Cosmological evolution of a ghost scalar field

Sergey V. Sushkov
Department of Mathematics, Kazan State Pedagogical University, Mezhlauk 1 st., Kazan 420021, Russia

Sung-Won Kim
Department of Science Education, Ewha Womans University, Seoul 120-750, Korea

We consider a scalar field with a negative kinetic term minimally coupled to gravity. We obtain an exact non-static spherically symmetric solution which describes a wormhole in cosmological setting. The wormhole is shown to connect two homogeneous spatially flat universes expanding with acceleration. Depending on the wormhole’s mass parameter \( m \), the acceleration can be constant (the de Sitter case) or infinitely growing.

1 Basic equations

Consider a real scalar field \( \phi \) minimally coupled to general relativity with the action given as follows

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{8\pi G} R - \varepsilon (\nabla \phi)^2 - 2V(\phi) \right],
\]  

where \( g_{\mu\nu} \) is a metric, \( g = \det(g_{\mu\nu}) \), and \( R \) is the scalar curvature. We will take the potential \( V(\phi) \) in the exponential form:

\[
V = V_0 \exp(-k\phi).
\]

The exponential potential has been considered in numerous papers devoted to cosmological models with scalar fields (see, for instance, \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 17\]). It arises as an effective potential in some supergravity theories or in Kaluza-Klein theories after dimensional reduction to an effective four-dimensional theory \[11\]. The exponential potential also arises in higher-order gravity theories after a transformation to the Einstein frame \[2, 3, 4, 5\]. We will assume that a parameter \( \varepsilon \) in the action \( \[1\] \) can take two values \( \pm 1 \). The choice \( \varepsilon = 1 \) corresponds to an ordinary scalar field. The static spherically symmetric solution of the theory \( \[1\] \) with \( \varepsilon = 1 \) was obtained in \[13, 14\]. With a view of cosmological applications this theory was studied in \[15, 16, 17\]. The choice \( \varepsilon = -1 \) gives a scalar field with a negative kinetic term or so-called ghost scalar field. Scalar fields with the
opposite sign of their kinetic terms have been previously considered in the literature. They appear in certain models of inflation [18, 19], they have been proposed as dark energy candidates [20, 21], and they also appear in certain unconventional supergravity theories which admit de Sitter solutions [22]. Moreover, ghost scalar fields have been shown to allow wormhole solutions [23, 24, 25, 26, 27].

In this work we will find an exact cosmological solution with a ghost scalar field in the theory (1). Therefore hereinafter we will assume \( \varepsilon = -1 \).

In this case varying the action (1) gives the Einstein equations

\[ R_{\mu\nu} = 8\pi G \left[ -\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu} V_0 \exp(-k\phi) \right], \]  

(3)

and the equation of motion of the scalar field

\[ \nabla^\alpha \nabla_\alpha \phi = kV_0 \exp(-k\phi). \]  

(4)

2 Static solution

In the case \( V_0 = 0 \) (no potential term) the static solution to Eqs. (3), (4) was obtained by Bronnikov [24] and more recently by Armendariz-Picón [26]. Adopting the result of [26] we can write down the solution as follows

\[ ds^2 = -e^{2u(r)}dt^2 + e^{-2u(r)} \left[ dr^2 + (r^2 + r_0^2)d\Omega^2 \right], \]  

(5)

\[ \phi(r) = (4\pi G\alpha^2)^{-1/2} u(r), \]  

(6)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \) is the metric of the unit sphere, the radial coordinate \( r \) varies from \( -\infty \) to \( \infty \), and

\[ u(r) = \frac{m}{r_0} \arctan \frac{r}{r_0}, \]  

(7)

\[ \alpha^2 = \frac{m^2}{m^2 + r_0^2}, \]  

(8)

with \( m \) and \( r_0 \) being two free parameters. Taking into account the following asymptotical behavior:

\[ e^{2u(r)} = \exp \left( \pm \frac{\pi m}{r_0} \right) \left[ 1 - \frac{2m}{r} \right] + O(r^{-2}) \]  

(9)

in the limit \( r \to \pm \infty \), we may see that the spacetime with the metric (5) possesses by two asymptotically flat regions. These regions are connected by the throat whose radius corresponds to the minimum of the radius of
two-dimensional sphere, \( R^2(r) = e^{-2u(r)}(r^2 + r_0^2) \). The minimum of \( R(r) \) is achieved at \( r = m \) and equal to
\[
R_0 = \exp\left(-\frac{m}{r_0} \arctan \frac{m}{r_0}\right) (m^2 + r_0^2)^{1/2}.
\] (10)

Asymptotical masses, corresponding to \( r \to \pm \infty \), are equal to \( m_\pm = \pm m \exp(\pm \pi m/2r_0) \). Note that the masses have both different values and different signs. A behavior of the scalar field is given by Eq. (6); it is seen that the scalar field smoothly varies between two asymptotical values \( \phi_\pm = \pm (\pi/(16G))^{1/2}[1 + (m/r_0)^2]^{1/2} \). Thus, we may summarize that the metric (5) describes a static spherically symmetric wormhole with the throat’s radius \( R_0 \) and two different asymptotical masses \( m_\pm \). When \( m = 0 \), the static solution (5), (6) reduces to
\[
ds^2 = -dt^2 + dr^2 + (r^2 + r_0^2)d\Omega^2,
\] (11)
\[
\phi(r) = \frac{1}{\sqrt{4\pi G}} \arctan \frac{r}{r_0}.
\] (12)

It is worth noting that the metric (11) was proposed \textit{a priori} by Morris and Thorne in the pioneering work [28] as a simple example of the wormhole spacetime metric.

### 3 Non-static solution

Let us now consider a non-static spherically symmetric solution of the equations (4), (3) in the case \( V_0 \neq 0 \). Such the solution is given by the following statement:

**Statement.** A time-dependent spherically symmetric solution of (4), (3) exists if and only if \( V_0 > 0 \) and, in this case, the general solution has the following form:
\[
ds^2 = -\exp(-2\alpha^2 aT + 2u) \, dT^2 + \exp(2aT - 2u) \left[ dr^2 + (r^2 + r_0^2) d\Omega^2 \right],
\] (13)
\[
\phi(T, r) = (4\pi G \alpha^2)^{-1/2} \left[ u - \alpha^2 aT \right],
\] (14)
where \( u(r) \) and \( \alpha \) are given by Eqs. (7), (8), and the parameters \( a \) and \( \alpha \) are related to the parameters of the potential as follows:
\[
k = 4\alpha(\pi G)^{1/2}, \quad V_0 = \frac{a^2(3 + \alpha^2)}{8\pi G}.
\] (15)

**Proof.** Let \( g_{\mu\nu} \) and \( \Phi \) be the ‘old’ static solutions (5), (6). Now consider the conformal transformation of the metric
\[
g_{\mu\nu} = \exp(2\mu(t))\overline{g}_{\mu\nu},
\] (16)
and suppose that at the same time the scalar field transforms as follows
\[ \phi = \phi - (4\pi G)^{-1/2}\alpha \mu(t), \]  
where \( \mu(t) \) is a new indefinite function of \( t \). Using the corresponding transformational properties of the Ricci tensor:
\[ R_{00} = \bar{R}_{00} - 3\ddot{\mu}, \]
\[ R_{0i} = \bar{R}_{0i} + \dot{\mu}\partial_i \ln(g_{00}), \]
\[ R_{ij} = \bar{R}_{ij} - (\ddot{\mu} + 2\dot{\mu}^2)g_{ij}g^{00}, \]
and taking into account that \( g_{\mu\nu} \) and \( \phi \) satisfy the Einstein equations
\[ R_{\mu\nu} = -8\pi G \phi_{,\mu} \phi_{,\nu}, \]  
(19)

it is easy to check that the metric tensor (16) and the scalar field (17) satisfy the equation (3) provided
\[ k = 4\alpha(\pi G)^{1/2} \]
and the function \( \mu(t) \) obeys the following two equations:
\[ \ddot{\mu} = (1 + \alpha^2)\dot{\mu}^2, \]
\[ (3 + \alpha^2)\dot{\mu}^2 = 8\pi GV_0 \exp \left[ 2(1 + \alpha^2)\mu \right]. \]

The solution of these equations is
\[ \mu(t) = -(1 + \alpha^2)^{-1} \ln |(1 + \alpha^2)aT|, \]
(22)
where \( a \) is a free parameter. Substituting \( \mu(t) \) given by Eq. (22) into (16), (17), we get
\[ ds^2 = |(1 + \alpha^2)aT|^{-2/(1 + \alpha^2)} \left\{ -e^{2\mu}dt^2 + e^{-2\mu} \left[ dr^2 + (r^2 + r_0^2)d\Omega^2 \right]\right\}, \]
(23)
\[ \phi(t, r) = (4\pi G \alpha^2)^{-1/2} \left[ u + \alpha^2(1 + \alpha^2)^{-1} \ln |(1 + \alpha^2)aT| \right]. \]  
(24)

And, at last, redefining the time coordinate,
\[ \frac{1}{(1 + \alpha^2)aT} = \pm \exp((1 + \alpha^2)aT), \]
(25)
we arrive at (13), (14).

To complete the proof we consider the scalar field equation (14). Substituting the expression (17) into (14) and taking into account that \( \phi \) satisfies the equation \( \nabla^\alpha \nabla_\alpha \phi = 0 \) we find
\[ \ddot{\mu} + 2\dot{\mu}^2 = \frac{\sqrt{4\pi G}}{\alpha}kV_0 \exp \left( 2 + \frac{k\alpha}{\sqrt{4\pi G}} \right) \mu. \]
(26)
As is easy to check straightforwardly, this equation is valid for \( \mu(t) \), \( k \), and \( V_0 \) given by the relations (22) and (15).³

³Notice also that the equation (26) coincides, if \( k = 4\alpha(\pi G)^{1/2} \), with the (1,1)-component of Einstein’s equations.
4 Wormhole in cosmological setting

In this section we shall analyze the solution (13), (14) found in the preceding section. First, it is a solution of the Einstein-minimal coupling scalar field equations with the potential
\[ V(\phi) = W(T, r) = \frac{a^2(3 + \alpha^2)}{8\pi G} \exp(-2u + 2\alpha^2aT). \] (27)
The solution depends on three parameters \( m, r_0 \) and \( a \). When \( a = 0 \), we get the static solution (5), (6) obtained in [24, 26]. Depending on a value of \( m \) we have two qualitatively different cases:

A. The case \( m = 0 (\alpha = 0) \). The solution (13), (14) now takes the simple form:
\[ ds^2 = -dT^2 + e^{2aT} dr^2 + (r^2 + r_0^2) d\Omega^2, \] (28)
\[ \phi(r) = \frac{1}{\sqrt{4\pi G}} \arctan \frac{r}{r_0}. \] (29)

Note that in this case the scalar field \( \phi \) does not depend on the time coordinate \( T \), though the metric (28) is non-static. The potential (27) becomes to be constant:
\[ W(r, T) \equiv \frac{3a^2}{8\pi G}, \] (30)
and corresponds, in fact, to the positive cosmological constant \( \Lambda = 3a^2 \) in the action (1). It is easy to see that at each moment of time the metric (28) coincides asymptotically (i.e. in the limit \( r \to \pm \infty \)) with the de Sitter one, and an intermediate region represents a throat connecting these asymptotically de Sitter regions. Thus, the spacetime (28) is a wormhole joining two de Sitter universes. The instant radius of the throat is equal to the minimal radius of two-dimensional sphere, \( R_0 = e^{aT}r_0 \); we see that it grows exponentially with time. Let us calculate now the scalar curvature:
\[ R = 12a^2 - \frac{2r_0^2 e^{-2aT}}{(r^2 + r_0^2)^2}. \] (31)

In the limit \( r \to \pm \infty \) as well as in the limit \( T \to \infty \) the scalar curvature has the De-Sitter value \( R_{DS} = 12a^2 \), while at \( T = -\infty \) the scalar curvature is singular. This singularity has a clear geometrical interpretation. Namely, at each moment of time the throat is represented as the 2D sphere of minimal radius. In the limit \( T \to -\infty \) the radius of sphere \( R_0 = e^{aT}r_0 \) tends to zero, the curvature of sphere goes to infinity, and the corresponding spacetime scalar curvature \( R \) becomes to be singular.

It is worth also noting that a metric of the kind of (28) was first introduced \textit{a priori} by Roman [29], who explored the possibility that inflation
might provide a mechanism for the enlargement of submicroscopic, i.e., Planck scale wormholes to macroscopic size.

B. The case \( m \neq 0 \) (\( \alpha \neq 0 \)). In this case the solution is described by the general formulas \((13), (14)\), and the potential \( V(\phi) = W(r, T) \) is given by \((27)\). The scalar curvature calculated in the metric \((13)\) reads

\[
R = 6a^2(2 + \alpha^2)e^{-2u + 2\alpha^2aT} - \frac{2(m^2 + r_0^2)e^{2u - 2\alpha T}}{(r^2 + r_0^2)^2}.
\]

(32)

As in the case \( m = 0 \) the scalar curvature is singular at \( T = -\infty \), and in addition it is now blowing up at \( T \to \infty \). To characterize the last singularity it will be convenient to introduce the proper time \( \tau \) as

\[
-\alpha^2a\tau = \exp(-\alpha^2aT),
\]

(33)

so that \( \tau \) runs from \(-\infty\) to \( 0_- \) while \( T \) varies from \(-\infty\) to \(+\infty\). Now we may rewrite the metric \((13)\) as follows

\[
ds^2 = -e^{2u}dr^2 + |\alpha^2a\tau|^{-2/\alpha^2}e^{-2u} [dr^2 + (r^2 + r_0^2)d\Omega^2].
\]

(34)

In the limit \( r \to \pm\infty \) the last metric describes an homogeneous spatially flat universe:

\[
ds^2 = -d\tilde{\tau}^2 + |\tilde{\tau}|^{-2/\alpha^2} [d\tilde{r}^2 + \tilde{r}^2d\Omega^2],
\]

(35)

with the scale factor \( a(\tilde{\tau}) = |\tilde{\tau}|^{-1/\alpha^2} \) and the scalar curvature

\[
R = \frac{6(2 + \alpha^2)}{\alpha^4\tilde{\tau}^2}.
\]

The corresponding Hubble parameter \( \dot{a}/a \) is equal to \(|\alpha^2\tilde{\tau}|^{-1} \), and the acceleration parameter \( \ddot{a}/a \) is \((1 + \alpha^2)(\alpha^4\tilde{\tau}^2)^{-1} \). It is seen that the universe is expanding with an acceleration into a “final” singularity at \( \tilde{\tau} = 0_- \), and the Hubble and acceleration parameters are infinitely growing in the course of expansion.

Thus, in the case \( m \neq 0 \), the metric \((13)\) represents a wormhole connecting two homogeneous spatially flat universes expanding with infinitely growing acceleration.

## 5 Summary

In this paper we have obtained the exact non-static spherically symmetric solution \((13), (14)\) in the theory of gravity with the ghost scalar field.\(^4\)

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\(^4\)In order to obtain Eq. \((35)\) we must take into account the asymptotical formula \((9)\) and make the rescaling \( \tilde{\tau} = \tau \exp(\pm\pi m/2r_0) \) and \( \tilde{r} = r[\alpha^2a \exp(\pm\pi m/2r_0)]^{-1/\alpha^2} \).
possessing the exponential potential. The spacetime described by the metric \( (13) \) represents two asymptotically homogeneous spatially flat universes connected by a throat. In the other words, one may interpret such the spacetime as a wormhole in cosmological setting. It is important to notice that both the universes and the throat of the wormhole are simultaneously expanding with acceleration. The character of acceleration qualitatively depends on the wormhole’s mass parameter \( m \). In case \( m = 0 \) the acceleration is constant, so that the corresponding spacetime configuration, given by the metric \( (28) \), represents two de Sitter universes joining by the throat. Note that Roman \cite{29} has considered such the spacetime as an example of inflating wormholes. In case \( m \neq 0 \) the acceleration turns out to be infinitely growing, so that the metric \( (13) \) describes now the inflating wormhole connecting two homogeneous spatially flat universes expanding with infinitely growing acceleration into the final singularity.

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