Rate Region of Scheduling a Wireless Network with Discrete Propagation Delays

Jun Ma, Yanxiao Liu and Shenghao Yang

Abstract

We study wireless networks where signal propagation delays are multiples of a time interval. Such a network can be modelled as a weighted hypergraph. The link scheduling problem of such a wireless network is closely related to the independent sets of the periodic hypergraph induced by the weighted hypergraph. As the periodic graph has infinitely many vertices, existing characterizations of graph independent sets cannot be applied to study link scheduling efficiently. To characterize the rate region of link scheduling, a directed graph of finite size called scheduling graph is derived to capture a certain conditional independence property of link scheduling over time. A collision-free schedule is equivalent to a path in the scheduling graph, and hence the rate region is equivalent to the convex hull of the rate vectors associated with the cycles of the scheduling graph. With the maximum independent set problem as a special case, calculating the whole rate region is NP hard and also hard to approximate. We derive two algorithms that benefit from a partial order on the paths in the scheduling graph, and can potentially find schedules that are not dominated by the existing cycle enumerating algorithms running in a given time. The first algorithm calculates the rate region incrementally in the cycle lengths so that a subset of the rate region corresponding to short cycles can be obtained efficiently. The second algorithm enumerates cycles associated with a maximal subgraph of the scheduling graph. In addition to scheduling a wireless network, the independent sets of periodic hypergraphs also find applications in some operational research problems.

I. INTRODUCTION

Wireless communication media, e.g., radio, light and sound, all have nonzero signal propagation delays between two communication devices with a nonzero distance. In the traditional wireless network scheduling problem that focused on radio based communications [1]–[3], the propagation delay is ignored as the time frame for transmitting signals is much longer than the signal propagation delay between devices. For example, most modern terrestrial wireless communications are done within couple of kilometers, and hence the propagation delay of radio wave is within tens of microseconds, while the radio frame length is typically a couple of milliseconds. We refer to the scheduling with a long signal frame length as framed scheduling in this paper.

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J. Ma, Y. Liu and S. Yang are with the School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen, Shenzhen, China. S. Yang is also with the Shenzhen Research Institute of Big Data, Shenzhen, China. Emails: junma@link.cuhk.edu.cn, yanxiaoliu@link.cuhk.edu.cn, shyang@cuhk.edu.cn

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In outer space and underwater acoustic communications, the propagation delay can be longer than seconds. For example, the sound speed in underwater acoustic is about 1500 meters per second, and hence the delay of sound for propagating 3 kilometers is about 2 seconds. Guided by the traditional framed scheduling approach, the frame length should be tens or hundreds of seconds, as used in some existing underwater acoustic communications [4]. Researchers have studied network scheduling taking propagation delay into consideration [5]–[10], and have seen significant performance advantages in terms of energy consumption and throughput by allowing much smaller frame length, comparable with the propagation delay between communication devices.

In contrast to the extensive researches on the traditional wireless network scheduling problem, the scheduling problem with propagation delay still lacks a systematic, theoretical framework. Early works [5], [6] tried to extend the traditional approaches to the problem with propagation delays. The collision constraints between network nodes in traditional framed scheduling can be described as a finite collision graph. With propagation delays, collisions can occur at different time and hence in general cannot be described by finite constraints. These works [5], [6] discussed special cases of the problem that can be modelled with finite constraints. In a network model where any two links can generate collisions to each other, Chitre, Montani and Shahabudeen [7] showed that periodic scheduling can achieve the maximum total rate and discussed a dynamic programming approach for solving the scheduling optimization problem. Based on their approach, line networks were further investigated in [9].

In this paper, we consider the link scheduling problem with propagation delays under a more general network model formed by three components: a link set, a collision profile, and a delay matrix. The link set is a finite set where elements are called links. The collision profile defines the collision relations among links. A link may have collision with certain subsets of the link set. A collision profile is said to be binary if a link only has collisions with individual links. At last, the delay matrix is an integer valued matrix that tells the propagation delay between two links. This delay matrix can be used to model the propagation delay that are multiples of a certain time interval. Graphically, this network model is a weighted hypergraph, which becomes a graph when the collision profile is binary.

The network link scheduling problem is closely related to the independent sets of the periodic hypergraph induced by the network model. As the periodic hypergraph is infinite and has a periodic structure, general algorithms about independent sets cannot solve the link scheduling problem efficiently and completely. We introduce a graphical approach to study the set of achievable scheduling rate vectors, called the scheduling rate region, together with the link schedules. Our results provide a fundamental theoretical guidance for network scheduling research, and may motivate many further researches. For example, a scheduling rate region can be applied in the utility maximization framework for wireless network protocol design [11], and can also be used to evaluate the performance of decentralized link scheduling algorithms. Moreover, our results are also of independent interest as a characterization of the independent sets of periodic hypergraphs, which may find applications in certain operational research problems [12], [13].

In particular, we observe that the collision constraints in the periodic hypergraph has the property that if two segments of the periodic hypergraph have a big enough distance over time, the schedules of these two segments do not affect each other. To capture this property, we derive a sequence of directed graphs of a finite size, called the
scheduling graphs, from the network model. We show that a collision-free schedule can be equivalent to a path in certain scheduling graphs of a sufficiently large size, and the scheduling rate region is equivalent to the convex hull of the rate vectors associated with the cycles of the scheduling graphs. For networks with a binary collision profile, a smaller scheduling graph can be used. For framed scheduling, the rate region is determined by the length-1 cycles of a scheduling graph.

Enumerating the cycles in a scheduling graph becomes the essential problem for calculating the rate region and finding optimal schedules. For a general directed graph of \( n \) vertices and \( m \) edges, all the cycles can be found in \( O((\eta + 1)(m + n)) \) times, where \( \eta \) is the number of cycles [14]. The problem is essentially hard as the size of a scheduling graph is exponential in terms of the number of links in the network. So we study how to find a larger subset of the rate region in a given time. Due to the special structure of scheduling graphs, we introduce a partial order on paths of a scheduling graph of the same length. We show that it is sufficient to enumerate all the maximal cycles of the scheduling graphs for calculating the rate region. Benefiting from the partial order property, two algorithms are derived and can potentially find schedules that are not dominated by the existing cycle enumerating algorithm when running in a given time. The first algorithm enumerates the maximal cycles incrementally in cycle lengths, so that we can calculate a subset of the rate region up to a certain cycle length more efficiently. This algorithm shares some properties of the cycle enumerating algorithm in [16], where a general graph is considered. Our second algorithm enumerates cycles associated with a maximal subgraph of the scheduling graph, and can find some cycles of longer length than that of the cycles found by the first algorithm in a given time.

The remainder of the paper is organized as follows. The network model and the basic properties of scheduling rate region are introduced in Section II. The traditional framed scheduling is a special case of our scheduling problem and is discussed in Section III. Scheduling graphs and rate region characterizations are discussed in Section IV. Partial order properties and rate region algorithms are studied in Section V. The isomorphism and connectivity properties of periodic graphs are reviewed in Appendix A.

II. NETWORK MODEL AND SCHEDULING RATE REGION

In this section, we introduce a network model with integer propagation delays, called a discrete network model, and discuss the scheduling problem of the network. Denote \( \mathbb{Z} \) as the set of integers and \( \mathbb{Z}^+ \) as the set of nonnegative integers. For two matrices \( A \) and \( B \) of the same size, we write \( A \preceq B \) if all the entries of \( A \) are not larger than the corresponding entries of \( B \) at the same positions. We similarly define \( A \succeq B \). For a matrix \( A \) and a scalar \( a \), we write \( A + a \) as the matrix obtained by adding each entry of \( A \) by \( a \). We similarly define \( A - a \).

A. Discrete Network Model

Suppose time is slotted and each timeslot is indexed by an integer \( t \in \mathbb{Z} \). Consider a network of \( N \) nodes indexed by \( 1, 2, \ldots, N \). Each node can transmit and receive a certain communication signal in a timeslot. From node \( i \) to node \( j \), the signal propagation delay is denoted by \( D(i, j) \in \mathbb{Z}^+ \). If a signal is transmitted by node \( i \) in timeslot

\[T^{\text{max}}(i, j)\]

\[T^{\text{max}}(i, j) = T^{\text{col}}(i, j) \text{ or } T^{\text{fl}}(i, j)
\]

1The maximum independent sets problem, i.e., enumerating length-1 cycles a scheduling graph, is NP hard and also hard to approximate [15].
The delay matrix \( D \) or binary model \([2], [17]\). In this case, we also write with propagation delay, the single collision domain model studied in \([7]\) has

To model half-duplex communications, the collision set \( I \) scenario that links \( l \) and \( l' \) with \( s_l = s_{l'} \) cannot be active in the same timeslot, we set \( \{l'\} \in \mathcal{I}(l) \) and \( \{l\} \in \mathcal{I}(l') \). To model half-duplex communications, the collision set \( \mathcal{I}(l) \) should include all subsets \( \{l'\} \) such that \( s_{l'} = r_l \).

When the collision set \( \mathcal{I}(l) \) has the property that for any \( \phi \in \mathcal{I}(l) \), \( |\phi| = 1 \), the collision set is called the protocol or binary model \([2], [17]\). In this case, we also write \( \mathcal{I}(l) \) as a subset of \( \mathcal{L} \) to simplify the notation. For scheduling with propagation delay, the single collision domain model studied in \([7]\) has \( \mathcal{I}(l) = \mathcal{L} \setminus \{l\} \), which is a special case of the binary model. As we will show later, some of our results specified for the binary collision model can be further refined.

**Example 1** (Multihop line network). Consider a multihop line network model with two integer parameters \( L \geq 1 \) and \( K \geq 1 \). The network has \( L + 1 \) nodes and the link set

\[
\mathcal{L} = \{l_i \triangleq (i, i + 1), i = 1, \ldots, L\}.
\]

The delay matrix \( D \) has \( D(i, j) = |i - j| \) for \( 1 \leq i, j \leq L + 1 \). We consider a binary collision model called the \( K \)-hop model where the reception of a node can only have collisions from nodes within \( K \) hops distance. For \( i = 1, \ldots, L \), the collision set of link \( l_i \) is

\[
\mathcal{I}(l_i) = \{l_j : j \neq i, |j - i - 1| \leq K\}.
\]  (1)
When \( K \geq 1 \), we have for \( i = 1, \ldots, L - 1 \), \( l_{i+1} \in \mathcal{I}(l_i) \), which means that node \( i + 1 \) is half-duplex, i.e., it cannot transmit and receive signals in the same timeslot. When \( L = 4 \) and \( K = 1 \), we have

\[
\begin{align*}
\mathcal{I}(l_1) &= \{l_2, l_3\} \\
\mathcal{I}(l_2) &= \{l_3, l_4\} \\
\mathcal{I}(l_3) &= \{l_4\} \\
\mathcal{I}(l_4) &= \emptyset.
\end{align*}
\]

When \( L = 4 \) and \( K = 2 \), we have

\[
\begin{align*}
\mathcal{I}(l_1) &= \{l_2, l_3, l_4\} \\
\mathcal{I}(l_2) &= \{l_1, l_3, l_4\} \\
\mathcal{I}(l_3) &= \{l_2, l_4\} \\
\mathcal{I}(l_4) &= \{l_3\}.
\end{align*}
\]

### B. Link-wise Network Model and Link Schedule

Now we describe a link-based network model to be used mainly in this paper. Define an \(|\mathcal{L}| \times |\mathcal{L}|\) matrix \( D_{\mathcal{L}} \) with \( D_{\mathcal{L}}(l, l') = D(s_{l'}, r_i) - D(s_{l'}, r_i) \), called a link-wise delay matrix. The definition of the link-wise delay matrix does not depend on the collision sets. Entries of \( D_{\mathcal{L}} \) can be negative. It is sufficient for us to check collision using \( D_{\mathcal{L}} \): link \( l \) active in a timeslot \( t \) has a collision if for a certain \( \phi \in \mathcal{I}(l) \), each link \( l' \in \phi \) is active in the timeslot \( t + D_{\mathcal{L}}(l, l') \). As we see, if \( l' \notin \bigcup_{\phi \in \mathcal{I}(l)} \phi \), then \( D_{\mathcal{L}}(l, l') \) is not involved in collision checking and hence the value of \( D_{\mathcal{L}}(l, l') \) is not necessary to be specified. For this case, we may mark \( D_{\mathcal{L}}(l, l') \) as \( * \) in a link-wise delay matrix.

When the context is clear, we also call \( D_{\mathcal{L}} \) the delay matrix.

Let \( \mathcal{I} = (\mathcal{I}(l), l \in \mathcal{L}) \) be the collision profile of the network. When all the collision sets \( \mathcal{I}(l) \) are binary, the collision profile is said to be binary. Our (link-based) network model, denoted by \( \mathcal{N} = (\mathcal{L}, \mathcal{I}, D_{\mathcal{L}}) \), is specified by the link set \( \mathcal{L} \), the collision profile \( \mathcal{I} \), and the link-wise delay matrix \( D_{\mathcal{L}} \). The network \( \mathcal{N} \) can be regarded as a weighted directed (hyper)graph with the vertex set \( \mathcal{L} \): When all the collision profile is binary, \( \mathcal{N} \) is a directed graph where \((l, l')\) is a directed edge if and only if \( l' \in \mathcal{I}(l) \) and the weight of \((l, l')\) is \( D_{\mathcal{L}}(l, l') \). When the collision profile is non-binary, \( \mathcal{N} \) is a hypergraph where \((l, \phi)\) is a directed edge if and only if \( \phi \in \mathcal{I}(l) \) and for each \( l' \in \phi \), the weight of \((l, l')\) is \( D_{\mathcal{L}}(l, l') \).

**Example 2.** Following Example 1 the link-wise delay matrix \( D_{\mathcal{L}} \) of the \( L \)-length, \( K \)-hop collision line network has

\[
D_{\mathcal{L}}(l_i, l_j) = D(i, i + 1) - D(j, i + 1) = 1 - |j - i - 1|.
\]

(2)

The network is denoted by \( \mathcal{N}_{L,K}^{\text{line}} = (\mathcal{L}, \mathcal{I}, D_{\mathcal{L}}) \), where \( \mathcal{L} = \{l_1, \ldots, l_L\} \), \( \mathcal{I} \) is defined in 1 and \( D_{\mathcal{L}} \) is defined in 2. The graphical representation of \( \mathcal{N}_{4,1}^{\text{line}} \) and \( \mathcal{N}_{4,2}^{\text{line}} \) are given in Fig. 1 and Fig. 2 respectively.
Fig. 1. The graphical representation of $N_{4,1}^{line}$. In this graph, as well as the following graphical representation of our discrete network models, the vertices in the graph represent links in the network.

One fundamental question related to a discrete network $N = (\mathcal{L}, \mathcal{I}, D_L)$ is the efficiency of link activation scheduling. A (link) schedule $S$ is a matrix of binary digits indexed by pairs $(l, t) \in \mathcal{L} \times \mathbb{Z}$, where $S(l, t) = 1$ indicates that $l$ is active in timeslot $t$, and inactive when $S(l, t) = 0$. Though defined for $t < 0$, in practice, we are interested only $S(l, t)$ with $t \geq 0$.

**Definition 1.** For a schedule $S$ and $(l, t) \in \mathcal{L} \times \mathbb{Z}$, we say $S(l, t)$ has a collision in network $N$ if for certain $\phi \in \mathcal{I}(l)$, $S(l', t + D_L(l, l')) = 1$ for every $l' \in \phi$. Otherwise, we say $S(l, t)$ is collision free, i.e., if for all $\phi \in \mathcal{I}(l)$, $S(l', t + D_L(l, l')) = 0$ for certain $l' \in \phi$. A schedule $S$ is said to be collision free if $S(l, t)$ is collision free for all $(l, t) \in \mathcal{L} \times \mathbb{Z}$ with $S(l, t) = 1$.

The (directed) periodic hypergraph induced by $N = (\mathcal{L}, \mathcal{I}, D_L)$, denoted by $N^\infty$, has the vertex set $\mathcal{L} \times \mathbb{Z}$, where there is a hyper-edge from $(l, t)$ to a subset $\{(l_i, t_i), i = 1, \ldots, k\}$ of $k$ vertices if and only if $\{l_i, i = 1, \ldots, k\} \in \mathcal{I}(l)$ and $t_i = t + D_L(l, l_i)$ for $i = 1, \ldots, k$. For a directed hypergraph with the vertex set $\mathcal{V}$ and edge set $\mathcal{E} \subset \mathcal{V} \times 2^{\mathcal{V}}$, a subset $\mathcal{A}$ of $\mathcal{V}$ is said to be independent if for any $(v, \mathcal{U}) \in \mathcal{E}$, $\{v\} \cup \mathcal{U} \not\subseteq \mathcal{A}$. An independent set of $N^\infty$ can be denoted by a binary $|\mathcal{L}|$-row matrix $A$ with columns indexed by integers. By this representation, a collision-free schedule of $N$ is equivalent to an independent set of $N^\infty$.

When all the collision sets are binary, the above definitions can be simplified as $N^\infty$ becomes to a graph with edges from $(l, t)$ to $(l', t + D(l, l'))$ for all $l' \in \mathcal{I}(l)$. Some properties of periodic graphs, including isomorphism and connectivity have been studied in literature [18], but the independent set problem has not been well understood. In the literature of periodic graphs, our network $(\mathcal{L}, \mathcal{I}, D_L)$ is also called a static graph.

In Section IV and V, we will introduce our technique for the scheduling problem, which can also help to solve the independent set problem in periodic (hyper)graphs. In Appendix A we will discuss some general properties of
periodic (hyper)graphs in literature that can help with our scheduling problem.

C. Scheduling Rate Region

For a network \( N = (\mathcal{L}, \mathcal{I}, D_L) \), denote for each schedule \( S \) and link \( l \)

\[
R^N_S(l) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \eta(S(l, t) = 1 \text{ and is collision free}),
\]

where \( \eta(A_1, A_2, \ldots) \) is the indicator function with value 1 if the sequence of conditions \( A_i \) are all true and 0 otherwise. To be consistent with the practice of network scheduling, we only use \( S(l, t) \) with \( t \geq 0 \) to define \( R^N_S(l) \). If the limit on the RHS of (3) exists, we say \( R^N_S(l) \) exists. When \( R^N_S(l) \) exists, we call \( R^N_S(l) \) the (scheduling) rate of link \( l \).

If \( R^N_S(l) \) exists for all \( l \in \mathcal{L} \), we call \( R^N_S = (R^N_S(l), l \in \mathcal{L}) \) the rate vector of \( S \) for \( N \). We may omit the superscript in \( R^N \) and \( R^N_S(l) \) when the network \( N \) is implied.

**Definition 2.** For a network \( N = (\mathcal{L}, \mathcal{I}, D_L) \), a rate vector \( R = (R(l), l \in \mathcal{L}) \) is said to be achievable if for any \( \epsilon > 0 \), there exists a schedule \( S \) such that \( R_S \succ R - \epsilon \). The collection \( \mathcal{R}^N \) of all the achievable rate vectors is called the rate region of \( N \).

Define the character of the network \( N \) as

\[
D^*_N = \max_{l \in \mathcal{L}} \max_{c \in \mathcal{I}(l)} \max_{l' \in c} |D_L(l, l')|.
\]

(4)

When \( N \) is known from the context, we also write \( D^*_N \) as \( D^* \). Note that \( D^* \leq \max_{1 \leq i, j \leq N} D(i, j) \), the maximum propagation delay in the network. When the network has a binary collision model, the formula of \( D^* \) can be simplified as

\[
D^*_N = \max_{l \in \mathcal{L}} \max_{l' \in \mathcal{I}(l)} |D_L(l, l')|,
\]

**Example 3.** For \( N^\text{line}_{L,K} \) defined in Example 2 by (1) and (2),

\[
D^* = \max_{1 \leq i \neq j \leq L, |j - i| \leq K} |1 - |j - i - 1||
= \max\{\min\{L, K\} - 1, 1\}.
\]

So, when \( K = 1 \), \( D^* = 1 \), and when \( L \geq K \geq 2 \), \( D^* = K - 1 \).

A schedule \( S \) has a period \( T_p \) if \( S(l, t) = S(l, t + T_p) \) for any \((l, t) \in \mathcal{L} \times \mathbb{Z}\). Similar to Definition 2 we say a rate vector \( R \) is achievable by collision-free, periodic schedules if for any \( \epsilon > 0 \), there exists a collision-free, periodic schedule \( S \) such that \( R_S \succ R - \epsilon \). It is sufficient to consider collision-free, periodic schedules only when studying the rate region \( \mathcal{R}^N \), which is justified by the next theorem proved in Appendix B.

**Theorem 1.** For a network \( N \), the rate region \( \mathcal{R}^N \) can be achieved using only collision-free, periodic schedules.

The following lemma is proved in Appendix B.
Lemma 2. The rate region $\mathcal{R}^N$ of a network $N$ is convex.

Define $N^T$ as the subgraph of $N^\infty$ induced by the vertex set $L \times \{0, 1, \ldots, T - 1\}$. An independent set of $N^T$ can be denoted by a binary $|L| \times T$ matrix. Then the all zero $|L| \times T$ matrix denotes an independent set of $N^T$. The rate vector of a independent set of $N^T$ is the vector formed by the sum of all the columns of the matrix presentation of the independent set, normalized by $T$. Define $\tilde{\mathcal{R}}^{NT}$ as the convex hull of the rate vectors of all the independent sets of $N^T$.

Theorem 3. For a discrete network $N$, $\mathcal{R}^N$ is equal to the closure of $\cup_{T=1,2,\ldots,T+D^*} \tilde{\mathcal{R}}^{NT}$.

Proved in Appendix B, Theorem 3 characterizes the rate region $\mathcal{R}^N$ using the independent sets of the subgraphs of the periodic (hyper)graph $N^\infty$. This characterization involves the union of infinitely many sets and hence is not explicit.

Example 4. For $N_{1,1}^{\text{line}}$ defined in Example 2 let $a_T$ be the maximum value such that $[a_T, \ldots, a_T] \in \frac{T}{T+D^*} \tilde{\mathcal{R}}^{NT}$. We can calculated that $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{3}{10}$, $a_4 = \frac{3}{19}$, $a_5 = \frac{3}{12}$, $\ldots$. We know that $\lim_{T \to \infty} a_T = \frac{1}{2}$.

III. FRAMED SCHEDULING

We discuss a special scheduling scheme called framed scheduling, which is motivated by the network scheduling schemes extensively used in the existing wireless networks. In a framed schedule, a link is active for a long consecutive sequence of timeslots, which is usually much longer than $D^*$.

Fix a network $(L, I, D_L)$. Let $T_F \geq D^* + 1$ be an integer called frame length. For framed scheduling, the timeslots $t \geq 0$ are separated into groups (also called frames) each of $T_F$ consecutive timeslots, i.e., the frame $k$ $(k = 0, 1, \ldots)$ includes the timeslots $kT_F, kT_F + 1, \ldots, (k + 1)T_F - 1$. A framed schedule $S$ of frame length $T_F$ satisfies the following properties:

- For each frame, all the links are inactive for the last $D^*$ timeslots. In other words, $S(l, kT_F + i) = 0$ for $i = T_F - D^*, \ldots, T_F - 1$ and $k = 0, 1, \ldots$.
- For each frame, if a link is active in one timeslot, it is active in all the first $T_F - D^*$ timeslots. In other words, for all $i = 0, 1, \ldots, T_F - D^* - 1$, either $S(l, kT_F + i) = 1$ or $S(l, kT_F + i) = 0$.

The rate region achieved by framed scheduling can be characterized using independent sets of directed graph $(L, I)$ [2]. Recall $\tilde{\mathcal{R}}^{(L, I)}$ is the convex hull of the rate vectors of all the independent sets of $(L, I)$.

A. Binary Collision

We first discuss networks with a binary collision profile, i.e., $I(l)$ is a subset of $L$.

Lemma 4. Consider a network $(L, I, D_L)$ with a binary collision profile. A framed schedule $S$ of frame length $T_F \geq 2D^* + 1$ is collision free if and only if for any link $l$ that is active in a frame, all links $l' \in I(l)$ are inactive in the same frame.
Proved in Appendix C, Lemma 4 says that for a collision-free, framed schedule $S$ of frame length $T_F \geq 2D^* + 1$, the set of all the active links in each frame is an independent set of $(\mathcal{L}, \mathcal{I})$. The next theorem is proved in Appendix C.

**Theorem 5.** Consider a network $(\mathcal{L}, \mathcal{I}, D_L)$ with a binary collision profile. For the rate vector of a collision-free, framed schedule of frame length $T_F \geq 2D^* + 1$, if it exists, it is in $(1 - D^*/T_F)\bar{R}^{(\mathcal{L}, \mathcal{I})}$ and hence in $\bar{R}^{(\mathcal{L}, \mathcal{I})}$. Moreover, any rate vector in $\bar{R}^{(\mathcal{L}, \mathcal{I})}$ can be achieved by collision-free, framed schedules.

**Example 5** (Multihop line network). We give an example of $N_{\text{line}}^{4,1}$ with 4 links. According to Lemma 4, the set of all the active links in each frame is an independent set of $(\mathcal{L}, \mathcal{I})$. Here we list the rate vectors corresponding to maximal independent sets:

$$r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, r_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, r_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$  \hspace{1cm} (5)

**B. General Collision**

Now we consider networks with a general collision profile. The lemma below is similar to Lemma 4 except that the minimum requirement of $T_F$ is larger.

**Lemma 4.** A framed schedule $S$ of frame length $T_F \geq 3D^* + 1$ is collision free if and only if for any link $l$ that is active in a frame, for all $\phi \in \mathcal{I}(l)$, certain $l' \not\in \phi$ is inactive in the same frame.

Lemma 4 is proved in Appendix C. Using the same argument as proving Theorem 5 except for applying Lemma 4 in place of Lemma 4, we obtain the next theorem.

**Theorem 5.** Consider a network $(\mathcal{L}, \mathcal{I}, D_L)$. For the rate vector of a collision-free, framed schedule of frame length $T_F \geq 3D^* + 1$, if it exists, it is in $(1 - D^*/T_F)\bar{R}^{(\mathcal{L}, \mathcal{I})}$ and hence in $\bar{R}^{(\mathcal{L}, \mathcal{I})}$. Moreover, any rate vector in $\bar{R}^{(\mathcal{L}, \mathcal{I})}$ can be achieved by collision-free, framed schedules.

For special cases such as $D(i, j) = 0$ for all $i, j$, $\bar{R}^{(\mathcal{L}, \mathcal{I})} = \mathcal{R}^N$. But in general, $\bar{R}^{(\mathcal{L}, \mathcal{I})} \subset \mathcal{R}^N$. Theorem 1 characterizes the rate region $\mathcal{R}^N$ in terms of the collision-free schedules, which are equivalent to the independent sets of the periodic graph $N^{\infty}$ induced by $\mathcal{N}$. However, the number of independent sets of $N^{\infty}$ is infinite, and hence this characterization is not explicit. In the following sections, we will provide an explicit characterization of the rate region $\mathcal{R}^N$, and derive an algorithm to find collision-free schedules and calculate (a subset of) the rate region.

**IV. Schedule and Rate Region Characterization**

In this section, we characterize the collision-free schedules and hence rate region $\mathcal{R}^N$ of a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D_L)$ using a graphical approach. Recall that the network scheduling problem here is equivalent to the independent set problem of a periodic (hyper)graph induced by $\mathcal{N}$. So the approach here can also help to understand
the independent set of periodic graphs which find applications in some operational research problems. Omitted proofs in this section can be found in Appendix D.

We need some further concepts about directed graphs: In a directed graph \( G \), a path of length \( k \) is a sequence of vertices \( v_0, v_1, \ldots, v_k \) where \( (v_i, v_{i+1}) \) (\( i = 0, 1, \ldots, k-1 \)) is a directed edge in \( G \). A path of length 0 is a vertex and a path of length 1 is an edge. A path is said to be closed if \( v_k = v_0 \). A path \( (v_0, v_1, \ldots) \) of infinite length has a period \( T \) if \( v_i = v_{i+T} \) for any \( i \geq 0 \). For a path of period \( T \), the sub-path \( (v_0, v_1, \ldots, v_T) \) is closed. A cycle in \( G \) is a closed path \( (v_0, v_1, \ldots, v_k) \) where \( v_i \neq v_j \) for any \( 0 \leq i \neq j \leq k-1 \), i.e., the first and the last vertices are the only repeated vertices. A cycle of length \( k \) is also called a \( k \)-cycle.

A. Scheduling Graphs

Recall that a schedule \( S \) is a matrix with columns indexed by \( t \in \mathbb{Z} \). For a schedule \( S \), integers \( T \geq 1 \) and \( k \in \mathbb{Z} \), denote \( S[T,k] \) as the submatrix of \( S \) with columns \( kT, kT + 1, \ldots, (k+1)T - 1 \). For a submatrix \( S' \) formed by \( T \) columns of a schedule \( S \), the columns in \( S' \) are indexed by \( 0, 1, \ldots, T-1 \).

**Definition 3** (Scheduling Graph). For a network \( \mathcal{N} \) and an integer \( T \geq 1 \), a scheduling graph is a directed graph denoted by \((\mathcal{M}_T, \mathcal{E}_T)\) defined as follows: the vertex set \( \mathcal{M}_T \) includes all the \( |\mathcal{L}| \times T \) binary matrices \( A \) such that \( A = S[T,0] \) for a certain collision-free schedule \( S \). The edge set \( \mathcal{E}_T \) includes all the vertex pairs \((A,B)\) such that \( A = S[T,0] \) and \( B = S[T,1] \) for a certain collision-free schedule \( S \).

**Example 6** (Multihop line network). We give \((\mathcal{M}_1, \mathcal{E}_1)\) of \( \mathcal{N}^\text{line}_{4,1} \) as an example. Here \( \mathcal{M}_1 \) includes the \( 4 \times 1 \) matrices \( v \) such that \( v \) can be a column of a certain collision-free schedule \( S \) of \( \mathcal{N}^\text{line}_{4,1} \). We have \( \mathcal{M}_1 = \{v_0, v_1, \ldots, v_8\} \), where

\[
\begin{align*}
v_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_8 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.
\end{align*}
\]

\( \mathcal{E}_1 \) includes all the pairs \((v,v')\) such that \(|v,v'|\) is equal to two consecutive columns of a certain collision-free
schedule, and can be denoted as the adjacent matrix:

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

(6)

The following three theorems build the connection of a path in \((M_T, E_T)\) and a collision-free schedule of a network \(N\).

**Theorem 6.** Consider a network \(N\) and a schedule \(S\). If \(S\) is collision-free, then for any integer \(T \geq 1\), the sequence \((S[T, k], k = 0, 1, \ldots)\) forms a path in \((M_T, E_T)\).

The converse of Theorem 6 can be proved when \(T\) is sufficiently large. We first discuss the case of binary collision.

**Theorem 7.** Consider a network \(N\) with a binary collision profile and a schedule \(S\). If for certain integer \(T \geq D^*\), the sequence \((S[T, k], k = 0, 1, \ldots)\) forms a path in \((M_T, E_T)\), then \(S\) is collision free.

For \(N_{4,1}^{\text{line}}\) discussed in Example 6, \(D^* = 1\). Then any schedule \(S\) that forms a path in \((M_1, E_1)\) as characterized in Example 6 is collision free. The next example show that when collision is not binary, for \(T < 2D^*\), a schedule \(S\) that forms a path in \((M_T, E_T)\) may not be collision free. For general collision, the converse of Theorem 6 needs a larger \(T\) as in Theorem 7.

**Example 7.** Consider a network \(N_4 = (L, I, D_L)\), where \(L = \{l_1, l_2, l_3, l_4\}\). The collision sets of each link are

\[
\begin{align*}
I(l_1) &= \emptyset \\
I(l_2) &= \{l_1, l_3\} \\
I(l_3) &= \{l_2, l_4\} \\
I(l_4) &= \emptyset.
\end{align*}
\]
The link-wise delay matrix $D_L$ is

$$
D_L = \begin{bmatrix}
* & * & * & * \\
1 & * & * & * \\
* & 1 & * & * \\
* & * & 1 & *
\end{bmatrix}.
$$

(7)

For this network, the character

$$
D^* = \max_{l \in L} \max_{\phi \in \mathcal{F}(l)} \max_{l' \in \phi} |D_L(l, l')| = 1.
$$

(8)

We illustrate that a schedule $S$ that forms a path in $(\mathcal{M}_1, \mathcal{E}_1)$ may not be collision free. First, we see that $(\mathcal{M}_1, \mathcal{E}_1)$ is a complete graph with the vertices set $\{0, 1\}^4$. Consider a schedule $S$ with a submatrix $S$ formed by three consecutive columns:

$$
S' = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
$$

Because $S'(l_1, 0) = S'(l_3, 2) = 1$, $S'(l_2, 1)$ has a collision. As $S'(l_2, 1) = 1$, $S$ is not collision free.

**Theorem 7.** Consider a network $\mathcal{N}$ and a schedule $S$. If for certain integer $T \geq 2D^*$, the sequence $(S[T, k], k = 0, 1, \ldots)$ forms a path in $(\mathcal{M}_T, \mathcal{E}_T)$, then $S$ is collision free.

The above theorems show that a collision-free schedule is equivalent to a directed path in a scheduling graph $(\mathcal{M}_T, \mathcal{E}_T)$ with $T$ sufficiently large, and hence it is enabled to investigate the rate region $\mathcal{R}^N$ using scheduling graphs. Observe that the all-zero $|\mathcal{L}| \times T$ matrix $0$ is in $\mathcal{M}_T$, and for any $A, B \in \mathcal{M}_T$, we have a path $(A, 0, B)$ in the scheduling graph as $(A, 0), (0, B) \in \mathcal{E}_T$. So the scheduling graph $(\mathcal{M}_T, \mathcal{E}_T)$ for a network $\mathcal{N}$ is strongly connected, i.e., there exists a path between any two vertices.

**B. Rate Region by Cycles**

Now we give a characterization of $\mathcal{R}^N$ using a scheduling graph. Denote $1$ as a column vector of all 1’s with length to be known in the context. Define the rate vector of a closed path $P = (A_0, A_1, \ldots, A_k)$ of $(\mathcal{M}_T, \mathcal{E}_T)$ as

$$
R_P = \frac{1}{k} \sum_{i=0}^{k-1} A_i 1.
$$

Denote $\text{cycle}(\mathcal{G})$ as the collection of all the cycles of a directed graph $\mathcal{G}$. Define

$$
\mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)} = \text{conv}(\{R_C : C \in \text{cycle}(\mathcal{M}_T, \mathcal{E}_T)\}).
$$

As $(\mathcal{M}_T, \mathcal{E}_T)$ is finite, $\text{cycle}(\mathcal{M}_T, \mathcal{E}_T)$ is finite and hence $\mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$ is a closed set.

**Lemma 8.** For a network $\mathcal{N}$ and a collision-free schedule $S$ of period $K$, $(S[T, i], i = 0, 1, \ldots, K)$ is a closed path in $(\mathcal{M}_T, \mathcal{E}_T)$, and $R_S \in \mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$.

**Theorem 9.** For a network $\mathcal{N}$, we have $\mathcal{R}_N \subset \mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$. 
The converse of the above theorem is discussed for two cases: binary collision and general collision.

**Lemma 10.** Consider a network $\mathcal{N}$ with a binary collision profile and an integer $T \geq D^*$. Let $(A_0, A_1, \ldots, A_k)$ be a closed path in the scheduling graph $(\mathcal{M}_T, \mathcal{E}_T)$. Define a schedule $S$ with period $kT$ such that $S[T, i] = A_i$ for $i = 0, 1, \ldots, k - 1$. Then $S$ is collision free for $\mathcal{N}$.

**Theorem 11.** For a network $\mathcal{N}$ with a binary collision profile and any integer $T \geq D^*$, $\mathcal{R}^N \supseteq \mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$.

By Theorem 9 and 11 we have for $T \geq D^*$, $\mathcal{R}^N = \mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$ when $\mathcal{N}$ has a binary collision profile. The general result is given by the following lemma and theorem.

**Lemma 10.** For a network $\mathcal{N}$ and an integer $T \geq 2D^*$, define a schedule $S$ with period $kT$ such that $S[T, i] = A_i$ for $i = 0, 1, \ldots, k - 1$, where $(A_0, A_1, \ldots, A_k)$ is a closed path in $(\mathcal{M}_T, \mathcal{E}_T)$. Then $S$ is collision free for $\mathcal{N}$.

**Proof:** Same as the proof of Lemma 10 except for applying Theorem 7 instead of Theorem 7.

**Theorem 11.** For a network $\mathcal{N}$ with a general collision profile and any integer $T \geq 2D^*$, $\mathcal{R}^N \supseteq \mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$.

**Proof:** Same as the proof of Theorem 11 except for applying Lemma 10 instead of Lemma 10.

Theorem 9 and Theorem 11 (or Theorem 11 for binary collision) together give an explicit characterization of $\mathcal{R}^N$, i.e., when $T \geq 2D^*$ (or $T \geq D^*$ for binary collision), $\mathcal{R}^N = \mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$. Compared with the characterization in Theorem 3 $\mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}$ is explicitly determined by the cycles in $(\mathcal{M}_T, \mathcal{E}_T)$. Moreover, by Lemma 8 a periodic, collision-free schedule is a closed path in $(\mathcal{M}_T, \mathcal{E}_T)$, which can be decomposed into a sequence of (not necessarily distinct) cycles (see, e.g., [19]). By Lemma 10 (or Lemma 10 for binary collision), cycles in $(\mathcal{M}_T, \mathcal{E}_T)$ can also help to construct periodic, collision-free schedules. In the next section, we will study how to find cycles in $(\mathcal{M}_T, \mathcal{E}_T)$, which is sufficient for us to construct all the periodic, collision-free schedules and calculate $\mathcal{R}^N$ when $T$ is sufficiently large.

Last in this section, we show that the rate region of framed scheduling can also be characterized by using scheduling graphs. Putting framed scheduling in the same picture, we can see why a general scheduling with delay can be better than framed scheduling. The next theorem summarizes the result, together with inner and outer bounds on $\mathcal{R}^N$. The outer bound $\mathcal{R}^{(\mathcal{M}_1, \mathcal{E}_1)}$ can also be calculated using the approaches to be discussed in the next section.

When calculating the rate region is difficult due to a large character, the outer bound can be simpler to evaluate.

**Theorem 12.** For a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D_\mathcal{L})$,

$$\bar{\mathcal{R}}^{(\mathcal{L}, \mathcal{I})} = \text{conv}\{R_C : C \text{ is a 1-cycle in } (\mathcal{M}_1, \mathcal{E}_1)\}$$

and

$$\bar{\mathcal{R}}^{(\mathcal{L}, \mathcal{I})} \subset \mathcal{R}^N \subset \mathcal{R}^{(\mathcal{M}_1, \mathcal{E}_1)}.$$
TABLE I
The computation time for enumerating all the cycles using Johnson’s algorithm.

|          | $N^{\text{line}}_{4,1}$ | $N^{\text{line}}_{5,1}$ | $N^{\text{line}}_{6,1}$ |
|----------|--------------------------|--------------------------|--------------------------|
| Number of vertices in $(M_1, E_1)$ | 9 | 15 | 25 |
| Number of edges in $(M_1, E_1)$     | 56 | 144 | 357 |
| Computation time                   | 0.17s | 330s | 610588s |

TABLE II
Run Johnson’s algorithm for $(M_1, E_1)$ of $N^{\text{line}}_{6,1}$. The table lists the cycle length range of a major fraction of cycles found during different running periods.

| Running time period              | 0 ~ 2500000s | 250001 ~ 400000s | 400001 ~ 610588s |
|---------------------------------|--------------|-----------------|-----------------|
| Cycle length range for 99% of cycles | 19 ~ 25    | 11 ~ 18        | 1 ~ 11          |

(e.g., Johnson’s algorithm [14]) to enumerate all the cycles in $(M_T, E_T)$ in $O((\eta + 1)(|M_T| + |E_T|))$ time, where $\eta$ is the number of cycles in $(M_T, E_T)$. See more about algorithms for enumerating cycles in a graph in [20]. In the worst case that $(M_T, E_T)$ is a complete graph, $\eta = 2^{|M_T|}$, the computational complexity is double exponential in the network of size $|L| \times T$. Therefore enumerating all the cycles is not efficient when the network size is large. In Table II we show the running time of Johnson’s algorithm for some line networks, and observe that the computation time increases fast with the network size. For all the numerical evaluations in this section, we use a computer with an Xeon E-2100 CPU and 4G RAM, running Ubuntu 16.04. All the algorithms are implemented using Python. Only one CPU core is used when evaluating algorithm execution time.

Due to the hardness nature of the problem, the existing algorithms can only give a subset of $R^{(M_T, E_T)}$ in a reasonable running time. Instead of enumerating all the cycles, it is reasonable to stop the searching process of Johnson’s algorithm in a given time. Table II shows the cycle length range of a major fraction of cycles found in different periods of the running time of Johnson’s algorithm for $(M_1, E_1)$ of $N^{\text{line}}_{6,1}$.

Henceforth in this section, we discuss algorithms that potentially find rate vectors that cannot be found by Johnson’s algorithm in a reasonable time. We observe that it is not necessary to enumerate all the cycles in $(M_T, E_T)$ due to the special structure of scheduling graphs. We introduce a partial order on paths of a scheduling graph of the same length, and show that finding all the maximal cycles is sufficient to calculate $R^{(M_T, E_T)}$. Based on the dominance property, we derive two algorithms for calculating rate vectors that are not obtained by the existing cycle enumerating algorithms in a given time. The first algorithm enumerates cycles incrementally in cycle length, so that we can find a subset of $R^{(M_T, E_T)}$ determined by the cycles in $(M_T, E_T)$ up to a certain length bounded by the computational costs. The second algorithm enumerates cycles associated with a maximal subgraph of $(M_T, E_T)$ and enables us to find longer cycles that potentially have a larger rate vector.

Omitted proof in this section can be found in Appendix E.
A. Dominance Properties

We first discuss some dominance properties of the scheduling graph \((\mathcal{M}_T, \mathcal{E}_T)\) which can help us to simplify the calculations. A sequence of matrices \(A = (A_0, A_1, \ldots)\) where \(A_i \in \{0, 1\}^{\mathcal{L} \times T}\) can be regarded as a matrix obtained by juxtaposing \(A_0, A_1, \ldots\). Therefore, the relation \(\succ\) and \(\preceq\) can be applied on a pair of sequences of the same length. For two sequences \(A = (A_0, A_1, \ldots)\) and \(B = (B_0, B_1, \ldots)\) of the same length (which can be unbounded) with \(A_i, B_i \in \{0, 1\}^{\mathcal{L} \times T}\), we say \(A\) dominates \(B\) if \(A \succ B\).

**Lemma 13.** For any \(k \geq 0\), if \(A = (A_0, A_1, \ldots, A_k)\) is a path in \((\mathcal{M}_T, \mathcal{E}_T)\), then any \(B = (B_0, B_1, \ldots, B_k)\) with \(A \succeq B\) is a path in \((\mathcal{M}_T, \mathcal{E}_T)\).

*Proof:* For any edge \((A', A'') \in \mathcal{E}_T\), we see any \((B', B'') \preceq (A', A'')\) is also an edge of \((\mathcal{M}_T, \mathcal{E}_T)\). The lemma can then be proved by checking \((B_i, B_{i+1}) \preceq (A_i, A_{i+1})\) for \(i = 0, 1, \ldots, k - 1\).

For a set \(A\) with partial order \(\succ\), we write \(\max_{\succ} A\) as the smallest subset \(B\) of \(A\) such that any element of \(A\) is dominated by certain elements of \(B\). Denote

\[
\text{cycle}^*(\mathcal{M}_T, \mathcal{E}_T) = \max_{\succ} \text{cycle}(\mathcal{M}_T, \mathcal{E}_T).
\]

The elements in \(\text{cycle}^*(\mathcal{M}_T, \mathcal{E}_T)\) are called the maximal cycles. It is sufficient to use \(\text{cycle}^*(\mathcal{M}_T, \mathcal{E}_T)\) to characterize all the cycles. The following theorem justifies the sufficiency of using maximal cycles to calculate the rate region.

**Theorem 14.** For a network \(\mathcal{N}\), a vector \(R\) of length \(|\mathcal{L}|\) is in \(\mathcal{R}^{(\mathcal{M}_T, \mathcal{E}_T)}\) if and only if there exists an \(R' \in \text{conv}(\{R_C : C \in \text{cycle}^*(\mathcal{M}_T, \mathcal{E}_T)\})\) such that \(R \preceq R'\).

Motivated by the dominance property, instead of \(\mathcal{E}_T\) we can find \(\mathcal{E}^* = \max_{\succ} \mathcal{E}_T\), which is the collection of maximal independent sets of \(\mathcal{N}^{2T}\). When the collision model is binary, we can use the Bron–Kerbosch algorithm [21] to enumerate all the maximal independent sets of \(\mathcal{N}^{2T}\), where \(T\) can be as small as \(D^*\). When it comes to the general collision model, \(\mathcal{N}^{2T}\) is a hypergraph, and the corresponding maximal independent set problem has been discussed in [22]. Let \(\mathcal{M}^*_L\) (resp. \(\mathcal{M}^*_R\)) be the collection of \(B\) such that \((B, B') \in \mathcal{E}^*\) (resp. \((B', B) \in \mathcal{E}^*)\) for certain \(B'\). As \(\mathcal{E}^* \subset \mathcal{M}^*_L \times \mathcal{M}^*_R\), we denote the directed graph \((\mathcal{M}^*_L, \mathcal{E}^*)\) as \((\mathcal{M}^*_L, \mathcal{E}^*, \mathcal{M}^*_R)\), where \(\mathcal{E}^*\) can be represented using an adjacent matrix with rows and columns indexed by elements in \(\mathcal{M}^*_L\) and \(\mathcal{M}^*_R\), represent.

**Example 8.** For \((\mathcal{M}_1, \mathcal{E}_1)\) in Example 6, \(\mathcal{M}^*_L\) and \(\mathcal{M}^*_R\) have the same vertices:

\[
v_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_8 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]
and \( E^* \) can be represented by the adjacent matrix:

\[
\begin{pmatrix}
  v_5 & v_6 & v_7 & v_8 \\
  v_5 & 1 & 0 & 0 & 1 \\
  v_6 & 1 & 0 & 0 & 0 \\
  v_7 & 0 & 1 & 0 & 0 \\
  v_8 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

(9)

After calculating \((M^*_L, E^*, M^*_R)\), we find the scheduling graph \((M_T, E_T)\). First, by definition, \( E^* \subseteq E_T \) and for any \( B \in E_T \), \( B \) is dominated by a certain \( B' \in E^* \). Second, by Lemma 13, if \( B \) is dominated by a certain \( B' \in E^* \), then \( B \in E_T \). By checking Example 6 and 8 we see that the adjacent matrix (9) is a submatrix of (6), and other entries of (6) can be decided by examining the entries of (9).

### B. Two Algorithms

#### 1) Algorithm A: Incremental Approach for Maximal Cycles

A path is said to be maximal if it is not dominated by any other path of the same length in \((M_T, E_T)\). Denote \( P^*_k \) as the set of maximal paths in \((M_T, E_T)\) of length \( k \). Note that \( E^* = P^*_1 \). By Lemma 13 all \( k \)-cycles are dominated by certain paths in \( P^*_k \). Our incremental approach for maximal cycles considers \( k = 1, 2, \ldots \) sequentially. For each value of \( k \), a superset \( G_k \) of \( P^*_k \) is first constructed using \( G_{k-1} \) (see the definition of \( G_k \) in Section \( \text{V-C} \)), and then for each path \( P \) in \( G_k \), all the \( k \)-cycles dominated by the path are explored (see the algorithm in Section \( \text{V-D} \)). If \( C_1 \prec C_2 \), the algorithm may enumerate \( C_2 \) only.

Let’s first use a length-1 path as an example to illustrate how to find cycles dominated by a path. For two binary matrices \( A \) and \( B \) of the same size, denote \( A \land B \) as the component-wise AND of these two matrices. We first observe that for any \((B_0, B_1) \in E^*\), as both \( B_0 \) and \( B_1 \) dominate \( B_0 \land B_1 \), \((B_0 \land B_1, B_0 \land B_1) \in E_T\) is a 1-cycle. Moreover, any 1-cycle \((C_0, C_0) \in E_T\) is dominated by certain \((B_0, B_1) \in E^*\) and hence \( C_0 \preceq B_0 \land B_1 \). Therefore, if we enumerate all the elements \((B_0, B_1) \) of \( P^*_1 \) to find all the maximal 1-cycles dominated by \((B_0, B_1) \), all the maximal 1-cycles can be found.

In general, for each path in \((M_T, E_T)\), we have a procedure Path2Cycles (see Algorithm 1) for finding all the \( k \)-cycles dominated by the path. If there are two cycles \( C_1 \prec C_2 \), the algorithm may output \( C_2 \) only. To verify that all the maximal \( k \)-cycles can be found, consider a \( k \)-cycle \( C \). There must exist \( B \in P^*_k \) such that \( B \succ C \). If we can go through all the paths \( B \) in \( P^*_k \), then Path2Cycles finds either \( C \) or a cycle \( C' \preceq C \).

Algorithm A enables us to enumerate cycles up to a certain length, but it is not as efficient as Johnson’s algorithm for enumerating all the cycles. Algorithm A shares some properties of the cycle enumerating algorithm in [16], where a general graph is considered.

#### 2) Algorithm B: Cycles associated with \((M^*_L \cup M^*_R, E^*)\)

Fix an integer \( k > 0 \). We can use an existing algorithm (e.g. depth-first search) to enumerate all the paths of \((M^*_L \cup M^*_R, E^*)\) up to length \( k \). For each path, we also use the procedure Path2Cycles (Algorithm 1) to find cycles dominated by the path. This algorithm finds maximal cycles only. Moreover, compared with Algorithm A, this algorithm can find longer cycles.
C. Algorithm for Enumerating $P_k^*$

Exactly characterizing $P_k^*$ is a hard problem. Here we present an iterative approach for enumerating a superset of $P_k^*$. For any $(A_0, A_1, \ldots, A_k) \in P_k^*$, there exist $(B_0, B_1, \ldots, B_{k-1}) \in P_{k-1}^*$ and $(B'_1, B_k) \in E^*$ such that

$$(B_0, B_1, \ldots, B_{k-1}) \succ (A_0, A_1, \ldots, A_{k-1}),$$

$$(B'_1, B_k) \succ (A_{k-1}, A_k).$$

We see $(B_0, \ldots, B_{k-2}, B_{k-1} \land B'_1, B_k)$ is a path of length $k$ and dominates $(A_0, A_1, \ldots, A_k)$. As the latter is maximal, we further have $B_i = A_i$ for $i = 0, 1, \ldots, k-2, k$ and $B_{k-1} \land B'_1 = A_{k-1}$. Therefore, $P_k^*$ is a subset of

$$\{(B_0, \ldots, B_{k-2}, B_{k-1} \land B'_1, B_k) : (B'_1, B_k) \in E^*,$$

$$(B_0, \ldots, B_{k-1}) \in P_{k-1}^*\}.$$

Motivated by the above discussion, we describe an iterative approach to enumerate a superset of $P_k^*$. A $k$-partite graph ($k \geq 2$) is denoted by $(V_i, U_i, i = 0, 1, \ldots, k-2, V_{k-1})$, where $V_i, i = 0, 1, \ldots, k-1$ are sets of vertices, and $U_i \subset V_i \times V_{i+1}, i = 0, 1, \ldots, k-2$ are sets of edges. Here we do not require that $V_i$ and $V_j$ ($i \neq j$) are disjoint. We say $(P_0, P_1, \ldots, P_{k-1})$ is a path in $(V_i, U_i, i = 0, 1, \ldots, k-2, V_{k-1})$ if $P_i \in V_i$ for $i = 0, 1, \ldots, k-1$ and $(P_i, P_{i+1}) \in U_i, i = 0, \ldots, k-2$.

Define first a directed bipartite graph $G_1 = (M_L^1, U_0, M_R^1)$ where $U_0 = E^*$. We see that any path in $P_1^*$ is an edge in $G_1$ and vice versa. For $k > 1$, we define a directed $(k+1)$-partite graph:

$$G_k = (M_L^k, U_0, V, U_1, \ldots, U_{k-2}, V, U_{k-1}^{'}, M_R^k),$$

where $V = \{B \land B' : B \in M_R^k, B' \in M_L^k\}$, and $U_{k-2}$ and $U_{k-1}^{'}, M_R^k$ are determined recursively as follows. Let

$$F_k = \{(A, B \land B', C) : (A, B) \in U_{k-2}, (B', C) \in E^*\},$$

$$F_k^* = \{(A, B, C) \in F_k : \forall B' > B, (A, B', C) \notin F_k\}. \quad (10)$$

Then,

$$U_{k-2} = \{(A, B) : (A, B, C) \in F_k^* \text{ for certain } C\},$$

$$U_{k-1}^{' - 1} = \{(B, C) : (A, B, C) \in F_k^* \text{ for certain } A\}. \quad (11)$$

**Lemma 15.** Any path in $P_k^*$ is a path of length $k$ in $G_k$ for $k = 1, 2, \ldots$.

In practice, $F_k$ is not necessarily calculated before $F_k^*$, and $F_k^*$ can be calculated directly by enumerating $U_{k-2}$ and $E^*$. The next lemma justifies the using of $F_k^*$, instead of $F_k$, in the definition of $U_{k-1}^{' - 1}$.

**Lemma 16.** For a scheduling graph $(M_T, E_T)$, for $i = 1, 2, \ldots$, $U_i^' \subset \tilde{U}$ and $U_i \subset \tilde{U}$, where

$$\tilde{U} = \{(B, C) : (A, B, C) \in F_2 \text{ for certain } A\},$$

$$\tilde{F} = \{(A, B \land B', C) : (A, B) \in \tilde{U}, (B', C) \in E^*\},$$

$$\tilde{U} = \{(A, B) : (A, B, C) \in \tilde{F} \text{ for certain } C\}.$$
We illustrate the above lemma using the following example.

**Example 9.** For the scheduling graph \((\mathcal{M}_1, \mathcal{E}_1)\) of \(N_{4,1}^{\text{line}}\), \(G_1 = (\mathcal{M}_L^*, \mathcal{E}^*, \mathcal{M}_L^*)\) is characterized in Example 8 where \(\mathcal{M}_L^* = \mathcal{M}_R^* = \{v_5, v_6, v_7, v_8\}\). For this example, we have \(\mathcal{V} = \mathcal{M}_1\), \(\mathcal{U}_0\) has an adjacent matrix

\[
\begin{bmatrix}
  v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
  v_5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
  v_6 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
  v_7 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
  v_8 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

\(\mathcal{U}_1^*\) and \(\mathcal{U}_2^*\) have, respectively, the adjacent matrices

\[
\begin{bmatrix}
  v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
  v_5 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
  v_6 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
  v_7 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
  v_8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\text{ and }
\begin{bmatrix}
  v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
  v_5 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
  v_6 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  v_7 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  v_8 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Moreover, we have the adjacent matrix of \(\mathcal{U}^*:\)

\[
\begin{bmatrix}
  v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
  v_5 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
  v_6 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
  v_7 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  v_8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

From the adjacent matrices, we see that \(\mathcal{U}_1^*, \mathcal{U}_2^* \subset \mathcal{U}^*\).

Algorithm A inductively calculates \(G_k\). Note that not all the paths in \(G_k\) are maximal, but we do not exclude them from calling Path2Cycles. If the number of paths in \(G_k\) is much less than the total number of length-\(k\) paths
in \((M_T, E_T)\), this approach can have a lower computational cost than that of the approach of directly enumerating the cycles.

**Example 10.** Consider the network \(A \text{line}_{4,1}^\text{line} \) defined in Example 2. For \(k = 1, \ldots, 4\), we list the number of paths in \(G_k\), the total number of length-\(k\) paths in \((M_1, E_1)\) and the total number of length-\(k\) paths in \((M^*, E^*)\) in the following table:

| \(k\) | \(k = 1\) | \(k = 2\) | \(k = 3\) | \(k = 4\) |
|-------|-----------|-----------|-----------|-----------|
| Number of length-\(k\) paths in \((M_1, E_1)\) | 56 | 363 | 2357 | 152633 |
| Number of paths in \(G_k\) | 6 | 16 | 64 | 180 |
| Number of length-\(k\) paths in \((M^*, E^*)\) | 6 | 9 | 15 | 25 |

**D. From Path to Cycles**

At last, given a path of length \(k\), we use Algorithm \(\text{[I]}\) to find all the cycles dominated by this path. If two cycles \(C_1 \prec C_2\), the algorithm may output \(C_2\) only. The algorithm mainly calls the \(\text{DISTINCT}\) function recursively, which checks whether \(B_0, B_1, \ldots, B_p\) are distinct to each other or not. If so, a cycle is found; otherwise, \(\text{DISTINCT}\) is called recursively with the argument obtained by modifying one entry of the two identical matrices.

To reduce the possibility that a path is searched for multiple times, matrices \(F_0, F_1, \ldots, F_p\) are included in the argument of \(\text{DISTINCT}\). Similar techniques have been used in [14]. Initially, \(F_i = 0\) for \(i = 0, 1, \ldots, p\). If the condition in Line 8 holds, the algorithm executes Line 10-12 for each \((l, t)\) such that the \((l, t)\) entry of \((B_j, B_i)\) is 1 and the \((l, t)\) entry of \((F_j, F_i)\) is 0. For each such \((l, t)\), we first generate \((B'_j, B'_i)\) same as \((B_i, B_j)\) except that the \((l, t)\) entry is 0, and call \(\text{DISTINCT}\) with \(B'_i\) and \(B'_j\) in places of \(B_i\) and \(B_j\), respectively (Line 10 and 11). We then set the \((l, t)\) entry in \((F_j, F_i)\) to 1 so that the \((l, t)\) entry of \((B_j, B_i)\) in the following calls of \(\text{DISTINCT}\) function would not be assigned to 0.

Suppose that Algorithm \(\text{[I]}\) is applied on \(B_0, \ldots, B_k\), and we have a maximal cycle \(C = (C_0, \ldots, C_{k-1}, C_0)\) dominated by \(B = (B_0, \ldots, B_k)\). To complete the justification of the algorithm, we show that the algorithm is guaranteed to find the cycle. First, as \(C_0 \not\prec B_0\) and \(C_0 \not\prec B_k\), we have \(C_0 \not\prec B_0 \land B_k\). Therefore, the argument of the first call of \(\text{DISTINCT}\) dominates \(C\). Now suppose that for a certain level of recursive calling of \(\text{DISTINCT}\), the argument \((A_0, A_1, \ldots, A_{k-1})\) dominates \((C_0, C_1, \ldots, C_{k-1})\). If all the \(k\) matrices are distinct, \((A_0, \ldots, A_{k-1}, A_0)\) is a cycle and dominates \(C\). As \(C\) is a maximal cycle dominated by \(B\), we have \(C = (A_0, \ldots, A_{k-1}, A_0) \in \mathcal{C}\). If \(A_i = A_j\), there must exists \((A'_i, A'_j)\) that dominating \((C_i, C_j)\) obtained by converting a certain entry 1 of \((A_i, A_j)\) to 0. Hence \(C\) is dominated by the argument of one of the next level’s call of \(\text{DISTINCT}\) function.
we list the time used for enumerating all the cycles of given lengths in most of which cannot be found by Johnson’s algorithm within 4 days.

Example 12. When $T = 4$, by using the approximation $a_4 = \frac{4}{10}$, to achieved $\frac{1}{2}$, a sufficiently large $T$ is in need. Moreover, $T$ can be much larger than 4.

E. Algorithm Evaluations

Here we provide some numerical results of evaluating Johnson’s algorithm, our Algorithm A and B. In Table IV we list the time used for enumerating all the cycles of given lengths in $N_{4,1}^{\text{line}}$, $N_{5,1}^{\text{line}}$ and $N_{6,1}^{\text{line}}$ using Algorithm A. We see that the computation time increases fast when the cycle length increases. Compared with the evaluation of Johnson’s algorithm on $N_{4,1}^{\text{line}}$ (see Table II), Algorithm A enumerates all the cycles up to length 9 within 2 hours, most of which cannot be found by Johnson’s algorithm within 4 days.

**Algorithm 1** Find cycles dominated by a path.

| Procedure PATH2CYCLES($B_0, B_1, \ldots, B_k$) |
|---|
| 1: procedure PATH2CYCLES($B_0, B_1, \ldots, B_k$) |
| 2: initialize $C = \emptyset$. |
| 3: call DISTINCT($B_0 \land B_k, 0, B_1, 0, \ldots, B_{k-1}, 0$). |
| 4: return $C$. |
| 5: function DISTINCT($B_0, F_0, B_1, F_1, \ldots, B_p, F_p$) |
| 6: initialize AllDistinct as TRUE. |
| 7: for $1 \leq j < i \leq p$ do |
| 8: if $B_j = B_i$ then |
| 9: for each $(l, t)$ such that the $(l, t)$ entry of $(B_j, B_i)$ is 1 and the $(l, t)$ entry of $(F_j, F_i)$ is 0 do |
| 10: construct $(B'_j, B'_i)$ same as $(B_j, B_i)$ except that the $(l, t)$ entry is 0. |
| 11: call DISTINCT with the argument $(B_0, \ldots, F_{j-1}, B'_j, F_j, B_{j+1}, \ldots, F_{i-1}, B'_i, F_i, B_{i+1}, \ldots, F_p)$. |
| 12: update $(F_j, F_i)$ by flipping the $(l, t)$ entry of $(F_j, F_i)$.
| 13: set AllDistinct as FALSE. |
| 14: break |
| 15: if AllDistinct is TRUE then |
| 16: add $(B_0, B_1, \ldots, B_p, B_0)$ to $C$. |

**Example 11.** For $N_{4,1}^{\text{line}}$ defined in Example 2, after evaluating maximal cycles up to length 4, we get the following four rate vectors in $R(M_3, \xi_3)$:

$$r_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad r_4 = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}.$$

The rate vectors achieved by framed scheduling in Example 5 are all dominated by $r_1, r_2$ and $r_3$, but $r_4$ is not achievable by the framed scheduling. Hence the rate region achieved by framed scheduling is a strict subset of our scheduling rate region of $N_{4,1}^{\text{line}}$.

**Example 12.** When $T = 4$, by using the approximation $a_4 = \frac{4}{10}$, to achieved $\frac{1}{2}$, a sufficiently large $T$ is in need. Moreover, $T$ can be much larger than 4.
TABLE IV
Evaluations of the cost of enumerating all the cycles of given lengths with Algorithm A. We do not apply parallel computing algorithms and multi-core computing.

| Cycle length | $N_{line}^{4,1}$ | $N_{line}^{5,1}$ | $N_{line}^{6,1}$ |
|--------------|------------------|------------------|------------------|
| 1            | 0.00012          | 0.0006           | 0.0008           |
| 2            | 0.0003           | 0.0012           | 0.0014           |
| 3            | 0.0011           | 0.0008           | 0.018            |
| 4            | 0.005            | 0.019            | 0.018            |
| 5            | 0.007            | 0.022            | 0.042            |
| 6            | 0.027            | 0.085            | 0.077            |
| 7            | 0.25             | 1.08             | 0.4              |
| 8            | 2.76             | 16.1             | 10.1             |
| 9            | 29.2             | 378.1            | 626.7            |
| 10           | -                | 8588.6           | 6378.1           |

In addition to finding different cycles, Algorithm A can also find new rate vectors that are not in the subset of the rate region found by Johnson’s algorithm in a limited time period. In the following evaluations, given a network, the cycles we enumerate using an algorithm may have the same rate vectors and some rate vectors may also have dominance relation. For each algorithm, we just keep a maximal subset of the rate vectors. For each network to be evaluated, we denote by $R_{1h}^{Johnson}$ and $R_{1h}^{Alg-A}$ the maximal subsets of the rate vectors of the cycles found by Johnson’s algorithm and Algorithm A within 1 hour, respectively. For a set $A$ of rate vectors, denote by $\mathcal{A}$ as the collection of all the rate vectors that are dominated by the convex hull of $A$. In other words, $R_{1h}^{Johnson}$ and $R_{1h}^{Alg-A}$ are the subsets of the rate region found by Johnson’s algorithm and Algorithm A, respectively.

In Table V we give the results of evaluating Johnson’s algorithm and Algorithm A for 1 hour on $N_{line}^{6,1}$, $N_{line}^{7,1}$ and $N_{line}^{8,1}$. We see from Table V that nearly all the rate vectors found by the Johnson’s algorithm are in the subset of the rate region found by Algorithm A, and more than half of the maximal rate vectors found by Algorithm A are not in the subset of the rate region found by the Johnson’s algorithm.

Algorithm B can also find new rate vectors that are not in the subset of the rate region found by Johnson’s algorithm and Algorithm A in a much shorter time period. In Table VI we give the results of evaluating Algorithm B in 1 minute on $N_{line}^{6,1}$, $N_{line}^{7,1}$ and $N_{line}^{8,1}$. For each network, we denote by $R_{1m}^{Alg-B}$ the maximal subset of the rate vectors of the cycles found by Algorithm B and denote by $\mathcal{A}$ the subset of the rate region found by Algorithm B within 1 minute. We see from Table VI that all the rate vectors found by the Johnson’s algorithm are in the subset of the rate region found by Algorithm B, and more than 48% of the rate vectors found by Algorithm B are not in the subset of the rate region found by the Johnson’s algorithm. More than 25% of the rate vectors found by algorithm A are not in the subset of the rate region found by Algorithm B, and more than 40% of the rate vectors

TABLE V
Evaluations of the rate of cycles in $N_{line}^{6,1}$, $N_{line}^{7,1}$ and $N_{line}^{8,1}$ by Johnson’s algorithm and Algorithm A in 1 hour.

|        | $N_{line}^{6,1}$ | $N_{line}^{7,1}$ | $N_{line}^{8,1}$ |
|--------|------------------|------------------|------------------|
| $|R_{1h}^{Johnson} \cap R_{1h}^{Alg-A}|$ | 453            | 94              | 8                |
| $|R_{1h}^{Alg-A} \setminus R_{1h}^{Johnson}|$ | 312            | 202             | 121              |
| $|R_{1h}^{Alg-A} \setminus R_{1h}^{Johnson}|$ | 0              | 0               | 1                |

In Table VI we give the results of evaluating Algorithm B in 1 minute on $N_{line}^{6,1}$, $N_{line}^{7,1}$ and $N_{line}^{8,1}$. For each network, we denote by $R_{1m}^{Alg-B}$ the maximal subset of the rate vectors of the cycles found by Algorithm B and denote by $\mathcal{A}$ the subset of the rate region found by Algorithm B within 1 minute. We see from Table VI that all the rate vectors found by the Johnson’s algorithm are in the subset of the rate region found by Algorithm B, and more than 48% of the rate vectors found by Algorithm B are not in the subset of the rate region found by the Johnson’s algorithm. More than 25% of the rate vectors found by algorithm A are not in the subset of the rate region found by Algorithm B, and more than 40% of the rate vectors
TABLE VI
EVALUATIONS OF THE RATE OF CYCLES IN $\mathcal{N}_{6,1}^{\text{line}}$, $\mathcal{N}_{7,1}^{\text{line}}$ AND $\mathcal{N}_{8,1}^{\text{line}}$ BY ALGORITHM B IN 1 MINUTE, JOHNSON’S ALGORITHM AND ALGORITHM A IN 1 HOUR.

|                      | $\mathcal{N}_{6,1}^{\text{line}}$ | $\mathcal{N}_{7,1}^{\text{line}}$ | $\mathcal{N}_{8,1}^{\text{line}}$ |
|----------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| $|\mathcal{R}_{\text{1m}}^{\text{Alg-B}}\backslash\mathcal{R}_{\text{1h}}^{\text{Alg-A}} |$ | 235 | 118 | 99 |
| $|\mathcal{R}_{\text{1h}}^{\text{Alg-B}}\backslash\mathcal{R}_{\text{1m}}^{\text{Alg-A}} |$ | 0 | 0 | 0 |
| $|\mathcal{R}_{\text{1m}}^{\text{Alg-B}}\backslash\mathcal{R}_{\text{1h}}^{\text{Alg-B}} |$ | 129 | 67 | 48 |
| $|\mathcal{R}_{\text{1h}}^{\text{Alg-A}}\backslash\mathcal{R}_{\text{1m}}^{\text{Alg-B}} |$ | 95 | 55 | 40 |
| $|\mathcal{R}_{\text{1h}}^{\text{Alg-A}}\backslash\mathcal{R}_{\text{1m}}^{\text{Alg-B}} |$ | 87 | 102 | 40 |

found by Algorithm B are not in the subset of the rate region found by Johnson’s algorithm.

From the above evaluations, we see that for some networks, nearly all the rate vectors found by Johnson’s algorithm in a limited time period are in the subset of the rate region found by Algorithm A and Algorithm B in a similar or shorter time. Algorithm A and Algorithm B both contribute some rate vectors that are not in the subset of the rate region found by each other. Hence combining Algorithm A and B can be more efficient for finding a subset of the rate region than Johnson’s algorithm.

VI. CONCLUDING REMARKS

We studied the link scheduling problem of wireless networks where signal propagation delays are multiples of certain time interval. We derived a graphical approach to completely characterize the rate region, and present algorithms to calculate a subset of the rate region more efficiently. The rate region provides a theoretical guideline for further researches on scheduling algorithms. For example, decentralized scheduling algorithms are preferred in practice. The achievable rate vectors can be compared with the rate region to see the optimality.

One direction to further study the region is to consider special network topologies that of practical or theoretical interest, for example, line/grid networks and cellular networks. For special network models, it is possible to obtain better results than the ones obtained in this paper, and find simpler algorithms for calculating the rate regions.

Though developed from networks with long propagation delays like underwater acoustic networks and satellite networks, scheduling with propagation delay is a general phenomenon. In terrestrial radio networks, if we allow radio frame length of tenth of a microsecond, the scheduling problem should take delay into consideration, and the scheduling rate region is larger than the existing framed scheduling rate region. To achieve the gain, however, we face new problems in communication with sub-microsecond frame length.

APPENDIX A

USEFUL PROPERTIES OF PERIODIC GRAPHS

As discussed in Section II, a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D_{\mathcal{L}})$ induces a periodic (hyper)graph $\mathcal{N}^{\infty}$, and the scheduling problem studied in this paper is closely related to the independent sets of $\mathcal{N}^{\infty}$. Here we discuss some isomorphism
and connectivity properties of periodic (hyper)graphs, which can help to simplify the scheduling problem in general.

A. Isomorphism

A vertex assignment for a network \( \mathcal{N} = (\mathcal{L}, \mathcal{I}, D_\mathcal{L}) \) is an integer-valued vector \( b = (b_l, l \in \mathcal{L}) \). Each vertex assignment \( b \) induces a new link-wise delay matrix \( D_\mathcal{L}^b = (D_\mathcal{L}^b(l, l')) \) where

\[
D_\mathcal{L}^b(l, l') = D_\mathcal{L}(l, l') + b_l - b_{l'},
\]

and hence a new network \( \mathcal{N}_b = (\mathcal{L}, \mathcal{I}, D_\mathcal{L}^b) \). According to [18], \( \mathcal{N}^\infty \) and \( \mathcal{N}_b^\infty \) are isomorphic with respect to the bijection \( f : \mathcal{L} \times \mathbb{Z} \to \mathcal{L} \times \mathbb{Z} \) with \( f(l, t) = (l, t + b_l) \). In other words, \( \mathcal{N}_b^\infty \) is obtained by shifting all the vertices in the row \( l \) of \( \mathcal{N}^\infty \) by \( b_l \), which implies the next proposition.

**Proposition 17.** For a network \( \mathcal{N} \) and a vertex assignment \( b \), \( \mathcal{R}^\mathcal{N} = \mathcal{R}^{\mathcal{N}_b} \).

**Example 13.** Consider a network \( \mathcal{N} = (\mathcal{L}, \mathcal{I}, D_\mathcal{L}) \) with the link set \( \mathcal{L} = \{l_1, l_2, l_3, l_4\} \), the collision sets

\[
\mathcal{I}(l_1) = \{l_2, l_3, l_4\}, \quad \mathcal{I}(l_2) = \{l_1, l_3, l_4\}, \quad \mathcal{I}(l_3) = \{l_2, l_4\}, \quad \mathcal{I}(l_4) = \{l_3\},
\]

and the link-wise propagation delay matrix

\[
D_\mathcal{L} = \begin{bmatrix}
  * & 0 & -2 & -4 \\
  0 & * & 0 & -2 \\
  * & 0 & * & 0 \\
  * & * & 0 & *
\end{bmatrix}.
\]

The character \( D_\mathcal{N}^* = 4 \). According to the characterization in Section [15] the rate region of \( \mathcal{N} \) can be determined by the cycles in the scheduling graph \( (\mathcal{M}_4, \mathcal{E}_4) \) of \( \mathcal{N} \), which includes 674 vertices.

For the vertex assignment \( b = (4, 3, 2, 1) \), the link-wise delay matrix is

\[
D_\mathcal{L}^b = \begin{bmatrix}
  * & 1 & 0 & -1 \\
-1 & * & 1 & 0 \\
* & -1 & * & 1 \\
* & * & -1 & *
\end{bmatrix}.
\]

The character of \( \mathcal{N}_b = (\mathcal{L}, \mathcal{I}, D_\mathcal{L}^b) \) becomes to 1, and the rate region of \( \mathcal{N}_b \) can be determined by the cycles in the scheduling graph \( (\mathcal{M}_1, \mathcal{E}_1) \) of \( \mathcal{N}_b \), which includes only 9 vertices. Therefore, it is possible to find an isomorphism of a network to simplify the calculation of the rate region.
As illustrated by the above example, an isomorphism with a smaller character may potentially reduce the computational cost of calculating rate region if an isomorphism with a smaller character can be found. Therefore, we may consider to find a vertex assignment $b$ that minimizes the following objective:

$$\max_{l \in L} \max_{\phi \in I(l)} \max_{l' \in \phi} (D_L(l, l') + b_l - b_{l'}).$$

How to solve the above optimization efficiently is beyond the scope of this paper.

**B. Connectivity**

In a directed graph, two vertices are said to be weakly connected if there exists an undirected path between these two vertices. Exploring the connectivity of $N^\infty$ can potentially simplify the calculation of scheduling rate region by considering each component of $N^\infty$ individually. We first discuss the connectivity of $N = (L, I, D_L)$ with a binary collision model, which has been studied in [18]. Let $g_N$ be the greatest common divisor of $D_L(l, l')$ for all $l \in L$ and $l' \in I(l)$. So $D_L/g_N$ is a well-defined delay matrix. According to [18], $N$ has $g_N$ disjoint, isomorphic subgraphs. The following proposition can be proved by applying their results.

**Proposition 18.** For a network $N = (L, I, D_L)$ of a binary collision model, $R_N = R^{(L, I, D_L/g_N)}$.

Suppose $N$ is connected. Subject to isomorphism (which does not change the rate region), there exists a spanning tree of $N$ consisting of only edges with weight 0. Then the number of weakly connected components of $N^\infty$ is the greatest common divisor (GCD) $g$ of $D_L(l, l')$ for all $l \in L$ and $l' \in I(l)$ [18].

**Example 14** (Using connectivity to reduce complexity). Consider a network $N = (L, I, D_L)$ with the link set $L = \{l_1, l_2, l_3, l_4\}$, the collision sets

$I(l_1) = \{l_2, l_3\}$,

$I(l_2) = \{l_3, l_4\}$,

$I(l_3) = \{l_4\}$,

$I(l_4) = \emptyset$

and the link-wise propagation delay

$$D_L = \begin{bmatrix}
* & 1 & 2 & *
* & * & 1 & 5
* & * & * & 1
* & * & * & *
\end{bmatrix}.$$  

As $D_{N}^* = 5$, the rate region of $N$ can be characterized using the scheduling graph $(M_5, E_5)$ of $N$. Given a vertex assignment $b = [0, 1, 2, 3]$, we get a new network $N_b = (L, I, D_L^b)$, where

$$D_{L}^b = \begin{bmatrix}
* & 0 & 3 & *
* & * & 0 & 3
* & * & * & 0
* & * & * & *
\end{bmatrix}.$$
As the GCD of $D^b_2$ is 3, by Proposition 17 and 18, $R^N = R^{N_b} = R^{N'}$, where $N' = (L, I, D^b_2/3)$. The character $D^b_3$ is 1, and hence the scheduling graph $(M_1, E_1)$ of $N'$ can be used to determine the rate region of $N$.

Now we discuss network $N = (L, I, D_L)$ with a general collision model, where $N$ is a hypergraph. For $l \in L$, define

$$T'(l) = \bigcup_{\phi \in I(l)} \phi.$$ 

Let $T' = (T'(l), l \in L)$. Then $N' = (L, T', D_L)$ is a new network with a binary collision model. By [23], two vertices in $N'^\infty$ are weakly connected if and only if the two corresponding vertices in $(N')^\infty$ are weakly connected. Let $g_{N'}$ be the GCD of $D_L(l, l')$ for all $l \in L$ and $l' \in I(l)$. Similar with Proposition 18, we have $R^N = R^{(L, I, D_L/g_{N'})}$.

**APPENDIX B**

**PROOFS ABOUT BASIC RATE REGION PROPERTIES**

*Proof of Theorem 7* Fix $R \in R$ and $\epsilon > 0$. By Definition 2, there exists a schedule $S$ such that

$$R_S(l) \geq R(l) - \epsilon/2 \tag{12}$$

for every $l \in L$. Define a schedule $S'$ such that

$$S'(l, t) = \begin{cases} 
1 & S(l, t) = 1 \text{ and is collision free} \\
0 & \text{otherwise.}
\end{cases}$$

We see that $S'$ is collision free and $R_{S'} = R_S$.

By definition, there exists a sufficiently large $T_0$ such that for all $T \geq T_0$ and all $l \in L$ (which is finite),

$$\left| R_{S'}(l) - \frac{1}{T} \sum_{t=0}^{T-1} S'(l, t) \right| \leq \frac{\epsilon}{4} \tag{13}$$

Fix any $T^* \geq \max\{T_0 + D^*, 4D^*/\epsilon\}$. Define a schedule $S^*$ with period $T^*$:

$$S^*(l, t) = \begin{cases} 
S'(l, t) & t = 0, 1, \ldots, T^* - 1 - D^*, \\
0 & t = T^* - D^*, \ldots, T^* - 1.
\end{cases}$$

Now we argue that $S^*$ is collision free.

Consider $(l, t)$ with $S^*(l, t) = 1$. According to the definition of $S^*$, there exists $t_0 \in \{0, 1, \ldots, T^* - 1 - D^*\}$ such that $t = kT^* + t_0$. We show $S^*(l, t)$ is collision free by contradiction. Assume for $\phi \in I(l)$, $S^*(l', t + D_L(l, l')) = 1$ for every $l' \in \phi$, i.e., $S^*(l, t)$ has a collision.

Let $t' = t + D_L(l, l')$. As $|D_L(l, l')| \leq D^*$, we have $kT^* - D^* \leq t' \leq kT^* + T^* - 1$. We discuss the possible range of $t'$ in three cases:

- When $kT^* - D^* \leq t' < kT^*$, by the definition of $S^*$, $S^*(l', t') = 0$.
- Similarly, when $(k + 1)T^* - D^* \leq t' \leq (k + 1)T^* - 1$, $S^*(l', t') = 0$.

Therefore, to satisfy our assumption, we must have

$$kT^* \leq t' \leq (k + 1)T^* - 1 - D^*.$$
Let \( t' = kT^* + t'_0 \). Due to the periodical property of \( S^* \), \( S^*(l', t') = S^*(l', t'_0) \). As \( t'_0 \in \{0, 1, \ldots, T^* - 1 - D^*\} \), by the definition of \( S^* \), \( S^*(l', t'_0) = S'(l', t'_0) \). Similarly, we have \( S'(l, t_0) = S^*(l, t_0) = S^*(l, t) = 1 \). As \( S'(l, t_0) \) is collision free, for certain \( l' \in \phi \), \( S'(l', t'_0) = 0 \), i.e., \( S^*(l', t') = 0 \). We get a contradiction to the assumption that \( S^*(l, t) \) has a collision.

We further have

\[
R_{S^*}(l) = \frac{1}{T^*} \sum_{t=0}^{T^*-1} S^*(l, t) \\
= \left(1 - \frac{D^*}{T^*}\right) \frac{1}{T^* - D^*} \sum_{t=0}^{T^* - 1 - D^*} S'(l, t) \\
\geq (1 - D^*/T^*) (R_{S^*}(l) - \epsilon/4) \\
\geq R_{S^*}(l) - \epsilon/4 - D^*/T^* \\
\geq R_{S^*}(l) - \epsilon/2 \\
\geq R(l) - \epsilon,
\]

where the first inequality follows from \( T^* \geq T_0 + D^* \) and (13), the third inequality follows from \( T^* \geq 4D^*/\epsilon \), and the last inequality is obtained by substituting (12). The proof of the theorem is complete.

**Proof of Lemma 2.** Fix \( R_1 \) and \( R_2 \) in \( \mathbb{R}^N \). Let \( R = \alpha R_1 + (1 - \alpha) R_2 \) where \( 0 < \alpha < 1 \). The lemma is proved by showing \( R \in \mathbb{R}^N \). Fix \( \epsilon > 0 \). By Theorem 1, there exists a collision-free schedule \( S_1 \) of period \( T_1 \) such that \( R_{S_1} \succ R_1 - \epsilon/2 \), and a collision-free schedule \( S_2 \) of period \( T_2 \) such that \( R_{S_2} \succ R_2 - \epsilon/2 \).

For an integer \( k_1 \), let \( k_2 = \lceil \frac{1 - \alpha}{\epsilon/2} \rfloor k_1 \). Construct a schedule \( S \) of period \( k_1T_1 + k_2T_2 + 2D^* \) such that \( S(l, t) = S_1(l, t) \) for \( t \in \{0, 1, \ldots, k_1T_1 - 1\} \), \( S(l, t) = S_2(l, t-k_1T_1-D^*) \) for \( t \in k_1T_1 + D^* + \{0, 1, \ldots, k_2T_2 - 1\} \), and \( S(l, t) = 0 \) for other values of \( t \) in the first period. Similar to the proof of Theorem 1, we can argue that the schedule \( S \) is collision free. The rate vector \( R_S \) satisfies

\[
R_S = \frac{k_1T_1R_{S_1} + k_2T_2R_{S_2}}{k_1T_1 + k_2T_2 + 2D^*} \\
\geq \frac{k_1T_1R_{S_1} + \frac{1 - \alpha}{\epsilon/2} k_1R_{S_2}}{k_1T_1 + \frac{1 - \alpha}{\epsilon/2} T_1k_1 + T_2 + 2D^*} \\
= \alpha R_{S_1} + (1 - \alpha) R_{S_2} \\
= \frac{1 + \alpha(T_2 + 2D^*)/(T_1k_1)}{R - \epsilon/2} \\
\geq \frac{R - \frac{\alpha(T_2 + 2D^*)/(T_1k_1)}{1 + \alpha(T_2 + 2D^*)/(T_1k_1)} + \epsilon/2}{1 + \alpha(T_2 + 2D^*)/(T_1k_1)}.
\]

Therefore, when \( k_1 \) is sufficiently large, \( R_S \succ R - \epsilon \), and hence \( R \in \mathbb{R}^N \). The proof is completed.

**Proof of Theorem 2.** Fix an integer \( T \geq 1 \). For any independent set \( A \) of \( N^T \), we can define a schedule \( S \) with period \( T + D^* \):

\[
S(l, t) = \begin{cases} 
A(l, t) & t = 0, 1, \ldots, T - 1, \\
0 & t = T, \ldots, T + D^* - 1.
\end{cases}
\]
The schedule $S$ has the rate vector $T/(T + D^*)A1$. By the convexity of $\mathcal{R}^N$, we get $\frac{T}{T + D^*}\mathcal{R}^{NT} \subset \mathcal{R}^N$.

For any $R \in \mathcal{R}^N$ and $\epsilon > 0$, by Theorem $[\text{1}]$ there exists a collision-free schedule $S$ such that

$$R_S(l) \geq R(l) - \epsilon/2$$

for every $l \in \mathcal{L}$. Similar with the proof of Theorem $[\text{1}]$ we construct a periodic schedule $S^*$ as follows. There exists a sufficiently large $T_0$ such that for all $T \geq T_0$ and all $l \in \mathcal{L}$,

$$\left| R_S(l) - \frac{1}{T} \sum_{t=0}^{T-1} S(l, t) \right| \leq \frac{\epsilon}{4}.$$  

Fix any $T^* \geq \max\{T_0 + D^*, 4D^*/\epsilon\}$. Define a schedule $S^*$ with period $T^*$:

$$S^*(l, t) = \begin{cases} 
S(l, t) & t = 0, 1, \ldots, T^* - 1 - D^*, \\
0 & t = T^* - D^*, \ldots, T^* - 1.
\end{cases}$$

Same as the proof of Theorem $[\text{1}]$ we can argue that $S^*$ is collision free and $R(l) - \epsilon \leq R_S^*(l)$.

By definition, $R_{S^*}(l) = \frac{1}{T} \sum_{t=0}^{T^*-1-D^*} S(l, t)$. As $S$ is collision free, $S(l, t), l \in \mathcal{L}$ and $t \in \{0, 1, \ldots, T^* - 1 - D^*\}$ specify an independent set of $NT^* - D^*$. So $R_{S^*} \in \frac{T}{T + D^*}\mathcal{R}^{NT^* - D^*}$. Note that if $R \in \mathcal{R}^{NT^*}$, any $R'$ with $R'(l) \leq R(l)$ is also in $\mathcal{R}^{NT^*}$. Hence, for any $\epsilon > 0$, $R - \epsilon \in \cup_{T=1,2,\ldots} \frac{T}{T + D^*}\mathcal{R}^{NT^*}$. Therefore, $R$ is in the closure of $\cup_{T=1,2,\ldots} \frac{T}{T + D^*}\mathcal{R}^{NT^*}$.

\section*{Appendix C}
\textbf{Proofs about Framed Scheduling}

\textbf{Proof of Lemma $[\text{2}]$} Suppose link $l$ is active in frame $k$, i.e., $S(l, kTF_i + i) = 1$ for $i = 0, 1, \ldots, TF - D^* - 1$. To prove the sufficient condition, suppose all $l' \in \mathcal{I}(l)$ are inactive in frame $k$. For any $l' \in \mathcal{I}(l)$ and $t \in kTF + \{0, 1, \ldots, TF - D^* - 1\}$, let $t' = t + D_C(l, l')$. If $S(l', t') = 0$ for all $l' \in \mathcal{I}(l)$, $S(l, t)$ is collision free. As $|D_C(l, l')| \leq D^*$, we have $t' \in [t - D^*, t + D^*) \subset [kTF - D^*, kTF + TF - 1]$. Similar with the proof of Theorem $[\text{1}]$ we can discuss this range of $t'$ in three disjoint parts:

- When $kTF - D^* \leq t' \leq kTF$, by the definition of the framed schedule $S$, $S(l', t') = 0$.
- When $(k + 1)TF - D^* \leq t' \leq (k + 1)TF - 1$, by the definition of the framed schedule $S$, $S(l', t') = 0$.
- When $kTF \leq t' < (k + 1)TF - D^* - 1$, by the condition of the claim, $S(l', t') = 0$.

Therefore $S(l, t)$ is collision free for $t \in kTF + \{0, 1, \ldots, TF - D^* - 1\}$.

To prove the necessary condition, consider certain $l' \in \mathcal{I}(l)$ is active in the $k$th frame. As $|D_C(l, l')| \leq D^*$, there exists $t_0 \in \{0, 1, \ldots, D^*\}$ such that $t_0 + D_C(l, l') \in \{0, 1, \ldots, D^*\}$. As $TF \geq 2D^* + 1$, $S(l, kTF + t_0) = 1$ and $S(l', kTF + t_0 + D_C(l, l')) = 1$, and hence $S(l, kTF + D^*)$ has a collision. The proof is complete.

\textbf{Proof of Theorem $[\text{3}]$} Let $S$ be a collision-free, framed schedule with a rate vector $R_S$. Define $r_k = (r_k(l), l \in \mathcal{L})$ as

$$r_k(l) = \frac{1}{TF - D^*} \sum_{i=0}^{TF-D^*-1} S(l, kTF_i + i).$$
By Lemma 4, all the active links in a frame form an independent set, and hence \( r_k \in \mathcal{R}^{(L,T,0)} \) for any \( k = 0, 1, \ldots \).

As the rate vector exists, we have

\[
R_S = \frac{T_F - D^*}{T_F} \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} r_j,
\]

where the latter is in \((1 - D^*/T_F)\mathcal{R}^{(L,T,0)}\).

Consider vector \( R \in \mathcal{R}^{(L,T,0)} \). We know that \( R = \sum_{i=1}^{n} \alpha_i r_i \), where \( r_i \) is the indicator vector of the \( i \)th independent set of \( L \), \( n \) is the number of independent sets of \( L \), \( \alpha_i \geq 0 \) and \( \sum_i \alpha_i = 1 \). For any \( \epsilon > 0 \) and integer \( n > 0 \), define a framed schedule of frame length \( T_F \) and period \( nT_F \): in each period, we have \([\alpha_i n] \) frames with active links in the \( i \)th independent set, and no active links in other frames. By Lemma 4, \( S \) is collision free. So we have

\[
R_S = \frac{T_F - D^*}{nT_F} \sum_{i=1}^{n} \alpha_i n |r_i|
\]

\[
\geq \frac{T_F - D^*}{nT_F} \sum_{i=1}^{n} (\alpha_i n - 1) r_i
\]

\[
= R - \frac{D^*}{T_F} - \frac{T_F - D^*}{nT_F} \sum_{i=1}^{n} r_i
\]

\[
\geq R - \epsilon
\]

when \( T_F \) and \( n \) are sufficiently large. So \( R \) is achievable by framed scheduling.

**Proof of Lemma 4** Suppose link \( l \) is active in frame \( k \), i.e., \( S(l, kT_F + i) = 1 \) for \( i = 0, 1, \ldots, T_F - D^* - 1 \).

To prove the sufficient condition, consider for all \( \phi \in \mathcal{I}(l) \), certain \( l' \in \phi \) is inactive in frame \( k \). Fix a \( \phi \in \mathcal{I}(l) \) and an \( l' \in \phi \) that is inactive in frame \( k \). Let \( t' = t + D_{\mathcal{L}}(l, l') \). Similar as the proof of Lemma 4, we can discuss the range of \( t' \) in three cases and get the conclusion that \( S(l', t') = 0 \). Therefore \( S(l, t) \) is collision free for \( t \in kT_F + [0, 1, \ldots, T_F - D^* - 1] \).

To prove the necessary condition, consider for certain \( \phi \in \mathcal{I}(l) \), all \( l' \in \phi \) are active in the \( k \)th frame. As \(|D_{\mathcal{L}}(l, l')| \leq D^* \) for all \( l' \in \phi \), we have \( D^* + D_{\mathcal{L}}(l, l') \in \{0, 1, \ldots, 2D^*\} \) for all \( l' \in \phi \). As \( T_F \geq 3D^* + 1 \), \( S(l, kT_F + D^*) = 1 \) and for all \( l' \in \phi \), \( S(l', kT_F + D^* + D_{\mathcal{L}}(l, l')) = 1 \), \( S(l, kT_F + D^*) \) has a collision. The proof is completed.

**Appendix D**

**Proofs about Scheduling Graphs**

Here are some proofs about scheduling graphs.

**Proof of Theorem 2** Suppose a schedule \( S \) is collision free, fix an integer \( T \geq 1 \). To show that \((S[T,k], k = 0, 1, \ldots)\) is a path in \((\mathcal{M}_T, \mathcal{E}_T)\), we need to verify that for \( k = 0, 1, \ldots \),

- \( S[T,k] \in \mathcal{M}_T \), and
- \( (S[T,k], S[T,k+1]) \in \mathcal{E}_T \).

Let \( S_k \) be a schedule with \( S_k(l, t) = S(l, t + kT) \). We see that \( S_k \) is a collision-free schedule. As \( S[T, k] = S_k[T, 0] \), we have \( S[T, k] \in \mathcal{M}_T \). Moreover, as \( S[T, k+1] = S_k[T, 1] \), we have \((S[T,k], S[T,k+1]) \in \mathcal{E}_T \), which completes the proof.
Proof of Theorem \[\text{Consider a schedule } S \text{ and fix an integer } T \geq D^*, \text{ such that the sequence } (S[T, k], k = 0, 1, \ldots) \text{ forms a path in } (M_T, \mathcal{E}_T). \text{ Fix } l \in \mathcal{L} \text{ and } t = kT + t_0 \text{ where } 0 \leq t_0 < T \text{ such that } S(l, t) = 1. \text{ For any } l' \in \mathcal{I}(l), \text{ by the definition of } D^* \text{ in } [\text{4}], \text{ we have:}
\]
\[
t' = D_{\mathcal{L}}(l, l') + t \\
\in [t - D^*, t + D^*] \\
\subset [t - T, t + T] \\
= [(k - 1)T + t_0, (k + 1)T + t_0].
\]

Note that \( S(l, t) \) has a collision if \( S(l', t') = 1 \). Consider three cases of \( t' \):

- **Case 1:** \( t' \in [kT, (k + 1)T - 1] \). Note that \( t \in [kT, (k + 1)T - 1] \) too. As \( S[T, k] \in \mathcal{M}_T \), we have \( S[T, k] = S'[T, 0] \) for certain collision-free schedule \( S' \). Therefore, as \( S(l, t) = S'(l, t_0) = 1 \), \( S(l', t') = S'(l', t_0 + D_{\mathcal{L}}(l, l')) = 0 \).

- **Case 2:** \( t' \in [(k - 1)T, kT - 1] \). Note that \( t \in [(k - 1)T, kT - 1] \) too. As \( (S[T, k - 1], S[T, k]) \in \mathcal{E}_T \), we have \( S[T, k - 1] = S'[T, 0] \) and \( S[T, k] = S'[T, 1] \) for certain collision-free schedule \( S' \). Therefore, as \( S(l, t) = S'(l, t_0 + T) = 1 \), \( S(l', t') = S'(l', t_0 + T + D_{\mathcal{L}}(l, l')) = 0 \).

- **Case 3:** \( t' \in [(k + 1)T, (k + 2)T - 1] \). Note that \( t \in [(k + 1)T, (k + 2)T - 1] \) too. As \( (S[T, k], S[T, k + 1]) \in \mathcal{E}_T \), we have \( S[T, k] = S'[T, 0] \) and \( S[T, k + 1] = S'[T, 1] \) for certain collision-free schedule \( S' \). Similar with the above case, \( S(l', t') = 0 \).

For all the above three cases, \( S(l', t') = 0 \). As the above analysis applies to any \( l' \in \mathcal{I}(l) \), we have \( S(l, t) \) is collision free.

\[\text{Proof of Theorem } [\text{5}] \text{ Consider a schedule } S \text{ and fix an integer } T \geq 2D^*, \text{ such that the sequence } (S[T, k], k = 0, 1, \ldots) \text{ forms a path in } (M_T, \mathcal{E}_T). \text{ Fix } l \in \mathcal{L} \text{ and } t = kT + t_0 \text{ where } 0 \leq t_0 < T \text{ such that } S(l, t) = 1. \text{ For certain } \phi \in \mathcal{I}(l), \text{ we will show that it is impossible that } S(l', t + D_{\mathcal{L}}(l, l')) = 1 \text{ for all } l' \in \phi.

For any \( l' \in \phi \),
\[
t' = D_{\mathcal{L}}(l, l') + t \\
\in [t - D^*, t + D^*] \\
\subset [t - T/2, t + T/2] \\
= [kT + t_0 - T/2, kT + t_0 + T/2].
\]

Here we assume \( T/2 \) is an integer. Otherwise, we may take \( [T/2] \). Consider two cases of \( t_0 \):

- **Case 1:** \( t_0 < T/2 \). For any \( l' \in \mathcal{I}(l) \), \( t' \in [kT - T, kT + T - 1] \). As \( (S[T, k - 1], S[T, k]) \in \mathcal{E}_T \), we have \( S[T, k - 1] = S'[T, 0] \) and \( S[T, k] = S'[T, 1] \) for certain collision-free schedule \( S' \). Therefore, as \( S(l, t) = S'(l, t_0 + T) = 1 \), we have \( S(l', t') = S'(l', t_0 + T + D_{\mathcal{L}}(l, l')) = 0 \) for certain \( l' \in \phi \).

- **Case 2:** \( T/2 \leq t_0 \leq T \). For any \( l' \in \mathcal{I}(l) \), \( t' \in [kT, kT + 2T - 1] \). As \( (S[T, k - 1], S[T, k]) \in \mathcal{E}_T \), we have \( S[T, k] = S'[T, 0] \) and \( S[T, k + 1] = S'[T, 1] \) for certain collision-free schedule \( S' \). Therefore, as \( S(l, t) = S'(l, t_0) = 1 \), we have \( S(l', t') = S'(l', t_0 + D_{\mathcal{L}}(l, l')) = 0 \) for certain \( l' \in \phi \).
For both cases, \( S(l', t') = 0 \) for certain \( l' \in \phi \). Therefore, we have \((l, t)\) is collision free for \( S \).

We prove a lemma that will be used later.

**Lemma 19.** For any integer \( T > 0 \), consider a closed path \((A_0, A_1, \ldots, A_k)\) in the scheduling graph \((M_T, E_T)\). Let \( a_{i:T+j} \) be the \( j \)th column of \( A_i \), where \( j = 0, 1, \ldots, T - 1 \) and \( i = 0, 1, \ldots, k \). Then \((a_0, a_1, \ldots, a_{kT})\) is a closed path in \((M_1, E_1)\).

**Proof.** To show that \((a_0, a_1, \ldots, a_{kT})\) is a closed path in \((M_1, E_1)\), we only need to verify that \( a_i \in M_1 \), \((a_i, a_{i+1}) \in E_1 \) and \( a_0 = a_{kT} \). For each \( a_i \), it is a column of \( A_i \) where \( A_i = S[T, 0] \) for a collision-free schedule \( S \), then \( a_i \) must be an \(|L| \times 1\) binary matrix in a collision-free schedule \( S \), and hence in \( M_1 \). Similarly, \((a_i, a_{i+1}) \in E_1 \) because consecutive columns in a collision-free schedule \( S \) fits definition of \((M_1, E_1)\).

**Proof of Lemma 8** By Theorem 6, \((S[T, i], i = 0, 1, \ldots)\) forms a path in \((M_T, E_T)\). As \( KT \) is also a period of \( S \), \( S[T, i] = S[T, i + K] \). Therefore, the path \((S[T, i], i = 1, 2, \ldots)\) has period \( K \) and hence \((S[T, i], i = 0, 1, \ldots, K)\) is a closed path in \((M_T, E_T)\).

A closed path can be decomposed into a sequence of (not necessarily distinct) cycles (see, e.g., 19). Suppose \((S[T, i], i = 0, 1, 2, \ldots, K)\) has the decomposition of cycles \( C_1, \ldots, C_K \) in cycle \((M_T, E_T)\), where \( C_i \) is of length \( k_i \). According the decomposition of the closed path,

\[
R_S = \frac{1}{KT} \sum_{k=0}^{K-1} S[T, i]1
\]

\[
= \frac{1}{KT} \sum_{i=1}^{K'} k_i R_{C_i}
\]

\[
\in R^{(M_T, E_T)}.
\]

So we have \( R_S \in R^{(M_T, E_T)} \).

**Proof of Theorem 9** To show \( R \in R^{(M_T, E_T)} \), consider \( R \in R^N \). By Theorem 1 for any \( \epsilon > 0 \), there exists a collision-free, periodic schedule \( S \) such that \( R_S \triangleright R - \epsilon \). By Lemma 8 \( R_S \in R^{(M_T, E_T)} \). As \( R^{(M_T, E_T)} \) is closed, we have \( R \in R^{(M_T, E_T)} \).

**Proof of Lemma 10** Fix any integer \( i = ak + b \) where \( a \geq 0 \) and \( 0 \leq b \leq k - 1 \). As \( S[T, i] = S[T, b] = A_b \in M_T \) and \((S[T, i], S[T, i + 1]) = (S[T, b], S[T, b + 1]) = (A_b, A_{b+1}) \in E_T \), we have the sequence \((S[T, i], i = 1, 2, \ldots)\) forming a path in \((M_T, E_T)\). According to Theorem 7 \( S \) is collision free.

**Proof of Theorem 11** When \( T \geq D^* \), to show \( R^{(M_T, E_T)} \subset R^N \), fix \( R \in R^{(M_T, E_T)} \). We can write

\[
R = \sum_{C \in \text{cycle}(M_T, E_T)} \alpha_C R_C,
\]

where \( \alpha_C \geq 0 \) and \( \sum_{C \in \text{cycle}(M_T, E_T)} \alpha_C = 1 \). For a cycle \( C = (C_0, C_1, \ldots, C_k) \) in \((M_T, E_T)\), we define a schedule \( S \) with period \( kT \) such that \( S[T, i] = C_i \) for \( i = 0, 1, \ldots, k - 1 \). By Lemma 10 \( S \) is collision free and hence \( R_C = R_S \in R^N \). As \( R^N \) is convex (see Lemma 2), we have \( R \in R^N \).
Proof of Theorem 12. For any 1-cycle (C₀, C₀) in (M₁, E₁), define schedule S with period 1 and S[1, i] = C₀. As \( R_{(C₀, C₀)} = R_S \in \mathcal{R}(\mathcal{L}, \mathcal{I}, 0) \), we have
\[
\text{conv}\{\{R_C : C \text{ is a 1-cycle in } (M₁, E₁)\}\} \subset \mathcal{R}(\mathcal{L}, \mathcal{I}, 0).
\]

On the other hand, for any independent set of (\mathcal{L}, \mathcal{I}, 0), the indicator vector can form a 1-cycle in (M₁, E₁). Therefore,
\[
\mathcal{R}(\mathcal{L}, \mathcal{I}, 0) \subset \text{conv}\{\{R_C : C \text{ is a 1-cycle in } (M₁, E₁)\}\}.
\]

By Lemma 19 any cycle in (M₆, E₆) has a corresponding closed path in (M₁, E₁) of the same rate. Therefore, by Theorem 9 \( R^{N} \subset \mathcal{R}(\mathcal{M₆}, \mathcal{E₆}) \subset \mathcal{R}(\mathcal{M₁}, \mathcal{E₁}) \). As framed schedules also have rates in \( R \), we have by Theorem 5
\[
\mathcal{R}(\mathcal{L}, \mathcal{I}, 0) \subset R^{N}.
\]

**APPENDIX E**

**Proofs about Cycles in Scheduling Graph**

Proof of Theorem 12. To simplify the notation, let \( C^* = \text{cycle}'(\mathcal{M₆}, \mathcal{E₆}) \). First, if \( R \in \mathcal{R}(\mathcal{M₆}, \mathcal{E₆}) \), then \( R = \sum_{C \in \text{cycle}(\mathcal{M₆}, \mathcal{E₆})} \alpha_C R_C \), where \( \alpha_C \geq 0 \) and \( \sum_{C \in \text{cycle}(\mathcal{M₆}, \mathcal{E₆})} \alpha_C = 1 \). For each \( C \in \text{cycle}(\mathcal{M₆}, \mathcal{E₆}) \), there exists \( C' \in C^* \) such that \( C' \succ C \), then \( R_C \succ R_{C'} \). Therefore, \( R = \sum_{C \in \text{cycle}(\mathcal{M₆}, \mathcal{E₆})} \alpha_C R_C \leq \sum_{C \in \text{cycle}(\mathcal{M₆}, \mathcal{E₆})} \alpha_C R_{C'} \in \text{conv}\{\{R_C : C \in C^*\}\} \).

Second, suppose \( R \not\leq R' \in \text{conv}\{\{R_C : C \in C^*\}\} \) and \( R' = \sum_{C' \in C^*} \alpha_{C'} R_{C'} \), where \( \alpha_{C'} \geq 0 \) and \( \sum_{C' \in C^*} \alpha_{C'} = 1 \). Assume \( R(l) < R'(l) \) for certain \( l \in \mathcal{L} \). For each \( C' \in C^* \), construct cycle \( C \) by setting the entries of the matrices in \( C' \) indexed by \( l \) to zero. Let \( \alpha_l = \frac{R(l)}{R'(l)} \) and
\[
R'' = \alpha_l \sum_{C' \in C^*} \alpha_{C'} R_{C'} + (1 - \alpha_l) \sum_{C' \in C^*} \alpha_{C'} R_{C'}.
\]

We can check that \( R'' \in \mathcal{R}(\mathcal{M₆}, \mathcal{E₆}) \), \( R''(l) = R(l) \) and \( R''(l') = R'(l') \) for \( l' \neq l \). By repeating the above procedure for other links \( l' \) with \( R''(l') > R(l') \), we can get \( R'' = R \in \mathcal{R}(\mathcal{M₆}, \mathcal{E₆}) \).

Proof of Lemma 15. First, the lemma is true for \( P_1^* \) as \( P_1^* = \mathcal{E}^* \). Assume the lemma holds for \( P_{k-1}^* \) and \( G_{k-1} \), where \( k \geq 2 \). Consider \( (A₀, A₁, \ldots, Aₖ) \in Pₖ^* \). There exist \( B_k \) and \( B'_k \) such that \( (A₀, \ldots, A_{k-2}, B_{k-1}) \in P_{k-1}^* \), \( (B'_{k-1}, A_k) \in \mathcal{E}^* \) and \( (B_{k-1}, A_k) \in \mathcal{E} \). By the induction hypothesis, \( A_i \in \mathcal{V}_i \) for \( i = 0, \ldots, k-2 \) form a path in \( G_{k-1} \), and \( (A_{k-2}, B_{k-1}) \in U_{k-2} \). Therefore, \( (A_{k-2}, A_{k-1}, A_k) \in \mathcal{F}_k \) defined in (10). As \( (A_{k-2}, A_{k-1}, A_k) \) is a maximal path, we have \( (A_{k-2}, A_{k-1}, A_k) \in \mathcal{F}_k^* \) defined in (11). Hence \( (A₀, A₁, \ldots, Aₖ) \) is a path of length \( k \) in \( G_k \).

Proof of Lemma 16. We first show that
\[
\tilde{U}' = U' \triangleq \{(B, C) : (A, B, C) \in \tilde{F} \text{ for certain } A\}. \tag{14}
\]

On one direction, for any \( (B, C) \in \mathcal{E}^* \), there exists \( B' \not\leq B \) such that \( (B', C) \in \tilde{U}' \). Hence, if \( (A, B, C) \in \mathcal{F}_2 \), then there exists \( A' \not\leq A \) such that \( (A', B, C) \in \tilde{F} \). Therefore, \( \tilde{U}' \subset U' \). On another direction, for any \( (B', C) \in \tilde{U}' \), there exists \( B \succ B' \) such that \( (B, C) \in \mathcal{E}^* \). Hence, if \( (A', B, C) \in \tilde{F} \), there exists \( A \succ A' \) such that \( (A, B, C) \in \mathcal{F}_2 \). Therefore, \( \tilde{U}' \supset U' \).
Now we prove $U'_i \subset \tilde{U}'$ by induction. As $F'_2 \subset F_2$, we have $U'_1 \subset \tilde{U}'$. Assume $U'_i \subset \tilde{U}'$ for a certain $i \geq 1$. We write

$$U'_{i+1} = \{(B, C) : (A, B, C) \in F'_{i+2} \text{ for certain } A\}$$

$$\subset \{(B, C) : (A, B, C) \in F_{i+2} \text{ for certain } A\}$$

$$= \{(B, C) : (A, D) \in U'_i, (D', C) \in E^*, D \land D' = B\}$$

$$\subset \{(B, C) : (A, D) \in \tilde{U}', (D', C) \in E^*, D \land D' = B\}$$

$$= \tilde{U},$$

where the second inclusion follows from the induction hypothesis, and the last equality follows from (14).

To prove $U_i \subset \tilde{U}$, we write

$$U_i = \{(A, B) : (A, B, C) \in F'_{i+2} \text{ for certain } C\}$$

$$\subset \{(A, B) : (A, B, C) \in F_{i+2} \text{ for certain } C\}$$

$$= \{(A, B) : (A, D) \in U'_i, (D', C) \in E^*, D \land D' = B\}$$

$$\subset \{(A, B) : (A, D) \in \tilde{U}', (D', C) \in E^*, D \land D' = B\}$$

$$= \tilde{U}.$$

The proof is complete.

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