SOME BOUNDS ON RELATIVE COMMUTATIVITY DEGREE

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Abstract. The relative commutativity degree of a subgroup $H$ of a finite group $G$, denoted by $\Pr(H, G)$, is the probability that an element of $G$ commutes with an element of $H$. In this article we obtain some lower and upper bounds for $\Pr(H, G)$ and their consequences. We also study an invariance property of $\Pr(H, G)$ and its generalizations, under isoclinism of pairs of groups.

1. Introduction

Let $H$ be a subgroup of a finite group $G$. Let $Z(G)$ denote the center of $G$ and define $Z(H, G) := \{ h \in H : hg = gh \forall g \in G \} = Z(G) \cap H$. The commutator $xyx^{-1}y^{-1}$ of any two elements $x, y \in G$ is denoted by $[x, y]$. By $K(G, H)$, we denote the set $\{ [x, y] : x \in G, y \in H \}$. The subgroup generated by $K(G, H)$ is denoted by $[G, H]$. Notice that $[G, G]$ is the commutator subgroup of $G$, which is also denoted by $G'$. For any element $x \in G$, by $C_G(x)$ we denote the conjugacy class of $x$ in $G$ and centralizer of $x$ in $G$ respectively. By $K(G)$ we denote $K(G, G)$.

The relative commutativity degree of $H$, denoted by $\Pr(H, G)$, is the probability that an element of $G$ commutes with an element of $H$. This notion has been introduced and studied in [5]. It is clear that if $H = G$, then $\Pr(H, G)$ coincides with $\Pr(G)$ which is known as the commutativity degree or commuting probability of $G$ (see [6, 7, 14, 15, 17]). Also, $\Pr(H, G) = 1$ if and only if $Z(H, G) = H$.

Importance of studying lower bound for $\Pr(G)$ goes back to 1973, where W. H. Gustafson [7] emphasized on getting this bound for an arbitrary finite group. Such a bound has been studied, under certain conditions on the group, in [5, Theorem 3.5] and [16, Corollary 2.3]. We obtain a better lower bound for $\Pr(G)$ as a corollary (Corollary 3.3) of the following theorem, which we prove in Section 3.

Theorem A. Let $H$ be a subgroup of a finite group $G$. Then

$$\Pr(H, G) \geq \frac{1}{|K(G, H)|} \left( 1 + \frac{|K(G, H)| - 1}{|H : Z(H, G)|} \right).$$

In particular, if $Z(H, G) \neq H$ then $\Pr(H, G) > \frac{1}{|K(G, H)|}$.

Following [4], for $g \in G$, we define $\Pr_g(H, G)$ to be $\frac{|\{(x, y) \in H \times G : xyx^{-1}y^{-1} = g\}|}{|H||G|}$. Notice that if $H = G$, then $\Pr_g(G) := \Pr_g(G, G)$ measures the probability that the commutator of two group elements is equal to a given element, which is studied extensively in [16] by M. R. Pournaki and R. Sobhani. In [16, Proposition 3.1], $\Pr_g(G)$ is computed for groups $G$ with $|G'| = p$, a prime (not necessarily the smallest one), and $G' \leq Z(G)$, using character theory of finite groups. In the following theorem, which we prove in Section 3, we generalize this result and prove it without using character theory.

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Theorem B. Let $H$ be a subgroup of a finite nilpotent group $G$. If $|[G,H]| = p$, a prime (not necessarily the smallest one), and $g \in [G,H]$ then
\[
\Pr_g(H,G) = \left\{ \begin{array}{ll}
\frac{1}{p} \left( 1 + \frac{p-1}{|H/[H,H]|} \right) & \text{if } g = 1 \\
\frac{1}{p} \left( 1 - \frac{1}{|H/[H,H]|} \right) & \text{if } g \neq 1.
\end{array} \right.
\]

In Section 2, following [18], we define isoclinism on the class of all pairs of groups $(G, H)$, where $H$ is any subgroup of $G$. Then we show that $\Pr_g(H,G)$ is invariant under isoclinism of pairs of groups. In Section 4, we obtain lower and upper bounds for $\Pr(H,G)$ in terms of largest and smallest class sizes of elements of $H - Z(H,G)$ respectively. In the last section we observe that for a finite group $G$ with given information on the conjugacy class sizes, the information about $\Pr(G)$ can be used to say something regarding the solvability and supersolvability of $G$.

2. ISOCLINIC PAIRS OF GROUPS

In 1940, P. Hall [8] introduced the following concept of isoclinism on the class of all groups:

Let $X$ be a finite group and $\bar{X} = X/Z(X)$. Then commutation in $X$ gives a well defined map $a_X : \bar{X} \times \bar{X} \to X'$ such that $a_X(xZ(X),yZ(X)) = [x,y]$ for $(x,y) \in X \times X$. Two finite groups $G$ and $H$ are called isoclinic if there exists an isomorphism $\alpha$ of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism $\beta$ of the subgroup $G'$ onto $H'$ such that the following diagram is commutative
\[
\begin{array}{ccc}
\bar{G} \times \bar{G} & \xrightarrow{a_{\bar{G}}} & G' \\
\downarrow{\alpha \times \alpha} & & \downarrow{\beta} \\
\bar{H} \times \bar{H} & \xrightarrow{a_{\bar{H}}} & H'.
\end{array}
\]
The resulting pair $(\alpha, \beta)$ is called an isoclinism of $G$ onto $H$. Notice that isoclinism is an equivalence relation among finite groups.

In 2008, M. R. Pournaki and R. Sobhani [16] proved that if $(\alpha, \beta)$ is an isoclinism of $G$ onto $H$ and $g \in G'$, then $\Pr_g(G) = \Pr_{\beta(g)}(H)$. Recently, Salemkar et. al [18] have introduced the concept of isoclinism on the class of all groups $(G,H)$, where $H$ is a normal subgroup of $G$. Notice that isoclinism on pairs of groups $(G,H)$ can be defined for any subgroup $H$ of $G$ in the following way:

Let $X$ be a finite group and $Y$ be a subgroup of $X$. Let $\bar{Y} = Y/Z(Y,X)$ and $\bar{X} = X/Z(Y,X)$. Then the map $a_{(Y,X)} : \bar{Y} \times \bar{X} \to [Y,X]$ defined by $a_{(Y,X)}(yZ(Y,X),xZ(Y,X)) = [y,x]$, is well defined.

Let $G_1$ and $G_2$ be two groups with subgroups $H_1$ and $H_2$ respectively. A pair of groups $(G_1,H_1)$ is said to be isoclinic to a pair of groups $(G_2,H_2)$ if there exists an isomorphism $\alpha$ from $\bar{G}_1 = G_1/Z(H_1,G_1)$ onto $\bar{G}_2 = G_2/Z(H_2,G_2)$ such that $\alpha(H_1/Z(H_1,G_1)) = H_2/Z(H_2,G_2)$, and an isomorphism $\beta : [H_1,G_1] \to [H_2,G_2]$, such that the following diagram commutes
\[
\begin{array}{ccc}
\bar{H}_1 \times \bar{G}_1 & \xrightarrow{a_{(H_1,G_1)}} & [H_1,G_1] \\
\downarrow{\alpha \times \alpha} & & \downarrow{\beta} \\
\bar{H}_2 \times \bar{G}_2 & \xrightarrow{a_{(H_2,G_2)}} & [H_2,G_2],
\end{array}
\]
where $\bar{H}_1 = H_1/Z(H_1,G_1)$ and $\bar{H}_2 = H_2/Z(H_2,G_2)$. Notice that isoclinism is an equivalence relation among pairs of finite groups.
The following result extends [19, Theorem 3.3] and [16, Lemma 3.5].

**Theorem 2.1.** Let \((G_1, H_1)\) and \((G_2, H_2)\) be two pairs of groups and \((\alpha, \beta)\) be an isoclinism from \((G_1, H_1)\) to \((G_2, H_2)\). If \(g \in \{H_1, G_1\}\) then \(\Pr_g(H_1, G_1) = \Pr_{\beta(g)}(H_2, G_2)\).

**Proof.** Let us set \(Z_1 = Z(H_1, G_1)\) and \(Z_2 = Z(H_2, G_2)\). Since \((\alpha, \beta)\) is an isoclinism from \((G_1, H_1)\) to \((G_2, H_2)\), diagram (11) commutes. Let \(g \in \{H_1, G_1\}\) be the given element. Consider the sets \(S_g = \{(h_1, z_1, g_1, z_1) \in H_1 \times Z_1 \times G_1 \times Z_1 : [h_1, g_1] = g\}\) and \(S_{\beta(g)} = \{(h_2, z_2, g_2, z_2) \in H_2 \times Z_2 \times G_2 \times Z_2 : [h_2, g_2] = \beta(g)\}\). Since diagram (11) commutes, it follows that \(|S_g| = |S_{\beta(g)}|\). Since the map \(a_i(H_i, G_i)\) for \(i = 1, 2\) is well defined, it follows that \(|\{(h_1, g_1) \in H_1 \times G_1 : [h_1, g_1] = g\}| = |Z_1| |S_g|\) and \(|\{(h_2, g_2) \in H_2 \times G_2 : [h_2, g_2] = \beta(g)\}| = |Z_2| |S_{\beta(g)}|\). Also, notice that \(|H_1 : Z_1| = |H_2 : Z_2|\) and \(|G_1 : Z_1| = |G_2 : Z_2|\). Hence

\[
\Pr_g(H_1, G_1) = \frac{|S_g|}{|H_1 : Z_1||G_1 : Z_1|} = \frac{|S_{\beta(g)}|}{|H_2 : Z_2||G_2 : Z_2|} = \Pr_{\beta(g)}(H_2, G_2).
\]

This completes the proof. \(\square\)

The concept of isoclinism can be generalised in the following way:

Let \(H\) be a finite group and \(H_1, H_2, \ldots, H_{m+1}\) be any \(m + 1\) subgroups of \(H\). Let \(Z_m(H_i)\) denote the subgroup \(H_i \cap Z_m(H)\) for \(1 \leq i \leq m + 1\), where \(Z_m(H)\) is the \(m\)th term in the upper central series of \(H\). Set \(\times_{i=1}^{m+1} H_i = H_1 \times H_2 \times \cdots \times H_{m+1}\). It follows from [2, IV, 7.6] that the map \(a_i(H_1, H_2, \ldots, H_{m+1})\) from \(\times_{i=1}^{m+1} Z_m(H_i)\) onto \([H_1, H_2, \ldots, H_{m+1}] = [\cdots [H_1, H_2], \ldots, H_{m+1}]\) defined by

\[
a_{i_1, i_2, \ldots, i_m}(h_1 Z_m(H_1), h_2 Z_m(H_2), \ldots, h_{m+1} Z_m(H_{m+1})) = [h_1, h_2, \ldots, h_{m+1}]
\]

is well defined.

Let \(K\) another group and \(K_1, K_2, \ldots, K_{m+1}\) be any \(m + 1\) subgroups of \(K\). Again set \(Z_m(K_i) = K_i \cap Z_m(K)\). Then \((H_1, H_2, \ldots, H_{m+1})\) is said to be \((m + 1)\)-tuple isoclinic to \((K_1, K_2, \ldots, K_{m+1})\) if there exist an isomorphism \(\alpha\) from \(\frac{H_i}{Z_m(H_i)}\) onto \(\frac{K_i}{Z_m(K_i)}\) such that \(a_i(\frac{H_i}{Z_m(H_i)}) = \frac{K_i}{Z_m(K_i)}\), where \(\alpha_i = \alpha_i\) for \(1 \leq i \leq m + 1\), and an isomorphism \(\beta\) from \([H_1, H_2, \ldots, H_{m+1}]\) onto \([K_1, K_2, \ldots, K_{m+1}]\) such that the following diagram commutes

\[
\begin{array}{ccc}
\times_{i=1}^{m+1} H_i & \xrightarrow{a_{i_1, i_2, \ldots, i_m}} & [H_1, H_2, \ldots, H_{m+1}] \\
\downarrow_{\times_{i=1}^{m+1} \alpha} & & \\
\times_{i=1}^{m+1} K_i & \xrightarrow{a_{i_1, i_2, \ldots, i_m}} & [K_1, K_2, \ldots, K_{m+1}].
\end{array}
\]

The pair \((\alpha, \beta)\) is called an \((m + 1)\)-tuple isoclinism of \((H_1, H_2, \ldots, H_{m+1})\) onto \((K_1, K_2, \ldots, K_{m+1})\).

Let \(g \in \{H_1, H_2, \ldots, H_{m+1}\}\). Then we can define

\[
\Pr_g(H_1, H_2, \ldots, H_{m+1}) = \frac{1}{\prod_{i=1}^{m+1} |H_i|} |\{(h_1, h_2, \ldots, h_{m+1}) \in \times_{i=1}^{m+1} H_i : [h_1, h_2, \ldots, h_{m+1}] = g\}|.
\]

**Remark 2.2.** Let \((\alpha, \beta)\) be an \((m + 1)\)-tuple isoclinism of \((H_1, H_2, \ldots, H_{m+1})\) onto \((K_1, K_2, \ldots, K_{m+1})\). Then using arguments as in the proof of Theorem 2.1 one can prove that

\[
\Pr_g(H_1, H_2, \ldots, H_{m+1}) = \Pr_{\beta(g)}(K_1, K_2, \ldots, K_{m+1}).
\]
3. LOWER BOUNDS FOR $\Pr(H, G)$

Notice that $yxy^{-1} = yx^{-1}x^{-1}y^{-1}x$ for all $y \in G$ and $x \in H$. So, it follows that

$$C_G(x) \subseteq K(G, H)x$$

for all $x \in H$.

The following lemma can be derived from [4, Theorem 2.3]. However, for completeness, we give a slightly modified proof here.

**Lemma 3.1.** Let $H$ be a subgroup of a finite group $G$ and $g \in G$. Then

$$\Pr_g(H, G) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|G : C_G(x)|}$$

**Proof.** Notice that $\{(x, y) \in H \times G : [x, y] = g\} = \bigcup_{x \in H} \{(x) \times T_x\}$, where $T_x = \{y \in G : [x, y] = g\}$. Further notice that, for any $x \in H$, the set $T_x$ is non-empty if and only if $g^{-1}x \in C_G(x)$. Suppose that $T_x$ is non-empty for some $x \in H$. Fix an element $t \in T_x$. It is easy to see that $T_x = tC_G(x)$. Hence

$$\Pr_g(H, G) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|G : C_G(x)|}$$

and the lemma follows. □

The following lemma is an easy exercise.

**Lemma 3.2.** Let $H$ be a subgroup of a finite group $G$. Then

$$\frac{1}{n} \left(1 + \frac{n-1}{|H : Z(H, G)|}\right) \geq \frac{1}{m} \left(1 + \frac{m-1}{|H : Z(H, G)|}\right)$$

for any two positive integers $m, n$ such that $m \geq n$. If $Z(H, G) \neq H$, then the equality holds if and only if $m = n$.

Now we are ready to prove Theorem A.

**Proof of Theorem A.** Let $G$ be a group and $H$ be a subgroup of $G$. Then, by putting $g = 1$ in Lemma 3.1 we get

$$\Pr(H, G) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|C_G(x)|}$$

$$= \frac{1}{|H|} \left(|Z(H, G)| + \sum_{x \in H - Z(H, G)} \frac{1}{|C_G(x)|}\right)$$

$$\geq \frac{1}{|H|} \left(|Z(H, G)| + \frac{|H| - |Z(H, G)|}{|K(G, H)|}\right), \text{ using } (2).$$

$$= \frac{1}{|K(G, H)|} \left(1 + \frac{|K(G, H)| - 1}{|H : Z(H, G)|}\right).$$

This completes the proof. □

We would like to remark here that Erfanian et. al [5, Theorem 3.5] proved

$$\Pr(H, G) \geq \frac{|Z(H, G)|}{|H|} + \frac{p(|H| - |Z(H, G)|)}{|H||G|},$$

where $p$ is a positive integer.
Corollary 3.3. If $K$ and equality holds if and only if $p$ where $K$ is the smallest prime dividing $|G|$. Further, Salemkar et. al [19] Theorem 2.2 (vi) proved

$$
\Pr(H, G) \geq \frac{1}{|G, H|} \left( 1 + \frac{|G, H| - 1}{|H : Z(H, G)|} \right)
$$

for any normal subgroup $H$ of $G$ with $|G, H| \leq Z(H, G)$.

However, if $[G, H] \neq G$ and $H \neq Z(H, G)$, then it can be checked easily that

$$
\frac{1}{|G, H|} \left( 1 + \frac{|G, H| - 1}{|H : Z(H, G)|} \right) \geq \frac{|Z(H, G)|}{|H|} + \frac{p(|H| - |Z(H, G)|)}{|H||G|}
$$

with equality if and only if $|G : [G, H]| = p$.

It follows from Lemma 3.2 that

$$
\frac{1}{|K(G, H)|} \left( 1 + \frac{|K(G, H)| - 1}{|H : Z(H, G)|} \right) \geq \frac{1}{|G, H|} \left( 1 + \frac{|G, H| - 1}{|H : Z(H, G)|} \right)
$$

and equality holds if and only if $K(G, H) = [G, H]$. This shows that the lower bound obtained in Theorem A is better than the known bounds mentioned above.

Examples of groups $G$ such that $|K(G)| < |G'|$ can be found in [13]. If we take $H$ as a maximal subgroup of such a group $G$, then $K(G, H)$ is properly contained in $[G, H] = G'$.

Putting $H = G$ in Theorem A and noticing that $Z(G) = Z(G, G)$, we get the following corollary.

**Corollary 3.3.** If $G$ is a finite group then

$$
\Pr(G) \geq \frac{1}{|K(G)|} \left( 1 + \frac{|K(G)| - 1}{|G : Z(G)|} \right).
$$

In particular, if $G$ is non-abelian, then $\Pr(G) > \frac{1}{|K(G)|}$.

Pournaki et. al [16, Corollary 2.3] obtained

$$
(3) \quad \Pr(G) \geq \frac{1}{|G'|} \left( 1 + \frac{|G'| - 1}{|G : Z(G)|} \right)
$$

if $G$ has only two complex irreducible character degrees. Recently, the first author together with A. K. Das obtained the same lower bound for $\Pr(G)$ without any restriction on $G$ (see [15, Theorem 1]). However, using Lemma 3.2 it is easy to see that the lower bound obtained in Corollary 3.3 is better than the bound in equation (3).

The following corollary gives some equivalent necessary and sufficient conditions for equality to hold in Theorem A.

**Corollary 3.4.** Let $H$ be a subgroup of a finite group $G$. If $Z(H, G) \neq H$, then the following statements are equivalent:

(i) $\Pr(H, G) = \frac{1}{|K(G, H)|} \left( 1 + \frac{|K(G, H)| - 1}{|H : Z(H, G)|} \right)$.

(ii) $\text{Ccl}_G(x) = K(G, H)x$ for all $x \in H - Z(H, G)$.

(iii) $K(G, H) = \{ yxy^{-1}x^{-1} : y \in G \}$ for all $x \in H - Z(H, G)$.

**Proof.** It is easy to see that (ii) and (iii) are equivalent. The equivalence of (i) and (ii) follows from the proof of Theorem A and equation (4). \hfill \Box

A finite group $G$ is said to be a Camina group if $\text{Ccl}_G(x) = G'x$ for all $x \in G - G'$. Let $H = G$ in Corollary 3.4 then $G$ has only two conjugacy class sizes, namely 1 and $|K(G)|$. Now it follows from [12] that $G$ is a direct product of a $p$-group and an abelian group. It then follows from [11] that the nilpotency class of $G$ is at most 3. We claim that in our case the nilpotency class is at most 2. Suppose that the nilpotency class is 3. Then there exists an element $u \in G' - Z(G)$. Then
by (iii) \( G' = \langle [G, u] \rangle \leq [G, G'] \), which is a contradiction. Thus the equivalent conditions in this corollary are satisfied if and only if \( G \) is isoclinic to some Camina \( p \)-group of class 2, for some prime \( p \).

There are also examples in the case when \( H \neq G \).

1. Let \( G \) be any Camina \( p \)-groups of class 2 and \( H \) be a maximal subgroup of \( G \). Then \( G' = [G, H] = K(G, H) \), \( Z(H, G) = Z(G) \) and for all \( x \in H - Z(G) \), \( \text{Cl}_G(x) = K(G, H)x \).

2. Let \( G \) be any Camina \( p \)-groups of class 3 and \( H = G' \). Then \( [G, H] = [H, G] = [G', G] = K(G, H) \), \( Z(H, G) = Z(G) \) and for all \( x \in H - Z(G) \), \( \text{Cl}_G(x) = K(G, H)x \).

3. Our next example is a finite group of order \( p^3 \), which is not a Camina \( p \)-group, \( p \) is an odd prime. Let

\[
G = \langle a_1, a_2, b, c_1, c_2 : [a_1, a_2] = b, [a_1, b] = c_1, [a_2, b] = c_2, \quad a_1^p = a_2^p = b^p = c_1^p = c_2^p = 1 \rangle.
\]

It follows from [10] that \( Z(G) < G' \), \( |Z(G)| = p^2 \), \( |G'| = p^3 \) and \( |\text{Cl}_G(x)| = p^2 \) for all \( x \in G - Z(G) \).

Now take \( H = G' \). Then \( [G, H] = [H, G] = [G', G] = K(G, H) \), \( Z(H, G) = Z(G) \) and for all \( x \in H - Z(G) \), \( \text{Cl}_G(x) = K(G, H)x \).

Now, we compute \( \Pr_g(H, G) \), as an application of the above results, for some classes of finite groups. Recall that for any two groups \( G_1 \) and \( G_2 \) with subgroups \( H_1 \) and \( H_2 \) respectively, we have

\[
(4) \quad \Pr(H_1 \times H_2, G_1 \times G_2) = \Pr(H_1, G_1) \Pr(H_2, G_2).
\]

**Lemma 3.5.** Let \( H \) be a subgroup of a finite group \( G \) and \( p \) be the smallest prime dividing \( |G| \). If \( ||G, H|| = p \) and \( g \in [G, H] \), then

\[
\Pr_g(H, G) = \begin{cases} \frac{1}{p} \left( 1 + \frac{p-1}{|H : Z(H, G)|} \right) & \text{if } g = 1 \\ \frac{1}{p} \left( 1 - \frac{1}{|H : Z(H, G)|} \right) & \text{if } g \neq 1. \end{cases}
\]

**Proof.** Since \( ||G, H|| = p \), it follows that \( K(G, H) = [G, H] \), and \( |\text{Cl}_G(x)| = ||G, H|| = p \) and \( \text{Cl}_G(x) = K(G, H)x \) for all \( x \in H - Z(H, G) \). Hence, if \( g = 1 \) then, by Corollary 3.4 we have

\[
\Pr_g(H, G) = \frac{1}{p} \left( 1 + \frac{p-1}{|H : Z(H, G)|} \right).
\]

Now consider that \( g \neq 1 \). Since \( \text{Cl}_G(x) = K(G, H)x \) for all \( x \in H - Z(H, G) \) and \( K(G, H) \) is a subgroup, it follows that \( g^{-1}x \in \text{Cl}_G(x) \). Notice that if \( x \in Z(H, G) \) and \( g^{-1}x \in \text{Cl}_G(x) \), then \( g = 1 \). Thus by Lemma 3.4 we have

\[
\Pr_g(H, G) = \frac{1}{|H|} \sum_{x \in H - Z(H, G)} \frac{1}{|\text{Cl}_G(x)|} = \frac{1}{p} \left( 1 - \frac{1}{|H : Z(H, G)|} \right).
\]

This completes the proof. \( \square \)

Now we are ready to prove Theorem B.
Proof of Theorem B. Let \( H \) be a subgroup of a finite nilpotent group \( G \) and \( p \) be a prime integer such that \(|[G, H]| = p\). Let \( P \) and \( Q \) be the Sylow \( p \)-subgroups of \( G \) and \( H \). Then there exist Hall \( p^t \)-subgroups \( A \) and \( B \) of \( G \) and \( H \) respectively such that \( G = P \times A \) and \( H = Q \times B \). Since \(|[G, H]| = p\), it follows that \(|[P, Q]| = p \) and \(|A, B| = 1\). Therefore \([G, H] = [P, Q]\). Let \( g \in [G, H] = [P, Q]\). Notice that \(|\{(x, y) \in H \times G : [x, y] = g\}| = |A||B|\{(u, v) \in Q \times P : [u, v] = g\}|. Hence \( \text{Pr}_g(H, G) = \text{Pr}_g(Q, P) \). Since \( P \) is a \( p \)-group such that \(|[P, Q]| = p\), hypothesis of Lemma 3.5 is satisfied. Hence

\[
\text{Pr}_g(H, G) = \text{Pr}_g(Q, P) = \begin{cases} 
\frac{1}{p} \left( 1 + \frac{p-1}{|Q:Z(Q, P)|} \right) & \text{if } g = 1 \\
\frac{1}{p} \left( 1 - \frac{1}{|Q:Z(Q, P)|} \right) & \text{if } g \neq 1.
\end{cases}
\]

Notice that \(|Q : Z(Q, P)| = |H : Z(H, G)|\). This completes the proof of the theorem.

We would like to remark that the hypothesis of Theorem B is naturally satisfied in many cases, namely (1) if \( G \) is an extraspecial \( p \)-group and \( H \) is any non-central subgroup of \( G \); (2) if \( G \) is a \( p \)-group of maximal class of order \( p^n \) and \( H = \gamma_{n-1}(G) \), where \( n \) is a positive integer and \( \gamma_{n-1}(G) \) denotes the \((n - 1)\)th term of the lower central series of \( G \). More generally, if we take \( G \) to be any finite \( p \)-group of nilpotency class \( c \) such that \( |\gamma_c(G)| = p \) and \( H = \gamma_{c-1}(G) \), then \(|[G, H]| = p\).

4. SOME BOUNDS ON \( \text{Pr}(H, G) \)

For a given finite group \( G \) and a subgroup \( H \) of \( G \) such that \( Z(H, G) \neq H \), let cs\((G, H)\) denote \( \{|\text{Cl}_G(x)| : x \in H\} \). Clearly, cs\((G) := \text{cs}(G, G)\) is the set of conjugacy class sizes of \( G \).

In the following theorem we give some bounds for \( \text{Pr}(H, G) \) in terms of the largest and smallest conjugacy class size of the elements of \( H - Z(H, G) \). Let \( s_H = \min\{|\text{Cl}_G(x)| : x \in H - Z(H, G)\} \) and \( l_H = \max\{|\text{Cl}_G(x)| : x \in H - Z(H, G)\} \).

**Theorem 4.1.** If \( H \) is a subgroup of a finite group \( G \) such that \( Z(H, G) \neq H \) then

\[
\frac{1}{l_H} \left( 1 + \frac{l_H - 1}{|H : Z(H, G)|} \right) \leq \text{Pr}(H, G) \leq \frac{1}{s_H} \left( 1 + \frac{s_H - 1}{|H : Z(H, G)|} \right)
\]

with equality if and only if cs\((G, H)\) = \( \{1, s_H = l_H\} \).

**Proof.** By Lemma 3.1 we have

\[
\text{Pr}(H, G) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|\text{Cl}_G(x)|}
\leq \frac{1}{|H|} \left( |Z(H, G)| + \frac{|H| - |Z(H, G)|}{s_H} \right)
= \frac{1}{s_H} \left( 1 + \frac{s_H - 1}{|H : Z(H, G)|} \right).
\]

The equality holds if and only if cs\((G, H)\) = \( \{1, s_H = l_H\} \). Similarly the other bound can be obtained. \( \square \)

If \( p \) is the smallest prime dividing \(|G|\) and \( H \) a subgroup of \( G \) then it has also been shown in \[3\] Theorem 3.5] that

\[
\text{Pr}(H, G) \leq \frac{1}{2} \left( 1 + \frac{1}{|H : Z(H, G)|} \right).
\]
However, Theorem 3.8 of [4] gives
\[ \Pr(H, G) \leq \frac{1}{p} \left( 1 + \frac{p - 1}{|H : Z(H, G)|} \right). \]
This bound is also obtained by Salemkar et. al for any normal subgroup \( H \) of \( G \) (see [19, Theorem 2.2(v)]).

By Lemma 3.2 we have, for any positive integer \( n \) such that \( 2 \leq n \leq s_H \),
\[ \frac{1}{s_H} \left( 1 + \frac{s_H - 1}{|H : Z(H, G)|} \right) \leq \frac{1}{n} \left( 1 + \frac{n - 1}{|H : Z(H, G)|} \right) \]
with equality if and only if \( s_H = n \).

Let \( p \) be the smallest prime dividing \( |G| \). Then notice that \( p \leq s_H \). Hence for \( n = p \), we get
\[ \frac{1}{s_H} \left( 1 + \frac{s_H - 1}{|H : Z(H, G)|} \right) \leq \frac{1}{p} \left( 1 + \frac{p - 1}{|H : Z(H, G)|} \right). \]
Which shows that the upper bound obtained in Theorem 4.1 is an improvement on the known upper bounds for \( \Pr(H, G) \).

Putting \( H = G \) in Theorem 4.1, we get the following result.

**Corollary 4.2.** Let \( G \) be a finite non-abelian group. Then
\[ \frac{1}{l_G} \left( 1 + \frac{l_G - 1}{|G : Z(G)|} \right) \leq \Pr(G) \leq \frac{1}{s_G} \left( 1 + \frac{s_G - 1}{|G : Z(G)|} \right) \]
with equality if and only if \( cs(G) = \{1, l_G = s_G\} \).

We conclude this section with the analogous bounds in terms of smallest and largest character degrees. Let \( \text{Irr}(G) \) denote the set of complex irreducible characters of \( G \) and \( \text{cd}(G) = \{\chi(1) : \chi \in \text{Irr}(G)\} \). In the last decades a number of results have been proved concerning \( \text{cd}(G) \). Let \( d \) and \( m \), respectively denote the smallest and largest degree of non-linear complex irreducible characters of \( G \). There are some bounds for \( \Pr(G) \) given in [6, Lemma 2(vi)] in terms of the smallest degree of non-linear complex irreducible characters of \( G \). Precisely, if \( G \) is non-abelian then
\[ \frac{1}{|G'|} \leq \Pr(G) \leq \frac{1}{|G'|} \left( 1 + \frac{|G'| - 1}{d^2} \right). \]
The equality holds in the right most inequality if and only if \( \text{cd}(G) = \{1, d = m \} \). The lower bound here can be improved by replacing \( |G : Z(G)| \) by \( m^2 \) in the proof of [15, Theorem 1], which is given in the following equation.
\[ \frac{1}{|G'|} \left( 1 + \frac{|G'| - 1}{m^2} \right) \leq \Pr(G). \]
The equality holds in (5) if and only if \( \text{cd}(G) = \{1, m = d \} \). Since \( \chi(1)^2 \leq |G : Z(G)| \) for all \( \chi \in \text{Irr}(G) \) we have the lower bound obtained in (5) is better than (3).

5. Some observations on class sizes

The study of the structure of a finite group \( G \) by imposing conditions on the set \( cs(G) \), the set of conjugacy class sizes of \( G \), has been studied by many authors in the literature. Recently A. Camina and R. D. Camina [3] gave an impressive survey on this topic. The following questions, which we state in our language, were posed in Section 3.2 of [3]:
Question. Let $G$ be a finite group.

1. If all conjugacy class sizes of $G$, including their multiplicities are known, then what can be said about the solvability of $G$?

2. If all conjugacy class sizes of $G$ are known, then what can be said about the solvability of $G$?

Let $A_n$ denote the alternating group of degree $n$. In 2006, R. M. Guralnick and G. R. Robinson \[6\] proved the following result.

Theorem 5.1 (Theorem 11). Let $G$ be a finite group such that $\Pr(G) > 3/40$. Then either $G$ is solvable, or else $G \cong A_5 \times T$ for some abelian group $T$, in which case $\Pr(G) = 1/12$.

Again in 2006, F. Barry et. al \[1\] proved the following result.

Theorem 5.2 (Theorem 4.10). Let $G$ be a finite group such that $\Pr(G) > 1/3$. Then $G$ is supersolvable.

Theorem 5.3 (Theorem 4.12). Let $G$ be a finite group of odd order such that $\Pr(G) > 11/75$. Then $G$ is supersolvable.

As a consequence of the preceding theorems, we observe the following information regarding above questions:

Regarding Question 1. Suppose that all conjugacy class sizes of $G$, including their multiplicities are known. Then $\Pr(G)$ can be computed in various ways, e.g., by putting $g = 1$ and $H = G$ in Lemma 3.1. Then Theorems 5.1, 5.2 and 5.3 provide interesting information on the solvability and supersolvability of the group $G$.

Regarding Question 2. Following N. Ito \[12\], we define conjugate type vector of a finite groups $G$ by $(1, n_1, n_2, \ldots, n_r)$, where $1 < n_1 < n_2 < \cdots < n_r$. Also the conjugate rank of $G$, which is denoted by $\text{crk}(G)$, is given by $r$. By putting $g = 1$ and $H = G$ in Lemma 3.1 we get

\[
\Pr(G) = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|C_G(x)|} \geq \frac{1}{|G|} \left( |Z(G)| + \text{crk}(G) + \frac{|G| - |Z(G)| - \sum_{i=1}^{r} n_i}{n_r} \right).
\]

Given the center and the conjugate type vector of a group $G$ of given order, one can compute $\Pr(G)$. Hence using (6), Theorems 5.1, 5.2 and 5.3 provide interesting information on the solvability and supersolvability of the group $G$.

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