Remez-Type Inequality for Discrete Sets

Y. Yomdin

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

e-mail: yosef.yomdin@weizmann.ac.il

Abstract

The classical Remez inequality bounds the maximum of the absolute value of a polynomial \( P(x) \) of degree \( d \) on \([-1, 1]\) through the maximum of its absolute value on any subset \( Z \) of positive measure in \([-1, 1]\). Similarly, in several variables the maximum of the absolute value of a polynomial \( P(x) \) of degree \( d \) on the unit cube \( Q_1^1 \subset \mathbb{R}^n \) can be bounded through the maximum of its absolute value on any subset \( Z \subset Q_1^1 \) of positive \( n \)-measure.

The main result of this paper is that the \( n \)-measure in the Remez inequality can be replaced by a certain geometric invariant \( \omega_d(Z) \) which can be effectively estimated in terms of the metric entropy of \( Z \) and which may be nonzero for discrete and even finite sets \( Z \).
1 Introduction

The classical Remez inequality ([32]) bounds the maximum of the absolute value of a polynomial $P(x)$ of degree $d$ on $[-1,1]$ through the maximum of its absolute value on any subset $Z$ of positive measure in $[-1,1]$. More accurately:

Let $P(x)$ be a polynomial of degree $d$. Then for any measurable $Z \subset [-1,1]$
\[
\max_{[-1,1]}|P(x)| \leq T_d\left( \frac{4-\mu}{\mu} \right) \max_Z |P(x)|,
\]
where $\mu = \mu_1(Z)$ is the Lebesgue measure of $Z$ and $T_d(x) = \cos(d \arccos(x))$ is the $d$-th Chebyshev polynomial.

In several variables the generalization of (1.1) was obtained in [10]:

**Theorem 1.1** Let $B \subset \mathbb{R}^n$ be a convex body and let $\Omega \subset B$ be a measurable set. Then for any real polynomial $P(x) = P(x_1, \ldots, x_n)$ of degree $d$ we have
\[
\sup_B |P| \leq T_d\left( \frac{1 + (1-\lambda)\frac{n}{n}}{1 - (1-\lambda)\frac{n}{n}} \right) \sup_{\Omega} |P|.
\]

Here $\lambda = \frac{\mu_n(\Omega)}{\mu_n(B)}$, with $\mu_n$ being the Lebesgue measure on $\mathbb{R}^n$. This inequality is sharp and for $n = 1$ it coincides with the classical Remez inequality.

It is well known that the inequality of the form (1.1) or (1.2) may be true also for some sets $Z$ of measure zero and even for certain discrete or finite sets $Z$. Let us mention here only a couple of the most relevant results in this direction: in [2, 3, 12, 26, 31, 38, 39] such inequalities are provided for $Z$ being a regular grid in $[-1,1]$ (in [3] trigonometric polynomials are considered). Let us mention [1] where a “dual” problem is considered of interpolation by polynomials of degree higher than the number of the nodes. In [14] discrete sets $Z \subset [-1,1]$ are studied (see Section 2.2 below). An invariant $\phi_Z(d)$ is defined and estimated in some examples, which is the best constant in the Remez-type inequality of degree $d$ for the couple $(Z \subset [-1,1])$. Below in Definition 1.1 we call this invariant (extended to any dimension) the Remez $d$-span of $Z$ and denote it $R_d(Z)$.

On the other hand, recently in [9] Remez inequality has been extended (for complex polynomials of $n$ variables) to subsets $Z$ of positive Hausdorff
s-measure, \( s > 2n - 2 \). Here \( 2n - 2 \) is the real dimension of a zero set of such a polynomial, so the result has a natural geometric interpretation: Remez-type inequalities are true for \( Z \) having Hausdorff dimension larger than the dimension of the corresponding zero sets. For real polynomials of \( n \) variables, under the above assumption on \( Z \), an integral version of the Remez inequality was proved in [9], and a question was posed of the existence of a “strong” Remez-type inequality (of the form (1.2)).

In [7] estimates have been obtained for covering numbers of sub-level sets of families of analytic functions depending analytically on a parameter. Using these estimates strong Remez type inequalities have been proved for the restrictions of analytic functions to certain fractal sets. The existence of such inequalities was conjectured in [9].

In [6], [28]-[30] analytic and quasi-analytic functions have been studied from a similar point of view.

For one complex variable results similar to Remez inequality are provided by the classical Cartan lemma (see, for example, [17, 13] and references therein):

Let \( P(z) \) be a monic polynomial of one complex variable of degree \( d \). For any given \( \epsilon > 0 \) consider \( V_{\epsilon,d}(P) = \{ z \in \mathbb{C}, |P(z)| \leq \epsilon^d \} \). Then \( V_{\epsilon,d}(P) \) can be covered by at most \( d \) complex discs \( D_j \) with radii \( r_j, j = 1, \ldots, d \) such that

\[
\sum_{j=1}^{d} r_j \leq 2\epsilon^d.
\]

In [40] (see also [41]) a generalization of the Cartan lemma to plurisub-harmonic functions was obtained which leads, in particular, to the bounds on the size of sub-level sets similar to those obtained in [9].

In the present paper we would like to address a general problem of characterizing sets \( Z \) for which Remez-type inequality is valid. Having this in mind let us give the following definition:

**Definition 1.1** A set \( Z \subset Q_1^n \subset \mathbb{R}^n \) is called \( d \)-definite if any real polynomial \( P(x) = P(x_1, \ldots, x_n) \) of degree \( d \) bounded in absolute value by 1 on \( Z \) is bounded in absolute value by a certain constant \( C(d, Z) \) (not depending on \( P \)) on \( Q_1^n \). The minimum \( R_d(Z) \) of all such constants \( C(d, Z) \) is called the Remez \( d \)-span of \( Z \).

In view of the above-mentioned results the following problem looks natural and important:
Characterize (through their metric geometry) all the sets $Z \subset Q^n_1 \subset \mathbb{R}^n$ with the finite Remez $d$-span $R_d(Z)$ and compute $R_d(Z)$ for such $Z$ in “geometric” terms.

In principle, there is a very simple answer to this question: $R_d(Z) = \infty$ if and only if $Z$ is contained in a zero set of a certain nonzero polynomial $P$ of degree $d$. Indeed, in the opposite case $\sup_Z |P|$ and $\sup_{Q^n_1} |P|$ both are norms on the finite dimensional space of polynomials of degree $d$. However, in general it is not easy to reformulate this condition in “effective geometric terms” and to provide explicit bounds on $R_d(Z)$ starting with an explicitly given $Z$. For finite sets $Z$ it is possible (in principle) to write an explicit answer through the “interpolation systems” (see, for example, [18, 27, 16] and references therein). But to analyze, for instance, the asymptotic behavior of $R_d(Z)$ as $d \to \infty$ for a “fractal” $Z$ may be a tough problem (compare [14] and Section 2.2 below).

In the present paper we construct for subsets $Z \subset Q^n_1$ a simple geometric invariant $\omega_d(Z)$ which we call a metric $d$-span of $Z$. The metric $d$-span $\omega_d(Z)$ can be effectively estimated in terms of the metric entropy of $Z$ and it may be nonzero for discrete and even finite sets $Z$. Our main result is that the $n$-measure in the Remez-type inequality (1.2) can be replaced by $\omega_d(Z)$.

To define $\omega_d(Z)$ let us recall that the covering number $M(\epsilon, A)$ of a metric space $A$ is the minimal number of closed $\epsilon$-balls covering $A$ (see [19, 21, 22, 25]). Below $A$ will be subsets of $\mathbb{R}^n$ equipped with the $l^\infty$ metric. So the $\epsilon$-balls in this metric are the cubes $Q^n_\epsilon$.

For a polynomial $P$ on $\mathbb{R}^n$ let us consider the sub-level set $V_\rho(P)$ defined by $V_\rho(P) = \{x \in Q^n_1, |P(x)| \leq \rho\}$. The following result is provided by ([33, 34, 20]):

**Theorem 1.2 (Vitushkin’s bound)** For $V = V_\rho(P)$ as above

$$M(\epsilon, V) \leq \sum_{i=0}^{n-1} C_i(n, d) (\frac{1}{\epsilon})^i + \mu_n(V) (\frac{1}{\epsilon})^n,$$

(1.3)

with $C_i(n, d) = C'_i(n)(2d)^{(n-i)}$. For $n = 1$ we have $M(\epsilon, V) \leq d + \mu_1(V) (\frac{1}{\epsilon})$, and for $n = 2$ we have

$$M(\epsilon, V) \leq (2d - 1)^2 + 8d (\frac{1}{\epsilon}) + \mu_2(V) (\frac{1}{\epsilon})^2.$$
For $\epsilon > 0$ we denote by $M_{n,d}(\epsilon)$ (or shortly $M_d(\epsilon)$) the polynomial of degree $n - 1$ in $\frac{1}{\epsilon}$ as appears in (1.3):

$$M_d(\epsilon) = \sum_{i=0}^{n-1} C_i(n,d)(\frac{1}{\epsilon})^i. \quad (1.4)$$

In particular,

$$M_{1,d}(\epsilon) = d, \quad M_{2,d}(\epsilon) = (2d - 1)^2 + 8d(\frac{1}{\epsilon}).$$

Now for each subset $Z \subset Q^n_1$ (possibly discrete or finite) we introduce the metric $(n,d)$-span of $Z$ via the following definition:

**Definition 1.2** Let $Z$ be a subset in $Q^n_1 \subset \mathbb{R}^n$. Then the metric $(n,d)$-span (or shortly $d$-span) $\omega_d(Z)$ is defined as

$$\omega_d(Z) = \sup_{\epsilon > 0} \epsilon^n [M(\epsilon, Z) - M_d(\epsilon)]. \quad (1.5)$$

Now we are ready to state the main result of this paper which, in particular, provides a partial answer to the above-stated general problem of describing sets with finite Remez $d$-span:

**Theorem 1.3** If $\omega_d(Z) = \omega > 0$ then $R_d(Z)$ is finite and satisfies

$$R_d(Z) \leq R(\omega) := T_d\left(\frac{1 + (1 - \omega)^\frac{1}{n}}{1 - (1 - \omega)^\frac{1}{n}}\right), \quad (1.6)$$

where $T_d(x) = \cos(d \arccos(x))$ is the $d$-th Chebyshev polynomial.

This theorem is proved in Section 2 via a combination of inequalities (1.2) and (1.3), the last being reinterpreted as in Theorem 2.1 below. The bound in (1.6) is finite if and only if $\omega > 0$. As an immediate corollary of Definition 1.2 and Theorem 1.3 we obtain the following general condition for positivity of $\omega_d(Z)$:

**Corollary 1** The $d$-span $\omega_d(Z)$ is positive if and only if for certain $\epsilon > 0$ we have $M(\epsilon, Z) > M_d(\epsilon)$. 

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Corollary 1 establishes finiteness of $R_d(Z)$ for a large class of sets. First of all, this is true for $Z$ having a Hausdorff dimension $\dim_H(Z)$ greater than $n - 1$. Together with an explicit bound given in Section 3 below this provides a partial answer to the question posed in [9] of existence in this case of a Remez-type inequality of the form (1.2).

In fact, we can replace the Hausdorff dimension by the entropy (or the box) dimension $\dim_e(Z)$. The entropy dimension is defined in terms of the asymptotic behavior of the covering number $M(\epsilon, Z)$ as $\epsilon$ tends to 0. It is larger (and often strictly larger) than the Hausdorff dimension and it may take any value up to $n$ also for countable subsets of $\mathbb{R}^n$.

However, Corollary 1 shows finiteness of $R_d(Z)$ also for sufficiently large sets of dimension exactly $n - 1$:

**Corollary 2** Let $Z$ be a $C^1$-hypersurface in $Q^n_1$ with $\mu_{n-1}(Z) > C_{n-1}(n, d)$. Then $Z$ is $d$-definite.

By the virtue of the covering number also sufficiently dense finite subsets of $d$-definite sets are themselves $d$-definite. Thus Corollary 1 provides a class of examples of finite $d$-definite subsets $Z \subset B^n_1$: roughly, those are sufficiently dense finite subsets of sets of dimension $n - 1$ or higher.

It is important to analyze the behavior of the Remez $d$-span $R_d(Z)$ for finite (and general) sets $Z$ in terms of their metric structure. Here an appropriate invariant may be the so-called $\beta$-spread, introduced in [36, 37]. A very closely related notion is the “$\beta$-weight of a minimal spanning trees” (see [23] and references therein). Some initial results in this direction are given in Section 3.2.6 below.

Let us stress that the sufficient condition for a set $Z$ to be $d$-definite provided by Corollary 1 is not necessary in general: any small piece $W'$ of an irreducible algebraic hypersurface $W$ in $\mathbb{R}^n$ of degree $d_1 > d$ is $d$-definite, since it is not contained in any algebraic hypersurface of degree $d$. The same is true for transcendental hypersurfaces $W$ as well as for transcendental (or algebraic of high degree) sets $W$ of smaller dimensions, in particular, for curves.

On the other hand, in each of these situations, if we take a sufficiently small piece $W'$ of $W$ inside the unit cube, the $(n - 1)$-area $\mu_{n-1}(W')$ of $W'$ is much less than $C_{n-1}(n, d)$. Consequently, $M(\epsilon, W') \asymp \mu_{n-1}(W')(\frac{1}{\epsilon})^{n-1}$ is
always strictly less than \( C_{n-1}(n, d)(\frac{1}{n})^{n-1} \). Therefore by Corollary 1 we have \( \omega_d(W') = 0 \).

This stresses once more the importance (and apparent difficulty) of the problem of a “geometric characterization” of the \( d \)-definite sets.

The behavior of polynomials on discrete sets plays an important role in the Whitney problem of extension of differentiable functions from closed sets ([35]). In particular, there is an apparent relation of the Remez type inequalities with the problem of extending “finite differences” to higher dimensions. See [5, 11, 15] for some results representing recent progress in the Whitney problem.

The paper is organized as follows: in Section 2 we prove Theorem 1.3 and state (in a simplified form) some of the main results describing the behavior of the \( d \)-span in specific situations. The proofs of these results are postponed till Section 3, where they are given with all the required details and accurate (but somewhat cumbersome) constants.

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2 Proof of Theorem 1.3 and some basic examples of \( d \)-definite sets

The following theorem relates the \( n \)-volume of the sub-level sets \( V_\rho \) of a polynomial of degree \( d \) with the metric \((d, n)\)-span \( \omega_d(Z) \) of subsets \( Z \subset V_\rho \):

**Theorem 2.1** Let \( P(x) = P(x_1, \ldots, x_n) \) be a polynomial of degree \( d \) and let \( Z \subset Q^n_1 \) be a given set. Then if \( Z \subset V_\rho(P) \) for a certain \( \rho \geq 0 \) then we have

\[
\mu_n(V_\rho(P)) \geq \omega_d(Z),
\]

where \( \mu_n \) denotes, as above, the Lebesgue \( n \)-measure.

**Proof:** This fact follows directly from the Vitushkin bound on the covering number of the sub-level sets \( V_\rho \) given in Theorem 1.2 above: for any polynomial \( P(x) = P(x_1, \ldots, x_n) \) of degree \( d \), for \( Z \subset V_\rho(P) \), and for any \( \epsilon > 0 \) we
have
\[ M(\epsilon, Z) \leq M(\epsilon, V_\rho(P)) \leq M_d(\epsilon) + \mu_n(V_\rho(P))(\frac{1}{\epsilon})^n. \]

Consequently, for any \( \epsilon > 0 \) we have
\[ \mu_n(V_\rho(P)) \geq \epsilon^n[M(\epsilon, Z) - M_d(\epsilon)], \tag{2.1} \]
and we can take a supremum with respect to \( \epsilon \). This completes the proof. ■

Proof of Theorem 1.3 Assume that \( P \) is bounded in absolute value by 1 on \( Z \). Then we have \( Z \subset V_1(P) \). By Theorem 2.1 \( \mu_n(V_1(P)) \geq \omega_d(Z) = \omega \). Now since \( P \) is bounded in absolute value by 1 on \( V_1(P) \) by definition, we can apply the Yu. Brudnyi-Ganzburg inequality (Theorem 1.1 above) with \( B = Q_1^n \) and \( \Omega = V_1(P) \). This completes the proof. ■

Let us now study some specific classes of \( Z \).

Theorem 2.2 A set \( Z \subset Q_1^n \) of positive \( s \)-Hausdorff measure, \( s > n - 1 \), is \( d \)-definite for any \( d \).

Proof: This follows directly from Theorem 3.2, Section 3 below, where also a lower bound for \( \omega_d(Z) \) is given. ■

The invariant \( \omega_d \) is strong enough to prove that sets of dimension exactly \( n - 1 \) are definite, assuming their \((n - 1)\)-Hausdorff measure is big enough.

Theorem 2.3 A set \( Z \subset Q_1^n \) with \((n - 1)\)-Hausdorff measure \( H_{n-1}(Z) \) satisfying
\[ H_{n-1}(Z) > 2^nC_{n-1}(n, d) \]
is \( d \)-definite. In particular, any curve \( Z \subset Q_1^2 \) of the length \( l(Z) \) satisfying \( l(Z) > 16\sqrt{2}d \) is \( d \)-definite.

Proof: This follows directly from Corollary 4, Section 3 below, where also a lower bound for \( \omega_d(Z) \) is given. ■

2.1 Bounding Remez \( d \)-span via Minimal spanning trees

Let us now consider finite sets \( Z \). By virtue of the definitions any sufficiently dense finite subset of a set with positive \( d \)-span also has positive \( d \)-span. See Theorem 3.4 of Section 3 below specifying the choice of such a dense finite subset in each of the cases considered above. One result addressing the specific geometry of \( Z \) is the following (the distance below is with respect to the \( l^\infty \)-norm on \( \mathbb{R}^n \)): 
Theorem 2.4 Let a degree $d$ and a finite subset $Z \subset Q_1^n$ be given, and let $\epsilon_0$ be the minimal distance between the points of $Z$. Assume that $|Z| = p > M_d(\epsilon_0)$. Then the set $Z$ is $d$-definite. In particular, any set with more than $d$ points in $Q_1^n$ is $d$-definite. Any set $Z \subset Q_2^n$ with the number of points larger than $(2d - 1)^2 + 8d(\frac{1}{\epsilon_0})$ is $d$-definite.

Proof: This follows from Definition 1.2 since $M(\epsilon_0, Z) = |Z| = p$. ■

Remark Let us stress the importance of the assumption $Z \subset Q_1^n$ in Theorem 2.4. Without it we could take all the points of $Z$ on the same straight line. However, inside the cube $Q_1^n$ the points of $Z$ must form an “essentially $n$-dimensional” configuration in order to satisfy the inequality $|Z| > M_d(\epsilon_0)$.

Following the direction of Theorem 2.4 we can analyze in a more systematic way the behavior of $d$-span of finite (and general) sets $Z$ in terms of the mutual distances between the points of $Z$. This can be done in terms of the so-called $\beta$-spread, introduced in [36, 37]. A very closely related notion is the “$\beta$-weight of minimal spanning trees” (see [23] and references therein). Some results in this direction are given in Section 3.2.6 below.

2.2 Examples in one dimension and the Favard bound

We complete the present section with writing down explicitly the resulting bounds for some one-dimensional sets $Z$. Let us start with a regular grid.

Theorem 2.5 Let $d$ be given and let $G_1^s = \{x_1 = -1, x_2, \ldots, x_s = 1\}$ be a regular grid in $[-1, 1]$, $s > d$. Then $R_d(G_1^s) \leq T_d(\frac{r \mu}{s})$ where $\mu = \frac{2(s-d)}{(s-1)}$. In particular, $R_d(G_1^s)$ is finite for $s > d$ and it tends to 1 for $s \to \infty$.

Proof: It follows from the bounds on $\omega_d(G_1^s)$ computed in Example 1, Section 3. The result of Theorem 2.5 was obtained by a different method in [39, 38]. ■

In Section 3.1 below we compute $\omega_d(Z)$ for $Z = Z_r = \{1, \frac{1}{2r}, \frac{1}{3r}, \ldots, \frac{1}{kr}, \ldots\}$. We get

$$\omega_d(Z_r) \asymp \frac{r^r}{(r+1)^{r+1}} \frac{1}{d^r}.$$ 

In particular, for $r = 1$ i.e for $Z_1 = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}, \ldots\}$ we get $\omega_d(Z_1) \asymp \frac{1}{4d}$. 

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Now for for 0 < q < 1 and for $Z(q) = \{1, q, q^2, q^3, \ldots, q^m, \ldots\}$ computations in Section 3.1 give $\omega_d(Z(q)) \simeq \frac{q^d}{\log(q^2)}$.

Substituting these expressions for the $d$-span into the expression of Theorem 1.3 we obtain:

**Theorem 2.6** For the sets $Z_r$ and $Z(q)$ as above

$$R_d(Z_r) \leq R_d(\omega) = T_d\left(\frac{2 - \omega}{\omega}\right),$$

where $\omega \simeq \frac{r^d}{(r+1)^{d+1}} \frac{1}{d^d}$ or $\omega \simeq \frac{q^d}{\log(q^2)}$, respectively. For $d \to \infty$ we have $R_d(\omega) \simeq (\frac{4(r+1)^{r+1}}{r})^d d^d$ or $R_d(\omega) \simeq (4 \log \frac{1}{q})^d (\frac{1}{q})^d$, respectively. In particular,

$$R_d(Z_1) \leq 2^{4d} d^d, \quad R_d(Z(\frac{1}{2})) \leq 2^{d^2 + 2d}.$$

In [14] slightly better bounds are given in the last two examples:

$$R_d(Z_1) \leq (2d)^d, \quad R_d(Z(\frac{1}{2})) \leq (d + 1)2^{\frac{d^2 + 4d - 2}{2}}.$$

Favard’s method for bounding $R_d(Z)$ is to fix $d + 1$ points $Z$ and to estimate the corresponding interpolation polynomial of degree $d$. This produces the following general bound ([14]):

$$R_d(Z) \leq \inf_{x_1, \ldots, x_{d+1} \in Z} \sum_{i=1}^{d+1} \frac{1}{A'_{x_1, \ldots, x_{d+1}}(x_i)}, \quad (2.2)$$

where $A_{x_1, \ldots, x_{d+1}}(x) = \prod_{j=1}^{d+1}(x - x_j)$.

Unfortunately, we cannot expect Favard’s approach to produce realistic bounds on $R_d(Z)$ for general one-dimensional $Z \subset [-1, 1]$. The problem is that considering polynomials of degree $d$ we analyze the finite subsets in $Z$ containing exactly $d + 1$ points, and therefore we cannot take into account the influence of the rest of the set $Z$. In the examples $Z_r$ and $Z(q)$ considered above this method works well since for each $d$ the first $d + 1$ points of these sets give a sufficiently accurate approximation of the entire set. However, for a uniform grid $G^1_s$, $s \gg d$, a straightforward application of the Favard estimate gives $R_d(G^1_s) \leq (2e)^d$ (the minimum in (2.2) being achieved on the
approximately uniform sub-grid formed by \( d+1 \) points in \( G_s^1 \)), and this bound does not depend on \( s \) at all. Our bound given by Theorem 2.6 (which in this case is sharp up to a constant) shows that \( R_d(G_s^1) \) for any fixed degree \( d \) indeed tends to 1 as \( s \) increases.

**Remark.** It is an interesting problem to investigate the asymptotic behavior of \( R_d(Z) \) as \( d \to \infty \) for “fractal” sets \( Z \) in one and several dimensions. The examples above give some hope that the metric \( d \)-spread, being rather a coarse metric invariant, still provides an adequate tool for this problem. On the other hand, as it was mentioned above, there are \( d \)-definite sets \( Z \) for which \( \omega_d(Z) = 0 \).

As for a regular grid \( G_s^n \) with the step \( \frac{1}{s} \) in the unit cube \( Q_1^n \) we notice that the following inequality is true:

**Lemma 2.1** For each \( n \) \( R_d(G_s^n) \leq [R_d(G_s^1)]^n. \)

**Proof:** Induction by the dimension.

### 3 More examples of \( d \)-definite sets

In this section we consider in somewhat more details properties of the \( d \)-span and present more examples, stressing the question of positivity of \( \omega_d(Z) \). In particular, we provide the proofs of Theorems 2.2 and 2.3 and of some results used in Section 2.2 above.

#### 3.1 Some one-dimensional examples

For \( n = 1 \) the sub-level set \( V = V_{\rho}(P) \subset [-1, 1] \) is just a finite union of closed intervals. The maximal possible number of these intervals is \( d = \deg P \). Clearly, the covering number \( M(\epsilon, V) \) satisfies \( M(\epsilon, V) \leq d + \mu_1(V) \frac{1}{\epsilon} \), in agreement with Theorem 1.2 above. We get

**Proposition 3.1** For a set \( Z \subset [-1, 1] \), \( \omega_d(Z) = \sup_{\epsilon>0} \epsilon (M(\epsilon, Z) - d) \).

This immediately implies

**Corollary 3** For \( |Z| \leq d \) we have \( \omega_d(Z) = 0 \). For \( |Z| > d \) the \( d \)-span \( \omega_d(Z) \) is strictly positive.
In fact, the following more accurate bound can be given:

**Proposition 3.2** Let $|Z| = p > d$ and let $\epsilon_0$ be the minimal distance between the points of $Z$. Then the $d$-span $\omega_d(Z)$ satisfies the inequality $\omega_d(Z) \geq \epsilon_0(p - d)$.

**Proof:** We have $M(\epsilon_0, Z) = p$. ■

In Section 3.3.5 below we generalize this last remark to higher dimensions.

Let us give now some initial specific examples where the $d$-span can be explicitly estimated.

**Example 1.** Let $G_s = \{x_1 = -1, x_2, \ldots, x_s = 1\}$ be a regular grid in $[-1, 1]$. The covering number $M(\epsilon, G_s)$ is $\lceil \frac{1}{\epsilon} \rceil + 1$ for $\epsilon \geq \frac{2}{s - 1}$, and it is $s$ for $\epsilon < \frac{2}{s - 1}$. Therefore the function $\epsilon(M(\epsilon, G_s) - d)$ behaves as $2 - d\epsilon$ for $\epsilon \geq \frac{2}{s - 1}$, and it is $\epsilon(s - d)$ for $\epsilon < \frac{2}{s - 1}$. As Corollary 3 above shows, for $s \leq d$ we get $\omega_d(G_s) = 0$. For $s > d$ the supremum is achieved for $\epsilon = \frac{2}{s - 1}$ and we get $\omega_d(G_s) = \frac{2(s - d)}{s - 1}$. Notice that $\omega_d(G_s)$ tends to the total length of $[-1, 1]$ as $s$ grows (or as the “density” of the set $G_s$ inside $[-1, 1]$ increases).

**Example 2.** Let $Z_r = \{1, \frac{1}{2r}, \frac{1}{3r}, \ldots, \frac{1}{kr}, \ldots\}$. An easy computation shows that $M(\epsilon, Z_r) \approx \left(\frac{1}{\epsilon}\right)^{\frac{1}{r + 1}}$. Hence

$$\omega_d(Z_r) = \sup_{\epsilon > 0} \epsilon(M(\epsilon, Z_r) - d) \approx \sup_{\epsilon > 0} \epsilon\left(\frac{1}{\epsilon}\right)^{\frac{1}{r + 1}} - d) = \frac{r^r}{(r + 1)^{r + 1}} \frac{1}{d^r},$$

the supremum being attained for $\epsilon = \left(\frac{r}{r + 1}\right)^r \frac{1}{d^r}$. In particular, for $Z_1 = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}, \ldots\}$ we get $\omega_d(Z_1) \approx \frac{1}{4d}$, the supremum being attained for $\epsilon = \frac{1}{4d}$.

**Example 3.** Let for $0 < q < 1$, $Z(q) = \{1, q, q^2, q^3, \ldots, q^m, \ldots\}$. Computations as above give $\omega_d(Z(q)) \approx \frac{q^d}{\log(q)}$.

As for sharpness of these bounds, we show it via Theorem 2.1 above, which claims that for a sublevel set $V = V_\rho(P)$ containing $Z$ we have $\mu(V) \geq \omega_D(Z)$. Now the sets of the form $V_\rho(P)$ are exactly all the sets containing at most $d$ intervals. Therefore if we can cover $Z$ by $d$ intervals of a total length $a$ then by Theorem 2.1 we have $\omega_D(Z) \leq a$.

In Example 1 let us cover the grid $G_s$ with $d$ intervals, each containing $\left\lceil \frac{s}{d} \right\rceil$ consecutive points. There are $d - 1$ gaps of the length $\frac{2}{s - 1}$ between these
intervals, so their total length is $2 - \frac{2(d-1)}{s-1} = \frac{2(s-d)}{s-1}$. So the bound above is sharp.

In Example 2 we can easily find a covering of the set $Z_r$ with $d$ intervals of the total length $\frac{1}{d^r}$. Indeed, take first $d-1$ intervals of a small length, each covering exactly one point from $\{1, \frac{1}{2^r}, \frac{1}{3^r}, \ldots, \frac{1}{(d-1)^r}\}$. The rest of the set $Z_r$ we cover by one interval of the length $\frac{1}{d^r}$. So also here the bound above is sharp, up to a constant.

The same is true also in Example 3.

3.2 Higher dimensions

Let us start with some simple general properties of the $d$-span. Certainly, this geometric invariant is “stronger” than the usual $n$-measure $\mu_n$.

**Proposition 3.3** For a measurable subset $Z \subset B^n_1$ the $d$-span $\omega_d(Z)$ satisfies $\omega_d(Z) \geq \mu_n(Z)$.

**Proof:** Take $\epsilon \to 0$ in Definition 1.2, notice that $M_d(\epsilon)$ grows at most as $(\frac{1}{\epsilon})^{n-1}$, and use the fact that if we can cover $Z$ by $M(\epsilon, Z)$ disjoint $\epsilon$-cubes then $M(\epsilon, Z) \geq \frac{\mu(Z)}{\epsilon^n}$. ■

3.2.1 Sets of positive $s$-Hausdorff measure, $n-1 < s < n$

The result above can be generalized to sets of fractal Hausdorff measures. Let us recall that for $\beta > 0$ the $\beta$-Hausdorff measure of $Z$ is defined as

$$H_\beta(Z) = \lim_{\alpha \to 0} H_\beta^\alpha(Z),$$

where $H_\beta^\alpha(Z)$ is the lower bound of all the sums of the form $\sum_{i=1}^\infty r_i^\beta$, $r_i \leq \alpha$ and $Z \subset \cup_{i=1}^\infty A_i$, with the diam $A_i \leq r_i$. (See e.g. [19]).

However, in case $s < n$ we need more geometric information on our set $Z$ (and not only the positivity of its $s$-Hausdorff measure $H_s(Z)$) to conclude that the volume of any simple semi-algebraic set containing $Z$ is large. Indeed, think about a long but rapidly oscillating curve inside a small ball in the plane.

What we need is a kind of an “injectivity radius” $\epsilon^0$ of $Z$ for which the covering $\epsilon^0$ balls are almost disjoint. Let us give the following definition:
Definition 3.1 Let $H_s(Z) > 0$. We define the s-injectivity radius $\alpha_s^0(Z)$ as the maximal $\alpha$ such that $H_s^{\alpha}(Z) \geq \frac{1}{2} H_s(Z)$ for all $\alpha' \leq \alpha$.

Now we can compare the covering number and the s-Hausdorff measure:

Proposition 3.4 For $\epsilon \leq \hat{\alpha} = \frac{1}{\sqrt{n}} \alpha_s^0(Z)$ we have $M(\epsilon, Z) \geq \frac{1}{2\sqrt{n}} H_s(Z)(\frac{1}{\epsilon})^s$.

Proof: By definition of $\alpha_s^0(Z)$ and of $H_s^0(Z)$ we have for any covering of $Z$ by $M(\epsilon, Z)$ $\epsilon$-cubes

$$M(\epsilon, Z)(\sqrt{n}\epsilon)^s \geq \frac{1}{2} H_s(Z).$$

Hence $M(\epsilon, Z) \geq \frac{1}{2\sqrt{n}} H_s(Z)(\frac{1}{\epsilon})^s$. ■

Let us fix $n$ and $d$. We fix also a certain $s = n - 1 + \sigma$, $\sigma > 0$. We can prove now a general lower bound for the $d$-span of sets $Z$ with positive Hausdorff s-measure.

Let us introduce some notations. As above, we have $M_d(\epsilon) = C_0(n, d) + C_1(n, d)(\frac{1}{\epsilon}) + \ldots + C_{n-1}(n, d)(\frac{1}{\epsilon})^{n-1}$, where the constants $C_0(n, d), \ldots, C_{n-1}(n, d)$ depending only on $n, d$ have been defined in Theorem 1.2 above. For small $\epsilon$ the leading term of degree $n - 1$ in $\frac{1}{\epsilon}$ in $M_d(\epsilon)$ determines the asymptotic behavior of this expression, so let us define $\epsilon_1 = \epsilon_1(d)$ as the maximal $\epsilon$ such that $M_d(\epsilon') \leq 2C_{n-1}(n, d)(\frac{1}{\epsilon'})^{n-1}$ for all $\epsilon' \leq \epsilon$. Finally, for any $H > 0$ let us put $\epsilon_2 = \epsilon_2(H, d) = \frac{H}{8C_{n-1}(n, d)\sqrt{n}}$. Now we are ready to state the result.

Theorem 3.1 Let $s = n - 1 + \sigma$, $\sigma > 0$, and let $Z \subset Q_1^n$ satisfy $H_s(Z) = H > 0$. Then

$$\omega_d(Z) \geq \frac{1}{4} \epsilon_1^{-\sigma} H_s(Z).$$

Here $\hat{\epsilon} = \min\{\hat{\alpha}, \epsilon_1(1), \epsilon_2(H, d)\}$.

Proof: By definition

$$\omega_d(Z) = \sup_{\epsilon > 0} \epsilon [M(Z, \epsilon) - M_d(\epsilon)] \geq \hat{\epsilon} [M(Z, \hat{\epsilon}) - M_d(\hat{\epsilon})].$$

By the choice of $\hat{\epsilon}$ and by Proposition 3.4 we have $M(\hat{\epsilon}, Z) \geq \frac{1}{2\sqrt{n}} H_s(Z)(\frac{1}{\epsilon})^s$, while $M_d(\hat{\epsilon}) \leq 2C_{n-1}(n, d)(\frac{1}{\epsilon})^{n-1}$ since $\hat{\epsilon} \leq \epsilon_1(d)$. Therefore

$$M(Z, \hat{\epsilon}) - M_D(\hat{\epsilon}) \geq \frac{1}{2\sqrt{n}} H_s(Z)(\frac{1}{\epsilon})^s - 2C_{n-1}(n, d)(\frac{1}{\epsilon})^{n-1}. \hspace{1cm} (3.2)$$

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Finally, the condition that \( \hat{\epsilon} \leq \epsilon_2(H, d) = \left[ \frac{H}{8C_{n-1}(n, d) \sqrt{n}} \right]^\frac{1}{s} \) implies that the right-hand side of (3.2) is not smaller than \( \frac{1}{4\sqrt{n}} H_s(Z) (\frac{1}{\epsilon})^s \). Combining this last inequality with (3.1) and (3.2) we obtain the required bound. ■

**Remark.** An important feature of Theorem 3.1 is that we do not need to assume that the \( s \)-Hausdorff measure of \( Z \) is “large”. Just the fact that \( H_s(Z) > 0 \) implies \( \omega_d(Z) > 0 \). To stress the dependence of the bound of Theorem 3.1 on \( s \) and \( H_s(Z) \) let us assume that the radius of injectivity \( \alpha_0(Z) \) is large while the measure \( H_s(Z) \) is small. Then \( \hat{\epsilon} = \min\{\hat{\alpha}, \epsilon_1(d), \epsilon_2(H, d)\} = \epsilon_2(H, d) = \tilde{C}(d, s) H_\frac{1}{s} \), and therefore by Theorem 3.1 \( \omega_d(Z) \sim [C_1 H]^{\frac{1}{s}} \). This bound blows up as \( \sigma \to 0 \) or \( s \to n - 1 \).

Being quite effective, the bound of Theorem 3.1 is not sharp. Compare [7, 8].

However, for \( s \) exactly equal to \( n - 1 \) there is still a possibility to bound \( \omega_d(Z) \) from below if \( H_s(Z) \) is strictly greater than \( 2\sqrt{n^{n-1}} C_{n-1}(n, d) \). This bound is obtained in Corollary 4 in Section 3.2.4 below.

### 3.2.2 Sets with large covering number

The following result is parallel to Theorem 3.1, but it replaces the assumption of positivity of \( H_s(Z) \) with the assumption that the covering number \( M(Z, \epsilon) \) grows as \( C_s(\frac{1}{\epsilon})^s \), \( s > n - 1 \), for \( \epsilon \) sufficiently small. We preserve essentially the same notation as in Theorem 3.1: define the \( s \)-covering injectivity radius \( \epsilon_0^s(Z) \) as the maximal \( \epsilon \) such that \( M(Z, \epsilon') \geq \frac{1}{2} C_s(\frac{1}{\epsilon'})^s \) for all \( \epsilon' \leq \epsilon \). The parameter \( \epsilon_1 = \epsilon_1(d) \) is defined exactly as above, and we put \( \epsilon_2^s(C_s, d) = [\frac{C_s}{C_{n-1}(n, d)}]^\frac{1}{s} \).

**Theorem 3.2** Let \( s = n - 1 + \sigma, \sigma > 0 \), and let \( Z \subset Q_1^s \) satisfy \( M(Z, \epsilon) \geq C_s(\frac{1}{\epsilon})^s \), for all sufficiently small \( \epsilon \). Then

\[
\omega_d(Z) \geq \frac{1}{4} \epsilon^{1-\sigma} C_s.
\]

Here \( \hat{\epsilon} = \min\{\epsilon_0^s(Z), \epsilon_1(d), \epsilon_2^s(C_s, d)\} \).

**Proof:** Exactly the same as for Theorem 3.1. ■

**Remark.** As above, if \( \hat{\epsilon} = \epsilon_2^s(C_s, d) = [\frac{C_s}{C_{n-1}(n, d)}]^\frac{1}{s} \) then we get

\[
\omega_d(Z) \geq \frac{1}{4} (\frac{1}{8C_{n-1}(n, d)})^{\frac{1-\sigma}{s}} C_s^{\frac{1}{s}}.
\]
Notice also that Theorem 3.2 formally implies Theorem 3.1 because of Proposition 3.4. However, since the Hausdorff measure is probably a somewhat more natural invariant than the covering number, it looks preferable to separate these two statements.

3.2.3 Entropy and Hausdorff dimension

We recall here the notions of the entropy and the Hausdorff dimensions.

**Definition 3.2** Let $A \subset X$ be a bounded subset in a metric space $X$.

1. $\dim_e A = \inf\{\beta, \exists K, \text{ such that for each } \epsilon > 0, N(\epsilon, A) \leq K(\frac{1}{\epsilon})^\beta\}$ is called the entropy dimension of $A$.

2. $\dim_H A = \inf\{\beta, S_\beta(A) < \infty\}$ is called the Hausdorff dimension of $A$.

The notion of the entropy dimension appears in fractal geometry under many different names, in particular: “Minkowski dimension” - probably, the most justified historically, - “capacity dimension”, “box dimension”.

It is well known (see, for example, [19]) that for any set $A$ we have $\dim_e A \geq \dim_H A$. In particular, for countable sets $A$ always $\dim_H A = 0$ while $\dim_e A$ may take any value. The bounds of Theorems 3.1 and 3.2 imply the following:

**Proposition 3.5** For any $d$ and for any subset $Z \subset B^n_1$ if $\dim_e Z > n - 1$ then $\omega_d(Z) > 0$. In particular, this is true if \( \dim_H Z > n - 1 \).

3.2.4 Sets of dimension $n - 1$

Now we consider the case $s = n - 1$. Here we start with the covering number and obtain the corresponding result for the Hausdorff measure as a corollary. Let $M(Z, \epsilon) \geq C(\frac{1}{\epsilon})^{n-1}$ for $\epsilon \leq \epsilon_{n-1}^0$, with $C > C_{n-1}(n, d)$. We define $\epsilon'(d, C)$ as the largest $\epsilon$ for which $M_d(\epsilon'') \leq Q(C, d)(\frac{1}{\epsilon''})^{n-1}$ for all $\epsilon'' \leq \epsilon$. Here $Q(C, d) = [C_{n-1}(n, d) + \frac{1}{2}(C - C_{n-1}(n, d))]$.

**Theorem 3.3** Let $Z \subset Q^n_1$ satisfy $M(Z, \epsilon) \geq C(\frac{1}{\epsilon})^{n-1}$ for $\epsilon \leq \epsilon_{n-1}^0$, with $C > C_{n-1}(n, d)$. Then

$$\omega_d(Z) \geq \frac{1}{2} \hat{\epsilon}(C - C_{n-1}(n, d)),$$

where $\hat{\epsilon} = \min\{\epsilon_{n-1}^0, \epsilon'(C, d)\}$. 

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Proof: Exactly as in Theorem 3.1. ■

Via Proposition 3.4 we obtain:

**Corollary 4** Let $Z \subset B^n_1$ satisfy $H_{n-1}(Z) > 2\sqrt{n^{n-1}}C_{n-1}(n,d)$. Then

$$\omega_d(Z) \geq \frac{1}{2}\hat{\epsilon}(C - C_{n-1}(n,d)),$$

where $C = \frac{1}{2\sqrt{n^{n-1}}}H_{n-1}(Z)$ and $\hat{\epsilon} = \min \{ \frac{1}{\sqrt{n}}\alpha^0_{n-1}(Z), \epsilon'_1(C,d) \}$.

3.2.5 Dense finite subsets in “massive” sets

Each of the results above produces, in particular, a finite subset $Z' \subset Q^n_1$ with $\omega_d(Z') > 0$. Indeed, in each of the situations covered by Theorems 3.1-3.3 and Corollary 4 let us define $Z' \subset Z$ as the set of the centers of all the $\frac{1}{2}\hat{\epsilon}$-cubes providing a covering of $Z$ with $M(Z, \frac{1}{2}\hat{\epsilon})$ elements. We have

**Theorem 3.4** In each of the situations covered by Theorems 3.1-3.3 and Corollary 4 the $d$-span of the finite set $Z' \subset Z$ satisfies $\omega_d(Z') \geq (\frac{1}{2})^nK$ where $K$ is the appropriate lower bound for $\omega_d(Z)$.

**Proof:** If certain $\frac{1}{2}\hat{\epsilon}$-cubes cover $Z'$ then the corresponding $\hat{\epsilon}$-cubes cover $Z$. Therefore $M(Z', \frac{1}{2}\hat{\epsilon}) \geq M(Z, \frac{1}{2}\hat{\epsilon})$. The rest of the proof goes exactly as in the results above. ■

3.2.6 Bounding $D$-span via Minimal spanning trees

Theorem 3.4 provides a class of examples of finite subsets $Z \subset B^n_1$ with positive $d$-span: roughly, those are sufficiently dense finite subsets of sets of dimension $n - 1$ or higher. It is important to analyze the behavior of $d$-span of finite (and general) sets $Z$, given by themselves, with no relation to an underlying “large” set, in terms of their metric structure. Here an appropriate invariant may be the so-called $\beta$-spread, introduced in [36, 37]. A very closely related notion is the “$\beta$-weight of minimal spanning trees” (see [23] and references therein). The main reason for us to relate the $d$-span with the $\beta$-spread and minimal spanning trees is that a lot of information is available today in this direction (for some initial references see [23]), and we can hope to ultimately incorporate this information in our study of polynomial and smooth interpolation problems.

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Let’s recall a definition of $\beta$-spread. Let $G_p$ be the set of all connected non-oriented trees with $p$ vertices. We write $(i, j) \in g$, for $g \in G_p$, if the vertices $i$ and $j$ are connected by the edge in $g$.

**Definition 3.3** Let $X$ be a metric space, $\beta > 0$. For each $x_1, \ldots, x_p \in X$ and $g \in G_p$ let $\rho_\beta(g, x_1, \ldots, x_p) = \sum_{(i,j) \in g} d(x_i, x_j)^\beta$, where $d$ is a distance in $X$. Define $\rho_\beta(x_1, \ldots, x_p)$ as $\inf_{g \in G_p} \rho_\beta(g, x_1, \ldots, x_p)$. The tree $g$ on which the infimum is achieved is called the $\beta$-minimal spanning tree. Now let $A \subset X$. We define the $\beta$-spread of $A$, $V_\beta(A)$, by

$$V_\beta(A) = \sup_{p, x_1, \ldots, x_p \in A} \rho_\beta(x_1, \ldots, x_p).$$

For $x_1, \ldots, x_p \in X$, $\rho_\beta(x_1, \ldots, x_p)$ is called a $\beta$-weight of the minimal spanning tree $g$ on $(x_1, \ldots, x_p)$. Notice that the 1-minimal tree is also minimal for any $\beta$ (see [23]).

Under a different name $\beta$-spread for subsets of a real line has been studied in [4]. A notion of $\beta$-weight has appeared earlier in geometric combinatorics and in fractal geometry. Compare [23, 24], [19] and references therein. However, we are not aware of any appearance of the general notion of $\beta$-spread in metric spaces, as defined above.

Let us also notice that as a function of $\beta$ the spread $V_\beta(A)$ is a kind of a zeta-function. For $A = \{0, 1, 3, 6, 10, \ldots, \frac{1}{2}n(n + 1), \ldots\}$ the spread $V_\beta(A)$ is exactly the Riemann $\zeta$-function $\zeta(-\beta)$, while for $A = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$ with $\alpha_0 = 0$, $\alpha_1 = 1, \ldots, \alpha_n = \sum_{i=1}^{n} \frac{1}{i}$, $n \geq 1$, we have $V_\beta(A) = \zeta(\beta)$. So it may be a good idea to substitute into $V_\beta(A)$ complex values of $\beta$. See [24] for a detailed treatment of fractal geometry from this point of view.

We shall not touch here general properties of $\beta$-spread, as well as its relations to the geometry of critical values of smooth functions. Instead we give a lower bound for the $d$-span in terms of $\beta$-spread. Let us first provide an immediate generalization to higher dimensions of Proposition 3.2 above. We have to consider here the $l^\infty$ distance instead of the usual Euclidean distance in $\mathbb{R}^n$.

**Proposition 3.6** Let $d$ and a finite subset $Z \subset Q_1^n$ be given, and let $\epsilon_0$ be the minimal distance between the points of $Z$. Assume that $|Z| = p > M_d(\epsilon_0)$. Then the $d$-span $\omega_d(Z)$ satisfies the inequality $\omega_d(Z) \geq \epsilon_0^n (p - M_d(\epsilon_0)) > 0$.  

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For any $Z$, finite or infinite, we can apply Proposition 3.6 to finite subsets of $Z$. Let us introduce some convenient notations (see [37]).

**Definition 3.4** Let $X$ be a metric space. For $x_1, \cdots, x_p \in X$, let $\nu(x_1, \cdots, x_p) = \min_{i \neq j} d(x_i, x_j)$. For $A \subset X$ define $\eta_A(p)$ for any natural $p \geq 2$ by

$$
\eta_A(p) = \sup_{x_1, \cdots, x_p \in A} \nu(x_1, \cdots, x_p).
$$

Proposition 3.6 implies

**Corollary 5** Let $d$ and a finite subset $Z \subset Q_1^n$ be given. If for a certain $p$ we have $M_d(\eta_Z(p)) < p$ then $\omega_d(Z) \geq \eta_Z^n(p)(p - M_d(\eta_Z(p))) > 0$.

Let us remind that $M_d(\epsilon) = \sum_{j=0}^{n-1} C_j(n,d)(\frac{1}{\epsilon})^j \leq C'(n,d)(\frac{1}{\epsilon})^{n-1}$ for $\epsilon \leq 1$. Hence we have a weaker but simpler version of Corollary 5:

**Corollary 6** Let $d$ and a finite subset $Z \subset Q_1^n$ be given. If for a certain $p$ we have $C'(n,d)(\eta_Z(p))^{1-n} < p$ then

$$
\omega_d(Z) \geq \eta_Z^n(p)(p - C'(n,d)(\eta_Z(p))^{1-n}) > 0. \quad (3.3)
$$

Now we can give a criterion of positivity of $\omega_d(Z)$ in terms of the $\beta$-spread of $Z$:

**Theorem 3.5** Let $d$ and a subset $Z \subset Q_1^n$ be given. If for a certain $\beta$, with $n - 1 < \beta \leq n$, we have $V_\beta(Z) > C'(n,d)^{\frac{\alpha}{n-1}} \zeta(\frac{\beta}{n-1})$, where $\zeta(x) = \sum_{p=1}^{\infty} \frac{1}{p^x}$, then $\omega_d(Z) > 0$.

**Proof:** Assume that $\omega_d(Z) = 0$. Then by Corollary 5 we have for each $p = 2, 3, \ldots$ that $C'(n,d)(\eta_Z(p))^{1-n} \geq p$. Hence $\eta_Z(p) \leq (\frac{C'(n,d)}{p})^{\frac{1}{n-1}}$ and for each $\beta$ we have

$$
\sum_{p=2}^{\infty} \eta_Z^\beta(p) \leq (C'(n,d))^{\frac{\beta}{n-1}} \sum_{p=1}^{\infty} \left(\frac{1}{p}\right)^{\frac{\beta}{n-1}} = C'(n,d)^{\frac{\beta}{n-1}} \zeta(\frac{\beta}{n-1}). \quad (3.4)
$$

Now, the following result relates $\eta_Z(p)$ and $V_\beta(Z)$:

**Proposition 3.7** For any $\beta > 0$

$$
\sup_{p \geq 2}(p-1)\eta_Z^\beta(p) \leq V_\beta(Z) \leq \sum_{j=2}^{\infty} \eta_Z^\beta(j). \quad (3.5)
$$
The proof of Proposition 3.7 is given in [36] (see also [37]). Combining this result with (3.4) we complete the proof of Theorem 3.5. □

In analogy with the Hausdorff and entropy dimensions let us define the $V$-dimension as follows: $\dim_V A = \inf\{\beta, V_\beta(A) < \infty\}$. It turns out that always $\dim_V A = \dim_e A$ (see [36, 37, 23]). Now Theorem 3.5 provides another proof of Proposition 3.5 above. Indeed, if $\dim_e Z = s > n - 1$, we fix some $\beta$ such that $n - 1 < \beta < s = \dim_V Z$. By definition of $\dim_V Z$ we have $V_\beta(Z) = \infty$ while $\zeta(\frac{\beta}{n-1})$ is finite since $\beta > n - 1$. Theorem 3.5 implies now that $\omega_d(Z) > 0$.

There are limit cases where $\beta$-spread is more sensitive to certain subtle geometric properties of $Z$ than the covering number (see [36, 37, 23] and references therein). It is also related with some important notions in Potential Theory, like transfinite diameter. We plan to present some results in this direction separately.

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