Notes about collapse in magnetohydrodynamics

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We discuss a problem about magnetic collapse as a possible process for singularity formation of the magnetic field in a finite time within ideal magneto-hydrodynamics for incompressible fluids. This process is very important from the point of view of various astrophysical applications, in particular, as a mechanism of magnetic filaments formation in the convective zone of the Sun. The collapse possibility is connected with compressibility of continuously distributed magnetic field lines. A well-known example of the formation of magnetic filaments in the kinematic dynamo approximation with a given velocity field, first considered by Parker in 1963 [1], rather indicates that the increase in the magnetic field is exponential in time. In the case of the kinematic approximation for the induction equation, the magnetic filaments formation is shown to occur in areas with a hyperbolic velocity profile.

I. INTRODUCTION

Collapse as a process of the singularity formation for smooth initial conditions represents one of the key issues for understanding nature for both hydrodynamic turbulence and MHD turbulence. The Kolmogorov-Obukhov theory [2, 3] of developed hydrodynamic turbulence at large Reynolds numbers, $Re \gg 1$, in inertial interval predicts the divergence of vorticity fluctuations $\langle \delta \omega \rangle$ with scale $\ell$ at small $\ell$ like $\ell^{-2/3}$, which indicates the connection of Kolmogorov turbulence with collapse.

Numerous executed in the late 90s numerical experiments seem to indicate the observation of collapse, with a more accurate examination showed its absence (a discussion of these issues can be found in [4, 5]). This problem still remains open, although there are numerical experiments that show the singularities formation on a solid wall in the framework of the three-dimensional Euler equations [6].

In two-dimensional Euler hydrodynamics, collapse - the appearance of a singularity in a finite time - is forbidden [7-9]. But this, however, does not exclude the singularity appearance in infinite time with exponential growth, as evidenced in numerical experiments [10], in which the formation of the vorticity quasi-shocks is accompanied by exponential in time narrowing their widths. In the three-dimensional Euler hydrodynamics numerical experiments also show an exponential increase in time of the vorticity $\omega$ in the pancake-type vortex structures for which the exponential in time narrowing of the pancake thickness $\ell$ was also observed [11-13]. The formation of such structures is possible, as shown in [14-16], for the frozen-in vorticity in the three-dimensional Euler equations and the vorticity rotor $B$ for two-dimensional flows [10]. Due to this property, frozen-in vector fields turn out to be compressible. Moreover, it was found that the formation of such structures can be considered as a folding process when the maximal values of $\omega_{max}$ and $B_{max}$ are evaluated proportionally to their widths as $\ell^{-2/3}$ [11-13, 17]. In MHD at high values of magnetic Reynolds numbers, $Re_m \gg 1$, the magnetic field can be considered also as a frozen-in field. Therefore one can expect, that the exponential in time growth also should be observed due to compressibility of magnetic field lines. Compressibility of magnetic field lines as applied to this problem was discussed for the first time in the works [15, 18]. In particular, [18] suggested that frozenness of magnetic field may cause collapse. In these notes, being a short review, we discuss such possibility for MHD equations in the so-called kinematic approximation when the magnetic field is relatively small and the inverse influence of a growing magnetic field on the velocity field can be neglected. This model is very popular in the theory of turbulent dynamo (see [14-21] and references there) when
a random velocity field is given. In this work we consider mainly the case when the velocity field is regular. Such situation is realized in the convective zone of the Sun. Observations of convective cells showed that the magnetic field in this area is heavily filamentated. This was first addressed attention in Parker’s pioneer work [1] (see also his book [22] and the references therein). In particular, in his first work, Parker found the behavior of a magnetic field in two-dimensional velocity field for convective flow in the case of periodic cell lattices in the form of rolls.

We discuss this problem statement in our work. The main summary that can be made is that for stationary two-dimensional flows magnetic field filamentation and its exponential growth are due to the presence in hyperbolic flows areas in the Okubo-Weiss meaning [23, 24]. In these areas, the magnetic field is gathered due to frozen-in magnetic field in a small neighborhood near a stationary hyperbolic point where velocity vanishes. This process has exponential in time character and stops due to frozenness destruction because of finite magnetic viscosity. As a result the magnetic field is saturated amplifying with respect to the initial field in $R_{m}^{1/2}$ times, that was established in a number of papers (see [23, 25, 26]).

The outline of this review is the following. Firstly, we discuss the parameters of the region of convective cells in the Sun, which occupies the upper part of the convective zone including the lower part of the photosphere. Parameters of flows in convective cells allow one to consider the problem of filamentation of magnetic fields in this area of the Sun in the kinematic approximation. In the third section we turn to the magnetic lines representation [15, 18], analogous to the vortex lines representation, introduced first for Euler equations in [14] (see also [16]). This representation clearly demonstrates compressibility of magnetic field lines. The velocity of field lines coincides with the fluid velocity component normal to the magnetic field direction. Divergence of this velocity component in a general situation is not zero, which ultimately leads to compressibility continuously distributed magnetic field lines. Based on this fact only it is possible qualitatively to show how filamentation of magnetic field develops in a convective cell. This process happens in the hyperbolic flow regions, and respectively the focusing of magnetic field lines takes place in a small neighborhood of a stationary hyperbolic points. In the three-dimensional geometry, this process in convective cells should lead to the formation of filaments flattened relative to interfaces between convection cells. In the last section we discuss the question of saturation of the magnetic field due to the finiteness of magnetic viscosity.

II. CONVECTION IN ASTROPHYSICS

Numerous observations of the magnetic field spatial distribution in solar convective cells (see, for example, data SOHO [27] missions, as well as the first observational data from the most powerful solar telescope DKIST [28]) indicate a very inhomogeneous distribution of magnetic fields already within the same convective cell: the magnetic field is concentrated in the form of magnetic filaments (often called magnetic flux tubes), the field in which exceeds significantly the average magnetic field $B_0$ on the Sun. Especially it is seen in areas of dark spots [29]. According to numerous data, $B_0$ has a value of several gauss (see, for example, [30] and references therein). (In this paper, we will assume $B_0 \sim 10$ G for estimates.) In each convective cell in its center - in the region of the upward flow - the magnetic field is practically absent, it is concentrated in areas of downward flows in the form of filaments with a magnetic field of the order one kilogauss or more. It should be noted that reconnection of magnetic flux tubes and various related phenomena in the form of flares happen higher - in the upper layers of the solar atmosphere, mainly in the chromosphere and corona (see, for example, the book [31] and references there). From this point of view, the question of the appearance of magnetic filaments seems to us very important and relevant.

As for convective cells, their horizontal size $L$ according to observations is about 500-1000 km. In accordance with theoretical and experimental data about laboratory convection [32] (see also [33]) vertical size of the cells is the same order as their horizontal size, which we will consider completed for solar convec-
tive cells, that is the most common assumption in these research. It should also be said about speeds and densities in area of convective cells. According to measurements [27], as well as to many other data (see, for example, [34]), the velocity $v$ in a cell is of the order of $1000 \text{ m/s}$, and density $\rho$ in the photosphere is of the order of $10^{-7} \text{ g/cm}^3$. Note also that between the convective zone and the photosphere there is no sharp gradient in density: the density changes smoothly (see, for example, review [35] and references therein). The density value in the convective zone, of course, is larger $10^{-6} \text{ g/cm}^3$. In literature the value of density at the boundary between convective zone and photosphere is commonly accepted to have the order of $10^{-6} - 10^{-5} \text{ g/cm}^3$. Nevertheless, within a convective cell the density can be considered practically uniform and, accordingly, the flow in the cell itself could be considered incompressible: $\text{div} \mathbf{v} = 0$.

The main issue addressed in this review is a qualitative explanation of the observation facts, namely, (i) why the magnetic field filamentation occurs near interfaces of convective cells corresponding to regions with downward flows and (ii) why in the central cell regions - areas of upward flows - the magnetic field is practically missing. In this sense, downward convective flows for magnetic field lines act as peculiar attractors. The main reason for this phenomenon, as will be shown in this work, is connected with frozenness of magnetic field lines into plasma which is a property for zero magnetic viscosity. According to all known data (see, e.g. [20, 21] and references there) in convective cells the magnetic Reynolds number $Re_m$ is of order $10^6$, which allows ones to neglect in the main approximation by magnetic viscosity $\eta_m = e^2/4\pi\sigma$ term in the MHD equations, where $\sigma$ is conductivity.

Another important parameter of solar convection is ratio $\Theta$ between kinetic energy density $\rho v^2/2$ and magnetic energy density $B^2/(8\pi)$. For example, for the photosphere with a density of $10^{-7} \text{ g/cm}^3$ this ratio turns out to be of the order of $10^2$, where for estimates we took $B_0 = 10 \text{ G}$, and the velocity is $1000 \text{ m/s}$. When approaching the convective zone $\Theta$ becomes about $10^3 - 10^4$. For such ratios energy densities the magnetic field weakly affects convection and accordingly, the velocity field can be considered the given. In what follows, we will consider purely stationary flows, moreover, two-dimensional ones, that, in our opinion, is unprincipled for explaining the effect itself.

Further, we restrict our consideration by purely stationary flows, moreover, two-dimensional flows, such that in our opinion, is unprincipled to explain the effect itself.

Using only these two assumptions, i.e. high magnetic Reynolds number and the weak influence of the magnetic field on convection, it will be shown that the magnetic field in the cell only due to convective flow tends to a filamentous state in the form magnetic flux tubes that are formed in the downward flow region and parallel to this flow. In the region of upward flow, the magnetic field has a tendency to be vanished; it is pushed to the periphery of convective cells. The main convection cell model, analytically and numerically investigated in this paper, is the two-dimensional one with flow in the form of periodic set of rolls [32, 33]. In this case, the magnetic field is shown to condensate in the region of downward flow, i.e., filamentation takes place on the interface between cells. Moreover, the magnetic field $B$ due to frozenness only, grows exponentially with time with simultaneous exponential narrowing of the filament itself. Magnetic field growth and accordingly, the narrowing of the magnetic filament, as shown in this paper, stops due to the destruction of the magnetic field frozenness because of magnetic viscosity. As a result, the magnetic field is saturated in filament at the level of $B_0 Re_m^{1/2}$. With $B_0 = 10 \text{ G}$ and $Re_m = 10^6$, the saturation field is of $10^3 \text{ G}$, which corresponds to observational data. Mechanism that leads to the formation of magnetic filaments for two-dimensional convective flows, qualitatively remains almost the same for three-dimensional convection cells. Magnetic filaments in this case should flatten while their growing in the vicinity of the downward flow. Numerical experiments performed in [36] for hexagonal cells, as well as recent observations [28] support the filamentation mechanism discussed here.

It is worth noting the importance of other magnetohydrodynamic processes arising in astrophysics, and their relationship with convection. Magnetic fields play an important role
for accretion disks formed near massive objects, such as black holes, neutron stars, white dwarfs, etc [37, 39]. The effect of convection on the magnetic field evolution has been discussed for a long time [40, 41] and it shows the relation between convective motions and the occurrence of magnetorotational instability in accretion disks. Possibility of suppressing magnetic field reversals in accretion disks due to accretion was discussed in [42]. Also we mention works [43, 44] devoted to modeling the relationship between convection and magnetic field in accretion disks. The effect of convective flows is essential for the evolution of magnetic fields in galaxies. Usually, convective flows are directed perpendicular to the equatorial plane. In [45–47], magnetic field generation was investigated taking into account the advection of the magnetic field helicity, which is an integral of motion in ideal magnetic hydrodynamics. In [48], there were performed studies of the effect of convection on the magnetic flux while star formation. All of these examples indicate the importance of studying convective flows and their effect on the magnetic field from the astrophysical point of view.

III. COMPRESSION OF MAGNETIC LINES AND ATTRACTOR

As formulated in the Introduction, in the case of a large density of kinetic energy versus magnetic energy density magnetic field is described by the equation of induction in the MHD approximation for a given velocity field:

\[ \frac{\partial \mathbf{B}}{\partial t} = \text{rot} \left( \mathbf{v} \times \mathbf{B} \right) + \eta_m \Delta \mathbf{B}, \quad \text{div} \mathbf{v} = 0. \]  

(1)

In this case, all the equations are written in dimensionless units: distances are measured in their typical values of \( L \), velocities - in characteristic values of \( V \). In the case of large magnetic Reynolds numbers \( R e_m \gg 1 \) this equation transforms into the frozenness equation:

\[ \frac{\partial \mathbf{B}}{\partial t} = \text{rot} \left( \mathbf{v} \times \mathbf{B} \right), \quad \text{div} \mathbf{v} = 0. \]  

(2)

Due to the vector product in the right-hand side of (2), only velocity component \( \mathbf{v}_n \), normal to the magnetic field line, can change the magnetic field. Tangential component \( \mathbf{v}_t \) in this case plays a passive role, providing an incompressibility condition \( \text{div} (\mathbf{v}_n + \mathbf{v}_t) = 0 \). Remind that frozenness of magnetic field means that each Lagrangian particle is pasted to its own field line and cannot leave it. A particle, therefore, has only one degree of freedom - motion along a magnetic field that obviously does not change the field. Hence it immediately follows that \( \mathbf{v}_n \) is the velocity of motion for the magnetic field line itself. This fact has simple geometric explanation. If one considers an arbitrary curve, then any deformation along the curve, obviously, does not change it; only the deformations normal to this curve are responsible for its shift.

On the other hand, in the general situation \( \text{div} \mathbf{v}_n \neq 0 \), according to [18], this is the reason that continuously distributed magnetic field lines are compressible objects. In particular, it follows if one considers the Lagrangian trajectories specified by the velocity \( \mathbf{v}_n, \mathbf{v}_n \), namely by the velocity of magnetic field lines:

\[ \frac{d\mathbf{r}}{dt} = \mathbf{v}_n(\mathbf{r}, t) \quad \mathbf{r}|_{t=0} = \mathbf{a}. \]

The solution of these equations gives a compressible mapping. The latter follows from the equation for Jacobian \( J = \det(\partial x_i/\partial a_k) \):

\[ \frac{dJ}{dt} = \text{div} \mathbf{v}_n \cdot J. \]  

(3)

It is necessary to emphasize that the equation (2) can be integrated in terms of \( \mathbf{r} = \mathbf{r}(\mathbf{a}, t) \) (see [14, 16, 18]):

\[ \mathbf{B}(\mathbf{r}, t) = \frac{(-\nabla \Phi_0)J}{J} \]  

(4)

Here \( \Phi_0(\mathbf{a}) \) is the distribution of the magnetic field at \( t = 0 \). \( \Phi_0(\mathbf{a}) \) plays the same role as the Cauchy invariant in ideal hydrodynamics [16, 18, 49, 50].

From the equation (3) it follows that the Jacobian \( J \) in the general situation, \( \text{div} \mathbf{v}_n \), can take arbitrary values, in particular vanishes, when according to (4) the magnetic field becomes infinite large. Due to this property, as well as the fact that the \( \mathbf{v}_n \) represents the velocity of the magnetic field lines, it becomes clear that magnetic field transfer with the speed \( \mathbf{v}_n \) will be carried out until the normal component is converted to zero, i.e. when
**B|v.** Areas where \( B|v \) should represent for magnetic field peculiar attractors. In three-dimensional space, as will be discussed below, these attractors must be two-dimensional. If the velocity field is stationary, then on attractor, the magnetic field can reach large values, perhaps even infinite if magnetic viscosity is neglected.

Figure 1 shows schematically the movement of magnetic field lines (marked in red) in the convection cell velocity field. The speed lines are blue and the arrows indicate the direction of flow. At the intersection of red and blue lines black arrows are drawn showing direction of the movement of magnetic lines. These arrows are perpendicular to the magnetic lines, show the direction of the normal velocity component \( v_n \). It immediately follows from this simple drawing that in the strip \( 0 \geq y \geq -\pi/2 \) all the magnetic lines from the right cell move closer to the line \( x = 0 \) on the left, and all magnetic lines from the left cell - to this line to the right. Thus, convective flow gathers all the magnetic field lines on the line \( x = 0 \). We emphasize that such a process is possible due to the compressibility of continuously distributed magnetic lines. Convective flow rakes all the magnetic lines, forming a magnetic filament. In this case, as is easily seen from this figure, the largest magnetic field should arise in the vicinity of the point \( x = y = 0 \). This point for convective flow is hyperbolic. Just because of hyperbolicity, as it will be shown in the next section, filament formation becomes possible. Note that raking the magnetic field occurs due to the magnetic field frozenness.

**FIG. 1: Magnetic field lines in convective cell.**

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**IV. CONVECTIVE CELL AND BOUNDARY CONDITIONS**

Let us discuss how the velocity field is arranged in one convective cell. In the center of the cell there is an upward flow, and at its border - moving stream down. In the simplest two-dimensional geometry, the velocity in the cell can be represented through the stream function \( \psi: v_x = -\partial_y \psi, v_y = \partial_x \psi \). For periodic chain of rolls with circulation, changing its sign when moving from one cells to another, \( \psi \) can be written as a product of two sines:

\[
\psi = C \sin (k_1 x) \sin (k_2 y)
\]

For a stationary system for Benard convection \( k_1 \neq k_2 \), but they are of the same order \( 52 \) (see also \( 53 \)). As it will be seen below, for everything subsequent it is unprincipled. Further therefore for simplicity put \( k_1 = k_2 = 1 \), and the constant \( C = 1 \) (this corresponds to the transition to dimensionless variables). As a result

\[
\psi = \sin x \cdot \sin y, \quad (5)
\]

and velocity components

\[
\begin{align*}
v_x &= -\sin x \cdot \cos y, \\
v_y &= \cos x \cdot \sin y.
\end{align*}
\]

Moreover, the line \( y = 0 \) will be considered the upper boundary convection cells. The velocity along the boundary in this case is parallel surface, the normal component is respectively zero.

In numerical experiments, we present the results for two cells with one common interface \( x = 0 \) along which the cooled fluid moves down. Two cells correspond to a rectangular area \( [-\pi \leq x \leq \pi, 0 \geq y \geq -\pi] \). Along the lines \( x = \pm \pi, 0 \geq y \geq -\pi \) fluid pops up (upflows), and along the center line \( x = 0, 0 \geq y \geq -\pi \) goes down (downflow). If in the initial moment of time the magnetic field \( B_0 \) is directed along \( y \)-axis, then one can see that the normal velocity component relative to \( B_0 \) will have positive \( x \)-projections in the upper right corner in the left cell, i.e. to lines \( x = 0, 0 \geq y \geq \pi/2 \) and, respectively, negative
V. FILAMENTATION

We turn now to finding the magnetic field depending on time and coordinates. This problem is in the general formulation for equation (2) as follows. In two-dimensional geometry when the magnetic field lies in the plane of convective flow (x, y), we introduce the magnetic potential, presenting it in the form

\[ A = -B_0 x + a, \]  

(6)

where the magnetic field \( B_0 \) is assumed to be uniform, directed along the y axis, and the fluctuation of \( a \) is periodic coordinate function both in \( x \) and in \( y \). The latter automatically will retain the full flux of the magnetic field across the boundary \( y = 0 \). For initial time, we can assume no fluctuations, \( a(t = 0) = 0 \), i.e. we start with a uniform magnetic field.

As explained above (see Fig. 1), magnetic field filamentation should appear in the upper right corner of the left cell near the top of the down stream at \( x = y = 0 \). The growth of the magnetic field at the maximum point should be observed precisely near this point.

The equation for the magnetic potential \( A \) is obtained by integrating equations for frozen-in field (2):

\[ \frac{\partial A}{\partial t} + (\mathbf{v} \cdot \nabla)A = 0, \]  

(7)

where the magnetic field can be expressed as follows

\[ B_x = \frac{\partial A}{\partial y}, \quad B_y = -\frac{\partial A}{\partial x}. \]  

(8)

From these relations it follows that the lines of constant value of \( A \) coincide with a magnetic field line. From the equation (7), with on the other hand, by the scalar product \( (\mathbf{v} \cdot \nabla)A \) it can be seen that (7) includes only the normal velocity component to the equipotential \( A = \text{const} \), i.e. to magnetic field line.

The equation for \( A \) is simply integrated using the method of characteristics. The equation for the characteristic has the form

\[ \frac{dr}{dt} = \mathbf{v}(r) \]

with the initial conditions \( r|_{t=0} = \mathbf{a} \). This equation in the component version represents the Hamilton equations

\[ \frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}, \]  

(9)

with the initial conditions \( x(t = 0) = a_x \) and \( y(t = 0) = a_y \). The coordinates \( x \) and \( y \) in these equations are canonically conjugate quantities, and the stream function \( \psi(x,y) \) is Hamiltonian. Since the velocity field is independent of time, then \( \psi(x,y) \) is the conserved quantity. Wherein the magnetic potential on the characteristic does not change with time.

Thus, the dynamics of the system (9) is determined by the properties of the Hamilton function \( \psi(x,y) \). For a finite area the function \( \psi(x,y) \), as a two-dimensional relief, is characterized by its extremums - minima, maxima and saddle points. At extremum points the gradient from \( \psi(x,y) \) is equal to zero, which corresponds to zero velocity. Which kind of extreme point is determined from the expansion of \( \psi(x,y) \) in the vicinity of the extremum \( \mathbf{r} = \mathbf{r}_0 \):

\[ \psi(r) = \psi(r_0) + \frac{1}{2} D_{ij} \Delta x_i \Delta x_j + ..., \]  

(10)

where \( \Delta r = r - r_0 \).

\[ D_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}|_{r=r_0}. \]

At maximum or minimum, the quadratic form \( D_{ij} \Delta x_i \Delta x_j \) is sign-definite. At this point the eigenvalues of the matrix \( D_{ij} \) are sign-definite if the following inequality is fulfilled

\[ \psi_{xx} \psi_{yy} - \psi_{xy}^2 > 0. \]  

(11)

According to [23, 24] such points are called elliptic, respectively, the region where the inequality (11) takes place is elliptic. With a opposite sign in the inequality (11), the stationary point becomes hyperbolic, and the corresponding region with the opposite sign in (11) is called hyperbolic.
Along the characteristic $x(t), y(t)$, the potential $A$ is constant, representing a passive scalar. Depending on the region (elliptic or hyperbolic), the potential $A$ will behave differently.

So, for example, for the stream function $\psi$, the point $x = y = 0$ is obviously hyperbolic; to this point, according to Fig. 1, the magnetic field should be attracted, which leads to filamentation of the magnetic field. The point $x = y = -\pi/2$, on the contrary, is an elliptical point around which the vector potential will rotate.

Now consider the solution to the problem for the initial condition $\psi$ for the stream function (5). Due to the fact that the magnetic potential fluctuations $a$ at $t = 0$ are absent, the magnetic potential on the characteristic will be equal to $A = -B_0 a_x$, i.e. depends only on the initial values of the $x$-coordinate of the fluid particles $a_x$.

Equations (9) are simply integrated. From the equality $\psi(x, y) = \psi(a_x, a_y)$ one can find dependence $y = y(x, a_x, a_y)$ and then substitute it in the right-hand side of the first equations (9). As a result, the equation for $x$

$$\frac{dx}{dt} = v_x(x, a_x, a_y)$$

is integrated trivially. Thus, we come to the general solution of the Cauchy problem for equation (7); this solution is written implicitly.

We will be interested in the behavior of the maximal magnetic field. The qualitative considerations given above show that the maximal magnetic field should be in a small neighborhood of point $x = y = 0$, i.e. for the center of the beginning of the downward flow, at the border between cells. We can take small deviations from the point $x = 0, y = 0$, considering $x$ and $y$ small. For such values of $x$ and $y$, the stream function, according to (5), can be approximately written as

$$\psi = xy.$$  

The initial condition for $\psi$ is the following

$$\psi = a_x a_y.$$  

For such a stream function, the equation for $x$

$$\frac{dx}{dt} = -x,$$

transforms into the linear one:

$$\frac{dx}{dt} = -x,$$

whose solution gives an exponential narrowing of the scale

$$x = a_x e^{-t}.$$  

For $y$ we have exponential growth: $y = a_y e^t$. From these asymptotics one can find the magnetic field behavior in this area. To do this, express derivatives with respect to $x$ and $y$ in the equations (8) through derivatives with respect to variables $a_x, a_y$. The easiest way to do this is based on the Jacobian technique. Note that due to the Hamiltonian equations (9) Jacobian

$$\frac{\partial(x, y)}{\partial(a_x, a_y)} = 1.$$  

In particular, for $B_x$ we have the following chain of transformations:

$$B_x = \frac{\partial(A, x)}{\partial(y, x)} =$$

$$\frac{\partial(A, x)}{\partial(a_y, a_x)} = \frac{\partial A}{\partial a_y} \frac{\partial x}{\partial a_x} - \frac{\partial A}{\partial a_x} \frac{\partial x}{\partial a_y}$$

Similar calculations for $B_y$ give

$$B_y = \frac{\partial(A, y)}{\partial(x, y)} = \frac{\partial(A, y)}{\partial(a_x, a_y)} =$$

$$- \frac{\partial A}{\partial a_x} \frac{\partial y}{\partial a_y} + \frac{\partial A}{\partial a_y} \frac{\partial y}{\partial a_x}.$$  

Hence for asymptotically as $x \to 0$ and $y \to 0$ we find

$$B_x = 0, B_y = B_0 e^t.$$  

If at the initial moment of time the $x$-component of the magnetic field is not equal to zero, then $B_x$ decays exponentially with time (see below). Thus, the maximum value of the magnetic field increases during time exponentially in the neighborhood of hyperbolic points. It is important that the maximal field is directed along downstream.
VI. NUMERICAL MODELLING

In the numerical integration of the MHD equations, at first magnetic potential fluctuations \((a)\) for \(a\) were determined from equations \((7)\):

\[
\frac{\partial a}{\partial t} + (\mathbf{v} \cdot \nabla) a = v_x B_0.
\]

This problem was solved in the region \(-\pi < x < \pi, -\pi < y < \pi\) with periodic boundary conditions along both coordinates, taking further only the lower strip \((-\pi < y < 0)\) corresponding to convective cells. For known magnetic potential we calculated a magnetic field by means of formulas \((8)\). For convenience, the average magnetic field \(B_0 = 1\) was chosen.

Numerically, the equation \((16)\) was solved on a grid 2000x2000 using an explicit numerical scheme with a small step, providing algorithm stability \([51]\). The time step was selected in accordance with a spatial step; in most cases, \(\Delta t = 2.5 \cdot 10^{-5}\).

Figures 2–4 show the results of integration for the vector potential \(A(x, y)\) at three moments in time. Recall that level lines \(A(x, y)\) coincide with the magnetic field lines, which in the figures correspond to the boundaries between the regions of one color. One can see the twist of the magnetic field lines inside the cells over time, which indicates that the corresponding region is elliptical. Moreover, in the neighborhood of the \(y\)-axis isolines of \(A(x, y)\) become with time more dense that corresponds to an increase of the magnetic field in the center, i.e. in the region of downward flow.

VII. INFLUENCE OF MAGNETIC VISCOSITY ON THE MAGNETIC FIELD EVOLUTION

The growth of the magnetic field in the filament is associated with its frozenness, which is destroyed due to finite magnetic viscosity. In this case, in the equation for \(a\) one needs to add the term responsible for magnetic viscosity:

\[
\frac{\partial a}{\partial t} + (\mathbf{v} \cdot \nabla) a = B_0 v_x + \frac{1}{Re_m} \Delta a,
\]

where \(Re_m\) – is the magnetic Reynolds number (in this equation we use dimensionless variables).

The corresponding spatial distribution of the vector potential for the case \(Re_m = 10^2\) is shown in Fig. 8. No cardinal changes occur.
FIG. 5: Magnetic field on $x$ axis. Black line shows case $t = 1$, red – $t = 2$.

FIG. 6: Maximal magnetic field evolution. Black line shows ideal case, red – $Re_m = 10$, blue – $Re_m = 10^2$, green – $Re_m = 10^3$.

compared with the non-dissipative case, except a little less twist.

However, studying evolution of the magnetic field with time for different magnetic Reynolds numbers and its comparison with the case of zero magnetic viscosity is much more interesting. At the early stage, when frozenness works, the magnetic field increases exponentially, and then, with a decrease in the filament thickness, saturation occurs reaching the stationary value $B_{sat}$ due to the frozenness destruction. Moreover, with the magnetic Reynolds number decrease this value becomes smaller (Fig. 6).

To estimate $B_{sat}$, we return to the original equation for the magnetic field (1), written in the dimensionless form:

$$\frac{\partial B}{\partial t} = \text{rot}[v, B] + \frac{1}{Re_m} \Delta B \quad (17)$$

Both the numerical experiment and analytical calculations show, that on the line $y = 0$ the magnetic field is maximal at the point $x = 0$. In the vicinity of this point, a magnetic field (having only one vertical component) obeys the equation that follows from (17):

$$\frac{\partial B_y}{\partial t} = \frac{1}{R_{em}} \frac{\partial}{\partial x} (xB_y) + \frac{1}{R_{em}} \frac{\partial^2 B_y}{\partial x^2}.$$\

Stationary solution to this equation is found from integration

$$\frac{\partial}{\partial x} (B_y x) + \frac{1}{R_{em}} \frac{\partial^2 B_y}{\partial x^2} = 0,$$

that gives

$$B(x) = B_{max} \exp \left(-\frac{(x - x_0)^2}{2R_{em}}\right). \quad (18)$$

Obviously, the integration constant $x_0$ should be set to zero, and the value of $B_{max}$ should be determined from the conservation of magnetic field flux. From (18) we have

$$\Phi \approx B_{max} \sqrt{\frac{2\pi}{R_{em}}} \quad (19)$$

At the initial moment, $\Phi = 2\pi B_0$, whence the estimate for maximal amplitude is of

$$B_{max} = B_0 \sqrt{2\pi R_{em}}.$$

It is worth noting that a similar estimate was obtained in a number of works [23, 25, 26]. Unfortunately, it is rather difficult to determine when such estimate was first received. Note that the estimates for the magnetic field are close to the results of numerical simulation (Fig. 6, 9). The width $\delta$ of this distribution determined only by magnetic viscosity. In dimensional variables $\delta \sim L/\sqrt{R_{em}}$. Saturation time of passing to stationary value can be evaluated as $T \sim \frac{1}{4} \ln R_{em}$. In dimensional variables saturation time

$$t_0 = T \frac{L^2}{\eta_m} = \frac{1}{2} \frac{L^2}{\eta_m} \ln \left(\frac{LV}{\eta_m}\right).$$
VIII. CONCLUDING REMARKS

Thus, for two-dimensional flows, filamentation of magnetic field occurs in the downward flow region. An exponential increase of the magnetic field is observed in neighborhood of the flow hyperbolicity, which in our opinion is the main criterion of the magnetic field filamentation. In the case of convective rolls, considered in this paper, this occurs in the vicinity of the hyperbolic point $x = y = 0$, where velocity components $v_x = -x$, $v_y = y$ respectively. Perpendicular the $y$ axis the velocity component $v_x$ due to frozenness gathers magnetic field, leading to its exponential growth. The question arises, how sensitive is the influence of the third component of magnetic field, namely, for a real three-dimensional problem. As noted magnetic field growth in a two-dimensional situation is determined by hyperbolicity flows near the interface of two convective cells. In the three-dimensional case, this hyperbolicity persists: when approaching the interface the flow can be considered flat, i.e. it has the very structure as in the two-dimensional case:

$$v_x = -x, \ v_y = y, \ v_z = 0.$$  \hfill (20)

In this case, according to (2) the behavior of the third component of magnetic field parallel to the interface in its vicinity will be determined from the equation

$$\frac{\partial B_z}{\partial t} + (\mathbf{v} \nabla) B_z = 0. \hfill (21)$$

Thus, in a neighborhood of the interface plane, $B_z$ represents the Lagrangian invariant unchanged when moving with plasma.

When magnetic viscous dissipation is taken into account, $B_z$ will attenuate. Concerning two other components of the magnetic field, at $\eta = 0$ they are found from the equations

$$\frac{\partial B_x}{\partial t} + (\mathbf{v} \nabla) B_x = -B_x,$$

$$\frac{\partial B_y}{\partial t} + (\mathbf{v} \nabla) B_y = B_y,$$

where the velocity is given by (20). From the second equation it follows that component $B_y$ grows exponentially (compare with (15)), and $B_x$, on the contrary, falls exponentially $\sim e^{-t}$. This process continues until frozenness is destroyed and the magnetic field in the filament comes out saturation due to magnetic viscosity. It follows from this fact that magnetic field filaments should be flattened relative to
the plane interface. It is also important to note that the difference between vertical and horizontal components of magnetic field occurs large enough.

We now discuss how the variability of convective cells affects the mechanism of formation of magnetic filaments as well as the magnetic field of filaments influences on the convective flow.

Characteristic variability time of convective cells can be estimated as the ratio of the cell size $L \sim 10^8$ cm to the characteristic speed $V \sim 10^5$ cm/s, which is consistent with the observational data. This is the time of the order of the inverse growth rate $\gamma^{-1}$, i.e. the magnetic field amplification in the "quiet" mode is roughly of the same order as $L/V$, but the convective motion does not stop and therefore pushing out magnetic field from the center of the cell to the periphery does not stop. The second question is to take into account the inverse effect of the magnetic field on the convective flow characteristics. For a typical speed of the order of $10^5$ cm/s and characteristic density $\sim 10^{-5}$ g/cm$^3$ it is possible to get that up to a magnetic field of the order of 1 kG it makes sense to speak about a weak influence of magnetic fields on flow characteristics. For larger fields, in our opinion, this mechanism should nevertheless work, since in the center of the cell due to the expulsion of the magnetic field to the periphery, the flow is practically independent of the magnetic field, but in the downward flow region, where magnetic filament is formed, magnetic field being parallel to the flow should be practically stationary.

We also note that the results presented in this paper are generally confirmed by numerical simulations \[36, 52, 53\] and observational data \[54, 55\], indicating a correlation of the amplifying magnetic fields with downward flows. In addition, it is worth mentioning a number of numerical results \[36, 56\] obtained for convective hexagonal cells in which the indicated correlation was observed.

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