On the Stanley depth of a special class of Borel type ideals
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Abstract

We give sharp bounds for the Stanley depth of a special class of monomial ideals of Borel type.

Keywords: monomial ideals, ideals of Borel type, Stanley depth.

MSC 2010: Primary: 13C15, Secondary: 13P10, 13F20, 05C07.

Introduction

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over $K$. Let $M$ be a $\mathbb{Z}^n$-graded $S$-module. A Stanley decomposition of $M$ is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ of $\mathbb{Z}^n$-graded $K$-vector spaces, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1, \ldots, x_n\}$ such that $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of $M$. We define $sdepth(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and $sdepth(M) = \max\{\mathcal{D} = 0 | D \text{ is a Stanley decomposition of } M\}$. The number $sdepth(M)$ is called the Stanley depth of $M$.

Stanley conjectured in [17] that $sdepth(M) \geq \text{depth}(M)$ for any $M$. The conjecture was disproved in [9] for $M = S/I$, where $I \subset S$ is a monomial ideal, but remains open in the case $M = I$. Herzog, Vladoiu and Zheng showed in [13] that $sdepth(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [16], Rinaldo gave a computer implementation for this algorithm, in the computer algebra system CoCoA [8]. For an introduction in the thematic of Stanley depth, we refer the reader to [10].

We say that a monomial ideal $I \subset S$ is of Borel type, see [12], if it satisfies the following condition: $(I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty)$, $(\forall) 1 \leq j \leq n$. The Mumford-Castelnuovo regularity of $I$ is the number $\text{reg}(I) = \max\{j - i : \beta_{ij}(I) \neq 0\}$, where $\beta_{ij}$'s are the graded Betti numbers. The regularity of the ideals of Borel type was extensively studied, see for instance [12], [1] and [5]. In the first section, we study the invariant $sdepth(I)$, for an ideal of Borel type. In the general case, we note some bounds for $sdepth(I)$, see Proposition 1.2 and we give some tighter ones, when $I$ has a special form, see Theorem 1.6.

1 Main results

First, we recall the construction of the sequential chain associated to a Borel type ideal $I \subset S$, see [12] for more details. Assume that $\text{Ass}(S/I) = \{P_0, \ldots, P_m\}$ with $P_i = (x_1, \ldots, x_{n_i})$, where $n \geq n_0 > n_1 > \cdots > n_m \geq 1$. Also, assume that $I = \bigcap_{i=0}^m Q_i$ is the reduced primary decomposition of $I$, with $P_i = \sqrt{Q_i}$, for all $0 \leq i \leq m$.

We define $I_k := \bigcap_{j=k}^m Q_j$, for all $0 \leq k \leq m$. One can easily check that $I_k = (I_{i-1} : x_{n_{i-1}}^\infty)$, for all $1 \leq i \leq m$. The sequence of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_m \subset I_{m+1} := S$ is

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called the sequential sequence of $I$. Let $J_i$ be the monomial ideal generated by $G(I_i)$ in $S_i := K[x_1, \ldots, x_n]$, for all $0 \leq i \leq m$. Then, the saturation $J_i^{\text{sat}} = (J_i : (x_1, \ldots, x_n)^\infty) = J_{i+1}S_i$, for all $0 \leq i \leq m$, where $J_{m+1} := S_{m+1}$. One has $I_{i+1}/I_i \cong (J_i^{\text{sat}}/J_i)[x_{n+1}, \ldots, x_n]$. If $M = \bigoplus_{t \geq 0} M_t$ is an Artinian graded $S$-module, we denote $s(M) = \max\{t : M_t \neq 0\}$. We recall the following result.

**Proposition 1.1.** ([12, Corollary 2.7]) $\text{reg}(I) = \max\{s(J_0^{\text{sat}}/J_0), \ldots, s(J_m^{\text{sat}}/J_m)\} + 1$.

**Proposition 1.2.** With the above notations, the following assertions hold:

1. $\text{sdepth}(S/I_i) = \text{depth}(S/I_i) = n - n_i$, for all $0 \leq i \leq n$.
2. $\text{sdepth}(J_0) \leq \text{sdepth}(I_1) \leq \cdots \leq \text{sdepth}(I_m)$.
3. $\text{depth}(I_i) = n - n_i + 1 \leq \text{sdepth}(I_i) \leq \text{sdepth}(P_i) = n - \left\lceil \frac{n}{2} \right\rceil$, $(\forall) 0 \leq i \leq m$.

**Proof.** (1) From [13, Lemma 3.6] it follows that $\text{sdepth}(S/I_i) = \text{depth}(S/I_i) = n - n_i$. Also, we have $\text{sdepth}(S/I_i) = \text{depth}(S/I_i) + n - n_i$. Since $P_iS_i = (x_1, \ldots, x_n)S_i \in \text{Ass}(S/I_i)$, it follows that $\text{depth}(S/I_i) = 0$ and thus, by [6, Theorem 1.4] or [10, Proposition 18], we get $\text{sdepth}(S/I_i) = \text{depth}(S/I_i) = 0$.

(2) Since $I_i = (I_{i-1} : x_{n_i-1}^\infty)$, by [14, Proposition 1.3] (see arXiv version), we get $\text{sdepth}(I_{i-1}) \leq \text{sdepth}(I_i)$, for all $1 \leq i \leq m$.

(3) Since $I_i = J_iS_i$ by [13, Lemma 3.6], it follows that $\text{sdepth}(I_i) = n - n_i + \text{sdepth}_{S_i}(J_i) \geq n - n_i + 1$. Since $P_i \in \text{Ass}(I_i)$, it follows that there exists a monomial $v \in S_i$ such that $P_i = (I_i : v)$. Therefore, by [14, Proposition 1.3] (see arXiv version), it follows that $\text{sdepth}(P_i) \geq \text{sdepth}(I_i)$. On the other hand, $P_i$ is generated by variables. Thus, by [13, Lemma 3.6] and [2, Theorem 1.1], it follows that $\text{sdepth}(P_i) = n - \left\lceil \frac{\text{ht}(P_i)}{2} \right\rceil = n - \left\lceil \frac{n}{2} \right\rceil$. \hfill $\square$

**Lemma 1.3.** Let $r \leq n$ and $a_1, \ldots, a_r$ be some positive integers. If $Q = (x_1^{a_1}, \ldots, x_r^{a_r}) \subset S$, then $\text{reg}(Q) = a_1 + \cdots + a_r - r + 1$.

**Proof.** Let $\bar{Q} = Q \cap S' \subset S'$, where $S' = K[x_1, \ldots, x_r]$. As a particular case of Proposition 1.1, we get $\text{reg}(Q) = \text{reg}(\bar{Q}) = s(S'/\bar{Q}) + 1 = a_1 + \cdots + a_r - r + 1$. \hfill $\square$

We recall the following result from [1].

**Proposition 1.4.** ([11, Corollary 3.17]) If $I \subset S$ is an ideal of Borel type with the irredundant irreducible decomposition $I = \bigcap_{i=1}^r C_i$, then $\text{reg}(I) = \max\{\text{reg}(C_i) : 1 \leq i \leq r\}$.

Let $n \geq n_0 > n_1 > \cdots > n_m \geq 1$ be some integers. Let $a_{ij}$ be some positive integers, where $0 \leq i \leq m$ and $1 \leq j \leq n_i$. We consider the monomial irreducible ideals $Q_i = (x_1^{a_{i1}}, \ldots, x_{n_i}^{a_{in_i}})$, for $0 \leq i \leq m$. Let $I_i := \bigcap_{j=1}^{n_i} Q_j$ and denote $I = I_0$. Since $P_i = (x_1, \ldots, x_{n_i}) = \sqrt{Q_i}$ for all $0 \leq i \leq m$, by [11, Proposition 5.2] or [5, Corollary 1.2], it follows that $I$ is an ideal of Borel type. As a direct consequence of Lemma 1.3 and Proposition 1.4, we get the following corollary.

**Corollary 1.5.** If $a_{ij} \geq a_{i+1j}$ for all $j \leq n_{i+1}$ and $i < m$, then $\text{reg}(I_i) = \text{reg}(Q_i) = a_{i1} + a_{i2} + \cdots + a_{im_i} - n_i + 1$, for all $0 \leq i \leq m$. 

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Lemma 2.2, we get:
\[ J \subset \text{sdepth}(I_i) \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_i. \]

**Proof.** The first inequality follows from Proposition 1.2(3). In order to prove the second one, we use induction on \( i \leq m \). If \( i = m \), then \( I_m = Q_m \) is an irreducible ideal, and therefore, by [7, Theorem 1.3], \( \text{sdepth}(I_m) = n - \left\lceil \frac{n_m}{2} \right\rceil = n + \left\lceil \frac{n_m}{2} \right\rceil - n_m. \)

Assume \( i < m \). We can write \( Q_i = U_i + V_i \), where \( U_i = (x_1^{a_1}, \ldots, x_{n_{i+1}}^{a_{n_{i+1}}}) \) and \( V_i = \left( x_{n_{i+1}}^{a_{n_{i+1}+1}}, \ldots, x_{n_m}^{a_m} \right) \). Since \( a_{ij} \geq a_{i+1,j} \) for all \( j \leq n_{i+1} \), it follows that \( U_i \subset Q_{i+1} \). Therefore, \( I_i = (U_i + V_i) \cap I_{i+1} = (U_i \cap I_{i+1}) + (V_i \cap I_{i+1}) \). Note that \( J := U_i \cap I_{i+1} = U_i \cap I_{i+2} \) is a Borel type ideal with the irreducible irredundant decomposition \( J = U_i \cap Q_{i+2} \cap \cdots \cap Q_m \), and, therefore, of the same class as \( I_{i+1} \). Thus, by induction hypothesis, it follows that \( \text{sdepth}(J) \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_{i+1}. \)

On the other hand, by [4, Remark 1.3] and the induction hypothesis, \( \text{sdepth}(V_i) + \text{sdepth}(I_{i+1}) - n = \text{sdepth}(I_{i+1}) - \left\lceil \frac{n_m}{2} \right\rceil. \)

Let \( V_i \subseteq S' = K[x_{n_{i+1}+1}, \ldots, x_n] \) be the monomial ideal generated by \( G(V_i) \) and let \( J' \subseteq S'' = K[x_1, \ldots, x_{n_{i+1}}, x_{n_{i+1}+1}, \ldots, x_n] \) be the monomial ideal generated by \( G(J) \). Since \( J \subseteq I_{i+1} \), it follows that \( I_i = (J' \otimes_K (S''/V_i)) \oplus (V_i \cap I_i) \). By [3, Proposition 2.10] and [15, Lemma 2.2], we get:
\[ \text{sdepth}(I_i) \geq \min\{\text{sdepth}(J) - n_i + n_{i+1}, \text{sdepth}(I_{i+1}) - \left\lceil \frac{n_i - n_{i+1}}{2} \right\rceil \} \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_i, \]
as required. \( \square \)

**Question:** What can we say about the case when the condition \( a_{ij} \geq a_{i+1,j} \) is removed? Of course, the method used in the proof of the Theorem 1.6 do not work. However, our computer experiments in CoCoa [8] suggested that the conclusion of the Theorem 1.6 might be true. Unfortunately, we are not able to give either a proof, or a counterexample.

The next example shows that the bounds given in Theorem 1.6 are sharp.

**Example 1.7.** Let \( I = Q_0 \cap Q_1 \), where \( Q_0 = (x_1^3, x_2^2, x_3^2, x_4, x_5) \) and \( Q_1 = (x_1, x_2, x_3, x_4) \) are ideals in \( S = K[x_1, \ldots, x_5] \). Then \( I_1 = Q_1 \) and \( \text{sdepth}(I_1) = 5 - \left\lceil \frac{1}{2} \right\rceil = 3 \). Also \( n = 5 \), \( n_0 = 5 \) and \( n_1 = 4 \). Using CoCoa, we get \( \text{sdepth}(I) = 2 = n - \left\lceil \frac{n_1}{2} \right\rceil - n_0 \). Let \( Q'_0 = (x_1^2, x_2^3, x_3, x_4, x_5) \subseteq S \) and \( I' = Q'_0 \cap Q_1 \). Using CoCoa [8], we get \( \text{sdepth}(I') = 3 = n - \left\lceil \frac{n_1}{2} \right\rceil \).

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