Complexity and quenches in models with three and four spin interactions

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Abstract. We study information theoretic quantities in models with three and four spin interactions. These models show distinctive characteristics compared to their nearest neighbour (NN) counterparts. Here, we quantify these in terms of the Nielsen complexity (NC) in static and quench scenarios, the Fubini–Study complexity (FSC), and the entanglement entropy (EE). The models that we study have a rich phase structure, and we show how the difference in the nature of phase transitions in these, compared to ones with NN interactions, result in different behaviour of information theoretic quantities, from ones known in the literature. For example, the derivative of the NC does not diverge but shows a discontinuity near continuous phase transitions, and the FSC may be regular and continuous across such transitions. We also study multiple quench scenarios in these models and contrast these with quenches in the transverse XY model. The EE shows a novel discontinuity both at first and second order quantum phase transitions.

Keywords: complexity, quantum phase transitions, quantum quenches

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1. Introduction

Over the last few years, study of information theory in quantum many body systems have fast gained popularity. A primary reason for this is that apart from being the foundations of what is perhaps the technology of the future, such studies point to the deep connection between diverse areas of physics. Indeed, various information theoretic measures such as the quantum information metric (QIM) [1–3] and its associated Fubini–Study complexity (FSC) [4], Nielsen complexity (NC) [5, 6], Loschmidt echo (LE) [7, 8] and entanglement entropy (EE) [9–12], find extensive applications in different areas of physics, from statistical systems to black holes, the connections often being realised in quantum field theory via the gauge-gravity duality [13–22].

Spin systems provide an ideal laboratory for studying information theory in a quantum mechanical setting, primarily because these are often amenable to analytical results and provide deep physical insights regarding the behaviour of information geometry near zero temperature quantum phase transitions (QPTs). As of now, these are well studied in, for example, the transverse XY model, a quasi-free fermionic model with nearest neighbour (NN) interactions [23, 24], and the Lipkin–Meshkov–Glick model, which is an exactly solvable model with infinite range interactions [25, 26], in both time-independent and time-dependent (quench) scenarios. The purpose of the present paper is to go beyond the physics of NN interactions and study these information theoretic quantities in spin models in the presence of three and four spin interactions. Fortunately, an exactly solvable model with such interactions is known, and was discussed in [27] (see also [28, 29]). Here, we study this model with some specific simplifying choice of parameters (that nonetheless preserves a rich phase structure), and

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show that there are some remarkable differences (as well as similarities) in the behaviour of information theoretic quantities when short range interactions are added to the NN one.

Let us elaborate upon the main motivations for this study. Firstly, to the best of our knowledge, spin systems apart from the ubiquitous short range NN models, and the infinite range Lipkin–Meshkov–Glick model, have not been studied in the current literature. Indeed, as we show in sequel, the model considered here shows several distinctive features compared to those, as far as information theoretic quantities are concerned. Secondly, as is known, the nature of the phase transitions in the model of [27] are different from the ones in say, the transverse $XY$ model. Specifically, the zero temperature second order phase transitions that this model exhibit are not due to the closing of an energy gap as happens in the latter one. The model studied here exhibits both first and second order phase transitions as two branches of the energy spectrum lie above or below zero, resulting in different magnetisations. Now, information geometry has been well studied in scenarios where second order QPTs occur due to the closing of the energy gap, and the scaling of the gap is an useful tool to compute the nature of both the NC and the FSC close to criticality [23]. Here on the other hand, we explore the situation where such scaling behaviour is absent in the phase diagram and in this sense this model offers a new perspective, and complements existing results on information theoretic quantities in spin systems. We should add here that information theoretic studies on quantum first order phase transitions are relatively rare in the literature, compared to its second order cousin. The simplicity and solvability of the model considered here provides a possible situation where such an analysis can be performed. Overall, our results indicate that some of the well known features of models with NN interactions are non-trivially modified here, the main reason being the entirely mechanism of the phase transition alluded to before.

With these motivations, in this paper, we first compute the QIM and the structure of geodesics for all the phases of the theory in the presence of four spin interactions. Our analysis indicates that contrary to some known examples, the Ricci scalar of the QIM and the FSC do not capture the entire phase structure of the theory. Interestingly, these show their ‘typical’ behaviours (known from statistical models where the second order phase transitions result from the closure of an energy gap) only at a particular type of second order phase transition, namely the gapped ferromagnetic to the gapless ferrimagnetic phases, and not between two ferrimagnetic phases (with different magnetisations). Next, we compute the NC for this model and show that its derivative does not diverge at second order phase transitions, but rather shows a discontinuity, again in contrast to known examples in the literature. A quench scenario is considered thereafter, where we couple our model to a spin half system, and compute the LE. Thereafter, we also consider multiple global quench scenarios in the model and contrast this with such quenches in the transverse $XY$ model. Finally, using the density matrix renormalisation group (DMRG) methods [2, 30–32], we compute the EE, and find sharp jumps across both first order and second order phase transitions, a result that is in contradiction to the ones known in the literature, namely that the EE shows discontinuity at a first order QPT, while a cusp or a kink in the EE indicates the second-order QPT [33, 34].

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In the following sections 2–6, we elaborate upon the above points, and the paper ends with our conclusions in section 7. This paper also includes two appendices A and B where some mathematical details are relegated to. Units with $\hbar = 1$ are used throughout.

2. The model

A quantum spin models with alternating NN couplings, and three spin and four spin exchange interactions was studied by [27]. This will be used extensively in the paper, and we first recall their main results. More details can be found in [28, 29]. The model Hamiltonian is given by

$$H = -H \sum_n (\mu_1 S_{n,1}^z + \mu_2 S_{n,2}^z) - J_1 \sum_n (S_{n,1}^x S_{n,2}^x + S_{n,1}^y S_{n,2}^y)$$

$$- J_2 \sum_n (S_{n,2}^x S_{n+1,1}^x + S_{n,2}^y S_{n+1,1}^y) - J_{13} \sum_n (S_{n,1}^x S_{n,2}^x S_{n+1,1}^x + S_{n,1}^y S_{n,2}^y S_{n+1,1}^y)$$

$$- J_{23} \sum_n (S_{n,2}^x S_{n+1,1}^x S_{n+1,2}^x + S_{n,2}^y S_{n+1,1}^y S_{n+1,2}^y)$$

$$- J_{14} \sum_n (S_{n,1}^x S_{n,2}^x S_{n+1,1}^x S_{n+1,2}^x + S_{n,1}^y S_{n,2}^y S_{n+1,1}^y S_{n+1,2}^y)$$

$$- J_{24} \sum_n (S_{n,2}^x S_{n+1,1}^x S_{n+1,2}^x S_{n+2,1}^x + S_{n,2}^y S_{n+1,1}^y S_{n+1,2}^y S_{n+2,1}^y).$$

(1)

Here, the indices 1 and 2 indicate two sub-lattices with NN spins occupying two different sub-lattices (the total number of cells will be denoted by $N$ in what follows, where we will assume $N$ to be odd) and $S_{n,1,2}^{x,y,z}$ are the spin-$\frac{1}{2}$ operators for sub-lattices 1 and 2 at the $n$th site. Also, $\mu_{1,2}$ are the Bohr magnetons in sub-lattices 1 and 2, $H$ is the external magnetic field along the $z$ axis, $J_{1,2}$ are the alternating coupling constants between NN spins, $J_{13,23}$ are the alternating coupling constants for three-spin interaction, and similarly, $J_{14,24}$ are the alternating coupling constants for four-spin interactions.

The large number of interaction parameters make a general analysis of the model cumbersome even if one chooses an overall scale, and we will make some simplifying choices. For ease of computation, we will in this paper, set $J_{24} = 0$, and it is further convenient to re-parametrise the Hamiltonian in terms of new couplings, $h$, $J$, and $J_3$ to get rid of some unwanted fractions, as : 

$$\mu_1 = 3\mu, \mu_2 = \mu, \mu H = h/2, J_1 = 2J, J_2 = -1, J_{13} = 5J, J_{23} = J_3, \text{ and } J_{14} = 4.$$ 

Then the Hamiltonian equation (1) in terms of these new couplings is

$$H = -\frac{h}{2} \sum_n (3S_{n,1}^z + S_{n,2}^z) - 2J \sum_n (S_{n,1}^x S_{n,2}^x + S_{n,1}^y S_{n,2}^y) + \sum_n (S_{n,2}^x S_{n+1,1}^x + S_{n,2}^y S_{n+1,1}^y).$$
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Figure 1. The phase diagram $h$ versus $J_3$ for the spin chain model of equation (2), with $J = 1$.

\[-J_3 \sum_n \left( 5S_{n,1}^x S_{n+1,1}^y S_{n+1,1}^x + 5S_{n,1}^y S_{n+1,1}^z S_{n+1,1}^y + S_{n,2}^x S_{n+1,2}^x S_{n+1,2}^y + S_{n,2}^y S_{n+1,2}^z S_{n+1,2}^y \right) \]
\[-4 \sum_n \left( S_{n,1}^x S_{n,2}^z S_{n+1,2}^x + S_{n,1}^y S_{n,2}^z S_{n+1,2}^y \right). \quad (2)\]

The Hamiltonian equation (2) can be diagonalised, and dispersion relations can be obtained by using the Jordan–Wigner, Fourier and Bogoliubov transformations detailed in appendix A, and after these transformations, we have

\[ \mathcal{H} = \sum_k \left( \mathcal{E}_{k,1} b_{k,1}^\dagger b_{k,1} + \mathcal{E}_{k,2} b_{k,2}^\dagger b_{k,2} \right) - Nh , \quad (3)\]

where we obtain the dispersion relation,

\[ \mathcal{E}_{k,1,2} = \left( h - \frac{3J_3}{2} \cos k \right) \mp \Lambda_k , \quad (4)\]

with $\Lambda_k = \sqrt{\left( \frac{h}{2} - J_3 \cos k \right)^2 + J^2 + \sin^2 k}$.

2.1. The ground state and phase diagram

Our model equation (2), inspite of various simplifications, has a rich phase diagram which we show in figure 1 (here, we have set $J = 1$). The ground state of any fermionic system corresponds to the situation where all possible states with positive energies are empty, and all states with negative energies are occupied. This ground state filling of two branches of equation (4) naturally depends on the values of the model parameters. Consider the non-negative values of the magnetic field parameter $h \geq 0$ (from here and onwards, the magnetic field is specified by $h$ instead of $H$). There are four critical values of magnetic field, denoted by $h_{1,2,3}$ and $h_{13}$, and the explicit expressions of $h_{1,2,3}$ can be evaluated by putting $\mathcal{E}_{k,1,2} = 0$ for $k = 0$ and $\pi$, while for $h_{13}$, we need to maximise the

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solution of $E_{k,1,2} = 0$ with respect to $k$. We find

$$h_{1,2} = \frac{1}{3} \left( \pm \sqrt{12J^2 + J_3^2 + 4J_3} \right),$$

$$h_3 = \frac{1}{3} \left( \sqrt{12J^2 + J_3^2 - 4J_3} \right),$$

and

$$h_{13} = \frac{\sqrt{24J^2 + J_3^2 \cos(2k_m) + J_3^2 - 12 \cos(2k_m) + 12}}{3\sqrt{2}} + \frac{4}{3}J_3 \cos(k_m),$$

where the expression of $k_m$ is lengthy, and is given in appendix B. Here, the critical line $h_{13}$ touches the line $h_1$ at $J_3 = 0.76$ for $J = 1$, and $h_{2,3}$ intersect the $J_3$-axis at $J_3 = (2\sqrt{J^2})/\sqrt{5}$. First, consider the case with a small magnetic field, $0 < h < h_3$, and values of three spin-exchange interactions, $0 < J_3 < 2/\sqrt{5}$. The first branch, $E_{k,1}$ is negative, and the second branch, $E_{k,2}$ is positive for any $k$. This domain is denoted by $I$ in the phase diagram and is referred to as the ferrimagnetic phase, as can be gleaned from the ground state magnetisation per cell in the thermodynamic limit given by [28, 29]

$$m_1^* = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{\partial E_{k,1}}{\partial h} \right) dk.$$  

In the region $h_{2,3} < h < h_1$, the first branch, $E_{k,1} < 0$ for $|k| < k_{c_1}$ and $E_{k,1} > 0$ for $|k| > k_{c_1}$, while the second branch, $E_{k,2} > 0$ for any $k \in (-\pi, \pi)$. This region is denoted as $II$ in the phase diagram, and is a ferrimagnetic phase with gapless excitations. Similarly, the region between $h_1 < h < h_{13}$ for $0 < J_3 < 0.76$ denoted by $III$ in the phase diagram is a ferrimagnetic phase in which the first branch $E_{k,1} < 0$, for $k \in (-k_{c_1}, -k_{c_2}) \cup (k_{c_2}, k_{c_1})$, while the second branch, $E_{k,2} > 0$ for any $k \in (-\pi, \pi)$ with

$$k_{c_{1,2}} = \cos^{-1} \left( \frac{4hJ_3}{5J_3^2 + 4} \pm \frac{\sqrt{h^2J_3^2 - 12h^2 + 20J^2J_3^2 + 16J^2 + 20J_3^2 + 16}}{5J_3^2 + 4} \right).$$

Consider now higher values of three spin-exchange interaction $J_3 > 2/\sqrt{5}$, and magnetic field in the range $0 < h < h_2$. Here, the first branch, $E_{k,1} < 0$ for $|k| < k_{c_1}$, and the second branch $E_{k,2} < 0$ for $|k| < k_{c_2}$. The system is in the ferrimagnetic state with gapless excitations and is denoted by $IV$ in the phase diagram. Finally, for $J_3 < 0.76$, both the branches are positive for $h > h_{13}$, while for $J_3 > 0.76$, both branches are positive for $h > h_1$. This region is denoted by $V$ in the phase diagram, and the system will show
ferromagnetic behaviour with gapped excitations. Note that the lines $h_{1,2,3,13}$ all indicate second order QPTs in the phase diagram of figure 1.

The ground state of the model of equation (2) varies with parameters $h$ and $J_3$, and they are classified with distinct ranges of $k$ values in different regions of the phase diagram. The ground state of phase I can be described as a vacuum state with the first branch fully occupied and the second branch empty, i.e. $b_{k,1}|\Psi\rangle_1 = 0$ and $b_{k,2}^\dagger|\Psi\rangle_1 = 0$, where $|\Psi\rangle_1 = \prod_k |\Psi_{k,1,2}\rangle_1$,

$$|\Psi_{k,1,2}\rangle_1 = -u_k |1\rangle_{k,1} |0\rangle_{k,2} + \frac{v_k (J - i \sin k)}{\sqrt{J^2 + \sin^2 k}} |0\rangle_{k,1} |1\rangle_{k,2},$$  \hspace{1cm} (9)

with $k \in (-\pi, \pi)$, $u_k = \cos(\frac{h}{2})$, $v_k = \sin(\frac{h}{2})$, and $\cos \theta_k = (\frac{h}{2} - J_3 \cos k)/\Lambda_k$. In region II of the phase diagram, $b_{k,1}|\Psi\rangle_II = 0$ for $|k| < k_{c_1}$ and $b_{k,2}^\dagger|\Psi\rangle_II = 0$ for $k \in (-\pi, \pi)$. This implies that the ground state in phase II is $|\Psi\rangle_II = \prod_k |\Psi_{k,1,2}\rangle_1$ for $|k| < k_{c_1}$ and is trivial for other values of $k$ in this region. Similarly, in the region III and IV of the phase diagram, the ground state is given by $|\Psi\rangle_{III,IV} = \prod_k |\Psi_{k,1,2}\rangle_1$, where $k \in (-k_{c_1}, -k_{c_2}) \cup (k_{c_2}, k_{c_3})$. Note that the limits of $k$ in the ground states of the regions III and IV are the same, but the values of $h$ and $J_3$ are different. The ground state of region V i.e. the ferromagnetic phase of the phase diagram is trivial since both the energy eigenvalue branches are positive for $k \in (-\pi, \pi)$, and is given by $|\Psi\rangle_V = \prod_k |1\rangle_{k,1} |1\rangle_{k,2}$.

By the same method, we also draw the phase diagram for the simpler case up to three spin interactions with parameters $\mu_1 = \mu_2 = 1$, $J_1 = 2$, $J_2 = 1$, $J_{13} = J_{23} = J_3 \geq 0$, and $J_{14} = J_{24} = 0$. The ground-state phase diagram is shown in figure 2, where F/AF denotes the ferromagnetic/antiferromagnetic phases, and 1 and 2 denote ferrimagnetic phases. The lines $H_{c_1,2}$ and $H_s$ are the lines of second-order QPT with critical values

$$H_{c_1} = \frac{1 - J_3}{2}, \hspace{0.5cm} H_{c_2} = \frac{J_3 - 3}{2}, \hspace{0.5cm} H_s = \frac{J_3 + 3}{2}. \hspace{1cm} (10)$$
Also, the line $H = 0, J_3 > 1$ is the line of first order QPT, i.e. the spontaneous magnetisation changes from zero to a finite value.

### 3. The QIM and complexity

The ground state wavefunction equation (9) is defined on the parameter space $h$ and $J_3$ for a fixed value of $J = 1$. The QIM, introduced in the context of quantum mechanics in [35] measures the ‘distance’ between two quantum states $|\Psi(\vec{\lambda})\rangle$ and $|\Psi(\vec{\lambda} + d\vec{\lambda})\rangle$, where $\lambda = \{h, J_3\}$ and $d\vec{\lambda}$ is the infinitesimal separation in parameter space. As shown in [35], the distance can be expressed as

$$
d\tau^2 = 1 - |\langle \Psi(\vec{\lambda}) | \Psi(\vec{\lambda} + d\vec{\lambda}) \rangle|^2 = g_{ij}d\lambda^i d\lambda^j + O(|d\vec{\lambda}|^3),
$$

where the QIM $g_{ij}$ is the real symmetric part of the more general structure—the quantum geometric tensor (QGT), denoted by $\chi_{ij}$, whose imaginary part is related to the Berry curvature. The expression of the QGT for the quantum state $|\Psi(\vec{\lambda})\rangle$ is

$$
\chi_{ij} = \langle \partial_i \Psi | \partial_j \Psi \rangle - \langle \partial_i \Psi | \Psi \rangle \langle \Psi | \partial_j \Psi \rangle,
$$

with $\partial_i \equiv \frac{\partial}{\partial \lambda^i}$, $i = 1, 2, \ldots, m$, and $m$ being the dimension of parameter the space, which in our case is 2. We now find the QIM in the regions of the ground state phase diagram.

In region I, the QIM takes the following form:

$$
g_{hh} = \frac{1}{4} \sum_k \left( \frac{\partial \theta_k}{\partial h} \right)^2, \quad g_{J_3 J_3} = \frac{1}{4} \sum_k \left( \frac{\partial \theta_k}{\partial J_3} \right)^2,
$$

$$
g_{h J_3} = g_{J_3 h} = \frac{1}{4} \sum_k \left( \frac{\partial \theta_k}{\partial h} \right) \left( \frac{\partial \theta_k}{\partial J_3} \right),
$$

where $k \in (-\pi, \pi)$ and $\theta_k$ is defined in section 2 after equation (9).

Analytic expressions for the metric components of equation (13) are cumbersome, and so is the expression for the resulting Ricci scalar (each of these typically result in pages of a standard Mathematic output, and are not very illuminating). These will therefore not be presented here for brevity. Instead, we compute the Ricci scalar, $R$ numerically, as shown in figure 3, for $J_3 = 0.5$ with the system sizes $N = 51$ (dot-dashed brown), $N = 101$ (dashed blue), and $N = 1001$ (solid red). Importantly, there is no divergence on the critical line $h_3 = 0.5$ obtained after fixing $J = 1$ and $J_3 = 0.5$.

In region II, the components of the QIM take the same form as given in equation (13), but here $k \in (-k_c, k_c)$. In figures 4(a) and (b) the oscillatory behaviour of $R$ as a function of $h$ is shown for $J_3 = 0.2$ and $J_3 = 2$, respectively with system size $N = 51$. These oscillations in $R$ are not particularly interesting, as these are due to the discontinuity in the metric tensor components, since $k$ is not summed over the entire range $(-\pi, \pi)$.
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Figure 3. Ricci scalar $R$ as a function of $h$ of region I, for $J_3 = 0.5$, with the system sizes $N = 51$ (dot-dashed brown), $N = 101$ (dashed blue), and $N = 1001$ (solid red).

in the metric tensor: the summation here runs for $k$ in the range $\left(-\frac{2\pi \lambda_{c_1}}{N}, \frac{2\pi \lambda_{c_1}}{N}\right)$ where $\lambda_{c_1} = \frac{N k_{c_1}}{2\pi}$. The integer values of $\lambda_{c_1}$ are different for distinct values of $h$ and $J_3$. For example, if $(J_3, h) = (2, 1.2)$, then $\lambda_{c_1} = 14.05$, and the summation runs over 0 to 14, while for $(J_3, h) = (2, 1.4)$, $\lambda_{c_1} = 13.45$, and summation runs over 0 to 13. This creates discontinuities in the nature of the metric tensor, which is reflected as oscillatory nature in the Ricci scalar plots. The interesting thing to observe here is the Ricci scalar plot in figure 4(b) is divergent on approaching the critical line $h_1 = 4$ (obtained for $J_3 = 2$ and $J = 1$), while there are no such divergences in figure 4(a) for the critical line $h_1 = 1.42$ (obtained for $J_3 = 0.2$ and $J = 1$). This difference in the plots of $R$ arises because the critical line $h_1$ separates the two ferrimagnetic phases for $J_3 < 0.76$ and the ferrimagnetic from the ferromagnetic phase for $J_3 > 0.76$. We remind the reader that with our choice of parameters, $J_3 = 0.76$ is the point where the critical lines $h_1$ and $h_{13}$ touch each other.

Similarly, in region III, the summation in QIM tensor in equation (13) runs over $(-k_{c_1}, -k_{c_2}) \cup (k_{c_2}, k_{c_1})$. In this region also, the Ricci scalar shows an apparently oscillatory nature, and has a clear divergence on approaching the critical line $h_{13}$. The behaviour of $R$ as a function of $h$ is shown in figure 5 for $J_3 = 0.2$, and $N = 101$. In the same way, we have computed $R$ in region IV. Here, it turns out that there is no divergence on the critical line $h_2$. Overall, the picture that emerges is that the geometry of the ground state of our model is regular everywhere except on the critical lines $h_{13}$, and $h_1$ for $J_3 > 0.76$. Hence, we conclude that the Ricci scalar can detect only those QPTs where the phase change occurs between the ferrimagnetic and ferromagnetic phases and not between two ferrimagnetic phases.
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Figure 4. Ricci scalar $R$ as a function of $h$ of region II with system size $N = 51$ for $J_3 = 0.2$ (a), $J_3 = 2$ (b). The vertical thick dashed black lines represent the critical line $h_1$ in the above figures.

Figure 5. Ricci scalar $R$ as a function of $h$ of region III, for $J_3 = 0.2$, with system size $N = 101$. The vertical thick dashed black line represents the critical line $h_{13} = 1.67$.

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3.1. The geodesics and FSC

The FSC, denoted by $C_{FS}$ hereafter, is defined as the geodesic distance on the parameter space of the quantum states on which the induced Riemannian metric is the QIM, i.e.

$$C_{FS} = \int_\mathcal{P} d\tau,$$

where $\mathcal{P}$ specifies the geodesic path between two points on the parameter space of the Hamiltonian and $d\tau$ is given by equation (11). As mentioned in [23], the behaviour of the FSC in spin models with NN interactions are generically non-analytic at QPTs, and its derivative diverges. Now we numerically compute the geodesics and FSC in the ground state phase diagram to observe how the expected behavior of these quantities modifies in the presence of multiple spin interactions. The method is standard and has been described in details for spin systems in [23]. Namely, to obtain the geodesics, we parametrise $h$ and $J_3$ as a function of an affine parameter $\tau$, and then we solve the two coupled second order differential equations given from

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^{i}_{jl} \frac{dx^j}{d\tau} \frac{dx^l}{d\tau} = 0,$$

where $x^i = \{h(\tau), J_3(\tau)\}$, and $\Gamma^{i}_{jl}$ are Christoffel symbols. We solve these two equations numerically in each region of the phase diagram by specifying the values of four initial conditions $h(\tau = 0)$, $J_3(\tau = 0)$, $\dot{h}(\tau = 0)$, $\dot{J}_3(\tau = 0)$, and one normalisation condition $g_{ij} \dot{x}^i \dot{x}^j = 1$, with dot implies a derivative with respect to $\tau$. This then determines $h(\tau)$ and $J_3(\tau)$. In regions I and IV, the geodesics do not show any special behaviour since the metric and the Ricci scalar are regular throughout these regions. We have inverted the solution of geodesics to find the FSC, i.e. $\tau(h)$, using a standard root finding procedure in Mathematica. These do not have any special behaviour up to the phase boundary $h_2,3$, reflecting that the ground state manifold is regular on the critical line $h_{2,3}$.

The behaviour of geodesics in region II is shown in figure 6. We have chosen two sets of initial conditions: One is $h(\tau = 0) = J_3(\tau = 0) = 1$, $J_3(\tau = 0) = 0.04$, and $\dot{h}(\tau = 0) = 6.78$ is determined from the normalisation condition for the geodesic plotted in red. And the second is: $h(\tau = 0) = J_3(\tau = 0) = 1$, $J_3(\tau = 0) = -4$, and $\dot{h}(\tau = 0) = 1.32$ for the geodesic plotted in blue in figure 6. These initial conditions are chosen by keeping in mind that the geodesic start from a particular region of the phase diagram. Then, we choose $J_3(\tau = 0)$, while $\dot{h}(\tau = 0)$ is obtained from the normalization condition to see geodesic evolution near the phase boundary. The critical line $h_1$ separates the two ferrimagnetic phases for $J_3 < 0.76$ and the ferrimagnetic from the ferromagnetic phase for $J_3 > 0.76$. Depending on the value of $J_3$, the geodesics may or may not show any interesting behaviour. For example, the geodesic (drawn in blue) approach to the phase boundary $h_1$ (for $J_3 < 0.76$) does not show any unusual behaviour, while the geodesic (drawn in red) keeps approaching the phase boundary line $h_1$ (for $J_3 > 0.76$), attains a minimum distance but never crosses it.

The FSC and its derivative in the region II are numerically evaluated by inverting the solution of the geodesics. The derivative of FSC, $\partial C_{FS}/\partial h$ (obtained by inverting
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Figure 6. Geodesics on $h - J_3$ plane in region II for system size $N = 51$ and $h(\tau = 0) = J_3(\tau = 0) = 1$. The solid red geodesic is plotted by choosing $J_3(\tau = 0) = 0.04$, $h(\tau = 0) = 6.78$, and the solid blue geodesic corresponds to $J_3(\tau = 0) = -4$, $h(\tau = 0) = 1.32$. The critical lines are shown in solid black.

Figure 7. The derivative $\partial C_{FS}/\partial h$ as a function of $h$ of region II, for system size $N = 51$ obtained by inverting the solution of geodesic plotted in red in figure 6.

the solution of the geodesic plotted in blue in figure 6) does not show any interesting behaviour. However, the derivative (obtained by inverting the geodesic solution plotted in red in figure 6) approaches zero when the distance between the geodesic and phase boundary line is minimal, as shown in figure 7. This can be explained by a simple relation that we have obtained previously in [23], namely,
\[
\frac{\partial \tau}{\partial h} = \frac{\partial C_{FS}}{\partial h} = (g_{hh})^\frac{1}{2}.
\]

Here, \( g_{hh} \to 0 \) at \( h \to h_1 \) for \( J_3 > 0.76 \) and \( J = 1 \) is fixed. The geodesics in region III have a similar kind of behaviour, i.e. it attains a minimum distance with the critical line \( h_{13} \). The derivative of FSC in region III, \( \frac{\partial C_{FS}}{\partial h} \to 0 \) at \( h \to h_{13} \).

The above analysis indicates how the geodesics and the FSC behave near the phase boundary lines of our model. Our results here should be contrasted with a similar study in the transverse XY model \([23]\). There, it was found that the derivative of the FSC diverges at the QPTs. On the other hand, here we do not find any such non-analytic behaviour of the FSC for our model with multiple spin interactions.

### 3.2. The NC

In quantum information theory, the circuit complexity is defined as the minimal number of universal gates needed to implement a unitary transformation \( U \), that acts on a reference state \( |\Psi_R\rangle \) to arrive at a target state \( |\Psi_T\rangle \). Nielsen and collaborators \([5, 6]\) proposed a geometric method to find this optimal circuit and showed that the minimal number of gates required in constructing the unitary transformation is equivalent to the geodesic distance between the identity operator and the unitary transformation. This is the NC, hereafter denoted by the symbol \( C_N \). In this geometric approach, the first step is to parametrize the unitary transformation as a path-ordered exponential of a time-dependent Hamiltonian \( \mathcal{H} \),

\[
U(\tau) = \hat{P} \exp \left( \int_0^\tau d\tau' \mathcal{H}(\tau') \right),
\]

with \( \mathcal{H}(\tau') = \sum_I Y^I(\tau') M_I \), \( Y^I(\tau') \) specifies a particular circuit in the unitary space, and \( M_I \) are the generators of the underlying symmetry group. The explicit form of \( Y^I(\tau) \) can be evaluated by using these generators and equation (17),

\[
Y^I(\tau) = -\text{Tr} \left[ (\partial_\tau U(\tau)) U^{-1}(\tau) M^T_I \right].
\]

where the superscript \( T \) denotes the transpose. We can define a cost functional for various paths (see, e.g. \([36]\)) to achieve the desired unitary operator \( U \) from identity as

\[
\mathcal{D}[U(\tau)] = \int_0^1 d\tau' \sum_I |Y^I(\tau')|^2.
\]

The minimal value of \( \mathcal{D}[U(\tau)] \) is the required NC, and is obtained by evaluating it on a geodesics in the space of unitaries, with boundary conditions \( U(\tau = 0) = 1 \), \( U(\tau = 1) = U \).

For a fixed value of \( J = 1 \), the ground state of our model can be expressed in the form:

\[
|\Psi\rangle = \prod_k \left[ -u_k |1\rangle_{k,1} |0\rangle_{k,2} + v_k e^{-i\phi_k} |0\rangle_{k,1} |1\rangle_{k,2} \right],
\]

\[\text{https://doi.org/10.1088/1742-5468/acd2e5}\]
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Figure 8. The derivative of NC with respect to $h^T$ versus $h^T$ for $J = 1$, $N = 101$, $J^R_3 = 0.1$, and $J^T_3 = 0.3$. Here, $h^R = 0.2, 1.05, 1.06$ in regions I, II, and III, respectively. The vertical thick dashed black lines mark the second-order QPTs.

where $e^{-i\phi_k} = (1 - i \sin k)/\sqrt{1 + \sin^2 k}$. By following [36], the NC of our model can be computed straightforwardly, i.e. $C_N = \sum_k |\Delta \theta_k|^2$, where $\Delta \theta_k = (\theta^T_k - \theta^R_k)/2$, and

$$\cos \theta^R_k = \frac{h^R - J^R \cos k}{\sqrt{(h^R - J^R \cos k)^2 + 1 + \sin^2 k}}. \quad (21)$$

Hereafter, the superscripts $R$ and $T$ on a variable are used to denote the reference and the target values respectively, of that variable. The derivative of NC with respect to $h^T$ is plotted in figure 8 as a function of $h^T$, with system size $N = 101$, $J^R_3 = 0.1$, and $J^T_3 = 0.3$. The values of $h^R$ are 0.2, 1.05, and 1.06 in regions I, II, and III, respectively. It can be seen that the derivative is discontinuous on the critical lines $h^T_3 = 0.76$ and $h^T_1 = 1.56$, and approaches zero on the critical line $h^T_{13} = 1.73$, in region III of the phase diagram. This behaviour can be explained using the relation [23]

$$\frac{\partial C_N}{\partial h^T} \sim \int \sum_k \left( \frac{\partial \theta^T_k}{\partial h^T} \right)^2 \, dh^T = \int g_{h^T} \, dh^T. \quad (22)$$

Since $g_{h^T} \rightarrow 0$ on $h^T \rightarrow h^T_{13}$, so the derivative of NC goes to zero on this critical line. A similar behaviour is observed as $h^T \rightarrow h^T_{1}$ for $J^T_3 > 0.76$ in region II.

Here, for the case of multiple spin interactions, we observe that interestingly, the derivative of the NC shows a discontinuity at the location of the QPTs in contrast to spin chains with NN interactions, where we encounter a divergence of the derivative of the NC at such phase transitions. The detailed analysis of the NC for spin chains with NN interactions, such as the transverse XY model, compass model, and compass model with the transverse magnetic field, was performed in [23], and for the Kitaev model in [36]. Contrary to what we obtain here, in all those cases, the behaviour of the NC at the QPTs were found to be non-analytic, i.e. its derivative diverges in all these models. As we have mentioned before, the behaviour here can be traced back to the different

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mechanism of the QPTs—namely that the phase transitions here do not correspond to the closing of an energy gap.

4. The time-dependent NC and Loschmidt Echo

We now consider the model equation (2) with a quantum quench, assuming the sudden quench approximation. Following the work of [37], in our quench protocol, we prepare the model equation (2) in the ground state, and at time $t=0$, we suddenly quench the external magnetic field parameter $h$ to $h + \delta$ by coupling our model to a two-level central spin-1/2 system. We then time evolve this system with the total Hamiltonian $H_F = H + H_I$, with $H$ given in equation (2), and the interaction Hamiltonian $H_I$ is

$$H_I = -\frac{\delta}{2}\langle e\rangle\sum_n (3S_{n,1}^z + S_{n,2}^z) ,$$

where $\delta$ is the interaction strength, and $|e\rangle$ is the excited state of the two-level system. The motivation behind this particular quench is to explore the dynamic relationship between the LE and NC in the long time limit, and to study the dynamics of the NC close to the critical lines $h_{1,2,3,13}$. By a standard method, we write the ground state $|\Psi\rangle_{h,J_3}$ of $H$ in terms of that of $H_F$, labeled $|\Psi_{k,1,2}\rangle_{h+\delta,J_3}$ for the $k$th Fourier mode. This gives

$$|\Psi\rangle_{h,J_3} = \prod_k \left[ \cos \Omega_k + \sin \Omega_k \chi_{k,1} \chi_{k,2} \right] |\Psi_{k,1,2}\rangle_{h+\delta,J_3} ,$$

where we have defined

$$\begin{align*}
\Omega_k &= \frac{1}{2} \left[ \theta_k(h,J_3) - \theta_k(h+\delta,J_3) \right], \\
\chi_{k,1} &= e^{-i\phi_k u_k(h+\delta,J_3)} d_{k,1} + u_k(h+\delta,J_3) d_{k,2}, \\
\chi_{k,2} &= -e^{-i\phi_k u_k(h+\delta,J_3)} d_{k,1} + v_k(h+\delta,J_3) d_{k,2} ,
\end{align*}$$

with the operators $d_{k,1}$ and $d_{k,2}$ being the Fourier operators, and $\theta_k$, $u_k$, and $v_k$ are defined in equation (A5) with the arguments appropriately shifted.

To compute the complexity, the reference and the target states are chosen to be $|\Psi\rangle_{h,J_3}$ and $|\Psi_e(t)\rangle$ respectively, where, $|\Psi_e(t)\rangle = e^{-iH_F(h+\delta,J_3)t}|\Psi\rangle_{h,J_1}$, i.e.

$$|\Psi_e(t)\rangle = \prod_k \left[ e^{-iE_{k,2}(h+\delta,J_3)t} \cos(\Omega_k) + e^{-iE_{k,1}(h+\delta,J_3)t} \sin(\Omega_k) \chi_{k,1} \chi_{k,2} \right] |\Psi_{k,1,2}\rangle_{h+\delta,J_3} .$$

The computation of the NC is now standard (see, e.g. [36]). The final result after the quench is

$$C_N(t) \equiv \sum_k C_{N_k} = \sum_k \Phi_k^2(h+\delta,J_3,t) ,$$

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Figure 9. Comparison between time-dependent NC (solid lines) and LE (scattered plots). The red (black squares), green (purple triangles), and blue (brown stars) correspond to $h = 0.5$, 1, and 1.4, respectively, for fixed values of $N = 101$, $\delta = 0.1$, $J_3 = 0.2$, and $J = 1$.

where

$$\Phi_k = \arccos \left( \sqrt{1 - \sin^2(2\Omega_k)\sin^2(\Lambda_k(h + \delta, J_3)t)} \right),$$

with the single particle excitations $\Lambda_k(h + \delta, J_3) = \sqrt{(h+\delta - J_3 \cos k)^2 + 1 + \sin^2 k}$. A particularly interesting quantity in a system after quench is the Loschmidt Echo. The system is initially prepared in the ground state $|\Psi\rangle_{h,J_3}$ of the Hamiltonian $\mathcal{H}$, and it is then quenched to evolve according to the Hamiltonian $\mathcal{H}_F$. The LE is defined as

$$\mathcal{L} = |\langle h, J_3 | e^{-i\mathcal{H}_F(h+\delta, J_3)t} | h, J_3 \rangle|^2,$$

$$= \prod_k \left[ 1 - \sin^2(2\Omega_k)\sin^2(\Lambda_k(h + \delta, J_3)t) \right].$$

(29)

As discussed in [24], the relation between LE and NC for small times, $\mathcal{L} \simeq e^{-CN}$ is also valid for our spin chain model. In fact, figure 9 shows that this relation continues to hold for large times as well. We have plotted the dynamical behavior of the time-dependent NC per system size, $C_N(t)/N$ along with negative of logarithm of LE per system size, $-\log(\mathcal{L})/N$ for $N = 101$, $\delta = 0.1$, $J_3 = 0.2$, and $J = 1$. The solid red (scattered black squares), solid green (scattered purple triangles), and solid blue (scattered brown stars) shows the numerical results for $C_N(t)/N$ ($-\log(\mathcal{L})/N$) corresponding to $h = 0.5$, 1, and 1.4, for regions I, II, and III of the phase diagrams respectively.

As anticipated, the time-dependent NC first increases linearly and then starts oscillating [24, 36]. However, in our model, the oscillations do not die out to approach
a time-independent value, contrary to what is observed in those works. This oscillatory behavior is observed here in all the regions of the phase diagram for any finite or large times and is well explained by the results in [24]. That is, the time-dependent NC and LE are oscillatory functions of \( k \) due to the nature of \( \Omega_k \) and \( \Lambda_k \). The oscillations die out rapidly in the cases considered in [24, 36] because a large number of Fourier modes start contributing with the increase in time, and they interfere destructively. But in our four-spin interaction model, the Fourier modes are asymmetrical in \( k \)-space. As a result, even if there are a large number of maxima and minima interfering with one another with increase in time, some parts of the contributing modes still cannot interfere in a completely destructive manner. Hence, temporal oscillations are always present in the system. Secondly, the relation \( \mathcal{L} = e^{-\mathcal{C}_N} \) is valid for all times and in all the regions of the phase diagram. This is because the term \( \sin^2(2\Omega_k) \ll 1 \) everywhere in the phase diagram and even on the critical lines \( h_{1,2,3,13} \), unlike in [24], where the relation is found to be invalid for the critical lines or when the quench protocol is on the critical lines.

Our second motivation for the above quenched magnetic field protocol was to analyze the role of time-dependent NC and LE to identify the critical lines. For our model of equation (2), we find that there is no non-analyticity or discontinuity in these quantities whenever the phase change occurs contrary to what we observed in [24, 37]. This reflects the fact that, like the NC, these two dynamical quantities cannot signal any particular characteristics of the critical behaviour of the model.

5. Multiple quench scenarios

In this section, we consider multiple quench scenarios [38, 39] in our model, extending the analysis of section 4. We will consider here a series of four sudden quenches in the model. The multiple quench protocol is as follows. At time \( t = 0 \), the external magnetic field \( h \) is quenched from \( h_0 \) to \( h_0 + \delta \) for a time \( t_1 = 15 \) s, where \( \delta \) is the interaction strength. At time \( t_1 \), the external magnetic field after quench i.e. \( h_0 + \delta \) is changed back to \( h_0 \) for a time \( t_2 = 15 \) s, which is defined as no quench in the system. This sequence has been repeated four times. The multiple quench protocol involves the interaction Hamiltonian to be \( \mathcal{H}_I^{\text{multi}} \), so that similar to equation (23), the total Hamiltonian is \( \mathcal{H}_F^{\text{multi}} = \mathcal{H} + \mathcal{H}_I^{\text{multi}} \), with

\[
\mathcal{H}_I^{\text{multi}} = -\frac{\delta}{2} |e\rangle\langle e| \sum_n \left( 3S_{z,n,1} + S_{z,n,2}^z \right) F(t),
\]

(30)

where we define

\[
F(t) = \begin{cases} 
\Theta \left( \sin \left( \frac{\pi t}{T} \right) \right), & 0 < t < 90 \\
1, & t > 90
\end{cases}
\]

Here, the quench time is taken as \( T = 15 \) s, \( \Theta \) is the Heaviside function, and \( |e\rangle \) is the excited state of the two-level system as discussed in the single quench scenario. The ground state of the system and the time-dependent NC after time \( t = T \) have
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Figure 10. The NC as a function of time is plotted for multiple (four) sudden quenches with system size $N = 501$, $J_3 = 1.5$, and $T = 15$ s. The blue, orange and green curves corresponds to $h_0 = 1, 1.3$, and $\delta = -0.4, -0.2, 0.2$, respectively.

been discussed in details in the single quench case. The ground state after time $t = 2T$, i.e. when $h_0 + \delta$ is changed back to $h_0$ is

$$|\Psi_e(t_1,t_2)\rangle = \prod_k \left[ e^{-i\xi_{k,2}(h,J_3)t_2}Y_k + e^{-i\xi_{k,1}(h,J_3)t_2}W_k R_{k,1}^\dagger R_{k,2} \right]|\Psi_{k,1,2}(h,J_3)\rangle,$$

where we have defined

$$Y_k = e^{-i\xi_{k,2}(h+\delta,J_3)t_1}\cos^2 \Omega_k + e^{-i\xi_{k,1}(h+\delta,J_3)t_1}\sin^2 \Omega_k,$$

$$W_k = \left( -e^{-i\xi_{k,2}(h+\delta,J_3)t_1} + e^{-i\xi_{k,1}(h+\delta,J_3)t_1} \right) \sin \Omega_k \cos \Omega_k,$$

$$R_{k,1} = e^{-i\theta_k}v_k(h,J_3)d_{k,1} + u_k(h,J_3)d_{k,2},$$

$$R_{k,2} = -e^{-i\theta_k}u_k(h,J_3)d_{k,1} + v_k(h,J_3)d_{k,2},$$

with the operators $d_{k,1}$ and $d_{k,2}$ being the Fourier operators, and $\theta_k$, $u_k$, and $v_k$ are defined earlier with the arguments appropriately shifted.

To compute the complexity, the reference and the target states are chosen to be $|\Psi_{h,J_3}\rangle$ and $|\Psi_e(t_1,t_2)\rangle$ respectively. The final result of NC after $t = 2T$ is

$$C_N(t_1,t_2) \equiv \sum_k C_{Nk}(t_1,t_2) = \sum_k \left( \arccos \left( \sqrt{Y_k^\dagger Y_k} \right) \right)^2.$$

The second and subsequent quenches become difficult to handle analytically, and we have performed numerical analysis for the same. The behaviour of the NC with time after four quenches is shown in figure 10, for system size $N = 501$, $J_3 = 1.5$, and $T = 15$ s. The time evolution of the NC exhibits an oscillating behaviour whenever the system is quenched, and the magnitude of oscillations decreases with time. Also, the magnitude of NC is found to be constant whenever $h_0 + \delta$ is changed back to $h_0$. In figure 10,
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Figure 11. The NC as a function of time is plotted for multiple (four) sudden quenches for $J_3 = 1.5$, and $T = 15$ s with $h_0 = 1$, and $\delta = -0.2$. The blue and orange lines correspond to system sizes $N = 501$ and 1001, respectively. The blue curve corresponds to $h_0 = 1$ and $\delta = -0.4$; the orange curve corresponds to $h_0 = 1$ and $\delta = -0.2$, and the green curve corresponds to $h_0 = 3$ and $\delta = 0.2$. Note that the green curve is plotted by choosing parameter values such that the system comes on the critical line $h_1$ after the quench. As concluded in the single quench case, the NC in multiple quench scenarios also does not indicate any critical behaviour of the system. In figure 11, we plot the NC at large times for $h_0 = 1$, and $\delta = -0.2$, for system size $N = 501$ (blue) and $N = 1001$ (orange). We observe here that for late times and for large system sizes, in our four spin interaction model, oscillations are regular, and their maximum amplitude decreases initially with time and later acquires nearly a constant value. These results are similar to what was observed in [39, 40] for integrable models.

We have also performed the same analysis of multiple quenches in the transverse spin-1/2 XY model, where the quench parameters are the anisotropic coupling and transverse magnetic field. When we quench the value of the magnetic field, the behaviour of the NC is similar to the one we encountered before—like in the four-spin exchange interaction model, the NC here first increases linearly and then starts oscillating over time. The oscillations are more rapid when the system is quenched on one of the critical lines. However, no other signal of QPT is observed from the complexity versus time curve, even for the transverse spin-1/2 XY model.

6. The DMRG and EE

We now compute the EE of the system using DMRG methods, which we first recall. The DMRG is a variational method that uses the matrix product states (MPS) and matrix product operator (MPO) representations to find the lowest energy state of a quantum many-body system. An MPS represents a quantum state with a product of matrices. The Hilbert space of the one-dimensional spin-1/2 chain has $2^N$ degrees of freedom for $N$ spins. The MPS can be manipulated to form a subspace of the larger $2^N$ dimensional Hilbert space which captures the essential physics. The search for the ground state using
DMRG is an iterative solution of the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$. In the first step, the Hamiltonian of the system has to be constructed as an MPO. To solve the Schrödinger equation for $|\psi\rangle$, we start with a guess of the MPS. The DMRG algorithm minimises the ground state energy in the space of the MPS, where matrix elements are treated as variational parameters. The minimisation is performed by a sweeping procedure where the matrix state at a site is optimised at one time, keeping all others fixed, then optimising the next matrix, and so on. When all matrices are optimised, we again sweep back and forth until convergence is achieved. The convergence is guaranteed as the energy goes down at each iteration step.

The DMRG uses the entanglement properties of a bipartite system to produce a state with minimal energy. In order to quantify the entanglement or quantum information, the most common measure is the Von Neumann EE denoted by $S$. It is defined in a standard fashion, by bipartition of a composite system into two subsystems, $A$ and $B$,

$$S = -Tr[\rho_A \log(\rho_A)] = -Tr[\rho_B \log(\rho_B)] ,$$

where $\rho_A$ and $\rho_B$ are the reduced density matrices of subsystems $A$ and $B$ respectively.

### 6.1. EE with three-spin interactions

For the model with three-spin interactions, we numerically calculate the ground state EE in the simplest case $\mu_1 = \mu_2 = 1$, $J_{13} = J_{23} = J_3$, and $J_{14} = J_{24} = 0$ using DMRG method. We split the spin chain of $N$ cells at the centre into two subsystems. Using singular value decomposition, the MPS is divided into left and right bipartition. The DMRG calculations are performed using the library [41] on $N = 51$ cells with maximum bond dimension $\chi = 300$. Consequently, the total number of sites in the spin chain is $2N = 102$, i.e. the left and right bipartitions each include 51 sites. The numerical results of EE as a function of $J_3$ is plotted in figure 12, for $H = 0$, $J_1 = 1.2$, and $J_2 = 0.8$. When $H = 0$, there are two QPTs. One occurs at $J_3 > (J_1 - J_2) = 0.4$, which is the onset of the first order QPT. Another occurs at $J_3 = J_1 + J_2 = 2$, when the line of second-order QPT, i.e. $H_{c2}$ intersects the first order QPT line ($J_3 > 0.4, H = 0$). From figure 12, we see that the EE shows a sharp jump whenever the critical point is approached, irrespective of the order of the QPT.

### 6.2. EE with four-spin interactions

Similar to the case of three-spin interactions, we have also plotted the numerical results of EE of our model equation (2) as a function of $J_3$ in figure 13. The blue and red curves in figure 13 correspond to $h = 0.25$ and $h = 2.5$, respectively, for a fixed value of $J = 1$. When $h = 0.25$, there are two critical points at $J_3 = 0.69$ and $J_3 = 1.09$. When $h = 2.5$, the critical point is observed at $J_3 = 0.97$. The EE clearly shows sharp jumps to indicate these QPTs, which are of second order as mentioned in section 2.1 (see discussion after equation (8)).
In the absence of four-spin interactions, the Von Neumann EE $S$ as a function of $J_3$ for $H = 0$, $J_1 = 1.2$, $J_2 = 0.8$, $J_{13} = J_{23} = J_3$, and $J_{14} = J_{24} = 0$. The vertical thick dashed black lines at $J_3 = 0.4$ and 2.0 represent the first and second-order QPT lines, respectively.

The von Neumann EE $S$ as a function of $J_3$ for the Hamiltonian of equation (2). The blue and red curves correspond to $h = 0.25$ and $h = 2.5$ respectively, for fixed $J = 1$. All the vertical thick dashed black lines mark the second-order QPTs.

We conclude that for both the first and second-order QPTs, the EE shows a discontinuous nature when the parameters takes their critical values. For the first order QPT, the result is in agreement with [42], in which the first order QPT is associated with the discontinuity in the density matrix elements. However, the discontinuity in EE for the second-order QPT in our case is contrary to what is observed in [42]. The latter suggests that the discontinuity or divergence should be present in the first derivative of the EE for second-order QPTs. On the contrary, what we find here is a discontinuity...
in EE itself. The reason for this result is that our model behaves differently from the other known exactly solvable spin chains.

As we have mentioned before, there are two branches of the energy spectrum, $E_{k,1,2}$. Depending on the magnetic field $h$ and the three spin interaction parameter $J_3$, these branches completely or partially lie above or below the zero level, resulting in different magnetisations $m^z$. The different phases in the ground state are thus determined by the values of $m^z = 0/1$ (antiferromagnetic/ferromagnetic) and $0 < m^z < 1$ (ferrimagnetic). Hence all the phase diagram characteristics of the model are determined by the signs of $E_{k,1,2}$, which can be positive, negative, or zero. This behaviour is different from spin models with NN interactions, in which QPTs usually occur whenever the energy gap between the branches of energy spectra vanishes.

For those models, the discontinuity is present in the derivative of EE for second-order QPTs, as concluded in [42]. But in our case, for different phases in the phase diagram, the ground state wavefunction $|\Psi\rangle$ takes distinct values, as the summation over the Fourier modes $k$ runs over specific ranges. The reduced density operator $\rho_{A,B}$ in equation (34) is the partial trace of the outer product of the ground state wavefunction $|\Psi\rangle$. So it becomes discontinuous whenever the phase change occurs, irrespective of the order of QPTs. The DMRG, a numerical technique for finding the ground state energy and wavefunction, confirms our analytical results of section 2 by showing the discontinuous nature of EE for both the first and second-order QPTs in figures 12 and 13.

7. Conclusions and discussions

In this paper, we have carried out a comprehensive analysis of information theoretic geometry in a spin model with three and four spin interactions along with the NN coupling. In particular, we computed the QIM and the related FSC, the NC in static as well as quench scenarios and the EE in these models. The motivation and importance of this study lies in the fact that the nature of phase transitions here are very different from the usual ones in models with NN interactions, such as the transverse field XY model, and it is important and interesting to quantify information theoretic measures and contrast these with the ones arising out of NN interactions.

Here, we find a number of distinctive features, which are very different from ones in models with NN interactions. The derivative of the NC does not diverge across a continuous phase transition, and the FSC also does not show any special behaviour across two ferrimagnetic phases. The EE shows a discontinuity across both first and second order phase transitions. These are in sharp contrast to the ones reported in the literature, in models with only NN interactions. Our analysis points to the fact that several features of information geometry that are valid in NN interaction models cease to be valid when three and four spin interactions are included.

Appendix A

In this appendix, we list the transformations to diagonalise the Hamiltonian of equation (2).
(i) Jordan–Wigner transformation

\[
\begin{align*}
S_{n,1}^z &= \frac{1}{2} \sigma_{n,1}^z - \frac{1}{2} a_{n,1}^\dagger a_{n,1}, \\
S_{n,2}^z &= \frac{1}{2} \sigma_{n,2}^z - \frac{1}{2} a_{n,2}^\dagger a_{n,2}, \\
S_{n,1}^+ &= S_{n,1}^z + i S_{n,1}^y = \prod_{m<n} \sigma_{m,1}^z \sigma_{m,2}^z a_{n,1}, \\
S_{n,1}^- &= S_{n,1}^z - i S_{n,1}^y = \prod_{m<n} \sigma_{m,1}^z \sigma_{m,2}^z, \\
S_{n,2}^+ &= \prod_{m<n} \sigma_{m,1}^z \sigma_{m,2}^z \sigma_{n,1}^z a_{n,2}, \\
S_{n,2}^- &= a_{n,2}^\dagger \prod_{m<n} \sigma_{m,1}^z \sigma_{m,2}^z \sigma_{n,1}^z ,
\end{align*}
\]  

(A1)

where \(a_{n,1,2}^\dagger\) and \(a_{n,1,2}\) are creation and annihilation operators which satisfy usual anti-commutation relations. After the Jordan–Wigner transformation, we obtain

\[
\begin{align*}
\mathcal{H} &= -\frac{\hbar}{2} \sum_n \left( 2 - 3a_{n,1}^\dagger a_{n,1} - a_{n,2}^\dagger a_{n,2} \right) - J \sum_n \left( a_{n,2}^\dagger a_{n,1} + a_{n,1}^\dagger a_{n,2} \right) \\
&\quad + \frac{1}{2} \sum_n \left( a_{n+1,1}^\dagger a_{n,2} + a_{n,2}^\dagger a_{n+1,1} \right) \\
&\quad - \frac{J_3}{4} \sum_n \left( 5a_{n+1,1}^\dagger a_{n,1} + 5a_{n,1}^\dagger a_{n+1,1} + a_{n+1,2}^\dagger a_{n,2} + a_{n,2}^\dagger a_{n+1,2} \right) \\
&\quad - \frac{1}{2} \sum_n \left( a_{n+1,2}^\dagger a_{n,1} + a_{n,1}^\dagger a_{n+1,2} \right). 
\end{align*}
\]  

(A2)

(ii) Fourier transformation

\[
\begin{align*}
a_{n,1} &= \frac{1}{\sqrt{N}} \sum_k d_{k,1} e^{i k n}, \\
a_{n,1}^\dagger &= \frac{1}{\sqrt{N}} \sum_k d_{k,1}^\dagger e^{-i k n},
\end{align*}
\]  

(A3)

and similarly for \(a_{n,2}^\dagger\) and \(a_{n,2}\). The Hamiltonian obtained after performing the Fourier transform is given by

\[
\begin{align*}
\mathcal{H} &= \sum_k \left[ \left( \frac{3\hbar}{2} - \frac{5J_3}{2} \cos k \right) d_{k,1}^\dagger d_{k,1} + \left( \frac{\hbar}{2} - \frac{J_3}{2} \cos k \right) d_{k,2}^\dagger d_{k,2} - (J + i \sin k) d_{k,1}^\dagger d_{k,2} \\
&\quad - (J - i \sin k) d_{k,2}^\dagger d_{k,1} - N \hbar \right].
\end{align*}
\]  

(A4)
where with imposition of periodic boundary conditions, the quasi-momentum $k$ takes the values

$$k = \frac{2\pi \lambda}{N}, \lambda = \frac{N-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{N-1}{2}.$$

(iii) **Bogoliubov transformation**

$$d_{k,1} = \frac{(J + i \sin k)}{\sqrt{J^2 + \sin^2 k}} \left( v_k b_{k,1} - u_k b_{k,2} \right),$$

$$d_{k,2} = u_k b_{k,1} + v_k b_{k,2},$$

$$d^{\dagger}_{k,1} = \frac{(J - i \sin k)}{\sqrt{J^2 + \sin^2 k}} \left( v_k b^{\dagger}_{k,1} - u_k b^{\dagger}_{k,2} \right),$$

$$d^{\dagger}_{k,2} = u_k b^{\dagger}_{k,1} + v_k b^{\dagger}_{k,2}. \quad (A5)$$

The coefficients $u_k$ and $v_k$ is defined in section 2.

**Appendix B**

Here, we will provide the value of $k_m$ for which the critical line $h_{13}$ has been found out in equation (6). This reads

$$k_m = -2 \tan^{-1} \left( \sqrt{\frac{a+b}{c}} \right), \quad (B1)$$

where we define

$$a = -64J^2 J^2_3 + 5J^4_3 - 120J^2_3 - 48$$

$$b = 16 \sqrt{(J^2 + 1)(56J^4_3 + 48J^2_3 - 5J^2_3)}$$

$$c = 64J^2 J^2_3 + 5J^4_3 + 8J^2_3 - 48. \quad (B2)$$

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