Kerr combs excited in nonlinear optical microresonators hold promise as chip scale generators of octave spanning optical frequency combs with unique characteristics \[7, 8\]. They also reveal a rich, and as yet not well understood, variety of nonlinear dynamical phenomena. Different regimes of comb generation have been observed and several theoretical models developed. While the main focus of the research in these studies is related to understanding the spectral properties of the Kerr comb, less work has been devoted to its time domain behavior. Several experimental \[2–5\] and theoretical \[6–9\] studies of mode locking regimes of this nonlinear process have been published very recently, but the problem of temporal growth of comb harmonics was addressed only theoretically \[7, 8\]. In this Letter we report on experimental study of transient dynamics of Kerr frequency combs and also propose a theoretical explanation for the observed results. We also suggest a numerical model, backed by the experiment, that predicts a different dynamics for Kerr comb growth compared with earlier theoretical predictions. Our research clearly explains why harmonics of the comb are modelocked.

Kerr combs are generated from electromagnetic vacuum fluctuations due to modulation instability of a continuous wave (cw) light confined in an externally pumped nonlinear dispersive resonator. When the power of the cw optical pump that is nearly resonant with one of the modes of the resonator exceeds a certain threshold, the cw field inside the resonator becomes unstable, and multiple frequency harmonics are generated in the modes. The harmonics are equally spaced due to energy and photon number conservation laws, imposed by four-wave mixing process (FWM), which is responsible for comb generation \[10, 11\].

The growth of the comb is not instantaneous. It was found \[7, 8\] that formation of a fully developed comb can take up to a hundred ring-down periods of the resonator mode. The goal of the present contribution is to measure this time interval directly. We performed a numerical simulation and found that the DC power of light exiting the resonator depends on the degree of comb formation. The effective intrinsic quality factor of the pumped optical mode suddenly drops long after the steady state amplitude of the circulating light in the mode is reached. The reduction of the intrinsic quality factor changes the attenuation of the pump mode since the balance between intrinsic loss and coupling loss in the resonator changes. Hence, the time delay between the moment the light enters the pump mode and the moment when the frequency comb is generated can be directly measured by detecting the light escaping the resonator. It is also possible to track the time dependence of the power of the RF signal generated by the comb to measure the delay. We performed such a measurement with a Kerr frequency comb produced in a high-Q CaF\textsubscript{2} whispering gallery mode (WGM) resonator, and experimentally confirmed the theoretical prediction of \[7, 8\] as well as the result of our numerical simulations. In what follows we describe our simulations and the experiment in detail.

To simulate the transient regime of the Kerr comb we use the theoretical model developed in \[8, 12\]. We introduce an interaction Hamiltonian

\[
\hat{\mathcal{V}} = -\frac{\hbar g}{2} (\hat{c}^\dagger)^2 \hat{c}^2,
\]

where

\[
g = \frac{\hbar \omega_0^2 c n_2}{V n_0^2}
\]

is the coupling parameter obtained under assumption of complete space overlap of the resonator modes, $\omega_0$ is the value of the optical frequency of the externally pumped mode, $c$ is the speed of light in vacuum, $n_2$ is the cubic nonlinearity of the material, $V$ is the effective geometrical volume occupied by the optical modes in the resonator, and $n_0$ is the linear index of refraction of the resonator host material. The operator $\hat{c}$ is given by the sum of annihilation operators of the electromagnetic field for 41 interacting resonator modes that we took into consideration:

\[
\hat{c} = \sum_{j=1}^{41} \hat{a}_j.
\]

The external cw pump is applied to the central mode of the group, so the Kerr frequency comb is expected to have twenty red- and twenty blue-detuned harmonics with respect to the frequency of the pumped mode. We take into account only the second order frequency dispersion that is recalculated for the frequency of the modes.

Equations describing the evolution of the field in the resonator modes are generated using Hamiltonian \[11\] and input-output formalism developed for ring resonators \[13\]

\[
\dot{\hat{a}}_j = - (\gamma_0 + i \omega_j) \hat{a}_j + \frac{i}{\hbar} [\hat{\mathcal{V}}, \hat{a}_j] + F_0 e^{-i \omega t} \delta_{21,j},
\]

where $\gamma_0$ is the intrinsic loss rate of the photonic mode, $\omega_j$ is the optical frequency of the resonator mode, $\hat{\mathcal{V}}$ is the interaction Hamiltonian, $F_0$ is the external cw pump amplitude, and $\delta_{21,j}$ is the Kronecker delta function.
where $\delta_{21,j}$ is the Kronecker’s delta; $\gamma_0 = \gamma_0c + \gamma_0i$ is the half width at the half maximum for the optical modes, assumed to be the same for the all modes involved; and $\gamma_0c$ and $\gamma_0i$ stand for coupling and intrinsic loss. The external optical pumping is given by

$$F_0 = \sqrt{\frac{2\gamma_0cP}{h\omega_0}}, \quad (5)$$

where $P$ is the value of the cw pump light. We neglect the quantum effects and do not take into account corresponding Langevin noise terms.

The set (4) should be supplied with an equation describing the light leaving the resonator. Assuming the pump power does not depend on time, the relative amplitude of the output field is given by [13]

$$\frac{\hat{e}_{out}}{\hat{e}_{in}} = \sqrt{1 - 2\gamma_0c\gamma_0i} - \sqrt{1 - 2\gamma_0c\gamma_0i \frac{2\gamma_0c - \gamma_0i}{\gamma_0c + \gamma_0i} F_0}, \quad (6)$$

where $\tau_0 = 2\pi Rn_0/c$ is the light round trip time for the resonator, and $R$ is the radius of the resonator. It is assumed that $\hat{e}(t - \tau_0) \approx \hat{e}(t)$.

We performed a numerical simulation of the growth of the first five harmonics of the frequency comb Fig. (1b) revealing several important features of comb generation. The sidebands grow exponentially with different growth rates. The first harmonic ((1) in Fig. (1) has the slowest growth rate of $\gamma_0$, the second harmonic ((2) in Fig. (1) has twice faster growth rate, $2\gamma_0$, the third harmonic $3\gamma_0$. The first harmonic starts to grow much earlier than others, the second harmonic starts before the third, etc. Such a temporal behavior shows that the particular realization of the Kerr comb is initiated by hyper-parametric oscillation [12] that involves only two optical sidebands closest in frequency to the pump. The next order of optical harmonics is generated in the stimulated comb due to FWM of the already generated sidebands and the pump light [14]. This stimulated process does not have a threshold. In other words, the threshold of comb generation coincides with the threshold of the hyper-parametric oscillation. The lowest order harmonics also determine the phase as well as the frequency of the rest of the harmonics. Such a comb is always phase locked and optical pulses are formed in the resonator (Fig. [2]).

An important observation is related to the temporal behavior of the pump light. Since the growth of the comb harmonics is not instantaneous, the pump light is not influenced by the comb growth initially. The pump power is impacted only after the frequency harmonics approach their saturation values. This behavior is clearly seen at the phase diagram (Fig. 3), where the pump mode has two attractors. The first one ((I) in Fig. 3a) corresponds to the steady state solution for the pump light in the nonlinear resonator with no harmonics generated, and the second attractor ((II) in Fig. 3b) corresponds to the steady state solution with the saturated comb. The duration of the transition process depends on the nonlinearity parameter $g/\gamma_0$ which defines the maximal number of photons generated in the comb harmonics. The larger $g$ is, the smaller is the number. On the other hand, the initial pho-
ton number is equal to unity. The growth rate of the comb is on the order of $\gamma_0$. Therefore, the transient process is longer when $g/\gamma_0$ is smaller. We have calculated the delay for two cases: $(g/\gamma_0)^{1/2} = 5 \times 10^{-5}$ (estimated for a small resonator, e.g. [15]) and $(g/\gamma_0)^{1/2} = 10^{-8}$ (estimated for a much larger resonator). The result of the calculation is shown in Fig. (3b).

The behavior of the DC power of light leaving the resonator has a certain peculiarity. The value of the power exiting the resonator initially decreases and then increases (Fig. 3b). The phenomenon can be explained from the standpoint of critical coupling [13]. There is no light at the output of a linear resonator if $\gamma_0/t = \gamma_0c$ and the steady state is reached; all the pump light is absorbed in the resonator. That is why the amplitude of the exiting light drops after the pump is on. A number of the pump photons is redistributed between harmonics of the comb, as the comb is generated. Those harmonics leave the resonator reducing the absorption of the pumping light. The interference phenomenon resulting in the critical coupling is also deteriorated since the pump light confined in the corresponding mode changes its amplitude and phase due to the nonlinear process.

\[
\langle a_{21}(t) \rangle = 5 \times 10^{-5} \quad \text{and} \quad \langle a_{22}(t) \rangle = 10^{-8}
\]

The temporal dependencies shown in Fig. (1b) and Fig. (3b) can be verified experimentally if one measures the power of the comb harmonics exiting the resonator. Instead of the direct measurement of the optical harmonics, the power of the radio frequency (RF) signal generated on a fast photodiode by the comb can be measured. We performed such experiments.

We used a calcium fluoride (CaF$_2$) whispering gallery mode (WGM) microresonator with loaded Q-factor $2.2 \times 10^9$ (full width at the half maximum of the mode is 90 kHz, and corresponding ring down time 1.75 $\mu$s). The intrinsic Q-factor of the resonator was $5.5 \times 10^9$ ($\gamma_0c = 1.5\gamma_0c$), which means that the attenuation of light in the modes was primarily given by the interaction with the evanescent field coupler (a glass prism), and not by the scattering and loss of the resonator host material. The resonator was pumped using a 1545 nm distributed feedback (DFB) semiconductor laser, self-injection locked to a selected resonator mode [16]. The laser was emitting approximately 6 mW of power, 30% of which entered the resonator.

![FIG. 4: Schematic of the experimental setup. The WGM resonator is pumped with a DFB laser, self injection locked to the selected mode. The output light is analyzed with an optical spectrum analyzer (OSA) showing the spectrum of the generated Kerr comb. Part of the light is sent to a fast photodiode (PD). The photocurrent, modulated with a frequency equal to the comb repetition rate, is directed to an RF power detector (RFD), and a fast oscilloscope. This signal is proportional to the convolution of the harmonics of the optical frequency comb. Some of the light is also detected with a slow photodiode (PD) and the photocurrent from this photodiode is forwarded to another channel of the same oscilloscope. This signal shows the integral DC power leaving the resonator. Comparing the signals with the fast oscilloscope we are able to measure the time delay between the generation of the Kerr comb and the moment the pump light enters the corresponding mode.](image-url)
It may appear that conditions of the numerical simulation and the experiment described above are not exactly the same. In the experiment we used a large CaF$_2$ resonator (6.72 mm in diameter) with free spectral range of 10 GHz. The GVD of modes of an ideal spheroidal resonator of this size is much smaller compared with the value of GVD we utilized in the numerical simulations. Moreover, GVD is normal for the fundamental mode sequence of the CaF$_2$ resonator (assuming its ideal spheroidal shape). Nevertheless, the optical frequency comb observed in the experiment has a spectral shape similar to the one obtained with the simulation, and the transient processes measured in the experiment have good correspondence with the theory. This contradiction is resolved if we take into account the notion that the morphology of a monolithic resonator allows changing the sign and value of GVD[11]. In the experiment we did not use the ideal spheroidal resonator or the fundamental mode sequence. We rather sent the light to a higher-order mode of the resonator. The observed Kerr frequency comb was generated for light tuned nearly at the top of the WGM resonance ($\omega \approx \omega_0$). This is possible only if GVD is anomalous and large for a locally selected mode family, in accordance with the theory of hyper-parametric oscillation[12]. That is why the numerical simulation for a resonator characterized with large anomalous dispersion is applicable for the description of the experiment performed with a resonator seemingly having normal dispersion.

To conclude, we have studied the transient regimes of Kerr frequency comb formation in a nonlinear monolithic optical resonator. We found that the well developed comb generation is delayed by tens of ring down intervals of the resonator. Kerr combs created in larger and/or less nonlinear resonators have longer transient period compared with those generated in smaller and/or more nonlinear resonators. We noted that Kerr combs generated in resonators possessing comparably large anomalous GVD are always mode-locked. Finally, we experimentally found that there exist mode families with large anomalous GVD even in WGM resonators made out of a material with normal GVD. We experimentally validated the results of our numerical simulations with a suitable mode family.

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