A construction of quotient \(A_\infty\)-categories

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October 29, 2018

Abstract

We construct an \(A_\infty\)-category \(D(\mathcal{C}|\mathcal{B})\) from a given \(A_\infty\)-category \(\mathcal{C}\) and its full subcategory \(\mathcal{B}\). The construction is similar to a particular case of Drinfeld’s construction of the quotient of differential graded categories [Dri04]. We use \(D(\mathcal{C}|\mathcal{B})\) to construct an \(A_\infty\)-functor of K-injective resolutions of a complex, when the ground ring is a field. The conventional derived category is obtained as the 0-th cohomology of the quotient of the differential graded category of complexes over acyclic complexes. This result follows also from Drinfeld’s theory of quotients of differential graded categories [Dri04].

In [Dri04] Drinfeld reviews and develops Keller’s construction of the quotient of differential graded categories [Kel99] and gives a new construction of the quotient. This construction consists of two parts. The first part replaces given pair \(\mathcal{B} \subset \mathcal{C}\) of a differential graded category \(\mathcal{C}\) and its full subcategory \(\mathcal{B}\) with another such pair \(\tilde{\mathcal{B}} \subset \tilde{\mathcal{C}}\), where \(\tilde{\mathcal{C}}\) is homotopically flat over the ground ring \(\mathbb{k}\) (K-flat) [Dri04, Section 3.3], and there is a quasi-equivalence \(\tilde{\mathcal{C}} \to \mathcal{C}\) [Dri04, Section 2.3]. The first step is not needed, when \(\mathcal{C}\) is already homotopically flat, for instance, when \(\mathbb{k}\) is a field. In the second part a new differential graded category \(\mathcal{C}/\mathcal{B}\) is produced from a given pair \(\mathcal{B} \subset \mathcal{C}\), by adding to \(\mathcal{C}\) new morphisms \(\varepsilon_U : U \to U\) of degree \(-1\) for every object \(U\) of \(\mathcal{B}\), such that \(d(\varepsilon_U) = \text{id}_U\).

In the present article we study an \(A_\infty\)-analogue of the second part of Drinfeld’s construction. Namely, to a given pair \(\mathcal{B} \subset \mathcal{C}\) of an \(A_\infty\)-category \(\mathcal{C}\) and its full subcategory \(\mathcal{B}\) we associate another \(A_\infty\)-category \(D(\mathcal{C}|\mathcal{B})\) via a construction related to the bar resolution of \(\mathcal{C}\). The \(A_\infty\)-category \(D(\mathcal{C}|\mathcal{B})\) plays the role of the quotient of \(\mathcal{C}\) over \(\mathcal{B}\) in some cases, for instance, when \(\mathbb{k}\) is a field. When \(\mathcal{C}\) is a differential graded category, \(D(\mathcal{C}|\mathcal{B})\) is precisely the category \(\mathcal{C}/\mathcal{B}\) constructed by Drinfeld [Dri04, Section 3.1].

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There exists another construction of quotient $A_\infty$-category $q(C|B)$, see [LM04]. It enjoys some universal property, and is significantly bigger in size than $D(C|B)$. However, when $C$ is unital, the two quotient constructions turn out to be $A_\infty$-equivalent. When $C$ is strictly unital, so is $D(C|B)$, while $q(C|B)$ is unital, but not strictly unital.

We apply our construction to the case of practical interest: $C = C(A)$ is the differential graded category of complexes in an Abelian category $A$, and $B = A(A)$ is the full subcategory of acyclic complexes. When the full subcategory formulas from [Lyu03].

Outline of the article with comments. In the first section we describe conventions and notations used in the article. In particular, we recall some conventions and useful formulas from [Lyu03].

In the second section we describe a construction of the quotient $A_\infty$-category $D(C|B)$, departing from an $A_\infty$-category $B$ fully embedded into an $A_\infty$-category $C$. The underlying quiver of $D(C|B)$ is described in Definition 2.1. Its particular case $D(C|C)$ is $s^{-1}T^+sC = \oplus_{n>0}s^{-1}T^nsC$, where $sC = C[1]$ stands for the suspended quiver $C$. We introduce two $A_\infty$-category structures for $s^{-1}T^+sC$. The first, $\bar{C} = (s^{-1}T^+sC, \bar{b})$ uses the differential $\bar{b}$, whose components all vanish except $\bar{b}_1 = b : T^+sC \to T^+sC$, which is the $A_\infty$-structure of $C$. The second, $\overline{C} = (s^{-1}T^+sC, \overline{b})$ is isomorphic to the first via a coalgebra automorphism $\mu : T(T^+sC) \to T(T^+sC)$, whose components are multiplications in the tensor algebra $T^+sC$. The resulting differential $\overline{b} = \overline{\mu}_b\mu^{-1} : T(T^+sC) \to T(T^+sC)$ is described componentwise in Proposition 2.2. The subquiver $D(C|B) \subset \overline{C}$ turns out to be an $\overline{A}_\infty$-subcategory (Proposition 2.2). $\overline{C}$ and $\overline{C}$ are, in a sense, trivial (they are contractible if $C$ is unital), but $D(C|B)$, in general, is not trivial. If $C$ is strictly unital, then so are $\overline{C}$ and $D(C|B)$ and their units are identified (Section 2.3). Notice that $\overline{C}$ is never strictly unital except when $C = 0$. Nevertheless, $\overline{C}$ can be unital. When $B$, $C$ are differential graded categories, we show in Section 2.6, that $D(C|B)$ coincides with the category $C/B$ defined by Drinfeld [Dri04, Section 3.1].

In the third section we construct functors between the obtained $A_\infty$-categories. When $B \subset C$, $J \subset I$ are full $A_\infty$-subcategories, and $i : C \to I$ is an $A_\infty$-functor which maps objects of $B$ into objects of $J$, we construct a strict $A_\infty$-functor $\bar{i} : \overline{C} \to \overline{I}$, whose first component $\bar{i}_1 = i : T^+sC \to T^+sJ$ is given by $i$ itself. The components of the conjugate $A_\infty$-functor $\bar{i} = \overline{\mu}_i\mu^{-1} : \overline{C} \to \overline{I}$ are described in Proposition 3.1. It turns out that $\bar{i}$ restricts to an $A_\infty$-functor $D(i) = \bar{i} : D(C|B) \to D(I|J)$ (Proposition 3.1). If $C$ is strictly unital, then $\overline{C}$ is unital (and contractible) and its unit transformation is computed in Section 3.3.

In the fourth section we construct $A_\infty$-transformations between functors obtained in the third section. When $B \hookrightarrow C$ and $J \hookrightarrow I$ are full $A_\infty$-subcategories, $f, g : C \to I$ are $A_\infty$-functors, which map objects of $B$ into objects of $J$, and $r : f \to g : C \to I$ is an
$A_\infty$-transformation, we construct an $A_\infty$-transformation $r : f \to g : C \to D$, whose only non-trivial components are $r_0 = r_0$ and $r_1 = r_1$. The components of the conjugate $A_\infty$-transformation $\overline{r} = \mu r \mu^{-1} : f \to g : C \to D$ are computed in Proposition 4.3. It turns out that $\overline{r}$ restricts to an $A_\infty$-transformation $D(r) : D(f) \to D(g) : D(C) \to D(D)$ (Proposition 4.1). Thus, $r$ and $\overline{r}$ are defined as maps on objects, 1-morphisms and 2-morphisms of the 2-category $A_\infty$ of $A_\infty$-categories. Actually, they are strict 2-functors $A_\infty \to A_\infty$ (Corollary 4.5, Corollary 4.7). We prove more: they are strict $\mathcal{K}$-2-functors $\mathcal{K}A_\infty \to \mathcal{K}A_\infty$, where the 2-category $\mathcal{K}A_\infty$ is enriched in $\mathcal{K} -$ the category of differential graded $\mathbb{k}$-modules, whose morphisms are chain maps modulo homotopy (Proposition 4.9). Compatibility of $\overline{r}$ with the composition of 2-morphisms is expressed via explicit homotopy (4.7). Components of this homotopy are found in Proposition 4.8. It turns out that this homotopy restricts to subcategories $D(\cdot\cdot\cdot)$ (Proposition 4.8). Therefore, $D$ is a $\mathcal{K}$-2-functor from the non-2-unital $\mathcal{K}$-2-category of pairs ($A_\infty$-category, full $A_\infty$-subcategory) to $\mathcal{K}A_\infty$ (Corollary 4.9). It can be viewed also as a 2-functor $D$ from the non-2-unital 2-category of pairs ($A_\infty$-category, full $A_\infty$-subcategory) to $A_\infty$ (Corollary 4.10).

In the fifth section we consider unital $A_\infty$-categories and prove that some of our $A_\infty$-categories are contractible. If $B$ is a full subcategory of a unital $A_\infty$-category $C$, then $D(C) \subset B$ is unital as well, and $D(C)$ is its unit transformation (Proposition 5.1). In particular, for a unital $A_\infty$-category $C$, both $\mathcal{C}$ and $\mathcal{C}$ are unital with the unit transformation $\mathcal{C}$ (resp. $\mathcal{C}$) (Corollaries 5.2, 5.3). If $i : C \to J$ is a unital $A_\infty$-functor, then $i : \mathcal{C} \to \mathcal{J}$ and (whenever defined) $D(i) : D(C) \to D(J)$ are unital as well (Corollaries 5.3, 5.4). When we restrict $r$, $\overline{r}$ or $D$ to unital $A_\infty$-categories (and unital $A_\infty$-functors), we get strict 2-functors of (usual 1-2-unital) ($\mathcal{K}$-)2-categories.

In the sixth section we consider contractible $A_\infty$-categories and $A_\infty$-functors. A unital $A_\infty$-functor $f : A \to B$ is called contractible if many equivalent conditions hold, including contractibility of complexes $(sB(Xf, Y), b_1), (sB(Y, Xf), b_1)$ for all $X \in \text{Ob} A, Y \in \text{Ob} B$ (Propositions 6.3, 6.4). A unital $A_\infty$-category $A$ is called contractible if several equivalent conditions hold, including contractibility of complexes $(sA(X, Y), b_1)$ for all objects $X, Y$ of $A$ (Definition 6.4, Proposition 6.4). If $C$ is a unital $A_\infty$-category, then $\mathcal{C}, \overline{\mathcal{C}}$ are contractible (Example 6.5). Nevertheless, in general, the subcategories $D(C) \subset \overline{\mathcal{C}}$ are not contractible. Contractible $A_\infty$-categories $B$ may be considered as trivial, because in this case any natural $A_\infty$-transformation $r : f \to g : A \to B$ is equivalent to 0 (Corollary 6.8). Moreover, non-empty contractible categories are equivalent to the 1-object-1-morphism $A_\infty$-category $\mathbb{1}$, such that $\text{Ob} \mathbb{1} = \{ \ast \}$ and $\mathbb{1}(\ast, \ast) = 0$ (Proposition 6.7, Remark 6.9).

In the seventh section we consider the case of a contractible full subcategory $\mathcal{F}$ of a unital $A_\infty$-category $\mathcal{E}$. In this case the canonical strict embedding $\mathcal{E} \to D(\mathcal{E}|\mathcal{F})$ is an equivalence (Proposition 7.4).

In the eighth section we prepare to construct the $K$-injective resolution $A_\infty$-functor. This concrete construction is deferred until the next section. In the eighth section we consider an abstract version of it. Given an $A_\infty$-functor $f : B \to C$, a map $g : \text{Ob} B \to
Ob $\mathcal{C}$ and cycles $r_X \in \mathcal{C}^0(Xf, Xg)$, $X \in \text{Ob} \mathcal{B}$, producing certain quasi-isomorphisms, we make $g$ into an $A_\infty$-functor $g : \mathcal{B} \to \mathcal{C}$ and $r_X$ into 0-th component $x r_0 s^{-1}$ of a natural $A_\infty$-transformation $r : f \to g : \mathcal{B} \to \mathcal{C}$ (Proposition 5.2). Next we prove the uniqueness of so constructed $g$ and $r$. Assuming that the initial data $(g : \text{Ob} \mathcal{B} \to \text{Ob} \mathcal{C}, (r_X)_{x \in \text{Ob} \mathcal{B}})$ give rise to two $A_\infty$-functors $g, g' : \mathcal{B} \to \mathcal{C}$ and two natural $A_\infty$-transformations $r : f \to g : \mathcal{B} \to \mathcal{C}$, we construct another natural $A_\infty$-transformation $p : g \to g' : \mathcal{B} \to \mathcal{C}$, such that $r'$ is the composition $f \overset{r}{\longrightarrow} g \overset{p}{\longrightarrow} g'$ in the 2-category $A_\infty$ (Proposition 5.4). Moreover, such $p$ is unique up to an equivalence (Proposition 5.5). If, in addition, $\mathcal{C}$ is unital, then the constructed $p$ is invertible (Corollary 5.6). If $f$ is unital, then the constructed $A_\infty$-functor $g$ is unital as well (Proposition 5.7).

In the ninth section we consider categories of complexes. Let $\mathbb{k}$ be a field, let $\mathcal{A}$ be an Abelian $\mathbb{k}$-linear category, and let $\mathcal{C} = \mathcal{C}(\mathcal{A})$ be the differential graded category of complexes in $\mathcal{A}$. Let $\mathcal{B} = \mathcal{A}(\mathcal{A})$ be its full subcategory of acyclic complexes, $\mathcal{I} = \mathcal{I}(\mathcal{A})$ denotes K-injective complexes, $\mathcal{J} = \mathcal{A}(\mathcal{A})$ denotes acyclic K-injective complexes. We assume that each complex $X \in \text{Ob} \mathcal{C}$ has a right K-injective resolution $r_X : X \to XI$ (a quasi-isomorphism with K-injective $XI \in \text{Ob} \mathcal{J}$). We notice that quasi-isomorphisms from $\mathcal{C}$ become “invertible modulo boundary” in the differential graded category $\mathcal{D}(\mathcal{C}|\mathcal{B})$ (Section 9.1). From the identity functor $f = \text{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$, a map $g : \text{Ob} \mathcal{C} \to \text{Ob} \mathcal{C}$, $X \to XI$ and quasi-isomorphisms $r_X$ we produce an $A_\infty$-functor $g : \mathcal{C} \to \mathcal{C}$, which factors as $g = (\mathcal{C} \overset{i}{\longrightarrow} \mathcal{J} \overset{\epsilon}{\longrightarrow} \mathcal{C})$, into “K-injective resolution” unital $A_\infty$-functor $i$ and the full embedding $\epsilon$ (Section 9.2). The unital $A_\infty$-functor $i : \mathcal{D}(\mathcal{C}|\mathcal{B}) \to \mathcal{D}(\mathcal{J}|\mathcal{J})$ and the faithful differential graded functor $\pi : \mathcal{D}(\mathcal{J}|\mathcal{J}) \to \mathcal{D}(\mathcal{C}|\mathcal{B})$ are $A_\infty$-equivalences quasi-inverse to each other. Due to contractibility of $\mathcal{J}$ the natural embedding $\mathcal{J} \to \mathcal{D}(\mathcal{J}|\mathcal{J})$ is an equivalence. Since $\mathcal{D}(\mathcal{C}|\mathcal{B})$ and $\mathcal{J}$ are $A_\infty$-equivalent, their 0-th cohomology categories are equivalent as usual $\mathbb{k}$-linear categories. That is, $H^0(\mathcal{D}(\mathcal{C}|\mathcal{B})) = H^0(\mathcal{D}(\mathcal{C}(\mathcal{A})|\mathcal{A}(\mathcal{A})))$ is equivalent to $H^0(\mathcal{J}) = H^0(\mathcal{I}(\mathcal{A}))$ - homotopy category of K-injective complexes, which is equivalent to the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$. This result (Section 9.2) motivated our studies. It follows also from Drinfeld’s theory of quotients of differential graded categories [Dri04]. This agrees with Bondal and Kapranov’s proposal to produce triangulated categories as homotopy categories of some differential graded categories [BK90].

1. Conventions

We keep the notations and conventions of [Lyu03], sometimes without explicit mentioning. Some of the conventions are recalled here.

We assume as in [Lyu03] that quivers, $A_\infty$-categories, etc. are small with respect to some universe $\mathcal{U}$.

The ground ring $\mathbb{k} \in \mathcal{U}$ is a unital associative commutative ring.

We use the right operators: the composition of two maps (or morphisms) $f : X \to Y$ and $g : Y \to Z$ is denoted $fg : X \to Z$; a map is written on elements as $f : x \mapsto xf = (x)f$. However, these conventions are not used systematically, and $f(x)$ might be used instead.
If $C$ is a $\mathbb{Z}$-graded $\mathbb{k}$-module, then $sC = C[1]$ denotes the same $\mathbb{k}$-module with the grading $(sC)^d = C^{d+1}$, the suspension of $C$. The shift “identity” map $C \to sC$ of degree $-1$ is also denoted $s$. Getzler and Jones demonstrated in [GJ90] that the suspension $s$ and the shift map $s$ are useful in the theory of $A_\infty$-algebras. We follow the Koszul sign convention:

$$(x \otimes y)(f \otimes g) = (-)^{gf} xf \otimes yg = (-1)^{\deg y - \deg f} xf \otimes yg.$$ 

A chain complex is called contractible if its identity endomorphism is homotopic to zero.

The category $\mathcal{Q}/S$ of $\mathcal{U}$-small graded $\mathbb{k}$-linear quivers with fixed set of objects $S$ admits a monoidal structure with the tensor product $A \times B \mapsto A \otimes B$, $(A \otimes B)(X,Y) = \bigoplus_{Z \in S} A(X,Z) \otimes_{\mathbb{k}} B(Z,Y)$. Thus, we have tensor powers $T^n A = A \otimes^n$ of a given graded $\mathbb{k}$-quiver $A$, such that $\text{Ob } T^n A = \text{Ob } A$. Explicitly,

$$T^n A(X,Y) = \bigoplus_{X_1,\ldots,X_{n-1} \in \text{Ob } A} A(X_0, X_1) \otimes_{\mathbb{k}} A(X_1, X_2) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} A(X_{n-1}, X_n),$$

where $X_0 = X$ and $X_n = Y$. In particular, $T^0 A$ denotes the unit object $\mathbb{k}S$, where $\mathbb{k}S(X,Y) = \mathbb{k}$ if $X = Y$ and vanishes otherwise.

As in any monoidal category, there is a notion of coassociative coalgebras $(\mathcal{B}, \Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B})$ in $\mathcal{Q}/S$ (in general, without counit). For $A_\infty$-category theory, we need only coalgebras $(\mathcal{B}, \Delta)$ in $\mathcal{Q}/S$ that satisfy an additional requirement: for all $X,Y \in S$.

$$\mathcal{B}(X,Y) = \bigcup_{k=2}^\infty \text{Ker}(\Delta^{(k)} : \mathcal{B}(X,Y) \to \mathcal{B}^{\otimes k}(X,Y)),$$

where $\Delta^{(2)} = \Delta$, $\Delta^{(3)} = \Delta(1 \otimes \Delta) = \Delta(\Delta \otimes 1) : \mathcal{B} \to \mathcal{B}^{\otimes 3}$, etc. Such coalgebras are named cocomplete cocategories by Keller [Kel06]. A counital coassociative coalgebra $(\mathcal{A} = T^0 B \oplus \mathcal{B}, \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \varepsilon : \mathcal{A} \to T^0 \mathcal{A})$ in $\mathcal{Q}/S$ is associated with $(\mathcal{B}, \Delta)$, namely:

$$\Delta|_{T^0 \mathcal{B}} = (T^0 B \xrightarrow{T} T^0 B \otimes T^0 B \xrightarrow{\text{in}_0 \otimes \text{in}_0} A \otimes A),$$

$$\Delta|_{\mathcal{B}} = (\mathcal{B} \xrightarrow{T} \mathcal{B} \otimes \mathcal{B} \xrightarrow{\text{in}_1 \otimes \text{in}_1} \mathcal{A} \otimes \mathcal{A}) + (\mathcal{B} \xrightarrow{T^0 B} \mathcal{B} \otimes \mathcal{B} \xrightarrow{\text{in}_0 \otimes \text{in}_0} \mathcal{A} \otimes \mathcal{A}),$$

or simply $f \Delta = f \otimes 1 + f \Delta + 1 \otimes f$ for $f \in \mathcal{B}(X,Y)$, and $\varepsilon = \text{pr}_0 : \mathcal{A} \to T^0 \mathcal{B} = T^0 \mathcal{A}$. The triple $(\mathcal{A}, \Delta, \varepsilon)$ is a cocategory in the sense of [Lyu03]. In the present article, we shall use only one kind of cocategory associated with quivers $\mathcal{C}$, namely, the cocomplete cocategory $\mathcal{B} = T^0 \mathcal{C} = \bigoplus_{n=1}^\infty T^n \mathcal{C}$, equipped with the comultiplication $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$, $(h_1 \otimes h_2 \otimes \cdots \otimes h_n) \Delta = \sum_{k=1}^{n-1} h_1 \otimes \cdots \otimes h_k \otimes h_{k+1} \otimes \cdots \otimes h_n$, gives rise to the cocategory $\mathcal{A} = T^0 \mathcal{C} = \bigoplus_{n=0}^\infty T^n \mathcal{C}$, equipped with the cut comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $(h_1 \otimes h_2 \otimes \cdots \otimes h_n) \Delta = \sum_{k=0}^{n} h_1 \otimes \cdots \otimes h_k \otimes h_{k+1} \otimes \cdots \otimes h_n$, and with the counit $\varepsilon = \text{pr}_0 : \mathcal{A} \to T^0 \mathcal{C} = T^0 \mathcal{A}$. 

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By definition, cocategory homomorphisms (in particular, $A_\infty$-functors) respect the cut comultiplication $\Delta$, and $A_\infty$-transformations are coderivations with respect to $\Delta$ (see e.g. \cite{Lyu03}).

We use the following standard equations for a differential $b$ in an $A_\infty$-category

$$
\sum_{r+n+t=k} (1^{Sr} \otimes b_n \otimes 1^{St})b_{r+1+t} = 0 : T^k sA \to sA. \quad (1.0.1)
$$

Since $b$ is a differential and a coderivation, it may be called a codifferential. Commutation relation $fb = bf$ for an $A_\infty$-functor $f : A \to B$ expands to the following

$$
\sum_{l>0, i_1 + \cdots + i_l = k} (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l})b_l = \sum_{r+n+t=k} (1^{Sr} \otimes b_n \otimes 1^{St})f_{r+1+t} : T^k sA \to sB. \quad (1.0.2)
$$

Given $A_\infty$-functors $f, g, h : B \to C$ and coderivations $f \xrightarrow{r} g \xrightarrow{p} h : B \to C$ of arbitrary degree we construct a map $\theta : TsB \to TsC$ as in Section 3 of \cite{Lyu03}. We view $\theta$ as a bilinear function $(r \otimes p)\theta$ of $r, p$. Its components $\theta_{kl} = \theta|_{T^k sB} : T^k sB \to T^l sC$ are given by formula (3.0.1) of \cite{Lyu03}

$$
\theta_{kl} = \sum f_{a_1} \otimes \cdots \otimes f_{a_\alpha} \otimes r_{j_1} \otimes g_{c_1} \otimes \cdots \otimes g_{c_\beta} \otimes p_t \otimes h_{e_1} \otimes \cdots \otimes h_{e_\gamma}, \quad (1.0.3)
$$

where the summation is taken over all terms with

$$
\alpha + \beta + \gamma + 2 = l, \quad \alpha_1 + \cdots + \alpha_\alpha + j + c_1 + \cdots + c_\beta + t + e_1 + \cdots + e_\gamma = k.
$$

The same formula can be presented as

$$
\theta_{kl} = \sum_{\alpha+\beta+\gamma+2=l} f_{a_\alpha} \otimes r_{j_1} \otimes g_{c_\beta} \otimes p_t \otimes h_{e_\gamma}, \quad (1.0.4)
$$

where $f_{a_\alpha} : T^{a_\alpha}A \to T^{a_\alpha}B$ are matrix elements of $f$ and similarly for $g, h$. By Proposition 3.1 of \cite{Lyu03} the map $\theta$ satisfies the equation

$$
\theta \Delta = \Delta(f \otimes \theta + r \otimes p + \theta \otimes h).
$$

Given $A_\infty$-categories $A$ and $B$, one constructs an $A_\infty$-category $A_\infty(A, B)$ of $A_\infty$-functors $A \to B$, equipped with a differential $B$ \cite{Fuk02, Kel06, KS06, KS, LH03, Lyu03, Section 5].

The category of graded $k$-linear quivers admits a symmetric monoidal structure with the tensor product $\mathcal{A} \otimes \mathcal{B}$, where $\text{Ob}\mathcal{A} \otimes \text{Ob}\mathcal{B} = \text{Ob}\mathcal{A} \times \text{Ob}\mathcal{B}$ and $(\mathcal{A} \otimes \mathcal{B})(\langle X, U \rangle, \langle Y, V \rangle) = \mathcal{A}(X, Y) \otimes_k \mathcal{B}(U, V)$. The same tensor product was denoted $\otimes$ in \cite{Lyu03}. Given $A_\infty$-categories $A, B, C$, there is a graded cocategory morphism of degree 0

$$
M : TsA_\infty(A, B) \otimes TsA_\infty(B, C) \to TsA_\infty(A, C),
$$

which satisfies equation (1 $\otimes B + B \otimes 1)M = MB$ \cite{Lyu03, Section 6].
2. An $A_\infty$-category

Let $\mathcal{B} \to \mathfrak{C}$ be a full $A_\infty$-subcategory. It means that $\text{Ob}\ \mathcal{B} \subset \text{Ob}\ \mathfrak{C}$, $\mathcal{B}(X,Y) = \mathfrak{C}(X,Y)$ for all $X,Y \in \text{Ob}\ \mathcal{B}$, and the operations for $\mathcal{B}$ coincide with those for $\mathfrak{C}$. Let us define another $A_\infty$-category $D(\mathfrak{C}|\mathcal{B})$. If $\mathcal{B}$, $\mathfrak{C}$ are differential graded categories, then $D(\mathfrak{C}|\mathcal{B})$ is differential graded as well and it coincides with the category $\mathfrak{C}/\mathcal{B}$ defined by Drinfeld in [Dri04, Section 3.1].

2.1 Definition. Let $T^+ \mathfrak{C} = \oplus_{n>0} T^n s\mathfrak{C}$ and $\mathcal{E} = D(\mathfrak{C}|\mathcal{B})$ be the following graded $k$-quivers: the class of objects is $\text{Ob} T^+ \mathfrak{C} = \text{Ob} \mathcal{E} = \text{Ob} \mathfrak{C}$, the morphisms for $X,Y \in \text{Ob} \mathcal{E}$ are

$$T^+ \mathfrak{C}(X,Y) = \oplus_{C_1,\ldots,C_{n-1} \in \text{Ob} \mathfrak{C}} s\mathfrak{C}(X,C_1) \otimes \cdots \otimes s\mathfrak{C}(C_{n-2},C_{n-1}) \otimes s\mathfrak{C}(C_{n-1},Y),$$

$$s\mathcal{E}(X,Y) = \oplus_{C_1,\ldots,C_{n-1} \in \text{Ob} \mathfrak{C}} s\mathcal{E}(X,C_1) \otimes \cdots \otimes s\mathcal{E}(C_{n-2},C_{n-1}) \otimes s\mathcal{E}(C_{n-1},Y),$$

where in the second case summation extends over all sequences of objects $(C_1,\ldots,C_{n-1})$ of $\mathcal{B}$. To the empty sequence $(n = 1)$ corresponds the summand $s\mathfrak{C}(X,Y)$.

Let us endow $s^{-1}T^+ \mathfrak{C}$ with a structure of $A_\infty$-category, given by $b : T(T^+ \mathfrak{C}) \to T(T^+ \mathfrak{C})$, with the components $b_0 = 0$, $b_1 = b : T^+ \mathfrak{C} \to T^+ \mathfrak{C}$, $b_k = 0$ for $k > 1$. This $A_\infty$-category is denoted $\mathfrak{C} = (s^{-1}T^+ \mathfrak{C},b)$. There is an $A_\infty$-functor $j : \mathfrak{C} \to (s^{-1}T^+ \mathfrak{C},b)$, specified by its components $j_k : T^k \mathfrak{C} \to T^+ \mathfrak{C}$, $k \geq 1$, where $j_k$ is the canonical embedding of the direct summand. The property $b_j = jb$ or

$$\sum_{r+k+t=n} (1^r \otimes b_k \otimes b_t) j_{r+k+t} = j_k b : T^n \mathfrak{C} \to T^+ \mathfrak{C},$$

is clear – this is just the expression of $b$ in terms of its components.

There is a coalgebra automorphism $\mu : TT^+ \mathfrak{C} \to TT^+ \mathfrak{C}$, specified by its components $\mu_k = \mu^{(k)} : T^k T^+ \mathfrak{C} \to T^+ \mathfrak{C}$, $k \geq 1$, where $\mu : T^+ \mathfrak{C} \otimes T^+ \mathfrak{C} \to T^+ \mathfrak{C}$ is the multiplication in the tensor algebra, $\mu^{(k)} = 0$ for $k \leq 0$, $\mu^{(1)} = 1 : T^+ \mathfrak{C} \to T^+ \mathfrak{C}$, $\mu^{(2)} = \mu$, $\mu^{(3)} = (\mu \otimes 1) \mu : (T^+ \mathfrak{C}) \otimes T^+ \mathfrak{C} \to T^+ \mathfrak{C}$ and so on. Its inverse is the coalgebra automorphism $\mu^{-1} = \mu^- : TT^+ \mathfrak{C} \to TT^+ \mathfrak{C}$, specified by its components $\mu_k^- = (-)^{k-1} \mu^{(k)} : T^k T^+ \mathfrak{C} \to T^+ \mathfrak{C}$. The fact that $\mu$ and $\mu^-$ are inverse to each other is proven as follows:

$$(\mu \mu^-)_n = \sum_{l_1 + \cdots + l_k = n} (\mu_{l_1} \otimes \cdots \otimes \mu_{l_k}) \mu_k^- = \sum_{l_1 + \cdots + l_k = n} (-)^{k-1} \mu^{(n)}$$

$$= \mu^{(n)} \sum_{k=1}^n (-1)^{k-1} \binom{n-1}{k-1} = \mu^{(n)}(1 - 1)^{n-1},$$

which equals id for $n = 1$ and vanishes for $n > 1$. Similarly, $\mu^- \mu = \text{id}$. 
2.2 Proposition. The conjugate codifferential \( \bar{b} = \mu b \mu^{-1} : T(T^+ s\mathcal{C}) \to T(T^+ s\mathcal{C}) \) has the following components: \( \bar{b}_0 = 0, \bar{b}_1 = b \) and for \( n \geq 2 \)

\[
\bar{b}_n = \mu^{(n)} b - (1 \otimes \mu^{(n-1)} b) \mu - (\mu^{(n)} b \otimes 1) \mu + (1 \otimes \mu^{(n-2)} b \otimes 1) \mu^{(3)} : (T^+ s\mathcal{C})^\otimes n \to T^+ s\mathcal{C},
\]

(2.2.1)

\[
\bar{b}_n = \mu^{(n)} b - (1 \otimes \mu^{(n-1)} b) \mu - (\mu^{(n-1)} b \otimes 1) \mu + (1 \otimes \mu^{(n-2)} b \otimes 1) \mu^{(3)} : (T^+ s\mathcal{C})^\otimes n \to T^+ s\mathcal{C},
\]

(2.2.2)

for all \( n \geq 0 \). The operations \( \bar{b}_n \) restrict to maps \( s\mathcal{E}^\otimes n \to s\mathcal{E} \) via the natural embedding \( s\mathcal{E} \subset T^+ s\mathcal{C} \) of graded \( k \)-quivers. Hence, \( \bar{b} \) turns \( \mathcal{E} \) and \( \bar{\mathcal{E}} \) into an \( A_\infty \)-category.

Proof. Let us define a \((1,1)\)-coderivation \( \bar{b} \) of degree 1 by its components (2.2.2). Substituting the definition of \( b \) via its components, we get formula (2.2.1). Clearly, \( \mu b \mu^{-1} \) is also a \((1,1)\)-coderivation of degree 1. Let us show that it coincides with \( \bar{b} \), that is, \( \bar{b} \mu = \mu b \). This equation expands to

\[
\sum_{r+k+t=n} (1^\otimes r \otimes b_k \otimes 1^\otimes t) \mu^{(r+1+t)} = \mu^{(n)} b = \mu^{(n)} \bar{b}_1,
\]

which follows immediately from (2.2.1) and from the standard expression of \( b \) via its components.

Clearly, \( \bar{b}^2 = \mu b \mu^{-1} = 0 \), hence, \( \bar{\mathcal{E}} = (s^{-1} T^+ s\mathcal{C}, \bar{b}) \) is an \( A_\infty \)-category. Map (2.2.1) is a sum of maps of the form

\[
1^\otimes q \otimes b_m \otimes 1^\otimes t : s\mathcal{E}(X, C_1) \otimes \cdots \otimes s\mathcal{E}(C_{q-1}, C_q) \otimes s\mathcal{E}(C_q, C_{q+1}) \otimes \cdots \otimes (T^+ s\mathcal{C})^\otimes n - 2 \otimes \cdots \otimes s\mathcal{E}(D_{l-t}, Y) \otimes s\mathcal{E}(D_l, D_l+1) \otimes \cdots \otimes s\mathcal{E}(D_{l-1}, Y),
\]

\[
\to s\mathcal{E}(X, C_1) \otimes \cdots \otimes s\mathcal{E}(C_{q-1}, C_q) \otimes s\mathcal{E}(C_q, D_{l-t}) \otimes s\mathcal{E}(D_{l-t}, D_{l-t+1}) \otimes \cdots \otimes s\mathcal{E}(D_{l-1}, Y),
\]

where \( C_0 = X \) for \( q = 0 \) and \( D_t = Y \) for \( t = 0 \). If the source is contained in \( (s\mathcal{E})^\otimes n(X, Y) \), then \( C_i, D_j \) are in \( \text{Ob} \mathcal{B} \) for all \( 0 < i < k, 0 < j < l \). Therefore, the target is a direct summand of \( s\mathcal{E}(X, Y) \). Thus the maps \( \bar{b}_n \) restrict to maps \( \bar{b}_n : (s\mathcal{E})^\otimes n(X, Y) \to s\mathcal{E}(X, Y) \). The obtained \((1,1)\)-coderivation \( \bar{b} : Ts\mathcal{E} \to Ts\mathcal{E} \) also satisfies \( \bar{b}^2 = 0 \). Thus it makes \( \mathcal{E} \) into an \( A_\infty \)-category.

In particular, (2.2.2) gives

\[
\bar{b}_2 = \mu b - (1 \otimes b + b \otimes 1) \mu,
\]

\[
\bar{b}_3 = \mu^{(3)} b - (1 \otimes \mu^{(2)} b) \mu - (\mu b \otimes 1) \mu + (1 \otimes b \otimes 1) \mu^{(3)},
\]

\[
\bar{b}_4 = \mu^{(4)} b - (1 \otimes \mu^{(3)} b) \mu - (\mu^{(3)} b \otimes 1) \mu + (1 \otimes \mu b \otimes 1) \mu^{(3)}.
\]

2.3 Remark. Let \( \mathcal{A} \) be an \( A_\infty \)-category, defined by a codifferential \( b : Ts\mathcal{A} \to Ts\mathcal{A} \), let \( \mathcal{B} \) be a graded \( k \)-quiver and let \( f : Ts\mathcal{A} \to Ts\mathcal{B} \) (resp. \( g : Ts\mathcal{B} \to Ts\mathcal{A} \) be an isomorphism of graded cocategories. Then the codifferential \( f^{-1} b f \) (resp. \( g f g^{-1} \)) is the unique codifferential on \( Ts\mathcal{B} \), which turns \( f \) (resp. \( g \)) into an invertible \( A_\infty \)-functor between \( \mathcal{A} \) and \( \mathcal{B} \).
2.4 Corollary. The coalgebra isomorphism \( \mu^{-1} : \overline{\mathcal{C}} = (s^{-1}T^+s\mathcal{C}, b) \to \overline{\mathcal{E}} = (s^{-1}T^+s\mathcal{E}, \bar{b}) \) is an \( A_\infty \)-functor. Its composition with \( j \) is a strict \( A_\infty \)-functor \( \overline{\mathcal{E}} = j\mu^{-1} : \mathcal{E} \to \mathcal{D}(\mathcal{E}|\mathcal{B}), X \mapsto X \), whose components are the direct summand embedding \( \overline{j}_1 : s\mathcal{E}(X,Y) = T^1s\mathcal{E}(X,Y) \hookrightarrow s\mathcal{E}(X,Y) \) and \( \overline{j}_n = 0 \) for \( n > 1 \).

Indeed,

\[
\overline{j}_n = \sum_{l_1 + \ldots + l_k = n} (\overline{j}_{l_1} \otimes \ldots \otimes \overline{j}_{l_k})(-1)^{k-1}\mu^{(k)} = \sum_{k=1}^{n} (-1)^{k-1}\binom{n-1}{k-1} \overline{j}_n = (1-1)^{n-1} \overline{j}_n,
\]

which equals \( \overline{j}_1 \) for \( n = 1 \) and vanishes for \( n > 1 \).

2.5. Strict unitality. Assume that \( A_\infty \)-category \( \mathcal{E} \) is strictly unital. It means that for each object \( X \) of \( \mathcal{E} \) there is an element \( 1_X \in \mathcal{E}^0(X,X) \), such that the map \( i^0_\mathcal{E} : k \to (s\mathcal{E})^{-1}(X,X) \), \( 1 \mapsto 1_Xs \) of degree \(-1\) satisfies equations \((1 \otimes i^0_\mathcal{E})\mu_2 = 1 : s\mathcal{E}(Y,X) \to s\mathcal{E}(Y,X) \) and \((i^0_\mathcal{E} \otimes s)\mu_2 = -1 : s\mathcal{E}(X,Z) \to s\mathcal{E}(X,Z) \) for all \( Y, Z \in \text{Ob} \mathcal{E} \), and \((\cdots \otimes 1_Xs \otimes \ldots)\mu_n = 0 \) if \( n \not= 2 \). Since \( \mathcal{E} \) is strictly unital, its full \( A_\infty \)-subcategory \( \mathcal{B} \) is strictly unital as well.

Let us show that in these assumptions \( \mathcal{E} = \mathcal{D}(\mathcal{E}|\mathcal{B}) \) is also strictly unital. We take the same elements \( 1_X \in \mathcal{E}^0(X,X) \subset \mathcal{E}^0(X,X) \), and \( \text{strict units of} \mathcal{E} \). We have \( 1_Xsb_1 = 1_Xsb_1 = 0 \). Explicit formulas give \((\cdots \otimes 1_Xs \otimes \ldots)\mu_n = 0 \) for \( n > 2 \). The map \( \overline{\mu}_2 : T^k\mathcal{E}(Y,X) \otimes \mathcal{E}(X,Z) \to T^+\mathcal{E}(Y,X) \) is the sum of maps

\[
1^{\otimes k-t} \otimes b_{t+1} : \mathcal{E}(Y,C_1) \otimes \ldots \otimes \mathcal{E}(C_{k-t},C_{k-t+1}) \otimes \ldots \otimes \mathcal{E}(C_{k-1},X) \otimes \mathcal{E}(X,X)
\]

\[
\to \mathcal{E}(Y,C_1) \otimes \ldots \otimes \mathcal{E}(C_{k-t},X)
\]

everover \( t > 0 \). Therefore, the map \( i^0_\mathcal{E} : k \to (s\mathcal{E})^{-1}(X,X) \), \( 1 \mapsto 1_Xs \) satisfies equations

\((1 \otimes i^0_\mathcal{E})\mu_2 = (1^{\otimes k} \otimes i^0_\mathcal{E})(1^{\otimes k-1} \otimes b_2) = 1^{\otimes k-1} \otimes 1 = 1 \).

Similarly, for \( \overline{\mu}_2 : \mathcal{E}(X,Z) \to T^+\mathcal{E}(Y,X) \) we have \((i^0_\mathcal{E} \otimes 1)\mu_2 = (i^0_\mathcal{E} \otimes 1^{\otimes k}(b_2 \otimes 1^{\otimes k-1}) = -1 \otimes 1^{\otimes k-1} = -1 \).

Therefore, \( \mathcal{E} \) and \( \overline{\mathcal{E}} \) are strictly unital with the unit \( i^\mathcal{E} \).

2.6. Differential graded categories. If \( b_k = 0 \) for \( k > 2 \), then explicit formulae in the case of \( \mathcal{E} \) show that we also have \( \overline{b}_k = 0 \) for \( k > 2 \). Combining this fact with the above unitality considerations, we see that if \( \mathcal{E} \) is a differential graded category, then so is \( \mathcal{D}(\mathcal{E}|\mathcal{B}) \). The differential graded category \( \mathcal{E} = \mathcal{D}(\mathcal{E}|\mathcal{B}) = \mathcal{E}/\mathcal{B} \) was constructed by Drinfeld \cite{Dri04}, Section 3.1]. This construction was a starting point of the present article. Let us describe it in detail.

Write down elements of \( \mathcal{E}(X,Y) \) as sequences \( f_1 \varepsilon_{C_1} f_2 \ldots \varepsilon_{C_{n-1}} f_n \), where \( f_i \in \mathcal{E}(C_{i-1},C_i) \), \( C_0 = X \), \( C_n = Y \), and \( C_i \in \text{Ob} \mathcal{B} \) for \( 0 < i < n \). The symbol \( \varepsilon_C \) for \( C \in \text{Ob} \mathcal{B} \) is assigned
degree $-1$. Its differential is set equal to $\varepsilon_C d = 1_C$. The graded Leibniz rule gives
\begin{align*}
(f_1 \varepsilon_C f_2 \cdots \varepsilon_{C_{n-1}} f_n)d &= \sum_{q+1+t=n} (-)^{f_{n-1}+1} f_1 \varepsilon_C f_2 \cdots \varepsilon_{C_q} (f_{q+1} m_1) \varepsilon_C f_{q+2} \cdots \varepsilon_{C_{n-1}} f_n \\
+ \sum_{q+2+t=n} (-)^{f_{n-2}+2} f_1 \varepsilon_C f_2 \cdots \varepsilon_{C_q} (f_{q+1} \cdot f_{q+2}) \varepsilon_C f_{q+3} \cdots \varepsilon_{C_{n-1}} f_n,
\end{align*}
where $f_{q+1} \cdot f_{q+2} = (f_{q+1} \otimes f_{q+2}) m_2$ is the composition. Introduce a degree $-1$ map
\[ s : \mathcal{E} \to s \mathcal{E} \subset T^+ \mathcal{E}, \quad f_1 \varepsilon_C f_2 \cdots \varepsilon_{C_{n-1}} f_n \mapsto f_1 s \otimes f_2 s \otimes \cdots \otimes f_n s. \]
One can check that $ds = s \tilde{b}_1$, where, naturally, $\tilde{b}_1 = b = \sum_{q+1+t=n} 1^\otimes q \otimes b_1 \otimes 1^\otimes t + \sum_{q+2+t=n} 1^\otimes q \otimes b_2 \otimes 1^\otimes t$.

The composition $\tilde{m}_2$ in $\mathcal{E}$ consists of the concatenation and the composition $m_2$ in $\mathcal{C}$:
\[ (f_1 \varepsilon_C \cdots f_{n-1} \varepsilon_{C_{n-1}} f_n \otimes g_1 \varepsilon_D g_2 \cdots \varepsilon_{D_{m-1}} g_m) \tilde{m}_2 \]
\[ = f_1 \varepsilon_C \cdots f_{n-1} \varepsilon_{C_{n-1}} (f_n \cdot g_1) \varepsilon_D g_2 \cdots \varepsilon_{D_{m-1}} g_m. \]
One can check that $\tilde{m}_2 s = (s \otimes s) \tilde{b}_2$; here $\tilde{b}_2 = 1^\otimes n^{-1} \otimes b_2 \otimes 1^\otimes m^{-1}$.

Specifically this construction applies to the case of the differential graded category $\mathcal{C} = C(A)$ of complexes of objects of an abelian category $A$. One may take for $\mathcal{B}$ the subcategory of acyclic complexes $\mathcal{B} = A(A)$.

### 3. An $A_\infty$-functor

Let $\mathcal{B} \hookrightarrow \mathcal{C}, \mathcal{I} \hookrightarrow \mathcal{J}$ be full $A_\infty$-subcategories. Let $i : \mathcal{C} \to \mathcal{J}$ be an $A_\infty$-functor, such that $X i \in \text{Ob} \mathcal{J}$ for $X \in \text{Ob} \mathcal{B}$. Then it restricts to an $A_\infty$-functor $\mathcal{B} \to \mathcal{I}$, denoted by $i'$. We are going to construct an extension of this functor to the $A_\infty$-categories $\mathcal{E} = D(\mathcal{C}|\mathcal{B})$ and $\mathcal{F} = D(\mathcal{J}|\mathcal{I})$.

Let us begin with a strict $A_\infty$-functor $\hat{i} : \underline{\mathcal{C}} \to \underline{\mathcal{J}}$, given by its components $\hat{i}_k = i : T^+ s \mathcal{C} \to T^+ s \mathcal{J}$ and $\hat{i}_k = 0$ for $k > 1$. The equation $\hat{i} b = b \hat{i}$ reduces to familiar $ib = bi$.

Therefore, $\hat{i} \equiv \mu \hat{j} \hat{j}^{-1} : \underline{\mathcal{C}} \to \underline{\mathcal{J}}$ is an $A_\infty$-functor as well.

The following diagram of $A_\infty$-functors commutes
\[ \begin{array}{ccc}
\mathcal{B} & \longrightarrow & \mathcal{C} \\
\downarrow i' & & \downarrow i \\
\mathcal{I} & \longrightarrow & \mathcal{J}
\end{array} \]
\[ \begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C} \\
\downarrow \hat{i} & & \downarrow \hat{i} \\
\mathcal{J} & \longrightarrow & \mathcal{J}
\end{array} \]
\[ \begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C} \\
\downarrow \mu^{-1} & & \downarrow \mu^{-1} \\
\mathcal{J} & \longrightarrow & \mathcal{J}
\end{array} \]

Indeed, $\hat{j} \hat{i} \hat{j} = i \hat{j} \hat{j}$ expands to
\[ \hat{j} \hat{i} \hat{j} = \sum_{i_1 \cdots i_k = n} (i_1 \otimes \cdots \otimes i_k) \hat{j} \hat{j} : T^+ s \mathcal{C} \to T^+ s \mathcal{J}, \]
which expresses $i$ in terms of its components.
3.1 Proposition. The $A_\infty$-functor $\bar{i}$ has the following components:

$$\bar{i}_n = \sum_{l_1 + \ldots + l_t = n} (-)^{k-1}(\mu^{(l_t)} \otimes \ldots \otimes \mu^{(l_k)})\bar{i}^{\otimes k}\mu^{(k)} : (T^+s\mathcal{C})^\otimes n \to T^+s\mathcal{J}. \quad (3.1.1)$$

The restriction of this map to $T^{k_1}s\mathcal{C} \otimes \ldots \otimes T^{k_n}s\mathcal{C}$ is

$$\bar{i}_n = \mu^{(n)} \sum_{(l_1, \ldots, l_t) \in L(k_1, \ldots, k_n)} (i_{l_1} \otimes \ldots \otimes i_{l_t}) : T^{k_1}s\mathcal{C} \otimes \ldots \otimes T^{k_n}s\mathcal{C} \to T^+s\mathcal{J}, \quad (3.1.2)$$

where

$$L(k_1, \ldots, k_n) = \cup_{t>0}\{(l_1, \ldots, l_t) \in \mathbb{Z}_{>0}^t | \forall q, s \in \mathbb{Z}_{>0}, q \leq t, s \leq n \mid l_1 + \ldots + l_q = k_1 + \ldots + k_s \iff q = t, s = n\}.$$ 

These maps restrict to maps $\bar{i}_n : T^n s\mathcal{D}(\mathcal{C} | \mathcal{B}) \to s\mathcal{D}(\mathcal{J} | \mathcal{J})$, which are components of an $A_\infty$-functor $\mathcal{D}(i) = \bar{i} : \mathcal{D}(\mathcal{C} | \mathcal{B}) \to \mathcal{D}(\mathcal{J} | \mathcal{J})$. The restriction of $\bar{i}_n$ to $T^n s\mathcal{C} \xrightarrow{\bar{i}^n} T^n s\mathcal{D}(\mathcal{C} | \mathcal{B})$ equals $T^n s\mathcal{C} \xrightarrow{\bar{i}^n} s\mathcal{J} \xrightarrow{\bar{j}^n} s\mathcal{D}(\mathcal{J} | \mathcal{J})$.

Proof. Let us prove (3.1.2). Let $\bar{i}$ denote the cocategory homomorphism $\bar{i} : \mathcal{C} \to \mathcal{J}$, defined by its components (3.1.2). We are going to prove that it satisfies $\bar{i}\mu = \mu\bar{i}$. Indeed, this equation expands to the following

$$\sum_{n_1 + \ldots + n_a = p} (\bar{i}_{n_1} \otimes \ldots \otimes \bar{i}_{n_a})\mu^{(a)} = \mu^{(p)}\bar{i} : T^c s\mathcal{C} \otimes \ldots \otimes T^{c_p} s\mathcal{C} \to T^+s\mathcal{J}, \quad (3.1.3)$$

which has to be proven for all $p \geq 1$. The right hand side is the sum of terms $i_{m_1} \otimes \ldots \otimes i_{m_t}$ such that $m_1 + \ldots + m_t = c_1 + \ldots + c_p$. Consider a set of positive integers

$$\mathbb{N} = \{m_1, m_1 + m_2, \ldots, m_1 + \ldots + m_t\} \cap \{c_1, c_1 + c_2, \ldots, c_1 + \ldots + c_p\}.$$ 

It contains $c_1 + \ldots + c_p$. Clearly, $i_{m_1} \otimes \ldots \otimes i_{m_t}$ will appear in the term $\bar{i}_{n_1} \otimes \ldots \otimes \bar{i}_{n_a}$ if and only if $\mathbb{N} = \{n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_a\}$. Since any finite subset $\mathbb{N} \subset \mathbb{Z}_{>0}$ has a unique presentation of this form via $n_1, \ldots, n_a$, equation (3.1.3) holds.

Let $X, Z_j, Y$ be objects of $\mathcal{C}$ and let $C^i_j$ be objects of $\mathcal{B}$. When $\bar{i}_n$ is applied to the $k$-module

$$s\mathcal{C}(X, C^1_1) \otimes \ldots \otimes s\mathcal{C}(C^1_{k_1-1}, Z_1) \otimes \ldots \otimes s\mathcal{C}(Z_{n-2}, C^{n-1}_1) \otimes \ldots \otimes s\mathcal{C}(C^m_{k_n-1}, Y),$$

the target space for the term $i_{m_1} \otimes \ldots \otimes i_{m_t}$ has the form

$$s\mathcal{J}(Xi, C^1_1i) \otimes \ldots \otimes s\mathcal{J}(C^1_{k_1-1}i, Z_1i) \otimes \ldots \otimes s\mathcal{J}(C^m_{k_n-1}i, Yi).$$
where $C^\bullet$ are objects of $\mathcal{B}$ (no $Z_j$ will appear!). Since $X_i, Y_i \in \text{Ob} \mathcal{J}$ and $C^\bullet_i \in \text{Ob} \mathcal{J}$, the above space is a direct summand of $s\mathcal{D}(\mathcal{J}|\mathcal{J})(X_i, Y_i)$. Therefore, the required map $i_n : T^n s\mathcal{D}(\mathcal{C}|\mathcal{B}) \to s\mathcal{D}(\mathcal{J}|\mathcal{J})$ is constructed.

The last statement is a particular case of (3.1.2). Indeed, if $k_1 = \cdots = k_n = 1$, then $L(1, \ldots, 1)$ consists of only one sequence $(n)$ of the length $t = 1$.

Since $\bar{i}$ is strict and $\bar{i}_1 = i$, equation (3.1.1) is the expansion of the definition $\bar{i} = \mu i \mu^{-1}$.

For example,

$\bar{i}_1 = i,$

$\bar{i}_2 = \mu i - (i \otimes i)\mu,$

$\bar{i}_3 = \mu(i - (i \otimes \mu i)\mu - (\mu i \otimes i)\mu + (i \otimes i \otimes i)\mu(i),$

$\bar{i}_4 = \mu(i - (i \otimes \mu(i \otimes i)\mu - (\mu(i \otimes i)\mu - (i \otimes i \otimes i)\mu(i)^3) + (i \otimes \mu(i \otimes i)\mu(i)^3) + (\mu(i \otimes i \otimes i)\mu(i)^3) - (i \otimes i \otimes i \otimes i)\mu(i)^4)\mu(i)^3.$

3.2 Corollary. We have a commutative diagram of $A_\infty$-functors

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{f} & \mathcal{C} \\
\downarrow i & & \downarrow i \\
\mathcal{J} & \xrightarrow{g} & \mathcal{D}
\end{array}
$$

When $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ are $A_\infty$-functors, then $f g = f g : \mathcal{A} \xrightarrow{A} \mathcal{C}$. This implies $\overline{fg} = \overline{fg} : \overline{A} \xrightarrow{\overline{A}} \overline{C}$. Assume that $\mathcal{A}' \hookrightarrow \mathcal{A}$, $\mathcal{B}' \hookrightarrow \mathcal{B}$, $\mathcal{C}' \hookrightarrow \mathcal{C}$ are full $A_\infty$-subcategories such that $(\text{Ob} \mathcal{A}') f \subset \text{Ob} \mathcal{B}'$, $(\text{Ob} \mathcal{B}'){g} \subset \text{Ob} \mathcal{C}'$. Denote $f' = f|_{\mathcal{A}'}$, $g' = g|_{\mathcal{B}'}$. Since $D(f)$ and $D(g)$ are just the restrictions of $\overline{f}$ and $\overline{g}$, we conclude that

$$D(f)D(g) = D(f g) : D(\mathcal{A}|\mathcal{A}') \to D(\mathcal{C}|\mathcal{C}'). \quad (3.2.1)$$

3.3. Strict unitality. Assume that the $A_\infty$-category $\mathcal{C}$ is strictly unital. As we know from Section 2.5, $D(\mathcal{C}|\mathcal{B})$ and $\overline{\mathcal{C}}$ are strictly unital with the unit transformation $\overline{1}$. Since $\mu^{-1} : \mathcal{C} \to \overline{\mathcal{C}}$ is an invertible $A_\infty$-functor, $\mathcal{C}$ is unital (see [Lyu03, Section 8.12]). Notice that $\overline{\mathcal{C}}$ is never strictly unital except when $\mathcal{C} = 0$, because $\overline{\mathcal{C}} = 0$. The transformation $i^C = \mu^{-1} i^C \mu : id_{\mathcal{C}} \to id_{\overline{\mathcal{C}}} : \mathcal{C} \to \overline{\mathcal{C}}$, whose components are

$$\begin{align*}
i_0^C &= i_0^C, \\
i_1^C &= (i_0^C \otimes 1 + 1 \otimes i_0^C)\mu, \\
i_2^C &= (1 \otimes i_0^C \otimes 1)\mu(3), \\
i_k^C &= 0 \quad \text{for } k > 2,
\end{align*}$$

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is a unit transformation of $\mu$. Indeed, let us define $i^\omega_\mu$ by the above components and let us prove that $\mu i^\omega_\mu = i^\omega_\mu \mu$. Clearly, $(\mu i^\omega_\mu)_0 = i^\omega_\mu = (i^\omega_\mu \mu)_0$. For $n > 0$ we have

$$(\mu i^\omega_\mu)_n = \mu^{(n)} i^\omega_\mu + \sum_{k+l=n} (\mu^{(k)} \otimes \mu^{(l)}) i^\omega_\mu$$

$$= (i^\omega_\mu \otimes 1^{\otimes n}) \mu^{(n+1)} + (1^{\otimes n} \otimes i^\omega_\mu) \mu^{(n+1)} + \sum_{k,l>0;k+l=n} (1^{\otimes k} \otimes i^\omega_\mu \otimes 1^{\otimes l}) \mu^{(n+1)}$$

$$= \sum_{k,l>0;k+l=n} (1^{\otimes k} \otimes i^\omega_\mu \otimes 1^{\otimes l}) \mu^{(n+1)} = (i^\omega_\mu \mu)_n.$$

4. **An $A_\infty$-transformation**

Let $\mathcal{B} \hookrightarrow \mathcal{C}$ and $\mathcal{J} \hookrightarrow \mathcal{L}$ be full $A_\infty$-subcategories. Let $f, g : \mathcal{C} \to \mathcal{L}$ be two $A_\infty$-functors such that $(\text{Ob}\mathcal{B})f \subset \text{Ob}\mathcal{J}$, $(\text{Ob}\mathcal{B})g \subset \text{Ob}\mathcal{J}$, and let $r : f \to g : \mathcal{C} \to \mathcal{L}$ be an $A_\infty$-transformation. Denote by $r' : f' \to g' : \mathcal{B} \to \mathcal{J}$ the restriction of $r$ to $\mathcal{B}$. We already have $\mu$ and $\overline{\mu}$ for $A_\infty$-categories $\mathcal{C}$, $f$ and $g$ for $A_\infty$-functors $f$. Now let us proceed with $A_\infty$-transformations.

Let us define an $A_\infty$-transformation $r : f \to g : \mathcal{C} \to \mathcal{L}$ via its components

$$r_0 = r_0_{\mathcal{J}}; \quad r_0 = [k \xrightarrow{r_0} (s\mathcal{J})(Xf, Xg) \xleftarrow{j_{\mathcal{J}}} (s\mathcal{J})(Xf, Xg)];$$

$$r_1 = r; \quad r_1 = r|_{T^+s\mathcal{C}} : T^+s\mathcal{C} = s\mathcal{C} \to T^+s\mathcal{J} = s\mathcal{J};$$

$$r_k = 0 \quad \text{for } k > 1.$$

Let us check that $\mu$ maps the $\omega$-globular set $A_\omega$ [[Lyu03, Definition 6.4]] into itself (so that sources and targets are preserved). It suffices to notice that the correspondence $r \mapsto \mu$ is additive, and if $r = [v, b]$, then $\mu = [v, b]$. Indeed,

$$[v, b]_0 = v_0 b = v_0 b = v_0 b = r_0_{\omega};$$

$$[v, b]_1 = v_1 b - (-)^v b_1 v_1 = v b - (-)^v b v = r = r_1;$$

$$[v, b]_k = v_k b - (-)^v b_k v_1 = 0 \pm 0 = 0 = r_k \quad \text{for } k > 1.$$

In particular, a natural $A_\infty$-transformation $r : f \to g : \mathcal{C} \to \mathcal{L}$ goes to the natural $A_\infty$-transformation $r : f \to g : \mathcal{C} \to \mathcal{L}$, and equivalent natural $A_\infty$-transformations $r, p$ go to equivalent $r, p$.

We claim that

$$r_j^p = \overline{r_j}^\omega : f j = j f \to g j = j g : \mathcal{C} \to \mathcal{L}.$$

Indeed, $(r j)_0 - (\overline{r j})_0 = r_0 j_1 - r_0 = 0$, and for $n > 0$

$$(r j)_n - (\overline{r j})_n = \sum_{a_1 + \cdots + a_1 + k + c_1 + \cdots + c_m = n} (f_{a_1} \otimes \cdots \otimes f_{a_1} \otimes r_k \otimes g_{c_1} \otimes \cdots \otimes g_{c_m}) j_{a_1 + k + c_1 + \cdots + c_m} - j_{n} r_k$$

$$= r|_{T^a \mathcal{C}} - r|_{T^a \mathcal{C}} = 0.$$
We define also the $A_\infty$-transformation conjugate to $\tau$:

$$\tau = \mu_\tau^{-1} : \jmath = \mu_\jmath^{-1} \to \gamma = \mu_\gamma^{-1} : \mathcal{C} \to \mathcal{J}$$

(not necessarily natural). Summing up, we have a commutative cylinder

$$\begin{array}{c}
\mathcal{B} \xrightarrow{g} \mathcal{C} \xrightarrow{\tilde{\mathcal{Z}}^e} \mathcal{C} \xrightarrow{\mu^{-1}} \mathcal{C} \\
\mathcal{J} \xrightarrow{\mathcal{J}} \mathcal{J} \xrightarrow{\tilde{\mathcal{L}}^e} \mathcal{J} \xrightarrow{\mu^{-1}} \mathcal{J}
\end{array}$$

(4.0.3)

The correspondence $\tau$ also maps the $\omega$-globular set $A_\omega$ into itself. Indeed, if $r = [v, b]$, then

$$\tau = \mu_\tau^{-1} = \mu_{[v, b]}^{-1} = [\mu_v^{-1}, \mu_b^{-1}] = [\tau, \bar{b}].$$

4.1 Proposition. The $A_\infty$-transformation $\tau$ has the following components

$$\tau_n = \sum_{0 \leq q \leq t} (-1)^q (\mu^{(l_1)} f \otimes \cdots \otimes \mu^{(l_q)} f \otimes r_0 j_{-1} \otimes \mu^{(l_{q+1})} g \otimes \cdots \otimes \mu^{(l_t)} g) \mu^{(t+1)}$$

$$+ \sum_{0 \leq q \leq t} (-1)^q (\mu^{(l_1)} f \otimes \cdots \otimes \mu^{(l_{q-1})} f \otimes \mu^{(l_q)} r \otimes \mu^{(l_{q+1})} g \otimes \cdots \otimes \mu^{(l_t)} g) \mu^{(t)}.$$  (4.1.1)

Explicitly, $\tau_0 = \tau_0 j_{-1}$ and for $n > 0$ the restriction of $\tau_n$ to $T^{k_1} s \mathcal{C} \otimes \cdots \otimes T^{k_n} s \mathcal{C}$ is

$$\tau_n = \mu^{(n)} \sum_{(a_1, \ldots, a_n ; k_1, \ldots, k_s) \in P(k_1, \ldots, k_n)} (f_{a_1} \otimes \cdots \otimes f_{a_n} \otimes r_k \otimes g_{c_1} \otimes \cdots \otimes g_{c_s}) j_{\alpha + 1 + \beta} :$$

$$T^{k_1} s \mathcal{C} \otimes \cdots \otimes T^{k_n} s \mathcal{C} \to T^+ s \mathcal{J},$$  (4.1.2)

where

$$P(k_1, \ldots, k_n) = \sqcup_{a, \beta \geq 0} \{(l_1, \ldots, l_\alpha; l_{\alpha + 1}; l_{\alpha + 2}, \ldots, l_{\alpha + 1 + \beta}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid \forall q \in \mathbb{Z}_{\geq 0}, q \leq \alpha + 1 + \beta \forall s \in \mathbb{Z}_{\geq 0}, s \leq n, l_1 + \cdots + l_q = k_1 + \cdots + k_s \iff q = \alpha + 1 + \beta, s = n \}.$$

These maps restrict to maps $\tau_n : T^s s \mathcal{D}(\mathcal{C} | \mathcal{B}) \to s \mathcal{D}(\mathcal{J} | \mathcal{J})$, which are components of an $A_\infty$-transformation $D(r) = \tau : \jmath \to \gamma : D(\mathcal{C} | \mathcal{B}) \to D(\mathcal{J} | \mathcal{J})$. The restriction of $\tau_n$ to $T^s s \mathcal{C} \xrightarrow{\tau_n} s \mathcal{J} \xrightarrow{\tau_0} s \mathcal{J}$ equals $T^s s \mathcal{C} \xrightarrow{\tau_n} s \mathcal{J} \xrightarrow{\tau_0} s \mathcal{J} \xrightarrow{\tau_0} s \mathcal{J} = s \mathcal{J}$.

Proof. Similarly to the case of $A_\infty$-functors, discussed in Proposition 3.3, let us define an $A_\infty$-transformation $\tau : \jmath \to \gamma : \mathcal{C} \to \mathcal{J}$ by its components (4.1.2) and prove that the equation $\tau_\mu = \mu_\tau$ holds. Clearly, $(\tau_\mu)_0 = \tau_0 = (\mu_\tau)_0$. We have to prove that for $n > 0$

$$\sum_{i_1 + \cdots + i_t = n} (f_{i_1} \otimes \cdots \otimes f_{i_{q-1}} \otimes f_{i_q} \otimes f_{i_{q+1}} \otimes \cdots \otimes f_{i_t}) \mu^{(t)} :$$

$$T^{k_1} s \mathcal{C} \otimes \cdots \otimes T^{k_n} s \mathcal{C} \to T^+ s \mathcal{J}.$$  (4.1.3)
The right-hand side is the sum of terms \( f_{a_1} \otimes \cdots \otimes f_{a_n} \otimes r_k \otimes g_{c_1} \otimes \cdots \otimes g_{c_\beta} \), such that \( a_1 + \cdots + a_\alpha + k + c_1 + \cdots + c_\beta = k_1 + \cdots + k_n \). Denote \((l_1, \ldots, l_{\alpha+1+\beta}) = (a_1, \ldots, a_\alpha, k, c_1, \ldots, c_\beta)\). Consider the subsequence \( \mathcal{N} \) of the sequence \( L = (0, l_1, l_1 + l_2, \ldots, l_1 + \cdots + l_{\alpha+1+\beta}) \) consisting of all elements which belong to the set \( \{0, k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_n\} \). The term \( f_{a_1} \otimes \cdots \otimes f_{a_\alpha} \otimes r_k \otimes g_{c_1} \otimes \cdots \otimes g_{c_\beta} \) will appear as a summand of the term \( \overline{\mathcal{F}}_{i_1} \otimes \cdots \otimes \overline{\mathcal{F}}_{i_{q-1}} \otimes \overline{\mathcal{T}}_{i_q} \otimes \overline{\mathcal{T}}_{i_{q+1}} \otimes \cdots \otimes \overline{\mathcal{T}}_{i_t} \) if and only if

\[
N = (0, i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_t), \\
\forall 1 \leq y \leq t \quad i_y = 0 \implies y = q.
\]

Let us prove that for a given sequence \((a_1, \ldots, a_\alpha; k; c_1, \ldots, c_\beta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_0^\beta\) there exists exactly one sequence \((i_1, \ldots, i_t; q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\) such that conditions (4.1.4)–(4.1.6) are satisfied. Indeed, the sequence \( N \) determines uniquely a sequence \((i_1, \ldots, i_t)\) of non-negative integers such that (4.1.4) holds. If \( k > 0 \) or \( a_1 + \cdots + a_\alpha \) does not belong to \( \mathcal{N} \), then all \( i_y \) are positive. This implies that the interval \([a_1 + \cdots + a_\alpha, a_1 + \cdots + a_\alpha + k]\) is contained in a unique interval of the form \([i_1 + \cdots + i_{q-1}, i_1 + \cdots + i_q]\).

If \( k = 0 \) and \( a_1 + \cdots + a_\alpha \) belongs to \( \mathcal{N} \), then \( a_1 + \cdots + a_\alpha = a_1 + \cdots + a_\alpha + k \) is repeated in \( \mathcal{N} \). Hence, there exists \( q > 0 \) such that \( a_1 + \cdots + a_\alpha = i_1 + \cdots + i_{q-1}, i_q = 0 \). Since \( \mathcal{M} \) and \( \mathcal{N} \) may contain no more than one repeated element, \( i_y \) is positive for \( y \neq q \). Therefore, conditions (4.1.4)–(4.1.6) are satisfied.

We conclude that (4.1.3) holds. Formula (4.1.1) is the expansion of the proven property \( \tau = \mu \mu \mu^{-1} \).

The target space for map (4.1.2) applied to \( \mathcal{L} \)-module (3.1.4) has the form

\[
s\mathcal{J}(X f, C_i^* f) \otimes \cdots \otimes s\mathcal{J}(C_i^* f, C_i^* g) \otimes \cdots \otimes s\mathcal{J}(C_i^* g, Y g),
\]

where \( C_i^* \) are objects of \( \mathcal{B} \) (and no \( Z_j \) will appear). Since objects \( C_i^* f \) and \( C_i^* g \) belong to \( \mathcal{J} \), the above space is a direct summand of \( s\mathcal{D}(\mathcal{J}|\mathcal{J})(X f, Y g) \). Therefore, the required map \( \overline{\mathcal{T}}_n : T^n s\mathcal{D}(\mathcal{C}|\mathcal{B}) \rightarrow s\mathcal{D}(\mathcal{J}|\mathcal{J}) \) is constructed.

The last statement is a particular case of (4.1.2). Indeed, if \( k_1 = \cdots = k_n = 1 \), then \( P(1, \ldots, 1) \) consists of only one element \((n; n)\), that is, \( \alpha = \beta = 0, i_1 = n \in \mathbb{Z}_{\geq 0} \).

In particular, the correspondence \( r \mapsto D(r) = \tau \) maps natural \( A_{\infty} \)-transformations to natural ones, and equivalent \( r, p : f \rightarrow g : \mathcal{B} \rightarrow \mathcal{C} \) are mapped to equivalent

\[
D(r), D(p) : \overline{f} \rightarrow \overline{g} : D(\mathcal{C}|\mathcal{B}) \rightarrow D(\mathcal{J}|\mathcal{J}).
\]

For example,

\[
\tau_1 = r - (f \otimes r_0 + r_0 \otimes g)\mu, \\
\tau_2 = \mu r - (f \otimes r + r \otimes g)\mu - (f \otimes r_0 + r_0 \otimes g)\mu + (f \otimes f \otimes r_0 + f \otimes r_0 \otimes g + r_0 \otimes g \otimes g)\mu^{(3)}.
\]
4.2 Corollary. We have a commutative cylinder

\[
\begin{array}{c}
\mathcal{B} \\ \downarrow f' \\
\mathcal{C} \\ \downarrow g \\
D(\mathcal{C}|\mathcal{B}) \\
\downarrow \tau
\end{array} \quad \begin{array}{c}
r' \\ \downarrow \\
r \\ \downarrow \\
\tau
\end{array}
\]

4.3. **K-2-categories and K-2-functors.** Let \( K \) denote the category \( K(\mathfrak{k}\text{-mod}) = H^0(\mathfrak{C}(\mathfrak{k}\text{-mod})) \) of differential graded complexes of \( \mathfrak{k} \)-modules, whose morphisms are chain maps modulo homotopy. A 1-unital, non-2-unital \( K \)-2-category \( \mathcal{K}A_\infty \) of \( A_\infty \)-categories is described in [Lyu03, Proposition 7.1]. Instead of the complex of 2-morphisms \( (A_\infty(\mathcal{A}, \mathcal{B})(f, g), m_1), m_1 = sB_1s^{-1}, \) we work with the shifted complex \( (sA_\infty(\mathcal{A}, \mathcal{B})(f, g), B_1) \). There is an obvious notion of a strict \( K \)-2-functor between such \( K \)-2-categories – a map of objects, maps of 1-morphisms and chain maps of 2-morphisms, which preserve all operations. The operations involving 2-morphisms are subject to equations in \( K \), which mean equations between chain maps up to homotopy.

We have applied the “underline” construction \( _{-} \) to three kinds of arguments \(-:\): \( A_\infty \)-categories, \( A_\infty \)-functors and \( A_\infty \)-transformations. Let us summarize the properties of this construction.

4.4 Proposition. The following assignment defines a strict \( K \)-2-functor \( _{-} : \mathcal{K}A_\infty \to \mathcal{K}A_\infty : \) an \( A_\infty \)-category \( \mathcal{A} \) is mapped to \( \underline{\mathcal{A}} \), an \( A_\infty \)-functor \( f : \mathcal{A} \to \mathcal{B} \) is mapped to \( \underline{f} : \underline{\mathcal{A}} \to \underline{\mathcal{B}} \), and the chain map of complexes of 2-morphisms is

\[
\_ : (sA_\infty(\mathcal{A}, \mathcal{B})(f, g), B_1) \to (sA_\infty(\underline{\mathcal{A}}, \underline{\mathcal{B}})(\underline{f}, g), B_1), \quad r \mapsto r,
\]

where \( B \) denotes the codifferential in \( TsA_\infty(\mathcal{A}, \mathcal{B}) \), in particular, \( vB_1 = [v, B] \).

Proof. We have seen in (1.0.2) that \( rB_1 = [r, B] = [r, \underline{B}] = \underline{rB} \), thus, \( r \mapsto \underline{r} \) is a chain map. The composition of \( A_\infty \)-functors is preserved, \( fg = \underline{fg} \). The right action of a 1-morphism \( h \) on a 2-morphism \( r \) is preserved, since \( rh = (\underline{r}h)(\underline{j}) = \underline{rh} \). The left action of a 1-morphism \( e \) on a 2-morphism \( r \) is preserved, since \( er = (\underline{e}(\underline{r})(\underline{j})) = \underline{er} \). The identity \( A_\infty \)-functor \( \text{id}_{\mathcal{A}} \) is mapped to the identity \( A_\infty \)-functor \( \underline{\text{id}}_{\mathcal{A}} = \text{id}_{\underline{\mathcal{A}}} \).

It remains to prove that the vertical composition of 2-morphisms

\[
m_2 = (s \otimes s)B_2s^{-1} : A_\infty(\mathcal{A}, \mathcal{B})(f, g) \otimes A_\infty(\mathcal{A}, \mathcal{B})(g, h) \to A_\infty(\mathcal{A}, \mathcal{B})(f, h)
\]

is preserved, that is, the diagram

\[
\begin{array}{c}
A_\infty(\mathcal{A}, \mathcal{B})(f, g) \otimes A_\infty(\mathcal{A}, \mathcal{B})(g, h) \\ m_2 \\
\downarrow \\
A_\infty(\mathcal{A}, \mathcal{B})(f, h)
\end{array} \quad \begin{array}{c}
A_\infty(\mathcal{A}, \mathcal{B})(f, g) \otimes A_\infty(\mathcal{A}, \mathcal{B})(g, h) \\ m_2 \\
\downarrow \\
A_\infty(\mathcal{A}, \mathcal{B})(f, h)
\end{array}
\]

\[
\begin{array}{c}
s_2s^{-1} \\
A_\infty(\mathcal{A}, \mathcal{B})(f, g) \otimes A_\infty(\mathcal{A}, \mathcal{B})(g, h) \\
\downarrow \\
A_\infty(\mathcal{A}, \mathcal{B})(f, h)
\end{array} \quad \begin{array}{c}
s_2s^{-1} \\
A_\infty(\mathcal{A}, \mathcal{B})(f, g) \otimes A_\infty(\mathcal{A}, \mathcal{B})(g, h) \\
\downarrow \\
A_\infty(\mathcal{A}, \mathcal{B})(f, h)
\end{array}
\]

\[
\begin{array}{c}
s_2s^{-1} \\
A_\infty(\mathcal{A}, \mathcal{B})(f, g) \otimes A_\infty(\mathcal{A}, \mathcal{B})(g, h) \\
\downarrow \\
A_\infty(\mathcal{A}, \mathcal{B})(f, h)
\end{array}
\]
is commutative in \( \mathcal{K} \). Here \( s \_ s^{-1} \) denotes the composition
\[
A_\infty(A, B)(f, h) \xrightarrow{s} s A_\infty(A, B)(f, h) \xrightarrow{s^{-1}} s A_\infty(A, B)(f, h) \xrightarrow{s} A_\infty(A, B)(f, h).
\]
Since \( b_k = 0 \) for \( k \geq 2 \), we have \( B_2 = 0 \) due to \cite[Equation (5.1.3)]{Lyu03}, hence, \( m_2 = 0 \).
Let us prove that \( m_2(s \_ s^{-1}) \sim 0 \). The homotopy is sought in the form \( (s \otimes s)Hs^{-1} \),
where
\[
H : s A_\infty(A, B)(f, g) \otimes s A_\infty(A, B)(g, h) \to s A_\infty(A, B)(f, h)
\]
is a \( k \)-linear map of degree 0. It has to satisfy the equation
\[
B_2 = HB_1 - (1 \otimes B_1 + B_1 \otimes 1)H,
\]
that is, for each \( r \in s A_\infty(A, B)(f, g), p \in s A_\infty(A, B)(g, h) \)
\[
(r \otimes p)B_2 = [(r \otimes p)H, b] - [(r \otimes p)(1 \otimes B_1 + B_1 \otimes 1)]H.
\]
(4.4.1)
A candidate for \( H \) is chosen similarly to definition (4.0.1). We choose the components of
the \((f, h)\)-coderivation \((r \otimes p)H\) as follows:
\[
[(r \otimes p)H]_0 = (r_0 \otimes p_0)j_2,
[(r \otimes p)H]_1 = (r \otimes p)\theta : sA(X, Y) \to sB(Xf, Yh),
[(r \otimes p)H]_k = 0 \quad \text{for } k > 1.
\]
Let us verify equation (4.4.1) for this \( H \). Both sides of (4.4.1) are \((f, h)\)-coderivations.
It suffices to check that all their components coincide. The 0-th component of the right hand side of (4.4.1) is
\[
[(r \otimes p)H]_0 b_1 - [(r \otimes p, b) + (-)^p[r, b] \otimes p)H]_0
= (r_0 \otimes p_0)j_2 b - (r_0 \otimes p_0 b_1 + (-)^p r_0 b_1 \otimes p_0)j_2
= (r_0 \otimes p_0)(b - 1 \otimes b_1 - b_1 \otimes 1) = (r_0 \otimes p_0)b_2 j_2,
\]
which equals \((r \otimes p)B_2\). Due to \cite[Equation (5.1.2)]{Lyu03} the first component of the right hand side of (4.4.1) equals
\[
[(r \otimes p)H]_1 b_1 - (-)^rp_1 [(r \otimes p)H]_1 - [(r \otimes p)(1 \otimes B_1 + B_1 \otimes 1)H]_1
= (r \otimes p)\theta b - (-)^rp(r \otimes p)\theta - [(r \otimes p)(1 \otimes B_1 + B_1 \otimes 1)]\theta
= (r \otimes p)B_2,
\]
which is \((r \otimes p)B_2\). The \( k \)-th component of the right hand side of (4.4.1) vanishes for \( k > 1 \), and so does \((r \otimes p)B_2\). Therefore, (4.4.1) and the proposition are proven.

4.5 Corollary. The same assignment \( A \mapsto A_\infty, f \mapsto f, r \mapsto r \) as in Proposition 4.4 gives
a strict 2-functor \( \_ : A_\infty \to A_\infty \) of non-2-unital 2-categories.
This is obtained by taking the 0-th cohomology of $\mathcal{K}A_\infty$ in Proposition 4.4.

Similarly, the “overline” construction $\overline{-}$, applied to three kinds of arguments, $A_\infty$-categories, $A_\infty$-functors and $A_\infty$-transformations, gives a strict $\mathcal{K}$-2-functor.

**4.6 Corollary.** The following assignment defines a strict $\mathcal{K}$-2-functor $\overline{-}: \mathcal{K}A_\infty \to \mathcal{K}A_\infty$: an $A_\infty$-category $A$ is mapped to $\overline{A}$, an $A_\infty$-functor $f: A \to B$ is mapped to $\overline{f}: \overline{A} \to \overline{B}$, and the chain map of complexes of 2-morphisms is

$$\overline{\cdot} : (sA_\infty(A, B)(f, g), B_1) \to (sA_\infty(\overline{A}, \overline{B})(\overline{f}, \overline{g}), \overline{B}_1), \quad r \mapsto \overline{r},$$

where $\overline{B}$ denotes the codifferential in $TsA_\infty(\overline{A}, \overline{B})$, in particular, $\overline{B}_1 = [v, \overline{b}]$. There is an invertible strict $\mathcal{K}$-2-transformation $\mu : \overline{-} \to \overline{-}, \mu : \overline{A} \to A$.

**Proof.** Starting with a $\mathcal{K}$-2-functor $\overline{-}$, a mapping $\text{Ob} \mathcal{K}A_\infty \to \text{Ob} \mathcal{K}A_\infty, A \mapsto \overline{A}$, and a family of invertible $A_\infty$-functors $\mu_A : \overline{A} \to A$, one may construct another $\mathcal{K}$-2-functor $\overline{-}$, which maps an $A_\infty$-category $A$ to $\overline{A}$, so that $\mu$ is a strict $\mathcal{K}$-2-transformation. Since $\mu$ is strict and invertible, the values of $\overline{f}$ and $\overline{r}$ are fixed by the requirements $\overline{f}\mu = \mu f$, $\overline{r}\mu = \mu \overline{r}$ for each $A_\infty$-functor $f$ and $A_\infty$-transformation $r$. \(\square\)

The detailed definition of strict $\mathcal{K}$-2-transformations is left to the interested reader.

**4.7 Corollary.** The same assignment $A \mapsto \overline{A}$, $f \mapsto \overline{f}$, $r \mapsto \overline{r}$ as in Corollary 4.6 gives a strict 2-functor $\overline{-}: A_\infty \to A_\infty$ of non-2-unital categories.

This is obtained by taking the 0-th cohomology of $\mathcal{K}A_\infty$ in Corollary 4.6.

It is instructive to find the homotopy which forces $\overline{-}$ to preserve the vertical composition of 2-morphisms. Denote $\text{ad} \mu$ the maps $sA_\infty(A, B)(f, g) \to sA_\infty(\overline{A}, \overline{B})(\overline{f}, \overline{g})$, $v \mapsto \mu v \mu^{-1}$. The following diagram commutes modulo homotopy:

```
\begin{tikzcd}
\end{tikzcd}
```

The right homotopy commutative square is obtained from [Lyu03, Equation (7.1.2)]:

$$(\rho \otimes \pi)B_2 \mu^{-1} - (\rho \mu^{-1} \otimes \pi \mu^{-1})B_2 = (\rho \otimes \pi)M_{20}B_1 - [(\rho \otimes \pi)(1 \otimes B_1 + B_1 \otimes 1)]\mu^{-1}M_{20}$$
for all $\rho \in sA_\infty(A, B)(f, g)$, $\pi \in sA_\infty(A, B)(g, h)$. Recall that $B_2 = 0$ and compose with $\mu$ to get
\[-(\mu \rho \mu^{-1} \otimes \mu \pi \mu^{-1})B_2 = [\mu(\rho \otimes \pi | \mu^{-1})]M_{20}[B_1 - \mu[(\rho \otimes \pi)(1 \otimes B_1 + B_1 \otimes 1)]\mu^{-1}]M_{20}.\]
In particular, for $\rho = \underline{\rho}$, $\pi = \underline{\pi}$ we have
\[-(\overline{r} \otimes \overline{p})B_2 = [\mu(\overline{r} \otimes \overline{p} | \mu^{-1})]M_{20}[B_1 - \mu[(\overline{r} \otimes \overline{p})(1 \otimes B_1 + B_1 \otimes 1)]\mu^{-1}]M_{20}.\]
The left homotopy commutative square, that is, (4.4.1) composed with ad $\mu$ gives
\[
\overline{(r \otimes p)B_2} = [\mu(r \otimes p)H\mu^{-1}]B_1 - \mu[(r \otimes p)(1 \otimes B_1 + B_1 \otimes 1)]H\mu^{-1}.
\]
We conclude that the exterior of diagram (4.7.1) is commutative modulo homotopy
\[
R : sA_\infty(A, B)(f, g) \otimes sA_\infty(A, B)(g, h) \rightarrow sA_\infty(\overline{A}, \overline{B})(\overline{f}, \overline{h}),
\]
\[
(r \otimes p)R = [\mu(r \otimes p)H\mu^{-1}]M_{20},
\]
that is,
\[
\overline{(r \otimes p)B_2} - (\overline{r} \otimes \overline{p})B_2 = (r \otimes p)R\overline{B_1} - [(r \otimes p)(1 \otimes B_1 + B_1 \otimes 1)]R. \tag{4.7.2}
\]

4.8 Proposition. The $(\overline{f}, \overline{h})$-transformation $(r \otimes p)R$ has the following components: $[(r \otimes p)R]_0 = 0$, and for $n > 0$ the restriction of $[(r \otimes p)R]_n$ to $T^{k_1}sA \otimes \cdots \otimes T^{k_n}sA$ is
\[
[(r \otimes p)R]_n = \mu^{(n)}(\sum_{(a, k; \epsilon; t; e)} (f_{a_1} \otimes \cdots \otimes f_{a_n} \otimes r_k \otimes g_{c_1} \otimes \cdots \otimes g_{c_\ell} \otimes p_t \otimes h_{e_1} \otimes \cdots \otimes h_{e_\ell}) \cdot \overline{j}_{a_\alpha + \beta + \gamma + 2} : T^{k_1}sA \otimes \cdots \otimes T^{k_n}sA \rightarrow T^+sB, \tag{4.8.1}
\]
where $(\alpha; k; \epsilon; t; e) = (a_1, \ldots, a_\alpha; k; c_1, \ldots, c_\beta; t; e_1, \ldots, e_\gamma)$ and
\[
Q(k_1, \ldots, k_n) = \bigcup_{\alpha, \beta, \gamma \geq 0} \{ (l_1, \ldots, l_\alpha; l_{\alpha + 1}; l_{\alpha + 2}, \ldots, l_{\alpha + \beta + 1}; l_{\alpha + \beta + 2}; l_{\alpha + \beta + 3}, \ldots, l_{\alpha + \beta + \gamma + 2}) \in \mathbb{Z}_{\geq 0}^\alpha \times \mathbb{Z}_{\geq 0}^\beta \times \mathbb{Z}_{\geq 0}^\gamma \times \mathbb{Z}_{\geq 0}^\gamma \mid \forall q \in \mathbb{Z}_{\geq 0}, q \leq \alpha + \beta + \gamma + 2 \forall s \in \mathbb{Z}_{\geq 0}, s \leq n
\]
\[
L_1 + \cdots + L_q = k_1 + \cdots + k_s \Leftrightarrow q = \alpha + \beta + \gamma + 2, s = n \}.
\]
If $\mathcal{A}' \subset \mathcal{A}$, $\mathcal{B}' \subset \mathcal{B}$ are full $A_{\infty}$-subcategories and $(\text{Ob}\mathcal{A}')f \subset \text{Ob}\mathcal{B}'$, $(\text{Ob}\mathcal{A}')g \subset \text{Ob}\mathcal{B}'$, $(\text{Ob}\mathcal{A}')h \subset \text{Ob}\mathcal{B}'$, then $[(r \otimes p)R]_n$ restrict to maps
\[
[(r \otimes p)R]_n : T^n\text{D}(\mathcal{A} | \mathcal{A}')(X, Y) \rightarrow s\text{D}(\mathcal{B} | \mathcal{B}')(Xf, Yh),
\]
which are components of an $A_{\infty}$-transformation
\[
(r \otimes p)R \in sA_\infty(\text{D}(\mathcal{A} | \mathcal{A}'), \text{D}(\mathcal{B} | \mathcal{B}'))(\overline{f}, \overline{h}).
\]
Proof. Denote by $R'$ an $(\bar{f}, \bar{h})$-coderivation, whose components are $R'_a = 0$ and $R'_n$ is given by the right hand side of (1.8.1). We want to prove that $(r \otimes p)R = R'$. This is equivalent to the equation

$$R'\mu = \mu(r \otimes p)H + \mu(\underline{\mu} \otimes \underline{p} \mid \underline{\mu}^{-1})M_{20}\mu.$$  \hspace{1cm} (4.8.2)$$

Let us transform the last term. Applying the identity $(1 \otimes M)M pr_1 = (M \otimes 1)M pr_1$ [Lyu03, Proposition 4.1] to an element

$$1 \otimes r \otimes p \otimes 1 \in T^0 sA_\infty(\bar{A}, \bar{A})(\mu, \mu) \otimes T^2 sA_\infty(\bar{A}, \bar{B})(f, h) \otimes T^0 sA_\infty(\bar{B}, \bar{B})(\mu^{-1}, \mu^{-1})$$

we find from

$$(1 \otimes r \otimes p \otimes 1)(1 \otimes M)M pr_1 = [1 \otimes (r \otimes p \mid \mu^{-1})M_{20} + 1 \otimes r\mu^{-1} \otimes p\mu^{-1}]M pr_1 = \mu(r \otimes p \mid \mu^{-1})M_{20},$$

$$(1 \otimes r \otimes p \otimes 1)(M \otimes 1)M pr_1 = (\mu_L \otimes \mu_P \otimes 1)M pr_1 = (\mu_L \otimes \mu_P \mid \mu^{-1})M_{20},$$

that

$$\mu(r \otimes p \mid \mu^{-1})M_{20} = (\mu_L \otimes \mu_P \mid \mu^{-1})M_{20}.$$  

Applying the same identity to an element

$$\rho \otimes \pi \otimes 1 \otimes 1 \in T^2 sA_\infty(\bar{A}, \bar{B})(\mu f, \mu h) \otimes T^0 sA_\infty(\bar{B}, \bar{B})(\lambda, \lambda) \otimes T^0 sA_\infty(\bar{B}, \bar{B})(\mu, \mu)$$

we find from

$$(\rho \otimes \pi \otimes 1 \otimes 1)(1 \otimes M)M pr_1 = (\rho \otimes \pi \otimes 1)M pr_1 = (\rho \otimes \pi \mid \lambda \mu)M_{20},$$

$$(\rho \otimes \pi \otimes 1 \otimes 1)(M \otimes 1)M pr_1 = [(\rho \otimes \pi \mid \lambda)M_{20} \otimes 1 + \rho \lambda \otimes \pi \lambda \otimes 1]M pr_1 = (\rho \otimes \pi \mid \lambda)M_{20}\mu + (\rho \lambda \otimes \pi \lambda \mid \mu)M_{20},$$

that

$$\mu(r \otimes p \mid \mu^{-1})M_{20} = (\rho \otimes \pi \mid \lambda)M_{20}\mu + (\rho \lambda \otimes \pi \lambda \mid \mu)M_{20}. $$

When $\lambda = \mu^{-1}$, the left hand side vanishes. Indeed, for each $k \geq 0$

$$[(\rho \otimes \pi \mid \id_{\bar{B}})M_{20}]_k = \sum_{l \geq 2} (\rho \otimes \pi)\theta_{kl} \id_l = 0.$$ 

Hence,

$$(\rho \otimes \pi \mid \mu^{-1})M_{20}\mu = - (\rho \mu^{-1} \otimes \pi \mu^{-1} \mid \mu)M_{20}. $$

In particular, for $\rho = \mu_L$, $\pi = \mu_P$ we have

$$(\mu_L \otimes \mu_P \mid \mu^{-1})M_{20}\mu = - (\tau \otimes \bar{p} \mid \mu)M_{20}. $$

Therefore, equation (1.8.2) can be rewritten as follows:

$$\mu(r \otimes p)H = R'\mu + (\tau \otimes \bar{p} \mid \mu)M_{20}. $$  \hspace{1cm} (4.8.3)
Both sides are \((\mu_f, \mu_h)\)-coderivations, or \((\overline{\mu}, \overline{\mu})\)-coderivations, which is the same thing. Let us prove that all their components coincide.

The 0-th components coincide, since
\[
[(r \otimes p)H]_0 = (r_0 \otimes p_0)j_2 = (\overline{r} \otimes \overline{p})\theta_0\mu_2 = [(\overline{r} \otimes \overline{p} | \mu)M_{20}]_0.
\]
For \(n > 0\) we have to verify the following equation for \(n\)-th components
\[
\mu^{(n)}(r \otimes p)\theta = \sum_{i_1 + \cdots + i_n = n} (\overline{f}_{i_1} \otimes \cdots \otimes \overline{f}_{i_{q-1}} \otimes \overline{R}_{i_q} \otimes \overline{h}_{i_{q+1}} \otimes \cdots \otimes \overline{h}_{i_n}) \mu^{(x)}
\]
\[
+ \sum_x (\overline{r} \otimes \overline{p})\theta_{nx} \mu^{(x)} : T^{k_1}sA \otimes \cdots \otimes T^{k_n}sA \rightarrow T^+sB.
\]

The left hand side is
\[
\sum_{a_1 + \cdots + a_k + c_1 + \cdots + c_\beta + t + e_1 + \cdots + e_\gamma = n} f_{a_1} \otimes \cdots \otimes f_{a_k} \otimes r_k \otimes g_{c_1} \otimes \cdots \otimes g_{c_\beta} \otimes p_t \otimes h_{e_1} \otimes \cdots \otimes h_{e_\gamma},
\]
(4.8.4)

Both sums in the right hand side consist of some of the above summands. Let us verify that each summand of (4.8.4) will occur exactly once either in
\[
\sum (\overline{f}_{i_1} \otimes \cdots \otimes \overline{f}_{i_{q-1}} \otimes \overline{R}_{i_q} \otimes \overline{h}_{i_{q+1}} \otimes \cdots \otimes \overline{h}_{i_n}) \mu^{(x)},
\]
or in
\[
\sum_x (\overline{r} \otimes \overline{p})\theta_{nx} \mu^{(x)}.
\]

Let us rewrite the sequence \((a_1, \ldots, a_k; k; c_1, \ldots, c_\beta; t; e_1, \ldots, e_\gamma)\) as
\[
(l_1, \ldots, l_\alpha; l_{\alpha+1}; l_{\alpha+2}, \ldots, l_{\alpha+\beta+1}; l_{\alpha+\beta+2}; l_{\alpha+\beta+3}, \ldots, l_{\alpha+\beta+\gamma+2}).
\]

Consider the subsequence \(N\) of the sequence \(L = (0, l_1, l_1 + l_2, \ldots, l_1 + \cdots + l_{\alpha+\beta+\gamma+2})\) consisting of all elements which belong to the set \(\{0, k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_n\}\). The term
\[
f_{i_1} \otimes \cdots \otimes f_{i_k} \otimes r_{i_{q+1}} \otimes g_{i_{q+2}} \otimes \cdots \otimes g_{i_{q+\beta+2}} \otimes p_{i_{q+\beta+2}} \otimes h_{i_{q+\beta+3}} \otimes \cdots \otimes h_{i_{q+\beta+\gamma+2}}
\]
(4.8.5)
will appear as a summand of
\[
(\overline{f}_{i_1} \otimes \cdots \otimes \overline{f}_{i_{q-1}} \otimes \overline{R}_{i_q} \otimes \overline{h}_{i_{q+1}} \otimes \cdots \otimes \overline{h}_{i_n}) \mu^{(x)},
\]
(4.8.6)
if and only if
\[
N = (0, i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_x),
\]
(4.8.7)
\[
i_1 + \cdots + i_{q-1} \leq l_1 + \cdots + l_\alpha, \quad l_1 + \cdots + l_{\alpha+\beta+2} \leq i_1 + \cdots + i_q,
\]
(4.8.8)
\[
\forall 1 \leq \gamma \leq x \quad i_\gamma = 0 \implies \gamma = q.
\]
(4.8.9)

The term (4.8.3) will appear as a summand of
\[
(\overline{f}_{i_1} \otimes \cdots \otimes \overline{f}_{i_{q-1}} \otimes \overline{R}_{i_q} \otimes \overline{g}_{i_{q+1}} \otimes \cdots \otimes \overline{g}_{i_{q-1}} \otimes \overline{p}_{i_q} \otimes \overline{h}_{i_{q+1}} \otimes \cdots \otimes \overline{h}_{i_n}) \mu^{(x)}
\]
(4.8.10)
(which is a term of $(\mathbf{r} \otimes \mathbf{p})\theta_{\mathbf{r} \mathbf{p}}(x)$) if and only if (4.8.7) holds and

\begin{align}
i_1 + \cdots + i_{y-1} &\leq l_1 + \cdots + l_\alpha, \quad l_1 + \cdots + l_{\alpha+1} \leq i_1 + \cdots + i_y, \tag{4.8.11} \\
i_1 + \cdots + i_{z-1} &\leq l_1 + \cdots + l_{\alpha+\beta+1}, \quad l_1 + \cdots + l_{\alpha+\beta+2} \leq i_1 + \cdots + i_z, \tag{4.8.12} \\
y < z \text{ and } \forall 1 \leq \gamma < x \quad i_\gamma = 0 \implies \gamma \in \{y, z\}. \tag{4.8.13}
\end{align}

A given non-decreasing sequence $N$ determines uniquely a sequence $(i_1, \ldots, i_x)$ of non-negative integers such that (4.8.7) holds.

If $l_{\alpha+1} > 0$ or $l_1 + \cdots + l_\alpha$ does not belong to $N$, then there exists exactly one element $y = y'$ such that (4.8.11) holds. If $l_{\alpha+1} = 0$ and $l_1 + \cdots + l_\alpha$ belongs to $N$, then there are at least 2 such elements. Denote by $y' > 0$ the least of them. Then $l_1 + \cdots + l_\alpha = i_1 + \cdots + i_{y'-1}$ and $i_{y'} = 0$. If $y$ satisfies both (4.8.11) and (4.8.13), then $y < y'$, hence, $y = y'$ is the only solution.

If $l_{\alpha+\beta+2} > 0$ or $l_1 + \cdots + l_{\alpha+\beta+1}$ does not belong to $N$, then there exists exactly one element $z = z'$ such that (4.8.12) holds. If $l_{\alpha+\beta+2} = 0$ and $l_1 + \cdots + l_{\alpha+\beta+1}$ belongs to $N$, then there are at least 2 such elements. Denote by $z'$ the biggest of them. Then $l_1 + \cdots + l_{\alpha+\beta+1} = i_1 + \cdots + i_{z'-1}$ and $i_{z'} = 0$. If $z$ satisfies both (4.8.12) and (4.8.13), then $z' \leq z$, hence, $z = z'$ is the only solution.

Since $[i_1 + \cdots + i_{y'-1}, i_1 + \cdots + i_{y'}]$ is the leftmost interval with ends in $N$ containing $[l_1 + \cdots + l_\alpha, l_1 + \cdots + l_{\alpha+1}]$, and the latter lies to the left of $[l_1 + \cdots + l_{\alpha+\beta+1}, l_1 + \cdots + l_{\alpha+\beta+2}]$, contained in the rightmost interval $[i_1 + \cdots + i_{z'-1}, i_1 + \cdots + i_z]$, we deduce that $y' \leq z'$.

If $y' = z'$, then $y' \leq y < z \leq z'$ can not be satisfied, hence, (4.8.11)–(4.8.13) has no solutions $(y, z)$. On the other hand, for $y = y' = z'$ the interval $[l_1 + \cdots + l_\alpha, l_1 + \cdots + l_{\alpha+\beta+2}]$ is contained in $[i_1 + \cdots + i_{q'-1}, i_1 + \cdots + i_q]$, that is, (4.8.8) holds. Only $l_{\alpha+1}$ and $l_{\alpha+\beta+2}$ might vanish, both are contained in $[l_1 + \cdots + l_\alpha, l_1 + \cdots + l_{\alpha+\beta+1}]$, hence, $i_\gamma$ might vanish only for $\gamma = q$, that is, (4.8.9) holds. Therefore, $y = y'$ satisfies conditions (4.8.8)–(4.8.9).

This solution is unique, since if (4.8.8) is satisfied for $q = q'$, then (4.8.11) holds for $y = q'$.

If $y' < z'$, then $y = y'$, $z = z'$ is the only solution of system of conditions (4.8.11)–(4.8.13). This is proved by examining the four cases which arise from the alternatives in the two paragraphs that follow (4.8.13). Let us prove that there are no solutions $q$ of the system of conditions (4.8.8)–(4.8.9). Suppose $q$ satisfies these conditions, then $y = q$ satisfies (4.8.11) and $z = q$ satisfies (4.8.13). Therefore, $y' \leq q \leq z'$, $i_1 + \cdots + i_{y'-1} = i_1 + \cdots + i_{q'-1}$ and $i_1 + \cdots + i_q = i_1 + \cdots + i_{z'}$. Due to (4.8.9) there exists no more than one $\gamma$ such that $i_\gamma = 0$. Thus, two possibilities exist: either $y' = q < q + 1 = z'$, or $y' = q - 1 < q = z'$. In the first case (4.8.12) and (4.8.8) imply

\begin{align*}
i_1 + \cdots + i_q &\leq l_1 + \cdots + l_{\alpha+\beta+1} \leq l_1 + \cdots + l_{\alpha+\beta+2} \leq i_1 + \cdots + i_q,
\end{align*}

hence, $i_1 + \cdots + i_q = l_1 + \cdots + l_{\alpha+\beta+1}$ and $l_{\alpha+\beta+2} = 0$. It follows that $i_{q+1} = 0$, which contradicts to (4.8.9). In the second case (4.8.8) and (4.8.11) imply

\begin{align*}
i_1 + \cdots + i_{q-1} &\leq l_1 + \cdots + l_\alpha \leq l_1 + \cdots + l_{\alpha+1} \leq i_1 + \cdots + i_{q-1},
\end{align*}

hence, $i_1 + \cdots + i_{q-1} = l_1 + \cdots + l_\alpha$ and $l_{\alpha+1} = 0$. It follows that $i_q = 0$. From (4.8.8) we deduce that $l_{\alpha+2} + \cdots + l_{\alpha+\beta+2} = 0$, which implies $i_{q+1} = 0$ and this contradicts (4.8.9).
We conclude that each term \((4.8.8)\) either occurs in a unique term \((4.8.10)\) or in a unique term \((4.8.1)\). Therefore, \((4.8.3)\) is proven.

Since \((4.8.1)\) is proven, it implies the statement for the transformation \((r \otimes p)R\). □

Denote by \(\mathcal{K}A'_{\infty}\) the non-2-unital \(\mathcal{K}\)-2-category, whose objects are pairs \((\mathcal{A}, \mathcal{A}')\), consisting of an \(A_{\infty}\)-category \(\mathcal{A}\) and a full \(A_{\infty}\)-subcategory \(\mathcal{A}' \subset \mathcal{A}\); 1-morphisms \((\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}')\) are \(A_{\infty}\)-functors \(f : \mathcal{A} \to \mathcal{B}\) such that \((\text{Ob} \mathcal{A}')f \subset \text{Ob} \mathcal{B}'\);

\[
\mathcal{K}A'_{\infty}((\mathcal{A}, \mathcal{A}'), (\mathcal{B}, \mathcal{B}'))(f, g) = \left( \mathcal{A}_{\infty}(\mathcal{A}, \mathcal{B})(f, g), m_1 \right),
\]

and the operations are induced by those of \(\mathcal{K}A_{\infty}\).

4.9 Corollary. The following assignment defines a strict \(\mathcal{K}\)-2-functor

\[
D : \mathcal{K}A'_{\infty} \longrightarrow \mathcal{K}A_{\infty},
(\mathcal{A}, \mathcal{A}') \longmapsto D(\mathcal{A}|\mathcal{A}'),
\]

\[
f : (\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}') \longmapsto \overline{f} : D(\mathcal{A}|\mathcal{A}') \to D(\mathcal{B}|\mathcal{B}'),
\]

\[
(sA_{\infty}((\mathcal{A}, \mathcal{A}'), (\mathcal{B}, \mathcal{B}'))(f, g), B_1) \longmapsto (sA_{\infty}(D(\mathcal{A}|\mathcal{A}'), D(\mathcal{B}|\mathcal{B}'))(\overline{f}, \overline{g}), \overline{B}_1),
\]

\(r \mapsto \overline{r}\).

Proof. Since the coderivation \((r \otimes p)R : Ts\overline{A} \to Ts\overline{B}\) restricts to a coderivation \((r \otimes p)R : TsD(\mathcal{A}|\mathcal{A}') \to TsD(\mathcal{B}|\mathcal{B}')\) by Proposition 4.8, \(D\) preserves the vertical composition of 2-morphisms modulo homotopy by \((4.7.2)\).

\(\Box\)

4.10 Corollary. Let \(A'_{\infty}\) be a non-2-unital 2-category, whose objects and 1-morphisms are the same as for \(\mathcal{K}A'_{\infty}\), and 2-morphisms are equivalence classes of natural \(A_{\infty}\)-transformations:

\[
A'_{\infty}((\mathcal{A}, \mathcal{A}'), (\mathcal{B}, \mathcal{B}'))(f, g) = H^0(\mathcal{A}_{\infty}(\mathcal{A}, \mathcal{B})(f, g), m_1),
\]

and the operations are induced by those of \(\mathcal{A}_{\infty}\). Then the following assignment defines a strict 2-functor

\[
D : A'_{\infty} \longrightarrow A_{\infty},
(\mathcal{A}, \mathcal{A}') \longmapsto D(\mathcal{A}|\mathcal{A}'),
\]

\[
f : (\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}') \longmapsto \overline{f} : D(\mathcal{A}|\mathcal{A}') \to D(\mathcal{B}|\mathcal{B}'),
\]

\[
r : f \to g : (\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}') \longmapsto \overline{r} : D(\mathcal{A}|\mathcal{A}') \to D(\mathcal{B}|\mathcal{B}').
\]

The corollary follows from Corollary \((1.9)\) by taking the 0-th cohomology.

5. Unitality

5.1 Proposition. Let \(\mathcal{B}\) be a full subcategory of a unital \(A_{\infty}\)-category \(\mathcal{C}\). Then the \(A_{\infty}\)-category \(D(\mathcal{C}|\mathcal{B})\) is also unital. If \(i^\mathcal{C}\) is a unit transformation of \(\mathcal{C}\), then \(D(i^\mathcal{C})\) is a unit transformation of \(D(\mathcal{C}|\mathcal{B})\).
Proof. The idempotent property \((i^c \otimes i^c)B_2 \equiv i^c\) implies by Corollary 1.9 that
\[
(D(i^c) \otimes D(i^c))\overline{B}_2 \equiv D((i^c \otimes i^c)B_2) \equiv D(i^c),
\]
so \(D(i^c)\) is an idempotent as well. Consider its 0-th component
\[
x D(i^c)_0 = \left[ k \xrightarrow{x_i^c} \mathcal{C}(X, X) \right] \rightarrow \mathcal{D}(\mathcal{E}|\mathcal{B})(X, X)\] .
We have to prove that
\[
(x D(i^c)_0 \otimes 1)\overline{b}_2, \quad (1 \otimes x D(i^c)_0)\overline{b}_2 : \mathcal{D}(\mathcal{E}|\mathcal{B})(X, Y) \rightarrow \mathcal{D}(\mathcal{E}|\mathcal{B})(X, Y)
\]
are homotopy invertible.
Consider the following \(\mathbb{Z}_{>0}\)-grading of the \(A_\infty\)-category \(\mathcal{D}(\mathcal{E}|\mathcal{B})\)
\[
G^k = T^k \mathcal{C} \cap \mathcal{D}(\mathcal{E}|\mathcal{B}), \quad k \geq 1, \\
G^k(X, Y) = \oplus_{C_1, \ldots, C_n \in \text{Ob } \mathcal{C}} \mathcal{C}(X, C_1) \otimes \mathcal{C}(C_1, C_2) \otimes \cdots \otimes \mathcal{C}(C_{k-1}, Y).
\]
Denote also \(C_0 = X, C_k = Y\). The corresponding increasing filtration
\[
0 = \Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_n \subset \Phi_{n+1} \subset \cdots \subset \mathcal{D}(\mathcal{E}|\mathcal{B})
\]
is made of \(\Phi_n = \oplus_{k=1}^n G^k\). The \(k\)-linear maps
\[
\overline{b}_1 = b, \quad (x i^c_0 \otimes 1)\overline{b}_2, \quad (1 \otimes x i^c_0)\overline{b}_2 : \mathcal{D}(\mathcal{E}|\mathcal{B})(X, Y) \rightarrow \mathcal{D}(\mathcal{E}|\mathcal{B})(X, Y)
\]
preserve the filtration. Consider the \(\mathbb{Z}_{>0} \times \mathbb{Z}\)-graded quiver, associated with this filtration. The above maps induce on graded components \(G^k\) the following maps:
\[
d_k = \sum_{\alpha+1+\beta=k} 1^{\otimes \alpha} \otimes b_1 \otimes 1^{\otimes \beta} : G^k(X, Y) \rightarrow G^k(X, Y), \\
(\alpha+1+\beta=k)
\]
\[
(1 \otimes x i^c_0)\overline{b}_2 = 1 + h_{C_{k-1}, Y} b_1 + b_1 h_{C_{k-1}, Y} : \mathcal{D}(C_{k-1}, Y) \rightarrow \mathcal{D}(C_{k-1}, Y),
\]
\[
(1 \otimes x i^c_0 \otimes 1)\overline{b}_2 = -1 + h'_{X, C_1} b_1 + b_1 h'_{X, C_1} : \mathcal{E}(X, C_1) \rightarrow \mathcal{E}(X, C_1).
\]
Using them we will present map \((5.1.3)\) restricted to \(\mathcal{D}(X, C_1) \otimes \cdots \otimes \mathcal{D}(C_{k-2}, C_{k-1}) \otimes \mathcal{D}(C_{k-1}, Y)\) as follows:
\[
1^{\otimes k-1} \otimes (1 \otimes y i^c_0)\overline{b}_2 = 1^{\otimes k-1} \otimes (1 + h b_1 + b_1 h)
\]
\[
= 1 + (1^{\otimes k-1} \otimes h) \sum_{\alpha+1+\beta=k} 1^{\otimes \alpha} \otimes b_1 \otimes 1^{\otimes \beta} + \left( \sum_{\alpha+1+\beta=k} 1^{\otimes \alpha} \otimes b_1 \otimes 1^{\otimes \beta} \right) (1^{\otimes k-1} \otimes h)
\]
\[
= 1 + (1^{\otimes k-1} \otimes h) d_k + d_k (1^{\otimes k-1} \otimes h).
\]
Let us define a $k$-linear map $H : sD(\mathcal{C}|\mathcal{B})(X,Y) \to sD(\mathcal{C}|\mathcal{B})(X,Y)$ of degree $-1$ as a direct sum of maps

$$1^\otimes k-1 \otimes h_{C_{k-1},Y} : s\mathcal{C}(X,C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-2}, C_{k-1}) \otimes s\mathcal{C}(C_{k-1}, Y)$$

$$\to s\mathcal{C}(X,C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-2}, C_{k-1}) \otimes s\mathcal{C}(C_{k-1}, Y).$$

Since $H$ preserves the subquivers $G^k$, it preserves also the filtration $\Phi_n$. Therefore, the chain (with respect to $b_1$) map

$$1 + N \overset{\text{def}}{=} (1 \otimes \gamma i_0^c) b_2 - H b_1 - b_1 H : sD(\mathcal{C}|\mathcal{B})(X,Y) \to sD(\mathcal{C}|\mathcal{B})(X,Y)$$

preserves the filtration and the associated map of graded complexes is identity. Hence, $N$ has a strictly lower triangular matrix with respect to the decomposition $sD(\mathcal{C}|\mathcal{B}) = \oplus_{k \geq 1} G^k$. Therefore, the map $1 + N$ is invertible with an inverse $\sum_{i=0}^\infty (-N)^i$. Hence, $(1 \otimes \gamma i_0^c) b_2$ is homotopy invertible.

Similarly,

$$\left(\chi i_0^c \otimes 1\right) b_2 \otimes 1^\otimes k-1 = (-1 + h'b_1 + b_1 h') \otimes 1^\otimes k-1$$

$$= -1 + (h' \otimes 1^\otimes k-1) \sum_{\alpha + 1 + \beta = k} 1^\otimes \alpha \otimes b_1 \otimes 1^\otimes \beta + \left( \sum_{\alpha + 1 + \beta = -k} 1^\otimes \alpha \otimes b_1 \otimes 1^\otimes \beta \right) (h' \otimes 1^\otimes k-1)$$

$$= -1 + (h' \otimes 1^\otimes k-1) d_k + d_k (h' \otimes 1^\otimes k-1).$$

Define a map $H'$ as a direct sum of maps

$$h'_{X,C_1} \otimes 1^\otimes k-1 : s\mathcal{C}(X,C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-1},Y)$$

$$\to s\mathcal{C}(X,C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-1},Y).$$

Then the chain map

$$-1 + N' \overset{\text{def}}{=} (\chi i_0^c \otimes 1) b_2 - H' b_1 - b_1 H' : sD(\mathcal{C}|\mathcal{B})(X,Y) \to sD(\mathcal{C}|\mathcal{B})(X,Y)$$

preserves the filtration and gives $-1$ on the diagonal. Hence, $N'$ is strictly lower triangular, and $-1 + N'$ is invertible with an inverse $-\sum_{i=0}^\infty (N')^i$. Therefore, $(\chi i_0^c \otimes 1) b_2$ is homotopy invertible. 

5.2 Corollary. If an $A_\infty$-category $\mathcal{C}$ is unital, then $\overline{\mathcal{C}}$ is unital with a unit transformation $\overline{i^c}$.

Indeed, $\overline{\mathcal{C}} = D(\mathcal{C}|\mathcal{E})$.

5.3 Corollary. If an $A_\infty$-category $\mathcal{C}$ is unital, then $\overline{\mathcal{C}}$ is unital with a unit transformation $\overline{i^c}$.

Proof. The $A_\infty$-functor $\mu^{-1} : \mathcal{C} \to \overline{\mathcal{C}}$ is invertible and $\overline{\mathcal{C}}$ is unital. Hence, by [Lyu03, Section 8.12] $\overline{\mathcal{C}}$ is unital and $\mu^{-1} \overline{i^c} \mu = \mu^{-1} \overline{i^c} \mu = \overline{i^c}$ is its unit transformation. \qed
5.4 Remark. Functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}(\mathcal{E}|\mathcal{B}), \mathcal{G} : \mathcal{C} \to \mathcal{G}, j^\mathcal{C} : \mathcal{C} \to \mathcal{C}$ are unital. This follows from Corollary 4.2 for $r = i^\mathcal{C}$ and from commutative diagram (4.0.3).

5.5 Corollary. Let $i : \mathcal{C} \to \mathcal{J}$ be a unital $A_\infty$-functor. Then the $A_\infty$-functors $\bar{i} : \mathcal{C} \to \mathcal{J}$ and $\bar{i} : \mathcal{C} \to \mathcal{J}$ are unital as well.

Proof. Since $ii^\mathcal{J} \equiv i^\mathcal{C}$, we have $\bar{i}ii^\mathcal{J} \equiv i^\mathcal{C}$ by Corollary 4.7. Therefore, $\bar{i}$ is unital by Corollary 5.2. We have also $ii^\mathcal{J} \equiv i^\mathcal{C}$ by Corollary 4.3. Hence, $\bar{i}$ is unital by Corollary 5.3. □

5.6 Corollary. Let $i : \mathcal{C} \to \mathcal{J}$ be a unital $A_\infty$-functor, which maps objects of a full $A_\infty$-subcategory $\mathcal{B} \subset \mathcal{C}$ to objects of a full $A_\infty$-subcategory $\mathcal{J} \subset \mathcal{J}$. Then the $A_\infty$-functor $D(i) : D(\mathcal{C}|\mathcal{B}) \to D(\mathcal{J}|\mathcal{J})$ is unital as well.

Proof. Since $ii^\mathcal{J} \equiv i^\mathcal{C}$, we have $D(i)D(i^\mathcal{J}) \equiv D(i^\mathcal{C})D(i)$ by Corollary 4.10. Therefore, $D(i)$ is unital by Proposition 5.1. □

Summing up, when we restrict $\mathcal{C}$, $\mathcal{B}$ or $D$ to unital $A_\infty$-categories, we get strict 2-functors of 1-unital left-2-unital ($K$-)2-categories (see Definition A.3 and Corollary 7.11 arXiv:math.CT/0210047). When we restrict $\mathcal{C}$, $\mathcal{B}$ or $D$ further to unital $A_\infty$-categories and unital $A_\infty$-functors, we get strict 2-functors of ordinary 1-2-unital ($K$-)2-categories.

6. Contractibility

A chain complex $C$ is contractible if $id_C$ is null-homotopic. We say that an $A_\infty$-category is contractible if all its complexes of morphisms are contractible. Such $A_\infty$-categories behave like categories with zero morphisms only, although contractibility might not be obvious. An example of this kind is provided by $\mathcal{C}$ and $\mathcal{C}$, when $\mathcal{C}$ is unital. In this section we also collect various notions of contractibility for $A_\infty$-functors. For unital $A_\infty$-functors all these definitions become equivalent.

6.1 Proposition. Let $\mathcal{B}$ be a unital $A_\infty$-category. Let $f : \mathcal{A} \to \mathcal{B}$ be an $A_\infty$-functor. Then the following conditions are equivalent:

(C1) For any $X \in \text{Ob}\mathcal{A}$ and any $V \in \text{Ob}\mathcal{B}$ the complex $(s\mathcal{B}(Xf,V), b_1)$ is contractible;

(C2) For any $U \in \text{Ob}\mathcal{B}$ and any $Y \in \mathcal{A}$ the complex $(s\mathcal{B}(U,Yf), b_1)$ is contractible;

(C3) For any object $X$ of $\mathcal{A}$ the complex $(s\mathcal{B}(Xf,Xf), b_1)$ is acyclic;

(C4) For any object $X$ of $\mathcal{A}$ there is an element $Xv \in (s\mathcal{B})^{-2}(Xf,Xf)$ such that $Xf^i_0 = Xvb_1$;

(C5) $fi^\mathcal{B} \equiv 0 : f \to f : \mathcal{A} \to \mathcal{B}$.
Proof. Clearly, (C1) ⇒ (C3) ⇒ (C4), (C2) ⇒ (C3) and (C5) ⇒ (C4).

(C4) ⇒ (C1): Consider a $k$-linear map $(xv \otimes 1)b_2 : sB(Xf, V) \to sB(Xf, V)$ of degree $-1$. Its commutator with $b_1$ is

$$(xv \otimes 1)b_2b_1 + b_1(xv \otimes 1)b_2 = -(xvb_1 \otimes 1)b_2 = -(xfi_0^B \otimes 1)b_2 \sim 1 : sB(Xf, V) \to sB(Xf, V)$$

by [Lyu03, Lemma 7.4]. Therefore, $sB(Xf, V)$ is contractible.

(C4) ⇒ (C2): Consider a $k$-linear map $(1 \otimes xv)b_2 : sB(U, Xf) \to sB(U, Xf)$ of degree $-1$. Its commutator with $b_1$ is

$$(1 \otimes xv)b_2b_1 + b_1(1 \otimes xv)b_2 = -(1 \otimes xv b_1)b_2 = -(1 \otimes xfi_0^B)b_2 \sim 1 : sB(U, Xf) \to sB(U, Xf)$$

by [Lyu03, Lemma 7.4]. Therefore, $sB(U, Xf)$ is contractible.

(C1) ⇒ (C5): We look for an $(f, f)$-coderivation $v$ of degree $-2$ such that $vb - bv = fi^v$. We choose its 0-th component as $xv_0 : k \to (sB)^{-2}(Xf, Xf)$, $1 \mapsto xv$, where $xv$ satisfies condition (C4).

Let $n$ be a positive integer. Assume that $(v_0, v_1, \ldots, v_{n-1})$ are already found such that an $(f, f)$-coderivation $\tilde{v} = (v_0, v_1, \ldots, v_{n-1}, 0, 0, \ldots)$ of degree $-2$ satisfies equations $\lambda_m = 0$ for $m < n$, where the $(f, f)$-coderivation $\lambda$ of degree $-1$ is $\lambda = fi^v - \tilde{v}b + b\tilde{v}$. To make the induction step, we look for a map

$$v_n : sA(X_0, X_1) \otimes \cdots \otimes sA(X_{n-1}, X_n) \to sB(X_0f, X_nf),$$

such that

$$v_n b_1 - \sum_{q+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})v_n = \lambda_n.$$  

(6.1.1)

The identity $\lambda b + b\lambda = 0$ implies

$$\lambda_n d = \lambda_n b_1 + \sum_{q+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\lambda_n = 0,$$

where $d$ is the differential in the complex

$$\text{Hom}(sA(X_0, X_1) \otimes \cdots \otimes sA(X_{n-1}, X_n), sB(X_0f, X_nf))$$  

(6.1.2)

(all complexes are equipped with the differential $b_1$). Since $sB(X_0f, X_nf)$ is contractible, so is complex (6.1.2). As it is acyclic, there exists $v_n$ such that $v_n d = \lambda_n$, that is, (6.1.1) holds. Induction finishes the construction of $v$.

**6.2 Proposition.** Let $A$ be a unital $A_\infty$-category. Let $f : A \to B$ be an $A_\infty$-functor. Then the following conditions are equivalent:

(C6) For all objects $X, Y$ of $A$ the chain map $f_1 : (sA(X, Y), b_1) \to (sB(Xf, Yf), b_1)$ is homotopic to 0;
(C7) For any object X of \( \mathcal{A} \) the chain map \( f_1 : (sA(X, X), b_1) \to (sB(X f, X f), b_1) \) is homotopic to 0;

(C8) For any object X of \( \mathcal{A} \) we have \( H^*(f_1) = 0 : H^*(sA(X, X), b_1) \to H^*(sB(X f, X f), b_1) \);

(C9) For any object X of \( \mathcal{A} \) there is an element \( x w \in (sB)^{-2}(X f, X f) \) such that \( x i_0^A f_1 = x wb_1 \).

**Proof.** Clearly, (C6) \( \implies \) (C7) \( \implies \) (C8) \( \implies \) (C9).

(C9) \( \implies \) (C6): Since \( f \) and \( b \) commute, we have

\[
(1 \otimes i_0^A)f_2b_1 + (1 \otimes i_0^A)(f_1 \otimes f_1)b_2 = (1 \otimes i_0^A)b_2f_1 + (1 \otimes i_0^A)(1 \otimes b_1 + b_1 \otimes 1)f_2,
\]

\[
b_1(1 \otimes i_0^A)f_2 + (1 \otimes i_0^A)f_2b_1 - (f_1 \otimes y w)b_2b_1 - b_1(f_1 \otimes y w)b_2 = (1 \otimes i_0^A)b_2f_1 \sim f_1
\]

by [Lyu03, Lemma 7.4]. Therefore, \( f_1 \) is homotopic to 0. \( \square \)

**6.3 Proposition.** Let \( \mathcal{A}, \mathcal{B} \) be unital \( A_\infty \)-categories. Let \( f : \mathcal{A} \to \mathcal{B} \) be a unital \( A_\infty \)-functor. Then conditions (C1)–(C9) are equivalent to the following conditions:

(C10) There is an isomorphism of \( A_\infty \)-functors \( f \simeq \mathcal{O}^f : \mathcal{A} \to \mathcal{B} \), where \( \mathcal{O}^f \) is defined as follows: \( X \mathcal{O}^f = X f \), \( \mathcal{O}^f_k = 0 \) for all \( k \geq 1 \);

(C11) \( i^A f \equiv 0 : f \to f : \mathcal{A} \to \mathcal{B} \).

**Proof.** Unitality implies that (C5) and (C11) are equivalent, and that (C4) and (C9) are equivalent.

(C5) \( \implies \) (C10): Consider zero natural \( A_\infty \)-transformations \( 0 : f \to \mathcal{O}^f : \mathcal{A} \to \mathcal{B} \) and \( 0 : \mathcal{O}^f \to f : \mathcal{A} \to \mathcal{B} \). Their composition in one order \( 0 \cdot 0 = 0 : f \to f : \mathcal{A} \to \mathcal{B} \) is equivalent to \( f i_B^B = f 1_s \) by (C5). Their composition in the other order \( 0 \cdot 0 = 0 : \mathcal{O}^f \to \mathcal{O}^f : \mathcal{A} \to \mathcal{B} \) is equivalent to \( \mathcal{O}^f i_B^B \). Indeed, there exists an \((f, f)\)-coderivation \( w \) of degree \(-2\) such that \( fi_B^B = wb - bw \). In particular, \( x(fi_B^B)_0 = xfi_0^B = xwb_1 \). Consider the \((\mathcal{O}^f, \mathcal{O}^f)\)-coderivation \( v \) of degree \(-2\), given by its components \( v_0 = w_0 \) and \( v_k = 0 \) for \( k > 0 \). Then \( x(\mathcal{O}^f i_B^B)_0 = xfi_0^B = xvb_1 \) and

\[
(\mathcal{O}^f i_B^B)_n = 0 = v_n b_1 - \sum_{q+k+t=n} (1^\otimes q \otimes b_k \otimes 1^\otimes t)v_{q+k+t} = (vb - bw)_n
\]

for \( n > 0 \). Therefore, \( \mathcal{O}^f i_B^B = vb - bw \).

(C10) \( \implies \) (C4): Since \( f \) is unital, isomorphic to it \( A_\infty \)-functor \( \mathcal{O}^f \) is unital. Thus,

\[
\mathcal{O}^f i_B^B \equiv i^A \mathcal{O}^f = 0 : \mathcal{O}^f \to \mathcal{O}^f : \mathcal{A} \to \mathcal{B}.
\]

Therefore, there exists an \((\mathcal{O}^f, \mathcal{O}^f)\)-coderivation \( v \) of degree \(-2\) such that \( \mathcal{O}^f i_B^B = vb - bw \). In particular, \( xfi_0^B = x(\mathcal{O}^f i_B^B)_0 = xvb_1 \), hence, (C4) holds. \( \square \)
6.4 Definition. Let $A$ be a unital $A_{\infty}$-category. An $A_{\infty}$-functor $f : A \to B$ is contractible if it satisfies equivalent conditions (C6)–(C9) of Proposition 6.2. An $A_{\infty}$-category $A$ is contractible if complexes $(sA(X,Y), b_1)$ are contractible for all objects $X, Y$ of $A$.

A contractible $A_{\infty}$-category $A$ is unital. Indeed, $x i_0^A = 0$ are unit elements of $A$. The identity $A_{\infty}$-functor $id : A \to A$ is contractible if and only if $A$ is contractible. A unital $A_{\infty}$-functor $f$ is contractible if and only if equivalent conditions (C1)–(C11) hold.

6.5 Example. If $C$ is a unital $A_{\infty}$-category, then $\overline{C}, \overline{C}$ are contractible. Indeed, by Corollaries 5.2 and 5.3 these categories are unital. In particular, for all objects $X, Y$ of $C$ the chain map

$0 = \left( 1 \otimes Y i_0^C \right) b_0 : s\overline{C}(X, Y) \to s\overline{C}(X, Y)$

is homotopy invertible. Hence, $(s\overline{C}(X, Y), b_1) = (s\overline{C}(X, Y), b_0) = \overline{C}$ is contractible. By Proposition 6.1 (C2) $A_{\infty}$-categories $\overline{C}$ and $\overline{C}$ are contractible.

6.6 Example. Let $B$ be a full subcategory of a unital $A_{\infty}$-category $C$. Then the $A_{\infty}$-functor $\overline{f} = \left( \overline{B} \hookrightarrow \overline{C} \overrightarrow{f} D(C|B) \right)$ is contractible according to criterion (C4): for any object $X$ of $B$

$(x i_0^C \otimes x i_0^C) b_1 = (x i_0^C \otimes x i_0^C) b_1 = x i_0^C + x v_0 b_1 = x i_0^C + x v_0 b_1.$

6.7 Proposition. A unital $A_{\infty}$-category $A$ is contractible if and only if the following equivalent conditions hold:

(C0) $A$ is equivalent in $A_{\infty}^u$ to an $A_{\infty}$-category $O$, such that $O(U, V) = 0$ for all objects $U, V$ of $O$;

(C1') For all objects $X, Y$ of $A$ the complex $(sA(X,Y), b_1)$ is contractible;

(C2') For any object $X$ of $A$ the complex $(sA(X), b_1)$ is contractible;

(C3') For any object $X$ of $A$ the complex $(sA(X), b_1)$ is acyclic;

(C4') For any object $X$ of $A$ there is an element $x v \in (sA)^{-2}(X, X)$ such that $x i_0^A = x v b_1$;

(C5') $i^A \equiv 0 : id_A \to id_A : A \to A$;

(C10') There is an isomorphism of $A_{\infty}$-functors $id_A \simeq O^{id} : A \to A$, where $X O^{id} = X$, $O_k^{id} = 0$ for all $k \geq 1$.

Proof. Conditions (C1')–(C5'), (C10') are just conditions (C1)–(C10) for $f = id_A$, hence, they are equivalent to contractibility of $A$. Notice that any $O$ as in (C0) is strictly unital. (C10') $\implies$ (C0): Denote by $O^A$ the $A_{\infty}$-category, whose class of objects is $Ob A$, and $O^A(X, Y) = 0$ for all objects $X, Y \in Ob A = Ob O^A$. Let $\phi : O^A \to A$, $\psi : A \to O^A$ be the unique $A_{\infty}$-functors such that $X \phi = X$, $X \psi = X$ for all $X \in Ob A$. Then $\phi \psi = id_A$ and $\psi \phi = O^{id,A} \simeq id_A$ by (C10').

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7. The case of a contractible subcategory

Taking the quotient $D(\mathcal{E}|\mathcal{F})$ can be interpreted as contracting the full $A_\infty$-subcategory $\mathcal{F} \subset \mathcal{E}$. If $\mathcal{F}$ were already contractible, one would expect that no further contracting is required. And, in fact, if $\mathcal{E}$ is unital, we shall prove below that $D(\mathcal{E}|\mathcal{F})$ is equivalent to $\mathcal{E}$. In the proof we shall construct inductively a new $A_\infty$-structure on $\mathcal{E}$. So first of all, we consider direct limits of $A_\infty$-structures on a given graded $k$-linear quiver.

7.1 Lemma. Let $\mathcal{B}$ be a graded $k$-quiver. Let $\mathcal{A}_k$, $k \geq 1$, be a sequence of $A_\infty$-categories, whose underlying graded $k$-quiver is $\mathcal{B}$. Let $\mathcal{A}_1 \xrightarrow{f_1^{i_1}} \mathcal{A}_2 \xrightarrow{f_2^{i_2}} \ldots$ be a sequence of $A_\infty$-functors, such that $f_1^{i_1} = \text{id}_{\mathcal{A}_k}$ for all $k$, and let $N_i$, $i \geq 2$ be an increasing sequence of positive integers, such that $f_1^{i_1} = 0$ for $k \geq N_i$. Then there exists a direct (2-)limit $\mathcal{A} = \lim_{\rightarrow} f_i \mathcal{A}_i$ of this diagram, and the structure $A_\infty$-functors $p_k : \mathcal{A}_k \to \mathcal{A}$ are invertible.

Proof. If $g : \mathcal{D} \to \mathcal{E}$ is an $A_\infty$-functor, such that $g_1 = \text{id}_{\mathcal{D}}$ and $g_i = 0$ for $i = 2, \ldots, k$, then for any such $i$ there exists a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}^{\otimes i} & \xrightarrow{\text{id}_{\mathcal{D}} \otimes^i} & \mathcal{E}^{\otimes i} \\
\downarrow b_i & & \downarrow b_i \\
\mathcal{D} & \xrightarrow{\text{id}_{\mathcal{D}}} & \mathcal{E}
\end{array}
$$

which allows us to identify the $A_\infty$-operations $b_i$, $i = 1, \ldots, k$ on $\mathcal{D}$ and $\mathcal{E}$.

Due to this remark, we take $\mathcal{B}$ as the underlying $k$-quiver of $\mathcal{A}$, we set $b_1 : s\mathcal{A} \to s\mathcal{A}$ to be $b_1 : s\mathcal{A}_1 \to s\mathcal{A}_1$ and we set $b_1 : s\mathcal{A}^{\otimes i} \to s\mathcal{A}$ equal $b_1 : s\mathcal{A}_{N_i}^{\otimes i} \to s\mathcal{A}_{N_i}$. This equips $\mathcal{A}$ with...
an $A_\infty$-structure. We define $p^k$ setting its $i$-th component equal to $p^k_i = (f^k f^{k+1} \ldots f^l)$, for $l = \max(k, N_i)$.

Given an $A_\infty$-category $\mathcal{C}$ and $A_\infty$-functors $\pi^k : A_k \to \mathcal{C}$, $k = 1, \ldots$, such that $\pi^k = f^k_{\pi^{k+1}}$, then there exists the unique $A_\infty$-functor $\pi : A \to \mathcal{C}$, such that $\pi^k = p^k \pi$, defined by $\pi_i = \pi_i^{N(i)}$, $i \geq 1$. It shows that the constructed $A$ is a direct limit of the diagram $(A_i, f^i, i \geq 1)$.

\section{7.2 Lemma.} Let $\mathcal{E}$ be an $A_\infty$-category and let $\mathcal{F}$ be its full $A_\infty$-subcategory such that the complex of $k$-modules $(s\mathcal{E}(X,Y), b_1)$ is contractible provided at least one of $X, Y$ belongs to $\text{Ob} \mathcal{F}$. Denote by $D_n(\mathcal{E}|\mathcal{F})$, $n = 2,3, \ldots$ the $k$-submodule in $(s\mathcal{E})^\otimes n$, which is a sum of $s\mathcal{E}(X_0, X_1) \otimes s\mathcal{E}(X_1, X_2) \otimes \cdots \otimes s\mathcal{E}(X_{n-2}, X_{n-1}) \otimes s\mathcal{E}(X_{n-1}, X_n)$, such that at least for one $i = 0, \ldots, n$ object $X_i$ belongs to $\text{Ob} \mathcal{F}$. Then there exists an invertible $A_\infty$-functor $g : \mathcal{E} \to \mathcal{E}_{\mathcal{F}}$, such that $A_\infty$-category $(\mathcal{E}_{\mathcal{F}}, b')$ coincides with $\mathcal{E}$ as a graded differential (with respect to $b_1$) $k$-quiver (that is, $g_1 = \text{id}_\mathcal{E}$) and $(D_n(\mathcal{E}|\mathcal{F}))b'_n = 0$ for any $n > 1$.

\textbf{Proof.} We construct a chain of $A_\infty$-isomorphisms $f^i : \mathcal{E}_i \to \mathcal{E}_{i+1}$, $i = 1, \ldots$, $\mathcal{E}_1 = \mathcal{E}$ as in Lemma 7.1 and then we set $\mathcal{E}_{\mathcal{F}} = \varprojlim f^i \mathcal{E}_i$ and $g = p^1$. For the constructed $\mathcal{E}_j$ and $f^j$, $j = 1, \ldots$, the following will hold: 1) $b_k|_{D_k(\mathcal{E}|\mathcal{F})} = 0$ holds in $\mathcal{E}_j$ for $k = 2, \ldots, j$; 2) $f^i_j = 0$ for $i \neq 1, j + 1$.

Given an $A_\infty$-category $\mathcal{E}_k$ and a $k$-quiver morphism $f^k = (\text{id}_\mathcal{E}, 0, \ldots, 0, f^k_{k+1}, 0, \ldots)$ : $T \mathcal{E}_k \to s\mathcal{E}$ of degree zero, they define (following Remark 2.3) a unique $A_\infty$-category structure $\mathcal{E}_{k+1}$ on the graded $k$-quiver $\mathcal{E}$ such that $f^k$ turns into an $A_\infty$-functor $f^k : \mathcal{E}_k \to \mathcal{E}_{k+1}$. Assume $f^j : \mathcal{E}_j \to \mathcal{E}_{j+1}$, $j = 1, \ldots, k - 1$ are constructed (the reasoning is valid for $k = 1$ too).

Let us fix for any sequence $X_0, X_1, \ldots, X_n$ as in the hypothesis of the lemma an index $l(X_0, \ldots, X_n)$ such that $X_{l(X_0, \ldots, X_n)}$ belongs to $\text{Ob} \mathcal{F}$. Any choice of $f^k_{k+1} : T^{k+1} s\mathcal{E}_k \to s\mathcal{E}$ determines by Remark 2.3 an $A_\infty$-category $\mathcal{E}_{k+1}$ with the operations $b_p = b^\mathcal{E}_{k+1}_p$. Notice that the conditions $b^\mathcal{E}_{k+1}_p|_{D_2(\mathcal{E}|\mathcal{F})} = 0, \ldots, b^\mathcal{E}_{k+1}_p|_{D_{k}(\mathcal{E}|\mathcal{F})} = 0$ hold automatically: in view of $f^k_1 = \text{id}_\mathcal{E}$ in $\mathcal{E}_k$ and $f^k_j = 0$, $1 < j \leq i \leq k$ the $i$-th condition (1.0.2) shows, that $b_1, \ldots, b_k$ in $\mathcal{E}_k$ and $\mathcal{E}_{k+1}$ coincide. The $(k+1)$-th condition (1.0.1) for $\mathcal{E}_k$ on $D_{k+1}(\mathcal{E}|\mathcal{F})$ turns into

$$\sum_{r+1+t=k+1} (1^\otimes r \otimes b_1 \otimes 1^\otimes t) b_{k+1} + b_{k+1} b_1 = 0$$

(for any other summand in the sum $\sum_{r+n+t=k+1} (1^\otimes r \otimes b_n \otimes 1^\otimes t) b_{r+1+t}$ either the first or the second factor vanishes by induction). On the other hand, by the induction assumptions the $(k+1)$-th condition (1.0.2) turns on $D_{k+1}(\mathcal{E}|\mathcal{F})$ into

$$(f^k_1)^{(k+1)} b_{k+1} + f^k_{k+1} b_1 = \sum_{r+1+t=k+1} (1^\otimes r \otimes b_1 \otimes 1^\otimes t) f^k_{k+1} + b_{k+1} f^k_1 : sD_{k+1}(\mathcal{E}|\mathcal{F}) \to s\mathcal{E}_{k+1}. \quad (7.2.1)$$

We consider the condition $b_{k+1}|_{D_{k+1}(\mathcal{E}|\mathcal{F})} = 0$ holds in the $A_\infty$-category $\mathcal{E}_{k+1}$ as an equation with respect to $f^k_{k+1}$. Denote $l = \min\{l(X_0, \ldots, X_{k+1}), k\}$. Choose a contracting
homotopy, i.e. $k$-module morphism $h : s\mathcal{E}(X_i, X_{i+1}) \to s\mathcal{E}(X_i, X_{i+1})$ of degree $-1$ such that $h b_1 + b_1 h = \text{id}_{s\mathcal{E}(X_i, X_{i+1})}$. We define $f^{k}_{k+1}$ on $D_{k+1}(\mathcal{E}[F])(X_0, \ldots, X_{k+1})$ as

$$f^{k}_{k+1} = -(1 \otimes h \otimes 1 \otimes (k-l)) b_{k+1}.$$ 

On $s\mathcal{E}(X_0, X_1) \otimes \cdots \otimes s\mathcal{E}(X_{n-1}, X_n)$ such that all $X_i \notin \text{Ob} \mathcal{F}$, we set $f^{k}_{k+1} = 0$.

Compare equations (2.1) restricted to $D_{k+1}(\mathcal{E}[F])(X_0, \ldots, X_{k+1})$ with the following computation

$$f^{k}_{k+1} b_1 = -(1 \otimes h \otimes 1 \otimes (k-l)) b_{k+1} b_1 = (1 \otimes h \otimes 1 \otimes (k-l)) \sum_{r+1+t=k+1} (1 \otimes r \otimes b_1 \otimes 1 \otimes t) b_{k+1}$$

$$= \sum_{r+1+t=k+1} (1 \otimes r \otimes b_1 \otimes 1 \otimes t)(-1 \otimes h \otimes 1 \otimes (k-l)) b_{k+1}$$

$$+(1 \otimes b_1 h \otimes 1 \otimes (k-l)) b_{k+1} + (1 \otimes h b_1 \otimes 1 \otimes (k-l)) b_{k+1} = \sum_{r+1+t=k+1} (1 \otimes r \otimes b_1 \otimes 1 \otimes t) f^{k}_{k+1} + b_{k+1}.$$ 

We deduce that in $\mathcal{E}_{k+1}$ the restriction of $b_{k+1}$ to $D_{k+1}(\mathcal{E}[F])$ vanishes. Now the lemma follows from the definition of the limit morphism $g: \mathcal{E} \to \varinjlim_{F_i} \mathcal{E}_i$. \hfill \qedsymbol

**7.3 Lemma.** Let $\mathcal{F} \subset \mathcal{E}_\tau$ be a full $A_\infty$-subcategory such that $b_n|_{D_n(\mathcal{E}_\tau|\mathcal{F})}$ vanishes for all $n \geq 2$. Then the canonical strict embedding $\mathcal{F}: \mathcal{E}_\tau \to \mathcal{D}(\mathcal{E}_\tau|\mathcal{F})$ admits a splitting strict $A_\infty$-functor $\pi: \mathcal{D}(\mathcal{E}_\tau|\mathcal{F}) \to \mathcal{E}_\tau$, that is, $\mathcal{F}\pi = \text{id}_{\mathcal{E}_\tau}$. Its first component is the projection

$$\pi_1 = (s\mathcal{D}(\mathcal{E}_\tau|\mathcal{F}) \to T^+ s\mathcal{E}_\tau \xrightarrow{\text{pr}_1} s\mathcal{E}_\tau).$$

**Proof.** Denote $\mathcal{A} = \mathcal{D}(\mathcal{E}_\tau|\mathcal{F}) \cap (s^{-1} T_{\geq 2} s\mathcal{E}_\tau) = \text{Ker} \pi_1$. Then $\mathcal{D}(\mathcal{E}_\tau|\mathcal{F}) = \mathcal{E}_\tau \oplus \mathcal{A}$ as a graded $k$-quiver and $s\mathcal{A} \subset \oplus_{n \geq 2} D_n(\mathcal{E}|\mathcal{F})$. Let us check that $\pi$ is an $A_\infty$-functor, that is, $\pi_1 \otimes b_k = b_k \pi_1$ for all $k \geq 1$.

Notice that $\mathcal{A}(X, Y) = \oplus_{X_i \in \text{Ob} \mathcal{F}} s\mathcal{E}(X_i, X_1) \otimes \cdots \otimes s\mathcal{E}(X_{n-1}, Y)$, and each substring of such a tensor product of length $k \geq 2$ is in $D_k(\mathcal{E}|\mathcal{F})$. The restriction to $s\mathcal{E}_{\geq 2}$ of the equation $\pi_1 \otimes b_k = b_k \pi_1$ follows from (2.2.1) or Corollary 2.4. Both sides, restricted to $sX_i \otimes \cdots \otimes sX_k$ where $X_j$ is $\mathcal{E}_\tau$ or $\mathcal{A}$, vanish if at least one $A$ is present. Indeed, $b_k$ vanishes in that case if $k > 1$ due to (2.2.1). For $k = 1$, $(s\mathcal{A})b_1 = (s\mathcal{A})b \subset s\mathcal{A}$ as equation $\pi_1|_{\mathcal{A}} = b|_{\mathcal{A}} = \sum_{q+1+t=n} 1 \otimes q \otimes b_1 \otimes 1 \otimes t: s\mathcal{A} \to s\mathcal{A}$ shows. Other claims are clear. \hfill \qedsymbol

**7.4 Proposition.** Let $\mathcal{F}$ be a contractible full subcategory of a unital $A_\infty$-category $\mathcal{E}$. Then there exists a quasi-inverse of the canonical strict embedding $\mathcal{F}^\mathcal{E}: \mathcal{E} \to \mathcal{D}(\mathcal{E}|\mathcal{F})$ unital $A_\infty$-functor $\pi^\mathcal{E}: \mathcal{D}(\mathcal{E}|\mathcal{F}) \to \mathcal{E}$ such that $\mathcal{F}^\mathcal{E} \pi^\mathcal{E} = \text{id}_\mathcal{E}$. In particular, $\mathcal{D}(\mathcal{E}|\mathcal{F})$ is equivalent to $\mathcal{E}$.

**Proof.** First of all, we prove the statements for the full embedding $\mathcal{F} \subset \mathcal{E}_\tau$ constructed in Lemma 7.2. Since $\mathcal{E}$ is unital and $g: \mathcal{E} \to \mathcal{E}_\tau$ is invertible, the $A_\infty$-category $\mathcal{E}_\tau$ is unital by [Lyu03, Section 8.12]. Let us prove that the $A_\infty$-functor $\pi^\mathcal{E}_\tau = \pi: \mathcal{D}(\mathcal{E}_\tau|\mathcal{F}) \to \mathcal{E}_\tau$ from Lemma 7.3 is unital, that is, $\pi^\mathcal{E}_\tau = \pi^\mathcal{E}_\tau$. 

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We look for a 3-morphism

\[ v : \pi i^{E} \rightarrow D(i^{E}) \pi : \pi \rightarrow \pi : D(\mathcal{E}_\mathcal{F}) \rightarrow \mathcal{E}_\mathcal{F}. \]

Since \((\pi i^{E})_0 = i^{E}_0 = (D(i^{E}) \pi)_0\), we may take \(v_0 = 0\). Let us proceed by induction. Assume that we have already found components \((v_0, v_1, \ldots, v_{n-1})\) of \(v\) such that \(v_m\) vanishes on \((s\mathcal{E}_\mathcal{F})^{\otimes m}\) for all \(m < n\). Define a \((\pi, \pi)\)-transformation \(\tilde{v} = (v_0, v_1, \ldots, v_{n-1}, 0, 0, \ldots)\) by these components. Denote by \(\lambda\) the \((\pi, \pi)\)-transformation \(\pi i^{E} - D(i^{E}) \pi - \tilde{v}b + \overline{b}\tilde{v}\). Our assumption is that \(\lambda_m = 0\) for \(m < n\). Clearly, \(\lambda b + \overline{b}\lambda = 0\). This implies

\[ 0 = (\lambda b + \overline{b}\lambda)_n = \lambda_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes \overline{b}_1 \otimes 1^{\otimes t}) \lambda_n, \]

that is, \(\lambda_n \in \text{Hom}^{-1}((s\mathcal{E}_\mathcal{F})^{\otimes n}, s\mathcal{E}_\mathcal{F})\) is a cocycle. We wish to prove that it is a coboundary of an element \(v_n \in \text{Hom}^{-2}((s\mathcal{E}_\mathcal{F})^{\otimes n}, s\mathcal{E}_\mathcal{F})\), that is,

\[ \lambda_n = v_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes \overline{b}_1 \otimes 1^{\otimes t}) v_n = v_n d. \]

We have \((s\mathcal{E}_\mathcal{F})^{\otimes n} = \oplus_{X_i \in \{\mathcal{E}_\mathcal{F}, A\}} sX_1 \otimes \cdots \otimes sX_n\). If at least one \(A\) is present in \(sX_1 \otimes \cdots \otimes sX_n\), then this summand is contractible, hence, \(\text{Hom}(sX_1 \otimes \cdots \otimes sX_n, s\mathcal{E}_\mathcal{F})\) is contractible. Therefore, there are such \(v'_n \in \text{Hom}^{-2}(sX_1 \otimes \cdots \otimes sX_n, s\mathcal{E}_\mathcal{F})\) that \(v'_n d = \lambda_n\) on \(sX_1 \otimes \cdots \otimes sX_n\). It remains to look at the case \(sX_1 \otimes \cdots \otimes sX_n = (s\mathcal{E}_\mathcal{F})^{\otimes n}\). By restriction to this submodule, we have

\[ (\pi i^{E})_n = i^{E}_n = (D(i^{E}) \pi)_n : (s\mathcal{E}_\mathcal{F})^{\otimes n} \rightarrow s\mathcal{E}_\mathcal{F}, \]

\[ (\overline{v}b - b\overline{v})_n = 0 : (s\mathcal{E}_\mathcal{F})^{\otimes n} \rightarrow s\mathcal{E}_\mathcal{F}. \]

Therefore, \(\lambda_n\) vanishes on \((s\mathcal{E}_\mathcal{F})^{\otimes n}\), and \(v_n |_{(s\mathcal{E}_\mathcal{F})^{\otimes n}} = 0\) satisfies the equation and the induction assumptions.

Define the following natural \(A\)-transformations

\[ r : \text{id} \rightarrow \pi \overline{\mathcal{F}} : D(\mathcal{E}_\mathcal{F}|\mathcal{F}) \rightarrow D(\mathcal{E}_\mathcal{F}|\mathcal{F}), \]

\[ p : \pi \overline{\mathcal{F}} \rightarrow \text{id} : D(\mathcal{E}_\mathcal{F}|\mathcal{F}) \rightarrow D(\mathcal{E}_\mathcal{F}|\mathcal{F}) \]

via its components, restricted to \(sX_1 \otimes \cdots \otimes sX_n \subset (sD(\mathcal{E}_\mathcal{F}|\mathcal{F}))^{\otimes n}\), namely, \(r_k = i^{E}_k\) and \(p_k = i^{E}_k\) if \(sX_1 \otimes \cdots \otimes sX_n = (s\mathcal{E}_\mathcal{F})^{\otimes n}\), and \(r_k = 0\), \(p_k = 0\) otherwise. We have to check the equation \(r\overline{b} + \overline{b}r = 0\). Its restriction to \((s\mathcal{E}_\mathcal{F})^{\otimes n}\) holds because on this submodule \(\overline{b}\) can be replaced with \(b\) and \(r\) with \(i\). If \(\{X_1, \ldots, X_n\}\) contains \(A\), then all terms in the following sums vanish on \(sX_1 \otimes \cdots \otimes sX_n\), hence,

\[ \sum_{q+k+t=n} (1^{\otimes q} \otimes r_k \otimes (\pi \overline{\mathcal{F}})_1^{\otimes t}) \overline{b}_{q+1+t} + \sum_{q+k+t=n} (1^{\otimes q} \otimes \overline{b}_k \otimes 1^{\otimes t}) r_{q+1+t} = 0. \]
In the same way we prove that $p \overline{b} + \overline{b} p = 0$. Thus, $r$ and $p$ are, indeed, natural $A_\infty$-transformations.

Let us prove now that $r$ and $p$ are inverse to each other 2-morphisms, that is,

$$(r \otimes p) B_2 \equiv D(i^{E_x}) : \text{id} \to \text{id} : D(\mathcal{E}_x|\mathcal{F}) \to D(\mathcal{E}_x|\mathcal{F}),$$

$$(p \otimes r) B_2 \equiv \pi \overline{\mathcal{F}}D(i^{E_x}) : \pi \overline{\mathcal{F}} \to \pi \overline{\mathcal{F}} : D(\mathcal{E}_x|\mathcal{F}) \to D(\mathcal{E}_x|\mathcal{F}).$$

We look for a 3-morphism

$$v : (r \otimes p) B_2 \to D(i^{E_x}) : \text{id} \to \text{id} : D(\mathcal{E}_x|\mathcal{F}) \to D(\mathcal{E}_x|\mathcal{F}).$$

Let us look at the restriction of the equation

$$(r \otimes p) B_2 - D(i^{E_x}) = v \overline{b} - b v$$

(7.4.1) to $(s E F)^\otimes n$. First of all, $(\pi \overline{\mathcal{F}})_1 = 1$ on $s\mathcal{E}_x$. Summands $b_k$, contained in $B_2$, are applied to elements of $(s E F)^\otimes k$ only. Hence, they can be replaced with $b_k$. Therefore, $B_2 D(\mathcal{E}_x|\mathcal{F})$ is replaced with $B_2^{E_x}$ on $(s \mathcal{E}_x)^\otimes n$. The problem of finding a 3-morphism

$$w : (i^{E_x} \otimes i^{E_x}) B_2 \to i^{E_x} : \text{id} \to \text{id} : \mathcal{E}_x \to \mathcal{E}_x$$

is solvable. We set the restriction of $v_n$ to $(s \mathcal{E}_x)^\otimes n$ equal $v_n = w_n : (s \mathcal{E}_x)^\otimes n \to s\mathcal{E}_x$ and it solves (7.4.1) on this submodule. The restriction of equation (7.4.1) to $s\mathcal{X}_1 \otimes \cdots \otimes s\mathcal{X}_n$ that contains factor $\mathcal{A}$, can be solved by induction due to contractibility of $s\mathcal{X}_1 \otimes \cdots \otimes s\mathcal{X}_n$ as above. Thus $v$ is constructed.

Similarly a 3-morphism

$$u : (p \otimes r) B_2 \to \pi \overline{\mathcal{F}}D(i^{E_x}) : \pi \overline{\mathcal{F}} \to \pi \overline{\mathcal{F}} : D(\mathcal{E}_x|\mathcal{F}) \to D(\mathcal{E}_x|\mathcal{F})$$

is constructed.

The property $\pi \pi = \text{id}_{\mathcal{E}_x}$ is proved in Lemma 7.3.

Now we turn to the general case. The invertible $A_\infty$-functor $g : \mathcal{E} \to \mathcal{E}_x$, constructed in Lemma 7.2 is the identity on objects. Denoting $\pi^{E_x} = \pi$ as above, $g' = g|_{\mathcal{E}} : \mathcal{F} \to \mathcal{F}$, and $\pi^{E_x} = \overline{g}\pi^{E_x} g^{-1}$, we get a diagram

$\begin{array}{c}
\mathcal{F} \\
\downarrow^{g'} \\
\mathcal{E} \\
\downarrow^{g} \\
\downarrow^{\pi} \\
\overline{\mathcal{F}} \\
\overline{\mathcal{E}} \\
\overline{\mathcal{E}_x} \\
\overline{D(\mathcal{E}_x|\mathcal{F})} \\
\overline{D(\mathcal{E}_x|\mathcal{F})}
\end{array}$

All the required properties of $\pi^{E_x}$ follow immediately from those of $\pi^{E_x}$.  

\[ \square \]
7.5. Reducing a full contractible subcategory to 0. Let \( \mathcal{F} \) be a full contractible subcategory of a unital \( A_\infty \)-category \( \mathcal{E} \). Let us consider another \( A_\infty \)-category \( \mathcal{E}/_p \mathcal{F} \), whose class of objects is \( \text{Ob} \mathcal{E} \). Here \( _p \) stands for the plain quotient. The morphisms are \( \mathcal{E}/_p \mathcal{F}(X,Y) = \mathcal{E}(X,Y) \), if \( X,Y \in \text{Ob} \mathcal{E} - \text{Ob} \mathcal{F} \) and \( \mathcal{E}/_p \mathcal{F}(X,Y) = 0 \) otherwise. The component of the differential for \( \mathcal{E}/_p \mathcal{F} \)

\[
b_n : s\mathcal{E}/_p \mathcal{F}(X_0, X_1) \otimes \cdots \otimes s\mathcal{E}/_p \mathcal{F}(X_{n-1}, X_n) \rightarrow s\mathcal{E}/_p \mathcal{F}(X_0, X_n)
\]
equals \( b_n \) for \( \mathcal{E} \) if \( X_0, \ldots, X_n \notin \text{Ob} \mathcal{F} \), and vanishes otherwise.

There is a strict embedding \( e : \mathcal{E}/_p \mathcal{F} \rightarrow \mathcal{E} \), which is the identity on objects, \( e_1 = \text{id} : s\mathcal{E}/_p \mathcal{F}(X,Y) \rightarrow s\mathcal{E}(X,Y) \) if \( X,Y \notin \text{Ob} \mathcal{F} \) and vanishes otherwise. The identity \( e_1^{\otimes n} b_n^E = b_n^{\mathcal{E}/_p \mathcal{F}} e_1 \) is obvious.

If \( \mathcal{E} \) is strictly unital with the strict unit \( i^\mathcal{E} \), then \( \mathcal{E}/_p \mathcal{F} \) is strictly unital with the strict unit \( i^{\mathcal{E}/_p \mathcal{F}} \), defined as follows. Its 0-th component is \( x i_0^{\mathcal{E}/_p \mathcal{F}} = x i_0^\mathcal{E} \) if \( X \notin \text{Ob} \mathcal{F} \), and vanishes otherwise.

Let us consider the general case of a unital \( \mathcal{E} \). Each complex \( (s\mathcal{E}(X,Y), b_1) \) is contractible if \( X \) or \( Y \) is an object of \( \mathcal{F} \) due to Proposition \( \text{(C1), (C2)} \). Therefore, \( e_1 : s\mathcal{E}/_p \mathcal{F}(X,Y) \rightarrow s\mathcal{E}(X,Y) \) is homotopy invertible for all pairs \( X,Y \) of objects of \( \mathcal{E} \). Consider the following data: identity map \( h = \text{id} : \text{Ob} \mathcal{E} \rightarrow \text{Ob} \mathcal{E}/_p \mathcal{F} \) and \( \mathbb{k} \)-linear maps \( x r_0 = x p_0 = x i_0^\mathcal{E} : \mathbb{k} \rightarrow (s\mathcal{E})^{-1}(X,X) \). Clearly, \( x i_0^\mathcal{E} b_1 = 0 \) and \( (x i_0^\mathcal{E} \otimes x i_0^\mathcal{E}) b_2 - x i_0^\mathcal{E} \in \text{Im} b_1 \). Therefore, the hypotheses of Theorem 8.8 of \cite{Lyu03} are satisfied. By this theorem we conclude that \( \mathcal{E}/_p \mathcal{F} \) is unital and \( e : \mathcal{E}/_p \mathcal{F} \rightarrow \mathcal{E} \) is a unital \( A_\infty \)-equivalence.

8. An \( A_\infty \)-functor related by an \( A_\infty \)-transformation to a given \( A_\infty \)-functor

Given an \( A_\infty \)-functor \( f \) and the 0-th component \( r_0 \) of a natural \( A_\infty \)-transformation \( r : f \rightarrow g \), we construct the \( A_\infty \)-functor \( g \) and extend \( r_0 \) to the whole \( A_\infty \)-transformation \( r \). We do it under additional assumptions on \( r_0 \) which are satisfied, for instance, when \( r_0 \) is invertible. In the next section we apply this construction to the case of the quasi-isomorphisms \( r_0 \).

8.1. Assumptions. Let \( \mathcal{B}, \mathcal{C} \) be \( A_\infty \)-categories, let \( f : \mathcal{B} \rightarrow \mathcal{C} \) be an \( A_\infty \)-functor and let \( g : \text{Ob} \mathcal{B} \rightarrow \text{Ob} \mathcal{C} \) be a map. Assume that for each object \( X \in \text{Ob} \mathcal{B} \), there is an element \( r_X \in \mathcal{C}(X f, X g) \) such that \( r_X s b_1 = 0 \). For any object \( Y \in \text{Ob} \mathcal{B} \), this element determines a chain map

\[
(r_X s \otimes 1)b_2 : s\mathcal{C}(X g, Y g) \rightarrow s\mathcal{C}(X f, Y g), \quad p \mapsto (-)^p(r_X s \otimes p)b_2.
\]

Finally, we assume that if any chain complex of \( \mathbb{k} \)-modules of the form \( N = s\mathcal{B}(X_0, X_1) \otimes_k \cdots \otimes_k s\mathcal{B}(X_{n-1}, X_n), n \geq 0 \), the following chain map

\[
u = \text{Hom}(N, (r_X s \otimes 1)b_2) : \text{Hom}_k^-(N, s\mathcal{C}(X g, Y g)) \rightarrow \text{Hom}_k^-(N, s\mathcal{C}(X f, Y g)) \quad (8.1.1)
\]
is a quasi-isomorphism. For $n = 0$, we have $N = k$ and the 0-th condition means that $(r_X s \otimes 1)b_2$ is a quasi-isomorphism.

8.2 Proposition. Under the above assumptions, the map $g : \text{Ob} \mathcal{B} \to \text{Ob} \mathcal{C}$ extends to an $A_\infty$-functor $g : \mathcal{B} \to \mathcal{C}$. There exists a natural $A_\infty$-transformation $r : f \to g : \mathcal{B} \to \mathcal{C}$ such that its 0-th component is $r_0 : k \to s\mathcal{C}(X f, X g)$, $1 \mapsto r_X s$.

All statements in this section (existence of the $A_\infty$-functor $g$ and the natural $A_\infty$-transformation $r$, their uniqueness in a certain sense, unitality of $g$ and invertibility of $r$) are proved in a similar fashion, using acyclicity of the cone of the quasi-isomorphism $u$.

Proof. The components $g_0 = 0$ and $r_0$ are already known. Let us build the remaining components by induction. Assume that we have already found components $g_m, r_m$ of the sought for $g, r$ for $m < n$, such that the equations

$$
-g_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) g_n = \sum_{l>1; i_1 + \cdots + i_l = n} (g_{i_1} \otimes g_{i_2} \otimes \cdots \otimes g_{i_l})b_l - \sum_{k>1; q+k+t=n} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) g_{q+1+t}, \quad (8.2.1)
$$

and

$$
-r_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) r_n + (r_0 \otimes g_n)b_2 = - \sum_{k<n; (q,k,t) \neq (0,0,1)} (f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes r_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_l})b_{q+1+t} - \sum_{k>1; q+k+t=n} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) r_{q+1+t}. \quad (8.2.2)
$$

Let us prove that there exist $k$-linear maps $g_n : s\mathcal{B}(X_0, X_1) \otimes_k \cdots \otimes_k s\mathcal{B}(X_{n-1}, X_n) \to s\mathcal{B}(X_0g, X_ng)$, $r_n : s\mathcal{B}(X_0, X_1) \otimes_k \cdots \otimes_k s\mathcal{B}(X_{n-1}, X_n) \to s\mathcal{B}(X_0f, X_ng)$ which solve the above equations.

Since the map $u = \text{Hom}(N, (r_X s \otimes 1)b_2)$ from (8.1.1) is a quasi-isomorphism, $\text{Cone}(u)$ is acyclic. As a differential graded $k$-module

$$
\text{Cone}(u) = \text{Hom}_k^*(N, s\mathcal{C}(X_0f, X_ng)) \oplus \text{Hom}_k^*(N, s\mathcal{C}(X_0g, X_ng))[1],
$$

$$
(v, p)d = (vd + pu, -pd),
$$

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where \( N = sB(X_0, X_1) \otimes_\mathbb{k} \cdots \otimes_\mathbb{k} sB(X_{n-1}, X_n) \). Denote by \( \lambda_n \in \text{Hom}_k^1(N, s\mathbb{C}(X_0g, X_ng)) \) the right hand side of (8.2.1) and by \( \nu_n \in \text{Hom}_k^0(N, s\mathbb{C}(X_0f, X_ng)) \) the right hand side of (8.2.2). Equations (8.2.1) and (8.2.2) mean that \((r_n, g_n) \in \text{Cone}^0(u)\). Since \( \text{Cone}(u) \) is acyclic, such a pair \((r_n, g_n) \in \text{Cone}^1(u)\) exists if and only if \((\nu_n, \lambda_n) \in \text{Cone}^0(u)\) is a cycle, that is, equations \(-\lambda_n d = 0, \nu_n d + \lambda_n u = 0\) are satisfied. Let us verify them now.

Introduce a cocategory homomorphism \( \tilde{g} : TsB \rightarrow Ts\mathbb{C} \) by its components \((g_1, \ldots, g_{n-1}, 0, 0, \ldots)\) (these are already known). The map \( \lambda = \tilde{g}b - b\tilde{g} \) is a \((\tilde{g}, \tilde{g})\)-coderivation. Its components \((\tilde{g}b - b\tilde{g})_k \) vanish for \( 0 \leq k \leq n - 1 \), and

\[
(\tilde{g}b - b\tilde{g})_n = \sum_{l>1;i_1+\cdots+i_l=n} (g_{i_1} g_{i_2} \cdots g_{i_l})b_l - \sum_{k>1; q+k+t=n} (1^\otimes q \otimes b_k \otimes 1^\otimes t)g_{q+1+t} = \lambda_n
\]

is the right hand side of (8.2.1). This coderivation commutes with \( b \) since \((\tilde{g}b - b\tilde{g})b + b(\tilde{g}b - b\tilde{g}) = 0\). Applying this identity to \( T^n sB \) and composing it with \( \text{pr}_1 : Ts\mathbb{C} \rightarrow s\mathbb{C} \), we get an identity

\[
(\tilde{g}b - b\tilde{g})_n b_1 + \sum_{q+1+t=n} (1^\otimes q \otimes b_1 \otimes 1^\otimes t)(\tilde{g}b - b\tilde{g})_n = 0,
\]

which means precisely that \( \lambda_n d = 0 \).

Introduce a \((f, \tilde{g})\)-coderivation \( \tilde{r} : TsB \rightarrow Ts\mathbb{C} \) by its components \((r_0, r_1, \ldots, r_{n-1}, 0, 0, \ldots)\) (these are already known). The map \( \tilde{r}b + b\tilde{r} \) has the following property:

\[
(\tilde{r}b + b\tilde{r}) \Delta = \Delta [f \otimes (\tilde{r}b + b\tilde{r})] + (\tilde{r}b + b\tilde{r}) \otimes \tilde{g} + \tilde{r} \otimes (\tilde{g}b - b\tilde{g})].
\]

Let us construct a map \( \theta = [\tilde{r} \otimes (\tilde{g}b - b\tilde{g})] \theta : TsB \rightarrow Ts\mathbb{C} \) for the data \( f \xrightarrow{\tilde{r}} \tilde{g} \xrightarrow{\tilde{g}b-b\tilde{g}} \tilde{g} : TsB \rightarrow Ts\mathbb{C} \) as in Section 3 of [Lyu03] (see also Section []). Its components \( \theta_{kl} = \theta \big|_{T^k sB} \text{pr}_l : T^k sB \rightarrow T^l s\mathbb{C} \) are given by formula (1.4)

\[
\theta_{kl} = \sum_{a+b+c+\gamma+2=t} f_{ao} \otimes \tilde{r}_j \otimes \tilde{g}_{c\beta} \otimes (\tilde{g}b - b\tilde{g})_t \otimes \tilde{g}_{\epsilon\gamma}.
\]

By Proposition 3.1 of [Lyu03] the map \( \theta \) satisfies the equation

\[
\theta \Delta = \Delta [f \otimes \theta + \theta \otimes \tilde{g} + \tilde{r} \otimes (\tilde{g}b - b\tilde{g})].
\]

Therefore, \( \nu = -\tilde{r}b - b\tilde{r} + [\tilde{r} \otimes (\tilde{g}b - b\tilde{g})] \theta : TsB \rightarrow Ts\mathbb{C} \) is a \((f, \tilde{g})\)-coderivation. Since \( \theta_{k1} = 0 \) for all \( k \), the components \( \nu_k \) vanish for \( k < n \), and

\[
\nu_n = -\sum_{k<n; (q,k,t) \neq (0,0,1)} (f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes r_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) b_{q+1+t} - \sum_{k>1; q+k+t=n} (1^\otimes q \otimes b_k \otimes 1^\otimes t) r_{q+1+t}
\]

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which is the right hand side of (8.2.2). We have an obvious identity

$$\nu b - b \nu = (-\bar{r}b - \bar{r} \theta) b - b(-\bar{r}b - \bar{r} \theta) = \theta b - b \theta.$$  

Applying this identity to $T^m sB$ and composing it with $\text{pr}_1 : TsC \to sC$, we get an identity

$$\nu_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) \nu_n = [r_0 \otimes (\tilde{g}b - b\tilde{g})_n] b_2,$$

since $[\bar{r} \otimes (\tilde{g}b - b\tilde{g})] \theta_{nt}$ vanishes for $l \neq 2$, and equals $r_0 \otimes (\tilde{g}b - b\tilde{g})_n$ for $l = 2$. The above equation means precisely that $\nu_n d = -\lambda_n u$. Indeed, $(\tilde{g}b - b\tilde{g})_n(r_{X_0} \otimes 1) = -r_0 \otimes (\tilde{g}b - b\tilde{g})_n$. Thus, proposition is proved by induction. \hfill \Box

8.3. Transformations between the constructed $A_{\infty}$-functors. Let $\mathcal{B}, \mathcal{C}$ be $A_{\infty}$-categories, let $f : \mathcal{B} \to \mathcal{C}$ be an $A_{\infty}$-functor, let $g : \text{Ob} \mathcal{B} \to \text{Ob} \mathcal{C}$ be a map, and assume that for each object $X \in \text{Ob} \mathcal{B}$ there is a map $r_0 : k \to (s\mathcal{C})^{-1}(Xf, Xg)$ such that $r_0 b_1 = 0$. Let the assumptions of Section 8.1 hold. Let $g, g' : \mathcal{B} \to \mathcal{C}$ be two $A_{\infty}$-functors, whose underlying map is the given $g : \text{Ob} \mathcal{B} \to \text{Ob} \mathcal{C}$. Let $r : f \to g : \mathcal{B} \to \mathcal{C}$, $r' : f \to g' : \mathcal{B} \to \mathcal{C}$ be natural $A_{\infty}$-transformations, whose 0-th component $r_0 = r'_0$ is the given map $r_0 : k \to (s\mathcal{C})^{-1}(Xf, Xg)$.

8.4 Proposition. Under the above assumptions, there exists a natural $A_{\infty}$-transformation $p : g \to g' : \mathcal{B} \to \mathcal{C}$ such that $r' = (f \longrightarrow g \longrightarrow p)$ in the 2-category $A_{\infty}$.

Proof. Let us construct a $(g, g')$-coderivation $p : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree $-1$ and a $(f, g')$-coderivation $v : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree $-2$ such that

$$pb + bp = 0,$$

$$(r \otimes p)B_2 - r' = [v, b],$$

that is, $p : g \to g' : \mathcal{B} \to \mathcal{C}$ is a 2-morphism and $v : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}$ is a 3-morphism. Let us build the components of $p$ and $v$ by induction. We have $p_k = 0$ and $v_k = 0$ for $k < 0$. Given non-negative $n$, assume that we have already found components $p_m, v_m$ of the sought for $p, v$ for $m < n$, such that equations

$$(pb + bp) \text{pr}_1 = 0 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0g, X_mg),$$

$$\{(r \otimes p)B_2 - r' - [v, b]\} \text{pr}_1 = 0 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0f, X_mg)$$

are satisfied for all $m < n$. Under these assumptions we will find such $p_n, v_n$ that the above equations are satisfied for $m = n$. Notice that for $m = n = 0$ the source complexes reduce to $k$. Let us write down these equations explicitly. The terms which contain unknown maps $p_n, v_n$ are singled out on the left hand side. The right hand side consists
of already known terms:

\[- p_n b_1 - \sum_{q+1+t=n} (1^\otimes q \otimes b_1 \otimes 1^\otimes t)p_n \]

\[= \sum_{q+k+t=n} (1^\otimes q \otimes b_k \otimes 1^\otimes t)p_{q+1+t} + \sum_{i_1 + \cdots + i_q + k + j_1 + \cdots + j_t = n} (g_{i_1} \otimes \cdots \otimes g_{i_q} \otimes p_k \otimes g_{j_1}' \otimes \cdots \otimes g_{j_t}')b_{q+1+t}, \]

(8.4.1)

\[v_n b_1 - \sum_{q+1+t=n} (1^\otimes q \otimes b_1 \otimes 1^\otimes t)v_n - (r_0 \otimes p_n)b_2 \]

\[= \sum_{a_1 + \cdots + a_q + j + c_1 + \cdots + c_\beta + k + e_1 + \cdots + e_\gamma = n} (f_{a_1} \otimes \cdots \otimes f_{a_q} \otimes r_j \otimes g_{c_1} \otimes \cdots \otimes g_{e_1} \otimes \cdots \otimes g_{e_\gamma})b_{a+\beta+\gamma+2-r'_n} \]

\[- \sum_{i_1 + \cdots + i_q + k + j_1 + \cdots + j_t = n} (f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes v_k \otimes g_{j_1}' \otimes \cdots \otimes g_{j_t}')b_{q+1+t} + \sum_{q+k+t=n} (1^\otimes q \otimes b_k \otimes 1^\otimes t)v_{q+1+t}. \]

(8.4.2)

The components of \((r \otimes p)B_2\) are computed by formula (5.1.3) of [Lyu03]:

\[[(r \otimes p)B_2]_n = \sum_l (r \otimes p)\theta_n b_l. \]

Denote by \(\lambda_n \in \text{Hom}_k^0(N, s\mathcal{C}(X_0g, X_ng))\) the right hand side of (8.4.1) and by \(\nu_n \in \text{Hom}_k^{-1}(N, s\mathcal{C}(X_0f, X_ng))\) the right hand side of (8.4.2), where \(N = s\mathcal{B}(X_0, X_1) \otimes_k \cdots \otimes_k s\mathcal{B}(X_{n-1}, X_n)\). In particular, \(N = k\) for \(n = 0\). Equations (8.4.1) and (8.4.2) mean that \((\nu_n, p_n)d = (\nu_n, \lambda_n) \in \text{Cone}^{-1}(u)\). Since \(\text{Cone}(u)\) is acyclic, such a pair \((\nu_n, p_n)\) exists if and only if \((\nu_n, \lambda_n) \in \text{Cone}^{-1}(u)\) is a cycle, that is, equations \(-\lambda_n d = 0, \nu_n d + \lambda_n = 0\) are satisfied. Let us verify them now.

Introduce a \((g, g')\)-coderivation \(\tilde{p} : Ts\mathcal{B} \to Ts\mathcal{C}\) of degree \(-1\) by its components \((p_0, p_1, \ldots, p_{n-1}, 0, 0, \ldots)\). The commutator \(\lambda = \tilde{p}b + b\tilde{p}\) is also a \((g, g')\)-coderivation (of degree \(0\)). Its components \(\lambda_m\) vanish for \(m < n\). The component \(\lambda_n = (\tilde{p}b + b\tilde{p})_n\) is the right hand side of (8.4.1). Consider the identity

\[(\tilde{p}b + b\tilde{p})b - b(\tilde{p}b + b\tilde{p}) = 0.\]

Applying this identity to \(T^n s\mathcal{B}\) and composing it with \(\text{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}\), we get an identity

\[\lambda_n b_1 - \sum_{q+1+t=n} (1^\otimes q \otimes b_1 \otimes 1^\otimes t)\lambda_n = 0, \]

that is, \(\lambda_n d = 0\).
Introduce a \((f,g')\)-coderivation \(\tilde{v} : Tn\mathcal{B} \to Ts\mathcal{C}\) of degree \(-2\) by its components \((v_0, v_1, \ldots, v_{n-1}, 0, 0, \ldots)\). All summands of the map \(\nu = (r \otimes \tilde{p})B_2 - r' - [\tilde{v}, b]\) are \((f,g')\)-coderivations of degree \(-1\). Hence, the same holds for \(\nu\). The components \(\nu_m\) vanish for \(m < n\). The component \(\nu_n\) is the right hand side of (8.4.2). Consider the commutator

\[
[\nu, b] = \nu B_1 = (r \otimes \tilde{p})B_2B_1 - r'B_1 - \tilde{v}B_1B_1 = -(r \otimes \tilde{p})(1 \otimes B_1 + B_1 \otimes 1)B_2 \\
= -(r \otimes \tilde{p}B_1)B_2 = -(r \otimes \lambda)B_2.
\]

Applying this identity to \(T^ns\mathcal{B}\) and composing it with \(\text{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}\) we get an identity

\[
\nu_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\nu_n = -(r_0 \otimes \lambda_n)b_2,
\]

that is, \(\nu_d = -\lambda u\). Thus, the proposition is proved by induction. \(\square\)

**8.5 Proposition** (Uniqueness of the transformations). Let assumptions of Sections 8.1 and 8.3 hold. The natural \(A_\infty\)-transformation \(p : g \to g' : \mathcal{B} \to \mathcal{C}\), such that \(r' = (f \longrightarrow g \longrightarrow p \longrightarrow g')\) in the 2-category \(A_\infty\), is unique up to an equivalence.

**Proof.** Assume that we have two such 2-morphisms \(p,q : g \to g' : \mathcal{B} \to \mathcal{C}\) and two 3-morphisms \(v : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}\) and \(w : (r \otimes q)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}\). We are looking for a 3-morphism \(x : p \to q : g \to g' : \mathcal{B} \to \mathcal{C}\) and the following 4-morphism, whose source depends on \(x\). Assuming that \(p - q = xB_1\) we deduce that

\[
-(r \otimes x)B_2B_1 = (r \otimes xB_1)B_2 = (r \otimes p)B_2 - (r \otimes q)B_2.
\]

Since \((r \otimes q)B_2 - r' = wB_1\), we find out that \((r \otimes p)B_2 - r' = [w - (r \otimes x)B_2]B_1\). Thus, we have two 3-morphisms with the common source and target \(v, w - (r \otimes x)B_2 : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}\). We are looking for a 4-morphism

\[
z : w - (r \otimes x)B_2 \to v : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C},
\]

as well as for \(x\). In other terms, we have to find coderivations \(x\) of degree \(-2\) and \(z\) of degree \(-3\) such that the following equations hold:

\[
p - q = xb - bx,
\]

\[
w - (r \otimes x)B_2 - v = zb + bz.
\]

Let us build the components of \(x\) and \(z\) by induction. We have \(x_k = 0\) and \(z_k = 0\) for \(k < 0\). Given non-negative \(n\), assume that we have already found components \(x_m, z_m\) of the sought \(x, z\) for \(m < n\), such that equations

\[
(p - q) \text{pr}_1 = (xb - bx) \text{pr}_1 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0g, X_mg),
\]

\[
[w - (r \otimes x)B_2 - v] \text{pr}_1 = (zb + bz) \text{pr}_1 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0f, X_mg)
\]

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are satisfied for all \( m < n \). Under these assumptions, we will find such \( x_n, z_n \) that the above equations are satisfied for \( m = n \). Notice that for \( m = n = 0 \) the source complexes reduce to \( k \). Let us write down these equations explicitly. The terms which contain unknown maps \( x_n, z_n \) are singled out on the left hand side. The right hand side consists of already known terms:

\[
- x_nb_1 + \sum_{\alpha+1+\beta=n} (1^\otimes\alpha \otimes b_1 \otimes 1^\otimes\beta) x_n = q_n - p_n \\
+ \sum_{i_1 + \cdots + i_n + k + j_1 + \cdots + j_n = n} (g_{i_1} \otimes \cdots \otimes g_{i_n} \otimes x_{k} \otimes g'_{j_1} \otimes \cdots \otimes g'_{j_n}) b_{\alpha+1+\beta} - \sum_{k+1} (1^\otimes\alpha \otimes b_k \otimes 1^\otimes\beta) x_{\alpha+1+\beta},
\]

(8.5.1)

\[
z_nb_1 + \sum_{\alpha+1+\beta=n} (1^\otimes\alpha \otimes b_1 \otimes 1^\otimes\beta) z_n + (r_0 \otimes x_n) b_2 \\
= - \sum_{a_1 + \cdots + a_n + j + c_1 + \cdots + c_n = n} \sum_{k<n} \sum_{s} (f_{a_1} \otimes \cdots \otimes f_{a_n} \otimes r_j \otimes g_{c_1} \otimes \cdots \otimes g_{c_n} \otimes x_{k} \otimes g'_{j_1} \otimes \cdots \otimes g'_{j_n}) b_{\alpha+1+\beta+\gamma+2}
\\
+ w_n - v_n - \sum_{i_1 + \cdots + i_n + k + j_1 + \cdots + j_n = n} \sum_{k<n} (f_{i_1} \otimes \cdots \otimes f_{i_n} \otimes z_k \otimes g'_{j_1} \otimes \cdots \otimes g'_{j_n}) b_{\alpha+1+\beta}
\\
- \sum_{k+1} (1^\otimes\alpha \otimes b_k \otimes 1^\otimes\beta) z_{\alpha+1+\beta}. \quad (8.5.2)
\]

Denote by \( \lambda_n \in \text{Hom}_{k}^{-1}(N, s\mathcal{C}(X_0g, X_ng)) \) the right hand side of (8.5.1) and by \( \nu_n \in \text{Hom}_{k}^{-2}(N, s\mathcal{C}(X_0f, X_ng)) \) the right hand side of (8.5.2), where \( N = s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes \otimes k \) \( s\mathcal{B}(X_{n-1}, X_n) \). In particular, \( N = k \) for \( n = 0 \). Equations (8.5.1) and (8.5.2) mean that \( (z_n, x_n)d = (\nu_n, \lambda_n) \in \text{Cone}^{-2}(u) \). Since \( \text{Cone}(u) \) is acyclic, such a pair \( (z_n, x_n) \in \text{Cone}^{-3}(u) \) exists if and only if \( (\nu_n, \lambda_n) \in \text{Cone}^{-2}(u) \) is a cycle, that is, equations \(-\lambda_n d = 0, \nu_n d + \lambda_n u = 0 \) are satisfied. Let us verify them now.

Introduce a \((g, g')\)-coderivation \( \tilde{x} : Ts\mathcal{B} \to Ts\mathcal{C} \) of degree \(-2\) by its components \((x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots)\). The commutator \( \tilde{x}b - b\tilde{x} \) is also a \((g, g')\)-coderivation (of degree \(-1\)). Hence, the map \( \lambda = -p + q + \tilde{x}b - b\tilde{x} \) is also a \((g, g')\)-coderivation of degree \(-1\). Its components \( \lambda_m \) vanish for \( m < n \). The component \( \lambda_n \) is the right hand side of (8.5.1). Consider the identity

\[
\lambda B_1 = -pB_1 + qB_1 + \tilde{x}B_1 B_1 = 0.
\]

Applying this identity to \( T^n s\mathcal{B} \) and composing it with \( pr_1 : Ts\mathcal{C} \to s\mathcal{C} \) we get an identity

\[
\lambda_n b_1 + \sum_{\alpha+1+\beta=n} (1^\otimes\alpha \otimes b_1 \otimes 1^\otimes\beta) \lambda_n = 0,
\]

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that is, \( \lambda_n d = 0 \).

Introduce a \((f, g')\)-coderivation \( \tilde{z} : TsB \to TsC \) of degree \(-3\) by its components 
\( (z_0, z_1, \ldots, z_{n-1}, 0, 0, \ldots) \). All summands of the map \( \nu = w - (r \otimes \tilde{x})B_2 - v - [\tilde{z}, b] \) are 
\((f, g')\)-coderivations of degree \(-2\). Hence, the same holds for \( \nu \). The components \( \nu_m \) 
vanish for \( m < n \). The component \( \nu_n \) is the right hand side of (8.5.2). Consider the commutator

\[
[\nu, b] = \nu B_1 = wB_1 - (r \otimes \tilde{x})B_2B_1 - vB_1 - \tilde{z}B_1B_1
\]

\( = (r \otimes q)B_2 - r' + (r \otimes \tilde{x}B_1)B_2 - (r \otimes p)B_2 + r' = [r \otimes (q - p + \tilde{x}B_1)]B_2 = (r \otimes \lambda)B_2. \)

Applying this identity to \( T^n sB \) and composing it with \( pr_1 : TsC \to sC \) we get an identity

\[
\nu_n b_1 - \sum_{\alpha + 1 + \beta = n} (1^{\otimes \alpha} \otimes b_1 \otimes 1^{\otimes \beta}) \nu_n = (r_0 \otimes \lambda_n)b_2,
\]

that is, \( \nu_n d = -\lambda_n u \). Thus, the proposition is proved by induction. \( \square \)

**8.6 Corollary.** In the assumptions of Proposition 8.7, let \( C \) be unital. Then the constructed 2-morphism \( p : g \to g' : B \to C \) is invertible in \( A_\infty \).

**Proof.** Exchanging the pairs \((g, r)\) and \((g', r')\), we see that there is a 2-morphism \( t : g' \to g : B \to C \), such that \( r = (f \xrightarrow{t} g' \xrightarrow{t} g) \). Therefore, \( r = (f \xrightarrow{r'} g \xrightarrow{r't} g) \). Since \( C \) is unital, there is a unit 2-endomorphism \( 1_g s = g^tC : g \to g : B \to C \). It satisfies the equation \( r = (f \xrightarrow{r} g \xrightarrow{1_g s} g) \). The uniqueness proved in Proposition 8.5 implies that \( p \cdot t = 1_g s \). Similarly, \( t \cdot p = 1_g s \). \( \square \)

**8.7 Proposition (Unitality of \( A_\infty \)-functors).** Let the assumptions of Section 8.1 hold. If \( A_\infty \)-categories \( B, C \) are unital and \( A_\infty \)-functor \( f : B \to C \) is unital, then the \( A_\infty \)-functor \( g : B \to C \) constructed in Proposition 8.2 is unital as well.

**Proof.** We are given a 2-morphism \( r : f \to g : B \to C \) and a 3-morphism \( v : fiC \to iB f : f \to f : B \to C \). We are looking for a 3-morphism \( w : giC \to iB g : g \to g : B \to C \) and a 4-morphism \( x \), whose target depends on \( w \). Let us describe \( x \) now for the above \( w \).

We have the following 3-morphisms

\[
(r \otimes iC)M_{11} : (fiC \otimes r)B_2 \to (r \otimes giC)B_2 : f \to g : B \to C,
\]

\[
(v \otimes r)B_2 : (fiC \otimes r)B_2 \to (iB f \otimes r)B_2 : f \to g : B \to C,
\]

\[
(r \otimes w)B_2 : (r \otimes iB g)B_2 \to (r \otimes giC)B_2 : f \to g : B \to C,
\]

\[
(iB \otimes r)M_{11} : (r \otimes iB g)B_2 \to (iB f \otimes r)B_2 : f \to g : B \to C.
\]

Indeed, for the cocategory homomorphism \( M : TsA_\infty \otimes TsA_\infty \to TsA_\infty \) we have an equation \( MB = (1 \otimes B + B \otimes 1)M \), see Section 6 of [Lyu03]. It implies, in particular,
that
\[(r \otimes i^c)M_{11}B_1 - (fi^c \otimes r)B_2 + (r \otimes gi^c)B_2 = (r \otimes i^c)(1 \otimes B_1 + B_1 \otimes 1)M_{11} = 0,\]
\[(v \otimes r)B_2B_1 = (vB_1 \otimes r)B_2 = (fi^c \otimes r)B_2 - (i^g f \otimes r)B_2,\]
\[(r \otimes w)B_2B_1 = -(r \otimes wB_1)B_2 = (r \otimes i^B g)B_2 - (r \otimes gi^c)B_2,\]
\[(i^B \otimes r)M_{11}B_1 - (r \otimes i^B g)B_2 + (i^B f \otimes r)B_2 = (i^B \otimes r)(1 \otimes B_1 + B_1 \otimes 1)M_{11} = 0.\]

Linear combinations of the above maps form 3-morphisms with the same source and target
\[(r \otimes i^c)M_{11} - (v \otimes r)B_2 : (i^B f \otimes r)B_2 \rightarrow (r \otimes gi^c)B_2 : f \rightarrow g : B \rightarrow \mathcal{C},\]
\[(r \otimes w)B_2 - (i^B \otimes r)M_{11} : (i^B f \otimes r)B_2 \rightarrow (r \otimes gi^c)B_2 : f \rightarrow g : B \rightarrow \mathcal{C}.\]

We are looking for a 4-morphism between the above 3-morphisms
\[x : (r \otimes i^c)M_{11} - (v \otimes r)B_2 \rightarrow (r \otimes w)B_2 - (i^B \otimes r)M_{11} : (i^B f \otimes r)B_2 \rightarrow (r \otimes gi^c)B_2 : f \rightarrow g : B \rightarrow \mathcal{C},\]
as well as for \(w\).

In other words, we have to find a \((g, g)\)-coderivation \(w\) of degree \(-2\) and an \((f, g)\)-coderivation \(x\) of degree \(-3\) such that the following equations hold:
\[-wb + bw = i^B g - gi^c,\]
\[xb + bx = (r \otimes i^c)M_{11} - (v \otimes r)B_2 - (r \otimes w)B_2 + (i^B \otimes r)M_{11}.\]

Let us construct the components of \(w\) and \(x\) by induction. We have \(w_k = 0\) and \(x_k = 0\) for \(k < 0\). Given non-negative \(n\), assume that we have already found components \(w_m, x_m\) of the sought for \(x, z\) for \(m < n\), such that the above equations restricted to \(T^m s \mathcal{B}\) are satisfied for all \(m < n\). Under these assumptions, we will find such \(w_n, x_n\) for \(m = n\). Let us write down these equations explicitly. The terms which contain unknown maps \(w_n, x_n\) are singled out on the left hand side. The right hand side consists of already known terms:
\[-w_n b_1 + \sum_{q+1+t=n} (1^q \otimes b_1 \otimes 1^t)w_n = \sum_{i_1 + \cdots + i_q + k + j_1 + \cdots + j_t = n} (g_{i_1} \otimes \cdots \otimes g_{i_q} \otimes w_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_t})b_{q+1+t} + \sum_{q+k+t=n} (1^q \otimes i^B_k \otimes 1^t)w_{q+1+t} + \sum_{q+k+t=n} (1^q \otimes i^c_k \otimes 1^t)g_{q+1} - i^c_n,\]
\[(8.7.1)\]
\[x_nb_1 + \sum_{q+1+t=n} (1^\otimes q \otimes b_1 \otimes 1^\otimes t)x_n + (r_0 \otimes w_n)b_2\]

\[= - \sum_{k<n} \left( f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes x_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_t} \right) b_{q+1+t} - \sum_{q+k+t=n} (1^\otimes q \otimes b_k \otimes 1^\otimes t)x_{q+1+t} \]

\[- \sum_{k<n} \left( f_{a_1} \otimes \cdots \otimes f_{a_n} \otimes v_j \otimes f_{c_1} \otimes \cdots \otimes f_{c_{\beta}} \otimes r_{k} \otimes g_{e_1} \otimes \cdots \otimes g_{e_\gamma} \right) b_{a_1 + \beta + \gamma + 2} \]

\[- + \sum_{i_1 + \cdots + i_q + k + j_1 + \cdots + j_t = n} \left( f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes r_j \otimes g_{c_1} \otimes \cdots \otimes g_{c_{\beta}} \otimes w_k \otimes g_{e_1} \otimes \cdots \otimes g_{e_\gamma} \right) b_{a_1 + \beta + \gamma + 2} \]

\[+ \sum_{i_1 + \cdots + i_q + k + j_1 + \cdots + j_t = n} \left( f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes r_j \otimes g_{c_1} \otimes \cdots \otimes g_{j_t} \right) i_t^{\otimes} \]

\[+ \sum_{q+k+t=n} (1^\otimes q \otimes i_k^{\otimes} \otimes 1^\otimes t)r_{q+1+t}. \quad (8.7.2)\]

Denote by \(\lambda_n \in \text{Hom}_k^{-1}(N, s\mathcal{C}(X_0 g, X_n g))\) the right hand side of (8.7.1) and by \(\nu_n \in \text{Hom}_k^{-2}(N, s\mathcal{C}(X_0 f, X_n g))\) the right hand side of (8.7.2), where \(N = s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes \mathcal{B}(X_{n-1}, X_n)\). Equations (8.7.1) and (8.7.2) mean that \((x_n, w_n) = (\nu_n, \lambda_n) \in \text{Cone}^{-2}(u)\). Since \(\text{Cone}(u)\) is acyclic, such a pair \((x_n, w_n) \in \text{Cone}^{-3}(u)\) exists if and only if \((\nu_n, \lambda_n) \in \text{Cone}^{-2}(u)\) is a cycle, that is, equations \(-\lambda_n d = 0, \nu_n d + \lambda_n u = 0\) are satisfied. Let us verify them now.

Introduce a \((g, g)\)-coderivation \(\tilde{w} : Ts\mathcal{B} \to Ts\mathcal{C}\) of degree \(-2\) by its components \((w_0, w_1, \ldots, w_{n-1}, 0, 0, \ldots)\). Hence, the map \(\lambda = \tilde{w} b - b\tilde{w} + i_s^B g - g\tilde{i}^C\) is also a \((g, g)\)-coderivation of degree \(-1\). Its components \(\lambda_m\) vanish for \(m < n\). The component \(\lambda_n\) is the right hand side of (8.7.1). Consider the identity

\[\lambda B_1 = \tilde{w} B_1 B_1 + (i_s^B g) B_1 - (g\tilde{i}^C) B_1 = 0.\]

Applying this identity to \(T^n s\mathcal{B}\) and composing it with \(pr_1 : Ts\mathcal{C} \to s\mathcal{C}\), we get an identity

\[\lambda_n b_1 + \sum_{q+1+t=n} (1^\otimes q \otimes b_1 \otimes 1^\otimes t)\lambda_n = 0,\]

that is, \(\lambda_n d = 0\).

Introduce a \((f, g)\)-coderivation \(\tilde{x} : Ts\mathcal{B} \to Ts\mathcal{C}\) of degree \(-3\) by its components \((x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots)\). All summands of the map

\[\nu = -\tilde{x} B_1 + (r \otimes i_s^C) M_{11} - (v \otimes r) B_2 - (r \otimes \tilde{w}) B_2 + (i_s^B \otimes r) M_{11}\]

are \((f, g)\)-coderivations of degree \(-2\). Hence, the same holds for \(\nu\). The components \(\nu_m\) vanish for \(m < n\). The component \(\nu_n\) is the right hand side of (8.7.2). Consider its differential

\[\nu B_1 = -\tilde{x} B_1 B_1 + (r \otimes i_s^C) M_{11} B_1 - (v \otimes r) B_2 B_1 - (r \otimes \tilde{w}) B_2 B_1 + (i_s^B \otimes r) M_{11} B_1\]

\[= (f i_s^C \otimes r) B_2 - (r \otimes g i_s^C) B_2 - (v B_1 \otimes r) B_2 + (r \otimes \tilde{w} B_1) B_2 + (r \otimes i_s^B g) B_2 - \tilde{i}^B f \otimes r) B_2\]

\[= (r \otimes (\tilde{w} B_1 - g i_s^C + i_s^B g)) B_2 = (r \otimes \lambda) B_2.\]
Applying identity $\nu B_1 = (r \otimes \lambda) B_2$ to $T^n s B$ and composing it with $\text{pr}_1 : Ts C \to s C$ we get an identity

$$\nu_n b_1 - \sum_{q+1+t=n} (1^\otimes q \otimes b_1 \otimes 1^\otimes t) \nu_n = (r_0 \otimes \lambda_n) b_2,$$

that is, $\nu_n d = -\lambda_n u$. Thus, the proposition is proved by induction. \hfill \Box

### 8.8. Invertible transformations.

Let $B, C$ be unital $A_\infty$-categories, and let $f, g : \text{Ob } B \to \text{Ob } C$ be maps. Assume that for each object $X$ of $B$ there are $k$-linear maps

$$x r_0 : k \to (s C)^{-1}(Xf, Xg), \quad x p_0 : k \to (s C)^{-1}(Xg, Xf),$$

$$x w_0 : k \to (s C)^{-2}(Xf, Xf), \quad x v_0 : k \to (s C)^{-2}(Xg, Xg),$$

such that

$$(x r_0 \otimes x p_0) b_2 - x f^0_b = x w_0 b_1,$$

$$(x p_0 \otimes x r_0) b_2 - x g^0_b = x v_0 b_1.$$  \hfill (8.8.1)

**8.9 Proposition.** Let the assumptions of Section 8.8 hold and, moreover, let $f : B \to C$ be a unital $A_\infty$-functor. Then the map $g$ extends to a unital $A_\infty$-functor $g : B \to C$ and the given $r_0, p_0$ extend to natural $A_\infty$-transformations $r : f \to g : B \to C$, $p : g \to f : B \to C$, inverse to each other.

**Proof.** Propositions 8.2 and 8.7 imply the existence and unitality of $g$. Indeed, since $(r_0 \otimes 1^\otimes 1) b_2$ is a homotopy invertible chain map, the map $u = \text{Hom}(N, (r_0 \otimes 1^\otimes 1) b_2)$ is also homotopy invertible, hence a quasi-isomorphism. Existence of $r : f \to g : B \to C$ is shown in Proposition 8.2. Existence of $p : g \to f : B \to C$, inverse to $r$ is proven in [Lyu03, Proposition 7.15]. \hfill \Box

### 9. Derived categories

Let $\mathcal{A}$ be a $\mathcal{U}$-small Abelian $k$-linear category, and let $C = C(\mathcal{A})$ or $C = C^+(\mathcal{A})$ be the differential graded category of complexes (resp. bounded below complexes) of objects of $\mathcal{A}$. Denote by $B = A(\mathcal{A})$ its full subcategory of acyclic complexes. Let $D = D(C|B)$ be the constructed $\mathcal{U}$-small differential graded category. We observe first that quasi-isomorphisms in $C$ become (homotopy) invertible elements in $D$.

Assuming that the ground ring $k$ is a field, we turn to the procedure of finding a $K$-injective resolution (if they exist) into a unital $A_\infty$-functor. Under these assumptions, we also show that $D = D(C|B)$ is $A_\infty$-equivalent to $I \subset C$ – the full subcategory of $K$-injective complexes. Hence, $H^0(D)$ is equivalent to the derived category $D(\mathcal{A})$. 45
9.1. Invertibility of quasi-isomorphisms. Assume that $X$, $Y$ are objects of $\mathcal{C}$ and $q : X \to Y$ is a quasi-isomorphism. In particular, $q \in \mathcal{C}^0(X,Y)$, $qm_1 = 0$. Let us prove that $r = qsm_1 \in (s\mathcal{D})^{-1}(X,Y)$ is invertible in the sense of Section 3.3, that is, there are elements $p \in (s\mathcal{D})^{-1}(Y,X)$, $w \in (s\mathcal{D})^{-2}(X,X)$, $v \in (s\mathcal{D})^{-2}(Y,Y)$, such that
\[
\begin{align*}
    r\overline{b}_1 &= 0, \quad p\overline{b}_1 = 0, \\
    (r \otimes p)\overline{b}_2 - 1_X s &= w\overline{b}_1, \\
    (p \otimes r)\overline{b}_2 - 1_Y s &= v\overline{b}_1.
\end{align*}
\] (9.1.1)

Indeed, denote $C = \text{Cone}(q) = (Y \oplus X[1], d^C)$, where $(y, x)d^C = (yd^Y + xq, -xd^X)$ for $y \in Y^l$, $x \in X^{l+1}$. Since $q$ is a quasi-isomorphism, $C$ is acyclic. There is a standard exact sequence of complexes $0 \to Y \xrightarrow{n} C \xrightarrow{k} X[1] \to 0$ with the chain maps $n, k$, $yn = (y, 0)$, $(0, x)k = x$. From now on we denote by $n, k$ also the corresponding elements $n \in \mathcal{C}^0(Y, C)$, $k \in \mathcal{C}^1(C, X)$. Define $p$ as $p = ns \otimes ks \in (s\mathcal{C})^{-1}(Y, C) \otimes (s\mathcal{C})^0(C, X) \subset (s\mathcal{D})^{-1}(Y, X)$. Then
\[
p\overline{b}_1 = pb = (ns \otimes ks)(1 \otimes b_1 + b_1 \otimes 1 + b_2) = -(n \otimes k)m_2 s = -(nk)s = 0.
\]

Denote by $h \in \mathcal{C}^{-1}(X, C)$ the following $k$-linear embedding $X \to C$, $X^l \to C^{l-1} = Y^{l-1} \oplus X^l$, $x \mapsto (0, x)$. Define $w$ as $w = hs \otimes ks \in (s\mathcal{C})^{-2}(X, C) \otimes (s\mathcal{C})^0(C, X) \subset (s\mathcal{D})^{-2}(X, X)$. Then
\[
w\overline{b}_1 = wb = (hs \otimes ks)(1 \otimes b_1 + b_1 \otimes 1 + b_2) = hm_1 s \otimes ks - (hk)s = (qn)s \otimes ks - 1_X s
\]
\[
= (qs \otimes ns)b_2 \otimes ks - 1_X s = [qs \otimes (ns \otimes ks)]\overline{b}_2 - 1_X s = (r \otimes p)\overline{b}_2 - 1_X s.
\]

Indeed, $qn = hm_1 = hd + dh : X \to C$ as explicit computation shows:
\[
xqn = (xq, 0) = (xq, -xd^X) + (0, xd^X) = (0, xd) = x(hd + dh).
\]

Denote by $z \in \mathcal{C}^0(C, Y)$ the following $k$-linear projection $z : C \to Y$, $(y, x) \mapsto y$. Define $v$ as $v = -ns \otimes zs \in (s\mathcal{C})^{-1}(Y, C) \otimes (s\mathcal{C})^{-1}(C, Y) \subset (s\mathcal{D})^{-2}(Y, Y)$. Then
\[
v\overline{b}_1 = vb = -(ns \otimes zs)(1 \otimes b_1 + b_1 \otimes 1 + b_2) = -ns \otimes zm_1 s - (nz)s = ns \otimes (kq)s - 1_Y s
\]
\[
= ns \otimes (ks \otimes qs)b_2 - 1_Y s = [(ns \otimes ks) \otimes qs]\overline{b}_2 - 1_Y s = (p \otimes r)\overline{b}_2 - 1_Y s.
\]

Indeed, $-kq = zm_1 = zd - dz : C \to Y$ as explicit computation shows:
\[
-(y, x)kq = -xq = yd - yd - xq = (y, x)zd - (y, x)d^C z = (y, x)(zd - dz).
\]

Thus, equations (9.1.1) hold true.
9.2. K-injective complexes. A complex $A \in \text{Ob} \mathcal{C}$ is K-injective if and only if for every quasi-isomorphism $t : X \to Y \in \mathcal{C}$ the chain map $\mathcal{C}(t, A) : \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$ is a quasi-isomorphism [Spa88, Proposition 1.5]. Assume that each complex $X \in \mathcal{C}$ has a right K-injective resolution $r_X : X \to X_0$, that is, $r_X$ is a quasi-isomorphism and $X_0 \in \text{Ob} \mathcal{C}$ is K-injective. Moreover, if $X$ is K-injective, we assume that $X_0 = X$ and $r_X = 1_X$. By definition, $\mathcal{C}(r_X, A) : s\mathcal{C}(X_0, A) \to s\mathcal{C}(X, A)$, $fs \mapsto (r_X f)s$ is a quasi-isomorphism. The assumption is satisfied, when $\mathcal{A}$ has enough injectives and $\mathcal{C} = \mathbb{C}^+(\mathcal{A})$, or when $\mathcal{A} = \mathcal{R}-\text{mod}$ or when $\mathcal{O}$ is a sheaf of rings on a topological space, and $\mathcal{A}$ is the category of sheaves of left $\mathcal{O}$-modules, see [Spa88].

Assume now that $\kappa$ is a field. Then for any chain complex of $\kappa$-modules of the form $N = s\mathcal{C}(X_0, X_1) \otimes_{\kappa} s\mathcal{C}(X_1, X_2) \otimes_{\kappa} \cdots \otimes_{\kappa} s\mathcal{C}(X_{n-1}, X_n)$, $n \geq 0$, $X_i \in \text{Ob} \mathcal{C}$, for any quasi-isomorphism $r_X : X \to Y$ and for any K-injective $A \in \mathcal{C}$, the following chain map

$$u = \text{Hom}(N, \mathcal{C}(r_X, A)) : \text{Hom}_{\mathcal{C}}^*(N, s\mathcal{C}(Y, A)) \to \text{Hom}_{\mathcal{C}}^*(N, s\mathcal{C}(X, A)),$$

is a quasi-isomorphism (any $\kappa$-module complex is $\kappa$-projective). Therefore, we may apply the results of Section 5 to the differential graded category $\mathcal{C} = \mathcal{C}$ or $\mathcal{C}^+$, and its full subcategories $\mathcal{B} = A(\mathcal{A})$ (resp. $\mathcal{J} = \mathcal{I}(\mathcal{A})$, $\mathcal{J} = A(\mathcal{A})$) of acyclic (resp. K-injective, acyclic K-injective) complexes. Denote by $e : \mathcal{J} \to \mathcal{C}$ the full embedding. Starting with the identity functor $f = \text{id}_{\mathcal{C}}$, we get the existence of $g = ie$ simultaneously with the existence of a unital $A_\infty$-functor $i : \mathcal{C} \to \mathcal{J}$ – “K-injective resolution functor” and a natural $A_\infty$-transformation $r : \text{id} \to ie : \mathcal{C} \to \mathcal{C}$ (Propositions 8.2, 8.7). The said $i$, $r$ are unique in the sense of Propositions 8.4, 8.5 and Corollary 8.6. Moreover, while solving equations (8.2.1)-(8.2.2) we will choose the solutions

$$i_n = g_n = \text{id}_n : s\mathcal{C}(X_0, X_1) \otimes_{\kappa} \cdots \otimes_{\kappa} s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{C}(X_0, X_n),$$

$$r_n = \bar{i}_n : s\mathcal{C}(X_0, X_1) \otimes_{\kappa} \cdots \otimes_{\kappa} s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{C}(X_0, X_n),$$

if $X_0, \ldots, X_n$ are K-injective (recall that $X_0i = X_0$, $X_ni = X_n$).

Extending $e$, $i$ to $A_\infty$-functors between the constructed categories, we get a unital strict $A_\infty$-embedding (actually, a faithful differential graded functor) $\mathcal{T} : \mathcal{D}(\mathcal{J}[\mathcal{I}]) \to \mathcal{D}(\mathcal{C}[\mathcal{B}])$, which is injective on objects, and a unital $A_\infty$-functor $\mathcal{I} : \mathcal{D}(\mathcal{C}[\mathcal{B}]) \to \mathcal{D}(\mathcal{J}[\mathcal{I}])$. Let us prove that these $A_\infty$-functors are quasi-inverse to each other. First of all, $ei = \text{id}_\mathcal{J}$ implies $\mathcal{T}i = \text{id}_{\mathcal{D}(\mathcal{J}[\mathcal{I}])}$. Secondly, there is a natural $A_\infty$-transformation $\mathcal{T} : \text{id} \to i\mathcal{T} : \mathcal{D}(\mathcal{C}[\mathcal{B}]) \to \mathcal{D}(\mathcal{C}[\mathcal{B}])$. Let us prove that it is invertible.

The 0-th component is

$$v_0 = [x \mapsto (s\mathcal{C})^{-1}(X, xi) \xrightarrow{\mathcal{T}i} (s\mathcal{D}(\mathcal{C}[\mathcal{B}]))^{-1}(X, Xi)].$$

We have proved in Section 8.3 that since $r_X$ is a quasi-isomorphism, the above element is invertible modulo boundary in the sense of Section 8.3; there exist $p_0$, $v_0$, $w_0$ such that

---

1 If $R \in ^\wedge \mathcal{U} \in \mathcal{U}$, where $^\wedge \mathcal{U}$ is a smaller universe, then $\mathcal{A} = \mathcal{R}-\text{mod}$ is a $\mathcal{U}$-small $^\wedge \mathcal{U}$-category. Is it possible to replace $\mathcal{B} = A(\mathcal{A})$ with some $^\wedge \mathcal{U}$-small category $\mathcal{B}' \subset \mathcal{B}$ to get a $^\wedge \mathcal{U}$-category $\mathcal{D} = \mathcal{D}(\mathcal{C}[\mathcal{B}'])$ $A_\infty$-equivalent to $\mathcal{U}$-small $\mathcal{D} = \mathcal{D}(\mathcal{C}[\mathcal{B}])$?

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equations (8.8.1) hold. We conclude by Proposition 8.9 that \( \mathfrak{r} \) is invertible, hence, \( D(\mathbb{C}|\mathbb{B}) \) and \( D(J|J) \) are equivalent.

Each acyclic \( K \)-injective complex \( X \) is contractible. Indeed, \( K(A)(X, X) \simeq D(A)(X, X) = 0 \) by \( \text{[Spa88, Proposition 1.5]} \). Hence, \( J \) is a contractible subcategory of \( I \). Thus, \( \mathfrak{j} : J \to D(J|J) \) is an equivalence. We deduce that \( D(\mathbb{C}|\mathbb{B}) \) and \( J \) are equivalent in \( A_{\infty}^w \). Taking \( H^0 \) we get equivalent categories \( H^0(D(\mathbb{C}|\mathbb{B})) \) and \( H^0(J) \). The latter is a full subcategory of \( K(A) \), whose objects are \( K \)-injective complexes. It is equivalent to the derived category \( D(A) \) (e.g. by \( \text{[KS90, Proposition 1.6.5]} \)). Hence, \( H^0(D(\mathbb{C}|\mathbb{B})) \) is equivalent to the derived category \( D(A) \). This result follows also from Drinfeld’s theory \( \text{[Dri04]} \). It motivated our study of \( A_{\infty} \)-categories.

Let \( F : \mathcal{A} \to \mathcal{B} \) be an additive \( k \)-linear functor between Abelian categories. The standard recipe \( \text{[Spa88]} \) of producing its right derived functor can be formulated in terms of the \( K \)-injective resolution \( A_{\infty} \)-functor \( i \) as follows. Apply \( H^0 \) to the \( A_{\infty} \)-functor

\[
D(C(A)|A(A)) \xrightarrow{i} D(I(A)|A(A)) \xrightarrow{D(F)} D(C(B)|A(B))
\]

(when \( F(\text{Ob Al}(A(A))) \subset \text{Ob A}(B) \)). Some work is required to identify the obtained functor \( \text{[Lyu03, Section 8.13]} \). \( H^0(idF) : D(A) \simeq H^0(D(C(A)|A(A))) \to H^0(D(C(B)|A(B))) \simeq D(B) \) with \( RF \); however, we shall not consider this topic here.

Acknowledgements. We are grateful to B. L. Tsygan for drawing our attention to the subject. We thank V. G. Drinfeld for the possibility to become acquainted with his work on quotients of differential graded categories before it was completed. We are grateful to all the participants of the \( A_{\infty} \)-category seminar at the Institute of Mathematics, Kyiv, especially to Yu. Bespalov and O. Manzyuk. We thank the referee for useful suggestions, which allowed us to improve the exposition.

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