Neutrino flux ratios at neutrino telescopes: The role of uncertainties of neutrino mixing parameters and applications to neutrino decay

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Abstract

In this paper, we derive simple and general perturbative formulas for the flavor flux ratios $R_{\alpha\beta} = \phi_{\nu_{\alpha}} / \phi_{\nu_{\beta}}$ that could be measured at neutrino telescopes. We discuss in detail the role of the uncertainties of the neutrino mixing parameters showing that they have to be seriously taken into account in any realistic discussion about flavor measurements at neutrino telescopes. In addition, we analyze the impact of such uncertainties in telling the standard neutrino oscillation framework from scenarios involving, e.g., neutrino decay and we find that the ratio $R_{e\mu}$ is the most sensitive one to “new physics” effects beyond the Standard Model. We also compute the more realistic muon-to-shower ratio for a particular configuration of the IceCube experiment, observing that using this experimental quantity a clear separation between standard and non-standard neutrino physics cannot be obtained.

1 Introduction

The standard framework of neutrino oscillations is a successful description of data from recent neutrino experiments. However, there might be subleading effects that are not covered by neutrino oscillations, since the experimental data are still impaired by rather large uncertainties. Such subleading effects could be described by introducing an extended neutrino oscillation framework including so-called damping effects that could stem from, e.g., neutrino decay or neutrino decoherence. In this work, we will introduce damping factors, describing the damping effects, in a phenomenological way as additional factors to the ordinary terms in the formulas for the neutrino oscillation probabilities. A promising situation to look for such subleading effects would be from neutrino sources at large distances, since in this case the result of the damping factors will be most visible because of averaging. In addition, it could be an alternative way to measure the fundamental neutrino parameters, but not the neutrino mass-squared difference, since the averaged neutrino oscillation

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probabilities do not depend on them. Therefore, we will investigate neutrino oscillations with damping effects in general, and different scenarios in which damping effects arise from neutrino decay in particular. We will study these scenarios in the case of neutrinos coming from large distances. The most plausible way to measure such neutrinos would be at neutrino telescopes in ice [1] or water [2]. Many papers in the literature have been devoted to study the dependence of the neutrino fluxes on the neutrino mixing parameters, namely the three leptonic mixing angles $\theta_{12}$, $\theta_{13}$, and $\theta_{23}$ as well as the CP-violating phase $\delta$ (for an incomplete list of references, see Refs. [3–9]). The usual starting point is that neutrinos are produced via decays of pions and kaons created by hadronic ($i.e.$, $pp$ collisions) and photodisintegration ($p\gamma$) interactions [10], leading to the well-known flux ratios at the source $\phi_{\nu_e}^0 : \phi_{\nu_\mu}^0 : \phi_{\nu_\tau}^0 = 1 : 2 : 0$. However, other effects, like muon energy loss in strong magnetic fields [11, 12] can noticeably alter the flavor composition; in addition, a different flux ratio at the source can be obtained for the so-called “neutron beam sources” in which neutrinos are produced from neutron decays [13, 14]. In any case, on their way from the source to the Earth, neutrinos oscillate and the fluxes arriving at the detector acquire a dependence on the neutrino mixing parameters which can, in principle, be used to improve our knowledge on the fundamental neutrino parameters and/or used in connection with reactor experiments and neutrino beams to better the determination of $\delta$ and the neutrino mass hierarchy. However, it could happen at the time when neutrino telescopes will be operational that the uncertainties of the neutrino mixing parameters could be large enough not to be neglected in theoretical estimates of the arriving fluxes [15]. This implies that any possible dependence on the neutrino mixing parameters can be completely hidden, and what is also important, that any small deviation from the expected number of neutrinos can simply be the effect of our ignorance about the precise values of the neutrino mixing parameters and not due to any “new physics” effects in large distance neutrino oscillations, as those given, $e.g.$, by neutrino decay [16, 17], breakdown of fundamental symmetries [14, 18–20], or pseudo-Dirac nature of neutrinos [21, 22].

For this reason, in this work, we analyze in some detail the impact of the current neutrino mixing parameter errors on the determination of the fluxes arriving at neutrino telescopes. We explicitly illustrate the dependence on the deviation from the best-fit values of $\theta_{12}, \theta_{13}$, and $\theta_{23}$ up to second order, in the case that the fluxes at the source are in the ratios $\phi_{\nu_e}^0 : \phi_{\nu_\mu}^0 : \phi_{\nu_\tau}^0 = 1 : 2 : 0$. For comparison, we also show how the fluxes on Earth change if neutrino decay takes place and we discuss whether it is possible to distinguish them from the fluxes computed in the standard neutrino oscillation framework, and to what extent.

Our paper is organized as follows: In Sec. 2 we derive the averaged neutrino oscillation probabilities including damping factors. Next, in Sec. 3 we analyze the three-flavor flux ratios and study the impact of the uncertainties of the fundamental neutrino parameters. Then, in Sec. 4 we investigate the flux ratios with neutrino decay, present the different neutrino decay scenarios, as well as we discuss the possibility to detect the subleading effects at neutrino telescopes. Finally, in Sec. 5 we summarize our work as well as we present our conclusions.
2 Derivation of averaged neutrino oscillation probabilities including damping factors

In general, the neutrino oscillation probabilities in the standard three-flavor framework have the following form:

$$P_{\alpha\beta} = \sum_{i=1}^{3} \sum_{j=1}^{3} J_{i\alpha}^{ij} \exp(-i\Phi_{ij}),$$  \hspace{1cm} (1)$$

where $J_{i\alpha}^{ij} = U_{\alpha j}U_{\beta i}^* U_{\alpha i} U_{\beta i}$ and $\Phi_{ij} = \Delta m_{ij}^2 L / (2E)$. Here $U$ is the leptonic mixing matrix, $\Delta m_{ij}^2 = m_i^2 - m_j^2$ are the neutrino mass-squared differences, $L$ is the baseline length, and $E$ is the neutrino energy. The standard parameterization of the leptonic mixing matrix is given by

$$U = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\
-s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & s_{13}s_{23} \\
-s_{13}s_{23}c_{12} - c_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23}c_{12} - s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23}
\end{pmatrix},$$ \hspace{1cm} (2)

where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$, and $\delta_{CP}$ is the Dirac CP-violating phase. In the case that the ratio $L/E$ is large, i.e., $L/E \gg (2\Delta m_{ij}^2)^{-1}$, one obtains in vacuum\(^1\) the following averaged neutrino oscillation probabilities

$$\langle P_{\alpha\beta} \rangle = \sum_{i=1}^{3} |J_{i\alpha}^{i\beta}| = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2.$$ \hspace{1cm} (3)$$

Note that these probabilities depend only on the parameters $J_{i\alpha}^{ij}$, where $i = j$, as well as they are independent of the oscillation frequencies, since these have been averaged out to zero for large values of the ratio $L/E$.

Now, introducing damping factors $D_{ij}$ in Eq. (1), the neutrino oscillation probabilities can be written as

$$P_{\alpha\beta} = \sum_{i=1}^{3} \sum_{j=1}^{3} D_{ij} J_{i\alpha}^{ij} \exp(-i\Phi_{ij}),$$

$$= \sum_{i=1}^{3} D_{ii} J_{i\alpha}^{i\beta} + 2 \sum_{1 \leq i, j \leq 3} D_{ij} |J_{i\alpha}^{ij}| \cos(\Phi_{ij} + \arg J_{i\alpha}^{ij}),$$ \hspace{1cm} (4)$$

where the damping factors are given by

$$D_{ij} = \exp\left(-\alpha_{ij} \frac{|\Delta m_{ij}^2| L^2}{E^2} \right)$$ \hspace{1cm} (5)$$

with $\alpha_{ij}$ being elements in a non-negative damping coefficient matrix. We will assume $\alpha_{ij}$ to be independent of energy. Note that both the oscillation frequencies $\Phi_{ij}$ and the damping factors $D_{ij}$ are functions of the baseline length $L$ and the neutrino energy $E$. The oscillation frequencies are functions of the ratio $L/E$ only, whereas the damping factors are more general functions of the parameters $L$ and $E$, i.e., $D_{ij} = D_{ij}(L/E^2)$.

Similarly, as in the averaging procedure above for the standard framework, one can perform an averaging of the neutrino oscillation probabilities including damping factors for large values of the ratio $L/E$, i.e., $L/E \gg (2\Delta m_{ij}^2)^{-1}$. Note that it would be unnatural

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\(^1\)Matter effects inside the source can affect the neutrino transition probabilities, see Ref. [23].
to include damping effects after one has performed averaging. Thus, in the case when \( \beta = \gamma \), the damping factors are functions of \((L/E)^\beta\) only, and therefore, the following averaging (with \( \ell = L/E \)) can be introduced:

\[
\langle P_{\alpha\beta} \rangle = \lim_{x \to \infty} \frac{\int_0^x P_{\alpha\beta}(\ell) \, d\ell}{\int_0^x \, d\ell} = \lim_{x \to \infty} \frac{1}{x} \int_0^x P_{\alpha\beta}(\ell) \, d\ell. \tag{6}
\]

In this work, we will not consider the case when \( \beta \neq \gamma \). Partly, because this case has less obvious applications. In general, for an arbitrary value of the parameter \( \beta \), it is not possible to compute the averaged neutrino oscillation probabilities including damping factors, but it is possible for the cases when \( \beta = 1 \) and \( \beta = 2 \), which are the cases that are important for applications to scenarios that could arise in Nature, see Ref. [24].

In the case when \( \beta = 1 \), we obtain

\[
\langle P_{\alpha\beta} \rangle = \lim_{x \to \infty} \sum_{i=1}^3 \sum_{j=1}^3 \langle D_{ij} \rangle(x) J_{ij}^{\alpha\beta}, \tag{7}
\]

where the effective damping factors are given by

\[
\langle D_{ij} \rangle(x) = \frac{1 - \exp\left(-\frac{i\Delta m_{ij}^2 x - \alpha_{ij} |\Delta m_{ij}^2| \xi x}{i\Delta m_{ij}^2 x + \alpha_{ij} |\Delta m_{ij}^2| \xi x}\right)}{1 + \exp\left(-\frac{i\Delta m_{ij}^2 x - \alpha_{ij} |\Delta m_{ij}^2| \xi x}{i\Delta m_{ij}^2 x + \alpha_{ij} |\Delta m_{ij}^2| \xi x}\right)}, \tag{8}
\]

which can be written in a more simpler form as

\[
\langle D_{ij} \rangle(x) = \sum_{n=0}^\infty \frac{1}{n+1} \frac{(-a_{ij} x)^n}{n!}, \tag{9}
\]

where

\[ a_{ij} = i \frac{\Delta m_{ij}^2}{2} + \alpha_{ij} |\Delta m_{ij}^2| \xi. \tag{10} \]

For \( \xi \neq 0 \), the parameters \( a_{ij} \) are in general non-zero if \( i \neq j \), and therefore, one finds that \( \lim_{x \to \infty} \langle D_{ij} \rangle(x) = 0 \), whereas \( a_{ij} = 0 \) if \( i = j \), which means that \( \langle D_{ii} \rangle(x) = 1 \), and hence, one regains the formula in Eq. (3). On the other hand, for \( \xi = 0 \), \( a_{ij} = \alpha_{ij} + i \frac{\Delta m_{ij}^2}{2} \) if \( i \neq j \), and especially, \( a_{ij} = \alpha_{ij} \) if \( i = j \). Thus, in the case that \( \xi = 0 \) and in the limit \( x \to \infty \), we find that \( \langle P_{\alpha\beta} \rangle = 0 \), since \( \lim_{x \to \infty} \langle D_{ij} \rangle(x) = 0 \) for all \( x \) if \( i = j \) and \( \lim_{x \to \infty} \langle D_{ij} \rangle(x) = 0 \) if \( i \neq j \) and \( \alpha_{ij} \neq 0 \). However, if \( \alpha_{ij} = 0 \) for fixed \( i = j \), then \( \lim_{x \to \infty} \langle D_{ii} \rangle(x) = 1 \). Thus, in the case that \( \xi = 0 \) and \( \alpha_{ii} = 0 \) for fixed \( i \), the averaged neutrino oscillation probabilities become

\[
\langle P_{\alpha\beta} \rangle = \langle J_{\alpha\beta}^{ii} \rangle = |U_{\alpha i}|^2 |U_{\beta i}|^2. \tag{11}
\]

In general, if \( \alpha_{ii} = 0 \) for more than one fixed \( i \), then one obtains

\[
\langle P_{\alpha\beta} \rangle = \sum_{i \in N} \langle J_{\alpha\beta}^{ii} \rangle = \sum_{i \in N} |U_{\alpha i}|^2 |U_{\beta i}|^2, \quad \text{(12)}
\]

where \( N \) can be any of the sets \{1, 2\}, \{1, 3\}, \{2, 3\}, and \{1, 2, 3\}. In the case when \( N = \{1, 2, 3\} \), one recovers the standard framework formula in Eq. (3).

\footnote{Note that averaging with respect to large distances can be obtained by simply assuming \( E \) to be a constant, \( i.e., \), by defining \( \ell = \frac{E}{L} \), where \( E \) is a constant.}
Similarly, in the case when $\beta = 2$, we can again write the formula for the averaged neutrino oscillation probabilities including damping factors as in Eq. (7), but now the effective damping factors are given by

$$
\langle D_{ij}(x) \rangle = \frac{1}{2k_{ij}} \sqrt{\pi} \exp \left[ -\frac{(\Delta m^2_{ij})^2}{16\alpha_{ij} k_{ij}^2} \right] 
\times \left[ \text{erf} \left( k_{ij} x + i \frac{\Delta m^2_{ij}}{4k_{ij}} \right) - i \text{erfi} \left( \frac{\Delta m^2_{ij}}{4k_{ij}} \right) \right],
$$

where $k_{ij} = \sqrt{\alpha_{ij} |\Delta m^2_{ij}| \xi}$. Note that the argument of the imaginary error function is independent of $x$. Again, the situation is the same as in the case when $\beta = 1$, which means that non-zero averaged neutrino oscillation probabilities can only be obtained for $\xi = 0$ and $\alpha_{ii} = 0$ for fixed $i$.

### 3 Neutrino flux ratios

In the cases $\beta = \gamma = 1$ and $\beta = \gamma = 2$, the averaged neutrino oscillation probabilities with damping factors included that are described in Eq. (12) can effectively be written as

$$
\langle P_{\alpha\beta} \rangle = \sum_{i=1}^{3} d_i J^i_{\alpha\beta},
$$

where $d_i$ are the normalized damping factors, which can have the value 0 or 1. Of course, if $d_i = 1$ for $i = 1, 2, 3$, then one finds the averaged neutrino oscillation probabilities in the standard framework, which are given in Eq. (3).

Starting from the flux ratios at the source $\phi^0_{\nu_e} : \phi^0_{\nu_\mu} : \phi^0_{\nu_\tau} = 1 : 2 : 0$, the neutrino fluxes arriving at a detector are sensitive to neutrino oscillation in vacuum and are then computed as

$$
\phi_{\nu_e} = \langle P_{ee} \rangle + 2 \langle P_{\mu e} \rangle,
\phi_{\nu_\mu} = \langle P_{e\mu} \rangle + 2 \langle P_{\mu\mu} \rangle,
\phi_{\nu_\tau} = \langle P_{e\tau} \rangle + 2 \langle P_{\mu\tau} \rangle,
$$

(15)

Then, the three-flavor flux ratios analyzed in this work are defined as follows:

$$
R_{e\mu} = \frac{\phi_{\nu_e}}{\phi_{\nu_\mu}}, \quad R_{e\tau} = \frac{\phi_{\nu_e}}{\phi_{\nu_\tau}}, \quad R_{\mu\tau} = \frac{\phi_{\nu_\mu}}{\phi_{\nu_\tau}}.
$$

(16)

Although the flux ratios are not independent of each other, since a measurement of two of them will give the value of the third, we prefer to discuss them separately.

### 3.1 Standard neutrino framework

In this subsection, we analyze the analytical form of three-flavor flux ratios on Earth. These ratios depend on three mixing angles and one CP-violating phase, as implied by Eq. (15), and are, in principle, sensitive to subleading effects (encoded in the dependence of $\theta_{13}$) and the errors of the mixing angles $\theta_{12}$ and $\theta_{23}$. For maximal mixing of $\theta_{23}$, i.e., $\theta_{23} = \pi/4$, and vanishing $\theta_{13}$, the robust prediction of Eq. (15) is $\phi_{\nu_e} : \phi_{\nu_\mu} : \phi_{\nu_\tau} = 1 : 1 : 1$ [3],
independently of $\theta_{12}$. Going beyond this zero-order approximation allows us not only to study the role of non-maximal $\theta_{23}$ and non-vanishing $\theta_{13}$, but also to understand the impact of introducing the uncertainties of the neutrino mixing parameters.

Since the exact formulas for the flux ratios are quite cumbersome, we prefer to present our results in some useful approximations. From experimental data, we know that $\theta_{13}$ is small and $\theta_{23}$ is close to maximal mixing. For that reason, we can expand the ratios in $\theta_{13}$ and $\theta_{23}$, parametrizing the deviation from maximal 23-mixing [25]. At the same time, we can also expand for small $\delta_{12} = \theta_{12} - \bar{\theta}_{12}$, $\bar{\theta}_{12}$ being the best-fit value for $\theta_{12}$. No restrictions have been applied on the CP-violating phase $\delta$, which means that the following formulas are valid to all orders in $\delta$. Up to second order in the small quantities $\theta_{13}$, $\delta_{12}$, and $\delta_{23}$, they read

\begin{align*}
R_{e\mu} &= 1 + \frac{3}{4} \cos(\delta) \sin(4\theta_{12}) \theta_{13} - \frac{3}{2} \sin^2(2\theta_{12}) \delta_{23} \\
&+ \frac{1}{8} \cos^2(\delta) [3 \cos(4\theta_{12}) - 5] \sin^2(2\theta_{12}) \theta_{13}^2 \\
&+ \frac{1}{32} [-28 \cos(4\theta_{12}) + 3 \cos(8\theta_{12}) - 103] \delta_{23}^2 \\
&+ 3 \cos(\delta) \cos(4\theta_{12}) \delta_{12} \theta_{13} - 3 \sin(4\theta_{12}) \delta_{12} \delta_{23} \\
&+ \frac{1}{16} \cos(\delta) [3 \sin(8\theta_{12}) - 22 \sin(4\theta_{12})] \theta_{13} \delta_{23}, \quad (17) \\
R_{e\tau} &= 1 + \frac{3}{4} \cos(\delta) \sin(4\theta_{12}) \theta_{13} - \frac{3}{2} \sin^2(2\theta_{12}) \delta_{23} \\
&+ \frac{1}{8} \cos^2(\delta) [3 \cos(4\theta_{12}) + 11] \sin^2(2\theta_{12}) \theta_{13}^2 \\
&+ \frac{1}{32} [4 \cos(4\theta_{12}) + 3 \cos(8\theta_{12}) + 121] \delta_{23}^2 \\
&+ 3 \cos(\delta) \cos(4\theta_{12}) \delta_{12} \theta_{13} - 3 \sin(4\theta_{12}) \delta_{12} \delta_{23} \\
&+ \frac{1}{16} \cos(\delta) [10 \sin(4\theta_{12}) + 3 \sin(8\theta_{12})] \theta_{13} \delta_{23}, \quad (18) \\
R_{\mu\tau} &= 1 \\
&+ 2 \cos^2(\delta) \sin^2(2\theta_{12}) \theta_{13}^2 + [\cos(4\theta_{12}) + 7] \delta_{23}^2 \\
&+ 2 \cos(\delta) \sin(4\theta_{12}) \theta_{13} \delta_{23}. \quad (19)
\end{align*}

Note that another series expansion up to first order with a different parametrization of the initial neutrino flux ratios for the neutrino oscillation probabilities has been presented in Ref. [9,26]. Now, some comments of the above formulas are in order. First, we discuss the flux ratio $R_{e\mu}$. In Eq. (17), we have clearly separated first- and second-order terms in the series expansion. The approximate relation is obviously much more accurate if referred to small values of $\theta_{13}$, $\delta_{12}$, and $\delta_{23}$. In order to check how good our series expansion is, we show in Fig. 1 the ratio between the approximate formulas (at first and second order, $R_{e\mu}^{\text{first}}$ and $R_{e\mu}^{\text{second}}$, respectively) and the exact formula for $R_{e\mu}$, as a function of the small and unknown mixing angle $\theta_{13}$. Furthermore, note that the accidental sum-rule $R_{e\mu} - R_{e\tau} + R_{\mu\tau} = 1$ holds up to second order in perturbation theory.

In order to fix some values for the fundamental neutrino parameters, we observe that
the following 99 % C.L. limits hold [27]

\[
\begin{array}{ll}
30^\circ & < \theta_{12} < 38^\circ \\
36^\circ & < \theta_{23} < 54^\circ \\
0 & < \theta_{13} < 10^\circ
\end{array}
\] (20)

with best-fit values corresponding to \( \tilde{\theta}_{12} = 33^\circ \), \( \tilde{\theta}_{23} = 45^\circ \), and \( \tilde{\theta}_{13} = 0 \). Thus, in order to be sensitive to \( \delta_{12} \) and \( \delta_{23} \), we choose \( \theta_{12} = 38^\circ \) and \( \theta_{23} = 50^\circ \) (corresponding to \( \delta_{12} = \delta_{23} = 5^\circ \)).

It can be clearly seen that the second-order approximation reproduces the exact values at the level of 1 % to 3 %, the worst case being obtained for larger values of the mixing angle \( \theta_{13} \). The larger discrepancy between the first-order approximation and the reference value is to be mainly ascribed to both the relatively large \( \delta_{23} \), which means that second-order terms in this quantity cannot be safely neglected, and the absence of terms of \( \mathcal{O}(\delta_{12}) \).

At first order in the small parameters, one important feature of this ratio is that it is independent of the error of \( \theta_{12} \), which only enter in higher-order terms coupled with the error of the mixing angle \( \theta_{23} \). This means that, once you have fixed the value of the mixing angle \( \theta_{12} \), its uncertainties are not relevant for the determination of \( R_{e\mu} \). Thus, this ratio mainly depends on \( \theta_{13} \) and the uncertainty of \( \theta_{23} \).

In order to estimate which of these parameters encodes the largest part of the uncertainty of \( R_{e\mu} \), we can evaluate the maximum spread due to the combination \( x = \cos(\delta) \theta_{13} \) (\( x = \theta_{23} \)) at fixed \( \theta_{23} \) (\( \cos(\delta) \theta_{13} \)), namely

\[
\Delta R_{e\mu} = \frac{R_{e\mu}(x_{\text{max}}) - R_{e\mu}(x_{\text{min}})}{R_{e\mu}(x = 0)},
\] (21)

\( x_{\text{max, min}} \) being the maximum and minimum value of the variable \( x \). It is simple to derive \( \Delta R_{e\mu} \) in the approximations used above. For \( x = \cos(\delta) \theta_{13} \), \( x_{\text{max, min}} = \pm \pi/18 \) [see Eq. (20)], and for vanishing \( \delta_{12} \) and \( \delta_{23} \), we simply obtain

\[
\Delta R_{e\mu} = \frac{\pi}{12} \sin^2(4\theta_{12}) \sim 20 \%,
\] (22)
Figure 2: The spread of the three flux ratios considered in this work due to the present uncertainties of the neutrino mixing parameters [see Eq. (20)], around the best-fit values $\theta_{12} = 33^\circ$ and $\theta_{23} = 45^\circ$, as a function of the mixing angle $\theta_{13}$. From above to below: $R_{e\tau}$ (red), $R_{\mu\tau}$ (blue), and $R_{e\mu}$ (grey).

for $\theta_{12}$ at the best-fit value. On the other hand, for $x = \delta_{23}$, $x_{\text{max,min}} = \pm 9\pi/180$, we obtain

$$\Delta R_{e\mu} = \frac{3\pi}{20} \sin^2(2\theta_{12}) \sim 45\,$$

which means that the error of $\theta_{23}$ contains the largest part of the uncertainty of $R_{e\mu}$ [28].

Then, we discuss the flux ratio $R_{e\tau}$. Due to the approximate $\mu - \tau$ symmetry [29–33] (for an incomplete list of references, see Ref. [9]), the flux ratio $R_{e\tau}$ shares similar characteristics as $R_{e\mu}$, as it can be seen in Eq. (18). In particular, the first-order approximations are exactly the same, whereas differences appear in terms of $O(\theta_{13}^2)$, $O(\delta_{23}^2)$, and $O(\theta_{13}\delta_{23})$.

Finally, we discuss the flux ratio $R_{\mu\tau}$. This flux ratio is very peculiar. At first order in small parameters, it is exactly 1 and the deviation from 1 is only present when second-order terms are included. However, no dependence on the uncertainty $\delta_{12}$ is present at least up to $O(\delta_{12}^2)$, which means that $R_{\mu\tau}$ is mainly sensitive to the central value of the mixing angle $\theta_{12}$, but, once fixed, the uncertainty of it does not affect the flux ratio. What is also an important feature, $R_{\mu\tau}$ is always larger than 1, as it can be seen considering that the terms in Eq. (19) under the conditions in Eq. (20) cannot give negative corrections to 1. This feature is not lost in the exact evaluation, as we have numerically checked. In summary, the flux ratio $R_{\mu\tau}$ is very close to 1, even if the uncertainties of the neutrino mixing parameters are taken into account, since the deviation from its standard value is an effect of second order in small quantities. For that reason, if some “new physics” mechanism would be able to produce a large deviation from 1, the $R_{\mu\tau}$ flux ratio could give us an interesting possibility to study such a “new physics” scenario.

In order to summarize the previous discussion about uncertainties, we have evaluated the whole spread of the flux ratios $R_{e\mu}$, $R_{e\tau}$, and $R_{\mu\tau}$ due to the present lack of knowledge of the neutrino mixing parameters. We have computed the minimum and maximum values of each $R_{\alpha\beta}$ varying $\theta_{12}$, $\theta_{23}$, and $\delta$ in the 99 % C.L. of Eq. (20) and presented the results in Fig. 2 as a function of $\theta_{13}$. From above to below, colored areas contain the possible
values for $R_{e\tau}$ (red), $R_{\mu\tau}$ (blue), and $R_{e\mu}$ (grey), respectively. As it can be easily seen, $R_{e\mu}$ can be as large (small) as 1.18 (0.72), whereas $R_{e\tau}$ can assume values ranging from 0.83 to 1.40. On the other hand, $R_{\mu\tau}$ can deviate from unity by 25%.

Thus, it seems to be clear that even in the case that the flux ratios could be measured with infinite precision at neutrino telescopes, it will be very difficult to estimate or put any reasonable bounds on the neutrino mixing angles (and the CP-violating phase) in the standard three-flavor oscillation framework. In particular, the ratios do not depend in a dramatic way on $\theta_{13}$ (both Figs. 1 and 2 numerically confirm this statement) and the dependence on $\delta$ is completely overshadowed by the uncertainty of the mixing angle $\theta_{23}$. However, assuming that some long-baseline neutrino experiments gave us a very precise measurement of the mixing angles $\theta_{12}$ and $\theta_{23}$ (which will certainly not happen in ten years from now) and assuming also the quite unrealistic scenario in which the flux ratio $R_{e\mu}$ will be measured within 5% precision at future neutrino telescopes, $\theta_{13}$ and $\delta$ will not be strongly constrained. In Fig. 3 we quantify this statement showing the allowed pairs $(\theta_{13}, \delta)$, which give $R_{e\mu} = 1 \pm 5\%$, for different values of $\theta_{23}$, namely 43°, 45°, and 47°, represented by dotted, solid, and dashed curves, respectively. The admitted parameter space is the region on the left-hand side of the curves. In principle, some values of $\theta_{13}$ and $\delta$ can be excluded, depending on the assumed value of the mixing angle $\theta_{23}$, but the precision obtained cannot be comparable with that usually claimed for neutrino factories and $\beta$-beams. However, as already discussed in Ref. [15], the combination of the information coming from neutrino telescopes and those from future neutrino experiment could help in constraining some of the neutrino mixing parameters, but only in the optimistic hypothesis that one can clearly separate the different neutrino sources (i.e., if the flux of high-energy neutrinos is under control).

In Ref. [4], the possibility of measuring the CP-violating phase $\delta$ using neutrinos from far distance was discussed.

Note that a first criticism about the possibility to measure $\theta_{13}$ and $\delta$ at neutrino telescopes can be found in Ref. [34].
4 Flux ratios including neutrino decay

Neutrino decay has been invoked in the past to solve both the solar and atmospheric neutrino data problems (see e.g. Refs. [35–37]). Decay will deplete the flux of neutrinos on Earth by the exponential factor

\[ \exp \left( -\frac{t}{\tau_{\text{lab}}} \right) = \exp \left( -\frac{L}{E} \frac{m}{\tau} \right) \] (24)

in which \( \tau \) is the rest-frame neutrino life-time, \( L \) the distance between the source and the detector, and \( E \) and \( m \) are the neutrino energy and mass, respectively. Since neutrino decay scenarios have not been validated so far [38, 39], we can roughly estimate the order of magnitude of the allowed values of the ratio \( \tau/m \) requiring that \( L/E \tau \ll 1 \). Thus, giving a lower bound \( \tau/m \geq 10^{-4} \text{s/eV} \), obtained for a typical \( L/E \) solar neutrino [40]. Note that from the supernova SN1987A neutrino events, one would obtain a stronger lower bound at the level of \( \tau/m \geq \mathcal{O}(10^5) \text{s/eV} \). However, this bound is not reliable if, as it is the case, the value of \( \theta_{12} \) is large [41, 42]. Given the bounds previously discussed, we cannot eliminate the possibility of astrophysical neutrino decay.

Two-body modes of Majoron type [43, 44] are still viable neutrino decay processes, since they are not strongly constrained (contrary, for example, to radiative decay modes or three-body modes, see e.g. Refs. [16, 45] and references therein). Stringent bounds on this class of models have been derived in Ref. [46], although the robustness of their conclusion has been questioned in Ref. [47]. Thus, we retain the possibility for neutrino decay in Majoron models to be a possible mechanism for flux ratio modifications. We adopt here the simplifying assumptions that the neutrinos completely decay into the lightest mass eigenstate or the lightest and next-to-lightest mass eigenstates over very large \( L/E \) and that there are no detectable decay products. Two different neutrino decay scenarios (with three active neutrino flavors) will be investigated. In the first scenario, which will be called “neutrino decay I”, the two heaviest neutrino mass eigenstates can decay and only the lightest neutrino mass eigenstate is stable, whereas in the second scenario, which will be called “neutrino decay II”, only the heaviest neutrino mass eigenstate can decay and the two lightest neutrino mass eigenstates are stable. The first neutrino decay scenario, neutrino decay I, has been heavily discussed in the literature, see e.g. Ref. [48]. Note that both neutrino decay scenarios can be considered for both normal (NH) and inverted neutrino mass hierarchy (IH), i.e., for the cases, \( m_1 < m_2 < m_3 \), where \( m_1 \) is the mass of the lightest neutrino mass eigenstate, and \( m_3 < m_1 < m_2 \), where \( m_3 \) is the lightest neutrino mass eigenstate. The different neutrino decay scenarios are schematically shown in Fig. 4.

In the case of neutrino decay scenarios, the parameter \( \beta = 1 \), and therefore, we can use Eq. (13) in order to compute averaged neutrino oscillation probabilities (including damping factors). Note that in the case of neutrino decay, the damping coefficient matrix is described by neutrino decay parameters [24], which are defined as \( \alpha_{ij} = (\alpha_i + \alpha_j)/2 \), where \( \alpha_i = m_i/\tau_i \) is the decay rate with \( m_i \) being the mass of the \( i \)th mass eigenstate and \( \tau_i \) is its life-time (in its own rest frame) [48]. Thus, in the case of “neutrino decay I”, we have \( \alpha_{11} = 0 \) or \( \alpha_{33} = 0 \) and all other \( \alpha_{ij} \)'s are non-zero in general, since only \( \alpha_1 = 0 \) (NH) or \( \alpha_3 = 0 \) (IH), which means that \( d_1 = 1 \) or \( d_3 = 1 \), i.e., only one of the \( d_i \)'s is equal to 1, and the others are zero (or, in other words, that the exponential factor in Eq. (24) is unity for the lightest mass eigenstate and vanishing for the others). In the case of “neutrino decay II”, we have \( \alpha_{11} = \alpha_{22} = 0 \) or \( \alpha_{11} = \alpha_{33} = 0 \), since \( \alpha_1 = \alpha_2 = 0 \) (NH) or \( \alpha_1 = \alpha_3 = 0 \) (IH), which means that two of the \( d_i \)'s are different from zero, i.e., \( d_1 = d_2 = 1 \) or \( d_1 = d_3 = 1 \). It is now easy to obtain flux ratios on Earth for both scenarios.
4.1 Neutrino decay I

The transition probabilities needed to build the flux ratios for the “neutrino decay I” scenario, illustrated in the first row of Fig. 4, are obtained from Eq. (14) in which the terms \( d_i J_{ii} \alpha_{\beta} \) have to be evaluated in the unique index corresponding to the stable mass eigenstate (\( i = 1 \) for NH and \( i = 3 \) forIH).

For the sake of simplicity and with the aim of checking that our damping formalism reproduces the results obtained with the standard formulation of large \( L/E \) behavior of the flux ratios, we report here only the first-order approximations. Note that the flux ratios are independent of the initial flavor composition at the source, and according to Ref. [17], we find that

\[
\phi_{\nu_e} : \phi_{\nu_\mu} : \phi_{\nu_\tau} = |U_{e1}|^2 : |U_{\mu1}|^2 : |U_{\tau1}|^2 \text{ for the NH and } \phi_{\nu_e} : \phi_{\nu_\mu} : \phi_{\nu_\tau} = |U_{e3}|^2 : |U_{\mu3}|^2 : |U_{\tau3}|^2 \text{ for the IH.}
\]

First, let us discuss the NH case:

\[
R_{e\mu} = 2 \cot^2 (\theta_{12}) - 4 \cot (\theta_{12}) \csc^2 (\theta_{12}) \delta_{12} - 4 \cos(\delta) \cot^3 (\theta_{12}) \theta_{13} + 4 \cot^2 (\theta_{12}) \delta_{23}, \tag{25}
\]

\[
R_{e\tau} = 2 \cot^2 (\theta_{12}) - 4 \cot (\theta_{12}) \csc^2 (\theta_{12}) \delta_{12} + 4 \cos(\delta) \cot^3 (\theta_{12}) \theta_{13} - 4 \cot^2 (\theta_{12}) \delta_{23}, \tag{26}
\]

\[
R_{\mu\tau} = 1 + 4 \cos(\delta) \cot (\theta_{12}) \theta_{13} - 4 \delta_{23}. \tag{27}
\]

The main feature of these formulas are, except for the flux ratio \( R_{\mu\tau} \), that the zeroth-order approximation is different from 1, being dependent on \( \cot^2 (\theta_{12}) \). For the best-fit value used in this work, this means that \( \phi_{\nu_e} : \phi_{\nu_\mu} : \phi_{\nu_\tau} \sim 5 : 1 : 1 \) [17], a very large deviation compared to the standard neutrino framework. Moreover, unlike Eqs. (17) and (18), a dependence on \( \delta_{12} \) appears at first order, which means that the error of the mixing angle \( \theta_{12} \) cannot be

\[\text{In Table I the goodness of our approximate formulas can be found.}\]
neglected. Note that $R_{\mu\tau}$ obtains corrections with respect to 1 also at first order. Thus, we expect that the intrinsic uncertainty of the flux ratios (given by our ignorance on the neutrino mixing parameters) is much larger than the standard framework case.

Second, in the case of IH, the flux ratios assume the following simple structure

$$R_{e\mu} = R_{e\tau} = 0,$$

$$R_{\mu\tau} = 1 + 4\delta_{23}. \quad (29)$$

The signature of this particular case is extremely different from both the standard framework and the “neutrino decay I” with NH. Corrections to $R_{e\mu} = 0$ and $R_{e\tau} = 0$ appear at second order in small quantities and are dependent on $\theta_{13}$. Thus, we expect a substantial deviation from zero at large $\theta_{13}$, since corrections appear only at second order in $\theta_{13}$ [49].

4.2 Neutrino decay II

Similarly, the “neutrino decay II” scenario is depicted in the second row in Fig. 4. In this case, the two lightest neutrino mass eigenstates are stable instead of only the lightest in the “neutrino decay I” scenario. Therefore, we will have two $d_iJ_{\alpha\beta}^i$ terms in Eq. (14) instead of only one ($i = 1$ and $i = 2$ for NH and $i = 1$ and $i = 3$ for IH). In App. A.1 we present formulas for the flux ratios only up to first order in the small parameters $\theta_{13}, \delta_{12},$ and $\delta_{23}$, since the first-order formulas make a much better approximation for “neutrino decay II” than “neutrino decay I”. Note that for NH, the formulas are independent of $\delta_{12}$.

In addition, for both NH and IH, all formulas are dependent on $\delta_{23}$, which means that the uncertainty of $\theta_{23}$ is crucial. Furthermore, we observe that the structure of the formulas for NH and IH is very different. For IH, nearly all terms (except from the zeroth-order term of the flux ratio $R_{\mu\tau}$) include the factor $3 - \cos(2\theta_{12}) = 2 + 2\sin^2(\theta_{12})$ in the denominators, which has a value between 2 and 4 and is therefore not equal to zero at any time though. In addition, $R_{e\mu}$ is independent of the CP-violating phase $\delta$, whereas $R_{e\tau}$ is dependent on $\delta$, which actually spoils the approximate $\mu - \tau$ symmetry. In some sense, the values of the flux ratios are closer to the standard neutrino framework for “neutrino decay II” than “neutrino decay I”.

4.3 Discussion about the neutrino decay scenarios

The previous considerations have been summarized in Figs. 5 and 6, in which we present the flux ratios $R_{e\mu}$ and $R_{\mu\tau}$, respectively, using the two neutrino decay scenarios with both NH and IH.

In Fig. 5 for the flux ratio $R_{e\mu}$, we observe that the “neutrino decay I” scenario for both NH and IH can, in principle, be distinguished from the standard neutrino oscillation framework. As expected from the very different first-order approximations in Eqs. (25) and (28), even including the errors of the neutrino mixing parameters, the bands representing the total spread of $R_{e\mu}$ are very well separated, especially for the IH case in which a quadratic behavior of $\theta_{13}$ can be clearly seen (notice that we are using a log scale on the vertical axis). In addition, we observe the large dependence of $\theta_{13}$ on the uncertainty of the NH case, which is mainly to be ascribed to the term $4\cos(\delta)\cot^3(\theta_{12})\theta_{13}$.

A very different situation arises for the “neutrino decay II” scenario (solid and dotted curves for NH and IH in Fig. 5 respectively). Due to the fact that this scenario contains two

\footnote{Again, in Table 1 we present the numerical results on the flux ratios.}

\footnote{Note that the flux ratio $R_{e\tau}$ is very similar to the flux ratio $R_{e\mu}$, and we have therefore not presented this flux ratio in a separate figure.}
Figure 5: The flux ratio $R_{e\mu}$ including neutrino decay as a function of the mixing angle $\theta_{13}$. The red (middle) area corresponds to the standard neutrino oscillation framework, whereas the blue (upper) and green (lower) areas correspond to neutrino decay with the lightest neutrino mass eigenstate being stable only for NH and IH, respectively. The case of the two lightest neutrino mass eigenstates being stable is visualized by the solid (NH) and dotted (IH) curves, respectively. The uncertainties of the neutrino mixing angles are those given in Eq. (20), around the best-fit values $\bar{\theta}_{12} = 33^\circ$ and $\bar{\theta}_{23} = 45^\circ$. 
Figure 6: The flux ratio $R_{\mu\tau}$ including neutrino decay as a function of the mixing angle $\theta_{13}$. The red (innermost) area corresponds to the standard neutrino oscillation framework, whereas the blue (outermost) and green (middle) areas correspond to neutrino decay with the lightest neutrino mass eigenstate being stable only for NH and IH, respectively. The case of the two lightest neutrino mass eigenstates being stable is visualized by the solid (NH) and dotted (IH) curves, respectively. The uncertainties of the neutrino mixing angles are those given in Eq. (20), around the best-fit values $\bar{\theta}_{12} = 33^\circ$ and $\bar{\theta}_{23} = 45^\circ$.

In Fig. 6, the results of the flux ratio $R_{\mu\tau}$ are shown. As already outlined in the previous section, in the standard framework, $R_{\mu\tau}$ is always larger than 1, even including the uncertainties of the neutrino mixing angles. The neutrino decay scenarios allow $R_{\mu\tau}$ to be smaller than 1 (which could be a clear signature of “new physics” effects), but the large spread of the ratio at any value of $\theta_{13}$ does not allow an explicit separation between the standard and decay phenomenologies. Thus, this flux ratio is not a good candidate to find deviations from the standard neutrino oscillation framework.

In Table 1, we present numerical calculations of the flux ratios $\phi_{\nu_e}$ : $\phi_{\nu_\mu}$ : $\phi_{\nu_\tau}$ for the different neutrino decay scenarios, comparing the zeroth-, first-, and second-order results with exact calculations. In general, we observe that the second-order results contain all the features connected with non-maximal $\theta_{23}$ and non-vanishing $\theta_{13}$. Thus, it seems to be important to include second-order effects in any analytical treatment of flux ratios. However, note that the first-order results work especially well for the standard framework and “neutrino decay II”, whereas for “neutrino decay I”, the first-order results do not reproduce all features of the exact computations.

*Note that the zeroth-order values for the flux ratios in the case of “neutrino decay I” with both NH and IH can, e.g., also be found in Ref. [17].
| $\phi_{\nu_e} : \phi_{\nu_\mu} : \phi_{\nu_\tau}$ | $0^{\text{th}}$ | $1^{\text{st}}$ | $2^{\text{nd}}$ | exact |
|----------------------------------------------|----------------|----------------|----------------|--------|
| standard framework                          | 1:1:1          | 0.93:1:1       | 0.93:1.06:1    | 0.93:1.06:1 |
| neutrino decay I (NH)                       | 4.74:1:1       | 3.08:1.06:1    | 3.14:0.99:1    | 3.20:0.99:1 |
| neutrino decay I (IH)                       | 0:1:1          | 0:1.35:1       | 0.02:1.41:1    | 0.02:1.42:1 |
| neutrino decay II (NH)                      | 2:1:1          | 1.81:0.65:1    | 1.78:0.74:1    | 1.78:0.73:1 |
| neutrino decay II (IH)                      | 1.08:1:1       | 0.91:1.28:1    | 0.93:1.30:1    | 0.93:1.30:1 |

Table 1: Flux ratios $\phi_{\nu_e} : \phi_{\nu_\mu} : \phi_{\nu_\tau}$ for the different neutrino decay scenarios. $0^{\text{th}}$, $1^{\text{st}}$, and $2^{\text{nd}}$ represent the results obtained up to zeroth, first, and second order in small parameters, respectively, whereas the exact computation is given in the column exact. The values of the fundamental neutrino parameters used are: $\theta_{12} = 33^\circ$, $\theta_{13} = 5^\circ$, and $\delta = 40^\circ$. In addition, we have assumed $\delta_{12} = 5^\circ$ and $\delta_{23} = 5^\circ$.

4.4 Discussion about neutrino telescopes

In the previous sections, we have discussed the flux ratios from a theoretical point of view; thus taking into account the role of the uncertainties of the neutrino mixing parameters. Let us now address the question of what can really be measured at neutrino telescopes and how the statistics, i.e., the number of expected events, affects the results that can be obtained.

The measurement of neutrino flavor seems to be a very difficult task, which means that the three flux ratios previously studied are unlikely to be directly accessible. However, they can be inferred from distinct classes of events: showers, muon tracks, and unique $\nu_\tau$ signatures (see, e.g., Ref. [50]). All neutrino flavors undergo neutral-current interactions (inside or nearby the detector), which result in hadronic showers in a detector. On the other hand, $\nu_e$ charged-current interactions also produce electromagnetic showers that, in principle, could be distinguished from the hadronic ones due to the different muon content (which is absent in the electromagnetic showers). The shower rate also includes $\nu_\tau$ charged-current events, since at least for energies below the order of PeV the tau track cannot be separated from the shower. For energies above the order of PeV, double-bang [51, 52] and lollipop events are also possible, but we do not discuss these tau signatures in this work. Muon tracks originate from $\nu_\mu$ charged-current interactions; muons always emerge from the shower, which means that this kind of process can be distinguished from a typical shower event.

Neutrino telescopes seem to be the ideal place where to seek for high-energy neutrino interactions. In deep ice or water, neutrinos are detected by observation of the Cherenkov light emitted by charged particles produced in charged-current and neutral-current interactions. Such detectors are mainly sensitive to TeV-PeV neutrino energies; thus opening the possibility of studying this still unexplored energy regime. In order to observe a hadronic event inside a telescope, a neutrino must first survive as it crosses the atmosphere and then the ice (or water) above the detector. Its typical interaction length in a medium of density $\rho$ is $L_0 = 1/(\rho N_A \sigma^{\nu N})$, where $\sigma^{\nu N} = \sigma^{\nu N}_{\text{CC}} + \sigma^{\nu N}_{\text{NC}}$ is the total cross-section to have
an interaction that depletes the neutrino energy. It is usual to express this length in terms of its depth: $x_0 = \rho L_0$ (i.e., one meter of water has a depth of 100 g/cm²).

A neutrino from a zenith angle $\theta_z$ must cross a column density of material $x(\theta_z) = \int_0^{\theta_z} dl \rho(l, \theta_z)$. In practice, the path in the atmosphere is negligible and $x(\theta_z)$ is just the depth of the ice or water above the detector. The probability that it does not interact before reaching the detector is then $P_{\text{surv}}(E_\nu, \theta_z) = e^{-x/x_0}$. Once in the detector, the probability of an event is $P_{\text{int}}(E_\nu) \approx 1 - e^{-L\rho L \sigma_{\nu N}^\text{int}}$, where $L$ is the linear dimension of the detector and $\sigma_{\nu N}^\text{int}$ is the cross-section for the interaction taken into account (charged or neutral). Therefore, the number of shower events in the telescope in an observation time $T$ for a fixed flavor is

$$N_{\text{sh}} = 2\pi A T \int dE_\nu \frac{d\Phi_{\nu_i}}{dE_\nu} \int d\cos\theta_z P_{\text{surv}} \int_{y_{\text{min}}}^{y_{\text{max}}} dy \frac{1}{\sigma_{\nu N}} \frac{d\sigma_{\nu N}}{dy} P_{\text{int}},$$

where $A$ is the detector’s cross-sectional area, $d\Phi_{\nu_i}/dE_\nu$ is the neutrino flux, and $\theta_z$ is the zenith angle. The variable $y$ is the usual inelasticity parameter in deep-inelastic scattering and represents the fraction of the neutrino energy going into hadronic channels, which means that $E_{\text{sh}} = y E_\nu$.

According to our previous discussion about flavor detectability, for neutral-current interactions and $\nu_\tau$ charged-currents (for energies below a few PeV), we should consider $y_{\text{min}} = y_{\text{thr}} = E_{\text{sh}}^{\text{th}}/E_\nu$ and $y_{\text{max}} = 1$, where $E_{\text{sh}}^{\text{th}}$ is the threshold energy for shower detection. For $\nu_e$ charged-currents, $y_{\text{min}} = 0$ and $y_{\text{max}} = 1$, since the outgoing electron also shower. Moreover, for $\nu_\tau$, Eq. (30) should be modified to include the effects of regeneration. In fact, because of the short lifetime of tau, the $\nu_\tau \rightarrow \tau \rightarrow \nu_\tau$ conversion takes place, with the result of softening the neutrino energy. In principle, one should perform a dedicated analysis to determine the average energy a $\nu_\tau$ has when it reaches the detector. Such a study is beyond the scope of our work (but see, e.g., Refs. [53, 54]); we approximate this fact assuming that, in neutral-current events, the final neutrino energy is about one half of the initial energy and that, in charged-current events, the $\nu_\tau$ energy is reduced to about one fifth [53]. Thus, we can evaluate these integrals to obtain the total number of shower events:

$$N_{\text{tot}} = \sum_{\nu_\mu, \nu_e, \nu_\tau} N_{\text{sh}}^{\nu_\mu} + \sum_{\nu_\mu, \nu_e} N_{\text{sh}}^{\nu_\tau}. \tag{31}$$

Equation (30) is a simple form to evaluate the shower rates at neutrino telescopes. Even if an exhaustive discussion of the problems related to the evaluation of rates is beyond the scope of this work, we should mention two main sources of uncertainties. Obviously, we do not know the neutrino fluxes reaching the Earth’s surface, neither the normalization nor the energy behavior. Thus, we can assume the form

$$\frac{d\Phi_{\nu_i}}{dE_\nu} = c \phi_{\nu_i} E_\nu^{-2}, \tag{32}$$

where $c$ is a normalization constant and $\phi_{\nu_i}$’s are fluxes entering in the definition of the flux ratios in Eq. (10). When calculating the ratios of rates, the normalization factor cancels out, but the spectral index is still a crucial ingredient. In addition, the neutrino-nucleon cross-section is also affected by large uncertainties, mainly due to the extrapolation to very small $x$ required to describe the extremely high-energy regime involved in the processes under discussion.

A high-energy muon undergoes energy loss propagating in the medium, and thus, generating showers along its track by bremsstrahlung, pair production, and photonuclear interactions. These showers are sources of Cherenkov light and can be easily distinguished
from showers previously discussed. Equation (30) applies, but now
\[ P_{\text{int}}(E_\nu) \approx 1 - e^{-R_\mu \rho N A \sigma_{\text{int}}^N}, \]  
(33)

where
\[ R_\mu = \frac{1}{\beta} \ln \left( \frac{\alpha + \beta E_\mu}{\alpha + \beta E_{\text{thr}}^\mu} \right), \]  
(34)

with \( \alpha = 2 \text{ MeV cm}^2/\text{g}, \beta = 4.2 \times 10^{-6} \text{ cm}^2/\text{g}, E_\mu = (1 - y) E_\nu, \) and \( E_{\text{thr}}^\mu \) is the value of the muon energy to be detected. Note that the \( y \) integration in Eq. (30) is modified in such a way that \( y_{\text{min}} = 0 \) and \( y_{\text{max}} = y_{\text{thr}}^\mu = 1 - E_{\text{thr}}^\mu / E_\nu. \)

From the previous discussion, it seems to be clear that a viable experimental quantity measurable at neutrino telescopes is the muon tracks \( N_{\text{tr}} \) to shower ratio. Factorizing out \( \phi_\nu \) from Eq. (32) and calculating the expected rates according to Eq. (30), we obtain
\[ R = \frac{N_{\text{tr}}}{N_{\text{sh}} = \frac{\phi_{\nu_\mu} N_{\text{tr}}}{\sum_{i=e,\tau} \phi_{\nu_i} (N_{\text{CC}}^i_{\text{sh},\nu_i} + N_{\text{NC}}^i_{\text{sh},\nu_i}) + \phi_{\nu_\mu} N_{\text{NC}}^\mu_{\text{sh},\nu_\mu}}}. \]  
(35)

Note that the ratio \( R \) depends on the neutrino mixing parameters through the analogous dependence in \( \phi_\nu \).

In order to verify these statements, we can use Eq. (35) once the energy thresholds for muons track and shower detection have been specified. To give an example, we use the IceCube detector setup and assume a quite conservative energy threshold \( E_{\text{thr}}^\mu = E_{\text{thr}}^\nu = 500 \text{ TeV} \) for both types of processes in order to consider event rates well above the atmospheric neutrino backgrounds [55], although for up-going neutrinos the threshold could be sizably smaller due to the screening effect of the Earth. For this setup, the result of \( R \) as a function of \( \theta_{13} \) is shown in Fig. 7, in which we have evaluated ratios \( R \) for the standard neutrino oscillation scenario and “neutrino decay I” (both hierarchies). The error bars represent the statistical error associated with the number of muon tracks and shower events [thus, including also the uncertainties of the mixing parameters given in Eq. (20)].

For neutrino fluxes in the ratio \( \phi_{\nu_e} : \phi_{\nu_\mu} : \phi_{\nu_\tau} = 1 : 1 : 1 \) (standard scenario), we expect about 15 muon tracks against about 11 shower events in one year of data taking. As shown in the left plot of Fig. 7 this event rate is not enough to distinguish among the three scenarios considered in the text. A large overlap between the standard scenario and “neutrino decay I” is present for almost any value of \( \theta_{13} \). We have verified that a large contribution to the ratio errors is due to poor statistics. If we increase the number of expected events by a factor of 10 (that could be achieved in ten years of data taking), we observe a strong reduction of the uncertainty of the ratios \( R \), as illustrated in the right plot of Fig. 7. Especially, at smaller \( \theta_{13} \), the ratios corresponding to the three scenarios are quite separated; thus the sensitivity of IceCube seems to be enough to tell standard from new physics scenarios involving neutrino decay apart. Note that the effect we have observed increasing the event rates cannot be mimicked by a reduction of the uncertainties of the neutrino mixing parameters. We have calculated the ratios \( R \) assuming a 10 % error for both \( \theta_{12} \) and \( \theta_{23} \); we have clearly observed a reduction of the errors of \( R \), but not sufficient to avoid the overlap among different scenarios, which means that, even for very precise measurements of the solar and atmospheric mixing angles, the error is dominated by statistics. In the opposite limit, that of infinite statistics, the right plot suggests (which we have numerically verified) that the distinction between the scenarios under study is feasible even with the current uncertainties of \( \theta_{12} \) and \( \theta_{23} \). Note that this statement has nothing to do with the fact that we can learn something on the neutrino mixing angles at neutrino telescopes, as we have extensively discussed in the previous sections.
Figure 7: The muon-to-shower ratio \( R \) as a function of the mixing angle \( \theta_{13} \), evaluated for three of the scenarios discussed in the main text: standard neutrino framework (black boxes), “neutrino decay I”, NH (blue diamonds), and “neutrino decay I”, IH (red circles). Left plot: Results obtained considering one year of data taking at IceCube. Right plot: Results obtained considering ten years of data taking at IceCube. In both plots, the uncertainties of the neutrino mixing angles are those given in Eq. (20).

5 Summary and conclusions

The next generation of neutrino telescopes will be able to observe many high-energy events, thus opening the very exciting possibility to measure neutrino flux ratios. However, many uncertainties from both theoretical and experimental point of view, can spoil the capability of extracting useful information about astrophysical sources and fundamental neutrino properties. In this work, we have studied in detail the analytical behavior of the three flux ratios \( R_{e\mu} \), \( R_{e\tau} \), and \( R_{\mu\tau} \), focusing on how the current errors of the neutrino mixing parameters can affect the theoretical expectations of \( R_{\alpha\beta} \) for any \( \alpha \) and \( \beta \) (for flux ratios at the source \( \phi_{\nu_e} : \phi_{\nu_{\mu}} : \phi_{\nu_{\tau}} = 1 : 2 : 0 \)). Due to the enormous distances traveled by neutrinos, the transition probabilities \( P_{\alpha\beta} \) among flavors generally assume a simple analytical structure. We have derived them starting from a general treatment of \( P_{\alpha\beta} \) which includes damping parameters to account for possible “new physics” effects in neutrino oscillations. We have showed that the only non-vanishing large \( L \) averaged transitions are those in which the non-diagonal damping factors completely disappear from the theory.

We have then expanded the flux ratios in terms of small parameters, namely the deviation from maximal \( \theta_{23} \), the deviation of \( \theta_{13} \) from zero, and the deviation of \( \theta_{12} \) from its best-fit value. Furthermore, we have studied in detail the uncertainties of \( R_{\alpha\beta} \) connected with our ignorance of the fundamental neutrino physics. We have observed that the largest indetermination on \( R_{e\mu} \) (at the level of 45%) is due to the fact that we do not know the \( \theta_{23} \) octant, whereas the uncertainty associated with the product \( \cos(\delta) \theta_{13} \) is at least a factor of two smaller. Due to the approximate \( \mu - \tau \) symmetry, the same conclusion can be drawn for \( R_{e\tau} \). On the other hand, \( R_{\mu\tau} \) behaves in a very peculiar way, since the corrections from the standard value \( R_{\mu\tau} = 1 \) are positive and of \( \mathcal{O}(\delta_{12}^2) \). These considerations have been summarized in Fig. 1 in which the total spread of the flux ratios caused by the current uncertainties of mixing parameters seems to be too large to admit the possibility of any realistic measurement of fundamental parameters in the neutrino sector, (e.g., \( \theta_{13} \), \( \delta \), and the octant of \( \theta_{23} \)).
We have performed the same detailed study on two different scenarios involving neutrino decay, considering the lightest mass eigenstate (neutrino decay I) or the lightest and next-to-lightest (neutrino decay II) as a stable particle(s). Once the errors of the mixing parameters are included, we have observed that the most promising ratio to measure deviations from the standard framework is \( R_{e\mu} \), especially for the “neutrino decay I” scenario with inverted hierarchy. In that case, especially for smaller values of \( \theta_{13} \), the difference between the ratios can be as large as two orders of magnitude. On the other hand, the “neutrino decay II” scenario can be hardly distinguished from the standard result.

Finally, we have studied the muon tracks to shower ratio \( R \) at IceCube, as a physical observable directly connected to the experiments. We have evaluated \( R \) in the standard picture of neutrino oscillations and the “neutrino decay I” scenario for both hierarchies. We have found that, contrary to the case in which \( R \) is computed with the neutrino mixing parameters fixed to their best-fit values, the inclusion of parameter uncertainties reduces the possibility to distinguish “new physics” effects in neutrino oscillations. Thus, we believe that any realistic analysis of physics reach of neutrino telescopes concerning flux ratios should carefully include parameters uncertainties.

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**A Series expansion formulas**

**A.1 First-order series expansions of neutrino flux ratios with two non-zero normalized damping factors**

The neutrino flux ratios using two of the normalized damping parameters equal to 1, \( d_1 = d_2 = 1 \), and the last one equal to zero:

\[
R_{e\mu}(d_1 = d_2 = 1) = 2 + 2 \cos(\delta) \sin(4\theta_{12})\theta_{13} + 2[3 + \cos(4\theta_{12})] \delta_{23},
\]

\[
R_{e\tau}(d_1 = d_2 = 1) = 2 + 2 \cos(\delta) \sin(4\theta_{12})\theta_{13} - 4 \sin^2(2\theta_{12}) \delta_{23},
\]

\[
R_{\mu\tau}(d_1 = d_2 = 1) = 1 - 4 \delta_{23}.
\]

The neutrino flux ratios using two of the normalized damping parameters equal to 1, \( d_1 = d_3 = 1 \), and the last one equal to zero:

\[
R_{e\mu}(d_1 = d_3 = 1) = \frac{4 \cos^2(\theta_{12})}{3 - \cos(2\theta_{12})} - \frac{16 \sin(2\theta_{12})}{[3 - \cos(2\theta_{12})]^2} \delta_{12} - \frac{32 \cos^2(\theta_{12})}{[3 - \cos(2\theta_{12})]^2} \delta_{23},
\]

\[
R_{e\tau}(d_1 = d_3 = 1) = \frac{4 \cos^2(\theta_{12})}{3 - \cos(2\theta_{12})} - \frac{16 \sin(2\theta_{12})}{[3 - \cos(2\theta_{12})]^2} \delta_{12}
\]

\[
+ \frac{32 \cos(\delta) \cos^3(\theta_{12}) \sin(\theta_{12})}{[3 - \cos(2\theta_{12})]^2} \theta_{13} - \frac{8 \sin^2(2\theta_{12})}{[3 - \cos(2\theta_{12})]^2} \delta_{23},
\]

\[
R_{\mu\tau}(d_1 = d_3 = 1) = 1 - 4 \delta_{23}.
\]
\[ R_{\mu \nu}(d_1 = d_3 = 1) = 1 + \frac{4 \cos(\delta) \sin(2\theta_{12})}{3 - \cos(2\theta_{12})} \theta_{13} + \frac{8 \cos^2(\theta_{12})}{3 - \cos(2\theta_{12})} \delta_{23}. \] (41)

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