Homology Groups of the Curvature Sets of $\mathbb{S}^1$

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Abstract

For $n \geq 2$, the $n$-th curvature set of a metric space $X$ is the set of all $n$-by-$n$ distance matrices of points sampled from $X$. We give a geometric and topological study of the curvature sets of the unit circle equipped with the geodesic metric. We show that the $n$-th curvature set is a quotient of the $(n-1)$-torus by the diagonal action of the reflection. We also characterize the homotopy type of several notable subsets of the curvature set, allowing us to compute its homology groups (which often have torsion) using the Mayer-Vietoris sequence. We also construct an abstract simplicial complex, which we call the state complex, that can be constructed using relative positions of points rather than their distances. We prove that this simplicial complex is homeomorphic to the $n$-th curvature set of the circle. We also discuss a connection between elliptopes (a well known notion from semidefinite optimization) and curvature sets of spheres of increasing dimension.

Contents

1 Introduction. ................................................................. 2

2 Notation. ................................................................. 4

3 Homology groups of $K_n(\mathbb{S}^1)$.
   3.1 $K_n(\mathbb{S}^1)$ as a quotient of $\mathbb{T}^n$ .......................... 5
   3.2 Homotopy types of the subsets $K_n^{(t)}(\mathbb{S}^1)$ ................. 6
   3.3 A cellular structure for $K_n(\mathbb{S}^1)$ and its chain groups. 8
   3.4 The computation of $H_m(K_n(\mathbb{S}^1))$ .......................... 12

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1 Introduction.

In this paper, we study the topological structure of the curvature sets of $\mathbb{S}^1$ equipped with the geodesic metric. For $n \geq 2$, the $n$-th curvature set of a metric space $(X, d_X)$ is the set

$$K_n(X) := \left\{ M \in \mathbb{R}^{n \times n} : \exists \{x_1, \ldots, x_n\} \subset X \text{ such that } M_{ij} = d_X(x_i, x_j) \right\}.$$

We also define the distance matrix map $D_n : X^n \to K_n(X)$ by setting $M := D_n(x_1, \ldots, x_n)$ to be the matrix such that $M_{ij} := d_X(x_i, x_j)$. In other words, $K_n(X) = \text{Im}(D_n)$ is the set of $n$-by-$n$ distance matrices generated by points of $X$. Similar objects have been studied before. Perhaps the most well known are configuration spaces, the sets

$$\text{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X : \text{ where } x_i \neq x_j \text{ if } i \neq j\},$$

where $X$ is a topological space. In words, the configuration space $\text{Conf}_n(X)$ is the set of $n$-point tuples of distinct points in $X$. Another similar space is the Ran space $\exp_n(X)$, which is the collection of nonempty subsets of $X$ of cardinality at most $n$ equipped with the quotient topology induced by the natural map $X^n \to \exp_n(X)$ defined by $(x_1, \ldots, x_n) \mapsto \{x_1\} \cup \cdots \cup \{x_n\}$. There is a substantial body of research on configuration spaces which includes applications to physics and robotics. There are also studies specifically on the Ran spaces of the circle (see the work of Bott [Bot52] and Tuffley [Tuf02]), including the characterization of the homotopy types of $\exp_n(\mathbb{S}^1)$ as a sphere $\mathbb{S}^n$ if $n$ is odd or $\mathbb{S}^{n-1}$ if $n$ is even [Tuf02]. In fact, the complement of $\exp_{k-2}(\mathbb{S}^1)$ in $\exp_k(\mathbb{S}^1)$ has the homotopy type of a $(k-1, k)$-torus knot complement [Tuf02].

The study of configuration and Ran spaces has a strong geometric flavor and even allows for very concrete results such as Theorems 4 through 7 in [Tuf02]. As for curvature sets, the article [Mém12] has a few illustrative characterizations of $K_2(X)$ and $K_3(X)$; we describe $K_3(\mathbb{S}^1)$ and $K_3(\mathbb{S}^2)$ here. Notice that a 3-by-3 distance matrix is symmetric and determined by the entries $d_{12}, d_{23}, d_{31}$, so we can visualize any $K_3(X)$ as a subset of $\mathbb{R}^3$. In the case of $\mathbb{S}^1$, three points are either contained in a semicircle or not. If they are, one of the distances is the sum of the other two, so $K_3(\mathbb{S}^1)$ contains the three 2-simplices $d_{\sigma(1), \sigma(3)} = d_{\sigma(1), \sigma(2)} + d_{\sigma(2), \sigma(3)}$, where $\sigma$ ranges over all permutations in $S_3$. If the points are not contained in the same semicircle, then $d_{12} + d_{23} + d_{31} = 2\pi$. That adds one more 2-simplex, so that $K_3(\mathbb{S}^1)$ is homeomorphic to the boundary of the 3-simplex with vertices $(0, 0, 0)$, $(\pi, \pi, 0)$, $(\pi, 0, \pi)$, and $(0, \pi, \pi)$; see Figure 1. As for $K_3(\mathbb{S}^2)$, any 3 points in general position lie on a plane.
whose intersection with $S^2$ is a circle $C$ with radius at most 1. While the restriction of the metric of $S^2$ to $C$ is not always geodesic, we do have that $d_{ij} \leq |\gamma_{ij}|$ where $\gamma_{ij}$ is the shortest path in $C$ that joins $x_i$ and $x_j$. With a bit more work, we see that $K_3(S^2)$ actually equals the union of $\lambda \cdot K_3(S^1)$, where $0 < \lambda \leq 1$, and that $K_3(S^2)$ is the convex hull of $K_3(S^1)$.

![Figure 1: A visualization of $K_3(S^1)$ in $\mathbb{R}^3$. The three simplices incident on the origin correspond to configurations $x_1, x_2, x_3 \in S^1$ contained in a semicircle; all other configurations correspond to the remaining 2-simplex. The set $K_3(S^2)$ is the full 3-simplex.](image)

Perhaps the fundamental result regarding curvature sets, though, was given by Gromov. He proved that curvature sets are a complete invariant of compact metric spaces, which means that two compact metric spaces $X$ and $Y$ are isometric if and only if $K_n(X) = K_n(Y)$ for all $n \geq 2$ [Gro07, Section 3.27]. After that, [Mém12] proved that curvature sets are stable under the Gromov-Hausdorff distance, that is,

$$d_{H}^{\mathbb{R}^{n\times n}}(K_n(X), K_n(Y)) \leq 2 \cdot d_{GH}(X, Y),$$

where $d_{H}^{\mathbb{R}^{n\times n}}$ is the Hausdorff distance between subsets of $\mathbb{R}^{n\times n}$ equipped with the $\ell^\infty$ metric and $d_{GH}$ is the Gromov-Hausdorff distance between metric spaces. That paper also has polynomial-time lower bounds for the Gromov-Hausdorff distance that can be computed from curvature sets.

Other than that, however, it doesn’t seem that the geometric study of curvature sets has gathered much attention and this paper is a first step in that direction. We give several characterizations and simplicial models of the curvature sets of $S^1$. The main model is as a quotient of the torus $T^n$ by the diagonal action of the group $O(2)$ of isometries of the circle. We also characterize the homotopy type of several subsets of $K_n(S^1)$ and their intersection. In particular, $K_n(S^1)$ is the union of two open subsets each of which is homotopy equivalent to $K_{n-1}(S^1)$ – a fact which allows us to use the Mayer-Vietoris sequence to compute the homology groups of $K_n(S^1)$.

**Theorem 1.1.**

$$H_m(K_n(S^1), \mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0, \\ \mathbb{Z}^{(n-1)}_{m-1} \oplus (\mathbb{Z}/2\mathbb{Z})^{\sum_{i=0}^{m-2} \binom{n-1}{i}} & m \text{ even, and } 2 \leq m \leq n-1, \\ 0 & \text{else.} \end{cases}$$

Note that when $n = 3$ the homology groups coincide with those of $S^2$, which is consistent with the fact that $K_3(S^1) \cong S^2$ as described above. When $n > 3$, the number of generators
of the torsion subgroups can range from 1 to $2^n - 1$ (minus a quadratic polynomial in $n$).

We also construct an abstract simplicial complex that we call the state complex $\text{St}_n(S^1)$ which does not depend directly on the distances between points, but is instead defined using the relative positions of the points in the circle. It is then (perhaps) surprising that the geometric realization $|\text{St}_n(S^1)|$ is homeomorphic to $K_n(S^1)$. We conclude the paper by identifying a precise relationship between $K_n(S^1)$ and the $n$-th elliptope, a basic object studied in semidefinite optimization. Our results about the homology groups of $K_n(S^1)$ provide a measure of complexity of the elliptopes.

Organization.

We start by setting up the relevant notation in Section 2. After that, the paper has two main sections: Section 3 on the homology groups of $K_n(S^1)$ and Section 4 on the state complex $\text{St}_n(S^1)$. Section 3 shows that the distance matrix map $D_n : \mathbb{T}^n \to K_n(S^1)$ induces the quotient topology induced by the diagonal action of $O(2)$ on $\mathbb{T}^n$. In Section 3.2, we study the homotopy type of certain subsets of $K_n(S^1)$ that we later use to compute the homology groups $H_m(K_n(S^1))$ in Section 3.3. Before the final computation, we give $K_n(S^1)$ a CW structure and compute the action induced by several maps in Section 3.3 in chain groups and homology. In the second part, we construct an abstract simplicial complex $\text{St}_n(S^1)$ (Section 4) and a map $K_n(S^1) \to \text{St}_n(S^1)$ (Section 4.3) that we later show induces a homeomorphism $|\text{St}_n(S^1)| \cong K_n(S^1)$ (Section 4.3). In order to do that, we show that the state matrix $\pi(x_1, \ldots, x_n)$ induces a clustering on $\{x_1, \ldots, x_n\}$ where $x_i$ and $x_j$ are in the same cluster if they are equal or antipodal. We also find the number of simplices and the Euler characteristic of $\text{St}_n(S^1)$ in Section 4.2. We conclude by mentioning the connections with elliptopes and possible future directions in Section 5. We also include an Appendix with an explicit calculation of $\text{St}_3(S^1)$ to illustrate the constructions in Section 4.

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2 Notation.

Given a topological space $X$, a boldface letter $\mathbf{x}$ represents a vector $\mathbf{x} := (x_1, \ldots, x_n) \in X^n$. Given a function $f : X \to X$, we also use $f$ to denote the product map $X^n \to X^n$ that sends $\mathbf{x} = (x_1, \ldots, x_n)$ to $f(\mathbf{x}) := (f(x_1), \ldots, f(x_n))$. We use $\mathbb{R}^{n \times n}$ to denote the space of $n$-by-$n$ real matrices with the usual topology. Given $M \in \mathbb{R}^{n \times n}$, we write $M_{ij} \in \mathbb{R}^n$ for the $i$-th row of and $M_{kj} \in \mathbb{R}^n$ for the $j$-th column of $M$.

For concreteness, we view $S^1$ as the unit circle in $\mathbb{C}$. We write $\arg : \mathbb{C} \setminus \{0\} \to [0, 2\pi)$ for the argument function. Observe that $\arg(z)$ increases in the anticlockwise direction. We
denote the geodesic metric on $S^1$ simply as $d$, where
\[ d(x, y) := \min \left( \arg \left( \frac{x}{y} \right), \arg \left( \frac{y}{x} \right) \right). \]

Let $\rho : S^1 \to S^1$ be $\rho(z) := z^{-1}$, the reflection along the real axis in $\mathbb{C}$. We denote the group of orthogonal transformations of $S^1$ as $O(2)$. The $n$-torus is the product space $\mathbb{T}^n := (S^1)^n$.

For the rest of the paper, the map $D_n$ will denote the map $D_n : \mathbb{T}^n \to K_n(S^1)$ such that $M = D_n(x)$ is the matrix that satisfies $M_{ij} := d(x_i, x_j)$.

## 3 Homology groups of $K_n(S^1)$.

The objective of this section is to compute the homology groups of $K_n(S^1)$. We start by showing that the defining map $D_n : \mathbb{T}^n \to K_n(S^1)$ is a quotient map under the action of $O(2)$.

### 3.1 $K_n(S^1)$ as a quotient of $\mathbb{T}^n$.

Observe that the distance matrices $D_n(x)$ and $D_n(\tau(x))$ are equal for any isometry $\tau \in O(2)$. We show the converse in this section: if $D_n(x) = D_n(y)$, then there exists $\tau \in O(2)$ such that $y = \tau(x)$.

**Lemma 3.1.** Let $x_1, x_2, x_3 \in S^1$ and $x'_1, x'_2, x'_3 \in S^1$ such that $d(x_i, x_j) = d(x'_i, x'_j)$. Suppose $d(x_1, x_2) \neq 0, \pi$. Let $\tau \in O(2)$ such that $\tau(x_1) = x'_1$ and $\tau(x_2) = x'_2$. Then $\tau(x_3) = x'_3$.

**Proof.** First, observe that any $\tau \in O(2)$ is determined by its action on a basis of $\mathbb{C}$ as an $\mathbb{R}$-vector space. Since $d(x_1, x_2) \neq 0, \pi$, the set $\{x_1, x_2\}$ is a basis, so $\tau$ is determined by $\tau(x_1) = x'_1$ and $\tau(x_2) = x'_2$. Now, observe that $d(x_i, x_j) = d(x'_i, x'_j)$ if and only if $\bar{x_i} \cdot x_j = \bar{x'_i} \cdot x'_j$ or $\bar{x'_i} \cdot x_j = \bar{x'_i} \cdot x'_j$, then $\tau(z) := \frac{x'_i}{x_i} \cdot z$ is unique such that $\tau(x_1) = x'_1$ and $\tau(x_2) = x'_2$. We claim that $\bar{x'_2} = \bar{x_2}$ and $\frac{x'_2}{x_2} = \frac{x_1}{x_2}$. In fact, the equation $\bar{x_3} \cdot \frac{x_1}{x_2} \cdot \frac{x_1}{x_2} = 1 = \frac{x'_3}{x'_1} \cdot \frac{x_1}{x_2} \cdot \frac{x_1}{x_2}$, which simplifies to $\bar{x_3} \cdot \frac{x_1}{x_2} = \frac{x'_3}{x'_1} \cdot \frac{x_1}{x_2}$, implies that $\frac{x'_1}{x_2} = \frac{x_3}{x_2}$ if and only if $\frac{x'_1}{x_1} = \frac{x_3}{x_1}$. If we had the opposite equalities $\frac{x'_2}{x_2} = \frac{x_3}{x_2}$ and $\frac{x'_3}{x_3} = \frac{x_1}{x_3}$, then the equation
\[ \frac{x_3}{x_2} \cdot \frac{x_1}{x_2} = \frac{x'_3}{x'_2} \cdot \frac{x'_1}{x'_2} = \frac{x_2}{x_3} \cdot \frac{x_1}{x_2} \]
gives $(\frac{x_2}{x_3})^2 = 1$. However, $x_2 \neq \pm 1$, so we must have $\frac{x'_2}{x_2} = \frac{x_2}{x_3}$ and $\frac{x'_3}{x_3} = \frac{x_1}{x_2}$. Therefore, $\tau(x_3) = \frac{x'_3}{x'_1} \cdot x_3 = x'_3$.

If we had $\frac{x'_2}{x_2} = \frac{x'_3}{x'_3}$ instead, the conditions of the lemma still apply to $1/x'_3$ and $\rho \circ \tau$ because $\rho$ is an isometry. The paragraph above implies that $\rho \circ \tau(x_3) = 1/x'_3$, which gives $\tau(x_3) = x'_3$. \hfill \Box

**Lemma 3.2.** For all $x, y \in \mathbb{T}^n$ such that $D_n(x) = D_n(y)$, there exists some $\tau \in O(2)$ such that $\tau(x) = y$. 


Proof. We first consider the case of \( n = 2 \). \( D_2(x) = D_2(y) \) implies \( d(x_1, x_2) = d(y_1, y_2) \) and, with that, \( \frac{x_1}{x_2} = \frac{y_1}{y_2} \) or \( \frac{x_1}{x_2} = \frac{y_2}{y_1} \). In the first case, \( \tau(z) := \frac{y_1}{y_2} \cdot z \) satisfies \( \tau(x) = \tau(y) \). If \( \frac{x_1}{x_2} = \frac{y_2}{y_1} \), define \( \tau(z) := \frac{1}{y_1} \cdot z \). This time, \( \rho \circ \tau \) satisfies \( \rho \circ \tau(x) = \rho \circ \tau(y) \).

Now, let \( n > 2 \). If all the entries of \( D_n(x) = D_n(y) \) are either 0 or \( \pi \), we must have \( x_i = \varepsilon_i x_1 \) and \( y_i = \varepsilon_i y_1 \), where \( \varepsilon_i \in \{+1, -1\} \). Then \( \tau(z) := \frac{x_1}{x_i} \cdot z \) satisfies \( \tau(x) = y \). Otherwise, pick \( i_1, i_2 \) such that \( d(x_{i_1}, x_{i_2}) \neq 0, \pi \). By the case \( n = 2 \), there exists \( \tau \in O(2) \) such that \( \tau(x_{i_1}) = y_{i_1} \) and \( \tau(x_{i_2}) = y_{i_2} \). Then by Lemma 3.1, for any \( i \neq i_1, i_2 \) we have \( \tau(x_i) = y_i \).

With these lemmas, we can show that the quotient by \( D_n \) is induced by \( O(2) \).

**Proposition 3.3.** \( D_n : \mathbb{T}^n \to K_n(S^1) \) is the open quotient map under the diagonal action of \( O(2) \). Hence \( K_n(S^1) \cong \mathbb{T}^n/O(2) \).

**Proof.** Since every coordinate function \( d(x_i, x_j) \) is continuous, \( D_n \) is as well and, by Lemma 3.2 \( D_n(x) = D_n(y) \) if and only if there exists \( \tau \in O(2) \) such that \( \tau(x) = y \). Since \( D_n \) is also surjective, it is a quotient map and \( K_n(S^1) \cong \mathbb{T}^n/O(2) \). The quotient is induced by a group action, so \( D_n \) is open.

In fact, we can see \( K_n(S^1) \) as a quotient of \( \mathbb{T}^{n-1} \) instead by modding out the translations in \( O(2) \). For concreteness, define \( i_n : \mathbb{T}^{n-1} \to \mathbb{T}^n \) by \( i_n(x_1, \ldots, x_{n-1}) := (1, x_1, \ldots, x_{n-1}) \), and let \( \widehat{D}_n := D_n \circ i_n : \mathbb{T}^{n-1} \to K_n(S^1) \).

**Proposition 3.4.** For \( x, y \in \mathbb{T}^{n-1} \), \( \widehat{D}_n(x) = \widehat{D}_n(y) \) implies either \( y = x \) or \( y = \rho(x) \). As a consequence, \( \widehat{D}_n \) is an open quotient map and \( K_n(S^1) \cong \mathbb{T}^{n-1}/\langle \rho \rangle \).

**Proof.** Since \( D_n \circ i_n(x) = D_n \circ i_n(y) \), Lemma 3.2 gives a \( \tau \in O(2) \) such that \( \tau(i_n(x)) = i_n(y) \). The first coordinates of \( i_n(x) \) and \( i_n(y) \) are +1, so we must have \( \tau = \text{id}_{S^1} \) or \( \tau = \rho \). Since \( D_n \) is surjective, it is the open quotient map \( \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}/\langle \rho \rangle \). Hence, \( K_n(S^1) \cong \mathbb{T}^{n-1}/\langle \rho \rangle \).

### 3.2 Homotopy types of the subsets \( K_n(t)(S^1) \).

Given \( t \in [0, \pi] \), define \( K_n(t)(S^1) \) as the subspace of \( K_n(S^1) \) of elements \( M \) such that \( M_{1n} = t \). Clearly, \( K_n(S^1) \) is the union of the two subsets \( K_n(S^1) \setminus K_n(0)(S^1) \) and \( K_n(S^1) \setminus K_n(\pi)(S^1) \). The objective of this section is to determine the homotopy type of these sets and that of their intersection in order to compute the homology groups of \( K_n(S^1) \) using the Mayer-Vietoris sequence in Section 3.3.

**Lemma 3.5.** If \( t = 0 \) or \( \pi \), \( K_n(t)(S^1) \) is homeomorphic to \( K_{n-1}(S^1) \).

**Proof.** Let \( t = 0 \) first, and consider the diagram:

\[
\begin{array}{ccc}
\mathbb{T}^{n-2} & \xrightarrow{i} & K_{n-1}(S^1) \\
\downarrow & & \downarrow \rho \\
\mathbb{T}^{n-2} \times \{+1\} & \xrightarrow{\widehat{D}_n} & K_n(0)(S^1) \\
\end{array}
\]
Notice that \( \mathbb{T}^{n-2} \times \{+1\} = \tilde{D}_n^{-1}(K_n^{(0)}(S^1)) = \tilde{D}_n^{-1} \circ \tilde{D}_n(\mathbb{T}^{n-2} \times \{+1\}) \), that is, the closed set \( \mathbb{T}^{n-2} \times \{+1\} \subset \mathbb{T}^{n-1} \) is saturated with respect to \( \tilde{D}_n \). Thus, the restriction of \( \tilde{D}_n \) to \( \mathbb{T}^{n-2} \times \{+1\} \to K_n^{(0)}(S^1) \) is a quotient map.

Now, let \( x, y \in \mathbb{T}^{n-2} \). We claim that \( \tilde{D}_{n-1}(x) = \tilde{D}_{n-1}(y) \) if and only if \( \tilde{D}_n \circ i(x) = \tilde{D}_n \circ i(y) \). Indeed, \( \tilde{D}_{n-1}(x) = \tilde{D}_{n-1}(y) \) if and only if there exists \( \tau \in \langle \rho \rangle \) such that \( \tau(x) = y \) by Proposition 3.6. Since \( \tau \) fixes \(+1\), the last condition is also equivalent \( \tau(i(x)) = i(y) \) which, in turn, is equivalent to \( \tilde{D}_n \circ i(x) = \tilde{D}_n \circ i(y) \) by Proposition 3.4. Then, the universal property of the quotient map \( \tilde{D}_{n-1} \) produces a unique continuous map \( \iota_0 : K_{n-1}(S^1) \to K_n^{(0)}(S^1) \) that makes the diagram above commute. Similarly, the universal property of \( \tilde{D}_n \) applied to \( \tilde{D}_{n-1} \circ i^{-1} \) produces a map \( K_n^{(0)}(S^1) \to K_{n-1}(S^1) \). This map has to be \( i_0^{-1} \) because \( \tilde{D}_n(x) = \tilde{D}_n(y) \) if and only if \( \tilde{D}_{n-1} \circ i^{-1}(x) = \tilde{D}_{n-1} \circ i^{-1}(y) \) for any \( x, y \in \mathbb{T}^{n-2} \times \{+1\} \). Thus, \( \iota_0 \) is a homeomorphism. The analogous argument shows that \( K_n^{(\pi)}(S^1) \) is homeomorphic to \( K_{n-1}(S^1) \) because \( \langle \rho \rangle \) also fixes \(-1 \in S^1 \).

**Lemma 3.6.** If \( t = 0 \) or \( t = \pi \), then \( K_n^{(t)}(S^1) \) is a deformation retract of \( K_n(S^1) \setminus K_n^{(\pi-t)}(S^1) \).

**Proof.** Let \( H_{-1} := S^1 \setminus \{+1\} \), the set of points in \( S^1 \) of the form \( e^{it} \) for \( 0 < r < 2\pi \). Define \( \xi : H_{-1} \times [0, 1] \to \{ -1 \} \) to be the deformation retraction given by \( \xi(e^{it}, r) := e^{i[(1-r)+\pi t]} \), where \( 0 < r < 2\pi \). Consider \( I := Id_{\mathbb{T}^{n-2}} \) and the deformation retraction \( \pi \times \xi \) from \( \mathbb{T}^{n-2} \times H_{-1} \) onto \( \mathbb{T}^{n-2} \times \{ -1 \} \). Observe that \( \mathbb{T}^{n-2} \times H_{-1} \) is an open subset of \( \mathbb{T}^{n-1} \) that is saturated with respect to \( \tilde{D}_n \), so the restriction of \( \tilde{D}_n \) to \( \mathbb{T}^{n-2} \times H_{-1} \to K_n(S^1) \setminus K_n^{(0)}(S^1) \) is a quotient map. Also, observe that \( \rho(e^{it}) = e^{i(2\pi-r)} \), so

\[
\xi(\rho(e^{it}), t) = \xi(e^{i(2\pi-r)}, t) = e^{i[(1-t)(2\pi-r)+\pi t]} = e^{i[2\pi-(1-t)r-\pi t]} = \rho \circ \xi(e^{it}, t).
\]

Hence, the map \( I \times \xi \) commutes with \( \rho \circ Id_{[0,1]} \) and thus descends to a homotopy defined on the quotient \( \mathbb{T}^{n-2} \times H_{-1} / \langle \rho \rangle \cong K_n(S^1) \setminus K_n^{(0)}(S^1) \). Composing with \( \tilde{D}_n \) gives a deformation retraction from \( \tilde{D}_n \circ (I \times \xi)(\mathbb{T}^{n-2} \times H_{-1}, 0) = \tilde{D}_n(\mathbb{T}^{n-2} \times H_{-1}) = K_n(S^1) \setminus K_n^{(0)}(S^1) \) onto \( \tilde{D}_n \circ (I \times H)(\mathbb{T}^{n-2} \times H_{-1}, 1) = \tilde{D}_n(\mathbb{T}^{n-2} \times \{ -1 \}) = K_n^{(\pi)}(S^1) \). The case of \( K_n(S^1) \setminus K_n^{(\pi)}(S^1) \) and \( K_n^{(0)}(S^1) \) is analogous.

**Corollary 3.7.** If \( t = 0 \) or \( t = \pi \), then \( K_n(S^1) \setminus K_n^{(t)}(S^1) \) is homotopy equivalent to \( K_{n-1}(S^1) \).

**Proof.** By Lemma 3.6 \( K_n(S^1) \setminus K_n^{(t)}(S^1) \) is homotopy equivalent to \( K_n^{(\pi-t)}(S^1) \) which, by Lemma 3.5, is homeomorphic to \( K_{n-1}(S^1) \).

**Lemma 3.8.** \( \left( K_n(S^1) \setminus K_n^{(\pi)}(S^1) \right) \cap \left( K_n(S^1) \setminus K_n^{(0)}(S^1) \right) \cong \mathbb{T}^{n-2} \times (0, \pi) \cong \mathbb{T}^{n-2} \). In particular, \( K_n^{(t)}(S^1) \) is homeomorphic to \( \mathbb{T}^{n-2} \) whenever \( t \neq 0, \pi \).

**Proof.** Define \( H^+ := \{ e^{it} : 0 < t < \pi \}, \quad H^- := \rho(H^+) \), and \( H := H^+ \cup H^- \). Let \( \tilde{K}_n := \left( K_n(S^1) \setminus K_n^{(\pi)}(S^1) \right) \cap \left( K_n(S^1) \setminus K_n^{(0)}(S^1) \right) \). Observe that \( \tilde{D}_n^{-1}(\tilde{K}_n) = \mathbb{T}^{n-2} \times H \). Since \( \rho(\mathbb{T}^{n-2} \times H^+) = \mathbb{T}^{n-2} \times H^- \), Proposition 3.4 implies that \( \tilde{D}_n^{-1}(M) \cap (\mathbb{T}^{n-2} \times H^+) \) has exactly one element for every \( M \in \tilde{K}_n \). Hence, the restriction of \( \tilde{D}_n \) to \( \mathbb{T}^{n-2} \times H^+ \to \tilde{K}_n \) is bijective. Since \( \tilde{D}_n \) is continuous and open by Proposition 3.4, it is a homeomorphism
from \(\mathbb{T}^{n-2} \times H^+ \cong \mathbb{T}^{n-2} \times (0, \pi)\) to \(\tilde{K}_n\). Furthermore, since \(H^+\) is contractible, \(\tilde{K}_n\) is homotopy equivalent to \(\mathbb{T}^{n-2}\). In particular, the restriction \(\mathbb{T}^{n-2} \times \{t\} \cong \mathbb{T}^{n-2} \xrightarrow{\tilde{D}_n} K^{(t)}_n(S^1)\) is a homeomorphism for any specific \(0 < t < \pi\).

\[\text{Remark 3.9.}\] The proof of Lemma 3.3 cannot be extended to show that \(K^{(t)}_n(S^1) \cong \mathbb{T}^{n-2}\) when \(t = 0, \pi\) because the restriction of \(\tilde{D}_n\) to \(\mathbb{T}^{n-2} \times \{t\} \to K^{(t)}_n(S^1)\) is 2-to-1. However, combining Lemmas 3.5 and 3.8 gives yet another representation of \(K_n(S^1)\) as the double mapping cylinder \(\mathbb{T}^{n-2} \times [0, \pi]/\sim\) where \((x, i) \sim (\rho(x), i)\) for \(i = 0, 1\).

### 3.3 A cellular structure for \(K_n(S^1)\) and its chain groups.

In this section, we occasionally write \(K_n = K_n(S^1)\) to reduce notational overload. We start by giving \(K_n(S^1)\) a CW-complex structure induced from the quotient \(K_n(S^1) = \mathbb{T}^{n-1}/\langle \rho \rangle\) in Proposition 3.4. Give \(S^1\) the CW-complex structure with two 0-dimensional cells \(v_1\) and \(v_2\) and two 1-dimensional cells \(e_1\) and \(e_2\) with boundary \(v_2 - v_1\). Consider the characteristic maps into \(S^1 \subset \mathbb{C}\) that send \(v_1 \mapsto +1, v_2 \mapsto -1, e_1 \mapsto (t \in [0, 1] \mapsto e^{it})\), and \(e_2 \mapsto (t \in [0, 1] \mapsto e^{-it})\). Under this structure, the reflection \(\rho\) interchanges \(e_1\) and \(e_2\) while preserving their orientation – in other words, \(\rho\) is a cellular map. This CW-complex induces a product CW structure on the torus \(\mathbb{T}^{n-1} = (S^1)^{n-1}\) along with a cellular action of \(\rho\). We know from the Künneth Theorem that the cellular chain groups and the homology of \(\mathbb{T}^{n-1}\) satisfy

\[
C_m(\mathbb{T}^{n-1}) = \bigoplus C_{i_1}(S^1) \otimes \cdots \otimes C_{i_m}(S^1)
\]

\[
H_m(\mathbb{T}^{n-1}) = \bigoplus H_{i_1}(S^1) \otimes \cdots \otimes H_{i_m}(S^1),
\]

where each sum runs over all indices \(0 \leq i_j \leq 1\) such that \(i_1 + \cdots + i_{n-1} = m\). Both groups are equipped with an action of the symmetric group \(S_{n-1}\).

Since \(\rho\) is a cellular map, Proposition 3.4 implies that \(K_n(S^1)\) inherits a CW structure from \(\mathbb{T}^{n-1}\) wherein a \(k\)-cell in \(K_n(S^1)\) is the equivalence class \([a] \in C_k(\mathbb{T}^{n-1})/\langle \rho \rangle\) of a \(k\)-cell from \(\mathbb{T}^{n-1}\). Under this identification, the quotient map \(\tilde{D}_n\) is cellular, so it induces the following structure on the chain groups:

\[
C_m(K_n(S^1)) \cong C_m(\mathbb{T}^{n-1})/(a \sim \rho^* (a)).
\]

By abuse of notation, we write \(a\) both for an element of \(C_m(\mathbb{T}^{n-1})\) and its equivalence class \((\tilde{D}_n)^* (a) \in C_m(K_n(S^1))\). However, we explicitly write \(a \sim b\) to indicate that \(a\) and \(b\) in \(C_m(\mathbb{T}^{n-1})\) represent the same class in \(C_m(K_n(S^1))\).

Additionally, the action of \(S_{n-1}\) on \(\mathbb{T}^{n-1}\) commutes with \(\rho\), so it also descends to an action of \(S_{n-1}\) on \(K_n\) and its chain complex. In particular, we can use the action of \(S_{n-1}\) to describe the generators of \(H_m(\mathbb{T}^{n-1})\) in terms of the 1-cycle \(e_1 - e_2\) that generates \(H_1(S^1)\). Going forward, the notation \((e_1 - e_2)^\otimes m \otimes v_1^\otimes (n-m-1)\) represents the \(m\) cycle \((e_1 - e_2) \otimes \cdots \otimes (e_1 - e_2) \otimes v_1 \otimes \cdots v_1\), where \((e_1 - e_2)\) is repeated \(m\) times and \(v_1, n-m-1\) times. Then

\[
I_m^{n-1} := \{ \sigma \cdot ((e_1 - e_2)^\otimes m \otimes v_1^\otimes (n-m-1)) : \sigma \in S_{n-1}\}
\]

is a generating set for \(H_m(\mathbb{T}^{n-1})\), where

\[
\sigma \cdot (b_1 \otimes \cdots \otimes b_{n-1}) = b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n-1)}.
\]
Proposition 3.10. Let \( A := K_n(S^1) \setminus K_n(\pi) (S^1) \) and \( B := K_n(S^1) \setminus K_n(0) (S^1) \). The homology groups of \( K_n \) obey the following Mayer-Vietoris sequence:

\[
\cdots \xrightarrow{\partial} H_m(\mathbb{T}^{n-2}) \xrightarrow{(i_*j_*)} H_m(K_{n-1}) \oplus H_m(K_{n-1}) \xrightarrow{k_*-l_*} H_m(K_n) \xrightarrow{\partial} H_{m-1}(\mathbb{T}^{n-2}) \xrightarrow{} \cdots
\]

where \( i_* \), \( j_* \), \( k_* \), and \( l_* \) are the maps induced by the inclusions \( i : A \cap B \hookrightarrow A \), \( j : A \cap B \hookrightarrow B \), \( k : A \hookrightarrow K_n \), \( l : B \hookrightarrow K_n \).

Proof. The conditions to use the Mayer-Vietoris sequence are satisfied because the sets \( A \) and \( B \) are open as they are the image of the open sets \( \mathbb{T}^{n-2} \times (S^1 \setminus \{1\}) \) and \( \mathbb{T}^{n-2} \times (S^1 \setminus \{+1\}) \), respectively, under the open map \( \hat{D}_n \). Also, \( K_n = A \cup B \), \( A \cong B \cong K_{n-1} \) by Corollary 3.7 and \( A \cap B \cong \mathbb{T}^{n-2} \) by Lemma 3.8.

Thanks to the Lemmas in the last section, we can give an explicit description of the action of some of the maps in the sequence. We calculate that explicitly for \( \partial \) in Lemma 3.11.

\[
\hat{D}_n(x) := \hat{D}_n(x \times (+1)) \text{ and } \hat{D}_n(x) := \hat{D}_n(x \times (-1)) \text{ for } k : A \hookrightarrow K_n \text{ and } l : B \hookrightarrow K_n \text{ by Lemma 3.6.}
\]

The actions induced by these maps in the chain groups are as follows:

- \( i_*(a) = a \otimes v_1 \in C_m(K_{n-1}) \) for \( a \in C_m(\mathbb{T}^{n-2}) \),
- \( j_*(a) = a \otimes v_2 \in C_m(K_{n-1}) \) for \( a \in C_m(\mathbb{T}^{n-2}) \),
- \( k_*(a \otimes v_1) = a \otimes v_1 \in C_m(K_n) \) for \( a \in C_m(K_{n-1}) \), and
- \( l_*(a \otimes v_2) = a \otimes v_2 \in C_m(K_n) \) for \( a \in C_m(K_{n-1}) \).

To compute the homology groups \( H_m(K_n) \) via the Mayer-Vietoris sequence, we need to know the action of some of the maps in the sequence. We calculate that explicitly for \( i_* \), \( j_* \) and the boundary map \( \partial_* \). We start with:

Lemma 3.11. The restriction of \( \hat{D}_n_* : C_m(\mathbb{T}^{n-1}) \to C_m(K_n) \) to \( L_m^{-1} \) is 0 when \( m \) is odd and induces multiplication by 2 when \( m \) is even.

Proof. Let \( a_0 := (e_1 - e_2)^{\otimes m} \otimes \cdots \otimes v_1^{\otimes n-m-1} \in L_m^{-1} \). Then

\[
\hat{D}_n_*(a_0) = e_1 \otimes (e_1 - e_2)^{\otimes (m-1)} \otimes v_1^{\otimes (n-m-1)} = \begin{cases} 0 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases}
\]

Hence, \( \hat{D}_n_*(a_0) = 0 \) if \( m \) is odd and \( \hat{D}_n_*(a_0) = 2 \cdot e_1 \otimes (e_1 - e_2)^{\otimes (m-1)} \otimes v_1^{\otimes (n-m-1)} \) if \( m \) is even. Since the action of \( \sigma \in S_{n-1} \) commutes with \( \hat{D}_n_* \), the analogous result holds for the rest of the terms \( \sigma \cdot a \in L_m^{-1} \).
As a consequence of the previous lemma, the action of the maps $i_*, j_* : H_m(\mathbb{T}^{n-2}) \to H_m(K_{n-1})$ is trivial if $m$ is odd, but we can't yet say that it is injective if $m$ is even. To show that, we first study the boundary map $\partial_* : H_{m+1}(K_n) \to H_m(\mathbb{T}^{n-2})$. We need to subdivide the CW structure on $\mathbb{S}^1$ as shown in Figure 2 in order to evaluate $\partial_*$. Notice that this subdivision is still compatible with the action of $\rho$. 

Figure 2: Subdivision of the CW structure of the circle.

**Lemma 3.12.** If $m \geq 2$ is even, then the set $\frac{1}{2}(\hat{D}_n)_*(L_m^{n-1})$ is linearly independent in $H_m(K_n)$.

**Proof.** Let us write $\partial_*$ for the boundary map in the Mayer-Vietoris sequence of Proposition 3.10 and $\partial$ for the boundary map in a chain complex (which complex will be clear from context). Recall that to find $\partial_*(c)$ for $c \in H_m(K_n)$, we have to write $c = a + b$, where $a \in C_m(A) = C_m(K_n^{(0)}(\mathbb{S}^1))$ and $b \in C_m(B) = C_m(K_n^{(\pi)}(\mathbb{S}^1))$ such that $\partial(a) = -\partial(b)$. Then we set $\partial_*(c) := \partial(a) = -\partial(b) \in H_m(A \cap B)$.

Let $a_0 := e_1 \otimes (e_1 - e_2) \otimes v_1^{(n-m-1)} \otimes e_0^{(m-1)}$. The set $\frac{1}{2}(\hat{D}_n)_*(L_m^{n-1})$ is generated by elements of the form $\sigma \cdot a_0$ by Lemma 3.11. Observe that the $i$-th component of $\sigma \cdot a_0$ is $v_1$ if and only if $\sigma(i) > m$. In particular, the last coordinate of $\sigma \cdot a_0$ is $v_1$ when $\sigma(n) > m$. In that case, $\sigma \cdot a_0 = \sigma \cdot a_0 + 0$, where $\sigma \cdot a_0 \in A, 0 \in B$, and thus, $\partial_*(\sigma \cdot a_0) = -\partial(0) = 0$. However, if $\sigma(n) \leq m$, we can write $\sigma \cdot a_0 = v_1^{\otimes k} \otimes e_1 \otimes a_0' \otimes (e_1 - e_2)$, where $k := \min\{i : \sigma(i + 1) \leq m\}$ and $a_0' \in L_m^{n-k-2} \subset C_m(\mathbb{T}^{m-k-2})$. Write $E_1$ for the set of $\sigma \cdot a_0 \in \frac{1}{2}(\hat{D}_n)_*(L_m^{n-1})$ such that $\sigma(n) > m$ and $E_2$ for those $\sigma \cdot a_0$ for which $\sigma(n) \leq m$.

Suppose then that $\sigma \cdot a_0 \in E_2$. Since $e_1 = e_1^+ + e_1^-$ and $e_2 = e_2^+ + e_2^-$, we can write

$$\sigma \cdot a_0 = v_1^{\otimes k} \otimes e_1 \otimes a_0' \otimes (e_1^+ - e_2^+) + v_1^{\otimes k} \otimes e_1 \otimes a_0' \otimes (e_1^- - e_2^-),$$

where the first term is in $C_m(K_n^{(0)}(\mathbb{S}^1))$ and the second, in $C_m(K_n^{(\pi)}(\mathbb{S}^1))$. Then, calculating in $C_m(K_n^{(0)}(\mathbb{S}^1)) \subset C_m(K_n)$, we have

$$\partial \left( v_1^{\otimes k} \otimes e_1 \otimes a_0' \otimes (e_1^+ - e_2^+) \right) = v_1^{\otimes k} \otimes \partial(e_1) \otimes a_0' \otimes (e_1^+ - e_2^+) - v_1^{\otimes k} \otimes e_1 \otimes \partial(a_0') \otimes (e_1^+ - e_2^+) + (-1)^{m-1} v_1^{\otimes k} \otimes e_1 \otimes a_0' \otimes \partial(e_1^+ - e_2^+)$$

$$= v_1^{\otimes k} \otimes (v_2 - v_1) \otimes a_0' \otimes (e_1^+ - e_2^+) - v_1^{\otimes k} \otimes e_1 \otimes 0 \otimes (e_1^+ - e_2^+) + (-1)^{m-1} v_1^{\otimes k} \otimes e_1 \otimes a_0' \otimes (v_4 - v_3)$$

$$= v_1^{\otimes k} \otimes (v_2 - v_1) \otimes a_0' \otimes (e_1^+ - e_2^+) + (-1)^{m-1} v_1^{\otimes k} \otimes e_1 \otimes a_0' \otimes (v_4 - v_3).$$ (1)
Observe that \( a'_0 \) has \((m - 2)\) factors \( e_1 - e_2 \), so \( \rho_* (a'_0) = (-1)^{m-2} a'_0 \), which equals \( a'_0 \) because \( m \) is even. Then the first term of Equation (1) equals

\[
v_1 \otimes (v_2 - v_1) \otimes a'_0 \otimes (e_1^+ - e_2^+)
\]

\[
\sim v_1 \otimes (v_2 - v_1) \otimes a'_0 \otimes e_1^+ - \rho_* (v_1 \otimes (v_2 - v_1) \otimes a'_0 \otimes e_2^+)
\]

\[
= v_1 \otimes (v_2 - v_1) \otimes a'_0 \otimes e_1^+ - v_1 \otimes (v_2 - v_1) \otimes a'_0 \otimes e_1^+
\]

\[
= 0.
\]

As for the second term of Equation (1),

\[
v_1 \otimes e_1 \otimes a'_0 \otimes (v_4 - v_3) = v_1 \otimes e_1 \otimes a'_0 \otimes v_4 - v_1 \otimes e_1 \otimes a'_0 \otimes v_3
\]

\[
\sim v_1 \otimes e_1 \otimes a'_0 \otimes v_4 - \rho_* (v_1 \otimes e_1 \otimes a'_0 \otimes v_3)
\]

\[
= v_1 \otimes e_1 \otimes a'_0 \otimes v_4 - v_1 \otimes e_1 \otimes a'_0 \otimes v_4
\]

\[
= v_1 \otimes (e_1 - e_2) \otimes a'_0 \otimes v_4.
\]

Lastly, notice that

\[
\partial (v_1 \otimes (e_1 - e_2) \otimes a'_0 \otimes e_2) = v_1 \otimes (e_1 - e_2) \otimes a'_0 \otimes v_4 - v_1 \otimes (e_1 - e_2) \otimes a'_0 \otimes v_1.
\]

Putting the last three equations together shows that (1) equals \((-1)^{m-1} v_1 \otimes (e_1 - e_2) \otimes a'_0 \otimes v_1 \) plus a boundary in \( C_{m-1}(K^{(0)}) \). Thus,

\[
\partial_* ([\sigma \cdot a_0]) = [\partial (v_1 \otimes e_1 \otimes a'_0 \otimes (e_1^+ - e_2^+))]
\]

\[
= [(-1)^{m-1} v_1 \otimes (e_1 - e_2) \otimes a'_0],
\]

which is an element of \( L^{n-2}_{m-1} \subset H_{m-1}(\mathbb{T}^{n-2}) \).

Clearly, the assignments \( a_0 \mapsto a'_0 \) and \([\sigma \cdot a_0] \mapsto \partial_* ([\sigma \cdot a_0]) \) are injective. Thus, \( \partial_* \left( \frac{1}{2}(\widetilde{D}_n)_* (L^{n-1}_m) \right) = \partial_* (E_2) \) has the same amount of elements as the subset of \( L^{n-1}_m \) whose last coordinate is \( e_1 - e_2 \), which is \( \binom{n-2}{m} \). However, that number is also the rank of \( H_{m-1}(\mathbb{T}^{n-2}) \), from which we conclude \( \partial_* \left( \frac{1}{2}(\widetilde{D}_n)_* (L^{n-1}_m) \right) = (-1)^{m-1} L^{n-2}_{m-1} \). Thus, a zero linear combination of elements of \( E_2 \) induces a zero linear combination in \( H_{m-1}(\mathbb{T}^{n-2}) \) which must be trivial because \( L^{n-2}_{m-1} \) is a generating set of \( H_{m-1}(\mathbb{T}^{n-2}) \). Hence, the starting linear combination must be trivial in \( H_m(K_n) \) and it follows that \( E_2 \) is linearly independent.

What about the elements of \( E_1 \)? Notice that \( (\widetilde{D}_n)_* (2 \cdot E_1) = L^{n-2}_m \otimes \{v_1\} \subset H_m(\mathbb{T}^{n-2} \times \{+1\}) \). Using the notation of Lemma 3.5,

\[
(i_0^{-1})_* (E_1) = \frac{1}{2} (i_0^{-1} \circ \widetilde{D}_n)_* (L^{n-2}_m \otimes \{v_1\}) = \frac{1}{2} (\widetilde{D}_{n-1} \circ i^{-1})_* (L^{n-2}_m \otimes \{v_1\}) = \frac{1}{2} (\widetilde{D}_{n-1})_* (L^{n-2}_m).
\]

Hence, an inductive argument on \( n \) shows that \( E_1 \) is linearly independent because the set \( \frac{1}{2} (\widetilde{D}_{n-1})_* (L^{n-2}_m) \subset H_m(K_{n-1}) \) already is.

So far we have that both \( E_1 \) and \( E_2 \) are linearly independent. To finish the proof of this lemma, observe that \( \langle E_1 \rangle \cap \langle E_2 \rangle = 0 \) in \( H_m(K_n) \) because the last coordinate in every element of \( E_1 \) is \( v_1 \), while the last coordinate in \( E_2 \) is \( e_1 - e_2 \). Hence, the set \( \frac{1}{2} (\widetilde{D}_n)_* (L^{n-1}_m) = E_1 \cup E_2 \) is linearly independent in \( H_m(K_n) \). \( \square \)
Corollary 3.13. The map \((\hat{D}_n)_* : H_m(\mathbb{T}^{n-1}) \to H_m(K_n)\) is 0 whenever \(m\) is odd and injective when \(m\) is even, in which case it induces coordinate-wise multiplication by 2.

**Proof.** By Lemma 3.11 \((\hat{D}_n)_*\) is 0 or multiplication by 2 at the level of chains. Hence, the map in homology satisfies the same property. If additionally \(m\) is even, Lemma 3.12 shows that \((\hat{D}_n)_*\) is injective because it sends the generating set \(L_{n-1}^m\) to the linearly independent set \((\hat{D}_n)_*(L_{n-1}^m)\).

\[\square\]

3.4 The computation of \(H_m(K_n(S^1))\).

Now, we have all the tools to prove our main result.

**Theorem 1.1.**

\[H_m(K_n(S^1), \mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0, \\ \mathbb{Z}^{(n-1)} \oplus \left(\mathbb{Z}/2\mathbb{Z}\right) & m \text{ even, and } 2 \leq m \leq n - 1, \\ 0 & \text{else.} \end{cases} \]

**Proof.** Recall the shorthand notation \(K_n = K_n(S^1)\). We start by giving a recursive formula to obtain the homology groups of \(K_n\) in terms of those of \(K_{n-1}\). First of all, \(K_2\) consists only of the edge \(e_1 \sim e_2\) pointing from \(v_1\) to \(v_2\), so its reduced homology groups are all 0. Additionally, \(H_0(K_n) = \mathbb{Z}\) because \(K_n\) is the image of the connected space \(\mathbb{T}^{n-1}\) under the continuous map \(\hat{D}_n\). As a consequence, the terms at the right end of the Mayer-Vietoris sequence (Proposition 3.10)

\[H_1(K_n) \xrightarrow{\partial} H_0(\mathbb{T}^{n-2}) \to H_0(K_{n-1}) \oplus H_0(K_{n-1}) \to H_0(K_n) \to 0 \]

are \(H_1(K_n) \xrightarrow{\partial} \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0\). The last 4 terms are part of the short-exact sequence

\[0 \to \mathbb{Z} \xrightarrow{(\pm 1)} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}} \mathbb{Z} \to 0, \text{ so } \partial = 0.\]

By Corollary 3.13 \(i_*\) and \(j_*\) are 0 in odd dimensions and induce coordinate-wise multiplication by 2 in even non-zero dimensions. Then, in

\[H_{2k+1}(K_n) \xrightarrow{\partial} H_{2k}(\mathbb{T}^{n-2}) \xrightarrow{\begin{pmatrix} i_* \\ j_* \end{pmatrix}} H_{2k}(K_{n-1}) \oplus H_{2k}(K_{n-1}), \]

we have \(\text{Im } \partial = \ker(i_*, j_*) = 0\). Moving to the left in the sequence above gives

\[H_{2k+1}(\mathbb{T}^{n-2}) \xrightarrow{0} H_{2k+1}(K_{n-1}) \oplus H_{2k+1}(K_{n-1}) \to H_{2k+1}(K_n) \xrightarrow{\partial = 0} H_{2k}(\mathbb{T}^{n-2}).\]

Thus, \(H_{2k+1}(K_n) = H_{2k+1}(K_{n-1}) \oplus H_{2k+1}(K_{n-1})\). Since \(H_{2k+1}(K_2) = 0\) for all \(k \geq 0\), we obtain \(H_{2k+1}(K_n) = 0\).

Now we only need to consider the even-dimensional terms in the sequence. Let’s rename the maps as follows:

\[0 \to H_{2k}(\mathbb{T}^{n-2}) \xrightarrow{a} H_{2k}(K_{n-1}) \oplus H_{2k}(K_{n-1}) \xrightarrow{b} H_{2k}(K_n) \xrightarrow{c} H_{2k-1}(\mathbb{T}^{n-2}) \to 0.\]
Write \( H_{2k}(K_n) = \mathbb{Z}^{\beta_{n,2k}} \oplus T_{n,2k} \), where \( T_{n,2k} \) is the torsion subgroup. Let \( \beta_{n,m} \) be the rank of \( H_m(K_n) \) and \( \beta^{(2)}_{n,m} \) be the number of generators of \( T_{n,2k} \). Then the above sequence becomes:

\[
0 \to \mathbb{Z}^{(n-2)} \xrightarrow{a} \mathbb{Z}^{2\beta_{n-1,2k}} \oplus T_{n-1,2k}^2 \xrightarrow{b} \mathbb{Z}^{\beta_{n,2k}} \oplus T_{n,2k} \xrightarrow{c} \mathbb{Z}^{(n-2)} \to 0.
\]

Here we make several observations. First, \( a(\mathbb{Z}^{(n-2)}) = (2 \cdot \mathbb{Z})^{(n-2)} \subset \mathbb{Z}^{2\beta_{n-1,2k}} \) because \( a \) is injective. Since \( b \circ a = 0 \), if \( \sigma \) is a generator of \( H_{2k}(\mathbb{T}^{n-2}) \), then \( b \) maps \( \frac{1}{2} a(\sigma) \) into \( T_{n,2k} \). This is because \( 2 \cdot b(\lambda(\sigma)) = b(a(\sigma)) = 0 \). Also, \( b \) is injective on \( T_{n-1,2k}^2 \) since this subgroup doesn’t intersect \( \text{Im}(a) = \ker(b) \) except at 0, so \( b(T_{n-1,2k}^2) \subset T_{n,2k} \). No other element from \( H_{2k}(K_{n-1})^2 \) maps to \( T_{n,2k} \) under \( b \), so \( T_{n,2k} \cong T_{n-1,2k}^2 \oplus \frac{1}{2} \text{Im}(a) \). Since the elements in \( \frac{1}{2} \text{Im}(a) \) have order 2, by induction, \( T_{n,2k} \) consists solely of copies of \( \mathbb{Z}/2\mathbb{Z} \). Moreover, the coefficients \( \beta^{(2)}_{n,2k} \) satisfy

\[
\beta^{(2)}_{n,2k} = 2 \cdot \beta^{(2)}_{n-1,2k} + (n-2)_{2k}.
\]

A simple induction verifies that \( \beta^{(2)}_{n,2k} = \sum_{i=0}^{n-2k-2} \binom{n-2}{i} \).

Regarding the free terms, since \( \ker(b) = \text{Im}(a) \), \( b \) is injective on \( H_{2k}(K_{n-1})/\text{Im}(a) \), and in particular, \( b(\mathbb{Z}^{2\beta_{n-1,2k}}/\frac{1}{2} \text{Im}(a)) \subset \mathbb{Z}^{\beta_{n,2k}} \). Also, since \( c \) is surjective,

\[
H_{2k}(K_n)/\text{Im}(b) \to H_{2k-1}(\mathbb{T}^{n-2})
\]

is an isomorphism. In particular, \( b \) restricted to \( \mathbb{Z}^{2\beta_{n-1,2k}}/\frac{1}{2} \text{Im}(a) \) must be an isomorphism onto a free subgroup of \( \mathbb{Z}^{\beta_{n,2k}} \). These facts imply the relation

\[
\beta_{n,2k} = \text{rk}(\mathbb{Z}^{2\beta_{n-1,2k}}/\frac{1}{2} \text{Im}(a)) + \text{rk}(H_{2k-1}(\mathbb{T}^{n-2})) = 2 \beta_{n-1,2k} - \binom{n-2}{2k} + \binom{n-2}{2k-1}.
\]

The closed formula \( \beta_{n,2k} = \binom{n-1}{2k} \) can then be obtained by induction using the relation above together with the identity \( \binom{n-2}{2k-1} + \binom{n-2}{2k} = \binom{n-1}{2k} \).

\[
4 \quad \textbf{The state complex of } \mathbb{S}^1.
\]

In this section we construct an abstract simplicial complex that we call the \( n \)-th state complex \( \text{St}_n(\mathbb{S}^1) \) of \( \mathbb{S}^1 \). We then show that the geometric realization of \( \text{St}_n(\mathbb{S}^1) \) is homeomorphic to \( K_n(\mathbb{S}^1) \). This homeomorphism has the peculiarity that it gives \( K_n(\mathbb{S}^1) \) a topological structure that does not strictly depend on distances. Instead, the state complex depends on whether the shortest path from \( x \in \mathbb{S}^1 \) to \( x' \in \mathbb{S}^1 \) has clockwise or anti-clockwise orientation.

\textbf{Definition 4.1.} Define the \textit{chirality function} \( \sigma : \mathbb{S}^1 \to \{-1, +1\} \) by

\[
\sigma(x) := \begin{cases} +1, & \text{if } \arg(x) \in [0, \pi), \\ -1, & \text{if } \arg(x) \in [\pi, 2\pi). \end{cases}
\]

We define the \textit{chiral state matrix} \( \pi(x) \) (or \textit{state matrix} for short) as the \( n \)-by-\( n \) matrix such that \( \pi_{ij}(x) = \sigma(x_j/x_i) \). We call the map \( \pi : \mathbb{T}^n \to \{-1, +1\}^{n \times n} \) the \textit{chiral state projection}.

\textbf{Notation:} Given \( a \in \{+1,-1\}^n \), let \( \text{supp}_+(a) := \{i : a_i = +1\} \).
As we mentioned above, the state matrix does not depend on the distance between the individual points. Instead, each entry \( \pi_{ij}(x) = \sigma(x_j/x_i) \) is +1 if the shortest path from \( x_i \) to \( x_j \) is anticlockwise and \(-1\) if not. This description is not completely accurate when \( x_i = \pm x_j \), however. If \( x_i = x_j \), the shortest path between \( x_i \) and \( x_j \) is constant, while if \( x_i = -x_j \), the two paths in opposite orientations have the same length. In these cases, we break the tie by setting \( \sigma(e^0) = +1 \) and \( \sigma(e^{i\pi}) = -1 \).

**Remark 4.2.** We say that \( \sigma \) is continuous in the clockwise direction to refer to the fact that \( \sigma(x) \to \sigma(x_0) \) whenever \( \arg(x) \searrow \arg(x_0) \).

**Example 4.3.** For brevity, we will only write the signs \( \pm \) instead of \( \pm 1 \). Consider the square \( x = (e^0, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2}) \in \mathbb{T}^4 \). Adding indices modulo 4, we see that \( \pi(x_j/x_i) = 1 \) if \( k = j, j+1 \) and \(-1\) if \( k = j+2, j+3 \). Then

\[
\pi(x) = \begin{pmatrix} + & + & - & - \\ - & + & + & - \\ - & - & + & + \\ + & - & - & + \end{pmatrix}.
\]

**Definition 4.4.** Let \( \mathcal{R} : \mathbb{R}^{n \times n} \to \mathcal{P}(\mathbb{R}^n) \) be the map that sends a matrix \( M \) to its set of row vectors. Consider the equivalence relation on \( \mathbb{R}^n \) given by \( x \sim -x \) and let \( q : \mathbb{R}^n \to \mathbb{R}^n/\sim \) be the quotient map. Let \( \alpha := q \circ \mathcal{R} \). We define the achiral cell map to be the composition \( \alpha \circ \pi : \mathbb{T}^n \to \mathbb{R}^n/\sim \).

**Example 4.5.** For the square in Example 4.3, we have \( \pi_{3*}(x) = -\pi_{1*}(x) \) and \( \pi_{4*}(x) = -\pi_{2*}(x) \). Hence, \( \alpha \circ \pi(x) = \{(++--) \sim (---+), (+--+ \sim (--+--))\} \).

**Definition 4.6.** The \( n \)-th State Complex of \( \mathbb{S}^1 \) is the image \( \text{St}_n(\mathbb{S}^1) := \alpha \circ \pi(\mathbb{T}^n) \).

See the appendix for the full calculation of \( \text{St}_3(\mathbb{S}^1) \) together with its Hasse diagram.

**Lemma 4.7.** Let \( \tau \in O(2) \) be a rotation. Then \( \pi(\tau(x)) = \pi(x) \).

*Proof.* Observe that \( \tau(x) = z \cdot x \) for some \( z \in \mathbb{S}^1 \), so \( \pi(\tau(x))_{ij} = \sigma(\tau(x)/\tau(x_i)) = \sigma(x_j/x_i) = \pi_{ij}(x) \).

**Lemma 4.8.** For any \( x \in \mathbb{T}^n \), \( x_i = \epsilon x_j \) for \( \epsilon \in \{+1,-1\} \) if and only if \( \pi_{ij}(x) = \pi_{ji}(x) = \epsilon \) and \( q(\pi_{is}(x)) = q(\pi_{js}(x)) \).

*Proof.* \((\Rightarrow)\) The equations \( \pi_{ij}(x) = \pi_{ji}(x) = \epsilon \) and \( q(\pi_{is}(x)) = q(\pi_{js}(x)) \) are immediate if \( x_i = x_j \). If \( x_j = -x_i \), then \( \pi_{ij}(x) = \pi_{ij}(x) = \sigma(-1) = -1 \). Also, for any \( 1 \leq k \leq n \), \( \pi_{jk}(x) = \sigma(x_k/x_j) = \sigma(-x_k/x_i) = -\sigma(x_k/x_i) = -\pi_{ik}(x) \). Hence, \( \pi_{is}(x) = -\pi_{js}(x) \) and, thus, \( q(\pi_{is}(x)) = q(\pi_{js}(x)) \).

\((\Leftarrow)\) Let \( \gamma_{ab} \) be the anticlockwise path from \( x_a \) to \( x_b \). Observe that if \( x_i \) and \( x_j \) are neither equal nor antipodal, then \( \pi_{ij}(x) \neq \pi_{ji}(x) \). This is because the union of \( \gamma_{ij} \) and \( \gamma_{ji} \) is \( \mathbb{S}^1 \), and one path has length strictly less than \( \pi \) while the other has length strictly larger than \( \pi \). For example, \( \pi_{ij}(x) = 1 \) implies \( |\gamma_{ij}| < \pi < |\gamma_{ji}| \) and \( \pi_{ji}(x) = -1 \neq \pi_{ij}(x) \). As a consequence, \( \pi_{ij}(x) = \pi_{ji}(x) \) implies that \( x_i \) and \( x_j \) are either equal or antipodal. The value of \( \epsilon \) follows similarly to the first part of the proof.

14
The equality $q(\pi_{is}(x)) = q(\pi_{is}(x))$ involves two cases: $\pi_{is}(x) = \pi_{js}(x)$ and $\pi_{is}(x) = -\pi_{js}(x)$. Observe that $\pi_{kk}(x) = 1$ for any $k$, so in case $\pi_{is}(x) = \pi_{js}(x)$, we have $\pi_{ij}(x) = \pi_{jj}(x) = 1$ and $\pi_{ji}(x) = \pi_{ii}(x) = 1$. By the previous paragraph, $x_i$ and $x_j$ are equal. If $\pi_{is}(x) = -\pi_{js}(x)$, we have $\pi_{ij}(x) = -\pi_{jj}(x) = -1$ and $\pi_{ji}(x) = -\pi_{ii}(x) = -1$ instead. Thus, $x_i$ and $x_j$ are antipodal. □

**Theorem 4.9.** $\text{St}_n(S^1)$ is an abstract simplicial complex with vertex set $\{+1, -1\}^n / \sim \subset \mathbb{R}^n / \sim$.

**Proof.** First, observe that elements of $\text{St}_n(S^1)$ are subsets of $\{+1, -1\}^n / \sim$ by definition. Conversely, for an element $[v] \in \{+1, -1\}^n / \sim$, choose any representative $v$ and set $x = v$. Then the row $\pi_{is}(x)$ is $v$ if $v_i = +1$ and $-v$ if $v_i = -1$, so $\alpha \circ \pi(x) = [v]$. Hence, the vertex set of $\text{St}_n(S^1)$ is $\{+1, -1\}^n / \sim$.

Let $x \in \mathbb{T}^n$, and $V = \{[v_1], \ldots, [v_m]\} \subset \alpha \circ \pi(x)$. Now we show that there exists $y \in \mathbb{T}^n$ such that $\alpha \circ \pi(y) = V$. Choose any $1 \leq i \leq n$ such that $q \circ \pi_{is}(x) = [v_j]$. Here, we choose $v_j$ to be the representative with $v_{ji} = +1$. Define $Y := \{\pm x_{i_1}, \ldots, \pm x_{i_m}\}$. For each $1 \leq i \leq n$, let $y_i$ be the first point in $Y$ that is reached clockwise from $x_i$, and define $y := (y_1, \ldots, y_n) \in \mathbb{T}^n$. We claim that $\pi_{ij}(y) = \pi_{ij}(x)$ for all $1 \leq j \leq m$.

Fix $1 \leq j \leq m$, and notice that $y_{ij} = x_{ij}$ by definition. Let $1 \leq i \leq n$. If $x_i = \pm x_{ik}$ for some $1 \leq k \leq m$, then $y_i = x_i$ and $\pi_{ij}(y) = \sigma(y_i/y_{ij}) = \sigma(x_i/x_{ij}) = \pi_{ij}(x)$. If $x_i \neq \pm x_{ik}$ for all $1 \leq k \leq m$, we have two cases depending on the sign of $\pi_{ij}(x) = \sigma(x_i/x_{ij})$. Let $\gamma_y$ and $\gamma'$ be the shortest paths from $x_{ij}$ to $x_i$, from $y_{ij}$ to $y_i$, and from $x_i$ to $y_i$, respectively. By definition of $y_i$, $\gamma'$ is clockwise. If $\sigma(x_i/x_{ij}) = +1$, then $\gamma_y$ is anticlockwise. Since $y_i$ is the point in $Y$ closest to $x_i$ in the clockwise direction, $y_i$ must be contained in $\gamma_y$. Then $\gamma_y$ must be anticlockwise, so $\sigma(y_i/y_{ij}) = +1$. If $\sigma(x_i/x_{ij}) = -1$ instead, $\gamma_y$ is clockwise. By definition of $y_i$, the path $\gamma'$ does not contain $-y_{ij}$ unless $y_i = -y_{ij}$. Then, the concatenated path $\gamma' \cdot \gamma_y$ goes from $y_{ij}$ to $y_i$ in clockwise direction and does not contain $-y_{ij}$ unless $y_i = -y_{ij}$. Hence, $\gamma_y = \gamma' \cdot \gamma_x$ is clockwise too and has length at most $\pi$. Thus, $\sigma(y_i/y_{ij}) = -1$. All in all, $\pi_{ij}(y) = \pi_{ij}(x)$ for all $1 \leq i \leq n$.

Now we finish proving that $\alpha \circ \pi(y) = V$. By the previous paragraph and the definition of $i_j$, $v_j = \pi_{is}(x) = \pi_{is}(y)$, so $[v_j] \in \alpha \circ \pi(y)$. Hence, $V \subset \alpha \circ \pi(y)$. Conversely, notice that for all $i$, $y_i = y_i$ or $y_i = -y_i$ for some $j$. Lemma 4.8 then gives $q \circ \pi_{is}(y) = q \circ \pi_{is}(y) \in V$. □
4.1 Cluster structures in $\mathbb{T}^n$.

We can interpret Lemma 4.8 by saying that $\alpha \circ \pi(x)$ induces a clustering of $x$ where $x_i$ and $x_j$ are in the same cluster if they are equal or antipodal or, equivalently, if $q \circ \pi_i(x) = q \circ \pi_j(x)$.

We take this idea as inspiration for the following construction.

**Definition 4.10.** Given $x \in \mathbb{T}^n$, define $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{T}^n$ where $\tilde{x}_i := x_i$ if $\pi_{1i}(x) = +1$ and $\tilde{x}_i := -x_i$ if $\pi_{1i}(x) = -1$.

**Definition 4.11.** Set $k_1 := 1$ and $a_1 := \pi_{k_1}(x)$. Inductively define $k_{j+1}$ to be any index such that $\tilde{x}_{k_{j+1}}$ is the first point that is reached from $\tilde{x}_{k_j}$ in anticlockwise direction for which $q \circ \pi_{k_{j+1}}(x) \in A \setminus \{[a_1], \ldots, [a_j]\}$. We also set $a_{j+1} := \pi_{k_{j+1}}(x)$ so that $A = \{[a_j] : 1 \leq j \leq m\}$, where $m \leq n$. Define the function $\xi : \{1, \ldots, n\} \rightarrow \{\pm 1, \ldots, \pm m\}$ by $\xi(i) := j \cdot \pi_{1i}(x)$, where $1 \leq j \leq m$ satisfies $q \circ \pi_{is}(x) = [a_j]$. We call $\xi$ the *cluster structure* induced by $x$.

**Remark 4.12.** At any point in Definition 4.11 there could exist $k'_j \neq k_j$ such that $\tilde{x}_{k'_j} = \tilde{x}_{k_j}$. However, $[a_j]$ is independent on the choice between $k_j$ and $k'_j$ and the rest of the definition carries through. Indeed, $\tilde{x}_{k'_j} = \tilde{x}_{k_j}$ implies $x_{k'_j} = \pm x_{k_j}$, so Lemma 4.8 gives $[a_j] = \alpha \circ \pi_{k'_j}(x) = \alpha \circ \pi_{k_j}(x)$.

The cluster structure $\xi$ induces the same clustering that we described at the start of the section, that is, $|\xi(i)| = |\xi(j)|$ if and only if $x_i$ and $x_j$ are equal or antipodal. The next Lemma gives a list of other equivalent conditions.

**Lemma 4.13.** Let $x \in \mathbb{T}^n$ and $m := |\alpha \circ \pi(x)|$. Let $k_1, \ldots, k_m$ be as in Definition 4.11. Then, for any $1 \leq i, j \leq n$, the following are equivalent:

1. $\tilde{x}_i = \tilde{x}_j$.

2. There exist $1 \leq \ell \leq m$ and $\epsilon_i, \epsilon_j \in \{+1, -1\}$ such that $x_i = \epsilon_i \tilde{x}_{k_\ell}$ and $x_j = \epsilon_j \tilde{x}_{k_\ell}$.

3. $q \circ \pi_{is}(x) = q \circ \pi_{js}(x)$.

4. For some $1 \leq \ell \leq m$, $|\xi(i)| = |\xi(j)| = \ell$.

**Proof.** By definition of $\tilde{x}$, item 1 is equivalent to $x_i = \pm x_j$. Then Lemma 4.8 gives 1 $\iff$ 3 while the equivalences 2 $\iff$ 4 and 3 $\iff$ 4 follow from Definition 4.11.

Even though Definition 4.11 relies on both $\pi(x)$ and the geometry of $\tilde{x}$, we can actually determine $\xi$ from $\pi(x)$ only. We prove this fact in two lemmas.

**Lemma 4.14.** Let $I_i$ be the $n$-by-$n$ diagonal matrix defined by $(I_i)_{jj} = +1$ if $j \neq i$ and $(I_i)_{ii} = -1$. Let $J := \text{diag}(\pi_{1i}(x))$. Then $\pi(x(-x_i;i)) = I_i \cdot \pi(x) \cdot I_i$ and, as a consequence, $
abla(x) = J \cdot \pi(x) \cdot J$.

**Proof.** Observe that the entries that differ in $\pi(x)$ and $\pi(x(-x_i;i))$ must be in the $i$-th row or the $i$-th column. Out of these, $\pi_{ii}(x(-x_i;i)) = +1$ for any $x \in \mathbb{T}^n$. As for $j \neq i$, $\arg(-x_i/x_j) = \arg(x_i/x_j) + \pi \mod 2\pi$. Then $\sigma(-x_i/x_j) = -\sigma(x_i/x_j)$ and $\pi_{ij}(x(-x_i;i)) = -\pi_{ij}(x)$. Similarly, $\pi_{ji}(x(-x_i;i)) = -\pi_{ji}(x)$. These observations give $\pi(x(-x_i;i)) = I_i \cdot \pi(x) \cdot I_i$. 

16
To prove $\pi(\tilde{x}) = J \cdot \pi(x) \cdot J$, let $1 < i_1 < \cdots < i_m \leq n$ be the indices such that $\pi_{i,j}(x) = -1$. Define $x_0 := x$ and, for $1 \leq j \leq m$, $x_j := x_{j-1}(-x_{i_j}, i_j)$. Since $\tilde{x} = x_m$ and $J = I_{i_1} \cdots I_{i_m}$, the previous paragraph implies that $\pi(\tilde{x}) = J \cdot \pi(x) \cdot J$.

A consequence of Lemma 4.14 is that $\pi_{1*}(\tilde{x}) = (+1, \ldots , +1)$ because we replaced every $x_i$ such that $\pi_{i*}(x) = -1$ with $-x_i$. This implies that $\tilde{x}$ is contained in a semicircle. While this follows from the definition of $\tilde{x}$, we can infer that with nothing more than $\pi(\tilde{x})$.

**Lemma 4.15.** For any $x \in \mathbb{T}^n$ and $1 \leq i \leq n$,

$$\text{supp}_+(\pi_{i*}(\tilde{x})) = \{1 \leq t \leq n : |c_x(i)| \leq |c_x(t)|\}.$$

As a consequence, $\text{supp}_+(\pi_{i*}(\tilde{x})) \subseteq \text{supp}_+(\pi_{j*}(\tilde{x}))$ if and only if $|c_x(i)| \geq |c_x(j)|$.

**Proof.** Let $\gamma$ be the anticlockwise path from $x_1$ to $-x_1$. We defined $\tilde{x}$ so that every $\tilde{x}_i$ is contained in $\gamma$ so, in particular, the shortest path between any $\tilde{x}_i$ and $\tilde{x}_j$ is contained in $\gamma$. Then $t \in \text{supp}_+(\pi_{i*}(\tilde{x}))$, i.e. $\pi_{i*}(\tilde{x}) = +1$, if and only if $\gamma$ reaches $\tilde{x}_i$ before $\tilde{x}_j$. By definition of $c_x$, this is equivalent to $|c_x(i)| \leq |c_x(t)|$.

With Lemma 4.15 at hand, we describe how to reconstruct $|c_x|$ from $\pi(\tilde{x})$. Consider the set $S := \{\text{supp}_+(\pi_{i*}(x)), \ldots , \text{supp}_+(\pi_{n*}(x))\}$ ordered by inclusion, and let $m := |S|$. Let $k_1 = 1$ and for each $1 < j \leq m$, select $1 < k_j \leq n$ so that $\text{supp}_+(\pi_{k_j*}(x)) \subset \cdots \subset \text{supp}_+(\pi_{k_2*}(x)) \subset \text{supp}_+(\pi_{k_1*}(x))$. Lemma 4.15 now gives $|c_x(k_j)| = j$ for $1 \leq j \leq m$. Furthermore, for any $1 \leq i \leq n$, $\text{supp}_+(\pi_{i*}(x)) = \text{supp}_+(\pi_{k_j*}(x))$ for some $1 \leq j \leq m$, and this is equivalent to $\pi_{i*}(x) = \pi_{k_j*}(x)$. Then $|c_x(i)| = j(i)$ by Lemma 4.13. This shows that $\pi(x)$ has enough information to reconstruct $c_x$. Lemma 4.14 shows how to obtain $\pi(\tilde{x})$ from $\pi(x)$, the former gives $|c_x|$ as above, and the signs of $c_x$ are given by $\pi(x)$ by definition.

Now we show the opposite: $\pi(x)$ can be reconstructed from $c_x$.

**Lemma 4.16.** Let $x \in \mathbb{T}^n$. Let $\epsilon_i := \text{sgn}(c_x(i))$. For all $1 \leq i, j \leq n$, $\pi_{ij}(x) = \epsilon_i \epsilon_j$ if $|c_x(i)| = |c_x(j)|$ and $\pi_{ij}(x) = \epsilon_i \epsilon_j \cdot \text{sgn}(|c_x(j)| - |c_x(i)|)$ otherwise.

**Proof.** Recall that $x_i = \epsilon_i \cdot \tilde{x}|_{c_x(i)}$ for all $i$. If $|c_x(i)| = |c_x(j)|$ then $\tilde{x}|_{c_x(i)} = \tilde{x}|_{c_x(j)}$, which in turn implies that either $x_i = x_j$ or $x_i = -x_j$. Thus, $\pi_{ij}(x) = +1$ if and only if $\epsilon_i = \epsilon_j$. This proves the first claim.

For the second claim, suppose $|c_x(i)| < |c_x(j)|$ so that the shortest path from $\tilde{x}|_{c_x(i)}$ to $\tilde{x}|_{c_x(j)}$ is anticlockwise and $\text{sgn}(|c_x(j)| - |c_x(i)|) = +1$. We have to cover four cases depending on the signs of $\epsilon_i$ and $\epsilon_j$. If $\epsilon_i = \epsilon_j = +1$, then $x_i = \tilde{x}|_{c_x(i)}$ and $x_j = \tilde{x}|_{c_x(j)}$, so $\pi_{ij}(x) = +1$.

If $\epsilon_i = \epsilon_j = -1$, we have instead $x_i = -\tilde{x}|_{c_x(i)}$ and $x_j = -\tilde{x}|_{c_x(j)}$. In $\mathbb{S}^1$, the antipodal map $z \mapsto -z$ is a rotation, so the shortest path from $x_i$ to $x_j$ has the same direction as the path from $\tilde{x}|_{c_x(i)}$ to $\tilde{x}|_{c_x(j)}$. Thus, $\pi_{ij}(x) = +1$.

If $\epsilon_i = +1$ and $\epsilon_j = -1$, the shortest path from $\tilde{x}|_{c_x(i)}$ to $\tilde{x}|_{c_x(j)}$ is $x_j$ is anticlockwise. This forces the shortest path from $x_j$ to $x_i$ to be anticlockwise. Similarly if $\epsilon_i = -1$ and $\epsilon_j = +1$, the shortest paths from $\tilde{x}|_{c_x(i)}$ to $\tilde{x}|_{c_x(j)}$ are both anticlockwise. In either case, $\pi_{ij}(x) = -1$.

We record the overall consequence in the following corollary.
Corollary 4.17. For any $x, y \in \mathbb{T}^m$, $\pi(x) = \pi(y)$ if and only if $c_x = c_y$.

Proof. The discussion between Lemmas 4.11 and 4.16 shows that $\pi(x) = \pi(y)$ implies $c_x = c_y$. The converse is given in Lemma 4.16 \qed

Now we consider the cluster structure $c^*_x$ of $\rho(x)$. Even though a pair $x_i$ and $x_j$ that is equal or antipodal remains so after applying $\rho$ to $x$, the point closest to $x_1$ might be different. We record the changes in the next Proposition.

Proposition 4.18. Let $c$ and $c^T$ be the cluster structures of $x$ and $\rho(x)$, respectively, and let $m = |\alpha \circ \pi(x)|$. Then:

$$c^T(i) = \begin{cases} -\text{sgn}(c(i)) \left(m + 2 - |c(i)|\right), & \text{if } |c(i)| \neq 1, \\ c(i), & \text{if } |c(i)| = 1. \end{cases}$$

The intuition behind the formula for $c^T$ is that $\rho$ exchanges the anticlockwise and clockwise directions, so the choices involved in Definition 4.11 for $c$ and $c^T$, with the exception of $k_1 = 1$, would be opposite. In fact, the function $c \mapsto m+2-c$ sends the sequence $2, \ldots, m$ to $m, \ldots, 2$. This argument is not a proof on its own, however, because $\rho(\bar{x})$ is usually different from $\rho(x)$. We work out the details next.

Proof. Let $\bar{X} := \{\bar{x}_1, \ldots, \bar{x}_n\}$. By Lemma 4.13 $q \circ \pi_+(x) = q \circ \pi_+(\rho(x))$ is equivalent to $\bar{x}_i = \bar{x}_j$, so $|\bar{X}| = m$. Let $k_1 := 1$ and for $1 < j \leq m$, choose indices $1 < k_j \leq n$ such that $|c(k_j)| = j$. Define $y := (x_{k_1}, \ldots, x_{k_m}) \in \mathbb{T}^m$ and let $c'$ and $(c')^T$ be the cluster structures of $y$ and $\rho(y)$. By definition of $c$, the points $x_1 = \bar{x}_{k_1}, \bar{x}_{k_2}, \ldots, \bar{x}_{k_m}$ are all distinct and appear in anticlockwise order, so $|c'(i)| = i$. As a consequence, $\pi_{ij}(\bar{y}) = +1$ if and only if $i \leq j$. We remark that the choice of $k_j$ is irrelevant as long as $\bar{x}_{k_j}$ is unique. Additionally, $(\bar{y})_j = \bar{x}_{k_j} = (\bar{x})_{k_j}$ because even though the definitions of $\bar{x}$ and $\bar{y}$ depend on $(\bar{x})_1$ and $(\bar{y})_1$, both points equal $x_1$.

Now we show that $c'$ and $(c')^T$ satisfy Equation (2) up to signs. Let $S_i := \text{supp}_+(\pi_+(\rho(y)))$ and $S_i^T := \text{supp}_+(\pi_+(\rho(y)))$. By Lemma 4.13 $S_j \subset S_i$ if and only if $i \leq j$. As suggested by the discussion preceding this proof, we will show that $S_i^T \subset S_i^T$ whenever $1 < i \leq j$. We already know $S_i^T \subset S_i^T = \{1, \ldots, m\}$ for $1 \leq i \leq m$. If $i > 1$, observe that $\bar{x}_{k_i}/x_{k_i} = \bar{x}_{k_i} = \bar{x}_{k_i}/x_{k_i}$ and $x_{k_i}^2 = \bar{x}_{k_i}^2$. Then, for $1 < i < j$, we have

$$\pi_{ij}(\rho(y)) = \sigma(\rho(x_{k_i})/\rho(x_{k_i})) = \sigma(\bar{x}_{k_i}/\bar{x}_{k_i} \cdot x_{k_i}/x_{k_i} \cdot \rho(x_{k_i})/\rho(x_{k_i}))$$

$$= \sigma(\bar{x}_{k_i}/\bar{x}_{k_i} \cdot (x_{k_i}/x_{k_i})^2) = \sigma(\bar{x}_{k_i}/\bar{x}_{k_i}) = \pi_{ij}(\bar{y}) = -\pi_{ij}(\bar{y}).$$

Hence, if $S_i^c$ denotes the complement of $S_i$ in $\{1, \ldots, m\}$, we have $S_i^c \subset S_i^c$ and $S_i^T = S_i^c \cup \{i\}$ (recall that $\pi_{ij}(x) = 1$ for any $x$, so $i \in S_i^T$). Then $S_i^T \subset S_i^T$ because $\pi_{ij}(\bar{y}) = -1$ implies $i \in S_i^c$ and, thus, $S_i^T = S_i^c \cup \{i\} \subset S_i^c \subset S_i^T$. Overall, we proved that $S_2^c \subset \cdots \subset S_m^c \subset S_1^T$, and from there, Lemma 4.13 gives $|c'(1)| = 1$, $|c'(m)| = 2$, $\ldots$, $|c'(2)| = m$. Since $|c'(i)| = i$, Equation (2) holds up to signs.

To show that $c$ and $c^T$ satisfy Equation (2), for every $1 \leq i \leq n$, let $1 \leq j_i \leq m$ such that
\(\bar{x}_i = \bar{x}_{k_j} = (\bar{y})_{j_i}\). Since the numbers \(k_j\) can be used in the definition of \(c\) and \(c^\top\), we get

\[|c(i)| = |c'(j_i)| = j_i\] and

\[|c^\top(i)| = |(c')^\top(j_i)|\] for all \(1 \leq i \leq n\). If \(j_i > 1\),

\[|c^\top(i)| = |(c')^\top(j_i)| = m + 2 - |c'(j_i)| = m + 2 - |c(i)|.\]

Furthermore, \(j_i \neq 1\) implies that \(x_i\) and \(x_1\) are neither equal nor antipodal, so \(\pi_{1i}(\rho(x)) = -\pi_{1i}\rho(x) = -\pi_{1i}(\rho(x))\) by Lemmas 4.18 and 4.26. In other words, \(\text{sgn}(c^\top(i)) = -\text{sgn}(c(i))\), so the first case of Equation (2) holds. The second case is equivalent to \(j_i = |c(i)| = 1\) and, thus, to \(x_i = \epsilon_i x_1\) for some \(\epsilon_i \in \{+1, -1\}\). In that case, \(\rho(x_i) = \epsilon_i x_1\) and we obtain \(c^\top(i) = c(i) = \epsilon_i.\)

**4.2 Face numbers of \(\text{St}_n(S^1)\) and Euler characteristic.**

Lemma 4.17 implies that \(\alpha \circ \pi(x)\) is invariant under translations, and we will later show that \(\alpha \circ \pi(\rho(x)) = \alpha \circ \pi(x)\) (see Lemmas 4.25 and 4.26); in other words, \(\alpha \circ \pi(x)\) is invariant under the action of \(O(2)\). However, \(c_X\) is only invariant under rotations. In fact, we will also show that every \(A = \alpha \circ \pi(x)\) corresponds to two cluster structures \(c_X\) and \(c^\top_X\) (see Remark 4.31). For now, we use this observation to find the number of simplices in each dimension in \(\text{St}_n(S^1)\).

For each \(k \leq n - 1\), let \(f(n, k)\) denote the number of \(k\) dimensional simplices in \(\text{St}_n(S^1)\). Let \(S(\cdot, \cdot)\) denote the Stirling numbers of the second kind. These are the number of ways we can partition the set \(\{1, \ldots, n\}\) into \(k\) unordered clusters.

**Proposition 4.19.** \(f(n, 0) = 2^{n-1}\) and for \(m \geq 1\), \(f(n, m) = 2^{n-2} \cdot m! \cdot S(n, m + 1)\).

**Proof.** We already showed that the vertex set of \(\text{St}_n(S^1)\) is \(\{+1, -1\}^n/\sim\) in Theorem 4.19 so \(f(n, 0) = 2^{n-1}\). By Remark 4.31, each \(A \in \text{St}_n(S^1)\) with \(m + 1\) elements corresponds to two cluster structures \(c_X\) and \(c^\top_X\). To count the number of functions \(c_X : \{1, \ldots, n\} \rightarrow \{-1, \ldots, -1, m + 1\}\), observe that the function \(|c_X|\) assigns each \(1 \leq i \leq n\) into one of \((m + 1)\) ordered clusters. Since we always have \(c_X(1) = 1\), the first cluster is already determined. To obtain \(c_X\) from \(|c_X|\) we have to choose \(n - 1\) signs. Thus, the number of functions \(c_X\) is \(2^{n-1} \cdot m! \cdot S(n, m + 1)\). The fact that each \(A\) corresponds to two functions \(c_X\) and \(c^\top_X\) gives the result. \[\square\]

These formulas can be used to calculate the Euler characteristic of \(K_n(S^1)\) for any \(n\).

**Proposition 4.20.** \(\chi(\text{St}_n(S^1)) = 2^{n-2}\).

**Proof.** We have \(\chi(\text{St}_n(S^1)) = \)

\[= 2^{n-1} + \sum_{k=1}^{n-1} (-1)^k S(n, k + 1)2^{n-2}k!\]

\[= 2^{n-2} + \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)2^{n-2}k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]

\[= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k S(n, k + 1)k!\]
\[
= 2^{n-2} + 2^{n-2} \sum_{k=0}^{n-1} (-1)^k \frac{k!}{(k+1)!} \sum_{j=0}^{k+1} (-1)^{k-j+1} j^n \binom{k+1}{j}
\]
\[
= 2^{n-2} \left[ 1 + \sum_{k=0}^{n-1} \frac{1}{k+1} \sum_{j=0}^{k+1} (-1)^{k-j+1} j^n \binom{k+1}{j} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{k} (-1)^{j} j^n \binom{k}{j} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{j} j^{-1} \binom{k}{j-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{j} j^{-1} \binom{k}{j-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{l=1}^{j} \frac{k-1}{j-1} \binom{j}{l} (-1)^{j+l} \sum_{m=1}^{l} (-1)^{l} \binom{l}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{l=1}^{j} \frac{k-1}{j-1} \binom{j}{l} (-1)^{j+l} \sum_{m=1}^{l} (-1)^{l} \binom{l}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{k}{j} \binom{k}{j} \sum_{l=1}^{j} \frac{(-1)^j}{(j-1)!} \frac{(k-l)!}{l!(k-l)!} \sum_{m=1}^{l} (-1)^{l} \binom{l}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{k}{j} \binom{k}{j} \sum_{l=1}^{j} \frac{(-1)^j}{(j-1)!} \frac{(k-l)!}{l!(k-l)!} \sum_{m=1}^{l} (-1)^{l} \binom{l}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{l=1}^{j} \frac{1}{k} \binom{k}{l} \sum_{j=0}^{k-l} (-1)^{j} \binom{k-l}{j} \sum_{m=1}^{l} (-1)^{m} \binom{l}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{l=1}^{j} \frac{1}{k} \binom{k}{l} \left[ l \delta_{0(k-l)} - \delta_{1(k-l)} \right] \sum_{m=1}^{l} (-1)^{m} \binom{l}{m} m^{n-1} \right] \tag{3}
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{l=1}^{j} \frac{1}{k} \binom{k}{l} l \delta_{0(k-l)} \sum_{m=1}^{l} (-1)^{m} \binom{l}{m} m^{n-1} + \sum_{k=1}^{n} \sum_{l=1}^{j} \frac{1}{k} \binom{k}{l} \delta_{1(k-l)} \sum_{m=1}^{l} (-1)^{m} \binom{l}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \frac{1}{k} \binom{k}{k} \sum_{m=1}^{k} (-1)^{m} \binom{k}{m} m^{n-1} + \sum_{k=1}^{n} \frac{1}{k} \binom{k}{k-1} \sum_{m=1}^{k-1} (-1)^{m} \binom{k-1}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{k=1}^{n} \sum_{m=1}^{k} (-1)^{m} \binom{k}{m} m^{n-1} + \sum_{k=1}^{n-1} \sum_{m=1}^{k} (-1)^{m} \binom{k}{m} m^{n-1} \right]
\]
\[
= 2^{n-2} \left[ 1 - \sum_{m=1}^{n} (-1)^m \binom{n}{m} m^{n-1} \right] \\
= 2^{n-2} \left[ 1 + \delta_{1n} \right]
\]

where the collapse of the central factor in equation 3 follows from the identity:

\[
\sum_{i=0}^{n} (-1)^i (i+m) \binom{n}{i} = m \sum_{i=0}^{n} (-1)^i \binom{n}{i} + \sum_{i=1}^{n} (-1)^i i \binom{n}{i} \\
= m\delta_{0n} + \sum_{i=1}^{n} (-1)^i n \binom{n-1}{i-1} \\
= m\delta_{0n} - n \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} = m\delta_{0n} - \delta_{1n}
\]

and where the last equality follows from the observation that the summation in the second to last line is (up to sign) the inclusion-exclusion sum for the number of ways of painting \(n-1\) labeled balls with exactly \(n\) colors, which is 0, while in the edge case that \(n = 1\), the value it takes (1) can be ascertained by inspection.

\textbf{Remark 4.21.} The Euler characteristic of \(K_n(S^1)\) can be computed via Theorem 1.1 and coincides with that of \(St_n(S^1)\). This is to be expected in light of the homeomorphism \(|St_n(S^1)| \cong K_n(S^1)\) given in Theorem 4.33.

\section*{4.3 A map from \(K_n(S^1)\) to \(St_n(S^1)\).}

We now prove a series of technical lemmas in order to construct a map \(g: K_n(S^1) \to St_n(S^1)\) making the following diagram commute:

\[
\begin{array}{ccc}
\mathbb{T}^n & \xrightarrow{D_n} & K_n(S^1) \\
\alpha \circ \pi & \downarrow & \downarrow g \\
& St_n(S^1) &
\end{array}
\]

For the first proof, let \(x \in \mathbb{T}^n\) and define \(x(t; i) := (x_1, \ldots, t, \ldots, x_n)\) as the vector where the \(i\)-th entry is replaced with \(t \in S^1\).

\textbf{Lemma 4.22.} Let \(x \in \mathbb{T}^n\). Given \(a, b \in S^1\), let \(\gamma_{ab}\) be the anticlockwise path from \(a\) to \(b\). Suppose that \(\gamma_{ab}\) contains \(x_i\) but not any other point \(x_j\) or its antipode \(-x_j\) for \(j \neq i\). Then \(\alpha \circ \pi(x(t; i)) = \alpha \circ \pi(x)\) for all \(t \in \gamma_{ab}\).

\textbf{Proof.} Consider the functions \(t \mapsto \sigma(t/x_j)\) and \(t \mapsto \sigma(x_j/t)\) for \(j \neq i\). These are piecewise constant functions with critical points at \(t = x_j\) and \(t = -x_j\). Similarly, the function
For all $t \mapsto \pi(x(t;i))$ is piecewise constant and has critical points at $t = x_j$ and $t = -x_j$ for all $j \neq i$. Since $\gamma_{ab}$ contains none of the critical points but does contain $x_i$, $\pi(x(t;i)) = \pi(x)$ for all $t \in (a, b)$.

**Remark 4.23.** Observe that, under the conditions of Lemma 4.22, the antipodal path $-\gamma_{ab}$ contains $-x_i$ but no other point $x_j, -x_j$. Hence, the same argument shows that $\alpha \circ \pi(x(t;i)) = \alpha \circ \pi(x(-x_i;t;i))$ for all $t \in -\gamma_{ab}$.

**Remark 4.24.** Let $a_i := q \circ \pi_{is}(x)$. Combining Lemmas 4.25 and 4.22 with Remark 4.2 shows that $\alpha \circ \pi(x(t;i)) = \alpha \circ \pi(x) \setminus \{a_i\}$ if $t \rightarrow x_i$ clockwise, and $\alpha \circ \pi(x(t;i)) = \alpha \circ \pi(x) \setminus \{a_{i-1}\}$ if $t \rightarrow x_i$ anticlockwise.

The next two lemmas show that $\alpha \circ \pi(x)$ is invariant under $\rho$ by showing that $\rho$ induces a transposition in $\pi(x)$ and that $\alpha$ is invariant under transpositions.

**Lemma 4.25.** Let $x \in \mathbb{T}^n$. Then $\pi(\rho(x)) = \pi(x)^{\top}$.

**Proof.** Observe that $\pi_{ij}(\rho(x)) = \sigma(x_j^{-1}/x_i^{-1}) = \sigma(x_i/x_j) = \pi_{ji}(x)$.

**Lemma 4.26.** For any $x \in \mathbb{T}^n$, $\alpha(\pi(x)) = \alpha(\pi(x)^{\top})$.

**Proof.** The strategy for this proof will be to show that, for each $1 \leq i \leq n$, there exists a $j$ such that $\pi_{is}(x)$ equals either $(\pi(x)^{\top})_{js}$ or $-(\pi(x)^{\top})_{js}$. In other words, for every $i$ there exists $j$ such that $q(\pi_{is}(x)) = q(\pi(\pi(x)^{\top})_{js})$, so $\alpha(\pi(x)) \subset \alpha(\pi(x)^{\top})$. It turns out that this inclusion is enough, because we can then write $\rho(x)$ in place of $x$ and use Lemma 4.25 to obtain $\alpha(\pi(x)^{\top}) = \alpha(\pi(\rho(x))) \subset \alpha(\pi(x)^{\top}) = \alpha(\pi(x))$. To wit, fix $1 \leq i \leq n$ and define

$$M_i := \max(\{\arg(x_j) \mid 1 \leq j \leq n\} \cup \{\arg(-x_j) \mid 1 \leq j \leq n\}).$$

We have two cases depending on which set realizes the maximum.

**Case 1:** $M_i = \arg(x_j)$ for some $j$.

**Case 2:** $M_i = -\arg(x_j)$ for some $j$. For any other point $x_k$, $\sigma(x_j/x_k)$ and $\sigma(x_j)$ are different.

**Figure 4: Left (Case 1):** $M_i = \arg(x_j)$ for some $j$. For any other point $x_k$, $\sigma(x_j/x_k)$ and $\sigma(x_j)$ are different. **Right (Case 2):** $M_i = -\arg(x_j)$ for some $j$. For any other point $x_k$, $\sigma(x_j/x_k) = \sigma(x_j)$. In both cases, the shaded regions do not contain any other point.

We claim that $\pi_{is}(x) = -(\pi(x)^{\top})_{js}$. Let $1 \leq k \leq n$. Since $\frac{x_j}{x_k} = \frac{x_j}{x_k} \frac{x_k}{x_k}$, we have $\arg(x_j) + \arg(\frac{x_j}{x_k}) \equiv \arg(x_j) \mod 2\pi$. However, all three of $\arg(x_j), \arg(x_j/x_k)$ and $\arg(x_j/x_k)$ are in the
interval \([0, 2\pi]\), and \(\arg(\frac{\pi}{x_i})\) is larger than or equal to the other two, so \(\arg(\frac{\pi}{x_j}) + \arg(\frac{\pi}{x_k}) = \arg(\frac{\pi}{x_i})\). Additionally, since the sum of \(\arg(\frac{\pi}{x_j})\) and \(\arg(\frac{\pi}{x_k})\) is less than \(2\pi\), at least one of them must be less than \(\pi\). However, if both were less than \(\pi\),

\[
M_i = \arg(\frac{\pi}{x_j}) = \arg(\frac{\pi}{x_k}) + \arg(\frac{\pi}{x_i}) \leq \arg(\frac{\pi}{x_k}) + \pi = \arg(\frac{\pi}{x_k}),
\]

which is a contradiction. Therefore, exactly one of \(\arg(\frac{\pi}{x_j})\) and \(\arg(\frac{\pi}{x_k})\) must be in \([0, \pi]\) while the other is in \([\pi, 2\pi]\). In other words, \(\sigma(\frac{\pi}{x_k}) \neq \sigma(\frac{\pi}{x_i})\), so \(\pi_{ik}(x) = \sigma(\frac{\pi}{x_k}) = -\sigma(\frac{\pi}{x_j}) = \pi_{kj}(x)\).

Since this equation holds for all \(k\), \(\pi_{is}(x) = -(\pi(x)^T)_{js}\).

**Case 2:** \(M_i = \arg(-\frac{\pi}{x_i})\) for some \(j\).

Now we claim that \(\pi_{is}(x) = (\pi(x)^T)_{js}\). First, observe that since \(\arg(-\frac{\pi}{x_i}) \equiv \arg(\frac{\pi}{x_i}) + \pi\) mod \(2\pi\), the assumption \(M_i = \arg(-\frac{\pi}{x_i}) \geq \arg(\frac{\pi}{x_i})\) forces \(\arg(\frac{\pi}{x_i}) < \pi\). Now, for each \(k\), the equation \(\arg(\frac{\pi}{x_k}) + \arg(\frac{\pi}{x_i}) \equiv \arg(\frac{\pi}{x_i})\) mod \(2\pi\) can imply one of two cases.

**Case 2a:** \(\arg(\frac{\pi}{x_j}) + \arg(\frac{\pi}{x_k}) = \arg(\frac{\pi}{x_i})\).

Since both \(\arg(\frac{\pi}{x_j})\) and \(\arg(\frac{\pi}{x_k})\) are nonnegative, \(\arg(\frac{\pi}{x_j}) + \arg(\frac{\pi}{x_k}) = \arg(\frac{\pi}{x_i}) < \pi\). Similarly, \(\arg(\frac{\pi}{x_k}) < \pi\), so \(\sigma(\frac{\pi}{x_k}) = \sigma(\frac{\pi}{x_j}) = 1\).

**Case 2b:** \(\arg(\frac{\pi}{x_j}) + \arg(\frac{\pi}{x_i}) = \arg(\frac{\pi}{x_k}) + 2\pi\).

Since \(\arg(\frac{\pi}{x_k}) < 2\pi\), we must have \(\arg(\frac{\pi}{x_i}) > \arg(\frac{\pi}{x_k})\). If in addition \(\arg(\frac{\pi}{x_k}) < \pi\), we would have \(\arg(-\frac{\pi}{x_i}) = \arg(\frac{\pi}{x_k}) + \pi > \arg(\frac{\pi}{x_j}) + \pi = \arg(-\frac{\pi}{x_j}) = M_i\). This is a contradiction, so \(\arg(\frac{\pi}{x_k}) \geq \pi\). Analogously we obtain \(\arg(\frac{\pi}{x_j}) \geq \pi\), and with that, \(\sigma(\frac{\pi}{x_k}) = \sigma(\frac{\pi}{x_i}) = -1\).

In summary, Cases 2a and 2b say that \(\sigma(\frac{\pi}{x_j}) = \sigma(\frac{\pi}{x_k})\) for every \(1 \leq k \leq n\). Thus, \(\pi_{ik}(x) = \pi_{kj}(x)\) for all \(j\), and \(\pi_{is}(x) = (\pi(x)^T)_{js}\). This is what we wanted.

**Lemma 4.27.** For all \(x \in \mathbb{T}^n\) and \(\tau \in O(2)\), \(\alpha \circ \pi(x) = \alpha \circ \pi(\tau(x))\).

**Proof.** If \(\det(\tau) = 1\), there exists \(\lambda \in S^1\) such that \(\tau(z) = \lambda z\). Then \([\pi(\tau(x))]_{ij} = \lambda x_j/x_i = x_j/x_i = [\pi(x)]_{ij},\) and \(\pi(x) = \pi(\tau(x))\). If \(\det(\tau) = -1\), then \(\det(\rho \circ \tau) = 1\) and \(\alpha \circ \pi(x) = \alpha \circ \pi(\rho \circ \tau(x))\). By Lemmas 4.25 and 4.26 we have \(\alpha \circ \pi(\tau(x)) = \alpha([\pi(\rho \circ \tau(x))]^T) = \alpha \circ \pi(\rho \circ \tau(x))\).

Now that we verified that \(D_n\) and \(\alpha \circ \pi\) are invariant under the action of \(O(2)\), we can construct the desired map from \(K_n(S^1)\) to the state complex.

**Theorem 4.28.** There exists a map \(g : K_n(S^1) \to St_n(S^1)\) such that for \(M \in K_n(S^1)\), \(g(M) = \alpha \circ \pi(x)\) for every \(x \in D_n^{-1}(M)\).

**Proof.** We only have to check that \(g\) is well defined. Indeed, for any \(x, y \in D_n^{-1}(M)\), Lemma 3.2 gives a \(\tau \in O(2)\) such that \(\tau(x) = y\), and by Lemma 4.27 \(\alpha \circ \pi(x) = \alpha \circ \pi(y)\). Thus, \(g(M) = \alpha \circ \pi(x)\) is well defined.

### 4.4 The geometric realization of \(St_n(S^1)\) is homomorphic to \(K_n(S^1)\).

In this section, we prove that the geometric realization of the state complex \(St_n(S^1)\) is homomorphic to \(K_n(S^1)\). In order to do that, for every \(A \in St_n(S^1)\) such that \(|A| = m\), we
would like to construct a function $\Phi_A : \Delta_{m-1} \to K_n(S^1)$ that is a homeomorphism onto its image and satisfies $\Phi_A(\partial\Delta_{m-1}) = \cup_{B=A\{a_i\}}\Phi_B(\Delta_{m-2})$. As an intermediate step, we define $\Phi_\epsilon$ for the cluster structure $c := c_\pi$ of any given $x \in \mathbb{T}^n$. We model $\Delta_m$ as

$$\Delta_m := \{ t \in \mathbb{R}^m : 0 \leq t_i \leq \pi \text{ and } t_1 + \ldots + t_n \leq \pi \}.$$  

Recall from the construction of $c$ that the argument of every point $\tilde{x}_{k_j}$ lies on the interval $[0, \pi)$, so

$$d_{k_1,k_2} + \ldots + d_{k_{m-1},k_m} < \pi.$$  

For $t \in \Delta_m$, let $S_1(t) := 0$ and $S_j(t) := t_1 + \ldots + t_{j-1}$. Define

$$\Phi_\epsilon : \Delta_{m-1} \to \{+1\} \times \mathbb{T}^{n-1}$$

$$t \mapsto (x_1, \ldots, x_n),$$

where $x_i = \text{sgn}(c(i)) \cdot \exp(i \cdot S_i(0))$. Notice that for $x = \Phi_\epsilon(t)$, $d(\tilde{x}_{k_j}, \tilde{x}_{k_{j+1}}) < \pi$, so $\Phi_\epsilon$ is injective and thus, a homeomorphism onto its image.

It is tempting to define $\Phi_A := \Phi_\epsilon$ where $c$ is the cluster structure $c_\pi$ of any $x$ such that $\alpha \circ \pi(x) = A$. The problem is that the cluster structure $\epsilon'$ of a point $x'$ such that $\alpha \circ \pi(x) = \alpha \circ \pi(x')$ can be different from $c$, and thus define distinct functions $\Phi_\epsilon$ and $\Phi_{\epsilon'}$. Instead, we will show that if $\alpha \circ \pi(x) = A$, then $\pi(x)$ can only be $\pi(x_0)$ or $\pi(x_0)^\top$ for a fixed $x_0 \in \mathbb{T}^n$. From the correspondence between $\pi(x)$ and cluster structures $c$ in Corollary 4.17, this means that the definition of $\Phi_A$ requires the choice between two cluster structures $c$ and $c^\top$. We will then show a way to make a consistent choice in order to define a global $\Phi : |St_n(S^1)| \to K_n(S^1)$.

We will use the following notation in the proof of the next two lemmas. Given $I \subset \{1, \ldots, m\}$, let $x_I$ be the vector with entries $x_i$ for $i \in I$. Define $A_I := \{ [a] : [a] \in A \}$ for any $A \subset \{+1,-1\}^m / \sim$. By abuse of notation, if $I = \{i_1, \ldots, i_r\}$ and $y = x_I$, we will denote the $j$-th component of $y_I = (x_{i_1}, \ldots, x_{i_r})$ as $y_{i_j}$ instead of the formally correct expression $y_j$. We apply the same convention to $\pi_{ij}(x_I)$ and $A_I$.

Lemma 4.29. Let $x \in \mathbb{T}^n$ and $A = \alpha \circ \pi(x)$. Let $I, J \subset \{1, \ldots, m\}$ such that $I \cup J = \{1, \ldots, m\}$, $I \setminus J \neq \emptyset$, and $J \setminus I \neq \emptyset$. Then $\alpha \circ \pi(x_I) = A_I$ and $\alpha \circ \pi(x_J) = A_J$.

Proof. For any $j \in J \setminus I$, let $i_j \in I$ be the index of the first point among all $x_i$ and $-x_i$ with $i \in I$ that is reached clockwise from $x_j$. Denote that point by $y_j := \epsilon_jx_{i_j}$ with $\epsilon_j \in \{+1,-1\}$ chosen accordingly. Let $\gamma : [0,1] \to S^1$ be any parametrization of the clockwise path from $x_j$ to $y_j$. Notice that $\gamma$ might contain points $\pm x_g$ with $g \in J$, while the only $\pm x_i$ with $i \in I$ that it contains is $y_j$. In other words, the critical points of the function $\xi \mapsto \pi(\gamma(\xi) ; j)$ are $y_j$ and possibly $\pm x_g$ for some $g \in J$. Then, for $s \neq j$, the entries $\pi_{gs}(x) = \pi_{gs}(\pi(\gamma(0); j))$ and $\pi_{gs}(\pi(y_j ; j)) = \pi_{gs}(\pi(\gamma(1); j))$ that are different from each other must have $g \in J$. If $s = j$ and $t \in I \setminus J$, the only critical point involved is $\gamma(1) = \epsilon_jx_{i_j}$, so $\pi_{jt}(\pi(\gamma(1); j)) = \pi_{jt}(\gamma(\xi); j) = \pi_{jt}(x)$ for $0 < t < 1$. Additionally, $\sigma$ is clockwise continuous and $\gamma$ goes clockwise (see Remark 4.2), so $\pi_{jt}(\pi(y_j ; j)) = \sigma(\gamma(1) ; x_I) = \lim_{\xi \to 1} \sigma(\gamma(\xi); x_I) = \pi_{jt}(x)$. In summary, $(\pi_{sj}(x) ; j))_{s = 1}^{n-1} \pi_{jt}(x) = \lim_{\xi \to 1} \sigma(\gamma(\xi); x_I) = \pi_{jt}(x)$. Thus, $(\pi_{js}(x) ; j))_{s = 1}^{n-1} \pi_{jt}(x) = \pi_{jt}(x)_{I = 1}^{n-1} \pi_{j}(x) I = \pi_{jt}(\pi(x_j ; j))_{I = 1}^{n-1} \pi_{jt}(x) I$. Define $z \in \mathbb{T}^n$ by $z_i := x_i$ if $i \in I$ and $z_j := y_j$ if $j \in J \setminus I$. The facts in the previous
paragraph imply that $A_I \subset \alpha \circ \pi(x_I)$. Indeed, for every $[a] \in A$, $[a] = q(\pi_{s*}(x))$ for some $s$. If $s \in I$, then clearly $[a] = q(\pi_{s*}(x_I)) \in \alpha \circ \pi(x_I)$. If $s \in J \setminus I$, $[a] = q(\pi(\pi_{s*}(x)_J)) = q((\pi_{s*}(x)_J) = q((\epsilon_s \pi_{s*}+(x_I)) \in \alpha \circ \pi(x_I)$. Thus, $A_I \subset \alpha \circ \pi(x_I)$. The converse $\alpha \circ \pi(x_I) \subset A_I$ is immediate because $x_I \subset x$ (where we abuse notation to think of $x_I$ and $x$ as sets), and the proof of $\alpha \circ \pi(x_I) = A_I$ is analogous.

**Lemma 4.30.** Let $A \in \text{St}_n(S^1)$ and fix $x_0 \in \mathbb{T}^n$ such that $\alpha \circ \pi(x_0) = A$. For any $x \in \mathbb{T}^n$ such that $\alpha \circ \pi(x) = A$, $\pi(x)$ equals $\pi(x_0)$ or $\pi(\rho(x_0)) = \pi(x_0)^T$.

**Remark 4.31.** Let $A$ and $x_0$ as above, and let $c$ be the cluster structure of $x_0$. By the correspondence between $\pi(x)$ and $c_x$ (Corollary 4.17), Lemma 4.30 implies that the cluster structure of every $x \in A$ such that $\alpha \circ \pi(x) = A$ has to be $c$ or $c^T$.

**Proof.** Let $m := |A|$. We first reduce the proof to the case $n = m$. For each $1 \leq k \leq m$, choose $1 \leq s_k \leq n$ such that $A = \{q(\pi_{s_1*}(x)), \ldots, q(\pi_{s_m*}(x))\}$ and define $a_k := \pi_{s_k*}(x)$. Let $I := \{s_1, \ldots, s_m\}$. For every $1 \leq i \leq n$, there exists $1 \leq k(i) \leq m$ such that $q(\pi_{a_i}(x)) = q(\pi_{s_k(i)}(x))$, so $A_I = \alpha \circ \pi(x_I)$. Thus, if the Lemma holds when $n = m$, then $\pi(x_I)$ equals $\pi((x_0)_I)$ or $\pi(\rho((x_0)_I))$. Additionally, Lemma 4.8 gives $x_i = \pi(x_I)$ for some $t_i \in \{1, -1\}$. Because of this equality we have $\pi_{a_i}(x) = \pi_{s_k(i)}(x)$ and $\pi_{a_j}(x) = \pi_{s_k(j)}(x)$. Hence, $\pi_{ij}(x) = \pi_{s_k(i)}(x) = \pi_{s_k(j)}(x) = \pi_{s_k(j)}(x_I)$. The same equalities hold for $x_0$, so if $\pi(x_I)$ equals $\pi((x_0)_I)$ (resp. $\pi(\rho((x_0)_I))$, then $\pi(x)$ equals $\pi(x_0)$ (resp. $\pi(\rho(x_0))$).

Now we move on to prove the case when $m = n$. Notice that $|A| = n$ implies that $q(\pi_{a_i}(x)) \neq q(\pi_{a_j}(x))$ for $i \neq j$, so by Lemma 4.8, $x_i$ and $x_j$ cannot be equal or antipodal. Then, if $i \neq j$, $\pi_{ij}(x) = \sigma(x_i/x_j) = -\sigma(x_i/x_j) = -\pi_{ji}(x)$. We will use these facts repeatedly during the proof. Besides that, we proceed by induction, so the general strategy will be to find conditions under which $\pi(y_I)$ determines $\pi(y)$ for some $I \subset \{1, \ldots, m\}$. Then if $\pi(x_I)$ equals $\pi((x_0)_I)$ or $\pi(\rho((x_0)_I))$, then $\pi(x)$ must equal $\pi(x_0)$ or $\pi(\rho(x_0))$, respectively. For now, let’s cover the base case $n = 2$. The matrix $\pi(x)$ can be $\Pi := \left( \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right)$ or $\Pi^t := \left( \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right)$, but $\alpha \circ \pi(x) = \{[(+1, +1)], [(+1, -1)]\}$. Hence, for any two points $x, x_0 \in \mathbb{T}^2$, $\alpha \circ \pi(x) = \alpha \circ \pi(x_0)$, but $\pi(x)$ may equal $\pi(x_0)$ or $\pi(\rho(x_0))$.

Now suppose that $A$ contains $\{(+1, \ldots, +1)\}$. Since $\alpha \circ \pi(x) = A$ and the diagonal entries of $\pi(x)$ are $+1$, there exists $1 \leq t_1 \leq n$ such that $\pi_{t_1*}(x) = (+1, \ldots, +1)$. This means that the shortest path from $x_{t_1}$ to any other point $x_j$ is anticlockwise and, thus, $x$ is contained in a semicircle. Suppose that the points appear in anticlockwise order as $x_{t_1}, \ldots, x_{t_n}$. Observe that the $t_1$ component of $\pi_{t_2*}(x)$ is $-1$ while all others are $+1$. Similarly, only the $t_m$ component of $\pi_{t_m*}(x)$ is $+1$, and any other $\pi_{t_1*}(x)$ must have at least two $+1$ components. Then, $A$ contains two elements $[a_0]$ and $[a'_0]$ whose representatives $a_0$ and $a'_0$ have exactly one entry different from the rest. Let $\ell$ (resp. $\ell'$) be the index of the entry in $a_0$ (resp. $a'_0$) that is different from the others. Suppose that $\ell' < \ell$. Then $\{t_1, t_n\} = \{\ell, \ell'\}$, although it is not clear whether $\ell = t_1$ or $\ell = t_n$. Regardless, let $I := \{1, \ldots, n\} \setminus \{\ell\}$. We claim that $\pi(x)$ is determined by $\pi(x_I)$ and the value of $\ell$. Indeed, $\pi_{ij}(x) = \pi_{ij}(x_I)$ for all $i, j \in I$, so we only need to find $\pi_{i\ell}(x)$ and $\pi_{\ell j}(x)$. If $\ell = t_1$, the shortest path from $x_{t_1}$ to any $x_j$ is anticlockwise, so $\pi_{ij}(x) = +1$. If $\ell = t_m$, the shortest path from $x_{t_m}$ to any $x_j$ is clockwise instead, so $\pi_{ij}(x) = -1$ for $j \neq \ell$ and $\pi_{\ell j}(x) = +1$. Furthermore, recall that $x$ has no pair of equal or antipodal points, so $\pi_{ij}(x) = -\pi_{ij}(x)$ for $i \neq \ell$. Thus, the claim is proved.

On the other hand, $\alpha \circ \pi(x_0) = A$ by hypothesis, so some of the observations above apply
to $x_0$ as well. In particular, $\ell$ and $\ell'$ depend only on $A$, not on $x$ or $x_0$. In contrast, the points of $x_0$ might be contained in the clockwise path from $(x_0)_{t_1}$ to $(x_0)_{t_0}$ (as opposed to the anticlockwise path, as is the case with $x$). Depending on whether the points are contained in the anticlockwise or the clockwise path, the matrix $\pi(x_0)$ satisfies $\pi_{t_1}(x_0) = (+1, \ldots, +1)$ or $\pi_{t_0}(x_0) = (+1, \ldots, +1)$ respectively. The latter is equivalent to $\pi_{t_1}(\rho(x_0)) = (+1, \ldots, +1)$ because $\rho$ sends clockwise paths to anticlockwise paths and, as such, it reverses the order of the points in $x_0$. In fact, the same argument shows that exactly one of $\pi_{t_0}(x_0) = (+1, \ldots, +1)$ or $\pi_{t_1}(\rho(x_0)) = (+1, \ldots, +1)$ holds. Thus, regardless of whether $\ell'$ is $t_1$ or $t_0$, we also have either $\pi_{t_1}(x_0) = (+1, \ldots, +1)$ or $\pi_{t_0}(\rho(x_0)) = (+1, \ldots, +1)$. Since both $\alpha \circ \pi(x_I)$ and $\alpha \circ \pi((x_0)_I)$ equal $A_I$, the induction hypothesis implies that either $\pi(x_I) = \pi((x_0)_I)$ or $\pi(x_I) = \pi(\rho((x_0)_I))$. The correct choice is the matrix whose $\ell'$-th row equals that of $\pi(x_I)$. Moreover, the row $\pi_{t_1}(x_0)$ determines the order of $x_0$ as we saw above, and it coincides with the order of $x$ if and only if $\pi(x_I) = \pi((x_0)_I)$. Now that $x_I$ has the same matrix $\pi(\cdot)$ and $x$ has the same order as either $x_0$ or $\rho(x_0)$, the paragraph above implies that $\pi(x)$ equals $\pi(x_0)$ or $\pi(\rho(x_0))$.

The last case is when $[(+1, \ldots, +1)] \notin A$, which means that the points of $x$ are not contained in a semicircle. In particular, $n \geq 3$. The strategy for this case will be to show that $\pi_{t_0}(x)$ can be either $\pi_{t_0}(x_0)$ or $\pi_{t_1}(\rho(x_0))$, and that the choice between $\pi(x_0)$ and $\pi(\rho(x_0))$ is the same for all $i_0$. Fix $1 \leq i_0 \leq n$. Let $x_{i_1}$ be the closest point to $-x_{i_0}$ in the clockwise direction, and let $I$ be the set of indices of all points in the clockwise path from $x_{i_1}$ to $-x_{i_1}$. Define $x_{j_1}$ and $J$ analogously, except that the directions are anticlockwise. Notice that $i_0 \in I \cap J$, $I \cup J = \{1, \ldots, n\}$, $I \setminus J \neq \emptyset$, and $J \setminus I \neq \emptyset$. Furthermore, since all the points in $x_I$ are contained in a semicircle, there exists $i \in I$ such that $\text{supp}_+(\pi_{i_0}(x)) = I$. Similarly, $\text{supp}_+(\pi_{j_1}(x)) = J$.

On the contrary, we claim that the set $X := \{x_{i_0}, x_{i_1}, x_{j_1}\}$ is not contained in a semicircle. This claim is equivalent to saying that $X$ is contained in a path from $x$ to $-x$ for some $x \in X$. We know that $x = x_{i_0}$ does not satisfy the claim by definition of $i_0$ and $j_1$. As for $x = x_{i_1}$, every point in $x_I$, including $x_{i_0}$, lies in the clockwise path from $x_{i_1}$ to $-x_{i_1}$. However, $x_{j_1}$ does not because otherwise, $x_J$ would also be contained in that path. That would imply that $x$ is contained in a semicircle, contradicting the assumption that $[(+1, \ldots, +1)] \notin A$. The proof that $x = x_{j_1}$ does not work is similar and, with that, we obtain the claim. As a consequence, there is no $[a] \in A$ for which $X \subset \text{supp}_+(a)$. Conversely, since $\alpha \circ \pi(x_0) = A$, we have that $X' := \{(x_0)_{i_0}, (x_0)_{i_1}, (x_0)_{j_1}\}$ is not contained in a semicircle either. If it were, there would exist $1 \leq s \leq n$ such that $a := \pi_{s_0}(x_0)$ has either $X' \subset \text{supp}_+(a)$ or $X' \subset \text{supp}_-(-a)$ which, as we know, is impossible.

By Lemma 4.29, $A_I = \alpha \circ \pi(x_I) = \alpha \circ \pi((x_0)_I)$ and, by induction hypothesis, $\pi(x_I)$ equals $\pi((x_0)_I)$ or $\pi(\rho((x_0)_I))$. The analogous statement holds for $J$, and we claim that $\pi(x_I) = \pi((x_0)_I)$ if and only if $\pi(x_J) = \pi((x_0)_J)$. To wit, since $X'$ is not contained in a semicircle, we must have $\pi_{i_0i_1}(x_0) = -\pi_{i_0j_1}(x_0)$ and $\pi_{i_1i_0}(\rho(x_0)) = -\pi_{i_1j_1}(\rho(x_0))$. Also, $|A| = n$ so $\pi_{i_0j_0}(x_0) = -\pi_{i_0j_0}(x_0) = -\pi_{i_0j_0}(\rho(x_0))$. The last equality follows by Lemma 4.25. Thus, if we had $\pi(x_I) = \pi((x_0)_I)$ and $\pi(x_J) = \pi(\rho((x_0)_J))$, then $\pi_{i_0i_1}(x_I) = \pi_{i_0i_1}(\rho((x_0)_I)) = -\pi_{i_0j_1}(\rho((x_0)_I)) = \pi_{i_0j_1}(x_J)$. This contradicts the definition of $i_0$ and $j_1$, which requires $\pi_{i_0i_1}(x_I) = +1 = -\pi_{i_0j_1}(x_J)$. Hence, we must have $\pi(x_I) = \pi((x_0)_I)$ and $\pi(x_J) = \pi((x_0)_J)$, or $\pi(x_I) = \pi(\rho((x_0)_I))$ and $\pi(x_J) = \pi(\rho((x_0)_J))$. In the first case, since $i_0 \in I \cap J$, for any $i \in I$, we can write $\pi_{i_0i_1}(x_I) = \pi_{i_0i_1}(x_J) = \pi_{i_0i_1}(x_0) = \pi_{i_0i_1}(x_0)$. Analogously,
if \( j \in J \), \( \pi_{i_0 j}(x) = \pi_{i_0 j}(x_0) \). Hence, \( \pi_{i_0 *}(x) = \pi_{i_0 *}(x_0) \) because \( I \cup J = \{1, \ldots, n\} \). If \( \pi(x_j) = \pi(\rho(x_0)) \) and \( \pi(x_J) = \pi(\rho(x_0)) \) hold instead, we get \( \pi_{i_0 *}(x) = \pi_{i_0 *}(\rho(x_0)) \).

In summary, we know that \( \pi_{i_0 *}(x) \) equals either \( \pi_{i_0 *}(x_0) \) or \( \pi_{i_0 *}(\rho(x_0)) \) for every \( 1 \leq i_0 \leq n \). The only fact left to prove is that this choice is the same for all \( i_0 \). Suppose that \( \pi_1(x) = \pi_1(x_0) \). Then for any \( i_0 > 1 \), \( \pi_{i_0 1}(x) = -\pi_{i_0 1}(x) = -\pi_{i_0 1}(x_0) = \pi_{i_0 1}(x_0) \). Since \( \pi_{i_0 *}(x) \) must equal \( \pi_{i_0 *}(x_0) \) or \( \pi_{i_0 *}(\rho(x_0)) \) and \( \pi_{i_1}(x_0) \neq \pi_{i_1}(\rho(x_0)) \) whenever \( i_0 > 1 \), the equality in the previous sentence forces \( \pi_{i_0 *}(x) = \pi_{i_0 *}(x_0) \) for all \( i_0 > 1 \). Hence, \( \pi(x) = \pi(x_0) \). If \( \pi_1(x) = \pi_1(\rho(x_0)) \) instead, the analogous argument shows \( \pi(x) = \pi(\rho(x_0)) \).

The lemma above says that if \( \alpha \circ \pi(x) = A \), then the image of both \( \pi(x) \) and \( \pi(\rho(x)) \) under \( \alpha \) is \( A \). What this means is that, if \( c \) and \( c' \) are the cluster structures of \( x \) and \( \rho(x) \), then there are two options to define \( \Phi_A \): either \( \Phi_c \) or \( \Phi_{c'} \). We will now study how these choices affect \( \Phi_{\alpha} \). Define \( \rho_{\Delta} : \Delta_m \to \Delta_m \) to be

\[
\rho_{\Delta}(t_1, \ldots, t_{m-1}) = (\pi - (t_1 + \cdots + t_{m-1}), t_{m-1}, \ldots, t_2).
\]

The fact that \( \rho_{\Delta} \) is an involution can be verified by inspection.

**Proposition 4.32.** Let \( x \in \mathbb{T}^n \), and let \( c \) and \( c' \) be the cluster structures of \( x \) and \( \rho(x) \). Then \( \rho \circ \Phi_c = \Phi_{c'} \circ \rho_{\Delta} \).

**Proof.** Recall that \( S_i(t) = 0 \). If \( |c(j)| = 1 \),

\[
(\Phi_{c'} \circ \rho_{\Delta}(t))_j = \text{sgn}(c'(j)) \exp(i \cdot S_{|c'(j)|}(\rho_{\Delta}(t)))
\]

\[
= \text{sgn}(c(j)) \exp(i \cdot S_{1}(\rho_{\Delta}(t)))
\]

\[
= \text{sgn}(c(j)) = \rho(\text{sgn}(c(j))) = (\rho \circ \Phi_c)(t)_j.
\]

Suppose now that \( |c(j)| \neq 1 \). Let \( S_m(t) := \pi \), and notice that \( S_{m+2-k}(\rho_{\Delta}(t)) = \pi - S_k(t) \) for \( 2 \leq k \leq m \). Then

\[
(\Phi_{c'} \circ \rho_{\Delta}(t))_j = \text{sgn}(c'(j)) \exp(i \cdot S_{|c'(j)|}(\rho_{\Delta}(t)))
\]

\[
= (- \text{sgn}(c(j))) \cdot \exp(i \cdot S_{m+2-|c(j)|}(\rho_{\Delta}(t)))
\]

\[
= (- \text{sgn}(c(j))) \cdot \exp(i \cdot [\pi - S_{|c(j)|}(t)])
\]

\[
= \rho(\text{sgn}(c(j)) \cdot \exp(i \cdot S_{|c(j)|}(t)))
\]

\[
= (\rho \circ \Phi_c)(t)_j.
\]

\( \square \)

The previous proposition shows that the worst that can happen in the definition of \( \Phi_c \) if we select a different \( x \) is that we need to conjugate with an involution and \( \rho \). However, \( K_n(S^1) \) is invariant under \( \rho \), so we can define a global map \( \Phi : |St_n(S^1)| \to K_n(S^1) \).

**Theorem 4.33.** \(|St_n(S^1)| \) is homeomorphic to \( K_n(S^1) \).
Proof. Let $A \in \text{St}_n(S^1)$ and choose $x \in \mathbb{T}^n$ such that $\alpha \circ \pi(x) = A$. Let $m = |A|$. In order for the choice of $\Phi_A$ to be consistent, replace $x$ with $\rho(x)$ if $\pi_{1,2}(x) = -1$. Let $c_A$ be the cluster structure of $x$ and, for $1 \leq j \leq m$, choose $1 \leq k_j \leq n$ as in Definition 4.11. In particular, for every $1 \leq i \leq n$ there exists $1 \leq j \leq m$ such that $x_i = \pm x_{k_j}$. Notice that $\Phi_{c_A}(t) = x$ where $t_j = d(x_{k_j}, x_{k_{j+1}})$, i.e. $x$ is in the image of $\Phi_{c_A}$. Define $\Phi_A : \Delta_{m-1} \to K_n(S^1)$ by $\Phi_A := D_n \circ \Phi_{c_A}$.

Recall the map $g : K_n(S^1) \to \text{St}_n(S^1)$ from Theorem 4.28. We claim that the image of $\Phi_A$ is the whole set $g^{-1}(A)$. By Lemma 4.30, for all $y \in g^{-1}(A)$, $\pi(y)$ equals $\pi(x)$ or $\pi(x)^\top$. If $c$ is the cluster structure of $y$, the correspondence between $\pi(y)$ and $c$ in Corollary 4.17 gives that $c = c_A$ or $c = c_A^\top$. Analogously to the case of $x$, $y$ is in $\text{Im}(\Phi_t)$ which equals either $\text{Im}(\Phi_{c_A})$ or, by Proposition 4.18 if $c = c_A^\top$, $\text{Im}(\rho \circ \Phi_{c_A})$. Regardless, we have $D_n(y) \in \text{Im}(\Phi_A)$, so $g^{-1}(A) \subset \text{Im}(\Phi_A)$. The inclusion $\text{Im}(\Phi_A) \subset g^{-1}(A)$ follows from the definition of $\Phi_A$.

We now verify that $\Phi_A(\partial \Delta_{m-1}) = \bigcup_{B \subset A} \Phi_B(\Delta_{m-2})$. Let $t \in \Delta_{m-1}$, $y = \Phi_A(t)$, and $c$ the cluster structure of $y$. Notice that if $t_j = 0$ for some $j$ or $t_1 + \cdots + t_{m-1} = \pi$, at least two points $y_{k_i}, y_{k_j}$ are equal or antipodal. Then, by Lemma 4.18 and Remark 4.24, $B := \alpha \circ \pi(y) \subset A$. By the first paragraph, $y \in \text{Im}(\Phi_t)$, so $\Phi_A(t) = D_n(y) \in \text{Im}(D_n \circ \Phi_t) = \text{Im}(\Phi_B)$. Conversely, let $B = A \setminus \{a_j\}$ and $y \in \text{Im}(\Phi_B)$. If $j \neq 1, m$, we have $y = \Phi_{c}(t)$ where $t_i = d(\tilde{y}_{k_i}, \tilde{y}_{k_{i+1}})$ and, by Remark 4.24, either $t_j = 0$ or $t_{j-1} = 0$. If $j = 1$, we have either $t_1 + \cdots + t_{m-1} = \pi$ or $t_2 = 0$ and, if $j = m$, we have $t_{m-1} = 0$ or $t_1 + \cdots + t_{m-1} = \pi$. Thus, $\text{Im}(\Phi_B) \subset \text{Im}(\Phi_A)$.

All in all, the collection of maps $\Phi_A : \Delta_m \to K_n(S^1)$ for $A \in \text{St}_n(S^1)$ induce a continuous map $\Phi : |\text{St}_n(S^1)| \to K_n(S^1)$ on the geometric realization of $\text{St}_n(S^1)$. Each $\Phi_A$ is injective because $\Phi_\varepsilon$ is and, by the first paragraph, is also surjective. Thus, $\Phi$ is a continuous bijection from $|\text{St}_n(S^1)|$, which is compact because $\text{St}_n(S^1)$ is finite, into a Hausdorff space $K_n(S^1)$. Hence, $\Phi$ is a homeomorphism.

5 Connection with elliptopes

Let $\mathcal{S}^n$ be the set of $n$-by-$n$ symmetric matrices. Define the $n$-th elliptope as:

$$\mathcal{E}_n := \{M \in \mathcal{S}^n : M \text{ is positive semidefinite and } M_{ii} = 1 \text{ for } i = 1, \ldots, n\}.$$ 

Many problems in convex optimization can be seen as the optimization of an objective function over the set $\mathcal{E}_n$. One case where these problems are interesting is as a semidefinite relaxation of possibly NP-hard problems such as binary quadratic optimizations. In other words, it is possible to obtain approximate solutions to a binary quadratic optimization using a semidefinite program in a much reasonable time. See [BPT12] for a more complete treatment on optimization, including elliptopes. In particular, Section 2.2.2 contains more thorough examples of semidefinite relaxations.

There is a connection between $\mathcal{E}_n$ and the curvature sets $K_n(S^m)$. Given two points $x, y \in \mathbb{S}^m$ thought of as column vectors in $\mathbb{R}^{m+1}$, the geodesic distance in $\mathbb{S}^m$ is given by $d_m(x, y) := \arccos(x^\top \cdot y)$. In fact, for $x \in (\mathbb{S}^m)^n$ (thought of as an $n$-by-$(m+1)$ matrix with columns $x_i \in \mathbb{S}^m \subset \mathbb{R}^{m+1}$), then $(D_n(x))_{ij} = \arccos(x_i^\top \cdot x_j)$. Then the matrix obtained by applying $\cos(\cdot)$ entry-wisely to $D_n(x)$ is the Gram matrix $x^\top \cdot x$, which is positive semidefinite. Hence, $\cos(D_n(x)) \in \mathcal{E}_n$. Conversely, every $M \in \mathcal{E}_n$ is a Gram matrix of a set of
points in $\mathbb{R}^{n+1}$ where $m = \text{rank}(M)$ and $0 \leq m \leq n-1$. Since $M_{ii} = 1$, these points actually belong to $S^m$. We then have the following result.

**Proposition 5.1.** The chain of inclusions $S^1 \subset S^2 \subset \cdots \subset S^{n-1}$ induces a filtration

$$\cos(K_n(S^1)) \subset \cdots \subset \cos(K_n(S^{n-1})) = \mathcal{E}_n$$

This proposition becomes suggestive in light of the computation of the homology groups of $K_n(S^1)$ (Theorem 1.1): it implies that the convex set $\mathcal{E}_n$ has a subset $\cos(K_n(S^1)) \cong K_n(S^1)$ which not only has non-trivial homology but also exhibits torsion. Additionally, Mémoli showed that $K_3(S^2)$ is the convex hull of $K_3(S^1)$ [Mém12]. In the case $n = 3$, Theorem 5.1 means that $E_3$ is the convex hull of $\cos(K_3(S^1))$. In fact, we can show a similar statement. Let $S^n_E$ be the $n$-sphere in $\mathbb{R}^{n+1}$ equipped with the Euclidean metric, and denote with $D^m_{n, E} : (S^m_E)^n \to K^m_n(S^n_E)$ the distance matrix map (see Section 1).

**Proposition 5.2.** $K_{n+1}(S^n_E)$ is the cone joining $K_{n+1}(S^{n-1}_E)$ to the origin in $\mathbb{R}^{(n+1) \times (n+1)}$.

**Proof.** Let $x \in (S^n_E)^{n+1}$. Any $n+1$ points in $\mathbb{R}^{n+1}$ generate an $m$-hyperplane, with $m < n+1$, that intersects $S^n_E$ in a rescaled sphere $\lambda \cdot S^{m-1}_E$ with $0 < \lambda \leq 1$. Then, $D^m_{n+1}(x) \in \lambda \cdot K_{n+1}(S^{m-1}_E) \subset \lambda \cdot K_{n+1}(S^{n-1}_E)$. Thus, $K_{n+1}(S^n_E) = \bigcup_{0 < \lambda \leq 1} \lambda \cdot K_{n+1}(S^{n-1}_E)$. \[\square\]

Let $d_{m,E}$ be the Euclidean metric on $S^n_E$, and observe that the function $f_E(d) := 2 \arcsin(d/2)$ satisfies $d_{m,E} = f_E \circ d_m$ and $K_n(S^n_E) = f_E(K_n(S^n))$. Propositions 5.1 and 5.2 imply that $\mathcal{E}_n$ is the convex hull of $\cos(f^{-1}_E(K_n(S^{n-2})))$. It seems interesting to explore these and other consequences of the connection between elliptopes and curvature sets.

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A Explicit computation of $\text{St}_3(\mathbb{S}^1)$.

Example A.1. Here we show the chiral state matrix, achiral cell maps and cluster structures for several configurations of three points in the circle. We also include the Hasse diagram of $\text{St}_3(\mathbb{S}^1)$, from where it can be seen that $|\text{St}_3(\mathbb{S}^1)| \simeq \mathbb{S}^2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Hasse_diagram.pdf}
\caption{Hasse diagram of $\text{St}_3(\mathbb{S}^1)$.}
\end{figure}

Table 1: The 2-simplices of $\text{St}_3(\mathbb{S}^1)$. No pair of points is equal or antipodal. Notice that, for all $k$, $\alpha \circ \pi(x)$ is equal in rows $f_{2k-1}$ and $f_{2k}$, while $c$ in row $2k$ equals $c^\top$ in row $2k - 1$.

| No. | $x \in \mathbb{T}^3$ | $\pi(x)$ | $\alpha \circ \pi(x)$ | $c(x)$ |
|-----|-----------------|-----------|---------------------|-------|
| $f_1$ | ![Diagram](https://example.com/diagram_f1.png) | $\begin{pmatrix} + & - & + \\ + & + & - \\ - & + & + \end{pmatrix}$ | $\begin{pmatrix} (+-+) \sim (-+-) \\ (++-) \sim (-++) \\ (+-+) \sim (+-) \end{pmatrix}$ | $(+1, -2, +3)$ |
| $f_2$ | ![Diagram](https://example.com/diagram_f2.png) | $\begin{pmatrix} + & + & - \\ - & + & + \\ + & - & + \end{pmatrix}$ | $\begin{pmatrix} (+-+) \sim (-+-) \\ (-++) \sim (+--) \\ (+-) \sim (-++) \end{pmatrix}$ | $(+1, -3, +2)$ |
Table 1: The 2-simplices of $\text{St}_3(\mathbb{S}^1)$. No pair of points is equal or antipodal. Notice that, for all $k$, $\alpha \circ \pi(x)$ is equal in rows $f_{2k-1}$ and $f_{2k}$, while $c$ in row $2k$ equals $c^T$ in row $2k-1$.

| No. | $x \in \mathbb{T}^3$ | $\pi(x)$ | $\alpha \circ \pi(x)$ | $c(x)$ |
|-----|----------------------|-----------|------------------------|--------|
| $f_3$ | $x_3 \xrightarrow{\alpha} x_2 \xrightarrow{} x_1$ | $(+++) \sim (-+-) \quad (-++) \sim (+-) \quad (-++) \sim (+--) \sim (++)$ | $(+1,+2,+3)$ |
| $f_4$ | $x_3 \xrightarrow{} x_1 \xrightarrow{\alpha} x_2$ | $(+-+) \sim (+++) \quad (+++) \sim (---)$ | $(1,-3,-2)$ |
| $f_5$ | $x_2 \xrightarrow{} x_3 \xrightarrow{\alpha} x_1$ | $(+++) \sim (---) \quad (-++) \sim (+-) \quad (-++) \sim (+--) \sim (++)$ | $(+1,+3,+2)$ |
| $f_6$ | $x_2 \xrightarrow{} x_3 \xrightarrow{\alpha} x_1$ | $(+-+) \sim (+++) \quad (+++) \sim (---) \quad (+++) \sim (+--) \sim (++)$ | $(1,-2,-3)$ |
| $f_7$ | $x_2 \xrightarrow{} x_1 \xrightarrow{\alpha} x_3$ | $(++-) \sim (-++) \quad (-++) \sim (+-) \quad (-++) \sim (+--) \sim (++)$ | $(+1,+2,-3)$ |
| $f_8$ | $x_3 \xrightarrow{} x_1 \xrightarrow{\alpha} x_2$ | $(+-+) \sim (+++) \quad (+++) \sim (---) \quad (-++) \sim (+--) \sim (++)$ | $(+1,+3,-2)$ |
Table 2: The 1-simplices of St$_3$(S$^1$). Two points are equal or antipodal to each other, while the third is not.

| No. | $\mathbf{x} \in \mathbb{T}^3$ | $\pi(\mathbf{x})$ | $\alpha \circ \pi(\mathbf{x})$ | $\mathbf{c}(\mathbf{x})$ |
|-----|-------------------------------|------------------|-------------------------------|------------------|
| $e_1$ | $x_3 = -x_1$ | (+ + +) \sim (- - +) \sim (++) | (+ +) \sim (- -) \sim (++) | (+1, +2, -1) |
| $e_2$ | $x_3 = -x_1$ | (+ - -) \sim (+ + +) \sim (- + +) | (+ - -) \sim (+ + +) \sim (- + +) | (+1, -2, -1) |
| $e_3$ | $x_2 = -x_1$ | (+ - +) \sim (- + +) \sim (+ - +) | (+ - +) \sim (- + +) \sim (+ - +) | (+1, -1, +2) |
| $e_4$ | $x_2 = -x_1$ | (+ - +) \sim (- + +) \sim (+ - +) | (+ - +) \sim (- + +) \sim (+ - +) | (+1, -1, -2) |
| $e_5$ | $x_3 = -x_2$ | (+ + -) \sim (- - -) \sim (+ + -) | (+ + -) \sim (- - -) \sim (+ + -) | (+1, +2, -2) |
| $e_6$ | $x_3 = -x_2$ | (+ - +) \sim (- + -) \sim (+ - +) | (+ - +) \sim (- + -) \sim (+ - +) | (+1, -2, +2) |
Table 2: The 1-simplices of $\text{St}_3(S^1)$. Two points are equal or antipodal to each other, while the third is not.

| No. | $x \in \mathbb{T}^3$ | $\pi(x)$ | $\alpha \circ \pi(x)$ | $c(x)$ |
|-----|----------------|---------|----------------|-------|
| $e_7$ | $x_2 = x_3$ | $\begin{pmatrix} + & + & + \\ - & + & + \\ - & + & + \end{pmatrix}$ | $(+++ \sim (---))$ | $(+1, +2, +2)$ |
| $e_8$ | $x_2 = x_3$ | $\begin{pmatrix} + & - & - \\ + & + & + \\ + & + & + \end{pmatrix}$ | $(+-+ \sim (+-+))$ | $(+1, -2, -2)$ |
| $e_9$ | $x_3$ | $\begin{pmatrix} + & + & + \\ + & + & + \\ - & - & + \end{pmatrix}$ | $(++- \sim (-+-))$ | $(+1, +1, +2)$ |
| $e_{10}$ | $x_3$ | $\begin{pmatrix} + & + & - \\ + & + & + \\ + & + & + \end{pmatrix}$ | $(++- \sim (-+-))$ | $(+1, +1, -2)$ |
| $e_{11}$ | $x_3$ | $\begin{pmatrix} + & + & + \\ - & - & - \\ + & + & + \end{pmatrix}$ | $(++- \sim (-+-))$ | $(+1, +2, +1)$ |
| $e_{12}$ | $x_3$ | $\begin{pmatrix} + & - & + \\ + & + & + \\ + & - & + \end{pmatrix}$ | $(+-+ \sim (+-+))$ | $(+1, -2, +1)$ |
Table 3: The 0-simplices of $\text{St}_3(\mathbb{S}^1)$. All points are either equal or antipodal to one another.

| No. | $x \in \mathbb{T}^3$ | $\pi(x)$ | $\alpha \circ \pi(x)$ | $c(x)$ |
|-----|---------------------|----------|------------------------|--------|
| $v_1$ | $x_2 = x_3 \rightarrow x_1$ | $\begin{pmatrix} + & - & - \\ - & + & + \\ - & + & + \end{pmatrix}$ | $(+-+) \sim (-++)$ | $(+1, -1, -1)$ |
| $v_2$ | $x_3 \rightarrow x_1 = x_2$ | $\begin{pmatrix} + & + & - \\ + & + & - \\ - & - & + \end{pmatrix}$ | $(++-) \sim (-+-)$ | $(+1, +1, -1)$ |
| $v_3$ | $x_2 \rightarrow x_1 = x_3$ | $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ | $(+-+) \sim (-+-)$ | $(+1, -1, +1)$ |
| $v_4$ | $x_1 = x_2 = x_3$ | $\begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}$ | $(+++) \sim (-+-)$ | $(+1, +1, +1)$ |