Anomalous correlators, “ghost” waves and nonlinear standing waves in the $\beta$-FPUT system

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We investigate the $\beta$-Fermi-Pasta-Ulam-Tsingou (FPUT) chain with periodic boundary conditions and establish numerically and theoretically the existence of the second-order anomalous correlator. The anomalous correlator manifests in the frequency-wave number Fourier spectrum as a presence of “ghost” waves with negative frequencies, in addition to positive ones predicted by the linear dispersion relation. We explain theoretically the existence of anomalous correlators and ghost waves by nonlinear interactions between waves. Namely, generalizing the classical Wick’s decomposition to include the second-order anomalous correlator, we show that the latter is responsible for the presence of such “ghost” waves. From a physical point of view, the development of the anomalous correlator is related to formation of nonlinear standing waves. We predict that similar phenomenon might occur in nonlinear systems dominated by nonlinear interactions, including surface gravity waves.

Nonlinear waves | Fermi-Pasta-Ulam-Tsingou | Anomalous correlators

Abbreviations: Nonlinear waves — FPU recurrence — resonant interactions

Introduction

Wave Turbulence theory has led to successful predictions on the wave spectrum in many fields of physics [1, 2]. In this framework the system is represented as a superposition of a large number of weakly interacting waves with the complex amplitude, $a_k = a(k,t)$. In its essence, the classical Wave Turbulence theory is a perturbation expansion in the amplitude $a_k$ of the nonlinearity, yielding, at the leading order, to a system of quasi-linear waves whose amplitudes are slowly modulated by resonant nonlinear interactions [3, 4, 5, 6, 7]. This modulation leads to a redistribution of the spectral energy density among length-scales, and is described by a wave kinetic equation. One way to derive the wave kinetic equation is to use the random phase and amplitude approach developed in [2, 8, 9]. The initial state of the system can be always prepared so that the assumption of random phases and amplitudes is true. Whether the phases remain random in the evolution of the system has been an issue of intense discussions. In Wave Turbulence theory, the standard object to look at is the second-order correlator, $\langle a_k(t) a_l^*(t) \rangle$, where $\langle \ldots \rangle$ is an average over an ensemble of initial conditions with different random phases and amplitudes. As will be clear later on, under the homogeneity assumption, the second order correlator is related to the wave action spectral density function, i.e. the wave spectrum, $n_k = n(k,t)$. However, one should note that the complex wave amplitude, as defined in the the Wave Turbulence theory, is a complex function also in physical space. Therefore, the second-order statistics are not fully determined by the above correlator. Here we will look at a different second-order correlator, called “anomalous correlator”, $\langle a_k(t) a_l(t) \rangle^*$, see [10, 11] which, under the hypothesis of homogeneity, will be related to the anomalous spectrum, $n_k = m(k,t)$, to be defined in the next Section. Indeed, if phases are totally random, this quantity would be zero. We will show that, in the nonlinear evolution of the system, this is not the case. Far from it, this quantity is strongly nonzero and, in the limit of weak nonlinearity, we predict analytically and verify numerically its value.

These ideas are tested on a simple, but non trivial, system, i.e. the celebrated $\beta$-Fermi-Pasta-Ulam-Tsingou (FPUT) chain. The chain model was introduced in the fifties to study the thermal equipartition in crystals [12]; it consists of $N$ identical masses each one connected by a nonlinear spring; the elastic force can be expressed as a power series in the spring deformation. Fermi, Pasta, Ulam and Tsingou integrated numerically the equations of motion and conjectured that, after many iterations, the system would exhibit a thermalization, i.e. a state in which the influence of the initial modes disappears and the system becomes random, with all modes excited equally (equipartition of energy) on average. Successful predictions on the time scale of equipartition have been recently obtained in [13, 14, 15] using the Wave Turbulence approach.

In this paper we perform extensive numerical simulations of the system and look at the possible excitations, once a thermalized state has been reached. This is all done by analysing the spatial -temporal $(k-\Omega)$ spectrum, i.e. the square of the space-time Fourier transform of the wave amplitudes. Analyses of the effective dispersion relation in the nonlinear system is a well known and widely used theoretical and numerical tool, see for example [16]. We demonstrate numerically that in addition to the “normal” waves with frequency $\omega$ predicted by the linear dispersion relation for wave number $k$, there are the “ghost” waves with the negative frequencies. Our theoretical analysis reveals that the origin of those “ghost” waves resides on the nonzero values of the second-order anomalous correlator.

The Model

We consider the Hamiltonian for a chain of $N$ identical particles:

$$H = H_2 + H_4$$

with

$$H_2 = \sum_{j=1}^{N} \left( \frac{1}{2} p_j^2 + \frac{1}{2} (q_j - q_{j+1})^2 \right),$$

$$H_4 = \frac{\beta}{4} \sum_{j=1}^{N} (q_j - q_{j+1})^4.$$
$q_j(t)$ is the displacement of the particle $j$ from its equilibrium position and $p_j(t)$ is the associated momentum; $\beta$ is the nonlinear spring coefficient (without loss of generality, we have set the masses and the linear spring constant equal to 1). The Newton’s law in physical space is given by:

$$\ddot{q}_j = (q_{j+1} + q_{j-1} - 2q_j) + \beta\left((q_{j+1} - q_j)^3 - (q_j - q_{j-1})^3\right)$$ \[3\]

We assume periodic boundary condition; our approach is developed in Fourier space and the following definitions of the direct and inverse Discrete Fourier Transforms are adopted:

$$Q_k = \frac{1}{N} \sum_{j=0}^{N-1} q_j e^{-i2\pi kj/N}, \quad q_j = \sum_{k=-N/2+1}^{N/2} Q_k e^{i2\pi kj/N},$$ \[4\]

where $k$ are discrete wave numbers and $Q_k$ are the Fourier amplitudes. The displacement $q_j$ and momentum $p_j$ of the $j$ particle are linked by canonically conjugated Hamilton equations

$$\dot{p}_j = \frac{\partial H}{\partial q_j}, \quad \dot{q}_j = -\frac{\partial H}{\partial p_j}.$$

We then perform the Fourier transformation to $Q_k$, $P_k$, and then additional canonical transformation to complex amplitude $a_k$ given by

$$a_k = \frac{1}{\sqrt{2\pi N}}(\omega_k Q_k + iP_k),$$ \[5\]

where $\omega_k = 2|\sin(\pi k/N)| > 0$ and $Q_k$, $P_k$ are the Fourier amplitudes of $q_j$ and $p_j$, respectively. In terms of $a_k$ the equation of motion reads:

$$i\frac{da_k}{dt} = \omega_k a_k + \sum_{i=1}^{N-1} \left( T^{(1)}_{1234} a_{i2} a_{34} a_{23}^{*} + T^{(2)}_{1234} a_{34}^{*} a_{23} a_{12}^{*} + T^{(3)}_{1234} a_{23}^{*} a_{34} a_{12}^{*} + T^{(4)}_{1234} a_{12}^{*} a_{34} a_{23}^{*} \right),$$ \[6\]

where all wave numbers $k_2$, $k_3$ and $k_4$ are summed from 0 to $N - 1$ and $\delta_{ik} = \delta(k_a + k_b + -.. - k_c - k_d - ...)$ is the generalized Kronecker Delta that accounts for a periodic Fourier space, i.e. its value is one when the argument is equal to 0 (mod $N$). Furthermore,

$$T^{(1)}_{1234} = -\frac{3}{4} B e^{i\pi(-k_1+k_2+k_3+k_4)/N} \frac{2\sin(\pi k/N)}{\sqrt{4\pi^2}}, \quad T^{(2)}_{1234} = -3T^{(1)}_{1234}, \quad T^{(3)}_{1234} = -T^{(4)}_{4231}, \quad T^{(4)}_{1234} = -T^{(1)}_{1234}. \[7\]

The $(k - \Omega)$ spectrum. The main statistical object discussed in this paper is the wave number-frequency $(k - \Omega)$ spectrum. Given the complex wave amplitudes as a function of wave number $k$ and frequency $\Omega$, we define the $(k - \Omega)$ second-order correlator as

$$\langle a(k_1, \Omega_1) a(k_2, \Omega_2) \rangle = N(k_1, \Omega_2) \delta(k_1 - k_2) \delta(\Omega_2 - \Omega_1),$$ \[8\]

where $\langle\ldots\rangle$ implies averages over initial conditions with different random phases. The two $\delta$s are the Kronecker one, over wave numbers, and the Dirac one, over frequency. Their presence is the result of the assumption of statistical homogeneity and stationarity. The $(k - \Omega)$ spectrum, $N(k, \Omega)$, is defined as follows:

$$N^{(a)}(k, \Omega) = \frac{1}{2\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(l, \tau) e^{-i2\pi kl/N} e^{-i\Omega \tau} d\tau,$$ \[9\]

with $R(l, \tau) = \langle a_j(t)^* a_{j+l}(t + \tau) \rangle$ is the space-time autocorrelation function.

The linear $(k - \Omega)$ spectrum - Before diving into the nonlinear dynamics, we discuss the predictions in the linear regime. Therefore, we start by neglecting the nonlinearities in equation (6) and find the solution in the form

$$a_k(t) = a_k(t_0) e^{-i\omega_k t}.$$ \[10\]

where $t_0$ is a time at which the solution is known or an initial condition. We then take Fourier transform in time

$$a(k, \Omega) = (a(k, t_0) \delta(\Omega - \omega_k))$$ \[11\]

After multiplication by its complex conjugate and after taking averages, we get:

$$N^{(a)}(k, \Omega) = n^{(a)}(k, t_0) \delta(\Omega - \omega_k),$$ \[12\]

where $n^{(a)}(k, t_0)$ is the standard wave spectrum at time $t_0$ related to the second-order correlator as

$$\langle a(k_1, t_0) a(k_2, t_0)^* \rangle = n^{(a)}(k_1, t_0) \delta(k_1 - k_2),$$ \[13\]

and defined via the autocorrelation function as

$$n^{(a)}(k_1, t_0) = \frac{1}{N} \sum_l (a_j(t_0) a_{j+l}(t_0)^*) e^{-i2\pi kl/N}.$$ \[14\]

In the linear regime $n^{(a)}(k_1, t_0)$ does not evolve in time.

Equation (12) implies that in the linear case the $(k - \Omega)$ spectrum is different from zero only for those values of $\Omega$ and $k$ for which the dispersion relation is satisfied. Note that in this formulation $\omega_k$ is defined as a positive quantity; therefore, only the positive branch of the dispersion relation curve appears in the linear regime.

Numerical results for the $(k - \Omega)$ spectrum. We now test the predictions from equation (12) both in the linear and nonlinear regime. We perform numerical simulations of the equations (3) using a symplectic algorithm, see [17]. We use 32 modes in the simulation; such choice is completely uninformative for the results presented below. In the linear regime, we just prescribe a thermalized spectrum with some initial random phases of the wave amplitudes $a_k$, and evolve the system in time up to a desired final time; a Fast Fourier Transform in time is then taken to build the $(k - \Omega)$ spectrum. In the nonlinear regime we perform long simulations up to a thermalized spectrum. For a given nonlinearity, about 1000 realizations characterized by different random phases are made and ensemble averages are considered to compute the $(k - \Omega)$ spectrum. All simulations have the same initial linear energy and, from an operative point of view, the only difference between them is the value of $\beta$. To characterize the strength of the nonlinearity, we use the following ratio between nonlinear and linear Hamiltonians at the beginning of each simulation:

$$\epsilon = \frac{H_t}{H_0} \propto \beta.$$ \[15\]

Results are shown in Figure 1, where, for different values of the nonlinear parameter $\epsilon$, the spectrum $N^{(\epsilon)}(k, \Omega)$ is plotted using a coloured logarithmic scale. We first focus our attention on the linear regime, $\epsilon = 0$: results are shown in Figure 1(a).
Fig. 1. \((k - \Omega)\) spectrum, \(N^{(a)}(k, \Omega)\), for different values of \(\epsilon\): (a) \(\epsilon = 0\), (b) \(\epsilon = 0.0089\) (c) \(\epsilon = 0.089\) (d) \(\epsilon = 1.12\). In the linear case, (a), the \(N^{(a)}(k, \Omega)\) is different from 0 only when the frequency \(\Omega\) matches the linear dispersion relation. As the nonlinearity is increased, (b - d), a frequency shift, a broadening of the frequencies and a lower branch less intense than the upper one are visible. Waves with negative frequencies are named “ghost” waves.

As well predicted by the theory, the plot shows dots in the positive frequency plane, where the frequencies \(\Omega\) and wave numbers \(k\) satisfy the linear dispersion curve \(\omega_k\). Increasing the nonlinearity, Figures 1 -(b,c,d), two well known effects appears: the first one is a shift of the frequencies, due to nonlinearity (this is more evident in Figures 1 -(c,d) where the frequency scale in the vertical axes has been changed). The second one is the broadening of the frequencies; this is related to the fact that the amplitude for each wave number is not constant in time; therefore, the amplitude-dependent frequencies are not constant in time and they oscillate around a mean value with some fluctuations. Those results are well understood, at least in the weakly nonlinear regime, and can be predicted using Wave Turbulence tools, see [2, 15]. Besides these two effects, starting from Figure 1-(b), the presence of a lower branch, whose intensity is much less than the upper one, starts to be visible. The lower curve becomes more important and, when the nonlinearity is of order one, is of the same order of magnitude of the upper one. The total number of waves

Fig. 2. Ratio between the number of “ghost” waves, \(N_{\text{ghost}}\), over the total number of waves, \(N_{\text{tot}}\), as a function of the nonlinearity.
in the simulation, \( N_{\text{tot}} \), is given by the integral over \( \Omega \) and the sum over all \( k \) of the function \( N^{(a)}(k, \Omega) \). In the weakly nonlinear regime, \( N_{\text{tot}} \) is an adiabatic invariant of the equation of motion (3); the plot highlights the existence of waves with negative frequencies, which will be named “ghost” waves. One of the scopes of the present paper is the understanding of the origin of such waves. Before entering into the discussion, we show in Figure 2 the number of “ghost” waves, \( N_{\text{ghost}} \), i.e. \( N^{(a)}(k, \Omega) \) integrated over negative frequencies and summed over all wave numbers, divided by the total number of waves, \( N_{\text{tot}} \). As can be seen from the plot, there is a monotonic growth of the “ghost” waves that, for very large nonlinearity, can reach values up to 25% of the total number.

**Anomalous correlators**

To explain the presence of “ghost” waves, we introduce the so-called second-order anomalous correlator [10, 11, 18]:

\[
\langle a_k(t) a_j(t) \rangle = m_k(t) \delta(k + j)
\]

[16]

with the anomalous spectrum defined as

\[
m^{(a)}_k(t) = \frac{1}{N} \sum_i \langle a_j a_{j+l} \rangle e^{-i2\pi kl/N}.
\]

[17]

Similarly, we also introduce the second-order \((k - \Omega)\) anomalous correlator:

\[
\langle a(k_i, \Omega_l) a(k_j, \Omega_m) \rangle = M^{(a)}(k_i, \Omega_l) \delta(k_i + k_j) \delta(\Omega_l + \Omega_m)
\]

[18]

where

\[
M^{(a)}(k, \Omega) = \frac{1}{2\pi N} \int_{-\infty}^{\infty} \sum_{l=1}^{N} S(l, \tau) e^{-i2\pi kl/N} e^{-i\Omega \tau} d\tau
\]

[19]

and \( S(l, \tau) = \langle a_j(t) a_{j+l}(t + \tau) \rangle \). The presence in equations (16) and (18) of the Kronecker \( \delta \) over wave numbers and the Dirac \( \delta \) over frequency, are related to the hypothesis of homogeneity and stationarity, respectively. Note that \( M^{(a)}(k, \Omega) \) is not the Fourier transform in time of \( m^{(a)}_k(t) \) and in general both can be complex functions.

To verify numerically that the anomalous correlator is indeed nonzero, we measure numerically the real part of the second-order correlator \( \langle a_k(t) a_k(t) \rangle \) as a function of \( k_1 \) and \( k_2 \). Results are plotted in Figure 3 where we show the results of two numerical simulations characterized by two different values of the nonlinear parameter, (a) \( \epsilon = 0.0089 \) and (b) \( \epsilon = 1.12 \). In both cases, a diagonal contribution is visible, pointing out the existence of anomalous correlators in the beta-FPUT model.

**Generalization of the Wick’s decomposition** - It is now straightforward to extend the Wick’s decomposition by taking into account the anomalous correlators as follows:

\[
\langle a^*_k a^*_p a_m a_n \rangle = n_k n_m (\delta^k_p \delta^*_m + \delta^k_m \delta^*_p) + m_k m_p \delta^k_p \delta^*_m,
\]

\[
\langle a^*_k a_p a_m n_n \rangle = n_k m_p (\delta^k_m \delta^*_p + \delta^k_p \delta^*_m) + n_k n_p \delta^k_p \delta^*_m,
\]

\[
\langle a_k a_p a_m n_n \rangle = m_k m_p (\delta^k_m \delta^*_p + \delta^k_p \delta^*_m) + n_k n_p \delta^k_p \delta^*_m.
\]

[20]

The above relations will be fundamental for making a natural closure of the moments when calculating analytically the \((k - \Omega)\) spectrum.
We show in Figure 4 the real part of the fourth-order correlator \( \langle a_{k_1} a_{k_2} a_{k_3}^* a_{k_2 + k_3 - k_1} \rangle \) with \( k_3 = 20 \), computed from numerical simulations for (a) \( \epsilon = 0.0089 \) and (b) \( \epsilon = 1.12 \). The diagonal lines in both figures, highlighting the contribution from the second-order anomalous correlator, are noticeable. The vertical and horizontal lines correspond to the trivial resonances in which two wave numbers are equal (mod \( N \)).

**Theoretical prediction for the anomalous correlator in the weakly nonlinear regime.** A key step for the development of a theory for the anomalous correlator is the change of variable (near identity transformation) which allows one to remove bound modes, i.e., those modes that are phase locked to free modes and do not obey the linear dispersion relation. The procedure is well known in Hamiltonian mechanics and documented for example in [1]. The transformation from variable \( a_k(t) \) to \( b_k(t) \) has the following form:

\[
a_1 = b_1 + \sum_{k_2, k_3, k_4} \left[ B_{1234}^{(1)} b_2 b_3 b_4^* \delta_{1234} + B_{1234}^{(3)} b_2^* b_4 b_3^* \delta_{1234} \right] + B_{1234}^{(4)} b_2^* b_4 b_3^* \delta_{1234}. \tag{21}
\]

The coefficients \( B_{1234}^{(i)} \) are selected in such a way to remove non resonant terms in the original Hamiltonian. The transformation is asymptotic in the sense that the small amplitude approximation is made and the terms in the sums on the right hand side are much smaller than the leading order term \( b_1 \). The evolution equation for variable \( b_k(t) \) contains resonant interactions and take the following standard form:

\[
\frac{d b_1}{dt} = \omega_1 b_1 + \sum_{k_2, k_3, k_4} T_{1234}^{(2)} b_2 b_3 b_4^* \delta_{1234} + \text{h.o.t.}
\]

where higher order terms arising from the transformation have been neglected. Starting from the transformation in (21) and the generalized Wick’s decomposition in (20), we can now build the anomalous spectrum:

\[
m_{k}^{(a)}(t) = m_{k}^{(b)}(t) + 2 \left( n_{k}^{(b)}(t) + n_{-k}^{(b)}(t) \right) \times \sum_{j} B_{k,j,j}^{(3)} n_{j}^{(b)}(t), \tag{22}
\]

with

\[
B_{1234}^{(3)} = \frac{T_{1234}^{(2)}}{\omega_4 - \omega_1 - \omega_2 - \omega_3}. \tag{23}
\]

where higher order terms in \( m_k \) have been neglected. The next step consists in building the evolution equation for \( m_{k}^{(b)}(t) \) from equation (22). Interestingly, the evolution equation for \( m_{k}^{(b)}(t) \) appears as a deterministic dispersive non homogeneous wave evolution equation

\[
\frac{d m_{k}^{(b)}}{dt} = 2 \omega_k m_{k}^{(b)} + \left( n_{k}^{(b)} + n_{-k}^{(b)} \right) \sum_{j} T_{k-j,k-j} m_{j}^{(b)} + 4 m_{k}^{(b)} \sum_{j} T_{k,j,j} m_{j}^{(b)} \tag{24}
\]

The leading order solution is given by

\[
m_{k}^{(b)}(t) = m_{k}^{(b)}(t_0) e^{-i 2 \omega_k t} + \text{higher order terms.}
\]
Putting things together, and assuming that the spectrum \( n_k \) is in stationary conditions, we get

\[
m_k^{(a)}(t) = m_k^{(b)}(t_0)e^{-i\omega_k t} + 2 \left( n_k^{(a)}(t_0) + n_{-k}^{(a)}(t_0) \right) \times \sum_j B_{k,j,j}^{(3)}(t_0).
\]

Note that we have used the fact that at the leading order \( n_k^{(b)}(t) \approx n_k^{(a)}(t_0) \). Averaging over time, in the limit of large times, assuming that the wave spectrum \( n_k^{(a)}(t) \) is stationary, the oscillating term vanishes:

\[
\langle m_k^{(a)}(t) \rangle_\tau = 2 \left( n_k^{(a)} + n_{-k}^{(a)} \right) \sum_j B_{k,j,j}^{(3)} n_j^{(a)},
\]

where \( \langle \ldots \rangle_\tau \) implies averaging over time. The prediction is shown in figures 7 against numerical simulations: the results are in good agreement in the weakly nonlinear regime.

### Theoretical prediction for "ghost" waves

We have now developed all the tools for predicting analytically the \((k-\Omega)\) spectrum as defined in equation (8). We start by writing the transformation in equation (21) in the frequency domain as follows:

\[
a_k,\Omega_p = b_k,\Omega_p + \int_{j,k,l} B_{ijkl}^{(1)} b_{j\Omega_p} b_{k\Omega_p} d\Omega_p d\Omega_l + \int_{j,k,l} B_{ijkl}^{(3)} b_{j\Omega_p} b_{k\Omega_p} d\Omega_p d\Omega_l,
\]

and build the correlator \( \langle a(k_1,\Omega_1) a(k_2,\Omega_2) \rangle \).

Using the generalized Wick’s decomposition and the hypothesis of stationarity and homogeneity, we get at leading order:

\[
N^{(a)}(k,\Omega) = N^{(b)}(k,\Omega) + 4\text{Re}[M^{(b)}(k,\Omega)]F(k)
\]

with

\[
N^{(b)}(k,\Omega) \approx n^{(a)}(k,t_0) \delta(\Omega - \omega_k),
M^{(b)}(k,\Omega) \approx m^{(a)}(k,t_0) \delta(\Omega + \omega_k),
F(k) = \int \sum_i B_{kkl}^{(3)} N^{(b)}(l,\Omega_p) d\Omega_p,
\]

where we have used the fact that at the leading order \( m^{(b)}(k,t) \approx m^{(a)}(k,t) \) and \( n^{(b)}(k,t_0) \approx n^{(a)}(k,t_0) \). The equations (28) and (29) predict the presence of the upper and lower branch in the \((k-\Omega)\) plane. The presence of "ghost" waves is clearly related to the second-order anomalous correlator. We can now predict the percentage of "ghost" waves as

\[
\frac{N_{\text{ghost}}}{N_{\text{tot}}} = \frac{4 \sum_k \text{Re}[m^{(a)}(k,t_0)]F(k)}{\sum_k (n^{(a)}(k,t_0) + 4\text{Re}[m^{(a)}(k,t_0)]F(k))}
\]

In figure (5) we plot the ratio [30] for several values of \( \epsilon \). We find that for small nonlinearity the results agree.

### Nonlinear standing waves

The development of a regime characterized by an anomalous spectrum \( \langle m_k^{(a)}(t) \rangle \), corresponds to a tendency for the system to develop standing waves in the original displacement \( q_j(t) \). Indeed, the existence of an an anomalous spectrum implies a correlation between positive and negative wave numbers. We illustrate such behavior by initializing the system in Fourier space with initial conditions with all particles at rest and displaced from the equilibrium as a single sine wave. In figure [6] we plot the displacement for all masses as a function of time as system reaches thermal equilibrium. The existence of several regions of standing wave behavior is clearly visible.

### Conclusion

i) Numerical simulations of the FPUT system with periodic boundary conditions shows that there is a nonzero correlation between positive and negative wave numbers.

ii) A nonzero value of the anomalous spectrum manifests itself as an existence of the "ghost" waves, i.e. waves with wave number \( k \) that oscillate at a negative frequency.

iii) In physical space the anomalous correlator manifest itself as standing waves, i.e. waves that do not propagate, but are localized at a particular points in physical space.

iv) These effects are purely nonlinear, and are increasing as the level of nonlinearity increases.

v) We test our theory on the celebrated FPUT model, and get an excellent agreement between theory and numerical experiments, for small nonlinearities.

vi) We predict that anomalous spectra, "ghost" and standing waves will be present in many other nonlinear systems, including surface gravity waves and the Nonlinear Schrodinger equation.
Fig. 5. The ratio between the number of “ghost” waves, $N_{\text{ghost}}$, over the total number of waves, $N_{\text{tot}}$, as a function of the nonlinearity. Numerical values are blue; theoretical values, red triangles, are given by [30].

Fig. 6. Color map of the displacement $q_j(t)$ for the system with $\epsilon = 4.74$ initialized with particles at rest with initial positions as a single sine wave. A nonlinear standing wave pattern is visible.

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1. Falkovich G, Lvov V, Zakharov VE (1992) Kolmogorov spectra of turbulence. (Springer, Berlin).
2. Nazarenko S (2011) Wave turbulence. (Springer) Vol. 825.
3. Benney J, Newell A (1969) Random wave closure. Studies in Appl. Math. 48:1.
4. Newell A (1968) The closure problem in a system of random gravity waves. Review of Geophysics 6:1.
5. Benney D, P-Saffmann (1966) Nonlinear interaction of random waves in a dispersive medium. Proc Royal. Soc 289:301–320.
6. Kadomtsev B (1965) Plasma Turbulence. (Academic Press, New York).
7. Zakharov V, L’vov V, G Falkovich (1992) Kolmogorov Spectra of Turbulence. (Springer-Verlag).
8. Choi Y, Lvov Y, Nazarenko S, Pokorni B (2005) Anomalous probability of large amplitudes in wave turbulence. Physics Letters A 339:361.
9. Choi Y, Lvov Y, Nazarenko S (2004) Probability densities and preservation of randomness in wave turbulence. Physics Letters A 332:230.
10. L’vov VS (2012) Wave turbulence under parametric excitation: applications to magnets. (Springer Science & Business Media).
11. Zakharov VE, L’vov V, Starobinets S (1975) Spin-wave turbulence beyond the parametric excitation threshold. Physics-Uspekhi 17(6):896–919.
12. Fermi E, Pasta J, Ulam S (1955) Studies of nonlinear problems. (I. Los Alamos Scientific Laboratory Report No. LA-1940), Technical report.
13. Onorato M, Vozella L, Proment D, Lvov Y (2015) Route to thermalization in the $\beta$-Fermi-Pasta-Ulam system. Proceedings of the National Academy of Sciences of the United States of America 112(14).
14. Pistone L, Onorato M, Chibbaro S (2018) Thermalization in the discrete nonlinear klein-gordon chain in the wave-turbulence framework. EPL (Europhysics Letters) 121(4):44003.
15. Lvov YV, Onorato M (2018) Double scaling in the relaxation time in the $\beta$-Fermi-Pasta-Ulam-Tsingou model. Physical review letters 120(14):144301.
16. Lee W, Kovacic G CD (2013) Generation of dispersion in nondispersive nonlinear waves in thermal equilibrium. Proceedings of the National Academy of Sciences of the United States of America 110(8):3237–3241.
17. Yoshida H (1990) Construction of higher order symplectic integrators. Physics Letters A 150(5):262–268.
18. Guasoni M, et al. (2017) Incoherent fermi-pasta-ulam recurrences and unconstrained thermalization mediated by strong phase correlations. Physical Review X 7(1):011025.