UNIVERSALITY OF NEWTON’S METHOD

A. G. RAMM

1. Introduction

In many cases one is interested in solving operator equation

\[ F(u) = h \]

where \( F \) is a nonlinear operator in real Hilbert space \( H \). Let us assume that equation (1.1) has a solution \( y \),

\[ F(y) = f, \]

that the Fréchet derivative \( F'(y) \) exists and is boundedly invertible:

\[ \| [F'(y)]^{-1} \| \leq m, \quad m = \text{const} > 0. \]

Let us also assume that \( F'(u) \) exists in the ball \( B(y, R) := \{ u : \| u - y \| \leq R \} \), depends continuously on \( u \), and \( \omega(R) \) is its modulus of continuity in the ball \( B(y, R) \):

\[ \sup_{u, v \in B(y, R), \| u - v \| \leq r} \| F'(u) - F'(v) \| = \omega(r). \]

The function \( \omega(r) \geq 0 \) is assumed to be continuous on the interval \([0, 2R]\), strictly increasing, and \( \omega(0) = 0 \).

A widely used method for solving equation (1.1) is the Newton method:

\[ u_{n+1} = u_n - [F'(u_n)]^{-1}F(u_n), \quad u_0 = z, \]

where \( z \) is an initial approximation. Sufficient condition for the convergence of the iterative scheme (1.5) to the solution \( y \) of equation (1.1) are proposed in [1], [2], [3], [4], and references therein. These conditions in most cases require a Lipschitz condition for \( F'(u) \), a sufficient closeness of the initial approximation \( u_0 \) to the solution \( y \), and other conditions (see, for example, [1], p.157).

In [4] a general method, the Dynamical Systems Method (DSM) is developed for solving equation (1.2).

This method consists of finding a nonlinear operator \( \Phi(t, u) \) such that the Cauchy problem

\[ \dot{u} = \Phi(t, u), \quad u(0) = u_0, \]

has a unique global solution \( u = u(t; u_0) \), there exists \( u(\infty) = \lim_{t \to \infty} u(t; u_0) \), and \( F(u(\infty)) = f \):

\[ \exists! u(t), \quad \forall t \geq 0; \quad \exists u(\infty); \quad F(u(\infty)) = f. \]

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Many examples of the possible choices of $\Phi(t, u)$ are given in [4]. Theoretical applications of the DSM are proposed in [6], [7]. A particular choice of $\Phi$, namely, $\Phi = -\left[F'(u)\right]^{-1}(F(u) - h)$, leads to a continuous analog of the Newton method:

$$\dot{u}(t) = -\left[F'(u(t))\right]^{-1}(F(u(t)) - h), \quad u(0) = u_0; \quad \dot{u}(t) = \frac{du(t)}{dt}.$$  

The question of general interest is: under what assumptions on $F$, $h$ and $u_0$, can one establish the conclusions (1.7), that is, the global existence and uniqueness of the solution to problem (1.8), the existence of $u(\infty)$, and the relation $F(u(\infty)) = h$?

The usual condition, sufficient for the local existence and uniqueness of the solution to (1.8) is the local Lipschitz condition on the right-hand side of (1.8). Such condition can be satisfied if $F'(u)$ satisfies a Lipschitz condition.

Our goal is to develop a novel approach to a study of equation (1.8). This approach does not require a Lipschitz condition for $F'(u)$, and it leads to a justification of the conclusion (1.7) (with $h$ replacing $f$) for the solution to problem (1.8) under natural assumptions on $h$ and $u_0$.

Apparently for the first time a proof of convergence of the continuous analog (1.8) of the Newton method and of the usual Newton method (1.5) is given without any smoothness assumptions on $F'(u)$, only the local continuity of $F'(u)$ is assumed, see (1.4).

The Newton-type methods are widely used in theoretical, numerical and applied research, and by this reason our results are of general interest for a wide audience.

Our results demonstrate the universality of the Newton method in the following sense: we prove that any operator equation (1.1) can be solved by either the usual Newton method (1.5) or by the DSM Newton method (1.8), provided that conditions (1.2)-(1.4) hold, the initial approximation $u_0$ is sufficiently close to $y$, where $y$ is the solution of equation (1.2), and the right-hand side $h$ in (1.1) is sufficiently close to $f$. Precise formulation of the results is given in three Theorems.

The basic tool in this paper is a new version of the inverse function theorem. The novelty of this version is in a specification of the region in which the inverse function exists in terms of the modulus of continuity of the operator $F'(u)$ in the ball $B(y, R)$.

In Section 2 we formulate and prove this version of the inverse function theorem. The result is stated as Theorem 1.

In Section 3 we justify the DSM for equation (1.8). The result is stated in Theorem 3.

In Section 4 we prove convergence of the usual Newton method (1.5). The result is stated in Theorem 6.

2. Inverse function theorem

Consider equation (1.1).

Let us make the following Assumptions A):

1. Equation (1.2) and estimates (1.3), (1.4) hold in $B(y, R)$,
2. $h \in B(f, \rho)$, \quad $\rho = \frac{(1-q)R}{m}$, \quad $q \in (0, 1)$,
3. $m\omega(R) = q$, \quad $q \in (0, 1)$.

Assumption (3) defines $R$ uniquely because $\omega(r)$ is assumed to be strictly increasing. We assume that equation $m\omega(R) = q$ has a solution. This assumption is always satisfied if $q \in (0, 1)$ is sufficiently small. The constant $m$ is defined in (1.3).
Our first result, Theorem 1, says that under Assumptions A) equation (1.1) is uniquely solvable for any \( h \) in a sufficiently small neighborhood of \( f \).

**Theorem 1.** If Assumptions A) hold then equation (1.1) has a unique solution \( u \) for any \( h \in B(f, \rho) \), and

\[
\|F'(u)^{-1}\| \leq \frac{m}{1-q}, \quad \forall u \in B(y, R).
\]

**Proof.** Let us denote \( Q := [F'(y)]^{-1}, \|Q\| \leq m. \)

Then equation (1.1) is equivalent to

\[
u = T(u), \quad T(u) := u - Q(F(u) - h).
\]

Let us check that \( T \) maps the ball \( B(y, R) \) into itself:

\[
TB(y, R) \subset B(y, R),
\]

and that \( T \) is a contraction mapping in this ball:

\[
\|T(u) - T(v)\| \leq q\|u - v\|, \quad \forall u, v \in B(y, R),
\]

where \( q \in (0, 1) \) is defined in Assumptions A).

If (2.2) and (2.3) are verified, then the contraction mapping principle guarantees existence and uniqueness of the solution to equation (2.2) in \( B(y, R) \), where \( R \) is defined by condition 3) in Assumptions A).

Let us check the inclusion (2.3). One has

\[
J_1 := \|u - y - Q(F(u) - h)\| = \|u - y - Q[F(u) - F(y) + f - h]\|,
\]

and

\[
F(u) - F(y) = \int_{0}^{1} F'(y + s(u - y))ds(u - y)
\]

\[
= F'(y)(u - y) + \int_{0}^{1} [F'(y + s(u - y)) - F'(y)]ds(u - y).
\]

Note that

\[
\|Q(f - h)\| \leq m\rho,
\]

and

\[
\sup_{s \in [0,1]} \|F'(y + s(u - y)) - F'(y)\| \leq \omega(R).
\]

Therefore, for any \( u \in B(y, R) \) one gets from (1.3), (2.3) and (2.5) the following estimate:

\[
J_1 \leq m\rho + m\omega(R)R \leq (1 - q)R + qR = R,
\]

where the inequalities

\[
\|f - h\| \leq \rho, \quad \|u - y\| \leq R,
\]

and assumptions 2) and 3) in Assumptions A) were used.

Let us establish inequality (2.4):

\[
J_2 := \|T(u) - T(v)\| = \|u - v - Q(F(u) - F(v))\|
\]

\[
F(u) - F(v) = F'(y)(u - v) + \int_{0}^{1} [F'(v + s(u - v)) - F'(y)]ds(u - v).
\]
Note that

\[ \|v + s(u - v) - y\| = \|(1 - s)(v - y) + s(u - y)\| \leq (1 - s)R + sR = R. \]

Thus, from (2.9) and (2.10) one gets

\[ \text{(2.11)} \]

\[ J_2 \leq m\omega(R)\|u - v\| \leq q\|u - v\|, \quad \forall u, v \in B(y, R). \]

Therefore, both conditions (2.3) and (2.4) are verified. Consequently, the existence of the unique solution to (1.1) in \( B(y, R) \) is proved. \( \square \)

Let us prove estimate (2.1). One has

\[ (2.12) \quad [F'(u)]^{-1} = [F'(y) + F'(u) - F'(y)]^{-1} \]

\[ = [I + (F'(y))^{-1}(F'(u) - F'(y))]^{-1}[F'(y)]^{-1}, \]

and

\[ (2.13) \quad \|(F'(y))^{-1}(F'(u) - F'(y))\| \leq m\omega(R) \leq q, \quad u \in B(y, R). \]

It is well known that if a linear operator \( A \) satisfies the estimate \( \|A\| \leq q \), where \( q \in (0, 1) \), then the inverse operator \((I + A)^{-1}\) does exist, and \( \|(I + A)^{-1}\| \leq \frac{1}{1-q} \). Thus, the operator \([I + (F'(y))^{-1}(F'(u) - F'(y))]^{-1}\) exists and its norm can be estimated as follows:

\[ (2.14) \quad \|[I + (F'(y))^{-1}(F'(u) - F'(y))]^{-1}\| \leq \frac{1}{1-q}. \]

Consequently, (2.12) and (2.14) imply (2.1). \( \square \)

Theorem 1 is proved.

Remark 2. If \( h = h(t) \in C^1([0, T]) \), then the solution \( u = u(t) \) of equation (1.1) is \( C^1([0, T]) \) provided that Assumptions A) hold.

Indeed, if \( h = h(t) \), then a formal differentiation of equation (1.1) with respect to \( t \) yields:

\[ (2.15) \quad F'(u(t))\dot{u}(t) = \dot{h}(t). \]

Since \( u(t) \in B(y, R) \), the operator \( F'(u(t)) \) is boundedly invertible and depends continuously on \( t \) because \( u(t) \) does. Thus,

\[ \dot{u}(t) = [F'(u(t))]^{-1}\dot{h}(t), \]

so \( \dot{u}(t) \) depends on \( t \) continuously.

The formal differentiation is justified if one proves that \( u(t) \) is differentiable at any \( t \in [0, T] \), that is,

\[ (2.16) \quad u(t + k) - u(t) = A(t)k + o(k), \quad k \to 0, \quad t \in [0, T], \]

where \( A(t) \in H \) does not depend on \( k \) and at the ends of the interval \([0, T]\) the derivatives are understood as one-sided.

To establish relation (2.16) one uses equation (1.1) and the assumption \( h \in C^1([0, T]) \). One has:

\[ (2.17) \quad F(u(t + k)) - F(u(t)) = h(t + k) - h(t) = \dot{h}(t)k + o(k), \quad k \to 0, \]

and

\[ (2.18) \quad F(u(t + k)) - F(u(t)) = \int_{0}^{1} F'(u(t) + s(u(t + k) - u(t))) ds(u(t + k) - u(t)). \]
The operator \( \int_0^1 F'(u(t) + s(u(t + k) - u(t))) \) is boundedly invertible (uniformly with respect to \( k \in (0, k_0) \), where \( 0 < k_0 \) is a sufficiently small number) as long as
\[
\sup_{s \in [0, 1]} \| u(t) + s(u(t + k) - u(t)) - y \| \leq R,
\]
see (2.11). This inequality holds, as one can easily check:
\[
\| u(t) + s(u(t + k) - u(t)) - y \| = \|(1-s)(u(t) - y) + s(u(t + k) - y) \| \leq (1-s)R + sR = R.
\]
Therefore, (2.16) follows from (2.17) and (2.18). Remark 2 is proved.

\[\square\]

3. Convergence of the DSM (1.8)

Consider the following equation
\[
(3.1) \quad F(u) = h + v(t),
\]
where
\[
(3.2) \quad u = u(t), \quad v(t) = e^{-t}v_0, \quad v_0 := F(u_0) - h, \quad r = \| v_0 \|.
\]
At \( t = 0 \) equation (3.1) has a unique solution \( u_0 \).

Let us make the following Assumptions B):
1. Assumptions A) hold,
2. \( h \in B(f, \delta) \), \( \delta + r \leq \rho := \frac{(1-q)R}{m} \).

**Theorem 3.** If Assumptions B) hold, then conclusions (1.7), with \( f \) replaced by \( h \), hold for the solution of problem (1.8).

**Proof.** 1. Proof of the global existence and uniqueness of the solution to problem (1.8).

One has
\[
\| h + v(t) - f \| \leq \| h - f \| + \| v_0 e^{-t} \| \leq \delta + r \leq \rho, \quad \forall t \geq 0.
\]
Thus, it follows from Theorem 1 that equation (3.1) has a unique solution
\[
u = u(t) \in B(y, R)
\]
defined on the interval \( t \in [0, \infty) \), and \( u(t) \in C^1([0, \infty)) \).

Differentiation of (3.1) with respect to \( t \) yields
\[
(3.3) \quad F'(u) \dot{u} = \dot{v} = -v = -(F(u(t)) - h).
\]
Since \( u(t) \in B(y, R) \), the operator \( F'(u(t)) \) is boundedly invertible, so equation (3.3) is equivalent to (1.8). The initial condition \( u(0) = u_0 \) is satisfied, as was mentioned below (3.2). Therefore, the existence of the unique global solution to (1.8) is proved. \[\square\]

2. Proof of the existence of \( u(\infty) \).

From (3.1), (3.2), and (1.8) it follows that
\[
(3.4) \quad \| \dot{u} \| \leq \frac{mr}{1-q} e^{-t}, \quad q \in (0, 1).
\]
This and the Cauchy criterion for the existence of the limit \( u(\infty) \) imply that \( u(\infty) \) exists.

Integrating (3.4), one gets
\[
(3.5) \quad \| u(t) - u_0 \| \leq \frac{mr}{1-q},
\]
and
\[ \|u(\infty) - u(t)\| \leq \frac{mr}{1-q} e^{-t}. \]

3. Proof of the relation \( F(u(\infty)) = h \).

Let us now prove that
\[ F(u(\infty)) = h. \]
Relation \( (3.7) \) follows from \( (3.1) \) and \( (3.2) \) as \( t \to \infty \), because \( v(\infty) = 0 \), \( u(t) \in B(y, R) \), and \( F \) is continuous in \( B(y, R) \).

\( \square \)

Theorem 3 is proved.

\( \square \)

Remark 4. Let us explain why there is no assumption on the location of \( u_0 \) in Theorem 3. The reason is simple: in the proof of Theorem 3 it was established that \( u(t) \in B(y, R) \) for all \( t \geq 0 \). Therefore, \( u(0) \in B(y, R) \).

4. The Newton method

The main goal in this Section is to prove convergence of the Newton method
\[ u_{n+1} = u_n - [F'(u_n)]^{-1}(F(u_n) - f), \quad u_0 = z, \]
to the solution \( y \) of equation \( (1.2) \) without any additional assumptions on the smoothness of \( F'(u) \). By \( z \in H \) we denote an initial approximation.

**Theorem 5.** Assume that \( (1.2) - (1.4) \) and Assumptions A hold, and that
\[ m\omega(R) = q \in (0, \frac{1}{2}), \quad q_1 \|z - y\| \leq R, \quad q_1 := \frac{q}{1-q}. \]

Then process \( (4.1) \) converges to \( y \).

**Proof.** One has
\[ u_{n+1} - y = u_n - y - [F'(u_n)]^{-1}\int_0^1 F'(y + s(u_n - y))ds(u_n - y) \]
\[ = -[F'(u_n)]^{-1}\int_0^1 [F'(y + s(u_n - y)) - F'(u_n)]ds(u_n - y) \]
Let
\[ a_n := \|u_n - y\|, \quad a_0 = \|z - y\|. \]
Then \( (4.2), (4.3) \) and \( (2.1) \) imply
\[ a_{n+1} \leq \frac{m\omega(R)}{1-q} a_n \leq \frac{q}{1-q} a_n := q_1 a_n. \]
From the assumption \( q \in (0, \frac{1}{2}) \) one derives that \( q_1 \in (0, 1) \). Thus, using \( (4.2) \), one gets:
\[ \|u_1 - y\| := a_1 \leq q_1 a_0 \leq R. \]
By induction one obtains:
\[ \|u_n - y\| \leq R, \quad \forall n = 1, 2, 3, \ldots. \]
Consequently, \( u_n \in B(y, R) \) for all \( n \), and estimates \( (2.1) \) and \( (4.3) \) are applicable for all \( n \). Therefore, \( (4.4) \) implies
\[ a_n \leq q_1^{n-1} R, \quad \forall n = 1, 2, 3, \ldots. \]
Therefore,

\[ \lim_{n \to \infty} a_n = 0. \]

Theorem 5 is proved.

\[ \square \]

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MATHEMATICS DEPARTMENT, KANSAS STATE UNIVERSITY, MANHATTAN, KS 66506-2602, USA
E-mail address: ramm@math.ksu.edu