QUATERNIONIC FOURIER-MELLIN TRANSFORM

ECKHARD HITZER

ABSTRACT. In this contribution we generalize the classical Fourier Mellin transform \[3\], which transforms functions \(f\) representing, e.g., a gray level image defined over a compact set of \(\mathbb{R}^2\). The quaternionic Fourier Mellin transform (QFMT) applies to functions \(f : \mathbb{R}^2 \to \mathbb{H}\), for which \(|f|\) is summable over \(\mathbb{R}^*_+ \times S^1\) under the measure \(d\theta\,dr\). \(\mathbb{R}^*_+\) is the multiplicative group of positive and non-zero real numbers. We investigate the properties of the QFMT similar to the investigation of the quaternionic Fourier Transform (QFT) in \[5\, 6\].

1. QUATERNIONS

Gauss, Rodrigues and Hamilton introduced the 4D quaternion algebra \(\mathbb{H}\) over \(\mathbb{R}\) with three imaginary units:

(1) \[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1. \]

Every quaternion

(2) \[ q = q_r + q_i i + q_j j + q_k k \in \mathbb{H}, \quad q_r, q_i, q_j, q_k \in \mathbb{R} \]

has quaternion conjugate (reversion in \(\text{Cl}_3^{+}\))

(3) \[ \tilde{q} = q_r - q_i i - q_j j - q_k k, \]

This leads to norm of \(q \in \mathbb{H}\), and an inverse of every non-zero \(q \in \mathbb{H}\)

(4) \[ |q| = \sqrt{\tilde{q}q} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \quad |pq| = |p||q|, \quad q^{-1} = \tilde{q}/|q|^2 = \frac{\tilde{q}}{qq}. \]

The scalar part of quaternions is symmetric

(5) \[ \text{Sc}(q) = q_r = \frac{1}{2}(q + \tilde{q}), \quad \text{Sc}(pq) = \text{Sc}(qp). \]

The inner product of quaternions defines orthogonality

(6) \[ \text{Sc}(pq) = p_rq_r + p_iq_i + p_jq_j + p_kq_k \in \mathbb{R}. \]
1.1. The (2D) orthogonal planes split (OPS) of quaternions. We consider an arbitrary pair of pure quaternions \( f, g \), \( f^2 = g^2 = -1 \). The orthogonal 2D planes split (OPS) is then defined with respect to a pair of pure unit quaternions \( f, g \) as

(7) \[ q^{\pm} = \frac{1}{2}(q \pm fg) \]

We thus observe, that

(8) \[ fg = q_+ - q_- \]

i.e. under the map \( f()g \) the \( q_+ \) part is invariant, but the \( q_- \) part changes sign. Both parts are two-dimensional, and span two completely orthogonal planes. For \( f \neq \pm g \) the \( q_+ \) plane is spanned by the orthogonal quaternions \( \{ f - g, 1 + fg \} \), and the \( q_- \) plane is spanned by \( \{ f + g, 1 - fg \} \). Vice versa, in general any two 2D orthogonal planes in \( \mathbb{H} \) determine a corresponding pair \( f, g \).

**Lemma 1.1 (Orthogonality of two OPS planes).** Given any two quaternions \( q, p \) and applying the OPS with respect to two linearly independent pure unit quaternions \( f, g \) we get zero for the scalar part of the mixed products

(9) \[ Sc(p_+q_-) = 0, \quad Sc(p_-q_+) = 0 \implies |q|^2 = |q_+|^2 + |q_-|^2 \]

1.2. Geometric interpretation of map \( f()g \). The map \( f()g \) rotates the \( q_- \) plane by \( 180^\circ \) around the \( q_+ \) axis plane, see Fig[2]. This interpretation of the map \( f()g \) is in perfect agreement with Coxeter’s notion of half-turn in [2].

We obtain the following important identities:

(10) \[ e^{\alpha f}q_+e^{\beta g} = q_+e^{(\beta+\alpha)g} = e^{(\alpha+\beta)f}q_\pm, \quad q \in \mathbb{H}, \alpha, \beta \in \mathbb{R} \]

For \( g \neq \pm f \) the set \( \{ f - g, 1 + fg, f + g, 1 - fg \} \) forms an orthogonal basis of \( \mathbb{H} \) interpreted as \( \mathbb{R}^4 \). We can therefore use the following representation for every \( q \in \mathbb{H} \)

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2Color versions of the figures (showing maxima and minima in red and blue colors, respectively), can be found in the electronic copy of this contribution to appear on [http://sinai.mech.fukui-u.ac.jp/gcj/pubs.html](http://sinai.mech.fukui-u.ac.jp/gcj/pubs.html).
by means of four real coefficients $q_1, q_2, q_3, q_4 \in \mathbb{R}$

\begin{align}
q &= q_1(1 + fg) + q_2(f - g) + q_3(1 - fg) + q_4(f + g), \\
q_1 &= Sc(q(1 + fg)^{-1}), \quad q_2 = Sc(q(f - g)^{-1}), \\
q_3 &= Sc(q(1 - fg)^{-1}), \quad q_4 = Sc(q(f + g)^{-1}).
\end{align}

In the case of $f = i, g = j$ we obtain the coefficients

\begin{align}
q_1 &= \frac{1}{2}(q_r + q_k), \quad q_2 = \frac{1}{2}(q_i - q_j), \quad q_3 = \frac{1}{2}(q_i - q_k), \quad q_4 = \frac{1}{2}(q_i + q_j).
\end{align}

The OPS with respect to a single pure unit quaternion, e.g., $f = g = i$ gives

\begin{align}
q_{\pm} &= \frac{1}{2}(q \pm iq\hat{i}), \quad q_+ = q_jj + q_kk = (q_j + q_k\hat{i})j, \quad q_- = q_r + q_i\hat{i},
\end{align}

where the $q_+$ plane is 2D and manifestly orthogonal to the 2D $q_-$ plane. The above corresponds to the simplex/perplex split of [4], see an application in Fig. 2 from [11].

2. The Quaternionic Fourier Mellin Transformations (QFMT)

2.1. Robert Hjalmar Mellin (1854–1933). Robert Hjalmar Mellin (1854–1933) [9], Fig. 3 was a Finnish mathematician, a student of G. Mittag-Leffler and K. Weierstrass. He became the director of the Polytechnic Institute in Helsinki, and in 1908 first professor of mathematics at Technical University of Finland. He was a fervent fennoman with fiery temperament, and co-founder of the Finnish Academy of Sciences. He became known for the Mellin transform with major applications to the evaluation of integrals, see [10], which lists 1624 references. During his last 10 years he tried to refute Einstein’s theory of relativity as logically untenable.
**Definition 2.1** (Definition: Classical Fourier Mellin transform (FMT)).

\[
\forall (v, k) \in \mathbb{R} \times \mathbb{Z}, \quad \mathcal{M}\{h\}(v, k) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} h(r, \theta) r^{-iv} e^{-ik\theta} d\theta \frac{dr}{r},
\]

where \( h : \mathbb{R}^2 \to \mathbb{R} \) denotes a function representing, e.g., a gray level image defined over a compact set of \( \mathbb{R}^2 \).

Well known applications are to shape recognition (independent of rotation and scale), image registration, and similarity.

### 2.2. Inner product, symmetric part, norm of quaternion-valued functions.

The quaternion \( \mathbb{H} \)-valued inner product for quaternion-valued functions \( h, m : \mathbb{R}^2 \to \mathbb{H} \) is given by

\[
(h, m) = \int_{\mathbb{R}^2} h(x) \overline{m(x)} d^2 x,
\]

with \( d^2 x = dx dy \).

It has a symmetric real scalar part

\[
\langle h, m \rangle = \frac{1}{2} [\langle h, m \rangle + \langle m, h \rangle] = \int_{\mathbb{R}^2} \text{Sc}(h(x) \overline{m(x)}) d^2 x,
\]

which allows to define a \( L^2(\mathbb{R}^2; \mathbb{H}) \)-norm

\[
\|h\|^2 = (h, h) = \langle h, h \rangle = \int_{\mathbb{R}^2} |h(x)|^2 d^2 x \quad \implies \quad \|h\|^2 = \|h_+\|^2 + \|h_-\|^2.
\]

A quaternion module can be defined as \( L^2(\mathbb{R}^2; \mathbb{H}) \) by

\[
L^2(\mathbb{R}^2; \mathbb{H}) = \{ h | h : \mathbb{R}^2 \to \mathbb{H}, \|h\| < \infty \}.
\]

We now define the generalization of the FMT to quaternionic signals.

**Definition 2.2** (Quaternionic Fourier Mellin transform (QFMT)). Let \( f, g \in \mathbb{H} : f^2 = g^2 = -1 \) be any pair of pure unit quaternions. The quaternionic Fourier Mellin transform (QFMT) is given by

\[
\forall (v, k) \in \mathbb{R} \times \mathbb{Z}, \quad \hat{h}(v, k) = \mathcal{M}\{h\}(v, k) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} r^{-fv} h(r, \theta) e^{-gk\theta} d\theta \frac{dr}{r},
\]

where \( h : \mathbb{R}^2 \to \mathbb{H} \) denotes a function from \( \mathbb{R}^2 \) into the algebra of quaternions \( \mathbb{H} \), such that \( |h| \) is summable over \( \mathbb{R}_+^* \times S^1 \) under the measure \( d\theta \frac{dr}{r} \). \( \mathbb{R}_+^* \) is the multiplicative group of positive and non-zero real numbers.
For $f = i, g = j$ we have the special case

$$(20) \forall (v, k) \in \mathbb{Z} \times \mathbb{R}, \quad \hat{h}(v, k) = \mathcal{M}\{h\}(v, k) = \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} r^{-iv} h(r, \theta) e^{-2jk\theta} dr d\theta.$$  

Note, that the $\pm$ split and the QFMT commute:

$$\mathcal{M}\{h\pm\} = \mathcal{M}\{h\} \pm.$$  

**Theorem 2.3** (Inverse QFMT). The QFMT can be inverted by

$$(21) \quad h(r, \theta) = \mathcal{M}^{-1}\{h\}(r, \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k \in \mathbb{Z}} r^{fv} \hat{h}(v, k) e^{qk\theta} dv.$$  

The proof uses

$$(22) \quad \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{qk(\theta-\theta')} = \delta(\theta-\theta'), \quad r^{fv} = e^{fv \ln r}, \quad \frac{1}{2\pi} \int_{0}^{2\pi} e^{f v (\ln(r)-s)} dv = \delta(\ln(r)-s).$$  

We now investigate the basic properties of the QFMT. First, left linearity: For $\alpha, \beta \in \{q \mid q = q_r + q_f f, q_r, q_f \in \mathbb{R}\},$

$$(23) \quad m(r, \theta) = \alpha h_1(r, \theta) + \beta h_2(r, \theta) \implies \hat{m}(v, k) = \alpha \hat{h}_1(v, k) + \beta \hat{h}_2(v, k).$$  

Second, right linearity: For $\alpha', \beta' \in \{q \mid q = q_r + q_f q_r, q_r, q_f \in \mathbb{R}\},$

$$(24) \quad m(r, \theta) = h_1(r, \theta) \alpha' + h_2(r, \theta) \beta' \implies \hat{m}(v, k) = \hat{h}_1(v, k) \alpha' + \hat{h}_2(v, k) \beta'.$$

The linearity of the QFMT leads to

$$(25) \quad \mathcal{M}\{h\}(v, k) = \mathcal{M}\{h_- + h_+\}(v, k) = \mathcal{M}\{h_-\}(v, k) + \mathcal{M}\{h_+\}(v, k),$$  

which gives rise to the following theorem.

**Theorem 2.4** (Quasi-complex FMT like forms for QFMT of $h\pm$). The QFMT of $h\pm$ parts of $h \in L^2(\mathbb{R}^2, \mathbb{H})$ have simple quasi-complex forms

$$(26) \quad \mathcal{M}\{h\pm\} = \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} h_\pm r^{gv} e^{-qk\theta} d\theta dr + \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} r^{jv} e^{qf k\theta} h_\pm d\theta dr.$$  

Theorem 2.4 allows to use discrete and fast software to compute the QFMT based on a pair of complex FMT transformations.

For the two split parts of the QFMT, we have the following lemma.

**Lemma 2.5** (Modulus identities). Due to $|q| = |q_-|^2 + |q_+|^2$ we get for $f : \mathbb{R}^2 \to \mathbb{H}$ the following identities

$$(27) \quad |\mathcal{M}\{h\}(v, k)|^2 = |\mathcal{M}\{h_-\}(v, k)|^2 + |\mathcal{M}\{h_+\}(v, k)|^2.$$  

Further properties are scaling and rotation: For $m(r, \theta) = h(ar, \theta + \phi), a > 0, 0 \leq \phi \leq 2\pi,$

$$(28) \quad \hat{m}(v, k) = a^{fv} \hat{h}(v, k) e^{qk\phi}.$$
Moreover, we have the following magnitude identity:

\[ |\hat{m}(v,k)| = |\hat{h}(v,k)|, \]

i.e. the magnitude of the QFMT of a scaled and rotated quaternion signal \( m(r, \theta) = h(ar, \theta + \phi) \) is identical to the magnitude of the QFMT of \( h \). Equation (29) forms the basis for applications to rotation and scale invariant shape recognition and image registration. This may now be extended to color images, since quaternions can encode colors RGB in their \( i, j, k \) components.

The reflection at the unit circle \((r \rightarrow \frac{1}{r})\) leads to

\[ m(r, \theta) = h(\frac{1}{r}, \theta) \quad \implies \quad \hat{m}(v,k) = \hat{h}(-v,k). \]

Reversing the sense of rotation \((\theta \rightarrow -\theta)\) yields

\[ m(r, \theta) = h(r, -\theta) \quad \implies \quad \hat{m}(v,k) = \hat{h}(v,-k). \]

Regarding radial and rotary modulation we assume

\[ m(r, \theta) = r^{v_0} h(r, \theta) e^{k_0 \theta}, \quad v_0 \in \mathbb{R}, k_0 \in \mathbb{Z}. \]

Then we get

\[ \hat{m}(v,k) = \hat{h}(v - v_0, k - k_0). \]

2.3. QFMT derivatives and power scaling. We note for the logarithmic derivative that \( \frac{d}{d \ln r} = r \frac{d}{dr} = r \partial_r \),

\[ \mathcal{M}\{ (r \partial_r)^n h \}(v,k) = (fv)^n \hat{h}(v,k), \quad n \in \mathbb{N}. \]

Applying the angular derivative with respect to \( \theta \) we obtain

\[ \mathcal{M}\{ \partial_\theta^n h \}(v,k) = \hat{h}(v,k)(gk)^n, \quad n \in \mathbb{N}. \]

Finally, power scaling with \( \ln r \) and \( \theta \) leads to

\[ \mathcal{M}\{ (\ln r)^m \theta^n h \}(v,k) = f^m \partial_v^n \partial_k \hat{h}(v,k) g^n, \quad m, n \in \mathbb{N}. \]

2.4. QFMT Plancherel and Parseval theorems. For the QFMT we have the following two theorems.

**Theorem 2.6** (QFMT Plancherel theorem). The scalar part of the inner product of two functions \( h, m : \mathbb{R}^2 \rightarrow \mathbb{H} \) is

\[ \langle h, m \rangle = \langle \hat{h}, \hat{m} \rangle. \]

**Theorem 2.7** (QFMT Parseval theorem). Let \( h : \mathbb{R}^2 \rightarrow \mathbb{H} \). Then

\[ \| h \| = \| \hat{h} \|, \quad \| h \|^2 = \| \hat{h} \|^2 = \| \hat{h}_+ \|^2 + \| \hat{h}_- \|^2. \]
Figure 4. Left: 2D FT is intrinsically 1D. Right: QFT is intrinsically 2D. Source: [1].

3. Symmetry and kernel structures of 2D FMT, FT, QFT, QFMT

The QFMT of real signals analyzes symmetry. The following notation will be used. The function $h_{ee}$ is even with respect to (w.r.t.) $r \rightarrow \frac{1}{r} \iff \ln r \rightarrow -\ln r$, i.e. w.r.t. the reflection at the unit circle, and even w.r.t. $\theta \rightarrow -\theta$, i.e. w.r.t. reversing the sense of rotation (reflection at the $\theta = 0$ line of polar coordinates in the $(r, \theta)$-plane). Similarly we denote by $h_{eo}$ even-odd symmetry, by $h_{oe}$ odd-even symmetry, and by $h_{oo}$ odd-odd symmetry.

Let $h$ be a real valued function $\mathbb{R}^2 \rightarrow \mathbb{R}$. The QFMT of $h$ results in

$$\hat{h}(v,k) = \hat{h}_{ee}(v,k) + \hat{h}_{eo}(v,k) + \hat{h}_{oe}(v,k) + \hat{h}_{oo}(v,k) .$$

The QFMT of a real signal therefore automatically separates components with different combinations of symmetry w.r.t. reflection at the unit circle and reversal of the sense of rotation. The four components of the QFMT kernel differ by radial and angular phase shifts, see the left side of Fig. 7. The symmetries of $r \rightarrow 1/r$ (reflection at yellow unit circle), and $\theta \rightarrow -\theta$ (reflection at green line) can be clearly seen on the right of Fig. 7.

Figure 8 shows real the component of the QFMT kernel for various values of $v, k$, demonstrating various angular and radial resolutions. Figure 9 shows the real component of the QFMT kernel for $v = k = 4$ at three different scales. Similar patterns appear at all scales. Figure 1 shows the kernels of complex 2D Fourier transform (FT) $e^{-i(ux + vy)}$, $i \in \mathbb{C}$, and the QFT $e^{-ixu}e^{-jvy}$, $i, j \in \mathbb{H}$, taken from [1], which treats applications to 2D gray scale images. Corresponding applications to color images can be found in [11]. The 2D FT is intrinsically 1D, the QFT is intrinsically 2D, which makes it superior in disparity estimation and 2D texture segmentation, etc.

In this section we assume $g \neq \pm f$, but a similar study is possible for $g = \pm f$. 

[1]
Figure 5. Left: Kernel of FMT. Right: Kernel of QFMT. $k, v \in \{0, 1, 2, 3, 4\}$.

Figure 6. Left: QFT kernel, right: QFMT kernel. Top row: $q_+$ parts: $1 + fg$ and $f - g$ components. Bottom row: $q_-$ parts: $1 - fg$ and $f + g$ components.

Figure 5 compares the kernels (real parts) of 2D complex FMT and the QFMT. Obviously the 2D QFMT can analyze genuine 2D textures better than the 2D complex FMT. Finally, Fig. 6 compares the kernels of the QFT (left) and the QFMT (right). The scale invariant feature of the QFMT is obvious. Compared with the left side of Fig. 5, the QFMT is obviously the linear superposition of two quasi-complex FMTs with opposite winding sense, as shown in Theorem 2.4.

4. Conclusion

The algebra of quaternions allows to construct a variety of quaternionic Fourier-Mellin transformations (QFMT), dependent on the choice of $f, g \in \mathbb{H}$, $f^2 = g^2 = -1$. Further variations would be to place both kernel factors initially at the left or right of the signal $h(r, \theta)$. The whole QFMT concept can easily be generalized to Clifford algebras $Cl(p, q)$, based on the general theory of square roots of $-1$ in $Cl(p, q)$. 
The modulus of the transform is scale and rotation invariant. Preceeded by 2D FT or by QFT, this allows translation, scale and rotation invariant object description. A diverse range of applications can therefore be imagined: Color object shape recognition, color image registration, application to evaluation of hypercomplex integrals, etc.

Future research may be on extensions to Clifford algebras $Cl(p,q)$, to windowed and wavelet transforms, discretization, and numerical implementations.

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Zechariah after the birth of his son John: And you, my child, ... will go on before the Lord to prepare the way for him, to give his people the knowledge of salvation through the forgiveness of their sins, because of the tender mercy of our God, by which the rising sun will come to us from heaven to shine on those living in darkness and in the shadow of death, to guide our feet into the path of peace. (Bible, Luke 1:76-79)

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Figure 7. Left: Four components of the QFMT kernel ($v = 2, k = 3$). Right: Symmetries of four components of the QFMT kernel ($v = 2, k = 3$).

Figure 8. Left: High angular resolution. Center: High radial resolution. Right: High radial and angular resolution.

Figure 9. Illustration of QFMT scaling.