A CLOSED FORMULA FOR SUBEXPONENTIAL CONSTANTS IN THE MULTILINEAR BOHNEBBLUST–HILLE INEQUALITY

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Abstract. For the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the multilinear Bohnenblust–Hille inequality asserts that there exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty}$ such that
\[
\left( \sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1}, \ldots, e_{i_m}) \right|^{2m/m+1} \right)^{m+1} \leq C_{\mathbb{K},m} \sup_{z_1,\ldots,z_m \in \mathbb{D}^N} |U(z_1, \ldots, z_m)|
\]
for all $m$-linear form $U : \mathbb{K}^N \times \cdots \times \mathbb{K}^N \to \mathbb{K}$ and every positive integer $N$, where $(e_i)_{i=1}^{N}$ denotes the canonical basis of $\mathbb{K}^N$ and $\mathbb{D}^N$ represents the open unit polydisk in $\mathbb{K}^N$. Since its proof in 1931, the estimates for $C_{\mathbb{K},m}$ have been improved in various papers. In 2012 it was shown that there exist constants $(C_{\mathbb{K},m})_{m=1}^{\infty}$ with subexponential growth satisfying the Bohnenblust-Hille inequality. However, these constants were obtained via a complicated recursive formula. In this paper, among other results, we obtain a closed (non-recursive) formula for these constants with subexponential growth.

1. Introduction

The complex multilinear Bohnenblust–Hille inequality asserts that for every positive integer $m \geq 1$ there exists a sequence of positive scalars $C_{\mathbb{K},m} \geq 1$ such that
\[
\left( \sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1}, \ldots, e_{i_m}) \right|^{2m/m+1} \right)^{m+1} \leq C_{\mathbb{K},m} \sup_{z_1,\ldots,z_m \in \mathbb{D}^N} |U(z_1, \ldots, z_m)|
\]
for all $m$-linear form $U : \mathbb{K}^N \times \cdots \times \mathbb{K}^N \to \mathbb{K}$ and every positive integer $N$, where $(e_i)_{i=1}^{N}$ is the canonical basis of $\mathbb{K}^N$ and $\mathbb{D}^N$ is the open unit polydisk in $\mathbb{K}^N$. This inequality was overlooked for some decades but it was rediscovered some years ago and, since then, several works and applications have appeared (see \[\text{[2,4,5,6,7,10,11]}\]). It is well-known (since the original proof of H.F. Bohnenblust and E. Hille) that the power $2m/m+1$ is sharp; on the other hand the optimal values of the constants $C_{\mathbb{K},m}$ are not known. In the case of real scalars the Bohnenblust–Hille inequality is also valid, but with different constants. In fact it is known that in the real case
\[
C_{\mathbb{R},2} = \sqrt{2}
\]
is optimal (see \[\text{[7]}\]) and, in the complex case,
\[
C_{\mathbb{C},2} \leq \frac{2}{\sqrt{\pi}}.
\]
The estimates for these constants are becoming more accurate along the time. For the complex case we have:

- \( C_{C,m} \leq m_{m+1}^{2m-1} \) (1931 - Bohnenblust and Hille [1]),
- \( C_{C,m} \leq 2^{m-1} \) (70’s - Kaijser [9] and Davie [2]),
- \( C_{C,m} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} \) (1995 - Queffélec [12]).

Although the optimal constants \( C_{K,m} \) are not known, some recent papers have investigated their asymptotical growth (see [6, 10]). Very recently, quite better estimates, with a surprising subexponential growth, were obtained in [6, 11] but the recursive way that these constants were obtained make the presentation of a closed formula a quite difficult task. One of the main goals of this paper is to present a closed formula for the constants with subexponential growth obtained in [6, 11].

2. First remarks

We begin by recalling the Khinchin inequality:

For any \( p > 0 \), there are constants \( A_p, B_p > 0 \) such that

\[
A_p \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{p}} \leq \left( \int_0^1 \left| \sum_{n=1}^{\infty} a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{p}},
\]

regardless of the \( (a_n)_{n=1}^{\infty} \in l_2 \). Above, \( r_n \) represents the \( n \)-th Rademacher function.

From [8] we know that the best values of \( A_p \) are

\[
A_p = \begin{cases} 
\sqrt{2} \left( \frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p}, & \text{if } p > p_0 \\
2^{p-1}, & \text{if } p < p_0.
\end{cases}
\]

where \( \Gamma \) denotes the Gamma Function and \( 1 < p_0 < 2 \) is so that

\[
\Gamma \left( \frac{p_0 + 1}{2} \right) = \frac{\sqrt{\pi}}{2}.
\]

Numerical calculations estimate

\( p_0 \approx 1.847 \).

The following result appears in [10]:

**Theorem 1.** For all positive integers \( n \),

\[
C_{R,2} = 2^\frac{1}{2},
\]

\[
C_{R,3} = 2^\frac{1}{3}
\]

and

\[
C_{R,n} = 2^{\frac{1}{2}} \left( \frac{C_{R,n-2}^{\frac{a}{a-2}}}{A_{a-1}^{\frac{a}{a-1}}} \right) \text{ for } n > 3.
\]

In particular, if \( 2 \leq n \leq 14 \)

\[
C_{R,n} = \begin{cases} 
2^{\frac{2+a-2}{a}}, & \text{if } n \text{ is even} \\
2^{\frac{2+a-2}{a}}, & \text{if } n \text{ is odd}
\end{cases}
\]
The above theorem allows to obtain a closed formula for the constants. It is shown in [10] that for an even positive integer $n > 14$,

$$C_{R,n} = 2^{\frac{n+2}{4}} r_n,$$

for a certain $r_n$ for which numerical computations show that it tends to a number close to 1.44. The formula for $r_n$ from [10] contains a slight imprecision which affects some decimals of the first constants. Below we show a correct formula for $r_n$.

**Proposition 1.** If $n > 14$ is even, then

$$C_{R,n} = 2^{\frac{n+2}{4}} r_n,$$

with

$$r_n = \frac{\pi^{\frac{(n+14)(n-14)}{8n}} \prod_{k=7}^{n-2} \left( \Gamma \left( \frac{6k+1}{4k+2} \right) \right)^{2k+1}}{2^{\frac{(n+12)(n-14)-24}{4n}}.}$$

**Proof.** Using the estimates from Theorem 1 we have

$$C_{R,4} = 2^\frac{3}{4} \left( \frac{C_{R,2}}{A^2_{\frac{3}{4}}} \right)^{\frac{3}{4}},$$

$$C_{R,6} = 2^\frac{3}{4} \left( \frac{2^\frac{3}{4} \left( \frac{C_{R,4}}{A^2_{\frac{3}{4}}} \right)^{\frac{3}{4}}}{A^2_{\frac{3}{4}}} \right)^{\frac{3}{4}} = \left( 2^\frac{3}{4} + \frac{3}{4} \right) \left( \frac{C_{R,2}}{A^2_{\frac{3}{4}}} \right)^{\frac{3}{4}} \left( \frac{A^2_{\frac{3}{4}}}{A^2_{\frac{3}{4}}} \right)^{\frac{3}{4}},$$

$$C_{R,8} = \frac{2^\frac{3}{4} \left( \frac{1}{2} + \frac{1}{4} \right)^{\frac{3}{4}} \left( \frac{C_{R,2}}{A^2_{\frac{3}{4}}} \right)^{\frac{3}{4}}}{\left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}} \left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}} \left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}}},$$

and so on. Hence

$$C_{R,n} = \frac{d_n}{s_n}$$

with

$$s_n = \left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}} \left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}} \left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}} \left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}} \left( A^2_{\frac{3}{4}} \right)^{\frac{3}{4}}$$

and

$$d_n = 2^\frac{3}{4} \left( \frac{1}{2} + \frac{1}{4} \right)^{\frac{3}{4}} \left( \frac{1}{2} + \frac{1}{4} \right)^{\frac{3}{4}} \left( \frac{1}{2} + \frac{1}{4} \right)^{\frac{3}{4}} \left( \frac{1}{2} + \frac{1}{4} \right)^{\frac{3}{4}} \left( \frac{1}{2} + \frac{1}{4} \right)^{\frac{3}{4}}.$$

For $p = \frac{2n-4}{n-1}$ and $2 \leq n \leq 14$, we have $p < 1.847$. So

$$A_p = 2^\frac{3}{4} - \frac{1}{4},$$

and, for $n > 14$, we have $p > p_0$ and

$$A_p = 2^\frac{3}{4} \left( \frac{\Gamma ((p+1)/2)}{\sqrt{\pi}} \right)^{\frac{3}{4}}.$$
We thus have
\[ s_n = \left( A_{\frac{2^4}{n}} \right)^{\frac{3}{2}} \cdots \left( A_{\frac{2^4}{n}} \right)^{\frac{n-2}{n}} \cdots \left( A_{\frac{2^4}{n}} \right)^{\frac{2}{n}} \times \left( 2^{-\frac{1}{2}} \right)^{\frac{3}{2}} \left( 2^{-\frac{1}{2}} \right)^{\frac{5}{2}} \cdots \left( 2^{-\frac{1}{2}} \right)^{\frac{n-2}{n}} \times \left( \Gamma \left( \frac{6k+1}{4k+2} \right) \right)^{\frac{2k+1}{2}} \left( \Gamma \left( \frac{2n+1}{n-1} \right) \right) \frac{1}{r_n} \]
\[ = 2^{\left( \frac{n+12}{4n} \right) - 24} \left( \prod_{k=7}^{\frac{n-2}{2}} \Gamma \left( \frac{6k+1}{4k+2} \right) \right) \frac{2k+1}{2} \left( \Gamma \left( \frac{2n+1}{n-1} \right) \right) \frac{1}{r_n} \]
\[ = 2^{\left( \frac{n+12}{4n} \right) - 24} \left( \prod_{k=7}^{\frac{n-2}{2}} \Gamma \left( \frac{6k+1}{4k+2} \right) \right) \frac{2k+1}{2} \left( \pi \left( \frac{2n+1}{n-1} \right) \right)^{-1} \frac{1}{r_n} \]

On the other hand a simple calculation shows that
\[ d_n = 2^{\frac{n+2}{2}} \]
and from (2.4) we obtain
\[ C_{n,n} = 2^{\frac{n+2}{2}} r_n. \]

Below we compare the values of the \( r_n \) from (2.3) and the \( r_n \) from [10]:

| \( n \) | \( r_n (2.3) \) | \( r_n ([10]) \) |
|-------|-----------------|-----------------|
| 30    | 1.387           | 1.375           |
| 50    | 1.404           | 1.397           |
| 100   | 1.420           | 1.416           |
| 250   | 1.431           | 1.429           |
| 500   | 1.435           | 1.434           |
| 1,000 | 1.4374          | 1.4371          |
| 10,000| 1.43989         | 1.43986         |
| 100,000| 1.44021       | 1.44021         |

Hence, although the formulas for \( r_n \) are different its values are very close and, as in [10], numerical estimates indicate that
\[ \lim_{n \to \infty} r_n \approx 1.44025. \]
We conjecture that
\[
\lim_{n \to \infty} r_n = \frac{e^{1 - \frac{1}{2} \gamma}}{\sqrt{2}},
\]
where \(\gamma\) denotes the Euler constant.

3. Main results

In [6] it was shown that there is a constant \(D\) (probably very close to 1.44) so that the sequence \((C_n)_{n=1}^{\infty}\) given by
\[
\begin{align*}
C_{2n} &= DC_n \\
C_{2n+1} &= D \left( C_n \right)^{\frac{2n}{n+1}} \left( C_{n+1} \right)^{\frac{2n}{n+1}},
\end{align*}
\]
with
\[
C_1 = 1 \quad \text{and} \quad C_2 = \sqrt{2}
\]
in the real case and
\[
C_1 = 1 \quad \text{and} \quad C_2 = \frac{2}{\sqrt{\pi}}
\]
in the complex case, satisfies the Bohnenblust–Hille inequality and, moreover, this sequence is subexponential. From now on \(C_n\) will denote the numbers given by the above formulas.

In this section we present a closed formula for these constants. Given a positive integer \(n\), it is plain that it can be written (in an unique way) as
\[
n = 2^k - l,
\]
where \(k\) is the smaller positive integer such that \(2^k \geq n\) and \(0 \leq l < 2^k - 1\).

**Theorem 2.** If \(n \geq 3\) is written as (3.1), then
\[
C_n = D^{k-1}C_2^{\frac{n-l}{n}}, \text{ if } l \leq 2^{k-2}
\]
and
\[
C_n = D^{n(k-1)+2^{k-2}-2l}C_2^{\frac{2^{k-1}-2l}{n}}, \text{ if } 2^{k-2} < l < 2^{k-1}
\]
where
\[
C_2 = \sqrt{2}, \quad \text{for real scalars}
\]
\[
C_2 = \frac{2}{\sqrt{\pi}}, \quad \text{for complex scalars}
\]

**Proof.** Since \(n \geq 3\), note that \(k \geq 2\).

We proceed by induction. Suppose the result valid for all \(m \leq n\).

Let
\[
n + 1 = 2^k - l
\]
with \(l\) and \(k\) so that \(k\) is the smaller positive integer such that \(2^k \geq n + 1\) and \(0 \leq l < 2^{k-1}\).

- **First Case:** \(l\) is even.
In this case $n + 1$ is even and
\begin{equation}
C_{n+1} = D \left( C_{\frac{n+1}{2}} \right)
\end{equation}
with
\[ \frac{n + 1}{2} = 2^{k-1} - \frac{l}{2}. \]
By induction hypothesis, the result is valid for $C_{\frac{n+1}{2}}$. We have two possible subcases for $\frac{l}{2}$:

**Subcase 1a** -
\begin{equation}
\frac{l}{2} \leq 2^{(k-1)-2} = 2^{k-3}.
\end{equation}

**Subcase 1b** -
\[ 2^{(k-1)-2} < \frac{l}{2} < 2^{(k-1)-1}, \]
i.e.,
\begin{equation}
2^{k-3} < \frac{l}{2} < 2^{k-2}.
\end{equation}

If (3.5) occurs, note that $l \leq 2^{k-2}$, from (3.4) we have
\begin{align*}
C_{n+1} &= D \left( D^{k-2}C_{\frac{n+1}{2}} \right) \\
&= D^{k-1}C_{\frac{n+1}{2}}^{(n+1)-l},
\end{align*}
and this is what we need.

If (3.6) occurs, note that $2^{k-2} < l < 2^{k-1}$. From (3.4) we have
\begin{align*}
C_{n+1} &= D \left( C_{\frac{n+1}{2}} \right) \\
&= D \left( D^{\frac{(n+1)(k-1)+2^{k-1-2l}}{n+1}}C_{\frac{n+1}{2}}^{k-1} \right) \\
&= D^{\frac{(n+1)(k-1)+2^{k-1-2l}}{n+1}}C_{\frac{n+1}{2}}^{2k-1},
\end{align*}
and again we get the desired result.

**Second Case**: $l$ is odd.

In this case $n + 1$ is odd and
\begin{equation}
C_{n+1} = D \left( C_{\frac{n+1}{2}} \right) \left( C_{\frac{n+1}{2}+1} \right) \left( C_{\frac{n+1}{2}+2} \right) \\
= D \left( C_{\frac{n}{2}} \right)^{\frac{n+2}{n+1}} \left( C_{\frac{n+2}{n+1}} \right).
\end{equation}

Since $n + 1 = 2^k - l$, we have
\[ n = 2^k - (l + 1), \]
and
\[ n + 2 = 2^k - (l - 1). \]
Since $0 \leq l < 2^{k-1}$, and $l$ is odd, then
\[ 0 \leq l + 1 < 2^{k-1}, \text{ or } l + 1 = 2^{k-1}. \]
and we have two subcases:

**Subcase 2a -**

\[(3.8)\]

\[n = 2^{k-1} - 0, \quad n + 2 = 2^k - (l - 1),\]

with

\[l = 2^{k-1} - 1\]

**Subcase 2b -**

\[(3.9)\]

\[n = 2^k - (l + 1), \quad n + 2 = 2^k - (l - 1),\]

with

\[l < 2^{k-1} - 1.\]

If (3.8) holds, then \[l = 2^{k-1} - 1.\] Since

\[\frac{n}{2} = 2^{k-2},\]

then \(C_{\frac{n}{2}}\) is of the form (3.2) and, since

\[\frac{n + 2}{2} = 2^{k-1} - \frac{(l - 1)}{2} \quad \text{and} \quad 2^{k-3} < \frac{l - 1}{2} < 2^{k-2},\]

then \(C_{\frac{n+2}{2}}\) is of the form (3.3). We have

\[C_{\frac{n}{2}} = D^{(k-2)-1}C_2\]

and

\[C_{\frac{n+2}{2}} = D^{\frac{n+2}{2} \cdot (k-2) + 2(k-1) - 2(l+1)} C_2^{\frac{n+2}{2} \cdot (k-2) + 2(k-1) - 2(l+1) - 2(l-1)} .\]

From (3.7), we get

\[C_{\frac{n}{2}} = D \left(C_{\frac{n}{2}}\right)^{\frac{n}{2}} \left(C_{\frac{n+2}{2}}\right)^{\frac{n+2}{2}} .\]

\[= D \left(D^{k-3}C_2\right)^{\frac{n}{2}} \left(D^{\frac{n+2}{2} \cdot (k-2) + 2(k-1) - 2(l+1)} C_2^{\frac{n+2}{2} \cdot (k-2) + 2(k-1) - 2(l+1) - 2(l-1)} .\]

and we have the desired result.

In the case that (3.9) holds, we have

\[l + 1 < 2^{k-1},\]

\[\frac{n}{2} = 2^{k-1} - \frac{(l + 1)}{2}\]

and

\[\frac{n + 2}{2} = 2^{k-1} - \frac{(l - 1)}{2}.\]

We have three sub-subcases:

**Sub-subcase 2ba -**

\[(3.10)\]

\[2^{k-2} < l + 1 < 2^{k-1}, \quad \text{and} \quad 2^{k-2} < l - 1 < 2^{k-1}\]
Sub-subcase 2bb -

(3.11) \[ 2^{k-2} < l + 1 < 2^{k-1}, \text{ and } l - 1 = 2^{k-2} \]

Sub-subcase 2bc -

(3.12) \[ l - 1 < l + 1 \leq 2^{k-2}. \]

If (3.10) holds, note that

\[ 2^{k-2} < l - 1 < l < l + 1 < 2^{k-1}, \]

and this \( C_{n+1} \) is of the form (3.3). Therefore

\[ 2^{k-3} \frac{l + 1}{2} < l^{2k-2}, \text{ and } 2^{k-3} \frac{l - 1}{2} < 2^{k-2} \]

and this \( C_{\frac{n}{2}} \) and \( C_{\frac{n+2}{2}} \) are written in the form (3.3); now, from (3.7) we have

\[
C_{n+1} = D \left( C_{\frac{n}{2}} \right) \left( C_{\frac{n+2}{2}} \right)
\]

\[
= D \left( D \left( \frac{\binom{k-2}{2} + 2^{k-2} - 2 \binom{l + 1}{2}}{C_2} \right) \right) \left( D \left( \frac{\binom{n+2}{2} - 2 \binom{l + 1}{2}}{C_2} \right) \right)
\]

If (3.11) holds, note that

\[ 2^{k-2} = l - 1 < l < l + 1 < 2^{k-1}, \]

and we need to obtain a formula like (3.3). Since

\[ 2^{k-3} \frac{l + 1}{2} < 2^{k-2}, \text{ and } 2^{k-3} \frac{l - 1}{2} = 2^{k-3} \]

then \( C_{\frac{n}{2}} \) is represented by (3.3) and \( C_{\frac{n+2}{2}} \) is of the form (3.2). So, from (3.7), we have

\[
C_{n+1} = D \left( D \left( \frac{\binom{k-2}{2} + 2^{k-2} - 2 \binom{l + 1}{2}}{C_2} \right) \right) \left( D \left( \frac{\binom{n+2}{2} - 2 \binom{l + 1}{2}}{C_2} \right) \right)
\]

Since

\[ 2^{k-1} - 2l = 2^{k-1} - 2 \left( 2^{k-2} + 1 \right) = -2, \]

and

\[ \frac{n + 2}{2} = 2^{k-1} - \left( \frac{l - 1}{2} \right) = 2^{k-1} - 2^{k-3}, \]

then

\[
C_{n+1} = D \left( D \left( \frac{\binom{k-2}{2} + 2^{k-2} - 2 \binom{l + 1}{2}}{C_2} \right) \right) \left( D \left( \frac{\binom{n+2}{2} - 2 \binom{l + 1}{2}}{C_2} \right) \right)
\]

Finally, if we have (3.12), note that

\[ l - 1 < l < l + 1 \leq 2^{k-2}, \]
and then $C_{n+1}$ must be of the form $\left(\frac{3}{2}\right)$. Hence

$$\frac{l - 1}{2} < \frac{l + 1}{2} \leq 2^{k-3},$$

and $C_{\frac{n}{2}}$ and $C_{\frac{n+2}{2}}$ are written in the form of $\left(\frac{3}{2}\right)$. Thus, again using $\left(\frac{3}{2}\right)$ we have

$$C_{n+1} = D \left( D^{k-2} C_{\frac{n}{2}}^{\frac{n+1}{n}} \right)^{\frac{n}{n+1}} \left( D^{k-2} C_{\frac{n+2}{2}}^{\frac{n+1}{n}} \right)^{\frac{n}{n+1}}$$

$$= D \cdot D^{(k-2) \frac{n}{n+1}} \cdot D^{(k-2) \frac{n+2}{n+1}} \cdot C_{\frac{n}{2}}^{\frac{n+1}{n+1}} \cdot C_{\frac{n+2}{2}}^{\frac{n+1}{n+1}}$$

$$= D \cdot D^{(k-2) \frac{n}{n+1}} \cdot C_{\frac{n}{2}}^{\frac{n+1}{n+1}} \cdot C_{\frac{n+2}{2}}^{\frac{n+1}{n+1}}$$

$$= D^{k-1} C_{\frac{n+1}{n+1}}^{\frac{n+1}{n+1}}$$

and the proof is done. \qed

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