Hilbert bundles as quantum-classical continua

Dedicated to Sanda Cleja-Tīgoiū on the occasion of her 70th birthday

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Abstract
A hybrid quantum-classical model is proposed whereby a micro-structured (Cosserat-type) continuum is construed as a principal Hilbert bundle. A numerical example demonstrates the possible applicability of the theory.

1 Motivation
Graphene-based qubits constitute one of the most promising avenues for the realization of quantum computers. By rolling a graphene sheet to produce a nanotube, it is possible to enhance both the mechanical stability of the sheet and the temporal coherence of the qubits. As a thought experiment, imagine a rectangular sheet reinforced at two of its opposite edges with rigid bars, as shown in Figure 1. As the two bars are moved in parallel towards each other, the sheet attains a curved shape, as shown. Suppose that a uniform distribution of qubits (say μ qubits per unit area) has been initially prepared uniformly over the whole sheet. If the sheet were to remain undeformed, the evolution would also be uniform and would abide by the Hamiltonian $H_0 = \mu \hat{\sigma}_x$, where $\hat{\sigma}_x$ is the Pauli matrix in the $x$-direction. Remarking that this example is not representative of an actual physical system or a technology to realize it, we will assume that the Hamiltonian is affected by the local curvature according to $H_R = \mu(1 + L/R)\hat{\sigma}_x$, where $R$ is the local radius of curvature. This radius depends on position and, by varying the distance between the rigid edges, it can be made to depend on time as well, thus achieving a kind of modulation.

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Examples of this kind, where the qubits can be replaced with harmonic oscillators or more involved quantum units, make us think of the theory of continuous media with internal microstructure. It goes back to the pioneering work of the Cosserat brothers [5], who proposed to enrich the kinematic description of a deformable body $B$ by considering, as part of its constitution, additional degrees of freedom supplied by one or more vector fields defined over $B$. A granular medium, for example, can be conceived as a matrix (or macromedium) endowed at each point with a triple of linearly independent vectors representing the grains (or micromedium). The triad is permitted to undergo strictly affine deformations, implying that the grains are small enough so that they sustain states of constant strain. In modern terminology, one has replaced the body $B$ (a 3-dimensional differentiable manifold) with its principal frame bundle $FB$. A configuration of $FB$ is a principal frame-bundle morphism $K : FB \rightarrow F\mathbb{E}^3$, where $F\mathbb{E}^3$ is the natural frame bundle of the ordinary Euclidean space of classical mechanics. Accordingly, we obtain the commutative diagram

$$
\begin{array}{ccc}
FB & \xrightarrow{K} & F\mathbb{E}^3 \\
\pi_B \downarrow & & \downarrow \pi_E \\
B & \xrightarrow{\kappa} & \mathbb{E}^3
\end{array}
$$

where $\kappa$ is an ordinary configuration of the macromedium, that is, an embedding of $B$ into $\mathbb{E}^3$, and where $\pi_B$ and $\pi_E$ are the respective bundle projections of $FB$ and $F\mathbb{E}^3$. We notice that, since bundle morphisms are, by definition, fibre preserving, the map $\kappa$ is already implied by the morphism $K$.
Other kinds of continua can be successfully modelled in this spirit [3]. Thus, some theories of nematic liquid crystals use a single vector field to represent the internal structure, while A-smectics can be represented by means of a differential form, whose lack of local exactness indicates the presence of defects [9]. What is common to these models is that the corresponding microstructured bodies are represented by associated bundles\(^1\) of \(FB\). A more general microstructural apparatus, in which the body is an arbitrary fibre bundle not necessarily associated with the principal frame bundle of an ordinary body manifold, has been introduced in [10, 1], on a purely theoretical basis.

In this paper, a model is proposed that can possibly accommodate a coupling between a classical continuum \(B\) and a quantum microstructure. This hybrid model is a fibre bundle over \(B\) whose typical fibre is a (complex) Hilbert space. As already remarked above, a technological motivation for such hybrids can be gathered from the many applications of graphene sheets, with a thickness consisting of a single atomic layer. Beyond such applications, however, is the idea of producing a viable geometric setting for a more general quantum-classical continuum. Although the Cosserat continuum has been invoked in a number of articles that include quantum-mechanical effects [16, 17, 4, 2], the essential geometric structure consisting of a fibre bundle with a Hilbert-space fibre appears to be new and promising. It is perhaps worthwhile at the outset to state that, if any, this paper constitutes a contribution to continuum mechanics rather than to quantum physics. Interesting quantum-classical hybrids have been proposed (see, e.g. [8]) for discrete systems. Our objective here is only to provide a framework for incorporating some elements of the quantum paradigm into the classical continuum\(^2\).

## 2 Hilbert bodies

Recall that a (smooth) fibre bundle consists of a differentiable manifold \(C\) (the total space), a base manifold \(B\), and a typical fibre manifold \(F\). These three manifolds are related in the following way:

1. There is a distinguished differentiable surjective projection map \(\pi : C \rightarrow \)

\(^1\)A rigorous treatment of the concept of associated bundles can be found, for example, in [13], pp 54-57.

\(^2\)Hilbert bundles have been used to provide alternative formulations of quantum mechanics and quantum field theory. See [14, 7, 12].
of everywhere maximal rank. In other words, the projection map is a surjective submersion. At each point \( b \in B \), the set \( \pi^{-1}(\{b\}) \) is called the fibre over \( b \).

2. Every point \( b \in B \) has an open neighbourhood \( V \subset B \) and a diffeomorphism \( \phi_V : \pi^{-1}(V) \to V \times F \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\pi^{-1}(V) & \xrightarrow{\phi_V} & V \times F \\
\downarrow{\pi} & & \downarrow{pr_1} \\
V & & 
\end{array}
\]

where \( pr_1 \) stands for the first projection of a Cartesian product. The pair \((V, \phi_V)\) is called a local trivialization.

3. Whenever two local trivializations, \((V_1, \phi_1)\) and \((V_2, \phi_2)\), have a non-empty intersection \( V = V_1 \cap V_2 \neq \emptyset \), the transition map \( \phi_{1,2} = \phi_2 \circ \phi_1^{-1} : \phi_1(V) \to \phi_2(V) \) is a diffeomorphism. We notice that at each point \( b \in V \) the transition map is a diffeomorphism of the typical fibre onto itself. We require each of these fibre diffeomorphisms to belong to a Lie group of transformations \( G \), known as the structure group of the bundle. This group is part and parcel of the definition.

More intuitively, a fibre bundle looks locally (that is, chunk-wise) as the product of two manifolds, namely, \( V \times F \). This identification is called local triviality. Once a trivialization is given, however, any other trivialization must be compatible with it, in the sense that the relation between the trivializations is restricted to a group of transformations \( G \) of the typical fibre \( F \). For example, if \( F \) is a vector space, we may require that \( G \) be the group of linear automorphisms of \( F \).

**Definition 2.1** A Hilbert bundle \( H \) is a fibre bundle over an ordinary body \( B \) whose typical fibre is a (separable) complex Hilbert space \( H \) and whose structure group is the unitary group \( U_H \) of \( H \).

We can construct the principal bundle \( P \) over \( B \) associated with \( H \) in the standard fashion \cite{15} by construing its fibre as the unitary group \( U_H \). As in any principal bundle, this group has a right action on the bundle itself. We will refer to this principal bundle as a Hilbert body. The fibre at \( X \in B \) will be denoted by \( P_X \).
3 Configurations and deformations

Our Hilbert body manifests itself in the Cartesian product $S = E^3 \times U_H$, in the sense suggested by the following definition.

**Definition 3.1** A configuration of a Hilbert body $P$ is a fibre-preserving embedding $K : P \to S$.

Given two configurations, $K_0$ and $K$, the deformation from the first to the second is the composition

$$\Xi = K \circ K_0^{-1} : K_0(P) \to K(P).$$

(3)

As a fibre-preserving map, $\Xi$ implies also an ordinary deformation $\xi$ of the base manifold $B$. A deformation $\Xi$ can be clearly regarded as a fibre-bundle morphism. The fibre-wise maps are assumed to be unitary transformations. Thus, the time evolution of a Hilbert body consists of an ordinary classical mechanics deformation of the body $B$ supplemented with a quantum field riding on the fibres. The interaction between these two mechanisms is the subject of the constitutive equations that define the particular nature of each system.

In a first-grade theory (for the so-called simple bodies) the material response depends only on the local 1-jet of the deformation. At each point $X \in B$, this 1-jet consists of 4 elements: (i) the macroscopic deformation $\xi(X)$, which we discard given the standard assumed invariance under spatial translations; (ii) the classical deformation gradient, which is a linear isomorphism $F(X) : T_X B \to T_{\xi(X)} E^3$; (iii) a unitary transformation $U(X) : P_X \to P_{\xi(X)}$; (iv) the referential gradient $\nabla U(X)$, which, in a coordinate system $X^I(I = 1, 2, 3)$ of $B$ in the reference configuration $K_0$, can be expressed as the 3 partial derivatives $U_I$.

4 Constitutive equations and time evolution

The kinematic setting introduced above appears to be general enough to accommodate the most sophisticated theories, including quantum field theories and entanglement, but this is not the intention of the present work. Rather, the objective is to provide a geometric framework to allow for a modicum of coupling between a classical macromedium and a quantum micromedium.
With this idea in mind, we can assume that the fibre at each point abides by the Schrödinger equation, except that the parameters of the system at hand depend on the present state of deformation of the macromedium at that point. Thus, the evolution is that of a system with time-dependent parameters\(^3\) although the time-dependence is not explicit, but mediated by the current value the deformation gradient \(\mathbf{F}(X)\) via, for instance, the invariants of the tensor \(\mathbf{C} = \mathbf{F}^T \mathbf{F}\).

Concomitantly, we may be at liberty of assuming that the deformation of the macromedium is also affected by the present state of the micromedium. This issue, however, may be problematic, since even an indirect measurement of the state of a quantum system leads to a collapse to an eigenvector. A milder, perhaps permissible, coupling could be obtained through the gradient \(\nabla U\). For each value of the index \(I\), the product \(W_I = U^\dagger U_I\) is an antihermitian operator, which can be used to mediate the coupling by means, for example, of the norm of the resulting vector of the traces or other invariants of \(W_I\).

Perhaps the simplest example of a Hilbert body consists of identifying the typical fibre \(H\) with the configuration space of a qubit, thus rendering a finite-dimensional fibre. A time-dependent coupling can be achieved by multiplying each of the Pauli matrices by a real function of the tensor \(\mathbf{C}\), while, if necessary, the elasticity of the underlying classical body can be made to depend on the norm alluded to above. Applicability of such systems can be substantiated experimentally, as shown in [6]. Vibrations of a graphene sheet can better be modelled with the infinite dimensional fibre of the harmonic oscillator.

5 Numerical example

We imagine (Figure 2) a long ribbon of unit cross sectional area with a periodic qubit density function \(\mu(X) = 1 + 0.2 \sin^2(\pi X)\), where \(X\) is the referential axial coordinate along the length of the ribbon. This may perhaps suggest the presence of a quantum dot per some unit of length. The ribbon is subjected to a uniform axial force \(f\) that varies in time \(t\) according to \(f = e^{-1/t}\), starting smoothly at zero and increasing to attain asymptotically

\(^3\)See, for example [11] for a detailed treatment of the time-dependent quantum harmonic oscillator.
a unit value. The qubit Hamiltonian (density) is assumed to be given by

\[ H(X, t) = 0.5 \mu(X) g(X, t) \sigma_z, \quad (4) \]

where \( \sigma_z \) is the third Pauli matrix, and where \( g(X, t) \) is a function involving the present and local value of the axial strain.

\[ f \leftarrow \begin{array}{cccccccc}
\text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \text{gray} & \text{darker gray} & \text{lighter gray} & \end{array} \rightarrow f \]

Figure 2: Varying density of qubits in ribbon

Denoting by \( U(X, t) \) the solution to the time-dependent Schrödinger equation at \( X \), we can evaluate the (real) determinant \( \Delta(X, t) \) of the anti-Hermitian operator \( W = U(X)U^\dagger \). For the sake of this (admittedly non-physical) example, we assume the elastic stiffness of the ribbon to be given by the expression

\[ E(X, t) = 50(2 - e^{-0.1|\Delta|}). \quad (5) \]

The interaction factor \( g(X, t) \) is assumed to be given by

\[ g(X, t) = 1 + \frac{f}{E(X, t)}. \quad (6) \]

Inertia effects are disregarded.

To solve for \( U(X, t) \), we impose cyclic boundary conditions at \( X = 0 \) and \( X = 1 \), and adopt the unit matrix as the initial condition at \( t = 0 \). The resulting system of first-order PDEs is solved numerically using the Mathematica® software. Figure 3 shows the final distribution of strains in the typical unit. The concentration of strains near the end and the middle is due to the large value of the determinant that this particular example delivers far from these regions. Thus, the appearance of localized inhomogeneities or defects is associated with the location of the maxima and minima of the density function \( \mu \).
Figure 3: Strain distribution in unit

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