Topological and holonomic quantum computation based on second-order topological superconductors

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Majorana fermions feature non-Abelian exchange statistics and promise fascinating applications in topological quantum computation. Recently, second-order topological superconductors (SOTSs) have been discovered to host Majorana fermions as localized quasiparticles with zero excitation energy, pointing out a new avenue to facilitate topological quantum computation. We provide a minimal model for SOTSs and systematically analyze the features of Majorana zero modes with analytical and numerical methods. We further construct the fundamental fusion principles of zero modes stemming from a single or multiple SOTS islands. Finally, we propose concrete schemes in different setups formed by SOTSs, enabling us to exchange and fuse the zero modes for non-Abelian braiding and holonomic quantum gate operations. We therefore show that universal quantum computing is possible in hardware formed by SOTSs.

I. INTRODUCTION

Majorana fermions are self-conjugate fermions\(^1\). They can arise as zero-energy Bogoliubov quasiparticles in condensed matter\(^2\)-\(^5\), such as vortex bound states in \(p\)-wave superconductors\(^6\),\(^7\), Majorana bound states in Josephson junctions\(^8\),\(^9\), and end states of nanowires with Rashba spin-orbit coupling or of ferromagnetic atomic chains\(^10\)-\(^15\). These bound states have zero excitation energy and are commonly coined Majorana zero modes (MZMs). MZMs are essentially one half of ordinary complex fermions and always come in pairs. When more than two MZMs are present, the braiding (exchange) operations on them correspond to non-Abelian rotations in the ground-state manifold spanned by them. They can thus serve as basic building blocks for topological quantum computation\(^3\),\(^7\),\(^16\)-\(^18\). If fusions between MZMs are adiabatically tunable, then they could also be exploited for holonomic quantum gates\(^19\),\(^20\). Hence, how to nucleate, fuse and braid MZMs in solid-state systems is one of the main focuses in modern condensed matter physics and quantum computer science.

Recently, second-order topological superconductors (SOTSs) have been discovered in various candidate systems and predicted to host localized MZMs in two dimensions lower than the gapped bulk\(^21\)-\(^43\). This opens up a new avenue towards Majorana-mediated topological quantum computation. Preliminary attempts have been made in this direction\(^13\)-\(^45\), which are, however, limited to only two MZMs. A comprehensive study of creating, fusing and braiding MZMs in SOTSs is still lacking. Importantly, the fusion and braiding of more than two MZMs are essential for a successful implementation of (topological) quantum gates\(^17\),\(^46\).

In this article, we show for the first time that the desired non-Abelian braiding operations of MZMs as well as a full set of holonomic gates can be achieved in the SOTS platform. To elucidate this, we provide a minimal model for SOTSs and discuss the features and behaviors of individual MZMs in a disk geometry, both analytically and numerically. We find that a finite chemical potential breaks an effective mirror symmetry and thus gives rise to tunable spin polarizations of the MZMs. Interestingly, the positions of the MZMs can be controlled by chemical potential and applied in-plane magnetic field.

Figure 1. Schematics of the SOTS-based setups for braiding (a) two, (b) four, and (c) more MZMs. The setups are scalable in a straightforward way. \(B_i\) with \(i = 1, 2, \ldots\) are in-plane magnetic fields applied to the triangle islands. The blue dots indicate the positions of the MZMs.
We systematically analyze the tunneling interaction between adjacent MZMs stemming from a single or multiple SOTS islands and identify the fundamental fusion principles between them. As an illustration of these principles, we demonstrate how to manipulate the fusion of MZMs in an incomplete disk by tuning magnetic field and chemical potential.

We put forward a number of striking setups formed by SOTSs, as sketched in Fig. 1, and present in detail corresponding schemes for braiding the MZMs based on variations of chemical potential, applied magnetic field, and geometry engineering. In a simple triangle setup, we can exchange two MZMs located at the vertices by rotating the magnetic field. In a trijunction setup constructed by three triangle islands, we are able to exchange any two of four MZMs by alternately rotating the magnetic fields applied to the islands. We could extend our theory to a ladder structure which is formed by elementary triangle islands and hosts any number of MZM pairs for braiding performance, see Fig. 1(c). Hence, our proposal is scalable in a straightforward way. Moreover, we propose a shamrock-like trijunction setup constituted by three incomplete disks. This trijunction hosts three MZMs that fuse exclusively with a fourth one. The fusion strengths are smoothly adjustable providing a feasible platform for holonomic quantum gates. Our schemes could be tested are smoothly adjustable providing a feasible platform for
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The article is organized as follows. In Sec. II, we introduce the minimal model for SOTSs, derive an effective boundary Hamiltonian and the wavefunctions and spin polarizations of MZMs. Next, we analyze the fusion properties of the MZMs in Sec. III. We proceed to describe the setups and schemes for braiding two or more MZMs, and discuss the important relevant physics in Sec. IV. We devote Sec. V to our proposal of holonomic gates. Finally, we discuss the experimental implementation and measurement, and summarize the results in Sec. VI.

II. MODEL HAMILTONIANS AND MAJORANA ZERO MODES

A. Effective boundary Hamiltonian

We start with considering simple two-dimensional SOTSs which is achieved in QSHIs in presence of superconductivity and moderate in-plane magnetic fields. The minimal model for the SOTSs can be written as

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{ex}} \]

where

\[ \mathcal{H}_0 = m(k)\tau_x \sigma_z + v \sin k_x s_x \sigma_x + v \sin k_y \tau_y \sigma_y - \mu \tau_z, \]

\[ \mathcal{H}_{\text{ex}} = \Delta(k) \tau_y s_y + g \mu B (\cos \theta \tau_x s_x + \sin \theta \sigma_0 \sigma_z), \]

and \[ m(k) = m_0 - 2m(2 - \cos k_x - \cos k_y). \]

Without loss of generality, we assume positive parameters \( v, m \) and \( m_0 \), and set the lattice constant \( a \) to unity. \( s, \sigma \) and \( \tau \) are Pauli matrices acting on spin, orbital, and Nambu spaces, respectively. \( \mu \) is the chemical potential. The pairing interaction can be s-wave or \( s_\pm \)-wave. However, the exact form of \( \Delta(k) \) is not important for our main results. We take it as a constant \( \Delta(k) = \Delta_0 > 0 \) for simplicity. \( B \equiv B(\cos \theta, \sin \theta) \) is the magnetic field with strength \( B \) and direction \( \theta \). The effective g-factors of the two orbitals are the same in the \( x \) direction and opposite in the \( y \) direction. This model could effectively describe QSHIs, e.g., an inverted HgTe quantum well with proximity-induced superconductivity, or a monolayer FeTe\(_{1-x}\)Se\(_x\) under in-plane magnetic fields.

The bulk model (1), in general, has only one crystalline symmetry, inversion symmetry. The existence of MZMs is not restricted to any specific geometry. Moreover, the MZMs appear as low-energy quasiparticles at open boundaries. Thus, to explore the MZMs, we start from the \( k \cdot \mathbf{p} \) limit of the model (1) and consider the SOTS in a large disk geometry. In the absence of \( B \), this low-energy model respects an emergent in-plane rotational symmetry. We first derive the boundary states of \( \mathcal{H}_0 \) which is decoupled into four blocks representing Dirac Hamiltonians. In the disk, the angular momentum \( v \) is a good quantum number. It is thus convenient to work in polar coordinates: \( r = \sqrt{x^2 + y^2} \) and \( \varphi = \arctan(y/x) \). The application of \( B \) will break this symmetry. However, in a large disk (with radius \( R \gg m/v \)), we can approximate a small segment of boundary at an arbitrary angle \( \varphi \) as a straight line. Define an effective coordinate \( s \equiv R \varphi \) along the segment and treat the corresponding momentum \( p_s \equiv v/R \) as a quasi-good quantum number. Then, the energy bands of the boundary states of the four blocks can be derived as

\[ E_{e,\pm}(p_s) = \mp vp_s - \mu, \]

\[ E_{h,\pm}(p_s) = \mp vp_s + \mu, \]

respectively. The boundary states are helical with velocity \( v \). Correspondingly, the wavefunctions can be written as

\[ \Psi_{e,\pm,p_s} = e^{ip_s \varphi} K(r)(1, -ie^{ip_s \varphi}, 0, 0, 0, 0, 0, 0)^T, \]

\[ \Psi_{e\pm,p_s} = is_y \Psi_{e,\pm,p_s}^* \text{ and } \Psi_{h,\pm,p_s} = \tau_2 \Psi_{e,\pm,p_s}, \]

where \( K(r) = N e^{ip_s \varphi} [e^{\lambda_1 (r-R)} - e^{\lambda_2 (r-R)}], \lambda_1/2 = v/2m \pm [(v/2m)^2 - m_0/m + p_s^2]^{1/2} \) and \( N \) is the normalization factor. We provide more details of this derivation in the Supplemental Materials.

With \( (\Psi_{e\pm,p_s}, \Psi_{e\pm,p_s}^*, \Psi_{h\pm,p_s}, \Psi_{h\pm,p_s}) \) as the basis, we next project the full Hamiltonian onto the boundary states. The resulting effective Hamiltonian on the boundary can be written as

\[ \tilde{\mathcal{H}} = \begin{pmatrix} -vp_s - \mu & ie^{-i\varphi} B & 0 & -\Delta_0 \\ -ie^{i\varphi} B & vp_s - \mu & \Delta_0 & 0 \\ 0 & -\Delta_0 & -vp_s + \mu & ie^{i\varphi} B \\ -\Delta_0 & 0 & -ie^{-i\varphi} B & vp_s + \mu \end{pmatrix}, \]
where $\tilde{B} = B \sin(\varphi - \theta)$ and $\gamma_B = 1$ has been chosen for convenience. The effective pairing potential $\Delta_0$ felt by the boundary states is constant and independent of the boundary orientation. It couples electrons to holes with opposite spins and preserves time-reversal and in-plane rotational symmetries. Thus, the energy spectrum is fully gapped in the absence of $B$. This implies that the pairing interaction alone cannot lead to a second-order topological phenomenon. The scenario becomes different when we turn on $B$. The magnetic field couples states with opposite spins and breaks the rotational symmetry. The effective magnetic field $\tilde{B}$ depends substantially on the angular position $\varphi$ (or equivalently, the boundary orientation which is parallel to the azimuthal direction at $\varphi$). Consequently, interesting physics including controllable MZMs arise, which we discuss in detail below.

**B. MZMs and their positions**

When $\mu = 0$, through a unitary transformation $U(\varphi) = e^{-i\pi \tau_z/4} e^{i \varphi \tau_x s_z/2}$, where $s$ and $\tau$ are Pauli matrices acting on spin and Nambu spaces of boundary states, respectively, Eq. (4) can be brought into block-diagonal form

$$U(\varphi) \tilde{H}(\varphi) U^{-1}(\varphi) = h_u \oplus h_d,$$

where $h_u/d = -\nu p_s s_z + (\tilde{B} \pm \Delta_0) s_y$. The two blocks $h_u/d$ are essentially one-dimensional Dirac Hamiltonians with the masses given by $\tilde{B} \pm \Delta_0$, respectively. They can be connected not only by inversion $\mathcal{P}$ but also by an effective mirror symmetry $\mathcal{M}$ with the mirror line in the field direction. Note that $\mathcal{P}$ and $\mathcal{M}$ act nonlocally on the boundary model, i.e., $\tilde{P} \tilde{H}(p_v, \varphi) \tilde{P}^{-1} = \tilde{H}(p_v, \varphi + \pi)$ and $\tilde{M} \tilde{H}(p_v, \varphi - \theta) \tilde{M}^{-1} = \tilde{H}(p_v, -\theta - \varphi)$, where the overhead tilde (‘) indicates boundary-state space. For $h_d$, the inclusion of $B > \Delta_0$ changes the sign of the Dirac mass at $\varphi_1 = \theta + \arcsin(\Delta_0/B)$ and $\varphi_2 = \theta - \arcsin(\Delta_0/B) + \pi$. Thus, two localized Majorana states with zero energy appear, which we label as $\gamma_1$ and $\gamma_2$. If inversion symmetry is present, then another pair of MZMs, labeled as $\gamma_3$ and $\gamma_4$, can be found from $h_u$. They are located at the positions different from $\varphi_1$ and $\varphi_2$ by an angle $\pi$.

When $\mu \neq 0$, the transformed Hamiltonian (5) is no longer block diagonal. Nevertheless, the four bands of the Hamiltonian can still be analytically derived:

$$E = \pm \sqrt{\tilde{B}^2 + A^2 p_v^2 + \Delta^2} \pm 2 \sqrt{\tilde{B}^2 \Delta^2 + A^2 \mu^2 p_v^2},$$

where $\Delta = \sqrt{\Delta_0^2 + \mu^2}$. Increasing $|\tilde{B}|$ from 0 to a value larger than $\Delta$, we observe band inversions happening at $p_v = 0$. Suppose that $B$ is sufficiently large, $B^2 > \Delta^2$, then, due to the oscillatory behavior of $\tilde{B}$, the band order changes when moving along the disk boundary. This indicates the appearance of MZMs. The positions of the MZMs are determined by the closing points of the bands.

Thus, the positions $\varphi_i$ of MZMs $\gamma_i$ (with $i \in \{1, 2, 3, 4\}$) are generally given by

$$\varphi_{1/4} = \theta \pm \arcsin(\Delta/B),$$

$$\varphi_{2/3} = \theta \mp \arcsin(\Delta/B) + \pi,$$

where $0 \leq \varphi_1 - \theta < \pi/2$. Similar to the MZMs in the $\mu = 0$ limit, $\gamma_1$ ($\gamma_2$) and $\gamma_3$ ($\gamma_4$) are always separated by an angle $\pi$. They are related to each other by inversion symmetry. In contrast, the separations between the neighboring MZMs $\gamma_1$ ($\gamma_2$) and $\gamma_4$ ($\gamma_3$) and $\gamma_1$ ($\gamma_3$) and

![Figure 2. Majorana zero modes in a disk geometry.](image-url)
Table I. Wavefunctions and spin polarizations of MZMs in an SOTS island. The basis for the wavefunctions is the same as the one of the model (1). The factors $F_{\pm}$ account for the spatial distribution and normalization of the wavefunctions. The magnitude of the spin polarizations $S^{(e/h)}_{\psi_i} \equiv (S^{(e/h)}_x, S^{(e/h)}_y, S^{(e/h)}_z)$ is given by $S_0 = \hbar \mu/(4\Delta)$. The subscripts $(e/h)$ stand for the electron and hole parts, respectively. The spin polarizations of the electron and hole parts are opposite, $S^{(h)}_{\psi_i} = -S^{(e)}_{\psi_i}$, as required by the Majorana nature of $\gamma_i$.

$\gamma_2$ ($\gamma_4$) are respectively given by

$$\varphi_{s1} = 2\arcsin(\Delta/B), \quad \text{and} \quad \varphi_{s2} = \pi - \varphi_{s1}. \quad (8)$$

These separations are independent of the field direction $\theta$. However, they are controllable by chemical potential $\mu$ and field strength $B$, via the ratio $\Delta/B$. Increasing $B$ or decreasing $\mu$, $\varphi_{s1}$ is monotonically increased whereas $\varphi_{s2}$ is decreased, as shown in Fig. 2(e).

Interestingly, the positions of $\varphi_i$ depend not only on the ratio $\Delta/B$ but also on the field direction, according to Eq. (7). When rotating $\mathbf{B}$, the MZMs $\psi_i$ move around the disk boundary. To confirm these features, we employ the tight-binding model (1) and define the disk by $i_x^2 + i_y^2 \leq N_r^2$, where $N_r$ is the radius of the disk and $i_x, i_y$ label the lattice sites in the $x$ and $y$ directions, respectively. We use the Kwant package to plot the wavefunctions. For a large magnetic field $B > \Delta$ applied in the $x$ direction, we clearly identify four MZMs centered at the symmetric positions obeying $\varphi_1 = -\varphi_4$ and $\varphi_{2/3} = \varphi_{2/3} + \pi$, see Fig. 2(a). When rotating $\mathbf{B}$ from $x$ to $-x$ direction, all the MZMs are moving anticlockwise. However, their separations are unchanged, see Fig. 2(a)–(d). These observations perfectly agree with our analytical results. Moreover, we find that although the positions of the MZMs are quite different for different $\theta$, the energy spectra of the disks are almost the same. There is always an energy gap protecting the MZMs from excited modes, as long as the MZMs are well separated.

### C. Wavefunctions and spin polarizations of MZMs

With the help of the boundary Hamiltonian (4), the wavefunctions of the MZMs $\psi_i$ (with $i \in \{1, 2, 3, 4\}$) can be analytically derived. We summarize the results in Table I and provide the detailed derivations in the Supplemental Material. In the table, the factors $F_{\pm} = $ $N e^{i \int s \xi(s) ds'}$ with $\xi = \Delta_0 - (\mathcal{B}^2 - \mu^2)^{1/2}$ account for the spatial distribution and normalization of the wavefunctions $\psi_i$. These factors indicate that the wavefunctions decay exponentially away from the corresponding centers $\varphi_i$.

| Position $\varphi_i$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ |
|---------------------|------------|------------|------------|------------|
| $\theta + \arcsin(\Delta/B)$ | $\theta - \arcsin(\Delta/B) + \pi$ | $\theta + \arcsin(\Delta/B) + \pi$ | $\theta - \arcsin(\Delta/B) + \pi$ |
| $\mathbf{F}_+$ | $e^{-(i\varphi_1 - \theta + \pi)/2}$ | $e^{-(i\varphi_2 + \theta + \pi)/2}$ | $e^{-(i\varphi_3 + \theta + \pi)/2}$ | $e^{-(i\varphi_4 + \theta + \pi)/2}$ |
| $\mathbf{F}_-$ | $e^{-(i\varphi_1 + \theta + \pi)/2}$ | $e^{-(i\varphi_2 - \theta + \pi)/2}$ | $e^{-(i\varphi_3 - \theta + \pi)/2}$ | $e^{-(i\varphi_4 - \theta + \pi)/2}$ |

| Polarization $S^{(e/h)}_{\psi_i}$ | $\pm S_0(\sin \varphi_1 - \cos \varphi_1, 0)$ | $\pm S_0(\sin \varphi_2 - \cos \varphi_2, 0)$ | $\pm S_0(-\sin \varphi_3, \cos \varphi_3, 0)$ | $\pm S_0(-\sin \varphi_4, \cos \varphi_4, 0)$ |

Let us now look at the spin polarizations of the MZMs. When $\mu = 0$, the disk system respects the effective mirror symmetry, as mentioned before. Then, the MZMs are spinless. However, in the presence of $\mu \neq 0$, the mirror symmetry is no longer preserved. Hence, the MZMs acquire finite spin polarizations in the $x$-$y$ plane, see Table I. The polarization magnitudes are given by $h \mu/(4\Delta)$. In the limit $|\mu| \gg \Delta_0$, the MZMs become fully polarized. The polarization directions depend on $\varphi_i$. They always point parallel or anti-parallel to the boundary orientation, as illustrated in Fig. 2(f). Thus, when moving the MZMs, their spin polarizations rotate. Since $\gamma_1$ and $\gamma_3$ are separated by $\pi$, they have the same polarization. Similarly, $\gamma_2$ and $\gamma_4$ have also an identical polarization. Note that the spin polarizations of $\gamma_i$ could be detected by magnetic tunneling experiments.
III. FUSION BETWEEN MZMS

We proceed to discuss the fusion between the MZMs. Let us first consider the disk geometry with vanishing $\mu$. Two adjacent MZMs can be brought close to each other by tuning $\mu$ or $B$, according to Eq. (8). If the two MZMs belong to the same block of Eq. (5), e.g., $\gamma_1$ and $\gamma_2$ (or $\gamma_3$ and $\gamma_4$), then they fuse with each other and acquire a hybridization energy which depends exponentially on their distance in space. In contrast, $h_u$ and $h_d$ do not interact with each other due to the effective mirror symmetry, $\gamma_1/2$ stemming from $h_u$ cannot fuse with $\gamma_3/4$ which stem from the other block $h_d$, even if they sit at the same position and have strong overlap in wavefunction.

A finite $\mu$, however, couples the two blocks. Therefore, the fusion between $\gamma_{1/4}$ ($\gamma_{2/3}$) is allowed. In the SOTSs, the fusion of MZMs is realized by the hopping interaction which depends on the momentum-orbital coupling. According to Eq. (1), the corresponding operators in the $x$ and $y$ directions can be found as $\hat{T}_x = i v s_x \sigma_x/2 + m r_z \sigma_z$ and $\hat{T}_y = i v t_y \sigma_y/2 + m r_z \sigma_z$, respectively. Thus, the fusion (also called coupling) strength between two MZMs, say $\gamma_i$ and $\gamma_j$ (with $i, j \in \{1, 2, 3, 4\}$), can be calculated as $\langle \Psi_i | (\hat{T}_x + \hat{T}_y) | \Psi_j \rangle$. We summarize the results in Table II. In a homogeneous system, the fusion between $\gamma_{1/2}$ ($\gamma_{3/4}$) is proportional to $\cos \theta$, while the one between $\gamma_{1/4}$ ($\gamma_{2/3}$) is linear in $\sin \theta$, where $\theta = \arctan(\mu/\Delta_0)$. The proportionality is determined by the overlap of the wavefunctions of the two involved MZMs. In contrast, in a single SOTS island, $\gamma_1$ and $\gamma_3$ ($\gamma_2$ and $\gamma_4$) are well separated in space. Moreover, they are related by inversion symmetry. Notice that $\hat{T}_x$ and $\hat{T}_y$ anti-commute with the inversion operator $P$. The fusion between $\gamma_{1/3}$ ($\gamma_{2/4}$) is prohibited even in the presence of finite $\mu$. These results are generic and also apply to the case where the two MZMs belong to different connected SOTS islands with the same pairing phase. However, if two islands have a pairing phase difference $2\Phi \neq 2\pi n$ with $n \in \{0, \pm 1, \ldots\}$, then $\gamma_1$ ($\gamma_2$) stemming from one island can also fuse with $\gamma_3$ ($\gamma_4$) from the other island, see Table II. The fusion induced by a phase difference can be exploited to realize the braiding of more than two MZMs, which we demonstrate below.

In a full disk, all angular positions $\varphi$ are available. Thus, the four MZMs related by inversion symmetry are:

| $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ |
|-----------|-----------|-----------|-----------|
| $\gamma'_1$ | $\times$ | $\cos \theta \cos \Phi$ | $\sin \Phi$ | $\sin \theta \cos \Phi$ |
| $\gamma'_2$ | $\cos \theta \cos \Phi$ | $\times$ | $\sin \theta \cos \Phi$ | $\sin \Phi$ |
| $\gamma'_3$ | $\sin \Phi$ | $\sin \theta \cos \Phi$ | $\times$ | $\cos \theta \cos \Phi$ |
| $\gamma'_4$ | $\sin \theta \cos \Phi$ | $\sin \Phi$ | $\cos \theta \cos \Phi$ | $\times$ |

Table II. Fusion of adjacent MZMs located in the same or different connected SOTS islands. The pairing phase difference between the two islands is $2\Phi$. For simplicity, we assume the same $\mu$ in all islands. The factors that depend on the wavefunction overlap are not displayed. $\times$ means that the fusion is impossible. The results for the case of a single island are obtained by $\gamma'_i = \gamma_i$ and $\Phi = 0$. We provide more details about the table in the Supplemental Material. 

Figure 3. Manipulation of the fusion of MZMs in an incomplete disk. (a) Controlling the fusion by tuning $\theta$. At the field direction $\theta = \theta_1 \in \{0, \pi/4, \pi/2, 3\pi/4\}$ modulo $\pi$, two adjacent MZMs collide into the sharp vertex and fuse. Tuning $\theta$ away from $\theta_1$ splits the two MZMs from the vertex and weakens their fusion. (b) Controlling the fusion by tuning $B$. The MZMs $\gamma_1$ and $\gamma_2$ are well separated for $B > B_{c1}$, whereas they collide and fuse (at the vertex) for $B < B_{c1}$, where $B_{c1} = \Delta/\sin(\alpha/2)$. (c) Controlling the fusion by tuning $\mu$. $\gamma_1$ and $\gamma_2$ are separated for $|\mu| < \mu_{c1}$ with $\mu_{c1} = [B^2 \sin^2(\alpha/2) - \Delta^2]^{1/2}$, whereas they collide and fuse for $|\mu| > \mu_{c1}$. $\mu = 0.05m_0$ in (a) and 0 in (b), $B = 0.6m_0$ and $\theta = 0$ in (a,c), and, for all panels, all other parameters are the same as those in Fig. 2.
pear or disappear simultaneously. To better understand the fusion behaviors, it is instructive to consider a geometry that breaks inversion symmetry. For concreteness, we consider in the following an incomplete disk made by cutting off two symmetric pieces normal to two radial directions and with a vertex angle $\pi - \alpha$ between the cutting lines, as illustrated in Fig. 3. This simple setup enables us to manipulate the fusion of adjacent MZMs in different ways.

We are able to control the fusion by tuning the field direction $\theta$, depending on the value of $\alpha$ which measures the range of angles missing in the setup. If $\alpha > \max(\varphi_{s1}, \varphi_{s2})$, then we can move any two adjacent MZMs into the vertex and fuse them by tuning $\theta$, see Fig. 3(a). Take the reflection-symmetry line of the incomplete disk in the $x$ direction for instance. At $\theta = 0, \pi/4, \pi/2$ and $3\pi/4$ (modulo $\pi$), respectively, the adjacent pairs of $\gamma_{1/2}$, $\gamma_{4/1}$, $\gamma_{3/4}$, and $\gamma_{2/3}$ are maximally fused. Note that the fusion between $\gamma_{1/4}$ ($\gamma_{2/3}$) always requires a finite $\mu$. If $\min(\varphi_{s1}, \varphi_{s2}) < \alpha < \max(\varphi_{s1}, \varphi_{s2})$, then only the adjacent pairs with angular separation given by $\min(\varphi_{s1}, \varphi_{s2})$ can be fused. Whereas the other adjacent pairs with angular separation $\max(\varphi_{s1}, \varphi_{s2})$ can never be pushed to the vertex together and their fusions are suppressed. Finally, if $\alpha < \min(\varphi_{s1}, \varphi_{s2})$, then we cannot fuse the MZMs at all.

Since the separations $\varphi_{s1}$ and $\varphi_{s2}$ depend on the ratio $\Delta/B$, we can also modulate the fusion by tuning $B$ or $\mu$. As an illustration, we first tune $B$ with the field direction set at $\theta = 0$, such that two MZMs, say $\gamma_1$ and $\gamma_2$, are brought close to the vertex, see Fig. 3(b). When $B > B_{c1} \equiv \Delta/\sin(\alpha/2)$, the separation $\varphi_{s1}$ between $\gamma_{1/2}$ is larger than $\alpha$. Thus, $\gamma_1$ and $\gamma_2$ remain well separated. In contrast, when $B < B_{c1}$, we find $\varphi_{s1} < \alpha$. Then, $\gamma_1$ and $\gamma_2$ collide and fuse at the vertex. At the critical point $B = B_{c1}$, $\gamma_1$ and $\gamma_2$ extend along the cutting lines, see the middle panel of Fig. 3(b). In all the cases, the other MZMs remain well separated and stay at zero energy. Similarly, we can set $B$ to other proper directions, e.g., $\theta = \pi/4, \pi/2$ or $3\pi/4$, and control the fusion of other adjacent pairs by tuning $B$. For the pair of $\gamma_{1/4}$ (or $\gamma_{2/3}$), the critical field is given by $B_{c2} = \Delta/\cos(\alpha/2)$. The fusion is achieved when $B > B_{c2}$. Alternatively, we can fix $B$ and $\theta$ and instead use $\mu$ to electrically control the fusion. For the pair of $\gamma_{1/2}$ (or $\gamma_{3/4}$), the fusion is enhanced when $|\mu| > \mu_{c1}$ and suppressed when $|\mu| < \mu_{c1}$, where $\mu_{c1} = [B^2 \sin^2(\alpha/2) - \Delta_0^2]^{1/2}$, see Fig. 3(c). For the pair of $\gamma_{1/4}$ (or $\gamma_{2/3}$), the critical $\mu$ reads $\mu_{c2} = [B^2 \cos^2(\alpha/2) - \Delta_0^2]^{1/2}$ and the fusion is reduced when $|\mu| > \mu_{c2}$ whereas revived when $|\mu| < \mu_{c2}$.

IV. BRAIDING MAJORANA ZERO MODES AND NON-ABELIAN STATISTICS

A. Braiding two Majorana zero modes

We have shown that there are four MZMs on the boundary of a disk. By geometry engineering, i.e., line cutting, and making certain angular positions unavailable, we are able to selectively squeeze two MZMs into the same position and fuse them there. In this section, we show that a single triangle setup with a finite $\mu$ enables us to fuse two out of the four MZMs, while the other two modes remain at zero energy. Such a setup allows us to exchange the remaining two MZMs by rotating the magnetic field $B$, without interference from the other modes. The three angles of the triangle are assumed to be smaller than $\min(\varphi_{s1}, \varphi_{s2})$. We label the remaining MZMs by $\gamma_\alpha$ and $\gamma_\beta$, respectively, and discuss the exchange process based on numerical simulations below.

Let us start from the state with $B$ applied in the $x$ direction, i.e., $\theta = 0$. In the initial state, $\gamma_\alpha$ sits at the left vertex and $\gamma_\beta$ sits at the right vertex, see Fig. 4(a). We adiabatically turn $\theta = 0 \rightarrow \pi/2$. As a consequence, the following movements occur in sequence. First, $\gamma_\beta$ smoothly shifts to the top vertex, while $\gamma_\alpha$ stays at the left vertex, see Fig. 4(b). Then, $\gamma_\beta$ stays at the top vertex.
while $\gamma_{\alpha}$ shifts to the right vertex, where $\gamma_{\beta}$ was sitting in the initial state. Finally, $\gamma_{\alpha}$ stays at the right vertex, while $\gamma_{\beta}$ shifts down to the left vertex. We can clearly trace the movements of each MZM\(^4\). Apparently, $\gamma_{\alpha}$ and $\gamma_{\beta}$ effectively exchange their positions. It is important to note that, during the entire process, $\gamma_{\alpha}$ and $\gamma_{\beta}$ stay robustly at zero energy and they are protected from mixing with other modes by a finite energy gap, as shown by the instantaneous spectrum in Fig. 4(c). It indicates that the degeneracy of the ground states remains unchanged, which is a necessary condition for an adiabatic operation.

We now analyze the initial and final quantum states formed by $\gamma_{\alpha}$ and $\gamma_{\beta}$ in the triangle island. It is instructive to view the vertices of the triangle on the boundary of a disk. We can then identify the process of braiding with a movement of the two MZMs around this boundary. As a result, one of the MZMs, say $\gamma_{\alpha}$, must pass the cut and change sign due to the antiperiodic boundary conditions. Therefore, the exchange rule of the two MZMs is given by: $\gamma_{\alpha} \mapsto \gamma_{\beta}$ and $\gamma_{\beta} \mapsto -\gamma_{\alpha}$. The exchange operator in terms of Majorana operators can then be constructed as $T = (1 + \gamma_{\alpha}\gamma_{\beta})/\sqrt{2}\,\,^2$. The two MZMs may define a complex fermion with the operator as $f = (\gamma_{\alpha} + i\gamma_{\beta})/2$. Thus, there are two ground states, $|0\rangle$ and $|1\rangle \equiv f^\dagger|0\rangle$, which correspond to the absence and presence of the complex fermion, respectively. The exchange operation can be rewritten as $T_{\alpha\beta} = \exp[i\pi(f^\dagger f - f f^\dagger)/4]$. It acts diagonally in the ground-state space $\{|0\rangle, |1\rangle\}$, and adds opposite Berry phases $\pm\pi/4$ to the two ground states.

**B. Braiding more Majorana zero modes**

To exploit the non-Abelian statistics of braiding operations for topological quantum computation, more than two MZMs are required\(^6\). To this end, we consider a trijunction made by connecting three triangle islands in the center, as illustrated in Fig. 5. We apply a finite chemical potential but different pairing phases in the three islands. As we have shown before, the triangle islands would each host a couple of MZMs at the vertices if they were not connected. However, as the islands have their vertices connected in the center, two MZMs are fused. Therefore, totally only four MZMs remain in the setup: two in one island and the other two in the other two islands, respectively. We denote the islands as $T_1$, $T_2$ and $T_3$, and the MZMs $\gamma_{T_1}$, $\gamma_{T_2}$, $\gamma_{T_3}$ and $\gamma_{T_d}$, respectively. Assume that three magnetic fields can be independently tuned in the three islands. With these preconditions, we show that this trijunction allows to braid any two of the four MZMs while keeping the other MZMs untouched. The two MZMs in the same island can be braided by rotating the corresponding magnetic field in the same way as described in the previous section. Thus, we focus on the braiding of two MZMs from different islands in the following.

Let us start by considering the situation where the magnetic fields $B_i$ are applied uniformly, namely, $B_i = B_0$ and $\theta_i = -\pi/3$ (with $i \in \{1, 2, 3\}$). At this moment, $\gamma_{\alpha}$ is located in $T_1$, $\gamma_{T_2}$ and $\gamma_{T_2}$, and $\gamma_{T_d}$ in $T_2$, as shown in Fig. 5(a). We braid $\gamma_{\alpha}$ and $\gamma_{T_2}$ in three steps in sequence. In the first step, we turn $\theta_2 = -\pi/3 \rightarrow -\pi/2$ and move $\gamma_{T_2}$ slowly to the center. After this step, $T_1$ has two MZMs, $\gamma_{\alpha}$ and $\gamma_{T_2}$, see Fig. 5(b). In the second step, we turn $\theta_1 = -\pi/3 \rightarrow \pi/2$. This results in the exchange of $\gamma_{\alpha}$ and $\gamma_{T_2}$ inside $T_1$, see Fig. 5(b)-(c). In the last step, we rotate $\theta_2$ back to $-\pi/3$. Thus, $\gamma_{T_d}$ is moved smoothly to the top vertex of $T_2$. Therefore, the positions of $\gamma_{\alpha}$ and $\gamma_{T_2}$ are exchanged. Importantly, during the entire process the four MZMs stay robustly at zero energy and they are also protected from high-energy modes by a finite energy gap, see Fig. 5(g). In a similar way, we can braid $\gamma_{T_d}$ and $\gamma_{T_2}$ by rotating $\theta_2$ and $\theta_3$ alternatively. The three exchanges $\gamma_{\alpha} \leftrightarrow \gamma_{T_2}$, $\gamma_{T_2} \leftrightarrow \gamma_{T_3}$ and $\gamma_{T_3} \leftrightarrow \gamma_{T_d}$ generate the whole braid group of the four MZMs.

Next, we discuss the interpretation of braiding $\gamma_{\alpha}$ and $\gamma_{T_2}$ in terms of quantum gates à la Ivanov\(^7\). There are totally four MZMs in the setup. In the initial state, both $\gamma_{\alpha}$ and $\gamma_{T_2}$ are in $T_2$. We combine $\gamma_{\alpha}$ and $\gamma_{T_2}$ to define a complex fermion $f_{bc} = (\gamma_{\alpha} + i\gamma_{T_2})/2$. Then, we have two degenerate ground states $|0_{bc}\rangle$ and $|1_{bc}\rangle = f_{bc}^\dagger|0_{bc}\rangle$ of $T_2$ with different fermion parity. Analogously, we can define another complex fermion as $f_{da} = (\gamma_{T_d} + i\gamma_{T_3})/2$. Then, $|0_{da}\rangle$ and $|1_{da}\rangle = f_{da}^\dagger|0_{da}\rangle$ would correspond to two ground states of $T_1$ and $T_3$ with different total fermion parity. Considering the three islands together, for a fixed global fermion parity, there are only two ground states which span the computational space of a single non-local qubit. Without loss of generality, we assume even fermion parity and write the two ground states as $|0\rangle \equiv |0_{bc}, 0_{da}\rangle$ and $|1\rangle \equiv |1_{bc}, 1_{da}\rangle$. The exchange operator $T_{ab} = (1 + \gamma_{\alpha}\gamma_{T_2})/\sqrt{2}$ of $\gamma_{\alpha}$ and $\gamma_{T_2}$ can be written as $T_{ab} = (\sigma_0 - i\sigma_2)/\sqrt{2}$ in the the basis $\{|0\rangle, |1\rangle\}$. It corresponds to a $\pi/2$ rotation in the Bloch sphere around the $x$ axis and thus mixes the qubit. The same result is obtained by exchanging $\gamma_{T_3}$ and $\gamma_{T_d}$. Similarly, the exchange $\gamma_{\alpha} \leftrightarrow \gamma_{T_3}$ (or $\gamma_{\alpha} \leftrightarrow \gamma_{T_d}$) corresponds to a $\pi/2$ rotation around the $z$ axis. Instead, they act diagonally on the qubit. These rotations are clearly non-commutative, consistent with the non-Abelian nature of the MZMs.

The triangle setups and schemes described above can be further generalized to the case with more pairs of MZMs. As a natural generalization, we consider a ladder structure formed by $N_{\text{tri}}$ elementary triangle islands $T_i$ with finite chemical potential, as sketched in Fig. 1(c). $N_{\text{tri}}$ is assumed to be an odd integer. Three islands connected by the same point have different pairing phases. For example, we may use $\Phi_{N_{\text{tri}}-1} = -\Phi_{N_{\text{tri}}} = 2\pi/3$ with $n \in \{1, 2, \ldots\}$. For uniform magnetic fields, say, with directions $\theta_i = -\pi/3$, this ladder setup supports $N_{\text{tri}} + 1$ MZMs in total. The setup can be scaled up by adding extra islands. We can braid any two of the MZMs by alternately varying the magnetic fields in an analogous way as described before in the triangle setups. Notably, since the braiding operations are topological, the choice of chemical potential and pairing phases and the controlling of magnetic fields do not need
Figure 5. Braiding four MZMs by rotating in-plane magnetic fields. Positions of the four MZMs at (a) \((\theta_1, \theta_2, \theta_3) = -\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)\), (b) \((\theta_1, \theta_2, \theta_3) = -\left(\frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}\right)\), (c) \((\theta_1, \theta_2, \theta_3) = -\left(\frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}\right)\), (d) \((\theta_1, \theta_2, \theta_3) = -\left(0, \frac{\pi}{2}, \frac{\pi}{3}\right)\), (e) \((\theta_1, \theta_2, \theta_3) = \left(\frac{\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{3}\right)\), and (f) \((\theta_1, \theta_2, \theta_3) = \left(\frac{\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{3}\right)\), respectively. (g) Energy flow during the braiding process. The three light colored areas indicate the three steps of the braiding process, as discussed in the main text. (a)-(f) correspond to the moments labeled by I-VI, respectively. There are always four MZMs (red flat bands at zero energy) and separated by a large energy gap from other modes (blue curves). An animation for the process is provided in the Supplemental Material. The dimension of the triangles are \(N_x = N_y = 30\), the pairing phases are \(\Phi_1 = 0\), \(\Phi_2 = -\Phi_3 = \frac{2\pi}{3}\), and all other parameters are the same as those in Fig. 4.

to be fine tuned.

V. HOLONIMIC GATES

According to the Gottesman-Knill theorem, the topological quantum gates provided by Majorana braiding are not sufficient for universal quantum computation. The latter requires indeed at least one non-Clifford single-qubit gate, such as the T gate (also known as the magic gate), which cannot be implemented just by exchanging MZMs. In general, non-topological procedures for realizing a Majorana magic gate require a very precise control of the system parameters and, as a result, they heavily rely on conventional error-correction schemes. In this respect, an interesting role is played by the so-called holonomic gates which, while not being topologically protected, can still feature a good degree of robustness against errors, thus reducing the amount of hardware necessary for subsequent state distillation.

The minimal model for Majorana-based holonomic gates has been proposed by Karzig et al. and consists of four “active” MZMs, \(\gamma_x\), \(\gamma_y\), \(\gamma_z\) and \(\gamma_0\). To successfully implement a holonomic gate, one has to adiabatically vary the three coupling strengths \(t_i\) between \(\gamma_i\) (with \(i \in \{x, y, z\}\)) and \(\gamma_0\). Indeed, for every closed loop in the three-dimensional parameter space spanned by \((t_x, t_y, t_z)\), a difference between the Berry phases picked up by the two states is developed. By properly designing the loop, it is possible to implement a T gate and take advantage of a universal geometrical decoupling in order to suppress the effect of finite control accuracy on the couplings \(t_i\). One of the main sources of errors is represented by the parasitic couplings \(t_{ij}\) between \(\gamma_i\) and \(\gamma_j\) (with \(i, j \in \{x, y, z\}\) and \(i \neq j\)). These couplings, which are basically unavoidable in a quantum wire-based setup, introduce additional and unwanted dynamical phases, which reduce the gate fidelity (despite error mitigation techniques based on conventional echo schemes).

Remarkably, SOTS-based setups can naturally guar-
energy spectra are displayed in the insets. Bγ and θ and Bnomic gates in a shamrock-like trijunction. The positions of the MZMs are controllable by the applied magnetic fields, as discussed before. For concreteness, we consider the incomplete disks with the same vertex angle of π/2 and evenly distributed. By properly tuning the magnetic fields B1 and B2, i.e., B1, B2 ∼ √2Δ0, θ1 ∼ −π/6 and θ2 ∼ 7π/6, we move the MZMs γ1 and γ4 of S1 and γ1 of S2 close to or in the center. Due to the effective mirror symmetry, these three adjacent MZMs cannot interact with each other even if they have strong overlap in their wavefunctions. We thus identify them with γx, γy and γz, respectively. Whether S1 and S2 have the same pairing phase or not is not important. We set both at zero Φ1 = Φ2 = 0 for concreteness. On the other hand, we adopt a different pairing phase Φ3 ≠ {0, π} in S3, and fix the MZM γ2 or γ3 of S3 in the center by tuning B3. This MZM would fuse with γi (with i ∈ {x, y, z}) if they are brought close to the center. We denote it as γ0. The couplings ti between γi and γ0 are closely determined by their distance. Recalling that these distances are controllable by B1 or B2, the couplings are thus smoothly adjustable.

To test our analysis and demonstrate the basic conditions for holonomic gates, we first consider S1 and S2 together and move γx, γy and γz all to the center, see Fig. 6(c). From the energy spectrum [the inset of Fig. 6(c)], we find that there is no splitting of MZMs from zero energy. This clearly indicates that γx, γy and γz can never interact with each other. We next consider S3 together with S1 or S2 and move γx, γy and γz to the center, respectively, see Fig. 6(d)-(f). In contrast to Fig. 6(c), evident energy splittings of two original MZMs can be observed (see the green dots in the insets). This signifies the couplings between γ0 and γx, γy and γz, respectively. The splitting energies are smooth functions of B1 or B2. Thus, by tuning B1 and B2, we are able to adiabatically adjust the couplings ti. For illustration, we start with the state, where γx and γ0 are in the center, while the other two not. In this initial state, we have |t2| ≫ |t3|, |t4|. We slowly move γx also into the center by carefully rotating B2 and then move γz away from the center by rotating B1. In the final state, we arrive at |t3| ≫ |t2|, |t4|. The instantaneous spectrum for this process is displayed in Fig. 6(b). Two finite energy levels resulting from the coupling of γ0 and γx or γz are always present (see the green curves), while the other MZMs stay robust at zero energy. This clearly shows that t3 and t2 are smoothly varied. Similarly, the variation of B1 enables us to adjust the other coupling t4 in a smooth manner.

Finally, we note that the shamrock-like trijunction can realize the full parameter space of (t2, t3, t4). In this sense, a full set of holonomic gates defined by the four

![Figure 6. Holonomic gates based on the MZMs.](image-url)

(a) Schematics of the four MZMs γx, γy, γz and γ0 relevant for the holonomic gates in a shamrock-like trijunction. The positions of these modes are controllable by the magnetic fields B1, B2 and B3. (b) Energy flow of the process that first moves γx to the center (during t1 → t2) and then moves γ0 away from the center (during t2 → t3). Insets are the positions of MZMs in the initial (θ1 = −π/3 and θ2 = 7π/6), middle (θ1 = −π/3 and θ2 = 4π/3), and final (θ1 = −π/6 and θ2 = 4π/3) states. The green densities are for the fused modes (i.e., two excited modes with lowest energy). (c-f) Coupling between different MZMs, i.e., γx, γy and γz in (c); γy and γ0 in (d); γx and γ0 in (e); and γz and γ0 in (f). The corresponding energy spectra are displayed in the insets. B1 = 0.54m0 in (a,c-f) and 0.65m0 in (a). In (c) θ1 = −π/6, θ2 = 4π/3, (d) θ1 = 0, (e) θ1 = −π/3, (f) θ2 = 4π/3, for all panels, Φ1 = Φ2 = 0, Φ3 = 2π/3, B2 = B3 = 0.54m0, θ3 = −π/3, μ = 0, Δ0 = 0.4m0, m0 = 2, N = 20, while all other parameters are the same as those in Fig. 2.

Due to vanishing parasitic couplings, thus providing a novel and convenient platform to study and implement Majorana-based holonomic gates. By exploiting the effective mirror symmetry featured by the SOTs at μ = 0, one can indeed have tij = 0 while still being able to smoothly and freely tune all the other three couplings ti. In the following, we demonstrate this remarkable feature

with a concrete example.
coupled MZMs can be achieved. There are many alternative ways of choosing the four relevant MZMs, which, however, share the essential physics. Other MZMs that exist in the setup but are not relevant to the problem can be safely ignored since they are always far away from the center.

VI. DISCUSSION

There are various candidate systems to implement our schemes. Particularly, the inverted HgTe quantum well in proximity to conventional superconductors and monolayer FeTe$_{1-x}$Se$_x$ have the right properties. The superconducting proximity effect has been realized in HgTe quantum wells$^{37,58}$. The application of high in-plane magnetic fields is also feasible in experiments$^{58}$. Monolayer FeTe$_{1-x}$Se$_x$ can host a quantum spin Hall phase coexisting with intrinsic superconductivity$^{49,59,60}$. Importantly, the superconducting gap is comparably large (up to 16.5 meV)$^{61}$, and it can sustain a large in-plane magnetic field (up to 45 T)$^{62}$. The localization length of the MZMs can be estimated by $\xi = \text{max}(v/\Delta_0, m/v)$. For typical parameters $v \approx 1.0$ eVÅ, $\Delta_0 \approx 1.0$ meV and $m \approx 10$ eVÅ$^2$, we derive $\xi \approx 10^3$ Å. Therefore, the length scales for our setups should be larger than 100 nm. For a g-factor close to $g \approx 2$, a magnetic field of around $\Delta_0/g\mu_B \approx 10$ T would be sufficient to induce the MZMs.

The MZMs in our setups can be locally probed, for example, by a scanning-tunneling-microscopy tip. Their existence may be signified as zero-bias peaks in the tunneling conductance$^{63}$. To read out the qubits in our setups, we may need to convert parity ground states into charge states in isolated islands. In the associated SOTS islands, we can fuse a pair of MZMs by varying the magnetic fields and then perform the parity measurement of the islands, e.g., by applying charge sensors$^{64,65}$. The parity information encoded in the MZMs could be alternatively measured via the Josephson effect. To do so, we approach an MZM to another one stemming from another island. Then, the Josephson current between the islands can be used to deduce the information$^8$.

In summary, we have provided a minimal model for SOTSs and analyzed the MZMs in a disk geometry. We have systematically investigated the fusion of the MZMs. Importantly, we have proposed different setups of SOTSs which allow to implement topological and holonomic gates via braiding and fusion of MZMs. Our results establish SOTSs as an ideal platform for scalable and fault-tolerant quantum information technologies.

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