Non-Markovian Dynamics of Discrete-Time Quantum Walks

Subhashish Banerjee,1 N. Pradeep Kumar,1 R. Srikanth,2 Vinayak Jagadish,3 and Francesco Petruccione3

1Indian Institute of Technology Jodhpur, Jodhpur 342011, India
2Poornaprajna Institute of Scientific Research, Bangalore 560 080, India
3Quantum Research Group, School of Chemistry and Physics, University of KwaZulu-Natal, Durban 4001, South Africa, and National Institute for Theoretical Physics (NITheP), KwaZulu-Natal, South Africa

In the case of the discrete time coined quantum walk the reduced dynamics of the coin shows non-Markovian recurrence features due to information back-flow from the position degree of freedom. Here we study how this non-Markovian behavior is modified in the presence of open system dynamics. In the process, we obtain useful insights into the nature of non-Markovian physics. In particular, we show that in the case of (non-Markovian) random telegraph noise (RTN), a further discernible recurrence feature is present in the dynamics. Moreover, this feature is correlated with the localization of the walker. On the other hand, no additional recurrence feature appears for other non-Markovian types of noise (Ornstein-Uhlenbeck and Power Law noise). We propose a power spectral method for comparing the relative strengths of the non-Markovian component due to the external noise and that due to the internal position degree of freedom.

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Introduction — Discrete-time quantum Walk (DTQW) is a quantum analogue of “classical random walk” (CRW) and describes the evolution of a quantum particle on a given topological structure. The simplest instance of DTQW is that of a quantum system with two levels translating on a one dimensional discrete position space \( \mathbb{Z} \), a topology which we use in this work. Any practical implementation of quantum walk demands taking into account the effects of the ambient environment, resulting in the phenomena of decoherence and dissipation.

In the Markovian regime, the environmental time scale is much smaller than the system time scale \( \mathcal{O} \), and the back-action by system on the environment, in terms of generating system-environment entanglement, is negligible. In contrast, non-Markovian noise features back-action, and furthermore also “back-flow” of information from the environment to the system, which can show up as a recurrence in the correlations between two initial system states. A particular manifestation of recurrence is a resonance like phenomena, and is responsible for the Anderson localization observed in quantum walks.

With the advancement of technologies, one is now able to go beyond Markovian phenomena and enter into the non-Markovian regime, which we undertake in this work. Unlike previous approaches to non-Markovianity, our approach will distinguish between different sources of non-Markovianity, in particular, non-Markovian back-flow. As a concrete application of our approach, we study coined discrete-time quantum walk (DTQW) on a line, subjected to (non-)Markovian dynamics. This is especially interesting because the reduced dynamics of the coin manifests non-Markovian behavior due to the “endemic” source given by the position degree of freedom. Our method will be able to disambiguate the non-Markovianity of such an endemic origin versus one due to environmental decoherence. Here we make use of a local dephasing non-Markovian noise model [9] modelled on the random telegraph noise (RTN) process [10] as well as the modified Ornstein-Uhlenbeck (OU) [11, 12] and the power law noise (PLN) [13].

Localization, which may be considered as an aspect of non-Markovian backflow, was observed in [14], in the context of a one dimensional continuous time quantum walk, under the influence of RTN noise, in the presence of disorder. This behavior is also observed here, in the context of DTQW on a line, under the influence of RTN noise, which we report elsewhere [17]. Also observed are the revival of quantum correlations in the transition from quantum to classical random walks, under the considered non-Markovian noise.

The essential ingredients of DTQW are the coin and the position [15], which describes the internal and external degrees of freedom of the particle, respectively. The state of total system is described by the Hilbert space \( \mathcal{H}_w = \mathcal{H}_c \otimes \mathcal{H}_p \) where span \( \mathcal{H}_c = \{ \langle 0 \rangle, \langle 1 \rangle \} \) and span \( \mathcal{H}_p = \{ \langle i \rangle \} \), \( i \in \mathbb{Z} \) representing the number of lattice sites available to the walker. To implement the DTQW, we initialize a quantum state \( \rho \) and evolve it using the coin and conditional shift (i.e., translation in a spatial dimension) operators. The coin operator is usually a two dimensional rotation matrix. The shift operator \( \hat{S} \) that translates the particle to either left or right is conditioned on the outcome of the coin operator. The general form of the shift operator is given as

\[
\hat{S} = |0\rangle \langle 0| \otimes \sum_{i \in \mathbb{Z}} |i\rangle - 1\rangle \langle i| + |1\rangle \langle 1| \otimes \sum_{i \in \mathbb{Z}} |i + 1\rangle \langle i|.
\]

The non-Markovian Noise — We will denote by \( \Omega(t) \) the random variable describing the noise fluctuation in each of the three cases and by \( M \) the mean. The autocorre-
In RTN, \( \gamma = 1/2\tau \), \( \tau \) being the time scale in which the RTN noise changes its phase. The function \( \Lambda(\nu) \) corresponds to two regimes; the purely damping regime, with Markovian behavior, where \( 2a/\gamma < 1 \), and damped oscillations, with non-Markovian behaviour, for \( 2a/\gamma > 1 \), \( a \) having the significance of the strength of the system-environment coupling. The regime of “minimal non-Markovian” corresponds to \( 2a/\gamma = 1 \), for which \( \mu = 0 \). For the modified OU noise, henceforth referred to as OUN, \( \gamma^{-1} = \tau_c \), where \( \tau_c \) is the finite correlation time of the environment (Table I). In Table I, we also indicate properties of power-law noise (PLN), under which the DTQW behaves similar to the OUN [7].

A key feature of Markovian (memoryless) open-system dynamics \( \Lambda \) is that given two distinct states \( \rho \) and \( \sigma \), distance measures \( \Delta \) (such as relative entropy or trace distance) satisfy \( \mathcal{D}[\Lambda(\rho), \Lambda(\sigma)] \leq \mathcal{D}[\rho, \sigma] \), while correlation measures \( \mathcal{C} \) (such as fidelity or mutual information) satisfy \( \mathcal{C}[\Lambda(\rho), \Lambda(\sigma)] \geq \mathcal{C}[\rho, \sigma] \). By contrast, non-Markovian dynamics can violate the above monotonicity property. In this work, we will use trace-distance (TD) based indicators, and correlate it with other features, such as oscillations in walk variance. Non-Markovianity has been also been studied using fidelity [16], relative entropy [17] and mutual information [17].

Anderson Localization — In his seminal work on transport properties of particles in a random media, Anderson showed that the systems with quenched disorder exhibits the phenomena of localization [18]. In the quantum walk scenario, Anderson Localization (AL) has been studied extensively in which the disorder is introduced either via broken links in the lattice [14] or by randomizing the coin operation [19]. AL can be interpreted as a memory property of the particle, as it remembers and localizes near its initial position when it is coupled to a disordered system. Here we show that the AL phase can be observed in the non-Markovian (memory) regime of RTN.

Figures (a) and (b) displays the probability distribution and the corresponding variance as a function of the noise amplitude \( a \) of the quantum walk under different noise regimes. In the absence of noise the variance of quantum walk shows the typical bimodal distribution, figure (a) and the variance evolves quadratically (ballistic transport : \( \text{var} \propto t^2 \)). Allowing for the interaction between RTN and coin in the Markovian regime (\( \gamma = 5.0 \)) the variance decays monotonically to the classical random walk limit (diffusive transport: \( \text{var} \propto t \)), this is shown as green dashed line in Figure (b). The non-Markovian regime of RTN leads to interesting features in the quantum walk dynamics. By tuning the correlation time and amplitude of RTN (\( \gamma = 0.001, a = 1.0 \)) we can observe AL phase, this is shown as solid line in Figure (b). In addition to AL, simply by tuning the noise amplitude \( a \) of RTN we observe that the quantum walk alternates between three different phases namely, the ballistic, diffusive and localization. In contrast to RTN, AL is absent in both OUN and PLN.

Distinguishing non-Markovian features of noise using Trace distance — Trace Distance (TD) [20] is a measure of distinguishability between two states, defined as \( D(\rho_1, \rho_2) = \frac{1}{2} Tr[\rho_1 - \rho_2] \), where \( \|A\| \) is the operator norm given by \( \sqrt{A^\dagger A} \). For non-Markovian processes, owing to the backflow of information from the environment to the system, there could be an increase in the distinguishability, causing a deviation from the monotonic decrease of \( D(\Phi(t)[\rho_1], \Phi(t)[\rho_2(t)]) \), where \( \Phi(t) \) is the noise superoperator. This idea has been exploited in an effort to witness non-Markovianity in [21]. Another measure of non-Markovianity, introduced in recent times, makes use of the deviation from CP of the intermediate dynamics [23].

Initially, we consider the noiseless (unitary) evolution of the quantum walk. To study the reduced dynamics of the coin state, we compute the trace distance by initializing...
FIG. 1. (Color online) (a) Probability distribution of the QW in the unitary regime (bimodal), Markovian (outer Gaussian, \( a = 0.4, \gamma = 5 \)) and non-Markovian (inner Gaussian, \( a = 1, \gamma = 0.001 \)) regimes of RTN, using \( t = 100 \) steps. (b) Log-linear plot of variance as a function of RTN amplitude \( a \). The solid (resp., dashed) line represent the corresponding non-Markovian (resp., Markovian) cases with \( \gamma = 0.001 \) (resp. \( \gamma = 5 \)). The flat dashed line represents the classical case. We note that the former alternates between localization (subclassical variance) and super-classical variance. The Markovian case shows plain decoherent behavior, characterized by monotonic decay of variance towards the classical value.

FIG. 2. (Color online) Plot of TD evolution, under the influence of RTN and OUN, with respect to the number of walk steps. (a) RTN: The trace distance plot in the noiseless quantum walk (top curve), in the Markovian (middle curve; \( \gamma = 1, a = 0.05 \)) and non-Markovian regime (bottom curve; \( \gamma = 0.001, a = 0.05 \)). The pure walk case, i.e., the QW in the absence of an external noise, shows rapid recurrences due to interaction with the position “environment” (primary component of non-Markovianity), while the bottom curve shows an additional oscillatory term (secondary component of non-Markovianity) attributed to the non-Markovianity in the environment-induced decoherence. (b) OUN: In the Non-Markovian regime (\( \gamma = 0.01, \Gamma = 0.1 \)), TD decays without the additional recurrent feature seen in the RTN case in relation to the Markovian case (\( \gamma = 1.0, \Gamma = 0.1 \)). PLN is similar to OUN in this respect.

The power spectrum of the evolution is computed in the time domain at step \( n \).

**Power spectral analysis** — In addition to the oscillations present in the noiseless evolution of quantum walk, a further backflow or recurrence structure arises when the coin is exposed to an external noise such as RTN in the non-Markovian regime. In order to disambiguate these two distinct sources of non-Markovianity, we compute the power spectrum of the time evolution \( D(\rho_1(t), \rho_2(t)) \) of correlation-like quantities such as trace distance or mutual information, minus the function \( \delta(t) \), the Monotonically Falling Best Fit (MFBF) function, which is the monotonically falling function that is closest to \( D(\rho_1(t), \rho_2(t)) \), according to a suitable distance measure. This allows us to usually locate in the frequency domain the different sources of the backflow aspect of non-Markovianity. In this case, the position degree of freedom serves as one source, while RTN producing environment serves as the other source.

The probability distribution of the evolution is computed for time \( N = 100 \) steps, as the absolute squared of the Discrete Fourier Transform (DFT), i.e., \( S(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-2\pi i k n / N} \), where \( x_n \) is TD or mutual information in the time domain at step \( n \).

**Case of RTN** — Figure 2.(a) depicts time evolution of the DTQW in the noiseless case (top plot), with RTN in the Markovian regime (middle plot) and with RTN in the non-Markovian regime (bottom plot). The high-frequency “primary” oscillation (which corresponds to non-Markovianity according to the Breuer measure \( \mathcal{B} \)) in the top plot is due to the interaction of the position and coin degrees of freedom \( \mathcal{B} \). Note the ring-down of the oscillations. In the middle plot, Markovian decoherence with RTN is seen to cause a monotonic fall, with an overlay of the position-induced oscillation. Finally, in the bottom-most plot, RTN in the non-Markovian regime introduces a new, “secondary”, lower-frequency oscillation component.

Figure 2.(b) depicts the corresponding situation with OUN, where the secondary oscillatory component, due to backflow from the external environment, and as depicted in the middle plot, is missing even in the non-Markovian regime. Instead non-Markovianity manifests as a back-action of the system on the reservoir, producing a slowing down of the decoherence rate. This phenomenon is essentially due to the low bandwidth of the reservoir frequency, \( \gamma \), resulting in a large reservoir correlation time in relation to the system correlation time.

In Figure 3, we present the power spectrum of the bottom plot of Figure 2.(a), from which we have subtracted the MFBF. The rationale is, as noted earlier, that Marko-
vian dynamics can only generate a monotonic decrease of a distance measure. Any departure from monotonicity should be attributed, then, to non-Markovianity, in particular backflow. The filtered plot can thus be considered as a measure of the lower bound on non-Markovian behavior. In the present case, the subtracted part turns out to be a simple power law (represented as a straight line in the log-normal plot). More generally, this subtracted part corresponds to a problem of monotonic curve fitting, which can be formulated as a semidefinite program \[ T \leq 0 \]. The peak around \( f = 0.27 \) (resp. \( f = 0.025 \)) corresponds to the primary (resp., secondary) non-Markovian source, namely the position degree of freedom (resp., the environment). The information backflow is a resonance like phenomena, producing the secondary peak. Thus, our approach provides a tool to disambiguate two sources of non-Markovian backflow.

**Detecting non-Markovianity using Mutual Information** — Quantum correlations as quantified by mutual information (MI) has been used to quantify non-Markovianity \[ 24 \]. Let \( \rho_1 \) and \( \rho_2 \) be the density matrices representing the system and the ancillary state, respectively. Given density operators \( \rho_1 \) and \( \rho_2 \), their mutual information is \( I_m(\rho) = S(\rho_1) + S(\rho_2) - S(\rho_{12}) \), where \( S(\cdot) \) is the von Neumann entropy \( S(\rho) := -\text{tr} \log_2 \rho \). Similar to the TD measure, MI is also a monotonically decreasing function when the dynamics is Markovian. Figure 4(a) presents the MI equivalent of Figure 2(a), while 4(b) presents the power spectrum of the plots of 4(a). We note the primary peak around \( f = 0.27 \). Apart from this, the spectrum of the Markovian noise is smooth, whereas that of the non-Markovian case shows signatures of secondary peaks, indicating non-Markovianity of environmental origin. Note that the spectral filtering method can also be employed for MI by finding a suitable MFBF function.

**Conclusion** — Information backflow from the environment to the system is an important aspect of non-Markovianity absent in Markovian dynamics. In a system such as a coined quantum walk, information backflow appears in the reduced coin system dynamics due to both environmental decoherence and the coin-position interaction. Here we have developed tools to disambiguate these two sources of non-Markovianity. We identify backflow by peaks in the power spectrum of time evolution of distance or nearness measures of a pair of quantum states. All known measures of non-Markovian behavior are incapable of making this distinction. This work thus presents novel insights into the nature and detection of non-Markovian evolution.

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\[ 1 \] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, in *Proceedings of the thirty-third annual
[2] C. Chandrashekar, S. Banerjee, and R. Srikanth, Phy. Rev. A 81, 062340 (2010).

[3] H.-P. Breuer and F. Petruccione, The theory of open quantum systems (Oxford University Press on Demand, 2002).

[4] C. Chandrashekar, R. Srikanth, and S. Banerjee, Phy. Rev. A 76, 022316 (2007).

[5] S. Banerjee, R. Srikanth, C. Chandrashekar, and P. Rungta, Phys. Rev. A 78, 052316 (2008).

[6] I. de Vega and D. Alonso, Rev. Mod. Phys. 89, 015001 (2017).

[7] N. Pradeep Kumar, V. Jagadish, S. Banerjee, R. Srikanth, and F. Petruccione, Draft under preparation.

[8] M. Hinarejos, C. D. Franco, A. Romanelli, and A. Perez, Phys. Rev. A 81, 014101 (2010).

[9] S. Daffer, K. Wódkiewicz, J. D. Cresser, and J. K. McIver, Phys. Rev. A 70, 010304 (2004).

[10] S. O. Rice, Stochastic Processes in Physics and Chemistry (Elsevier, Amsterdam, 1992).

[11] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev 36, 823 (1930).

[12] T. Yu and J. Eberly, Opt. Commun 283, 676 (2010).

[13] W. S. Kendall and B. Jørgensen, Phys. Rev. E 84, 066120 (2011).

[14] C. Benedetti, F. Buscemi, P. Bordone, and M. G. Paris, Phys. Rev. A 93, 042313 (2016).

[15] J. Kempe, Cont. Phy 44, 307 (2003).

[16] A. Rajagopal, A. U. Devi, and R. Rendell, Phys. Rev. A 82, 042107 (2010).

[17] A. R. Usha Devi, A. K. Rajagopal, and Sudha, Phys. Rev. A 83, 022109 (2011).

[18] P. W. Anderson, Phys. Rev. 109, 1492 (1958).

[19] C. Chandrashekar, arXiv preprint arXiv:1212.5984 (2012).

[20] E.-M. Laine, J. Piilo, and H.-P. Breuer, Phys. Rev. A 81, 062115 (2010).

[21] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett 103, 210401 (2009).

[22] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys 88, 021002 (2016).

[23] A. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett 105, 050403 (2010).

[24] S. Luo, S. Fu, and H. Song, Phys. Rev. A 86, 044101 (2012).