Bound on the curvature of the Isgur-Wise function of the baryon semileptonic decay $\Lambda_b \to \Lambda_c \ell \nu_{\ell}$

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Abstract

In the heavy quark limit of QCD, using the Operator Product Expansion, the formalism of Falk for hadrons of arbitrary spin, and the non-forward amplitude, as proposed by Uraltsev, we formulate sum rules involving the Isgur-Wise function $\xi_{\Lambda}(w)$ of the baryon transition $\Lambda_b \to \Lambda_c \ell \nu_{\ell}$, where the light cloud has $j^P = 0^+$ for both initial and final baryons. We recover the lower bound for the slope $\rho_{\Lambda}^2 = -\xi_{\Lambda}'(1) \geq 0$ obtained by Isgur et al., and we generalize it by demonstrating that the IW function $\xi_{\Lambda}(w)$ is an alternate series in powers of $(w - 1)$, i.e. $(-1)^n \xi_{\Lambda}^{(n)}(1) \geq 0$. Moreover, exploiting systematically the sum rules, we get an improved lower bound for the curvature in terms of the slope, $\sigma_{\Lambda}^2 = \xi_{\Lambda}''(1) \geq \frac{3}{4} \rho_{\Lambda}^2 + (\rho_{\Lambda}^2)^2$. This bound constrains the shape of the Isgur-Wise function and it will be compelling in the analysis of future precise data on the differential rate of the baryon semileptonic decay $\Lambda_b \to \Lambda_c \ell \nu_{\ell}$, that has a large measured branching ratio, of about 5\%.

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1 Introduction.

Let us first briefly review the situation of meson semileptonic decays \( \overline{B} \rightarrow D^{(*)}(\ell \nu) \), and next shift to the topic of the present paper, the baryon decay \( \Lambda_b \rightarrow \Lambda_c(\ell \nu) \). The meson case will illuminate some aspects of the baryon case. In the leading order of the heavy quark expansion of QCD, Bjorken sum rule (SR) \([1, 2]\) relates the slope of the elastic heavy meson Isgur-Wise (IW) function \( \xi(w) \), to the IW functions of the transition between the ground state \( j^P = \frac{1}{2}^- \) and the \( j^P = \frac{1}{2}^+, \frac{3}{2}^+ \) excited states, \( \tau_{1/2}^{(n)}(w), \tau_{3/2}^{(n)}(w) \), at zero recoil \( w = 1 \) (\( n \) is a radial quantum number). This SR leads to the lower bound \( \rho^2 = -\xi'(1) \geq \frac{1}{4} \). A new SR was formulated by Uraltsev in the heavy quark limit \([3]\), involving also \( \tau_{1/2}^{(n)}(w), \tau_{3/2}^{(n)}(w) \), that implies, combined with Bjorken SR, the much stronger lower bound

\[
\rho^2 \geq \frac{3}{4} \tag{1}
\]

A basic ingredient in deriving this bound was the consideration of the non-forward amplitude \( \overline{B}(v_i) \rightarrow D^{(n)}(v_f) \rightarrow \overline{B}(v_f) \), allowing for general \( v_i, v_f, v'_f \) and where \( B \) is a ground state meson. In refs. \([4, 5, 6]\) we have developed, in the heavy quark limit of QCD, a manifestly covariant formalism within the Operator Product Expansion (OPE), using the whole tower of heavy meson states \([7]\). We did recover Uraltsev SR plus a general class of SR that allow to bound also higher derivatives of the IW function. In particular, we found two bounds for the curvature \( \sigma^2 = \xi''(1) \) in terms of \( \rho^2 \), namely

\[
\sigma^2 \geq \frac{5}{4} \rho^2 \tag{2}
\]

\[
\sigma^2 \geq \frac{1}{5} \left[ 4\rho^2 + 3(\rho^2)^2 \right] \tag{3}
\]

that both reduce to

\[
\sigma^2 \geq \frac{15}{16} \tag{4}
\]

for the lower limit \((1)\).

On the other hand, we found also lower bounds for all higher derivatives, namely

\[
(-1)^L \xi^{(L)}(1) \geq \frac{(2L + 1)!!}{2^{2L}} \tag{5}
\]

that reduce to \((1)\) and \((4)\) for the slope and the curvature.
In the baryon “elastic” case, several types of transitions can be considered, namely $\Lambda_b \to \Lambda_c$, $\Sigma_b \to \Sigma_c$ (and the related transitions to the latter, namely $\Sigma_b \to \Sigma^{*}_c$, $\Sigma^{*}_b \to \Sigma^{*}_c$), where $\Lambda_Q$ has isospin $I = 0$ and $J^P = \frac{1}{2}^+$, and $\Sigma_Q$ has $I = 1$, $J^P = \frac{1}{2}^+$ ($I = 1$, $J^P = \frac{3}{2}^+$ for $\Sigma^{*}_Q$), as well as the corresponding transitions for $\Xi_b$, $\Omega_b$, etc.

We will here concentrate on the semileptonic decay $\Lambda_b \to \Lambda_c \ell \nu_\ell$. Some decay modes have already been measured for the $\Lambda_b$, in particular this semileptonic decay, $BR(\Lambda_b \to \Lambda_c \ell \nu_\ell) \cong 5\%$, a large fraction of the inclusive decay $BR(\Lambda_b \to \Lambda_c \ell \nu_\ell + \text{anything}) \cong 10\%$. Hopefully, in the near future, at the LHC-b program, the exclusive semileptonic decay $\Lambda_b \to \Lambda_c \ell \nu_\ell$ will be measured in detail, in particular its differential rate. Therefore, it is of interest to study the properties of the corresponding elastic IW function, that we denote here by $\xi_\Lambda(w)$.

In the heavy quark limit for $b$ and $c$ quarks, the baryons $\Lambda_b$ and $\Lambda_c$ have a light cloud with $j = 0$. Isgur et al. [8] formulated the equivalent of Bjorken sum rule for this case,

$$\rho_\Lambda^2 = -\xi_\Lambda'(1) = \sum_{n \geq 0} \left[ \tau^{(n)}_1(1) \right]^2$$

obtaining therefore

$$\rho_\Lambda^2 \geq 0$$

The quantities $\tau^{(n)}_1(1)$ denote the $0^+ \to 1^-$ ($j^P$ of the light cloud, $n$ denotes a radial quantum number) IW functions at zero recoil. We use this notation to keep track of the analogy with the meson case, and we will make below the link with the notation of Falk [7]. Let us still underline that only intermediate states $\Lambda^{(n)}_c$ with isospin $I = 0$ can contribute.

Baryon semileptonic decays have aroused some interest. New symmetries in these decays were formulated by Isgur and Wise [9]. Baryons of arbitrary spin were considered in detail in [7], whose formalism we use below. On the other hand, the IW function for $\Lambda_b \to \Lambda_c \ell \nu_\ell$ was studied in the large $N_c$ limit by Jenkins et al. [10] and by Chow [11]. Power corrections to baryon form factors were studied by Georgi et al. [12], Falk and Neubert [13] and Mannel and Roberts [14]. Within the dispersive approach in QCD, baryon form factors have been studied at finite mass by C. Boyd et al. [15, 16]. The slope for the IW was computed within the QCD Sum Rules approach by Huang et al. [17]. Extensive and useful review papers on heavy
baryons are those of Körner et al. [18] and Falk [19]. Finally, a study of semileptonic decays of $\Lambda_b$, $\Lambda_c$ baryons in quark models has been performed by Pervin et al. [20], where it has been pointed out that the extension to baryons of our meson bound (3) was lacking. The present paper answers to this need.

Thus, our aim is to extend the program realized in the meson case to the baryon transition $\Lambda_b \to \Lambda_c \ell \nu_\ell$ and formulate the equivalent relations to the results of the meson case, in particular (3) and (5).

Let us underline that there is an important difference between the meson case $B \to D^{(*)}$ and the baryon case $\Lambda_b \to \Lambda_c$. In the sum rules (SR) of the meson case we had contributions where the light cloud has, for a given orbital angular momentum $L$, two possible values $j^P = (L \pm \frac{1}{2})^P$, $P = (-1)^{L+1}$, where $S = \frac{1}{2}$ stands for the spin of the spectator light quark. In the baryon transition $\Lambda_b \to \Lambda_c$, since the two spectator quarks have total spin and isospin $S = I = 0$, we have, for a given $L$, a single type of intermediate states, with $j^P = L^P$, $P = (-1)^L$.

There is another important difference between the meson and the baryon cases. This concerns the $1/m_Q$ corrections. Georgi et al. [12] have demonstrated that the first order $1/m_Q$ corrections in $\Lambda_b \to \Lambda_c \ell \nu_\ell$ are given simply in terms of the IW function $\xi_\Lambda(w)$ and the constant $\overline{\Lambda} = m_{\Lambda_b} - m_b$. This is to be distinguished from the meson case, where the $1/m_Q$ corrections depend on the IW function $\xi(w)$, the constant $\overline{\Lambda} = m_B - m_b$ and on another function $\xi_3(w)$, as shown for example by Falk and Neubert [21]. Thus, the decay $\Lambda_b \to \Lambda_c \ell \nu_\ell$ has $1/m_Q$ corrections much better controlled than the meson case.

The paper is organized as follows. In Section 2 we recall the SR for mesons within the framework of the non-forward amplitude. In Section 3 we formulate the SR in the baryon case and underline the differences with the meson case. In Section 4 we generalize the inequality (7) for all the derivatives, demonstrating that the baryon IW function $\xi_\Lambda(w)$ is an alternate series in powers of $(w - 1)$. In Section 5, exploiting systematically all the SR that can be obtained for the baryon case, we get an improved bound for the curvature of the IW function, that is the equivalent of the inequality (3) for the meson case. In Section 6 we overview further tasks to be performed and compare our results with previous work on heavy baryon form factors. Finally, in Section 7 we conclude.
2 Recall of the sum rules in the meson case.

In the case of mesons, the general SR obtained from the OPE can be written in the compact way [4]

\[ L_{\text{Hadrons}}(w_i, w_f, w_{if}) = R_{\text{OPE}}(w_i, w_f, w_{if}) \] (8)

where the l.h.s. is the sum over the intermediate \( D \) states, while the r.h.s. is the OPE counterpart. This expression writes, in the heavy quark limit [4]:

\[
\sum_{D=P,V} \sum_n Tr \left[ \overline{B}_f(v_f) \Gamma_f D^{(n)}(v') \right] Tr \left[ D^{(n)}(v') \Gamma_i B_i(v_i) \right] \xi^{(n)}(w_i) \xi^{(n)}(w_f) \\
+ \text{Other excited states} = -2 \xi(w_{if}) Tr \left[ \overline{B}_f(v_f) \Gamma_f P'_+ \Gamma_i B_i(v_i) \right] (9)
\]

where

\[ w_i = v_i \cdot v' \quad w_f = v_f \cdot v' \quad w_{if} = v_i \cdot v_f \] (10)

\[ P'_+ = \frac{1 + \beta'}{2} \] is the positive energy projector on the intermediate \( c \) quark. We assume that the IW functions are real and the \( \overline{B} \) meson is the pseudoscalar ground state \((j^P, J^P) = \left( \frac{1}{2}^-, 0^- \right)\), \( j \) is the angular momentum of the light cloud and \( J \) the spin of the bound state. The heavy quark currents considered in the preceding expression are

\[ \overline{h}_{v'} \Gamma_i h_{v_i} \quad \overline{h}_{v_f} \Gamma_f h_{v_f} \] (11)

In (9) \( B(v), D(v) \) are the \( 4 \times 4 \) matrices representing the \( \overline{B}, D \) states [7], and \( \overline{B} = \gamma^0 B' + \gamma^0 \) denotes the Dirac conjugate matrix. The domain for the variables \((w_i, w_f, w_{if})\) is [4]:

\[ w_i \geq 1 \quad w_f \geq 1 \]
\[ w_i w_f - \sqrt{(w_i^2 - 1)(w_f^2 - 1)} \leq w_{if} \leq w_i w_f + \sqrt{(w_i^2 - 1)(w_f^2 - 1)} \] (12)

Taking \( w_i = w_f = w \), the domain becomes

\[ w \geq 1 \quad 1 \leq w_{if} \leq 2w^2 - 1 \] (13)

In [4] the following SR were established. Taking \( \Gamma_i = \beta_i \) and \( \Gamma_f = \beta_f \) one finds the so-called Vector SR

\[(w + 1)^2 \sum_{L \geq 0} \frac{L + 1}{2L + 1} S_L(w, w_{if}) \sum_n \left[ \tau_{L+1/2}^{(L)}(w_i) \right]^2 + \sum_{L \geq 1} S_L(w, w_{if}) \sum_n \left[ \tau_{L-1/2}^{(L)}(w_i) \right]^2\]
\( = (1 + 2w + w_{if}) \xi(w_{if}) \) (14)

and for \( \Gamma_i = \gamma_i \gamma_5 \) and \( \Gamma_f = \gamma_f \gamma_5 \) one finds the Axial SR

\[
\sum_{L \geq 0} S_{L+1}(w, w_{if}) \sum_n \left[ \tau_{L+1/2}^{(L)}(w) \right]^2 + (w - 1)^2 \sum_{L \geq 1} \frac{L}{2L - 1} S_{L-1}(w, w_{if}) \sum_n \left[ \tau_{L-1/2}^{(L)}(w) \right]^2 \\
= -(1 - 2w + w_{if}) \xi(w_{if})
\] (15)

In the precedent equations the IW functions \( \tau_{L \pm 1/2}^{(L)}(w) \) correspond to the transitions \( \frac{1}{2} - \rightarrow \left( L \pm \frac{1}{2} \right)^P, P = (-1)^{L+1} \), and the function \( S_L(w, w_{if}) \) is given by the Legendre polynomial

\[
S_L(w, w_{if}) = \sum_{0 \leq k \leq L/2} C_{L,k} (w^2 - 1)^{2k} (w^2 - w_{if})^{L-2k}
\] (16)

with

\[
C_{L,k} = (-1)^k \frac{(L!)^2}{(2L)!} \frac{(2L - 2k)!}{k!(L-k)!(L-2k)!}
\] (17)

Differentiating \( n \) times both SR (14), (15) with respect to \( w \) and \( w_{if} \) and going to the border of the domain \( (13) \ w_{if} = w = 1 \), one gets the bounds (1)-(5).

To be complete in this recall of the meson case, let us remind that, on the other hand, Uraltsev [22] did propose a special limit of HQET, namely the so-called BPS limit, that implies

\[
\rho^2 = \frac{3}{4}
\] (18)

among other interesting consequences for subleading quantities. In [23] it was demonstrated, using the above SR, that if the slope reaches its lower bound (1), as happens in the BPS limit, then all derivatives reach their lower bounds (5), and then the IW function is completely fixed, namely

\[
\xi(w) = \left( \frac{2}{w + 1} \right)^{3/2}
\] (19)

3 Sum rules for the baryon case.

As explained in [8], the following fields can be written for the \( J = j \pm \frac{1}{2} \) baryons, where \( j \) is the spin of the light cloud.
One defines the tensor-spinor (we change Falk’s notation $A^{\mu_1 \cdots \mu_j}$ by $\varepsilon^{\mu_1 \cdots \mu_j}$ to underline what is technically common with the meson case)

$$\psi^{\mu_1 \cdots \mu_j} = \varepsilon^{\mu_1 \cdots \mu_j} u_h$$  (20)

where $\varepsilon^{\mu_1 \cdots \mu_j}$ is symmetric, and the following transversity and tracelessness conditions are fulfilled

$$v_{\mu_k} \varepsilon^{\mu_1 \cdots \mu_j} = 0 \quad (k = 1, \cdots j)$$

$$g_{\mu_i \mu_k} \varepsilon^{\mu_1 \cdots \mu_j} = 0 \quad (i, k = 1, \cdots j) \quad (i \neq k)$$  (21)

Then, there are two baryon fields corresponding to $J = j \pm \frac{1}{2}$

$$\psi_{j-1/2}^{\mu_1 \cdots \mu_j} = \frac{1}{2j + 1} \left[ (\gamma^{\mu_1} + v^{\mu_1}) \gamma_{\nu_1} g^{\mu_2}_{\nu_2} \cdots g^{\mu_j}_{\nu_j} + \cdots + g^{\mu_1}_{\nu_1} \cdots g^{\mu_{j-1}}_{\nu_{j-1}} (\gamma^{\mu_j} + v^{\mu_j}) \gamma_{\nu_j} \right] \varepsilon^{\nu_1 \cdots \nu_j} u_h$$  (22)

$$\psi_{j+1/2}^{\mu_1 \cdots \mu_j} = \left\{ g^{\mu_1}_{\nu_1} \cdots g^{\mu_j}_{\nu_j} ight\}$$

$$- \frac{1}{2j + 1} \left[ (\gamma^{\mu_1} + v^{\mu_1}) \gamma_{\nu_1} g^{\mu_2}_{\nu_2} \cdots g^{\mu_j}_{\nu_j} + \cdots + g^{\mu_1}_{\nu_1} \cdots g^{\mu_{j-1}}_{\nu_{j-1}} (\gamma^{\mu_j} + v^{\mu_j}) \gamma_{\nu_j} \right] \varepsilon^{\nu_1 \cdots \nu_j} u_h$$  (23)

where $u_h$ is the spin $\frac{1}{2}$ field of the heavy quark.

It follows, from (21),

$$v_{\mu_k} \psi_{j+1/2}^{\mu_1 \cdots \mu_j} = 0 \quad (k = 1, \cdots j)$$

$$g_{\mu_i \mu_k} \psi_{j+1/2}^{\mu_1 \cdots \mu_j} = 0 \quad (i, k = 1, \cdots j) \quad (i \neq k)$$  (24)

The tensor (23) is the generalization to all $j$ of the Rarita-Schwinger vector-spinor field, that satisfies another condition,

$$\gamma_{\mu_k} \psi_{j+1/2}^{\mu_1 \cdots \mu_j} = 0 \quad (k = 1, \cdots j)$$  (25)

The matrix elements of a heavy quark current

$$\mathcal{F}_{\nu} \Gamma h_{\nu}$$  (26)

for the transition $j \pm \frac{1}{2} \rightarrow j' \pm \frac{1}{2}$ writes [7]

$$\langle H_{j' \pm 1/2}(v')|J(q)|H_{j \pm 1/2}(v)\rangle = \overline{\psi}_{j' \pm 1/2}^{\mu_1 \cdots \nu_{j'}} \Gamma \psi_{j \pm 1/2}^{\mu_1 \cdots \mu_j} \zeta_{\nu_1 \cdots \nu_{j'}, \mu_1 \cdots \mu_j}$$  (27)
where $j' \geq j$ is assumed and the tensor $\zeta_{\nu_1 \cdots \nu_j', \mu_1 \cdots \mu_j}$ is given by the expression

$$
\zeta_{\nu_1 \cdots \nu_j', \mu_1 \cdots \mu_j} = (-1)^j (v' - v)_{\nu_{j+1}} \cdots (v' - v)_{\nu_{j'}} C_{0}^{(j', 0)} (w) g_{\nu_1 \mu_1} \cdots g_{\nu_j \mu_j}
$$

We find, in an analogous way to the meson SR, and general currents

$$
\bar{h}_v \Gamma_i h_{v_i} \quad \text{and} \quad \bar{h}_{v_f} \Gamma f h_{v_f}
$$

We find, in an analogous way to the meson SR,

$$
\sum_n \sum_L \tau_{L}^{(n)} (w_i) \tau_{L}^{(n)} (w_f) v_{f \mu_1} \cdots v_{f \mu_L} v_{i \nu_1} \cdots v_{i \nu_L} \left( \bar{\psi}_{L+1/2} \psi_{L-1/2} \Gamma_i u_{h_i} + \bar{\psi}_{L-1/2} \psi_{L+1/2} \Gamma_i u_{h_i} \right) = \xi_\Lambda (w_f) \left[ \bar{\psi}_{h_f} \Gamma f N'_+ \Gamma_i u_{h_i} \right]
$$

We find, in an analogous way to the meson SR,

$$
\sum_n \sum_L \tau_{L}^{(n)} (w_i) \tau_{L}^{(n)} (w_f) v_{f \mu_1} \cdots v_{f \mu_L} v_{i \nu_1} \cdots v_{i \nu_L} \left( \bar{\psi}_{L+1/2} \psi_{L-1/2} \Gamma_i u_{h_i} + \bar{\psi}_{L-1/2} \psi_{L+1/2} \Gamma_i u_{h_i} \right) = \xi_\Lambda (w_f) \left[ \bar{\psi}_{h_f} \Gamma_f N'_+ \Gamma_i u_{h_i} \right]
$$

Let us underline that in the case of the SR $\Lambda_b(v_i) \rightarrow \Lambda_c^{(n)}(v') \rightarrow \Lambda_b(v_f)$, only intermediate states $\Lambda_c^{(n)}$ with isospin $I = 0$ can contribute and, in general, all $j^P = L^P$, $P = (-1)^L$, contribute. Hence the notation $\tau_{L}^{(n)} (w)$ for the transition IW functions.
3.1 Vector sum rule.

Adopting vector currents aligned along the intermediate four-velocity $v'$

$$\Gamma_i = \mathcal{I}_i = \varphi'$$
$$\Gamma_f = \mathcal{I}_f = \varphi'$$

(36)

and denoting the tensor

$$T^{\rho_1 \ldots \rho_L, \sigma_1 \ldots \sigma_L} = \sum_{\lambda} \varepsilon^\lambda(\lambda^*)^{\rho_1 \ldots \rho_L} \varepsilon^\lambda(\lambda)_{\sigma_1 \ldots \sigma_L}$$

(37)

where the sum is carried over the $2L + 1$ polarizations $\lambda = -L, \ldots, +L$, we find, using the symmetry properties of the tensor (37),

$$\sum_n \sum_L \tau^{(n)}_L (w_i) \tau^{(n)}_L (w_f) T^{\rho_1 \ldots \rho_L, \sigma_1 \ldots \sigma_L} \left\{ v_{f\rho_1} \cdots v_{f\rho_L}, v_{i\sigma_1} \cdots v_{i\sigma_L} \left[ \overline{\Lambda}_{h_f, L+} u_{h_i} \right] \right\}$$

$$= \frac{L}{2L + 1} v_{f\rho_2} \cdots v_{f\rho_L}, v_{i\sigma_2} \cdots v_{i\sigma_L}$$

$$\left( (w_f + 1) v_{i\sigma_1} \left[ \overline{\Lambda}_{h_f, \gamma_1, L+} u_{h_i} \right] + (w_i + 1) v_{f\rho_1} \left[ \overline{\Lambda}_{h_f, \gamma_1, L+} u_{h_i} \right] \right)$$

$$+ 2 \left( \frac{L}{2L + 1} \right)^2 (w_i + 1) (w_f + 1) v_{f\rho_2} \cdots v_{f\rho_L}, v_{i\sigma_2} \cdots v_{i\sigma_L} \left[ \overline{\Lambda}_{h_f, \gamma_1, L+} u_{h_i} \right] \right\}$$

$$= \xi_{\lambda} (w_{if}) \left[ \overline{\Lambda}_{h_f, L+} u_{h_i} \right]$$

(38)

and $\Lambda'_{L+}$ denotes the positive energy projector on the intermediate spinors $\Lambda'_{L+} = \sum_s u_{\rho'}^{(s)} \overline{u}_{\rho'}^{(s)}$.

In some aspects, this expression seems simpler than the one for the meson case. However, now the SR depends not only on the quantity [4]

$$S_L(w_i, w_f, w_{if}) = T^{\rho_1 \ldots \rho_L, \sigma_1 \ldots \sigma_L} v_{f\rho_1} \cdots v_{f\rho_L}, v_{i\sigma_1} \cdots v_{i\sigma_L}$$

(39)

but also on the quantities

$$T_L^{(1)}(w_i, w_f, w_{if}) = T^{\rho_1 \ldots \rho_L, \sigma_1 \ldots \sigma_L} v_{f\rho_2} \cdots v_{f\rho_L}, v_{i\sigma_1} \cdots v_{i\sigma_L} \left[ \overline{\Lambda}_{h_f, \gamma_1, L+} u_{h_i} \right]$$

$$T_L^{(2)}(w_i, w_f, w_{if}) = T^{\rho_1 \ldots \rho_L, \sigma_1 \ldots \sigma_L} v_{f\rho_1} \cdots v_{f\rho_L}, v_{i\sigma_2} \cdots v_{i\sigma_L} \left[ \overline{\Lambda}_{h_f, \gamma_1, L+} u_{h_i} \right]$$

$$U_L(w_i, w_f, w_{if}) = T^{\rho_1 \ldots \rho_L, \sigma_1 \ldots \sigma_L} v_{f\rho_2} \cdots v_{f\rho_L}, v_{i\sigma_1} \cdots v_{i\sigma_L} \left[ \overline{\Lambda}_{h_f, \gamma_1, L+} u_{h_i} \right]$$

(40)

To compute [40] we need the tensor

$$S_L^{\rho_1, \sigma_1} = \frac{\partial^2}{\partial v_{\rho_1} \partial v_{i\sigma_1}} S_L(w_i, w_f, w_{if})$$

(41)
A calculation using the expression [4]

\[ S_L(w_i, w_f, w_{if}) = \sum_{0 \leq k \leq \frac{L}{2}} C_{L,k} \left( w_i^2 - 1 \right)^k \left( w_f^2 - 1 \right)^k (w_i w_f - w_{if})^{L-2k} \]  (42)

that reduces to (16) for \( w_i = w_f = w \), and \( C_{L,k} \) is given by (17), yields

\[ S_{\rho_1,\sigma_1}^L = \frac{1}{L^2} \sum_{0 \leq k \leq \frac{L}{2}} C_{L,k} \left( w_i^2 - 1 \right)^k \left( w_f^2 - 1 \right)^k (w_i w_f - w_{if})^{L-2k} \]

\[ \left[ (L - 2k)(w_i w_f - w_{if})^{-1} (v'^{\rho_1} v'^{\sigma_1} - g^{\rho_1 \sigma_1}) + 2k(L - 2k) \left( w_i^2 - 1 \right)^{-1} (w_i v'^{\rho_1} - v'^{\rho_1}) \right] \]

A straightforward but lengthy calculation gives

\[ T^{(1)}_L(w_i, w_f, w_{if}) = \frac{1}{L} \sum_{0 \leq k \leq \frac{L}{2}} C_{L,k} \left( w_i^2 - 1 \right)^k \left( w_f^2 - 1 \right)^k (w_i w_f - w_{if})^{L-2k} \]

\[ \{ \left[ (w_i w_f - w_{if})^{-1} (w_f + 1) (L - 2k) + (w_i + 1)^{-1} 2k \right] \left[ \pi_{h_i} \Lambda'_+ u_{h_i} \right] \]

\[ - (w_i w_f - w_{if})^{-1} (w_f + 1) (L - 2k) \left[ \pi_{h_i} u_{h_i} \right] \} \]  (44)

\[ T^{(2)}_L(w_i, w_f, w_{if}) = T^{(1)}_L(w_f, w_i, w_{if}) \]  (45)

\[ U_L(w_i, w_f, w_{if}) = \frac{1}{L^2} \sum_{0 \leq k \leq \frac{L}{2}} C_{L,k} \left( w_i^2 - 1 \right)^k \left( w_f^2 - 1 \right)^k (w_i w_f - w_{if})^{L-2k} \]

\[ \{ \left[ (L - 2k)(w_i w_f - w_{if})^{-1} (3 + 4k) + (L - 2k)(L - 2k - 1)(w_i w_f - w_{if})^{-2} \right. \]

\[ (w_i w_f + w_i + w_f - 1 - 2w_{if}) + 4k^2 (w_i + 1)^{-1} (w_f + 1)^{-1} \left[ \pi_{h_i} \Lambda'_+ u_{h_i} \right] \]

\[ - (w_i w_f - w_{if})^{-1} (L - 2k)(2L + 1) \left[ \pi_{h_i} u_{h_i} \right] \} \]  (46)

Using these expressions in the SR [33] we find that the coefficient of the bilinear \( [\pi_{h_i} u_{h_i}] \) vanishes identically, and the coefficient of \( [\pi_{h_i} \Lambda'_+ u_{h_i}] \), non-vanishing in general, gives the SR

\[ \sum_{n} \sum_{L} \tau^{(n)}_L(w_i) \tau^{(n)}_L(w_f) \sum_{0 \leq k \leq \frac{L}{2}} C_{L,k} \left( w_i^2 - 1 \right)^k \left( w_f^2 - 1 \right)^k \{ (w_i w_f - w_{if})^{L-2k} \}

10
\[- \frac{2}{2L+1} \left[ (L-2k)(w_i+1)(w_f+1)(w_i w_f - w_{if})^{L-2k-1} + 2k(w_i w_f - w_{if})^{L-2k} \right] \]
\[+ \frac{2}{(2L+1)^2} \left[ (L-2k)(3+4k)(w_i+1)(w_f+1)(w_i w_f - w_{if})^{L-2k-1} + (L-2k)(L-2k-1)(w_i+1)(w_f+1)(w_i w_f + w_i + w_f - 1 - 2w_{if}) (w_i w_f - w_{if})^{L-2k-2} + 4k^2(w_i w_f - w_{if})^{L-2k} \right] \] = \xi_{\Lambda}(w_{if}) \quad (47)

Recall that in the sum of the l.h.s. the IW functions
\[\xi_{\Lambda}^{(n)}(w) = \tau_0^{(n)}(w) \quad \xi_{\Lambda}(w) = \tau_0^{(0)}(w) \quad (48)\]
also appear, and \(C_{L,k}\) is given by (17).

As we did for mesons, for our purpose it is enough to take \(w_i = w_f = w\), that gives the simpler expression
\[\sum_{L \geq 0} \sum_{n \geq 0} \left[ \tau_L^{(n)}(w) \right]^2 \sum_{0 \leq k \leq L} C_{L,k} \left( w^2 - 1 \right)^2 \left\{ \left( w^2 - w_{if} \right)^{L-2k} \right\} \]
\[- \frac{2}{2L+1} \left[ (L-2k)(w+1)^2 \left( w^2 - w_{if} \right)^{L-2k-1} + 2k \left( w^2 - w_{if} \right)^{L-2k} \right] \]
\[+ \frac{2}{(2L+1)^2} \left[ (L-2k)(3+4k)(w+1)^2 \left( w^2 - w_{if} \right)^{L-2k-1} + (L-2k)(L-2k-1)(w+1)^2 \left( w^2 + 2w - 1 - 2w_{if} \right)^{L-2k-2} + 4k^2 \left( w^2 - w_{if} \right)^{L-2k} \right] \] = \xi_{\Lambda}(w_{if}) \quad (49)

### 3.2 Axial sum rule.

Following the analogy with the calculation in the meson case, if we take, instead of the vector currents \((36)\), the axial currents aligned also along the intermediate four-velocity \(v'\)
\[\Gamma_i = \Gamma_i = \not p' \gamma_5 \quad \Gamma_f = \Gamma_f = \not p' \gamma_5 \quad (50)\]
we obtain, instead of \((49)\), the expression
\[\sum_{L \geq 0} \sum_{n \geq 0} \left[ \tau_L^{(n)}(w) \right]^2 \sum_{0 \leq k \leq L} C_{L,k} \left( w^2 - 1 \right)^2 \left\{ \left( w^2 - w_{if} \right)^{L-2k} \right\} \]
\[- \frac{2}{2L+1} \left[ (L-2k)(w+1)^2 \left( w^2 - w_{if} \right)^{L-2k-1} + 2k \left( w^2 - w_{if} \right)^{L-2k} \right] \]
\[+ \frac{2}{(2L+1)^2} \left[ (L-2k)(L+2+2k)(w-1)^2 \left( w^2 - w_{if} \right)^{L-2k-1} \right] \]
\[-(L - 2k)(L - 2k - 1)(w - 1)^2 (2w + w_{if} + 1) \left( w^2 - w_{if} \right)^{L-2k-2} \\
+ 4k^2 \left( w^2 - w_{if} \right)^{L-2k} \right\} = \xi_\Lambda (w_{if}) \quad (51)\]

4 Bounds on the derivatives of the Isgur-Wise function.

We will now exploit the Vector SR (49), by computing its derivatives and going to the frontier of the domain (13) \( w \to 1, w_{if} \to 1 \)

\[ \left( \frac{d^{p+q}}{dw_{if}^p dw_{if}^q} \right)_{w_{if}=w=1} \quad (52) \]

For arbitrary \( p \) and \( q = 0 \), one finds

\[ \xi_\Lambda^{(p)} (1) = (-1)^p p! \sum_{n\geq 0} \left[ \tau_\Lambda^{(n)} (1) \right]^2 \quad (53) \]

We recover therefore the result of Isgur et al. [8] for the slope (6) and generalize it for any derivative, giving

\[ (-1)^p \xi_\Lambda^{(p)} (1) \geq 0 \quad (54) \]

that demonstrates that the IW function \( \xi_\Lambda (w) \) is an alternate series in powers of \((w - 1)\).

We find the same result (54) from the Axial SR using (50)-(51).

5 Improved bound on the curvature.

5.1 Vector sum rule.

Moreover, an improved bound can be found on the curvature, similar to the one found in the meson case [3].

To obtain it, let us consider the Vector SR (49) and (52) for different values for \( p, q \) satisfying \( p + q \leq 2 \).

For \( p = q = 0 \) we obtain \([\xi_\Lambda (1)]^2 = \xi_\Lambda (1)\), i.e. \( 1 = 1 \). For both cases \( p = 1, q = 0 \) or \( p = 0, q = 1 \) we obtain

\[ \xi_\Lambda' (1) = -\sum_{n\geq 0} \left[ \tau_1^{(n)} (1) \right]^2 \quad (55) \]
i.e. eq. (6). For $p = 2, q = 0$ we get equation (53) for $p = 2,$

$$\xi''_\Lambda(1) = 2 \sum_{n \geq 0} \left[ \tau^{(n)}_2(1) \right]^2$$

(56)

while for $p = 1, q = 1$ one gets

$$2 \sum_{n \geq 0} \left[ \tau^{(n)}_2(1) \right]^2 + \sum_{n \geq 0} \tau^{(n)}_1(1) \tau''^{(n)}_1(1) = 0$$

(57)

and finally for $p = 0, q = 2,$

$$\sum_{n \geq 0} \left[ \tau^{(n)}_1(1) \right]^2 + \frac{8}{3} \sum_{n \geq 0} \left[ \tau^{(n)}_2(1) \right]^2 + \sum_{n \geq 0} \left[ \xi''^{(n)}(1) \right]^2$$

$$+ 4 \sum_{n \geq 0} \tau^{(n)}_1(1) \tau''^{(n)}_1(1) + \xi''_\Lambda(1) = 1$$

(58)

where we have used the notation (48). Eliminating the unknown $\sum_{n \geq 0} \tau^{(n)}_1(1)\tau''^{(n)}_1(1)$ between eqs. (57) and (58), and using (55) we obtain finally for the curvature,

$$\sigma^2_\Lambda = \xi''(1) = \frac{3}{5} \left\{ \rho^2_\Lambda + (\rho^2_\Lambda)^2 + \sum_{n \neq 0} \left[ \xi''(1) \right]^2 \right\}$$

(59)

that implies the improved bound

$$\sigma^2_\Lambda = \xi''_\Lambda(1) \geq \frac{3}{5} \left[ \rho^2_\Lambda + (\rho^2_\Lambda)^2 \right]$$

(60)

5.2 Axial sum rule.

Let us now consider the Axial SR (51) and use (52) for different values for $p, q$ satisfying $p + q \leq 3.$

For $p = q = 0$ we obtain a trivial result. For $(p = 1, q = 0), (p = 0, q = 1),$ $(p = 2, q = 0), (p = 1, q = 1)$ and $(p = 0, q = 2)$ we get respectively the same equations (53)-(58) as for the Vector SR.

Notice that in the meson case we got different SR for the vector and the axial currents. This corresponds to the fact underlined above that, in the meson case, in the SR we have contributions that for a given $L,$ the light cloud has two possible values $j^P = (L \pm \frac{1}{2})^P,$ $P = (-1)^L.$

For the baryon transition $\Lambda_b \rightarrow \Lambda_c,$ since the two spectator quarks have total spin and isospin $S = I = 0,$ for a given $L$ we have a single type of intermediate
states, with \(j^P = L^P\). This explains why we obtain less information in the baryon case than in the meson case for the elastic IW function.

Although not providing new information, the consideration of the present case with the axial current remains interesting as a check of the results found for the vector current.

6 Prospects and comparison with previous work.

Some work should be pursued on this subject. On the one hand, within our approach, bounds on higher derivatives may be obtained and could be useful.

On the other hand, one should include the radiative corrections to the slope and curvature of \(\xi_\Lambda(w)\) and \(1/m_Q\) corrections within Heavy Quark Effective Theory, as well as the Wilson coefficients that make the matching with the physical form factors. This program was performed in the meson case by Dorsten [24], and should be carried out in the baryon case.

These improvements should be accomplished in order to make a realistic comparison with experiment and with models for the different \(\Lambda_c \to \Lambda_c \ell \nu_\ell\) form factors at finite mass.

However, within rigorous QCD methods, there is one theoretical scheme that can be directly confronted with our bound, namely the IW function computed in the heavy quark and large \(N_c\) limits [10]. Notice that our bounds hold in the heavy quark limit at the actual value \(N_c = 3\).

The result at large \(N_c\) [10] is a simple exponential form

\[
\xi_\Lambda(w) = exp \left[ -\rho_\Lambda^2 (w - 1) \right]
\]  
with

\[
\rho_\Lambda^2 = \lambda N^{3/2} \quad \lambda = O(1)
\]  
(62)

The bound for the curvature (60) implies for the slope of the exponential form (61)

\[
\rho_\Lambda^2 \geq 3/2
\]  
(63)
that, from [62], is trivially satisfied in the large $N_c$ limit. However, the phenomenological guess (3.8) from [10], $\rho_{\Lambda}^2 = 1.3$, slightly violates the bound. But we should keep in mind that a guess obtained from the large $N_c$ limit not necessarily should satisfy a constraint obtained in the physical case $N_c = 3$.

The other work on baryon form factors $\Lambda_b \rightarrow \Lambda_c \ell \nu_\ell$ that is based on QCD is the dispersive approach of Boyd et al. [15, 16], that uses, at finite mass, crossing symmetry, dispersion relations and pertubative QCD evaluated far from the physical resonances. This method, that has received a number of improvements in the meson case, gives in principle a model independent description of the various form factors in terms of a finite number of parameters. In the most favorable case, it allows for baryons to describe one of the helicity amplitudes in terms of two unknown constants [16]. Notice that, unlike appealing to the crossed channel $\Lambda_b \overline{\Lambda}_c$, our results hold directly in the semileptonic region.

Since the dispersive approach of Boyd et al. is formulated directly at finite mass, the comparison with the present work would require to carefully obtain the relation between the IW function $\xi_{\Lambda}(w)$ and the physical form factors along the lines exposed at the beginning of the present Section. On the other hand, ref. [16] does not give, unlike in the meson case (cf. the relation between the curvature and the slope, formula (7.7)), simple formulae for the baryon form factors, and one would need to carefully extract the results for baryons from the general formalism. Because of these two reasons, we postpone to future work the comparison of the results of the present paper with the ones of [16].

7 Conclusion.

In conclusion, from the OPE sum rules obtained in the heavy quark limit of QCD using the non-forward amplitude, we get an expansion of the elastic Isgur-Wise function $\xi_{\Lambda}(w)$ for the process $\Lambda_b \rightarrow \Lambda_c \ell \nu_\ell$, up to order $(w - 1)^2$,

$$\xi_{\Lambda}(w) = 1 - \rho_{\Lambda}^2 (w - 1) + \frac{\sigma_{\Lambda}^2}{2} (w - 1)^2 + O[(w - 1)^3]$$

with the constraints

$$\rho_{\Lambda}^2 \geq 0 \quad \sigma_{\Lambda}^2 \geq \frac{3}{5} \left[ \rho_{\Lambda}^2 + \left( \rho_{\Lambda}^2 \right)^2 \right]$$

(64)

(65)
While the first inequality (65) for the slope was known [8], the second one is new, the main result of this paper. On the other hand, we have demonstrated that $\xi_{\Lambda}(w)$ is an alternate series in powers of $(w - 1)$.

Notice the important point that if the slope is not small, of $O(1)$, owing to the available phase space the curvature will have a measurable effect, and will be important in the extrapolation of the differential rate at $w \to 1$.

On the experimental side, hopefully the LHC-b program would provide information on the shape of this function, and help to further constrain the CKM matrix element $V_{cb}$. Let us recall that there is a small tension between the exclusive $B \to D^{(*)} \ell \nu_{\ell}$ and the inclusive $B \to X_c \ell \nu_{\ell}$ determinations of $V_{cb}$, although the error of the former determination is rather large.

On the theoretical side, one should include the radiative corrections to the slope and curvature of $\xi_{\Lambda}(w)$ and $1/m_Q$ corrections, as well as the Wilson coefficients that make the matching with the physical form factors. This would allow to compare with future data and with other theoretical or phenomenological schemes of baryon form factors at finite mass. Also, once these necessary improvements are realized, any future fit to the differential distribution of the process $\Lambda_b \to \Lambda_c \ell \nu_{\ell}$ using the Isgur-Wise function $\xi_{\Lambda}(w)$ should take into account our constraints.

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