ERGODIC PROPERTIES OF SKEW PRODUCTS IN INFINITE MEASURE

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Abstract. Let $(\Omega, \mu)$ be a shift of finite type with a Markov probability, and $(Y, \nu)$ a non-atomic standard measure space. For each symbol $i$ of the symbolic space, let $\Phi_i$ be a measure-preserving automorphism of $(Y, \nu)$. We study skew products of the form $(\omega, y) \mapsto (\sigma \omega, \Phi_{\omega_0}(y))$, where $\sigma$ is the shift map on $(\Omega, \mu)$. We prove that, when the skew product is conservative, it is ergodic if and only if the $\Phi_i$'s have no common non-trivial invariant set.

In the second part we study the skew product when $\Omega = \{0, 1\}^\mathbb{Z}$, $\mu$ is a Bernoulli measure, and $\Phi_0, \Phi_1$ are $\mathbb{R}$-extensions of a same uniquely ergodic probability-preserving automorphism. We prove that, for a large class of roof functions, the skew product is rationally ergodic with return sequence asymptotic to $\sqrt{n}$, and its trajectories satisfy the central, functional central and local limit theorem.

1. Introduction and statement of results

Let $T = \mathbb{R}/\mathbb{Z}$ and $A = T \times \mathbb{R}$. Of course, for any $\alpha \in T$ the transformation $\Phi_0 : (x, t) \in A \mapsto (x + \alpha, t)$ is not ergodic wrt the Lebesgue measure on $A$. Now let $\beta \in T$, $\phi : T \to \mathbb{R}$ a $L^1$-function with zero mean, and $\Phi_1 : (x, t) \in A \mapsto (x + \beta, t + \phi(x))$. $\Phi_1$ also preserves the Lebesgue measure on $A$. There are clear obstructions for its ergodicity, e.g. when the equation $\phi(x) = \psi(x + \beta) - \psi(\beta)$ has a solution $\psi$.

In this paper we study ergodic properties of random iterations of such transformations. Because the invariant foliations of $\Phi_0$ and $\Phi_1$ are different, it can happen that the random dynamical system is ergodic. The theorem below gives, in terms of $\phi$, checkable conditions for ergodicity. Given a probability space $(X, \nu)$, let $L_0^1(X, \nu)$ denote the set of $L^1$-integrable functions $\phi : X \to \mathbb{R}$ with zero mean, and let the essential image of $\phi \in L_0^1(X, \nu)$ be the set of $t \in \mathbb{R}$ for which $\phi^{-1}[t - \varepsilon, t + \varepsilon]$ has positive $\nu$-measure for any $\varepsilon > 0$.

**Theorem 1.1.** Let $\mu$ be a Bernoulli measure on $\{0, 1\}^\mathbb{Z}$, let $T_0, T_1$ be probability-preserving automorphisms of a non-atomic standard probability space $(X, \nu)$, with $T_0$ ergodic, and let $\phi \in L_0^1(X, \nu)$. Then

$$F : \{0, 1\}^\mathbb{Z} \times X \times \mathbb{R} \longrightarrow \{0, 1\}^\mathbb{Z} \times X \times \mathbb{R}$$

$$(\omega, x, t) \longrightarrow (\sigma \omega, T_{\omega_0} x, t + \omega_0 \phi(x))$$

is ergodic iff the closed subgroup generated by the essential image of $\phi$ is $\mathbb{R}$.

Above and henceforth, we endow skew products with the product measure. Theorem 1.1 is consequence of a more general statement. Let $\Omega \subset \{0, \ldots, k-1\}^\mathbb{Z}$.

*Date:* May 3, 2014.

*2010 Mathematics Subject Classification.* Primary: 37A25, 37A40. Secondary: 60F05.

*Key words and phrases.* infinite ergodic theory, local limit theorem, random dynamical system, rational ergodicity, skew product.
Theorem 1.2. Let \((\Omega, \mu)\) be a shift of finite type with a Markov probability, and \(\Phi_0, \ldots, \Phi_{k-1}\) measure-preserving automorphisms of a non-atomic standard measure space \((Y, \nu)\). Assume that \(F : (\omega, y) \in \Omega \times Y \mapsto (\sigma \omega, \Phi_{\omega_0}(y))\) is conservative. Then \(F\) is ergodic iff \(\Phi_0, \ldots, \Phi_{k-1}\) have no common non-trivial invariant set.

Corollary 1.3. Let \((\Omega, \mu)\) be a shift of finite type with a Markov probability, and \(T_0, \ldots, T_{k-1}\) probability-preserving automorphisms of a non-atomic standard probability space \((X, \nu)\) with no common non-trivial invariant sets. Let \(\phi_i \in L_0^1(X, \nu)\) and \(\Phi_i : (x, t) \in X \times \mathbb{R} \mapsto (T_i(x), t + \phi_i(x))\), \(i = 0, \ldots, k-1\). Then the skew product \((\omega, x, t) \mapsto (\sigma \omega, \Phi_{\omega_0}(x, t))\) is ergodic iff \(\Phi_0, \ldots, \Phi_{k-1}\) have no common non-trivial invariant set.

Theorem 1.2 is related to a result of Kakutani \([11]\). Let \((S, \rho)\) be a probability space, \((\Omega, \mu) = (S^n, \rho^n)\), and \(\sigma : (\Omega, \mu) \to (\Omega, \mu)\) be the shift map. Kakutani proved that if \((Y, \nu)\) is a probability space and \(\{\Phi_s\}_{s \in S}\) is a measurable family of probability-preserving automorphisms of \((Y, \nu)\), then \(F : (\omega, y) \mapsto (\sigma \omega, \Phi_{\omega_0}(y))\) is ergodic iff \(\{\Phi_s\}_{s \in S}\) have no common non-trivial invariant set. Observe that, in this case, \(F\) is automatically conservative.

The first version of Kakutani’s theorem for infinite measures appeared in a paper of Wos \([17]\), also for Bernoulli systems of the form \((S^n, \rho^n)\). Thus Theorem 1.2 does not follow either from Kakutani’s neither from Wos’ results. We would like to thank David Sauzin for pointing us reference \([17]\). Indeed, he has a strong application of such result for the context of standard maps \([13]\).

Some classical theorems in ergodic theory are not valid for infinite measures. E.g. Birkhoff’s averages converge to zero almost surely, provided the transformation is conservative and ergodic. This leads the following question: what is a candidate for Birkhoff-type theorem? One attempt was made by Aaronson, who introduced the notion of rational ergodicity (see \([2,3]\) for the definition). Given a function \(f\), denote its Birkhoff sums by \(S_nf\). Rationally ergodic maps possess a sort of Cesàro-averaged version of convergence in measure: there is a sequence \(\{a_n\}_{n \geq 1}\) such that, for every \(L^1\)-function \(f\) and every sequence \(\{n_k\}_{k \geq 1}\) of positive integers, there exists a subsequence \(\{n_{k_l}\}_{l \geq 1}\) such that \(S_{n_{k_l}}f / a_{n_{k_l}}\) converges to \(\int f\) almost everywhere. This latter property is called weak homogeneity and the sequence \(\{a_n\}_{n \geq 1}\) is called a return sequence.

Many authors investigated ergodic transformations of \(\mathbb{A}\) \([7,8,12,15,16]\), but few established rational ergodicity. Aaronson and Keane \([3]\) considered “deterministic” random walks driven by irrational rotations of \(\mathbb{T}\), and showed that the associated skew product on \(\mathbb{A}\) is rationally ergodic. In \([6]\) we constructed, for almost every \(\alpha \in \mathbb{R}\), skew products of the form \((x, t) \in \mathbb{A} \mapsto (x + \alpha, t + \phi(x))\) that are rationally ergodic along a subsequence of iterates. Here we consider a special case of Theorem 1.2 and prove that the associated skew product is rationally ergodic.

Theorem 1.4. Let \(\Omega = \{0, 1\}^\mathbb{Z}\), \(\mu\) a Bernoulli measure on \(\Omega\), \(T\) a uniquely ergodic probability-preserving automorphism of a non-atomic standard probability space \((X, \nu)\), and \(\phi : X \to \mathbb{R}\) a non-zero continuous function with
\[
\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi(T^i x) \to 0 \quad \text{uniformly in } x. \tag{1.1}
\]
Then \((\omega, x, t) \mapsto (\sigma \omega, Tx, t + \omega_0 \phi(x))\) is rationally ergodic with return sequence \(\sqrt{n}\), and its trajectories satisfy central, functional central and local limit theorem.
Assumption \([1.1]\) is natural for obtaining limit theorems, because the speed of growth of \(\sum_{i=0}^{n-1} \phi(T^i x)\) has to be lower than in simple random walks of \(\mathbb{Z}\). It holds e.g. when \(\phi\) is a coboundary for \(T\).

Let \(F\) denote the skew product \((\omega, x, t) \mapsto (\sigma \omega, Tx, t + \omega_0 \phi(x))\). The third coordinate of \(F^n\) is
\[
 t + \sum_{i=0}^{n-1} \omega_i \phi(T^i x) = t + \frac{1}{2} \sum_{i=0}^{n-1} \phi(T^i x)(2\omega_i - 1) + \frac{1}{2} \sum_{i=0}^{n-1} \phi(T^i x). \tag{1.2}
\]
Let \(X_1, X_2, \ldots\) be independent identically distributed random variables, each with law \(P[X_n = 1] = P[X_n = -1] = \frac{1}{2}\). For each \(x \in X\), let \(\{S_n^x\}_{n \geq 1}\) be the martingale
\[
 S_n^x = \phi(x) \cdot X_0 + \phi(Tx) \cdot X_1 + \cdots + \phi(T^{n-1} x) \cdot X_{n-1}.
\]
Then \([1.2]\) equals \(t + \frac{1}{2}(S_n^x + \sum_{i=0}^{n-1} \phi(T^i x))\). Because \(\phi\) satisfies \([1.1]\) and \(\{S_n^x\}_{n \geq 1}\) is a martingale with bounded increments, the sequences \(\sum_{i=0}^{n-1} \omega_i \phi(T^i x), n \geq 1\), satisfy both the central and functional central limit theorem. Rational ergodicity does not follow from these theorems. For that we need a local limit theorem.

**Theorem 1.5.** Under the conditions of Theorem \([1.4]\) let \(\{s_n^x\}_{n \geq 1} \subseteq \mathbb{R}\) with
\[
 \lim_{n \to \infty} s_n^x / \sqrt{n} = 0 \quad \text{uniformly in } x.
\]
Given \(t > 0\), there are \(K, n_0 > 0\) such that
\[
 K^{-1} \leq \sqrt{n} \cdot P[S_n^x \in [-t, t] - s_n^x] \leq K, \quad \forall n > n_0, \forall x \in X.
\]

Theorem \([1.5]\) is a uniform local limit theorem with moving targets, where the increments are independent but not identically distributed. Its proof uses Fourier analysis. See e.g. \(\S 10.4\) of \([5]\) for Fourier analytical proofs of limit theorems.

Now consider a special case of Theorem \([1.4]\) let \(T_0, T_1\) be irrational rotations of \(\mathbb{T}\). When \(\phi\) has small variation, \(\Phi_1\) is a conservative perturbation of \(\Phi_0\), a particular situation that naturally appears in the phenomenon called *Arnold diffusion*. In \([14]\), the author proposed that a small perturbation in the Gevrey category of a non-degenerate integrable Hamiltonian system gives rise to a dynamics that can be reduced to a skew product extension of integrable transformations of \(\mathbb{A}\) over \(\{0, 1\}^\mathbb{Z}\), and proved that the trajectories of the skew product satisfy the functional central limit theorem.

Our results apply to a slight variation of the model proposed in \([14]\), when the integrable transformations of \(\mathbb{A}\) are \(\mathbb{R}\)-extensions of rotations of \(\mathbb{T}\), and we also obtain a uniform local limit theorem with moving targets (Theorem \([1.5]\)), and that the skew product is rationally ergodic (Theorem \([1.4]\)). We believe these results can be extended to the case treated in \([14]\).

The paper is organized as follows. In \(\S 2\) we establish the necessary preliminaries. In \(\S 3\) we prove Theorem \([1.2]\) and Corollary \([1.3]\). Section \(4\) encloses the first part of the paper, where we prove Theorem \([1.1]\). The second part consists of \(\S \S 5\) and \(6\) in \(\S 5\) we prove Theorem \([1.6]\) and in \(\S 6\) we prove Theorem \([1.4]\).

### 2. Notation and preliminaries

**Definition 2.1.** Let \(f, g : \mathbb{N} \to \mathbb{R}\). We write \(f \lesssim g\) if there is \(C > 0\) such that
\[
 |f(n)| \leq C \cdot |g(n)|, \quad \forall n \in \mathbb{N}.
\]
If \(f \lesssim g\) and \(g \lesssim f\), we write \(f \sim g\).
Given an irreducible stochastic matrix $P = (p_{ij})_{0 \leq i, j < k}$, $\Omega = \Omega(P)$ is the \textit{shift of finite type} with transition matrix $P$:

$$\Omega = \{(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \{0, \ldots, k-1\}^\mathbb{Z} : p_{\omega_i, \omega_{i+1}} > 0 \text{ for all } i \in \mathbb{Z}\}.$$ 

$\omega = (\omega_n)_{n \in \mathbb{Z}}$ denotes an element of $\Omega$. Let $\sigma : \Omega \to \Omega$ be the left shift, i.e. $(\sigma \omega)_n = \omega_{n+1}$. Given $\underline{\omega} \in \Omega$, a \textit{cylinder} containing $\underline{\omega}$ is a set of the form

$$[\omega_n = \underline{\omega}_n, \ldots, \omega_m = \underline{\omega}_m]\ = \{\omega \in \Omega : \omega_n = \underline{\omega}_n, \ldots, \omega_m = \underline{\omega}_m\}.$$ 

Given a probability vector $\pi = (\pi_0, \ldots, \pi_{k-1})$, let $\mu$ be the probability on $\Omega$ defined as

$$\mu[\omega_n = \underline{\omega}_n, \ldots, \omega_m = \underline{\omega}_m] = \pi_{\underline{\omega}_n}p_{\underline{\omega}_n, \underline{\omega}_{n+1}} \cdots p_{\underline{\omega}_{m-1}, \underline{\omega}_m}.$$ 

$\mu$ is called a \textit{Markov probability}. Clearly, it is invariant under $\sigma$.

The proof of Theorem 1.2 will use the hyperbolic structure of $\Omega$. We now setup the tools that will be needed.

### 2.1. s-sets and u-sets.

Let $\underline{\omega} \in \Omega$. A \textit{s-set} is a set of the form

$$[\omega_i = \underline{\omega}_i, i \geq n] = \{\omega \in \Omega : \omega_i = \underline{\omega}_i \text{ for all } i \geq n\}$$

and a \textit{u-set} is a set of the form

$$[\omega_i = \underline{\omega}_i, i \leq n] = \{\omega \in \Omega : \omega_i = \underline{\omega}_i \text{ for all } i \leq n\}.$$

A cylinder $[\omega_n = \underline{\omega}_n]$ can be seen either as a union of s-sets or of u-sets:

$$[\omega_n = \underline{\omega}_n] = \bigcup_{\omega \in [\omega_n = \underline{\omega}_n]} [\omega_i = \underline{\omega}_i, i \geq n] = \bigcup_{\omega \in [\omega_n = \underline{\omega}_n]} [\omega_i = \underline{\omega}_i, i \leq n].$$

Furthermore, for any $\tilde{\omega} \in [\omega_n = \underline{\omega}_n]$ the intersections

$$[\omega_i = \underline{\omega}_i, i \geq n] \cap [\omega_i = \tilde{\omega}_i, i \leq n] \quad \text{and} \quad [\omega_i = \tilde{\omega}_i, i \geq n] \cap [\omega_i = \underline{\omega}_i, i \leq n]$$

consist of single points $[\underline{\omega}, \tilde{\omega}]$ and $[\tilde{\omega}, \underline{\omega}]$.

![Local cylinder coordinates](image)

Thus the map

$$[\omega_n = \underline{\omega}_n] \quad \longrightarrow \quad [\omega_i = \underline{\omega}_i, i \geq n] \times [\omega_i = \underline{\omega}_i, i \leq n]$$

$$\tilde{\omega} \quad \longmapsto \quad ([\underline{\omega}, \tilde{\omega}], [\tilde{\omega}, \underline{\omega}]).$$

is a bijection. Call it a \textit{local cylinder coordinate} of $[\omega_n = \underline{\omega}_n]$.

Each s-set $[\omega_i = \underline{\omega}_i, i \geq n]$ is isomorphic to a one-sided symbolic space. Its sigma-algebra is generated by the infinite cylinders of the form

$$[\omega_i = \tilde{\omega}_i, i \geq n - k], \quad \text{where} \ \tilde{\omega} \in [\omega_i = \underline{\omega}_i, i \geq n] \text{ and } k \geq 0.$$
Call them \( s \)-cylinders of the \( s \)-set \( [\omega_i = \omega_i, i \geq n] \), and \( k \) the length of the \( s \)-cylinder. Of course, \( s \)-sets are \( s \)-cylinders of themselves, and even more: a \( s \)-set is a \( s \)-cylinder of infinitely many \( s \)-sets. Define \( u \)-cylinders in a similar way.

### 2.2. \( s \)-measures and \( u \)-measures

Endow each \( s \)-set \( [\omega_i = \omega_i, i \geq n] \) with a \( s \)-measure \( \mu^s \), defined on its \( s \)-cylinders by

\[
\mu^s[\omega_i = \hat{\omega}_i, i \geq n-k] = \pi_{\omega_{n-k}} \cdots \pi_{\omega_0}.
\]

Similarly, define a \( u \)-measure \( \mu^u \) on \( [\omega_i = \omega_i, i \leq n] \) by

\[
\mu^u[\omega_i = \hat{\omega}_i, i \leq n+k] = \pi_{\omega_{n+k}} \cdots \pi_{\omega_0}.
\]

\( \mu^s \) and \( \mu^u \) are one-sided Markov probabilities. A local cylinder coordinate \( [\omega_n = \omega_n] \rightarrow [\omega_i = \omega_i, i \geq n] \times [\omega_i = \omega_i, i \leq n] \) sends the restriction \( \mu[\omega_n = \omega_n] \) to the product measure \( \mu^s \times \mu^u \).

One-sided Markov probabilities satisfy a ratio preserving property: if \( A, B \) are subsets of a cylinder of length \( k \), then the quotient of the measures of their \( k \)-th iterates is preserved. This is the content of the next lemma. Let \( [\omega_i = \omega_i, i \geq n-k] \) be a \( s \)-cylinder of length \( k \) of the \( s \)-set \( [\omega_i = \omega_i, i \geq n] \). Observe that

\[
\sigma^{-k}[\omega_i = \omega_i, i \geq n-k] = [\omega_i = \omega_{i-k}, i \geq n]
\]

is another \( s \)-set, and thus can be endowed with a \( s \)-measure \( \mu^s \).

**Lemma 2.2.** Let \( [\omega_i = \omega_i, i \geq n] \) be a \( s \)-set, and let \( [\omega_i = \omega_i, i \geq n-k] \) be a \( s \)-cylinder of length \( k \). If \( A, B \subset [\omega_i = \omega_i, i \geq n-k] \), then

\[
\frac{\mu^s(\sigma^{-k}A)}{\mu^s(\sigma^{-k}B)} = \frac{\mu^s(A)}{\mu^s(B)}.
\]

(2.1)

Analogously, if \( A, B \) are contained in a \( u \)-cylinder of length \( k \) of a \( u \)-set, then

\[
\frac{\mu^u(\sigma^kA)}{\mu^u(\sigma^kB)} = \frac{\mu^u(A)}{\mu^u(B)}.
\]

**Proof.** The sigma-algebra on \( [\omega_i = \omega_i, i \geq n-k] \) is generated by \( s \)-cylinders of length \( \geq k \). Thus we can assume \( A \) and \( B \) are both \( s \)-cylinders of length \( \geq k \). Take \( \hat{\omega}, \hat{\omega} \in [\omega_i = \omega_i, i \geq n-k] \), take \( l, m \geq k \), and let

\[
A = [\omega_i = \hat{\omega}_i, i \geq n-l] \quad \text{and} \quad B = [\omega_i = \hat{\omega}_i, i \geq n-m].
\]

We have

\[
\sigma^{-k}A = [\omega_i = \hat{\omega}_{i-k}, i \geq n-l+k] \quad \text{and} \quad \sigma^{-k}B = [\omega_i = \hat{\omega}_{i-k}, i \geq n-m+k].
\]

As \( s \)-cylinders of \( [\omega_i = \omega_i, i \geq n] \), the quotient of their \( \mu^s \)-measures is

\[
\frac{\mu^s(\sigma^{-k}A)}{\mu^s(\sigma^{-k}B)} = \frac{\pi_{\omega_{n-k}} \cdots \pi_{\omega_0}}{\pi_{\omega_{n-m-k}} \cdots \pi_{\omega_0}} = \frac{\pi_{\omega_{n-k}} \cdots \pi_{\omega_0}}{\pi_{\omega_{n-m-k}} \cdots \pi_{\omega_0}} = \frac{\mu^s(A)}{\mu^s(B)}.
\]

where in the second equality we used that \( \hat{\omega}_i = \hat{\omega}_i = \omega_i \) for \( n-k \leq i \leq n \). The other statement is proved similarly. \( \square \)
The above lemma constitutes the first of three properties of s-sets, u-sets, s-measures and u-measures we will need. The second is that non-trivial subsets of cylinders cannot be simultaneously saturated by s-sets and u-sets.

**Lemma 2.3.** Let \( A \subset [\omega_n = \varpi_n] \) with positive \( \mu \)-measure. If for \( \mu \)-almost every \( \tilde{\omega} \in A \) both

\[
[\omega_i = \tilde{\omega}_i, i \geq n] \text{ and } [\omega_i = \tilde{\omega}_i, i \leq n] \subset A,
\]

then \( A = [\omega_n = \varpi_n] \).

**Proof.** Let \( A' \) be the image of \( A \) under the local cylinder coordinates \([\omega_n = \varpi_n] \rightarrow [\omega_i = \varpi_i, i \geq n] \times [\omega_i = \varpi_i, i \leq n] \). Because \([\omega_i = \tilde{\omega}_i, i \geq n] \subset A \) for \( \mu \)-almost every \( \tilde{\omega} \in A \), \( A' \) is a product set of the form \([\omega_i = \varpi_i, i \geq n] \times U \). Because \([\omega_i = \tilde{\omega}_i, i \leq n] \subset A \) for \( \mu \)-almost every \( \tilde{\omega} \in A \), \( A' \) is also a product set of the form \( S \times [\omega_i = \varpi_i, i \leq n] \). This clearly implies that \( A' = [\omega_i = \varpi_i, i \geq n] \times [\omega_i = \varpi_i, i \leq n] \), and then \( A = [\omega_n = \varpi_n] \). □

The third property is a Lebesgue differentiation theorem.

**Lemma 2.4.** Let \( A \subset [\omega_n = \varpi_n] \). Then for \( \mu \)-almost every \( \tilde{\omega} \in A \)

\[
\lim_{k \to \infty} \frac{\mu^*(A \cap [\omega_i = \tilde{\omega}_i, i \geq n - k])}{\mu^*[\omega_i = \tilde{\omega}_i, i \geq n - k]} = \lim_{k \to \infty} \frac{\mu^*(A \cap [\omega_i = \tilde{\omega}_i, i \leq n + k])}{\mu^*[\omega_i = \tilde{\omega}_i, i \leq n + k]} = 1.
\]

**Proof.** Fix a s-set \([\omega_i = \varpi_i, i \geq n] \), and let

\[
\mathcal{P}_k = \{[\omega_i = \tilde{\omega}_i, i \geq n - k] : \tilde{\omega} \in [\omega_i = \varpi_i, i \geq n]\}
\]

be its partition into s-cylinder of length \( k \). \( \bigcup_{k \geq 0} \mathcal{P}_k \) equals the sigma-algebra on \([\omega_i = \varpi_i, i \geq n]\). For each \( k \geq 0 \), let \( \mathcal{F}_k \) be the sigma-algebra generated by \( \mathcal{P}_k \).

For \( \omega \in [\omega_i = \varpi_i, i \geq n] \), \( \alpha_k(\omega) = [\omega_i = \tilde{\omega}_i, i \geq n - k] \) is the element of \( \mathcal{P}_k \) containing \( \omega \). For any measurable bounded function \( f : [\omega_i = \varpi_i, i \geq n] \to \mathbb{R} \), the sequence of functions \( \{\mathbb{E}[f|\mathcal{F}_k]\}_{k \geq 0} \) converges pointwise \( \mu^* \)-almost surely to \( f \), by the martingale convergence theorem. When \( f = \chi_A \),

\[
\mathbb{E}[f|\mathcal{F}_k](\omega) = \frac{1}{\mu^*(\alpha_k(\omega))} \int_{\alpha_k(\omega)} f \, d\mu^* = \frac{\mu^*(A \cap [\omega_i = \tilde{\omega}_i, i \geq n - k])}{\mu^*[\omega_i = \tilde{\omega}_i, i \geq n - k]},
\]

and so

\[
\lim_{k \to \infty} \frac{\mu^*(A \cap [\omega_i = \tilde{\omega}_i, i \geq n - k])}{\mu^*[\omega_i = \tilde{\omega}_i, i \geq n - k]} = \chi_A(\tilde{\omega}) \tag{2.2}
\]

for \( \mu^* \)-almost every \( \omega \in [\omega_i = \varpi_i, i \geq n] \).

By a similar argument,

\[
\lim_{k \to \infty} \frac{\mu^*(A \cap [\omega_i = \tilde{\omega}_i, i \leq n + k])}{\mu^*[\omega_i = \tilde{\omega}_i, i \leq n + k]} = \chi_A(\tilde{\omega}) \tag{2.3}
\]

for \( \mu^* \)-almost every \( \omega \in [\omega_i = \varpi_i, i \leq n] \). Because the local cylinder coordinates send \( \mu|_{[\omega_n = \varpi_n]} \) to \( \mu^* \times \mu^* \), relations (2.2) and (2.3) give the result. □
2.3. **Infinite ergodic theory.** Let $\Phi$ be an ergodic measure-preserving automorphism of a non-atomic standard measure space $(Y, \nu)$. Assume that $\Phi$ is conservative: $\nu(A) = 0$ for any measurable $A \subset Y$ such that $\{\Phi^{-n}A\}_{n \geq 0}$ are pairwise disjoint.

As stated in the introduction, for every $f \in L^1(Y, \nu)$ the Birkhoff averages $S_n f(y)/n$ converge to zero $\nu$-almost everywhere. Nevertheless, Hopf’s ratio ergodic theorem is an indication that some sort of regularity might exist and it might still be possible, for a specific sequence $\{a_n\}_{n \geq 1}$, to smooth out the fluctuations of $S_n f/a_n$ by means of a summability method.

One attempt to obtain this was made by Aaronson, who introduced the notion of rational ergodicity (see §3.3 of [2]). Given a measurable set $A \subset Y$, let $R_n : A \to \mathbb{N}$ be the return function of $A$ with respect to $\Phi$:

$$R_n(y) = \# \{1 \leq i \leq n : \Phi^i(y) \in A\}.$$  

**Definition 2.5.** A conservative ergodic measure-preserving automorphism $\Phi$ of a non-atomic standard measure space $(Y, \nu)$ is called rationally ergodic if there is a measurable set $A \subset Y$ with $0 < \nu(A) < \infty$ such that the return function $R_n : A \to \mathbb{N}$ satisfies a Renyi inequality:

$$\int_A R_n^2 d\nu \lesssim \left( \int_A R_n d\nu \right)^2.$$

Aaronson [1] (see also Theorem 3.3.1 of [2]) proved that every rationally ergodic automorphism is weakly homogeneous: if $\{a_n\}_{n \geq 1}$ is defined by

$$a_n = \frac{1}{\nu(A)^2} \int_A R_n d\nu = \frac{1}{\nu(A)^2} \sum_{i=1}^{n} \nu(A \cap \Phi^{-i}A),$$

then every sequence $\{n_k\}_{k \geq 1}$ of positive integers can be refined to a subsequence $\{n_{k_l}\}_{l \geq 0}$ such that for all $f \in L^1(Y, \nu)$ it holds

$$\frac{1}{N} \sum_{l=1}^{N} \frac{1}{a_{n_{k_l}}} S_{n_{k_l}} f(y) \to \int_Y f d\nu \text{ a.e.}$$

$\{a_n\}_{n \geq 1}$ is called a return sequence of $\Phi$ and it is unique up to asymptotic equality.

We conclude these preliminaries stating a result that will be used in the next section.

**Theorem 2.6 (Atkinson [4]).** Let $T$ be an ergodic probability-preserving automorphism of a non-atomic standard probability space $(X, \nu)$, and let $\phi \in L^0_1(X, \nu)$. Then $\nu$-almost every $x \in X$ has the following property: for any measurable set $A \subset X$ containing $x$ with $\nu(A) > 0$ and any $\varepsilon > 0$, the set

$$\{n \geq 1 : T^n x \in A \text{ and } |S_n \phi(x)| < \varepsilon\}$$

is infinite.

In other words, the $\mathbb{R}$-extension $(x, t) \mapsto (T x, t + \phi(x))$ is conservative.

3. Kakutani’s theorem: proof of Theorem [1.2]

Call $\{\Phi_0, \ldots, \Phi_{k-1}\}$ an ergodic system if $\Phi_0, \ldots, \Phi_{k-1}$ have no common non-trivial invariant set: every measurable set $A \subset Y$ such that

$$A = \Phi_0^{-1} A = \Phi_1^{-1} A = \cdots = \Phi_{k-1}^{-1} A$$

are disjoint.
either has zero or full \( \nu \)-measure. Alternatively, any \( g \in L^\infty(Y, \nu) \) such that \( g \circ \Phi_0 = \cdots = g \circ \Phi_{k-1} = g \) is constant almost everywhere.

Here we assume the skew product

\[
F : \Omega \times Y \rightarrow \Omega \times Y \\
(\omega, y) \mapsto (\sigma \omega, \Phi_\omega(y))
\]

is conservative and we want to prove that \( F \) is ergodic if and only if \( \{\Phi_0, \ldots, \Phi_{k-1}\} \) is an ergodic system. Clearly, if \( F \) is ergodic then also is \( \{\Phi_0, \ldots, \Phi_{k-1}\} \). For instance, if \( g(y) \) is invariant simultaneously for \( \Phi_0, \ldots, \Phi_{k-1} \), then \( f(\omega, y) = g(y) \) is \( F \)-invariant.

We claim the converse is equivalent to prove that any bounded \( F \)-invariant function \( f(\omega, y) \) does not depend on the first coordinate, i.e. there is a bounded function \( g(y) \) such that

\[
f(\omega, y) = g(y) \quad \text{a.e.} \tag{3.1}
\]

Indeed, if we assume this and let \( f(\omega, x, t) = g(x, t) \) be \( F \)-invariant, then whenever \( \omega_0 = i \) we get

\[
(g \circ \Phi_i)(y) = f(\sigma \omega, \Phi_i(y)) = (f \circ F)(\omega, y) = f(\omega, y) = g(y)
\]

and so \( g \) is \( \Phi_t \)-invariant. By assumption, \( g \) is constant almost everywhere and thus also is \( f \).

Fix a set \( A \subset \Omega \times Y \) of positive measure, invariant under \( F \). In terms of characteristic functions, condition (3.1) translates to saying that \( A = \Omega \times B \) for some \( B \subset Y \). Alternatively, we define \( A_y \subset \Omega \) by

\[
A = \bigcup_{y \in Y} A_y \times \{y\}
\]

and want to prove that \( A_y = \Omega \) for almost every \( (\omega, y) \in A \). We prove this using the tools developed in \([2]\).

**Lemma 3.1.** If \( \mu(A_y \cap [\omega_n = \bar{\omega}_n]) > 0 \), then \( [\omega_n = \bar{\omega}_n] \subset A_y \).

Assume Lemma 3.1 has been proved. Each non-trivial \( A_y \) intersects some cylinder \( [\omega_0 = \bar{\omega}_0] \), and then \( [\omega_0 = \bar{\omega}_0] \subset A_y \). Because \( \Omega = \Omega(P) \) and \( P \) is an irreducible matrix, there is \( n \geq 1 \) such that

\[
\mu([\omega_0 = \bar{\omega}_0] \cap [\omega_n = \bar{\omega}_n]) > 0 \quad \text{for any } \bar{\omega}, \bar{\omega} \in \Omega.
\]

In particular, \( \nu(A_y \cap [\omega_n = \bar{\omega}_n]) > 0 \) for any \( \bar{\omega} \in \Omega \). Again by Lemma 3.1, it follows that \( [\omega_n = \bar{\omega}_n] \subset A_y \) for any \( \bar{\omega} \in \Omega \), and so

\[
\Omega = \bigcup_{\bar{\omega} \in \Omega} [\omega_n = \bar{\omega}_n] \subset A_y,
\]

thus proving that \( A_y = \Omega \).

**Proof of Lemma 3.1** According to Lemma 2.3 it is enough to prove that

\[
[\omega_i = \bar{\omega}_i, i \geq n] \quad \text{and} \quad [\omega_i = \bar{\omega}_i, i \leq n] \subset A_y \tag{3.2}
\]

for almost every \( \bar{\omega} \in A_y \). Define measurable functions \( \{f_k\}_{k \geq 1} \) on \( A \) by

\[
f_k(\bar{\omega}, \bar{y}) = \frac{\mu^n(A_{\bar{y}} \cap [\omega_i = \bar{\omega}_i, i \leq n + k])}{\mu^n([\omega_i = \bar{\omega}_i, i \leq n + k])}.
\]
By Lemma 2.4,
\[
\lim_{k \to \infty} f_k(\tilde{\omega}, \tilde{y}) = 1 \quad \text{a.e.} \quad (\tilde{\omega}, \tilde{y}) \in A.
\] (3.3)

Assume first that (3.3) holds uniformly in \( A \). Fix \( \delta > 0 \) and let \( k_0 \geq 1 \) for which \( f_k > 1 - \delta \) for all \( k > k_0 \). Because \( F \) is conservative, for almost every \( (\hat{\omega}, y) \in A \) there is \( k > k_0 \) such that \( (\tilde{\omega}, \tilde{y}) = F^{-k}(\hat{\omega}, y) \in A \), and then
\[
\frac{\mu^u(A_y \cap [\omega_i = \tilde{\omega}, i \leq n + k])}{\mu^u[\omega_i = \tilde{\omega}, i \leq n + k]} > 1 - \delta.
\] (3.4)

Figure 2. The saturation of \( \Omega \times \{y\} \).

Because \( F^k([\omega_i = \tilde{\omega}, i \leq n + k] \times \{y\}) = [\omega_i = \tilde{\omega}, i \leq n] \times \{y\} \), Lemma 2.2 and relation (3.4) give that
\[
\mu^u(A_y \cap [\omega_i = \hat{\omega}, i \leq n]) = \frac{\mu^u(A_y \cap [\omega_i = \tilde{\omega}, i \leq n])}{\mu^u[\omega_i = \tilde{\omega}, i \leq n]} = \frac{\mu^u(A_y \cap [\omega_i = \tilde{\omega}, i \leq n])}{\mu^u[\omega_i = \tilde{\omega}, i \leq n+k]} > 1 - \delta.
\]

Both \((\hat{\omega}, y) \in A \) and \( \delta > 0 \) are arbitrary, and thus \([\omega_i = \hat{\omega}, i \leq n] \subset A_y \) for almost every \((\hat{\omega}, y) \in A \). Anologously, \([\omega_i = \hat{\omega}, i \geq n] \subset A_y \) for almost every \((\hat{\omega}, y) \in A \), and this establishes (3.2).

In general, the convergence in (3.3) is not uniform. Instead, do the following: for each \( A' \subset A \) with finite measure and each \( \varepsilon > 0 \), Egorov’s theorem assures the existence of \( A'' \subset A' \) such that
(1) \( (\mu \times \nu)(A' \setminus A'') < \varepsilon \), and
(2) \{\( f_k \}_{k \geq 1} \) converges uniformly in \( A'' \).

By the previous argument, (3.2) holds almost everywhere in \( A'' \). This concludes the proof of the lemma. □

Remark 3.2. In [17], Woś proved a random ergodic theorem for sub-Markovian operators in \( L^\infty \). Because Koopman-von Neumann operators of measure-preserving automorphisms of non-atomic standard probability spaces are always sub-Markovian, his result characterizes ergodicity for random dynamical systems over Bernoulli
systems. It should be interesting to mix our tools with Woś’ in order to extend his theorem to skew products over shifts of finite type.

It is not clear to us under which conditions \( F \) is conservative. For instance, it can happen that each \( \Phi_i \) is conservative and \( F \) is not. Here is an example communicated by Jon Aaronson: let \( Y = \{-1,1\}^\mathbb{Z} \times \mathbb{Z}^3 \) and \( \nu = \text{Bernoulli measure on } \{-1,1\}^\mathbb{Z} \times \text{counting measure on } \mathbb{Z}^3 \), and let \( \Phi_0, \Phi_1, \Phi_2 \) be measure-preserving transformations on \( (Y,\nu) \) given by

\[
\begin{align*}
\Phi_0(\theta, y) &= (\varphi(\theta), y + \theta_0(1, 0, 0)) \\
\Phi_1(\theta, y) &= (\varphi(\theta), y + \theta_0(0, 1, 0)) \\
\Phi_2(\theta, y) &= (\varphi(\theta), y + \theta_0(0, 0, 1)),
\end{align*}
\]

where \( \varphi \) is the shift map on \( \{-1,1\}^\mathbb{Z} \) and \( \theta \in \{-1,1\}^\mathbb{Z} \). Each \( \Phi_i \) is isomorphic to a random walk on \( \mathbb{Z} \), and so is conservative. But \( F \) is a random walk on \( \mathbb{Z}^3 \), which is not conservative.

Corollary 1.3 considers a class of conservative transformations for which the skew product is conservative, as we’ll now see.

**Proof of Corollary 1.3** By Theorem 1.2, we just need to prove that \( F \) is conservative. Consider the skew product

\[
H : \Omega \times X \rightarrow \Omega \times X \quad \left( \omega, x \right) \mapsto \left( \sigma \omega, T_{\omega_0} x \right).
\]

\( H \) is a measure-preserving transformation in the probability space \( (\Omega \times X, \mu \times \nu) \). In particular, it is conservative. By assumption, \( \{T_0, \ldots, T_{k-1}\} \) is an ergodic system. Thus, Theorem 1.2 implies that \( H \) is ergodic.

Now note that \( F(\omega, x, t) = (H(\omega, x), t + \phi_{\omega_0}(x)) \) is a skew product over \( H \) and

\[
\int_{\Omega \times X} \phi_{\omega_0}(x) d\mu(\omega) d\nu(x) = \sum_{i=0}^{k-1} \mu(\{\omega_0 = i\}) \int_X \phi_i(x) d\nu(x) = 0. \tag{3.5}
\]

By Theorem 2.6, it follows that \( F \) is conservative, and the proof is finished.  \( \square \)

Corollary 1.3 holds whenever the \( \phi_i \)'s satisfy equality (3.5). This is also a necessary condition. For example, let \( \phi_0 = 0 \) and \( \phi_1 \) without zero mean such that the closed subgroup generated by the essential image of \( \phi_1 \) is \( \mathbb{R} \). By Theorem 1.1 (to be proved in §4), \( \{\Phi_0, \Phi_1\} \) is an ergodic system. By Theorem 2.6, \( F \) is not conservative. If \( F \) is also ergodic, then it is isomorphic to the translation \( n \mapsto n + 1 \) on the integers (see Proposition 1.2.1 of [2]). But we can choose \( \phi_1 \) properly such that this is not the case.

### 4. Proof of Theorem 1.1

Let \( G \) be the closed subgroup generated by the essential image of \( \phi \). \( G \) is either equal to \( \alpha \mathbb{Z} \) or \( \mathbb{R} \). Assume \( G = \alpha \mathbb{Z} \). If \( \alpha = 0 \), then \( F \) is clearly not ergodic. If \( \alpha \neq 0 \), then

\[
A = \Omega \times X \times (\alpha \mathbb{Z} + [0, \alpha/4])
\]

is a non-trivial \( F \)-invariant set, and again \( F \) is not ergodic.

Now assume \( G = \mathbb{R} \). We want to prove that \( F \) is ergodic. Let

\[
\Phi_0(x, t) = (T_0 x, t) \quad \text{and} \quad \Phi_1(x, t) = (T_1 x, t + \phi(x)).
\]
By Corollary 1.3 it is enough to prove that \( \{\Phi_0, \Phi_1\} \) is an ergodic system. Let 
\( g(x, t) \) be a bounded function, invariant under \( \Phi_0 \) and \( \Phi_1 \). Then

\[
g(T_0 x, t) = (g \circ \Phi_0)(x, t) = g(x, t).
\]

Because \( T_0 \) is ergodic, \( g \) does not depend on the first coordinate, i.e. there is \( h(t) \) such that \( g(x, t) = h(t) \) almost everywhere. It remains to prove that \( h \) is constant

almost everywhere. Note that

\[
h(t + \phi(x)) = g(T_1 x, t + \phi(x)) = (g \circ \Phi_1)(x, t) = g(x, t) = h(t)
\]

and so \( h(t + \phi(x)) = h(t) \) for almost every \( t \in \mathbb{R} \) and almost every \( x \in X \). Thus the set

\[
P = \{ s \in \mathbb{R} : h(t + s) = h(t) \text{ for almost every } t \in \mathbb{R} \}
\]

contains the essential image of \( \phi \).

We claim that \( P \) is a closed subgroup of \( \mathbb{R} \). It is clearly a subgroup. By the

Riesz representation theorem,

\[
P = \left\{ s \in \mathbb{R} : \int \mathbb{R} h(t + s)u(t)dt = \int \mathbb{R} h(t)u(t)dt \text{ for every } u \in C_c(\mathbb{R}) \right\},
\]

where \( C_c(\mathbb{R}) \) is the set of continuous functions \( u : \mathbb{R} \to \mathbb{R} \) of compact support. By

the dominated convergence theorem, \( P \) is closed. Thus \( P = \mathbb{R} \), i.e. \( h \) is constant

almost everywhere. This concludes the proof.

5. LOCAL LIMIT THEOREM: PROOF OF THEOREM 1.5

We now prove Theorem 1.5. To simplify notation, denote \( c^x_i = \phi(T^ix) \) and

\[
s_n^x = \sum_{i=0}^{n-1} \phi(T^ix) = \sum_{i=0}^{n-1} c^x_i.
\]

As we have seen in the introduction, the third coordinate of \( F^n(\omega, x, t) \) is equal to

\[
t + \sum_{i=0}^{n-1} \omega_i c^x_i = t + \frac{1}{2} \sum_{i=0}^{n-1} c^x_i X_i + \frac{s_n^x}{2} = t + \frac{1}{2} S_n^x + \frac{1}{2} s_n^x,
\]

where \( \{S_n^x\}_{n \geq 1} \) is the martingale defined by

\[
S_n^x = c_n^x X_0 + \cdots + c_{n-1}^x X_{n-1}
\]

and \( \{X_n\}_{n \geq 1} \) are independent identically distributed random variables, each with

law \( \mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = \frac{1}{2} \).

Because \( \phi \) satisfies (1.1) and \( \{S^x_n\}_{n \geq 1} \) is a martingale with bounded increments, the

sequences \( \sum_{i=0}^{n-1} \omega_i \phi(T^ix), n \geq 1, \) satisfy both the central and functional central

limit theorem \cite{10}. Thus the trajectories of \( F \) have a normal diffusion. Furthermore,

because the trajectories of \( T \) equidistribute in \( X \), \( \{S^x_n\}_{n \geq 1} \) satisfies the local limit

theorem

\[
\lim_{n \to \infty} \sqrt{2\pi n} \cdot \mathbb{P}[S_n^x \in [a, b]] = b - a
\]

and even the local limit theorem with moving targets

\[
\lim_{n \to \infty} \sqrt{2\pi n} \cdot \mathbb{P}[S_n^x \in [a, b] - s_n] = b - a \tag{5.1}
\]

where \( \{s_n\}_{n \geq 1} \) is a sequence such that \( s_n/\sqrt{n} \to 0 \). The proof is similar to those

in §10.4 of [3].
The local limit theorems above do not imply rational ergodicity, because different $x$’s may give different rates of convergence. Rational ergodicity does not take into account multiplicative constants, so what we need is to bound the expression in the limit (5.1) away from zero and infinity, uniformly in both $x$ and $n$. This is the content of Theorem 1.5, which we’ll now prove.

We assume, after a proper dilation, that $t = 1$. The proof proceeds as follows: firstly, we use the unique ergodicity of $T$ to estimate the characteristic function of $S_X$, uniformly in $x$ and $n$. Secondly, we use Fourier analysis and this estimate to establish the result.

Given a random variable $Y$, let $\varphi_Y : \mathbb{R} \to \mathbb{C}$ be its characteristics function:

$$\varphi_Y(t) = \mathbb{E}[\exp(itY)].$$

**Lemma 5.1.** There exist $\delta, a, b, n_0 > 0$ such that for every $n > n_0$ and every $x \in X$

$$\exp(-at^2) \leq \varphi_{S_X^t}\left(\frac{t}{\sqrt{n}}\right) \leq \exp(-bt^2), \quad \forall |t| \leq \delta \sqrt{n}.$$  

**Proof.** We have

$$\varphi_{X_0}(t) = \cos t = 1 - \frac{t^2}{2} + O(t^4)$$

and so, for $|t|$ small,

$$\log \varphi_{X_0}(t) \leq \log \left(1 - \frac{t^2}{4}\right) \leq -\frac{t^2}{8}$$

and

$$\log \varphi_{X_0}(t) \geq \log(1 - t^2) \geq -2t^2.$$  

Let $C = \sup_{x \in X} |\phi(x)|$ and take $\delta > 0$ small so that

$$\exp(-2t^2) \leq \varphi_{X_0}(t) \leq \exp(-t^2/8), \quad \forall |t| < \delta C. \quad (5.2)$$

Because

$$\varphi_{S_X^t}\left(\frac{t}{\sqrt{n}}\right) = \varphi_{X_0}\left(\frac{c_{1}^n-t}{\sqrt{n}}\right) \cdots \varphi_{X_0}\left(\frac{c_{n}^n-t}{\sqrt{n}}\right),$$

(5.2) implies that, for every $|t| < \delta \sqrt{n},$

$$\exp\left(-\sum_{i=0}^{n-1}(c_i^t)^2 \cdot t^2\right) \leq \varphi_{S_X^t}\left(\frac{t}{\sqrt{n}}\right) \leq \exp\left(-\frac{\sum_{i=0}^{n-1}(c_i^t)^2}{8n} \cdot t^2\right).$$

By Birkhoff’s ergodic theorem, there is $n_0 > 0$ such that

$$\frac{1}{2} \int \phi^2 d\nu \leq \frac{1}{n} \sum_{i=0}^{n-1}(c_i^x)^2 \leq 2 \int \phi^2 d\nu, \quad \forall n \geq n_0, \forall x \in X.$$  

Take

$$a = 4 \int \phi^2 d\nu \quad \text{and} \quad b = \frac{1}{16} \int \phi^2 d\nu$$

to conclude the proof of the lemma. \[\square\]

Let $\chi_{[-1,1]}$ denote the indicator function of the interval $[-1,1]$, and fix functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $\hat{g}, \hat{h}$ denote the Fourier transforms of $g, h$.\footnote{\hat{g}, \hat{h} denote the Fourier transforms of $g, h$.}
(i) $g \leq \chi_{[-1,1]} \leq h$,
(ii) $\hat{g}(0) > 0$, and
(iii) $\hat{g}, \hat{h}$ are continuous with support contained in $[-\Delta, \Delta]$, for some $\Delta > 0$.

It is not hard to exhibit such functions. Take, for example,

$$g = \frac{1}{12} \left[ \left( \frac{-1 - 4\epsilon}{2} \right)^4 - \left( \frac{1 - 4\epsilon}{2} \right)^2 \right] \quad \text{and} \quad h = \chi_{[-1,1]}^2.$$ 

By (ii) and (iii), we can assume that $\delta > 0$ satisfies

(iv) $\hat{g}|_{[-\delta, \delta]} > \hat{g}(0)/2$.

**Proof of Theorem 1.5.** We want to estimate

$$\sqrt{n} \cdot \mathbb{P}[S_n^x \in [-1,1] - s_n^x] = \sqrt{n} \cdot \mathbb{E}[\chi_{[-1,1]}(S_n^x + s_n^x)].$$

Because

$$\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)] \leq \sqrt{n} \cdot \mathbb{E}[\chi_{[-1,1]}(S_n^x + s_n^x)] \leq \sqrt{n} \cdot \mathbb{E}[h(S_n^x + s_n^x)],$$

it is enough to estimate $\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)]$ away from zero and $\sqrt{n} \cdot \mathbb{E}[h(S_n^x + s_n^x)]$ away from infinity.

**Part 1.** Bound of $\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)]$ away from zero.

By the Fourier inverse theorem,

$$\sqrt{n} \cdot \mathbb{E}[g(S_n^x + s_n^x)] = \sqrt{n} \cdot \mathbb{E} \left[ \int_{\mathbb{R}} \hat{g}(t) \exp(it(S_n^x + s_n^x)) dt \right]$$

$$= \sqrt{n} \int_{\mathbb{R}} \hat{g}(t) \mathbb{E} [\exp(it(S_n^x + s_n^x))] dt$$

$$= \sqrt{n} \int_{\mathbb{R}} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt$$

$$= \sqrt{n} \int_{-\delta}^{\delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt + \sqrt{n} \int_{\delta < |t| < \Delta} \hat{g}(t) \varphi_{S_n^x + s_n^x}(t) dt.$$ 

We claim that there are $\lambda < 1$ and $n_0 \geq 1$ such that

$$|\varphi_{S_n^x + s_n^x}(t)| \leq \lambda^n, \quad \forall x \in X, \forall n > n_0, \forall \delta < |t| < \Delta. \quad (5.3)$$

To prove this, take $\epsilon, \rho < 1$ such that $[\epsilon, 2\epsilon] \subset \phi(X)$ and

$$|\cos(st)| < \rho, \quad \forall s \in [\epsilon, 2\epsilon], \forall \delta < |t| < \Delta.$$  

Because $\phi$ is continuous and $T$ is uniquely ergodic, there is $n_0 > 0$ such that

$$\frac{\# \{0 \leq i < n : T^ix \in \phi^{-1}[\epsilon, 2\epsilon] \}}{n} > \alpha, \quad \forall x \in X, \forall n > n_0,$$

where $2\alpha = \nu(\phi^{-1}[\epsilon, 2\epsilon]) > 0$. Thus, for every $x \in X, n > n_0$ and $\delta < |t| < \Delta$
\[ |\varphi_{S_n^x+s_n^x}(t)| = |\varphi_{S_n^x}(t)| \]
\[ = |\cos(c_0^x t) \cdots \cos(c_{n-1}^x t)| \]
\[ \leq \prod_{0 \leq i < n} |\cos(c_i^x t)| \]
\[ < \rho^\# \{0 \leq i < n: c_i^x \in [\varepsilon, 2\varepsilon]\} \]
\[ = \rho^\# \{0 \leq i < n: T^i x \in \phi^{-1} [\varepsilon, 2\varepsilon]\} \]
\[ < \rho^n \alpha_n = \lambda^n, \]

where \( \lambda = \rho^n < 1 \). This establishes (5.3). Then

\[ \left| \sqrt{n} \int_{\delta < |t| < \Delta} \hat{g}(t)\varphi_{S_n^x+s_n^x}(t)dt \right| < 2\Delta \|\hat{g}\|_\infty \cdot \sqrt{n} \cdot \lambda^n. \quad (5.4) \]

To estimate the integral close to zero, first apply a change of variables:

\[ \sqrt{n} \int_{-\delta}^{\delta} \hat{g}(t)\varphi_{S_n^x+s_n^x}(t)dt = \int_{-\delta}^{\delta} \hat{g} \left( \frac{t}{\sqrt{n}} \right) \varphi_{S_n^x+s_n^x} \left( \frac{t}{\sqrt{n}} \right) dt \]
\[ = \int_{-\delta}^{\delta} \hat{g} \left( \frac{t}{\sqrt{n}} \right) \varphi_{S_n^x} \left( \frac{t}{\sqrt{n}} \right) \exp \left( \frac{it \cdot s_n^x}{\sqrt{n}} \right) dt \]
\[ = \int_{-\delta}^{\delta} \hat{g} \left( \frac{t}{\sqrt{n}} \right) \varphi_{S_n^x} \left( \frac{t}{\sqrt{n}} \right) \cos \left( \frac{t \cdot s_n^x}{\sqrt{n}} \right) dt. \]

Let \( \beta > 0 \) such that \( \cos \left| [-\beta, \beta] \right| > 1/2 \), let

\[ m_n^x = \min \left\{ \frac{\beta \sqrt{n}}{s_n^x}, \frac{\delta \sqrt{n}}{2} \right\}, \]

and divide the former integral into two parts accordingly to \( m_n^x \):

\[ \sqrt{n} \int_{-\delta}^{\delta} \hat{g}(t)\varphi_{S_n^x+s_n^x}(t)dt = \int_{|t| < m_n^x} \hat{g} \left( \frac{t}{\sqrt{n}} \right) \varphi_{S_n^x} \left( \frac{t}{\sqrt{n}} \right) \cos \left( \frac{t \cdot s_n^x}{\sqrt{n}} \right) dt + \]
\[ \int_{m_n^x < |t| < \delta} \hat{g} \left( \frac{t}{\sqrt{n}} \right) \varphi_{S_n^x} \left( \frac{t}{\sqrt{n}} \right) \cos \left( \frac{t \cdot s_n^x}{\sqrt{n}} \right) dt \]
\[ = I_1 + I_2. \]

By the choice of \( g \) and \( \beta \), the fact that \( m_n^x \to \infty \) as \( n \to \infty \), and Lemma 5.1 we have

\[ I_1 \geq \frac{\hat{g}(0)}{4} \int_{|t| < m_n^x} \exp(-at^2)dt \geq \frac{\hat{g}(0)}{4} \int_{-1}^{1} \exp(-at^2)dt \]
for every sufficiently large $n$ and arbitrary $x$. Similarly,

$$I_2 \geq - \int_{m_n^* < |t| < \delta \sqrt{n}} \left| \hat{g} \left( \frac{t}{\sqrt{n}} \right) \varphi S_n^x \left( \frac{t}{\sqrt{n}} \right) \cos \left( \frac{t x_n^e}{\sqrt{n}} \right) \right| dt$$

$$\geq - \int_{m_n^* < |t| < \delta \sqrt{n}} \| \hat{g} \|_\infty \exp(-bt^2) dt \geq - \| \hat{g} \|_\infty \int_{|t| > m_n^*} \exp(-bt^2) dt.$$ 

Thus

$$\sqrt{n} \int_{-\delta}^{\delta} \hat{h}(t) \varphi S_n^x + s_n^x(t) dt \geq \frac{\hat{g}(0)}{4} \int_{-1}^{1} \exp(-at^2) dt - \| \hat{g} \|_\infty \int_{|t| > m_n^*} \exp(-bt^2) dt$$

is bounded away from zero if $n$ is large, uniformly in $x$. This, together with (5.4), proves Part 1.

**Part 2.** Bound of $\sqrt{n} \cdot \mathbb{E}[h(S_n^x + s_n^x)]$ away from infinity.

Analogously as in Part 1, inequality (5.3) gives

$$\left| \sqrt{n} \int_{|t| < \Delta} \hat{h}(t) \varphi S_n^x + s_n^x(t) dt \right| < 2\Delta \| \hat{h} \|_\infty \cdot \sqrt{n} \cdot \lambda^n$$

and Lemma 5.1 gives

$$\sqrt{n} \int_{-\delta}^{\delta} \hat{h}(t) \varphi S_n^x + s_n^x(t) dt = \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} \hat{h} \left( \frac{t}{\sqrt{n}} \right) \varphi S_n^x + s_n^x \left( \frac{t}{\sqrt{n}} \right) dt \leq \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} \hat{h} \left( \frac{t}{\sqrt{n}} \right) \varphi S_n^x \left( \frac{t}{\sqrt{n}} \right) dt \leq \| \hat{h} \|_\infty \int_{-\delta \sqrt{n}}^{\delta \sqrt{n}} \exp(-bt^2) dt \leq \| \hat{h} \|_\infty \int_{\mathbb{R}} \exp(-bt^2) dt,$$

which is finite. \(\square\)

The above proof is robust: given a compact set $\Lambda$, there are constants $K, n_0 > 0$ such that Theorem 1.5 is valid for any sequence $\{s_n^x + t\}_{n \geq 1}, t \in \Lambda$.

**6. Rational ergodicity: proof of Theorem 1.4**

Let $R_n$ be the return function of $\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]$ with respect to $F$:

$$R_n(\omega, x, t) = \# \left\{ 1 \leq i \leq n : F^i(\omega, x, t) \in \Omega \times X \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\}$$

$$= \sum_{i=1}^{n} \chi_{[-1,1]}(S_i^x(\omega) + s_i^x + 2t).$$
We will show that
\[
\int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R_n^2 \lesssim n \lesssim \left( \int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R_n \right)^2.
\]

(6.1)

Fix \( x \in X \) and \( t \in [-\frac{1}{2}, \frac{1}{2}] \). By Theorem 1.5 we have
\[
\int_{\Omega} R_n(\omega, x, t) = \sum_{i=1}^{n} \int_{\Omega} \chi_{[-1,1]}(S^x_i(\omega) + s^x_i + 2t)
\]
\[
= \sum_{i=1}^{n} P[S^x_i \in [-1,1] - (s^x_i + 2t)]
\]
\[
\sim \sum_{i=n_0+1}^{n} P[S^x_i \in [-1,1] - (s^x_i + 2t)]
\]
\[
\sim \sum_{n_0 < i \leq n} i^{-1/2}
\]
\[
\sim \int_{1}^{n} x^{-1/2} dx
\]
\[
\sim \sqrt{n}
\]
and thus
\[
\int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R_n \gtrsim \sqrt{n}.
\]

(6.2)

Now
\[
\int_{\Omega} R_n(\omega, x, t)^2 = \sum_{i=1}^{n} \int_{\Omega} \chi_{[-1,1]}(S^x_i(\omega) + s^x_i + 2t)
\]
\[
+ 2 \sum_{i<j} \int_{\Omega} \chi_{[-1,1]}(S^x_i(\omega) + s^x_i + 2t) \cdot \chi_{[-1,1]}(S^x_j(\omega) + s^x_j + 2t)
\]
\[
= \int_{\Omega} R_n(\omega, x, t)
\]
\[
+ 2 \sum_{i<j} P[S^x_i \in [-1,1] - (s^x_i + 2t), S^x_j \in [-1,1] - (s^x_j + 2t)]
\]
\[
\sim \sqrt{n} + \sum_{i<j} P[S^x_i \in [-1,1] - (s^x_i + 2t), S^x_j \in [-1,1] - (s^x_j + 2t)].
\]

Observe that \( S^x_j(\omega) = S^x_i(\omega) + S^{T_{j-i}}_{T_{j-i}}(\sigma^i \omega) \). Because \( S^x_i(\omega) \) depends on the coordinates \( \omega_0, \ldots, \omega_{i-1} \) and \( S^{T_{j-i}}_{T_{j-i}}(\sigma^i \omega) \) depends on the coordinates \( \omega_i, \ldots, \omega_{j-1} \), the random variables \( S^x_i(\omega) \) and \( S^{T_{j-i}}_{T_{j-i}}(\sigma^i \omega) \) are independent. Thus, whenever \( i > n_0 \)
and $j - i > n_0$

\[
\mathbb{P}\left[ S^x_i \in [-1, 1] - (s^x_i + 2t), \quad S^x_j \in [-1, 1] - (s^x_j + 2t) \right] \leq \mathbb{P}\left[ S^x_i \in [-1, 1] - (s^x_i + 2t), \quad S^{Tx}_{j-i} \in [-2, 2] - (s^x_i + s^x_j) \right] \n
= \mathbb{P}[S^x_i \in [-1, 1] - (s^x_i + 2t)] \times \mathbb{P}[S^{Tx}_{j-i} \in [-2, 2] - (s^x_i + s^x_j)]
\]

\[
\lesssim (j - i)^{-1/2}.
\]

It follows that

\[
\sum_{i<j} \mathbb{P}\left[ S^x_i \in [-1, 1] - (s^x_i + 2t), \quad S^x_j \in [-1, 1] - (s^x_j + 2t) \right] \lesssim \sum_{1 \leq i < j \leq n, i, j > n_0} (j - i)^{-1/2}
\]

\[
\leq \left( \sum_{i=1}^{n} i^{-1/2} \right)^2 \sim n.
\]

This implies

\[
\int_{\Omega} R^2_n(\omega, x, t)^2 \lesssim n
\]

for every $x \in X$ and every $t \in [-\frac{1}{2}, \frac{1}{2}]$. Thus

\[
\int_{\Omega \times X \times [-\frac{1}{2}, \frac{1}{2}]} R^2_n \lesssim n
\]

which, together with (6.2), establishes (6.1). This concludes the proof of Theorem 1.4.

**Remark 6.1.** A consequence of Theorem 1.4 is that $F$ has generalized laws of large numbers (see §3.3 of [2]). A natural step in the program is to prove that it satisfies a second-order ergodic theorem in the sense of [9].

7. Acknowledgements

P.C. is supported by Fapesp-Brazil. Y.L. is supported by the European Research Council, grant 239885. E.P. is supported by CNPq-Brazil.

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