed system of equations for the acoustic quantities. The acoustic propagation operator is applied in Section 6 to identify the true sound source in Lighthill’s right-hand side. In Section 7 we discuss the structure of hyperbolic conservation laws. For this purpose we use a symmetry argument to prove an energy conservation result. Moreover, we propose a definition of sound intensity which reduces to the classical definition for a medium at rest.

### 2 Preliminaries

In the following it is assumed that all functions are sufficiently smooth such that all further operations are valid. Let us consider the compressible Navier-Stokes equations for a fixed time $\tau > 0$

$$\frac{D \rho}{D t} + \rho \nabla \cdot u = 0 \quad \text{in} \ (0, \tau) \times \mathbb{R}^d, \quad (2)$$

$$\rho \frac{D u}{D t} + \nabla p = \nabla \cdot \tau \quad \text{in} \ (0, \tau) \times \mathbb{R}^d, \quad (3)$$

$$\rho T \frac{D s}{D t} = -\nabla \cdot q + \tau : \nabla u \quad \text{in} \ (0, \tau) \times \mathbb{R}^d, \quad (4)$$

$$\left(\rho, u^\top, s\right)^\top = g \quad \text{in} \ \{0\} \times \mathbb{R}^d, \quad (5)$$

where $g : \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ is a known function and $\rho, p, s : [0, \tau) \times \mathbb{R}^d \to \mathbb{R}$, $u, q : [0, \tau) \times \mathbb{R}^d \to \mathbb{R}^d$, $\tau : [0, \tau) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ denote the density, pressure, entropy, velocity, heat flux and the stress tensor, respectively. Here and in what follows, the symbol $\frac{D}{D t} = \partial_t + u \cdot \nabla$ denotes the material derivative. Furthermore, $I \in \mathbb{R}^{d \times d}$ denotes the identity matrix. As usual, we use Fourier’s law $q = \lambda \nabla T$, and for a Newtonian medium Stokes’ hypothesis

$$\tau = \mu \left(\nabla u + (\nabla u)^\top - \frac{2}{3} \nabla \cdot u I\right),$$
with \( \lambda, \mu, T : [0, \tau) \times \mathbb{R}^d \to \mathbb{R} \) the heat conductivity, dynamic viscosity and the temperature. In order to close the system \((2) - (4)\), we consider two equations of state (cf. [Bat02 (1.5.14)])

\[
dv = \alpha_p v dT - \beta_T v dp,
\]

\[
ds = \frac{c_p}{T} dT - \alpha_p v dp,
\]

where

\[
\alpha_p = \frac{1}{v} \left( \frac{\partial v}{\partial T} \right)_p, \quad \beta_T = -\frac{1}{v} \left( \frac{\partial v}{\partial p} \right)_T, \quad c_p = T \left( \frac{\partial s}{\partial T} \right)_p, \quad v = \frac{1}{\rho},
\]

denotes the volume expansivity, isothermal compressibility, specific heat at constant pressure and the specific volume, respectively. If we can express a thermodynamic property as a function of two other variables of state, the fluid is called divariant. Consequently, the thermodynamic state of a general divariant fluid can be determined by the three measurable quantities \( \alpha_p, \beta_T \) and \( c_p \). Moreover, we can write equivalent equations of state under consideration of the speed of sound \( c = (\partial p/\partial \rho)^{1/2} \). To do so, using (cf. [CHS90])

\[
c_v = c_p - \frac{\alpha_p^2 v T}{\beta_T}, \quad c^2 = \frac{c_p}{c_v \beta_T},
\]

and interpreting \( dp, dp \) and \( ds \) along the pathlines, we conclude

\[
\frac{D \rho}{Dt} = \frac{1}{c^2} \frac{D \rho}{Dt} - \frac{\rho \alpha_p T}{c_p} \frac{D s}{Dt},
\]

\[
\frac{D T}{Dt} = \frac{T \alpha_p \rho}{c_p} \frac{D p}{Dt} + \frac{T c_p}{D s}{Dt}.
\]

In the following we restrict our attention to isentropic flows with \( \frac{D c}{Dt} = 0 \). As a consequence, the energy equation \((4)\) reduces to \( \frac{D s}{Dt} = 0 \). Thus we obtain the closed set of equations

\[
\frac{D \rho}{Dt} + \rho \nabla \cdot u = 0 \quad \text{in} \ (0, \tau) \times \mathbb{R}^d,
\]

\[
\rho \frac{D u}{Dt} + \nabla p = \nabla \cdot \tau \quad \text{in} \ (0, \tau) \times \mathbb{R}^d,
\]

\[
\frac{D \rho}{Dt} = \frac{1}{c^2} \frac{D p}{Dt} \quad \text{in} \ (0, \tau) \times \mathbb{R}^d,
\]

\[
(p, u^\top, \rho)^T = g \quad \text{in} \ \{0\} \times \mathbb{R}^d.
\]

The dimensionless form of the isentropic Navier-Stokes equation is obtained by using the following non-dimensional variables. For simplicity of notation, we replace

\[
\frac{t c_R}{x_R} \to t, \quad \frac{x}{x_R} \to x, \quad \frac{\rho}{\rho_R} \to \rho, \quad \frac{p}{p_R c_R^2} \to p, \quad \frac{\mu}{\mu_R} \to \mu, \quad \frac{c}{c_R} \to c, \quad \frac{x_R}{c_R} \partial_t \to \partial_t, \quad x_R \nabla \to \nabla.
\]

Consequently, it follows that

\[
\frac{D \rho}{Dt} + \rho \nabla \cdot u = 0 \quad \text{in} \ (0, \tau) \times \mathbb{R}^d,
\]

\[
\frac{D u}{Dt} + \nabla p = \frac{Ma}{Re} \nabla \cdot \tau \quad \text{in} \ (0, \tau) \times \mathbb{R}^d,
\]

\[
\frac{D \rho}{Dt} = \frac{1}{c^2} \frac{D p}{Dt} \quad \text{in} \ (0, \tau) \times \mathbb{R}^d,
\]

\[
(p, u^\top, \rho)^T = g \quad \text{in} \ \{0\} \times \mathbb{R}^d,
\]

where \( Ma = \frac{u c}{c_R} \) and \( Re = \frac{\rho_R u_R x_R}{\mu_R} \) are the global Mach and Reynolds number, respectively.
3 Conservation laws in the presence of uniform mean flow

In this section we give a brief introduction on conservation laws which can be considered as the most fundamental principles in physics. Using standard arguments, it is shown that the fluctuations around an uniform mean flow can be formulated as a conservation law.

Let $U$ be an intrinsic quantity of a principle under consideration and $F_i(U), i = 1, \ldots, d$ a vector field such that the physical system satisfies (with the help of Einstein’s summation convention)

$$\partial_t U + \partial_x F_i(U) = 0.$$  \hfill (1)

Integrating over an arbitrary but fixed domain $\Omega$ and applying the divergence Theorem, we see that the rate of change of the quantity $U$ inside $\Omega$ is equal to the flux across the boundary $\partial \Omega$

$$\frac{d}{dt} \int_\Omega U \, dx = - \int_{\partial \Omega} F(U) \cdot n \, ds,$$

where $n$ denotes the unit outward normal vector. If the equations also include dissipation effects the conservation law is called balance law. For instance, the compressible Navier-Stokes equations of the previous section are balance laws which involves acoustic effects as well as other compressible pressure fluctuations.

Assuming the simplest case where the medium is at rest ($\bar{u} = 0$) and carrying out the steps 1.-3. for the isentropic fluctuations ($p' = c^2 \rho'$) we satisfy the conservation law by setting

$$U = \begin{pmatrix} \frac{p'}{\rho^2 c^2} \\ \bar{\rho} \bar{u}_1 \\ \vdots \\ \bar{\rho} \bar{u}_d \end{pmatrix}, \quad F_i(U) = \begin{pmatrix} \frac{U_i}{\rho} \\ \bar{\rho} \bar{c}^2 U_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad F_d(U) = \begin{pmatrix} \frac{U_d}{\rho} \\ 0 \\ 0 \\ \bar{\rho} \bar{c}^2 U_1 \end{pmatrix},$$

where $\langle \cdot \rangle$ denotes the time mean value. Next, we can eliminate the velocity fluctuations and deduce the wave equation

$$\bar{D}^2 \left( \frac{p'}{c^2 \rho} \right) - \nabla \cdot \left( \frac{1}{\rho} \nabla p' \right) = 0.$$  \hfill (10)

Now, we consider a moving medium with mean flow ($\bar{u} = \text{const}$) together with a moving frame with coordinates $X \in \mathbb{R}^d$ and a fixed reference frame with coordinates $x(t) = X + \bar{u}t$. Since the medium is at rest in the moving frame, it follows that the wave equation

$$\partial_t^2 \left( \frac{P'}{c^2 \rho} \right) - \nabla_X \cdot \left( \frac{1}{\rho} \nabla_X P' \right) = 0$$  \hfill (11)

is still valid for $P'(t, X) = p'(t, x(t, X))$. In order to perform the transformation of (11) into the fixed frame, we differentiate $F(t, X) = f(t, x(t, X))$. That is, we have

$$\partial_t F = \partial_t f + \bar{u} \cdot \nabla f, \quad \partial_X F = \partial_{x_i} f \partial_{X_i} x_j = \partial_{x_j} f.$$  \hfill (12)

Introducing the notation $\frac{D}{Dt} = \partial_t + \bar{u} \cdot \nabla$ we can write the convective wave equation

$$\frac{D^2}{Dt} \left( \frac{p'}{c^2 \rho} \right) - \nabla \cdot \left( \frac{1}{\rho} \nabla p' \right) = 0.$$  \hfill (13)
Since $\partial X, x_j$ is not a constant diagonal matrix in general, (12) shows that one cannot generalise this strategy. On the other hand, we can derive a conservation law with

$$U = \begin{pmatrix} \rho' \\ \rho'c' \\ \vdots \\ \rho'n' \end{pmatrix}, \quad F_1(U) = \begin{pmatrix} \tilde{u}_1 U_1 + \frac{U_2}{\rho} \\ \tilde{u}_1 U_2 + \rho c^2 U_1 \\ \tilde{u}_1 U_3 \\ \vdots \\ \tilde{u}_1 U_{d+1} \end{pmatrix}, \ldots, \quad F_d(U) = \begin{pmatrix} \tilde{u}_d U_1 + \frac{U_{d+1}}{\rho} \\ \tilde{u}_d U_2 \\ \vdots \\ \tilde{u}_d U_d \\ \tilde{u}_d U_{d+1} + \rho c^2 U_1 \end{pmatrix},$$

and (13) resp. directly from the compressible Navier-Stokes equations, if the mass conservation for the mean flow $\bar{u} \cdot \nabla \bar{p} = -\bar{\rho} \nabla \bar{u} = 0$ is considered. Notice, that for $\bar{\rho} = \text{const}$ plane waves satisfy the impedance relation $p'/u' \cdot n = \bar{\rho}c$. Hence, together with $p' = c^2 \rho'$ we have

$$\frac{|\rho'|}{\bar{\rho}} = \frac{|u' \cdot n|}{c} \leq \frac{\|u'\|}{c} = \text{Ma'}.$$

Thus, for general solutions of the wave equation the fluctuation mach number $\text{Ma'}$ is a measure for the error in the linearisation step. Let us finally point out, that the differential operators of the conservation laws (10) and (13) are also the differential operators of the analogies of Lighthill (Lig52) and Möhring (Mö99).

### 4 Definition of the base flow

In order to consider the base flow, we recall some properties of time averaging. Let $f : (0, t) \to \mathbb{R}$ be a given function. Then, the time average of $f$ may be interpreted as the solution of the following ordinary differential equation

$$d_t f_0 = 0, \quad \text{in } (t_0, t_0 + T),$$

$$f_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \, dt, \quad \text{in } \{t_0\}.$$  \hspace{1cm} (14)

Moreover, consider

$$d_t g = \frac{f}{T}, \quad \text{in } (t_0, t_0 + T),$$

$$g = 0, \quad \text{in } \{t_0\},$$  \hspace{1cm} (15)

and applying the fundamental Theorem of calculus the above integral can be written as $g(t_0 + T)$. In the following we introduce the pathline-averaged base flow by time-averaging of a function which characterise a fluid particle $X$. To do so, let $x = (x_0, x^\top) \in \mathbb{R}^d$ denote a point in the time-space domain at position $x = (x_1, x_2, \ldots, x_d)^\top$ and time $x_0 = t$. Now, for every $x$ and every velocity field $u = (1, u^\top) \in \mathbb{R}^d$ one can find a fluid particle such that the movement of the particle $x_X(t) = (t, x(t)^\top)^\top$ defined by

$$\dot{x}_X(t) = u(x_X(t)), \quad x_X(0) = (0, X^\top)^\top,$$

satisfies $x = x_X(t)$. It remains to consider functions of the form $f, g : (0, t) \times \mathbb{R}^d \to \mathbb{R}$. From the lagrangian point of view we have to change $f_0 \to f_0 \circ x_X$ and $g \to g \circ x_X$ in (13) and (15) resp.

$$d_t (g \circ x_X) = \partial_t g + u \cdot \nabla g = \frac{f}{T}, \quad \text{in } (t_0, t_0 + T) \times \mathbb{R}^d,$$

$$g \circ x_X = 0, \quad \text{in } \{t_0\} \times \mathbb{R}^d.$$  \hspace{1cm} (17)
which is equivalent to the following backward problem for the time variable 

\[ 0 < \epsilon \ll 1. \]

Summarizing the above considerations, we have to solve (17) and (21) to determine the base flow form the given flow variable \( f \).

### 5 Propagation of sound

In this section we investigate the propagation of sound for high-unsteady flows. Namely, a system of perturbation equations in conservation form and, as a consequence, a closed system of equations for the acoustic quantities are derived. These results are consequences of an asymptotic analysis of the base flow and the isentropic Navier-Stokes equations followed by a projection onto the acoustic perturbations.

The perturbations are the first order fluctuations around the pathline-averaged base flow \( \bar{f}(x(t)) \), which satisfies property (1). We start with an uniform asymptotic expansion (cf. [Joh05, p. 20])

\[
\begin{align*}
\bar{f}(x(t); \epsilon) &= \bar{f}_0(x(t)) + f_1(x(t))\epsilon + O(\epsilon^2), \\
&= \bar{f}(x(t)) + \bar{f}'(x(t)) + O(\epsilon^2),
\end{align*}
\]

where \( \bar{f} = f_0 \) and \( f_1 \) are independent of \( \epsilon \). One important property of asymptotic expansions is that there is no need for a convergent-series representation of \( f \). Therefore, the following is assumed:

**Assumption 1.** The solutions \( \rho, u \) and \( p \) of (16) - (19) can be decomposed by an uniform asymptotic expansion around the pathline-averaged base flow

\[
\begin{align*}
\rho(t,x; \epsilon) &= \bar{\rho}(t,x) + \rho'(t,x) + O(\epsilon^2), \\
\bar{u}(t,x; \epsilon) &= \bar{u}(t,x) + \bar{u}'(t,x) + O(\epsilon^2), \\
\bar{p}(t,x; \epsilon) &= \bar{p}(t,x) + \bar{p}'(t,x) + O(\epsilon^2),
\end{align*}
\]
Then, by substituting the decompositions (23) in (6) – (9), we can apply a well known result from the asymptotic analysis which leads to a hierarchy of equations for the terms multiplied by the same power of $\epsilon$. Hence, from $\frac{D\rho}{Dt} = 0$ it follows that (8) can be solved by setting
\[ \rho_i = \frac{p_i}{c^2}, \quad i \in \mathbb{N}_0. \] (24)
Moreover, (6) and (7) reduce to
\[ \frac{D\rho'}{Dt} c^2 + \bar{\rho} \nabla \cdot \bar{u}' + \rho' \nabla \cdot u' = -\bar{\rho} \nabla \cdot \bar{u} + O(\epsilon^2), \] (25)
\[ (\bar{p} + \rho') \frac{Du'}{Dt} + \nabla p' - \frac{Ma}{Re} \nabla \cdot \tau' = -\nabla \bar{p} + \frac{Ma}{Re} \nabla \cdot \bar{\tau} + O(\epsilon^2). \] (26)
Now, (24) immediately implies
\[ \frac{D\rho'}{Dt} c^2 + \bar{\rho} \nabla \cdot \bar{u}' + \rho' \nabla \cdot u' = -\bar{\rho} \nabla \cdot \bar{u} - \frac{p'}{c^2} \nabla \cdot u' + O(\epsilon^2), \] (27)
\[ \bar{p} \frac{Du'}{Dt} + \nabla p' - \frac{Ma}{Re} \nabla \cdot \tau' = -\nabla \bar{p} + \frac{Ma}{Re} \nabla \cdot \bar{\tau} - \frac{p'}{c^2} \frac{Du'}{Dt} + O(\epsilon^2), \] (28)
resp.
\[ \partial_t \left( \frac{p' y}{c^2} \right) + \nabla \cdot \left( \frac{p' y}{c^2} u + p' I - \frac{Ma}{Re} \tau' \right) = -\nabla \bar{p} + \frac{Ma}{Re} \nabla \cdot \bar{\tau} - \frac{p'}{c^2} \frac{Du'}{Dt} + \bar{\rho} u' \nabla \cdot u + O(\epsilon^2), \] (29)
\[ \partial_t \left( \bar{\rho} u' \right) + \nabla \cdot \left( \bar{\rho} u' \otimes u + p' I - \frac{Ma}{Re} \tau' \right) = -\nabla \bar{p} + \frac{Ma}{Re} \nabla \cdot \bar{\tau} - \frac{p'}{c^2} \frac{Du'}{Dt} + \bar{\rho} u' \nabla \cdot u + O(\epsilon^2), \] (30)
where we have used the notation $\nabla \cdot (a \otimes b) = \partial_i (a_j b_i), i, j = 1, \cdots, d$.

**Lemma 1.** Let Assumption 1 be satisfied, $\frac{D\rho}{Dt} = 0$ and $\frac{Ma}{Re} = O(\epsilon^k), k \geq 1$. Then, for the base components of (23) it holds that
\[ (i) \quad \nabla \cdot \bar{u} = 0, \quad \bar{\rho} = \text{constant}, \] (31)
\[ (ii) \quad \frac{D}{Dt} \bar{u} = 0, \quad u' \cdot \nabla \bar{u} = 0, \] (32)
\[ (iii) \quad \nabla \bar{u} : (\nabla \bar{u})^\top = \partial_i \bar{u}_j \partial_j \bar{u}_i = 0. \] (33)

**Proof.** In the following we are performing an asymptotic analysis again. To do so, the first assertion follows by the consideration of leading order terms in (29) and (30) and the definition of the pathline-averaged pressure. Next, the definition of the pathline-averaged velocity implies the second statement. Finally, the identity
\[ \nabla \cdot \frac{D}{Dt} \bar{u} = \frac{D}{Dt} \nabla \cdot \bar{u} + \nabla \bar{u} : (\nabla \bar{u})^\top \] completes the proof.

**Lemma 2.** Let Assumption 1 be satisfied, $c = \text{constant}$ and $\frac{Ma}{Re} = O(\epsilon^k), k \geq 1$. Then, for the base components of (23) it holds that
\[ \bar{\rho} = \frac{\bar{p}}{c^2} = \text{constant}. \] (34)
Proof. The desired result follows by using (24) and Lemma 1.

Note that for the aforementioned Lemmas no assumption on a slowly varying flow is necessary. Furthermore, the approach under consideration can be interpreted as a perturbation ansatz about an incompressible base flow.

Now, we obtain a conservation law for the perturbations around the base flow.

**Theorem 2.** Let Assumption \( \frac{Ma}{Re} = \mathcal{O}(\epsilon^k), k \geq 2 \) be satisfied and let \( g : \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d \) be a known function. Then, the first order fluctuation components of (23) are solutions of

\[
\begin{align*}
\partial_t \left( \frac{p'}{\rho c^2} \right) + \nabla \cdot \left( \frac{p'}{\rho c^2} \bar{u} + u' \right) &= 0 \quad \text{in} \ (0, \tau) \times \mathbb{R}^d, \\
\partial_t (\bar{\rho} u') + \nabla \cdot (\bar{\rho} u' \otimes \bar{u} + p' I) &= 0 \quad \text{in} \ (0, \tau) \times \mathbb{R}^d, \\
(p', u')^T &= g \quad \text{in} \ \{0\} \times \mathbb{R}^d.
\end{align*}
\]

**Proof.** The proof is obtained by using Lemma 1 and the equations for the first order terms.

**Remark 1.** The assumption \( \frac{Ma}{Re} = \mathcal{O}(\epsilon^k), k \geq 2 \) may be changed to \( \frac{Ma}{Re} = \mathcal{O}(\epsilon) \) by adding \( \frac{Ma}{Re} \nabla \cdot \bar{\tau} \) on the right hand side of (36). Moreover, \( \frac{Ma}{Re} \) can be interpreted as a linearisation error.

**Theorem 3 (Splitting Theorem).** Let Assumption \( \frac{Ma}{Re} = \mathcal{O}(\epsilon^k), k \geq 2 \) be satisfied. Then, the first order fluctuation components of (23) can be split uniquely into an acoustic variable, which is radiated by a \( c \)-dependent velocity, and into the hydrodynamic variable

\[
\begin{align*}
(p', u') &= \begin{pmatrix} p' \\ u' \end{pmatrix} + \begin{pmatrix} 0 \\ u^a \end{pmatrix}, \quad \nabla \times u^a = 0, \quad \nabla \cdot u^v = 0.
\end{align*}
\]

Moreover, the acoustic variables satisfy independent equations

\[
\begin{align*}
\frac{\bar{D}}{Dt} \left( \frac{p'}{\rho c^2} \right) + \nabla \cdot u^a &= 0, \\
\frac{\bar{D}}{Dt} \nabla \cdot u^a + \frac{1}{\rho} \Delta p' &= 0, \\
\nabla \times \frac{\bar{D}}{Dt} u^v &= -\nabla \times \frac{\bar{D}}{Dt} u^a.
\end{align*}
\]

**Proof.** Using the notation \( U = (p', u'^T)^T \) and denote the vectors of unity by \( e_j, j = 1, \cdots, d + 1 \), the equations (35) and (36) directly give

\[
L(U) = \partial_t U + A_j \partial_{e_j} U = -\partial_{e_j} s_j,
\]

where

\[
A_j = \frac{\rho c^2}{\rho c} e_1 e_{j+1}^T + \frac{1}{\rho} e_{j+1} e_1^T,
\]

are constant matrices and

\[
s_j = (p' \bar{u}_j) e_1 + \sum_{k=1}^{d} \frac{u'_{k}}{\rho_{j+1}} \bar{u}_j e_{k+1} = S_{kj}.
\]
Therefore, we can apply the Laplace / Fourier transformation \( \hat{f}(\bar{\omega}, \alpha) = LF[f(t,x)](\bar{\omega}, \alpha) \), that is

\[
\hat{f}(\bar{\omega}, \alpha) = \frac{1}{(2\pi)^{d+1}} \int_0^\infty \int_{\mathbb{R}^d} f(t,x) e^{-i(\alpha \cdot x - \bar{\omega} t)} \, dx \, dt.
\]

The inverse transformation is defined by

\[
f(t,x) = LF^{-1}[\hat{f}(\bar{\omega}, \alpha)](t,x) = \int_{\mathbb{R}^d} \hat{f}(\bar{\omega}, \alpha) e^{i(\alpha \cdot x - \bar{\omega} t)} \, d\alpha \, d\bar{\omega}, \quad \sigma > \sigma_0,
\]

where \( \sigma \) is a constant number such that the path of integration is in the region of convergence of \( \hat{f}(\bar{\omega}, \alpha) \).

Using the following properties of the Laplace / Fourier transformation

\[
LF[\lambda f + \mu g] = \lambda LF[f] + \mu LF[g], \quad \lambda, \mu \in \mathbb{R},
\]

\[
LF[\partial_t f] = -i\bar{\omega} LF[f] - \frac{1}{2\pi} f_0,
\]

\[
LF[\partial_{x_j} f] = i\alpha_j LF[f], \quad j = 1, \ldots, d,
\]

where

\[
f_0(\alpha) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(0, x) e^{i\alpha \cdot x} \, dx,
\]

it immediately implies

\[
LF[L(U)] = -i (\bar{\omega} I - \alpha_j A_j) \hat{U} - \frac{1}{2\pi} U_0 = -i\alpha_j \hat{s}_j.
\]

In order to separate the \( c \)-dependent acoustic terms from the hydrodynamic terms, we consider the decomposition \( A = RAR^{-1} \), with the matrices given by

\[
R = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
\alpha_d & 0 & \cdots & 0 \\
0 & \alpha_d & \ddots & \vdots \\
\|\alpha\| c \rho & \|\alpha\| c \rho & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_d \\
-\alpha_1 - \alpha_2 - \cdots - \alpha_{d-1}
\end{pmatrix}
\]

and

\[
\Lambda = \begin{pmatrix}
-\|\alpha\| c + \omega & 0 & 0 & 0 & \cdots & 0 \\
0 & \|\alpha\| c + \omega & 0 & 0 & \cdots & 0 \\
0 & 0 & \omega & 0 & \cdots & 0 \\
0 & 0 & 0 & \omega & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0 & \omega
\end{pmatrix}.
\]

Then, the decoupled system of equations follow as

\[
\Lambda R^{-1} \hat{U} = iR^{-1} \left( -i\alpha_j \hat{s}_j + \frac{1}{2\pi} U_0 \right).
\]
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Now, by the concrete structure of $R$ and $\Lambda$, we obtain the projected acoustic equations

$$RI_aAR^{-1}\dot{U} = RI_aR^{-1}\left(\alpha_j\dot{s}_j + \frac{i}{2\pi}U_0\right)$$

(42)

and the projected hydrodynamic equations

$$RI_vAR^{-1}\dot{U} = RI_vR^{-1}\left(\alpha_j\dot{s}_j + \frac{i}{2\pi}U_0\right),$$

(43)

where $I_a = e_1e_1^\top + e_2e_2^\top$ and $I_v = I - I_a$. Using the notation $B = (\alpha\alpha^\top)/\|\alpha\|^2$, it is not difficult to see that

$$RI_aAR^{-1} = \begin{pmatrix}
\omega & -\alpha_1c^2\rho & \cdots & -\alpha_dc^2\rho \\
-\frac{\alpha_1}{\rho} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
-\frac{\alpha_d}{\rho} & \cdots & \ddots & \ddots \\
\omega B & \cdots & \cdots & \cdots 
\end{pmatrix}$$

and

$$RI_vR^{-1} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}$$

are equal in the first row to $A$ and $I$, resp. Therefore, we conclude that (35) is a purely acoustic equation. The remaining acoustic equations lead to

$$-\frac{\alpha}{\rho}\dot{p}' + \omega B\dot{u}' = B\left(\dot{S}\alpha + \frac{i}{2\pi}u_0\right),$$

which reduce to the scalar equation

$$-\frac{\|\alpha\|^2}{\rho}\dot{p}' + \omega\alpha^\top\dot{u}' = \alpha^\top\dot{S}\alpha + \frac{i}{2\pi}\alpha^\top u_0.$$  

Now, the back transformation yields

$$\partial_t\nabla \cdot u' + \frac{1}{\rho}\Delta p' + \nabla \cdot \nabla \cdot (u' \otimes u) = 0.$$  

(44)

Applying the product rule, it follows that

$$\nabla \cdot (\dot{u} \otimes u') = \nabla \cdot \nabla \cdot (u' \otimes \dot{u}).$$

Moreover, we obtain from (32)

$$\nabla \cdot (\dot{u} \otimes u') = \dot{u} \nabla \cdot u',$$

which leads to

$$\partial_t\nabla \cdot u' + \frac{1}{\rho}\Delta p' + \nabla \cdot (\dot{u} \nabla \cdot u') = 0.$$  

(45)

By the Helmholtz decomposition (cf. [Ach73, Chapter 3.5]) we can decouple the irrotational and solenoidal parts of the velocity perturbation such that

$$u' = u^a + u^v = \nabla \varphi + \nabla \times \psi, \quad \nabla \cdot \psi = 0.$$  

Consequently,

$$\nabla \times u^a = 0, \quad \nabla \cdot u^v = 0.$$
proves the assertion (40).

Second, concerning the hydrodynamic projection we have

\[
RI_v \Lambda R^{-1} = \begin{pmatrix}
0 & \cdots & 0 \\
\omega C \\
0
\end{pmatrix}, \quad RI_v R^{-1} = \begin{pmatrix}
0 & \cdots & 0 \\
C \\
0
\end{pmatrix}, \quad C = \frac{\|\alpha\|^2 I - \alpha \alpha^T}{\|\alpha\|^2}.
\]

Thus, we obtain

\[
\omega \left( \alpha^T \alpha I - \alpha \alpha^T \right) \dot{u}' = \left( \alpha^T \alpha I - \alpha \alpha^T \right) \left( S\alpha + \frac{i}{2\pi} u_0 \right).
\] (46)

In order to consider the back-transformation we separate the two cases

\[
d = 2 : \quad \alpha^T \alpha I - \alpha \alpha^T = \begin{pmatrix}
\alpha_2^2 & -\alpha_1 \alpha_2 \\
-\alpha_1 \alpha_2 & \alpha_1^2
\end{pmatrix} = \begin{pmatrix}
\alpha_2 \\
-\alpha_1
\end{pmatrix} \left( \alpha_2, -\alpha_1 \right),
\]

\[
d = 3 : \quad \alpha^T \alpha I - \alpha \alpha^T = \begin{pmatrix}
\alpha_2^2 + \alpha_3^2 & -\alpha_1 \alpha_2 & -\alpha_1 \alpha_3 \\
-\alpha_1 \alpha_2 & \alpha_1^2 + \alpha_3^2 & -\alpha_2 \alpha_3 \\
-\alpha_1 \alpha_3 & -\alpha_2 \alpha_3 & \alpha_1^2 + \alpha_2^2
\end{pmatrix} = \begin{pmatrix}
0 & -\alpha_3 & \alpha_2 \\
\alpha_3 & 0 & -\alpha_1 \\
-\alpha_2 & \alpha_1 & 0
\end{pmatrix}.
\]

Applying the Helmholtz decomposition, we conclude that the Laplace / Fourier transformation of (41) satisfies (46).

\[\square\]

**Remark 2.** The equations (39) and (40) answer the question raised in the introduction: Vorticity fluctuations and their derivatives are always non-acoustic perturbations for the flow under consideration. Let us highlight the key ingredients:

- The pathline-averaged base flow and their properties from Lemma 1 and 2;
- The divergence structure of conservation law (35) – (37);
- The separation of the divergence free velocity perturbations from the irrotational part.

The divergence structure of the conservation law was essential to formulate an equation without the divergence free velocity (see (40)). Therefore, the author believes that it is not possible to derive an equation for \( u' \) without the irrotational velocity (see (41)).

Moreover, the propagation of sound can also be formulated as follows:

**Corollary 1.** Under the assumptions of Theorem 3 the acoustic perturbations satisfy

\[
\frac{1}{c^2} \left( \frac{\bar{D}}{Dt} \right)^2 p' - \Delta p' = 0 \quad \text{in} \ (0, \tau) \times \mathbb{R}^d,
\]

\[
p' = p'_0 \quad \text{in} \ \{0\} \times \mathbb{R}^d.
\] (47) (48)

**Proof.** Taking \( \frac{\bar{D}}{Dt} \) (39) – (40) immediately implies the result. \[\square\]

**Corollary 2.** Under the assumptions of Theorem 3 the acoustic perturbations equipped with suitable initial conditions satisfy in \( (0, \tau) \times \mathbb{R}^d \)

\[
\partial_t \left( \frac{p'}{\rho c^2} \right) + \nabla \cdot \left( \frac{p'}{\rho c^2} u + u^a \right) = 0,
\] (49)
and one of the following equations

\[
\begin{align*}
\partial_t (\bar{\rho} u^a) + \nabla \cdot (\bar{\rho} u^a \otimes \bar{u} + p'I) &= - \nabla \cdot (\bar{\rho} u^a \otimes \bar{u}) , \\
\partial_t (\bar{\rho} u^a) + \nabla (\bar{\rho} u^a \cdot \bar{u}) + \nabla p' &= \bar{\rho} u^a \cdot (\nabla \bar{u}) - \nabla \cdot (\bar{\rho} u^a \otimes \bar{u}) , \\
\partial_t (\bar{\rho} u^a) + \nabla \cdot (\bar{u} \otimes \bar{\rho} u^a + p'I) &= \bar{\rho} u^a \cdot \nabla \bar{u} .
\end{align*}
\]

(50)\hline
(51)\hline
(52)

Proof. Using the splitting (38) the first equation for the acoustic velocity follows directly from (36). Then, the identity

\[
\nabla \cdot (u^a \otimes \bar{u}) = \bar{u}_i \partial_i u^a_j = \bar{u}_i \partial_j u^a_i = \nabla (u^a \cdot \bar{u}) - u^a \cdot (\nabla \bar{u})^T
\]

implies (51). Finally, after taking the divergence, (52) satisfies (40).

\]

6 Sources of sound

In the previous section it was demonstrated that the propagation of sound waves are determined by the homogeneous convective wave equation. Now, we define the sources of sound as the deviations from the aforementioned equation. To do so, the connection of the acoustic fluctuation along the pathline-averaged flow to the Lighthill approach is discussed. Therefore, the current understanding of Lighthill’s solution \( \rho'_L \) and \( p'_L \), resp. is repeated [Fed00, p. 746]: “There is no reason for regarding \( \rho'_L \) and \( p'_L \) as the acoustic components because no accurate definition of acoustic components in a high-unsteady flow has been given in reference [Lig52], so that \( \rho'_L \) and \( p'_L \) represent merely the differences between the local values of flow variables \( \rho, p \) and arbitrary constants \( \rho_0, p_0 \).”

In this context let us highlight that the acoustic pressure perturbation satisfies Lighthill’s analogy:

\[\text{Corollary 3. Let the assumptions of Theorem 3 be satisfied. Then, the acoustic perturbations satisfy}
\]

\[
\begin{align*}
\frac{1}{c^2} \partial_t^2 p' - c^2 \Delta p' &= \nabla \cdot \nabla \cdot (\rho u \otimes u) + O(\epsilon^2) \quad \text{in } (0, \tau) \times \mathbb{R}^d , \\
p' &= p'_0 \quad \text{in } \{0\} \times \mathbb{R}^d .
\end{align*}
\]

(53)\hline
(54)

Proof. As usual, the Navier-Stokes equations (6) – (11) implies

\[
\begin{align*}
\partial_t^2 \rho' - c^2 \Delta \rho &= \nabla \cdot \nabla \cdot (\rho u \otimes u + (p - c^2 \rho) I - \frac{Ma}{Re} \tau) .
\end{align*}
\]

(55)

Using (24) and \( \frac{Ma}{Re} \in O(\epsilon^2) \) we obtain

\[
\begin{align*}
\frac{1}{c^2} \partial_t^2 p' - \Delta p' &= \nabla \cdot \nabla \cdot (\rho u \otimes u) + O(\epsilon^2) .
\end{align*}
\]

Furthermore, applying (31) and the decomposition (23) it follows that

\[
\begin{align*}
\frac{1}{c^2} \partial_t^2 p' - \Delta p' &= \nabla \cdot \nabla \cdot (\rho u \otimes u) + O(\epsilon^2) .
\end{align*}
\]

By the Splitting Theorem we conclude that \( p' \) is an acoustic quantity.

It is a well known fact, that the right-hand side depends on the solution variable \( p' \). Therefore, the right-hand side has to be interpreted / modelled as an equivalent source. This interpretation was often criticised because both sides are differential expressions of equal standing. Now, the previous results can be used to attack this problem and identify the true sources of sound.
Theorem 4. Let the assumptions of Theorem 3 be satisfied. Then, the acoustic perturbations satisfy

\[ \frac{1}{c^2} \left( \frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \right)^2 \rho' = \nabla \cdot \nabla \cdot \left( \rho \mathbf{u}' \otimes \mathbf{u}' \right) + O(\epsilon^2) \text{ in } (0, \tau) \times \mathbb{R}^d, \tag{56} \]

\[ \rho' = \rho'_0 \text{ in } \{0\} \times \mathbb{R}^d. \tag{57} \]

Proof. By the definition of the material derivative and (31), we have

\[ \left( \frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \right)^2 \rho = \partial_t^2 \rho + \partial_i \bar{u} \cdot \nabla \rho + 2 \bar{u} \cdot \nabla \partial_i \rho + \nabla \cdot \nabla \cdot \left( \rho \bar{u} \otimes \bar{u} \right) - \nabla \bar{u} : \nabla \left( \rho \bar{u} \right)^\top. \]

Then, taking into consideration (33) and (2), it follows

\[ -\nabla \bar{u} : \nabla \left( \rho \bar{u} \right)^\top + 2 \bar{u} \cdot \nabla \partial_t \rho = -\left( \bar{u} \cdot \nabla \bar{u} \right) \cdot \nabla \rho - 2 \bar{u} \cdot \nabla \nabla \cdot \left( \rho \mathbf{u} \right). \]

Moreover, the relations (31) and (32) yield

\[ -\bar{u} \cdot \nabla \nabla \cdot \left( \rho \mathbf{u} \right) = -\nabla \cdot \nabla \cdot \left( \rho \bar{u} \otimes \mathbf{u} \right) + \nabla \cdot \left( \rho \bar{u} \cdot \nabla \bar{u} \right). \]

Hence, applying (31) and (32) again, we have

\[ \left( \frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \right)^2 \rho = \partial_t^2 \rho - \nabla \cdot \nabla \cdot \left( \rho \mathbf{u} - \rho \mathbf{u}' \right), \]

which completes the proof by using the analog arguments as in the proof of Corollary 3. \qed

From the point of view of hybrid methods, we have a known hydrodynamic solution \( \mathbf{u} \) and after the computation of the pathline-averaged base flow the right-hand side is known. Hence, (56) is a closed equation with the correct propagation of sound and a non-linear sound source of second order.

7 Hyperbolic conservation laws

In the following, it is demonstrated that the first order perturbations of (23) satisfy an energy conservation law. This energy identity which is based on a symmetry argument is an analogous result to the energy Theorem of Möhring’s analogy [Mö99]. As a consequence, we can define the sound intensity which is a generalisation of the classical definition for high-unsteady flows.

Notice that, the system of Theorem 2 can be written in conservation form

\[ L(U) = \partial_t U + \partial_{x \mathbf{i}} F_i(U) = 0, \text{ in } (0, \tau) \times \Omega, \]

\[ U = U_0, \text{ in } \{0\} \times \Omega. \]

Here, the solution vector and the flux functions are defined by

\[ U = \left( \frac{\rho'}{\rho c^2}, \bar{\rho} \mathbf{u}' \right)^\top, \quad F_i(U) = \bar{u}_i U + \frac{U_{i+1}}{\rho} e_1 + \bar{\rho} c^2 U_1 e_{i+1}, \]

where \( e_i, i = 1, \cdots, d+1 \) are the vectors of unity. Defining the coefficient mapping \( A_i : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1,d+1} \) by

\[ A_i = \bar{u}_i I + \frac{1}{\rho} e_i e_{i+1}^\top + \bar{\rho} c^2 e_{i+1} e_1^\top, \]

we obtain the identity \( F_i(U) = A_i U \). Using the product rule we find \( \partial_{x \mathbf{i}} F_i(U) = A_i \partial_{x \mathbf{i}} U + \partial_{x \mathbf{i}} A_i U \).
Now, setting \( A(\nu) = A_1 \nu \),
for all \( \nu \in \mathbb{R}^d \), we can show that the system is hyperbolic. In other words, the matrix \( A(\nu) \) can be diagonalised as
\[
A(\nu) = RAR^{-1}, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{d+1}), \quad \lambda_1 \leq \cdots \leq \lambda_{d+1} \in \mathbb{R},
\]
where the eigenvalues are given by
\[
\lambda_1 = \bar{u} \cdot \nu - c, \quad \lambda_2 = \cdots = \lambda_d = \bar{u} \cdot \nu, \quad \lambda_{d+1} = \bar{u} \cdot \nu + c.
\]

### 7.1 The conservation law in symmetry variables

Let \( S_0 : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1, d+1} \) be a symmetric positive definite mapping which simultaneously symmetrises all \( A_i \) from left, that is \( S_0 A_i = (S_0 A_i)^\top \). It follows that \( A_i S_0^{-1} = S_0^{-1} S_0 A_i S_0^{-1} \) is also symmetric and consequently,
\[
L(U) = S_0^{-1} \partial_t (S_0 U) + A_i S_0^{-1} \partial_{x_i} (S_0 U) + \partial_{x_i} A_i U - S_0^{-1} \partial_t S_0 U - A_i S_0^{-1} \partial_{x_i} S_0 U.
\]

Moreover, setting \( \hat{A}_0 = S_0^{-1}, \hat{A}_i = A_i \hat{A}_0 \) and \( V = \hat{A}_0^{-1} U \), we get
\[
\hat{L}(V) = L(U(V)) = \hat{A}_0 \partial_t V + \hat{A}_i \partial_{x_i} V + (\partial_t \hat{A}_0 + \partial_{x_i} \hat{A}_i) V.
\]

Now, we consider \( V = (\bar{\rho} p', \bar{\rho} u'^\top)^\top = (\bar{\rho}^2 c^2 U_1, U_2, \ldots, U_{d+1})^\top \) and a short calculation gives
\[
\hat{A}_0 = \frac{1}{\bar{\rho}^2 c^2 e_1 e_1^\top + \sum_{j=2}^{d+1} e_j e_j^\top}, \quad \hat{A}_i = \frac{\bar{u}_i}{\bar{\rho}^2 c^2} e_1 e_1^\top + \sum_{j=2}^{d+1} \bar{u}_i e_j e_j^\top + \frac{1}{\bar{\rho}} e_1 e_{i+1} + \frac{1}{\bar{\rho}} e_{i+1} e_1^\top,
\]
which satisfies the symmetry conditions on \( \hat{A}_0 \) and \( \hat{A}_i \). Setting
\[
\hat{F}_i(V) = F_i(U(V)) = \hat{A}_i V = \bar{u}_i \left( \frac{V_1}{\bar{\rho}^2 c^2}, V_2, \ldots, V_{d+1} \right)^\top + \frac{V_{i+1}}{\bar{\rho}} e_1 + \frac{V_i}{\bar{\rho}} e_{i+1},
\]
we have the symmetric hyperbolic system in conservation form:
\[
\hat{A}_0 \partial_t V + \partial_{x_i} \hat{F}_i(V) + \partial_t \hat{A}_0 V = 0, \text{ in } (0, \tau) \times \Omega,
\]
\[
V = V_0, \text{ in } \{0\} \times \Omega.
\]

Finally, the framework of symmetry variables for hyperbolic conservation laws allows us to prove the energy conservation law.

**Theorem 5.** Let the assumptions of Theorem[3] be satisfied. Then, the fluctuation components of \( q_1(U) \) admit an additional conservation law
\[
\partial_t \eta(U) + \partial_{x_i} q_i(U) = 0, \text{ in } (0, \tau) \times \Omega,
\]
\[
U = U_0, \text{ in } \{0\} \times \Omega,
\]
where
\[
\eta(U) = \frac{1}{2} V(U)^\top \hat{A}_0 V(U) = \frac{1}{2} \left( \bar{\rho}^2 c^2 U_{1}^2 + U_2^2 + \cdots + U_{d+1}^2 \right) = \frac{1}{2} \left( \frac{p'^2}{c^2} + \bar{\rho}^2 u'^\top u' \right),
\]
\[
q_i(U) = \frac{1}{2} V(U)^\top \hat{A}_i V(U) = \eta(U) \bar{u}_i + \bar{\rho}^2 U_{i} U_{i+1} = \eta(U) \bar{u}_i + \bar{\rho} p' u'_i.
\]
Under the assumption that $U$ vanishes for large but finite $\|x\|$, there exists the energy
\[
\int_{\mathbb{R}^d} \eta(U(t, x)) \, dx = \int_{\mathbb{R}^d} \eta(U(0, x)) \, dx, \quad \forall t \in [0, \tau],
\]
and the acoustic energy identity, resp.
\[
\frac{1}{2} \int_{\mathbb{R}^d} \rho'(t, x)^2 + \rho^2 (u^n(t, x))^\top u^n(t, x) \, dx = \int_{\mathbb{R}^d} \eta(U(0, x)) \, dx - \frac{1}{2} \int_{\mathbb{R}^d} \rho^2 (u^v(t, x))^\top u^v(t, x) \, dx, \quad \forall t \in [0, \tau].
\]

**Proof.** By the symmetry of (60) it follows that
\[
V^\top \hat{L}(V) = \partial_t \left( \frac{1}{2} V^\top \hat{A}_0 V \right) + \partial_{x_i} \left( \frac{1}{2} V^\top \hat{A}_i V \right)
= \partial_t \eta(U) + \partial_{x_i} q_i(U) = 0.
\]

Now, if $U$ vanishes for large but finite $\|x\|$, then it follows that
\[
\int_{\mathbb{R}^d} V^\top \hat{L}(V) \, dx = \frac{d}{d\tau} \int_{\mathbb{R}^d} \eta(U(\tau, x)) \, dx = 0.
\]

Moreover, the orthogonality
\[
\int_{\mathbb{R}^d} (u^a)^\top u^v \, dx = -\int_{\mathbb{R}^d} \varphi \nabla \cdot u^v \, dx = 0
\]
finishes the proof.

Finally, as a direct consequence of Theorem 5, the sound intensity can be generalised for high-unsteady flows. By denoting
\[
\bar{f}^i(x) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(t, x) \, dt
\]
the temporal average, we have

**Corollary 4.** Let the assumptions of Theorem 3 be satisfied. Then, the fluctuation components of (23) admit an additional conservation law
\[
\partial_t E(U) + \partial_{x_i} \left( E(U) \bar{u}_i + p' u_i \right) = 0, \quad \text{in } (0, \tau) \times \Omega,
\]
\[
U = U_0, \quad \text{in } \{0\} \times \Omega,
\]
where
\[
E(U) = \frac{1}{2} \left( \frac{p'^2}{\bar{\rho} c^2} + \bar{\rho} u^\top u' \right).
\]

Under the assumption that $E$ is bounded, the perturbed intensity $I = E \bar{u} + p' u'$ satisfies $\nabla \cdot I = 0$. 
8 Conclusions

In this paper we have done a mathematical analysis of vibrations along the fluid particle path. It shall be outlined to note that this physically meaningful definition of sound and pseudosound fulfills a conservation law. In the presence of high-unsteady flows, the paper presented the first closed convective wave equation for the propagation of sound without approximations. Moreover, the acoustic fluctuation under consideration shedded some new light on the understanding of Lighthill’s wave equation, i.e. an inhomogeneous convective wave equation with a known sound source was derived. We also derived a conservation law for the perturbed energy, which is capable to generalise the sound intensity.

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