Necessity of macroscopic operation for the creation of superpositions of macroscopically distinct states

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We consider the creation of superpositions of macroscopically distinct states by a completely-positive (CP) operation on a subsystem. We conclude that the subsystem on which the CP operation acts must be macroscopically large if the success probability of the CP operation does not vanish in the thermodynamic limit. In order to obtain this conclusion, we show two inequalities each of which represents a trade-off relation among the magnitude of an indicator for superpositions of macroscopically distinct states, the success probability of a CP operation, and the volume of the subsystem on which the CP operation acts.

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I. INTRODUCTION

The change of a physical properties of the total system under a certain operation on a subsystem has long been attracting much special attentions in quantum physics. For example, in quantum measurement theory [1, 2], one of the most important quantities is the change of the probability distribution of an observable under a projective measurement of another observable on the apparatus. Recently, the change of the magnitude of entanglement under local operations on subsystems and classical communications among them has also been intensively studied in quantum information and quantum computation [3, 4].

In this paper, we consider the change of the magnitude of two indicators, \( p \) and \( q \), for superpositions of macroscopically distinct states under a completely-constant (CP) operation on a subsystem, in order to quantitatively investigate the creation of superpositions of macroscopically distinct states \( \mathcal{E} \) by such an operation. In other words, we first prepare a state which does not contain a superposition of macroscopically distinct states, and next perform a CP operation on a subsystem so that the state changes into another state which contains a superposition of macroscopically distinct states. The existence of a superposition of macroscopically distinct states, if any, is identified by calculating the indicators \( p \) or \( q \). By deriving two trade-off relations among the magnitude of the indicators, the success probability of a CP operation, and the volume of the subsystem on which the CP operation acts, we generally show that if the success probability of the CP operation does not vanish in the thermodynamic limit, the subsystem on which the CP operation acts must be macroscopically large, i.e., of the order of the size of the total system. This means the necessity of a macroscopic operation for the creation of a superposition of macroscopically distinct states.

The outline of this paper is as follows. Firstly, we consider a CP operation which maps a pure state to a pure state. As the indicator for superposition of macroscopically distinct states in pure states, we use the index \( p \) [8, 10, 11, 14], which will be briefly reviewed in the next section for the convenience of the reader. In Sec. III we will derive a trade-off relation among the magnitude of the index \( p \), the success probability of a CP operation, and the volume of the subsystem on which the CP operation acts. From this trade-off relation, we conclude that it is necessary to access a macroscopically large subsystem to create a superposition of macroscopically distinct states by a CP operation with a non-vanishing success probability in the thermodynamic limit.

Secondly, we consider more general CP operation which maps a mixed state to a mixed state. In this case, we use another indicator for superpositions of macroscopically distinct states, which is called the index \( q \) [9], since the index \( p \) is not an indicator for superpositions of macroscopically distinct states if the state is mixed. After briefly explaining the definition of the index \( q \) in Sec. IV, we derive a similar trade-off relation as that for the index \( p \) in Sec. V. This trade-off relation again means the necessity of a macroscopic operation for the creation of a superposition of macroscopically distinct states with a non-vanishing success probability in the thermodynamic limit.

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II. PURE STATES: INDEX p

As the indicator for superpositions of macroscopically distinct states in pure states, we use the index $p$. For the convenience of the reader, we here briefly review the definition of the index $p$. For more details, see Refs. [8, 10, 11, 14].

Let us consider an $N$-site lattice, where $N$ is large but finite ($1 \ll N < \infty$). For example, a ring of $N$ spin-1/2 particles, which is often studied in condensed matter physics, is one example of such systems. Throughout this paper, we assume that the dimension of the Hilbert space on each site is an $N$-independent constant.

We use two symbols, $O$ and $o$, in order to represent asymptotic behaviors of a function $f(N)$ in the thermodynamic limit $N \to \infty$. Firstly, $f(N) = O(N^n)$ means

$$\lim_{N \to \infty} \frac{f(N)}{N^n} = \text{const.} \neq 0.$$  

Secondly, $f(N) = o(N^n)$ means

$$\lim_{N \to \infty} \frac{f(N)}{N^n} = 0.$$  

For example, if $f(N) = 3N^5 + 2N^2 + 9$, we denote $f(N) = O(N^5)$ or $f(N) = o(N^6)$.

For a given pure state $|\psi\rangle$, the index $p$ ($1 \leq p \leq 2$) is defined by

$$\max_A \left[ \langle \psi | \hat{A}^2 | \psi \rangle - \langle \psi | \hat{A} | \psi \rangle^2 \right] = O(N^p),$$

where the maximum is taken over all Hermitian additive operators $\hat{A}$. Here, an additive operator \cite{16, 17}

$$\hat{A} = \sum_{i=1}^{N} \hat{a}(l)$$

is a sum of local operators $\{\hat{a}(l)\}_{i=1}^{N}$, where

$$\hat{a}(l) \equiv \hat{1}(1) \otimes \cdots \otimes \hat{1}(l-1) \otimes \hat{a}(l) \otimes \hat{1}(l+1) \otimes \cdots \otimes \hat{1}(N)$$

is a local operator acting on site $l$. For example, if the system is a ring of $N$ spin-1/2 particles, $\hat{a}(l)$ is a linear combination of three Pauli operators, $\hat{\sigma}_x(l), \hat{\sigma}_y(l), \hat{\sigma}_z(l)$, and the identity operator $\hat{1}(l)$. In this case, the $x$-component of the total magnetization

$$\hat{M}_x \equiv \sum_{l=1}^{N} \hat{\sigma}_x(l)$$

and the $z$-component of the total staggered magnetization

$$\hat{M}_z^t \equiv \sum_{l=1}^{N} (-1)^l \hat{\sigma}_z(l)$$

are, for example, additive operators. In order to make $\hat{A}$ additive, we assume that each $\hat{a}(l)$ is independent of $N$. Since multiplying $\hat{a}(l)$ by an $N$-independent constant does not change the essential results of this paper, we henceforth assume that $\hat{a}(l)$ is normalized as $\|\hat{a}(l)\|_\infty \leq 1$ without loss of generality, where $\|X\|_\infty$ is the operator norm of $X$.

The index $p$ takes the minimum value 1 for any “product state”

$$|\psi\rangle = \bigotimes_{l=1}^{N} |\phi_l\rangle,$$

where $|\phi_l\rangle$ is a state of site $l$. If $p$ takes the maximum value 2, on the other hand, $|\psi\rangle$ contains a superposition of macroscopically distinct states, because in this case a Hermitian additive operator has a “macroscopically large” fluctuation in the sense that

$$\lim_{N \to \infty} \frac{\sqrt{\langle \psi | \hat{A}^2 | \psi \rangle - \langle \psi | \hat{A} | \psi \rangle^2}}{N} \neq 0,$$
and because the fluctuation of an observable in a pure state means the existence of a superposition of eigenvectors of the observable corresponding to different eigenvalues \([10,11]\). Here, we say that two eigenvectors \(A_1\) and \(A_2\) of an additive operator \(\hat{A}\) corresponding to eigenvalues \(A_1\) and \(A_2\), respectively, are macroscopically distinct with each other if \(A_1 - A_2 = O(N)\).

For example, the “Cat state”

\[
|C\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle^N + |1\rangle^N \right)
\]

has \(p = 2\), since

\[
\langle C|\hat{M}_2^2|C\rangle - \langle C|\hat{M}_2|C\rangle^2 = O(N^2),
\]

and therefore contains a superposition of macroscopically distinct states.

For the derivation of the trade-off relation in the next section, it is useful to point out another representation of the definition of the index \(p\). Let

\[
\|\hat{X}\|_k \equiv \left( \sum_{j=1}^{r} |e_j|^k \right)^{1/k}
\]

be the \(k\)-norm of an operator \(\hat{X}\), where \(e_j\) is \(j\)-th eigenvalue of \(\hat{X}\) and \(r\) is the rank of \(\hat{X}^{12}\). Then, as will be shown below, the relation

\[
\max_{\hat{A}} \left\| [\hat{A}, |\psi\rangle \langle \psi|] \right\|_k = O(N^{\frac{p}{2}}) \tag{1}
\]

holds for any \(k\), where the maximum is taken over all Hermitian additive operators \(\hat{A}\). This equation gives another physical meaning of the index \(p\): the index \(p\) quantifies the “noncommutativity” between the state and an additive operator \([12]\). In other words, a superposition of macroscopically distinct states can also be detected by finding a “macroscopically large” noncommutativity.

In particular, if we consider the operator norm, i.e., \(k = \infty\),

\[
\max_{\hat{A}} \left\| [\hat{A}, |\psi\rangle \langle \psi|] \right\|_\infty = O(N^{\frac{p}{2}})
\]

is satisfied. This relation will be used in the next section in order to derive the trade-off relation.

**Proof of Eq. (1):** Let us rewrite \(\hat{A}|\psi\rangle\) as

\[
\hat{A}|\psi\rangle = \langle \psi|\hat{A}|\psi\rangle|\psi\rangle + \sqrt{\langle \psi|\hat{A}^2|\psi\rangle - \langle \psi|\hat{A}|\psi\rangle^2}|\phi\rangle,
\]

where

\[
|\phi\rangle \equiv \frac{\hat{A} - \langle \psi|\hat{A}|\psi\rangle}{\sqrt{\langle \psi|\hat{A}^2|\psi\rangle - \langle \psi|\hat{A}|\psi\rangle^2}}|\psi\rangle
\]

is a normalized vector orthogonal to \(|\psi\rangle\). Then, we obtain

\[
i[\hat{A}, |\psi\rangle \langle \psi|] = i \sqrt{\langle \psi|\hat{A}^2|\psi\rangle - \langle \psi|\hat{A}|\psi\rangle^2} \left( |\phi\rangle \langle \psi| - |\psi\rangle \langle \phi| \right),
\]

which means that the eigenvalues of the Hermitian operator \(i[\hat{A}, |\psi\rangle \langle \psi|]\) are

\[
\pm \sqrt{\langle \psi|\hat{A}^2|\psi\rangle - \langle \psi|\hat{A}|\psi\rangle^2}.
\]

Hence Eq. (1) has been shown.
III. TRADE-OFF RELATION FOR PURE STATES

Let us consider a CP operation on a subsystem $S$ which maps a pure state $|\psi_1\rangle$ to a pure state $|\psi_2\rangle$. By considering the normalization of the state after the CP operation, the state change is generally written as

$$|\psi_1\rangle \rightarrow |\psi_2\rangle = \frac{\hat{E}|\psi_1\rangle}{\sqrt{\langle\psi_1|\hat{E}^\dagger\hat{E}|\psi_1\rangle}},$$

where $\hat{E}$ is a Kraus operator acting on $S$. For example, if the total system is a ring of $N$ spin-1/2 particles and we consider the spin-flip operation on even sites, the subsystem $S$ is the set of even sites and the Kraus operator acting on $S$ is

$$\hat{E} = \prod_{l={\text{even}}} \hat{\sigma}_x(l).$$

Since $\langle\psi|\hat{E}^\dagger\hat{E}|\psi\rangle \leq 1$ for any state $|\psi\rangle$, we obtain $\|\hat{E}\|_\infty \leq 1$. By using the triangle inequality of the operator norm and the fact that

$$\left\| \left[ \hat{A}, \hat{E} \right] \right\|_\infty = \left\| \sum_{l \in S} [\hat{\sigma}(l), \hat{E}] \right\|_\infty \leq 2|S|,$$

where $|S|$ is the volume of $S$ (i.e., the number of sites belonging to the subsystem $S$), we obtain

$$\left\| \left[ \hat{A}, |\psi_2\rangle\langle\psi_1| \right] \right\|_\infty = \frac{1}{G} \left\| \left[ \hat{A}, \hat{E} \right]|\psi_2\rangle\langle\psi_1| \hat{E}^\dagger \right\|_\infty$$

$$= \frac{1}{G} \left\| \left[ \hat{A}, \hat{E} \right]|\psi_2\rangle\langle\psi_1| \hat{E}^\dagger + \hat{E} \left[ \hat{A}, |\psi_1\rangle\langle\psi_1| \right] \hat{E}^\dagger + \hat{E} |\psi_1\rangle \langle\psi_1| \left[ \hat{A}, \hat{E}^\dagger \right] \right\|_\infty$$

$$\leq \frac{1}{G} \left( \left\| \left[ \hat{A}, \hat{E} \right] \right\|_\infty + \left\| \left[ \hat{A}, |\psi_1\rangle\langle\psi_1| \right] \right\|_\infty + \left\| \left[ \hat{A}, \hat{E}^\dagger \right] \right\|_\infty \right)$$

$$\leq \frac{1}{G} \left( 4|S| + \left\| \left[ \hat{A}, |\psi_1\rangle\langle\psi_1| \right] \right\|_\infty \right)$$

for any additive operator $\hat{A}$, where $G \equiv \langle\psi_1|\hat{E}^\dagger\hat{E}|\psi_1\rangle$ is the success probability of the CP operation.

Let us assume that the value of $p$ for $|\psi_1\rangle$ is $p_1$ and that for $|\psi_2\rangle$ is $p_2$. Then, the above inequality gives

$$O(N^{p_2/2}) \leq \frac{1}{G} \left( 4|S| + O(N^{p_1/2}) \right), \tag{2}$$

which represents the trade-off relation among the magnitude of the index $p$, the success probability $G$ of the CP operation, and the volume $|S|$ of the subsystem $S$ on which the CP operation acts.

In the following two subsections, let us investigate this trade-off relation for the most important case $p_1 < p_2 = 2$, i.e., the case where a state which contains a superposition of macroscopically distinct states is created from a state which does not contain a superposition of macroscopically distinct states.

A. If $G = O(N^0)$

Firstly, if the success probability of the CP operation does not vanish in the thermodynamic limit, i.e., $G = O(N^0)$, Eq. (2) gives

$$|S| = O(N),$$

which means that it is necessary to access a macroscopically large subsystem in order to create a superposition of macroscopically distinct states from a state without such superpositions by a pure-to-pure CP operation with a non-vanishing success probability as $N \to \infty$. This is one of the two main results of this paper (the other is given in Sec. IV).

Inversely, $|S| = O(N)$ is also a sufficient condition for the creation of a superposition of macroscopically distinct states with a non-vanishing success probability as $N \to \infty$. For example, it is easy to verify that the state

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle^\otimes |S\rangle + |1\rangle^\otimes |S\rangle \right) \otimes |\phi\rangle,$$
which obviously has $p = 2$ if $|S| = O(N)$, can be created from any state $|\psi_1\rangle$ by a deterministic CP operation (i.e., $G = 1$) in the following manner. Let

$$|\psi_1\rangle = \sum_{i=1}^{v} \lambda_i |\xi_i\rangle \otimes |\phi_i\rangle$$

be a Schmidt decomposition of the given state $|\psi_1\rangle$, where $v$ is the Schmidt rank, $\sum_{i=1}^{v} |\lambda_i|^2 = 1$, $|\xi_i\rangle$’s are states of the subsystem $S$, and $|\phi_i\rangle$’s are states of other sites. Then, the application of the operator

$$\frac{1}{\sqrt{2}} \left( |0\rangle \otimes |S\rangle + |1\rangle \otimes |S\rangle \right) \sum_{i=1}^{v} |\xi_i\rangle,$$

which acts on the subsystem $S$, to the state $|\psi_1\rangle$ deterministically creates $|\psi_2\rangle$.

**B. If $G = o(N^0)$**

Secondly, if $G = o(N^0)$, an access to a macroscopically large subsystem is no longer necessary. For example, let us consider the state

$$\frac{1}{N^\alpha}|1^\otimes N\rangle + \sqrt{1 - \frac{1}{N^{2\alpha}}}|0^\otimes N\rangle$$

with $0 < \alpha \leq \frac{1}{2}$. By using the VCM method, which is a method of efficiently calculating the magnitude of the index $p$, it is straightforward to show that this state has $p = 2 - 2\alpha$, and therefore does not contain a superposition of macroscopically distinct states. However, this state can be changed into the state

$$|\phi\rangle \otimes \frac{1}{\sqrt{2}} \left( |0^\otimes N-1\rangle + |1^\otimes N-1\rangle \right),$$

which obviously has $p = 2$, by the local projection

$$\left( \sqrt{1 - \frac{1}{N^{2\alpha}}}|1\rangle + \frac{1}{N^\alpha}|0\rangle \right) \left( \sqrt{1 - \frac{1}{N^{2\alpha}}}|1\rangle + \frac{1}{N^\alpha}|0\rangle \right)$$

on the single site with the success probability $G = O(N^{-2\alpha})$. Here, $|\phi\rangle$ is a state of a single site. Indeed, in this case, the right-hand side of the inequality is $O(N^{1+\alpha})$, and therefore the inequality is not violated.

**IV. MIXED STATES: INDEX $q$**

In the previous sections, we have studied the change of the magnitude of the index $p$ under a pure-to-pure CP operation. If we consider mixed states, however, the naive generalization of the index $p$ to mixed states:

$$\max_{\hat{A}} \left[ \text{Tr}(\hat{\rho} \hat{A}^2) - \text{Tr}(\hat{\rho} \hat{A})^2 \right] = O(N^p)$$

is no longer a good indicator for superpositions of macroscopically distinct states, since a fluctuation is not necessarily equivalent to a superposition if the state is mixed. For example, let us consider a “classical mixture” of two macroscopically distinct states:

$$\hat{\rho} = \frac{1}{2} |0^\otimes N\rangle \langle 0^\otimes N| + \frac{1}{2} |1^\otimes N\rangle \langle 1^\otimes N|.$$ 

Although it is obvious that this state contains no superposition of macroscopically distinct states, this state has $p = 2$ in terms of the above generalization since

$$\text{Tr}(\hat{\rho} \hat{M}_z^2) - \text{Tr}(\hat{\rho} \hat{M}_z)^2 = O(N^2).$$

This example shows that we must use another indicator if the state is mixed.
A good indicator, the index \( q \), for superpositions of macroscopically distinct states in mixed states was recently proposed in Ref. [9]. For a given state \( \hat{\rho} \), the index \( q \) (\( 1 \leq q \leq 2 \)) is defined by

\[
\max\left(N, \max_{\hat{A}, \hat{\eta}} \langle \hat{\rho}[\hat{A}, [\hat{A}, \hat{\eta}]] \rangle \right) = O(N^q),
\]

where \( \hat{\eta} \) is a projection operator and \( \hat{A} \) is a Hermitian additive operator. As detailed in Ref. [9], \( q \) takes the minimum value 1 for any mixture of product states:

\[
\sum_i \lambda_i \left( \bigotimes_{l=1}^N \langle \phi_i^l | \right) \left( \bigotimes_{l=1}^N \langle \phi_i^l | \right),
\]

where \( 0 \leq \lambda_i \leq 1 \), \( \sum_i \lambda_i = 1 \), and \( |\phi_i^l \rangle \) is a state of site \( l \). On the other hand, if \( q \) takes the maximum value 2, \( \hat{\rho} \) contains a superposition of macroscopically distinct states [18].

For later convenience, let us give another representation of the definition of the index \( q \). Let \( \hat{X} \) be any traceless operator. Then,

\[
\| \hat{X} \|_1 = 2 \max_{\hat{\eta}} \text{Tr}(\hat{\eta} \hat{X})
\]

holds, where \( \hat{\eta} \) is a projection operator [13]. By using this relation and the fact

\[
\text{Tr}\left(\hat{\rho}[\hat{A}, [\hat{A}, \hat{\eta}]]\right) = \text{Tr}\left(\hat{\eta}[\hat{A}, [\hat{A}, \hat{\rho}]]\right),
\]

it is easy to verify that the index \( q \) is also defined by

\[
\max\left(N, \max_{\hat{A}} \| [\hat{A}, [\hat{A}, \hat{\rho}]] \|_1 \right) = O(N^q).
\]

This expression gives a clear physical meaning of the index \( q \): the index \( q \) quantifies the “noncommutativity” between the state and an additive operator.

V. TRADE-OFF RELATION FOR MIXED STATES

Let us consider a CP operation on a subsystem \( S \) which maps a state \( \hat{\rho}_1 \) to a state \( \hat{\rho}_2 \). By considering the normalization of the state after the operation, the state change is generally written as

\[
\hat{\rho}_1 \rightarrow \hat{\rho}_2 = \frac{\sum_{k=1}^M \hat{E}_k \hat{\rho}_1 \hat{E}_k^\dagger}{\text{Tr}\left(\sum_{k=1}^M \hat{E}_k \hat{\rho}_1 \hat{E}_k^\dagger\right)},
\]

where \( \hat{E}_k \) is a Kraus operator acting on \( S \) [3].

As is shown in Appendix, the inequality

\[
\| [\hat{A}, [\hat{A}, \hat{\rho}_2]] \|_1 \leq \frac{1}{G} \left( \| [\hat{A}, [\hat{A}, \hat{\rho}_1]] \|_1 + 16|S|^2 N + 4|S|^2 G + 12|S|^2 \right)
\]

holds for any additive operator \( \hat{A} \), where

\[
G = \text{Tr}\left(\sum_{k=1}^M \hat{E}_k^\dagger \hat{E}_k \hat{\rho}_1\right)
\]

is the success probability of the CP operation.

As in the case of the index \( p \), this inequality means the trade-off relation among the magnitude of the index \( q \), the success probability \( G \) of the CP operation, and the volume \( |S| \) of the subsystem \( S \) on which the CP operation acts. In particular, if we consider the creation of a state having \( q = 2 \) from a state having \( q < 2 \) by a CP operation with a non-vanishing success probability as \( N \rightarrow \infty \) (i.e., \( G = O(N^0) \)), this trade-off relation gives the same necessary condition:

\[
|S| = O(N)
\]

for \( S \). In other words, it is necessary to access a macroscopically large subsystem in order to create a superposition of macroscopically distinct states by a mix-to-mix CP operation with a non-vanishing success probability as \( N \rightarrow \infty \). This is the other main result of this paper.
VI. CONCLUSION

In this paper, we have studied the change of the magnitude of two indicators, \( p \) and \( q \), for superpositions of macroscopically distinct states under a creation of such superpositions by a CP operation on a subsystem. By deriving two trade-off relations among the magnitude of the indicators, the success probability of a CP operation, and the volume of the subsystem on which the CP operation acts, we have generally shown that it is necessary to access a macroscopically large subsystem to create a superposition of macroscopically distinct states by a CP operation with a non-vanishing success probability in the thermodynamic limit.

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APPENDIX

In this appendix, we will show the inequality given in Sec. V. We assume that the dimension of the Hilbert space on each site is an \( N \)-independent constant and \( \| \hat{a}(l) \|_\infty \leq 1 \).

Let us first mention a useful lemma.

Lemma: For any Hermitian operator \( \hat{X} \),

\[
\left\| \sum_{k=1}^{M} \hat{E}_k \hat{X} \hat{E}_k^\dagger \right\|_1 \leq \| \hat{X} \|_1.
\]

Proof: Since

\[
\text{Tr} \left( \sum_{k=1}^{M} \hat{E}_k^\dagger \hat{E}_k \hat{\rho} \right) \leq 1
\]

for any \( \hat{\rho} \), we obtain

\[
\sum_{k=1}^{M} \hat{E}_k^\dagger \hat{E}_k \leq 1.
\]

Let

\[
\hat{X} = \sum_{i=1}^{r} x_i |i\rangle \langle i|
\]

be a spectral decomposition of \( \hat{X} \). Then,

\[
\left\| \sum_{k=1}^{M} \hat{E}_k \hat{X} \hat{E}_k^\dagger \right\|_1 \leq \sum_{k=1}^{M} \sum_{i=1}^{r} |x_i| \left\| \hat{E}_k |i\rangle \langle i| \right\|_1
\]

\[
= \sum_{k=1}^{M} \sum_{i=1}^{r} |x_i| \langle i| \hat{E}_k^\dagger \hat{E}_k |i\rangle
\]

\[
\leq \sum_{i=1}^{r} |x_i| = \| \hat{X} \|_1.
\]

Hence the lemma has been shown.
By using this lemma, let us next show the inequality. Consider

$$\| [\hat{A}, [\hat{A}, \hat{\rho}_1]] \|_1 = \frac{1}{G} \left\| \sum_{k=1}^{M} [\hat{A}, [\hat{A}, \hat{E}_k \hat{\rho}_1 \hat{E}_k^\dagger]] \right\|_1 \leq \frac{1}{G} \sum_{j=1}^{3} \| \hat{\xi}_j \|_1,$$

where

$$\hat{\xi}_1 = \sum_{k=1}^{M} \hat{E}_k [\hat{A}, [\hat{A}, \hat{\rho}_1]] \hat{E}_k^\dagger,$$

$$\hat{\xi}_2 = 2 \sum_{k=1}^{M} \left( [\hat{A}, \hat{E}_k] [\hat{A}, \hat{\rho}_1] \hat{E}_k^\dagger + \hat{E}_k [\hat{A}, \hat{\rho}_1] [\hat{A}, \hat{E}_k^\dagger] \right),$$

$$\hat{\xi}_3 = \sum_{k=1}^{M} \left( [\hat{A}, \hat{E}_k] \hat{\rho}_1 \hat{E}_k^\dagger + \hat{E}_k \hat{\rho}_1 [\hat{A}, \hat{E}_k^\dagger] + 2 [\hat{A}, \hat{E}_k] \hat{\rho}_1 [\hat{A}, \hat{E}_k^\dagger] \right).$$

Since $[\hat{A}, [\hat{A}, \hat{\rho}_1]]$ is Hermitian,

$$\| \hat{\xi}_1 \|_1 \leq \| [\hat{A}, [\hat{A}, \hat{\rho}_1]] \|_1$$

from the lemma.

Let us define

$$\hat{A}_S = \sum_{l \in S} \hat{a}(l).$$

Then,

$$\| \hat{\xi}_2 \|_1 = 2 \left\| \sum_{k=1}^{M} \left( [\hat{A}_S, \hat{E}_k] [\hat{A}, \hat{\rho}_1] \hat{E}_k^\dagger + \hat{E}_k [\hat{A}, \hat{\rho}_1] [\hat{A}_S, \hat{E}_k^\dagger] \right) \right\|_1$$

$$= 2 \left\| [\hat{A}_S, \sum_{k=1}^{M} \hat{E}_k [\hat{A}, \hat{\rho}_1] \hat{E}_k^\dagger] + \sum_{k=1}^{M} \hat{E}_k [\hat{A}, \hat{\rho}_1] [\hat{A}_S, \hat{E}_k^\dagger] \right\|_1$$

$$\leq 16 |S| N$$

and

$$\| \hat{\xi}_3 \|_1 = \left\| \sum_{k=1}^{M} \left( [\hat{A}_S, [\hat{A}_S, \hat{E}_k]] \hat{\rho}_1 \hat{E}_k^\dagger + \hat{E}_k \hat{\rho}_1 [\hat{A}_S, \hat{E}_k^\dagger] + 2 [\hat{A}_S, \hat{E}_k] \hat{\rho}_1 [\hat{A}_S, \hat{E}_k^\dagger] \right) \right\|_1$$

$$= \left\| [\hat{A}_S, \sum_{k=1}^{M} \hat{E}_k \hat{\rho}_1 \hat{E}_k^\dagger] + \sum_{k=1}^{M} \hat{E}_k [\hat{A}_S, \hat{\rho}_1] \hat{E}_k^\dagger + 2 [\hat{A}_S, \sum_{k=1}^{M} \hat{E}_k \hat{\rho}_1, \hat{A}_S \hat{E}_k^\dagger] \right\|_1$$

$$\leq 4 |S|^2 G + 12 |S|^2.$$

Therefore, we finally obtain

$$\| [\hat{A}, [\hat{A}, \hat{\rho}_2]] \|_1 \leq \frac{1}{G} \left( \| [\hat{A}, [\hat{A}, \hat{\rho}_1]] \|_1 + 16 |S| N + 4 |S|^2 G + 12 |S|^2 \right).$$

Hence the inequality has been shown.

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