K-theory of group Banach algebras and Banach property RD

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Abstract

We investigate Banach algebras of convolution operators on the $L^p$ spaces of a locally compact group, and their K-theory. We show that for a discrete group, the corresponding K-theory groups depend continuously on $p$ in an inductive sense. Via a Banach version of property RD, we show that for a large class of groups, the K-theory groups of the Banach algebras are independent of $p$.

1 Introduction

Let $G$ be a locally compact group. Denote $EG$ its classifying space for proper actions, and $K_i^{EG}(EG)$ the equivariant K-homology groups. The Baum-Connes conjecture claims that the associated Baum-Connes assembly map $\mu : K_i^{EG}(EG) \to K_i(C^*_r(G))$ is an isomorphism between abelian groups, where $K_i(C^*_r(G))$ is the $i$-th K-theory group of the reduced $C^*$ algebra $C^*_r(G)$ of $G$. The conjecture has been verified for a large class of groups, including groups with Haagerup property [5], hyperbolic groups [12, 11], reductive Lie groups over local fields and cocompact lattices in $SL_3$ over a local field [11].

If one replaces $C^*_r(G)$ by the Banach algebra $L^1(G)$ of integrable functions on $G$, this is the so-called Bost’s conjecture, which has been proved for a much larger class of groups, including all lattices in a reductive Lie group over a local field [11].

Motivated by a theorem due to the second named author that hyperbolic groups act isometrically properly on some $L^p$ space [19, 13], G. Kasparov and he introduced an $L^p$ version of Baum-Connes assembly map, where one replaces $C^*_r(G)$ by the Banach algebra of convolution operators on the space of $p$-integrable functions $L^p(G)$ on $G$ [4]. This algebra is denoted by $B^p_r(G)$ in this article, which
in more precise terms is the Banach algebra obtained by completing the convolution algebra of compactly supported continuous functions $C_c(G)$ with respect to the operator norm on $L^p(G)$.

These algebras of operators in $L^p$-spaces also appear in the work of N. C. Phillips [15] on the study on $L^p$ cross-product and Cuntz algebras.

The aim of this work is to investigate the relations among the K-theory groups of these different group Banach algebras $B_p^r(G)$.

**Theorem 1.1.** For a locally compact group $G$, the K-theory groups of $B_p^r(G)$ and $B_q^r(G)$ are canonically isomorphic, where $p \in [1, \infty]$ and $q$ is the dual number of $p$, namely $1/q + 1/p = 1$.

To compare K-theory groups of algebras $B_p^r(G)$ for different $p$, a natural way is to consider a subalgebra $B_p^{*,r}(G)$ in $B_p^r(G)$ which is closed under involution. This is the involutive Banach algebra obtained by completing $C_c(G)$ with respect to the norm

$$\|f\|_{B_p^{*,r}(G)} = \max\{\|f\|_{B(L^p(G))}, \|f^*\|_{B(L^p(G))}\}$$

where $f^*(g) = \overline{f(g^{-1})}$. Under the condition that the group has the property of rapid decay (property RD) [6, 2, 15, 17], we are able to show that this procedure does not lose K-theoretic information.

**Theorem 1.2.** Let $p \in [1, \infty]$. Let $G$ be a locally compact group with property RD. The K-theory groups of $B_p^r(G)$ and $B_p^{*,r}(G)$ are canonically isomorphic.

In fact, the statement holds under a weaker condition that the group has a Banach version of property RD, which we call property (RD)$_q$ where $1/q + 1/p = 1$ (Theorem 4.4 and Theorem 4.9). We prove this by showing that there exists a subalgebra which is closed under holomorphic functional calculus in both $B_p^r(G)$ and $B_p^{*,r}(G)$ (Proposition 4.6).

In the case when the group acts properly on an $L^p$ space, this phenomena of $*$-independence is also verified in [4]. We conjecture that it is a general phenomenon for groups.

The advantage of $B_p^{*,r}(G)$ is that it allows to use interpolation to give a canonical morphism

$$i_{p',p} : B_{p'}^{*,r}(G) \rightarrow B_p^{*,r}(G)$$

for $2 \leq p < p' \leq \infty$. Letting $p$ vary from 1 to 2, we obtain a family of continuous inclusions

$$L^1(G) \xrightarrow{i_{1,p}} B_p^{*,r}(G) \xrightarrow{i_{p,2}} C_r^*(G).$$

When the group is discrete, we show the following semi-continuity result for $p \to 2$. 

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Theorem 1.3. Let $G$ be a finitely generated group. $K_*(C^*_r(G))$ is the inductive limit of the system $\{K_*(B^p_*(G)), p > 2\}$. Namely,

$$K_*(C^*_r(G)) = \bigcup_{p > 2} i^*_{p,2}(K_*(B^p_*(G))).$$

The following corollary gives us an alternative criterion for the $K$-theory of $C^*_r(G)$: it is either equal to that of $B^p_*(G)$ for some $p > 2$, or it is dramatically larger than that of every $B^p_*(G), p > 2$.

Corollary 1.4. Let $G$ be a finitely generated group. Then there exists $p > 2$ such that

$$i^*_{p,2}(K_*(B^p_*(G))) = K_*(C^*_r(G)),$$

unless for every $p > 2$ (in particular $K_*(l^1(G))$) $i^*_{p,2}(K_*(B^p_*(G))) \subset K_*(C^*_r(G))$ is of infinite index.

For some large class of groups listed below, via a Banach version of property RD that we call property $(RD)_q$ (see Section 3), and unconditional completion [11], we are able to show that their $K$-theory groups indeed do not depend on the parameter $p \in [1, \infty]$.

Theorem 1.5. Let $G$ be either
- a semi-simple Lie group over a local field, or a cocompact lattice in it with property RD, or
- a hyperbolic group, or
- a finitely generated group of polynomial growth.

$K_*(B^p_r(G))$ are isomorphic for $p \in [1, \infty]$.

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2 The Banach algebras $B^p_r(G)$ and $B^{p,*}_r(G)$

In this section, we introduce the $L^p$ convolution algebra $B^p_r(G)$ and its involutive counterpart $B^{p,*}_r(G)$, and investigate their basic properties.

Let $G$ be a locally compact group, $p \in [1, \infty]$. Denote $B^p_r(G)$ the Banach algebra obtained by completing $C_c(G)$ with the operator norm $\|f\|_{B^p_r(G)}, f \in C_c(G)$. Denote $B^{p,*}_r(G)$ the involutive version of $B^p_r(G)$, namely the completion of $C_c(G)$ with respect to the norm

$$\|f\|_{B^{p,*}_r(G)} = \max\{\|f\|_{B^p_r(G)}, \|f^*\|_{B^p_r(G)}\},$$
where \( f^*(x) = \overline{f(x^{-1})} \).

Except for the abelian case, these two algebras are usually not identical.

**Lemma 2.1.** If \( G \) is a countable discrete group and \( H \subset G \) a subgroup, then for any \( f \in \mathbb{C}(G) \) supported in \( H \) we have \( \|f\|_{B^p_r(G)} = \|f\|_{B^p_r(H)} \).

**Proposition 2.2.** If \( G \) is a non-elementary hyperbolic group, or a non-abelian linear group, then
\[
B^p_r(G) \neq B^p_r(H)
\]
for any \( p \in (1, 2) \cup (2, \infty) \).

**Proof:** By Tit’s alternative, a non-abelian linear group contains a non-abelian free subgroup. The same holds as well for a non-elementary hyperbolic group. The statement then follows from Lemma 2.1 and the main Theorem in [16].

The argument obviously works for any discrete group containing a non-abelian free group as a subgroup.

**Proposition 2.3.** There exists an amenable group \( G \) such that
\[
B^p_r(G) \neq B^p_r(H)
\]
for \( p = 4 \).

**Proof:** The construction of \( G \) is based on results in [14]. We thank G. Pisier for pointing us to [14] and generously sharing his observations.

**Lemma A.** [14] There exists a finite group \( G_0 \) and a function \( f \in \mathbb{C}(G_0) \) such that
\[
\|f^*\|_{B(L^4(G_0))} > \|f\|_{B(L^4(G_0))}.
\]

In what follows, we construct the amenable group \( G \) as in the statement of the proposition, and functions \( f_n \in \mathbb{C}(G) \) such that
\[
\|f_n\|_{B(L^4(G))} = 1
\]
and
\[
\|f_n^*\|_{B(L^4(G))} \to \infty.
\]

Define
\[
G_n := \prod_{i=1}^{n} G_0,
\]
where $G_0$ is as in Lemma A, and

\[ G = \{ (g_i \in G_0), \text{ only finitely many } g_i \text{ are not the neutral element} \} \subset \prod_{i=1}^{\infty} G_0. \]

The following map

\[ G_n \to G \]

\[ (g_1, ..., g_n) \mapsto (g_1, ..., g_n, e, ..., e, ...) \]

is an embedding of groups. We have $G_n \subset G_{n+1}$. Define $f'_n : G_n \to \mathbb{C}$ by

\[ f'_n(g_1, ..., g_n) = f(g_1) ... f(g_n)/M^n \]

where

\[ M = \|f\|_{B(l^1(G_0))}. \]

Extend the definition from $G_n$ to $G$ by zero outside of $G_n$, we get a finitely supported function $f_n \in \mathbb{C}(G)$.

**Lemma B.** Let $H_1, H_2$, be discrete groups, and $\phi_i \in \mathbb{C}(H_i)$. The tensor product $\phi = \phi_1 \otimes \phi_2$ (namely $\phi(h_1, h_2) = \phi(h_1)\phi(h_2)$) is finitely supported on $H = H_1 \times H_2$, and we have for all $p \in [1, \infty]$

\[ \|\phi\|_{B(l^p(H))} = \|\phi_1\|_{B(l^p(H_1))}\|\phi_2\|_{B(l^p(H_2))}. \]

\[ \Box \]

Let $G'_n$ be the subgroup in $G$ such that $G = G_n \times G'_n$. Since $f_n = f'_n \otimes \delta_{e_{G'_n}}$, we have $\|f_n\|_{B(l^1(G))} = 1$ and

\[ \|f'_n\|_{B(l^1(G))} = (\|f^*\|_{B(l^1(G_0))}/M)^n \to \infty. \]

This terminates the proof. \[ \Box \]

**Proposition 2.4.** Let $G$ be a locally compact group. For $p < p' < \infty$, the identity map on $C_c(G)$ extends to a continuous (contractive) injective morphism of Banach algebras

\[ i_{p',p} : B_{p'}^p(G) \to B_p^{p'}(G). \]

**Remark.** $i_{p',p}$ fails to be surjective in general. Indeed, it is shown in [16] that there exists a function $f$ on a non-abelian free group $G$ such that $\|f\|_{C^*_r(G)}$ is finite and $\|f\|_{B(l^{p'}(G))}$ is infinite for any $p' \neq 2$, which means that $i_{p',2}$ is never surjective on $G$.

**Proof.** Denote $q, q'$ the duals of $p, p'$ respectively : $1/p + 1/q = 1$, $1/p' + 1/q' = 1$. Let $\theta \in [0,1]$ such that $1/p = \theta/p' + (1-\theta)/q'$, as
a consequence $1/q = (1 - \theta)/p' + \theta/q'$. By complex interpolation we have
\[
\|f\|_{B(L^p(G))} \leq \|f\|_{i}^{\theta} \|f\|_{B(L^{p'}(G))}^{1-\theta},
\]
and
\[
\|f\|_{B(L^q(G))} \leq \|f\|_{i}^{\theta} \|f\|_{B(L^{q'}(G))}^{1-\theta}.
\]
Consequently, we have
\[
\|f\|_{B_p^q(G)} \leq \|f\|_{B_{p'}^{q'}(G)}.
\]
So $i_{p',p}$ is continuous and contractive.

Now prove that $i_{p',p}$ is injective. Suppose $F \in B_{p'}^{q'}(G)$ such that $i_{p',p}(F) = 0$. Prove that $F(\xi) = 0$ for any $\xi \in L^{p'}(G)$ and $F(\eta) = 0$, $\forall \eta \in L^{q'}(G)$. Since $C_c(G)$ is a dense subset in both $L^{p'}(G)$ and $L^{q'}(G)$, by continuity of $F$, it suffices to prove that $\langle \phi, F(f) \rangle = 0$, $\forall f, \phi \in C_c(G)$. Let $F_n \in C_c(G)$ such that $F_n \to F$ in $B_{p'}^{q'}(G)$. By continuity of $i_{p',p}$ we have $F_n \to 0$ in $B_{p'}^{q'}(G)$. Thus $\langle \phi, F(f) \rangle = \lim_n \langle \phi, F_n(f) \rangle = 0$. \qed

3 Semi-continuity of $K_*(B_{p'}^{q'}(G))$

In this section, we investigate K-theoretic properties of $B_p^q(G)$ and $B_{p'}^{q'}(G)$. We first show Theorem 1.3.

**Proof of Theorem 1.3.** Observe that
\[
i_{p',p} : B_p^q(G) \to B_{p'}^{q'}(G), f \mapsto [g \mapsto \overline{f}(g^{-1})]
\]
is an isometric anti-isomorphism between Banach algebras. Thus it sends idempotents onto idempotents, invertibles onto invertibles, and preserves equivalence relation. \qed

Next we prove Theorem 1.4 which is merely a special case of the following theorem.

**Theorem 3.1.** Let $G$ be a finitely generated group. For $j = 0, 1$, $i_{p',p}^*(K_j(B_{p'}^{q'}(G)))$ is a subgroup in $K_j(B_p^q(G))$, and we have
\[
K_j(B_p^q(G)) = \bigcup_{p' > p} i_{p',p}^*(K_j(B_{p'}^{q'}(G))). \quad (*)
\]

**Proof.** The following argument is inspired by [11].

Let $\alpha \in [0, 1]$ such that $1/q' = (1 - \alpha) + \alpha/q$ (consequently $1/p' = \alpha/p$). We show that for $F \in M_n(\mathbb{C}(G))$
\[
\|F\|_{M_n(\mathbb{C}(G))} \leq e^{\frac{\lambda(1-\alpha)}{q}'m_F} \|F\|_{M_n(\mathbb{C}(G))}, \quad (**)\]

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where \( \lambda > 0 \) is such that \( |B_m(0)| \leq e^{\lambda n} \) and \( m_F \) is the smallest \( m \) such that all \( F_{i,j} \in \mathbb{C}(G) \), \( 1 \leq i, j \leq n \), are supported in \( B_m(0) \subset G \). It suffices to show inequality (**) for \( n = 1 \). Let \( f \in \mathbb{C}(G) \) with support in \( B_m(0) \). By complex interpolation, we have

\[
\|f\|_{B(\ell^r(G))} \leq \|f\|_{\ell^r(G)} \|f\|_{B(\ell^1(G))}^{1-\alpha} \|f\|_{B(\ell^1(G))}^\alpha.
\]

We have for any \( s > 1 \), \( \|f\|_{\ell^s(G)} \leq \|f\|_{B(\ell^1(G))} \), and by Holder’s inequality, \( \|f\|_{\ell^1(G)} \leq e^{\lambda n/s} \|f\|_{\ell^1(G)} \) where \( 1/t + 1/s = 1 \). Apply \( s = p, q \) and combine with the previous two inequalities, we obtain the desired (**) (for \( n = 1 \)).

Prove (*) for \( j = 0 \). It suffices to show that, for \( n \in \mathbb{N} \) and \( e \in Idem(M_n(B^{p,*}_p(G))) \), there exist \( p' > p \) and \( e' \in Idem(M_n(B^{p',*}_{p'}(G))) \) such that

\[
\|e - e'\|_{M_n(B^{p'}(G))} < 1/\|1 - 2e\|_{M_n(B^{p,*}_p(G))},
\]

where \( \|F\|_{M_n(B)} \) is defined to be \( \max_{i,j} \|F_{i,j}\|_B \) for a Banach algebra \( B \). Let \( e \in Idem(M_n(B^{p,*}_p(G))) \). By density, for any \( \varepsilon > 0 \) there exists \( f = f_\varepsilon \in M_n(\mathbb{C}(G)) \) such that \( \|e - f\|_{M_n(B^{p,*}_p(G))} < \varepsilon \). Denote \( M = \|e\|_{M_n(B^{p,*}_p(G))} \). Let \( m \in \mathbb{N} \) such that \( B_m(0) \) contains the support of all \( (f^2 - f)_{i,j} \). Since \( f^2 - f = (f - e)f + (e - f)(f - e) \) we have \( \|f^2 - f\|_{M_n(B^{p,*}_p(G))} < \varepsilon (2M + \varepsilon + 1) \). Consider those \( \varepsilon > 0 \) such that \( \varepsilon (2M + \varepsilon + 1) < 1/4 \). By inequality (**) there exists \( p' > p \) (depending on \( m \) and \( M \)) such that \( (1 - \alpha > 0 \) is small enough and consequently

\[
\|f^2 - f\|_{M_n(B^{p',*}_{p'}(G))} < 1/4.
\]

Let \( D_0, D_1 \) be the open disk of radius \( 1/2 \) and centered at \( 0, 1 \) respectively on the complex plane, and denote by \( \varphi \) the analytic function on \( U = D_0 \cup D_1 \) that sends \( D_0 \) to \( 0 \) and \( D_1 \) to \( 1 \). We have that \( Spec_{M_n(B^{p,*}_p(G))}(f) \subset U \), and \( \varphi(f) \) is an idempotent in \( M_n(B^{p,*}_p(G)) \).

By continuity of the inverse map on \( B^{p,*}_p(G) \), we have that \( \varphi(f_\varepsilon) \to \varphi(e) = e \) when \( \varepsilon \to 0 \). Take \( \varepsilon > 0 \) small enough such that \( \|\varphi(f_\varepsilon) - e\|_{M_n(B^{p,*}_p(G))} < 1/\|1 - e\|_{M_n(B^{p,*}_p(G))} \) and we are done.

Now Prove (*) for \( j = 1 \). Let \( F \in GL_n(B^{p,*}_p(G)) \) be an invertible element. Choose \( f \in M_n(\mathbb{C}(G)) \) close to \( F \) and \( \phi \in M_n(\mathbb{C}(G)) \) close to \( F^{-1} \) such that \( \|\phi f - 1\|_{M_n(B^{p,*}_p(G))} \leq 1/3 \) and \( f \) lies in the same connected component as \( F \) in \( GL_n(B^{p,*}_p(G)) \). By inequality (**) there exists \( p' > p \) such that \( \|\phi f - 1\|_{M_n(B^{p',*}_{p'}(G))} \leq 1/2 \). Thus \( \phi f \) is invertible in \( M_n(B^{p',*}_{p'}(G)) \) and consequently \( f^{-1} = (\phi f)^{-1} \phi \in M_n(B^{p'*}_{p'}(G)) \).
4 A Banach version of property RD

In this section, we define a Banach version of property RD, and prove Theorem 1.5 in the introduction.

Definition 4.1. Let $q \in [1, \infty]$ and $G$ be a locally compact group. Say that $G$ has $(RD)_q$ (with respect to a measurable length function $l$), if there exists a polynomial $P$ such that for any continuous function $f$ with support in $B_n(e)$, we have

$$\|f\|_{B(L^q(G))} \leq P(n)\|f\|_{L^q(G)}.$$ 

It is obvious that every group has $(RD)_1$.

Proposition 4.2. Let $G$ be a locally compact group. If $G$ is of polynomial growth with respect to some length function $l$, then it has $(RD)_q$ for all $q \in [1, \infty]$.

Proof. By Holder’s inequality, we have for any continuous function $f$ supported in $B_n(e)$

$$\|f\|_{B(L^q(G))} \leq \|f\|_{L^1(G)} \leq |B_n(e)|^{1/p}\|f\|_{L^q(G)}.$$ 

A polynomial bound on $|B_n(e)|$ clearly yields $(RD)_q$ for $G$. \hfill \Box

The following proposition suggests that the notion of $(RD)_q$ is more interesting when $q \in (1, 2]$.

Proposition 4.3. A countable discrete group having $(RD)_p$ for some $p \in (2, \infty]$ with respect to a length function $l$ is of polynomial growth in $l$.

Proof. Let $q \in [1, 2)$ be the dual number of $p$. Let $P$ be the polynomial for $(RD)_p$ of $G$. We have for $f \in C(G)$ supported in $B_n(e)$

$$\|f\|_{L^q(G)} \leq \|f\|_{B(L^q(G))} = \|f^*\|_{B(l^p(G))} \leq P(n)\|f^*\|_{l^p(G)} = P(n)\|f\|_{l^p(G)}.$$ 

Take $f_n = \chi_{B_n}$, we have

$$|B_n|^{1/q} \leq P(n)|B_n|^{1/p}.$$ 

Since $1/q - 1/p > 0$, it is immediate that the group has polynomial growth. \hfill \Box

The following result is due to V. Lafforgue. His original proof involves a combinatorial characterization of RD. Here we give a proof
in the discrete case based on an idea using Mazur map, and another proof due to G. Pisier based on complex interpolation. We think that these two proofs are different and have their own merits, and include them both in the article. B. Nica informed us that he also has a proof of this result.

**Theorem 4.4.** (V. Lafforgue) If $G$ is a locally compact group with property $(RD)_q$ for some $q > 1$, then it has $(RD)_{q'}$ for any $q' \in (1, q)$. In particular, RD implies $(RD)_q$ for $q \in (1, 2)$.

**First proof** (in the discrete case). Suppose that $G$ is a discrete group having $(RD)_q$ with respect to a polynomial $P$. Let $\phi \in C(G)$. Set

$$\phi_\alpha(g) := |\phi(g)|^\alpha, \alpha = q'/q < 1.$$ 

We then have $\|\phi\|_q = \|\phi_\alpha\|_{q'}^{q'/q}$. Since $a_1^n + \ldots + a_n^n \geq (a_1 + \ldots + a_n)^\alpha$ for $a_i \geq 0$ and $\alpha < 1$, we have

$$f_\alpha \phi_\alpha(g) \geq (\|f\|\phi_\alpha)(g).$$

Therefore,

$$\|f\|\phi_\alpha\|_{q'} = \|f\|\phi_\alpha\|_{q'}^{q'/q} \leq \|f_\alpha \phi_\alpha\|_{q'}_{q'}^{q'/q} \leq P(n)\|f\|\phi_\alpha\|_{q'}^{q'/q}.$$

This completes the proof since the left hand side is $\geq \|f\|\phi_\alpha\|_{q'}$. □

**Second proof** (G. Pisier). Let $\theta \in (0, 1)$ such that $1/q' = (1 - \theta) + \theta/q$. Let $f \in C_c(G)$ be a function supported in the ball of radius $n$ and define

$$F_z(g) = |f(g)|^{(1 - \theta + z/q')} \phi(g)$$

where $\phi$ is the phase function of $f$, namely $\phi(g) = f(g)/|f(g)|$ whenever $f(g) \neq 0$ and zero otherwise. We have obvious relations

$$F_\theta(g) = f(g), F_0(g) = |f(g)|^{q'/q} \phi(g), F_1(g) = |f(g)|^{q'/q} \phi(g).$$

and $\|F_0\|_1 = \|F_1\|_1 = \|F_\theta\|_{q'/q'}$.

Let $x, y \in C_c(G)$ be arbitrary functions. Define $X_z, Y_z$ for $x, y$ by the same formula as $F_z$ for $f$. Now the function $\{z \in \mathbb{C}, 0 \leq Re(z) \leq 1\} \rightarrow \mathbb{C}, z \mapsto \langle X_z, F_y Y_z \rangle$ is analytic on the interior and continuous on the boundary. By Hadamard three-lines lemma, we have

$$|\langle x, f y \rangle| = |\langle X_\theta, F_\theta Y_\theta \rangle| \leq \sup_{t \in \mathbb{R}} \langle X_{it}, F_{it} Y_{it} \rangle^{1 - \theta} \sup_{t \in \mathbb{R}} \langle X_{1 + it}, F_{1 + it} Y_{1 + it} \rangle^\theta \leq (\|x\|_{q'} \|f\|_{q'} \|y\|_{q'})^{(1 - \theta)q'} P(n)^\theta (\|x\|_{q'} \|f\|_{q'} \|y\|_{q'})^{\theta q'/q}.$$
where $P$ is the polynomial function in the definition $(RD)_q$. This implies $\|f\|_{B(L^q(G))} \leq P(n)\theta \|f\|_{L^q(G)}$.

Analogous to the $L^2$ case, we have the following result for amenable groups with rapid decay.

Proposition 4.5. Let $G$ be a compactly generated amenable group. If $G$ has $(RD)_q$ for some $q \in (1, 2]$, then $G$ is of polynomial growth.

Proof. The following argument is well-known.

Lemma. Let $G$ be amenable and $f \in C^c(G)$ a non negative function. Then

$$\|f\|_{B(L^p(G))} = \|f\|_{L^1(G)}$$

for any $p \in [1, \infty]$.

Proof. The statement follows from applying Folner sets.

Now take $f = \chi_{B_n}$. By similar arguments as in the proof of Proposition 4.3 one sees that $G$ is of polynomial growth.

Proposition 4.6. Let $p \in [1, \infty]$ and $q$ its dual number. Let $G$ be a locally compact group with property $(RD)_q$ with respect to a continuous length function $L$. Then for sufficiently large $t > 0$, the space $S_q^t(G)$ of elements $f \in L^q(G)$ such that

$$\|f\|_{S_q^t} := \|g \mapsto (1 + L(g))^t f(g)\|_{L^q(G)} < \infty$$

is a Banach algebra for the norm $\| \cdot \|_{S_q^t}$. It is contained in $B_{p,s}^q(G) \subset B_{p}^q(G), B_{p}^q(G)$, and stable under holomorphic functional calculus in each of these three algebras.

Proof. Here we generalize the argument in [10] to the case of locally compact groups and $L^p$ spaces.

Suppose $G$ has $(RD)_q$ with polynomial $n \mapsto Cn^D$ for some $C, D > 0$. First prove containment, namely

$$\|f\|_{B(L^p(G))} \leq K\|f\|_{S_q^t}, t \geq D + 1$$

(by duality the inequality for $\|f\|_{B(L^p(G))}$ follows from this one). For this let $S_n := B_n(e) \setminus B_{n-1}(e), f_n := f 1_{S_n}$

$$\|f\|_{B(L^p)} \leq \sum_n \|f_n\|_{B(L^p)} \leq \sum_n Cn^D \|f_n\|_{L^p}$$
by Holder’s inequality
\[ \leq C \left( \sum n^{-p} \right)^{1/p} \left( \sum n^{(D+1)q} \| f_n \|_q^q \right)^{1/q} = K \| f \|_{S_q^{D+1}}. \]

Now we prove that it is an algebra. By the inequality \((x + y)^s \leq 2^{s-1}(x^s + y^s), s \geq 1\) and the triangular inequality for \(L\), we have for non-negative \(f_1, f_2 \in C_c(G), x \in G\)

\[ f_1 \ast f_2(x)(1 + L(x))^s \leq 2^{s-1} \left( f_1 \ast (f_2(1 + L)^s)(x) + (f_1(1 + L)^s) \ast f_2(x) \right). \]

Therefore,
\[ \| f_1 \ast f_2 \|^q_{S_q^s} \leq 2^{q(s-1)+q-1} \left( \| f_1 \ast (f_2(1 + L)^s) \|^q_{L^q} + \| (f_1(1 + L)^s) \ast f_2 \|^q_{L^q} \right) \]
\[ \leq 2^{qs-1} K^q \left( \| f_1 \|^q_{S_q^{D+1}} \| f_2 \|^q_{S_q^s} + \| f_2 \|^q_{S_q^{D+1}} \| f_1 \|^q_{S_q^s} \right) \leq 2^{qs} K^q \| f_1 \|^q_{S_q^s} \| f_2 \|^q_{S_q^s}, \]

namely
\[ \| f_1 \ast f_2 \|^q_{S_q^s} \leq 2^s K \| f_1 \|^q_{S_q^s} \| f_2 \|^q_{S_q^s} \]

for \(s \geq D + 1\). The inequality still holds for \(f_1, f_2 \in C_c(G)\) without the non-negativity assumption by triangular inequality.

Now prove that it is stable under holomorphic functional calculus in both \(B^p_c(G)\) and \(B^p(G)\) (and consequently it is so in \(B^{p,*}_p\) as well). In fact, for \(B^p_c(G)\) it suffices to show for \(f \in C_c(G)\)

\[ \lim_n \| f^n \|^{1/n}_{S_q^s} = \lim_n \| f^n \|^{1/n}_{B(L^q(G))}. \]

First of all, by an argument as before we have
\[ \| f^{n+1} \|^q_{S_q^s} \leq 2^{qs-1} \left( \| f \|^q_{B(L^q(G))} \| f^n \|^q_{S_q^s} + \| f^n \|^q_{B(L^q(G))} \| f \|^q_{S_q^s} \right). \]

By an inductive argument we have \(\forall n\)
\[ \| f^n \|^q_{S_q^s} \leq 2^{ns} \| f \|^n_{S_q^s} \| f \|^{n-1}_{B(L^q(G))}, \]

which implies
\[ \lim_n \| f^n \|^{1/n}_{S_q^s} \leq 2^s \| f \|_{B(L^q(G))}. \]

Replacing \(f\) by \(f^p\) and let \(p \to \infty\) we have proved \(\leq\) part in the inequality. The other part \(\geq\) follows from the inequality concerning the containment \(S_q^s(G) \subset B^p_c(G)\) as in the beginning of this proof. The proof for \(B^p(G)\) is similar and so is omitted.

Let \(G\) be as before a locally compact group and \(L : G \to \mathbb{R}\) a continuous length function. Denote \(C_u(G)\) the translation algebra,
namely $C_c(G, L^\infty(G))$, and $B^p_u(G), p \in [1, \infty]$ its operator completion on $L^p(G)$, and $B^{p,u*}_u(G)$ its involutive counterpart.

Let $\delta$ be the densely defined derivation on $B^p_u(G)$ defined by the formula

$$\delta(T) = [l, T] = lT - TL,$$

where by abuse of notation, $l$ is the densely defined operator on $L^p(G)$ by point-wise multiplication $l : f \mapsto (x \mapsto l(x)f(x))$. $\delta$ is also a densely defined derivation on $B^{p,u*}_u(G)$ by the same formula.

The following proposition is well-known when $p = 2$, yet we cannot find a proof in the literature.

**Proposition 4.7.** $\delta$ is a closed derivation on both $B^p_u(G)$ and $B^{p,u*}_u(G)$.

**Proof.** Following [1] we say that a one-parameter group of automorphisms $a_t : \mathbb{R} \to \text{Aut}(B)$ on a Banach algebra is strongly continuous if $[t \mapsto a_tb] \in C(\mathbb{R}; B)$ is a continuous mapping for any $b \in B$. Its generator is denoted by $\delta_a$.

**Lemma.** ([1] Lemma 4.2.1) Let $a_t : \mathbb{R} \to \text{Aut}(B)$ be a strongly continuous as above. Then $x \in \text{Dom}(\delta_a), y = \delta_a x \in B$ if and only if

$$a_t x = x + \int_0^t a_s y ds, \forall t \in \mathbb{R}.$$ 

As a consequence, $\delta_a$ is a closed operator. \hfill $\blacksquare$

Recall that $l : G \to \mathbb{R}$ is a continuous length function. Define the following algebraic automorphism

$$a_t : B^p_u(G) \to B^p_u(G), T \mapsto e^{ilt}Te^{-ilt}, t \in \mathbb{R}$$

where $e^{ilt}$ acts on $L^p(G)$ unitarily by point-wise multiplication of the function $x \mapsto e^{ilt(x)}$. It is clear that $\delta_a = i\delta$.

By a density argument, it suffices to prove continuity on $C_u(G)$. First

$$|e^{ilt} - 1|^2 = (\cos t - 1)^2 + (\sin t)^2 \leq C^2 t^2$$

for some $C > 0$. Therefore,

$$(a_t e_g - e_g)v(x) = (e^{ilt(x)-ilt(g^{-1}x)} - 1)v(g^{-1}x),$$

and

$$\|a_t e_g - e_g\|_p \leq Ctl(g)\|v\|_p$$

This implies that for $T = \int e_g f_g dg$ with finite propagation

$$\|a_t(T) - T\|_{B(L^p(G))} \leq Ct \int l(g)\|f_g\|_{\infty} dg \to 0, t \to 0.$$ 

Therefore, $\delta$ is a closed derivation on both $B^p_u(G)$ and $B^{p,u*}_u(G)$. \hfill $\blacksquare$
Proposition 4.8. Let \( p \in [1, \infty] \) and \( q \) be its dual number. When \( G \) is discrete, \( S_q^\infty(G) := \cap_{t > 0} S_{q}^t(G) \) is the space of smooth vectors in \( B_p^r(G), B_q^r(G) \) and \( B_p^{\ast r}(G) \) with respect to \( \delta \).

Proof. The following argument is well-known. We include it here for the reader’s convenience.

First of all, we have
\[
(\delta^k(T)\xi)(g) = \sum_h T_{g,h}\xi(h)(l(g) - l(h))^k,
\]
which implies that for \( g \in G \)
\[
|(|\delta^k(f)\xi)(g)| \leq (|l^k|f|) \ast |\xi|)(g)
\]
and therefore, by \((RD)_q\)
\[
\|\delta^k f\|_{B(l^r(G))} \leq \|l^k|f|\|_{B(l^r(G))} \leq \|f(1 + l)^{k+s}\|_{B(l^r(G))}, r = p, q
\]
for sufficiently large \( s > 0 \). Thus \( S_q^\infty \) is contained in the intersection of domains of \( \delta^k \) in \( B_p^r(G), B_q^r(G) \) and also \( B_p^{\ast r}(G) \).

For the other inclusion, notice that
\[
\|fl^k\|_{B(l^r(G))} = \|\delta^k f(\delta_e)\|_{B(l^r(G))} \leq \|\delta^k f\|_{B(l^r(G))},
\]
\[
\|fl^k\|_{B(l^r(G))} = \|f^*l^k\|_{B(l^r(G))} = \|\delta^k f^*(\delta_e)\|_{B(l^r(G))} \leq \|\delta^k f\|_{B(l^r(G))},
\]
where \( \delta_e \) denotes the Dirac function at \( e \in G \).

In summary, \( S_q^\infty(G) \) is exactly the space of smooth vectors in \( B_p^r(G), B_q^r(G) \) and \( B_p^{\ast r}(G) \).

Theorem 1.2 in the introduction is a special case of the following statement.

Corollary 4.9. Let \( q_o \in [1, \infty], q \in [1, q_o], p \) be the dual of \( q \). Let \( G \) be a locally compact group with property \((RD)_{q_o} \). Then the canonical algebraic morphism \( B_p^{\ast r}(G) \rightarrow B_p^r(G) \) induces isomorphism between their K-theory groups. So does \( B_p^{\ast r}(G) \rightarrow B_q^r(G) \).

Proof. The statement follows from Theorem 4.4 and Proposition 4.6.

Theorem 1.5 is a special case of the following Corollary.

Corollary 4.10. Let \( G \) be a locally compact group in Lafforgue’s class \( C' \) having property \((RD)_q \), \( q \in [1, 2] \). \( i_{p',p} : B_p^{\ast r}(G) \rightarrow B_q^{\ast r}(G) \) induces isomorphism in K-theory for any \( 2 \leq p < p' \leq +\infty \), where \( p \) is the dual number of \( q \).
Proof. Apply the assembly maps for unconditional completions \([11]\) to \(S^t_q(G)\) and \(S^t_q'(G)\) for sufficiently large \(t > 0\), where \(q'\) is dual to \(p'\). The claim follows from Proposition \([4.6]\) the injectivity \([9, 7, 8]\) and surjectivity \([12, 11]\) of the assembly maps \([11]\).

5 Open problems

In this last section, we list several interesting open problems.

1. Is the algebra \(B^p_r(G)\) non-involution for every (discrete) non-amenable group? Namely, is it true that on a non-amenable group, there exists a function \(f : G \to \mathbb{C}\) that acts as a bounded operator by convolution on \(L^p(G)\), but its involution \(f^*\) does not?

2. Does the canonical morphism \(B^{p,*}_r(G) \to B^p_r(G), p \geq 2\), always induces an isomorphism in K-theory? We already know that it is true for groups with \((RD)_q\) where \(1/p + 1/q = 1\), and for groups acting properly isometrically on \(L^p\) spaces.

3. Is it true that \(K_*(B^p_r(G))\) and \(K_*(B^{p,*}_r(G))\) are independent of \(p\) for any locally compact group \(G\)?

4. Is there a locally compact group with property \(RD\) but fails \((RD)_{q}\) for some \(q \in (1, 2)\)?

5. Does semi-continuity (Theorem \([13]\)) hold for a locally compact group? What about crossed product? See \([3]\) for a discussion on \(L^p\) cross product.

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