Contact 5-manifolds with $SU(2)$-structure

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Abstract

We consider 5-manifolds with a contact form arising from a hypo structure [9], which we call hypo-contact. We provide existence conditions for such a structure on an oriented hypersurface of a 6-manifold with a half-flat $SU(3)$-structure. For half-flat manifolds with a Killing vector field $X$ preserving the $SU(3)$-structure we study the geometry of the orbits space. Moreover, we describe the solvable Lie algebras admitting a hypo-contact structure. This allows us to exhibit examples of Sasakian $\eta$-Einstein manifolds, as well as to prove that such structures give rise to new metrics with holonomy $SU(3)$ and $G_2$.

1 Introduction

Recently, Conti and Salamon introduced in [9] hypo structures on 5-manifolds as a generalization in dimension 5 of Sasakian-Einstein metrics; indeed, Sasakian-Einstein metrics correspond to Killing spinors and hypo structures are induced by generalized Killing spinors. In terms of differential forms, a hypo structure on a 5-manifold $N$ is determined by a quadruplet $(\eta, \omega_i, 1 \leq i \leq 3)$ of differential forms, where $\eta$ is a nowhere vanishing 1-form and $\omega_i$ are 2-forms on $N$ satisfying certain relations (see (3) in Section 2). If the forms $\eta$ and $\omega_i$ satisfy

$$d\eta = -2\omega_3, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_2 = -3\eta \wedge \omega_1,$$

then $N$ is a Sasakian-Einstein manifold, that is, a Riemannian manifold such that $N \times \mathbb{R}$ with the cone metric is Kähler and Ricci flat [3]. Thus $N \times \mathbb{R}$ has holonomy contained in $SU(3)$ or, equivalently, it has an integrable $SU(3)$-structure which means that there is an almost Hermitian structure, with Kähler form $F$, and a $(3,0)$-form $\Psi = \Psi_+ + i\Psi_-$ on $N^5 \times \mathbb{R}$ satisfying $dF = d\Psi_+ = d\Psi_- = 0$. In the general case of a hypo structure, in [9] it is proved that a real analytic hypo structure on a real analytic 5-manifold $N$ can be lifted to an integrable $SU(3)$-structure on $N \times I$, for some open interval $I$ or equivalently that $(\eta, \omega_i, 1 \leq i \leq 3)$ belongs to a one-parameter family of hypo structures $(\eta(t), \omega_i(t), 1 \leq i \leq 3)$ satisfying the evolution equations [9] given in Section 2. Without assuming real analyticity no general result is known. Conversely, any oriented hypersurface of a 6-manifold with an integrable $SU(3)$-structure is naturally endowed with a hypo structure (see Section 2 for details).

In general, for a hypo 5-manifold the 1-form $\eta$ is not a contact form. In this paper we deal with 5-manifolds $N$ having a hypo-contact structure, that is, a hypo structure $(\eta, \omega_1, \omega_2, \omega_3)$ such that $d\eta = -2\omega_3$, so $\eta$ is a contact form on $N$. Such structures were considered by Conti in [8] and by Bedulli and Vezzoni in [2], where an explicit expression for the Ricci and scalar curvature is given in terms of torsion forms and its derivatives.

If we weaken the integrability condition of the $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ on the 6-manifold $M$ to be half-flat in the sense of [7], i.e. $F \wedge F$ and $\Psi_+$ are closed, Hitchin in [13] proved that there is a $G_2$-structure on $M \times I$ with holonomy contained in $G_2$ if the half-flat structure $(F, \Psi_+, \Psi_-)$ is such that certain evolution equations admit a solution $(F(t), \Psi_+(t), \Psi_-(t))$, for all real parameter $t$ lying in some interval $I$, with $F(0) = F$, $\Psi_+(0) = \Psi_+$ and $\Psi_-(0) = \Psi_-$. Regarding hypo-contact structures, in Theorem [25] we provide conditions which imply that there is a hypo-contact structure on any oriented hypersurface $f: N \to M$ of a half-flat manifold $M$; and when
M has a Killing vector field preserving the $SU(3)$-structure, we study the geometry of the orbits space. Moreover, in Proposition 2.2 we show how to lift a hypo structure on a 5-manifold $N$ to a half-flat structure on the total space of a $G$-circle bundle over $N$.

Our main results concern solvable Lie groups of dimension 5 with a left-invariant hypo-contact structure. In particular, using such structures and solving the corresponding evolution equations, we construct new metrics with holonomy $SU(3)$ and $G_2$. In Section 3 the classification of solvable Lie algebras with a hypo-contact structure is given, showing the following theorem.

**Theorem 1.1** A 5-dimensional solvable Lie algebra admits a hypo-contact structure if and only if it is isomorphic to one of the following:

- $\mathfrak{b}_1 : [X_1, X_4] = [X_2, X_3] = X_5$;
- $\mathfrak{b}_2 : \frac{1}{5}[X_1, X_5] = [X_2, X_4] = X_1, [X_2, X_5] = X_2, [X_3, X_5] = X_3, [X_4, X_5] = -3X_4$;
- $\mathfrak{b}_3 : \frac{1}{7}[X_1, X_4] = [X_2, X_3] = X_1, [X_2, X_4] = [X_3, X_5] = X_2, [X_2, X_5] = -[X_3, X_4] = -X_3$;
- $\mathfrak{b}_4 : [X_1, X_4] = X_1, [X_2, X_5] = X_2, [X_3, X_4] = [X_3, X_5] = -X_3$;
- $\mathfrak{b}_5 : [X_1, X_5] = [X_2, X_4] = X_1, [X_3, X_4] = X_2, [X_3, X_5] = -X_3, [X_4, X_5] = X_4$.

Therefore, all of them are irreducible and $\mathfrak{b}_1$ is the unique nilpotent Lie algebra having a hypo-contact structure. In [11] Diatta gives a list of solvable contact Lie algebras in dimension 5 and he shows that, up to isomorphism, there are three nilpotent contact Lie algebras of dimension 5. By [9] only two of these nilpotent Lie algebras have hypo structures. Since the center of the Lie algebras $\mathfrak{b}_2, \ldots, \mathfrak{b}_5$ is trivial, we conclude that there are many 5-dimensional solvable contact Lie algebras with no hypo-contact structures.

In [9 Theorem 14] it is proved that a hypo structure is Sasakian if and only if it is $\eta$-Einstein [5, 15]. The Lie algebras described in Theorem 1.1 cannot be Einstein because they are solvable and contact [11]. In Section 3 we study which of these Lie algebras are $\eta$-Einstein or, equivalently, Sasakian. We show that the only 5-dimensional solvable Lie algebras admitting a hypo-contact $\eta$-Einstein structure, are $\mathfrak{b}_1$ and $\mathfrak{b}_3$ (Proposition 4.2). Concerning contact Calabi-Yau structures recently introduced in [16], in Proposition 4.6 it is proved that there are no 5-dimensional solvable non-nilpotent Lie algebras admitting such a structure.

In Section 4 we solve the Conti-Salamon evolution equations for the left-invariant hypo-contact structure on the simply connected solvable Lie group $H_i$ ($1 \leq i \leq 5$) whose Lie algebra is $\mathfrak{b}_i$. More concretely we obtain the following result.

**Theorem 1.2** Any left-invariant hypo-contact structure on the 5-dimensional solvable Lie group $H_i$ ($1 \leq i \leq 5$) gives rise to a metric with holonomy $SU(3)$ on $H_i \times I$, for some open interval $I$.

This theorem is an existence result; in fact, our metric is explicit only for the left-invariant hypo-contact structure on the nilpotent Lie group $H_1$, recovering in this way the well-known example obtained in [12].

Finally, Section 6 is devoted to show the existence of new metrics with holonomy $G_2$. To this end, using Proposition 2.2 we consider the circle bundles over $H_i$ ($1 \leq i \leq 5$) whose total space $K_i$ has a half-flat structure induced by the left-invariant hypo-contact structure on $H_i$. Solving the Hitchin evolution equations, we prove the following theorem.

**Theorem 1.3** The half-flat structure on $K_i$ ($i = 1, 4, 5$) gives rise to a metric with holonomy $G_2$ on $K_i \times I$, for some open interval $I$.

We must notice that the above metric on $K_1 \times I$ agrees with the one obtained in [6]. However, as far as we know, the other metrics on $K_i \times I$ ($i = 4, 5$) are new and, as we explain in Section 6 they can be considered as a “deformation” of the metric with holonomy $G_2$ found in [6].

2 Hypo-contact structures

In this section, we study 5-manifolds with a *hypo-contact* structure, that is, a hypo structure in the sense of [9] carrying a contact form. We prove that there exists such a structure on any oriented hypersurface
of an special half-flat manifold, namely, such that the Kähler form is preserved by the normal vector field and its differential is equal two times the real part of the (3,0)-form. First we need to recall some properties of hypo structures on 5-manifolds.

Let $N$ be a 5-manifold with an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$, that is to say \cite{9}, $\eta$ is a nowhere vanishing 1-form and $\omega_i$ are 2-forms on $N$ satisfying

\begin{equation}(1)\quad \omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0,
\end{equation}

for some nowhere vanishing 4-form $v$, and

\begin{equation}(2)\quad i_X \omega_3 = i_Y \omega_1 \Rightarrow \omega_2(X,Y) \geq 0,
\end{equation}

where $i_X$ denotes the contraction by $X$.

An $SU(2)$-structure determined by $(\eta, \omega_1, \omega_2, \omega_3)$ is called hypo if the following equations

\begin{equation}(3)\quad d\omega_3 = 0, \quad d(\eta \wedge \omega_1) = 0, \quad d(\eta \wedge \omega_2) = 0
\end{equation}

are satisfied \cite{9}.

**Definition 2.1** We say that an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on a manifold $N$ is hypo-contact if it satisfies

\begin{equation}\label{eq:def21} d\eta = -2\omega_3, \quad d(\eta \wedge \omega_1) = 0, \quad d(\eta \wedge \omega_2) = 0. \end{equation}

Regarding the intrinsic torsion of these $SU(2)$-structures, we recall that in Proposition 10 of \cite{9}, it is proved that the hypo structures are the $SU(2)$-structures whose intrinsic torsion takes values in the space $2\mathbb{R} \oplus \Lambda^2(\mathbb{R}^4)^* \oplus 3\Lambda^2(\mathbb{R}^4)^*$. Now, one can check that the hypo-contact structures are the $SU(2)$-structures whose intrinsic torsion lies in the $SU(2)$-module $2\mathbb{R} \oplus 2\Lambda^2(\mathbb{R}^4)^*$.

An $SU(2)$-structure on $N$ induces an $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ on $N \times \mathbb{R}$ defined by

\begin{equation}(4)\quad F = \omega_3 + \eta \wedge dt, \quad \Psi = \Psi_+ + i\Psi_- = (\omega_1 + i\omega_2) \wedge (\eta + i dt),
\end{equation}

where $t$ is a coordinate on $\mathbb{R}$. Vice versa, let $f : N \longrightarrow M$ be an oriented hypersurface of a 6-manifold $M$ with an $SU(3)$-structure $(F, \Psi_+, \Psi_-)$, and denote by $U$ the unit normal vector field. Then $N$ has an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ given by

\begin{equation}(5)\quad \eta = -i_U F, \quad \omega_3 = f^* F, \quad \omega_1 = i_U \Psi_-, \quad \omega_2 = -i_U \Psi_+.
\end{equation}

If $M$ has holonomy contained in $SU(3)$, that is, if the $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ is integrable or, equivalently,

\begin{equation}\label{eq:def21} dF = d\Psi_+ = d\Psi_- = 0,
\end{equation}

any oriented hypersurface $N$ of $M$ is naturally endowed with a hypo structure \cite{9}. Indeed, the conditions $dF = d\Psi_+ = d\Psi_- = 0$ imply that the induced $SU(2)$-structure on $N$ defined by \ref{eq:def21} satisfies \ref{eq:def21}. If in addition the Lie derivative $L_U F$ is equal to $2 f^* (F)$, then the induced $SU(2)$-structure is hypo-contact.

Concerning the converse, Conti and Salamon \cite{9} prove that a real analytic hypo structure on a real analytic 5-manifold $N$ can be lifted to an integrable $SU(3)$-structure on $N \times I$, for some open interval $I$. More precisely, they show that if $(\eta, \omega_1, \omega_2, \omega_3)$ belongs to a one-parameter family of hypo structures $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ satisfying the evolution equations

\begin{equation}(6)\quad \begin{cases} \partial_t \omega_3 = -d\eta, \\
\partial_t (\omega_2 \wedge \eta) = d\omega_1, \\
\partial_t (\omega_1 \wedge \eta) = -d\omega_2, 
\end{cases}
\end{equation}

for all $t$ lying in some open interval $I$, then the $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ on $N \times I$ given by

\begin{equation}\label{eq:def21} F = \eta(t) \wedge dt + \omega_3(t), \quad \Psi = \Psi_+ + i\Psi_- = (\omega_1(t) + i\omega_2(t)) \wedge (\eta(t) + i dt))
\end{equation}
is integrable.

In Section 5 we shall back to the equations (10). Now, we weaken the integrability condition of the $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ on $M$ to be half-flat in the sense of [7], that is $d(F \wedge F) = d\Psi_+ = 0$. First we show how to lift a hypo structure on a 5-manifold $N$ to a half-flat structure on the total space of a circle bundle over $N$.

**Proposition 2.2** Let $N$ be a 5-manifold equipped with a hypo structure $(\eta, \omega_1, \omega_2, \omega_3)$. For any integral closed 2-form $\Omega$ on $N$ annihilating both $\omega_3$ and $\cos \theta \omega_1 + \sin \theta \omega_2$ for some $\theta$, there is a principal circle bundle $\pi: M \to N$ with connection form $\rho$ such that $\Omega$ is the curvature of $\rho$ and such that the $SU(3)$-structure $(F^\theta, \Psi_+^\theta, \Psi_-^\theta)$ on $M$ given by

\[
F^\theta = \pi^*(\cos \theta \omega_1 + \sin \theta \omega_2) + \pi^*(\eta) \wedge \rho,
\]

\[
\Psi_+^\theta = \pi^*((-\sin \theta \omega_1 + \cos \theta \omega_2) \wedge \eta) - \pi^*(\omega_3) \wedge \rho,
\]

\[
\Psi_-^\theta = \pi^*(- \sin \theta \omega_1 + \cos \theta \omega_2) \wedge \rho + \pi^*(\omega_3) \wedge \pi^*(\eta),
\]

is half-flat.

**Proof:** Since $d\rho = \pi^*(\Omega)$, a simple calculation shows that

\[
d(F^\theta \wedge F^\theta) = -2\pi^*(\eta) \wedge \pi^*((\cos \theta \omega_1 + \sin \theta \omega_2) \wedge \Omega) = 0,
\]

and

\[
d(\Psi_+^\theta) = -\pi^*(\omega_3 \wedge \Omega) = 0.
\]

The existence of a principal circle bundle in the conditions above follows from a well known result by Kobayashi [14].

**Remark 2.3** Notice that $\Omega = 0$ satisfies the hypothesis in the previous proposition for each $\theta$ and one gets the trivial circle bundle $M = N \times \mathbb{R}$ with the half-flat structure which is the natural extension to $M$ of the hypo structure on $N$. In Section 6 we show non-trivial solutions on circle bundles over solvable Lie groups with a hypo-contact structure.

As a consequence of Proposition 2.2 we have

**Corollary 2.4** Let $N$ be a 5-manifold with a hypo-contact structure $(\eta, \omega_1, \omega_2, \omega_3)$. For any $\theta$, let us consider the half-flat structure on $N \times \mathbb{R}$ defined in Proposition 2.2. Then,

\[
i_U(dF^\theta - 2\Psi_+^\theta) = 0,
\]

where $U$ denotes the vector field on $\mathbb{R}$ dual to $\rho = dt$.

**Proof:** Clearly $dF^\theta = (\cos \theta d\omega_1 + \sin \theta d\omega_2) - 2\omega_3 \wedge dt$, since $d\eta = -2\omega_3$. So, $i_U dF^\theta = -2\omega_3 = 2i_U \Psi_+^\theta$, which proves that $i_U (dF^\theta - 2\Psi_+^\theta) = 0$.

**Theorem 2.5** Let $M$ be a 6-dimensional manifold endowed with a half-flat structure $(F, \Psi_+, \Psi_-)$. Let $f: N \to M$ be an oriented hypersurface of $M$. Denote the unit normal vector field by $U$. Suppose that

\[
dF = 2\Psi_+, \quad \mathcal{L}_U F = 0,
\]

where $\mathcal{L}$ denotes the Lie derivative. Then, the forms $(\eta, \omega_1, \omega_2, \omega_3)$ on $N$ given by

\[
\eta = -i_U F, \quad \omega_1 = -i_U \Psi_-, \quad \omega_2 = f^* F, \quad \omega_3 = -i_U \Psi_+,
\]

define a hypo-contact structure on $N$. 


Proof: Equations (8) imply $f^* (\Psi_+) = -\omega_1 \wedge \eta$, so that $\omega_1 \wedge \eta$ is closed if the $SU(3)$-structure is half-flat.

Using again (8), we have

$$d\eta = -d(i_U F) = i_U dF - \mathcal{L}_U F = i_U dF = 2i_U \Psi_+ = -2\omega_3,$$

since $\mathcal{L}_U F = 0$ and $dF = 2\Psi_+$.

To complete the proof, we notice that $d\omega_2 = f^* (dF) = 2f^* (\Psi_+) = -2\omega_1 \wedge \eta$. Therefore, $d(\omega_2 \wedge \eta) = d\omega_2 \wedge \eta + \omega_2 \wedge d\eta = 0$.

An example of a 6-manifold satisfying the conditions of the Theorem 2.5 is the compact nilmanifold defined by the equations

$$de^i = 0 \quad (1 \leq i \leq 4), \quad de^5 = -2e^{14} - 2e^{23}, \quad de^6 = -2e^{13} + 2e^{24},$$

with the half-flat structure $(F, \Psi_+, \Psi_-)$ given by

$$F = e^{12} + e^{34} + e^{56}, \quad \Psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \quad \Psi_- = e^{136} + e^{145} + e^{235} - e^{246}.$$

Consider the 5-submanifold whose unit normal vector field is the dual to $-e^6$, that is, the 5-dimensional compact submanifold determined by the equations $de^i = 0 \ (1 \leq i \leq 4)$, $de^5 = -2e^{14} - 2e^{23}$. Then, the equations $dF = 2\Psi_+$ and $\mathcal{L}_U F = 0$ are satisfied.

Proposition 2.6 Let $M$ be a 6-dimensional manifold endowed with a half-flat structure $(F, \Psi_+, \Psi_-)$, and let $f : N \rightarrow M$ be an oriented hypersurface of $M$. Denote the unit normal vector field by $U$.

Suppose that

$$(9) \quad g(\nabla_U U, X) = 0, \quad \mathcal{L}_U \Psi_+ = 0,$$

for any vector field $X$ on $N$. Then, the forms $(\eta, \omega_1, \omega_2, \omega_3)$ on $N$ given by (8) define a hypo structure on $N$.

Proof: Proceeding as in Theorem 2.5 we see that $d(\omega_1 \wedge \eta) = 0$. Moreover, taking account (8), we have

$$d\omega_3 = -d(i_U \Psi_+) = i_U d\Psi_+ - \mathcal{L}_U \Psi_+ = 0$$

because both terms vanish. Therefore, only it remains to prove that $d(\omega_2 \wedge \eta) = 0$.

Denote by $\rho$ the 1-form on $M$ dual to the normal vector field $U$, and by $\mathfrak{X}(M)$ the Lie algebra of the vector fields on $M$. Then, the restriction $\mathfrak{X}(M)|_N$ to $N$ of $\mathfrak{X}(M)$ is the direct sum

$$\mathfrak{X}(M)|_N = \mathfrak{X}(N) \oplus U.$$

Firstly, we see that, for any vector fields $X, Y$ on $N$, $d\rho(X, Y) = d\rho(U, X) = 0$. In fact, we have

$$(10) \quad d\rho(X, Y) = X\rho(Y) - Y\rho(X) - \rho[X, Y] = 0.$$

Also, for any vector field $X$ on $N$, we get

$$(11) \quad d\rho(U, X) = \mathbb{U}\rho(X) - X\rho(U) - \rho[\mathbb{U}, X] = -\rho[\mathbb{U}, X] = 0,$$

since the normal component of $[U, X]$ is

$$g(U, [U, X]) = g(U, \nabla_U X - \nabla_X U) = g(U, \nabla_U X) = g(\nabla_U U, X) = 0.$$

From equations (8) it follows that $F = \omega_2 + \eta \wedge \rho$. Now from (10), (11) and using that $\omega_2 \wedge d\omega_2 = 0$, we get

$$0 = d(F \wedge F) = 2(\omega_2 \wedge d\omega_2 + d(\omega_2 \wedge \eta) \wedge \rho - \omega_2 \wedge \eta \wedge d\rho) = 2d(\omega_2 \wedge \eta) \wedge \rho,$$

which implies that $d(\omega_2 \wedge \eta) = 0$.

To finish this section, we consider $SU(3)$-structures on a manifold with a Killing vector field $X$ preserving the $SU(3)$-structure, and we study the conditions under which the $SU(3)$-structure induces
a hypo-contact structure \((\eta, \omega_i)\) on the 5-submanifold \(N\) determined by \(X\) as follows. Let \(M\) be a 6-dimensional manifold endowed with an \(SU(3)\)-structure \((F, \Psi_+, \Psi_-)\), and let \(X \in \mathfrak{X}(M)\) be a Killing vector field on \(M\) which preserves the \(SU(3)\)-structure, that is \(X\) is an infinitesimal isometry satisfying

\[
\mathcal{L}_X F = 0, \quad \mathcal{L}_X \Psi_+ = 0, \quad \mathcal{L}_X \Psi_- = 0.
\]

In a suitable neighborhood of any point \(p\) of \(M\) where \(X_p \neq 0\), let us denote by \(N\) the 5-dimensional manifold formed from the orbits of \(X\).

Let \(x\) be the function given by

\[
x = g(X, X)^{1/2},
\]

where \(g\) denotes the Riemannian metric on \(M\) determined by the \(SU(3)\)-structure. Since \(X\) is a Killing vector field, we have that \(\mathcal{L}_X (x) = 0\), so the function \(x\) descends to a function on \(N\) which we denote again by \(x\).

On the other hand, let us define a 1-form \(\alpha\) on \(M\) by

\[
\alpha(Z) = \frac{1}{x^2} g(Z, X),
\]

for any \(Z \in \mathfrak{X}(M)\). Observe that \(\alpha(X) \equiv 1\). The form \(\alpha\) is also invariant by \(X\); in fact, since \(\mathcal{L}_X \alpha = i_X d\alpha + d_i X\alpha\), it suffices to see that \((i_X d\alpha)(Z) = 0\) for any vector field \(Z \in \mathfrak{X}(M)\). But

\[
(i_X d\alpha)(Z) = d\alpha(X, Z) = X(\alpha(Z)) - \alpha([X, Z]) = \mathcal{L}_X \left( \frac{1}{x^2} g(Z, X) \right) - \frac{1}{x^2} g(\mathcal{L}_X Z, X) = \mathcal{L}_X \left( \frac{1}{x^2} \right) g(Z, X) + \frac{1}{x^2} \mathcal{L}_X (g(Z, X)) - \frac{1}{x^2} g(\mathcal{L}_X Z, X) = 0,
\]

because \(X\) is Killing and \(dx(X) = 0\). Therefore, \(\alpha\) descends to a 1-form on \(N\) which again we denote by the same letter.

**Lemma 2.7** In the above conditions, the quadruplet of differential forms \((\eta, \omega_1, \omega_2, \omega_3)\) given by

\[
(14) \quad \eta = -i_X F, \quad \omega_1 = x i_X (F \wedge \alpha), \quad \omega_2 = i_X \Psi_-, \quad \omega_3 = -i_X \Psi_+,
\]

defines an \(SU(2)\)-structure on \(N\), where \(x\) and \(\alpha\) are the function and the 1-form on \(N\) induced by (12) and (13), respectively.

**Proof:** First we show that the Lie derivative of the forms \(i_X F, x i_X (F \wedge \alpha), i_X \Psi_-\) and \(i_X \Psi_+\) with respect to \(X\) is zero, so these forms descend to forms on \(N\). In fact, since \(X\) preserves the \(SU(3)\)-structure, we have

\[
\mathcal{L}_X (i_X F) = i_X (d_i X F) = i_X (\mathcal{L}_X F) = 0, \\
\mathcal{L}_X (x i_X (F \wedge \alpha)) = (\mathcal{L}_X x) i_X (F \wedge \alpha) + x(\mathcal{L}_X i_X (F \wedge \alpha)) = 0, \\
\mathcal{L}_X (i_X \Psi_\pm) = i_X (d_i X \Psi_\pm) = i_X (\mathcal{L}_X \Psi_\pm) = 0.
\]

Now it remains to see that \((\eta, \omega_i)\) defines an \(SU(2)\)-structure. Let \(E_6 = \frac{1}{x} X\) be the unitary vector field in the direction of \(X\). We can consider a local orthonormal basis \(E_1, \ldots, E_6\) such that the \(SU(3)\)-structure expresses in terms of the dual basis \(e^1, \ldots, e^6\) as follows

\[
F = e^{12} + e^{34} + e^{56}, \quad \Psi_+ = (e^{13} + e^{42}) e^5 - (e^{14} + e^{23}) e^6, \quad \Psi_- = (e^{14} + e^{23}) e^5 + (e^{13} + e^{42}) e^6.
\]

Notice that \(\alpha = \frac{1}{x} e^6\). Therefore, locally we have

\[
\eta = -i_X F = -i_X E_6 (e^{12} + e^{34} + e^{56}) = xe^5, \\
\omega_1 = x i_X (F \wedge \alpha) = i_X E_6 (e^{12} + e^{34} + e^{56}) = xe^{12} + e^{34}, \\
\omega_2 = i_X (\Psi_-) = i_X E_6 ((e^{14} + e^{23}) e^5 + (e^{13} + e^{42}) e^6) = xe^{13} + e^{42}, \\
\omega_3 = -i_X (\Psi_+) = -i_X E_6 ((e^{13} + e^{42}) e^5 - (e^{14} + e^{23}) e^6) = xe^{14} + e^{23},
\]

and thus \((\eta, \omega_i)\) is an \(SU(2)\)-structure.
**Theorem 2.8** Let $M$ be a 6-manifold in the conditions of Lemma 2.7. Suppose that $X$ is a Killing vector field of constant length, preserving a half-flat $SU(3)$-structure on $M$ and satisfying $d\alpha \wedge i_X \Psi_+ = 0$. Then the structure on $N$ given by (14) is hypo-contact.

**Proof:** First we notice that for any $SU(3)$-structure on $M$, the $SU(2)$-structure on $N$ defined by (14) satisfies

$$\omega_1 \wedge \eta = -\frac{x}{2} i_X(F \wedge F), \quad \omega_2 \wedge \eta = x^2 i_X(\alpha \wedge \Psi_+ + \rho^2(\Psi_+ - \alpha \wedge i_X \Psi_+)).$$

Therefore, we get

$$-2d(\omega_1 \wedge \eta) = dx \wedge i_X(F \wedge F) - x i_X d(F \wedge F),$$

$$d(\omega_2 \wedge \eta) = 2xdx \wedge (\Psi_+ - \alpha \wedge i_X \Psi_+) + x^2(d(\Psi_+ - \alpha \wedge i_X \Psi_+)).$$

Now, let us consider a Killing vector field $X$ of constant length such that it preserves a half-flat $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ on $M$ and satisfies $d\alpha \wedge i_X \Psi_+ = 0$, then the structure on $N$ given by (14) is hypo since

$$d\omega_3 = 0, \quad d(\omega_1 \wedge \eta) = 0, \quad d(\omega_2 \wedge \eta) = -x^2 d\alpha \wedge i_X \Psi_+ = 0.$$  

Moreover, if $i_X(dF - 2\Psi_+ = 0$, then $d\eta = -2\omega_3$, and so the $SU(2)$-structure on $N$ is hypo-contact. QED

The previous study is done in the same vein of the papers [11] and [10] where $S^1$-bundles with a $U(1)$-invariant $SU(3)$-structure (or $G_2$-structure) are considered.

**Remark 2.9** We must notice that in the conditions of Lemma 2.7, if $X$ is a Killing vector field on $M$ preserving the $SU(3)$-structure (not necessarily half-flat) and satisfying $i_X(dF - 2\Psi_+ = 0$, then the 1-form $\eta$ is a contact form on $N$.

## 3 Solvable Lie algebras with a hypo-contact structure

The purpose of this Section is to prove Theorem 1.1. First, we need to show the following propositions.

**Proposition 3.1** Let $\mathfrak{g}$ be a solvable Lie algebra of dimension 5 with a hypo-contact structure $(\eta, \omega_1, \omega_2, \omega_3)$. Then, there is a basis $e^1, \ldots, e^5$ for $\mathfrak{g}^*$ such that

$$\eta = e^5, \quad \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} + e^{42}, \quad \omega_3 = e^{14} + e^{23},$$

and

$$\begin{cases}
  de^1 = Ae^{14} + Ae^{23}, \\
  de^2 = B_{12}e^{12} + B_{13}e^{13} + B_{14}e^{14} + B_{15}e^{15} - B_{14}e^{24} + (2A + B_{13})e^{24} \\
  + B_{25}e^{25} + B_{34}e^{34} + B_{35}e^{35}, \\
  de^3 = (3A + B_{13})e^{12} + C_{13}e^{13} + C_{14}e^{14} + C_{15}e^{15} - C_{14}e^{23} - (B_{12} + B_{34} - C_{13})e^{24} \\
  + C_{25}e^{25} - (A + B_{13})e^{34} - B_{25}e^{35}, \\
  de^4 = B_{14}e^{12} + C_{14}e^{13} + (B_{34} - C_{13})e^{14} + D_{15}e^{15} + (B_{12} + C_{13})e^{23} + C_{14}e^{24} \\
  + C_{15}e^{25} - B_{14}e^{34} - B_{15}e^{35}, \\
  de^5 = -2e^{14} - 2e^{23},
\end{cases}$$

where the coefficients $A, B_{12}, B_{13}, B_{14}, B_{15}, B_{25}, B_{34}, B_{35}, C_{13}, C_{14}, C_{15}, C_{25}$ and $D_{15}$ satisfy the conditions

$$d(de^i) = 0$$

for $i = 1, 2, 3, 4, 5$. 


Proof: Let $V$ be the subspace of $\mathfrak{g}^*$ orthogonal to $\eta$. Since $\mathfrak{g}$ is solvable, there is a nonzero element $\alpha \in \mathfrak{g}^*$ which is closed. Thus,

$$\alpha = \beta + \lambda \eta,$$

where $\beta \in V$ and $\lambda \in \mathbb{R}$. Now, $d\alpha = 0$ is equivalent to $d\beta = -\lambda d\eta$. Therefore, $\gamma = \frac{1}{\lambda} d\beta$ is a unit element in $V = \langle \eta \rangle^\perp$ satisfying

$$d\gamma = \tau \ d\eta,$$

with $\tau = -\lambda/\|\beta\|$. From [9 Corollary 3], there is a basis $\epsilon^1, \ldots, \epsilon^5$ for $\mathfrak{g}^*$ satisfying (15) with $\epsilon^1 = \gamma$.

Therefore, $d\epsilon^i = d\eta = -2\omega_3 = -2e^{14} - 2e^{23}$ and $d\epsilon^1 = d\gamma = d\eta = A e^{14} + A e^{23}$, where $A = -2\tau$, so the differentials of $\epsilon^1, \ldots, \epsilon^5$ are given by

$$d\epsilon^1 = A e^{14} + A e^{23},$$
$$d\epsilon^2 = B_{12} e^{12} + B_{13} e^{13} + B_{14} e^{14} + \cdots + B_{34} e^{34} + B_{35} e^{35} + B_{45} e^{45},$$
$$d\epsilon^3 = C_{12} e^{12} + C_{13} e^{13} + C_{14} e^{14} + \cdots + C_{34} e^{34} + C_{35} e^{35} + C_{45} e^{45},$$
$$d\epsilon^4 = D_{12} e^{12} + D_{13} e^{13} + D_{14} e^{14} + \cdots + D_{34} e^{34} + D_{35} e^{35} + D_{45} e^{45},$$
$$d\epsilon^5 = -2e^{14} - 2e^{23} ,$$

where the coefficients must satisfy the Jacobi identity $d(\epsilon^i) = 0$, $1 \leq i \leq 5$, and the additional conditions

$$d(\eta \wedge \omega_1) = d(\eta \wedge \omega_2) = 0 \text{ in order to have a hypo-contact structure.}$$

By imposing that $d(e^{12} + e^{34}) = d(\eta \wedge \omega_1) = 0$, $d(e^{14} - e^{23}) = d(\eta \wedge \omega_2) = 0$, and $d(e^{14} + e^{23}) = -(1/2)d(\epsilon^5) = 0$, the coefficients in (15) satisfy the following relations:

$$B_{23} = -B_{14}, \ B_{24} = 2A + B_{13}, \ B_{25} = 0, \ C_{12} = 3A + B_{13}, \ C_{23} = -C_{14}, \ C_{24} = -B_{12} - B_{34} + C_{13},$$
$$C_{34} = -A - B_{13}, \ C_{35} = -B_{25}, \ C_{45} = 0, \ D_{12} = B_{14}, \ D_{13} = C_{14}, \ D_{14} = B_{34} - C_{13},$$
$$D_{23} = B_{12} + C_{13}, \ D_{24} = C_{14}, \ D_{25} = C_{15}, \ D_{34} = -B_{14}, \ D_{35} = -B_{15}, \ D_{45} = 0.$$

This completes the proof of (16). Notice that the coefficients must also satisfy (17). \[\square\]

Let $E_1, \ldots, E_5$ be the basis for $\mathfrak{g}$ dual to the basis $\epsilon^1, \ldots, \epsilon^5$ and let us denote by $c^l_{ijk}$ the component in $E_l$ of $[E_i, [E_j, E_k]] + [[E_i, E_j], E_k] + [[E_i, E_k], E_j]$. It is clear that the Jacobi identity is satisfied if and only if $c^l_{ijk} = 0$ for $1 \leq i < j < k \leq 5$ and $1 \leq l \leq 5$.

A direct calculation shows that $c^3_{134} = c^3_{134} = c^2_{134} = 0$ if and only if

$$2B_{15} = B_{12}B_{14} + B_{14}B_{34} + 2B_{13}C_{14},$$
$$2B_{25} = B_{12}B_{13} + 4AB_{34} + 3B_{13}B_{34} - 2B_{13}C_{14} - 2B_{14}C_{14},$$
$$2B_{35} = 2AB_{13} + 2B_{13}^2 + 2B_{14}^2 - B_{12}B_{34} + B_{34}^2,$$

respectively. Moreover, $c^3_{123} = c^3_{124} = c^4_{123} = 0$ if and only if

$$2C_{15} = -4AB_{14} - 2B_{13}B_{14} + 3B_{12}C_{14} + B_{34}C_{14},$$
$$2C_{25} = -12A^2 - B_{13}^2 - 10AB_{13} - 2B_{12}B_{34} - 2B_{13}B_{34} + 3B_{12}C_{13} + 3B_{34}C_{13} - 2C_{13}^2 - 2C_{14}^2,$$
$$2D_{15} = -B_{12}^2 - B_{13}^2 + B_{12}B_{34} - 3B_{12}C_{13} + B_{34}C_{13} - 2C_{13}^2 - 2C_{14}^2,$$

respectively.

Corollary 3.2 Let $\mathfrak{g}$ be a solvable Lie algebra with a hypo-contact structure $(\eta, \omega_1, \omega_2, \omega_3)$. Then, there is a basis $\epsilon^1, \ldots, \epsilon^5$ of $\mathfrak{g}^*$ satisfying (15), (16), (19), (20) and where the seven remaining coefficients $A, B_{12}, B_{13}, B_{14}, B_{34}, C_{13}, C_{14}$ satisfy the Jacobi identity (17).
Proposition 3.3 Let \( \mathfrak{g} \) be a solvable Lie algebra with a basis \( e^1, \ldots, e^5 \) for \( \mathfrak{g}^* \) in the conditions of Proposition 3.4. Then, the structure equations (10) reduce to one of the following six families:

\[
\begin{align*}
\begin{cases}
de^1 &= 0, \\
de^2 &= re^{12}, \\
de^3 &= re^{13}, \\
de^4 &= -r e^{14} - 3 r^2 e^{15} + 2 r e^{23}, \\
de^5 &= -2 e^{14} - 2 e^{23},
\end{cases}
\end{align*}
\]

where \( r \in \mathbb{R}^* \); moreover, \( \mathfrak{g}^1 = \langle E_2, E_3, E_4, E_5 \rangle \), \( \mathfrak{g}^2 = \langle r E_4 - E_5 \rangle \) and \( \mathfrak{g}^3 = 0 \).

\[
\begin{align*}
\begin{cases}
de^1 &= 0, \\
de^2 &= re^{12} + 3 r e^{34} + 3 r^2 e^{35}, \\
de^3 &= re^{13} - 3 r e^{24} - 3 r^2 e^{25}, \\
de^4 &= -r de^5, \\
de^5 &= -2 e^{14} - 2 e^{23},
\end{cases}
\end{align*}
\]

where \( r \in \mathbb{R}^* \); moreover, \( \mathfrak{g}^1 = \langle E_2, E_3, r E_4 - E_5 \rangle \), \( \mathfrak{g}^2 = \langle r E_4 - E_5 \rangle \) and \( \mathfrak{g}^3 = 0 \).

\[
\begin{align*}
\begin{cases}
de^1 &= 0, \\
de^2 &= re^{14} - re^{23} - ar e^{25} + r^2 e^{35}, \\
de^3 &= \frac{a}{r} de^2, \\
de^4 &= re^{12} + ae^{13} - (a^2 + r^2) e^{15} + ae^{24} - re^{34}, \\
de^5 &= -2 e^{14} - 2 e^{23},
\end{cases}
\end{align*}
\]

where \( a \in \mathbb{R} \) and \( r \in \mathbb{R}^* \); moreover, \( \mathfrak{g}^1 = \langle r E_2 + a E_3, E_4, E_5 \rangle \) and \( \mathfrak{g}^2 = 0 \).

\[
\begin{align*}
\begin{cases}
de^1 &= de^2 = 0, \\
de^3 &= ae^{13} + be^{14} - be^{23} + ae^{24} - (a^2 + b^2) e^{25}, \\
de^4 &= be^{13} - ae^{14} - (a^2 + b^2) e^{15} + ae^{23} + be^{24}, \\
de^5 &= -2 e^{14} - 2 e^{23},
\end{cases}
\end{align*}
\]

where \( a, b \in \mathbb{R} \); moreover, if \( a \) or \( b \) is nonzero then \( \mathfrak{g}^1 = \langle E_3, E_4, E_5 \rangle \) and \( \mathfrak{g}^2 = 0 \).

\[
\begin{align*}
\begin{cases}
de^1 &= 0, \\
de^2 &= re^{34} + \frac{r^2}{2} e^{35}, \\
de^3 &= re^{13}, \\
de^4 &= -\frac{r^2}{2} e^{15} + r e^{23}, \\
de^5 &= -2 e^{14} - 2 e^{23},
\end{cases}
\end{align*}
\]

where \( r \in \mathbb{R}^* \); moreover, \( \mathfrak{g}^1 = \langle E_2, E_3, E_4, E_5 \rangle \), \( \mathfrak{g}^2 = \langle E_2, r E_4 - 2 E_5 \rangle \) and \( \mathfrak{g}^3 = 0 \).

\[
\begin{align*}
\begin{cases}
de^1 &= 0, \\
de^2 &= re^{12} + ae^{13} + ae^{24} + \frac{a}{r} (r^2 + a^2) e^{25} + \frac{a^2}{r^2} e^{34} + \frac{r^2}{2} (r^2 + a^2) e^{35}, \\
de^3 &= re^{12} + \frac{a^2}{r} e^{13} - re^{24} - \frac{1}{r} (r^2 + a^2) e^{25} - ae^{34} - \frac{a}{2} (r^2 + a^2) e^{35}, \\
de^4 &= -\frac{a^2}{2} e^{15} + \frac{r^2 + a^2}{r} e^{23}, \\
de^5 &= -2 e^{14} - 2 e^{23},
\end{cases}
\end{align*}
\]

where \( r \in \mathbb{R}^* \) and \( a \in \mathbb{R} \); moreover, \( \mathfrak{g}^1 = \langle E_2, E_3, E_4, E_5 \rangle \), \( \mathfrak{g}^2 = \langle a E_2 - r E_3, (r^2 + a^2) E_4 - 2r E_5 \rangle \) and \( \mathfrak{g}^3 = 0 \).
\textbf{Proof}: We divide the proof in two cases: $A = 0$ and $A \neq 0$.

Let us suppose first that $A = 0$. By Corollary 3.2 the coefficients $B_{15}, B_{25}, B_{35}, C_{15}, C_{25}, D_{15}$ are determined by $B_{12}, B_{13}, B_{14}, B_{34}, C_{13}, C_{14}$, and they are explicitly given by (19) and (20). A direct calculation shows that

\[ c_{125}^4 = 0 \iff B_{13}(-3B_{12} - B_{34} - 2C_{13})B_{14} + (3B_{12}^2 + 2B_{13}^2 + B_{12}B_{34})C_{14} = 0; \]
\[ c_{135}^4 = 0 \iff (-2B_{13}^2 - B_{12}C_{13} - B_{34}C_{13} - 2C_{13}^2)B_{14} + B_{13}(3B_{12} + B_{34} + 2C_{13})C_{14} = 0; \]
\[ c_{145}^4 = 0 \iff B_{13}(B_{34} - C_{13})B_{14} + (B_{13}^2 - B_{12}B_{34})C_{14} = 0; \]
\[ c_{145}^4 = 0 \iff (B_{13}^2 - B_{12}C_{13} - B_{34}C_{13} + C_{13}^2)B_{14} + B_{13}(B_{34} - C_{13})C_{14} = 0. \]

Let us denote by $\rho_{ij}$ the determinant of the system given by the equations $i$ and $j$ above, i.e.

\[ \rho_{12} = (B_{13}^2 - B_{12}C_{13})(-3B_{12}^2 + 4B_{13}^2 - 4B_{12}B_{34} - B_{34}^2 - 6B_{12}C_{13} - 2B_{34}C_{13}), \]
\[ \rho_{13} = -3B_{13}(B_{12} + B_{34})(B_{13}^2 - B_{12}C_{13}), \]
\[ \rho_{14} = -(B_{13}^2 - B_{12}C_{13})(3B_{12}^2 + 2B_{13}^2 + 4B_{12}B_{34} + B_{34}^2 - 3B_{12}C_{13} - B_{34}C_{13}), \]
\[ \rho_{23} = -(B_{13}^2 - B_{12}C_{13})(2B_{13}^2 + B_{12}B_{34} + B_{34}^2 + 2B_{34}C_{13}), \]
\[ \rho_{24} = -3B_{13}(B_{12} + B_{34})(B_{13}^2 - B_{12}C_{13}) = \rho_{13}, \]
\[ \rho_{34} = -(B_{13}^2 - B_{12}C_{13})(B_{13}^2 - B_{12}B_{34} - B_{34}^2 + B_{34}C_{13}). \]

\textbf{Case 1}: At least one of the determinants $\rho_{ij}$ is nonzero. In this case, $B_{14} = C_{14} = 0$. Moreover,

\[ c_{125}^2 = B_{13}(2B_{12}^2 + B_{12}B_{34} + B_{34}^2 - 3B_{12}C_{13} - B_{34}C_{13}), \]

and

\[ c_{125}^3 = -2B_{13}(B_{12}^2 - B_{12}B_{34} - B_{34}^2 + B_{34}C_{13}), \]

which implies that $B_{13}(B_{12}^2 - B_{12}C_{13}) = 0$ in order for the Jacobi identity be satisfied. Since $\rho_{ij} \neq 0$ for some $i, j$, it is necessary that $B_{13} = 0$.

Since $B_{13} = B_{14} = C_{14} = 0$ the equations (16) reduce to

\[
\begin{aligned}
d e^1 &= 0, \\
d e^3 &= B_{12}e^{12} + B_{34}e^{34} + \frac{1}{2}B_{34}(B_{34} - B_{12})e^{35}, \\
d e^5 &= C_{13}e^{13} - (B_{12} + B_{34} - C_{13})e^{24} - \frac{1}{2}(B_{12} + B_{34} - 2C_{13})e^{25}, \\
d e^7 &= (B_{34} - C_{13})e^{14} - \frac{1}{2}(B_{12} + C_{13})(B_{12} - B_{34} + 2C_{13})e^{15}, \\
d e^9 &= -2e^{14} - 2e^{23},
\end{aligned}
\]

because $B_{15} = B_{25} = C_{15} = 0$, $B_{35} = \frac{1}{2}B_{34}(B_{34} - B_{12})$, $C_{25} = -\frac{1}{2}(B_{12} + B_{34} - 2C_{13})(B_{12} + B_{34} - C_{13})$ and $D_{15} = -\frac{1}{2}(B_{12} + C_{13})(B_{12} - B_{34} + 2C_{13})$. Now, the Jacobi identity is satisfied if and only if $B_{12}(B_{34} - 3C_{13})(B_{12} + B_{34} - C_{13}) = 0$, $B_{34}C_{13}(2B_{12} - B_{34} + C_{13}) = 0$ and $B_{34}B_{13} = (B_{12} + B_{34} - C_{13}) = 0$. But, the nonvanishing of some $\rho_{ij}$ implies that $B_{12}$ and $C_{13}$ cannot be zero, so

\[ (B_{34} - 3C_{13})(B_{12} + B_{34} - C_{13}) = 0, \quad B_{34}(2B_{12} - B_{34} + C_{13}) = 0, \quad B_{34}(B_{12} - C_{13})(B_{12} + B_{34} - C_{13}) = 0. \]

If $B_{34} = 0$ then (28) implies $C_{13} = B_{12} \neq 0$ and from (27) we get (21) with $r = B_{12} \in \mathbb{R}^*$. Otherwise, $B_{34} \neq 0$ implies that $B_{34} = 3B_{12} = C_{13} = B_{12}$, and equations (27) reduce to (22) with $r = B_{12} \in \mathbb{R}^*$.

\textbf{Case 2}: All the determinants $\rho_{ij}$ vanish. First, we prove that $B_{13}^2 = B_{12}C_{13}$.

In fact, if $B_{13}^2 \neq B_{12}C_{13}$ then all the determinants $\rho_{ij}$ vanish if and only if $B_{13}(B_{12} + B_{34}) = 0$ and

\[ 3B_{12}^2 - 4B_{13}^2 + 4B_{12}B_{34} + B_{34}^2 + 6B_{12}C_{13} + 2B_{34}C_{13} = 0, \]

\[ 3B_{12}^2 + 2B_{13}^2 + 4B_{12}B_{34} + B_{34}^2 - 3B_{12}C_{13} - B_{34}C_{13} = 0, \]

\[ 2B_{13}^2 + B_{12}B_{34} + B_{34}^2 = 0, \quad B_{13}B_{12}B_{34} - B_{34}^2 + B_{34}C_{13} = 0. \]
Notice that $B_{13}$ must be zero, because otherwise $B_{34} = -B_{12}$ and the equations \((29)\) would reduce to $B_{13}^2 - B_{12}C_{13} = 0$, contradicting our assumption. Since $B_{13} = 0$ we have that $B_{12}, C_{13} \neq 0$ and \((24)\) become

\[
\begin{align*}
3B_{12}^2 + 6B_{12}C_{13} + B_{34}(4B_{12} + B_{34} + 2C_{13}) &= 0, \\
3B_{12}^2 - 3B_{12}C_{13} + B_{34}(4B_{12} + B_{34} - C_{13}) &= 0, \\
B_{34}(B_{12} + B_{34} + 2C_{13}) &= 0, \\
B_{34}(B_{12} + B_{34} - C_{13}) &= 0.
\end{align*}
\]

From the last two equations we have that $B_{34} = 0$, because $C_{13}$ is nonzero. But in such case the first two equations are satisfied if and only if $B_{12}C_{13} = 0$, which is again a contradiction. Therefore, we conclude that there are no solutions if $B_{13}^2 \neq B_{12}C_{13}$.

**Case 2.1:** If $B_{13}^2 = B_{12}C_{13}$ and $B_{12} = 0$, then $B_{13} = 0$ and the Jacobi identity is satisfied if and only if

\[
B_{14}B_{34} = B_{13}C_{13} = B_{34}(B_{34}C_{13} - C_{13}^2 - C_{14}^2) = 0.
\]

So, if $B_{14} \neq 0$ then $B_{34} = C_{13} = 0$, and the equations \((16)\) reduce to \((23)\) with $a = C_{14} \in \mathbb{R}$ and $r = B_{14} \in \mathbb{R}^*$. On the other hand, if $B_{14} = B_{34} = 0$ then we obtain equations \((21)\) with $a = C_{13}$ and $b = C_{14} \in \mathbb{R}$.

Finally, let us suppose that $B_{14} = 0$ and $B_{34} \neq 0$, which implies that $C_{13} + C_{14} - B_{34}C_{13} = 0$. A long but direct calculation shows that the corresponding Lie algebra is solvable only for $C_{13} \neq 0$ and $C_{14} = 0$, and in this case the equations \((16)\) reduce to \((25)\) with $r = C_{13} \in \mathbb{R}^*$.

**Case 2.2:** If $B_{13}^2 = B_{12}C_{13}$ and $B_{12} \neq 0$, then $C_{13} = B_{13}^2/B_{12}$ and

\[
\begin{align*}
c_{125}^1 &= 0 \iff (3B_{12}^2 + 2B_{13}^2 + B_{12}B_{34})(B_{13}B_{14} - B_{12}C_{14}) = 0, \\
c_{145}^2 &= 0 \iff (B_{13}^2 - B_{12}B_{34})(B_{13}B_{14} - B_{12}C_{14}) = 0,
\end{align*}
\]

which implies that

\[
(B_{12}^2 + B_{13}^2)(B_{13}B_{14} - B_{12}C_{14}) = 0.
\]

Since $B_{12} \neq 0$ we get $C_{14} = B_{13}B_{14}/B_{12}$. A direct calculation shows that the Jacobi identity holds if and only if

\[
(B_{12} + B_{34})(B_{13}^2 + B_{14}^2 - B_{12}B_{34}) = 0.
\]

We distinguish two cases:

(i) $B_{34} = -B_{12} \neq 0$: in this case the equations \((16)\) reduce to \((23)\). In fact, take $\theta \in \left(0, \frac{2\pi}{3}\right)$ such that $\cos 3\theta = B_{14}(B_{12}^2 + B_{13}^2 + B_{14}^2)^{-\frac{1}{2}}$, $\sin 3\theta = (B_{13}^2 + B_{14}^2)^{\frac{1}{2}}(B_{12}^2 + B_{13}^2 + B_{14}^2)^{-\frac{1}{2}}$. Then, from \((16)\) we have that the new basis

\[
\begin{align*}
f^1 &= \cos \theta e^1 + \sin \theta B_{13}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^2 - \sin \theta B_{12}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^3, \\
f^2 &= -\sin \theta B_{13}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^1 + \cos \theta e^2 - \sin \theta B_{12}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^4, \\
f^3 &= \sin \theta B_{12}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^1 + \cos \theta e^3 - \sin \theta B_{13}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^4, \\
f^4 &= \sin \theta B_{13}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^2 + \sin \theta B_{13}(B_{12}^2 + B_{13}^2)^{-\frac{1}{2}}e^3 + \cos \theta e^4, \\
f^5 &= e^5,
\end{align*}
\]

satisfies \((23)\) for $a = B_{13}/B_{12} \sqrt{B_{12}^2 + B_{13}^2 + B_{14}^2}$ and $r = \epsilon \sqrt{B_{12}^2 + B_{13}^2 + B_{14}^2}$ with $\epsilon = \pm 1$. Moreover, $f^{12} + f^{34} = e^{12} + e^{34}$, $f^{13} + f^{42} = e^{13} + e^{42}$ and $f^{14} + f^{23} = e^{14} + e^{23}$.

(ii) If $B_{34} \neq -B_{12}$ then $B_{34} = (B_{12}^2 + B_{13}^2)/B_{12}$. Now, a long but direct calculation shows that the corresponding Lie algebra is solvable only for $B_{14} = 0$, and in this case the equations \((16)\) reduce to \((20)\) with $r = B_{12} \neq 0$ and $a = B_{13} \in \mathbb{R}$. This completes the proof of the case $A = 0$. 

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Moreover, the proof of Proposition 3.3 shows that if such a Lie algebra admits hypo-contact structure

\[ A, B, C \]

(34)

Notice that the condition

\[ f, B \]

remaining coefficients \( A, B, C, B_{13}, B_{34}, C_{13} \) and \( C_{14} \). Moreover, from (19) and (20) we get \( B_{15} = C_{15} = D_{15} = 0 \), thus the differentials of \( e^i \) have the form

\[
\begin{aligned}
    de^1 &= Ae^{14} + Ae^{23}, \\
    de^2 &= (B_{34} - 2C_{13})e^{12} + B_{13}e^{13} + (2A + B_{13})e^{24} + B_{25}e^{35} + B_{34}e^{34} + B_{35}e^{35}, \\
    de^3 &= (3A + B_{13})e^{12} + C_{13}e^{13} - (2B_{34} - 3C_{13})e^{24} + C_{25}e^{35} - (A + B_{13})e^{34} - B_{25}e^{35}, \\
    de^4 &= (B_{34} - C_{13}) (e^{14} + e^{23}), \\
    de^5 &= -2e^{14} - 2e^{23}.
\end{aligned}
\]

(30)

It can be proved directly that the Jacobi identity for (30) is satisfied if and only if

\[
(3A + B_{13})B_{35} - B_{13}C_{25} = 0, \quad ( -2B_{34} + 3C_{13})B_{35} - B_{34}C_{25} = 0,
\]

and

\[
\begin{aligned}
    B_{25} - 2A B_{34} - 2B_{13}B_{34} + 2B_{13}C_{13} &= 0, \\
    6A^2 + 7A B_{13} + B_{13}^2 + 2B_{34}^2 - 7B_{34}C_{13} + 6C_{13}^2 + C_{25} &= 0, \\
    -6A B_{25} - 2B_{13}B_{25} + B_{34}C_{25} - 3C_{13}C_{25} &= 0,
\end{aligned}
\]

(32)

\[
\begin{aligned}
    A B_{13} + B_{13}^2 - B_{35} + B_{34}C_{13} &= 0, \\
    2B_{13}B_{25} - B_{34}B_{35} + 3B_{34}C_{13} &= 0, \\
    4B_{25}B_{34} - 6B_{25}C_{13} + 3A C_{25} + 2B_{13}C_{25} &= 0, \\
    2B_{25}B_{34} - 3A B_{35} - 2B_{13}B_{35} &= 0.
\end{aligned}
\]

Let \( \rho = -3 (AB_{34} + B_{34}B_{13} - B_{13}C_{13}) \) be the determinant of the linear system (81). If \( \rho \neq 0 \) then \( B_{35} = C_{25} = 0 \), and the first equation in (32) implies that \( B_{25} = -\frac{2}{3} \rho \neq 0 \), so in order to be satisfied the remaining equations in (32) we must have \( B_{13} = B_{34} = C_{13} = 0 \), which is in contradiction with the second equation in (32). Therefore, \( \rho = 0 \) and there is \( a \in \mathbb{R} \) such that \(-2B_{34} + 3C_{13} = 3a A + a B_{13} \) and \( B_{34} = a B_{13} \). From the first equation in (32) we get

\[
(33)
\]

\[
B_{34} = a B_{13}, \quad C_{13} = a(A + B_{13}), \quad B_{25} = 0,
\]

and (81)-(82) reduce to

\[
\begin{aligned}
    3A B_{35} + B_{13}B_{35} - B_{13}C_{25} &= 0, \\
    C_{25} + (6A^2 + 5A B_{13} + B_{13}^2)(1 + a^2) &= 0, \\
    B_{35} - B_{13}(A + B_{13})(1 + a^2) &= 0, \\
    (3A + 2B_{13})C_{25} &= 0, \\
    (3A + 2B_{13})B_{35} &= 0.
\end{aligned}
\]

(34)

Notice that \( B_{13} \neq -\frac{3}{2} A \) implies \( B_{35} = C_{25} = 0 \), and from the third equation in (34) it follows that \( B_{13} = 0, -A \), which does not solve the second equation in (34). Therefore, \( B_{13} = -\frac{3}{2} A \) and the solution to (34) is \( B_{35} = \frac{3}{2} A^2 (1 + a^2) \) and \( C_{25} = -B_{35} \). From (30) and (33) we get that the new basis \( f^1 = \nu(a e^1 + e^4), f^2 = \nu(a e^2 - e^3), f^3 = \nu(e^2 + a e^3), f^4 = \nu(-e^4 + a e^3), f^5 = e^5 \), where \( \nu = -(1 + a^2)^{-1/2} \), satisfies (22) with \( r = -\nu^{-1} A/2 \) and \( f^{12} + f^{34} = e^{12} + e^{34}, f^{13} + f^{42} = e^{13} + e^{42}, f^{14} + f^{23} = e^{14} + e^{23} \). That is to say, the case \( A \neq 0 \) reduces to (22) and the proof of the proposition is complete.

Remark 3.4 Notice that the condition \( [g, g] \neq g \) implies that \( g^* \) has a nonzero element which is closed, so the proof of Proposition 3.1 still holds for Lie algebras satisfying \( [g, g] \neq g \), even when they are not solvable. Moreover, the proof of Proposition 3.3 shows that if such a Lie algebra admits hypo-contact structure then it belongs to Case 2.1 with \( B_{34} \neq 0 = B_{14} = C_{13} = C_{14} \), Case 2.1 with \( B_{34}C_{13}C_{14} \neq 0 = B_{14} \), or Case 2.2 with \( (B_{34} + B_{12})B_{14} \neq 0 \).
Now, using Proposition 3.3 we obtain the classification of solvable hypo-contact Lie algebras.

**Proof of Theorem 1.1:** A solvable Lie algebra with a hypo-contact structure belongs, by Proposition 3.3 to one of the six families (21)–(26). Therefore, in order to prove the theorem, it suffices to show that \(\mathfrak{h}_1, \ldots, \mathfrak{h}_5\) are the Lie algebras underlying these families. For the family (21), the new basis
\[
\alpha^1 = 2e^4 - 3re^5, \quad \alpha^2 = 5e^3, \quad \alpha^3 = 2re^2, \quad \alpha^4 = -3e^4 - 3re^5, \quad \alpha^5 = re^1
\]
satisfies
\[
d\alpha^1 = -2a^{15} - \alpha^{23}, \quad d\alpha^2 = -\alpha^{25}, \quad d\alpha^3 = -\alpha^{35}, \quad d\alpha^4 = 3a^{45}, \quad d\alpha^5 = 0.
\]
Therefore, any Lie algebra in the family (21) is isomorphic to \(\mathfrak{h}_2\).

Any Lie algebra in the family (22) is isomorphic to \(\mathfrak{h}_3\). In fact, with respect to the new basis
\[
\alpha^1 = re^4, \quad \alpha^2 = \sqrt{2}re^3, \quad \alpha^3 = \sqrt{2}re^2, \quad \alpha^4 = re^1, \quad \alpha^5 = 3re^4 + 3r^2e^5,
\]
the equations (22) become
\[
d\alpha^1 = -2a^{14} - \alpha^{23}, \quad d\alpha^2 = -\alpha^{24} - \alpha^{35}, \quad d\alpha^3 = \alpha^{25} - \alpha^{34}, \quad d\alpha^4 = 4a^{35} = 0.
\]
Any Lie algebra in the family (23) is isomorphic to \(\mathfrak{h}_4\), because with respect to the new basis
\[
\alpha^1 = 2e^2 + re^5, \quad \alpha^2 = \frac{\sqrt{3}a}{r} e^1 - \frac{\sqrt{a^2 + r^2}}{r} e^2 + \sqrt{3} e^4 + \sqrt{a^2 + r^2} e^5, \\
\alpha^3 = -\frac{\sqrt{3}a}{r} e^1 - \frac{\sqrt{a^2 + r^2}}{r} e^2 - \sqrt{3} e^4 + \sqrt{a^2 + r^2} e^5, \\
\alpha^4 = -2a e^2 + 2re^3, \\
\alpha^5 = -\sqrt{3} \sqrt{a^2 + r^2} e^1 + a e^2 - re^3,
\]
the equations (23) become
\[
d\alpha^1 = -a^{14}, \quad d\alpha^2 = -a^{25}, \quad d\alpha^3 = a^{34} + a^{35}, \quad d\alpha^4 = d\alpha^5 = 0.
\]
It is clear that \(\mathfrak{h}_1\) is obtained when \(a = b = 0\) in the family (24). If \((a, b) \neq (0, 0)\) then, after the change of basis \(f^1 = e^2, f^2 = e^1, f^3 = e^4, f^4 = e^3, f^5 = e^5\) if necessary, we can suppose that \(b \neq 0\).

Now, let us fix a pair \((a, b)\) with \(b \neq 0\). Let us consider equations (24) for the pair \((a, r = b \neq 0)\) in terms of \(e^1, \ldots, e^5\). Then, the new basis given by
\[
f^1 = \sin \sigma e^1 - \cos \sigma (\cos \theta e^2 - \sin \theta e^3), \\
f^2 = \cos \sigma e^1 + \sin \sigma (\cos \theta e^2 - \sin \theta e^3), \\
f^3 = \sin \sigma (\sin \theta e^2 + \cos \theta e^3) + \cos \sigma e^4, \\
f^4 = -\cos \sigma (\sin \theta e^2 + \cos \theta e^3) + \sin \sigma e^4, \\
f^5 = e^5,
\]
where \(\theta \in (0, 2\pi)\) is such that \(\cos \theta = a/\sqrt{a^2 + b^2}, \sin \theta = b/\sqrt{a^2 + b^2}\) and \(\sigma = (\theta - \pi)/3\), satisfies equations of the form (24) for the given pair \((a, b)\). Therefore, the Lie algebras underlying (24) are all isomorphic to \(\mathfrak{h}_4\).

Any Lie algebra in the family (25) is isomorphic to \(\mathfrak{h}_5\), because with respect to the new basis
\[
\alpha^1 = e^4 - \frac{r}{2} e^5, \quad \alpha^2 = -\sqrt{2} e^2, \quad \alpha^3 = -e^4 - \frac{r}{2} e^5, \quad \alpha^4 = \sqrt{2} re^3, \quad \alpha^5 = re^1,
\]
the equations (25) transform into
\[
d\alpha^1 = -\alpha^{15} - \alpha^{24}, \quad d\alpha^2 = -\alpha^{34}, \quad d\alpha^3 = \alpha^{35}, \quad d\alpha^4 = -\alpha^{45}, \quad d\alpha^5 = 0.
\]
Also, the Lie algebras underlying (23) are all isomorphic to \(\mathfrak{h}_5\). In fact, let us fix a pair \((a, r)\) with \(r \neq 0\), and let us consider equations (24) for \(s = (a^2 + r^2)/r\) in terms of \(e^1, \ldots, e^5\). Then, the new basis given by

\[
\begin{align*}
    f^1 &= e^1, \\
    f^2 &= \cos \theta e^2 + \sin \theta e^3, \\
    f^3 &= -\sin \theta e^2 + \cos \theta e^3, \\
    f^4 &= e^4, \\
    f^5 &= e^5.
\end{align*}
\]

where \(\theta \in (0, 2\pi)\) is such that \(\cos \theta = a/\sqrt{a^2 + r^2}\) and \(\sin \theta = r/\sqrt{a^2 + r^2}\), satisfies equations of the (24) for the given pair \((a, r)\). Therefore, the Lie algebras underlying (24) are all isomorphic to \(\mathfrak{h}_5\).

Diatta obtains in [11] a list of solvable contact Lie algebras in dimension 5 and many of them have non-trivial center. Notice that \(\mathfrak{h}_1, \ldots, \mathfrak{h}_5\) correspond to the Lie algebras 1, 4\((p = 1, q = -3)\), 22, 18\((p = q = -1)\) and 15\((p = -1)\), respectively, in Diatta’s list and that the center of the solvable Lie algebras \(\mathfrak{h}_2, \ldots, \mathfrak{h}_5\) is trivial.

**Definition 3.5** Let \(\mathfrak{g}\) and \(\tilde{\mathfrak{g}}\) be Lie algebras endowed with hypo structures \((\eta, \omega_1)\) and \((\tilde{\eta}, \tilde{\omega}_1)\), respectively. We say that the hypo structures are **equivalent by rotation** if there is an isomorphism of Lie algebras \(F: \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}\) such that \(\eta = F^*\tilde{\eta}, \omega_3 = F^*\tilde{\omega}_3, \omega_1 = \cos \theta F^*\tilde{\omega}_2 - \sin \theta F^*\tilde{\omega}_1\) and \(\omega_2 = \sin \theta F^*\tilde{\omega}_1 + \cos \theta F^*\tilde{\omega}_2\), for some \(\theta\).

If two hypo structures are equivalent by rotation via \(F\) then \(F\) preserves the induced metrics. In the following result we show which families of hypo-contact structures given in Proposition 3.3 are equivalent by rotation.

**Proposition 3.6** Any hypo-contact structure in the family (23) (respectively, (20)) is equivalent by rotation to a hypo-contact structure in the family (24) for some \((a, b) \neq (0, 0)\) (respectively, 25).

**Proof :** Let us consider equations (23) for \((a, r \neq 0)\) in terms of \(e^1, \ldots, e^5\), and equations (24) for \((a, b = r \neq 0)\) in terms of \(f^1, \ldots, f^5\). Then, (35) defines an isomorphism of Lie algebras such that \(e^5 = f^5, e^{14} + e^{23} = f^{14} + f^{23}\) and

\[
e^{12} + e^{34} = \cos(-\theta)(f^{12} + f^{34}) - \sin(-\theta)(f^{13} + f^{42}), \quad e^{13} + e^{42} = \sin(-\theta)(f^{12} + f^{34}) + \cos(-\theta)(f^{13} + f^{42}).
\]

This shows that any hypo-contact structure in family (23) is equivalent by rotation to the hypo-contact structure in the family (24) for \((a, b = r \neq 0)\).

Let us consider equations (26) for \((a, r \neq 0)\) in terms of \(f^1, \ldots, f^5\), and equations (25) for \(s = (a^2 + r^2)/r\) in terms of \(e^1, \ldots, e^5\). Then, (36) defines an isomorphism of Lie algebras such that \(f^5 = e^5, f^{14} + f^{23} = e^{14} + e^{23}\) and

\[
f^{12} + f^{34} = \cos \theta(e^{12} + e^{34}) - \sin \theta(e^{13} + e^{42}), \quad f^{13} + f^{42} = \sin \theta(e^{12} + e^{34}) + \cos \theta(e^{13} + e^{42}).
\]

This shows that any hypo-contact structure in family (26) is equivalent by rotation to a hypo-contact structure in the family (25).

**Remark 3.7** A direct calculation shows that any equivalence by rotation between hypo-contact structures in the families (23) and (24) must have \(\theta \neq 0\). The same holds for any equivalence by rotation between hypo-contact structures in the families (25) and (26).

### 4 \(K\)-contact and \(\eta\)-Einstein structures

The Lie algebras described in Proposition 3.3 cannot be Einstein [11]. In this section, we show that any Lie algebra of (22) and the Lie algebra of (23) for \(a = b = 0\) with the hypo-contact structure defined by (15) are the only ones which are \(\eta\)-Einstein [11] or, equivalently [9], Sasakian. Moreover, we prove that these Lie algebras are also \(K\)-contact [3].

Consider an odd-dimensional manifold \(M\) with a contact form \(\eta\) and associated metric \(g\). Denote by \(\xi\) the vector field on \(M\) dual of \(\eta\). Recall that \((M, g, \eta)\) is said to be \(K\)-contact if \(\xi\) is a Killing vector field;
(M, g, η) is called $\eta$-Einstein \[15\] if there exist smooth functions $\tau, \nu$ on M such that the Ricci curvature tensor satisfies
\[
\text{Ric}(X, Y) = \tau g(X, Y) + \nu \eta(X)\eta(Y),
\]
for any vector fields $X, Y$ on $M$. The functions $\tau$ and $\nu$ are uniquely determined by
\[
s = \tau \dim M + \nu, \quad \text{Ric}(\xi, \xi) = \tau + \nu,
\]
where $s$ denotes the scalar curvature of $g$. When $\nu = 0$ we have the well-known Einstein condition. In our situation, $\tau$ and $\nu$ are constant and $\xi$ is the vector dual to $\eta = e^3$.

In the following proposition we distinguish the solvable Lie algebras of Proposition 3.3 for which the hypo-contact structure defined by (15) is $K$-contact.

**Proposition 4.1** Let $g$ be a solvable Lie algebra with a $K$-contact hypo structure $(\eta, \omega_1, \omega_2, \omega_3)$ and with a basis $e^1, \ldots, e^5$ for $g^*$ which satisfies the conditions of Proposition 3.4. Then, its structure equations reduce to (22) or (24) for $a = b = 0$. Moreover, they are Sasakian and $\eta$-Einstein.

**Proof:** Denote by $e_1, \ldots, e_5$ the dual basis of $e^1, \ldots, e^5$. The vector $e_5$ is a Killing vector if and only if $g(\nabla_Y e_5, Z) + g(\nabla_Z e_5, Y) = 0$ for any $Y, Z \in g$ or, equivalently, $g([e_5, e_i], e_j) + g([e_5, e_j], e_i) = 0$, for any $i, j$. Since the basis $e_1, \ldots, e_5$ is orthonormal, the latter condition is equivalent to
\[
d\eta = -2\omega_3, \quad d\omega_1 = \lambda \omega_2 \wedge \eta, \quad d\omega_2 = -\lambda \omega_1 \wedge \eta,
\]
for any $i, j$. In particular, for $(i, j) = (1, 4)$, since $e^1$ is closed, from Proposition 3.3 we get that $d\eta = 0$ for the families (22), (23), (24) unless $a = b = 0$. Therefore, these families are not $K$-contact.

On the other hand, it is clear that (22) and (24) for $a = b = 0$ satisfy (37). Moreover, the equations
\[
d\eta = -2\omega_3, \quad d\omega_1 = \lambda \omega_2 \wedge \eta, \quad d\omega_2 = -\lambda \omega_1 \wedge \eta,
\]
are satisfied for these families, where $\lambda = -3r^2$ for the family (22), and $\lambda = 0$ for (24) with $a = b = 0$. Therefore, $(\omega, g)$ is $\eta$-Einstein (see [5]). But Theorem 14 of [9] asserts that a hypo structure is Sasakian if and only if it is $\eta$-Einstein, which completes the proof.

Furthermore, concerning the $\eta$-Einstein property we have:

**Proposition 4.2** The families (22) and (24) for $a = b = 0$ with the hypo-contact structure defined by (15) are the only ones which are $\eta$-Einstein.

**Proof:** First, notice that the $\eta$-Einstein condition is preserved under equivalence by rotation. Therefore, from Proposition 3.6 it suffices to study the families (21), (22), (24) and (25). By direct computation one can check that the nonzero components of the Ricci tensor for these four families are given respectively by
\[
\text{Ric}(e_i, e_i) = \begin{cases} 
-\frac{1}{2}(g_{44} + 18r^2 + 4), & i = 1, \\
-(3r^2 + 2), & i = 2, 3, \\
\frac{1}{2}(g_{44} + 6r^2 - 4), & i = 4, \\
-\frac{1}{2}(g_{44} - 8), & i = 5.
\end{cases}
\]
\[
\text{Ric}(e_i, e_i) = \begin{cases} 
-2(3r^2 + 1), & i = 1, \ldots, 4, \\
4, & i = 5.
\end{cases}
\]
\[
\text{Ric}(e_i, e_i) = \begin{cases} 
-\frac{1}{2}((a^2 + b^2)^2 + 8(a^2 + b^2) + 4), & i = 1, 2, \\
\frac{1}{2}((a^2 + b^2)^2 - 4), & i = 3, 4, \\
4 - (a^2 + b^2)^2, & i = 5.
\end{cases}
\]
\[
\text{Ric}(e_i, e_i) = \begin{cases} 
-\frac{1}{8} (r^4 + 16r^2 + 16), & i = 1, 3, \\
\frac{1}{8} (r^4 - 16), & i = 2, 4, \\
-\frac{1}{4} (r^4 - 16), & i = 5.
\end{cases}
\]

Therefore, \((\eta = e^5, g)\) is \(\eta\)-Einstein only for the families (22) and (24) for \(a = b = 0\). Notice that

\[
\text{Ric}(X, Y) = -2(1 + 3r^2)\eta(X, Y) + 6(1 + r^2)\eta(X)\eta(Y)
\]

for the family (22), and

\[
\text{Ric}(X, Y) = -2g(X, Y) + 6\eta(X)\eta(Y)
\]

for (24) with \(a = b = 0\).

\[\square\]

**Remark 4.3** We must notice that for the families (22) and (24) with \(a = b = 0\), \(\eta\)-Einstein condition implies \(K\)-contact property, which in general is not true.

As a consequence of the above propositions and Theorem 1.1, we conclude

**Corollary 4.4** The only 5-dimensional solvable Lie algebras admitting a \(K\)-contact hypo structure or a hypo-contact \(\eta\)-Einstein structure are \(\mathfrak{h}_1\) and \(\mathfrak{h}_2\).

Hypo-contact structures are related to the contact Calabi-Yau structures introduced recently in [16]. A contact Calabi-Yau structure on a \((2n + 1)\)-dimensional manifold \(M\) is a triple \((\eta, J, \epsilon)\), where \((\eta, J)\) is a Sasakian structure and \(\epsilon \in \Lambda_{n+1}^n(\ker \eta)\) is a nowhere vanishing basic form on \(\ker \eta\) such that \(d\epsilon = 0\) and \(\epsilon \wedge \overline{\epsilon} = (-1)^{\frac{n(n+1)}{2}} i^n(d\eta)^n\). In dimension 5, if \((\eta, J, \epsilon)\) is a contact Calabi-Yau structure then the quadruplet \((-\eta, \omega_1 = \Re \epsilon, \omega_2 = 3\iota \epsilon, \omega_3 = -\frac{1}{7}d\eta\) defines a hypo-contact structure on \(M\) for which \(d\omega_1 = d\omega_2 = 0\) and the metric induced by \((\eta, J)\) is \(\eta\)-Einstein with \(\tau = -2\) and \(\nu = 6\) [16 Corollary 3.7].

Tomassini and Vezzoni classify 5-nilmanifolds admitting invariant contact Calabi-Yau structure and prove that, up to isomorphism, the only (non-trivial) 5-dimensional nilpotent Lie algebra admitting hypo-contact structure is \(\mathfrak{h}_1\). This result also follows directly from the fact that \(\mathfrak{h}_1\) is the only 5-dimensional nilpotent Lie algebra admitting a Sasakian structure [17 Corollary 5.5]. Next we show that there are no 5-dimensional solvable non-nilpotent Lie algebras admitting contact Calabi-Yau structure.

**Proposition 4.5** Let \(\mathfrak{g}\) be a 5-dimensional Lie algebra such that \([\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}\) admitting a contact Calabi-Yau structure. Then, \(\mathfrak{g}\) is isomorphic to the nilpotent Lie algebra \(\mathfrak{h}_1\).

**Proof:** It is sufficient to prove that if \(\mathfrak{g}\) admits a hypo-contact structure \((\eta, \omega_1)\) with \(\omega_1\) and \(\omega_2\) closed, then \(\mathfrak{g}\) is isomorphic to \(\mathfrak{h}_1\). If \(\mathfrak{g}\) is solvable then from Proposition 3.3 one can see directly that \(\omega_1\) and \(\omega_2\) are both closed only if \(a = b = 0\) in the family (24), which corresponds to \(\mathfrak{h}_1\). Finally, when \(\mathfrak{g}\) is not solvable we apply Remark 3.4 and a direct calculation shows that \(\omega_1\) and \(\omega_2\) are also nonclosed in this case.

\[\square\]

### 5 Metrics with holonomy \(SU(3)\)

The purpose of this section is to prove Theorem 1.2, that is, any left-invariant hypo-contact structure \((\eta, \omega_1, \omega_2, \omega_3)\) on a solvable Lie group \(N\) gives rise to a metric with holonomy \(SU(3)\) via the Conti-Salamon evolution equations [16]. From now on, to avoid confusion, we denote the exterior differential on \(N\) by \(d\), and the exterior differential on \(N \times I\) by \(\hat{d}\). Then, the (hypo) evolution equations (38) are written as follows

\[
\begin{align*}
\partial_t \omega_3(t) &= -\hat{d}\eta(t), \\
\partial_t (\omega_2(t) \wedge \eta(t)) &= \hat{d}\omega_1(t), \\
\partial_t (\omega_1(t) \wedge \eta(t)) &= -\hat{d}\omega_2(t).
\end{align*}
\]
In order to prove Theorem 1.2 we first observe the following fact. Let \((\eta, \omega_i)\) and \((\tilde{\eta}, \tilde{\omega}_i)\) be two hypo-contact structures on a Lie algebra \(g\) which are equivalent by rotation in the sense of Definition 3.3. If \((\tilde{\eta}(t), \tilde{\omega}_i(t))\) is a solution of the evolution equations (38) for \((\tilde{\eta}, \tilde{\omega}_i)\), then \(\eta(t) = F^*(\tilde{\eta}(t))\), \(\omega_i(t) = F^*(\tilde{\omega}_i(t))\), \(\omega_1(t) = \cos \theta F^*(\tilde{\omega}_1(t)) - \sin \theta F^*(\tilde{\omega}_2(t))\) and \(\omega_2 = \sin \theta F^*(\tilde{\omega}_1(t)) + \cos \theta F^*(\tilde{\omega}_2(t))\) is a solution of (38) for the hypo-contact structure \((\eta, \omega_i)\). Therefore, it suffices to prove the theorem up to equivalence by rotation of the hypo-contact structure.

Proof of Theorem 1.2: From the observation above, Theorem 1.1 and Propositions 3.3 and 3.6 we shall concentrate on the families (24), (22), (24) and (25), showing for each case the existence of a solution of the evolution equations for which the metric associated to the corresponding integrable \(SU(3)\)-structure has holonomy group equal to \(SU(3)\).

We consider first the \(\eta\)-Einstein case in detail. In this case we have that \(d\omega_1 = \lambda \omega_2 \wedge e^5\) and \(d\omega_2 = -\lambda \omega_1 \wedge e^5\), where \(\lambda = -3r^2\) for the family (22) and \(\lambda = 0\) for the nilpotent Lie algebra corresponding to (24) for \(a = b = 0\). A solution of the hypo evolution equations is given by

\[
\eta(t) = \frac{1}{2} f'(t) e^5, \quad \omega_1(t) = f(t) (e^{12} + e^{34}), \quad \omega_2(t) = f(t) (e^{13} - e^{24}), \quad \omega_3(t) = f(t) (e^{14} + e^{33}),
\]

where \(f(t)\) is a function such that \(f(0) = 1, f'(0) = 2\) and satisfies the ordinary differential equation

\[
ff'' + (f')^2 - 2\lambda f = 0.
\]

For \(\lambda = 0\) one has the explicit solution \(f(t) = (1 + 4t)^{1/2}\) and the Riemannian metric with \(SU(3)\)-holonomy that one gets is the one obtained in [12], namely

\[
g = (1 + 4t)^{1/2}((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{1}{4} r^2 (e^5)^2 + dt^2.
\]

If \(\lambda = -3r^2 < 0\) then, after performing a first integration, one obtains the first order differential equation

\[
f'(t) = \frac{2}{f(t)} \left(1 + r^2 - r^2 f^3(t)\right)^{1/4}
\]

with initial condition \(f(0) = 1\). Therefore, there exists a unique solution \(f(t)\) defined on some open neighbourhood \(I\) around \(t = 0\). The basis of 1-forms on the manifold \(H_3 \times I\) given by

\[
\eta^1 = \sqrt{f(t)} e^1, \quad \eta^2 = \sqrt{f(t)} e^2, \quad \eta^3 = \sqrt{f(t)} e^3, \quad \eta^4 = \sqrt{f(t)} e^4, \quad \eta^5 = \frac{f'(t)}{2} e^5, \quad \eta^6 = dt,
\]

is orthonormal with respect to the Riemannian metric associated to the corresponding integrable \(SU(3)\)-structure on \(H_3 \times I\). By computing the curvature forms \(\Omega^1\) and applying the Ambrose-Singer theorem, one can see that the holonomy group is actually \(SU(3)\). In fact, a direct calculation shows that, for each \(r\), the curvature forms

\[
\Omega^1 = -\Omega^3 = -\frac{4r^2 f(t) + (f'(t))^2}{4f^3(t)} (\eta^{12} - \eta^{34}),
\]

\[
\Omega^1 = \Omega^2 = \frac{4r^2 f(t) + (f'(t))^2}{4f^3(t)} (\eta^{13} + \eta^{24}),
\]

\[
\Omega^1 = \Omega^4 = -\frac{4r^2 f(t) + (f'(t))^2}{2f^2(t)} (2\eta^{14} + \eta^{23}) + \frac{4(f'(t))^2 - 2f(t)f''(t)}{2f^2(t)} \eta^{56},
\]

\[
\Omega^1 = \Omega^6 = \frac{(f'(t))^2 - 2f(t)f''(t)}{4f^2(t)} (\eta^{15} + \eta^{46}),
\]

\[
\Omega^1 = -\Omega^5 = \frac{(f'(t))^2 - 2f(t)f''(t)}{4f^2(t)} (\eta^{16} - \eta^{45}),
\]

\[
\Omega^2 = -\Omega^3 = \frac{4r^2 f(t) + (f'(t))^2}{2f^2(t)} (\eta^{14} + 2\eta^{23}) + \frac{4(f'(t))^2 - 2f(t)f''(t)}{2f^2(t)} \eta^{56},
\]

\[
\Omega^2 = \Omega^3 = \frac{(f'(t))^2 - 2f(t)f''(t)}{4f^2(t)} (\eta^{25} + \eta^{36}),
\]

\[
\Omega^2 = -\Omega^5 = \frac{(f'(t))^2 - 2f(t)f''(t)}{4f^2(t)} (\eta^{26} - \eta^{35}),
\]

\[
\Omega^2 = -\Omega^6 = \frac{(f'(t))^2 - 2f(t)f''(t)}{4f^2(t)} (\eta^{35} - \eta^{26}),
\]

\[
\Omega^3 = -\Omega^4 = \frac{(f'(t))^2 - 2f(t)f''(t)}{4f^2(t)} (\eta^{36} - \eta^{25}),
\]

\[
\Omega^3 = \Omega^5 = \frac{4r^2 f(t) + (f'(t))^2}{4f^3(t)} (\eta^{15} - \eta^{46}).
\]
are linearly independent at \( t = 0 \), since \( f(0) = 1, f'(0) = 2, f''(0) = -2(3r^2 + 2) \) and \( f'''(0) = 24(r^2 + 1) \). Therefore, any \( \eta \)-Einstein hypo-contact structure gives rise to a metric whose holonomy group is equal to \( SU(3) \).

For the family (24) a solution for the hypo evolution equations is given by
\[
\eta(t) = \frac{1}{2} f'(t) e^5, \quad \omega_1(t) = \frac{1}{2} f'(t) f''(t) e^{12} + \frac{2}{f(t)} e^{34}, \\
\omega_2(t) = f(t) (e^{13} - e^{24}), \quad \omega_3(t) = f(t) (e^{14} + e^{23}),
\]
where \( f(t) \) satisfies the differential equation
\[
4\rho + f'^3 + ff'f'' = 0
\]
with initial conditions \( f(0) = 1 \) and \( f'(0) = 2 \), where \( \rho = a^2 + b^2 \). After performing a first integration, one obtains the first order differential equation
\[
f'(t) = \left( 8 + 4\rho - 4\rho f^3(t) \right)^{1/3}
\]
with initial conditions \( f(0) = 1 \). Therefore, there exists a unique solution \( f(t) \) defined on some open neighbourhood \( I \) around \( t = 0 \).

A similar computation as in the \( \eta \)-Einstein case above, shows that the holonomy group of the metric associated to the corresponding integrable \( SU(3) \)-structure on \( H_3 \times I \) is also equal to \( SU(3) \). In fact, the curvature forms \( \Omega_1, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8 \) and \( \Omega_9 \) take the following values when \( t = 0 \):
\[
(\Omega_1)_{t=0} = -\frac{(\rho-2)^2}{4}(e^{12} - e^{34}), \\
(\Omega_1)_t = \frac{(\rho-2)^2-8}{4}(e^{13} + e^{24}) + bp(e^{15} + e^{46}) + a\rho(e^{25} + e^{36}), \\
(\Omega_2)_t = \frac{(\rho-2)^2-12}{2}(e^{14} - e^{23}) + b\rho(e^{15} + e^{46}) - 2e^{23} + b\rho(e^{25} + e^{36}) - \frac{(\rho-2)(\rho-6)}{2} e^{56}, \\
(\Omega_3)_t = b\rho(e^{13} + e^{24}) - a\rho(e^{14} - e^{23}) - \frac{(\rho+2)(\rho-6)}{4}(e^{15} + e^{46}), \\
(\Omega_4)_t = 3(\rho+2)^2(4e^{16} - e^{45}), \\
(\Omega_5)_t = -2e^{14} + a\rho(e^{15} + e^{46}) + \frac{(\rho-2)^2-12}{2} e^{23} - b\rho(e^{25} + e^{36}) - \frac{(\rho+2)(\rho-6)}{2} e^{56}, \\
(\Omega_6)_t = a\rho(e^{13} + e^{24}) + b\rho(e^{14} - e^{23}) - \frac{(\rho+2)(\rho-6)}{2}(e^{25} + e^{36}), \\
(\Omega_7)_t = 3(\rho+2)^2(e^{26} - e^{35}),
\]
where \( e^6 \) denotes the 1-form \( dt \) evaluated at \( t = 0 \), and they are linearly independent if and only if \( \rho \neq 2, 6 \). Moreover, if \( \rho = a^2 + b^2 = 2 \) then \( (\nabla_{\omega_1} \nabla_{\omega_2} \Omega_{12})_{t=0} = -288(e^{12} - e^{34}) \) and
\[
(\nabla_{\omega_1} \Omega_{3})_{t=0} = 12b(e^{13} + e^{24}) - 12a(e^{14} - e^{23}) - 96(e^{15} + e^{46}), \\
(\nabla_{\omega_2} \Omega_{3})_{t=0} = 12a(e^{13} + e^{24}) + 12b(e^{14} - e^{23}) - 96(e^{25} + e^{36}),
\]
which implies that \( \nabla_{\omega_1} \nabla_{\omega_2} \Omega_{12}, \Omega_{3}, \nabla_{\omega_1} \Omega_{12}, \Omega_{3}, \Omega_{14}, \nabla_{\omega_2} \Omega_{5}, \Omega_{6}, \Omega_{3}, \nabla_{\omega_2} \Omega_{5} \) and \( \Omega_{12} \) are linearly independent at \( t = 0 \).

For the remaining case \( \rho = a^2 + b^2 = 6 \), since
\[
(\nabla_{\omega_1} \Omega_{12})_{t=0} = 536e^{14} - 96a(e^{15} + e^{46}) + 40e^{23} + 96b(e^{25} + e^{36}) - 576e^{56},
\]
we have that the forms \( \Omega_{12}, \Omega_{3}, \nabla_{\omega_1} \Omega_{12}, \Omega_{3}, \Omega_{14}, \nabla_{\omega_2} \Omega_{5}, \Omega_{6}, \Omega_{3}, \nabla_{\omega_2} \Omega_{5} \) and \( \Omega_{12} \) are independent at \( t = 0 \). Therefore, any left-invariant hypo-contact structure on the Lie group \( H_3 \) gives rise to a metric with holonomy \( SU(3) \).

For the family (24) a solution of the hypo evolution equations is given by
\[
\eta(t) = \frac{1}{2} f'(t) e^5, \quad \omega_1(t) = f(t) (e^{12} + e^{34}), \\
\omega_2(t) = \frac{1}{2} f^2(t) f''(t) e^{13} - \frac{2}{f(t)} e^{24}, \quad \omega_3(t) = f(t) (e^{14} + e^{23}),
\]
where $f(t)$ satisfies the differential equation

$$2r^2 + f'^3 + ff'' = 0$$

with initial conditions $f(0) = 1$ and $f'(0) = 2$. After performing a first integration, one obtains the first order differential equation

$$f'(t) = \frac{(8 + 2r^2 - 2r^2 f^3(t))^{1/3}}{f(t)}$$

with initial conditions $f(0) = 1$, so there exists a unique solution $f(t)$ defined on some open neighbourhood $I$ around $t = 0$. A similar computation as above shows that the holonomy group of the metric associated to the corresponding integrable $SU(3)$-structure on $H_5 \times I$ is again equal to $SU(3)$. In fact

$$\begin{align*}
(\Omega_2^1)_{t=0} &= \frac{(r^2+4)(r^2-4)}{16}(e^{12} - e^{34}), \\
(\Omega_1^1)_{t=0} &= \frac{2}{4}(r^2-8e^{14} - e^{23} - e^{12})e^{56}, \\
(\Omega_6^1)_{t=0} &= \frac{3(r^2+4)^2}{16}(e^{16} - e^{45}), \\
(\Omega_5^2)_{t=0} &= \frac{3}{16}(e^{25} + e^{36}), \\
(\Omega_5^3)_{t=0} &= -\frac{2}{16}(e^{14} - \frac{r^2-8e^{23} + e^{12}}{4}e^{56}),
\end{align*}$$

where again $e^6$ denotes the 1-form $dt$ evaluated at $t = 0$. Moreover,

$$\begin{align*}
(\nabla_\omega \Omega_2^1)_{t=0} &= \frac{(r^2+4)(r^4-2r^2+16)}{8}(e^{12} - e^{34}), \\
(\nabla_\omega \Omega_1^1)_{t=0} &= -\frac{(r^2+4)(r^4-8r^2+48)}{8}(e^{15} + e^{46}), \\
(\nabla_\omega \Omega_6^1)_{t=0} &= -\frac{2}{4}(r+4)(r-4)e^{14} - (r^4 - 3r^2 + 32)e^{23} + (r^4 - 4r^2 + 48)e^{56}, \\
(\nabla_\omega \Omega_5^2)_{t=0} &= -\frac{2}{4}(r+4)(r^4-8r^2+48)(e^{25} - e^{35}).
\end{align*}$$

A direct calculation shows that, for each $r$, eight of the twelve 2-forms above are linearly independent.

Finally, for the the family (21), a solution of the hypo evolution equations is given by

$$\eta(t) = \frac{1}{2}f(t)e^5, \quad \omega_1(t) = \frac{1}{4}f^2(t)f'(t)e^{12} + \frac{2}{f(t)}e^{34},$$

$$\omega_2(t) = \frac{1}{2}f^2(t)f'(t)e^{13} - \frac{2}{f(t)}e^{24}, \quad \omega_3(t) = f(t)(e^{14} + e^{23}),$$

where $f(t)$ satisfies the differential equation

$$12r^2 + f't' + f'^2f'' = 0$$

with initial conditions $f(0) = 1$ and $f'(0) = 2$. Equivalently, $f(t)$ must satisfy the first order differential equation

$$f'(t) = \frac{2}{f(t)}\sqrt{1 + r^2 - r^2f^3(t)}$$

with initial conditions $f(0) = 1$, so there exists a unique solution $f(t)$ defined on some open neighbourhood $I$ around $t = 0$. One can prove that the holonomy of the resulting metric on $H_2 \times I$ is again $SU(3)$. $\textbf{QED}$

6 Metrics with holonomy $G_2$

Let $H$ be a simply connected solvable Lie group of dimension 5 with a left-invariant hypo-contact structure. In order to prove Theorem 1.3, we study first the induced half-flat structures on the total space of a circle bundle over $H$. In particular, we will show that many hypo-contact structures on $H$ define not only the natural half-flat structure on the trivial bundle $H_5 \times \mathbb{R}$ but also another half-flat structure on a non-trivial $S^4$-bundle, which allows us to construct a metric with holonomy $G_2$. $\textbf{QED}$
Let us recall that Hitchin in [13] proved that if $M$ is a 6-manifold with a half-flat structure $(F, \Psi_+, \Psi_-)$ which belongs to a family $(F(t), \Psi_+(t), \Psi_-(t))$ of half-flat structures on $M$, for some real parameter $t$ lying in some interval $I = (t_-, t_+)$, satisfying the evolution equations

\[
\begin{align*}
\partial_t \Psi_+(t) &= \hat{d}F(t), \\
F(t) \wedge \partial_t (F(t)) &= -\hat{d}\Psi_-(t),
\end{align*}
\]

then $M \times I$ has a Riemannian metric whose holonomy is contained in $G_2$. In fact, it is easy to check that the 4-forms $\varphi$ and $*\varphi$ given by

$$\varphi = F(t) \wedge dt + \Psi_+(t), \quad *\varphi = \psi_-(t) \wedge dt + \frac{1}{2}F(t)^2,$$

are closed.

Next, we show that a solution of (hypo) evolution equations produces a solution of Hitchin evolution equations. Let $N$ be a 5-manifold with a hypo structure $(\eta, \omega)$ which belongs to a one-parameter family of hypo structures $(\eta(t), \omega(t))$, for some real parameter $t \in I$, satisfying the (hypo) evolution equations (38). Then, we know that an integrable $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ on $M = N \times I$ is given by

$$F = \eta(t) \wedge dt + \omega_3(t), \quad \Psi = (\omega_1(t) + i\omega_2(t)) \wedge (\eta(t) + idt).$$

On the other hand, Proposition 2.2 implies that the $SU(3)$-structure $(F, \Psi_+, \Psi_-)$ on $M = N \times \mathbb{R}$ given by

$$F = \lambda \omega_1 + \mu \omega_2 + \eta \wedge e^6, \quad \Psi_+ = (-\mu \omega_1 + \lambda \omega_2) \wedge \eta - \omega_3 \wedge e^6, \quad \Psi_- = (-\mu \omega_1 + \lambda \omega_2) \wedge e^6 + \omega_3 \wedge \eta,$$

is half-flat for $\lambda, \mu \in \mathbb{R}$ with $\lambda^2 + \mu^2 = 1$. Moreover, using again Proposition 2.2 we have the one-parameter family of half-flat structures $(F(t), \Psi_+(t), \Psi_-(t))$ on $M = N \times \mathbb{R}$ defined by

\[
\begin{align*}
F(t) &= \lambda \omega_1(t) + \mu \omega_2(t) + \eta(t) \wedge e^6, \\
\Psi_+(t) &= (-\mu \omega_1(t) + \lambda \omega_2(t)) \wedge \eta(t) - \omega_3(t) \wedge e^6, \\
\Psi_-(t) &= (-\mu \omega_1(t) + \lambda \omega_2(t)) \wedge e^6 + \omega_3(t) \wedge \eta(t),
\end{align*}
\]

where $e^6(t) = e^6$, for any $t$.

**Proposition 6.1** The family $(F(t), \Psi_+(t), \Psi_-(t))$ of half-flat structures on $M = N \times \mathbb{R}$ given by (40) is a solution of the Hitchin evolution equations (39).

**Proof:** Clearly, $\hat{d}F(t) = \lambda \hat{d}\omega_1(t) + \mu \hat{d}\omega_2(t)$ and from equations (38) we have $\partial_t \Psi_+(t) = \hat{d}F(t)$. Moreover, since $\hat{d}\Psi_-(t) = (-\mu \hat{d}\omega_1(t) + \lambda \hat{d}\omega_2(t)) \wedge e^6 + \hat{d}(\omega_3(t) \wedge \eta(t))$ and

$$F(t) \wedge \partial_t F(t) = \frac{1}{2} \partial_t ((\lambda \omega_1(t) + \mu \omega_2(t))^2) + [\lambda \partial_t (\omega_1(t) \wedge \eta(t)) + \mu \partial_t (\omega_2(t) \wedge \eta(t))] \wedge e^6,$$

the second equation in (39) is satisfied if and only if

$$\hat{d}(\omega_3(t) \wedge \eta(t)) = -\frac{1}{2} \partial_t ((\lambda \omega_1(t) + \mu \omega_2(t))^2).$$

But, from (11) and $\lambda^2 + \mu^2 = 1$ we get $(\lambda \omega_1(t) + \mu \omega_2(t))^2 = \omega_3(t) \wedge \omega_3(t)$, and therefore

$$\frac{1}{2} \partial_t ((\lambda \omega_1(t) + \mu \omega_2(t))^2) = \frac{1}{2} \partial_t (\omega_3(t) \wedge \omega_3(t)) = \omega_3(t) \wedge \partial_t \omega_3(t) = -\omega_3(t) \wedge \hat{d} \eta(t) = -\hat{d}(\omega_3(t) \wedge \eta(t)).$$

We must notice that this result, which is also used in (10), implies that the holonomy of the resulting $G_2$-metric on $M \times I$ is contained in $SU(3)$, because it is actually a product metric. This fact justifies our study of half-flat structures on non-trivial circle bundles (see Remark 6.2 below).
Let \( \mathfrak{h} \) be a solvable 5-dimensional Lie algebra with a hypo structure \((\eta, \omega_1, \omega_2, \omega_3)\). Consider the extension \( \mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}e_6 \), with \( e_6 \) such that the Jacobi identity is satisfied. The \( SU(3) \)-structure on \( \mathfrak{k} \) defined by

\[
F = \lambda \omega_1 + \mu \omega_2 + e^{56}, \quad \Psi_+ = (-\mu \omega_1 + \lambda \omega_2) \wedge e^{5} - \omega_3 \wedge e^{6}, \quad \Psi_- = (-\mu \omega_1 + \lambda \omega_2) \wedge e^{6} + \omega_3 \wedge e^{5},
\]

with \( \lambda^2 + \mu^2 = 1 \), is half-flat if and only if \( d(F \wedge F) = 2(\lambda \omega_1 + \mu \omega_2) \wedge e^{5} \wedge (de^6) = 0 \) and \( d(\Psi_+) = -\omega_3 \wedge (de^6) = 0 \). From these equations one has that

\[
de^6 = a_1 e^{12} + a_2 e^{13} + a_3(\epsilon^{14} - \epsilon^{23}) + a_5 e^{24} + a_6 e^{34},
\]

with \( \lambda(a_1 + a_6) + \mu(a_2 - a_5) = 0 \). Then \( d(de^6) = 0 \) only in the following cases:

1. \( de^6 = 0 \) for all the families;
2. \( de^6 = a_1 e^{12} + a_2 e^{13} \), with \( \lambda a_1 + \mu a_2 = 0 \) for the family (21);
3. \( de^6 = a_2(-\frac{a_2}{\lambda} e^{12} + e^{13}) \) for the family (23) with \( \mu = \frac{a_2}{\lambda} \);
4. \( de^6 = a_1 e^{12} \) for the family (24) with \( \lambda = 0 \);
5. \( de^6 = a_2 e^{13} \) for the family (25) with \( \mu = 0 \);
6. \( de^6 = a_1 e^{12} + \frac{a_2}{\lambda} e^{13} \) for the family (26) with \( \lambda = -\frac{a_2}{\lambda} \).

**Remark 6.2** Notice that the previous cases 2–6 give a classification of the half-flat structures on \( \mathfrak{k} \) which are a non-trivial extension of the hypo structure on \( \mathfrak{h} \).

**Proof of Theorem 1.3**: For the non-trivial \( S^1 \)-bundle \( K \) associated to the family (24) with \( \lambda = 0, \mu = 1 \) and \( de^6 = a_1 e^{12} \), one has that a solution of the evolution equations (39) is given by

\[
F(t) = f(t)(e^{13} - e^{24}) + k(t)h(t)e^{56},
\]

\[
\Psi_+(t) = -f(t)^2 k(t)^2 e^{125} - e^{345} - f(t)h(t)(e^{146} + e^{236}),
\]

\[
\Psi_-(t) = -f(t)^2 h(t)k(t)e^{126} - \frac{h(t)}{k(t)} e^{346} + k(t)f(t)(e^{145} + e^{235}),
\]

where \( f(t), k(t), h(t) \) are functions satisfying the system of ordinary differential equations

\[
\begin{cases}
(fh)' = 2kh, \\
(f^2k^2)' = a_1 k - 2(a^2 + b^2)f, \\
f'f = 2kf + \frac{a_1 h}{2k}f,
\end{cases}
\]

and the initial conditions \( f(0) = k(0) = h(0) = 1 \). This system is easily seen to be equivalent to

\[
f' = 2k + \frac{a_1 h}{2k}f, \quad h' = -\frac{a_1 h^2}{2k f^2}, \quad k' = -\frac{a^2 + b^2 + 2k^3}{kf},
\]

and thus by the theorem on existence of solutions for a system of ordinary differential equations, there exists an open interval \( I \) containing \( t = 0 \) on which the previous system admits a unique solution \( (f(t), k(t), h(t)) \) satisfying the initial condition \( f(0) = k(0) = h(0) = 1 \).

For \( a = b = 0 \), the 5-dimensional hypo-contact Lie algebra is the nilpotent Lie algebra \( \mathfrak{h}_1 \) and a solution in this case is given by

\[
a_1 = 2, \quad f(t) = (1 + 5t)^{\frac{a}{b}}, \quad h(t) = (1 + 5t)^{\frac{a}{b}}, \quad k(t) = (1 + 5t)^{-\frac{a}{b}}.
\]

The corresponding metric with holonomy \( G_2 \) that we obtain is the one found in [6].

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For \( a^2 + b^2 \neq 0 \) and \( a_1 = 2 \), the corresponding metric \( g(t) \) on \( K \times I \), where \( K \) has structure equations

\[
\begin{align*}
    d\epsilon^1 &= d\epsilon^2 = 0, \\
    d\epsilon^3 &= a e^{13} + b e^{14} - b e^{23} + a e^{24} - (a^2 + b^2) e^{25}, \\
    d\epsilon^4 &= b e^{13} - a e^{14} - (a^2 + b^2) e^{15} + a e^{23} + b e^{24}, \\
    d\epsilon^5 &= -2 e^{14} - 2 e^{23}, \\
    d\epsilon^6 &= 2 e^{12},
\end{align*}
\]

is given by

\[
g(t) = f(t)^2 k(t)((\epsilon^1)^2 + (\epsilon^2)^2) + \frac{1}{k(t)}((\epsilon^3)^2 + (\epsilon^4)^2) + k(t)^2(e^5)^2 + h(t)^2(e^6)^2 + (dt)^2.
\]

The metric \( g(t) \) has holonomy \( G_2 \) for \((a, b)\) in a small neighbourhood around \((0, 0)\), since the solution \((f(t), k(t), h(t))\) of the system \((\text{[11]})\) depends continuously on the parameters \(a\) and \(b\), and for \(a = b = 0\), the holonomy of the corresponding metric is \( G_2 \).

For the non-trivial extension on the Lie group \( \tilde{K} \) associated to the family \((\text{[20]})\) with \(\mu = 0, \lambda = 1\) and \(de^6 = a_2 e^{13}\), one has that a solution of the evolution equations \((\text{[39]})\) is given by

\[
\begin{align*}
    F(t) &= f(t)(e^{12} + e^{34}) + k(t)h(t)e^{56}, \\
    \Psi_+(t) &= f(t)^2 k(t)e^{135} - e^{245} - f(t)h(t)(e^{146} + e^{236}), \\
    \Psi_-(t) &= f(t)^2 h(t)k(t)e^{136} - \frac{h(t)}{k(t)}e^{246} + f(t)k(t)(e^{145} + e^{235}),
\end{align*}
\]

where \(f(t), k(t), h(t)\) are functions satisfying the system of ordinary differential equations

\[
(42) \quad f' = 2k - \frac{a_2 h}{2k f}, \quad h' = \frac{a_2 h^2}{2k f^2}, \quad k' = -\frac{r^2 + 4k^3}{2k f},
\]

and the initial conditions \(f(0) = k(0) = h(0) = 1\). Thus by the theorem on existence of solutions for a system of ordinary differential equations, there is an open interval \(I\) containing \(t = 0\) on which the previous system has a unique solution \((f(t), k(t), h(t))\) satisfying the initial condition \(f(0) = k(0) = h(0) = 1\).

Since the system \((\text{[12]})\) for \(r = 0\) and \(a_2 = -2\) coincides with the system \((\text{[11]})\) for \(a = b = 0\) and \(a_1 = 2\), we can use the same argument as for the previous family to prove that in a small neighbourhood around \(0\) the corresponding metric \(\tilde{g}(t)\) on \(\tilde{K} \times I\) has holonomy \(G_2\). In this case, \(\tilde{K}\) has structure equations

\[
\begin{align*}
    de^1 &= 0, \\
    de^2 &= re^{34} + \frac{r^2}{2} e^{35}, \\
    de^4 &= -\frac{r^2}{2} e^{15} + r e^{23}, \\
    de^5 &= -2 e^{14} - 2 e^{23}, \\
    de^6 &= -2 e^{13},
\end{align*}
\]

and the metric \(\tilde{g}(t)\) is given, in terms of the basis \((e^1, \ldots, e^6, dt)\) by

\[
\tilde{g}(t) = f(t)^2 k(t)((e^1)^2 + (e^2)^2) + \frac{1}{k(t)}((e^3)^2 + (e^4)^2) + k(t)^2(e^5)^2 + h(t)^2(e^6)^2 + (dt)^2.
\]

**Remark 6.3** Note that the 6-dimensional solvable Lie groups \(K\) (with \(a^2 + b^2 \neq 0\)) and \(\tilde{K}\) (with \(r \neq 0\)) are not isomorphic, since \(t^2 = 0\) for the first family while \(\tilde{t}^2 \neq 0\) for the second one. For \(a = b = 0\) and \(r = 0\) one gets the same 6-dimensional nilpotent Lie group. Moreover, taking into account the explicit isomorphisms given in the proof of Theorem \((\text{[11]})\) one can see that for any \((a, b) \neq (0, 0)\) the solvable Lie algebra \(\mathfrak{f}\) is isomorphic to

\[
\begin{align*}
    d\alpha^1 &= -\alpha^{14}, & d\alpha^2 &= -\alpha^{25}, & d\alpha^3 &= \alpha^{34} + \alpha^{35}, & d\alpha^4 &= \alpha^5 = 0, & d\alpha^6 &= \alpha^{45},
\end{align*}
\]

and that for any \(r \neq 0\) the Lie algebra \(\tilde{\mathfrak{f}}\) is isomorphic to the product \(\mathfrak{h}_5 \times \mathbb{R}, \mathfrak{h}_5\) being the solvable Lie algebra of Theorem \((\text{[11]})\).
Remark 6.4 From the proof of Theorem 1.3 above, we see that one can ensure that the holonomy of our examples equals $G_2$ when the parameters $a, b, r$ are sufficiently close to 0. To our knowledge, there is no similar result in the literature about existence of metrics of holonomy equal to $G_2$ neither on $K \times I$ nor on $\tilde{K} \times I$, so in this sense our result provides new spaces of $G_2$ holonomy.

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