Localisation of Interacting Power-Law Random Banded Fermions

S. J. Thomson 1, 2, ∗ and M. Schiró 3, †

1Centre de Physique Théorique, CNRS, Institut Polytechnique de Paris, Route de Saclay, F-91128 Palaiseau, France
2Centre de Physique Théorique, Université Paris-Saclay, CNRS, CEA, F-91191 Gif-sur-Yvette, France
3JEIP, USR 3573 CNRS, Collège de France, PSL Research University,
11 Place Marcelin Berthelot, 75321 Paris Cedex 05, France

(Dated: May 1, 2020)

We study a model of one-dimensional fermions with random long-range hoppings and interaction, both Gaussian-distributed with power-law widths which decay with different exponents, and on-site disorder. This model realizes an interacting many-body version of the well-studied power-law banded random matrix model. Using a truncated flow equation approach, we study static and dynamical properties as a function of hopping and interaction exponents. We show that, at large on-site disorder and for short-range interactions, a transition from a delocalised phase to quasi many-body-localised (MBL) behavior exists upon decreasing the hopping range. This quasi-MBL phase is characterized by intriguing properties such as algebraically decaying l-bit interactions. Surprisingly we find that a crossover survives, albeit broadened, upon increasing the range of the interactions.

Introduction - The combination of disorder and interactions in many-body quantum systems can lead to a rich variety of physics far from equilibrium, from many-body localisation (MBL) [1–5] to quantum glasses [6–11]. MBL is by now rather well understood for one-dimensional models with short-range interactions, where a set of mutually commuting local integrals of motion (LIOMs, or locators) provides at most a sufficient condition for localisation (MBL) [40] and the delocalization of non-interacting fermions with power-law hopping [41], we show that for rapidly decaying power laws (α, β > 2) the system at large on-site disorder is in a quasi-MBL phase [70] characterized by algebraically decaying l-bit interactions [33, 42] that we explicitly construct. This quasi-MBL phase turns delocalised upon decreasing α. Interestingly, this transition survives upon decreasing the range of the interactions β.

The model - We start from a Hamiltonian describing a one-dimensional chain of interacting fermions given by:

\[ H = \sum_i h_i n_i + \frac{1}{2} \sum_{ij} V_{ij} n_i n_j + \sum_{ij} J_{ij} c_i^\dagger c_j \]  

(1)

where the on-site disorder is drawn from a box distribution \( h_i \in [0,W] \). The couplings \( J_{ij} \) and \( V_{ij} \) are also random and drawn from Gaussian distributions with standard deviations which decay with distance as \( \sigma_J = J_0 / |i - j|^\alpha \) and \( \sigma_V = V_0 / |i - j|^\beta \) respectively. We fix \( J_0 = 0.5 \) and \( V_0 = 0.1 \) and \( W = 5 \), such that the model with short-ranged hopping and interactions (respectively
\(\alpha = \beta = \infty\) would be in the MBL phase, and vary the power-law exponents \(\alpha\) and \(\beta\) only.

In absence of interactions, the model reduces to the Power-Law Random Banded Matrix (PRBM) model, previously studied in Refs. [17–25]. In this case, it is known that long-range hoppings favour the delocalised phase and give rise to a delocalization transition even in \(d = 1\), for \(\alpha \leq 1\). In the following we discuss what happens to the PRBM model in presence of interactions and study the interplay/competition between power-law hoppings and power-law interactions. We first consider the two effects separately, fixing \(\alpha = \infty\) and varying \(\beta\) and vice versa, while later we present a complete phase diagram in the \((\alpha, \beta)\) plane.

**Method** - We attack the problem using the flow equation approach [43–53] which we have recently used to study MBL in the short ranged case [40] as well as the non-interacting PRBM model [41]. The main idea is to diagonalise the Hamiltonian through a series of infinitesimal unitary transforms parametrised by a fictitious ‘flow time’ \(l\) which runs from \(l = 0\) (initial basis) to \(l \rightarrow \infty\) (diagonal basis). The Hamiltonian flow reads

\[
\frac{d\mathcal{H}}{dl} = [\eta(l), \mathcal{H}(l)],
\]

where \(\eta(l)\) is the generator of the flow and the initial condition at \(l = 0\) is given by the Hamiltonian in Eq. (1). While for quadratic problems the approach is exact, in presence of interactions the flow generates higher-order couplings not present in the original microscopic model. We use here a truncated scheme, originally introduced in Ref [40], that captures the essential physics of the MBL phase. We make an ansatz for the form of the running Hamiltonian \(\mathcal{H}(l) = \mathcal{H}_0(l) + V(l)\), with

\[
\mathcal{H}_0(l) = \sum_i h_i(l) : c_i^\dagger c_i : + \frac{1}{2} \sum_{ij} \Delta_{ij}(l) : c_i^\dagger c_j^\dagger c_j c_i : + \sum_{ij} J_{ij}(l) : c_i^\dagger c_j ;
\]

\[
V(l) = \sum_{ij} J_{ij}(l) : c_i^\dagger c_j ;
\]

which allows us to close the hierarchy of flow equations, discarding all newly generated higher-order terms. In Eq. (3) the : \(\mathcal{O}:\) notation signifies normal-ordering [71]. Given the above ansatz, here we use Wegner’s choice for the generator and choose it to be the commutator of the diagonal and off-diagonal parts of the Hamiltonian, \(\eta(l) = [\mathcal{H}_0(l), V(l)]\). The flow equations for the running couplings can be read off from \(d\mathcal{H}/dl = [\eta(l), \mathcal{H}(l)]\) [54]. This choice of generator, although not unique [41, 48, 55], guarantees [43, 56] that the off-diagonal terms vanish in the \(l \rightarrow \infty\) limit; other choices of generator are also possible [41, 48, 55]. We can quantify the accuracy of our truncation by computing quantities which are preserved by unitary transforms, such as traces of integer powers of the Hamiltonian. By computing how precisely they are preserved by this ansatz we can get a measure for the error in this truncation scheme - see [54] for details.

**Decay of l-bit interactions and real-space support** - In the \(l \rightarrow \infty\) limit, the off-diagonal terms will vanish and we will obtain a diagonal Hamiltonian given by \(\mathcal{H} = \sum_i h_i n_i + \frac{1}{2} \sum_{ij} \Delta_{ij} n_i n_j\). In all of the following, the tilde notation indicates quantities in the \(l \rightarrow \infty\) diagonal l-bit basis. By computing the effective Hamiltonian, we can extract two different aspects of its long-range behaviour. First, we extract the distance-dependence of the coefficients \(\Delta_{ij}\). These coefficients, which decay exponentially in short-range systems [40, 57, 58], are strongly modified by the existence of long-range couplings.

We can also compute the real-space support of the l-bit operators directly. Starting from a local density operator \(\tilde{n}_i\) defined in the diagonal \(l \rightarrow \infty\) basis with support only on a single site, we can transform it back into the physical (i.e. real space) basis by inverting the unitary transform used to diagonalise the Hamiltonian. This operator can be transformed according to:

\[
\frac{d\tilde{n}_i(l)}{dl} = [\eta(l), \tilde{n}_i(l)]
\]

here starting with \(l = \infty\) and flowing backwards to \(l = 0\). To parameterise the flow of this operator, we make the following ansatz for the running number operator:

\[
\tilde{n}_i(l) = \sum_j A^{(i)}(l) n_j + \sum_{jk} B^{(i)}(l) c_j^\dagger c_k
\]

Note that higher-order terms cannot be consistently included at this order of the truncation scheme. The normal-ordering procedure employed as part of this construction [54] does, however, allow us to take into account the leading effects of the interactions even at this order. In Fig. 1, we show these quantities in the case of power-law hopping and nearest-neighbor interactions (corresponding to \(\beta = \infty\)). The \(\Delta_{ij}\) retain their exponentially-decaying nature at short distances, but acquire power-law tails at long range, with a decay exponent \(\zeta \approx 2\alpha\) for \(\alpha \geq 1\). This follows immediately from the structure of the eigenstates of the PRBM problem, which are indeed exponentially localised at short distance with power-law tails [19]. The real-space support of the l-bits also show power-law tails characteristic of delocalisation, after an initial exponential decay at short range. The precise distance where the decay crosses from exponential to power-law depends on the exponent, as well as both the disorder and interaction strength. As \(\alpha \rightarrow \infty\), the real-space support of the l-bits decays exponentially over a larger range before the power-law tail appears, and the resulting l-bits closely match the nearest-neighbour case (black dashed line). In Fig. 2, we show the case of power-law interactions and nearest-neighbor hopping (corresponding to \(\alpha = \infty\)). The \(\Delta_{ij}\) retain their initial power-law distribution at all distances and at all stages during the flow.
procedure. Surprisingly, we find that the real-space support of the $l$-bits is essentially unmodified by the range of the interactions: they retain their exponentially decaying character even in the limit of $\beta = 0$, with only an extremely small extended ‘tail’ appearing following the strong initial exponential decay. This may be an effect of the truncation in Eq. 3 suppressing degrees of freedom responsible for delocalisation, or it may be that delocalisation is only seen in (weak) higher-order contributions to Eq. 5, corresponding to multipole processes.

Dynamics of Imbalance and Phase Diagram - We now move on to study the effect of power-law couplings on the quantum dynamics of the system. This can be accessed using the FE approach [40, 54]. We set up an initial charge density wave (CDW) state and see how it relaxes under its own quantum dynamics. To monitor this, we define the imbalance as:

$$I(t) = \frac{2}{L} \sum_i (-1)^i \langle n_i(t) \rangle$$

(6)

The long time behavior of the imbalance is often used as a proxy for the MBL transition, since in a localised phase any initial inhomogeneity persists at long time due to enhanced memory of initial conditions while in a thermal, delocalised phase the imbalance is expected to decay to zero as a power law with a disorder-dependent exponent, vanishing at the transition [59, 60]. Using the time-dependent mean-field decoupling on the effective $l$-bit Hamiltonian, the results for the relaxation dynamics of the imbalance are shown in Figure 3, for chains of length $L = 64$ in the cases of power-law hopping with nearest-neighbour interactions (panel a), and nearest-neighbour hopping with power-law interactions (panel b). In Fig. 3a, we see that for $\alpha \geq 1$ the system remains localised as for the short-range model, while upon decreasing $\alpha$ the imbalance continuously decrease toward zero, a behavior that is reminiscent of the PRBM model. For $\alpha = 0$, the decay of the imbalance is approximately exponential, while for $\alpha > 0$ it is consistent with a power-law. On the contrary, Fig. 3b shows that decreasing $\beta$, i.e. making the range of interactions larger, has little to no effect on the long-time imbalance and the system remains localized, with small values of $\beta$ leading to the appearance of a short plateau that vanishes at longer times.

Having examined their effects separately, we now compute the imbalance in the presence of both long-ranged interactions and long-range hopping, and obtain a qualitative phase diagram shown in Fig. 4 where we show the imbalance $I(t)$ at a time $t^* = 100$ after the quench as a function of $\alpha, \beta$ and super-impose lines at fixed imbalance as guide to the eye. In the upper-right corner, corresponding to fast decaying hopping and interactions ($\alpha, \beta \geq 2$), the system is in a quasi-MBL phase, with a finite and large imbalance. Keeping $\beta \geq 2$ and decreasing the hopping exponent $\alpha$ the imbalance displays a sharp crossover from localized to delocalized behavior, the interacting many-body analog of the PRBM previously studied with FE techniques in Ref. [41]. We can
FIG. 3: Relaxation of the imbalance following a quench from a CDW state with a) power-law hopping $\alpha \in [0, 2.5]$ in increments of 0.25 (bottom to top) and $\beta = \infty$ (nearest-neighbour interactions) and b) power-law interactions $\beta \in [0, 2.5]$ in increments of 0.25 (top to bottom) with $\alpha = \infty$ (nearest-neighbour hopping). Decreasing $\alpha$ makes the long-time imbalance go to zero (as a power law in time for small $\alpha$, see top inset) whereas changing $\beta$ has almost no effect on the long-time dynamics of the imbalance which approaches a finite plateau almost exponentially (see inset). Chain size $L = 64$, disorder realisations $N_s = 256$.

now ask what happens to those two phases as we increase the range of the interaction, i.e. decreases $\beta$ toward zero. The ergodic phase is expected to be robust to long-range interactions, and indeed we see that the imbalance for $\alpha < 1$ remains constant and close to zero upon decreasing $\beta$ (see the almost vertical contour lines). On the other hand, and quite surprisingly, we find the imbalance to remain strongly unaffected by long range interactions even for $\alpha \gtrsim 1$, consistently with the results of Figure 3 for the $\alpha = \infty$ case. However the lines at fixed imbalance bends toward right for small $\beta$, suggesting that the lower right corner of the phase diagram is less localised than the upper right corner.

Discussion - Our results show that upon increasing the range of the hopping, a transition from delocalisation to quasi-MBL exists, both for short ranged interactions as well as for $\beta < 2$, in a regime where perturbative arguments based on locator expansion would exclude it. While one could suspect of an artefact of our truncation scheme, we have performed extensive checks to validate our approach in this regime, including comparison with exact numerics for small system sizes and monitoring the flow invariant, a sensitive probe of the validity of our scheme (see [54] for details). Still we cannot exclude that certain processes leading to delocalisation are not captured by our scheme. This quasi-MBL phase could also be metastable for finite size and finite time: recent works suggest that in the intermediate regime $1 < \beta < 2$ an infinitely large system would be delocalised while finite-size systems will see a localisation transition as a function of increasing system size $L$ (or equivalently, exhibit a size-dependent critical disorder $W_c(L)$) [26, 28, 61, 62].

Our results show [54] that the quasi-MBL phase shrinks as the system size is increased. The Gaussian distribution of couplings could also play a role, as long-range random sign couplings, commonly studied in quantum spin models, exhibit enhanced delocalization [54]. Finally, it is worth noticing that in the $\alpha, \beta \rightarrow 0$ limit, Eq. (1) reduces to a model of fermions with all-to-all random couplings, reminiscent of the maximally chaotic Sachdev-Ye-Kitaev model [63]. Increasing $\alpha$ and $\beta = 0$ induces a finite range hopping and interaction which could lead, in addition to the random onsite field, to an increased localised behavior [64], at least for finite systems.

Conclusion - We have used the flow equation method to study a model of one dimensional fermions with gaussian distributed, power-law decaying, hoppings and interactions and diagonal box disorder, an interacting many body version of the celebrated PRBM model. For large diagonal disorder, compared to typical scales of interactions and hoppings, we have provided evidence of a transition from a delocalised ergodic phase to a quasi MBL phase upon increasing the exponent $\alpha$ controlling the range of hopping. Such a crossover survives even for slowly decaying interactions, $\beta < 2$, although it appears to become less sharp. This quasi-MBL phase, although possibly metastable, appears to have intriguing properties such as algebraically decaying $l$-bit interactions. An open question is the stability of such a phase to the propagation of ergodic bubbles: avalanche arguments suggest that MBL should disappear in any dimension for
interactions decaying slower than exponential [65] and our model and approach could provide insights into this largely unexplored question, for example by studying the coupling of this quasi-MBL phase to an ergodic bath [66].

Acknowledgements - The computations were performed on the Collège de France IPH computer cluster. We acknowledge use of the QuSpin exact diagonalisation library for benchmarking our FE code [67, 68], and support from the grant DynDisQ from DIM SRTEQ.

* Electronic address: steven.thomson@polytechnique.edu
† Electronic address: marco.schiro@ipht.fr; On Leave from: Institut de Physique Théorique, Université Paris Saclay, CNRS, CEA, F-91191 Gif-sur-Yvette, France

1. D. M. Basko, I. L. Aleiner, and B. L. Altschuler, Annals of physics 321, 1126 (2006).
2. E. Altman and R. Vosk, Annu. Rev. Condens. Matter Phys. 6, 383 (2015).
3. R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. 6, 15 (2015).
4. F. Alet and N. Laflorencie, Comptes Rendus Physique 19, 498 (2018).
5. D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Rev. Mod. Phys. 91, 021001 (2019).
6. M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B 40, 546 (1989).
7. L. F. Cugliandolo and G. Lozano, Phys. Rev. Lett. 80, 4970 (1998).
8. G. Biroli and L. F. Cugliandolo, Phys. Rev. B 64, 014206 (2001).
9. G. Biroli and O. Parcollet, Phys. Rev. B 65, 094414 (2002).
10. L. Berthier and G. Biroli, Rev. Mod. Phys. 83, 587 (2011).
11. S. J. Thomson, P. Urbani, and M. Schiro, arXiv e-prints arXiv:1904.03147 (2019), 1904.03147.
12. M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. 111, 127201 (2013).
13. D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B 90, 174202 (2014).
14. J. Z. Imbrie, Journal of Statistical Physics 163, 998 (2016), ISSN 1572-9613.
15. J. Z. Imbrie, V. Ros, and A. Scardicchio, Annalen der Physik 529, 1600278 (2017).
16. P. W. Anderson, Physical review 109, 1492 (1958).
17. C. Yeung and Y. Oono, EPL (Europhysics Letters) 4, 1061 (1987).
18. L. S. Levitov, Physical review letters 64, 547 (1990).
19. A. D. Mirlin, Y. V. Fyodorov, F.-M. Dittes, J. Quezada, and T. H. Seligman, Physical Review E 54, 3221 (1996).
20. L. Levitov, Annalen der Physik 8, 697 (1999).
21. I. Varga and D. Braun, Physical Review B 61, R11859 (2000).
22. A. D. Mirlin and F. Evers, Physical Review B 62, 7920 (2000).
23. F. Evers and A. D. Mirlin, Physical Review Letters 84, 3690 (2000).
24. V. E. Kravtsov, O. Yevtushenko, and E. Cuevas, Journal of Physics A: Mathematical and General 39, 2021 (2006).
25. F. Evers and A. D. Mirlin, Reviews of Modern Physics 80, 1355 (2008).
26. A. L. Burin (2006), cond-mat/0611387.
27. N. Y. Yao, C. R. Laumann, S. Gopalakrishnan, M. Knap, M. Mueller, E. A. Demler, and M. D. Lukin, Physical review letters 113, 243002 (2014).
28. A. L. Burin, Physical Review B 92, 104428 (2015).
29. D. B. Gutman, I. V. Protopopov, A. L. Burin, I. V. Gornyi, R. A. Santos, and A. D. Mirlin, Physical Review B 93, 245427 (2016).
30. R. M. Nandkishore and S. L. Sondhi, Physical Review X 7, 041021 (2017).
31. L. F. Santos, F. Borgonovi, and G. L. Celardo, Physical review letters 116, 250402 (2016).
32. S. Roy and D. E. Logan, SciPost Phys. 7, 42 (2019).
33. X. Deng, G. Masella, G. Pupillo, and L. Santos (2019), 1912.08131.
34. X. Deng, V. E. Kravtsov, G. V. Shlyapnikov, and L. Santos, Phys. Rev. Lett. 120, 110602 (2018).
35. P. A. Nosov, I. M. Khaymovich, and V. E. Kravtsov, Phys. Rev. B 99, 104203 (2019).
36. G. A. Álvarez, D. Suter, and R. Kaiser, Science 349, 846 (2015), ISSN 0036-8075.
37. J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, Nature Physics 12, 907 (2016).
38. J. Zeiher, J.-y. Choi, A. Rubio-Abadal, T. Pohl, R. van Bijnen, I. Bloch, and C. Gross, Phys. Rev. X 7, 041063 (2017).
39. S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, et al., Nature 543, 221 (2017).
40. S. J. Thomson and M. Schiró, Physical Review B 97, 060201(R) (2018).
41. S. J. Thomson and M. Schiró, Eur. Phy. J. B (2020).
42. G. De Tomasi, Phys. Rev. B 99, 054204 (2019).
43. S. Kehrein, The flow equation approach to many-particle systems, vol. 217 (Springer, 2007).
44. M. Moeckel and S. Kehrein, Phys. Rev. Lett. 100, 175702 (2008).
45. A. Hackl and S. Kehrein, Phys. Rev. B 78, 092303 (2008).
46. A. Hackl and S. Kehrein, Journal of Physics: Condensed Matter 21, 015601 (2009).
47. M. Eckstein, A. Hackl, S. Kehrein, M. Kollar, M. Moeckel, P. Werner, and F. Wolf, The European Physical Journal Special Topics 180, 217 (2009).
48. C. Monthus, Journal of Physics A: Mathematical and Theoretical 49, 305002 (2016).
49. V. L. Quito, P. Titum, D. Pekker, and G. Refael, Phys. Rev. B 94, 104202 (2016).
50. D. Pekker, B. K. Clark, V. Oganesyan, and G. Refael, Physical Review Letters 119, 075701 (2017).
51. S. Savitz, C. Peng, and G. Refael, Phys. Rev. B 100, 094201 (2019).
52. X. You, D. Pekker, and B. K. Clark (2019), arXiv:1909.11097.
53. P. S. Kelly, R. Nandkishore, and J. Marino, Nuclear Physics B 951, 114886 (2020).
54. See Supplementary Material for more details on the flow equations and for comparison with exact diagonalization.
55. S. Savitz and G. Refael, Physical Review B 96, 115129 (2017).
56. F. Wegner, Annalen der Physik 506, 77 (1994).
[57] L. Rademaker and M. Ortuño, Phys. Rev. Lett. 116, 010404 (2016).
[58] L. Rademaker, M. Ortuño, and A. M. Somoza, Annalen der Physik pp. 1600322-n/a (2017), ISSN 1521-3889, 1600322.
[59] D. J. Luitz, N. Laflorencie, and F. Alet, Phys. Rev. B 93, 060201(R) (2016).
[60] G. Biroli and M. Tarzia, Phys. Rev. B 96, 201114(R) (2017).
[61] K. S. Tikhonov and A. D. Mirlin, Physical Review B 97, 214205 (2018).
[62] S. Gopalakrishnan and D. A. Huse, Physical Review B 99, 134205 (2019).
[63] S. Sachdev and J. Ye, Phys. Rev. Lett. 70, 3339 (1993).
[64] A. M. García-García and M. Tezuka, Physical Review B 99, 054202 (2019).
[65] W. De Roeck and F. Huveneers, Phys. Rev. B 95, 155129 (2017).
[66] P. J. D. Crowley and A. Chandran, arXiv e-prints arXiv:1910.10812 (2019), 1910.10812.
[67] P. Weinberg and M. Bukov, SciPost Phys. 2, 003 (2017).
[68] P. Weinberg and M. Bukov, SciPost Phys. 7, 20 (2019).
[69] F. Wegner, Journal of Physics A: Mathematical and General 39, 8221 (2006).
[70] We use the term “quasi-MBL” because we cannot rule out the disappearance of localisation at longer times, or at larger system sizes, due to the effects of avalanche instabilities - see Ref. [65] and the Discussion at the end of our manuscript for more details.
[71] We adopt normal ordering using the : ˆO : notation in order to i) ensure a consistent ordering of operators when computing commutation relations, and ii) efficiently resum contributions from higher-order terms to turn the flow equation method into a non-perturbative scheme - see Refs. [43, 69] and the Supplementary Material for details.
Supplementary Material to “Localisation of Interacting Power-Law Random Banded Fermions”

S. J. Thomson\textsuperscript{1, 2, ∗} and M. Schirò\textsuperscript{3, 4, †}
\textsuperscript{1}Centre de Physique Théorique, CNRS, Institut Polytechnique de Paris, Route de Saclay, F-91128 Palaiseau, France
\textsuperscript{2}Institut de Physique Théorique, Université Paris-Saclay, CNRS, CEA, F-91191 Gif-sur-Yvette, France
\textsuperscript{3}JEIP, USR 3573 CNRS, Collège de France, PSL Research University,
11 Place Marcelin Berthelot, 75321 Paris Cedex 05, France

(Dated: May 1, 2020)

Normal-ordering

A key ingredient in the calculation is the adoption of a normal-ordering procedure [1–3], which allow us to consistently group together terms at each order of the Hamiltonian, and to incorporate corrections from higher-order terms which are then discarded from our variational manifold. We will assume all contractions will be computed with respect to a product state, and the relevant contractions will be denoted:

\[
\{c_i^\dagger, c_j\} = G_{ij} + \bar{G}_{ji} = \delta_{ij} \quad (1)
\]

\[
G_{ij} = \langle c_i^\dagger c_j \rangle = \delta_{ij} \langle n_i \rangle \quad (2)
\]

\[
\bar{G}_{ji} = \langle c_j c_i^\dagger \rangle = \delta_{ij} - \langle c_i^\dagger c_j \rangle = \delta_{ij}(1 - \langle n_i \rangle) \quad (3)
\]

To calculate the commutators of normal-ordered strings of operators, we need to use the following theorem [2]:

\[
: O_1(A) : O_2(A') : = : \exp \left( \sum_{ij} G_{ij} \frac{\partial^2}{\partial A_j^i \partial A_i} \right) O_1(A) O_2(A') :
\]

which, for example, leads to the following commutation relation for pairs of fermion operators:

\[
:\{ c_i^\dagger c_\beta : : c_j^\dagger c_\gamma \} = \{ G_{\gamma \beta} + \bar{G}_{\beta \gamma} \} : c_i^\dagger c_\beta : - (G_{\alpha \delta} + \bar{G}_{\delta \alpha}) : c_j^\dagger c_\beta : + (G_{\alpha \delta} \bar{G}_{\beta \gamma} - G_{\gamma \beta} \bar{G}_{\delta \alpha})
\]

\[
= \delta_{\beta \gamma} : c_i^\dagger c_\delta : - \delta_{\alpha \delta} : c_j^\dagger c_\beta : + (G_{\alpha \delta} \bar{G}_{\beta \gamma} - G_{\gamma \beta} \bar{G}_{\delta \alpha})
\]

which is just the regular commutator plus a constant. All necessary commutators can be computed from Eq. 4, though the calculation is extremely tedious and will not be shown here: for further details, see Refs. [1–3]. In principle, one should define an \( l \)-dependent state and recompute the normal-ordering corrections at each flow timestep accordingly, however to capture the main physics it is sufficient to simply pick a target state and compute the corrections with respect to that state [2]. In the main text, we compute the contractions with respect to an infinite-temperature product state such that \( \langle n_i \rangle = 0.5 \forall i \). This has the advantage that many of the normal-ordering corrections (e.g. the final terms in Eq. 6 above) vanish identically.

Generator

The generator \( \eta (l) \) is scale-dependent and changes throughout the flow. Here we use Wegner’s choice for the generator such that it is given by the commutator of the diagonal and off-diagonal parts of the Hamiltonian, \( \eta (l) = [H_0(l), V(l)] \). This choice guarantees [2, 4] that the off-diagonal terms vanish in the \( l \to \infty \) limit; other choices of generator with different convergence properties are also possible [5–7]. With this choice, the generator is given by:

\[
\eta = \sum_{ij} F_{ij} : c_i^\dagger c_j : + \sum_{ijk} \zeta^k_{ij} : c_k c_k^\dagger c_i^\dagger c_j : \quad (7)
\]

with \( F_{ij} \equiv J_{ij} \left[ (h_i - h_j) - \Delta_{ij}(\langle n_i \rangle - \langle n_j \rangle) \right] \) and \( \zeta^k_{ij} \equiv J_{ij}(\Delta_{ik} - \Delta_{jk}) \), where the scale-dependence of the coefficients has been suppressed for clarity. The interaction term will lead to the generation of new higher-order terms in the Hamiltonian during the flow. In practice, the successive generation of these higher-order terms quickly renders the calculation analytically intractable, so we make the truncation specified in the main text and discard all newly generated terms outside of this variational manifold.
Flow Equations

The flow of the Hamiltonian coefficients can be read off from \( \frac{d\mathcal{H}}{dl} = [\eta(l), \mathcal{H}(l)] \), following a lengthy calculation. Explicit expressions for the flow equations are as follows:

\[
\frac{dh_i(l)}{dl} = 2 \sum_j J_{ij}^2 (h_i - h_j) - 4 \sum_j J_{ij}^2 \Delta_{ij} (\langle n_i \rangle - \langle n_j \rangle) + \sum_{jk} J_{jk}^2 (\Delta_{ik} - \Delta_{ij}) (\langle n_k \rangle - \langle n_j \rangle)
\]

\[
\frac{dJ_{ij}(l)}{dl} = -J_{ij} (h_i - h_j)^2 - \sum_k J_{ik} J_{kj} (2h_k - h_i - h_j) + 2J_{ij} \Delta_{ij} (h_i - h_j) (\langle n_i \rangle - \langle n_j \rangle)
\]

\[
- J_{ij} \Delta_{ij}^2 (\langle n_i \rangle + \langle n_j \rangle - 2\langle n_i \rangle \langle n_j \rangle) - \frac{1}{2} \sum_k J_{ij} (\Delta_{ik} - \Delta_{jk}^2) (\langle n_k \rangle (1 - \langle n_k \rangle))
\]

\[
+ \sum_{k} J_{ik} J_{kj} [(\Delta_{ij} - 2\Delta_{jk})(\langle n_j \rangle - \langle n_k \rangle) + (\Delta_{ij} - 2\Delta_{ik})(\langle n_i \rangle - \langle n_k \rangle)]
\]

\[
\frac{d\Delta_{ij}(l)}{dl} = 2 \sum_{k \neq i,j} [J_{ik}^2 (\Delta_{ij} - \Delta_{kj}) + J_{jk}^2 (\Delta_{ij} - \Delta_{ik})]
\]

We numerically integrate these equations until the off-diagonal elements have decayed to the required accuracy, typically using \( l_{\text{max}} \approx 10^3 \), discarding couplings which have reached zero below some cutoff (typically \( 10^{-6} \) or less). Note that in the low-disorder, large-interaction limit where the system may become delocalised, the neglected higher-order terms may carry significant weight. In this case, the normal-ordering corrections due to these neglected higher-order terms can become large enough to overpower the dominant Wegner decay terms, i.e. the \(-J_{ij} (h_i - h_j)^2\) term in Eq. 9. In order to obtain physically reasonable flows in this regime, one must neglect the normal-ordering corrections whenever they threaten to overwhelm the dominant Wegner rules. Retaining them will result in an unphysical and uncontrolled flow. The normal-ordering corrections should always be sub-leading to the main terms in the flow equations. Also note that in the same low-disorder, large-interaction regime, Eq. 10 can exhibit spurious divergences which must be handled carefully in order to obtain physically reasonable results. We emphasise, however, that when these problems are encountered, the validity of the ansatz will likely have already broken down.

Accuracy: Eigenvalue Comparison with Exact Diagonalisation

In order to benchmark the accuracy of our results, we compared the static properties (i.e. the eigenvalues) with Exact Diagonalisation (ED) results obtained using the QuSpin package [8, 9]. We define the averaged relative error as:

\[
\delta \varepsilon = \frac{1}{N} \sum_i^N \frac{|\varepsilon_i^{FE} - \varepsilon_i^{ED}|}{\varepsilon_i^{ED}}
\]

where the sum runs over states in the many-body Hilbert space. We can compute this quantity, here restricting ourselves to the half-filled states in the centre of the spectrum, for a variety of power-law exponents \( \alpha \) and \( \beta \) in order to benchmark the accuracy of our results. The results are summarised in Fig. 1, where we show the average relative error across the phase diagram shown in Fig. 4 of the main text, here for a system size of \( L = 12 \) and with \( N_s = 512 \) disorder realisations. We also verified that the error decreases rapidly with increasing disorder strength, as expected, though we do not reproduce this data here.

We do not attempt to compute level spacing distributions in this work - while it is possible to resolve exponentially small level spacings using flow equation methods (see Ref. [7]), it requires integration to extremely large flow times, and in any case the truncation of the running Hamiltonian to only a polynomial number of couplings likely prevents us from resolving exponentially small features, as previously found in Ref [3] for a short-range MBL system.

Accuracy: Invariants of the Flow

As with any other unitary transform, there are a variety of conserved quantities of the flow equation formalism. Specifically, traces of integer powers of the Hamiltonian \( I_p = \text{Tr}[\mathcal{H}^p] \) are commonly known as ‘invariants of the flow’,
and are preserved by an exact implementation of the flow equation formalism. As we have seen, however, in order for the calculation to remain tractable we must make an approximation for the running Hamiltonian of the system. The neglect of any terms not contained within the ansatz Hamiltonian introduces an error: this error may be quantified by computing the invariants of the flow at the start and end of the procedure, and then computing the difference between them. This difference is zero if the unitary transform is exact, and non-zero if the truncation has introduced an error. Here we focus on the second invariant \( p = 2 \) and define the truncation error as:

\[
\delta I_2 = \left| \frac{I_2(l = 0) - I_2(l = \infty)}{I_2(l = 0) + I_2(l = \infty)} \right|
\]  

The main source of error in this scheme is the strength of the interactions, which contribute to the generation of higher-order terms not included in our variational manifold. In the present case, as the truncated higher-order terms scale approximately with integer powers of the interaction strength \( \Delta_0 \ll 1 \), the high order terms are typically small and the accuracy very good. For example, in the results shown in Figs. 1 and 2 of the main text, \( \delta I_2 \leq 0.015 \) at all times. However, in the limit of \( \beta \to 0 \), there are a large number of interaction terms and the neglected terms can begin to become significant. To get an idea of the accuracy of our results, we can compute this quantity across the phase diagram in Fig. 4 of the main text: the result is shown in Fig. 2. We find that the transform is almost perfectly unitary across the entire phase diagram, with the main deviations away from unitarity occurring close to \( \beta = 0 \).

**Dynamics**

We can compute the dynamics of any operator by transforming the operator into the basis which diagonalises the Hamiltonian, time-evolving with respect to the diagonal Hamiltonian, and then flowing the operator back into the physical basis. We demonstrate this with the number operator \( n_i(t) \). We make the following ansatz for the flow of this operator:

\[
n_i(l, t = 0) = \sum_j A_j^{(i)}(l) : c_j^\dagger c_j : + \sum_{jk} B_{jk}^{(i)}(l) : c_j^\dagger c_k : \]

where the coefficients are explicit functions of both the fictitious flow time \( l \) and the real time \( t \).
The flow equations for this operator can be obtained by computing \( n'_i(l) = [\eta(l), n_i] \) and are given by:

\[
\frac{dA^i_j}{dl} = -2 \sum_k J_{jk}(h_k - h_j)B_{kj}, \tag{14}
\]

\[
\frac{dB_{jk}}{dl} = -J_{jk}(h_k - h_j)(A^i_k - A^j_i) - \sum_n [J_{nj}(h_n - h_j)B_{nk} + J_{nk}(h_n - h_k)B_{nj}], \tag{15}
\]

After transforming \( n_i(t = 0) \) into the diagonal basis, we can time-evolve it with respect to the diagonal Hamiltonian \( \hat{H} = \sum_k \tilde{h}_k \tilde{n}_k + \frac{1}{2} \sum_{jk} \tilde{\Delta}_{jk} \tilde{n}_j \tilde{n}_k \) by solving the Heisenberg equation of motion. As the Hamiltonian is still interacting, despite being diagonal, a closed form cannot be obtained. To close the equation, we choose a time-dependent decoupling of the interaction term, and the time-evolved operator is given by:

\[
\tilde{n}_i(l = \infty, t) = \sum_j A^{(i)}_j(l) n_j + \sum_{jk} B^{(i)}_{jk}(l) e^{i\phi_{jk}(t)} c_j^\dagger c_k, \tag{16}
\]

\[
\phi_{jk}(t) = \int_0^t dt' \left[ (\tilde{h}_k - \tilde{h}_j) + \sum_m (\tilde{\Delta}_{km} - \tilde{\Delta}_{jm})(n_m(t')) \right], \tag{17}
\]

where the expectation values are calculated self-consistently at each timestep. We then use the flow equations (Eqs. 15) to transform the number operator back into the original basis, where it will take the form:

\[
n_i(l = 0, t) = \sum_j A^{(i)}_j(t) n_j + \sum_{jk} B^{(i)}_{jk}(t) c_j^\dagger c_k, \tag{18}
\]

At this point, the expectation value of this operator may be computed with respect to the desired initial state. In the main text, our initial state is always a product state of the form \( |0101...\rangle \). Note that this procedure also gives us a very natural way to understand operator spreading [10–14], as the real-space support of the operator is encoded in the time-dependent coefficients \( A^{(i)}_j(t) \) and \( B^{(i)}_{jk}(t) \), which can be straightforwardly extracted [7].
In order to verify the accuracy of our method, we benchmarked the dynamic results with exact diagonalisation (ED). For this, we again employed the QuSpin package [8, 9]. Sample results for the density dynamics on a single site are shown in Fig. 3 for a variety of values of $\alpha$ and $\beta$ across the phase diagram. The agreement in all cases is excellent, with flow equations differing only very slightly from the exact results.

FIG. 3: The density dynamics on the central site of a chain of length $L = 12$ when quenched from a CDW initial state and averaged over 512 disorder realisations, comparing ED (blue) with FE (orange). a) $\alpha = 0.5, \beta = 2.0$, b) $\alpha = 2.0, \beta = 2.0$, c) $\alpha = 0.5, \beta = 0.5$, d) $\alpha = 3.0, \beta = 0.5$. In all cases, the results are close, but the FE method slightly overestimates the localisation. In the more strongly localised phases for $\alpha, \beta \gg 1$, the FE and ED results agree very closely. The insets show the decay of fluctuations around their long-time mean value, with $\delta n = \sigma^2(\langle n_{L/2} \rangle - \bar{n})$ and $\bar{n} = \langle n_{L/2}(t) \rangle_{t \to \infty}$: note the power-law decay in the ED data which is not seen in the FE data, due to the mean-field decoupling employed for the dynamics.

Despite this excellent agreement, it is interesting to note that the results from the flow equation method do not capture the decay of fluctuations around their mean values (shown in the insets of Fig. 3). The reason for this is due to the mean-field decoupling, which does not allow for the slow build-up of correlations that leads to the power-law decay of fluctuations (or to the logarithmic growth of entanglement entropy). Similar results are seen in the quantum Fisher information (not shown), a proxy for the entanglement entropy, which does not display the expected slow increase with time due to the nature of the mean-field decoupling used here in computing the dynamics.
FIG. 4: The same quantity as in Fig. 4 of the main text, here for system size $L = 36$ and averaged over $N_{100}$ disorder realisations. The solid white line represents $I(t^* = 100) = 0.25$ (half the maximum value) for the $L = 36$ system, and is a rough indicator of the position of the transition, while the dashed white line is the same quantity for the $L = 64$ system shown in Fig. 4 of the main text. There is a clear drift of the boundary towards larger values of $\alpha$ as we increase the system size, however the main features are robust.

Phase Diagram: Effect of System Size

To verify our conclusions, we have also computed the phase diagram for a chain of $L = 36$ sites averaged over $N_s = 100$ disorder realisations, shown in Fig. 4. The phase boundary moves, as expected, but the general conclusion is the same. This demonstrates that the main features of the phase diagram presented in the main text are robust. The flow invariant remains below a maximum value of $\delta I_2^{\text{max}} = 0.012$ at all points in this figure. This data suggests that, all other things being equal, there is a slow growth of the number of resonances as the system size is increased, consistent with the resonance counting arguments in the existing literature. Our results are an indication that even for large system sizes, localisation still persists over a large region of the phase diagram. Note however that the reversal of curvature seen in Fig. 4 of the main text for $\alpha > 1$ is not present in this data, and the $L = 36$ system is more localised in this region, with a larger imbalance. This is consistent with the idea that larger systems exhibit more delocalising resonances, destabilising the localised phase.

Random-Sign Disorder

Previous works on long-range couplings in spin chains have considered so-called ‘random sign disorder’, in which the couplings are fixed in magnitude but allowed to vary in sign, i.e. $J_{ij} = \pm J_0/|i-j|^\alpha$ and $V_{ij} = \pm V_0/|i-j|^\beta$ where the signs are chosen randomly. These works have predicted the absence of a localised phase in the regime $d \leq \beta \leq 2d$, whereas we find clear signs of localisation in this regime. While this could be a finite-size effect, or equivalently we may simply be below the critical disorder threshold for this system size, we have nonetheless simulated this type of disorder as well in order to compare with our (zero mean) Gaussian-distributed random couplings. The results are shown in Fig. 5.

Remarkably, we find that the case of Gaussian-distributed random couplings is indeed significantly more localised than the pure random-sign disorder, both quantitatively and qualitatively. This difference, while striking at first sight, can be explained simply by the typical magnitude of the coupling terms being large (and, crucially, non-zero) in the case of random-sign disorder, while the typical value is identically zero for the Gaussian-distributed disorder considered in the main text.
FIG. 5: Various static properties of the fixed-point Hamiltonian with random-sign disorder, rather than Gaussian-distributed disorder. All data here is taken for system sizes $L = 64$ with $N_s = 128$ disorder realisations, and the colour schemes are the same as in the main text. The left column shows data for long-range hopping, while the right column shows data for long-range interactions. 

a) Fixed-point couplings $\Delta_{ij}$ in the case of power-law hopping and nearest-neighbour interactions $\beta \to \infty$, again with $\alpha \in [0,5]$ as in the main text. The black dashed lines are the same as in the main text. 
b) The same quantity plotted for the case of power-law interactions (with $\alpha \to \infty$ and $\beta \in [0,5]$ as before). 
c) The real-space support of the $l$-bits in the case of long-range hopping. The black dashed line is the same as in the main text (the $\alpha, \beta \to \infty$ limit with Gaussian distributed disorder), while the blue dots show the $\alpha, \beta \to \infty$ limit of the random-sign disorder. 
d) The same quantity plotted for long-range interactions, with $\alpha \to \infty$ and $\beta \in [0,5]$. The black dashed line is again the same as in the main text, while the red dots show the $\alpha, \beta \to \infty$ limit of the random-sign disorder.

* Electronic address: steven.thomson@polytechnique.edu
† Electronic address: marco.schiro@ipht.fr
[1] F. Wegner, Journal of Physics A: Mathematical and General 39, 8221 (2006).
[2] S. Kehrein, The flow equation approach to many-particle systems, vol. 217 (Springer, 2007).
[3] S. Thomson and M. Schiró, Physical Review B 97, 060201 (2018).
[4] F. Wegner, Annalen der Physik 506, 77 (1994).
[5] C. Monthus, Journal of Physics A: Mathematical and Theoretical 49, 305002 (2016).
[6] S. Savitz and G. Refael, Physical Review B 96, 115129 (2017).
[7] S. J. Thomson and M. Schiró, Eur. Phys. J. B (2020).
[8] P. Weinberg and M. Bukov, SciPost Phys. 2, 003 (2017).
[9] P. Weinberg and M. Bukov, SciPost Phys. 7, 20 (2019).
[10] M. Foss-Feig, Z.-X. Gong, C. W. Clark, and A. V. Gorshkov, Phys. Rev. Lett. 114, 157201 (2015).
[11] A. Nahum, S. Vijay, and J. Haah, Phys. Rev. X 8, 021014 (2018).
[12] V. Khemani, A. Vishwanath, and D. A. Huse, Phys. Rev. X 8, 031057 (2018).
[13] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, Phys. Rev. B 98, 220303 (2018).
[14] D. J. Luitz and Y. Bar Lev, Phys. Rev. A 99, 010105 (2019).