**Abstract**

We show that a radiative modification of a recently proposed model by Altarelli and Feruglio with softly broken $A_4$ symmetry leads naturally to nonvanishing $U_{e3}$ with $\theta_{13} \simeq 2^\circ - 4^\circ$. The observed mass squared differences for solar and atmospheric neutrinos are reproduced, whereas the predicted solar neutrino mixing angle is brought down from the tri-bimaximal prediction to be in better agreement with the latest global analysis including experimental data from KamLAND and SNO.
Experimental measurements on neutrino oscillations are consistent with nearly maximal atmospheric neutrino mixing angle ($\theta_{23} \simeq 45^\circ$), large but less than maximal solar neutrino mixing angle ($\theta_{12} \simeq 34^\circ$), and a small “CHOOZ” angle ($\theta_{13} < 10^\circ$). Whereas the three neutrino masses could be quasi-degenerate or hierarchical with no normal or inverted ordering, these values of the mixing angles are found to be remarkably close to the conjectured tri-bimaximal mixing ansatz of Harrison, Perkins, and Scott (HPS) \cite{1},

$$U_{\text{tb}} = \begin{pmatrix}
\sqrt{2} / 6 & \sqrt{1} / 6 & 0 \\
-\sqrt{1} / 6 & \sqrt{2} / 6 & -\sqrt{1} / 6 \\
\sqrt{1} / 6 & -\sqrt{2} / 6 & \sqrt{1} / 6
\end{pmatrix}.$$  

The importance of the non-Abelian discrete symmetry group $A_4$ in understanding why charged leptons may have very different masses and yet a symmetry exists for the neutrino mass matrix has been discussed by Ma and collaborators in recent papers \cite{2, 3, 4}. In particular, it was shown in \cite{4} how the HPS ansatz may be realized. In an interesting development, Altarelli and Feruglio (AF) have proposed the simplest such model with only two parameters in supersymmetric as well as non-supersymmetric cases \cite{5}. The relevant $A_4$ symmetric Lagrangian, after spontaneous symmetry breaking, becomes

$$L_{\text{AF}} = v_d v_T / \Lambda (y_e e^c e + y_\mu \mu^c \mu + y_\tau \tau^c \tau) + x_a v_u^2 (u / \Lambda^2) (\nu_e \nu_e + 2 \nu_\mu \nu_\tau) + x_b v_S^2 2 v_S / 3 \Lambda^2 (\nu_e \nu_e + \nu_\mu \nu_\mu + \nu_\tau \nu_\tau - \nu_e \nu_\mu - \nu_\mu \nu_\tau - \nu_\tau \nu_e) + h.c. \quad (1)$$

Here $\Lambda$ is the scale of new physics ($\equiv$ seesaw scale) and the Higgs content with their vacuum expectation values are presented in Table 1 where all fields except $\chi_i^+$ have been used in Ref. \cite{5}. Then the neutrino mass matrix takes the form

$$M_{\nu}^{\text{AF}} = m_0 \begin{pmatrix}
a + 2d/3 & -d/3 & -d/3 \\
-d/3 & 2d/3 & a - d/3 \\
-d/3 & a - d/3 & 2d/3
\end{pmatrix}, \quad (2)$$

where $a = 2x_a u / \Lambda$, $d = 2x_b v_S / \Lambda$, $m_0 = v_u^2 / \Lambda$. This is exactly diagonalized \cite{4} by the HPS matrix, resulting from the underlying $A_4$ symmetry with

$$\sin \theta_{12}^0 = \frac{1}{\sqrt{3}}, \quad \sin \theta_{23}^0 = -\frac{1}{\sqrt{2}}, \quad \sin \theta_{13}^0 = 0. \quad (3)$$

and mass eigenvalues

$$m_1^0 = a + d, \quad m_2^0 = a, \quad m_3^0 = d - a. \quad (4)$$
Lepton | $SU(2)_L$ | $A_4$
--- | --- | ---
$(\nu_i, l_i)$ | 2 | 3
$l^c_i$ | 1 | 1

| Scalar | VEV |
--- | --- |
h$_u$ | 2 | $< h^0_u > = v_u$ |
h$_d$ | 2 | $< h^0_d > = v_d$ |
$\xi$ | 1 | $< \xi^0 > = u$ |
$\phi_S$ | 1 | 3 | $< \phi^0_S > = (v_S, v_S, v_S)$ |
$\phi_T$ | 1 | 3 | $< \phi_T > = (v_T, 0, 0)$ |
$\chi^+_i$ | 1 | 3 |

Table 1: List of fermion and scalar fields used in this model.

where the common mass factor $m_0$ has been absorbed into $a$ and $d$.

Although this model prediction of $\theta^0_{13} = 0$ is consistent with the CHOOZ - Palo Verde upper bound, $\sin^2 \theta^0_{13} < 0.16$ ($\theta_{13} < 10^\circ$) \cite{6,7}, the actual value of the mixing angle is of considerable theoretical and experimental interest as more accurate values on the parameter are expected to emerge from long baseline and future reactor experiments such as Double CHOOZ, Triple CHOOZ, Nova and others \cite{8}. Thus, it is worthwhile to study some modification of the AF model which may predict $\theta_{13} \neq 0$, which is necessary for CP violation in neutrino oscillations. The second observation is that the tri-bimaximal mixing matrix predicts $\tan^2 \theta^0_{12} = 0.5$ corresponding to $\theta^0_{12} = 35.3^\circ$ whereas a recent global analysis including the KamLAND and the latest SNO data gives $(\tan^2 \theta_{12})_{\text{expt.}} = 0.45 \pm 0.05$ corresponding to $(\theta^0_{12})_{\text{expt.}} = 33.8^\circ \pm 1.5^\circ$ \cite{7}. Within 1\(\sigma\) the tri-bimaximal mixing prediction just touches the upper limit of the solar neutrino mixing angle and it is desirable to have a model prediction where the mixing angle is in accord with the central value obtained from global analysis.

Further, while fitting the available neutrino data the following relations between $|a|$ and $|d|$ have been found useful. In the AF parametrisation both $\Delta m^2_{\text{atm}}$ and $\Delta m^2_{\odot}$ have been fitted with an ansatz,

$$|d| = -2|a|(1 - 2R) \cos \phi. \quad (5)$$

where $\phi = \text{arg}(d) - \text{arg}(a)$ and $R = \Delta m^2_{\odot}/\Delta m^2_{\text{atm}}$ and this relation manifests a moderate fine tuning of the two parameters. On the other hand, one may assume a simpler relationship,

$$|d| = -2|a| \cos \phi, \quad (6)$$

in which case $|m_1| = |m_2|$ at the seesaw scale. In such a scenario, while the value of $\Delta m^2_{\text{atm}}$ is
fitted, $\Delta m^2_{32}$ is expected to be generated by radiative corrections. However, such corrections are negligible for small neutrino masses $|m_i| << 0.3$ eV in the Standard Model (SM) or in the Minimal Supersymmetric Standard Model (MSSM). New particles and interactions (i.e. new physics) are then required.

In the present work we modify the AF model with an aim to accommodate the above desirable features with a parametric relation given by Eq. (6). [We note that deviations from tri-bimaximal mixing have already been considered in the AF model, using higher-dimensional operators. We take the view of starting with the tri-bimaximal form of the neutrino mass matrix given by Eq. (2), however it may arise, and then modifying it with a particular radiative mechanism.] The model then predicts new values of the mixing angles $\theta_{12}$ and $\theta_{13}$ while bringing the former closer to the central value of the experimental data and lifting the latter to nonvanishing values which are within the accessible limits of planned experiments [8]. Because of its structure, any nonvanishing value of $\theta_{13}$ in this model is found to be constrained by the solar and atmospheric mixing angles, hence we predict only small values of the CHOOZ angle in the end.

To generate the desirable new radiative contributions to $M_\nu$, we introduce three singlet charged scalars $\chi^+_i (i = 1, 2, 3)$ transforming as an $A_4$-triplet in the nonsupersymmetric version of the AF model with two Higgs doublets $h_u$ and $h_d$. The Lagrangian of the present model has three parts,

$$\mathcal{L} = \mathcal{L}_{AF} + \mathcal{L}_1 + \mathcal{L}_2. \quad (7)$$

Here $\mathcal{L}_{AF}$ is already given in Eq. (1) and in Ref. [5], and $\mathcal{L}_1$ is the additional contribution of the $\chi^+_i$ scalars that respects $A_4$ symmetry. The term $\mathcal{L}_2$ is introduced to break the $A_4$ symmetry softly and in conjunction with $\mathcal{L}_1$, it gives rise to new radiative contributions known often as the Zee mechanism [9], as depicted in Fig. 1. Explicitly $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\mathcal{L}_1 = f (L L \chi_i) \subset (3 \times 3 \times 3)$$

$$= f (\nu_\mu \tau \chi^+_1 + \nu_\tau e \chi^+_2 + \nu_e \mu \chi^+_3 - \nu_\tau \mu \chi^+_1 - \nu_e \tau \chi^+_2 - \nu_\mu e \chi^+_3). \quad (8)$$

$$\mathcal{L}_2 = c_{12} h^T_u \tau_2 h_d (\chi^+_1 + \chi^+_2 + \chi^+_3). \quad (9)$$

It is to be noted that $L$ in $\mathcal{L}_1$ denotes lepton doublets. The neutrino mass matrix comes out as

$$M_\nu = \begin{pmatrix}
     a + 2d/3 & -d/3 & -d/3 - \epsilon \\
     -d/3 & 2d/3 & a - d/3 + \epsilon \\
     -d/3 - \epsilon & a - d/3 + \epsilon & 2d/3
  \end{pmatrix}, \quad (10)$$

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where the $a$ and $d$ terms are obtained in the same way as in the AF model due to higher dimensional operators. The $\epsilon$ terms are the additional contributions from the one-loop radiative diagram as shown in Fig. 1,

$$\epsilon = f m^2 \frac{\bar{c}_{12} v_u}{v_d} F(m_{\chi}^2, m_{h_u}^2),$$

(11)

with the definition,

$$F(M_1^2, M_2^2) = \frac{1}{16\pi^2 (M_1^2 - M_2^2)} \ln \frac{M_1^2}{M_2^2}.$$  

(12)

Figure 1: One loop radiative $\nu_{e,\mu} - \nu_\tau$ mass due to charged Higgs exchange.

Apart from the presence of the scalar singlets $\chi^+_1$, the effective theory below the seesaw scale in this case is analogous to the nonsupersymmetric two-Higgs doublet model (2HDM). The masses $m_{\chi_i}$ are degenerate and have been generically represented as $m_\chi$. Furthermore, the interactions written in Eqs. (8) and (9) can generate corrections to all the off-diagonal entries of $M_{\nu}$, but we retain only the dominant terms proportional to $m_{\tau}^2$. Diagonalizing the neutrino mass matrix given in Eq. (10) with the assumption that $\epsilon$ is small, we obtain the three mass eigenvalues as

$$m_1 = a + d + \epsilon, \quad m_2 = a, \quad m_3 = d - a - \epsilon,$$

(13)

where the parameters are, in general, complex. Model predictions for neutrino oscillations with $U_{13} = 0$ have been discussed for complex values of $a$ and $d$ in Ref. \[4\,5\]. For the sake of simplicity and economy of parameters, we discuss model predictions by treating all the three parameters to be real and then, more generally, by treating only $d$ as complex.

**A) Real parameters**

In this case by treating all the three parameters as real and by solving the eigenvalue equation the following expressions are obtained in the leading approximation with $|\epsilon| << |a|, |d|$, 

$$\sin \theta_{12} = \frac{1}{\sqrt{3}} + \delta_1, \quad \sin \theta_{23} = -\left(\frac{1}{\sqrt{2}} + \delta_2\right), \quad \sin \theta_{13} = \delta_3,$$

(14)
\[\delta_1 = \frac{\epsilon}{d\sqrt{3}}, \quad \delta_2 = \frac{1}{3} \left[ \frac{\epsilon\sqrt{2}}{4a} - \frac{\epsilon}{\sqrt{2}(2a-d)} \right], \]
\[\delta_3 = \frac{1}{3} \left[ \frac{\epsilon\sqrt{2}}{2a} + \frac{\epsilon}{\sqrt{2}(2a-d)} \right].\]  

(15)

We find that it is possible to fit \(\Delta m^2_\odot = m_2^2 - m_1^2\) and \(\Delta m^2_{\text{atm}} = m_3^2 - m_2^2\) if the two parameters are related as
\[d = -\kappa a,\]  

(16)

where \(\kappa\) is a positive rational number. Then
\[m_1 = (1 - \kappa)a + \epsilon, \quad m_2 = a, \quad m_3 = -(1+\kappa)a - \epsilon,\]  

\[\Delta m^2_{\text{atm}} \approx (\kappa^2 + 2\kappa) a^2,\]  

(17)

\[\Delta m^2_\odot = \left[ (2 - \kappa)/(2 + \kappa) \right] \Delta m^2_{\text{atm}} + 2\epsilon(\kappa - 1) \frac{\Delta m^2_{\text{atm}}}{\kappa^2 + 2\kappa},\]  

(18)

where we have used Eq. (17) to determine the parameter \(a\),
\[a = \sqrt{\frac{\Delta m^2_{\text{atm}}}{\kappa^2 + 2\kappa}}.\]  

(19)

In order to estimate the model predictions, we now express \(\epsilon\), mass eigenvalues, and mixing angles in terms of \(\Delta m^2_{\text{atm}}, \Delta m^2_\odot\), and the positive number \(\kappa\),
\[\epsilon = \left( \frac{(\kappa^2 + 2\kappa)\Delta m^2_{\text{atm}}}{2(\kappa - 1)} \right)^{1/2} \left[ \frac{\Delta m^2_\odot}{\Delta m^2_{\text{atm}}} - \frac{2 - \kappa}{2 + \kappa} \right],\]  

(20)

\[m_1 = -(\kappa - 1) \sqrt{\frac{\Delta m^2_{\text{atm}}}{\kappa^2 + 2\kappa}} + \epsilon, \quad m_2 = \sqrt{\frac{\Delta m^2_{\text{atm}}}{\kappa^2 + 2\kappa}}, \quad m_3 = -(\kappa + 1) \sqrt{\frac{\Delta m^2_{\text{atm}}}{\kappa^2 + 2\kappa}} - \epsilon,\]  

(21)

\[\sin \theta_{13} = \frac{\kappa(\kappa + 3)}{6\sqrt{2}(\kappa - 1)} \left[ \frac{\Delta m^2_\odot}{\Delta m^2_{\text{atm}}} - \frac{2 - \kappa}{2 + \kappa} \right] = \frac{1}{\sqrt{3}} - \frac{\sqrt{6}(\kappa + 2)}{\kappa(\kappa + 3)} \sin \theta_{13},\]  

\[\sin \theta_{12} = \frac{1}{\sqrt{3}} - \frac{\kappa + 2}{2\sqrt{3}(\kappa - 1)} \left[ \frac{\Delta m^2_\odot}{\Delta m^2_{\text{atm}}} - \frac{2 - \kappa}{2 + \kappa} \right] = \frac{1}{\sqrt{3}} - \frac{\sqrt{6}(\kappa + 2)}{\kappa(\kappa + 3)} \sin \theta_{13},\]  

\[\tan^2 \theta_{23} = 1 + \frac{\kappa^2}{3(\kappa - 1)} \left[ \frac{\Delta m^2_\odot}{\Delta m^2_{\text{atm}}} - \frac{2 - \kappa}{2 + \kappa} \right] = 1 + \frac{\sqrt{2}\kappa}{(\kappa + 3)} \sin \theta_{13}.\]  

(22)

It is evident from Eqs. (20) to (22) that \(\epsilon\) and corrections to the mixing angles depend upon \(\kappa\) apart from the experimentally determined quantities like \(\Delta m^2_\odot, \Delta m^2_{\text{atm}},\) and the ratio, \(R = \Delta m^2_\odot/\Delta m^2_{\text{atm}}\). While a positive \(\epsilon\) would predict \(\sin \theta_{13} > 0, \sin \theta_{12} < 1/\sqrt{3},\) and \(\tan^2 \theta_{23} > 1,\) a negative value would give \(\sin \theta_{13} < 0, \sin \theta_{12} > 1/\sqrt{3},\) and \(\tan^2 \theta_{23} < 1.\) Since
the tri-bimaximal prediction corresponding to $\sin \theta_{12} = 1/\sqrt{3}$ is just on the border line of the maximal value allowed by the recent global analysis \[4\], the negative values of $\epsilon$ and $\sin \theta_{13}$ which shift $\sin \theta_{12}$ further away are strongly disfavored. We thus search for small and positive values of $\epsilon$ to predict the masses and mixing angles.

We note that the number $\kappa$ is not arbitrary. It is clear from Eq. (18) that the smallness of $\Delta m^2_\odot$ compared to $\Delta m^2_{\text{atm}}$ requires $\kappa \simeq 2$. For larger values of $\kappa > 2.5$ the leading term dominance condition, $|\epsilon| << |a|, |d|$, breaks down. For values of $2.2 < \kappa < 2.5$, the solar mixing angle prediction falls below the present 99% confidence limit and results in $\theta_{12} < 30^\circ$. Thus we use the most plausible value $\kappa = 2$ (i.e. Eq. (6) with $\phi = 0$ and corresponding to the symmetry limit $|m_1| = |m_2|$) and all the parameters in Eqs. (20) to (22) are determined in terms of $\Delta m^2_\odot$, $\Delta m^2_{\text{atm}}$ and their ratio $R$ with

\begin{align*}
\epsilon &= \sqrt{2} \sqrt{\Delta m^2_{\text{atm}} R}, \\
m_1 &= -\frac{1}{2\sqrt{2}} \sqrt{\Delta m^2_{\text{atm}}} + \sqrt{2} \sqrt{\Delta m^2_{\text{atm}}} R, \\
m_2 &= \frac{1}{2\sqrt{2}} \sqrt{\Delta m^2_{\text{atm}}}, \\
m_3 &= -\frac{3}{2\sqrt{2}} \sqrt{\Delta m^2_{\text{atm}}} - \sqrt{2} \sqrt{\Delta m^2_{\text{atm}}} R, \\
\sin \theta_{13} &= \frac{5}{3\sqrt{2}} R, \\
\sin \theta_{12} &= \frac{1}{\sqrt{3}} - \frac{2\sqrt{6}}{5} \sin \theta_{13}, \\
\tan^2 \theta_{23} &= 1 + \frac{4\sqrt{2}}{5} \sin \theta_{13}. 
\end{align*}

Using the allowed range for $\Delta m^2_\odot = (7.2 - 8.9) \times 10^{-3}$ eV$^2$, $\Delta m^2_{\text{atm}} = (1.7 - 3.3) \times 10^{-3}$ eV$^2$, we find $R = (2.2 - 5.2) \times 10^{-2}$, and for $\kappa = 2$, we obtain

\begin{align*}
m_1 &= -0.015 \text{ eV}, \\
m_2 &= 0.017 \text{ eV}, \\
m_3 &= -0.055 \text{ eV}. 
\end{align*}

Thus the mass eigenvalues are normally ordered \[4\]. While there is hierarchy between $m_{1,2}$ and $m_3$ the masses $m_1$ and $m_2$ are nearly quasi-degenerate. The kinematical neutrino mass $|m_{\nu_e}|$ and the effective neutrino mass $|m_{ee}|$ contributing to neutrinoless double beta decay are also small,

\begin{align*}
|m_{\nu_e}| &\simeq |m_{1,2}| \simeq 0.016 \text{ eV}, \\
|m_{ee}| &\simeq 0.01 \text{ eV}, 
\end{align*}

which are beyond the detection limits of planned experiments in near future \[10\] \[11\]. The predictions for mixing angles are

\begin{align*}
\theta_{12} &= 31.13^\circ - 33.5^\circ, \\
\theta_{13} &= 3.5^\circ - 1.5^\circ, \\
\theta_{23} &= 45.5^\circ - 46^\circ. 
\end{align*}

In Eq. (27), the smaller (larger) value of $\theta_{13}$ is correlated with larger (smaller) value of $\theta_{12}$. Although still larger values of $\theta_{13}$ even closer to the CHOOZ upper limit are permitted by
the model they are correlated with smaller values of $\theta_{12} < 30^\circ$ and hence are ruled out even at 99% confidence level. For example with $\kappa = 2.25$ we obtain $\theta_{13} = 6^\circ$, but $\theta_{12} = 29.2^\circ$ which is below the range allowed at 99% level and hence ruled out. Thus the prediction of the angle up to $\theta_{13} \simeq 4^\circ$ is quite natural in this model. The value of $\nu_\mu - \nu_\tau$ mixing angle is also found to increase slightly beyond the tri-bimaximal prediction.

**(B) One complex and two real parameters**

In this case we treat $a$ and $\epsilon$ to be real but $d$ complex with its phase $\phi (= \arg(d))$. In order to maintain the experimentally observed smallness of $\Delta m^2_\odot$ compared to $\Delta m^2_{\text{atm}}$ we use the relation 

$$|d| = -2a \cos \phi. \tag{28}$$

Then in the leading approximation,

$$|m_1|^2 = a^2 - 2\epsilon a(2\cos^2 \phi - 1), \quad |m_2|^2 = |a|^2,$$

$$|m_3|^2 = (1 + 8\cos^2 \phi)|a|^2 + 2\epsilon a(2\cos^2 \phi - 1), \tag{29}$$

$$\Delta m^2_{\text{atm}} = |m_3|^2 - |m_2|^2 \simeq 8a^2 \cos^2 \phi,$$

$$\Delta m^2_\odot = |m_2|^2 - |m_1|^2 = 2\epsilon a(2\cos^2 \phi - 1). \tag{30}$$

Eq. (28) suggests that $\phi$ lies in the second quadrant for positive values of $a$. Eqs. (28) to (30) give

$$|a| = -\sqrt{\Delta m^2_{\text{atm}}/(2\sqrt{2}\cos \phi)}, \quad \frac{\epsilon}{\sqrt{a}} = \frac{4\cos^2 \phi}{2\cos^2 \phi - 1}R, \tag{31}$$

Thus, the masses $|m_1|$, $|m_2|$, $|m_3|$, and the parameters $|d|$, $|a|$ and $\epsilon$ are expressed in terms of $\Delta m^2_{\text{atm}}$, $\Delta m^2_\odot$, $R$ and the phase angle $\phi$. For example with $\phi = 180^\circ$, $\Delta m^2_{\text{atm}} = 2.5 \times 10^{-3}$ eV$^2$ and $\Delta m^2_\odot = 8 \times 10^{-5}$ eV$^2$, $R = 3.2 \times 10^{-2}$ and we obtain the same values of masses as in Eqs. (25) and (26) with normal ordering. With the general expressions for mass eigenvalues given in Eq. (11) with complex $d$, we solve the eigenvalue equation and use Eqs. (28) to (31) to derive the following expressions involving the mixing angles,

$$|\sin \theta_{13}| = \frac{1}{6\sqrt{2}} \frac{|\epsilon|}{|a|} \left( \frac{9 + 16\cos^2 \phi}{1 + 3\cos^2 \phi} \right)^{1/2}$$

$$= \frac{\sqrt{2}}{3} \left( \frac{9 + 16\cos^2 \phi}{1 + 3\cos^2 \phi} \right)^{1/2} \frac{\cos^2 \phi}{|2\cos^2 \phi - 1|}R, \tag{32}$$

$$\sin \theta_{12} = \frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{3}} \frac{|\epsilon|}{|a|}$$
\[ \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \left| 2 \cos^2 \phi - 1 \right| R, \quad (33) \]

\[ \tan^2 \theta_{23} = 1 + \frac{4 \epsilon}{3 |a|} \frac{\cos^2 \phi}{1 + 3 \cos^2 \phi} \]

\[ 1 + \frac{16}{3} \frac{\cos^4 \phi}{|2 \cos^2 \phi - 1| (1 + 3 \cos^2 \phi)} R. \quad (34) \]

In the suitable limit of \(|\cos \phi| \to 1\), Eqs. (32) to (34) go over, as they should, to expressions given in Eq. (22) for the real case with \(\kappa = 2\). We find that interesting solutions bringing down the solar neutrino mixing from the tri-bimaximal limit with \(\theta_{12} < 35.3^\circ\) while increasing \(|\sin \theta_{13}|\) substantially from its zero limit are possible if the phase of the complex parameter \(d\) is in the second quadrant. While \(\cos \phi = -1\) gives predictions on mixing angles as in Eq. (27), Eqs. (32) to (34) can provide substantially different values of mixings for certain other values of the parameter,

\[
\begin{align*}
\cos \phi &= -0.575 : \quad \theta_{12} = 33.5^\circ - 31.2^\circ, \quad \theta_{23} = 45.5^\circ - 46^\circ, \quad \theta_{13} = 1.7^\circ - 3.7^\circ; \\
\cos \phi &= -0.643 : \quad \theta_{12} = 31.5^\circ - 27^\circ, \quad \theta_{23} = 45.7^\circ - 46.6^\circ, \quad \theta_{13} = 4^\circ - 8.5^\circ.
\end{align*}
\]

Thus, the prediction for CHOOZ angle could be larger than 4° if \(\theta_{12} < 31.5^\circ\). For example \(\theta_{13} = 5^\circ\) would require \(\theta_{12} = 30^\circ\) which is already near the 2.5σ limit of global analysis.

The values of light neutrino masses and mixing angles obtained by matching the experimentally observed values of \(\Delta m^2_\odot\) and \(\Delta m^2_{\text{atm}}\) are not likely to change significantly by radiative corrections through renormalization-group (RG) effects \([12]\). This is due to the fact that the mass eigenvalues are small \(|m_i| \simeq O(10^{-2})\) eV. Further \(m_1\) and \(m_2\) have opposite signs and that prevents significant change of \(\sin \theta_{12}\) by RG effects. Although \(m_1\) and \(m_3\) have the same sign, they are not so close to produce significant changes in the predicted values of \(\sin \theta_{13}\). In the 2HDM, the radiative corrections are expected to be further reduced in the region of small values of \(\tan \beta = v_u/v_d \simeq O(1)\). limit of planned reactor and long baseline neutrino experiments \([8]\). The prediction of solar neutrino mixing angle few degrees below the tri-bimaximal value could be tested by precision experiments in near future. Prediction of \(\theta_{13} > 5^\circ\) is possible if the solar neutrino mixing angle is below the 2.5σ limit of the current global analysis. We find that the present model successfully explains the existing solar, atmospheric and CHOOZ experimental results including those from KamLAND and SNO.

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