Research Article

Generalizing Certain Analytic Functions Correlative to the $n$-th Coefficient of Certain Class of Bi-Univalent Functions

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1. Introduction

The structural properties and information about Geometric Function Theory depends on the estimation of coefficient of analytic functions. For example, the second coefficient estimation ($|a_2|$) in the set of univalent functions gives the growth, distortion bounds, and also covering theorems. The study of coefficient bounds for the classes of bi-univalent functions was first investigated by Levin [1] in 1967, but the interest sparks among the researchers when Brannan and Taha [2] conjectured the coefficient bounds for the classes of bi-univalent functions. A series of coefficients investigation have been carried out recently in [3–11] and coefficient properties by Rehman et al. [12]. In fact, the work of Srivastava et al. [13] has made a huge impact on the development of bi-univalent functions and appeared frequently in the literature ever since the publication of their pioneering work. In a recent development, Srivastava et al. [14] have made the use of Faber polynomial expansions with $q$-analysis to determine the bounds for the $n$-th coefficient in the Taylor–Maclaurin series expansions. In addition to this, Srivastava et al. [15] made the use of a linear combination of three functions $((f(z)/z), f'(z),$ and $zf''(z))$ with the technique involving the Faber polynomials and determined the coefficient estimates for the general Taylor–Maclaurin functions belonging to the bi-univalent function.

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk, $\{U = z \in \mathbb{C}: |z| < 1\}$ of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Furthermore, we represent by $\mathcal{T}$ the class of univalent functions, which is defined as a function $f \in \mathcal{A}$ is called univalent on $U$ (or schlicht or one-to-one) if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in U$ with $z_1 \neq z_2$. The leading member of this class is the famous Koebe’s function of the form $K(z) = z/(1-z)^2$. Some other examples are $v(z) = z$, $t(z) = z/(1-z)$, and $u(z) = (z/1-z^2)$ (see [16]). Next, we denote by $\Sigma$ the class of bi-univalent functions that states the class of functions $f \in \mathcal{A}$ is said to be bi-univalent in the unit disk $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$. For example, $f(z) = (z/1-z)$, $d(z) = -\log(1-z)$, and $j(z) = (1/2)\log(1+z/1-z)$ (for details, see [10,13]). According to
Köbe’s One Quarter Theorem (see [17], p. 31), the range of every function of class $\mathcal{S}$ contains the disk $\{w: |w| < 1/4\}$.

Köbe’s theorem ensures that the image of a unit disc $\mathbb{U}$, under every univalent function $f \in \mathcal{S}$, contains a disk of radius $(1/4)$. Therefore, every univalent function in $f \in \mathcal{S}$ has an inverse $f^{-1}$ defined by

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}),$$

$$f^{-1}(w) = w, \quad \left| w - r_o(f); r_o(f) \right| \geq \frac{1}{4}.$$

These type inverse functions can easily be verified by

$$\varphi(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_3^3 - 5a_3a_3 + a_4)w^4$$

$$+ (14a_4^4 - 21a_3^2a_3 + 3a_3^2 + 6a_2a_3 - a_3)w^5 \ldots,$$

where $\varphi(w) = f^{-1}(w)$.

Let $f(z)$, $g(z)$, and $w(z)$ be analytic functions in the open unit disc $\mathbb{U}$. The function $f(z)$ is said to be a subordinate to $g(z)$, expressed as $f(z) \prec g(z)$, if there exists a Schwartz function $w$, that is, $w(0) = 0$, $|w(z)| < 1$, and $f(z) = g(w(z))$. Particularly, if the function $g(z)$ is univalent in the unit disc $\mathbb{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. [18, 19]).

$$K_{n+1} = \frac{(-n)!}{(2n+1)(n-1)!}a_n^2 + \frac{(-n)!}{(2n)(n-1)!}a_n^2 a_3$$

$$+ \frac{(-n)!}{(2n+3)(n-1)!}a_n^4 + \frac{(-n)!}{(2n+2)(n-1)!}a_n^4 a_5$$

$$+ \frac{(-n)!}{(2n+1)(n-1)!}a_n^6 + \sum_{j \geq 7} a_n^{j-7} S_j,$$

with $(-n)! \equiv \Gamma(1-n) = (-n)(-n-1) \ldots$ such that $(n \in \mathbb{N} \cup \{0\}, \{n! \in \{1, 2, 3, \ldots\})$ and $S_j (7 \leq j \leq n)$ is a homogeneous polynomial in variables $(a_2, a_3, \ldots, a_n)$. For more details, one may refer to the expansion of $K_{n+1}$, and for the coefficients of its inverse functions, see Theorem 6.1 in [25] (p. 209) and [6]. Likewise, using the general term $K_{n+1}$, we can compute the first six terms as follows:

$$K_2^2 = -2a_2,$$

$$K_3^3 = 3(2a_3^2 - a_3),$$

$$K_4^4 = -4(5a_4^3 - 5a_4a_3 + a_4),$$

$$K_5^5 = -5(14a_5^4 - 21a_4^2a_3 + 3a_3^2 + 6a_2a_3 - a_3),$$

$$K_6^6 = -6(42a_6^4 - 84a_5^2a_4 + 28a_2a_4^3 - 7a_4a_3 - 28a_2a_3^2 - 7a_2a_4 - a_4).$$

The Faber polynomials introduced in 1903 by Faber [26] (also see [27]) play an important role in various areas of mathematical sciences, in particular the geometric function theory (Gong [28], Chapter III, and Schiffer [29]). The

2. Discussion

Löwner [20] and Pommerenke [19] proved that the inverse of Köbe’s function concedes the best bounds for all $|a_n|$. However, new techniques have been adopted recently to determine the peculiar behavior of coefficients $a_n$ for various subclasses of $\mathcal{S}$ (see, for example, [21–24]).

The series expansion of the inverse of $f \in \mathcal{S}$ in some disk about the origin is given by

$$f^{-1}(w) = w + a_2w^2 + a_3w^3 + a_4w^4 + a_5w^5 + \cdots$$

Moreover, a function $f(z)$ which is univalent in a neighborhood about the origin with its inverse satisfies the condition $f(f^{-1}(w)) = w$, and then, equation (4) can be written as

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3$$

$$+ a_4(f^{-1}(w))^4 + a_5(f^{-1}(w))^5 + \cdots,$$

where $f^{-1}(w) = w + \sum_{n=2}^{\infty} (1/n)K_{n+1} a_n w^n$ is the Faber polynomial expansion of functions $f \in \mathcal{S}$ of the form (1), where

$$\mathcal{S}(z) = 1 + B_1 z + B_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

for all $(B_n \in \mathbb{R})$ with $(B_1 > 0, n \in \mathbb{N} = \{1, 2, 3, \ldots\})$. 

The calculus of Faber polynomials gets more importance, especially when it was found useful in the study of inverse functions ($f^{-1}$) (see for details [25, 30–32]). Based on the implication of Faber polynomial expansions in determining the coefficient estimations of the bi-univalent functions and following the work of [1, 3, 9, 10, 13–15, 33, 34] and [35], we are motivated to derive new type of polynomials that collaborate with the Faber polynomial expansion to estimate the coefficient bounds for a certain class of bi-univalent functions beyond $|a_n| \geq 2$: $n \in \mathbb{N} = \{2, 3, 4, \ldots\}$. Throughout the article, we consider \( \mathcal{S}(z) \) to be analytic with its positive real part on the unit disk $\mathbb{U}$; obeying the conditions \( \mathcal{S}(0) = 1, \mathcal{S}'(0) > 0 \), and \( \mathcal{S}(\mathbb{U}) \) is symmetric with respect to real axis. This type of function can be expressed as series expansion of the following form:

$$\mathcal{S}(z) = 1 + B_1 z + B_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

for all $(B_n \in \mathbb{R})$ with $(B_1 > 0, n \in \mathbb{N} = \{1, 2, 3, \ldots\})$. 

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We now define the class $\Sigma \mathcal{H}(\alpha, \tau, \gamma)(\mathcal{B})$ with the following conditions.

**Definition 1.** Let $0 \leq \gamma \leq 1$, $\alpha \in \mathbb{C}, \{0\}$, and $0 \leq \alpha \leq 1$. A function $f \in \Sigma$ is in the class $\Sigma \mathcal{H}(\alpha, \tau, \gamma)(\mathcal{B})$ if the following subordinations hold:

$$\alpha + \frac{1}{t^\alpha} \left[ (\alpha - \gamma) \frac{f(z)}{z} + \gamma f^\prime(z) - a \right] \prec \mathcal{G}(z), \quad (z \in \mathbb{U}),$$  

(9)

$$\alpha + \frac{1}{t^\alpha} \left[ (\alpha - \gamma) \frac{\mathcal{G}(w)}{w} + \gamma \mathcal{H}^\prime(w) - a \right] \prec \mathcal{G}(w), \quad (w \in \mathbb{U}),$$  

(10)

where $\mathcal{G}(w) = f^{-1}(w)$.

**Remark 1.** If we set $\alpha = 1$ in $\Sigma \mathcal{H}(\alpha, \tau, \gamma)(\mathcal{B})$, the class interacts with the class introduced in [3]. Next, if we set $\alpha = \tau = 1, \gamma = 0$, and $\mathcal{G}(z) = (1 + Lz/1 + \mathcal{F}z), (-1 \leq \mathcal{F} \leq 1; z \in \mathbb{U})$ in Definition 1, we get the following class $\Sigma \mathcal{H}^\prime (\mathcal{L}, \mathcal{F})$.

**Definition 2 (see [3]).** A function $f \in \Sigma$ is in the class $\Sigma \mathcal{H}^\prime (\mathcal{L}, \mathcal{F})$, if the following subordinations hold:

$$\frac{f(z)}{z} + \frac{1 + Lz}{1 + \mathcal{F}z} \prec \mathcal{G}(z), \quad (z \in \mathbb{U}),$$  

(11)

In fact, the class we introduced is mainly inspired by Bansal and Sokol [3], Kumar et al. [7], and [13, 33]. For example, for $\alpha = 0$ in $\Sigma \mathcal{H}(\alpha, \tau, \gamma)(\mathcal{B})$, the class interacts with [7]; for $\alpha = \tau = 1, \gamma = 0$, $\mathcal{G}(z) = (1 + z/1 - z)^{\eta}$, $(0 \leq \eta < 1)$, the class joins to the function class given in [33].

In order to prove our main result, we need the following lemma.

**Lemma 1 (see [17]).** Let the function $b \in \mathcal{F}$ be given by the series

$$b(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots, \quad (z \in \mathbb{U}),$$  

(12)

then the sharp estimate $|h_n| \leq 2, (n \in \mathbb{N})$, holds.

The two important functions that frequently appear in the literature of bi-univalent functions are considered by many authors for determining the initial coefficient bounds of certain class of bi-univalent functions. Functions $b_1(z)$ and $b_2(w)$, respectively, are defined by

$$b_1(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots + q_n z^n,$$  

(13)

$$b_2(w) = \frac{1 + t(w)}{1 - t(w)} = 1 + v_1 w + v_2 w^2 + v_3 w^3 + \cdots + v_n w^n,$$  

(14)

where by Lemma 1, $|q_n|$ and $|v_n| \leq 2$.

3. Preliminary Results

In order to accomplish the intended formula, we introduce two polynomials and define them as $\mathcal{R}$- and $\mathcal{X}$- polynomials, respectively, as follows:

$$s(z) = R_1 z + R_2 z^2 + R_3 z^3 + \cdots = \sum_{n=1}^{\infty} R_n z^n, \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$  

(15)

$$t(w) = X_1 w + X_2 w^2 + X_3 w^3 + \cdots = \sum_{n=1}^{\infty} X_n w^n, \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$  

(16)

where in equation (15),

$$\mathcal{R}_1 = \frac{1}{2} q_1,$$

$$\mathcal{R}_2 = \frac{1}{2} (q_2 - \mathcal{R}_1 q_1),$$

$$\mathcal{R}_3 = \frac{1}{2} (q_3 - \mathcal{R}_1 q_2 - \mathcal{R}_2 q_1),$$

$$\mathcal{R}_4 = \frac{1}{2} (q_4 - \mathcal{R}_1 q_3 - \mathcal{R}_2 q_2 - \mathcal{R}_3 q_1),$$

$$\mathcal{R}_5 = \frac{1}{2} (q_5 - \mathcal{R}_1 q_4 - \mathcal{R}_2 q_3 - \mathcal{R}_3 q_2 - \mathcal{R}_4 q_1),$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\mathcal{R}_n = \frac{1}{2} (q_n - \mathcal{R}_1 q_{n-1} - \mathcal{R}_2 q_{n-2} - \mathcal{R}_3 q_{n-3} - \mathcal{R}_4 q_{n-4} - \cdots - \mathcal{R}_{n-1} q_1),$$  

(17)

and in equation (16),

$$\mathcal{X}_1 = \frac{1}{2} v_1,$$

$$\mathcal{X}_2 = \frac{1}{2} (v_2 - \mathcal{X}_1 v_1),$$

$$\mathcal{X}_3 = \frac{1}{2} (v_3 - \mathcal{X}_1 v_2 - \mathcal{X}_2 v_1),$$

$$\mathcal{X}_4 = \frac{1}{2} (v_4 - \mathcal{X}_1 v_3 - \mathcal{X}_2 v_2 - \mathcal{X}_3 v_1),$$

$$\mathcal{X}_5 = \frac{1}{2} (v_5 - \mathcal{X}_1 v_4 - \mathcal{X}_2 v_3 - \mathcal{X}_3 v_2 - \mathcal{X}_4 v_1),$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\mathcal{X}_n = \frac{1}{2} (v_n - \mathcal{X}_1 v_{n-1} - \mathcal{X}_2 v_{n-2} - \mathcal{X}_3 v_{n-3} - \mathcal{X}_4 v_{n-4} - \cdots - \mathcal{X}_{n-1} v_1).$$  

(18)

The triangular arrays in (17) and (18) lead us to form the general term of $\mathcal{R}_n$ and $\mathcal{X}_n$, respectively, as follows:
\[ R_n = \frac{1}{2} \left( q_n - \sum_{j=1}^{n-1} R_j q_{n-j} \right), \quad (n \in \mathbb{N}) = \{1, 2, 3, \ldots\}, \]  

(19)

\[ X_n = \frac{1}{2} \left( v_n - \sum_{j=1}^{n-1} X_j v_{n-j} \right), \quad (n \in \mathbb{N}) = \{1, 2, 3, \ldots\}. \]  

(20)

Therefore, equation (15) becomes

\[ s(z) = \frac{1}{2} \left( q_n - \sum_{j=1}^{n-1} R_j q_{n-j} \right) z^n, \quad (n \in \mathbb{N}) = \{1, 2, 3, \ldots\}, \]  

(21)

and equation (16) yields \( t(w) \):

\[ t(w) = \frac{1}{2} \sum_{n=1}^{\infty} \left( v_n - \sum_{j=1}^{n-1} X_j v_{n-j} \right) w^n, \quad (n \in \mathbb{N}) = \{1, 2, 3, \ldots\}. \]  

(22)

Now, using the above series of expressions in (17) and (18), we enlist below some of the terms such as \( R_1, R_2, R_3, R_4, R_5, R_6 \), etc. and \( X_1, X_2, X_3, X_4, X_5, X_6 \), etc., respectively:

\[ R_1 = \frac{1}{2} q_1, \]
\[ R_2 = \frac{1}{2} \left[ q_2 - \frac{1}{2} q_1^2 \right], \]
\[ R_3 = \frac{1}{2} \left[ \left( q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1^3 \right], \]
\[ R_4 = \frac{1}{2} \left[ \left( q_4 - \frac{1}{2} q_1 q_3 \right) - \frac{1}{2} q_2 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{2} q_1 \left( \frac{1}{2} q_1 (q_2 - q_1) \right) \right], \]
\[ R_5 = \frac{1}{2} \left[ \left( q_5 - \frac{1}{2} q_1 q_4 \right) - \frac{1}{2} q_2 \left( q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_3 \left( q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left( \frac{1}{2} q_1 (q_2 - q_1) \right) \right], \]
\[ R_6 = \frac{1}{2} \left[ \left( q_6 - \frac{1}{2} q_1 q_5 \right) - \frac{1}{2} q_2 \left( q_2 - \frac{1}{2} q_1^2 \right) - \frac{1}{2} q_3 \left( q_3 - \frac{1}{2} q_1 q_2 \right) - \frac{1}{2} q_1 \left( \frac{1}{2} q_1 (q_2 - q_1) \right) \right]. \]

and these expressions go on.

Upon using equation (18), we obtain the following list of \( X_n; n \in \mathbb{N} = \{1, 2, 3, \ldots\} \):
\[ x_1 = \frac{1}{2} v_1, \]
\[ x_2 = \frac{1}{2} \left[ v_2 - \frac{1}{2} v_1^2 \right], \]
\[ x_3 = \frac{1}{2} \left[ \left( v_3 - \frac{1}{2} v_1 v_2 \right) - \frac{1}{2} v_1 \left( v_2 - \frac{1}{2} v_1^2 \right) \right], \]
\[ x_4 = \frac{1}{2} \left[ \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) - \frac{1}{2} v_1 \left[ \left( v_3 - \frac{1}{2} v_1 v_2 \right) - \frac{1}{2} v_1 \left( v_2 - \frac{1}{2} v_1^2 \right) \right] \right], \]
\[ x_5 = \frac{1}{2} \left[ \left( v_5 - \frac{1}{2} v_1 v_4 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) - \frac{1}{2} v_3 \left[ \frac{1}{2} v_1 \left( v_3 - \frac{1}{2} v_1 v_2 \right) - \frac{1}{2} v_1 \left( v_2 - \frac{1}{2} v_1^2 \right) \right] \right] \]
\[ - \frac{1}{2} v_1 \left[ \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) - \frac{1}{2} v_3 \left[ \frac{1}{2} v_1 \left( v_3 - \frac{1}{2} v_1 v_2 \right) - \frac{1}{2} v_1 \left( v_2 - \frac{1}{2} v_1^2 \right) \right] \right] \]
\[ x_6 = \frac{1}{2} \left[ \left( v_6 - \frac{1}{2} v_1 v_5 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) - \frac{1}{2} v_4 \left[ \frac{1}{2} v_1 \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) - \frac{1}{2} v_4 \left[ \frac{1}{2} v_1 \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) \right] \right] \right], \]
\[ - \frac{1}{2} v_2 \left[ \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) - \frac{1}{2} v_4 \left[ \frac{1}{2} v_1 \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) \right] \right] \]
\[ - \frac{1}{2} v_4 \left[ \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) - \frac{1}{2} v_4 \left[ \frac{1}{2} v_1 \left( v_4 - \frac{1}{2} v_1 v_3 \right) - \frac{1}{2} v_2 \left( v_2 - \frac{1}{2} v_1^2 \right) \right] \right] \],

and the expressions go on.

**Note 1.** The following notations will be used in coming sections:

1. \( ^n \mathcal{R}_t_i \): the set of all possible \( \mathcal{R} \)'s for \( t_i = \{1, 2, 3, \ldots \} \)
2. \( ^\mu \mathcal{R}_t_i \): list of usable \( \mathcal{R} \)'s
3. \( ^\nu B_t_i \): list of all possible \( B' \)'s, where \( B_t_i \) is chosen from (8)
4. \( ^p B_t_i \): number of previous \( B' \)'s for \( t_i = \{1, 2, 3, \ldots \} \)
5. \( \binom{t_k}{m} \): \( m \), shows the degree of each combination of \( ^p \mathcal{R}_t_i \) and \( t_k \), represents the number of \( ^p \mathcal{R}_t_i \), in a single combination
6. \( B_t_i \{ ^\mu \mathcal{R}_t_i \} \): the multipliers of \( B_t_i \) in terms of \( ^\mu \mathcal{R}_t_i \).

**Remark 2.** The relation to determine \( ^\mu \mathcal{R}_t_i \), associated with \( B_t_i \), is given by

\[ ^\mu \mathcal{R}_t_i = n - ^p B_t_i, \]  \hspace{1cm} (25)

where \( n \) is taken from the coefficient of \( z^n; n = \{1, 2, 3, \ldots \} \). For example, referring to (6) in Note 1, if we need to find \( B_3 \{ ^\mu \mathcal{R}_t_i \} \) for the coefficient of \( z^3 \), then \( ^p B_3 = \{ B_1, B_2 \} = 2B' \) s. This implies \( \{ ^\mu \mathcal{R}_t_i \} = 4 - 2 = \{ \mathcal{R}_1, \mathcal{R}_2 \} \). Therefore, for the coefficient of \( z^4 \), the multipliers of \( B_3 = B_3 \{ \mathcal{R}_1, \mathcal{R}_2 \} \).

### 4. Main Results

The study of bi-univalent functions shows that most authors use the comparison method between their newly introduced bi-univalent classes and the analytic functions under certain conditions to estimate the coefficient bounds for their function classes. The same type of studies can be found in [3, 7, 10] and many others. The main crux to estimate the \( n \)-th coefficient bounds of such bi-univalent functions lies in the generalization of the two analytic functions defined in (13) and (14) and the generalization of the class of certain bi-univalent functions. Hence, we present the generalization of two analytic functions that assist in structuring the formula for estimating the \( n \)-th coefficient bound for certain class of bi-univalent functions.

#### 4.1. Analytic Functions Correlative to Bi-Univalent Functions

**Theorem 1.** If \( \{ t_i, t_j, t_k, (t_k - n) > 0 \} \) and the function \( b_1(z) = (1 + s(z)/1 - s(z)) \) is analytic in the unit disk, then for a certain bi-univalent function, the comparative coefficients of \( \{ z^m; n \in \mathbb{N}, z \in \mathbb{C} \} \) may be represented by the following expansions:

\[
\frac{(\alpha + ny)}{r^n} a_{n+1} = \sum_{i=1}^{n} B_{t_i} \left( \sum_{t_i = 1}^{n} \mathcal{R}_{t_i}^{-(t_i-1)} \mathcal{R}_{n-(t_i-1)t_i} \right) + \sum_{t_i \neq t_j} \mathcal{R}_{t_i}^{-(t_i-2)} \mathcal{R}_{t_j} \mathcal{R}_{n-(t_i-2)t_i-t_j} \binom{t_k}{m}.
\]  \hspace{1cm} (26)

**Proof:** The given function \( b_1(z) = (1 + s(z)/1 - s(z)) \) is analytic in \( \mathbb{U} \) as \( b_1(0) = 1 \), and thus, function \( b_1 \) has the following Taylor–Maclaurin series expansion:
\begin{equation}
\frac{b_1(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \ldots. \quad (27)
\end{equation}

Since \( s \colon \mathbb{U} \rightarrow \mathbb{U} \), the analytic function \( b_1 \) has a positive real part on \( \mathbb{U} \), and in view of Lemma 1, we have \( |q_n| \leq 2 \cdot n \in \mathbb{N} \). Hence, solving (27) for \( s(z) \), we get

\begin{equation}
s(z) = \frac{b_1(z) - 1}{b_1(z) + 1} = \frac{q_1 z + q_2 z^2 + \ldots + q_n z^n}{2 + (q_1 z + q_2 z^2 + \ldots + q_n z^n)} = R_1 z + R_2 z^2 + \ldots + R_n z^n. \quad (28)
\end{equation}

Now, utilizing equations (8) and (9) and equations (27) and (28), we obtain

\begin{equation}
\alpha + \frac{1}{r^\alpha} \left( (a + \gamma) \frac{f(z)}{z} + \gamma f'(z) - a \right) = \mathcal{G} \left( \frac{b_1(z) - 1}{b_1(z) + 1} \right) = 1 + B_1 R_1 z + (B_1 R_2 + B_2 R_1) z^2 + (B_1 R_3 + 2B_2 R_1 R_2 + B_3 R_1^3) z^3
\end{equation}

\begin{align*}
&+ (B_1 R_4 + B_2 (R_2^3 + 2R_3 R_1)) + 3B_3 R_1^3 R_2 + B_4 R_1^4) z^4 + (B_1 R_5 + 2B_2 (R_2 R_3 + R_1 R_4)) \end{align*}

\begin{align*}
&+ (3B_4 (R_2 R_4 + R_3 R_2) + 5B_5 R_1^5) z^5 + \ldots. \quad (29)
\end{align*}

Since a function \( f \) belonging to the class of bi-univalent functions is in the form of (1), a calculation shows that the \( f^{-1}(z) = \mathcal{G}(w) \) has similar expansion as in expression (3). So, by using (1) in (29), we obtain

\begin{equation}
\alpha + \frac{2\alpha}{r^\alpha} + \left\{ \frac{\alpha + \gamma}{r^\alpha} a_1 z + \frac{\alpha + 2\gamma}{r^\alpha} a_2 z^2 + \ldots + \frac{\alpha + ny}{r^\alpha} a_{n+1} z^n \right\}
\end{equation}

\begin{align*}
&+ (B_1 R_4 + B_2 (R_2^3 + 2R_3 R_1)) + 3B_3 R_1^3 R_2 + B_4 R_1^4) z^4 + (B_1 R_5 + 2B_2 (R_2 R_3 + R_1 R_4)) \end{align*}

\begin{align*}
&+ (3B_4 (R_2 R_4 + R_3 R_2) + 5B_5 R_1^5) z^5 + \ldots. \quad (30)
\end{align*}

\begin{remark}
In the above equation, the left-hand side is the generalized form of the class \( \Sigma \mathcal{H}(\alpha, \beta, \gamma)(\mathcal{S}) \), while the right-hand side is the newly induced polynomial function associated with (8), (27), and (28).
\end{remark}

\begin{remark}
The polynomial on the right-hand side of (30) is extendable to \( z^n \) and thus needs to determine the coefficient of \( z^n \).

Now, by comparing the coefficients of \( z, z^2, z^3, \ldots, z^n \), we have the following expressions:

[Remainder of the text would continue with mathematical expressions and proofs]
\[
\frac{(\alpha + \gamma)}{r^2} a_2 = B_1 R_1,
\]
\[
\frac{(\alpha + 2\gamma)}{r^2} a_3 = B_1 R_2 + B_2 R_1^2,
\]
\[
\frac{(\alpha + 3\gamma)}{r^2} a_4 = B_1 R_3 + 2B_2 R_2 R_1 + B_3 R_1^3,
\]
\[
\frac{(\alpha + 4\gamma)}{r^2} a_5 = B_1 R_4 + B_2 R_3 R_1 + 2B_1 R_1 R_2^2 + 3B_2 R_2^2 R_1 + B_4 R_1^4,
\]
\[
\frac{(\alpha + 5\gamma)}{r^2} a_6 = B_1 R_5 + B_2 (R_3 R_2 R_1 + R_4 R_1) + 3B_1 (R_1 R_3^2 + R_2^3 R_1) + 4B_2 R_2^3 R_2 R_1 + B_4 R_1^6,
\]
\[
\frac{(\alpha + 6\gamma)}{r^2} a_7 = B_1 R_6 + B_2 (R_2 R_3 R_2 + 2R_4 R_3 + 2B_1 R_2 R_4 + 3B_2 R_3 R_1 + 3B_3 R_1^3 R_2 + R_4^2 R_1^3) + B_6 R_1^8,
\]
\[
\vdots
\]
\[
\frac{(\alpha + n\gamma)}{r^2} a_n = B_1 R_n + \sum_{i=2}^{n-1} B_i \left( \sum_{t_i = 1}^{t_n} R_{t_1}^{t_2} R_{n-(t_2-1)t_1} + \sum_{t_i \neq t_j}^{t_n} R_{t_i}^{t_2} R_{t_j} R_{n-(t_2-1)t_1} \right).\]

Thus, by iteration, we deduce a general formula that relates (8) and (9) with \( R \)–polynomial defined in (15). So the acquired formula that finds the series of expressions asserted in (31) is given as follows:

\[
\frac{(\alpha + n\gamma)}{r^2} a_n = B_1 R_n + \sum_{i=2}^{n-1} B_i \left( \sum_{t_i = 1}^{t_n} R_{t_1}^{t_2} R_{n-(t_2-1)t_1} + \sum_{t_i \neq t_j}^{t_n} R_{t_i}^{t_2} R_{t_j} R_{n-(t_2-1)t_1} \right).\]

This proves our assertion in (26).

Note 2. Observe that the above equation consists of two parts in which the first part of \( t_i \) takes single values, while the second part takes simultaneously two different values such that \( t_i \neq t_j \) for \( i \neq j \). In the case of \( t_i, t_j, t_k, (t_k - n) < 0 \), the whole term is omitted. Also, \( m \in \mathbb{N} \) shows the degree of each combination of \((\alpha R_{t_i}; t_i = \{1, 2, 3 \ldots \})\) such that the values of \( R_n \) are chosen from expression (17) or from (19). For numerical explanation of (32), see Example 1 in Section 4.

Theorem 2. If \( \{t_i, t_j, t_k, (t_k - n)\} > 0 \) and the function \( b_2(w) = (1 + t(w)/1 - t(w)) \) is analytic in the unit disk, then for a certain bi-univalent function, the comparative coefficients of \( \{w^n; n \in \mathbb{N}, w \in \mathbb{C}\} \) may be represented by the following expansions:

\[
\frac{(\alpha + n\gamma) K_n^{(n+1)}}{r^2} \frac{\alpha^{(n+1)}}{n+1} = \sum_{i=1}^{m} B_i \left( \sum_{t_i = 1}^{t_n} R_{t_1}^{t_2} R_{n-(t_2-1)t_1} + \sum_{t_i \neq t_j}^{t_n} R_{t_i}^{t_2} R_{t_j} R_{n-(t_2-1)t_1} \right) \binom{t_k}{m}.
\]

Proof. The given function \( b_2(w) = (1 + t(w)/1 - t(w)) \) is analytic in \( \mathbb{U} \) as \( b_2(0) = 1 \), and thus, function \( b_2(w) \) has the following Taylor–Maclaurin series expansion:

\[
b_2(w) = \frac{1 + t(w)}{1 - t(w)} = 1 + v_1 w + v_2 w^2 + v_3 w^3 + \cdots.
\]

Since \( t: \mathbb{U} \rightarrow \mathbb{U} \), the analytic function \( b_2 \) has a positive real part on \( \mathbb{U} \), and in the view of Lemma 1, we have \(|v_\alpha| \leq 2; n \in \mathbb{N}\). Thus, solving (34) for \( t(w) \), we obtain

\[
t(w) = \frac{b_2(w) - 1}{b_2(w) + 1} = \frac{v_1 w + v_2 w^2 + \cdots + v_n w^n}{2 + (v_1 w + v_2 w^2 + \cdots + v_n w^n)}
\]

\[
= \mathcal{R}_1 w + \mathcal{R}_2 w^2 + \cdots + \mathcal{R}_n w^n,
\]
so, from equations (8) and (10) and equations (34) and (35), we get

\[
\alpha + \frac{1}{\tau} \left[ (\alpha - \gamma) \frac{G(w)}{w} + \gamma H'(w) - \alpha \right] = G \left( \frac{b_2(w) - 1}{b_2(w) + 1} \right) = 1 + B_1 \mathcal{X}_1 \omega + (B_1 \mathcal{X}_2 + B_2 \mathcal{X}_1^3) \omega^2 + (B_1 \mathcal{X}_3 + 2B_3 \mathcal{X}_2 \mathcal{X}_2 + B_3 \mathcal{X}_1) \omega^3 \\
+ (B_1 \mathcal{X}_4 + B_4 \mathcal{X}_2^2 + 2\mathcal{X}_2 \mathcal{X}_1) + 3B_4 \mathcal{X}_4 \mathcal{X}_2 + B_4 \mathcal{X}_1^5) \omega^4 \\
+ \left( \begin{array}{c}
B_7 \mathcal{X}_5 + 2B_7 \mathcal{X}_2 \mathcal{X}_3 + 3B_7 \mathcal{X}_3 \mathcal{X}_2 + B_7 \mathcal{X}_1^5 \\
+ 3B_7 \mathcal{X}_4 \mathcal{X}_2^2 + \mathcal{X}_2^2 \mathcal{X}_1^4 + 4B_7 \mathcal{X}_1^3 \mathcal{X}_2 + B_7 \mathcal{X}_1^5 \end{array} \right) \omega^5 + \ldots
\]

(36)

Since a function \( f \) belonging to the class of bi-univalent functions has the Maclaurin series given by (1), a calculation shows that the inverse of \( G(w) = f^{-1}(w) \) has similar expansion as in (3).

Now, by using (2), (5), and (7) in (36), we get

\[
\alpha + \frac{\alpha + 2\gamma K^{-2}_1}{r^a} \omega + \frac{\alpha + 2\gamma K^{-3}_2}{r^a} \omega^2 + \ldots + \frac{\alpha + ny K^{-n+1}_n}{r^a} \omega^n = 1 + B_1 \mathcal{X}_1 \omega \\
+ (B_1 \mathcal{X}_2 + B_2 \mathcal{X}_1^2) \omega^2 + (B_1 \mathcal{X}_3 + 2B_3 \mathcal{X}_2 \mathcal{X}_2 + B_3 \mathcal{X}_1^3) \omega^3 + (B_1 \mathcal{X}_4 + B_4 \mathcal{X}_2^2 + 2\mathcal{X}_2 \mathcal{X}_1) \mathcal{X}_2 + B_4 \mathcal{X}_1^5) \omega^4 \\
+ (B_1 \mathcal{X}_5 + 2B_5 \mathcal{X}_3 \mathcal{X}_2 + 3B_5 \mathcal{X}_3 \mathcal{X}_2^2 + 4B_5 \mathcal{X}_3 \mathcal{X}_2 + B_5 \mathcal{X}_1^5) \omega^5 + \ldots
\]

(37)

Thus, by comparing the coefficients of \( \omega, \omega^2, \omega^3, \ldots, \omega^n \), we have

\[
\begin{align*}
\frac{(\alpha + \gamma) K^{-2}_1}{2} &= B_1 \mathcal{X}_2 + B_2 \mathcal{X}_1^2, \\
\frac{(\alpha + 2\gamma) K^{-3}_2}{3} &= B_1 \mathcal{X}_3 + 2B_3 \mathcal{X}_2 \mathcal{X}_2 + B_3 \mathcal{X}_1^3, \\
\frac{(\alpha + 3\gamma) K^{-4}_3}{4} &= B_1 \mathcal{X}_4 + 3B_3 \mathcal{X}_3 \mathcal{X}_2 + 3B_4 \mathcal{X}_4, \\
\frac{(\alpha + 4\gamma) K^{-5}_4}{5} &= B_1 \mathcal{X}_5 + 4B_4 \mathcal{X}_4 \mathcal{X}_2 + 4B_5 \mathcal{X}_5, \\
\frac{(\alpha + 5\gamma) K^{-6}_5}{6} &= B_1 \mathcal{X}_6 + 5B_5 \mathcal{X}_5 \mathcal{X}_2 + 5B_6 \mathcal{X}_6, \\
\frac{(\alpha + 6\gamma) K^{-7}_6}{7} &= B_1 \mathcal{X}_7 + 6B_6 \mathcal{X}_6 \mathcal{X}_2 + 6B_7 \mathcal{X}_7, \\
\frac{(\alpha + ny) K^{-n+1}_n}{n+1} &= B_1 \mathcal{X}_n + \sum_{i=2}^{n-1} B_i \left( \sum_{l=1}^{t_2} \mathcal{X}_i^{l-1} \mathcal{X}_1^{n-(l+1)} + \sum_{l \neq j}^{t_{n-1}} \mathcal{X}_i^{l-2} \mathcal{X}_j \mathcal{X}_1^{n-(l+1)} \right) + \sum_{l \neq j}^{t_{n-1}} \mathcal{X}_i^{l-2} \mathcal{X}_j \mathcal{X}_1^{n-(l+1)} \right) \left( t_k \right) + B_n \mathcal{X}_n.
\end{align*}
\]
Similar to (32), we obtain the following formula that relates (8) and (10) with $\tilde{X}$-polynomial defined in (16):

\[
\frac{(\alpha + n\gamma) K_n^{-n(\alpha + 1)}}{t^\alpha} = \sum_{i=1}^{n} B_i \left( \sum_{t_i=1}^{t_k} \prod_{t_j=1}^{t_{i-1}} \tilde{X}_{-n(t_j-t_i)}^i \tilde{X}_{-n(t_i-2)}^i \tilde{X}_{-(t_i-t_j)}^i \right) \left( \frac{t_k}{m} \right),
\]

(39)

where $m$ represents the degree of each combination of $\tilde{X}_i$ such that the values of $(X_n; n \in \mathbb{N} = \{1, 2, 3, \ldots\})$ are chosen from expression (18) and values of $(\frac{K_n^{-n(\alpha + 1)}}{n + 1}; n \in \mathbb{N} = \{1, 2, 3, \ldots\})$ are obtained from (7) or in general from (6). This statement completes the proof.

\[\square\]

4.2. Applicability of $R$- and $\tilde{X}$-Polynomials. In this section, we employ $R$- and $\tilde{X}$-polynomials on our class stated in Definition 2. Here, the aim is to test our newly constructed relation defined in (30) on our class $\Sigma (\alpha, \tau, \gamma) (\mathcal{G})$ in order to obtain the initial coefficient estimation.

**Theorem 3.** Let $f(z) \in \Sigma (\alpha, \tau, \gamma) (\mathcal{G})$ be of the form (2), then

\[
|a_1| \leq \frac{|\gamma| B_1(3/2)}{\sqrt{|\gamma^2 B_1(\alpha + 2\gamma) + (\alpha + \gamma)^2 (B_1 - B_2)}}
\]

(40)

\[
|a_2| \leq B_1 \left( \frac{\gamma^2 B_1}{(\alpha + \gamma)^2 + \frac{1}{(\alpha + 2\gamma)}} \right).
\]

(41)

\[
\frac{(\alpha + \gamma)}{\tau^\alpha} a_{1+1} = \sum_{i=1}^{n} B_i \left( \sum_{t_i=1}^{1} \prod_{t_j=1}^{t_{i-1}} \mathcal{R}_{t_i-j}^i \mathcal{R}_{t_i-1}^i \right) \left( \frac{1}{m} \right) = B_1 (\mathcal{R}_1) \left( \frac{1}{1} \right),
\]

(44)

For the next corresponding expression, utilizing the formula in (39) by putting $n = 1$, we get

\[
^{n}B_1 = \{B_1\},
\]

\[
^{n} \mathcal{R}_1 = \{\mathcal{R}_1\},
\]

\[
^{n+1} \mathcal{R}_1 = n - B_1 \Rightarrow 1 - 0 = 1 = \{\mathcal{R}_1\}.
\]

(45)

\[
\frac{(\alpha + \gamma) K_1^{-n(\alpha + 1)}}{t^\alpha} = \sum_{i=1}^{n} B_i \left( \sum_{t_i=1}^{1} \prod_{t_j=1}^{t_{i-1}} \tilde{X}_{-n(t_j-t_i)}^i \tilde{X}_{-n(t_i-2)}^i \tilde{X}_{-(t_i-t_j)}^i \right) \left( \frac{1}{m} \right) = B_1 (\tilde{X}_1) \left( \frac{1}{1} \right),
\]

(46)

Once again in view of Note 3, the second part of the formula in (39) is dropped as it needs simultaneously two values (see Note 1). Thus, putting $n = 1$ in (39), we obtain
such that the value of $\left( K_{\frac{3}{2}} \right)$ is provided by the Faber polynomial expansions given in (7) or could be calculated from (6).

**Remark 5.** Note that, by comparing (44), (46), and (7) together with (23) and (24), we get $R_1 = -\mathcal{X}_1$, that leads to $q_1 = -v_1$.

Now, by squaring (44), we obtain the following initial value of $a_2^2$:

$$ a_2^2 = \frac{B_1^2 r_{2 \alpha}}{(\alpha + \gamma)^2} R_2^2 = \frac{B_1^2 r_{2 \alpha}}{(\alpha + \gamma)^2} X_1^2, \quad (47) $$

$$ a_2^2 = \frac{1}{4} \frac{B_1^2 r_{2 \alpha}}{(\alpha + \gamma)^2} v_1^2. $$

For the next expression, utilizing the formula in (39) by putting $n = 2$, we get

$$ \frac{\alpha + 2\gamma}{r^n} d_{z_1} = \sum_{i=1}^{\infty} B_i \left( \sum_{j=1}^{i} R_{j-1} X_{2-j} \right) \left( \frac{t_k}{m} \right) = B_1 R_1 \left( \frac{1}{m} \right) + B_2 R_2 \left( \frac{2}{m} \right). $$

$$ \frac{\alpha + 2\gamma}{r^n} d_{z_1} = B_1 R_1 + B_2 R_2. $$

Again, for the conclusive value of $a_2^2$, we compare the coefficients of $z^2$ and using the formula in (32) by putting $n = 2$, we get

$$ \begin{align*}
\text{where the value of } & (K_{\frac{3}{2}})^3/3) \text{ is given by the Faber polynomial expansion defined in (7) or could be calculated from (6). Now, adding (49) and (50) and then using (17), (18), (23), and (24) together with the value of } v_1^2 \text{ from (47), we conclude the following value of } a_2^2: \\
& (\alpha + 2\gamma) \frac{K_{\frac{3}{2}}}{2 + 1} = B_1 X_2 \left( \frac{1}{m} \right) + B_2 X_1^2 \left( \frac{2}{m} \right), \quad (50) \\
& (\alpha + 2\gamma) \frac{K_{\frac{3}{2}}}{3} = B_1 X_2 + B_2 X_1^2, \\
& 2(\alpha + 2\gamma) \frac{a_2^2}{r} = B_1 R_2 + B_2 R_1^2 + B_1 X_2 + B_2 X_1^2 = B_1 R_2 + B_1 X_2 + 2 B_2 X_1^2 = \frac{1}{2} B_1 (q_2 + v_2) - \frac{1}{2} v_1^2 (B_1 - B_2). \\
& a_2^2 = \frac{B_1^2 r_{2 \alpha}^2 \left[ q_2 + v_2 \right]}{4 \left( B_1^2 r_{2 \alpha} (\alpha + 2\gamma) + (\alpha + \gamma)^2 (B_1 - B_2) \right)}. \quad (51) \\
\end{align*} $$

Next, subtracting (49) from (50) and then using (17), (18), (23), and (24) together with the value of $v_1^2$ from (47), we obtain the following value of $a_3$:

$$ \begin{align*}
& \frac{2(\alpha + 2\gamma)}{r^n} (a_3 - a_2^2) = B_1 R_2 + B_2 R_1^2 - B_1 X_2 - B_2 X_1^2 = B_1 (R_2 - X_2) = \frac{1}{2} B_1 (q_2 - v_2), \\
& a_3 = \frac{1}{4} B_1 r_{\alpha} \frac{r \left[ B_1 (q_2 - v_2) \right]}{(\alpha + \gamma)^2 (\alpha + 2\gamma)} \cdot (52) \\
\end{align*} $$
If we set out the value of Remark 6, proof asserted in (40) and (41).

Remark 6. If we set out the value of \( \alpha = 1 \), we receive Theorem 1 of [3].

Remark 7. If we put the value of \( \alpha = 1, \gamma = 1, \) and \( \tau = 1, \) we receive Theorem 1 of [33].

4.3. Certain Coefficient of \( z^n \). In this subsection, we offer an example that demonstrates the calculation of the certain coefficient of \( z^n \), and here, we take \( n = 4 \).

Example 1. How to determine the expression by comparing the coefficient of \( z^4 \), with the corresponding class of bi-univalent function?

Manipulating (31) for \( n = 4 \), two extreme terms are \( B_4 \mathfrak{R}_4 \) and \( B_4 \mathfrak{R}_4^3 \), so we only work for finding the mean terms that is \( B_2 \{ \mu \mathfrak{R}_j \} \) and \( B_3 \{ \mu \mathfrak{R}_j \} \) (also refer to Remark 1):

\[
\frac{(a + 4\tau)}{\tau^a} a_5 = B_1 \mathfrak{R}_4 + B_2 \{ \mu \mathfrak{R}_4 \} + B_3 \{ \mu \mathfrak{R}_j \} + B_4 \mathfrak{R}_4^3.
\]

The expression in (53) means that \( \mathfrak{R}_j \) involves only \( \mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \) and \( \mathfrak{R}_4 \) in its final estimation. So, on further exploration, we obtain the following expressions:

\[
\frac{(a + 4\tau)}{\tau^a} a_5 = B_1 \mathfrak{R}_4 + B_2 \{ \mu \mathfrak{R}_4 \} + B_3 \{ \mu \mathfrak{R}_j \} + B_4 \mathfrak{R}_4^3.
\]

\[
\frac{(a + 4\tau)}{\tau^a} a_5 = B_1 \mathfrak{R}_4 + B_2 \{ \mu \mathfrak{R}_4 \} + B_3 \{ \mu \mathfrak{R}_j \} + B_4 \mathfrak{R}_4^3.
\]

Note that some of the terms in above equation violates the conditions \( (t_k < 0) \) stated in Theorem 1. Hence, by neglecting those terms, we get

\[
\frac{(a + 4\tau)}{\tau^a} a_5 = B_1 \mathfrak{R}_4 + B_2 \{ \mu \mathfrak{R}_4 \} + B_3 \{ \mu \mathfrak{R}_j \} + B_4 \mathfrak{R}_4^3
\]

\[
\frac{(a + 4\tau)}{\tau^a} a_5 = B_1 \mathfrak{R}_4 + B_2 \{ \mu \mathfrak{R}_4 \} + B_3 \{ \mu \mathfrak{R}_j \} + B_4 \mathfrak{R}_4^3
\]

\[
\frac{(a + 4\tau)}{\tau^a} a_5 = B_1 \mathfrak{R}_4 + B_2 \{ \mu \mathfrak{R}_4 \} + B_3 \{ \mu \mathfrak{R}_j \} + B_4 \mathfrak{R}_4^3
\]

Note 4. It is worth mentioning that \( n \) preserves its value in each product combination of \( \mu \mathfrak{R}_j \). For example, in equation (55), the multipliers of \( B_4 \) are given by \( \mathfrak{R}_2^2 \mathfrak{R}_2 \), where \( n = 2 \times 1 + 1 \times 2 = 4 \). Also, observe that some of the terms are omitted in Example 1 and see the conditions stated in Note 2.

5. Corollaries and Consequences

This section discusses some interesting corollaries and other results.

Firstly, if we set \( \mathcal{B} (z) = (1 + \mathcal{L}z/a + \mathcal{F}z) \), \((-1 < \mathcal{L} \leq 1; z \in \mathbb{U})\), then equation (8) becomes
\[ G(z) = \frac{1 + \mathcal{L}z}{1 + \mathcal{F}z} = 1 + (\mathcal{L} - \mathcal{F}) \left\{ z - \mathcal{F}z^2 + \mathcal{F}^2z^3 - \cdots \right\}, \]

\[ = 1 + (\mathcal{L} - \mathcal{F}) \sum_{n=1}^{\infty} (\mathcal{F})^{-n-1} z^n. \]

(56)

Then, we can write \( B_n = (\mathcal{F})^{-n-1}(\mathcal{L} - \mathcal{F})z^n \), so by putting the comparing values of \( B_1 = (\mathcal{L} - \mathcal{F}) \) and \( B_2 = -\mathcal{F}(\mathcal{L} - \mathcal{F}) \) in Theorem 3, we obtain the following corollary.

**Corollary 1.** Let \( f(z) \in \Sigma \mathcal{A}(\alpha, \tau, \gamma)(\mathcal{F})(1 + \mathcal{L}z/1 + \mathcal{F}z) \) be of the form (1), then

\[ |a_2| \leq \frac{\tau^2 (\mathcal{L} - \mathcal{F})}{\sqrt{\tau^2 (\mathcal{L} - \mathcal{F}) (\alpha + 2\gamma) + (\alpha + \gamma)^2 (1 + \mathcal{F})}}, \]

\[ |a_3| \leq (\mathcal{L} - \mathcal{F}) |a_2| \left\{ \frac{\tau^3 (\mathcal{L} - \mathcal{F})}{(\alpha + \gamma)^2 + (\alpha + 2\gamma)} \right\}. \]

(57)

**Remark 8.** If we put the value of \( \alpha = 1, \gamma = 1, \tau = 1, \) and \( \mathcal{L} = 1 - 2\mu, (0 \leq \mu < 1) \) and \( \mathcal{F} = -1 \) in Corollary 1, we receive Theorem 2 of [13].

**Remark 9.** Setting \( \alpha = 1, \tau = 1, \) and \( \gamma = 0 \) in Corollary 1, we obtain Corollary 2.3 of [3].

Secondly, if we set \( \mathcal{G}(z) = (1 + z/1 - z)^n, (0 \leq \eta < 1, z \in \mathbb{U}) \), then equation (8) becomes

\[ \mathcal{G}(z) = \left( \frac{1 + z}{1 - z} \right)^n = \mathcal{C}_0 + \mathcal{C}_1 z + \mathcal{C}_2 z^2 + \mathcal{C}_3 z^3 \]

\[ + \mathcal{C}_4 z^4 + \cdots + \mathcal{C}_n z^n \]

\[ = 1 + 2\eta z + 2\eta^2 z^2 + \frac{2}{3} (2\eta^3 + \eta) z^3 + \frac{2}{3} \eta^2 (2 + \eta^2) z^4 \]

\[ + \mathcal{O}[z^n] = 1 + \sum_{n=1}^{\infty} \mathcal{C}_n z^n, \]

(58)

where \( \mathcal{O}[z^n] \) represents the sign of “Big-O Notation” (for details, see [36]) and \( \mathcal{C}_n \) is the \( n \)-th coefficient for the above series that may be generalized in the following manner:

\[ \mathcal{C}_0 = 1, \]

\[ \mathcal{C}_1 = 2\eta, \]

\[ \mathcal{C}_2 = 2\eta^2, \]

\[ \mathcal{C}_3 = \frac{2}{3} \eta (1 + 2\eta^2), \]

\[ \mathcal{C}_4 = \frac{2}{3} \eta^2 (2 + \eta^2), \]

\[ \mathcal{C}_5 = \frac{2}{15} \eta (3 + 10\eta^2 + 2\eta^4), \]

\[ \vdots \]

\[ \mathcal{C}_n = \frac{1}{n} (2\eta \mathcal{C}_{n-1} + (n - 2) \mathcal{C}_{n-2}). \]

We then write (59), in a general term, \( \mathcal{B}_n = (1/n) (2\eta \mathcal{C}_{n-1} + (n - 2) \mathcal{C}_{n-2}), \) so by putting the comparing values of \( B_1 = \mathcal{C}_1 = 2\eta \) and \( B_2 = \mathcal{C}_2 = 2\eta^2 \) in Theorem 3, we obtain the following corollary.

**Corollary 2.** Let \( f(z) \in \Sigma \mathcal{A}(\alpha, \tau, \gamma)(\mathcal{G})(1 + z/1 - z)^n \) be of the form (1), then

\[ |a_2| \leq \frac{2|\tau^2| \eta}{\sqrt{2\eta \tau^2 (\alpha + 2\gamma) + (\alpha + \gamma)^2 (1 - \eta)}, \]

\[ |a_3| \leq 2\eta |\tau^3| \left\{ \frac{2|\tau^2| \eta^2}{(\alpha + \gamma)^2 + (\alpha + 2\gamma)} \right\}. \]

(60)

**Remark 10.** If we put the value of \( \alpha = 1, \gamma = 1, \) and \( \tau = 1, \) in Corollary 2, we receive Theorem 1 of [13].

Finally, if we set \( \mathcal{G}(z) = (1 + (1 - 2\mu)z/1 - z), (0 \leq \mu < 1, z \in \mathbb{U}) \), then equation (8) becomes

\[ \mathcal{G}(z) = \frac{1 + (1 - 2\mu)z}{1 - z} = 1 + 2(1 - \mu)z + 2(1 - \mu)z^2 \]

\[ + \cdots + 2(1 - \mu)z^n = 1 + 2(1 - \mu) \sum_{n=1}^{\infty} z^n. \]

(61)

Hence, the general term for \( \mathcal{B}_n = 2(1 - \mu) \), and by putting the comparing values of \( B_1 = B_2 = B_3 = 2(1 - \mu) \) in Theorem 3, we obtain the following corollary.
Corollary 3. Let \( f(z) \in \Sigma(\alpha, \tau, \gamma)(\mathcal{S})(1 + (1 - 2\mu)z/1 - z) \) be of the form (1), then
\[
|a_2| \leq \frac{2\tau^2(1 - \mu)}{(\alpha + 2\mu)}.
\]
\[
|a_3| \leq 2(1 - \mu)\tau^2\left\{\frac{2|\tau|}{(\alpha + \mu)^2(\alpha + 2\mu)}\right\}.
\]
Lastly, by putting \( \alpha = 0, \tau = 1, \) and \( \gamma = 1 \) in Remark 1, we obtain the following new result.

Corollary 4. Let \( f(z) \in \Sigma(\mathcal{L}, \mathcal{F}) \) be of the form (1), then
\[
|a_2| \leq \frac{\mathcal{L} - \mathcal{F}}{\sqrt{2\mathcal{F} + 1}}.
\]
\[
|a_3| \leq (\mathcal{L} - \mathcal{F})(\mathcal{L} - \mathcal{F} + \frac{1}{2}).
\]

6. Conclusions

We presented an idea to generalize the analytic functions that are correlative to certain bi-univalent functions. By utilizing the \( \mathcal{R} \)- and \( \mathcal{X} \)-polynomials, we derived the initial estimates for our newly defined class of a bi-univalent function \( \Sigma(\mathcal{R}, \mathcal{X})(\mathcal{S}) \) (see Definition 1 and Theorem 3). Furthermore, we provided an example to calculate the coefficients of \( z^n \) in terms of \( \mathcal{R} \)’s (see Example 1). In addition to this, we have given the \( n \)-th term of several classes (see (56), (59), and (61)). The \( \mathcal{R} \)- and \( \mathcal{X} \)-coefficients defined in (32) and (39) allow researchers to compare the \( n \)-th coefficients of \( (z) \) or \( (u) \) with Faber polynomial expansions to explore the \( n \)-th coefficient estimate for a certain bi-univalent function.

Data Availability

All data generated or analysed during the study are included within the submitted article.

Disclosure

All the authors agreed with the content of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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