Further results on delay-dependent robust $H_\infty$ control for uncertain systems with interval time-varying delays

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ABSTRACT

This work focuses on the delay-dependent robust $H_\infty$ control for uncertain linear systems with interval time-varying delays. The key features of the method include employment of a tighter integral inequality and construction of an appropriate Lyapunov–Krasovskii functionals (LKF). Meanwhile, the delay-dependent conditions with less conservatism are derived owing to the consideration of the information of the lower bound of time delay. Based on the obtained criteria, robust $H_\infty$ controller design and performance analysis for the uncertain system are presented. Some numerical examples are also provided to show the effectiveness and advantage of the main results.

1. Introduction

The phenomenon of time delays and parameter uncertainty frequently appear in many practical systems such as communication systems, engineering systems, chemical engineering systems, nuclear reactors, population ecology and other fields (Park et al., 2015; Zhang et al., 2019; Zhou, 2016). It is well recognized that the existence of delays and uncertainty often causes poor performance, oscillation or even system instabilities (Qian et al., 2017; Wang et al., 2017). Therefore, the stability analysis of uncertain systems with time-varying delays has become an important issue in theory and application, and many related problems also have been researched, such as robust stability (Cheng et al., 2016; Li & Liao, 2018; Lu et al., 2019), $H_\infty$ performance (Ding et al., 2017; Kwon et al., 2016; Meng et al., 2020), passivity (Peng & Jian, 2018), control (Niamsup & Phat, 2020). During the past decades, lots of fruitful results have been achieved. See the references Lee et al. (2014), Cheng et al. (2016), Bai et al. (2016), Qian et al. (2019) and Zheng et al. (2020).

As is well-known, $H_\infty$ control is a worst-case design which is suitable and effective for solving system robustness against disturbances with no prior knowledge other than being energy bounded and parametric uncertainties. The maximum delay and $H_\infty$ performance level are two key indexes to judge the conservatism of the obtained criterion. In order to continuously reduce the conservatism, the considerable research efforts have been made on two aspects, one is the selection of appropriate Lyapunov–Krasovskii functional, and the other is to estimate the derivative of LKF more accurately, such as delay partitioning approach (Ding et al., 2017), free-matrix-based integral inequality (Wang et al., 2017), the augmented LKF approach (Meng et al., 2020; Peng & Jian, 2018), Wirtinger-based inequality (Li & Liao, 2018), reciprocally convex combination (Kwon et al., 2016; Li & Xue, 2016; Zhang et al., 2017), auxiliary function-based integral inequality (Zhang et al., 2018) and Bessel-Legendre inequality (Seuret & Gouaisbaut, 2018).

Recently, considering the disturbances and parameter uncertainty in modelling, the robust $H_\infty$ control has been studied in Bai et al. (2016), Sun et al. (2018), Meng et al. (2020) and Niamsup and Phat (2020). In Meng et al. (2020), based on the sliding mode observer framework, a sufficient condition and novel integral sliding surface function with the compensator were proposed to investigate robust $H_\infty$ asymptotic stabilization for uncertain neutral-type systems. A new delay-dependent sufficient condition for admissibility of the system with non-differentiable delay was presented, and state feedback controllers was designed which ensure the descriptor closed-loop system admissible with a maximum $H_\infty$ disturbance attenuation level in Niamsup and Phat (2020). In Bai et al. (2016), the free weighting matrix approach was used to obtain some delay-dependent robust $H_\infty$ performance analysis for uncertain linear system. In Seuret and Gouaisbaut (2018), in order to integrate the LKF with the estimating technique effectively, a new augmented vector and LKF with triple integral terms were constructed. However, it should be noted that, the criteria derived
Based on the improved stability criteria, a delay-time-varying delay systems with disturbances. The main contributions of the paper can be summarized as follows:

1. For estimating the derivative of LKF more accurately, the derivatives of triple integrals are separated elaborately to use the delay information ignored in existing methods and are estimated by employing improved Wirtinger inequality and reciprocally convex method, which can lead to less conservative results.

2. Based on the improved stability criteria, a delay-dependent condition for the existence of a state feedback controller is obtained, which ensures asymptotic stability and a prescribed $H_{\infty}$ performance level of the closed-loop system for all admissible uncertainties.

### 2. Problem formulations

Consider the following uncertain systems with time-varying delays and disturbance:

\[
\begin{align*}
\dot{x}(t) &= (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - h(t)) + (B + \Delta B(t))u(t) + B_{\omega} \omega(t) \\
z(t) &= Cx(t) + C_\omega x(t - h(t)) + Du(t) + D_\omega \omega(t) \\
x(t) &= \phi(t), t \in [-h_2, 0]
\end{align*}
\]

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, $\omega(t) \in \mathbb{R}^p$ is the disturbance input such that $\omega(t) \in L_2[0, \infty)$; $z(t) \in \mathbb{R}^r$ denotes the output, the initial condition $\phi(t)$ is a continuously differentiable vector-valued function; $A, A_d, B, C, C_\omega, D$ and $D_\omega$ are known constant matrices with appropriate dimensions; $\Delta A(t), \Delta A_d(t)$ and $\Delta B(t)$ are the uncertainties of system matrices of form

\[
[\Delta A(t) \quad \Delta A_d(t) \quad \Delta B(t)] = H\Phi(t) \begin{bmatrix} E_a & E_d & E_b \end{bmatrix}
\]

in which $E_a, E_d, E_b$ and $H$ are known constant matrices and the time-varying nonlinear function $F(t)$ satisfies $F^T(t) F(t) \leq I$. The delay $h(t)$, is time-varying continuous function that satisfies

\[
h_1 \leq h(t) \leq h_2, \quad |\dot{h}(t)| \leq \mu < 1
\]

where $0 \leq h_1 < h_2$, and $\mu$ are constant values.

The purpose of this paper is to study the robust stability analysis and $H_{\infty}$ performance for systems (1). In order to obtain our main results, we need the following definition and lemmas.

**Definition 1:** [Qian et al., 2019]. Given a scalar $\gamma > 0$, system (1) is said to be asymptotically stable with the $H_{\infty}$ performance level $\gamma$, if it is asymptotically stable and satisfies the $H_{\infty}$-norm constraint

\[
||z(t)||_2 \leq \gamma ||\omega(t)||_2
\]

for all nonzero $\omega(t) \in L_2[0, \infty)$ under zero initial condition.

**Lemma 1:** For any constant symmetric matrix $S \in \mathbb{R}^{n \times n}$, real scalars $a, b$ satisfying $a < b$, and vector-valued function $\omega \in [a, b] \to \mathbb{R}^p$, the following integral inequality holds

\[
\int_a^b \omega^T(s) S \omega^T(s) ds \geq \frac{1}{b - a} \left( v_0^T S v_0 + 3 v_1^T S v_1 + 5 v_2^T S v_2 \right)
\]

where

\[
\begin{align*}
v_0 &= \omega(b) - \omega(a) \\
v_1 &= \omega(b) + \omega(a) - 2 \int_a^b \omega(s) ds \\
v_2 &= \omega(b) - \omega(a) + 6 \int_a^b \omega(s) ds - \frac{12}{(b - a)^2} \int_a^b \int_s^b \omega(u) du ds
\end{align*}
\]

**Lemma 2:** [Qian et al., 2019]. For any positive definite matrix $S \in \mathbb{R}^{n \times n}$, real scalars $a, b$ satisfying $a < b$, and vector-valued function $x \in [a, b] \to \mathbb{R}^p$, the following integral inequality holds

\[
\begin{align*}
\int_a^b \int_a^b x^T(\alpha) S x(\alpha) d\alpha d\beta &\geq 2 \chi_1^T S \chi_1 + 4 \chi_2^T S \chi_2 \\
\int_a^b \int_a^b x^T(\alpha) S x(\alpha) d\alpha d\beta &\geq 2 \chi_3^T S \chi_3 + 4 \chi_4^T S \chi_4
\end{align*}
\]

where

\[
\begin{align*}
\chi_1 &= x(b) - \frac{1}{b - a} \int_a^b x(\alpha) d\alpha, \quad \chi_2 = x(b) + 2 \int_a^b x(\alpha) d\alpha - \frac{6}{(b - a)^2} \int_a^b \int_a^b x(\alpha) d\alpha d\beta, \\
\chi_3 &= x(a) - \frac{1}{b - a} \int_a^b x(\alpha) d\alpha, \quad \chi_4 = x(a) - \frac{4}{(b - a)^2} \int_a^b \int_a^b x(\alpha) d\alpha d\beta.
\end{align*}
\]
Lemma 3: [Zhang et al., 2018]. Let $R_1, R_2 \in S_m^+$, $\xi_1, \xi_2 \in R^m$, and a scalar $\alpha \in (0, 1)$. If there exist matrices $X_1, X_2 \in S^n$ and $Y_1, Y_2 \in R^{m \times m}$ such that
\[
\begin{bmatrix}
R_1 - X_1 & Y_1 \\
* & R_2
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
R_1 & Y_2 \\
* & R_2 - X_2
\end{bmatrix} \geq 0
\]
the following inequality holds
\[
\frac{1}{\alpha} \xi_1^T R_1 \xi_1 + \frac{1}{1 - \alpha} \xi_2^T R_2 \xi_2 \geq \xi_1^T [R_1 + (1 - \alpha) X_1] \xi_1 \\
+ \xi_2^T [R_2 + \alpha X_2] \xi_2 + 2 \xi_1^T \alpha Y_1 + (1 - \alpha) Y_2 \xi_2
\]

3. Main results

In this section, by constructing a newly augmented LKF with triple integral term and applying improved Wirtinger inequality and extended reciprocally convex, the improved stability criteria and $H_{\infty}$ controller design for system (1) are given. Consider the following uncertain system (1) with $\omega(t) = 0$
\[
\begin{cases}
\dot{x}(t) = (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - h(t)) + (B + \Delta B(t))u(t) \\
z(t) = Cx(t) + C_d x(t - h(t)) + Du(t) \\
x(t) = \phi(t), t \in [-h_2, 0]
\end{cases}
\]
For simplicity, some matrices and vectors are defined as follows:
\[
\begin{align*}
\varphi_1(t) &= \frac{1}{h(t)} \int_{t-h(t)}^{t} x(s) ds, \\
\varphi_2(t) &= \frac{1}{h(t) - h_1} \int_{t-h(t)}^{t-h_1} x(s) ds, \\
\varphi_3(t) &= \frac{1}{h_2 - h} \int_{t-h}^{t-h_2} x(s) ds, \\
\varphi_4(t) &= \frac{1}{h^2} \int_{-h}^{0} \int_{t-h}^{t} x(s) ds db, \\
\varphi_5(t) &= \frac{1}{(h(t) - h_1)^2} \int_{-h_1}^{0} \int_{t-h_1}^{t-h_1} x(s) ds db, \\
\varphi_6(t) &= \frac{1}{(h_2 - h)^2} \int_{-h_2}^{0} \int_{t-h_2}^{t-h_2} x(s) ds db,
\end{align*}
\]
\[
\xi(t) = \operatorname{col}[x(t), x(t - h(t)), x(t - h_1), x(t - h_2), \dot{x}(t), \\
\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t), \varphi_5(t), \varphi_6(t)].
\]

Theorem 1: Given scalars $0 \leq h_1 \leq h_2, \delta > 0$, the system (4) with $u(t) = 0$ is robustly asymptotically stable if there exist positive definite matrices $P \in R^{n \times n}, Q_1, Q_2, Q_3, U_1, U_2, R_1, R_2 \in R^{n \times n}, X_i, Y_i \in R^{n \times n}$, $i = 1, 2, 3, 4$, and any matrix $M_1, M_2, M_3 \in R^{n \times n}$ such that the following LMIs hold for $h(t) = (h_1, h_2)$ and $\dot{h}(t) = \{ -\mu, \mu \}$
\[
\Xi = \begin{bmatrix}
\Xi_1 & \Xi_2 \\
* & \Xi_3
\end{bmatrix} < 0
\]
\[
\begin{bmatrix}
\tilde{U}_1 - X_1 & Y_1 \\
* & \tilde{U}_1
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
\tilde{U}_1 & Y_2 \\
* & \tilde{U}_1 - X_2
\end{bmatrix} \geq 0
\]
\[
\begin{bmatrix}
\tilde{U}_2 - X_3 & Y_3 \\
* & \tilde{U}_2
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
\tilde{U}_2 & Y_4 \\
* & \tilde{U}_2 - X_4
\end{bmatrix} \geq 0
\]
where
\[
\Xi_1 = \Sigma + H e(W_1^T W_2) - \Omega_1 - \Omega_2 + \Upsilon,
\]
\[
\Xi_2 = \begin{bmatrix}
\Pi_1 & \Pi_2
\end{bmatrix},
\]
\[
\Xi_3 = \text{diag}\{ -\delta l, -\delta l \},
\]
\[
\Sigma = H e(G_1^T P G_2) + Q + h_2^2 G_1^T \tilde{U}_1 G_2 + h_2^2 G_2^T \tilde{U}_2 G_2
\]
\[
+ h_2 G_2^T (\tilde{R}_1 + \tilde{R}_2) G_2,
\]
\[
G_1 = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (h(t) - h_1) I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
G_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(1 - h(t)) I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]
\[
\Omega_1 = G_1^T (\tilde{U}_1 + (1 - \alpha) X_1) G_1 + 2 G_1^T (\alpha Y_1 + (1 - \alpha) Y_2) G_2
\]
\[
+ G_2^T (\tilde{U}_1 + \alpha X_2) G_2,
\]
\[
\Omega_2 = G_1^T (\tilde{U}_2 + (1 - \beta) X_3) G_1 + 2 G_1^T (\beta Y_3 + (1 - \beta) Y_4) G_3
\]
\[
+ G_3^T (\tilde{U}_2 + \beta X_2) G_3,
\]
\[
\Gamma_1 = \text{col}[e_2 - e_4, e_2 + e_4 - 2e_8, e_2 - e_4 - 6e_8 + 12e_{11}]
\]
\[
\Gamma_2 = \text{col}[e_1 - e_2, e_1 + e_2 - 2e_6, e_1 - e_2 - 6e_6 + 12e_9],
\]
\[
\Gamma_3 = \text{col}[e_3 - e_2, e_3 + e_2 - 2e_7, e_3 - e_2 - 6e_7 + 12e_{10}],
\]
\[
Q = \text{diag}\{ -(1 - h(t)) Q_3, Q_2 - Q_1 + Q_3, \\
-Q_2, 0, 0, 0, 0, 0, 0, 0 \}.
\]
\[ W_1 = \begin{bmatrix} M_1^T & M_2^T & 0 & 0 & M_3^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]
\[ W_2 = \begin{bmatrix} A & A_d & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]
\[ \Upsilon = \chi_1(e_3, e_7, e_{12}, R_1) + \chi_2(e_2, e_8, e_{11}, R_1) + \chi_3(e_4, e_8, e_{12}, R_2) + \chi_4(e_2, e_7, e_{10}, R), \]
\[ \chi_1(e_3, e_7, e_{10}, R_1) = -2[e_3 - e_7]^T R_1[e_3 - e_7] - 4[e_3 + 2e_7 - 6e_{10}]^T R_1[e_3 + 2e_7 - 6e_{10}], \]
\[ \chi_2(e_2, e_8, e_{11}, R_1) = -2[e_2 - e_8]^T R_1[e_2 - e_8] - 4[e_2 + 2e_8 - 6e_{11}]^T R_1[e_2 + 2e_8 - 6e_{11}], \]
\[ \chi_3(e_4, e_8, e_{12}, R_2) = -2[e_4 - e_8]^T R_2[e_4 - e_8] - 4[e_4 + 2e_8 - 6e_{12}]^T R_2[e_4 + 2e_8 - 6e_{12}], \]
\[ \chi_4(e_2, e_3, e_4, e_7, e_8, R) = -2[e_2 - e_3 - e_4 - 2e_7]^T R[e_2 - e_3 - e_4 - 2e_7], \]
\[ \hat{U}_i = \text{diag}(U_i, 0, 0, 0), \quad \tilde{U}_i = \text{diag}(U_i, 3U_i, 5U_i), \]
\[ \hat{R}_i = \text{diag}(R_i, 0, 0, 0), \quad \tilde{R}_i = \text{diag}(R_i, 3R_i), i = 1, 2, \]
\[ \Pi_1 = \begin{bmatrix} H^T M_1^T & 0 & 0 & H^T M_2^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \]
\[ \Pi_2 = \begin{bmatrix} \delta E_{\alpha} & \delta E_{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \]

**Proof:** Define the following LKF candidate
\[ V(t) = \sum_{i=1}^{4} V_i(t) \]
where
\[ V_1(t) = \xi^T(t) R \xi(t) \]
\[ V_2(t) = \int_{t-h_1}^{t-h_2} x^T(s) Q_1 x(s) ds + \int_{t-h_2}^{t} x^T(s) Q_2 x(s) ds \]
\[ V_3(t) = h_2 \int_{t-h_2}^{t} \dot{x}^T(s) U_1 \dot{x}(s) ds \]
\[ V_4(t) = \int_{t-h_2}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \]
with
\[ \xi(t) = \text{col} \left\{ x(t), \int_{t-h(t)}^{t} x(s) ds, \int_{t-h(t)}^{t-h_1} x(s) ds, \int_{t-h_2}^{t-h(t)} x(s) ds \right\}, \]
\[ h_{12} = h_2 - h_1. \]

Differentiating \( V(t) \) along the trajectories of system (4) yields
\[ \dot{V}_1(t, x(t)) = \xi^T(t) (G_1^T P G_2 + G_2^T P G_1) \xi(t) \]
\[ \dot{V}_2(t, x(t)) = \xi^T(t) Q \xi(t) \]
\[ \dot{V}_3(t, x(t)) = \xi^T(t) G_2^T (h_2^2 \tilde{U}_1 + h_2^2 \tilde{U}_2) G_2 \xi(t) \]
\[ \dot{V}_4(t, x(t)) = \xi^T(t) G_2^T h_2 \int_{t-h_2}^{t} \dot{x}^T(s) U_1 \dot{x}(s) ds \]
we know that
\[ - h_2 \int_{t-h_2}^{t} \dot{x}^T(s) U_1 \dot{x}(s) ds = - h_2 \int_{t-h(t)}^{t} \dot{x}^T(s) U_1 \dot{x}(s) ds \]
\[ - h_2 \int_{t-h_2}^{t} \dot{x}^T(s) U_1 \dot{x}(s) ds \]
By using Lemma 1, one can obtain
\[ h_2 \int_{t-h(t)}^{t} \dot{x}^T(s) U_1 \dot{x}(s) ds \geq \frac{1}{\alpha} (\Gamma_1 \xi(t))^T \tilde{U}_1 (\Gamma_1 \xi(t)) \]
\[ h_2 \int_{t-h(t)}^{t} \dot{x}^T(s) U_1 \dot{x}(s) ds \geq \frac{1}{1-\alpha} (\Gamma_2 \xi(t))^T \tilde{U}_1 (\Gamma_2 \xi(t)) \]
where \( \alpha = (h_2 - h(t))/h_2 \), and it follows from Lemma 3 that

\[
-h_2 \int_{t-h_2}^{t} \dot{x}^T(s)U_1\dot{x}(s)ds \leq -\zeta^T(t)
\]

\[
\times [\Gamma_1^T (\dot{U}_1 + (1 - \alpha)X_1)\Gamma_1 + 2\Gamma_1^T (\alpha Y_1 + (1 - \alpha)Y_2)\Gamma_2 + \Gamma_2^T (\dot{U}_2 + \alpha X_2)\Gamma_2] \zeta(t)
\]

\[
= -\zeta^T(t)\Omega_1 \zeta(t) \quad (12)
\]

Similarly, for \(-h_{12} \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U_2\dot{x}(s)ds\), we have

\[
-h_{12} \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U_2\dot{x}(s)ds = -h_{12} \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U_2\dot{x}(s)ds
\]

\[
- h_{12} \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)U_2\dot{x}(s)ds
\]

\[
\leq -\zeta^T(t) [\Gamma_1^T (\dot{U}_2 + (1 - \beta)X_2)\Gamma_1 + 2\Gamma_1^T (\beta Y_2 + (1 - \beta)Y_3)\Gamma_3 + \Gamma_3^T (\dot{U}_3 + \beta X_3)\Gamma_3] \zeta(t)
\]

\[
= -\zeta^T(t)\Omega_2 \zeta(t) \quad (13)
\]

where \( \beta = (h_2 - h(t))/(h_2 - h_1) \), \( \Gamma_1 = \text{col}[e_2 - e_4, e_2 + e_4 - 2e_8, e_2 - e_4 - 6e_9 + 12e_1], \)

\( \Gamma_2 = \text{col}[e_1 - 2e_2, e_1 + e_2 - 2e_6, e_1 - e_2 - 2e_6 + 12e_9], \)

\( \Gamma_3 = \text{col}[e_3 - e_2, e_2 + e_3 - 2e_7, e_3 - e_2 - 2e_7 + 12e_10]. \)

\[
\check{V}_4(t,x(t)) = \zeta^T(t)G_1^T (h_2 R_1 + h_2 R_2)G_2 \zeta(t)
\]

\[
- \int_{h_2}^{t-h(t)} \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta
\]

\[
- \int_{h_2}^{t-h(t)} \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta
\]

\[
= (h_2 - h(t)) \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)R_1\dot{x}(s)ds
\]

\[
- \int_{h_2}^{t-h(t)} \int_{h(t)}^{t-h(t)} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta
\]

\[
- \int_{h_2}^{t-h(t)} \int_{h(t)}^{t-h(t)} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta
\]

\[
= (h(t) - h_1) \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)R_2\dot{x}(s)ds \quad (14)
\]

where \( h_3 = (h_2 - h_2^2)/2 \), and by applying Lemma 2

\[
- \int_{h(t)}^{t-h(t)} \dot{x}^T(s)R_3\dot{x}(s)dsd\theta
\]

\[
\leq -2 \left[ x(t - h_1) - \frac{1}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right]^T
\]

\[
\times R_1 \left[ x(t - h_1) - \frac{1}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right]^T
\]

\[
- 4 \left[ x(t - h_1) + \frac{2}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right] R_1^T
\]

\[
\times \frac{6}{(h(t) - h_1)^2} \int_{t-h(t)}^{t-h(t)} x(s)d\theta
\]

\[
= \zeta^T(t) \chi_1(e_2, e_7, e_{10}, R_1) \zeta(t)
\]

\[
\int_{h_2 - h(t)}^{t-h(t)} \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)R_1\dot{x}(s)ds d\theta
\]

\[
\leq -2 \left[ x(t - h(t)) - \frac{1}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right]^T
\]

\[
\times R_1 \left[ x(t - h(t)) - \frac{1}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right]^T
\]

\[
- 4 \left[ x(t - h(t)) + \frac{2}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right] R_1^T
\]

\[
- \frac{6}{(h(t) - h_1)^2} \int_{t-h(t)}^{t-h(t)} x(s)d\theta
\]

\[
= \zeta^T(t) \chi_2(e_2, e_8, e_{11}, R_1) \zeta(t)
\]

\[
\int_{h_2 - h(t)}^{t-h(t)} \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)R_2\dot{x}(s)ds d\theta
\]

\[
\leq -2 \left[ x(t - h(t)) - \frac{1}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right]^T
\]

\[
\times R_2 \left[ x(t - h(t)) - \frac{1}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right]^T
\]

\[
- 4 \left[ x(t - h(t)) - \frac{4}{h(t) - h_1} \int_{t-h(t)}^{t-h(t)} x(s)ds \right] R_2^T
\]

\[
+ \frac{6}{(h(t) - h_1)^2} \int_{t-h(t)}^{t-h(t)} x(s)d\theta
\]

\[
= \zeta^T(t) \chi_3(e_2, e_7, e_{10}, R_2) \zeta(t)
\]
where \( R = \xi \) and the system (1) with \( u(t) = 0 \) robust asymptotically stability. Then using Lemma 4, assume scalar \( \delta > 0 \) such that the following inequality holds

\[
\Xi_1 + \delta \Pi_1 \Pi_1^T + \delta^{-1} \Phi^T \Phi < 0
\]

By Schur complements, it is easily seen that inequality (18) is equivalent to condition (5). This completes the proof.

**Remark 1:** As is known, an appropriate LKF is crucial to reduce the conservatism of the system. A newly augmented LKF containing single, double and triple forms is constructed, and combined the improved Wirtinger inequality and extended reciprocally convex. Since the information of lower bound of time delay is considered in LKF, Theorem 1 can deal with the case that the lower bound of time delay is not restricted to be 0, but some recent results cannot deal with this case (An et al., 2014; Qian et al., 2018). It will be shown Theorem 1 can provide less conservatism results through some numerical examples.

**Remark 2:** It should be noted that, auxiliary-function-based double integral inequalities are used to estimate \( \int_{-h_2}^{0} \int_{-t}^{0} \hat{x}(s) ds d\theta \) and \( \int_{-h_2}^{0} \int_{-h_1}^{0} \hat{x}(s) ds d\theta \) in the \( V_3(t) \), and a new extended relaxed integral inequality is employed to tackling with the single integral terms. Meanwhile, by utilizing the improved Wirtinger inequality in Lemma 1 and extended reciprocally convex inequalities, a much tighter lower bound is obtained and the conservatism of the proposed method is further reduced efficiently.

Further, considering the external disturbance, the robust \( H_\infty \) performance analysis and controller for the system (1) are developed.

**Theorem 2:** Given scalars \( 0 \leq h_1 < h_2, \delta > 0 \) and \( \gamma > 0 \), the system (1) with \( u(t) = 0 \) is robust asymptotically stable with \( H_\infty \) performance \( \gamma \), if there exist positive definite matrices \( P \in \mathbb{R}^{n \times n}, Q_1, Q_2, Q_3, U_1, U_2, R_1, R_2 \in \mathbb{R}^{n \times n}, X_i, Y_i \in \mathbb{R}^{3n \times 3n}, i = 1, 2, 3, 4 \), and any matrix \( M_1, M_2, M_3 \in \mathbb{R}^{n \times n} \) such that the following LMIs hold for \( h(t) = [h_1, h_2] \) and \( h(t) = \)
where
\[ \hat{\Sigma} = \begin{bmatrix} \hat{\Sigma}_1 & \hat{\Sigma}_2 & \hat{\Sigma}_3 & \hat{\Sigma}_4 \\ * & \hat{\Sigma}_3 & 0 & 0 \\ * & * & \hat{\Sigma}_3 & 0 \\ * & * & * & \hat{\Sigma}_3 \end{bmatrix} < 0 \]

for a prescribed \( \gamma > 0 \), the system (1) is robustly asymptotically stable with memoryless feedback controller gain \( K \) and \( H \).

**Theorem 3:** Given scalars \( 0 \leq h_1 < h_2, \sigma > 0 \) and \( \gamma > 0 \), the system (1) is robustly asymptotically stable with memoryless feedback controller gain \( K \) and \( H \), if there exist positive definite matrices \( \hat{P} \in \mathbb{R}^{n \times 4n}, \hat{Q}_1, \hat{Q}_2 \), \( \hat{Q}_3, \hat{U}_1, \hat{U}_2, \hat{R}_1, \hat{R}_2 \in \mathbb{R}^{n \times 3n}, i = 1,2,3,4, \) and any matrix \( W, S \in \mathbb{R}^{n \times k} \) such that the following LMIs hold

\[ \tilde{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_1 & \tilde{\Sigma}_2 & \tilde{\Sigma}_3 & \tilde{\Sigma}_4 & \tilde{\Sigma}_5 \\ * & \tilde{\Sigma}_3 & 0 & 0 & 0 \\ * & * & \tilde{\Sigma}_3 & 0 & 0 \\ * & * & * & \tilde{\Sigma}_3 & 0 \end{bmatrix} < 0 \]

\[ \tilde{\Sigma}_1 = \Sigma + \text{He}(\hat{P}_1 F(t) \hat{P}_1) < 0, \tilde{\Sigma}_2 = \Sigma - \tilde{\Sigma}_2 + \tilde{\Sigma}_2 \]

when \( \Theta + \text{He}(\hat{P}_1 F(t) \hat{P}_1) < 0, J_{2\omega} < 0 \). Similar to the proof of Theorem 1, \( \Theta + \text{He}(\hat{P}_1 F(t) \hat{P}_1) < 0 \) is equivalent to inequality (19). Therefore, we can conclude that if inequality (19) holds, \( J_{2\omega} < 0 \) and \( \| z(t) \| \leq \gamma \| \omega(t) \| \) is satisfied for any nonzero \( \omega(t) \in \mathbb{R}^3 \). This completes the proof.

Now, we will present the robust \( H_\infty \) controller design result for the system (1). Assume that a proportional feedback controller is employed, \( u(t) = Kx(t) \).

**Proof:** For the disturbance \( \omega(t) \in \mathbb{R}^3 \), \( \Lambda \) could be converted to

\[ \hat{\Lambda} = 2[x^T(t)M_1 + x^T(t - h(t))M_2 + x(t)M_3] \]

for a prescribed \( \gamma > 0 \), the following cost function is defined

\[ J_{2\omega} = \int_0^\infty [\zeta^T(t)\zeta(t) - \gamma^2 \omega(t)\omega(t)]dt \]

Considering the same LKF (8), for any nonzero \( \omega(t) \in \mathbb{R}^3 \) and \( t > 0 \), we have

\[ J_{2\omega} \leq \int_0^\infty [\zeta^T(t)\zeta(t) - \gamma^2 \omega(t)\omega(t)]dt + \hat{\Lambda}_1 \int_0^\infty \hat{\Lambda}_2(t)^T[\Theta + \text{He}(\hat{P}_1 F(t) \hat{P}_1)]\hat{\Lambda}_2(t)]dt \]

where

\[ \tilde{\zeta} = [\zeta^T(t) \omega^T(t) ]^T, \Theta = \begin{bmatrix} \hat{\Sigma}_1 & \hat{\Sigma}_3 & \hat{\Sigma}_5 \\ * & \hat{\Sigma}_3 & 0 \\ * & * & \hat{\Sigma}_3 \end{bmatrix}, \hat{\Pi}_1 = \begin{bmatrix} \Pi_1 & 0 \end{bmatrix}, \hat{\Phi} = [\Phi \ 0]. \]
Proof: Replace $A$ and $Du(t)$ with $A + BK$ and $DKx(t)$, respectively. Moreover, we denote
\[ \sigma = \delta^{-1}, \hat{P} = \text{diag}(M^{-1}, M^{-1}, M^{-1}, M^{-1}) \]
\[ \times P\text{diag}(M^{-T}, M^{-T}, M^{-T}, M^{-T}), \]
\[ \hat{Q}_i = M^{-1}Q_iM^{-T}, \quad i = 1, 2, 3, \quad \hat{U}_i = M^{-1}U_iM^{-T}, \]
\[ \hat{R}_i = \text{diag}(M^{-1}, M^{-1}, M^{-1}, M^{-1})X(\text{diag}(M^{-T}, M^{-T}, M^{-T})), \]
\[ i = 1, 2, W = M^{-1}, S = KM. \]

Then pre-multiplying (5) by $\text{diag}(M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, \hat{M}, \hat{M}, \hat{M})$ and post-multiplying both sides of (6) and (7) by $\text{diag}(M^{-1}, M^{-1}, M^{-1}, M^{-1})$ and $\text{diag}(M^{-T}, M^{-T}, M^{-T}, M^{-T})$ respectively, and LMI (24) can be derived. Similarly, post-multiplying and post-multiplying both sides of (6) and (7) by $\text{diag}(M^{-1}, M^{-1}, M^{-1}, M^{-1})$ and $\text{diag}(M^{-T}, M^{-T}, M^{-T}, M^{-T})$ respectively, we can obtain LMI (25) and (26). This completes the proof. \[ \Box \]

Remark 3: It is worth pointing out that, although Wirtinger inequality and LKF with triple integral terms have been used to obtain less conservative stability results of time-delay systems (Qian et al., 2017; Wang et al., 2017; Zhang et al., 2018), the most of existing results only discussed stability analysis, not consider in controller design for time-delay systems. Therefore, our method is expected more effective than the existing methods in (Kwon et al., 2016; Sun et al., 2018).

For a given $h_2$, the minimum $\gamma$ that satisfies (19) for Theorem 2 can be obtained by solving a constrained optimization problem. For example, the minimum $\gamma$ with parametric uncertainties can be obtained by solving the following constrained optimization problem
\[ \text{Min } \gamma^2 \]
\[ \text{S.t. } \hat{P} > 0, \hat{Q}_1 > 0, \hat{U}_1 > 0, \hat{U}_2 > 0, \hat{R}_1 > 0, \hat{R}_2 > 0, \sigma > 0, \]
\[ \text{LMI}s \text{in (24), (25) and (26)}. \]

\[ (27) \]

4. Numerical simulations

In this section, we use some numerical examples to demonstrate the effectiveness and advantages of the main results in this paper.

Example 1: Consider the following system
\[ \dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)) \]
The above system has been used in the literature for concerning delay-dependent stability analysis to check the conservatism of methods. For various $h_1$ and $\mu = 0.9$, maximum allowable delay bounds $h_2$ obtained by Theorem 1 and some recent ones are listed in Table 1. From Table 1, one can see that the proposed method in Theorem 1 can produce the larger upper bound $h_2$ than those obtained in Bai et al. (2016), An et al. (2014) and Hien and Trinh (2015). Furthermore, it can be seen that our result has a smaller number of variables than that of An et al. (2014).

Example 2: Consider the uncertain system (4) with $u(t) = 0$
\[ A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \]
\[ H = \text{diag}(1, 1), \quad E_o = \text{diag}(1.6, 0.05), \quad E_d = \text{diag}(0.1, 0.3). \]
The purpose of this example is to find the admissible upper bounds $h_2$, which ensure robust asymptotic stability of the above system when the different lower bound $h_1$ and $\mu$ are given. The maximum upper bounds $h_2$ for different $h_1$ and $\mu$ obtained from Theorem 1 are shown in Table 2. For comparison, Table 2 also lists the upper bounds obtained from the criteria in Wu et al. (2014), Yan, Zhang and Meng (2016) and Sun et al. (2018). From Table 2, it is clear that our results are less conservative. Moreover, with the increase of $\mu$, the maximum admissible upper bound $h_2$ decreases gradually. When the lower delay is not restricted to 0, the main results in this paper can also solve the problem, while [15] fails to be used in this case.

Example 3: Consider the uncertain system (1) with $u(t) = 0$
\[ A = \begin{bmatrix} -3.0242 & 2.7527 \\ 0.8104 & -4.3988 \end{bmatrix}, \quad A_d = \begin{bmatrix} 2.8409 & -1.2355 \\ -9.8952 & -0.1443 \end{bmatrix}, \]
\[ B_{u0} = \begin{bmatrix} -0.9043 & 0.4325 \\ -0.7774 & 0.1846 \end{bmatrix}, \]
\[ C = \begin{bmatrix} -0.9647 & -1.6555 \\ 0.8245 & -0.8378 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1.2723 & 0.2718 \\ 0.4810 & -0.2368 \end{bmatrix}. \]

| $\mu$ | Methods | $h_1 = 0$ | $h_1 = 0.5$ | $h_1 = 1$ |
|-------|---------|---------|-----------|---------|
| 0     | Wu et al. (2014) | 1.1490 |            | 1.1490 |
|       | Yan et al. (2016) | 1.1490 | 1.4311 | 1.4311 |
|       | Sun et al. (2018) | 1.62   | 1.68   | 1.76   |
| 0.5   | Wu et al. (2014) | 0.9247 |            | 1.0931 |
|       | Yan et al. (2016) | 0.9428 | 0.9630 | 1.2256 |
|       | Sun et al. (2018) | 1.0949 | 1.1139 | 1.2256 |
| 0.9   | Wu et al. (2014) | 0.6954 |            | 1.0930 |
|       | Yan et al. (2016) | 0.8189 | 0.9292 | 1.2240 |
|       | Sun et al. (2018) | 1.0848 | 1.0955 | 1.2240 |
|       | Theorem 1 | 1.41   | 1.52   | 1.74   |
Table 1. Maximum allowable delay bounds $h_2$ for given $h_1$ and $\mu = 0.9$.

| Methods         | $h_1 = 2$ | $h_1 = 3$ | $h_1 = 4$ | $h_1 = 5$ | Number of variables |
|-----------------|-----------|-----------|-----------|-----------|--------------------|
| An et al. (2014)| 3.0240    | 3.6616    | 4.3788    | 5.1453    | 25$m^2 + 9n$      |
| Hien and Trinh (2015)| 3.1634    | 3.6648    | 4.4467    | 5.2147    | 6.5$m^2 + 4.5n$   |
| Liu (2014)      | 3.2745    | 4.0059    | 4.7783    | 5.5817    | 16.5$m^2 + 7.5n$  |
| Bai et al. (2016)| 3.3348    | 4.1756    | 5.0203    | 5.8951    | 10$m^2 + 7n$      |
| Theorem 1       | 3.4588    | 4.3263    | 5.3572    | 6.0949    | 22.5$m^2 + 5.5n$  |
| Theorem 2       | 3.0240    | 3.6616    | 4.3788    | 5.1453    | 25$m^2 + 9n$      |
| Theorem 3       | 3.326     | 0.3802    | 0.3831    | 0.4019    | 0.4137             |

$D_\omega = \begin{bmatrix} 0.1352 & -1.0236 \\ -0.0125 & 0.3368 \end{bmatrix},
H = \begin{bmatrix} 1.5 \\ 0.8 \end{bmatrix},
E_a = [0.2, 0.3],
E_d = [0.1, 0.2].$

This example was borrowed from Sun et al. (2018). Our purpose is to compare the admissible upper bounds on the delay for different $h_1$ and $\gamma > 0$, which guarantee the system (1) with $u(t) = 0$ is robust $H_\infty$ stable with disturbance. By using Matlab LMI control toolbox, the admissible upper bound $h_2$ for different $h_1$ and $\gamma > 0$ obtained by Theorem 2 are shown in Table 3. For comparison, Table 3 also lists the upper bounds obtained from the criteria in (7). Obviously, the maximum upper bounds obtained from Theorem 2 are larger than those of Sun et al. (2018) in the same conditions, which further implies that the main results in this paper are less conservative than the existing results.

**Example 4:** To verify the effectiveness of robust $H_\infty$ controller design results derived by Theorem 3, consider the following uncertain system

$A = \begin{bmatrix} 4 & 0.1 & -0.3 \\ -0.2 & 3 & -0.2 \\ 0.2 & -0.3 & 2 \end{bmatrix},
A_d = \begin{bmatrix} -0.4 & 0.1 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix},
B = \begin{bmatrix} -4 & 0.2 & 0 \\ 0 & 3 & 1 \\ 0.1 & 0 & 3 \end{bmatrix},
C = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},
C_d = C, D_\omega = 0.1.$

$B_\omega = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \\ 0 & 0.2 \end{bmatrix},
D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},
E_a = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix},
E_d = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix},
E_b = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}.$

Similar to Sun et al. (2018), this paper assumes $h(t)$ satisfies $1.2 \leq h(t) \leq 1.8$ and $\mu = 0.3$. The minimum allowable $\gamma = 0.2465$ for a prescribed delay bound could be derived by solving the convex optimization problem (27), and the robust $H_\infty$ controller gain $K$ can be calculated by

$K = \begin{bmatrix} 1.6452 & -0.1595 & -0.0370 \\ 0.7934 & 3.4723 & 0.6513 \\ -0.0235 & 0.0148 & -1.0759 \end{bmatrix}.$

By applying the approach in Sun et al. (2018), the minimum allowable $\gamma = 0.3854$ for a prescribed delay bound could be derived in the same condition, and the robust $H_\infty$ controller gain $\hat{K}$ can be calculated by

$\hat{K} = \begin{bmatrix} 1.5314 & -0.1415 & -0.0466 \\ 1.0515 & 2.5836 & 0.5568 \\ -0.0366 & 0.0068 & -1.1274 \end{bmatrix}.$

It can be seen that compared with the reference Sun et al. (2018), the optimal disturbance attenuation level $\gamma$ obtained by Theorem 3 is smaller. Therefore, it can be concluded that Theorem 3 is less conservative than the results derived in Sun et al. (2018).

**Table 3.** Admissible upper bound $h_2$ with $u(t) = 0$ for given $\gamma > 0$ and $h_1$.

| $h_1$ | Methods         | $\gamma = 2.5$ | $\gamma = 3.0$ | $\gamma = 3.5$ | $\gamma = 4.0$ | $\gamma = 5.5$ | $\gamma = 6.0$ |
|-------|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0     | Yan et al. (2016)| 0.0961         | 0.1677         | 0.2111         | 0.2406         | 0.2917         | 0.3024         |
|       | Xu et al. (2017) | 0.0961         | 0.1677         | 0.2111         | 0.2406         | 0.2917         | 0.3024         |
|       | Sun et al. (2018)| 0.1511         | 0.2334         | 0.2722         | 0.2969         | 0.3396         | 0.3490         |
|       | Theorem 3        | 0.309          | 0.3768         | 0.3826         | 0.4107         | 0.4135         | 0.4186         |
| 0.15  | Yan et al. (2016)| Unstable       | 0.1677         | 0.2112         | 0.2407         | 0.2918         | 0.3025         |
|       | Sun et al. (2018)| 0.1544         | 0.2345         | 0.2727         | 0.2969         | 0.3396         | 0.3490         |
|       | Theorem 3        | 0.326          | 0.3802         | 0.3831         | 0.4019         | 0.4137         | 0.4189         |
5. Conclusions

In this paper, we have investigated the delay-dependent robust $H_\infty$ control for uncertain linear system with interval time-varying delays and disturbance. By utilizing the improved Wirtinger inequality and reciprocally convex approach, the improved robust stability criteria are obtained based on the new augmented Lyapunov–Krasovskii functional. Since the proposed method contains more time-delay terms and the information of the lower bound of time delay, the main results have less conservative. Based on the improved criteria, robust $H_\infty$ controller design and performance analysis for uncertain system have been presented. The effectiveness and advantage of the proposed method have been demonstrated through numerical examples.

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