On the compatibility of Nonlinear Electrodynamics models with Robinson–Trautman geometry

T. Tahan
d
Institute of Theoretical Physics, Faculty of Mathematics and Physics,
Charles University, Prague, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
(Dated: October 6, 2020)

Robinson–Trautman solutions with Nonlinear Electrodynamics are investigated for both $L(F)$ and $L(F, G)$ Lagrangians and presence of electric and magnetic charges as well as electromagnetic radiation is assumed. Particular interest is devoted to models representing regular black holes for spherically symmetric situations. The results show clear uniqueness of Maxwell electrodynamics with respect to compatibility with Robinson–Trautman class. Additionally, regular black hole models are clearly not suited to this class while famous Born–Infeld model illustrates important distinction between $L(F)$ and $L(F, G)$ for obtained electric field when magnetic field is nontrivial.

PACS numbers: 04.20.Jb, 04.70.Bw

I. INTRODUCTION

Nonlinear electrodynamics (NE) is a generalization of linear Maxwell theory to nonlinear theory. Such theory was initially developed to solve the problem of divergent field of a point charge (see e.g. [1]) also providing reasonable self-energy of charged particle. The best-known and frequently used model of NE was proposed by Born and Infeld in 1934 [2]. Excellent overview of nonlinear electrodynamics and its main features was given in a book by Plebański [3]. Einstein gravity coupled with nonlinear electrodynamics has attracted intensive attentions in the literature and this carried over to modified theories as well (for example [4]). Considerable interest is specifically devoted to gravitating NE models which provide resolution (removal) of spacetime singularity in the center of black hole. Such solutions are called regular black holes and their association with NE started with the model proposed in [5].

Generally, the Lagrangian $L$ of nonlinear electrodynamics is supposed to be a scalar function of the invariant $F = F_{\mu\nu}F^{\mu\nu}$ and $G = F_{\mu\nu}F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$ (in fact one should consider only $G^2$ to eliminate pseudoscalar nature of $G$). If one applies NE theories in static spherically symmetric spacetimes then the form of Lagrangian reduces to $L(F)$ which is as well the most frequently used one in the literature. $L(F, G)$ form of Lagrangian is used for studying light propagation in the geometric optics approximation [6], particularly for comparison with linear Maxwell theory. For example in [7], by analyzing Fresnel equations of wave propagation, they showed that there is no birefringence in the Born-Infeld model, but the velocity of light (as waves of NE) is different from $c$ and always less than or equal to $c$ (in Maxwell limit). In [8] conditions for causal propagation in $L(F, G)$ theories were derived and it was shown that in case of $L(F)$ form of Lagrangian any theory other than the standard Maxwell vacuum one necessarily violates the causality conditions for some allowed background fields.

Apart from studying general physical properties of NE ($L(F, G)$) there are not many nontrivial ($F \neq 0, G \neq 0$) exact solutions of Einstein gravity coupled with these models of nonlinear electrodynamics. Although solution in flat spacetime for Born–Infeld model using Newman–Penrose formalism was obtained already in [9].

This lack of solutions for $L(F, G)$ type model is the initial motivation for our research. Since this means stepping out of the spherically symmetric situation we decided to consider Robinson–Trautman class which essentially contains deformations of spherically symmetric situations. Additionally, this class contains generically dynamical solutions which settle down to symmetric situation by radiation of gravitational waves and electromagnetic radiation for the Maxwell theory as well. This dynamical nature of selected geometry has additional benefit of making it potentially suitable for nonlinear stability studies of the NE solutions. Knowledge of stability of a given static or stationary solution leads to better understanding of these solutions. Unstable solutions have usually less physical significance than stable ones since they are likely to decay to a stable configuration.

One of the most frequent methods of stability analysis is however linear stability. In [10] C. Moreno and O. Sarbach presented a study of dynamical stability of black hole solutions in self-gravitating nonlinear electrodynamics with respect to arbitrary linear fluctuations of the metric and electromagnetic field. They showed the stability for several specific models of NE and particularly those corresponding to regular black holes such as Bardeen black hole based on some conditions on the electromagnetic Lagrangian. Based on the pulsation master equations obtained in [10] the fundamental Quasi-Normal (QN) modes associated with the gravitational and electromagnetic perturbations of black holes in NE theories were computed in [11]. Parallel to this line of investigation there are several other studies of QN modes for NE [12] but their results apply only to test fields propagating...
on the fixed geometry of such black holes, unlike in [11].

Alternative study of linear stability based on the sign of the effective energy shows that except for Born–Infeld model most of other models (and especially those for regular black holes) are unstable [12].

This disagreement concerning linear stability when different methods are used or the imposed conditions differ is exacerbated by additional issue for regular black holes. The stability analyses are predominantly involving the region above horizon. This is crucial for regular black holes since the 'removed' singularity is below the horizon and outer solutions tend to be quite similar to other charged black hole solutions. It is important to specifically understand the stability below the horizon in case of regular black holes.

These issues might be resolved by finding exact solutions of Einstein gravity coupled to NE corresponding to dynamical spacetime with no symmetries and proceed with nonlinear stability analysis based on them. From our past experiences [13, 14] the Robinson–Trautman (RT) spacetimes provide potentially suitable candidate for such study.

Vacuum Robinson–Trautman spacetime [17–19] represent deformations of spherically symmetric geometries and by radiating gravitational waves they asymptotically settle down to Schwarzschild black hole. We hope to see similar behavior when coupled to NE, this time settling down to the spherically symmetric NE configuration after the possible emission of gravitational and electromagnetic radiation. Such an approach was already successful for showing nonlinear stability of the Schwarzschild thin shell wormhole [16]. In [13], we have found very limited (with electric charge only and without radiation) Robinson–Trautman solutions with NE sources for several specific models of NE Lagrangian (both with Maxwell limit and without). The solutions were generated from the spherically symmetric ones. In all cases of NE models considered the singularity of the electromagnetic field is resolved as in the static spherically symmetric cases. However, the models resolving the curvature singularity in spherically symmetric spacetimes could not be generalized to the Robinson–Trautman geometry using the approach therein. Here we want to consider this problem in full generality and prove the impossibility of the generalization more rigorously.

II. VACUUM ROBINSON–TRAUTMAN METRIC

The vacuum Robinson–Trautman spacetime can be described by the line element [17, 18]

$$\text{d}s^2 = -2H \text{d}u^2 - 2 \text{d}u \text{d}r + \frac{r^2}{p^2} (\text{d}x^2 + \text{d}y^2),$$

(2.1)

where

$$2H = K - 2r(\ln P), u - 2m/r$$

(2.2)

where $K = \Delta(\ln P)$ and

$$\Delta \equiv P^2(\partial_{xx} + \partial_{yy}).$$

(2.3)

The metric generally contains two functions, $P(u, x, y)$ and $m(u)$. The function $m(u)$ might be set to a constant by suitable coordinate transformation [19] and we consider this is the case for the coordinates of [24]. Einstein equations then reduce to a single nonlinear PDE — Robinson–Trautman equation

$$\Delta(\ln P) + 12 m(\ln P), u = 0.$$

(2.4)

These spacetimes are then of algebraic type II.

As required by the definition of Robinson–Trautman family the spacetime admits a geodesic, shearfree, twistfree and expanding null congruence generated by $l = \partial_x$.

$$\kappa = \sigma = 0, \rho \neq 0$$

(2.5)

with $r$ being an affine parameter along this congruence, $u$ is a retarded time and $u = \text{const}$ hypersurfaces are null. It shows the propagation of the tetrad components of the Weyl tensor in the $u$-direction, from null surface to null surface. Spatial coordinates $x, y$ span transversal 2-space which has Gaussian curvature (for $r = 1$)

$$K(u, x, y) \equiv \Delta(\ln P).$$

(2.6)

For general $r = \text{const}$ and $u = \text{const}$, the Gaussian curvature is $K/r^2$ so that, as $r \to \infty$, these 2-spaces become locally flat. As usual we will assume that the transversal 2-spaces are compact and connected which leads to a subclass that contains the Schwarzschild solution (considering vanishing cosmological constant for simplicity) corresponding to $K = 1$ (consistent with spherical symmetry). This subclass thus represents its generalization to nonsymmetric dynamical situation. In addition, the Ricci tensor components picked out by the null congruence $l$ and the (complex) tetrad vector $m$ are assumed to satisfy

$$R_{11} = R_{13} = R_{44} = 0$$

(2.7)

For analysis of Robinson–Trautman equation (2.4) it is useful to introduce the following parametrization

$$P = f(u, x, y) P_0,$$

(2.8)

where $f$ is a function on a 2-sphere $S^2$, corresponding to $P_0 = 1 + \frac{4}{3}(x^2 + y^2)$ (such choice gives $K = 1$). By rigorous analysis of equation (2.4) Chruściel [20, 21] proved that, for an arbitrary, sufficiently smooth initial data $f(x, y, u_i)$ on an initial hypersurface $u = u_i$, Robinson–Trautman type II vacuum spacetimes (2.1) exist globally for all $u \geq u_i$. Moreover, they asymptotically converge to the Schwarzschild–(anti-)de Sitter metric with the corresponding mass $m$ and cosmological constant $\Lambda$ as $u \to +\infty$. This convergence is exponentially fast because $f$ behaves asymptotically as

$$f = \sum_{i,j \geq 0} f_{i,j} u^i e^{-2ju/m}$$

(2.9)
where \( f_{i,j} \) are smooth functions of the spatial coordinates \( x, y \). For large retarded times \( u \), the function \( P \) given by \( 2.2 \) exponentially approaches \( P_0 \) which describes the corresponding spherically symmetric solution.

This dynamical black hole radiates exact gravitational waves that carry away the nonlinear deformations from sphericity.

### III. ROBINSON–TRAUTMAN SOLUTION COUPLED TO ELECTROMAGNETISM

We consider the following action, describing a electromagnetic field in form of nonlinear electrodynamic minimally coupled to gravity,

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R + \mathbb{L}(F, G) \right], \tag{3.1}
\]

where \( R \) is the Ricci scalar for the metric \( g_{\mu\nu} \) (we use units in which \( c = \hbar = 8\pi G = 1 \)). \( \mathbb{L}(F, G) \) is the Lagrangian of the nonlinear electromagnetic field which we assume to be an arbitrary function of the invariant \( F \) and \( G \) constructed from a closed Maxwell 2-form. By applying the variation with respect to the metric for the action \( 3.1 \), we get Einstein equations

\[
G_{\mu\nu} = T_{\mu\nu}, \tag{3.2}
\]

For keeping the original form of Robinson–Trautman spacetime we assume the following metric function

\[
d\tau^2 = -(2H + Q) du^2 - 2 du dr + \frac{R^2}{F^2} (dx^2 + dy^2), \tag{3.3}
\]

where \( Q(u, r, x, y) \) and \( R(u, r) \) and subsequently we assume \( u, r, x, y \) coordinate ordering. The metric solution \( H \) is presented in \( 2.2 \), for simplicity in equations we assume the \( m \) parameter in metric solution is zero. To mentioned that one can always find the \( m/r \) in \( Q \).

In the next section we will find all the field equations for general nonlinear electrodynamic Lagrangian but as an example we will study Born–Infeld model. To do that we will first obtained the modified Maxwell equations in the following subsection and then in the other part we will summarized Einstein equations.

#### A. Modified Maxwell equations

In a general case Lagrangian \( \mathbb{L} \) of nonlinear electrodynamics is supposed to be the scalar function of the invariants \( F = F_{ab} F^{ab} \) and \( G = G_{ab} F^{*ab} \). The electromagnetic fields are obeying the (generally) modified Maxwell (NE) field equations. The source–free nonlinear Maxwell equations are obtained in standard way from variational principle and it would be

\[
F_{\mu\nu,\lambda} = 0 \tag{3.4}
\]

\[
\left( \sqrt{-g} \mathbb{L}_F F^{\mu\nu} + \sqrt{-g} \mathbb{L}_G F^{*\mu\nu} \right)_{,\mu} = 0 \tag{3.5}
\]

we use the abbreviations \( \mathbb{L}_F = \frac{\partial \mathbb{L}}{\partial F} \), \( \mathbb{L}_G = \frac{\partial \mathbb{L}}{\partial *F} \), etc. We consider \( F^{\mu\nu} \) to be the only fundamental variable as shown by Plebański \( 2 \).

The nonzero electromagnetic field components in Robinson–Trautman class are \( F_{ur}, F_{ux}, F_{uy} \) and \( F_{xy} \). Since the metric \( 3.3 \) can accommodate only the outgoing rays aligned with the principle null direction we assume that \( F_{ux}, F_{uy} \) or \( F_{ux}, F_{uy} \) are zero since they would otherwise correspond to rays in the other null direction (ingoing). This is related to fixing the initial conditions for the evolution of Robinson–Trautman geometry. The electromagnetic field invariant is then

\[
F = F_{\mu\nu}, \quad F^{\mu\nu} = 2 (f_m - f_c) \]

where

\[
f_m = g^{xx} g^{yy} (F_{xy})^2, \quad f_c = (F_{ur})^2
\]

and the second invariant would be

\[
G = F_{\mu\nu} \star F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}
\]

\( \varepsilon^{\mu\nu\alpha\beta} \) is the totally antisymmetric Levi-Civita tensor with \( \varepsilon^{urxy} = \frac{1}{\sqrt{-g}} \). Therefore \( G \) can be written in the following form

\[
G = - \frac{4}{\sqrt{-g}} F_{ur} F_{xy}
\]

The duality relations between the electromagnetic fields take this form

\[
F^{xy} = \frac{F_{ur}}{\sqrt{-g}}, \quad F^{yr} = \frac{F_{ux}}{\sqrt{-g}} \tag{3.6}
\]

The Maxwell equation \( 3.4 \) has the following components in our case

\[
\partial_u F_{xy} + \partial_y F_{ux} + \partial_x F_{ru} = 0 \tag{3.7}
\]

\[
\partial_r F_{xy} = 0 \tag{3.8}
\]

\[
\partial_x F_{ru} + \partial_r F_{ux} = 0 \tag{3.9}
\]

\[
\partial_y F_{ru} + \partial_r F_{uy} = 0 \tag{3.10}
\]

It is possible to simplify the form of electromagnetic field using the above equations. From \( 3.8 \), we can integrate (we selected a convenient form of expressing the result)

\[
F_{xy} = \frac{B(u, x, y)}{P(u, x, y)^2} \tag{3.11}
\]

and also we introduce a notation \( F_{ur} = -E(u, r, x, y) \). If we take \( r \) derivative of \( 3.7 \) we obtain another useful relation

\[
(F_{ux})_{yr} = (F_{uy})_{xr}. \tag{3.12}
\]
The second modified Maxwell equation (3.33) for the metric (3.33) has the following components (introducing notation \( \Xi = \sqrt{-g} L_F \) and \( \tilde{\Xi} = \sqrt{-\tilde{g}} \tilde{L}_G \)):

\[
\left( \Xi F^{ru} + \tilde{\Xi} \ast F^{ru} \right)_{,r} = 0 \quad (3.13)
\]

\[
\left( \Xi F^{rx} + \tilde{\Xi} \ast F^{rx} \right)_{,r} + \left( \Xi F^{ry} + \tilde{\Xi} \ast F^{ry} \right)_{,y} = 0 \quad (3.14)
\]

\[
\left( \Xi F^{yu} + \tilde{\Xi} \ast F^{yu} \right)_{,r} + \left( \Xi F^{yx} + \tilde{\Xi} \ast F^{yx} \right)_{,x} = 0 \quad (3.15)
\]

and the last equation is

\[
\left( \Xi F^{xu} + \tilde{\Xi} \ast F^{xu} \right)_{,u} + \left( \Xi F^{xv} + \tilde{\Xi} \ast F^{xv} \right)_{,v} + \left( \Xi F^{yu} + \tilde{\Xi} \ast F^{yu} \right)_{,y} = 0 \quad (3.16)
\]

The above equations are not yet simplified to do that we start with equation (3.12). By using the duality relations between electromagnetic fields in (3.10), the equation (3.16) can be written as following

\[
\Xi F_{ur} + \tilde{L}_G F_{xy} = \tilde{\mathcal{C}}(u, x, y) \quad (3.17)
\]

Now with the above expression, we simplify equation (3.10) further

\[
\left( \frac{\partial}{\partial r} F_{ux} + \tilde{L}_G F_{uy} \right)_{,x} + \left( \frac{\partial}{\partial r} F_{uy} + \tilde{L}_G F_{ux} \right)_{,y} - \left( \tilde{\mathcal{C}} \right)_{,u} = 0 \quad (3.18)
\]

with \( \tilde{\mathcal{C}} \) being \( r \)-independent. Taking \( r \) derivative of the above equation (3.18) we get the following constraint relation which will be useful later on for finding the expressions for radiative fields,

\[
\left( \frac{\partial}{\partial r} F_{ux} + \tilde{L}_G F_{uy} \right)_{,r} + \Omega_{,y} = 0 \quad (3.19)
\]

\[
\left( \frac{\partial}{\partial r} F_{uy} + \tilde{L}_G F_{ux} \right)_{,r} - \Omega_{,x} = 0 \quad (3.20)
\]

where \( \Omega = \left( \tilde{L}_G E + \frac{\partial}{\partial r} \frac{\tilde{L}_G B}{L_F} \right) \).

From the above equations, (3.19) and (3.20), we can find the radiative fields \( F_{ux}(u, r, x, y) \) and \( F_{uy}(u, r, x, y) \) valid for any general nonlinear electrodynamic Lagrangian in the following form

\[
F_{ux} = \frac{\frac{\partial}{\partial r} \left( \epsilon_0 (\Omega_{,x} dr - \epsilon_0) + \tilde{L}_G (\Omega_{,y} dr + \epsilon_1) \right)}{\tilde{L}_F - L_G^2} \quad (3.21)
\]

\[
F_{uy} = \frac{\frac{\partial}{\partial r} \left( \tilde{L}_F (\Omega_{,x} dr + \epsilon_1) + \tilde{L}_G (\Omega_{,y} dr - \epsilon_0) \right)}{\tilde{L}_F - L_G^2} \quad (3.22)
\]

where \( \epsilon_0 (u, x, y) \) and \( \epsilon_1 (u, x, y) \) are integration constants.

One can show that the above expression can be simplified into \( R = r \) with coordinate transformation, which we will assume from now on. It has been shown in (22) that the above condition on energy momentum tensor components leads to such result for any static spherically symmetric spacetime of four and more dimensions and obviously we provided its generalization to the dynamical spacetime under consideration.

The nonzero components of energy momentum tensor for electromagnetism fields are

\[
T^u_u = T^r_r = \frac{\left( \tilde{L}_G \right)_{,u} + 2 f_c \tilde{L}_F}{2} \quad (3.27)
\]

\[
T^x_x = T^y_y = \frac{\left( \tilde{L}_G \right)_{,y} - 2 f_m \tilde{L}_F}{2} \quad (3.28)
\]
are nonzero off diagonal terms; i.e.,
\[
T^r_u = -2 (F_{u\lambda} F^{\lambda r}) \mathbb{L}_F
\]
(3.29)
\[
= 2 \mathbb{L}_F \frac{P^2}{r^2} \{(F_{ux})^2 + (F_{uy})^2 \}
\]
\[
T^r_x = -2 (F_{x\lambda} F^{\lambda r}) \mathbb{L}_F
\]
(3.30)
\[
= 2 \mathbb{L}_F \left\{ F_{ur} F_{ux} + F_{uy} F_{xy} \frac{P^2}{r^2} \right\}
\]
\[
T^r_y = -2 (F_{y\lambda} F^{\lambda r}) \mathbb{L}_F
\]
(3.31)
\[
= 2 \mathbb{L}_F \left\{ F_{ur} F_{uy} + F_{ux} F_{xy} \frac{P^2}{r^2} \right\}
\]
and also the following relations between other energy momentum components,
\[
T^x_u = -\frac{P^2}{r^2} T^r_x, \quad T^y_u = -\frac{P^2}{r^2} T^r_y.
\]
(3.32)

The geometrical part of the rest of Einstein field equations is given by the following expressions
\[
G^r_r = G^u_u = \frac{Q_r}{r} + \frac{Q}{r^2}
\]
(3.33)
\[
G^x_x = G^y_y = \frac{Q_{rx}}{2} + \frac{Q_r}{r}
\]
(3.34)
and the off diagonal terms are
\[
G^r_x = -\frac{1}{2} Q_{xr}
\]
(3.36)
\[
G^r_y = -\frac{1}{2} Q_{yr}
\]
(3.37)
together with the relations similar to (3.32)
\[
G^x_u = -\frac{P^2}{r^2} G^r_x, \quad G^y_u = -\frac{P^2}{r^2} G^r_y
\]
(3.38)
the last off diagonal term which is known in vacuum case as Robinson–Trautman equation is
\[
G^u_u = -\frac{1}{2r^2} \Delta (K + Q) - \frac{1}{r} ([\ln P]_u (r Q, r - 2Q) + Q, u).
\]
(3.39)

All these field equations are obtained for the most general set up which can be used for any particular nonlinear electrodynamic Lagrangian.

**IV. CONSISTENCY OF THE FIELD EQUATIONS**

In this section we check the consistency of Einstein equations with general NE in Robinson–Trautman class described in section 11. Using one of the Einstein equations (3.2), \(G^u_u = T^u_u\) with \(G^u_u\) being given by (3.33), we obtain
\[
(r Q)_{,r} = r^2 T^u_u
\]
(4.1)
Employing the expression for \(T^u_u\) from (3.27) and substituting the resulting equation in \(G^x_x = T^r_x\) with the respective sides expressed using (3.34) and (3.25) we arrive at
\[
G^2 \mathbb{L}_{GG} \left( \frac{r E_x - 2E}{4E} \right) - G \mathbb{L}_{FG} (2r E_r E + F) = -4E^2 \mathbb{L}_{FF} \left( \frac{r E_r E + \frac{2B^2}{r^4}}{L} \right) + \mathbb{L}_F (r E_r E + 2E^2)
\]
(4.2)
which can be alternatively expressed as
\[
r E_r E \left[ 2F \mathbb{L}_{GG} + 4E^2 (\mathbb{L}_{FF} - \mathbb{L}_{GG}) - (2G \mathbb{L}_{FG} + \mathbb{L}_F) \right] + G^2 \left[ \mathbb{L}_{FF} - \mathbb{L}_{GG} \right] - 2FG \mathbb{L}_{FG} - 2E^2 \mathbb{L}_F = 0
\]
(4.3)
The above expression is equivalent to a component of Maxwell equations (3.13). If the Lagrangian would be in the form \(L(F)\) this equivalence is still satisfied and the form of the relation (4.3) simplifies considerably
\[
\mathbb{L}_F = \frac{L_F (1 + \zeta)}{4 \left( \frac{P^2}{r^2} + E^2 \zeta \right)}
\]
(4.4)
where \(\zeta = \frac{r E_x}{2E}\). One can arrive at this result by simply putting all \(G\) derivatives of Lagrangian to zero in (4.3).

If we apply the same procedure for equation \(G^x_x = T^r_x\) and substitute for \(Q\) using (4.1), we obtain an expression that, in the case of \(L(F)\) model, corresponds to a component of the Maxwell equation (3.39). But for \(L(F, G)\) model these two are not equivalent and equation \(G^x_x = T^r_x\) has to be satisfied independently. We show the relevant computations in Appendix A.

**A. Example I: Maxwell theory**

When studying nonlinear electrodynamics it is always worth to first review the results for linear theory, namely the Maxwell Lagrangian, in order to provide a comparison. Maxwell theory corresponds to Lagrangian \(L = -F\) which is also frequently considered as a desirable weak-field limit (when \(F\) and \(G\) are small) of a general nonlinear Lagrangian. Note that the overview of a complete solution for Maxwell theory in Robinson–Trautman class is contained in 11. The analysis therein was performed in NP formalism. Here, we are briefly reviewing the Maxwell solution in RT class by using tensorial formalisms as obtained in previous sections.

The independence of Lagrangian on \(G\) significantly simplifies most of the Maxwell equations. The magnetic field is still the same as (3.11), \(B(u, x, y) = P^2 F_{xy}\). To find the electric field we use the equation (4.17) and obtain the following expression for electric field \(E(u, r, x, y)\)
\[
E(u, r, x, y) = \frac{A(u, x, y)}{r^2}
\]
(4.5)
where \( A(u, x, y) = P^2 \tilde{C} \). For finding the radiative fields we use equations (3.21) and (3.22).

\[
F_{ux} = \frac{B_y}{r} - \epsilon_0 \tag{4.6}
\]

\[
F_{uy} = -\frac{B_x}{r} - \epsilon_1 \tag{4.7}
\]

From equations (3.18) and (3.12) we obtain the following

\[
\tilde{C}_u, u - \epsilon_0, x = \epsilon_1, y = 0 \tag{4.8}
\]

\[
B_{xx} + B_{yy} = 0 \tag{4.9}
\]

Since we are interested in black hole solutions we assume the transversal spaces to be compact, then \( \Delta B = 0 \) (see (4.9)) means that \( B \) should be a constant in \( x \) and \( y \) which further leads to \( F_{ux} \) and \( F_{uy} \) being \( r \)-independent due to (4.6) and (4.7). Using this result together with (3.9) and (4.10) we conclude that \( E_x = 0 \) and \( E_y = 0 \). Therefore we have \( B(u) \), \( E(u, r) \) and the Maxwell Lagrangian is independent of \( x \) and \( y \).

From (4.5) and the electromagnetic energy momentum tensor components (3.27) (3.28), the diagonal energy momentum components reduce to

\[
T^u_u = T^r_r = -\left( \frac{A^2 + B^2}{r^4} \right) \tag{4.10}
\]

\[
T^r_x = = T^u_y \left( \frac{A^2 + B^2}{r^4} \right) \tag{4.11}
\]

and the off diagonal energy momentum components to

\[
T^r_u = -2 \frac{P^2}{r^2} \{ (F_{ux})^2 + (F_{uy})^2 \} \tag{4.12}
\]

\[
T^r_x = 2 \frac{r^2}{r^2} \{ A F_{ux} - B F_{uy} \} \tag{4.13}
\]

\[
T^r_y = 2 \frac{r^2}{r^2} \{ A F_{uy} + B F_{ux} \} \tag{4.14}
\]

Using \( G^u_u = T^u_u \) and \( G^r_x = T^r_x \) the metric function \( Q \) takes the form

\[
Q(u, r, x, y) = \frac{A(u)^2 + B(u)^2}{r^2} - 2 \frac{m(u, x, y)}{r} \tag{4.15}
\]

where we have denoted the constant of integration as \(-2m\) to reach a proper Schwarzschild limit. By substituting the metric solution in \( G^r_x - T^r_x = 0 \) while using (3.36) and (4.13) we get the following result

\[
m_{,x} = 2(-A F_{ux} + B F_{uy}) \tag{4.16}
\]

and similarly for \( G^r_y - T^r_y \) by using (3.37) and (4.14), we obtain

\[
m_{,y} = -2(A F_{uy} + B F_{ux}) \tag{4.17}
\]

Using these equations together with the relation for the fields in (3.7) and equation (4.8) we can get to the following expression for \( m \)

\[
\Delta m = 2P^2 (B(F_{xy}),_u + A \tilde{C},_u)
\]

It can be written more explicitly as

\[
\Delta m = 2(B B, u - 2(\ln P),_u (A^2 + B^2) + A A, u) \tag{4.18}
\]

And finally, the last equation \( G^r_u = T^r_u \) (see (3.39) and (4.12)) can be split into two equation for terms of different orders in \( r \)

\[
\Delta K + 12m (\ln P, u - 4m, u = 4P^2 \{ (\epsilon_0)^2 + (\epsilon_1)^2 \} \tag{4.19}
\]

\[
\Delta m + 4(\ln P, u (A^2 + B^2) = (A^2 + B^2),_u \tag{4.20}
\]

where equation (4.20) is identical to (4.18). All the equations for Maxwell theory in RT class are satisfied provided we solve the evolution equation (4.19). It is clear that if one considers vanishing electric charge (which corresponds to \( A = 0 \)) the resulting solution can still be nontrivial radiative one and the same holds when \( B = 0 \).

\section*{B. Example II: Born–Infeld theory}

Born–Infeld Lagrangian is one of the most attractive NE models which in the weak limit goes to Maxwell case and in the strong regime goes to square root model (for NE solutions in closely related Kundt class of geometries for such Lagrangian see [23]). This model was used in many different areas, from flat spacetime with the aim to remove a point-charge singularity up to the string theory where it features as a low energy model. Also there are many studies of its properties in GR such as stability, quasi normal modes, etc. and most recently interaction with scalar field [24]. In [9] they studied Born–Infeld model in flat spacetime in NP formalism while considering second electromagnetic invariant \( G \). We aim to find the exact solution for Born–Infeld model in dynamical spacetime (Robinson–Trautman class). Special attention is given to the radiative fields for BI model in this geometry. Note that most of the studies in nonlinear electrodynamics is done for static or stationary spacetime and studying dynamical behavior of this theory in exact form is absent in the literature. In this subsection we use the field equations from previous sections to see whether such dynamical solutions exist in RT class or not.

Born–Infeld Lagrangian has the following form

\[
\mathbb{L}(F, G) = 4 \beta^2 \left( 1 - \sqrt{1 + \frac{F}{2 \beta^2} - \frac{G^2}{16 \beta^4}} \right) \tag{4.21}
\]

where \( \beta \) is a constant which has the physical interpretation of a critical field strength. For solving modified Maxwell equations for this specific Lagrangian we start with (3.17) and obtain the expression for electric field \( E(u, r, x, y) \) in terms of magnetic field \( B(u, x, y) \) as following

\[
E(u, r, x, y) = \pm \frac{C \beta}{\sqrt{\beta^2 r^4 + B^2 + C^2}} \tag{4.22}
\]
where $C = P^2 \tilde{C}$ together with $B$ (3.11) are function of $u, x, y$. In comparison with previous results concerning only $L(F)$ Lagrangian, the above electric field reproduces them when $B = 0$, for example in paper [15] which was investigating this model in RT class. Curiously, when both $E$ and $B$ are nonzero and the $L(F)$ Lagrangian model is considered the resulting electric field differs from the result obtained for $L(F, G)$ and moreover the field is no longer regular at the origin. This illustrates the importance of inclusion of invariant $G$ when considering magnetic field as well. The next step is to find the radiative fields using (3.21) and (3.22) for BI Lagrangian

$$F_{ux} = \sum \left[ \Gamma (\int \Omega_y dr - \epsilon_0) - C B (\int \Omega_x dr + \epsilon_1) \right]$$ \hspace{1cm} (4.23)

$$F_{uy} = -\sum \left[ \Gamma (\int \Omega_x dr + \epsilon_1) - C B (\int \Omega_y dr - \epsilon_0) \right]$$ \hspace{1cm} (4.24)

where

$$\Omega = -\frac{\beta B}{\sqrt{3^2 r^4 + B^2 + C^2}}$$

$$\Gamma = \frac{\beta^2 r^2}{B^2 (3^2 r^4 - C^2) + \beta^2 r^4 (3^2 r^4 + C^2)}$$

$$\Sigma = \frac{\beta^2 r^4 + B^2}{B^2 (3^2 r^4 - C^2) + \beta^2 r^4 (3^2 r^4 + C^2)}$$

After obtaining electromagnetic fields we obtain the metric solution (3.3), which is primarily determined by metric function $Q$, using (4.1) which takes the following form

$$Q(u, r, x, y) = -\frac{m}{r} + \frac{2 \beta^2}{3} \frac{1}{r^2} - \frac{2 \beta}{r} \int \sqrt{3^2 r^4 + B^2 + C^2}$$ \hspace{1cm} (4.25)

where $B$, $C$ as before and also $m$ are function of $(u, x, y)$. Not surprisingly the above solution is of the same form (apart from dependencies on $x, y$) as found in [15] with the only difference being the presence of $B$. Since both electric field here and those obtained in [15] are also similar, one would have expected the similarity in metric solution as well. However, one should as well note that we have a nontrivial radiative fields present in our current investigation which was not the case in [15]. With all the electromagnetic fields and the metric solution known one should check the fulfillment of the rest of the field equations. Arriving at the equation $G_{r x} = T_{r x}$ we realize that it cannot be satisfied. Therefore it is not possible to have a nontrivial electric $E$ and magnetic $B$ fields together with radiative fields for Born–Infeld model in RT class.

Note that with the assumption of vanishing radiative fields, one finds exact solution in this theory with both electric and magnetic point charges, similar to [25]. Moreover, it is clear that then functions $C$ and $B$ appearing in (1.22) can only be $u$-dependent as a result.

C. Other models

Besides the Born–Infeld model the above result is valid for several other models of NE — it is not possible to find consistent solutions for NE in RT class with nontrivial radiative terms. Although some Einstein equations are equivalent to Maxwell equations (as was shown at the beginning of section IV) the additional field equations containing radiative terms cannot be satisfied in RT class.

V. ROBINSON–TRAUTMAN SOLUTIONS FOR L(F)

Due to the absence of nontrivial exact solutions of Einstein gravity coupled with NE Lagrangian when we assumed electric and magnetic fields in RT class, in this section we study the case when there exist only a magnetic (or electric) field. Note that considering “pure magnetic field” (the same would apply to “pure electric field”) means the second invariant $G \sim E \cdot B$ vanishes identically, so the form of Lagrangian is effectively $L(F)$. We already found solutions for several models of NE for $L(F)$ Lagrangian in [15] in this spacetime while considering only electric point charge without having electromagnetic radiation. In this section we concentrate on solutions for magnetic charge. First we find general formula for unknown metric function $Q$ in our metric ansatz (4.23) in the presence of ‘magnetic field’. Then we evaluate conditions for finding regular black holes in RT class and extend the solution to contain electromagnetic radiation as well. Furthermore in the subsequent section we use the same method to find the exact solution for ‘electric field’.

A. Magnetic field

Recently, studying magnetic fields in NE became subject of substantial interest as a means to find regular black holes in static spherically symmetric spacetimes. By regular black holes we mean geometries lacking singularity at the center of a black hole determined by the presence of a horizon. Thus all scalar invariants are regular everywhere in the spacetime. The first regular black hole was introduced by Bardeen [26] as a solution generated by certain stress energy tensor without clear physical interpretation. More recently, number of models for regular black holes have been proposed together with physically motivated matter content needed for their explanation [7].

The idea of constructing regular black holes by nonlinear electrodynamics as a source was introduced by Ayón-Beato and García [2]. The same authors showed that the corresponding source of Bardeen black hole can be associated with a specific model of nonlinear electrodynamics Lagrangian coupled to gravity [28]. Soon after Bronnikov [29] proved a theorem which says that there
is no spherically symmetric solution with a globally regular metric coupled to nonlinear electrodynamics satisfying weak field limit and having nonzero electric charge. Note that all these solution and most of the nonlinear electrodynamics models representing regular black hole spacetimes presented in literature are static spherically symmetric \cite{30}. Some of these regular black hole solutions have been extended to stationary spacetimes \cite{31}.

Here we are mainly interested in the possibility of having regular black holes in RT class by utilizing suitable NE model as a source. As we already mentioned earlier based on \cite{29}, regular black holes in static spherically symmetric (SSS) situations need only magnetic field. Therefore we study the same situation here although our spacetime is not static (not even stationary) nor spherically symmetric. But in this way we will see the possibility of extending those SSS solutions to the dynamical spacetime.

In general, no matter what model of NE is applied the equation \eqref{5.3} holds which means magnetic field \( F_{xy} \) is always independent of \( r \). Since we are studying the case when there is only a magnetic field we put electric field \( E(u, r, x, y) \) and the radiative terms, \( F_{ux}, F_{uy} \), to zero. The modified Maxwell equations are getting significantly shorter and simpler.

Using \eqref{5.7} we conclude that \( F_{xy} \) is \( u \)-independent and from \eqref{5.14} and \eqref{5.15} we obtain the following expression

\[
F_{xy}(x, y) = \frac{r^2 C_0(u, r)}{P^2 \mathbb{L}_P} \tag{5.1}
\]

With no radiative terms, \( F_{ux} = 0 \) and \( F_{uy} = 0 \), all the off diagonal energy momentum tensor components are vanishing and then Einstein equations \( G^r_x = T^r_x \) and \( G^r_y = T^r_y \) mean that \( Q \) can not be a function of \( x, y \). By checking the remaining Einstein equations (e.g., \( G^u_u = T^u_u \)) we have

\[
(rQ)_r = r^2 \mathbb{L}_/2 \tag{5.2}
\]

which can be written in the following form

\[
Q(u, r) = \frac{1}{2r} \int r^2 \mathbb{L}(F)\,dr - \frac{m(u)}{r} \tag{5.3}
\]

where " - \( m " is an integration constant. It is clear from above equation that the Lagrangian should be \( \mathbb{L}(u, r) \). Therefore \( F \), the electromagnetic scalar invariant, should be \( F(u, r) \). Since from definition we have \( F = 2 \frac{F_{x}^{2}}{F_{xy}} \frac{P^4}{r^4} \) then the function \( P \) has necessarily a separated form, namely

\[
P = \frac{q_m(u)}{\sqrt{F_{xy}}} \tag{5.3}
\]

Then the electromagnetic scalar invariant is \( F = 2 \frac{q_m^2}{r^4} \) and is obviously singular at \( r = 0 \). The above expression for \( \mathbb{P} \) is also consistent with \eqref{5.1}.

Let us recall that the aim of using only magnetic field here is to find regular black hole solutions defined by regularity of scalar curvature invariants such as Ricci and Kretschmann scalars. These two scalar quantities for our metric \eqref{5.3} can be expressed as

\[
Kretschmann = (Q_{rr})^2 + \left( \frac{2Q_r}{r} \right)^2 + \left( \frac{2Q}{r^2} \right)^2 \tag{5.4}
\]

\[
Ricci = -Q_{rr} - \frac{4Q_r}{r} - \frac{2Q}{r^2} \tag{5.5}
\]

Let us note that the form of these two quantities is not changing even for more general case when \( Q \) is a function of \( x, y \) as well. For having regular solution the mass term \( m/r \) must be excluded from \( Q(u, r) \). The main equation in Robinson–Trautman class which determines its dynamics is so-called RT equation \eqref{5.3} (with \( T^r_u = 0 \) now)

\[
G^r_u = -\frac{1}{2r^2} \Delta K - \frac{1}{r} \left[ (\ln P)_{,u} (rQ_{,r} - 2Q) + Q_{,u} \right] \tag{5.6}
\]

where \( K(u, x, y) \) is \( K = \Delta (\ln P) \) as in vacuum RT. Using \eqref{5.3} it can be written as following

\[
\frac{q_m^2 \kappa_0}{2r} + (\ln q_m)_{,u} (rQ_{,r} - 2Q) + Q_{,u} = 0 \tag{5.7}
\]

where \( \kappa_0 = \Delta K |_{q_m=1} \) is necessarily a constant. The solution for the above equation is

\[
Q = \frac{q_m^2}{2} \left( 2f(r/q_m) - \frac{q_m \kappa_0}{r} \int q_m \,du \right) \tag{5.8}
\]

where \( f \) is an arbitrary function. If we look at this solution for \( Q \) with nonzero \( \kappa_0 \) (when \( \kappa_0 \) is zero then the solution reduces from algebraic type II to type D — in effect reducing to only spherically symmetric solutions) and plug it into scalar invariants \eqref{5.6} and \eqref{5.5} we see that it is not possible to have a regular solution. Although there can be certain solutions for arbitrary nonlinear electrodynamics Lagrangian but none of them can correspond to regular black hole.

We can impose some assumptions on the magnetic field or the corresponding metric solution to see whether in some special cases it is possible to have regular solution. One such assumption is that the so-called "magnetic charge", \( q_m \), is a constant. Then from \( F = \frac{2q_m^2}{r^4} \) the Lagrangian is now only \( r \)-dependent. So the metric solution \( Q \) \eqref{5.2} can be written in the following form

\[
Q(u, r) = \mathcal{R}(r) - \frac{m(u)}{r} \tag{5.9}
\]

For checking the regularity we again compute two scalar invariant quantities, \eqref{5.4} and \eqref{5.5}. It is clear that parameter \( m \) in \eqref{5.9} has to vanish as previously. Let us assume that there exists some form of \( \mathcal{R}(r) \) that makes
The two scalar invariant quantities regular. But now we need to make sure that with this assumption the space-time is still of type II, namely not spherically symmetric. From the main equation determining the dynamics of RT spacetime (3.39), we have

\[ G^r_u = -\frac{1}{2\gamma^2} \Delta \Delta (\ln \rho) = 0 \]  

(5.10)

which obviously shows that the space-time is no longer type II but rather type D only and thus spherically symmetric.

If we consider an electric charge instead of a magnetic one we will arrive at a form of type II but rather type D only and thus spherically symmetric. From the main equation determining the dynamics of RT spacetime (5.18), and obtain type of solutions already discussed in [15]. These cannot give rise to spherically symmetric regular black hole solutions due to already discussed results of 29.

The above negative results for regular black holes stem from two crucial facts. First, the necessity of assuming vanishing mass parameter \( m \) to preserve regularity and from the absence of radiative terms that lead to restricted form of \( Q \). This means that there are no relevant sources for nontrivial Gaussian curvature (\( K \neq \text{const.} \)) in RT equation (3.39).

**B. Magnetic charge and Radiation**

In this section we continue the study with same assumptions as in previous subsection with the difference that we allow for radiative terms, i.e., \( E = 0 \) but \( F_{xy} \), \( F_{uy} \) are assumed nonzero.

From (3.10) and (3.11), the radiative field components \( F_{ux} \) and \( F_{uy} \) are independent of \( r \) like for \( F_{xy} \). Using the definition (3.11) and equations (3.19) and (3.20) we get the following relations for these two radiative fields

\[ F_{ux} = -\left(\frac{\Lambda_F B}{\Lambda_F B}\right)_y F_{uy} \]  

(5.11)

Recalling that the electromagnetic scalar invariant is \( F = \frac{2B^2}{r^4} \), the above equation will simplify to

\[ F_{ux} = -\frac{B_y}{B_x} F_{uy} \]  

(5.12)

and from (3.13) and also (5.12) we get

\[ (F_{ux})_x = -(F_{uy})_y \]  

(5.13)

The solution (5.12) for Einstein equations from the previous subsection still holds here and we can write

\[ \Lambda = \frac{2}{r^2} (r Q)_r \]  

(5.14)

By taking "x" derivative from both sides and knowing that electromagnetic scalar invariant is \( F = \frac{2B^2}{r^4} \) we get

\[ 2\Lambda_F B_x B = r^2 (r Q)_{xx} \]  

(5.15)

and similar by using y derivative. From \( G^r_x = T^r_x \) (3.36), \( G^r_y = T^r_y \) (3.37), \( G^r_u = T^r_u \) (3.38), we further obtain

\[ -4B \Lambda_F F_{uy} = r^2 Q_{,y} \]  

(5.16)

\[ -4B \Lambda_F F_{ux} = r^2 Q_{,x} \]  

(5.17)

With the help of (5.13) and the y-version of it we arrive at

\[ -2(r Q)_r \frac{F_{uy}}{B_x} = Q_{,x} \]  

(5.18)

\[ -2(r Q)_r \frac{F_{ux}}{B_y} = Q_{,y} \]  

(5.19)

Combining these two pair of equations together and applying the (5.12) we get

\[ 2r \frac{F_{ux}}{B_y} (Q_x - Q_y) - (Q_x + Q_y) = C_0 \]  

(5.20)

\[ 2r \frac{F_{ux}}{B_y} (Q_x + Q_y) - (Q_x - Q_y) = C_1 \]  

(5.21)

where \( C_0(u, x, y) \) and \( C_1(u, x, y) \) are integration constants. By solving for \( Q_x \) and \( Q_y \) from the above equations and assuming \( Q_x = Q_y \) one can show that \( Q \) should be independent of \( x, y \). By further using Einstein field equations, for example \( G^r_x - T^r_x = 0 \) and \( G^r_y - T^r_y = 0 \), we immediately conclude that the radiative fields must vanish. Therefore we end up with the case where there exists only magnetic field like in the previous section.

Note that we are generalizing the solutions based on vacuum RT solution (containing \( m(u) \)). If we consider \( -\frac{m(u, x, y)}{r^4} \) term in vacuum RT solution with the new assumption that our Lagrangian is of the form \( L(u, r) \) then from Eqs (5.16) and (5.17) (instead of \( Q \) one should put the vacuum metric function \( H (2.2) \) now), we will see that only for Maxwell case \( \Lambda_F = -1 \) it is possible to have radiation consistent with \( m(u, x, y) \neq 0 \).

**C. Electric charge and Radiation**

For comparison with the previous sections we study when there exists only electric field and electromagnetic radiation terms. In [13] we have found several solutions for different models of nonlinear electrodynamics for electric point charge without considering electromagnetic radiation. This part would be a generalization of [13]. As usual we start with modified Maxwell equations, from (3.19) we have

\[ \Lambda_F F_{ur} = A \frac{Q}{r^2} \]  

(5.22)

where \( A(u, x, y) = P^2 \tilde{C} \) and \( \tilde{C}(u, x, y) \) is an integration constant. From (3.19) and (5.20) (note that we assume
\[ F_{xy} = 0 \] we obtain

\[ \mathbb{L}_F F^x_u = \hat{C}_1 \]
\[ \mathbb{L}_F F^{uy} = \hat{C}_2 \]

where \( \hat{C}_1 \) and \( \hat{C}_2 \) are integration constants. From \( 3.30 \) we get

\[ (F_{ux})_{,y} = (F_{uy})_{,x} \]

By checking the Einstein equations component \( G^a_u = T^a_u \) we arrive at

\[ (r Q)_{,r} = r^2 T^u_u \]

which means that

\[ Q(u, r, x, y) = \frac{1}{r} \int r^2 T^u_u \, dr - \frac{2 m(u, x, y)}{r} \]

since we are looking for possibility to have radiative terms we check equations \( G^x_x = T^x_x \) (see \( 3.36 \)) and \( G^y_y = T^y_y \) (see \( 3.31 \)) using \( 5.22 \) to arrive at the following expressions for the radiative fields

\[ -4 A F_{ux} = r^2 Q_{,xr} \]
\[ -4 A F_{uy} = r^2 Q_{,yr} \]

By taking \( x, r \) derivatives of \( Q \) (see \( 5.28 \)) namely \( Q_{,xr} \), and putting them back to the above equations we find (plus equivalent expression using \( y, r \) derivatives)

\[ -4 A F_{ux} = r^2 (T^u_u)_{,x} - \int r^2 (T^u_u)_{,x} \, dr + 2 m_{,x} \]

We can express the relevant energy momentum tensor component from \( 3.27 \)

\[ (T^u_u)_{,x} = \frac{\mathbb{L}_F}{2} F_{,x} + \frac{2}{r^2} \left[ A_x F_{ur} + A (F_{ur})_{,x} \right] \]
\[ F_{,x} = -4 (F_{ux}) (F_{ur})_{,x} \]

where \( f_c = (F_{ur})^2 \). Finally we arrive at

\[ (T^u_u)_{,x} = \frac{2 A_x}{r^2} F_{ur} \]

Now \( 5.29 \) would be (using \( 5.23 \) and also \( 5.22 \))

\[ \left( \frac{2 \hat{C}_1}{A_x} r^2 + r \right) F_{ur} = \int F_{ur} \, dr - \frac{m_{,x}}{A_x} \]

and by \( r \) derivative of the above equation while remembering \( F_{ur} = -E \) we obtain

\[ E(u, r, x, y) = -\frac{q_c (u, x, y)}{(2 \sqrt{A_x} r + 1)^2} \]

and similarly from the same procedure, starting with \( y, r \) derivatives before \( 5.29 \), we get alternative form of \( E \). Comparing these two we get this constraint equation

\[ \frac{\hat{C}_1}{A_x} = \frac{\hat{C}_2}{A_y} \]

From the expression for \( E \) and using \( 3.36 \) we can express radiative term in the following way

\[ F_{ux} = -\int E_{,r} \, dr + \hat{c}_0 \]

where \( \hat{c}_0(u, x, y) \) is an integration constant. On the other hand by substituting \( \mathbb{L}_F \) from \( (5.22) \) into \( (5.23) \), one expects that both expressions \( (5.23) \) and \( (5.35) \) for \( F_{ux} \) are the same to arrive at

\[ -\frac{A}{r^2} \left( -\int E_{,r} \, dr + \hat{c}_0 \right) = \hat{C}_1 \]

From this equality and the expression for electric field \( 5.34 \) we get the following relations

\[ q_c (u, x, y) = q_c (u), \quad \hat{c}_0 = \frac{q_c A_x^2}{4 \sqrt{C_1 A \sqrt{A}}} \]

\[ \hat{C}_1 = \frac{a A_y}{\sqrt{A}} \]

where \( a(u) \) is an integration constant. Applying this procedure to \( F_{uy} \) we find the relations

\[ \hat{c}_1 = \frac{q_c A_y^2}{4 C_2 A \sqrt{A}} \]
\[ \hat{C}_2 = \frac{a A_y}{\sqrt{A}} \]

where \( \hat{c}_1(u, x, y) \) is an integration constant coming from \( 5.10 \) in a similar way to \( \hat{c}_0 \) in \( 5.35 \).

Using the above results we can re-express the electric field from \( 5.34 \) in a simpler way

\[ E(u, r, x, y) = -\frac{q_c}{(2 \sqrt{A} r + 1)^2} \]

One expects that this electric field corresponds to some specific NE Lagrangian. To find the Lagrangian we first find \( r \) in terms \( F (F = -2 E^2) \) and substitute it in \( 5.22 \) to obtain the following expression

\[ \mathbb{L}_F = \frac{4 a^2}{q_c} \frac{1}{\left(-1 + \left(-\frac{r}{2a}\right)^{1/4}\right)^2} \]

Such form clearly means that both \( q_c \) and \( a \) have to be constants and represent parameters of the Lagrangian model rather than, e.g., interpreting \( q_c \) as a charge which is anyway not its proper interpretation.

Then we can integrate to obtain corresponding Lagrangian

\[ L = -32 q_c a^2 \left[ 3 \ln (1 - u) - \frac{u^3 + 3 u^2 - 4 u - 2}{2 (1 - u)} + L_0 \right] \]

where \( u = \sqrt{\frac{2 - F}{2 a^2}} \) and \( L_0 \) is an integration constant.

For having exact Maxwell limit one should set \( L_0 = -1 \) but we consider \( L_0 = 0 \) in the following calculations. Due to the logarithmic term in the Lagrangian we are limited to field strength corresponding to \( u < 1 \) which can be fine-tuned using \( q_c \) to accommodate high values of \( F \).
Since the electric field and the Lagrangian are clear we can find the metric solution by using $G^u_u = T^u_u$ and $G^r_r = T^r_r$, which leads to

$$Q(u, r, x, y) = \frac{\mu}{r} + \tilde{K}(u, x, y) - 8\epsilon_0 a \sqrt{A(u, x, y)} r
+ 16\alpha r^2 + 48\Delta r^2 \ln \left( \frac{2a r}{2a r + \sqrt{A(u, x, y)}} \right)$$

(5.40)

where $\tilde{K} = 2\epsilon_0 A$, $\Lambda = -\frac{2^2}{3}$ and the modified mass term is

$$\mu(u, x, y) = -2m(u, x, y) + \frac{q}{3\epsilon_0} A^2 \left( 7 - 6 \ln(2\sqrt{A}) \right)$$

This solution has the following form asymptotically (for $r \rightarrow \infty$)

$$Q \rightarrow 16\Lambda r^2 + \frac{3\alpha \mu + 2q}{3ar} - \frac{q}{4a^2 r^2} + \frac{q}{2a^2 r^3} + O\left( \frac{1}{r^4} \right)$$

(5.41)

To satisfy further Einstein equations $G^r_y = T^r_y$ and $G^r_x = T^r_x$ the function $\mu(u, x, y)$ term should not be $x, y$ dependent ($\mu(u)$). The last equation to satisfy is $G^r_u = T^r_u$ (expressed using (5.31) and (5.29)) which leads to these dynamical equations

$$\Delta (K + \tilde{K}) - 6\mu \ln(P), u + 2\mu, u = 0$$

(5.42)

$$\Delta A + P^2 \tilde{\Delta} (\ln(A)) + \frac{A^2}{a q_e} (2\tilde{K} (\ln(P), u - \tilde{K}, u) = 0$$

where we have introduced a new Laplace operator (compare with (2.33))

$$\tilde{\Delta} \equiv A^2 (\partial_{xx} + \partial_{yy})$$

Provided we solve the above two coupled nonlinear parabolic type PDE (5.42) for unknown functions $P$ and $A$ we have a complete specific solution for our model. Such a solution is generally radiative (both in terms of electromagnetic and gravitational waves), has Maxwell-type limit and contains a term mimicking the cosmological constant (see (5.21)). The metric solution is singular at $r = 0$ while the electric field is not, provided $a$ is positive. Strangely, this NE model seems to be a unique one compatible with general solutions containing electromagnetic radiation in RT class.

If we assume from the beginning that electric field has restricted dependence on coordinates ($E(u, r)$) then using (3.9) and (3.10) we derive that $F_{ux}$ (and similarly $F_{uy}$) is $r$-independent and immediately from (5.28) (or (5.22)) one sees that $L_F$ has to be a constant leading to Maxwell theory.

### VI. CONCLUSION AND DISCUSSION

The main purpose of this paper was to derive conditions for existence of NE solutions in RT class that would contain electromagnetic radiation. Since general RT solutions represent nonlinear deformations of spherically symmetric situations this would provide potential basis for investigation of nonlinear stability of spherically symmetric NE solutions. This would be especially interesting for the regular black hole solutions sourced by NE. The presented results show that Maxwell theory is the unique one providing general solutions including electromagnetic radiation while NE models suffer from serious restrictions with models providing regular black holes affected in particular. The only partial exception to this is a NE model derived in part (5.4) which provides solution with electric charge and radiation while having Maxwell limit and providing singular geometry but regular electromagnetic field.

Our results specifically mean that even Born-Infeld model cannot support magnetic and electric charge accompanied by radiative terms in the general RT geometry. This model also offers interesting justification for considering $L(F, G)$ Lagrangian instead of just $L(F)$ when $B \neq 0$ and $E \neq 0$ are considered. Namely the regularity of the electric field is destroyed for $L(F)$.

Further results for magnetic charge only (facilitating effective transition $L(F, G) \rightarrow L(F)$) with or without radiative terms explicitly show the impossibility of generalizing regular black hole solutions to RT class. This means one cannot hope to perform nonlinear stability analysis within this class but it also raises question whether this means that such solutions are only limited to highly symmetric situations and thus do not represent astrophysically relevant situations arising from generic conditions. This represents additional complications for regular black hole models which already suffer from non-Maxwellian weak field limit and in the strong field limit both the value of their Lagrangian and energy momentum tensor attain constant values (this applies to, e.g., Lagrangians corresponding to Bardeen and Hayward models).

Naturally, one should pursue investigation in more general geometries to confirm the above results there. Especially spacetimes where the radiation is not confined dominantly along one direction. This might prove to be essential for NE since their electromagnetic perturbations do not propagate along null direction generally (5.3). This is also connected to a broader issue of well-possessed of NE (5.4) within the RT geometry.

### Appendix: $L(F)$ vs. $L(F, G)$ model

Here we will consider the distinction between $L(F)$ and $L(F, G)$ models with respect to Einstein equations component $G^r_x = T^r_x$ and its relation to Maxwell equations. This equation can be cast in the following form

$$- \frac{1}{2} Q_{,xx} = 2L_F \left\{ F_{ur} F_{ux} + F_{uy} \frac{B}{r^2} \right\}$$

(A.1)
with $T^r{}_x$ and $G^r{}_x$ being given in (3.30) and (3.30). From (3.31), we can find
\[Q_{xr} = r (T^u{}_{ux}) - \frac{1}{r^2} \int r^2 (T^u{}_{ux})_x \, dr \tag{A.2}\]
and by substituting the above expression into (A.1) we obtain
\[
r (T^u{}_{ux}) = -4 \mathbb{L}_F \left\{ F_{ux} F_{ux} + F_{uy} \frac{B}{y^2} \right\} \tag{A.3}\]
For removing the integral we take $r$-derivative of this equation
\[
r^3 (T^u{}_{ux}) + 2r^2 (T^u{}_{ux})_x = -4 \left\{ A F_{ux} + F_{uy} \mathbb{L}_F B \right\}_x \tag{A.4}\]
Upon further simplification, using (3.27), we have the following
\[
r^3 (T^u{}_{ux}) + 2r^2 (T^u{}_{ux})_x = \mathbb{L}_F \left( r^2 F_{xx} - \frac{8}{r^2} B B_{xx} \right) - \frac{4}{r^2} B^2 F_{xx} \mathbb{L}_F = \mathbb{L}_F \tag{A.5}\]
and by substituting the above expression into (A.4) we get
\[
\mathbb{L}_F \left( r^2 F_{xx} - \frac{4}{r^2} B B_{xx} \right) = -4 \left\{ A F_{ux} \right\}_x \tag{A.6}\]
where we used (3.22) and (5.22). With further straightforward simplification we obtain a result equivalent to one component of Maxwell equations (3.31). Thus for $L(F)$ it is straightforward to satisfy both the considered Einstein equations component and Maxwell equations component since one is an integrability condition of the other.

If we repeat the same procedure for $L(F, G)$ model we obtain the following expression
\[
F_{ux} = \left( \frac{y^5}{B} \right) \frac{r^2 E L_F (E_{xx} + (F_{ux})_x)}{B r^2 L_{GG} (2E - rE_{xx}) + B L_{FG} (2B^2 + r^5 EE_{xx})} \tag{A.7}\]
The presence of terms related to $G$-derivatives of Lagrangian that in this case $G^r{}_x = T^r{}_x$ is completely independent from Maxwell equations which results in serious constraint on finding solutions to a coupled system. The final expression clearly significantly simplifies for $L(F)$ model and only the boxed terms survive leading to previous result.

\section*{Acknowledgments}

The author thanks Otakar Svítek for helpful discussions.
laers and E. Herlt, *Exact Solutions of the Einstein’s Field Equations*, 2nd edn (CUPress Cambridge, 2002).

[20] P. T. Chruściel, Commun. Math. Phys. **137**, 289 (1991).
[21] P. T. Chruściel, Proc. Roy. Soc. Lond. A **436**, 299 (1992).
[22] T. Jacobson, *Class. Quant. Grav.* **24**, 5717 (2007).
[23] T. Tahamtan and O. Svítek, Eur. Phys. J. C **77** 384 (2017).
[24] T. Tahamtan, Phys. Rev. D **101** 124023 (2020); J. Barrientos, P. A. González and Y. Vásquez, Eur. Phys. J. C. **76**, 677 (2016).
[25] M. Demianski, Foundations of Physics, VoL **16**, No. 2, 187 (1986).
[26] J. Bardeen, presented at GR5, Tbilisi, U.S.S.R., published in the conference proceedings (U.S.S.R., 1968).
[27] I. Dymnikova, Gen. Relativ. Gravit. **24**, 235 (1992); I. Dymnikova and B. Soltysek, Gen. Relativ. Gravit. **30**, 12 (1998); M. Mars, M. M. Martín-Prats, and J. M. M. Senovilla, Classical Quantum Gravity **13**, L51 (1996); A. Borde, Phys. Rev. D **55**, 7615 (1997); S. A. Hayward, Phys. Rev. Lett. **96**, 031103 (2006); V. P. Frolov, Phys. Rev. D **94**, 104056 (2016); A. B. Balakin, J. P. S. Lemos, and A. E. Zayats, Phys. Rev. D **93**, 024008 (2016).

[28] E. Ayón-Beato, A. García, Phys. Lett. B **493**, 149 (2000).
[29] K. A. Bronnikov, Phys. Rev. D **63**, 044005 (2001).
[30] E. Ayón-Beato, A. García, Phys. Lett. B **464**, 25 (1999); I. Dymnikova, Class. Quantum Grav. **21**, 4417 (2004); L. Balart and E. C. Vagenas, Phys. Rev. D **63**, 124045 (2014); Z. Y. Fan and X. Wang, *Phys. Rev. D* **94**, 124027 (2016); K. A. Bronnikov, *Phys. Rev. D* **96**, 128501 (2017); J. P. de Leon, Phys. Rev. D **95**, 124015 (2017); B. Toshmatov, Z. Stuchlík and B. Ahmedov, Phys. Rev. D **98**, 028501 (2018).
[31] C. Bambi and L. Modesto, Phys. Lett. B **721**, 329 (2013); S. G. Ghosh and S. D. Maharaj, Eur. Phys. J. C. **75**, 7 (2015); R. Torres and F. Fayos, Gen. Relativ. Gravit. **49**, 2 (2017); B. Toshmatov, Z. Stuchlík and B. Ahmedov, Phys. Rev. D **95**, 084037 (2017).
[32] F. Abalos, F. Carrasco, É. Goulart and O. Reula, Phys. Rev. D **92**, 084024 (2015); C. Kozameh, O. Reula and H. Kreiss, Classical Quantum Gravity **25**, 025004 (2008).