The normalized Laplacians and random walks of the parallel subdivision graphs

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Abstract. The \(k\)-parallel subdivision graph \(S_k(G)\) is generated from \(G\) which each edge of \(G\) is replaced by \(k\) parallel paths of length 2. The \(2k\)-parallel subdivision graph \(S_{2k}(G)\) is constructed from \(G\) which each edge of \(G\) is replaced by \(k\) parallel paths of length 3. In this paper, the normalized Laplacian spectra of \(S_k(G)\) and \(S_{2k}(G)\) are given. They turn out that the multiplicities of the corresponding eigenvalues are only determined by \(k\). As applications, the expected hitting time, the expected commute time and any two-points resistance distance between vertices \(i\) and \(j\) of \(S_k(G)\), \(S_k(G)\) and \(S_{2k}(G)\) with \(r\) iterations are given. Moreover, the multiplicative degree Kirchhoff index, Kemeny’s constant and the number of spanning tress of \(S_k(G)\), \(S_{k}(G)\), \(S_{2k}(G)\) and \(S_{2k}(G)\) are respectively obtained. Our results have generalized the previous works in Xie et al. and Guo et al. respectively.

Keywords: Subdivision graph, Normalized Laplacian, Expected hitting time, Multiplicative degree Kirchhoff index, Kemeny’s constant, Spanning tress

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1. Introduction

Our convention in this paper is that the \((n, m)\)-graph \(G = (V(G), E(G))\) is the simple and connected graph with \(n\) vertices and \(m\) edges. For the associated notation of the graph theory, we follow [1]. One of the most critical fields in the graph theory is spectral graph theory. In particular, spectral graph theory is consist of algebraic spectral graph theory and analytic spectral graph theory. Spectral graph theory not only have pure mathematics properties, but also several vital applications in practical, examples include information-theoretic hashing of 3D objects [2], threshold selection in gene co-expression networks [3], pattern recognition, data mining and image matching [4]. Thus, it is of great interests to determine the spectra of some graphs, especially for those large graphs which resulted from the small graph by graph operations in past two decades. For adjacency and Laplacian spectra of certain graph operations, see [5–18] and references therein. Alone this line, we consider the normalized Laplacian spectra of two parallel subdivision graphs in this paper, see Figure 1.

It is worthy to mentioned that \(S_1(G)\) is the subdivision graph \(S(G)\) if \(k = 1\) for \(S_k(G)\) that considered in [19]. Besides, \(S_2(G)\) is such diamond hierarchial graph which was constructed by Guo et al. [20] if \(k = 2\) for \(S_k(G)\). Moreover, while \(k = 1\) and \(G\) is regular for \(S_{2k}(G)\), then the graph \(S_2(G)\) is the equivalent graph.generated by using clique-star transformation) of the stellated regular graphs [21].

In what follows, we will recall the definition of the normalized Laplacian. The formal definition is defined as

\[
\mathcal{L}(G) = \begin{cases} 
\frac{-w(i,j)}{\sqrt{d(i)d(j)}}, & i \text{ is adjacent to } j; \\
\frac{d(i) - w(i,i)}{d(i)}, & i = j, d(i) \neq 0; \\
0, & \text{otherwise}. 
\end{cases}
\]
In the definition defined as above, \( w(i, j) \) is the weight of the edge \( ij \) and \( w(i, u) \) the vertex weight. In particular, \( w(i, j) = 1 \) and \( w(i, i) = 0 \) if vertex weight and edge weight are respectively 0 and 1. This yields

\[
\mathcal{L}(G) = \begin{cases} 
\frac{-1}{\sqrt{d(i)d(j)}} & i \text{ is adjacent to } j; \\
1, & i = j, d(i) \neq 0; \\
0, & \text{else.} 
\end{cases}
\]

(1.1)

Apparently, one has \( \mathcal{L}(G) = D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}} \).

Recent years, Xie and Zhang \[19, 22\] respectively determined the normalized Laplacians of subdivisions and iterated triangulations graph. In 2017, the normalized Laplacians for quadrilateral graphs are considered by Li and Hou \[23\]. Pan et al. \[24\] computed the normalized Laplacians of some graphs which involve graph transformations. For details, see \[25–30\] and references therein.

A topic of a good deal of attention in network science is the notion of network criticality. Some measures for network criticality are depended on the paths in networks. Thus the shortest paths in networks are the focus of attention while we compute network criticality. But the shortest paths are not considering the impacts of all the paths. The concepts of resistance distance and Kirchhoff index are such measures which involving all the information among paths in networks, one may refer to \[31\].

Resistance distance \[32\] is defined by Klein and Randić. Denote \( \Omega_{ij} \) the resistance distance between vertices \( i \) and \( j \). In \[33\], Klein et al. defined the resistance distance sum rules, which is Kirchhoff index \( Kf(G) \). The resistance distance based graph invariants have attracted some experts due to these applications in harmonic analysis, random walks, chemical and computer science. Considering the impacts of the vertex degree, Chen et al. \[34\] defined the multiplicative degree Kirchhoff index \( Kf^*(G) \) and found that there is a surprising relation between \( Kf^*(G) \) and the eigenvalues of \( \mathcal{L}(G) \).

The hitting time or first passage time \( T_j \) of the vertex \( j \) is defined as the minimum steps of the random walk needs to reach that vertex. The expected hitting time or mean first passage time when the walk starts at \( i \) is denoted by \( E_i T_j(G) \). The expected commute time \( C_{uv}(G) \) is defined as the expected time it takes for a particle to start from vertex \( i \), reach vertex \( j \), and then return to vertex \( i \) in the graph \( G \). The Kemeny’s constant \( Ke(G) \) is the expected number of steps needed for the transition from a starting vertex to a destination vertex, which is related to the finite ergodic Markov chains. Spanning trees are the subgraphs of \( G \) which contain each edge in \( G \), also an important quantity characterizing the reliability of a network, one may refer to \[35, 36\].
For a general graph, it is hard enough to determine the resistance distance between any points, unless the graph is very small or one knows the complete information of the graph. A. K. Chandra et al. in [37] built a relation between the expected hitting time and resistance distance of a graph. This provides a available method to calculate any two-points resistance distance, see [38–42]. The main results of this paper are presented as below.

1.1. The normalized Laplacian of $S_k(G)$

In this subsection, we will give the complete information for the normalized Laplacian of the $k$-parallel subdivision graph. Before proceeding, we shall disgress to define two functions as below.

$$f_1(x) = \frac{2 + \sqrt{4 - 2x}}{2}, \quad f_2(x) = \frac{2 - \sqrt{4 - 2x}}{2}.$$

**Theorem 1.1.** Assume that $G$ is a $(n, m)$-graph. Thus the eigenvalues of $\mathcal{L}(S_k(G))$ are as below.

- $f_1(\lambda)$ and $f_2(\lambda)$ are eigenvalues of $\mathcal{L}(S_k(G))$, if $\lambda(\lambda \neq 0)$ is an eigenvalue of $\mathcal{L}(G)$. Moreover, the multiplicities of $f_1(\lambda)$ and $f_2(\lambda)$ are same as $\lambda$.
- 0 and 2 are the eigenvalues of $\mathcal{L}(S_k(G))$ with the multiplicity 1.
- $\mathcal{L}(S_k(G))$ has the eigenvalue 1. Its multiplicity is $km - n + 2$, if $G$ is bipartite and $km - n$.

1.2. The normalized Laplacian of $S_{2k}(G)$

In this subsection, the normalized Laplacian eigenvalues of the $2k$-parallel subdivision graph are proposed. In addition, let $\mu_1, \mu_2$ and $\mu_3$ be the roots of the equation $4\mu^3 - 12\mu^2 + 9\mu - \lambda = 0$.

**Theorem 1.2.** Assume that $G$ is a $(n, m)$-graph. Therefore the eigenvalues of $\mathcal{L}(S_{2k}(G))$ are as follows.

- $\mu_1, \mu_2$ and $\mu_3$ are eigenvalues of $\mathcal{L}(S_{2k}(G))$, if $\lambda(\lambda \neq 0, 2)$ is an eigenvalue of $\mathcal{L}(G)$. The multiplicities of $\mu_1, \mu_2$ and $\mu_3$ are same as $\lambda$.
- $\mathcal{L}(S_{2k}(G))$ has an eigenvalue 0 with the multiplicity 1. $\mathcal{L}(S_{2k}(G))$ has an eigenvalue 2 and its multiplicity is 1, if $G$ is bipartite.
- $\mathcal{L}(S_{2k}(G))$ has the eigenvalues $\frac{1}{2}$ and $\frac{3}{2}$. Their multiplicities are respectively $km - n$ and $km - n + 2$, if $G$ is non-bipartite.
- $\mathcal{L}(S_{2k}(G))$ has the eigenvalues $\frac{1}{2}$ and $\frac{3}{2}$, if $G$ is non-bipartite.

The rest parts are summarized as below. We propose some lemmas in section 2. The proofs of the main results are provided in section 3. The spectra of $\mathcal{L}(S_k(G))$ and $\mathcal{L}(S_{2k}(G))$ are determined in sections 4 and 5. As byproduct, the formulas for $E_i T_j(S_k(G))$, $r_{ij}(S_k(G))$, $K f(S_k(G))(K f(S_{2k}(G))$ resp.), $K e(S_k(G))(K f(S_{2k}(G))$ resp.) and $\tau(S_k(G))(\tau(S_{2k}(G))$ resp.) are respectively given.

2. Preliminaries

In this section, we put some lemmas that used in applications section. Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the normalized Laplacian eigenvalues of $\mathcal{L}(G)$.

In 2007, Chen and Zhang found that $K f^*(G)$ is closely related to the eigenvalues of $\mathcal{L}(S_k(G))$ in $G$.

**Lemma 2.1.** [34] Assume that $G$ is a $(n, m)$-graph. Then $K f^*(G) = 2m \sum_{i=2}^{n} \frac{1}{\lambda_i}$. 


Kemeny and Snell in 1960 introduced the graph invariant which is called Kemeny’s constant. It also related to the eigenvalues of \( \mathcal{L}(S_k(G)) \).

**Lemma 2.2.** Assume that \( G \) is a \((n, m)\)-graph. Then \( Ke(G) = \sum_{i=2}^{n} \frac{1}{\lambda_i} \).

Combining Lemmas 2.1 and 2.2, one finds \( Kf^*(G) = 2m \cdot Ke(G) \). Let \( B(G) \) denote the vertex-edge incident matrix and \( r(B(G)) \) the rank of \( B(G) \), one has the following.

**Lemma 2.3.** Assume that \( G \) is a \((n, m)\)-graph. Thus \( r(B(G)) = \begin{cases} n, & \text{if } G \text{ is non-bipartite;} \\ n-1, & \text{if } G \text{ is bipartite.} \end{cases} \)

**Lemma 2.4.** Assume that \( G \) is a \((n, m)\)-graph. One obtains
\[
\prod_{i=1}^{n} d(v_i) \prod_{i=2}^{n} \lambda_i = 2m \tau(G),
\]
where \( \tau(G) \) is the number of spanning trees of \( G \).

At this point, one considers the normalized adjacency matrix \( N(G) \), namely
\[
N(G) = D(G)^{-\frac{1}{2}} A(G) D(G)^{-\frac{1}{2}} = D(G)^{-\frac{1}{2}} (I - \mathcal{L}(G)) D(G)^{-\frac{1}{2}},
\]
where \( I \) is the identity matrix.

The above equation leads that matrix \( N(G) \) has the same eigenvalues as matrix \( I - \mathcal{L}(G) \). Assume that the eigenvalues of \( I - \mathcal{L}(G) \) has the orthogonal eigenvectors \( v_1, v_2, \ldots, v_n \). Namely
\[
v_i = (v_{i1}, v_{i2}, \ldots, v_{in})^T, \quad i = 1, 2, \ldots, n.
\]

One knows that
\[
E = (v_1, v_2, \ldots, v_n) \text{ is the orthogonal matrix, then}
\]
\[
\sum_{k=1}^{n} v_{ik} v_{jk} = \sum_{k=1}^{n} v_{ki} v_{kj} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{else.} \end{cases}
\]

In particular,
\[
v_1 = \left( \sqrt{d_1/2m}, \sqrt{d_2/2m}, \ldots, \sqrt{d_n/2m} \right).
\]

Let \( G \) be a bipartite graph with a vertex partition \( V(G) = V_1 \cup V_2 \). Thus
\[
v_{ni} = \sqrt{d_i/2m}, i \in V_1; v_{nj} = -\sqrt{d_j/2m}, j \in V_2.
\]

**Lemma 2.5.** Assume that \( G \) is a \((n, m)\)-graph. One obtains
\[
E_i T_j(G) = 2m \sum_{a=2}^{n} \frac{1}{1-\lambda_a} \left( \frac{v_{ai}^2}{d_j} + \frac{v_{ai} v_{aj}}{\sqrt{d_i d_j}} \right).
\]

**Lemma 2.6.** Assume that \( G \) is a \((n, m)\)-graph. One gets
\[
E_i T_j(G) + E_j T_i(G) = 2m \Omega_{ij}(G).
\]

The following lemma built a connection between the expected hitting time and expected commute time.

**Lemma 2.7.** Assume that \( G \) is a \((n, m)\)-graph. One gets
\[
E_i T_j(G) + E_j T_i(G) = C_{uv}(G).
\]

Certainly, one reaches \( C_{uv}(G) = 2m \Omega_{ij}(G) \).
3. The proofs of Theorems 1.1 and 1.2

The most focuses in this section are on determining the eigenvalues of \( \mathcal{L}(S_k(G)) \) and \( \mathcal{L}(S_{2k}(G)) \). Before we begin, we turn to prove two lemmas as basic but necessary in the proofs.

**Lemma 3.1.** \( \mathcal{L}(G) \) has an eigenvalue \( 4\sigma - 2\sigma^2 \) and its multiplicity is same as \( \sigma \), if \( \sigma(\sigma \neq 1) \) is an eigenvalue of \( \mathcal{L}(S_k(G)) \).

**Proof.** According to the construction of \( S_k(G) \), let \( N_0 = \{\mu_1^0, \mu_2^0, \ldots, \mu_m^0\} \), \( N_1 = \{\mu_1^1, \mu_2^1, \ldots, \mu_m^1\} \), \ldots, \( N_{k-1} = \{\mu_1^{k-1}, \mu_2^{k-1}, \ldots, \mu_m^{k-1}\} \). Evidently, \( N = N_0 \cup N_1 \cup \cdots \cup N_{k-1} \). At this place, we recall that

\[
d_w(S_k(G)) = \begin{cases} k \cdot d_u(G), & w \in V(G); \\ 2, & w \in N. \end{cases}
\]

(3.2)

Set \( \varepsilon = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+km}\} \) as an eigenvector corresponding to the eigenvalue \( \sigma \) of \( S_k(G) \). Together the definition of normalized Laplacian with Eq.(1.1), one gets the following equation for any vertex \( w \) in \( S_k(G) \).

\[
(1 - \sigma)\varepsilon_w = \sum_{u \sim w} \frac{1}{d_w(S_k(G))} \varepsilon_u,
\]

(3.3)

where denote \( S_k(G) \) by \( S_k \) for simplify.

Suppose that \( N_l(u) \) is the set which consists of all neighbors of \( u \) in \( N_l \) \((l = 0, 1, \ldots, k - 1)\). The set of all neighbors of \( u \) in \( G \) is denoted by \( N_G(u) \). For any vertex \( u \) in \( G \) and Eqs.(3.2) and (3.3), one has

\[
(1 - \sigma)\varepsilon_u = \sum_{l=0}^{k-1} \sum_{u_l' \in N_l(u)} \frac{1}{d_u(S_k(G))} \varepsilon_{u_l'},
\]

(3.4)

For any vertex \( u_l' \in N_l(u) \), one obtains

\[
(1 - \sigma)\varepsilon_{u_l'} = \frac{1}{\sqrt{d_u(S_k(G))d_{u_l'}(S_k)}} \varepsilon_u + \frac{1}{\sqrt{d_v(S_k(G))d_{u_l'}(S_k)}} \varepsilon_v = \frac{1}{\sqrt{2k \cdot d_u(G)} \varepsilon_u} + \frac{1}{\sqrt{2k \cdot d_v(G)} \varepsilon_v},
\]

(3.5)

where vertex \( v \) is adjacent to vertex \( u \) in \( G \).

Combining Eq.(3.4) and Eq.(3.5), we have

\[
(1 - \sigma)^2 \varepsilon_u = \sum_{l=0}^{k-1} \sum_{u_l' \in N_l(u)} \left( \frac{1}{2k \cdot d_u(G)} \varepsilon_u + \frac{1}{2k \cdot d_v(G)} \varepsilon_v \right)
\]

\[
= k \cdot \sum_{v \in N_G(u)} \left( \frac{1}{2k \cdot d_u(G)} \varepsilon_u + \frac{1}{2k \cdot d_v(G)} \varepsilon_v \right)
\]

\[
= \frac{1}{2} \varepsilon_u + \frac{1}{2} \sum_{v \in N_G(u)} \frac{1}{d_u(G)d_v(G)} \varepsilon_v,
\]

(3.6)

for \( \sigma \neq 1 \).

By a straightforward calculation of Eq.(3.6), it gives

\[
[1 - (4\sigma - 2\sigma^2)] \varepsilon_u = \sum_{v \in N_G(u)} \frac{1}{d_u(G)d_v(G)} \varepsilon_v,
\]

for \( \sigma \neq 1 \).

Based on Eq.(3.3), one knows \( 4\sigma - 2\sigma^2 \) is an eigenvalue of \( \mathcal{L}(G) \). Furthermore, there exits a bijection between \( \sigma \) and \( 4\sigma - 2\sigma^2 \), thus the multiplicity of \( 4\sigma - 2\sigma^2 \) is same as that of \( \sigma \).

This has completed the proof.
Lemma 3.2. $\mathcal{L}(G)$ has an eigenvalue $\zeta(4\zeta^2 - 12\zeta + 9)$ and its multiplicity is same as $\zeta$, if $\zeta(\zeta \neq \frac{1}{2}, \frac{3}{2})$ is an eigenvalue of $\mathcal{L}(S_{2k}(G))$.

Proof. For the graph $S_{2k}(G)$, let $M_0 = \{\mu_0^0, \mu_1^0, \ldots, \mu_{m_1}^0, \mu_{m_2}^0\}$, $N_1 = \{\mu_1^1, \mu_1^1, \ldots, \mu_{m_1}^1, \mu_{m_2}^1\}$, $\ldots$, $N_{k-1} = \{\mu_{k-1}^{k-1}, \mu_{k-1}^{k-1}, \ldots, \mu_{m_1}^{k-1}, \mu_{m_2}^{k-1}\}$. Obviously, $M = M_0 \cup M_1 \cup \cdots \cup M_{k-1}$. At this point, we review that

$$d_w(S_{2k}(G)) = \begin{cases} k \cdot d_w(G), & w \in V(G); \\ 2, & w \in M. \end{cases}$$ (3.7)

Set $\xi = \{\xi_1, \xi_2, \ldots, \xi_{n+2km}\}$ as an eigenvector corresponding to the eigenvalue $\zeta$ of $S_{2k}(G)$. Together the definition of normalized Laplacian with Eq.(1.1), one gets the following equation for any vertex $s$ in $S_{2k}(G)$.

$$(1 - \zeta)\xi_s = \sum_{v \sim s} \frac{1}{\sqrt{d_s(S_{2k})d_v(S_{2k})}} \xi_v,$$ (3.8)

where denote $S_{2k}(G)$ by $S_{2k}$ for short.

Assume that $M_l(u)$ is the set which consists of all neighbors of $u$ in $M_l$ ($l = 0, 1, \ldots, k - 1$). Denote the set of all neighbors of $u$ in $G$ by $N_G(u)$. Any vertex $u$ in $G$ and Eq.(3.7) imply that

$$(1 - \zeta)\xi_u = \sum_{l=0}^{k-1} \sum_{u_{l+1}^l \in M_l(u)} \frac{1}{\sqrt{d_{u_{l+1}^l}(S_{2k})d_u(S_{2k})}} \xi_{u_{l+1}^l} = \sum_{l=0}^{k-1} \sum_{u_{l+1}^l \in M_l(u)} \frac{1}{\sqrt{2k \cdot d_u(G)}} \xi_{u_{l+1}^l}.$$ (3.9)

For any $u_{l+1}^l \in M_l(u)$, we obtain

$$(1 - \zeta)\xi_{u_{l+1}^l} = \frac{1}{\sqrt{d_{u_{l+1}^l}(S_{2k})d_u(S_{2k})}} \xi_u + \frac{1}{\sqrt{d_{u_{l+1}^l}(S_{2k})d_{u_{l+2}^l}(S_{2k})}} \xi_{u_{l+2}^l} = \frac{1}{\sqrt{2k \cdot d_u(G)}} \xi_u + \frac{1}{2} \xi_{u_{l+2}^l},$$ (3.10)

where vertex $u_{l+2}^l$ in $M_l$ is adjacent to vertex $u_{l+1}^l$.

Similarly, for any $u_{l+2}^l \in M_l$, this yields

$$(1 - \zeta)\xi_{u_{l+2}^l} = \frac{1}{\sqrt{d_{u_{l+1}^l}(S_{2k})d_{u_{l+2}^l}(S_{2k})}} \xi_{u_{l+1}^l} + \frac{1}{\sqrt{d_{u_{l+2}^l}(S_{2k})d_{u_{l+3}^l}(S_{2k})}} \xi_{u_{l+3}^l} = \frac{1}{2} \xi_{u_{l+1}^l} + \frac{1}{\sqrt{2k \cdot d_u(G)}} \xi_v.$$ (3.11)

Combining Eq.(3.11) and Eq.(3.10), one arrives at

$$2(\zeta - \frac{1}{2})(\zeta - \frac{3}{2})\xi_{u_{l+1}^l} = \frac{2(1 - \zeta)}{\sqrt{2k \cdot d_u(G)}} \xi_u + \frac{1}{\sqrt{2k \cdot d_u(G)}} \xi_v.$$ (3.12)

Substituting Eq.(3.12) to Eq.(3.9), one gets

$$2(\zeta - \frac{1}{2})(\zeta - \frac{3}{2})(1 - \zeta)\xi_u = k \cdot \sum_{v \in N_G(u)} \left( \frac{1 - \zeta}{k \cdot d_u(G)} \xi_u + \frac{1}{2k \sqrt{d_u(G)}} \xi_v \right) = (1 - \zeta)\xi_u + \frac{1}{2} \sum_{v \in N_G(u)} \frac{1}{\sqrt{d_u(G)d_v(G)}} \xi_v,$$ (3.13)
where $\zeta \neq \frac{1}{2}, \frac{3}{2}$.

By a explicit analysis of Eq.(3.13), that is

$$[1 - \zeta(4\zeta^2 - 12\zeta + 9)]\xi_u = \sum_{v \in N_G(u)} \frac{1}{\sqrt{d_u(G)d_v(G)}} \xi_v,$$

for $\zeta \neq \frac{1}{2}, \frac{3}{2}$.

According to Eq.(3.8), one gets $\zeta(4\zeta^2 - 12\zeta + 9)$ is an eigenvalue of $L(G)$. Moreover, there exits a bijection between $\mu$ and $\zeta \neq \frac{1}{2}, \frac{3}{2}$, thus the multiplicity of $\zeta \neq \frac{1}{2}, \frac{3}{2}$ is same as that of $\zeta$.

This completes the proof.

Now, we proceed by going into more details on the proofs of the normalized Laplacians of $S_k(G)$ and $S_{2k}(G)$. Our first goal in the following is determined the eigenvalues of $L(S_k(G))$.

3.1. The proof of Theorem 1.1

At this point, we slightly to observe those two functions that defined in the subsection 1.2 as follows.

$$f_1(x) = \frac{2 + \sqrt{4 - 2x}}{2}, \quad f_2(x) = \frac{2 - \sqrt{4 - 2x}}{2}.$$  \hspace{1cm} (3.14)

Assume that $\sigma(\sigma \neq 1)$ is an eigenvalue of $L(S_k(G))$ and $\lambda$ an eigenvalue of $L(G)$, then by Lemma 3.1, one has $\lambda = 4\sigma - 2\sigma^2$. Put it in another way, one obtains $\sigma = \frac{2 \pm \sqrt{1 - 2\lambda}}{2}$, $\sigma \neq 1$.

It is routine to check that $L(G)$ has an eigenvalue 0 and its multiplicity is 1. Then substituting 0 into Eq.(3.14), we get $\sigma = 0, 2$ with the multiplicity 1.

The graph $S_k(G)$ is bipartite whether $G$ is bipartite or not. By Lemma 3.1, one obtains the rest of the eigenvalues of $S_k(G)$ are 1. Further, 1 is with the multiplicity $km - n + 2$ if $G$ is bipartite and $km - n$ otherwise.

3.2. The proof of Theorem 1.2

Assume that $L(S_{2k}(G))$ has an eigenvalue $\zeta(\zeta \neq \frac{1}{2}, \frac{3}{2})$ and $L(G)$ has an eigenvalue $\lambda$, then by Lemma 3.2, one has $\lambda = \zeta(4\zeta^2 - 12\zeta + 9)$. Namely, one gets

$$4\zeta^3 - 12\zeta^2 + 9\zeta - \lambda = 0, \quad \zeta \neq \frac{1}{2}, \frac{3}{2}. \hspace{1cm} (3.15)$$

It is worthy to mention that 0 is a normalized Laplacian eigenvalue of $G$ with the multiplicity 1. Combining Eq.(3.15) and $\lambda = 0$, this gives $\zeta = 0, \frac{1}{2}, \frac{3}{2}$.

Note that $L(G)$ has an eigenvalue 2 with the multiplicity 1, when $G$ is bipartite. Substituting $\lambda = 2$ into Eq.(3.15), then one arrives at $\zeta = 2, \frac{1}{2}, \frac{1}{2}$.

Assume that $G$ is non-bipartite. Combining Eq.(3.12) and $\zeta = \frac{1}{2}$, one gets

$$\frac{\xi_u}{\sqrt{d_u(G)}} = -\frac{\xi_v}{\sqrt{d_v(G)}. \hspace{1cm} (3.16)}$$

There must be an odd cycle in $G$ due to $G$ is non-bipartite. Assume that the length of that odd cycle is $p$ and those vertices are named $t_1, t_2, \ldots, t_p$. Thus

$$\frac{\xi_{t_1}}{\sqrt{d_{t_1}(G)}} = -\frac{\xi_{t_2}}{\sqrt{d_{t_2}(G)}} = \ldots = \frac{\xi_{t_p}}{\sqrt{d_{t_p}(G)}} = -\frac{\xi_{t_1}}{\sqrt{d_{t_1}(G)}}$$
From the above equation, one has $\xi_u = 0$, $u \in V(G)$. Alone with Eq.(3.9), we have
\[
\sum_{l=0}^{k-1} \sum_{u'_{l+1} \in M_l(u)} \frac{1}{\sqrt{2k \cdot d_u(G)}} \xi_{u'_{l+1}} = 0,
\]
namely,
\[
\sum_{q \in M(u)} \xi_q = 0, \quad M(u) = \bigcup_{l=0}^{k-1} M_l(u). \tag{3.17}
\]
Substituting $\zeta = \frac{1}{2}$ and $\xi_u = 0$, $u \in V(G)$ into Eq.(3.10), one obtains $\xi_{u_{l+1}} - \xi_{u_{l+2}} = 0$, $l = 0, 1, \ldots, k-1$.
\[
\begin{align*}
\xi_{u_{1+1}} &= \xi_{u_{1+2}} = y_1, \\
\xi_{u_{2+1}} &= \xi_{u_{2+2}} = y_1, \\
\xi_{u_{3+1}} &= \xi_{u_{3+2}} = y_1, \\
&\vdots \\
\xi_{u_{m+1}} &= \xi_{u_{m+2}} = \xi_{u_{m+3}} = \xi_{u_{m+4}} = \xi_{u_{k-1}} = y_{k-1}.
\end{align*}
\]
Let
\[
y_0 = (y_1^0, y_2^0, \ldots, y_m^0)^T, \quad y_1 = (y_1^1, y_2^1, \ldots, y_m^1)^T, \quad \ldots, \quad y_{k-1} = (y_1^{k-1}, y_2^{k-1}, \ldots, y_m^{k-1})^T.
\]
Assume that $B(G) = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ is the incident matrix of $G$ and $y = (y_0, y_1, \ldots, y_{k-1})^T$. According to Eq.(3.17), one can write it by another way, that is
\[
Cy = 0, \tag{3.18}
\]
where $C = (\beta_1, \beta_2, \ldots, \beta_n)^T$, $\beta_1 = (\alpha_1, \alpha_2, \ldots, \alpha_k), \beta_2 = (\alpha_2, \alpha_3, \ldots, \alpha_k), \ldots, \beta_n = (\alpha_n, \alpha_n, \ldots, \alpha_k), \ldots, \beta_n = (\alpha_k, \alpha_k, \ldots, \alpha_k)$.

Evidently, $r(B(G)) = r(C)$. Thus, Eq.(3.18) has $km - n$ linearly independent solutions based on Lemma (2.3). Put it in another way, the multiplicity of the eigenvalue $\frac{3}{2}$ of $\mathcal{L}(S_{2k}(G))$ is $km - n$. Therefore, the multiplicity of the eigenvalue $\frac{3}{2}$ of $\mathcal{L}(S_{2k}(G))$ is $km - n + 2$.

The graph $S_{2k}(G)$ is bipartite when $G$ is bipartite, then . As we know, the eigenvalues of $\mathcal{L}(S_{2k}(G))$ are symmetric about 1, thus the multiplicities of $\frac{1}{2}$ and $\frac{3}{2}$ are equal to $km - n + 2$.

4. Applications of Theorem 1.1

In what follows, the expected hitting time and any two-points resistance distance between any vertices $i$ and $j$ of $S_k(G)$ are given. At here, we disregard to propose the adjacency and degree diagonal matrices of $S_k(G)$.

\[
A(S_k(G)) = \begin{pmatrix}
0 & B(G) & \cdots & B(G) \\
B(G)^T & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
B(G)^T & 0 & \cdots & 0
\end{pmatrix}, \quad D(S_k^r(G)) = \begin{pmatrix}
kD(G) & 0 & \cdots & 0 \\
0 & 2I_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 2I_m
\end{pmatrix}.
\]
Lemma 4.1. Assume that in terms of Theorem 1.1 of $\mathbb{N}$ has the eigenvalues $\pm \sqrt{\frac{1+\lambda_i}{2}}$, $a = 1, 2, \ldots, n$ and $0$ with multiplicities respectively 1 and $km - n$, if $G$ is non-bipartite graph. Then the corresponding orthonormal eigenvectors are

$$
\frac{1}{\sqrt{2}}\begin{pmatrix}
\pm \frac{1}{\sqrt{6(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i \\
\vdots \\
\pm \frac{1}{\sqrt{6(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i
\end{pmatrix}, \quad i = 1, 2, \ldots, n;
$$

where $(y_1, y_2, \ldots, y_{m-n})$ is an orthonormal basis of the kernel space of the matrix $B(G)$.

- $N(S_k(G))$ has the eigenvalues $\pm \sqrt{\frac{1+\lambda_i}{2}}$, $a = 1, 2, \ldots, n - 1$ and $0$ with multiplicities respectively 1 and $km - n + 2$, if $G$ is bipartite graph. Then the corresponding orthonormal eigenvectors are

$$
\frac{1}{\sqrt{2}}\begin{pmatrix}
\pm \frac{1}{\sqrt{6(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i \\
\vdots \\
\pm \frac{1}{\sqrt{6(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i
\end{pmatrix}, \quad i = 1, 2, \ldots, n - 1;
$$
Proof. According to the properties of orthogonal matrix and \( B(G)B(G)^T = A(G) + D(G) \), then

- If \( G \) is non-bipartite graph, then

\[ y^T_i y_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases} \]

On the one hand,

\[
N(S_k(G)) \begin{pmatrix} \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \\ \vdots \\ \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \end{pmatrix} = \pm \sqrt{1 + \lambda_i} \begin{pmatrix} \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \\ \vdots \\ \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \end{pmatrix},
\]

\[
N(S_k(G)) \begin{pmatrix} \sqrt{\frac{k(k-1)}{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \\ \sqrt{\frac{k(k-1)}{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \\ \sqrt{\frac{k(k-1)}{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \\ \vdots \\ \sqrt{\frac{k(k-1)}{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, N(S_k(G)) \begin{pmatrix} 0 \\ yz \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

On the other hand,

\[
\begin{pmatrix} \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \\ \vdots \\ \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_i \end{pmatrix}^T \begin{pmatrix} \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_j \\ \vdots \\ \pm \frac{1}{\sqrt{k(1 + \lambda_j)}} B^T D^{-\frac{1}{2}} v_j \end{pmatrix} = \begin{cases} 2, & \text{if } i = j \text{ and the signs are the same; } \\ 0, & \text{else.} \end{cases}
\]
The Lemma 4.5 has given the orthonormal eigenvectors of the corresponding eigenvalues in terms of Theorem 1.1 of $N(S_k(G))$. At this point, we will put the orthonormal eigenvectors of $N(S_k(G))$ in another way.

Moreover, one obtains that

$$
\left(\begin{array}{c}
\pm \frac{1}{\sqrt{k(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i \\
\pm \frac{1}{\sqrt{k(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i \\
\vdots \\
\pm \frac{1}{\sqrt{k(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i
\end{array}\right)^T
\left(\begin{array}{c}
\pm \frac{1}{\sqrt{k(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i \\
\pm \frac{1}{\sqrt{k(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i \\
\vdots \\
\pm \frac{1}{\sqrt{k(1+\lambda_i)}} B^T D^{-\frac{1}{2}} v_i
\end{array}\right) = 0,
$$

Hence, those orthonormal eigenvectors of the corresponding eigenvalues for the graph $S_k(G)$ as shown in theorem.

- If $G$ is non-bipartite graph, the proofs of this case are similar to the above procedure. Hence, we omit here.

The result as desired. 

The Lemma 4.5 has given the orthonormal eigenvectors of the corresponding eigenvalues in terms of Theorem 1.1 of $N(S_k(G))$. At this point, we will put the orthonormal eigenvectors of $N(S_k(G))$ in another way.
1. The orthonormal eigenvectors of the corresponding eigenvalues $\pm \sqrt{1 + \lambda_a} = \pm 1$ are as follows.

\[
\begin{pmatrix}
\sqrt{\frac{kd_1}{4km}} & \sqrt{\frac{kd_2}{4km}} & \cdots & \sqrt{\frac{kd_n}{4km}} \\
\sqrt{2/4km} & \sqrt{2/4km} & \cdots & \sqrt{2/4km}
\end{pmatrix}^T,
\]

respectively.

2. The eigenvectors $u_j$ of the corresponding eigenvalue $\pm \sqrt{\frac{1 + \lambda_a}{2}}$, $a = 2, 3, \ldots, n$ are as follows.

\[u_j = \begin{cases} 
\frac{\sqrt{2}}{2} v_{kj}, & \text{if } j \in V(G); \\
\mp \frac{1}{2k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right), & \text{if } j \in N \text{ with } N_{S_k(G)}(j) = \{s,t\}.
\end{cases}\]

Based on the properties of the orthogonal matrix, yields

\[
\sum_{l=1}^{m-n} y_{lj}^2 = 1 - \frac{1}{km} \sum_{a=2}^{n} \frac{1}{k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 - \sum_{a=1}^{n} \frac{k-1}{k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2
\]

if $G$ is non-bipartite graph.

3. The eigenvectors $u_j$ of the corresponding eigenvalue $\pm \sqrt{\frac{1 + \lambda_a}{2}}$, $a = 2, 3, \ldots, n-1$ are as follows.

\[u_j = \begin{cases} 
\frac{\sqrt{2}}{2} v_{kj}, & \text{if } j \in V(G); \\
\mp \frac{1}{2k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right), & \text{if } j \in N \text{ with } N_{S_k(G)}(j) = \{s,t\}.
\end{cases}\]

Based on the properties of the orthogonal matrix, yields

\[
\sum_{l=1}^{m-n+1} y_{lj}^2 = 1 - \frac{1}{km} \sum_{a=2}^{n-1} \frac{1}{k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 - \sum_{a=1}^{n-1} \frac{k-1}{k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2
\]

if $G$ is bipartite graph.

According to the structures of the graph $S_k(G)$, the selections of vertices $i$ and $j$ can be divided into three cases while determine the hitting time between vertices $i$ and $j$.

**Case 1.** $i, j \in V(G)$ and $G$ is non-bipartite graph, then

\[
E_i T_j (S_k(G)) = 4km \sum_{a=2}^{n} \left( \frac{1}{1 - \sqrt{\frac{1 + \lambda_a}{2}}} + \frac{1}{1 + \sqrt{\frac{1 + \lambda_a}{2}}} \right) \left( \frac{v_{a1}^2 v_{as}}{2kd_j} + \frac{v_{aj} v_{as}}{2k \sqrt{d_j d_l}} \right)
\]

\[
= 4km \sum_{a=2}^{n} \frac{4}{1 - \lambda_a} \left( \frac{v_{a1}^2}{2kd_j} + \frac{v_{aj} v_{as}}{2k \sqrt{d_j d_l}} \right)
\]

\[
= 8m \sum_{a=2}^{n} \frac{1}{1 - \lambda_a} \left( \frac{v_{a1}^2}{d_j} + \frac{v_{aj} v_{as}}{\sqrt{d_j d_l}} \right)
\]

\[
= 4E_i T_j (G),
\]
while $G$ is bipartite graph, then

$$E_iT_j(S_k(G)) = 4km \left[ \sum_{a=2}^{n-1} \left( \frac{1}{1 - \sqrt{1 + \lambda_a}} + \frac{1}{1 + \sqrt{1 + \lambda_a}} \right) \left( \frac{v_{aj}^2}{2kd_j} + \frac{v_{aj}v_{ai}}{k\sqrt{d_id_j}} \right) + \frac{v_{aj}v_{ni}}{k\sqrt{d_id_j}} \right]$$

$$= 4km \left[ \sum_{a=2}^{n-1} \frac{2}{k} \left( \frac{1}{1 - \lambda_a} \left( \frac{v_{aj}^2}{d_j} + \frac{v_{aj}v_{ai}}{\sqrt{d_id_j}} \right) + \frac{v_{aj}v_{ni}}{k\sqrt{d_id_j}} \right) \right]$$

$$= 8m \sum_{a=2}^{n} \frac{1}{1 - \lambda_a} \left( \frac{v_{aj}^2}{d_j} + \frac{v_{aj}v_{ai}}{\sqrt{d_id_j}} \right)$$

$$= 4E_iT_j(G).$$

Hence, $E_iT_j(S_k(G)) = 4E_iT_j(G)$ holds whatever $G$ is bipartite graph or not.

Case 2. $i \in N, j \in V(G)$ and $N_{S_k(G)}(i) = \{s, t\}$, then

$$E_iT_j(S_k(G)) = 1 + \frac{1}{2} (E_iT_j(S_k(G)) + E_jT_i(S_k(G))) = 1 + 2E_iT_j(G) + 2E_jT_i(G).$$

For $E_jT_i(S_k(G))$, one obtains the following equation based on Lemma 2.5,Enumerates 1 and 2.

$$E_jT_i(S_k(G)) = 4km \left[ \frac{1}{4km} \sum_{a=2}^{n} \left( \frac{1}{2 - 2\sqrt{1 + \lambda_a}} + \frac{1}{2 + 2\sqrt{1 + \lambda_a}} \right) \frac{1}{2k(1 + \lambda_a)} \left( \frac{v_{as}v_{ai}}{\sqrt{d_s}} + \frac{v_{at}v_{aj}}{\sqrt{d_t}} \right)^2 \right.$$

$$- \sum_{a=2}^{n} \left( \frac{1}{2k - 2k\sqrt{1 + \lambda_a}} - \frac{1}{2k + 2k\sqrt{1 + \lambda_a}} \right) \frac{v_{aj}}{\sqrt{2(1 + \lambda_a)d_j}} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 \right]$$

$$+ \sum_{i=1}^{m-n} \frac{y_i^2}{2} + \sum_{a=1}^{n} \frac{k - 1}{2k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 \right]$$

$$= 4km \left[ \sum_{a=2}^{n} \frac{1}{k(1 + \lambda_a)(1 - \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 \right.$$

$$- \sum_{a=2}^{n} \frac{1}{k(1 - \lambda_a)} \left( \frac{v_{as}v_{aj}}{\sqrt{d_s}} + \frac{v_{at}v_{aj}}{\sqrt{d_t}} \right)^2$$

$$+ \frac{1}{4km} + 1 - \frac{1}{m} - \sum_{a=2}^{n} \frac{1 + \lambda_a}{\sqrt{d_s}} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2$$

$$+ \sum_{a=2}^{n} \frac{k - 1}{2k(1 + \lambda_a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 \right]$$

$$= 2km - 1 + 4m \sum_{a=2}^{n} \frac{1}{1 - \lambda_a} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 - \frac{v_{as}v_{aj}}{\sqrt{d_s}} - \frac{v_{at}v_{aj}}{\sqrt{d_t}}$$

$$= 2km - 1 + 2[E_jT_i(G) + E_jT_i(G)] - [E_iT_s(G) + E_iT_s(G)],$$

if $G$ is non-bipartite graph.
In the same way, one arrives at

\[ E_j T_i(S_k(G)) = 4km \left\{ \frac{1}{4km} + \sum_{a=2}^{n} \left( \frac{1}{2 - 2\sqrt{1 + \lambda a}} + \frac{1}{2 + 2\sqrt{1 + \lambda a}} \right) \frac{1}{2k(1 + \lambda a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 \right\} \]

\[ - \sum_{a=2}^{n-1} \left( \frac{1}{2k - 2k\sqrt{1 + \lambda a}} - \frac{1}{2k + 2k\sqrt{1 + \lambda a}} \right) \frac{v_{aj}}{\sqrt{2(1 + \lambda a)d_j}} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right) \]

\[ + \sum_{l=1}^{m-n+1} \frac{y_{il}^2}{2} + \sum_{a=1}^{n} \frac{k-1}{2k(1 + \lambda a)} \left( \frac{v_{as}}{\sqrt{d_s}} + \frac{v_{at}}{\sqrt{d_t}} \right)^2 \]

\[ = 2km - 1 + 4m \sum_{a=2}^{n} \frac{1}{1 - \lambda a} \left( \frac{v_{as}^2}{d_s} + \frac{v_{at}^2}{d_t} - \frac{v_{as}v_{at}}{\sqrt{d_s}d_t} - \frac{v_{as}v_{at}}{\sqrt{d_t}d_s} - \frac{v_{as}^2}{2d_s} - \frac{v_{at}^2}{2d_t} + \frac{v_{as}v_{at}}{\sqrt{d_s}d_t} \right) \]

\[ = 2km - 1 + 2 \left[ E_j T_t(G) + E_j T_i(G) \right] - \left[ E_i T_s(G) + E_j T_i(G) \right], \]

if \( G \) is bipartite graph.

**Case 3.** \( i, j \in N, N_{S_k(G)}(i) = \{s, t\} \) and \( N_{S_k(G)}(j) = \{p, q\} \). According to Case 2, then

\[ E_i T_j(S_k(G)) = 1 + \frac{1}{2} \left[ E_j T_j(S_k(G)) + E_i T_j(S_k(G)) \right] \]

\[ = 1 + \frac{1}{2} \left[ 2km - 1 + 2 \left( E_i T_p(G) + E_i T_q(G) \right) - \left( E_q T_p(G) + E_p T_q(G) \right) \right] \]

\[ + \frac{1}{2} \left[ 2km - 1 + 2 \left( E_i T_p(G) + E_i T_q(G) \right) - \left( E_q T_p(G) + E_p T_q(G) \right) \right] \]

\[ = 2km + E_i T_p(G) + E_i T_q(G) + E_i T_p(G) + E_i T_q(G) - E_q T_p(G) - E_p T_q(G). \]

In the same way, one obtains

\[ E_j T_i(S_k(G)) = 2km + E_p T_s(G) + E_q T_s(G) + E_p T_i(G) + E_q T_i(G) - E_q T_p(G) - E_p T_q(G). \]

Combining Cases 1, 2 and 3, we get the following theorem.

**Theorem 4.2.** Assume that \( G \) is a \((n, m)\)-graph, then the expected hitting time between vertices \( i \) and \( j \) in \( S_k(G) \) are as follows.

- \( i, j \in V(G) \), then
  \[ E_j T_i(S_k(G)) = 4E_i T_j(S_k(G)). \]

- \( i \in N, j \in V(G) \) and \( N_{S_k(G)}(i) = \{s, t\} \), then
  \[ E_i T_j(S_k(G)) = 1 + 2E_i T_j(G) + 2E_i T_j(G), \]

  \[ E_j T_i(S_k(G)) = 2km - 1 + 2 \left[ E_i T_s(G) + E_j T_i(G) \right] - \left[ E_i T_k(G) + E_i T_i(G) \right]. \]

- \( i \in N, N_{S_k(G)}(i) = \{s, t\} \) and \( N_{S_k(G)}(j) = \{p, q\} \), then
  \[ E_i T_j(S_k(G)) = 2km + E_p T_p(G) + E_q T_p(G) + E_i T_p(G) + E_i T_q(G) - E_q T_p(G) - E_p T_q(G), \]

  \[ E_j T_i(S_k(G)) = 2km + E_p T_s(G) + E_q T_s(G) + E_p T_i(G) + E_q T_i(G) - E_i T_p(G) - E_p T_s(G). \]

Based on Lemma 2.6 and above theorem, leads
Corollary 4.3. Assume that $G$ is a $(n,m)$-graph, then any two-points resistance distance between vertices $i$ and $j$ in $S_k(G)$ are as below.

- $i, j \in V(G)$, then
  $$\Omega_{ij}(S_k(G)) = \frac{2}{k} \Omega_{ij}(G).$$

- $i \in N$, $j \in V(G)$ and $N_{S_k(G)}(i) = \{s, t\}$, then
  $$\Omega_{ij}(S_k(G)) = \frac{k + 2\Omega_{st}(G) + 2\Omega_{ij}(G) - \Omega_{st}(G)}{2k}.$$

- $i, j \in N$, $N_{S_k(G)}(i) = \{s, t\}$ and $N_{S_k(G)}(j) = \{p, q\}$, then
  $$\Omega_{ij}(S_k(G)) = \frac{2k + \Omega_{sp}(G) + \Omega_{sq}(G) + \Omega_{tp}(G) + \Omega_{tq}(G) - \Omega_{pq}(G) - \Omega_{st}(G)}{2k}.$$

Remark 1. While $k = 2$, we can get the Theorem 4.2 of [20]. Indeed, one can compute the multiplicative degree-Kirchhoff index of $S_k(G)$ through any two-points resistance distance between vertices $i$ and $j$ in $S_k(G)$. But in this paper, we prefer to use the normalized Laplacian spectra of $S_k(G)$ to calculate the multiplicative degree-Kirchhoff index of $S_k(G)$ (see Eq. (4.20)). This seems much easier.

According to the relation between the expected hitting time and expected commute time, the following corollary can be immediately obtained.

Corollary 4.4. Assume that $G$ is a $(n,m)$-graph, then the expected commute time between vertices $i$ and $j$ in $S_k(G)$ are as below.

- $i, j \in V(G)$, then
  $$C_{ij}(S_k(G)) = \frac{4m}{k} \Omega_{ij}(G).$$

- $i \in N$, $j \in V(G)$ and $N_{S_k(G)}(i) = \{s, t\}$, then
  $$C_{ij}(S_k(G)) = m \cdot \frac{k + 2\Omega_{st}(G) + 2C_{ij}(G) - C_{st}(G)}{k}.$$

- $i, j \in N$, $N_{S_k(G)}(i) = \{s, t\}$ and $N_{S_k(G)}(j) = \{p, q\}$, then
  $$C_{ij}(S_k(G)) = m \cdot \frac{2k + C_{sp}(G) + C_{sq}(G) + C_{tp}(G) + C_{tq}(G) - C_{pq}(G) - C_{st}(G)}{k}.$$

It remains to now look at the relation between the normalized Laplacians of $G$ and $S_k(G)$. First goal is to determine the eigenvalues of $L^r(S_k(G))$ with $r$ iterations, i.e., the eigenvalues of $L(S_k^r(G))$. Second, we provide in this part the explicit formulas for $Kf^r(S_k(G))(Kf^r(S_k^r(G))$ resp.), $Ke(S_k(G))(Ke(S_k^r(G))$ resp.) and $\tau(S_k(G))|\tau(S_k^r(G))$ resp.).

Let $S^0_k(G) = G$, $S^1_k(G) = S_k(S^0_k(G))$, ..., $S^r_k(G) = S_k(S^{r-1}_k(G))$. Then denote $|E(S^r_k(G))|$ and $|V(S^r_k(G))|$ the edge set and vertex set of $S^r_k(G)$, $|E_r|$ and $|V_r|$ for simplify. According to the construction of $S^r_k(G)$, one has

$$|E_r| = 2k|E_{r-1}|, |V_r| = |V_{r-1}| + k|E_{r-1}|.$$  

It is easy to obtain

$$|E_r| = m(2k)^r, |V_r| = n + km \frac{(2k)^r - 1}{2k - 1}.$$
Assume that $U$ is a finite multiset of real numbers. To obtain the normalized Laplacians of $S_k^r(G)$, we also have to define two new multisets in the line with Eq. (3.14) as follows.

$$f_1(U) = \bigcup_{x \in U} \{f_1(x)\}, \quad f_2(U) = \bigcup_{x \in U} \{f_2(x)\}.$$ 

With those notation in hand, the normalized Laplacians of $S_k^r(G)$ can be determined immediately by Theorem 1.1 and the structure of $S_k^r(G)$.

**Theorem 4.5.** Assume that $S_k^r(G)$ is the $r$-th iterations of the graph $S_k(G)$, then the normalized Laplacian spectra of $S_k^r(G)$ are as follows.

$$
\begin{align*}
&f_1(\Gamma(G) \setminus \{0,2\}) \bigcup f_2(\Gamma(G) \setminus \{0,2\}) \bigcup \{1,1,\ldots,1\}, \quad r = 1 \text{ and } G \text{ is bipartite;} \\
&f_1(\Gamma(G)) \bigcup f_2(\Gamma(G)) \bigcup \{1,1,\ldots,1\}, \quad r = 1 \text{ and } G \text{ is nonbipartite;} \\
&f_1(\Gamma(S_k^{r-1}(G)) \setminus \{0,2\}) \bigcup f_2(\Gamma(S_k^{r-1}(G)) \setminus \{0,2\}) \bigcup \{0,2\} \bigcup \{1,1,\ldots,1\}, \quad r > 1,
\end{align*}
$$

where $\Gamma(S_k^{r-1}(G))$ is the normalized Laplacian spectrum of $S_k^{r-1}(G)$, the multiplicities are respectively $2r^{-1}(n-1) + \sum_{j=2}^{r-2} (k|E_r \setminus |V_r|) + 2$ and $\sum_{j=2}^{r-2} (k|E_r \setminus |V_r|) + 2$ of $f_1(\Gamma(S_k^{r-1}(G)) \setminus \{0,2\})$ and $\{1,1,\ldots,1\}$ for $r > 1$.

According to the normalized Laplacians of $S_k^r(G)$, we derive the explicit formulas for $Kf^*(S_k^r(G))$, $Ke(S_k^r(G))$ and $\tau(S_k^r(G))$.

**Theorem 4.6.** Assume that $G$ is a $(n,m)$-graph. One has

$$Kf^*(S_k^r(G)) = (8k)^rKf^*(G) + \frac{(2k)^r(4^r - 1)}{3} (m - 2mn) + \frac{k(4k)^r(4^r - 2^r)}{k - 2} \frac{m^2}{k - 2} - \frac{k(2k)^r[4^r - 2k(4^r - 1) + 3(2k)^r - 4]}{3(k - 2)(2k - 1)} m^2,$$

where $k > 2$, $r \geq 1$.

**Proof.** According to Theorem 1.1 and $\lambda_i \neq 2$, one has $f_1(\lambda_i)$ and $f_2(\lambda_i)$ are two roots of $-2\sigma^2 + 4\sigma - \lambda_i = 0$, then

$$\frac{1}{f_1(\lambda_i)} + \frac{1}{f_2(\lambda_i)} = \frac{4}{\lambda_i}, \quad f_1(\lambda_i) f_2(\lambda_i) = \lambda_i^2,$$

(4.19)

In addition, $1 + 1 = 2, 1 \times 1 = 2 = \frac{2}{2} = 1$, thus $\lambda_i = 2$ also satisfy Eq.(4.19). Based on Lemma (2.1), one obtains

$$Kf^*(S_k^1(G)) = 4km \left( \sum_{i=2}^{n} \left( \frac{1}{f_1(\lambda_i)} + \frac{1}{f_2(\lambda_i)} \right) + \frac{1}{2} + km - n \right) = 4km \left( \sum_{i=2}^{n} \frac{4}{\lambda_i} + \frac{1}{2} + km - n \right) = 8k \cdot Kf^*(G) + 2km(1 + km - n).$$

(4.20)
By Eq. (4.20) and the construction of $S_k^r(G)$, we get

$$Kf^*(S_k^r(G)) = 8k \cdot Kf^*(S_k^{r-1}(G)) + 2k|E_r-1|(1 + 2k|E_{r-1}| - 2|V_{r-1}|)$$

$$= (8k)^rKf^*(G) + 2k \sum_{i=0}^{r-1} (8k)^{r-1-i}|E_i|(1 + 2k|E_i| - 2|V_i|)$$

$$= (8k)^rKf^*(G) + \frac{(2k)^r(4^r - 1)}{3}(m - 2mn) + \frac{k(4k)^r(k^r - 2^r)}{k - 2}m^2$$

$$- \frac{k(2k)^r[4^r - 2k(4^r - 1) + 3(2k)^r - 4]}{3(k - 2)(2k - 1)}m^2.$$

This completes the proof.

**Remark 2.** When $k = 1$, it is

$$Kf^*(S(G)) = 8 \cdot Kf^*(G) + 2m(1 + 2m - 2n).$$

This coincides the Theorem 4.1 of [19].

**Remark 3.** When $k = 2$, it satisfies

$$Kf^*(S_2^1(G)) = 16 \cdot Kf^*(G) + 4m(1 + 4m - 2n).$$

This equals the Theorem 4.1 of [20].

For $S_2^r(G)$, the multiplicative degree-Kirchhoff index is as below.

$$Kf^*(S_2^r(G)) = 16 \cdot Kf^*(S_2^{r-1}(G)) + 4|E_{r-1}|(1 + 4|E_r-1| - 2|V_{r-1}|)$$

$$= (16)^rKf^*(G) + 4 \sum_{i=0}^{r-1} (16)^{r-1-i}|E_i|(1 + 4|E_i| - 2|V_i|)$$

$$= (16)^rKf^*(G) + \frac{4^r(4^r - 1)}{3}m - \frac{2 \cdot 4^r(4^r - 1)}{3}mn$$

$$+ \frac{2 \cdot 4^r[2(4^r - 1) + 3r \cdot 4^r]}{9}m^2.$$

**Theorem 4.7.** Assume that $G$ is a $(n, m)$-graph. One has

$$Ke(S_k^r(G)) = 4^rKe(G) + \frac{4^r - 1}{6}(1 - 2n) + \frac{2^r - 1}{k - 2}m - \frac{k[4^r - 2k(4^r - 1) + 3(2k)^r - 4]}{6(k - 2)(2k - 1)}m,$$

where $k > 2$, $r \geq 1$.

**Proof.** By Lemma (2.2) and the relation between $Ke(G)$ and $Kf^*(G)$, one gets

$$Ke(S_k^1(G)) = \frac{1}{4km}Kf^*(S_k^1(G)) = 4Ke(G) + \frac{1}{2}(1 + 2km - 2n).$$

Further,

$$Ke(S_k^r(G)) = \frac{1}{2m(2k)^r}Kf^*(S_k^r(G))$$

$$= \frac{(8k)^r}{2m(2k)^r}Kf^*(G) + \frac{(2k)^r(4^r - 1)}{6m(2k)^r}(m - 2mn) + \frac{k(4k)^r(k^r - 2^r)}{2m(2k)^r(k - 2)}m^2$$

$$- \frac{k(2k)^r[4^r - 2k(4^r - 1) + 3(2k)^r - 4]}{6m(2k)^r(k - 2)(2k - 1)}m^2$$

$$= 4^rKe(G) + \frac{4^r - 1}{6}(1 - 2n) + \frac{2^r - 1}{k - 2}m - \frac{k[4^r - 2k(4^r - 1) + 3(2k)^r - 4]}{6(k - 2)(2k - 1)}m.$$
The proof has completed.

**Remark 4.** When \( k = 2 \), this yields

\[
Ke(S^r_k(G)) = \frac{1}{2m \cdot 4^r} K f^*(S^r_k(G)) \\
= 4^r Ke(G) + \frac{4^r - 1}{6} (1 - 2n) + \frac{2(4^r - 1) + 3r \cdot 4^r}{9} m.
\]

While \( r = 1 \), we can directly obtain the theorem 4.1 in [20].

In the latest part of this section, the formula for \( \tau(S^r_k(G)) \) is determined.

**Theorem 4.8.** Assume that \( G \) is a \((n,m)\)-graph. One has

\[
\tau(S^r_k(G)) = 2^{km} \frac{1 - (2k)^r}{1 - 2k} \cdot k^{nr + km} (\lambda_2)^{r-1} \cdot \tau(G),
\]

where \( k \geq 1 \), \( r \geq 1 \).

**Proof.** Based on Eq.(3.2), it is routine to check

\[
\prod_{i=1}^{n+km} d_i(S_k(G)) = 2^{km} \cdot \prod_{i=1}^{n} kd_i(G) = 2^{km} \cdot k^n \cdot \prod_{i=1}^{n} d_i(G).
\]

By Eq.(4.21) and Lemma (2.4), we obtain

\[
\tau(S^1_k(G)) = 2 \prod_{i=1}^{n+km} d_i(S_k(G)) \prod_{i=2}^{n} f_1(\lambda_i) f_2(\lambda_i)
\]

\[
= \frac{2^{km} \cdot k^n \cdot \prod_{i=1}^{n} d_i(G) \prod_{i=2}^{n} \lambda_i}{4km}
\]

\[
= 2^{km-1} \cdot k^{n-1} \cdot \tau(G).
\]

Hence, one arrives at

\[
\tau(S^r_k(G)) = 2^{k|E_r-1|-1} \cdot k^{|V_r-1|-1} \cdot \tau(S^r_{k-1}(G))
\]

\[
= 2^{\sum_{i=0}^{\lambda_1}(k|E_i|-1)} \cdot k^{\sum_{i=0}^{\lambda_1-1} (|V_i|-1)} \tau(G)
\]

\[
= 2^{km \cdot \frac{1 - (2k)^r}{1 - 2k} \cdot r} \cdot k^{nr + km} \frac{(\lambda_2)^{r-1} \cdot (1 - 2k)^{r-1}}{1 - 2k} \tau(G).
\]

The result as desired.

\[
\tau(S^r_k(G)) = 2^{km \cdot \frac{1 - (2k)^r}{1 - 2k} \cdot r} \cdot k^{nr + km} \frac{(\lambda_2)^{r-1} \cdot (1 - 2k)^{r-1}}{1 - 2k} \tau(G).
\]

\[
\tau(S^r_k(G)) = 2^{km \cdot \frac{1 - (2k)^r}{1 - 2k} \cdot r} \cdot k^{nr + km} \frac{(\lambda_2)^{r-1} \cdot (1 - 2k)^{r-1}}{1 - 2k} \tau(G).
\]

\[
\tau(S^r_k(G)) = 2^{km \cdot \frac{1 - (2k)^r}{1 - 2k} \cdot r} \cdot k^{nr + km} \frac{(\lambda_2)^{r-1} \cdot (1 - 2k)^{r-1}}{1 - 2k} \tau(G).
\]

\[
\tau(S^r_k(G)) = 2^{km \cdot \frac{1 - (2k)^r}{1 - 2k} \cdot r} \cdot k^{nr + km} \frac{(\lambda_2)^{r-1} \cdot (1 - 2k)^{r-1}}{1 - 2k} \tau(G).
\]

\[
\tau(S^r_k(G)) = 2^{km \cdot \frac{1 - (2k)^r}{1 - 2k} \cdot r} \cdot k^{nr + km} \frac{(\lambda_2)^{r-1} \cdot (1 - 2k)^{r-1}}{1 - 2k} \tau(G).
\]

The result as desired.

**5. Applications of Theorem 1.2**

Let \( S^r_{2k}(G) = G \), \( S^r_{2k-1}(G) = S_{2k}(S^r_{2k-1}(G)) \), \ldots, \( S^r_{2k}(G) = S_{2k}(S^r_{2k-1}(G)) \). Then denote \(|E(S^r_{2k}(G))|\) and \(|V(S^r_{2k}(G))|\) the edge set and vertex set of \( S^r_{2k}(G) \), \(|E'_r|\) and \(|V'_r|\) for simplify. According to the construction of \( S^r_{2k}(G) \), one has

\[
|E'_r| = 3k|E_{r-1}|, \quad |V'_r| = |V_{r-1}| + 2k|E_{r-1}|.
\]

It is easy to obtain

\[
|E'_r| = m(3k)^r, \quad |V'_r| = n + 2km (3k)^r - 1.
\]
Denote \( g_1(\lambda_i), g_2(\lambda_i) \) and \( g_3(\lambda_i) \) the roots of the equation \( 4\zeta^3 - 12\zeta^2 + 9\zeta - \lambda_i = 0 \). We at here define three new multiset as below.

\[
g_1(U) = \bigcup_{x \in U} \{g_1(x)\}, \quad g_2(U) = \bigcup_{x \in U} \{g_2(x)\}, \quad g_3(U) = \bigcup_{x \in U} \{g_3(x)\}.
\]

In what follows, the normalized Laplacians of \( S_{2k}(G) \) are given directly based on those notation and the structure of \( S_{2k}(G) \).

**Theorem 5.1.** Assume that \( S_{2k}(G) \) is the \( r \)-th iterations of the graph \( S_{2k}(G) \), then the normalized Laplacian spectra of \( S_{2k}(G) \) are as follows.

\[
\begin{aligned}
&\ 
\left\{\begin{array}{ll}
g_1(\Gamma(S_{2k}^{-1}(G))\setminus\{0, 2\}) \cup g_2(\Gamma(S_{2k}^{-1}(G))\setminus\{0, 2\}) \cup g_3(\Gamma(S_{2k}^{-1}(G))\setminus\{0, 2\}) \\
\{1, 2 \} \cup \left\{ \frac{1}{2} \right\} \cup \left\{ \frac{1}{2} \right\},
\end{array}\right. \\
&\ 
\left\{\begin{array}{ll}
g_1(\Gamma(S_{2k}^{-1}(G))\setminus\{0\}) \cup g_2(\Gamma(S_{2k}^{-1}(G))\setminus\{0\}) \cup g_3(\Gamma(S_{2k}^{-1}(G))\setminus\{0\}) \\
\{1, 2 \} \cup \left\{ \frac{1}{2} \right\} \cup \left\{ \frac{1}{2} \right\},
\end{array}\right.
\end{aligned}
\]

Based on the normalized Laplacians of \( S_{2k}^r(G) \), the closed-form formulas for \( Kf^*(S_{2k}^r(G)) \), \( Ke(S_{2k}^r(G)) \) and \( \tau(S_{2k}^r(G)) \) are given.

**Theorem 5.2.** Assume that \( G \) is a \((n, m)\)-graph. One has

\[
Kf^*(S_{2k}^r(G)) = (27k^r)^r Kf^*(G) + \frac{16k^2(9k)^{r-1}(k^r - 3^r)}{k - 3} m^2 + \frac{11k^2(3k)^{r-1}(9^r - 1)}{8} m
\]

\[
-6k^2(3k)^{r-2}(9^r - 1) n + \frac{4k^3(3k)^{r-2}(9 - 9^r + 3k(9^r - 1) - 8(3k)^{r-1})}{(3k - 1)(k - 3)} m^2.
\]

where \( k \) is integer and \( k \neq 3 \), \( r \geq 1 \).

**Proof.** According to Theorem 1.2 and \( \lambda_i \neq 0, 2 \), one has \( \zeta_1, \zeta_2 \) and \( \zeta_3 \) are three roots of \( 4\zeta^3 - 12\zeta^2 + 9\zeta - \lambda_i = 0 \), then

\[
\frac{1}{\zeta_1} + \frac{1}{\zeta_2} + \frac{1}{\zeta_3} = \frac{9}{\lambda_i}, \quad \zeta_1 \zeta_2 \zeta_3 = \frac{\lambda_i}{4} \tag{5.22}
\]

In addition, \( 2 + 2 = \frac{3}{2}, \ 2 \times \frac{3}{2} = 2 \times \frac{3}{2} = 1 \), hence \( \lambda_i = 2 \) also satisfy Eq.(5.22). Based on the definition of \( Kf^*(G) \), one obtains

\[
Kf^*(S_{2k}^r(G)) = 6km \left( \sum_{i=2}^{n} \left( \frac{1}{\zeta_1} + \frac{1}{\zeta_2} + \frac{1}{\zeta_3} \right) + 2(km - n) + \frac{2}{3}(km - n + 2) + \frac{1}{2}\right)
\]

\[
= 6km \left( \sum_{i=2}^{n} \frac{9}{\lambda_i} + 2(km - n) + \frac{2}{3}(km - n + 2) + \frac{1}{2}\right)
\]

\[
= 27k \cdot Kf^*(G) + 16k^2 m^2 - 16kmn + 11km. \tag{5.23}
\]
By Eq.(5.23) and the construction of $S_{2k}^r(G)$, one has

$$K f^*(S_{2k}^r(G)) = 27k \cdot K f^*(S_{2k}^{r-1}(G)) + 16k^2|E'_{r-1}|^2 - 16k|E'_{r-1}||V'_{r-1}| + 11k|E'_{r-1}|$$

$$= (27k)^r K f^*(G) + 16k^2 \sum_{i=0}^{r-1} (27k)^{r-1-i}|E'_i|^2$$

$$- 16k \sum_{i=0}^{r-1} (27k)^{r-1-i}|E'_i||V'_i| + 11k \sum_{i=0}^{r-1} (27k)^{r-1-i}|E'_i|$$

$$= (27k)^r K f^*(G) + \frac{16k^2(9k)^{r-1}(k^r - 3^r)}{k-3} m^2 + \frac{11k^2(3k)^{r-1}(9^r - 1)}{8} m$$

$$- 6k^2(3k)^{r-2}(9^r - 1)n + \frac{4k^3(3k)^{r-2}[9^r - 3k(9^r - 1) - 8(3k)^r]}{(3k-1)(k-3)} m^2.$$ 

This completes the proof.

**Remark 5.** When $k = 3$, one has

$$K f^*(S_3^1(G)) = 81 \cdot K f^*(G) + 144m^2 - 48mn + 33m.$$ 

For $S_{2k}^r(G)$, the multiplicative degree-Kirchhoff index is as follows.

$$K f^*(S_{2k}^r(G)) = 81 \cdot K f^*(S_{2k}^{r-1}(G)) + 144|E'_{r-1}|^2 - 48|E'_{r-1}||V'_{r-1}| + 33|E'_{r-1}|$$

$$= 81^r \cdot K f^*(G) + 144 \sum_{i=0}^{r-1} (81)^{r-1-i}|E'_i|^2 - 48 \sum_{i=0}^{r-1} (81)^{r-1-i}|E'_i||V'_i| + 33 \sum_{i=0}^{r-1} (81)^{r-1-i}|E'_i|$$

$$= 81^r \cdot K f^*(G) + \frac{11 \cdot 3^{2r-1}(9^r - 1)}{8} m - 2 \cdot 3^{2r-1}(9^r - 1)mn$$

$$+ \frac{3^{2r-1}[3(9^r - 1) + 8r \cdot 9^r]}{2} m^2.$$ 

**Theorem 5.3.** Assume that $G$ is a $(n,m)$-graph. One has

$$Ke(S_{2k}^r(G)) = 9^r Ke(G) + \frac{8k \cdot 3^{r-2}(k^r - 3^r)}{k-3} m - \frac{9^r - 1}{3m} n + \frac{11k(9^r - 1)}{48}$$

$$+ \frac{2k[9^r - 9^r + 3k(9^r - 1) - 8(3k)^r]}{9(3k - 1)(k-3)} n,$$

where $k$ is integer and $k \neq 3$, $r \geq 1$.

**Proof.** According to the relation between $Ke(G)$ and $K f^*(G)$, one gets

$$Ke(S_{2k}^r(G)) = \frac{1}{6km} K f^*(S_{2k}^r(G)) = 9Ke(G) + \frac{1}{6}(16km - 16n + 11).$$

Additionally,

$$Ke(S_{2k}^r(G)) = \frac{1}{2m(3k)^r} K f^*(S_{2k}^r(G))$$

$$= \frac{(27k)^r}{2m(3k)^r} K f^*(G) + \frac{16k^2(9k)^{r-1}(k^r - 3^r)}{2m(3k)^r(k-3)} m^2 + \frac{11k^2(3k)^{r-1}(9^r - 1)}{16m(3k)^r}$$

$$- \frac{6k^2(3k)^{r-2}(9^r - 1)}{2m(3k)^r} n + \frac{4k^3(3k)^{r-2}[9^r - 3k(9^r - 1) - 8(3k)^r]}{2m(3k)^r(3k - 1)(k-3)} m^2$$

$$= 9^r Ke(G) + \frac{8k \cdot 3^{r-2}(k^r - 3^r)}{k-3} m - \frac{9^r - 1}{3m} n + \frac{11k(9^r - 1)}{48}$$

$$+ \frac{2k[9^r - 9^r + 3k(9^r - 1) - 8(3k)^r]}{9(3k - 1)(k-3)} n.$$
As desired. \hfill \blacksquare

**Remark 6.** While $k = 3$, this leads

\[
K_e(S'_6(G)) = \frac{1}{2m} \cdot 9^r K f^*(S'_6(G)) = 9^r K e(G) + \frac{9^r - 1}{48} (11 - 16n) + \frac{3(9^r - 1) + 8r \cdot 9^r}{12} m.
\]

Before proceeding, we shall give an equation and define a function that will use in the following results. On the one hand, by Eq.(3.7), then

\[
\prod_{i=1}^{n+2km} d_i(S_{2k}(G)) = 2^{2km} \cdot \prod_{i=1}^{n} k d_i(G) = 2^{2km} \cdot k^n \cdot \prod_{i=1}^{n} d_i(G).
\] \hspace{1cm} (5.24)

On the other hand, we define

\[
\varphi(r) = \frac{2km(r - 1 - 3kr + 3^r k^r)}{(3k - 1)^2}.
\]

**Theorem 5.4.** Assume that $G$ is a $(n, m)$-graph. One has

\[
\tau(S'_{2k}(G)) = \left(\frac{1}{2}\right)^{r(1-n)-\varphi(r)} \left(\frac{3}{2}\right)^{r(2-n)-\varphi(r)} \frac{3^{km(3^r k^r)-1} k^{n r-1} + \varphi(r) \tau(G)}{3^{km-1} k^{n-1} \cdot \tau(G)}
\]

where $k \geq 1$, $r \geq 1$.

**Proof.** By Eq.(5.24) and Lemma (2.4), one obtains

\[
\tau(S'_{2k}(G)) = \frac{2 \prod_{i=1}^{n+2km} d_i(S_{2k}(G)) \prod_{i=2}^{n} \zeta_1 \zeta_2 \zeta_3 \left(\frac{1}{3}\right)^{km-n} \left(\frac{3}{2}\right)^{km-n+2}}{6km} \cdot \frac{2^{2km}}{6km} \cdot k^n \cdot \prod_{i=1}^{n} d_i(G) \prod_{i=2}^{n} \lambda_i \left(\frac{1}{3}\right)^{km-n+1} \left(\frac{3}{2}\right)^{km-n+2}
\]

Moreover, the above equation yields

\[
\tau(S'_{2k}(G)) = \left(\frac{1}{2}\right)^{1-|V'_{r-1}|} \left(\frac{3}{2}\right)^{2-|V'_{r-1}|} \frac{3^k |E'_{r-1}| - 1} {k |V'_{r-1}| - 1} \cdot \tau(S'_{2k-1}(G))
\]

\[
= \left(\frac{1}{2}\right)^{\sum_{i=0}^{r-1} (1-|V'_i|)} \left(\frac{3}{2}\right)^{3 \sum_{i=0}^{r-1} (2-|V'_i|)} \frac{3^k \sum_{i=0}^{r-1} (|E'_i| - 1) k \sum_{i=0}^{r-1} (|V'_i| - 1) \tau(G)}{3^{km(3^r k^r)-1} k^{n r-1} + \varphi(r) \tau(G)}
\]

The result as desired. \hfill \blacksquare

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