RING ISOMORPHISMS OF \(*\)-SUBALGEBRAS OF MURRAY–VON NEUMANN FACTORS

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Abstract. The present paper is devoted to study of ring isomorphisms of \(*\)-subalgebras of Murray–von Neumann factors. Let \(M, N\) be von Neumann factors of type II\(_1\), and let \(S(M), S(N)\) be the \(*\)-algebras of all measurable operators affiliated with \(M\) and \(N\), respectively. Suppose that \(A \subset S(M), B \subset S(N)\) are their \(*\)-subalgebras such that \(M \subset A, N \subset B\). We prove that for every ring isomorphism \(\Phi : A \rightarrow B\) there exist a positive invertible element \(a \in B\) with \(a^{-1} \in B\) and a real \(*\)-isomorphism \(\Psi : M \rightarrow N\) (which extends to a real \(*\)-isomorphism from \(A\) onto \(B\)) such that \(\Phi(x) = a\Psi(x)a^{-1}\) for all \(x \in A\). In particular, \(\Phi\) is real-linear and continuous in the measure topology. In particular, noncommutative Arens algebras and noncommutative \(L_{\log}\)-algebras associated with von Neumann factors of type II\(_1\) satisfy the above conditions and the main Theorem implies the automatic continuity of their ring isomorphisms in the corresponding metrics. We also present an example of a \(*\)-subalgebra in \(S(M)\), which shows that the condition \(M \subset A\) is essential in the above mentioned result.

1. Introduction

In 1930’s, motivated by the geometry of lattice of the projections of type II\(_1\) factors, von Neumann built the theory on the correspondence between complemented orthomodular lattices and regular rings. Let us recall one of his achievements [20, Part II, Theorem 4.2], applied to the case of \(*\)-regular rings. Let \(R, R'\) be \(*\)-regular rings such that their lattices of projections \(L_R\) and \(L_{R'}\) are lattice-isomorphic. If \(R\) has order \(n \geq 3\) (which means that it contains a ring of matrices of order \(n\)), then there exists a ring isomorphism of \(R\) and \(R'\) which generates given lattice isomorphism between \(L_R\) and \(L_{R'}\). One of important classes of \(*\)-regular rings are the \(*\)-algebra of operators affiliated with a finite von Neumann algebra. Let \(M\) be a von Neumann algebra and let \(S(M)\) (respectively, \(LS(M)\)) be the \(*\)-algebra of all measurable (respectively, locally measurable) operators affiliated with \(M\). Note that if \(M\) is a finite von Neumann algebra then every operator affiliated with \(M\) is automatically measurable and hence the \(*\)-algebras \(S(M)\) and \(LS(M)\) coincide. Applied to the case of arbitrary type II\(_1\) von Neumann algebras, the above von Neumann isomorphism theorem is formulated as follows: If \(M\) and \(N\) are von Neumann algebras of type II\(_1\) and \(\Phi : P(M) \rightarrow P(N)\) is a lattice isomorphism then there exists a unique ring isomorphism \(\Psi : S(M) \rightarrow S(N)\) such that \(\Phi(l(x)) = l(\Psi(x))\) for any \(x \in S(M)\), where \(l(a)\) is the left support of the element \(a\).

Note that in the case of commutative regular rings the picture is completely different. Let us recall a problem of isomorphisms for an important class of commutative regular rings with an atomic Boolean algebra of idempotents, namely, so-called Tychonoff
semifields. Given an arbitrary set $\Delta$, a Tychonoff semifield $\mathbb{R}^\Delta$ is defined as the product of $|\Delta|$ copies of the real field, equipped with the pointwise algebraic operations, natural partial order and the Tychonoff’s topology. These operations make $\mathbb{R}^\Delta$ a topological regular ring. The set of all idempotents of the semifield $\mathbb{R}^\Delta$ with the induced topology and order is topologically isomorphic to $\{0, 1\}^\Delta$. For $g \in \Delta$ denote by $1_g$ an atom from $\{0, 1\}^\Delta$ defined as $1_g(g) = 1$ and $1_g(g') = 0$ for $g \neq g'$ ($g' \in \Delta$) and $1_\Delta$ is identity of $\mathbb{R}^\Delta$.

The following two questions are equivalent (see [5], [6]):

(a) Does there exist an algebraic homomorphism $\psi : \mathbb{R}^\Delta \to \mathbb{R}$ satisfying the condition $\psi(1_g) = 0$ for all $g \in \Delta$, such that $\psi(1_\Delta) = 1$?

(b) Does there exist a non trivial two-valued countably additive measure $\mu : \{0, 1\}^\Delta \to \mathbb{R}$ satisfying the condition $\mu(1_g) = 0$ for all $g \in \Delta$?

The second question is the famous Ulam Problem [25] which is connected with the properties of cardinal $|\Delta|$.

Returning to the noncommutative case recall that in the recent paper [17] M. Mori characterized lattice isomorphisms between projection lattices $P(M)$ and $P(N)$ of arbitrary von Neumann algebras $M$ and $N$, respectively, by means of ring isomorphisms between the algebras $LS(M)$ and $LS(N)$. In this connection he investigated the following problem.

**Question 1.1.** Let $M, N$ be von Neumann algebras. What is the general form of ring isomorphisms from $LS(M)$ onto $LS(N)$?

In [17] Theorem B] M. Mori himself gave an answer to the above Question [1.1] in the case of von Neumann algebras of type $I_\infty$ and III. Namely, any ring isomorphism $\Phi$ from $LS(M)$ onto $LS(N)$ has the form

$$\Phi(x) = y\Psi(x)y^{-1}, \ x \in LS(M),$$

where $\Psi$ is a real $*$-isomorphism from $LS(M)$ onto $LS(N)$ and $y \in LS(N)$ is an invertible element. Note that in the case where $\Phi$ is an algebraic isomorphism of type $I_\infty$ von Neumann algebras, the above presentation was obtained in [4].

In [17] the author conjectured that the representation of ring isomorphisms, mentioned above for type $I_\infty$ and III cases holds also for type II von Neumann algebras. In [7] we have answered affirmatively to the above Question [1.1] of Mori in the case of von Neumann algebras of type $\Pi_1$. Namely, it was shown [7] Theorems 1.3 and 1.4] that if $M, N$ are von Neumann algebras of type $\Pi_1$ any ring isomorphism $\Phi : S(M) \to S(N)$ is continuous in the measure topology and there exist an invertible element $a \in S(N)$ and a real $*$-isomorphism $\Psi : M \to N$ (which extends to a real $*$-isomorphism from $S(M)$ onto $S(N)$) such that $\Phi(x) = a\Psi(x)a^{-1}$ for all $x \in S(M)$. As a corollary we also obtained that for von Neumann algebras $M$ and $N$ of type $\Pi_1$ the projection lattices $P(M)$ and $P(N)$ are lattice isomorphic, if and only if the von Neumann algebras $M$ and $N$ are Jordan $*$-isomorphic. The present paper can be considered as an extension of the results from [7].

In Section 2 we give preliminaries on Murray-von Neumann algebras and its special subalgebras – so-called noncommutative Arens algebras and noncommutative $L_{log}$-algebras.

The following Theorem which is the main result of the present paper we shall prove in Section 3.

**Theorem 1.2.** Let $M, N$ be von Neumann factors of type $\Pi_1$ and let $A \subset S(M)$, $B \subset S(N)$ be $*$-subalgebras such that $M \subset A$, $N \subset B$. Suppose that $\Phi : A \to B$ is a
ring isomorphism. Then there exist a positive invertible element \( a \in B \) with \( a^{-1} \in B \) and a real \(*\)-isomorphism \( \Psi : M \to N \) (which extends to a real \(*\)-isomorphism from \( A \) onto \( B \)) such that \( \Phi(x) = a\Psi(x)a^{-1} \) for all \( x \in A \). In particular, \( \Phi \) is real-linear and continuous in the measure topology.

In Section 4 we show that there is a \(*\)-regular subalgebra of algebra of all measurable operators with respect to a hyperfinite factor of type II\(_1\) which admits an algebra automorphism, discontinuous in the measure topology. In particular, since Theorem 1.2 gives us automatic continuity in the measure topology of ring isomorphisms, the mentioned example shows that the condition \( M \subset A \) is essential in Theorem 1.2.

2. Preliminaries

For \(*\)-algebras \( A \) and \( B \), a (not necessarily linear) bijection \( \Phi : A \to B \) is called

- a ring isomorphism if it is additive and multiplicative;
- a real algebra isomorphism if it is a real-linear ring isomorphism;
- an algebra isomorphism if it is a complex-linear ring isomorphism;
- a real \(*\)-isomorphism if it is a real algebra isomorphism and satisfies \( \Phi(x^*) = \Phi(x)^* \) for all \( x \in A \);
- a \(*\)-isomorphism if it is a complex-linear real \(*\)-isomorphism.

2.1. Murray-von Neumann algebra. Let \( H \) be a Hilbert space, \( B(H) \) be the \(*\)-algebra of all bounded linear operators on \( H \) and let \( M \) be a von Neumann algebra in \( B(H) \).

Denote by \( P(M) \) the set of all projections in \( M \). Recall that two projections \( e, f \in P(M) \) are called equivalent (denoted as \( e \sim f \)) if there exists an element \( u \in M \) such that \( u^*u = e \) and \( uu^* = f \). For projections \( e, f \in M \) notation \( e \leq f \) means that there exists a projection \( q \in M \) such that \( e \sim q \leq f \). A projection \( p \in M \) is said to be finite, if it is not equivalent to its proper sub-projection, i.e. the conditions \( q \leq p \) and \( q \sim p \) imply that \( q = p \) (for details information concerning von Neumann algebras see [14, 22]).

A densely defined closed linear operator \( x : \text{dom}(x) \to H \) (here the domain \( \text{dom}(x) \) of \( x \) is a dense linear subspace in \( H \)) is said to be affiliated with \( M \) if \( xy \subset xy \) for all \( y \) from the commutant \( M' \) of the algebra \( M \).

A linear operator \( x \) affiliated with \( M \) is called measurable with respect to \( M \) if \( e_{(\lambda, \infty)}(|x|) \) is a finite projection for some \( \lambda > 0 \). Here \( e_{(\lambda, \infty)}(|x|) \) is the spectral projection of \( |x| \) corresponding to the interval \((\lambda, +\infty)\). We denote the set of all measurable operators by \( S(M) \).

Let \( x, y \in S(M) \). It is well known that \( x+y \) and \( xy \) are densely-defined and preclosed operators. Moreover, the (closures of) operators \( x+y, xy \) and \( x^* \) are also in \( S(M) \). When equipped with these operations, \( S(M) \) becomes a unital \(*\)-algebra over \( \mathbb{C} \) (see [16, 23]). It is clear that \( M \) is a \(*\)-subalgebra of \( S(M) \). In the case of finite von Neumann algebra \( M \), all operators affiliated with \( M \) are measurable and the algebra \( S(M) \) is referred to as the Murray-von Neumann algebra associated with \( M \) (see [15]).

Let \( M \) be a von Neumann algebra with a faithful normal finite trace \( \tau \). Consider the topology \( t_\tau \) of convergence in measure or measure topology [18] on \( S(M) \), which is defined by the following neighborhoods of zero:

\[ N(\varepsilon, \delta) = \{ x \in S(M) : \exists e \in P(M), \tau(1-e) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon \}, \]

where \( \varepsilon, \delta \) are positive numbers. The pair \( (S(M), t_\tau) \) is a complete topological \(*\)-algebra.
We define the so-called rank metric $\rho$ on $S(\mathcal{M})$ by setting

$$\rho(x, y) = \tau(r(x - y)) = \tau(l(x - y)), \ x, y \in S(\mathcal{M}).$$

In fact, the rank-metric $\rho$ was firstly introduced in a general case of regular rings in \cite{12}, where it was shown it is a metric. By \cite{10} Proposition 2.1, the algebra $S(\mathcal{M})$ equipped with the metric $\rho$ is a complete topological $*$-ring.

Let $\mathcal{M}$ be a finite von Neumann algebra. A $*$-subalgebra $\mathcal{A}$ of $S(\mathcal{M})$ is said to be regular, if it is a regular ring in the sense of von Neumann, i.e., if for every $a \in \mathcal{A}$ there exists an element $b \in \mathcal{A}$ such that $aba = a$.

Given $a \in S(\mathcal{M})$ let $a = v|a|$ be the polar decomposition of $a$. Then $l(a) = vv^*$ and $r(a) = v^*v$ are left and right supports of the element $a$, respectively. The projection $s(a) = l(a) \lor r(a)$ is the support of the element $a$. It is clear that $r(a) = s(|a|)$ and $l(a) = s(|a^*|)$. There is a unique element $i(a)$ in $S(\mathcal{M})$ such that $ai(a) = l(a)$, $i(a)a = r(a)$, $ai(a)a = a$, $i(a)l(a) = i(a)$ and $r(a)i(a) = i(a)$. The element $i(a)$ is called the partial inverse of the element $a$. Therefore $S(\mathcal{M})$ is a regular $*$-algebra (see \cite{8}, \cite{21}).

Let $e \in S(\mathcal{M})$ be an idempotent, i.e., $e^2 = e$. Recall the following properties of the left projection \cite{7}:

$$l(e)e = e, \ el(e) = l(e). \quad (1)$$

It is clear that the left support $l(e)$ of the idempotent $e$ is uniquely determined by the above two equalities.

2.2. Noncommutative Arens algebras and noncommutative $L_{\log}$-algebras.

In this subsection we present two classes of $*$-subalgebras in $S(\mathcal{M})$ which satisfy the conditions of Theorem \ref{thm1.2}.

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace $\tau$. Given $p \geq 1$ denote by $L_p(\mathcal{M}, \tau)$ the set of all elements $x$ from $S(\mathcal{M})$ such that $\tau(|x|^p) < +\infty$. It is well-known that $L_p(\mathcal{M}, \tau)$ is a Banach space with respect to the norm

$$||x||_p = (\tau(|x|^p))^{1/p}, \ x \in L_p(\mathcal{M}, \tau).$$

The intersection

$$L^\omega(\mathcal{M}, \tau) = \bigcap_{p \geq 1} L_p(\mathcal{M}, \tau).$$

is a $*$-subalgebra in $S(\mathcal{M})$ \cite{13}. The algebra $L^\omega(\mathcal{M}, \tau)$ is called a noncommutative Arens algebra and it is a locally convex complete metrizable $*$-algebra with respect to the topology generated by the family of norms $\{|| \cdot ||_p\}_{p \geq 1}$ (see \cite{13}). Note that in the commutative (functional space) case the algebra $L^\omega[0, 1]$ was introduced by R. Arens in \cite{2}.

Let $L_{\log}(\mathcal{M}, \tau)$ be the set of all elements $x$ from $S(\mathcal{M})$ which satisfy

$$||x||_{\log} = \tau(\log(1 + |x|)) < +\infty.$$ 

It is known that \cite{11} Theorem 4.9 the pair $(L_{\log}(\mathcal{M}, \tau), || \cdot ||_{\log})$ is a topological $*$-algebra with respect to a complete metric space topology.

Note that by \cite{11} Proposition 4.7, it follows that the Arens algebra $L^\omega(\mathcal{M}, \tau)$ is $*$-subalgebra of $L_{\log}(\mathcal{M}, \tau)$. It clear that if $\tau$ is finite, then $\mathcal{M}$ is a $*$-subalgebra in both algebras $L^\omega(\mathcal{M}, \tau)$ and $L_{\log}(\mathcal{M}, \tau)$. It should be noted that $L_{\log}$-algebras were considered as certain invariant subspaces in order to obtain upper-triangular-type decompositions of unbounded operators (see \cite{12}).
3. Proof of the main result

Let $\mathcal{M}$ and $\mathcal{N}$ be arbitrary type II$_1$ von Neumann factors with faithful normal normalised traces $\tau_\mathcal{M}$ and $\tau_\mathcal{N}$, respectively and let $\Phi: \mathcal{A} \to \mathcal{B}$ be a ring isomorphism.

The following is a well-known result which is crucial in our construction. For convenience, we include the full proof.

By $\rho_\mathcal{M}$ we denote the rank-metric on $S(\mathcal{M})$.

Lemma 3.1. $\mathcal{M}$ is $\rho_\mathcal{M}$-dense in $S(\mathcal{M})$.

Proof. Let $x \in S(\mathcal{M})$ and let $x = v|x|$ be the polar decomposition of $x$. Consider the spectral resolution $|x| = \int_0^\infty \lambda d\varepsilon_\lambda$ of $|x|$ and let $e_n = e_{(0,n]}(|x|)$ be the spectral projection of $|x|$ corresponding to the interval $(0,n]$. Set $x_n = xe_n$, $n \in \mathbb{N}$. Since $|x|e_n \leq ne_n \in \mathcal{M}$, it follows that $x_n = v|x|e_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Further,

$$\rho_\mathcal{M}(x,x_n) = \tau(r(x-x_n)) = \tau(r(v|x|(1-e_n))) \leq \tau(1-e_n) \to 0,$$

because $e_n \uparrow 1$. This means that $x_n \xrightarrow{\rho_\mathcal{M}} x$, and therefore $\mathcal{M}$ is $\rho_\mathcal{M}$-dense in $S(\mathcal{M})$. \hfill \Box

In the proof of the next Lemma by $t_\mathcal{M}$ and $t_\mathcal{N}$ we denote the measure topologies on $S(\mathcal{M})$ and $S(\mathcal{N})$, respectively.

Lemma 3.2. $\Phi$ is continuous in the topology generated by the rank-metric.

Proof. Consider the mapping

$$p \in P(\mathcal{M}) \mapsto l(\Phi(p)) \in P(\mathcal{N}),$$

(2)

where $l(x)$ is the left support of the element $x$.

Let us show that this mapping is a lattice-isomorphism from $P(\mathcal{M})$ onto $P(\mathcal{N})$.

For $p, q \in P(\mathcal{M})$ with $p \leq q$ we have that

$$l(\Phi(p)) = l(\Phi(qp)) = l(\Phi(q)\Phi(p)) \leq l(\Phi(q)).$$

Let $p, q \in P(\mathcal{M})$ be projections such that $l(\Phi(p)) \leq l(\Phi(q))$. Then

$$\Phi(p) = l(\Phi(p)) \Phi(p) = l(\Phi(q)) l(\Phi(p)) \Phi(p) = \Phi(q)l(\Phi(q)) l(\Phi(p)) \Phi(p) \equiv \Phi(q)l(\Phi(q)) \Phi(p) = \Phi(qp).$$

Since $\Phi$ is a bijection, it follows that $p = qp$, i.e. $p \leq q$. In particular, if $l(\Phi(p)) = l(\Phi(q))$, where $p, q \in P(\mathcal{M})$, then $p = q$.

Let $f \in P(\mathcal{N})$. Take an element $x \in S(\mathcal{M})$ such that $\Phi(x) = f$. Then $x$ is an idempotent, and hence $x = e + w$, where $e = l(x)$ and $w \in eS(\mathcal{M})(1-e)$. We have that

$$\Phi(e) = \Phi(x) - \Phi(w) = f - \Phi(e)\Phi(w)(1-\Phi(e)).$$

Therefore

$$\Phi(e)f = f^2 - \Phi(e)\Phi(w)(1-\Phi(e))f = f - \Phi(e)\Phi(w)(1-\Phi(e))\Phi(x)$$

$$= f - \Phi(e)\Phi(w)(\Phi(x) - \Phi(ex)) = f - \Phi(e)\Phi(w)(\Phi(x) - \Phi(x)) = f.$$

Further,

$$f\Phi(e) = \Phi(x)\Phi(e) = \Phi(xe) = \Phi((e+w)e) = \Phi(e),$$

because $we = w(1-e)e = 0$. The above two equalities show $l(\Phi(e)) = f$. 
So, the mapping defined by (2) is an order-isomorphism from \( P(M) \) onto \( P(N) \). By [9, Page 24, Lemma 2], this mapping is a lattice-isomorphism from \( P(M) \) onto \( P(N) \), that is,

\[
l (\Phi(p \lor q)) = l (\Phi(p)) \lor l (\Phi(q)) \quad \text{and} \quad l (\Phi(p \land q)) = l (\Phi(p)) \land l (\Phi(q))
\]

for all \( p, q \in P(M) \).

Since \( S(M) \) and \( S(N) \) are regular rings containing the matrix ring over the field of complex numbers of order bigger than 3, by [20, Part II, Theorem 4.2], the lattice isomorphism of \( P(M) \) and \( P(N) \) defined as (2) is generated by a ring isomorphism \( \Theta \) from \( S(M) \) onto \( S(N) \), i.e., \( l(\Theta(p)) = l(\Phi(p)) \) for all \( p \in P(M) \). By [7, Theorem 1.3] the ring isomorphism \( \Theta \) is continuous in the measure topology.

Let \( x_n \xrightarrow{\rho_M} 0 \). Then \( \tau_M(l(x_n)) = \rho_M(x_n, 0) \to 0 \), and therefore \( l(x_n) \xrightarrow{\tau_M} 0 \). Since \( \Theta \) is continuous in the measure topology, it follows that \( \Theta(l(x_n)) \xrightarrow{\tau_N} 0 \). By [7, Lemma 2.2], it follows that \( l(\Theta(l(x_n))) \xrightarrow{\tau_N} 0 \), because \( \Theta(l(x_n)) \) is an idempotent for all \( n \). Further, we have that

\[
l (\Phi(x_n)) = l(\Phi(l(x_n)x_n)) = l(\Phi(l(x_n))\Phi(x_n)) \leq l(\Phi(l(x_n)))
\]

Thus \( \rho_N(\Phi(x_n), 0) = \tau_N(l(\Phi(x_n))) \to 0 \). This means that \( \Phi(x_n) \xrightarrow{\rho_N} 0 \).

**Lemma 3.3.** A ring isomorphism \( \Phi : A \to B \) extends to a ring isomorphism \( \widetilde{\Phi} \) from \( S(M) \) onto \( S(N) \).

**Proof.** By Lemma 3.1 the \( * \)-subalgebra \( M \) is \( \rho_M \)-dense in \( S(M) \), and therefore \( A \) is also \( \rho_M \)-dense in \( S(M) \). Using this observation we can define a mapping \( \widetilde{\Phi} \) from \( S(M) \) into \( S(N) \) as

\[
\widetilde{\Phi} = \rho_N - \lim_{n \to \infty} \Phi(x_n), \tag{3}
\]

where \( \{x_n\} \subset A \) is a sequence such that \( x_n \xrightarrow{\rho_M} x \).

Let us show the mapping \( \widetilde{\Phi} \) is a well-defined ring isomorphism.

Firstly, we shall show that this mapping is well-defined. Let \( \{x_n\} \subset A \) be a sequence such that \( x_n \xrightarrow{\rho_M} x \). Then \( x_n - x_m \xrightarrow{\rho_M} 0 \) as \( n, m \to \infty \). Since by Lemma 3.2 \( \Phi \) is continuous in the topology generated by the rank-metric, it follows that \( \Phi(x_n) - \Phi(x_m) \xrightarrow{\rho_N} 0 \) as \( n, m \to \infty \). Since \( S(N) \) is \( \rho_N \)-complete, it follows that there exists \( \rho_N - \lim_{n \to \infty} \Phi(x_n) \in S(N) \). So, the limit on the right side of (3) exists.

Now suppose that \( \{x_n\}, \{x'_n\} \subset A \) are sequences such that \( x_n \xrightarrow{\rho_M} x \) and \( x'_n \xrightarrow{\rho_M} x \). Then \( x_n - x'_n \xrightarrow{\rho_M} 0 \) as \( n \to \infty \), and again by \( \rho_M \)-\( \rho_N \)-continuity of \( \Phi \) we obtain that \( \Phi(x_n) - \Phi(x'_n) \xrightarrow{\rho_N} 0 \) as \( n \to \infty \). Thus

\[
\widetilde{\Phi}(x) = \rho_N - \lim_{n \to \infty} \Phi(x_n) = \rho_N - \lim_{n \to \infty} \Phi(x'_n).
\]

So, \( \widetilde{\Phi} \) is a well-defined mapping.

Let us show the additivity and multiplicity of \( \widetilde{\Phi} \). For \( x, y \in S(M) \) take sequences \( \{x_n\}, \{y_n\} \subset A \) such that \( x_n \xrightarrow{\rho_M} x \) and \( y_n \xrightarrow{\rho_M} y \). Then

\[
\widetilde{\Phi}(x) + \widetilde{\Phi}(y) = \lim_{n \to \infty} \Phi(x_n) + \lim_{n \to \infty} \Phi(y_n) = \lim_{n \to \infty} \Phi(x_n + y_n) = \widetilde{\Phi}(x + y).
\]

By a similar argument we get the multiplicativity of \( \widetilde{\Phi} \).
The next step is the proof of the surjectivity of $\tilde{\Phi}$.

Let $y \in S(\mathcal{N})$ and let $\{y_n\} \subset \mathcal{B}$ be a sequence such that $y_n \xrightarrow{\rho_N} y$. Then $y_n - y_m \xrightarrow{\rho_N} 0$ as $n, m \to \infty$. Since by Lemma 3.2, $\Phi^{-1}$ is $\rho_N$-continuous, it follows that $\Phi^{-1}(y_n) - \Phi^{-1}(y_m) \xrightarrow{\rho_M} 0$ as $n, m \to \infty$. Since $S(\mathcal{M})$ is $\rho_M$-complete, it follows that there exists $x = \rho_M - \lim_{n \to \infty} \Phi^{-1}(y_n) \in S(\mathcal{M})$. Then

$$
\tilde{\Phi}(x) = \rho_N - \lim_{n \to \infty} \Phi(\Phi^{-1}(y_n)) = \rho_N - \lim_{n \to \infty} y_n = y.
$$

The final step of the proof is the injectivity of $\tilde{\Phi}$.

Let $x \in S(\mathcal{M})$ and suppose that $\tilde{\Phi}(x) = 0$. Let $x = v|x|$ be the polar decomposition of $x$. Consider the spectral resolution $|x| = \int_0^\infty \lambda d\epsilon_\lambda$ of $|x|$ and let $e_n = e_{(0,n)}(|x|)$ be the spectral projection of $|x|$ corresponding to the interval $(0, n]$. Note that $xe_n \in \mathcal{M} \subset \mathcal{A}$ for all $n \in \mathbb{N}$. Further,

$$
0 = \tilde{\Phi}(x)e_n = \tilde{\Phi}(xe_n) = \Phi(xe_n).
$$

Since $\Phi$ is a ring isomorphism, it follows that $xe_n = 0$ for all $n \in \mathbb{N}$. From $e_n \uparrow 1$, we have that $xe_n x^* \uparrow xx^*$. Thus $xx^* = 0$, and hence $x = 0$. The proof is complete. \qed

Proof of Theorem 1.2 By Lemma 3.3 a ring isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ extends to a ring isomorphism $\tilde{\Phi}$ from $S(\mathcal{M})$ onto $S(\mathcal{N})$. Then by [7, Theorem 1.4] there exist an invertible element $a \in S(\mathcal{N})$ and a real $*$-isomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ (which extends to a real $*$-isomorphism from $S(\mathcal{M})$ onto $S(\mathcal{N})$) such that $\tilde{\Phi}(x) = a\Psi(x)a^{-1}$ for all $x \in S(\mathcal{M})$.

Let us first to show that $a \in \mathcal{B}$ and $a^{-1} \in \mathcal{B}$.

Let $a = v|a|$ be the polar decomposition of $a$. Since $a$ is invertible, it follows that $v$ is unitary. Since

$$
a\Psi(x)a^{-1} = v|a|v^*v\Psi(x)v^*v|a|^{-1}v^*,
$$

replacing, if necessary, $a$ to $v|a|v^*$ and $\Phi$ to $v\Psi(\cdot)v^*$, we can assume that $a$ is a positive invertible element in $S(\mathcal{N})$.

Consider the spectral resolution $a = \int_0^\infty \lambda d\epsilon_\lambda$ of $a$ and let $e_\lambda = e_{(0,\lambda]}(a)$ be the spectral projection of $a$ corresponding to the interval $(0, \lambda]$, $\lambda > 0$.

If $a \in \mathcal{N}$, then $a \in \mathcal{B}$, because $\mathcal{N} \subset \mathcal{B}$. So, we need to consider the case $a \in S(\mathcal{N}) \setminus \mathcal{N}$. Then there exists a positive number $\lambda$ such that $\tau_\mathcal{N}(1 - e_\lambda) \leq \frac{1}{2}$. Since $\tau_\mathcal{M}$ is a normalised trace, it follows that

$$
\tau_\mathcal{N}(e_\lambda) \geq \frac{1}{2} \geq \tau_\mathcal{N}(1 - e_\lambda).
$$

Since $\mathcal{N}$ is a von Neumann factor of type $\Pi_1$, it follows that $1 - e_\lambda \lessgtr e_\lambda$. Take a partial isometry $u \in \mathcal{N}$ such that $uu^* = 1 - e_\lambda$ and $u^*u \leq e_\lambda$. By the choice of the spectral projection $e_\lambda$ we obtain that

$$
ae_\lambda \leq \lambda e_\lambda.
$$

Further, let $w$ be an element in $\mathcal{A}$ such that $\Psi(w) = u$. Note that $w$ is also a partial isometry in $\mathcal{M} \subset \mathcal{A}$. We have that

$$
\Phi(w) = \tilde{\Phi}(w) = a\Psi(w)a^{-1} = aua^{-1} = a(uu^*)ua^{-1} = a(1 - e_\lambda)ua^{-1}.
$$
Multiplying the last equality from the right side by the element $ae_{\lambda}u^*$ we obtain
\[
\Phi(w)ae_{\lambda}u^* = (a(1 - e_{\lambda})uu^{-1})ae_{\lambda}u^* = a(1 - e_{\lambda})ue_{\lambda}u^*
\]
\[
= a(1 - e_{\lambda})uu^*e_{\lambda}u^* = a(1 - e_{\lambda})uu^*uu^* = a(1 - e_{\lambda}),
\]
because $uu^* = 1 - e_{\lambda}$ and $u^*u \leq e_{\lambda}$. Taking into account (4) and the inclusions $u^*, \Phi(w) \in \mathcal{B}$, from the last equality, we conclude that $a(1 - e_{\lambda}) \in \mathcal{B}$. Hence
\[
a = ae_{\lambda} + a(1 - e_{\lambda}) \in \mathcal{N} + \mathcal{B} \subset \mathcal{B}.
\]
By a similar argument we can show that $a^{-1} \in \mathcal{B}$.

Finally, we show that the restriction $\Psi|_{\mathcal{A}}$ of the real $*$-isomorphism $\Psi$ onto $\mathcal{A}$ maps $\mathcal{A}$ onto $\mathcal{B}$. Indeed, since $a, a^{-1} \in \mathcal{B}$, it follows that
\[
\Psi(x) = a^{-1}\Phi(x)a = a^{-1}\Phi(x)a \in \mathcal{B}
\]
for all $x \in \mathcal{A}$. Further, considering the inverse map $\Phi^{-1}$ which acts as
\[
\Phi^{-1}(y) = \Psi^{-1}(a^{-1}) \Psi^{-1}(x)\Psi^{-1}(a), \ y \in \mathcal{B},
\]
we conclude that both $\Psi^{-1}(a)$ and $\Phi^{-1}(a)^{-1}$ are in $\mathcal{A}$, and that $\Psi^{-1}$ maps $\mathcal{B}$ onto $\mathcal{A}$. So, $\Psi(\mathcal{A}) = \mathcal{B}$. The proof is complete. \hfill \Box

**Corollary 3.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-subalgebras from Theorem 3.2. Suppose that these algebras are equipped with metrics $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ respectively, such that both $(\mathcal{A}, \rho_{\mathcal{A}})$, $(\mathcal{B}, \rho_{\mathcal{B}})$ are complete topological $*$-algebras and additionally, convergence with their metrics implies the convergence in measure. Then any ring isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is $\rho_{\mathcal{A}}$-$\rho_{\mathcal{B}}$-continuous.

**Proof.** Take a sequence $\{x_n\} \subset \mathcal{A}$ such that $x_n \xrightarrow{\rho_{\mathcal{A}}} 0$ and $\Phi(x_n) \xrightarrow{\rho_{\mathcal{B}}} y \in \mathcal{B}$; in particular, $\Phi(x_n) \rightarrow y$ in the measure topology in $S(\mathcal{N})$. Since $\rho_{\mathcal{A}}$-convergence implies the convergence in measure in $S(\mathcal{M})$, it follows that $x_n \rightarrow 0$ in the measure topology. Further, the continuity of $\Phi$ in the measure topology implies that $\Phi(x_n) \rightarrow 0$ in the measure topology in $S(\mathcal{N})$. Thus $y = 0$, and hence by the closed graph theorem (see [27, Page 79]), we conclude that $\Phi$ is $\rho_{\mathcal{A}}$-$\rho_{\mathcal{B}}$-continuous. \hfill \Box

**Remark 3.5.** Note that the noncommutative Arens algebras and noncommutative $L_{log}$-algebras associated with von Neumann factors of type $II_1$ satisfy the conditions of Corollary 3.4. Indeed, in the following series identical imbeddings are continuous
\[
(L_{\omega}(\mathcal{M}, \tau), \{|| \cdot ||_{p,1} \}_{p \geq 1}) \subset (L_1(\mathcal{M}, \tau), || \cdot ||_1) \subset (L_{log}(\mathcal{M}, \tau), || \cdot ||_{log}) \subset (S(\mathcal{M}), t_\tau).
\]
The continuity of the first imbedding immediately follows from the definition, the continuity of the second and the third imbeddings follow from [11 Proposition 4.7 and Remark 4.8].

4. **DISCONTINUOUS ALGEBRA AUTOMORPHISM OF A $*$-REGULAR ALGEBRA**

In this Section we show that there is a $*$-regular subalgebra of algebra of all measurable operators with respect to the hyperfinite factor of type $II_1$ which admits an algebra automorphism, discontinuous in the measure topology.

Let $\mathcal{R}$ be the hyperfinite $II_1$-factor with the faithful normal normalised trace $\tau$. There is a system of matrix units $\mathcal{E} = \{e_{ij}^{(n)} : n = 0, 1, \ldots, i, j = 1, \ldots, 2^n\}$ in $\mathcal{R}$ (here $e_{1,1}^{(0)} = 1$) such that [26]
\[
(a) \ e_{ij}^{(n)} e_{k,l}^{(n)} = \delta_{jk} e_{il}^{(n)};
\]
Lemma 4.2.

Proof. Since $\Phi : R_{\infty} \rightarrow R_{\infty}$ is a *-regular algebra as a sum of increasing sequence of matrix algebras (see e.g. 24, Theorem 3).

Now we begin to construct a discontinuous algebra automorphism of $R_{\infty}$.

Define the sequences $\{a_n : n = 1, 2, \ldots\}$ and $\{c_n : n = 1, 2, \ldots\} (a_n, c_n \in R_n)$ by the rule

$$c_n = 2^n \sum_{k=1}^{2^n-1} e_{2k-1,2k-1}^{(n)} + \sum_{k=1}^{2^n-1} e_{2k,2k}^{(n)}, n \in \mathbb{N}$$

and

$$a_n = \prod_{k=1}^{n} c_k, n \in \mathbb{N}.$$ 

Note that all $a_n, c_n$ are invertible in $R_n$.

For $n \geq 1$ define an algebra automorphism $\Phi_n$ of $R_n$ as follows

$$\Phi_n(x) = a_n x a_n^{-1}, x \in R_n.$$ 

Lemma 4.1. $\Phi_n|_{R_{n-1}} = \Phi_{n-1}$ for all $n \in \mathbb{N}$.

Proof. Since $a_{n-1}^{-1} a_n = a_n a_{n-1}^{-1} = c_n$, it suffices to show that

$$[c_n, R_{n-1}] = 0.$$ 

Using the property (d) of matrix units, for fixed $1 \leq i, j \leq 2^{n-1}$ we have that

$$\begin{bmatrix} c_n, e_{i,j}^{(n-1)} \end{bmatrix} (d) = 2^n \sum_{k=1}^{2^{n-1}} e_{2k-1,2k}^{(n)} + \sum_{k=1}^{2^{n-1}} e_{2k,2k}^{(n)}, e_{2i-1,2j-1}^{(n)} + e_{2i,2j}^{(n)}$$

$$= 2^n \sum_{k=1}^{2^{n-1}} e_{2k-1,2k}^{(n)} + e_{2i-1,2j-1}^{(n)} + e_{2i,2j}^{(n)} + e_{2i,2j}^{(n)} = 0.$$ 

Lemma 4.2. There exists an algebra isomorphism $\Phi : R_{\infty} \rightarrow R_{\infty}$ such that $\Phi|_{R_n} = \Phi_n$ for all $n = 1, 2, \ldots$. 

Note that all $a_n, c_n$ are invertible in $R_n$. For $n \geq 1$ define an algebra automorphism $\Phi_n$ of $R_n$ as follows

$$\Phi_n(x) = a_n x a_n^{-1}, x \in R_n.$$

Lemma 4.1. $\Phi_n|_{R_{n-1}} = \Phi_{n-1}$ for all $n \in \mathbb{N}$. 

Proof. Since $a_{n-1}^{-1} a_n = a_n a_{n-1}^{-1} = c_n$, it suffices to show that

$$[c_n, R_{n-1}] = 0.$$ 

Using the property (d) of matrix units, for fixed $1 \leq i, j \leq 2^{n-1}$ we have that
Proof. Define the mapping $\Phi : \bigcup_{n=1}^{\infty} R_n \to \bigcup_{n=1}^{\infty} R_n$ by setting $\Phi|_{R_n} = \Phi_n$. By Lemma 4.1 we have $\Phi_n|_{R_{n-1}} = \Phi_{n-1}$, and therefore, $\Phi$ is a well-defined mapping. It is clear that $\Phi$ is an algebra automorphism of $R_\infty$. \hfill \Box

Lemma 4.3. For each $n \geq 1$ the element $a_n$ can be represented as

$$a_n = \sum_{k=1}^{2^n} \gamma_k^{(n)} e_{k,k}$$

where

$$\gamma_{2k-1}^{(n)} = 2^n \gamma_{2k}^{(n)}$$

for all $n \geq 1$ and $k = 1, \ldots, 2^{n-1}$.

Proof. The proof is by the induction on $n$. For $n = 1$ we have that

$$a_1 = 2^1 e_{1,1}^{(1)} + e_{2,2}^{(1)}$$

and therefore $\gamma_1^{(1)} = 2 \gamma_2^{(1)}$. Suppose that we have proved the required assertion for $n - 1$. Taking into account that $a_n = a_{n-1}c_n$ and the equality $e_{k,k}^{(n-1)} = e_{2k-1,2k-1}^{(n)} + e_{2k,2k}^{(n)}$ we have that

$$a_n = a_{n-1}c_n = \sum_{l=1}^{2^{n-1}} \gamma_l^{(n-1)} e_{l,l}^{(n-1)} \left(2^n \sum_{k=1}^{2^{n-1}} e_{2k-1,2k-1}^{(n)} + \sum_{k=1}^{2^{n-1}} e_{2k,2k}^{(n)} \right)$$

$$= \left(\sum_{l=1}^{2^n} \gamma_l^{(n-1)} e_{2l-1,2l-1}^{(n)} + \sum_{l=1}^{2^n} \gamma_l^{(n-1)} e_{2l,2l}^{(n)} \right) \left(2^n \sum_{k=1}^{2^{n-1}} e_{2k-1,2k-1}^{(n)} + \sum_{k=1}^{2^{n-1}} e_{2k,2k}^{(n)} \right)$$

$$= \sum_{k=1}^{2^n} \left(2^n \gamma_k^{(n-1)} e_{2k-1,2k-1}^{(n)} + \gamma_k^{(n-1)} e_{2k,2k}^{(n)} \right).$$

Thus

$$a_n = \sum_{k=1}^{2^n} \gamma_k^{(n)} e_{k,k}^{(n)}$$

where

$$\gamma_{2k-1}^{(n)} = 2^n \gamma_{2k}^{(n-1)}, \quad \gamma_{2k}^{(n)} = \gamma_k^{(n-1)}.$$

Hence $\gamma_{2k-1}^{(n)} = 2^n \gamma_{2k}^{(n)}$ for all $k = 1, \ldots, 2^{n-1}$. \hfill \Box

Lemma 4.4. The algebra isomorphism $\Phi$ is discontinuous in the measure topology.

Proof. For each $n \geq 2$ take a partial isometry

$$v_n = \sum_{i=1}^{2^{n-1}} e_{2i-1,2i}^{(n)}.$$
By (5) we have that \( a_n^{-1} = \sum_{j=1}^{2^n} \frac{1}{\gamma_j} e_{j,j}^{(n)} \). Using the last equality we obtain that

\[
\Phi(v_n) = \Phi_n \left( \sum_{i=1}^{2^n-1} e_{2^{i-1},2i}^{(n)} \right) = \sum_{k=1}^{2^n} \gamma_k^{(n)} e_{k,k}^{(n)} \sum_{i=1}^{2^n-1} e_{2^{i-1},2i}^{(n)} \sum_{j=1}^{2^n} \frac{1}{\gamma_j} e_{j,j}^{(n)}
\]

\[
= \sum_{i=1}^{2^n-1} \frac{\gamma_{2i-1}^{(n)}}{\gamma_{2i}^{(n)}} e_{2i-1,2i}^{(n)} = 2^n \sum_{i=1}^{2^n-1} e_{2^{i-1},2i}^{(n)} = 2^n v_n,
\]

and therefore

\[
|\Phi(v_n)| = 2^n \sum_{i=1}^{2^n} e_{2i,2i}^{(n)}. \tag{7}
\]

Since \(||v_n||_M = 1\) for all \(n\), it follows that \(2^{-n}v_n \rightarrow 0\) in measure. But (7) show that the sequence \(\{\Phi\left(2^{-n}v_n\right)\}\) does not converge to zero in measure, because

\[
\tau\left(l(\Phi(v_n))\right) = \tau\left(\sum_{i=1}^{2^n} e_{2^{i-1},2i}^{(n)}\right) = \frac{1}{2}.
\]

for all \(n \in \mathbb{N}\). This means that \(\Phi\) is discontinuous in the measure topology. \(\square\)

So, we have proved the following result.

**Theorem 4.5.** The algebra \(R_\infty\) admits an algebra automorphism, which is discontinuous in the measure topology.

**Remark 4.6.** It is clear that Theorem 1.2 is an extension of [7, Theorem 1.4] and they provide the automatic continuity of ring isomorphisms in the measure topology. Thus the above Theorem 4.5 shows that the condition \(M \subset A\) is essential in both of these theorems.

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