Massless DKP field in a Lyra manifold

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Abstract

Massless scalar and vector fields are coupled to the Lyra geometry by means of the Duffin–Kemmer–Petiau (DKP) theory. Using the Schwinger variational principle, the equations of motion, conservation laws and gauge symmetry are implemented. We find that the scalar field couples to the anholonomic part of the torsion tensor, and the gauge symmetry of the electromagnetic field does not break by the coupling with torsion.

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1. Introduction

The existence and nature of the interaction of gravitational torsion with other fields is one of the most intriguing questions of the current research in the classical theory of gravitation. The usual geometries applied to model the gravitational interaction give results that are not in complete accordance with those physically expected. For instance, the Riemann geometry completely ignores the presence of spin as a fundamental characteristic of the matter field [1], but the quantum experiences demonstrate that it is on the same footing as the mass. The Riemann–Cartan geometry incorporates torsion as the geometric counterpart of spin, in the same sense as curvature is manifested by the presence of mass–energy. However, in the Einstein–Cartan theory [2] torsion is not a propagating quantity, remaining confined to the interior of matter, which is not in complete accord with the idea of a \textit{dynamic} spacetime.

Here, we propose using the Lyra geometry [3, 4] to model the gravitational content as a natural way to incorporate a dynamic torsion. In the Lyra geometry, scale transformations are incorporated in the structureless manifold, and all the differential geometry is constructed in harmony with these scale transformations. In the following section a brief introduction to the Lyra geometry is given.

Our main objective here is to check if the hypothesis of the Lyra geometry as a model of spacetime provides good physical expectation about the torsion behaviour. To do it, we couple the Lyra manifold with massless bosonic fields of spins 0 and 1 through the Duffin–Kemmer–Petiau theory [5, 6], which has the advantage of an algebraically unified description.
of both fields. The study of the coupling, equations of motion, conservation laws and gauge 
transformations is performed by means of the Schwinger variational principle [7, 8], which is 
a powerful tool to classical as well as quantum calculations.

Finally, some observations about the results found will be made.

2. The Lyra geometry

The Lyra manifold [3] is defined giving a tensor metric $g_{\mu\nu}$ and a positive definite scalar 
function $\phi$ which we call the scale function. In the Lyra geometry one can change the scale 
and a coordinate system in an independent way, to compose what is called a reference system 
transformation: let $M \subseteq \mathbb{R}^N$ and $U$ be an open ball in $\mathbb{R}^n$, $(N \geq n)$ and let $\chi : U \cap M$. The 
pair $(\chi, U)$ defines a coordinate system. Now, we define a reference system by 
$(\chi, U, \phi)$, where $\phi$ transforms as

$$\bar{\phi}(\bar{x}) = \phi(\chi(\bar{x})); \phi(x(\bar{x}))), \quad \frac{\partial \bar{\phi}}{\partial \phi} \neq 0 \quad (1)$$

under a reference system transformation.

In Lyra’s manifold, vectors transform as

$$\bar{A}_\nu = \frac{\bar{\phi}}{\phi} \frac{\partial \bar{x}_\nu}{\partial x^\mu} A^\mu. \quad (2)$$

In this geometry, the affine connection is

$$\tilde{\Gamma}^\rho_{\mu\nu} = \frac{\phi}{\bar{\phi}} \left[ \Gamma^\rho_{\lambda\sigma} \frac{\partial x^\lambda}{\partial x^\mu} \frac{\partial x^\sigma}{\partial x^\nu} + \frac{1}{\phi} \frac{\partial \bar{x}^\rho}{\partial x^\mu} \frac{\partial \bar{x}^\sigma}{\partial x^\nu} + \frac{1}{\phi} \frac{\partial \bar{x}^\rho}{\partial x^\nu} \frac{\partial \bar{x}^\sigma}{\partial x^\mu} - \delta^\rho_{\nu} \right] \ln \left( \frac{\bar{\phi}}{\phi} \right). \quad (3)$$

One can define the covariant derivative for a vector field as

$$\nabla_\mu A^\nu = \frac{\partial A^\nu}{\partial x^\mu} + \tilde{\Gamma}^\nu_{\mu\alpha} A^\alpha, \quad \nabla_\mu A_\nu = \frac{1}{\phi} \partial_\mu A_\nu - \tilde{\Gamma}^\rho_{\mu\nu} A_\rho.$$  

The metricity condition $\nabla_\mu g_{\nu\alpha} = 0$ provided

$$\tilde{\Gamma}^\rho_{\mu\nu} = \frac{1}{\phi} \left[ \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{[\nu|\mu]} \right] g^\alpha_{\sigma\rho} - \tilde{\Gamma}^\sigma_{[\nu]} g_{\sigma\mu} g^\rho_{\lambda},$$

where

$$\tilde{\Gamma}^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \quad (5)$$

is analogous to the Levi-Civita connection and $\tilde{\Gamma}^\rho_{[\nu|\mu]}$ is the antisymmetric part of the 
connection:

$$2\tilde{\Gamma}^\rho_{[\nu|\mu]} = \tilde{\Gamma}^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\nu\mu} = \frac{1}{\phi} \left( \delta^\rho_{\nu} \partial_\mu - \delta^\rho_{\mu} \partial_\nu \right) \ln \left( \frac{\bar{\phi}}{\phi} \right). \quad (6)$$

The richness of Lyra’s geometry is demonstrated by the curvature [4]

$$\tilde{R}^\rho_{\mu\sigma\nu} = \frac{1}{\phi^2} \left( \frac{\partial^2 (\phi \tilde{\Gamma}^\rho_{\sigma\nu})}{\partial x^\rho} - \frac{\partial (\phi \tilde{\Gamma}^\rho_{\sigma\nu})}{\partial x^\sigma} + \phi \tilde{\Gamma}^\rho_{\rho\lambda} \phi \tilde{\Gamma}^\lambda_{\sigma\nu} - \phi \tilde{\Gamma}^\rho_{\rho\lambda} \phi \tilde{\Gamma}^\lambda_{\mu\sigma} \right) \quad (7)$$

and the torsion

$$\tilde{\tau}^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\nu\mu} - \frac{1}{\phi} \left( \delta^\rho_{\nu} \partial_\mu - \delta^\rho_{\mu} \partial_\nu \right) \ln \phi. \quad (8)$$
where the second term is the anholonomic contribution; thus, we get
\[ \tilde{\tau}_{\mu}^{\rho} = -\frac{1}{\phi} \left( \delta_{\mu}^{\rho} \partial_{\nu} - \delta_{\nu}^{\rho} \partial_{\mu} \right) \ln \bar{\phi}, \]
which has an intrinsic link with the scale functions and whose trace is given by
\[ \tilde{\tau}_{\mu} = \frac{3}{\phi} \partial_{\mu} \ln \bar{\phi}. \]

In the following section we introduce the behaviour of the massless DKP field in the Lyra geometry.

3. The massless DKP field in the Lyra manifold

The massless DKP theory cannot be obtained as a zero mass limit of the massive DKP case, so we consider the Harish-Chandra Lagrangian density for the massless DKP theory in Minkowski spacetime \( M^4 \), given by [5]
\[ L_M = i \bar{\psi} \gamma^a \partial^a \psi - i \bar{\partial} \bar{\psi} \beta^a \gamma^a \psi - \bar{\psi} \gamma \psi, \]
where the \( \beta^a \) matrices satisfy the usual DKP algebra
\[ \beta^a \beta^b \beta^c + \beta^c \beta^b \beta^a = \beta^a \eta^{bc} + \beta^c \eta^{ba}, \]
and \( \gamma \) is a singular matrix satisfying
\[ \beta^a \gamma + \gamma \beta^a = \beta^a, \quad \gamma^2 = \gamma. \]

From the above Lagrangian follows the massless DKP wave equation:
\[ i \beta^a \partial^a \psi - \gamma \psi = 0. \]

As was known, the Minkowskian Lagrangian density (11) in its massless spin 1 sector reproduces the electromagnetic or Maxwell theory with its respective \( U(1) \) local gauge symmetry.

To construct the covariant derivative of a massless DKP field in the Lyra geometry, we follow the standard procedure of analysing the behaviour of the field under local Lorentz transformations:
\[ \psi(x) \rightarrow \psi'(x) = U(x) \psi(x), \]
where \( U \) is a spin representation of the Lorentz group characterizing the DKP field. Now we define a spin connection \( S_{\mu} \) in a such way that the object
\[ \nabla_{\mu} \psi = \frac{1}{\phi} \partial_{\mu} \psi + S_{\mu} \psi \]
transforms like a DKP field in (15); thus, we set
\[ \nabla_{\mu} \psi \rightarrow (\nabla_{\mu} \psi)' = U(x) \nabla_{\mu} \psi \]
and therefore \( S \) transforms like
\[ S_{\mu}' = U(x) S_{\mu} U^{-1}(x) - \frac{1}{\phi} (\partial_{\mu} U) U^{-1}(x). \]

From the covariant derivative of the DKP field (16) and remembering that \( \tilde{\psi} \psi \) must be a scalar under the transformation (15), it follows that
\[ \nabla_{\mu} \tilde{\psi} = \frac{1}{\phi} \partial_{\mu} \tilde{\psi} - \bar{\psi} S_{\mu}. \]

We choose a representation in which \( \beta^0 = \beta^0, \beta^1 = -\beta^1 \) and \( \gamma^1 = \gamma. \)
Then, we use the covariant derivative of the DKP current

\[
\nabla_\mu (\bar{\psi} \beta^\nu \psi) = \frac{1}{\phi} \partial_\mu (\bar{\psi} \beta^\nu \psi) + \Gamma^\nu_{\mu\lambda} (\bar{\psi} \beta^\lambda \psi) \\
= (\nabla_\mu \bar{\psi}) \beta^\nu \psi + \bar{\psi} (\nabla_\mu \beta^\nu) \psi + \bar{\psi} \beta^\nu (\nabla_\mu \psi)
\]

to get the following expression for the covariant derivative of \(\beta^\nu\):

\[
\nabla_\mu \beta^\nu = \frac{1}{\phi} \partial_\mu \beta^\nu + \Gamma^\nu_{\mu\lambda} \beta^\lambda + S_\mu \beta^\nu - \beta^\nu S_\mu.
\]

(19)

To solve the equation above we introduce the tetrad field \(e^{\mu}_a\) and its inverse \(e^{\mu}_a\) related to the spacetime metric by the following equations:

\[
g^{\mu\nu}(x) = \eta^{ab} e^{\mu}_a(x) e^{\nu}_b(x),
\]

\[
g^{\mu\nu}(x) = \eta^{ab} e^{\mu}_a(x) e^{\nu}_b(x),
\]

\[
e^{\mu}_a(x) = g^{\mu\nu}(x) \eta^{ab} e^{\nu}_b(x).
\]

(20)

In the Lyra geometry the metricity condition, \(\nabla_\mu g^{\alpha\beta} = 0\), is held, as well as the absolute parallelism of the tetrad:

\[
\nabla_\mu e^{\nu}_a = \frac{1}{\phi} \partial_\mu e^{\nu}_a + \Gamma^\nu_{\mu\lambda} e^{\lambda}_a + \omega_{\mu a} e^{\nu}_b = 0,
\]

(21)

where \(\omega_{\mu a b}\) is the spin connection coefficients. Expressing \(\beta^\nu\) in terms of the tetrad fields, \(\beta^\nu = e^{\nu}_{a \beta a}\), in equation (19) we get

\[
\omega_{\mu a} e^{\nu}_b \beta^a = S_\mu \beta^\nu - \beta^\nu S_\mu = [S_\mu, \beta^\nu]
\]

(22)

from which we found that

\[
S_\mu = \frac{1}{2} \omega_{\mu a b} S^{a b}
\]

(23)

with

\[
S^{a b} = [\beta^a, \beta^b].
\]

(24)

With a covariant derivative of the DKP field well defined, we can consider the Lagrangian density (11) of the massless DKP field minimally coupled [9, 10] to the Lyra manifold; thus, the action reads

\[
S = \int_\Omega d\phi^4 e (i \bar{\psi} \gamma^\mu e^{\mu}_a \beta^a \nabla_\mu \psi - i \nabla_\mu \bar{\psi} \beta^a e^{\mu}_a \gamma \psi - \bar{\psi} \gamma \psi),
\]

(25)

where \(\nabla_\mu\) is the Lyra covariant derivative of the DKP field defined above.

4. Dynamics of the massless DKP field coupled to the Lyra manifold

In the following we use a classical version of the Schwinger action principle such as the one which was treated in the context of classical mechanics by Sudarshan and Mukunda [8]. The Schwinger action principle is the most general version of the usual variational principles. It was proposed originally at the scope of the quantum field theory [7], but its application goes beyond this area. In the classical context, the basic statement of the Schwinger principle is

\[
\delta S = \delta \int_\Omega d\phi^4 e \mathcal{L} = \int_{\partial\Omega} d\sigma G^\mu,
\]

(26)

where \(S\) is the classical actions and \(G^\mu\)'s are the generators of the canonical transformations. The Schwinger principle can be employed, choosing suitable variations in each case, to obtain
commutation relations in the quantum context or canonical transformations in the classical one, as well as equations of motion or still perturbative expansions.

Here, we will apply the action principle to derive the equations of motion of the massless DKP field in an external Lyra background and its conservations laws associated with translations and rotations in such space.

Making the total variation of the action integral (25) we get

\[
\delta S = \int_{\Omega} \left[ 4\mathcal{L} - \frac{i}{\phi} \bar{\psi} \gamma \beta^\mu \partial_\mu \psi + \frac{i}{\phi} \partial_\mu \bar{\psi} \beta^\mu \gamma \psi \right] \left( \frac{\delta \phi}{\phi} \right) + \int_{\Omega} d\phi^4 \left( \frac{\delta e}{e} \right) \mathcal{L}
\]

where \( \mathcal{L} \) is the Lagrangian density in (25). Choosing different specializations of the variations, one can easily obtain the equations of motion and the energy–momentum and spin density tensors.

4.1. Equations of motion

To begin, we choose to make functional variations only in the massless DKP field, thus we set

\[
\delta \phi = \delta e^\mu_b = \delta \omega^\mu_{ab} = 0
\]

and considering \( [\delta, \partial_\mu] = 0 \), from (27) we have, after an integration by parts,

\[
\delta S = \int_{\Omega} d\sigma_\mu e^4 \left[ \bar{\psi} \gamma \beta^\mu (\delta \psi) - (\delta \bar{\psi}) \beta^\mu \gamma \psi \right]
\]

where \( \bar{\psi} \) is the trace torsion, and it is given by

\[
\mathring{\tau}_\mu = \bar{\tau}_{\mu} \rho = \frac{3}{\phi} \partial_\mu \ln \phi.
\]

Following the action principle we get the generator of the variations of the massless DKP field,

\[
G_{\delta \psi} = \int_{\Omega} d\sigma_\mu e^4 \left[ \bar{\psi} \gamma \beta^\mu (\delta \psi) - (\delta \bar{\psi}) \beta^\mu \gamma \psi \right].
\]

and its equations of motion in the Lyra manifold

\[
\begin{align*}
i\beta^\mu (\nabla_\mu + \mathring{\tau}_\mu \gamma) \psi - \gamma \psi &= 0 \\
i\nabla_\mu \bar{\psi} \beta^\mu + i\mathring{\tau}_\mu \bar{\psi} \gamma \beta^\mu + \bar{\psi} \gamma &= 0.
\end{align*}
\]
4.2. Local gauge symmetry

We set \( \delta \phi = \delta e^\mu_{\beta b} = \delta \omega^\mu_{ab} = 0 \) in (27), thus we get

\[
\delta S = \int_{\Omega} dx \, e^4 \delta \bar{\psi} \left( i \gamma \beta^\mu \nabla_\mu \psi - \gamma \psi + i \bar{\psi} \gamma \beta^\mu \gamma \psi \right)
- \int_{\Omega} dx \, e^4 \left( i \nabla_\mu \bar{\psi} \beta^\mu \gamma + \bar{\psi} \gamma - i \bar{\psi} \gamma \beta^\mu \gamma \psi \right) \delta \psi
+ \int_{\Omega} dx \, e^4 \left[ \frac{1}{\Phi} \bar{\psi} \gamma \beta^\mu \delta \nabla_\mu \psi - \frac{i}{\Phi} \delta \bar{\psi} \beta^\mu \gamma \psi \right].
\]

(32)

and choosing the local variation of the fields as being

\[
\delta \psi = (1 - \gamma) \Phi, \quad \delta \bar{\psi} = \bar{\Phi}(1 - \gamma)
\]

then the variation (32) reads

\[
\delta S = \int_{\Omega} dx \, e^4 \left( \bar{\psi} i \gamma \beta^\mu \nabla_\mu \psi - \bar{\psi} \gamma \beta^\mu \gamma \psi \right) - e^4 \left[ i \nabla_\mu \bar{\psi} \beta^\mu \gamma \psi \right] \delta e_{\mu}^a.
\]

If we impose that the variations (33) give rise to a symmetry, \( \delta S = 0 \), then the fields \( \Phi \) and \( \bar{\Phi} \) must satisfy

\[
i \beta^\mu \nabla_\mu (1 - \gamma) \Phi = 0 \quad i \nabla_\mu \bar{\Phi} (1 - \gamma) \beta^\mu = 0.
\]

(34)

Under such conditions the local transformation

\[
\psi \rightarrow \psi + (1 - \gamma) \Phi \quad \bar{\psi} \rightarrow \bar{\psi} + \bar{\Phi}(1 - \gamma)
\]

is a local gauge symmetry of the action (25) and the equations of motion (31).

4.3. Energy–momentum tensor and spin tensor density

Now, we vary only the background manifold and we assume that \( \delta \omega^\mu_{ab} \) and \( \delta e^\mu_{\alpha a} \) are independent variations; the general variation (27) reads

\[
\delta S = \int_{\Omega} dx \, e^4 \left[ i \bar{\psi} \gamma \beta^\mu \nabla_\mu \psi - \bar{\psi} \gamma \beta^\mu \gamma \psi + e^a_{\mu} \mathcal{L} \right] \delta e_{\mu}^a
+ \int_{\Omega} dx \, e^4 \left( \bar{\psi} \gamma \beta^\mu \nabla_\mu \psi + \bar{\psi} \beta^\mu \gamma \psi \right) \frac{1}{2} \delta \omega^\mu_{ab},
\]

(36)

where we have used \( \delta e = - e^a_{\mu} \delta e_{\mu}^a \).

First, holding only the variations in the tetrad field, \( \delta \omega^\mu_{ab} = 0 \), we found for the variation of the action

\[
\delta S = \int_{\Omega} dx \, e^4 \left[ i \bar{\psi} \gamma \beta^\mu \nabla_\mu \psi - i \nabla_\mu \bar{\psi} \beta^\mu \gamma \psi - e^a_{\mu} \mathcal{L} \right] \delta e_{\mu}^a.
\]

Defining the energy–momentum density tensor as

\[
T^a_{\mu} \equiv \frac{1}{\phi^4 e} \frac{\delta S}{\delta e_{\mu}^a} = i \bar{\psi} \gamma \beta^\mu \nabla_\mu \psi - i \nabla_\mu \bar{\psi} \beta^\mu \gamma \psi - e^a_{\mu} \mathcal{L},
\]

(37)

which can be written in coordinates as \( T^a_{\mu} = e^\beta\mu T_\mu^a \).

Now, making functional variations only in the components of the spin connection, \( \delta e_{\mu}^a = 0 \), we found for the action variation

\[
\delta S = \int_{\Omega} dx \, e^4 \left[ \frac{1}{2} \delta \omega^\mu_{ab} \bar{\psi} \left( \gamma \beta^\mu S^ab + S^ab \beta^\mu \gamma \psi \right) \right].
\]

(38)

We define the spin density tensor as being

\[
S^a_{ab} \equiv \frac{2}{\phi^4 e} \frac{\delta S}{\delta \omega^\mu_{ab}} = i \bar{\psi} \gamma \beta^\mu S^ab + S^ab \beta^\mu \gamma \psi.
\]

(39)
4.4. Functional scale variations and the trace relation

Under a pure infinitesimal variation $\delta \phi$ of the scale function the action variation (27) is expressed as

$$\delta S = - \int_{\Omega} dx \, e \phi^4 \left( T^{\mu}_{\ a} e^{\mu}_{\ a} - \frac{1}{2} S^{\mu}_{a b} \omega_{\mu a b} \right) \frac{\delta \phi}{\phi},$$

where we have used the definition of the energy–momentum and the spin density tensors. From it we obtain the following algebraic property satisfied by the energy–momentum tensor trace:

$$T^{\mu}_{\ a} e^{\mu}_{\ a} - \frac{1}{2} S^{\mu}_{a b} \omega_{\mu a b} = 0,$$

which is the so-called trace relation. Such identity can be used to constraint the form of the connection $\omega$ in a given content of matter.

4.5. Conservation laws

As another application of the Schwinger action principle, let us derive the conservation laws associated with local Lorentz transformations and infinitesimal general coordinate transformations.

4.5.1. Local Lorentz transformations. Under local Lorentz transformations, the functional variations of the tetrad and the spin connection are given by

$$\delta e^{\mu}_{\ a} = \delta e^{b}_{\ a} e^{\mu}_{\ b},$$

$$\delta \omega_{\mu a b} = \omega_{\mu c} e^{c}_{b} \delta e^{a}_{c} - \omega_{\mu a} e^{c}_{b} \delta e^{c}_{b} - \frac{1}{\phi} \partial_{\mu} \delta \epsilon_{a b},$$

with $\delta \epsilon_{a b} = - \delta \epsilon_{b a}$, where the first variation expresses the vectorial character of the tetrad in the Minkowski spacetime, and the second one comes from (17) with

$$U = 1 + \frac{1}{2} \delta \epsilon_{a b} S^{a b}.$$

The general expression (36) can be written using the definitions of energy–momentum and spin tensors as being

$$\delta S = \int_{\Omega} dx \, e \phi^4 \left( T^{\mu}_{\ a} \delta e^{\mu}_{\ a} + S^{a b} \frac{1}{2} \delta \omega_{a b} \right),$$

Substituting variations (42) and (43) and after some integration by parts and algebraic manipulations we get

$$\delta S = - \int_{\delta \Omega} d\sigma \phi^{3} e S^{a b} \frac{1}{2} \delta \epsilon_{a b} + \int_{\Omega} dx \, \phi^4 e \left( \nabla_{\mu} S^{a b} + \tilde{\tau}_{\mu} S^{a b} - T^{a b} + T^{b a} \right) \frac{1}{2} \delta \epsilon_{a b},$$

where $\tilde{\tau}_{\mu}$ is the trace torsion defined in (10). Thus, from the action principle, we get $G_{\delta \epsilon}$, the generator of infinitesimal changes in the DKP field under local Lorentz transformations,

$$G_{\delta \epsilon} = - \int_{\delta \Omega} d\sigma \phi^{3} e S^{a b} \frac{1}{2} \delta \epsilon_{a b},$$

and the conservation law for the spinning content of the theory

$$\nabla_{\mu} S^{a b} + \tilde{\tau}_{\mu} S^{a b} = T^{a b} - T^{b a}.$$

The most important aspect here is the coupling among the spin density tensor and the torsion.
4.5.2. General coordinate transformation. Now let us perform a general coordinate transformation in the action. From the Lyra transformation rule for vectors and the infinitesimal displacement $\delta x^\mu = x^\mu + \delta x^\mu$, we have that the tetrad field transforms as

$$\bar{e}^\mu_a(\bar{x}) = \frac{\phi(\bar{x})}{\phi(x)} \frac{\delta \bar{x}^\mu}{\delta x^\nu} e^\nu_a(x),$$

(49)

with its form variation $\delta e^\mu_a(x) = \bar{e}^\mu_a(x) - e^\mu_a(x)$ given by

$$\delta e^\mu_a(x) = \frac{\phi(x)}{\phi(\bar{x})} \left[ e^\mu_a(x) + e^\nu_a(x) \partial_\nu \delta x^\mu - \delta x^\nu \partial_\nu e^\mu_a(x) - e^\mu_a(x) \ln \phi(x) \right] - e^\mu_a(x).$$

(50)

While the spin connection transforms as

$$\bar{\omega}^{\mu a}_{\nu b}(\bar{x}) = \frac{\phi(x)}{\phi(\bar{x})} \partial_\mu \left[ \frac{\phi(x)}{\phi(\bar{x})} \omega^{\nu b}_{\nu a}(x) \right]$$

(51)

with its form variation $\delta \omega^{\mu a}_{\nu b}(x) = \bar{\omega}^{\mu a}_{\nu b}(x) - \omega^{\mu a}_{\nu b}(x)$ given as

$$\delta \omega^{\mu a}_{\nu b}(x) = \frac{\phi(x)}{\phi(\bar{x})} \left[ \omega^{\mu a}_{\nu b}(x) - \omega^{\nu a}_{\nu b}(x) \partial_\mu \delta x^\nu - \delta x^\nu \partial_\nu \omega^{\mu a}_{\nu b}(x) - \omega^{\mu a}_{\nu b}(x) \ln \phi \right] - \omega^{\mu a}_{\nu b}(x).$$

(52)

Substituting expressions (50) and (52) in (45),

$$\delta S = \int d\sigma_{\Omega} \epsilon \phi^4 \left( \frac{e^\nu_a T^\mu_{\nu a}}{\phi} \frac{1}{2} \omega^{\mu a}_{\nu b} S^{\mu a}_{\nu b} \right) \delta x^\nu + \int d\sigma_{\Omega} \epsilon \phi^4 \left( \frac{\phi}{\phi} - 1 \right) \left( T^\mu_{\nu a} e^\nu_a - \phi \frac{1}{2} \omega^{\mu a}_{\nu b} S^{\mu a}_{\nu b} \right)$$

$$- \int d\sigma_{\Omega} \epsilon \phi^4 \left[ \frac{\phi}{\phi} (\nabla^\mu T^\nu_{\mu} + \tilde{\tau}^\mu_{\nu a} T^\nu_{\mu} + \tilde{\tau}^\nu_{\mu a} T^\nu_{\mu}) \right] + \frac{\phi}{\phi^2} \left( \partial_\mu \ln \phi \right) T^\mu_{\nu a} + \left( \frac{\phi}{\phi} - \frac{\phi}{\phi} \right) \omega^{\mu a}_{\nu b} T^{\nu a}$$

$$+ \frac{1}{2} \phi \left( \partial_\mu \ln \phi \right) S^{\nu a}_{\mu b} \omega^{\mu a}_{\nu b} + \frac{1}{2} \phi \left( \omega^{\mu a}_{\mu b} R^\nu_{\nu a} \right) \delta x^\nu.$$

(53)

According to the Schwinger principle, we get the generator $G_{\delta x}$,

$$G_{\delta x} = \int d\sigma_{\Omega} \epsilon \phi^4 \left( \frac{\phi}{\phi} T^\nu_{\nu a} - \phi \frac{1}{2} S^{\nu a}_{\mu b} \omega^{\mu a}_{\nu b} \right) \delta x^\nu.$$

(54)

which establishes the form of the variations under infinitesimal coordinate transformations.

Next, from the third integral in (53) and due to the invariance of the action under general coordinate transformations we obtain the conservation law of energy–momentum:

$$\frac{\phi}{\phi} (\nabla^\mu T^\nu_{\mu} + \tilde{\tau}^\mu_{\nu a} T^\nu_{\mu} + \tilde{\tau}^\nu_{\mu a} T^\nu_{\mu}) + \frac{1}{2} \phi S^{\nu a}_{\mu b} \omega^{\mu a}_{\nu b} + \left( \frac{\phi}{\phi} - \frac{\phi}{\phi} \right) \omega^{\mu a}_{\nu b} T^{\nu a}$$

$$+ \frac{1}{2} \phi \left( \partial_\mu \ln \phi \right) \left[ \frac{\phi^2}{\phi^2} T^\nu_{\nu a} + \frac{1}{2} S^{\nu a}_{\mu b} \omega^{\mu a}_{\nu b} \right] = 0.$$ 

(55)

By imposing the invariance of the action under general coordinate transformations, the second integral in (53) allows us to get a new relation; we named it as the trace symmetry. Such identity for $\phi \neq \phi$ can be written as

$$T^\mu_{\nu a} e^\nu_a - \frac{\phi}{\phi} \frac{1}{2} S^{\nu a}_{\mu b} \omega^{\mu a}_{\nu b} = 0.$$ 

(56)
It can be considered as the generalization of the trace relation shown in (41) and which relates the different scale functions with the geometry and the field content. For \( \bar{\phi} \equiv \phi \) it reduces to (41).

5. The spin content

The massless DKP theory describes in an algebraically unified way the spin 0 and 1 fields. However, scalar and vector fields have different particular behaviour on couplings and gauge transformations. To see how this can be implemented in the equations above, we use the projectors of Umezawa [11, 12] to select these distinct spin sectors.

5.1. Spin 0 sector

The spin 0 projectors \( P \) and \( P^\mu = e^\mu_a P_a \) are such that \( P \psi \) and \( P^\mu \psi \) transform, respectively, as a scalar and a vector in the Lyra spacetime. Then, first we apply the scalar projector \( P \) in the equation of motion (31) and we get

\[
P \rightarrow i \nabla_\mu (P^\mu \psi) + i \bar{\tau}_\mu (P^\mu \gamma \psi) - P \gamma \psi = 0.
\]

We multiply it by \((1 - \gamma)\) and get

\[
i(\nabla_\mu + \bar{\tau}_\mu)(P^\mu \gamma \psi) = 0.
\]

(57)

Next, by applying the vector projector \( P^\mu \) we obtain

\[
P^\mu \rightarrow i \nabla^\mu (P \psi) + i \bar{\tau}^\mu (P \gamma \psi) - P^\mu \gamma \psi = 0
\]

and remembering that \( P \gamma = \gamma P \), we obtain

\[
P^\mu \gamma \psi = i(\nabla^\mu + \bar{\tau}^\mu \gamma)(P \psi).
\]

(58)

By mixing equations (57) and (58), we find the equation of motion for the scalar field \( P \psi \) in the Lyra spacetime:

\[
(\nabla_\mu + \bar{\tau}_\mu)(\nabla^\mu + \bar{\tau}^\mu \gamma) P \psi = 0.
\]

(59)

We use a specific representation of the DKP algebra in which the singular \( \gamma \) matrix is

\[
\gamma = \text{diag}(\lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda),
\]

such that the condition \( \gamma^2 - \gamma = 0 \) implies that \( \lambda \in \{0, 1\} \). In Minkowski spacetime, the value \( \lambda = 0 \) reproduces the massless Klein–Gordon–Fock field [6]. Thus, we restrict our attention to the \( \lambda = 0 \) case only. The scalar sector of the massless DKP theory can be explicitly worked out by using the five-dimensional representation of the massless DKP algebra. Thus, the field \( \psi \) is given by a five-component column vector

\[
\psi = (\varphi, \psi^0, \psi^1, \psi^2, \psi^3)^T,
\]

(60)

where \( \varphi \) and \( \psi^\mu (\mu = 0, 1, 2, 3) \) behave, respectively, as a scalar and a 4-vector under Lorentz transformations on the Minkowski space. And the other projections are

\[
P \psi = (\varphi, 0, 0, 0, 0)^T, \quad P^\mu \psi = (\psi^\mu, 0, 0, 0, 0)^T,
\]

\[
\gamma \psi = (0, \psi^0, \psi^1, \psi^2, \psi^3)^T
\]

\[
P \gamma \psi = 0, \quad P^\mu \gamma \psi = (\psi^\mu, 0, 0, 0, 0)^T.
\]

Expressing the massless DKP action (25) in this representation we obtain

\[
S_0 = \int dx \, \phi^4 e^{i[\psi^\mu \nabla_\mu \varphi - i\psi^\mu \nabla_\mu \psi^a - \psi^\mu \psi_a]}.
\]

(61)
and from equation (58) we get the vectorial component of the DKP field

$$\psi^\mu = i \nabla^\mu \varphi,$$

and together with equation (57) we get the equation of motion for the scalar field $\varphi$:

$$(\nabla_\mu + \tilde{\tau}_\mu)\nabla^\mu \varphi = 0,$$  \hspace{1cm} (63)

or

$$\frac{1}{\varphi} (\partial_\mu \partial^\mu \varphi + \dot{\varphi}/\Gamma_1 \partial_\mu \varphi) + \frac{2}{\varphi} (\partial^\mu \varphi) \partial_\mu \ln(\varphi) = 0.$$  \hspace{1cm} (64)

Thus, in contrast to what happens in the Riemann–Cartan case [6], we conclude that the spin 0 sector of the massless DKP field does interact with the anholonomic component of the Lyra torsion.

Finally, in this representation the action for the spin 0 sector of the massless DKP field in the Lyra spacetime is obtained by substituting equation (62) into the action (61), which reduces to the usual one obtained from the Minkowski Klein–Gordon–Fock Lagrangian density minimally coupled to the Lyra spacetime:

$$S_0 = \int dx \, \varphi^4 \nabla_\mu \varphi^* \nabla^\mu \varphi.$$  \hspace{1cm} (65)

The gauge transformation (35) reads

$$\varphi' = \varphi + \varphi \Phi_1, \quad \psi' = \psi^\mu,$$

while condition (34) becomes $\nabla_\mu \varphi \Phi_1 = \frac{1}{\varphi} \partial_\mu \varphi \Phi_1 = 0$, i.e., $\varphi \Phi_1$ must be a constant.

### 5.2. Spin 1 sector

The spin 1 projectors $R^\mu = e^\mu_a R_a$ and $R^{\mu\nu} = e^\mu_a e^{\nu_b} R_{ab}$ are such that $R^\mu \psi$ and $R^{\mu\nu} \psi$ transform, respectively, as a vector and a second rank tensor under general coordinate transformations. Thus, using the projector $R^\mu$ in the equation of motion (31) we have

$$i(\nabla_\nu + \tilde{\tau}_\nu)(R^\mu \psi) + i\tilde{\tau}_\nu (R^{\mu\nu} \gamma \psi) - R^\mu \gamma \psi = 0.$$  \hspace{1cm} (66)

Multiplying by $(1 - \gamma)$ we get

$$i(\nabla_\nu + \tilde{\tau}_\nu)(R^{\mu\nu} \gamma \psi) = 0,$$

where we have used $R^\mu \gamma = \gamma R^\mu$. Next, applying the projector $R^{\mu\nu}$ we obtain

$$i(\nabla_\nu + \tilde{\tau}_\nu)(R^{\mu\nu} \beta^\nu \psi) + i\tilde{\tau}_\nu (R^{\mu\nu} \beta^{\nu\gamma} \psi) - R^{\mu\nu} \gamma \psi = 0.$$  \hspace{1cm} (67)

Using the properties $R^{\mu\nu} \beta^\nu = R^{\mu\nu} g^{\nu\rho} - R^\nu g^{\mu\rho}$ and $R^{\mu\nu} \gamma = (1 - \gamma) R^{\mu\nu}$, we get

$$R^{\mu\nu} \gamma \psi = i(\nabla_\nu + \tilde{\tau}_\nu) [g^{\mu\nu}(R^\mu \psi) - g^{\mu\rho} (R^\rho \psi)].$$  \hspace{1cm} (68)

From both equations (66) and (67) we obtain the equation of motion for the massless vector field $R^\mu \psi$:

$$\nabla_\nu (\nabla_\mu + \tilde{\tau}_\mu) [g^{\mu\nu}(R^\mu \psi) - g^{\mu\rho} (R^\rho \psi)] = 0.$$  \hspace{1cm} (69)

We use a specific representation of the DKP algebra in which the singular $\gamma$ matrix is

$$\gamma = \text{diag}(\lambda, \lambda, \lambda, \lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda),$$

from the condition $\gamma^2 - \gamma = 0$ implies that $\lambda = 0$ or 1. In the Minkowski spacetime, the value $\lambda = 0$ reproduces the electromagnetic field [6]; thus, we restrict our attention to the $\lambda = 0$ case only. Then in this representation the DKP field $\psi$ is now a ten-component column vector,

$$\psi = (\psi^0, \psi^1, \psi^2, \psi^3, \psi^{23}, \psi^{31}, \psi^{12}, \psi^{10}, \psi^{20}, \psi^{30})^T.$$
where $\psi^a$ ($a = 0, 1, 2, 3$) and $\psi^{ab}$ behave, respectively, as a 4-vector and an antisymmetric tensor under Lorentz transformations on the Minkowski space. And we also get

$$R^\mu \psi = (\psi^\mu, 0, 0, 0, 0, 0, 0, 0)^T$$
$$R^{\mu\nu} \psi = (\psi^{\mu\nu}, 0, 0, 0, 0, 0, 0, 0)^T$$
$$\gamma \psi = (0, 0, 0, 0, \psi^{23}, \psi^{31}, \psi^{12}, \psi^{10}, \psi^{30})^T$$
$$R^\mu \gamma \psi = 0, \quad R^{\mu\nu} \gamma \psi = (\psi^{\mu\nu}, 0, 0, 0, 0, 0, 0)^T.$$

Then, we get the following relations among $\psi$ components:

$$i \psi_{\mu\nu} = \nabla_\mu \psi_\nu - \nabla_\nu \psi_\mu, \quad (69)$$

which leads to the equation of motion for the spin 1 sector of the massless DKP field in the Lyra spacetime:

$$(\nabla_\mu + \tau_\mu)(\nabla_\mu \psi_\nu - \nabla_\nu \psi_\mu) = 0, \quad (70)$$

In terms of these components the DKP action (25) is written as

$$S_1 = \int d^4x \phi^4 e(\nabla_\mu \psi_\nu^* - \nabla_\nu \psi_\mu^*)(\nabla_\mu \psi_\nu - \nabla_\nu \psi_\mu), \quad (71)$$

Then, using equation (69) in the action above we obtain the action for the spin 1 sector of the massless DKP field in the Lyra manifold:

$$S_1 = -\frac{1}{2} \int d^4x \phi^4 e(\nabla_\mu \psi_\nu^* - \nabla_\nu \psi_\mu^*)(\nabla_\mu \psi_\nu - \nabla_\nu \psi_\mu), \quad (72)$$

where $\nabla_\mu \psi_\nu - \nabla_\nu \psi_\mu$ is written as

$$\nabla_\mu \psi_\nu - \nabla_\nu \psi_\mu = \frac{1}{\phi} (\partial_\mu \psi_\nu - \partial_\nu \psi_\mu) - \frac{1}{\phi} (\psi_\mu \partial_\nu - \psi_\nu \partial_\mu) \ln \left( \frac{\phi}{\bar{\phi}} \right).$$

5.3. On the solution of the gauge condition for the vectorial field

The minimal coupling prescription leads to a coupling among the electromagnetic field and torsion in such a way that the gauge symmetry is broken in the Einstein–Cartan spacetime. The first attempt to solve this problem was achieved by Hojman and collaborators [15], by means of introducing a new scalar field, the so-called laplon field, and a consequent modification in the gauge transformations. This turns the torsion into a propagating quantity [16], but in such a way that the propagation is not an intrinsic phenomenon of the spacetime structure, but rather induced by the presence of gauge fields. Several other proposals ad hoc of the construction of a propagating torsion are present in the literature [1, 17], but until now no one got a direct consequence of the geometry adopted to describe the spacetime. Here, we show that the gauge invariance and the propagating character of the torsion can be viewed as a very effect of the structure of the Lyra manifold.

The gauge transformation (35) now reads

$$\psi'^\mu = \psi^\mu + \Phi^\mu, \quad \psi'^{\mu\nu} = \psi^{\mu\nu}$$

and the gauge condition for the massless DKP field is given by

$$\nabla_\mu \Phi_\nu - \nabla_\nu \Phi_\mu = 0,$$

or, explicitly,

$$\frac{1}{\phi} (\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu) - 2 \Phi_\mu \tilde{\Gamma}^{\rho}_{\mu\nu} = 0.$$
Using
\[
2\tilde{\Gamma}^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\nu\mu} = \frac{1}{\phi} \left( \delta^\rho_\mu \partial_\nu - \delta^\rho_\nu \partial_\mu \right) \ln \left( \frac{\phi}{\bar{\phi}} \right),
\]
we have
\[
\partial\nu \Phi_\nu - \partial\mu \Phi_\mu - (\Phi_\mu \partial_\nu - \Phi_\nu \partial_\mu) \ln \left( \frac{\phi}{\bar{\phi}} \right) = 0.
\]
(74)

The gauge covariance of the system is assured since this equation has solution. Let us begin looking for the simplest case when \( \Phi_\mu = \partial_\mu \Lambda \); then (74) becomes
\[
(\partial_\mu \Lambda \partial_\nu - \partial_\nu \Lambda \partial_\mu) \ln \left( \frac{\phi}{\bar{\phi}} \right) = 0.
\]
(75)
Therefore, if we choose
\[
\Lambda(x) = \ln \left( \frac{\phi}{\bar{\phi}} \right).
\]
(76)

(75) is satisfied.

However, this is not the most general solution. Another possible solution is found by solving
\[
\partial_\mu \Phi_\nu + \Phi_\nu \partial_\mu \ln \left( \frac{\phi}{\bar{\phi}} \right) = 0,
\]
(77)
or
\[
\partial_\mu \Phi_\nu - \Lambda_\mu \Phi_\nu = 0, \quad \Lambda_\mu \equiv \partial_\mu \ln \left( \frac{\phi}{\bar{\phi}} \right),
\]
(78)
where \( \Lambda_\mu \) is considered as a prescribed function (part of the external background). The solution of equations (77) or (78) is
\[
\Phi_\nu(x) = \Phi_\nu(x_0) \exp \left( \int_{\gamma} \gamma^\mu \Lambda_\mu \right) = \Phi_\nu(x_0) \left( \frac{\phi}{\bar{\phi}} \right),
\]
(79)
where \( \gamma \) is a particular trajectory joining the points \( x \) and \( x_0 \), and \( \Phi_\nu(x_0) \) is a constant.

A specialization to \( \phi / \bar{\phi} \equiv \text{cte} \) gives
\[
\Phi_\nu(x) = \left( \frac{\phi(x)}{\bar{\phi}(x)} \right) \Phi_\nu(x_0) \equiv \text{cte}.
\]
(80)

But, if \( \phi, \bar{\phi} \equiv \text{cte} \), then (74) is
\[
\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu = 0,
\]
(81)
which admits a solution of the form
\[
\Phi_\nu(x) = \partial_\nu \Lambda(x),
\]
(82)
with \( \Lambda \) an arbitrary scalar function.

On the other hand, if \( \phi = \bar{\phi} \), (74) again simplify in
\[
\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu = 0
\]
(83)
and an analogous argument implies that
\[
\Phi_\nu(x) = \partial_\nu \Lambda(x)
\]
(84)
and \( \Lambda \) arbitrary. Thus, we show that in the Lyra spacetime the coupling between electromagnetic field and torsion preserves the local \( U(1) \) gauge symmetry.
6. Conclusions and perspectives

We have shown how to couple massless, scalar and vector, fields to the Lyra manifold by means of the DKP theory. The equations of motion, conservation laws and gauge symmetry were obtained using the Schwinger action principle. Projecting to select the two distinct spin sectors, we find that the scalar field couples to the trace of the torsion, and the gauge symmetry of the vectorial sector is maintained if one uses the scale function to perform the gauge transformation. These are completely new and unexpected results, which do not occur in other geometries like the Riemann–Cartan one [6].

From the Lyra geometry we can obtain the Riemannian one when the scale functions are set $\phi = \bar{\phi} \equiv 1$, obtaining from (55) and (48) the conservation laws for the DKP field in the torsionless geometry:

$$\nabla_\mu T^\mu_\nu + \frac{1}{2} S^{\mu ab} R_{\nu\mu ab} = 0,$$

(85)

$$\nabla_\mu S^{\mu ab} = T_{ab} - T^{ba}.$$  

(86)

It is possible to show that in the spin 0 sector the spin density tensor $S^{\mu ab} = 0$; thus (85) is reduced to $\nabla_\mu T^\mu_\nu = 0$ while the second equation, (86), gives that the energy–momentum tensor is symmetric, the well-known results for the Klein–Gordon–Fock field. Also, for the spin 1 sector the conservation laws (85) and (86) reproduce the electromagnetic case.

It is worth noting that the structure of the conservation laws (55), (48), the trace relation (41) and the trace symmetry (56) is independent of the matter fields, irrespective of the mass or spin content. Thus, for example, such equations have the same form for the spin 1/2 field [13] or for the massive DKP field [14].

The trace relation (41) and the trace symmetry (56) in the scalar sector of the DKP field give a traceless energy–momentum tensor. This is just related to the scale invariance of the Lyra geometry, even in the massive situation. The results obtained here seem to indicate that the Lyra geometry would be a useful tool to implement a kind of conformal symmetry for massive fields. To make precise this reason, a deeper study of the relation among conformal invariance and Lyra scale transformations is necessary as suggested by the observed relation for the vacuum gravitational field equations [18]. One of the most interesting consequences of relations (41) and (56) is that both are not Lorentz covariants, despite the fact that they were obtained from an action principle. This is not so surprising, since the variation operation involved is defined only on a section of the fibre bundle of the theory, which is not related to the tangent space of the manifold as in the usual Riemannian case. There are others similar known cases of such effects [19], and perhaps the calculations presented here could be investigated by means of a recently developed technique of redefining the Lie derivative [20].

The investigation of the back reaction of the fields (at least for spins 0, 1/2 and 1) upon the geometry would show us how a propagation equation for the scale function can be obtained. These studies are currently under discussion.

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