Some Results Related to Soft Topological Spaces

E. Peyghan, B. Samadi and A. Tayebi

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Abstract

The notion of soft sets is introduced as a general mathematical tool for dealing with uncertainty. In this paper, we consider the concepts of soft compactness, countably soft compactness and obtain some results. We study some soft separation axioms that have been studied by Min and Shabir-Naz. By constructing a special soft topological space, show that some classical results in general topology are not true about soft topological spaces, for instance every compact Hausdorff spaces need not be normal.

Keywords: Soft closed, Soft compact space, Soft open, Soft topological spaces.

1 Introduction

During recent years General Topology was developed by many mathematicians. The theory of generalized topological spaces, which was founded by Á. Császár is one of these developments [6]. Recently, in [16] Shabir-Naz introduced and studied the concepts of soft topological spaces and some related concepts. The generalized topology is different from topology by its axioms (A collection of subsets of X is a generalized topology on X if and only if it contains empty set and arbitrary union of its elements). But the soft topology is based on soft sets theory and not sets.

Some notions in Mathematics can be considered as mathematical tools for dealing with uncertainties, namely theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of rough sets and etc. But all of these theories have their own difficulties. In [11], Molodtsov introduced the concept of a soft set in order to solve complicated problems in the economics, engineering, and environmental areas because no mathematical tools can successfully deal with the various kinds of uncertainties in these problems. He successfully applied the soft theory in several directions, such as game theory, probability, Perron integration, Riemann integration and theory of measurement [11] [12].

In [9], Maji-Biswas-Roy defined and studied operations of soft sets. Then Pei-Miao [14] and Chen [5] improved the work of Maji-Biswas-Roy [8] [9]. The properties and applications of soft set theory have been studied increasingly in [4]. In [5], Çağman-Enginoglu redefined the operations of the soft sets and constructed a multi-int decision making method by using these new operations,

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and developed soft set theory. Then to make easy compaction with the operations of soft sets, they presented the soft matrix theory and set up the soft maxmin decision making method \[4\]. These decision making methods can be successfully applied to many problems that contain uncertainties. In \[16\], the authors studied some concepts related to soft spaces such as soft interior, soft subspace and soft separation axioms. Recently, Aygunoglu-Aygun introduced the soft product topology and defined the version of compactness in soft spaces named soft compactness \[2\].

In this paper, we consider the concepts of soft compactness and countably soft compact and get some results. Then, we study some soft separation axioms that have studied by Min and Shabir-Naz. By constructing some examples we show that some classical results in general topology are not true about soft topological spaces, for instance every compact Hausdorff spaces need not be normal.

2 Preliminaries

In this section, we recall some definitions and concepts discussed in \[7\] \[10\] \[16\] \[17\]. Let \(U\) be an initial universe and \(E\) be a set of parameters. Let \(\mathcal{P}(U)\) denotes the power set of \(U\) and \(A\) be a nonempty subset of \(E\). A pair \((F, A)\) is called a soft set over \(U\), where \(F\) is a mapping given by \(F : A \rightarrow \mathcal{P}(U)\). For two soft sets \((F, A)\) and \((G, B)\) over common universe \(U\), we say that \((F, A)\) is a soft subset \((G, B)\) if \(A \subseteq B\) and \(F(e) \subseteq G(e)\), for all \(e \in A\). In this case, we write \((F, A) \subseteq (G, B)\) and \((G, B)\) is said to be a soft super set of \((F, A)\). Two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\) are said to be soft equal if \((F, A) \subseteq (G, B)\) and \((G, B) \subseteq (F, A)\). A soft set \((F, A)\) over \(U\) is called a wall soft set, denoted by \(\Phi_A\), if for each \(e \in A\), \(F(e) = \emptyset\). Similarly, it is called absolute soft set, denoted by \(\tilde{U}\), if for each \(e \in A\), \(F(e) = U\).

The union of two soft sets \((F, A)\) and \((G, B)\) over the common universe \(U\) is the soft set \((H, C)\), where \(C = A \cup B\) and for each \(e \in C\),

\[
H(e) = \begin{cases} 
F(e) & e \in A \setminus B \\
G(e) & e \in B \setminus A \\
F(e) \cup G(e) & e \in A \cap B
\end{cases}
\]

We write \((F, A) \cup (G, B) = (H, C)\). Moreover, the intersection \((H, C)\) of two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), denoted by \((F, A) \cap (G, B)\), is defined as \(C = A \cap B\) and \(H(e) = F(e) \cap G(e)\) for each \(e \in C\). The difference \((H, E)\) of two soft sets \((F, E)\) and \((G, E)\) over \(X\), denoted by \((F, E) \setminus (G, E)\), is defined as \(H(e) = F(e) \setminus G(e)\), for each \(e \in E\). Let \(Y\) be a nonempty subset of \(X\). Then \(\tilde{Y}\) denotes the soft set \((Y, E)\) over \(X\) where \(Y(e) = Y\), for each \(e \in E\). In particular, \((X, E)\) will be denoted by \(\tilde{X}\). Let \((F, E)\) be a soft set over \(X\) and \(x \in X\). We say that \(x \in (F, E)\), whenever \(x \in F(e)\), for each \(e \in E\) \[15\].

The relative complement of a soft set \((F, A)\) is denoted by \((F, A)'\) and is defined by \((F, A)' = (F', A)\) where \(F : A \rightarrow \mathcal{P}(U)\) is defined by following

\[F'(e) = U \setminus F(e), \quad \forall e \in A\.]
Let $\tau$ be the collection of soft sets over $X$. Then $\tau$ is called a soft topology on $X$ if $\tau$ satisfies the following axioms:

(i) $\Phi, \tilde{X}$ belong to $\tau$.
(ii) The union of any number of soft sets in $\tau$ belongs to $\tau$.
(iii) The intersection of any two soft sets in $\tau$ belongs to $\tau$.

The triple $(X, \tau, E)$ is called a soft topological space over $X$. The members of $\tau$ are said to be soft open in $X$, and the soft set $(F, E)$ is called soft closed in $X$ if its relative component $(F', E)'$ belongs to $\tau$.

The proof of the following proposition is an easy application of De Morgan’s laws with the definition of a soft topology on $X$ (see Proposition 3.3 of [17]).

**Proposition 2.1.** Let $(X, \tau, E)$ be a soft space over $X$. Then

1) $\Phi, \tilde{X}$ are closed soft set over $X$;
2) The intersection of any number of soft closed sets is a soft closed set over $X$;
3) The union of any two soft closed sets is a soft closed set over $X$.

### 3 Soft Compactness

In this section, we are going to introduce the concept of soft compactness about soft topological spaces and study some properties related to these spaces (also, see [17]).

A family $A = \{ (F_\alpha, E) \}_{\alpha \in J}$ of soft sets is a cover of a soft set $(F, E)$ if $(F, E) \subseteq \bigcup_{\alpha \in J} (F_\alpha, E)$. It is a soft open cover if each member of $A$ is a soft open set. A subcover of $A$ is a subfamily of $A$ which is also a cover. A soft topological space $(X, \tau, E)$ is said to be soft compact if each soft open cover of $(X, E)$ has a finite subcover.

Let $(X, \tau_1, E)$ and $(X, \tau_2, E)$ be soft topological spaces. If $\tau_1 \subseteq \tau_2$, then $\tau_2$ is soft finer than $\tau_1$. If $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$, then $\tau_1$ is soft comparable with $\tau_2$.

Then, we have the following.

**Proposition 3.1.** Let $(X, \tau_2, E)$ be a soft compact space and $\tau_1 \subseteq \tau_2$. Then $(X, \tau_1, E)$ is soft compact.

**Proof.** Let $\{ (F_\alpha, E) \}_{\alpha \in J}$ be a soft open cover of $\tilde{X}$ by soft open sets of $(X, \tau_1, E)$. Since $\tau_1 \subseteq \tau_2$, then $\{ (F_\alpha, E) \}_{\alpha \in J}$ is a soft open cover of $\tilde{X}$ by soft open sets of $(X, \tau_2, E)$. But $(X, \tau_2, E)$ is soft compact. Therefore

$$(X, E) \subseteq (F_{\alpha_1}, E) \cup \ldots \cup (F_{\alpha_n}, E),$$

for some $\alpha_1, \ldots, \alpha_n \in J$. Hence $(X, \tau_1, E)$ is soft compact.

In this paper, for convenience, let $SS(X)_E$ be the family of soft sets over $X$ with set of parameters $E$. We will apply two next propositions so much in the proofs.

**Proposition 3.2.** Let $(F, E)$, $(G, E)$, $(H, E)$ and $(I, E)$ be soft sets in $SS(X)_E$. Then the following hold.
(i) \((F, E) \subseteq (G, E)\) if and only if \((F, E) \cap (G, E) = (F, E)\);

(ii) \((F, E) \subseteq (G, E), (H, E)\) if and only if \((F, E) \subseteq (G, E) \cap (H, E)\);

(iii) If \((F, E) \subseteq (H, E)\) and \((G, E) \subseteq (I, E)\), then \((F, E) \cup (G, E) \subseteq (H, E) \cup (I, E)\);

(iv) \((F, E) \cap (F, E)' = \Phi_E\);

(v) \((F, E) \cap (G, E) = \Phi_E\) if and only if \((F, E) \subseteq (G, E)'\);

(vi) \((F, E) \subseteq (G, E)\) if and only if \((G, E)' \subseteq (F, E)\).

Proof. Here, we only prove the (iii). Let \((F, E) \cup (G, E) = (J, E)\) and \((H, E) \cup (I, E) = (K, E)\). Since \((F, E) \subseteq (H, E)\) and \((G, E) \subseteq (I, E)\); then

\[ F(e) \subseteq H(e) \quad \text{and} \quad G(e) \subseteq I(e), \quad \forall e \in E. \]

Therefore

\[ J(e) = F(e) \cup G(e) \subseteq H(e) \cup I(e) = K(e). \]

Hence \((J, E) \subseteq (K, E)\). \(\square\)

Also we can obtain the following easily.

**Proposition 3.3.** Let \((F, E)\) be a soft set and \(\{(F_\alpha, E)\}_{\alpha \in J}\) be a family of soft sets in \(SS(X)_E\). Then the following hold.

(i) \((F, E) \cap (F, E)' = \Phi_E\);

(ii) \((F, E) \cup \Phi_E = (F, E)\);

(iii) \((F, E) \cap (\cup_{\alpha \in J}(F_\alpha, E)) = \cup_{\alpha \in J}((F, E) \cap (F_\alpha, E))\);

(iv) \(\Phi_E' = \tilde{X}\);

(v) \(\tilde{X}' = \Phi_E\).

Let \((F, E)\) be a soft set over \(X\) and \(Y\) be a nonempty subset of \(X\). Then the sub-soft set of \((F, E)\) over \(Y\) denoted by \((Y, F, E)\) is defined as follows

\[ ^YF(e) = Y \cap F(e), \]

for each \(e \in E\). In other word \((^YF, E) = \tilde{Y} \cap (F, E)\). Now, suppose that \((X, \tau, E)\) be a soft topological space over \(X\) and \(Y\) be a nonempty subset of \(X\). Then

\[ \tau_Y = \{(^YF, E) | (F, E) \in \tau\}, \]

is said to be soft relative topology on \(Y\) and \((Y, \tau_Y, E)\) is called a soft subspace of \((X, \tau, E)\). Here, we exhibit a criterion that applies \(\tilde{Y}\) is soft compact by soft open covers of \(\tilde{Y}\), that all of members are soft open sets in \(X\).

**Theorem 3.4.** Let \((Y, \tau_Y, E)\) be a soft subspace of a soft space \((X, \tau, E)\). Then \((Y, \tau_Y, E)\) is soft compact if and only if every cover of \(\tilde{Y}\) by soft open sets in \(X\) contains a finite subcover.
Proof. Let \((Y, \tau_Y, E)\) be a soft compact subspace of a soft Hausdorff space \((X, \tau, E)\). Let \(x \in (X, E) \setminus (Y, E)\). Then for all \(y \in (Y, E)\), \(x \neq y\). Therefore, there exist soft open sets \((U_y, E)\) and \((U_{xy}, E)\) containing \(x\) and \(y\), respectively such that \((U_y, E) \cap (U_{xy}, E) = \Phi_E\). Obviously, \(\{(U_{xy}, E)\}_{y \in Y}\) is a cover of \(\tilde{Y}\) by soft open sets in \(X\). By Theorem 3.4 we have \((Y, E) = (U_{xy}, E) \cup \ldots \cup (U_{y_n}, E)\) for some \(y_1, \ldots, y_n \in Y\). Now, \(x \in (U_{y_1}, E) \cap \ldots \cap (U_{y_n}, E) = (U_x, E)\) and Proposition 3.3 implies that \((U_x, E) \cap (Y, E) = \Phi_E\). Hence \(x \in (U_x, E) \subseteq (X, E) \setminus (Y, E)\). Then \((X, E) \setminus (Y, E) = \bigcup_{x \in X \setminus Y} (U_x, E)\). Therefore \((X, E) \setminus (Y, E)\) is soft open. Hence \((Y, E)\) is soft closed. \(\Box\)

Using Propositions 3.2 and 3.3, we are going to prove that every soft closed subspace of a soft compact space is soft compact.

**Theorem 3.6.** Every soft closed subset of a soft compact space is soft compact.

Proof. Let \((Y, \tau_Y, E)\) be a soft compact subspace of a soft compact space \((X, \tau, E)\) such that \((Y, E)\) is a soft closed in \(X\). Let \(\{(F_\alpha, E)\}_{\alpha \in J}\) be a cover of \(\tilde{Y}\) by soft open sets in \(X\). \((Y, E)'\) is a soft open set in \(X\). Propositions 3.2 and 3.3 show that \(\{(F_\alpha, E)\}_{\alpha \in J} \cup \{(Y', E)\}\) form a soft open cover of \(\tilde{X}\). Therefore

\[
(X, E) \subseteq (F_{\alpha_1}, E) \cup \ldots \cup (F_{\alpha_n}, E) \cup (Y', E),
\]

for some \(\alpha_1, \ldots, \alpha_n \in J\). Applying the previous proposition we can see that \(\{(F_\alpha, E)\}_{\alpha \in J}\) is a subcover of \(Y\). This completes the proof. \(\Box\)

Let \((X, \tau, E)\) be a soft topological spaces and \(\mathcal{B} \subseteq \tau\). If every element of \(\tau\) can be written as a union of elements of \(\mathcal{B}\), then \(\mathcal{B}\) is called a soft basis for the soft topology \(\tau\). Each element of \(\mathcal{B}\) is called a soft basis element.

We can characterize soft compact spaces in term of basis elements as follows:

**Theorem 3.7.** A soft topological space \((X, \tau, E)\) is soft compact if and only if there is a soft basis \(\mathcal{B}\) for \(\tau\) such that every cover of \(\tilde{X}\) by elements of \(\mathcal{B}\) has a finite subcover.
Proof. Let $(X, \tau, E)$ be soft compact. Obviously, $\tau$ is a soft basis for $\tau$. Therefore, every cover of $\tilde{X}$ by elements of $\tau$ has finite subcover. Conversely, let $\{(U_\alpha, E)\}_{\alpha \in J}$ be a soft open cover of $\tilde{X}$. We can write $(U_\alpha, E)$ as a union of basis elements, for each $\alpha \in J$. These elements form a soft open cover of $\tilde{X}$ such as $\{(F_B, E)\}_{B \in I}$. Therefore $\tilde{X} = (F_{B_1}, E) \cup \ldots \cup (F_{B_n}, E)$, for some $B_1, \ldots, B_n \in I$. Let $(F_{B_i}, E) \subseteq (U_{\alpha_i}, E)$, for each $1 \leq i \leq n$. This implies that $\{(U_\alpha, E)\}_{\alpha = 1}^n$ is a finite subcover of $\tilde{X}$. Hence, $(X, \tau, E)$ is soft compact.

Remark 3.8. Clearly, a soft set is not a set. Indeed, the differences between soft topological spaces and topological spaces arise from this fact. In a sense, when $|E| = 1$, a soft set $(F, E)$ behaves similar to a set. In fact, in this case the soft set $(F, E)$ is the same as the set $F(e)$, where $E = \{e\}$. Therefore when $|E| = 1$, the soft topological spaces are the same as topological spaces. Nevertheless, in this paper we will see some differences between these two concepts when $|E| \geq 2$.

Now, we consider a countably soft compact space constructed around a soft topology. A soft topological space $(X, \tau, E)$ is said to be countably soft compact if every countable soft open cover of $\tilde{X}$ contains a finite subcover of $\tilde{X}$. Obviously, every soft compact space is countably soft compact but the following example shows that the converse is not true in general.

Example 3.9. We consider the (topological) space $S\Omega$, the minimal uncountable well-ordered set with order topology (see [12]). Let $X = S\Omega$, $E = \{e\}$ and $\tau = \{(F, E) | F(e) \text{ is open in } S\Omega\}$. Considering Remark 3.8, the soft topological space $(X, \tau, E)$ is countably soft compact but not soft compact.

There is a criterion for a soft space to be countable soft compact in term of soft closed sets rather than soft open sets. First we have a definition.

A collection $\mathcal{A}$ of soft set is said to have the finite intersection property if for every finite sub-collection $\{(A_1, E) \cap \ldots \cap (A_n, E)\}$ of $\mathcal{A}$, the intersection $(A_1, E) \cap \ldots \cap (A_n, E)$ is non-null.

Theorem 3.10. A soft topological space is countably soft compact if and only if every countable family of soft closed sets with the finite intersection property has a nonnull intersection.

Proof. Let the soft space $(X, \tau, E)$ be countably soft compact. Let the family $\{(F_n, E)\}_{n=1}^\infty$ of soft closed sets have the finite intersection property. If $\cap_{n=1}^\infty (F_n, E) = \phi_E$ by Proposition 3.3, $\{(F_n, E)\}_{n=1}^\infty$ is a countable soft open cover of $\tilde{X}$. Therefore $\tilde{X} = (F_{n_1}, E) \cup \ldots \cup (F_{n_k}, E)$, for some $n_1, \ldots, n_k \in N$. Now, De Morgan’s laws and Proposition 3.3 imply that $(F_{n_1}, E) \cap \ldots \cap (F_{n_k}, E) = \phi_E$. This is a contradiction. Conversely, let $\{(F_n, E)\}_{n=1}^\infty$ be a countable soft open cover of $\tilde{X}$ without any subcover. Then $\{(F_n, E)\}_{n=1}^\infty$ is a family of soft closed sets over $X$ such that $\cap_{n=1}^\infty (F_n, E)' = \phi_E$. Let $n_1, \ldots, n_k$ be arbitrary positive integers. If $(F_{n_1}, E)' \cap \ldots \cap (F_{n_k}, E)' = \phi_E$ then $\tilde{X} = (F_{n_1}, E) \cup \ldots \cup (F_{n_k}, E)$, that is impossible. Therefore $(F_{n_1}, E)' \cap \ldots \cap (F_{n_k}, E)' \neq \phi_E$, for each $n_1, \ldots, n_k \in N$. This shows that $\{(F_n, E)\}_{n=1}^\infty$ have the finite intersection property. Therefore $\cap_{n=1}^\infty (F_n, E)' \neq \phi_E$. This is a contradiction.

An immediate result of previous theorem is the following.
Corollary 3.11. A soft space \((X, \tau, E)\) is countably soft compact if and only if every nested sequence \((F_1, E) \supseteq (F_2, E) \supseteq \ldots \) of nonnull soft closed sets over \(X\) has a nonnull intersection.

Proof. Let \((X, \tau, E)\) is countably soft compact. The collection \(\{(F_n, E)\}_{n=1}^{\infty}\) have the finite intersection property. Therefore \(\cap_{n=1}^{\infty} (F_n, E) \neq \emptyset_E\). Conversely, let \(\{(C_n, E)\}_{n=1}^{\infty}\) be a collection of soft closed sets with the finite intersection property. By Proposition 2.1, we construct nested sequence \((F_1, E) \supseteq (F_2, E) \supseteq \ldots \) of nonnull soft closed sets by setting \((F_n, E) = (C_1, E) \cap \ldots \cap (C_n, E)\), for each positive integer \(n\). By the hypothesis \(\cap_{n=1}^{\infty} (F_n, E) = \cap_{n=1}^{\infty} (C_n, E) \neq \emptyset_E\). Now, Theorem 3.8 implies that \((X, \tau, E)\) is countably soft compact. □

4 Soft Separation Axioms

In this section, we will study some soft separation axioms that have studied in \([10, 13]\). First, we recall the definitions.

A soft topological space \((X, \tau, E)\) over \(X\) is called a soft \(T_i\)-space if for each pair of distinct points, at least one has neighborhood not containing the other, and a soft \(T_i\)-space if for each pair of distinct points, each one has a neighborhood not containing the other. Also, the soft space \((X, \tau, E)\) is said to be soft \(T_\omega\)-space (or soft Hausdorff) if for each pair \(x, y\) of distinct points of \(X\), there exist disjoint soft open sets containing \(x\) and \(y\), respectively.

Obviously, every soft \(T_i\)-space \((i = 1, 2)\) is a soft \(T_{i-1}\)-space. But by Remark 3.6 and general topology the converse is not true. In \([10]\), the authors have shown that if \((x, E)\) is a soft closed set in set \((X, \tau, E)\), for all \(x \in X\), then \((X, \tau, E)\) is soft \(T_1\), but the converse does not hold in general.

The soft space \((X, \tau, E)\) over \(X\) is called soft regular if for each soft closed set \((G, E)\) and \(x \in X\) such that \(x \notin (G, E)\) there exist soft open sets \((F_1, E)\) and \((F_2, E)\) such that \(x \in (F_1, E), (G, E) \subseteq (F_2, E)\) and \((F_1, E) \cap (F_2, E) = \emptyset_E\). The soft space \((X, \tau, E)\) is said to be soft \(T_3\)-space if it is soft regular and soft \(T_1\)-space.

Before proceeding, we introduce the concept of soft closure of a soft set (see \([7]\)). Let \((X, \tau, E)\) be a soft topological space and \((F, E)\) be a soft set over \(X\). Then the soft closure of \((F, E)\), denoted by \((\overline{F}, E)\), is the intersection of all soft closed super sets of \((F, E)\). First, we prove the following.

Lemma 4.1. Let \((X, \tau, E)\) be a soft topological space and \((F, E)\) be a soft set over \(X\). If \(x \in (\overline{F}, E)\), then every soft open set \((G, E)\) containing \(x\) intersects \((F, E)\).

Proof. Let \(x \in (\overline{F}, E)\). Let there is a soft open set \((G, E)\) containing \(x\) such that \((F, E) \cap (G, E) = \emptyset_E\). By Proposition 3.2, we have \((F, E) \subseteq (G, E)\). Therefore \((\overline{F}, E) \subseteq (G, E)\). Hence \(x \in (G, E) \cap (G, E)\). This is a contradiction. Therefore \((F, E) \cap (G, E) \neq \emptyset_E\), for each soft open set \((G, E)\) containing \(x\). □

The following example shows that the converse of Lemma 4.1 is not true.

Example 4.2. Suppose that the following sets are given: \(X = \{h_1, h_2, h_3\}, E = \{e_1, e_2\}\) and \(\tau = \{\emptyset_E, \overline{X}, (F_1, E), (F_2, E), \ldots, (F_{30}, E)\}\) where \(F_1, F_2, \ldots, F_{30}\)
are given in Example 9 of [16]. Then \((X, \tau)\) is a soft topological space over \(X\). We consider the soft set \((F_{25}, E)\), where
\[
F_{25}(e_1) = \{h_2\}, \quad F_{25}(e_2) = X.
\]
It is easy to see that the following hold
\[
(F_{25}, E) = (F_{25}, E), \quad h_1 \notin (F_2, E).
\]
But for every soft open set \((F, E)\) containing \(h_1\), we have \((F, E) \cap (F_{25}, E) \neq \phi_E\).

**Proposition 4.3.** Let \((X, \tau, E)\) be a soft regular space. Then, for each point \(x\) of \(X\) and a soft open set \((F, E)\) containing \(x\), there is a soft open set \((G, E)\) containing \(x\) such that \((G, E) \subseteq (F, E)\).

**Proof.** \((F, E)'\) is a soft closed set not containing \(x\). Therefore, there exist soft open sets \((G, E)\) and \((H, E)\) such that \(x \in (G, E), (F, E)' \subseteq (H, E)\) and \((G, E) \cap (H, E) = \phi_E\). Proposition [4.2] implies that \((G, E) \subseteq (H, E)'\). Therefore \((G, E) \subseteq (H, E)' \subseteq ((F, E)'\)' = (F, E).

The following example shows that the converse of Proposition 4.3 does not hold in general.

**Example 4.4.** Let \(X = \{h\}, E = \{e_1, e_2\}\) and \(\tau = \{\phi_E, \tilde{X}, (F_1, E), (F_2, E)\}\), where
\[
F_1(e_1) = \{h\}, \quad F_1(e_2) = \emptyset \quad \& \quad F_2(e_1) = \emptyset, F_2(e_2) = \{h\}
\]

It is easy to see that \((X, \tau, E)\) is not soft regular Nevertheless, for \(h \in X\) and soft open set \(\tilde{X}\) containing \(h\), \(\tilde{X}\) itself is a soft open set containing \(h\) such that \(h \in \tilde{X} \subseteq \tilde{X}\).

Now, we exhibit a necessary and sufficient condition for a soft space to be a soft regular space.

**Theorem 4.5.** A soft space \((X, \tau, E)\) is soft regular if and only if for each \(x \in X\) and soft closed set \((F, E)\) not containing \(x\), there is a soft open set \((G, E)\) containing \(x\) such that \((G, E) \cap (F, E) = \phi_E\).

**Proof.** Let \((X, \tau, E)\) be soft regular. There exist soft open sets \((G, E)\) and \((H, E)\) such that \(x \in (G, E), (F, E) \subseteq (H, E)\) and \((G, E) \cap (H, E) = \phi_E\). Then \((G, E) \subseteq (H, E)' \subseteq (F, E)'\). This implies that \((G, E) \subseteq (H, E)' \subseteq (F, E)'\). Therefore \((G, E) \cap (F, E) = \phi_E\).

Conversely, Proposition 3.2 implies that \((F, E) \subseteq (G, E)'\). Therefore there is a soft open set \((G, E)'\) containing \((F, E)\) such that \((G, E) \cap (G, E)' = \phi_E\). This completes the proof.

A soft space topological space \((X, \tau, E)\) is said to be soft normal if for each soft closed sets \((F, E)\) and \((G, E)\) over \(X\) with null intersection there exist soft open sets \((F_1, E)\) and \((F_2, E)\) containing \((F, E)\) and \((G, E)\) respectively, such that \((F_1, E) \cap (F_2, E) = \phi_E\). Also, a soft topological space \((X, \tau, E)\) is said to be a soft \(T_1\)-space if it is soft normal and soft \(T_1\)-space.
Theorem 4.6. Let \((X, \tau, E)\) be a soft space. Let for each soft closed set \((F, E)\) and soft open set \((G, E)\) containing \((F, E)\) there is a soft open set \((H, E)\) containing \((F, E)\) such that \((\overline{F}, E) \subseteq (G, E)\). Then \((X, \tau, E)\) is soft normal.

Proof. For each soft closed sets \((F, E)\) and \((I, E)\) with null intersection \((I, E)'\) is a soft open set containing \((F, E)\). Therefore there is a soft open set \((H, E)\) containing \((F, E)\) such that \((H, E) \subseteq (I, E)'\). By Proposition 3.2 \((I, E)' \subseteq (H, E)'\). Since \((H, E) \subseteq ((H, E)'\)'\), we have \((H, E) \cap (H, E)' = \emptyset E\). Hence \((X, \tau, E)\) is soft normal.

There is an obvious question to ask at this point. Is a soft \(T_{4}\)-space a soft \(T_{3}\)-space? The soft space \((X, \tau, E)\) in Example 4.4, shows that the answer is "NO". In fact it is easy to see that \((X, \tau, E)\) is a soft \(T_{4}\)-space and not a soft \(T_{3}\)-space.

Remark 4.7. In Theorem 3.17 of [10], the following is proved:

Theorem. ([10]) Let \((X, \tau, E)\) be a soft topological space over \(X\) and \(x \in X\). Then the following are equivalent:

1. \((X, \tau, E)\) is a soft regular space;
2. For each soft closed set \((G, E)\) such that \((x, E) \cap (G, E) = \emptyset E\).

There exist soft two open sets \((F_1, E)\) and \((F_2, E)\) such that \((x, E) \subseteq (F_1, E), (G, E) \subseteq (F_2, E)\) and \((F_1, E) \cap (F_2, E) = \emptyset E\).

By Example 4.3 we can see that this theorem is incorrect. In fact the soft space \((X, \tau, E)\) in this example satisfies in (2), but it is not soft regular. We note that \((x, E) \cap (G, E) = \emptyset E\) is not equivalent to \(x \notin (G, E)\). But \((x, E) \subseteq (G, E)\) is. Therefore, we must replace the condition \((x, E) \subseteq (G, E) = \emptyset E\) in Theorem 3.17 of [10].

Remark 4.8. In Theorem 3.25 of [10], the following is proved:

Theorem. ([10]) Let \((X, \tau, E)\) be a soft topological space over \(X\). If \((X, \tau, E)\) is a soft normal space and if \((x, E)\) is a soft closed set for each \(x \in X\), then \((X, \tau, E)\) is a soft \(T_{3}\)-space.

This theorem is incorrect. The soft space \((X, \tau, E)\) in Example 4.4 satisfies in the conditions of the theorem, but it is not a soft \(T_{3}\)-space.

There are some familiar results on the applications of compactness in separation axioms in General Topology such as: Every compact Hausdorff space is normal. But it is not true about soft topological spaces. Consider the following example.

Example 4.9. Let \(X = \{h\}, E = \{e_i\}_{i=1}^{5}\) and \(\tau = \{\emptyset E, \bar{X}, (F_1, E), (F_2, E), (F_3, E)\}\), where

\[F_1(e_1) = \emptyset, F_1(e_2) = X, F_1(e_3) = \emptyset, F_1(e_4) = X, F_1(e_5) = \emptyset;\]
\[ F_2(e_1) = X, \; F_2(e_2) = X, \; F_2(e_3) = X, \; F_2(e_4) = \emptyset, \; F_2(e_5) = X; \]
\[ F_3(e_1) = \emptyset, \; F_3(e_2) = X, \; F_3(e_3) = \emptyset, \; F_3(e_4) = \emptyset, \; F_3(e_5) = \emptyset. \]

It is easy to see that \((X, \tau, E)\) is not soft normal. Nevertheless, it is soft compact.

It is remarkable that every compact Hausdorff space is not normal, even if we consider \((X, \tau, E)\) as a soft regular space. Indeed, the Example 4.4 is a counterexample.

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Esmaeil Peyghan and Babak Samadi  
Department of Mathematics, Faculty of Science  
Arak University  
Arak 38156-8-8349, Iran  
Email: epeyghan@gmail.com

Akbar Tayebi  
Department of Mathematics, Faculty of Science  
University of Qom  
Qom, Iran  
Email: akbar.tayebi@gmail.com