A test of the g–ology model for one-dimensional interacting Fermi systems

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Bosonization predicts that the specific heat, \(C(T)\), of a one-dimensional interacting Fermi system is a sum of the specific heats of free collective charge and spin excitations, plus the term with the running backscattering amplitude which flows to zero logarithmically with decreasing \(T\). We verify whether this result is reproduced in the g–ology model. Of specific interest are the anomalous terms in \(C(T)\) that depend on the bare backscattering amplitude. We show that these terms can be incorporated into a renormalized spin velocity. We do this by proving the equivalence of the results for \(C(T)\) obtained within the g–ology model and by bosonization with velocities obtained by the numerical solution of the Bethe-ansatz equations for the Hubbard model.

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One-dimensional interacting fermionic systems are believed to be described by an effective low-energy theory: the g–ology model \([1, 2, 3]\). This model involves a small number of interaction vertices describing small momentum scattering of fermions near the same and opposite Fermi points \((g_4 \text{ and } g_2, \text{ correspondingly})\) and \(2k_F\) scattering \(g_1\) [for fermions on a lattice, there is also an umklapp vertex \((g_6)\)]. The effective vertices are, in principle, obtained by integrating out high-energy fermions in microscopic models, e.g., the Hubbard model. To first order in \(U\), \(g_i = U/(2\pi v_F)\); beyond first order, all \(g_i\) are different, and each of them is represented by a series in \(U\).

A powerful way to treat the g–ology model is bosonization, which transforms interacting fermions into the collective bosonic excitations in the charge and spin channels \([3, 5]\). For the case of a repulsive interaction between original fermions, considered in this paper, the bosonization shows that the charge sector is a free Gaussian theory, while the spin sector becomes asymptotically free in the low-energy limit, when the coupling of the marginally irrelevant process \((2k_F\) scattering of fermions of opposite spins) flows to zero \([6, 7, 8, 9]\). As a consequence, bosonization predicts that at the lowest temperatures the specific heat of interacting 1D fermions is the same as the specific heat of two systems of acoustic 1D phonons, i.e.,

\[
C(T) = \frac{\pi T}{3} \left( \frac{1}{v_\rho} + \frac{1}{v_\sigma} \right),
\]

(1)

where \(v_\rho\) and \(v_\sigma\) are the charge and spin velocities, correspondingly. Eq. (1) was verified by a weak-coupling renormalization group (RG) treatment of the original fermionic model, in which all non-logarithmic corrections to the couplings were neglected \([1, 3, 5, 10]\). However, it has never been proven explicitly that the bosonization result in Eq. (1) is valid to all orders in the interaction.

Recent results call for a further study of the validity of Eq. (1) beyond the weak-coupling limit. In particular, two of us have obtained \([11]\) the specific heat of interacting 1D fermions to order \(g_4^2\) and found that the effect of \(2k_F\) scattering in the spin channel is more involved than it had been previously thought – in addition to terms that depend on the running coupling \(g_1(T)\), the free energy also contains terms that depend on the bare coupling \(g_1\) (see Eq. (2) below).

If, from the field-theoretical point of view, these terms should regarded as anomalies, i.e., they can be viewed equivalently as low- or high-energy contributions. These anomalous terms cannot be simply absorbed into the Gaussian part of the bosonic Hamiltonian, and it is not a priori clear whether such terms can be incorporated into the renormalized charge and spin velocities.

The goal of this paper is to prove that the answer to the question formulated above is affirmative, at least within perturbation theory. We show this by comparing the specific heat obtained from second-order perturbation theory in the g–ology model, calculated in Ref. \([11]\), with Eq. (1), where the velocities are obtained from the Bethe ansatz solution of the Hubbard model \([4]\).

Our starting point is the Hamiltonian of the 1D Hubbard model

\[
\mathcal{H} = \sum_{k,\sigma} (\epsilon_k - \mu) c_{k,\sigma}^\dagger c_{k,\sigma} + U \sum_{k,\sigma} \sum_{l,\sigma} c_{k,\sigma}^\dagger c_{k,\sigma}^\dagger c_{l,\sigma} c_{l,\sigma}^\dagger,
\]

(2)

where \(\epsilon_k = -2t \cos k\) and \(\mu = -2t \cos k_F\) is the chemical potential (the lattice spacing is set to unity). For \(U = 0\) the Fermi velocity is related to the Fermi momentum \(k_F\) by \(v_F = 2t \sin k_F\).

Away from half-filling, umklapp scattering requires collisions of more than two fermions and is therefore neglected in our treatment. The spin and charge velocities can be obtained by solving certain linear integral equations obtained from the Bethe ansatz solution of the Hubbard model \([4]\). At strong coupling, the spin velocity is \(v_s = (\pi/2)(4t^2/U)(1 - \sin 2k_F/2k_F)\) \([12]\). The weak
coupling limit has been studied in \[13\]. It is possible to derive a small $U$ expansion for the spin and charge velocities analytically by solving the integral equations driven in Ref. \[13\] by Wiener-Hopf methods. As we are interested in the $O(U^2)$ terms of the expansions, the resulting calculations are somewhat involved. We therefore have solved the integral equations numerically for small $U$ and found that the results are well fit by the series

$$v_\rho \simeq v_F \left(1 + \frac{U}{2 \pi v_F} + c \left(\frac{U}{2 \pi v_F}\right)^2\right), \quad (3a)$$

$$v_\sigma \simeq v_F \left(1 - \frac{U}{2 \pi v_F} + c' \left(\frac{U}{2 \pi v_F}\right)^2\right). \quad (3b)$$

Based on the analytic result for the spin velocity at half-filling \[4\] and the fact that the only marginally irrelevant operator (the interaction of spin currents) is the same at and below half-filling we do not expect logarithmic terms to appear in these expansions. We have determined the coefficients $c$ and $c'$ for a number of different densities. The results are shown in Table I. As we will see below, we will only need the difference $c' - c$ to verify the validity of Eq. \[11\]. In all cases we find that within the numerical accuracy of our computation

$$c' - c = 1. \quad (4)$$

The deviation from 1 is less than one percent in all cases. The spin and charge velocities can alternatively be calculated in perturbation theory for the $g$-ology model \[11\], \[14\]. An explicit calculation of $C(T)$ within this model is somewhat involved as the low-energy $g$-ology model contains two momentum cutoffs, $\Lambda_f$ and $\Lambda_b$, constraining the integration over the fermionic dispersion and over the momentum transfers near $2k_F$, respectively. [The $g$-ology model is only valid when $\Lambda_f > \Lambda_b$, i.e., when the interaction vanishes at the cutoff set by the dispersion.] Some terms in $C(T)$ are cutoff-independent while some depend logarithmically on the ratio $\Lambda_f/\Lambda_b$. Fortunately, at least to second order in $g$, all cutoff-dependent renormalizations can be absorbed into the renormalized backscattering amplitude $\tilde{g}_1 = g_1 - 2g_2^2 \log(\Lambda_f/\Lambda_b)$, so that the specific heat is expressed in terms of $g_4$, $g_2$, and $\tilde{g}_1$ without any explicit dependence on the cutoffs \[11\]. To second order in $g$, $C(T)$ is given by

$$C(T) = \frac{2 \pi T}{3v_F} \left[1 + (\tilde{g}_1 - g_4) + (\tilde{g}_1 - g_4)^2 + g_4^2 + \left(g_2 - \frac{1}{2} \tilde{g}_1\right)^2 + \frac{3}{4} \tilde{g}_1^2 + O(g^3)\right], \quad (5)$$

where all vertices are measured in the units of $2 \pi v_F$.

To compare Eq. \[6\] with Eqs. \[11\] where $v_\rho$ and $v_\sigma$ given by Eqs. \[3a,3b\], we first note that the $g$-ology model can be bosonized by expressing the operators of right- and left-moving fermions, $R_\alpha$ and $L_\alpha$ ($\alpha = \uparrow, \downarrow$) as

$$R_\alpha(x), L_\alpha(x) = \frac{1}{\sqrt{2 \pi b}} \exp \left[\pm i (\phi_\alpha(x) \mp \theta_\alpha(x))\right], \quad (6)$$

where $b$ is a short-distance cutoff related to the fermionic momentum cutoff ($\Lambda_f$) of the $g$-ology model. Under bosonization, the terms in the fermionic Hamiltonian parameterized by the couplings $g_4$ and $g_2$ are mapped onto the free, Gaussian part of the bosonized Hamiltonian. The $2k_F$ term, parameterized by $g_1$, leads to non-linear, cosine terms in the bosonic Hamiltonian, which give rise to interactions in the spin channel.

To first order in $\tilde{g}_1$, backscattering just renormalizes the prefactors in the Gaussian part of the bosonized Hamiltonian \[15\], \[16\], so that the $g$-ology model can be reduced to a gas of free acoustic bosons with $H_G = H_G^{(\rho)} + H_G^{(\sigma)}$, where

$$H_G^{(\rho)} = \frac{1}{2} \int dx \left(1 + 2g_4 + 2g_2 - 2\tilde{g}_1\right) \left(\partial_x \phi_\rho\right)^2 + \left(1 + 2g_4 - 2g_2\right) \left(\partial_x \phi_\rho\right)^2,$$

$$H_G^{(\sigma)} = \frac{1}{2} \int dx \left(1 - 2\tilde{g}_1\right) \left(\partial_x \phi_\sigma\right)^2 + \left(\partial_x \theta_\sigma\right)^2, \quad (7)$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$\rho$ & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 \\
\hline
$c$ & -0.51 & -0.56 & -0.64 & -0.74 & -0.87 & -1.01 & -1.16 & -1.31 \\
$c'$ & 0.49 & 0.44 & 0.36 & 0.26 & 0.13 & -0.01 & -0.16 & -0.31 \\
\hline
\end{tabular}
\caption{Coefficients $c$ and $c'$ for different densities in the 1D Hubbard model. The density $n = 2k_F/\pi$; $n = 1$ corresponds to half-filling.}
\end{table}
and the charge and spin bosons are defined as $\phi_{\rho,\sigma} = (\phi_1 \pm \phi_2)/\sqrt{2}$ and $\theta_{\rho,\sigma} = (\theta_1 \pm \theta_2)/\sqrt{2}$.

If this were the only effect of backscattering, the specific heat is still given by Eq. (9) can be cast into Eq. (8) with the effective spin and charge velocities $\tilde{v}_\sigma$ and $\tilde{v}_\rho$ read off from Eq. (7):

$$\tilde{v}_\rho^2 = v_F^2 \left(1 + 2g_4 - \tilde{g}_1\right)^2 - (2g_2 - \tilde{g}_1)^2$$

$$\tilde{v}_\sigma^2 = v_F^2 \left(1 - \tilde{g}_1\right)^2 - (\tilde{g}_1^2)$$

Re-expressing Eq. (5) in terms of $\tilde{v}_\sigma$ and $\tilde{v}_\rho$, we obtain

$$C(T) = \frac{\pi T}{3} \left(\frac{1}{\tilde{v}_\rho} + \frac{1}{\tilde{v}_\sigma}\right) + \frac{\pi T}{3v_F} \tilde{g}_1^2 + O(\tilde{g}_1^3).$$

We see that this expression differs from Eq. (11). This is not surprising because the effect of backscattering cannot be simply absorbed into the Gaussian part of the bosonized Hamiltonian, beyond the first order in $g_1$.

The issue therefore is whether the extra $\tilde{g}_1^2$ term in Eq. (9) can be absorbed into renormalization of the velocities $\tilde{v}_\sigma 
\rightarrow \nu_\sigma$, so that the specific heat is still given by Eq. (11) with the renormalized velocities $\nu_\rho$ and $\nu_\sigma$. A simple extension of the previous analysis to a non-SU(2) symmetric case shows that, beyond the first order, $2k_F$ scattering contributes only to the spin part of the bosonized Hamiltonian. The real issue then is renormalization of the spin velocity $\tilde{v}_\sigma \rightarrow \nu_\sigma$. The charge given by Eq. (8) must be the same as the exact one, i.e., $\tilde{v}_\rho = v_\rho$.

A straightforward way to check this is to compare Eqs. (1) and (9) with Eqs. (8a) and (8b). Quite generally, one can write $v_\rho = \tilde{v}_\rho + \alpha v_F \tilde{g}_1^2$, where $\alpha$ is a dimensionless constant. Only if $\alpha = -1$, the extra $\tilde{g}_1^2$ term in Eq. (9) can be absorbed into $v_\sigma$, i.e., Eq. (9) can be cast into Eq. (11). Using Eq. (8), we obtain

$$v_\rho = v_F \left(1 + 2g_4 - \tilde{g}_1 - \frac{1}{2}(2g_2 - \tilde{g}_1)^2\right),$$

$$v_\sigma = v_F \left(1 - \tilde{g}_1 - \frac{1}{2} - \frac{2a}{2} \tilde{g}_1\right)$$

These two expressions should be the same as Eqs. (8a) and (8b).

To compare Eqs. (8a) and (10a), we need to evaluate renormalizations of $g_4$ and $g_1$ to second order in $U$ and to select the contributions which comes from high energies. By construction, such contributions are absorbed into the bare couplings of the $g$–ology model.

There are four second-order diagrams for the $g_4$ amplitude to order $U^2$ (see Fig. 1). For a constant $U$, diagrams a) and b) cancel each other. Diagram c) contains the polarization bubble $\Pi(0) = (1/(2\pi)^2 \int dk d\omega/(i\omega - \epsilon_k + \mu)^2$. Renormalizations due to $\Pi(0)$ are within the low-energy theory, as one can evaluate $\Pi(0)$ in such a way that the result $\Pi(0) = -1/(\pi v_F)$ is determined entirely by the states near the Fermi energy (11). The remaining diagram d) describes renormalization in the Cooper channel. Up to a prefactor, it is given by

$$\int dk d\omega \frac{1}{(i\omega - \epsilon_k + \mu)(i\omega + \epsilon_k - \mu)}.$$ (11)

The momentum integration in Eq. (11) is not confined to the Fermi surface, i.e., this diagram does contribute to high-energy renormalization of $g_4$. In the Hubbard model, the momentum integration is limited from above by the Brillouin zone. The lower limit $\Lambda_f$ can be safely set to zero as the integral is infrared-finite, i.e., $\int \frac{dk}{2\pi}$. Integrating over $\omega$ and then over $k$, we obtain

$$g_4 = \frac{U}{2\pi v_F} \left(1 - \frac{U}{2\pi v_F}\right).$$ (12)

Note that the only dependence on the density is through $v_F = 2t \sin k_F$. We verified that a model of fermions in a continuum with dispersion $k^2/2m$ gives the same result.

Renormalization of $g_1$ is more involved because $g_1$ is a running coupling, and the second-order result depends logarithmically on the upper cutoff of the low-energy theory, which is the lower limit of the integration for “high-energy” renormalization. The diagrams are the same as in Fig. 1. As for $g_4$, diagrams a) and b) cancel each other while diagram c) contains $\Pi(0)$. Therefore, only Cooper diagram d) contributes to the “high-energy” renormalization. Evaluating Cooper diagram for backscattering, we obtain

$$g_1 = \left(\frac{U}{2\pi v_F}\right) \left(1 - L \frac{U}{2\pi v_F}\right),$$ (13)

where

$$L = \frac{v_F P}{2\pi} \int_{-\pi}^{\pi} dk \int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2 + (\epsilon_k - \mu)^2}.$$ (14)

And the symbol $P \int$ here implies that momentum integral does not include the regions near the Fermi points of width $2\Lambda_f$. Integrating over frequency and then over momentum, we obtain

$$L = 2 \log \frac{2\sin k_F}{\Lambda_f}.$$ (15)

Substituting Eq. (15) into Eq. (13), we then obtain

$$g_1 = \frac{U}{2\pi v_F} \left(1 - \frac{U}{\pi v_F} \log \frac{2\sin k_F}{\Lambda_f}\right).$$ (16)
In addition, \( g_1 \) is renormalized within the low-energy g-ology model. This renormalization depends logarithmically on the ratio of the bosonic and fermionic cutoff of the \( g \)-ology model: \( \tilde{g}_1 = g_1 - 2g_1^2 \log (\Lambda_f/\Lambda_b) \) [11]. Adding up this result with Eq. (17) we find that the combination of the high-energy and low-energy renormalizations just replaces \( \Lambda_f \) by \( \Lambda_b \) under the logarithm, i.e.,

\[
\tilde{g}_1 = \frac{U}{2\pi v_F} \left( 1 - \frac{U}{\pi v_F} \log \frac{2\sin k_F}{\Lambda_b} \right).
\] (17)

Alternatively, one can obtain the full renormalization of \( g_1 \) by excluding the regions of width \( \Lambda_b \) near \( \pm 2k_F \) from the integration over the momentum transfer \( k \). This gives the same result as in Eq. (17).

The value of \( \Lambda_b \) is unknown: as we said earlier, the \( g \)-ology model assumes that the low-energy properties of the original system of 1D fermions with a short-range interaction are the same as in the model where interactions are artificially restricted to narrow regions of momentum transfers either near zero (for \( g_1 \) and \( g_2 \)) or \( 2k_F \) (for \( g_1 \)). We can only realistically expect that \( v_F \Lambda_b \) is substantially smaller than a half of the fermionic band-width \( W/2 = 2t \). Still, we have two pairs of equations to compare [Eqs. (10a, 10b) and (11a, 11b)] and two unknown parameters: \( \Lambda_b \) and \( a \). Solving for the unknowns, we obtain

\[
\Lambda_b = 2\sin k_F \exp \left( -\frac{5}{4} - \frac{c}{2} \right),
\]

\[
a = c' - c - 2.
\] (18)

By virtue of Eq. (14), we conclude that \( a = -1 \), which is precisely the value of \( a \) one needs to cast Eq. (13) into Eq. (11). This, we believe, is a “numerical proof” of the statement that at very low temperatures the specific heat of 1D interacting fermions is the same as two system of acoustic phonons with certain spin and charge velocities.

We also find that for most of densities \( v_F \Lambda_b/(2t) = \Lambda_b \sin k_F \) is smaller than one, as it should be, otherwise the \( g \)-ology model cannot be justified. In particular, at quarter-filling, \( \Lambda_b \sin k_F \sim 0.44 \). Near half-filling, however, \( \Lambda_b \sin k_F \) becomes larger than one, which questions the validity of the \( g \)-ology model. Note also that in the opposite limit of small density, \( \Lambda_b \) goes to zero as it indeed should as at vanishing \( k_F \) the linearized dispersion no longer holds.

To conclude, in this paper we obtained expressions for spin and charge velocities for interacting 1D fermions in terms of the couplings of the \( g \)-ology model. These can be related to the microscopic parameters of the 1D Hubbard model via a comparison to weak coupling expansions of the velocities obtained from the Bethe ansatz solution. Using these results, we have shown that all terms in the specific heat in the \( g \)-ology model that do not flow under RG are absorbed into the specific heat of two free gases of massless bosons. As a result, the full specific heat of a system of interacting fermions in 1D is a sum of the specific heats of two free massless Bose gases of charge and spin excitations, and the true interaction term, which contains the running backscattering amplitude and logarithmically flows to zero with decreasing \( T \).

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