Article

Mathematical Modelling of Transient Processes in a Three Phase Electric Power System for a Single Phase Short-Circuit

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Abstract: Field approaches are employed to develop a mathematical model of a power network section. The facility consists of two electric power subsystems described with ordinary differential equations and presented as concentrated parameter systems connected with a three-phase power supply line, presented as a distributed parameter system. The model of the electric power line is described with partial differential equations. Mathematically, the supply line model is described utilizing a mixed problem with explicitly indefinite boundary conditions. All electromagnetic state equations of the integrated system are introduced in their matrix-vector forms. The equation of the three-phase long supply line is expressed untraditionally as a system of two first-order differential equations as a function of long line voltage. Since the power supply line is part of the integrated system that includes two subsystems, the boundary conditions at the line’s start and end are implicitly defined, avoiding the traditional application of the Dirichlet first-type boundary condition. An expanded system of ordinary differential equations that describe physical processes in both the supply and loading subsystems is used to calculate the boundary conditions. To this end, third-type boundary conditions, or Poincaré’s conditions, serve to describe the wave equation of the electric power line. Such an integrated model of an electric power system helps analyse transient processes across the supply line when the electric power system is switched on and is single-phase short-circuited at the final point of the electric power line. A comparison of computer simulation results with well-known software packages shows a convergence of approx. 96%.

Keywords: mathematical modelling; long line equation; three-phase power grid; electric network; distributed parameters; transient electromagnetic processes

1. Introduction

The mathematical modelling of transient states in complicated electricity transmission systems is important to stages of both design and operation. Among the many states surveyed in electricity transmission systems, emergency modes must be distinguished connected with short-circuiting, particularly with single-phase short-circuiting. These processes cause interruptions of power supply to users and consequently interference with business operations. Transient electromagnetic processes in single-phase short-circuited electric power networks are often studied as this type of short-circuiting accounts for more than 80% of all single-phase short-circuiting in electric power networks.

Application of mathematical modelling apparatus that does not require costly experimentation on real electric power systems is a possible method of analysing electromagnetic processes. Hundreds of kilometers are analysed in high-voltage power grids. Wave processes are more pronounced in these lines than in lower-voltage networks and thus have considerable effects on physical phenomena. Transient processes across such systems are...
studied in two ways: using circuit-based approaches with equivalent chain diagrams or field-based approaches using wave line equations in the spatiotemporal domain. The field-based method is obviously superior as it helps describe transient processes based on basic laws of applied electrodynamics. In practice, the field-based models are more complex as they are described not only with ordinary differential equations but also partial differential equations. These equations require the determination of boundary conditions for their subsequent integration, which is not always a simple problem. Therefore, we propose here a method of finding third-type boundary conditions for the long-line equation in its matrix-vector form. This approach helps calculate complex physical processes in an electric power system to a high degree of adequacy, on the one hand, and provides for the use of the line model as an autonomous physical object in complicated electric power systems on the other.

To this end, a model of the power supply line is represented as a second-order equation as a function of line voltage. Contrary to the traditional method, where equations comprise two functions (line voltage and current), this approach allows for operations of one function only, considering Poincare’s conditions. An integrated system of equations of an electric network section was solved by discretising spatial derivatives in the line equation using the straight-line method, including successive integration of equations of the electric power line, supply, and loading subsystems using the 2nd order implicit Giro method.

Transient processes in long electric power lines are commonly studied in the literature. Some of them, closest to this paper, will be discussed here.

The article [1] analyses transient states in the overvoltages of two-wire cable lines caused by lightning. The cable line was modelled based on first-order long line partial differential equations without regard to cable resistance and conductance and introduced equivalent quantities of these parameters. Computer simulation was conducted in the CST Studio Suite 2019 environment.

Paper [2] presents a mathematical model of a transmission line based on the methods of “lost elements” and “space state of the line”. The proposed method is a simple and practical procedure for modelling a three-phase transmission line directly in the time domain without explicitly using inverse transforms. The line model also considers frequency-dependent parameters, taking into account the impact of the soil. In addition, the paper suggests applying the procedure of analytical integration of equations of electromagnetic state, which makes it possible to study the transient and steady-state modes of the line operation. The results were verified employing the EMTP-RV software suite.

The authors of [3] suggest using the PSCAD software suite to model a three-phase power supply line with distributed parameters. To reproduce processes in the transmission line, a software tool applies the method of “travelling waves”. This model can be used to model transients during commutations, short circuits, and other modes of line operation. The modelling of various modes is carried out by additional involvement of R, L, C elements.

Implementation of the frequency model of the line in real-time is described in [4]. The stages are described of obtaining and integrating differential equations that model electromagnetic processes in a multiphase power supply line, taking into account parameters dependent on frequency. The model was implemented in the MatLab/Simulink software suite through the S-function programmed in C language.

Analysis of transient processes with the help of MatLab software suite using the Simulink library is also worth noting. For example, a simulation model of a three-phase electric power transmission system was developed in [5]. It can operate in both symmetric and asymmetric states. The line model integrated in the mentioned library was used. An imitation model of an electric power supply system was developed [6]. An original model, included in the general software library, was used to model the transmission line.

Articles [7,8] introduce supply line mathematical models that include cascades of equivalent line circuits in series which address complicated line parameters. The electro-
magnetic state of each cascade is described with ordinary first-order differential equations. Their results were verified using a reverse algorithm of Laplace transformations.

Voltage rise in a high-voltage line is analysed in [9]. The three-phase supply line is represented as a series of equivalent circuits. Computer simulation employed complex software ATP. The same subject matter is addressed in [10] that uses Dynast software.

Paper [11] presents a mathematical model of an ideally transposed three-phase power supply line. The model enabled the calculation of phase non-stationary currents and voltages along the line. The currents and voltages were thus expressed as state equations of an electric equivalent circuit, including phase and interphase line parameters. The line model is analysed in EMTP-RV software. Simulation results were analysed as the supply line was switched on and idle. The article [12] models voltage rises across a 400 kV line at lighting. Transient processes across equipment were analysed to limit the voltage increases. The simulations are undertaken in EMTP-RV using a line model contained in the EMTP-RV software library.

The article [13] presents a mathematical model of a power supply system whose key element is a long line supplied from a power transformer. Two first-order equations, including first-type Dirichlet boundary conditions, assumed a priori as knowns, were employed to model transient electromagnetic processes across the line.

The authors of [14] model a power supply line in the Momentum Simulator. The line was described with first-order partial differential equations, including set boundary conditions. Another line model is suggested in [15], where the mathematical model of a supply line is based on a second-order partial differential line equations. Contrary to ours, that model requires specified boundary conditions.

A mathematical model of a high-voltage supply line is developed in [16], emphasizing the modelling of transmission line supports. Transient processes across the supply line are analysed using two first-order line equations without considering dispersed phase and interphase conductivity. Results of computer simulations, even in the case of a lightning discharge into the supports, are presented. The transient processes were computed in PSCAD, and [17] also uses the PSCAD to model voltage rises in a 500 kV power supply line regarding short-circuiting the electric power grid.

Summarizing the above, we can say that insufficient attention has been paid to mathematical modeling of transients in actual long power transmission lines, while modeling these processes based on simplified approaches is described in sufficient detail. However, these approaches require well-defined boundary conditions regarding the long transmission line equation [3,13–15]. The conditions were determined using analytical integration [2] or equivalent substitutions [7–10]. Concerning the MatLab software package, the distributed line model built into the Simulink library was also simplified. This model does not consider all the distributed parameters of the line [18] to simplify the calculations using the d’Alembert method [1,4–6,11,12,16,17], which may affect the results.

Thus, this paper aims to develop methods for the modeling and analysis of transient electromagnetic processes in long three-phase power supply lines in normal modes and single-phase short circuits.

2. Mathematical Model of an Electric Power Network

A series of input parameters must be taken into maximum possible consideration for the most adequate determination of transient processes in high and ultra-high voltage supply lines. Realising such a goal is obviously not easy, especially if mixed (boundary) problems are solved, that is, systems of ordinary and partial differential equations where a supply line is represented as a distributed parameter system. Calculation of boundary conditions for a line equation not explicitly presented in the general case is the key difficulty. Calculation of boundary conditions obviously requires the complicated apparatus of applied mathematics. Therefore, simplified approaches are used to solve electric power problems, including equivalent circuits of power supply subsystems, including concentrated parameter circuits in series—chain circuits. On the one hand, such an ap-
proach substantially simplifies analysis of transient processes in a power supply line; on the other, the simplifications fail to address an important physical process—electromagnetic wave motion along the line. Electromagnetic waves are known to impede spatiotemporal changes of voltages, currents, powers, etc., in mechanical systems [19–21]. We believe, therefore, that field-based approaches to the analysis of transient processes in supply lines give maximum adequacy.

In the general case of a power supply line operating as an element connecting two subsystems of an electric power network, boundary conditions on line ends are unknown. We suggest applying a method of computing boundary conditions for the equation of a second-order three-phase line described by a system of equations represented in their matrix-vector form. The line model is analysed as autonomous, which allows its use in any analysis of electric power network systems.

Figure 1 shows a general structural diagram of a fragment of an electric power network consisting of two subsystems connected with a supply line of distributed electric parameters.

![Figure 1](image1.png)

Figure 1. Structural diagram of an electric power network supply system.

Calculation of boundary conditions for the supply line equation will be presented in the example of a part of an electric power network with 750 kV default voltage, shown in Figure 2.

![Figure 2](image2.png)

Figure 2. Computing diagram of an electric network power supply system.

The electric power network in Figure 1 consists of an equivalent supply subsystem (electric power system 1), distributed parameter transmission line, and equivalent resistance-induction load (electric power system 2). Parameters of the supply subsystem comprise: electromotive forces emf—$e_S^{(A)}$, $e_S^{(B)}$, $e_S^{(C)}$, inductances—$L_S^{(A)}$, $L_S^{(B)}$, $L_S^{(C)}$, and resistances—$r_S^{(A)}$, $r_S^{(B)}$, $r_S^{(C)}$. The three-phase transmission line and the equivalent line load encompass resistances—$r_L^{(A)}$, $r_L^{(B)}$, $r_L^{(C)}$, and inductances—$L_L^{(A)}$, $L_L^{(B)}$, $L_L^{(C)}$.

Based on Figure 2 and the second Kirchhoff law, equations of voltage and electromotive force balance as well as a differential equation of the power supply line as a function of line voltage [22] are presented in their matrix and vector forms:

\[ \frac{di_S}{dt} = L_S^{-1}(e_S - r_S i_S - u_{BL}) \]
\[ \frac{di_L}{dt} = L_L^{-1}(u_{EL} - r_L i_L) \]
\[ \frac{\partial^2 u}{\partial t^2} = (L_0 C_0)^{-1} \left( \frac{\partial^2 u}{\partial x^2} - (L_0 g_0 + r_0 C_0) \frac{\partial u}{\partial t} - r_0 g_0 u \right) \]
\[ r_S = \text{diag}\left(r_S^{(A)}_S, r_S^{(B)}_S, r_S^{(C)}_S\right); \quad r_L = \text{diag}\left(r_L^{(A)}_L, r_L^{(B)}_L, r_L^{(C)}_L\right); \]
\[ L_S = \text{diag}\left(L_S^{(A)}_S, L_S^{(B)}_S, L_S^{(C)}_S\right); \quad L_L = \text{diag}\left(L_L^{(A)}_L, L_L^{(B)}_L, L_L^{(C)}_L\right) \]
\[ \mathbf{e}_S = \text{colon}\left(e_S^{(A)}_S, e_S^{(B)}_S, e_S^{(C)}_S\right); \quad \mathbf{i}_S = \text{colon}\left(i_S^{(A)}_S, i_S^{(B)}_S, i_S^{(C)}_S\right); \quad \mathbf{i}_L = \text{colon}\left(i_L^{(A)}_L, i_L^{(B)}_L, i_L^{(C)}_L\right) \]
\[ \mathbf{u}_{BL} = \text{colon}\left(u_{BL}^{(A)}, u_{BL}^{(B)}, u_{BL}^{(C)}\right); \quad \mathbf{u}_{EL} = \text{colon}\left(u_{EL}^{(A)}, u_{EL}^{(B)}, u_{EL}^{(C)}\right) \]

where \( i_S^{(A)}, i_S^{(B)}, i_S^{(C)}, i_L^{(A)}, i_L^{(B)}, i_L^{(C)} \)—currents across branches of the supply system and of the equivalent load; \( u_{BL}^{(A)}, u_{BL}^{(B)}, u_{BL}^{(C)}, u_{EL}^{(A)}, u_{EL}^{(B)}, u_{EL}^{(C)} \)—phase voltages on starting and ending points of the line where matrix elements are symmetrical to the central diagonal (for an ideally transposed line); \( r_0, g_0, C_0, L_0 \)—resistance, conductivity, capacitance, and inductance per unit length of the line, respectively; \( g, C \)—phase-to-phase conductivity and capacitance per unit length, respectively; \( M \)—mutual inductance per unit length; \( r_G \)—earth resistance per unit length.

Equation (2) is employed to model transient processes in a three-phase power supply line. This enables handling of only one variable, the supply line voltage \( u(x,t) \).

The most important problem for solving Equation (2) is determining boundary conditions. In the simplest case, the line voltage can be assumed to be known (Dirichlet’s first-type boundary conditions) and measured, e.g., with measurement equipment. Where the line is part of a power system, like in our case in Figure 2, Dirichlet’s conditions are unknown because the voltage function dependent on the state of the power system is not determined. Therefore, the concept of second-type, Poincare conditions [23] is used in this study. This means an integrated system of power system differential equations should be applied to calculate the conditions. Such an approach greatly complicates the modelling, namely, the integration of state equations, especially in the case of matrix-vectorial line equations. For symmetrical states, where the line is part of a single-linear scheme, the method of solving the boundary problem is presented, for example, in [24–26].

### 3. Determinations of Boundary Conditions

The equation of voltage balance for distributed parameter systems [22] is employed in the determination of boundary conditions:

\[ -\frac{\partial u}{\partial x} = L_0 \frac{\partial i}{\partial t} + r_0 i \]  \hspace{1cm} (7)

By discretizing (2) and (7) using the straight-line method and the notion of the central derivative [27], we obtain:

\[ \frac{dv_j}{dt} = (L_0 C_0)^{-1} \left( \frac{1}{\Delta x^2} (u_{j-1} - 2u_j + u_{j+1}) - (L_0 g_0 + r_0 C_0)v_j - r_0 g_0 u_j \right), \]  \hspace{1cm} (8)

\[ \frac{di_j}{dt} = L_0^{-1} \frac{u_{j-1} - u_{j+1}}{2\Delta x} - L_0^{-1} r_0 i_j, \quad j = 1, \ldots, N, \]  \hspace{1cm} (9)

where \( \Delta x \)—discretisation step, \( N \)—number of discretisation nodes.
Equations (8) and (9) for the first discretisation node of a three-phase power supply line are as follows:

\[
\frac{dv_1}{dt} = (L_0C_0)^{-1} \left( \frac{u_0 - 2u_1 + u_2}{(\Delta x)^2} - (L_0g_0 + r_0C_0)v_1 - r_0g_0u_1 \right), \quad \frac{dv_1}{dt} = v_1, \quad (10)
\]

\[
\frac{di_1}{dt} = L_0^{-1}u_0 - u_2 - r_0^{-1}r_0i_1. \quad (11)
\]

An unknown function \(u_0\) in (10) and (11) represents voltage across a fictitious (virtual) discretisation node of the supply line, which prevents a solution to the mixed problem. The computing diagram in Figure 3 serves to find this function.

Figure 3 depicts a phase A computing diagram of an equivalent emf connection with the first discretisation node of a three-phase power supply line. If \(\Gamma\)-similar equivalent circuit of the supply line is employed, the voltage at the line’s start is identical with voltage of the first discretisation node: \(u_{BL} = u_1\).

Starting from the first Kirchhoff law [22], cf. Figures 2 and 3, this will be expressed in the matrix-vector form:

\[
i_S - i_1 - \Delta i = 0; \quad \Delta i = \text{colon} \left( \Delta i^{(A)}, \Delta i^{(B)}, \Delta i^{(C)} \right) \quad (12)
\]

\[
\Delta i^{(A)} = \Delta i^{(AB)} + \Delta i^{(AC)} + \Delta i^{(AG)}; \quad \Delta i^{(B)} = \Delta i^{(AB)} + \Delta i^{(BC)} + \Delta i^{(BG)}; \quad \Delta i^{(C)} = \Delta i^{(AC)} + \Delta i^{(BC)} + \Delta i^{(CG)} \quad (13)
\]

where \(\Delta i^{(m,n)}\) — leakage currents between phases \((m, n)\) (line phases, \(m \neq n\)), \(\Delta i^{(m,G)}\) — leakage currents between phases and earth.

The leakage currents will be produced by the following, based on the first Kirchhoff law [22], cf. Figure 3:

\[
\Delta i = \Delta xC_0 \frac{dv_1}{dt} + \Delta xg_0u_1 \quad (14)
\]

The first equation in (12) and (14) will be differentiated relative to time and considering the initial conditions [27]:

\[
\frac{di_S}{dt} - \frac{di_1}{dt} - \frac{d\Delta i}{dt} = 0; \quad \frac{d\Delta i}{dt} = \Delta xC_0 \frac{dv_1}{dt} + \Delta xg_0v_1 \quad (15)
\]

The first equation from (1), (11), and the second equation from (15) will be substituted to the first equation of (15) to produce:

\[
L_S^{-1}(e_S - r_5i_5 - u_{BL}) - \left( L_0^{-1} \frac{1}{2\Delta x} (u_0 - u_2) - L_0^{-1}r_0i_1 \right) - \left( \Delta xC_0 \frac{dv_1}{dt} + \Delta xg_0v_1 \right) = 0 \quad (16)
\]
The first Equation from (10) written for the first discretisation node of the supply line \((j = 1)\) will be substituted to (16) to produce:

\[
L_S^{-1}(e_S - r_S i_S - u_{BL}) - \left( L_0^{-1} \frac{1}{\Delta x} (u_0 - u_2) - L_0^{-1} r_0 i_1 \right) - \\
\left( \Delta x C_0 \left( (u_0 - 2u_1 + u_2) - (L_0 g_0 + r_0 C_0) v_1 - r_0 g_0 u_1 \right) \right) + \Delta x g_0 v_1 = 0
\]

(17)

Considering \(u_{BL} = u_1\), the end result is:

\[
u_0 = \frac{2}{3} \Delta x \left( \frac{2}{\Delta x} + \Delta x r_0 g_0 - L_0 L_S^{-1} \right) u_1 - \frac{1}{2} u_2 + \\
+ \frac{2}{3} \Delta x \left[ \Delta x r_0 C_0 v_1 + r_0 i_1 + L_0 L_S^{-1} (e_S - r_S i_S) \right] ,
\]

(18)

\(\Delta x C_0 (L_0 C_0)^{-1} - C_0 C_0^{-1} L_0^{-1} = L_0^{-1}\)

This (18) helps to find \(u_0\) of the fictitious discretization node of (8) and (9).

A similar problem is then approached for the line’s end (Figure 2). Since an asymmetrical \(\Gamma\)-similar equivalent line circuit is applied to solving the mixed problem and calculating boundary conditions, methods for calculating these conditions at the line’s start and end are different. A method of computing boundary conditions at the line’s end needs to be developed. (8) and (9) for the last discretisation node, \(j = N\), will be written in the discrete space:

\[
\frac{d v_N}{d t} = (L_0 C_0)^{-1} \left( \frac{u_{N-1} - 2u_N + u_{N+1}}{\Delta x^2} - (L_0 g_0 + r_0 C_0) v_N - r_0 g_0 u_N \right), \quad \frac{d u_N}{d t} = v_N; \tag{19}
\]

\[
\frac{d i_N}{d t} = L_0^{-1} \frac{u_{N-1} - u_{N+1}}{2 \Delta x} - L_0^{-1} r_0 i_N. \tag{20}
\]

Equations (19) and (20) indicate a comparable problem of calculating a function of virtual voltage across a fictitious node: \(u_{N+1}\). The method of finding this voltage is described and tested, but is imperfect because when mathematical models are formulated given different configurations of line connections with other elements of electric power systems, the fictitious voltage \(u_{N+1}\) will constantly change. This, in turn, creates certain inconvieniences in the further organization of the model in the form of a program code for analysis of transient processes. Therefore, we suggest introducing \(u_{EL}\) voltage to universalise and autonomise the mathematical model of the line (see Figures 2 and 4).

![Figure 4](image)

Figure 4. Computing diagram for voltage determination across a fictitious node at the line’s end.

Figure 4 shows an equivalent circuit of the final discrete line section and equivalent active induction load for phase A.

The fictitious voltage \(u_{N+1}\) is found as follows. For the last discrete contour of the line (see Figure 4), we write the equation according to the second Kirchhoff law:

\[
\frac{d i_N}{d t} = \Delta x L_0^{-1} (u_N - \Delta x R_0 i_N - u_{EL}) \tag{21}
\]
Let us view Equation (20) written for the last discretization node \((j = N)\) and (21).

\[
\mathbf{L}_0^{-1} \left( \frac{1}{2\Delta x} (\mathbf{u}_{N-1} - \mathbf{u}_{N+1}) - \mathbf{R}_0 \mathbf{i}_N \right) = [\Delta x \mathbf{L}_0]^{-1} (\mathbf{u}_N - \Delta x \mathbf{R}_0 \mathbf{i}_N - \mathbf{u}_{EL}) \tag{22}
\]

By expressing the voltage function of the fictitious node \(\mathbf{u}_{N+1}\) from (22), we can write:

\[
\mathbf{u}_{N+1} = \mathbf{u}_{N-1} - 2(\mathbf{u}_N - \mathbf{u}_{EL}) \tag{23}
\]

Thus, using (23) as a function of the fictitious voltage \(\mathbf{u}_{N+1}\), it is possible to avoid its replacement when changing the configuration of the power supply line connection circuit with other elements of the electric power system, which makes the mathematical model of the line autonomous and universal. However, \(\mathbf{u}_{EL}\) needs to be found now.

The current \(\mathbf{i}_N\) is identical to \(\mathbf{i}_L\) (see Figure 4). Therefore, considering the initial conditions, we can write (see the second expression in (1) and (21)) as follows:

\[
\mathbf{i}_L = \mathbf{i}_N \Rightarrow \frac{d\mathbf{i}_L}{dt} = \frac{d\mathbf{i}_N}{dt} \Rightarrow \mathbf{L}_L^{-1} (\mathbf{u}_{EL} - \mathbf{r}_L \mathbf{i}_L) = [\Delta x \mathbf{L}_0]^{-1} (\mathbf{u}_N - \Delta x \mathbf{r}_0 \mathbf{i}_N - \mathbf{u}_{EL}) \tag{24}
\]

The phase voltage at the line's end is finally produced in its matrix-vector form:

\[
\mathbf{u}_{EL} = \left[ \mathbf{L}_L^{-1} + [\Delta x \mathbf{L}_0]^{-1} \right]^{-1} \left[ [\Delta x \mathbf{L}_0]^{-1} (\mathbf{u}_N - \Delta x \mathbf{r}_0 \mathbf{i}_N) + \mathbf{L}_L^{-1} \mathbf{r}_L \mathbf{i}_L \right] \tag{25}
\]

The system of differential Equations (1), (8) and (9) is subject to joint integration noting (3)–(6), (13), (18), (23) and (25).

This method of calculating Poincare boundary conditions for the long line equation in its matrix-vector form, combined with searching for an input and output function of the line voltages, significantly expands applications of the line model as an autonomous object. This approach helps analyse complicated transient processes in electric power supply systems across a wide range of system states, including all asymmetrical states, with very high accuracy.

4. Computer Simulation Results

A computer simulation was performed to study transient electromagnetic processes in the electrical system of power transmission presented in Figure 2, where the supply line is switched on for asymmetric loading and further operation in the mode of a single-phase short circuit at the end of the line. The simulation was performed as follows: at the time \(t = 0\) s, a voltage was applied to the initial section of the line given appropriate smf values. After the system switched to the steady-state mode, a single-phase short circuit to ground was simulated at the end of the power supply line in phase \(A\) at \(t = 0.1\) s.

We investigated short-circuiting at the end of the line in two cases: a short circuit 18 km away from the line load and directly at the end of the line. With these two different short circuits, different processes take place in the line. In the first case, zeros were assigned to elements of the load branch of phase \(A\) (short circuit on the load, \(r_L^{(A)} = 0\) Ω, \(L_L^{(A)} = 0\) H). In the other case, the short circuit was simulated by equating the phase voltage (phase \(A\)) at the end of the line to zero (\(u_{EL} = 0\) kV), with \(r_L^{(A)}\), \(L_L^{(A)}\) not equal to zero.

The following parameters of the equivalent circuit elements (Figure 2) correspond to the parameters of a real 360 km long transmission line 750 kV which connects the Substation Western Ukrainian with the Substation Vinnytsia. The parameters of the long line and the equivalent load are listed in Table 1.
Table 1. The parameters of the long line and the equivalent load.

| Parameter                                      | Value                          |
|------------------------------------------------|-------------------------------|
| Line length, $l$                               | 360 km                        |
| Resistance per unit length of the line, $r_0$ | $1.9 \times 10^{-5} \Omega/m$ |
| Inductance per unit length of the line, $L_0$  | $1.665 \times 10^{-6} H/m$    |
| Capacitance per unit length of the line, $C_0$ | $1.0131 \times 10^{-11} F/m$  |
| Conductivity per unit length, $g_0$            | $3.25 \times 10^{-11} Sm/m$   |
| Capacitance per unit length, $C$               | $1.0122 \times 10^{-12} F/m$  |
| Phase-to-phase conductivity per unit length, $g$| $3.25 \times 10^{-13} Sm/m$   |
| Earth resistance per unit length, $r_G$        | $5 \times 10^{-7} \Omega/m$   |
| Mutual inductance per unit length, $M$         | $7.41 \times 10^{-7} H/m$     |
| Resistances of the supply subsystem, $r_L^{(A)}$, $r_L^{(B)}$, $r_L^{(C)}$ | $400 \Omega, 350 \Omega, 300 \Omega$ |
| Inductances of the supply subsystem, $L_L^{(A)}$, $L_L^{(B)}$, $L_L^{(C)}$ | $0.8 \text{ H}, 0.71 \text{ H}, 0.61 \text{ H}$ |

Computer simulation is performed with the following mode parameters: $e_S^{(A)} = 643 \sin(\omega t) \text{kV}$, $e_S^{(B)} = 643 \sin(\omega t - 120^\circ) \text{kV}$, $e_S^{(C)} = 643 \sin(\omega t + 120^\circ) \text{kV}$. The step of spatial discretization of partial differential equations using the straight-line method is equal to $\Delta x = l/20 = 360/20 = 18 \text{ km}$. Conventional differential equations were integrated with the Gear method with a step of $\Delta t = 27 \mu s$.

Since we analyze three-phase modes, all distributions of voltage and current functions considered in this paper are denoted as follows: phase $A$—yellow lines, phase $B$—green lines, and phase $C$—red lines.

Spatial distributions of phase current in the line at times $t = 0.001 \text{s}$, are presented in Figure 5.

![Figure 5](image-url)  
**Figure 5.** Space distribution of current changes at $t = 0.001 \text{s}$.

We can see from Figure 5 that at $t = 0.001 \text{s}$, the current of phase $A$ at the beginning of the line is 600 A, gradually decreasing along the line to zero. The current at the end of the line is zero because the electromagnetic wave in such a short time had not yet reached the end of the line. The currents of phases $B$ and $C$ were the same (overlapping green and red lines) and present in the line only due to electromagnetic induction because the voltages of these phases at the beginning of the line were still zero (phases $B$ and $C$ are not yet switched on).

Spatial distributions of phase current in the line at $t = 0.004 \text{s}$ are presented in Figure 6.
Figure 6. Space distribution of current changes across the line at $t = 0.004$ s.

Figure 6 shows at $t = 0.004$ s, the current of phase A at the beginning of the line is 1000 A, closer to the middle of the line with the minimum value of 750 A, and at the end of the line 800 A. The current of phase C at the beginning of the line is $-400$ A, in the middle of the line it is almost zero, and at the end of the line $-150$ A. Since at $t = 0.004$ s phase B is not yet on, the current of phase B is already induced by phases A and C and has an almost linear spatial distribution 150 A at the beginning of the line, and $-150$ A at the end of the line.

Spatial distributions of phase current functions in the line at $t = 0.008$ s are presented in Figure 7.

Figure 7. Space distribution of current changes across the line at $t = 0.008$ s.

In Figure 7, we see that at $t = 0.008$s, spatial distributions of the phase currents in the line are almost linear. The current of phase A at the beginning of the line is 0.4 kA, increasing to 1.5 kA at the end. The current of phase B at the beginning of the line has a value of 1 kA, gradually decreasing along the line to zero, because after switching on the phase, the electromagnetic wave had not yet reached the end of the line. The current of
phase C at the beginning of the line is \(-1\) kA, and in the middle and at the end of the line \(-1.2\) kA.

Figures 8 and 9 show the transient processes of the currents in the middle of the line and phase voltages at the last discrete node of the line (18 km away from the end of the line) during its switch-on, and subsequent transition to a single-phase short circuit mode in phase A at the end of the line for the first case.

Since the line loads are asymmetric, amplitudes in the steady-state of the phase currents have different values: the current of phase A has an amplitude of \(1.11\) kA, phase B \(-1.22\) kA, and phase C \(-1.36\) kA. After a single-phase short circuit in phase A at the end of the line \((t = 0.1\) s\), the shock current of this phase in the middle of the line reaches \(6.6\) kA, and the steady-state amplitude of the current is \(4.25\) kA. Due to the short circuit in phase A, the steady-state amplitude of the current of phase C increases to \(1.86\) kA, and the current of phase B remains unchanged.
When we analyze transients of phase voltages on the last node of the line discretization (see Figure 9), we see that after the studied system enters the steady-state of the voltage on the last discrete node, the lines have the following amplitudes: phase $A$ $-618$ kV, phase $B$ $-590$ kV, phase $C$ $-566$ kV. After the short circuit in phase $A$, the steady-state amplitude of the voltage of this phase decreases to $33$ kV, and the voltages of the other phases increase to $643$ kV and $727$ kV of phases $B$ and $C$, respectively. The single-phase short circuit at the end of the line in phase $A$ causes an overvoltage in phase $C$, reaching 1.13 of the maximum voltages compared to phase $A$.

Figures 10 and 11 show the same transient as Figures 8 and 9, but for the second case.

**Figure 10.** Transient currents in the middle line section for the second experiment.

**Figure 11.** Transient phase voltages in the final discretisation node (18 km away from the line’s end) for the second experiment.

It does not make sense to analyze transient processes when the line is switched on for the second case because they are identical to the first case until the short circuit, therefore, we present an analysis of transients after the single-phase short circuit in phase $A$. 
As Figure 10 shows, the shock current of phase A in the middle of the line reaches 8.73 kA, the steady-state amplitude is 5.31 kA. In the second case, aperiodic components occur in the currents of the “healthy” phases, which produces shock currents. Thus, the shock current of phase B reaches 2.6 kA, and after attenuation of the aperiodic component, its amplitude returns to its level before the emergency. The shock current of phase C reaches the value of 4.14 kA, and the steady-state amplitude of the current after a short circuit increases to reach 3.07 kA.

When we analyze the phase voltages on the last discrete node of the line (see Figure 11), we see that the steady-state amplitude of the voltage of phase A after the short circuit for the second case is the same as for the first one, while the voltage amplitudes of the remaining phases decrease to −520 kV and 587 kV of phases B and C, respectively.

Figure 12 shows the time-space distribution of the phase A voltage when the line is switched on, \( t \in (0; 0.02) \) s.

![Figure 12. Space-temporal distribution of phase A voltage at the time \( t \in (0; 0.02) \) s.](image)

The voltage of phase A has maximum oscillations at the end of the line, while there are no voltage oscillations at the beginning of the line. Figures 13 and 14 show the space-time distributions of phase B voltage and current when the line is switched on, \( t \in (0; 0.02) \) s.

![Figure 13. Space-temporal distribution of phase B voltage at the time \( t \in (0; 0.02) \) s.](image)

![Figure 14. Space-temporal distribution of phase B current at the time \( t \in (0; 0.02) \) s.](image)
We see that while phase B is switched off, its voltage is absent, but due to electromagnetic induction in this phase, there is a small voltage (see Figure 13). The same applies to the current (see Figure 14).

We can see from the Figures that the voltage of phase B has the largest oscillations at the end of the line (see Figure 13), identical to the phase B current in Figure 14—at the beginning of the line. It is expedient to analyse Figures 12–14 and Figures 5–11.

5. Verification of the Results

The results obtained based on the mathematical model were compared with those derived from the well-known software RE, developed at the Department of Power Engineering and Control System of the National University «Lviv Polytechnic». The RE software package was developed at the Lviv Polytechnic State University. It is based on equivalent circuits of a long line. Differential equations were integrated using Giro second-order method [28].

In the RE software package, the investigated line is presented as twenty-three-phase \( \Gamma \)-substituted schemes of elementary sections in series. The differential equation of the line was discretised in twenty parts. The first case of computer simulation was considered during the verification. In Figure 15, yellow and red lines represent the time distributions of phase A and C currents in the middle of the line, which were obtained based on our mathematical model.

![Figure 15. Temporal distributions of phase A and C current in the middle line section.](image)

Phase B current is not shown for the sake of clarity of Figure 15. Black lines show the same currents as those obtained from RE software. Analysis of the presented results (Figure 15) proves the simulations using the RE software and our mathematical model are nearly identical. Analytical calculations of reliability showed the results converged 96% of the time, therefore, the model can be concluded to give adequate results.

6. Conclusions

Application of the laws of electrodynamics makes it possible to build mathematical models of long power supply lines at the field level, which, in contrast to equivalent circuit diagrams widely used in the literature, addresses wave processes in facilities with a high degree of adequacy.

Application of Dirichlet first-type boundary conditions to the differential equation of the second-order long line does not allow for solving this equation since voltages at the line start and end as part of the electric power network are unknown. Poincaré third-type
boundary conditions were used to solve this problem and the line equation, which also helps describe transient processes in an electric power network.

To solve the differential equation of a distributed parameter long line, the second and third-type Neumann and Poincare boundary conditions were used, which improved the efficiency of the quantitative picture of transients in an object, thus avoiding time-consuming and complex search procedures for unknown functions at the beginning and end of the line, meaning the search for boundary conditions of the first Dirichlet kind.

The proposed method of searching for functions of input and output voltages at the beginning and the end of a long line significantly expands the scope of the line model as an autonomous object of the general model of an electrical power transmission system.

Based on a comparative analysis of our computer simulation results of mathematical models with the practical results obtained using the well-known software RE, we may conclude they show a sufficient similarity of 96%.

The subject matter of this paper will be used in further research on the joint operation of turbogenerators, unit transformers, switching facilities, and ultra-high voltage long transmission lines.

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