Monopole Dynamics from the M-fivebrane

N.D. Lambert
and
P.C. West⋆

Department of Mathematics
King’s College, London
England
WC2R 2LS

ABSTRACT

We study the BPS states of the M-fivebrane which correspond to monopoles of $N = 2 \, SU(2)$ gauge theory. Far away from the centres of the monopoles these states may be viewed as solitons in the Seiberg-Witten effective action. It is argued that these solutions are smooth and some properties of their moduli space are discussed.

KEYWORDS: M-Theory, Fivebrane, Monopoles

PACS: 11.15.Tk, 11.25.Sq, 11.30.Pb

⋆ lambert, pwest@mth.kcl.ac.uk
1. Introduction

One of the most unexpected and detailed relations to come out the various string theory dualities has been the connection between $p$-branes, viewed as supergravity solitons, and Yang-Mills theories, obtained from the perturbative D-brane description. In particular, this paper was motivated by the observation that a single M-fivebrane is capable of understanding some complex details of quantum non-Abelian gauge theory. More precisely it was observed in [1] that the M-theory picture of $N$ D-fourbranes suspended between two NS-fivebranes in type IIA string theory is just a single M-fivebrane wrapped on a Riemann surface. It was then further argued that the Riemann surface in question is none other than the Seiberg-Witten curve for the corresponding $N = 2 \, SU(N)$ gauge theory on the D-fourbranes.

Let us consider this configuration from the M-fivebrane point of view. We can denote this configuration by listing the worldvolume directions of each M-fivebrane

\[ M5 : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]
\[ M5 : \quad 1 \quad 2 \quad 3 \quad 6 \quad 7 \] \hspace{1cm} (1.1)

where we have suppressed the time dimension since this is common to all branes. To make contact with type IIA string theory and D-branes one compactifies on $x^7$. In this case the first M-fivebrane becomes the two parallel NS-fivebranes and the second M-fivebrane becomes $N$ D-fourbranes. From the point of view of the first M-fivebrane the second M-fivebrane appears as a threebrane soliton with worldvolume coordinates $(x^0, x^1, x^2, x^3)$. By solving the field equations for this configuration one sees that the threebrane can be viewed as simply the first M-fivebrane wrapped on a Riemann surface [3]. Moreover it was shown in [2,4] that not just the elliptic curve but in fact the entire low energy Seiberg-Witten effective action can obtained as the classical effective action for this threebrane soliton. Thus, a single M-fivebrane is capable of predicting an infinite number of instanton coefficients in the four dimensional non-Abelian gauge theory. Therefore one is naturally lead to the
expectation that the M-fivebrane contains more information on non-Abelian fields than might naively be expected.

To explore this possibility one is led to study the M-fivebrane description of BPS states in Seiberg-Witten theory, or more precisely, since we will obtain results outside of the Seiberg-Witten effective description, low energy $N = 2$ super-Yang-Mills theory in the presence of BPS states. These states are of quite some interest and have been studied from many points of view. Not least because of subtleties in the predicted spectrum [6,7]. We will be particularly interested in monopoles since these states are intimately connected to the non-Abelian gauge structure and quantum dynamics.

The configurations in question can be pictured as

\[
\begin{align*}
M5 : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
M5 : & \quad 1 \quad 2 \quad 3 \quad 6 \quad 7 \\
M2 : & \quad 4 \quad 6 \\
M2 : & \quad 5 \quad 7
\end{align*}
\]

(1.2)

Note that although there are two M-twobranes and two M-fivebranes there are only three independent supersymmetry projectors, corresponding to 1/8 of spacetime supersymmetry or 1/4 of the M-fivebrane worldvolume supersymmetry. In other words after adding the first M-twobrane to the configuration (1.1) we find that the second M-twobrane can appear without breaking any more supersymmetry. We include it to obtain the most general configuration. Indeed the appearance of two M-twobranes is crucial for our analysis.

One problem with self-dual strings obtained by intersecting an M-twobrane with an M-fivebrane is that they have infinite tension, due to the infinite length of the M-twobrane [8,9,10]. Clearly this is an unwanted feature when trying to compare with the smooth BPS states in a Yang-Mills gauge theory. One way to avoid the infinite energy is to place the M-twobrane suspended between two parallel M-fivebranes. Then, despite the fact that the proper distance between the
M-fivebranes diverges, the resulting self-dual string has finite tension [11]. Unfortunately, such a direct approach is unavailable since there is no known description of two parallel M-fivebranes as this involves some kind of non-Abelian tensor multiplet.

However, we wish to consider configurations with both self-dual strings and threebranes on the M-fivebrane worldvolume. In such a situation we expect to find finite energy solutions for the strings since they can be stretched between the different branches of the same M-fivebrane. In other words we wish to consider self-dual strings which are, in some sense, wrapped around the Riemann surface which is embedded in spacetime. In this case the M-twobrane has a finite worldvolume in spacetime and its boundary coincides with some cycle of the Riemann surface. Furthermore the M-fivebrane worldvolume theory just contains an Abelian tensor multiplet.

Another issue that we must consider is whether or not the classical M-fivebrane equations provide an accurate description of the configuration (1.2), especially where the M-twobranes intersect the M-fivebranes. In order to ensure that the classical M-fivebrane’s equations of motion are valid we must keep the curvatures small. We therefore need to look for smooth solutions in \( x^1, x^2, x^3 \) and \( x^4, x^5 \). In addition we require that the space derivatives \( \partial_1, \partial_2, \partial_3 \) are small. Indeed, following the spirit of effective actions, we will only keep terms that are second order in spacetime derivatives. The justification for this approximation, and the use of the classical M-fivebranes equations, therefore rests on the existence of well-behaved solutions to the effective equations of motion we will derive.

It is natural to consider the resulting moduli space of these solutions and compare it to that of monopoles in \( N = 2 \) Yang-Mills theory. In fact we will argue below that there are finite energy solutions and that the two moduli spaces agree, thus providing an Abelian, M-theory description of monopole moduli space. This would mean that the low energy scattering of monopoles in Yang-Mills theory could be reproduced from the M-fivebrane equations of motion describing self-dual
strings scattering on a Riemann surface. We note here that the moduli space of self-dual strings has been studied earlier in a different context [12]. There the infinite tension strings of [2] (as obtained by the intersection of an M-twobrane with a single, flat M-fivebrane) were considered and a hyper-Kähler with torsion metric was found.

The M-theory realisation of Seiberg-Witten theory and its BPS states have already been well studied in [13,14,15]. However these papers treated the M-fivebrane as infinitely heavy as compared to the M-twobranes. It was therefore assumed that their geometry would be unaffected by the presence of the self-dual strings. This assumption seems reasonable from the macroscopic supergravity picture, however, from the M-fivebrane point of view the presence of self-dual strings dramatically alters the geometry and even topology of the worldvolume [8,9,10]. Thus we may expect the simple picture of an M-fivebrane wrapped around a Riemann surface, with M-twobranes attached, to break down near the self-dual strings. We therefore expect to see significant departures to the Seiberg-Witten dynamics in the low energy effective action. Our approach then offers a new method for studying these states in addition to the standard field theory methods and the M-theory methods of [13,14,15].

In the next section we discuss the Bogomol’yi conditions and resulting field equations for the M-fivebrane configuration (1.2). In section three we consider a limit where the equations of motion coincide with what can be obtained from the Seiberg-Witten effective action. In the fourth section we turn our attention to the full equations and argue for the existence of smooth solutions and consider the moduli space. Finally we conclude with some comments in section five.
2. Self-dual Strings on a Riemann Surface

The Bogomol’nyi Equations

In this paper we consider the worldvolume theory of an M-fivebrane with coordinates \(x^0, x^1, x^2, x^3, x^4, x^5\). We label the six dimensional world indices by \(m, n, p, \ldots\) and four-dimensional ones by \(\mu, \nu, \rho, \ldots = 0, 1, 2, 3\), of which we denote the three spatial coordinates by \(i, j, k, \ldots = 1, 2, 3\), or collectively as \(x\). Tangent indices are denoted by \(a, b, c, \ldots\). The M-fivebrane worldvolume theory possesses a self-dual three-form \(h_{mnp}\) and five scalar modes \(X^6, \ldots, X^{10}\) for bosonic fields and has eight fermion degrees of freedom. The equations of motion are invariant under six dimensional \((2, 0)\) supersymmetry [16] and an internal symmetry \(\text{Spin}(1,5) \times \text{Spin}(5)\) where \(\text{Spin}(1,5)\) is the Lorentz group and the \(\text{Spin}(5)\) is an R-symmetry. The six-dimensional \(\Gamma^a\)-matrices are written in terms of matrices \(\gamma^a\),

\[
\Gamma^a = \begin{pmatrix} 0 & \gamma^a \\ \tilde{\gamma}^a & 0 \end{pmatrix},
\]

with \(\gamma^0 = -\tilde{\gamma}^0 = 1\) and \(\gamma^a = \tilde{\gamma}^a\), \(a = 1, \ldots, 5\) are five-dimensional Euclidean \(\Gamma\)-matrices. In addition to these the worldvolume theory inherits a set of Euclidean \(\Gamma\)-matrices from the five-dimensional space transverse to the M-fivebrane. We denote these matrices by \(\gamma_{a'}\), whose unique irreducible representation is a spinor transforming under the \(\text{Spin}(5)\) R-symmetry. It is important to note that the \(\gamma^a\) and \(\gamma_{a'}\) matrices act on different spinor indices and so commute with each other.

The reader is referred to [16,4,17] for more details of the notation.

The configuration of an M-twobrane intersecting two M-fivebranes has also been discussed in [17] from the viewpoint of generalised calibrated geometries. Two intersecting M-fivebranes in the \((x^0, x^1, x^2, x^3, x^4, x^5)\) and \((x^0, x^1, x^2, x^3, X^6, X^7)\) planes reduce the supersymmetry to spinors such that [9]

\[
\epsilon \gamma^{45} \gamma_{67} = -\epsilon.
\]

The configuration (1.2) also has an M-twobrane in the \((x^0, x^5, X^7)\) plane with the
corresponding projector [3]

\[ \epsilon \gamma^{05} \gamma^7 = \eta \epsilon , \]

where \( \eta = \pm 1 \). These two projectors actually imply that another M-twobrane can be introduced in the \((x^0, x^4, X^6)\) plane with the projector \( \epsilon \gamma^{04} \gamma^6 = -\eta \epsilon \), without breaking any additional supersymmetries. It is helpful to introduce complex notation

\[ z = (x^4 + ix^5) \Lambda^2 , \]

where \( \Lambda \) is a mass scale introduced for later convenience. The active M-fivebrane scalars are denoted by

\[ s = (X^6 + iX^7)/R , \]

where we treat \( X^7 \) as a compact coordinate with period \( R \). Thus by taking the limit of small \( R \) we obtain a perturbative description in terms of type IIA string theory as discussed in [1]. For clarity in this paper we will largely suppress the constants \( \Lambda \) and \( R \). In complex notation these projectors can simply be written as

\[ \epsilon \gamma_0 z = \eta \epsilon \gamma \bar{s} , \]

\[ \epsilon \gamma^z \gamma s = 0 . \]

(2.1)

In total this configuration preserves one quarter of the M-fivebrane’s worldvolume supersymmetry, i.e. it preserves four supersymmetries, the equivalent of \( N = 4 \), \( D = 1 \) supersymmetry.

Before proceeding with the equations of motion it is necessary to consider the self-dual three-form \( h_{abc} \) on the M-fivebrane worldvolume. This can be decomposed into a four-dimensional vector \( v_a \) and anti-symmetric tensor \( F_{ab} \) as follows (all indices are in the tangent frame)

\[ h_{abz} = \kappa F_{ab} , \quad h_{abz} = \bar{\kappa} \bar{F}_{ab} , \]

\[ h_{azz} = i v_a , \]

(2.2)

where self-duality implies that \( h_{abc} = 2 \epsilon_{abcd} v^d \) and \( F_{ab} = \frac{i}{2} \epsilon_{abcd} \mathcal{F}^{cd} \). Here we have introduced \( \kappa \) which is an arbitrary function and can be removed by a redefinition
of $F_{\mu\nu}$. Later we will make contact with the Seiberg-Witten effective action where we write $F_{ab} = F_{ab} + \frac{i}{2} \varepsilon_{abcd} F^{cd}$. There we will need to choose $\kappa$ so that $F_{ab}$ satisfies the standard Bianchi identity. As discussed in [16] $h_{abc}$ is not a closed three-form. Rather it is related to a closed three-form $H_{abc}$ via

$$H_{abc} = (m^{-1})_{a}^{d} h_{dbc} ,$$

where $m_{a}^{b} = \delta_{a}^{b} - 2k_{a}^{b}$ and $k_{a}^{b} = h_{acd}h^{bcd}$. One can also derive the useful relation [16]

$$(m^{-1})_{a}^{b} = Q^{-1}(\delta_{a}^{b} + 2k_{a}^{b}) ,$$

where $Q = 1 - \frac{2}{3}k_{a}^{b}k_{b}^{a}$. Finally we will use the worldvolume metric $g_{mn}$ which is just the standard induced metric

$$g_{mn} = \eta_{mn} + \frac{1}{2} \partial_{m}s\partial_{n}\bar{s} + \frac{1}{2} \partial_{n}s\partial_{m}\bar{s} ,$$

$$= \epsilon_{a}^{m} \epsilon_{n}^{b} \eta_{ab} .$$

Since we are interested in the equations of motions at low energy we only consider expressions up to second order in spacetime derivatives. We also look for static solutions. For the convenience of the reader we list the components of the veilbein $e_{m}^{a}$, to second order in spacetime derivatives, for the geometry used in this paper

$$e_{\mu}^{a} = \delta_{\mu}^{a} - \frac{1}{2} \left( \frac{1}{\text{det}e} \right)^{2} (\bar{s}\partial_{s}\partial_{\mu}\bar{s}\partial_{a}\bar{s} + \partial_{s}\bar{s}\partial_{\mu}s\partial_{a}s) ,$$

$$+ \frac{1}{4} \left( 1 + |\partial_{s}|^{2} + |\bar{\partial}_{s}|^{2} \right) (\bar{s}\partial_{\mu}\bar{s} + \partial_{\mu}s\partial_{a}s) ,$$

$$e_{\mu}^{\bar{z}} = \frac{(X^{2} - |\partial_{s}|^{2})\bar{s}\partial_{\mu}\bar{s} + (X^{2} - |\bar{\partial}_{s}|^{2})\bar{s}\partial_{\mu}s}{X \text{det}e} ,$$

$$e_{\mu}^{z} = \frac{(X^{2} - |\partial_{s}|^{2})s\partial_{\mu}s + (X^{2} - |\bar{\partial}_{s}|^{2})s\partial_{\mu}\bar{s}}{X \text{det}e} ,$$

$$e_{z}^{\bar{z}} = \frac{\bar{s}\partial_{s}\bar{s}}{X} ,$$

$$e_{\bar{z}}^{z} = \frac{\bar{s}\partial_{s}s}{X} ,$$

$$e_{z}^{\bar{z}} = e_{\bar{z}}^{z} = X .$$
where
\[ X^2 = \frac{1}{2} \left[ (1 + |\partial s|^2 + |\bar{\partial}s|^2) + \text{det} e \right], \]
\[ \text{det} e = \sqrt{(1 + |\partial s|^2 + |\bar{\partial}s|^2)^2 - 4|\partial s|^2|\bar{\partial}s|^2}. \]

For a more detailed discussion of the worldvolume fields and equations of motion for the M-fivebrane we refer the reader to [16].

The full non-linear supersymmetry variation for worldvolume fermions is derived in detail in [17,18]

\[ \hat{\delta} \Theta = \frac{1}{2} \epsilon \left\{ \text{det}(e^{-1}) \partial_m X_c' (\gamma^m)(\gamma_c') - \frac{1}{3!} \text{det}(e^{-1}) \partial_m X_c' \partial_{n} X_c' \partial_{m} X_c' (\gamma^{m}_{m_{1}m_{2}m_{3}}) (\gamma_{c_{1}c_{2}c_{3}}) \right. \]
\[ \left. + \frac{1}{5!} \text{det}(e^{-1}) \partial_m X_c' \ldots \partial_m X_c' (\gamma^{m_{1} \ldots m_{5}})_{\alpha \beta} (\gamma_{c_{1} \ldots c_{5}}) \right. \]
\[ \left. - h^{m_{1}m_{2}m_{3}} \partial_m X_c' \partial_{m_{2}} X_c' \partial_{m_{3}} X_c' (\gamma_{m_{1}})_{\alpha \beta} (\gamma_{c_{1}'c_{2}'c_{3}'}) \right. \]
\[ \left. - \frac{1}{3} h^{m_{1}m_{2}m_{3}} (\gamma_{m_{1}m_{2}m_{3}}) \delta_{i}^{j} \right\}, \]

where \( \gamma_{mnp} = \gamma_{[m} \tilde{\gamma}_{n} \gamma_{p]} \) is anti-self-dual and we have adopted the convention that the \( \gamma_m \) matrices always appear with tangent indices (i.e. \( \gamma_m = \delta^a_m \gamma_a \)). Specialising to the case with two scalars yields, in complex notation,

\[ \hat{\delta} \Theta = \epsilon \left[ \frac{1}{2} \text{det} e \gamma^m \partial_m s \gamma_s + \frac{1}{2} \text{det} e \gamma^m \partial_m \bar{s} \gamma_{\bar{s}} - \frac{1}{2} h^{mnp} \gamma_m \partial_n \bar{s} \gamma_{s} \bar{s} - \frac{1}{3!} h^{mnp} \gamma_{mnp} \right], \]

(2.5)

Note that the projectors (2.1) imply that there are four independent terms appearing in (2.5) proportional to

\[ \epsilon \gamma_0 iz, \quad \epsilon \gamma_0 z \bar{z}, \quad \epsilon \gamma_{i} z \bar{z}, \quad \epsilon \gamma_0, \]

and their complex conjugates. Thus we may obtain the Bogomol'nyi equations by setting the corresponding coefficients to zero. Using the decomposition (2.2) this
yields

\[ \kappa F_{0i} = \frac{1}{8} \eta \left( \frac{1 + |s|^2 - |\bar{s}|^2}{X^2 - |\bar{s}|^2} \right) \left( \frac{X^2 \bar{s} \partial s + \partial \bar{s} \partial s \partial i \bar{s}}{X \det e} \right), \]

\[ v_0 = -\frac{i}{16} \eta \left( \frac{1 + |s|^2 - |\bar{s}|^2}{(X^2 - |\bar{s}|^2)^2} \right) \left[ (1 + |s|^2 + |\bar{s}|^2) \frac{\bar{s} \partial i \bar{s} \partial \bar{s}}{(X - |\bar{s}|^2)^2} \right] + \frac{i}{4} \eta \frac{\bar{s}}{X^2 - |\bar{s}|^2}, \]

\[ v_i = \frac{1}{16} \eta \bar{s} \left( \frac{1 + |s|^2 - |\bar{s}|^2}{(X^2 - |\bar{s}|^2)^2} \right) \frac{\epsilon_{ijk} \partial j s \partial k \bar{s}}{\det e}, \]

\[ \bar{s} = -\partial \bar{s}, \]

respectively.

Lastly we need to calculate the three-form \( H \) which most naturally appears in the equations of motion. To this end we calculate the matrix \( m^{-1} = Q^{-1}(1 + 2k) \) as

\[ m^{-1} = Q^{-1} \begin{pmatrix} \delta_{\mu}^{\nu} + 2k_{\mu}^{\nu} & 32i\kappa v_0 \bar{F}_{\mu}^{0} & -32i\kappa v_0 F_{\mu}^{0} \\ -16i\kappa v_0 \bar{F}^{\nu 0} & 1 - 16v_0^2 & 4\kappa^2 \bar{F}^{2} \\ 16i\kappa v_0 \bar{F}^{\nu 0} & 4\kappa^2 \bar{F}^{2} & 1 - 16v_0^2 \end{pmatrix}, \]

where

\[ k_{\mu}^{\nu} = 8v_0^2 \delta_{\mu}^{\nu} + 16v_0 v_0^2 + 4|\kappa|^2 F_{\mu\lambda} \bar{F}^{\nu\lambda} + 4|\kappa|^2 \bar{F}_{\mu\lambda} \bar{F}^{\nu\lambda}, \]

\[ Q = 1 - 256v_0^2 (v_0^2 - 2|\kappa|^2 F_{0i} \bar{F}^{0i}) , \]

and then use the definition (2.3). Despite the complicated form of these Bogomoln’yi equations one finds after a lengthy calculation that the three-form \( H \) takes on a relatively simple form. In particular, in the world frame we find
\[ H_{i\bar{z}z} = 0 \, , \]
\[ H_{0ij} = 0 \, , \]
\[ H_{0iz} = \frac{1}{8} \eta \partial_i s \, , \quad H_{0i\bar{z}} = \frac{1}{8} \eta \partial_i \bar{s} \, , \]
\[ H_{ijz} = \frac{i}{8} \eta \epsilon_{ijk} \partial^k s \, , \quad H_{ij\bar{z}} = -\frac{i}{8} \eta \epsilon_{ijk} \partial^k \bar{s} \, , \]
\[ H_{0\bar{z}z} = -\frac{1}{4} \eta \partial \bar{s} \, , \]
\[ H_{ijk} = -\frac{i}{8} \eta \epsilon_{ijk} \left( \frac{4\partial s + 2\partial s \partial_i s \partial^i \bar{s} + \partial s \partial_i \bar{s} \partial^i s - \partial s \partial_i s \partial^i \bar{s}}{1 + |\partial s|^2 - |\partial \bar{s}|^2} \right) . \tag{2.7} \]

Note that in contrast to the case with no self-dual strings, \( s \) is no longer a holomorphic function. Rather it only satisfies \( \partial s = -\partial \bar{s} \), which can be thought of as one of the Cauchy-Riemann equations (the other Cauchy-Riemann equation is \( \partial \bar{s} = \partial s \)). This non-holomorphicity is not unexpected in light of the observations in [14,15] that the M-fivebrane wraps a Riemann surface with one complex structure while the M-twobrane wraps a Riemann surface with a different complex structure (note that the embedding space \( \mathbb{R}^3 \times S^1 \) is hyper-Kähler and has three complex structures). Here we see that the flux of \( H \) through the surface \( \Sigma \) measures the non-holomorphicity in \( s \), i.e. the self-dual strings distort the holomorphic structure of the Riemann surface.

The Equation of Motion

Now we must turn our attention to solving the M-fivebrane’s equation of motion. Since the Bogomoln’yi equation relates the scalars to the three-form we need only check the equation of motion for \( H_{mnp} \). This in turn is equivalent to the condition that \( \partial_{[m} H_{npq]} = 0 \) [16]. Most of the equations obtained from this condition are identically true on behalf of the Bogomoln’yi equations. The only non-trivial equation is \( \partial_{[i} H_{ijk]} = 0 \) which yields

\[ \partial_i \partial^i s + R^2 \Lambda^4 \partial \left[ \frac{4R^{-2} \partial \bar{s} + 2\partial s \partial_i s \partial^i \bar{s} + \partial s \partial_i \bar{s} \partial^i s - \partial s \partial_i s \partial^i s}{1 + R^2 \Lambda^4 |\partial s|^2 - R^2 \Lambda^4 |\partial \bar{s}|^2} \right] = 0 \, , \tag{2.8} \]

where we have reintroduced the constants \( R \) and \( \Lambda \) for future reference. Alterna-
This document discusses the properties and applications of a three-form and its potential. It focuses on the choice of potential components and their implications for solving certain equations. The document also explores the behavior of strings wrapped around a cycle of a manifold, specifically considering the conditions under which the string solutions asymptotically approach zero as certain variables tend to infinity.
make contact with four-dimensional $SU(2)$ gauge theory we impose precisely the same considerations as in [1]. Namely, upon compactification on $X^7$ to type IIA string theory in ten dimensions, the M-fivebranes should reduce to two parallel NS-fivebranes with two D-fourbranes suspended between them. The appropriate curve $s_0$ is then [1]

$$e^{-s_0} = z^2 - u \pm \sqrt{(z^2 - u)^2 - 1},$$  \hspace{1cm} \text{(2.11)}

where $u(x)$ is the modulus of the curve and takes the value $u_0$ as $|x| \to \infty$. Let us denote the resulting Riemann surface by $\Sigma_0$, which is of course just the Seiberg-Witten curve. We therefore impose the boundary condition that $s \to s_0$ as $|x| \to \infty$ where $s_0$ is given by (2.11).

The above boundary conditions impose smoothness of the solutions at spatial infinity. Let us now look for smooth solutions to the equations of motion (2.9) and (2.10) in the interior. We will limit our discussion here to a single self-dual string depending only on $z, \bar{z}$ and $|x|$. Since we want $H_{0iz}$ to be well defined at the origin we will look for solutions with $\partial_i s \to 0$ as $|x| \to 0$. In this case we find, for small $|x|$,

$$\partial^k b_k = -\frac{4i\bar{s}(0)}{1 + |\partial s(0)|^2 - |ar{\partial}s(0)|^2},$$

where $s(0) = s(z, \bar{z}, 0)$. If we now write $b^k = b^k(|x|)$ we find, near $|x| = 0$,

$$b^k = -\frac{4}{3} i x^k \frac{\bar{s}(0)}{1 + |\partial s(0)|^2 - |ar{\partial}s(0)|^2} + O(|x|^3),$$

$$s(z, \bar{z}, |x|) = s(0) - \frac{2}{3} |x|^2 \partial \left[ \frac{\bar{s}(0)}{1 + |\partial s(0)|^2 - |ar{\partial}s(0)|^2} \right] + O(|x|^4).$$ \hspace{1cm} \text{(2.12)}

Thus we find a smooth non-trivial solution, so long as $\bar{s} \neq 0$. In the holomorphic case we shall see below that there is no finite solution except $\partial_i s \equiv 0$. To show that there are indeed solutions which are smooth everywhere requires that (2.12) can be extended smoothly to all $|x|$ and furthermore that it satisfies the boundary conditions imposed above.
As an aside we note what happens when $\partial_i s \equiv 0$, $\bar{\partial}s \neq 0$. In this case the closure of $H$ leaves us with the condition

$$\frac{\bar{\partial}s}{1 + |\partial s|^2 - |\bar{\partial}s|^2} = i \theta ,$$

(2.13)

where $\theta$ is any real constant. However, in contrast to the solutions of interest in the rest of this paper, it follows from (2.13) that $\bar{\partial}s$ must be everywhere non-vanishing. In addition one can check that although (2.13) is a first order condition it automatically implies the second order equation

$$0 = g^{mn} \nabla_m \nabla_n s
= -\frac{1}{2} \frac{1}{\text{det} e} \left\{ \partial \left[ \frac{1 - |\partial s|^2 + |\bar{\partial}s|^2}{\text{det} e} \partial s \right] + \bar{\partial} \left[ \frac{1 + |\partial s|^2 - |\bar{\partial}s|^2}{\text{det} e} \partial s \right] \right\},$$

(2.14)

which this is just the familiar minimal area equations for the surface defined by $s(z, \bar{z})$

$$S = \int d^2 z \, \text{det} e .$$

3. The Large Distance, Seiberg-Witten Limit

In this section we will analyse the case $\bar{\partial}s = 0$. Since in the large $|x|$ limit, $s$ becomes a holomorphic function $s_0(z)$, the analysis in this section applies to this asymptotic regime. More precisely we expect this to be a suitable approximation when $|x| >> l$, where $l$ is the size of the cycles on the Riemann surface. In the interior $s$ will be non-holomorphic and the dynamics will differ significantly. However this Seiberg-Witten limit is sufficient to evaluate the charges as seen at infinity.

For $\bar{\partial}s_0 = 0$ the equation of motion becomes simply

$$\partial_i \bar{\partial}^i s_0 + \partial \left[ \frac{\partial s_0 \partial_i \bar{\partial}^i s_0 - \bar{\partial}s_0 \partial_i s_0 \bar{\partial}^i s_0}{1 + |\partial s_0|^2} \right] = 0 .$$

Furthermore from (2.11) one can see that $\partial_i s_0 = \partial_i u s_0/du = \partial_i u \lambda_z$ where $\lambda = \lambda_z dz$ is the holomorphic one form on the Riemann surface $\Sigma_0$ described by $s_0$. We
may follow the method of [4,5] to reduce this equation of motion to four dimensions. To be more precise we consider

\[
\int \left\{ \partial_i \partial^i \sigma_0 + \partial \left[ \frac{\partial \sigma_0 \partial_i \sigma_i \partial \sigma_0 - \partial \sigma_0 \partial_i \sigma_i \partial \sigma_0}{1 + |\partial \sigma_0|^2} \right] \right\} dz \wedge \bar{\lambda} = 0 ,
\]

and express the resulting four dimensional equations in terms of the periods [6]

\[
a = \oint_A \sigma_0 dz , \quad a_D = \oint_B \sigma_0 dz , \quad (3.1)
\]

where \(A\) and \(B\) are a basis of one cycles of the Riemann Surface. The equation of motion can be expanded into the form

\[
\partial^i \partial_i uI + \partial^i u \partial_i u \frac{dI}{du} - \partial_i \bar{u} \partial^i \bar{u} J + \partial_i \bar{u} \partial^i \bar{u} K = 0 ,
\]

where we have introduced the integrals

\[
I = \int \lambda \wedge \bar{\lambda} = (\tau - \bar{\tau}) \frac{da d\bar{a}}{du d\bar{u}} ,
\]

\[
J = \int \partial \left( \frac{\lambda^2 \partial \sigma_0}{1 + |\partial \sigma_0|^2} \right) dz \wedge \bar{\lambda} = 0 ,
\]

\[
K = \int \partial \left( \frac{\bar{\lambda}^2 \partial \sigma_0}{1 + |\partial \sigma_0|^2} \right) dz \wedge \bar{\lambda} = -\frac{d\tau}{da} \left( \frac{d\bar{a}}{d\bar{u}} \right)^3 ,
\]

which were evaluated in [4,5]. In this way we arrive at the four dimensional equation of motion

\[
\partial_i \partial^i a(\tau - \bar{\tau}) + \partial^i a \partial_i a \frac{d\tau}{da} - \partial^i \bar{a} \partial_i \bar{a} \frac{d\bar{\tau}}{da} = 0 ,
\]

or more simply

\[
\text{Im } \partial_i \partial^i a_D = 0 , \quad \text{Im } \partial_i \partial^i a = 0 . \quad (3.2)
\]

These equations are nothing more than a special case of the equation of motion
obtained from the Seiberg-Witten action

\[ S_{SW} = \int d^4 x \Im (\tau \partial_{\mu} a \partial^{\mu} \bar{a} + 16 \tau F_{\mu\nu} F^{\mu\nu}) , \]  

(3.3)

and subsequently imposing the Bogomoln’yj condition

\[ \partial_i a = 8 \eta F_{0i} , \]  

(3.4)

Comparing this with the first equation in (2.6)

\[ \partial_i s_0 = 8 \eta \kappa \det(e) F_{0i} , \]  

for the case \( \bar{s} = 0 \) requires that \( \kappa = \det(e^{-1}) \frac{ds_0}{da} \). Indeed this is precisely the normalisation found in [4,5] where it was needed to ensure that \( F_{\mu\nu} \) is a curl and so can be identified with the field strength of a gauge field, namely the superpartner of the scalar \( a \). The second equation in (3.2) is then just the Bianchi identity for \( F_{\mu\nu} \).

The Laplace equations (3.2) have the general solution for point sources given by

\[ \Im a = \langle \Im a \rangle + \sum_n \frac{Q_n}{|x - y_n|} , \quad \Im a_D = \langle \Im a_D \rangle + \sum_n \frac{P_n}{|x - y_n|} , \]  

(3.5)

where \( Q_n \) and \( P_n \) are constants. Here \( y_n \) the centres of the solitons and \( \langle \Im a \rangle \) and \( \langle \Im a_D \rangle \) are the imaginary parts of the vacuum expectation values of \( a \) and \( a_D \) respectively. In addition to (3.5) one needs the exact form for \( a_D \) as a function of \( a \) in order to find \( \text{Re}(a) \) and \( \text{Re}(a_D) \). This can be obtained from (2.11) and (3.1) leading to an expansion [6]

\[ a_D = i \frac{a}{\pi} + i \frac{a}{\pi} \text{Im} a^2 + i a \sum_{k=1}^{\infty} \frac{c_k}{a^{4k}} , \]  

(3.6)

where the coefficients are \( c_k \) are real. From the Bogomoln’yj condition (3.4) we
find

\[ E_i = \frac{1}{8} \eta \partial_i \text{Re}(a), \quad B_i = -\frac{1}{8} \eta \partial_i \text{Im}(a) \]  

(3.7)

where \( E_i = F_{0i} \) and \( B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \). Thus magnetic states correspond to imaginary \( a \) and electric states to real \( a \).

For a magnetic state, where \( \text{Re}(a) \) is constant, the asymptotic Magnetic field is precisely what one expects

\[ B^i = -\frac{1}{8} \eta \partial^i \text{Im}(a) = \frac{1}{8} \eta \sum_n \frac{Q_n (x^i - y^i_n)}{|x - y_n|^3}. \]

From (3.6) we may (in principle) determine \( a_D \). Note that the presence of the logarithm and the reality of the coefficients \( c_k \) in (3.6) imply that \( \text{Im}(a_D) = -\text{Im}(a) \), which is consistent with (3.2). One then sees that the \( Q_n \) are the magnetic charges of the monopoles.

The situation is rather different for electric states where \( \text{Im}(a) \) is constant. In this case the asymptotic form for \( \text{Re}(a) \) is determined by inverting the function \( a_D(a) \), which is a very subtle system to solve. Noting that \( \partial_i \text{Im}(a) = 0 \) and \( \tau = da_D/da \) we find

\[ E^i = \frac{1}{8} \eta \partial^i \text{Re}(a) = -\frac{1}{8} \eta \frac{1}{\text{Im}\tau} \sum_n \frac{P_n (x^i - y^i_n)}{|x - y_n|^3}. \]

Thus from the leading order behaviour at infinity we see that \(-P_n/\text{Im}\tau(< a >)\) can be identified with the electric charges of the solitons. However, the presence of the non-trivial terms in \( \tau \) alters the behaviour of the electric fields in the interior, corresponding to the (anti-) screening of electric charge.

Finally we note that if we follow the Seiberg-Witten equations (3.2) into the centre of a soliton we find \( \text{Im}(a), \text{Im}(a_D) \to \infty \). Thus we are pushed into a regime where only the perturbative terms (i.e. the first two terms in (3.6)) in the effective action are important. This presumably reflects the fact that the underlying Yang-Mills theory is asymptotically free.
We can evaluate the energy of these configurations from (3.3) to be

\[ E = \int d^3x \text{ Im } \tau \left( \partial_i a \partial_i \bar{a} + 32E_i E^i + 32B_i B^i \right) = \frac{3}{2} \int d^3x \left( \partial_i \text{Re}(a) \partial_i \text{Im}(a_D) - \partial_i \text{Re}(a) \partial_i \text{Im}(a) \right), \tag{3.8} \]

where we have used the fact that the integrand is a total derivative to express \( E \) as a surface integral. One can easily see that \( E \) diverges due to the behaviour near the centres of the solitons. In particular, from (3.5) one sees that the measures \( dS^i \partial_i \text{Im}(a) \) and \( dS^i \partial_i \text{Im}(a_D) \) remain finite near the centre of a soliton. However, because \( |a| \to \infty \), only the first two terms in (3.6) are important so that \( \text{Re}(a) \) and \( \text{Re}(a_D) \) diverge, rendering \( E \) infinite. In other words if we set \( \partial_s = 0 \) everywhere then we obtain solitons to the M-fivebrane equations of motion which can be viewed as solutions of the Seiberg-Witten action but with a divergent mass. Of course this divergence near the self-dual strings contradicts the low energy approximation we have used and therefore does not accurately describe the smooth BPS states of the M-fivebrane that we wish to study.

One might think that this divergence is specific to the form of the function \( a_D(a) \) in the Seiberg-Witten solution. For example one might consider the curves corresponding to field theories with a positive \( \beta \)-function. Let us suppose that instead of (2.11) we had a curve for which \( a_D(a) \) was finite as \( a \to \infty \). There would therefore be no divergence in the energy at the soliton core. Such a theory would be unphysical however since \( \tau = da_D/da \) and hence \( \tau(\infty) = 0 \). Thus effective action would vanish at \( a = \infty \) and furthermore the holomorphicity of \( \tau \) would lead to a violation of \( \text{Im}\tau \geq 0 \). In fact this is impossible given the construction of \( \tau \) as the period matrix of a Riemann surface. Alternatively, if one has a smooth solution then the energy \( E \) can be evaluated by integrating over the sphere at infinity but this leads to an expression which is not in general positive definite, contradicting the manifest positive definite nature of \( E \).
Thus we find that by imposing the condition $\bar{\partial}s = 0$ we do not obtain smooth, finite energy solutions. Indeed these solutions do not seem physical from the point of view of the M-fivebrane/M-twobrane system either. To see this consider a self-dual string which is wrapped around the $B$-cycle of the Riemann surface (since $\bar{\partial}s = 0$ there is a clear notion of a Riemann surface). According to the solution (3.5), as we approach the string at $|x| = 0$, $a$ diverges. Thus $u \propto a^2$ also diverges which corresponds to the two D-fourbranes in the type IIA picture moving infinity far apart. On the Riemann surface this means that the length of the $B$-cycle grows without bound. In other words the self-dual string is choosing to wrap around a cycle of infinite length. This helps to explain why the corresponding four-dimensional solution has a divergent energy, even though we have argued in the introduction that the self-dual string has a finite tension. Since the condition $\bar{\partial}s = 0$ ignores most of the details of where the self-dual string lies on the Riemann surface, it is perhaps not surprising that we obtain unphysical solutions. In the next section we shall analysis the full non-holomorphic equations, including the effects of self-dual strings. Intuitively one sees that there might be finite energy solutions since we would expect the self-dual string to wrap around cycles with as small a length as possible. Thus at $|x| = 0$ one might suppose that the cycle which the self-dual string wraps shrinks to zero size. In the Seiberg-Witten description these are the strong coupling singularities where $\text{Re}(a)$ and $\text{Re}(a_D)$ are finite, leading to a convergent form for $E$ above. Of course these arguments are highly speculative, nevertheless they are indications that finite energy solutions do exist.
4. Moduli Space

We now wish to analyse the full equation of motion for self-dual strings on a Riemann surface, i.e. without assuming that $s$ is holomorphic. As mentioned above we will use the boundary condition that at spatial infinity $s$ just the Seiberg-Witten curve (2.11). The full equations of motion then describe how $s(z, \bar{z})$ behaves as a function of $x$. Note that, even though $s$ is no longer holomorphic, the the manifold $\Sigma$ is still topologically a Riemann surface. This is because, by virtue of the boundary condition and the equation of motion, it is just a smooth (although non-holomorphic) deformation of the Seiberg-Witten curve. In other words we may still think of $\Sigma$ as a genus one Riemann surface, but one that is not embedded holomorphically in spacetime.

First we note from (2.8) that, from the four-dimensional point of view, the term

$$\Lambda^4 \partial \left[ \frac{4\tilde{\partial}s}{1 + R^2 \Lambda^4 |\partial s|^2 - R^2 \Lambda^4 |\tilde{\partial}s|^2} \right]$$

acts as a source for $\partial_i \partial^i s$. To help understand this term let us expand $s = s_0 + R^2 s_1$, where $\tilde{\partial}s_0 = 0$, and consider the type IIA string theory description by letting $R \rightarrow 0$, keeping $R\Lambda^2$ fixed. The equation of motion for $s$ now becomes

$$\partial_i \partial^i s_0 + R^2 \Lambda^4 \partial \left[ \frac{\partial s_0 \partial \bar{s}_0 \partial^i s_0 - \tilde{\partial}s_0 \partial_i s_0 \partial^i \bar{s}_0}{1 + R^2 \Lambda^4 |\partial s_0|^2} \right] = -4R^2 \Lambda^4 \partial \left[ \frac{\tilde{\partial}s_1}{1 + R^2 \Lambda^4 |\partial s_0|^2} \right]. \tag{4.1}$$

We may proceed to obtain four-dimensional equations for the BPS states as follows. We write (4.1) as

$$E_z = \partial_i \partial^i s_0 + \partial(T - \bar{T}) + 4R^2 \Lambda^4 \partial \left[ \frac{\tilde{\partial}s_1}{1 + R^2 \Lambda^4 |\partial s_0|^2} \right] = 0,$$

where

$$T = R^2 \Lambda^4 \frac{\partial s_0 \partial \bar{s}_0 \partial^i \bar{s}_0}{1 + R^2 \Lambda^4 |\partial s_0|^2}.$$
Next we follow \([4,5]\) and consider
\[
0 = \int_A \left( E_z dz - \bar{E}_z d\bar{z} \right)
= \int_A \left( \partial_i \partial^i s dz - \partial_i \bar{\partial}^i s d\bar{z} + d(T - \bar{T}) + 4R^2 \Lambda^4 d \left[ \frac{\bar{\partial}s_1}{1 + R^2 \Lambda^4 |\partial s_0|^2} \right] \right).
\]

Here \(A\) is the \(A\)-cycle of the Riemann surface defined by the curve \(s_0(z)\). In particular \(s_0\) is given in (2.11). It is therefore possible to chose the \(A\)-cycle to avoid any singular points of \(T - \bar{T}\). Since \(T\) is single valued the contribution of \(T\) in the four-dimensional equation of motion vanishes and we arrive at the modified Seiberg-Witten equation
\[
\partial_i \partial^i \text{Im}(a) = 2iR^2 \Lambda^4 \int_A d \left[ \frac{\bar{\partial}s_1}{1 + R^2 \Lambda^4 |\partial s_0|^2} \right]. \tag{4.2}
\]

Similarly we may reduce over the \(B\)-cycle to obtain
\[
\partial_i \partial^i \text{Im}(a_D) = 2iR^2 \Lambda^4 \int_B d \left[ \frac{\bar{\partial}s_1}{1 + R^2 \Lambda^4 |\partial s_0|^2} \right]. \tag{4.3}
\]

Clearly this method of dimensional reduction agrees with the one in section three when \(s_1 = 0\).

In the above \(\bar{\partial}s_1 \propto H_{0z\bar{z}}\) also appears as a total derivative. However, since we do not know the form of \(\bar{\partial}s_1\) we can not say that it is single-valued and hence that the integrals vanish. In fact a self-dual string wrapped around a Riemann surface acts as a domain wall in the Riemann surface. Thus in particular if the self-dual string wraps around the \(B\)-cycle then, as the \(A\)-cycle is traversed, \(H_{0z\bar{z}}\) will increase by one unit of charge when the self-dual string is crossed. Since the Riemann surface is compact, one sees that \(H_{0z\bar{z}}\), and hence \(\bar{\partial}s_1\), must be multi-valued. Therefore we expect that the \(A\)-cycle integral will be non-zero. A similar situation occurs for self-dual strings wrapped around the \(A\)-cycle. Thus we find
there is a source for $\text{Im}(a)$ if the self-dual string wraps the $B$-cycle and a source for $\text{Im}(a_D)$ if the self-dual string wraps the $A$-cycle. In particular examining (3.7) shows that if the self-dual string wraps the $B$-cycle we obtain magnetic sources and if it wraps the $A$-cycle we obtain electric sources. This agrees with previous studies [13,14,15] which identified the electric and magnetic states as corresponding to self-dual strings wrapped around $A$ or $B$ cycles.

This is a detailed analysis of the equations of motion that we have been able to obtain. In the above we have argued that there are smooth solutions to the equations of motion so let us now assume this to be the case. We can then, in principle, construct the low energy equations of motion for the solitons by allowing their moduli to become time-dependent. The natural generalisation of (2.8) to time-dependent configurations is

$$
\partial_\mu \partial^\mu s + \partial \left[ \frac{4\ddot{s} + 2s \partial_\mu s \partial^{\mu} \ddot{s} + 2s \partial_\mu \dddot{s} + \ddot{s} s \partial_\mu \partial^{\mu} s - \dddot{s} s \partial_\mu \partial^{\mu} s}{1 + |\partial s|^2 - |\ddot{s}|^2} \right] = 0 . \tag{4.4}
$$

In the case that $\ddot{s} = 0$ one can check that this is indeed what the M-fivebrane equations of motion yield by comparing with the equations of motion in [4]. However we have not checked this for the general case, although clearly it is the only possibility compatible with Lorentz invariance. After substituting in the general solution one would then integrate the equations of motion over the $x^i, z, \bar{z}$ coordinates. The resulting equations may also be viewed as arising from a one dimensional sigma model with the the moduli space of solutions for a target space.

Let us illustrate this for the case $\ddot{s} = 0$, even though for this case we are not able to obtain smooth low energy behaviour. Here one can follow precisely the same steps for equation (4.4) that we did for (2.8) in the last section. In this way we arrive at the four-dimensional equation

$$
\partial_\mu \partial^\mu a(\tau - \bar{\tau}) + \partial^\mu a \partial_\mu a \frac{d\tau}{da} - \partial^\mu \bar{a} \partial_\mu \bar{a} \frac{d\bar{\tau}}{d\bar{a}} = 0 .
$$

If we now substitute in the solution to the Bogomoln’yi equations (3.2) with time-
dependent moduli we obtain
\[ \ddot{a}(\tau - \bar{\tau}) + a^2 \frac{d\tau}{da} - \dot{a}^2 \frac{d\bar{\tau}}{d\bar{a}} = 0, \]
where a dot denotes a time derivative. These equations can now be viewed as arising from the effective action
\[ S = \int dt d^3 x \text{Im} \tau |\dot{a}|^2. \]
The last step is to write $\dot{a} = \sum_\alpha \frac{\partial a}{\partial y^\alpha} \dot{y}^\alpha$, where $y^\alpha$ are the moduli, and integrate over space. If we had smooth solutions this would then lead to an effective action
\[ S = \int dt g_{\alpha\beta} \dot{y}^\alpha \dot{y}^\beta. \]
Here $g_{\alpha\beta}$ is an induced metric on the moduli space of solutions. For the rest of this section we will try to consider some properties of this metric, for the general case $\partial s \neq 0$ where we expect smooth solutions to exist.

Since these solutions preserve one quarter of the sixteen worldvolume supersymmetries, this sigma model must admit $N = 4$, $D = 1$ supersymmetry. However there are two types of multiplet in one dimension with four supercharges. The first, $N = 4A$, is essentially the dimensional reduction of two dimensional $(2,2)$ supersymmetry and requires that the moduli space metric is Kähler. The second, $N = 4B$, is related to the reduction of two-dimensional $(4,0)$ supersymmetry and this requires that the moduli space metric is Hyper-Kähler, or hyper-Kähler with torsion [19]. Actually there is a subtlety here in that the three complex structures need not be covariantly constant, hence they need not be Hyper-Kähler in the strict sense of the word.

First let us recall the situation for monopoles in $N = 2$ super Yang-Mills gauge theory. It is well known that in a monopole background the only fermion zero modes are chiral in a certain Euclidean sense (see for example [20]). Thus
the supersymmetry is of the $N = 4B$ type. This means that the moduli space metric, which again must admit four supersymmetries, is hyper-Kähler (possibly with torsion), rather than just Kähler (although the complex structures need not be covariantly constant). In fact it is well known and can be proved directly that the monopole moduli space metric is hyper-Kähler (in the strict sense of the word) [21].

To check the chirality of the preserved supersymmetries in our solution we must construct the four-dimensional $\Gamma$-matrices. Unfortunately, the reduction we considered in section two to obtain six-dimensional $\Gamma$-matrices is not very useful here. Instead, let us denote by $\Gamma(D)$ the $\Gamma$-matrices in $D$ dimensions. For $d$ even we can always consider the decomposition from $D$ to $d$ dimensions given by

$$
\Gamma(D) = \begin{cases} 
\Gamma^a_d \otimes \mathbb{I} & a = 0, 1, \ldots, d-1 \\
\Gamma^{d+1}_d \otimes \Gamma^a_{(D-d)} & a' = d, \ldots, n-1
\end{cases},
$$

where $\Gamma^{d+1}_d = c \Gamma^{012\ldots d-1}_d$ and $c$ is chosen so that $(\Gamma^{d+1}_d)^2 = 1$. Thus for the case in hand we may set

$$
\Gamma^0,\ldots,5_{(11)} = \Gamma^0,\ldots,5_{(6)} \otimes \mathbb{I}, \quad \Gamma^6,\ldots,10_{(11)} = \Gamma^7_{(6)} \otimes \Gamma^1,\ldots,5_{(5)},
$$

where $\Gamma^7_{(6)} = \Gamma^{012345}_{(6)}$. The Euclidean five-dimensional $\Gamma$-matrices arise from the transverse space to the M-fivebrane and can be further decomposed as

$$
\Gamma^{6,7}_{(5)} = \tau^{1,2} \otimes \mathbb{I}, \quad \Gamma^{8,9,10}_{(5)} = \tau^3 \otimes \Sigma^{1,2,3},
$$

reflecting the presence of the second M-fivebrane. Next we can further reduce to four dimensions

$$
\Gamma^{0,1,2,3}_{(6)} = \Gamma^{0,1,2,3}_{(4)} \otimes \mathbb{I}, \quad \Gamma^{4,5}_{(6)} = \Gamma^5_{(4)} \otimes \sigma^{1,2},
$$

Here and above $\Sigma^i, \sigma^i$ and $\tau^i, \ i = 1, 2, 3$ are three sets of Pauli matrices and $\Gamma^5_{(4)} = -i \Gamma^{0123}_{(4)}$. Under this decomposition the spinors $\epsilon$ now carry four indices:
\( \epsilon^{\alpha,r,r',i} \), where \( \alpha = 1, 2, 3, 4 \), \( r = 1, 2 \), \( r' = 1, 2 \) and \( i = 1, 2, 3 \). The first index is just the four-dimensional \( \text{Spin}(1, 3) \) index. The last index carries a representation of \( SO(3) \cong SU(2) \) which can be identified with the R-symmetry of the four-dimensional \( N = 2 \) superalgebra of the threebrane soliton. The other two indices represent internal symmetries which are broken by the intersecting M-fivebranes.

The projectors of the two M-fivebranes reduce the supersymmetries to \( N = 2 \) in four dimensions. In eleven dimensions the projections are \( \epsilon \Gamma^{012345}_{(11)} = \epsilon \Gamma^{012367}_{(11)} = \epsilon \), which are expressed in four-dimensions as

\[
-\epsilon \Gamma^5_4 \otimes \sigma^3 \otimes I \otimes I = -\epsilon \Gamma^5_4 \otimes I \otimes \tau^3 \otimes I = \epsilon ,
\]

respectively. Thus the four-dimensional \( N = 2 \) supersymmetries have their chirality correlated with internal \( r, r' \) indices. Next we must construct the M-twobrane projector \( \epsilon \Gamma^{057}_{(11)} = \eta \epsilon \), where again \( \eta = \pm 1 \). In four-dimensions this becomes

\[
-i \epsilon \Gamma^0_4 \otimes \sigma^1 \otimes \tau^2 \otimes I = \eta \epsilon .
\]

To see that this projects \( \epsilon \) on to a set of chiral supersymmetries we consider the Euclidean four-dimensional \( \Gamma \)-matrices defined by

\[
\bar{\Gamma}^{1,2,3} \equiv \Gamma^{1,2,3}_{(4)} \otimes I \otimes \tau^2 \otimes I , \quad \bar{\Gamma}^4 \equiv \Gamma^5_4 \otimes \sigma^1 \otimes I \otimes I .
\]

One can easily check that \( \bar{\Gamma}^5 \equiv \bar{\Gamma}^{1,2,3}_4 = \Gamma^{057}_{(11)} \). Thus the supersymmetries preserved by the solitons are chiral with respect to this Euclidean four-dimensional \( \Gamma \)-matrix algebra

\[
\epsilon \bar{\Gamma}^5 = \eta \epsilon .
\]

It then follows that the moduli space sigma model has \( N = 4B \) supersymmetry in one dimension. Another way to see this is to note that the preserved supersymmetries transform non-trivially under the \( SO(3) \) R-symmetry. This also forces
the one-dimensional sigma model effective theory to have $N = 4B$ supersymmetry. By the same reasoning as above it follows that the moduli space of solutions is again hyper-Kähler or hyper-Kähler with torsion. However it seems reasonable to assume that, as in the Yang-Mills case, there is no torsion and the complex structures are indeed covariantly constant. In addition, due to overall translational and rotational symmetries of the configurations, the soliton solutions constructed here have exactly the same symmetry properties as monopoles in the Yang-Mills theory, i.e., translational symmetry of the centre of mass and the action of the $SO(3)$ rotation group. We are therefore led to the conjecture that the two moduli space metrics agree.

It follows from the $4B$ supersymmetry that their must be $4k$ bosonic zero modes for a given soliton solution. Again this is precisely the same as the number of bosonic zero modes of a $k$-monopole in $SU(2)$ Yang-Mills theory. As with monopoles it is clear that $3k$ of these zero modes come from the locations of the centres of the solitons, i.e. the $y_n$ in (3.5). However, the other $k$ zero modes are less obvious. Their origin is well-known though and they arise as non-trivial gauge transformations of the vector field at infinity [21].

It is instructive to recall the role of these other zero modes in the standard treatment of monopoles in $N = 2$ gauge theory. Namely dyonic states are obtained by turning on their conjugate momentum. Furthermore these zero modes are periodic and hence electric charge is discrete in the quantum theory. However, from the point of view of the M-fivebrane we have seen (see also [13,14,15]) that dyons correspond to wrapping the self-dual string around a combination of $A$ and $B$ cycles. This suggests an interesting interpretation for the periodic zero modes in terms of the geometry of the surface $\Sigma$.

In the situation considered here this leads to a slight puzzle. Namely, if the one gets $k$ bosonic zero modes from non-trivial gauge transformations at infinity, then what happened to the fourth translational zero mode of a self-dual string in six dimensions? In fact it is not hard to see that the presence of the Riemann
surface, i.e. the two M-fivebranes, removes the fourth zero mode. More precisely, a self-dual string will only have a translational zero mode if there is an isometry in a particular direction. However, the metric on the Riemann surface ensures that there is a minimum length cycle, about which the self-dual string will wrap. Moving the self-dual string off this cycle will then cost energy as the string length is increased.

In the limit where $\bar{\partial}s = 0$ we can determine to some extent where the string must wrap. One can easily see from (2.11) that there is a discrete symmetry $s_0 \leftrightarrow -s_0$, corresponding to interchanging the two NS-fivebranes in the type IIA picture. From the point of view of the Riemann surface this corresponds to interchanging the two sheets which cover the plane. Let us assume that there is a unique minimal length cycle, for each homology class, which must therefore be invariant under this symmetry. Thus a minimum length curve must lie on both sheets of the $z$ plane and therefore, since it is connected, it must pass through the branch cuts. For the $A$-cycle this means that the curve must run between the two branch points.

For the $B$-cycle one finds that, to respect the $s \leftrightarrow -s$ symmetry, the curve must double-up unless it too passes through the branch points. But the branch points $z = \pm \sqrt{u \pm 1}$ are precisely the points where $s_0 = 0$. So we see that the self-dual string must wrap a cycle running between the zeros of $s_0$. This interpretation of the zeros of the Seiberg-Witten differential $s_0 dz$ has arisen before within string theory studies of the Seiberg-Witten solution [22,23].

5. Conclusion

In this paper we have studied BPS states of the M-fivebrane which, under type IIA/ M-theory duality, correspond to monopole states in $N = 2 SU(2)$ super-Yang-Mills theory. In particular we discussed a differential equation for the solitons and the relation of these solutions to Bogomoln’yi states in the Seiberg-Witten effective theory. We saw that the M-fivebrane theory led to significant corrections to Seiberg-Witten dynamics and suggested the existence of smooth non-singular
solutions. Thus we argued that there is a smooth moduli space of solutions to the M-fivebrane Bogomol'nyi equations. We also argued that the metric on this space is hyper-Kähler and hence it is natural to relate it to the monopole moduli space metric. Let us conclude now with some additional comments on our work.

In general we have suppressed the dependence on the parameters $R$ and $\Lambda$. Indeed for the case that $\bar{\partial}s = 0$ the low energy dynamics are in fact independent of both $R$ and $\Lambda$ [1,4]. This is a crucial point which leads to the expectation that the low energy dynamics are precisely the same as for the perturbative Yang-Mills description [1] (obtained from the type IIA string theory picture as $R \rightarrow 0$). In the $\bar{\partial}s \neq 0$ case, however, one does see a non-trivial dependence on the parameter $R$. However this also leads to an extra parameter in the low energy theory which may also enter into the moduli space metric. We have argued that this moduli space has the same symmetries as monopole moduli space. With the exception of the one and two monopole moduli spaces, these symmetries do not uniquely specify the monopole metric [21]. Thus it is possible that this extra parameter is associated to deformations of the monopole moduli space which preserve the symmetries.

A final point to consider is the stability of the BPS states. It was shown in [6] that the spectrum of BPS states is non-trivial and indeed stable BPS states can be made unstable as one varies the vacuum expectation value $<a>$. In particular, at weak coupling the theory contains dyons with arbitrary integer electric charge and unit magnetic charge and the $W^\pm$ bosons. However at strong coupling only the monopole and dyon with unit electric charge are stable [7]. Note that in the case $\bar{\partial}s = 0$ the BPS states are given by self-dual strings wrapped around the Riemann surface [13,14,15]. Modular invariance of the Riemann surface presumably leads to a complete $SL(2,\mathbb{Z})$ spectrum of monopole/dyon states. However we have seen that in the full M-fivebrane description the Riemann surface is no longer holomorphically embedded in spacetime and hence there is no $SL(2,\mathbb{Z})$ modular symmetry. Therefore the M-fivebrane does not immediately predict a full $SL(2,\mathbb{Z})$ spectrum of monopole/dyon states. It would be interesting to see if the correct BPS states can be predicted by the M-fivebrane approach presented here.
We would like to thank Jerome Gauntlett for helpful discussions.

REFERENCES

1. E. Witten, Nucl. Phys. B500 (1997) 3, hep-th/9703166
2. P.S. Howe, N.D. Lambert and P.C. West, Phys. Lett. B (1998), hep-th/9710034
3. P.S. Howe, N.D. Lambert and P.C. West, Phys. Lett. B419 (1998) 79, hep-th/9710033
4. N.D. Lambert and P.C. West, Nucl. Phys. B524 (1998) 141, hep-th/9711040
5. N.D. Lambert and P.C. West, Phys. Lett. B424 (1998) 281, hep-th/9801104
6. N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087
7. F. Ferrari and A. Bilal, Nucl. Phys. B469 (1996) 387, hep-th/9602082
8. C.G. Callan. Jr. and J.M. Maldecena, Nucl. Phys. B513 (1998) 198, hep-th/9708147
9. P.S. Howe, N.D. Lambert and P.C. West, Nucl. Phys. B515 (1998) 203, hep-th/9709014
10. G.W. Gibbons, Nucl. Phys. B514 (1998) 603, hep-th/9709027
11. E. Bergshoeff, J. Gomis and P.K. Townsend, Phys. Lett. B421 (1998) 109, hep-th/9711043
12. J. Gutowski and G. Papadopoulos, Phys. Lett. B432 (1998) 97, hep-th/9802180
13. A. Fayyazuddin and M. Spalinski, Nucl. Phys. B508 (1997) 219, hep-th/9706087
14. M. Henningson and P. Yi, Phys. Rev. D57 (1998) 1291, hep-th/9707251
15. A. Mikhailov, BPS States and Minimal Surfaces, hep-th/9708068
16. P.S. Howe and E. Sezgin, Phys. Lett. B394 (1997) 62, hep-th/9611008; P.S. Howe, E. Sezgin and P.C. West, Phys. Lett. B399 (1997) 49, hep-th/9702008; P.S. Howe, E. Sezgin and P.C. West, Phys. Lett. B400 (1997) 255, hep-th/9702111

17. J.P. Gauntlett, N.D. Lambert and P.C. West, Supersymmetric Fivebrane Solitons, hep-th/9811024

18. J.P. Gauntlett, N.D. Lambert and P.C. West, hep-th/9803216, to appear in Comm. Math. Phys.

19. G. Gibbons, G. Papadopoulos and K. Stelle, Nucl. Phys. B508 (1997) 623, hep-th/9706207

20. J.M. Figueroa-O’Farrill, Electromagnetic Duality for Children, unpublished

21. M. Atiyah and N. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton University Press, Princeton, 1988

22. A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. B477 (1996) 746, hep-th/9604034

23. J. Schulze and N. Warner, Nucl. Phys. B498 (1997) 101, hep-th/9702012