Spectral Analysis of the Kohn Laplacian on Lens Spaces

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Joint work with Elena Kim (MIT) and Yunus E. Zeytuncu (Dearborn).
Funded by the National Science Foundation (DMS-1950102 and DMS-1659203).
\( \Omega \subseteq \mathbb{R}^d \) is a bounded domain.

\( N(\lambda) \) is the number of eigenvalues less than \( \lambda \) (counting their multiplicities) of the standard Laplacian

\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}.
\]
Weyl’s Law: Spectrum (Eigenvalues) and Geometry

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- Weyl’s law:

**Theorem (Weyl-1911)**

\[
\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{\text{vol}(\Omega)}{2^d \pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right)}.
\]

- One can generalize Weyl’s law to Riemannian manifolds.
CR Manifold

- CR stands for either Cauchy-Riemann or complex-real.
- Smooth manifolds but with some complex structure:

\[ T_p(M) = H_p(M) \oplus X_p(M), \]

where \( H_p(M) \) is the complex tangent space and \( X_p(M) \) is the real tangent space.
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- Roughly,

**Definition**

Let \( M \) be a smooth manifold. \( M \subseteq \mathbb{C}^n \) is a CR manifold if and only if \( \dim H_p(M) \) is independent of \( p \).

- **Example**: any hypersurface in \( \mathbb{C}^n \), like \( S^{2n-1} \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n} \).
- **Example**: any complex manifold.
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- **Example**: any complex manifold.
- Every CR manifold comes with a Kohn Laplacian, \( \Box_b \) (CR version of standard Laplacian).
Goal: Analog of Weyl’s law for the Kohn Laplacian on spheres $S^{2n-1}$,

$$\Box_b : L^2(S^{2n-1}) \to L^2(S^{2n-1}).$$
Goal: Analog of Weyl's law for the Kohn Laplacian on spheres \( S^{2n-1} \),

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\]

\( L^2 \left( S^{2n-1} \right) \) has spectral decomposition,

\[
L^2 \left( S^{2n-1} \right) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}_{p,q} \left( S^{2n-1} \right).
\]

Folland: Eigenvalue associated with \( \mathcal{H}_{p,q} \left( S^{2n-1} \right) \) is \( 2q \left( p + n - 1 \right) \).

\[
\dim \mathcal{H}_{p,q} \left( S^{2n-1} \right) = \binom{n+p-1}{p} \binom{n+q-1}{q} - \binom{n+p-2}{p-1} \binom{n+q-2}{q-1}.
\]
Theorem (BGS$^+21$)

Let \( N(\lambda) \) be the eigenvalue counting function (with multiplicity) for \( \Box_b \) on \( L^2(S^{2n-1}) \). Then,

\[
\lim_{n \to \infty} \frac{N(\lambda)}{\lambda^n} = \text{vol}(S^{2n-1}) \frac{n-1}{n(2\pi)^n \Gamma(n+1)} \int_{-\infty}^{\infty} \left( \frac{x}{\sinh x} \right)^n e^{-(n-2)x} \, dx.
\]
**Theorem (BGS+21)**

Let $N(\lambda)$ be the eigenvalue counting function (with multiplicity) for $\Box_b$ on $L^2(S^{2n-1})$. Then,

$$\lim_{n \to \infty} \frac{N(\lambda)}{\lambda^n} = \text{vol}(S^{2n-1}) \frac{n - 1}{n (2\pi)^n \Gamma(n + 1)} \int_{-\infty}^{\infty} \left(\frac{x}{\sinh x}\right)^n e^{-(n-2)x} \, dx.$$ 

In comparison to the standard Laplacian:

**Theorem (Weyl-1911)**

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n-1/2}} = \frac{\text{vol}(S^{2n-1})}{2^{2n-1} \pi^{n-1/2} \Gamma(n + \frac{1}{2})}.$$
A lens space is a quotient of an odd-dimensional sphere by the action of a particular type of matrix. Example: $\mathbb{R}P^3$. 

Let $k \in \mathbb{N}$, $\zeta = e^{2\pi i/k}$.

There exist $l_1, \ldots, l_n$ relatively prime to $k$.

$g \in U(n)$, $g\zeta_j g^{-1} = \zeta_{l_j} z_j g$.

The lens space denoted by $L(k; l_1, \ldots, l_n) = L(\vec{l})$ is the quotient of $S^{2n-1}$ by $G = \langle g \rangle$. 

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Lens Spaces

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- $l_1, \ldots, l_n$ relatively prime to $k$.

- $g \in U(n), g z_j = \zeta^{l_j} z_j$

$$g = \begin{bmatrix}
\zeta^{l_1} & 0 & \cdots & 0 \\
0 & \zeta^{l_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \zeta^{l_n}
\end{bmatrix}.$$

- The lens space denoted by $L(k; l_1, \ldots, l_n) = L(k; \vec{l})$ is the quotient of $S^{2n-1}$ by $G = \langle g \rangle$. 
Given the lens space $L(k; l_1, \ldots, l_n)$, we denote the eigenvalue counting function for $\Box_b$ on the lens space by $N_L(\lambda)$, and the eigenvalue counting function for $S^{2n-1}$ by $N(\lambda)$. We have

$$\lim_{\lambda \to \infty} \frac{N_L(\lambda)}{N(\lambda)} = \frac{1}{k}.$$
$G$ acts naturally on $L^2(S^{2n-1})$ by precomposition

$$g \ast f = f \circ g.$$ 

- $g \ast \mathcal{H}_{p,q}(S^{2n-1}) \subseteq \mathcal{H}_{p,q}(S^{2n-1}).$
- Denote the set of elements of $\mathcal{H}_{p,q}(S^{2n-1})$ that are fixed under the action of $G$ by $\mathcal{H}^G_{p,q}.$
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- Denote the set of elements of $\mathcal{H}_{p,q}(S^{2n-1})$ that are fixed under the action of $G$ by $\mathcal{H}_{p,q}^G$.  
- We have

$$L^2(L(k; l_1, \ldots, l_n)) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}_{p,q}^G.$$  

- The eigenvalue for $\mathcal{H}_{p,q}^G$ for $\Box_b$ on the lens space is the same as the eigenvalue on the sphere, $2q(p + n - 1)$.  
- Want to compute $\dim \mathcal{H}_{p,q}^G$.  

$\mathcal{H}_{p,q}^G$
For $\alpha \in \mathbb{N}^n$, define $|\alpha| = \sum_{j=1}^{n} \alpha_j$.

Denote, for $\alpha, \beta \in \mathbb{N}^n$

$$D^\alpha = \frac{\partial |\alpha|}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \quad D^\beta = \frac{\partial |\beta|}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}}.$$
Basis for $\mathcal{H}_{p,q}(S^{2n-1})$

- For $\alpha \in \mathbb{N}^n$, define $|\alpha| = \sum_{j=1}^{n} \alpha_j$.

- Denote, for $\alpha, \beta \in \mathbb{N}^n$

$$\overline{D}^\alpha = \frac{\partial |\alpha|}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \quad D^\beta = \frac{\partial |\beta|}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}}.$$ 

- For $p, q \in \mathbb{N}$

$$\left\{ \overline{D}^\alpha D^\beta |z|^{2-2n} : |\alpha| = p, |\beta| = q, \alpha_1 = 0 \text{ or } \beta_1 = 0 \right\}$$

is a basis for $\mathcal{H}_{p,q}(S^{2n-1})$. 
Invariant Basis Elements of $\mathcal{H}_{p,q}(S^{2n-1})$

- Let

$$f_{\alpha,\beta} = \overline{D}^\alpha D^\beta |z|^{2-2n}.$$
Invariant Basis Elements of $\mathcal{H}_{p,q}(S^{2n-1})$

- Let
  
  $f_{\alpha,\beta} = \overline{D^\alpha} D^\beta |z|^{2-2n}$.

- Each $f_{\alpha,\beta}$ is an eigenvector for the group action of $G$
  
  $g * f_{\alpha,\beta} = \zeta \sum_{j=1}^{n} l_j (\alpha_j - \beta_j) f_{\alpha,\beta}$. 
Invariant Basis Elements of $\mathcal{H}_{p,q}(S^{2n-1})$

- Let
  \[ f_{\alpha,\beta} = D^\alpha D^\beta |z|^{2-2n}. \]
- Each $f_{\alpha,\beta}$ is an eigenvector for the group action of $G$
  \[ g * f_{\alpha,\beta} = \zeta \sum_{j=1}^{n} l_j (\alpha_j - \beta_j) f_{\alpha,\beta}. \]
- So the dimension of $\mathcal{H}_{p,q}^G$ is equal to the number of solutions to the following system:
  \[
  |\alpha| = p, \ |\beta| = q; \\
  \alpha_1 = 0 \text{ or } \beta_1 = 0; \\
  \sum_{j=1}^{n} l_j (\alpha_j - \beta_j) \equiv 0 \mod k.
  \]
The Problem

- Eigenvalue of $\mathcal{H}_{p,q}(S^{2n-1})$ for $\Box_b$ is $2q(p+n-1)$.
- $N(\lambda), \ N_L(\lambda)$ number of positive eigenvalues of $\Box_b$ on $S^{2n-1}$, $L(k; \vec{l})$ (counting multiplicities) less than $\lambda$

$$N_L(\lambda) = \sum_{p \geq 0, q > 0, \ 0 < 2q(p+n-1) \leq \lambda} \dim \mathcal{H}^G_{p,q}.$$
The Problem

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$$N_L(\lambda) = \sum_{p \geq 0, q > 0, \atop 0 < 2q(p+n-1) \leq \lambda} \dim \mathcal{H}^G_{p,q}.$$ 

- So $N_L(\lambda)$ is equal to the number of solutions to the following system

$$\alpha, \beta \in \mathbb{N}^n$$

$$0 < 2|\alpha|(|\beta| + n - 1) \leq \lambda$$

$$\alpha_1 = 0 \text{ or } \beta_1 = 0;$$

$$\sum_{j=1}^m l_j (\alpha_j - \beta_j) \equiv 0 \text{ mod } k.$$
The Problem

- Eigenvalue of $\mathcal{H}_{p,q}(S^{2n-1})$ for $\Box_b$ is $2q(p+n-1)$.
- $N(\lambda)$, $N_L(\lambda)$ number of positive eigenvalues of $\Box_b$ on $S^{2n-1}$, $L(k; \vec{l})$ (counting multiplicities) less than $\lambda$

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- So $N_L(\lambda)$ is equal to the number of solutions to the following system

$$\alpha, \beta \in \mathbb{N}^n$$
$$0 < 2|\alpha|(|\beta| + n - 1) \leq \lambda$$
$$\alpha_1 = 0 \text{ or } \beta_1 = 0;$$

$$\sum_{j=1}^{m} l_j (\alpha_j - \beta_j) \equiv 0 \mod k.$$ 

- Our calculations yield:

$$\lim_{\lambda \to \infty} \frac{N_L(\lambda)}{N(\lambda)} = \frac{1}{k}.$$
Theorem (Ikeda-Yamamoto)

Two lens spaces $L(k; l_1, \ldots, l_n)$ and $L(k'; l'_1, \ldots, l'_n)$ are isometric as Riemannian manifolds if and only if

- $k = k'$, and
- there exists an integer $a$ and a permutation $\sigma$ such that $(l'_1, l'_2, \ldots, l'_n) \equiv (\pm al_{\sigma(1)}, \pm al_{\sigma(2)}, \ldots, \pm al_{\sigma(n)}) \pmod{k}$.
Can You Hear the Shape of a Lens Space?

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Theorem (Ikeda-Yamamoto, Main Theorem)

Two 3-dimensional lens spaces have the same spectrum for the real Laplacian if and only if they are isometric as Riemannian manifolds.

Example: $L(3; 1, 1)$ and $L(3; 1, 2)$ have the same real spectrum.
Can You Hear the Shape of a CR Lens Space?

- CR isometry is a stronger condition than Riemannian isometry.

**Theorem (2021)**

Let $L(k; l_1, \ldots, l_n)$ and $L(k', l'_1, \ldots, l'_n)$. If

- $k = k'$, and

- there exists an integer $a$ and a permutation $\sigma$ such that $(l'_1, l'_2, \ldots, l'_n) \equiv (al_{\sigma(1)}, al_{\sigma(2)}, \ldots, al_{\sigma(n)}) \pmod{k}$.

then the spaces are CR isometric.
Can You Hear the Shape of a CR Lens Space?

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Let \( L(k; l_1, \ldots, l_n) \) and \( L(k'; l'_1, \ldots, l'_n) \). If

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- there exists an integer \( a \) and a permutation \( \sigma \) such that
  \[
  (l'_1, l'_2, \ldots, l'_n) \equiv (al_{\sigma(1)}, al_{\sigma(2)}, \ldots, al_{\sigma(n)}) \pmod{k}.
  \]

then the spaces are CR isometric.

What happens with \( L(3; 1, 1) \) and \( L(3; 1, 2) \)?

Spectrum of \( \Box_b \) on \( L(3; 1, 1) = \{0, 4, 6, 10, 12, 16, 18, \ldots\} \)

Spectrum of \( \Box_b \) on \( L(3; 1, 2) = \{0, 4, 6, 8, 10, 12, 14, 16, \ldots\} \)

**Goal:** Characterize 3-dimensional lens spaces up to CR isometries via spectra.
Can You Hear the Shape of a CR Lens Space?

Conjecture

Two 3-dimensional lens spaces have the same Kohn spectrum if and only if they are CR isometric.

This would mean the Kohn Laplacian is “more sensitive” than the standard Laplacian.
Proposition (2021)

If $L(k; l_1, l_2, \ldots, l_n)$ is isospectral to $L(k'; l'_1, l'_2, \ldots, l'_n)$ with respect to $\Box_b$, then $k = k'$.

- Analog of Weyl’s law
Some Partial Results

**Proposition (2021)**

If $L(k; l_1, l_2, \ldots, l_n)$ is isospectral to $L(k'; l_1', l_2', \ldots, l_n')$ with respect to $\Box_b$, then $k = k'$.

- Analog of Weyl's law

**Theorem (2021)**

Let $k$ be a prime. If $L(k; l_1, l_2)$ and $L(k; l_1', l_2')$ are isospectral with respect to $\Box_b$, then they are CR isometric.

- Generating function approach
Generating Function Approach

Given a lens space \( L(k; l_1, l_2, \ldots, l_n) \), we define a generating function

\[
F(z, w) = \sum_{p,q \geq 0} \left( \dim H_{p,q}^G \right) z^p w^q.
\]

Using tools from group representation theory, we obtain the closed form

\[
F(z, w) = \frac{1}{k} \sum_{m=0}^{1} \frac{1}{1 - zw \prod_{i=1}^{n} (1 - \zeta_m \ell_i z)(1 - \zeta_m^{-1} \ell_i w)}
\]

where \( \zeta = e^{2\pi i/k} \).

This equivalence relates spectral information \( \dim H_{p,q}^G \) to information about the geometry of the lens space \( (k, \vec{l}) \).
Generating Function Approach

- Given a lens space $L(k; l_1, l_2, \ldots, l_n)$, we define a generating function

$$F(z, w) = \sum_{p, q \geq 0} (\dim \mathcal{H}^G_{p,q}) z^p w^q.$$ 

- Using tools from group representation theory, we obtain the closed form

$$F(z, w) = \frac{1}{k} \sum_{m=0}^{k-1} \frac{1 - zw}{\prod_{i=1}^{n} (1 - \zeta ^{m l_i} z)(1 - \zeta ^{-m l_i} w)}$$

where $\zeta = e^{2\pi i / k}$.
Generating Function Approach

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where $\zeta = e^{2\pi i / k}$.

- This equivalence relates spectral information (dim $\mathcal{H}_{p,q}^G$) to information about the geometry of the lens space ($k$ and $\vec{l}$).
Outline of Argument

\[ \dim \mathcal{H}^G_{p,q} = \dim \mathcal{H}^{G'}_{p,q} \text{ for all } p, q \]

\[ (l_1, \ldots l_n) \equiv (a l'_{\sigma(1)}, \ldots, a l'_{\sigma(n)}) \]

- Fix two lens spaces \( L(k, \ell_1, \ldots, \ell_n) \) and \( L(k, \ell'_1, \ldots, \ell'_n) \).
- We would like to show that if two lens spaces are CR isospectral, they are CR isometric.
Outline of Argument

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\[ (l_1, \ldots, l_n) \equiv (a l'_{\sigma(1)}, \ldots, a l'_{\sigma(n)}) \]

- These directions quickly follow from the definitions.
Outline of Argument

We can show this direction by extending a result from Ikeda and Yamamoto.
This equivalence comes from the generating function

$$\sum_{p,q \geq 0} (\dim H^G_{p,q}) z^p w^q = \frac{1}{k} \sum_{m=0}^{k-1} \frac{1 - zw}{\prod_{i=1}^n (1 - \zeta^{ml_i} z)(1 - \zeta^{-ml_i} w)}.$$

\[ \text{dim } H^G_{p,q} = \text{dim } H^{G'}_{p,q} \text{ for all } p, q \leftrightarrow (l_1, \ldots l_n) \equiv (a l'_{\sigma(1)}, \ldots, a l'_{\sigma(n)}) \]
We have shown the dotted arrow in the case when $n = 2$ and $k$ is prime.

We have conjectured that it is true for all $k$ when $n = 2$. 

**Outline of Argument**

\[
\dim \mathcal{H}^G_{p,q} = \dim \mathcal{H}^{G'}_{p,q} \text{ for all } p, q \iff (l_1, \ldots, l_n) \equiv (al'_{\sigma(1)}, \ldots, al'_{\sigma(n)})
\]
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