Gravitational S-Duality Realized on NUT - Schwarzschild and NUT - de Sitter Metrics

U. Ellwanger

Laboratoire de Physique Théorique*, Université Paris XI, Bâtiment 210, 91405 Orsay Cedex, France
E-mail: Ulrich.Ellwanger@th.u-psud.fr

Abstract

Gravitational S-duality is defined by the contraction of two indices of the Riemann tensor with the ε tensor. We review its realization in linearized gravity, and study its generalization to full non-linear gravity by means of explicit examples: Up to a rescaling of the coordinates, it relates two Taub-NUT-Schwarzschild metrics by interchanging $m$ with $\ell$, provided both parameters are non-zero. In the presence of a cosmological constant gravitational S-duality can be implemented at the expense of the introduction of a three-form field, whose value turns out to be dual to the cosmological constant.

LPT Orsay 01-117
December 2001

*Unité Mixte de Recherche CNRS - UMR N° 8627
1 Introduction

$S$-dualities play an important role in relating different configurations in string- and $M$-theory. $S$-dualities are best understood between theories of free abelian $n$-form gauge fields in arbitrary dimensions. In four dimensional Maxwell theory, for example, $S$-duality replaces electric charges by magnetic charges and vice versa. After quantification Dirac’s quantization condition then allows to relate the strong and weak coupling phases of the theory.

Given the non-trivial gravitational backgrounds of many configurations in string- and $M$-theory it becomes important to learn more about $S$-duality in gravity (even without quantification, i.e. without the analog of a Dirac quantization condition). An important example is the Taub-NUT solution of Einstein’s equations \[1,2\], which is characterized by two parameters: a “Schwarzschild” mass $m$ and a “NUT” parameter $\ell$. The interpretation of $\ell$ as the gravitational analog of a magnetic charge has a long history \[3\]. This suggests the existence of a duality-like transformation in pure gravity, and various proposals for such transformations have been made \[4–9\]. Not all of them \[7\], however, correspond to $S$-duality in the sense that the parameters $m$ and $\ell$ of the Taub-NUT solution get interchanged.

At the level of linearized gravity, where the non-abelian structure of the Lorentz algebra does not yet become apparent, $S$-duality has been considered recently in \[8,9\]. In full non-linear gravity the most natural definition of gravitational $S$-duality is obtained using the method of differential forms \[10\], where duality acts on the Riemann tensor with indices in a flat tangent space. This approach has been used widely for the search for Riemann tensors which are (anti-) self-dual, since these are automatically solutions of the (Euclidean) Einstein equations \[11,10\]. The Euclidean Taub-NUT solution with $\ell = m$ is a corresponding example \[11,10\].

The interesting question is in how far $S$-duality exists in full non-linear gravity in the
following sense: Whereas the dual of an arbitrary Riemann tensor can always be constructed, it is not clear whether there exists always a dual metric associated to it. (This can only be proven in linearized gravity, see [8,9] and below.) In order to find out under which conditions such a dual metric exists it is helpful to have at one’s disposal as many explicit examples as possible.

It is the purpose of the present paper to provide two families of such examples. First we consider Taub-NUT spaces with $\ell \neq m$. We find that a dual metric can be constructed explicitly, which allows us to discuss its properties. We find indeed that, up to a rescaling of the metric (or the coordinates), the dual metric is again a Taub-NUT space with $\ell$ and $m$ interchanged. The required rescaling, however, involves the ratio $\ell/m$ and becomes singular in the limits $\ell \to 0$ or $m \to 0$. Hence the dual of a “pure” Schwarzschild metric does not exist. This result is very different from $S$-duality in four dimensional Maxwell theory, which acts without problems on pure electric or pure magnetic charges.

Next we consider Taub-NUT-de Sitter spaces with $\ell \neq m$ and a Ricci tensor proportional to a cosmological constant $\Lambda$. We find that a duality transformation can still be defined, at the price of introducing an additional three-form field $A_{abc} = A_{[abc]}$ with a field strength $F_{abcd} = F_{[abcd]}$ (here, for convenience, the latin letters denote indices in flat tangent space).

Such three-form fields do not constitute dynamical degrees of freedom in four dimensions, but their vacuum configuration $F_{abcd} = \Sigma \varepsilon_{abcd}$ (with $\Sigma = \text{const.}$) contributes to and possibly cancels a cosmological constant \[\Sigma\]. Here, however, their role is not to cancel a cosmological constant, but we find that they are dual to a cosmological constant in the sense that the duality transformation interchanges $\Sigma$ and $\Lambda$. This result may give a new twist to the cosmological constant problem.

The rest of the paper is organized as follows: In the next chapter we review the dual of linearized gravity. Here a dual metric can always be defined thanks to a generalization of the
Poincaré lemma (trivial cohomology) valid for irreducible tensor fields with mixed symmetries as in the case of the Riemann tensor. In chapter three we switch to full non-linear gravity using duality in differential geometry, and compute and discuss the duals of Taub-NUT metrics. In chapter four we introduce a generalization of the duality transformation for non-vanishing Ricci tensor and in the presence of a three-form field, and generalize the previous results to Taub-NUT-de Sitter metrics. In chapter five we conclude with an outlook.

2 The Dual of Linearized Gravity

In linearized gravity the deviation of the metric tensor $g_{\mu\nu}$ from the flat Minkowski metric $\eta_{\mu\nu}$ is considered to be small:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \ |h_{\mu\nu}| \ll 1 \ . \quad (2.1)$$

To first order in $h_{\mu\nu}$ the Riemann tensor $R_{\mu\nu\rho\sigma}$ reads

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\mu\rho,\nu\sigma} - h_{\nu\sigma,\mu\rho}) \ . \quad (2.2)$$

$R_{\mu\nu\rho\sigma}$ has the symmetry properties

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = +R_{\rho\sigma\mu\nu} \ . \quad (2.3)$$

It satisfies the first Bianchi identity (or cyclic identity)

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0 \quad (2.4)$$

and the second Bianchi identity

$$\partial_\lambda R_{\mu\nu\rho\sigma} + \partial_\rho R_{\mu\nu\sigma\lambda} + \partial_\sigma R_{\mu\nu\lambda\rho} = 0 \ . \quad (2.5)$$
In addition we will impose, to start with, the vacuum equations of motion which imply the vanishing of the Ricci tensor:

\[ R^\nu_\rho \equiv R^{\mu \nu} \rho_\mu = 0 \tag{2.6} \]

Now we define the dual Riemann tensor \( \tilde{R}_{\mu \nu \rho \sigma} \) by

\[ \tilde{R}_{\mu \nu \rho \sigma} = \frac{1}{4} \left[ \varepsilon_{\mu \nu \alpha \beta} R^{\alpha \beta} \rho_\sigma + R_{\mu \nu}^{\alpha \beta} \varepsilon_{\alpha \beta \rho \sigma} \right] . \tag{2.7} \]

In order to treat metrics with Euclidean and Lorentzian signatures simultaneously in the following, we introduce the sign \( \sigma \) with

\[ \sigma = +1 \quad (\text{Euclidean signature}) \]
\[ \sigma = -1 \quad (\text{Lorentzian signature}) . \tag{2.8} \]

Then the \( \varepsilon \) tensor satisfies

\[ \varepsilon_{\mu \nu \rho \sigma} \varepsilon^{\mu \nu \rho \sigma} = 24\sigma . \tag{2.9} \]

The duality transformation \( (2.7) \) ensures that the dual Riemann tensor is symmetric,

\[ \tilde{R}_{\mu \nu \rho \sigma} = \tilde{R}_{\rho \sigma \mu \nu} . \tag{2.10} \]

If we apply the duality transformation \( (2.7) \) twice, we would like to obtain the identity (up to a sign depending on the signature of the metric):

\[ \tilde{\tilde{R}}_{\mu \nu \rho \sigma} \overset{!}{=} \sigma R_{\mu \nu \rho \sigma} . \tag{2.11} \]

One finds, however, that \( (2.11) \) holds only if \( R_{\mu \nu \rho \sigma} \) satisfies
\[ R_{\mu\nu\rho\sigma} = \frac{\sigma}{4} \epsilon_{\mu\nu\alpha\beta} R^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta\rho\sigma} . \]  
(2.12)

This equation is not an identity; if one replaces \( R \) by \( \tilde{R} \) on its left-hand side, it corresponds to what has been denoted duality transformation in [7].

Instead of (2.7) we could have defined dual Riemann tensors by

\[ \tilde{R}^{(1)}_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} R^{\alpha\beta}_{\rho\sigma} \]  
(2.13)

or

\[ \tilde{R}^{(2)}_{\mu\nu\rho\sigma} = \frac{1}{2} R^{\alpha\beta}_{\mu\nu} \epsilon_{\alpha\beta\rho\sigma} . \]  
(2.14)

In both cases (2.11) is always satisfied; however, the symmetry property (2.11) holds for \( \tilde{R}^{(1)} \) or \( \tilde{R}^{(2)} \) only if \( R \) satisfies (2.12). Thus, if we require that both eqs. (2.10) and (2.11) hold, we have to restrict ourselves to metrics whose Riemann tensor satisfies (2.12), and now all duality transformations (2.7), (2.13) and (2.14) are equivalent. For definiteness, however, we will subsequently refer to eq. (2.7) as our definition of the duality transformation.

The properties of \( \tilde{R}_{\mu\nu\rho\sigma} \) have previously been discussed by Hull [9]. Its first Bianchi identity follows from the vanishing of the Ricci tensor (2.6):

\[ \tilde{R}_{\mu\nu\rho\sigma} + \tilde{R}_{\mu\sigma\nu\rho} + \tilde{R}_{\mu\rho\sigma\nu} = 0 . \]  
(2.15)

Its second Bianchi identity follows from the second Bianchi identity of \( R_{\mu\nu\rho\sigma} \), (2.5), if (2.6) holds:

\[ \partial_\lambda \tilde{R}_{\mu\nu\rho\sigma} + \partial_\rho \tilde{R}_{\mu\nu\sigma\lambda} + \partial_\sigma \tilde{R}_{\mu\nu\lambda\rho} = 0 . \]  
(2.16)
Finally the first Bianchi identity for $R_{\mu\nu\rho\sigma}$, eq. (2.4), implies the vanishing of the dual Ricci tensor:

$$\tilde{R}^\mu_\rho \equiv \tilde{R}^\mu_{\rho\mu} = 0 .$$ (2.17)

$\tilde{R}_{\mu\nu\rho\sigma}$ has thus the same properties as $R_{\mu\nu\rho\sigma}$. Its symmetries together with the Bianchi identities (2.15) and (2.16) are sufficient to prove that $\tilde{R}_{\mu\nu\rho\sigma}$ can be written in terms of a dual linearized metric $\tilde{h}_{\mu\nu}$ [13] (the vanishing of the dual Ricci tensor is not needed to this end) as

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \tilde{h}_{\mu\sigma,\nu\rho} + \tilde{h}_{\nu\rho,\mu\sigma} - \tilde{h}_{\mu\rho,\nu\sigma} - \tilde{h}_{\nu\sigma,\mu\rho} \right) .$$ (2.18)

An explicit formula for $\tilde{h}_{\mu\nu}$ in terms of $\tilde{R}_{\mu\nu\rho\sigma}$ is given in [13] in the coordinate gauge $x^\mu \tilde{h}_{\mu\nu} = x^\nu \tilde{h}_{\mu\nu} = 0$, which reads in our convention (2.2)

$$\tilde{h}_{\mu\nu}(x) = - \int_0^1 dt \int_0^t dt' x^\sigma x^\rho \tilde{R}_{\mu\rho\sigma}(t' x) .$$ (2.19)

Thus the $S$-dual of linearized gravity can be constructed explicitly. Let us close this short section with a note on gauge symmetries. $R_{\mu\nu\rho\sigma}$ is invariant under

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_{\mu} \Lambda_\nu + \partial_\nu \Lambda_{\mu} ,$$ (2.20)

and $\tilde{R}_{\mu\nu\rho\sigma}$ is invariant under

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_{\mu} \tilde{\Lambda}_\nu + \partial_\nu \tilde{\Lambda}_{\mu} .$$ (2.21)

At this level the gauge parameters $\Lambda_\mu$ and $\tilde{\Lambda}_\mu$ are independent (as the “electric” and “magnetic” U(1) gauge symmetries in the context of electromagnetic duality) and, moreover, not
seemingly related to general coordinate transformations. Clearly this is an artefact of linearized gravity. In full gravity one requires that both $R_{\mu\nu\rho\sigma}$ and $\tilde{R}_{\mu\nu\rho\sigma}$ transform as tensors under general coordinate transformations, and the transformations of $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ (or better $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$) get linked through the link (2.7) between the Riemann tensors (which remains, however, to be covariantized).

3 The Dual of Full Gravity and Taub-NUT Spaces

An attempt to covariantize the duality transformation (2.7) and to preserve the properties of the dual Riemann tensor $\tilde{R}_{\mu\nu\rho\sigma}$ meets the following obstacles:

a) in order to render the $\varepsilon$ tensor in (2.7) covariant we have to multiply it by $\sqrt{g}$ (or $\sqrt{\tilde{g}}$ under the assumption that $\tilde{g}$ exists and transforms as $g$); in order to raise/lower indices we have to contract them with $g^{\mu\nu}$ (or $\tilde{g}^{\mu\nu}$), notably in order to derive the first Bianchi identity (2.15) for $\tilde{R}_{\mu\nu\rho\sigma}$ from (2.6). Either choice between $g^{\mu\nu}$ or $\tilde{g}^{\mu\nu}$ destroys the symmetry corresponding to $g \leftrightarrow \tilde{g}$ together with $R \leftrightarrow \tilde{R}$.

b) the derivatives in the second Bianchi identities (2.5) and (2.16) have to be covariantized with the help of Christoffel symbols which are both related to the same metric $g_{\mu\nu}$ (or $\tilde{g}_{\mu\nu}$), if (2.16) and (2.5) are required to follow from each other. Hence it is not possible to maintain the standard form of both covariantized Bianchi identities (2.5) and (2.16), and to derive (2.16) from (2.5) after covariantization.

The most convenient way to circumvent the obstacles a) is to define the duality transformation (2.7) in terms of differential forms [10]: One expresses the metric $g_{\mu\nu}$ in terms of an orthonormal base of vierbeins $e^a_{\mu}$ (with inverses $e^a_{\mu}$),

$$g_{\mu\nu} = \eta_{ab} \, e^a_{\mu} \, e^b_{\nu}, \quad \eta_{ab} = g_{\mu\nu} \, e^a_{\mu} \, e^b_{\nu},$$

and defines the Riemann tensor with (latin) indices in the flat tangent space:
\[ R_{abcd} = e_a{}^\mu e_b{}^\nu e_c{}^\rho e_d{}^\sigma R_{\mu\nu\rho\sigma}. \] (3.2)

Instead of eq. (2.7) we define now the dual Riemann tensor by

\[ \tilde{R}_{abcd} = \frac{1}{4} \left[ \varepsilon_{abef} R^{ef}_{\phantom{ef}cd} + R_{ab}^{\phantom{ef}ef} \varepsilon_{efcd} \right]. \] (3.3)

Since now indices are raised and lowered with the flat metric \( \eta_{ab} \) it is straightforward to show that the first Bianchi identity for \( \tilde{R}_{abcd} \), the analog of eq. (2.15), still follows from the vanishing Ricci tensor (and vice versa).

The obstacle b), however, has not been removed: The covariant second Bianchi identity for \( R_{abcd} \) involves the spin connection associated to the metric \( g_{\mu\nu} \), and it is not possible to prove a covariant second Bianchi identity for \( \tilde{R}_{abcd} \) involving a spin connection associated to a dual metric \( \tilde{g}_{\mu\nu} \) (which is not even defined at this stage).

Hence, although a dual Riemann tensor \( \tilde{R}_{abcd} \) can always be constructed from eq. (3.3), it is not clear whether it can always be derived from a dual metric \( \tilde{g}_{\mu\nu} \). (An iterative procedure which allows to extend the result (2.19) of linearized gravity to higher orders in \( h_{\mu\nu} \) (or \( \tilde{h}_{\mu\nu} \)) may possibly be derived along the lines in ref. [14].)

Therefore it is of interest to find specific examples of metrics \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \), whose Riemann tensors are related through (3.3) without having to resort to a weak field expansion. Subsequently we will consider Taub-NUT spaces for general \( m \) and \( \ell \), which contain the Schwarzschild metric in the limit \( \ell \to 0 \). For \( m = \ell \), in Euclidean space-time, they are well-known to be self-dual in the sense of eq. (3.3) [11,10].

The Taub-NUT metric can be written as follows [1,2] (we recall the definition of the sign \( \sigma \) in eq. (2.8)):
\begin{equation}
\begin{aligned}
ds^2 &= \sigma f^2(r) \left( dt + 4\ell \sin^2 \frac{\theta}{2} d\phi \right)^2 + f^{-2}(r) dr^2 + \left( r^2 - \sigma \ell^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\end{aligned}
\end{equation}

with

\begin{equation}
\begin{aligned}
f^2(r) &= 1 - \frac{2(mr - \sigma \ell^2)}{r^2 - \sigma \ell^2}.
\end{aligned}
\end{equation}

The non-vanishing components of the Riemann tensor $R_{abcd}$, as defined in eq. \eqref{eq:riemann}, have been computed by Misner \cite{misner}:

\begin{equation}
\begin{aligned}
R_{0101} &= -2A_{m,\ell}(r) \\
R_{0202} &= R_{0303} = A_{m,\ell}(r) \\
R_{1212} &= R_{1313} = \sigma A_{m,\ell}(r) \\
R_{2323} &= -2\sigma A_{m,\ell}(r) \\
R_{0312} &= -R_{0213} = D_{m,\ell}(r) \\
R_{0123} &= -2D_{m,\ell}(r)
\end{aligned}
\end{equation}

with

\begin{equation}
\begin{aligned}
A_{m,\ell}(r) &= \frac{-\sigma mr^3 + 3\ell^2 r^2 - 3ml^2 r + \sigma \ell^4}{(r^2 - \sigma \ell^2)^3}, \\
D_{m,\ell}(r) &= \frac{\sigma \ell r^3 - \sigma mr^2 + 3r \ell^3 - ml^3}{(r^2 - \sigma \ell^2)^3}.
\end{aligned}
\end{equation}

One easily verifies that $R_{abcd}$ satisfies the analog of eq. \eqref{eq:analog} (with greek indices replaced by latin indices), hence eqs. \eqref{eq:analog1} and \eqref{eq:analog2} are both satisfied, and we can replace eq. \eqref{eq:replace} by the analogs of eqs. \eqref{eq:analog3} or \eqref{eq:analog4}. Subsequently we have to specify one component of the tensor $\varepsilon_{abcd}$: we use, both for Euclidean and Lorentzian signatures,

\begin{equation}
\varepsilon_{0123} = +1.
\end{equation}
Then we obtain the following non-vanishing components of $\tilde{R}_{abcd}$:

\[ \begin{align*}
\tilde{R}_{0101} &= -2D_{m,\ell}(r) \\
\tilde{R}_{0202} &= D_{m,\ell}(r) \\
\tilde{R}_{1212} &= \tilde{R}_{1313} = \sigma D_{m,\ell}(r) \\
\tilde{R}_{2323} &= -2\sigma D_{m,\ell}(r) \\
\tilde{R}_{0312} &= -\tilde{R}_{0213} = \sigma A_{m,\ell}(r) \\
\tilde{R}_{0123} &= -2\sigma A_{m,\ell}(r). 
\end{align*} \]

(3.9)

The search for a dual metric, which is associated to the dual Riemann tensor $\tilde{R}_{abcd}$, is greatly simplified by the following properties of the functions $A_{m,\ell}(r)$ and $D_{m,\ell}(r)$: If one defines

\[ m' = \sigma \frac{\ell^2}{m} \]  

(3.10)

one has

\[ \begin{align*}
A_{m',\ell}(r) &= -\sigma \frac{\ell}{m} D_{m,\ell}(r) , \\
D_{m',\ell}(r) &= \frac{\ell}{m} A_{m,\ell}(r).
\end{align*} \]

(3.11)

Furthermore a rescaling of the Riemann tensor corresponds to a rescaling of the metric:

\[ \gamma R_{abcd} = R_{abcd} \bigg|_{g_{\mu\nu} \to \gamma^{-1} g_{\mu\nu}}. \]  

(3.12)

Inserting eqs. (3.11) and (3.12) into (3.9) one finds that the following metric generates a Riemann tensor whose components are given by (3.9):

\[ \tilde{ds}^2 = \frac{\ell}{m} \left\{ -\tilde{f}^2(r) \left( dt + 4\ell \sin^2 \frac{\theta}{2} d\phi \right)^2 - \sigma \left[ \tilde{f}^{-2}(r) dr^2 + (r^2 - \sigma \ell^2) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \right\} \]

(3.13)
with
\[ \tilde{f}^2(r) = 1 + \frac{2\sigma\ell^2(m - r)}{m(r^2 - \sigma\ell^2)}. \] (3.14)

In order to bring the metric (3.13) into the form of the metric (3.4) one has to rescale the coordinates \( t \) and \( r \),
\[ t = \sqrt{\frac{m}{\ell}} t', \quad r = \sqrt{\frac{m}{\ell}} r', \] (3.15)
and define
\[ \tilde{m} = \sigma \ell^{5/2} m^{-3/2} \quad \text{and} \quad \tilde{\ell} = \ell^{3/2} m^{-1/2}. \] (3.16)

Now \( \tilde{d}s^2 \) becomes
\[ \tilde{d}s^2 = -\tilde{f}^2(r') \left( dt' + 4\tilde{\ell} \sin^2 \theta d\phi \right)^2 - \sigma \left[ \tilde{f}^{-2}(r') dr'^2 + (r'^2 - \sigma \tilde{\ell}^2) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] (3.17) \]
with
\[ \tilde{f}^2(r') = 1 - \frac{2(\tilde{m}r' - \sigma\tilde{\ell}^2)}{r'^2 - \sigma\tilde{\ell}^2}. \] (3.18)

Now \( \tilde{f}^2 \) is of the same form as \( f^2 \) in eq. (3.13), but \( \tilde{d}s^2 \) still differs by a factor \(-\sigma\) from \( ds^2 \) in eq. (3.4). In the Euclidean case \((\sigma = +1)\) we find that, for \( m = \ell \), the metric and hence the Riemann tensor are anti-self-dual. Here we have tacitely assumed that \( m \) and \( \ell \) are positive. For \( \ell \) negative and \( m = -\ell \), however, we have to replace \( \ell \) by \(-\ell\) in eqs. (3.13) and (3.16), and one finds that the metric and the Riemann tensor are self-dual. In the Lorentzian case \((\sigma = -1)\) eqs. (3.16) imply that \( \tilde{m} \) is negative if \( \ell \) and \( m \) are real and positive. These are the well-known (anti-) self-duality properties of Taub-NUT metrics with \(|m| = |\ell|\).
Actually, up to rescalings of the $t$ and $r$ coordinates, only the ratio $|m/\ell|$ has a physical significance in Taub-NUT spaces. From eqs. (3.16) one finds immediately

$$\left| \frac{m}{\ell} \right| = \left| \frac{\tilde{r}}{\tilde{m}} \right|,$$

hence $\ell$ and $m$ indeed exchange their role under the duality transformation (3.3). This agrees with the interpretation of $\ell$ as a “magnetic” mass [3–7], and with the interpretation of (3.3) as the analog of $S$-duality in Maxwell theory.

However, the presence of the factors $\ell$ in eqs. (3.13) and (3.15) shows that the analogy breaks down in the limit $\ell \to 0$: Now $\tilde{ds}^2$ vanishes, or the rescaling (3.15) becomes singular. Hence the duals of the Schwarzschild metric (and, similarly, the dual of the Taub-NUT metric with $m = 0$) do not exist.

One could be tempted to confine oneself to the asymptotic regime $r \to \infty$, and to apply the explicit formula (2.19) for $\tilde{h}_{\mu\nu}$ in the pure Schwarzschild case. Here one has to use Cartesian coordinates instead of spherical coordinates such that $|h_{\mu\nu}| \ll 1$ for $r \to \infty$. Then one finds indeed that

$$x^\rho x^\sigma \tilde{R}_{\rho\sigma\mu\nu}(t'x) = 0$$

for $\ell = 0$ in eq. (2.19), in agreement with the results above.

Hence the analogy between gravitational and Maxwell $S$-duality holds only for “dyonic” Taub-NUT metrics with both $m$ and $\ell \neq 0$.

4 Duality in Taub-NUT-de Sitter Spaces

In Taub-NUT-de Sitter spaces the Ricci tensor vanishes no longer, but is proportional to a cosmological constant:
\[ R^b_c \equiv R^{ab}_{\ ca} = \Lambda \delta^b_c \] (4.1)

instead of eq. (2.4). (We continue to work with tensors with indices in flat tangent space, which are related to the tensors with indices in ordinary space-time through the vierbeins \( e_{\mu}^a \). Furthermore we find it more convenient to define the cosmological constant in terms of the Ricci tensor instead of the more conventional definition in terms of the Einstein tensor.)

Of course the first and second Bianchi identities for \( R_{abcd} \) remain intact, but now one has to wonder how one can obtain the validity of the first Bianchi identity (2.15) for a dual Riemann tensor which previously required the vanishing of the Ricci tensor.

In some sense the cosmological constant \( \Lambda \) represents a (trivial) "matter" degree of freedom, and quite generally duality in the presence of matter (if it exists at all) requires some mixing between the "gauge fields" and "matter".

First, the dual of a cosmological constant (in a sense specified below) turns out to be a three-form field \( A_{abc} = A_{[abc]} \), with a field strength

\[ F_{abcd} = \partial_{[a} A_{bcd]} . \] (4.2)

The equations of motion for \( A_{abc} \) read

\[ \partial^a F_{abcd} = 0 , \] (4.3)

and the only solutions respecting Lorentz covariance are of the form

\[ F_{abcd} = \Sigma \varepsilon_{abcd} , \quad \Sigma = \text{const.} \] (4.4)

Now we consider the following generalization of the duality transformation (3.3):
\[
\tilde{R}_{abcd} = \frac{1}{4} \left[ \varepsilon_{abef} \left( R^{ef}_{\phantom{ef}cd} + F^{ef}_{\phantom{ef}cd} \right) + \left( R^{\phantom{ef}ef}_{ab} + F^{\phantom{ef}ef}_{ab} \right) \varepsilon_{efcd} \right] + \frac{1}{12} \varepsilon_{abcd} R,
\]

(4.5a)

\[
\tilde{F}_{abcd} = -\frac{1}{12} \varepsilon_{abcd} R,
\]

(4.5b)

where

\[
R \equiv R^{ab}_{\phantom{ab}ba}.
\]

(4.6)

Let us discuss the properties of \( \tilde{R}_{abcd} \). First, \( \tilde{R}_{abcd} \) still has the same symmetry properties (2.3) as \( R_{abcd} \). Next, the first Bianchi identity still holds:

\[
\tilde{R}_{abcd} + \tilde{R}_{adbc} + \tilde{R}_{acdb} = 0
\]

(4.7)

where one has to use eq. (4.4) for \( F_{abcd} \) (i.e. the equation of motion for \( A_{abc} \)), and the last term \( \sim R \) in (4.5a) serves to cancel the contributions proportional to the cosmological constant. Also, the second Bianchi identity still holds at the linearized level:

\[
\partial_e \tilde{R}_{abcd} + \partial_c \tilde{R}_{abde} + \partial_d \tilde{R}_{abec} = 0
\]

(4.8)

where one has to use the linearized second Bianchi identity for \( R_{abcd} \), and the fact that both the Ricci tensor \( R^a_b \) and \( F_{abcd} \) are constant. Eqs. (4.7) and (4.8) are already sufficient to prove that, at the linearized level, \( \tilde{R}_{abcd} \) can again be expressed in terms of a dual metric \( \tilde{h}_{ab} \) as in eq. (2.19) (the distinction between latin and greek indices is meaningless at the linearized level).

For the dual Ricci tensor one obtains

\[
\tilde{R}^a_b = -3\sigma \Sigma \delta^a_b
\]

(4.9)
with the help of the first Bianchi identity for $R_{abcd}$, and eq. (4.4) for $F_{abcd}$. Hence $\tilde{R}^a_b$ is proportional to a dual cosmological constant $\tilde{\Lambda}$ with

$$\tilde{\Lambda} = -3\sigma \Sigma .$$  \hfill (4.10)

$\tilde{F}_{abcd}$ always satisfies the Bianchi identity

$$\partial_a [\tilde{F}_{bcde}] = 0 \quad (4.11)$$

which is a trivial identity in $d = 4$. The dual equations of motion

$$\partial^a \tilde{F}_{abcd} = 0 \quad (4.12)$$

follow from the constancy of the Riemann scalar $R$: together with (4.1) eq. (4.5b) gives evidently

$$\tilde{F}_{abcd} = -\frac{1}{3} \varepsilon_{abcd} \Lambda . \quad (4.13)$$

Eq. (4.11) shows that $\tilde{F}_{abcd}$ can be written as

$$\tilde{F}_{abcd} = \partial_a [\tilde{A}_{bcde}] \quad (4.14)$$

and the solution of the equation of motion (4.12) for $\tilde{A}_{abc}$ gives

$$\tilde{F}_{abcd} = \varepsilon_{abcd} \tilde{\Sigma} . \quad (4.15)$$

with, from (4.13),

$$\tilde{\Sigma} = -\frac{1}{3} \Lambda . \quad (4.16)$$
Equations (4.10) and (4.16) clarify in what sense $A_{abc}$ is dual to the cosmological constant: Up to a factor $(-3)$ (and the sign $\sigma$) the duality transformations (4.5) lead to an interchange of $\Sigma$, the parameter characterizing the solution of the equation of motion of $A_{abc}$, with the cosmological constant $\Lambda$.

The effect of a double duality transformation on $F_{abcd}$ is easily obtained from eqs. (4.13) and (4.10):

$$\tilde{F}_{abcd} = \sigma F_{abcd}.$$  \hfill (4.17)

After some calculation one finds that the effect of a double duality transformation on $R_{abcd}$ is the same as before:

$$\tilde{R}_{abcd} = \sigma R_{abcd}$$  \hfill (4.18)

if $R_{abcd}$ satisfies

$$R_{abcd} = \frac{\sigma}{4} \varepsilon_{abcdef} R^{efgh} \varepsilon_{ghcd}.$$  \hfill (4.19)

Hence, on metrics which satisfy (4.19), our generalized duality transformation (4.5) has all desirable properties. As before, however, the validity of a second Bianchi identity for $\tilde{R}_{abcd}$ cannot be proven beyond the linearized level.

Let us now study the effect of (4.5) on Taub-NUT-de Sitter metrics. These metrics can be written in the same form as the Taub-NUT metric (3.4); it suffices to replace the function $f^2(r)$ by

$$f^2(r) = 1 - \frac{2(mr - \sigma \ell^2) - \Lambda \left(\frac{1}{3}r^4 - 2\sigma \ell^2 r^2 - \ell^4\right)}{r^2 - \sigma \ell^2}.$$  \hfill (4.20)
Now the non-vanishing components of the Riemann tensor are, instead of eqs. (3.6),

\begin{align*}
R_{0101} &= -2A_\Lambda(r) \\
R_{0202} = R_{0303} &= C_\Lambda(r) \\
R_{1212} = R_{1313} &= \sigma C_\Lambda(r) \\
R_{2323} &= -2\sigma A_\Lambda(r) \\
R_{0312} = -R_{0213} &= D_\Lambda(r) \\
R_{0123} &= -2D_\Lambda(r)
\end{align*}

(4.21)

with

\begin{align*}
A_\Lambda(r) &= \left(1 - \frac{4}{3}\sigma \Lambda \ell^2\right) A_{\bar{m},\ell}(r) + \frac{\sigma}{6}\Lambda \\
C_\Lambda(r) &= \left(1 - \frac{4}{3}\sigma \Lambda \ell^2\right) A_{\bar{m},\ell}(r) - \frac{\sigma}{3}\Lambda \\
D_\Lambda(r) &= \left(1 - \frac{4}{3}\sigma \Lambda \ell^2\right) D_{\bar{m},\ell}(r)
\end{align*}

(4.22)

where \( A_{\bar{m},\ell} \) and \( D_{\bar{m},\ell} \) are the functions defined in eq. (3.7) and

\[
\bar{m} = m \left(1 - \frac{4}{3}\sigma \Lambda \ell^2\right)^{-1}.
\]

(4.23)

Constructing the components of the dual Riemann tensor from eq. (4.5a) one obtains now additional contributions from the terms \( \sim F_{abcd} \) and \( \sim R \). One finds

\begin{align*}
\tilde{R}_{0101} &= -2D_\Lambda + \Sigma \\
\tilde{R}_{0202} = \tilde{R}_{0303} &= D_\Lambda + \Sigma \\
\tilde{R}_{1212} = \tilde{R}_{1313} &= \sigma D_\Lambda + \sigma \Sigma \\
\tilde{R}_{2323} &= -2\sigma D_\Lambda + \sigma \Sigma \\
\tilde{R}_{0312} = \tilde{R}_{0213} &= \sigma C_\Lambda + \frac{1}{3}\Lambda \\
\tilde{R}_{0123} &= -2\sigma A_\Lambda + \frac{1}{3}\Lambda .
\end{align*}

(4.24)
Again the properties (3.11) of the functions \(A(r), D(r)\) help to find a metric which reproduces the same components of the Riemann tensor:

\[
\tilde{ds}^2 = -\frac{\ell}{m + 4\sigma \Sigma \ell^3} \left\{ \tilde{f}^2(r) \left( dt + 4\ell \sin^2 \frac{\theta}{2} d\phi \right)^2 + \sigma \left[ \tilde{f}^2(r) dr^2 + \left( r^2 - \sigma \ell^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \right\}
\]  

(4.25)

with

\[
\tilde{f}^2(r) = 1 + \frac{2\sigma \ell^2 \left( m + 4\sigma \Sigma \ell^3 - r \left( 1 - \frac{4}{3} \sigma \Lambda \ell^2 \right) \right) + 3\Sigma \ell \left( \frac{1}{3} r^4 - 2\sigma r^2 \ell^2 - \ell^4 \right)}{\left( m + 4\sigma \Sigma \ell^3 \right) \left( r^2 - \sigma \ell^2 \right)}.
\]  

(4.26)

A rescaling similar to eqs. (3.13),

\[
t = \sqrt{\frac{m + 4\sigma \Sigma \ell^3}{\ell}} t', \quad r = \sqrt{\frac{m + 4\sigma \Sigma \ell^3}{\ell}} r',
\]  

(4.27)

and dual parameters similar to eqs. (3.16),

\[
\tilde{m} = \sigma \left( 1 - \frac{4}{3} \sigma \Lambda \ell^2 \right) \left( m + 4\sigma \Sigma \ell^3 \right)^{-3/2} \ell^{5/2}, \quad \tilde{\ell} = \ell^{3/2} \left( m + 4\sigma \Sigma \ell^2 \right)^{-1/2},
\]  

\[
\tilde{\Lambda} = -3\sigma \Sigma,
\]  

(4.28)

allow to write the dual metric again in the form (3.17) with

\[
\tilde{f}^2(r') = 1 - \frac{2 \left( \tilde{m} r' - \sigma \tilde{\ell}^2 \right) + \sigma \tilde{\Lambda} \left( \frac{1}{3} r'^4 - 2\sigma r'^2 \tilde{\ell}^2 - \tilde{\ell}^4 \right)}{r'^2 - \sigma \tilde{\ell}^2}.
\]  

(4.29)

Thus, up to signs (cf. the remarks below eq. (3.18)) the metric dual to a Taub-NUT-de Sitter metric is again of the Taub-NUT-de Sitter form. Now the limit \(m \to 0\) actually exists, but the limit \(\ell \to 0\) (pure Schwarzschild-de Sitter) is still singular.
Let us close this section with some remarks on a dual action. In the present approach we have obtained simple duality relations between the cosmological constant $\Lambda$ and $\Sigma$, cf. eqs. (4.10), (4.16) and (4.28). Clearly the equation of motion for $A_{abc}$, eq. (4.3), follows from a quadratic “matter” action $\sim \lambda F_{abcd}F^{abcd}$. Then the cosmological constant $\Lambda$ is actually equal to $24\lambda \sigma \Sigma^2$ plus an arbitrary additional contribution $\Lambda'$. The same holds for the dual cosmological constant $\tilde{\Lambda}$. It is then straightforward to write down duality relations between $\Lambda'$, $\Sigma$ and $\tilde{\Lambda}'$, $\tilde{\Sigma}$ (which depend on the couplings $\lambda$, $\tilde{\lambda}$), but we did not find these relations very illuminating. The previous relations show in a much more direct way that, e.g., $\tilde{\Lambda} = 0$ iff $\Sigma = 0$ (and vice versa) which may have some interesting applications.

5 Conclusions and Outlook

In the present paper we tried to push the concept of gravitational $S$-duality beyond linearized gravity. We have seen that the duality relation between the first Bianchi identity and the equations of motion can be maintained. It is then of interest to find out, under which conditions the dual Riemann tensor satisfies the second Bianchi identity or, equivalently, under which conditions a dual metric exists. At least to this end it is very helpful to study explicitly the properties of metrics related by $S$-duality.

In the case of Taub-NUT metrics we have seen that, as expected, $S$-duality corresponds to an exchange of the parameters $m$ and $\ell$. However we have also seen that, due to the required rescaling of the coordinates, the metric has to be “dyonic” in the sense that both parameters $m$ and $\ell$ are non-vanishing.

Then we managed to generalize the concept of $S$-duality to Taub-NUT-de Sitter spaces with cosmological constant $\Lambda$. Now a three-form $A_{abc}$ has to be introduced. Somewhat unexpectedly it is not needed to cancel the cosmological constant as proposed in [12], but it turns out to be dual to the cosmological constant.
This phenomenon may pave the way towards a new solution of the cosmological constant problem: If, for some reason, this three-form field vanishes such that $\Sigma = 0$ and, simultaneously, matter couples to the dual metric $\tilde{g}_{\mu\nu}$, the universe as described by the dual metric has automatically a vanishing cosmological constant $\tilde{\Lambda} \sim \Sigma = 0$ (or vice versa).

However, many properties of gravitational S-duality have to be better understood before such a speculation can be supported more seriously. The first open question is evidently for which metrics, which solve the vacuum Einstein equations, a dual metric exists (beyond the linearized level). The next open question concerns the coupling of gravity to matter. We have already emphasized that, in the presence of matter, the duality transformation rules certainly have to be modified. Our results in the presence of a cosmological constant and/or a three-form field indicate a first step in this direction. It may turn out, however, that the framework of Riemannian geometry (with, e.g., the absence of torsion) is too restrictive in order to allow for general gravitational S-duality beyond a few particular configurations of metrics and matter fields.

In the present approach to gravitational S-duality four space-time dimensions evidently play a particular role: in $d \neq 4$ the dual of the Riemann tensor has no longer the same number of indices. In linearized gravity it can still be written in terms of a gauge field with mixed symmetry properties [9], but the relation of this field to a Riemannian metric is not clear. If this relation could be better understood, gravitational duality in the presence of matter in $d = 4$ could be obtained from pure gravitational duality in $D > 4$ after compactification.

Acknowledgement

It is a pleasure to thank B. Carter and M. Dubois-Violette for stimulating discussions.
References

[1] A. Taub, Ann. Math. 53 (1951) 472;
   E. Newman, L. Tamburino, T. Unit, J. Math. Phys. 4 (1963) 915.

[2] C. Misner, J. Math. Phys. 4 (1963) 924.

[3] J. Dowker, J. Roche, Proc. Phys. Soc. 92 (1967) 1;
   R. Mignani, Lett. Nuovo Cim. 22 (1978) 597;
   S. Ramaswamy, A. Sen, J. Math. Phys. 22 (1981) 2612;
   G. Murphy, Int. J. Theor. Phys. 22 (1983) 477;
   A. Zee, Phys. Rev. Lett. 55 (1985) 2379, Erratum ibid. 56 (1986) 1101;
   D. Lynden-Bell, M. Nouri-Zonoz, Rev. Mod. Phys. 70 (1998) 427.

[4] R. Geroch, J. Math. Phys. 12 (1971) 918.

[5] A. Magnon, J. Math. Phys. 28 (1987) 2149; Gen. Rel. Grav. 19 (1987) 809.

[6] H. Garcia-Compean, O. Obregon, J. Plebanski, C. Ramirez, Phys. Rev. D57 (1998) 7501;
   H. Garcia-Campean, O. Obregon, C. Ramirez, Phys. Rev. D58 (1998) 104012.

[7] N. Dadhich, Mod. Phys. Lett. A14 (1999) 337; ibid. 759;
   M. Nouri-Zonoz, N. Dadhich, D. Lynden-Bell, Class. Quant. Grav. 16 (1999) 1021.

[8] J. Nieto, Phys. Lett. A262 (1999) 274.

[9] C. Hull, Nucl. Phys. B583 (2000) 237;
   C. Hull, JHEP 0109 (2001) 27.

[10] T. Eguchi, P. Gilkey, A. Hanson, Phys. Rept. 66 (1980) 213.
[11] S. Hawking, Phys. Lett. 60A (1977) 81.

[12] A. Aurilia, H. Nicolai, P. Townsend, Nucl. Phys. B176 (1980) 509;
    M. Duff, P. van Nieuwenhuizen, Phys. Lett. B94 (1980) 179;
    S. Hawking, Phys. Lett. B134 (1984) 403;
    J. Brown, C. Teitelboim, Phys. Lett. B195 (1987) 177;
    M. Duff, Phys. Lett. B226 (1989) 36.

[13] M. Dubois-Violette, M. Henneaux, Lett. Math. Phys. 49 (1999) 245;
    M. Dubois-Violette, M. Henneaux, math-qa/0110088.

[14] S. Deser, Gen. Rel. Grav. 1 (1970) 9.