Influence of higher orders of Neumann expansion on accuracy of stochastic linear elastic finite element method with random physical parameters

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Abstract
The objective of this study is to quantify the influence of higher orders of expansion in the formulation of stochastic finite elements method on the linear elastic response in 2-dimensional problems with random physical parameters in the left hand side term. Neumann expansion was used to get an explicit expression of the result. Young’s modulus was considered as a random variable following normal distribution. The coefficient of variance (COV) of this input parameter ranged in this study up to 0.3 (30%), and mainly 20% of COV was analyzed. The displacement was selected as the quantity of interest. The difference in distribution function of the displacement for different orders of expansion was observed in the tail distribution. A fundamental example revealed the limitation of the applicability of first, second and third orders being approximately 3%, 12% and 20% of COV of input parameter. In the analysis of 2-phase composite material, the influence of geometrical random morphology was larger than that of physical parameter, but the latter was not negligible in the microscopic response.

Keywords: Stochastic finite element method, Neumann expansion, Higher order expansion, Random physical parameter, Coefficient of variance

1. Introduction

During the last decades, the consideration of the uncertainties in the definition of physical systems has become a topic of huge interest (Sudret and Der Kiureghian, 2000; Kubo and Matsuda, 2016; Yoshimura, A. et al., 2016). This topic has been studied by a large number of engineering communities, from geo-engineering (Ghiocel and Ghanem, 2002; Haba et al., 2018) to biomechanics (Sansalone et al., 2013). Studies have led to consider the physical parameters of a problem as random variables or random processes. In order to solve those new stochastic problems, different methods have been presented as concisely described below.

The most commonly used method is Monte Carlo simulation, which has been shown to be accurate enough if a large number of samples are used. However, when the number of random parameters increases, the computational cost can be very high. It has been pointed out in a series of studies on multi-step Monte Carlo simulation (Akimoto and Takano, 2016) that the result can lack of accuracy to characterize with precision the tail distribution. Nevertheless, Monte Carlo simulation remains a good reference in the development of new stochastic computational methods.

As one of the stochastic finite element method, one of the authors has focused on first order perturbation (Wen et al., 2016). This approach consists in approximating the different random parameters as first order random variables. The result is then separated into a mean value and a deviation part, and is supposed to have the same distribution as the initial random parameter. Also, the mean value of the stochastic result is supposed to be equal to the result obtained in a
deterministic calculation with the mean value of the different parameters. This method has shown great results in some applications, such as fiber reinforced composites materials (Sakata et al., 2010; Hoang and Takano, 2020) and porous materials including trabecular bone (Basaruddin et al., 2013). However, the first order perturbation approach has a serious limitation that only low variations can be covered. Therefore, in the analysis of inter-individual differences in mechanical property of bone with very large variability, a heuristic correction factor was used after calibration by experimental data (Basaruddin et al., 2013). The first order perturbation approach is easy to be implemented and has been extended to a nonlinear multiscale problem (Hoang et al., 2020), but the above limitation should be quantitatively and furthermore discussed.

Another approach in recent studies is the spectral decomposition of the random processes. The random processes are expanded along a basis of random parameters. The proper generalized decomposition spectral method was used to represent the result in an explicit form, using a finite number of random variables (Legrain et al., 2017).

In the same way, a method using Karhunen-Loeve expansion and polynomial chaos has been presented. These methods consist in discretizing random parameters in a Fourier-like series, and solve the problem along a basis of orthogonal random variable (Ghanem and Spanos, 1991). In order to improve the readability of the result, polynomial chaos can be applied to get a new basis of orthonormal polynomials, along which the result can be expanded. This result was first demonstrated using Gaussian distributions, before being extended to non-Gaussian distributions (Sakamoto and Ghanem, 2002). In this direction, we can find many papers that described the use of Karhunen-Loeve expansion (Shang and Yun, 2013; Liu et al., 2017), the influence of the numerical approximation of the expansion (Huang et al., 2001), and the formulation of polynomial chaos (Nouy, 2010).

The merit of these methods over first order perturbation approach lies in the consideration of higher orders of expansion for stochastic processes, which enables us to cover large variability. However, still in this case, it is needed to show the guideline to determine the necessary order to be expanded depending on the level of variability.

Neumann expansion focusses on random variables instead of random processes. Thus, it is simpler to be implemented than Karhunen-Loeve expansion, with less terms. Neumann expansion can be adopted to many industrial situations because random variable are more often used than random processes. The polynomial chaos is not a solution by itself, but a mathematical way to improve Karhunen-Loeve readability.

Therefore, in this paper, by using Neumann expansion to get an explicit expression of the result and for fundamental 2-dimensional linear elastic problems with the randomness in Young’s modulus, the accuracy of different orders of expansion is presented. The random parameter is considered in the formulation of the stiffness matrix of the system. Hence, the displacement is used as the quantity of interest. To put focus on the tail distribution, the cumulative probability of the displacement was plotted. In the demonstrative example of two-phase composite material, in addition to the physical parameter, the random morphology of the 2nd phase was considered by taking 10 random samples. The influence of physical and geometrical randomness on the microscopic displacement and macroscopic one is discussed.

2. Theoretical formulation
2.1 Random variables

We consider a system made of isotropic material where the Young’s modulus, $E$, is a random variable following normal distribution. It is not necessarily uniform in the whole system, and can take different values in different positions in the applications to composite materials or multi-material system. For this sake, the domain is separated into sub-domains where material constants are uniform, from 1 to $n$. The values of the parameters in these domains are defined by the problems settings. In each sub-domain, the Young’s is represented by an independent normally distributed random variable. The other parameters are considered to be deterministic in this study. Because of the normal distribution of the Young’s modulus, for each area, we can write:

$$E_i = E_{mi} + \xi_i E_{ai} \quad i = 1, ..., n$$

(1)

Here, $E_{mi}$ is the mean value of each Young’s modulus, $E_{ai}$ is the associated standard deviation, and $\xi_i$ is an independent random variable with standard normal distribution, with mean value 0 and standard deviation 1.

2.2 Formulation of the stiffness matrix

In a 2-dimensional static problem with the thickness of $t$, the equation to be solved by finite element method is of the form:

$$Ku = f$$

(2)
where $K$ is the stiffness matrix, $u$ is nodal displacement and $f$ is equivalent nodal force. It is constructed element by element with the expression:

$$K = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} w_i w_j B^T D B \text{det}(J) t$$

(3)

where $n_g$ is the number of Gauss points, $B$ is the strain-displacement matrix and $\text{det}(J)$ is Jacobian. In later numerical analyses, 4-noded element with 4 Gauss points was used. When the material constants have variability, we can use Eq. (1) to write:

$$D_i = D_{mi} + \xi_i D_{\sigma i}$$

(4)

We can also see that this matrix is linear in Eq. (3). Thus, we can extract the first part of the right-hand side term and assemble all the elementary stiffness matrices to get a global stiffness matrix that we can call mean stiffness matrix, referred as $K_m$. Next, to consider the random part of $D_i$ for each sub-domain $i$, calculated stiffness matrix is related to only one random variable. We now have a set of $n$ global stiffness matrices referred as $K_{\sigma i}$. Finally, Eq. (2) becomes:

$$K_m + \sum_{i=1}^{n} \xi_i K_{\sigma i} u = f$$

(5)

### 2.3 Explicit formulation of the result

We can take the invert of the mean stiffness matrix to obtain a new expression:

$$\left[ I + \sum_{i=1}^{n} \xi_i Q_i \right] u = g$$

(6)

with:

$$Q_i = K_m^{-1} K_{\sigma i}$$

$$g = K_m^{-1} f$$

(7)

(8)

Equation (6) is fitted to apply Neumann expansion. With this expansion, we can get an explicit expression of the result:

$$u = \left( \sum_{k=0}^{\infty} (-1)^k \left[ \sum_{i=1}^{n} \xi_i Q_i \right]^k \right) g$$

(9)

Using this expression, the influence of the orders on the accuracy of calculated displacements is investigated in the following chapter, under 2 different conditions. The calculated displacements at problem-dependent specific nodal points are used as the quantity of interest. The strain, stress and reaction force are not calculated in this fundamental study. The coefficient of variance (COV) is used to describe the variability of $E$. The different values of material parameters were chosen in the second example.

### 3. Numerical examples

#### 3.1 A plate of single material under tension

The first example consists in a 2-dimensional flat plate made of single isotropic material shown in Fig. 1, perfectly fixed at one end, and under traction at the other end. The parameters used in the linear elastic analysis are also shown. No temperature change is considered. Note that the mean value of Young’s modulus was determined to be 30 GPa as described in Fig. 1, but its COV was varied up to 33%.

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Fig. 1 Isothermal tensile problem of a 2-dimensional flat plate made of linear elastic isotropic single material.
Figure 2 shows the standard deviation of the displacement in x direction at point A, which is close to the fixed end. To compare among different orders from first order to 4th order, results were divided by the value obtained with the first order of expansion.

It was found that the first order expansion can provide the same results with higher orders up to input COV of 3%. The second order expansion works well up to approximately 12%, and the 3rd order could cover up to approximately 20% as input COV. Since most of the high-quality engineering materials have small variability in their material constants, the first order perturbation approach may provide us with reliable prediction. However, in the procedure of new development of advanced materials by new manufacturing methods, the test specimens may have relatively large variability or uncertainty. For those problems, higher order expansion should be used. Depending on the assumption put on the input COV, this result can become a guideline to determine which order should be employed.

For deeper discussion, the differences of the distribution functions of displacement at point A among different orders were investigated. As a reference, the result by Monte Carlo simulation with 1000 samples was used hereafter. The COV of the input parameter was fixed to 20%.

First, the accuracy of the first order expansion is shown by plotting the value of \( 1.0 - \text{(cumulative probability)} \), which is often used in reliability engineering, in Fig. 3(a). It was found that the inaccurate displacements appear in the tail distribution only. In the same way, the result of the 3rd order expansion is shown in Fig. 3(b). The result obtained by 2nd order didn’t perfectly agree with that of Monte Carlo simulation. We tested 4th order expansion in the same way, and confirmed that the tail probability was improved. Also the displacements at other points in the plate were investigated carefully, but all gave the same result with Fig. 3.

This fundamental example revealed that the difference lies only in the tail distribution even in the case of first order expansion with 20% COV of input parameter. When we remind the accuracy and reliability in the distribution function of the input parameter, especially in the tail distribution, we can hardly emphasize the problem of inaccurate prediction of the tail distribution in most of the engineering problems. Higher orders than 2nd order expansion approach gave reliable enough stochastic prediction.

### 3.2 Composite material under compression with randomly distributed 2nd phase material

The aim of the second example is to demonstrate that the presented formulation can consider multiple sub-domains by analyzing a problem of two-phase composite material. Moreover, the following two issues are of greater interest. Firstly, by including the randomness of the geometrical distribution of the 2nd phase, the influences of physical...
Fig. 3 Comparison of distribution function of the displacement in x direction at point A between Neumann expansion and Monte Carlo simulation with 1000 samples in the case of input COV of 20%

parameters and geometrical parameters on the results are discussed. Also, the influences of the random variables on the microscopic response and macroscopic response are compared.

A 2-dimensional domain shown in Fig. 4(a) with perfectly fixed bottom edge is under compression. Note that both side edges of the domain are not constrained. The volume fraction of the 2nd phase is around 0.2. The distribution of the second phase follows 2-dimensional uniform distribution to define the position of the inclusion points, in a way that would keep the volume fraction around 0.2. In the computer code, a random number was generated to obtain the models with different geometry. But, the physical parameters are considered to be deterministic. On the other hand, the Young’s modulus of matrix material is assumed to have variability, whose mean value is $E_{ml}$ and COV is 20%. To model the random distribution of the 2nd phase, 10 numerical models shown in Fig. 4(b) are used.

To observe the macroscopic response, Fig. 5 shows the cumulative probability of the displacement in y direction at the corner point B, in the same way with the previous example. The results of 10 numerical models are shown for first, second and third order expansions. The results of the first order expansion were different from others in the tail distribution. However, the scattering due to the geometrical randomness was more significant. Since the Young’s modulus of the 2nd phase is much larger than that of matrix material, the 2nd phase mainly supports the load and can form the load path. Since the overall stiffness is influenced by the load path formation (Takano, et al. 2008; Yoshiwara et al. 2011), it is reasonable that the influence of geometrical randomness is larger than the influence of Young’s modulus of the matrix material, although its COV is very large. Except the tail distribution, the scattered results mainly due to the morphology are almost the same among different orders of expansion. In practical problem of composite material, the properties of matrix material may include uncertainty while molding or curing. Even in that case, this example implies that the distribution of the 2nd phase may give greater influence on the process parameters such as the mold pressure.

To observe the microscopic displacement in the composite material, the displacements on the cross-sections were investigated. Figure 6 shows the displacement in x direction on cross sectional line A ($x=40 \, \mu m$). Note that interpolation was used if the nodal point is not exactly on the cross-sectional line. Only two cases using first order expansion (blue lines) and second order expansion (black lines) are shown. In Fig. 6, we can recognize the difference between blue and black lines. That is, the absolute value of the compressive displacement is larger in the results by second order expansion. To this end, it was considered that the influence of random physical variable of matrix material appeared more in the microscopic response, whereas the scattering in the macroscopic response was governed by the microscopic morphology in the two-phase composite material.

In this analysis, 10 models with different morphology of the 2nd phase, which is almost homogenously dispersed, were analyzed. The above described findings may be true even if more number of models are analyzed.
(a) Problem setting

(b) Random distribution of the 2nd phase

Fig. 4 Isothermal compression problem of a 2-dimensional domain of composite material, where the Young’s modulus matrix material has variability with 20% of COV and the 2nd phase has geometrical randomness.

Fig. 5 Distribution functions of the displacement in $y$ direction at point B for 10 numerical models with different geometrical morphology for each order expansion.

Fig. 6 Displacement in $x$ direction along cross-sectional line A for 10 numerical models with different geometrical morphology for each order expansion.
4. Conclusion

This paper presented the formulation of stochastic finite element method using Neumann expansion considering the variability in Young’s modulus and two demonstrative examples aiming at quantifying the influence of higher orders of expansion on the accuracy of the calculated displacements. The first order expansion could be applied to problems with 3% COV of input parameter, and second order could work until 12% COV, but third or fourth orders were necessary when COV of input parameter was 20%. In the case of systems with high variability, the use of Neumann expansion to get higher orders of approximation becomes interesting when compared with first order approximation method, which can only represent low variations. It was also found that the differences among expansion orders appeared in the tail distribution of the displacement. For composite materials with microscopic heterogeneity, the geometrical morphology gave significant influence and the influence of random physical parameters was observed more in the microscopic displacement. The influence of the different orders on the displacement appeared in this example also in the tail distribution. Many studies on the stochastic finite element method so far assumed COV of input parameters. With the growing need for stochastic simulation of practical problems with statistically measured data of input parameters, this paper could show the guideline in the choice of moderate order of expansion.

With this study, we could use Neumann expansion with less terms than Karhunen-Loeve expansion, as random variables were used instead of random processes. With Neumann expansion, the number of terms also increases when the number of random parameters increases when higher orders are used, but this total is still less than the number of terms to be created by Karhunen-Loeve expansion for the same number of random parameters.

This study has some limitations. Only 2-dimensional linear elastic problem was solved and the displacement was considered as the quantity of interest. For Young’s modulus, normal distribution was assumed because it is the most fitted for expansion and give the most readable formulation. Other kinds of distribution will be able to be used, but further study is needed on this point. The applications are limited to simple problems and high gradient of strain or stress field was not discussed. Finally, the prototype program in this study was not good enough to discuss deeply about the computational time. One of the merits of the use of first order perturbation (Wen et al., 2016) lies in the short computational time compared with the Monte Carlo simulation for problems with geometrical random parameters. When higher order expansion is used, the computational cost should be investigated furthermore.

The computational cost was not discussed in this study. Focus was given only to the accuracy of the distribution function, and the results were compared with Monte Carlo simulation. The computational time for the present method was much shorter than that for Monte Carlo simulation, but is still not short enough for practical use. Further study is needed to reduce the computational time.

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