ON A CONVEX LEVEL SET OF A PLURISUBHARMONIC FUNCTION AND THE SUPPORT OF THE MONGE-AMPERE CURRENT

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Abstract. In this paper, we study a geometric property of a continuous plurisubharmonic function which is a solution of the Monge-Ampère equation and has a convex level set. To prove our main theorem, we show a minimum principle of a maximal plurisubharmonic function. By using our results and Lempert’s results, we show a relation between the supports of the Monge-Ampère currents and complex $k$-extreme points of closed balls for the Kobayashi distance in a bounded convex domain in $\mathbb{C}^n$.

1. Introduction

Let $D$ be a domain in $\mathbb{C}^n$. Let $\text{Psh}(D)$ be plurisubharmonic functions in $D$. In [2] and [3], the Monge-Ampère current $(dd^c u)^k$ ($k = 1, \ldots, n$) is defined for $u \in \text{Psh}(D) \cap C^0(D)$. The function $u \in \text{Psh}(D) \cap C^0(D)$ such that $(dd^c u)^n = 0$ in $D$ can be characterized as the maximal plurisubharmonic function (c.f. [12]). Here we say $u \in \text{Psh}(D) \cap C^0(D)$ is maximal if for every relatively compact open subset $G$ of $D$, and for each upper semicontinuous function $v$ on $G$ such that $v \in \text{Psh}(G)$ and $v \leq u$ on $\partial G$, we have $v \leq u$ in $G$. If $u \in \text{Psh}(D) \cap C^3(D)$ such that $(dd^c u)^k = 0$ in $D$, there exists a foliation on the interior of the support of $(dd^c u)^{k-1}$ by complex $(n-k+1)$-dimensional submanifolds such that the restriction of $u$ to any leaf of the foliation is pluriharmonic (see [1]). The geometric properties of a solution of $(dd^c u)^k = 0$ are more complicated in lower regularity. For example, Sibony showed that there exist a $C^{1,1}$ plurisubharmonic function $u$ in the Euclidean unit ball $B$ in $\mathbb{C}^2$ and a point $p \in B$ such that $(dd^c u)^2 = 0$ in $B$, $u$ attains its minimum at $p$ and there exists no holomorphic disk through $p$ on which $u$ is harmonic (see [4] for Sibony’s example and more examples).

In this paper, we study a geometric property of $u \in \text{Psh}(D) \cap C^0(D)$ which is a solution of the Monge-Ampère equation $(dd^c u)^k = 0$ ($k = 1, \ldots, n-1$) and has a convex level set.

Let

$$\overline{B}_u(t) = \{ z \in D; u(z) \leq t \},$$
$$S_u(t) = \{ z \in D; u(z) = t \}$$

for $t \in \mathbb{R}$. Our main result is the following.

**Theorem 1.** Let $D$ be a domain in $\mathbb{C}^n$ and $u \in \text{Psh}(D) \cap C^0(D)$. Let $k \in \{1, \ldots, n-1\}$ and $r \in \mathbb{R}$. Assume that $(dd^c u)^k = 0$ in $D$ and that $\overline{B}_u(r)$ is convex. Then any point of $S_u(r)$ lies in the relative interior of a complex $(n-k)$-dimensional convex set contained in $S_u(r)$.

Here a complex $(n-k)$-dimensional convex set is a convex set whose affine hull is a complex $(n-k)$-dimensional affine space. We do not know whether there exists a
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complex \((n - k + 1)\)-dimensional locally closed submanifold without boundary through a point of \(S_u(r)\) on which \(u\) is pluriharmonic. However, the converse holds (see Proposition 1).

To prove Theorem 1, we show the following minimum principle of a maximal plurisubharmonic function.

**Theorem 2.** Let \(D \subset \mathbb{C}^n\) be a domain and let \(u \in \text{Psh}(D) \cap C^0(D)\) be a non-negative maximal plurisubharmonic function, that is, \(u \geq 0\) and \((dd^c u)^n = 0\) in \(D\). Assume that \(u^{-1}(0) \subset D\) is non-empty and convex. Then any point of \(u^{-1}(0)\) lies in the relative interior of a complex one-dimensional convex set contained in \(u^{-1}(0)\).

The condition that \(u^{-1}(0)\) is convex can not be removed because of Sibony’s example.

Assume that \(D \subset \mathbb{C}^n\) is a bounded convex domain. Lempert [13] showed the relation between the Kobayashi distance and the pluricomplex Green function in \(D\). Recall that the pluricomplex Green function of \(D\) with logarithmic pole at \(x \in D\) is defined as

\[
K_{D,x}(z) = \sup\{u(z); u \in \text{Psh}(D), u < 0, \limsup_{w \to x} u(w) - \log \|w - x\| < \infty\}
\]

and \(K_{D,x}\) is the unique solution for the following homogeneous Monge-Ampère equation:

\[
\begin{cases}
  u \in \text{Psh}(D) \cap L^\infty_{\text{loc}}(\overline{D} \setminus \{x\}), \\
  (dd^c u)^n = 0 \text{ in } D \setminus \{x\}, \\
  u(z) - \log \|z - x\| = O(1) \text{ for } z \to x, \\
  \lim_{z \to p} u(z) = 0 \text{ for all } p \in \partial D
\end{cases}
\]

(see [13], [5]). Let \(k_D\) be the Kobayashi distance on \(D\). Lempert [13] showed that

\[
K_{D,x}(y) = \log \tanh k_D(x, y) \quad \text{for } x, y \in D, x \neq y.
\]

Hence the balls centered at \(x\) for the Kobayashi distance of \(D\) coincide with the level sets of \(K_{D,x}\). It was also shown in [13] that the balls for the Kobayashi distance of \(D\) are convex. If \(D\) is smoothly bounded and strictly convex, then \(K_{D,x}\) is a smooth function in \(D \setminus \{x\}\) such that \((dd^c K_{D,x})^{n-1} \neq 0\) in \(D \setminus \{x\}\) and the balls centered at \(x\) for the Kobayashi distance are strictly convex (see [13] and [14]). However, if \(D\) is a general bounded convex domain, balls for the Kobayashi distance are not always strictly (pseudo)convex and \(K_{D,x}\) might be pluriharmonic in an open subset of \(D\). In Section 3, we show a relation between the supports of Monge-Ampère currents \((dd^c K_{D,x})^k, (dd^c e^{K_{D,x}})^k\) and complex \(k\)-extreme points of balls centered at \(x\) for the Kobayashi distance by using our results (Theorem 1 and Proposition 1) and Lempert’s results [13] (see Section 3 for the definition of complex \(k\)-extreme points).

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2. **Proof of Theorem 1 and Theorem 2**

Before giving the proof, we recall a notion of Jensen measure. The reader will find a much more study about Jensen measure in [4], [17]. Let \(\Omega \subset \mathbb{C}^n\) be a bounded domain
and let \( \mu \) be a positive, regular Borel measure on \( \overline{\Omega} \). We say that \( B \subseteq \overline{\Omega} \) for continuous plurisubharmonic functions, if \( u(\overline{z}) \leq \int_{\overline{\Omega}} u \, d\mu \) for every function \( u \in \text{Psh}(\Omega) \cap C^0(\overline{\Omega}) \). We denote by \( J^c_\overline{z} \) the set of Jensen measures for continuous plurisubharmonic functions having barycenter \( \overline{z} \). The following theorem is a consequence of Edwards’ theorem.

**Theorem 3** ([9]). Let \( \Omega \subset \mathbb{C}^n \) be a bounded domain, and let \( v \) be a real valued lower semicontinuous function on \( \overline{\Omega} \). Then, for every \( z \in \overline{\Omega} \),

\[
\sup\{u(\overline{z}); u \in \text{Psh}(\Omega) \cap C^0(\overline{\Omega}), u \leq v \text{ in } \overline{\Omega}\} = \inf \left\{ \int_{\overline{\Omega}} v \, d\mu; \mu \in J^c_\overline{z} \right\}.
\]

We first prove the following lemma.

**Lemma 1.** Let \( D \) be a domain in \( \mathbb{C}^n \). Let \( u \in \text{Psh}(D) \cap C^0(D) \). Then \((dd^c u)^n = 0 \) in \( D \) if and only if, for any \( z \in D \) and any connected open neighborhood \( U \subset \subset D \) of \( z \), there exists a Jensen measure \( \mu \in J^c_z \) on \( U \) such that \( u(z) = \int u \, d\mu \) and \( \mu \neq \delta_z \) where \( \delta_z \) is the Dirac mass at \( z \).

**Proof.** Let \( x \in D \) and take the Euclidean open ball \( B(x) \subset \subset D \) of a small radius centered at \( x \). Let

\[
v(z) = \sup\{\phi(z); \phi \in \text{Psh}(B(x)) \cap C^0(\overline{B(x)}), \phi \leq u \text{ on } \partial B(x)\}
\]

for \( z \in \overline{B(x)} \). Then \( v \in \text{Psh}(B(x)) \cap C^0(\overline{B(x)}) \) and \( v \) is the unique solution of the Monge-Ampère equation \((dd^c v)^n = 0 \) in \( B(x) \) such that \( v = u \) on \( \partial B(x) \) (see [2]).

Assume that \((dd^c u)^n = 0 \) in \( D \). Then \( u = v \) in \( \overline{B(x)} \). Take the continuous function \( h \) in \( \overline{B(x)} \) which is harmonic in \( B(x) \) and \( h = u \) on \( \partial B(x) \). Then

\[
u(z) = u(z) = \sup\{\phi(z); \phi \in \text{Psh}(B(x)) \cap C^0(\overline{B(x)}), \phi \leq h \text{ in } \overline{B(x)}\}.
\]

By Theorem 3

\[
u(z) = \inf \left\{ \int_{\overline{B(x)}} h \, d\mu; \mu \in J^c_z \right\}
\]

where \( J^c_z \) is the set of Jensen measures for continuous plurisubharmonic functions having barycenter \( z \) in \( \overline{B(x)} \). For any \( z \), \( J^c_z \) is a compact set in the weak topology and there exists \( \mu \in J^c_z \) such that \( u(x) = \int_{\overline{B(x)}} h \, d\mu \). Since \( u(x) \leq \int_{\overline{B(x)}} u \, d\mu \leq \int_{\overline{B(x)}} h \, d\mu \), we have \( u(x) = \int_{\overline{B(x)}} u \, d\mu \). If \( \mu = \delta_x \), \( u(x) = h(x) \) and \( u = h \) in \( \overline{B(x)} \) by the strong maximum principle of a subharmonic function \( u - h \). Then \( u \) is harmonic in \( B(x) \). Let \( \tau \) be the probability measure invariant under the rotation on \( \partial B(x) \). Then \( \tau \in J^c_z \) and \( u(y) = \int u \, d\tau \). We complete the proof of the necessity.

We next show the sufficiency. Assume that \( u \neq v \) in \( B(x) \). Then

\[
m = \sup_{z \in \overline{B(x)}} \{v(z) - u(z)\}
\]

is positive. Let \( A = \{z \in \overline{B(x)}; v(z) - u(z) = m\} \). Let \( d \) be the Euclidean distance on \( \mathbb{C}^n \) and let \( s = d(A, \partial(\overline{B(x)}) = \inf\{d(z, w); z \in A, w \in \partial B(x)\} \). Since \( u = v \) on \( \partial B(x) \), \( s > 0 \). Take \( y \in A \) such that \( d(y, \partial B(x)) = s \). By the hypothesis, there exists a Jensen measure \( \mu \in J^c_y \) on \( \overline{B(x)} \) such that \( u(y) = \int u \, d\mu \) and \( \mu \neq \delta_y \). Then \( m \geq \int (v - u) \, d\mu \geq v(y) - u(y) = m \). Hence the support of \( \mu \) is contained in \( A \). By
Proposition 4.4 of [8], there exists a positive current $T \neq 0$ of bidimension $(1, 1)$ with compact support such that $dd^c T = \mu - \delta_y$. We define $\psi(z) = |z_1 - x_1|^2 + \cdots + |z_n - x_n|^2$. It follows that 

$$0 < \langle dd^c T, \psi \rangle = \int \psi d\mu - \psi(y).$$

However, the right hand side of the above equation is non-positive by the choice of $y$ and this is a contradiction. Therefore $u = v$ in $B(x)$ and $(dd^c u)^n = 0$. \hfill \Box

**Proof of Theorem 2.** Let $x \in u^{-1}(0)$. By Lemma 1, there exists $\mu \in \mathcal{J}^n_x$ on a small neighborhood of $x$ such that $u(x) = \int u d\mu$ and $\mu \neq \delta_x$. Since $u \geq u(x) = 0$, we have $u = 0$ on $|\mu|$ where $|\mu|$ is the support of $\mu$ and $|\mu| \subset u^{-1}(0)$. Let $\text{ch}(|\mu|)$ be the convex hull of $|\mu|$. Since $u^{-1}(0)$ is convex, $\text{ch}(|\mu|) \subset u^{-1}(0)$. By Proposition 4.4 of [8], there exists a positive current $T \neq 0$ of bidimension $(1, 1)$ with compact support in $D$ such that $dd^c T = \mu - \delta_y$.

Assume that $x$ does not lie in the relative interior of $\text{ch}(|\mu|)$. The Hahn-Banach Theorem implies that there exist a linear function $L : \mathbb{C}^n \to \mathbb{R}$ and $t \in \mathbb{R}$ such that $L > t$ in the relative interior of $\text{ch}(|\mu|)$ and $L(x) \leq t$. Then $0 = \langle dd^c T, L \rangle = \int L d\mu - L(x) > t - t = 0$ and this is a contradiction. Therefore $x$ lies in the relative interior of $\text{ch}(|\mu|)$.

We show that $\text{ch}(|\mu|)$ contains a complex one-dimensional convex set. Assume that $\text{ch}(|\mu|)$ is contained in a totally real convex set. There exists a holomorphic affine coordinate $(w_1, \ldots, w_n)$ such that $\text{ch}(|\mu|) \subset \{w \in \mathbb{C}^n; \Re w_1 = \cdots = \Re w_n = 0\}$. Define a non-negative strictly plurisubharmonic function $g = (\Re w_1)^2 + \cdots + (\Re w_n)^2$. (This is the special case of Lemma 1.2 of [10]). Then $0 < \langle T, dd^c g \rangle = \int g d\mu - g(x)$ since $T \neq 0$. However, the right hand side of the above equation is equal to 0 (note that $x \in \text{ch}(|\mu|)$). This is a contradiction. Hence $x$ lies in the relative interior of a complex one-dimensional convex set contained in $u^{-1}(0)$.

**Remark 1.** In the above proof, let $|T|$ be the support of $T$ and let $\text{ch}(|T|)$ be the convex hull of $|T|$. Then $\text{ch}(|\mu|) = \text{ch}(|T|)$ since $|T| \subset \widetilde{\mu}|_D = \{z \in D; |f(z)| \leq \sup_{|\mu|} |f| \text{ for any } f \in \mathcal{O}(D)\}$ (see [11], Proposition 4.3 of [8]) and 

$$\text{ch}(|\mu|) \subset \text{ch}(|T|) \subset \text{ch}(|\widetilde{\mu}|_D) = \text{ch}(|\mu|).$$

Assume that $u \in C^2(D)$. Then $\langle T, dd^c u \rangle = 0$. Conversely, let $S \neq 0$ be a positive current of bidimension $(1, 1)$ with compact support in $D$ such that $dd^c S + \delta_x$ is a positive measure and $\langle S, dd^c u \rangle = 0$. Then the convex hull of the support of $S$ contains the required complex one-dimensional convex set.

We need the following lemma to prove Theorem 2.

**Lemma 2.** Under the same hypotheses as in Theorem 2 let $x \in S_u(r)$ which is not contained in the interior of $\overline{B}_u(r)$. Let $M \subset \mathbb{C}^n$ be a complex $k$-dimensional affine subspace through $x$ such that $M$ is tangent to $\overline{B}_u(r)$, that is, the intersection of $M$ and the interior of $\overline{B}_u(r)$ is empty. Then $x$ lies in the relative interior of a complex one-dimensional convex set contained in $S_u(r) \cap M$.

Note that the above $M$ always exists since $\overline{B}_u(r)$ is convex.

**Proof.** By taking a suitable holomorphic affine coordinate $(z_1, \ldots, z_n)$, we may assume that $x = (0, \ldots, 0)$ and $M$ is defined by $z_{k+1} = \cdots = z_n = 0$. Let $z' = (z_1, \ldots, z_k)$ and $z'' = (z_{k+1}, \ldots, z_n)$. Take open balls $0 \in U \subset \mathbb{C}^k$ and $0 \in V \subset \mathbb{C}^{n-k}$ of small
radii such that $U \times V$ is contained in $D$. We show that $(dd^cu|_{M\cap D})^k = 0$ in $U \times \{0\}$. Let $a \in V$ and let $M(a)$ be the intersection of $D$ and the complex $k$-dimensional affine subspace defined by $z'' = a$. Let $\phi$ be a non-negative test function in $U$ and $\psi$ be a non-negative test function in $V$ such that $\psi(0) > 0$. We take a decreasing sequence of smooth plurisubharmonic functions $\{u_i\}$ on a neighborhood of the support of $\phi(z')\psi(z'')$ in $U \times V$ such that $u_i$ converges to $u$. Then $(dd^cu_i)^k \to (dd^cu)^k$ ($i \to \infty$) in the weak topology on the space of currents (see [3]). By Fubini’s theorem, we have that

$$0 = \int_{U \times V} \phi(z')\psi(z'')(dd^cu)^k \bigg|_{z'' = 0} \bigg\{ \frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j \bigg\} = \lim_{i \to \infty} \int_{U \times V} \phi(z')\psi(z'')(dd^cu_i)^k \bigg|_{z'' = 0} \bigg\{ \frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j \bigg\} = \lim_{i \to \infty} \int_{V} \psi(z'') \bigg\{ \frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j \bigg\} \int_{M(z'')} \phi(z')(dd^cu_i|_{M(z'')})^k.$$  

Since $u$ is continuous in $D$, $u$ is uniformly continuous in a neighborhood of the support of $\phi(z')\psi(z'')$. If we consider $(dd^cu_i|_{M(z'')})^k$ as a current in $U$, $(dd^cu|_{M(z'')})^k \to (dd^cu|_{M(z'')})^k$ when $z'' \to z''_0 \in V$ in the weak topology (see Chapter III of [6]). Therefore $\int_{M(z'')} \phi(z')(dd^cu|M(z''))^k$ is a non-negative continuous function of $z''$ and, by Lebesgue’s dominated convergence theorem, we have that

$$\int_{V} \psi(z'') \bigg\{ \frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j \bigg\} \int_{M(z'')} \phi(z')(dd^cu|M(z''))^k = 0.$$  

Since $\psi(0) > 0$, $\int_{M\cap D} \phi(z')(dd^cu|_{M\cap D})^k = 0$ for any non-negative test function $\phi$. Hence $(dd^cu|_{M\cap D})^k = 0$ in $U \times \{0\}$.

Since $\overline{B}_u(r)$ is convex and $M$ is tangent to $\overline{B}_u(r)$, $u|_{M\cap D}$ attains its minimum $r$ at $x$ and $u|_{M\cap D}(r)$ is convex. By Theorem [7], there exists a complex one-dimensional convex set in $S_u(r) \cap M$ which contains $x$ in its relative interior. This completes the proof. \hfill $\square$

**Proof of Theorem [7]** We prove Theorem [1] by induction on $l = n - k$ ($l = 1, \ldots, n - 1$). Let $x \in S_u(r)$. If $x$ lies in the interior of $\overline{B}_u(r)$, the theorem holds. Hence we may assume that $x$ is not contained in the interior of $\overline{B}_u(r)$. If $l = 1$, then $k = n - 1$ and the statement follows by Lemma [2] For $l > 1$, the inductive hypothesis implies that there exists a complex $(n - k - 1)$-dimensional convex set $C_1$ in $S_u(r)$ which contains $x$ in its relative interior. Let $H$ be a complex affine hyperplane through $x$ and which is tangent to $\overline{B}_u(r)$. Take a complex $k$-dimensional affine subspace $M$ of $H$ such that $x \in M$ and $M \cap C_1 = \{x\}$. By Lemma [2] there exists a complex one-dimensional convex set $C_2 \subset S_u(r)$ which contains $x$ in its relative interior. Then the convex hull $\text{ch}(C_1 \cup C_2)$ is contained in $\overline{B}_u(r)$ since $\overline{B}_u(r)$ is convex and $\text{ch}(C_1 \cup C_2)$ is contained in $S_u(r)$ by the maximum principle. \hfill $\square$

We now show the following proposition announced in Section 1.

**Proposition 1.** Let $D$ be a domain in $\mathbb{C}^n$ and $u \in \text{Psh}(D) \cap C^0(D)$. Let $k \in \{1, \ldots, n\}$. Assume that any point of $D$ lies in a complex $(n - k + 1)$-dimensional locally closed
submanifold of $D$ without boundary on which $u$ is pluriharmonic. Then $(dd^c u)^k = 0$ in $D$.

Proof. We show

$$
\int_D \phi(z)(dd^c u)^k \prod_{i=1}^{n-k} \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_i = 0
$$

for any non-negative test function $\phi$ in $D$ and any $1 \leq j_1 < \cdots < j_{n-k} \leq n$. Without loss of generality, we may assume that $j_1 = k + 1, \ldots, j_{n-k} = n$. Let $U \subset \mathbb{C}^k, V \subset \mathbb{C}^{n-k}$ be open subsets such that $U \times V \subset D$. We may assume that $D = U \times V$. Let $z' = (z_1, \ldots, z_k)$ and $z'' = (z_{k+1}, \ldots, z_n)$ and let $p : \mathbb{C}^n \to \mathbb{C}^{n-k}$ be the projection map such that $p(z) = z''$. Put $\tilde{U}(z'') = (U \times V) \cap p^{-1}(z'')$ for $z'' \in V$. Then, by the same argument as in the proof of Lemma 2 it follows that

$$
\int_{U \times V} \phi(z)(dd^c u)^k \prod_{i=k+1}^{n} \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_i = \int_{V} \prod_{i=k+1}^{n} \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_i \int_{\tilde{U}(z'')} \phi(z', z'') |(dd^c u)|_{\tilde{U}(z'')}^k.
$$

Hence it is enough to show that $(dd^c u)|_{\tilde{U}(z'')}^k = 0$ in $U(z'')$ for any $z'' \in V$. Let $x \in U(z'')$. Then $x$ lies in a complex $(n-k+1)$-dimensional locally closed submanifold $N \subset D$ without boundary on which $u$ is pluriharmonic. It holds that $N \cap U(z'')$ contains an irreducible complex curve $R$ through $x$. Let $\pi : \tilde{R} \to R$ be a resolution of singularities of $R$. Then $\pi^* u$ is harmonic in $\tilde{R}$. Let $\tilde{x} \in \tilde{R}$ such that $\pi(\tilde{x}) = x$. Let $\varphi$ be a continuous function from the closed unit disk $\overline{\Delta}$ in $\mathbb{C}$ to a small neighborhood of $\tilde{x}$ in $\tilde{R}$ such that $\varphi$ is holomorphic in the unit disk $\Delta$ and $\varphi(0) = \tilde{x}$. Let $\lambda$ be the probability measure invariant under rotations on $\partial \Delta$. Then $\mu = \pi_* \varphi_* \lambda$ is a Jensen measure with barycenter $x$ on $U(z'')$ such that $\int u d\mu = u(x)$. By Lemma 1 we have $(dd^c u)|_{\tilde{U}(z'')}^k = 0$ in $U(z'')$.

Corollary 1. Let $\Omega \subset \mathbb{C}^n$ be a tube domain and let $\omega \subset \mathbb{R}^n$ be a domain such that $\Omega = \{z \in \mathbb{C}^n; \Re z \in \omega\}$. Let $u \in \Psh(\Omega) \cap C^0(\Omega)$ such that $u(z)$ is independent of $\Im z$ and let $\tilde{u} \in C^0(\omega)$ such that $\tilde{u}(\Re z) = u(z)$. Let $k \in \{1, \ldots, n\}$.

1. Assume that $(dd^c u)^k = 0$ in $\Omega$. Then, for any point of $x \in \omega$, there exists a real $(n-k)$-dimensional convex set $C$ in $\omega$ such that the relative interior of $C$ contains $x$ and $\tilde{u}$ is constant on $C$.

2. Assume that any point of $\omega$ lies in the relative interior of a real $(n-k+1)$-dimensional convex set on which $\tilde{u}$ is affine. Then $(dd^c u)^k = 0$ in $\Omega$.

Proof. Since the statements are local, we may assume that $\Omega$ and $\omega$ are convex. By the hypothesis, $u$ and $\tilde{u}$ are convex functions. Hence (1) and (2) follow immediately from Theorem 1 and Proposition 1.

Let $D$ be a Reinhardt domain in $(\mathbb{C}^*)^n$. Then we have statements analogous to Corollary 1 for $u \in \Psh(D) \cap C^0(D)$ such that $u(z)$ is independent of $\arg(z)$ for $z \in D$.

Example 1. Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and $u \in \Psh(D) \cap C^0(D)$. Define

$$
U(z_1, \ldots, z_n) = \sup_{0 \leq \theta_1, \ldots, \theta_n \leq 2\pi} u(e^{\sqrt{-1}\theta_1} z_1, \ldots, e^{\sqrt{-1}\theta_n} z_n)
$$

for all $z_1, \ldots, z_n \in D$. Then $U$ is a convex function on $D$ and $(dd^c U)^k = 0$ in $D$.
for $z \in D$. Then $U \in \text{Psh}(D) \cap C^0(D)$ and $U(z)$ is independent of $\arg(z)$. Assume that $(dd^c U)^k = 0$ in $D$. Then, for any $x \in D$ such that $x_i \neq 0$ for $1 \leq i \leq n$, there exists a real $(n - k + 1)$-dimensional logarithmically convex set $C \subset \{(|z_1|, \ldots, |z_n|) \in \mathbb{R}^n; z \in D\}$ such that the relative interior of $C$ contains $(|x_1|, \ldots, |x_n|)$ and $U$ is constant on $\{(z_1, \ldots, z_n) \in D; (|z_1|, \ldots, |z_n|) \in C\}$.

3. BALLS FOR THE KOBYASHI DISTANCE IN A BOUNDED CONVEX DOMAIN

Throughout this section $D$ will denote a bounded convex domain in $\mathbb{C}^n$. We define the complex $k$-extreme point of a convex set.

**Definition 1.** Let $C \subset \mathbb{C}^n$ be a convex set. A point $z \in C$ is complex $k$-extreme point $(0 \leq k \leq n)$ if $z$ lies in the relative interior of a complex $k$-dimensional convex set within $C$, but not a complex $(k + 1)$-dimensional convex set within $C$. We denote by $E^k(C)$ the set of complex $k$-extreme points of $C$.

Let $\overline{B}_D(x,r) \subset D$ denote the closed ball of radius $r \geq 0$ centered at $x$ for the Kobayashi distance of $D$. Recall that $\overline{B}_D(x,r)$ is convex. In this section, we show a relation between the supports of Monge-Ampère currents $(dd^c K_{D,x})^k, (dd^c e^{K_{D,x}})^k$ and $E^k(\overline{B}_D(x,r))$. We define

$$E^k_{x} = \bigcup_{i=0}^{k-1} \bigcup_{r \geq 0} E^i(\overline{B}_D(x,r))$$

for $k = 1, \ldots, n$.

For example, if $D = \{z \in \mathbb{C}^n; |z_1| < 1, \ldots, |z_n| < 1\}$ is a polydisk, $E^0_{x}$ is the set of points $z \in D$ where there exists positive integers $1 \leq i(1) < \cdots < i(n-k+1) \leq n$ such that $|z_{i(1)}| = \cdots = |z_{i(n-k+1)}| = \max\{|z_1|, \ldots, |z_n|\}$. Let $k_D$ be the Kobayashi distance on $D$. We note that

$$D \setminus E^k_x = \{z \in D; z \text{ lies in the relative interior of a complex } k\text{-dimensional convex set contained in } \overline{B}_D(x,k_D(x,z))\}.$$

**Remark 2.** The complex $k$-dimensional convex set appearing in the right hand side of the above equation is contained in the Kobayashi sphere of radius $k_D(x,z)$ centered at $x$ by the maximum principle.

Let $T$ be a current in $D$. We denote by $|T|$ the support of $T$. By results of [3] and [4], currents $(dd^c K_{D,x})^k, (dd^c e^{K_{D,x}})^k$ $(1 \leq k \leq n)$ are well-defined in $D$.

**Theorem 4.** Let $D \subset \mathbb{C}^n$ be a bounded convex domain. Then:

(i) $|(dd^c e^{K_{D,x}})^{n-k+1}| \subset E^k_x \cap D \subset |(dd^c K_{D,x})^n|$ for $1 \leq k \leq n - 2$.

(ii) $|dd^c K_{D,x}| = E^{(n-1)}_x \cap D$.

**Remark 3.** A current $dK_{D,x} \wedge d^c K_{D,x} \wedge (dd^c K_{D,x})^k$ $(1 \leq k \leq n - 1)$ is well-defined in $D \setminus \{x\}$. Since

$$(dd^c e^{K_{D,x}})^k = e^{K_{D,x}}(dd^c K_{D,x})^k + ke^{K_{D,x}}dK_{D,x} \wedge d^c K_{D,x} \wedge (dd^c K_{D,x})^{k-1},$$

it follows $|(dd^c e^{K_{D,x}})^k| = |(dd^c K_{D,x})^k| \cup |dK_{D,x} \wedge d^c K_{D,x} \wedge (dd^c K_{D,x})^{k-1}|$ in $D \setminus \{x\}$. 
A CONVEX LEVEL SET AND THE MONGE-AMPERE CURRENT

Proof. By Theorem 1, it follows that $D \setminus (dd^c K_{D,x})^{n-k} \subset D \setminus E_x^<k$ for $1 \leq k \leq n - 1$. For $z \in D \setminus E_x^<k$, there exists a complex $k$-dimensional convex set which contains $z$ in its relative interior on which $e^{K_{D,x}}$ is constant. Then $D \setminus E_x^<k \subset D \setminus (dd^c e^{K_{D,x}})^{n-k+1}$ for $1 \leq k \leq n - 2$ by Proposition 1. This completes the proof of (i).

Next, we prove (ii). Since we have already proved that $E_x^{< (n-1)} \cap D \subset |dd^c K_{D,x}|$, it is enough to show $D \setminus E_x^{< (n-1)} \subset D \setminus |dd^c K_{D,x}|$.

Before giving the proof, we recall that a complex geodesic (for the Kobayashi distance) of a convex domain $D$ is a holomorphic map $\varphi$ from the unit disk $\Delta \subset \mathbb{C}$ into $D$ such that $k_\Delta(\zeta_1, \zeta_2) = k_D(\varphi(\zeta_1), \varphi(\zeta_2))$ for every pair of points $\zeta_1, \zeta_2 \in \Delta$. For any two points in a convex domain $D$, there exists a complex geodesic which through those (see [13] and [15]).

Let $y \in D \setminus E_x^{< (n-1)}$. Let $\varphi : \Delta \to D$ be a complex geodesic which through $x$ and $y$ such that $\varphi(0) = x$. Then there exists Lempert’s projection $\rho$, that is, a holomorphic retraction $\rho : D \rightarrow \varphi(\Delta) \subset D$ such that $\rho$ is the identity map on $\varphi(\Delta)$ and $\rho^{-1}(z)$ is the intersection of $D$ and the complex affine hyperplane which is tangent to $\overline{BD(x, k_D(x, z))}$ for $z \in \varphi(\Delta)$ (see [13] and [15]). Let $W \subset D \setminus E_x^{< (n-1)}$ be a small open neighborhood of $y$ such that $\rho^{-1}(z) \cap W$ is connected for any $z \in \varphi(\Delta) \cap W$. Let $S_D(x, t)$ be the Kobayashi sphere of radius $t > 0$ centered at $x$. We show that

$$\rho^{-1}(z) \cap W \subset S_D(x, k_D(x, z))$$

for any $z \in \varphi(\Delta) \cap W$. Since $\rho^{-1}(z) \cap W \cap S_D(x, k_D(x, z))$ is non-empty and closed in $\rho^{-1}(z) \cap W$, it is enough to show that $\rho^{-1}(z) \cap W \cap S_D(x, k_D(x, z))$ is open in $\rho^{-1}(z) \cap W$. Let $\gamma \in \rho^{-1}(z) \cap W \cap S_D(x, k_D(x, z))$. We have $k_D(x, z) = k_D(x, \gamma)$. There exists a complex $(n - 1)$-dimensional convex set $C \subset S_D(x, k_D(x, z))$ which contains $\gamma$ in its relative interior (see Remark 2). Note that $\varphi$ is a biholomorphic map from $\Delta$ to $\varphi(\Delta)$. By the distance decreasing property of the Kobayashi distance, $\varphi^{-1}(\rho(C)) \subset \{ \zeta \in \mathbb{C} ; |\zeta| \leq \tanh k_D(x, z) \}$. Since $|\varphi^{-1}(\rho(\gamma))| = \tanh k_D(x, z)$, the maximum principle implies that $\varphi^{-1}(\rho(C))$ is a point and $C$ is contained in $\rho^{-1}(z) \cap S_D(x, k_D(x, z))$. Hence $\rho^{-1}(z) \cap W \cap S_D(x, k_D(x, z))$ is open in $\rho^{-1}(z) \cap W$ and $\rho^{-1}(z) \cap W \subset S_D(x, k_D(x, z))$. Then $K_{D,x}$ is constant on $\rho^{-1}(z) \cap W$. Therefore $K_{D,x}(y) = K_{D,x}(\rho(y))$ for $y \in W$. Since $K_{D,x}$ is harmonic on $\varphi(\Delta) \setminus \{ x \}$, $K_{D,x} \circ \rho$ is pluriharmonic and $dd^c K_{D,x} = 0$ in $W$. This completes the proof of (ii). \qed

4. THE CASE OF A CONVEX BALANCED DOMAIN

Let $D \subset \mathbb{C}^n$ be a bounded convex balanced domain, that is, let $D$ be a bounded convex domain such that $\lambda z \in D$ for any $z \in D$ and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. Let $\delta : \mathbb{C}^n \to [0, \infty)$ be the Minkowski function of $D$. Then it is known that $\log \delta \in \text{Psh}(D)$ and $\log \delta$ is the pluricomplex Green function of $D$ with logarithmic pole at $\{0\}$.

Proposition 2. Let $D$ be a bounded convex balanced domain in $\mathbb{C}^n$. Then $E_0^{< k} \cap D = |(dd^c \log \delta)^{n-k}|$ for $1 \leq k \leq n - 1$.

Proof. By Theorem 4 and Remark 3, it follows that

$$|d \log \delta \wedge d^c \log \delta \wedge (dd^c \log \delta)^{n-k}| \subset E_0^{< k} \cap D \subset |(dd^c \log \delta)^{n-k}|$$
for \(1 \leq k \leq n - 1\). By the same argument as that used on the proof of Proposition 2 of [16], \(|(dd^c \log \delta)^{n-k}| = |d \log \delta \wedge d^c \log \delta \wedge (dd^c \log \delta)^{n-k}|\). This completes the proof. \(\square\)

**Remark 4.** We can also derive Proposition 2 from Proposition 1 and Theorem 4.

Let \(A(\overline{D})\) be the uniform algebra consists of all continuous functions on \(\overline{D}\) which can be approximated uniformly on \(\overline{D}\) by continuous functions which are holomorphic on \(D\). Let \(\partial S\overline{D}\) denote the Shilov boundary of \(\overline{D}\). Note that, if \(D\) is a bounded convex domain, \(A(\overline{D})\) is equal to the uniform algebra consists of all continuous functions on \(\overline{D}\) which can be approximated uniformly on \(\overline{D}\) by continuous functions which are holomorphic on a neighborhood of \(\overline{D}\). If \(D\) is a bounded convex balanced domain, \(|(dd^c \log \delta)^{n-1}|\) is equal to \(\bigcup_{0 \leq r < 1} r \partial S\overline{D}\) by Proposition 2 of [16]. Hence Proposition 2 implies the fact that, in a complex normed vector space with finite dimension, the closure of the set of the complex extreme points of the closed unit ball is equal to the Shilov boundary of it.

**Example 2.** Let \(P = \{z \in \mathbb{C}^n; |z_1| < 1, \ldots, |z_n| < 1\}\) be a polydisk and let \(\delta(z) = \max\{|z_1|, \ldots, |z_n|\}\) be the Minkowski function of \(P\). Then \(|(dd^c \log \delta)^i| (1 \leq i \leq n-1)\) is the set of points \(z \in P\) where there exist positive integers \(1 \leq i(1) < \cdots < i(k+1) \leq n\) such that \(|z_{i(1)}| = \cdots = |z_{i(k+1)}| = \max\{|z_1|, \ldots, |z_n|\}\).

**Example 3.** Let \(1 < n_1 < n_2\) be positive integers and let \(B_j\) be the Euclidean unit ball in \(\mathbb{C}^{n_j}\) \((j = 1, 2)\). Let \(D = B_1 \times B_2\) and let

\[
\delta(z, w) = \max \left\{ \sqrt{|z_1|^2 + \cdots + |z_{n_1}|^2}, \sqrt{|w_1|^2 + \cdots + |w_{n_2}|^2} \right\} \quad (z \in B_1, w \in B_2)
\]

be the Minkowski function of \(D\). Then

\[
|(dd^c \log \delta)| = \cdots = |(dd^c \log \delta)^{n_1-1}| = D,
\]

\[
|(dd^c \log \delta)^{n_1}| = \cdots = |(dd^c \log \delta)^{n_2-1}| = \{(z, w) \in B_1 \times B_2; \|z\| \leq \|w\|\}, \quad \text{and}
\]

\[
|(dd^c \log \delta)^{n_2}| = \cdots = |(dd^c \log \delta)^{n_1+n_2-1}| = \{(z, w) \in B_1 \times B_2; \|z\| = \|w\|\}.
\]

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