THE AREA OF EXPONENTIAL RANDOM WALK
AND PARTIAL SUMS OF UNIFORM ORDER STATISTICS

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Abstract. Let $S_i$ be a random walk with standard exponential increments. We call $\sum_{i=1}^{k} S_i$ its $k$-step area. The random variable $\inf_{k \geq 1} \frac{2}{k(k+1)} \sum_{i=1}^{k} S_i$ plays important role in the study of so-called one-dimensional sticky particles model. We find the distribution of this variable and prove that

$$\mathbb{P}\left\{ \inf_{k \geq 1} \frac{2}{k(k+1)} \sum_{i=1}^{k} S_i \geq t \right\} = \mathbb{P}\left\{ \inf_{k \geq 1} \sum_{i=1}^{k} (S_i - it) \geq 0 \right\} = \sqrt{1 - te^{-t/2}}$$

for $0 \leq t \leq 1$. We also show that

$$\lim_{n \to \infty} \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2n}{n} \sum_{i=1}^{k} U_{i,n} \geq t \right\} = \sqrt{1 - te^{-t/2}},$$

where $U_{i,n}$ are the order statistics of $n$ i.i.d. random variables uniformly distributed on $[0, 1]$.

Key words and phrases: area of random walk, exponential random walk, partial sums of order statistics, ruin probability, sticky particles.

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1. Introduction

Let $S_i$ be a positive random walk. We call $\sum_{i=1}^{k} S_i$, where $k \geq 1$, its $k$-step area. We are interested in $k$-step areas of an exponential random walk $S_i$, that is a walk with standard exponential increments.

Let us normalize each $k$-step area dividing it by its expectation $E \sum_{i=1}^{k} S_i = \frac{k(k+1)}{2}$. Now introduce the random variable $\inf_{k \geq 1} \frac{2}{k(k+1)} \sum_{i=1}^{k} S_i$, which is the main object of study in this paper. Our goal is to find the probabilities

$$G(t) := \mathbb{P}\left\{ \inf_{k \geq 1} \frac{2}{k(k+1)} \sum_{i=1}^{k} S_i \geq t \right\}.$$

We can also write the right-hand side in a more pleasant form

$$G(t) = \mathbb{P}\left\{ \inf_{k \geq 1} \sum_{i=1}^{k} (S_i - it) \geq 0 \right\},$$

which resembles a ruin probability.

The function $G(t)$ arises in the study of so-called one-dimensional sticky particles model. Let us briefly describe the model. We consider a system of $n$ identical particles, each one of mass $n^{-1}$. At time zero the immobile particles are randomly distributed on the real line. The particles begin to move under the forces of mutual attraction. When two or more particles collide, they stick together forming a new particle ("cluster") whose characteristics are defined by the laws of

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mass and momentum conservation. Between collisions particles move according to the laws of Newtonian mechanics.

We suppose that the force of mutual attraction does not depend on distance and equals the product of masses; this is very natural for one-dimensional models. Thus at any moment, the acceleration of a particle is equal to difference of masses to the right and to the left of the particle.

There are two natural and well-known models of random initial positions of particles. In the uniform model, \( n \) particles are uniformly and independently spread on \([0,1]\). In the Poisson model, the particles are located at the points of first \( n \) jumps of a Poisson process with intensity \( n \) (i.e., a standard Poisson process multiplied by \( n^{-1} \)).

For more information about systems of sticky particles, see [1], [3], [4], [5], and references therein.

Let us agree that the term “particle” refers to the initial particles only, and let a “cluster” be a product of a collision as well as an initial particle that has not experienced any collisions. Thus at any moment \( t > 0 \), the system consists of clusters and each cluster contains one or more particles. As time goes, clusters aggregate and became larger and larger while the number of clusters decreases. Finally, at some moment all clusters merge into a single cluster containing all (initial) particles.

By \( K_n(t) \) denote the number of clusters at time \( t \) in the system of \( n \) particles. This quantity is a random step function, which decreases (in \( t \)) from its initial value \( n \) to 1. The problem, which leads to the study of the function \( G(t) \), is to describe the asymptotics of \( K_n(t) \) as \( n \to \infty \). This problem was introduced in [5], where the author proved the following statement. Both in the uniform and the Poisson models of initial positions, for any \( t \geq 0 \), we have

\[
\frac{K_n(t)}{n} \overset{D}{\to} K(t), \quad n \to \infty,
\]

where \( K(t) \) is a deterministic function satisfying \( K(t) = e^{t^2 (G(t^2))} \). It was conjectured on the basis of numerical simulations that \( K(t) = 1 - t^2 \) for \( 0 \leq t \leq 1 \).

The main result of the current paper, the formula

\[
G(t) = \mathbb{P}\left\{ \inf_{k \geq 1} \sum_{i=1}^{k} (S_i - it) \geq 0 \right\} = \sqrt{1 - t} e^{-t^2/2}, \quad 0 \leq t \leq 1,
\]

shows that the conjecture is true. Our study of the problem was motivated by the wish to prove the weird formula [1] as well as by the necessity to verify some properties of \( K(t) \) the author needed in his further investigation of \( K_n(t) \). New results on the number of clusters will be soon published in [6].

We also note that in view of the well-know connection between exponential random walks and order statistics, \( G(t) \) could be represented in the form

\[
G(t) = \lim_{n \to \infty} \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2n}{k(k + 1)} \sum_{i=1}^{k} U_{i,n} \geq t \right\},
\]

where \( U_{i,n} \) are the order statistics of \( n \) i.i.d. random variables uniformly distributed on \([0,1]\). This equality will be proved rigorously in the end of Sec. 4. Thus \( G(t) \) is closely related to partial sums of uniform order statistics.
The proof of (1) is organized as follows. In Sec. 2 and 3 we study properties of the functions
\[ G_n(t) := \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2}{k(k+1)} \sum_{i=1}^{k} S_i \geq t \right\}, \]
which converge to \( G(t) \). We show that \( G_n(t) \) are continuously differentiable and that \( G'_n(t) \) converge uniformly; consequently, \( G(t) \) has a continuous derivative. We also obtain an ordinary differential equation for \( G(t) \), but the right-hand side of this equation will be represented as the sum of a series with unknown coefficients. In Sec. 4 we find these coefficients, solve the differential equation, and get (1).

2. “Partial densities” and continuity of \( G(t) \)

We will use the bold type for multi-dimensional variables. Let us indicate the dimension of these variables with subscript, e.g., \( \mathbf{x}_n \in \mathbb{R}^n \); we will omit these subscripts as often as possible. The coordinates of \( \mathbf{x}_n \) will be always denoted by \( x_1, \ldots, x_n \). By \( \mathbf{0} = 0_n \) and \( \mathbf{1} = 1_n \) denote \((0, \ldots, 0)^\top \in \mathbb{R}^n \) and \((1, \ldots, 1)^\top \in \mathbb{R}^n \), respectively.

Let \( X_i \) be increments of the exponential random walk \( S_i = X_1 + \cdots + X_i \). By definition, \( X_i \) are i.i.d. standard exponential random variables. Put \( Y_k := \frac{2}{k(k+1)} \sum_{i=1}^{k} S_i \); clearly, we have \( Y_k = \frac{2}{k(k+1)} \sum_{i=1}^{k} (k - i + 1)X_i \). Hence \( \mathbf{Y}_n = (Y_1, \ldots, Y_n)^\top \) is a linear function of \( \mathbf{X}_n = (X_1, \ldots, X_n)^\top \), that is \( \mathbf{Y}_n = \mathbf{A}_n \mathbf{X}_n \), where

\[
\mathbf{A}_n := \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
2/3 & 1/3 & 0 & \cdots & 0 \\
3/6 & 2/6 & 1/6 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
\frac{2n}{n(n+1)} & \frac{2(n-1)}{n(n+1)} & \frac{2(n-2)}{n(n+1)} & \cdots & \frac{2}{n(n+1)}
\end{pmatrix}.
\]

This matrix is nonsingular; by \( \mathbf{L}_n := \mathbf{A}_n^{-1} \) denote the inverse matrix. As far as for every \( k \geq 3 \),

\[
X_k = S_k - S_{k-1} = \left( \sum_{i=1}^{k} S_i - \sum_{i=1}^{k-1} S_i \right) - \left( \sum_{i=1}^{k-1} S_i - \sum_{i=1}^{k-2} S_i \right) = \sum_{i=1}^{k} S_i - 2 \sum_{i=1}^{k-1} S_i + \sum_{i=1}^{k-2} S_i = \frac{k(k+1)}{2} Y_k - (k-1)k Y_{k-1} + \frac{(k-2)(k-1)}{2} Y_{k-2},
\]

we conclude that

\[
\mathbf{L}_n = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & -6 & 6 & 0 & \cdots & 0 & 0 & 0 \\
0 & 3 & -12 & 10 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \mathbf{l}_{n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -2\mathbf{l}_{n-2} & \mathbf{l}_{n-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & \mathbf{l}_{n-2} & -2\mathbf{l}_{n-1} & \mathbf{l}_n
\end{pmatrix}, \quad \mathbf{l}_k := \frac{k(k+1)}{2}.
\]

This matrix has three nonzero diagonals; note that the sum of elements of each row equals one and the sum of elements of each column except the last two equals zero.
The distribution of $Y_n = AX_n$ is concentrated on the $n$-dimensional cone $\{y = y_n : Ly \ge 0\} \subset \mathbb{R}_+^n$, because $LY_n = X_n \ge 0$ ($x \ge y$ denotes coordinate-wise inequalities). For the density $p_{Y_n}(y) = p(y)$ of $Y_n$, we have $p_{Y_n}(y) = |\det A|^{-1}p_{X_n}(A^{-1}y) = \frac{n!(n+1)!}{2^n} \exp\{-Ly, 1\} I_{\{LY \ge 0\}}$. For each of $n - 2$ first columns of $L$, the sum of its elements equals zero; the sum of elements in the columns $n - 1$ and $n$ equals $-l_{n-1} = \frac{-(n-1)n}{2}$ and $l_n = \frac{n(n+1)}{2}$, respectively. Thus we conclude

$$p(y_n) = \frac{n!(n+1)!}{2^n} e^{-\frac{n(n+1)}{2} y_o + \frac{(n-1)n}{2} y_{n-1}} \cdot I_{\{LY \ge 0\}}. \tag{2}$$

This is not surprise that the density of $Y_n = (Y_1, \ldots, Y_n)^T$ depends only on $Y_{n-1}$ and $Y_n$. In fact, we have $\frac{n(n+1)}{2} y_n - \frac{(n-1)n}{2} Y_{n-1} = S_n$, and the density of independent exponential random variables $X_n = (X_1, \ldots, X_n)^T$ depends on $S_n$ only.

Now we write

$$G_n(t) = \mathbb{P}\left\{ \min_{1 \le k \le n} Y_k \ge t \right\} = \sum_{k=1}^n \mathbb{P}\left\{ \min_{1 \le i \le n} Y_i = Y_k, Y_k \ge t \right\}, \tag{3}$$

and for any $1 \le k \le n$ introduce

$$g_n^{(k)}(t) := \int_{\{y = y_n, y \ge t\}} p(y) d\lambda_{n-1}(y), \tag{4}$$

where $\lambda_{n-1}$ denotes the Lebesgue measure on the $(n-1)$-dimensional set $\{y = y_n : y \ge t, y_k = t\}$. Then by the Lebesgue theorem, for a.e. $t$,

$$g_n^{(k)}(t) = -\mathbb{P}\left\{ \min_{1 \le i \le n} Y_i = Y_k, Y_k \ge t \right\}'.$$ \tag{5}

where the derivative exists a.e. Consequently, $G_n(t)$ is differentiable a.e., and for almost every $t$, we have

$$G_n'(t) = -\sum_{k=1}^n g_n^{(k)}(t). \tag{6}$$

Let us call $g_n^{(k)}(t)$ the $k$th partial density of the random variable $\min_{1 \le k \le n} Y_k$. In the next section we will show that partial densities are continuous, hence $[5] \text{ and } [6]$ hold for every $t$ and $G_n(t)$ is differentiable.

Finally, making the change of variables $y = t(z + 1)$ in $[4]$ and using $[2]$, we get

$$g_n^{(k)}(t) = \frac{n!(n+1)!}{2^n} e^{-nt} \int_{\{z = z_n, Lz \ge -1, z \ge 0, z_k = 0\}} e^{-\frac{n(n+1)}{2} t_0 + \frac{(n-1)n}{2} t_{n-1}} d\lambda_{n-1}(z). \tag{7}$$

Indeed, the inequality $LY \ge 0$ transforms to $Lz \ge -1$ because for each row of $L$, the sum of its elements equals one.

We finish the section with the following statement.

**Proposition 1.** The functions $G_n(t)$ are continuous. For every $\varepsilon \in (0, 1)$, $G_n(t)$ converge to $G(t)$ uniformly on $[0, 1 - \varepsilon]$. The function $G(t)$ is continuous on $[0, 1]$.

**Proof.** As

$$G_n(t) = \mathbb{P}\left\{ \min_{1 \le k \le n} \sum_{i=1}^k (S_i - it) \ge 0 \right\},$$
we can write
\[ G_n(t) - G(t) = \mathbb{P}\left\{ \min_{1 \leq k \leq n} \sum_{i=1}^{k} (S_i - it) \geq 0, \inf_{k>n} \sum_{i=1}^{k} (S_i - it) < 0 \right\} \]
\[ < \mathbb{P}\left\{ \inf_{k>n} \sum_{i=n+1}^{k} (S_i - it) < 0 \right\} \]
\[ < \mathbb{P}\left\{ \exists \ i > n : S_i - it < 0 \right\}. \]

Then
\[ \sup_{0 \leq t \leq 1 - \varepsilon} |G_n(t) - G(t)| < \sup_{0 \leq t \leq 1 - \varepsilon} \mathbb{P}\left\{ \inf_{i>n} \frac{S_i}{i} < t \right\} = \mathbb{P}\left\{ \inf_{i>n} \frac{S_i}{i} < 1 - \varepsilon \right\}, \]
and the last expression tends to zero by the strong law of large numbers.

By (1) and (3), \( G_n(t) \) are (absolutely) continuous; hence \( G(t) \) is continuous on \([0, 1] \) as a uniform limit of continuous functions.

3. Properties of “partial densities” and differentiability of \( G(t) \)

Here we prove several important properties of partial densities \( g_n^{(k)}(t) \). Let us first state the auxiliary Lemmata (1), (2), and (3) and then prove the differentiability of \( G(t) \) in Proposition (2). The lemmata will be proved afterwards.

**Lemma 1.** For every \( n \geq 1 \),
\[ g_{n+1}^{(1)}(t) = G_n(t)e^{-t}. \]

**Lemma 2 (Chaining property).** For every \( n \geq 2 \) and \( 1 \leq k \leq n - 1 \),
\[ g_{n}^{(k)}(t) = c_k (te^{-t})^{k-1} g_{n-k+1}^{(1)}(t), \]
where \( \{c_k\}_{k \geq 1} \) are some positive constants. These constants satisfy \( c_k = O(\sqrt{k} e^k). \)

**Lemma 3.** The functions \( g_n^{(n)}(t) \) are continuous. For every \( \varepsilon \in (0, 1) \), \( g_n^{(n)}(t) \) converge to 0 uniformly on \([0, 1 - \varepsilon] \).

**Proposition 2.** The functions \( G_n(t) \) are continuously differentiable. For every \( \varepsilon \in (0, 1) \), \( G'_n(t) \) converge to \(-G(t)e^{-t} \sum_{k=1}^{\infty} c_k (te^{-t})^{k-1} \) uniformly on \([0, 1 - \varepsilon] \). The function \( G(t) \) is continuously differentiable on \([0, 1] \), and
\[ G'(t) = -G(t)e^{-t} \sum_{k=1}^{\infty} c_k (te^{-t})^{k-1}, \quad t \in [0, 1). \]

**Proof of Proposition 2.** By (3), (1), and (5), we get continuous differentiability of \( G_n(t) \) if we show that partial densities are continuous. The last partial density \( g_n^{(n)}(t) \) is continuous by Lemma (3). Using Lemmata (1) and (2),
\[ g_n^{(k)}(t) = c_k (te^{-t})^{k-1} G_{n-k}(t)e^{-t}, \quad 1 \leq k \leq n - 1; \] (8)
but \( G_{n-k}(t) \) are continuous by Proposition (1) thus \( g_n^{(k)}(t) \) are continuous. Note that now we know that (5) and (4) are true for every \( t \).
Further, \((te^{-t})' = (1-t)e^{-t} \geq 0\) on \([0,1]\), thus \(te^{-t} < 1 \cdot e^{-1} = e^{-1}\) for \(t \in [0,1]\). Therefore in view of the estimate on the rate of growth of \(c_k\) from Lemma 2, \(\sum_{k=1}^{\infty} c_k (te^{-t})^{k-1}\) converges on \([0,1]\). Clearly, this convergence is uniform on \([0,1-\varepsilon]\).

Recall that \(1 \geq G_n(t) \geq 0\) for every \(t\). Then by (6) (which holds for every \(t\)) and (8),

\[
\left| -G(t)e^{-t} \sum_{k=1}^{\infty} c_k (te^{-t})^{k-1} - G'(n) \right| \\
\leq e^{-t} \sum_{k=1}^{n-1} c_k (te^{-t})^{k-1} (G_{n-k}(t) - G(t)) + e^{-t} G(t) \sum_{k=n}^{\infty} c_k (te^{-t})^{k-1} + g^{(n)}(t)
\]

\[
\leq (G_{n/2}(t) - G(t)) \sum_{k=1}^{n/2} c_k (te^{-t})^{k-1} + \sum_{k=n/2+1}^{\infty} c_k (te^{-t})^{k-1} + g^{(n)}(t).
\]

The last expression tends to zero uniformly in \(t \in [0,1-\varepsilon]\). Indeed, for the third term, use Lemma 3; the second one is a remainder of the uniformly converging series; and the first term tends to zero by Proposition 1. \hfill \square

**Proof of Lemma 1.** The \(Y_1 = X_1\) is a standard exponential random variable, therefore

\[
g^{(1)}_{n+1}(t) = -\mathbb{P}\left\{ \min_{1 \leq k \leq n+1} Y_k = Y_1, Y_1 \geq t \right\} = -\mathbb{P}\left\{ \min_{2 \leq k \leq n+1} Y_k \geq Y_1, Y_1 \geq t \right\}
\]

\[
= - \left( \int_t^{\infty} \mathbb{E}\left\{ \min_{2 \leq k \leq n+1} Y_k \geq Y_1 \right| Y_1 = s \right\} d\mathbb{P}\{Y_1 < s\} \right)^t
\]

\[
= \mathbb{E}\left\{ \min_{2 \leq k \leq n+1} Y_k \geq Y_1 \right| Y_1 = t \} e^{-t}
\]

for a.e. \(t\). After simple transformations

\[
\mathbb{E}\left\{ \min_{2 \leq k \leq n+1} Y_k \geq Y_1 \right| Y_1 = t \}
\]

\[
= \mathbb{E}\{ \forall 2 \leq k \leq n+1, \frac{2}{k(k+1)} (kX_1 + (k-1)X_2 + \cdots + X_k) \geq X_1 \left| X_1 = t \right\}
\]

\[
= \mathbb{E}\{ \forall 2 \leq k \leq n+1, \frac{2}{k(k+1)} ((k-1)X_2 + \cdots + X_k) \geq (1 - \frac{2}{k+1})X_1 \left| X_1 = t \right\},
\]

we use the independence of \(X_i\) and find

\[
\mathbb{E}\left\{ \min_{2 \leq k \leq n+1} Y_k \geq Y_1 \right| Y_1 = t \} = \mathbb{P}\{ \forall 2 \leq k \leq n+1, \frac{2}{(k-1)k} ((k-1)X_2 + \cdots + X_k) \geq t \}
\]

The right-hand side equals \(G_n(t)\). It remains to note that the conditional expectation is continuous because \(G_n(t)\) is continuous (by Proposition 1), therefore our argument is true for every \(t\). \hfill \square

**Proof of Lemma 2.** The case \(k = 1\) is trivial, we put \(c_1 := 1\). Now suppose that \(2 \leq k \leq n-1\).

Let us introduce the following notations. For an \(l \times m\) matrix \(M\), by \(M^{(i_1, \ldots, i_r, j_1, \ldots, j_s)}\) (where \(r \leq l, s \leq m\) and \(1 \leq i_1 < \cdots < i_r \leq l, 1 \leq j_1 < \cdots < j_s \leq m\) denote the \((l-r) \times (m-s)\) matrix obtained from \(M\) by deleting the rows \(i_1, \ldots, i_r\) and the columns \(j_1, \ldots, j_s\). For multi-dimensional variables, we will use the analogous notation.

Consider the integration set \(\{ y = y_{n} : Ly \geq -1, y \geq 0, y_k = 0 \}\) from (7). We claim that the first \(k-1\) and the last \(n-k\) coordinates of any element of this set satisfy independent
and combine (9) with Fubini’s theorem to rewrite (7) in a simpler form. For here is a block matrix. This implies that the constraints for \( y \) is independent, i.e., (9) holds true.

But by Lemma 1, \( L \) is the \((n \times n)\) matrix consisting of the elements from the top left “corner” of \( L_n \). Take an \( y \in \{ y = y_m : Ly \ge -1, y \ge 0, y_k = 0 \} \), then \( y^{(k)} \) satisfies \( L_n^{(\varnothing:k)} y^{(k)} \ge -1 \), and \( y^{(k)} \ge 0 \). In view of \( y^{(k)} \ge 0 \), the \((k+1)\)th of the inequalities \( L_n^{(\varnothing:k)} y^{(k)} \ge -1 \), namely, \( \frac{1}{2} y_{k+1} + \frac{(k+1)(k+2)}{2} y_{k+1} \ge -1 \), holds automatically. Therefore we can delete the row \( k+1 \) from \( L_n^{(\varnothing:k)} \), that is

\[
\{ L_n^{(\varnothing:k)} y^{(k)} \ge -1, y^{(k)} \ge 0 \} = \{ L_n^{(k+1):k} y^{(k)} \ge -1, y^{(k)} \ge 0 \}.
\]

Since \( L_n \) has three nonzero diagonals,

\[
L_n^{(k+1):k} = \begin{pmatrix} L_n^{(\varnothing:k)} & 0 \\ 0 & L_n^{(1, \ldots, k+1; 1, \ldots, k)} \end{pmatrix}
\]

is a block matrix. This implies that the constraints for \( y_1, \ldots, y_{k-1} \) and \( y_{k+1}, \ldots, y_n \) are independent, i.e., (9) holds true.

Now define

\[
P_m := \{ y = y_{m-1} : L_m^{(\varnothing:m)} y \ge -1, y \ge 0 \}, \quad m \ge 2
\]

\[
v_m := \lambda_{m-1}(P_m),
\]

\[
c_m := 2^{-m} m!(m+1)! v_m,
\]

and combine (10) with Fubini’s theorem to rewrite (7) in a simpler form. For \( k = n-1 \) we get

\[
g_n^{(n-1)}(t) = \frac{n! (n+1)!}{2^n} t^{n-1} e^{-nt} v_{n-1} \int_{\{y \ge 0\}} e^{-\frac{n(n+1)}{2} y d\lambda_1(y)},
\]

and for \( 2 \le k \le n-2 \) we get

\[
g_n^{(k)}(t) = \frac{n! (n+1)!}{2^n} t^{n-1} e^{-nt} v_k \int_{\{y = y_{n-k} : L_n^{(1, \ldots, k+1; 1, \ldots, k)} y \ge -1, y \ge 0\}} e^{-\frac{n(n+1)}{2} y_{n-k} + \frac{(n-1)n}{2} y_{n-k-1} d\lambda_{n-k}(y)}.
\]

For the simpler case \( k = n-1 \), we integrate in (12) and find

\[
g_n^{(n-1)}(t) = c_n t^{n-2} e^{-nt}.
\]

But by Lemma 1, \( g_2^{(1)}(t) = e^{-2t} \), and there is nothing to prove.
For the harder case $2 \leq k \leq n - 2$, we use (17) and write
\[
g_{n-k+1}^{(1)}(t) = \frac{(n - k + 1)!}{2^{n-k+1} \lambda^{n-k+1}} (n - k + 2)! e^{-(n-k+1)t} \int_{\{z=z_{n-k+1}: Lz \geq -1, z \geq 0, z_1 = 0\}} e^{-\frac{(n-k+1)(n-k+2)}{2} \sum_{l \neq 1} z_{n-k} + \frac{1}{2} \sum_{l \neq 1} t z_{n-k} - 1, \lambda_{n-k}}(z). \tag{14}
\]

Take an element $z$ of the integration set, then $z^{(1)}$ satisfies $L_{n-k+1}^{(2;1)} z^{(1)} \geq -1_{n-k}$ and $z^{(1)} \geq 0_{n-k}$. But the first of the inequalities $L_{n-k+1}^{(2;1)} z^{(1)} \geq -1_{n-k}$, that is $0 \geq -1$, is always true, while the second one, $3z_2 \geq -1$, follows from $z^{(1)} \geq 0_{n-k}$. Therefore
\[
\{z = z_{n-k+1}: L_{n-k+1} z \geq -1, z \geq 0, z_1 = 0\} = \{0\} \times \{z = z_{n-k}: L_{n-k+1}^{(1;2;1)} z \geq -1_{n-k}, z \geq 0_{n-k}\},
\]
and transforming the integral in (14), we get
\[
g_{n-k+1}^{(1)}(t) = \frac{(n - k + 1)!}{2^{n-k+1} \lambda^{n-k+1}} (n - k + 2)! e^{-(n-k+1)t} \int_{\{z=z_{n-k+1}: L_{n-k+1}^{(1;2;1)} z \geq -1_{n-k}, z \geq 0_{n-k}\}} e^{-\frac{(n-k+1)(n-k+2)}{2} \sum_{l \neq 1} z_{n-k} + \frac{1}{2} \sum_{l \neq 1} t z_{n-k} - 1, \lambda_{n-k}}(z). \tag{15}
\]

Compare (12) and (15). The integrals in these formulas do not look nice, but the point is that one could be obtained from the other by a very simple change of variables. Recall that
\[
L_{n-k+1}^{(1;2;1)} = \begin{pmatrix}
-6 & 6 & 0 & \ldots & 0 & 0 \\
3 & -12 & 10 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & l_{n-k} & 0 \\
0 & 0 & 0 & \ldots & -2l_{n-k} & l_{n-k+1}
\end{pmatrix}
\]
and
\[
L_{n}^{(1;...,k+1;...,k)} = \begin{pmatrix}
-2l_{k+1} & l_{k+2} & 0 & \ldots & 0 & 0 \\
l_{k+1} & -2l_{k+2} & l_{k+3} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & l_{n-1} & 0 \\
0 & 0 & 0 & \ldots & -2l_{n-1} & l_{n}
\end{pmatrix}
\]
By $J_m^{s \rightarrow i}$, where $i, s, m \geq 1$, we denote the $m \times m$ diagonal matrix with the elements $\frac{L_{i}}{l_{i}}, \frac{l_{i+1}}{l_{i}}, \ldots, \frac{l_{i+m-1}}{l_{i+m-1}}$ on the diagonal (counting from the top left corner). Then
\[
L_{n-k+1}^{(1;2;1)} = L_{n}^{(1;...,k+1;...,k)} J_{n-k}^{k+1 \rightarrow 2}
\]
because the right-sided multiplication by $J_{n-k}^{k+1 \rightarrow 2}$ multiplies the first column of $L_{n}^{(1;...,k+1;...,k)}$ by $\frac{L_{k+1}}{l_{k+1}} = \frac{3}{l_{k+1}},$ the second column by $\frac{l_{k+2}}{l_{k+2}} = \frac{6}{l_{k+2}}$, etc.

Hence the change of variables $z = (J_{n-k}^{k+1 \rightarrow 2})^{-1}$ transforms the integral from (15) to the integral in (12) times $|\det(J_{n-k}^{k+1 \rightarrow 2})|^{-1}$. But
\[
\det J_m^{s \rightarrow i} = \frac{(s - 1)!}{(i - 1)!} s! (i + m - 1)! (i + m)! (s + m - 1)! (s + m)! \tag{17}
\]
and from (12) and (15),
\[ g_{n-k+1}^{(1)}(t) = \frac{n!(n+1)!}{2^n} t^{n-1} e^{-nt} v_k \]
\[ g_{n-k}^{(1)}(t) = \frac{2^n}{(n-k)!} (n-k+1)! e^{-(n-k+1)t} \]

It remains to check that \( c_k = O(\sqrt{k} e^k) \) to finish the proof of Lemma. By (13), \( g_{k+1}(t) = c_k t^{k-1} e^{-(k+1)t} \), and integrating from 0 to \( \infty \), we get \( \mathbb{P} \left\{ \min_{1 \leq i \leq k+1} Y_i = Y_k \right\} = c_k \frac{(k-1)!}{(k+1)^k}. \) Then \( c_k < \frac{(k+1)^k}{(k-1)!} \), and by Stirling’s formula,
\[
c_k \leq C \frac{(k+1)^k e^{k-1}}{(k-1)^k \sqrt{2\pi(k-1)}} = C \left( \frac{k+1}{k-1} \right)^{k-1} \sqrt{\frac{2\pi}{k(k-1)}} e^{k-1} = O(\sqrt{k} e^k).
\]

**Proof of Lemma 3.** Using notations (10), formula (7) could be written as
\[ g_n^{(n)}(t) = \frac{n!(n+1)!}{2^n} t^{n-1} e^{-nt} \int_{P_n} e^{\frac{(n-1)n}{2} y_{n-1} d\lambda_{n-1}(y_{n-1})}.
\]
(18)

Recall that, first, \( y_{n-1} \in P_n \) implies \( y_{n-1} \geq 0 \), and second, from the proof of Lemma 2 we know that \( P_n \subset \mathbb{R}^{n-1} \) has finite volume. Then \( P_n \) is bounded since it is an intersection of half-spaces; note that we give another proof of boundedness of \( P_n \) while proving Lemma 3 from the next section.

Now it is clear that \( g_n^{(n)}(t) \) is continuous. Further, as far as \( y_{n-1} \geq 0 \) for \( y_{n-1} \in P_n \), the integral from (18) increases in \( t \). By \( (t^{n-1} e^{-nt})' = t^{n-2} e^{-nt} ((n-1) - nt) \), we conclude that \( g_n^{(n)}(t) \) increases (in \( t \)) at least on \([0, \frac{n-1}{n}]\). Now suppose \( n \) is such that \( \frac{n-1}{n} \geq 1 - \varepsilon/2 \); then \( g_n^{(n)} \) is increasing on \([0, 1 - \varepsilon/2]\), and in view of (5) (which holds for every \( t \)),
\[
\sup_{0 \leq t \leq 1 - \varepsilon} g_n^{(n)}(t) = g_n^{(n)}(1 - \varepsilon) \leq \frac{2}{\varepsilon} \int_{1-\varepsilon}^{1-\varepsilon/2} g_n^{(n)}(s) ds \leq \frac{2}{\varepsilon} \mathbb{P}\{Y_n \leq 1 - \varepsilon/2\}.
\]
The last expression tends to zero as \( n \to \infty \). Indeed, \( Y_n = \frac{2}{n(n+1)} \sum_{i=1}^{n} S_i \to 1 \) a.s. because \( S_n \to 1 \) a.s. by the strong law of large numbers.

**4. The differential equation for \( G(t) \)**

By Proposition 2 we know that \( G(t) \) satisfies
\[
\begin{cases}
G'(t) = -G(t) t^{-1} f(te^{-t}), & t \in [0, 1) \\
G(0) = 1,
\end{cases}
\]
(19)

where \( f(x) := \sum_{k=1}^{\infty} c_k x^k \) is the generating function of \( c_k \). By Lemma 2, this series converges for \(|x| < e^{-1}\). Let us formulate Lemma 4 which is indispensable for finding \( f(x) \), and then find \( G(t) \) in Proposition 3. The lemma will be proved afterwards.

**Lemma 4.** For \( n \geq 2 \), we have
\[
c_n = \frac{n(n+1)}{n-1} \sum_{k=1}^{n-1} \frac{c_k c_{n-k}}{(k+1)(n-k+1)}.
\]
and \( c_1 = 1 \).

**Proposition 3.** The function \( G(t) \) satisfies the differential equation
\[
\begin{cases}
G''(t) = \frac{t-2}{2(1-t)} G(t), & t \in [0, 1) \\
G(0) = 1,
\end{cases}
\]
which has a unique on \([0, 1)\) solution \( \sqrt{1 - te^{-t/2}} \).

**Proof of Proposition 3.** Define the variables \( b_n := \frac{n}{n+1} \). The generating function \( h(x) := \sum_{k=1}^{\infty} b_k x^k \) of these variables satisfies
\[
(xh(x))' = f(x), \quad |x| < e^{-1}
\]
and \( h(0) = 0 \); we recall that the sum of a power series could be differentiated termwise inside the circle of its convergence.

Using Lemma 4, we find that \( b_n = \frac{n}{n+1} \sum_{k=1}^{n-1} b_k \), \( b_1 = 1/2 \). Then for \( |x| < e^{-1} \),
\[
h^2(x) = (b_1 x + b_2 x^2 + \ldots)^2 = \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} b_k b_{n-k} x^n = \sum_{n=1}^{\infty} \frac{n-1}{n} b_n x^n = \sum_{n=1}^{\infty} b_n x^n - \sum_{n=1}^{\infty} \frac{b_n}{n} x^n,
\]
and by differentiation,
\[
2h(x)h'(x) = h'(x) - \frac{h(x)}{x}, \quad |x| < e^{-1}.
\]

Now we have
\[
\begin{cases}
\frac{2h-1}{h} = \frac{1}{x}, & |x| < e^{-1} \\
h'(0) = b_1 = 1/2,
\end{cases}
\]
and by taking into account that \( h > 0 \) for \( x > 0 \),
\[
2h - \ln h = - \ln x + C, \quad 0 \leq x < e^{-1};
\]
then
\[
\ln 2h - 2h = \ln x + C', \quad 0 \leq x < e^{-1},
\]
\[
2he^{-2h} = C'' x, \quad 0 \leq x < e^{-1}.
\]
Using \( h'(0) = 1/2 \), we find that \( C'' = 1 \). The function \( q(x) := xe^{-x} \) is invertible on \([0, 1]\), and we obtain
\[
h(x) = \frac{q^{-1}(x)}{2}, \quad 0 \leq x < e^{-1}.
\]

Finally, by (21),
\[
f(te^{-t}) = f(q(t)) = h(q(t)) + q(t)h'(q(t)) = h(q(t)) + q(t) \frac{h(q(t))'}{q'(t)}
\]
\[
= \frac{t}{2} + te^{-t} \frac{1/2}{(1-t)e^{-t}} = \frac{2-t}{2(1-t)}, \quad t \in [0, 1),
\]
and applying (19), we see that \( G(t) \) satisfies (20).

It remains to solve (20). As far as \( \frac{t-2}{2(1-t)} = -\frac{1}{2} - \frac{1}{2(1-t)} \), we get
\[
\begin{cases}
\ln G(t) = -\frac{t}{2} + \frac{1}{2} \ln(1-t) + C, & t \in [0, 1) \\
G(0) = 1.
\end{cases}
\]
Then \( C = 0 \), and \( G(t) = \sqrt{1 - te^{-t/2}} \) on \([0,1]\). Note that this equality holds on \([0,1]\). \(\square\)

**Proof of Lemma 4.** We recall that \( c_n = 2^{-n} n!(n+1) v_n \) for \( n \geq 2 \), where \( v_n \) is the volume of

\[
P_n = \{ y = y_{n-1} : L_n^{(\emptyset; n)} y \geq -1_{n-1}, \ y \geq 0_{n-1} \},
\]

see (10). That is why our goal is to find \( v_n \). As far as \( c_1 = 1 \), we define \( v_1 := 1 \) to satisfy \( c_1 = 2^{-1} n!(1+1)v_1 \).

Let us study properties of \( P_n \). We will temporary forget about geometric intuition and use algebraic arguments only. Evidently,

\[
P_n = \{ y = y_{n-1} : L_n^{(1:n)} y \geq -1_{n-1}, \ y \geq 0_{n-1} \}
\]

because the first of the inequalities \( L_n^{(\emptyset; n)} y \geq -1_{n-1} \), namely, \( y_1 \geq -1 \), follows from \( y \geq 0_{n-1} \).

We claim that (for every \( n \geq 2 \)) the matrix \( L_n^{(1:n)} \) is nonsingular. By \( I_k^{\Delta} \) denote the \( k \times k \) upper triangular matrix with all its \( k(k+1)/2 \) nonzero elements equal 1, and denote \( I_k^\nabla := (I_k^{\Delta})^\top \), which is lower triangular. For the matrix \( L_n^{(1:n)} \), the sum of elements of each column except the first and the last ones equals zero. For the matrix \( I_{n-1}^\nabla L_n^{(1:n)} \), the sum of elements of each column except the first one equals zero. That is why we easily get

\[
I_{n-1}^\nabla I_{n-1}^\nabla L_n^{(1:n)} = \begin{pmatrix}
-2 & 0 & 0 & \ldots & 0 & 0
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots
-2 & 0 & 0 & \ldots & -l_{n-2} & 0
-1 & 0 & 0 & \ldots & 0 & -l_{n-1}
\end{pmatrix},
\]

and thus \( \det L_n^{(1:n)} = \det(I_{n-1}^\nabla I_{n-1}^\nabla L_n^{(1:n)}) \neq 0 \).

Let us now show that \( L_n^{(1:n)} y \geq 0 \) implies \( y \leq 0 \). The proof is by induction. For \( n = 2 \), the statement is trivial. Assume that the statement is true for an \( n \geq 2 \); then check that it holds for \( n+1 \). Since \( I_n^\Delta P_n \) has positive elements only, \( L_n^{(1:n+1)} y \geq 0 \) implies \( I_n^\Delta P_n L_n^{(1:n+1)} y \geq 0 \), and by (22) we conclude that \( y_1 \leq 0 \). Further, \( L_n^{(1:n+1)} y \geq 0 \) implies \( L_n^{(1,2,1:n+1)} y^{(1)} \geq 0_{n-1} \). In fact, we just get rid of the first of the inequalities \( L_n^{(1:n+1)} y \geq 0 \) and replace \( y_1 - 6y_2 + 6y_3 \geq 0 \), which is the second inequality, by the less restrictive \( -6y_2 + 6y_3 \geq 0 \) (recall that \( y_1 \leq 0 \) !). Then

\[
L_n^{(1,2,1:n+1)} = L_n^{(1:n)} J_n^{1-2}_{n-1} \quad \text{(see the comment to analogous statement (16) and the definition of } J_m^{(n)}), \text{ therefore } L_n^{(1,2,1:n+1)} y^{(1)} \geq 0_{n-1} \text{ is equivalent to } L_n^{(1:n)} z \geq 0_{n-1}, \text{ where } z := J_n^{1-2} y^{(1)}.\n\]

By the assumption, \( z \leq 0_{n-1} \); hence \( y^{(1)} \leq 0_{n-1} \) because \( z \) is obtained from \( y^{(1)} \) by the tension with positive coefficients. Finally, we get \( y \leq 0_n \).

By (22) we easily find that a unique solution of \( L_n^{(1:n)} y = -1 \) is

\[
y_{n-1}^* := \left( \frac{n-1}{2}, \frac{n-2}{3}, \ldots, \frac{2}{n-1}, \frac{1}{n} \right)^\top.
\]

We see that \( y^* \in P_n \), and since \( L_n^{(1:n)} (y - y^*) \geq 0_{n-1} \) for every \( y \in P_n \), we have \( y \leq y^* \).
Now it is clear that $P_n$ is an $(n - 1)$-dimensional convex polyhedron with $2(n - 1)$ faces. Denote by $O_n$ the point of intersection of $n - 1$ hyperplanes (faces) $L^{\{1, n\}}_n y = -1_{n-1}$, and put $F^{(k)}_n := P_n \cap \{y = y_{n-1} : y_k = 0\}$, where $1 \leq k \leq n - 1$, see the figure. Naturally, $O_n = y^*_{n-1}$, and by $y_{n-1} \in P_n \implies y_{n-1} \leq y^*_{n-1}$, the $P_n$ is a disjoint union of $n - 1$ simplexes with the common vertex $O_n$ and the bases $F^{(k)}_n$ (to be pedantic, the simplexes themselves are not disjoint, but their interiors are). Recalling (23),

$$v_n = \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{n-k}{k+1} \cdot \lambda_{n-2}(F^{(k)}_n).$$

(24)

Thus we reduced the problem of finding the volume of $P_n$ to finding the volumes of $F^{(k)}_n$.

For $2 \leq k \leq n - 2$, we have

$$F^{(k)}_n = \{y = y_{n-1} : L^{\{1, n\}}_n y \geq -1_{n-1}, y \geq 0_{n-1}, y_k = 0\} \quad \text{is a statement about } L_n. \text{ Therefore the proof is a verbatim copy of the proof of (9). Then}

$$F^{(k)}_n = P_k \times \{0\} \times \{y = y_{n-k-1} : L^{\{1, k+1, \ldots, k, n\}}_n y \geq -1_{n-k-1}, y \geq 0_{n-k-1}\},

$$

and in view of $L^{\{1, n-k\}}_{n-k} = L^{\{1, \ldots, k+1, \ldots, k, n\}}_n J^{k+1 \rightarrow 1}_{n-k-1}$ (see the comment to analogous statement (16) and the definition of $J^{m \rightarrow i}_{n-k}$), we make the change of variables $y = J^{k+1 \rightarrow 1}_{n-k-1} z$ and obtain

$$F^{(k)}_n = P_k \times \{0\} \times J^{k+1 \rightarrow 1}_{n-k-1} \{z = z_{n-k-1} : L^{\{1, n-k\}}_{n-k-1} z \geq -1_{n-k-1}, z \geq 0_{n-k-1}\}.

$$

Finally, $F^{(k)}_n = P_k \times \{0\} \times J^{k+1 \rightarrow 1}_{n-k-1} P_{n-k}$, and by (17),

$$\lambda_{n-2}(F^{(k)}_n) = \frac{k!(k+1)!(n-k-1)!(n-k)!}{(n-1)!n!} v_k v_{n-k}.

(25)

For $k = 1$, we could repeat word by word the argument we used in the case $2 \leq k \leq n - 2$. At the final step, we get $F^{(1)}_n = \{0\} \times J^{2 \rightarrow 1}_{n-2} P_{n-1}$, and since $v_1 = 1$, (25) also holds for $k = 1$.

For $k = n - 1$, it is readily seen that $F^{(n-1)}_n = P_{n-1} \times \{0\}$, hence (25) holds for $k = n - 1$.

Thus (25) is true for $1 \leq k \leq n - 1$. It remains to take a look at (24) and (25) to get

$$v_n = \frac{1}{(n-1)(n-1)!n!} \sum_{k=1}^{n-1} (n-k)!^2 v_k v_{n-k}.

$$

The application of $v_n = \frac{2^n}{n(n+1)!} \sum$ finishes the proof of Lemma. □
Let us finish the paper proving the equality

\[ G(t) = \lim_{n \to \infty} \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2n}{k(k+1)} \sum_{i=1}^{k} U_{i,n} \geq t \right\}, \]

where \( U_{i,n} \) are the order statistics of \( n \) i.i.d. random variables uniformly distributed on \([0, 1]\).

It is well-known (see, for example, [2, Sec. 9.1]) that

\[ (U_{1,n}, U_{2,n}, \ldots, U_{n,n}) \overset{D}{=} \left( \frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}} \right) \]

for every \( n \geq 1 \). Then

\[ \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2n}{k(k+1)} \sum_{i=1}^{k} U_{i,n} \geq t \right\} = \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2}{k(k+1)} \sum_{i=1}^{k} S_i \geq \frac{S_{n+1}}{n} t \right\}, \]

and combining the law of large numbers with the pointwise convergence of \( G_n \) to \( G \), for every \( \varepsilon \in (0, 1) \), we have

\[ G((1 + \varepsilon)t) \leq \lim_{n \to \infty} \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2n}{k(k+1)} \sum_{i=1}^{k} U_{i,n} \geq t \right\} \leq \lim_{n \to \infty} \mathbb{P}\left\{ \min_{1 \leq k \leq n} \frac{2n}{k(k+1)} \sum_{i=1}^{k} U_{i,n} \geq t \right\} \leq G((1 - \varepsilon)t). \]

We complete the proof proceeding to the limit \( \varepsilon \searrow 0 \) and using the continuity of \( G(t) \).

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