Abstract

This paper develops an analytic methods for investigating uniform hypergraphs. Its starting point is the spectral theory of 2-graphs, in particular, the largest and the smallest eigenvalues of 2-graphs. On the one hand, this simple setup is extended to weighted $r$-graphs, and on the other, the eigenvalues-numbers $\lambda$ and $\lambda_{\text{min}}$ are generalized to eigenvalues-functions $\lambda^{(p)}$ and $\lambda_{\text{min}}^{(p)}$, which encompass also other graph parameters like Lagrangians and number of edges. The resulting theory is new even for 2-graphs, where well-settled topics become challenges again.

The paper covers a multitude of topics, with more than a hundred concrete statements to underpin an analytic theory for hypergraphs. Essential among these topics are a Perron-Frobenius type theory and methods for extremal hypergraph problems.

Many open problems are raised and directions for possible further research are outlined.
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This paper outlines an analytical method in hypergraph theory, shaped after the spectral theory of 2-graphs. After decades of polishing, spectral methods for 2-graphs reside on a solid ground, with traditions settled both in tools and problems. Naturally, we want similar comfort and convenience for spectra of hypergraphs. Recently, several researchers have contributed to this goal, but the endeavor is far from completed, and even the central concepts are not in stone yet. To give results that remain relevant in these dynamic times, we take a somewhat conservative viewpoint and focus only on two fundamental concepts, very likely to be of top interest in the nearest future. More precisely, we study parameters similar to the largest and the smallest eigenvalues of 2-graphs, which are by far the most studied graph eigenvalues anyway. We define these parameters variationally, like the Rayleigh principle defines the extremal eigenvalues of Hermitian matrices. This approach goes along the work of Lim [23] on eigenvalues of hypermatrices, but historically, the same idea has been suggested back in 1930 by Lusternik and Schnirelman [24]. We show that our concepts fit well also with the algebraic definitions of eigenvalues proposed by Qi [32]. These references are for orientation only, as the paper is mostly self-contained.

More important, we parametrize our eigenvalues with a real parameter; thus instead of a pair of eigenvalues-numbers, with each graph $G$ we associate a pair of real functions $\lambda(p)(G)$ and $\lambda_{\min}(G)$. On the one hand, this choice covers the extremal Z-eigenvalues of Qi, but more importantly the function $\lambda(p)$ naturally brings together other fundamental parameters, like the Lagrangian and the number of edges. These ideas extend the approach of Keevash, Lenz and Mubayi [20], which in turn builds upon Friedman and Wigderson [12]. Other relevant contributions in a similar vein are by Cooper and Dutle [3] and by Pearson and Zhang [31].

Some of the problems presented below extend well-known problems for 2-graphs to hypergraphs, sometimes with similar solutions as well. But we also present a few completely new topics for which 2-graphs do not suggest even the slightest clue; most likely such topics will prevail in the future study of hypergraphs. Interestingly, the fundamental linear or multilinear algebraic concepts shape the landscape of the new theory, but the proof methods are essentially nonlinear. Thus, real analysis is more usable than linear algebraic techniques. Very likely, differential manifolds theory will provide important new tools.

We proceed with the outline of the individual sections.

In Section 1, we define the parameters $\lambda(p)(G)$ and $\lambda_{\min}(G)$ for a graph $G$ and a real number $p \geq 1$. We introduce eigenvectors and extend the definitions weighted graphs, which are essentially equivalent to nonnegative symmetric hypermatrices.

Section 2 starts with calculations of concrete $\lambda(p)$ and $\lambda_{\min}$ and then continues with extensions of well-known results for 2-graphs. We also initiate a systematic study of $\lambda(p)(G)$ and $\lambda_{\min}(G)$ as functions of $p$ for any fixed graph $G$.

Section 3 investigates systems of nonlinear equations for $\lambda(p)$ and $\lambda_{\min}$ arising using Lagrange multipliers. It is shown that $\lambda(p)$ and $\lambda_{\min}$ comply with the eigenvalue definitions of Qi. Also, we discuss $\lambda(p)$ of regular graphs, which turn out to be a difficult problem for some hypergraphs.

Section 4 is intended to prepare the reader for Perron-Frobenius type theorems for hypergraphs. A careful selection of examples should fend off hasty expectations arising from 2-graphs. In particular, it is shown that graphs as simple as cycles pose difficult problems about $\lambda(p)$. In fact, the analysis of the $r$-cycles answers in the negative a question of Pearson and Zhang [31].
Section 5 studies Perron-Frobenius type questions for $\lambda^{(p)}$. We examine in detail the traditional topics for symmetric nonnegative matrices and offer extensions for hypergraphs. Most solutions are new and rather complicated, e.g., we introduce the new notion of graph tightness, which extends graph connectedness. Very likely the results proved in this section are precursors of corresponding results for nonnegative hypermatrices.

Section 6 presents relations between $\lambda^{(p)}$, $\lambda^{(p)}_{\text{min}}$ and various graph operations like blow-up, sum, and join. We state and prove simple analogs of the celebrated Weyl’s inequalities for sums of Hermitian matrices and give several applications. We also discuss some results and problems of Nordhaus-Stewart type.

Section 7 is dedicated to relations of $\lambda^{(p)}$ to partiteness, chromatic number, degrees, and graph linearity. Some of these relations are well-known for 2-graphs, but others are specific to hypergraphs. The section ends up with a few bounds on the minimum and maximum entries of eigenvectors to $\lambda^{(p)}$, useful in applications.

Section 8 focuses on bounds on $\lambda^{(p)}_{\text{min}}$. First we give essentially best possible bounds on $\lambda^{(p)}_{\text{min}}$ in terms of the graph order and size. Next we establish for which graphs $G$ the equality $\lambda^{(p)}_{\text{min}} (G) = -\lambda^{(p)} (G)$ holds. For 2-graphs these are the bipartite graphs; for hypergraphs the relevant property is “having an odd transversal.” We answer also a question of Pearson and Zhang about symmetry of the algebraic spectrum of a hypergraph.

Section 9 is dedicated to extremal problems for hypergraphs, a topic that has been developed in a recent paper by the author. The main theorem here is that spectral extremal and edge extremal problems are asymptotically equivalent. This is a new result even for 2-graphs.

Section 10 is a very brief excursion in random hypergraphs. Two theorems are stated about $\lambda^{(q)}$ and $\lambda^{(q)}_{\text{min}}$ of the random graph $G^r (n, p)$ for fixed $p > 0$.

Section 11 is a summary of the main topics of the paper; it outlines directions for further research, and raises several problems and questions.

Section 12 contains reference material and a glossary of hypergraph terms. There is basic information on classical inequalities and on polynomial forms. Parts of the paper may seem easier if the reader skims throughout this section beforehand.
1 The basic definitions

Given a nonempty set $V$, write $V^{(r)}$ for the family of all $r$-subsets of $V$. An $r$-uniform hypergraph ($= r$-graph) consists of a set of vertices $V = V(G)$ and a set of edges $E(G) \subset V^{(r)}$. For convenience we identify $G$ with the indicator function of $E(G)$, that is to say, $G : V^{(r)} \to \{0, 1\}$ and $G(e) = 1$ iff $e \in E(G)$. Further, $v(G)$ stands for the number of vertices, called the order of $G$; $|G|$ stands for the number of edges, called the size of $G$. If $v(G) = n$ and $V(G)$ is not defined explicitly, it is assumed that $V(G) = [n] = \{1, \ldots, n\}$; this assumption is crucial for our notation.

In this paper “graph” stands for “uniform hypergraph”; thus, “ordinary” graphs are referred to as “2-graphs”. If $r \geq 2$, we write $G^r$ for the family of all $r$-graphs, and $G^r(n)$ for the family of all $r$-graphs of order $n$.

Given a $G \in G^r(n)$, the polynomial form (= polyform) of $G$ is a function $P_G(x) : \mathbb{R}^n \to \mathbb{R}$ defined for any vector $[x_i] \in \mathbb{R}^n$ as

$$P_G([x_i]) := r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} \cdots x_{i_r}.$$

If $r = 2$, the polyform $P_G([x_i])$ is the well-known quadratic form

$$2 \sum_{\{i, j\} \in E(G)} x_i x_j,$$

so polyforms naturally extend quadratic forms to hypergraphs.

Note that $P_G(x)$ is a homogenous polynomial of degree $r$ and has a continuous derivative in each variable. More details about $P_G(x)$ can be found in Section 12.3. Let us note that the coefficient $r!$ makes our results consistent with a large body of work on hypermatrices.

1.1 The largest and the smallest eigenvalues of an $r$-graph

Let $G \in G^r(n)$. Define the largest eigenvalue $\lambda(G)$ of $G$ as

$$\lambda(G) := \max_{|x|_r = 1} P_G(x),$$

and the smallest eigenvalue $\lambda_{\min}(G)$ as

$$\lambda_{\min}(G) := \min_{|x|_r = 1} P_G(x).$$

If $G$ has no edges, we let $\lambda(G) = \lambda_{\min}(G) = 0$.

Note that the condition $|x|_r = 1$ describes $S_r^{(n-1)}$, the $(n-1)$-dimensional unit sphere in the $l^r$ norm in $\mathbb{R}^n$; see Section 12.1 for more details. Since $S_r^{(n-1)}$ is a compact set, and $P_G(x)$ is continuous, $P_G(x)$ attains its minimum and maximum on $S_r^{(n-1)}$, hence $\lambda(G)$ and $\lambda_{\min}(G)$ are well defined. Also, note that if $r = 2$, the Rayleigh principle states that the above equations indeed define the largest and the smallest eigenvalues of $G$. 
1.2 Introduction of $\lambda^{(p)}$ and $\lambda^{(p)}_{\min}$

For an $r$-graph $G$ the parameters $\lambda(G)$ and $\lambda_{\min}(G)$ are special in many ways; however, great insight comes from the study of the functions $\lambda^{(p)}(G)$ and $\lambda^{(p)}_{\min}(G)$ defined for any real number $p \geq 1$ as

$$
\lambda^{(p)}(G) := \max_{|x|_p = 1} P_{G}(x), \quad \text{(1)}
$$

$$
\lambda^{(p)}_{\min}(G) := \min_{|x|_p = 1} P_{G}(x). \quad \text{(2)}
$$

Here the condition $|x|_p = 1$ describes the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}_p$ in the $l^p$ norm, which is compact; since $P_{G}(x)$ is continuous, $\lambda^{(p)}(G)$ and $\lambda^{(p)}_{\min}(G)$ are well defined.

Note that $\lambda^{(r)}(G) = \lambda(G)$ and $\lambda^{(r)}_{\min}(G) = \lambda_{\min}(G)$. Also note that $\lambda^{(1)}(G)$ is another much studied graph parameter, known as the Lagrangian$^1$ of $G$. So $\lambda^{(p)}(G)$ links $\lambda(G)$ to a body of previous work on hypergraph problems. The parameter $\lambda^{(p)}(G)$ has been introduced by Keevash, Lenz and Mubayi [20], although they require $p > 1$. It seems that little is known about $\lambda^{(p)}(G)$ and $\lambda^{(p)}_{\min}(G)$ even for 2-graphs.

For any $p \geq 1$, if $x$ is a vector such that $|x|_p = [x_i] = 1$ and $\lambda^{(p)}(G) = P_{G}(x)$, then the vector $x' = [||x_i||]$ satisfies $|x'|_p = 1$ and so

$$
\lambda^{(p)}(G) = P_{G}(x) \leq P_{G}(x') \leq \lambda^{(p)}(G),
$$

implying that $\lambda^{(p)}(G) = P_{G}(x')$. Therefore, there is always a nonnegative vector $x$ such that $|x|_p = 1$ and $\lambda^{(p)}(G) = P_{G}(x)$. This implies also the following observations.

**Proposition 1.1** If $p \geq 1$ and $G \in \mathcal{G}^r(n)$, then

$$
\lambda^{(p)}(G) = \max_{|x|_p = 1} |P_{G}(x)|.
$$

In particular, $\lambda^{(p)}(G) \geq |\lambda^{(p)}_{\min}(G)|$ or, equivalently, $\lambda^{(p)}_{\min}(G) \geq -\lambda^{(p)}(G)$.

If $r$ is odd, then $P_{G}(x)$ is odd, and so $\lambda^{(p)}_{\min}(G) = -\lambda^{(p)}(G)$; thus $\lambda^{(p)}_{\min}(G)$ can give new information only if $r$ is even.

1.3 Eigenvectors

If $G \in \mathcal{G}^r(n)$ and $[x_i]$ is an $n$-vector such that $|[x_i]|_p = 1$ and $\lambda^{(p)}(G) = P_{G}([x_i])$, then $[x_i]$ will be called an eigenvector to $\lambda^{(p)}(G)$. For $\lambda^{(p)}_{\min}(G)$ eigenvectors are defined the same way. For convenience we write $\mathbb{S}^{n-1}_{p,+}$ for the set of nonnegative $n$-vectors $x$ with $|x|_p = 1$. Thus, $\lambda^{(p)}(G)$ always has an eigenvector in $\mathbb{S}^{n-1}_{p,+}$. The following inequalities relate $\lambda^{(p)}(G)$ and $\lambda^{(p)}_{\min}(G)$ to arbitrary vectors $x$.

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Let us note that this use of the name Lagrangian is at odds with the tradition. Indeed, names as Laplacian, Hessian, Gramian, Grassmanian, etc., usually denote a structured object like matrix, operator, or manifold, and not just a single number.
Proposition 1.2  Let $p \geq 1$. If $G \in \mathcal{G}^r (n)$ and $\mathbf{x}$ is a real vector, then

$$P_G (\mathbf{x}) \leq \lambda^{(p)} (G) |\mathbf{x}|_p^p,$$

with equality if and only if $\mathbf{x} = 0$ or $|\mathbf{x}|_p^{-1} \mathbf{x}$ is an eigenvector to $\lambda^{(p)} (G)$. Also,

$$P_G (\mathbf{x}) \geq \lambda^{(p)}_{\min} (G) |\mathbf{x}|_p^p,$$

with equality if and only if $\mathbf{x} = 0$ or $|\mathbf{x}|_p^{-1} \mathbf{x}$ is an eigenvector to $\lambda^{(p)}_{\min} (G)$.

Note that in our definition eigenvectors to $\lambda^{(p)} (G)$ always have length 1 in the $l^p$ norm. This seems to be a strong restriction of the traditional concept in Linear Algebra; in fact this restriction gives convenience at a negligible loss, because invariant subspaces are irrelevant to the study of $\lambda^{(p)} (G)$. Indeed, for 2-graphs the eigenvectors of $\lambda (G)$ or $\lambda_{\min} (G)$ span invariant subspaces, but if $r \geq 3$ and $G \in \mathcal{G}^r$, the set of eigenvectors to $\lambda (G)$ or to $\lambda_{\min} (G)$ could be just a finite set. The situation with $\lambda^{(p)} (G)$ for $p \neq r$ could be even more rigid. We shall consider many examples below.

1.4 Putting weights on edges

We introduce here weighted $r$-graphs, a natural and useful extension of $r$-graphs. Thus, a weighted $r$-graph $G$ with set of vertices $V$ is a nonnegative real function $G : V^{(r)} \mapsto [0, \infty)$. The set of edges $E (G)$ of $G$ is defined $E (G) = \{ e : e \in V^{(r)} \text{ and } G (e) > 0 \}$, that is to say, $E (G)$ is the support of $G$. The order $v (G)$ of $G$ is the cardinality of $V$ and the size is defined as $|G| = \sum \{ G(e) : e \in V^{(r)} \}$. Weighted $r$-graphs provide the natural setup for many a statement about graphs, say for Weyl’s inequalities in Proposition 6.2 or interlacing inequalities.

We write $\mathcal{W}^r$ for the family of all weighted $r$-graphs and $\mathcal{W}^r (n)$ for the family of all weighted $r$-graphs of order $n$. As usual, $|G|_p$ stands for the $l^p$-norm of $G$ and $|G|_{\infty}$ is the maximum of $G$. Clearly $\mathcal{W}^r (n)$ is a complete metric space in any $l^p$ norm, $1 \leq p \leq \infty$. Also, if $H \in \mathcal{W}^r$ and $G \in \mathcal{W}^r$, $H$ is a called a subgraph of $G$, if $V (H) \subset V (G)$, and $e \in E (H)$ implies $H (e) = G (e)$; a subgraph $H$ of $G$ is called induced if $e \in E (G)$ and $e \subset V (H)$ implies $H (e) = G (e)$.

Given a vector $[x_i] \in \mathbb{R}^n$, the polyform of $G$ is defined as

$$P_G ([x_i]) := r! \sum_{\{i_1, \ldots, i_r\} \in E (G)} G \{i_1, \ldots, i_r\} x_{i_1} \cdots x_{i_r},$$

and the definitions of $\lambda (G)$, $\lambda_{\min} (G)$, $\lambda^{(p)} (G)$, $\lambda^{(p)}_{\min} (G)$ and eigenvectors are the same as above.

Proposition 1.3  If $G \in \mathcal{W}^r (n)$ and $H \in \mathcal{W}^r (n)$, then

$$E (G + H) = E (G) \cup E (H) \quad \text{and} \quad E (G \cdot H) = E (G) \cap E (H).$$

For every $\mathbf{x} \in \mathbb{R}^n$,

$$P_{G+H} (\mathbf{x}) = P_G (\mathbf{x}) + P_H (\mathbf{x}).$$

Many results in this paper smoothly extend from graphs to weighted graphs at no additional cost. However, for simplicity we shall avoid a systematic extension, leaving it to the interested reader; e.g., one can see that Proposition 1.2 holds with no change for weighted graphs as well.
2 Basic properties of $\lambda^{(p)}$ and $\lambda^{(p)}_{\text{min}}$

For a start, let us find $\lambda^{(p)}(K^r_r)$ and $\lambda^{(p)}_{\text{min}}(K^r_r)$ and their eigenvectors, where $K^r_r$ is the $r$-graph of order $r$ consisting of a single edge. In this case, $P_{K^r_r}(x) = r!x_1 \cdots x_r$. Letting $|x_1|^p + \cdots + |x_r|^p = 1$, the AM-GM inequality (71) implies that

$$x_1 \cdots x_r \leq |x_1| \cdots |x_r| \leq \left(\frac{|x_1|^p + \cdots + |x_r|^p}{r}\right)^{r/p} = r^{-r/p},$$

with equality holding if and only if $x_1 = \pm r^{-1/p}, \ldots, x_r = \pm r^{-1/p}$, and $x_1 \cdots x_r > 0$. Therefore,

$$\lambda^{(p)}(K^r_r) = r!/r^{r/p}. \tag{3}$$

The eigenvectors to $\lambda^{(p)}(K^r_r)$ are all vectors of the type $(\pm r^{-1/p}, \ldots, \pm r^{-1/p})$, with the product of the entries being positive. Likewise we see that $\lambda^{(p)}_{\text{min}}(K^r_r) = -r!/r^{r/p}$ and the eigenvectors to $\lambda^{(p)}_{\text{min}}(K^r_r)$ are all vectors of the type $(\pm r^{-1/p}, \ldots, \pm r^{-1/p})$, with the product of the entries being negative.

Hence, for $r \geq 3$ we get a situation, which is impossible for 2-graphs; let us summarize these findings.

**Fact 2.1** If $r \geq 3$, then both $\lambda^{(p)}(K^r_r)$ and $\lambda^{(p)}_{\text{min}}(K^r_r)$ have $r$ linearly independent eigenvectors.

Trivially, $\lambda^{(p)}$ is monotone with respect to edge weights:

**Proposition 2.2** Let $p \geq 1$, $r \geq 2$, let $G \in \mathcal{W}^r$ and $H \in \mathcal{W}^r$. If $V(G) = V(H)$ and $H(U) \leq G(U)$ for each $U \in (V(G))^{(r)}$, then $\lambda^{(p)}(H) \leq \lambda^{(p)}(G)$.

Next, note that $\lambda^{(p)}(G)$ is monotone with respect to edge addition and $\lambda^{(p)}_{\text{min}}(G)$ is monotone with respect to vertex addition.

**Proposition 2.3** If $G \in \mathcal{W}^r$, $H \in \mathcal{W}^r$, and $H$ is a subgraph of $G$, then

$$\lambda^{(p)}(H) \leq \lambda^{(p)}(G). \tag{4}$$

If $H$ is an induced subgraph of $G$, then

$$\lambda^{(p)}_{\text{min}}(G) \leq \lambda^{(p)}_{\text{min}}(H).$$

Hence, $\lambda^{(p)}_{\text{min}}(G) < 0$, unless $G$ has no edges.

Let us note that the conditions for strict inequality in (4) are not obvious; we postpone the discussion to Corollary 5.4 below. Somewhat unexpected is the fact that if $G \in \mathcal{G}^r$ has no isolated vertices and $p > r$, then $\lambda^{(p)}(G)$ is always larger than $\lambda^{(p)}$ of each of its proper subgraphs; this statement is stated in detail in Theorem 5.5 and Corollary 5.6.

Further, Propositions 2.3 and 1.2 and the convexity of $x^s$ for $x \geq 0$, $s \geq 1$, imply the following facts, which are as expected.
Proposition 2.4 Let $1 \leq p \leq r$, $G_1, \ldots, G_k$ be pairwise vertex disjoint $r$-graphs. If $G$ is their union, then

$$\lambda^{(p)}(G) = \max \left\{ \lambda^{(p)}(G_1), \ldots, \lambda^{(p)}(G_k) \right\}$$

$$\lambda^{(p)}_{\min}(G) = \min \left\{ \lambda^{(p)}_{\min}(G_1), \ldots, \lambda^{(p)}_{\min}(G_k) \right\}.$$

Again, if $p > r$ and $G \in \mathcal{G}^r$, the statements are different.

Theorem 2.5 Let $p > r \geq 2$ and let $G_1, \ldots, G_k$ be the components of an $r$-graph $G$. If $G$ has no isolated vertices, then

$$\lambda^{(p)}(G) = \left( \sum_{i=1}^{k} (\lambda^{(p)}(G_i))^{p/(p-r)} \right)^{(p-r)/p}$$

and

$$\lambda^{(p)}_{\min}(G) = - \left( \sum_{i=1}^{k} |\lambda^{(p)}_{\min}(G_i)|^{p/(p-r)} \right)^{(p-r)/p}.$$

Proof We shall prove only (5). Let $G_1, \ldots, G_k$ and $G$ be as required. Let $x \in S^{n-1}$ be an eigenvector to $\lambda^{(p)}(G)$ and let $y_i$ be the restriction of $x$ to the set $V(G_i)$. Now, by Proposition 1.2

$$\lambda^{(p)}(G) = P_G(x) = \sum_{i=1}^{k} P_{G_i}(y_i) \leq \sum_{i=1}^{k} \lambda^{(p)}(G_i) |y_i|^r_{p}.$$

Letting $s = p/r$, $t = p/(p-r)$, we have $1/s + 1/t = r/p + (p-r)/p = 1$, and applying Hölder’s inequality (67), we get

$$\sum_{i=1}^{k} \lambda^{(p)}(G_i) |y_i|^r_{p} \leq \left( \sum_{i=1}^{k} (\lambda^{(p)}(G_i))^t \right)^{1/t} \left( \sum_{i=1}^{k} |y_i|^s_{p} \right)^{1/s}$$

$$= \left( \sum_{i=1}^{k} (\lambda^{(p)}(G_i))^{p/(p-r)} \right)^{(p-r)/p} \left( \sum_{i=1}^{k} |y_i|^p_{p} \right)^{r/p} = \left( \sum_{i=1}^{k} (\lambda^{(p)}(G_i))^{p/(p-r)} \right)^{(p-r)/p}.$$

To prove equality in (5) for each $i = 1, \ldots, k$, choose an eigenvector $z_i$ to $\lambda^{(p)}(G_i)$; then scale each $z_i$ so that $\sum_{i=1}^{k} |z_i|^p_{p} = 1$ and $(|z_i|^s, \ldots, |z_k|^s)$ is collinear to $\left( (\lambda^{(p)}(G_1))^t, \ldots, (\lambda^{(p)}(G_k))^t \right)$. Now, letting $u$ be equal to $z_i$ within $V(G_i)$ for $i = 1, \ldots, k$, we see that $|u|^p_{p} = 1$ and

$$\lambda^{(p)}(G) \geq P_G(u) = \left( \sum_{i=1}^{k} (\lambda^{(p)}(G_i))^t \right)^{1/t} \left( \sum_{i=1}^{k} |z_i|^s_{p} \right)^{1/s} = \left( \sum_{i=1}^{k} (\lambda^{(p)}(G_i))^{p/(p-r)} \right)^{(p-r)/p},$$

completing the proof of (5). □
2.1 Bounds on $\lambda^{(p)}$ in terms of order and size

For 2-graphs it is known that $\lambda(G) \leq n - 1$, with equality holding for complete graphs. Write $(n)_r$ for the falling factorial $n (n - 1) \cdots (n - r + 1)$. Maclaurin’s and the PM inequalities (68) and (69) imply an absolute upper bound on $\lambda^{(p)}(G)$; the conditions for equality in these inequalities imply that the bound is attained precisely for complete graphs.

**Proposition 2.6** If $p \geq 1$ and $G \in \mathcal{W}^r(n)$, then $\lambda^{(p)}(G) \leq |G|_\infty (n)_r/n^r/p$. Equality holds if and only if $G$ is a constant.

Let $n > r$, and $G \in \mathcal{G}^r(n)$ be a complete graph. If $r$ is even, then $\pm n^{-1/p}j_n$ are the only eigenvectors to $\lambda^{(p)}(G)$, and if $r$ is odd, the only eigenvector is $n^{-1/p}j_n$.

Although the above proposition elucidates the absolute maximum of $\lambda^{(p)}$, the following fundamental bounds are more flexible and usable.

**Theorem 2.7** Let $G \in \mathcal{W}^r(n)$. If $p \geq 1$, then

$$\lambda^{(p)}(G) \geq r! |G| / n^r/p. \quad (6)$$

If $p > 1$, then

$$\lambda^{(p)}(G) \leq \left( (n)_r / n^r \right)^{1/p} |r!G|_{p/(p-1)}. \quad (7)$$

If $p = 1$, then

$$\lambda^{(1)}(G) \leq (n)_r / n^r |G|_\infty. \quad (8)$$

**Proof** Indeed, setting $x = n^{-1/p}j_n$ in (11), we obtain

$$\lambda^{(p)}(G) \geq r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} G(\{i_1, \ldots, i_r\}) n^{-r/p} = \frac{r! |G|}{n^{r/p}},$$

proving (6). Let now $x = [x_i]$ be a eigenvector to $\lambda^{(p)}(G)$. Hölder’s inequality (67) with $p = 1$, $q = p$ and $k = |G|$ implies that

$$\sum_{\{i_1, \ldots, i_r\} \in E} G(\{i_1, \ldots, i_r\}) x_{i_1} \cdots x_{i_r} \leq \left( \sum_{e \in E} (G(e))^{p/(p-1)} \right)^{(p-1)/p} \left( \sum_{\{i_1, \ldots, i_r\} \in E} |x_{i_1}|^p \cdots |x_{i_r}|^p \right)^{1/p}$$

$$= |G|_{p/(p-1)} \left( \sum_{\{i_1, \ldots, i_r\} \in E} |x_{i_1}|^p \cdots |x_{i_r}|^p \right)^{1/p}.$$ 

Now, letting $y := (|x_1|^p, \ldots, |x_n|^p)$ and applying Maclaurin’s inequality (70), we see that

$$S_r(y) \left( \frac{n}{r} \right) \leq \left( \frac{S_1(y)}{n} \right)^{r} = \left( \frac{|x_1|^p + \cdots + |x_n|^p}{n} \right)^{r} = n^{-r}.$$
Therefore,
\[ \lambda^{(p)} (G) \leq r ! \left( \left( \frac{n}{r} \right) / n^r \right)^{1/p} |G|_{p/(p-1)}^{1/p} \leq \left( \left( (n)_r / n^r \right)^{1/p} \right) r ! |G|_{p/(p-1)}, \]
proving (7).

Finally, if \( p = 1 \), inequality (8) follows by Maclaurin’s inequality. \( \square \)

Let us point out that the application of the PM and Maclaurin’s inequalities in the proof of (7) is rather typical and works well for similar upper bounds on \( \lambda^{(p)} (G) \).

Simpler versions of inequality (6) have been proved in [3] and [31]. Note that (6) generalizes the inequality of Collatz and Sinogovits [5] \( \lambda (G) \geq 2 |G| / n \) for 2-graphs. Likewise, (7) generalizes the inequality of Wilf [40] for 2-graphs:

\[ \lambda^{(p)} (G) \leq \sqrt{2} (1 - 1/n) |G|. \]

Since \( 0 < (n)_r / n^r < 1 \), from (7) we obtain a weaker, but simple and very usable inequality, involving just the \( l^{1-1/p} \) norm of \( G \).

**Corollary 2.8** If \( p > 1 \) and \( G \in \mathcal{W}^r (n) \), then
\[ \lambda^{(p)} (G) \leq |r ! G|_{p/(p-1)}. \]

If \( |G|_\infty > 0 \), the inequality is strict. In particular, if \( G \in \mathcal{G}^r (n) \), then
\[ \lambda^{(p)} (G) \leq r ! |G|^{1-1/p}. \quad (9) \]

Note that inequality (9) has been proved by Keevash, Lenz and Mubayi in [20]. This useful bound is essentially tight as explained below.

**Proposition 2.9** Let \( p > 1 \) and \( n \geq r \). If \( m \leq \binom{n}{r} \), there exists a \( G \in \mathcal{G}^r (n) \) such that
\[ \lambda^{(p)} (G) = (1 - o (m)) (r ! |G|)^{1-1/p}. \]

Indeed, given a natural \( m \), let \( k \) satisfy \( \binom{k-1}{r-1} < m \leq \binom{k}{r} \), and let \( G \in \mathcal{G}^r (n) \) be a graph with \( |G| = m \), such that \( K_{k-1}^r \subset G \subset K_k^r \), i.e., \( G \) has \( n - k \) isolated vertices. Now, by (3) and Proposition 2.3
\[ (k-1)_r / (k-1)^{r/p} = \lambda^{(p)} (K_{k-1}^r) \leq \lambda^{(p)} (K_k^r) = \lambda^{(p)} (K_k^r) = (k)_r / k^{r/p}. \]

### 2.2 \( \lambda^{(p)} (G) \) as a function of \( p \)

Let us note that the introduction of the parameter \( p \) in \( \lambda^{(p)} \) and \( \lambda^{(p)}_{\min} \) is not for want of complications. Not only this parametrization is a constant source of new insights, but also it plays a unification role. Indeed, assume that \( G \) is a fixed \( r \)-graph and consider \( \lambda^{(p)} (G) \) as a function of \( p \).
Since $\lambda^{(p)}(G)$ always has an eigenvector in $S_{p,r+1}$, we can change the variables in (11) obtaining the following equivalent definition of $\lambda^{(p)}(G)$:

$$
\lambda^{(p)}(G) := \max_{|y_1|+\ldots+|y_n|=1} r! \sum_{\{i_1,\ldots,i_r\} \in E(G)} |y_{i_1}|^{1/p} \ldots |y_{i_r}|^{1/p}.
$$

(10)

Now, this definition helps to see very clearly some essential features of $\lambda^{(p)}(G)$, like the fact that $\lambda^{(p)}(G)$ is increasing in $p$. The mean value theorem implies that the inequality

$$
|y_{i_1}|^{1/q} \ldots |y_{i_r}|^{1/q} - |y_{i_1}|^{1/p} \ldots |y_{i_r}|^{1/p} \leq q - p
$$

whenever $q \geq p \geq 1$. Therefore,

$$
|\lambda^{(q)}(G) - \lambda^{(p)}(G)| \leq |q - p| r! |G|,
$$

whenever $q \geq 1$, $p \geq 1$ and so $\lambda^{(p)}(G)$ is a Lipshitz function in $p$. We get the following summary:

**Proposition 2.10** If $G$ is a fixed $r$-graph and $p \geq 1$, then $\lambda^{(p)}(G)$ is increasing and continuous in $p$. Also $\lambda^{(1)}(G) < 1$ and

$$
\lim_{p \to \infty} \lambda^{(p)}(G) = r! |G|.
$$

So, as a function of $p$, $\lambda^{(p)}(G)$ seamlessly encompasses three fundamental graph parameters - the Lagrangian, the spectral radius and the number of edges. Let us observe though that for some graphs $\lambda^{(p)}(G)$ is not continuously differentiable in $p$. Indeed, if $G$ consists of two disjoint complete $r$-graphs of order $n$, then

$$
\lambda^{(p)}(G) = \begin{cases} 
(n)_r / n^{r/p} & \text{if } p \leq r; \\
2(n)_r / (2n)^{r/p} & \text{if } p \geq r.
\end{cases}
$$

Note that the value of $\lambda^{(p)}(G)$ for $p \geq r$ follows from Proposition 3.7 below. Differentiating $\lambda^{(p)}(G)$ for $p > r$ and $p < r$, and taking the limits as $p \to r$, we see that

$$
\lim_{p \to r^+} \frac{d}{dp} (\lambda^{(p)}(G)) = \lim_{p \to r^+} \frac{-2r (n)_r \log (2n)}{p^2 (2n)^{r/p}} = -\frac{(n)_r \log (2n)}{rn},
$$

$$
\lim_{p \to r^-} \frac{d}{dp} (\lambda^{(p)}(G)) = \lim_{p \to r^-} \frac{-r (n)_r \log (n)}{p^2 (n)^{r/p}} = -\frac{(n)_r \log n}{rn}.
$$

Hence $\lambda^{(p)}(G)$ is not continuously differentiable at $r$. This situation is more general than it seems.

**Proposition 2.11** Let $0 \leq k \leq r - 2$ and $G \in \mathcal{G}_r^e(2r - k)$. It $G$ is a union of two edges sharing exactly $k$ vertices, then $\lambda^{(p)}(G)$ is not continuously differentiable at $p = r - k$.

However the following open questions seem relevant.

**Question 2.12** Suppose that $G \in \mathcal{G}_r^e$. Is $\lambda^{(p)}(G)$ continuously differentiable for $p > r$? Is $\lambda^{(p)}(G)$ continuously differentiable for $p \neq k$, $k = 2, \ldots , r$?
In the following propositions we shall estimate how fast $\lambda^{(p)} (G)$ increases.

**Proposition 2.13** If $p \geq 1$ and $G \in \mathcal{G}^r (n)$, then the function

$$h_G (p) := \frac{\lambda^{(p)} (G)}{n^{r/p}}$$

is nonincreasing in $p$, and

$$\lim_{p \to \infty} h_G (p) = r! |G| .$$

**Proof** Let $p > q \geq 1$ and let $[x_i] \in S^{n-1}$ be an eigenvector to $\lambda^{(q)} (G)$. By the PM inequality we have $\| [x_i] \|_p \geq n^{1/p-1/q}$, and Proposition 1.2 implies that

$$\lambda^{(p)} (G) \geq P_G ([x_i]) / \| [x_i] \|^r_p \geq \lambda^{(q)} (G) n^{r/q-r/p} .$$

□

Here is a similar statement involving the number of edges of $G$, which can be proved applying the PM inequality to the definition (10).

**Proposition 2.14** If $p \geq 1$ and $G \in \mathcal{G}^r$, then the function

$$f_G (p) := \left( \frac{\lambda^{(p)} (G)}{r! |G|} \right)^p$$

is nonincreasing in $p$.

### 2.3 $\lambda^{(p)}_{\min} (G)$ as a function of $p$

Some of the above properties of $\lambda^{(p)}$ can be proved also for $\lambda^{(p)}_{\min}$. Thus, assume that $G$ is a fixed $r$-graph and consider $\lambda^{(p)} (G)$ as a function of $p$. Taking an eigenvector $[x_i] \in S^{n-1}_p$ and changing the variables by the one-to-one correspondence $x_i \to |y_i|^{1/p-1}$, we see that

$$\lambda^{(p)}_{\min} (G) = \min_{\|y_1\|+\cdots+\|y_r\|=1} r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} y_{i_1} \cdots y_{i_r} |y_{i_1} \cdots y_{i_r}|^{1/p-1} .$$

Some algebra gives that

$$\left| \lambda^{(q)}_{\min} (G) - \lambda^{(p)}_{\min} (G) \right| < |q-p| r! |G| ,$$

which implies also the following proposition.

**Proposition 2.15** If $G$ is a fixed $r$-graph and $p \geq 1$, then $\lambda^{(p)}_{\min} (G)$ is decreasing and continuous in $p$. Also, $\lambda^{(1)}_{\min} (G) \geq -1$ and the limit $\lim_{p \to \infty} \lambda^{(p)}_{\min} (G)$ exists.
One can figure out a description of the limit \( \lim_{p \to \infty} \lambda_{\min}^{(p)} (G) \), but its combinatorial significance is not completely clear. If \( G \) has an odd transversal, then \( \lim_{p \to \infty} \lambda_{\min}^{(p)} (G) = -r! |G| \), see Theorem 8.5 below. For 2-graphs this is equivalent to \( G \) being bipartite.

Like for \( \lambda^{(p)} (G) \), we have the following estimate for the rate of change of \( \lambda_{\min}^{(p)} (G) \).

**Proposition 2.16** If \( p \geq 1 \) and \( G \in G^r (n) \), then the function \( g_G (p) := \lambda_{\min}^{(p)} (G) / n^{r/p} \) is nondecreasing in \( p \).

Let \( 0 \leq k \leq r - 2 \). Taking \( G \in G^r (2r - k) \) to be the union of two edges sharing exactly \( k \) vertices, we get a graph with \( \lambda_{\min}^{(p)} (G) = -\lambda^{(p)} (G) \), and, as above, we see that \( \lambda^{(p)} (G) \) is not continuously differentiable at \( p = r - k \). Here is a natural question.

**Question 2.17** Suppose that \( G \in G^r \). Is \( \lambda^{(p)} (G) \) continuously differentiable for \( p > r \)? Is \( \lambda^{(p)} (G) \) continuously differentiable for \( p \neq k, k = 2, \ldots, r \)?

### 3 Eigenequations

The Rayleigh principle and the Courant-Fisher inequalities allow to define eigenvalues of Hermitian matrices as critical values of quadratic forms over the unit sphere \( S^{n-1}_2 \). From this variational definition the standard definition via linear equations can be recovered using Lagrange multipliers. We follow the same path for eigenvalues of hypergraphs; in our case it is particularly simple because we are interested mostly in the largest and the smallest eigenvalues. Thus, next we shall show that the variational definitions (1) and (2) lead to systems of equations arising from Lagrange multipliers.

#### 3.1 The system of eigenequations

Suppose that \( G \in W^r (n) \) and let \( [x_i] \in S^{n-1}_p \) be an eigenvector to \( \lambda^{(p)} (G) \). If \( p > 1 \), the function

\[
g (y_1, \ldots, y_n) := |y_1|^p + \ldots + |y_n|^p
\]

has continuous partial derivatives in each variable (see 12.1). Thus, using Lagrange’s method (Theorem 12.1), there exists a \( \mu \) such that for each \( k = 1, \ldots, n \),

\[
\mu p x_k |x_k|^{p-2} = \frac{\partial P_G ([x_i])}{\partial x_k} = r! \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} G (\{k,i_1,\ldots,i_{r-1}\}) x_i_1 \cdot \cdot \cdot x_i_{r-1}.
\]

Now, multiplying the \( k \)'th equation by \( x_k \) and adding them all, we find that

\[
\mu p = \mu p \sum_{k \in V(G)} |x_k|^p = \sum_{k \in V(G)} x_k \frac{\partial P_G ([x_i])}{\partial x_k} = r P_G ([x_i]) = r \lambda^{(p)} (G).
\]

Hence, we arrive at the following theorem.
Theorem 3.1 Let $G \in \mathcal{W}^r(n)$ and $p > 1$. If $[x_i] \in S^{n-1}_p$ is an eigenvector to $\lambda^{(p)}(G)$, then $x_1, \ldots, x_n$ satisfy the equations

$$\lambda^{(p)}(G) x_k |x_k|^{p-2} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n. \quad (11)$$

For $p > 1$ equations $(11)$ are a powerful tool in the study of $\lambda^{(p)}(G)$, but since $|x|$ is not differentiable at 0, they are not always available for $p = 1$.

A similar argument for $\lambda_{\min}^{(p)}(G)$ leads to the following theorem.

Theorem 3.2 Let $G \in \mathcal{W}^r(n)$ and $p > 1$. If $[x_i] \in S^{n-1}_p$ is an eigenvector to $\lambda_{\min}^{(p)}(G)$, then $x_1, \ldots, x_n$ satisfy the equations

$$\lambda_{\min}^{(p)}(G) x_k |x_k|^{p-2} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n. \quad (12)$$

Note that if $G$ is a 2-graph with adjacency matrix $A$, then $(11)$ and $(12)$ reduce to the familiar equations

$$Ax = \lambda(G)x \quad \text{and} \quad Ay = \lambda_{\min}(G)y.$$ 

Therefore, we shall call equations $(11)$ and $(12)$ the *eigenequations* for $\lambda^{(p)}(G)$ and for $\lambda_{\min}^{(p)}(G)$. In general, given $G \in \mathcal{W}^r(n)$, there may be many different real or complex numbers $\lambda$ and $n$-vectors $[x_i]$ with $|[x_i]|_p = 1$ satisfying the equations

$$\lambda x_k |x_k|^{p-2} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n. \quad (13)$$

This multiplicity may remain even if we impose additional restrictions, like $[x_i] \geq 0$ or $[x_i] > 0$, or $G$ being a connected $r$-graph. Nevertheless, having a unique solution $(\lambda, [x_i])$ to $(13)$ is highly desirable; the Perron-Frobenius type theory developed in Section 5 provides some conditions that guarantee this property.

### 3.2 Algebraic definitions of eigenvalues

In this subsection we discuss some algebraic definitions of hypergraph eigenvalues along the work of Qi [32], who proposed to define eigenvalues of hypermatrices using equations similar to $(13)$. If the definition of Qi is applied to a graph $G \in \mathcal{W}^r(n)$, then an eigenvalue of $G$ is a complex number $\lambda$ which satisfies the equation

$$\lambda x_k^{-1} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n. \quad (14)$$

for some nonzero vector $[x_i] \in \mathbb{C}^n$. When $x_1, \ldots, x_n$ are real, $\lambda$ is also real and is called an $H$-eigenvalue.
Another definition suggested by Friedman and Wigderson and developed by Qi defines a graph eigenvalues as solutions $\lambda$ of the system
\[
\lambda x_k = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,
\]
for some complex $x_1, \ldots, x_n$ with $|x_1|^2 + \cdots + |x_k|^2 = 1$. In this case, $\lambda$ is called an $E$-eigenvalue of $G$; if $x_1, \ldots, x_n$ are real, then $\lambda$ is called a $Z$-eigenvalue of $G$.

It is not hard to see that these definitions fit with our setup for $\lambda^{(p)}(G)$ and $\lambda^{(p)}_{\min}(G)$. Indeed, since $\lambda(G)$ satisfies with a vector $[x_i] \in S^{n-1}_{p,+}$, it is an $H$-eigenvalue in the definition of Qi; moreover, $\lambda(G)$ has the largest absolute value among all eigenvalues defined by $\lambda^{(p)}$.

**Proposition 3.3** Let $G \in W^r(n)$. If the complex number $\lambda$ satisfies the equations (14) for some nonzero complex vector $[x_i]$, then $|\lambda| \leq \lambda(G)$.

**Proof** Suppose that $\lambda \in \mathbb{C}$ and $[x_i] \in \mathbb{C}^n$ satisfy the system (14). For $k = 1, \ldots, n$ the triangle inequality implies that
\[
|\lambda| |x_k|^{r-1} = \frac{1}{r} \left| \frac{\partial P_G([x_i])}{\partial x_k} \right| \leq (r-1)! \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} |x_{i_1}| \cdots |x_{i_{r-1}}|, \quad k = 1, \ldots, n.
\]

Multiplying the $k$'th inequality by $|x_k|$ and adding them all, we obtain
\[
|\lambda| (|x_1|^r + \cdots + |x_n|^r) \leq P_G((|x_1|, \ldots, |x_n|)) \leq \lambda(G) (|x_1|^r + \cdots + |x_n|^r),
\]

implying the assertion. \(\square\)

Similarly $\lambda^{(2)}(G)$ is unique among all $E$-eigenvalues; in fact, the proof is valid for all $p > 1$: If the complex number $\lambda$ satisfies the equations
\[
\lambda x_k |x_k|^{p-2} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,
\]
for some complex vector $[x_i]$ with $|[x_i]|_p = 1$, then $|\lambda| \leq \lambda^{(p)}(G)$.

In the same spirit, one can show that $\lambda^{(p)}_{\min}(G)$ is the smallest real solution to (14); we extend this fact for $\lambda^{(p)}_{\min}(G)$.

**Proposition 3.4** Let $G \in W^r(n)$ and $p > 1$. If the real number $\lambda$ and $[x_i] \in S^{n-1}_p$ satisfy the equations
\[
\lambda^{(p)}_{\min}(G) x_k |x_k|^{p-2} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,
\]
then $\lambda^{(p)}_{\min}(G) \leq \lambda$. In particular, if $p = r$, and $\lambda$ and $[x_i] \in S^{n-1}_p$ satisfy the equations (14), then $\lambda^{(p)}_{\min}(G) \leq \lambda$. 

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3.3 Regular graphs and $\lambda^{(p)}$

A weighted graph $G \in W^r(n)$ is called (vertex) regular if all vertex degrees are equal, i.e., all vertex degrees are equal to $r |G| / n$. It is easy to see that for every regular graph $G \in W^r(n)$, there is a positive $\lambda$ satisfying the eigen-equations for $\lambda^{(p)}(G)$.

**Proposition 3.5** If $G \in W^r(n)$ is regular, then for every $p > 1$ the number $\lambda = r! |G| n^{-r/p}$ and the vector $[x_i] = n^{-1/p} j_n$ satisfy the equations

$$\lambda x_k^{p-1} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n. \quad (15)$$

Conversely if for some $p > 1$ there is a number $\lambda > 0$ such that $[x_i] = n^{-1/p} j_n$, satisfy the equations $(15)$, then $G$ is regular.

We can easily relate these simple observations to $\lambda^{(p)}(G)$.

**Proposition 3.6** If $G \in W^r(n)$ and $\lambda^{(p)}(G) = r! |G| n^{-r/p}$ for some $p > 1$, then $G$ is regular.

Moreover, if $p \geq r$, the converse of this statement is true as well, and Proposition 4.8 shows that the cycles $C_n^r$ are counterexamples if $1 < p < r$.

**Proposition 3.7** If $p \geq r$ and $G \in W^r(n)$ is regular, then $\lambda^{(p)}(G) = r! |G| n^{-r/p}$.

We omit this proof as the statement is easy to prove, first for $p = r$, and then the general case by Proposition 2.13. An important open problem here is the following one:

**Problem 3.8** Let $r \geq 2$, $1 < p < r$. Characterize all regular graphs $G \in G^r(n)$ such that

$$\lambda^{(p)}(G) = r! |G| n^{-r/p}. \quad (16)$$

For example, if $G$ is a complete or a complete multipartite $r$-graph, or $(r-1)$-set regular, then equality holds in $(16)$. On the other hand, if $G$ is relatively sparse, like $|G| = o(n^{r/p})$, then $(16)$ fails for sure. Indeed, if $|G| = o(n^{r/p})$, then for $n$ sufficiently large,

$$\lambda^{(p)}(G) \geq \lambda^{(p)}(K_r) = r! n^{r/p} > r! |G| n^{-r/p}.$$

But note that $(16)$ may fail even if $G$ is quite dense; e.g., if $G$ is the disjoint union of two complete $r$-graphs of order $n$, then for $n$ sufficiently large,

$$\lambda^{(p)}(G) = (n) r^{-1} n^{-r/p} > \frac{2(n)_r}{2^r r! n^{r/p}} = r! \cdot 2 \left( \frac{n}{r} \right)^r (2n)^{-r/p} = r! |G| (2n)^{-r/p}.$$

Therefore, it seems that Problem 3.8 is quite important, insofar that its complete solution would most certainly relate $\lambda^{(p)}(G)$ to the local edge density of $G$. 

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3.4 Symmetric vertices and eigenvectors

Let \( G \in \mathcal{W}^r \) and let \( u, v \in V(G) \). For practical calculations we wish to have structural conditions on \( G \), which would guarantee that \( x_u = x_v \) for any eigenvector \([x_i]\) to \( \lambda^{(p)}(G) \). Thus, we say that \( u \) and \( v \) are equivalent in \( G \), in writing \( u \sim v \), if transposing \( u \) and \( v \) and leaving the remaining vertices intact we get an automorphism of \( G \). Obviously, \( u \sim v \) if every edge \( e \in E(G) \) such that \( e \cap \{u, v\} \neq \emptyset \) satisfies

\[
\{u, v\} \subset e \text{ or } (e \setminus \{v\}) \cup \{u\} \in E(G) \text{ or } (e \setminus \{u\}) \cup \{v\} \in E(G).
\]

Now equations (11) imply the following lemma.

**Lemma 3.9** Let \( G \in \mathcal{W}^r(n) \) and let \( u \sim v \). If \( p > 1 \) and \([x_i]\) is an eigenvector to \( \lambda^{(p)}(G) \) or to \( \lambda^{(p)}_{\text{min}}(G) \), then \( x_u = x_v \).

**Proof** Write \( \lambda \) for \( \lambda^{(p)}(G) \) or \( \lambda^{(p)}_{\text{min}}(G) \) and let \([x_i]\) be an eigenvector to \( \lambda \). We have

\[
\lambda x_u |x_u|^{p-2} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_u} \quad \text{and} \quad \lambda x_v |x_v|^{p-2} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_v}.
\]

Hence, using that \( u \) and \( v \) are equivalent, we see that

\[
\lambda x_u |x_u|^{p-2} - \lambda x_v |x_v|^{p-2} = (x_v - x_u) (r - 1)! \sum_{\{u, v, i_1, \ldots, i_{r-2}\}} G(\{u, v, i_1, \ldots, i_{r-2}\}) x_{i_1} \cdots x_{i_{r-2}}.
\]

Since the function \( f(x) := x |x|^{p-2} \) is increasing in \( x \) for every real \( x \), we see that \( x_u - x_v = 0 \), completing the proof. \( \square \)

Note that the symmetric vertices in general do not have equal entries, see Proposition 4.8 below. Lemma 3.9 implies a practical statement very similar to Corollary 12, in [20].

**Corollary 3.10** Let \( G \in \mathcal{W}^r(n) \). If \( V(G) \) be partitioned into equivalence classes by the relation \( \sim \), then every eigenvector \([x_i]\) to \( \lambda^{(p)}(G) \) or to \( \lambda^{(p)}_{\text{min}}(G) \) is constant within each equivalence class.

The above corollary can be quite useful in calculating or estimating \( \lambda^{(p)}(G) \); for instance, to calculate \( \lambda^{(p)} \) and \( \lambda^{(p)}_{\text{min}} \) of a \( \beta \)-star.

**Proposition 3.11** Let \( G \in \mathcal{G}^r((r-1)k + 1) \) and let \( G \) consist of \( k \) edges sharing a single vertex. If \( p \geq r-1 \), then \( \lambda^{(p)}(G) = (r!/r^r/p) k^{1-(r-1)/p} \). Also, \( \lambda^{(p)}_{\text{min}}(G) = -(r!/r^r/p) k^{1-(r-1)/p} \).
4 Some warning illustrations

Let us note again that best-known case of $\lambda^{(p)}(G)$ of a graph $G \in \mathcal{W}^{r}$, $(r \geq 2, p \geq 1)$, is the largest eigenvalue of a 2-graph. However, expectations based on eigenvalues of 2-graphs can collide with the real properties of $\lambda^{(p)}(G)$ if $r \geq 3$ or if $r = 2$ and $p \neq 2$. The purpose of the examples below is to deflect some wrong expectations, and at the same time to outline limitations to Perron-Frobenius type properties for $\lambda^{(p)}$.

4.1 Zero always satisfies the eigenequations

Let start with a simple observation. If $r \geq 3$, any vector with at most $r-2$ nonzero entries satisfies the eigenequations (13) with $\lambda = 0$; this follows trivially as every edge consists of $r$ distinct vertices.

**Proposition 4.1** Let $n \geq r \geq 3$ and $p > 1$. For every $G \in \mathcal{W}^{r}(n)$, there are $n$ linearly independent nonnegative solutions $[x_i]$ to the equations

$$0 \cdot x_k^{p-1} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n.$$ 

This fact is impossible for 2-graphs with edges, but it is unavoidable for any uniform hypergraph.

4.2 Strange eigenvectors in graphs with two edges

In this subsection we discuss $r$-graphs formed by two edges with exactly $k$ vertices in common. This simple construction will give examples of eigenvectors to $\lambda^{(p)}(G)$, which are abnormal from the viewpoint of 2-graphs, but natural for hypergraphs. These examples also outline the scope of validity of some theorems in Section 5. Finally the reader can practice simple methods for evaluating $\lambda^{(p)}$ and $\lambda^{(p)}_{\min}$.

First we shall discuss in some detail the following 3-graph.

**Proposition 4.2** Let $G \in \mathcal{G}_3(5)$ and $G$ consist of two edges sharing a single vertex. We have

$$\lambda^{(2)}(G) = \frac{2}{\sqrt{3}},$$

and $\lambda^{(2)}(G)$ has infinitely many positive eigenvectors and two nonnegative ones with zero entries.

If $1 < p < 2$, then

$$\lambda^{(p)}(G) = 6 \cdot 3^{-3/p},$$

and $\lambda^{(p)}(G)$ has no positive eigenvector. There exists a positive $\lambda < \lambda^{(p)}(G)$ and a positive vector $[x_i] \in \mathbb{S}_p^4$ satisfying the eigenequations for $\lambda^{(p)}(G)$.

**Proof** Let $V(G) = [5]$ and let the two edges of $G$ be $\{1, 2, 3\}$ and $\{3, 4, 5\}$. Suppose that $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{S}_p^4$ is an eigenvector to $\lambda^{(2)}(G)$. Theorem 3.9 implies that $x_1 = x_2$ and $x_4 = x_5$. Setting $x_3 = x$, we have

$$\lambda^{(2)}(G) = 3! \max (x_1 x_2 x_3 + x_3 x_4 x_5) = 3! \cdot \max (x_1^2 + x_4^2) x = 3! \cdot \max_{0 < x \leq 1} \left( \frac{1 - x^2}{2} \right) x = \frac{2}{\sqrt{3}}.$$
Clearly, the maximum is attained for any vector \((s, s, 3^{-1/2}, t, t)\), with \(s^2 + t^2 = 1/3\).

Let now \(1 < p < 2\) and \((x_1, x_2, x_3, x_4, x_5) \in \mathbb{S}_{p,+}^4\) is an eigenvector to \(\lambda^{(p)}(G)\). We see that

\[
\lambda^{(p)}(G) = 3! \max_{|x|_p = 1} (x_1 x_2 x_3 + x_3 x_4 x_5) = 3! \max_{|x|_p = 1} (x_1^2 + x_4^2) x_3.
\]

If \(x_3\) is fixed, then \(\max x_1^2 + x_4^2 \) subject to \(x_1 + x_4 = (1 - x_3^p) / 2\) is attained if \(x_1 = 0\) or \(x_4 = 0\) because \(f(y) = y^{2/p}\) is a convex function. Therefore, \(\lambda^{(p)}(G)\) has no positive eigenvector. By (3) we find that

\[
\lambda^{(p)}(G) = 3! \cdot 3^{3-3/p} = \frac{6}{3^{3/p}}.
\]

We notice that in the above proposition \(G\) is connected, but \(\lambda^{(2)}(G)\) has infinitely many positive eigenvectors, and if \(1 < p < 2\), then \(\lambda^{(p)}(G)\) has no positive eigenvector at all. This is impossible for the largest eigenvalue of a connected 2-graph.

For \(r \geq 3\) the example of Proposition 4.2 can be generalized as follows.

**Proposition 4.3** Let \(r \geq 3\), \(1 \leq k \leq r - 2\), let \(G \in \mathcal{G}^r(2r - k)\) and let \(G\) consists of two edges sharing precisely \(k\) vertices. We have

\[
\lambda^{(r-k)}(G) = (r - 1)!/r^{(r-1)},
\]

and the set of eigenvectors to \(\lambda^{(r-k)}(G)\) contains a circle; in particular, infinitely many positive vectors and two eigenvectors with 0 entries.

If \(1 < p < r - k\), then

\[
\lambda^{(p)}(G) = r!p^{-r/p}
\]

and \(\lambda^{(p)}(G)\) has no positive eigenvector. Moreover, there exists a positive \(\lambda < \lambda^{(p)}(G)\) and a positive \(x \in \mathbb{S}_{p,+}^{2r-k-1}\), satisfying the eigenequations for \(\lambda^{(p)}(G)\).

The weird properties of the examples in Proposition 4.2 seem due more to the fact that \(p \leq r - 1\) than to the fact \(r \geq 3\). However, a similar phenomenon is observed also for \(\lambda(G) = \lambda^{(r)}(G)\) as described below.

**Proposition 4.4** Let \(r \geq 3\) and \(G \in \mathcal{G}^r(r + 2)\). If \(G\) consists of two edges sharing precisely \(r - 2\) vertices, then \(\lambda = (r - 1)!\) and the vector \(x = (r^{-1/r}, \ldots, r^{-1/r}, 0, 0)\) satisfy the eigenequations for \(\lambda(G)\)

\[
\lambda x_k^{r-1} = \frac{1}{r} \frac{\partial P_G(x)}{\partial x_k}, \quad k = 1, \ldots, r + 2,
\]

but \(\lambda(G) > (r - 1)!\).
4.3 Strange eigenvectors in cycles

The purpose of this subsection is to show that regular, connected graphs as simple as cycles can also have strange eigenvectors of $\lambda^{(p)}$. Along this example we also answer a question of Pearson and Zhang.

Let us begin with raising a question about $\lambda^{(p)}$ of 2-cycles:

**Question 4.5** *If $C_n$ is the 2-cycle of order $n$ and $1 < p < 2$, what is $\lambda^{(p)}(C_n)$?*

To answer this question one has to find

$$\max_{x_1^p + \cdots + x_n^p} x_1x_2 + \cdots + x_{n-1}x_n + x_nx_1,$$

which is quite challenging for $n > 4$. However, for $n = 4$ there is a definite answer.

**Proposition 4.6** *If $p \geq 1$, then $\lambda^{(p)}(C_4) = 2^{3-4/p}$. If $p > 1$, the only nonnegative eigenvector to $\lambda^{(p)}(C_4)$ is $4^{-1/p}j_4$.*

Although we do not know $\lambda^{(p)}(C_n)$ precisely, we still can draw a number of puzzling conclusions. Since $C_n$ is connected it is not hard to see that every nonnegative eigenvector to $\lambda^{(p)}(C_n)$ is positive as shown in Theorem 5.2. However, if $1 < p < 2$, then for $n$ sufficiently large the vector $n^{-1/p}j_n$ is not an eigenvector to $\lambda^{(p)}(C_n)$, because

$$\lambda^{(p)}(C_n) \geq \lambda^{(p)}(K_2) = 2 \cdot 2^{-2/p} > 2n^{1-2/p} = P_{C_n}(n^{-1/p}j_n).$$

Therefore, if $n$ is sufficiently large, any nonnegative eigenvector $(x_1, \ldots, x_n)$ to $\lambda^{(p)}(C_n)$ has at least two distinct entries; hence, $(x_1, x_2, \ldots, x_n) \neq (x_n, x_1, \ldots, x_{n-1})$, and there are at least two positive eigenvectors to $\lambda^{(p)}(C_n)$. These findings are summarized in the following proposition.

**Proposition 4.7** *For every $p \in (1, 2)$ there exists an $n_0(p)$, such that if $n > n_0(p)$, then $\lambda^{(p)}(C_n)$ has at least two distinct positive eigenvectors, different from $n^{-1/p}j_n$. In addition, the value $\lambda = 2n^{1-2/p}$ and the vector $x = n^{-1/p}j_n$ satisfy the eigenequations (11) for $r = 2$ and $G = C_n$.*

To extend this proposition to $r$-graphs, define the $r$-cycle $C_n^r$ of order $n$ as: $v(C_n^r) = \mathbb{Z}/n\mathbb{Z}$, the additive group of the integer remainders mod $n$; the edges of $C_n^r$ are all sets of the type $\{i + 1, \ldots, i + r\}$, $i \in \mathbb{Z}/n\mathbb{Z}$. In other words, the vertices of $C_n^r$ can be arranged on a circle so that its edges are all segments of $r$ consecutive vertices along the circle.

It is not hard to generalize the previous proposition as follows.

**Proposition 4.8** *For every $p \in (1, r)$ there exists an $n_0(p)$, such that if $n > n_0(p)$, then $\lambda^{(p)}(C_n^r)$ has at least two distinct positive eigenvectors, different from $n^{-1/p}j_n$. In addition, the value $\lambda = r!n^{1-r/p}$ and the vector $x = n^{-1/p}j_n$ satisfy the eigenequations (11) for $G = C_n^r$.*

For $p = 2$ and $r \geq 3$ this example yields a negative answer to Question 4.9 of Pearson and Zhang [31].
4.4 \( \lambda^{(p)} \) and \( \lambda^{(p)}_{\text{min}} \) of \( \beta \)-stars

Recall that a graph \( G \in \mathcal{G}^r((r-1)k+1) \) consisting of \( k \) edges sharing a single vertex is called a \( \beta \)-star. Proposition 3.11 gives \( \lambda^{(p)}(G) \) and \( \lambda^{(p)}_{\text{min}}(G) \) whenever \( p \geq r - 1 \). Using the method of Proposition 1.2, it is not hard to obtain a more complete picture which sheds light on the possible structure of eigenvectors of a simple \( r \)-graph.

Proposition 4.9 If \( p > r - 1 \), then

\[
\lambda^{(p)}(G) = (r!/r^r)^{-1}(r-1)/p, \quad \lambda^{(p)}_{\text{min}}(G) = -(r!/r^r)^{-1}(r-1)/p, \]

and \( \lambda^{(p)}(G) \) has a single eigenvector \( x \in S^{(r-1)k}_{p,+} \).

If \( p < r - 1 \), then

\[
\lambda^{(p)}(G) = r!/r^r/p, \quad \lambda^{(p)}_{\text{min}}(G) = -r!/r^r/p. \]

Each eigenvector \( x \) to \( \lambda^{(p)}(G) \) or to \( \lambda^{(p)}_{\text{min}}(G) \) has \( r \) entries of modulus \( r^{-1}/p \) belonging to a single edge and is zero elsewhere.

Finally, if \( p = r - 1 \), then \( \lambda^{(p)}(G) = (r-1)!/r^{1(r-1)} \) and \( \lambda^{(p)}_{\text{min}}(G) = -(r-1)!/r^{1(r-1)} \); each of the sets of eigenvectors to \( \lambda^{(p)}(G) \) and to \( \lambda^{(p)}_{\text{min}}(G) \) contains a \((k-1)\)-dimensional sphere.

5 Elemental Perron-Frobenius theory for \( r \)-graphs

The Perron-Frobenius theory of nonnegative matrices is extremely useful in the study of the largest eigenvalue of 2-graphs. Unfortunately, before the work of Friedland, Gaubert and Han \( [10] \), and of Cooper and Dutle \( [3] \), the literature on nonnegative hypermatrices totally missed the point for hypergraphs, as the adjacency hypermatrix of an \( r \)-graph is always reducible if \( r \geq 3 \). The papers \( [10] \) and \( [3] \) put the study of \( \lambda(G) \) on a more solid ground, but none of these papers gave a complete picture. The situation is additionally complicated with the introduction of \( \lambda^{(p)}(G) \), where the dependence on \( p \) has not been studied even for \( r = 2 \). In this section we make several steps in laying down a Perron-Frobenius type theory for \( \lambda^{(p)}(G) \). The emerging complex picture is essentially combinatorial; this is not surprising, as the Perron-Frobenius theory for matrices builds on the combinatorial property “strong connectedness” of the matrix digraph.

Let us first state the starting point for 2-graphs. Theorem 5.1 below captures the three essential ingredients of what we refer to as the Perron-Frobenius theory for 2-graphs.

Theorem 5.1 Let \( G \) be a connected 2-graph with adjacency matrix \( A \).

(a) If \( Ax = \lambda(G)x \) for some nonzero \( x \), then \( x > 0 \) or \( x < 0 \);

(b) There exists a unique \( x > 0 \) such that \( Ax = \lambda(G)x \);

(c) If \( Ay = \mu y \) for some number \( \mu \) and vector \( y \geq 0 \), then \( \mu = \lambda(G) \).

We want to extend Theorem 5.1 to \( \lambda^{(p)}(G) \) of an weighted \( r \)-graph \( G \) for all \( p > 1 \), \( r \geq 2 \). Just a cursory inspection of the examples in Section 4 shows that a literal extension of Theorem 5.1 would fail in many points. First, Fact 2.1 shows that even very simple connected graphs may have
eigenvectors to $\lambda (G)$ with entries of different sign. But as it turns out the sign of the eigenvector entries is a nonissue for hypergraphs; we postpone the discussion to Subsection 5.6 and meanwhile focus only on nonnegative eigenvectors. Each of the three clauses of Theorem 5.1 is extended in a separate subsections below.

5.1 Positivity of eigenvectors to $\lambda (p)$

Our first goal in this subsection is to extend clause (a) of Theorem 5.1. A serious obstruction to our plans comes from the example in Proposition 4.3, which shows that if $1 < p \leq r - 1$, then this clause cannot be literally extended to $\lambda (p)$. We give a conditional extension in Theorem 5.3 below; however, we start with a simpler case, which already contains the main idea. Note that for graphs our theorem is stronger than Theorem 1.1 of [10].

Theorem 5.2 Let $r \geq 2$, $p > r - 1$, $G \in \mathcal{W}^r (n)$, and $[x_i] \in S_{p,+}^{n-1}$. If $G$ is connected and $[x_i]$ satisfies the equations

$$\lambda (p) (G) x_k^{p-1} = \frac{1}{r} \partial P_G ([x_i]) \frac{\partial}{\partial x_k}, \quad k = 1, \ldots, n,$$

then $x_1, \ldots, x_n$ are positive.

Proof Our proof refines an idea of Cooper and Dutle [3], Lemma 3.3. Assume that $p$, $G$, and $[x_i]$ are as required. Write $G_0$ for the graph induced by the vertices with zero entries in $[x_i]$, and assume for a contradiction that $G_0$ is nonempty. Since $G$ is connected, there exists an edge $e$ such that

$$U = V (G_0) \cap e \neq \emptyset \quad \text{and} \quad W = e \backslash V (G_0) \neq \emptyset.$$

To finish the proof we shall construct a vector $y \in S_{p,+}^{n-1}$ such that $P_G (y) > P_G ([x_i]) = \lambda (p) (G)$, which is the desired contradiction. Let $u \in W$ and for every sufficiently small $\varepsilon > 0$, define a $\delta := \delta (\varepsilon)$ by

$$\delta := x_u - \sqrt[x_u - |U| \varepsilon^p].$$

Clearly,

$$|U| \varepsilon^p + (x_u - \delta)^p = x_u^p,$$

and $\delta (\varepsilon) \to 0$ as $\varepsilon \to 0$. Since for each $v \in W$, the entry $x_v$ is positive, we may and shall assume that

$$\delta < \min_{v \in W} \{x_v\} / 2 \quad \text{and} \quad \varepsilon < \min_{v \in W} \{x_j\} - \delta.$$

Now, define the vector $y = [y_i]$ by

$$y_i := \begin{cases} x_i + \varepsilon, & \text{if } i \in U; \\ x_i - \delta, & \text{if } i = u; \\ x_i, & \text{if } i \notin U \cup \{u\}. \end{cases}$$

First, (18) and (19) imply that $|y|_p = |x|_p = 1$ and $y \geq 0$; hence, $y \in S_{p,+}^{n-1}$. Also, by Bernoulli’s inequality (72), $x_u^p - (x_u - \delta)^p > p\delta (x_u - \delta)^{p-1}$ and so,

$$r \varepsilon^p > |U| \varepsilon^p = x_u^p - (x_u - \delta)^p > p\delta (x_u - \delta)^{p-1} > p\delta \left( \frac{x_u}{2} \right)^{p-1},$$

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implying that
\[ \delta < r \frac{2^{p-1}}{x_u^{p-1}} \varepsilon^p. \]

Further, set for short
\[ D := \frac{\partial P_G ([x_i])}{\partial x_u} = r! \sum_{\{u, i_1, \ldots, i_{r-1}\} \in E(G)} G (\{u, i_1, \ldots, i_{r-1}\}) x_{i_1} \cdots x_{i_{r-1}}, \]
and note that
\[ P_G (y) - P_G (x) \geq r! G (e) \prod_{i \in e} y_i - r! \delta \frac{\partial P_G ([x_i])}{\partial x_u} \geq r! (x_u - \delta) \varepsilon^{r-1} - \delta D \]
\[ \geq r! G (e) \left( \frac{x_u}{2} \right) \varepsilon^{r-1} - r \frac{2^{p-1}}{x_u^{p-1}} D \varepsilon^p \]
\[ = \left( r! G (e) \left( \frac{x_u}{2} \right) - \left( r \frac{2^{p-1}}{x_u^{p-1}} D \right) \varepsilon^{p-r+1} \right) \varepsilon^{r-1}. \]

In view of \( p - r + 1 > 0 \), if \( \varepsilon \) is sufficiently small, then \( P_G (y) - P_G ([x_i]) > 0 \), contradicting that \( P_G (y) \leq P_G ([x_i]) \) and completing the proof. \( \square \)

The examples in Proposition \ref{prop1.3} show that the assertion of Theorem \ref{thm5.2} cannot be extended for \( p \leq r - 1 \). However we can force such extensions by requiring stronger connectedness of \( G \), which we define next:

Let \( 1 \leq k \leq r - 1 \) and let \( G \in G^r \). \( G \) is called \( k \)-tight, if \( E(G) \neq \emptyset \) and for any proper set \( U \subset V(G) \) containing edges, there is an edge \( e \) such that \( k \leq |e \cap U| \leq r - 1 \).

Note that a graph is 1-tight if and only if it is connected. Also if \( G \) is \( p \)-tight, then it is \( q \)-tight for \( 1 \leq q \leq p \). If \( G \in G^r (2r - k) \) consists of two edges with exactly \( k \) vertices in common, then \( G \) is \( k \)-tight but not \( (k + 1) \)-tight; hence one can anticipate that the properties of the graphs in Proposition \ref{prop1.3} have something to do with their tightness; such connections do exist indeed.

**Theorem 5.3** Let \( 1 \leq k \leq r - 1 \), \( p > r - k \), \( G \in W^r (n) \), and \([x_i] \in S_{p,i}^{n-1} \). If \( G \) is \( k \)-tight and \([x_i]\) satisfies the equations
\[ \lambda^{(p)} (G) x_k^{p-1} = \frac{\partial P_G ([x_i])}{\partial x_k}, \quad k = 1, \ldots, n, \]
then \( x_1, \ldots, x_n \) are positive.

**Proof** Our proof is similar to the proof of Theorem \ref{thm5.2}, so we omit some details. Write \( G_0 \) for the graph induced by the vertices with zero entries in \([x_i]\), and assume for a contradiction that \( G_0 \) is nonempty. Note that \( V(G) \setminus V(G_0) \) contains an edge, as \( \lambda^{(p)} (G) > 0 \); since \( G \) is \( k \)-tight, there is an edge \( e \) one with \( e \cap V(G_0) \neq \emptyset \) and \( |e \setminus V(G_0)| \geq k \) let
\[ U = V(G_0) \cap e \quad \text{and} \quad W = e \setminus V(G_0). \]
As \(|W| \geq k\), we have
\[ |U| = r - |W| \leq r - k. \]

To finish the proof we shall construct a vector \(y \in S^{n-1}_{p,+}\) such that \(P_G(y) > P_G([x_i]) = \lambda^{(p)}(G)\), which is the desired contradiction. Let \(u \in W\) and for every sufficiently small \(\varepsilon > 0\), define a \(\delta := \delta(\varepsilon)\) by
\[ \delta := x_u - \sqrt{x_u^p - \varepsilon}. \]

Clearly, \(|U|\varepsilon + (x_u - \delta)^p = x_u^p\), and \(\delta(\varepsilon) \to 0\) as \(\varepsilon \to 0\). Since for each \(v \in W\), the entry \(x_v\) is positive, we may and shall assume that \(\delta < \min_{v \in W} \{ x_v \}/2\) and 
\[ \varepsilon < \min_{v \in W} \{ x_j \} - \delta. \]

Now, define the vector \(y = [y_i]\) by
\[ y_i := \begin{cases} 
  x_i + \varepsilon, & \text{if } i \in U, \\
  x_i - \delta, & \text{if } i = u, \\
  x_i, & \text{if } i \notin U \cup \{ u \},
\end{cases} \]
and note that \(y \in S^{n-1}_{p,+}\). Also, as in Theorem 5.2, we find that
\[ \delta < r \frac{2p-1}{x_u^p-1} \varepsilon^p. \]

Further, set for short
\[ C := \prod_{i \in W \setminus \{ u \}} x_i, \]
\[ D := \sum_{\{u,i_1,\ldots,i_{r-1}\} \in E(G)} G(\{u,i_1,\ldots,i_{r-1}\}) x_{i_1} \cdots x_{i_{r-1}}, \]
and note that
\[ P_G(y) - P_G(x) \geq r!G(e) \prod_{i \in e_j} y_i - r! \delta \frac{\partial P_G([x_i])}{\partial x_u} = r! \prod_{i \in W} y_i \prod_{i \in U} y_i - \delta D \]
\[ \geq r!G(\varepsilon) (x_u - \delta) C \varepsilon^{r-k} - \delta D \geq r! \left( \frac{x_u}{2} \right) C \varepsilon^{r-k} - r \frac{2p-1}{x_u^p-1} D \varepsilon^p \]
\[ = \left( r!G(\varepsilon) \left( \frac{x_u}{2} \right) - \left( \frac{2p-1}{x_u^p-1} D \right) \varepsilon^{p-k} \right) \varepsilon^{r-k}. \]

In view of \(p - r + k > 0\), if \(\varepsilon\) is sufficiently small, then \(P_G(y) - P_G([x_i]) > 0\), contradicting that \(P_G(y) \leq P_G([x_i])\) and completing the proof. \(\square\)

Armed with Theorem 5.3, we can find how \(\lambda^{(p)}(G)\) changes when taking subgraphs.
**Corollary 5.4** Let \( r \geq 2, r - 1 \geq k \geq 1, p > r - k, \) and \( G \in \mathcal{W}^r. \) If \( G \) is \( k \)-tight and \( H \) is a subgraph of \( G, \) then
\[
\lambda^{(p)}(H) < \lambda^{(p)}(G),
\]
unless \( H = G. \) In particular, if \( p > r - 1 \) and \( G \) is connected, then \( \lambda^{(p)}(H) < \lambda^{(p)}(G) \) for every proper subgraph \( H \) of \( G. \)

The examples of Proposition 4.3 show that Theorem 5.3 is as good as one can get, but they do not shed enough light on the case \( p > r, \) which is somewhat surprising, as the following theorem shows.

**Theorem 5.5** Let \( p > r \geq 2, G \in \mathcal{W}^r(n), \) and \( [x_i] \in \mathbb{S}_{p,+}^{n-1}. \) If \( G \) is nonzero and \( [x_i] \) is an eigenvector to \( \lambda^{(p)}(G), \) then \( x_u > 0 \) for each non-isolated vertex \( u. \)

We omit the proof which is almost the same as the proof of Theorem 5.2. Instead, let us make the following observation.

**Corollary 5.6** Let \( p > r \geq 2 \) and let \( G \in \mathcal{W}^r \) and \( H \in \mathcal{W}^r. \) If \( H \) is subgraph of \( G, \) then
\[
\lambda^{(p)}(H) < \lambda^{(p)}(G),
\]
unless \( G \) has no edges or \( H = G. \)

In the light of the examples in Proposition 4.3, the notion of \( k \)-tightness gives a pretty strong sufficient condition for the eigenvectors of \( \lambda^{(p)}(G) \) to have only nonzero entries. Also, for 2-graphs, one can easily see the following characterization.

**Proposition 5.7** Let \( G \in \mathcal{W}_2 \) and \( 1 < p \leq 2. \) There is an eigenvector to \( \lambda^{(p)}(G) \) with nonzero entries if and only if \( \lambda^{(p)}(G) = \lambda^{(p)}(G') \) for every component \( G' \) of \( G. \)

However for \( r \geq 3, \) the corresponding problems are far from resolved:

**Problem 5.8** Let \( r \geq 3 \) and \( 1 < p \leq r. \) Characterize all \( G \in \mathcal{W}^r(n), \) such that all eigenvectors to \( \lambda^{(p)}(G) \) have only nonzero entries.

**Problem 5.9** Let \( r \geq 3 \) and \( 1 < p \leq r. \) Characterize all \( G \in \mathcal{W}^r(n), \) such that there is an eigenvector to \( \lambda^{(p)}(G) \) with all entries nonzero.
5.2 Uniqueness of the positive eigenvector to $\lambda^{(p)}$

In this subsection we generalize clause (b) of Theorem 5.1. The main obstruction in this task is exemplified by the $r$-cycle $C^n_r$: as Proposition 4.8 shows if $1 < p < r$, then $\lambda^{(p)}(C^n_r)$ always has at least two positive eigenvectors. Finding precisely for which graphs $G \in \mathcal{G}^r$ there is a unique positive eigenvector to $\lambda^{(p)}(G)$ is currently an open problem. Note that tightness is not relevant in this characterization, as the cycles $C^n_r$ are $(r - 1)$-tight. We give a limited solution below, leaving the general problem for future study.

Here is the proposed generalization of clause (b).

Theorem 5.10 If $p \geq r \geq 2$ and $G \in \mathcal{W}^r(n)$, if $G$ is connected, there exists a unique $[x_i] \in S_{p+}^{n-1}$ satisfying the equations

$$\lambda^{(p)}(G) x_k^{p-1} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n.$$  

For the proof of the theorem we shall need the following proposition.

Proposition 5.11 Let $p \geq 1$, $G \in \mathcal{W}^r(n)$, and $[x_i] \in S_{p+}^{n-1}$. If $[x_i]$ satisfies the inequalities

$$\lambda^{(p)}(G) x_k^{p-1} \leq \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,$$  

then $[x_i]$ is an eigenvector to $\lambda^{(p)}(G)$ and equality holds in (20) for each $k = 1, \ldots, n$.

Proof Multiplying both sides of (20) by $x_k$ and adding all inequalities, we obtain

$$\lambda^{(p)}(G) \sum_{k=1}^{n} x_k^p \leq \frac{1}{r} \sum_{k=1}^{n} x_k \frac{\partial P_G(x)}{\partial x_k} = P_G(x) \leq \lambda^{(p)}(G) \sum_{k=1}^{n} x_k^p.$$  

Therefore, equalities hold in (20) and $x$ is an eigenvector to $\lambda(G)$.

Proof of Theorem 5.10 Let $x = [x_i] \in S_{p+}^{n-1}$ and $y = [y_i] \in S_{p+}^{n-1}$ be two positive eigenvectors to $\lambda^{(p)}(G)$; we have to prove that $x = y$. Define a vector $z = [z_i] \in S_{p+}^{n-1}$ by the equations

$$z_k := \sqrt{\frac{x_k^p + y_k^p}{2}}, \quad k = 1, \ldots, n.$$  

Now, for each $k = 1, \ldots, n$, add the two equations

$$\lambda^{(p)}(G) x_k^p = (r - 1)! \sum_{\{k, i_1 \cdots i_{r-1}\} \in E(G)} x_k x_{i_1} \cdots x_{i_{r-1}},$$

and

$$\lambda^{(p)}(G) y_k^p = (r - 1)! \sum_{\{k, i_1 \cdots i_{r-1}\} \in E(G)} y_k y_{i_1} \cdots y_{i_{r-1}}.$$
possibly hold if \( 1 < p < r \).

Propositions 4.3, 4.4 and 4.8 show that there are \( r \) vertices. Lemma 5.11 implies that equalities hold in (21) such that \((x, y)\) and so \(y\) is collinear to \(x\). Since \(G\) is connected, this assertion can be put simply as: for every vertex \(x\) such that \((x, y) = c(x, y)\). Finally, this equation implies that \(y = (y_1/x_1) x_i\), for \(i = 1, \ldots, n\), and so \(y\) is collinear to \(x\), and so \(x = y\). □

Having had the experience with Theorem 5.5 we easily come up with the following theorem.

**Theorem 5.12** If \( p > r \geq 2 \) and \( G \in \mathcal{G}^r (n) \), there is a unique \([x_i] \in S_{p,r+1}^{n-1}\) satisfying the equations

\[
\lambda(p) (G) \, x_k^p = \frac{\lambda(p) (G) \, x_k^p + \lambda(p) (G) \, y_k^p}{2} = (r-1)! \sum_{\{k, i_1, \ldots, i_{r-1}\} \in E(G)} G \{k, i_1, \ldots, i_{r-1}\} \frac{x_k x_{i_1} \cdots x_{i_{r-1}} + y_k y_{i_1} \cdots y_{i_{r-1}}}{2}.
\]

Applying the generalized Cauchy-Schwarz inequality (66) to the vectors \((x, y)\) and \((x_i, y_i)\), \(1 \leq s \leq r - 1\), and the PM inequality implies that

\[
\frac{x_k x_{i_1} \cdots x_{i_{r-1}} + y_k y_{i_1} \cdots y_{i_{r-1}}}{2} \leq \sqrt{\frac{x_k^p + y_k^p}{2}} \prod_{s=1}^{r-1} \sqrt{\frac{x_{i_s}^p + y_{i_s}^p}{2}} \leq \sqrt{\frac{x_k^p + y_k^p}{2}} \prod_{s=1}^{r} \sqrt{\frac{x_{i_s}^p + y_{i_s}^p}{2}} = \sum_{\{k, i_1, \ldots, i_{r-1}\} \in E(G)} z_k z_{i_1} \cdots z_{i_{r-1}}.
\]

Therefore,

\[
\lambda (G) \, z_k^{p-1} \leq \sum_{\{k, i_1, \ldots, i_{r-1}\} \in E(G)} G \{k, i_1, \ldots, i_{r-1}\} \, z_{i_1} \cdots z_{i_{r-1}} = \frac{1}{r} \frac{\partial P_G (z)}{\partial z_k} \quad k = 1, \ldots, n.
\]

and Lemma 5.11 implies that equalities hold in (21). By the condition for equality in (66), if the vertices \(i\) and \(j\) are contained in the same edge of \(G\), then there is a \(c\), such that \((x_i, y_i) = c(x_j, y_j)\). Since \(G\) is connected, this assertion can be put simply as: for every vertex \(i\) of \(G\), there is a \(c\), such that \((x_i, y_i) = c(x_1, y_1)\). Finally, this equation implies that \(y_i = (y_1/x_1) x_i\), for \(i = 1, \ldots, n\), and so \(y\) is collinear to \(x\), and so \(x = y\).



5.3 **Uniqueness of \( \lambda(p) \)**

In this subsection we shall extend clause (c) of Theorem 5.1. Note first that a literal extension is impossible in view of Proposition 4.1 if \( r \geq 3 \), for every \( r\)-graph, the value \( \lambda = 0 \) satisfies (17) with many nonnegative eigenvectors. Since this situation is unavoidable, it seems reasonable to consider only positive \( \lambda \) and nonnegative \([x_i] \in S_{p,r+1}^{n-1}\) satisfying (17). But even with these restriction, Propositions 4.3, 4.4 and 4.8 show that there are \( r\)-graphs for which clause (c) cannot possibly hold if \( 1 < p < r \). In fact, the following problems is open.
Problem 5.13 Given $1 < p < r$, characterize all graphs $G \in \mathcal{G}^r (n)$ for which there is a unique $\lambda = \lambda^{(p)} (G)$ is the only positive number satisfying the equations

$$\lambda x_k^{p-1} = \frac{1}{r} \frac{\partial P_G ([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,$$

for some $[x_i] \in S_{p,+}^{n-1}$.

However, clause (c) is an important practical issue, so we shall establish necessary and sufficient conditions for its validity for $p \geq r$.

Theorem 5.14 Let $p \geq r \geq 2$, $G \in \mathcal{G}^r (n)$ and $[x_i] \in S_{p,+}^{n-1}$. If $G$ is $(r-1)$-tight and $[x_i]$ satisfies the equations

$$\lambda x_k^{p-1} = \frac{1}{r} \frac{\partial P_G ([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,$$

for some $\lambda > 0$, then $x_1, \ldots, x_n$ are positive and $\lambda = \lambda^{(p)} (G)$.

Before starting the proof, let us note the following proposition, which is useful in its own right.

Proposition 5.15 Let $r \geq 2$, $p > 1$, $G \in \mathcal{G}^r (n)$, and $[x_i] \in S_{p,+}^{n-1}$. If $G$ is $(r-1)$-tight and $[x_i]$ satisfies the equations

$$\lambda x_k^{p-1} = \frac{1}{r} \frac{\partial P_G ([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,$$

for some $\lambda > 0$, then $x_1, \ldots, x_n$ are positive.

The proof follows from the observation that if $\{i_1, \ldots, i_r\} \in E (G)$ satisfies $x_{i_1} \cdots x_{i_{r-1}} > 0$, then $x_{i_r} > 0$.

Proof of Theorem 5.14 We adapt an idea from [6]. Let $[y_i] \in S_{p,+}^{n-1}$ be an eigenvector to $\lambda^{(p)} (G)$. Proposition 5.15 implies that $[x_i] > 0$ and $[y_i] > 0$. Let

$$\sigma = \min \{x_1/y_1, \ldots, x_n/y_n\} = x_k/y_k.$$ 

Clearly $\sigma > 0$; also $\sigma \leq 1$, for otherwise $|[x_i]|_p > |[y_i]|_p$, a contradiction. Further,

$$\lambda x_k^{p-1} = \frac{1}{r} \frac{\partial P_G ([x_i])}{\partial x_k} \geq \frac{1}{r} \sigma^{r-1} \frac{\partial P_G ([y_i])}{\partial y_k} = \sigma^{r-1} \lambda^{(p)} (G) y_k^{p-1} = \sigma^{r-p} \lambda^{(p)} (G) x_k^{p-1}.$$ 

implying that $\lambda^{(p)} (G) \leq \lambda$. But $\lambda = P_G ([x_i]) \leq \lambda^{(p)} (G)$, and so $\lambda = \lambda^{(p)} (G)$, completing the proof. \hspace{1cm} \Box

One can think that the requirement $G$ to be $(r-1)$-tight in Theorem 5.14 is too strong. However, it is best possible, as the following theorem and its corollary suggest.

Theorem 5.16 Let $r \geq 2$, $p > 1$, and $G \in \mathcal{G}^r (n)$. If $G$ is not $(r-1)$-tight, there are $\lambda > 0$ and $[x_i] \in S_{p,+}^{n-1}$ such that

$$\lambda x_k^{p-1} = \frac{1}{r} \frac{\partial P_G ([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,$$

but $[x_i]$ is not positive.
Sketch of a proof Let \( e_1 \in E(G) \) and let \( U \) be a set of vertices containing an edge, but no edge \( e \in E(G) \) satisfies \( |e \cap U| = r - 1 \). Let \( G_1 = G[U] \) and set \( \lambda = \lambda^{(p)}(G_1) \); clearly \( \lambda > 0 \). To get \( [x_i] \in \mathbb{S}^{n-1}_{p,+} \), take a nonnegative eigenvector to \( \lambda^{(p)}(G_1) \) and set \( x_v = 0 \) for all \( x_v \in V(G) \setminus V_1 \). □

If \( r \geq 3 \), we can see the decisive role of \((r - 1)\)-tightness as laid down in the following statement.

**Corollary 5.17** Let \( p \geq r \geq 3 \) and \( G \in \mathcal{G}^r(n) \). If \( G \) is connected, but not \((r - 1)\)-tight, there exists \([x_i] \in \mathbb{S}^{n-1}_{p,+} \) satisfying the equations

\[
\lambda x_k^{p-1} = \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,
\]

for some positive \( \lambda < \lambda^{(p)}(G) \).

### 5.4 The Collatz-Wielandt function of \( \lambda^{(p)} \)

In this subsection we shall deduce Collatz-Wielandt type characterizations of \( \lambda^{(p)}(G) \). The following theorem is analogous to the Collatz-Wielandt minimax theorem.

**Theorem 5.18** If \( p > 1 \) and \( G \in \mathcal{G}^r(n) \), then

\[
\lambda^{(p)}(G) = \max_{[x_i] \in \mathbb{S}^{n-1}_{p,+}} \min_{x_k > 0} \frac{\partial P_G([x_i])}{\partial x_k} x_k^{-p+1}.
\]

**Proof** Since \( \mathbb{S}^{n-1}_{p,+} \) is compact, there exists \([y_i] \in \mathbb{S}^{n-1}_{p,+} \) such that

\[
\lambda = \min_{y_k > 0} \frac{1}{r} \frac{\partial P_G([y_i])}{\partial y_k} y_k^{-p+1} = \max_{[x_i] \in \mathbb{S}^{n-1}_{p,+}} \min_{x_k > 0} \frac{\partial P_G([x_i])}{\partial x_k} x_k^{-p+1}.
\]

This equation clearly implies that

\[
\lambda y_k^{p-1} \leq \frac{1}{r} \frac{\partial P_G([y_i])}{\partial y_k}, \quad k = 1, \ldots, n. \tag{22}
\]

Let \([z_i] \in \mathbb{S}^{n-1}_{p,+} \) be an eigenvector to \( \lambda^{(p)}(G) \). Clearly, if \( z_k > 0 \), then

\[
\lambda^{(p)}(G) = \frac{\partial P_G([z_i])}{\partial z_k} z_k^{-p+1} = \min_{z_j > 0} \frac{\partial P_G([z_i])}{\partial z_j} z_j^{-p+1}
\]

and therefore \( \lambda \geq \lambda^{(p)}(G) \). Substituting \( \lambda^{(p)}(G) \) for \( \lambda \) in (22), we get

\[
\lambda^{(p)}(G) y_k^{p-1} \leq \frac{1}{r} \frac{\partial P_G([y_i])}{\partial y_k}, \quad k = 1, \ldots, n.
\]

Now, Proposition 5.11 implies that equality holds for each \( k \in [n] \) and so \( \lambda^{(p)}(G) = \lambda \). □

Note that no special requirements about \( G \) are needed in Theorem 5.18. Usually such theorems require that \( G \) is connected, but there is no justification for such weakening of the statement.

Next we use the proof of Theorem 5.14 to get a more flexible theorem.
Theorem 5.19 Let $p \geq r \geq 2$, $G \in \mathcal{W}^r(n)$, and $[x_i] \in S_{p,+}^{n-1}$. If $[x_i] > 0$ and $[x_i]$ satisfies the inequalities

$$\lambda x_k^{p-1} \geq \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,$$

(23)

for some real $\lambda$, then $\lambda \geq \lambda^{(p)}(G)$. If $\lambda^{(p)}(A) = \lambda$, then equality holds in (23) for all $k \in [n]$ unless $p = r$ and $G$ is disconnected.

Proof Let $[y_i] \in S_{p,+}^{n-1}$ be an eigenvector to $\lambda^{(p)}(G)$. Let $\sigma := \min \{x_i/y_i : y_i > 0\}$. Clearly $\sigma > 0$; also $\sigma \leq 1$, for otherwise $|[x_i]|_p > |[y_i]|_p$, a contradiction. Note that $x_i \geq \sigma y_i$ for every $i \in [n]$. Since $x_k = \sigma y_k$ for some $k \in [n]$, we see that

$$\lambda x_k^{p-1} \geq \frac{1}{r} \frac{\partial P_G([x_i])}{\partial x_k} \geq \frac{1}{r} \sigma^{r-1} \frac{\partial P_G([y_i])}{\partial y_k} = \sigma^{r-1} \lambda^{(p)}(G) y_k^{p-1} = \sigma^{r-p} \lambda^{(p)}(G) x_k^{p-1}.$$

implying that $\lambda^{(p)}(G) \leq \lambda$. If $\lambda^{(p)}(G) = \lambda$, then $\sigma = 1$ or $r = p$. If $\sigma = 1$, then $[x_i] = [y_i]$ and so equalities hold in (23) for all $k \in [n]$, so assume that $\sigma < 1$ and $r = p$. We see that $x_j = \sigma y_j$ for every vertex $j$ which is contained in an edge together with $k$. Therefore $x_i = \sigma y_i$ for the component of $G$ containing $x_k$ and so $G$ is disconnected.

It is not hard to see that the disconnected $r$-graph $G = K_{r}^+ \cup K_{r+1}^+$ satisfies inequalities (23) with $\lambda = \lambda^{(r)}(G)$ and some vector $[x_i] \in S_{r,+}^{n-1}$ such that not all inequalities (23) are equalities. Another point to make here is that for $p < r$ the assertion may not be true: indeed the cycle $C_{n}^p$ has a unique positive eigenvector to vector $\lambda^{(p)}(G)$ for every $p > 1$, which is different from $n^{-1/p}j_n$. The vector $[x_i] = n^{-1/p}j_n$ together with $\lambda = |r!C_{n}^r|/n^r = r!/n^{1-rp}$ satisfies the inequalities (23), but $\lambda^{(p)}(C_{n}^p) > r!/n^{1-rp}$.

The above theorem helps to prove another theorem, related to the Collatz-Wielandt function.

Theorem 5.20 Let $p \geq r \geq 2$ and $G \in \mathcal{G}^r(n)$. If $G$ is connected, then

$$\lambda^{(p)}(G) = \inf_{[x_i] > 0, |[x_i]|_p = 1} \max_{j \in [n]} \frac{\partial P_G([x_i])}{\partial x_j} x_j^{-p+1}.$$

Proof Let $[x_i] \in S_{p,+}^{n-1}$ be an eigenvector to $\lambda^{(p)}(G)$; by Theorem 5.2 $[x_i] > 0$. Further,

$$\lambda^{(p)}(G) = \frac{\partial P_G([x_i])}{\partial x_k} x_k^{-p+1}, \quad k = 1, \ldots, n,$$

and so

$$\inf_{[x_i] \in S_{p,+}^{n-1}} \max_{k \in [n]} \frac{\partial P_G([x_i])}{\partial x_k} x_k^{-p+1} \leq \lambda^{(p)}(G).$$

Let $[y_i] > 0$ and $|[y_i]|_p = 1$. Clearly, the value

$$\lambda = \max_k \frac{\partial P_G([y_i])}{\partial y_k} y_k^{-p+1}$$

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satisfies
\[ \lambda y_k^{p-1} \geq \frac{\partial P_G([y_k])}{\partial y_k}, \quad k = 1, \ldots, n, \]
and by Theorem 5.19 \( \lambda \geq \lambda^{(p)}(G) \); hence,
\[ \inf_{[x_i] > 0, \{[x_i]_p\}=1} \max_{j \in [n]} \frac{\partial P_G([x_i])}{\partial x_j} x_j^{-p+1} \geq \lambda^{(p)}(G), \]
proving the theorem.
\[ \square \]

As above, the cycle \( C_n^r \) shows that Theorem 5.20 cannot be extended for \( p < r \) even for \((r-1)\)-tight graphs.

5.5 A recap for 2-graphs

Let us note that the concept of \( k \)-tightness is essentially irrelevant for 2-graphs because \( k \) can only take the value 1, and 1-tight graph is the same as connected. Thus, for reader’s sake, we shall summarize the above results for 2-graphs in a single theorem parallel to Theorem 5.1.

Theorem 5.21 Let \( G \) be a connected 2-graph.

(a) If \( p > 1 \) and \([x_i] \in \mathbb{S}^{n-1}_{p,+}\) satisfies the equations
\[ \mu x_k^{p-1} = \sum_{\{k,i\} \in E(G)} x_i, \quad k = 1, \ldots, n, \]
for \( \mu = \lambda^{(p)}(G) \), then \( x_1, \ldots, x_n \) are positive;

(b) If \( p \geq 2 \), there is a unique positive eigenvector \( \mathbf{x} \) to \( \lambda^{(p)}(G) \);

(c) If \( p \geq 2 \) and a vector \([x_i] \in \mathbb{S}^{n-1}_{p,+}\) satisfies equations (24) for some \( \mu \), then \( \mu = \lambda^{(p)}(G) \).

5.6 Eigenvectors to \( \lambda^{(p)} \) and even transversals

In this subsection we address the situation outlined in Fact 2.1: eigenvectors to \( \lambda^{(p)}(G) \) of a connected \( r \)-graph \( G \) may have entries of different sign. It turns out that this property is related to the existence of even transversals in \( G \). Recall that an even transversal in a graph \( G \) is a nonempty set of vertices intersecting each edge in an even number of vertices. A proper even transversal is a proper subset of \( V(G) \).

Theorem 5.22 Let \( r \geq 2, p > r - 1, \) and \( G \in G^r(n) \). If \( G \) is connected, then \( \lambda^{(p)}(G) \) has an eigenvector \([x_i] \in \mathbb{S}^{n-1}_p\) with entries of different signs if and only if \( G \) has a proper even transversal.
Proof Let \([x_i] \in S^{n-1}_p\) be an eigenvector to \(\lambda^{(p)}(G)\) with entries of different sign. Note that \(\lambda^{(p)}(G) = P_G([x_i]) = P_G([|x_i|]) = \lambda^{(p)}(G)\). Hence \([|x_i|]\) is an eigenvector to \(\lambda^{(p)}(G)\), and by Theorem 5.3 \([|x_i|] > 0\). Let \(V^-\) be the set of vertices with negative entries in \([x_i]\). We shall prove that \(V^-\) is a proper even transversal of \(G\). By the assumption \(V^- \neq \emptyset\) and \(V(G) \setminus V^- \neq \emptyset\). Also every edge \(\{i_1, \ldots, i_r\} \in E(G)\) intersects \(V^-\) in an even number of vertices, otherwise the product \(x_{i_1} \cdots x_{i_r}\) is negative and so \(P_G([x_i]) < P_G([|x_i|])\), a contradiction.

Suppose now that \(G\) has a proper even transversal \(U\), and let \([x_i] \in S^{n-1}_p\) be a positive eigenvector to \(\lambda^{(p)}(G)\). Define \([y_i] \in S^{n-1}_p\) by

\[
y_i = \begin{cases} 
-x_i, & \text{if } i \in U, \\
x_i, & \text{if } i \notin U.
\end{cases}
\]

Clearly \(P_G([x_i]) = P_G([y_i])\), and so \([y_i]\) is an eigenvector to \(\lambda^{(p)}(G)\) with entries of different signs.

\(\square\)

6 Relations of \(\lambda^{(p)}\) and \(\lambda^{(p)}_{\min}\) to some graph operations

We consider here only simple graph operations like union of graphs, graph blow-up and star-like graphs. Some of the results are shaped after useful spectral results for 2-graphs.

6.1 \(\lambda^{(p)}\) and \(\lambda^{(p)}_{\min}\) of blow-ups of graphs

Given a graph \(G \in \mathcal{G}^r(n)\) and positive integers \(k_1, \ldots, k_n\), write \(G(k_1, \ldots, k_n)\) for the graph obtained by replacing each vertex \(v \in V(G)\) with a set \(U_v\) of size \(x_v\) and each edge \(\{v_1, \ldots, v_r\} \in E(G)\) with a complete \(r\)-partite \(r\)-graph with vertex classes \(U_{v_1}, \ldots, U_{v_r}\). The graph \(G(k_1, \ldots, k_n)\) is called a blow-up of \(G\).

 Blow-ups are very useful in studying graphs, in particular for studying their spectra. Let us note that if \(G \in \mathcal{G}^r\) and \(k \geq 1\), then \(|G(k, \ldots, k)| = k^r |G|\); in the following proposition a similar property is proved for \(\lambda^{(p)}\) and \(\lambda^{(p)}_{\min}\) also.

Proposition 6.1 If \(p \geq 1\), \(k \geq 1\) and \(G \in \mathcal{G}^r(n)\), then

\[
\begin{align*}
\lambda^{(p)}(G(k, \ldots, k)) &= k^{r-p} \lambda^{(p)}(G), \\
\lambda^{(p)}_{\min}(G(k, \ldots, k)) &= k^{r-p} \lambda^{(p)}_{\min}(G)
\end{align*}
\]  \hspace{1cm} (25)

Proof We shall prove only (25). By definition, \(V(G(k, \ldots, k))\) can be partitioned into \(n\) disjoint sets \(U_1, \ldots, U_n\) each consisting of \(k\) vertices such that if \(\{i_1, \ldots, i_r\} \in E(G)\), then \(\{j_1, \ldots, j_r\} \in E(G(k, \ldots, k))\) for every \(j_1 \in U_{i_1}, j_2 \in U_{i_2}, \ldots, j_r \in U_{i_r}\). First we shall prove that

\[
\lambda^{(p)}_{\min}(G(k, \ldots, k)) \leq k^{r-p} \lambda^{(p)}(G).
\]  \hspace{1cm} (26)
Let \( [x_i] \in \mathbb{S}_p^{n-1} \) be an eigenvector to \( \lambda_{\min}^{(p)}(G) \). We define a new eigenvector \( [y_i] \in \mathbb{S}_p^{kn-1} \) as follows: for each \( i \in V(G(k, \ldots, k)) \), set \( y_i := k^{-1/p}x_j \), where \( j \) is the unique value satisfying \( i \in U_j \). Clearly, \( [y_i] \in \mathbb{S}_p^{nk-1} \), and therefore,

\[
\lambda_{\min}^{(p)}(G(k, \ldots, k)) \leq P_{G(k, \ldots, k)}([y_i]) = r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} \left( \sum_{j \in U_{i_1}} x_j \right) \cdots \left( \sum_{j \in U_{i_r}} x_j \right)
\]

\[
= \frac{1}{k^{r/p}} k^r P_G([x_i]) = k^{r-r/p} \lambda_{\min}^{(p)}(G),
\]

proving (26). To complete the proof of (26) we shall show that

\[
\lambda_{\min}^{(p)}(G(k, \ldots, k)) \leq k^{r-r/p} \lambda_{\min}^{(p)}(G).
\]

Let \( [x_i] \in \mathbb{S}_p^{nk-1} \) be an eigenvector to \( \lambda_{\min}^{(p)}(G(k, \ldots, k)) \). By definition,

\[
P_{G(k, \ldots, k)}([x_i]) = r! \sum_{\{i_1, \ldots, i_r\} \in E(G(k, \ldots, k))} x_{i_1} \cdots x_{i_r}
\]

\[
= r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} \left( \sum_{j \in U_{i_1}} x_j \right) \cdots \left( \sum_{j \in U_{i_r}} x_j \right)
\]

Next, Corollary 3.10 implies that \( x_i \) are the same within each class \( U_j \). Now, setting for each \( s \in [n] \), \( y_s := k^{1/p}x_j \), where \( x_j \in U_s \), we get a vector \( y = [y_i] \in \mathbb{S}_p^{n-1} \). Also, \( y \) satisfies

\[
\lambda_{\min}^{(p)}(G(k, \ldots, k)) = P_{G(k, \ldots, k)}([x_i])
\]

\[
= r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} \left( \sum_{j \in U_{i_1}} x_j \right) \cdots \left( \sum_{j \in U_{i_r}} x_j \right)
\]

\[
= r! k^{r-r/p} \sum_{\{i_1, \ldots, i_r\} \in E(G)} y_{i_1} \cdots y_{i_r} = k^{r-r/p} P_G(y)
\]

\[
\geq k^{r-r/p} \lambda_{\min}^{(p)}(G).
\]

This completes the proof of (27); in view of (26), the proof of (25) is completed as well.

\[\Box\]

### 6.2 Weyl type inequalities and applications

The following inequalities are shaped after Weyl’s inequalities for Hermitian matrices and have numerous applications.

**Proposition 6.2** If \( G_1 \in \mathcal{W}^r(n) \), \( G_2 \in \mathcal{W}^r(n) \), then

\[
\lambda_{\min}^{(p)}(G_1 + G_2) \leq \lambda_{\min}^{(p)}(G_1) + \lambda_{\min}^{(p)}(G_2).
\]

\[
\lambda_{\min}^{(p)}(G_1) + \lambda_{\min}^{(p)}(G_2) \leq \lambda_{\min}^{(p)}(G_1 + G_2) \leq \lambda_{\min}^{(p)}(G_1) + \lambda_{\min}^{(p)}(G_2).
\]

(28)
Proof If \([x_i] \in S_p^n, [y_i] \in S_p^n, \text{ and } [z_i] \in S_p^n\) are eigenvectors to \(\lambda^{(p)}(G), \lambda_{\min}^{(p)}(G)\) and \(\lambda_{\min}^{(p)}(G_2)\), then
\[
\begin{align*}
\lambda^{(p)}(G_1 + G_2) &= P_{G_1+G_2}([x_i]) = P_{G_1}([x_i]) + P_{G_2}([x_i]) \leq \lambda^{(p)}(G_1) + \lambda^{(p)}(G_2), \\
\lambda_{\min}^{(p)}(G_1 + G_2) &= P_{G_1+G_2}([y_i]) = P_{G_1}([y_i]) + P_{G_2}([y_i]) \geq \lambda_{\min}^{(p)}(G_1) + \lambda_{\min}^{(p)}(G_2), \\
\lambda_{\min}^{(p)}(G_2) &= P_{G_2}([z_i]) = P_{G_1+G_2}([z_i]) - P_{G_1}([z_i]) \geq \lambda_{\min}^{(p)}(G_1 + G_2) - \lambda^{(p)}(G_1).
\end{align*}
\]
This completes the proof of the proposition. \(\square\)

Next, we shall deduce several useful applications, starting with perturbation bounds on \(\lambda^{(p)}(G)\) and \(\lambda_{\min}^{(p)}(G)\).

**Proposition 6.3** Let \(p \geq 1, k \geq 1\) and \(G_1 \in \mathcal{G}^c(n), G_2 \in \mathcal{G}^r(n)\). If \(G_1\) and \(G_2\) differ in at most \(k\) edges, then
\[
\begin{align*}
|\lambda^{(p)}(G_1) - \lambda^{(p)}(G_2)| &\leq (r!k)^{1/p} \\
|\lambda_{\min}^{(p)}(G_1) - \lambda_{\min}^{(p)}(G_2)| &\leq (r!k)^{1/p}
\end{align*}
\]
(29)

Proof We shall prove only (29). Assume that \(\lambda_{\min}^{(p)}(G_1) - \lambda_{\min}^{(p)}(G_2) \geq 0\) and write \(G_1 \setminus G_2, G_2 \setminus G_1, G_2 \cap G_1\), for the graphs with vertex set \([n]\) and edge sets \(E(G_1) \setminus E(G_2), E(G_2) \setminus E(G_1), E(G_2) \cap E(G_1)\). Now, inequalities (28) imply that
\[
\begin{align*}
\lambda_{\min}^{(p)}(G_1) &\leq \lambda^{(p)}(G_1 \setminus G_2) + \lambda_{\min}^{(p)}(G_2 \cap G_1) \\
\lambda_{\min}^{(p)}(G_2) &\geq \lambda_{\min}^{(p)}(G_2 \cap G_1) + \lambda_{\min}^{(p)}(G_2 \setminus G_1)
\end{align*}
\]
and so,
\[
\lambda_{\min}^{(p)}(G_1) - \lambda_{\min}^{(p)}(G_2) \leq \lambda^{(p)}(G_1 \setminus G_2) - \lambda_{\min}^{(p)}(G_2 \setminus G_1) \leq \lambda^{(p)}(G_1 \setminus G_2) + \lambda^{(p)}(G_2 \setminus G_1).
\]
Defining \(G'\) by \(v(G') = [n], E(G') = (E(G_1) \setminus E(G_2)) \cup (E(G_2) \setminus E(G_1))\), we see that \(|G'| \leq k\), and so
\[
\lambda_{\min}^{(p)}(G_1) - \lambda_{\min}^{(p)}(G_2) \leq \lambda^{(p)}(G') \leq (r!k)^{1/p}.
\]
This completes the proof of (29). \(\square\)

The set \(\mathcal{W}^r(n)\) is a complete metric space in any \(l^q\) norm, \(1 \leq q \leq \infty\); Many graph parameters like \(|G|\) are continuous functions of \(G \in \mathcal{W}^r(n)\). Weyl’s inequalities imply that for fixed \(p \geq 1\), both \(\lambda^{(p)}(G)\) and \(\lambda_{\min}^{(p)}(G)\) are also continuous functions of \(G\).

**Proposition 6.4** If \(p \geq 1, G_1 \in \mathcal{W}^r(n), \text{ and } G_2 \in \mathcal{W}^r(n), \) then
\[
\begin{align*}
|\lambda^{(p)}(G_1) - \lambda^{(p)}(G_2)| &\leq \lambda^{(p)}(|G_1 - G_2|) \leq |G_1 - G_2|_{p(p-1)} \\
|\lambda_{\min}^{(p)}(G_1) - \lambda_{\min}^{(p)}(G_2)| &\leq \lambda^{(p)}(|G_1 - G_2|) \leq |G_1 - G_2|_{p/(p-1)}
\end{align*}
\]
In particular, if \(p\) is fixed and \(G \in \mathcal{W}^r(n), \) then \(\lambda^{(p)}(G)\) and \(\lambda_{\min}^{(p)}(G)\) are continuous functions of \(G\).
Note that Proposition 6.4 makes sense for $p = 1$ with the proviso $p / (p - 1) = \infty$.

Another consequence from Weyl’s inequalities are two Nordhaus-Stewart type bounds about $\lambda^{(p)}$. obtained following the footprints of Nosal [30]:

**Proposition 6.5** Let $G \in \mathcal{G}^r(n)$ and $\overline{G}$ be its complement. If $p \geq 1$, then

$$\lambda^{(p)}(G) + \lambda^{(p)}(\overline{G}) \leq 2^{1/p} (n)^{1 - 1/p}$$

and

$$\lambda^{(p)}(G) + \lambda^{(p)}(\overline{G}) \geq (n)^{r} / n^{r/p}$$

If equality holds in (31), then $G$ is regular. If $p \geq r$, and $G$ is regular, then equality holds in (31).

Note that the upper and lower bounds are close within a multiplicative factor of $2^{1/p}$; however, unlike (31), the upper bound (30) seems not too tight. This observation prompts the following Nordhaus-Stewart type problems.

**Problem 6.6** If $p > 1$, find

$$\max_{G \in \mathcal{G}^r(n)} \lambda^{(p)}(G) + \lambda^{(p)}(\overline{G}),$$

and

$$\min_{G \in \mathcal{G}^r(n)} \lambda^{(p)}_{\min}(G) + \lambda^{(p)}_{\min}(\overline{G}).$$

### 6.3 Star-like $r$-graphs

In this subsection we discuss $r$-graphs with certain intersection properties. The possible variations are indeed numerous but we shall focus on two constructions only. Our interest is motivated by certain extremal problems discussed later.

Let $r \geq 3$ and $G \in \mathcal{G}^{r-1}(n - 1)$. Choose a vertex $v \notin V(G)$ and define the graph $G \vee K_1 \in \mathcal{G}^r(n)$ by

$$V(G \vee K_1) := V(G) \cup \{v\}, \quad E(G \vee K_1) := \{e \cup \{v\} : e \in E(G)\}.$$

**Proposition 6.7** For every $p \geq 1$ and every $G \in \mathcal{G}^{r-1}$,

$$\lambda^{(p)}(G \vee K_1) = r^{1 - r/p} (r - 1)^{(r-1)/p} \lambda^{(p)}(G),$$

$$\lambda^{(p)}(G \vee K_1) = -r^{1 - r/p} (r - 1)^{(r-1)/p} \lambda^{(p)}(G).$$

**Proof** Take a nonnegative eigenvector $x = (x_0, \ldots, x_n) \in \mathbb{S}^n_{p, +}$ to $\lambda^{(p)}(G \vee K_1)$; suppose that $x_2, \ldots, x_n$ are the entries corresponding to vertices in $V(G)$ and $x_1$ is the entry corresponding to $v$.

$$\lambda^{(p)}(G \vee K_1) = \max_{x_0^p + \cdots + x_n^p = 1} P_{G \vee K_1}(x) = \max_{x_0^p + \cdots + x_n^p = 1} x_0 P_G(x')$$

$$= \max_{0 \leq x_1 \leq 1} x_1 \max_{x_0^p + \cdots + x_n^p = 1} P_G(x') = \max_{0 \leq x_1 \leq 1} x_1 \lambda^{(p)}(G)(1 - x_1^{p(r-1)/p})$$

$$= r \lambda^{(p)}(G) \max_{0 \leq x_1 \leq 1} x_1 (1 - x_1^{p(r-1)/p})$$
Using calculus, we find that the maximum above is attained at \( x_1 = r^{-1/p} \) and the desired result follows.

In particular if \( G \) is \( K_n^{r-1} \), the complete \((r - 1)\)-graph of order \( n \), we obtain

\[
\lambda^{(p)} \left( K_n^{r-1} \lor K_1 \right) = r^{1-1/p} (r - 1)^{(r-1)/p} (n)_{r-1} n^{-(r-1)/p}.
\]

The above construction can be generalized as follows: Let \( r \geq 3 \) and \( G \in \mathcal{G}^{r-1} (n - t) \). Choose a set of \( t \) vertices \( T \) with \( T \cap V (G) = \emptyset \) and define the graph \( G \lor tK_1 \in \mathcal{G}^r (n) \) by

\[
V (G \lor tK_1) := V (G) \cup T, \quad E (G \lor tK_1) := \{ e \cup \{ v \} : v \in T, \ e \in E (G) \}.
\]

Exactly as in the previous proposition we obtain the following relations.

**Proposition 6.8** Let \( r \geq 3, \ t \geq 1 \) and \( G \in \mathcal{G}^{r-1} \). For every \( p \geq 1 \),

\[
\lambda^{(p)} (G \lor tK_1) = t^{1-1/p} r^{1-r/p} (r - 1)^{(r-1)/p} \lambda^{(p)} (G),
\]

\[
\lambda^{(p)}_{\min} (G \lor tK_1) = -t^{1-1/p} r^{1-r/p} (r - 1)^{(r-1)/p} \lambda^{(p)} (G).
\]

Here is another construction similar to the above: Let \( r \geq 3, \ r > t \geq 1 \) and \( G \in \mathcal{G}^{r-t} (n - t) \). Choose a set of \( t \) vertices \( T \) with \( T \cap V (G) = \emptyset \) and define \( G \lor K^r_t \in \mathcal{G}^r (n) \) by

\[
V (G \lor K^r_t) := V (G) \cup T, \quad E (G \lor K^r_t) := \{ e \cup T : \ e \in E (G) \}.
\]

A graph with the structure of \( G \lor K^r_t \) is called a \textit{t-star}. The \textit{t-star} \( K_n^{r-1} \lor K^r_t \) is called a complete \textit{t-star} of order \( n \) and is denoted by \( S_{t,n}^r \).

**Proposition 6.9** Let \( r \geq 3, \ r > t \geq 1 \) and \( G \in \mathcal{G}^{r-t} \). For every \( p \geq 1 \),

\[
\lambda^{(p)} (G \lor K^r_t) = \frac{r! (r - t)^{(r-t)/p}}{r^{r/p} (r - t)!} \lambda^{(p)} (G),
\]

\[
\lambda^{(p)}_{\min} (G \lor K^r_t) = -\frac{r! (r - t)^{(r-t)/p}}{r^{r/p} (r - t)!} \lambda^{(p)} (G).
\]

In particular,

\[
\lambda^{(p)} (S_{t,n}^r) = \frac{(r)_t (r - t)^{(r-t)/p} (n - t)_{r-t}}{r^{r/p} (n - t)^{(r-t)/p}}, \quad (32)
\]

\[
\lambda^{(p)}_{\min} (S_{t,n}^r) = -\frac{(r)_t (r - t)^{(r-t)/p} (n - t)_{r-t}}{r^{r/p} (n - t)^{(r-t)/p}},
\]

and eigenvectors to \( \lambda^{(p)} (S_{t,n}^r) \) and \( \lambda^{(p)}_{\min} (S_{t,n}^r) \) have only nonzero entries.

Note that equation (32) has been proved in [20], Lemma 13.
7 More properties of $\lambda^{(p)}$

In this section we present results on $\lambda^{(p)} (G)$ if $G$ is a graph with some special property. Our first goal is to improve the bound $\lambda^{(p)} (G) \leq (r! |G|)^{1-1/p}$ in (9), using extra information about $G$.

7.1 $k$-partite and $k$-chromatic graphs

If $G$ is a $k$-partite 2-graph of order $n$, then Cvetković showed that $\lambda (G) \leq (1 - 1/k) n$ and Edwards and Elphick [8] improved that to $\lambda (G) \leq \sqrt{2(1 - 1/k)} |G|$. The following theorems extend these inequalities in several directions. First, using the proof of inequality (7) one can verify the following upper bounds for $k$-partite $r$-graphs.

**Theorem 7.1** Let $k > r$, $p \geq 1$ and $G \in G^r$. If $G$ is $k$-partite, then

$$\lambda^{(p)} (G) \leq ((k)_r / k^r)^{1/p} (r! |G|)^{1-1/p}$$

(33)

If $p > 1$ and $G$ has no isolated vertices, equality holds if and only if $G$ is a complete $k$-partite, with equal vertex classes. Furthermore, if $G$ is of order $n$, then

$$\lambda^{(p)} (G) \leq ((k)_r / k^r) n^{r-r/p}.$$ 

Equality holds if and only if $G$ is a complete $k$-partite graph with equal vertex classes.

Note that inequality (7) follows from this more general theorem because $(k)_r / k^r \leq (n)_r / n^r < 1$.

If $k = r$ Theorem the above inequalities become particularly simple, but more cases of equality arise.

**Proposition 7.2** Let $p \geq 1$ and $G \in G^r$. If $G$ is $r$-partite, then

$$\lambda^{(p)} (G) \leq (r! / r^{r/p}) |G|^{1-1/p}.$$ 

(34)

If $p > 1$, equality holds if and only if $G$ is a complete $r$-partite.

In particular, if $G \in G^r$ is $r$-partite and $k_1, \ldots, k_r$ are the sizes of its vertex classes, then

$$\lambda^{(p)} (G) \leq (r! / r^{r/p}) (k_1 \cdots k_r)^{1-1/p},$$

Equality holds if and only if $G$ is a complete $r$-partite $r$-graph.

In the following proposition we deduce bounds on $\lambda^{(p)}$ of the Turán 2-graph. A cruder form of these bounds has been given in [20], Lemma 13.

**Proposition 7.3** If $T_r (n)$ is the Turán 2-graph of order $n$, then

$$\lambda^{(1)} (T_k (n)) = 1 - 1/k,$$ 

(35)

and for every $p > 1$,

$$2 |T_k (n)| n^{-2/p} \leq \lambda^{(p)} (T_k (n)) \leq 2 |T_k (n)| n^{-2/p} \left(1 + k / (4pn^2)\right)$$

(36)
Sketch of the proof  The equality (35) follows from (33) and (34). The lower bound in (36) follows by (6). The upper bound follows from (33) and (34), using the fact that \(2|T_k(n)| \geq (1 - 1/k)n^2 - k/4\) and Bernoulli’s inequality.

Eigenvalues of 2-graphs have a lot of fascinating relations with the chromatic number and such seems to be the case with hypergraphs as well. We state here a results similar to the above mentioned bound of Edwards and Elphick. The proof method is described in 9.4.

**Theorem 7.4**  If \(G \in \mathcal{G}^r(n)\) and \(\chi(G) = k\), then

\[
\lambda^{(p)}(G) \leq (1 - k^{-r+1})^{1/p} (r! |G|)^{1-1/p}
\]

and

\[
\lambda^{(p)}(G) \leq (1 - k^{-r+1}) n^{r-r/p}.
\]

These bounds are essentially best possible as shown by the complete \(k\)-chromatic graph with chromatic classes of sizes \([n/k]\) and \([n/k]\).

### 7.2 A coloring theme of Szekeres and Wilf

One of the most appealing results in spectral graph theory is the inequality \(\chi(G) \leq \lambda(G) + 1\), proved for 2-graphs by Wilf in [39]. Somewhat later Szekeres and Wilf[38] showed that this inequality belongs to the study of a fundamental parameter called graph degeneracy. Similar results hold for hypergraphs as well, but we need a few definitions first: a \(\beta\)-star with vertex \(v\) is a graph such that the intersection of every two edges is \(\{v\}\). If \(G\) is a graph and \(v \in V(G)\), the \(\beta\)-degree \(d^\beta(v)\) of \(v\) is the size the maximum \(\beta\)-star with vertex \(v\); \(\delta^\beta(G)\) is the smallest \(\beta\)-degree of \(G\).

Berge [1], p.116, generalized the result of Wilf and Szekeres proving that for every graph \(G\)

\[
\chi(G) \leq \max_{H \subseteq G} \delta^\beta(H) + 1,
\]

which implies for 2 graphs that \(\chi(G) \leq \lambda(G) + 1\). Moreover, Cooper and Dutle [3] observed that for every \(G \in \mathcal{G}^r(n)\),

\[
\chi(G) \leq \lambda(G) / (r - 1)! + 1;
\]

however, for \(r \geq 3\), there is a certain incongruity in this bound, as the left side never exceeds \(n / (r - 1)\) while almost surely \(\lambda(G) = \Theta(n^{r-1})\). We propose a tight generalization of Wilf’s bound of a different kind. Recall that the 2-section \(G(2)\) of a graph \(G\) is a 2-graph with \(V(G(2)) = V(G)\) and \(E(G(2))\) consisting of all pairs of vertices that belong to an edge of \(G\). Clearly,

\[
\lambda(G(2)) \geq \delta(G(2)) \geq (r - 1) \delta^\beta(G).
\]

and this, together with (37), gives the following generalization of Wilf’s bound.
Proposition 7.5 If \( G \in \mathcal{G}^r \), then
\[
\chi(G) \leq \frac{\lambda(G(2))}{(r-1) + 1}.
\]

It will be interesting to prove that for every \( G \in \mathcal{G}^r(n) \),
\[
\lambda(G(2)) \leq \frac{\lambda(G)}{(r-2)!}
\]
which would imply the results of Cooper and Dutle; also, this inequality suggests a more general problem.

Problem 7.6 If \( G \in \mathcal{G}^r(n), 2 \leq k < r, \) and \( p \geq 1 \), find tight upper and lower bounds on \( \lambda(p)(G(k)) \).

7.3 \( \lambda^{(p)} \) and vertex degrees

For the largest eigenvalue of a 2-graph there is a tremendous variety of bounds using the degrees of \( G \). Unfortunately the situation with \( \lambda^{(p)} \) is more subtle even for 2-graphs. First, we saw in Theorem 2.7 that the inequality \( \lambda(G) \geq 2 |G|/n \) for 2-graphs generalizes seamlessly for \( \lambda^{(p)}(G) \) of an \( r \)-graph \( G \) and any \( p \geq 1 \), but the condition for equality becomes quite intricate, even for \( r = 2 \); see the discussion in Subsections 3.3, 4.3 and 7.5 for a number of special cases. In general, Problem 3.8 captures the main difficulty of this topic.

Another cornerstone bound on \( \lambda(G) \) for a 2-graph \( G \) with maximum degree \( \Delta \), is the inequality \( \lambda(G) \leq \Delta \). This inequality also generalizes to \( r \)-graphs, but not so directly.

Proposition 7.7 Let \( G \in \mathcal{W}^r(n) \) and \( \Delta(G) = \Delta \).

(i) If \( p \geq r \), then
\[
\lambda^{(p)}(G) \leq \frac{(r-1)!\Delta}{n^{r/p-1}}.
\]

(ii) If \( p \) is a positive integer, equality holds if and only if \( G \) is regular. If \( p = r \), equality holds if and only if \( G \) contains a \( \Delta \)-regular component.

(iii) If \( p \) is a positive integer, \( \lambda^{(p)}(G) \) is non-decreasing for 1 < \( p < r \), and
\[
\lambda^{(p)}(G) < (r-1)!\Delta^{(1-1/p)/(r-1/r)}.
\]

Proof Let \([x_i] \in S^{n-1}_{p, +}\) be an eigenvector to \( \lambda^{(p)}(G) \). Assume that \( p \geq r \) and let \( x_k = \max \{x_1, \ldots, x_n\} \).

The eigenvalues for \( \lambda^{(p)}(G) \) and the vertex \( k \) implies that
\[
\frac{\lambda^{(p)}(G)}{(r-1)!} x_k^{p-1} = \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} G(\{k,i_1,\ldots,i_{r-1}\}) x_{i_1} \cdots x_{i_{r-1}} \leq \Delta x_k^{r-1}\]
Since \( x_k \geq n^{-1/p} \) and \( p \geq r \), we find that

\[
\frac{\lambda^{(p)} (G)}{(r-1)!} \leq \Delta x_k^{r-p} \leq \Delta \left( n^{-1/p} \right)^{r-p} = \frac{\Delta}{n^{r/p-1}},
\]

proving (38). Now if \( p > r \) and we have equality in (38), then \( x_k = n^{-1/p} \) and so \( x_1 = \cdots = x_n = n^{-1/p} \). Thus, equations (11) show that all degrees are equal to \( \lambda^{(p)} (G) / (r-1)! = \Delta \), and \( G \) is regular. On the other hand,

\[
\frac{|G|}{n^{r/p}} = \frac{1}{(r-1)!} P_G \left( \frac{n^{-1/p}}{j_n} \right) \leq \frac{\lambda^{(p)} (G)}{(r-1)!} \leq \frac{\Delta}{n^{r/p-1}},
\]

and so if \( G \) is regular, then \( \lambda^{(p)} (G) / (r-1)! = \Delta n^{1-r/p} \), completing the proof of (i) for \( p > r \). We leave the case of equality for \( p = r \) to the reader.

To prove (ii) let \( s = p (r-1) / (r-p) \), and note that \( s > 1 \). The PM inequality implies that

\[
\frac{\lambda^{(p)} (G)}{(r-1)!} x_k^{p-1} = \sum_{\{k, i_1, \ldots, i_{r-1}\} \in E(G)} x_1^{p} \cdots x_{i_{r-1}} x^{(r-1)(s-p)} \leq \Delta^{1-1/s} \left( \sum_{\{k, i_1, \ldots, i_{r-1}\} \in E(G)} x_1^{p} \cdots x_{i_{r-1}} \right)^{1/s} \leq \Delta^{1-1/s} \left( \sum_{\{k, i_1, \ldots, i_{r-1}\} \in E(G)} x_1^{p} \cdots x_{i_{r-1}} \right)^{1/s},
\]

Hence,

\[
\frac{\lambda^{(p)} (G)}{(r-1)!} \leq \Delta^{1-1/s} \left( \sum_{\{k, i_1, \ldots, i_{r-1}\} \in E(G)} x_1^{p} \cdots x_{i_{r-1}} \right)^{1/s}.
\]

Maclaurin’s inequality and the fact that \( x_1^{p} + \cdots + x_1^{p} = 1 \) imply that

\[
\frac{\lambda^{(p)} (G)}{(r-1)!} < \Delta^{1-1/s} = \Delta^{(1-1/p)/(1-1/r)},
\]

completing the proof of (ii).

To prove (iii), let \( k \) be the maximal integer such that

\[
\binom{k-1}{r-1} \leq \Delta
\]

and let \( G \) be union of disjoint \( K_{k}^{r} \). Now, Propositions 2.6 and 2.4 together with an easy calculation, give the result. \( \square \)

For a 2-graph \( G \) it is known also that \( \lambda (G) \geq \sqrt{\Delta (G)} \). This bound also can be extended to \( r \)-graphs as follows.
Proposition 7.8 If $p \geq 1$, $G \in \mathcal{G}^r(n)$ and $\Delta(G) = \Delta$, then

$$\lambda_p(G) \geq \left(\frac{r!}{r^{r/p}}\right) \Delta^{1-(r-1)/p}.$$ 

If $G$ is the $\beta$-star $S_{\Delta}^r$, then equality holds.

**Sketch of a proof** We can suppose that $G$ has precisely $\Delta$ edges all sharing a common vertex $u$. Let $n = V(G)$ and note that $\Delta (r - 1) \geq (n - 1)$. Construct an $n$-vector $[y_i]$ by letting $y_u = r^{-1/p}$ and $y_i = ((r - 1)/r(n - 1))^{1/p}$ for the remaining entries. Clearly $[y_i] \in S_{n-1}^{p-1}$ and

$$\lambda_p(G) \geq P_G([y_i]) = \left(\frac{r! \Delta (r - 1)^{(r-1)/p}}{r^{r/p} \Delta (r-1)/p}\right) = \left(\frac{r!}{r^{r/p}}\right) \Delta^{1-(r-1)/p} = \lambda_p(S_{\Delta}^r),$$

completing the proof. □

A very useful bound for 2-graphs is the inequality of Hofmeister [16]

$$\lambda(G) \geq \left(\frac{1}{n} \sum d^2(u)\right)^{1/2}.$$ 

We have no clues how this inequality can be generalized to $r$-graphs, but we shall outline a limitation to possible generalizations. In the concluding section we shall return to this topic.

Proposition 7.9 If $r \geq 2$ and $\varepsilon > 0$, there is a $G \in \mathcal{G}^r(n)$ such that

$$\lambda(G) < (r - 1)! \left(\frac{1}{n} \sum d^{r/(r-1)+\varepsilon}(u)\right)^{1/(r/(r-1)+\varepsilon)}.$$ 

**Proof** Take $G$ to be the complete $r$-partite $r$-graph with vertex classes $V_1, \ldots, V_r$, where $|V_1| = 1$, and $|V_2| = \cdots = |V_r| = k$. Clearly, $v(G) = k(r - 1) + 1$ and

$$\frac{1}{k(r - 1) + 1} \sum_{u \in V(G)} d^{r/(r-1)+\varepsilon}(u) > \frac{1}{k(r - 1) + 1} (k^{r-1})^{r/(r-1)+\varepsilon} = \frac{1}{k(r - 1) + 1} k^{r+\varepsilon(r-1)}.$$ 

On the other hand,

$$\left(\frac{\lambda(G)}{(r - 1)!}\right)^{r/(r-1)+\varepsilon} = \left(r\left|G^{1-1/r}\right|^{r/(r-1)+\varepsilon}\right) = \left(r\left|k^{r-1}\right|^{1-1/r}\right)^{r/(r-1)+\varepsilon} = r^{r/(r-1)+\varepsilon} k^{r-1+\varepsilon} (r-1)^2/r,$$

and a short calculation gives the desired inequality for $k$ sufficiently large. □
7.4 $\lambda^{(p)}$ and set degrees

For hypergraphs the concept of degree can be extended from vertices to sets, and these set degrees are at least as important as vertex degree for 2-graphs. Thus, given a graph $G$ and a set $U \subset V(G)$, the set degree $d(U)$ of $U$ is defined as

$$d(U) = \sum_{e \in E(G), U \subset e} G(e).$$

$G$ is called $k$-set regular if the degrees of all $k$-subsets of $V(G)$ are equal. Note that if $G \in W^r$ is $k$-set regular, then it is $l$-set regular for each $l \in [k]$, and for any $k$-set $U \subset V(G)$, we have

$$d(U) = |G| \left( \frac{r}{k} \right) / \left( \frac{n}{k} \right) = |G| (n)_k / (r)_k. \quad (39)$$

In the same vein, let us also define $\Delta_k (G) = \max \{ d(U) : U \subset V(G), |U| = k \}$. It turns out that $k$-set regularity goes quite well with $\lambda^{(p)}$ if $k \geq 2$.

**Theorem 7.10** Let $r > k \geq 2$. If a graph $G \in W^r (n)$ is $k$-set regular, then

$$\lambda^{(p)} (G) = r! |G| / n^{r/p}$$

and $j_n$ is an eigenvector to $\lambda^{(p)} (G)$.

**Sketch of a proof** We know that $\lambda^{(p)} (G) \geq r! |G| / n^{r/p}$, so we shall prove the opposite inequality. Taking an eigenvector $[x_i] \in S_{kp+1}$, Maclaurin’s and the PM inequalities imply that

$$\lambda^{(p)} (G) = r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} G (\{i_1, \ldots, i_r\}) x_{i_1} \cdots x_{i_r} \leq r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} G (\{i_1, \ldots, i_r\}) \left( S_k (x_{i_1}^p, \ldots, x_{i_r}^p) / \left( \begin{array}{c} r \\ k \end{array} \right) \right)^{r/(kp)} \leq r! |G| \left( \frac{1}{|G|} \left( \begin{array}{c} r \\ k \end{array} \right) \right)^{-1} \sum_{\{i_1, \ldots, i_r\} \in E(G)} G (\{i_1, \ldots, i_r\}) \left( S_k (x_{i_1}^p, \ldots, x_{i_r}^p) \right)^{r/(kp)} = r! |G|^{1-r/(kp)} \left( \frac{1}{|G|} \left( \begin{array}{c} r \\ k \end{array} \right) \right)^{-1} \sum_{\{i_1, \ldots, i_k\} \in V^{(r)}} d (\{i_1, \ldots, i_k\}) x_{i_1}^p \cdots x_{i_k}^p \left( \frac{x_1^p + \cdots + x_n^p}{n} \right)^{r/p}.\quad (39)$$

Now, (39), and again Maclaurin’s inequality give

$$\lambda^{(p)} (G) = r! |G| \left( \frac{n}{k} \right)^{-1} S_k (x_1^p, \ldots, x_n^p)^{r/(kp)} \leq r! |G| \left( \frac{x_1^p + \cdots + x_n^p}{n} \right)^{r/p},$$

$$= r! |G| / n^{r/p}$$

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Theorem 7.11 Let $r > k \geq 2$ and $p \geq r/k$. If $G \in \mathcal{W}^r(n)$ and $\Delta_k = \Delta_k(G)$, then
\[
\frac{r! |G|}{n^{r/p}} \leq \lambda^{(p)}(G) \leq \frac{r! |G|}{n^{r/p}} \left( \frac{(n)_k \Delta_k}{(r)_{k-1} |G|} \right)^{r/(kp)}.
\]

In turn, Theorem 7.11 can be used to estimate the largest eigenvalue of random $r$-graphs, see, e.g., Section 7.11.

7.5 $k$-linear graphs and Steiner systems

If $k \geq 1$, an $r$-graph is called $k$-linear if every two edges share at most $k$ vertices; for short, 1-linear graphs are called linear. Clearly all 2-graphs are linear, so the concept makes sense only for hypergraphs. In fact, linearity is related to Steiner systems; recall that a Steiner $(k, r, n)$-system is a graph in $\mathcal{G}^r(n)$ such that every set of $k$ vertices is contained in exactly one edge.

Theorem 7.12 Let $1 \leq k \leq r - 2$ and $G \in \mathcal{G}^r(n)$. If $G$ is $k$-linear, then
\[
\lambda^{(r/(k+1))}(G) \leq r! \left( \frac{r}{k+1} \right)^{-1} \frac{n}{k+1} / n^{k+1}.
\]
Equality holds if and only if $G$ is a Steiner $(k+1, r, n)$-system.

Proof Let $\{x_i\} \in S_{r/(k+1),+}^{n-1}$ be an eigenvector to $\lambda^{(r/(k+1))}(G)$. If $\{i_1, \ldots, i_r\} \in E(G)$, by the AM-GM inequality we have
\[
x_{i_1} \cdots x_{i_r} \leq S_{k+1} \left( \left( x_{i_1}^{r/(k+1)}, \ldots, x_{i_r}^{r/(k+1)} \right) \right) / \left( \frac{r}{k+1} \right);
\]
hence
\[
\sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} \cdots x_{i_r} \leq \left( \frac{r}{k+1} \right)^{-1} \sum_{\{i_1, \ldots, i_r\} \in E(G)} S_{k+1} \left( \left( x_{i_1}^{r/(k+1)}, \ldots, x_{i_r}^{r/(k+1)} \right) \right)
\]
In the right side we have a sum of monomials of the type $x_{i_1}^{r/(k+1)} \cdots x_{i_{k+1}}^{r/(k+1)}$ where $\{i_1, \ldots, i_{k+1}\}$ is a $(k+1)$-subset of some edge of $G$. Since every $(k+1)$-subset $\{i_1, \ldots, i_{k+1}\}$ belongs to at most one edge, Maclaurin’s inequality implies that
\[
\lambda^{(r/(k+1))}(G) \leq r! S_{k+1} \left( \left( x_1^{r/(k+1)}, \ldots, x_n^{r/(k+1)} \right) \right)
\]
\[
\leq r! \left( \frac{r}{k+1} \right)^{-1} \frac{n}{k+1} S_{k+1} \left( \left( x_1^{r/(k+1)}, \ldots, x_n^{r/(k+1)} \right) \right)^{k+1}
\]
\[
= r! \left( \frac{r}{k+1} \right)^{-1} \frac{n}{k+1} / n^{k+1}.
\]
If equality holds in (40), then the condition for equality in Maclaurin’s inequality implies that 
\[ x_i = n - (k+1)/r \] 
and so equality holds in (40). □

Since \( \lambda(p)(G) \) is increasing in \( p \), we obtain the following more applicable bound.

**Proposition 7.13** Let \( 1 \leq k \leq r - 2 \), \( 1 \leq p \leq r/(k+1) \), and \( G \in \mathcal{G}^r(n) \). If \( G \) is \( k \)-linear, then
\[ \lambda(p)(G) \leq r!/(r_{k+1}). \]

Let us make also an easy observation.

**Proposition 7.14** Let \( 1 \leq k \leq r - 2 \) and \( G \in \mathcal{G}^r(n) \). If \( G \) is \( k \)-linear and \( 1 \leq p \leq r/(k+1) \), then the vector \( n^{-1/p} j_n \) is not an eigenvector to \( \lambda(p)(G) \). If \( G \) is a Steiner \( (k+1,r,n) \)-system, then \( n^{-(k+1)/r} j_n \) is an eigenvector to \( \lambda^{(r/(k+1))}(G) \).

**Proof** Every \( (k+1) \)-set of vertices belongs to at most one edge; therefore,
\[ \binom{n}{k+1} \geq |G| \binom{r}{k+1}. \]
Hence, if \( p < r/(k+1) \), then
\[ \frac{r! |G|}{n^{r/p}} \leq \frac{r! (n)_{k+1}}{n^{r/p} (r)_{k+1}} = o(p). \]
The second statement is immediate from Theorem 7.12. □

**Question 7.15** Is it true that if \( G \in \mathcal{G}^r(n) \) and \( G \) is \( k \)-linear and \( n^{-(k+1)/r} j_n \) is an eigenvector to \( \lambda^{(r/(k+1))}(G) \), then \( G \) is a Steiner \( (k+1,r,n) \)-system?

### 7.6 Bounds on the entries of a \( \lambda(p) \) eigenvector

Information about the entries of eigenvector can be quite useful in calculations. Papendieck and Recht [34] showed that if \( G \in \mathcal{G}_2(n) \) and \( [x_i] \in \mathbb{S}_2^{n-1} \) is an eigenvector to \( \lambda(G) \), then \( x_k^2 \leq 1/2 \) for any \( k \in V(G) \). This bound easily extends to \( r \)-graphs.

**Proposition 7.16** Let \( G \in \mathcal{G}^r(n) \) and \( [x_i] \in \mathbb{S}^{n-1} \). If \( [x_i] \) is an eigenvector to \( \lambda(p)(G) \), then \( |x_k|^p \leq 1/r \) for any \( k \in V(G) \). If \( G \) is a \( \beta \)-star, then equality holds.

In fact, the result easily generalizes to star-like subgraphs. This seems new even for 2-graphs.
Proposition 7.17 If $G \in \mathcal{G}^r$ and $U \subset V(G)$ is such that $|e \cap U| \leq 1$ for every $e \in E(G)$, then
\[
\sum_{k \in U} |x_k|^p \leq \frac{1}{r}.
\]

If $G$ is a star-like graph of the type $K_s \vee tK_{r-1}$, then equality holds above.

It would be interesting to determine all cases of equality in the above two propositions.

A useful result in spectral extremal theory for 2-graphs is the following bound from [27]:

Let $G$ be an 2-graph with minimum degree $\delta$, and let $x = (x_1, \ldots, x_n)$ be a nonnegative eigenvector to $\lambda(G)$ with $|x|_2 = 1$. If $x = \min \{x_1, \ldots, x_n\}$, then
\[
x^2 (\lambda(G)^2 + \delta n - \delta^2) \leq \delta.
\]

The bound (41) is exact for many graphs, and it has been crucial in proving upper bounds on $\lambda(G)$ by induction on the number of vertices of $G$. Similar bounds for hypergraphs are useful as well. Below we state and prove such a result; despite its awkward form, for $r = p = 2$ it yields precisely (41).

Theorem 7.18 Let $1 \leq p \leq r$, $G \in \mathcal{G}^r(n)$, $\delta(G) = \delta$, $\lambda^{(p)}(G) = \lambda$, and $[x_i] \in S_p^{n-1}$. If $[x_i]$ is an eigenvector to $\lambda^{(p)}(G)$, then the value $\sigma := \min \{|x_1|^p, \ldots, |x_n|^p\}$ satisfies
\[
\left( \left( \frac{\lambda n^{r/p-1}}{(r-1)!} \right)^p - \delta^p \right)^{\sigma - 1} \leq \left( \frac{n-1}{r-1} \right) \delta^{p-1} \left( \frac{1 - \sigma}{(n-1)^{r-1} - \sigma^{r-1}} \right).
\]

Proof Set for short $V = V(G)$ and let $k \in V$ be a vertex of degree $\delta$. Since $|[x_i]|$ is also an eigenvector to $\lambda^{(p)}(G)$ we can assume that $|x_i| \geq 0$. The eigenequation for $\lambda^{(p)}(G)$ and the vertex $k$ implies that
\[
\lambda \sigma^{1/p} \leq \lambda x_k^{p-1} = (r-1)! \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} x_{i_1} \ldots x_{i_{r-1}}.
\]

Now, dividing by $(r-1)!$ and applying the PM inequality to the right side, we find that
\[
\left( \frac{\lambda \sigma^{1/p}}{(r-1)!} \right)^p \leq \delta^{p-1} \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} x_{i_1}^p \ldots x_{i_{r-1}}^p.
\]
Our next goal is to bound the quantity $A = \sum_{i=1}^{X} x_i^p \cdots x_{i_{r-1}}^p$ from above. First, let $X_k = (V \setminus \{v_k\})^{(r-1)}$ and note that

$$A = \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} x_{i_1}^p \cdots x_{i_{r-1}}^p \leq \sum_{\{i_1,\ldots,i_{r-1}\} \in X_k} x_{i_1}^p \cdots x_{i_{r-1}}^p - \left(\sum_{\{i_1,\ldots,i_{r-1}\} \in X_k} x_{i_1}^p \cdots x_{i_{r-1}}^p \right)$$

$$\leq \sum_{\{i_1,\ldots,i_{r-1}\} \in X_k} x_{i_1}^p \cdots x_{i_{r-1}}^p - \left(\sum_{\{i_1,\ldots,i_{r-1}\} \in X_k} \sigma^{r-1} \right)$$

$$= \sum_{\{i_1,\ldots,i_{r-1}\} \in X_k} x_{i_1}^p \cdots x_{i_{r-1}}^p - \left(\frac{n-1}{r-1} - \delta\right)\sigma^{r-1}. \quad (44)$$

Next, applying Maclaurin’s inequality for the $(r-1)$’th symmetric function of the variables $x_i^p$, $i \in V \setminus \{v_k\}$, we find that

$$\frac{1}{(n-1)^{r-1}} \sum_{\{i_1,\ldots,i_{r-1}\} \in X_k} x_{i_1}^p \cdots x_{i_{r-1}}^p \leq \left(\frac{1}{n-1} \sum_{i \in V \setminus \{k\}} x_i^p\right)^{r-1} = \frac{1}{(n-1)^{r-1}} (1 - \sigma)^{r-1}. \quad (43)$$

Hence, replacing in (44), we obtain the desired bound on $A$:

$$A = \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} x_{i_1}^p \cdots x_{i_{r-1}}^p \leq \left(\frac{n-1}{r-1} - \delta\right)\sigma^{r-1}.$$

Returning back to (43), we see that

$$\left(\frac{\lambda}{(r-1)!}\right)^p \sigma^{p-1} \leq \left(\frac{n-1}{r-1}\right)\delta^{p-1} \left(\frac{(1 - \sigma)^{r-1}}{(n-1)^{r-1}} - \sigma^{r-1}\right) + \delta^{p-1}\sigma^{r-1}.$$

Since $p \leq r$ and $\sigma \leq 1/n$, we see that

$$\frac{\lambda n^{r/p-1}}{(r-1)!} \sigma^{r/p-1} \leq \frac{\lambda}{(r-1)!} \sigma^{1-1/p}.$$

Hence,

$$\left(\frac{\lambda n^{r/p-1}}{(r-1)!}\right)^p \sigma^{r-1} \leq \left(\frac{\lambda}{(r-1)!}\right)^p \sigma^{p-1} \leq \left(\frac{n-1}{r-1}\right)\delta^{p-1} \left(\frac{(1 - \sigma)^{r-1}}{(n-1)^{r-1}} - \sigma^{r-1}\right) + \delta^{p-1}\sigma^{r-1},$$

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and so,
\[
\left( \frac{\lambda n^{r/p-1}}{(r-1)!} - \delta \right)^p \sigma^{r-1} \leq \left( \frac{n-1}{r-1} \right) \delta^{p-1} \left( \frac{(1-\sigma)^{r-1}}{(n-1)^{r-1}} - \sigma^{r-1} \right),
\]
completing the proof of Theorem 7.18.

A weaker but handier form of Theorem 7.18 can be obtained by first proving that
\[
\left( \frac{\lambda n^{r/p-1}}{(r-1)!} - \delta \right)^p \geq \left( \frac{\lambda n^{r/p-1}}{(r-1)!} - \delta \right) p \delta^{p-1}
\]
using Bernoulli’s inequality, and then rearranging (42) to get the following corollary.

**Corollary 7.19** Under the assumptions of Theorem 7.18 we have
\[
\frac{\lambda n^{r/p-1}}{(r-1)!} - \delta \leq \left( \frac{n-1}{r-1} \right) \left( \frac{(1/\sigma - 1)^{r-1}}{(n-1)^{r-1}} - 1 \right).
\]
Finally using elements of the proof of Theorem 7.18 we obtain the following simple bounds.

**Proposition 7.20** Let \( G \in G^r(n) \) and \( [x_i] \in S^{n-1}_p \). If \( [x_i] \) is an eigenvector to \( \lambda^{(p)}(G) \), then for every \( k \in V(G) \),
\[
|x_k|^p \leq (r-1)! d(k) / \left( \lambda^{(p)}(G) \right)^{p/(p-1)}.
\]

**Proof** Since \([x_i]\) is also an eigenvector to \( \lambda^{(p)}(G) \) we can assume that \( [x_i] \geq 0 \). The eigenequation for \( \lambda^{(p)}(G) \) and the vertex \( k \) implies that
\[
\frac{\lambda x_k^{p-1}}{(r-1)!} = \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} x_{i_1} \ldots x_{i_{r-1}}.
\]
Applying the PM inequality to the right side we find that
\[
\left( \frac{\lambda x_k^{p-1}}{(r-1)!} \right)^p \leq d(k)^{p-1} \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E(G)} x_{i_1}^p \ldots x_{i_{r-1}}^p \leq d(k)^{p-1} \left( \frac{n-1}{r-1} \right) \left( \frac{1}{n-1} \right)^{r-1}
\]
and the assertion follows by simple algebra.

8 More properties of \( \lambda_{\min}^{(p)} \)

The study of \( \lambda_{\min}^{(p)} \) is considerably harder than of \( \lambda^{(p)} \); e.g., for even \( r \geq 4 \) we do not know \( \lambda_{\min}^{(p)} \) of the complete \( r \)-graph of order \( n \). Since \( \lambda_{\min}^{(p)}(G) = -\lambda^{(p)}(G) \) if \( r \) is odd and \( G \in G^r(n) \), in this section we shall assume that \( r \) is even. However, bipartite 2-graphs show that \( \lambda_{\min}^{(p)}(G) = -\lambda^{(p)}(G) \) can hold for even \( r \) as well; this interesting situation is fully investigated below, in subsection 8.2, where symmetry of eigenvalues is explored in general and a question of Pearson and Zhang is answered.
8.1 Lower bounds on $\lambda_{\min}$ in terms of order and size

The first question that one may ask about $\lambda_{\min}$ is: how small can be $\lambda_{\min}(G)$ of an $r$-graph $G$ of order $n$. For 2-graphs there are several well-known bounds, like $\lambda_{\min}(G) \geq -\sqrt{|G|}$ and $\lambda_{\min}(G) > -n/2$. The purpose of this subsection is to extend these two bounds to $r$-graphs if $r$ is even.

**Theorem 8.1** If $r$ is even, $p \geq 1$, and $G \in G^r(n)$, then

$$\lambda_{\min}^p(G) \geq -(r! |G|)^{\frac{1}{1-1/p}} / 2^{1/p}.$$ 

For the proof of this theorem we need a simple analytical bound stated as follows.

**Proposition 8.2** If $r$ is even, then

$$\sum_{s=0}^{r/2-1} \frac{(1-x)^{2s+1}(1+x)^{r-2s+1}}{(2s+1)!(r-2s-1)!} \leq \frac{2^{r-1}}{r!}$$

and therefore if $0 \leq a \leq 1$, then

$$r! \sum_{s=0}^{r/2-1} \frac{a^{2s+1}(1-a)^{r-2s+1}}{(2s+1)!(r-2s-1)!} \leq \frac{1}{2}.$$

**Proof of Theorem 8.1** Let $[x_i] \in S_p^n$ be an eigenvector to $\lambda_{\min}(G)$, let $V_1$ be the set of vertices $v$ for which $x_v < 0$ and let $V_2 = V(G) \setminus V_1$. Let $G'$ be a subgraph of $G$ of order $n$ such that $e \in E(G')$ whenever $|e \cap V_1|$ is odd. Note that if $\{i_1, \ldots, i_r\} \in E(G')$, then $x_{i_1} \cdots x_{i_r} \leq 0$; conversely if $\{i_1, \ldots, i_r\} \in E(G) \setminus E(G')$, then $x_{i_1} \cdots x_{i_r} \geq 0$. We conclude that

$$\lambda_{\min}(G') \leq P_{G'}(x) \leq P_G(x) = \lambda_{\min}(G),$$

The PM inequality implies that

$$\lambda_{\min}(G') \geq P_{G'}(x) \geq -r! |G'|^{\frac{1}{1-1/p}} \left( \sum_{\{i_1, \ldots, i_r\} \in E(G')} |x_{i_1}|^p \cdots |x_{i_r}|^p \right)^{1/p}.$$

Our next purpose is find an upper bound on the value

$$\sum_{\{i_1, \ldots, i_r\} \in E(G')} |x_{i_1}|^p \cdots |x_{i_r}|^p.$$

Set first $|V_1| = k$, let $y = (y_1, \ldots, y_k)$ and $z = (z_1, \ldots, z_{n-k})$ be the restrictions of $(x_1^p, \ldots, x_n^p)$ to $V_1$ and to $V_2$. Let $G''$ be the $r$-graph such that $V(G'') = V(G)$ and $E(G'')$ is the set of all $r$-subsets $e \subset V(G)$ for which $|e \cap V_1|$ is odd. Clearly $G'$ is a subgraph of $G''$ and so

$$\sum_{\{i_1, \ldots, i_r\} \in E(G')} |x_{i_1}|^p \cdots |x_{i_r}|^p \leq \sum_{\{i_1, \ldots, i_r\} \in E(G'')} |x_{i_1}|^p \cdots |x_{i_r}|^p,$$

$$= S_1(y) S_{r-1}(z) + S_3(y) S_{r-3}(z) + \cdots + S_{r-1}(y) S_1(z),$$

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where \( S_k(w) \) is the \( k \)'th symmetric function of the entries of a vector \( w \). Maclaurin’s inequality implies that that for every \( k = 1, 3, \ldots, r - 1 \),

\[
S_k(y) S_{r-k}(z) \leq \left( \frac{p}{k} \right) \left( \frac{a}{p} \right)^k \left( \frac{n-p}{r-k} \right) \left( \frac{1-a}{n-p} \right)^{r-k} \leq \frac{a^k (1-a)^{r-k}}{k! (r-k)!}.
\]

where \( a = \sum_{i \in V_1} y_i \) and \( \sum_{i \in V_2} z_i = 1 - a \). Proposition 8.2 implies that

\[
\frac{r}{2} - 1 \sum_{s=0}^{r/2-1} \frac{a^{2s+1} (1-a)^{r-2s+1}}{(2s+1)! (r-2s-1)!} \leq \frac{1}{2r!},
\]

and so,

\[
\lambda_{\min}(G) \geq -r! |G'|^{1-1/p} \left( \sum_{\{i_1, \ldots, i_r\} \in E(G')} |x_{i_1}|^p \cdots |x_{i_r}|^p \right)^{1/p} \geq -r! |G'|^{1-1/p} \left( 2r! \right)^{1/p}
\]

as required.

Following the proof of the previous theorem, one can show that

\[
r! |G''| \leq n^r/2,
\]

obtaining the following absolute bound.

**Theorem 8.3** If \( r \) is even and \( G \in G^r(n) \), then \( \lambda^{(p)}_{\min}(G) \geq -n^{r-r/p}/2. \)

To show that the last theorem is essentially tight let us give an example.

**Proposition 8.4** If \( r \) is even, there exists \( G \in G^r(n) \) such that

\[
\lambda^{(p)}_{\min}(G) \leq -n^{r-r/p}/2 + r^2 n^{r-1-r/p}.
\]

**Proof** Let \( V(G) := [n] \) and let \( V_1 = [\lfloor n/2 \rfloor], V_2 = [n] \setminus V_1 \). Let \( E(G) \) be the set of all \( r \)-subsets of \( [n] \) that intersect \( V_1 \) in an odd number of vertices. We claim that \( G \) satisfies the requirement. Indeed, define a vector \( x = [x_i] \in \mathbb{S}^{n-1}_r \) by

\[
x_i := \begin{cases} 
-\frac{1}{n^{r-1/p}} & \text{if } i \in V_1 \\
\frac{1}{n^{r-1/p}} & \text{if } i \in V_2.
\end{cases}
\]

Clearly,

\[
\lambda_{\min}(G) \leq P_G(x) = r! \sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} \cdots x_{i_r} = \frac{r!}{n^{r/p}} \sum_{\{i_1, \ldots, i_r\} \in E(G)} -1 = -\frac{r!}{n^{r/p}} |G|.
\]
To complete the proof note that
\[
r!|G| = r! \left( \left( \frac{|V_1|}{1} \right) \left( \frac{|V_2|}{r-1} \right) + \left( \frac{|V_1|}{3} \right) \left( \frac{|V_2|}{r-3} \right) + \cdots + \left( \frac{|V_1|}{r-1} \right) \left( \frac{|V_2|}{1} \right) \right)
\]
\[
\geq \left( \frac{n}{2} - r \right)^r \left( \frac{r!}{1!(r-1)!} + \frac{r!}{3!(r-3)!} + \cdots + \frac{r!}{(r-1)!!} \right)
\]
\[
> \left( \left( \frac{n}{2} \right)^r - r^2 \left( \frac{n}{2} \right)^{r-1} \right)^{2^{r-1}} = \frac{n^r}{2^r} - r^2 n^{r-1}
\]

implying the required bound. \(\square\)

### 8.2 Odd transversals and symmetry of the algebraic spectrum

One of the best-known theorems in spectral graph theory is the following one: If a 2-graph \(G\) is bipartite, then its spectrum is symmetric with respect to 0. If \(G\) is connected and \(\lambda(G) = -\lambda_{\min}(G)\), then \(G\) is bipartite.

Not surprisingly there have been attempts to generalize this statement for hypergraphs, but they seem overly algebraic and not too convincing. We offer here another, rather natural generalization, replacing “bipartite” by “having odd transversal”. Note that “having odd transversal” is a monotone graph property, inherited by subgraphs, just like subgraphs of bipartite 2-graphs are also bipartite.

Our main interest is in \(\lambda(p)(G)\) and \(\lambda_{\min}(G)\), but at the end of the subsection we outline a general statement about other possible eigenvalues. Here is our first theorem.

**Theorem 8.5** If \(G \in \mathcal{W}^r\) and \(G\) has an odd transversal, then \(\lambda(p)(G) = -\lambda_{\min}(G)\) for every \(p \geq 1\).

**Proof** Let \(U \subset V(G)\) be an odd transversal of \(G\). Let \([x_i] \in \mathbb{S}^{n-1}_{p^+}\) be an eigenvector to \(\lambda(p)(G)\); negate \(x_i\) whenever \(i \in U\), and write \(y\) for the resulting vector. Clearly \(y \in \mathbb{S}^{n-1}_{p}\) and

\[
P_G(y) = -P_G([x_i]) = -\lambda(p)(G);
\]

hence, in view of Proposition 1.1, \(P_G(y) = \lambda_{\min}(G) = -\lambda(p)(G)\), completing the proof. \(\square\)

To elucidate the picture let us state an immediate corollary from Theorem 8.5.

**Corollary 8.6** If \(G\) is an \(r\)-partite \(r\)-graph, then \(\lambda(p)(G) = -\lambda_{\min}(G)\) for every \(p \geq 1\).

A converse of Theorem 8.5 can be proved only if \(\lambda(p)(G)\) has a positive eigenvector, so we give the following statement.

**Theorem 8.7** Let \(G \in \mathcal{W}^r\). If \(G\) is connected, and \(\lambda(p)(G) = -\lambda_{\min}(G)\) for some \(p > r - 1\), then \(G\) has an odd transversal.
Theorem 8.8 Let $G \in \mathcal{W}^r (n)$. If $G$ has an odd transversal, $p > 1$ and $\lambda$ is a complex number satisfying
\[
\lambda x_k |x_k|^{p-2} = \frac{1}{r} \frac{\partial P_G (\|x\|)}{\partial x_k}, \quad k = 1, \ldots, n, \tag{45}
\]
for some $x \in \mathbb{C}^n$ with $\|x\|_p = 1$, then $-\lambda$ also satisfies (45) for some $y \in \mathbb{C}^n$ with $\|y\|_p = 1$.

From Theorem 8.7 we see that the converse of Theorem 8.8 also holds for connected graphs. In a similar vein, Pearson and Zhang asked in [31], Question 4.10, what conditions would guarantee that the set of algebraic eigenvalues determined by equations (14) is symmetric with respect to 0. We answer this question in the following two statements.

Theorem 8.9 If $r$ is odd and $G \in \mathcal{W}^r (n)$, then $-\lambda (G)$ never satisfies the equations
\[
-\lambda (G) x_k^{r-1} = \frac{1}{r} \frac{\partial P (\|x\|)}{\partial x_k}, \quad k = 1, \ldots, n, \tag{46}
\]
for a nonzero vector $x \in \mathbb{C}^n$.

Proof Assume for a contradiction that $x \in \mathbb{C}^n$ is a nonzero vector satisfying (46). Clearly we can assume that $\|x\|_r = 1$. As $\|x\|$ is an eigenvector to $\lambda (G)$, we see that
\[
\lambda (G) (\|x\|^r + \cdots + \|x\|_r^r) = \frac{1}{r} \sum x_k \frac{\partial P (\|x\|)}{\partial x_k} = P (\|x\|).
\]

This implies that for each $k \in [n]$,
\[
\sum_{\{k, i_1, \ldots, i_{r-1}\} \subseteq E (G)} G (\{k, i_1, \ldots, i_{r-1}\}) x_k x_{i_1} \cdots x_{i_{r-1}} = \sum_{\{k, i_1, \ldots, i_{r-1}\} \subseteq E (G)} G (\{k, \ldots, i_{r-1}\}) x_k |x_{i_1}| \cdots |x_{i_{r-1}}|
\]

\[53\]
and so for each $k \in [n]$, the value $\arg x_{i_1} \cdots x_{i_{r-1}}$ is the same for each edge $\{k, i_1, \ldots, i_{r-1}\}$. Hence, (46) implies that
\[
\pi + r \arg x_{i_1} = \arg x_{i_1} \cdots x_{i_r} \pmod{2\pi}
\]
for every edge $\{i_1, \ldots, i_r\}$. By symmetry we obtain
\[
r\pi + r \arg x_{i_1} + \cdots + r \arg x_{i_r} = r \arg x_{i_1} \cdots x_{i_r} \pmod{2\pi} = r \arg x_{i_1} + \cdots + r \arg x_{i_r} \pmod{2\pi},
\]
and so $r\pi = 0 \pmod{2\pi}$, a contradiction, as $r$ is odd.

So the algebraic spectrum of an $r$-graph can be symmetric only for even $r$. The following proposition gives a sufficient condition for symmetry if $r$ is even.

**Proposition 8.10** Let $r$ be even and $G \in \mathcal{W}^r(n)$. If $G$ has an odd transversal and $\lambda$ satisfies the equations
\[
\lambda x_k^{r-1} = \frac{1}{r} \frac{\partial P([x_i])}{\partial x_k}, \quad k = 1, \ldots, n,
\]
for some nonzero vector $[x_i] \in \mathbb{C}^n$, then $-\lambda$ satisfies the same equations for some $[x_i] \in \mathbb{C}^n$.

Finally, from Theorem 8.7 we already know that if $G$ is connected, and $-\lambda(G)$ satisfies the equations (46), then $G$ has an odd transversal. This completely answers the question of Pearson and Zhang.

## 9 Spectral extremal hypergraph theory

The statements of Proposition and Theorem discuss problems of the following type: how large can be $\lambda^{(p)}(G)$ of an $r$-graph $G$ with some particular property. Such problems belong to extremal graph theory and have been extensively studied for 2-graphs. Past experience shows that the variations of these problems are practically infinite even for 2-graphs. Since extremal problems for hypergraphs are overwhelmingly diverse and hard, we shall base the study of the extrema of $\lambda^{(p)}$ and $\lambda^{(\min)}$ on the following two principles: first, we shall focus on hereditary properties of graphs; second, we shall seek asymptotic solutions foremost and shall deduce exact ones only afterwards, whenever possible.

### 9.1 Hereditary properties of hypergraphs

A *property* of graphs is a family of $r$-graphs closed under isomorphisms. A property is called *monotone* if it is closed under taking subgraphs, and *hereditary*, if it is closed under taking induced subgraphs. For example, given a set of $r$-graphs $\mathcal{F}$, the family of all $r$-graphs that do not contain any $F \in \mathcal{F}$ as a subgraph is a monotone property, denoted by $\text{Mon}(\mathcal{F})$. Likewise, the family of all $r$-graphs that do not contain any $F \in \mathcal{F}$ as an induced subgraph is a hereditary property, denoted as $\text{Her}(\mathcal{F})$. When $\mathcal{F}$ consists of a single graph $F$, we shall write $\text{Mon}(F)$ and $\text{Her}(F)$ instead of $\text{Mon}([F])$ and $\text{Her}([F])$. Given a property $\mathcal{P}$, we write $\mathcal{P}_n$ for the set of all graphs in $\mathcal{P}$ of order $n$. 54
The typical extremal hypergraph problem is the following one: Given a hereditary property $P$ of $r$-graphs, find

$$ex(P, n) := \max_{G \in P_n} |G|.$$ (47)

If $r = 2$ and $P$ is a monotone property, asymptotic solutions are given by the theorem of Erdős and Stone; for a general hereditary property $P$ an asymptotic solution was given in [29]. For $r \geq 3$ the problem has turned out to be generally hard and is solved only for very few properties $P$; see [18] for an up-to-date discussion.

An easier asymptotic version of the same problem arises from the following fact, established by Katona, Nemetz and Simonovits [21]: If $P$ is a hereditary property of $r$-graphs, then the sequence

$$\left\{ \frac{ex(P, n)}{n^r} \right\}_{n=1}^{\infty}$$

is nonincreasing and so the limit

$$\pi(P) := \lim_{n \to \infty} \frac{ex(P, n)}{n^r}$$

always exists.

One of the most appealing features of $\lambda^{(p)}$ is that under the same umbrella it covers three graph parameters, all important in extrema problems - the graph Lagrangian, the largest eigenvalue and the number of edges. So, in analogy to (17), given a hereditary property $P$ of $r$-graphs, we set

$$\lambda^{(p)}(P, n) := \max_{G \in P_n} \lambda^{(p)}(G).$$

Let us begin with a theorem about $\lambda^{(p)}(G)$, which is similar to the above mentioned result of Katona, Nemetz and Simonovits.

**Theorem 9.1** Let $p \geq 1$. If $P$ is a hereditary property of $r$-graphs, then the limit

$$\lambda^{(p)}(P) := \lim_{n \to \infty} \lambda^{(p)}(P, n) n^{r/p - r}$$ (48)

exists. If $p = 1$, then $\lambda^{(1)}(P, n)$ is nondecreasing, and so

$$\lambda^{(1)}(P, n) \leq \lambda^{(1)}(P).$$ (49)

If $p > 1$, then $\lambda^{(p)}(P)$ satisfies

$$\lambda^{(p)}(P) \leq \lambda^{(p)}(P, n) n^{r/p} / (n)_r.$$ (50)

For a proof of Theorem 9.1 we refer the reader to [28]; it is very similar to the proof of Theorem 9.2 below. Before exploring some of the consequences of Theorem 9.1 we shall extend the above setup to $\lambda^{(p)}_{\min}$ as well. Thus, if $P$ is a hereditary property of $r$-graphs, we define

$$\lambda^{(p)}_{\min}(P, n) := \min_{G \in P_n} \lambda^{(p)}_{\min}(G).$$

and prove the following statement:
Theorem 9.2 Let $p \geq 1$. If $\mathcal{P}$ is a hereditary property of $r$-graphs, then the limit
\[
\lambda_{\min}^{(p)} (\mathcal{P}) := \lim_{n \to \infty} \lambda_{\min}^{(p)} (\mathcal{P}, n) n^{r/p-r}
\] (51)
exists. If $p = 1$, then $\lambda^{(1)} (\mathcal{P}, n)$ is nonincreasing, and so
\[
\lambda_{\min}^{(1)} (\mathcal{P}) \leq \lambda_{\min}^{(1)} (\mathcal{P}, n). \tag{52}
\]
If $p > 1$, then $\lambda^{(p)} (\mathcal{P})$ satisfies
\[
\lambda_{\min}^{(p)} (\mathcal{P}) \geq \lambda_{\min}^{(p)} (\mathcal{P}, n) n^{r/p} / (n^r). \tag{53}
\]

Proof Set for short $\lambda_n^{(p)} = \lambda_{\min}^{(p)} (\mathcal{P}, n)$. Let $G \in \mathcal{P}_n$ be such that $\lambda_n^{(p)} = \lambda_{\min}^{(p)} (G)$ and let $x = [x_i] \in \mathbb{S}^{n-1}_p$ be an eigenvector to $\lambda_{\min}^{(p)} (G)$. If $p = 1$, we obviously have $\lambda_n^{(1)} \leq \lambda_{n-1}^{(1)}$, and in view of
\[
\lambda_n^{(1)} = P_G (x) \geq -r! \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \ldots x_{i_r} > -(x_1 + \cdots + x_n)^r = -1,
\]
the sequence $\{\lambda_n^{(1)}\}_{n=1}^{\infty}$ is converging to some $\lambda$. We have
\[
\lambda = \lim_{n \to \infty} \lambda_n^{(1)} n^{r-r} = \lambda_{\min}^{(1)} (\mathcal{P}),
\]
proving (52).

Suppose now that $p > 1$. Since $|x|_p = 1$, there is a vertex $k \in V (G)$ such that $|x_k|^p \leq 1/n$. Write $G - k$ for the $r$-graph obtained from $G$ by omitting the vertex $k$, and let $x'$ be the $(n - 1)$-vector obtained from $x$ by omitting the entry $x_k$. Now the eigenvalue for $\lambda_{\min}^{(p)} (G)$ and the vertex $k$ implies that
\[
P_{G-k} (x') = P_G (x) - r! x_k \sum_{\{k,i_1,\ldots,i_{r-1}\} \in E (G)} x_{i_1} \ldots x_{i_{r-1}} =
\]
\[
= \lambda_{\min}^{(p)} (G) - r x_k \left( \lambda_{\min}^{(p)} (G) |x_k|^{p-2} \right) = \lambda_n^{(p)} (1 - r |x_k|^p)
\]
Since $\mathcal{P}$ is a hereditary property, $G - k \in \mathcal{P}_{n-1}$, and therefore,
\[
P_{G-k} (x') \geq \lambda_{\min}^{(p)} (G-k) |x'|_p^r = \lambda_{\min}^{(p)} (G - k) (1 - |x_k|^p)^{r/p} \geq \lambda_{n-1}^{(p)} (1 - |x_k|^p)^{r/p}.
\]
Thus, we obtain
\[
\lambda_n^{(p)} \geq \lambda_{n-1}^{(p)} \frac{(1 - |x_k|^p)^{r/p}}{(1 - r |x_k|^p)}.
\] (54)

Note that the function
\[
f (y) := \frac{(1 - y)^{r/p}}{1 - ry}
\]
is nondecreasing in \( y \) for \( 0 \leq y \leq 1/n \) and \( n \) sufficiently large. Indeed,

\[
\frac{d}{dy} f(y) = -\frac{1}{p} \frac{(1-y)^{r/p-1} (1-ry) + r (1-y)^{r/p}}{(1-ry)^2}
\]

\[
= \left( -\frac{1}{p} (1-ry) + (1-y) \right) \frac{r (1-y)^{r/p-1}}{(1-ry)^2}
\]

\[
= \left( -\frac{1}{p} - 1 \right) + \left( \frac{r}{p} - 1 \right) y \frac{r (1-y)^{r/p-1}}{(1-ry)^2} \geq 0
\]

Here we use the fact that \( 1/p - 1 > 0 \) and that \( (r/p - 1) y \) tends to 0 when \( n \to \infty \).

Hence, in view of (54) and \( \lambda_{n-1}^{(p)} < 0 \), we find that for \( n \) large enough,

\[
\lambda_n^{(p)} \geq \lambda_{n-1}^{(p)} f \left( \frac{|x_k|^p}{n} \right) \geq \lambda_{n-1}^{(p)} f \left( \frac{1}{n} \right) = \lambda_{n-1}^{(p)} \frac{n (1-1/n)^{r/p}}{n-r},
\]

and so,

\[
\frac{\lambda_n^{(p)} n^{r/p}}{n (n-1) \cdots (n-r+1)} \geq \frac{\lambda_{n-1}^{(p)} (n-1)^{r/p}}{(n-1) (n-2) \cdots (n-r)}.
\]

Therefore, the sequence

\[
\left\{ \lambda_n^{(p)} n^{r/p} / (n)_r \right\}_{n=1}^{\infty}
\]

is nondecreasing, and so it is converging, completing the proof of (51) and (53) for \( p > 1 \). \( \square \)

### 9.2 Asymptotic equivalence of \( \lambda^{(p)} (\mathcal{P}) \) and \( \pi (\mathcal{P}) \)

Let \( \mathcal{P} \) be a hereditary property of \( r \)-graphs. If \( G \in \mathcal{P}_n \) is such that \(|G| = ex(\mathcal{P}, n)\), then Theorems 2.7 and 9.1 imply that

\[
|G| \leq \lambda^{(p)} (G) = \lambda^{(p)} (\mathcal{P}, n) n^{r/p-r}
\]

and therefore

\[
\lambda^{(p)} (\mathcal{P}, n) n^{r/p} / (n)_r \geq ex (\mathcal{P}, n) / \binom{n}{r},
\]

Letting \( n \to \infty \), we find that

\[
\lambda^{(p)} (\mathcal{P}) \geq \pi (\mathcal{P}).
\]

In fact, there is almost always equality in this relation as stated in the following theorem, proved in [28].

**Theorem 9.3** If \( \mathcal{P} \) is a hereditary property of \( r \)-graphs and \( p > 1 \), then

\[
\lambda^{(p)} (\mathcal{P}) = \pi (\mathcal{P}).
\]
Since a result of this scope is not available in the literature, some remarks are due here. First, using \ref{57}, every result about \(\pi(P)\) of a hereditary property \(P\) gives a result about \(\lambda(p)(P)\) as well, so we readily obtain a number of asymptotic results about \(\lambda(p)\). As we shall see in Subsection 9.4 in many important cases such asymptotic results can be converted to explicit non-asymptotic ones. But equality \ref{57} is more significant, as finding \(\pi(P)\) now can be reduced to maximization of a smooth function subject to a smooth constraint, and so \(\lambda(p)(G)\) offers advantages compared to \(|G|\) in finding \(\pi(P)\).

9.3 Forbidden blow-ups

It is well-known (see, e.g., \cite{18}, Theorem 2.2) that if \(H(k_1, \ldots, k_h)\) is a fixed blow-up of \(H \in G^r(h)\), then

\[\pi(\text{Mon}(H)) = \pi(\text{Mon}(H(k_1, \ldots, k_h))}\]  

(58)

Theorem 9.3 obviously implies a similar result for \(\lambda(p)\).

**Theorem 9.4** If \(p > 1\) and \(H(k_1, \ldots, k_h)\) is a fixed blow-up of \(H \in G^r(h)\), then

\[\lambda(p)(\text{Mon}(H)) = \lambda(p)(\text{Mon}(H(k_1, \ldots, k_h)))\]

As seen below, a similar statement exists for \(\lambda(p)_{\min}\) as well. However, we have no statement similar to Theorem 9.3 for \(\lambda(p)_{\min}\) and thus our proof of Theorem 9.5 uses the Hypergraph Removal Lemma and other fundamental results about \(r\)-graphs.

**Theorem 9.5** If \(p > 1\) and \(H(k_1, \ldots, k_h)\) is a fixed blow-up of \(H \in G^r(h)\), then

\[\lambda(p)_{\min}(\text{Mon}(H)) = \lambda(p)_{\min}(\text{Mon}(H(k_1, \ldots, k_h)))\]

**Proof** For the purposes of this proof write \(k_H(G)\) for the number of subgraphs of \(G\) which are isomorphic to \(H\). We start by recalling the Hypergraph Removal Lemma, one of the most important consequences of the Hypergraph Regularity Lemma, proved independently by Gowers \cite{14} and by Nagle, Rödl, Schacht and Skokan \cite{25}, \cite{36}.

**Removal Lemma** Let \(H\) be an \(r\)-graph of order \(h\) and let \(\varepsilon > 0\). There exists \(\delta = \delta_H(\varepsilon) > 0\) such that if \(G\) is an \(r\)-graph of order \(n\), with \(k_H(G) < \delta n^h\), then there is an \(r\)-graph \(G_0 \subset G\) such that \(|G| - |G_0| = \varepsilon n^r\) and \(k_H(G_0) = 0\).

In \cite{9} Erdös showed that for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(G\) is an \(r\)-graph with \(|G| \geq \varepsilon n^r\), then \(K_r(k, \ldots, k) \subset G\) for some \(k \geq \delta (\log n)^{1/(r-1)}\). As noted by Rödl and Schacht \cite{35} (also by Bollobás, unpublished) this result of Erdös implies the following general assertion.

**Theorem A** Let \(H\) be an \(r\)-graph of order \(h\) and let \(\varepsilon > 0\). There exists \(\delta = \delta_H(\varepsilon) > 0\) such that if \(G\) is an \(r\)-graph of order \(n\), with \(k_H(G) \geq \varepsilon n^h\), then \(H(k, \ldots, k) \subset G\) for some \(k = \left\lceil \delta (\log n)^{1/(h-1)} \right\rceil\).
Suppose now that $H$ is an $r$-graph of order $h$, let $H(k_1, \ldots, k_h)$ be a fixed blow-up of $H$, and set $k = \max \{k_1, \ldots, k_h\}$. Take $G \in \text{Mon} (H(k_1, \ldots, k_h))_n$ such that

$$\lambda^{(p)}_{\min} (G) = \lambda^{(p)}_{\min} (\text{Mon} (H(k_1, \ldots, k_h)))_n$$

For every $\varepsilon > 0$, choose $\delta = \delta_H (\varepsilon)$ as in the Removal Lemma. Since $H(k_1, \ldots, k) \not\in G$, Theorem A implies that if $n$ is sufficiently large, then $k_H (G) < \delta n^h$. Now the Removal Lemma implies that there is an $r$-graph $G_0 \subset G$ such that $|G| \geq |G| - \varepsilon n^r$ and $k_H (G_0) = 0$. Clearly, we can assume that $V (G_0) = V (G)$.

By Proposition 6.3 we see that

$$\lambda^{(p)}_{\min} (G) \geq \lambda^{(p)}_{\min} (G_0) - (\varepsilon r! n^r)^{1/p},$$

and hence,

$$\frac{\lambda^{(p)}_{\min} (\text{Mon} (H(k_1, \ldots, k)))_n}{(n - 1)_{r-1}} \geq \frac{\left[ \lambda^{(p)}_{\min} (G_0) - (\varepsilon r! n^r)^{1/p} \right] n^{r/p-1}}{(n - 1)_{r-1}}$$

$$\geq \lambda^{(p)}_{\min} (\text{Mon} (H)) - o(1) - \frac{(\varepsilon r!)^{1-1/p} n^{r-r/p} n^{1-r/p}}{n_{r-1}^{1/p}}$$

$$= \lambda^{(p)}_{\min} (\text{Mon} (H)) + o(1) + (\varepsilon r!)^{1-1/p}.$$

Since $\varepsilon$ can be made arbitrarily small, we see that

$$\lambda^{(p)}_{\min} (\text{Mon} H(k_1, \ldots, k_h)) \geq \lambda^{(p)}_{\min} (\text{Mon} (H)),$$

completing the proof of Theorem 9.5. \hfill \Box

### 9.4 Flat properties

According to Theorem 9.3, every hereditary property $P$ satisfies either $\lambda^{(1)} (P) > \pi (P)$ or $\lambda^{(1)} (P) = \pi (P)$; if the latter case we shall call $P$ flat. Flat properties possess truly remarkable features with respect to extremal problems, some of which are presented below.

Let us note that, in general, $\pi (P)$ alone is not sufficient to estimate $ex (P, n)$ for small values of $n$ and for arbitrary hereditary property $P$. However, flat properties allow for tight, explicit upper bounds on $ex (P, n)$ and $\lambda^{(p)} (P)$. Let us first outline a class of flat properties, whose study has been started by Sidorenko [37]:

A graph property $P$ is said to be multiplicative if $G \in P_n$ implies that $G(k_1, \ldots, k_n) \in P$ for every vector of positive integers $k_1, \ldots, k_n$. This is to say, a multiplicative property contains the blow-ups of all its members.

Multiplicative properties are quite convenient for extremal graph theory, and they are ubiquitous as well. Indeed, following Sidorenko [37], call a graph $F$ covered if every two vertices of $F$ are contained in an edge. Clearly, complete $r$-graphs are covered, and for $r = 2$ these are the only covered graphs, but for $r \geq 3$ there are many noncomplete ones. For example, the Fano plane
3-graph $F_7$ is a noncomplete covered graph. Obviously, if $F$ is a covered graph, then $\text{Mon}(F)$ is both a hereditary and a multiplicative property.

Below we illustrate Theorems 9.7 and 9.8 using $F_7$ as forbidden graph because it is covered and $\pi(\text{Mon}(F_7))$ is known. There are other graphs with these features, e.g., Keevash in [18], Sec. 14, lists graphs, like “expanded triangle”, “3-book with 3 pages”, “4-book with 4 pages” and others. Using these and similar references, the reader may easily come up with other illustrations.

Another example of hereditary, multiplicative properties comes from vertex colorings. Let $C(k)$ be the family of all $r$-graphs $G$ with $\chi(G) \leq k$. Note first that $C(k)$ is hereditary and multiplicative property. This statement is more or less obvious, but it does not follow by forbidding covered subgraphs. The following proposition summarizes the principal facts about $C(k)$.

**Proposition 9.6** For all $k$ the class $C(k)$ is a hereditary and multiplicative property, and $\pi(C(k)) = (1 - k^{-r+1})$.

The following theorem has numerous applications. It is shaped after a result of Sidorenko [37].

**Theorem 9.7** If $\mathcal{P}$ is a hereditary, multiplicative property, then it is flat; that is to say, $\lambda^{(p)}(\mathcal{P}) = \pi(\mathcal{P})$ for every $p \geq 1$.

To illustrate the usability of Theorem 9.7 note that the 3-graph $F_7$ is covered, and, as determined in [13] and [19], $\pi(\text{Mon}(F_7)) = 3/4$, so we immediately get that if $p \geq 1$, then

$$\lambda^{(p)}(\text{Mon}(F_7)) = 3/2.$$ 

However, the great advantage of flat properties is that they allow tight upper bounds on $\lambda^{(p)}(G)$ and $|G|$ for every graph $G$ that belongs to a flat property, as stated below.

**Theorem 9.8** If $\mathcal{P}$ is a flat property, and $G \in \mathcal{P}_n$, then

$$|G| \leq \pi(\mathcal{P}) n^r/r!.$$ 

and for every $p \geq 1$,

$$\lambda^{(p)}(G) \leq \pi(\mathcal{P}) n^{r-r/p}.$$ 

Both inequalities (59) and (60) are tight.

Taking again the Fano plane as an example, we obtain the following tight inequality:

**Corollary 9.9** If $G$ is a 3-graph of order $n$, not containing the Fano plane, then for all $p \geq 1$,

$$\lambda^{(p)}(G) \leq \frac{3}{4} n^{3-3/p}.$$ 

This inequality is essentially equivalent to Corollary 3 in [20], albeit it is somewhat less precise. We believe however, that Theorem 9.8 shows clearly why such a result is possible at all.

With respect to chromatic number, an early result of Cvetković [2] states: if $G$ is a 2-graph of order $n$ and chromatic number $\chi$, then

$$\lambda(G) \leq (1 - 1/\chi) n.$$ 

This bound easily generalizes for hypergraphs.
Corollary 9.10 Let $G$ be an $r$-graph of order $n$ and let $p \geq 1$. If $\chi(G) = k$, then
\[
\lambda^{(p)}(G) \leq (1 - k^{-r+1}) n^{r-r/p};
\]

Furthermore, recalling that complete graphs are the only covered 2-graphs, it becomes clear that the bound (60) is analogous to Wilf’s bound [40]: if $G$ is a 2-graph of order $n$ and clique number $\omega$, then
\[
\lambda(G) \leq (1 - 1/\omega) n.
\]
This has been improved in [26], namely: if $G$ is a 2-graph with $m$ edges and clique number $\omega$, then
\[
\lambda(G) \leq \sqrt{2(1 - 1/\omega)} m.
\]
It turns out that the proof of (63) generalizes to hypergraphs, giving the following theorem, which strengthens (60) exactly as (63) strengthens (62).

Theorem 9.11 If $\mathcal{P}$ is a flat property, and $G \in \mathcal{P}$, then
\[
\lambda^{(p)}(G) \leq \pi(G)^{1/p} (r! |G|)^{1-1/p}.
\]

Let us emphasize the peculiar fact that the bound (64) does not depend on the order of $G$, but it is asymptotically tight in many cases. In particular, for 3-graphs with no $F_7$ we obtain the following tight bound:

Corollary 9.12 If $G \in \mathcal{G}^r$ and $G$ does not contain the Fano plane, then
\[
\lambda^{(p)}(G) \leq 3 \cdot 2^{1-3/p} |G|^{1-1/p}.
\]

There are flat properties $\mathcal{P}$ of 2-graphs that are not multiplicative, e.g., let $\mathcal{P} = Her(C_4)$, i.e., $\mathcal{P}$ is the class of all graphs with no induced 4-cycle. Trivially, all complete graphs belong to $\mathcal{P}$ and so
\[
\lambda^{(1)}(\mathcal{P}) = \pi(G) = 1.
\]
However, obviously $\mathcal{P}$ is not multiplicative, as $C_4 = K_2(2,2)$ and $\pi(Her(C_4)) = 0$. As a consequence of this example, we come up with the following sufficient condition for flat properties.

Theorem 9.13 If $\mathcal{F}$ is a set of $r$-graphs each of which is a blow-up of a covered graph, then $Her(\mathcal{F})$ is flat.

Apparently Theorem 9.13 greatly extends the range of flat properties, however further work is needed to determine the limits of its applicability.
9.5 Intersecting families

We are interested in the following question of classical flavor: let $n$ be sufficiently large and $G \in \mathcal{G}^r(n)$ be such that every two edges share at least $t$ vertices. How large $\lambda^{(p)}(G)$ can be?

Erdős, Ko and Rado have shown that if $G \in \mathcal{G}^r(n)$ satisfies the premise, then $|G| < \binom{n-t}{r-t}$ unless $G = S_{t,n}^r$. Moreover Erdős, Ko and Rado a stronger stability result: there is $c = c(r, t) > 0$ such that if $|G| > c\binom{n-t}{r-t-1}$, then $G \subset S_{t,n}^r$. This fact is enough to prove the following result, first shown by Keevash, Lenz and Mubayi [20] in a more complicated setup.

**Theorem 9.14** Let $G \in \mathcal{G}^r(n)$. If $n$ is sufficiently large and every two edges of $G$ share at least $t$ vertices, then $\lambda^{(p)}(G) < \lambda^{(p)}(S_{t,n}^r)$, unless $G = S_{t,n}^r$.

**Sketch of the proof** Assume that $\lambda^{(p)}(G) \geq \lambda^{(p)}(S_{t,n}^r)$. By the bounds (9) and (32) one has

$$r! |G| \geq \left(\lambda^{(p)}(G)\right)^{p/(p-1)} \geq \left(\lambda^{(p)}(S_{t,n}^r)\right)^{p/(p-1)}$$

$$= \left(\frac{(r)_t (r-t)^{(r-t)/p} (n-t)^{r-t}}{r^r p (n-t)^{(r-t)/p}}\right)^{p/(p-1)} \geq (n-t)^{r-t}$$

$$= \left(\frac{(r)_t (r-t)^{(r-t)/p}}{r^r p}\right)^{p/(p-1)} (n-t)^{r-t}.$$ 

Hence, if $n$ is sufficiently large, then $|G| > c\binom{n-t}{r-t-1}$ and by the stability result of Erdős, Ko and Rado $G \subset S_{t,n}^r$ and so $\lambda^{(p)}(G) \leq \lambda^{(p)}(S_{t,n}^r)$. Since the eigenvectors to $\lambda^{(p)}(S_{t,n}^r)$ have no zero entries, if $G \neq S_{t,n}^r$, then $\lambda^{(p)}(G) < \lambda^{(p)}(S_{t,n}^r)$.

\[\square\]

10 Random graphs

A random $r$-graph $G^r(n, p)$ is an $r$-graph of order $n$, in which any $r$-set $e \in V^{(r)}$ belongs to $E(G)$ with probability $p$, independently of other members of $V^{(r)}$. In this definition, $p$ is not necessarily constant and may depend on $n$.

In $G^r(n, p)$ the distribution of the set degrees is binomial, e.g., the distribution of the the $(r-1)$-set degrees is

$$\mathbb{P}(d(U) = k) = \binom{n-r+1}{k} p^k (1-p)^{n-r+1-k}$$

where $U \in V^{(r-1)}$. This fact, together with inequality (6), Theorem 7.11 and Proposition 6.3, leads to the following estimate.

**Theorem 10.1** If $0 < p < 1$ is fixed and $q > 1$, then almost surely,

$$\lambda^{(q)}(G^r(n, p)) = r! p \left(\frac{n}{r}\right)^{n^r/q} = (p + o(1)) n^{r-r/q}$$

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Fro $q = 2$ Friedman and Wigderson in [12] proved a stronger statement requiring only that $p = \Omega(\log n/n)$. In fact, Theorem 10.2 can also be strengthened for variable $p$, but we leave this for future exploration.

Clearly, for odd $r$ we have $\lambda_{\min}^{(p)}(G^r(n,p)) = -\lambda^{(q)}(G^r(n,p))$, but for even $r$ the picture is completely different.

**Theorem 10.2** If $r$ is odd, $0 < p < 1$ is fixed and $q > 1$, then almost surely,

$$\left| \lambda_{\min}^{(q)}(G^r(n,p)) \right| \leq n^{(r+1)/2-r/q+o(1)}$$

For $r = 2$ and $q \geq 2$, Theorem 10.2 follows from well-known results but all other cases require an involved new proof which will be given elsewhere.

Let us note that the parameter “second largest eigenvalue”, defined by Friedman and Wigderson in [12] has nothing to do with eigenvalues in the sense of the present paper.

**11 Concluding remarks**

In this section we offer a final discussion of the approach taken in this paper and outline some areas for further research. First, we showed above that the fundamental parameters $\lambda^{(p)}$ and $\lambda_{\min}^{(p)}$ comply well with most definitions of hypergraph and hypermatrix eigenvalues. This fact is important, because for odd rank the algebraic eigenvalues of Qi are different from the variational eigenvalues of Lim, and their blending is not in sight. For hypergraphs Cooper and Dutle [3] and Pearson and Zhang [31] adopted the algebraic approach of Qi, but unfortunately did not get much further than the largest eigenvalue. Indeed, the authors of [3] have have put significant effort in computing the mind-boggling algebraic spectra of some simple hypergraphs, but even for complete 4-graphs the work is still unfinished; see, Dutle [7] for some spectacular facts. One gets the impression that finding the algebraic eigenvalues of a hypergraph require involved computations, but very few of them are of any use to hypergraphs.

On the other hand, $\lambda^{(p)}$ and $\lambda_{\min}^{(p)}$ alone give rise to a mountain of hypergraph-relevant problems. As we saw, $\lambda^{(p)}$ blends seamlessly various spectral and nonspectral parameters, thus becoming a cornerstone of a general analytic theory of hypergraphs. So there is a good deal of work to be done here, before the definitions settle and one could pursue the study of the “second largest” and any other eigenvalues of hypergraphs.

There are several areas deserving intensive further exploration. First, this is a Perron-Frobenius theory for hypergraphs, with possible extension to cubical nonnegative hypermatrices. We state two problems motivated by Section 5.

**Problem 11.1** Given $1 < p < r$, characterize all $G \in \mathcal{W}^r$ such that $\lambda^{(p)}(G)$ has a unique positive eigenvector.

**Problem 11.2** Given $1 < p < r$, characterize all $G \in \mathcal{W}^r$ such that if $\lambda > 0$ and $[x_i] \in S_{p+1}^{n-1}$ satisfy the eigenvalues equation for $\lambda^{(p)}(G)$, then $\lambda = \lambda^{(p)}(G)$.
Second, \( \lambda(p) \) seems a powerful device for studying extremal problems for hypergraphs. So far the relation is one directional: solved extremal problems for edges are transformed into bounds for \( \lambda(p) \), in many cases stronger than the original results. However, it is likely that the other direction may work as well if closer relations between \( \lambda(p) \) and the graph structure are established.

Third, it is challenging to study the function \( f_G(p) := \lambda(p)(G) \) for any fixed \( G \in \mathcal{G}^r \). In particular, the following questions seem to be not too difficult.

**Question 11.3** Is the function \( f_G(p) \) differentiable for \( p > r \)? Is \( f_G(p) \) analytic for \( p > r \)?

Also, the example of non-differentiable \( f_G(p) \) given in subsection 2.2 suggests the following question.

**Question 11.4** For which graphs \( G \in \mathcal{G}^r \) the function \( f_G(p) \) is differentiable for every \( p > 1 \)?

Fourth, for 2-graphs relations of \( \lambda(2) \) and degrees have proved to be extremely useful for applications. However, for hypergraphs little is known in this vein at present, particularly regarding set degrees. Of the many possible questions we choose only the following, rather challenging, one.

**Question 11.5** If \( r \geq 3 \) and \( G \in \mathcal{G}^r(n) \) is it always true that \( \lambda(G) \geq \left( \frac{1}{n} \sum_{u \in V(G)} q^{r/(r-1)}(u) \right)^{1-1/r} \)?

Finally, we need more powerful algebraic and analytic methods to calculate and estimate \( \lambda(p) \) and \( \lambda_{\min}^{(p)} \). It is exasperating that \( \lambda(p) \) of the cycle \( C_n^r \) is not known for \( 1 < p < r \), and even the following natural question is difficult to answer.

**Question 11.6** For even \( r \geq 4 \) what is \( \lambda_{\min} \) of the complete \( r \)-graph of order \( n \)?

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### 12 Notation and some basic facts

#### 12.1 Vectors, norms and spheres

The entries of vectors considered in this paper are either real or complex numbers and their type is specified if necessary. A vector with entries \( x_1, \ldots, x_n \) will be denoted by \([x_i]\) and sometimes by \((x_1, \ldots, x_n)\) or by lower case bold letters. In particular, \(j_n\) stands for the all ones vector of size \( n \). A nonnegative (positive) vector is a real vector \( x \) with nonnegative (positive) entries, in writing \( x \geq 0 \) (\( x > 0 \)). Given a real number \( p \) such that \( p \geq 1 \), the \( l^p \) norm of a complex vector \([x_i]\) is defined as

\[
||[x_i]||_p := \left( |x_1|^p + \cdots + |x_n|^p \right)^{1/p}.
\]
Recall that $|x|_p + |y|_p \geq |x + y|_p$ and $|\beta x|_p = |\beta||x|_p$ for any vectors $x$ and $y$, and any number $\beta$. Given $p \geq 1$, the $(n-1)$-dimensional unit sphere $S^{n-1}_p$ in the $l^p$ norm is the set of all real $n$-vectors $[x_i]$ satisfying

$$|x_1|^p + \cdots + |x_n|^p = 1. \quad (65)$$

Clearly $S^{n-1}_p$ is a compact set. For convenience, we write $S^{n-1}_{p,+}$ for the set of the nonnegative vectors in $S^{n-1}_p$, which is compact as well.

If $p = 1$, then $S^{n-1}_1$ is not smooth, but $S^{n-1}_p$ is smooth for any $p > 1$. Indeed, note that

$$\frac{d}{dx}|x|^p = \begin{cases} p x^{p-1} & \text{if } x > 0 \\ -p (-x)^{p-1} & \text{if } x < 0 \end{cases},$$

and so

$$\frac{d}{dx}|x|^p = px|x|^{p-2}.$$

Therefore the partial derivatives of the left side of (65) exist and are continuous.

Moreover, note that if $p > 1$, the function $f(x) := x|x|^{p-2}$ is increasing in $x$ for all real $x$ and therefore is bijective.

### 12.2 Classical inequalities

Below we summarize some classical inequalities, but we shall start with a simple version of the Lagrange multiplier theorem, widely used for proving inequalities in general.

**Theorem 12.1** Let $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ be real functions of the real variables $x_1, \ldots, x_n$, and suppose that the partial derivatives of $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ exist and are continuous. If $c$ is a fixed number and $(x_1^*, \ldots, x_n^*)$ satisfies

$$f(x_1^*, \ldots, x_n^*) = \max_{g(x_1, \ldots, x_n) = c} f(x_1, \ldots, x_n),$$

then there exists a number $\mu$ such that

$$\frac{\partial f}{\partial x_i}(x_1^*, \ldots, x_n^*) = \mu \frac{\partial g}{\partial x_i}(x_1^*, \ldots, x_n^*)$$

for every $i = 1, \ldots, n$.

Note first that Theorem 12.1 gives only a necessary condition for a maximum; second, it remains true if max is replaced by min.

Classical inequalities are extremely useful for eigenvalues of hypergraphs, so we list a few of them for ease of use; see [15] for more details. Let us start with a generalization of the Cauchy-Schwarz inequality for more than two vectors: If $k \geq 2$, and $x_1 := [x_1^{(1)}], x_2 := [x_2^{(2)}], \ldots, x_k := [x_k^{(k)}]$ are nonnegative $n$-vectors, then

$$\sum_{i=1}^n \prod_{j=1}^k x_i^{(j)} \leq |x_1|^k \cdots |x_k|^k. \quad (66)$$
Equality holds if and only if all vectors are collinear to one of them.

Another generalization of the Cauchy-Schwarz inequality is Hölder’s inequality: Let \( x = [x_i] \) and \( y = [y_i] \) be real nonzero vectors. If the positive numbers \( s \) and \( t \) satisfy \( 1/s + 1/t = 1 \), then

\[
x_1y_1 + \cdots + x_ny_n \leq |x|_s |y|_t.
\]

If equality holds, then \((|x_1|^s, \ldots, |x_n|^s)\) and \((|y_1|^t, \ldots, |y_n|^t)\) are collinear.

Next, we give the Power Mean or the PM inequality: Let \( x_1, \ldots, x_k \) be nonnegative real numbers. If \( 0 < p < q \), then

\[
\left( \frac{x_1^p + \cdots + x_k^p}{k} \right)^{1/p} \leq \left( \frac{x_1^q + \cdots + x_k^q}{k} \right)^{1/q}
\]

with equality holding if and only if \( x_1 = \cdots = x_k \).

Let \( x = [x_i] \) be a real vector and \( S_r(x) \) be the \( r \)'th symmetric function of \( x_1, \ldots, x_n \). In particular,

\[
S_1(x) := x_1 + \cdots + x_n \quad S_n(x) := x_1 \cdots x_n.
\]

Here is Maclaurin’s inequality, which is very useful for hypergraphs: If \( x = [x_i] \) is a nonnegative real vector, then

\[
\frac{S_1(x)}{n} \geq \cdots \geq \left( \frac{S_r(x)}{\binom{n}{r}} \right)^{1/r} \geq \cdots \geq (S_n(x))^{1/n}.
\]

The cases of equality in (69) are somewhat tricky, so we formulate only the case which is actually needed in the paper: If \( x = [x_i] \) is a nonnegative vector and

\[
\frac{S_1(x)}{n} = \left( \frac{S_r(x)}{\binom{n}{r}} \right)^{1/r}
\]

for some \( 1 < r \leq n \), then \( x_1 = \cdots = x_n \).

The inequality

\[
\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n}
\]

is a particular case of (69), with equality if and only if \( x_1 = \cdots = x_n \). We shall refer to (71) as the arithmetic mean - geometric mean or the AM-GM inequality.

We finish this subsection with Bernoulli’s inequality: If \( a \) and \( x \) are real numbers satisfying \( a > 1 \), \( x > -1 \) and \( x \neq 0 \), then

\[
(1 + x)^a > 1 + ax.
\]
12.3 Polyforms

For reader’s convenience we give here some remarks about polyforms of $r$-graphs. First, if $G$ is an weighted $r$-graph, then $P_G (x)$ is homogenous of degree $r$, that is to say, for any real $s$, $P_G (sx) = s^r P_G (x)$ ; also, $P_G (x)$ is even for even $r$ and odd for odd $r$. 

Crucial to many calculations are the following observations used in the paper without explicit reference.

**Proposition 12.2** If $G \in \mathcal{W}^r (n)$ and $[x_i]$ is an $n$-vector, then for each $k = 1, \ldots , n$

$$\frac{\partial P_G ([x_i])}{\partial x_k} = r! \sum_{\{ k, i_1, \ldots , i_{r-1} \} \in E(G)} G (\{ k, i_1, \ldots , i_{r-1} \}) x_{i_1} \cdots x_{i_{r-1}}.$$ 

This implies also that

$$r P_G ([x_i]) = \sum_{k=1}^{n} x_k \frac{\partial P_G ([x_i])}{\partial x_k}.$$ 

If $G$ is a complete $r$-graph, then $P_G (x) = r! S_r (x)$. Likewise, we see the following proposition.

**Proposition 12.3** If $G \in \mathcal{G}^r$ is a complete $k$-partite graph, with vertex sets $V_1, \ldots , V_k$, then

$$P_G ([x_i]) = r! S_r (y),$$

where $y = (y_1, \ldots , y_k)$, is defined by

$$y_s := \sum_{i \in V_s} x_i, \quad s = 1, \ldots , k.$$ 

12.4 A mini-glossary of hypergraphs

The reader is referred to \[1\] for introductory material on hypergraphs. We reiterate that in this paper “graph” stands for “uniform hypergraph”; thus, “ordinary” graphs are referred to as “2-graphs”. Graphs extend naturally to weighted $r$-graphs, as explained in Section 1.4. 

We write $\mathcal{G}^r$ for the family of all $r$-graphs and $\mathcal{G}^r (n)$ for the family of all $r$-graphs of order $n$. Likewise, $\mathcal{W}^r$ stands for the family of all weighted $r$-graphs and $\mathcal{W}^r (n)$ for the family of all weighted $r$-graphs of order $n$. Given a weighted graph $G$, we write:

- $V (G)$ for the vertex set of $G$;
- $E (G)$ for the edge set of $G$;
- $|G|$ for $\sum \{ G (e) : e \in E (G) \}$ ;
- $G [U]$ for the graph induced by a set $U \subset V (G)$.

In the following definitions, if not specified otherwise, “graph” stands for weighted graph.

- **$k$-chromatic graph** - the vertices can be partitioned into $k$ sets so that each edge intersects at least two sets. An edge maximal $k$-chromatic graph $G$ is called **complete $k$-chromatic**; the complement of a complete $k$-chromatic $G$ is a union of $k$ disjoint complete graphs;
chromatic number of a graph $G$, in writing $\chi (G)$, is the smallest $k$ for which $G$ is $k$-chromatic. Using colors, $\chi (G)$ is the smallest number of colors needed to color the vertices of $G$ so that no edge is monochromatic;

complement of a graph - the complement of a graph $G \in \mathcal{G}$ is the graph $\overline{G} \in \mathcal{G}$ with $V (\overline{G}) = V (G)$ and $E (\overline{G}) = (V (G))^{(r)} \setminus E (G)$;

complete graph - a graph having all possible edges; $K^r_n$ stands for the complete $r$-graph of order $n$;

connected graph - for any partition of the vertices into two sets, there is an edge that intersects both sets;

degree - given a graph $G$ and $u \in V (G)$, the degree of $u$ is $d (u) = \sum \{G (e) : e \in E (G) \text{ and } u \in e\}$; $\delta (G)$ and $\Delta (G)$ denote the minimum and maximum vertex degrees of $G$; more generally, if $U \subset V (G)$, the degree of $U$ is $d (U) = \sum \{G (e) : e \in E (G) \text{ and } U \subset e\}$;

$\beta$-star with vertex $v$ - a set of edges such that the intersection of every two edges is $v$;

$\beta$-degree of a vertex - given a graph $G$ and $v \in V (G)$, the $\beta$-degree $d^\beta (v)$ of $v$ is the maximum size of a $\beta$-star with vertex $v$;

$\Delta^\beta (G)$, $\delta^\beta (G)$ - the maximum and the minimum $\beta$-degrees of the vertices of $G$;

graph property - a family of graphs closed under isomorphisms; a property closed under taking subgraphs is called monotone; a property closed under taking induced subgraphs is called hereditary;

isolated vertex - a vertex not contained in any edge;

$k$-linear graph - a graph $G$ is $k$-linear if every two edges of $G$ share at most $k$ vertices; a $1$-linear graph is called linear;

order of a graph - the number of its vertices;

$k$-partite graph - a graph whose vertices can be partitioned into $k$ sets so that no edge has two vertices from the same set. An edge maximal $k$-partite graph is called complete $k$-partite;

rank of a graph - the cardinality of its edges; e.g., $r$-graphs have rank $r$;

regular graph - each vertex has the same degree;

$k$-set regular graph - each set of $k$ vertices has the same degree;

$k$-section of a graph $G$ is the $k$-graph $G (k) \in \mathcal{G}_k$ with $V (G (k)) = V (G)$ and $E (G (k))$ is the set of all $k$-subsets of edges of $G$;

size of a graph $G$ - the number $|G| = \sum \{G (e) : e \in E (G)\}$;

Steiner $(k, r, n)$-system is a graph in $\mathcal{G}^r (n)$ such that any set of $k$ vertices is contained in exactly one edge; a Steiner triple system is a Steiner $(2, 3, n)$-system;

subgraph - if $H \in \mathcal{W}$ and $G \in \mathcal{W}$, $H$ is a subgraph of $G$, if $V (H) \subset V (G)$, and $e \in E (H)$ implies that $H (e) = G (e)$; a subgraph $H$ of $G$ is called induced if $e \in E (G)$ and $e \subset V (H)$ implies that $H (e) = G (e)$;

$k$-tight graph - a graph $G \in \mathcal{W}$ is $k$-tight if $E (G) \neq \emptyset$ and for any proper $U \subset V (G)$ containing edges, there is an edge $e$ such that $k \leq |e \cap U| \leq r - 1$; a graph is 1-tight if and only if it is connected;

transversal of a graph - a set of vertices intersecting each edge; odd (even) transversal - a set of vertices intersecting each edge in an odd (even) number of vertices; even transversals may have empty intersections with edges;
union of graphs - if $G \in \mathcal{G}$ and $H \in \mathcal{G}$, their union $G \cup H$ is an $F \in \mathcal{G}$ defined by $V(F) = V(G) \cup V(H)$, and $E(F) = E(G) \cup E(H)$. In particular, $tG$ denotes the union of $t$ vertex disjoint copies of $G$.

13 List of references

References

[1] C. Berge, Hypergraphs, combinatorics of finite sets, North-Holland, 1989.

[2] D. Cvetković, Chromatic number and the spectrum of a graph, *Publ. Inst. Math. (Beograd)* **14** (28) (1972), 25-38.

[3] J. Cooper and A. Dutle, Spectra of Hypergraphs, *Linear Algebra Appl.* **436** (2012) 3268-3292.

[4] D. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980, 368 pp.

[5] L. Collatz and U. Sinogowitz, Spektren endlicher Grafen, *Abh. Math. Sem. Univ. Hamburg* **21** (1957) 63-77.

[6] K.C. Chang, K. Pearson, and T. Zhang, Perron–Frobenius theorem for nonnegative tensors, *Commun. Math. Sci.* **6** (2008), 507–520.

[7] A. Dutle, Computational results for spectra of hypergraphs, preprint at 
http://www.math.sc.edu/~dutle/spectraresults.pdf

[8] C. Edwards and C. Elphick, Lower bounds for the clique and the chromatic number of a graph, *Discrete Appl. Math.* **5** (1983) 51-64.

[9] P. Erdős, Extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964),183-190.

[10] S. Friedland, S. Gaubert and L. Han, Perron–Frobenius theorem for nonnegative multilinear forms and extensions, *Linear Algebra Appl.* **438** (2013), 738-749.

[11] J. Friedman, Some graphs with small second eigenvalue, *Combinatorica* **15** (1995), 31-42.

[12] J. Friedman and A. Wigderson, On the second eigenvalue of hypergraphs, *Combinatorica* **15** (1995) 43-65.

[13] Z. Füredi and M. Simonovits, Triple systems not containing a Fano configuration, *Combin. Probab. Comput.* **14** (2005), 467-484.

[14] W.T. Gowers, Hypergraph regularity and the multidimensional Szemerédi theorem, *Ann. of Math.* **166** (2007), 897-946.
[15] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities, 2nd edition*, Cambridge University Press, 1988.

[16] M. Hofmeister, Spectral radius and degree sequence, Math. Nachr. 139(1988), 37-44.

[17] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[18] P. Keevash, Hypergraph Turán problems, *Surveys in Combinatorics*, Cambridge University Press, 2011, 83–140.

[19] P. Keevash and B. Sudakov, The Turán number of the Fano plane, *Combinatorica* 25 (2005), 561-574.

[20] P. Keevash, J. Lenz, and D. Mubayi, Spectral extremal problems for hypergraphs, preprint available at [arXiv:1304.0050](https://arxiv.org/abs/1304.0050).

[21] G. Katona, T. Nemetz and M. Simonovits, On a problem of Turán in the theory of graphs, *Mat. Lapok* 15 (1964), 228–238.

[22] J. Lenz, and D. Mubayi, Eigenvalues and quasirandom hypergraphs, preprint available at [arXiv:1208.4863v3](https://arxiv.org/abs/1208.4863v3).

[23] L.H. Lim, Singular values and eigenvalues of hypermatrices: a variational approach, *in Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP ’05)* 1 (2005), pp. 129–132.

[24] L. Lusternik and L. Schnirelman, Topological methods in variational problems (in Russian), *Inst. Mat. Mech., Moscow State Univ.*, 1930.

[25] B. Nagle, V. Rödl and M. Schacht, The counting lemma for regular k-uniform hypergraphs, *Random Structures Algorithms* 28 (2006), 113-179.

[26] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph. *Combin. Probab. Comput.* 11 (2002), 179–189.

[27] V. Nikiforov, A spectral condition for odd cycles, *Linear Algebra Appl.* 428 (2008), 1492-1498.

[28] V. Nikiforov, An analytic theory of extremal hypergraph problems preprint available at [arXiv:1305.1073](https://arxiv.org/abs/1305.1073).

[29] V. Nikiforov, Some extremal problems for hereditary properties of graphs, preprint available at [arXiv:1305.1072v1](https://arxiv.org/abs/1305.1072v1).

[30] E. Nosal, Eigenvalues of Graphs, Master’s thesis, University of Calgary, 1970.

[31] K. Pearson and T. Zhang - On spectral hypergraph theory of the adjacency tensor, to appear in *Graphs and Combinatorics*.

[32] L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symbolic Comput.* 40 (2005) 1302–1324.
[33] L. Qi, Rank and eigenvalues of a supersymmetric tensor, a multivariate homogeneous polyno-
momial and an algebraic surface defined by them, J. Symbolic Comput. 41 (2006) 1309–1327.

[34] B. Papendieck and P. Recht, On maximal entries in the principal eigenvector of graphs, Linear
Algebra and Appl. 310 (2000), 129–138.

[35] V. Rödl and M. Schacht, Complete partite subgraphs in dense hypergraphs, Random Structures
Algorithms, 41 (2012), 557-573.

[36] V. Rödl and J. Skokan, Regularity lemma for uniform hypergraphs, Random Structures Al-
gorithms 25 (2004), 1-42.

[37] A.F Sidorenko, On the maximal number of edges in a uniform hypergraph with no forbidden
subgraphs (in Russian), Mat. Zametki, 41 (1987), 433-455; (English translation in Math Notes
41 (1987), 247–259.)

[38] G. Szekeres and H.S. Wilf, An inequality for the chromatic number of a graph, J. Combin.
Theory Ser. B 4 (1968), 1-3.

[39] H.S. Wilf, The eigenvalues of a graph and its chromatic number, J. London Math. Soc. 42
(1967), 330-332.

[40] H.S. Wilf, Spectral bounds for the clique and independence numbers of graphs, J. Combin.
Theory Ser. B 40 (1986), 113-117.