Internal resonances in nonlinear fractionally damped plates of the Uflyand-Mindlin type

M V Shitikova¹,², E I Osipova¹, E Y Rossikhina¹

¹Research Center on Dynamics of Solids and Structures, Voronezh State Technical University, 20-letiya Oktyabrya Street 84, Voronezh 394006, Russia
²RAASN Research Institute of Structural Physics, Lokomotivnuy Proezd 21, Moscow 127238, Russia

E-mail: mvs@vgasu.vrn.ru

Abstract. In the present paper, the non-linear free vibrations of fractionally damped plates are investigated for the case when its equations of motion involve the rotary inertia and shear deformations via five coupled nonlinear differential equations in terms of three mutually orthogonal displacements and two angles of rotation. The damping features of the surrounding medium are described by the fractional derivative Kelvin-Voigt model. The procedure resulting in decoupling linear parts of governing equations has been proposed, what allows one to utilize effectively the fractional derivative expansion method for solving the non-linear equations of motion. The occurrence of the internal or combinational resonances has been revealed and classified for plates of the Uflyand-Mindlin type.

1. Introduction

Last few decades the interest to dynamic non-linear behaviour of viscoelastic plates or elastic plates embedded in a viscoelastic medium does not fall off due to new advanced materials exhibiting nonlinear behavior [1-3]. The damping forces are usually introduced on the basis of the Rayleigh's hypothesis [4], or the modal damping [5]. With the purpose of describing the damping properties of plates, the Kelvin-Voigt model [2,4] is the most popular in engineering analysis by using linear or nonlinear springs in viscoelastic elements [3,4].

The study of free nonlinear vibrations is highly important when it is a need to define the dynamic properties of a mechanical system, especially in the case when nonlinear vibrations are accompanied by the internal resonance, since it results in strong coupling between the modes of vibrations involved [6-9].

Nowadays fractional calculus is considered to be one of the most powerful tools for solving linear and nonlinear dynamic problems of structural mechanics, what is supported by numerous examples, which are collected and described in the survey articles [10,11], in so doing the fractional derivative Kelvin-Voigt, Maxwell and standard linear solid models are widely used for the analysis of the dynamic response of different structural elements.

Thus, linear vibrations of Kirchhoff-Love plates with Kelvin-Voigt fractional damping were considered for rectangular plates using a single equation for vertical vibrations in [12] and three equations of in-plane and transverse vibrations in [13]. It has been shown [11] that if viscoelastic properties of plates are described by the Kelvin-Voigt model assuming the Poisson’s ratio as the time-
independent value (though for real viscoelastic materials the Poisson's ratio is always a time-dependent function [14]), then this case coincides with that of the dynamic response of elastic bodies embedded in a viscoelastic medium. The attenuation of vibratory motions of fractionally damped simply supported plates under harmonic loading has been studied in [15] with the purpose of minimization of plate’s deflection via modelling the attached multiple absorbers as Kelvin-Voigt fractional oscillators.

The analysis of nonlinear vibrations of plates utilising the Kelvin-Voigt model involving fractional derivatives was investigated in [16,17]. Thus, von Karman plate equation was considered in [16] using fractional derivative damping to study the cases of primary, subharmonic and superharmonic resonance conditions but without the occurrence of the internal resonance. Nonlinear random vibrations of the same plate were analyzed in [18].

In the given paper, the procedure proposed in [9] for treating internal resonances during free nonlinear vibrations of elastic plates in a fractional derivative viscoelastic surrounding medium, when the damped motion is governed by a system of three equations, has been generalized for the case of a fractionally damped nonlinear elastic plate of Uflyand-Mindlin type, the motion of which is governed by five equations involving shear deformations and rotary inertia.

2. Problem formulation

The equations of nonlinear free damped vibrations of an elastic rectangular plate of the Uflyand-Mindlin type in a viscoelastic surrounding medium could be obtained via the generalization of the equations, which are presented in [19] with due account for shear deformations and rotary inertia, by adding the forces of resistance of the surrounding medium in terms of fractional derivatives as it was done in [8,9]. These equations in the dimensionless form could be written as

\begin{align*}
\ddot{u}_{xx} + \frac{1-\mu}{2} \beta_1^2 \dot{v}_{yy} + \frac{1+\mu}{2} \beta_1 \dot{u}_{xy} + w_{xx} \left( \ddot{w}_{xx} + \frac{1-\mu}{2} \beta_1^2 \ddot{w}_{yy} \right) + \frac{1+\mu}{2} \beta_1^2 w_{xy} = \ddot{u} + \chi_1 D \ddot{u}, \\
\ddot{v}_{yy} + \frac{1-\mu}{2} \beta_1 \dot{v}_{xx} + \frac{1+\mu}{2} \beta_1 \dot{v}_{xy} + w_{yy} \left( \ddot{w}_{yy} + \frac{1-\mu}{2} \beta_1^2 \ddot{w}_{xx} \right) + \frac{1+\mu}{2} \beta_1 w_{xx} = \ddot{v} + \chi_2 D \ddot{v}, \\
\ddot{w}_{xx} + \frac{1+\mu}{2} \beta_1 \dot{w}_{xx} + \frac{1+\mu}{2} \beta_1 \dot{w}_{xy} + \frac{1+\mu}{2} \beta_1 \dot{w}_{xy} = \ddot{w} + \chi_3 D \ddot{w}, \\
\ddot{\psi}_{x,xx} + \frac{1-\mu}{2} \beta_1^3 \dot{\psi}_{x,yy} + \frac{1+\mu}{2} \beta_1 \dot{\psi}_{x,xy} - 6 k^2 \frac{1-\mu}{\beta_2^2} \left( \dot{\psi}_{x} + \dot{\psi}_{y} \right) = \ddot{\psi}_{x} + \chi_4 D \ddot{\psi}_{x}, \\
\ddot{\psi}_{x,xy} + \frac{1-\mu}{2} \beta_1 \dot{\psi}_{x,xx} + \frac{1+\mu}{2} \beta_1 \dot{\psi}_{x,yy} - 6 k^2 \frac{1-\mu}{\beta_2^2} \left( \dot{\psi}_{x} + \dot{\psi}_{y} \right) = \ddot{\psi}_{y} + \chi_5 D \ddot{\psi}_{y},
\end{align*}

where \( u = u(x,y,t) \), \( v = v(x,y,t) \) and \( w = w(x,y,t) \) are dimensionless displacements of plate's middle surface in the \( x \), \( y \), and \( z \) directions, respectively, \( \psi_x(x,y,t) \) and \( \psi_y(x,y,t) \) are the angles of rotation of the normal to the middle surface and in the plane tangent to the lines \( y \) and \( x \), \( k \) is the
shear coefficient, $\mu$ is the Poisson's ratio, $\beta_1 = a / b$ and $\beta_2 = h / a$ are the parameters defining the dimensions of the plate, $a$ and $b$ are the plate's dimensions along the $x$- and $y$-axes, respectively, $h$ is its thickness, and $t$ is the dimensionless time, $\chi_i$ ($i = 1, 2, \ldots, 5$) are damping coefficients, overdots denote time-derivatives, lower indices after a comma label the derivatives with respect to the corresponding coordinates, and $D^\gamma$ is Riemann-Liouville fractional derivative $[20]$

$$D^\gamma F = \frac{\partial}{\partial t} \int_0^t \frac{F(t-t')dt'}{(1-\gamma)t'^{\gamma-1}}. \quad (6)$$

In equations (1)-(5), dimensionless values are connected with the corresponding dimension values marked by asterisks by the following relationships:

$$u = \frac{u^*}{a}, \quad v = \frac{v^*}{a}, \quad w = \frac{w^*}{a}, \quad x = \frac{x^*}{a}, \quad y = \frac{y^*}{b}, \quad t = \frac{t^*}{a} \sqrt{\frac{E}{(1-\mu^2)\rho}}. \quad (7)$$

3. Method of solution

The solution of equations (1)-(5) for a simply-supported plate are sought as expansions in terms of eigen modes of vibrations

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{1mn}(t) \eta_{1mn}(x,y), \quad v(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{2mn}(t) \eta_{2mn}(x,y), \quad (8)$$

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{3mn}(t) \eta_{3mn}(x,y),$$

$$\phi_x(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{4mn}(t) \eta_{4mn}(x,y), \quad \phi_y(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{5mn}(t) \eta_{5mn}(x,y),$$

where $x_{imn}(t)$ ($i = 1, 2, \ldots, 5$) are the generalized displacements corresponding to plate's in-plane displacements, its deflection and angles of rotation using the eigen forms

$$\eta_{1mn}(x,y) = \eta_{4mn}(x,y) = \cos \pi mx \sin \pi ny,$$

$$\eta_{2mn}(x,y) = \sin \pi mx \cos \pi ny,$$

$$\eta_{3mn}(x,y) = \eta_{5mn}(x,y) = \sin \pi mx \sin \pi ny. \quad (9)$$

Substituting (8) and (9) in equations (1)-(5), multiplying then (1)-(5) by $\eta_{i,mn}(x,y)$, respectively, integrating over $x$ and $y$, and applying the condition of orthogonality within the regions $0 \leq x, y \leq 1$, we obtain a system of coupled nonlinear second-order differential equations in $x_{i,mn}(t)$

$$\ddot{x}_{1mn} + \chi_1 D^\gamma x_{1mn} + x_{1mn} S_{11}^{nu} + x_{2mn} S_{12}^{nu} = -F_{1mn}, \quad (10)$$

$$\ddot{x}_{2mn} + \chi_2 D^\gamma x_{2mn} + x_{1mn} S_{21}^{nu} + x_{2mn} S_{22}^{nu} = -F_{2mn}, \quad (11)$$

$$\ddot{x}_{3mn} + \chi_3 D^\gamma x_{3mn} + x_{3mn} S_{31}^{nu} + x_{4mn} S_{34}^{nu} + x_{5mn} S_{35}^{nu} = -F_{3mn}, \quad (12)$$

$$\ddot{x}_{4mn} + \chi_4 D^\gamma x_{4mn} + x_{3mn} S_{43}^{nu} + x_{4mn} S_{44}^{nu} + x_{5mn} S_{45}^{nu} = 0, \quad (13)$$

$$\ddot{x}_{5mn} + \chi_5 D^\gamma x_{5mn} + x_{3mn} S_{53}^{nu} + x_{4mn} S_{54}^{nu} + x_{5mn} S_{55}^{nu} = 0, \quad (14)$$

where
Nonlinear parts of equations (10)-(14) have the form

$$\delta_{mn} = k^2 \frac{1 - \mu}{2} \pi^2 \left( m^2 + \beta_1 n^2 \right), \quad \delta_{m4} = k^2 \frac{1 - \mu}{2} \pi m, \quad \delta_{43} = 6 k^2 \frac{1 - \mu}{\beta_2^2} \pi m, \quad \delta_{45} = \pi^2 \left( m^2 + \frac{1 - \mu}{2} \beta_1 n^2 \right) + 6 k^2 \frac{1 - \mu}{\beta_2^2},$$

(15)

Nonlinear parts of equations (10)-(14) have the form

$$F_{1mn} = \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} x_{3m_1 n_1} x_{3m_2 n_2} A_{mn}^{(n_1 n_2 n_3)}, \quad F_{2mn} = \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} x_{3m_1 n_1} x_{3m_2 n_2} B_{mn}^{(n_1 n_2 n_3)},$$

$$F_{3mn} = \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} \sum_{n_5} x_{3m_1 n_1} x_{3m_2 n_2} C_{mn}^{(n_1 n_2 n_3 n_4)} + x_{3m_1 n_1} x_{3m_2 n_2} D_{mn}^{(n_1 n_2 n_3 n_4)},$$

where coefficients $A_{mn}^{(n_1 n_2 n_3)}$, $B_{mn}^{(n_1 n_2 n_3)}$, $C_{mn}^{(n_1 n_2 n_3 n_4)}$, and $D_{mn}^{(n_1 n_2 n_3 n_4)}$ are presented in [9].

The analysis of the structure of equations (10)-(14) shows that equations (10) and (11) are coupled with each other via linear terms and with equation (12) in terms of nonlinear terms $F_{jmn}$ ($j = 1, 2, 3$). Equations (13) and (14) are coupled with each other and with equation (12) only via linear terms. Thus, the linearized equations (10)-(14) are decoupled in two linear subsystems.

3.1. Solution of the eigen value problem and decoupling the equations of motion

To determine the natural frequencies of linear vibrations $\omega_{i mn}$ ($i = 1, 2, 3, 4, 5$), it is a need to solve the linear eigen value problem. The characteristic equation of the linearized equations (10) and (11) has the form

$$\omega_{i}^{2} - \omega_{i}^{2} \delta_{mn} \left( S_{i 11} + S_{i 12} \right) + S_{i 11} S_{i 22} - S_{i 12} S_{i 21} = 0,$$

(17)

and its solution provides the natural frequencies of in-plane vibrations

$$\omega_{1}^{2} = \pi^2 \left( m^2 + \beta_1 n^2 \right), \quad \omega_{2}^{2} = \frac{1 - \mu}{2} \pi^2 \left( m^2 + \beta_1 n^2 \right),$$

(18)

which coincide with those obtained in [8]. The linearized set of equations (12)-(14) provides the following frequency equation:

$$\omega_{i}^{2} + \omega_{i}^{2} + \omega_{i}^{2} + \omega_{i}^{2} + \omega_{i}^{2} = 0,$$

(19)

where

$$\omega_{i}^{2} = S_{i 11} S_{i 22} - S_{i 12} S_{i 21}, \quad \omega_{i}^{2} = S_{i 11} S_{i 22} - S_{i 12} S_{i 21},$$

(20)

The solution of equation (19) results in three sets of natural frequencies, $\omega_{i 1 mn}$, $\omega_{i 2 mn}$, and $\omega_{i 3 mn}$, and the least of them, $\omega_{i 3 mn}$, corresponds to the frequency of flexural vibrations. It is defined as
\[
\omega_{mn}^2 = \frac{1}{4\beta_2^2} \left\{ 12k^2(1 - \mu) + \beta_2^2 \pi^2 \left( 2 + k^2(1 - \mu) \right) \left( m^2 + \beta_2^2 n^2 \right) \\
- \left[ 12k^2(1 - \mu) + \beta_2^2 \pi^2 \left( 2 + k^2(1 - \mu) \right) \left( m^2 + \beta_2^2 n^2 \right) \right]^2 - 8\beta_2^4 k^2(1 - \mu) \pi^4 \left( m^2 + \beta_2^2 n^2 \right)^2 \right\}^{1/2}. \tag{20}
\]

The other two roots of equation (19) correspond to the high frequency vibrations and have the form
\[
\omega_{4mn}^2 = \frac{1 - \mu}{2} \left[ \frac{12}{\beta_2^2} k^2 + \pi^2 \left( m^2 + \beta_2^2 n^2 \right) \right], \tag{21}
\]
\[
\omega_{5mn}^2 = \frac{1}{4\beta_2^2} \left\{ 12k^2(1 - \mu) + \beta_2^2 \pi^2 \left( 2 + k^2(1 - \mu) \right) \left( m^2 + \beta_2^2 n^2 \right) \\
+ \left[ 12k^2(1 - \mu) + \beta_2^2 \pi^2 \left( 2 + k^2(1 - \mu) \right) \left( m^2 + \beta_2^2 n^2 \right) \right]^2 - 8\beta_2^4 k^2(1 - \mu) \pi^4 \left( m^2 + \beta_2^2 n^2 \right)^2 \right\}^{1/2}. \tag{22}
\]

The natural frequencies correspond to mutually orthogonal eigen vectors
\[
\{L_1^{\text{mn}}, \ L_2^{\text{mn}}, \ L_3^{\text{mn}}, \ L_4^{\text{mn}}, \ L_5^{\text{mn}}\} \quad (i = 1, 2),
\]
\[
\{L_3^{\text{mn}}, \ L_4^{\text{mn}}, \ L_5^{\text{mn}}, \ L_6^{\text{mn}}, \ L_7^{\text{mn}}\} \quad (i = 3, 4, 5).
\]

Following [9], let us expand the matrices \(S_{ij}^{\text{mn}} \) \((i, j = 1, 2)\), \(S_{ij}^{\text{mn}} \) \((i, j = 3, 4, 5)\) and generalized displacements \(x_{\text{imn}}\) entering in equations (10)-(14) in terms of the eigen vectors
\[
x_{\text{imn}}^{\text{mn}} = X_{1\text{imn}} L_1^{\text{imn}} + X_{2\text{imn}} L_2^{\text{imn}} \quad (i = 1, 2),
\]
\[
x_{\text{imn}}^{\text{mn}} = X_{3\text{imn}} L_3^{\text{imn}} + X_{4\text{imn}} L_4^{\text{imn}} + X_{5\text{imn}} L_5^{\text{imn}} \quad (i = 3, 4, 5). \tag{25}
\]

Now substituting expansions (23)-(25) in equations (10)-(14) and then multiplying (10)-(11) successively by \(L_1^{\text{imn}}, L_2^{\text{imn}}\), and (12)-(14) successively by \(L_3^{\text{imn}}, L_4^{\text{imn}}, L_5^{\text{imn}}\), and finally by \(L_6^{\text{imn}}, L_7^{\text{imn}}\) considering the conditions of eigen vectors’ orthogonality: \(L_1^{\text{imn}} L_2^{\text{imn}} = 0\) at \(K \neq N\) and \(L_1^{\text{imn}} L_3^{\text{imn}} = 1\) \((K, N=I,II,III,IV,V)\) yield the following system of equations of motion:
\[
\ddot{X}_{1\text{imn}} + \chi_1 \dot{D} X_{1\text{imn}} + \omega_{1\text{imn}}^2 X_{1\text{imn}} = -\sum_i F_{i\text{imn}} L_1^{\text{imn}}, \tag{26}
\]
\[
\ddot{X}_{2\text{imn}} + \chi_2 \dot{D} X_{2\text{imn}} + \omega_{2\text{imn}}^2 X_{2\text{imn}} = -\sum_i F_{i\text{imn}} L_2^{\text{imn}}, \tag{27}
\]
\[
\ddot{X}_{3\text{imn}} + \chi_3 \dot{D} X_{3\text{imn}} + \omega_{3\text{imn}}^2 X_{3\text{imn}} = -\sum_i F_{i\text{imn}} L_3^{\text{imn}}, \tag{28}
\]
\[
\ddot{X}_{4\text{imn}} + \chi_4 \dot{D} X_{4\text{imn}} + \omega_{4\text{imn}}^2 X_{4\text{imn}} = 0, \tag{29}
\]
\[
\ddot{X}_{5\text{imn}} + \chi_5 \dot{D} X_{5\text{imn}} + \omega_{5\text{imn}}^2 X_{5\text{imn}} = 0, \tag{30}
\]
in terms of new generalized displacements $X_{jmn}$

$$X_{1mn} = x_{1mn} L_{1mn}^1 + x_{2mn} L_{2mn}^1, \quad (31)$$

$$X_{2mn} = x_{1mn} L_{1mn}^2 + x_{2mn} L_{2mn}^2, \quad (32)$$

$$X_{3mn} = x_{3mn} L_{3mn}^3 + x_{4mn} L_{2mn}^3 + x_{5mn} L_{5mn}^3, \quad (33)$$

$$X_{4mn} = x_{3mn} L_{3mn}^4 + x_{4mn} L_{2mn}^4 + x_{5mn} L_{5mn}^4, \quad (34)$$

$$X_{5mn} = x_{3mn} L_{3mn}^5 + x_{4mn} L_{2mn}^5 + x_{5mn} L_{5mn}^5. \quad (35)$$

Note that the left-hand side parts of (26)-(30) are linear and independent, and equations (26)-(28) are coupled only by right-hand side non-linear terms.

Moreover, the set of equations (26)-(30) is decoupled into three subsystems, namely: the first subset compiles three nonlinear fractional derivative equations (26)-(28), the second and the third subsystems involve one linear fractional derivative equation each, i.e. equations (29) and (30), respectively. Thus, for finding a solution it is necessary to examine each subsystem.

3.2. Analysis of the reduced equations of motion

Equations (29) and (30) describe free damped vibrations of a linear oscillator with a viscoelastic resistance force modelled in terms of the fractional derivative Kelvin-Voigt model [10]. In the case of weak damping, i.e. when $\chi_i = \epsilon \omega_i$ or $\chi_i = \epsilon^2 \omega_i$ with $0 < \epsilon \ll 1$, approximate analytical solutions of equations similar to (29) and (30) have been found in [10,21] utilizing the fractional derivative expansion method [22], which is the extension of the multiple time scales procedure [23].

Free damped vibrations of a linear fractional derivative Kelvin-Voigt oscillator in a medium with finite viscosity, i.e. without any restrictions on the magnitude of the damping coefficient $\chi_i$, have been studied analytically in [10] utilizing the Green function [24].

As for the first subsystem (26)-(28) involving three nonlinear equations with fractional derivative terms, then it has the similar structure as the set of three governing equations considered previously but ignoring the influence of the rotary inertia and shear deformations [19].

Following [9] it could be shown that the solution of equations (26)-(28) could be constructed using the generalized method of multiple time scales [22]. We will not repeat this procedure, since it is described in detail in [22], and it could be easily adopted to equations (26)-(28) within an accuracy of coefficients.

Thus, it has been revealed that nonlinear vibrations of the plate could be accompanied by different types of the internal resonance when two or more modes are coupled. Moreover, its type depends on the order of smallness of the viscosity involved into consideration. Thus, it has been found that at the $\epsilon$-order, damped vibrations could be accompanied by the following types of the internal resonance:

- the two-to-one internal resonance (2:1)
  $$\omega_1 = 2 \omega_2 \left( \omega_i \neq \omega_j, \ 2 \omega_i \neq \omega_j \right), \quad (36)$$
  $$\omega_2 = 2 \omega_3 \left( \omega_i \neq \omega_j, \ 2 \omega_i \neq \omega_j \right), \quad (37)$$

and the one-to-one-to-two internal resonance (1:1:2)
  $$\omega_1 = \omega_2 = 2 \omega_3; \quad (38)$$

while at the $\epsilon^2$-order, the following types of the internal resonance could occur:

- the one-to-one internal resonance (1:1)
The one-to-one-to-one internal resonance (1:1:1)

\[ \omega_1 = \omega_2 = \omega_3, \]

and the combinational resonance of the additive-difference type

\[ \omega_1 = \omega_2 + 2\omega_3, \]
\[ \omega_3 = 2\omega_1 - \omega_2, \quad \omega_2 = \omega_1 - 2\omega_3, \]

where \( \omega_1 \) and \( \omega_2 \) are the frequencies of certain modes of in-plane vibrations in the \( x \)- and \( y \)-axes, respectively, and \( \omega_3 \) is the frequency of a certain mode of out-of-plane vibrations.

For each type of the resonance, the nonlinear sets of resolving equations in terms of amplitudes and phase differences could be obtained using the same procedure as in [9]. The effect of viscosity on the energy exchange is revealed by the fact that each mode is characterized by its damping coefficient connected with the natural frequency by the exponential relationship with a negative fractional exponent. Thus, during free vibrations of the plate with internal resonances three regimes could be observed: stationary vibrations occurring without damping at \( \gamma = 0 \), quasistationary vibrations when damping is governed by an ordinary derivative at \( \gamma = 1 \), and transient vibrations when damping is defined by a derivative of the fractional order at \( 0 < \gamma < 1 \).

4. Analysis of spectra of natural frequencies

In order to show that the phenomenon of internal resonance is highly important, it is a need to analyze the spectra of natural frequencies.

Thus, natural frequencies of vibrations \( \omega_{mn} \) \( (i=1,2,...,5) \) calculated according to (18) and (20)-(22), as well as frequency of vertical flexural vibrations without shear deformations and rotary inertia calculated via the formula [9]

\[ \bar{\omega}_{mn}^2 = \frac{\beta_x^2}{12} \pi^4 \left( m^2 + \beta_y^3 n^2 \right)^2 \]  

are given in tables 1 and 2 for a square plate, i.e. at \( \beta_1 = a \) \( / \) \( b = 1 \), at \( \beta_2 = h / a = 0.1 \) and 0.05, respectively.

Reference to tables 1 and 2 shows the effect of the shear deformations and rotary inertia on the frequencies of flexural vibrations, in so doing the thicker the plate, the more difference between the frequencies \( \omega_x \) and \( \bar{\omega}_x \). Thus, for example, for the square plate the frequency of the fundamental mode at \( m=1; n=1 \) calculated by the classical theory at \( \beta_2 = 0.1 \) and 0.05 is reduced, respectively, by 3.51 and 1.05 \% as compared with that calculated by the refined theory.

From tables 1 and 2 it is seen that internal resonances of all types (36)-(43) could take place, and the occurrence of this or that case depends on the plate’s dimensions, i.e. on magnitudes of coefficients \( \beta \) and \( \beta_2 \).

As soon as the case of the internal resonance is revealed, then the further treatment of nonlinear Eqs. (26)-(30) could be carried out by the procedure developed in [9] within an accuracy of the coefficients.
Table 1. Natural frequencies of vibrations $\omega_{\text{mn}} (i=1,2,...,5)$ at $\beta_1 = 1$ and $\beta_2 = 0.1$.

| $m$ | $n$ | $\omega_{\text{mn}}$ | $\omega_{\text{2mn}}$ | $\omega_{\text{3mn}} / \bar{\omega}_{\text{3mn}}$ | $\omega_{\text{4mn}}$ | $\omega_{\text{5mn}}$ |
|-----|-----|---------------------|---------------------|---------------------------------|---------------------|---------------------|
| 1   | 1   | 4.443               | 2.628               | 0.550/0.570                     | 18.892              | 19.370              |
| 1   | 2   | 7.023               | 4.156               | 1.313/1.425                     | 19.164              | 20.298              |
| 2   | 1   | 7.023               | 4.156               | 1.313/1.425                     | 19.164              | 20.298              |
| 2   | 2   | 8.886               | 5.257               | 2.017/2.279                     | 19.433              | 21.164              |
| 1   | 3   | 9.935               | 5.877               | 2.458/2.849                     | 19.610              | 21.715              |
| 3   | 1   | 9.935               | 5.877               | 2.458/2.849                     | 19.610              | 21.715              |
| 2   | 3   | 11.327              | 6.701               | 3.080/3.704                     | 19.873              | 22.500              |
| 3   | 3   | 13.329              | 7.885               | 4.043/5.128                     | 20.302              | 23.731              |
| 1   | 4   | 12.953              | 7.663               | 3.857/4.843                     | 20.217              | 23.491              |
| 2   | 4   | 14.050              | 8.312               | 4.405/5.698                     | 20.472              | 24.198              |
| 3   | 4   | 15.708              | 9.293               | 5.263/7.123                     | 20.889              | 25.318              |
| 4   | 4   | 17.772              | 10.514              | 6.368/9.117                     | 21.460              | 26.784              |
| 1   | 5   | 16.019              | 9.471               | 5.427/7.408                     | 20.972              | 25.534              |
| 2   | 5   | 16.918              | 10.009              | 5.907/8.262                     | 21.217              | 26.169              |
| 3   | 5   | 18.319              | 10.831              | 6.667/9.687                     | 21.621              | 27.184              |
| 4   | 5   | 20.116              | 11.901              | 7.660/11.681                    | 22.173              | 28.531              |
| 5   | 5   | 22.214              | 13.142              | 8.838/14.246                    | 23.510              | 30.155              |

Table 2. Natural frequencies of vibrations $\omega_{\text{mn}} (i=1,2,...,5)$ at $\beta_1 = 1$ and $\beta_2 = 0.05$.

| $m$ | $n$ | $\omega_{\text{mn}}$ | $\omega_{\text{2mn}}$ | $\omega_{\text{3mn}} / \bar{\omega}_{\text{3mn}}$ | $\omega_{\text{4mn}}$ | $\omega_{\text{5mn}}$ |
|-----|-----|---------------------|---------------------|---------------------------------|---------------------|---------------------|
| 1   | 1   | 4.443               | 2.628               | 0.282/0.285                     | 37.509              | 37.755              |
| 1   | 2   | 7.023               | 4.156               | 0.697/0.712                     | 37.647              | 38.253              |
| 2   | 2   | 8.886               | 5.257               | 1.101/1.140                     | 37.784              | 38.740              |
| 1   | 3   | 9.935               | 5.877               | 1.365/1.423                     | 37.915              | 39.060              |
| 2   | 3   | 11.327              | 6.701               | 1.753/1.852                     | 38.012              | 39.530              |
| 3   | 3   | 13.329              | 7.885               | 2.381/2.564                     | 38.238              | 40.296              |
| 1   | 4   | 12.953              | 7.663               | 2.257/2.422                     | 38.193              | 40.145              |
| 2   | 4   | 14.050              | 8.312               | 2.627/2.849                     | 38.329              | 40.596              |
| 3   | 4   | 15.708              | 9.293               | 3.224/3.561                     | 38.553              | 41.332              |
| 4   | 4   | 17.772              | 10.514              | 4.030/4.559                     | 38.866              | 42.329              |
| 1   | 5   | 16.019              | 9.471               | 3.341/3.704                     | 38.598              | 41.476              |
| 2   | 5   | 16.918              | 10.009              | 3.689/4.131                     | 38.732              | 41.906              |
| 3   | 5   | 18.319              | 10.831              | 4.253/4.843                     | 39.264              | 42.607              |
| 4   | 5   | 20.116              | 11.901              | 5.017/5.841                     | 39.658              | 43.560              |
| 5   | 5   | 22.214              | 13.142              | 5.956/7.123                     | 39.658              | 44.743              |

5. Conclusion
The nonlinear free vibrations of fractionally damped plates of Uflyand-Mindlin type have been studied utilizing five coupled nonlinear differential equations in terms of three displacements and two rotational angles. The generalized method of multiple time scales has been applied for solving nonlinear governing equations of motion via expanding the amplitude functions into power series in terms of the small parameter and different time scales. Numerical analysis of the natural frequency spectra reveals the possibility of the occurrence of different internal and combinational resonances.

References
[1] Sathyamoorty M 1987 Nonlinear vibration analysis of plates: A review and survey of current developments Appl. Mech. Rev. 40 1553–1561
[2] Amabili M 2008 Nonlinear Vibrations and Stability of Shells and Plates (London: Cambridge University Press)
[3] Amabili M 2018 Nonlinear damping in large-amplitude vibrations: modelling and experiments Nonlinear Dyn. 93 5–18
[4] Rossikhin Yu A and Shitikova M V 2019 Encyclopedia of Continuum Mechanics ed H Altenbach and A Ochsner vol 3 (Berlin, Heidelberg: Springer) pp 2512–2518
[5] Clough R W and Penzien J 1975 Dynamics of Structures (New York: McGraw-Hill)
[6] Chang S I, Bajaj A K, Krousgrill C M 1993 Non-linear vibrations and chaos in harmonically excited rectangular plates with one-to-one internal resonance Nonlinear Dyn. 4 433–460
[7] Hao Y X, Zhang W, Ji X L 2010 Nonlinear dynamic response of functionally graded rectangular plates under different internal resonances Math. Problems Eng. 2010 738648
[8] Rossikhin Yu A and Shitikova M V 2003 Free damped non-linear vibrations of a viscoelastic plate under the two-to-one internal resonances Materials Science Forum 440-441 pp 29–36
[9] Rossikhin Yu A, Shitikova M V and Ngenzi J Cl 2015 A new approach for studying nonlinear dynamic response of a thin plate with internal resonance in a fractional viscoelastic medium Shock Vibr. 2015 795606
[10] Rossikhin Yu A and Shitikova M V 2010 Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results Appl. Mech. Rev. 63 ID 01081
[11] Rossikhin Yu A and Shitikova M V 2019 Handbook of Fractional Calculus with Applications vol 7, Part A., ed D Baleanu and A M Lopes (Berlin: De Gruyter) pp 159–192
[12] (Stevanovic) Hedrih K 2005 Partial fractional differential equations of creeping and vibrations of plate and their solutions (First part) J. Mech. Behavior Mat. 16 305–314
[13] Rossikhin Yu A and Shitikova M V 2006 Analysis of damped vibrations of linear viscoelastic plates with damping modeled with fractional derivatives Signal Processing 86 2703–2711
[14] Hilton H H 2001 Implications and constraints of time-independent Poisson ratios in linear isotropic and anisotropic viscoelasticity J. Elast. 63 221–251
[15] Ari M, Faal R T and Zayernouri M 2020 Vibrations suppression of fractionally damped plates using multiple optimal dynamic vibration Int. J. Comput. Math. 97 851–874
[16] Permoon M R, Haddadpour H and Javadi M 2018 Nonlinear vibration of fractional viscoelastic plate: primary, subharmonic, and superharmonic response Int. J. Non-Linear Mech. 99 154–164
[17] Babouskos N G and Katsikadelis J T 2010 Nonlinear vibrations of viscoelastic plates of fractional derivative type: An AEM solution The Open Mech. J. 4 8–20
[18] Malara G and Spanos P D 2018 Nonlinear random vibrations of plates endowed with fractional derivative elements Prob. Eng. Mech. 54 2–8
[19] Volmir A S 1972 Nonlinear Dynamics of Plates and Shells (Moscow: Nauka)
[20] Samko S G, Kilbas A A and Marichev O I 1993 Fractional Integrals and Derivatives. Theory and Applications (Amsterdam: Gordon and Breach Science Publishers)
[21] Rossikhin Yu A and Shitikova M V 2009 New approach for the analysis of damped vibrations of fractional oscillators Shock Vibr. 16 365–387
[22] Shitikova M V 2020 The fractional derivative expansion method in nonlinear dynamic analysis of structures Nonlinear Dyn. 99 109–122
[23] Nayfeh A H 1973 Perturbation Methods (New York: Wiley)
[24] Meshkov S I, Pachevskaja G N, Postnikov V S and Rossikhin Yu A 1971 Integral representation of $\gamma$, -functions and their application to problems in linear viscoelasticity Int. J. Eng. Sci. 9 387–398

Acknowledgement
This study has been carried out within the Project # 7.4.4 in the 2020 Plan of Fundamental Research coordinated jointly by the Russian Academy of Architecture and Civil Engineering and Ministry of Civil Engineering and Public Utilities of the Russian Federation.