Hypergeometric SLE with $\kappa = 8$:
Convergence of UST and LERW in Topological Rectangles

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Abstract

We consider uniform spanning tree (UST) in topological rectangles with alternating boundary conditions. The Peano curves associated to the UST converge weakly to hypergeometric SLE$_8$, denoted by hSLE$_8$. From the convergence result, we obtain the continuity and reversibility of hSLE$_8$ as well as an interesting connection between SLE$_8$ and hSLE$_8$. The loop-erased random walk (LERW) branch in the UST converges weakly to SLE$_2(-1,-1;-1,-1)$. We also obtain the limiting joint distribution of the two end points of the LERW branch.

Keywords: uniform spanning tree, loop-erased random walk, Schramm Loewner evolution.

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1 Introduction

In [Sch00], O. Schramm introduced a random process—Schramm Loewner Evolution (SLE)—as a candidate for the scaling limit of interfaces in two-dimensional critical lattice models. The setup is as follows. We consider a bounded simply connected domain $\Omega \subseteq \mathbb{C}$ such that $\partial \Omega$ is locally connected. Let $\phi$ be any conformal map from the unit disk $U$ onto $\Omega$. As $\partial \Omega$ is locally connected, the conformal map $\phi$ can be extended continuously to $U$ and $\partial \Omega$ is a curve (see [Pom92, Theorem 2.1]). A Dobrushin domain $(\Omega; x, y)$ is a bounded simply connected domain $\Omega$ with two boundary points $x, y$ such that $\partial \Omega$ is locally connected. We denote by $(xy)$ the boundary arc going from $x$ to $y$ in counterclockwise order. Suppose that $(\Omega_\delta; x_\delta, y_\delta)$ is an approximation of $(\Omega; x, y)$ on $\delta \mathbb{Z}^2$. Consider a critical lattice model on $\Omega_\delta$ with Dobrushin boundary conditions, for instance, Ising model, percolation, uniform spanning tree etc. In these examples, there is an interface in $\Omega_\delta$ connecting $x_\delta$ to $y_\delta$. It is conjectured that such interface has a conformally invariant scaling limit which can be identified by $\text{SLE}_\kappa$ where the parameter $\kappa$ varies for different models. Since the introduction of SLE, there are several models for which the conjecture is proved: the Peano curve in uniform spanning tree converges to $\text{SLE}_8$ (with further assumption that $\partial \Omega$ is $C^1$ and simple, see Theorem 4.1) and the loop-erased random walk converges to $\text{SLE}_2$ [LSW04], the interface in percolation converges to $\text{SLE}_6$ [Smi01], the level line of discrete Gaussian free field converges to $\text{SLE}_4$ [SS09], the interface in Ising model converges to $\text{SLE}_3$ and the interface in FK-Ising model converges to $\text{SLE}_{16/3}$ [CDCH14]. The proof requires two inputs: 1st. tightness of interfaces; 2nd. discrete martingale observable. With the tightness, there are always subsequential limits of interfaces; and then one uses the observable to show that all the subsequential limits are the same and also identify the unique limit.

Dobrushin boundary conditions are the simplest boundary condition. It is then natural to consider critical lattice models with more complicated boundary conditions. One possibility is to consider the model in polygons with alternating boundary conditions. In general, a (topological) polygon $(\Omega; x_1, \ldots, x_p)$ is a bounded simply connected domain $\Omega$ with distinct boundary points $x_1, \ldots, x_p$ such that $\partial \Omega$ is locally connected and $x_1, \ldots, x_p$ lies on $\partial \Omega$ in counterclockwise order. In this article, we focus on topological rectangles $(\Omega; a, b, c, d)$, i.e. a polygon with four marked points on the boundary. We call it a quad. Suppose that $(\Omega_\delta; a_\delta, b_\delta, c_\delta, d_\delta)$ is an approximation of $(\Omega; a, b, c, d)$ on $\delta \mathbb{Z}^2$. Consider a critical lattice model on $\Omega_\delta$ with alternating boundary conditions, it turns out that the scaling limit of interfaces in this case becomes hypergeometric SLE, denoted by $\text{hSLE}$, which is a variant of SLE process whose drift term involves a hypergeometric function. Note that variant of SLE whose driving function involves hypergeometric functions are considered earlier in [Zha10], [Qia18] and [Wu20], see Appendix D for their connection. We stick to the definition in [Wu20] where the author relates $\text{hSLE}$ to critical lattice model in quad. For instance, the interface in critical Ising model in quad converges to $\text{hSLE}_3$, see [Izy15] and [Wu20]; the interface in critical FK-Ising model in quad converges to $\text{hSLE}_{16/3}$, see [KS18] and [BPW21].
In this article, we focus on uniform spanning tree in quad with alternating boundary conditions. Not surprisingly, the associated Peano curve converges to $\text{hSLE}_8$ process (see precise setup in Theorem 1.4). The third author of this article studies $\text{hSLE}_\kappa$ process in $\text{Wu20}$ with $\kappa \in (0,8)$. She only treats the process with $\kappa \in (0,8)$ due to technical difficulty: the analysis there does not apply to $\kappa = 8$. The first goal of this article is to address $\text{hSLE}_\kappa$ process with $\kappa = 8$.

1.1 Hypergeometric SLE with $\kappa = 8$

Hypergeometric SLE is a two-parameter family of random curves in quad. The two parameters are $\kappa > 0$ and $\nu \in \mathbb{R}$, and we denote it by $\text{hSLE}_\kappa(\nu)$. An $\text{hSLE}_\kappa(\nu)$ in quad $(\Omega; a, b, c, d)$ is a process in $\Omega$ from $a$ to $d$ with marked points $(b, c)$. When describing scaling limit of interfaces in quad, the parameter $\kappa$ corresponds to different underlying model and the parameter $\nu$ corresponds to different boundary conditions on the boundary are $(bc)$. For instance, we consider critical Ising model in $(\Omega; a_\delta, b_\delta, c_\delta, d_\delta)$ with boundary conditions $\otimes$ on $(a_\delta b_\delta) \cup (c_\delta d_\delta)$ and $\otimes$ on $(d_\delta a_\delta)$ and $\xi \in \{\otimes, \text{free}\}$ on $(b_\delta c_\delta)$ and denote by $\eta_\delta$ the interface from $a_\delta$ to $d_\delta$. Then the scaling limit of $\eta_\delta$ has the law of $\text{hSLE}_3(\nu = -3/2)$ if $\xi = \otimes$ and has the law of $\text{hSLE}_3(\nu = 3/2)$ if $\xi = \text{free}$. See $\text{Wu20}$, Proposition 1.6 or $\text{Izy15}$. $\text{hSLE}_\kappa(\nu)$ is also related to $\text{SLE}_\kappa(\rho)$ process: the time-reversal of $\text{SLE}_\kappa(\rho)$ is an $\text{hSLE}_\kappa(\rho - 2)$ process for $\rho \geq \kappa/2 - 2$. See $\text{Wu20}$, Theorem 1.1. The continuity and reversibility of $\text{hSLE}_\kappa(\nu)$ are addressed in $\text{Wu20}$ for $\kappa \in (0,8)$. Our first main result is about the continuity and reversibility of $\text{hSLE}_\kappa(\nu)$ with $\kappa = 8$.

**Theorem 1.1.** Fix $\nu \geq 0$ and $x_1 < x_2 < x_3 < x_4$. The process $\text{hSLE}_8(\nu)$ in the upper half-plane $\mathbb{H}$ from $x_1$ to $x_4$ with marked points $(x_2, x_3)$ is almost surely generated by a continuous curve denoted by $\eta$. Furthermore, the process $\eta$ enjoys reversibility: the time-reversal of $\eta$ has the law of $\text{hSLE}_8(\nu)$ in $\mathbb{H}$ from $x_4$ to $x_1$ with marked points $(x_3, x_2)$.

The process $\text{hSLE}_8(\nu)$ in general quad is defined via conformal image: For a general quad $(\Omega; a, b, c, d)$, let $\phi$ be any conformal map from $\Omega$ onto $\mathbb{H}$ such that $\phi(a) < \phi(b) < \phi(c) < \phi(d)$. We define $\text{hSLE}_8(\nu)$ in $\Omega$ from $a$ to $d$ with marked points $(b, c)$ to be $\phi^{-1}(\eta)$ where $\eta$ is an $\text{hSLE}_8(\nu)$ in $\mathbb{H}$ from $\phi(a)$ to $\phi(d)$ with marked points $(\phi(b), \phi(c))$.

In Section 2, we will give preliminaries on SLE; and in Section 3, we will give definition of $\text{hSLE}_\kappa(\nu)$. In fact, we will address $\text{hSLE}_\kappa(\nu)$ for $\kappa \geq 8$ in Section 3. As the case of $\kappa > 8$ is less relevant, we omit the corresponding conclusion in the introduction. In literature, the continuity of $\text{SLE}_\kappa$ is first proved in $\text{RS05}$ for $\kappa \neq 8$ and then proved in $\text{LSW04}$ for $\kappa = 8$, because the technical analysis in the continuum $\text{RS05}$ does not apply for $\kappa = 8$ whose continuity is proved using convergence of Peano curve in UST $\text{LSW04}$. We encounter a similar situation for hSLE as well. The continuity of $\text{hSLE}_\kappa(\nu)$ with $\kappa \in (0,8)$ is proved in $\text{Wu20}$ using continuity of $\text{SLE}_\kappa$ and $\text{SLE}_\kappa(\rho)$ and analysis in the continuum. However, such analysis does not apply to the case with $\kappa = 8$. The full continuity and reversibility results of $\text{hSLE}_8(\nu)$ process are proved using the convergence of Peano curves in UST. Note that, in the definition of $\text{hSLE}_8(\nu)$, the parameter $\nu$ may take values in $\mathbb{R}$, and the continuity and reversibility are believed to hold for all $\nu > -2$, but we are only able to prove them for $\nu \geq 0$. Only the case when $\nu = 0$ is relevant for UST in quad. We denote $\text{hSLE}_8(\nu)$ by $\text{hSLE}_8$ when $\nu = 0$, and the rest of the introduction will focus on $\text{hSLE}_8$.

First of all, we may find $\text{hSLE}_8$ inside SLE$_8$.

**Proposition 1.2.** Fix $x < y$ and suppose $\eta \sim \text{SLE}_8$ in $\mathbb{H}$ from $x$ to $\infty$. Let $T_y$ be the first time that $\eta$ swallows $y$, and define $\gamma := \partial(\eta[0, T_y]) \cap \overline{\mathbb{H}}$ (here we view $\eta[0, T_y]$ as a compact set) and we view $\gamma$ as a continuous simple curve starting from $y$ and terminating at some point in $(-\infty, x)$. Let $\tau$ be any stopping time for $\gamma$ before the terminating time. Then the conditional law of $(\eta(t), 0 \leq t \leq T_y)$ given $\gamma[0, \tau]$ is $\text{hSLE}_8$ in $\mathbb{H} \setminus \gamma[0, \tau]$ from $x$ to $y^-$ with marked points $(y^+, \infty)$ conditional that its first hitting point on $\gamma[0, \tau]$ is given by $\gamma(\tau)$.

The proof for Proposition 1.2 depends on the domain Markov property for UST which will be given in Section 5. Moreover, the calculation in Section 5 also gives the following consequence Proposition 1.3.
Recall that given a quad \((\Omega; a, b, c, d)\), there exists a unique \(K > 0\) and a unique conformal map from \(\Omega\) onto the rectangle \([0, 1] \times [0, iK]\) which sends \(a, b, c, d \in \partial \Omega\) to the four corners \(0, 1, 1 + iK, iK\) respectively. We call \(K\) the conformal modulus of the quad \((\Omega; a, b, c, d)\).

**Proposition 1.3.** Fix a quad \((\Omega; a, b, c, d)\). Let \(K > 0\) be the conformal modulus of the quad \((\Omega; a, b, c, d)\), and let \(f\) be the conformal map from \(\Omega\) onto the rectangle \((0, 1) \times (0, iK)\) which sends \((a, b, c, d)\) to \((0, 1, 1 + iK, iK)\). Suppose \(\eta \sim \text{hSLE}_8\) in \(\Omega\) from \(a\) to \(d\) with marked points \((b, c)\). Then we have

\[
P [z \notin \eta] = \text{Re} f(z), \quad \forall z \in \Omega.
\] (1.1)

We remark that we do not derive the probability in (1.1) from standard Itô’s calculus. The proof bases on the analysis from UST: we will first prove that (1.1) holds in the discrete setting and then obtain (1.1) by proving the convergence of the corresponding discrete harmonic functions with mixed boundary conditions.

### 1.2 Uniform spanning tree (UST)

Let us come back to uniform spanning tree. We start with definitions and notations. The square lattice \(\mathbb{Z}^2\) is the graph with vertex set \(V(\mathbb{Z}^2) := \{(m, n) : m, n \in \mathbb{Z}\}\) and edge set \(E(\mathbb{Z}^2)\) given by edges between nearest neighbors. This is our primal lattice. Its dual lattice is denoted by \((\mathbb{Z}^2)^*\). The medial lattice \((\mathbb{Z}^2)^\circ\) is the graph with centers of edges of \(\mathbb{Z}^2\) as vertex set and edges connecting nearest vertices. For a finite subgraph \(G = (V(G), E(G)) \subset \mathbb{Z}^2\), we denote by \(\partial G\) the inner boundary of \(G\): \(\partial G = \{x \in V(G) : \exists y \notin V(G) \text{ such that } \{x, y\} \in E(\mathbb{Z}^2)\}\). In this article, when we add the subscript or superscript \(\delta\), we mean scaling subgraphs of the lattices \(\mathbb{Z}^2, (\mathbb{Z}^2)^*, (\mathbb{Z}^2)^\circ\) by \(\delta\).

**Uniform spanning tree.** Suppose that \(G = (V(G), E(G))\) is a finite connected graph. A forest is a subgraph of \(G\) that has no cycles. A tree is a connected forest. A subgraph of \(G\) is spanning if it covers \(V(G)\). A uniform spanning tree on \(G\) is a probability measure on the set of all spanning trees of \(G\) in which every tree is chosen with equal probability. Given a disjoint sequence \((\alpha_k : 1 \leq k \leq N)\) of trees of \(G\), a spanning tree with \((\alpha_k : 1 \leq k \leq N)\) wired is a spanning tree \(T\) such that \(\alpha_k \subset T\) for \(1 \leq k \leq N\). A uniform spanning tree with \((\alpha_k : 1 \leq k \leq N)\) wired is a probability measure on the set of all spanning trees of \(G\) with \((\alpha_k : 1 \leq k \leq N)\) wired under which every tree is chosen with equal probability.

**Space of curves.** A path is defined by a continuous map from \([0, 1]\) to \(\mathbb{C}\). Let \(C\) be the space of unparameterized paths in \(\mathbb{C}\). Define the metric on \(C\) as follows:

\[
d(\gamma_1, \gamma_2) := \inf \sup_{t \in [0, 1]} |\hat{\gamma}_1(t) - \hat{\gamma}_2(t)|,
\] (1.2)

where the infimum is taken over all the choices of parameterizations \(\hat{\gamma}_1\) and \(\hat{\gamma}_2\) of \(\gamma_1\) and \(\gamma_2\). The metric space \((C, d)\) is complete and separable, see [KST16]. Let \(P\) be a family of probability measures on \(C\). We say \(P\) is tight if for any \(\epsilon > 0\), there exists a compact set \(K_\epsilon\) such that \(P[K_\epsilon] \geq 1 - \epsilon\) for any \(P \in P\). We say \(P\) is relatively compact if every sequence of elements in \(P\) has a weakly convergent subsequence. As the metric space is complete and separable, relative compactness is equivalent to tightness.

**Convergence of discrete polygons.** A sequence of discrete polygons \((\Omega_\delta; x_1^\delta, \ldots, x_p^\delta)\) on \(\delta\mathbb{Z}^2\) is said to converge to a polygon \((\Omega; x_1, \ldots, x_p)\) in the Carathéodory sense if there exist conformal maps \(\phi_\delta\) from \(\mathbb{U}\) onto \(\Omega_\delta\) and conformal map \(\phi\) from \(\mathbb{U}\) onto \(\Omega\) such that \(\phi_\delta \to \phi\) as \(\delta \to 0\) uniformly on compact subsets of \(\mathbb{U}\) and \(\phi_\delta^{-1}(x_j^\delta) \to \phi^{-1}(x_j)\) for \(1 \leq j \leq p\).

We will encounter another type of convergence of polygons: a sequence of discrete polygons \((\Omega_\delta; x_1^\delta, \ldots, x_p^\delta)\) on \(\delta\mathbb{Z}^2\) converge to a polygon \((\Omega; x_1, \ldots, x_p)\) in the following sense: there exists a constant \(C > 0\), such that

\[
d((x_i^\delta, x_{i+1}^\delta), (x_i x_{i+1})) \leq C\delta, \quad \text{for } i \in \{1, \ldots, p\},
\] (1.3)
where $d$ is the metric (1.2) and we use the convention that $x_{p+1} = x_1$. Note that such convergence implies the convergence in the Carathéodory sense. These two types of convergence apply to polygons in the medial lattice in a similar way.

Suppose a sequence of medial quads $(\Omega^\delta; a_\delta^e, b_\delta^e, c_\delta^e, d_\delta^e)$ on $\delta(\mathbb{Z}^2)^e$ converges to a quad $(\Omega; a, b, c, d)$ in the sense (1.3). See details in Section 4. Let $\Omega_\delta \subset \delta\mathbb{Z}^2$ be the corresponding graph on the primal lattice. We consider uniform spanning tree (UST) on $\Omega_\delta$ with alternating boundary conditions: $(a_\delta b_\delta)$ is wired and $(c_\delta d_\delta)$ is wired (but $(a_\delta b_\delta)$ and $(c_\delta d_\delta)$ are not wired together). Let $T_\delta$ be the UST on $\Omega_\delta$ with such alternating boundary conditions. Then there exists a triple of curves $((\eta^L_\delta; \gamma^M_\delta; \eta^R_\delta))$ such that $\eta^L_\delta$ runs along the tree $T_\delta$ from $a_\delta$ to $d_\delta$, and $\gamma^M_\delta$ is the unique simple path in $T_\delta$ connecting $(a_\delta b_\delta)$ to $(c_\delta d_\delta)$, and $\eta^R_\delta$ runs along the tree $T_\delta$ from $b_\delta$ to $c_\delta$, see Figure 1.1 and see detail in Section 4.2. We have the following convergence of the triple $(\eta^L_\delta; \gamma^M_\delta; \eta^R_\delta)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The solid edges in black are wired boundary arcs $(ab)$ and $(cd)$, the solid edges in red are dual-wired boundary arcs $(b^*c^*)$ and $(d^*a^*)$. The thin edges are in the tree $T$. The solid edges in green are in the branch $\gamma^M$ in $T$. This branch intersects $(ab)$ at $X^M$ and intersects $(cd)$ at $Y^M$. The two curves in orange are the Peano curves $\eta^L$ (from $a$ to $d$) and $\eta^R$ (from $b$ to $c$).}
\end{figure}

**Theorem 1.4.** Fix a quad $(\Omega; a, b, c, d)$ such that $\partial\Omega$ is $C^1$ and simple. Suppose that a sequence of medial quads $(\Omega^\delta; a_\delta^e, b_\delta^e, c_\delta^e, d_\delta^e)$ converges to $(\Omega; a, b, c, d)$ in the sense (1.3). Consider the UST in $(\Omega_\delta; a_\delta, b_\delta, c_\delta, d_\delta)$ with alternating boundary conditions and consider the triple of curves $((\eta^L_\delta; \gamma^M_\delta; \eta^R_\delta))$ as described above. Then the triple $((\eta^L_\delta; \gamma^M_\delta; \eta^R_\delta))$ converges weakly to a triple of continuous curves $((\eta^L; \gamma^M; \eta^R))$ in metric (1.2). The law of $((\eta^L; \gamma^M; \eta^R))$ is characterized by the following properties: the marginal law of $\eta^L$ is hSLE$_8$ in $\Omega$ from $a$ to $d$ with marked points $(b, c)$; given $\eta^L$, the conditional law of $\eta^R$ is SLE$_8$ in $\Omega \setminus \eta^L$ from $b$ to $c$; and $\gamma^M = \eta^L \cap \eta^R$. Furthermore, given $\gamma^M$, denote by $\Omega^L$ and $\Omega^R$ the two connected components of $\Omega \setminus \gamma^M$ such that $\Omega^L$ has $a, d$ on the boundary and $\Omega^R$ has $b, c$ on the boundary, then the conditional law of $\eta^L$ is SLE$_8$ in $\Omega^L$ from $a$ to $d$ and the conditional law of $\eta^R$ is SLE$_8$ in $\Omega^R$ from $b$ to $c$, and $\eta^L$ and $\eta^R$ are conditionally independent given $\gamma^M$.

In Section 4 we prove the convergence of the Peano curve and complete the proof of Theorem 1.4. The work [Dub06] predicts the limiting distribution of a Peano curve in UST in general polygon with a special alternating boundary condition. However, the proof there lacks the complete construction of discrete observable and the analysis of the corresponding limit of the observable. Our proof in Section 4 focuses
on the Peano curve in UST in quad and follows the standard strategy: we first derive the tightness of the Peano curves and construct a discrete martingale observable, and then identify the subsequential limits through the observable. This part is a generalization of the work in [Sch00] and [LSW04], and we proceed following a simplification advocated by Smirnov, who provided a different observable from the one in [LSW04]. Smirnov’s observable is more suitable for the setup in quad. We remark that the observable here does not have an explicit closed form and the identification of the driving process is a non-trivial step, see Lemma 1.12. As a byproduct, we obtain the continuity and reversibility of hSLE_8 and complete the proof of Theorem 1.1.

We emphasize that the simple and C^1 regularity on \( \partial \Omega \) in the assumption of Theorem 1.4 is crucial in the proof of the tightness. We derive the tightness following the argument in [Sch00, Theorem 11.1], where the simple and C^1 regularity is assumed. See Appendix [C]

1.3 Loop-erased random walk (LERW)

From Theorem 1.4, we see that the Peano curve \( \eta_8^8 \) converges weakly to hSLE_8 and the limit of \( \gamma_8^M \) is part of the boundary of hSLE_8. In the following theorem, we provide an explicit characterization of the limiting distribution of \( \gamma_8^M \).

**Theorem 1.5.** Fix a quad \( (\Omega; a, b, c, d) \) such that \( \partial \Omega \) is C^1 and simple. Let \( K > 0 \) be the conformal modulus of the quad \( (\Omega; a, b, c, d) \), and let \( f \) be the conformal map from \( \Omega \) onto the rectangle \((0, 1) \times (0, iK)\) which sends \((a, b, c, d)\) to \((0, 1, 1 + iK, iK)\). Assume the same setup as in Theorem 1.4. Then the law of \( \gamma_8^M \) converges weakly to a continuous curve \( \gamma^M \) whose law is characterized by the following properties.

Denote by \( X^M = \gamma^M \cap (ab) \) and by \( Y^M = \gamma^M \cap (cd) \).

1. The law of the point \( f(X^M) \) is uniform in \((0, 1)\).

2. Given \( X^M \), the conditional law of \( \gamma^M \) is an SLE_2(−1, −1; −1, −1) in \( \Omega \) from \( X^M \) to \( (cd) \) with force points \((d, a; b, c)\) stopped at the first hitting time of \((cd)\).

Furthermore, denote by \( x^M = f(X^M) \) and \( y^M = \text{Ref}(Y^M) \), the joint density of \((x^M, y^M)\) is given by

\[
\rho_K(x, y) = \frac{\pi}{4K} \sum_{n \in Z} \left( \frac{1}{\cosh^2 \left( \frac{\pi}{2K} (x - y - 2n) \right)} + \frac{1}{\cosh^2 \left( \frac{\pi}{2K} (x + y - 2n) \right)} \right), \quad \forall x, y \in (0, 1).
\]

In Section 5, we work on the LERW branch and complete the proof of Theorem 1.5. The section has three parts.

- We first derive the joint distribution of the pair \((X^M, Y^M)\) in Section 5.1. We derive the formula (1.4) through discrete observable. The analysis on discrete harmonic function from [CW21] plays an important role.

- We then derive the conditional law of \( \gamma^M \) given \( X^M \) in Section 5.2. In fact, this part of the conclusion is already solved in [Zha08c] in an implicit form with more generality. The derivation follows the standard strategy: showing the tightness and constructing discrete martingale observable. We provide details of the proof in a self-contained way for our particular setting and derive the explicit answer in Section 5.2.

- In Section 5.3, we use the domain Markov property of the UST and complete the proof of Proposition 1.2. We also remark that the proof for the law of \( \gamma^M \) in Theorem 1.5 also provides an alternative proof for the duality result of SLE_8, see Corollary 1.6. Such duality relation was previously proved in [Zha08a] and [MS16a] in the continuous setting for general \( \kappa \) which is substantially more involved. Our proof in Section 5.3 is specific for \( \kappa = 8 \) because we use the convergence of UST and LERW.\textsuperscript{1}

\textsuperscript{1}H.W. learned this observable from a master course delivered by Smirnov in 2015, but we are not able to identify a published reference.
Corollary 1.6. Fix \( x < y \) and suppose \( \eta \sim \text{SLE}_8 \) in \( \mathbb{H} \) from \( x \) to \( \infty \). Let \( T_y \) be the first time that \( \eta \) swallows \( y \), and define \( \gamma := \partial(\eta[0,T_y]) \cap \mathbb{H} \) (here we view \( \eta[0,T_y] \) as a compact set) and we view \( \gamma \) as a continuous simple curve starting from \( y \) and terminating at some point in \((-\infty,x)\). Then the law of \( \gamma \) is the same as \( \text{SLE}_2(-1,-1; -1,-1) \) in \( \mathbb{H} \) from \( y \) to \((-\infty,x)\) with force points \((x,y^-; y^+,\infty)\).

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2 Preliminaries on SLE

Notations. For \( z \in \mathbb{C} \) and \( r > 0 \), we denote by \( B(z,r) \) the ball with center \( z \) and radius \( r \) and by \( B(z,r)^c \) the complement of \( B(z,r) \) in \( \mathbb{C} \). In particular, we denote \( B(0,1) \) by \( \mathbb{U} \).

Loewner chain. An \( \mathbb{H} \)-hull is a compact subset \( K \) of \( \overline{\mathbb{H}} \) such that \( \mathbb{H} \setminus K \) is simply connected. By Riemann’s mapping theorem, there exists a unique conformal map \( g_K \) from \( \mathbb{H} \setminus K \) onto \( \mathbb{H} \) with normalization \( \lim_{z \to \infty} |g_K(z) - z| = 0 \), and we call \( a(K) := \lim_{z \to \infty} z(g_K(z) - z) \) the half-plane capacity of \( K \) seen from \( \infty \). Loewner chain is a collection of \( \mathbb{H} \)-hulls \((K_t, t \geq 0)\) associated to the family of conformal maps \((g_t, t \geq 0)\) which solves the following Loewner equation: for each \( z \in \mathbb{H} \),

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,
\]

where \((W_t, t \geq 0)\) is a one-dimensional continuous function which we call the driving function. For \( z \in \overline{\mathbb{U}} \), the swallowing time of \( z \) is defined to be \( \sup \{ t \geq 0 : \min_{z \in [0,t]} |g_t(z) - W_s| > 0 \} \). Let \( K_t \) be the closure of \( \{ z \in \mathbb{H} : T_z \leq t \} \). It turns out that \( g_t \) is the unique conformal map from \( \mathbb{H} \setminus K_t \) onto \( \mathbb{H} \) with normalization \( \lim_{z \to \infty} |g_t(z) - z| = 0 \). Since the half-plane capacity of \( K_t \) is \( \lim_{z \to \infty} z(g_t(z) - z) = 2t \), we say that the process \((K_t, t \geq 0)\) is parameterized by the half-plane capacity. We say that \((K_t, t \geq 0)\) can be generated by continuous curve \((\eta(t), t \geq 0)\) if, for any \( t \), the unbounded connected component of \( \mathbb{H} \setminus \eta[0,t] \) is the same as \( \mathbb{H} \setminus K_t \).

Schramm Loewner evolution. Schramm Loewner evolution \( \text{SLE}_\kappa \) is the random Loewner chain driven by \( W_t = \sqrt{\kappa}B_t \) where \( \kappa > 0 \) and \((B_t, t \geq 0)\) is one-dimensional Brownian motion starting from \( 0 \). \( \text{SLE}_\kappa \) process is almost surely generated by continuous curve \( \eta \). The continuity for \( \kappa \neq 8 \) is proved in [RS05], and the continuity for \( \kappa = 8 \) is proved in [LSW04]. Moreover, the curve \( \eta \) is almost surely transient: \( \lim_{t \to \infty} |\eta(t)| = \infty \). When \( \kappa \in (0,4] \), the curve is simple; when \( \kappa \in (4,8) \), the curve is self-touching; when \( \kappa \geq 8 \), the curve is space-filling.

In the above, \( \text{SLE}_\kappa \) is in \( \mathbb{H} \) from \( 0 \) to \( \infty \), we may define it in any Dobrushin domain \((\Omega; x, y)\) via conformal image: let \( \phi \) be any conformal map from \( \Omega \) onto \( \mathbb{H} \) such that \( \phi(0) = 0 \) and \( \phi(\infty) = \infty \). We define \( \text{SLE}_\kappa \) in \( \Omega \) from \( x \) to \( y \) to be \( \phi^{-1}(\eta) \) where \( \eta \) is an \( \text{SLE}_\kappa \) in \( \mathbb{H} \) from \( 0 \) to \( \infty \). When \( \kappa \in (0,8) \), \( \text{SLE}_\kappa \) enjoys reversibility: suppose \( \eta \) is an \( \text{SLE}_\kappa \) in \( \Omega \) from \( x \) to \( y \), the time-reversal of \( \eta \) has the same law as \( \text{SLE}_\kappa \) in \( \Omega \) from \( y \) to \( x \), proved in [Zha08b], [MS16b], [MS16c]. When \( \kappa > 8 \), the time-reversal of \( \text{SLE}_\kappa \) is no longer \( \text{SLE}_\kappa \), it becomes \( \text{SLE}_\kappa(\kappa/2-4; \kappa/2-4) \), see [MS17] Theorem 1.19.

\( \text{SLE}_\kappa(\rho) \) process. \( \text{SLE}_\kappa(\rho) \) process is a variant of \( \text{SLE}_\kappa \) where one keeps track of multiple marked points. Suppose \( y^L = (y^{L,1} < \cdots < y^{L,1}) \leq 0 \), \( y^R = (0 \leq y^{R,1} < y^{R,2} < \cdots < y^{R,r}) \) and \( \rho^L = (\rho^{L,1}, \ldots, \rho^{L,1}) \), \( \rho^R = (\rho^{R,1}, \ldots, \rho^{R,r}) \) with \( \rho^{L,i}, \rho^{R,i} \in \mathbb{R} \). An \( \text{SLE}_\kappa(\rho^L; \rho^R) \) process with force points \((y^L; y^R)\) is the
Loewner chain driven by $W_t$ which is the solution to the following system of SDEs:

$$
\begin{align*}
  \frac{dW_t}{\sqrt{\kappa}} &= dB_t + \sum_{i=1}^{l} \frac{\rho^{L,i} dt}{W_t - V^{L,i}_t} + \sum_{i=1}^{r} \frac{\rho^{R,i} dt}{W_t - V^{R,i}_t}, & W_0 = 0; \\
  \frac{dV^{L,i}_t}{V^{L,i}_t - W_t} &= \frac{2dt}{V^{L,i}_t - W_t}, & V^{L,i}_0 = y^{L,i}, & 1 \leq i \leq l; \\
  \frac{dV^{R,i}_t}{V^{R,i}_t - W_t} &= \frac{2dt}{V^{R,i}_t - W_t}, & V^{R,i}_0 = y^{R,i}, & 1 \leq i \leq r;
\end{align*}
$$

where $(B_t, t \geq 0)$ is one-dimensional Brownian motion starting from 0. We define the continuation threshold of $\text{SLE}_\kappa(\rho^L; \rho^R)$ to be the infimum of the time $t$ for which

$$
either \sum_{i: V^{L,i}_t = W_t} \rho^{L,i} \leq -2, or \sum_{i: V^{R,i}_t = W_t} \rho^{R,i} \leq -2.$$

$\text{SLE}_\kappa(\rho^L; \rho^R)$ process is well-defined up to the continuation threshold and it is almost surely generated by continuous curve up to and including the continuation threshold, see [MST16].

The law of $\text{SLE}_\kappa(\rho^L; \rho^R)$ is absolutely continuous with respect to $\text{SLE}_\kappa$, and we will give the Radon-Nikodym derivative below, see also [SW05]. To simplify the notation for the Radon-Nikodym derivative, we focus on $\text{SLE}_\kappa(\rho)$ process when all force points are located to the same side of the process. Consider $\text{SLE}_\kappa(\rho)$ with force points $y$ where $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ and $y = (0 \leq y_1 < \cdots < y_n)$. The law of $\text{SLE}_\kappa(\rho)$ with force points $y$ is absolutely continuous with respect to $\text{SLE}_\kappa$ up to the first time that $y_1$ is swallowed, and the Radon-Nikodym derivative is $M_t/M_0$ where

$$
M_t = \prod_{1 \leq i \leq n} \left( g_i(y_i) \rho_i (\rho_i + 4 - \kappa)/4(\kappa) (g_i(y_i) - W_t)^\rho_i/\kappa \right) \times \prod_{1 \leq i < j \leq n} (g_i(y_j) - g_i(y_i))^{\rho_i/\rho_j}. 
$$

$\text{SLE}_\kappa(\rho^L; \rho^R)$ process can be defined in general polygons. Suppose $(\Omega; y^{L,i}, \ldots, y^{L,1}, x, y^{R,1}, \ldots, y^{R,r}, y)$ is a polygon with $l+r+2$ marked points. Let $\phi$ be any conformal map from $\Omega$ onto $\mathbb{H}$ such that $\phi(x) = 0$ and $\phi(y) = \infty$. We define $\text{SLE}_\kappa(\rho^L; \rho^R)$ in $\Omega$ from $x$ to $y$ with force points $(y^{L,i}, \ldots, y^{L,1}, y^{R,1}, \ldots, y^{R,r})$ to be $\phi^{-1}(\eta)$ where $\eta$ is an $\text{SLE}_\kappa(\rho^L; \rho^R)$ in $\mathbb{H}$ from 0 to $\infty$ with force points $(\phi(y^{L,d}), \ldots, \phi(y^{L,1}); \phi(y^{R,1}), \ldots, \phi(y^{R,r}))$.

$\text{SLE}_2(-1, -1; -1, -1)$ process. Let us discuss $\text{SLE}_2(-1, -1; -1, -1)$ mentioned in Theorem 1.5. Suppose $(\Omega; d, a, x, b, c, y)$ is a polygon with six marked points and consider $\text{SLE}_2(-1, -1; -1, -1)$ in $\Omega$ from $x$ to $y$ with force points $(d, a; b, c)$. Note that the total of the force point weights is $-4$ which is $\kappa = 6$. Such process is target independent in the following sense: for distinct $y_1, y_2 \in (cd)$, let $\eta_i$ be the $\text{SLE}_2(-1, -1; -1, -1)$ in $\Omega$ from $x$ to $y_i$ with force points $(d, a; b, c)$, and let $T_i$ be the first time that $\eta_i$ hits $(cd)$ (in fact, $T_i$ is the continuation threshold of $\eta_i$ for $i = 1, 2$. Then the law of $(\eta_1(t), 0 \leq t \leq T_1)$ is the same as the law of $(\eta_2(t), 0 \leq t \leq T_2)$. See [SW05] for the target-independence for a general setup. As the law of $(\eta_i(t), 0 \leq t \leq T_i)$ does not depend on the location of the target point, we say that it is an $\text{SLE}_2(-1, -1; -1, -1)$ in $\Omega$ from $x$ to $(cd)$ with force points $(d, a; b, c)$.

3 Hypergeometric SLE with $\kappa \geq 8$

For $\kappa > 0$ and $\nu > (-4) \lor (\kappa/2 - 6)$, define the hypergeometric function (see Appendix A):

$$
F(z) := 2F_1 \left( \frac{2\nu + 4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{2\nu + 8}{\kappa}; z \right).
$$

Set

$$
h = \frac{6 - \kappa}{2\kappa}, a = \frac{\nu + 2}{\kappa}, b = \frac{(\nu + 2)(\nu + 6 - \kappa)}{4\kappa}.
$$
For $x_1 < x_2 < x_3 < x_4$, define partition function
\[
Z_{\kappa,\nu}(x_1, x_2, x_3, x_4) = (x_4 - x_1)^{-2\kappa}(x_3 - x_2)^{-2\kappa} z^\nu F(z), \quad \text{where} \quad z = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)}. \tag{3.3}
\]

The process $h\text{SLE}_\kappa(\nu)$ in $\mathbb{H}$ from $x_1$ to $x_4$ with marked points $(x_2, x_3)$ is the Loewner chain driven by $W_t$ which is the solution to the following SDEs:
\[
\begin{cases}
    dW_t = \sqrt{\kappa} dB_t + \kappa (\partial_t \log Z_{\kappa,\nu})(W_t, V_t^1, V_t^3, V_t^4) dt, \quad W_0 = x_1; \\
    dV_t^i = \frac{2dt}{V_t^i - V_t^r}, \quad V_0^i = x_i, \quad \text{for } i = 2, 3, 4;
\end{cases} \tag{3.4}
\]

where $(B_t, t \geq 0)$ is one-dimensional Brownian motion starting from 0. Combining (3.4) and (A.2), the law of $h\text{SLE}_\kappa(\nu)$ is the same as $\text{SLE}_\kappa$ in $\mathbb{H}$ from $x_1$ to $\infty$ weighted by the following local martingale:
\[
M_t = g_t(x_2)^h g_t(x_3)^h g_t(x_4)^h Z_{\kappa,\nu}(W_t, g_t(x_2), g_t(x_3), g_t(x_4)). \tag{3.5}
\]

It is clear that the solution to (3.4) is well-defined up to the swallowing time of $x_2$. We denote by $T_{x_3}$ the swallowing time of $x_3$. To fully understand solutions to (3.4), we will address the following two questions:

- Is there a unique solution (in law) to (3.4) up to and including $T_{x_3}$?
- Whether the Loewner chain is generated by a continuous curve up to and including $T_{x_3}$?

The answers to these questions are positive. The proof turns out to be very different for $\kappa \neq 8$ and for $\kappa = 8$. These questions are addressed in Chapter 20 for $\kappa \in (0, 8)$. A similar analysis applies to the case when $\kappa > 8$, see Section 3.1. The proof for $\kappa = 8$ uses analysis from UST and will be completed in Section 4.

In summary, for $\kappa \geq 8$, we will show that the process is well-defined up to $T_{x_3}$; moreover, it is generated by a continuous curve $\eta$ up to and including $T_{x_3}$. After $T_{x_3}$, we continue the process as a standard SLE$_\kappa$ from $\eta(T_{x_3})$ towards $x_4$ in the remaining domain. The reason for such choice comes from the observation in the discrete setup, see the second last paragraph in the proof of Theorem 4.2.

In the above, we have defined $h\text{SLE}$ in $\mathbb{H}$ and we may extend the definition to general quad via conformal image: For a quad $(\Omega; x_1, x_2, x_3, x_4)$, let $\phi$ be any conformal map from $\Omega$ onto $\mathbb{H}$ such that $\phi(x_1) < \phi(x_2) < \phi(x_3) < \phi(x_4)$. We define $h\text{SLE}_\kappa(\nu)$ in $\Omega$ from $x_1$ to $x_4$ with marked points $(x_2, x_3)$ to be $\phi^{-1}(\eta)$ where $\eta$ is an $h\text{SLE}_\kappa(\nu)$ in $\mathbb{H}$ from $\phi(x_1)$ to $\phi(x_4)$ with marked points $(\phi(x_3), \phi(x_4))$. Moreover, if the marked points $x_1, x_2, x_3, x_4$ lie on sufficiently regular boundary segments (e.g. $C^{1+\epsilon}$ for some $\epsilon > 0$), we may extend the partition function $Z_{\kappa,\nu}$ via conformal image:
\[
Z_{\kappa,\nu}(\Omega; x_1, x_2, x_3, x_4) = \phi'(x_1)^{-h} \phi'(x_2)^{-b} \phi'(x_3)^{-b} \phi'(x_4)^{-h} Z_{\kappa,\nu}(\phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4)). \tag{3.6}
\]

### 3.1 Continuity of $h\text{SLE}_\kappa(\nu)$ with $\kappa \geq 8$

To derive the continuity of $h\text{SLE}_\kappa(\nu)$, it is more convenient to work in $\mathbb{H}$ with $x_1 = 0$ and $x_4 = \infty$. Consider $h\text{SLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to $\infty$ with marked points $(x, y)$ where $0 < x < y$. In this case, the SDEs (3.4) becomes the following:
\[
\begin{cases}
    dW_t = \sqrt{\kappa} dB_t + \frac{(\nu+2)dt}{W_t - V_t^r} + \frac{\nu+2)dt}{W_t - V_t^r} - \kappa \frac{F'(Z_t)}{F(Z_t)} \left( \frac{1-Z_t}{V_t^r - W_t} \right) dt, \quad W_0 = 0; \\
    dV_t^x = \frac{2dt}{V_t^x - W_t}, \quad dV_t^y = \frac{2dt}{V_t^y - W_t}, \quad V_0^x = x, V_0^y = y; \quad \text{where} \quad Z_t = \frac{V_t^r - W_t}{V_t^r - V_t^r}.
\end{cases} \tag{3.7}
\]

We denote by $T_x$ the swallowing time of $x$ and by $T_y$ the swallowing time of $y$. The main result of this section is the continuity of the process up to and including $T_y$. 


From (3.7), it is clear that the Loewner chain is well-defined up to $T_x$. As in (3.5), the process has the same law as SLE$_\kappa$ in $\mathbb{H}$ from 0 to $\infty$ weighted by the following local martingale:

$$M_t = g_t'(x)g_t'(y)^b(g_t(y) - g_t(x))^{-2b}Z_t^\nu F(Z_t).$$

(3.8)

In particular, it is generated by continuous curve up to $T_x$. However, the continuity of the process around $T_x$ or $T_y$ can be problematic in general. The behavior of the process near $T_x$ or $T_y$ depends on the asymptotic of the hypergeometric function $F$ as $t \to T_x$. See Lemma A.1. We see that the asymptotic of $F$ is generated by a continuous curve up to and including Lemma 3.1.

**Lemma 3.1.** Fix $\kappa \geq 8, \nu > -2$ and $0 < x < y$. Suppose $\eta \sim \text{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to $\infty$ with marked points $(x, y)$. Then $\eta$ is generated by a continuous curve up to $T_y$.

**Proof.** We compare the law of $\eta$ with SLE$_\kappa(\nu + 2, \kappa - 6 - \nu)$ in $\mathbb{H}$ from 0 to $\infty$ with force points $(x, y)$. By (2.1) and (3.8), the Radon-Nikodym derivative is given by $R_t/R_0$ where

$$R_t = F(Z_t)(V_t^y - W_t)^{4/\kappa - 1}.$$

When $\kappa > 8$, we write

$$R_t = F(Z_t)(1 - Z_t)^{1 - 8/\kappa}(V_t^y - V_t^x)^{8/\kappa - 1}(V_t^y - W_t)^{-4/\kappa}.$$

By (A.6), the function $F(z)(1 - z)^{1 - 8/\kappa}$ is uniformly bounded for $z \in [0, 1]$. Define, for $n \geq 1$,

$$S_n = \inf\{t : V_t^y - V_t^x \leq 1/n\}.$$

Then $R_t$ is bounded up to $S_n$. Since SLE$_\kappa(\nu + 2, \kappa - 6 - \nu)$ is generated by a continuous curve, the process $\eta$ is generated by a continuous curve up to $S_n$. This holds for any $n$, thus $\eta$ is generated by a continuous curve up to $T_y = \lim_n S_n$.

When $\kappa = 8$, we write

$$R_t = \frac{F(Z_t)}{\log \frac{1}{1 - Z_t}} \left(\log \frac{1}{1 - Z_t}\right)(V_t^y - W_t)^{4/\kappa - 1}.$$

By (A.7), we know that $F(z)/\left(\log \frac{1}{1 - z}\right)$ is uniformly bounded for $z \in [0, 1]$. Define $S_n$ in the same way as before. Then $R_t$ is bounded up to $S_n$. Similarly, the process $\eta$ is continuous up to $T_y = \lim_n S_n$. \square

In Lemma 3.1, we obtain the continuity of hSLE$_\kappa(\nu)$ up to $T_y$ by showing that the process is absolutely continuous with respect to SLE$_\kappa(\nu + 2, \kappa - 6 - \nu)$. However, the absolute continuity is no longer true when the process approaches $T_y$. In the following, we will derive the continuity of the process as $t \to T_y$. From Lemma A.1, we see that the asymptotic of $F(z)$ as $z \to 1$ is very different between $\kappa > 8$ and $\kappa = 8$. We will treat the two cases separately: we prove the continuity of hSLE with $\kappa > 8$ in Lemma 3.2, and the continuity with $\kappa = 8$ in Corollary 4.13.

**Lemma 3.2.** Fix $\kappa > 8, \nu \geq \kappa/2 - 6$ and $0 < x < y$. Suppose $\eta \sim \text{hSLE}_\kappa(\nu)$ in $\mathbb{H}$ from 0 to $\infty$ with marked points $(x, y)$. Then $\eta$ is generated by a continuous curve up to and including $T_y$.

**Proof.** Since hSLE$_\kappa(\nu)$ is scaling invariant, we may assume $y = 1$ and $x \in (0, 1)$. We denote $T_y$ by $T$. In this lemma, we discuss the continuity of the process $(K_t, 0 \leq t \leq T)$ as $t \to T$. We need to zoom in around the point 1. To this end, we perform a standard change of coordinate and parameterize the process according to the capacity seen from the point 1. See [SW05, Theorem 3].

Set $\varphi(z) = z/(1 - z)$, this is the M"obius transformation of $\mathbb{H}$ that sends the triple $(0, 1, \infty)$ to $(0, \infty, -1)$. Denote by $\tilde{\varphi} = \varphi(x) = x/(1 - x) > 0$. Denote the image of $(K_t, 0 \leq t \leq T)$ under $\varphi$ by $(\tilde{K}_s, 0 \leq s \leq \tilde{T})$ where we parameterize this process by its capacity seen from $\infty$. Let $(\tilde{g}_s, s \geq 0)$ be the corresponding
family of conformal maps and \((\tilde{W}_s, s \geq 0)\) be the driving function. From (3.5) and (3.6), the law of \(\hat{K}_s\) is the same as \(\text{SLE}_\kappa\) in \(\mathbb{H}\) from 0 to \(\infty\) weighted by the following local martingale
\[
\tilde{M}_s = \tilde{g}_s(-1)^b \tilde{g}_s(\tilde{x}) \tilde{g}_s(-1)^{-2b} \tilde{Z}_s^a F(\tilde{Z}_s), \quad \text{where} \quad \tilde{Z}_s = \frac{\tilde{g}_s(\tilde{x}) - \tilde{W}_s}{\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1)}.
\]

Compare the law of \(\hat{K}\) with respect to \(\text{SLE}_\kappa(2; \nu + 2)\) in \(\mathbb{H}\) from 0 to \(\infty\) with force points \((-1; \tilde{x})\). The Radon-Nikodym derivative is given by \(\tilde{R}_s/\tilde{R}_0\) where
\[
\tilde{R}_s = (1 - \tilde{Z}_s)^{1 - 8/\kappa} F(\tilde{Z}_s)(\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1))^{1 - 8/\kappa - 2a}.
\]

When \(\kappa > 8\) and \(\nu \geq \kappa / 2 - 6 > -2\), the function \((1 - z)^{1 - 8/\kappa} F(z)\) is uniformly bounded on \(z \in [0, 1]\) due to (A.6). The process \(\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1)\) is increasing in \(s\), thus \(\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1) \geq 1/(1 - x)\). Since \(\nu \geq \kappa / 2 - 6\), the exponent of the term \(\tilde{g}_s(\tilde{x}) - \tilde{g}_s(-1)\) is \(1 - 8/\kappa - 2a \leq 0\). Therefore, \(\tilde{R}_s\) is bounded. This implies that the law of \(\hat{K}_s\) is absolutely continuous with respect to the law of \(\text{SLE}_\kappa(2; \nu + 2)\) up to and including the swallowing time of \(-1\). Hence \((\hat{K}_s, 0 \leq s \leq \hat{S})\) is generated by a continuous curve up to and including \(\hat{S}\). In particular, this implies that the original process \((K_t, 0 \leq t \leq T)\) is generated by a continuous curve up to and including \(T\).

To sum up the results in this section for \(\kappa > 8\), we have the following continuity of \(\text{hSLE}_\kappa(\nu)\).

**Proposition 3.3.** Fix \(\kappa > 8, \nu \geq \kappa / 2 - 6\) and \(x_1 < x_2 < x_3 < x_4\). The process \(\text{hSLE}_\kappa(\nu)\) in \(\mathbb{H}\) from \(x_1\) to \(x_4\) is almost surely generated by a continuous curve.

**Proof.** Lemma 3.2 gives the continuity of \(\text{hSLE}_\kappa(\nu)\) up to and including \(T_{x_3}\). After \(T_{x_3}\), we continue the process by standard \(\text{SLE}_\kappa\) in the remaining domain from \(\eta(T_{x_3})\) to \(x_4\). Thus, the process \(\text{hSLE}_\kappa(\nu)\) is continuous for all time. \(\square\)

### 3.2 Non-reversibility of \(\text{hSLE}_\kappa(\nu)\) with \(\kappa > 8\)

The time-reversal of \(\text{SLE}_\kappa\) with \(\kappa > 8\) was fully addressed in [MS17 Theorem 1.19]: consider \(\text{SLE}_\kappa(p_1; p_2)\) with force points next to the starting point for \(p_1, p_2 \in (-2, \kappa / 2 - 2)\), its time-reversal is an \(\text{SLE}_\kappa(\kappa / 2 - 4 - p_2; \kappa / 2 - 4 - p_1)\) process with force points next to the starting point. In particular, the time-reversal of \(\text{SLE}_\kappa\) is an \(\text{SLE}_\kappa(\kappa / 2 - 4; \kappa / 2 - 4)\) process. This indicates that the time-reversal of \(\text{hSLE}_\kappa(\nu)\) with \(\kappa > 8\) is a variant of \(\text{SLE}_\kappa\) where one has four extra marked points. In particular, the time-reversal is no longer in the family of \(\text{hSLE}\) which is a variant of \(\text{SLE}\) with two extra marked points, see Lemma 3.4. Therefore, it is only reasonable to talk about reversibility of \(\text{hSLE}_\kappa(\nu)\) with \(\kappa \leq 8\). The reversibility of \(\text{hSLE}_\kappa(\nu)\) with \(\kappa < 8\) was addressed in [Wu20 Section 3.3]. We will discuss the reversibility of \(\text{hSLE}_\kappa(\nu)\) with \(\kappa = 8\) in Section 3.3.

**Lemma 3.4.** When \(\kappa > 8\) and \(\nu \geq \kappa / 2 - 6\). The time-reversal of \(\text{hSLE}_\kappa(\nu)\) is not an \(\text{hSLE}_\kappa(\tilde{\nu})\) for any value of \(\tilde{\nu}\).

**Proof.** Fix \(x_1 < x_2 < x_3 < x_4\). Suppose \(\eta\) is an \(\text{hSLE}_\kappa(\nu)\) in \(\mathbb{H}\) from \(x_1\) to \(x_4\) and let \(\tilde{\eta}\) be its time-reversal. Let \(\gamma\) be an \(\text{hSLE}_\kappa(\tilde{\nu})\) in \(\mathbb{H}\) from \(x_4\) to \(x_1\) with marked points \((x_3, x_2)\). We will compare the laws of \(\tilde{\eta}\) and \(\gamma\) in small neighborhood of \(x_1\).

- In the construction of \(\eta\), we know that the process almost surely hits the interval \((x_3, x_4)\) and after the hitting time, we continue the process as a standard \(\text{SLE}_\kappa\) in the remaining domain. Therefore, the initial segment of \(\tilde{\eta}\) is the time-reversal of a standard \(\text{SLE}_\kappa\). By [MS17 Theorem 1.19], we know that, in small neighborhood of \(x_4\), the law of \(\tilde{\eta}\) and the law of \(\text{SLE}_\kappa(\kappa / 2 - 4; \kappa / 2 - 4)\) are absolutely continuous with respect to each other.

- From the definition of \(\text{hSLE}_\kappa(\tilde{\nu})\), we know that, in small neighborhood of \(x_4\), the law of \(\gamma\) and the law of \(\text{SLE}_\kappa\) are absolutely continuous with respect to each other.

Combining the above two observations, we see that \(\tilde{\eta}\) cannot have the same law as \(\gamma\), because \(\text{SLE}_\kappa(\kappa / 2 - 4; \kappa / 2 - 4)\) and \(\text{SLE}_\kappa\) are not absolutely continuous with respect to each other. \(\square\)
3.3 Continuity and reversibility of hSLE$_{\kappa}(\nu)$ with $\kappa = 8$

The continuity and reversibility of hSLE$_8$ will be given in Corollary 4.13 in Section 4.4. The proof there is based on the convergence of the Peano curve for UST. Assuming this is true, we are able to prove the continuity and reversibility of hSLE$_8(\nu)$ for $\nu \geq 0$.

Proof of Theorem 1.1. We may assume $x_1 = 0 < x_2 = x < x_3 = y < x_4 = \infty$. Suppose $\eta \sim$ hSLE$_8(\nu)$ in $\mathbb{H}$ from 0 to $\infty$ with marked points $(x, y)$. Recall that $\eta$ has the same law as SLE$_8$ in $\mathbb{H}$ from 0 to $\infty$ weighted by the following local martingale:

$$M_t = g'_t(x)^b g'_t(y)^b (g_t(y) - g_t(x))^{-2b} Z_t^a F(Z_t),$$

where

$$h = \frac{-1}{8}, \quad a = \frac{\nu + 2}{8}, \quad b = \frac{(\nu + 2)(\nu - 2)}{32}, \quad F(z) = 2F_1 \left( 2a, \frac{1}{2}; 2a + \frac{1}{2}; z \right).$$

Suppose $\gamma \sim$ hSLE$_8$ in $\mathbb{H}$ from 0 to $\infty$ with marked points $(x, y)$. Then $\gamma$ has the same law as SLE$_8$ in $\mathbb{H}$ from 0 to $\infty$ weighted by the following local martingale:

$$N_t = g'_t(x)^h g'_t(y)^h (g_t(y) - g_t(x))^{-2h} Z_t^{1/4} G(Z_t), \quad \text{where} \quad G(z) = 2F_1 \left( \frac{1}{2}, \frac{1}{2}, 1; z \right).$$

Combining (3.9) and (3.10), we see that the law of $\gamma$ is the same as the law of $\eta$ weighted by the following local martingale:

$$R_t = \frac{M_t}{N_t} = \left( \frac{g'_t(x)g'_t(y)}{(g_t(y) - g_t(x))^2} \right)^{\nu^2/32} \times Z_t^{\nu/8} \times \frac{F(Z_t)}{G(Z_t)}.$$

The term $Z_t$ takes values in $[0, 1]$, the term $F(Z_t)/G(Z_t)$ is uniformly bounded due to (A.7). The term

$$\frac{g'_t(x)g'_t(y)}{(g_t(y) - g_t(x))^2}$$

is the boundary Poisson kernel of the domain $\mathbb{H} \setminus \gamma [0, t]$ and it is positive and bounded from above by $(y - x)^{-2}$. Thus $R_t$ is uniformly bounded and the law of $\eta$ is absolutely continuous with respect to the law of $\gamma$ up to and including $T_y$.

By Corollary 4.13 the process $\gamma$ is continuous up to and including $T_y$ and $\gamma \cap [x, y] = \emptyset$. By the absolute continuity, the process $\eta$ is continuous up to and including $T_y$ and $\eta \cap [x, y] = \emptyset$. After $T_y$, we continue the process by standard SLE$_8$ in the remaining domain from $\eta(T_y)$ to $\infty$. Thus $\eta$ is a continuous curve for all time.

It remains to show the reversibility. We denote by $D$ the connected component of $\mathbb{H} \setminus \gamma$ with $[x, y]$ on the boundary. Since $\gamma$ is continuous and $\gamma \cap [x, y] = \emptyset$, we have $Z_t \to 1$ as $t \to T_y$. Thus, from (A.7),

$$R_t \to H_D(x, y)^{\nu^2/32} \times \sqrt{\pi} \frac{(\nu + 2)\Gamma(2 + \nu)}{(\nu + 4)\Gamma\left(\frac{3}{2} + \frac{\nu}{4}\right)}, \quad \text{as} \quad t \to T_y.$$

From this, we find that the law of $\eta$ is the same as the law of $\gamma$ weighted by the boundary Poisson kernel $H_D(x, y)^{\nu^2/32}$. Since the law of $\gamma$ is reversible due to Corollary 4.13 and the boundary Poisson kernel is conformally invariant, we obtain the reversibility of $\eta$. This completes the proof.

4 Convergence of UST in quads

4.1 UST with Dobrushin boundary conditions

We first introduce Dobrushin domains. Informally speaking, a Dobrushin domain is a simply connected subgraph $\Omega$ of $\mathbb{Z}^2$ with two fixed boundary points $a, b$, and the boundary arc $(ab)$ is in $\mathbb{Z}^2$ and the boundary arc $(ba)$ is in $(\mathbb{Z}^2)^*$. 

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Consider the medial lattice \((\mathbb{Z}^2)^*\). We orient the edges of the medial lattice such that edges of a face containing a vertex in \(\mathbb{Z}^2\) are oriented counterclockwise and edges of a face containing a vertex in \((\mathbb{Z}^2)^*\) are oriented clockwise. Let \(a^\circ, b^\circ\) be two distinct medial vertices, and \((a^\circ b^\circ)\) and \((b^\circ a^\circ)\) be two paths of neighboring medial vertices satisfying the following conditions: (1) the edges along \((a^\circ b^\circ)\) point in clockwise way with the orientation inherited from the medial lattice; (2) the edges along \((b^\circ a^\circ)\) point in counterclockwise way with the orientation inherited from the medial lattice; (3) the two paths are edge-avoiding and \((a^\circ b^\circ) \cap (b^\circ a^\circ) = \{a^\circ, b^\circ\}\). See Figure 4.1.

Given \((a^\circ b^\circ)\) and \((b^\circ a^\circ)\), the medial Dobrushin domain \((\Omega^*; a^\circ, b^\circ)\) is defined as the subgraph of \((\mathbb{Z}^2)^*\) induced by the vertices enclosed by or on the path \((a^\circ b^\circ) \cup (b^\circ a^\circ)\). Let \(\Omega \subset \mathbb{Z}^2\) be the graph with edge set consisting of edges passing through end-points of medial edges in \(E(\Omega^*)\) \((b^\circ a^\circ)\) and with vertex set given by the endpoints of these edges. Here we always assume that \(\Omega\) is simply connected. The vertices of \(\Omega\) nearest to \(a^\circ, b^\circ\) are denoted by \(a, b\) and we call \((\Omega; a, b)\) primal Dobrushin domain. Let \((ab)\) be the set of edges corresponding to medial vertices in \((a^\circ b^\circ) \cap \partial \Omega^*\). Let \(\Omega^* \subset \mathbb{Z}^2\) be the graph with edge set consisting of edges passing through end-points of medial edges in \(E(\Omega^*)\) \((a^\circ b^\circ)\) and with vertex set given by the endpoints of these edges. Here we always assume that \(\Omega^*\) is simply connected. The vertices of \(\Omega^*\) nearest to \(a^\circ, b^\circ\) are denoted by \(a^*, b^*\). Let \((b^*a^*)\) be the set of edges corresponding to medial vertices in \((b^\circ a^\circ) \cap \partial \Omega^*\). Note that \(a\) is the vertex of \(\Omega\) that is nearest to \(a^\circ\) and \(a^*\) is the vertex of \(\Omega^*\) that is nearest to \(a^\circ\). See Figure 4.1.

![Diagram](image.png)

**Figure 4.1:** In the left panel, the solid edges in black are wired boundary arc \((ab)\), the solid edges in red are dual-wired boundary arc \((b^*a^*)\). The edges in blue are the boundary arcs \((a^\circ b^\circ)\) and \((b^\circ a^\circ)\) on the medial lattice. The thin edges are in the tree \(T\). In the right panel, the curve in orange is the Peano curve associated to \(T\).

Suppose that \(T\) is a spanning tree on some primal Dobrushin domain \((\Omega; a, b)\) with \((ab)\) wired. Consider its dual configuration \(T^* \subset E(\Omega^*)\) defined as follows: \(\mathbb{1}_{T^*\cdot e^*} = \mathbb{1}_{T\cdot e}\) for any \(e \in E(\Omega)\) where \(e^*\) is the dual edge corresponding to \(e\). It is clear that \(T^*\) is a spanning tree on the dual Dobrushin domain \((\Omega^*; a^*, b^*)\) with \((b^*a^*)\) wired. There exists a unique path, called Peano curve, on \((\mathbb{Z}^2)^*\), running between \(T\) and \(T^*\) from \(a^\circ\) to \(b^\circ\). The following theorem concerns the convergence of the Peano curve of UST.

**Theorem 4.1.** Fix a Dobrushin domain \((\Omega; a, b)\) such that \(\partial \Omega\) is \(C^1\) and simple. Suppose that a sequence of medial Dobrushin domains \((\Omega^*_n; a^\circ_n, b^\circ_n)\) converges to \((\Omega; a, b)\) in the sense of \((1.3)\). Consider UST on the primal domain \(\Omega_n\) with \((a_n b_n)\) wired. Denote by \(\eta_{b_n}\) the induced Peano curve. Then the law of \(\eta_{b_n}\) converges weakly to SLE\(_6\) in \(\Omega\) from \(a\) to \(b\).

This statement is proved in [LSW04] Theorems 4.7 and 4.8. The proof of tightness of Peano curves uses argument in [Sch00] where the notions of trunk and dual trunk play an important role. The trunk and
dual trunk are also important later in this article, and we will give its formal definition below. Roughly speaking, they are the limits of the UST $\mathcal{T}_\delta$ and its dual $\mathcal{T}_\delta^*$ as $\delta \to 0$.

Consider the UST $\mathcal{T}_\delta$ on the primal domain. For $\epsilon > 0$, we first define its $\epsilon$-trunk. For any $x, y \in \mathcal{T}_\delta$, there is a unique simple path on $\mathcal{T}_\delta$ from $x$ to $y$, which we denote by $\eta_{\delta}^{x,y}$. We denote by $x'$ the first point at which $\eta_{\delta}^{x,y}$ hits $\partial B(x, \epsilon)$ and $y'$ the last point at which $\eta_{\delta}^{x,y}$ hits $\partial B(y, \epsilon)$. We denote by $\mathcal{I}_\epsilon(x, y)$ the unique simple path on $\mathcal{T}_\delta$ connecting $x'$ and $y'$. If such $x'$ or $y'$ does not exist, we define $\mathcal{I}_\epsilon(x, y) = \emptyset$. Then, the $\epsilon$-trunk is defined to be

$$\text{trunk}_\delta(\epsilon) := \bigcup_{x, y \in \mathcal{T}_\delta} \mathcal{I}_\epsilon(x, y).$$

We can couple the configurations in the same probability space and choose $\delta_m \to 0$ such that $\text{trunk}_{\delta_m}(\frac{1}{n})$ converges in Hausdorff metric for every $n \in \mathbb{N}$ as $m \to \infty$ almost surely. We define

$$\text{trunk}_0\left(\frac{1}{n}\right) := \lim_{m \to \infty} \text{trunk}_{\delta_m}\left(\frac{1}{n}\right), \quad \text{and} \quad \text{trunk} = \bigcup_{n > 0} \text{trunk}_0\left(\frac{1}{n}\right).$$

Note that the wired boundary arc $(a_b d_b)$ belongs to the UST on $\Omega_\delta$. Thus, by definition, trunk contains $(ab)$. Similarly, we define $\text{trunk}_0^*(\epsilon)$, $\text{trunk}_n^*(\frac{1}{n})$ and trunk* for the dual UST $\mathcal{T}_\delta^*$. [Sch00, Theorem 11.1] guarantees that, if $\partial \Omega$ is $C^1$ and simple, we have

$$\mathbb{P}[\text{trunk} \cap \text{trunk}^* = \emptyset] = 1.$$

The strategy of the proof Theorem 4.1 can be summarized as follows: The first step is to show the tightness of the Peano curves. This is a consequence of the fact that $\text{trunk} \cap \text{trunk}^* = \emptyset$ almost surely. The second step is to show the convergence of driving function by martingale observable [LSW04, Theorem 4.4]. This step only requires $\Omega_\delta^* \to \Omega$ in the Carathéodory sense and does not need smoothness regularity assumption on $\partial \Omega$.

### 4.2 UST in quads: tightness

Now we introduce the discrete quad in Theorem 1.4. Informally speaking, this is a simply connected subgraph $\Omega$ of $\mathbb{Z}^2$ with four fixed boundary points $a, b, c, d$ in counterclockwise order, and the boundary arcs $(ab), (cd)$ are in $\mathbb{Z}^2$ and the boundary arcs $(bc), (da)$ are in $(\mathbb{Z}^2)^*$. Let $a^0, b^0, c^0, d^0$ be four distinct medial vertices, and $(a^0b^0), (b^0c^0), (c^0d^0)$ and $(d^0a^0)$ be four paths of neighboring medial vertices satisfying the following conditions: (1) the edges along $(a^0b^0)$ and $(c^0d^0)$ point in clockwise way with the orientation inherited from the medial lattice; (2) the edges along $(b^0c^0)$ and $(d^0a^0)$ point in counterclockwise way with the orientation inherited from the medial lattice; (3) all the paths are edge-avoiding and $(a^0b^0) \cap (b^0c^0) = \{b^0\}, (b^0c^0) \cap (c^0d^0) = \{c^0\}, (c^0d^0) \cap (d^0a^0) = \{d^0\}, (d^0a^0) \cap (a^0b^0) = \{a^0\}.$ See Figure 4.2.

Given $(a^0b^0), (b^0c^0), (c^0d^0)$ and $(d^0a^0)$, the medial quad $(\Omega^0; a^0, b^0, c^0, d^0)$ is defined as the subgraph of $(\mathbb{Z}^2)^0$ induced by the vertices enclosed by or on the path $(a^0b^0) \cup (b^0c^0) \cup (c^0d^0) \cup (d^0a^0)$. The inner boundary $\partial \Omega^0$ is the set of vertices of $\Omega^0$ with strictly fewer than four incident edges in $E(\Omega^0)$. Let $\Omega \subset \mathbb{Z}^2$ be the graph with edge set consisting of edges passing through end-points of medial edges in $E(\Omega^0) \setminus ((b^0c^0) \cup (d^0a^0))$ and with vertex set given by the endpoints of these edges. Here we always assume that $\Omega$ is simply connected. The vertices of $\Omega$ nearest to $a^0, b^0, c^0, d^0$ are denoted by $a, b, c, d$ and we call $(\Omega; a, b, c, d)$ the primal quad. Let $(ab)$ and $(cd)$ be the set of edges corresponding to medial vertices in $\partial \Omega^0$, which are also endpoints of medial edges in $(a^0b^0)$ and $(c^0d^0)$ respectively. One can define $(bc)$ and $(da)$ to be the two components of $\partial \Omega^0 \setminus ((ab) \cup (cd)).$ See Figure 4.2. Similarly, we can define the dual quad as follows. Let $\Omega^* \subset (\mathbb{Z}^2)^*$ be the graph with edge set consisting of edges passing through end-points of medial edges in $E(\Omega^0) \setminus ((a^0b^0) \cup (c^0d^0))$ and with vertex set given by the endpoints of these edges. Here we always assume that $\Omega^*$ is simply connected. The vertices of $\Omega^*$ nearest to $a^0, b^0, c^0, d^0$ are
denoted by $a^*, b^*, c^*, d^*$ and we call $(\Omega^*; a^*, b^*, c^*, d^*)$ the dual quad. Let $(b^*c^*)$ and $(d^*a^*)$ be the set of edges corresponding to medial vertices in $(b^*c^*)$ and $(d^*a^*)$, which are also in $\partial\Omega^*$. One can define $(a^*b^*)$ and $(c^*d^*)$ to be the two components of $\partial\Omega^* \setminus ((b^*c^*) \cup (d^*a^*))$. See Figure 4.2.

Suppose that $T$ is a spanning tree on some primal quad $(\Omega; a, b, c, d)$ with $(ab)$ wired and $(cd)$ wired respectively. Its dual configuration $T^*$ is a spanning forest with two trees in the dual quad $(\Omega^*; a^*, b^*, c^*, d^*)$ such that one tree contains the dual-wired arc $(b^*c^*)$ and the other tree contains the dual-wired arc $(d^*a^*)$. There exist two paths on $\mathbb{Z}^2^*$ running between $T$ and $T^*$ from $a^*$ to $d^*$ and from $b^*$ to $c^*$ respectively, see Figure 1.1. We still call them Peano curves as before. With the same notations as in Section 1, we denote by $\eta^L$ the Peano curve from $a^*$ to $d^*$, by $\eta^R$ the Peano curve from $b^*$ to $c^*$ and by $\gamma^M$ the unique simple path in $T$ from $(ab)$ to $(cd)$.

Figure 4.2: The solid edges in black are wired boundary arcs $(ab)$ and $(cd)$, the solid edges in red are dual-wired boundary arcs $(b^*c^*)$ and $(d^*a^*)$. The edges in blue are the boundary arcs $(a^*b^*)$, $(b^*c^*)$, $(c^*d^*)$, and $(d^*a^*)$ on the medial lattice. The thin edges are in the tree $T$.

**Theorem 4.2.** Fix a quad $(\Omega; a, b, c, d)$ such that $\partial\Omega$ is $C^1$ and simple. Suppose that a sequence of medial quads $((\Omega^*_\delta; a^*_\delta, b^*_\delta, c^*_\delta, d^*_\delta))$ converges to $(\Omega; a, b, c, d)$ in the sense of (1.3). Consider UST on the primal domain $\Omega_\delta$ with $(a^*_\delta b^*_\delta)$ wired and $(c^*_\delta d^*_\delta)$ wired. Denote by $\eta^L_{\delta}$ the Peano curve connecting $a^*_\delta$ and $d^*_\delta$. Then the law of $\eta^L_{\delta}$ converges weakly to $\mathsf{hSLE}_8$ in $\Omega$ from $a$ to $d$ with marked points $(b, c)$ as $\delta \to 0$.

The proof of Theorem 4.2 consists of two steps: the first step is showing the tightness of the Peano curves, see Proposition 4.3; the second step is constructing an holomorphic observable, see Lemmas 4.7 and 4.8. With these two steps at hand, we will complete the proof of Theorem 4.2 in Section 4.4. Although these steps are standard, the proof involves a non-trivial calculation, see Lemma 4.12.

**Proposition 4.3.** Assume the same setup as in Theorem 4.2. The family of Peano curves $\{\eta^L_{\delta}\}_{\delta>0}$ is tight. Furthermore, $\mathbb{P}[\eta^L \cap [bc] = \emptyset] = 1$ for any subsequential limit $\eta^L$ of $\{\eta^L_{\delta}\}_{\delta>0}$.

The proof of Proposition 4.3 has two parts. The proof of tightness follows the same strategy as [LSW04, Proposition 4.5, Lemma 4.6] where the property of trunk and dual trunk plays an essential role, see Lemma 4.4.
Lemma 4.4. Assume the same setup as in Theorem 4.2. Define $\text{trunk}_\delta(\epsilon)$, $\text{trunk}_0\left(\frac{1}{\delta}\right)$ and $\text{trunk}$ for the UST $T_\delta$ and $\text{trunk}_3(\epsilon)$, $\text{trunk}_0^*(\frac{1}{\delta})$ and $\text{trunk}^*$ for the corresponding dual forest $T_\delta^*$ as in Section 4.1. We have

$$\mathbb{P}[\text{trunk} \cap \text{trunk}^* \neq \emptyset] = 0. \quad (4.1)$$

In the proof of Lemma 4.4, we need that $\partial \Omega$ is $C^1$ and simple. To prove $\mathbb{P}[\etaL \cap \{bc\} = \emptyset] = 1$ in Proposition 4.3, we first show that $\mathbb{P}[\hat{\eta}_L \cap \{bc\} = \emptyset] = 1$, which is a consequence of Lemma 4.4. We then show that $\mathbb{P}[c \notin \etaL] = 1$. To this end, we need to estimate the probability $\mathbb{P}[\hat{\eta}_L \cap B(c_\delta, \epsilon) \neq \emptyset]$, see Lemma 4.5. Similarly, we have $\mathbb{P}[b \notin \etaL] = 1$.

Lemma 4.5. Assume the same setup as in Theorem 4.2. We have

$$\lim_{\epsilon \to 0} \lim_{\delta \to 0} \mathbb{P}[\eta_L \text{ hits } B(c_\delta, \epsilon)] = 0. \quad (4.2)$$

In the proof of Lemma 4.5, we do not need that $\partial \Omega$ is $C^1$ and simple. The details of the proofs of Proposition 4.3 and Lemma 4.4 and Lemma 4.5 will be given in Appendix C.

The following conclusion is about the tightness of the LERW branch in the UST. It will be useful in both this section and Section 5.

Proposition 4.6. Assume the same setup as in Theorem 1.4. Then $\{\gamma_\delta^M\}_{\delta > 0}$ is tight. Moreover, any subsequential limit is a simple curve in $\overline{\Omega}$ which intersects $\partial \Omega$ only at its two ends. Furthermore, one of the two ends is in $(ab)$ and the other one is in $(cd)$.

The proof of Proposition 4.6 is similar to [LSW04] Lemma 3.12 and we prove it in Appendix C.

4.3 UST in quads: holomorphic observable

A function $u : \mathbb{Z}^2 \to \mathbb{C}$ is called (discrete) harmonic at a vertex $x \in \mathbb{Z}^2$ if $\sum_{i=1}^4 u(x_i) = 4u(x)$, where $(x_i : i = 1, 2, 3, 4)$ are the four neighbors of $x$ in $\mathbb{Z}^2$. We say a function $u$ is harmonic on a subgraph of $\mathbb{Z}^2$ if it is harmonic at all vertices in the subgraph. A function $f : \mathbb{Z}^2 \cup (\mathbb{Z}^2)^* \to \mathbb{C}$ is said to be (discrete) holomorphic around a medial vertex $x^0$ if one has $f(n) - f(s) = i(f(e) - f(w))$, where $n, s, w, e$ are the vertices incident to $x^0$ in counterclockwise order. We say a function $f$ is holomorphic on a subgraph of $\mathbb{Z}^2 \cup (\mathbb{Z}^2)^*$ if it is holomorphic at all vertices in the subgraph. Note that, for a discrete holomorphic function $f$ on a subgraph of $\mathbb{Z}^2 \cup (\mathbb{Z}^2)^*$, its restriction on $\mathbb{Z}^2$ and its restriction on $(\mathbb{Z}^2)^*$ are both harmonic (see [DC13] Proposition 8.15)). We summarize the setup for discrete observable below.

- Consider the set of spanning trees in the primal domain $\Omega_\delta$ with $(a_\delta b_\delta)$ wired and $(c_\delta d_\delta)$ wired. Denote this set by $\text{ST}(\delta)$ and denote its cardinality by $|\text{ST}(\delta)|$. Let $T_\delta$ be chosen uniformly among these trees. Recall that $\eta^L_\delta$ is the Peano curve along $T_\delta$ from $a_\delta$ to $d_\delta$, and $\eta^R_\delta$ is the Peano curve along $T_\delta$ from $b_\delta$ to $c_\delta$, and $\gamma_\delta^M$ is the unique simple path in $T_\delta$ connecting $(a_\delta b_\delta)$ to $(c_\delta d_\delta)$. For a vertex $z^*$ in $\Omega^1_\delta$, define $u_\delta(z^*)$ to be the probability that $z^*$ lies to the right of $\gamma_\delta^M$, i.e. $z^*$ lies in the component of $\Omega^1_\delta \setminus \gamma_\delta^M$ with $b_\delta$ and $c_\delta$ on the boundary. We abuse notation here and regard $\Omega^1_\delta$ as a planar domain.

- Consider the set of spanning forests in the primal domain $\Omega_\delta$ with $(a_\delta b_\delta)$ wired and $(c_\delta d_\delta)$ wired such that it has only two trees: one of them contains the wired arc $(a_\delta b_\delta)$ and the other one contains the wired arc $(c_\delta d_\delta)$. Denote this set by $\text{SF}_2(\delta)$ and denote its cardinality by $|\text{SF}_2(\delta)|$. Let $F_\delta$ be chosen uniformly among these forests. For $z \in \Omega_\delta$, define $v_\delta(z)$ to be the probability that $z$ lies in the same tree as the wired arc $(c_\delta d_\delta)$ in $F_\delta$.

Lemma 4.7. Define

$$f_\delta(\cdot) := u_\delta(\cdot) + \frac{|\text{SF}_2(\delta)|}{|\text{ST}(\delta)|} v_\delta(\cdot).$$
We view it as a function on \( \Omega_\delta \cup \Omega^*_\delta \): it equals \( u_\delta \) on \( \Omega^*_\delta \) and it equals \( \frac{|SF_2(\delta)|}{|ST(\delta)|} v_\delta \) on \( \Omega_\delta \). The function \( f_\delta \) is discrete holomorphic on \((\Omega_\delta \cup \Omega^*_\delta)\setminus((a_\delta b_\delta) \cup (c_\delta d_\delta) \cup (b_\delta^* c_\delta^*) \cup (d_\delta^* a_\delta^*))\). Moreover, it has the following boundary data:

\[
\begin{align*}
\text{Re} f_\delta &= 0, \quad \text{on } (d_\delta^* a_\delta^*) ; \\
\text{Im} f_\delta &= 0, \quad \text{on } (a_\delta b_\delta) ; \\
\text{Re} f_\delta &= 1, \quad \text{on } (b_\delta^* c_\delta^*) ; \\
\text{Im} f_\delta &= \frac{|SF_2(\delta)|}{|ST(\delta)|} , \quad \text{on } (c_\delta d_\delta) .
\end{align*}
\]

**Proof.** For \( z^* \in \Omega^*_\delta \), denote by \( E(\mathcal{T}_\delta; z^*) \) the event that \( z^* \) lies to the right of \( \gamma^M_\delta \). For \( z \in \Omega_\delta \), denote by \( E(\mathcal{F}_\delta; z) \) the event that \( z \) lies in the same tree as the wired arc \((c_\delta d_\delta)\) in \( \mathcal{F}_\delta \). Assume \( \{n, s\} \) is a primal edge of \( \Omega_\delta \), and the corresponding dual edge is denoted by \( \{w, e\} \) such that \( w, s, e, n \) are in counterclockwise order (see Figure 4.3). Then we have

\[
u_\delta(e) - u_\delta(w) = \mathbb{P}[E(\mathcal{T}_\delta; e)] - \mathbb{P}[E(\mathcal{T}_\delta; w)]
= \mathbb{P}[E(\mathcal{F}_\delta; n) \cap E(\mathcal{F}_\delta; s)] - \mathbb{P}[E(\mathcal{F}_\delta; n) \cap E(\mathcal{F}_\delta; s)]
= \frac{|SF_2(\delta)|}{|ST(\delta)|} \mathbb{P}[E(\mathcal{F}_\delta; n)] - \frac{|SF_2(\delta)|}{|ST(\delta)|} \mathbb{P}[E(\mathcal{F}_\delta; s)]
= \frac{|SF_2(\delta)|}{|ST(\delta)|} (v_\delta(n) - v_\delta(s)).
\]

The third equal sign is due to the observation explained in Figure 4.3. This gives the discrete holomorphicity of \( f_\delta \). The boundary data is clear from the construction. \( \square \)

**Lemma 4.8.** Fix a quad \((\Omega; a, b, c, d)\). Suppose that a sequence of medial domains \((\Omega_\delta; a_\delta^*, b_\delta^*, c_\delta^*, d_\delta^*)\) converges to \((\Omega; a, b, c, d)\) in the Carathéodory sense as \( \delta \to 0 \). Let \( K > 0 \) be the conformal modulus of the quad \((\Omega; a, b, c, d)\), and let \( f \) be the conformal map from \( \Omega \) onto the rectangle \((0, 1) \times (0, iK)\) which sends \((a, b, c, d)\) to \((0, 1, 1 + iK, iK)\). Then the discrete holomorphic function \( f_\delta \) in Lemma 4.7 (regarded as a function on \( \Omega_\delta \) by interpolating linearly among vertices) converges to \( f \) locally uniformly as \( \delta \to 0 \).

We emphasize that, in Lemma 4.8, we do not need that \( \partial \Omega \) is \( C^1 \) and simple. We only assume that \( \partial \Omega \) is locally connected and we only require the convergence of polygons in the Carathéodory sense.

**Proof.** We claim that \( \left\{ \frac{|SF_2(\delta)|}{|ST(\delta)|} \right\}_{\delta > 0} \) is uniformly bounded. Assuming this, for any sequence \( \delta_n \to 0 \), there exists a subsequence, still denoted by \( \delta_n \), and a constant \( K \) such that

\[
u_\delta_n \to u, \quad \nu_\delta_n \to v, \quad \text{locally uniformly}; \quad \text{and} \quad \frac{|SF_2(\delta_n)|}{|ST(\delta_n)|} \to K, \quad \text{as } n \to \infty.
\]
From the boundary values of \( u_\delta \) and \( v_\delta \), the discrete harmonicity of \( u_\delta \) and \( v_\delta \), and Beurling estimate (for instance, see [CST11, Proposition 2.11]), it is clear that \( u = 1 \) on \((bc)\) and \( u = 0 \) on \((da)\), and that \( v = 1 \) on \((cd)\) and \( v = 0 \) on \((ab)\). By standard argument, the limit of discrete holomorphic function is holomorphic, see for instance the first step in the proof of Lemma B.2. In other words, \( g := u + iKv \) is holomorphic on \( \Omega \). Moreover, if we fix a conformal map \( \xi \) from \( \Omega \) onto \( \mathbb{U} \), then the boundary data of \( u \circ \xi^{-1} \) is as follows: \( u \circ \xi^{-1} = 1 \) on \((\xi(b)\xi(c))\), and \( u = 0 \) on \((\xi(d)\xi'(a))\), and \( \partial_u(u \circ \xi^{-1}) = 0 \) on \((\xi(a)\xi(b)) \cup (\xi(c)\xi(d))\). Such boundary data uniquely determines the bounded harmonic function \( u \), see Lemma B.3. Note that \( \text{Ref} \) has the same boundary data, thus \( u = \text{Ref} \). Consequently, \( g \) and \( f \) differ up to a constant. Since \( v \) and \( \text{Im} f \) both equal to 0 on \((ab)\), we have \( g = f \). This also implies that \( K = K \).

It remains to show that \( \left\{ \frac{|SF_2(\delta)|}{|ST(\delta)|} \right\}_{\delta > 0} \) is uniformly bounded. If this is not the case, there exists a sequence \( \delta_n \) to 0 such that \( |SF_2(\delta_n)| \rightarrow \infty \). Then, by the same argument as above, the function \( \left| \frac{|SF_2(\delta)|}{|ST(\delta)|} \right| \) goes to 0. This is a contradiction. Therefore, \( \left\{ \frac{|SF_2(\delta)|}{|ST(\delta)|} \right\}_{\delta > 0} \) is uniformly bounded and we complete the proof.

As a consequence of Lemmas 4.7 and 4.8, we see that \( \left| \frac{|SF_2(\delta)|}{|ST(\delta)|} \right| \rightarrow K \) as \( \delta \rightarrow 0 \). This is a special case of [KW11 Theorem 1.1] for the grove with two nodes.

### 4.4 Proof of Theorem 4.2

We fix the following notations in this section.

- Fix a quad \((\Omega; a, b, c, d)\). Let \( K \) be its conformal modulus and denote by \( f_{(\Omega;a,b,c,d)} \) the conformal map from \( \Omega \) onto \((0,1) \times (0, iK)\) sending \((a, b, c, d)\) to \((0, 1, 1 + iK, iK)\).

- Suppose that a sequence of medial quads \((\Omega^0_\delta; a^0_\delta, b^0_\delta, c^0_\delta, d^0_\delta)\) and \((\Omega; a, b, c, d)\) satisfy the assumptions in Theorem 4.2. We choose conformal maps \( \phi_\delta : \Omega^0_\delta \rightarrow \mathbb{H} \) with \( \phi_\delta(a^0_\delta) = 0, \phi_\delta(d^0_\delta) = \infty \) and \( \phi : \Omega \rightarrow \mathbb{H} \) with \( \phi(a) = 0, \phi(d) = \infty \) such that \( \phi_\delta^{-1} \) converges to \( \phi^{-1} \) uniformly on \( \mathbb{H} \). (See [Pom92 Corollary 2.4]. See also [LSW04] the last paragraph above Proposition 4.5]).

- For a primal quad \((\Omega^0_\delta; a^0_\delta, b^0_\delta, c^0_\delta, d^0_\delta)\), we denote by \( f^0_{(\Omega^0_\delta;a^0_\delta,b^0_\delta,c^0_\delta,d^0_\delta)} \) the discrete observable as defined in Lemma 4.7 and we linearly interpolate it among vertices. Recall that \( f^0_{(\Omega^0_\delta;a^0_\delta,b^0_\delta,c^0_\delta,d^0_\delta)} \) converges to \( f_{(\Omega;a,b,c,d)} \) locally uniformly due to Lemma 4.8.

- The uniform spanning tree \( T_\delta \) and the Peano curve \( \eta^0_\delta \) are defined in the same way as in Section 4.2. Define \( \eta^L_\delta := \phi_\delta(\eta^0_\delta) \) and parameterize \( \eta^L_\delta \) by the half-plane capacity and parameterize \( \eta^L_\delta \) so that \( \eta^L_\delta(t) = \phi_\delta(\eta^0_\delta(t)) \).

- For \( \eta^L_\delta \), define by \( (W^L_\delta, t \geq 0) \) the driving function of \( \eta^L_\delta \) and by \( (g^L_\delta, t \geq 0) \) the corresponding conformal maps. Define \( X^L_\delta := g^L_\delta(\phi_\delta(b_\delta)), Y^L_\delta := g^L_\delta(\phi_\delta(c_\delta)) \). Denote by \( F^L_\delta \) the filtration generated by \( (W^L_\delta, 0 \leq s \leq t) \). Let \( K^0_\delta(t) \) be the modulus of the quad \((\mathbb{H}; W^L_\delta, X^L_\delta, Y^L_\delta, \infty)\) and define \( F^L_\delta \) to be the conformal map from \( \mathbb{H} \) onto \((0,1) \times (0, iK^0_\delta(t))\) sending \((W^L_\delta, X^L_\delta, Y^L_\delta, \infty) \) to \((0, 1, 1 + iK^0_\delta(t), iK^0_\delta(t))\).

- For \( \eta^L_\delta \), let \( \tau_\delta \) be the first time that \( \eta^L_\delta \) hits \((c^0_\delta, d^0_\delta)\). For every \( t < \tau_\delta \), the slit domain \( \Omega^L_\delta(t) \) is defined as the component of \( \Omega^0_\delta \setminus \eta^L_\delta[0, t] \) that contains \( c^0_\delta \) and \( d^0_\delta \) on the boundary. We define \((\Omega^0_\delta(t); a^0_\delta(t), b^0_\delta(t), c^0_\delta(t), d^0_\delta(t))\) to be the primal discrete quad as follows: The domain \( \Omega^0_\delta(t) \) is the primal domain associated to \( \Omega^L_\delta(t) \). The point \( a^0_\delta(t) \) is the primal vertex nearest to \( \eta^L_\delta(t) \). The definition of the point \( b^0_\delta(t) \) is a little complicated: if \( \eta^L_\delta[0, t] \) does not hit \((b^0_\delta c^0_\delta)\), then \( b^0_\delta(t) = b^0_\delta \); if \( \eta^L_\delta[0, t] \) hits \((b^0_\delta c^0_\delta)\), then \( b^0_\delta(t) \) is the primal vertex nearest to \( \eta^L_\delta[0, t] \cap (b^0_\delta c^0_\delta) \). The boundary conditions for \((\Omega^0_\delta(t); a^0_\delta(t), b^0_\delta(t), c^0_\delta(t), d^0_\delta(t))\) are inherited from \((\Omega^0_\delta; a^0_\delta, b^0_\delta, c^0_\delta, d^0_\delta)\) and \( \eta^L_\delta[0, t] \) : the boundary arc \((a^0_\delta(t)b^0_\delta(t))\) is wired and the boundary arc \((c^0_\delta d^0_\delta)\) is wired.
Lemma 4.9. For $z \in \Omega_\delta \cup \Omega_\delta^c$, denote by $\tau_z^\delta := \inf\{t : z \notin \Omega_\delta^c(t)\}$. Then, the process given by the observables
\[
(f_{(\Omega_\delta(t) ; \varphi_\delta(t)), b_\delta(t), c_\delta, d_\delta}(z), t \geq 0)
\]
is a local martingale up to $\tau_2^\delta \wedge \tau_3$ with respect to the filtration $(F^\delta_t, t \geq 0)$. Moreover, for every $\epsilon > 0$ and for any compact subset $A$ on $\Omega$, one has almost surely the following convergence of the conformal maps:
\[
\mathbb{E}\left[f_{\tau_2^\delta}^\delta\left(g_{\tau_2^\delta}\left(\phi_\delta(z)\right)\right) - f_{\tau_1^\delta}^\delta\left(g_{\tau_1^\delta}\left(\phi_\delta(z)\right)\right) \mid F_{\tau_1^\delta}\right] \to 0,
\]
uniformly for $z \in A$ and uniformly for any stopping times $0 < \tau_1^\delta < \tau_2^\delta < \tau_3^\delta \wedge \tau_3^\delta$, where $\tau_{\delta,\epsilon} := \inf\{t : \text{dist}(\eta_\delta^\epsilon([0, t]), A) = \epsilon\}$ and $\tau_3^\delta$ is the first time that $\eta_\delta^\epsilon$ hits the $\epsilon$-neighborhood of $(b_\delta^0, d_\delta^0)$.

Proof. We first show that $(f_{(\Omega_\delta(t) ; \varphi_\delta(t)), b_\delta(t), c_\delta, d_\delta}(z), t \geq 0)$ is a local martingale and we will consider its real part and its imaginary part separately. We fix two stopping times $\tau_1^\delta < \tau_2^\delta < \tau_3^\delta \wedge \tau_3$. Define $u^{(i)}_\delta$ and $v^{(i)}_\delta$ similarly as $u_\delta$ and $v_\delta$ in the primal quad $(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)$ for $i = 1, 2$.

For the real part, for every $z \in \Omega_\delta^c$, we have
\[
u^{(1)}_\delta(z) = \mathbb{E}\left[z \text{ lies to the right of } \eta_\delta^L \mid F_{\tau_1^\delta}\right] = \mathbb{E}\left[\mathbb{E}\left[z \text{ lies to the right of } \eta_\delta^L \mid F_{\tau_1^\delta}\right] \mid F_{\tau_1^\delta}\right] = \mathbb{E}\left[u^{(2)}_\delta(z) \mid F_{\tau_1^\delta}\right].
\]
This implies that $(\text{Re} f_{(\Omega_\delta(t) ; \varphi_\delta(t)), b_\delta(t), c_\delta, d_\delta}(z), t \geq 0))$ is a martingale up to $\tau_z^\delta \wedge \tau_3$.

For the imaginary part, define $SF_2(\tau_1^\delta)$ and $ST(\tau_i^\delta)$ similarly as $SF_2(\delta)$ and $ST(\delta)$ in the primal quad $(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)$ for $i = 1, 2$. Define $SF_2(\tau_i^\delta; z)$ to be the subset of $SF_2(\tau_i^\delta)$ such that $z$ lies in the same tree as the wired arc $c_\delta d_\delta$ for $i = 1, 2$. Then, for every $z \in \Omega_\delta$, we have
\[
\frac{|SF_2(\tau_1^\delta)|}{|ST(\tau_1^\delta)|} \nu^{(1)}_\delta(z) = \frac{|SF_2(\tau_1^\delta; z)|}{|ST(\tau_1^\delta)|} = \mathbb{E}\left[\frac{|SF_2(\tau_2^\delta)|}{|ST(\tau_2^\delta)|} \nu^{(2)}_\delta(z) \mid F_{\tau_2^\delta}\right].
\]
This implies that $(\text{Im} f_{(\Omega_\delta(t) ; \varphi_\delta(t)), b_\delta(t), c_\delta, d_\delta}(z), t \geq 0))$ is a martingale up to $\tau_z^\delta \wedge \tau_3$. We get the first conclusion.

It remains to show (4.3). We will prove that $f_{\tau_2^\delta}^\delta \circ g_{\tau_2^\delta}^\delta \circ \phi_\delta - f_{(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)}^\delta$ converges to $0$ uniformly on $A$ for $i = 1, 2$. If this is not the case, there exists a sequence $\delta_n \to 0$ and a sequence $z_{\delta_n} \in A$ such that
\[
f_{\tau_2^\delta}^\delta\left(g_{\tau_2^\delta}(\phi_\delta(z_{\delta_n}))\right) - f_{(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)}^\delta(z_{\delta_n})
\]
does not converge to $0$. (4.4)

Note that, by Carathéodory kernel theorem (for instance, see [Pom92, Theorem 1.8]), there exists a subsequence, still denoted by $\delta_n$, such that $\lim_{\delta_n} (\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta) = \lim_{\delta_n} (\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)$ converges to a quad $(\Omega_i; a_i, b_i, c_i, d_i)$ in the Carathéodory sense for $i = 1, 2$. (Here we need definitions of $\tau_i^\delta$ to guarantee the limit is still a quad.) By Lemma 4.8 the sequence $f_{(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)}^\delta$ converges to $f_{(\Omega_i; a_i, b_i, c_i, d_i)}$ uniformly on $A$. Note that $f_{(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)}^\delta$ is the conformal map from $\Omega_\delta^c(\tau_i^\delta)$ to a rectangle which maps $(\eta^L_\delta(\tau_i^\delta), b_\delta^0, c_\delta^0, d_\delta^0)$ to the four corners. By the Carathéodory convergence and the description of $f_{(\Omega_i; a_i, b_i, c_i, d_i)}$ in Lemma 4.8, the sequence $f_{(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)}^\delta$ converges to $f_{(\Omega_i; a_i, b_i, c_i, d_i)}$ on $A$ as well. This is a contradiction with (4.4). To obtain (4.3), we still need to show that there exists a constant $M = M(\epsilon)$, such that
\[
\left|f_{\tau_2^\delta}^\delta\left(g_{\tau_2^\delta}(\phi_\delta(z))\right)\right| \leq M
\]
and
\[
\left|f_{(\Omega_\delta(\tau_i^\delta); a_\delta(\tau_i^\delta), b_\delta(\tau_i^\delta), c_\delta, d_\delta)}^\delta(z)\right| \leq M
\]
uniformly for $z \in A$ and $i = 1, 2$.

This can also be obtained by contradiction similarly as above argument. (Here we need the definitions of the stopping times: $0 < \tau_1^\delta < \tau_2^\delta < \tau_3^\delta \wedge \tau_3^\delta$.) Combining with (4.4) and dominated convergence theorem, we get (4.3). \qed
From Proposition 4.3 and Proposition 4.6, both the law of \( \{ \eta^L_{\delta>0} \} \) and the law of \( \{ \gamma^M_{\delta>0} \} \) are tight in metric (1.2). In the following, we will consider the distributions of the limits of \( \eta^L_\delta \) and \( \gamma^M_\delta \). Assume the same setup as in Theorem 1.4 and fix the following notations.

- Suppose \( (\eta^L; \gamma^M) \) is any subsequential limit of \( \{ (\eta^L_{\delta>0}; \gamma^M_{\delta>0}) \} \). We may assume \( \eta^L_{\delta_n} \to \eta^L \) in law and \( \gamma^M_{\delta_n} \to \gamma^M \) in law, both in metric (1.2). By Skorokhod’s representation theorem, we may couple them together such that \( \eta^L_{\delta_n} \to \eta^L \) and \( \gamma^M_{\delta_n} \to \gamma^M \) almost surely, both in metric (1.2). We will show that the conditional law of \( \eta^L \) given \( \gamma^M \) is SLE\(_8\) in \( \Omega^L \) in Lemma 4.10.

- Define \( \hat{\eta}^L = \phi(\eta^L) \) and parameterize \( \hat{\eta}^L \) by the half-plane capacity and parameterize \( \eta^L \) so that \( \hat{\eta}^L(t) = \phi(\eta^L(t)) \). We will show that \( \hat{\eta}^L \) has continuous driving function in Corollary 4.11 and denote its driving function by \( (W_t, t \geq 0) \). We will derive the law of \( (W_t, t \geq 0) \) in Lemma 4.12.

**Lemma 4.10.** The conditional law of \( \eta^L \) given \( \gamma^M \) is SLE\(_8\) in \( \Omega^L \) from \( a \) to \( d \).

**Proof.** We denote by \( \Omega^L \) the connected component of \( \Omega \setminus \gamma^M \) which contains \( \{ da \} \) on its boundary. This is well-defined since we have \( \gamma^M \cap [da] = \emptyset \) almost surely due to Proposition 4.6. Denote by \( \Omega^L_{\delta_n} \) the connected component of \( \Omega_{\delta_n} \setminus \gamma^M_{\delta_n} \) which contains \( \{ d_{\delta_n}, \partial_{\delta_n} \} \) on its boundary. Denote by \( \Omega^L_{\delta_n} \) the median graph associated with \( \Omega^L_{\delta_n} \). Since \( \gamma^M_{\delta_n} \to \gamma^M \) as curves, the medial (random) Dobrushin domains \( (\Omega^L_{\delta_n}; a_{\delta_n}^*, b_{\delta_n}^*, c_{\delta_n}^*, d_{\delta_n}^*) \) converges to \( (\Omega^L; a, d) \) in the following sense: there exists a constant \( C > 0 \),

\[
d((d_{\delta_n}^*, a_{\delta_n}^*), (da)) \leq C \delta_n, \quad d((a_{\delta_n}^*, d_{\delta_n}^*), (ad)) \to 0, \quad \text{as } n \to \infty,
\]

where \( d \) is the metric (1.2). Note that in the proof of Theorem 4.1, the conditions that \( \partial \Omega \) is \( C \) and simple and the convergence of boundaries are only used in [LSW04, Section 4.3] to get the uniform continuity, that is, the tightness of the discrete Peano curves in metric (1.2). See the begining of [LSW04, Section 4.3]. These conditions are used to ensure that \( \text{trunk} \cap \text{trunk}^* = \emptyset \) almost surely. We will summarize the proof of the proof in the proof of Proposition 4.3. Although we do not have \( C \) regularity on \( \partial \Omega^L \) neither the required convergence of domains, we already have \( \text{trunk} \cap \text{trunk}^* = \emptyset \) almost surely due to Lemma 4.4. This is because \( \text{trunk}_{\delta_n} \) in the primal graph associated with the medial Dobrushin domain \( (\Omega^L_{\delta_n}; a_{\delta_n}^*, b_{\delta_n}^*, c_{\delta_n}^*, d_{\delta_n}^*) \) is a subset of \( \text{trunk}_{\delta_n} \) in \( (\Omega^L_{\delta_n}; a_{\delta_n}, b_{\delta_n}, c_{\delta_n}, d_{\delta_n}) \) and \( \text{trunk}^*_{\delta_n} \) in the dual graph associated with the medial Dobrushin domain \( (\Omega^L_{\delta_n}; a_{\delta_n}^*, b_{\delta_n}^*, c_{\delta_n}^*, d_{\delta_n}^*) \) is a subset of \( \text{trunk}^*_{\delta_n} \) in \( (\Omega^L_{\delta_n}; a_{\delta_n}, b_{\delta_n}, c_{\delta_n}, d_{\delta_n}) \). In the part of the proof of Theorem 4.4 to obtain the convergence of the driving functions, the authors only need two requirements:

- the radius of the domains seen from an interior point is uniformly large if we rescale the discrete domain so that they are subgraphs of \( \mathbb{Z}^2 \);

- the harmonic measure of the wired arc seen from the same interior point belongs to \((\epsilon, 1 - \epsilon)\) for \( \epsilon > 0 \).

See more details in [LSW04, Theorem 4.4]. Note that in our case, we have the convergence of domains in the Carathéodory sense, which satisfies the above two assumptions. Thus, by the same proof of Theorem 4.1, we have that \( \eta^L \) is SLE\(_8\) in \( \Omega^L \) from \( a \) to \( d \).

**Corollary 4.11.** The process \( \hat{\eta}^L = \phi(\eta^L) \) has continuous driving function when parameterized by the half-plane capacity. Moreover, when parameterized by the half-plane capacity, \( \hat{\eta}^L_{\delta_n} \to \hat{\eta}^L \) locally uniformly and the driving functions \( (W_t^\delta, t \geq 0) \to (W_t, t \geq 0) \) locally uniformly.

**Proof.** We only need to show that \( \hat{\eta}^L \) has continuous driving function when parameterized by the half-plane capacity. The rest of the statement is true due to [LW23b, Proposition 4.10]. Recall that \( \phi \) is a conformal map from \( \Omega \) onto \( \mathbb{H} \). Since \( \partial \Omega \) is simple, \( \phi \) can be extended injectively and continuously to \( \partial \Omega \). This implies that \( \hat{\eta}^L = \phi(\eta^L) \) is a continuous curve. Recall that \( \Omega^L \) is the connected component of \( \Omega \setminus \gamma^M \).
which contains $[da]$ on its boundary. Fix a conformal map $\phi^L$ from $\Omega^L$ onto $\mathbb{H}$. We parameterize $\eta^L$ by $[0, +\infty)$ such that for every $t > 0$, the half-plane capacity of $\phi^L(\eta^L[0, t])$ equals $t$. Note that under this parameterization, $\eta^L$ is a continuous function on $[0, +\infty)$. From Lemma 4.10 $\phi^L(\eta^L)$ is an SLE$_8$. Thus, we have the following two observations:

- For every $0 < t < T$, we have $\eta^L(t, T)$ is contained in the closure of the connected component of $\Omega^L \setminus \eta^L(0, t)$. This implies that $\phi(\eta^L(t, T))$ is contained in the closure of the unbounded connected component of $\mathbb{H} \setminus \phi(\eta^L(0, t))$.

- The set $\{s \in (t, T) : \eta^L(s) \in \eta^L[0, t] \cup \partial\Omega^L\}$ has empty interior. In particular, this implies that the set $\{s \in (t, T) : \phi(\eta^L(s)) \in \phi(\eta^L[0, t]) \cup R\}$ has empty interior.

Thus, by [Law05, Proposition 4.3] and [MS16a, Proposition 6.12], almost surely, $\hat{\eta}^L = \phi(\eta^L)$ is driven by a continuous function after being reparameterized by the half-plane capacity. This completes the proof.

For $x < \gamma < w$, define

$$\Theta(x, y, w) := \frac{2}{w - x} + \frac{2}{w - y} - \frac{8}{(y - w)^2} \frac{\Phi'(x-w, y-w)}{\Phi'(w-w, y-w)}$$

where $F(z) = 2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; z\right)$. (4.5)

Note that $F$ is the hypergeometric function in (4.6) with $\kappa = 8, \nu = 0$.

**Lemma 4.12.** For every $\epsilon > 0$, we denote by $\tau_\epsilon$ the first time that $\eta^L$ hits the $\epsilon$-neighbourhood of $(bd)$. Then, the law of the driving function $(W_t, t \geq 0)$ of $\hat{\eta}^L$ is given by the following SDEs up to $\tau_\epsilon$:

$$dW_t = \sqrt{8d}B_t + \Theta(X_t, Y_t, W_t)dt, \quad W_0 = 0;$$

$$dX_t = \frac{2\epsilon t}{X_t - W_t}, \quad X_0 = 0;$$

$$dY_t = \frac{2\epsilon t}{Y_t - W_t}, \quad Y_0 = 0.$$ (4.6)

where $(B_t, t \geq 0)$ is one-dimensional Brownian motion starting from 0 and $\Theta$ is defined in (4.5).

**Proof.** Recall that $\hat{\eta}^L_{\delta_n} = \phi_{\delta_n}(\hat{\eta}^L_{\delta_n})$ and $\hat{\eta}^L = \phi(\eta^L)$ are parameterized by half-plane capacity. From Corollary 4.11 we see that $\hat{\eta}^L$ is driven by a continuous function $W$; and that $\hat{\eta}^L_{\delta_n}$ converges to $\hat{\eta}^L$ locally uniformly. We define $T_M := \inf\{t : \hat{\eta}^L[0, t] \cap \partial B(0, M) \neq \emptyset\}$ for every $M > 0$ and $\tau_{\epsilon, z} := \inf\{t : \hat{\eta}^L[0, t] \cap \partial B(z, \epsilon') \neq \emptyset\}$ for every $z \in \mathbb{H}$ and $\epsilon' > 0$. It suffices to prove that (4.6) holds up to $\tau_\epsilon \wedge T_M$ since we may get the result by letting $M \to \infty$. We define $T^\delta_{\delta_n}$ and $\tau^\delta_{\epsilon, z}$ similarly for $\hat{\eta}^L_{\delta_n}$. We may assume $T^\delta_{\delta_n} \to T_M$, $\tau^\delta_{\epsilon, z} \to \tau_{\epsilon, z}$, and $\tau^\delta_{\epsilon, z} \to \tau_\epsilon$ by considering a continuous modification, see details in [Kar19, Appendix B] and [Kar20]. Then, Lemma 4.9 implies that $M_t(z)$ is a martingale up to $\tau_\epsilon \wedge T_M \wedge \tau_{\epsilon, z}$.

First, we prove that $W_t$ is a semimartingale (similar argument already appeared in [Kar20]). Define $g(w, x, y, z) := f(z, 0, x-w, y-w, \infty)(z-w)$ on $\{(w, x, y, z) \in \mathbb{R}^3 : w < x < y\} \times \mathbb{H}$. Note that $\partial_w g(w, x, y, z)$ is also an analytic function on $\mathbb{H}$. We now show that the zero set of $\partial_w g(w, x, y, z)$ is isolated. Otherwise, $\partial_w g(w, x, y, z)$ equals a constant $C$ on $\mathbb{H}$. By letting $z \to w$, we have $C = 0$. Thus $g(w, x, y, z)$ is independent of $w$. This contradicts the fact that $g(w, x, y, z)$ is the conformal map from $\mathbb{H}$ onto $(0, 1) \times (0, iK)$ sending $(w, x, y, \infty)$ to $(0, 1, 1 + tK, iK)$. Thus, for every $w < x < y$, the zero set of $\partial_w g(w, x, y, z)$ is isolated. Consequently, there exists $z \in \mathbb{H}$ such that $\partial_w g(w, x, y, z) \neq 0$. By continuity, $(w, x, y, z) \neq 0$ on an interval containing $(w, x, y)$. Combining with implicit function theorem, there exists a smooth function $\psi$ such that $w = \psi(x, y, z, g)$ on an open neighborhood of $(x, y, z, g)$. Consequently, there exists a sequence of triple sets $\{(O_i, z_i, \psi_i)\}_{i \geq 1}$ satisfying:
• \( O_i \) is an open set of \( \mathbb{R}^3 \) for each \( i \geq 1 \);
• \( \cup_i O_i = \{(w, x, y) \in \mathbb{R}^3 : w < x < y\} \);
• For each \( O_i \), there exist \( z_i \in \mathbb{H} \) and a smooth function \( \psi_i \) such that
  \[
  w = \psi_i(x, y, z_i, g), \quad \text{for all} \ (w, x, y) \in O_i.
  \]

Define a sequence of stopping times \( \{T_i\}_{i \geq 1} \) as follows: Define \( T_1 := 0 \) and define \( (O_{T_1}; z_{T_1}, \psi_{T_1}) \) to be any element in \( \{(O_i; z_i, \psi_i)\}_{i \geq 1} \) such that \( O_{T_1} \) contains \( (0, x, y) \). Suppose that \( T_n \) and \( (O_{T_n}; z_{T_n}, \psi_{T_n}) \) are well-defined, we set
  \[
  T_{n+1} := \inf\{t > T_n : (W_t, X_t, Y_t) \not\in O_{T_n}\}
  \]
and define \( (O_{T_{n+1}}; z_{T_{n+1}}, \psi_{T_{n+1}}) \) to be any element in \( \{(O_i; z_i, \psi_i)\}_{i \geq 1} \) such that \( O_{T_{n+1}} \) contains the point \( (W_{T_{n+1}}, X_{T_{n+1}}, Y_{T_{n+1}}) \). Then, we have
  \[
  W_t = \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \leq t < T_{i+1}\}} \psi_{T_i}(X_t, Y_t, g_t(z_{T_i}), M_t(z_{T_i})).
  \]

This implies that \( W_t \) is a semimartingale.

Next, let us calculate the drift term of \( M_t(z) \). From Schwarz-Christoffel formula (see e.g. [Ahl78 Chapter 6-Section 2.2]), we have, for \( 0 < x < y \) and for \( z \in \mathbb{H} \),
  \[
  f_{(\mathbb{H};0,x,y,\infty)}(z) = \frac{\int_0^{2/x}(s(s-1)(s-\frac{y}{x}))^{-1/2}ds}{\int_0^1(s(s-1)(s-\frac{y}{x}))^{-1/2}ds}.\]

Denote by \( \mathcal{K} \) the elliptic integral of the first kind \([A.8]\). By changing of variable \( s = \sin^2 \theta \), we have
  \[
  f_{(\mathbb{H};0,x,y,\infty)}(z) = \frac{\mathcal{K}(\arcsin \sqrt{z/x}, x/y)}{\mathcal{K}(x/y)}. \tag{4.7}
  \]

Therefore,
  \[
  M_t(z) = \frac{\mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)}, \quad \text{where} \quad S_t = \arcsin \sqrt{\frac{g_t(z) - W_t}{X_t - W_t}}, \quad U_t = \frac{X_t - W_t}{Y_t - W_t}. \tag{4.8}
  \]

We first calculate \( dU_t \) and \( dS_t \):
  \[
  dU_t = \frac{1}{(Y_t - W_t)^2} \left( \frac{2}{U_t} - 2U_t \right) dt + (X_t - Y_t)dW_t - (1 - U_t)d\langle W \rangle_t, \tag{4.9}
  \]
  \[
  dS_t = \frac{\cot S_t}{(X_t - W_t)^2} \left( 2 + \cot^2 S_t \right) dt - \frac{1}{2} (X_t - W_t)dW_t - \frac{1}{8} (3 + \cot^2 S_t) d\langle W \rangle_t.
  \]

Applying Itô’s formula in (4.8), we have
  \[
  dM_t(z) = \frac{1}{\mathcal{K}(U_t)} d\mathcal{K}(S_t, U_t) - \frac{\mathcal{K}(S_t, U_t)}{\mathcal{K}^2(U_t)} d\mathcal{K}(U_t) - \frac{1}{\mathcal{K}^2(U_t)} d(\mathcal{K}(S, U), \mathcal{K}(U))_t + \frac{\mathcal{K}(S_t, U_t)}{\mathcal{K}^3(U_t)} d(\mathcal{K}(U))_t. \tag{4.10}
  \]
We denote by $L_t$ the drift term of $W_t$. Since the drift term of $M_t(z)$ is zero, plugging (4.9) into (4.10), we have

\[
0 = \frac{\partial_x \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} \frac{\cot S_t}{(X_t - W_t)^2} \left( (2 + \cot^2 S_t)dt - \frac{1}{2}(X_t - W_t)dL_t - \frac{1}{8}(3 + \cot^2 S_t)d\langle W \rangle_t \right) + \frac{\partial_x^2 \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} \frac{1}{(Y_t - W_t)^2} \left( \left( \frac{2}{U_t} - 2U_t \right) dt + (X_t - Y_t)dL_t - (1 - U_t)d\langle W \rangle_t \right)
+ \frac{1}{8} \frac{\partial_x^2 \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} (X_t - Y_t)^2 d\langle W \rangle_t + \frac{1}{2} \frac{\partial_x^2 \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} (X_t - Y_t)^2 \frac{\cot S_t}{(Y_t - W_t)^4} d\langle W \rangle_t
- \frac{1}{2} \frac{\partial_x^2 \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} (X_t - Y_t) \cot S_t \frac{\partial_x \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} (Y_t - W_t)^4 d\langle W \rangle_t
- \frac{1}{\mathcal{K}^2(U_t)} \frac{\partial_x \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} \frac{2}{(U_t - 2U_t)} \left( (X_t - Y_t)dL_t - (1 - U_t)d\langle W \rangle_t \right) - \frac{1}{2} \frac{\partial_x^2 \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} \frac{(X_t - Y_t)^2}{(Y_t - W_t)^4} \frac{\partial_x \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} (X_t - Y_t)^2 d\langle W \rangle_t + \frac{1}{2} \frac{\partial_x^2 \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} \frac{(X_t - Y_t)^2}{(Y_t - W_t)^4} \frac{\partial_x \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} (Y_t - W_t)^4 d\langle W \rangle_t
+ \frac{\partial_x \mathcal{K}(S_t, U_t)}{\mathcal{K}(U_t)} \frac{1}{(Y_t - W_t)^2} (X_t - W_t) d\langle W \rangle_t.
\]

Note that
\[
g_t(z) - W_t \to 0, \quad S_t \to 0, \quad \sqrt{g_t(z) - W_t} \cot S_t \to \sqrt{X_t - W_t}, \quad \text{as} \quad z \to \tilde{\eta}^L(t).
\]
Combining with (A.10) and the trivial facts $\partial_x \mathcal{K}(\varphi, x), \partial_x^2 \mathcal{K}(\varphi, x) \to 0$ as $\varphi \to 0$, we have
\[
\mathcal{K}(S_t, U_t) \to 0, \quad \partial_x \mathcal{K}(S_t, U_t) \to 1, \quad \partial_x^2 \mathcal{K}(S_t, U_t) \to 0,
\]
\[
\frac{\partial_x^2 \mathcal{K}(S_t, U_t)}{\sqrt{g_t(z) - W_t}} \to \frac{U_t}{\sqrt{X_t - W_t}}, \quad \frac{\partial_x \mathcal{K}(S_t, U_t)}{g_t(z) - W_t} \to \frac{1}{2(X_t - W_t)}, \quad \partial_x^2 \mathcal{K}(S_t, U_t) \to 0, \quad \text{as} \quad z \to \tilde{\eta}^L(t).
\]
In the right hand-side of (4.11), the leading term is of order $\cot^3 S_t$. Dividing (4.11) by $\cot^3 S_t$ and letting $z \to \tilde{\eta}^L(t)$, we have
\[
d\langle W \rangle_t = 8dt.
\]
Plugging (4.12) into (4.11), the leading term now is of order $\cot S_t$. Dividing (4.11) by $\cot S_t$ and letting $z \to \tilde{\eta}^L(t)$, we have
\[
- \frac{1}{X_t - W_t} \left( dt + \frac{1}{2}(X_t - W_t)dL_t \right) + \frac{U_t}{X_t - W_t} dt + \frac{4\partial_x \mathcal{K}(U_t)}{\mathcal{K}(U_t)} \frac{X_t - Y_t}{(Y_t - W_t)^2} dt = 0.
\]
By (A.9), we have
\[
dL_t = \Theta(X_t, Y_t, W_t)dt.
\]
Combining (4.12) and (4.13), we obtain the result.

Proof of Theorem 4.12: Recall that $\tilde{\eta}^L_{\delta_n} = \phi_{\delta_n}(\eta^L_n)$ and $\tilde{\eta}^L = \phi(\eta^L)$ are parameterized by the half-plane capacity. From Corollary 4.11, $\tilde{\eta}^L_{\delta_n} \to \tilde{\eta}^L$ locally uniformly as $n \to \infty$ almost surely. Recall the definitions of trunk$_{\delta}(\epsilon)$, trunk$_{0}(\frac{1}{n})$ and trunk for $T_{\delta}$ and trunk$_{\delta}(\frac{1}{n})$ and trunk$^*$ for $T^*$ in Lemma 4.4. We may assume that (by subtracting a further subsequence), in such coupling, the sequence of trunks $\{\text{trunk}_{\delta_n}(\frac{1}{m})\}_n$ converges to $\text{trunk}^*_n(\frac{1}{m})$ in Hausdorff distance for every $m \in \mathbb{N}$. We define $\tau := \inf\{t > 0 : \eta^L[0, t] \cap (cd) \neq \emptyset\}$. By Lemma 4.12 the driving function of $\tilde{\eta}^L$ has the same law as the one for hSLE$_{8}$. 

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up to $\tau$. (Note that $\eta^L \cap [bc] = \emptyset$ almost surely.) In order to show that $\hat{\eta}^L$ has the same law as hSLE$_8$ as a whole process, it remains to analyze the continuity of the process as $t \to \tau$ and to derive the limit after the time $\tau$.

Define $\tau_{\delta_n,\epsilon} := \inf\{t > 0 : \text{dist}(\eta^L_{\delta_n}(t), (c^2_{\delta_n}, d^2_{\delta_n})) = \epsilon\}$. Recall that $\tau_{\delta_n}$ is the first time that $\eta^L_{\delta_n}$ hits $(c^2_{\delta_n}, d^2_{\delta_n})$. Firstly, we will show

$$\lim_{n \to \infty} \tau_{\delta_n} = \tau \quad \text{almost surely.} \quad (4.14)$$

It is clear that

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \tau_{\delta_n,\epsilon} \leq \tau \leq \lim_{n \to \infty} \tau_{\delta_n} \leq \lim_{n \to \infty} \tau_{\delta_n} \quad \text{almost surely.}$$

Denote by $T = \lim_{\epsilon \to 0} \lim_{n \to \infty} \tau_{\delta_n,\epsilon}$. From Lemma 4.5, we have $\hat{\eta}^L(T) \in (cd)$. If (4.14) doesn’t hold, there exists $t$ with $T < t < \lim_{n \to \infty} \tau_{\delta_n}$ such that $\rho := \text{dist}(\eta^L(t), \partial \Omega) > 0$. By the locally uniform convergence, there exists a subsequence of $(\delta_n)_{n \geq 1}$ (still denoted by $(\delta_n)_{n \geq 1}$) such that dist($\eta^L(t), \partial \Omega_{\delta_n}$) > $\rho'$ and $t \in (\tau_{\delta_n,\epsilon}, \tau_{\delta_n})$ for a for some $\epsilon > 0$. In such case, we can choose $v \in \Omega^*_\delta$ adjacent to $\eta^L_{\delta_n}(t)$ on the right-hand side such that it is connected to $(b^\ast_{\delta_n}, c^\ast_{\delta_n})$ in the dual forest by a unique simple path which we denote by $T_{v,n}$. Then, we have diam($T_{v,n}$) > $\rho'_2$. Since the trunk $\{\text{trunk}^\ast_{\delta_n}(\frac{1}{m})\}_n$ converges to $\text{trunk}^\ast_0(\frac{1}{m})$ in Hausdorff distance for every $m \in \mathbb{N}$, this implies $\{\text{trunk} \cap \text{trunk}^\ast = \emptyset\}$. But Lemma 4.4 says that almost surely $\text{trunk} \cap \text{trunk}^\ast = \emptyset$. This implies (4.14).

Secondly, we see that the driving function of $\hat{\eta}^L$ solves (4.6) up to $\tau$ by Lemma 4.12. From Proposition 4.3, the curve $\eta^L$ does not hit $[bc]$. We define $x := \phi(b)$ and $y := \phi(c)$. By Lemma 4.12, we can couple $W$ and a one-dimensional Brownian motion $B$ starting from 0 together such that, for $t < \tau$,

$$W_t = \sqrt{8}B_t + \int_0^t \frac{2ds}{W_s - V^x_s} + \int_0^t \frac{-2ds}{W_s - V^y_s} - 8 \int_0^t \frac{F'(Z_s)}{F(Z_s)} \left(1 - \frac{Z_s}{V^y_s - W_s}\right) ds.$$

Thirdly, we prove that $W$ solves (4.6) up to and including $\tau$. We may assume that $\tau < \infty$. Note that, for any $t < \tau$,

$$W_t = \sqrt{8}B_t - 2 \int_0^t \frac{V^y_s - V^x_s}{(W_s - V^x_s)(W_s - V^y_s)} ds - 8 \int_0^t \frac{F'(Z_s)}{F(Z_s)} \left(1 - \frac{Z_s}{V^y_s - W_s}\right) ds.$$

By Corollary 4.11 we get that $W : [0, \infty) \to \mathbb{R}$ is a continuous function. Thus, we have

$$8 \int_0^t \frac{F'(Z_s)}{F(Z_s)} \left(1 - \frac{Z_s}{V^y_s - W_s}\right) ds \leq \max_{t \in [0,\tau]} |W_t - \sqrt{8}B_t| < \infty.$$

Then, by monotone convergence theorem, we have

$$\lim_{t \to \tau} \int_0^t \frac{F'(Z_s)}{F(Z_s)} \left(1 - \frac{Z_s}{V^y_s - W_s}\right) ds = \int_0^\tau \frac{F'(Z_s)}{F(Z_s)} \left(1 - \frac{Z_s}{V^y_s - W_s}\right) ds < \infty.$$

Take $z \in \mathbb{R}$ such that $z > \hat{\eta}^L(\tau)$. For any $t < \tau$, we have $g_t(y) \leq g_t(z)$. This implies

$$\int_0^t \frac{2ds}{V^y_s - W_s} \leq g_t(z).$$

By monotone convergence theorem, we have

$$\lim_{t \to \tau} \int_0^t \frac{2ds}{V^x_s - W_s} = \int_0^\tau \frac{2ds}{V^x_s - W_s} \leq \lim_{t \to \tau} \int_0^t \frac{2ds}{V^y_s - W_s} = \int_0^\tau \frac{2ds}{V^y_s - W_s} \leq g_\tau(z) < \infty.$$

Therefore, letting $t \to \tau$, we have

$$W_\tau = \sqrt{8}B_\tau + \int_0^\tau \frac{2ds}{W_s - V^x_s} + \int_0^\tau \frac{-2ds}{W_s - V^y_s} - 8 \int_0^\tau \frac{F'(Z_s)}{F(Z_s)} \left(1 - \frac{Z_s}{V^y_s - W_s}\right) ds.$$
In other words, the driving function $\hat{W}$ of $\hat{\eta}^L$ has the same law as the one for $h\text{SLE}_8$ up to and including $\tau$. In this step, it is important that $\hat{W}$ is continuous up to and including $\tau$ due to Corollary 4.11.

Finally, we will show that the driving function of $\hat{\eta}^L[\tau, \infty]$ given $\hat{\eta}^L[0, \tau]$ is $\sqrt{8}$ times a one-dimensional Brownian motion starting from 0. Denote by $\Omega(\tau)$ the connected component of $\Omega \setminus \hat{\eta}^L[0, \tau]$ having $d$ on its boundary. From above, we know that $\Omega_{\delta_n}(\tau_{\delta_n})$ converges to $\Omega(\tau)$ in the Carathéodory sense by Carathéodory kernel theorem. By the domain Markov property, conditioning on $\eta_{\delta_n}(0, \tau_{\delta_n})$, the remaining curve $\eta_{\delta_n}(\tau_{\delta_n}, \infty]$ has the same law as the Peano curve from $\eta_{\delta_n}(\tau_{\delta_n})$ to $d_\delta$ in $\Omega_{\delta_n}(\tau_{\delta_n})$ with Dobrushin boundary conditions. By [LSW04, Theorem 4.4], the driving function of $\hat{\eta}^L[\tau, \infty]$ has the same law as the driving function of $\text{SLE}_8$ in $\Omega(\tau)$ from $\eta(\tau)$ to $d$. It is important that the convergence of driving function only requires the convergence of domains in the Carathéodory sense and there is no smoothness regularity requirement on the boundary of the limiting domain. Thus, the driving function of $\hat{\eta}^L[\tau, \infty]$ given $\hat{\eta}^L[0, \tau]$ is $\sqrt{8}$ times a one-dimensional Brownian motion starting from 0.

In summary, the driving function of $\hat{\eta}^L$ is the same as the one for $h\text{SLE}_8$ as a whole process. This completes the proof. $\square$

As a consequence of Theorem 4.2, we have the following.

**Corollary 4.13.** Fix a quad $(\Omega; a, b, c, d)$. The process $\eta \sim h\text{SLE}_8$ in $\Omega$ from $a$ to $d$ with marked points $(b, c)$ has the following properties: It is almost surely generated by continuous curve and $\eta \cap [bc] = \emptyset$ almost surely. Moreover, it is reversible: the time-reversal of $\eta$ has the law of $h\text{SLE}_8$ in $\Omega$ from $d$ to $a$ with marked points $(c, b)$.

**Proof.** We may assume that $\partial \Omega$ is $C^1$ and simple. Note that, if the conclusion holds under such assumption, it would also hold for a general quad with locally connected boundary via conformal mapping.

First of all, we argue that there exists a unique solution in law to the SDE (4.6) up to and including $\tau$—the first time that $\phi(b)$ is swallowed. From the proof of Theorem 4.2, we see that there exists a version of solution $W$ to the SDE (4.6) up to $\tau$. Moreover, it is generated by a continuous curve $\eta$ up to and including $\tau$ and $\eta \cap [\phi(b), \phi(c)] = \emptyset$. Suppose $\hat{W}$ is another solution. Denote by $\tau_c$ the first time that the process gets within $\epsilon$-neighborhood of $(\phi(b), \infty)$. As the SDE (4.6) has a unique solution up to $\tau_c$, the two processes $W$ and $\hat{W}$ have the same law up to $\tau_c$. We may couple them so that $W_t = \hat{W}_t$ for $t \leq \tau_c$. We denote by $P_\epsilon$ the probability measure corresponding to this coupling. Then, the family $\{P_\epsilon\}_{\epsilon > 0}$ is consistent on the product space of continuous functions on $[0, +\infty)$. Thus, we can construct a new probability measure $Q$ such that under $Q$, two driving functions $W_t = \hat{W}_t$ for all $t < T$ where $T$ is the first hitting time of $[\phi(b), \infty)$. Since $\eta \cap [\phi(b), \phi(c)] = \emptyset$, we have $\tau = T$. This implies that the SDE (4.6) has a unique solution up to and including $\tau$, which is given by the limit of the Peano curve in Theorem 4.2. Then, the continuity of $h\text{SLE}_8$ is a consequence of Proposition 4.3. For the reversibility, we denote by $R(\eta^L_\delta)$ the time-reversal of $\eta^L_\delta$. By Theorem 4.2, the law of $R(\eta^L_\delta)$ converges to $h\text{SLE}_8$ in $\Omega$ from $d$ to $a$ with marked points $(c, b)$ as $\delta \to 0$. This implies the reversibility and completes the proof. $\square$

### 4.5 Proof of Theorem 1.4

In this section, we will complete the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Note that, the families $\{\eta^L_\delta\}_{\delta > 0}$ and $\{\eta^R_\delta\}_{\delta > 0}$ are tight due to Proposition 4.3 and the family $\{\gamma^M_\delta\}_{\delta > 0}$ is tight due to Proposition 4.6. Hence for any sequence $\delta_n \to 0$, there exists a subsequence, still denoted by $\delta_n$, such that $\{(\eta^L_{\delta_n}, \gamma^M_{\delta_n}, \eta^R_{\delta_n})\}$ converges in law as $n \to \infty$. We couple $\{(\eta^L_{\delta_n}, \gamma^M_{\delta_n}, \eta^R_{\delta_n})\}$ together such that $\eta^L_{\delta_n} \to \eta^L$ and $\gamma^M_{\delta_n} \to \gamma^M$ and $\eta^R_{\delta_n} \to \eta^R$ as curves almost surely. We will prove that the law of the triple $(\eta^L; \gamma^M; \eta^R)$ is the one in Theorem 1.4.

The law of $\eta^L$ is $h\text{SLE}_8$ in $\Omega$ from $a$ to $d$ with marked points $(b, c)$ due to Theorem 4.2. By Lemma 4.10, the conditional law of $\eta^L$ given $\gamma^M$ is $\text{SLE}_8$. Similarly, the conditional law of $\eta^R$ given $\gamma^M$ is $\text{SLE}_8$. These imply that $\eta^L$ and $\eta^R$ are conditionally independent given $\gamma^M$. Since $\text{SLE}_8$ is space-filling, we have
\( \gamma^M = \eta^L \cap \eta^R \). Therefore, the conditional law of \( \eta^R \) given \( \eta^L \) is the same as the conditional law of \( \eta^R \) given \( \gamma^M \) which is SLE\(_8\). This completes the proof. \( \square \)

**Corollary 4.14.** Consider the continuous curve \( \gamma^M \) in the triple of Theorem 1.4. We have \( \mathbb{P}[z \in \gamma^M] = 0 \) for any \( z \in \Omega \) and \( \text{Leb}(\gamma^M) = 0 \) almost surely.

**Proof.** By Theorem 4.2, the law of \( \eta^L \) is hSLE\(_8\) in \( \Omega \) from \( a \) to \( d \) with marked points \((b, c)\). From Theorem 1.4, the curve \( \gamma^M \) is the part of the boundary of \( \eta^L \) inside \( \Omega \). We parameterize \( \gamma^M \) so that \( \gamma^M(0) = X^M \) and \( \gamma^M(1) = Y^M \). Let \( \eta \) be an SLE\(_8\) in \( \Omega \) from \( a \) to \( d \) and denote by \( \tau \) the first time that \( \eta \) swallows \( b \). From (3.10), the law of \( \eta^L \) is absolutely continuous with respect to \( \eta \) up to \( \tau \). For every \( z \in \Omega \), the probability that \( z \) belongs to the frontier of SLE\(_8\) equals 0, thus \( \mathbb{P}[z \in \gamma^M[0, t]] = 0 \) for any \( t \in (0, 1) \). As \( \gamma^M = \cup_i \gamma^M[0, 1 - 1/n], \) we have \( \mathbb{P}[z \in \gamma^M] = 0 \). This implies that \( \mathbb{P}[\text{Leb}(\gamma^M)] = 0 \), which implies that almost surely, we have \( \text{Leb}(\gamma^M) = 0 \) as desired. \( \square \)

## 5 Convergence of LERW in quads

### 5.1 The pair of random points \((X^M, Y^M)\)

The goal of this section is to derive the limiting distribution of the pair of random points \((X^M, Y^M)\) in Theorem 1.5. We summarize the setup below.

- Fix a quad \((\Omega; a, b, c, d)\) such that \( \partial \Omega \) is \( C^1 \) and simple. Suppose that a sequence of medial quads \((\Omega^*_d; a_d^0, b_d^0, c_d^0, d_d^0)\) converges to \((\Omega; a, b, c, d)\) in the sense of 1.3. Assume the same setup as in Section 4.2. We consider the UST \( T_{\delta} \) in \( \Omega_{\delta} \) with \((a_\delta b_\delta)\) wired and \((c_\delta d_\delta)\) wired. There are two Peano curves running along \( T_{\delta} \), and we denote by \( \eta_\delta^L \) the one from \( a_\delta \) to \( d_\delta \) and by \( \eta_\delta^R \) the one from \( b_\delta \) to \( c_\delta \). There exists a unique simple path in \( T_{\delta} \), denoted by \( \gamma_\delta^M \), connecting \((a_\delta b_\delta)\) to \((c_\delta d_\delta)\).

- For the quad \((\Omega; a, b, c, d)\), let \( K \) be its conformal modulus and denote by \( f = f_{(\Omega; a, b, c, d)} \) the conformal map from \( \Omega \) onto \((0, 1) \times (0, iK)\) which sends \((a, b, c, d)\) to \((0, 1, 1 + iK, iK)\) and extend its definition continuously to the boundary.

- Consider the Poisson kernel for the rectangle \( f(\Omega) = (0, 1) \times (0, iK) \). Define, for all \( r \in (f(a)f(b)) \cup (f(c)f(d)) \) and for all \( z \in [(0, 1) \times [0, iK]) \setminus \{r\}, \)

\[
P_K(z, r) = \text{Im} \sum_{n \in \mathbb{Z}} \left( \frac{1}{\exp(\frac{K}{2}(2n - r + z)) - 1} + \frac{1}{\exp(\frac{K}{2}(2n - r - \bar{z})) - 1} \right).
\]

(5.1)

Note that \( P_K(\cdot, r) \) is continuous on \([0, 1] \times [0, iK]) \setminus \{r\}, \) and it is harmonic on \((0, 1) \times (0, iK)\) with the following boundary data:

\[
P_K(\cdot, r) = 0 \text{ on } (f(a)f(b)) \cup (f(c)f(d)) \setminus \{r\}, \quad \partial_n P_K(\cdot, r) = 0 \text{ on } (f(b)f(c)) \cup (f(d)f(a)).
\]

(5.2)

where \( n \) is the outer normal vector.

Before we proceed, let us explain how we figure out the formula for Poisson kernel in (5.1) as a harmonic solution with boundary data (5.2). The Poisson kernel \( P_K(z, r) \) with boundary data (5.2) can be regarded as the “renormalized probability” that the Brownian motion from \( z \) reflecting on the edges \((0, iK) \cup (1, 1 + iK)\) first hits \((0, 1) \cup (iK, 1 + iK)\) at \( r \). By considering the strip \( S := \mathbb{R} \times [0, iK] \), this “renormalized probability” is equal to the “renormalized probability” that the Brownian motion from \( z \) exits \( S \) at the points \( \{r + n, -r + n\}_{n \in \mathbb{Z}} \). Hence we have

\[
P_K(z, r) = \sum_{n \in \mathbb{Z}} (P_S(z, r + n) + P_S(z, -r + n)),
\]
where $P_S(z, q)$ is the Poisson kernel of the strip $S$ with Dirichlet boundary condition. Note that the Poisson kernel $P_S(z, q)$ can be obtained from the Poisson kernel of the upper half plane $\mathbb{H}$ by composing with the conformal map $\varphi(z) = e^z$. This gives the desired formula in (5.1).

Let us come back to the distribution of the limits of the pairs $(X^M_\delta, Y^M_\delta)_{\delta > 0}$.

**Proposition 5.1.** The pair $(X^M_\delta, Y^M_\delta)$ converges weakly as a pair of points in $\mathbb{R}^2$ to a random pair of points $(X^M, Y^M)$ as $\delta \to 0$. Denote by $x^M := f(X^M)$ and $y^M := \text{Ref}(Y^M)$. The law of the pair $(x^M, y^M)$ is characterized by the following.

(1) The law of $x^M$ is uniform on $(0, 1)$.

(2) The conditional density of $y^M$ given $x^M \in (0, 1)$ is the following:

$$
\rho_K(x^M, y) = \partial_n P_K(z, y + iK)\big|_{z = x^M}, \quad \forall y \in (0, 1).
$$

In particular, the joint density of $(x^M, y^M)$ is given by (1.4).

The proof of Proposition 5.1 is split into three lemmas. In Lemma 5.2, we first derive the limiting distribution of $X^M_\delta$. This step is immediate from the convergence of the observable in Lemmas 4.7 and 4.8. We then derive the conditional law of $y^M$ given $x^M$. To this end, we first analyze the conditional probability in the discrete setting in Lemma 5.3 and use good control on discrete harmonic functions proved in [CW21]; and then we derive the limit of the conditional probability in Lemma 5.4.

**Lemma 5.2.** The pair $(X^M_\delta, Y^M_\delta)$ converges weakly as a pair of points in $\mathbb{R}^2$ to a random pair of points $(X^M, Y^M)$ as $\delta \to 0$. Moreover, the law of $x^M = f(X^M)$ is uniform on $(0, 1)$.

**Proof.** From Theorem 1.1, the curve $\gamma^M_\delta$ converges weakly to $\gamma^M$ as $\delta \to 0$. Since $X^M_\delta = \gamma^M_\delta(0)$, $Y^M_\delta = \gamma^M_\delta(1)$ and $X^M = \gamma^M(0)$, $Y^M = \gamma^M(1)$, this implies that $(X^M_\delta, Y^M_\delta)$ converges weakly to $(X^M, Y^M)$ as a pair of points in $\mathbb{R}^2$ as $\delta \to 0$, where $X^M = \gamma^M \cap (ab)$ and $Y^M = \gamma^M \cap (cd)$. It remains to show that $f(X^M)$ is uniform in $(0, 1)$.

Recall from Lemma 4.7 that $u_\delta(z^*)$ is the probability that $z^*$ lies to the right of $\eta^L_\delta$ for every $z^* \in \Omega^*_\delta$ and that $u_\delta$ converges to $\text{Ref}$ locally uniformly due to Lemma 4.8. We denote by $\Omega^R$ the connected component of $\Omega \setminus \gamma^M$ which contains $[bc]$ on its boundary. It is also the connected component of $\Omega \setminus \eta^L$ which contains $[bc]$ on its boundary. For every $z \in \Omega$ and $z^*_n \in \Omega^*_\delta_n$ such that $z^*_n \rightarrow z$, we have

$$
\{ z \in \Omega^R \} \subset \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} \{ z^*_n \text{ lies to the right of } \eta^L_n \} \subset \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} \{ z^*_n \text{ lies to the right of } \eta^L_n \}
$$

This implies that

$$
P[z \in \Omega^R] \leq \lim_{n \to \infty} u_\delta(z^*_n) = \text{Ref}(z) = \lim_{n \to \infty} u_\delta(z^*_n) \leq P[z \in \overline{\Omega}^R].
$$

Note that $P[z \in \Omega^R] = P[z \in \overline{\Omega}^R]$ as $P[z \in \gamma^M] = 0$ due to Corollary 4.14. Therefore,

$$
P[z \in \Omega^R] = \text{Ref}(z), \quad \forall z \in \Omega.
$$

(5.4)

For every $\theta \in (ab)$, we choose $\{w_n\} \subset \Omega$ such that $w_n \to \theta$ as $n \to \infty$. Then, we have

$$
\{ X^M \in (a\theta) \} \subset \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} \{ w_n \in \Omega^R \} \subset \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} \{ w_n \in \Omega^R \} \subset \{ X^M \in (a\theta) \}.
$$

Since $f$ is continuous on $\overline{\Omega}$, we have

$$
P[X^M \in (a\theta)] \leq \lim_{n \to \infty} \text{Ref}(w_n) = \text{Ref}(\theta) \leq \lim_{n \to \infty} \text{Ref}(w_n) \leq P[X^M \in (a\theta)].
$$
Furthermore, for every $\tilde{\theta} \in (\theta b)$, we have
\[
\text{Ref}(\theta) \leq \mathbb{P}[X^M \in (a\theta)] \leq \mathbb{P}[X^M \in (a\tilde{\theta})] \leq \text{Ref}(\tilde{\theta}).
\]
By letting $\tilde{\theta} \to \theta$, we have
\[
\mathbb{P}[X^M \in (a, \theta)] = \text{Ref}(\theta).
\]
This implies that $\mathbb{P}[f(X^M) \in (0, \text{Ref}(\theta))] = \mathbb{P}[X^M \in (a, \theta)] = \text{Ref}(\theta)$ as desired. Note that $f(a, b) = (0, 1)$ and we complete the proof.

![Figure 5.1: The function $\lambda = \lambda_{\tilde{c}, d}$ is the unique bounded harmonic function on $(0, 1) \times (0, iK)$ with the following boundary data: $\lambda = 0$ on $(f(a)f(b)) \cup (f(c)f(\tilde{c})) \cup (f(\tilde{d})f(d))$; $\lambda = 1$ on $(f(\tilde{c})f(\tilde{d}))$; and $\partial_n \lambda = 0$ on $(f(b)f(c)) \cup (f(d)f(a))$ where $n$ is the outer normal vector.](image)

**Lemma 5.3.** Fix a polygon $(\Omega; a, \tilde{a}, \tilde{b}, b, c, \tilde{c}, d, d)$ with eight marked points. Suppose that a sequence of medial polygons $(\Omega^\delta; a^\delta, \tilde{a}^\delta, \tilde{b}^\delta, b^\delta, c^\delta, \tilde{c}^\delta, d^\delta, d^\delta)$ converges to $(\Omega; a, \tilde{a}, \tilde{b}, b, c, \tilde{c}, d, d)$ in the sense of (1.3). Denote by $\lambda_{\tilde{c}, d}$ the unique bounded harmonic function on $(0, 1) \times (0, iK)$ with the boundary data as shown in Figure 5.1. Denote by $\lambda_{c, d}$ when $\tilde{c} = c$ and $\tilde{d} = d$. Then, for every $\epsilon > 0$, there exist $\delta_0 > 0$ and $s > 0$ such that for all $\delta \leq \delta_0$ and $x^\delta \in (\tilde{a}^\delta \tilde{b}^\delta)$ and $x \in (\tilde{a} \tilde{b})$ with $\text{dist}(x^\delta, x) \leq s$, we have
\[
\left| \mathbb{P} \left[ X^M_{\delta} \in (\tilde{c}^\delta \tilde{d}^\delta) \mid X^M_{\delta} = x^\delta \right] - \frac{\partial_n \lambda_{\tilde{c}, d}(f(x))}{\partial_n \lambda_{c, d}(f(x))} \right| \leq \epsilon.
\]

**Proof.** First, we show that $\partial_n \lambda_{c, d}(f(x)) \neq 0$ for all $x \in (ab)$. Let $g$ be the bounded harmonic function on $(0, 1) \times (0, iK)$ with the following boundary data: $g = 0$ on $(f(d)f(c))$ and $g = 1$ on $(f(c)f(d))$. By maximum principle, we have $\lambda_{c, d}(y) \geq g(y)$ for every $y \in [0, 1] \times [0, iK]$. Thus, we have
\[
\partial_n \lambda_{c, d}(f(x)) \leq \partial_n g(f(x)) < 0, \quad \text{for all } x \in (ab).
\]

Next, we prove (5.5). To this end, Wilson’s algorithm allows us to use estimates on loop erased random walk to get estimations on the branch $\gamma^M_{\delta}$. To be precise, denote by $\Omega_{\delta}$ the graph obtained from $\Omega_{\delta}$ by regarding $(a_\delta b_\delta)$ as a single vertex (and keep all edges connecting $(a_\delta b_\delta)$ to interior vertices). Let $\tilde{\gamma}_{\delta}$ be the loop-erasure of the random walk on $\Omega_{\delta}$ starting from $(a_\delta b_\delta)$ and stopped whenever it hits $(c_\delta d_{\delta})$. Wilson’s algorithm [Pem91] tells that $\gamma^M_{\delta}$ (viewed in $\Omega_{\delta}$) has the same law as $\tilde{\gamma}_{\delta}$. Going back to $\Omega_{\delta}$, we obtain that, for $v_{\delta} \in \Omega_{\delta}$ with $v_{\delta} \sim (a_\delta b_\delta)$ and $\mathbb{P}[\tilde{\gamma}_{\delta}(1) = v_{\delta}] > 0$, it holds that $\mathbb{P}[\gamma^M_{\delta}(1) = v_{\delta}] = \mathbb{P}[\tilde{\gamma}_{\delta}(1) = v_{\delta}]$. Moreover, the conditional law of $\gamma^M_{\delta}$ given $\{\gamma^M_{\delta}(1) = v_{\delta}\}$ is the same as the loop-erasure of the random walk in $\Omega_{\delta}$ starting from $v_{\delta}$ conditioned to hit $(a_\delta b_\delta)$ at $(c_\delta d_{\delta})$.

For any $w_{\delta} \in \Omega_{\delta}$, denote by $\mathbb{P}^{u_{\delta}}$ the law of random walk $\mathcal{R}$ in $\Omega_{\delta}$ starting from $w_{\delta}$. Define
\[
u_{\delta}(w_{\delta}) := \mathbb{P}^{u_{\delta}} \left[ \mathcal{R} \text{ hits } (a_\delta b_\delta) \cup (c_\delta d_{\delta}) \text{ at } (c_\delta d_{\delta}) \right],
\]
\[
u_{\delta}(w_{\delta}) := \mathbb{P}^{u_{\delta}} \left[ \mathcal{R} \text{ hits } (a_\delta b_\delta) \cup (c_\delta d_{\delta}) \text{ at } (\tilde{c}^\delta \tilde{d}^\delta) \right].
\]
By the above relation between \( \gamma^M_\delta \) and the random walk, we have

\[
P \left[ Y^M_\delta \in (\tilde{c}_\delta \tilde{d}_\delta) \mid X^M_\delta = x_\delta \right] = \frac{\sum_{v_\delta \sim x_\delta} \mathbb{P}^{v_\delta} \left[ R \text{ hits } (a_\delta b_\delta) \cup (c_\delta d_\delta) \text{ at } (\tilde{c}_\delta \tilde{d}_\delta) \right]}{\sum_{v_\delta \sim x_\delta} \mathbb{P}^{v_\delta} \left[ R \text{ hits } (a_\delta b_\delta) \cup (c_\delta d_\delta) \text{ at } (c_\delta d_\delta) \right]} = \frac{\sum_{v_\delta \sim x_\delta} \tilde{u}_\delta(v_\delta)}{\sum_{v_\delta \sim x_\delta} u_\delta(v_\delta)}.
\]

The function \( \tilde{u}_\delta \) is a discrete harmonic function on \( \Omega_\delta \setminus (a_\delta b_\delta) \cup (c_\delta d_\delta) \) with the following boundary data: \( \tilde{u}_\delta = 0 \) on \( (a_\delta b_\delta) \cup (c_\delta \tilde{d}_\delta) \cup (\tilde{d}_\delta d_\delta) \) and \( \tilde{u}_\delta = 1 \) on \( (\tilde{c}_\delta \tilde{d}_\delta) \). Similarly, \( u_\delta \) is a discrete harmonic function on \( \Omega_\delta \setminus (a_\delta b_\delta) \cup (c_\delta d_\delta) \) with the following boundary data: \( u_\delta = 0 \) on \( (a_\delta b_\delta) \) and \( u_\delta = 1 \) on \( (c_\delta d_\delta) \). By [CW21 Corollary 3.8], for every \( \epsilon > 0 \), there exists \( s_1 > 0 \) such that

\[
1 - \epsilon \leq \frac{\tilde{u}_\delta(z_\delta)}{u_\delta(z_\delta)} \leq 1 + \epsilon, \quad \text{for all } z_\delta \in \Omega_\delta \cap B(x_\delta, s_1) \setminus \partial \Omega_\delta \text{ and all } x_\delta \in (\tilde{a}_\delta \tilde{b}_\delta).
\]

This implies that

\[
1 - \epsilon \leq P \left[ Y^M_\delta \in (\tilde{c}_\delta \tilde{d}_\delta) \mid X^M_\delta = x_\delta \right] \times \frac{u_\delta(z_\delta)}{\tilde{u}_\delta(z_\delta)} \leq 1 + \epsilon.
\]

We choose \( s < \frac{n}{4} \). Since \( \partial \Omega \) is a curve, we can choose a simple curve \( L \) such that \( \frac{\epsilon}{4} < \text{dist}(L, (\tilde{a}_\delta \tilde{b}_\delta)) < \frac{\epsilon}{2} \). Note that there exists \( \delta_1 > 0 \) such that for all \( \delta < \delta_1 \), we can choose a discrete simple curve \( L_\delta \subset \Omega_\delta \) with \( \frac{\epsilon}{4} < \text{dist}(L_\delta, (\tilde{a}_\delta \tilde{b}_\delta)) < \frac{\epsilon}{2} \) and \( L_\delta \to L \) as curves. By the same argument as in the proof of Lemma B.2, we have \( \tilde{u}_\delta \to \lambda_{\tilde{c}_\delta \tilde{d}_\delta} f \) and \( u_\delta \to \lambda_{\delta \delta} f \) locally uniformly in \( \Omega \). Then, there exists \( \delta_2 > 0 \) such that the following holds: for every \( \delta < \delta_2 \) and \( \text{dist}(x_\delta, x) < s \), there exists \( z \in B(x_\delta, s) \cap L \) such that

\[
\left| \frac{\tilde{u}_\delta(z_\delta)}{u_\delta(z_\delta)} - \frac{\lambda_{\tilde{c}_\delta \tilde{d}_\delta}(f(z_\delta))}{\lambda_{\delta \delta}(f(z))} \right| < \epsilon, \quad \forall z_\delta \in B(x_\delta, s) \cap L_\delta.
\]

Since \( f \) is continuous on \( \overline{\Omega} \), we have \( \text{diam}(f(B(x, s))) \to 0 \) as \( s \to 0 \). By Taylor expansion, we can choose \( s \) small enough such that

\[
\left| \frac{\lambda_{\tilde{c}_\delta \tilde{d}_\delta}(f(z))}{\lambda_{\delta \delta}(f(z))} - \frac{\partial_n \lambda_{\tilde{c}_\delta \tilde{d}_\delta}(f(x))}{\partial_n \lambda_{\delta \delta}(f(x))} \right| \leq \epsilon, \quad \text{for all } x \in (\tilde{a}_\delta \tilde{b}_\delta) \text{ and } z \in B(x, s) \cap L.
\]

This implies that, if \( \delta < \delta_1 \wedge \delta_2 \) and \( \text{dist}(x_\delta, x) < s \), we have

\[
(1 - 3\epsilon) \frac{\partial_n \lambda_{\tilde{c}_\delta \tilde{d}_\delta}(f(x))}{\partial_n \lambda_{\tilde{c}_\delta \tilde{d}_\delta}(f(x))} \leq \mathbb{P} \left[ Y^M_\delta \in (\tilde{c}_\delta \tilde{d}_\delta) \mid X^M_\delta = x_\delta \right] \leq (1 + 3\epsilon) \frac{\partial_n \lambda_{\tilde{c}_\delta \tilde{d}_\delta}(f(x))}{\partial_n \lambda_{\tilde{c}_\delta \tilde{d}_\delta}(f(x))}.
\]

This completes the proof.

**Lemma 5.4.** The conditional law of \( Y^M \) given \( X^M \) is given by

\[
P \left[ Y^M \in (\tilde{c} \tilde{d}) \mid X^M \right] = \frac{\partial_n \lambda_{\tilde{c} \tilde{d}}(f(X^M))}{\partial_n \lambda_{\tilde{c} \tilde{d}}(f(X^M))}.
\]

*Proof.* By conformal invariance, we may assume \( \Omega = (0, 1) \times (0, iK) \). We couple \((X^M_\delta, Y^M_\delta)\) and \((X^M, Y^M)\) together such that \( X^M_\delta \to X^M \) and \( Y^M_\delta \to Y^M \) almost surely. Fix a polygon \((\Omega; a, \tilde{a}, b, \tilde{b}, c, \tilde{c}, d, \tilde{d})\) with eight marked points. Suppose that a sequence of medial polygons \((\Omega^\delta; a^\delta, \tilde{a}^\delta, b^\delta, \tilde{b}^\delta, c^\delta, \tilde{c}^\delta, d^\delta, \tilde{d}^\delta)\) converges to \((\Omega; a, \tilde{a}, b, \tilde{b}, c, \tilde{c}, d, \tilde{d})\) in the sense of [1.3]. For any \( \delta_n \to 0 \), we have

\[
\{X^M \in (\tilde{a} \tilde{b}), Y^M \in (\tilde{c} \tilde{d})\} \subset \left\{ \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \{X^M_{\delta_n} \in (\tilde{a} \tilde{b} \delta_n), Y^M_{\delta_n} \in (\tilde{c} \tilde{d} \delta_n)\} \right\}
\]

\[
\subset \left\{ \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \{X^M_{\delta_n} \in (\tilde{a} \tilde{b} \delta_n), Y^M_{\delta_n} \in (\tilde{c} \tilde{d} \delta_n)\} \right\} \subset \{X^M \in [\tilde{a} \tilde{b}], Y^M \in [\tilde{c} \tilde{d}]\}.
\]
Therefore, have

By letting \( \delta \to 0 \), we choose \( s \) the same as in Lemma 5.3. Divide \((\tilde{a}, \tilde{b})\) into \( \cup_{j=0}^{m}[x^j, x^{j+1}] \) with \( x^0 = \tilde{a} \) and \( x^{m+1} = \tilde{b} \) such that the length of \([x^j, x^{j+1}]\) is less than \( s \). Denote by \( \{x^0_{\delta_n}, x^1_{\delta_n}, \ldots, x^m_{\delta_n} = \tilde{\delta}_n\} \subset (\tilde{a}_n, \tilde{b}_n) \) the corresponding discrete approximation. By Lemma 5.3, for \( n \) large enough, we have

By Lemma 5.2 the law of \( X^M \) is uniform on \((\tilde{a}, \tilde{b})\). This implies that

Thus, we have

By letting \( s \to 0 \) \((m \to \infty)\) and \( \epsilon \to 0 \), we obtain

where we used that \( X^M \) is uniform distributed on \((0, 1)\) (see Lemma 5.2). Plugging into (5.7) implies that

Note that the marginal law of \( X^M \) is uniform on \((\tilde{a}, \tilde{b})\) and the marginal law of \( Y^M \) is uniform on \((\tilde{c}, \tilde{d})\), we have

Therefore,

This gives (5.6) and completes the proof.

Proof of Proposition 5.7. The convergence of \((X^M_\delta, Y^M_\delta)\) and the law of \( x^M = f(X^M) \) is derived Lemma 5.2. The conditional law of \( Y^M \) given \( X^M \) is derived in Lemma 5.4. Since \( \lambda_{\tilde{c}, \tilde{d}}(\cdot) \) is a bounded harmonic function on \([0, 1] \times [0, iK]\) with boundary data illustrated in Figure 5.1, Poisson integral formula implies

Similarly,

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Let us calculate the outer normal derivative of $P_K$: for $x, y \in (0, 1)$,
\[
\partial_n P_K(z, y + iK)|_{z=x} = \frac{\pi}{4K} \sum_{n \in \mathbb{Z}} \left( \frac{1}{\cosh^2 \left( \frac{x}{2K} (x - y - 2n) \right)} + \frac{1}{\cosh^2 \left( \frac{x}{2K} (x + y - 2n) \right)} \right),
\]
where the right-hand side is the same as $\rho_K(x, y)$ defined in (1.4). Hence for $x \in (0, 1)$, we have
\[
\partial_n \lambda_{\tilde{c}, \tilde{d}}(x) = \int_{\text{Ref}(\tilde{d})}^{\text{Ref}(\tilde{c})} (\partial_n P_K(z, y + iK)|_{z=x}) dy = \int_{\text{Ref}(\tilde{d})}^{\text{Ref}(\tilde{c})} \rho_K(x, y)dy
\]
and
\[
\partial_n \lambda_{c, d}(x) = \int_0^1 (\partial_n P_K(z, y + iK)|_{z=x}) dy
\]
\[
= \frac{\pi}{4K} \sum_{n \in \mathbb{Z}} \left( \int_0^1 \frac{dy}{\cosh^2 \left( \frac{x}{2K} (x - y - 2n) \right)} + \int_0^1 \frac{dy}{\cosh^2 \left( \frac{x}{2K} (x + y - 2n) \right)} \right)
\]
\[
= \frac{\pi}{4K} \sum_{n \in \mathbb{Z}} \left( \int_{2n}^{2n+1} \frac{dr}{\cosh^2 \left( \frac{x}{2K} (x - r) \right)} + \int_{2n}^{2n-1} \frac{dr}{\cosh^2 \left( \frac{x}{2K} (x - r) \right)} \right)
\]
\[
= \frac{\pi}{4K} \int_{-\infty}^{+\infty} \frac{dr}{\cosh^2 \left( \frac{x}{2K} (x - r) \right)} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dr}{\cosh^2(r)} = 1.
\]

Therefore, combining with (5.6) we get
\[
P[Y^M \in (\tilde{c}d) | X^M] = \frac{\partial_n \lambda_{\tilde{c}, \tilde{d}}(f(X^M))}{\partial_n \lambda_{c,d}(f(X^M))} = \frac{\partial_n \lambda_{\tilde{c}, \tilde{d}}(f(X^M))}{\partial_n \lambda_{c,d}(f(X^M))} = \int_{\text{Ref}(\tilde{d})}^{\text{Ref}(\tilde{c})} \rho_K(x, y)dy.
\]

This gives the density in (1.4) and completes the proof. \qed

Corollary 5.5. Fix a polygon $(\Omega; a, x, b, c, \tilde{c}, \tilde{d}, d)$ with seven marked points. Suppose that a sequence of medial polygons $(\Omega_{\delta}; a_{\delta}, x_{\delta}, b_{\delta}, c_{\delta}, \tilde{c}_{\delta}, \tilde{d}_{\delta}, d_{\delta})$ converges to $(\Omega; a, x, b, c, \tilde{c}, \tilde{d}, d)$ in the sense of (1.3). Denote by $f = f_{(\Omega; a, b, c, d)}$ the conformal map from $\Omega$ onto $(0, 1) \times (0, iK)$ which sends $(a, b, c, d)$ to $(0, 1, 1 + iK, iK)$. We extend $f$ continuously to the boundary. Then,
\[
\lim_{\delta \to 0} \mathbb{P}[Y^M_{\delta} \in (\tilde{c}_{\delta}d_{\delta}) | X^M_{\delta} = x_{\delta}] = \int_{\text{Ref}(\tilde{d})}^{\text{Ref}(\tilde{c})} \rho_K(f(x), y)dy.
\]

Proof. First let $\delta \to 0$ and then let $\epsilon \to 0$ in (5.5), combining with (5.9), we obtain the conclusion. \qed

We emphasize that in Corollary 5.5 we only need the assumption that $\partial \Omega$ is locally connected and we do not require extra regularity.

5.2 Proof of Theorem 1.5

The joint law of $(X^M, Y^M)$ in Theorem 1.5 is given in Proposition 5.1, and to complete the proof of Theorem 1.5 it remains to show that the conditional law of $\gamma^M$ given $X^M$ is SLE$_2(-1, -1; -1, -1)$. We follow the strategy in [Zha08c]. We fix the following notation in this section.

- Fix $d = -\infty < a < b < c$. Denote by $K$ the conformal modulus of the quad $(\hat{\Omega}; a, b, c, \infty)$ and by $f(\cdot; a, b, c)$ the conformal map from $\hat{\Omega}$ onto $(0, 1) \times (0, iK)$ sending $(a, b, c, \infty)$ to $(0, 1, 1 + iK, iK)$.
• Fix $d = -\infty < a < w < b < c$. Define

$$P(z; a, w, b, c) := P_K(f(z; a, b, c), f(w; a, b, c)), \quad \forall z \in \mathbb{H},$$

(5.10)

where $P_K$ is given in (5.1). Note that $P(\cdot ; \cdot, \cdot, \cdot)$ is smooth on $\mathbb{H} \times \{(a, w, b, c) \in \mathbb{R}^4 : a < w < b < c\}$. It is the Poisson kernel on $\mathbb{H}$ with the boundary data:

$$P(\cdot; a, w, b, c) = 0, \quad \text{on } (a, w) \cup (w, b) \cup (c, \infty); \quad \partial_n P(\cdot; a, w, b, c) = 0, \quad \text{on } (-\infty, a) \cup (b, c);$$

(5.11)

and the normalization:

$$\int_{\mathbb{C}} (\partial_n P(z; a, w, b, c) |_{z = x}) dx = 1.$$  

(5.12)

The strategy is as follows: first, we show that the Poisson kernel satisfies a certain PDE in Lemma 5.6; then we show that the conditional density in (5.3) gives a martingale observable for $\gamma^M$ in Lemma 5.10. With these two lemmas at hand, we obtain the driving function from the martingale observable. This last step involves a non-trivial calculation where Lemma 5.6 plays a crucial role.

**Lemma 5.6.** For $a < w < b < c$ and $z = x + iy \in \mathbb{H}$, consider the function $P(z; a, w, b, c)$ in (5.10). Denote by $\partial_x$ the partial derivative with respect to the real part of the first (complex) variable and by $\partial_y$ the partial derivative with respect to the imaginary part of the first (complex) variable. Define

$$D := \frac{2}{a - w} \partial_a + \frac{2}{b - w} \partial_b + \frac{2}{c - w} \partial_c + 2f''(w; a, b, c) \partial_w + \partial^2 w + \text{Re} \left( \frac{2}{z - w} \right) \partial_x + \text{Im} \left( \frac{2}{z - w} \right) \partial_y.$$  

Then, we have $D P(z; a, w, b, c) = 0$.

To prove Lemma 5.6, we define

$$\mathcal{V}(z) := DP(z; a, w, b, c).$$

(5.13)

The goal is to show $\mathcal{V} \equiv 0$. To this end, we will show that $\mathcal{V}$ is harmonic in $\mathbb{H}$ and has the same boundary data as $P(\cdot; a, w, b, c)$ in Lemma 5.7. We will show that $\mathcal{V}$ is bounded near $a, b, c, \infty$ in Lemma 5.8. This implies that $\mathcal{V}$ is a bounded harmonic function such that

$$\mathcal{V} = 0, \quad \text{on } (a, b) \cup (c, \infty); \quad \partial_n \mathcal{V} = 0, \quad \text{on } (-\infty, a) \cup (b, c).$$

By the same argument used in the proof of Lemma B.1, we get that $\mathcal{V} \equiv 0$ and then complete the proof of Lemma 5.6.

**Lemma 5.7.** The function $\mathcal{V}$ in (5.13) is harmonic in $\mathbb{H}$ and has the same boundary data as $P(\cdot; a, w, b, c)$.

**Proof.** First, we show that $\mathcal{V}(\cdot)$ is harmonic in $\mathbb{H}$. Note that by the explicit form of $P(z; a, w, b, c)$ in (5.10) and $P_K$ in (5.1), the sum over terms $n \neq 0$ is smooth on $\mathbb{H} \times \{(a, w, b, c) \in \mathbb{R}^4 : a < w < b < c\}$ and it is analytic on $\mathbb{H} \setminus \{a, b, c\}$ when $a < w < b < c$ is fixed. For the term $n = 0$, by subtracting $\text{Im} \frac{K}{\pi f''(w; a, b, c)} \frac{1}{(z - w)}$, we get a function which is smooth on $\mathbb{H} \times \{(a, w, b, c) \in \mathbb{R}^4 : a < w < b < c\}$ and analytic on $\mathbb{H} \setminus \{a, b, c\}$ when $a < w < b < c$ is fixed. Therefore we may write

$$P(z; a, w, b, c) = \text{Im} \left( G(z; a, w, b, c) + \frac{K}{\pi f''(w; a, b, c)} \frac{1}{(z - w)} \right), \quad \forall z \in \mathbb{H}, w \in (a, b),$$

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where $G(\cdot, \cdot, \cdot, \cdot)$ is a smooth function function on $\mathbb{H} \times \{(a, w, b, c) \in \mathbb{R}^4 : a < w < b < c\}$. Moreover, $G$ is analytic on $\mathbb{H} \setminus \{a, b, c\}$ when $a < w < b < c$ is fixed. Then, we get

\[
\begin{align*}
\left( \operatorname{Re} \left( \frac{2}{z-w} \right) \partial_x + \operatorname{Im} \left( \frac{2}{z-w} \right) \partial_y \right) P(z; a, w, b, c) &= \operatorname{Im} \left( \frac{2G'(z; a, w, b, c)}{z-w} - \frac{2K}{\pi f'(w; a, b, c)(z-w)^3} \right), \\
\partial_w P(z; a, w, b, c) &= \operatorname{Im} \left( \partial_w G(z; a, w, b, c) + \frac{K}{\pi f'(w; a, b, c)} \frac{1}{(z-w)^2} - \frac{K}{\pi} \frac{f''(w; a, b, c)}{f'(w; a, b, c)^2} \frac{1}{z-w} \right), \\
\partial_w^2 P(z; a, w, b, c) &= \operatorname{Im} \left( \left( \partial_w^2 G(z; a, w, b, c) - \frac{K}{\pi} \frac{f'''(w; a, b, c) f'(w; a, b, c) - 2f''(w; a, b, c)^2}{f'(w; a, b, c)^3} \right) \right) + \operatorname{Im} \left( \frac{2K}{\pi f'(w; a, b, c)} \frac{1}{(z-w)^3} - \frac{2K}{\pi} \frac{f''(w; a, b, c)}{f'(w; a, b, c)^2} \frac{1}{(z-w)^2} \right).
\end{align*}
\]

Therefore, we can write

\[
\mathcal{V}(z) = \operatorname{Im} \left( G_1(z) + \frac{G_2(z)}{z-w} \right),
\]

where $G_1$ and $G_2$ are analytic functions on $\mathbb{H} \setminus \{a, b, c\}$. This implies that $\mathcal{V}$ is harmonic.

Next, we show that $\mathcal{V}(\cdot)$ has the same boundary data as $P(\cdot; a, w, b, c)$. From (5.11), we have $P(z; a, w, b, c) = 0$ for all $a < z \neq w < b < c$. Thus

\[
\ell P(z; a, w, b, c) = 0, \quad \forall a < z \neq w < b < c, \quad \text{for all } \ell = \partial_x, \partial_y, \partial_a, \partial_b, \partial_c, \partial_w, \partial_{w}^2, \partial_z.
\]

Therefore, $\mathcal{V}(\cdot) = 0$ on $(a, w) \cup (w, b)$. Similarly, since $P(z; a, w, b, c) = 0$ for all $a < w < b < c < z$, we have $\mathcal{V}(\cdot) = 0$ on $(c, +\infty)$. Since $\partial_n P(x; a, w, b, c) = 0$ for all $a < w < b < x < c$ or $x < a < w < b < c$, we have

\[
\partial_n \ell P(x; a, w, b, c) = \ell \partial_n P(x; a, w, b, c) = 0, \quad \text{for all } \ell = \partial_a, \partial_b, \partial_c, \partial_w, \partial_{w}^2, \partial_x,
\]

\[
\begin{align*}
\partial_n \left( \operatorname{Re} \frac{2}{z-w} \partial_x P(z; a, w, b, c) \right) &\bigg|_{z=x} = \frac{2}{x-w} \partial_x \partial_n P(x; a, w, b, c) = 0, \\
\partial_n \left( \operatorname{Im} \frac{2}{z-w} \partial_y P(z; a, w, b, c) \right) &\bigg|_{z=x} = -\frac{2}{(x-w)^2} \partial_n P(x; a, w, b, c) = 0,
\end{align*}
\]

\[
\forall a < w < b < x < c \text{ or } x < a < w < b < c.
\]

Here the interchange of $\partial_n$ and $\partial_a, \partial_b, \partial_c, \partial_w, \partial_x$ is legal due to the explicit form of $P(z; a, w, b, c)$ in (5.10). Thus, $\partial_n \mathcal{V}(\cdot) = 0$ on $(b, c) \cup (-\infty, a)$. This completes the proof.

\[\Box\]

**Lemma 5.8.** The function $\mathcal{V}$ in (5.13) is bounded near $a, b, c, \infty$.

**Proof.** We first investigate its behavior around $a$. From (4.7), we have

\[
f(z; a, b, c) = K \left( \arcsin \sqrt{\frac{z-a}{b-a}} \frac{b-a}{c-a} \right) / K \left( \frac{b-a}{c-a} \right),
\]

and

\[
K = K(a, b, c) = \operatorname{Im} K \left( \arcsin \sqrt{\frac{c-a}{b-a}} \frac{b-a}{c-a} \right) / K \left( \frac{b-a}{c-a} \right).
\]
Note that $f(\cdot, \cdot, \cdot)$ is smooth on $\mathbb{R} \setminus \{(a, b, c) \times \{(a, b, c) \in \mathbb{R}^3 : a < b < c\}$ and $K(\cdot, \cdot, \cdot)$ is smooth on \{(a, b, c) \in \mathbb{R}^3 : a < b < c\}. This implies that $\partial_w P$ and $\partial_w^2 P$ are continuous at $a$. Moreover, for all $\ell \in \{\partial_a, \partial_b, \partial_c, \partial_x, \partial_y\}$, we have

$$\ell P(z; a, w, b, c) = \text{Im} \sum_{n \in \mathbb{Z}} \frac{1}{\sinh^2 \left( \frac{\pi}{2K} (2n - f(w; a, b, c) + f(z; a, b, c)) \right)} \ell \left( \frac{\pi}{4K} (f(z; a, b, c) - f(w; a, b, c)) \right)$$

$$- \text{Im} \sum_{n \in \mathbb{Z}} \frac{1}{\sinh^2 \left( \frac{\pi}{2K} (2n - f(w; a, b, c) - f(z; a, b, c)) \right)} \ell \left( \frac{\pi}{4K} (f(z; a, b, c) + f(w; a, b, c)) \right).$$

Denote by $\tilde{z} := \text{arcsin} \left( \frac{\sqrt{z-a}}{\sqrt{b-a}} \right)$ and $s := (b-a)/(c-a)$. We have

$$\partial_b f(z; a, b, c) = -\frac{\sqrt{(c-a)(z-a)}}{2(b-a)K(s)\sqrt{(c-z)(b-z)}} + \frac{\partial_x K(\tilde{z}, s)}{(c-a)K(s)} - \frac{K(\tilde{z}, s)K'(s)}{(c-a)K^2(s)};$$

$$\partial_c f(z; a, b, c) = -\frac{(b-a)\partial_x K(\tilde{z}, s)}{(c-a)^2K(s)} + \frac{(b-a)K(\tilde{z}, s)K'(s)}{(c-a)^2K^2(s)}.$$ 

This implies that

$$\partial_b P(z; a, w, b, c) \to 0, \quad \partial_c P(z; a, w, b, c) \to 0, \quad \text{as } z \to a. \quad (5.15)$$

A direct calculation implies

$$\left| \partial_y P(z; a, w, b, c) \text{Im} \left( \frac{2}{z-w} \right) \right| = \left| \frac{\sqrt{c-a}}{2\sqrt{(b-z)(c-z)(z-a)K(s)}} \frac{\text{Im} \left( \frac{2}{z-w} \right)}{z-w} \right|$$

$$= \left| \frac{\sqrt{c-a} \times \text{Im} z}{\sqrt{(b-z)(c-z)(z-a)|K(s)|z-w|^2}} \right|.$$ 

Thus

$$\partial_y P(z; a, w, b, c) \text{Im} \left( \frac{2}{z-w} \right) \to 0, \quad \text{as } z \to a. \quad (5.16)$$

Note that

$$\left( \frac{2}{a-w} \partial_a + \text{Re} \left( \frac{2}{z-w} \partial_x \right) \right) f(z; a, w, b, c)$$

$$= \frac{\sqrt{c-a}}{\sqrt{(c-z)(b-z)(z-a)K(s)}} \left( \frac{z-a}{(b-a)(a-w)} - \text{Re} \frac{z-a}{(z-w)(a-w)} \right)$$

$$+ \frac{(b-c)\partial_x K(\tilde{z}, s)}{(c-a)^2K(s)} - \frac{(b-c)K(\tilde{z}, s)K'(s)}{(c-a)^2K^2(s)}.$$ 

Therefore we have

$$\left( \frac{2}{a-w} \partial_a + \text{Re} \left( \frac{2}{z-w} \partial_x \right) \right) P(z; a, w, b, c) \to 0, \quad \text{as } z \to a. \quad (5.17)$$

Recall that $\partial_w P$ and $\partial_w^2 P$ are continuous at $a$, combining with (5.15), (5.16) and (5.17), we see that $\mathcal{V}(z)$ remains bounded as $z \to a$. Similarly, we may show that $\mathcal{V}(z)$ is also bounded near $b, c, \infty$. 

**Proof of Lemma 5.6.** When there is no ambiguity, we write $\partial_n P(z; a, w, b, c)|_{z=x}$ as $\partial_n P(x; a, w, b, c)$. The goal is to show $\mathcal{V}(z) = 0$ for every $z \in \mathbb{H}$. To this end, we evaluate the value of $\int_c^\infty \partial_n \mathcal{V}(x)dx$. On the one hand, we consider the following function:

$$\tilde{\mathcal{V}}(\cdot) := \mathcal{V}(\cdot) - \frac{\pi}{K} G_2(w) f'(w; a, b, c) P(\cdot; a, w, b, c),$$

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where $G_2$ is defined as in (5.14). By Lemma 5.7, $\tilde{V}$ is harmonic on $\mathbb{H}$ and has the same boundary data as $P(:a, w, b, c)$. Moreover, by the construction $\tilde{V}$ is bounded near $w$ and by Lemma 5.8 $\tilde{V}$ is bounded near $a, b, c, \infty$. Thus, $\tilde{V}$ is bounded in $\mathbb{H}$ and hence $\tilde{V} \equiv 0$. This implies that

$$\int_c^{+\infty} \partial_n V(x)dx = \frac{\pi G_2(w)f'(w;a, b, c)}{K} \int_c^{+\infty} \partial_n P(x; a, w, b, c)dx = \frac{\pi}{K} G_2(w)f'(w;a, b, c), \quad (5.18)$$

where the second equality is due to (5.12). On the other hand, we have

$$\int_c^{\infty} \partial_n \ell P(x; a, w, b, c)dx = \ell \int_c^{\infty} \partial_n P(x; a, w, b, c)dx = 0, \quad \text{for } \ell = \partial_a, \partial_b, \partial_w, \partial_w^2; \quad \text{(by (5.12))}$$

$$\int_c^{\infty} \partial_n \frac{2}{c - w} \partial_c P(x; a, w, b, c)dx = \frac{2}{c - w} \int_c^{\infty} \partial_c \partial_n P(x; a, w, b, c)dx = \frac{2}{c - w} \partial_n P(c; a, w, b, c); \quad \text{(by (5.12))}$$

$$\int_c^{\infty} \left( \partial_n \left( \text{Re} \frac{2}{z - w} \partial_x P(z; a, w, b, c) \right) \right) \bigg|_{z=x} dx$$

$$= \int_c^{\infty} \frac{2}{x - w} \partial_x \partial_n P(x; a, w, b, c)dx = \frac{-2}{c - w} \partial_n P(c; a, w, b, c) + \int_c^{\infty} \frac{2}{x - w} \partial_x \partial_n P(x; a, w, b, c)dx;$$

$$\int_c^{\infty} \left( \partial_n \left( \text{Im} \frac{2}{z - w} \partial_y P(z; a, w, b, c) \right) \right) \bigg|_{z=x} dx = \int_c^{\infty} \frac{-2}{x - w} \partial_y \partial_n P(x; a, w, b, c)dx. \quad \text{(5.20)}$$

Therefore,

$$\int_c^{+\infty} \partial_n V(x)dx = \int_c^{\infty} \partial_n DP(z; a, w, b, c) \bigg|_{z=x} dx = 0.$$

Comparing with (5.18), we have $G_2(w) = 0$. Consequently, $V \equiv 0$ as desired. \hfill $\square$

**Corollary 5.9.** For $a < w < b < c < x$, define

$$F(x; a, w, b, c) = \partial_n P(z; a, w, b, c) \bigg|_{z=x}. \quad (5.19)$$

Then we have

$$\left( \frac{2}{a - w} \partial_a + \frac{2}{b - w} \partial_b + \frac{2}{c - w} \partial_c + \frac{2}{f'(w; a, b, c)} f''(w; a, b, c) \partial_w + \partial_w^2 + \frac{2}{x - w} \partial_x + \frac{-2}{(x - w)^2} \right) F = 0. \quad (5.20)$$

**Proof.** The PDE (5.20) can be obtained by taking $\partial_n$ in $DP = 0$ from Lemma 5.6. \hfill $\square$

**Lemma 5.10.** Assume the same setup as in Theorem 1.3. Choose a conformal map $\phi$ from $\Omega$ onto $\mathbb{H}$ such that $\phi(d) = \infty$ and $\phi(a) < \phi(b) < \phi(c)$. Denote by $(W_t, t \geq 0)$ the driving function of $\phi(\gamma^M)$ and by $(g_t, t \geq 0)$ the corresponding conformal maps. For any $x \in (\phi(c), +\infty)$, the process

$$(g_t'(x)F(g_t(x); g_t(\phi(a)), W_t, g_t(\phi(b)), g_t(\phi(c))), t \geq 0)$$

is a martingale up to the first time that $\gamma^M$ hits $(cd)$, where $F$ is defined in (5.19).

**Proof.** Fix two boundary points $\tilde{c}, \tilde{d}$ such that $a, b, c, \tilde{c}, \tilde{d}, d$ are in counterclockwise order. Choose a sequence of medial polygons $(\Omega_\delta^M; a^\delta, b^\delta, c^\delta, \tilde{c}^\delta, \tilde{d}^\delta, d^\delta)$ converges to $(\Omega; a, b, c, \tilde{c}, \tilde{d}, d)$ in the sense of (1.3) and choose a sequence of conformal maps $\phi_\delta : \Omega_\delta \rightarrow \mathbb{H}$ with $\phi_\delta(d_\delta) = \infty$ such that $\phi_\delta^{-1}$ converges to $\phi^{-1}$ uniformly on $\mathbb{H}$ as $\delta \rightarrow 0$. By Theorem 1.3, we have that $\gamma^M_\delta \rightarrow \gamma^M$ in law as $\delta \rightarrow 0$. Couple $\{\gamma^M_\delta\}_{\delta > 0}$ and $\gamma^M$ together such that $\gamma^M_\delta \rightarrow \gamma^M$ almost surely as $\delta \rightarrow 0$. Recall that $X^M = \gamma^M \cap (ab), Y^M = \gamma^M \cap (cd)$ and $X^M_\delta = \gamma^M_\delta \cap (a_\delta b_\delta), Y^M_\delta = \gamma^M_\delta \cap (c_\delta d_\delta).$
We parameterize \( \phi(\gamma^M) \) by the half-plane capacity and parameterize \( \gamma^M \) such that \( \phi(\gamma^M(t)) = \phi(\gamma^M(t)) \). Denote by \( T \) the first time that \( \gamma^M \) hits \( (cd) \). For \( \epsilon > 0 \), define \( T_\epsilon = \inf \{ t : \text{dist}(\gamma^M(t), (ba)) = \epsilon \} \). For \( t < T \), denote by \( K_t \) the conformal modulus of the quad \( (H; g_t(\phi(a)), g_t(\phi(b)), g_t(\phi(c)), \infty) \) and by \( f_t \) the conformal map from \( H \) onto \( (0, 1) \times (0, iK_t) \) sending \( (g_t(\phi(a)), g_t(\phi(b)), g_t(\phi(c)), \infty) \) to \( (0, 1, 1 + iK_t, iK_t) \).

We parameterize \( \gamma^M_\delta \) similarly, define \( T^\delta_\epsilon = \inf \{ t : \text{dist}(\gamma^M_\delta, (b_\delta a_\delta)) = \epsilon \} \). We may assume \( T^\delta_\epsilon \to T_\epsilon \) almost surely as \( \delta \to 0 \) by considering the continuous modification, see details in [Kar19] Appendix B and [Kar20]. For every \( t < T^\delta_\epsilon \) almost surely as \( \delta \to 0 \) by the half-plane capacity and parameterize \( \gamma^M_\delta \) on curves, we have

\[
\rho_{K_T} = \int_{f_1(\gamma^M_\delta(\phi(c)))} f_1(\gamma^M_\delta(\phi(\delta))) \rho_{K_{T_\epsilon}}(f_1(\gamma^M_\delta(W_{iK_T})), Re y)dy,
\]

where \( \rho_K \) is defined in (1.4). Thus, by bounded convergence theorem, we have

\[
\mathbb{E} \left[ \mathbf{1}_{\{ Y^M \in \{\bar{\delta}d\} \}} R(\gamma^M_\delta[0, t \wedge T^\delta_\epsilon]) \right] \to \mathbb{E} \left[ \mathbf{1}_{\{ Y^M \in \{\bar{\delta}d\} \}} R(\gamma^M[0, t \wedge T_\epsilon]) \right], \quad \text{as } \delta \to 0.
\]

From Corollary 5.5, we have

\[
P \left[ Y^M_\delta(t \wedge T^\delta_\epsilon) \in (\bar{\delta}d) \mid X^M_\delta(t \wedge T^\delta_\epsilon) = \gamma^M_\delta(t \wedge T^\delta_\epsilon) \right] \to \int_{f_1(\gamma^M_\delta(\phi(\bar{\delta})))} f_1(\gamma^M_\delta(\phi(\delta))) \rho_{K_{T_\epsilon}}(f_1(\gamma^M_\delta(W_{iK_T})), Re y)dy,
\]

where \( \rho_K \) is defined in (1.4). Thus, by bounded convergence theorem, we have

\[
\mathbb{E} \left[ P \left[ Y^M_\delta(t \wedge T^\delta_\epsilon) \in (\bar{\delta}d) \mid X^M_\delta(t \wedge T^\delta_\epsilon) = \gamma^M_\delta(t \wedge T^\delta_\epsilon) \right] R(\gamma^M_\delta[0, t \wedge T^\delta_\epsilon]) \right] \\
= \mathbb{E} \left[ R(\gamma^M[0, t \wedge T_\epsilon]) \int_{f_1(\gamma^M_\delta(\phi(\bar{\delta})))} f_1(\gamma^M_\delta(\phi(\delta))) \rho_{K_{T_\epsilon}}(f_1(\gamma^M_\delta(W_{iK_T})), Re y)dy \right], \quad \text{as } \delta \to 0.
\]

Combining (5.21) and (5.22), we have

\[
\mathbb{E} \left[ \mathbf{1}_{\{ Y^M \in \{\bar{\delta}d\} \}} R(\gamma^M([0, t \wedge T_\epsilon])) \right] \\
= \mathbb{E} \left[ R(\gamma^M([0, t \wedge T_\epsilon])) \int_{f_1(\gamma^M(\phi(\bar{\delta})))} f_1(\gamma^M(\phi(\delta))) \rho_{K_{T_\epsilon}}(f_1(\gamma^M(W_{iK_T})), Re y)dy \right].
\]

This implies that the process

\[
\left( \int_{f_1(\gamma^M(\phi(\bar{\delta})))} f_1(\gamma^M(\phi(\delta))) \rho_{K_{T_\epsilon}}(f_1(W_t), Re y)dy, \quad t \geq 0 \right)
\]

is a martingale up to \( T_\epsilon \). Thus, the process

\[
\left( (f_t \circ g_t)(x) \rho_{K_{T_\epsilon}}(f_t(W_t), Re f_t(g_t(x))), \quad t \geq 0 \right)
\]

is a martingale up to \( T_\epsilon \) for every \( x \in (\phi(c), +\infty) \). Combining (1.4), (5.8), (5.10) and (5.19), we have

\[
F(x; \phi(a), W_0, \phi(b), \phi(c)) = f'(x)\rho_K(f(W_0), Ref(x)).
\]
Thus, the process
\[(g_t(x)F(g_t(x); g_t(\phi(a)), W_t, g_t(\phi(b)), g_t(\phi(c))), t \geq 0)\]
is a martingale up to \(T\) for every \(x \in (\phi(c), +\infty)\). From Proposition 4.6, the curve \(\gamma^M\) intersects \(\partial \Omega\) only at its two ends almost surely. Thus \(T_\epsilon \to T\) as \(\epsilon \to 0\). This completes the proof. \(\square\)

**Proof of Theorem 1.5.** The joint law of \((X^M, Y^M)\) is derived in Proposition 5.1. It remains to show that the conditional law of \(\gamma^M\) given \(X^M\) is SLE\((-1, -1, -1, -1)\). To this end, we may assume \(\Omega = \mathbb{H}\) with \(d = \infty\) and \(a < b < c\) and parameterize \(\gamma^M\) by the half-plane capacity. Denote by \(T\) the first time that \(\gamma^M\) hits \((c, \infty)\). Denote by \((W_t, t \geq 0)\) the driving function of \(\gamma^M\) and by \((g_t, t \geq 0)\) the corresponding conformal maps. For \(a < w < b < c < x\), define \(F(x; a, w, b, c)\) as in (5.19). Lemma 5.10 tells that the process
\[(g_t(x)F(g_t(x); g_t(a), W_t, g_t(b), g_t(c)), t \geq 0)\]
is a semimartingale. Denote by \(L_t\) the drift term of \(W_t\). By Itô’s formula, we have
\[
\partial_w F \left( dL_t - 2 \frac{f''(W_t; g_t(a), g_t(b), g_t(c))}{f'(W_t; g_t(a), g_t(b), g_t(c))} dt + \frac{1}{2} \partial_w^2 F (d(W)_t - 2dt) \right) = 0.
\]
Combining with (5.20), we have
\[
\partial_w F \left( dL_t - \frac{1}{g_t(a) - W_t} + \frac{1}{g_t(b) - W_t} + \frac{1}{g_t(c) - W_t} \right) dt + \frac{1}{2} \partial_w^2 F (d(W)_t - 2dt) = 0.
\]
From (4.7) and (A.10), we have
\[
2f''(w; a, b, c) = \frac{1}{a - w} + \frac{1}{b - w} + \frac{1}{c - w}.
\]
Thus, it simplifies as
\[
\partial_w F \left( dL_t - \frac{1}{g_t(a) - W_t} + \frac{1}{g_t(b) - W_t} + \frac{1}{g_t(c) - W_t} \right) dt + \frac{1}{2} \partial_w^2 F (d(W)_t - 2dt) = 0. \tag{5.23}
\]
Note that almost surely, (5.23) holds for all \(x \in \mathbb{Q} \cap (c, +\infty)\). By the continuity, almost surely, it holds for all \(x \in (c, +\infty)\). Now we fix \((a, b, c)\) and \(t\). Define
\[
S_1(x) := \partial_w F(g_t(x); g_t(a), W_t, g_t(b), g_t(c)), \quad S_2(x) := \partial_w^2 F(g_t(x); g_t(a), W_t, g_t(b), g_t(c)).
\]
It suffices to prove
\[
\exists x, x' \in (c, \infty) \text{ such that } S_1(x)S_2(x') \neq S_2(x)S_1(x') \tag{5.24}
\]
Assume this is true, then we have
\[
dL_t = \left( \frac{1}{g_t(a) - W_t} + \frac{1}{g_t(b) - W_t} + \frac{1}{g_t(c) - W_t} \right) dt \quad \text{and} \quad d(W)_t = 2dt.
\]
This shows that \(\gamma^M\) is SLE\((-1, -1, -1, -1)\) as desired.

It remains to show (5.24). Denote \(f(\cdot) = f(\cdot; a, b, c)\). Note that
\[
S_1(x) = -2f'(g_t(x)) \frac{\pi}{4K} \times \frac{\pi}{2K} f'(W_t) \times \sum_{n \in \mathbb{Z}} \left( \sinh \left( \frac{\pi}{2K} f(W_t) - \text{Re} f(g_t(x)) - 2n \right) \right) + \sinh \left( \frac{\pi}{2K} f(W_t) + \text{Re} f(g_t(x)) - 2n \right) \frac{\pi}{2K} f'(W_t) \sinh \left( \frac{\pi}{2K} f(W_t) - \text{Re} f(g_t(x)) - 2n \right),
\]
\[
S_2(x) = \frac{f''(W_t)}{f'(W_t)} S_1(x)
\]
\[
+ \frac{2f'(g_t(x))}{4K} \times \left( \frac{\pi}{2K} f'(W_t) \right)^2 \times \sum_{n \in \mathbb{Z}} \left( \frac{2 \sinh^2 \left( \frac{\pi}{2K} f(W_t) - \text{Re} f(g_t(x)) - 2n \right) \sinh \left( \frac{\pi}{2K} f(W_t) + \text{Re} f(g_t(x)) - 2n \right)}{\cosh^4 \left( \frac{\pi}{2K} f(W_t) - \text{Re} f(g_t(x)) - 2n \right)} + \frac{2 \sinh^2 \left( \frac{\pi}{2K} f(W_t) + \text{Re} f(g_t(x)) - 2n \right) - 1}{\cosh^4 \left( \frac{\pi}{2K} f(W_t) + \text{Re} f(g_t(x)) - 2n \right)} \right).
\]

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Define
\[ R_1(z) := \sum_{n \in \mathbb{Z}} \left( \frac{\cosh \left( \frac{\pi}{2K} (f(W_t) - z - 2n) \right)}{\sinh^3 \left( \frac{\pi}{2K} (f(W_t) - z - 2n) \right)} + \frac{\cosh \left( \frac{\pi}{2K} (f(W_t) + z - 2n) \right)}{\sinh^3 \left( \frac{\pi}{2K} (f(W_t) + z - 2n) \right)} \right) \]
and
\[ R_2(z) := -\sum_{n \in \mathbb{Z}} \left( \frac{2 \cosh^2 \left( \frac{\pi}{2K} (f(W_t) - z - 2n) \right) + 1}{\sinh^4 \left( \frac{\pi}{2K} (f(W_t) - z - 2n) \right)} + \frac{2 \cosh^2 \left( \frac{\pi}{2K} (f(W_t) + z - 2n) \right) + 1}{\sinh^4 \left( \frac{\pi}{2K} (f(W_t) + z - 2n) \right)} \right). \]

If (5.24) is false, then \( S_1(x)/S_2(x) \) is constant for \( x > c \). Thus, there exists \( \lambda \) (which is random) such that \( (R_1 - \lambda R_2)(z) \mid_{z \in (iK,1+iK)} = 0 \). Since \( R_1 \) and \( R_2 \) are analytic functions in \( (0,1) \times (0,iK) \), this implies that \( R_1 = \lambda R_2 \) in \( (0,1) \times (0,iK) \). This is a contradiction by considering the asymptotic of \( R_1 \) and \( R_2 \) when \( z \to f(W_t) \):
\[
\lim_{z \to f(W_t)} R_1(z)(z - f(W_t))^3 = \left( \frac{-2K}{\pi} \right)^3, \quad \lim_{z \to f(W_t)} R_2(z)(z - f(W_t))^4 = -3 \left( \frac{2K}{\pi} \right)^4.
\]
This completes the proof.

5.3 Consequences

In this section, we complete the proof for Propositions 1.2 and 1.3 and Corollary 1.6.

Proof of Proposition 1.3 The conclusion is immediate from Theorem 1.4 and (5.4).

The proof for Proposition 1.2 and Corollary 1.6 bases on the following observation in the discrete for UST. Fix a Dobrushin domain \((\Omega; c, d)\) such that \( \partial \Omega \) is \( C^1 \) and simple. Suppose \((\Omega_\delta; c_\delta, d_\delta)\) is an approximation of \((\Omega; c, d)\) on \( \delta \mathbb{Z}^2 \) as in Section 4.1. Let \( T_\delta \) be the UST in \((\Omega_\delta; c_\delta, d_\delta)\) with \((c_\delta d_\delta)\) wired. Denote by \( \eta_\delta \) the associated Peano curve along \( T_\delta \) from \( d_\delta^0 \) to \( c_\delta^0 \). Fix \( a \in (dc) \) and let \( a_\delta \) be the medial vertex along \((c_\delta^0 a_\delta^0)\) nearest to \( a \). Let \( a_\delta \in V(\Omega_\delta) \) be the primal vertex in \( \Omega_\delta \) that is nearest to \( a_\delta^0 \). Let \( a_\delta^* \) (resp. \( b_\delta^* \)) be the dual vertex along \((d_\delta^* c_\delta^* )\) that is nearest to \( a_\delta^0 \) and is closer to \( d_\delta^0 \) (resp. closer to \( c_\delta^0 \)) along \((d_\delta^* c_\delta^* )\). See Figure 5.2. We divide the Peano curve \( \eta_\delta \) into two parts: denote by \( \eta_\delta^L \) the part of \( \eta_\delta \) from \( d_\delta^0 \) to \( a_\delta^0 \), and denote by \( \eta_\delta^R \) the part of \( \eta_\delta \) from \( a_\delta^0 \) to \( c_\delta^0 \). Denote by \( \eta_\delta^t \) the time-reversal of \( \eta_\delta^L \). There is a branch in \( T_\delta \) connecting \( a_\delta \) to \( (c_\delta d_\delta) \) and we denote it by \( \gamma_\delta \). We parameterize \( \gamma_\delta \) so that it starts from \( a_\delta \) and terminates when it hits \((c_\delta d_\delta)\). We have the convergence of the triple \((\eta_\delta^L; \gamma_\delta; \eta_\delta^R)\).

Lemma 5.11 Fix a polygon \((\Omega; d, a, c)\) with three marked points such that \( \partial \Omega \) is \( C^1 \) and simple. Suppose that a sequence of medial polygons \((\Omega_\delta^0; d_\delta^0, a_\delta, c_\delta)\) converges to \((\Omega; d, a, c)\) in the sense of (1.3). Then the triple \((\eta_\delta^L; \gamma_\delta; \eta_\delta^R)\) converges weakly to a triple of continuous curves \((\eta^L; \gamma; \eta^R)\) whose law is characterized as follows. Let \( \eta \) be an SLE\(_8\) in \( \Omega \) from \( d \) to \( c \) and let \( T_\eta \) be the first time that it swallows \( a \). Then, the joint distribution of \((\eta^L; \eta^R)\) is the same as \((\eta(T_\eta - t), 0 \leq t \leq T_\eta; \eta(t), t > T_\eta)\); and \( \gamma = \eta^L \cap \eta^R \).

Proof First of all, recall that the tightness of \( \{ \eta_\delta^L \}_{\delta > 0} \) and \( \{ \eta_\delta^R \}_{\delta > 0} \) are given in the proof of Theorem 4.1. The tightness of \( \{ \gamma_\delta \}_{\delta > 0} \) can be proved in the same way as in Proposition 4.6. Therefore, the triple \( \{ (\eta_\delta^L; \gamma_\delta; \eta_\delta^R) \}_{\delta > 0} \) is tight.

Next, we determine the law of subsequential limits. Suppose \((\eta^L; \gamma; \eta^R)\) is any subsequential limit. There exists \( \{ \delta_n \} \) with \( \delta_n \to 0 \) as \( n \to \infty \), such that \( \eta_\delta \to \eta^L \) and \( \gamma_\delta \to \gamma \) and \( \eta_\delta^R \to \eta^R \) in law as \( n \to \infty \). By Theorem 4.1, \( \{ \eta_\delta \}_{\delta > 0} \) converges weakly to \( \eta \) as \( n \to \infty \). Thus, \((\eta^L; \eta^R)\) has the same law as \((\eta(T_\eta - t), 0 \leq t \leq T_\eta; \eta(t), t > T_\eta)\). Since SLE\(_8\) is space filling, we have \( \gamma = \eta^L \cap \eta^R \). This completes the proof.

From the observation in Lemma 5.11 we arrive at the following lemma.
Figure 5.2: In the left panel, the solid edges in black are wired boundary arc \((cd)\), and the solid edges in red are dual-wired boundary arc \((d^*c^*)\). The thin edges are in the UST. The thin edges in red are the branch \(\gamma\) in the tree connecting \(a\) to \((cd)\). Suppose \(T\) is the hitting time of \(\gamma\) at \((cd)\) and \(\tau\) is any stopping time before \(T\). In the right panel, the solid edges in blue are \(\gamma[0, \tau]\) and the solid edges in green are \(\gamma[\tau, T]\). The two orange curves are \(\eta^L\) and \(\eta^R\).

**Lemma 5.12.** Fix a polygon \((\Omega; d, a, c)\) with three marked points such that \(\partial\Omega\) is \(C^1\) and simple. Let \(\eta\) be an SLE\(_8\) in \(\Omega\) from \(d\) to \(c\) and let \(T_a\) be the first time that it swallows \(a\). We define \(\gamma := \partial(\eta[0, T_a]) \cap \Omega\) (here we view \(\eta[0, T_a]\) as a compact set) and view \(\gamma\) as a continuous simple curve starting from \(a\) and terminating at some point in \((cd)\). Denote by \(\Omega^L\) and \(\Omega^R\) the two connected components of \(\Omega \setminus \gamma\) such that \(\Omega^L\) has \(d\) on the boundary and \(\Omega^R\) has \(c\) on the boundary. The joint law of the triple

\[
(\eta(T_a - t), 0 \leq t \leq T_a; \quad \gamma; \quad \eta(t), t \geq T_a)
\]

can be characterized as follows: \(\gamma\) is SLE\(_2\)(\(-1, -1; -1, -1\)) in \(\Omega\) from \(a\) to \((cd)\) with force points \((d, a^-; a^+, c)\); given \(\gamma\), the conditional law of \((\eta(T_a - t), 0 \leq t \leq T_a)\) is SLE\(_8\) in \(\Omega^L\) from \(a^-\) to \(d\) and the conditional law of \((\eta(t), t \geq T_a)\) is SLE\(_8\) in \(\Omega^R\) from \(a^+\) to \(c\), and \((\eta(T_a - t), 0 \leq t \leq T_a)\) and \((\eta(t), t \geq T_a)\) are conditionally independent given \(\gamma\).

**Proof.** First, we derive the marginal law of \(\gamma\). Choose a conformal map \(\phi\) from \(\Omega\) onto \(\mathbb{H}\) such that \(\phi(a) = 0\) and \(\phi(d) = \infty\). Denote by \((W_t, t \geq 0)\) the driving function of \(\phi(\gamma)\) and by \((g_t, t \geq 0)\) the corresponding conformal maps. By the argument in the proof of Lemma 5.10, for any \(x \in (\phi(c), +\infty)\), the process

\[
(g_t(x)F(g_t(x); g_t(0^-), W_t, g_t(0^+), g_t(\phi(c))), t > 0)
\]

is a martingale up to the first time that \(\gamma\) hits \((cd)\) where \(F\) is defined in (5.19). For \(\epsilon > 0\), define \(\tau_\epsilon := \inf\{t : \text{dist}(\phi(\gamma(t)), 0) \geq \epsilon\}\). By the argument in the proof of Theorem 1.5, the conditional law of \((\gamma(t), t \geq \tau_\epsilon)\) given \(\gamma[0, \tau_\epsilon]\) is SLE\(_2\)(\(-1, -1; -1, -1\)) in \(\Omega \setminus \gamma[0, \tau_\epsilon]\) from \(\gamma(\tau_\epsilon)\) to \((cd)\) with force points \((d, a^-; a^+, c)\). Let \(\epsilon \to 0\), the law of \(\gamma\) is SLE\(_2\)(\(-1, -1; -1, -1\)) in \(\Omega\) from \(a\) to \((cd)\) with force points \((d, a^-; a^+, c)\).

Second, the conditional law of \((\eta(T_a - t), 0 \leq t \leq T_a; \eta(t), t \geq T_a)\) given \(\gamma\) can be proved in the same way as in Theorem 1.4 thanks to the observation in Lemma 5.11 and Figure 5.2.

**Proof of Corollary 1.6.** The conclusion is immediate from Lemma 5.12.

**Proof of Proposition 1.2.** Assume the same notation as in Lemma 5.12. We parameterize \(\gamma\) so that it starts from \(a\) and terminates when it hits \((cd)\) at time \(T\). Let \(\tau\) be any stopping time of \(\gamma\) before \(T\).
Consider the conditional law of the following triple given $\gamma[0, \tau]$:

$$(\eta(T_a - t), 0 \leq t \leq T_a; \gamma(t), t \leq t \leq T; \eta(t), t \geq T_a).$$

From Lemma 5.12, the law of $(\gamma(t), \tau \leq t \leq T)$ is SLE$_2(-1, -1, -1, -1)$ in $\Omega \setminus \gamma[0, \tau]$ from $\gamma(\tau)$ to $(-\infty, x)$ with force points $(d, a^-; a^+, c)$; the conditional law of $(\eta(T_a - t), 0 \leq t \leq T_a)$ given $\gamma[\tau, T]$ is SLE$_8$ in $\Omega^L$ from $a^-$ to $d$, the conditional law of $(\eta(t), t \geq T_a)$ given $\gamma[\tau, T]$ is SLE$_8$ in $\Omega^R$ from $a^+$ to $c$, and $(\eta(T_a - t), 0 \leq t \leq T_a)$ and $(\eta(t), t \geq T_a)$ are conditionally independent given $\gamma[\tau, T]$. Comparing with Theorems 1.4 and 1.5, we see that the triple has the same law as the triple $(\eta^L; \gamma^M; \eta^R)$ in Theorem 1.4 in the quad $(\Omega \setminus \gamma[0, \tau]; d, a^-, a^+, c)$ conditional on $X^M = \gamma(\tau)$. In particular, the law of $(\eta(T_a - t), 0 \leq t \leq T_a)$ is hSLE in $\Omega \setminus \gamma[0, \tau]$ from $a^-$ to $d$ conditional that its last hitting point of $\gamma[0, \tau]$ is $\gamma(\tau)$. Combining with reversibility of hSLE$_8$ proved in Corollary 4.13 we obtain the conclusion. \hfill \Box

## A Hypergeometric function and elliptic integral

For $A, B, C \in \mathbb{R}$, the hypergeometric function is defined for $|z| < 1$ by the power series:

$$F(z) = {}_2F_1(A, B, C; z) = \sum_{n=0}^{\infty} \frac{(A)_n(B)_n}{(C)_n} \frac{z^n}{n!}, \quad (A.1)$$

where $(x)_n := x(x+1) \cdots (x+n-1)$ for $n \geq 1$ and $(x)_0 = 1$ for $n = 0$. The power series is well-defined when $C \notin \{0, -1, -2, -3, \ldots\}$. The hypergeometric function is a solution of Euler’s hypergeometric differential equation:

$$z(1-z)F''(z) + (C - (A + B + 1)z) F'(z) - ABF(z) = 0. \quad (A.2)$$

We collect some properties for hypergeometric functions here. For $z \in (-1, 1)$, we have (see AS92 Eq. 15.2.1 and Eq. 15.3.3)

$$\begin{align*}
_2F_1(A, B, C; z) &= (1 - z)^{C - A - B} \_2F_1(C - A, C - B, C; z), \\
\frac{d}{dz} \_2F_1(A, B, C; z) &= \frac{AB}{C} \_2F_1(A + 1, B + 1, C + 1; z). 
\end{align*} \quad (A.3, A.4)$$

The series $(A.1)$ is absolutely convergent on $z \in [0, 1]$ when $C > A + B$ and $C \notin \{0, -1, -2, \ldots\}$. In this case, we have (see AS92 Eq. 15.1.20)

$$\_2F_1(A, B, C; 1) = \frac{\Gamma(C)\Gamma(C - A - B)}{\Gamma(C - A)\Gamma(C - B)}, \quad (A.5)$$

where $\Gamma$ is Gamma Function.

We need the following lemma on hypergeometric functions in Section 3.1. Fix $\kappa \geq 8$ and $\nu > -2$. Recall from (3.1), we denote

$$F(z) := \_2F_1\left(\frac{2\nu + 4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{2\nu + 8}{\kappa}; z\right).$$

**Lemma A.1.** Fix $\kappa \geq 8$ and $\nu > -2$. Denote by $\Gamma$ the Gamma function. The function $F$ defined in (3.1) is increasing on $[0, 1)$ with $F(0) = 1$. Moreover, we have the following asymptotic.

- **When $\kappa > 8$ and $\nu > -2$, we have**

$$\lim_{z \to 1^-} (1 - z)^{1-8/\kappa} F(z) = \_2F_1\left(\frac{4}{\kappa}, \frac{2\nu + 12}{\kappa} - 1, \frac{2\nu + 8}{\kappa}; 1\right) = \frac{\Gamma(\frac{2\nu + 8}{\kappa})\Gamma(1 - \frac{8}{\kappa})}{\Gamma(\frac{2\nu + 4}{\kappa})\Gamma(1 - \frac{4}{\kappa})} \in (0, \infty). \quad (A.6)$$

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• When \( \kappa = 8 \) and \( \nu > -2 \), we have

\[
\lim_{z \to 1^{-}} \frac{F(z)}{-\log \frac{1}{1-z}} = \frac{1}{\sqrt{\pi}} \frac{(\nu + 2)\Gamma(2 + \frac{4}{\kappa})}{(\nu + 4)\Gamma(\frac{3}{2} + \frac{4}{\kappa})} (\nu + 2) \in (0, \infty).
\]

(A.7)

**Proof.** In this lemma, we set

\[A = \frac{2\nu + 4}{\kappa}, \quad B = 1 - \frac{4}{\kappa}, \quad C = \frac{2\nu + 8}{\kappa}.\]

When \( \kappa \geq 8 \) and \( \nu > -2 \), we have \( A > 0, B > 0, C > 0 \), thus \( F(z) \) is increasing on \([0, 1)\) by the definition in (A.1). It remains to derive the asymptotic.

When \( \kappa > 8 \) and \( \nu > -2 \), by (A.3), we have

\[F(z) = (1 - z)^{\frac{8}{\kappa} - 1} 2F_1\left(\frac{4}{\kappa}, \frac{2\nu + 12}{\kappa} - 1, \frac{2\nu + 8}{\kappa}; z\right).\]

By (A.5), we have

\[2F_1\left(\frac{4}{\kappa}, \frac{2\nu + 12}{\kappa} - 1, \frac{2\nu + 8}{\kappa}; 1\right) = \frac{\Gamma(2 + \frac{8}{\kappa})\Gamma(1 - \frac{8}{\kappa})}{\Gamma(\frac{2\nu + 12}{\kappa})\Gamma(1 - \frac{4}{\kappa})} \in (0, \infty).\]

This gives (A.6).

When \( \kappa = 8 \) and \( \nu > -2 \), we have

\[
\lim_{z \to 1^{-}} \frac{F(z)}{-\log \frac{1}{1-z}} = \lim_{z \to 1^{-}} (1 - z)F'(z)
\]

(by L'Hospital rule)

\[
= \lim_{z \to 1^{-}} \frac{AB}{C} (1 - z) 2F_1(A + 1, B + 1, C + 1; z)
\]

(by (A.4))

\[
= \lim_{z \to 1^{-}} \frac{AB}{C} 2F_1(C - A, C - B, C + 1; z)
\]

(by (A.3))

\[
= \frac{AB \Gamma(C + 1)\Gamma(A + B + 1 - C)}{C \Gamma(1 + A)\Gamma(1 + B)},
\]

(by (A.5))

as desired in (A.7).

Next, we introduce elliptic integral. Denote by \( K \) the elliptic integral of the first kind (see [AS92, Eq. 17.2.6]): for \( \varphi \in \mathbb{R} \) and \( x \in (0, 1) \),

\[
K(\varphi, x) := \int_{0}^{\varphi} \frac{d\theta}{\sqrt{1 - x \sin^2 \theta}}, \quad K(x) := \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - x \sin^2 \theta}}.
\]

(A.8)

There is a relation between complete elliptic integral and hypergeometric function (see [AS92, Eq. 17.3.9]):

\[
K(x) = \frac{\pi}{2} 2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; x\right), \quad \forall x \in (0, 1).
\]

(A.9)

Let us calculate the derivatives of \( K \):

\[
\partial_{\varphi} K(\varphi, x) = \frac{1}{\sqrt{1 - x \sin^2 \varphi}}, \quad \partial_{\varphi}^2 K(\varphi, x) = \frac{x \sin \varphi \cos \varphi}{\sqrt{1 - x \sin^2 \varphi}}, \quad \partial_{\varphi} \partial_{x} K(\varphi, x) = \frac{\sin^2 \varphi}{2\sqrt{1 - x \sin^2 \varphi}}.
\]

(A.10)

The derivatives \( \partial_{x} K \) and \( \partial_{x}^2 K \) involve the elliptic integral of the second kind, but luckily, we do not need them.
B Convergence of discrete harmonic functions

Discrete harmonic functions have been studied in many cases in proof of convergence of discrete observables to continuous observables, see for instance [LSW04], [Smi01], [SS09] and [CDCH14]. Some standard methods to study discrete harmonic functions and discrete holomorphic functions have been established, see for instance [Che16] and [CS11]. In this appendix, we collect some basic facts about harmonic functions with mixed boundary conditions.

Lemma B.1. Fix a quad \((\Omega; a, b, c, d)\) and fix a conformal map \(\xi\) from \(\Omega\) onto \(\mathbb{U}\) and extend its definition continuously to the boundary. There is a unique bounded harmonic function \(u\) on \(\Omega\) such that \(u \circ \xi^{-1}\) satisfies the following boundary data:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
u \circ \xi^{-1} = 1, & \text{on } (\xi(a)\xi(b)); \\
u \circ \xi^{-1} = 0, & \text{on } (\xi(c)\xi(d)); \\
\partial_n u \circ \xi^{-1} = 0, & \text{on } (\xi(b)\xi(c)) \cup (\xi(d)\xi(a)); \\
\end{array} \right.
\]

(B.1)

where \(n\) is the outer normal vector. Moreover, the harmonic function \(u\) can only obtain its maximum on \((ab)\) and obtain its minimum on \((cd)\).

Proof. The existence of such harmonic function is clear. We only need to show the uniqueness. Suppose there are two bounded harmonic functions \(u_1\) and \(u_2\) with the boundary data (B.1). Define \(\tilde{u} = (u_1 - u_2) \circ \xi^{-1}\). Then \(\tilde{u}\) is a bounded harmonic function with the following boundary data: \(\tilde{u} = 0\) on \((\xi(a)\xi(b)) \cup (\xi(c)\xi(d))\) and \(\partial_n \tilde{u} = 0\) on \((\xi(b)\xi(c)) \cup (\xi(d)\xi(a))\). It suffices to show \(\tilde{u} = 0\).

First, we extend \(\tilde{u}\) to \(\mathbb{C} \setminus ((\xi(a)\xi(b)) \cup (\xi(c)\xi(d)))\) harmonically as follows. Choose \(\tilde{\nu}\) to be a harmonic conjugate of \(\tilde{u}\). Since \(\partial_n \tilde{u} = 0\) on \((\xi(b)\xi(c)) \cup (\xi(d)\xi(a))\), we see that \(\tilde{\nu}\) is constant along \((\xi(b)\xi(c))\) and is constant along \((\xi(d)\xi(a))\). We may set \(\tilde{\nu} = 0\) on \((\xi(b)\xi(c))\). Define \(g = i\tilde{u} - \tilde{\nu}\) and this is a holomorphic function on \(\mathbb{U}\). We define \(g\) on \(\mathbb{C} \setminus \mathbb{U}\) by setting \(g(z) = \overline{g(1/z)}\). Since \(\partial_n \tilde{u} = 0\) on \((\xi(b)\xi(c)) \cup (\xi(d)\xi(a))\), by Schwarz reflection principle, the function \(g\) can be extended to a holomorphic function on \(\mathbb{C} \setminus ((\xi(a)\xi(b)) \cup (\xi(c)\xi(d)))\) which we still denote by \(g\). This implies that \(\tilde{u}\) can be extended to \(\mathbb{C} \setminus ((\xi(a)\xi(b)) \cup (\xi(c)\xi(d)))\) harmonically and we still denote its extension by \(\tilde{u}\).

Second, we show that \(\tilde{u}\) is continuous at \(\xi(a), \xi(b), \xi(c)\) and \(\xi(d)\). It suffices to show \(\lim_{z \to \xi(a)} \tilde{u}(z) = 0\) and the limit at the other three points can be derived similarly. Suppose \(|\tilde{u} \circ \xi^{-1}| \leq M\) for some \(M > 0\). Fix two small constants \(r \geq \epsilon > 0\). Define \(\tilde{u}_M\) to be the harmonic function on \(B(\xi(a), r) \setminus (\xi(a)\xi(b))\) with the following boundary data: \(\tilde{u}_M = 0\) on \(B(\xi(a), r) \cap (\xi(a)\xi(b))\) and \(\tilde{u}_M = M\) on \(\partial B(\xi(a), r)\). From maximum principle, we have \(|\tilde{u}_M(z)| \leq 1\) for all \(z \in B(\xi(a), r) \setminus (\xi(a)\xi(b))\). Combining with Beurling estimate, for every \(z \in B(\xi(a), \epsilon)\), we have

\[|\tilde{u}(z)| \leq |\tilde{u}_M(z)| \leq CM\sqrt{\epsilon/r},\]

for some universal constant \(C > 0\). This gives \(\lim_{z \to \xi(a)} \tilde{u}(z) = 0\) as desired.

From the first step, \(\tilde{u}\) is a bounded harmonic function on \(\mathbb{C} \setminus ((\xi(a)\xi(b)) \cup (\xi(c)\xi(d)))\), by maximum principle, it obtains its maximum and minimum on \([\xi(a)\xi(b)] \cup [\xi(c)\xi(d)]\). In particular, \(\tilde{u}\) obtains its maximum and minimum in \(\mathbb{U}\) on \([\xi(a)\xi(b)] \cup [\xi(c)\xi(d)]\). Recall that \(\tilde{u} = 0\) on \((\xi(a)\xi(b)) \cup (\xi(c)\xi(d))\) and it is continuous at \(\xi(a), \xi(b), \xi(c), \xi(d)\) as proved in the second step. Thus the maximum and the minimum of \(\tilde{u}\) are both zero on \([\xi(a)\xi(b)] \cup [\xi(c)\xi(d)]\). This gives \(\tilde{u} = 0\) as desired.

The last statement can be proved by similar reflection method and the maximum and minimum principle. This completes the proof. \(\square\)

Lemma B.2. Fix a quad \((\Omega; a, b, c, d)\) and fix a conformal map \(\xi\) from \(\Omega\) onto \(\mathbb{U}\) and extend \(\xi\) continuously to the boundary. Suppose a sequence of discrete domains \((\Omega_\delta; a_\delta, b_\delta, c_\delta, d_\delta)\) converges to \((\Omega; a, b, c, d)\) in the Carathéodory sense as \(\delta \to 0\). For \(z \in V(\Omega_\delta)\), let \(u_\delta(z)\) be the probability that a simple random walk in \(\Omega_\delta\) starting from \(z\) hits \((a_\delta b_\delta)\) before \((c_\delta d_\delta)\). Let \(u(z)\) be the bounded harmonic function in Lemma B.1. Then \(u_\delta\) converges to \(u\) locally uniformly as \(\delta \to 0\).

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Proof. This is a consequence of [LSW04 Proposition 4.2]. We summarize the proof here for concreteness. For any function $g_\delta$ on $V(\Omega_\delta)$ or $V(\Omega_\delta^*)$, define its discrete derivatives as
\[
\partial^\delta_{x} g_\delta := \delta^{-1}(g_\delta(v + \delta) - g_\delta(v)), \quad \partial^\delta_{y} g_\delta := \delta^{-1}(g_\delta(v + i\delta) - g_\delta(v)).
\]
We extend $u_\delta$ and all its derivatives to functions on $\Omega_\delta$ by linear interpolation.

First, we show that any subsequential limit of $u_\delta$ is harmonic on $\Omega$. Since $u_\delta$ is discrete harmonic and $0 \leq u_\delta \leq 1$, by [LSW04 Lemma 5.2], for any compact set $S \subset \Omega$ and $k \in \mathbb{N}$, there exists a constant $C > 0$ which depends on $S$ and $k$ such that
\[
|\partial^\delta_{a_1} \partial^\delta_{a_2} \cdots \partial^\delta_{a_k} u_\delta(v)| \leq C, \quad \text{for any } \partial^\delta_{a_1}, \ldots, \partial^\delta_{a_k} \in \{\partial^\delta_{x}, \partial^\delta_{y}\}, \text{ and } v \in S \cap \Omega_\delta.
\]
By Arzela-Ascoli theorem, for any sequence $\delta_n \to 0$, there exists a subsequence, still denoted by $\{\delta_n\}$, and continuous functions $u, u_x, u_y, u_{xx}, u_{yy}$ such that
\[
u_\delta \to u, \quad \partial^\delta_{x} u_{\delta_n} \to u_x, \quad \partial^\delta_{y} u_{\delta_n} \to u_y, \quad \partial^\delta_{x} \partial^\delta_{y} u_{\delta_n} \to u_{xx}, \quad \partial^\delta_{y} \partial^\delta_{x} u_{\delta_n} \to u_{yy}, \quad \text{locally uniformly}.
\]
This implies
\[
u_x = \partial_x u, \quad \nu_y = \partial_y u, \quad \nu_{xx} = \partial^2_x u, \quad \nu_{yy} = \partial^2_y u.
\]
Since $u_{\delta_n}$ is discrete harmonic, the function $u$ is harmonic. This shows that any subsequential limit of $u_\delta$ is harmonic.

Next, we show that any subsequential limit $u$ has the boundary data given in the statement. Note that such boundary data and harmonicity and boundedness uniquely determines the subsequential limit, hence gives the convergence of sequence $u_\delta$. From the definition of $u_\delta$ and the discrete Beurling estimate, it is clear that $u = 1$ on $(ab)$ and $u = 0$ on $(cd)$. To derive the boundary data along $(bc) \cup (ad)$, we need to introduce the discrete harmonic conjugate function $v^*_\delta$ of $u_\delta$.

Define $v^*_\delta = 0$ on $(b^*_\delta c^*_\delta)$. For every oriented edge $e^*_\delta = \{x^*_\delta, y^*_\delta\} \in E(\Omega^*_\delta)$, there is a unique oriented edge $e = \{x_\delta, y_\delta\} \in E(\Omega_\delta)$ which crosses $e^*_\delta$ from its right-side. Define $v^*_\delta(y^*_\delta) - v^*_\delta(x^*_\delta) := u_\delta(y_\delta) - u_\delta(x_\delta)$. Since $u_\delta$ is discrete harmonic, this is well-defined and $v^*_\delta$ is constant on $(d^*_\delta a^*_\delta)$. Moreover, $v^*_\delta$ takes its maximum, denoted by $L_\delta$, on $(d^*_\delta a^*_\delta)$.

We claim that $\{L_\delta\}_{\delta > 0}$ is uniformly bounded. Assume this is true, by the same argument as above, for any sequence $\delta_n \to 0$, there exists a subsequence, still denoted by $\delta_n$, and a constant $K$ and a harmonic function $v$ such that $v^*_\delta_n \to v$ and the other related derivatives also converge locally uniformly and that $L_n \to K$ as $n \to \infty$. From the construction and the discrete Beurling estimate, we have $v = K$ on $(da)$ and $v = 0$ on $(bc)$. By the definition of $v^*_\delta$, the function $f^*_\delta := u_\delta + iv^*_\delta$ is discrete holomorphic. Then, the convergence of discrete derivatives implies that $f := u + iv$ is holomorphic on $\Omega$. By Schwartz reflection principle, we can extend $f \circ \xi^{-1}$ to $\partial \Omega \setminus \{\xi(a), \xi(b), \xi(c), \xi(d)\}$ analytically. By Cauchy-Riemann equation, we have
\[
\partial_n(u \circ \xi^{-1}) = \partial_t(v \circ \xi^{-1}) = 0
\]
on $(\xi(b)\xi(c))$ and $(\xi(d)\xi(a))$ where $t$ is the tangential vector. This gives the required boundary data of $u$ along $(bc) \cup (da)$.

Finally, it remains to show that $\{L_\delta\}_{\delta > 0}$ is uniformly bounded. If this is not the case, there exists a sequence $\delta_n \to 0$ such that $L_{\delta_n} \to \infty$ as $n \to \infty$. By the same argument as above, the sequence $\frac{1}{L_{\delta_n}}f(\Omega_{\delta_n}, a_{\delta_n}, b_{\delta_n}, c_{\delta_n}, d_{\delta_n})$ converges to a holomorphic function $h$ locally uniformly. In such case, we have $\text{Re}h = 0$ on $\Omega$, thus $h$ is constant. But $\text{Im}h = 1$ on $(da)$ and $\text{Im}h = 0$ on $(bc)$, this gives a contradiction. Thus, $\{L_\delta\}_{\delta > 0}$ is uniformly bounded and we complete the proof.

\[\square\]

C Tightness

The goal of this section is to show the tightness of LERW branches (Proposition 4.6) and the Peano curves (Proposition 4.3). We first recall the tightness of LERW branches in the case of Dobrushin boundary
The sequence \(\{\tilde{\gamma}_\delta^0\}_{\delta>0}\) is tight. Moreover, any subsequential limit is a simple curve in \(\overline{\Omega}\) which intersects \([ab]\) only at its end.

**Proof.** The proof is similar to the proof of [LSW04, Lemma 3.12] and we will prove a similar result in Lemma C.2. Since the proof of Lemma C.1 is almost the same as the proof of Lemma C.2 which is the prerequisite of all of our main results, we choose to provide a detailed proof there.

### C.1 Tightness of loop-erased random walk

In this section, the goal is to show the tightness of LERW branches in quad. The setup is as follows. Given a quad \((\Omega; a, b, c, d)\) such that \((bc)\) and \((da)\) are simple and \(C^1\). Suppose there exists a simply connected domain \(\tilde{\Omega}\) such that \(\partial \tilde{\Omega}\) is \(C^1\) and simple with \(\Omega \subset \tilde{\Omega}\) and \(\partial \Omega \cap \partial \tilde{\Omega} = [bc] \cup [da]\). Fix a sequence of discrete quads \(\{(\Omega_\delta; a_\delta, b_\delta, c_\delta, d_\delta)\}_{\delta>0}\) satisfying that there exists a constant \(C > 0\) such that

\[
d((b_\delta a_\delta), (bc)) \leq C\delta, \quad d((a_\delta b_\delta), (da)) \leq C\delta, \quad d((a_\delta b_\delta), (ab)) \rightarrow 0, \quad d((c_\delta d_\delta), (cd)) \rightarrow 0, \quad \text{as } \delta \rightarrow 0,
\]

where \(d\) is the metric \((1.2)\). Fix \(o \in \Omega\) and denote by \(o_\delta\) the discrete approximation of \(o\) on \(\Omega_\delta\) and denote by \(\tilde{\gamma}_\delta^0\) the loop-erased random walk from \(o_\delta\) to \((a_\delta b_\delta)\) on \(\Omega_\delta\).

**Lemma C.1.** The sequence \(\{\tilde{\gamma}_\delta^0\}_{\delta>0}\) is tight. Moreover, any subsequential limit is a simple curve in \(\overline{\Omega}\) which intersects \([ab]\) only at its end.

**Lemma C.2.** The sequence \(\{\gamma_\delta^M\}_{\delta>0}\) is tight. Moreover, any subsequential limit is a simple curve in \(\overline{\Omega}\) which intersects \([ab] \cup [cd]\) only at its two ends.

**Lemma C.3.** There exist constants \(C = C(\Omega) > 0\) and \(\delta_0 = \delta_0(\Omega) > 0\), such that the following holds: for any \(\delta \leq \delta_1 \leq \delta_0\) and connected subgraph \(A\) of \(\Omega_\delta\) with \(\text{diam}(A) > \delta_1\), if \(z_\delta \in \Omega_\delta\) satisfies \(\text{dist}(z_\delta, A \cup (a_\delta b_\delta) \cup (c_\delta d_\delta)) \leq \delta_1\), then

\[
\mathbb{P}[R \text{ hits } \partial B(z_\delta, C\delta_1) \text{ before } A \cup (a_\delta b_\delta) \cup (c_\delta d_\delta)] \leq 1/2,
\]
where $\mathcal{R}$ is the simple random walk starting from $z_\delta$, reflecting at $(b_\delta c_\delta) \cup (d_\delta a_\delta)$, and stopping when it hits $(a_\delta b_\delta) \cup (c_\delta d_\delta)$. Similarly, for any discrete curve $\gamma_\delta$ on $\Omega_\delta$, if $z_\delta \in \Omega_\delta$ satisfies $\text{dist}(z_\delta, A \cup \gamma_\delta \cup (a_\delta b_\delta) \cup (c_\delta d_\delta)) \leq \delta_1$, then

$$P[\mathcal{R} \text{ hits } \partial B(z_\delta, C\delta_1) \text{ before } A \cup \gamma_\delta \cup (a_\delta b_\delta) \cup (c_\delta d_\delta)] \leq 1/2,$$

where $\mathcal{R}$ is the same random walk as above.

**Proof.** We only prove the first statement since the second is similar. By the assumptions, there exists a simply connected domain $\tilde{\Omega}$ such that.

By the assumptions, there exists a simply connected domain $\tilde{\Omega}$ such that $\partial \tilde{\Omega}$ is $C^1$ with $\Omega \subset \tilde{\Omega}$ and $\partial \Omega \cap \partial \tilde{\Omega} = [bc] \cup [da]$. Denote by $\Omega_\delta$ the discrete approximation of $\tilde{\Omega}$ on $\delta \mathbb{Z}^2$ such that $\partial \Omega_\delta \cap \partial \tilde{\Omega} = [b_\delta c_\delta] \cup [d_\delta a_\delta]$ and that there exists $C > 0$ such that,

$$d(\partial \Omega_\delta, \partial \tilde{\Omega}) \leq C\delta,$$

where $d$ is the metric $[1, 2]$. Denote by $\tilde{\mathcal{R}}$ the simple random walk on $\tilde{\Omega}_\delta$ starting from $z_\delta$ and reflecting at $\partial \tilde{\Omega}_\delta$. Define

$$P_1 := P[\tilde{\mathcal{R}} \text{ hits } \partial B(z_\delta, C\delta_1) \text{ before } A]$$

and

$$P_2 := P[\tilde{\mathcal{R}} \text{ hits } \partial B(z_\delta, C\delta_1) \text{ before } (a_\delta b_\delta)], \quad P_3 := P[\tilde{\mathcal{R}} \text{ hits } \partial B(z_\delta, C\delta_1) \text{ before } (c_\delta d_\delta)].$$

Then, we have

$$P[\mathcal{R} \text{ hits } \partial B(z_\delta, C\delta_1) \text{ before } A \cup (a_\delta b_\delta) \cup (c_\delta d_\delta)] \leq \min\{P_1, P_2, P_3\}.$$

The conclusion follows from [Sch00, Lemma 11.2]. The proof there relies crucially on the simple and $C^1$ regularity assumption and the stronger convergence. In the proof there, the author used estimate of the discrete Dirichlet energy to get estimates on discrete harmonic functions. To this end, the author counted the number of certain discrete paths. The $C^1$ and simple regularity condition and the stronger convergence make it possible to have a good control of the number of the discrete paths when $A$ is near $(b_\delta c_\delta)$ and $(d_\delta a_\delta)$.

**Lemma C.4.** For $z_0 \in \Omega$, $\alpha > 0$ and $0 < \beta \leq 1/4 \text{dist}((ab), (cd))$, let $A_\delta(z_0, \beta, \alpha; \gamma_\delta^M)$ denote the event that there are two points $o_1, o_2 \in \gamma_\delta^M$ with $o_1, o_2 \in B(z_0, \beta/4)$ and $|o_1 - o_2| \leq \alpha$, such that the subarc of $\gamma_\delta^M$ between $o_1$ and $o_2$ is not contained in $B(z_0, \beta)$. Then for any $\epsilon > 0$, there exists $\alpha = \alpha(\beta, \epsilon)$ such that

$$P[A_\delta(z_0, \beta, \alpha; \gamma_\delta^M)] \leq \epsilon, \quad \forall \delta > 0.$$

**Proof.** The proof is similar to the proof of [Sch00, Lemma 3.4]. By the choice of $\beta$, we may assume that $\text{dist}(B(z_0, \beta), (a_\delta b_\delta)) > 1/4 \text{dist}((ab), (cd))$. The case for $\text{dist}(B(z_0, \beta), (c_\delta d_\delta)) > 1/4 \text{dist}((ab), (cd))$ can be dealt similarly by the reversibility of loop-erased random walk and simple random walk. Let $\mathcal{R}_\delta$ be the simple random walk from $(a_\delta b_\delta)$ to $(c_\delta d_\delta)$ which generates the loop-erased random walk $\gamma_\delta^M$. Let $t_0 = 0$. For $j \geq 1$, define inductively

$$s_j := \inf\{t \geq t_{j-1} : \mathcal{R}_\delta(t) \in B(z_0, \beta/4)\} \quad \text{and} \quad t_j := \inf\{t \geq s_j : \mathcal{R}_\delta(t) \notin B(z_0, \beta)\}.$$

Let $\tau$ be the hitting time of $\mathcal{R}_\delta$ at $(c_\delta d_\delta)$. For every positive integer $t \in \mathbb{N}$, define $\text{LE}([\mathcal{R}_\delta[0, t]])$ to be the loop erasure of $\mathcal{R}_\delta[0, t]$. Define $A_\delta(z_0, \beta, \alpha; \text{LE}([\mathcal{R}_\delta[0, t]])$ similarly as $A_\delta(z_0, \beta, \alpha; \gamma_\delta^M)$. For $j \geq 0$, define the event $T_j := \{t_j \leq \tau\}$. For simplicity, we denote by $Y_j$ the event $A_\delta(z_0, \beta, \alpha; \text{LE}([\mathcal{R}_\delta[0, t]])$. Note that $Y_1 = \emptyset$ and

$$A_\delta(z_0, \beta, \alpha; \gamma_\delta^M) \subset \{\cup_{j=2}^{\infty}(Y_j \cap T_j)\} \subset \left\{\bigcup_{j=2}^{m} Y_j \bigcup \bigcup_{j=2}^{m} T_{m+1} \right\}, \quad \text{for every } m > 1.$$
Thus, we only need to control $\mathbb{P} [ \bigcup_{j=1}^{m} Y_j ]$ and $\mathbb{P} [ T_{m+1} ]$.

First, for every $j \geq 1$, denote by $Q_j$ the set consisting of connected components of $\mathcal{R}_0 [0, s_{j+1}] \cap B(z_0, \beta)$ that intersect $\partial B(z_0, \beta/4)$ and $\partial B(z_0, \beta)$ and do not contain $\mathcal{R}_0 (s_{j+1})$. Note that the cardinality $|Q_j|$ of $Q_j$ is at most $j$. Given $Y_j^c$, on the event $Y_{j+1}$, the random walk $\mathcal{R}_0$ first comes into the $\alpha$-neighbour of $Q_j$ in $B(z_0, \alpha)$, then it exits $B(z_0, \beta)$ without hitting any component in $Q_j$.

Combining with Lemma C.3, there exist constants $C > 0$ and $k > 0$, such that
\[
\mathbb{P} [ Y_{j+1} | Y_j^c ] \leq C |Q_j| \left( \frac{\alpha}{\beta} \right)^k \leq C j \left( \frac{\alpha}{\beta} \right)^k.
\]
This implies that
\[
\mathbb{P} [ \bigcup_{j=1}^{m} Y_j ] \leq \sum_{j=1}^{m-1} \mathbb{P} [ Y_{j+1} | Y_j^c ] \leq C m^2 \left( \frac{\alpha}{\beta} \right)^k.
\]
(C.1)

Second, for every $z_\delta \in \Omega_\delta$, define
\[
h_\delta (z_\delta) := \mathbb{P} [ \text{the simple random walk starting from } z_\delta \text{ hits } (c_\delta d_\delta) \text{ before } B(z_0, \beta/4) \cup (a_\delta b_\delta) ].
\]
Recall that the simple random walk will continue when it hits $(h_\delta a_\delta) \cup (d_\delta a_\delta)$ and thus $h_\delta$ is harmonic on $\Omega_\delta \setminus \{ (a_\delta b_\delta) \cup (c_\delta d_\delta) \cup B(z_0, \beta/4) \}$. Let $\partial B_\delta(z_0, \beta)$ be the collection of vertices on $\Omega_\delta$ closest to $\partial B(z_0, \beta)$. We claim that there exists $0 < q < 1$, such that for every $\delta > 0$,
\[
\min_{z_\delta \in \partial B_\delta(z_0, \beta)} h_\delta (z_\delta) \geq q \quad \text{(C.2)}
\]
Assuming this is true. Combining with the Markov property of random walk, we have
\[
\mathbb{P} [ T_{m+1} ] \leq (1 - q)^m. \quad \text{(C.3)}
\]
Combining (C.1) and (C.3), for every $\epsilon > 0$, by first choosing $m$ large enough and then choosing appropriate $\alpha = \alpha (\beta, \epsilon)$, we have
\[
\mathbb{P} [ A_\delta(z_0, \beta, \alpha; \gamma_\delta^M) ] \leq \epsilon.
\]
This completes the proof.

We are left to prove (C.2). Assuming (C.2) does not hold. Then, there exist $\delta_n \to 0$ and $z_\delta_n \in \partial B_\delta(z_0, \beta)$, such that
\[
\lim_{n \to \infty} h_\delta_n(z_\delta_n) = 0.
\]
By the minimal principle for discrete harmonic functions, we can find a path $r_\delta_n$ connecting $z_\delta_n$ to $B(z_0, \beta/4)$ such that
\[
\max_{v \in r_\delta_n} h_\delta_n(v) \leq h_\delta_n(z_\delta_n).
\]
By extracting a subsequence, we may assume $r_\delta_n \subset B(z_0, \beta)$ and $r_\delta_n$ converges to a compact set $K$ in Hausdorff metric and $h_\delta_n$ converges to a harmonic function $h$ locally uniformly. By Beurling estimate, we have $h$ equals 1 on $(cd)$ and equals 0 on $(ab) \cup \partial B(z_0, \beta/4)$. Since diam$(r_\delta_n) \geq \beta/4$, we have diam$(K) \geq \beta/4$. Choose a sequence of points $\{w_m\} \subset \Omega$ such that $w_m \to w \in K \cap B^c(z_0, \beta/4)$. Denote by $w^M_m$ the discrete approximation of $w_m$ on $\Omega_\delta_n$. By Lemma C.3, there exist constants $C > 0$ and $k > 0$, such that
\[
h_\delta_n(w^M_m) \leq h_\delta_n(z_\delta_n) + C \left( \frac{\text{dist}(w^M_m, r_\delta_n)}{\beta} \right)^k.
\]
By letting $n \to \infty$, we have
\[
h(w_m) \leq C \left( \frac{\text{dist}(w_m, K)}{\beta} \right)^k.
\]
By letting $m \to \infty$, we have
\[
h(w) = 0. \quad \text{(C.4)}
\]
There are two cases:
• If \( K \cap B^c(z_0, \beta/4) \cap \Omega \neq \emptyset \), we can choose \( w \in K \cap B^c(z_0, \beta/4) \cap \Omega \). Then \([C.4]\) contradicts the minimum principle since we have \( h \) equals 1 on \((cd)\).

• If \( K \cap B^c(z_0, \beta/4) \subset \partial \Omega \), in this case, \( \Omega_\delta \setminus \{(a_\delta b_\delta \cup B(z_0, \beta/4)\} \) is simply connected. Thus, we can construct the harmonic conjugate of \( h_\delta \) as in Lemma \([B.2]\). Then \( h \) only equals 0 on \((ab)\cup \partial B(z_0, \beta/4)\) by the same argument as in Lemma \([B.1]\). Since \( B(z_0, \beta) \cap (ab) = \emptyset \), we have that \( w \in (ba) \). This is a contradiction to \([C.4]\).

We complete the proof of \( (C.2) \).

**Proof of Lemma \([C.2]\)** The proof is similar to the proof of [LSW04, Lemma 3.12], which is essentially contained in [Sch00, Theorem 1.1]. Suppose \( \Upsilon : (0, \infty) \rightarrow (0, 1) \) is an increasing function. For every simply connected domain \( D \), denote by \( \chi_\Upsilon(D) \) the space of simple curves \( r : [0, 1] \rightarrow \overline{D} \), such that for every \( 0 \leq s_1 < s_2 \leq 1 \),

\[
\text{dist}(r[0, s_1], r[s_2, 1]) \geq \Upsilon(\text{diam}(r[s_1, s_2]))
\]

We claim that for \( \epsilon > 0 \), there exists \( \Upsilon \) such that

\[
\mathbb{P}[\gamma_\delta^M \in \chi_\Upsilon(\Omega)] \geq 1 - \epsilon, \quad \text{for all } \delta.
\]

Roughly speaking, this estimate says that \( \gamma_\delta^M \) does not create “almost bubble” with high probability. We choose \( R > 0 \) such that \( \Omega_\delta \subset B(0, R) \) for all \( \delta \). Then, we have

\[
\mathbb{P}[\gamma_\delta^M \in \chi_\Upsilon(B(0, R))] \geq 1 - \epsilon.
\]

By the same argument as in [LSW04, Lemma 3.10], the set \( \chi_\Upsilon(B(0, R)) \) is a compact set of simple curves. This completes the proof of tightness.

Second, we prove that any subsequential limit \( \gamma \) intersects \( [ab] \cup [cd] \) only at its two ends. We only prove that \( \gamma \) intersects \( [cd] \) at its one end. The proof for \( [ab] \) is similar by the reversibility of loop-erased random walk. Denote by \( \mathcal{R}_\delta \) the random walk which generates the loop-erased random walk \( \gamma_\delta^M \). For \( m \geq 1 \), define

\[
\tau_\delta^m := \inf \{ t : \text{dist}(\gamma_\delta^M(t), (c_\delta d_\delta)) = 2^{-2m} \}
\]

and let \( \tau_\delta \) be the hitting time of \( \gamma_\delta^M \) at \( (c_\delta d_\delta) \). Similarly, define

\[
\tilde{\tau}_\delta^m := \inf \{ t : \text{dist}(\mathcal{R}_\delta(t), (c_\delta d_\delta)) = 2^{-2m} \}
\]

and let \( \tilde{\tau}_\delta \) be the hitting time of \( \mathcal{R}_\delta \) at \( (c_\delta d_\delta) \). By Lemma \([C.3]\) there exist constants \( C > 0 \) and \( c > 0 \), such that for every \( m > 0 \), we have

\[
\mathbb{P}[\text{diam}(\mathcal{R}_\delta[\tilde{\tau}_\delta^m, \tilde{\tau}_\delta]) \geq 2^{-m}] \leq C 2^{-cm}.
\]

Since \( \gamma_\delta^M[\tau_\delta^m, \tau_\delta] \subset \mathcal{R}[\tilde{\tau}_\delta^m, \tilde{\tau}_\delta] \), we have

\[
\mathbb{P}[\text{diam}(\gamma_\delta^M[\tau_\delta^m, \tau_\delta]) \geq 2^{-m}] \leq C 2^{-cm}.
\]

Now, suppose \( \gamma_\delta^M \) converges to \( \gamma \) in law and we may couple them together such that \( \gamma_\delta^M \) converges to \( \gamma \) almost surely. We define \( \tau^m \) and \( \tau \) for \( \gamma \) similarly as before. Then, we have

\[
\{\text{diam}(\gamma[\tau^m, \tau]) \geq 2^{-m}\} \subset \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \{\text{diam}(\gamma_\delta^M[\tau_\delta^m, \tau_\delta]) \geq 2^{-m}\}.
\]

This implies that

\[
\mathbb{P}[\text{diam}(\gamma[\tau^m, \tau]) \geq 2^{-m}] \leq C 2^{-cm}.
\]

Thus, we have

\[
\sum_{m=1}^{\infty} \mathbb{P}[\text{diam}(\gamma[\tau^m, \tau]) \geq 2^{-m}] < +\infty.
\]
By Borel-Cantelli lemma, we have that \( \gamma \) hits \([cd]\) only at its end almost surely.

It remains to prove [C.5]. It suffices to show it for any sequence \( \delta_n \to 0 \). For \( \alpha, \beta > 0 \), denote by \( \mathcal{A}_{\delta_n}(\beta, \alpha) \) the event that there exist \( 0 \leq s_1 < s_2 \leq 1 \) such that \( \text{dist}(\gamma_{\delta_n}^M[0, s_1], \gamma_{\delta_n}^M[\beta, \alpha]) \leq \alpha \) but \( \text{diam}(\gamma_{\delta_n}^M[s_1, s_2]) \geq \beta \). Note that we can choose \( \{z_1, \ldots, z_k\} \subset \Omega \), where \( k = k(\beta) \), such that every ball with center in \( \Omega \) and radius \( 2\alpha \) is contained in one of the \( k \) balls \( B(z_j, \frac{\beta}{2^n}) \) for \( j = 1, \ldots, k \). Recall the definition of \( \mathcal{A}(z, \beta, \alpha; \gamma_{\delta_n}^M) \) in Lemma C.4. Note that \( \mathcal{A}_{\delta_n}(\beta, \alpha) \subset \bigcup_{j=1}^{k} \mathcal{A}(z_j, \beta, \alpha; \gamma_{\delta_n}^M) \). Thus, by Lemma C.4 for every \( m > 0 \), we can choose \( \alpha_m \) such that

\[
P[\mathcal{A}_{\delta_n}(2^{-m-1} \text{dist}((ab), (cd)), \alpha_m)] \leq \frac{\epsilon}{2^m}, \quad \text{for all } n.
\]

We choose \( \Upsilon \) such that \( \Upsilon(t) < \alpha_m \) for every \( t \leq 2^{-m-1} \text{dist}((ab), (cd)) \). Then, we have

\[
P[\gamma_{\delta_n}^M \in \chi_\Upsilon(\Omega_{\delta_n})] \geq P[(\bigcup_{m=1}^\infty \mathcal{A}_{\delta_n}(2^{-m-1} \text{dist}((ab), (cd)), \alpha_m))^c] \geq 1 - \epsilon.
\]

This gives [C.5] and completes the proof. \( \square \)

Fix \( o_1, o_2 \in \overline{\Omega} \) and suppose \( \delta^1, \delta^2 \) are the approximations of \( o_1 \) and \( o_2 \) (we allow \( o_1 \) and \( o_2 \) to be \((ab)\) and \((cd)\) and allow \( \delta^1 \) and \( \delta^2 \) to be \((a\delta b_\delta)\) and \((c\delta d_\delta)\)). Denote by \( \gamma_{\delta^1, \delta^2}^o \) the branch of the UST between \( \delta^1 \) and \( \delta^2 \). Recall that the wired boundary arcs \((a\delta b_\delta)\) and \((c\delta d_\delta)\) belong to the UST. Thus, if \( \gamma_{\delta^1, \delta^2}^o \) hits boundary arcs \((a\delta b_\delta)\) or \((c\delta d_\delta)\), it always contains subarcs of boundary arcs so that it is a continuous simple curve.

**Lemma C.5.** Fix \( o_1, o_2 \in \overline{\Omega} \) and suppose \( \delta^1, \delta^2 \) are the approximations of \( o_1 \) and \( o_2 \). Then \( \{\gamma_{\delta^1, \delta^2}^o\}_{\delta > 0} \) is tight. Moreover, any subsequential limit is a simple curve in \( \overline{\Omega} \).

**Proof.** It is equivalent to show that the law of \( \{\gamma_{\delta^1, \delta^2}^o\}_{\delta > 0} \) is relatively compact. We fix a subsequence \( \{\delta_n\}_{n \geq 0} \). From Lemma C.2, the sequence \( \{\gamma_{\delta_n}^M\}_{n \geq 0} \) is tight. By extracting a subsequences, we may assume that \( \{\gamma_{\delta_n}^M\}_{n \geq 0} \) converges to a continuous simple curve \( \gamma^M \) in law and therefore we can couple them together such that \( \gamma_{\delta_n}^M \) converges to \( \gamma^M \) almost surely. We only need to consider two cases.

- At least one of \( o_1 \) and \( o_2 \) equals \((ab)\) or \((cd)\).
- Both \( o_1 \) and \( o_2 \) belong to the interior of \( \Omega \).

For the first case, we may assume \( o_2 \) equals \((ab)\). We first run the loop-erased random walk \( \gamma_{\delta_n}^M \) and then run the loop-erased random walk from \( \delta^1_n \) to \((a\delta_n b_{\delta_n}) \cup \gamma_{\delta_n}^M \cup (c\delta_n d_{\delta_n}) \) which we denote by \( \tilde{\gamma}_{\delta_n}^{o_1} \). From Lemma C.1 the sequence \( \{\tilde{\gamma}_{\delta_n}^{o_1}\}_{n \geq 0} \) is tight and any subsequential limit \( \tilde{\gamma}^{o_1} \) of \( \{\tilde{\gamma}_{\delta_n}^{o_1}\}_{n \geq 0} \) hits \([ab] \cup \gamma^M \cup [cd]\) at \( \tilde{\gamma}^{o_1} \)'s end. We may couple them together such that \( \tilde{\gamma}_{\delta_n}^{o_1} \) converges to \( \tilde{\gamma}^{o_1} \) almost surely. Note that \( \gamma_{\delta_n}^{o_1, (ab)} \) may consist of subsegments of \( \gamma_{\delta_n}^M \) and \( \tilde{\gamma}_{\delta_n}^{o_1} \) and \((c\delta_n d_{\delta_n})\). From the fact that \( \gamma_{\delta_n}^M \) converges to \( \gamma^M \) and \( \gamma_{\delta_n}^{o_1} \) converges to \( \tilde{\gamma}^{o_1} \) almost surely and the fact that \( \tilde{\gamma}^{o_1} \) intersects \([ab] \cup \gamma^M \cup [cd]\) at \( \tilde{\gamma}^{o_1} \)'s end almost surely, we have that \( \gamma_{\delta_n}^{o_1, (ab)} \) converges to a continuous simple curve almost surely. This completes the proof of the first case.

For the second case, we first run \( \gamma_{\delta_n}^M \) and \( \tilde{\gamma}_{\delta_n}^{o_1} \) in the same way as above and then run the loop-erased random walk from \( \delta^2_n \) to \((a\delta_n b_{\delta_n}) \cup \gamma_{\delta_n}^M \cup \tilde{\gamma}_{\delta_n}^{o_1} \cup (c\delta_n d_{\delta_n}) \) which we denote by \( \tilde{\gamma}_{\delta_n}^{o_2} \). The remaining proof is similar to the first case (by replacing \( \gamma_{\delta_n}^M \) by \( \gamma_{\delta_n}^M \cup \tilde{\gamma}_{\delta_n}^{o_1} \) and replacing \( \gamma_{\delta_n}^{o_1} \) by \( \tilde{\gamma}_{\delta_n}^{o_2} \)). This completes the proof. \( \square \)
C.2 Proof of Propositions 4.3 and 4.6

In this section, we assume the following setup. Suppose $(\Omega; a, b, c, d)$ is a quad such that $\partial \Omega$ is $C^1$ and simple and suppose that a sequence of discrete quads $\{(\Omega_\delta; a_\delta, b_\delta, c_\delta, d_\delta)\}_{\delta>0}$ converges to $(\Omega; a, b, c, d)$ as in the sense of (1.3). Note that here we require that the boundary $\partial \Omega$ is $C^1$ and simple because we need to apply Lemma C.3 both on primal lattice and dual lattice. We consider the UST on $\Omega_\delta$ with alternating boundary conditions. We will complete the proof of Proposition 4.3 and Proposition 4.6. To this end, we first show Lemmas 4.3 and 4.5. Recall that trunk$_L^\delta (\epsilon)$, trunk$_R^\delta (\epsilon)$ and trunk for the UST and trunk$_L^\delta (\epsilon)$, trunk$_R^\delta (\epsilon)$ and trunk for dual forest are defined in Lemma 4.4.

Proof of Lemma 4.4. The proof of this lemma is almost the same as the proof of [Sch00, Theorem 11.1]. We summarize the strategy of the proof of [Sch00, Theorem 11.1] briefly. There are two main steps.

1. Showing that with large probability, the trunk$_\delta (\epsilon)$ is contained in the minimal subtree which contains a collection of vertices $V_\delta(\epsilon)$ in $\Omega_\delta$ that forms a kind of dense set in $\Omega$ and showing a similar result about trunk$_L^\delta (\epsilon)$. Here we require that $V_\delta$ satisfies the following condition: for a small constant $\delta_0$ which depends on $\epsilon$, for every $z_\delta \in \Omega_\delta$, there is a vertex $v_\delta \in V_\delta(\epsilon)$ such that $|v_\delta - z_\delta| < \delta_0$. Note that (4.1) is equivalent to the following equation:

$$\lim_{\epsilon \to 0, \delta \to 0} \lim_{\delta \to 0} \mathbb{P} \left[ \text{the Hausdorff distance between trunk}_l(\epsilon) \cap \text{trunk}_L^\delta (\epsilon) \right] = 0.$$

Thus, the results in this step reduce the estimate of the distance between trunk$_\delta (\epsilon)$ and trunk$_L^\delta (\epsilon)$ to an estimate of the distance between a subtree of the UST and a subforest of the dual forest.

2. Reducing the estimate about a subtree of the UST and a subforest of the corresponding dual forest in Step 1 to estimates about random walks on $\Omega_\delta$ and $\Omega_\delta^*$ and completing these estimates.

Now, we show how to carry out these steps in our setting.

Step 1. The proof, as being pointed out in the proof of [Sch00, Theorem 11.1(i)], is the same as the proof of [Sch00, Theorem 10.2], which only uses the discrete Beurling estimate. On the primal graph, we first generate the branch $\gamma_\delta^M$ between $(a_\delta b_\delta)$ and $(c_\delta d_\delta)$. Then, the Beurling estimate in $\Omega_\delta \setminus \gamma_\delta^M$ is from the second item of Lemma C.3. Recall that we denote by $\Omega_\delta^L$ and $\Omega_\delta^R$ the two connected components of $\Omega_\delta \setminus \gamma_\delta^M$. The boundary conditions of them are all Dobrushin boundary conditions. Denote by trunk$_L^\delta (\epsilon)$ the $\epsilon$–trunk of the UST in $\Omega_\delta^L$ and denote by trunk$_R^\delta (\epsilon)$ similarly. Thus, the proof of [Sch00, Theorem 10.2] works for trunk$_L^\delta (\epsilon)$ and trunk$_R^\delta (\epsilon)$ as well and implies the following: For every $\epsilon > 0$, there exists $\hat{\delta} > 0$ with the following property. For $i \in \{L, R\}$, let $V_\delta^i$ be a vertex set such that for every $w \in \Omega_\delta^i$, there exists a point $z \in V_\delta^i$ satisfying dist$(w, z) \leq \hat{\delta}$, then,

$$\mathbb{P} \left[ \text{trunk}_L^\delta (\epsilon) \subset T_\delta^i \right] \geq 1 - \epsilon,$$

where $T_\delta^i$ the minimal subtree of the UST which contains $V_\delta^i$. Denote by $V_\delta = V_\delta^L \cup V_\delta^R \cup \{a_\delta, b_\delta, c_\delta, d_\delta\}$. Recall that we always require the $\epsilon$–trunk contains the wired boundary arc. By Wilson’s algorithm, we can couple the USTs in $\Omega_\delta^L$ and $\Omega_\delta^R$ and $\Omega_\delta$ together, such that

$$\text{trunk}_\delta(\epsilon) \subset \text{trunk}_L^\delta (\epsilon) \cup \text{trunk}_R^\delta (\epsilon).$$

Denoted by $T_\delta^V$ the the minimal subtree of the UST on $\Omega_\delta$ which contains $V_\delta$. Then, we have

$$\mathbb{P} \left[ \text{trunk}_\delta (\epsilon) \subset T_\delta^V \right] \geq 1 - 2\epsilon.$$

On the dual graph $(\Omega_\delta^*, a_\delta^*, b_\delta^*, c_\delta^*, d_\delta^*)$, the dual forest can be obtained as follows. We denote by $\hat{\Omega}_\delta^*$ the graph obtained from $\Omega_\delta^*$ by wiring $(b_\delta^* c_\delta^*)$ and $(d_\delta^* a_\delta^*)$ as one vertex and keeping the incident relation with other vertices. We generate the UST on $\hat{\Omega}_\delta^*$ and then view its edges as edges on $\Omega_\delta^*$. Note that in this case, there are two connected components for the UST if we view it on $\hat{\Omega}_\delta^*$. This is exactly the dual forest. The discrete Beurling estimate for the dual forest is obtained from Lemma C.3 similarly by considering the random walk on $\hat{\Omega}_\delta^*$ and hence the conclusion about trunk$_L^\delta (\epsilon)$ is similar.
Step 2. As explained in the first paragraph of the proof of [Sch00, Theorem 11.1], the remaining proof is the same as the proof of [Sch00, Theorem 10.7]. As explained in the third paragraph of the proof of [Sch00, Theorem 10.7], we only need to prove the following estimate on the primal graph $\Omega_\delta$ and the same estimate on $\Omega_\delta^*$: For every $a_1, a_2 \in \Omega$, denote by $a^\delta_1, a^\delta_2$ the corresponding approximations on $\Omega_\delta$ and by $\beta_\delta$ the branch in the UST between $a^\delta_1$ and $a^\delta_2$, for every $v_\delta \in \Omega_\delta \setminus ((a_\delta b_\delta) \cup (c_\delta d_\delta))$, denote by $J(v_\delta, \beta_\delta)$ the branch in the UST from $v_\delta$ to $\beta_\delta$. Here, if $\beta_\delta$ intersects $(a_\delta b_\delta)$ or $(c_\delta d_\delta)$, we require that $(a_\delta b_\delta) \subset \beta_\delta$ or $(c_\delta d_\delta) \subset \beta_\delta$. This is natural by our definition of UST with $(a_\delta b_\delta)$ wired and $(c_\delta d_\delta)$ wired. Then, for every $s > 0$ and $t > 0$, we have

$$\lim_{h \to 0 \delta \to 0} \limsup \mathbb{P} \left[ \exists v_\delta \in B(a^\delta_1, s)^c \cap B(a^\delta_2, s)^c \cap \Omega_\delta \setminus ((a_\delta b_\delta) \cup (c_\delta d_\delta)) \text{ such that } \text{dist}(v_\delta, \beta_\delta) < h \text{ and } \text{diam}(J(v_\delta, \beta_\delta)) > t \right] = 0.$$ 

The proof of this equation is similar to the proof of [Sch00, Theorem 10.7]. In [Sch00], the main inputs of the proof of [Sch00, Theorem 10.7] are [Sch00, Lemma 10.8, Corollary 10.6] whose proof requires:

(a) discrete Beurling estimate;

(b) estimates of discrete harmonic function, see [Sch00, Lemma 10.9];

(c) the fact that any subsequential limit of the branches of the UST in Hausdorff metric are simple curves.

In our setup, the discrete Beurling estimate (a) is true due to Lemma C.3 the estimates of discrete harmonic function (b) comes from [Sch00, Lemma 10.9] directly (in particular, we do not need the boundary regularity in (b)); and the fact (c) that any subsequential limit of the branches of the UST is are simple curves also holds due to Lemma C.5 and its analogue for the dual forest. Therefore, the proof works in our setup as well. This completes the proof.

Next, we will prove Lemma 4.5. Recall that $\eta^L_\delta$ is the Peano curve from $a^\infty_\delta$ to $d^\infty_\delta$ and that $\gamma^M_\delta$ is the branch of the UST connecting $(a_\delta b_\delta)$ to $(c_\delta d_\delta)$. To estimate the probability in (4.2), we will use Wilson’s algorithm which relates $\gamma^M_\delta$ to loop-erased random walk, as described in the proof of Lemma 5.3.

Proof of Lemma 4.5. The proof is almost the same as the proof of [Sch00, Lemma 10.9]. We denote by $\hat{\Omega}_\delta$ the graph obtained from $\Omega_\delta$ by viewing $(a_\delta b_\delta)$ as one vertex and keeping the incident relation from $(a_\delta b_\delta)$ to other vertices. Note that here we do not view $(c_\delta d_\delta)$ as one vertex. We denote by $\hat{\mathcal{R}}$ the simple random walk on $\hat{\Omega}_\delta$ which ends at $(c_\delta d_\delta)$. Denote by $\mathbb{P}^{z_\delta}$ the law of the simple random walk starting from $z_\delta$. Define

$$h^{\mathcal{R}}_{\delta,\epsilon}(z_\delta) := \mathbb{P}^{z_\delta}[\hat{\mathcal{R}} \text{ hits } B(c_\delta, \epsilon) \text{ before } (c_\delta d_\delta)].$$

Note that the right boundary of $\eta^L_\delta$ equals $\gamma^M_\delta$. By Wilson’s algorithm, $\gamma^M_\delta$ can be generated by the loop-erasure of $\hat{\mathcal{R}}$. Thus, it suffices to show $h^{\mathcal{R}}_{\delta,\epsilon}((a_\delta b_\delta)) \to 0$ when letting $\delta \to 0$ and then letting $\epsilon \to 0$.

Fix $r > \epsilon$ and we denote by $\mathcal{R}$ the simple random walk on $\hat{\Omega}_\delta$ which ends at $(c_\delta d_\delta) \cap B^c(c_\delta, r)$. We define

$$\mathcal{T}^\delta_{\delta,\epsilon}(z_\delta) := \mathbb{P}[\mathcal{R} \text{ hits } B(c_\delta, \epsilon) \text{ before } (c_\delta d_\delta)].$$

Then, we have

$$h^\mathcal{R}_{\delta,\epsilon} \leq \mathcal{T}^\delta_{\delta,\epsilon} \text{ on } \hat{\Omega}_\delta.$$ 

Thus, it suffices to show that $\mathcal{T}^\delta_{\delta,\epsilon}((a_\delta b_\delta)) \to 0$ when letting $\delta \to 0$ and then letting $\epsilon \to 0$.

The remaining argument is same as the argument in [Sch00, Lemma 10.9]. Since $\mathcal{T}^\delta_{\delta,\epsilon}$ is a discrete harmonic function on $\hat{\Omega}_\delta \setminus \{B(c_\delta, \epsilon) \cup ((c_\delta d_\delta) \cap B^c(c_\delta, r))\}$ and equals 0 on $(c_\delta d_\delta) \cap B^c(c_\delta, r)$ and equals 1 on $B(c_\delta, \epsilon)$. By maximum principle, there is a path from $(a_\delta b_\delta)$ to $B(c_\delta, \epsilon)$, which we denote by $l_\delta$, such that for every $z_\delta \in l_\delta$, we have

$$\mathcal{T}^\delta_{\delta,\epsilon}(z_\delta) \geq \mathcal{T}^\delta_{\delta,\epsilon}((a_\delta b_\delta)).$$

50
Next, we will bound the discrete Dirichlet energy of \( \overline{h}_{\delta,\epsilon} \). Note that there exists a constant \( C > 0 \), which only depends on \( \Omega \), such that there are at least \( \frac{C}{\delta} \) paths connecting \((c_\delta d_\delta) \cap B^c(c_\delta, r)\) to \( l_\delta \) and each path has length less than \( \frac{C}{\delta} \). This implies that by Cauchy-Schwarz inequality, the discrete Dirichlet energy of \( \overline{h}_{\delta,\epsilon} \) satisfies the following bound

\[
\sum_{x_\delta, y_\delta \in \Omega_\delta} |\overline{h}_{\delta,\epsilon}(x_\delta) - \overline{h}_{\delta,\epsilon}(y_\delta)|^2 \geq C \overline{h}_{\delta,\epsilon}((a_\delta b_\delta)). \tag{C.6}
\]

Note that \( \overline{h}_{\delta,\epsilon} \) is the discrete function with minimal discrete Dirichlet energy over the set

\[\{r_\delta : \Omega \rightarrow \mathbb{R} : r_\delta \text{ equals 0 on } (c_\delta d_\delta) \cap B^c(c_\delta, r) \text{ and equals 1 on } B(c_\delta, \epsilon)\}.
\]

We define a special discrete function \( r_\delta \) as follows: \( r_\delta \) equals 1 on \( B(c_\delta, \epsilon) \) and equals 0 on \( B^c(c_\delta, r) \). For \( z_\delta \in B(c_\delta, r) \setminus B(c_\delta, \epsilon) \), we define \( r_\delta(z_\delta) := \frac{\log(|r|/|z_\delta - c|)}{\log(r/\epsilon)} \). As in [Sch00, Lemma 10.9], the discrete Dirichlet energy of \( r_\delta \) satisfies the following bound: there exists a constant \( C > 0 \),

\[
\sum_{x_\delta, y_\delta \in \Omega_\delta} |r_\delta(x_\delta) - r_\delta(y_\delta)|^2 \leq C \frac{1}{\log |r/\epsilon|}.
\]

Thus, combining with (C.6), we have

\[
\overline{h}_{\delta,\epsilon}((a_\delta b_\delta)) \leq C \frac{1}{\log |r/\epsilon|}.
\]

This completes the proof.

---

**Figure C.1:** The left-up figure corresponds to the event \( \{A_{\epsilon_0}^\delta \setminus \chi_{A_{\epsilon_0}^\delta}(s_1(\epsilon))\} \). On this event, the “almost bubble” is in the interior of \( \mathbb{H} \). The right-up figure corresponds to the event \( \{A_{\epsilon_0}^\delta \cap \chi_{A_{\epsilon_0}^\delta}(s_1(\epsilon))\} \setminus (\chi_{A_{\epsilon_0}^\delta}(s_0(\epsilon)) \cup \chi_{b,c,\epsilon_0}^\delta(s_0(\epsilon))) \). On this event, the “almost bubble” is near \( \mathbb{R} \setminus \{0,b,c\} \). The left-bottom figure corresponds to the event \( \{A_{\epsilon_0}^\delta \cap \chi_{A_{\epsilon_0}^\delta}(s_0(\epsilon))\} \). On this event, the “almost bubble” is near \( 0 \). The right-bottom figure corresponds to the event \( \{A_{\epsilon_0}^\delta \cap \chi_{b,c,\epsilon_0}^\delta(s_0(\epsilon))\} \). On this event, the “almost bubble” is near \( \{b,c\} \).
**Proof of Proposition 4.3.** The proof is similar to [LSW04 Proposition 4.5, Lemma 4.6]. We summarize the proof below and adjust it to our setting. By the convergence of domains, we can choose conformal maps $\phi_\delta : \Omega_\delta^1 \rightarrow \mathbb{H}$ with $\phi_\delta(a_\delta^1) = 0$ and $\phi_\delta(d_\delta^1) = \infty$ such that $\phi_\delta^{-1}$ converges uniformly on $\mathbb{H}$. Define $\hat{\eta}_\delta^L := \phi_\delta(\eta_\delta^L)$ and we parameterize $\hat{\eta}_\delta^L$ by the half-plane capacity. To prove the tightness, there are two parts.

- For every $t > 0, \epsilon > 0$, there exists $\epsilon_0 > 0$ such that
  \[
  \mathbb{P}[\sup\{\hat{\eta}_\delta^L(t_2) - \hat{\eta}_\delta^L(t_1) : 0 \leq t_1 < t_2 < t, |t_2 - t_1| \leq \epsilon_0\} \geq \epsilon] < \epsilon. \tag{C.7}
  \]
- The transience of curves at $d$: for any $\epsilon > 0$, there exist $\epsilon' \leq \epsilon$ and $\delta_0 > 0$ such that, for all $\delta \leq \delta_0$,
  \[
  \mathbb{P}[\hat{\eta}_\delta^L \text{ hits } \partial B(d, \epsilon) \text{ after hitting } \partial B(d, \epsilon')] < \epsilon. \tag{C.8}
  \]

We first derive (C.7). For $0 < t_1 < t_2 < +\infty$, let $(g_\delta^1, t \geq 0)$ be the corresponding conformal maps of $\hat{\eta}_\delta^L$ and let $Y_\delta(t_1, t_2) := \text{diam}(g_\delta^1(\hat{\eta}_\delta^L[t_1, t_2]))$. Combining with [LSW04 Lemma 2.1], to prove (C.7), it suffices to prove the following statement [LSW04 Lemma 4.6]: For every $\epsilon > 0$, there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that, for all $\delta < \delta_0$,

\[
\mathbb{P}\left[\sup\{\hat{\eta}_\delta^L(t_2) - \hat{\eta}_\delta^L(t_1) : 0 \leq t_1 < t_2 \leq \tau_\delta, Y_\delta(t_1, t_2) \leq \epsilon_0\} \geq \epsilon\right] < \epsilon \tag{C.9}
\]

where $\tau_\delta := \inf\{t \geq 0 : |\eta_\delta^L(t)| = \epsilon^{-1}\}$. For $\delta, \epsilon, \epsilon_0 > 0$, define

\[
\mathcal{A}_{\epsilon_0}^{\delta, \epsilon} := \{\exists \delta \leq t_2 \leq \tau_\delta \text{ such that } |\hat{\eta}_\delta^L(t_2) - \hat{\eta}_\delta^L(t_1)| \geq \epsilon, \text{ but } Y_\delta(t_1, t_2) \leq \epsilon_0\}.
\]

The event $\mathcal{A}_{\epsilon_0}^{\delta, \epsilon}$ implies the existence of "almost bubble" before time $\tau_\delta$. To prove (C.9), it suffices to show that, for a well-chosen positive function $\epsilon_0(\epsilon)$ such that $\epsilon_0(\epsilon) \to 0$ as $\epsilon \to 0$, we have

\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \sup_{\epsilon_0(\epsilon)} \mathbb{P}[\mathcal{A}_{\epsilon_0}^{\delta, \epsilon}] = 0. \tag{C.10}
\]

We then show (C.10). On the event $\mathcal{A}_{\epsilon_0}^{\delta, \epsilon}$, there exists a simple curve $\hat{\gamma}_{\delta, \epsilon_0}$ with short length, such that the interior surrounded by $\hat{\gamma}_{\delta, \epsilon_0} \cup \hat{\eta}_\delta^L[0, t_1]$ contains $\hat{\eta}_\delta^L[t_1, t_2]$ as follows. Denote by $Z$ the semicircle $2\epsilon^{-1}\partial U \cap \mathbb{H}$. On the one hand, there is a constant $C > 0$ such that $\text{dist}(g_\delta^1(Z), g_\delta^1(\hat{\eta}_\delta^L[t_1, t_2])) \geq C$ for all $t \leq \tau_\delta$. On the other hand, on the event $\mathcal{A}_{\epsilon_0}^{\delta, \epsilon}$, we have $Y_\delta(t_1, t_2) \leq \epsilon_0 \to 0$ as $\epsilon_0 \to 0$. These two facts guarantee that the extremal length of simple arcs in $\mathbb{H} \setminus g_\delta^1(\hat{\eta}_\delta^L[t_1, t_2])$ which separate $g_\delta^1(Z)$ from $g_\delta^1(\hat{\eta}_\delta^L[t_1, t_2])$ tends to $0$ as $\epsilon_0 \to 0$. By the conformal invariance of extremal length, there exists a simple curve $\hat{\gamma}_{\delta, \epsilon_0}$ in $\mathbb{H} \setminus \hat{\eta}_\delta^L[0, t_1]$ separating $\hat{\eta}_\delta^L[t_1, t_2]$ and $Z$ such that the length of $\hat{\gamma}_{\delta, \epsilon_0}$ tends to $0$ as $\epsilon_0 \to 0$.

For $s > 0$, we denote

\[
\chi_{0, \epsilon_0}^\delta(s) := \{\text{dist}(0, \gamma_{\delta, \epsilon_0}) < s\}, \quad \chi_{1, \epsilon_0}^\delta(s) := \{\text{dist}(\mathbb{R}, \gamma_{\delta, \epsilon_0}) < s\},
\]

\[
\chi_{\delta, \epsilon_0}^\delta(s) := \{\text{dist}[\{\phi_\delta(b_\delta^1), \phi_\delta(c_\delta^1)\}, \gamma_{\delta, \epsilon_0}) < s\}.
\]

We choose two function $s_1(\epsilon)$ and $s_0(\epsilon)$ such that $s_1(\epsilon) < s_0(\epsilon)$ and $\lim_{\epsilon \to 0} \frac{s_0(\epsilon)}{\epsilon} = 0$. Now, we divide $\mathcal{A}_{\epsilon_0}^{\delta, \epsilon}$ into four events

\[
\{\mathcal{A}_{\epsilon_0}^{\delta, \epsilon} \setminus \chi_{1, \epsilon_0}^\delta(s_1(\epsilon))\}, \quad \{\mathcal{A}_{\epsilon_0}^{\delta, \epsilon} \cap \chi_{0, \epsilon_0}^\delta(s_1(\epsilon)) \setminus (\chi_{0, \epsilon_0}^\delta(s_0(\epsilon)) \cup \chi_{\delta, \epsilon_0}^\delta(s_0(\epsilon)))\},
\]

\[
\{\mathcal{A}_{\epsilon_0}^{\delta, \epsilon} \cap \chi_{0, \epsilon_0}^\delta(s_0(\epsilon))\}, \quad \{\mathcal{A}_{\epsilon_0}^{\delta, \epsilon} \cap \chi_{\delta, \epsilon_0}^\delta(s_0(\epsilon))\}.
\]

These four events are classified by the location of the “almost bubble”. See Figure C.1. For the first two events, we may use the same argument as [LSW04 Eq. (4.9), Eq. (4.10)], and we have

\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \sup_{\epsilon_0(\epsilon)} \mathbb{P}[\mathcal{A}_{\epsilon_0}^{\delta, \epsilon} \setminus \chi_{1, \epsilon_0}^\delta(s_1(\epsilon))] = 0. \tag{C.11}
\]

\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \sup_{\epsilon_0(\epsilon)} \mathbb{P}[\mathcal{A}_{\epsilon_0}^{\delta, \epsilon} \cap \chi_{0, \epsilon_0}^\delta(s_1(\epsilon)) \setminus (\chi_{0, \epsilon_0}^\delta(s_0(\epsilon)) \cup \chi_{\delta, \epsilon_0}^\delta(s_0(\epsilon))) = 0. \tag{C.12}
\]
This part of the proof relies on (4.11). The third event can be estimated by the same argument for LSW04 Eq. (4.11), and we have

\[
\lim \lim_{\epsilon \to 0} \sup_{\delta \to 0} \sup_{\epsilon_0 \leq \epsilon_0(\delta)} \mathbb{P}[A^{\delta_0}_0 \cap \chi^{\delta}_0(s_0(\epsilon))] = 0. 
\]  
(C.13)

For the fourth event, from (4.2), we have

\[
\lim \lim_{\epsilon \to 0} \sup_{\delta \to 0} \sup_{\epsilon_0 \leq \epsilon_0(\delta)} \mathbb{P}[A^{\delta_0}_0 \cap \chi^{\delta}_0(s_0(\epsilon))] = \lim \lim_{\epsilon \to 0} \mathbb{P}[\eta^L_{\delta} \text{ hits } B(c_\delta, \epsilon)] = 0. \]  
(C.14)

To sum up, for the four events, we have (C.11), (C.12), (C.13) and (C.14). They imply (C.10) and complete the proof for (C.7).

Next, we show (C.8). We follow the same argument as in LSW04 The last paragraph of the proof of Lemma 4.6. Fix two constants \( r < \epsilon \) and \( C > 0 \). We choose \( v \in V(\Omega_\delta) \) which is adjacent to \( \partial B(d, r) \). Suppose \( v^* \in V(\Omega_\delta^2) \) which is adjacent to \( v \). We denote by \( \chi_v \) the LERW which starts from \( v \) and ends at \( (c_\delta d_\delta) \). Similarly, we denote by \( \chi^*_v \) the LERW which starts from \( v^* \) and ends at \( (d_\delta^2 c_\delta^2) \). We define \( A_1 := \{ \text{diam}(\chi_v) < C r \} \) and \( A_2 := \{ \text{dist}(d, \chi_v) > \frac{1}{C} r \} \). We also define \( A^*_1 \) and \( A^*_2 \) similarly by replacing \( \chi_v \) with \( \chi^*_v \). We choose \( C \) such that

\[
\mathbb{P}[A_1 \cap A_2 \cap A^*_1 \cap A^*_2] \geq 1 - \epsilon. \]

We choose \( r \) such that \( Cr < \epsilon \) and we choose \( \epsilon' < \frac{1}{r} \). Note that on the event \( A_1 \cap A_2 \cap A^*_1 \cap A^*_2 \), we have that \( \eta^L_{\delta} \) can not hit \( \partial B(d, \epsilon) \) after hitting \( \partial B(d, \epsilon') \). Since \( \eta^L_{\delta} \) can only go through the edge \( \{v, v^*\} \) once. This implies (C.8). Together with (C.7), we complete the proof of tightness.

Finally, suppose \( \eta^L \) is any subsequential limit of \( \{\eta^L_{\delta}\}_{\delta>0} \), we check that \( \mathbb{P}[\eta^L \cap [bc] = \emptyset] = 1 \). We have the following two observations.

- The event \( \{ \text{trunk } \cap \text{trunk}' \neq \emptyset \} \) implies \( \{ \eta^L \cap (bc) \neq \emptyset \} \). Thus \( \mathbb{P}[\eta^L \cap (bc) = \emptyset] = 1. \)
- From (4.2), we have \( \mathbb{P}[c \notin \eta^L] = 1. \) By symmetry, we have \( \mathbb{P}[b \notin \eta^L] = 1. \)

In summary, we have \( \mathbb{P}[\eta^L \cap [bc] = \emptyset] = 1 \) as desired. \( \square \)

**Remark C.6.** We treat the tightness of the LERW branches (Proposition 4.6) and the Peano curves (Proposition 4.3) for UST in topological rectangle in this appendix. Similar tightness argument also holds for the LERW branches and the Peano curves for UST in general topological polygons. These models will be analyzed in subsequent articles [LPW21] and [LW23a].

### D Hypergeometric SLE in literature

In this section, we compare “hypergeometric SLE” in the literature [Zha10], [Qia18] and [Wu20]. The variant of SLE whose driving function involves hypergeometric function is first considered in [Zha10]. This is a family of curves with two parameters \( \kappa \) and \( \rho \) and two marked points and the author calls them “intermediate SLE”. In [Zha10], the author considers the case when \( \kappa \in (0, 4) \) and \( \rho \geq \frac{\kappa-4}{2} \) and shows that the time-reversal of \( \text{SLE}_\kappa(\rho) \) can be described by intermediate \( \text{SLE}_\kappa(\rho) \) with two marked points.

In [Qia18], the author first uses the terminology “hypergeometric SLE” and the notion hSLE. The author extends “intermediate SLE” to more general setup: a family of curves with three parameters \( \lambda, \mu \) and \( \nu \), which is denoted by \( \text{hSLE}_\kappa(\lambda, \mu, \nu) \). The explicit form of driving function will be given below. The parameters of hypergeometric functions are as follows [Qia18 Eq.(4.2)]:

\[
A := \lambda + \mu + \nu + 2, \quad B := \mu + \nu - \lambda, \quad C := 2\nu + 3/2, \quad (D.1)
\]
with restrictions [Qia18, Eq.(4.5)]:
\[
\lambda > -3/4, \quad \mu \geq -1/4, \quad \nu > -3/4, \quad \mu < \lambda + \nu + 3/2.
\] (D.2)

The driving function \(W\) satisfies the following SDE [Qia18, Eq.(4.21)]:
\[
\begin{cases}
    dW_t = \sqrt{\kappa} dB_t + J(W_t - O_t, W_t - V_t) dt;
    \\
    dO_t = \frac{2dt}{\nu - W_t} dt;
    \\
    dV_t = \frac{2dt}{\nu - W_t} dt.
\end{cases}
\] (D.3)

Here \(J\) is defined by [Qia18, Eq.(4.17)]:
\[
J(p; q) = J(A, B, C, p; q) = \frac{\nu}{p} Q\left( A, B, C, \frac{p}{q - p} \right)
\] (D.4)
where (Qia18, the equation above Eq.(4.12), the equation above Eq.(4.15)):
\[
Q(A, B, C, z) := zG'(z)/G(z), \quad G(z) = z^\nu(1 + z)^\mu w(A, B, C; -z),
\] (D.5)
and \(w(A, B, C; z)\) is an analytic extension of \(2F_1(A, B, C; z)\). In [Qia18], the author mainly considers the case \(\kappa = 8/3\). This value is special since it is related to the conformal restriction measure [LSW03].

In [Wu20], the author uses the same terminology “hypercentric SLE” and the same notation \(hSLE\). In this paper, she extends the definition of the “intermediate SLE” in another direction, compared with [Qia18]. This is still a family of curves with two parameters \(\kappa\) and \(\rho\) and two marked points, which is called \(hSLE_{\kappa}(\rho - 2)\) (to avoid the ambiguity, we replace \(\nu\) there by \(\rho - 2\)). The author in [Wu20] extends it to the case when \(\kappa \in (0, 8)\) and \(\rho \in \mathbb{R}\):

- When \(\kappa \in (0, 8)\) and \(\rho > -2 \vee (\kappa/2 - 4)\), the process is defined in [Wu20, Eq.(3.2), Eq.(3.4)-(3.7)].
  In this case, the process is defined in the same way as (3.1)-(3.4) in this article. In particular, when \(\kappa \in (0, 4)\) and \(\rho > -2\), it is the same as the intermediate SLE\(_{\kappa}(\rho)\) [Zha10].

- When \(\kappa \in (0, 8)\) and \(\rho \leq -2 \vee (\kappa/2 - 4)\), the process is defined in [Wu20, Eq.(3.3), Eq.(3.4)-(3.7)].

In our Section 3, we further extend this definition to the case when \(\kappa \geq 8\) and \(\rho > \kappa/2 - 4\). In [Wu20], the author relates \(hSLE_{\kappa}(\rho - 2)\) to the scaling limit of interfaces of discrete models in quad, see [Wu20, Proposition 1.6] for Ising model and [KS18] for FK-Ising model. In our Section 4, we relates \(hSLE_{8}\) to the scaling limit of the Peano curve in UST in quad.

Now, let us compare the two notions of “hypercentric SLE” in [Qia18] and [Wu20]. The “hypercentric SLE” defined in [Wu20] is a subfamily of the “hypercentric SLE” defined in [Qia18] (as cited above (D.1)-(D.5)) only when \(\kappa = 8/3\) and \(\rho > -2\): the process \(hSLE_{8/3}(\rho - 2)\) defined in (3.1)-(3.4) is the same as \(hSLE_{8/3}(\rho - 2, 3\rho/8, 3\rho/8)\) defined in (D.1)-(D.5) (the restrictions (D.2) needs to be relaxed to include the case when \(\lambda = -2\)). For other \(\kappa\)’s, they are two different families of processes.

In order to get a more general relation, one has to change the constants in (D.1) to the following ones:
\[
A = \lambda + \mu + \nu + 8/\kappa - 1, \quad B = \mu + \nu - \lambda, \quad C = 2\nu + 4/\kappa,
\] (D.6)
and relaxes the restrictions in (D.2).

Lemma D.1. When \(\kappa > 0\) and \(\rho > (-2) \vee (\kappa/2 - 4)\), the process \(hSLE_{\kappa}(\rho - 2)\) defined in (3.1)-(3.4) is a special case of the process \(hSLE_{\kappa}(\lambda, \mu, \nu)\) defined in (D.3)-(D.6) for
\[
\mu = \nu = \frac{\rho}{\kappa} \quad \text{and} \quad \lambda = 1 - \frac{8}{\kappa}.
\] (D.7)
Proof. In [Qia18], the author considers the case $V_0 < O_0 = 0 \leq W_0$. In this case, $z = p/(q - p) < 0$, we use the following relation [AS92 Eq.15.3.4]
\[ w(A, B, C; -z) = (1 + z)^{-A}w\left(A, C - B, C; \frac{z}{z + 1}\right) \]
and write $\tilde{w}(\cdot) = w(A, C - B, C; \cdot)$. By the choice of $A, B, C$ in [D.6], we have
\[ \tilde{w}(\cdot) = 2F_1(\lambda + \mu + \nu + 8/\kappa - 1; \nu + \lambda + 4/\kappa - \mu; 2\nu + 4/\kappa; \cdot) \].
Setting $p = W_t - O_t$ and $q = W_t - V_t$, we find
\[ J(W_t - O_t, W_t - V_t) = \frac{\kappa \nu}{W_t - O_t} + \frac{\kappa(\mu - A)}{W_t - V_t} - \frac{\kappa(O_t - V_t)}{(W_t - V_t)^2} \tilde{w}'((W_t - O_t)/(W_t - V_t)) \cdot \frac{\tilde{w}((W_t - O_t)/(W_t - V_t))}{(W_t - O_t)/(W_t - V_t)} \cdot dt \].

To compare with (3.7), we reflect the hSLE$_\kappa(\mu, \nu, \kappa)$ with respect to $iR$, the drift term of the resulting driving function becomes
\[ dL_t := \frac{\kappa \nu}{W_t - O_t} dt - \frac{\kappa(A - \mu)}{W_t - V_t} dt - \frac{\kappa(V_t - O_t)}{(W_t - V_t)^2} \tilde{w}'((W_t - O_t)/(W_t - V_t)) \cdot \frac{\tilde{w}((W_t - O_t)/(W_t - V_t))}{(W_t - O_t)/(W_t - V_t)} dt \].

Compared with (3.7), we see that hSLE$_\kappa(\rho - 2)$ defined in (3.7) is a special case of the process defined in [D.3]-[D.6] with parameters (D.7). \qed

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