AXIOMATIC FOUNDATIONS OF
GALILEAN QUANTUM FIELD THEORIES

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Abstract

A realistic axiomatic formulation of Galilean Quantum Field Theories is presented, from which the most important theorems of the theory can be deduced. In comparison with others formulations, the formal aspect has been improved by the use of certain mathematical theories, such as group theory and the theory of rigged Hilbert spaces. Our approach regards the fields as real things with symmetry properties. The general structure is analyzed and contrasted with relativistic theories.

KEYWORDS: Axiomatization, Quantum Field Theories, Galilean Invariance.

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1 INTRODUCTION

There are different ways to present a physical theory. On the one hand, the informal way (for example, the historical approach) proceeds with the help of analogies and metaphors. These approaches are hardly appropriate for the study of the structure of the theory. In fact, in these formulations the structure emerges gradually, the assumptions are rarely exposed explicitly and the basic concepts are presented disorderly. Moreover, the theory is interpreted by analogies and heuristic clues. On the other hand, the formal way, or the axiomatic approach as it is most widely known, consists of a logical organization of the theory with both an adequate characterization of the physical meaning of the symbolism and an examination of the metatheoretical aspects. In our opinion, every formal analysis of the structure of a theory should be a consequence of the study of its axiomatic framework in such a way that all the presuppositions, theorems and interpretation rules are exhibited explicitly.

There are different informal approaches for the quantization of a nonrelativistic field theory. Redmond and Uretzky developed the in-out formalism in the case of second quantization. Dresden and Kahn studied nonrelativistic field theories assuming invariance with respect to the Euclidean group of transformations. Schweber studied the nonrelativistic Bethe-Salpeter equation. Lévy-Leblond investigated field theories based on the notion of field operators, local commutability and Galilean invariance, and Dadashev presented a system of axioms following the axiomatics of Wightman for the relativistic case.

In this article we present an axiomatic formulation of the Galilean quantum field theories (GQFT) and we analyze its conceptual and mathematical structure. In our axiomatization we follow the basic postulates adopted by many of the formulations of the relativistic theory but we adapt them to the Galilean framework, i.e. Galilean covariance, local commutability and irreducibility. The mathematical aspect has been improved by the use of certain mathematical theories, such as group theory and the theory of rigged Hilbert spaces. Differing from the bulk of the formulations, we assume a realistic ontology, namely that of references and we construct the theory from the notion of basic thing and system with properties. In the present case, these are: basic fields and systems of basic fields with different properties (symmetry properties among them), respectively. Besides, we base our interpretation of the theory in the rigorous semantics of references.
From these postulates we will obtain rigorous and general proofs that show the rich structure of the theory. For example, we will see that Galilean quantum field theory is less restrictive than its relativistic counterpart, mainly due to the restricted form of the commutation relations of the local field operators. As a consequence of this weaker condition, the powerful results of relativistic field theory, such as CPT and spin-statistics theorems, cannot be derived in GQFT. This shows the important role that commutation relations play in relativistic field theories. However, we must remark that there are several powerful results in Galilean field theory that do not exist in the relativistic one, such as the existence of a mass superselection rule.

The structure of the paper is the following. In the second section, we expound some ontological concepts that must be presupposed by any quantum field theory. In the third section we review the Galilei group, its Lie algebra and its physical representations. The axiomatic of GQFT is develop in the fourth section, along with remarks that clarify the axioms and some of the principal theorems. In the fifth section we discuss some of the consequences of GQFT such as crossing symmetry and spin-statistic theorems, and the final conclusions are given in section sixth.

2 ONTOLOGICAL BACKGROUND

It is undeniable that there are some ontological queries that must be carefully elucidated in the background of any field theory. In order to understand the physical concepts of “particle”, “field” and their mutual actions it is mandatory to have an accurate characterization of the concepts of system, interaction and state. Since we deal with physical systems, we are interested in a formal characterization of the ontological concept of a system first. We shall assume the realistic ontology of Bunge\(^8,9\) and in this section we shall summarize the results relevant for this paper.

The basic concept of this ontology i.e. that of a substantial individual, will be denoted by \(x\). Substantial individuals can associate to form new substantial individuals, and they differ from the fictional entities called bare individuals (i.e. individuals without properties) precisely in that they have a number of properties \(P\) in addition to their capability of association. We shall propose some useful definitions.

**Def. 1 (Substantial Property)** The substantial properties are those pos-
sessed by some real thing, i.e.

\[ P \in \mathcal{P} \leftrightarrow (\exists x)(x \in S \land Px). \]

Here \( \mathcal{P} \) is the set of all substantial properties, \( S \) is the set of all substantial individuals and \( Px \) designate ‘\( x \) has the property \( P \)’.

**Def. 2** The set of all the properties of a given individual is given by

\[ P(x) = \{ P \in \mathcal{P} | Px \}. \]

These properties can be *intrinsic* (\( P_i \)) or *relational* (\( P_r \)). The intrinsic properties (e.g. charge, spin) are inherent and they are represented by unary predicates or applications while relational properties (e.g. position, momentum) are represented by \( n \)-ary predicates (\( n > 1 \)), as long as nonconceptual arguments are considered.

We will need below the concepts of same and identical individuals:

**Def. 3** Two individual are the same if they have exactly the same properties, i.e.,

\[ \forall x, y \in S, P(x) = P(y) \Rightarrow x \equiv y. \]

**Def. 4** Two individual are identical if their intrinsic properties are the same, i.e.,

\[ \forall (x, y \in S)[P_i(x) = P_i(y) \Rightarrow x \ id \ y]. \]

For example, two spin-up electrons are identical because they have the same intrinsic properties. We now can define a concrete thing as made up from substantial individuals \( x \) together with their properties \( P(x) \).

**Def. 5 (Thing)** Denoting a thing by \( X \), we define it as the ordered pair \(^1\):

\[ X = \langle x, P(x) \rangle. \]

The set of all things will be denoted by \( \Theta \). Some things are complex, so we can introduce the notion of composition of things:

\(^1\)The brackets \( \langle, \rangle \) denote an ordered set.
Def. 6 (Absolute Composition) For every \( X \in \Theta \), the composition of \( X \) is:

\[
\mathcal{C}(X) = \{ Y \in \Theta | Y \sqsubset X \}
\]

where “\( Y \sqsubset X \)” designates “\( Y \) is a part of \( X \)”.

Let us introduce the concept of action:

Def. 7 (Action) A thing \( X \) acts on another thing \( Y \) if \( X \) modifies the behavior of \( Y \). (\( X \triangleright Y : X \) acts on \( Y \)). If the action is mutual it is said that they interact (\( X \bowtie Y \)).

Def. 8 (Connection) Two things are connected if at least one of them acts on the other.

Now we have all we need to define a concrete system:

Def. 9 (System) A system \( \sigma \) is a thing composed of at least two different connected things.

Two important features of a system are its composition and its environment,

Def. 10 (A-composition) The composition of a system \( \sigma \) at a given time \( t \in T \) with respect to a class \( A \) of things is the set of its \( A \)-parts at \( t \):

\[
\mathcal{C}_A(\sigma, t) = \{ X \in A | X \sqsubset \sigma \text{ at } t \in T \}
\]

Def. 11 (A-environment) The environment of a system \( \sigma \) at time \( t \) is the set of things of kind \( A \) that are not components of \( \sigma \) but that are connected with some or all the components of \( \sigma \), that is,

\[
\mathcal{E}_A(\sigma, t) = \{ X \in A | X \notin \mathcal{C}_A(\sigma, t) \land (\exists Y)(Y \in \mathcal{C}_A(\sigma, t) \land (X \triangleright Y \lor Y \triangleright X)) \}
\]

Def. 12 (Closed System) A system is closed at the instant \( t \) if and only if its environment is empty, i.e. \( \mathcal{E}_A(\sigma, t) = \emptyset \)

Theoretical physics does not characterize concrete things but rather concepts, in particular with conceptual schemes called models of things. We will assume that any property \( P \) of a thing \( X \) it is represented by a mathematical application \( F \), that is to say, \( F \triangleq P \). 5
Def. 13 (Functional schema) Let $\sigma$ be a system. A functional schema $b$ of $\sigma$ is a certain nonempty set $M$, together with a finite sequence of mathematical applications $F_i$ on $M$, each one of which represents a property of $\sigma$. Shortly:

$$b = \langle M, F \rangle$$

where $F = \langle F_1, F_2, \ldots, F_p \rangle : M \to V_1 \times V_2 \times \cdots \times V_p$.

Therefore, the system $\sigma$ will be represented by the functional schema $b$, that is to say, $b \triangleq \sigma$.

It is natural to assume that all things are in some state. The state of a system can be characterized as follows:

Def. 14 (State function) Let $\sigma$ be a system modeled by a functional schema $b = \langle M, F \rangle$. Then, each $F_i$ is a state function of $\sigma$. $F$ is the state vector of $\sigma$, and its value

$$F(m) = \langle F_1(m), F_2(m), \ldots, F_p(m) \rangle$$

for any $m \in M$ represents the state of $\sigma$ in the representation $b$.

In our axiomatic we shall be interested with basic fields and systems of basic field. These will be the specific physical systems to be characterized by our axiomatic frame.

3 MATHEMATICAL BACKGROUND

The symmetry group of nonrelativistic transformations from one inertial frame of reference to another is the proper Galilei group $G$, defined as

Def. 15 The set of all transformations in Galilean space-time consisting of space and time displacement, pure Galilean transformations and space rotations constitutes a continuous group $G$ of 10 parameters called the Galilei group.

The generic element will be denoted by $g = (b, a, \mathbf{v}, R)$, with $b \triangleq$ time translation, $a \triangleq$ space translation, $\mathbf{v} \triangleq$ pure Galilean transformation and $R \triangleq$ three-dimensional rotation. Its action on space and time is represented by the coordinate transformation

$$x \to x' = Rx + vt + a,$$
The multiplication law is given by
\[ g'g = (b', a', v', R')(b, a, v, R) \]
\[ = (b' + b, a' + R'a + bv', v' + R'v, R'R). \]

The identity element is
\[ e = (0, 0, 0, 1). \]

The inverse element is
\[ g^{-1} = (-b, -R^{-1}(a - bv), -R^{-1}v, R^{-1}). \]

Contrary to the case of the Poincaré group, İnönü and Wigner\textsuperscript{10} found that the basis vectors of the ordinary representation of the Galilei group cannot be interpreted as physical states of particles. On the other hand, Bargmann\textsuperscript{11} had shown that the physically meaningful representations needed to describe particles are the projective representations. In other words, we obtain a unitary representation of the symmetry group \( G \) if to each element \( g \in G \) (indeed of its universal covering group) we can associate a unitary operator \( \hat{U}(g) \) on the physical Hilbert space. However, the composition rule of the unitary operators \( \hat{U}(g) \) and \( \hat{U}(g') \) cannot be written as in the case of an ordinary representation, \( \hat{U}(g)\hat{U}(g') = \hat{U}(gg') \). Indeed, the group is represented projectively on physical states; that is, the unitary operators satisfy the composition rule, \( \hat{U}(g)\hat{U}(g') = e^{i\zeta(g,g')}\hat{U}(gg') \), with \( \zeta \) a real phase. Bargmann\textsuperscript{11} has shown that for the Galilei group the phases \( \zeta(g,g') \) cannot, in general, be made equal to zero by a redefinition of \( \hat{U}(g) \).

The presence of phases in a projective representation of a group has a counterpart in the Lie algebra of the group: it is the appearance of terms on the right-hand side of the commutation relations proportional to the unit element. These terms are called central charges. If the generators \( \hat{H}, \hat{P}_i, \hat{K}_i, \hat{J}_i \) denote time translations, space translations, pure Galilean transformations and rotations respectively, we can define:

**Def. 16** The Lie algebra \( \mathcal{G} \) of the Galilei group is given by the commutation rules of the generators \( \hat{H}, \hat{P}_i, \hat{K}_i, \hat{J}_i \) which satisfy the following relations:

\[
\begin{align*}
[\hat{J}_i, \hat{J}_j] &= i\hbar\epsilon_{ijk}\hat{J}_k, \\
[\hat{J}_i, \hat{K}_j] &= i\hbar\epsilon_{ijk}\hat{K}_k, \\
[\hat{J}_i, \hat{P}_j] &= i\hbar\epsilon_{ijk}\hat{P}_k, \\
[\hat{K}_i, \hat{H}] &= i\hbar\hat{P}_i, \\
[\hat{K}_i, \hat{P}_j] &= i\hbar\delta_{ij}m, \\
[\hat{J}_i, \hat{H}] &= 0, \\
[\hat{K}_i, \hat{K}_j] &= 0, \\
[\hat{P}_i, \hat{P}_j] &= 0, \\
[\hat{P}_i, \hat{H}] &= 0,
\end{align*}
\]
where \( \hbar \) is Planck’s constant and \( m \) is a real number (the central charge).

So, the Galilean algebra does allow for a central charge that cannot be removed by a redefinition of the generators.\(^{11} \) However, we can enlarge the Galilean group \( G \), by adding one more generator to its Lie algebra, which commutes with all the other generators, and whose eigenvalues coincide with the central charge. This generator will be denoted by \( \hat{M} \). This extension by an Abelian one-dimensional group is called central because \( \hat{M} \) commutes with all the other elements of the Lie algebra, and it is nontrivial because \( \hat{M} \) appears on the right side of some commutations relations. The expanded Lie algebra \( \hat{G} \) is free of central charges and the extended group \( \hat{G} \) has only ordinary representations. The space and time translations form an abelian subgroup of the Galilei group. The representations of this subgroup are known and will be denoted by their eigenvalues \((\mathbf{p}, E)\).

The following theorems can be deduced from the above definitions:

**Thm. 1** The algebra \( \hat{G} \) admits the following invariants:

\[
\begin{align*}
\hat{Q}_1 &= \hat{M}, \\
\hat{Q}_2 &= 2\hat{M}\hat{H} - \hat{P}^2 \equiv 2\hat{M}\hat{W}, \\
\hat{Q}_3 &= (\hat{M}\hat{J} - \hat{K} \times \hat{P})^2 = \hat{M}^2\hat{S}^2,
\end{align*}
\]

where \( \hat{W} \) is the internal energy operator and \( \hat{S} \) is the spin operator.

**Proof:** See references.\(^{12, 13} \)

**Thm. 2** a) The set of basis vectors \( \{|\mathbf{p}, E, \lambda\rangle\} \) spans a vector space which is invariant under Galilei group transformations. The action of an arbitrary Galilei transformation on these basis vectors is given by:

\[
\hat{U}(g)|\mathbf{p}, E, \lambda\rangle = \exp[-\frac{i}{\hbar}(E'b - \mathbf{p}'\cdot\mathbf{a})]\sum_{\lambda'}|\mathbf{p}', E', \lambda'\rangle D^{(s)}(\lambda')(R),
\]

\[
\begin{align*}
\mathbf{p}' &= R\mathbf{p} + m\mathbf{v}, \\
E' &= \frac{\mathbf{p}'^2}{2m} + W, \\
&= E + \mathbf{v} \cdot R\mathbf{p} + \frac{1}{2}m\mathbf{v}^2,
\end{align*}
\]

and \( D^{(s)}(R) \) is the representation matrix of the rotation group corresponding to the spin \( s \).
b) The resulting representation, labeled by \([m, W, s]\), is unitary and irreducible.

**Proof:** See references\(^{11,13}\)

**Theorem 3** Every irreducible representation of the translation subgroup \((p, E)\) satisfy the following condition:

\[
E - \frac{p^2}{2m} = W = \text{const.}
\]

**Proof:** See references\(^{13,14}\)

**Theorem 4** The invariant measure is given by,

\[
d\mu(p, E) = \delta\left(E - \frac{p^2}{2m} - W\right)d^3pdE
\]

**Proof:** From Thm 3

We shall use below the Peter-Weyl theorem, which is a generalization of the Fourier theorem and holds for every compact Lie group (see\(^{15}\)). Although the Galilean group is non-compact, it is valid for the compact part, i.e. the representation of the rotation group:

**Theorem 5** The irreducible representation basis functions form a complete basis in the space of (Lebesgue) square-integrable functions defined on the group manifold.

In our axiomatic formulation, the generators of the Lie algebra \(\mathcal{G}\) will be postulated and identified later on by means of semantic assumptions.

## 4 AXIOMATICS OF GQFT

In this section we shall exhibit the axiomatic structure of the theory based on the ontological background and making use of the symmetry properties presented above. Firstly, we shall present the two set of ideas that GQFT takes for granted: Formal and material background. The formal background consist of all the logical and mathematical ideas it employs. The material background consists of all the generic and specific theories it presupposes. \(^2\)

\(^2\)We shall display only a list of the main items.
4.1 Formal Background

1. Classical logic.
2. Formal semantics.$^{6,7}$
3. Mathematical analysis with its presuppositions, and the theory of generalized functions.$^{16}$
4. Group theory.

4.2 Material Background

1. Protophysics$^{17}$ (i.e. Physical probabilities, Chronology, Physical geometry, . . .).
2. Dimensional analysis.

Remark:

We do not use dimensional analysis explicitly in this paper (except in axiom $[\mathfrak{M}]$). However, dimensional analysis does play a very important, albeit silent, role in every physical theory: it classifies physical quantities in dimensionally homogeneous classes and so restricts the form of their mathematical representations.

4.3 Primitive Basis

The conceptual space of the theory is generated by the basis $\mathcal{B}$ of primitive concepts, where

$$\mathcal{B} = \langle E_3, T, \Sigma, \Sigma, \mathcal{H}_E, \mathcal{P}, A, \hbar, \mathcal{F}, G \rangle$$

The elements of this basis will be characterized mathematically $[\mathfrak{M}]$, physically $[\mathfrak{P}]$ and semantically $[\mathfrak{S}]$ by the axiomatic basis of the theory and the derived theorems.

4.4 Definitions

Def. 17 $K \overset{\text{Df}}{=} \text{set of physical reference systems}.$

Def. 18 $eiv \overset{\text{Df}}{=} \text{eigenvalues of } \hat{A}.$

Def. 19 $\langle \Phi_1 | \Phi_2 \rangle \overset{\text{Df}}{=} \text{scalar product of the vectors } \Phi_1 \text{ and } \Phi_2.$
4.5 Axioms

Group I: Space and Time

A. 1 (Space)
[M] $E^3 \equiv$ Euclidean three-dimensional space.
[S] $E^3 \triangleq$ physical space.

A. 2 (Time)
[M] $T \equiv$ interval of real line $\mathbb{R}$.
[S] $T \triangleq$ time interval.
[S] The relation $\leq$ that orders $T$ means “before” $\lor$ “simultaneous with”.

Remark:
These axioms only characterize our use of the notions of space and time. Ontological analysis of these notions can be seen in references$^8,18$

Group II: F-Systems and States

A. 3 (Systems)
[M] $\Sigma, \bar{\Sigma}$: nonempty numerable sets.
[S] $(\forall \sigma_i)_{\Sigma} (\sigma_i \triangleq$ a basic field).
[S] $(\forall \sigma)_{\Sigma} (C(\sigma) = \{\sigma_1, \ldots, \sigma_N\})(\sigma \triangleq f$-system).
[S] $(\forall \sigma)_{\Sigma} (\sigma \triangleq$ environment of some f-system ). In particular, $(\sigma_o \triangleq$ the empty environment).
[S] $(\forall \sigma)_{\Sigma} (\langle \sigma, \sigma_o \rangle \triangleq$ a closed f-system).
[P] $(\exists K)(K \subset \Sigma \land$ the configuration of $k \in K$ is independent of time).

A. 4 (State Space)
[M] $(\forall \sigma)_{\Sigma} (\exists H_E = \langle \mathcal{L}, \mathcal{H}, \mathcal{L}' \rangle \equiv$ rigged Hilbert space).
[P] There exists a one-to-one correspondence between physical states of $\sigma \in \Sigma$ and rays $\mathcal{R}_\sigma \subset \mathcal{H}$. 

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\[ [M] \ (\forall \sigma)_{\Sigma}(C(\sigma) = \{\sigma_1, \ldots, \sigma_N\}) \Rightarrow H_E = \otimes_{i=1}^N H_{E_i}. \]

\[ [S] \ |\Phi(\sigma, k)\rangle \in R_{\sigma} \overset{d}{=} \text{state vector that is the representative of the ray } R_{\sigma} \text{ that corresponds to the f-system } \sigma \text{ with respect to } k \in K. \]

\[ [P] \ (\exists|0\rangle)_{R_{\sigma}}(|0\rangle) \overset{d}{=} \text{the normalized state called the vacuum state }. \]

Remarks:

1. The third axiom must be explicitly postulated in order to state that the reference class of the GQFT is not an empty set. Moreover, the elements of this set are semantically interpreted as ‘fields’. These axioms are not trivial since that a different interpretation could claim that GQFT deals only with experimental results (i.e., a set of data not a physical system) or that the reference class are particles (not fields) because they are observables.

2. The rigged Hilbert space is an extension of the ordinary Hilbert space and it is introduced here in order to include the operators (as position and linear momentum operators) whose eigenfunctions does not have finite norm. Moreover, from the physical point of view, the axiom states that every physical state of the system is mapped on a subset \( \mathcal{H} \).

Group III: Operators and Physical Quantities

A. 5 (Physical Properties)

\[ [M] \ P \overset{\equiv}{=} \text{nonempty family of applications over } \Sigma. \]

\[ [M] \ A \overset{\equiv}{=} \text{ring of operators over } H_E. \]

\[ [P] \ (\forall P)_{\Sigma}(\exists \sigma)(P \in P(\sigma)). \]

\[ [P] \ (\forall P)_{\Sigma}(\exists \hat{A})(\hat{A} \triangleq P). \]

\[ [P] \ (\forall \sigma)_{\Sigma}(\forall \hat{A})(\forall a)(e^{iv\hat{A}} = a \land \hat{A} \triangleq P \Rightarrow a \text{ is the sole value that } P \text{ takes on } \sigma). \]

\[ [M] \ h \in \mathbb{R}^+. \]

\[^{3}\text{To avoid unnecessary complexity in notation we are not going to make explicit the dependence of the state on the system and on the reference system.} \]
\[\mathcal{P} \ [\hbar] = LMT^{-1}.\]

**A. 6 (Linearity and Hermiticity)**

\[\begin{align*}
\{M\} & \ (\forall \sigma)_{\Sigma}(\forall \hat{A})(\forall P)(\hat{A} \triangleq P \land |\Phi_1\rangle, |\Phi_2\rangle \in \mathcal{H}_E \Rightarrow \ \\
(a) & \ \hat{A}[\lambda_1|\Phi_1\rangle + \lambda_2|\Phi_2\rangle] = \lambda_1\hat{A}|\Phi_1\rangle + \lambda_2\hat{A}|\Phi_2\rangle, \ \text{with } \lambda_1, \lambda_2 \in \mathbb{C} \\
(b) & \ \hat{A}^\dagger = \hat{A}\end{align*}\]

**A. 7 (Probability Densities)**

\[\begin{align*}
\{P\} & \ (\forall \sigma)_{\Sigma}(\forall \hat{A})(\forall P)(\forall |a\rangle)_{\mathcal{H}_E}(\forall |\Phi\rangle)_{\mathcal{H}_E}(\hat{A} \triangleq P \land \hat{A}|a\rangle = a|a\rangle \Rightarrow \text{the probability density } \langle \Phi|a\rangle\langle a|\Phi\rangle \text{ corresponds to the property } P \text{ of the } f\text{-system } \sigma).
\]

**A. 8 (Unitary Operators)**

\[\begin{align*}
\{P\} & \ (\forall \sigma)_{\Sigma}(\forall \hat{A})(\forall P)(\forall \hat{U})(\hat{A} \triangleq P \land \hat{U} is an operator on \mathcal{H}_E \land \hat{U}^\dagger = \hat{U}^{-1} \Rightarrow \hat{U}\hat{A}\hat{U}^{-1} \triangleq P).\end{align*}\]

**Remark:**

The axioms of Group III are typical of a realistic formulation of Quantum Mechanics. In addition of the mathematical properties (attributes) of the operators, the axioms characterize semantically the properties of the systems, that is, every property of the system is represented by an operator. Note that the eigenvalues of these operators are not interpreted as the only possible values of a ‘measurement’ made on the system, since the relationship with measurement cannot be made at the level of principles.

**Group IV: Quantum Fields**

**A. 9 (Field Operators)**

\[\begin{align*}
\{M\} & \ \mathcal{F} \subset A \equiv \text{nonempty set of operators over } \mathcal{H}_E. \\
\{S\} & \ (\forall \sigma_i)_{\Sigma}(\exists \hat{\psi}_\lambda)_{\mathcal{F}}(\hat{\psi}_\lambda \overset{d}{=} \lambda\text{-component of the local field operators associated with the basic field } \sigma_i). \\
\{S\} & \ (\forall \sigma_i)_{\Sigma}(\forall (x, t))_{\mathbb{E}^3 \times T}(\hat{\psi}_\lambda(x, t) \overset{d}{=} \text{the amplitude of the basic field } \sigma_i \text{ at } x, t).\end{align*}\]
\[\forall \hat{\psi}_\lambda \forall (x, t) \in \mathbb{R}_3 \times T \langle \hat{\psi}_\lambda (x, t) | 0 \rangle = 0.\]

\[\forall \hat{\psi}_\lambda \forall (x, t) \in \mathbb{R}_3 \times T \langle 0 | \hat{\psi}_\lambda (x, t) = \langle x, t; \lambda \rangle \rangle.\]

A. 10 (Irreducibility)

[M] The set of field operators is irreducible, i.e.:

\[
(\forall \sigma)_{\sum} (\forall \hat{\mathcal{O}}) \{ \hat{\mathcal{O}} = \sum_{N, M = 0}^{\infty} \sum_{\lambda_1 \ldots N} \int \mathcal{D}x' \mathcal{D}x \hat{\Psi}_{\lambda'} (x', t) \hat{\Psi}_\lambda (x, t) C_{NM} \}
\]

where

\[
\mathcal{D}x' \mathcal{D}x \overset{d}{=} d^3x'_1 \ldots d^3x'_N d^3x_1 \ldots d^3x_M,
\]

\[
\hat{\Psi}_{\lambda'} (x', t) \overset{d}{=} \hat{\psi}_{\lambda'}^\dagger (x'_1, t) \ldots \hat{\psi}_{\lambda'}^\dagger (x'_N, t),
\]

\[
\hat{\Psi}_\lambda (x, t) \overset{d}{=} \hat{\psi}_\lambda (x_1, t) \ldots \hat{\psi}_\lambda (x_M, t),
\]

\[
C_{NM} \overset{d}{=} C_{\lambda_1 \ldots \lambda_N; \lambda_1 \ldots \lambda_M} (x'_1, \ldots, x'_N, x_1, \ldots, x_M),
\]

\[
C_{NM} = C_{MN}^*,
\]

and \( \hat{\psi}_\lambda \overset{d}{=} \lambda \)-component of the \( i \)-th field operator \( \hat{\psi} \).

A. 11 (Local Commutativity)

[M] The field operators satisfy the following commutation or anticommutation rules at equal times:

\[
[\hat{\psi}_\lambda (x, t), \hat{\psi}_{\lambda'} (y, t)]_\mp = [\hat{\psi}_{\lambda'}^\dagger (x, t), \hat{\psi}_\lambda^\dagger (y, t)]_\mp = 0
\]

\[
[\hat{\psi}_\lambda (x, t), \hat{\psi}_{\lambda'}^\dagger (y, t)]_\mp = \delta_{\lambda \lambda'} \delta^3 (x - y).
\]

Remarks:

1. \( \Sigma \times \bar{\Sigma} \) is the class of reference of GQFT. A member \( \langle \sigma, \bar{\sigma} \rangle \) denotes an arbitrary system (so called f-system) composed of basic fields \( \sigma_i \), eventually in interaction with the environment \( \bar{\sigma} \). In particular, we will deal with closed f-systems which are described by \( \langle \sigma, \bar{\sigma}_o \rangle \), with \( \sigma_o \) denoting the empty environment. These physical fields are characterized mathematically by operators acting over a region of space and time (see A.9) and with certain transformation properties
(see A.13). In other words, A.3 characterizes the physical system that the theory refers to, and A.9 characterizes the mathematical concept of a field. Indeed, not all mathematical field must represent a physical (real) entity. However, we will assume that the field operator represent a physical property of the basic field, that is, the amplitude of \( \sigma_i \).

2. We must emphasize that the \textit{primitive concept} of this field theory is the \textit{physical field} that will be characterized (its properties) by an operator. As we shall see, the concept of “particle” is a \textit{derived concept} as a the “quantum” of the field.

3. Note that A.9 states that \( \mathcal{F} \subset A \). Indeed, \( \mathcal{F} \) is a particular set that need to be considered independently of \( A \). Moreover, A.10 states that the properties of a system of fields will be represented as a polynomial of the field operators associated to each basic field.

4. A basic field \( \sigma_i \) is modeled by a functional schema \( b = \langle M, \mathbf{F} \rangle \), where we can identify the set \( M \) as the Cartesian product of certain set \( \Sigma \times \Sigma \times \mathbb{E}_3 \times T \times A \) and \( \mathbf{F} \) as the set of operators \( \langle \hat{\psi}_{-a}(x, t), \ldots , \hat{\psi}_a(x, t) \rangle \). Moreover, the physical space of accessible states is \( V_i = \mathcal{H}_S \oplus \mathcal{H}_A \subset \mathcal{H}_E \) (see A.4 and Thm.11).

5. Since the reference of nonrelativistic quantum field theories are basic fields, in order to obtain a unitary quantum theory of fields, we need only one equation of motion. This situation is different from that of the classical theories of fields, where one equation of motion for fields and another for particles are needed.

4.6 \textbf{Definitions}

\textbf{Def. 20 (Vector basis of an f-system } \sigma )

\[ |q_1, q_2, \ldots , q_N \rangle = |q_1 \rangle |q_2 \rangle \cdots |q_N \rangle , \]

\textit{where } \( |q_i \rangle \overset{d}{=} |x_i , t ; \lambda_i \rangle \).
Def. 21 (Symmetric Hilbert spaces)
\[ \mathcal{H}_S \equiv \{ |q_1, q_2, \ldots, q_N\rangle | |q_1, q_2, \ldots, q_N\rangle \in \mathcal{H}_E \wedge |q_1, q_2, \ldots, q_N\rangle = |q_{P_1}, q_{P_2}, \ldots, q_{P_N}\rangle \}. \]

Def. 22 (Antisymmetric Hilbert spaces)
\[ \mathcal{H}_A \equiv \{ |q_1, q_2, \ldots, q_N\rangle | |q_1, q_2, \ldots, q_N\rangle \in \mathcal{H}_E \wedge |q_1, q_2, \ldots, q_N\rangle = (-1)^\mu |q_{P_1}, q_{P_2}, \ldots, q_{P_N}\rangle \} \]
with \( \mu \) the number of permutations \( P \) of \( |q_i\rangle \).

4.7 Theorems

Thm. 6 The general basis vector of an f-system \( \sigma \) can be written in terms of operators \( \hat{\psi}^\dagger \) acting on the vacuum state as follows:
\[ |q_1, q_2, \ldots, q_N\rangle = \hat{\psi}_1^\dagger(q_1) \hat{\psi}_2^\dagger(q_2) \ldots \hat{\psi}_N^\dagger(q_N)|0\rangle, \]
where \( \hat{\psi}_i^\dagger(q_i) \equiv \hat{\psi}_\lambda^\dagger(x_i, t) \).

Proof: From A. 4 and A. 9

Thm. 7 The general basis vector \( |q_1, q_2, \ldots, q_N\rangle \) of an f-system \( \sigma \) can be written as:
\[ |q_1, q_2, \ldots, q_N\rangle = (\pm)^\mu |q_{P_1}, q_{P_2}, \ldots, q_{P_N}\rangle, \]
where \( \mu \) is the number of permutations of the operators \( \psi^\dagger \) associated to identical basic fields \( \sigma_i \).

Proof: From Thm. 6 and A. 11

Thm. 8 The field operator \( \hat{\psi}_\lambda(x, t) \) acts on the basis vector \( |x, t; \lambda\rangle \) as:
\[ \hat{\psi}_\lambda(x, t)|x, t; \lambda\rangle = |0\rangle. \]

Proof: Using A. 4, A. 9 and A. 11

Thm. 9 The action of operator \( \hat{\psi}(q) \equiv \hat{\psi}_\lambda(x, t) \) on a general basis vector \( |q_1, q_2, \ldots, q_N\rangle \) is given by,
\[ \hat{\psi}(q)|q_1, q_2, \ldots, q_N\rangle = \sum_{r=1}^{N} (\pm)^{r+1} \delta(q - q_r)|q_1, \ldots, q_{r-1}, q_{r+1}, \ldots, q_N\rangle \]
with +1 or −1 for \( |q_1, q_2, \ldots, q_N\rangle \in \mathcal{H}_S \) and \( \mathcal{H}_A \) respectively.
Proof: From Thm. 8 or A. 11

Thm. 10 (Orthonormality) The following orthonormality condition holds for a general basis vector:

\[ \langle q'_1, q'_2, \ldots, q'_N \mid q_1, q_2, \ldots, q_M \rangle = \delta_{NM} \sum_{\mu} (\pm)^P \prod \delta(q_i - q'_P) \]

Proof: From Thm. 6 Thm. 7 and A.11 and using the normalization of vacuum state.

Thm. 11 The space of accessible states to a f-system \( \sigma \in \Sigma \) is given by \( \mathcal{H}_S \oplus \mathcal{H}_A \subset \mathcal{H}_E \).

Proof: See ref. 22

Remark: From Thm. 11 we deduce that non-symmetric states just do not exist. This means that there is no arbitrary restriction to symmetric or antisymmetric states.

Group V: Symmetries and Group Structure

A. 12 (Galilei Group)

[S] \( G \triangleq \) Galilei Group.

[S] \( (\forall g)_G(g = (b, a, v, R) \triangleq \) the general element of the Galilei group).

[M] The structure of Lie algebra \( \tilde{G} \) of the extended group \( \tilde{G} \), is generated by \( \{ \hat{H}, \hat{P}_i, \hat{K}_i, \hat{J}_i, \hat{M} \} \subset A \)

[S] \( \hat{H} \triangleq \) the time translations generator.

[S] \( (\forall \sigma)_\Sigma(\text{eiv}_\hat{H} \triangleq E \triangleq \) the energy of \( \sigma \)).

[S] \( \hat{P}_i \triangleq \) the spatial translations generator.

[S] \( (\forall \sigma)_\Sigma(\text{eiv}\hat{P}_i \triangleq p_i \triangleq \) the \( i \)-th component of linear momentum of \( \sigma \)).

[S] \( \hat{K}_i \triangleq \) the generator of the pure transformations of Galilei.
[S] \( \hat{J}_i \triangleq \) the generator of spatial rotations.

[S] \((\forall \sigma)(e^{i v \hat{J}_i} = j_i) \triangleq \) the \(i\)-th component of angular momentum of \(\sigma\).

[M] \(\hat{M}\) has a discrete spectrum of real eigenvalues and its called mass operator.

[S] \((\forall \sigma)(e^{i v \hat{M}} \triangleq m) \triangleq \) mass of \(\sigma\).

[P] The vacuum state \(|0\rangle\) is the zero-mass state that is invariant under Galilean transformations (up to a possible phase factor accounting for a constant energy).

A. 13 (Field Transformations)

[M] Let \(\{\hat{\psi}_\lambda(x, t); \lambda = -s, \ldots, s\}\) be a set of field operators. They transform locally under Galilean transformations \(g = (b, a, v, R)\) as:

\[
\hat{U}(g)\hat{\psi}_\lambda(x, t)\hat{U}^{-1}(g) = \exp\left[i\frac{\hbar}{\sqrt{2}}m\gamma(g; x, t)\right] \sum_{\lambda'} D_{\lambda\lambda'}^{(s)}(R^{-1})\hat{\psi}_{\lambda'}(x', t')
\]

where \(D_{\lambda\lambda'}^{(s)}\) is the \((2s + 1)\)-dimensional unitary matrix representation of the rotation group.

\[
\gamma(g; x, t) = \frac{1}{2}v^2 t + v.Rx,
\]

and

\[
x' = Rx + vt + a,
\]

\[
t' = t + b.
\]

Remarks:

1. The [A.12] states the set of operators which are the generators of the Lie algebra \(\mathcal{G}\) of the Galilei group. The elements of this set are semantically interpreted as specific properties of the system.
2. The core of the GQFT is stated in A.13. This axiom characterizes the transformation properties of the systems, i.e. its Galilean symmetry properties. Note, moreover, that the equation given in A.13 may be seen like a finite interval field equation in comparison with its differential form (see Thm. 14).

4.8 Theorems

Thm. 12 (Mass conservation) The Galilean invariance of a system requires mass conservation.

Proof: Since the operator \( \hat{M} \) commutes with all the elements of the extended Lie algebra \( \tilde{G} \), in particular with the Hamiltonian \( \hat{H} \).

Thm. 13 (Bargmann superselection “rule”) \( \mathcal{H} \) decomposes into mutually orthogonal vectors are eigenvectors of \( \hat{M} \).

Proof: From the extended algebra \( \tilde{G} \) and A.12.

Thm. 14 The equation of motion satisfied by a given field operator is:

\[ i\hbar \frac{\partial}{\partial t} \hat{\psi}_\lambda = [\hat{\psi}_\lambda, \hat{H}] \]

Proof: Consequence of the invariance under time displacement.

Thm. 15 The field operator \( \hat{\psi}_\lambda^\dagger(\mathbf{x},t) \) transforms locally under a Galilean transformation as:

\[ \hat{U}(g)\hat{\psi}_\lambda^\dagger(\mathbf{x},t)\hat{U}^{-1}(g) = \exp[-\frac{i}{\hbar} m \gamma(g; \mathbf{x},t)] \sum_{\lambda'} D^{(s)}_{\lambda\lambda'}(R) \hat{\psi}_{\lambda'}^\dagger(\mathbf{x}',t'). \]

Proof: From A.13 and taking into account that \( D_{\lambda\lambda'}(R^{-1}) = D^\dagger_{\lambda\lambda'}(R) = (D_{\lambda\lambda'}(R))^* \) since \( D(R) \) is unitary.

Thm. 16 No Galilean field operator of non-zero mass can be hermitian.

Proof: Compare Thm. 15 with the transformation rule of A.13.

Remarks:
1. The simplest of the representations of the Galilei group is obtained for \( D(R) = 1 \). This is the scalar representation of the rotation group, which describes particles of zero spin.

2. Thm. 13 shows that the existence of a state which is a superposition of two states with different masses, is in conflict with Galilean invariance. This is a very strong constraint that does not have a counterpart in relativistic theories since it restricts the possible kind of process allowed by a Galilean theory.

3. Since the expanded Lie algebra is free of central charges and has only ordinary representations, we are not forced to “impose” any superselection “rule.”

4. From the commutation relations of the Lie algebra \( G \) given in Def. 16, we can see that the Hamiltonian \( \hat{H} \) does not appear on the right hand side. This means that in order to obtain a covariant theory, the interaction Hamiltonian must be invariant under a Galilean transformation, but the other generators will remain the same as the free theory. This is to be compared with the relativistic situation, where \( \hat{H} \) appears in the Lie algebra of Poincaré group, when the commutators of pure Lorentz and space translation generators are taken. Thus, when the Hamiltonian is modified by an interaction Hamiltonian, it requires a modification of the pure Lorentz transformation, while no modification is needed in the nonrelativistic case when the interaction Hamiltonian is Galilean invariant. This explain why is so easy to construct nontrivial Galilean theories.

We shall now give some formal definitions motivated by the axioms stated above:

4.9 Definitions

Def. 23 (Free particle state) The state of a basic field characterized by an irreducible representation of the Galilei group with null internal energy, denoted by \([m, W = 0, s]\), will be called free particle state.
Def. 24 (Total number operator) The total number of particles operator is given by

\[ \hat{N}(t) = \sum_{\lambda} \int d^3x \hat{\psi}^\dagger_{\lambda}(x, t) \hat{\psi}_{\lambda}(x, t). \]

Def. 25 (Free Hamiltonian) The free Hamiltonian is given by

\[ \hat{H}_0 = \sum_{\lambda} \int d^3x \hat{\psi}^\dagger_{\lambda}(x, t) \left( -\frac{\hbar^2}{2m} \nabla^2 \right) \hat{\psi}_{\lambda}(x, t). \]

Remark:

Usually, the “particle states” are called “particles”. We use the same convention in order to respect this widely used designation.

So far, the field operators describe only properties of free, non-interacting fields. The multi-particle states encountered in free field theories are eigenstates of the free Hamiltonian. This means that they remain unchanged as long as the system is left undisturbed. The following axiom relates the field operator associated to interacting fields with the field operator associated to free fields.

Group VI: Asymptotic Condition

A. 14

[P] Let \( \hat{H} \in A \) such that \( \hat{H} = \hat{H}_0 + \hat{V} \), with \( \hat{H} \to \hat{H}_0 \) as \( t \to \pm \infty \), where \( \hat{H}_0 \) = free Hamiltonian and \( \hat{V} \) = interaction Hamiltonian. Then, the field operators of \( \hat{\psi}_\lambda(x, t) \) evolving with \( \hat{H} \) behave as the free fields \( \hat{\psi}^{in}_\lambda(x, t) \) or \( \hat{\psi}^{out}_\lambda(x, t) \) for \( t \to \pm \infty \), i.e.:

\[ \hat{\psi}^{in}_\lambda(x, t) = \lim_{t \to -\infty} \hat{\psi}_\lambda(x, t), \]

\[ \hat{\psi}^{out}_\lambda(x, t) = \lim_{t \to +\infty} \hat{\psi}_\lambda(x, t). \]

Remarks:
1. This axiom is not valid for Coulombian potentials. To include this kind of interaction in the theory we need to redefine the operators $\hat{H}_0$ and $\hat{V}$ in order to satisfy the asymptotic condition.

2. Redmond and Uretsky,\textsuperscript{1} showed that in the case of the so-called “second quantization” the in-out operators cannot be defined for Coulomb potentials or potential which decrease more slowly that the Coulomb ones for large separations.

The $S$-matrix can be defined in the same manner as in the relativistic theory, from the following conventions:

### 4.10 Definitions

**Def. 26** Let $\hat{H}_0$ and $\hat{H} = \hat{H}_0 + \hat{V}$ be the free and full Hamiltonians. We define the Möller operator:

$$\hat{\Omega}(t) \overset{\text{DF}}{=} e^{i\hat{H}t}e^{-\frac{i}{\hbar}\hat{H}_0 t}.$$ 

**Def. 27** (Evolution operator)

$$\hat{U}(t, t_0) \overset{\text{DF}}{=} \hat{\Omega}(t)^\dagger\hat{\Omega}(t_0) = e^{\frac{i}{\hbar}\hat{H}_0 t_0}e^{\frac{i}{\hbar}(\hat{H}(t-t_0))}e^{-\frac{i}{\hbar}\hat{H}_0 t_0}.$$ 

**Def. 28** (Scattering operator)

$$\hat{S} \overset{\text{DF}}{=} \lim_{t_0 \to -\infty} \hat{U}(t, t_0).$$ 

**Def. 29** (Asymptotic states) Let $|\alpha\rangle$ be the free-particle states and $\hat{\Omega}(t)$ the Möller operator.

a) The in-states are defined as: $|\alpha\rangle_{\text{in}} \overset{\text{DF}}{=} \lim_{t \to -\infty} \hat{\Omega}(t)|\alpha\rangle$.

b) The out-states are defined as: $|\alpha\rangle_{\text{out}} \overset{\text{DF}}{=} \lim_{t \to +\infty} \hat{\Omega}(t)|\alpha\rangle$.

**Def. 30** (S-matrix) Let us consider a process in which an initial configuration of particles $|\alpha\rangle_{\text{in}}$ ends up as a final configuration $|\beta\rangle_{\text{out}}$. The elements of the $S$-matrix are given by

$$S_{\alpha\beta} \overset{\text{DF}}{=} \langle \beta | \alpha \rangle_{\text{in}},$$

and they represent the probability amplitude for the transition $|\alpha\rangle \to |\beta\rangle$. 

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Remark:

We shall agree to call process an evolution of field depending on time. However, a formal definition of a process can be found in.\(^8\)

### 4.11 Theorems

**Thm. 17 (Normalization)** The in-states and out-states are normalized like the free particle states; i.e.:

\[
\langle\alpha|\beta\rangle_{\text{in}} = \langle\alpha|\beta\rangle_{\text{out}} = \langle\alpha|\beta\rangle = \delta(\beta - \alpha).
\]

**Proof:** From Def. \(^{29}\) and taking into account that the states are time independent.

**Thm. 18** We can express the \(\hat{S}\) operator between free particle states \(\langle\beta|\) and \(|\alpha\rangle\) as:

\[
S_{\beta\alpha} = \langle\beta|\hat{S}\alpha\rangle.
\]

**Proof:** From Def. \(^{28}\) and Def. \(^{30}\)

### 5 GENERAL STRUCTURE OF GQFT

In order to study the general structure of GQFT we shall now deduce some of the specific theorems entailed by the axiomatic system. The following conclusions were partially derived in ref.\(^2,4\) The first two theorems restrain the possible kind of processes allowed by Galilean invariance.

#### 5.1 Theorems

**Thm. 19** Let \(\{\hat{\psi}_\lambda^\dagger(i,\mathbf{x}_i, t), i = 1 \ldots N\}\) and \(\{\hat{\psi}_\lambda(x_j, t), j = 1 \ldots M\}\) be the sets of field operator associated to basic fields with the same mass \(m\). In order that the operator \(\hat{O} = \hat{U}(g)\hat{O}\hat{U}^{-1}(g)\) given in A. \(^{10}\) be invariant under Galilean transformations, three conditions must be fulfilled:

a) There are as many \(\hat{\psi}\) as \(\hat{\psi}\) field operators, (i.e. \(N = M\)).

b) The \(\hat{\psi}\) and \(\hat{\psi}\) field operators act pairwise at the same point.
c) The coefficients $C_{NM}$ are Galilean invariant, that is:

$$C_{\lambda_1'...\lambda_N'\lambda_1...\lambda_M}(x'_1,...,x'_N,x_1,...,x_M) = \sum_{\lambda'_1...\lambda'_N} \sum_{\lambda_1...\lambda_M} D_{\lambda_1'\lambda_1}(R)...D_{\lambda_N'\lambda_N}(R)D_{\lambda_1'\lambda_1}(R^{-1})...D_{\lambda_M'\lambda_M}(R^{-1}) \times C_{\lambda_1'...\lambda_N'\lambda_1...\lambda_M}(x'_1,...,x'_N,x_1,...,x_M).$$

**Proof:** See Appendix.

**Thm. 20 (Conservation of the number of particles)** The operator $\hat{O}$ given in Thm. [19] commutes with the number operator $\hat{N}(t)$.

**Proof:** From a) and b) of Thm. [19] and Def. [24]

**Thm. 21** Let $\{\hat{\psi}_{i}(x,t), i = 1...N\}$ and $\{\hat{\psi}_{j}(x,t), j = 1...M\}$ be the sets of field operators associated to basic fields with different masses denoted by $m'_i$ and $m_j$ respectively and taken at the same point of space-time. The operator $\hat{O}$ will be invariant under Galilean transformations if

$$\sum_{i}^{N} m'_i = \sum_{j}^{M} m_j.$$

**Proof:** See Appendix.

**Thm. 22 (Non-conservation of the number of particles)** The operator $\tilde{O}$ given in Thm. [21] does not commute with the number operator.

**Proof:** From the fact that in general $N \neq M$.

Remarks:

1. As an example of Thm. [19] we have the “two-body” interaction Hamiltonian $\hat{V}$ which describes the mutual action between two “particles”,

$$\tilde{V} = \frac{1}{2} \int d^3x d^3x' \hat{\psi}^\dagger(x,t)\hat{\psi}^\dagger(x',t)\tilde{V}(|x - x'|)\hat{\psi}(x',t)\hat{\psi}(x,t)$$

In this way, second quantization appears as a particular case of another theory when certain subsidiary assumptions are adjoined.
2. From Thm. 19 and Thm. 20 we can see that in a theory with only one kind of particles of fixed mass, mass conservation necessarily implies conservation of the number of particles and forbids production processes.  

3. As we can see from Thm. 21 and Thm. 22 it is possible to have Galilean invariant theories that describe production processes in which the number of particles is not conserved but the total mass is conserved. In this case we have to include field operators associated to several kinds of basic fields of different masses and acting at the same point.  

4. Thm. 21 provides a rule for building a theory that describe production processes. It means that the field operators must be chosen in agreement with the mass conservation as expressed in Thm. 22. This is a strong constraint and it does not exist in relativistic field theory.  

5.2 Definitions  
We inquire now whether the spin-statistics relation and crossing symmetry are required by GQFT. In order to answer this question, we need to introduce the concept of an “antiparticle”. This concept is also motivated by the possibility of describing “pair production” processes, as can be seen from Thm. 21. Thus, this model allows us to interpret particles with negative mass eigenvalues as antiparticles, i.e.,  

Def. 31 (Free Antiparticle state) The state of a basic field characterized by an irreducible representation of the Galilei group with null internal energy, denoted by $[-m, -W = 0, s^*]$, where $s^*$ denote the complex conjugate representation of $D(R)$ will be called a free antiparticle state.  

According to the Peter-Weyl theorem, the functions $D^{(s)}_{\lambda\lambda'}(R)$ form a complete basis in the space of square integrable functions (or operator valued functions). For the non-compact part of the Galilei group we shall use the fact that we are dealing with a rigged Hilbert space $\mathcal{H}_E$, where the eigenfunctions of infinite norm form a complete set. In this way, we can take the next definition (usually referred as the plane wave expansion of the field operator) as “analogous” to the Peter-Weyl theorem for non-compact groups. Thus, we can define the following field operators:
Def. 32 (Annihilation field operator) The free field operator associated with a basic field $\sigma_i$ given by:

$$\hat{\psi}^-(x, t) \overset{Df}{=} (2\pi)^{-3/2} \sum \int_\mathcal{M} d\mu(p, E)e^{i\mu(Et - p \cdot x)} D^{(s)}_{\lambda\lambda'}(R^{-1}) \hat{a}(p, E, \lambda').$$

will be called the “annihilation” field operator of free particles.

We can define a new field operator by taking the hermitian conjugate of the field defined above,

Def. 33 (Creation field operator) The free field operator associated a basic field $\sigma_i$ given by:

$$\hat{\psi}^+(x, t) \overset{Df}{=} (2\pi)^{-3/2} \sum \int_\mathcal{M} d\mu(p, E)e^{-i\mu(Et - p \cdot x)} D^{(s)}_{\lambda\lambda'}(R) \hat{a}^+(p, E, \lambda').$$

will be called the “creation” field operator of free particles.

Remarks:

1. As it is well known, the mass-shell condition $p^2 + m^2 = 0$ in the relativistic case provides negative and positive solutions. This is related to the existence of an antiparticle state for every ordinary particle state, where we assume that one particle is the antiparticle of another if their masses and spin are equal and their charges are opposite. In the nonrelativistic case the eigenvalues of $\hat{M}$ can be positive or negative, and we can interpret to a particle with positive mass-eigenvalue and the antiparticle with the opposite mass-eigenvalue. So the mass behaves in nonrelativistic case as a kind of charge. This is due to the fact that there exists a “Bargmann’s superselection rule”.

2. We won’t assume that every particle has an antiparticle since this must be proved from our axioms. In the case of particles that are their own antiparticles we take $\hat{b}(q) = \hat{a}(q)$.

3. Since quantum field theory deals with transmutations of systems, the operators given in Def. 32 and Def. 33 conceptualize the change brought about by the decrement or increment in the number of particles of the system.
5.3 Theorems

**Thm. 23** The operator coefficients \( \hat{a}(k) \overset{d}{=} \hat{a}(p, E, \lambda) \), \( \hat{a}^\dagger(k) \overset{d}{=} \hat{a}^\dagger(p, E, \lambda) \) satisfy

\[
[\hat{a}(k'), \hat{a}^\dagger(k)]_\mp = \delta(k' - k),
\]
\[
[\hat{a}(k'), \hat{a}(k)]_\mp = [\hat{a}^\dagger(k'), \hat{a}^\dagger(k)]_\mp = 0.
\]

**Proof:** From A. 11 Def. 33 and Def. 32.

**Thm. 24** The operators \( \hat{a}^\dagger(k) \) and \( \hat{a}(k) \) that satisfy the above commutation relations act on a general state \( |k_1, k_2, \ldots, k_N\rangle \) as:

a) \( \hat{a}^\dagger(k) |k_1, k_2, \ldots, k_N\rangle = |k, k_1, k_2, \ldots, k_N\rangle \).

b) \( \hat{a}(k) |k_1, k_2, \ldots, k_N\rangle = \sum_{r=1}^{N} (\pm)^{r+1} \delta(k - k_r) |k_1, \ldots, k_{r-1}, k_{r+1}, \ldots, k_N\rangle \) with +1 or −1 for \( |k_1, k_2, \ldots, k_N\rangle \in \mathcal{H}_S \) and \( \mathcal{H}_A \) respectively.

**Proof:** From Thm. 6 Def. 32 and Def. 33.

**Thm. 25** The operators \( \hat{a} \) and \( \hat{a}^\dagger \) are mutually adjoint: \( (\hat{a}^\dagger(k))^\dagger = \hat{a}(k) \).

**Proof:** Using Def. 32 and Def. 33.

**Thm. 26** The operator \( \hat{a}^\dagger(p, E, \lambda) \) transforms under an arbitrary Galilei transformation according to:

\[
\hat{U}(g) \hat{a}^\dagger(p, E, \lambda) \hat{U}^{-1}(g) = e^{-\frac{i\hbar}{2}(E' b - p' a)} \sum_{\lambda'} D^{(s)}_{\lambda' \lambda}(R) \hat{a}^\dagger(p', E', \lambda')
\]

with \( p', E' \) and \( D^{(s)}(R) \) as given in Thm. 2.

**Proof:** From Thm. 24 and Thm. 2 (Alternatively: from A. 13 and Def. 33).

**Thm. 27** The operator \( \hat{a}(p, E, \lambda) \) transforms under a Galilei transformation as:

\[
\hat{U}(g) \hat{a}(p, E, \lambda) \hat{U}^{-1}(g) = e^{\frac{i\hbar}{2}(E' b - p' a)} \sum_{\lambda'} D^{(s)}_{\lambda' \lambda}(R^{-1}) \hat{a}(p', E', \lambda').
\]
Proof: Using Thm. 27 and Thm. 25

In order to put the transformation of $\hat{\psi}_\lambda^\dagger(x, t)$ in a similar form as $\hat{\psi}_\lambda(x, t)$, it will be convenient to consider the following theorem (we follow the same idea given by 19 for the relativistic case),

**Thm. 28** The operators $\hat{\alpha}(p, E, \lambda) \overset{Df}{\equiv} \sum_{\lambda'} D_{\lambda\lambda'}(R^{-1}) \hat{\alpha}(p', E', \lambda')$ and $\hat{\beta}^\dagger(p, E, \lambda) \overset{Df}{\equiv} \sum_{\lambda'} \{D(R^{-1})C^{-1}\}_{\lambda\lambda'} \hat{\beta}^\dagger(p', E', \lambda')$ transform as:

$$
\hat{U}(g) \hat{\alpha}(p, E, \lambda) \hat{U}^{-1}(g) = e^{i(E'b - p'a)} \sum_{\lambda'} D_{\lambda\lambda'}(R^{-1}) \hat{\alpha}(p', E', \lambda'),
$$

$$
\hat{U}(g) \hat{\beta}^\dagger(p, E, \lambda) \hat{U}^{-1}(g) = e^{-i(E'b - p'a)} \sum_{\lambda'} D_{\lambda\lambda'}(R^{-1}) \hat{\beta}^\dagger(p', E', \lambda').
$$

Proof: The transformation of $\hat{\alpha}(p, E, \lambda)$ follows from Thm. 27. To obtain $\hat{\beta}^\dagger(p, E, \lambda)$, use the Thm. 26 and the following properties of the unitary representation of $D(R)$:

$$(D(s)(R))^* = CD(s)(R)C^{-1}$$

so, we can write:

$$D_{\lambda\lambda'}^s(R) = (D_{\lambda\lambda'}^s(R^{-1}))^* = (CD_{\lambda\lambda'}^s(R^{-1})C^{-1})_{\lambda\lambda'}$$

where $C$ is a $(2s + 1) \times (2s + 1)$ matrix with

$$C^*[C = (-1)^{2s} \ C^\dagger C = 1$$

**Thm. 29** The creation field operator of antiparticles given by:

$$\hat{\psi}^-_{\lambda}(x, t) \overset{Df}{\equiv} (2\pi)^{-3/2} \int \mu(p, E) e^{-i(E'b - p'a)} \hat{\beta}^\dagger(p', E', \lambda)$$

transforms as required by A. 13, i.e.:

$$\hat{U}(g) \hat{\psi}^-_{\lambda}(x, t) \hat{U}^{-1}(g) = \exp\left[-\frac{i}{\hbar}(m \gamma(g; x, t)) \right] \sum_{\lambda'} D_{\lambda\lambda'}^s(R^{-1}) \hat{\psi}^-_{\lambda'}(x', t').$$

Proof: Using Thm. 28 (See Appendix for details).

In order to obtain Hermitian operators and field operators that satisfy the transformation rule of A.13 we construct a field operator by taking linear combinations of $\hat{\psi}^-_{\lambda}(x, t)$ and $\hat{\psi}^-_{\lambda}(x, t)$. That is, we need the following definition:
Def. 34 The local field operators constructed as linear combinations of particles annihilation field operator and antiparticles creation field operator are given by:

\[
\hat{\psi}_\lambda(x, t) = \xi \hat{\psi}^-_\lambda(x, t) + \eta \hat{\psi}^+\bar{\lambda}(x, t)
= (2\pi)^{-3/2} \int d\mu(p, E) \{\xi e^{i\bar{h}(E t - p \cdot x)} \hat{\alpha}(p, E, \lambda) + \eta e^{-i\bar{h}(E t - p \cdot x)} \hat{\beta}^\dagger(p, E, \lambda)\}.
\]

Thm. 30 The field operator \(\hat{\psi}_\lambda(x, t)\) as given in Def. 34 has the property of Galilei transformation given in A. 13.

Proof: From Def. 34 and using Thm. 28 and Thm. 29.

Thm. 31 The field operator \(\hat{\psi}_\lambda(x, t)\) as given in Def. 34 satisfies the commutation or anticommutation rule:

\[
[\hat{\psi}_\lambda(x, t), \hat{\psi}_\lambda^\dagger(y, t)] = (|\xi|^2 \mp |\eta|^2) \delta_{\lambda\lambda'} \delta^3(x - y).
\]

Proof: Using A. 11.

Thus, this theorem leads immediately to the two most important consequences of the Galilean theory,

Thm. 32 (Crossing and Statistics) Statistics: No statistics can be deduced from a Galilean theory field.

Crossing: A full crossing symmetry with \(|\xi| = |\eta|\) is not required.

Proof: The commutation rule of Thm. 31 satisfies the local commutability condition given in A. 11 for any value of \(\xi\) and \(\eta\).

Remarks:

1. It is easy to see that the field operator of Def. 34 cannot be built as a linear combination of \(\hat{a}\) and \(\hat{a}^\dagger\) because it does not transform as required by A. 13 since \(\hat{a}\) and \(\hat{a}^\dagger\) annihilate and create the same particle with mass \(m\).

2. The case of a particle without antiparticle is satisfied by \(\eta = 0\).
3. **Thm. 32** does not select a sign for the commutator (i.e., $\mp$) in order to describe a particle with integer or half-integer spin. In other words, from the axiom of local commutability (A. 11) we cannot decide whether a particle with integer (or half-integer) spin must be a boson or a fermion. Thus, Galilean field theories do not imply any relation between spin and statistics.

4. There is no symmetry in GQFT between particle and antiparticle. Indeed, any value of $\xi$ and $\eta$ are admissible, even the non existence of antiparticles. So, CPT theorem has no analogous in GQFT.

5. From **Thm. 32** we can appreciate the weakening of the commutation relations of Galilean quantum field theory since in relativistic quantum field theory, crossing symmetry and the relation between spin and statistic arises from the “causality” requirement, as has been shown by Weinberg.¹⁹

6 **CONCLUSIONS**

We have presented a formal axiomatization of the Galilean quantum field theories. These theories are generated by ten primitive concepts, among them the reference class, i.e., fields. The constraints are obtained from the choice of a symmetry group plus appropriate commutation rules. This means that we consider the fields as things with properties represented by operators that satisfy certain symmetry transformations. Fields are unobservable but they should not be regarded as auxiliary devices with no physical meaning. Indeed, due to its success, the program of explaining the behavior of matter in terms of fields should be considered as an approximately true model of reality. On the other hand, free particles are described as states of the basic fields characterized by an irreducible representation of the Galilei group with null internal energy. This does not imply that particles are consequences of symmetries since symmetries are symmetries of properties of things. In other words, no things, no symmetries.

As a comparison, second quantization is a generalization of quantum mechanics to the case of $n$ “particles” and differs of a quantum field theory in the primitive concepts. In this theory, the primitive concept is the quantum field as an “extended object” whose elementary excitations describe a
“quanton”. This last real object is the fundamental primitive of quantum mechanics of systems (see reference\textsuperscript{21,22}).

The existence of non-trivial Galilean quantum field theories is asserted by the “Gali-Lee” model and other examples studied by Lévy-Leblond.\textsuperscript{4} Moreover, non-relativistic second quantization itself can be interpreted as a Galilean quantum field theory, whose elementary excitations are “quanton-like” objects.

The general structure of GQFT has a number of interesting physical consequences that can be resumed as follows:

1. From the property of field transformations, the spin of the particles can be described in a Galilean frame.

2. Mass conservation is obtained from the assumption of Galilei group as symmetry group of the theory. Moreover, Bargmann’s superselection rule is obtained as a theorem that prevents the superposition of states of different masses.

3. Galilean invariant operators describing processes with number of particles conservation of the same mass must be constructed starting from creation and annihilation operators acting pairwise at the same point.

4. In spite of the fact that the Bargmann’s superselection rule imposes strong constraints, production process with non-conservation of number of particles are admitted in a Galilean theory. To this end, the field operators must be associated to particles of different mass.

5. In order to describe pair production, antiparticles with negative mass-eigenvalue can be defined. Thus, the counterpart to the particle-antiparticle relation is not lost in a nonrelativistic frame.

6. However, no crossing symmetry between particle and antiparticle as in a relativistic theory can be deduced.

7. Neither can the spin statistics relation be derived from local commutability. So, the CPT theorem has not counterpart in a Galilean theory.

Let us comment very briefly some philosophical consequences of the present axiomatization. Using the semantic tools defined in reference\textsuperscript{6} one can
identify from our axiom system the physical reference class of the theory, i.e., the set of physical objects it describes. The reference class is \( \mathcal{R} = (\Sigma \times \bar{\Sigma}) \), i.e. quantum field systems and their environments. There are no “observers” or “experiments” in the interpretation of either the axioms or the theorems of the theory although the latter (but not the former) can be analyzed within the theory using suitable additional hypothesis. This results shows the realistic character of the theory. Note that the reference class can be identified only when the axioms, background and primitive basis of the theory have been stated in a formal way.

To summarize, the benefit of constructing this axiomatization becomes clear since it provides a deeper understanding of Galilean Quantum Field Theories and shows that they have a very rich structure. We think that the consequences of this structure is a valuable tool to “measure” the stronger results coming from relativistic theory. However, this task cannot be make completely until its realistic axiomatization that will be presented in a future paper.

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Appendix

Proof of Thm. 19

The transformation of the operator can be written as:

\[
\hat{O} = \hat{U}(g)\hat{O}\hat{U}^{-1}(g) = \sum_{N,M=0}^{\infty} \sum_{x_1',...x_N',\lambda_1,...,\lambda_M} \int Dx'Dx\hat{U}(g)\hat{\Psi}_{x'}(x',t)U^{-1}(g) \times \\
\hat{U}(g)\hat{\Psi}_{x}(x,t)U^{-1}(g)\mathcal{C}_{NM}
\]

where

\[
\hat{\Psi}_{x}(x',t) = \hat{U}\hat{\Psi}_{x'}(x',t)\hat{U}^{-1} = \exp\left\{ -i\frac{\hbar}{m} \left[ \frac{1}{2} v^2 N t + v \cdot R(x_1' + ... + x_M') \right] \right\} \times 
\]
\[
\sum_{\lambda_1, \ldots, \lambda_N} D_{\lambda_1, \lambda_1}(R) \ldots D_{\lambda_N, \lambda_N}(R) \psi_{\lambda_1}^\dagger(x_1^\prime, t) \ldots \psi_{\lambda_N}^\dagger(x_N^\prime, t)
\]

\[
\hat{\Psi}(x, \bar{t}) = \hat{U}\hat{\Psi}_\lambda(x, t)\hat{U}^{-1} = \exp\left\{ \frac{i}{\hbar}[m_1^2 + \ldots + m_N^2]\right\} \times
\]

\[
\sum_{\lambda_1, \ldots, \lambda_M} D_{\lambda_1, \lambda_1}(R^{-1}) \ldots D_{\lambda_M, \lambda_M}(R^{-1}) \psi_{\lambda_1}^\dagger(x_1, \bar{t}) \ldots \psi_{\lambda_M}^\dagger(x_M, \bar{t})
\]

Replacing the last two expressions in \(\hat{O}\) we obtain:

\[
\hat{O} = \hat{U}(g)\hat{O}\hat{U}^{-1}(g) = \sum_{N, M=0}^{\infty} \sum_{\lambda_1, \ldots, \lambda_N} \sum_{\lambda_1, \ldots, \lambda_M} \int D\bar{x}^\prime D\bar{x} \hat{\Psi}_{\lambda}^\dagger(\bar{x}, \bar{t}) \hat{\Psi}_{\lambda}(\bar{x}, \bar{t}) \mathcal{C}_{NM}
\]

where (a) and (b) follow from the requirement that the phase factors cancels. (c) follows immediately.

**Proof of Thm. 21**

The transformed operator takes the form:

\[
\hat{O} = \sum_{N, M=0}^{\infty} \sum_{\lambda_1, \ldots, \lambda_N} \sum_{\lambda_1, \ldots, \lambda_M} \int D\bar{x}^\prime D\bar{x} \hat{\Psi}_{\lambda}^\dagger(\bar{x}, \bar{t}) \hat{\Psi}_{\lambda}(\bar{x}, \bar{t}) \mathcal{C}_{NM}
\]

with

\[
\hat{\Psi}_{\lambda}^\dagger(\bar{x}, \bar{t}) = \exp\left[-\frac{i}{\hbar}(m_1^2 + \ldots + m_N^2)(\frac{1}{2}v^2t + v.Rx)\right] D_{\lambda_1, \lambda_1}(R) \ldots
\]

\[
\ldots D_{\lambda_N, \lambda_N}(R) \psi_{\lambda_1}^\dagger(x_1, \bar{t}) \ldots \psi_{\lambda_N}^\dagger(x_N, \bar{t})
\]

\[
\hat{\Psi}_{\lambda}(\bar{x}, \bar{t}) = \exp\left[\frac{i}{\hbar}(m_1 + \ldots + m_M)(\frac{1}{2}v^2t + v.Rx)\right] D_{\lambda_1, \lambda_1}(R^{-1}) \ldots
\]

\[
\ldots D_{\lambda_M, \lambda_M}(R^{-1}) \psi_{\lambda_1}^\dagger(x_1, \bar{t}) \ldots \psi_{\lambda_M}^\dagger(x_M, \bar{t})
\]

The theorem follows from the above expressions with the requirement of phase factors cancels.
Proof of Thm. 29

By steps:

a) First, we prove the following expression:

\[ e^{-\frac{i}{\hbar}(Et-p.x)}e^{-\frac{i}{\hbar}(E'b-p'.a)} = e^{-\frac{i}{\hbar}(E't'-p'.x')}e^{-\frac{i}{\hbar}\tilde{m}\gamma} \]

with \( \gamma = \frac{1}{2}v^2t + v.Rx \). Using the definitions of \( E \) and \( x \), we can write the LHS as:

\[ \text{LHS} = e^{-\frac{i}{\hbar}[(E' - v.Rp - \frac{1}{2}\tilde{m}v^2)t - Rp.(x' - vt - a)]}e^{-\frac{i}{\hbar}(E'b-p'.a)} \]

then inserting the expression of \( t \) and arranging the terms, we obtain:

\[ \text{LHS} = e^{-\frac{i}{\hbar}(E'(t' - b) - v.Rp(t' - b) - \frac{1}{2}\tilde{m}v^2t - Rp.x' + Rp.v(t' - b) + Rp.a + E'b - p'.a]} \]

doing the same for \( p \) and \( x' \) and canceling we have,

\[ \text{LHS} = e^{-\frac{i}{\hbar}[E't' - \frac{1}{2}\tilde{m}v^2t - (p' - \tilde{m}v).x' + (p' - \tilde{m}v).a - p'.a]} = e^{-\frac{i}{\hbar}[E't' - p'.x' + \frac{1}{2}\tilde{m}v^2t + \tilde{m}v.(Rx + vt + a) - \tilde{m}v.a]} \]

and using the expression of \( \gamma \), we arrive to:

\[ \text{LHS} = e^{-\frac{i}{\hbar}(E't' - p'.x') + \frac{1}{2}\tilde{m}v^2t + \tilde{m}v.Rx] = e^{-\frac{i}{\hbar}(E't' - p'.x')}e^{-\frac{i}{\hbar}\tilde{m}\gamma} \]

b) Now, we prove the transformation rule:

\[ \hat{U}(g)\hat{\psi}^{-c\dagger}_\lambda(x, t)\hat{U}^{-1}(g) = (2\pi)^{-3/2} \int d\mu e^{-\frac{i}{\hbar}(Et-p.x)}\hat{U}(g)\hat{\beta}^\dagger(p, E, \lambda)\hat{U}^{-1}(g) \]

from Thm. 28

\[ \hat{U}(g)\hat{\psi}^{-c\dagger}_\lambda(x, t)\hat{U}^{-1}(g) = (2\pi)^{-3/2} \int d\mu e^{-\frac{i}{\hbar}(Et-p.x)}e^{-\frac{i}{\hbar}(E'b-p'.a)} \times \sum_{\lambda'} D^{(s)}_{\lambda\lambda'}(R^{-1})\hat{\beta}^\dagger(p', E', \lambda') \]

using the expression obtained previously and arranging, we have:

\[ \hat{U}(g)\hat{\psi}^{-c\dagger}_\lambda(x, t)\hat{U}^{-1}(g) = (2\pi)^{-3/2} \int d\mu e^{-\frac{i}{\hbar}(E't'-p'.x')}e^{-\frac{i}{\hbar}\tilde{m}\gamma} \times \sum_{\lambda'} D^{(s)}_{\lambda\lambda'}(R^{-1})\hat{\beta}^\dagger(p', E', \lambda') \]

34
\[ e^{-\frac{i}{\hbar} \tilde{m} \gamma} \sum_{\lambda'} D_{\lambda\lambda'}^{(s)}(R^{-1}) \int d\mu e^{-\frac{i}{\hbar} (E' - p' \cdot x')} \hat{\beta}^\dagger(p', E', \lambda') \]

using the fact that the antiparticle has mass \(\tilde{m} = -m\) and the definition of \(\hat{\psi}_{\lambda'}^{\text{c}}(x', t')\) we obtain finally,

\[ \hat{U}(g) \hat{\psi}_{\lambda}^{\text{c}}(x, t) \hat{U}^{-1}(g) = e^{-\frac{i}{\hbar} \tilde{m} \gamma} \sum_{\lambda'} D_{\lambda\lambda'}^{(s)}(R^{-1}) \hat{\psi}_{\lambda'}^{\text{c}}(x', t') \]

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