Trace operator and the Dirichlet problem for elliptic equations on arbitrary bounded open sets

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Abstract

We consider the Dirichlet problem on general, possibly nonsmooth bounded domain, for elliptic linear equation with uniformly elliptic divergence form operator. We investigate carefully the relationship between weak, soft and the Perron-Wiener-Brelot solutions of the problem. To this end, we extend the usual notion of the trace operator to Sobolev space $H^1(D)$ with $D$ being an arbitrary bounded open subset of $\mathbb{R}^d$. In the second part of the paper, we prove some existence results for the Dirichlet problem for semilinear equations with measure data on the right-hand side and $L^1$-data on the Martin boundary of $D$.

Keywords: Trace operator, elliptic equation, Dirichlet problem, Martin boundary.

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1 Introduction

Let $D$ be a bounded open subset of $\mathbb{R}^d$, $d \geq 2$. The main purpose of the first part of the present paper is to investigate the relationship between solutions of the weak Dirichlet problem: for given $\psi \in H^1(D)$ find $u \in H^1(D)$ such that

$$-Au = 0 \quad \text{in} \quad D, \quad u - \psi \in H^1_0(D)$$

(problem wDP($A, D, \psi$) for short) and solutions of the Dirichlet problem, which formally can be formulated as follows: for given measurable $\psi : \partial D \to \mathbb{R}$ find $u \in H^1_{\text{loc}}(D)$ such that

$$-Au = 0 \quad \text{in} \quad D, \quad u = \psi \quad \text{on} \quad \partial D$$

(problem DP($A, \partial D, \psi$) for short). We stress that, contrary to (1.1), in (1.2) the boundary data $\psi$ are given only on $\partial D$. In the paper we consider weak, soft and Perron-Wiener-Brelot (PWB-solutions for short) solutions to (1.2). In the case where $D$ is irregular, careful analysis of the relationship between these notions of solutions of (1.2) and solutions of (1.1) requires the study of more general then (1.2) Dirichlet problem

$$-Au = 0 \quad \text{in} \quad D, \quad u = \psi \quad \text{on} \quad \partial M D$$

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where $\psi : \partial_M D \to \mathbb{R}$ and $\partial_M D$ is the Martin boundary of $D$. Therefore, in fact, in the paper we also consider problem (1.3).

In the second part of the paper we apply the results of the first part to study the Dirichlet problem for semilinear equations with general measure data on the right-hand side and $L^1$-boundary data.

In (1.1) and (1.2), $A$ is a divergence form operator

$$Au = \sum_{i,j=1}^d (a_{ij}u_{x_i})x_j$$

(1.4)

and the equation $-Au = 0$ holds in the weak sense, i.e.

$$\mathcal{E}(u, v) := (a \nabla u, \nabla v)_{L^2(D;m)} = 0, \quad v \in C^1_c(D).$$

(1.5)

In the whole paper we assume that $a = (a_{ij})_{i,j=1,...,d} : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a bounded symmetric matrix-valued measurable function such that for some $\lambda > 0$,

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in \mathbb{R}^d, \xi = (\xi_2, \ldots, \xi_d) \in \mathbb{R}^d.$$  

(1.6)

Our problems concerning the linear Dirichlet problem are classical. The relationship between solutions of (1.1) and (1.2) is quite well understood in the case where $D$ has regular boundary $\partial D$ and $\psi \in C(\partial D)$. In the present paper we concentrate on the case where $D$ is an arbitrary open set. The second goal is to extend the existing theory to possibly discontinuous boundary data. To achieve our goals, we extend the usual notion of the trace operator to the space $H^1(D)$ with general bounded open subset $D$ of $\mathbb{R}^d$. In fact, in order to study semilinear equations with measure data, we extend the trace operator to even wider space $T$ defined later on.

To describe the content of the paper, we must first explain what we mean by a solution to (1.2) in case $\partial D$ is irregular. We start with recalling some classical results for $\psi \in C(\partial D)$. Let $m$ denote the Lebesgue measure on $\mathbb{R}^d$. It is well known that for each $\psi \in H^1(D)$ there exists a unique solution $u \in H^1(D)$ of the weak Dirichlet problem (1.1) and that $u$ has an $m$-version belonging to $C(D)$ (see [38]). Moreover, if $\psi_1 - \psi_2 \in H^1_0(D)$, then the solution of $wDP(A,D,\psi_1)$ is equal to the solution of $wDP(A,D,\psi_2)$. Therefore, we can define the positive linear operator

$$B : H^1(D)/H^1_0(D) \to H^1(D)$$

which assigns to each $\psi \in H^1(D)/H^1_0(D)$ the unique solution of (1.1). By [33], $B$ is continuous, and moreover, for $\psi \in (H^1(D)/H^1_0(D)) \cap C(\overline{D})$, we have

$$\|B\psi\| \leq C(\lambda, D) \max_{\partial D} |\psi|,$$

(1.7)

where for $u \in H^1_{loc}(D)$, we write

$$\|u\| = \sup_{D' \subset \subset D} \text{dist}(D', D)\|\nabla u\|_{L^2(D',m)} + \sup_D |u|.$$

Let $\mathcal{X} = \{u \in H^1_{loc}(D) \cap C(D) : \|u\| < \infty\}$. By (1.7), we can extend $B$ to a positive linear continuous operator

$$B : C(\partial D) \to \mathcal{X}.$$  

(1.8)
For given \( \psi \in C(\partial D) \), the function \( B\psi \in X \) is called a weak solution of (1.2). Of course, the solution \( u := B\psi \) to (1.2) satisfies the second condition in (1.2) only formally because in general, \( u \) is not continuous up to the boundary (unless \( D \) is regular).

To encompass broader class of boundary data, in the paper we propose a definition of a solution to (1.2) based on the notion of the harmonic measure. Solutions in the sense of this definition will be called soft solutions. To fix notation, recall that for a given \( x \in D \) the harmonic measure associated with \( x \), \( D \) and the operator \( A \) is the unique Borel measure on \( \partial D \) such that

\[
B\psi(x) = \int_{\partial D} \psi(y) \omega^A_{x,D}(dy), \quad \psi \in C(\partial D).
\]  

The measure \( \omega^A_{x,D}(dy) \) exists and is unique because \( B \) defined by (1.8) is positive and continuous. If there is no ambiguity, we drop \( D \) in the notation. Note that, by Harnack’s inequality (see [37]), for any \( x,y \) in the same connected component of \( D \), the measures \( \omega^A_x \) and \( \omega^A_y \) are mutually absolutely continuous and that it may happen that \( \omega^A_x \) is completely singular with respect to the surface Lebesgue measure \( \sigma \) on \( \partial D \) even if \( D \) is smooth (see [10, 36]).

Let \( \{G^D_\alpha \}, \alpha > 0 \) denote the resolvent operator of \( A \) on \( D \) (with zero Dirichlet condition), and for a positive \( f \in L^2(D;m) \), let \( G^D f = \sup_{\alpha>0} G^D_\alpha f \). Set \( \delta = G^D 1 \) and

\[
\tilde{H}^1_\delta(D) = \{ u \in H^1 \cap L^2(D;m) : \|u\|_{\tilde{H}^1_\delta} < \infty \},
\]

where \( \|u\|_{\tilde{H}^1_\delta} = \|u\|_{L^2(D;m)} + \|\sqrt{\delta} \nabla u\|_{L^2(D;m)} \). By a soft solution to (1.2) with \( \psi \in L^2(\partial D;\omega^A_m) \), where

\[
\omega^A_m(dy) = \int_D \omega^A_x(dy) m(dx),
\]

we mean a unique function \( u \in \tilde{H}^1_\delta(D) \cap C(D) \) such that

\[
\mathcal{E}(u,v) = 0, \quad v \in C^1_c(D),
\]

and for some (and hence every) increasing sequence of open sets \( \{D_n\} \subset D \) with \( \bigcup_{n \geq 1} D_n = D \), we have

\[
\int_{\partial D_n} u(y) \omega^A_{x,D_n}(dy) \rightarrow \int_{\partial D} \psi(y) \omega^A_{x,D}(dy)
\]

for \( m \)-a.e. \( x \in D \). In the paper we show that, for \( \psi \in C(\partial D) \), soft solutions of (1.2) are weak solution of (1.2).

Our definition of soft solution to (1.2) resembles the definition adopted in the literature in the case of regular domains and regular coefficients \( a_{ij} \) (see [11, 35]). The difference is that in these papers the harmonic measures in (1.12) are replaced by the surface measure \( \sigma \). In the case where the domain and the coefficients are regular, such a modification of the definition of a solution is possible because then the harmonic measures are absolutely continuous with respect to the surface measure (see [14, 17, 18]).

Another way of defining a solution to (1.2) is the method of sub and superharmonic functions (so called Perron-Wiener-Brelot method, see [4]). Recall that, for a given
measurable $\psi : \partial D \rightarrow \mathbb{R}$, a function $u$ is called a PWB-solution of (1.2) if

$$u = \sup \{ v : v \text{ is subharmonic and } \limsup_{y \to x} v(y) \leq \psi(x), \ x \in \partial D \}$$

$$= \inf \{ v : v \text{ is superharmonic and } \liminf_{y \to x} v(y) \geq \psi(x), \ x \in \partial D \}.$$ 

We prove that, if $\psi \in C(\partial D)$, then PWB-solution of (1.2) satisfies

$$u(x) = \int_{\partial D} \psi(y) \omega^A_x(dy), \ x \in D. \quad (1.13)$$

Therefore, if $\psi \in C(\partial D)$ then weak and PWB-solutions of (1.2) coincide. Our result generalizes the corresponding result from [1] proved in the case where $A = \Delta$ (see also [24, 45]). In fact, we show that, if $\psi \in \mathcal{B}(\partial D)$ and the right-hand side of (1.13) is finite for every $x \in D$, then the PWB-solution of (1.2) exists and is given by (1.13). It is worth mentioning that our result (and the corresponding one in [1]) follows easily from a result proved in [4] and the fact that

$$\omega^A_{x,D}(dy) = P_x(X_{\tau_D} \in dy), \ x \in D,$$

where $X = (\{X_t, \ t \geq 0\}, \{P_x, x \in \mathbb{R}^d\}, \{\mathcal{F}_t, \ t \geq 0\})$ is a diffusion process associated with the operator $A$, and

$$\tau_D = \inf\{t > 0 : X_t \in \mathbb{R}^d \setminus D\}. \quad (1.14)$$

However, our proof is much more elementary. In contrast to [4], it is not based on the abstract theory of balayage spaces.

We now briefly describe our main results on the relation between solutions of (1.1) and (1.2). We first assume that $D$ is a Lipschitz domain. Let $\text{Tr} : H^1(D) \rightarrow L^2(D; \sigma)$ denote the trace operator. In the paper we show that, if $\psi \in H^1(D)$, then the solution $u$ of (1.1) has the representation

$$u(x) = \int_{\partial D} \widetilde{\text{Tr}}(\psi)(y) \omega^A_x(dy), \quad (1.15)$$

where $\widetilde{\text{Tr}}(\psi)$ is a $\sigma$-version of the trace $\text{Tr}(\psi)$ of $\psi$ (determined $\omega^A_m$-a.e.) defined as

$$\widetilde{\text{Tr}}(\psi) := \tilde{\psi}|_{\partial D},$$

where $\tilde{\psi}$ is a quasi-continuous version of an extension of $\psi$ to $H^1(\mathbb{R}^d)$. In different words, if $u$ is a solution to wDP$(A,D,\psi)$, then $u$ is a PWB-solution to DP$(A,\partial D,\widetilde{\text{Tr}}(\psi))$. It is worth mentioning that a version $\tilde{\psi}$ does not depend on the operator $A$ since by (1.6),

$$\lambda \text{Cap}_\Delta \leq \text{Cap}_A \leq \|a\|_{\infty} \text{Cap}_\Delta.$$ 

Observe that in (1.15), one can not take any $\sigma$-version of $\text{Tr}(\psi)$ because as mentioned before, in general, $\omega^A_{x,D}$ is not absolutely continuous with respect to the surface measure $\sigma$. This is one of the main differences between PWB-solutions and weak solutions. Namely, contrary to PWB-solutions, weak solutions to the Dirichlet problem do not depend on the $\sigma$-version of the boundary data.
In the general case, when \( D \) is an arbitrary bounded open set, to establish a relation between weak Dirichlet problem and Dirichlet problem, we introduce the trace operator on \( H^1(D) \). It is well known that there exists a positive function \( g_D \in B(D \times D) \), called Green function for \( D \) (and \( A \)) such that

\[
G^D f = \int_D g_D(\cdot,y)f(y)\,m(dy) \quad m.a.e.
\]

for any positive \( f \in L^2(D; m) \) and the functions \( g_D(x,\cdot) \) and \( g_D(\cdot,y) \) are excessive for all \( x,y \in D \). Set \( \kappa(x,y) = g_D(x,y)\delta^{-1}(y) \) and define the metric \( \rho \) on \( D \) by

\[
g(x,y) = \sum_{n \geq 1} 2^{-n} \frac{|\hat{\kappa}f_n(x) - \hat{\kappa}f_n(y)|}{1 + |\hat{\kappa}f_n(x) - \hat{\kappa}f_n(y)|},
\]

where \( \hat{\kappa}f_n = \int \kappa(x,\cdot)f_n(x)\,m(dx) \) and \( \{f_n\} \) is a dense subset of \( C_0(D) \). Let \( D^* \) be the completion of \( D \) with respect to the metric \( \rho \), and let \( \partial_M D := D^* - D \) (\( \partial_M D \) is the so called Martin boundary of \( D \)). We define the harmonic measure on the Martin boundary by

\[
h^A_x(dy) = P_x(X_{\tau_D -} \in dy), \quad x \in D,
\]

where \( X_{\tau_D -} = \lim_{t \uparrow \tau_D} X_t \) and the limit is taken with respect to the metric \( \rho \). Similarly to (1.11), we put \( h^A_m(dy) = \int h^A_x(dy)\,m(dx) \). We prove that there exists a trace operator

\[
\gamma_A : H^1(D) \to L^2(\partial_M D; h^A_m),
\]

i.e. a continuous linear operator such that \( \gamma_A(\psi) = \psi|_{\partial D} \) for \( \psi \in C(\overline{D}) \cap H^1(D) \). The last equality is meaningful since we show that there exists an imbedding

\[
i_M : L^2(\partial D; \omega^A_m) \hookrightarrow L^2(\partial_M D; h^A_m). \quad (1.17)
\]

More precisely, for every \( \psi \in L^2(\partial D; \omega^A_m) \) there exists a unique function \( i_M(\psi) \in L^2(\partial_M D; h^A_m) \) such that

\[
\int_{\partial D} \psi(y)\,\omega^A_m(dy) = \int_{\partial_M D} i_M(\psi)(y)\,h^A_m(dy), \quad x \in D.
\]

We also prove that the trace theorem holds for the operator \( \gamma_A \), i.e.

\[
\gamma_A(\psi) = 0 \quad \text{if and only if} \quad \psi \in H^1_0(D).
\]

It is interesting that in spite of the fact that the trace operator \( \gamma_A \) depends on \( A \), the above result holds independently of \( A \).

In general, the embedding (1.17) is strict. We show that

\[
\gamma_A^{-1}(L^2(\partial D; \omega^A_m)) = H^1_c(D), \quad (1.18)
\]

where \( H^1_c(D) \) is the set of those \( u \in H^1(D) \) for which there exists \( \psi \in B(\partial D) \) such that

\[
[0,\tau_D] \ni t \mapsto (u1_D + \psi 1_{\partial D})(X_t) \quad \text{is continuous} \quad P_x\text{-a.s. for q.e. } x \in D,
\]

where q.e. is the abbreviation for quasi everywhere with respect to the capacity \( \text{Cap}_A \). The space \( H^1_c(D) \) is a closed subspace of \( H^1(D) \) and

\[
\text{cl}(H^1(D) \cap C(\overline{D})) \subset H^1_c(D).
\]
An immediate corollary to (1.18) is that $L^2(\partial D; \omega_m^A)$ is isomorphic to $L^2(\partial M D; h_m^A)$ if and only if $H^1_c(D) = H^1(D)$. Equivalently, there exists a trace operator on $H^1(D)$ to $L^2(\partial D; \omega_m^A)$ if and only if $H^1_c(D) = H^1(D)$. By using this trace operator we show that, if $u$ is a solution of $wDP(A, D, \psi)$, then it is a weak solution of the Dirichlet problem

$$-Au = 0 \quad \text{in} \quad D, \quad u = \gamma_A(\psi) \quad \text{on} \quad \partial_M D$$

(problem $DP(A, \partial_M D, \gamma_A(\psi))$ for short). More precisely, $u \in H^1(D)$ and

$$\mathcal{E}(u, v) = 0, \quad v \in H_0^1(D), \quad \gamma_A(u) = \gamma_A(\psi).$$

Boundary behaviour of a quasi-continuous version of a function from $H^1(D)$ was for the first time considered by Doob [16] in the case where $A = \Delta$ (see also [39, 47]). A substantial part of our paper is devoted to extension of Doob's results to operators of the form (1.4) and wider class of functions and to investigate properties of the trace operator $\gamma_A$. A similar in spirit (but purely analytic) approach to the definition of the trace operator on arbitrary bounded domains is considered in [44] for spaces $W^{1,p}(D)$ with $p > d$. In [44] the author also changes equivalently the Euclidean metric on $D$ to a metric $\tilde{\rho}$ in such a way that $u \in W^{1,p}(D)$ is uniformly continuous on $D$ with respect to $\tilde{\rho}$ (since $p > d$, $u \in C(D)$). This allows him to consider $u$ on $\partial D'$, where $D'$ is the completion of $D$ with respect to the metric $\tilde{\rho}$. In our paper we consider functions $u \in H^1(D)$ which, in general, are not continuous but only quasi-continuous, so the problem is more involved. To solve it requires us to use some notions and methods of the potential theory.

In the second part of the paper we treat the Dirichlet problem for semilinear equations of the form

$$-Au = f(\cdot, u) + \mu \quad \text{in} \quad D, \quad u = \psi \quad \text{on} \quad \partial_M D,$$

(1.19)

where $\psi \in L^1(\partial_M D; h_m^A)$ and $\mu$ is a Borel measure on $D$ such that $\int_D \delta d|\mu| < \infty$. As for $f$, we assume that it is continuous and nonincreasing with respect to $u$. To deal with (1.19), we first extend the trace operator to the set $T$ ($H^1(D) \subset T$) of all functions $\psi \in \mathcal{B}(D)$ for which there exists $g \in \mathcal{B}(\partial_M D)$ such that the process

$$[0, \tau_D] \ni t \mapsto (1_D \psi + 1_{\partial_M D} g)(X_t 1_{\{t < \tau_D\}} + X_{\tau_D} 1_{\{t = \tau_D\}})$$

is continuous at $\tau_D$ under the measure $P_x$ for $m$-a.e. $x \in D$. For $\psi \in T$, we put

$$\gamma_A(\psi) := g.$$

Let $\| \cdot \|_{q.u.}$ denote the metric of quasi-uniform convergence. We show that

$$\gamma_A : (T, \| \cdot \|_{q.u.}) \to L^0(\partial_M D; h_m^A)$$

is continuous.

Let $H_{D^p}$ be the space defined probabilistically as a class of harmonic functions $u$ on $D$ for which the family $\{u^p(X_{\tau_V}), V \subset D\}$ is uniformly integrable under the measure $P_x$ for $m$-a.e. $x \in D$. We equip $H_{D^p}$ with the metric induced by the norm

$$\|u\|_{D^p}^p := \int_D \sup_{V \subset D} E_x |u(X_{\tau_V})|^p m(dx).$$
For $p > 1$, we define $H_{\mathcal{D}^p}$ to be the space of all harmonic functions $u$ on $D$ such that
\[
\|u\|_{H_{\mathcal{D}^p}}^p := E_m \sup_{t<\tau_D} |u(X_t)|^p < \infty.
\]
We show that $H_{\mathcal{D}^p} \subset \mathcal{T}$, $p \geq 1$ and that
\[
\gamma_A : H_{\mathcal{D}^p} \rightarrow L^p(\partial_M D; h_{m}^A)
\]
is an isometric isomorphism. Moreover, for $p > 1$,
\[
\gamma_A : H_{\mathcal{D}^p} \rightarrow L^p(\partial_M D; h_{m}^A)
\]
is a homeomorphism which implies that $H_{\mathcal{D}^p} = H_{\mathcal{D}^p}$. We also show that similarly to the case where $p = 2$, for $p \geq 1$ we have
\[
L^p(\partial D; \omega_m^A) = L^p(\partial M D; h_m^A)
\]
if and only if $H^1_c(D) = H^1(D)$.

In the paper we propose two different but equivalent definitions of a solution to (1.19). In both definitions by a solution we mean a function $f$ such that $f(\cdot, u) \in L^1(E; \delta \cdot m)$. Additionally, in the first definition we require that the equality
\[
\begin{align*}
\int_D u(x) &= \int_D f(u(y)g_D(x, y) m(dy) + \int_D g_D(x, y) \mu(dy) + \int_{\partial M D} \psi(y)h_2^A(dy)
\end{align*}
\]
is satisfied for all $x \in D$. In the second definitoin, so called Stampacchia’s definition by duality, we require that
\[
\begin{align*}
\int_D (u - \gamma_A^{-1}(\psi))G^D \eta dm &= \int_D f(\cdot, u)G^D \eta dm + \int_D \hat{G^n} \eta d\mu, \quad \eta \in L^\infty(D; m),
\end{align*}
\]
where $\hat{G^n} \eta$ is a continuous $m$-version of $G^n \eta$ ($G^n$ is strongly Feller). In general, there is no solution to (1.19) (see [7]). Following [6, 7] and [27], for given $A, f$ denote by $\mathcal{G}_0$ the set of all good measures for (1.19), i.e the set of all bounded Borel measures $\mu$ on $D$ for which there exists a solution to (1.19) with $\psi \equiv 0$. Our main result says that for every $\mu \in \mathcal{G}_0$ and $\psi \in L^1(\partial M D; h_{m}^A)$ there exists a unique solution to (1.19) (this agrees with the results of [34], where it is proved, that the class of good measures does not depend on the boundary data in case of $A = \Delta$). We also prove that for every $k > 0$,
\[
\|T_k(u)\|_{H_1^k}^2 \leq 3k\lambda^{-1}(\|\psi\|_{L^1(\partial M D; h_{m}^A)} + \|f(\cdot, 0)\|_{L^1(\partial M D; h_{m}^A)} + \|\mu\|_{TV, \delta}),
\]
where $T_k(u)(x) = ((-k) \vee u(x)) \wedge k$. Furthermore, we show that for regular domains of class $C^{1,1}$, $u \in L^p(D; \delta \cdot m)$ with $p < \frac{d}{d-1}$. This, when combined with (1.20), implies that $u \in W^{1,q}$ with $q < \frac{2d}{2d-1}$. Finally, Let us note that, if $D$ is of class $C^{1,1}$, then there exist $c_1, c_2 > 0$ such that
\[
c_1 \rho(x) \leq \delta(x) \leq c_2 \rho(x), \quad x \in D
\]
where $\rho(x) = \text{dist}(x, \partial D)$. The theory of so called very weak solutions to elliptic equations with data in $L^1(D; \rho \cdot m)$ have attracted quite interest in recent years (see, e.g., [15, 40] and references therein).
2 Weak and soft solutions to the Dirichlet problem

Let $D$ be an arbitrary bounded open subset of $\mathbb{R}^d$. In this section, we provide a stochastic representation for weak solutions of $\text{DP}(A, \partial D, \psi)$ with $\psi \in C(\partial D)$. Based on this result, we give the definition of a soft solution to $\text{DP}(A, \partial D, \psi)$ with $\psi \in L^2(\partial D; \omega_m^A)$, where $\omega_m^A$ is defined by (1.11). We next show that, if $\psi \in C(\partial D)$, then soft solutions are weak solution to $\text{DP}(A, \partial D, \psi)$. In Section 3 we will show that soft solutions of $\text{DP}(A, \partial D, \psi)$ are solutions obtained via the Perron-Wiener-Brelot method.

In what follows, for given open bounded sets $V, U \subset \mathbb{R}^d$, we write $V \subset U$ if $\overline{V} \subset U$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the set of all Borel measurable functions on $\mathbb{R}^d$, and by $\mathcal{B}^+(\mathbb{R}^d)$ the subset of $\mathcal{B}(\mathbb{R}^d)$ consisting of positive functions. For a given positive Borel measure $\mu$ on $\mathbb{R}^d$ and $g \in \mathcal{B}^+(\mathbb{R}^d)$, we denote by $g \cdot \mu$ the Borel measure on $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} \eta d(g \cdot \mu) := \int_{\mathbb{R}^d} \eta g \, d\mu, \quad \eta \in \mathcal{B}^+(\mathbb{R}^d).$$

We denote by $g^D_D$ the Green function for $D$ and operator $A - \alpha I$ ($\alpha \geq 0$), and by $\text{Cap}_A$ the capacity associated with the operator $A$. Recall that a function $u$ on $\mathbb{R}^d$ is called quasi-continuous if and only if for every $\varepsilon > 0$ there exists a closed $F_\varepsilon \subset K$ such that $\text{Cap}_A(\mathbb{R}^d \setminus F_\varepsilon) \leq \varepsilon$ and $u|_{F_\varepsilon}$ is continuous. For a positive Borel measure $\mu$ on $D$ and $\alpha \geq 0$, we set

$$R_\alpha^{D} \mu(x) = \int_D g^D_D(x, y) \mu(dy), \quad x \in D.$$

Let $X$ be a diffusion on $\mathbb{R}^d$ associated with the operator $A$ (see [19, 42, 48]). It is well known that, if $\mu = f \cdot m$ for some $f \in \mathcal{B}^+(D)$, then

$$R_\alpha^{D} f(x) := R_\alpha^{D}(f \cdot m) = E_x \int_{\tau_D}^{\tau_D} e^{-\alpha t} f(X_t) \, dt, \quad x \in D, \tag{2.1}$$

where $\tau_D$ is defined by (1.14) and for $f \in L^p(D; m)$, $R_\alpha^{D} f$ is a quasi-continuous $m$-version of $G^D f$ and if $f \in L^\infty(D; m)$ then $R_\alpha^{D} f$ is a continuous $m$-version of $G^D f$ on $D$. Moreover, if $\mu$ is smooth in the sense of Dirichlet forms (see [19, Section 2.2] for the definition), then by [19, Lemma 2.2.10, Theorem 5.4.2] there exists a unique positive continuous additive functional $A^\mu$ of $X$ such that for a.e. $x \in D$,

$$R_\alpha^{D} \mu(x) = E_x \int_0^{\tau_D} e^{-\alpha t} \, dA^\mu_t. \tag{2.2}$$

We put $R^D := R_0^{D}$, $g^D := g^D_D$.

**Lemma 2.1.** If $\psi \in C(\partial D)$, then $E.\psi(X_{\tau_D}) \in C(D)$.

**Proof.** By Tietze’s extension theorem, we may assume that $\psi \in C_0(\mathbb{R}^d)$. Let $V \subset \mathbb{R}^d$ be a bounded open set such that $\overline{D} \subset\subset V$, and let $\psi_\alpha := \alpha R_\alpha^V \psi$. It is an elementary check that $\psi_\alpha \to \psi$ uniformly on $\overline{D}$. By the strong Markov property and (2.1),

$$E_x e^{-\alpha T_D} \psi_\alpha(X_{\tau_D}) = \alpha E_x e^{-\alpha T_D} E_{X_{\tau_D}} \int_0^{T_D} e^{-\alpha r} \psi(X_r) \, dr$$

$$= \alpha E_x e^{-\alpha T_D} \int_0^{T_D} e^{-\alpha (r - \tau_D)} \psi(X_r) \, dr = \alpha E_x \int_0^{T_D} e^{-\alpha r} \psi(X_r) \, dr = \alpha R_\alpha^V(x) \psi - \alpha R_\alpha^D \psi(x), \quad x \in D.$$
Since \((R^V_\alpha)_{\alpha>0}\) and \((R^D_\alpha)_{\alpha>0}\) are strongly Feller, \(Ee^{-\alpha\tau_D}\psi_\alpha(X_{\tau_D}) \in C(D)\), from which the desired assertion easily follows.

**Lemma 2.2.** Let \(u \in H^1_0(D)\) be quasi-continuous. Then for every \(x \in D\),

\[
u(X_t) \to 0 \quad P_x\text{-a.s. as } t \nearrow \tau_D.
\]

**Proof.** Let \(X^D\) denote the part of the process \(X\) on \(D\) (see [19, Section 4.4]). By Fukushima’s decomposition (see [19, Theorem 5.2.2]), for q.e. \(x \in D\),

\[
u(X^D_t) = u(x) + A_t + M_t, \quad t \geq 0, \quad P_x\text{-a.s.}
\]

where \(A\) is a continuous additive functional of \(X^D\) of zero energy and \(M\) is a continuous martingale additive functional of \(X^D\) of finite energy. From this we get

\[
u(X^D_t) = -(A_{\tau_D} - A_t) - (M_{\tau_D} - M_t), \quad t \leq \tau_D, \quad P_x\text{-a.s.},
\]

since \(u(X^D_{\tau_D}) = u(\Delta) = 0\), where \(\Delta\) is an extra point which is a one-point compactification of \(D\) (see [19, Theorem A.2.10]). Hence we get the assertion of the lemma for q.e. \(x \in D\). Now, set

\[
u(x) = P_x\left(\limsup_{t \nearrow \tau_D} |u(X_t)| > 0\right), \quad x \in D.
\]

Since \(\nu = 0\) q.e. on \(D\), we have \(\nu = 0\) on \(D\) by [5, Proposition II.3.2].

**Definition 2.3.** Let \(\psi \in C(\partial D)\). The function \(B\psi\), where \(B\) is defined by (1.8), is called a weak solution of \(\text{DP}(A, \partial D, \psi)\).

In Theorem 2.4 below, we give a probabilistic representation of a weak solution of \(\text{DP}(A, \partial D, \psi)\). Its proof is based solely upon Fukushima’s decomposition of additive functionals of \(X\). Then, we give another proof based upon results of [41], where the author used the theory of weak convergence of diffusion processes. The second proof gives us even stronger result then that formulated in Theorem 2.4.

**Theorem 2.4.** Let \(\psi \in C(\partial D)\) and let \(u\) be a weak solution to (1.2). Then

\[
u(x) = E_x\psi(X_{\tau_D}), \quad x \in D.
\]

**Proof.** We first assume that \(\psi \in H^1(D) \cap C(\overline{D})\). Let \(\{D_n\}\) be an increasing sequence of Lipschitz open sets such that \(D_n \subset D\) and \(\bigcup_{n \geq 1} D_n = D\). Since \(D_n\) is Lipschitz, there exists an extension of \(u|_{D_n}\) to \(\tilde{u}_n \in H^1(\mathbb{R}^d)\). By Fukushima’s decomposition (see
Letting \( n \rightarrow \infty \) in (2.7) and using (1.7) yields (2.3).

\[ u_n(x) = E_x u_n(x_{\tau_D}), \quad x \in D. \]  

Letting \( n \rightarrow \infty \) in (2.7) and using (1.7) yields (2.3).

**Corollary 2.5.** For every \( x \in D \),

\[ P_x(X_{\tau_D} \in dy) = \omega_x^A(dy). \]

**Remark 2.6.** Let \( D \) be connected, \( x \in D \) and \( p \geq 1 \). If \( \psi \in B^+(\partial D) \cap L^p(\partial D; \omega_x^A) \),

then \( \psi \in L^p(\partial D; \omega_y^A) \) for every \( y \in D \).

**Proof.** By (1.9) and Harnack’s inequality for every \( y \in D \) there exists \( c_y > 0 \) such that

\[ \omega_y^A \leq c_y \omega_x^A. \]  

Hence we get the result.

**Corollary 2.7.** Let \( D \) be connected. Assume that \( \psi \in B^+(\partial D) \cap L^p(\partial D; \omega_x^A) \) for some \( p \geq 1 \), \( x \in D \) and define \( u \) by (2.3). Then \( u \in C(D) \) and for every \( y \in D \),

\[ |u(y)| \leq \|\psi\|_{L^p(\partial D; \omega_y^A)}. \]  

**Proof.** Inequality (2.9) follows immediately from (2.3) and by Remark 2.6 the right-hand side of (2.9) is finite for every \( y \in D \). Choose \( r > 0 \) so that \( B(x, r) \subset D \). By Harnack’s inequality, there is \( c > 0 \) such that \( \omega_y^A \leq c \omega_x^A \) for \( y \in B(x, r) \). Choose \( \psi_n \in C(\partial D) \) so that \( \psi_n \rightarrow \psi \) in \( L^2(\partial D, \omega_x^A) \) and define \( u_n \) by (2.3) but with \( \psi \) replaced by \( \psi_n \). Then by (2.9),

\[ |u(y) - u_n(y)| \leq c \|\psi - \psi_n\|_{L^p(\partial D; \omega_y^A)}, \quad y \in B(x, r). \]

This implies that \( u \in C(D) \) because \( u_n \in C(D) \) by Lemma 2.1.

We will see in Section 3 that \( u \) defined by (2.3) is the Perron-Wiener-Brelot solution of the problem \( DP(A, \partial D, \psi) \). Therefore the next corollary generalizes to operators of the form (1.4) the corresponding result of [1] proved for \( A = \Delta \).
Corollary 2.8. Assume that $\psi \in C(\partial D)$ and there exists $\Psi \in H^1_{\text{loc}}(D) \cap C(\overline{D})$ such that $\Psi|_{\partial D} = \psi$ and $-A\Psi \in H^{-1}(D)$. Let $u \in H^1_0(D)$ be such that

$$-Au = A\Psi \quad \text{in} \quad H^{-1}(D).$$

Then

$$(u + \Psi)(x) = E_x\psi(X_{\tau_D}), \quad x \in D.$$  

Proof. Let $\{D_n\} \subset \subset D$ be an increasing sequence such that $\bigcup_{n \geq 1} D_n$, and let $w = u + \Psi$. Then $w \in C(D) \cap H^1_{\text{loc}}(D)$ and $\mathcal{E}(u,v) = 0$ for $v \in C^2_c(D)$, so $w$ is a weak solution to $\text{DP}(A, \partial D_n, w|_{\partial D_n})$. By Theorem 2.4,

$$w(x) = E_xw(X_{\tau_{D_n}}), \quad x \in D_n.$$  

By continuity of $\Psi$ and Lemma 2.2, $E_xw(X_{\tau_{D_n}}) \to E_x\psi(X_{\tau_D}), x \in D$, which proves the corollary.

Let $\bar{\sigma}$ denote the symmetric square root of $a$, i.e.

$$\bar{\sigma}(x) = a^{1/2}(x), \quad x \in \mathbb{R}^d. \quad (2.10)$$

Proposition 2.9. Let $\psi \in C(\partial D)$ and let $u$ be a weak solution to (1.2). Then for every $x \in D$,

$$u(X_t) = \psi(X_{\tau_D}) - \int_t^{\tau_D} \bar{\sigma}\nabla u(X_r) dB_r, \quad t \leq \tau_D, \quad P_x\text{-a.s.} \quad (2.11)$$

Proof. We first assume that $\psi \in C(\overline{D}) \cap H^1(D)$. Let $\{D_n\}$ be an increasing sequence of bounded open Lipschitz subsets of $D$ such that $D_n \subset \subset D$ and $\bigcup_{n \geq 1} D_n = D$. It is clear that $u$ is a weak solution to $\text{DP}(A, D_n, u|_{\partial D_n})$. By [41], there is a Wiener process $B$ such that

$$u(X_t) = u(X_{\tau_{D_n}}) - \int_t^{\tau_{D_n}} \bar{\sigma}\nabla u(X_r) dB_r, \quad t \leq \tau_{D_n}, \quad P_x\text{-a.s.}, \quad x \in D_n. \quad (2.12)$$

By (1.7), $u$ is bounded. Therefore, by using the Burkholder-Davis-Gundy inequality, it may be concluded from the above equation that there exists $c > 0$ depending only on $\|u\|_{\infty}$ such that

$$E_x \int_0^{\tau_{D_n}} |\bar{\sigma}\nabla u(X_r)|^2 dr \leq c, \quad x \in D_n.$$  

Applying Fatou’s lemma gives

$$E_x \int_0^{\tau_D} |\bar{\sigma}\nabla u(X_r)|^2 dr \leq c, \quad x \in D. \quad (2.13)$$

By Lemma 2.2 and regularity of $\psi$, we have $u(X_t) \to \psi(X_{\tau_D}), t \nearrow \tau_D$ $P_x\text{-a.s.}$ for $x \in D$. Therefore letting $n \to \infty$ in (2.12) we obtain (2.11).

We now assume that $\psi \in C(\partial D)$. Choose a sequence $\{\psi_n\} \subset C(\overline{D}) \cap H^1(D)$ so that $\psi_n \to \psi$ uniformly on $\partial D$. Let $u_n$ be a weak solution to (1.2) with $\psi$ replaced by $\psi_n$. By what has already been proved,

$$u_n(X_t) = \psi_n(X_{\tau_D}) - \int_t^{\tau_D} \bar{\sigma}\nabla u_n(X_r) dB_r, \quad t \leq \tau_D, \quad P_x\text{-a.s.} \quad (2.14)$$
By (1.7), \( u_n \to u \) pointwise and in \( H^1_{\text{loc}} \). Moreover, by Itô's formula,

\[
|u_n(x) - u_m(x)|^2 + E_x \int_0^{\tau_D} |\sigma \nabla (u_n - u_m)|^2(X_r) \, dr = E_x |\psi_n - \psi_m|^2(X_{\tau_D})
\]

(2.15)

for every \( x \in D \). Therefore letting \( n \to \infty \) in (2.14) and using the Burkholder-Davis-Gundy inequality we get the desired result. \( \square \)

**Remark 2.10.** Observe that by taking expectation in (2.11) with \( Gundy \) inequality we get the desired result.

Assertion (iv) follows from (2.17). As for (ii), we know from Theorem 2.4 that it holds for \( u_n \). By (v), \( u_n \to u \in H^1_{\text{loc}}(D) \), so (ii) holds for \( u \), too. Assertion (iii) follows easily from (vi). Inequality (vii) follows from Doob's \( L^2 \)-inequality. \( \square \)

**Definition 2.12.** We say that \( u \) is a soft solution of DP\((A, \partial D, \psi)\) if (i)–(iii) of Proposition 2.11 are satisfied.
Proposition 2.13. For every $\psi \in L^2(\partial D; \omega^A_m)$ there exists a unique soft solution to DP($A, \partial D, \psi$).

Proof. The existence part follows from Proposition 2.11. To prove uniqueness, suppose that $u_1, u_2$ are soft solutions to DP($A, \partial D, \psi$). Write $u = u_1 - u_2$ and consider an increasing sequence $\{D_n\}$ of open subsets of $D$ such that $D_n \subset \subset D$ and $\bigcup_{n \geq 1} D_n = D$. Since $u \in H^1(D_n) \cap C(\overline{D}_n)$,

$$u(x) = E_x u(X_{\tau D_n}), \quad x \in D_n$$

by Theorem 2.4. Therefore, by Corollary 2.5 and (1.12), the right-hand side of the above inequality converges to zero as $n \to \infty$. This proves that $u_1 = u_2$. \qed

Remark 2.14. By Theorem 2.4 and Proposition 2.11, the definitions of soft solution and weak solution agree for $\psi \in C(\partial D)$. However, be careful! When $D$ is a Lipschitz domain, then one can define a weak solution to DP($A, \partial D, \psi$) for $\psi \in H^{1/2}(\partial D)$ as a function $u \in H^1(D)$ such that (1.5) is satisfied and $\text{Tr}(u) = \psi$, where $\text{Tr}$ is the usual trace operator. In general, $u$ defined in this way does not have the property formulated in Proposition 2.11(iii). The reason is that the trace is defined $\sigma$-a.e. ($\sigma$ is the Lebesgue surface measure on $\partial D$), so the weak solution to DP($A, D, \psi$) defined via the operator $\text{Tr}$ does not depend on the $\sigma$-version of $\psi$. This is not true for soft solutions because they are defined by (2.3), and in general, the measure $\omega^A_m$ is not absolutely continuous with respect to $\sigma$ (in fact, it may be completely singular, see [10, 36]). In different words, if $\psi_1 = \psi_2$ $\sigma$-a.e., then weak solutions of DP($A, \partial D, \psi_1$) defined via the trace operator $\text{Tr}$ are equal, but it may happen that $u_1(x) \neq u_2(x)$, $x \in D$, where $u_1, u_2$ are defined by (2.3) with $\psi$ replaced by $\psi_1$ and $\psi_2$, respectively. The definition of soft solution is just more sensitive to the boundary values. In Section 4 we will show that, if $u$ is a weak solution to DP($A, \partial D, \psi$) defined via the trace operator $\text{Tr}$, then it is a soft solution of DP, but with some specially chosen $\sigma$-version of $\psi$ (defined $\omega^A_m$-a.e.).

3 Perron-Wiener-Brelot solutions

We begin with recalling some notions from [4]. Let $E$ be a locally compact separable metric space. For a given class of functions $F \subset \mathcal{B}^+(E)$, we set $S(F) = \{\sup_{n \geq 1} f_n : \{f_n, n \geq 1\} \subset F\}$. We say that $F$ is $\sigma$-stable if $S(F) = F$. Let $\mathcal{W}$ be a convex cone of positive numerical lower semi-continuous (l.s.c) functions on $E$. By $\mathcal{T}^f$ we denote the smallest topology under which all functions from $\mathcal{W}$ are continuous (so called fine topology). For $u \in \mathcal{B}(E)$, we denote by $\bar{u}^f$ its l.s.c. regularization with respect to the topology $\mathcal{T}^f$.

We say that a pair $(E, \mathcal{W})$ is the balayage space if

(B₁) $\mathcal{W}$ is $\sigma$-stable,

(B₂) for every $V \subset \mathcal{W}$, $\inf \bar{V}^f \in \mathcal{W}$,

(B₃) for all $u, v, w \in \mathcal{W}$ such that $u \leq v + w$ there exist $\bar{v}, \bar{w} \in \mathcal{W}$ such that $u = \bar{v} + \bar{w}$, $\bar{v} \leq v$, $\bar{w} \leq w$,

(B₄) there exists a function cone $\mathcal{P} \subset C^+(E)$ such that $S(\mathcal{P}) = \mathcal{W}$. 


Example 3.1. Let \( \{P_t, t \geq 0\} \) be the semigroup generated by the operator (1.4). By [4, Corollary II.4.9], the pair \((E_P, \mathbb{R}^d)\), where \(E_P\) is the set of excessive functions on \(\mathbb{R}^d\) defined as \(E_P = \{ f \in \mathcal{B}^+(\mathbb{R}^d) : \sup_{t>0} P_tf = f \}\), is a balayage space.

It is well known (see [4, Section III.2]) that every balayage space \((E, W)\) generates naturally the so called harmonic kernel \((H_U, U \subset E, U - \text{open})\). By \(*H(U)\) we denote the set of all hyperharmonic functions on \(U\), i.e.

\[ *H(U) = \{ u \in \mathcal{B}(E) ; u|_U \text{ is l.s.c., } -\infty < H_V(u)(x) \leq u(x), \ x \in V, V - \text{open, } \nabla \subset U \}. \]

The class \(H(U)\) of harmonic functions is defined by

\[ H(U) = *H(U) \cap (-*H(U)). \] (3.1)

For \( \psi \in \mathcal{B}^+(E) \), we set

\[ \mathcal{U}_\psi^U = \{ u \in *H(U) : u \text{ is lower bounded on } U \text{ and } \liminf_{y \to x} u(y) \geq \psi(x), \ x \in \partial U \}, \]

\[ \mathcal{L}_\psi^U = -\mathcal{U}_\psi^U, \]

and we define operators \(\mathcal{P}_\psi^U\) and \(H_\psi^U\) by

\[ \mathcal{P}_\psi^U \psi \equiv \inf \mathcal{U}_\psi^U, \quad H_\psi^U \psi \equiv \sup \mathcal{L}_\psi^U. \]

If \(\mathcal{P}_\psi^U \psi = H_\psi^U \psi\), we set \(H^U \psi = H_\psi^U \psi\).

3.1 Linear equations

From now on we consider the balayage space from Example 3.1. As in Section 2, \(D\) is an arbitrary bounded open subset of \(\mathbb{R}^d\). By [4, Theorem VI.3.16], for all bounded open \(V \subset \mathbb{R}^d\) and \(u \in \mathcal{B}^+(\mathbb{R}^d)\),

\[ H_V(u)(x) = E_x u(X_{tv}), \quad x \in V. \] (3.2)

Definition 3.2. Let \(\psi \in \mathcal{B}(\partial D)\). We say that \(u : D \to \mathbb{R}\) is a Perron-Wiener-Brelot solution of \(\text{DP}(A, \partial D, \psi)\) if \(u = \mathcal{P}_\psi^D \psi = H_\psi^D \psi\).

Proposition 3.3. Assume that \(u \in \mathcal{B}(D)\) is lower bounded and finite m-a.e. Then \(u \in *H(D)\) if and only if there exists a positive Borel measure \(\mu\) on \(D\) such that

\[ u - R^D \mu \in H^1_{loc}(D), \quad \mathcal{E}(u - R^D \mu, v) = 0, \quad v \in C^1_c(D). \] (3.3)

Proof. Assume that \(u \in *H(D)\). Since \(u\) is lower bounded, we may assume that \(u\) is positive. By [5, Corollary II.5.3], \(u\) is an excessive function with respect to the resolvent \((R^D)_{t \geq 0}\). Therefore, by Riesz’s decomposition theorem (see [22]), there exists a Borel positive measure \(\mu\) on \(D\) and a positive harmonic function \(h\) such that

\[ u = R^D \mu + h. \]

By the above, the definition of a harmonic function and (3.2),

\[ E_x (u - R\mu)(X_{\tau_V}) = u(x) - R\mu(x), \quad x \in V \]

for any \(V \subset D\). By Corollary 2.7, \(u - R\mu \in C(D)\) and by Proposition 2.11 (3.3) holds. Now assume that (3.3) is satisfied. By [38], \(u - R\mu \in C(D)\), so by Theorem 2.4 and (3.2), \(u \in *H(D)\). \(\square\)
Corollary 3.4. Function $u \in B(D)$ is harmonic iff $u \in H^1_{loc}(D)$ and

$$
E(u,v) = 0, \quad v \in C^1_c(D).
$$

Theorem 3.5. Assume that $\psi \in L^1(\partial D; \omega^A_m)$. Then there exists a unique PWB-solution $u$ to $\text{DP}(A, \partial D, \psi)$. Moreover,

$$
u(x) = E_x \psi(X_{\tau_D}), \quad x \in D.
$$

Proof. By Corollary 2.6, $\psi \in L^1(\partial D; \omega^A_x)$ for every $x \in D$. Therefore, by [4, Corollary VII.2.12], there exists a PWB-solution $u$ to $\text{DP}(A, \partial D, \psi)$ and $u = H_D \psi$. By virtue of (3.2), this proves the theorem.

The following corollary follows immediately from Proposition 2.13 and Theorem 3.5.

Corollary 3.6. If $\psi \in L^2(\partial D; \omega^A_m)$, then $u$ is a soft solution to $\text{DP}(A, \partial D, \psi)$ if and only if it is a PWB-solution. If $\psi \in C(\partial D)$, then $u$ is a weak solutions to $\text{DP}(A, \partial D, \psi)$ if and only if it is PWB-solution.

Let us recall that a bounded open set is called regular at $x \in \partial D$ for $A$ if each weak solution $u$ to $\text{DP}(A, \partial D, \psi)$ with $\psi \in C(\partial D)$ has the property that $u(y) \to \psi(x)$ as $y \to x$, $y \in D$. The following corollary is a consequence of Corollary 3.6 and [13, Theorem 1.23]. Note that [13, Theorem 1.23] is proved for Brownian motion, but its proof for the diffusion $X$ goes without any changes (the inequality preceding [13, Eq. (35)] holds true for $X$ by [48, Lemma II.1.2]).

Corollary 3.7. The following statements are equivalent:

(i) $D$ is regular at $x \in \partial D$ for $A$.

(ii) $P_x(\tau_D > 0) = 0$.

The following result was proved by different methods in [33].

Corollary 3.8. Let $D$ be a bounded open subset of $\mathbb{R}^d$ and let $x \in \partial D$. Then $x$ is regular for $A$ if and only if it is regular for $\Delta$.

Proof. Follows from Corollary 3.7 and [25, Corollary 1].

Remark 3.9. By [25, Remark 1], if the fine topologies $\mathcal{O}_1, \mathcal{O}_2$ generated by two operators $A_1, A_2$ are equal, then the regular points for $A_1$ and $A_2$ are the same for each open set $D \subset \mathbb{R}^d$. Since the fine topology is generated by excessive functions, the well known estimates for Green functions (see, e.g., [33]) imply that the fine topologies generated by $A$ and $\Delta$ are the same.

In Theorem 3.5 we obtained stochastic representation of PWB-solutions by using the theory of balayage spaces developed in [4]. This theory is rather abstract and advanced. In the next subsection we derive the stochastic representation (in more general context of semilinear equation) in more elementary way, without referring to the results of [4].
3.2 Semilinear equations

From now on, we treat formula (3.2) as the definition of the operator \( H_V \). If \( X \) is a Wiener process (i.e. \( A = \frac{1}{2} \Delta \)), it is rotationally invariant. Using this property, one can show the following formula

\[
H_{B(x,r)}(u)(x) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} u(y) d\sigma(y)
\]

(see [13, Proposition 1.21]).

**Remark 3.10.** It is worth mentioning that the proof of Proposition 3.3 is based on formula (3.2), and that in the proof we do not use the fact that \( \{H_V\} \) is a harmonic kernel. Therefore in Proposition 3.3 we have proved without referring to the theory of balayage spaces that a lower bounded finite \( m \)-a.e. \( u \in B(D) \) is hyperharmonic if and only if (3.3) is satisfied.

We will consider the following equation

\[
-Au = f(\cdot, u) + \mu \quad \text{on} \quad D, \quad u = \psi \quad \text{on} \quad \partial D,
\]

(3.4)

where \( \psi \in L^1(\partial D; \omega^A_m) \) and \( \mu \) is smooth measure (i.e. absolutely continuous with respect to Cap\(_A\)) such that \( R^D|\mu| < \infty \) q.e.

**Definition 3.11.** We say that \( u \) is the PWB-solution of (3.4) if \( R^D|f(\cdot, u)| < \infty \) q.e. and

\[
u - R^D(f(\cdot, u)) - R^D\mu = H^D\psi \quad \text{q.e.}
\]

**Definition 3.12.** We say that a Borel measurable function \( u \) on \( D \) is a subsolution (resp. supersolution) of (3.4) if \( u - R^D(f(\cdot, u)) - R^D\mu \in \mathcal{L}^D_\psi \) (resp. \( u - R^D(f(\cdot, u)) - R^D\mu \in \mathcal{U}^D_\psi \)).

We will see that the solution of (3.4) exists, and moreover, it is the supremum over all supersolutions of (3.4). To prove this, we will need the following two lemmas.

**Lemma 3.13.** Assume that \( u_i - R^D\mu_i \in \mathcal{L}^D_\psi, \ i = 1, 2, \) for some smooth measures \( \mu_1, \mu_2 \) such that \( R^D|\mu_i| < \infty \) q.e. Then

\[
u_1 \vee u_2 - R^D(1_{\{u_1 > u_2\}} \cdot \mu_1) - R^D(1_{\{u_1 \leq u_2\}} \cdot \mu_2) \in \mathcal{L}^D_{\psi_1 \vee \psi_2}.
\]

**Proof.** Write \( h_i = u_i - R^D\mu_i \). By [5, Corollary II.5.3] and [43, Excercise 4.20], \( -h_i(X) \) is a càdlàg supermartingale under \( P_x \) for q.e. \( x \in D \). Therefore, for q.e. \( x \in D \), there is an increasing predictable càdlàg process \( A^{x,i} \) and a local martingale \( M^{x,i} \) such that

\[
h_i(X_t) = h_i(X_0) + \int_0^t dA^{x,i}_r + \int_0^t dM^{x,i}_r, \quad t < \tau_D \quad P_x\text{-a.s.}
\]

Hence

\[
u_i(X_t) = \nu_i(X_0) - \int_0^t dA^{\mu_i}_r + \int_0^t dA^{x,i}_r + \int_0^t dM^{x,i}_r, \quad t < \tau_D \quad P_x\text{-a.s.}
\]
for some local martingale $\tilde{M}^{x,i}$. Therefore, applying the Itô-Meyer formula, we see that for q.e. $x \in D$,

$$(u_1 \lor u_2)(X_t) = (u_1 \lor u_2)(X_0) - \int_0^t 1_{\{u_1 > u_2\}}(X_r-) dA_r^{u_1}$$

$$- \int_0^t 1_{\{u_1 \leq u_2\}}(X_r-) dA_r^{u_2} + \int_0^t 1_{\{u_1 > u_2\}}(X_r-) dA_r^{u_2,1}$$

$$+ \int_0^t 1_{\{u_1 \leq u_2\}}(X_r-) dA_r^{x,2} + \int_0^t dL_r^x + \int_0^t dN_r^x, \quad t < \tau_D, \quad P_x\text{-a.s.}$$

for some local martingale $N^x$ and an increasing càdlàg process $L^x$. This implies the desired result. \hfill \Box

Let us consider the following assumptions:

(H1) $E_x|\psi(X_{\tau_D})| < \infty$ for q.e. $x \in D$.

(H2) There exists $g \in B^+(D)$ such that $R^D g < \infty$ and $f(x,y) \leq g(x)|y|$ for all $x \in D$, $y \in \mathbb{R}$.

(H3) $y \mapsto f(x,y)$ is continuous for every $x \in D$.

(H4) If $\underline{u}, \overline{u} \in B(D)$, $\underline{u} \leq \overline{u}$ and $R^D|f(\cdot, \underline{u})| + R^D|f(\cdot, \overline{u})| < \infty$ q.e., then there exists $h \in B^+(D)$ such that $R^D h < \infty$ q.e. and $|f(\cdot, u)| \leq h$ for every $u \in B(D)$ with $\underline{u} \leq u \leq \overline{u}$.

**Lemma 3.14.** Assume that $\underline{u}$ (resp. $\overline{u}$) is a subsolution (resp. supersolution) of (3.4) such that $\underline{u} \leq \overline{u}$. Let

$$\bar{f}(x,y) = \begin{cases} f(x,\overline{u}(x)), & y \geq \overline{u}(x), \\ f(x,y), & \underline{u}(x) < y < \overline{u}(x), \\ f(x,\underline{u}(x)), & y \leq \underline{u}(x), \end{cases}$$

(3.5)

and let $u$ be a solution of (3.4) with $f$ replaced by $\bar{f}$. Then $\underline{u} \leq u \leq \overline{u}$.

**Proof.** By the definitions of a solution and a supersolution of (3.4),

$$(u - \underline{u}) - (R^D \bar{f}_u - R^D f_{\underline{u}}) \in \mathcal{L}^D_0.$$

By Lemma 3.13,

$$(u - \overline{u}) + - R^D(1_{\{u > \overline{u}\}}(\bar{f}_u - f_{\overline{u}})) \in \mathcal{L}^D_0.$$

But $1_{\{u > \overline{u}\}}(\bar{f}_u - f_{\overline{u}}) = 0$, which implies that $(u - \overline{u})^+ \in \mathcal{L}^D_0$. Hence $(u - \overline{u})^+ = 0$, so $u \leq \overline{u}$. Similarly we prove that $u \leq \underline{u}$. \hfill \Box

**Corollary 3.15.** Assume that (H1)–(H4) are satisfied and there exists a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (3.4) such that $\underline{u} \leq \overline{u}$. Then there exists a solution $u$ of (3.4) such that $\underline{u} \leq u \leq \overline{u}$.

**Proof.** Observe that $u$ is a solution to (3.4) if and only if $w = u - H_D \psi$ is a solution to the problem

$$-Aw = f_{H_D \psi}(:,w) \quad \text{in} \quad D, \quad w_{|\partial D} = 0.$$
with
\[ f_{HD}(x, y) = f(x, y + H_D\psi(x)), \quad x \in D, \ y \in \mathbb{R}. \]
Therefore, by \cite[Theorem 3.4]{26} there exists a solution \( u \) of (3.4) with \( f \) replaced by \( \bar{f} \), where \( \bar{f} \) is defined by (3.5). By Lemma 3.14, \( u \leq u \leq \bar{u} \), so in fact \( u \) is a solution of (3.4). \qed

**Proposition 3.16.** Assume that (H1)–(H4) are satisfied and there exists a subsolution \( \underline{u} \) and a supersolution \( \bar{u} \) of (3.4) such that \( \underline{u} \leq \bar{u} \). Then
\[ w = \sup\{v \leq \underline{u} : v \text{ is a subsolution of (3.4)}\} \quad (3.6) \]
is a solution of (3.4).

**Proof.** Set \( C = \{v \leq \underline{u} : v \text{ is a subsolution of (3.4)}\} \). By the assumptions of the proposition, \( C \) is nonempty. By \cite[Theorem V.1.17]{5} and Lemma 3.13, there exists a nondecreasing sequence \( \{w_n\} \subset C \) such that \( w_n \to w \) q.e. It is clear by assumptions (H3), (H4) that
\[ H_V(w_n - R^D f_{w_n} - R^D \mu) \to H_V(w - R^D f_w - R^D \mu) \]
for every open set \( V \subset D \). Hence \( w \in C \). By Corollary 3.15 there exists a solution \( u \) of (3.4) such that \( w \leq u \leq \bar{u} \). But \( u \in C \), so \( u \leq w \), which implies that \( u = w \). \qed

4 Weak Dirichlet problem vs Dirichlet problem on Lipschitz domain

Throughout this section, we assume that \( D \) is a bounded open Lipschitz subset of \( \mathbb{R}^d \). It is well known (see, e.g., \cite{21, 23}) that then exists a linear continuous operator (trace operator)
\[ \text{Tr} : H^1(D) \to L^2(\partial D; \sigma) \]
such that
\[ \text{Tr}(u) = u|_{\partial D}, \quad u \in C(\overline{D}) \cap H^1(D). \]

**Definition 4.1.** Let \( \psi \in \gamma(H^1(D)) = H^{1/2}(\partial D) \). We say that \( u \in H^1(D) \) is a weak solution of DP\((A, \partial D, \psi)\) if (1.5) is satisfied and \( \text{Tr}(u) = \psi \).

**Remark 4.2.** It is clear that if \( \psi \in H^{1/2}(\partial D) \cap C(\partial D) \), then \( u \) a weak solution of DP\((A, \partial D, \psi)\) in the sense of Definition 4.1 if and only if it is a weak solution in the sense of Definition 2.3.

The aim of this section is to explain the relation between weak solutions of DP and PWB-solutions of DP. It is known that for every \( v \in H^1(D) \), \( \text{Tr}(v) = 0 \) if and only if \( v \in H^1_0(D) \). Consequently, if \( u \) is a weak solution to DP\((A, \partial D, \psi)\), then \( u \) is a solution to wDP\((A, D, \bar{\psi})\) with \( \bar{\psi} \in H^1(D) \) such that \( \text{Tr}(\bar{\psi}) = \psi \). Therefore the results of this section are the first step in describing a relation between solutions of wDP and DP.

Let us recall from the previous section, that each PWB-solution of DP\((A, \partial D, \psi)\) is of the form
\[ v(x) = \int_{\partial D} \psi(y) \omega^A_x(dy) \]
for some $\psi \in \mathcal{B}(\partial D)$. It is well known (see [10]) that the measure $\omega_m^A$ may be completely singular with respect to the surface measure $\sigma$ on $\partial D$. As a consequence, PWB-solutions depend on the $\sigma$-version of $\psi$. On the other hand, weak solutions of DP are not sensitive to the $\sigma$-version of $\psi$. For this reason the relation between weak and PWB-solutions to DP is a rather delicate matter.

**Proposition 4.3.** Let $\psi \in H^{1/2}(\partial D)$ and let $u$ be a weak solution of $\text{DP}(A, \partial D, \psi)$. Then for $m$-a.e. $x \in D$,

$$u(x) = \int_{\partial D} \tilde{\psi}(y) \omega_x^A(dy),$$

where $\tilde{\psi}$ is a quasi-continuous $m$-version of an extension $\tilde{\psi} \in H^1(\mathbb{R}^d)$ of $\psi$.

**Proof.** Let $\{D_n\}$ be an increasing sequence of open sets such that $D_n \subset D$ and $\bigcup_{n \geq 1} D_n = D$. Since $u$ is a weak solution to $\text{DP}(A, \partial D, \psi)$, $u \in H^1(D) \cap C(D)$. Therefore, $u$ is a weak solution to $\text{DP}(A, \partial D_n, u_{\partial D_n})$ for each $n \geq 1$. Consequently, by Theorem 2.4,

$$u(x) = E_x u(X_{\tau_{D_n}}), \quad x \in D. \quad (4.1)$$

By the definition of a weak solution, $\text{Tr}(u) = \psi$. Hence $u - \tilde{\psi} \in H^1_0(D)$, so by Lemma 2.2, for q.e. $x \in D$, $(u - \tilde{\psi})(X_t) \to 0$ $P_x$-a.s. as $t \nearrow \tau_D$. Since $\tilde{\psi}$ is quasi-continuous, $\tilde{\psi}(X_t) \to \tilde{\psi}(X_{\tau_D})$ $P_x$-a.s. as $t \nearrow \tau_D$. Hence, for q.e. $x \in D$, $u(X_t) \to \tilde{\psi}(X_{\tau_D})$ $P_x$-a.s. as $t \nearrow \tau_D$. By [26, Remark 2.13], the family $\{u(X_\tau), \tau \in T\}$ is uniformly integrable under the measure $P_x$ for q.e. $x \in D$. Hence, by the Vitali convergence theorem, $E_x u(X_{\tau_{D_n}}) \to E_x \tilde{\psi}(X_{\tau_D})$ q.e., which when combined with (4.1) and Corollary 2.5 gives the desired result. \hfill $\square$

**Corollary 4.4.** Let $\psi \in H^{1/2}(\partial D)$ and let $\tilde{\psi}$ be as in Proposition 4.3. If $u$ is a weak solution to $\text{DP}(A, \partial D, \psi)$, then $\tilde{\psi}$ is a soft solution and PWB-solution to $\text{DP}(A, \partial D, \tilde{\psi}|_{\partial D})$.

**Remark 4.5.** In general, the function $\tilde{\psi}|_{\partial D}$ appearing in Corollary 4.4 is not a $\sigma$-version of $\text{Tr}(\psi)$. In general, the measures $\sigma$ and $\omega_m^A$ determine different equivalent classes (no inclusion between equivalent classes). However, there exists an $\omega_m^A$-version of $\tilde{\psi}|_{\partial D}$ which is a $\sigma$-version of $\text{Tr}(\psi)$. The construction of such a version is as follows. Let $\varphi_\alpha := \alpha R_\alpha(T_\alpha(\tilde{\psi}))$. Then $\varphi_\alpha \in H^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, so by the definition of the trace operator,

$$\text{Tr}(\varphi_\alpha) = (\varphi_\alpha)|_{\partial D}.$$

It is known that $\varphi_\alpha \rightarrow \tilde{\psi}$ in $H^1(\mathbb{R}^d)$. Hence $\text{Tr}(\varphi_\alpha) \rightarrow \text{Tr}(\tilde{\psi})$ in $L^2(\partial D; \sigma)$ by continuity of the trace operator. Moreover, by [19, Theorem 2.1.4], we may assume that $\varphi_\alpha \rightarrow \tilde{\psi}$ q.e. Therefore, by Corollary 2.5, $(\varphi_\alpha)|_{\partial D} \rightarrow \tilde{\psi}|_{\partial D} \omega_m^A$-a.e. It follows that $g$ defined as $g = \lim \sup_{\alpha \rightarrow \infty} (\varphi_\alpha)|_{\partial D}$ has the property that $g = \text{Tr}(\psi)$ $\sigma$-a.e. and $g = \tilde{\psi}|_{\partial D} \omega_m^A$-a.e.

5 **Trace operator on $H^1(D)$ for arbitrary bounded open set $D$**

Let $q$ be the metric defined by (1.16), $D^*$ be the completion of $D$ with respect to $q$, and let $\partial M D = D^* - D$. By [12, Proposition 14.6], for every $x \in D$,

$$P_x(X_{\tau_D} := \lim_{t \nearrow \tau_D} X_t \text{ exists with respect to the metric } q) = 1.$$
From now on $X_{\tau_D-}$ denotes $\lim_{t \nearrow \tau_D} X_t$, where limit is taken in metric $\rho$. It is clear that for every $x \in D$,
$$P_x(X_{\tau_D-} \in \partial_M D) = 1.$$ Therefore, we can define a measure $h^A_x$ on $\partial D$ (called harmonic measure) as
$$h^A_x(dy) = P_x(X_{\tau_D-} \in dy). \quad (5.1)$$
Let $\mathbb{X}^D = (X^D, \{P_x, x \in D \cup \{\Delta\}\}, \mathcal{F}, \tau_D)$ denote the process $\mathbb{X}$ killed upon leaving $D$ (see, e.g., [5, 19] for the definition of the killed process). Let us recall that $\{\Delta\}$ is one-point compactification of $D$ and by convention $u(\Delta) = 0$. By $\mathcal{I}$ we denote the invariant $\sigma$-field for $\mathbb{X}^D$, i.e. $A \in \mathcal{I}$ if and only if $A \in \mathcal{F}$ and for every $t \geq 0$,
$$\theta_t^{-1}(A \cap \{\tau_D > 0\}) = A \cap \{\tau_D > t\}.$$ Let
$$X_t^* = 1_{[0,\tau_D]}(t)X_t + 1_{\{t=\tau_D\}}X_{\tau_D-}, \quad t \in [0,\tau_D].$$ By $\gamma_A$ we denote the linear operator
$$\gamma_A : H^1(D) \to L^2(\partial_M D; h^A_m)$$
which to every $\psi \in H^1(D)$ assigns the unique function $\gamma_A(\psi) \in L^2(\partial_M D; h^A_m)$ such that the process
$$[0,\tau_D] \ni t \mapsto (1_D \psi + \gamma_A(\psi)1_{\partial_M D})(X_t^*) \quad (5.2)$$
is continuous under the measure $P_x$ for $m$-a.e. $x \in D$. Uniqueness of the trace $\gamma_A(\psi)$ follows from the fact that by the definition of $\gamma_A$,
$$\gamma_A(\psi)(X_{\tau_D-}) = \lim_{t \nearrow \tau_D} \psi(X_t) \quad P_x\text{-a.s.} \quad (5.3)$$

**Remark 5.1.** Let $u_n$, $n \geq 1$, be a solution to wDP$(A,D,\psi_n)$ and $u$ be a solution to wDP$(A,D,\psi)$. By an elementary calculus, if $\psi_n \to \psi$ in $H^1(D)$, then $u_n \to u$ in $H^1(D)$.

**Theorem 5.2.** The operator $\gamma_A$ is well defined and continuous.

**Proof.** Let $\psi \in H^1(D)$. Then, by [33], there exists a unique solution $u \in H^1(D) \cap C(\bar{D})$ of wDP$(A,D,\psi)$. Let $\{D_n\}$ be an increasing sequence of open sets such that $D_n \subset \subset D$ and $\bigcup_{n \geq 1} D_n = D$. It is clear that $u$ is a weak solution to DP$(A,D_n,u|_{\partial D_n})$ for $n \geq 1$. Consequently, by Theorem 2.4,
$$u(x) = E_x u(X_{\tau_{D_n}}), \quad x \in D. \quad (5.4)$$
By Proposition 2.9, for every $x \in D_n$,
$$u(X_t) = u(X_{\tau_{D_n}}) - \int_t^{\tau_{D_n}} \tilde{\sigma} \nabla u(X_r) \, dr, \quad t \in [0,\tau_{D_n}], \quad (5.5)$$
where $\tilde{\sigma}$ is defined by (2.10). It follows that $u(X)$ is a martingale on $[0,\tau_{D_n}]$ for every $n \geq 1$, and hence a martingale on $[0,\tau_D)$. By Itô’s formula,
$$E_x|u(X_{\tau_{D_n}})|^2 = |u(x)|^2 + E_x \int_0^{\tau_{D_n}} |\tilde{\sigma} \nabla u(X_r)|^2 \, dr \leq |u(x)|^2 + \|a\|_{\infty} E_x \int_0^{\tau_D} |\nabla u(X_r)|^2 \, dr. \quad (5.6)$$
Since \( u \in C(D) \), the left-hand side of the above inequality is finite for \( m \)-a.e. \( x \in D \). The integral on the right-hand side is equal to \( R^D |\nabla u|^2 \). Since \( R^D |\nabla u|^2 \in L^1(D; m) \), the process \( u(X) \) is a uniformly integrable martingale on \([0, \tau_D]\) under the measure \( P_x \) for \( m \)-a.e. \( x \in D \). Let

\[
Y = \limsup_{t \nearrow \tau_D} u(X_t).
\]

Then by the martingale convergence theorem, for \( m \)-a.e. \( x \in D \), \( E_x|Y| < \infty \), \( Y = \lim_{t \nearrow \tau_D} u(X_t) P_x \)-a.s. and in \( L^1(\Omega, P_x) \). It is an elementary check that \( Y \in \mathcal{I} \). Hence, by [12, Theorem 14.10], there exists \( g \in \mathcal{B}(\partial M D) \) such that for \( m \)-a.e. \( x \in D \),

\[
Y = g(X_{\tau_D-}) \quad P_x\text{-a.s.} \tag{5.7}
\]

Using Fatou’s lemma, we deduce from (5.6) and (5.7) that

\[
\|g\|^2_{L^2(\partial M D; h^A_{\Omega})} \leq \|u\|^2_{L^2(D; m)} + \|a\|_{\infty} \|\delta\|_{\infty} \|\nabla u\|^2_{L^2(D; m)}. \tag{5.8}
\]

It is clear that the process \( (u1_D + g1_{\partial M D})(X^*) \) is continuous on \([0, \tau_D]\). By the definition of a solution to \( wDP(A, D, \psi) \), \( u - \psi \in H^1_0(D) \), so by Lemma 2.2, \( (u - \psi)(X_t) \to 0 \) \( P_x \)-a.s. as \( t \nearrow \tau_D \) for \( m \)-a.e. \( x \in D \). Hence \( \psi(X_t) \to g(X_{\tau_D-}) P_x\)-a.s. as \( t \nearrow \tau_D \) for \( m \)-a.e. \( x \in D \). This implies that the process \( (1_D\psi + g1_{\partial M D})(X^*) \) is continuous on \([0, \tau_D]\) under the measure \( P_x \) for \( m \)-a.e. \( x \in D \). Thus \( \gamma_A(\psi) = g \), so \( \gamma_A \) is well defined. Continuity of \( \gamma_A \) follows from (5.8) and Remark 5.1.

**Corollary 5.3.** If \( u \) is a solution to \( wDP(A, D, \psi) \) with \( \psi \in H^1(D) \), then \( \gamma_A(u) = \gamma_A(\psi) \) and

\[
u(x) = E_x\gamma_A(\psi)(X_{\tau_D-}) \quad m\text{-a.e.} \tag{5.9}
\]

**Proof.** In the proof of Theorem 5.2 we have shown that the processes \( (u1_D + g1_{\partial M D})(X^*) \) and \( (\psi1_D + g1_{\partial M D})(X^*) \) are continuous on \([0, \tau_D]\). This implies that \( g = \gamma_A(u) = \gamma_A(\psi) \). By [26, Remark 2.13], the family \( \{u(X_{\tau_D})\} \) is uniformly integrable under the measure \( P_x \) for \( m \)-a.e. \( x \in D \). From this, continuity of the process \( (u1_D + g1_{\partial M D})(X^*) \) and (5.4) we get (5.9) by the Vitali convergence theorem.

**Theorem 5.4.** There exists an isometric embedding

\[
i_M : L^2(\partial D; \omega^A_D) \hookrightarrow L^2(\partial M D; h^A_{\Omega})
\]

such that for \( m \)-a.e. \( x \in D \),

\[
i_M(\psi)(X_{\tau_D-}) = \psi(X_{\tau_D}) \quad P_x\text{-a.s.} \tag{5.10}
\]

**Proof.** Let \( \psi \in L^2(\partial D; \omega^A_D) \). Set \( u(x) = E_x\psi(X_{\tau_D}), x \in D \). By Corollary 2.7, \( u \in C(D) \cap H^1(D) \). By the Markov property,

\[
u(X_t) = E_x(\psi(X_{\tau_D})|\mathcal{F}_t), \quad t \leq \tau_D,
\]

and by the martingale convergence theorem, \( u(X_t) \to \psi(X_{\tau_D}) \) as \( t \nearrow \tau_D \). Set

\[
Y = \limsup_{t \nearrow \tau_D} u(X_t).
\]

As in the proof of Theorem 5.2, we show that there exists \( g \in \mathcal{B}(\partial M D) \) such that (5.7) is satisfied. Putting now

\[
i_M(\psi) = g, \tag{5.11}
\]

we get (5.10). Squaring (5.10), taking the expectation with respect to \( P_x \) and then integrating with respect to \( x \) on \( D \) shows that \( i_M \) is an isometry.
Remark 5.5. From Theorem 5.4 it follows in particular that $L^2(\partial D; \omega^A_m)$ is a closed subset of $L^2(\partial M \cap D; h^A_m)$.

By Theorem 5.4, we may think of $L^2(\partial D; \omega^A_m)$ as being a subset of $L^2(\partial M \cap D; h^A_m)$. This allows us to establish some standard properties of the trace operator $\gamma_A$.

Proposition 5.6. Let $\psi \in H^1(D)$. Then $\psi \in H^1_0(D)$ if and only if $\gamma_A(\psi) = 0$.

Proof. Assume that $\psi \in H^1_0(D)$. Then, by Lemma 2.2, $\psi(X_t) \to 0$ $P_x$-a.s. as $t \wedge \tau_D$ for $m$-a.e. $x \in D$. Therefore, by (5.3), $\gamma_A(\psi) = 0$. Conversely, suppose that $\gamma_A(\psi) = 0$. Let $u$ be a solution to wDP$(A, D, \psi)$. Then $u - \psi \in H^1_0(D)$. By Corollary 5.3, $u \equiv 0$. Thus $\psi \in H^1_0(D)$. $\square$

Proposition 5.7. Let $\psi \in H^1(D) \cap C(\overline{D})$. Then $\gamma_A(\psi) = \psi|_{\partial D}$.

Proof. Let $u(x) = E_x \psi(X_{\tau_D})$. By (5.11), $i_M(\psi|_{\partial D}) = \gamma_A(u)$. Therefore, it is enough to show that $\gamma_A(u) = \gamma_A(\psi)$. But by Theorem 2.4, $u - \psi \in H^1(D)$, which when combined with Proposition 5.6 gives the result. $\square$

Let $H^1_e(D)$ denote the set of all $u \in H^1(D)$ for which there exists $\psi \in B(\partial D)$ such that

$$[0, \tau_D] \ni t \mapsto (u1_D + \psi1_{\partial D})(X_t)$$

is continuous $P_x$-a.s. for $m$-a.e. $x \in D$.

Proposition 5.8. We have

$$\gamma_A^{-1}(L^2(\partial D; \omega^A_m)) = H^1_e(D).$$

Proof. Assume that $\psi \in \gamma_A^{-1}(L^2(\partial D; \omega^A_m))$. By the definition of the trace operator, the process

$$[0, \tau_D] \ni t \mapsto 1_{[0, \tau_D]}(t)\psi(X_t) + 1_{\{t = \tau_D\}}\gamma_A(\psi)(X_{\tau_D}^-)$$

is continuous under $P_x$-a.s. for $m$-a.e. $x \in D$. But $\gamma_A(\psi) \in L^2(\partial D; \omega^A_m)$, so there exists $\psi \in L^2(\partial D; \omega^A_m)$ such that $i_M(\psi) = \gamma_A(\psi)$. By (5.10) we get that $\psi \in H^1_e(D)$. Now assume that $\psi \in H^1_e(D)$. Let $g \in B(\partial D)$ be such that

$$[0, \tau_D] \ni t \mapsto (\psi1_D + g1_{\partial D})(X_t)$$

is continuous $P_x$-a.s. for a.e. $x \in D$.

Let $u$ be a solution to wDP$(A, D, \psi)$. By Corollary 5.3, for $m$-a.e. $x \in D$ we have

$$\gamma_A(\psi)(X_{\tau_D}^-) = \lim_{t \nearrow \tau_D} u(X_t), \quad P_x\text{-a.s.}$$

Since $(u - \psi) \in H^1_0(D)$ by Lemma 2.2, $\lim_{t \nearrow \tau_D} u(X_t) = \lim_{t \nearrow \tau_D} \psi(X_t)$ $P_x$-a.s. for $m$-a.e. $x \in D$. But $\lim_{t \nearrow \tau_D} \psi(X_t) = g(X_{\tau_D})$. Hence

$$\gamma_A(\psi)(X_{\tau_D}^-) = g(X_{\tau_D}) \quad P_x\text{-a.s.}$$

for $m$-a.e. $x \in D$, which implies that $i_M(g) = \gamma_A(\psi)$. $\square$

Corollary 5.9. $H^1_e(D)$ is a closed subspace of $H^1(D)$.

Theorem 5.10. For every bounded open set $D \subset \mathbb{R}^d$, $L^2(\partial D; \omega^A_m) = L^2(\partial M \cap D; h^A_m)$ if and only if $H^1_e(D) = H^1(D)$. 
Hence, by Corollary 2.7, 

\[ \text{cl}(H^1(D) \cap C(\overline{D})) \subset H^1_c(D) \]

since \( H^1_c(D) \) is a closed subspace of \( H^1(D) \). Hence, in particular, if \( \text{cl}(H^1(D) \cap C(\overline{D})) = H^1(D) \), then there exists the trace operator from \( H^1(D) \) to \( L^2(\partial D; \omega_m^A) \).

### 6 Weak Dirichlet problem vs Dirichlet problem on arbitrary domain

**Lemma 6.1.** Let \( \psi \in \mathcal{B}_b(\partial M D) \) and let \( u(x) = E_x \psi(X_{\tau_D^-}), \ x \in D \). Then \( u \in C(D) \).

**Proof.** Of course, \( u \in \mathcal{B}_b(D) \). Let \( V \subset D \). Then, by the strong Markov property,

\[ u(X_{\tau_V}) = E_{X_{\tau_V}} \psi(X_{\tau_D^-}) = E_x(\psi(X_{\tau_D^-})|\mathcal{F}_{\tau_V}). \]

Taking the expectation with respect to \( P_x \), we get

\[ u(x) = E_x u(X_{\tau_V}), \ x \in D. \]  \hspace{1cm} (6.1)

Hence, by Corollary 2.7, \( u \in C(V) \). Since \( V \subset \subset U \) was arbitrary, \( u \in C(D) \). \( \square \)

**Corollary 6.2.** Let \( D \) be a bounded domain. Then for every \( B(x_0, r) \subset D \) there exists \( c > 0 \) such that for all \( x, y \in B(x_0, r) \),

\[ c^{-1} h^A_x(dz) \leq h^A_y(dz) \leq c h^A_x(dz). \]

**Proof.** Let \( \psi \in \mathcal{B}_b(\partial M D) \) and let \( u(x) = E_x \psi(X_{\tau_D^-}), \ x \in D \). It is clear that \( u \) is bounded. Let \( B(x_0, r) \subset \subset V \subset \subset D \). By (6.1) and Proposition 2.11, \( u \in H^1_{\text{loc}}(V) \) and \( \mathcal{E}(u, v) = 0, \ v \in C^1_\text{c}(V) \). Hence, by Harnack’s inequality, there is \( c > 0 \) such that

\[ c^{-1} u(x) \leq u(y) \leq cu(x), \ x, y \in B(x_0, r). \]

Since \( c \) is independent of \( \psi \), the desired result follows. \( \square \)

**Corollary 6.3.** Let \( D \) be a bounded open domain. If \( \psi \in L^p(\partial M D; h^A_x) \) for some \( p > 0 \) and \( x \in D \), then \( \psi \in L^p(\partial M D; h^A_y) \) for every \( y \in D \).

**Corollary 6.4.** Let \( \psi \in L^1(\partial M D; h^A_m) \) and let \( u(x) = E_x \psi(X_{\tau_D^-}), \ x \in D \). Then \( u \in C(D) \).

**Proof.** By Corollary 6.3, \( \psi \in L^1(\partial M D; h^A_x), \ x \in D \). Set \( \psi_n = (\psi \wedge n) \vee (-n), \ u_n(x) = E_x \psi_n(X_{\tau_D^-}), \ x \in D \), and choose \( x_0 \in D, \ r > 0 \) so that \( B(x_0, r) \subset D \). By Corollary 6.2,

\[ |u_n(x) - u(x)| \leq c \int_{ \partial M D } |\psi_n(y) - \psi(y)| h^A_{x_0}(dy), \ x \in B(x_0, r). \]

Hence \( u_n \to u \) uniformly on compact subsets of \( D \). Since \( u_n \in C(D) \) by Lemma 6.1, \( u \in C(D) \). \( \square \)

In what follows, we set \( H^{1/2}(\partial M D) := \gamma_A(H^1(D)) \).
**Definition 6.5.** Let \( \psi \in H^{1/2}(\partial_M D) \). We say that \( u \in H^1(D) \) is a weak solution to the Dirichlet problem

\[-Au = 0 \text{ in } D, \quad u|_{\partial_M D} = \psi \text{ on } \partial_M D,\]

(DP\((A, \partial_M D, \psi)\) for short) if

\[\mathcal{E}(u,v) = 0, \quad v \in H^1_0(D), \quad \gamma_A(u) = \psi.\]

**Theorem 6.6.** Let \( \psi \in H^1(D) \). If \( u \in H^1(D) \cap C(D) \) is a solution to wDP\((A, D, \psi)\), then \( u \) is a weak solution to DP\((A, \partial_M D, \gamma_A(\psi))\). Moreover,

\[u(x) = E_x \gamma_A(\psi)(X_{\tau_D -}), \quad x \in D.\]  

*Proof.* By Corollary 6.4, it is sufficient to show that (6.2) holds for \( m \)-a.e. \( x \in D \). But this follows from Corollary 5.3. Furthermore, by the definition of a solution of wDP and Proposition 5.6 we get the first assertion of the theorem. \( \square \)

**Proposition 6.7.** Let \( \psi \in L^2(\partial_M D; h_m^A) \) and let \( u \) be defined by

\[u(x) = E_x \psi(X_{\tau_D -}), \quad x \in D.\]  

Then,

(i) \( u \in \tilde{H}_D^1(D) \cap C(D) \).

(ii) \( \mathcal{E}(u,v) = 0 \) for every \( v \in C_c^\infty(D) \).

(iii) \( \int_{\partial D_n} u(y) \omega_{x,D_n}^A(dy) \to \int_{\partial_M D} \psi(y) h_{x,D}^A(dy) \) for every \( x \) and for every increasing sequence \( \{D_n\} \) of bounded open subsets of \( \mathbb{R}^d \) such that \( \{D_n\} \subset \subset D \) and \( \bigcup_{n \geq 1} D_n = D \).

(iv) \( |u(x)| + \|\sqrt{g_D(x, \cdot)} \nabla u\|_{L^2(D;m)} \leq (1 + \lambda)^{-1/2} \|\psi\|_{L^2(\partial_M D; h_m^A)} \) for every \( x \in D \).

(v) \( \|u\|_{\tilde{H}_D^1} \leq (1 + \lambda)^{-1/2} \|\psi\|_{L^2(\partial_M D; h_m^A)} \).

(vi) \( u(X_t) = \psi(X_{\tau_D -}) - \int_0^{\tau_D} \sigma \nabla u(X_t) dB_t, t \leq \tau_D, P_x\text{-a.s. for every } x \in D \).

(vii) \( E_x \sup_{t \leq \tau_D} |u(X_t)|^2 \leq 4 \|\psi\|^2_{L^2(\partial_M D; h_m^A)} \) for every \( x \in D \).

*Proof.* By Corollary 6.4, \( u \in C(D) \). By the strong Markov property (see the reasoning in the proof of Lemma 6.1), \( u(x) = E_x u(X_{\tau_D -}) \) for \( x \in \overline{D} \), so by Theorem 2.4, \( u \) is also a solution to DP\((A, D_n, u|_{\partial D_n})\). Therefore, by Proposition 2.11, \( u \in H^1_{loc} \) and assertion (ii) holds true. Moreover, by Proposition 2.9, (2.12) is satisfied. By the Markov property,

\[u(X_t) = E_x(\psi(X_{\tau_D -})|\mathcal{F}_t), \quad t \leq \tau_D, \quad P_x\text{-a.s.} \]

for every \( x \in D \). Applying Doob’s \( L^2 \)-inequality, we obtain

\[E_x \sup_{t \leq \tau_D} |u(X_t)|^2 \leq 4E_x |\psi(X_{\tau_D -})|^2, \quad x \in D. \]
This gives (vii). By (2.12),
\[ [u(X)]_{\tau_{D_n}} = \int_0^{\tau_{D_n}} |\tilde{\sigma} \nabla u|^2(X_r) \, dr \quad P_x\text{-a.s.} \]

Using (vii) and the Burkholder-Davis-Gundy inequality, we obtain
\[ E_x \int_0^{\tau_{D_n}} |\nabla u(X_r)|^2 \leq c E_x |\psi(X_{\tau_{D_n}})|^2, \quad x \in D_n, \quad n \geq 1. \quad (6.6) \]

Letting \( n \to \infty \) in (6.6) yields
\[ E_x \int_0^{\tau_D} |\nabla u(X_r)|^2 \leq c E_x |\psi(X_{\tau_D})|^2, \quad x \in D. \quad (6.7) \]

By (6.4) and the martingale convergence theorem, for every \( x \in D \),
\[ u(X_t) \to \psi(X_{\tau_D}) \quad P_x\text{-a.s.} \quad (6.8) \]
as \( t \nearrow \tau_D \). Letting \( n \to \infty \) in (2.12) and using (6.7) and (6.8) we get (vi). By (6.5) and (6.8),
\[ E_x u(X_{\tau_{D_n}}) \to E_x \psi(X_{\tau_D}), \]
which yields (iii). By (vi) and Itô’s formula,
\[ |u(x)|^2 + E_x \int_0^{\tau_D} |\tilde{\sigma} \nabla u(X_r)|^2 \, dr = E_x |\psi(X_{\tau_D})|^2, \quad x \in D, \]
which implies (iv). Assertion (v) is a consequence of (iv). \( \square \)

**Definition 6.8.** Let \( \psi \in L^2(\partial_M D; h_m^A) \). We say that \( u : D \to \mathbb{R} \) is a soft solution to DP\((A, \partial_M D, \psi)\) if (i)–(iii) of Proposition 6.7 are satisfied.

**Proposition 6.9.** For every \( \psi \in L^2(\partial_M D; h_m^A) \) there exists a unique soft solution to DP\((A, \partial_M D, \psi)\).

**Proof.** The proof is analogous to the proof of Proposition 2.11. \( \square \)

### 7 Extension of the trace operator

Denote by \( \mathcal{T} \) the subset of \( \mathcal{B}(D) \) consisting of all functions \( \psi \) for which there exists \( g \in \mathcal{B}(\partial_M D) \) such that the process \([0, \tau_D] \ni t \mapsto (1_D \psi + 1_{\partial_M D} g)(X^*_t)\) is continuous at \( \tau_D \) under the measure \( P_x \) for \( m\text{-a.e.} \ x \in D \). Of course, such a function is unique up to the measure \( h_m^A \) because
\[ g(X_{\tau_D}) = \lim_{t \nearrow \tau_D} \psi(X_t) \quad P_x\text{-a.s.} \quad (7.1) \]
for \( m\text{-a.e.} \ x \in D \). We equip \( \mathcal{T} \) with the metric \( \|u\|_{q,u} \) defined by
\[ \|u\|_{q,u} = E_m \sup_{t < \tau_D} (|u(X_t)| \wedge 1), \]
where $P_m(dy) := \int_D P_x(dy) m(dx)$. By [32], if $u_n \to u$ quasi-uniformly, i.e.
\[
\lim_{N \to \infty} \text{Cap}_A \left( \bigcup_{n \geq N} \{ |u_n - u| > \varepsilon \} \right) = 0, \quad \varepsilon > 0.
\]
then $\|u - u_n\|_{q.u.} \to 0$. Conversely, if $\|u - u_n\|_{q.u.} \to 0$ then there exists subsequence $(n_k) \subset (n)$ such that $u_n \to u$ quasi-uniformly. By $L^0(\partial M; h_m^A)$ we denote the space $\mathcal{B}(\partial M)$ equipped with the metric of convergence in measure $h_m^A$.

Consider the trace operator
\[
\gamma_A : \mathcal{T} \to \mathcal{B}(\partial M), \quad \gamma_A(\psi) := g, \quad \psi \in \mathcal{T}.
\]

**Proposition 7.1.** The operator $\gamma_A : (\mathcal{T}, \| \cdot \|_{q.u.}) \to L^0(\partial M; h_m^A)$ is continuous.

**Proof.** Let $u \in \mathcal{T}$. Suppose that $\|u_n\|_{q.u.} \to 0$. For all $n \geq 1$ we have
\[
\int_{\partial M} (|\gamma_A(u_n)| \wedge 1)(y) h_m^A(dy) = E_m(|\gamma_A(u_n)| \wedge 1)(X_{\tau_D^c}) \leq \|u_n\|_{q.u.}
\]
from which we deduce that $\gamma_A(u_n) \to 0$ in measure $h_m^A$. \hfill \Box

For $p \geq 1$, we set
\[
\mathcal{T}_0 = \gamma_A^{-1}(0), \quad \mathcal{T}^p = \gamma_A^{-1}(L^p(\partial M; h_m^A)).
\]

Of course, $\mathcal{T}_0 \subset \mathcal{T}^p$ for every $p \geq 1$.

**Remark 7.2.** By Theorem 5.2, $H^1(D) \subset \mathcal{T}^2$.

We denote by $\mathcal{G}^p$, $p \geq 1$, the set of all quasi-continuous $u \in \mathcal{B}(D)$ having a finite norm
\[
\|u\|_{\mathcal{G}^p} = (E_m \sup_{t < \tau_D} |u(X_t)|^p)^{1/p},
\]
and by $\mathcal{D}^p$ we denote the set of all quasi-continuous $u \in \mathcal{B}(D)$ for which the family $\{|u|^p(X_{\tau_V}) : V \subset U\}$ is uniformly integrable under the measure $P_x$ for $m$-a.e. $x \in D$. We equip $\mathcal{D}^p$ with the norm
\[
\|u\|_{\mathcal{D}^p} = \left( \int_D \sup_{V \subset D} E_x |u(X_{\tau_V})|^p m(dx) \right)^{1/p}.
\]
Let us stress that the norms $\| \cdot \|_{\mathcal{D}^p}, \| \cdot \|_{\mathcal{G}^p}$ depend on the operator $A$. It is an elementary check that $(\mathcal{G}^p, \| \cdot \|_{\mathcal{G}^p})$ with $p \geq 1$ are Banach spaces. It is clear that $\mathcal{G}^p \subset \mathcal{D}^p$ for every $p \geq 1$.

**Proposition 7.3.** $(\mathcal{D}^p, \| \cdot \|_{\mathcal{D}^p})$ is a Banach space for every $p \geq 1$.

**Proof.** Let $\{u_n\} \subset \mathcal{D}^p$ be a Cauchy sequence in the norm $\| \cdot \|_{\mathcal{D}^p}$. By [32] (see comments following Lemma 1 in [32]), for every $n \geq 1$ there exists $\psi_n \in C_c(D)$ such that
\[
\|u_n - \psi_n| \wedge 1\|_{\mathcal{D}^p} \leq E_m \sup_{t < \tau_D} |u_n(X_t) - \psi_n(X_t)|^p \wedge 1 \leq 2^{-2pn}
\]
(the first inequality above is obvious). Set \( V_{n,k} = \{ |\psi_n - \psi_k| \wedge 1 \leq \varepsilon \} \cap D \), and let \( \{ D_l, l \geq 1 \} \) be an increasing sequence of open subsets of \( D \) such that \( D_l \subset \subset D \) and \( \bigcup_{l \geq 1} D_l = D \). Observe that \( \tau_{n,k} \cap D_l \rightarrow \tau_{n,k} \) as \( l \rightarrow \infty \). For \( \varepsilon \leq 1 \) and \( n \leq k \) we have

\[
(P_m(\sup_{t<\tau_D} |\psi_n - \psi_k|(X_t) > \varepsilon))^{1/p} = (P_m(\sup_{t<\tau_D} |\psi_n - \psi_k| \wedge 1(X_t) > \varepsilon))^{1/p} 
\]

\[
\leq (P_m(|\psi_n - \psi_k| \wedge 1(X_{\tau_{n,k}}) > \varepsilon))^{1/p} 
\]

\[
\leq \varepsilon^{-1}(E_m|\psi_n - \psi_k|^p \wedge 1(X_{\tau_{n,k}}))^{1/p} 
\]

\[
= \lim_{l \rightarrow \infty} \varepsilon^{-1}[E_m|\psi_n - \psi_k|^p \wedge 1(X_{\tau_{n,k}} \cap D_l)]^{1/p} 
\]

\[
\leq \varepsilon^{-1}||\psi_n - \psi_k| \wedge 1||_p 
\]

\[
\leq \varepsilon^{-1}(2^{-2n+1} + ||u_n - u_k||_p). 
\]

By the above inequality and (7.2),

\[
(P_m(\sup_{t<\tau_D} |u_n - u_k|(X_t) > \varepsilon))^{1/p} \leq \varepsilon^{-1}(2^{-2n+2} + ||u_n - u_k||_p). 
\]

(7.3)

Let \( \{ n_k \} \) be a sequence such that \( ||u_{n_k} - u_{n_{k+1}}||_p \leq 2^{-2k} \). Then by (7.3),

\[
P_m(\sup_{t<\tau_D} |u_{n_k} - u_{n_{k+1}}|(X_t) > 2^{-k}) \leq 2^{-p(k-3)}. 
\]

From this and the Borel-Cantelli lemma we deduce that

\[
P_m(\limsup_{k \geq 1} \Lambda_k) = 0, 
\]

(7.4)

where \( \Lambda_k = \{ \omega \in \Omega; \sup_{t<\tau_D} |u_{n_k} - u_{n_{k+1}}|(X_t) > 2^{-k} \} \). Set \( u = \liminf_{k \rightarrow \infty} u_{n_k} \). From (7.4) it follows that for every \( \varepsilon > 0 \),

\[
\lim_{k \rightarrow \infty} P_m(\sup_{t<\tau_D} |u_{n_k}(X_t) - u(X_t)| > \varepsilon) = 0. 
\]

(7.5)

Hence, in particular, \( u \) is quasi-continuous (see [32, Theorem 1]). Since \( \{ u_n \} \) is a Cauchy sequence in \( \mathcal{D}^p \), there exists \( m_0 \in \mathbb{N} \) such that for all \( k,l \geq m_0 \) and \( R \geq 0 \),

\[
\int_{D_{V \subset \subset D}} E_x(|u_{n_k}(X_{\tau_V}) - u_{n_l}(X_{\tau_V})|^p \wedge R) m(dx) \leq \varepsilon^p. 
\]

Letting \( l \rightarrow \infty \) in the above inequality and using (7.5) and the Lebesgue dominated convergence theorem, we get

\[
\int_{D_{V \subset \subset D}} E_x(|u_{n_k}(X_{\tau_V}) - u(X_{\tau_V})|^p \wedge R) m(dx) \leq \varepsilon^p. 
\]

Now, letting \( R \rightarrow \infty \), we get \( ||u_{n_k} - u||_p \leq \varepsilon \) for \( k \geq m_0 \), from which the desired result follows.

We set

\[
H_{\mathcal{D}^p} = H(D) \cap \mathcal{D}^p, \quad H_{\mathcal{G}^p} = H(D) \cap \mathcal{G}^p, 
\]

where \( H(D) \) is defined by (3.1).
Theorem 7.4. \( H_{\mathcal{D}^p} \subset \mathcal{T}^p \) for every \( p \geq 1 \). Moreover,

(i) \( \gamma_A : H_{\mathcal{D}^p} \to L^p(\partial_M D; h_m^A) \) is an isometric isomorphism,

(ii) for every \( p > 1, \gamma_A : H_{\mathcal{D}^p} \to L^p(\partial_M D; h_m^A) \) is a homeomorphism,

(iii) if \( u \in H_{\mathcal{D}^1}, \) then \( E_x(\int_0^{\tau_D} |\nabla u|^2(X_r) \, dr)^{p/2} \leq \lambda^{-q} c_q \| \gamma_A(u) \|_{L^1(\partial_M D; h_m^A)}^q < \infty \) for all \( x \in D \) and \( q \in (0, 1) \), and moreover, there exists a Wiener process \( B \) such that for every \( x \in D \),

\[
u(X_t) = \gamma_A(u)(X_{\tau_D -}) - \int_t^{\tau_D} \sigma \nabla u(X_r) \, dB_r, \quad t \leq \tau_D \quad P_x\text{-a.s.} \quad (7.6)
\]

(iv) if \( p > 1 \) and \( u \in H_{\mathcal{D}^p}, \) then \( E_x(\int_0^{\tau_D} |\nabla u|^2(X_r) \, dr)^{p/2} \leq \lambda^{-p} c_p \| \gamma_A(u) \|^p_{L^p(\partial_M D; h_m^A)} < \infty \) for every \( x \in D \),

(v) for every \( p > 1, H_{\mathcal{D}^p} = H_{\mathcal{D}^p} \) and \( \| u \|_{H_{\mathcal{D}^p}} \leq \| u \|_{H_{\mathcal{D}^p}} \leq \frac{p-1}{p} \| u \|_{H_{\mathcal{D}^p}} \) for any \( u \in H_{\mathcal{D}^p} \).

Proof. (i) Let \( u \in H_{\mathcal{D}^p} \) and \( \{D_n\} \) be an increasing sequence of open sets such that \( D_n \subset D \) and \( \bigcup_{n \geq 1} D_n = D \). Since \( u \in H(D), u(x) = E_x u(X_{\tau_{D_n}}), x \in D_n \). Hence, by Theorem 2.4 and Proposition 2.9,

\[
u(X_t) = u(x) + \int_0^t \sigma \nabla u(X_r) \, dB_r, \quad t < \tau_D, \quad P_x\text{-a.s.} \quad (7.7)
\]

for every \( x \in D \). Furthermore, since \( u \in \mathcal{D}^p, u(X) \) is a uniformly integrable martingale on \([0, \tau_D]\) under the measure \( P_x \) for \( m\)-a.e. \( x \in D \). Let

\[
Y = \limsup_{t \searrow \tau_D} u(X_t).
\]

Then, by the martingale convergence theorem, \( Y = \lim_{t \searrow \tau_D} u(X_t) \) \( P_m\text{-a.s.} \). Since \( |u|^p(X) \) is uniformly integrable under measure \( P_m \) (\( u \in H_{\mathcal{D}^p} \)) we have that \( Y = \lim_{t \searrow \tau_D} u(X_t) \) in \( L^p(\Omega, P_m) \). It is clear that \( Y \in \mathcal{I} \), so there exists \( g \in B(\partial_M D) \) such that \( g(X_{\tau_D -}) = Y \) \( P_m\text{-a.s.} \). Since \( Y \in L^p(\Omega, P_m) \), \( g \in L^p(\partial_M D; h_m^A) \). It is clear that \( u \in \mathcal{T}^p \) and \( \gamma_A(u) = g \). Hence

\[
u(x) = E_x \gamma_A(u)(X_{\tau_D -}) \quad (7.8)
\]

for \( m\)-a.e. \( x \in D \). Thus \( \gamma_A \) is an injection. Let \( \psi \in L^p(\partial_M D; h_m^A), \) and let

\[
u(x) = E_x \psi(X_{\tau_D -}), \quad x \in D. \quad (7.9)
\]

By the Markov property,

\[
u(X_t) = E_x(\psi(X_{\tau_D -})|\mathcal{F}_t), \quad t \leq \tau_D. \quad (7.10)
\]

From this and the martingale convergence theorem we conclude that \( \gamma_A(u) = \psi \). Furthermore, from (7.8) and the strong Markov property it follows that for every \( V \subset U, \)

\[
u(X_{\tau_V}) = E_x(\psi(X_{\tau_D -})|\mathcal{F}_{\tau_V}), \quad x \in D.
\]
(see the reasoning in the proof of Lemma 6.1). Taking the expectation shows that 
$u \in H(D)$. Moreover, by (7.10), $u(X)$ is a martingale on $[0, \tau_D]$, so $|u(X)|^p$ is a

submartingale on $[0, \tau_D]$ under the measure $P_x$ for every $x \in D$. Hence

$$
\|u\|^p_{\mathcal{D}^p} = \int_{D \cap C \subset D} \sup_{t \in \tau_D} E_x|u(X_{\tau_D})|^p m(dx) 
= \int_D |\psi(X_{\tau_D})|^p m(dx) = \|\psi\|^p_{L^p(\partial_M D; h^A_m)}, \tag{7.11}
$$

which completes the proof of (i).

(ii) Suppose that $u \in H_{\mathcal{D}^p}$ for some $p > 1$. Since $H_{\mathcal{D}^p} \subset H_{\mathcal{D}^q}$, from the proof of

part (i) we know that $H_{\mathcal{D}^p} \subset T^p$ and (7.8) is satisfied. Thus $\gamma_A$ is an injection. Assume

that $\psi \in L^p(\partial_M D; h^A_m)$ and define $u$ by (7.9). By the proof of part (i), $\gamma_A(u) = \psi$ and

$u \in H_{\mathcal{D}^p}$. By (7.10) and Doob's $L^p$-inequality,

$$
E_m|\psi(X_{\tau_D})|^p \leq E_m \sup_{t \leq \tau_D} |u(X_t)|^p \leq \frac{p}{p - 1} E_m|\psi(X_{\tau_D})|^p, \tag{7.12}
$$

which completes the proof of (ii).

(iii) By Remarks 6.3 and 6.4, equation (7.8) holds true for every $x \in D$. Hence, by

the Markov property, for every $x \in D$ we have

$$
u(X_t) = E_x(\gamma_A(u)(X_{\tau_D})|\mathcal{F}_t), \quad t \leq \tau_D, \quad P_x\text{-a.s.}
$$

It follows that in fact $u(X)$ is a uniformly integrable martingale on $[0, \tau_D]$ under $P_x$ for

every $x \in D$. Furthermore, by the martingale convergence theorem, $\gamma_A(u)(X_{\tau_D}) = \lim_{t \nearrow \tau_D} u(X_t)$ $P_x$-a.s. and in $L^1(\Omega, P_x)$ for every $x \in D$. By (7.7), [9, Lemma 6.1] and

the Burkholder-Davis-Gundy inequality,

$$
E_x(\int_0^{\tau_D} |\sigma \nabla u(X_t)|^2 dt)^{q/2} \leq c_q E_x \sup_{t \leq \tau_D} |u(X_t)|^q \leq \frac{c_q}{1 - q} (E_x|u(X_{\tau_D})|^q)^{q/\eta},
$$

for all $x \in D$ and $q \in (0, 1)$. Using now Fatou's lemma and Corollary 6.3, we get the

inequality appearing in (iii). Furthermore, letting $t \not\nearrow \tau_D$ in (7.7) we obtain (7.6). Part

(iv) follows immediately from (7.6), Corollary 6.3 and the Burkholder-Davis-Gundy inequality. Part (v)

follows from (i), (ii) and (7.12). \hfill \square

**Corollary 7.5.** Assume that $p \geq 1$ and

$$
u(x) = E_x \psi(X_{\tau_D}), \quad x \in D. \tag{7.13}
$$

Then $u \in H_{\mathcal{D}^p}$ if and only if $\psi \in L^p(\partial_M D; h^A_m)$, and if $u \in H_{\mathcal{D}^p}$, then $\psi = \gamma_A(u)$.

Moreover, any $u \in H_{\mathcal{D}^{p_1}}$ is of the form (7.13) with $\psi = \gamma_A(u)$.

**Proof.** If $u \in H_{\mathcal{D}^{p_1}}$, then taking the expectation with respect to $P_x$ in (7.6) with

t = 0 (this is possible since $u \in H_{\mathcal{D}^{p_1}}$) we get (7.7) with $\psi = \gamma_A(u)$. Let $u$ be of the

form (7.13). Assume that $u \in H_{\mathcal{D}^p}$. Then, by Theorem 7.4(iii), $u(X_{\tau_D}) = \gamma_A(u)(X_{\tau_D})$

$P_x$-a.s. for $x \in D$. On the other hand, by (7.13) and the Markov property, for every

$x \in D$,

$$
u(X_t) = E_x \psi(X_{\tau_D})|\mathcal{F}_t), \quad t \leq \tau_D, \quad P_x\text{-a.s.}
$$

Hence $\psi(X_{\tau_D}) = \gamma_A(u)(X_{\tau_D})$ $P_x$-a.s. for $x \in D$, which implies that $\gamma_A(u) = \psi$.

By Theorem 7.4, $\gamma_A(u) \in L^p(\partial_M D; h^A_m)$. Now suppose that $\psi \in L^p(\partial_M D; h^A_m)$. By

Theorem 7.4(i) there exists $\bar{u} \in H_{\mathcal{D}^p}$ such that $\gamma_A(\bar{u}) = \psi$. Moreover, by Theorem

7.4(iii), $\bar{u}(x) = E_x \gamma_A(\bar{u})(X_{\tau_D})$ for $x \in D$. Thus $u = \bar{u}$. \hfill \square
Remark 7.6. From the definition of the trace operator it follows that for every $p \geq 1$

$$\|\gamma_A(u)\|_{L^p(\partial M; \mathcal{H}_0^A)} \leq \|u\|_{\mathcal{D}^p}, \quad u \in \mathcal{D}^p \cap \mathcal{T}. $$

Definition 7.7. We say that a function $u$ on $D$ is of potential type if $u \in \mathcal{T}_0$.

Proposition 7.8. Each potential is a function of potential type.

Proof. Let $u$ be a potential. Without loss of generality, we may assume that $u$ is bounded. It is well known that $u_n := nR^n u \uparrow u$ quasi-uniformly as $n \to \infty$. Hence, by Proposition 7.1, $\gamma_A(u_n) \to \gamma_A(u)$ in measure $h_A^m$. Since $u_n \in H^1_0(D)$, $\gamma_A(u_n) = 0$, and consequently $\gamma_A(u) = 0$. $\blacksquare$

Proposition 7.9. For every $\psi \in \mathcal{T}_1$ there exist a unique harmonic function $h \in H_D$ and a function $p$ of potential type such that

$$\psi = h + p.$$

Proof. Define $h$ by $h(x) = E_x \gamma_A(u)(X_{\tau_D \cdots}), x \in D$, and set $p := \psi - h$. By Corollary 7.5 $h \in H_D$ and $\gamma_A(p) = 0$. Now, suppose that there is another harmonic function $h' \in H_D$ and a function $p'$ of potential type such that $u = h' + p'$. Then $p - p' = h' - h$, and hence by Proposition 7.8 $\gamma_A(h) = \gamma_A(h')$. By this and Theorem 7.4, $h = h'$, and, in consequence, $p = p'$. $\blacksquare$

8 Dirichlet problem for semilinear equations with measure data

Let $\mu$ be a Borel measure on $D$. In this section we are concerned with the problem of existence of solutions of the Dirichlet problem

$$-Au = f(\cdot, u) + \mu \quad \text{in} \quad D, \quad u = \psi \quad \text{on} \quad \partial M D. \quad (8.1)$$

In (8.1), $f : D \times \mathbb{R} \to \mathbb{R}$ is a measurable functions which is continuous and nonincreasing with respect to $u$, and $\mu$ is a Radon signed measure on $D$. It is known (see [20]) that $\mu$ admits unique decomposition of the form

$$\mu = \mu_c + \mu_d$$

into the measure $\mu_c$, which is singular with respect to $\text{Cap}_A$ (the concentrated part of $\mu$) and the measure $\mu_d$, which is absolutely continuous with respect to $\text{Cap}_A$ (the diffuse part of $\mu$). In the sequel, we set $L^p_\delta(D; m) := L^p(D; \delta \cdot m)$ for $p > 0$, and we denote by $W^{1,q}_\delta(D)$ the space of functions $u \in W^{1,q}_\text{loc}(D)$ such that

$$\|u\|_{W^{1,q}_\delta} := \|u\|_{L^q(D; m)} + \|
abla u\|_{L^q_\delta(D; m)} < \infty.$$ 

Semilinear problems of the form (8.1) with $\psi = 0$ were for the first time considered in the paper by Brezis and Strauss [8] in the case where $A = \Delta$ and $\mu \in L^1(D; m)$ (see also [30]). An important contribution to the theory was made in the paper [2], in which equations of the form (8.1) with zero boundary data but general bounded smooth measure and operator of the form (1.4) are considered. At present, in the case where
\( \psi = 0 \), existence, uniqueness and regularity results are known for (8.1) with general bounded smooth measure and general, possibly nonlocal, operator \( A \) corresponding to a Dirichlet form (see [28, 29]).

The case \( \mu_c \neq 0 \) is much more involved. In 1975 Bénilan and Brezis considered (8.1) with \( A = \Delta, \psi = 0 \) and \( \mu = \delta_a \) for some \( a \in D \). They showed that if \( d \geq 3 \) and \( f(u) = u|u|^{p-1} \) with \( p > \frac{d}{d-2} \), then there is no solution to (8.1) (see [3] for interesting historical comments on the problem). To analyze the nonexistence phenomena behind the semilinear Dirichlet problem, Brezis, Marcus and Ponce [6, 7] introduced the concept of good measure for (8.1), i.e. a measure for which there exists a solution to (8.1), and the concept of reduced measure for \( \mu \), i.e. the largest good measure which is less then or equal to \( \mu \). In [27] the notions of good and reduced measure were extended to (8.1) with general Dirichlet operator and \( \psi = 0 \).

In what follows, we concentrate on (8.1) with \( A \) defined by (1.4) and nonzero boundary condition \( \psi \). As for \( \mu \), we will assume that it belongs to the space \( \mathcal{M}_\delta \) of all signed Borel measures on \( D \) such that \( \|\mu\|_{TV,\delta} := \int_D \delta d|\mu| < \infty \), where \( |\mu| \) denotes the variation of \( |\mu| \). Note that \( \mathcal{M}_\delta \) includes all bounded Radon measures on \( D \).

Denote by \( G_\psi(A,f) \) the set of all good measures for \( A \) and \( f \), i.e. the set of all \( \mu \in \mathcal{M}_\delta \) for which there exists a solution to (8.1). By \( G_0(A,f) \) we denote the set \( G_\psi(A,f) \) with \( \psi = 0 \). We will show that, under some assumptions on \( f \),

\[ G_0(A,f) = G_\psi(A,f). \tag{8.2} \]

This extends in part the results of [27] where considered problem of the form (8.1) with \( \psi \equiv 0 \) and general Dirichlet operators, and [34] where it was shown (for the Laplace operator) that reduced measure (the biggest measure less than \( \mu \) for which there exists a solution to (8.1)) does not depend on the boundary conditions. Thanks to (8.2) we may apply to (8.1) the results of [27], where we proved some characterization of the set \( G_0(A,f) \).

To formulate our assumptions on \( f \) and prove (8.2), we will need the notions of quasi-integrable and quasi-bounded function, which we define below.

We say that \( u \in B(D) \) is quasi-integrable (\( u \in qL^1(D;m) \) in abbreviation) if for q.e. \( x \in D \),

\[ P_x\left( \int_0^{r_D} |f(X_r)| \, dr < \infty \right) = 1. \]

Note that, by [19, Theorem 4.2.5] if for every \( \varepsilon > 0 \) there exists a Borel set \( B_\varepsilon \subset D \) such that \( \text{Cap}_A(D \setminus B_\varepsilon) < \varepsilon \) and \( u \in L^1(B_\varepsilon;m) \) then \( u \in qL^1(D;m) \).

We say that \( u \in B(D) \) is quasi-bounded if for q.e. \( x \in D \),

\[ P_x(\sup_{t<t_D} |u(X_t)| < \infty) = 1. \]

By [19, Theorem 4.2.5] if for every \( \varepsilon > 0 \) there is a Borel set \( B_\varepsilon \subset D \) and a constant \( M_\varepsilon > 0 \) such that \( \text{Cap}(D \setminus B_\varepsilon) < \varepsilon \) and \( |u(x)| \leq M_\varepsilon \) for every \( x \in B_\varepsilon \) then \( u \) is quasi-bounded.

In the rest of this section, unless explicitly otherwise stated, we assume that \( f, \mu, \psi \) satisfy the following assumptions.

(A1) \( \mu \in \mathcal{M}_\delta \) and \( \psi \in L^1(\partial_M D; h_m^A) \).
(A2) \( f : D \times \mathbb{R} \to \mathbb{R} \) is a measurable function such that \( y \mapsto f(x,y) \) is continuous and non-increasing for every \( x \in D \), \( f(\cdot,y) \in qL^1(D;m) \) for every \( y \in \mathbb{R} \) and \( f(\cdot,0) \in L^1_\delta(D;m) \).

It is well known that for every \( \eta \in L^\infty(D;m) \), \( R^D \eta \in W^{1,\infty}_0(D) \cap C(D) \). It is also clear that \( |R^D \eta| \leq \|\eta\|_\infty \delta \).

**Definition 8.1.** We say that a \( u \in L^1(D;m) \) is a solution of the problem
\[
-Au = \mu \quad \text{in} \quad D, \quad u|_{\partial D} = 0 \quad (8.3)
\]
if
\[
\int_D u \eta \, dm = \int_D R^D \eta \, d\mu, \quad \eta \in L^\infty(D;m).
\]

**Remark 8.2.** By [27, Proposition 4.12], \( u \) is a solution to (8.3) if and only if \( u = R^D \mu \) \( m \)-a.e. Moreover by [19, Theorem 4.6.1] function \( R^D \mu \) is quasi-continuous and if \( \mu \) is bounded then by [46] \( u \in W^{1,q}_0 \) with \( q < \frac{d}{d-1} \) and
\[
\|u\|_{W^{1,q}_0} \leq c_0 \|\mu\|_{TV}.
\]

In the sequel, we consider the operator \( \gamma_A \) with domain \( H_{D^1} \).

**Definition 8.3.** We say that \( u \in L^1(D;m) \) is a solution to (8.1) if \( f(\cdot,u) \in L^1_\delta(D;m) \) and \( u - \gamma_A^{-1}(\psi) \) is a solution to (8.3) with \( \mu \) replaced by \( f(\cdot,u) \cdot m + \mu \).

**Proposition 8.4.** There exists at most one solution to (8.1).

**Proof.** By the definition, \( u \) is a solution of (8.1) if and only if \( v := u - \gamma_A^{-1}(\psi) \) is a solution of
\[
-Av = f_{\gamma_A^{-1}(\psi)}(\cdot,v) + \mu \quad \text{in} \quad D, \quad u = 0 \quad \text{on} \quad \partial D, \quad (8.4)
\]
where
\[
f_{\gamma_A^{-1}(\psi)}(x,y) = f(x,y + \gamma_A^{-1}(\psi)(x)).
\]

Therefore the desired result follows from [27, Corollary 4.3]. \( \square \)

**Remark 8.5.** By Corollary 7.5 \( \gamma_A^{-1}(\psi) = E_{\psi}(X_{\tau_D^+}) \). So, by Remark 8.2, \( u \) is a solution to (8.1) if and only if \( f(\cdot,u) \in L^1_\delta(D;m) \) and for \( m \)-a.e. \( x \in D \),
\[
u(x) = \int_D f(y,u(y))g_D(x,y)m(dy) + \int_D g_D(x,y) \mu(dy) + \int_{\partial D} \psi(y)h^A_x(dy).
\]

If \( \mu \) is smooth, then the above equation is equivalent to
\[
u(x) = E_x \int_0^{\tau_D} f(X_r,u(X_r)) \, dr + E_x \int_0^{\tau_D} dA^\mu_r + E_x \psi(X_{\tau_D^+})
\]
(see (2.1), (2.2) and (5.1)). Therefore, under the notation of Proposition 8.4, \( u \) is a solution to (8.1) with smooth \( \mu \) if and only if \( f(\cdot,u) \in L^1_\delta(D;m) \) and for \( m \)-a.e. \( x \in D \) we have
\[
u(x) = E_x \int_0^{\tau_D} f_{\gamma_A^{-1}(\psi)}(X_r,v(X_r)) \, dr + E_x \int_0^{\tau_D} dA^\mu_r. \quad (8.5)
\]

The above formula is in agreement with the probabilistic definition of a solution to (8.4) considered in [28] if we replace “\( m \)-a.e.” by “q.e.” In fact, (8.5) holds true for q.e. \( x \in D \) after replacing the left-hand side of (8.5) by its quasi-continuous \( m \)-version. Such a version exists by [28, Lemma 4.3] and is given by the right-hand side of (8.5), which is finite for q.e. \( x \in D \) by [28, Lemma 4.2].
Remark 8.6. By [27, Theorem 3.3], if \( u \) is a solution to (8.1) and \( f(\cdot, u) \in L^1(D; m) \), \( \|\mu\|_{TV} < \infty \), then \( T_k(u - \gamma_A^{-1}(\psi)) \in H^1_0(D) \) for every \( k \geq 0 \).

Proposition 8.7. If \( \mu \in \mathcal{M}_\delta \) is smooth, then there exists a unique solution of (8.1).

Proof. Uniqueness follows from Proposition 8.4. Write \( f_{n,m} = (f \wedge n) \vee (-m) \). From (8.4), (8.5) and [28, Theorem 4.7] it follows that for all \( n, m \geq 1 \) there exists a unique solution \( u_{n,m} \) of (8.1) with \( f \) replaced by \( f_{n,m} \). We may and will assume that each \( u_{n,m} \) is quasi-continuous. By (8.4), (8.5) and [28, Proposition 4.9], \( u_{n,m} \geq u_{n,m+1}, n, m \geq 1 \), q.e. By [28, Theorem 4.7], \( u_{n,m} \in \mathcal{D}^1 \) and there exists a martingale \( M^{n,m,x} \) such that for q.e. \( x \in D \),

\[
\begin{aligned}
    u_{n,m}(X_t) &= \psi(X_{\tau_D} - ) + \int_t^{\tau_D} f_{n,m}(X_r, u_{n,m}(X_r)) \, dr \\
    &\quad + \int_t^{\tau_D} dA^\mu_r - \int_t^{\tau_D} dM^{n,m,x}_r, \quad t \leq \tau_D, \quad P_x\text{-a.s.}
\end{aligned}
\]

Applying Tanaka's formula, we see that for q.e. \( x \in D \),

\[
    |u_{n,m}(X_t)| + \int_t^{\tau_D} dL_r = |\psi(X_{\tau_D} - )| + \int_t^{\tau_D} \text{sgn}(u_{n,m}(X_r)) f_{n,m}(X_r, u_{n,m}(X_r)) \, dr \\
    &\quad + \int_t^{\tau_D} \text{sgn}(u_{n,m}(X_r)) \, dA^\mu_r \\
    &\quad - \int_t^{\tau_D} \text{sgn}(u_{n,m}(X_r)) \, dM^{n,m,x}_r, \quad t \leq \tau_D, \quad P_x\text{-a.s.}, \quad (8.6)
\]

where \( L \) is the symmetric local time of \( u_{n,m}(X) \) at zero. Taking the expectation in (8.6) with \( t = 0 \) and using (A2) we get

\[
    |u_{n,m}(x)| + E_x \int_0^{\tau_D} |f_{n,m}(X_r, u_{n,m}(X_r))| \, dr \\
    \leq E_x|\psi(X_{\tau_D} - )| + E_x \int_0^{\tau_D} |f(X_r, 0)| \, dr + E_x \int_0^{\tau_D} \sum_{m=1}^{\infty} dA^m_r =: v(x). \quad (8.7)
\]

By the above inequality and (A2), for all \( n', m', n, m \geq 1 \) we have

\[
    |f_{n',m'}(x, u_{n,m}(x))| \leq |f(x, v(x))| + |f(x, -v(x))|, \quad x \in D. \quad (8.8)
\]

The functions \( E \int_0^{\tau_D} |f(X_r, 0)| \, dr \) and \( E \int_0^{\tau_D} |A^m_r| \), as potentials of a symmetric process, are quasi-continuous (see [32]). Moreover, by Proposition 7.8, they belong to \( \mathcal{T}_0 \). By Corollary 7.5, \( E|\psi(X_{\tau_D} - )| \in H_2 \) and belongs to \( \mathcal{T}^1 \). Therefore \( v \) is quasi-continuous and quasi-bounded. Set \( V_k = \{v > k\} \). Since \( v \) is quasi-bounded, the sequence \( \{\tau_k\} \) is a chain. Set

\[
    \sigma^k_t = \inf \{t \geq 0, \int_0^t |f(X_r, k)| \, dr + \int_0^t |f(X_r, -k)| \, dr \geq l\}, \quad \tau_{k,l} = \tau_k \wedge \sigma^k_l.
\]

By (A2), \( \{\sigma^k\} \) is a chain with respect to \( l \) (with fixed \( k \)). Let \( u_n = \inf_{m \geq 1} u_{n,m} \). By the construction of \( \{\sigma^k\} \) and the Lebesgue dominated convergence theorem,

\[
    E_x \int_0^{\tau^k} |f_{n,m}(X_r, u_{n,m}(X_r)) - f_n(X_r, u_n(X_r))| \, dr \to 0 \quad (8.9)
\]

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as \( m \to \infty \). By (8.6), for \( \text{q.e. } x \in D \),

\[
    u_{n,m}(x) = E_x u_{n,m}(X_{\sigma^k}) + E_x \int_0^{\sigma^k} f_n(X_r, u_{n,m}(X_r)) \, dr + E_x \int_0^{\sigma^k} dA^\mu_r,
\]

Letting \( m \to \infty \) in the above equality and using (8.9) we get

\[
    u_n(x) = E_x u_n(X_{\sigma^k}) + E_x \int_0^{\sigma^k} f_n(X_r, u_n(X_r)) \, dr + E_x \int_0^{\sigma^k} dA^\mu_r
\]

(8.10)

for \( \text{q.e. } x \in D \). It is clear that \( E_x \int_0^{\tau_D} |f(X_r, 0)| \, dr, E_x \int_0^{\tau_D} dA^{|\mu|}_r \in \mathcal{D}^1 \), so by Corollary 7.5, \( v \in \mathcal{D}^1 \). From this and (8.7) we conclude that the family \( \{u_n(X_\tau), \tau \in \mathcal{T}\} \) is uniformly integrable under the measure \( P_x \) for \( m \)-a.e. \( x \in D \). Therefore, by the properties of the sequence \( \{\sigma^k\} \), for \( m \)-a.e. \( x \in D \) we have

\[
    \lim_{k \to \infty} \lim_{l \to \infty} E_x u_n(X_{\sigma^k}) = E_x \psi(X_{\tau_D}^-).
\]

By (8.7), (A1), (A2) and Fatou’s lemma, \( f(\cdot, u_n) \in L^1_\delta(D; m) \). Hence, in particular, \( E_x \int_0^{\tau_D} |f_n(X_r, u_n(X_r))| \, dr < \infty \) for \( m \)-a.e. \( x \in D \). Therefore letting \( l \to \infty \) and then \( k \to \infty \) in (8.10) we see that for \( m \)-a.e. \( x \in D \),

\[
    u_n(x) = E_x \psi(X_{\tau_D}^-) + E_x \int_0^{\tau_D} f_n(X_r, u_n(X_r)) \, dr + E_x \int_0^{\tau_D} dA^\mu_r,
\]

(8.11)

which, by Remark 8.5, shows that \( u_n \) is a solution to (8.1) with \( f \) replaced by \( f_n \). Letting \( n \to \infty \) in (8.11) and using the arguments similar to those used above we show that \( u \) is a solution of (8.1).

\[\square\]

**Corollary 8.8.** For every \( h \in H_\mathcal{D} \) and every quasi-bounded \( v \in \mathcal{B}(D) \) such that \( f(\cdot, v) \in L^1_\delta(D; m) \) there exists \( g \in L^1_\delta(D; m) \) with the property that \( f(\cdot, v + h + R^D g) \in L^1_\delta(D; m) \).

**Proof.** By Theorem 7.4, there is \( \psi \in L^1(\partial M D; h^A_m) \) such that \( h = \gamma^{-1}_A(\psi) \). By the assumptions on \( v \), the function \( f_v \) has the same properties as \( f \). Therefore, by Proposition 8.7, there exists a unique solution to the problem

\[
    -Aw = f_v(\cdot, w) \quad \text{in } D, \quad w|_{\partial M D} = -\psi.
\]

Write \( g = f_v(\cdot, w) \). Then, by the very definition of a solution, \( g \in L^1_\delta(D; m) \) and

\[
    w = -\gamma^{-1}_A(\psi) + R^D g.
\]

For the proof of (8.2) it is convenient to define beforehand some subsets of the set of all measures \( \mu \in \mathcal{M}_\delta \) whose potential \( R^D \mu \) admits decomposition of the form

\[
    R^D \mu = R^D f_0 - h + v, \quad (8.12)
\]

where \( f_0 \in L^1_\delta(D; m) \), \( h \) is a harmonic function and \( v \) is a function from the space \( v \in L^1(D; m) \) such that \( f(\cdot, v) \in L^1_\delta(D; m) \).

By \( \mathcal{R}^p(A, f) \), \( p > 1 \) (resp. \( \mathcal{R}^1(A, f) \)) we denote the set of \( \mu \in \mathcal{M}_\delta \) such that \( R^D \mu \) admits decomposition (8.12) with \( h \in H_{\delta^0} \) (resp. \( h \in H_\mathcal{D} \)). By \( \mathcal{R}_0 \) we denote the set of those \( \mu \in \mathcal{R}^1 \) for which \( h = 0 \) in decomposition (8.12).

**Corollary 8.9.** \( \mathcal{R}^p(A, f) = \mathcal{R}_0(A, f) \) for every \( p \geq 1 \).
Proof. Of course, $\mathcal{R}_0 \subset \mathcal{R}^p$ for all $p \geq 1$. Let $\mu \in \mathcal{R}^1$. Then $R^D\mu$ admits decomposition (8.12). It is well known that $R^D|\mu|, R^D|f^0|$ are quasi-continuous as excessive function of the symmetric regular Dirichlet form (see, e.g., [19, Theorem 4.6.1]). Moreover, by Proposition 7.8, $R^D|\mu|, R^D|f^0| \in T_0$, so they are quasi-bounded. The function $h$ is also quasi-bounded as it is continuous on $D$ and belongs to $T_1$. Consequently, $v$ is quasi-bounded. By Corollary 8.8, there exists $g \in L^1_\delta(D;m)$ such that $f(., v + h + R^D g) \in L^1_\delta(D;m)$. This implies that $\mu \in \mathcal{R}_0$.

Remark 8.10. Assume additionally that there is $r \in L^1_\delta(D;m)$ such that $f^+(x,y) \leq r(x)$ for all $x \in D, y \in \mathbb{R}$. It is immediate that the results of [27, Section 5] are true under considered here assumptions on $f$. Let $g \in L^1_\delta(D;m)$ be such that $f(\cdot, \eta) \in L^1_\delta(D;m)$ with $\eta = \gamma^{-1}_A(\psi) + R^D g$. Observe that $\mu \in \mathcal{G}_\psi(A,f)$ if and only if $\mu - g \cdot m \in \mathcal{G}_0(A,f_\eta)$. By [27, Corollary 5.12], the last one is equivalent to $\mu \in \mathcal{G}_0(A,f_\eta)$. Therefore, by [27, Theorem 5.11], $\mu \in \mathcal{G}_\psi$ if and only if $\mu_c \in \mathcal{G}_\psi$.

Theorem 8.11. Assume that there is $r \in L^1_\delta(D;m)$ such that $f^+(x,y) \leq r(x), x \in D, y \in \mathbb{R}$. Then (8.2) holds true for every $\psi \in L^1(\partial_M D; h^a_k)$.

Proof. By [27, Corollary 5.12], $\mathcal{R}_0 = \mathcal{G}_0$, whereas by Corollary 8.9, $\mathcal{G}_0 = \mathcal{R}^1$. Therefore, it suffices to show that $\mathcal{R}^1 = \mathcal{G}_\psi$. It is clear that $\mathcal{G}_\psi \subset \mathcal{R}^1$. Let $\mu \in \mathcal{R}^1$. Then $\mu$ admits decomposition (8.12). By Corollary 8.8, there exists $g \in L^1_\delta(D;m)$ such that $f(\cdot, w) \in L^1_\delta(D;m)$, where $w = v - h + \gamma^{-1}_A(\psi) + R^D g$. Therefore $w$ is a solution to the problem

$$-Aw = f(\cdot, w) + \mu' \quad \text{in} \quad D, \quad w|_{\partial_M D} = \psi$$

with $\mu' = \mu - f_0 \cdot m + g \cdot m - F(\cdot, w) \cdot m$. Hence $\mu' \in \mathcal{G}_\psi$. Consequently, $\mu \in \mathcal{G}_\psi$ by Remark 8.10.

Proposition 8.12. Let $u$ be a solution to (8.1). Then for every $k > 0$,

$$\|u\|_{L^1(\partial D;m)} + \|T_k(u)\|_{L^1(D;m)} \leq 3k \lambda^{-1}(\|\psi\|_{L^1(\partial_M D; h^a_k)} + \|f(\cdot, 0)\|_{L^1_\delta(D;m)} + \|\mu\|_{TV, \delta}).$$

Proof. By Theorem 7.4(iii) and [27, Theorem 3.7], for q.e. $x \in D$ we have

$$u(X_t) = \psi(X_{\tau_D -}) + \int_{t}^{\tau_D} f(X_r, u(X_r)) \, dr + \int_{t}^{\tau_D} dA_r^{\mu_d} - \int_{t}^{\tau_D} \sigma \nabla u(X_r) \, dB_r, \quad t \leq \tau_D \quad P_x\text{-a.s.}$$

By the Tanaka formula,

$$|u(X_t)| = |\psi(X_{\tau_D -})| + \int_{t}^{\tau_D} \text{sgn}(u)(X_r) f(X_r, u(X_r)) \, dr + \int_{t}^{\tau_D} \text{sgn}(u)(X_r) dA_r^{\mu_d} - \int_{t}^{\tau_D} dL_r^0 - \int_{t}^{\tau_D} \text{sgn}(u)(X_r) \sigma \nabla u(X_r) \, dB_r, \quad t \leq \tau_D, \quad (8.13)$$

where $L^0$ is the symmetric local time of $u(X)$ at 0, and

$$|(u \land k)(X_t)| = \left|\psi \land k\right|(X_{\tau_D -}) + \int_{t}^{\tau_D} \mathbf{1}_{\{|u| \leq k\}} \text{sgn}(u)(X_r) f(X_r, u(X_r)) \, dr + \int_{t}^{\tau_D} \mathbf{1}_{\{|u| \leq k\}} \text{sgn}(u)(X_r) dA_r^{\mu_d} - \int_{t}^{\tau_D} dL_r^0 - \int_{t}^{\tau_D} d\tilde{L}_r^k$$

$$- \int_{t}^{\tau_D} \mathbf{1}_{\{|u| \leq k\}} \text{sgn}(u)(X_r) \sigma \nabla u(X_r) \, dB_r, \quad t \leq \tau_D.$$
where \( \bar{L}^k \) is the local time of \( |u(X)| \) at \( k \). Integrating by parts and using the fact that \( \bar{L}^k \) increases only when \( |u| = k \) and \( L^0 \) increases only when \( u = 0 \), we obtain

\[
(|u| \wedge k)(|u| - k)(x) + E_x \int_0^{\tau_{D}} |\bar{\sigma} \nabla T_k(u)|^2(X_r) \, dr \\
\leq E_x(|\psi| \wedge k)(|\psi| - k)(X_{\tau_{D} -}) + E_x \int_0^{\tau_{D}} \{ |u| \wedge k + 1_{(|u| \leq k)} (|u| - k) \} \text{sgn}(u)(X_r) f(X_r, u(X_r)) \, dr \\
+ E_x \int_0^{\tau_{D}} \{ |u| \wedge k + 1_{(|u| \leq k)} (|u| - k) \} \text{sgn}(u)(X_r) \, dA^{au}_r + kE_x \int_0^{\tau_{D}} dL^0_r.
\]

From the above equation we conclude that

\[
E_m \int_0^{\tau_{D}} |\bar{\sigma} \nabla T_k(u)|^2(X_r) \, dr \leq k (E_m |\psi|(X_{\tau_{D} -}) + \|u\|_{L^1} + E_m \int_0^{\tau_{D}} dL^0_r) \\
+ 2k (E_m \int_0^{\tau_{D}} |f(X_r, u(X_r))| \, dr + E_m \int_0^{\tau_{D}} dA^{au}_r).
\]

By (8.13), monotonicity of \( f \) and [27, Theorem 3.7],

\[
\|u\|_{L^1} + E_m \int_0^{\tau_{D}} dL^0_r + E_m \int_0^{\tau_{D}} |f(X_r, u(X_r))| \, dr \\
\leq E_m |\psi|(X_{\tau_{D} -}) + E_m \int_0^{\tau_{D}} |f(X_r, 0)| \, dr + \|\mu\|_{TV, \delta},
\]

which when combined with (8.14) proves the proposition. \( \square \)

In the rest of the section we assume that \( D \) is of class \( C^{1,1} \). It is well known (see [50, 51]) that under this assumption there exist \( c_1, c_2 > 0 \) such that

\[
c_1 \text{dist}(x, \partial D) \leq R^D 1(x) \leq c_2 \text{dist}(x, \partial D), \quad x \in D. 
\]

**Proposition 8.13.** Let \( u \in H_D \). Then \( u \in L^p_\delta(D; m) \) for \( p < \frac{d}{d - 1} \).

**Proof.** By Theorem 7.4, there is \( \psi \in L^1(\partial_M D; h^A_m) \) such that \( \gamma^{-1}_A(\psi) = u \). Denote by \( w_r \) the solution of the problem

\[-A w_r = -w_r |w_r|^{r - 1} \quad \text{in} \quad D, \quad (w_r)|_{\partial_M D} = \psi \]

with some \( r > 1 \). It exists by Proposition 8.7. Write \( g_r = -w_r |w_r|^{r - 1} \). By the definition of a solution, \( g_r \in L^1_{\delta}(D; m) \) and

\[
w_r = \gamma^{-1}_A(\psi) + R^D g_r.
\]

By [40], \( R^D g_r \in L^p_\delta(D; m) \) for \( p < \frac{d}{d - 1} \). Furthermore, since \( g_r \in L^1_{\delta}(D; m) \), \( w_r \in L^q_\delta(D; m) \) as well. Therefore, from (8.15) with \( r = p \), it follows that \( \gamma^{-1}_A(\psi) \in L^q_\delta(D; m) \) for \( p < \frac{d}{d - 1} \). This proves the proposition since \( u = \gamma^{-1}_A(\psi) \). \( \square \)

**Corollary 8.14.** Let \( u \) be a solution to (8.1). Then \( u \in W^{1,q}_\delta(D) \) for \( q < \frac{2d}{2d - 1} \).

**Proof.** By [40] and Proposition 8.13, \( u \in L^p_\delta(D; m) \) for \( p < \frac{d}{d - 1} \). Combining this with Proposition 8.12 and using standard argument (see [2, Lemma 4.2]) gives the desired result. \( \square \)
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