A Comparison of Group Criticality Notions for Simple Games

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Abstract

We define a notion of the criticality of a player for simple monotone games based on cooperation with other players, either to form a winning coalition or to break a winning one, with an essential role for all the players involved. We compare it with the notion of differential criticality given by Beisbart that measures power as the opportunity left by other players.

We prove that our proposal satisfies an extension of the strong monotonicity introduced by Young, which assigns no power to dummy players and free riders, and can easily be computed from the minimal winning and blocking coalitions. Our analysis shows that the measures of group criticality defined so far cannot weigh essential players while only remaining an opportunity measure. We propose a group opportunity test to reconcile the two views.

Keywords: Group criticality - Decisiveness - Essential players - Opportunity - Monotonicity - Power rankings

1 Introduction

The commonly accepted distinction in game theory between non-cooperative and cooperative games would lead one to think that the former always examines the actions of individual players against the other players, while the

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latter focuses exclusively on groups of players, or coalitions, acting together. The reality, however, is more subtle. The notion of Nash equilibrium, a pillar of non-cooperative games, has been strengthened to include joint deviations by groups in the strong Nash equilibrium (see Aumann [3]). Conversely, many research works in cooperative game theory devoted great deal of effort into defining indices that measure the importance of single agents as averages of their marginal relevance with respect to the other players (as in the Shapley value [23]) as if players were isolated in their action. In simple monotone games, these values count the occurrences in which single players are critical, i.e. they are capable of overturning an outcome. To underline this one-against-all approach, an alternative definition considers players critical if they have the simultaneous opportunity to make a coalition of other players win or lose (see Beisbart [6]). This opportunity must therefore only depend on the power that the other players concede to the single agent.

Focusing on cooperative games, interactions among groups of players take place in many phases: before playing the game in the process of coalition formation, and, once the coalitions are formed, interactions may continue among competing coalitions that define a partition of the players, and among players in the same coalition. The analysis of partition function form games was first considered in Thrall and Lucas [27] and Maschler [21], with important contributions by Aumann and Dreze [4] and Owen [24]. The topic has grown as a relevant subject for study in game theory and we refer to Kóczy [19] for a recent survey of the topic. Other notable examples of restricted interaction models are given by Myerson, [23] who defines communication through conference structures, i.e. finite sequences of subsets, and, again, Myerson, [22] who defines cooperate bilateral agreements through links in a graph. A survey of models with restricted communication is given by Borm et al. [7].

A less numerous but equally important body of work increased the number of players involved in the marginal increment analysis needed to measure player relevance. For simple monotone games, recent contributions have extended the notion of criticality to include players that may change an outcome of a game only through the help of other players, originating criticalities of higher order or rank. The first proposal, the rank of differential criticality (\(d\)-criticality for short), was given by the already mentioned work of Beisbart [6]. Central to this definition is keeping criticality as a measure of the opportunity left by other players that are still valid for the higher ranks. More recently, Dall’Aglio et al. [9] introduced the notion of the order of criticality of a player to characterize situations where players may cooperate to break a
winning coalition, with an essential role for all the players involved. In this work, the notion of the order of criticality provided by the latter proposal is extended to the losing coalitions to define the group essential criticality or \(g\)-criticality rank of a player. This creates a common setting for the comparison of the two notions of criticality. It turns out that both notions share a common core – that of the essential minimal critical (or \(m\)-critical) players. In turn, \(m\)-criticality determines the rank of \(d\)-criticality of the players that are not \(m\)-critical, so that the differences in the average rank of \(d\)-criticality between players are exclusively determined by their probability of not being minimally critical. Null players are never \(g\)-critical (and \(m\)-critical), while they are always \(d\)-critical in some rank. Moreover, \(g\)-criticality is the only notion that satisfies an extension of the strong monotonicity as introduced by Young, [30] measured by the first order of stochastic dominance between vectors of average probability for all ranks. On the other hand, \(d\)-criticality is the only notion that measures criticality as the opportunity left by the other players. We wonder whether this opportunity test is the valid benchmark for comparing the action of groups and we close by proposing a notion of group opportunity that is satisfied by \(g\)-criticality.

The paper is structured as follow: Section 2 recalls the definitions and results of previous works. In Section 3, the group essential criticality is introduced. In Section 4, some procedures for computing the various notions of group criticality are provided. In Section 5, several features of the competing definitions such as their monotonicity and their sensitivity to the dummy players are compared. Section 6 is devoted to a new look at the notion of opportunity. Section 7 concludes.

\section{Preliminary notions and available results}

A \emph{simple cooperative game with transferable utility} (TU-game) is a pair \((N, v)\), where \(N = \{1, 2, \ldots, n\}\) denotes the finite set of players and \(v : 2^n \to \{0, 1\}\) is the \emph{characteristic function}, with \(v(\emptyset) = 0\), \(v(S) \leq v(T)\) for all \(S, T\) subsets of \(N\) such that \(S \subseteq T\) and \(v(N) = 1\).

Given a coalition \(S \subseteq N\), if \(v(S) = 0\) then \(S\) is a \emph{losing} coalition, while if \(v(S) = 1\), then \(S\) is a \emph{winning} coalition. The marginal contribution of player
$i$ to $S$, is defined as:

$$v^i(S) = \begin{cases} v(S) - v(S \setminus \{i\}) & \text{if } i \in S; \\ v(S \cup \{i\}) - v(S) & \text{if } i \notin S. \end{cases}$$

The function $v^i$ is called the \emph{derivative} of $v$ with respect to (wrt hereafter) $i$.

We let $\mathcal{W} = \{S \subseteq N : v(S) = 1\}$ be the set of winning coalitions and let $\mathcal{W}_{\min} = \{W \in \mathcal{W} : \not\exists S \in \mathcal{W}, S \subset W\}$ be the set of minimal winning coalitions.

Beisbart \cite{6} proposes measures that “quantify the extent to which a voter can make a difference as a member of a group”.

First of all, the notion of criticality of a coalition is given.

\textbf{Definition 2.1.} (Definition 3.1 in \cite{6}) Let $G \subseteq N$. $G$ is \emph{critical} wrt a coalition $S$, if $S \cup G \in \mathcal{W}$ and $S \setminus G \notin \mathcal{W}$. If $G$ is critical wrt to $S$, $G$ is called \emph{critical inside (outside, resp.) $S$}, if $S \in \mathcal{W}$ ($S \notin \mathcal{W}$, resp.).

Turning to a measure of the single players, Beisbart provides the following:

\textbf{Definition 2.2} (Definitions 4.1 and 4.2 in \cite{6}). A player $i \in N$ is $d$-\emph{critical} of integer rank $\kappa^d_i(S)$ wrt $S \subseteq N$ if there is $G \subseteq N$ with $i \in G$ and $|G| = \kappa^d_i(S)$, such that $G$ is critical wrt $S$ and $G$ has minimal cardinality: no other coalition $G'$ with $|G'| < \kappa^d_i(S)$ and $i \in G'$ is critical wrt $S$.

\textbf{Remark 2.3.} Given any coalition $S$ and a critical coalition $G$, then each $i \in G$ is $d$-critical of some ranks $\kappa^d_i(S)$ wrt $S$ and for each $i, j \in G$ we can have $\kappa^d_i(S) \neq \kappa^d_j(S)$.

\textbf{Example 2.4.} Take $\mathcal{W} = \{\{1, 3\}, \{1, 2, 3\}\}$ and $S = \{1\}$. Player 2 is $d$-critical of rank $\kappa^d_2(S) = 2$ via the coalition $G = \{2, 3\}$. We can observe that $3 \in G$ but is $d$-critical of rank $\kappa^d_3(S) = 1$ via the coalition $G' = \{3\}$. Finally, player 1 is $d$-critical of order $\kappa^d_1(S) = 2$ via the coalition $G'' = \{1, 3\}$.

Based on the above definitions, the author defines a voting power index as the probability of a random coalition being critical of any given rank, thus extending the classical measurement of power based on the player’s solitary effort. Beisbart points out that the proposed indices do not depend on the player’s action, but only on the opportunity that the other players offer to the observed player.
Definition 2.5. A criticality notion passes the opportunity test, if whenever a player \( a \in N \) is critical wrt \( S \), then, the same player is also critical wrt \( S \cup \{ a \} \) and \( S \setminus \{ a \} \).

The proposed notion of criticality passes the test for any rank, while another proposal by the same author that requires a player to be indispensable in overturning the outcome fails the test, and for this reason is relegated to the work’s appendix.

Definition 2.6. Let \( (N, v) \) be a TU-cooperative game and fix a coalition \( S \subseteq N \).

1. Let \( G \) be critical wrt a \( S \). Player \( i \) is essential for \( G \) being critical (wrt \( S \)), if \( G \setminus \{ i \} \) is not critical wrt \( S \).

2. A player \( i \) is essential wrt \( S \) if there is at least one critical coalition \( G \) (wrt \( S \)) for which it is essential.

3. A player \( i \in N \) is e-critical of integer rank \( \kappa_i^e(S) \) wrt \( S \), if there is a coalition \( G \) with the following properties:
   
   (a) \( |G| = \kappa_i^e(S) \)
   
   (b) \( G \) is critical wrt \( S \)
   
   (c) \( i \) is essential for \( G \) being critical wrt \( S \).

More recently, Dall’Aglio et al. [9], unaware of Beisbart’s work, gave another definition of criticality that involves several players. This notion only considers the criticality inside \( S \), i.e. when the player acts with others to make the coalition \( S \) lose.

Definition 2.7. Let \( k \geq 0 \) be an integer, let \( S \subseteq N \), with \( |S| \geq k + 1 \), be a winning coalition. We say that a player \( i \) is negative critical of order \( k + 1 \) wrt a coalition \( S \), and write \( \rho_i^-(S) = k + 1 \), if \( k \) is the minimum integer such that there is a coalition \( K \subseteq S \setminus \{ i \} \) of cardinality \( k \) with

\[
v(S \setminus K) - v(S \setminus (K \cup \{ i \})) = 1.
\]

(2.1)

\(^{1}\)In the original work the player was simply referred to as critical. We add here the term “negative” to distinguish it from the complementary situation described in Definition 3.1.
The notion has been further investigated in the context of connection games in Dall’Aglio et al. \cite{11}, to define monotone indices of power (Dall’Aglio et al. \cite{12}), to rank the players according to a lexicographic criterion (Aleandri et al. \cite{1}) and as a dual definition of the outside criticality when the desirability relations is total (Aleandri et al. \cite{2}).

3 Essential criticality

In this section, we build upon the definitions given by Breisbart \cite{6} and by Dall’Aglio et al. \cite{9} to come up with a notion of group criticality that draws elements from both sources. We then make a comparison between the new proposal and that of \textit{d}-criticality used by Beisbart.

We reconsider Definition \ref{def:2.7} and notice that a similar definition can be given for the case when the player acts to turn the coalition \(S\) into a winner.

**Definition 3.1.** Let \(k \geq 0\) be an integer, let \(S \subseteq N\), with \(|S| \leq n - k - 1\), be a losing coalition. We say that a player \(i\) is positive critical of order \(k + 1\) wrt a coalition \(S\), and write \(\rho^+_i(S) = k + 1\), if \(k\) is the minimum integer such that there is a coalition \(K \subseteq N \setminus (S \cup \{i\})\) of cardinality \(k\) with

\[
v(S \cup (K \cup \{i\})) - v(S \cup K) = 1.
\]  

(3.1)

We now turn to the notion of \textit{e}-criticality of definition \ref{def:2.6} and note that it does not require all the players in \(G\) to be essential.

**Example 3.2.** (continue example \ref{ex:2.4}) Taking the coalition \(G = \{2, 3\}\), we observe that player 3 is essential for \(G\) being critical, but player 2 is not essential. On the other hand, take \(S' = \{2\}\), then players 1 and 3 are \textit{e}-critical of rank 2. Consider \(G = \{1, 3\}\) then \(G\) is critical for \(S\) and \(G \setminus \{1\}\), \(G \setminus \{3\}\) are not critical.

The previous example shows that players 1 and 3 are \textit{e}-critical of rank 3 too. Indeed, taking \(\widetilde{G} = \{1, 2, 3\}\) then \(\widetilde{G}\) is critical for \(S\) and \(\widetilde{G} \setminus \{1\}\), \(\widetilde{G} \setminus \{3\}\) are not critical. The \textit{e}-criticality rank of a player does not provide any information about the minimum number of players required to make the coalition \(G\) critical. Moreover, a player \(i \in N\) is always \textit{d}-critical of some order wrt a coalition, while it may fail to be \textit{e}-critical of any order.

**Example 3.3.** (continue example \ref{ex:2.4}) Take \(S = \{1, 3\}\), then player 2 is \textit{d}-critical of rank 2 via coalitions \(\{2, 1\}\) and \(\{2, 3\}\) but they are not \textit{e}-critical.
of rank 2 because they are not essential, given that $S$ is winning and player 2 has no role in making it lose.

The non-essential players in the critical coalition have no effective power, and it is natural to restrict our attention to essential players alone.

**Definition 3.4.** A coalition $G \subseteq N$ is called *essential critical*, or simply *essential*, wrt a coalition $S \subseteq N$, if each agent $i \in G$ is essential for $G$ being critical wrt $S$. Let us define $G^e(S)$ as the set of all essential coalitions wrt $S$.

When essential coalitions $G$ are considered, only two criticality scenarios, for player $i \in N$, come into play:

a) $S \notin \mathcal{W}$ and $i \notin S$. In this case, $G \subset S^c$, where $S^c = N \setminus S$, and we have outside or positive criticality.

b) $S \in \mathcal{W}$ and $i \in S$. In this case, $G \subset S$ and we have inside or negative criticality.

Let us consider the other two cases:

c) Suppose $S \notin \mathcal{W}$ and $i \in S$. Then, if a critical coalition $G$ contains $i$, that player is not essential, together with the whole coalition $G$. In fact, $S \cup G = S \cup (G \setminus \{i\}) \in \mathcal{W}$ and $S \setminus G \subset S \setminus (G \setminus \{i\}) \notin \mathcal{W}$.

d) Similarly, when $S \in \mathcal{W}$ and $i \notin S$ and a critical coalition $G$ contains $i$, that player is again not essential and neither is the coalition $G$. Indeed, $S \cup G \subset S \cup (G \setminus \{i\}) \subset S \in \mathcal{W}$ and $S \setminus (G \setminus \{i\}) = S \setminus G \notin \mathcal{W}$.

The essential coalitions form a minimal (in terms of cardinality) cover for the set of all critical coalitions for a given $S$.

**Proposition 3.5.** Let be $S \subseteq N$ and $G$ a critical coalition wrt $S$. If we remove the non-essential players from $G$, we still have a critical coalition. Moreover, any coalition $G'$ such that $G \subseteq G'$ is critical wrt $S$.

**Proof.** Let us write $G = E \cup nE$, where $E$ is the set of essential players and $nE$ is the set of non-essential players. According to the definition of a non-essential player, for all $i \in nE$, the coalition $G \setminus \{i\}$ is critical wrt $S$. The first part of the proposition is proved. For the second part it is sufficient to observe that $S \cup G' \supseteq S \cup G \in \mathcal{W}$ and $S \setminus G' \subseteq S \setminus G \notin \mathcal{W}$. 

Clearly, essential coalitions may have a different cardinality.
**Example 3.6.** Let us consider $W = \{\{1, 2, 3\}, \{1, 2, 4, 5\}\}$ and take $S = \{1\}$, then the coalition $\{2, 3\}$ and $\{2, 4, 5\}$ are both essential.

When essential coalitions are considered, we are able to define a notion of criticality that takes into account the fact that each player involved is essential and that, in doing so, a minimal number of players are involved.

**Definition 3.7.** A player $i \in N$ is group essential critical, or simply g-critical of integer rank $\kappa_i^g(S)$ wrt $S \subseteq N$ if there is an essential coalition $G \subseteq N$ of cardinality $|G| = \kappa_i^g(S)$, containing player $i$ such that no other coalition $G' \subseteq N$ with $i \in G'$ and $|G'| < \kappa_i^g(S)$ is essential critical wrt $S$.

**Example 3.8.** (continue example 3.6) Take the coalition $S = \{1\}$, then players 2 and 3 are g-critical of rank 2 and players 4 and 5 are g-critical of rank 3.

We are now able to create the link between Beisbart’s approach [6] and that of Dall’Aglio et al. [9].

**Proposition 3.9.** Given any coalition $S \subseteq N$, then player $i \in N$ is g-critical of rank $\kappa_i^g(S)$ if and only if player $i$ is either positive or negative critical of order $\kappa_i^g(S)$.

**Proof.** If player $i$ is not g-critical of any order then they are not essential. This implies that $v(S \setminus T) - v(S \setminus (T \cup \{i\})) = 0$ for all $T \subseteq S \setminus \{i\}$, then player $i$ is not critical of any order. The converse is straightforward.

Case 1: Player $i \in S$. Suppose that player $i$ is g-critical of order $\kappa_i^g(S) > 0$, then there is an essential critical coalition $G \subseteq S$ (wrt $S$) such that $|G| = \kappa_i^g(S)$. Define $K = G \setminus \{i\}$, then we observe that equation (2.1) holds and $\rho_i^-(S) \leq \kappa_i^g(S)$. Now suppose that $\rho_i^-(S) < \kappa_i^g(S)$ and there is $K' \subset S$ such that $|K'| < |K|$ and such that equation (2.1) is satisfied, then by the minimality of $\rho_i^-(S)$ the coalition $K' \cup \{i\}$ is essential critical wrt coalition $S$, implying that player $i$ is g-critical of rank $k' + 1 < \kappa_i^g(S) - 1$.

Conversely, suppose that player $i$ is negative critical of order $\kappa_i^g(S) > 0$, then there is a coalition $K \subseteq S \setminus \{i\}$ such that equation (2.1) is satisfied. By defining $G = K \cup \{i\}$, we can observe that $G$ is essential critical wrt $S$, then player $i$ is g-critical of order less or equal than $\kappa_i^g(S)$. Suppose there is an essential critical coalition $G'$ containing player $i$ such that $|G'| < \kappa_i^g(S)$ . This implies that $S \setminus G \notin W$ and $S \setminus (G \setminus \{i\}) \in W$; then taking $K' = G' \setminus \{i\}$, equation (2.1) holds and $\rho_i^-(S) < \kappa_i^g(S)$.

Case 2: The proof is analogous to Case 1 where in place of $S$ we use $S^c$. □

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4 Computing group criticality

We now focus on some computational techniques to derive the rank of criticality of a player according to the old and new definitions that we have reviewed so far. We begin with the newly introduced notion and show a straightforward method to derive the rank of $g$-criticality wrt to any coalition from the collection of minimal winning coalitions $W_{\text{min}}$ or the dual notions of minimal blocking ones. A similar procedure had already been used to compute the order of negative criticality in Aleandri et al. \[3\]. Given $\mathcal{C} \subseteq 2^N$ and $S \subseteq N$, we define $\mathcal{C} \setminus S = \{ C \setminus S : C \in \mathcal{C} \}$, and, for any player $i$, $(\mathcal{C})_i = \{ C : i \in C \in \mathcal{C} \}$. As acknowledged by Beisbart \[6\] (p.478), such a relationship between $d$-criticality and minimal winning coalitions (or minimal blocking ones $B_{\text{min}}$) cannot be found.\(^2\)

**Proposition 4.1.** Take $S \notin \mathcal{W}$, then player $i$ is $g$-critical if $(W_{\text{min}} \setminus S)_i$ is non-empty with the rank given by the minimal cardinality of the sets in $(W_{\text{min}} \setminus S)_i$. Taking $S \in \mathcal{W}$, player $i$ is $g$-critical if $(B_{\text{min}} \setminus S^c)_i$ is non-empty with the rank given by the minimal cardinality of the sets in $(B_{\text{min}} \setminus S^c)_i$.

**Proof.** Take $S \notin \mathcal{W}$. Using hypothesis $T \in (W_{\text{min}} \setminus S)_i$ with minimal cardinality, we show that $T$ is an essential critical coalition wrt $S$. By construction, $T$ is critical wrt $S$. Because of the criticality of $T$ there is $S_T \subseteq S$, such that $T \cup S_T \in W_{\text{min}}$. Now suppose that $T$ is not essential, then there is $j \in T$ such that $T \setminus \{ j \}$ is critical wrt $S$. This implies that $(T \setminus \{ j \}) \cup S_T \cup (S \setminus S_T) \in \mathcal{W}$. We know that $(T \setminus \{ j \}) \cup S_T \notin W_{\text{min}}$, then there is $A \subseteq (S \setminus S_T)$ such that $(T \setminus \{ j \}) \cup S_T \cup A \notin W_{\text{min}}$, but then we obtain $T \setminus \{ j \} \in (W_{\text{min}} \setminus S)_i$ that contradicts the minimality of $T$.

An analogous argument proves the second statement of the proposition. \(\Box\)

**Example 4.2.** Consider 5 players and $W_{\text{min}} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{4, 5, 6, 7\}\}$ and take $S = \{1\} \notin \mathcal{W}$. Now $W_{\text{min}} \setminus S = \{\{2, 3\}, \{3, 4, 5\}, \{4, 5, 6, 7\}\}$, so players 2 and 3 are $g$-critical of rank 1, players 4 and 5 are $g$-critical of rank 3 and players 6 and 7 are $g$-critical of rank 4. If $S = \{3, 4, 5, 6, 7\} \in \mathcal{W}$, $B_{\text{min}} = \{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 6\}, \{3, 7\}\}$ and $B_{\text{min}} \setminus S^c = \{\{4\}, \{5\}, \{3, 6\}, \{3, 7\}\}$, so players 4 and 5 are $g$-critical of rank 1, while players 3, 6 and 7 are $g$-critical of rank 2.

\(^2\)Beisbart actually mentions the collection of maximal losing coalitions, which are, set by set, the complements of the minimal blocking coalitions.
We now turn to $d$-criticality and note that a group of coalitions plays a special role in computing the rank of each player.

**Definition 4.3.** A coalition $G$ is essential minimal for $S$ if $G = \text{arg min}\{|G'| : G' \in \mathcal{G}^e(S)\}$. We denote the family of all minimal essential coalitions for $S$ as $\mathcal{G}_e^e(S)$.

Each coalition $S \subseteq N$ has a minimal order of criticality. We may therefore define

$$k^m(S) = \min\{|G' : G' \in \mathcal{G}_e^e S\}. \quad (4.1)$$

Players in the essential minimal coalitions play a special role and deserve a definition of their own.

**Definition 4.4.** A player $i$ is essential minimal critical, or simply $m$-critical of rank $k^m(S)$ if there is $G \in \mathcal{G}_e^e(S)$ such that $i \in G$ and $|G| = k^m(S)$. If a player does not belong to any minimal essential coalition wrt $S$, it is called non-minimal.

We observe that a $g$-critical player may fail to be $m$-critical.

**Example 4.5.** Consider example 3.6, player 4 is $g$-critical of rank 3, but $\mathcal{G}_e^m(S) = \{\{2, 3\}\}$.

The following result clarifies the importance of $m$-critical players in defining the ranks of $d$-criticality. Players that are $m$-critical of a certain rank are $d$-critical and $g$-critical of the same rank. All the other players become $d$-critical of the rank immediately above, while there is no restriction on $g$-criticality. Therefore, $d$-criticality is completely determined by $m$-criticality.

**Proposition 4.6.** Given a coalition $S \subseteq N$,

i. If player $i \in N$ is $m$-critical for $S$ with rank $k^m(S)$, then $i$ is $d$-critical and $g$-critical of the same rank;

ii. Otherwise, player $i$ is $d$-critical of rank $k^m(S) + 1$.

**Proof.** Denote as $\kappa^d_i(S)$ the rank for $d$-criticality of $i \in N$ wrt $S$ and suppose $i$ is $m$-critical with rank $k^m(S)$. Then the minimal essential coalition $\tilde{G}$ to which $i$ belongs is critical, therefore $\kappa^d_i(S) \leq k^m(S)$. The inequality cannot be strict, otherwise we would be able to extract a minimal essential coalition with a smaller cardinality than $k^m(S)$ from the critical coalition that defines
the rank of $d$-criticality. Since $\widetilde{G}$ is essential, a similar argument shows that player $i$ is also $g$-critical of the same rank.

For any other player $j \in N$, consider the coalition $\hat{G} = G_m \cup \{i\}$ with $G_m \in G_e(S)$. Now $\hat{G}$ is critical for $S$, with $|\hat{G}| = k^m(S) + 1$. No smaller coalition containing $j$ does the same, otherwise the coalition would be minimal and we would fall under the previous case. Player $j$ is therefore $d$-critical of rank $k^m(S) + 1$.

The previous definition shows that some players become $d$-critical without being essential for the corresponding critical coalition, gaining a rank that does not reflect the real impact on coalition formation. We call such players free riders.

**Definition 4.7.** A player $i \in N$ is a free rider for $d$-criticality if either i) $i$ is not $g$-critical of any rank or ii) its rank for $g$-critical is higher than that of $d$-criticality.

**Example 4.8.** Take $\mathcal{W}_{\text{min}} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{4, 5, 6, 7\}\}$ and $S = \{1\}$. Here $\{2, 3\}$ is the only minimal essential coalition so 2 and 3 are $m$-critical, $d$-critical and $g$-critical of rank 2. Players 4,5 are $d$-critical and $g$-critical of rank 3. Finally, players 6 and 7 are $g$-critical of rank 4, but $d$-critical of rank 3, so they are free riders.

Note player 4 may form an essential coalition with 3 and 5 or may join the minimal essential coalition $\{2, 3\}$, but this will not lower their rank. A similar argument occurs for player 5.

The previous example should convince the reader that free riding occurs when the rank for $g$ criticality is at least two units above the minimal rank.

## 5 A comparison of the properties

The computational techniques in the previous section already reveal many features of the two different notions of group criticality. We continue the comparison with other quantitative and qualitative benchmarks.

### 5.1 Monotonicity

We base our analysis on an extension to rankings of the notion of monotonicity introduced by Young, [30], which relies on criticality of the first order.
Definition 5.1. Let $\mathcal{T}^N$ be the set of cooperative games $(N, v)$ and let $\succeq$ be a preorder on $\mathbb{R}^d$. An allocation procedure, $\phi : \mathcal{T}^N \to (\mathbb{R}^d)^N$, satisfies the strong monotonicity wrt the preorder $\succeq$ if, for any two games $v, w$ and any player $i \in N$, $w^i(S) \geq v^i(S)$ for all $S$ implies $\phi_i(w) \succeq \phi_i(v)$.

Let $\mathcal{P}^N$ be the set of probability distributions on the power set of $N$. Take $p \in \mathcal{P}^N$, with $p(S)$ denoting the probability of coalition $S \subseteq N$ forming and we introduce new voting powers.

Definition 5.2. The essential minimal measure of voting power of order $\kappa$ for player $i \in N$, denoted $\beta^m_{i,\kappa}$ is the probability of $i$ of being $m$-critical wrt to a random coalition $S \subseteq N$ originating from probability $p$.

By replacing $m$-criticality with $g$-criticality we obtain the group essential measure of voting power of order $\kappa$, $\beta^g_{i,\kappa}$, while by replacing it with $d$-criticality, we obtain the differential measure of criticality, $\beta^d_{i,\kappa}$, always of order $\kappa$.

In Beisbart [6] the index $\beta^d_{i,\kappa}$ was simply referred to as the measure of voting power of rank $\kappa$. We add the term ”differential” to distinguish it from the other values defined here.

Given a game $v \in \mathcal{T}^N$ define, for each player $i \in N$, the sequence of indices

$$\beta^x_i(v) := (\beta^x_{i,1}(v), \ldots, \beta^x_{i,n}(v)),$$

where $x = m, x = d$ or $x = g$. For our purposes, we are interested in a global index that includes the whole sequence of ranks. We consider $\pi^x_{i,\kappa}$, the probability of being $x$-critical up to order $\kappa$,

$$\pi^x_{i,\kappa}(v) = \sum_{\ell=1}^{\kappa} \beta^x_{i,\ell}(v).$$

Definition 5.3. Let $v, w \in \mathcal{T}^N$ and a player $i \in N$. The vector of average ranks $\beta^x_i(w)$ dominates $\beta^x_i(v)$ in first-order stochastic dominance, and we write $\beta^x_i(w) \succeq_{\text{lsd}} \beta^x_i(v)$, whenever $\pi^x_{i,\kappa}(w) \geq \pi^x_{i,\kappa}(v)$ for every $\kappa \in N$.

We now examine Young’s strong monotonicity for the different notions of criticality that we are comparing. We start with $g$-criticality.

Lemma 5.4. Suppose $w^i(S) \geq v^i(S)$ for all $S \subseteq N$ and some $i \in N$. If $i$ is $g$-critical of rank $k^g_i(T)$ wrt to $T$ in the game $v$, then $i$ is $g$-critical wrt to the same coalition in the game $w$ with a rank smaller than or equal to $k^g_i(T)$. 

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Proof. Denote as $G^T$ the essential coalition of minimal cardinality that
defines the rank of $g$-criticality of $i$ wrt to $T$ in the game $v$. Since $w^i(S) \geq v^i(S)$, player $i$ remains essential for the criticality of $G^T$ wrt to $T$ under $w$, while other players may lose their essentiality. The rank in $w$ may therefore remain the same or become lower.

The strong monotonicity of $g$-criticality follows.

**Proposition 5.5.** $g$-criticality satisfies strong monotonicity under the first-order stochastic criterion.

Proof. Suppose $w^i(S) \geq v^i(S)$. By Lemma 5.3 if $i$ is $g$-critical under $v$, it is also $g$-critical under $w$ wrt to same coalition, with equal or smaller rank. Some other coalitions that failed to be $g$-critical under $v$ may become critical under $w$, but this can only increase the partial cumulative values of $\pi_{i}^{\pi,w}$. The first-order stochastic dominance of the average ranks vector of $w$ wrt to $v$ follows.

The following examples illustrate the proposition at work and, most importantly, show that neither $d$-criticality nor $m$-criticality share the same property.

**Example 5.6.** Consider two games $v$ and $w$ on $N = \{1, 2, 3, 4\}$ with minimal winning coalition families given, respectively, by $W^w_{\min} = \{\{1, 2, 3\}, \{4\}\}$ and $W^w_{\min} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$. It can easily be verified that $w^i(S) \geq v^i(S)$ for all $S \subseteq N$, with strict majorization for the coalitions $\{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}$ and $N$. The presence of criticality, together with the rank, are described for the two games in Table III.

It turns out that $\beta^{g,1}_1(w) = \beta^{g,1}_1(v) + p(\{2, 4\}) + p(\{1, 2, 4\}) + p(\{2, 3, 4\}) + p(\{1, 2, 3, 4\})$, $\beta^{g,2}_1(w) = \beta^{g,2}_1(v) + p(\{3, 4\}) + p(\{4\})$, and $\beta^{g,3}_1(w) = \beta^{g,3}_1(v)$. Therefore $\beta^g_1(w) \geq \beta^g_1(v)$ for any $p \in \mathcal{P}^N$. Conversely, $\pi^{d,2}_1(v) = 1$ and $\pi^{d,2}_1(w) = 1 - p(\emptyset) - p(\{1\})$. Therefore, $d$-criticality is not strongly monotone for every probability distribution $p \in \mathcal{P}^N$ such that either $p(\emptyset) > 0$ or $p(\{1\}) > 0$ (or both).

**Example 5.7.** Now consider 5 players and the simple games $v$ and $w$ defined respectively by the minimal winning coalition sets $W^w_{\min} = \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\}$ and $W^w_{\min} = \{\{1, 2, 3\}, \{3, 5\}, \{2, 4, 5\}\}$. Again, $w^i(S) \geq v^i(S)$ for all $S \subseteq N$, with strict majorization for the coalitions $\{2, 3\}, \{1, 2, 3\}$,
\begin{table}
\begin{center}
\begin{tabular}{|c|rr|rr|}
\hline
$S$ & $d$-criticality & & $g$-criticality & \\
\hline
$\emptyset$ & 2 & 3 & 3 & 3 \\
\{1\} & 2 & $\times$ & $\times$ & \\
\{2\} & 2 & 2 & 2 & 2 \\
\{3\} & 2 & 2 & 2 & 2 \\
\{4\} & 2 & 2 & $\times$ & 2 \\
\{1,2\} & 2 & 2 & $\times$ & $\times$ \\
\{1,3\} & 2 & 2 & $\times$ & $\times$ \\
\{1,4\} & 2 & 2 & $\times$ & $\times$ \\
\{2,3\} & 1 & 1 & 1 & 1 \\
\{2,4\} & 2 & 1 & $\times$ & 1 \\
\{3,4\} & 2 & 2 & $\times$ & 2 \\
\{1,2,3\} & 1 & 1 & 1 & 1 \\
\{1,2,4\} & 2 & 1 & $\times$ & 1 \\
\{1,3,4\} & 2 & 2 & $\times$ & $\times$ \\
\{2,3,4\} & 2 & 1 & $\times$ & 1 \\
\{1,2,3,4\} & 2 & 1 & $\times$ & 1 \\
\hline
\end{tabular}
\end{center}
\caption{Comparing the ranks of criticality for $v$ and $w$ in Example 5.6.}
\end{table}

\{2,3,5\},\{2,4,5\} and \{3,4,5\}. The change in the ranks for both group and minimal criticality for player 1 are given in Table 2.

All the coalitions which are group critical of some rank under $v$ remain critical under $w$, with the rank unaltered or decreased. Instead, some coalitions that are essential minimal critical under $v$, stop being critical under $w$. Any probability distribution assigning positive probability to those coalitions reveals that the notion of minimal criticality is non-monotone.

Remark 5.8. In [11], a weighted sum of indices was considered and the notion of monotonicity introduced by Turnovec [28] for weighted voting games was proved for the negative order of criticality when all the coalitions are equally likely to occur. The assumption of equal probability was crucial for the result and the use of weights, which might be considered arbitrary, make the present result more general.

In [1], another criterion is examined for single coalitions, the lexicographic order, defined as follows: $\beta^x_i(w) \geq \beta^x_i(v)$ if $\beta^x_{i,1}(w) > \beta^x_{i,1}(v)$.
or, if $\beta^{x,1}_i(w) = \beta^{x,1}_i(v)$, when $\beta^{x,2}_i(w) > \beta^{x,2}_i(v)$. When both indices of rank 1 and 2 are equal, the third rank is compared, and so on. It is easy to verify that $\beta^{x}_i(w) \succeq_{1sd} \beta^{x}_i(v)$ implies $\beta^{x}_i(w) \succeq_{lex} \beta^{x}_i(v)$.

### 5.2 Dummy players

Another problem with $d$-criticality comes from the fact that any player – including dummy ones – is critical of some order. Therefore, power indices based on $d$-criticality will always assign non-null power to dummy players, as long as the index is based on a probability distribution that supports any coalition. This is not the case with $g$-criticality, since dummy players are always inessential.

**Example 5.9.** Consider again $N = \{1, 2, 3, 4, 5\}$ and $W_{\text{min}} = \{\{1, 2\}, \{1, 3, 4\}\}$, player 5 is clearly a dummy player but is $d$-critical of rank 2 for any coalition $S \subseteq N$, while the same player is never $g$-critical.

For a more rigorous analysis, we turn to the total probability of being critical according to the various criteria.
Definition 5.10. We denote the probability of essential minimal (group essential, differential, resp.) criticality for a player \( i \in N \) as \( \pi_i^x = \pi_i^{x,m} \) where \( x = m, x = g, x = d \), resp.

Proposition 5.11. The following relationships hold for any \( i \in N \),

\[
\pi_i^m \leq \pi_i^g \leq \pi_i^d = 1 \tag{5.1}
\]

Proof. To prove (5.1), we simply note that minimal essential criticality is only a part of both essential and differential criticality, and therefore \( \pi_i^m \leq \pi_i^g \) and \( \pi_i^m \leq \pi_i^d \). The last equality holds because not all agents are essential critical of any order.

Proposition 5.12. The following relation holds for any \( i \in N \):

\[
\text{Player } i \text{ is a dummy } \iff \pi_i^g = 0 \iff \pi_i^m = 0. \tag{5.2}
\]

Proof. If player \( i \) is a dummy it is easy to show that \( \pi_i^g = 0 \), then by Proposition 5.11 \( \pi_i^m = 0 \).

Suppose that player \( i \) is not a dummy. Then they belong to some minimal winning coalition, \( S \). We observe that the coalition \( G = \{i\} \) is minimal essential wrt \( S \) and then \( \pi_i^m > 0 \), then \( \pi_i^g > 0 \).

5.3 Average ranks

In section 4, we have shown that the rank of \( d \)-criticality is determined by a restricted group of \( m \)-critical players. This dependence is confirmed by the analysis of the average rank of differential criticality \( \bar{\kappa}^d_i \), for a player \( i \in N \), defined as

\[
\bar{\kappa}^d_i := \sum_{\kappa=1}^{n} \kappa \beta_i^{d,\kappa}.
\]

Since every player is \( d \)-critical of some order, we have

\[
\bar{\kappa}^d_i = \sum_{S \subseteq N} k^d_i(S)p(S).
\]

We now show that \( \bar{\kappa}^d_i \) depends on two parameters of minimal criticality: the probability of being minimal essential critical and the global average order of minimal criticality defined as

\[
\mu = \sum_{S \subseteq N} k^m(S)p(S).
\]
Proposition 5.13. For any player \( i \in N \),
\[
\bar{\kappa}_i^d = 1 + \mu - \pi^m_i
\]  
(5.3)

Proof. For any \( i \in N \), we have
\[
\bar{\kappa}_i^d = \sum_{S \subseteq N} k_i^d(S)p(S) = \sum_{S \subseteq N} k_i^d(S)p(S) + \sum_{S \subseteq N} k_i^d(S)p(S)
\]
\[
= \sum_{S \subseteq N} k^m(S)p(S) + \sum_{S \subseteq N} (k^m(S) + 1)p(S)
\]
\[
= \sum_{S \subseteq N} k^m(S)p(S) + \sum_{S \subseteq N} p(S) = \mu + 1 - \pi^m_i.
\]

The following corollary shows that the difference between the average order of differential criticality of two players depends exclusively on the difference in their probability of essential minimal criticality.

Corollary 5.14. Given two players \( i, j \in N \), then
\[
\bar{\kappa}_i^d - \bar{\kappa}_j^d = \pi^m_j - \pi^m_i.
\]

Example 5.15. Consider 4 players and \( \mathcal{W}_{\min} = \{\{1, 2\}, \{3\}\} \). Take a uniform probability distribution \( p \in \mathcal{P}^N \), \( p(S) = \frac{1}{16} \). Then we have the following values:
\[
\bar{\kappa}_1^d = \bar{\kappa}_2^d = \frac{28}{16}, \quad \bar{\kappa}_3^d = \frac{20}{16}, \quad \bar{\kappa}_4^d = \frac{34}{16}
\]

We next verify Proposition 5.13. We note that \( k^m(S) = 1 \) for all the coalitions, but coalitions \( \{1, 2, 3\} \) and \( \{1, 2, 3, 4\} \), for which \( k^m(S) = 2 \). Therefore,
\[
\mu = \left( 1 - \frac{2}{16} \right) + 2 \frac{2}{16} = \frac{18}{16}.
\]

Then we have
\[
1 + \mu - \pi^m_1 = 1 + \frac{18}{16} - \frac{6}{16} = \frac{28}{48} = \bar{\kappa}_1^d; \quad 1 + \mu - \pi^m_2 = 1 + \frac{18}{16} - \frac{6}{16} = \frac{28}{48} = \bar{\kappa}_2^d;
\]
\[
1 + \mu - \pi^m_3 = 1 + \frac{18}{16} - \frac{14}{16} = \frac{20}{48} = \bar{\kappa}_3^d; \quad 1 + \mu - \pi^m_4 = 1 + \frac{18}{16} - 0 = \frac{34}{48} = \bar{\kappa}_4^d.
\]
6 A New Look at the Opportunity Test using essentiality

Beisbart has defined $d$-criticality for the purpose of creating a power index that measures the players’ influence in conjunction with others and satisfies the opportunity test of Definition 2.3. Clearly, $g$-criticality does not pass the same test. Take for instance $S = \{1\}$ in Example 4.8. While player 2 is $d$-critical of rank 2 for both $S \setminus \{2\} = S$ and $S \cup \{2\} = \{1, 2\}$, the same player is $g$-critical of the same rank for $S$ alone, while it is inessential for $S \cup \{2\}$, and therefore not $g$-critical of any order. This is not a coincidence and it takes place whenever a player is critical with the help of others.

**Proposition 6.1.** Suppose player $i \in N$ is $d$-critical of rank 2 or higher wrt $S$. Then $i$ is inessential for at least one of the critical coalitions that define the rank of $d$-criticality in $S \setminus \{i\}$ and $S \cup \{i\}$.

**Proof.** If and $i \notin S \in \mathcal{W}$, then $i$ is always inessential for the critical coalition that turns $S = S \setminus \{i\}$ into a losing coalition. The case when $i \in S \notin \mathcal{W}$ and $i$ is not essential for $S = S \cup \{i\}$ is similar.

Consider now $i \in S \in \mathcal{W}$. Since $i$ is $d$-critical with a rank greater than 1, then $S \setminus \{i\} \in \mathcal{W}$, with $i$ not essential for the critical coalition that turns $S \setminus \{i\}$ into a losing one. The case $a \notin S \notin \mathcal{W}$ is treated similarly. \qed

When a group of players is involved, $d$-criticality passes the opportunity test, but the price to pay is to make inessential players critical in at least half of the cases where their criticality rank is 2 or higher. The fraction of cases may be higher than that, since a player may be inessential for both $S \setminus \{i\}$ and $S \cup \{i\}$. Consider for instance player 6 in Example 4.8 who is inessential for both $S \setminus \{6\}$ and $S \cup \{6\}$ when $S = \{1\}$.

We question here the adequacy of the opportunity test for Definition 2.3 which was defined in a context where voting power measures “quantify the extent (denoted as $E$) to which voter $a$ has the opportunity to make a difference as to whether a bill passes or not”, in a situation where the focus is on “the extent (denoted as $E_k$) to which $a$ can be a member of a group of size $\kappa$ that has the opportunity to make a difference as to whether the bill passes”. In other words, the opportunity test should be modified in a way that takes into account the ability of the group to change the game outcome, instead of the ability of a single player.
In order to proceed, we examine the implications of the opportunity test, to verify whether a new test involving groups of players satisfies similar properties. The test has three important implications:

a. No matter what action a player takes, they will always be critical for the outcome;

b. depending on the action, the player will be either outside or inside critical;

c. if outside and inside criticality have the same importance, the power of the player is the same no matter what action is taken.

In defining an opportunity test for coalitions, we would like the test to show a coalition $K$ to be critical in some sense for a coalition $S$, independently of the behaviour of players in $K$ (implication $a$) and we would like players in $K$ capable of both outside and inside criticality wrt $S$ (implication $b$). We set implication $c$ aside for the moment. In a voting context, we require that, no matter how players in $K$ cast their vote, some players will always be critical – and every player is critical for some configuration. Turning to a more formal approach, we propose a coalition-wise opportunity test.

**Definition 6.2.** A criticality criterion for coalitions satisfies the opportunity test for coalitions if every time that coalition $K$ is critical for coalition $S$, we have the following:

i) For any $H \subseteq K$, there is a coalition $G_H \subseteq K$ that is critical wrt either $S \cup H$ or $S \setminus H$.

ii) Every player is critical for at least one configuration of players

$$\bigcup_{H \subseteq K} G_H = K$$

iii) There are two coalitions $H'$ and $H''$, such that $G_{H'}$ is critical outside and $G_{H''}$ is critical inside.

We now focus on the notion of essential criticality.

**Proposition 6.3.** If we replace the expression "criticality" with "essential criticality" in i) of Definition 6.2, essential critical coalitions pass the test and they are the only coalitions to do so.
Proof. If $K$ is an essential coalition for $S$, then $G_K = K$ and $G_H = K \setminus H$ for $H \neq K$. $G_K$ is essential critical for $S \cup H$ when $S \notin W$, and it is essential critical for $S \setminus H$ when $S \in W$. If $K$ is not critical, it does not satisfy property (iii) of Definition 6.2 while if it is not essential it does not satisfy property ii).

We now give examples in the context of voting. Suppose $S$ is a coalition of players voting “yes” on a motion that will not pass with their votes alone. Therefore $S$ is a losing coalition. Suppose, moreover, that $K$ is a coalition of players voting “yes” on a motion and every player in $K$ is inside $g$-critical of rank 1. Otherwise, the bill will not pass and only the players who voted “no” will now be critical – their rank being given by the cardinality of this smaller group. Conversely, if $S$ is winning and $K$ is essential critical inside $S$, players will be inside critical of rank 1 only if they jointly vote “no”. Conversely, only the players who voted “yes” will be jointly critical, their rank being given by their cardinality.

We remark that player $i$ is $g$-critical of rank $\kappa$ wrt $S$ if $\kappa$ is the minimum cardinality of the essential critical coalitions that contain player $i$. In extending the opportunity test notion, we set aside property (c) implied by the one-player opportunity test. When $g$-criticality is under scrutiny, the symmetry of criticality breaks down. We may construct power indices restricted to the different configurations of players in $K$. Suppose $p_K$ is a probability distribution on the subsets of $K$ that indicates the players voting “yes”. We may then consider a restricted power index à la Banzhof:

$$\beta_{K}^{i,r} = \sum_{H \subseteq K} p_K(H) \mathbb{1}_i(G_H)$$

where $\mathbb{1}_i(G_H) = \begin{cases} 1 & \text{if } i \in G_H \\ 0 & \text{otherwise} \end{cases}$

To make things simpler, we assume $S$ losing, $K$ essential critical with $|K| = k$ and $p(H) = 1/2^k$ for every $H \subseteq K$. Now

$$\beta_{K}^{i,1} = 1/2^{k-1}$$

and this is the sum of two critical events: inside criticality for $S \cup K$ and outside criticality for $(S \cup K) \setminus \{i\}$.

For higher ranks, only outside criticality is counted and, for any $r = 2, 3, \ldots, k$,

$$\beta_{K}^{i,r} = \frac{(r-1)}{2^k}$$

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The measure of outside criticality is not compensated by any degree of inside criticality for the same rank, and this is where symmetry breaks down.

We note that symmetry holds when first order rank is considered in line with what is already known about classical criticality. Higher order criticality, instead, contributes to a single side alone: outside criticality when $S \notin \mathcal{W}$ and inside criticality otherwise. This asymmetric part arises from the collaborative effort among players.

7 Conclusions

By extending the notion of order of criticality given in Dall’Aglio et al. [9] for inside players, we have defined the rank of group essential criticality of a player and we have compared this notion with that of differential criticality from Beisbart [6]. For any given coalition, the two notions coincide for those players who belong to the essential critical coalitions of minimal cardinality, who are labelled minimal critical, or $m$-critical players, but usually differ for the remaining players. Group criticality satisfies several properties that differential criticality lacks: it is strongly monotone in the sense defined by Young [30], it is never associated with dummy players and it is not deterministically determined by the set of $m$-critical players. Conversely, $d$-criticality is the only criterion that is compatible with the principle that a measure of criticality should evaluate the degree of opportunity that the other players give. When several players work together to overturn a game’s outcome, the two principles of opportunity and essentiality in their current definitions are incompatible and a decision must be taken on which one to save. We believe that this hiatus originates in an improper application of the opportunity principle in a context where players do not act alone, but work together to reach their goals. For this reason and with the goal of reconciling the two principles, we propose an asymmetric group opportunity test which is satisfied by the newly introduced notion of group criticality.

Regarding future research directions, we believe that investigating the relationship between indices that measure the importance of the single players by evaluating their criticality jointly with others and those that measure the importance of coalitions as a whole should be analysed in more detail. We refer to some extensions of power indices to include whole coalitions instead of single players. Both the Shapley and the Banzhaf values have been extended to measure the interaction among players in Grabisch and Roubens [14].
a similar vein, Hausken and Mohr [16] have defined the Shapley value of one player to another player and this work has been recently extended by Hausken [15] to define the Shapley value of one coalition to another coalition.

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