A COUNTEREXAMPLE FOR BOUNDEDNESS
OF PSEUDO-DIFFERENTIAL OPERATORS
ON MODULATION SPACES

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Abstract. We prove that pseudo-differential operators with symbols in the
class $S^{0,\delta}_1$ ($0 < \delta < 1$) are not always bounded on the modulation space $M^{p,q}$
($q \neq 2$).

1. Introduction

In the 1980s, by Feichtinger [2, 3], the modulation spaces $M^{p,q}$ were introduced
as a fundamental function space of time-frequency analysis, which originated in
signal analysis or quantum mechanics. See also [4] or Triebel [12]. The exact
definition will be given in the next section, but the main idea is to consider the
decaying property of a function with respect to the space variable and the vari-
able of its Fourier transform simultaneously. Based on the same idea, Sjöstrand [9]
independently introduced a symbol class which assures the $L^2$-boundedness of corre-
sponding pseudo-differential operators. Since this pioneering work, the modulation
spaces have also been recognized as a useful tool for the theory of pseudo-differential
operators (see Gröchenig [6]).

In this paper, we investigate the boundedness property of pseudo-differential
operators with symbols in $S^{m,\delta}_{p,q}$ on the modulation spaces $M^{p,q}$. There have already
been several papers on this subject. For example, Gröchenig and Heil [7], Tachizawa
[10], Toft [11] proved that pseudo-differential operators with symbols in $S^0_{0,0}$ are
$M^{p,q}$-bounded. On the other hand, Calderón and Vaillancourt [1] proved that
pseudo-differential operators with symbols in $S^0_{0,\delta}$ with $0 < \delta < 1$ (hence $S^0_{1,\delta}$) are
$L^2$-bounded (hence $M^{2,2}$-bounded) by reducing them to the case of $S^0_{0,0}$. In view of
these results, the class $S^0_{1,\delta}$ with $0 < \delta < 1$ appears to induce the $M^{p,q}$-boundedness
as well. The objective of this paper is to show that this is not true:

Theorem 1.1. Let $1 < p, q < \infty$, $m \in \mathbb{R}$ and $0 < \delta < 1$. If $m > -|1/q - 1/2|\delta n$,
then there exists a symbol $\sigma \in S^{m}_{1,\delta}$ such that $\sigma(X,D)$ is not bounded on $M^{p,q}(\mathbb{R}^n)$.

In particular, Theorem 1.1 actually says that symbols in the class $S^0_{1,\delta}$ ($0 < \delta < 1$)
do not always induce the $M^{p,q}$-boundedness in the case $q \neq 2$. We will prove this
fact by constructing a counterexample.

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We remark that we can admit the endpoints for $p$ and $q$ in the statement of Theorem 1.1. For example, if we assume that symbols in the class $S_{1,\delta}^m$ with $m > -\delta n/2$ induce the $M^{1,1}$-boundedness, then those with $m > -|1/q - 1/2|\delta n$ induce the $M^{0,q}$-boundedness ($1 < q < 2$) by interpolating it with the boundedness of the class $S_{1,\delta}^0$ on $M^{2,2} = L^2$. This contradicts Theorem 1.1 with $1 < p = q < 2$; hence Theorem 1.1 is still true for $p = q = 1$. This remark is due to Professor Hans G. Feichtinger.

2. Main result

Let $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ be the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in S(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \, d\xi.$$

Let $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$. The symbol class $S_{\rho,\delta}^m$ consists of all $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n = \{0,1,2,\ldots\}^n$. For $\sigma \in S_{\rho,\delta}^m$, we define the pseudo-differential operator $\sigma(X,D)$ by

$$\sigma(X,D)f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \hat{f}(\xi) \, d\xi$$

for $f \in S(\mathbb{R}^n)$. Given a symbol $\sigma \in S_{\rho,\delta}^m$ with $\delta < 1$, the symbol $\sigma^*$ defined by

$$\sigma^*(x,\xi) = \text{Os}(\xi)^{\frac{1}{2}} \int_{\mathbb{R}^n} e^{-iy\cdot\zeta} \sigma(x+y,\xi+\zeta) \, dy \, d\zeta$$

satisfies $\sigma^* \in S_{\rho,\delta}^m$ and

$$\sigma(X,D)f,g) = (f,\sigma^*(X,D)g) \quad \text{for all } f,g \in S(\mathbb{R}^n),$$

where $\chi \in S(\mathbb{R}^{2n})$ satisfies $\chi(0,0) = 1$ and $(\cdot,\cdot)$ denotes the inner product on $L^2(\mathbb{R}^n)$ ([8, Chapter 2, Theorem 2.6]). Note that oscillatory integrals are independent of the choice of $\chi \in S(\mathbb{R}^{2n})$ satisfying $\chi(0,0) = 1$ ([8, Chapter 1, Theorem 6.4]).

We introduce the modulation spaces based on Gröchenig [5]. Fix a function $\gamma \in S(\mathbb{R}^n) \setminus \{0\}$ (called a window function). Then the short-time Fourier transform $V_\gamma f$ of $f \in S'(\mathbb{R}^n)$ with respect to $\gamma$ is defined by

$$V_\gamma f(x,\xi) = (f, M_\xi T_x \gamma) \quad \text{for } x,\xi \in \mathbb{R}^n,$$

where $M_\xi \gamma(t) = e^{it\xi} \gamma(t)$ and $T_x \gamma(t) = \gamma(t-x)$. We can express this in a form of the integral

$$V_\gamma f(x,\xi) = \int_{\mathbb{R}^n} f(t) \overline{\gamma(t-x)} e^{-it\xi} \, dt,$$

which actually converges for appropriate functions $f$ on $\mathbb{R}^n$, say $f \in L^p(\mathbb{R}^n)$. 

We note that, for \( f \in \mathcal{S}'(\mathbb{R}^n) \), \( V_\gamma f \) is continuous on \( \mathbb{R}^{2n} \) and \( |V_\gamma f(x, \xi)| \leq C(1 + |x| + |\xi|)^N \) for some constants \( C, N \geq 0 \) ([5, Theorem 11.2.3]). Let \( 1 \leq p, q \leq \infty \). Then the modulation space \( M^{p,q}(\mathbb{R}^n) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\|f\|_{M^{p,q}} = \|V_\gamma f\|_{L^{p,q}} = \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_\gamma f(x, \xi)|^p \, dx \right)^{\frac{q}{p}} \, d\xi \right\}^{1/q} < \infty.
\]

We note that \( M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \) ([5, Proposition 11.3.1]) and \( M^{p,q}(\mathbb{R}^n) \) is a Banach space ([5, Proposition 11.3.5]). The definition of \( M^{p,q}(\mathbb{R}^n) \) is independent of the choice of the window function \( \gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \); that is, different window functions yield equivalent norms ([5, Proposition 11.3.2]).

We also introduce a special symbol which will act as the counterexample for the boundedness stated in the Introduction. Let \( \varphi, \eta \in \mathcal{S}(\mathbb{R}^n) \) be real-valued functions satisfying

\[
\varphi: \ \text{supp} \varphi \subset \{ \xi : |\xi| \leq 1/8 \}, \quad \int_{\mathbb{R}^n} \varphi(\xi) \, d\xi = 1,
\]

\[
\eta: \ \text{supp} \eta \subset \{ \xi : 2^{-1/2} \leq |\xi| \leq 2^{1/2} \}, \quad \eta = 1 \text{ on } \{ \xi : 2^{-1/4} \leq |\xi| \leq 2^{1/4} \}.
\]

Moreover, we assume that \( \varphi \) is radial. Then we define the symbol \( \sigma_\delta \) by

\[
(2.3) \quad \sigma_\delta(x, \xi) = \sum_{j = j_0}^{\infty} 2^{jm} \left( \sum_{0 < |k| \leq 2^{j+2}} e^{-ik \cdot (2^{j/2}x - k)} \Phi(2^{j/2}x - k) \right) \eta(2^{-j} \xi),
\]

where \( \Phi = \mathcal{F}^{-1} \varphi, 0 < \delta < 1 \) and \( j_0 \in \mathbb{Z}_+ \) is chosen to satisfy

\[
1 + 2^{j_0(\delta-1)+1} \leq 2^{1/4}, \quad 1 - 2^{j_0(\delta-1)+1} \geq 2^{-1/4}, \quad 2^{-j_0\delta/2} \sqrt{n} \leq 2^{-3}.
\]

The symbol \( \sigma_\delta^* \) is constructed from \( \sigma_\delta \) using the oscillatory integral (2.1).

Now, we state our main result, which is a precise version of Theorem 1.1 in the Introduction.

**Theorem 2.1.** Let \( 1 < p, q < \infty \), \( 0 < \delta < 1 \) and \( m > -|1/q - 1/2|\delta n \). Then the symbols \( \sigma_\delta \) and \( \sigma_\delta^* \) defined by (2.3) belong to the class \( S^{m}_{\infty, \delta} \). Moreover, if \( q \geq 2 \) (\( q \leq 2 \) resp.), then the corresponding operator \( \sigma_\delta(X, D) \) (\( \sigma_\delta^*(X, D) \) resp.) is not bounded on \( M^{p,q}(\mathbb{R}^n) \).

The proof of Theorem 2.1 will be given in the next section.

### 3. Proof

In what follows, we consider the symbol \( \tau_\delta \) instead of \( \sigma_\delta \) for the sake of simplicity. In order to avoid confusion, we repeat the notation in this context, and also introduce a family of functions \( \{f_{j, \epsilon, \delta}\}_j \). Let \( \varphi, \psi, \eta \in \mathcal{S}(\mathbb{R}^n) \) be real-valued functions satisfying

\[
\varphi: \ \text{supp} \varphi \subset \{ \xi : |\xi| \leq 1/8 \}, \quad \int_{\mathbb{R}^n} \varphi(\xi) \, d\xi = 1,
\]

\[
\psi: \ \text{supp} \psi \subset \{ \xi : |\xi| \leq 1/2 \}, \quad \psi = 1 \text{ on } \{ \xi : |\xi| \leq 1/4 \},
\]

\[
\eta: \ \text{supp} \eta \subset \{ \xi : 2^{-1/2} \leq |\xi| \leq 2^{1/2} \}, \quad \eta = 1 \text{ on } \{ \xi : 2^{-1/4} \leq |\xi| \leq 2^{1/4} \}.
\]
Moreover, we assume that \( \varphi \) and \( \psi \) are radial. This assumption implies that \( \Phi \) and \( \Psi \) are also real-valued functions, where \( \Phi = \mathcal{F}^{-1} \varphi \) and \( \Psi = \mathcal{F}^{-1} \psi \). For these \( \Phi, \Psi, \eta \), we define the symbol \( \tau_\delta \) and the functions \( f_{j,\epsilon,\delta} \) by

\[
\tau_\delta(x, \xi) = \sum_{j=j_0}^{\infty} 2^{jm} \left( \sum_{0 < |k| \leq 2^j \delta} e^{-ik' \cdot (2^j \delta x - k)} \Phi(2^j \delta x - k) \right) \eta(2^{-j} \xi)
\]

and

\[
f_{j,\epsilon,\delta}(x) = \sum_{0 < |k'| \leq 2^j \delta} |k'|^{-n/q-\epsilon} e^{ik' \cdot (x - k')} \Psi(x - k'),
\]

where \( k, k' \in \mathbb{Z}^n, \epsilon > 0, 0 < \delta < 1/2, \) and \( j_0 \in \mathbb{Z}_+ \) is chosen to satisfy

\[
1 + 2^{j_0(2\delta-1)+1} \leq 2^{1/4}, \quad 1 - 2^{j_0(2\delta-1)+1} \geq 2^{-1/4}, \quad 2^{-j_0 \delta} \sqrt{n} \leq 2^{-3}.
\]

Note that \( \sigma_{2\delta}(x, \xi) = \tau_\delta(x, \xi) \).

**Lemma 3.1.** The symbol \( \tau_\delta \) defined by (3.1) belongs to \( S_{1,2\delta}^m \).

**Proof.** Since \( \operatorname{supp} \eta(2^{-j} \cdot) \subset \{ 2^{-j-1/2} \leq |\xi| \leq 2^{j+1/2} \} \), we see that, for each \( \xi \in \mathbb{R}^n \), at most one term in the sum (3.1) is nonzero with respect to \( j \). Note that \( 2^j \sim |\xi| \sim 1 + |\xi| \) on \( \operatorname{supp} \eta(2^{-j} \cdot) \). Let \( \alpha, \beta \in \mathbb{Z}_+^n \) and \( \xi \in \operatorname{supp} \eta(2^{-j} \cdot) \). Using

\[
\partial_\xi^\alpha \partial_x^\beta \tau_\delta(x, \xi) = 2^{jm} \sum_{0 < |k| \leq 2^j \delta} \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} (2^j \delta k)^{\beta_1} e^{-ik \cdot (2^j \delta x - k)}

\times 2^j \delta |\beta_2| (\partial^{\beta_2} \Phi)(2^j \delta x - k) 2^{-j |\alpha|} (\partial^{\alpha} \eta)(2^{-j} \xi),
\]

we have

\[
|\partial_\xi^\alpha \partial_x^\beta \tau_\delta(x, \xi)| \leq C 2^j (m-|\alpha|) \left( \sum_{0 < |k| \leq 2^j \delta} (1 + |2^j \delta x - k|)^{-n-1} \right)

\times \left( \sum_{\beta_1 + \beta_2 = \beta} 2^j \delta \| (\partial^{\beta_2} \Phi) \|_{L^\infty} \right)

\leq C 2^j (m-|\alpha| + 2\delta |\beta|) \left( \sup_{y \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} (1 + |y - k|)^{-n-1} \right)

\leq C (1 + |\xi|)^{m-|\alpha|+2\delta |\beta|}.
\]

In the case \( \xi \notin \bigcup_{j \geq j_0} \operatorname{supp} \eta(2^{-j} \cdot) \), we have nothing to prove. \( \square \)

**Lemma 3.2.** Let \( 1 < p, q < \infty \), and let \( f_{j,\epsilon,\delta} \) be defined by (3.2). Then the following are true:

1. The Fourier transform of \( e^{i2^j(1-\delta)x_1} f_{j,\epsilon,\delta}(x) \) is

\[
\mathcal{F}[M_{2^j(1-\delta)e_1} f_{j,\epsilon,\delta}](\xi) = \sum_{0 < |k'| \leq 2^j \delta} |k'|^{-n/q-\epsilon} e^{-ik' \cdot (\xi - 2^j(1-\delta)e_1)} \psi(\xi - 2^j(1-\delta)e_1 - k'),
\]

where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \).

2. There exists a constant \( C > 0 \) such that

\[
\|f_{j,\epsilon,\delta}(2^j \delta \cdot)\|_{L^{p,q}} \leq C 2^{j \delta n (1/q-1)}\quad \text{for all } j \geq j_0.
\]
Proof. We consider only (2). Let $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Since

$$V_\gamma[f_{j,\epsilon,\delta}(2^j \xi)](2^{-j} \xi) \Phi(x) = \sum_{0 < |k'| \leq 2^j \delta} |k'|^{-n/q - \epsilon} e^{-ik' \cdot (x-k)} e^{i(2^j \xi) \cdot tx}$$

we have

$$\leq 2^{-j \delta n} \sum_{0 < |k'| \leq 2^j \delta} |k'|^{-n/q - \epsilon} e^{-ik' \cdot (x-k)} e^{i(2^j \xi) \cdot tx}$$

and

$$\leq 2^{-j \delta n} \sum_{0 < |k'| \leq 2^j \delta} |k'|^{-n/q - \epsilon} e^{-ik' \cdot (x-k)}$$

we have

$$\leq 2^{-j \delta n} \sum_{0 < |k'| \leq 2^j \delta} |k'|^{-n/q - \epsilon}$$

where $\widetilde{\Psi}(t) = (\Psi(-t))$. Hence, we get

$$\|V_\gamma[f_{j,\epsilon,\delta}(2^j \xi)]\|_{L^p,q}$$

$$= 2^{-j \delta n} \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_\gamma[f_{j,\epsilon,\delta}(2^j \xi)]| dx \right)^p \frac{d\xi}{q/p} \right\}^{1/q}$$

$$\leq C \left\{ \sum_{|\alpha_1 + \alpha_2| \leq 2n} \left\| \partial^{\alpha_1} \Psi \ast (\partial^{\alpha_2} \gamma)(2^{-j} \xi) \right\|_{L^p} \right\}^q \frac{1}{q}$$

$$\leq C \sum_{|\alpha_1 + \alpha_2| \leq 2n} \left\| \partial^{\alpha_1} \Psi \ast (\partial^{\alpha_2} \gamma)(2^{-j} \xi) \right\|_{L^p}$$

and

$$\leq C \left\{ \sum_{|\alpha_1 + \alpha_2| \leq 2n} \left\| \partial^{\alpha_1} \Psi \ast (\partial^{\alpha_2} \gamma)(2^{-j} \xi) \right\|_{L^p} \right\}^q \frac{1}{q}$$

The proof is complete. \[\square\]

Lemma 3.3. Let $\tau_\delta$ be defined by (3.1), and let

$$(3.3) \ g_{j,\epsilon,\delta}(x) = \sum_{0 < |k|, |k'| \leq 2^j \delta} |k'|^{-n/q - \epsilon} e^{-ik' \cdot (x-k)} e^{i(2^j \xi) \cdot tx} \Phi(x-k) \Psi(x-k').$$

Then

$$\tau_\delta(X, D)[(M_{f_{j,\epsilon,\delta}}(2^j \xi))] \Phi(x) = 2^{jm} e^{i2^l x} g_{j,\epsilon,\delta}(2^j x)$$

for all $j \geq j_0$ and $x \in \mathbb{R}^n$, where $f_{j,\epsilon,\delta}$ is defined by (3.2).
Proof. By Lemma 3.2 (1), we have
\[
\mathcal{F}[(M_{f_j,\epsilon,\delta})\psi, f_j, \epsilon, \delta)](2j^\delta)\xi) = 2^{-j\delta n} \mathcal{F}[M_{f_j,\epsilon,\delta}]\psi(2j^\delta \xi - (2j^{(1-\delta)}e_1 + k')).
\]

Since $2\delta < 1$, $j \geq j_0$, $1 + 2^{j_0(2\delta - 1) + 1} \leq 2^{1/4}$ and $1 - 2^{j_0(2\delta - 1) + 1} \geq 2^{-1/4}$, we see that
\[
\text{supp} \psi(2j^\delta \cdot -(2j^{(1-\delta)}e_1 + k')) \subset \{\xi : 2j - 2j^{2j^\delta + 1} \leq |\xi| \leq 2j + 2j^{2j^\delta + 1}\}
\]
for all $|k'| \leq 2j^\delta$. This implies
\[
\text{supp} \mathcal{F}[(M_{f_j,\epsilon,\delta})\psi, f_j, \epsilon, \delta)](2j^\delta)\xi) \subset \{\xi : 2j^{1-1/4} \leq |\xi| \leq 2j^{1+1/4}\}
\]
for all $j \geq j_0$. Hence, noting that $\text{supp} \eta(2j^{1-\delta}) \subset \{2j^{1-1/2} \leq |\xi| \leq 2j^{1+1/2}\}$ and $\eta(2j^{1-\delta}) = 1$ on $\{2j^{1-1/4} \leq |\xi| \leq 2j^{1+1/4}\}$, we obtain
\[
\tau_\delta(X, D)[(M_{f_j,\epsilon,\delta})\psi, f_j, \epsilon, \delta)](2j^\delta)\xi)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left\{ 2^{im} \left( \sum_{0 < |k| \leq 2j^\delta} e^{-ik\cdot(2j^\delta x - k)} \Phi(2j^\delta x - k) \right) \eta(2j^\delta)\xi) \right\} \mathcal{F}[(M_{f_j,\epsilon,\delta})\psi, f_j, \epsilon, \delta)](2j^\delta)\xi) d\xi
\]
\[
= 2^{im} \left( \sum_{0 < |k| \leq 2j^\delta} e^{-ik\cdot(2j^\delta x - k)} \Phi(2j^\delta x - k) \right) \mathcal{F}[(M_{f_j,\epsilon,\delta})\psi, f_j, \epsilon, \delta)](2j^\delta)\xi) d\xi
\]
\[
= 2^{im} \left( \sum_{0 < |k| \leq 2j^\delta} e^{-ik\cdot(2j^\delta x - k)} \Phi(2j^\delta x - k) \right) (M_{f_j,\epsilon,\delta})\psi, f_j, \epsilon, \delta)](2j^\delta)\xi)\]
\[
= 2^{im} e^{i2j^\delta x} g_j, \epsilon, \delta)](2j^\delta)\xi)
\]
for all $j \geq j_0$ and $x \in \mathbb{R}^n$. The proof is complete. \qed

Lemma 3.4 ([5, Corollary 11.2.7]). Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Then
\[
(f, g) = \frac{1}{\|\gamma\|_L^2} \int_{\mathbb{R}^n} V_\gamma f(x, \xi) \overline{V_\gamma g(x, \xi)} dx d\xi \quad \text{for all } g \in \mathcal{S}(\mathbb{R}^n).
\]

For $1 \leq p \leq \infty$, $p'$ is the conjugate exponent of $p$ (that is, $1/p + 1/p' = 1$).
Lemma 3.5 ([5, Proposition 11.3.4, Theorem 11.3.6]). If $1 \leq p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}(\mathbb{R}^n)$ and $M^{p,q}(\mathbb{R}^n)^* = M^{p',q'}(\mathbb{R}^n)$ under the duality

$$
\langle f, g \rangle = \frac{1}{\|\gamma\|_{L^2}^2} \int_{\mathbb{R}^{2n}} V_\gamma f(x, \xi) \overline{V_\gamma g(x, \xi)} \, dx \, d\xi
$$

for $f \in M^{p,q}(\mathbb{R}^n)$ and $g \in M^{p',q'}(\mathbb{R}^n)$, where $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$.

We denote by $B$ the tensor product of B-splines of degree 2; that is,

$$
B(t) = \prod_{i=1}^{n} \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(t_i),
$$

where $\chi_{[-1/2,1/2]}$ is the characteristic function of $[-1/2,1/2]$. Note that supp $B \subset [-1,1]^n$ and $\mathcal{F}^{-1}B(t) = (2\pi)^{-n} \prod_{i=1}^{n} \{(\sin(t_i/2))/(t_i/2)\}^2 \in M^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$. By Lemmas 3.4 and 3.5, if $1 < p, q < \infty$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$
(3.4) \quad \|f\|_{M^{p,q}} = \sup_{\|g\|_{M^{p',q'}} = 1} |\langle f, g \rangle| \geq \left| \left\langle f, \frac{\mathcal{F}^{-1}B}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \right\rangle \right|
$$

$$
= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \int_{\mathbb{R}^{2n}} V_\gamma f(x, \xi) \overline{V_\gamma \mathcal{F}^{-1}B(x, \xi)} \, dx \, d\xi \right|
$$

$$
= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \int_{\mathbb{R}^{2n}} f(t) \mathcal{F}^{-1}B(t) \, dt \right|
$$

where $\gamma \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\gamma\|_{L^2} = 1$.

Lemma 3.6. Let $1 < p, q < \infty$, and let $g_{j,\epsilon,\delta}$ be defined by (3.3). Then there exists a constant $C > 0$ such that

$$
\|g_{j,\epsilon,\delta}(2^{j\delta} \cdot)\|_{M^{p,q}} \geq C2^{-j\delta(n/q+\epsilon)} \quad \text{for all } j \geq j_0.
$$

Proof. Let $B$ be the tensor product of B-splines of degree 2. Note that $g_{j,\epsilon,\delta}(2^{j\delta} \cdot) \in \mathcal{S}(\mathbb{R}^n)$. By (3.4), we have

$$
\|g_{j,\epsilon,\delta}(2^{j\delta} \cdot)\|_{M^{p,q}} \geq \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \int_{\mathbb{R}^n} g_{j,\epsilon,\delta}(2^{j\delta} x) \mathcal{F}^{-1}B(x) \, dx \right|
$$

$$
= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left\{ \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} \Phi(2^{j\delta}x - k) \Psi(2^{j\delta}x - k) \mathcal{F}^{-1}B(x) \, dx \right\}
$$

$$
+ \left\{ \sum_{0 < |k| \leq 2^{j\delta}} \sum_{0 < |k'| \leq 2^{j\delta}} \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \int_{\mathbb{R}^n} \left( e^{-ik \cdot (2^{j\delta}x - k)} \Phi(2^{j\delta}x - k) \mathcal{F}^{-1}B(x) \right) e^{-ik' \cdot (2^{j\delta}x - k')} \Psi(2^{j\delta}x - k') \, dx \right| \right\}
$$

$$
= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} |I + II|.$$

We first consider I. Note that our assumptions \( \int_{\mathbb{R}^n} \varphi(\xi) d\xi = 1 \), \( \text{supp} \varphi \subseteq \{ |\xi| \leq 1/8 \} \) and \( \psi = 1 \) on \( \{ |\xi| \leq 1/4 \} \) give \( \varphi \ast \psi = 1 \) on \( \{ |\xi| \leq 1/8 \} \). Since \( \text{supp} B \subseteq \{ |\xi| \leq \sqrt{n} \} \) and \( 2^{-j^2} \sqrt{n} \leq 2^{-j^\delta} \sqrt{n} \leq 1/8 \), by Plancherel’s theorem, we see that
\[
\sum_{0 < |k| \leq 2^{j^\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} \Phi(2^{j^\delta} x - k) \Psi(2^{j^\delta} x - k) \mathcal{F}^{-1} B(x) \, dx
\]
\[
= \sum_{0 < |k| \leq 2^{j^\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} (\Phi \Psi)(2^{j^\delta} x - k) \mathcal{F}^{-1} B(x) \, dx
\]
\[
= \sum_{0 < |k| \leq 2^{j^\delta}} |k|^{-n/q-\epsilon} \frac{1}{(2\pi)^{2n}} \int_{|\xi| \leq \sqrt{n}} 2^{-j^\delta n} e^{-i(k \cdot (2^{-j^\delta} \xi))} \varphi \ast \psi(2^{-j^\delta} \xi) B(\xi) \, d\xi
\]
\[
= C_n 2^{-j^\delta n} \sum_{0 < |k| \leq 2^{j^\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} e^{-i(2^{-j^\delta} k \cdot \xi)} B(\xi) \, d\xi
\]
\[
= C_n 2^{-j^\delta n} \sum_{0 < |k| \leq 2^{j^\delta}} |k|^{-n/q-\epsilon} \prod_{i=1}^{n} \left( \frac{\sin k_i / 2^{j^\delta + 1}}{k_i / 2^{j^\delta + 1}} \right)^2.
\]
We next consider II. Using
\[
\mathcal{F}[T_k M_{-k} \Phi] = e^{i|k|^2} T_{-k} M_{-k} \varphi \quad \text{and} \quad \mathcal{F}[T_{k'} M_{-k} \Psi] = e^{i|k'|^2} T_{-k'} M_{-k'} \psi,
\]
we have
\[
\int_{\mathbb{R}^n} \left( e^{-i(k \cdot (2^{j^\delta} x - k))} \Phi(2^{j^\delta} x - k) \mathcal{F}^{-1} B(x) \right) \mathcal{F}^{-1} \left( e^{i(k' \cdot (2^{j^\delta} x - k'))} \Psi(2^{j^\delta} x - k') \right) \, dx
\]
\[
= \int_{\mathbb{R}^n} \left( T_k M_{-k} \Phi(x) 2^{-j^\delta n} \mathcal{F}^{-1} B(2^{-j^\delta} x) \right) \left( T_{k'} M_{-k} \Psi(x) \right) \, dx
\]
\[
= \frac{e^{i(|k|^2 - |k'|^2)}}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \left[ \left( T_k M_{-k} \varphi \right) \ast \left( B(2^{j^\delta} \cdot) \right) \right] \left( T_{k'} M_{-k} \psi \right) \, d\xi
\]
\[
= \frac{e^{i(|k|^2 - |k'|^2)}}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \left[ \left( M_{-k} \varphi \right) \ast \left( B(2^{j^\delta} \cdot) \right) \right] \left( \xi + k \right) \left[ M_{-k'} \psi \right] \left( \xi + k' \right) \, d\xi.
\]
Since \( \text{supp} \varphi \subseteq \{ |\xi| \leq 1/8 \} \), \( \text{supp} B(2^{j^\delta} \cdot) \subseteq \{ |\xi| \leq 2^{-j^\delta} \sqrt{n} \} \subseteq \{ |\xi| \leq 1/8 \} \), we see that \( \text{supp} (M_{-k} \varphi) \ast (B(2^{j^\delta} \cdot)) \subseteq \{ |\xi| \leq 1/4 \} \). On the other hand, \( \text{supp} M_{-k'} \psi \subseteq \{ |\xi| \leq 1/2 \} \). Hence, if \( k \neq k' \), then
\[
\frac{e^{i(|k|^2 - |k'|^2)}}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \left[ \left( M_{-k} \varphi \right) \ast \left( B(2^{j^\delta} \cdot) \right) \right] \left( \xi + k \right) \left[ M_{-k'} \psi \right] \left( \xi + k' \right) \, d\xi = 0;
\]
that is, \( II = 0 \). Therefore, since \( \prod_{i=1}^{n} (\sin x_i / x_i)^2 \geq C \) on \( [-1/2, 1/2]^n \), we get
\[
\frac{1}{\| \mathcal{F}^{-1} B \|_{\mathcal{M}^{p', q'}}} |I + II| = \frac{C_n 2^{-j^\delta n}}{\| \mathcal{F}^{-1} B \|_{\mathcal{M}^{p', q'}}} \sum_{0 < |k| \leq 2^{j^\delta}} |k|^{-n/q-\epsilon} \prod_{i=1}^{n} \left( \frac{\sin k_i / 2^{j^\delta + 1}}{k_i / 2^{j^\delta + 1}} \right)^2
\]
\[
= C_n 2^{-j^\delta n} \sum_{0 < |k| \leq 2^{j^\delta}} |k|^{-n/q-\epsilon} \prod_{i=1}^{n} \left( \frac{\sin k_i / 2^{j^\delta + 1}}{k_i / 2^{j^\delta + 1}} \right)^2
\]
\[
\geq C_n 2^{-j^\delta n} 2^{-j^\delta (n/q+\epsilon)} \sum_{0 < |k| \leq 2^{j^\delta}} 1 \geq C 2^{-j^\delta (n/q+\epsilon)}.
\]
The proof is complete. \( \square \).
We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assume that \(1 < p, q < \infty\), \(0 < \delta < 1\) and \(m > -|1/q - 1/2|\delta n\). Let \(\sigma_{\delta}\) be defined by (2.3). Then \(\sigma_{\delta}(x, \xi) = \tau_{\delta/2}(x, \xi)\), where \(\tau_{\delta/2}\) is defined by (3.1). By Lemma 3.1, we see that \(\sigma_{\delta} \in S_{1, \delta}^{m}\). This implies \(\sigma_{\delta}^* \in S_{1, \delta}^{m}\), where \(\sigma_{\delta}^*\) is defined by (2.1).

We first consider the case \(q \geq 2\). In this case, \(m > (1/q - 1/2)\delta n\). Since \(m > (1/q - 1/2)\delta n\), we can take \(\epsilon > 0\) such that \(m - \epsilon \delta/2 > (1/q - 1/2)\delta n\). We assume that \(\sigma_{\delta}(X, D)\) is bounded on \(M^{p, q}(\mathbb{R}^n)\). Then, by Lemmas 3.2, 3.3, 3.6 and the modulation invariance of the norm \(\| \cdot \|_{M^{p, q}}\), we see that

\[
C2^{j(m-\delta(n'/q)+\epsilon)/2} \leq 2^{jm}\|g_{j, \epsilon, \delta/2}(2^{j\delta/2})\|_{M^{p, q}} = \|2^{jm}e^{i2^jx}g_{j, \epsilon, \delta/2}(2^{j\delta/2})\|_{M^{p, q}}
\]

\[
= \|\sigma_{\delta}(X, D)[(M_{2^{j(1-\delta/2)})_1}f_{j, \epsilon, \delta/2}(2^{j\delta/2})]\|_{M^{p, q}}
\]

\[
\leq \|\sigma_{\delta}(X, D)\|_{L(L^{p, q})}(M^{p, q})\|((M_{2^{j(1-\delta/2)})_1}f_{j, \epsilon, \delta/2}(2^{j\delta/2})\|_{M^{p, q}}
\]

\[
= \|\sigma_{\delta}(X, D)\|_{L(L^{p, q})}(f_{j, \epsilon, \delta/2}(2^{j\delta/2})\|_{M^{p, q}} \leq C2^{j\delta n(1/q-1)/2}
\]

for all \(j \geq j_0\), where \(f_{j, \epsilon, \delta/2}\) and \(g_{j, \epsilon, \delta/2}\) are defined by (3.2) and (3.3). However, since \(m - \epsilon \delta/2 > (1/q - 1/2)\delta n\), this is a contradiction. Hence, \(\sigma_{\delta}\) belongs to \(S_{1, \delta}^{m}\), but \(\sigma_{\delta}(X, D)\) is not bounded on \(M^{p, q}(\mathbb{R}^n)\).

We next consider the case \(q \leq 2\). In this case, \(m > -(1/q - 1/2)\delta n\). Since \(q' \geq 2\) and \(m > (1/q' - 1/2)\delta n\), by Theorem 2.1 with \(q \geq 2\), we see that \(\sigma_{\delta}(X, D)\) is not bounded on \(M^{p', q'}(\mathbb{R}^n)\). By duality and (2.2), if \(\sigma_{\delta}^*(X, D)\) is bounded on \(M^{p, q}(\mathbb{R}^n)\), then \(\sigma_{\delta}^*(X, D)\) is bounded on \(M^{p', q'}(\mathbb{R}^n)\). Hence, \(\sigma_{\delta}^*\) belongs to \(S_{1, \delta}^{m}\), but \(\sigma_{\delta}^*(X, D)\) is not bounded on \(M^{p, q}(\mathbb{R}^n)\). The proof is complete. \(\square\)

References

[1] A.P. Calderón and R. Vaillancourt, A class of bounded pseudo-differential operators, Proc. Nat. Acad. Sci. U.S.A. 69 (1972), 1185-1187. MR0298480 (45:7532)

[2] H.G. Feichtinger, Banach spaces of distributions of Wiener’s type and interpolation, in: P. Butzer, B. Sz.-Nagy and E. Görlich (Eds.), Proc. Conf. Oberwolfach, Functional Analysis and Approximation, August 1980, Int. Ser. Num. Math., Vol. 60, Birkhäuser-Verlag, Basel, Boston, Stuttgart, 1981, pp. 153-165. MR650272 (83g:43005)

[3] H.G. Feichtinger, Modulation spaces on locally compact abelian groups, in: M. Krishna, R. Radha and S. Thangavelu (Eds.), Wavelets and Applications, Chennai, India, Allied Publishers, New Delhi, 2003, pp. 99-140, Updated version of a technical report, University of Vienna, 1983.

[4] H.G. Feichtinger, Modulation spaces: Looking back and ahead, Sampl. Theory Signal Image Process. 5 (2006), 109-140. MR2233968 (2007j:43003)

[5] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001. MR1843717 (2002h:42001)

[6] K. Gröchenig, Time-Frequency analysis of Sjöstrand’s class, Rev. Mat. Iberoamericana 22 (2006), 703-724. MR2294795

[7] K. Gröchenig and C. Heil, Modulation spaces and pseudodifferential operators, Integral Equations Operator Theory 34 (1999), 439-457. MR1702232 (2001a:47051)

[8] H. Kumano-go, Pseudo-Differential Operators, MIT Press, Cambridge, 1981.

[9] J. Sjöstrand, An algebra of pseudodifferential operators, Math. Res. Lett. 1 (1994), 185-192. MR1266757 (95b:47065)

[10] K. Tachizawa, The boundedness of pseudodifferential operators on modulation spaces, Math. Nachr. 168 (1994), 263-277. MR1282643 (95b:42038)
[11] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I, J. Funct. Anal. 207 (2004), 399-429. MR2032995 (2004j:35312)

[12] H. Triebel, Modulation spaces on the Euclidean $n$-space, Z. Anal. Anwendungen 2 (1983), 443-457. MR725159 (85i:46040)

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