G-ALGEBRAS, GROUP GRADED ALGEBRAS, AND CLIFFORD EXTENSIONS OF BLOCKS

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Abstract. Let $K$ be a normal subgroup of the finite group $H$. To a block of a $K$-interior $H$-algebra we associate a group extension, and we prove that this extension is isomorphic to an extension associated to a block given by the Brauer homomorphism. This may be regarded as a generalization and an alternative treatment of Dade’s results [5, Section 12].

1. Introduction

1.1. Let $p$ be a prime, and let $\mathcal{O}$ be a discrete valuation ring with residue field $k$ of characteristic $p$. We make no assumptions on the size of $\mathcal{O}$ and $k$, also allowing $\mathcal{O} = k$. Let $K$ be a normal subgroup of the finite group $H$, and we denote by $G$ the factor group $H/K$.

1.2. Under the assumption that $k$ is algebraically closed, Dade introduced in [5] the Clifford extension of a block $b$ of $K$, and proved that this extension (which is a central extension of $k^*$ by a certain stabilizer of $b$) is isomorphic to an extension associated to an irreducible modular character lying in a Brauer correspondent of $b$ (see [5, Section 12]). Let us briefly discuss the arguments and assumptions used in [5]. Two main ingredients for the isomorphism of the Clifford extensions were [5, Theorems 8.7 and 9.5]. Note that [5, Theorem 8.7] is essentially Brauer’s First Main Theorem, while [5, Theorem 9.5] extends the Brauer correspondence to the case of conjugacy classes of maximal ideals. Besides these two theorems there are two important sets of hypotheses, namely [5, 7.1] and [5, 10.1]. These conditions assure that the equalities [5, 10.4] hold, and give a situation which is a slightly more general case than of the group ring $\mathcal{O}H$.

1.3. Here we consider a unitary $K$-interior $H$-algebra $A_1$ over $\mathcal{O}$. The $K$-interior $H$-algebra $A_1$ gives rise to an $H$-interior algebra $A$, which is strongly $G$-graded (see [10, 9.1], or [7, 2.1 and 2.2] for a more general version of this construction). The $H$-interior algebra $A$ does not satisfy conditions [5, 7.1], and therefore the equality [5, 10.4 a)],

$$C_A(A_1) = A^K,$$

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it is also not satisfied. Still, one of the centralizers occurring in this equality is suitable for the construction of the extensions. We will not use a mix of two results analogous to [5] Theorems 8.7 and 9.5, because for that we would need more assumptions. Instead, we consider two blocks that arise from [2] Theorem 3.2.

1.4. We start by choosing a primitive idempotent \( b \) of \( A^K_1 \) which lies in the center of \( A_1 \). By the assumptions made on \( A_1 \), the idempotent \( b \) is actually a block of \( A_1 \). Instead of working with the centralizer \( C_A(A_1) \) we consider \( A^K \). As we have already mentioned, the equality \( C_A(A_1) = A^K \) holds if the algebra \( A \) satisfies assumptions [5] 7.1. In our case we only have the inclusion

\[
C_A(A_1) \subseteq A^K = C_A(K \cdot 1),
\]

because \( A_1 \) is \( K \)-interior. We work here with \( A^K \), and we consider, as in [6] Paragraph 2| the subgroup \( G_b \) of \( G \) fixing \( b \), and its normal subgroup \( G[b] \). Restricting \( A^K \) to the components indexed by \( G[b] \) gives a strongly \( G[b] \)-graded \( G_b \)-algebra, and this allows the construction in Section 2 of the Clifford extension associated to the block \( b \).

1.5. Let \( P \leq K \) be a defect group of the point \( \{ b \} \) of \( A^K_1 \). The Brauer quotient \( A(P) \) of the algebra \( A \) with respect to \( P \) is a \( G \)-graded algebra, and its restriction \( A'(P) \) to \( C_H(P)K/K \) is strongly graded. The Brauer quotient \( A_1(P) \) of \( A_1 \) is a \( C_K(P) \)-interior \( C_H(P) \)-algebra, and, as in [7] 2.1} again, it gives rise to a \( C_H(P) \)-interior \( C_H(P)/C_K(P) \)-graded algebra \( R \). We show in Section 4} that \( R \) and \( A'(P) \) are isomorphic as \( N_H(P) \)-algebras.

By [2] Theorem 3.2, to the point \( \{ b \} \) of \( A^K_1 \) it corresponds the point \( \{ \tilde{b} := \text{Br}_P(b) \} \) of \( A_1(P)^{N_K(P)} \). Since \( b \) is central then so is \( \tilde{b} \). Using the isomorphism stated in Proposition 3.4, we regard \( \tilde{b} \) as a primitive idempotent of \( R^{N_K(P)}_1 \). Next, we consider the centralizer

\[
C_R(C_K(P))^{N_K(P)} = R^{N_K(P)}.
\]

This is a \( C_H(P)/C_K(P) \)-graded \( N_H(P) \)-algebra, and we localize it by using \( \tilde{b} \) to the subgroup \( C_H(P)\tilde{b}/C_K(P) \). Further, we take the \( N_K(P) \)-fixed elements and we consider the group \( C_H(P)[\tilde{b}] \) for which \( \tilde{b}R^{N_K(P)} \) is strongly graded. We construct in Section 4 the first Clifford extension associated to \( \tilde{b} \).

1.6. Of course, the algebra \( A(P) \) is \( C_H(P) \)-interior. The centralizer

\[
C_A(P)(C_K(P)) = A(P)^{C_K(P)}
\]

is too large, so one must take

\[
(C_A(P)(C_K(P)))^{N_K(P)} = A(P)^{N_K(P)}.
\]

Going back to the \( G \)-graded Brauer quotient we see that

\[
\tilde{b}A(P)\tilde{b} = \bigoplus_{\sigma \in G_b} \tilde{b}A(P)_\sigma.
\]

We denote \( G[\tilde{b}] \) the subgroup of \( G_b \) such that

\[
E := \bigoplus_{\sigma \in G[\tilde{b}]} \tilde{b}A(P)^{N_K(P)}_\sigma
\]

is strongly graded. Since \( E_1 = \tilde{b}A_1(P)^{N_K(P)} \) is local, we consider the quotient

\[
\tilde{E} = E/\tilde{E}J(E_1)
\]
the crossed product of $\bar{E}_1$ with $G[\bar{b}]$ corresponding to the second Clifford extension of $\bar{b}$.

1.7. In [5, Section 11] a so-called “right kernel” of a bilinear map is introduced. That map exists because of the assumption that the field $k$ is algebraically closed, so the construction of the Clifford extensions yields twisted group algebras over $k$ whose components are one dimensional. In the present situation, we see that in general the two Clifford extensions correspond to crossed products over the skew fields

$$\bar{C}_1 := bA_1^K / J(bA_1^K)$$

and

$$\bar{E}_1 := \bar{b}(A_1(P))^{N_K(P)}/J(\bar{b}(A_1(P))^{N_K(P)})$$

extending $k$, respectively. Clearly $\bar{C}_1$ and $\bar{E}_1$ are trivially $N_K(P)$-acted so there is no need to work with this subgroup.

Our main result is Theorem 7.3 below, where we show that the above crossed products containing $\bar{C}_1$ and $\bar{E}_1$ as identity components are isomorphic as group-graded algebras, and the isomorphism preserves the action of the stabilizer of $b$. Moreover, we show that $G[b] = G[\bar{b}]$ is a subgroup in the normalizer of $P$ in $H_b$ and that these two isomorphic extensions contain the first Clifford extension of $\bar{b}$.

1.8. At the end of the paper we deal with the special case of the group algebra. In this case, our $K$-interior $H$-algebra is $A_1 = OK$, and we have the equalities

$$C_{O_H}(OK) = (OH)^K$$

and

$$(C_{kC_H(D)}(kC_K(D)))^{N_K(D)} = (kC_H(D))^{N_K(D)}.$$ 

For this particular $K$-interior $H$-algebra, the result tells more. For both algebras $O_{NP}$ and $O_{N_H(P)P}$ the Brauer quotient is the same, exactly $kC_H(P)$. So considering again the Clifford extensions as before one ends up with three isomorphic crossed products acted by the same $N_H(P)_b$. This generalizes [5, Corollary 12.6] (and see also [4] for a concise presentation) to the case of arbitrary base field $k$.

Our general assumptions and notations are standard. We refer the reader to [12] and [10] for Puig’s theory of $G$-algebras and pointed groups, and to [9] for facts on group graded algebras.

2. The Clifford extension of a block

2.1. As in the introduction, let $K$ be a normal subgroup of the finite group $H$, and let $G = H/K$. Let $A_1$ be an unitary $K$-interior $H$-algebra over the $O$. As is [7, 2.1], there exists a strongly $G$-graded algebra

$$A := \bigoplus_{\sigma \in G} A_\sigma$$

with structural homomorphisms

$$OH \to A$$

of $G$-graded algebras. This homomorphism endows $A$ with the structure of a $G$-graded $H$-interior algebra, and hence $A$ is also an $H$-algebra by conjugation:

$$a_h^h = h^{-1} \cdot a_\sigma \cdot h \in A_{\sigma h},$$

where $a_\sigma \in A_\sigma$.
where \( a_{\sigma} \in A_{\sigma}, \sigma \in G \), and \( h \in H \). Moreover, we have that
\[
A_{\sigma} = A_1 \otimes x \text{ and } A_1 \simeq A_1 \otimes 1,
\]
for each \( \sigma \in G \). Here \( x \) is a representative in \( \sigma \) and the tensor product is over \( k \).
The multiplication in \( A \) is defined in [7, 2.1], and see also [10, 9.1].

2.2. Let \( A_1^K \) denote the subalgebra of \( A_1 \) consisting of elements fixed under the
conjugation action of \( K \). The interior \( H \)-algebra is \( K \)-interior by restriction, hence
in the same manner we may consider
\[
C_A(K \cdot 1) = A^K,
\]
the fixed elements in \( A \) under the conjugation action of the same group \( K \). Because
\( K \) is a normal subgroup of \( H \) we clearly have that
\[
A^K = \bigoplus_{\sigma \in G} (A_\sigma)^K
\]
is a \( G \)-graded subalgebra of \( A \).

2.3. Let \( b \) be a primitive idempotent of \( A^K \), lying in \( Z(A_1) \). The idempotent \( b \) need
not necessarily be central in \( C_A(K \cdot 1) \), so it is convenient to consider the stabilizer
\( G_b \) of \( b \) in \( G \). In this situation, \( bA^K \) becomes a \( G_b \)-graded \( G_b \)-interior algebra, but
it is not strongly graded in general. As in [9], we consider the subset
\[
G[b] = \{ \sigma \in G_b \mid (bA_{\sigma})^K \cdot (bA_{\sigma - 1})^K = (bA_1)^K \}.
\]
of \( G_b \).

**Proposition 2.4.** The subset \( G[b] \) is a normal subgroup of \( G_b \).

**Proof.** For all \( \sigma, \tau \in G[b] \), we have
\[
(bA_1)^K = (bA_\sigma)^K \cdot (bA_{\sigma - 1})^K = (bA_\sigma)^K \cdot (bA_1)^K \cdot (bA_{\sigma - 1})^K
\]
\[
= (bA_\sigma)^K \cdot (bA_\tau)^K \cdot (bA_{\tau - 1})^K \cdot (bA_{\sigma - 1})^K
\]
\[
\subseteq (bA_{\sigma \tau})^K \cdot (bA_{(\sigma \tau - 1)})^K.
\]
This proves that \( G[b] \) is a subgroup of \( G_b \). Now consider the elements \( \sigma \in G[b] \) and
\( \theta \in G_b \). We shall prove the equality
\[
((bA_{\sigma})^K)^\theta = (bA_{\sigma^\theta})^K.
\]
For this, it suffices to take \( h \in \theta \) and prove that
\[
((bA_{\sigma})^K)^h = (bA_{\sigma^h})^K.
\]
Let \( a_{\sigma} \in (bA_\sigma)^K \), then \( a_{\sigma}^h \in A_{\sigma^h} \). For any \( l \in K \) we have
\[
(a_{\sigma}^h)^l = l^{-1} \cdot a_{\sigma}^h \cdot l = l^{-1}h^{-1} \cdot a_{\sigma} \cdot hl = h^{-1} \cdot a_{\sigma} \cdot h.
\]
Conversely, let \( a_{\sigma^h} \in (bA_\sigma)^K \). As before, we obtain that \( (a_{\sigma^h})^h \in (bA_\sigma)^K \), and
by applying the action of \( h \), the desired inclusion follows. We have
\[
((bA_{\sigma})^K)^\theta \cdot ((bA_{\sigma - 1})^K)^\theta = ((bA_1)^K)^\theta,
\]
or equivalently,
\[
(bA_{\sigma^h})^K \cdot (bA_{(\sigma^h) - 1})^K = (bA_1)^K,
\]
which proves the statement. \( \Box \)
2.5. We now denote
\[ C := \bigoplus_{\sigma \in G[b]} b(A_\sigma)^K = \bigoplus_{\sigma \in G[b]} C_\sigma \]
where for each \( \sigma \in G[b] \), we have denoted
\[ C_\sigma := b(A_\sigma)^K. \]
The proposition above implies that \( C \) is a strongly \( G[b] \)-graded \( G[b] \)-algebra. Its identity component
\[ C_1 = bA_1^K \]
is a local ring, so by [11, Lemma 1.1] it follows that \( C \) is a crossed product of \( C_1 \) with \( G[b] \).

2.6. We have the skew field
\[ \tilde{C}_1 := C_1/J(C_1) \]
whose center \( \tilde{k}_1 = Z(\tilde{C}_1) \) is a finite extension of \( k \). Consequently,
\[ \tilde{C} := C/CJ(C_1) \]
is a crossed product, and in the same time it is a \( G_b \)-algebra. For any \( \sigma \in G[b] \) we identify
\[ \tilde{C}_\sigma = C_\sigma/C_\sigma J(C_1). \]
The corresponding group extension
\[ (1) \quad 1 \to \tilde{C}_1^* \to \tilde{hU}(\tilde{C}) \to G[b] \to 1 \]
is the Clifford extension of the block \( b \). Here \( \tilde{hU} \) denotes the group of homogenous units.

3. The Brauer Quotient

By the Brauer quotient of a \( H \)-algebra with respect to a \( p \)-subgroup \( P \) of \( H \) we mean the structure of an \( N_H(P) \)-algebra as presented in [14, §11]. In general, there is no natural graded structure of the Brauer quotient, with respect to an arbitrary \( p \)-subgroup of \( H \), on our strongly \( G \)-graded \( H \)-interior algebra \( A \). Therefore, we consider the situation when \( P \) is a \( p \)-subgroup of \( K \).

3.1. Let \( P \) be a \( p \)-subgroup of \( K \). Since the restriction to \( K \) of the action of \( H \) on the algebra \( A \) leaves invariant each homogeneous component of \( A \), we have
\[ A^P = \bigoplus_{\sigma \in G} A_\sigma^P. \]
For any subgroup \( Q \) in \( P \) we have the equality \( A_Q^P \cap A_\sigma^P = (A_\sigma)_Q^P \). Using this equality, the Brauer quotient of \( A \) becomes
\[ A(P) = k \otimes \mathcal{O} \left( A^P / \sum_{Q < P} A_Q^P \right) \]
\[ = k \otimes \mathcal{O} \left( \bigoplus_{\sigma \in G} (A_\sigma)^P / \bigoplus_{\sigma \in G} \sum_{Q < P} (A_\sigma)_Q^P \right) \]
\[ \simeq k \otimes \mathcal{O} \bigoplus_{\sigma \in G} (A_\sigma)^P / \sum_{Q < P} (A_\sigma)_Q^P = \bigoplus_{\sigma \in G} A(P)_\sigma. \]
In this situation, the \( N_H(P) \)-algebra \( A(P) \) is \( G \)-graded, with identity component \( A(P)_1 \simeq A_1(P) \).
3.2. It is useful to consider the subalgebra

\[ A'(P) = \bigoplus_{\sigma \in C_H(P)K/K} A(P)_\sigma, \]

which is an \( N_H(P) \)-invariant subalgebra of \( A(P) \). With the notation of \[ \ref{2} \] the map \( a \mapsto a \otimes 1 \) is an isomorphism of \( H \)-algebras between \( A_1 \) and \( A_1 \otimes 1 \). This implies that

\[ A'_1 \simeq (A_1 \otimes 1)^P = A_1^P \otimes 1. \]

If \( \sigma \in C_H(P)K/K \) and \( a \in A_1^P \) and \( x \in \sigma \cap C_H(P) \), we can identify \( a \) with

\[ a \otimes 1 = (a \otimes x)(1 \otimes x^{-1}) \in A_\sigma^P \cdot A_{\sigma^{-1}}^P. \]

Then the equality \( A_1^P = A_\sigma^P \cdot A_{\sigma^{-1}}^P \) is valid, showing that \( A'(P) \) is a strongly \( C_H(P)K/K \)-graded \( N_H(P) \)-algebra. In fact, this also follows from the proof of Proposition \[ \ref{3.3} \] below.

3.3. We may also construct the Brauer quotient \( A_1(P) \) of the the \( K \)-interior \( H \)-algebra \( A_1 \), so \( A_1(P) \) is a \( C_K(P) \)-interior \( N_H(P) \)-algebra. We then regard \( A_1(P) \) to be a \( C_K(P) \)-interior \( C_H(P) \)-algebra. In this case, following \[ \ref{3.3} \] we obtain the strongly \( C_H(P)/C_K(P) \)-graded \( N_H(P) \)-interior \( k \)-algebra

\[ R := \bigoplus_{\tau \in C_H(P)/C_K(P)} R_\tau. \]

For each \( \tau \in C_H(P)/C_K(P) \) we have

\[ R_\tau = A_1(P) \otimes x \]

for some representative \( x \) in \( \tau \).

**Proposition 3.4.** The strongly graded \( k \)-algebras \( A'(P) \) and \( R \) are isomorphic as \( N_H(P) \)-algebras.

**Proof.** There is an obvious bijection between the sets of components of the two algebras, because of the natural isomorphism

\[ C_H(P)K/K \simeq C_H(P)/C_K(P). \]

Next, we fix \( \tau \in C_H(P)/C_K(P) \) and \( \sigma \in C_H(P)K/K \) such that \( \tau = \sigma \cap C_H(P) \), and we define

\[ \psi_\tau : R_\tau \to A'(P)_\sigma, \quad \bar{a} \otimes x \mapsto a \otimes x, \]

for \( x \in \tau \). We have denoted \( \bar{a} := Br_P(a) \) for some element \( a \) in \( A_1^P \), while \( x \) stands for a representative in \( \tau \). If \( a_1 \in \bar{a} \) then \( a_1 - a \in \text{Ker}(Br_P) \) which means \( a_1 - a = \text{Tr}_Q^P(c) \) for some \( c \in A_1^Q \). Then

\[ (a_1 - a) \otimes x = \text{Tr}_Q^P(c) \otimes x = \text{Tr}_Q^P(c \otimes x) \in \sum_{Q<P} (A_P)_Q^P, \]

since \( x \in C_H(P) \). So \( \psi_\tau \) is a well-defined map. Moreover, the direct sum of these maps gives the graded homomorphism

\[ \psi : R \to A'(P), \]

whose identity component is an isomorphism. Indeed, \( A'(P)_1 = A_1(P) \) and \( R_1 = A_1(P) \otimes 1 \simeq A_1(P) \), and one can easily prove that \( \psi_1 \) is both injective and surjective. The statement follows by applying \[ \ref{3.4} \] Proposition 2.12. \qed
4. The second extension

4.1. Assume that the subgroup $P$ of $K$ is a defect group of $b$. According to [12 §18], this means $b \in (A_1)_{N_K^P}$ and $B_{P}(b) \neq 0$. Since $b$ central and primitive in $A_1^P$, it is actually a point of $H$ on $A_1$ with defect group $P$ (see [12 §3]). Applying [2 Theorem 3.2], the element $\bar{b} = B_{P}(b)$ lies in the center of $A_1(P)^{N_K(P)}$ and forms a singleton, hence a point of $A_1(P)^{N_K(P)}$, also with defect group $P$. The identity component $R_1$ of the algebra $R$ constructed in [4.3] is isomorphic to $A_1(P)$. Using this isomorphism, we regard $\bar{b}$ as an element of $R_1$, so $\bar{b}$ is a point of $R_1^{N_K(P)}$ with defect group $P$.

4.2. Let $C_H(P)_{\bar{b}}$ denote the stabilizer of $\bar{b}$ in $C_H(P)$, and let

$$S := \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} R_\tau.$$ 

Moreover $\bar{b}$ is a central element of $S$ and one can check the equality

$$\bar{b}R\bar{b} := \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} \bar{b}R_\tau.$$ 

Then $\bar{b}R\bar{b}$ is $C_H(P)/C_K(P)$-graded $N_H(P)_{\bar{b}}$-invariant algebra.

4.3. Since $N_K(P)$ is a normal subgroup of $N_H(P)_{\bar{b}}$, we may consider the centralizer

$$C_S(C_K(P))_{N_K(P)} \cdot 1 = S^{N_K(P)}.$$ 

The normal subgroup $N_K(P)$ of $N_H(P)_{\bar{b}}$ centralizes $C_H(P)_{\bar{b}}/C_K(P)$, and $S^{N_K(P)}$ is a $N_H(P)_{\bar{b}}$-algebra in which $\bar{b}$ is central. Then

$$S^{N_K(P)} = \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} (R_\tau)^{N_K(P)},$$

and

$$\bar{b}S^{N_K(P)} = (\bar{b}S)^{N_K(P)} = (\bar{b}R\bar{b})^{N_K(P)} = \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} \bar{b}(R_\tau)^{N_K(P)}.$$ 

Of course we need

$$\overline{C_H(P)_{\bar{b}}} = \{ \tau \in C_H(P)_{\bar{b}}/C_K(P) \mid \bar{b}(R_\tau)^{N_K(P)} \cdot \bar{b}(R_{\tau^{-1}})^{N_K(P)} = \bar{b}(R_1)^{N_K(P)} \},$$

the normal subgroup of $C_H(P)_{\bar{b}}/C_K(P)$ that makes $\bar{b}S^{N_K(P)}$ strongly graded. Denote

$$D := \bigoplus_{\tau \in \overline{C_H(P)_{\bar{b}}}} \bar{b}(R_\tau)^{N_K(P)}$$

$$= \bigoplus_{\tau \in \overline{C_H(P)_{\bar{b}}}} D_\tau,$$

where for each $\tau \in \overline{C_H(P)_{\bar{b}}}$ we have

$$D_\tau = \bar{b}(R_\tau)^{N_K(P)}.$$

By construction, it follows that $D_\tau D_{\tau^{-1}} = D_1$ for all $\tau \in \overline{C_H(P)_{\bar{b}}}$. This makes $D$ a $\overline{C_H(P)_{\bar{b}}}$-strongly graded $N_H(P)_{\bar{b}}$-algebra. By the localness of $D_1$ and by [11 Lemma 1.1] the order $D$ is a crossed product of $D_1$ with $\overline{C_H(P)_{\bar{b}}}$.
4.4. Denote \( \bar{D}_1 := D_1 / J(D_1) \)
and
\( \bar{D} = D / DJ(D_1) \).
Since the identity component \( D_1 = \bar{b}R_{k^1}^N(P) \) of \( D \) is a local ring, we obtain \( \bar{D}_1 \)
a skew field whose center \( \bar{k}_2 = Z(\bar{D}_1) \) is a finite extension of \( k \). Moreover, \( \bar{D} \) is
an \( N_H(P)b \)-algebra, and in the same time a crossed product of \( \bar{D}_1 \) with \( \overline{C_H(P)}[\bar{b}] \),
corresponding to the group extension
\[
1 \rightarrow \bar{D}_1^* \rightarrow hU(\bar{D}) \rightarrow \overline{C_H(P)}[\bar{b}] \rightarrow 1.
\]
As usual, for \( \tau \in \overline{C_H(P)}[\bar{b}] \) we identify
\( \bar{D}_\tau = D_\tau / D_\tau J(\bar{D}_1) \).

5. The third extension

If, as above, \( N_H(D)b \) denotes the stabilizer of \( b \) in \( N_H(P) \) and \( N_H(D)\bar{b} \) denotes
the stabilizer of \( \bar{b} \) in \( N_H(P) \), one can easily show that these two stabilizers are
equal.

5.1. We return to our \( G \)-graded \( H \)-interior algebra
\[
A = \bigoplus_{\sigma \in G} A_\sigma,
\]
and to its \( H_b \)-subalgebra
\[
bA := \bigoplus_{\sigma \in G_b} bA_\sigma.
\]
Since the defect group \( P \) fixes \( b \) we have \( (bA)^P = bA^P \) and we may consider the
Brauer homomorphism in the following situation.
\[
\text{Br}_P : (bA)^P \rightarrow (bA)(P).
\]
Of course this is a morphism of \( N_H(P)b \)-algebras. Note that we have the direct
sums decompositions
\[
(bA)^P = \bigoplus_{\sigma \in G_b} (bA_\sigma)^P
\]
and
\[
(bA)(P) = \bigoplus_{\sigma \in G_b} \left( (bA_\sigma)^P / \sum_{Q < P} (bA_\sigma)_Q^P \right) := \bigoplus_{\sigma \in G_b} (bA)(P)_\sigma.
\]
This is actually the same situation as in Section 4, so we do not give any further
explanation on the structure of these algebras.

5.2. We are going to relate this Brauer quotient to the construction made in Section 4. For
that we need to make one replacement. Recall that \( \bar{b} \) denotes the image of \( b \) under the Brauer morphism evaluated on the identity components. We see that
for any \( \sigma \in G_b \) there is an isomorphism of \( N_H(P)b \)-invariant \( k \)-spaces, that is
\[
(bA_\sigma)^P / \sum_{Q < P} (bA_\sigma)_Q^P \cong \bar{b}((A_\sigma)^P / \sum_{Q < P} (A_\sigma)_Q^P).
\]
Thus we can reconsider the Brauer morphism in this way:
\[
\text{Br}_P : bA^P \rightarrow \bar{b}A''(P) = \bar{b}A(P)\bar{b},
\]
where
\[ A''(P) = \bigoplus_{\sigma \in G_b} A(P)_\sigma. \]

5.3. Next we are interested in the subgroups of \( G_b \) for which \( bA'' \) and \( \bar{b}A''(P) \) become strongly graded. So we consider
\[ T_1 = \{ \sigma \in G_b \mid bA''_\sigma \cdot bA''_{\sigma^{-1}} = bA'' \} \]
and
\[ T_2 = \{ \sigma \in G_b \mid \bar{b}A''(P)_\sigma \cdot \bar{b}A''(P)_{\sigma^{-1}} = \bar{b}A''(P)_{\sigma^{-1}} \}. \]

Since we are concerned mainly with subalgebras that are strongly graded, as we will see, all the subgroups of \( G_b \) that appear in our further constructions are included in \( T_1 \) or in \( T_2 \). An obvious remark is that the Brauer morphism carries the surjection componentwise, and this implies that the restriction
\[ \bigoplus_{\sigma \in T_1} (bA''_\sigma) \rightarrow \bigoplus_{\sigma \in T_2} bA''(P)_\sigma \]
is well defined. Indeed, if \( \sigma \in T_1 \) then we have
\[ \bar{b}A''(P)_1 = \text{Br}_P(bA''_1) = \text{Br}_P(bA''_\sigma \cdot bA''_{\sigma^{-1}}) = \bar{b}A''(P)_\sigma \cdot \bar{b}A''(P)_{\sigma^{-1}}. \]

This restriction could be surjective provided that the idempotent \( b \) remains primitive in \( A''_1 \), which in general it is not true.

Following the proof of Proposition 2.3 one can show that \( T_1 \) and \( T_2 \) are both respectively \( N_H(P)_b \)-invariant and \( N_H(P)_{\bar{b}} \)-invariant subgroups of \( G_b \). But since \( N_H(P)_b = N_H(P)_{\bar{b}} \) and \( T_1 \leq T_2 \) this restriction is a morphism of \( N_H(P)_b \)-algebras.

5.4. At the beginning of Section 4 we saw that \( \bar{b} \) is a primitive central idempotent of \( A(P)_1^{N_K(P)} \cong A_1(P)^{N_K(P)} \), hence \( \bar{b} \) is a central idempotent of
\[ \bar{b}A''(P)^{N_K(P)} = \bigoplus_{\sigma \in G_b} \bar{b}A''(P)_\sigma \]
and of
\[ (\bigoplus_{\sigma \in T_2} \bar{b}A''(P)_\sigma)^{N_K(P)} = \bigoplus_{\sigma \in T_2} \bar{b}A''(P)_\sigma. \]

In both cases we can introduce the subgroups
\[ G[\bar{b}] = \{ \sigma \in G_b \mid bA''_\sigma \cdot bA''_{\sigma^{-1}} = bA''(P)_1 \} \]
and
\[ T_2[\bar{b}] = \{ \sigma \in T_2 \mid \bar{b}A''(P)_\sigma \cdot \bar{b}A''(P)_{\sigma^{-1}} = \bar{b}A''(P)_1 \} \]
of \( G_b \), determining two strongly graded subalgebras. As expected, these subgroups coincide and are \( N_H(P)_b \)-invariant.

The last statement follows by using the localness of \( \bar{b}A''(P)_1^{N_K(P)} \), the inclusions
\[ (\bigoplus_{\sigma \in T_2} \bar{b}A''(P)_\sigma)^{N_K(P)} \hookrightarrow \bar{b}A''(P)^{N_K(P)} \hookrightarrow \bar{b}A''(P) \]
and \([11] \text{Lemma 1.1} \). Then for any \( \sigma \in T_2[\bar{b}] \) there is a unit \( \bar{a}_\sigma \in \bar{b}A''(P)_\sigma \cap U(\bar{b}A''(P)^{N_K(P)}) \) such that
\[ \bar{b}A''(P)_\sigma = \bar{a}_\sigma \bar{b}A''(P)_1^{N_K(P)} = \bar{b}A''(P)_1^{N_K(P)} \bar{a}_\sigma. \]
So \( T_2[\hat{b}] \) is a subgroup of \( G[\hat{b}] \). Now let \( \sigma \in G[\hat{b}] \). A similar argument as before gives a unit \( \bar{a}_\sigma \in \bar{b}A(P)_\sigma^{N_K(P)} \cap U(\bar{b}A''(P)^{N_K(P)}) \) such that

\[
\bar{b}A(P)_\sigma^{N_K(P)} = \bar{a}_\sigma \bar{b}A(P)_1^{N_K(P)} = \bar{b}A(P)_1^{N_K(P)} \bar{a}_\sigma,
\]
as a component of \( \bar{b}A''(P)^{N_K(P)} \). Using one of the above inclusion it follows that \( \bar{a}_\sigma \) is an invertible element in the biggest algebra. This implies that \( \sigma \) belongs to \( T_2 \). Keeping in mind that \( \bar{a}_\sigma \) is still an \( N_K(P) \)-invariant invertible element, we have \( \sigma \in T_2[\hat{b}] \). The \( N_H(P)_b \)-invariance follows again using the technique of the proof of Proposition [24].

5.5. In the previous paragraph we saw that regardless the starting point, we end up with the same subgroup that makes our \( N_K(P) \)-invariant subalgebra of \( \bar{b}A(P)'' \) strongly graded. Repeating the construction of Section 4 we obtain the algebra

\[
E := \bigoplus_{\sigma \in G[\hat{b}]} \bar{b}A(P)_\sigma^{N_K(P)} := \bigoplus_{\sigma \in G[\hat{b}]} E_\sigma,
\]
which is a crossed product of \( E_1 := \bar{b}A(P)_1^{N_K(P)} \) with \( G[\hat{b}] \) such that \( \bar{E}_1 := E_1/J(E_1) \) is a skew field whose center \( \hat{k}_3 = Z(\bar{E}_1) \) is a finite extension of \( k \). The quotient \( \bar{E} := E/EJ(E_1) \) is a \( N_H(P)_b \)-algebra, a crossed product of \( \bar{E}_1 \) with \( G[\hat{b}] \) corresponding to the Clifford extension

\[(3) \quad 1 \to \bar{E}_1 \to \bar{h}U(\bar{E}) \to G[\hat{b}] \to 1.\]

We should note that \( E \) does not contain any zero components, meaning that for any \( \sigma \in G[\hat{b}] \) we have \( \bar{b}A(P)_\sigma^{N_K(P)} \neq 0 \).

6. Remarks on the first extension

We go back to our first extension, since we have introduced the group \( T_1 \), and we need to relate it to \( G[\hat{b}] \).

6.1. First, let us observe that \( G[\hat{b}] \) is a subgroup of \( T_1 \). This follows from the fact that \( C \) is a crossed product (see [23], [24]) and because of the inclusions

\[
C = \bigoplus_{\sigma \in G[\hat{b}]} b(A_\sigma)^K \hookrightarrow \bigoplus_{\sigma \in G_b} b(A_\sigma)^K \hookrightarrow \bigoplus_{\sigma \in G_b} b(A_\sigma)^P.
\]

6.2. Conversely, as in the previous section, since

\[
\bigoplus_{\sigma \in T_1} b(A_\sigma)^K \hookrightarrow \bigoplus_{\sigma \in T_1} b(A_\sigma)^P,
\]
we have \( T_1[\hat{b}] = G[\hat{b}] \).

6.3. If \( \sigma \in G[\hat{b}] \) since \( bA_1^K = bA^K \cdot bA^K_{-1} \) then \( bA^K \neq 0 \) and \( bA^K \not\subseteq \sum_{Q < P}(bA_\sigma)_Q^K \). This easily follows from the definition of \( G[\hat{b}] \) and from the proof of Lemma [7.3] below. This remark forces the corresponding component in the Brauer quotient to satisfy \( \bar{b}A(P)_\sigma^{N_K(P)} \neq 0 \).
7. The isomorphism of two extensions

We keep the notations of the previous sections. The next lemma will be needed.

**Lemma 7.1.** The skew fields $\bar{C}_1$ and $\bar{E}_1$ are isomorphic as $N_H(P)_b$-algebras.

**Proof.** Recall that
$$\bar{C}_1 = C_1/J(C_1) = bA_1^K/J(bA_1^K)$$
and
$$\bar{E}_1 = E_1/J(E_1) = \tilde{b}(A_1(P))^{N_K(P)}/J(\tilde{b}(A_1(P))^{N_K(P)}).$$

We have $P$ a defect group of the point $\{b\} \subset A_1^K$, and $b \in (A_1)_b^K$. Arguments similar to those of [8, Lemma 3.4], together with the proof of [3, Lemma 1.12], show that the map
$$\text{Br}_P : bA_1^K \to (\text{Br}_P(bA_1(P)))^{N_K(P)} = \tilde{b}(A_1(P))^{N_K(P)}$$
is onto. Notice that
$$\text{Br}_P((A_1)_P^K) = A_1(P)^{N_K(P)}.$$  
This is true since for $\text{Tr}_P^K(a) \in (A_1)_P^K$, by using the Mackey decomposition, we get
$$\text{Br}_P(\text{Tr}_P^K(a)) = \text{Br}_P(\sum_{l \in P \cap K/P} \text{Tr}_{P_l/P_l}^P(a^l)) = \sum_{l \in N_K(P)/P} (\text{Br}_P(a))^l.$$  
The idempotent $b$ belongs to $(A_1)_P^K$, and the ideal $b(A_1)_P^{N_K(P)}$ is mapped onto the ideal $\tilde{b}A_1(P)^{N_K(P)}$, which contains the identity $\tilde{b}$ of $\tilde{b}A_1(P)^{N_K(P)}$. To finish the proof, it suffices to apply [10, Proposition 3.23].

Before stating the main result we introduce one more subgroup of $T_2$.

7.2. The $N_H(P)_b$-subalgebra
$$\bigoplus_{\sigma \in N_H(P)_b K/K} \tilde{b}A(P)^{N_K(P)}_{\sigma}$$
need not be a strongly graded subalgebra of $\tilde{b}A''(P)$. So we introduce the $N_H(P)_b$-invariant subgroup
$$\overline{N_H(P)[b]} = \{ \sigma \in N_H(P)_b K/K \mid \tilde{b}A(P)^{N_K(P)}_{\sigma} \cdot \tilde{b}A(P)^{N_K(P)}_{\sigma^{-1}} = \tilde{b}A(P)^{N_K(P)}_{1}\}$$
of $N_H(P)_bK/K$. The inclusions
$$\bigoplus_{\sigma \in \overline{N_H(P)[b]}} \tilde{b}A(P)^{N_K(P)}_{\sigma} \hookrightarrow \bigoplus_{\sigma \in N_H(P)_b K/K} \tilde{b}A(P)^{N_K(P)}_{\sigma} \hookrightarrow \tilde{b}A''(P),$$
show that $\overline{N_H(P)[b]}$ is a subgroup of $T_2$.

**Theorem 7.3.** The following statements hold.

1. $G_b$ equals $N_H(P)_b K/K = N_H(P)_b K/K$.
2. The groups $G[b]$ and $G[\tilde{b}]$ are equal and they both coincide with $\overline{N_H(P)[b]}$.
3. The extensions (1) and (3) are isomorphic.
4. The isomorphism between the extensions (1) and (3) is compatible with the identity isomorphism
$$G[b] \to \overline{N_H(P)[b]},$$
and preserves the action of
$$G_b \simeq N_H(P)_b/N_K(P)$$
on the two extensions.

(5) There is a monomorphism from extension (2) into extension (3), which is also compatible with the natural monomorphism

\[ C_H(P)[b] \to G[b], \]

and preserves the action of \( G_b \) on these extensions.

Proof. We know that the Brauer morphism is compatible with the \( N_H(P) \)-action. We also know that \( b = Br_P(b) \) and \( N_H(P)_b = N_H(P)_b \). Denote by \( H_b \) the inverse image of \( G_b \) in \( H \). Then \( H_b \) normalizes \( K \), and since all defect groups of \( b \) are conjugate under \( K \), \( H_b \) acts on the set of defect groups of \( b \). We have \( b \in (A_1)_K \) and \( Br_P(b) \neq 0 \). Then if \( h \in H_b \), we obtain \( b \in (A_1)_K \), and clearly \( Br_P(b) \neq 0 \), because otherwise \( b \in \sum_{Q \leq P} (A_1)_Q^{(n)} \), implying \( Br_P(b) = 0 \). So for any \( h \in H_b \) there is \( l \in K \) such that \( P^h = P^l \), and from this we obtain \( hl^{-1} \in N_{H_b}(P) = N_H(P)_b \). The inclusion \( N_H(P)_b K \subseteq H_b \) is trivial, proving the equality

\[ H_b = N_H(P)_b K = N_H(P)_b K. \]

This is our first statement.

We claim that the restriction of the Brauer homomorphism

\[ (bA)^K = \bigoplus_{\sigma \in G_b} (bA_\sigma)^K \to \bar bA''(P)^{N_K(P)} = \bigoplus_{\sigma \in G_b} \bar bA(P)^{N_K(P)} \]

is an epimorphism. Indeed, because \( P \) is a defect group of \( b \), this block belongs to the ideal \((A_1)_K^b \subseteq A_0^K\). The equality

\[ Br_P(bA^K) = \bar bA''(P)^{N_K(P)} \]

is well known. The sets \( bA^K \) and \( \bar bA''(P)^{N_K(P)} \) are ideals, one in \( bA^K \) and the other in \( \bar bA''(P)^{N_K(P)} \), both containing the identity element of the respective algebra, so the claim is proved.

If \( \sigma \in G[b] \) then \( bA^K = bA_\sigma^K \cdot bA_{\sigma^{-1}}^K \). Using the proof of Lemma \( 7.1 \) we obtain

\[ \bar bA(P)^{N_K(P)}_1 = Br_P(bA^K) = Br_P(bA_\sigma^K) Br_P(bA_{\sigma^{-1}}^K) \]

so the image of \( C \) is in \( E \), and \( G[b] \leq G[\bar b] \).

If \( \sigma \in G[b] \) we have

\[ E_\sigma \cdot E_{\sigma^{-1}} = E_1. \]

Then

\[ \text{Br}_P^{-1}(E_\sigma) \cdot \text{Br}_P^{-1}(E_{\sigma^{-1}}) \notin J(C_1), \]

because otherwise

\[ E_\sigma \cdot E_{\sigma^{-1}} = \text{Br}_P(\text{Br}_P^{-1}(E_\sigma) \cdot \text{Br}_P^{-1}(E_{\sigma^{-1}})) \subseteq \text{Br}_P(J(C_1)) \subseteq J(E_1) \]

which is false. We have

\[ \text{Br}_P^{-1}(E_\sigma) \cdot \text{Br}_P^{-1}(E_{\sigma^{-1}}) = bA^K_\sigma \cdot bA^K_{\sigma^{-1}} \subseteq C_1, \]

and we obtain

\[ J(C_1) + bA^K_\sigma \cdot bA^K_{\sigma^{-1}} = C_1. \]
Consequently, $\sigma \in G[b]$, and this proves the inclusion $G[b] \leq G[\bar{b}]$. Together with this inclusion we have shown that the restriction

$$\text{Br}_P : C \to E$$

is also an epimorphism of $N_H(P)_{b}$-algebras.

We have already used the inclusion $\text{Br}_P(J(C_1)) \subseteq J(E_1)$, which is a well-known result on $\mathcal{O}$-algebras related by an epimorphism. Using the fact that $C$ and $E$ are strongly graded algebras, the results in [3 1.5.A.] prove

$$\text{Br}_P(CJ(C_1)) \subseteq EJ(E_1).$$

In this way we get a new $N_H(P)_{b}$-algebra epimorphisms, namely

$$\text{Br}_P : \bar{C} \to \bar{E}, \text{ for } \bar{a} \in \bar{C} \text{ we have } \text{Br}_P(\bar{a}) = \text{Br}_P(a).$$

Statement (3) follows from this $G[\bar{b}]$-graded epimorphism, from the equality $G[b] = G[\bar{b}]$ and Lemma [7.1].

Using (1) the inclusion

$$\bigoplus_{\sigma \in N_H(P)_{b}K/K} \bar{b}A(P)_{\sigma} \hookrightarrow \bar{b}A''(P) = \bigoplus_{\sigma \in G_{b}} \bar{b}A(P)_{\sigma}$$

is now an equality. Forcing

$$\bigoplus_{\sigma \in N_H(P)_{b}K/K} \bar{b}A(P)^{N_{K}(P)}_{\sigma} \hookrightarrow \bar{b}A''(P)^{N_{K}(P)} = \bigoplus_{\sigma \in G_{b}} \bar{b}A(P)^{N_{K}(P)}_{\sigma}$$

to be an equality. The definitions of $G[\bar{b}]$ and $N_H(P)[\bar{b}]$, see [5.4] and [7.2] prove the equality of these two groups. This shows the last part of (2).

By the construction of the first extension $G[b]$ is a normal subgroup of $G_{b}$, so by using assertions (1) and (2), $G[\bar{b}]$ is a normal subgroup of $N_H(P)_{b}K/K$. Since the Brauer map is a morphism of $N_H(P)_{b}$-algebras, statement (4) of the theorem is immediate.

By Proposition [8.3] the $N_H(P)$-algebras

$$R := \bigoplus_{\tau \in C_H(P)/C_{K}(P)} R_{\tau}$$

and

$$A'(P) = \bigoplus_{\sigma \in C_H(P)_{b}K/K} A(P)_{\sigma}$$

are isomorphic as $N_H(P)$-algebras. So $R^{N_{K}(P)}$ and $A'(P)^{N_{K}(P)}$ are isomorphic as $N_H(P)$-algebras too. This implies that

$$\bar{bRb}^{N_{K}(P)} = \bar{bS}^{N_{K}(P)} \overset{\psi}{\cong} \bar{bA}'(P)^{N_{K}(P)}$$

and

$$\bar{bA}'(P)^{N_{K}(P)} = \bigoplus_{\sigma \in C_H(P)_{b}K/K} \bar{bA}(P)^{N_{K}(P)}_{\sigma}.$$
then \( \overline{C}_H(P)[\bar{b}]' \) is the largest subgroup for which the restriction of \( \bar{b}A'(P)^{N_K(P)} \) is strongly graded. If \( \tau \in \overline{C}_H(P)[\bar{b}] \) then

\[
\bar{b}A(P)^{N_K(P)} = \psi(\bar{b}R_{\tau}^{N_K(P)}) = \psi(\bar{b}R_{\tau-1}^{N_K(P)}) \\
= \bar{b}A(P)^{N_K(P)} \cdot \bar{b}A(P)^{N_K(P)},
\]

where \( \sigma \in C_H(P) \) and \( \tau \) belongs to \( \overline{C}_H(P)[\bar{b}]' \). Conversely, if \( \sigma \in \overline{C}_H(P)[\bar{b}]' \), as before

\[
\psi^{-1}(\bar{b}A(P)^{N_K(P)}) \cdot \psi^{-1}(\bar{b}A(P)^{N_K(P)}) \notin J(\bar{b}R_{1}^{N_K(P)}),
\]

since \( \bar{b}R_{1}^{N_K(P)} \) is local and \( \psi(J(\bar{b}R_{1}^{N_K(P)})) = J(\bar{b}A_1(P)^{N_K(P)}) \). Hence for the corresponding \( \tau = \sigma \cap C_H(P) \) we have

\[
\bar{b}R_{1}^{N_K(P)} = \bar{b}R_{\tau}^{N_K(P)} \cdot \bar{b}R_{\tau-1}^{N_K(P)}.
\]

By the definition of the action of \( N_H(P) \), on these algebras, the groups \( \overline{C}_H(P)[\bar{b}]' \) and \( \overline{C}_H(P)[\bar{b}] \) are both \( N_H(P)_{\bar{b}} \)-invariant hence normal subgroups of \( N_H(P)_{\bar{b}}K/\bar{b} \) and of \( N_H(P)_{\bar{b}}/C_K(\bar{b}) \) respectively. So \( \overline{C}_H(P)[\bar{b}] \) embeds into \( G[\bar{b}] \) and then \( D \) embeds in \( E \). The equalities

\[
J(D) = DJ(D_1), J(E) = EJ(E_1) \text{ and } J(E_1) = J(D_1)
\]

define a map \( \tilde{D} \to \tilde{E} \) such that all the vertical maps in the following commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \tilde{D}_1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \tilde{E}_1
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{D}_1 & \longrightarrow & \text{hU}(\tilde{D}) \\
\downarrow & & \downarrow \\
\text{hU}(\tilde{E}) & \longrightarrow & \overline{C}_H(P)[\bar{b}]
\end{array}
\]

\[
\begin{array}{ccc}
\overline{C}_H(P)[\bar{b}] & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
G[\bar{b}] & \longrightarrow & 1
\end{array}
\]

are injective.

\[ \square \]

8. The group algebra case

8.1. The main theorem shows that the group \( G[\bar{b}] \) defining the Clifford extension equals \( \overline{N}_H(P)_{\bar{b}}[\bar{b}] \) and it is at least \( \overline{C}_H(P)[\bar{b}] \). This situation is generated by the groups \( T_1 \) and \( T_2 \). They are the first that strongly graduate the Brauer domain and codomain, and they include all the others subgroups of the algebras we work with. Actually \( G[\bar{b}] \) equals \( \overline{C}_H(P)[\bar{b}] \) exactly when \( \overline{C}_H(P)[\bar{b}] = \overline{N}_H(P)[\bar{b}] \).

There are some cases of \( K \)-interior \( H \)-algebras for which \( T_2 \) is included or it equals the centralizer. This is the case of a group algebra. Let \( G \) denote the quotient \( H/K \), let \( A = \mathcal{O}H \) and let \( b \) be a block of \( \mathcal{O}K \) having defect group \( P \leq K \). The special case of the group algebra gives

\[
A = \bigoplus_{\sigma \in G} A_{\sigma}, \text{ where } A_{\sigma} = \mathcal{O}\sigma, \text{ for all } \sigma \in G
\]

and \( b \) is primitive in \( Z(\mathcal{O}K) = \mathcal{O}K^K \). For each \( \sigma \in G \) we denote by \( C_{\sigma} \) the intersection \( b(\mathcal{O}\sigma \cap A^K) \). As above we introduce \( G[\bar{b}] \) and \( G_b \), and then

\[
bAb = \bigoplus_{\sigma \in G_b} bA_{\sigma} = b\mathcal{O}H_b,
\]

while

\[
C = \bigoplus_{\sigma \in G[\bar{b}]} bC_{\sigma}
\]
is a strongly $G[b]$-graded $H_b$-algebra. Letting $\hat{k}_1 = C_1/J(C_1)$, the quotient

$$C/CJ(C_1) = \bigoplus_{\sigma \in G[b]} C_{\sigma}/C_{\sigma}J(C_1)$$

is the crossed product of $\hat{k}_1$ with $G[b]$ that corresponds to the Clifford extension

$$(1') \hspace{1cm} 1 \to \hat{k}_1^* \to hU(\hat{C}) \to G[b] \to 1.$$

$\textbf{8.2.}$ If $b_1$ is the Brauer correspondent of $b$, it also has defect $P$ and it lies in

$$Z(\mathcal{O}N_K(P)) = \mathcal{O}N_K(P)^{N_K(P)}.$$

The $N_K(P)$-interior $N_H(P)$-algebra $\mathcal{O}N_K(P)$ is the identity component of

$$\mathcal{O}N_H(P) = \bigoplus_{\sigma \in N_H(P)/N_K(P)} \mathcal{O}\sigma.$$

Moreover

$$b_1\mathcal{O}N_H(P)b_1 = b_1\mathcal{O}N_H(P)b_1 = \bigoplus_{\sigma \in N_H(P)b_1/N_K(P)} b_1\mathcal{O}\sigma.$$

As in the above construction, we have the normal subgroup $G[b_1]$ of $N_H(P)b_1/N_K(P)$ determining a strongly graded $N_H(P)_{b_1}$-subalgebra of

$$b_1\mathcal{O}N_H(P)_{b_1}^{N_K(P)} = \bigoplus_{\sigma \in N_H(P)b_1/N_K(P)} b_1(\mathcal{O}\sigma)^{N_K(P)};$$

more precisely,

$$E = \bigoplus_{\sigma \in G[b_1]} E_\sigma = \bigoplus_{\sigma \in G[b_1]} b_1(\mathcal{O}\sigma)^{N_K(P)}.$$

Then letting $\bar{k}_2 = E_1/J(E_1)$, the quotient $\bar{E} = E/EJ(E_1)$ is the crossed product of $\bar{k}_2$ with $G[b_1]$ associated to the extension

$$(2') \hspace{1cm} 1 \to \bar{k}_2^* \to hU(\bar{E}) \to G[b_1] \to 1.$$

$\textbf{8.3.}$ Let us take a look of the Brauer quotients of these two algebras. We have

$$\text{Br}^H_P : (\mathcal{O}H)^P \to (\mathcal{O}H)^P / \sum_{Q < P} (\mathcal{O}H)_Q^P \simeq kC_H(P)$$

and

$$\text{Br}^{N_H(P)}_P : (\mathcal{O}N_H(P))^P \to (\mathcal{O}N_H(P))^P / \sum_{Q < P} (\mathcal{O}N_H(P))_Q^P \simeq kC_H(P).$$

Then we have the maps

$$\text{Br}^H_P : (b\mathcal{O}H_b)^K \to \tilde{b}kC_H(P)^{N_K(P)}_b$$

and

$$\text{Br}^{N_H(P)}_P : (b_1\mathcal{O}N_H(P)b_1)^{N_K(P)}_b \to \tilde{b}_1kC_H(P)^{N_K(P)}_b,$$

where $\tilde{b} := \text{Br}^H_P(b), \tilde{b}_1 := \text{Br}^{N_H(P)}_P(b_1), N_H(P)_b = N_H(P)_{b_1}$ and $N_H(P)b_1 = N_H(P)_{b_1}$. Since $\bar{b} = b_1$ (see the proof of [11 Theorem 5.1]), the $N_H(P)_b$-algebra

$$D := \tilde{b}kC_H(P)^{N_K(P)}_b$$

determines a unique extension

$$(3') \hspace{1cm} 1 \to \hat{k}_3^* \to hU(\tilde{D}) \to \overline{C_H(P)[\bar{b}]} \to 1.$$
By applying twice the main result, once for each of the Brauer maps defined above, we get that the extensions $(1')$, $(2')$ and $(3')$ are isomorphic. Note that in the case of the group algebra we have

$$(bO\Omega)(P) = (bO\Omega')(P) = \bar{b}C_H(P),$$

and this is why $G[b] = G[\bar{b}] = G[b_1] = \mathcal{N}(P)[b] = C_H(P)[b]$.

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