One-Dimensional Lieb–Oxford Bounds

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We here prove a conjectured Lieb–Oxford bound in one dimension. It is established that
\[ I(\psi) \geq -C_1 \int_\mathbb{R} \rho_\psi(x)^2 dx \]
with \( C_1 = 1 \), where \( I(\psi) \) is the indirect Coulomb energy for interacting
electrons in one dimension modelled with a Dirac (contact) potential and \( \rho_\psi \) is the one-body particle
density of a wave function \( \psi \). This bound follows the general form \( I(\psi) \geq -C_d \int_\mathbb{R} \rho_\psi(x)^{1+\frac{d}{2}} dx \)
for a \( d \)-dimensional quantum system previously proven for \( d = 2, 3 \). The additional term \( \int_\mathbb{R} \rho_\psi(x)^{2} \ln(C/\epsilon \rho_\psi(x)) dx \)
for a Lieb–Oxford bound using a convex version of the soft Coulomb potential, where \( \epsilon \) denotes the softening parameter, is also derived.

I. INTRODUCTION

The Lieb–Oxford bound is an important result in quantum physics that gives a lower bound of the indirect interaction energy and is related to the stability of matter [1, 2]. It states for Coulomb systems that the indirect interaction energy is related to the stability of matter physics that gives a lower bound of the indirect interaction energy and is related to the stability of matter [1, 2]. It states for Coulomb systems that the indirect interaction energy and is related to the stability of matter physics that gives a lower bound of the indirect interaction energy and is related to the stability of matter. Crucial for the case \( d = 2 \) is the Lieb–Oxford bound [3]. Moreover, for two dimensions, the Lieb–Oxford bound follows the general form \( I(\psi) \geq -C_2 \int_\mathbb{R} \rho_\psi(x)^{3/2} dx \) has been proven by Lieb, Solovej and Yngvason [3]. In the work of Räsänen et al. [2], an argument based on universal scaling properties was used to conjecture that in general for dimensions \( d = 1, 2, 3 \),
\[ I(\psi) \geq -C_d \int_\mathbb{R} \rho_\psi(x)^{1+\frac{d}{2}} dx, \quad C_d > 0. \tag{1} \]
Note that Eq. (1) agrees with the proven results for \( d = 2, 3 \). Furthermore, based on the argument that a good estimate of \( C_d \) is obtained from the amount of correlation in the infinite \( d \)-dimensional homogeneous electron gas in the low-density limit, Ref. [3] provided improved bounds for \( d = 2, 3 \) and a proposal for a one-dimensional bound. Crucial for the case \( d = 1 \) is that the Coulomb potential \( v(r) = r^{-1} \) is too singular. Thus, for \( d = 1 \) an interaction potential first has to be chosen before defining the indirect interaction energy \( I \).

To the best of the author’s knowledge, a proof of a Lieb–Oxford bound of the form \( I(\psi) \geq -C_d \int_\mathbb{R} \rho_\psi(x)^{1+\frac{d}{2}} dx \) has not yet been presented in the one-dimensional case. Addressing this issue, we here prove the conjecture of Räsänen et al. [2] by demonstrating that Eq. (1) holds with \( C_1 = 1 \) when \( d = 1 \) for interactions modelled by a Dirac potential. In the following we also consider a convex version of the soft Coulomb potential and derive the term \( \int_\mathbb{R} \rho_\psi(x)^{2} \ln(C/\epsilon \rho_\psi(x)) dx \) for, respectively, the Dirac potential \( v(x) = \eta \delta(x) \) and the soft Coulomb potential \( v(x) = (x^2 + \epsilon^2)^{-1/2} \) modelling the interaction \( \sum_{i>j} v(|x_i - x_j|) \). This was also confirmed by Seidl, Räsänen and Gori–Giori [3] for finite homogeneous electron gas in the strong interaction limit.

The study of Lieb–Oxford bounds in one dimension is not only a fundamental question in itself but also useful for different models and applications in low-dimensional physics. For example, the local-density approximation of the exchange-correlation term in density-functional theory has been applied to reproduce features of the Luttinger liquid [12], which replaces Fermi liquid theory in one dimension, making rigorous one-dimensional density-functional constraints (as in three dimensions [13]) relevant. Moreover, one-dimensional Lieb–Oxford bounds can also be applicable for confined higher-dimensional systems [14] and, as noted in Ref. [3], there is a crossover between one- and two-dimensional bounds.

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II. LIEB–OXFORD BOUNDS IN ONE DIMENSION

A. Prerequisite

We will assume that the $N$-particle wave function $\psi(x) \in L^2((\mathbb{R} \times \{|\uparrow, \downarrow\})^N)$ is normalized, i.e., that $|||\psi|||_2 = 1$ holds. The one-body particle density associated with a wave function $\psi$ is obtained from

$$\rho_\psi(x) = N \sum_{q \in \{\uparrow, \downarrow\}} \int_{\mathbb{R}^{N-1}} |\psi(x, q_1, x_2, q_2, \ldots, x_N, q_N)|^2 \times dx_2 \cdots dx_N.$$ 

Another natural assumption is that the wave function $\psi$ has finite kinetic energy, i.e.,

$$\mathcal{K}(\psi) = \frac{1}{2} \sum_{q \in \{\uparrow, \downarrow\}} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx_1 \cdots dx_N$$

is finite. If for $\psi$ we have $\mathcal{K}(\psi)$ finite, then $\psi$ is an element of the Sobolev space $H^1$. By the Hoffmann–Ostenhof inequality, finite kinetic energy (and $L^2$ normalization) of $\psi$ implies $\rho_\psi^{1/2} \in H^1(\mathbb{R})$. The Sobolev inequality in one dimension (see e.g. Theorem 8.5 in [16])

$$2||f||_\infty^2 \leq ||f||_2^2 + \left|\frac{df}{dx}\right|_2^2,$$

gives with the choice $f = \rho_\psi^{1/2}$ that (by interpolation) $\rho_\psi \in L^p(\mathbb{R})$ for all $p \in [1, \infty]$. In particular, we have $||\rho_\psi||_1 = \int_\mathbb{R} \rho_\psi(x) dx = N$.

Let the mutual repulsion be modelled by a potential $v : \mathbb{R}^+ \to \mathbb{R}$. For an $N$-electron wave function $\psi(x_1, q_1, \cdots, x_N, q_N)$ defined on $(\mathbb{R} \times \{\uparrow, \downarrow\})^N$, the indirect interaction energy is defined by

$$I(\psi) = \langle \psi \xi \sum_{i<j} (v(|x_i - x_j|)\xi - D(\rho_\psi, \rho_\psi).$$

In Eq. (2), $D(\rho_\psi, \rho_\psi)$ denotes the direct part (or Hartree term) of the interaction energy, i.e.,

$$D(\rho_\psi, \rho_\psi) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\psi(x) \rho_\psi(y) v(|x - y|) dx dy.$$ 

The rest of this work will be dedicated to obtaining a constant $C_1$ (or sometimes constants) such that $I(\psi)$ is bounded below by $-C_1 \int_\mathbb{R} \rho_\psi(x)^2 dx$ (or variations thereof).

B. Conjectured Lieb–Oxford bound

Based on physical arguments, Ref. [8] conjectured that in one dimension the indirect interaction energy satisfies

$$I(\psi) \geq -C_1 \int_\mathbb{R} \rho_\psi(x)^2 dx$$

(3)

with $C_1 = 1/2$ for the choice of a Dirac potential $v = \eta \delta$, with $\eta > 0$ a constant. Their argument was based on that

$$I(\psi) \geq -\frac{C_1}{A_1} E^{\text{LDA}}_\infty(\rho),$$

(4)

where $E^{\text{LDA}}(\rho) = A_1 \int_\mathbb{R} \rho(x)^2 dx$ is the exchange energy for a homogeneous gas in one dimension. For $v = \eta \delta$, one has $A_1 = \eta/4$. (Note that $E^{\text{LDA}}(\psi)$ in the notation of Ref. [9] is the exchange-correlation energy that equals $I(\psi)$ plus the non-negative contribution from correlation kinetic energy, and our $I(\psi)$ is denoted $\mathcal{W}(\psi)$ in Ref. [9].) In the limit $r_\delta = 1/(2\eta) \to \infty$, it was argued that $I(\psi)/E^{\text{LDA}}_\infty(\rho)$ approaches $X_1 = 2$ and by Eq. (3) it then follows that $C_1 = \eta/2$. This was also confirmed for a finite homogeneous and strictly correlated electron gas in the limit $N \to \infty$ by Seidl, Räsänen and Gori–Giori [10].

In Refs. [9, 10], the bound in Eq. (3) was supported by the study of the soft Coulomb potential (as well as a regularized Coulomb potential in [10]). The soft Coulomb potential, with a softening parameter $\varepsilon > 0$, is given by

$$v_\varepsilon(r) = \frac{1}{\sqrt{r^2 + \varepsilon^2}}, \quad r \geq 0.$$ 

Using that $A_1 = 1/2$ for $v = v_\varepsilon$, Ref. [9] obtained by the same argument as for the Dirac potential a modified one-dimensional Lieb–Oxford bound

$$I(\psi) \geq -C_1 \int_\mathbb{R} \rho_\psi(x)^2 \left( \frac{\varepsilon K_1}{\varepsilon + \rho_\psi(x)} \right) dx,$$ 

(6)

with $C_1 = 1$, $K_1 = 3/2 - \mu$, and $K_2 = 2/\pi$ and where $\mu = 0.577$ is Euler’s constant (see also Refs. [10, 11]). Based on the physical arguments in both Refs. [9, 10], the constant $X_1 = C_1/A_1 = 2$ was obtained for Dirac and soft Coulomb potential. However, the potentials have different values of the exchange constant $A_1$.

We will in the next section prove Eq. (3) and a weaker version of Eq. (6), in both cases with larger constants than in Ref. [9]. For Eq. (3) we bound just the direct term of the interaction energy and obtain $C_1 = 1$, which by no means is proven to be optimal. To obtain Eq. (6), with $\int_\mathbb{R} \rho_\psi(x)^2 \ln(K_2/(\varepsilon \rho_\psi(x))) dx$ instead of $\int_\mathbb{R} \rho_\psi(x)^2 \ln(K_2/(\varepsilon \rho_\psi(x))) dx$, we use a bound of Hainzl and Seiringer [14] and apply it to a convex version of the soft Coulomb potential.

C. Proof of the conjectured Lieb–Oxford bound

We will in this section present our main result that Eq. (5) holds with $C_1 = 1$ for $v = \delta$. We begin by noting
some useful facts from Ref. \[14\] based on a generalization of the Fefferman–de la Llave decomposition. Let \(\chi_r(x) = \Theta(r - |x|)\) with \(\Theta\) denoting the Heaviside step function and introduce (following the notation of Ref. \[14\])

\[
\alpha_\psi(r, z) = \langle \psi \| \sum_{i=1}^{N} \chi_r(x_i - z)\| \psi \rangle = \int_{z-r}^{z+r} \rho_\psi(x) dx.
\]

**Remark 1.** In Ref. \[14\] the function \(\alpha_\psi\) is bounded by \(2r M \rho_\psi\), where

\[
(M f)(x) = \sup_{r>0} \left\{ \frac{1}{2r} \int_{|x-y|<r} f(y) dy \right\}
\]

is the Hardy–Littlewood maximal function. This bound is used to obtain Lemma 2 in Ref. \[14\].

**Lemma 1** (Hainzl and Seiringer). Let \(v(r), r \geq 0\), have distributional derivatives \(v'\) and \(v''\) and suppose \(v(r)\) and \(rv'(r) \rightarrow 0\) as \(r \rightarrow +\infty\). Then

\[
v(r) = 2 \int_0^\infty v''(2s)f_r(s)ds,
\]

where \(f_r(s) = \max(0, 2s - r)\). In particular, for \(v \in L^2\) with \(\|\psi\|_2 = 1\) we have

\[
D(\rho_\psi, \rho_\psi) = \int_0^\infty \int_\mathbb{R} v''(2r)\alpha_\psi(r, z)^2dzdr.
\]

**Proof.** By Lemma 1 in Ref. \[14\] and the discussion that follows directly after the lemma, we integrate by parts (where the derivatives are interpreted in the sense of distributions) to obtain Eq. (7). That the direct term \(D(\rho_\psi, \rho_\psi)\) is given by Eq. (8) follows from Eq. (13) in Ref. \[14\].

**Theorem 2** (Proof of the conjectured bound in Eq. \[9\] with \(C_1 = 1\)). Let \(v = \delta\), where \(\delta\) denotes the Dirac delta “function”. Then for \(\psi \in L^2\) with norm equal one, \(C_1 = 1\).

\[
I(\psi) \geq -C_1 \int_\mathbb{R} \rho_\psi(z)^2dx, \quad C_1 = 1.
\]

**Proof.** First, set \(b_r(x, z) = 1\) if \(|x - z| \leq r\), otherwise set \(b_r\) equal zero. Since \(b_r = b_r^2\), we have by the Cauchy–Schwarz inequality

\[
\alpha_\psi(z, r)^2 = \left( \int_\mathbb{R} \rho_\psi(x)b_r(x, z)^2dx \right)^2
\]

\[
\leq 2r \int_\mathbb{R} \rho_\psi(x)^2b_r(x, z)dx.
\]

From Eq. (2) we obtain \(I(\psi) \geq -D(\rho_\psi, \rho_\psi)\). Using Eq. (8) of Lemma 1 with \(v = \delta\) and Eq. (10), we obtain

\[
I(\psi) \geq -D(\rho_\psi, \rho_\psi)
\]

\[
\geq -\int_0^\infty v''(2r)2r \int_\mathbb{R} \rho_\psi(x)^2 \left( \int_\mathbb{R} f_r(x, z)dz \right) dxdr
\]

\[
= -\int_0^\infty v''(2r)(2r)^2 \int_\mathbb{R} \rho_\psi(x)^2dx.
\]

The calculation \(\int_0^\infty v''(r)r^2dr = 2\) gives the claim in Eq. (9). \(\square\)

**Remark 2.** Note that for the scaled Dirac potential \(v = \eta\delta\) the proof above gives the constant \(C_1 = \eta\), i.e., twice as large as the proposed constant of Räisänen et al. \[14\].

**Remark 3.** Theorem 2 holds also for approximate contact potential. For example, let \(v \in C^0\), be defined by

\[
v_\varepsilon(r) = \begin{cases} \frac{2}{\varepsilon} - \frac{2}{\varepsilon}r , & 0 \leq r \leq \varepsilon, \\ 0 , & \varepsilon < r, \end{cases}
\]

then for all \(\varepsilon > 0\) we have for \(v = v_\varepsilon\) the lower bound \(I(\psi) \geq -\int_\mathbb{R} \rho_\psi(x)^2dx\).

To obtain a bound of the form given by Eq. (6), we first make the soft Coulomb potential convex. Set for \(r \geq 0\)

\[
v_\varepsilon^{SC}(r) = v_\varepsilon^{SC}(r + r_\varepsilon), \quad \varepsilon > 0,
\]

where \(r_\varepsilon = \varepsilon/\sqrt{2}\) and \(v_\varepsilon^{SC}\) given by Eq. (5). A direct calculation shows that the derivatives satisfy

\[
(v_\varepsilon^{SSC}(r)')' = \frac{1}{r + r_\varepsilon} \frac{r + r_\varepsilon}{(r + r_\varepsilon)^2 + \varepsilon^2}^{3/2},
\]

\[
(v_\varepsilon^{SSC}(r)'')'' = \frac{2(r + r_\varepsilon)^2 - \varepsilon^2}{((r + r_\varepsilon)^2 + \varepsilon^2)^{5/2}}.
\]

Thus, \(v_\varepsilon^{SSC}(r)\) is convex.

**Theorem 3** (Convex soft Coulomb potential). Let \(v = v_\varepsilon^{SC}\), where \(v_\varepsilon^{SC}\) denotes the convex soft Coulomb potential with parameter \(\varepsilon\) given by Eq. (12). For \(\psi \in L^2\) with norm equal one and finite kinetic energy \(K(\psi)\), we have

\[
I(\psi) \geq -C_1 \int_\mathbb{R} \rho_\psi^2 \left( K_1 + \frac{1}{\varepsilon} \ln \left( \frac{K_2}{\varepsilon \rho_\psi} \right) \right) + dx,
\]

where \(C_1 = 16\), \(K_1 = 1 + \ln(2e^3 + 2) = 4.742\), and \(K_2 = 2\sqrt{2e} = 7.689\).

**Remark 4.** Theorem 3 establishes a weaker version of Eq. (6) with larger constants and \(\ln(\cdot)\) replaced with \([\ln(\cdot)]_+\).

**Proof.** We will here follow the proof of Theorem 3 in Ref. \[14\] closely. Suppose the inequality

\[
I(\psi) \geq -\int_\mathbb{R} M_{\rho_\psi} \left( \ln(2e) + \ln \left( \frac{\sqrt{2}}{\varepsilon \rho_\psi} \right) \right) dx,
\]

where \(Mf\) denotes the Hardy–Littlewood maximal function of \(f\) given in Remark 1. The argument from Eq. (22) to Eq. (25) in Ref. \[14\], with only the involved constants differing, then gives (it is here \(K(\psi)\) finite is used)

\[
I(\psi) \geq -16 \int_\mathbb{R} \rho_\psi^2 \left( \ln(2e) + \ln \left( e^3 + \frac{\sqrt{2}}{\varepsilon \rho_\psi} \right) \right) dx,
\]
with the factor 16 in front a consequence of the use of the Hardy–Littlewood maximal function. Since \(\ln(e^3 + f) \leq \ln(e^3 + 1) + [\ln f]_+\) with \(f = \sqrt{2}/(\epsilon \rho_\psi) \geq 0\), we obtain

\[
I(\psi) \geq -C_1 \int_\mathbb{R} \rho_\psi^2 \left( K_1 + [\ln \left( \frac{K_2}{\epsilon \rho_\psi} \right)]_+ \right) \, dx,
\]

where \(C_1, K_1,\) and \(K_2\) are given as in the formulation of the theorem. Thus, we are done if we can establish Eq. (13).

To meet that end, Lemma 2 in Ref. [14] applied to \(v = v_\epsilon^{\text{CSC}}\), that satisfies \(v\) is convex and \(\lim_{r \to \infty} v(r) = 0\), gives for any \(\beta(x) \geq 0\)

\[
I(\psi) \geq -\frac{1}{2} \int_\mathbb{R} \left( (\mathcal{M} \rho_\psi)(x) \right)^2 \int_0^{\beta(x)} v''(r)r^2 \, dr \\
+ (\mathcal{M} \rho_\psi)(x) \int_{\beta(x)}^\infty v''(r)rdr \, dx.
\]

Define the functions \(f_1(x) = \int_0^{\beta(x)} v''(r)r^2dr\) and \(f_2(x) = \int_{\beta(x)}^\infty v''(r)rdr\) with \(v(r) = v_\epsilon^{\text{CSC}}(r) = v_\epsilon^{\text{CSC}}(r + r_\epsilon)\). We first note that \(f_1(x) \leq 2 \int_0^{\beta(x)} v \, dr\) such that we obtain

\[
f_1(x) \leq 2 \left[ \ln(\sqrt{r^2 + e^2 + r_x}) \right]^{\beta(x)+r_x} \, \left( \frac{\beta(x) + r_x}{r_x} \right) \\
= 2 \ln \left( \frac{\beta(x) + r_x}{r_x} \right) = 2 \ln \left( \frac{\beta(x) + r_x}{r_x} \right) + \frac{1}{r_x} \int_{r_x}^{\beta(x)} v'(r) \, dr.
\]

Furthermore, it holds

\[
f_2(x) \leq \left( \frac{2}{\beta(x)} \right)^{1/2} \leq \frac{2}{\beta(x)}.
\]

For the choice \(\beta = (\mathcal{M} \rho_\psi)^{-1}\), set \(B_\psi = \{ x \in \mathbb{R} : (\mathcal{M} \rho_\psi)(x) \leq r^{-1}_\psi \}\) and denote its complement by \(\overline{B}_\psi\).

We have by Eq. (15)

\[
\int_{\overline{B}_\psi} \mathcal{M}_\psi^2 f_1dx \leq 2 \int_{\overline{B}_\psi} \mathcal{M}_\psi^2 \ln \left( \frac{2}{r_x \mathcal{M} \rho_\psi} \right) \, dx \\
+ 2 \ln(2) \int_{\overline{B}_\psi} \mathcal{M}_\psi^2 dx,
\]

where the first equality is a definition. Using the fact that \(f_{\overline{B}_\psi} = f_{\mathbb{R}} - f_{B_\psi}\) yields

\[
\frac{\hat{f}_1}{2} \leq \int_{\overline{B}_\psi} (\mathcal{M} \rho_\psi)(x)^2 \ln \left( \frac{1}{r_x \mathcal{M} \rho_\psi} \right) \, dx \\
+ \ln(2) \int_{\mathbb{R}} \mathcal{M} \rho_\psi^2 \, dx.
\]

On \(B_\psi\) the equality \(\ln(1/(r_x \mathcal{M} \rho_\psi)) = [\ln(1/(r_x \mathcal{M} \rho_\psi))]_+\) holds. Since for a positive function \(f\) the inequality \(\int_{\mathbb{R}} f \, dx \leq \int_{\overline{B}_\psi} f \, dx\) is true, we obtain

\[
\frac{\hat{f}_1}{2} \leq \int_{\mathbb{R}} \mathcal{M} \rho_\psi^2 \left( \ln(2) + \left[ \ln \left( \frac{\sqrt{2}}{\epsilon \mathcal{M} \rho_\psi} \right) \right]_+ \right) \, dx.
\]

Next, we have \(\hat{f}_2 = \int_{\mathbb{R}} \mathcal{M} \rho_\psi f_2 \, dx \leq 2 \int_{\mathbb{R}} (\mathcal{M} \rho_\psi)(x)^2 \, dx\) as a consequence of Eq. (10). Inserting the upper bounds for \(\hat{f}_1\) and \(\hat{f}_2\) into Eq. (10) yields Eq. (13).

\[\square\]

D. Lieb–Oxford bound for convex homogeneous potential

We will here complement the study of the Dirac and soft Coulomb potential with investigations of one-dimensional Lieb–Oxford bounds for the homogeneous potential \(v(r) = r^{2-\epsilon}\). We first note that the general result of Lundholm et al. (see Lemma 16 in [17]) gives, in particular, a one-dimensional Lieb–Oxford bound

\[
I(\psi) \geq -\frac{2(2-\epsilon)^2}{(1-\epsilon)} \int_{\mathbb{R}} \frac{\rho_\psi(x)^2 \, dx}{(1-\epsilon)}.
\]

The bound is arbitrarily close to the integral \(\int_{\mathbb{R}} \rho_\psi(x)^2 \, dx\) of the right-hand side of Eq. (3), but with a constant tending to infinity as \(\epsilon \to 0^+\). This is the same result one would get if Lemma 2 in Ref. [14] was applied to \(v = r^{2-\epsilon}\) (with their \(\beta(x)\) equal to the inverse of \(\mathcal{M} \rho_\psi\)). Both approaches make use of the Hardy–Littlewood maximal function that introduces an extra factor \(2(1-\epsilon)/\epsilon\).

(Di Marino has proven similar result in the setting of strictly correlated electrons [18].) We here note that we can instead use the Hardy–Littlewood–Sobolev inequality and, as in the case of the Dirac potential, just bound the direct term to obtain a Lieb–Oxford-type inequality with a better constant as \(\epsilon \to 0^+\) (see Remark [19]).

Theorem 4 (Homogeneous potential). Let \(v = v_\epsilon^H\), where \(v_\epsilon^H(r)\) for \(r \geq 0\) is given by

\[
v_\epsilon^H(r) = \frac{1}{r_\epsilon^{1-\epsilon}}, \quad 0 < \epsilon < 1.
\]

For \(\psi \in H^1\) with \(||\psi||_2 = 1\), we have

\[
I(\psi) \geq -C_\epsilon \left( \int_{\mathbb{R}} \frac{\rho_\psi(x)^2 - \frac{\epsilon}{r_\epsilon}}{dx} \right)^{1+\epsilon} \\
\geq -C_\epsilon N^{2\epsilon} \left( \int_{\mathbb{R}} \rho_\psi(x)^2 \, dx \right)^{1-\epsilon},
\]

where \(N\) is the particle number and

\[
C_\epsilon = \frac{1}{2} \frac{\sqrt{\pi}}{\pi^\epsilon} \frac{\Gamma(a)}{\Gamma(\frac{3}{2} + \frac{\epsilon}{2})}, \quad \Gamma(a) = \int_0^\infty t^{a-1}e^{-t} \, dt.
\]

In particular, the scaled version \(I^s\), corresponding to \(v = \epsilon v_\epsilon^H\), satisfies

\[
\lim_{\epsilon \to 0^+} I^s(\psi) \geq -\frac{1}{2} \int_{\mathbb{R}} \rho_\psi(x)^2 \, dx.
\]
Proof. We use the Hardy–Littlewood–Sobolev inequality (see e.g. Theorem 4.3 in [16]): For \( p, q > 1 \), \( 0 < \alpha < 1 \), \( 1/p + 1/q = 2 - \alpha \), we have

\[
\int f(x)|x-y|^{-\alpha}g(y)dx\leq C_{\text{HLS}}(p,q,\alpha)\|f\|_p\|g\|_q.
\]

Setting \( f = g = \rho \), \( \alpha = 1 - \varepsilon \) and \( p = q \), it follows that \( p = 2/(1 + \varepsilon) \in (1, 2) \) and moreover

\[
2C_\varepsilon = C_{\text{HLS}}(2 - \varepsilon, 2 - \varepsilon, 1 - \varepsilon) = \frac{\sqrt{\pi}}{\pi^\frac{2}{\alpha+2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)}.
\]

where the first equality defines the constant \( C_\varepsilon \). Thus, directly from the Hardy–Littlewood–Sobolev inequality

\[
I(\psi) \geq -D(\rho_\varepsilon) = -D(\rho_\varepsilon, \rho_\varepsilon) \geq -C_\varepsilon \left( \int \rho_\varepsilon(x)^2 \frac{dx}{r^\varepsilon} \right)^{1+\varepsilon}.
\]

Next, we note that by definition of \( I \) for a given \( v \), it follows that \( v \to \varepsilon v \) corresponds to \( I \to I' = \varepsilon I \). By Hölder’s inequality we obtain

\[
\int \rho_\varepsilon(x)^2 \frac{dx}{r^\varepsilon} \leq \int \rho_\varepsilon(x)^2 \frac{dx}{r^\varepsilon} \int \rho_\varepsilon(x)^\frac{1}{\varepsilon} dx \leq N^\frac{1}{\varepsilon} \left( \int \rho_\varepsilon(x)^2 dx \right)^{1-\frac{1}{\varepsilon}},
\]

and where we used \( \int \rho_\varepsilon(x) dx = N \). This gives

\[
\left( \int \rho_\varepsilon(x)^2 \frac{dx}{r^\varepsilon} \right)^{1+\varepsilon} \leq N^{2\varepsilon} \left( \int \rho_\varepsilon(x)^2 dx \right)^{1-\varepsilon}.
\]

Since \( a\Gamma(a) = \Gamma(a+1) \) and \( \lim_{\varepsilon \to 0^+} \Gamma(a + b) = \Gamma(a) \) for \( a, b > 0 \), we have

\[
\lim_{\varepsilon \to 0^+} \varepsilon C_\varepsilon N^2 \varepsilon = \lim_{\varepsilon \to 0^+} \frac{N^2 \varepsilon}{2} \left( \frac{\varepsilon}{\pi^\frac{2}{\alpha+2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)} \right) = \frac{1}{2}
\]

and the claim follows. \( \square \)

Remark 5. Note that for the two different bounds Eq. (19) and Eq. (17), the constants satisfy

\[
\lim_{\varepsilon \to 0^+} \frac{2\varepsilon(2-\varepsilon)}{\varepsilon(1-\varepsilon)} \left( \frac{N^2 \varepsilon}{2} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)} \right) = 2.
\]

Thus, for sufficiently small \( \varepsilon \), i.e., closer to the term \( \int \rho_\varepsilon(x)^2 dx \), the Hardy–Littlewood–Sobolev inequality provides a relatively better constant.

III. CONCLUSION

We have here showed that the conjectured Lieb–Oxford bound in one dimension \( f(v) \geq -C_1 \int \rho_\varepsilon(x)^2 dx \) can be obtained with \( C_1 = 1 \) for the Dirac potential \( v = \delta \). We have also derived bounds for a convex version of the soft Coulomb potential and the homogeneous potential \( r^{\varepsilon-1} \), both approximating the ill-defined Coulomb potential. For the convex soft Coulomb potential, the logarithmic term \( \int \rho_\varepsilon(x)^2 \ln(C/\varepsilon \rho_\varepsilon(x)) dx \) was proven in the Lieb–Oxford bound. To be able to obtain the integral \( \int \rho_\varepsilon(x)^2 dx \) for the homogeneous potential \( r^{\varepsilon-1} \), scaling with \( \varepsilon \) was needed and gave a constant \( 1/2 \) for \( \varepsilon \varepsilon^{\varepsilon-1} \) in the limit \( \varepsilon \to 0^+ \).

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