Characterization of Lipschitz Functions via the Commutators of the Fractional Maximal Function on Stratified Lie Groups

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Abstract

In this paper, the main aim is to consider the boundedness of the fractional maximal commutator $M_{\alpha,b}$ and the nonlinear commutator $[b, M_{\alpha}]$ on the Lebesgue spaces over some stratified Lie group $G$ when $b$ belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces on Lie group are given.

Keywords: stratified Lie group, fractional maximal function, Lipschitz function, commutator

2020 MSC: 42B35, 43A80

1 Introduction and main results

Stratified groups appear in quantum physics and many parts of mathematics, including several complex variables, Fourier analysis, geometry, and topology [1, 2]. The geometry structure of stratified groups is so good that it inherits a lot of analysis properties from the Euclidean spaces [3, 4]. Apart from this, the difference between the geometry structures of Euclidean spaces and stratified groups makes the study of function spaces on them more complicated. However, many harmonic analysis problems on stratified Lie groups deserve a further investigation since most results of the theory of Fourier transforms and distributions in Euclidean spaces cannot yet be duplicated.

Let $T$ be the classical singular integral operator. The commutator $[b, T]$ generated by $T$ and a suitable function $b$ is defined by

$$[b, T]f = bT(f) - T(bf).$$ (1.1)

It is known that the commutators are intimately related to the regularity properties of the solutions of certain partial differential equations (PDE), see [5–7].

The first result for the commutator $[b, T]$ was established by Coifman et al.[8], and the authors proved that $b \in \text{BMO}(\mathbb{R}^n)$ (bounded mean oscillation functions) if and only if the commutator (1.1) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson [9] generalized the results in [8] to functions belonging to a Lipschitz functional space and gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via commutator (1.1), and the author proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also [10]).

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Preprint submitted to Journal of \LaTeX Templates June 17, 2022
In addition, using real interpolation techniques, Milman and Schonbek[11] established a commutator result that applies to the Hardy-Littlewood maximal function as well as to a large class of nonlinear operators. In 2000, Bastero et al.[12] proved the necessary and sufficient condition for the boundedness of the nonlinear commutators \([b,M]\) and \([b,M^2]\) on \(L^p\) spaces. In 2009, Zhang and Wu[13] studied the same problem for \([b,M_\alpha]\). In 2017, Zhang[14] considered some new characterizations of the Lipschitz spaces via the boundedness of maximal commutator \(M_\alpha\) and the (nonlinear) commutator \([b,M]\) in Lebesgue spaces and Morrey spaces on Euclidean spaces. In 2018, Zhang et al.[15] gave necessary and sufficient conditions for the boundedness of the nonlinear commutators \([b,M_\alpha]\) and \([b,M^2]\) on Orlicz spaces when the symbol \(b\) belongs to Lipschitz spaces, and obtained some new characterizations of non-negative Lipschitz functions.

And Guliyev[16] recently gave necessary and sufficient conditions for the boundedness of the fractional maximal commutators in the Orlicz spaces \(L^p(\mathbb{G})\) on any stratified Lie group \(\mathbb{G}\) when \(b\) belongs to \(\text{BMO}(\mathbb{G})\) spaces, and obtained some new characterizations for certain subclasses of \(\text{BMO}(\mathbb{G})\) spaces.

Inspired by the above literature, the purpose of this paper is to study the boundedness of the fractional maximal commutator \(M_{\alpha,b}\) and the nonlinear commutator \([b,M_\alpha]\) on the Lebesgue spaces over some stratified Lie group \(\mathbb{G}\) when \(b \in \dot{A}_\beta(\mathbb{G})\), by which some new characterizations of the Lipschitz spaces are given.

Let \(0 \leq \alpha < Q\) and \(f : \mathbb{G} \to \mathbb{R}\) be a locally integrable function, the fractional maximal function is defined by

\[
M_\alpha(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_B |f(y)| \, dy,
\]

where the supremum is taken over all balls \(B \subset \mathbb{G}\) containing \(x\) with radius \(r > 0\), and \(|B|\) is the Haar measure of the \(\mathbb{G}\)-ball \(B\). And the fractional maximal commutator generated by the operator \(M_\alpha\) and a locally integrable function \(b\) is defined by

\[
M_{\alpha,b}(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_B |b(x) - b(y)| |f(y)| \, dy.
\]

If \(\alpha = 0\), then \(M_{0,b} \equiv M_b\) is the sublinear maximal commutator operator.

On the other hand, similar to (1.1), the commutator generated by the fractional maximal operator \(M_\alpha\) and a suitable function \(b\) is defined by

\[
[b,M_\alpha](f)(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x).
\]

Note that operators \(M_{\alpha,b}\) and \([b,M_\alpha]\) essentially differ from each other. For example, \(M_{\alpha,b}\) is positive and sublinear, but \([b,M_\alpha]\) is neither positive nor sublinear.

The first main result of this paper is to study the boundedness of \(M_{\alpha,b}\) when the symbol \(b\) belongs to a Lipschitz space. Some characterizations of the Lipschitz space via such commutator are given.

**Theorem 1.1** Let \(b\) be a locally integrable function and let \(0 < \beta < 1\), \(0 < \alpha < Q\) and \(0 < \alpha + \beta < Q\). Then the following statements are equivalent:

1. \(b \in \dot{A}_\beta(\mathbb{G})\).
2. \(M_{\alpha,b}\) is bounded from \(L^p(\mathbb{G})\) to \(L^q(\mathbb{G})\) for all \(p, q\) with \(1 < p < \frac{Q}{Q+\beta}\) and \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}\).
3. \(M_{\alpha,b}\) is bounded from \(L^p(\mathbb{G})\) to \(L^q(\mathbb{G})\) for some \(p, q\) with \(1 < p < \frac{Q}{Q+\beta}\) and \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}\).
4. There exists $q \in [1, \infty)$ such that
\[ \sup_B \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^q dx \right)^{1/q} < \infty. \tag{1.2} \]

5. For all $q \in [1, \infty)$ we have (1.2).

**Remark 1** For the case $\alpha = 0$ and $\mathbb{G} = \mathbb{R}^n$, similar results were given in [14] for Lebesgue spaces with constant exponents, and in [17, 18] for the variable case.

The second main result of this paper aims to study the mapping properties of the (nonlinear) commutator $[b, M_{\alpha}]$ when $b$ belongs to some Lipschitz space.

**Theorem 1.2** Let $0 < \beta < 1$, $0 < \alpha < Q$, $0 < \alpha + \beta < Q$ and let $b$ be a locally integrable function. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \geq 0$.
2. $[b, M_{\alpha}]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for all $p$ and $q$ satisfy $1 < p < \frac{Q}{\alpha + \beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta}{Q}$.
3. $[b, M_{\alpha}]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for some $p$ and $q$ such that $1 < p < \frac{Q}{\alpha + \beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta}{Q}$.
4. There exists $s \in [1, \infty)$ such that
\[ \sup_{B \ni x} \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)|^s dx \right)^{1/s} < \infty. \tag{1.3} \]

5. For all $s \in [1, \infty)$ we have (1.3).

**Remark 2** Let $b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \geq 0$ if and only if
\[ \sup_{B \ni x} \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)|^s dx \right)^{1/s} < \infty. \tag{1.4} \]

Compared with (1.4), (1.3) gives a new characterization for nonnegative Lipschitz functions.

This paper is organized as follows. In the next section, we recall some basic definitions and known results. In Section 3, we will prove Theorems 1.1 and 1.2.

Throughout this paper, the letter $C$ always stands for a constant independent of the main parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter $L^p$ $(1 \leq p \leq \infty)$ will always denote the standard $L^p$-space with respect to the Haar measure $dx$, with the $L^p$-norm $\| \cdot \|_p$. And let $W L^p$ be weak-type $L^p$-space. Denote by $\chi_E$ the characteristic function of a measurable set $E$ of $\mathbb{G}$. 


2 Preliminaries and lemmas

2.1 Lie group $G$

To prove the main results of this paper, we first recall some necessary notions and remarks. Firstly, we recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to [1, 3, 19].

**Definition 2.1** We say that a Lie algebra $G$ is stratified if there is a direct sum vector space decomposition

$$G = \bigoplus_{j=1}^{m} V_j = V_1 \oplus \cdots \oplus V_m$$  \((2.1)\)

such that $G$ is nilpotent of step $m$ if $m$ is the smallest integer for which all Lie brackets (or iterated commutators) of order $m + 1$ are zero, that is,

$$[V_1, V_j] = \begin{cases} V_{j+1} & 1 \leq j \leq m - 1 \\ 0 & j \geq m \end{cases}$$

holds.

It is not difficult to find that the above $V_1$ generates the whole of the Lie algebra $G$ by taking Lie brackets.

**Remark 3** [20] Let $G = G_1 \supset G_2 \supset \cdots \supset G_{m+1} = \{0\}$ denote the lower central series of $G$, and \{X_1, \ldots, X_N\} be a basis for $V_1$ of $G$.

(i) The direct sum decomposition (2.1) can be constructed by identifying each $G_j$ as a vector subspace of $G$ and setting $V_m = G_m$ and $V_j = G_j \setminus G_{j+1}$ for $j = 1, \ldots, m - 1$.

(ii) The dimension of $G$ at infinity as the integer $Q$ is given by

$$Q = \sum_{j=1}^{m} j \dim(V_j) = \sum_{j=1}^{m} \dim(G_j).$$

**Definition 2.2** A Lie group $G$ is said to be stratified when it is a connected simply-connected Lie group and its Lie algebra $G$ is stratified.

If $G$ is stratified, then its Lie algebra $G$ admits a canonical family of dilations \{\delta_r\}, namely, for $r > 0$, $X_k \in V_k$ ($k = 1, \ldots, m$),

$$\delta_r \left( \sum_{k=1}^{m} X_k \right) = \sum_{k=1}^{m} r^k X_k,$$

which are Lie algebra automorphisms.

By the Baker-Campbell-Hausdorff formula for sufficiently small elements $X$ and $Y$ of $G$ one has

$$\exp X \exp Y = \exp H(X, Y) = X + Y + \frac{1}{2} [X, Y] + \cdots$$

where $\exp : G \to G$ is the exponential map, $H(X, Y)$ is an infinite linear combination of $X$ and $Y$ and their Lie brackets, and the dots denote terms of order higher than two.

The following properties can be found in [21](see Proposition 1.1.1, or Proposition 2.1 in [22] or Proposition 1.2 in [1]).
Proposition 2.1 Let $\mathcal{G}$ be a nilpotent Lie algebra, and let $\mathbb{G}$ be the corresponding connected and simply-connected nilpotent Lie group. Then we have

1. The exponential map $\exp: \mathcal{G} \to \mathbb{G}$ is a diffeomorphism. Furthermore, the group law $(x, y) \mapsto xy$ is a polynomial map if $\mathbb{G}$ is identified with $\mathcal{G}$ via $\exp$.

2. Let $\lambda$ be a Lebesgue measure on $\mathcal{G}$, then $\exp\lambda$ is a bi-invariant Haar measure on $\mathbb{G}$ (or a bi-invariant Haar measure $dx$ on $\mathbb{G}$ is just the lift of Lebesgue measure on $\mathcal{G}$ via $\exp$).

Thereafter, $y^{-1}$ represents the inverse of $y \in \mathbb{G}$, $y^{-1}x$ stands for the group multiplication of $y^{-1}$ by $x$ and the group identity element of $\mathbb{G}$ will be referred to as the origin denotes by $e$.

A homogenous norm on $\mathbb{G}$ is a continuous function $x \to \rho(x)$ from $\mathbb{G}$ to $[0, \infty)$, which is $C^\infty$ on $\mathbb{G}\setminus\{0\}$ and satisfies

$$\begin{cases} \rho(x^{-1}) = \rho(x), \\ \rho(t x) = t \rho(x) \quad \text{for all } x \in \mathbb{G} \text{ and } t > 0, \\ \rho(e) = 0. \end{cases}$$

Moreover, there exists a constant $c_0 \geq 1$ such that $\rho(xy) \leq c_0(\rho(x) + \rho(y))$ for all $x, y \in \mathbb{G}$.

With the norm above, we define the $\mathbb{G}$ ball centered at $x$ with radius $r$ by $B(x, r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\}$, and by $\lambda B$ denote the ball $B(x, r)$ with $\lambda > 0$, let $B_e = B(e, r) = \{y \in \mathbb{G} : \rho(y) < r\}$ be the open ball centered at $e$ with radius $r$, which is the image under $\delta_r$ of $B(e, 1)$. And by $\overline{B}(x, r) = \mathbb{G} \setminus B(x, r) = \{y \in \mathbb{G} : \rho(y^{-1}x) \geq r\}$ denote the complement of $B(x, r)$. Let $|B(x, r)|$ be the Haar measure of the ball $B(x, r) \subset \mathbb{G}$, and there exists $c_1 = c_1(\mathbb{G})$ such that

$$|B(x, r)| = c_1 r^Q, \quad x \in \mathbb{G}, \quad r > 0.$$  

The most basic partial differential operator in a stratified Lie group is the sub-Laplacian associated with $X$ is the second-order partial differential operator on $\mathbb{G}$ given by

$$\mathcal{Q} = \sum_{i=1}^{n} X_i^2$$

The following lemma is known as the Hölder’s inequality on Lebesgue spaces over Lie groups $\mathbb{G}$, it can also be found in [23] or [16], when Young function $\Phi(t) = t^p$ and its complementary function $\Psi(t) = t^q$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.1 (Hölder’s inequality on $\mathbb{G}$) Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbb{G}$ be a measurable set and measurable functions $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then there exists a positive constant $C$ such that

$$\int_{\Omega} |f(x)g(x)|dx \leq C \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$  

By elementary calculations we have the following property. It can also be found in [16], when Young function $\Phi(t) = t^p$.

Lemma 2.2 (Norms of characteristic functions) Let $0 < p < \infty$ and $\Omega \subset \mathbb{G}$ be a measurable set with finite Haar measure. Then

$$\|\chi_{\Omega}\|_{L^p(\mathbb{G})} = \|\chi_{\Omega}\|_{W^1_{L^p(\mathbb{G})}} = |\Omega|^{1/p}.$$  

5
2.2 Lipschitz spaces on $G$

Next we give the definition of the Lipschitz spaces on $G$, and state some basic properties and useful lemmas.

**Definition 2.3 (Lipschitz-type spaces on $G$)**

(1) Let $0 < \beta < 1$, we say a function $b$ belongs to the Lipschitz space $\hat{\Lambda}_\beta(G)$ if there exists a constant $C > 0$ such that for all $x, y \in G$,

$$|b(x) - b(y)| \leq C(\rho(y^{-1}x))^\beta,$$

where $\rho$ is the homogenous norm. The smallest such constant $C$ is called the $\hat{\Lambda}_\beta$ norm of $b$ and is denoted by $\|b\|_{\hat{\Lambda}_\beta(G)}$.

(2) (see [24]) Let $0 < \beta < 1$ and $1 \leq p < \infty$. The space $\text{Lip}_{\beta,p}(G)$ is defined to be the set of all locally integrable functions $b$, i.e., there exists a positive constant $C$, such that

$$\sup_{B \ni x} \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{1/p} \leq C$$

where the supremum is taken over every ball $B \subset G$ containing $x$ and $b_B = \frac{1}{|B|} \int_B b(x)dx$. The least constant $C$ satisfying the conditions above shall be denoted by $\|b\|_{\text{Lip}_{\beta,p}(G)}$.

(3) (see [24]) Let $0 < \beta < 1$. When $p = \infty$, we shall say that a locally integrable functions $b$ belongs to $\text{Lip}_{\beta,\infty}(G)$ if there exists a constant $C$ such that

$$\text{ess sup}_{x \in B} \frac{|b(x) - b_B|}{|B|^{\beta/Q}} \leq C$$

holds for every ball $B \subset G$ with $b_B = \frac{1}{|B|} \int_B b(x)dx$. And $\|b\|_{\text{Lip}_{\beta,\infty}(G)}$ stand for the least constant $C$ satisfying the conditions above.

**Remark 4**

(i) Similar to the definition of Lipschitz space $\hat{\Lambda}_\beta(G)$ in (1), we also have the definition form as following (see [25–27] et al.)

$$\|b\|_{\hat{\Lambda}_\beta(G)} = \sup_{x, y \in G, x \neq y} \frac{|b(xy) - b(x)|}{(\rho(y))^\beta} = \sup_{x, y \in G, x \neq y} \frac{|b(x) - b(y)|}{(\rho(y^{-1}x))^\beta}.$$

And $\|b\|_{\hat{\Lambda}_\beta(G)} = 0$ if and only if $b$ is constant.

(ii) In (2), when $p = 1$, we have

$$\|b\|_{\text{Lip}_{\beta,1}(G)} = \sup_{B \ni x} \frac{1}{|B|^\beta/Q} \left( \frac{1}{|B|} \int_B |b(x) - b_B| dx \right) := \|b\|_{\text{Lip}_\beta(G)}$$

(iii) There are two basically different approaches to Lipschitz classes on the $n$-dimensional Euclidean space. Lipschitz classes can be defined via Poisson (or Weierstrass) integrals of $L^p$-functions, or, equivalently, by means of higher order difference operators (see [28]).

**Lemma 2.3** (see [24, 26, 29]) Let $0 < \beta < 1$ and the function $b(x)$ integrable on bounded subsets of $G$. 


(1) When $1 \leq p < \infty$, then 
\[
\|b\|_{\Lambda^p(G)} = \|b\|_{\text{Lip}_p(G)} \approx \|b\|_{\text{Lip}_{p,\beta}(G)}.
\]

(2) Let balls $B_1 \subset B_2 \subset G$ and $b \in \text{Lip}_{\beta,p}(G)$ with $p \in [1, \infty]$. Then there exists a constant $C$ depends on $B_1$ and $B_2$ only, such that
\[
|b_{B_1} - b_{B_2}| \leq C\|b\|_{\text{Lip}_{\beta,p}(G)}|B_2|^\beta/Q.
\]

(3) When $1 \leq p < \infty$, then there exists a constant $C$ depends on $\beta$ and $p$ only, such that
\[
|b(x) - b(y)| \leq C\|b\|_{\text{Lip}_{\beta,p}(G)}|B|^\beta/Q
\]
holds for any ball $B$ containing $x$ and $y$.

2.3 Maximal function

Let $f : G \rightarrow \mathbb{R}$ be a locally integrable function. The Hardy–Littlewood maximal function $M$ is given by
\[
M(f)(x) = \sup_{B \ni x} |B|^{-1} \int_B |f(y)|dy,
\]
where the supremum is taken over all balls $B \subset G$ containing $x$.

The fractional maximal function $M_\alpha(f)$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $M(f)(x) \equiv M_0(f)(x)$.

For a function $b$ defined on $G$, we denote
\[
b^-(x) := -\min\{b, 0\} = \begin{cases} 
0, & \text{if } b(x) \geq 0 \\
|b(x)|, & \text{if } b(x) < 0 
\end{cases}
\]
and $b^+(x) = |b(x)| - b^-(x)$. Obviously, $b(x) = b^+(x) - b^-(x)$.

Now, we give the following pointwise estimate for $[b, M_\alpha]$ on $G$.

Lemma 2.4 (pointwise estimates for $[b, M_\alpha]$) Let $0 \leq \alpha < Q$, $f : G \rightarrow \mathbb{R}$ be a locally integrable function.

(1) If $b$ is any non-negative locally integrable function on $G$, then
\[
\|[b, M_\alpha](f)(x)\| \leq M_{\alpha,b}(f)(x).
\]

(2) If $b$ is any locally integrable function on $G$, then
\[
\|[b, M_\alpha](f)(x)\| \leq M_{\alpha,b}(f)(x) + 2b^-(x)M_\alpha(f)(x).
\]

(3) Assume that $0 < \beta < 1$ and $0 < \alpha + \beta < Q$. If $b \in \dot{\Lambda}_\beta(G)$ and $b \geq 0$, then for arbitrary $x \in G$ such that $M_\alpha(f)(x) < \infty$, we have
\[
\|[b, M_\alpha](f)(x)\| \leq \|b\|_{\dot{\Lambda}_\beta(G)}M_{\alpha+\beta}(f)(x).
\]
Proof (1) For any fixed $x \in \mathbb{G}$ such that $M_{\alpha}(f)(x) < \infty$, since $b \geq 0$ then

$$
[b, M_{\alpha}](f)(x) = |b(x)M_{\alpha}(f)(x) - M_{\alpha}(bf)(x)|
$$

$$
= \left| \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_{B} b(x)|f(y)|dy \right|
\leq \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_{B} |b(x) - b(y)||f(y)|dy
= M_{\alpha,b}(f)(x).
$$

(2) Similar to the discussion in [30]. For any fixed $x \in \mathbb{G}$ such that $M_{\alpha}(f)(x) < \infty$, and any $b \in L^{1}_{\text{loc}}(\mathbb{G})$, we have

$$
[b, M_{\alpha}](f)(x) = |b(x)M_{\alpha}(f)(x) - M_{\alpha}(bf)(x)|
$$

$$
= \left| \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_{B} b(x)f(y)dy \right|
\leq \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_{B} |b(x) - b(y)||f(y)|dy
+ 2 \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_{B} b^{-}(x)f(y)dy
= M_{\alpha,b}(f)(x) + 2b^{-}(x)M_{\alpha}(f)(x).
$$

(3) Similar to the discussion of lemma 2.11 in [17]. For any fixed $x \in \mathbb{G}$ such that $M_{\alpha}(f)(x) < \infty$, if $b \in \Lambda_{\delta}(\mathbb{G})$ and $b \geq 0$, then we have

$$
[b, M_{\alpha}](f)(x) = |b(x)M_{\alpha}(f)(x) - M_{\alpha}(bf)(x)|
$$

$$
= \left| \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_{B} b(x)f(y)dy \right|
\leq \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_{B} |b(x) - b(y)||f(y)|dy
\leq ||b||_{\Lambda_{\delta}(\mathbb{G})} \sup_{B \ni x} \frac{1}{|B|^{1-(\alpha + \beta)/Q}} \int_{B} |f(y)|dy
\leq ||b||_{\Lambda_{\delta}(\mathbb{G})} M_{\alpha+\beta}(f)(x).
$$

In the case $\alpha = 0$, similar to Lemma 2.4, we can also get the following pointwise estimates for $[b, M]$ and ignore the proof.

Lemma 2.5 (pointwise estimates for $[b, M]$) Let $f : \mathbb{G} \to \mathbb{R}$ be a locally integrable function.
(1) If \( b \) is any non-negative locally integrable function on \( \mathbb{G} \), then
\[
\| [b, M](f)(x) \| \leq M_b(f)(x).
\]

(2) If \( b \) is any locally integrable function on \( \mathbb{G} \), then
\[
\| [b, M](f)(x) \| \leq M_b(f)(x) + 2b^-(x)M(f)(x).
\]

(3) Assume that \( 0 < \beta < 1 \). If \( b \in \mathring{\Lambda}_\beta(\mathbb{G}) \) and \( b \geq 0 \), then for arbitrary \( x \in \mathbb{G} \) such that \( M(f)(x) < \infty \), we have
\[
\| [b, M](f)(x) \| \leq \| b \|_{\mathring{\Lambda}_\beta(\mathbb{G})} M_\beta(f)(x).
\]

To prove our results, we recall the definition of the maximal operator with respect to a ball. For a fixed ball \( B_0 \), the maximal function with respect to \( B_0 \) of a function \( f \) is given by
\[
M_{B_0}(f)(x) = \sup_{B \supseteq B_0} |B|^{-1} \int_B |f(y)|dy,
\]
and
\[
M_{\alpha,B_0}(f)(x) = \sup_{B \supseteq B_0} \frac{1}{|B|^{1-\alpha/Q}} \int_B |f(y)|dy,
\]
where the supremum is taken over all the balls \( B \) with \( B \subset B_0 \) and \( x \in B \).

In the following results, (1) can be found in [16] (see lemma 3.2). And similar to the discussion in [13], by elementary calculations and derivations we can obtain the desired relations (2) on \( \mathbb{G} \).

**Lemma 2.6 (pointwise estimates)** Let \( 0 \leq \alpha < Q \), and \( f : \mathbb{G} \to \mathbb{R} \) be a locally integrable function.

1. If \( B_0 \) is a ball on \( \mathbb{G} \) with radius \( r_0 \), then \( |B_0|^{\alpha/Q} \leq M_\alpha(\chi_{B_0})(x) = M_{\alpha,B_0}(\chi_{B_0})(x) \) for every \( x \in B_0 \).
2. \( M_\alpha(f \chi_{B})(x) = M_{\alpha,B}(f)(x) \) and \( M_\alpha(\chi_{B})(x) = M_{\alpha,B}(\chi_{B})(x) = |B|^{\alpha/Q} \) for every \( x \in B \subset \mathbb{G} \).

The following propositions can be found in [31].

**Proposition 2.2** Let \( 0 \leq \alpha < Q \) and \( 1 < p < \gamma^{-1} = \frac{Q}{\alpha} \) with \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q} \). Then the following two conditions are equivalent:

1. There is a constant \( C > 0 \) such that for any \( f \in L^p(\mathbb{G}) \) the inequality
\[
\left( \int_{\mathbb{G}} \left( M_\gamma(f \omega)(x) \right)^q \omega(x)dx \right)^{1/q} \leq C \left( \int_{\mathbb{G}} |f(x)|^p \omega(x)dx \right)^{1/p}
\]
holds.
2. \( \omega \in A_{1+q/p'}(\mathbb{G}) \), \( p' = \frac{p}{p-1} \).
Proposition 2.3 Let $0 < \alpha < Q$, $\gamma = \alpha/Q$, $q = (1 - \gamma)^{-1}$, and $f \in L^q(G)$. Then the following two conditions are equivalent:

(1) $\omega \{x \in G : M_\gamma(f^\gamma)(x) > \lambda\} \leq C \lambda^{-q} \left( \int_G |f(x)|^q dx \right)^q$ with a constant $C > 0$ independent of $f$ and $\lambda > 0$.

(2) $\omega \in A_1(G)$.

The following strong and weak-type boundedness of $M_\alpha$ can be obtained from Propositions 2.2 and 2.3 when the weight $\omega = 1$, see [31] for more details. And the first part can also be obtained from ?? when the weights $\omega = 1$ and $\nu = 1$.

Lemma 2.7 Let $0 < \alpha < Q$, $1 \leq p \leq Q/\alpha$ with $1/q = 1/p - \alpha/Q$, and $f \in L^p(G)$.

(1) If $1 < p \leq Q/\alpha$, then there exists a positive constant $C$ such that

$$\|M_\alpha(f)\|_{L^q(G)} \leq C \|f\|_{L^p(G)}$$

(2) If $p = 1$, then there exists a positive constant $C$ such that

$$|\{x \in G : M_\alpha(f)(x) > \lambda\}| \leq C (\lambda^{-1} \|f\|_{L^1(G)})^{Q/(Q-\alpha)}$$

holds for all $\lambda > 0$.

Lemma 2.8 Let $0 < \beta < 1$, $0 < \alpha < Q$, $0 < \alpha + \beta < Q$ and $b \in \dot{A}_{\beta}(G)$. If (1.3) holds for some $s \in [1, \infty)$, then

$$\frac{1}{|B|^{1 + \beta/Q}} \int_B |b(x) - M_B(b)(x)| dx < \infty.$$

Proof We first consider the following inequality.

$$\frac{1}{|B|^{1 + \beta/Q}} \int_B |b(x) - M_B(b)(x)| dx \leq \frac{1}{|B|^{1 + \beta/Q}} \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)| dx$$

$$+ \frac{1}{|B|^{1 + \beta/Q}} \int_B ||B|^{-\alpha/Q} M_{\alpha,B}(b)(x) - M_B(b)(x)|| dx = I_1 + I_2.$$

For $I_1$, applying hypothesis (1.3) and Hölder’s inequality (see Lemma 2.1) we have

$$I_1 \leq \frac{1}{|B|^{1 + \beta/Q}} \left( \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)|^s dx \right)^{1/s} \left( \int_B \chi_B(x) dx \right)^{1/s'}$$

$$= \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)|^s dx \right)^{1/s} \leq C,$$

where the constant $C$ is independent of $B$. 

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Next, we consider $I_2$. From the definition of $M_{\alpha,B}$ and Lemma 2.6, it is not difficult to check that the pointwise estimates $M_{\alpha}(b\chi_B)(x) = M_{\alpha,B}(b)(x)$ and $M_{\alpha}(\chi_B)(x) = M_{\alpha,B}(\chi_B)(x) = |B|^{\alpha/Q}$ for any fixed ball $B \subset \mathbb{G}$ and all $x \in B$. Furthermore, when $\alpha = 0$, for all $x \in B$, we have (also see [12])

$$M(\chi_B)(x) = \chi_B(x) = 1 \quad \text{and} \quad M(b\chi_B)(x) = M(b)(x).$$

Then, for any $x \in B$,

$$\begin{aligned}
|B|^{-\alpha/Q} M_{\alpha,B}(b)(x) - M_B(b)(x) \\
\leq |B|^{-\alpha/Q} |M_{\alpha,B}(b)(x) - |B|^\alpha/Q |b(x)|| + |b(x)| - M_B(b)(x) \\
\leq |B|^{-\alpha/Q} |M_{\alpha}(b\chi_B)(x) - |b(x)||M_{\alpha}(\chi_B)(x)| \\
+ |b(x)||M(\chi_B)(x) - M(b\chi_B)(x)|| \\
\leq |B|^{-\alpha/Q} ||b|| M_{\alpha}(\chi_B)(x) + ||b|| M(\chi_B)(x)||.
\end{aligned}$$

(2.4)

Noting that $0 \leq |b| \in \hat{\Lambda}_\beta(\mathbb{G})$ since $b \in \hat{\Lambda}_\beta(\mathbb{G})$. Therefore, we can apply Lemma 2.4 to $||b|| M_{\alpha}$ and Lemma 2.5 to $||b|| M$ since $|b| \in \hat{\Lambda}_\beta(\mathbb{G})$ and $|b| \geq 0$. By Lemmas 2.4 to 2.6, for arbitrary $x \in B$, we have

$$||b|| M_{\alpha}(\chi_B)(x) \leq ||b|| \Lambda_{\alpha}(\mathbb{G}) M_{\alpha+\beta}(\chi_B)(x) \leq C ||b|| \Lambda_{\alpha}(\mathbb{G}) |B|^{|\alpha+\beta|/Q},$$

and

$$||b|| M(\chi_B)(x) \leq ||b|| \Lambda_{\beta}(\mathbb{G}) M_{\beta}(\chi_B)(x) \leq C ||b|| \Lambda_{\beta}(\mathbb{G}) |B|^{|\beta|/Q}.$$

By (2.4), we have

$$I_2 = \frac{1}{|B|^{1+\beta/Q}} \int_B |B|^{-\alpha/Q} M_{\alpha,B}(b)(x) - M_B(b)(x)| \, dx$$

$$\leq \frac{C}{|B|^{1+\alpha+\beta/Q}} \int_B ||b|| M_{\alpha}(\chi_B)(x) \, dx$$

$$+ \frac{C}{|B|^{1+\beta/Q}} \int_B ||b|| M(\chi_B)(x) \, dx$$

$$\leq C ||b|| \Lambda_{\beta}(\mathbb{G}).$$

Combined with the above estimation, putting $I_1$ and $I_2$ into (2.3), we can obtain

$$\frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - M_B(b)(x)| \, dx < C.$$

This completes the proof of Lemma 2.8.

3 Proof of the principal results

We now give the proof of the principal results. First, we prove Theorem 1.1.

Proof of Theorem 1.1 Since the implications $2 \implies 3$ and $5 \implies 4$ follows readily, and $2 \implies 5$ is similar to $3 \implies 4$, we only need to prove $1 \implies 2$, $3 \implies 4$, and $4 \implies 1$ (see Figure 1).
Figure 1: Proof structure

where \( w_{ij} \) denotes \( i \Rightarrow j \)

1 \( \Rightarrow \) 2: Let \( b \in \hat{\Lambda}_\beta(G) \), then, using (1) in Definition 2.3, we have

\[
M_{\alpha,b}(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_B |b(x) - b(y)||f(y)|dy \\
\leq C\|b\|_{\hat{\Lambda}_\beta(G)} \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_B |p(y^{-1}x)|\beta |f(y)|dy \\
\leq C\|b\|_{\hat{\Lambda}_\beta(G)} \sup_{B \ni x} \frac{1}{|B|^{1-(\alpha+\beta)/Q}} \int_B |f(y)|dy \\
\leq C\|b\|_{\hat{\Lambda}_\beta(G)} M_{\alpha+\beta}(f)(x).
\]

Therefore, assertion 2 follows from Lemma 2.7 and (3.1).

3 \( \Rightarrow \) 4: For any fixed ball \( B \subset G \), we have for all \( x \in B \),

\[
|b(x) - b_B| \leq \frac{1}{|B|} \int_B |b(x) - b(y)|dy \\
= \frac{1}{|B|} \int_B |b(x) - b(y)|\chi_B(y)dy \\
\leq \frac{1}{|B|^\alpha/Q} M_{\alpha,b}(\chi_B)(x).
\]

Then, for all \( x \in G \),

\[
|(b(x) - b_B)\chi_B(x)| \leq \frac{1}{|B|^\alpha/Q} M_{\alpha,b}(\chi_B)(x).
\]

Using assertion 3, there exists \( p, q \) satisfy \( 1 < p < \frac{Q}{\alpha+\beta} \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q} \) such that \( M_{\alpha,b} \) is bounded from \( L^p(G) \) to \( L^q(G) \). For any ball \( B \subset G \) containing \( x \), by Lemma 2.2, one obtains

\[
\frac{1}{|B|^\beta/Q} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^qdx \right)^{1/q} \leq \frac{1}{|B|^\alpha+\beta/Q} \left( \frac{1}{|B|} \int_B (M_{\alpha,b}(\chi_B)(x))^q dx \right)^{1/q} \\
\leq \frac{1}{|B|^1/q+\alpha+\beta/Q} \left( \int_B (M_{\alpha,b}(\chi_B)(x))^q dx \right)^{1/q} \\
\leq \frac{C}{|B|^1/q+\alpha+\beta/Q} \|\chi_B\|_{L^p(G)} \\
\leq C.
\]

Thus, this together with Lemma 2.3 gives \( b \in \hat{\Lambda}_\beta(G) \).
4 \implies 1: For any ball $B \subset \mathbb{G}$ containing $x$, it follows from Hölder’s inequality (see Lemma 2.1) and assertion 4 that
\[
\frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - b_B| \, dx \leq \frac{C}{|B|^{1+\beta/Q}} \left( \int_B |b(x) - b_B|^q \, dx \right)^{1/q} \left( \int_B \chi_B(x) \, dx \right)^{1/q} \\
\leq \frac{C}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^q \, dx \right)^{1/q} \\
\leq C.
\]

It follows from Lemma 2.3 that $b \in \dot{A}_\beta(\mathbb{G})$ since $B$ is an arbitrary ball in $\mathbb{G}$.

The proof of Theorem 1.1 is completed.

Now, we prove Theorem 1.2.

**Proof of Theorem 1.2** Since the implications $2 \implies 3$ and $5 \implies 4$ follows readily, and $2 \implies 5$ is similar to $3 \implies 4$, we only need to prove $1 \implies 2$, $3 \implies 4$, and $4 \implies 1$ (see Figure 1).

1 $\implies 2$: For any fixed $x \in \mathbb{G}$ such that $\mathcal{M}_a(f)(x) < \infty$, by Lemma 2.4 and $b \geq 0$, we have
\[
[|b, \mathcal{M}_a|f](x)] \leq \mathcal{M}_{a,b}(f)(x).
\]

It follows from Theorem 1.1 that $[b, \mathcal{M}_a]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ since $b \in \dot{A}_\beta(\mathbb{G})$.

3 $\implies 4$: From the definition of $\mathcal{M}_{a,B}$ and Lemma 2.6, we can obtain that the pointwise estimates $\mathcal{M}_{a}(b \chi_B)(x) = \mathcal{M}_{a,B}(b)(x)$ and $\mathcal{M}_{a}(\chi_B)(x) = \mathcal{M}_{a,B}(\chi_B)(x) = |B|^{\alpha/Q}$ for any fixed ball $B \subset \mathbb{G}$ and all $x \in B$.

Then for any $x \in B$, we have
\[
b(x) - |B|^{-\alpha/Q} \mathcal{M}_{a,B}(b)(x) = |B|^{-\alpha/Q} \left( b(x)|B|^{\alpha/Q} - \mathcal{M}_{a,B}(b)(x) \right) \\
= |B|^{-\alpha/Q} \left( b(x)\mathcal{M}_{a}(\chi_B)(x) - \mathcal{M}_{a,B}(\chi_B)(x) \right) \\
= |B|^{-\alpha/Q} |b, \mathcal{M}_{a}|(\chi_B)(x).
\]

Thus
\[
\left( b(x) - |B|^{-\alpha/Q} \mathcal{M}_{a,B}(b)(x) \right) \chi_B(x) = |B|^{-\alpha/Q} |b, \mathcal{M}_{a}|(\chi_B)(x) \chi_B(x).
\]

Using assertion 3, there exists $p, q$ satisfy $1 < p < \frac{Q}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}$ such that $[b, \mathcal{M}_a]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$. For any ball $B \subset \mathbb{G}$ containing $x$, by Lemma 2.2, we get
\[
\frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - |B|^{-\alpha/Q} \mathcal{M}_{a,B}(b)(x)|^q \, dx \right)^{1/q} \\
= |B|^{-\beta/Q} \left( \int_B |b(x) - |B|^{-\alpha/Q} \mathcal{M}_{a,B}(b)(x)|^q \, dx \right)^{1/q} \\
\leq |B|^{-(\alpha+\beta)/Q-1/q} \left( |b, \mathcal{M}_{a}|(\chi_B) \right)_{L^q(\mathbb{G})} \\
\leq C |B|^{-(\alpha+\beta)/Q-1/q} \|\chi_B\|_{L^p(\mathbb{G})} \\
\leq C,
\]

which gives (1.3) for $s = q$ since the ball $B \subset \mathbb{G}$ is arbitrary and $C$ is independent of $B$.  

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4 \implies 1: To prove \( b \in \hat{A}_\beta(G) \), by Lemma 2.3, it suffices to verify that there is a constant \( C > 0 \) such that for all balls \( B \subset G \),
\[
|B|^{-1-\beta/Q} \int_B |b(x) - b_B| \, dx \leq C.
\] (3.2)

For any fixed ball \( B \subset G \), let \( E = \{ x \in B : b(x) \leq b_B \} \) and \( F = \{ x \in B : b(x) > b_B \} \). The following equality is trivially true (see [12], page 3331):
\[
\int_E |b(x) - b_B| \, dx = \int_F |b(x) - b_B| \, dx.
\]

Noticing the obvious estimate \( b_B \leq |B|^{-\alpha/Q} M_{\alpha,B}(b)(x) \) for any \( x \in B \) and \( b(x) \leq b_B \) for any \( x \in E \), thus we have \( b(x) \leq b_B \leq |B|^{-\alpha/Q} M_{\alpha,B}(b)(x) \) for any \( x \in E \) (see [13]), then for any \( x \in E \), we have
\[
|b(x) - b_B| \leq |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)|.
\]

Therefore,
\[
\frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - b_B| \, dx = \frac{1}{|B|^{1+\beta/Q}} \int_{E \cup F} |b(x) - b_B| \, dx
\]
\[
= \frac{2}{|B|^{1+\beta/Q}} \int_E |b(x) - b_B| \, dx
\]
\[
\leq \frac{2}{|B|^{1+\beta/Q}} \int_E |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)| \, dx
\]
\[
\leq \frac{2}{|B|^{1+\beta/Q}} \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)| \, dx.
\] (3.3)

On the other hand, it follows from Hölder’s inequality (see Lemma 2.1) and (1.3) that
\[
\frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)| \, dx
\]
\[
\leq \frac{C}{|B|^{1+\beta/Q}} \left( \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)|^q \, dx \right)^{1/q} \left| B \right|^{1/q'}
\]
\[
\leq \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)|^q \, dx \right)^{1/q}
\]
\[
\leq C,
\]
which gives (3.2) for \( q = s \) together with (3.3). Thus we achieve \( b \in \hat{A}_\beta(G) \) from Lemma 2.3.

In order to prove \( b \geq 0 \), it suffices to show \( b^- = 0 \). For any fixed ball \( B \subset G \), observe that
\[
0 \leq b^+(x) = |b(x)| - b^-(x) \leq |b(x)| \leq M_B(b)(x)
\]
for \( x \in B \) and thus we have that, for \( x \in B \),
\[
0 \leq b^-(x) \leq M_B(b)(x) - b^+(x) \leq M_B(b)(x) - b^+(x) + b^-(x) = M_B(b)(x) - b(x).
\]

Using Lemma 2.8, then, for any ball \( B \subset G \), we obtain
\[
\frac{1}{|B|} \int_B b^-(x) \, dx \leq \frac{1}{|B|} \int_B |M_B(b)(x) - b(x)| \, dx
\]
\[ \frac{|B|^{\beta/Q}}{|B|^{1+\beta/Q}} \int_B |b(x) - M_B(b)(x)| \, dx \leq C |B|^{\beta/Q}. \]

Thus, \( b^- = 0 \) follows from Lebesgue’s differentiation theorem.

The proof of Theorem 1.2 is completed.

**Acknowledgments**

This work is supported partly by the National Natural Science Foundation of China (Grant No.11571160), Scientific Project-HLJ (No.2019- KYYWF-0909) and Youth Project-HLJ (No.2020YQ07).

**Data Availability Statement**

My manuscript has no associate data.

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