Regular and chaotic motion in softened gravitational systems

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ABSTRACT

The stability of the dynamical trajectories of softened spherical gravitational systems is examined, both in the case of the full \(N\)-body problem and that of trajectories moving in the gravitational field of non-interacting background particles. In the latter case, for \(N \geq 10000\), some trajectories, even if unstable, had exceedingly long diffusion times, which correlated with the characteristic e-folding timescale of the instability. For trajectories of \(N \approx 100000\) systems this timescale could be arbitrarily large — and thus appear to correspond to regular orbits. For centrally concentrated systems, low angular momentum trajectories were found to be systematically more unstable. This phenomenon is analogous to the well known case of trajectories in generic centrally concentrated non-spherical smooth systems, where eccentric trajectories are found to be chaotic. The exponentiation times also correlate with the conservation of the angular momenta along the trajectories. For \(N\) up to a few hundred, the instability timescales of \(N\)-body systems and their variation with particle number are similar to those of the most chaotic trajectories in inhomogeneous non-interacting systems. For larger \(N\) (up to a few thousand) the values of the these timescales were found to saturate, increasing significantly more slowly with \(N\). We attribute this to collective effects in the fully self-gravitating problem, which are apparent in the time-variations of the time dependent Liapunov exponents. The results presented here go some way towards resolving the long standing apparent paradoxes concerning the local instability of trajectories. This now appears to be a manifestation of mechanisms driving evolution in gravitational systems and their interactions — and may thus be a useful diagnostic of such processes.

Key words: Celestial mechanics, stellar dynamics - Instabilities - Gravitation - Galaxies: kinematic & dynamics

1 INTRODUCTION

A persisting and perplexing question regarding the dynamics of \(N\)-body systems concerns the interpretation of the local divergence of their trajectories and the origin of this instability. Does this phenomenon, first pointed out by Miller (1964), tell us anything non-trivial about the behaviour of gravitational systems? Does it, for example, have any relation with processes, whether collective or local, leading to evolution in these systems? Is it useful in understanding and characterising such processes?

A series of studies by Kandrup and collaborators (e.g., Kandrup et al. 1994) have shown the local instability of trajectories to be an extremely robust phenomenon — occurring for all initial conditions and perturbations investigated, for a variety
of initial macroscopic configurations, and with an exponential timescale that is roughly independent of these properties and of \( N \), whether the system is initially in dynamical equilibrium or not. In potentials resulting from collections of fixed point particles the exponential divergence and its constant e-folding time have been checked for \( N \) up to a million particles (Valluri & Merritt 1999). However, as might be expected, conservation of the action variables, for systems with separable smoothed out potential, improves with increasing \( N \). Thus, at least in some respects and over sufficiently short timescales, the trajectories of particles in high-\( N \) discrete systems resemble those in continuous ones — since, qualitatively, the motion is completely determined by those action variables. This implies that time averages of trajectories in steady state discrete systems will approach those of the corresponding smooth systems for times much larger than the exponentiation time-scale (which is a fraction of the dynamical time). Nevertheless, the latter do not exhibit exponential divergence of nearby trajectories in the case of separable potentials. Moreover, trajectories in fixed smooth potentials are characteristics of the time-independent collisionless Boltzmann equation (CBE). If therefore, as is often assumed, large \( N \)-body gravitational systems are adequately described by the steady state CBE for times long compared to their dynamical time, the phenomenon of the \( N \)-invariant e-folding time must be considered paradoxical.

It can be shown (e.g., Braun & Hepp 1977; Spohn 1980) that, for large large-\( N \) gravitational systems, the dynamics described by the full equations of motion, converge towards that resulting from solutions of the CBE, provided that the potential has bounded first and second derivatives (a condition which enters, for example, in the form of Eq. 2.52 of Spohn). This amounts to “softening” the potential to get rid of the singularity at the origin. Indeed, in potentials with a short cut-off, the force due to the mean field is always greater than the force between any two particles for large enough \( N \). For example, in a spherical system, this simply requires that for a test particle at radius \( R \)

\[
N > \frac{R^2}{\epsilon^2}, \tag{1}
\]

where \( \epsilon \) is the (Plummer) softening length. Or equivalently that \( \epsilon > R/\sqrt{N} \). This in fact justifies the mean field approach embodied in the CBE.

Because, in the case of point mass systems, the binary force between neighbours can be larger than the force due to the mean field, formally speaking, the dynamics of even a large-\( N \) system is not equivalent to the one described by CBE — and thus the exponential divergence does not in itself signal any inconsistency. Perhaps the only puzzling point then is why the timescale of the exponential divergence does not correlate with particle number while other dynamical properties of gravitational systems obviously do. (As noted above, this is even true for trajectories of systems of fixed point particles, where the exponential instability cannot be attributed to any time-dependence: Valluri & Merritt 1999).

Goodman, Heggie & Hut (GHH) conducted an extensive investigation of the exponential instability and concluded that it is precisely at the radius where the mean field force and the force due to near neighbours become comparable that the latter interactions have the greatest contribution to the process. Since the interparticle spacing, projected on two dimensions, decreases as \( 1/\sqrt{N} \), a test particle is likely to suffer one such encounter at every crossing of the system — independent of \( N \). For larger \( N \) these encounters last for shorter times and thus the deflection angle decreases (roughly as \( 1/\sqrt{N} \)). Nevertheless, the initial divergence between two trajectories (in the linear approximation) is independent of \( N \) (see GHH). Thus the trajectories of two particles, initially separated by a small distance compared to the impact parameter, will diverge exponentially with a timescale independent of \( N \). Consider however two trajectories separated in such a way that one is affected by an encounter with a field star at impact parameter \( \sim R/\sqrt{N} \) and the other not significantly affected — being at a distance \( \gg R/\sqrt{N} \) from the perturbing object. The affected test particle will be deflected by an angle \( \theta \sim 1/\sqrt{N} \). Thus, for large (compared to \( R/\sqrt{N} \)) separations and large \( N \), the divergence will be small.

In conventional dynamical systems where chaos is present for most initial conditions, it is usually either produced by the bulk properties of the potential or by local large angle scatterings. In both cases large deviations of trajectories, resulting from the non-integrable component of the potential, are expected in times comparable to the exponentiation times. In unsoftened gravitational systems on the other hand, the contribution to the potential responsible for exponential divergence on very short timescales independent of \( N \) comes from encounters whose range, duration, and deflection angles progressively become smaller at larger \( N \). Their effect on the actual trajectories becomes small, and the divergence does not lead to any macroscopic evolution on the divergence timescale — since it is only at small separation that the divergence is independent of \( N \). Any evolutionary effects due to the exponential instability, therefore, will have to come from larger range interactions, whether individual or collective.

In order to answer the questions posed in the opening paragraph of this paper one will therefore need to eliminate effects due to encounters in the interaction range which contributes to the \( N \)-invariant
e-folding time, but do not, for sufficiently large-
N, significantly change particle trajectories on such
timescales. When this is done, the mathematical
conditions for large N-body systems to be described
by the CBE, for times much longer than the dyna-
mical time, are then also satisfied. An important
consequence therefore is, if these large-N systems
are adequately described by steady state solutions of
the CBE, the divergence timescale in systems where
the smoothed out background potential is separa-
ble will have to increase with N. Moreover, since
in such a description there are no particle-particle
correlations and trajectories are independent, diver-
gence timescales in the full N-body problem and
their variation with N should be similar to those of
trajectories in compatible systems of non-interacting
background particles. It is the object of this paper
to examine the validity of such assertions.

By introducing a short range cutoff, GHH studied
the behaviour of the divergence timescale due
to random long (compared to \(N^{1/3}\)) range inter-
actions. They found it to increase as \(N\). Calcula-
tions of the Liapunov exponents from full N-
body simulations of softened particles over short
timescales confirmed this for low N (up to a few
hundred). One of the goals of the present paper is
the investigation of the variation of the exponen-
tiation times of softened N-body systems to higher
N and longer times (Section 3). We confine our-
selves to spherical systems which, being separable
in the continuum limit, should have, in that limit,
steady states completely characterised by regular or-
bits with no exponential divergence. In addition, in
an attempt to isolate effects that depend on the full
self-consistent N-body interaction from those asso-
ciated with discreteness noise resulting from rapid
spatial and time variations of the potential, we also
examine the divergence of single trajectories moving
in systems of non-interacting softened particles. We
do this for three different configurations: a collection
of non-interacting particles moving in a spherical
box (Section 3), a statistically homogeneous distri-
bution of fixed particles, and a distribution of par-
ticles with density decreasing with radius as \(1/r^2\)
(Section 4). This latter system is known to contain
chaotic orbits when non-spherical perturbations are
introduced, even in the continuum limit. We here ex-
amine whether this feature leads to different stabil-
ity properties for its trajectories in comparison with the
homogeneous case (where, in the aforementioned
limit, the potential is harmonic and all trajectories
are regular even in the triaxial case), in the hope of
exploring the effects of the smoothed out mass dis-
tribution on the stability of trajectories in discrete
systems. We summarise and attempt to interpret the
results in Section 4.

2 N-BODY SYSTEMS

2.1 Numerical evaluation of Liapunov exponents

Liapunov exponents measure the linear stability
along the trajectories of dynamical systems. For ex-
ample, let
\[
\dot{x}_i = f_i
\]
be the Newtonian equations of motions of an N-
body gravitational system (with \(i = 1, N\)), and
\[
\delta x_0 = \delta f_i
\]
the corresponding variational (i.e., linearised) equa-
tions. The latter measure the deviation of nearby
states to the one determined by the first set of equa-
tions. Next define
\[
X = (x_1, ..., x_N, \dot{x}_1, ..., \dot{x}_N)
\]
as the 6N-dimensional “vector field” of the system, and
\[
\xi = (\delta x_1, ..., \delta x_N, \delta \dot{x}_1, ..., \delta \dot{x}_N)
\]
as the corresponding vector field of variations. Then
the above equations can be written as
\[
\dot{X} = F
\]
and
\[
\dot{\xi} = \delta F.
\]
Now if we choose a particular trajec-
tory \(\bar{X} = \bar{X}(t, o, \bar{X}_o)\) against which we would like
to measure the deviation, with \(\bar{X}_o\) being the point
where we start measuring that deviation (i.e., the
initial conditions), then (8) can be rewritten as
\[
\dot{\xi} = D_\alpha F(\bar{X}(t, o, \bar{X}_o))\xi,
\]
where \(D_\alpha F\) is the Jacobian \(6N \times 6N\) matrix
\[
\partial F_i/\partial x_j \quad \text{and} \quad i, j = 1, 6N.
\]
Now let
\[
X_s = X_s(\bar{X}(t, o, \bar{X}_o))
\]
be the fundamental solution of this matrix with the
initial condition being the identity matrix, the solu-
tion of (8) is then given by (Wiggins 1991)
\[
\xi = X_s(t)\xi_0,
\]
which describes the evolution under the linearised
dynamics with initial conditions \(\xi_0\) in the space of
linear variations.

A Liapunov exponent is the infinite limit of the “time dependent Liapunov exponent” (Wiggins
1991) at \(\xi_0\) in the direction \(\xi_0\) at time \(t\) which is
given by
\[
\lambda(\xi, t) = \frac{\|\xi(t)\|}{\|\xi_0\|} = \frac{1}{t} \log \left( \frac{\|X_s(t)\xi_0\|}{\|\xi_0\|} \right).
\]
The Liapunov exponents are then defined as
\[ \sigma(\xi_0, X_0) = \lim_{t \to \infty} \lambda(\xi_0, X_0, t). \] (12)

Numerically of course only \( \lambda \) can be calculated. We will refer to the inverse of this time dependent Liapunov exponent as the “exponentiation time”, the “exponential timescale” or the e-folding time.

For a Hamiltonian system with \( f \) degrees of freedom, there are \( 2f \) linearly independent directions in phase space for the vector \( \xi_0 \) to point at, hence there are \( 2f \) Liapunov exponents. A positive Liapunov exponent indicates unstable behaviour characteristic of chaotic motion. Thus, determining the maximal exponent is sufficient for detecting the presence of such behaviour. The evaluation of the maximal exponent is straightforward enough. This is because exponential instability, if it is present, will cause almost all initial linear tangent space vectors to realign themselves along the subspace of maximal expansion. A numerical determination of a Liapunov exponent from almost any initial chosen direction for the linear variations will thus tend to give an evaluation of the maximal exponent (Wolf et al. 1985). The only complication that arises is that, when the exponentially increasing solutions of the linearised equations become too large, the calculation is slowed down (eventually leading to numerical overflow). This is easily remedied however by application of the “standard algorithm” of Benettin et al. (1976). This algorithm is based on the local averaging of the deviation ratio throughout the evolution being of the order of a few percent. Systems are enclosed using the method of single mass particles started from appropriate initial conditions and whose energy, mass and radius satisfy (El-Zant 1998) \[ ER/GM^2 > -0.335. \] (18)

In general, the values of the Liapunov exponents will depend on the initial conditions. If the phase space is “topologically transitive” however — that is if trajectories can visit any region of the phase space — for almost all initial conditions in the connected region all the Liapunov exponents will be equal. This however is not true of the time dependent Liapunov exponents which can be different at any given time. Convergence indicates that an invariant phase space distribution has been reached. The convergence time is therefore a measure of the transport properties of a given system’s phase space. The time dependent Liapunov exponents of systems with more complex phase space take longer to converge. Liapunov exponents of regular orbits converge to zero at a rate \( \sim e^{\frac{t}{\lambda}} \).

Strictly speaking, Liapunov exponents of open gravitational systems do not converge at all, because these systems are thermodynamically unstable and continuously evolve towards more and more concentrated states (e.g., Antonov 1962; Lynden Bell & Wood 1968; Padmanabhan 1990). Moreover, the evolution rate will depend on \( N \), making the comparison of systems with different \( N \) ambiguous. This is not the case however for enclosed \( N \)-body systems of single mass particles started from appropriate initial conditions and whose energy, mass and radius satisfy (El-Zant 1998) \[ \frac{ER}{GM^2} > -0.335. \] (18)

We define our system parameters such that \( G = M = R = 1 \), randomly distributing our particles inside \( R = 1 \). The quantity on the left hand side of condition (18) is then solely determined by the initial velocities. These are taken to be random in direction and their absolute values vary with radius as \( v = v_0 \exp(r) \), where \( v_0 \) is determined by the initial kinetic energy of the system. Since we have already determined the spatial distribution, this amounts to fixing the virial ratio. Our systems must have virial ratios greater than 0.5 if they are to satisfy relation (18) and be thermodynamically stable. At the same time the virial ratio should not be too large, so as not to modify the dynamics significantly. We choose a virial ratio of 0.69. Systems starting with these initial conditions were shown (El-Zant 1998) to quickly evolve towards quasisteady isothermal distributions, with the total change in the virial ratio throughout the evolution being of the order of a few percent. Systems are enclosed using the method also described in El-Zant(1998): particles venturing beyond \( R = 1 \) are subject to a force of the form \( mK(1 - r)^3 \), with \( K = 1300 \) and \( m = 1/N \) is the mass of a particle.

We numerically integrate the Newtonian equations of motion along with their variational counter-
Regular and chaotic motion in softened gravitational systems

Figure 1. Time variation of inverse of the time dependent Liapunov exponents (calculated from Eq. 11) for different particle numbers and random realisations of self gravitating $N$-body systems.

parts starting with random initial conditions for the latters (each component of the vectors in the tangent space of variations takes a value randomly chosen between zero and one and the vector is normalised so that its norm is one). In the simulations presented here the boundary force is not included in the variational equations, however its inclusion was not found to significantly affect the results. The integration is advanced using a highly accurate variable order variable stepsize Adams method, as implemented in the NAG routine D02CJF with tolerance $10^{-5}$. For systems consisting of a 100 particles, the dependence of the macroscopic evolution on the tolerance was checked and found not to vary significantly for tolerances of $10^{-3}$ to $10^{-10}$. For the tolerance used here, the energy was conserved to better than one part in 10000 for a hundred units and better than one percent for 20000. The energy conservation is much better for larger particle numbers. The renormalisation time for the variational equations was taken to be one unit. For the above parameters this is of the order of a dynamical time. More precisely, if one defines the crossing time of a system in virial equilibrium by

$$T_{\text{cross}} = M^{5/2}/(2E)^{3/2},$$

(19)

it is found to be about 4 time units for systems with the above parameters. In practice because our systems have a virial ratio of 0.69 instead of 0.5, the crossing time is 0.67 times shorter. In all simulations the (Plummer) softening of length is fixed at $0.1 \, (\geq R/\sqrt{N}$ for all $N$ investigated).

2.3 Results

For the full $N$-body problem, a total of 13 runs were investigated. Two different realisations for each $N = 100, 200, 400, 800$ system, three for $N = 1600$ and four with systems of $N = 3200$ particles. The $N = 100$ runs were integrated up to 20000 time units and so was one of the $N = 200$ runs, the other one was stopped at $N = 6600$ when it was clear the results were converging. The $N = 400$ runs were integrated up to 7000 time units and the $N = 800$ for 5000 and 10000 units. For $N = 1600$, two runs were pursued until 4400 units while one was stopped at 1500 units. Finally, two of the $N = 3200$ runs were integrated up to 700 units and the other two for an additional 400 units, for a total of 1100 time units.

As can be seen from Fig. 1, for simulations that were run up to 4400 time units, the exponential divergence timescales show good convergence,
Figure 2. Time variation of the inverse of the time dependent Liapunov exponents for two different realisations of $N = 3200$ self gravitating systems. After the initial evolution, leading to a more or less quasi-steady state, weakly damped modes can remain for at least a few hundred dynamical times.

Figure 3. Variation of the exponential timescales with particle number for self gravitating $N$-body systems. The plotted line, which fits well the results for runs with $N = 100, 200$ and $N = 400$ has slope of 0.4 although there seems to be a small systematic decrease in some of the higher-$N$ runs. This appears to be a result of slight increase in central concentration accompanying slow evolution towards the invariant equilibrium state (reached much quicker in the lower-$N$ runs). Also, in these runs, it is possible to detect small regular oscillations in the exponential timescales for early times (smaller than 800 units). These are eventually damped out (although variations on longer timescales persist for much longer).

For $N = 3200$ however, these oscillations are much larger and are not completely damped over the timescales investigated (Fig. 2). These oscillations reflect weakly damped global density oscillations. By inspection, at least three timescales can be detected. They are of the order of one, ten and few tens of time units.

Already from from Fig. 2 and Fig. 3, it is apparent that the exponentiation timescale increases with $N$, but the rate of this increase decreases for larger $N$. This is evident in Fig. 3, where we have plotted the final exponentiation times for all the runs as a function of $N$. The straight line in that figure fits the first three values of $N$. It has a slope of 0.4, which is roughly compatible with a slope of a third, pre-
Regular and chaotic motion in softened gravitational systems

Figure 4. Time variation of the inverse of the time dependent Liapunov exponents (calculated from Eq. [11]) for systems where one (test) particle interacts with all the other (background) particles, which are non-interacting. The upper panel shows results for random initial realisations of systems consisting of 1000 and 2000 particles, while the lower panel shows the corresponding results when $N = 4000, 8000$. Invariably, the final exponentiation times are longer for systems with larger $N$. Convergence however is much slower for larger systems (note the different time scales in the upper and lower plots).

predicted by GHH on the basis of their analysis of the divergence in a infinite homogeneous medium of softened particles, and confirmed by their simulations at lower $N$. One therefore naturally suspects that the discrepancy results from self gravitating modes leading to fluctuations in the mean field, as evidenced by the oscillation of the values of the exponential times for higher values of $N$ noted above. These global oscillations appear to be quickly washed out by discreteness noise in the low $N$ ($< 800$) limit.

For purposes of comparison, in order to investigate further the possibility that effects due to self-gravity are indeed responsible for the saturation of the exponentiation times, we consider in the next sections several systems where self gravity is not included.

3 NON-INTERACTING MOVING PARTICLES

In this approximation one particle interacts with all other particles, which do not interact with each other (but the motion of which is affected by the interaction with that particle). The system is enclosed as in the previous section and the total kinetic energy is scaled so as to equal that of the self-gravitating systems described there. The interacting particle is picked randomly from the spatially homogeneous initial particle distribution, and the full set of $12N$ first order differential equations, describing the motion of all $N$ particles and its variations, is integrated. The Liapunov exponents are then obtained in the same manner as in the previous section (with the force law now only containing the interactions of
Identical systems for 1100 time units. For we integrated 100, 88, and 38 initially statistically the same time interval. For $N$ and $N$, the velocities are chosen such that the total energy is smaller than the energy of the zero velocity. The Liapunov exponents (and the associated variances for pairs of runs with different particle numbers (1000, 2000, 4000 and 8000). Note that both the convergence and exponentiation timescales are much shorter for smaller $N$. This is what is to be expected if the divergence timescale is related to “mixing” and the loss of memory of initial conditions. Nevertheless, for the runs described in this section, convergence was always reached — which implies that all trajectories are chaotic and, for a given $N$, share the same region of phase space. As we will see in the next section, this does not appear to be always the case with trajectories of non-interacting softened systems.

The variation of the final exponential timescales as a function of particle number, for all the runs, is shown in Fig. 5. As can be seen, not only is there no clear sign of saturation in the increase of the exponentiation timescales for the larger $N$ runs, but the rate of increase in the exponentiation times is obviously very different from the self-gravitating case. The observed $N$-variation is very well fit by a power law. Best least square fits gave a slope of 0.64 (plus or minus 0.008). This is rather different from the slope of $\sim 1$ expected if there was direct correspondence with the $N$ variation of the relaxation time inferred from two-body relaxation theory — even though the situation here is clearly similar to the idealised systems that are the subject of this theory (see, e.g., Chandrasekhar 1942). It is however much larger than the slope of the corresponding relation for the exponentiation timescales in the full $N$-body problem — even for small $N$.

4 NON-INTERACTING FIXED PARTICLES

Integrating the full equations of motion, even for enclosed systems of non-interacting particles, for long intervals, is time consuming. To investigate the behaviour of the Liapunov exponents for larger $N$ we therefore revert to an even simpler situation, whereas we integrate individual trajectories moving in the potential of systems of fixed particles for short times but for a large number of initial conditions. For each initial condition, only six first order equations of motion along with their associated variational equations are integrated. The Liapunov exponents are then obtained by the same method as in the previous sections.

Two types of distributions are discussed: homogeneous and $p \sim 1/r^2$. All system parameters and units are the same as in the previous sections. The velocities are chosen such that the total energy is smaller than the energy of the zero veloc-
ity surface $V_{ZV}$ for a corresponding smooth distribution at $R = 1$ (which, for large $N$, roughly coincides with the zero velocity surface corresponding to the actual particle distribution). That is, each velocity component $i$ is determined from $v_i = \sqrt{-2(V + V_{ZV}) \times (1 - 2X)}$, where $X$ is a random number uniformly distributed between 0 and 1 and $V$ is the particle’s initial potential energy as determined by its initial spatial coordinates. The (Plummer) softening length is again fixed at 0.1.

4.1 Homogeneously distributed systems

Fig. 6 shows histograms displaying the distribution of the exponentiation timescales of samples of 500 trajectories integrated in homogeneous systems of 100 and 10000 softened particles, with the initial conditions described above and for 420 time units. As can be seen, the spread in exponentiation times increases significantly for larger $N$. No obvious correlation was found between the values of the Liapunov exponents for a given $N$ and the initial conditions of the trajectories. In fact, starting from the same initial conditions and randomly varying the positions of the background particles gave similar results. This is indicative of a very complex phase space structure that may well be worth studying in more detail in the future. Perhaps more important, it may also indicate that a transition from a highly chaotic regime with a single Liapunov exponent to a mixed phase space is taking place.

In the case of systems without discreteness noise, that is when the potential does not contain rapid spatial variations as is the case here, mixed phase spaces have sets of initial conditions from which orbits have zero Liapunov exponents (in the
infinite time limit). This appears to be the case for some of the trajectories of the $N = 100000$ systems. For these, monotonous decrease of the Liapunov exponents (roughly with the expected rate of $\log t/t$) was detected for times up to 4200 units. Thus it appears that some trajectories actually become regular for $N = 100000$, despite the discreteness of the system. Nevertheless, most exponents tended to converge to some non-zero value when the integration was continued for longer times. It is possible that Arnold diffusion (see, e.g., Lichtenberg & Lieberman 1983), highly ineffective in potentials without high frequency variations, is at work here. Since the associated diffusion timescales could be very long, convergence is expected to be slow. For higher $N$ convergence is slower — hence the increase in dispersion.

Fig. 7 displays the the changes in the exponentiation times as a function of $N$ for subsets of 50 randomly chosen trajectories of systems with $N = 100, 500, 1000, 5000, 10000, 100000$. As expected, one sees an increase of the average value of the exponentiation times accompanied by an increase in their scatter. Even though it appears there is some saturation in the average values as $N$ increases, this may be due to the relatively short integration times (which, as is by now clear, can prevent convergence of the Liapunov exponents to their final values). For example, in the case of $N = 100000$, we have integrated 50 randomly chosen initial conditions for ten times longer than the other trajectories (integrated for the standard time interval of 420 time units). The exponentiation timescales for these longer time integrations are shown slightly to the right. It is clear that they are, on average, larger. Best fits for the values of the exponentiation times as a function of $N$ are compatible with those found in the previous section for the non-interacting moving particles.

4.2 Systems with density proportional to $1/r^2$

As was mentioned in the previous subsection, in the case of the homogeneous systems studied there, no correlation was found between the values of the exponentiation times and the initial conditions. As we will now see, this is not the case for systems with centrally concentrated particle distributions.

Fig. 8 shows histograms for a sample of 500 trajectories integrated in particle distributions with $N = 100$ and $N = 10000$ and average densities varying as $\sim 1/r^2$. Evidently, the major difference from Fig. 9 is that, along with the increase in the average value and the scatter of the exponentiation times, their distribution becomes bimodal. These two groups turn out to be correlated with two distinct sets of initial conditions — which in turn can be related to stability characteristics of trajectories in smooth non-spherical potentials. Recall that in such potentials, in the presence of a central mass concentration or a central density cusp, elongated orbits passing near the centre tend to be chaotic. Most box orbits and elongated loops with low initial angular momentum (which may also include low angular momentum orbits in axisymmetric potentials; e.g., Caranicolas & Innanen 1991) will fall in this category (e.g., Gerhard & Binney 1985; Schwarschchild 1993; Merritt & Fridman 1996; El-Zant & Hassler 1998).

Fig. 9 shows the correlation between the initial angular momenta of the integrated trajectories and the corresponding exponentiation times. As is clear, for $N = 100$, little correlation exists: the dynamics is dominated by the discreteness noise and trajectories can diffuse freely from one distribution to another. However, for $N = 10000$ and $N = 50000$, a clear correlation emerges between the two quantities, with trajectories starting with higher values of the initial angular momentum being more regular. Thus, in these cases, the discreteness noise acts as a perturbation — having similar effects to those of non-spherical perturbations in smooth systems. In both situations, trajectories starting at higher angular momenta are relatively immune to the destabilising effect of the non-integrable perturbation, as compared to the ones starting from low angular momenta (for an explanation of this phenomenon see El-Zant & Hassler 1998).

In a smooth spherical system, the angular mo-
Figure 8. Distribution of the inverse of the Liapunov exponents after 420 time units for 500 trajectories moving in the potential of fixed particle systems with $\rho \propto 1/r^2$ and with $N = 100$ (upper panel) and $N = 10000$.

The two sets of trajectories with markedly different Liapunov numbers turn out to also have different variation of these numbers as a function of $N$. This is shown in Fig. 11. The two lines drawn are least square fits with slopes 0.28 and 0.72 — roughly consistent with those of $N$-body systems with relatively small $N$ and trajectories in homogeneous distributions of non-interacting softened particles respectively. One can also see that the upper
Figure 9. Exponentiation times as a function of initial angular momenta for trajectories integrated for 420 time units in systems of 100 (upper panel) 10000 (middle panel) and 50000 particles distributed so that $\rho \propto 1/r^2$. 
Regular and chaotic motion in softened gravitational systems

Figure 10. Relative dispersion in the angular momenta along trajectories of systems of 10000 (upper panel) and 50000 particles distributed such that $\rho \propto 1/r^2$. In the upper panels the little circles correspond to quantities averaged over 420 time units while the crosses correspond to ones averaged over 840 time units.

Figure 10. Relative dispersion in the angular momenta along trajectories of systems of 10000 (upper panel) and 50000 particles distributed such that $\rho \propto 1/r^2$. In the upper panels the little circles correspond to quantities averaged over 420 time units while the crosses correspond to ones averaged over 840 time units.

line fits the data only for $N < 10000$, for larger $N$ saturation appears to set in. Since this could be due to the short standard integration time of 420 units, we integrated the low Liapunov exponent trajectories in systems of $N = 10000, 25000, 50000$ for up to 4200 time units. The results are shown in the right hand panel of Fig. 11. Although for some trajectories the exponentiation time-scales continue to rise so as to be consistent with the growth observed for smaller values of $N$, some exponentiation times actually decrease — their values becoming consistent with those of the population with the smallest exponentiation times rather than with their original group! This is because, if one integrates long enough, trajectories can diffuse through phase space between the two distributions — fluctuations can eventually lead to changes in angular momenta even for the more regular orbits. When this decreases, relatively regular trajectories can diffuse into the highly chaotic region. On the other hand, it was found that for some of the trajectories, the time dependent Liapunov exponents continue to decrease as the integration proceeds (up to 4200 units) these are therefore candidates for being regular trajectories. Whether longer integrations would confirm this or whether they too would eventually escape from the regular regions (through Arnold diffusion or some other mechanism) is an open question.

Clearly the phase space structure of these systems is highly complex and it may also well be worth further investigation. It is clear however that for short enough timescales, of the order of hundreds of crossing time, a certain convergence towards the regular behaviour described by the continuum limit is apparent. This can be inferred, for example, from Fig. 11 where one can see that for $N = 50000$ relatively more trajectories have exponentiation times approximating that of the most regular orbits than
for $N = 10000$. Moreover, this exponential time is constant over a large range of initial momenta and is essentially fixed by the integration time — during that time, the behaviour of the time dependent Liapunov exponents of these trajectories indicates that they have properties of regular trajectories (i.e., the exponents monotonously decrease with time).

5 SUMMARY AND CONCLUDING REMARKS

In this paper the stability of the trajectories of softened gravitational systems was examined. The exponential instability characterising the divergence of nearby trajectories in the case of self-gravitating systems of point particles is also present in the corresponding softened systems. However, whereas in the case of unsoftened $N$-body systems, the associated e-folding times do not depend on the number of particles, in the case of softened systems, the timescales do vary with $N$. For $N$ up to a few hundred this variation is a power law $N^a$ with $a \approx 0.4$, roughly consistent with $a = 1/3$ predicted by GHH on the basis of analysing random encounters of softened particles in an infinite distribution. For larger $N$ (up to a few thousand) however, the exponentiation times increase much more slowly. This effect appears to be connected to large scale time dependent variations in the potential that are swamped at small $N$ by discreteness noise. If this is indeed the case, then one might expect the situation to be different if self gravity is not taken into account.

To see if the slow rate of increase in the exponential times of $N$-body systems did indeed arise from collective effects, we have integrated trajectories in centrally concentrated and homogeneous distributions of non-interacting particles, which were either moving independently or were kept fixed. In the homogeneous case, not only was there no clear sign of saturation in the $N$-dependence of the e-folding times, but the rate of increase was characterised by a power law with another, much larger, value of the exponent ($a \approx 0.7$) for $N$ up to 100000. In addition, the dispersion in the values of the Liapunov exponents integrated from different initial conditions increases with $N$. This is indicative of a complex phase space structure, perhaps a mixed phase space where regular and chaotic regions coexist. Indeed, for $N = 50000$ and $N = 100000$, what appeared to be completely regular trajectories — characterised by time dependent Liapunov exponents monotonously decreasing for thousands of dynamical times — were found. This phenomenon is, of course, completely absent in the case of point particles systems (Valluri & Merritt 1999).

For $N \geq 10000$, trajectories of systems of fixed particles that, instead of being homogeneously distributed, had radial density profiles varying as $\sim 1/r^2$ could be classified into two groups. High angular momentum trajectories had, in general, larger exponentiation times than low angular momentum.
ones. This situation is analogous to the case of smooth centrally concentrated non-spherical potentials, where eccentric trajectories can be chaotic. It therefore appears that, in the present case, discreteness noise replaces the effect of global non-spherical perturbations as the trigger of chaotic behaviour.

As apposed to the case of point particle systems, the divergence timescales of the trajectories correlate rather well with the conservation of the action variables (as evidenced by the conservation of the angular momenta) along them. That is, as may be expected if the exponents tell us something about the phase space diffusion, more chaotic trajectories (i.e., those with larger Liapunov exponents) conserve their action variables less efficiently than the more regular orbits (which should approximate ones in the smoothed out potential, where these quantities are exactly conserved).

In centrally concentrated systems, the most chaotic trajectories and the least chaotic ones also have different $N$-variations of the exponential timescales, with the former variation being a power law with $a = 0.28$ and the latter with $a = 0.72$. These values correspond to those of $N$-body systems of a few hundred particles and of trajectories of homogeneous non-interacting distributions respectively. This result is also analogous to what is found in smooth non-spherical: in the continuum limit, homogeneous systems have harmonic potentials and contain only regular orbits. The low Liapunov exponent trajectories in the centrally concentrated systems also correspond to orbits that remain regular when non-spherical perturbations are introduced. $N$-body systems, on the other hand, are inhomogeneous. The largest Liapunov exponent in an $N$-body system might then be expected to correspond to that of the most chaotic trajectories in a corresponding fixed particle system — provided there are no collective effects influencing the results. This expected correspondence, and its existence in small-$N$ systems, may be taken as further evidence that the saturation in the $N$-body case might be due to such effects: under the action of collective phenomena the self consistent field is time varying, and therefore the saturation does not contradict the possibility that such systems obey the CBE — even in spherical but time dependent smooth potentials trajectories can be chaotic. Weinberg (1998) has shown that long lived modes can be triggered by discreteness noise even for particle numbers far larger than those of the $N$-body systems described here, and can lead to evolutionary effects on a timescale shorter than that of the unamplified discreteness noise (see also Kandrup & Severne 1986).

From the above discussion it is clear that the local stability of trajectories of softened gravitational systems can tell us much more about their dynamical properties than in the case of systems consisting of point particles. Once the effect of the singularity in the potential is removed, many more interesting features become apparent, and obvious correlations with some important dynamical properties are then revealed. It thus appears that the investigation of the local stability of motion in this case should provide clues concerning the mechanisms driving chaotic behaviour in gravitational systems and their interaction. They include high frequency discreteness noise, larger scale collective phenomena and the effect of the shape of the global smoothed out potential. These are precisely the same phenomena that drive macroscopic evolution in gravitational systems. Since, in the absence of chaotic behaviour, orbital action variables are conserved, once “phase mixing” in the corresponding angle variables has taken place, no further evolution is possible. Therefore, as has been argued in El-Zant(1997), the macroscopic evolution of gravitational systems is necessarily driven by chaotic motion resulting from the aforementioned mechanisms and their interactions. The study of the local stability of trajectories of gravitational systems is thus not just interesting on intrinsic merit, but helps isolate and quantify evolutionary mechanisms.

Since actual astronomical systems interact through the singular Newtonian potential and not a softened version of it, one may wonder as to the applicability of the results described here to the orbital stability of real systems. Nevertheless, most studies of large dissipationless stellar systems assume (either explicitly or implicitly) that these systems are described by the CBE — an assumption that, as pointed out in the introduction, can only, strictly speaking, be justified in the case of softened systems. The study of softened systems therefore is important for testing the validity of this widely used assumption. In this paper, as far as I am aware, first evidence is given of the convergence towards the regular dynamics described by the CBE for steady state systems with separable potentials. The fact that this convergence is much faster in systems where self gravity has been artificially suppressed, and a statistical steady state forced, may be of crucial physical relevance. Nevertheless, if one assumes, as argued in the introduction, that the mechanism leading to the short $N$-invariant e-folding time in point particle systems is physically unimportant, then the results presented here may be seen as an important step in resolving the long standing apparent paradox concerning the exponential divergence of trajectories of gravitational systems.

In El-Zant (1997) as well as in a couple of other papers (El-Zant 1998b; El-Zant & Gurzadyan 1998) another, less direct (but perhaps more powerful), geometric method characterising the local divergence of geodesics on the (Lagrangian) configuration space of dynamical systems, was applied to dynamic and
static gravitating ones. There too, attempts to find relations between macroscopic evolution and microscopic instability were made, and correlations between some characteristic physical quantities (e.g., rotation, clustering and central concentration) were also found. In the more abstract geometric setting, however, it is more difficult to isolate the physical mechanisms at work than when applying the much more direct method used in this paper. Moreover, in the case of point particles, where in the geometric case the divergence is mainly due to the negative curvature of the configuration manifold, the relation between the methods is far from clear. Not only does the negative curvature of the configuration space of point particle systems measure time independent deviations between orbits, and not temporal trajectories as in the method used here, but also the terms giving rise to the exponential instability (on a timescale similar to that found using Liapunov exponents) involve sums of the first derivatives of the potential, and not second derivatives as the case when integrating the variational equations. In addition, highly technical questions on whether the local exponential divergence of geodesics on its own justifies the usual global conclusions required to infer macroscopic evolution (outside the idealised domain encountered in mathematical studies where this was shown to be the case: see, e.g., Szczesny & Dobrowolski 1999) are involved. This is especially relevant since point mass systems have singular potentials, which makes the metric also singular and the curvature undefined at some points of the configuration manifold, which then becomes “incomplete” (e.g., Abraham & Marsden 1978). The global structure of geodesics can be affected by these singular “boundaries”. When, by softening the potential, the singularity is removed, the curvature is no longer negative. Chaotic behaviour can then result only from fluctuations in the curvature along the motion, the amplitude of which will decrease with particle numbers. These fluctuations can also be related to the Liapunov exponents (see Casetti, Pettini & Cohen 2000; Latora, Rapisarda & Ruffo 1999 and the references therein). It is probably then in this regime that correspondence between the two methods can be most easily examined.

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