Quantum Singularities

D.A. Konkowski\textsuperscript{1},
T.M. Helliwell\textsuperscript{2}, and C. Wieland\textsuperscript{2}
\textsuperscript{1}U.S. Naval Academy, U.S.A. and Queen Mary, University of London, U.K.
\textsuperscript{2}Harvey Mudd College, U.S.A.

The definitions of classical and quantum singularities in general relativity are reviewed. The occurrence of quantum mechanical singularities in certain spherically symmetric and cylindrically symmetric (including infinite line mass) spacetimes is considered. A strong repulsive “potential” near the classical singularity is shown to turn a classically singular spacetime into a quantum mechanically nonsingular spacetime.

1 Introduction

In classical general relativity singularities are not part of the spacetime; they are boundary points indicated by incomplete geodesics in an otherwise maximal spacetime. Thus, at least for timelike and null geodesics, this incompleteness can be considered as the abrupt ending of classical particle paths. What happens if, instead of classical particles, quantum mechanical wave packets are used? This is the question G. Horowitz and D. Marolf \cite{1} set out to answer. We will review their definition here and use simple spherical and cylindrical spacetimes to illustrate the effects drawing a conclusion, in the end, about which classical singularities turn quantum mechanically nonsingular.

2 Classical Singularities

A spacetime is defined as a connected, $C^\infty$, paracompact, Hausdorff manifold $M$ with Lorentzian metric $g_{\mu\nu}$ \cite{2}. A classical singularity in a maximal spacetime is indicated by incomplete geodesics and/or incomplete curves of bounded acceleration \cite{3}.

Singularities in maximal spacetimes can be classified \cite{3} into three basic types: quasiregular, nonscalar curvature, and scalar curvature. The mildest is quasiregular and the strongest is scalar curvature. At a scalar curvature singularity, physical quantities such as energy density and tidal forces diverge in the frame of all observers who approach the singularity. For example, the center of a Schwarzschild black hole or the beginning of a Big Bang cosmology \cite{3} \cite{2}. At a nonscalar curvature singularity,
there exist curves through each point arbitrarily close to the singularity such that observers moving on these curves experience perfectly regular tidal forces. Whimper cosmologies are a good example [4]. For a quasiregular singularity, no observers see physical quantities diverge, even though their worldlines end at the singularity in a finite proper time. The canonical examples are conical singularities (e.g., 2D cones [3] and idealized cosmic strings [5]).

3 Quantum Singularities

To decide whether a spacetime is quantum mechanically singular we will use the criterion proposed by Horowitz and Marolf [1]. They call a spacetime quantum mechanically nonsingular if the evolution of a test wave packet in the spacetime is uniquely determined by the initial wave packet, without having to put arbitrary boundary conditions at the classical singularity. Their construction is restricted to static spacetimes.

According to Horowitz and Marolf, a static spacetime is quantum mechanically singular if the spatial portion of the Klein-Gordon wave operator is not essentially self-adjoint [6]. A relativistic scalar quantum particle with mass $M$ can be described by the positive frequency solution to the Klein-Gordon equation $\frac{\partial^2 \psi}{\partial t^2} = -A \psi$ in a static spacetime where the spatial operator $A$ is defined to be $A = -VD_i(VD_i) + V^2 M^2$ with $V = -\xi^\nu \xi^\nu$. Here $\xi^\nu$ is the timelike Killing field and $D_i$ is the spatial covariant derivative on the static slice $\Sigma$. The appropriate Hilbert space is $L^2(\Sigma)$, the space of square integrable functions on $\Sigma$.

If we initially define the domain of $A$ to be $C_0^\infty(\Sigma)$, $A$ is real, positive, symmetric operator and self-adjoint extensions always exist [4]. If there is only a single, unique extension $A_E$, the $A$ is essentially self-adjoint. In this case, the Klein-Gordon equation for a free scalar particle takes the form [1]: $i \frac{d\psi}{dt} = A_E^{1/2} \psi$ with $\psi(t) = exp(-it(A_E)^{1/2})\psi(0)$. These equations are ambiguous if $A$ is not essentially self adjoint. This fact led Horowitz and Marolf to define quantum mechanically singular spacetimes as those in which $A$ is not essentially self-adjoint. Examples are considered by Horowitz and Marolf [1].

A simple test for essential self-adjointness of the operator (i.e., quantum singularity of the spacetime) may be used [3] [6]. Essentially one takes the solutions to a test equation

$$(\nabla^2 \pm i) \Phi = 0 \quad (1)$$

and looks to see whether there is more than one $L^2$ solution for each $i$ and each choice of separation constant, near the singularity. If there is more than one $L^2$ solution the spacetime is quantum mechanically singular.
4  Spherical Spacetimes

A class of static, spherical spacetimes with timelike singularities were considered first in the Horowitz and Marolf paper [1]. Consider the metric

$$ds^2 = -dt^2 + dr^2 + r^{2p}(d\theta^2 + \sin^2 \theta d\phi^2).$$

This spacetime is geodesically incomplete and thus classically singular unless \(p = 1\). What about quantum mechanically singular? Take the test equation Eq.(1), separate variables, \(\Psi \sim f(r)Y(\theta, \phi)\), and consider the radial equation,

$$f'' + \frac{2p}{r}f' + \left[\pm i - \frac{c}{r^{2p}}\right]f = 0$$

where \(c\) is a constant. Next rewrite the radial equation in Schrodinger form by letting \(f = r^{-p}F\) and obtain

$$F'' + \left[\pm i - \frac{p(p-1)}{r^2} - \frac{c}{r^{2p}}\right]F = 0.$$

Near \(r = 0\), if \(0 < p < 1\) the “potential” is attractive, while if \(p \geq 1\) the “potential” is repulsive. Near \(r = 0\), one solution of the original equation goes like a constant (and is thus \(L^2\) using the appropriate measure) and the other goes like \(r^{1-2p}\) (and is thus \(L^2\) if \(p < \frac{3}{2}\)). We, therefore, see that these spherical spacetimes are quantum mechanically singular, if \(p < \frac{3}{2}\) (unless \(p = 1\), and quantum mechanically singular if \(p > \frac{3}{2}\) (or \(p = 1\)). The spacetimes are quantum mechanically nonsingular if the spacetime metric induces a very repulsive potential.

5  Cylindrical Spacetimes

Next consider a class of static, cylindrical metrics with timelike singularities,

$$ds^2 = -dt^2 + dr^2 + r^{2a}d\phi^2 + r^{2b}dz^2,$$

where \(r\) is a radial coordinate and \(\phi\) is an angular coordinate with the usual ranges. These metrics are geodesically incomplete and thus classically singular unless \(a = 1\) and \(b = 0\) (flat spacetime in cylindrical polar coordinates). What about quantum mechanically singular? Take the test equation Eq.(1), separate variables so \(\Psi \sim f(r)e^{im\phi}e^{ikz}\), and consider the radial equation,

$$f'' = \frac{a+b}{r}f' + \left[\pm i - \frac{m^2}{r^{2a}} - \frac{k^2}{r^{2b}}\right]f = 0.$$ 

We can rewrite the radial equation in Schodinger form by letting \(f = r^{(a+b)/2}F\) and obtain

$$F'' + \left[\pm i - \frac{(a+b)}{2} - \frac{m^2}{r^{2a}} - \frac{k^2}{r^{2b}}\right]F = 0.$$
Assume \( m = 0, k = 0 \), for simplicity. Near \( r = 0 \), if \( a + b < 2 \), the “potential” is attractive, while if \( a + b \geq 2 \), the “potential” is repulsive.

Near \( r = 0 \), one solution of the original equation goes like a constant (and is thus \( L^2 \) using the appropriate measure) and the other goes like \( r^{1-(a+b)} \) (and is thus \( L^2 \) if \( a + b < 3 \)). We, therefore, see that these cylindrical spacetimes are quantum mechanically singular if \( a + b < 3 \) (except for Minkowski spacetime) and quantum mechanically nonsingular if \( a + b \geq 3 \) (or Minkowski spacetime). These cylindrical spacetimes are thus quantum mechanically nonsingular if the spacetime metric induces a very repulsive potential.

## 6 “Infinite Line Mass” Spacetimes

Finally, consider another class of cylindrical spacetimes. For certain parameter values one can interpret the Levi-Civita metric,

\[
ds^2 = r^{4\sigma} dt^2 - r^{8\sigma^2 - 4\sigma} (dr^2 + dz^2) - \frac{r^{2-4\sigma}}{c^2} d\theta^2.
\]

as an “infinite line mass” spacetime. In fact, after some controversy in the literature (see, e.g., [7], [8]), the following interpretations have become somewhat accepted: \( \sigma = 0, 1/2 \) locally flat; \( \sigma = 0, c = 1 \) Minkowski spacetime; \( \sigma = 0, c \neq 1 \) cosmic string spacetime; \( 0 < \sigma < 1/2 \) infinite line mass spacetimes; \( \sigma = 1/2 \) Minkowski spacetime in accelerated coordinates (planar source).

The Levi-Civita metric is static, cylindrically symmetric and classically singular at \( r = 0 \) unless (a) \( \sigma = 0, c = 1 \) or (b) \( \sigma = 1/2 \). What about quantum mechanically singular? One can again use the test equation, separate variables and obtain a radial equation which can be written in Schrodinger form. (For brevity we will just give the results here; details can be found in Konkowski, Helliwell and Wieland [9]). We find that both linearly independent solutions to the second order ordinary differential radial equation are \( L^2 \) at \( r = 0 \) except: (a) \( \sigma = 0, c = 1 \), any \( m \), Minkowski spacetime, (b) \( \sigma = 0, |m|C \geq 1 \), cosmic string spacetime (thus cosmic string is “generically” singular for wave packets which are a combination of arbitrary modes [11] [9]), and \( \sigma = 1/2 \), any \( m \), Minkowski spacetime in accelerated coordinates. Again, a strong repulsive potential for certain modes shields the singularity (in the cosmic string case) from the conical singularity on the axis.

All Levi-Civita spacetimes are thus quantum mechanically singular except Minkowski \( (\sigma = 0, c = 1) \) and Minkowski in accelerated coordinates \( (\sigma = 1/2) \). One needs to interpret the ‘quantum singularity’ of the physically reasonable infinite line mass spacetimes as the need for one to put “boundary conditions” at the line mass itself or ‘round-off’ this \( \delta \)-function singularity and then put boundary conditions at the matter surface (a similar argument was used for the cosmic string case, see e.g., Kay and Studer [10]).
7 Conclusions

It is thus clear that if the repulsive barrier near the classical singularity is sufficiently strong, the probability of a quantum mechanical particle penetrating to the origin is sufficiently small, that the quantum mechanical particle doesn’t feel the singularity in some sense.

One can ask whether the “strength” of a classical singularity has any effect on the existence of a quantum mechanical singularity. The answer appears to be no: spacetimes with classical quasiregular singularities are as likely to be quantum mechanically singular as are classical scalar curvature singularities [11].

One can also ask whether the type of probing particle (scalar, null vector, spinor) has any effect on whether a singularity is or is not quantum mechanically singular. The answer appears to be generically no, although the wave packet modes producing the quantum singularity may differ depending on particle type [12].

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