DEGENERATE CHERN-WEIL THEORY AND EQUIVARIANT COHOMOLOGY

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Abstract. We develop a Chern-Weil theory for compact Lie group action whose generic stabilizers are finite in the framework of equivariant cohomology. This provides a method of changing an equivariant closed form within its cohomological class to a form more suitable to yield localization results. This work is motivated by our work [5] on reproving wall crossing formulas in Seiberg-Witten theory, where the Lie group is the circle. As applications, we derive two localization formulas of Kalkman type for $G = SU(2)$ or $SO(3)$-actions on compact manifolds with boundary. One of the formulas is then used to yield a very simple proof of a localization formula due to Jeffrey-Kirwan [15] in the case of $G = SU(2)$ or $SO(3)$.

Throughout this paper, $G$ will be a compact connected Lie group, with $\mathfrak{g}$ as its Lie algebra. Assume that $G$ acts freely on a smooth manifold $P$. Then the quotient map $P \rightarrow P/G = M$ gives $P$ a structure of principal $G$-bundle. The celebrated Chern-Weil theory gives us a homomorphism

$$cw : S(\mathfrak{g}^*)^G \rightarrow H^*(M),$$

called the Chern-Weil homomorphism. Here $S(\mathfrak{g}^*)^G$ is the algebra of polynomials on $\mathfrak{g}$ which is invariant under the adjoint representation of $G$ on $\mathfrak{g}$. The Chern-Weil construction uses a connection 1-form $\omega \in (\Omega^1(P) \times \mathfrak{g})^G$ and its curvature 2-form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. The equation $d\Omega = [\Omega, \omega]$ can be used to show that for any invariant polynomial $F \in S^n(\mathfrak{g}^*)^G$, $F(\Omega)$ is the pullback of a closed form on $M$. This defines the homomorphism (1). Furthermore, for two connections $\omega^0$ and $\omega^1$ with curvatures $\Omega^0$ and $\Omega^1$ respectively, there is a canonically defined differential form $T(\omega^0, \omega^1)F$ on $M$, called the transgression form, such that

$$dT(\omega^0, \omega^1)F = F(\Omega^1) - F(\Omega^0).$$

Therefore, the Chern-Weil homomorphism is independent of the choice of $\omega$. We call this Chern’s formulation. Cartan [6] presented Weil’s formulation, which we shall review in §1. Through Weil’s formulation, Cartan (§5 in [5]) discovered that the Chern-Weil homomorphism can be factored as

$$S(\mathfrak{g}^*)^G \xrightarrow{\phi} H^*_G(P) \xrightarrow{(r^G)_*} H^*(M),$$

where $H^*_G(P)$ is the equivariant cohomology of $P$, and $\phi$ is the homomorphism which gives $H^*_G(P)$ the structure of an $H^*(BG) \cong S(\mathfrak{g}^*)^G$-module. The homomorphism $(r^G)_*$ is induced from a homomorphism on the chain level obtained by a similar Chern-Weil construction.

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In this paper, we shall generalize the above picture to the case that the $G$-action on a smooth manifold $W$ is only locally free on a dense open set $W^0 \subset W$. Using a connection $\omega$ on $W^0$, and a cut-off function $f$, we shall construct homomorphisms
\[ cw^G_f : S(\mathfrak{g}^*)^G \to H^*_G(W), \]
and
\[ (r^G_f)_* : H^*_G(W) \to H^*_G(W), \]
such that $cw^G_f = (r^G_f)_* \circ \phi$. Here $(r^G_f)_*$ is induced from a homomorphism $r^G_f$ at the chain level in Cartan model for equivariant cohomology. We shall also construct transgression operator to show that $cw^G_f$ and $(r^G_f)_*$ are independent of the choices of connection $\omega$ and the cut-off function $f$. An important observation, pointed out to us by Professor Michèle Vergé, is that when one takes $f \equiv 0$, then our calculation shows that the homomorphism $(r^G_f)_*$ is the identity map. The main results of this paper are stated in Theorem 2.1-2.6. We call these results the degenerate Chern-Weil theory. We remark that our approach corresponds to Chern’s formulation. It depends on calculations by brute force. It is interesting to find a Weil’s formulation, which might make the argument simpler.

Even though the results of this paper provide an invariant for non-free group actions (which is interesting in its own respect), the main motivation is to give a method of choosing a nice representative for an equivariant cohomological class to obtain localization results. At the chain level, for suitable choice of $\omega$ and $f$, $r^G_f$ gives us a nice way to change an equivariant closed form $\alpha$ within its equivariant cohomological class to $r^G_f(\alpha)$, with the following property: in a neighborhood of the singular set of the group action, $r^G_f(\alpha) = \alpha$, outside a larger neighborhood, $r^G_f(\alpha)$ is the pullback of an ordinary differential form from the quotient. This provides a simple explanation for the localization phenomenon in equivariant cohomology. When $\text{deg}(\alpha) = \dim(X)$, one often considers integral $\int_X \alpha$. But we have
\[ \int_X \alpha = \int_X r^G_f(\alpha), \]
by dimension reason. However, $r^G_f(\alpha)$ vanishes outside a neighborhood of the singular set of the group (e.g., at where $f = 1$). So the only contribution to the integral is from near the singular set. Localization formula could then be obtained by shrinking the support of $1 - f$. This is in the same spirit as the proof of the localization formula given in Berline-Getzler-Vergne [4]. (It might be possible to reprove their formula along this line.) A similar argument explains why one can expect localization formula on manifolds with boundary, such as Kalkman’s formula [17]. For details, see §3. It would be interesting to compare our work with the theory of singular connections of Harvey-Lawson [12] which concerns characteristic classes and singularities of vector bundle homomorphisms. For other methods of obtaining localization formulas, see, e.g., Atiyah-Bott [1] and Witten [26].

In our earlier work Cao-Zhou [5], a localization formula for circle action due to Kalkman [17] is used to obtain wall crossing formulas in Seiberg-Witten theory due to Li-Liu [20] and Okonek-Teleman [23]. An important ingredient in [5] is the construction of degenerate first Chern class for a circle action. The results in this paper are nontrivial generalizations from circle group to compact Lie groups. As explained above, the application to localization formula is the main motivation for studying degenerate Chern-Weil theory.
As illustrations of our localization idea, we prove two nonabelian localization formulas (Theorem 3.1 and Theorem 3.2) of Kalkman type for \( G = SU(2) \) and \( SO(3) \). Theorem 3.1 should be very useful in the study of various wall crossing phenomenon. In a sequel [6], we apply Theorem 3.1 to study wall crossing phenomenon in symplectic reduction. On the other hand, though moduli spaces in Donaldson theory are in general noncompact and our results do not yet readily apply to the study of wall crossing phenomenon of Donaldson invariants, we believe suitable modifications should yield some results in this direction. Along the same line, a localization formula of this type for \( U(2) \)-action should shed some lights on the conjectured equivalence of Seiberg-Witten theory and Donaldson theory. We shall leave such issues for future investigations. As an application of Theorem 3.2, we shall give a very simple proof of the nonabelian localization formula of Jeffrey-Kirwan [15] in the case of Hamiltonian \( SU(2) \) or \( SO(3) \)-actions.

The rest of the paper is organized as follows. In §1 we review the equivariant cohomology and fix some notations. The degenerate Chern-Weil theory is presented in §2. In §3 we prove two nonabelian localization formulas of Kalkman type, Theorem 3.1 and Theorem 3.2. The application of Theorem 3.2 to symplectic reduction is given in §4.

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1. Preliminaries on equivariant cohomology

We will use two differential geometric models, the Weil model and the Cartan model, for equivariant cohomology. For the sake of completeness, we also discuss Borel model at the end of this section. We refer the reader to Atiyah-Bott [1], Cartan [7,8], Berline-Getzler-Vergne [4], Duflo-Kumar-Vergne [9], Kalkman [16], Lawson [19] and Mathai-Quillen [22] and the references therein for more details.

1.1. Weil algebra. The Weil algebra [1] is the Hopf algebra

\[ W(g) = \Lambda(g^*) \otimes S(g^*), \]

where elements in \( \Lambda^1(g^*) \) have degree 1, and elements in \( S^1(g^*) \) have degree 2. Let \( \{ \xi_a \} \) be a basis of \( g \), such that

\[ [\xi_a, \xi_b] = f_{ab}^c \xi_c, \]

where \( f_{ab}^c \)'s are the structure constants. Let \( \{ \theta^a \} \) be a dual basis in \( \Lambda^1(g^*) \), and \( \{ \Theta^a \} \) a dual basis in \( S^1(g^*) \). Define the Weil differential \( d_w : W(g) \to W(g) \) by
setting
\[ d_w \theta^a = - \frac{1}{2} f_{bc}^a \theta^b \theta^c + \Theta^a, \]
\[ d_w \Theta^a = - f_{bc}^a \Theta^b \]
and extending it as a derivation of degree 1. There are also contractions \( i_a \) and Lie derivatives \( L_a \) on \( W(\mathfrak{g}) \) defined by
\[ i_a \theta^b = \delta^b_a, \quad L_a \theta^b = - f_{bc}^a \theta^c. \]
Notice that \( G \) acts on \( W(\mathfrak{g}) \) by extending the co-adjoint representation. Its linearization can be identified with \( L_a \)'s. It is easy to verify the homotopy formula
\[ L_a = d_w i_a + i_a d_w. \]

1.2. Algebras with Weil structures. We need the following

Definition. An algebra with Weil structure over \( G \) is a graded differential algebra
\[ (A^* = \oplus_{j=0}^{+\infty} A^j, d) \]
over \( \mathbb{R} \), with a left representation
\[ L : G \to Aut(A^*, d) \]
of degree 0, and a \( G \)-equivariant linear map \( i : \mathfrak{g} \to End A^* \) of degree \(-1\), such that
1. \( i_\xi i_\eta + i_\eta i_\xi = 0 \), for \( \xi, \eta \in \mathfrak{g} \);
2. \( L_\xi = d \circ i_\xi + i_\xi \circ d \), where \( L : \mathfrak{g} \to Der(A^*, d) \) is the linearization of the representation \( L : G \to Aut(A^*, d) \).

A simple example of algebra with Weil structure is \( \Omega(X) \) with ordinary contractions, Lie derivatives and the exterior differential, for a \( G \)-manifold \( X \). Another example is the Weil algebra. Now, given an algebra \( (A^*, d, i, L) \) with Weil structure over \( G \), define the basic subalgebra by
\[ A^*_\text{basic} = \{ \phi \in (A^*)^G | i_\xi \phi = 0, \forall \xi \in \mathfrak{g} \}. \]
It is straightforward to verify the following

Lemma 1.1. \( A^*_\text{basic} \) is a graded differential subalgebra of \( A^* \).

The cohomology of the basic subalgebra is called the basic cohomology, and denoted by \( H^*(A)_{\text{basic}} \). It is well-known that the basic cohomology of \( W(\mathfrak{g}) \) is \( S(\mathfrak{g}^*)^G \). When \( \pi : P \to M \), the basic cohomology of \( \Omega(P) \) is the de Rham cohomology \( H^*(M) \).

Lemma 1.2. If \( \rho : A^* \to B^* \) is a homomorphism of algebras with Weil structures over \( G \), then \( \rho \) induces a homomorphism
\[ \rho_{\text{basic}} : (A^*_\text{basic}, d) \to (B^*_\text{basic}, d), \]
and therefore, a homomorphism
\[ (\rho_{\text{basic}})_* : H^*(A)_{\text{basic}} \to H^*(B)_{\text{basic}}. \]
The above definition and lemmas about algebras with Weil structures are taken from Lawson \cite{Lawson}. They appeared in Cartan \cite{Cartan} and Kamber-Tondeur \cite{KamberTondeur} with different terminologies.

If $P \to M$ is a principal $G$-bundle, a connection $\omega = \xi_a \omega^a$ with curvature $\Omega = \xi_a \Omega^a$ defines a homomorphism of algebras with Weil structures

$$W(g) \to \Omega(P),$$

by sending $\theta^a$ to $\omega^a$ and $\Theta^a$ to $\Omega^a$. Applying Lemma 1.2, one gets the Chern-Weil homomorphism

$$cw : S(g^*)^G \to H^*(M)$$

by identifying the basic cohomology of $\Omega(P)$ with $H^*(M)$. We call this way of getting the Chern-Weil homomorphism Weil’s formulation. See Cartan \cite{Cartan}. Notice that we can factor (2) as a composition of two homomorphisms

$$W(g) \to W(g) \otimes \Omega(P) \xrightarrow{cw^p} \Omega(P),$$

where the first one is the inclusion, and $cw^p$ is defined by extending (2). One can show that $(cw^p)_{basic}$ is an isomorphism. Indeed, if $i^w : \Omega(P) \hookrightarrow W(g) \otimes \Omega(P)$ is the inclusion, then $cw^p \circ i^w = 1$ on $\Omega(P)$ implies that $(i^w_{basic})_*$ is injective. Cartan’s proof to Theorem 3 in \cite{Cartan} shows that $(i^w_{basic})_*$ is also surjective, and hence an inverse to $(cw^p)_{basic}$.

1.3. **Equivariant cohomology: Weil model and Cartan model.** Let $X$ be a compact smooth $G$-manifold. The $G$-action on $X$ induces a homomorphism from the Lie algebra $g$ to the Lie algebra of vector fields on $X$. Denote by $\iota_a$ and $L_a$ the contraction and the Lie derivative by the vector field corresponding to $\xi_a \in g$ respectively. Consider the tensor product of algebras with Weil structures

$$W(g) \otimes \Omega(X),$$

where one uses the diagonal $G$-action, and the contraction $\iota_a \otimes 1 + 1 \otimes \iota_a$, the Lie derivative $L_a \otimes 1 + 1 \otimes L_a$, and differential $d_w \otimes 1 + 1 \otimes d$. The corresponding basic cohomology is called equivariant cohomology (via Weil model), and is denoted by $H_G^*(X)$.

Motivated by the work of Cartan \cite{Cartan}, one can also consider the Cartan model which is given by the complex $(\Omega_G(X), D_G)$, where $\Omega_G(X) = (S(g^*) \otimes \Omega(X))^G$, and $D_G = 1 \otimes d - \Theta^a \otimes \iota_a$, called the Cartan differential. When there is only one Lie group involved, we will use $D$ for $D_G$. Since $D$ is a $G$-equivariant operator on $S(g^*) \otimes \Omega(X)$, it then maps $\Omega_G(X)$ to itself. Furthermore, since $\Theta^a \otimes L_a$ acts as zero on $S(g^*)$, we have

$$D^2 = -\Theta^a \otimes L_a = -\Theta^a(L_a \otimes 1 + 1 \otimes L_a).$$

Therefore, $D^2 = 0$ on $\Omega_G(X) = (S(g^*) \otimes \Omega(X))^G$.

It is possible to identify $H_G^*(X) = H^*(\Omega_G(X), D)$ through an isomorphism

$$\Psi : W(g) \otimes \Omega(X) \to W(g) \otimes \Omega(X)$$

defined by

$$\Psi = \prod_a \exp(-\theta^a \otimes \iota_a) = \prod_a (1 - \theta^a \otimes \iota_a).$$

In fact, $\Psi^{-1}$ maps $((W(g) \otimes \Omega(X))_{basic}, d_w \otimes 1 + 1 \otimes d)$ to $((S(g^*) \otimes \Omega(X))^G, D)$. See Cartan \cite{Cartan}, Mathai-Quillen \cite{MathaiQuillen} and Kalkman \cite{Kalkman} for more details.
In the case of a principal $G$-bundle $\pi : P \to M$, one can define a homomorphism

$$r^G : S(\mathfrak{g}^*) \otimes \Omega(P) \to \Omega(P)$$

by $r^G = cu^p \circ \Psi$. It is easy to see that if $\alpha \in (S(\mathfrak{g}^*) \otimes \Omega(P))^G$ and $D\alpha = 0$, then $d^rG(\alpha) = 0$, and $r^G(\alpha)$ is the pullback of a form on $M = P/G$. As the Cartan model version of the fact that $(cu^p)^*_\alpha$ is an isomorphism, $r^G$ induces an isomorphism between $H^*_G(P)$ with $H^*(M)$. For a proof, see Duflo-Kumar-Vergne [4]. (We thank Michèle Vergne for bringing our attention to this reference.)

**Remark.** When the group action is locally free, i.e., all the isotropy subgroups are discrete, then $M/G$ is an orbifold [24]. The above discussions carry through if one uses de Rham theory for orbifolds.

1.4. **Reduction to the maximal torus.** Let $T$ be a maximal torus of $G$, with Lie algebra $\mathfrak{t}$. The inclusion $\mathfrak{t} \hookrightarrow \mathfrak{g}$ induces a map $\mathfrak{g}^* \to \mathfrak{t}^*$. Alternatively, if we endow $\mathfrak{g}$ with a $G$-invariant inner product, then one gets an orthogonal projection $\mathfrak{g}^* \to \mathfrak{t}^*$, which can be identified with the map above. This can be extended to a projection $p_1 : S(\mathfrak{g}^*) \to S(\mathfrak{t}^*)$. Similarly, if we endow $X$ with a $G$-invariant Riemannian metric, it then induces an inner product on $\Omega(X)$. So we get a projection $p_2 : \Omega(X) \to \Omega(X)^T$. Put $p_1$ and $p_2$ together, we get a projection

$$p_1 \otimes p_2 : S(\mathfrak{g}^*) \otimes \Omega(X) \to S(\mathfrak{t}^*) \otimes \Omega(X)^T,$$

which induces a projection

$$p : \Omega_G(X) \to \Omega_T(X).$$

It is an easy exercise to see that $pD_G = D_Tp$, hence $p$ induces a homomorphism $H^*_G(X) \to H^*_T(X)$. Let $W$ be the Weyl group, then $p$ induces an isomorphism $H^*_G(X) \cong H^*_T(X)^W$. For a proof, see e.g. Hsiang [13] or Duflo-Kumar-Vergne [4].

**Note.** In an earlier version, we falsely claim that the image of $p(\omega_G(X)) = \Omega_T(X)^W$. Michèle Vergne provided us with a counter-example. She also informed us about the references [13] and [4].

**Note.** We prefer to use $u^a$ instead of $\Theta^a$ when the Lie group is a torus, and $u$ in the case of a circle.

1.5. **Equivariant Euler class.** We will also need the notion of equivariant Euler class [1]. Let $F$ be a connected closed oriented manifold, and $\pi : E \to F$ be a smooth complex vector bundle over $F$. Assume that there is an $S^1$-action on $E$ by bundle homomorphisms, which covers an $S^1$-action on $F$. Then one can define the equivariant Euler class $\epsilon(E) \in H^*_G(F)$, which satisfies

$$\epsilon(E_1 \oplus E_2) = \epsilon(E_1)\epsilon(E_2)$$

for two $S^1$ bundles $E_1$ and $E_2$ over $F$. We will be concerned with the case when the action of $S^1$ on $F$ is trivial. In this case, $E$ has a decomposition as $S^1$ bundles

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r,$$

where each $L_j$ is a line bundle such that the action of $exp(2\pi \sqrt{-1}t)$ on $L_j$ is multiplication by $exp(2\pi \sqrt{-1}m_jt)$, for some weight $m_j \in \mathbb{Z}$. By formula (8.8) in Atiyah-Bott [6],

$$\epsilon(L_j) = m_ju + c_1(L_j).$$
Hence we have
\[ \epsilon(E) = \prod_{j=1}^{r}(m_ju + c_1(L_j)). \]

Here \( u \) is dual to an element \( \xi \) in the Lie algebra of \( S^1 \), such that if \( S^1 \) is given an invariant metric in which \( |\xi| = 1 \), then \( \text{vol}(S^1) = 1 \).

### 1.6. Borel model

Historically, equivariant cohomology was defined by Borel model (cf. Atiyah-Bott [1]). Let \( \pi : EG \to BG \) be a universal principal \( G \)-bundle (it is unique up to homotopy). The Borel construction of a \( G \)-manifold \( X \) is
\[ X_G := (X \times EG)/G, \]
where \( G \) acts on \( X \times EG \) diagonally. It can be shown that \( H^*(X_G, \mathbb{R}) \cong H^*_G(X) \) (see e.g. Lawson [19]). In particular, when \( X \) is a point, \( X_G = BG \), one has
\[ H^*(BG, \mathbb{R}) \cong H^*_G(pt) \cong S(g^*)^G. \]

The homomorphism \( H^*(BG, \mathbb{R}) \to H^*(X_G, \mathbb{R}) \) induced from the map of \( G \)-spaces \( X \to pt \) can be identified with the homomorphism
\[ S(g^*)^G \to H^*_G(X). \]

Since their relationships with differential forms, the Cartan model and Weil model became popular after the works of Berline-Vergne [2, 3], Atiyah-Bott [1], Mathai-Quillen [22], etc.

### 2. Degenerate Chern-Weil Theory

Our construction in this section is motivated by the equivariant Chern-Weil theory for equivariant principal bundles due to Berline-Vergne [2]. In the special case of \( G = S^1 \), the construction is used in Cao-Zhou [5] to prove wall crossing formulas in Seiberg-Witten theory. (In fact, the original construction in [5] was different and more complicated, we were led to the present version by consideration of generalization to the nonabelian case.)

Let \( W \) be a compact \( G \)-manifold, possibly with boundary \( \partial W \), such that the \( G \)-action on an open subset of \( W \), which contains \( \partial W \), is (locally) free. We will call \( W \) a degenerate principal \( G \)-bundle. Denote by \( W^0 \) the set of points in \( W \) whose stabilizers have dimension \( > 0 \) and set \( W^0 = W - W^s \). Let \( f : W \to [0, 1] \) be a \( G \)-invariant smooth function on \( W \) which vanishes on a tubular neighborhood of \( W^s \), and is identically 1 outside a larger tubular neighborhood. Let \( \omega \) be a connection of the principal bundle \( W^0 \to W^0/G \). We call \( \omega_f = f \cdot \omega \in g \otimes \Omega^1(W) \) a degenerate connection, and
\[ \Omega^G_f = d\omega_f + \frac{1}{2}[\omega_f, \omega_f] - (1 + f)\xi_a \Theta^a \in g \otimes \Omega^2_G(W) \]
the degenerate equivariant curvature of \( \omega_f \).

**Lemma 2.1.** We have \( D\Omega^G_f = [\Omega^G_f, \omega_f] \).

**Proof.** It suffices to prove it on \( W^0 \), on which we have \( t_a \omega = \xi_a, t_a df = 0 \). Furthermore,
\[ t_a dw = t_a (d\omega + \frac{1}{2}[\omega, \omega]) - \frac{1}{2}t_a [\omega, \omega] = -[\xi_a, \omega], \]
\[ [[\omega_f, \omega_f], \omega_f] = f^3[[\omega, \omega], \omega] = 0 \]
So on \(W^0\), we have

\[
D\Omega_f^G = d(d\omega_f + \frac{1}{2}[\omega_f, \omega_f]) - (-1 + f)\xi_a \Theta^a
- \Theta^a \xi_b d(\omega_f + \frac{1}{2}[\omega_f, \omega_f]) - (-1 + f)\xi_a \Theta^a
= [d\omega_f, \omega_f] - df \xi_a \Theta^a - \Theta^a \xi_a(df \wedge \omega + f d\omega + \frac{f^2}{2}[\omega, \omega])
= [d\omega_f + \frac{1}{2}[\omega_f, \omega_f], \omega_f] - df \xi_a \Theta^a + \Theta^a df \xi_a + (f - f^2)\Theta^a[\xi_a, \omega]
= [\Omega_f, \omega_f] + [(1 - f)\xi_a \Theta^a, f \omega] = [\Omega_f^G, \omega_f].
\]

In the remaining part of this section, we shall adopt the following

**Conventions.** If \(i_1, \cdots, i_q\) are indices, then \((i_1 \cdots i_q)\) means symmetrizing on these indices, and \([i_1 \cdots i_q]\) means antisymmetrize on these indices. Furthermore, notation like \(i_1 \cdots b \cdots i_q\) means \(b\) does not participate in the (anti-)symmetrization.

**Lemma 2.2.** Let \(F \in S^q(g^*)^G\), then

\[
qF([\Omega_f^G, \omega_f], \Omega_f^G, \cdots, \Omega_f^G) = 0.
\]

**Proof.** As in the ordinary case, this is equivalent to the invariance of \(F\). Let \(F = \sum_{a_1,\cdots,a_q} a_{i_1,\cdots,i_q} \Theta^{i_1} \cdots \Theta^{i_q}\), where \(a_{i_1,\cdots,i_q} = a_{(i_1,\cdots,i_q)}\). Since \(F\) is \(G\)-invariant, we have

\[
0 = L_b F = \sum_{a_1,\cdots,a_q} q a_{i_1,\cdots,i_q} (L_b \Theta^{i_1}) \Theta^{i_2} \cdots \Theta^{i_q} - \sum_{a_1,\cdots,a_q} q f_{b_{i_1,\cdots,i_q}} a_{i_1,\cdots,i_q} \Theta^{i_1} \Theta^{i_2} \cdots \Theta^{i_q}
= -q f_{b_{i_1,\cdots,i_q}}^i a_{i_1,\cdots,i_q} \Theta^{i_1} \Theta^{i_2} \cdots \Theta^{i_q},
\]

where the last term is obtained after symmetrizing the indices \(c, i_2, \cdots, i_q\). Therefore, \(q f_{b_{i_1,\cdots,i_q}}^i a_{i_1,\cdots,i_q} \Theta^{i_1} \Theta^{i_2} \cdots \Theta^{i_q} = 0\). Hence we have

\[
qF([\Omega_f^G, \omega_f], \Omega_f^G, \cdots, \Omega_f^G)
= -\omega^c_f q f_{b_{i_1,\cdots,i_q}}^i a_{i_1,\cdots,i_q} (\Omega_f^G)^c \wedge (\Omega_f^G)^{i_2} \wedge \cdots \wedge (\Omega_f^G)^{i_q}
= -\omega^c_f q f_{b_{i_1,\cdots,i_q}}^i a_{i_1,\cdots,i_q} (\Omega_f^G)^c \wedge (\Omega_f^G)^{i_2} \wedge \cdots \wedge (\Omega_f^G)^{i_q} = 0.
\]

**Lemma 2.3.** Let \(F \in S^q(g^*)^G\), then \(F(\Omega^G_f, \cdots, \Omega^G_f) \in (S(g^*) \otimes \Omega(W))^G\). Furthermore,

\[
DF(\Omega^G_f, \cdots, \Omega^G_f) = 0.
\]

**Proof.** The first statement is obvious. For the second,

\[
DF(\Omega^G_f, \cdots, \Omega^G_f) = qF(D\Omega^G_f, \Omega^G_f, \cdots, \Omega^G_f)
= qF([\Omega^G_f, \omega_f], \Omega^G_f, \cdots, \Omega^G_f) = 0.
\]

As a corollary to Lemma 2.3, we have
Theorem 2.1. Let $W$ be a degenerate principal $G$-bundle. Given a degenerate connection $\omega_f$ with equivariant degenerate curvature $\Omega_f^G$, there is a homomorphism (called degenerate Chern-Weil homomorphism)

$$cw_f : S(\mathfrak{g}^*)^G \to H^*_G(W),$$

which is induced from the homomorphism

$$CW_f : S(\mathfrak{g}^*) \to S(\mathfrak{g}^*) \otimes \Omega(W)$$

given by $F \in S(\mathfrak{g}^*) \mapsto F(\Omega_f^G)$.

Similar to Lemma 2.2, one can prove the following

Lemma 2.4. Let $F \in S^q(\mathfrak{g}^*)^G$, then

$$q(q-1)F(\alpha, [\Omega_f^G, \omega_f], \Omega_f^G, \cdots, \Omega_f^G) = qF([\alpha, \omega_f], \Omega_f^G, \cdots, \Omega_f^G)$$

for $\alpha \in \Omega^1(W) \otimes \mathfrak{g}$.

Theorem 2.2. The degenerate Chern-Weil homomorphism in Theorem 2.1 does not depend on the choice of the connection $\omega$ on $W^0$ or the cut-off function $f$.

Proof. Let $\omega_f^0$ and $\omega_f^1$ be two connections on $W$, with degenerate equivariant curvatures $(\Omega_f^G)^0$ and $(\Omega_f^G)^1$ respectively, and consider

$$\tilde{\omega}_f = (1-t)\omega_f^0 + t\omega_f^1.$$  

Then $\tilde{\omega}_f$ is a degenerate connection on $W \times I$, $I = [0,1]$. Denote by $\tilde{\Omega}_f^G$ the degenerate equivariant curvature of $\tilde{\omega}_f$, $\pi : W \times I \to W$ the projection, and let

$$\int_{\pi} : W(\mathfrak{g}) \otimes \Omega(W \times I) \to W(\mathfrak{g}) \otimes \Omega(W)$$

be defined by

$$\int_{\pi} \alpha(t) + dt \wedge \beta(t) = \int_0^1 \beta(t) dt,$$

where $\alpha(t)$ and $\beta(t)$ are families of equivariant differential forms on $W$ depending smoothly on $t$. For any $F \in S^q(\mathfrak{g}^*)^G$, define the degenerate transgression operator

$$T_{(\omega_f^0, \omega_f^1)} F = \int_{\pi} F(\tilde{\Omega}_f^G).$$

Then one can check that

$$DT_{(\omega_f^0, \omega_f^1)} F = F((\Omega_f^G)^1) - F((\Omega_f^G)^0).$$

Indeed, if we let $\delta = \omega_f^1 - \omega_f^0$ and $(\Omega_f^G)^t$ be the degenerate equivariant curvature of $\omega_f^t = \omega_f^0 + t\delta$, then

$$(\Omega_f^G)^t = (\Omega_f^G)^0 + t\delta + t(\omega_f^0, \delta) + \frac{t^2}{2} [\delta, \delta] + dt \wedge \delta.$$  

So we have

$$\frac{d}{dt}(\Omega_f^G)^t = d\delta + [\omega_f^t, \delta] = D\delta + [\omega_f^t, \delta],$$

$$T_{(\omega_f^0, \omega_f^1)} F = \int_0^1 qF(\delta, (\Omega_f^G)^t \cdots, (\Omega_f^G)^t) dt.$$
Hence,
\[ DT_{(\omega^0, \omega^1)} F \]
\[ = \int_0^1 (qF(D\delta, (\Omega^G_f)^t, \cdots, (\Omega^G_f)^t) dt + q(q-1)F(\delta, D(\Omega^G_f)^t, (\Omega^G_f)^t, \cdots, (\Omega^G_f)^t) dt \]
\[ = \int_0^1 (qF(D\delta, (\Omega^G_f)^t, \cdots, (\Omega^G_f)^t)
+ q(q-1)F(\delta, [(\Omega^G_f)^t, \omega^1], (\Omega^G_f)^t, \cdots, (\Omega^G_f)^t) dt \quad \text{(by Lemma 2.1)} \]
\[ = \int_0^1 qF(D\delta + [\omega^1, \delta], (\Omega^G_f)^t, \cdots, (\Omega^G_f)^t) dt \quad \text{(by Lemma 2.4)} \]
\[ = \int_0^1 qF_0(D\delta + [\omega^1, \delta], (\Omega^G_f)^t, \cdots, (\Omega^G_f)^t) dt \]
\[ = F((\Omega^G_f)^t, \cdots, (\Omega^G_f)^t) - F((\Omega^G_f)^0, \cdots, (\Omega^G_f)^0). \]

Therefore, the degenerate Chern-Weil homomorphism is independent of the choice of \( \omega_f \). Similarly, if \( f^0 \) and \( f^1 \) are two cut-off functions used to carry out the construction, then on \( W \times I \), setting \( \bar{f} = (1-t)f^0 + tf^1 \), \( \bar{\omega} = \bar{f} \omega \), and
\[
\bar{\Omega}^G_f = d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}] - (1 + \bar{f})\xi^a \Theta^a,
\]
we can define a similar transgression operator:
\[ T_{(f^0, f^1)} F = \int_\pi F(\bar{\Omega}^G_f). \]

Then the same proof as above shows that
\[ DT_{(f^0, f^1)} F = F(\Omega^G_f) - F(\Omega^G_f^0). \]

Hence the degenerate Chern-Weil homomorphism is also independent of the choice of \( f \).

Now consider the homomorphism \( CW^G_f : W(\mathfrak{g}) \otimes \Omega(W) \to S(\mathfrak{g}^*) \otimes \Omega(W) \) defined by extending the Chern-Weil construction \( \Theta^a \mapsto \omega^G_f, \Theta^a \mapsto (\Omega^G_f)^a \) as a \( \Omega(W) \)-module map. Define \( r^G_f : S(\mathfrak{g}^*) \otimes \Omega(W) \to S(\mathfrak{g}^*) \otimes \Omega(W) \) by
\[ r^G_f(\alpha) = CW^G_f(\Psi(\alpha)). \]

Let \( U \) be any open set on which \( f = 1 \), then on \( U \), we have \( \omega_f = \omega, \Omega^G_f = \Omega \). Therefore \( r^G_f = r^G \) on \( U \). So \( r^G_f \) is a generalization of \( r^G \). It is easy to see that \( r^G_f \) maps \( (\Theta(\mathfrak{g}^*) \otimes \Omega(W))^G \) to itself.

**Theorem 2.3.** The homomorphism \( r^G_f : (\Theta(\mathfrak{g}^*) \otimes \Omega(W))^G \to (\Theta(\mathfrak{g}^*) \otimes \Omega(W))^G \) satisfies \( Dr^G_f = r^G_f D \). Hence it induces a homomorphism of cohomologies:
\[ (r^G_f)_* : H^*_G(W) \to H^*_G(W). \]

**Theorem 2.4.** The homomorphism \( (r^G_f)_* \) in Theorem 2.3 does not depend on the choice of the connection \( \omega \) on \( W^0 \) or the choice of the cut-off function \( f \).
An important observation, pointed out to us by Michèle Vergne, is that, if one takes \( f \equiv 0 \), then \( \Omega^G_f = \xi \Theta^\omega \). Therefore, we have

**Theorem 2.5.** The homomorphism \((r^G_f)_*\) on cohomology is the identity map.

Let \( i^c \) denote the inclusion \( S(\mathfrak{g}^*) \to S(\mathfrak{g}^*) \otimes \Omega(W) \). It is obvious that \( \text{CW}_f = r^G_f \circ i^c \). This equality reveals that Theorem 2.1 and Theorem 2.2 are special cases of Theorem 2.3 and Theorem 2.4 respectively. Since \( \text{CW}_f = (r^G_f)_* \circ i^c_* \), we have the following

**Theorem 2.6.** We have \( \text{CW}_f = i^c_* \).

We now present the proof of Theorems 2.3 and 2.4. Theorem 2.4 is an easy consequence of Theorem 2.3 by a construction of transgression homomorphism. Our proof of Theorem 2.3 relies on calculations by brute force. It will be nice to find a more conceptual proof. To begin with, we have the following lemma which plays a similar role in the proof of Theorem 2.3 as Lemma 2.4 in the proof of Theorem 2.1.

**Lemma 2.5.** Let \( \alpha = \Theta^{i_1} \cdots \Theta^{i_q} \alpha_{i_1 \cdots i_q} \in \( S(\mathfrak{g}^*) \otimes \Omega(X) \)^G \), \( \alpha_{i_1 \cdots i_q} = \alpha_{(i_1 \cdots i_q)} \), then we have

\[
\mathcal{L}_b \alpha_{i_1 \cdots i_q} = d \tau_b \alpha_{i_1 \cdots i_q} + \tau_b d \alpha_{i_1 \cdots i_q} = q f_{(i_1}^p \alpha_{[p i_2 \cdots i_q]}).
\]

Furthermore, if \( \alpha \) is D-closed then we have

\[
d \alpha_{i_1 \cdots i_q} = \tau (\alpha_{i_1 \cdots i_q}).
\]

**Proof.** By \( G \)-invariance of \( \alpha \),

\[
0 = (L_b \otimes 1 + 1 \otimes L_b) \sum \Theta^{i_1} \cdots \Theta^{i_q} \alpha_{i_1 \cdots i_q} = \sum \Theta^{i_1} \cdots \Theta^{i_q} L_b \alpha_{i_1 \cdots i_q} + \sum L_b (\Theta^{i_1} \cdots \Theta^{i_q}) \alpha_{i_1 \cdots i_q} = \sum \Theta^{i_1} \cdots \Theta^{i_q} L_b \alpha_{i_1 \cdots i_q} + \sum q L_b (\Theta^{i_1}) \Theta^{i_2} \cdots \Theta^{i_q} \alpha_{i_1 \cdots i_q} = \sum \Theta^{i_1} \cdots \Theta^{i_q} (L_b \alpha_{i_1 \cdots i_q} - q f_{(i_1}^p \alpha_{[p i_2 \cdots i_q]}).
\]

The last equality is obtained by interchanging \( p \) with \( i_1 \). This proves (4). Similarly, from \( D \alpha = 0 \), we get

\[
0 = \sum \Theta^{i_1} \cdots \Theta^{i_q} d \alpha_{i_1 \cdots i_q} - \sum \Theta^{i_1} \cdots \Theta^{i_q} \Theta^b \alpha_{i_1 \cdots i_q} = \sum \Theta^{i_1} \cdots \Theta^{i_q} (d \alpha_{i_1 \cdots i_q} - \tau (\alpha_{i_1 \cdots i_q})).
\]

This proves (5). \( \square \)
Proof of Theorem 2.3. Let $\alpha = \Theta^1 \alpha^1 \cdots \Theta^q \alpha^q = \sum q \Theta^{i_1} \cdots \Theta^{i_q} \alpha_{i_1} \cdots i_q$, then using the summation convention, we have

$$\Psi(\alpha) = \prod_{a=1}^{k} (1 - \theta^a \otimes \iota_a) \alpha$$

$$= \sum_{j=0}^{k} (-1)^{j(j+1)/2} \sum_{a_1 < \cdots < a_j} \theta^{a_1} \cdots \theta^{a_j} \Theta^I \iota_{a_1} \cdots \iota_{a_j} \alpha_I$$

$$= \sum_{j=0}^{k} \frac{(-1)^{j(j+1)/2}}{j!} \theta^{a_1} \cdots \theta^{a_j} \Theta^I \iota_{a_1} \cdots \iota_{a_j} \alpha_I.$$ 

Applying the degenerate Chern-Weil construction $CW_f$, we get

$$r^G_f(\alpha) = \sum_{j=0}^{k} \frac{(-1)^{j(j+1)/2}}{j!} \omega_f^a \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega^G_f)^I \wedge \iota_{a_1} \cdots \iota_{a_j} \alpha_I.$$ 

Taking $D$ on both sides, we see that $Dr^G_f(\alpha)$ is equal to:

$$(6) \quad \sum_{j=0}^{k} \frac{(-1)^{j(j+1)/2}}{j!} D(\omega_f^a \wedge \cdots \wedge \omega_f^{a_j}) \wedge (\Omega^G_f)^I \wedge \iota_{a_1} \cdots \iota_{a_j} \alpha_I$$

$$+(7) \quad \sum_{j=0}^{k} \frac{(-1)^{j(j+3)/2}}{j!} \omega_f^a \wedge \cdots \wedge \omega_f^{a_j} \wedge D(\Omega^G_f)^I \wedge \iota_{a_1} \cdots \iota_{a_j} \alpha_I$$

$$+(8) \quad \sum_{j=0}^{k} \frac{(-1)^{j(j+3)/2}}{j!} \omega_f^a \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega^G_f)^I \wedge D(\iota_{a_1} \cdots \iota_{a_j} \alpha_I).$$

We will examine each of the above terms separately. To start with, recall that

$$D\omega_f^a = d\omega_f^a - f\Theta^a$$

$$= (\Omega^G_f)^a - \frac{1}{2} f_{bc} \omega_f^b \wedge \omega_f^c - \Theta^a,$$

$$D(\Omega^G_f)^a = f_{bc} (\Omega^G_f)^b \wedge \omega_f^c = -f_{bc} \omega_f^b (\Omega^G_f)^c.$$ 

Then (6) can be written as
By a renaming of the indices, it is easy to see that (6b) and (8) together yield

\[
\sum_{j=1}^{k} \frac{(-1)^{j(j+1)/2}}{j!} D(\omega_f^{a_1} \wedge \cdots \wedge \omega_f^{a_j}) \wedge (\Omega_f^G)^I \wedge \Theta^{a_1}_{\alpha_1} \cdots \Theta^{a_j}_{\alpha_j} = 0.
\]

Similarly, we rewrite (8) as

\[
\sum_{j=0}^{k} \frac{(-1)^{j(j+3)/2}}{j!} \omega_f^{a_1} \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega_f^G)^I \wedge D(\tau_{a_1} \cdots \tau_{a_j} \alpha_I) = 0
\]

By a renaming of the indices, it is easy to see that (6c) and (8b) together yield

\[
\sum_{b=1}^{k} (-1)^{k(k+3)/2} \omega_f^{a_1} \wedge \cdots \wedge \omega_f^{a_k} \wedge (\Omega_f^G)^I \wedge \Theta^{b}_{\alpha_1} \cdots \Theta^{b}_{\alpha_k} = 0.
\]

Now (7) can be written as summation for \( j = 0 \) to \( k \) of \( \frac{(-1)^{j(j+3)/2}}{j!} \omega_f^{a_1} \wedge \cdots \wedge \omega_f^{a_j} \) wedge the following terms.
\[
D(\Omega_f^G)^t \wedge \ell_{a_1} \cdots \ell_{a_l} \alpha_I \\
= D((\Omega_f^G)^{t_1} \wedge \cdots \wedge (\Omega_f^G)^{t_q}) \wedge \ell_{a_1} \cdots \ell_{a_l} \alpha_{i_1} \cdots i_q \\
= qD(\Omega_f^G)^{t_1} \wedge (\Omega_f^G)^{t_2} \wedge \cdots \wedge (\Omega_f^G)^{t_q} \wedge \ell_{a_1} \cdots \ell_{a_l} \alpha_{i_1} \cdots i_q \\
= -qf_{b_a}^{i_1} \wedge (\Omega_f^G)^{t_1} \wedge (\Omega_f^G)^{t_2} \wedge \cdots \wedge (\Omega_f^G)^{t_q} \wedge \ell_{a_1} \cdots \ell_{a_l} \alpha_{i_1} \cdots i_q \\
= -\omega_f^{j_1} \wedge (\Omega_f^G)^{t_1} \wedge (\Omega_f^G)^{t_2} \wedge \cdots \wedge (\Omega_f^G)^{t_q} \wedge \ell_{a_1} \cdots \ell_{a_l} (qf_{b_a}^{i_1} \alpha_{i_1} \cdots i_q). \\
= -\omega_f^{j_1} \wedge (\Omega_f^G)^{t_1} \wedge (\Omega_f^G)^{t_2} \wedge \cdots \wedge (\Omega_f^G)^{t_q} \wedge \ell_{a_1} \cdots \ell_{a_l} \ell_b \alpha_{i_1} \cdots i_q \\
= -\omega_f^{j_1} \wedge (\Omega_f^G)^{t_1} \wedge \ell_{a_1} \cdots \ell_{a_l} \ell_b \alpha_I.
\]

We have used (4) in the second to last equality. Renaming \( b \) by \( a_1, a_l \) by \( a_{l+1} \), one sees that (7) is equal to
\[
-\sum_{j=0}^{k} \frac{(-1)^{j+1/2}}{j!} \omega_f^{a_j} \wedge \cdots \wedge \omega_f^{a_{j+1}} \wedge (\Omega_f^G)^{t_1} \wedge \ell_{a_1} \cdots \ell_{a_{j+1}} \ell_a \alpha_I.
\]

The contribution from \( j = k \) is clearly zero, so by changing \( j \) to \( j - 1 \), (7) is equal to
\[
(7') \quad -\sum_{j=1}^{k} \frac{(-1)^{j-1/2}}{(j - 1)!} \omega_f^{a_j} \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega_f^G)^{t_1} \wedge \ell_{a_1} \cdots \ell_{a_j} \ell_a \alpha_I.
\]

Notice that in (6a), it won’t change the result if we take the summation for \( j = 1 \) to \( k + 1 \). Change the index \( a_1 \) to \( b, a_l \) to \( a_{l-1} \) for \( l > 1 \), one sees that (6a) is equal to
\[
(6a') \quad -\sum_{j=0}^{k} \frac{(-1)^{j+1/2}}{j!} \omega_f^{a_j} \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega_f^G)^{t_1} \wedge \ell_{a_1} \cdots \ell_{a_j} b \alpha_I.
\]

Similarly, (6b) is equal to
\[
(6b') \quad -\sum_{j=2}^{k} \frac{(-1)^{j-1/2}}{j!} \omega_f^{a_j} \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega_f^G)^{t_1} \wedge f_c^{\alpha_{a_2} a_3} \cdots \ell_{a_j} c \alpha_I.
\]

To summarize, we have \( Dr_f^G \alpha = (6a') + (6b') + (7') + (8a) \). On the other hand,
\[
r_f^G D \alpha
\]
\[
(9a) \quad \sum_{j=0}^{k} \frac{(-1)^{j+1/2}}{j!} \omega_f^{a_j} \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega_f^G)^{t_1} \wedge \ell_{a_1} \cdots \ell_{a_j} d \alpha_I
\]
\[
(9b) \quad -\sum_{j=0}^{k} \frac{(-1)^{j+1/2}}{j!} \omega_f^{a_j} \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega_f^G)^{t_1} \wedge (\Omega_f^G)^{t_2} \wedge \ell_{a_1} \cdots \ell_{a_j} b \alpha_I
\]

Since (6a') cancels (9b), \( Dr_f^G \alpha - r_f^G D \alpha = (8a) - (9a) + (7') + (6b') \). It is the summation for \( j = 0 \) to \( k \) of \( \sum_{j=0}^{k} \frac{(-1)^{j-1/2}}{j!} \omega_f^{a_j} \wedge \cdots \wedge \omega_f^{a_j} \wedge (\Omega_f^G)^{t_1} \wedge (\Omega_f^G)^{t_2} \wedge \ell_{a_1} \cdots \ell_{a_j} c \alpha_I \)
\[
(d\ell_{a_1} \cdots \ell_{a_j}) - (1)^{j} \ell_{a_1} \cdots \ell_{a_j} d - j \ell_{a_2} \cdots \ell_{a_j} \omega_{a_1} - \frac{1}{2} j(j - 1) \ell_{a_2} \cdots \ell_{a_j} \ell_c \alpha_I,
\]
which is easily shown to be zero by repeatedly using \( d\iota = \mathcal{L}_a - \iota a d \) and \( \mathcal{L}_a \iota_b = \iota_b \mathcal{L}_a - f_{ab}^c \iota_c \).

**Lemma 2.6.** Let \( D_W : \Omega_G(W) \rightarrow \Omega_G(W) \) and \( D_{W \times I} : \Omega_G(W \times I) \rightarrow \Omega_G(W \times I) \) be the Cartan differentials on \( W \) and \( W \times I \) respectively. If \( \tilde{\alpha} \in \Omega_G(W \times I) \) can be written as

\[
\tilde{\alpha} = \alpha(t) + dt \wedge \beta(t),
\]

where \( \alpha(t) \) and \( \beta(t) \) are families of equivariant differential forms on \( W \) which depend smoothly on \( t \), then

\[
\int_\pi D_{W \times I}(\tilde{\alpha}) + D_W \int_\pi \tilde{\alpha} = \alpha(1) - \alpha(0).
\]

**Proof.** Notice that \( D_{W \times I} = D_W + dt \wedge \frac{\partial}{\partial t} \). Then we have

\[
\int_\pi D_{W \times I}(\tilde{\alpha}) = \int_\pi D_W \alpha(t) + dt \wedge \frac{\partial}{\partial t} \alpha(t) - dt \wedge D_W \beta(t) = \int_0^1 \frac{d}{dt} \alpha(t) - D_W \beta(t) dt.
\]

On the other hand,

\[
D_W \int_\pi \tilde{\alpha} = D_W \int_0^1 \beta(t) dt = \int_0^1 [D_W \beta(t)] dt.
\]

So we have

\[
\int_\pi D_{W \times I}(\tilde{\alpha}) + D_W \int_\pi \tilde{\alpha} = \int_0^1 \frac{d}{dt} \alpha(t) dt = \alpha(1) - \alpha(0).
\]

**Proof of Theorem 2.4.** For any \( \alpha \in \Omega_G(W) \), we have

\[
D_{W \times I}(\pi^* \alpha) = \pi^* (D_W \alpha).
\]

For two connections \( \omega^0 \) and \( \omega^1 \) on \( W^0 \), use the notations in the proofs of Theorem 2.2 and Theorem 2.3, we define a degenerate transgression operator

\[
T_{(\omega^0, \omega^1)} \alpha = \int_\pi \sum_{j=0}^k (-1)^{j(j+1)/2} \bar{\omega}^1_{i_1} \wedge \cdots \wedge \bar{\omega}^1_{i_j} \wedge (\bar{\Omega}_f^G)^1 \wedge \pi^*(\iota_{a_1} \cdots \iota_{a_j} \alpha_I)
\]

where \( \bar{\omega}^G_f : \Omega_G(W \times I) \rightarrow \Omega_G(W \times I) \) is obtained by the degenerate Chern-Weil construction for \( \bar{\omega}_f \) and \( \bar{\Omega}_f^G \). Then by Lemma 2.6 and Theorem 2.3 for \( W \times I \), we
compute $\alpha$ is free, $W$ dimensional compact connected Lie group acting on $W$ as follows. Let why one can expect localization formula on manifolds. The general situation is coming paper [6]. Before we embarking on a proof of Theorem 3.1, let us explain also independent of the choice of $f$ where

$$\text{Proposition 3.1.}$$

We now state our first nonabelian generalization: formulas in Seiberg-Witten theory due to Li-Liu [20] and Okonek-Teleman [23].

1. In our earlier work [5], we have applied this formula to obtain wall crossing in symplectic reduction in a forth-coming paper [17]. Let us recall Kalkman’s localization formula for circle action [17] stated as follows:

$$\text{Theorem 3.1.}$$

A similar construction shows that it is also independent of the choice of $f$.

$$\text{3. Two nonabelian localization formulas}$$

In this section, we prove two nonabelian localization formulas of Kalkman type for $G = SU(2)$ and $SO(3)$. First, let us recall Kalkman’s localization formula for circle action [17] stated as follows:

$$\text{Proposition 3.1.}$$

Let $W$ be an $S^1$-manifold with an invariant boundary $\partial W$, such that the $S^1$-action on $\partial W$ is locally free and effective. Suppose that $F = \{P_k\}$ is a decomposition of the fixed point set into connected components. Denote by $\nu_k$ the normal bundle of $P_k$ in $W$, and $c(\nu_k)$ the equivariant Euler class of $\nu_k$. Then for any homogeneous $D_{S^1}$-closed form $\alpha$ on $W$ of total degree $\dim(W) - 2$, we have

$$\int_{\partial W/S^1} r^{S^1}(\alpha) = \sum_k \int_{P_k} \frac{\alpha u}{c(\nu_k)}.$$

In the above formula, we have used the normalization such that $S^1$ has volume 1. In our earlier work [5], we have applied this formula to obtain wall crossing formulas in Seiberg-Witten theory due to Li-Liu [20] and Okonek-Teleman [23]. We now state our first nonabelian generalization:

$$\text{Theorem 3.1.}$$

Assume that $G = SU(2)$ or $SO(3)$ acts on a compact manifold $W$ with boundary $\partial W$, such that the $G$-action on $\partial W$ is locally free and effective. Let $T \subset G$ be a circle subgroup, with fixed point set $F = \{P_k\}$. Suppose that $G$ is given a bi-invariant metric such that $\text{vol}(T) = 1$. Then for any homogeneous $D_G$-closed form $\alpha$ of total degree $\dim(W) - 4$, we have

$$\int_{\partial W/G} r^G(\alpha) = -\frac{1}{c(G)} \sum_k \int_{P_k} \frac{p(\alpha) u^2}{c(\nu_k)},$$

where $c(SU(2)) = 1$, $c(SO(3)) = 2$ and $p : \Omega_G(W) \to \Omega_T(W)$ is defined in (3).

We will apply Theorem 3.1 to wall crossing in symplectic reduction in a forthcoming paper [17]. Before we embarking on a proof of Theorem 3.1, let us explain why one can expect localization formula on manifolds. The general situation is as follows. Let $W$ be a compact $n$-dimensional manifold with boundary, $G$ a $m$-dimensional compact connected Lie group acting on $W$, such that he action on $\partial W$ is free, $\alpha$ an equivariant closed form of degree $n - m - 1$ on $W$. The problem is to compute $\int_{\partial W/G} r^G(\alpha|_{\partial W})$. Suppose that $g$ is given a $G$-invariant Euclidean metric,
\{\xi_1, \cdots, \xi_m\} an orthonormal basis with structure constants \(c_{ij}^k\)'s, \(\omega_f = \sum_a \omega_f^a \xi_a\) a degenerate connection on \(W\). Then by Stokes theorem and \(D r^G_f(\alpha) = 0\), we have

\[
\int_{\partial W/G} r^G(\alpha|_{\partial W}) = \frac{1}{\vol(G)} \int_{\partial W} \omega_1^a \wedge \cdots \wedge \omega_m^a \wedge r_f^G(\alpha) = \frac{1}{\vol(G)} \int_{\partial W} D(\omega_1^f \wedge \cdots \wedge \omega_m^f \wedge r_f^G(\alpha)) = \frac{1}{\vol(G)} \int_W \sum_{j=1}^m (-1)^{j-1} (\Omega_f^j \wedge \omega_1^f \wedge \cdots \wedge \omega_j^f \wedge \cdots \wedge \omega_m^f \wedge r_f^G(\alpha)) = \frac{1}{\vol(G)} \int_W \sum_{j=1}^m (-1)^{j-1} (\Omega_f^j \wedge \omega_1^f \wedge \cdots \wedge \omega_j^f \wedge \cdots \wedge \omega_m^f \wedge r_f^G(\alpha)) = \frac{1}{\vol(G)} \int_W \sum_{j=1}^m (-1)^{j-1} (\Omega_f^j \wedge \omega_1^f \wedge \cdots \wedge \omega_j^f \wedge \cdots \wedge \omega_m^f \wedge r_f^G(\alpha)).
\]

On a point \(x \in W\) where \(f = 1\), i.e. \(\omega_f\) is a connection, one gets a decomposition \(\partial W = V_x \oplus H_x\), then both \(\Omega_f^j\) and \(r^G_f(\alpha)\) are exterior forms on \(H_x\), i.e., contraction with any vector in \(V_x\) is zero. Now \(\Omega_f^j \wedge r^G_f(\alpha)\) is an exterior form of degree

\[2 + (n - m - 1) = (n - m) + 1 = \dim H_x + 1,\]

therefore, it must vanish. Hence the integral above concentrates near the set where the action of \(G\) is not locally free. Compare with the method Kalkman \[10\] used to prove his formula. The dimension counting argument will be used repeatedly below.

To prove Theorem 3.1, we take a basis \(\{\xi_1, \xi_2, \xi_3\}\) for \(g\) such that 

\[\langle [\xi_i, \xi_j], \xi_k \rangle = a(G) \delta_{ijk} \xi_k.\]

Here \(\delta_{ijk}\) is nonzero only if \(ijk\) is a permutation of 123, and when that is the case, equals the sign of the permutation. Furthermore, \(a(SU(2)) = 4\pi, a(SO(3)) = 2\pi\). For \(G = SU(2)\), one can take

\[
\xi_1 = 2\pi \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}, \quad \xi_2 = 2\pi \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \xi_3 = 2\pi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

For \(G = SO(3)\), one can take

\[
\xi_1 = 2\pi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \xi_2 = 2\pi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi_3 = 2\pi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

We also take a bi-invariant metric on \(G\) such that \(\{\xi_1, \xi_2, \xi_3\}\) defines an orthonormal basis at the identity. Then it is clear that \(\vol(T) = 1\), and

\[
\vol(SU(2)) = \vol(S^3(1/2\pi)) = 2\pi^2(1/2\pi)^3 = 1/(4\pi),
\]

\[
\vol(SO(3)) = \vol(S^3(1/\pi))/2 = \pi^2(1/\pi)^3 = 1/\pi.
\]

Let \(\omega = \omega^j \xi_j\) be a connection on \(W^0\).

**Lemma 3.1.** For \(G = SU(2)\) or \(SO(3)\), let \(\alpha\) be a homogeneous \(D_G\)-closed form of total degree \(\dim(W) - 4\). Then on \(\partial W\), we have \(\Omega^1 \wedge r^G(\alpha) = 0\). Furthermore, we have

\[
\int_{\partial W} \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge r^G(\alpha) = -\frac{1}{a(G)} \int_{\partial W} \omega^1 \wedge d\omega^2 \wedge r^G(\alpha).
\]
Proof. For each point \( x \in \partial W \), the connection gives a decomposition \( T_x \partial W = V_x \oplus H_x \). Both \( \Omega^1 \) and \( r^G(\alpha) \) are exterior forms on \( H_x \), i.e., contraction with any vector in \( V_x \) is zero. Then \( \Omega^1 \wedge r^G(\alpha) \) is an exterior form on \( H_x \) of degree

\[
2 + \dim(W) - 4 = \dim(\partial W) - 1 = \dim(H_x) + 2,
\]
hence it vanishes. Now \( \Omega^1 = d\omega^1 + \frac{1}{4} \mathbf{f}^b_{\mu} \omega^b \wedge \omega^c = d\omega^1 + a(G) \omega^2 \wedge \omega^3 \), so we have

\[
\omega^2 \wedge \omega^3 = \frac{1}{a(G)}(\Omega^1 - d\omega^1).
\]

Therefore,

\[
\int_{\partial W} \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge r^G(\alpha) = \frac{1}{a(G)} \int_{\partial W} \omega^1 \wedge (\Omega^1 - d\omega^1) \wedge r^G(\alpha)
= -\frac{1}{a(G)} \int_{\partial W} \omega^1 \wedge d\omega^1 \wedge r^G(\alpha).
\]

Proof of Theorem 3.1. Let \( \beta = (d(f \omega^1) - (-1 + f)u) \wedge p(r^G_j(r^G_2(\alpha))) \in \Omega_S(W) \). It is clear that \( d(f \omega^1) - (-1 + f)u = D_T \) is closed. By Theorem 2.6,

\[
D_T p(r^G_j(r^G_2(\alpha))) = D_T r^G_j(r^G_2(\alpha)) = D_T r^G_j D_T (r^G_2(\alpha)) = 0.
\]

Hence \( D_T \beta = 0 \). Furthermore, near \( \partial W \), since \( r^G(\alpha) \) is basic, \( r^G(r^G(\alpha)) = r^G(\alpha) \). So near \( \partial W \), we have

\[
r^T(\beta) = r^T(d\omega^1 \wedge r^G(\alpha)) = d\omega^1 \wedge r^G(\alpha).
\]

On the other hand, near each \( P_k \), \( f = 0 \) and \( r^G_j(\alpha) = \alpha \), so \( \beta = p(\alpha)u \). Therefore, by Lemma 3.1 and Kalkman’s formula for \( \beta \), we have

\[
\int_{\partial W/G} r^G(\alpha) = \frac{1}{\text{vol}(G)} \int_{\partial W} \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge r^G(\alpha)
= -\frac{1}{a(G) \text{vol}(G)} \int_{\partial W} \omega^1 \wedge d\omega^1 \wedge r^G(\alpha)
= -\frac{1}{c(G)} \int_{\partial W} \omega^1 \wedge r^T(\beta)
= -\frac{1}{c(G)} \sum_k \int_{P_k} \frac{\beta u}{c(\nu_k)} = -\frac{1}{c(G)} \sum_k \int_{P_k} \frac{p(\alpha) u^2}{c(\nu_k)}.
\]

Here \( c(G) = a(G) \text{vol}(G) \), \( c(SU(2)) = 4\pi \cdot 1/(4\pi) = 1 \), \( c(SO(3)) = 2\pi/\pi = 2 \).

Now let \( W \) be a compact, oriented \( n \)-dimensional \( G \)-manifold with \( \partial W = Y \times S^2 \) for some closed oriented manifold \( Y \), such that the action of \( G \) on \( \partial W \) is given by the diagonal action of a locally-free and effective action on \( Y \) and the coadjoint action of \( SU(2) \) or \( SO(3) \) on \( S^2 \subset g^* \). Assume that there is a \( G \)-equivariant map \( \psi : W \to S^2 \), such that \( \psi|_{\partial W} \) is the projection \( \pi_2 : Y \times S^2 \to S^2 \). Using the linear coordinates \( (x_1, x_2, x_3) \) in the basis dual to \( \{ \xi_1, \xi_2, \xi_3 \} \) on \( g^* \), the action of \( T = \{ \exp(tk_1) : t \in \mathbb{R} \} \subset G \) on \( S^2 = \{ x_1^2 + x_2^2 + x_3^2 = 1 \} \) is given by

\[
\exp(tk_1) \cdot (x_1, x_2, x_3) = (x_1, x_2 \cos(2\pi t) - x_3 \sin(2\pi t), x_2 \sin(2\pi t) + x_3 \cos(2\pi t))
\]
for \( G = SO(3) \), and

\[
\exp(tk_1) \cdot (x_1, x_2, x_3) = (x_1, x_2 \cos(4\pi t) - x_3 \sin(4\pi t), x_2 \sin(4\pi t) + x_3 \cos(4\pi t))
\]
for $G = SU(2)$. Denote by $F$ the fixed point set of the $T$-action on $W$. For any component $P \subset F$, since $\psi : W \to S^2$ is equivariant, $\psi(P)$ is a fixed point of the $T$-action on $S^2$, i.e., $\psi(P) = (\pm 1, 0, 0)$. Denote by $F_+$ the set of points fixed by $T$ which are mapped to $(1, 0, 0)$ by $\psi$.

**Theorem 3.2.** Let $W$ be as described above. Assume that $\alpha$ is an equivariantly closed $(n - 6)$-form on $W$, such that $r^G(\alpha|_{\partial W}) = \pi_1(\alpha_0)$ for some differential form $\alpha_0$ on $Y/G$, where $\pi_1/G : (Y \times S^2)/G \to Y/G$ is induced by the projection $\pi_1 : Y \times S^2 \to Y$. Then we have

$$\int_{Y/G} \alpha_0 = -\frac{b(G)}{c(G)} \sum_{P_k \subset F_+} \int_{P_k} \frac{w^3(p)}{c(P_k)},$$

where $c(G)$ is as in Theorem 3.1 $b(G) = 2$ for $G = SU(2)$, and $b(G) = 1$ for $G = SO(3)$.

**Proof.** Fix a connection 1-form $\omega$ on the principal bundle $Y \to Y/G$. The pullback of $\omega$ to $Y \times S^2$, which we still denote by $\omega$, is a connection for the principal bundle $Y \times S^2 \to (Y \times S^2)/G$. Now on $Y$, we have $\Omega^1 = d\omega^1 + 4\pi a^2 \wedge \omega^3$. Since $\Omega^1$ is the pullback of a 2-form on $Y/G$, $\Omega^1 \wedge \alpha_0$ is the pullback of a $(n - 4)$-form on $Y/G$. Hence $\Omega^1 \wedge \alpha_0 = 0$, since $\dim(Y/G) = n - 6$. It follows that

$$\int_{Y/G} \alpha_0 = \frac{1}{\text{vol}(G)} \int_Y \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \alpha_0$$

$$= \frac{1}{a(G) \text{vol}(G)} \int_Y \omega^1 \wedge (\Omega^1 - d\omega^1) \wedge \alpha_0$$

$$= -\frac{1}{c(G)} \int_Y \omega^1 \wedge d\omega^1 \wedge \alpha_0. \tag{10}$$

Now we endow a symplectic structure on $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ by

$v = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2.$

Then the action of $T$ on $S^2$ is a Hamiltonian action with moment map given by

$(x_1, x_2, x_3) \mapsto 2\pi b(G) x_1 u$. Hence $v_T = v + 2\pi(1 + b(G)x_1) u$ is $D_T$-closed on $S^2$.

Now consider $\psi^*(v_T)$. If we denote by $V$ the vector field on $W$ generated by the action of $T$, then

$$r^T(\psi^* v_T) = \psi^*(v) - \omega^1 \wedge \iota_V \psi^*(v) + 2\pi(1 + b(G)x_1) d\omega^1. \tag{11}$$

Now on $\partial W$, $d\omega^1 \wedge d\omega^1 \wedge r^G(\alpha|_{\partial W})$ must vanish, since it is the pullback of an $(n - 2)$-form on $Y/S^3$, which has dimension $n - 4$. So by (11) we have

$$\omega^1 \wedge d\omega^1 \wedge r^T(\psi^*(v_T)) \wedge r^G(\alpha|_{\partial W}) = \omega^1 \wedge d\omega^1 \wedge \psi^*(v) \wedge r^G(\alpha|_{\partial W}).$$

On $\partial W$, the integration of $\psi^*(v)$ on each fiber of $\partial W = Y \times S^2 \to Y$ is $4\pi$. So from equation (11), we get

$$\int_{Y/G} \alpha_0 = -\frac{1}{c(G)} \int_Y \omega^1 \wedge d\omega^1 \wedge \alpha_0$$

$$= -\frac{1}{4\pi c(G)} \int_{Y \times S^2} \omega^1 \wedge d\omega^1 \wedge \psi^*(v) \wedge r^G(\alpha|_{\partial W})$$

$$= -\frac{1}{4\pi c(G)} \int_{Y \times S^2} \omega^1 \wedge d\omega^1 \wedge r^T(\psi^*(v_T)) \wedge r^G(\alpha|_{\partial W}).$$
Let $\beta = (d(f\omega^{1})-(1+f)u)\wedge r_{T}^{*}(\psi^{*}(\nu_{T}))\wedge p(r_{T}^{*}(r_{T}^{*}(\alpha))) \in \Omega_{T}(W)$, then $D_{T}\beta = 0$. Near $\partial W$, we have

$$r_{T}(\beta) = dw^{1}\wedge r_{T}(\psi^{*}(\nu_{T})) \wedge r^{G}(\alpha|_{\partial W}).$$

On each component $P_{k}$ of $F_{+}$, $r_{T}^{*}(\alpha) = \alpha$, $r_{T}^{*}(\psi^{*}(\nu_{T})) = \psi^{*}(\nu_{T}) = \psi^{*}(\nu) + 2\pi(1+b(G)\psi* x_{1}) = 4\pi b(G)u$. Therefore on $F_{+}$, $\beta = 4\pi b(G)u^{2}p(\alpha)|_{P_{k}}$. Similarly, on each component of $F_{-}$, $\beta = 0$. Apply Kalkman’s formula then completes the proof. \hfill \Box

4. APPLICATION TO SYMPLECTIC REDUCTION

As a corollary to their general nonabelian localization formula \cite{44}, Jeffrey and Kirwan proved the following:

**Theorem 4.1.** (Jeffrey-Kirwan \cite{15}, Corollary 3.3) For $G = SU(2)$ or $SO(3)$, let $\mu : M \rightarrow \mathfrak{g}^{*}$ be the moment map of a Hamiltonian $G$-action on a closed symplectic manifold $(M, \omega)$ for $G = SU(2)$ or $SO(3)$. Suppose that $G$-action on $\mu^{-1}(0)$ is locally free and effective, so that one can obtain the symplectic reduction $(M_{0}, \omega_{0})$. For any $D_{G}$-closed $\eta \in \Omega_{G}(M)$, let $\eta_{0} := r^{G}(\eta|_{\mu^{-1}(0)}) \in \Omega(M_{0})$. Then

$$\int_{M_{0}} \eta_{0} e^{\omega_{0}} = -\frac{b(G)}{c(G)} \text{Res}_{0} \left( \frac{u^{2}}{P_{k}} \sum_{P_{k} \subset F_{+}} e^{\mu_{T}(P_{k})} \int_{P_{k}} \frac{p(\eta)e^{\omega}}{\epsilon(P_{k})} \right),$$

where $\text{Res}_{0}$ denote the coefficient of $1/u$, $b(G)$, $c(G)$ are the constants given in Theorem \cite{44}.

We will give an elementary proof of this result using the following theorem, which is derived from Theorem \cite{44}.

**Theorem 4.2.** Assume that $\mu : M \rightarrow \mathfrak{g}^{*}$ is as in Theorem \cite{44}. Suppose that $\dim(M) = 2n + 6$, then for any $D_{G}$-closed $2n$-form $\alpha \in \Omega_{G}(M)$, we have

$$\int_{M_{0}} r^{G}(\alpha|_{\mu^{-1}(0)}) = -\frac{b(G)}{c(G)} \sum_{P_{k} \subset \bar{F}_{+}} \int_{P_{k}} \frac{u^{3}p(\alpha)}{\epsilon(P_{k})},$$

where $\bar{F}_{+}$ (or $\bar{F}_{-}$) is the subset of the fixed point set $F$ of $T = U(1) \subset G$ consisting of those components on which $\mu_{T} > 0$ (< 0).

**Proof of Theorem \cite{44} by Theorem \cite{44}** By Berline-Vergne \cite{8} and Atiyah-Bott \cite{11}, one can regard the moment map $\mu : M \rightarrow \mathfrak{g}^{*}$ as an element of $(\mathfrak{g}^{*} \otimes \Omega^{0}(M))^{G}$, so that $\varpi + \mu$ is $D_{G}$-closed. Assume that $\deg(\eta) = 2s \leq 2n$. (The case of $\eta$ having odd degree is trivial.) Consider $\alpha := \eta(\varpi + \mu)^{n-s}/(n-s)!$. Then $\alpha$ is $D_{G}$-closed, and $r^{G}(\alpha|_{\mu^{-1}(0)}) = \eta_{0}\varpi_{0}^{n-s}/(n-s)!$. Applying Theorem \cite{44} to this $\alpha$, we get

$$\int_{M_{0}} \eta_{0} e^{\omega_{0}} = \int_{M_{0}} \eta_{0}\varpi_{0}^{n-s}/(n-s)!$$

$$= -\frac{b(G)}{c(G)} \sum_{P_{k} \subset \bar{F}} \int_{P_{k}} \frac{u^{3}p(\eta)(\varpi + \mu_{T}u)^{n-s}}{(n-s)!\epsilon(P_{k})}$$

(12)
Assume that dim(Pₖ) = 2lₖ, then the codimension of Pₖ in M is 2(n - lₖ + 3). Now we write
\[ p(\eta) = \sum_{a} p(\eta)_{a} u^{s-a}, \quad (\deg(p(\eta)_{a}) = 2a) \]
\[ (\omega + \mu_{T} u)^{n-s} = \sum_{b} \binom{n-s}{b} \omega^{b} \mu_{T}^{n-s-b} u^{n-s-b}, \]
\[ \frac{1}{\epsilon(P_{k})} = \frac{1}{u^{n-lₖ+3}} \sum_{c} \sigma_{c}(P_{k})/u^{c}, \quad (\deg(\sigma_{c}(P_{k})) = 2c) \]
where \( p(\eta)_{a} \) and \( \sigma_{c}(P_{k}) \) are differential forms on \( P_{k} \). From equation (12) we get
\[
\int_{M_{0}} \eta \epsilon^{a} \]
\[ = - \frac{b(G)}{c(G)} \sum_{P_{k} \subset F_{k} \ a,b,c} \frac{u^{3}}{(n-s)!u^{n-lₖ+3}} \binom{n-s}{b} u^{(s-a)+(n-s-b)-c} \]
\[ \cdot \int_{P_{k}} p(\eta)_{a} \omega^{b} \sigma_{c}(P_{k}) \mu_{T}^{n-s-b} \quad \text{(nonzero if and only if } a + b + c = lₖ) \]
\[ = - \frac{b(G)}{c(G)} \sum_{P_{k} \subset F_{k} \ a+b+c=lₖ} \frac{u^{n-a-b-c+3}}{(n-s)!u^{n-lₖ+3}} \binom{n-s}{b} \int_{P_{k}} p(\eta)_{a} \omega^{b} \sigma_{c}(P_{k}) \mu_{T}^{n-s-b} \]
\[ = - \frac{b(G)}{c(G)} \sum_{P_{k} \subset F_{k} \ a+b+c=lₖ} \frac{1}{(n-s)!} \binom{n-s}{b} \int_{P_{k}} p(\eta)_{a} \omega^{b} \sigma_{c}(P_{k}) \mu_{T}^{n-s-b}. \]
A similar computation shows that
\[
- \frac{b(G)}{c(G)} \text{Re} S_{0} \left( u^{2} \sum_{P_{k} \subset F_{k}} \int_{P_{k}} \frac{p(\eta)_{a} \omega^{b} \sigma_{c}(P_{k}) \mu_{T}^{n-s-b}}{\epsilon(P_{k})} \right) \]
gives the same answer. Notice now that on each \( P_{k} \), the moment map \( \mu_{T} \) is constant. This then completes the proof. \( \square \)

To prove Theorem 1.2, as in Jeffrey-Kirwan [14], one can use the following result from symplectic geometry:

**Proposition 4.1.** (Gotay [10], Guillemin-Sternberg [11], Marle [21]) Assume 0 is a regular value of \( \mu \) (so that \( \mu^{-1}(0) \) is a smooth manifold and \( G \) acts on \( \mu^{-1}(0) \) with finite stabilizers). Then there is a neighborhood \( \mathcal{O} \cong \mu^{-1}(0) \times \{ z \in g^{*}, |z| \leq h \} \) \( \subseteq \mu^{-1}(0) \times g^{*} \) of \( \mu^{-1}(0) \) on which the symplectic form is given as follows. Let \( P \cong \mu^{-1}(0) \overset{\tau}{\rightarrow} M_{0} \) be the orbifold principal \( G \)-bundle given by the projection map \( q: \mu^{-1}(0) \rightarrow M_{0} = \mu^{-1}(0)/G \), and let \( \omega \in \Omega^{1}(P) \otimes g \) be a connection for it. Let \( \omega_{0} \) denote the induced symplectic form on \( M_{0} \). Then if we define a 1-form \( \tau \) on \( \mathcal{O} \subset P \times g^{*} \) by \( \tau_{p,z} = \zeta(\theta) \) (for \( p \in P \) and \( z \in g^{*} \)), the symplectic form on \( \mathcal{O} \) is given by
\[
\omega = q^{*} \omega_{0} + d\tau.
\]
Further, the moment map on \( \mathcal{O} \) is given by \( \mu(p,z) = z \).

**Proof of Theorem 4.3** Let \( W \) be the real blow-up of \( M \) along \( \mu^{-1}(0) \), i.e. the result of replacing \( \mu^{-1}(0) \) by the unit normal bundle of \( \mu^{-1}(0) \) in \( M \). Then by Proposition 4.1, \( W \) is a compact manifold with boundary \( \partial W = \mu^{-1}(0) \times S^{2} \). The
action of $G$ on $M$ lifts to an action on $W$, which, on $\partial W$, is given by the diagonal action on $\mu^{-1}(0) \times S^2$. Similarly, let $\hat{g}^*$ be the real blowup of $g^*$ at 0. Then the moment map $\mu : M \to g^*$ lifts to an $G$-equivariant map $\hat{\mu} : W \to \hat{g}^*$. Since $\hat{g}^*$ can be identified with $S^2 \times \mathbb{R}_+$, we have a natural projection $\pi_1 : \hat{g}^* \to S^2$. Consider the composition $\psi = \pi_1 \circ \hat{\mu} : W \to S^2$ which on $\partial W$ is just the projection $\mu^{-1}(0) \times S^2 \to S^2$. Then the pair $W$ and $\psi$ satisfies the conditions in Theorem 3.2 and hence Theorem 4.2 follows.

\begin{flushright}
\Box
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Note. Professor Michèle Vergne has suggested us to find a proof without using any normal form theorem from symplectic geometry. See e.g. her note on Jeffrey-Kirwan-Witten formula [25]. It is actually possible in our context. Consider in general a $G$-equivariant map $\mu : M \to g^*$ on a $G$-manifold $M$, where $G = SU(2)$ or $SO(3)$, such that $0 \in g^*$ is a regular value. Then one has a trivialization of the normal bundle of $\mu^{-1}(0)$ by pulling back a basis of $g^*$. Applying this to the moment map of a Hamiltonian $SU(2)$ or $SO(3)$-action, we can proceed as above. We are saved the effort of finding the normal for the symplectic form (which is not used) in Proposition 4.1.

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