Precise Propagation of Upper and Lower Probability Bounds in System P

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Abstract
In this paper we consider the inference rules of System P in the framework of coherent imprecise probabilistic assessments. Exploiting our algorithms, we propagate the lower and upper probability bounds associated with the conditional assertions of a given knowledge base, automatically obtaining the precise probability bounds for the derived conclusions of the inference rules. This allows a more flexible and realistic use of System P in default reasoning and provides an exact illustration of the degradation of the inference rules when interpreted in probabilistic terms. We also examine the disjunctive Weak Rational Monotony of System P$^+$ proposed by Adams in his extended probability logic.

Keywords: Nonmonotonic reasoning, System P, Conditional probability bounds, Precise propagation, Coherence.

Introduction
In the applications of intelligent systems to automated uncertain reasoning the explicit knowledge of the agent is represented by a knowledge base $K$, constituted by a set of conditional assertions (i.e. defaults). The nonmonotonic inferential process is developed using a suitable set of rules. Among the many formalisms which have been proposed for default reasoning, the so-called System P developed in (Kraus, Lehmann, and Magidor 1990) is widely accepted and has a probabilistic semantics based on infinitesimal probabilities, see (Adams 1975, Pearl 1988). An extended probability logic has been proposed in (Adams 1986) by allowing disjunction of conditionals. The corresponding system $P^+$ has been studied in (Schurz 1998) where the perspectives of the infinitesimal probability semantics and that of a noninfinitesimal one, based on probabilistic inequality relations, have been unified. In (Hawthorne 1996) many logics of nonmonotonic conditionals, that behave like conditional probabilities at various levels of probabilistic support, have been examined. In the quoted paper the author shows that, for each given conditional $\rightarrow$, there is a probabilistic support level $r$ and a conditional probability $P$ such that, for all sentences $B, A$, it is $B \rightarrow A$ only if $P(A|B) \geq r$. We recall that an early examination of Adams' rules by means of imprecise probabilities has been given in (Dubois, and Prade 1991). In the quoted paper a semantic interpretation in terms of intervals has been given for the relations of negligibility, closeness and comparability examined in the system of qualitative reasoning proposed in (Raiman 1986). Moreover, an application to the inference rules of Adams has been given interpreting "$P(A|B)$ is large" by means of the relation of closeness to 1 (making the infinitesimal parameter $\epsilon$ explicit). In this way a quantification of the degradation of the validity of Adams' rules when reasoning with noninfinitesimal probabilities has been obtained. While in practical applications the semantics of infinitesimal probabilities may involve some difficulties, the approach based on lower (and possibly upper) probability bounds, proposed also in (Bourne, and Parsons 1998), is clearly more flexible and realistic. In this paper the propagation of probability bounds to the conditional assertions associated with the rules of System P is examined in the framework of the de Finetti's probabilistic methodology, based on the coherence principle. Notice that a coherent set of conditional probability assessments satisfies all the usual properties of conditional probabilities. A short examination of the logic of conditionals of Adams from the point of view of coherence has been given in (Gilio 1997), where the propagation of probabilistic bounds has not been considered. We point out that the peculiarity of our approach is given by the possibility of looking at the conditional probability $P(A|B)$ as a primitive concept, with no need of assuming that the probability of the conditioning event $B$ be positive. On the contrary, in the probabilistic approaches usually adopted in the literature, see, e.g., (Adams 1975, Hawthorne 1996, Schurz 1998), by definition the quantity $P(A|B)$ is the ratio of $P(AB)$ and $P(B)$ if $P(B) \neq 0$, with $P(A|B) = 1$ if $P(B) = 0$. Notice that a clear rationale of this latter assumption does not seem to exist. Indeed,
in the framework of coherence, this assumption is not made and the case of conditioning events of zero probability is managed without any problem: as an example, the condition \( P(A|B) + P(A^c|B) = 1 \), where \( A^c \) denotes the negation of \( A \), is satisfied also when \( P(B) = 0 \). We think that the opportunity, offered by the probabilistic approach based on coherence, of developing a completely general treatment of probabilistic default reasoning is important specially in the field of nonmonotonic reasoning where infinitesimal probabilities play a significant role. Moreover, exploiting the algorithms developed in our framework, the lower and upper probability bounds associated with the conditional assertions of a given knowledge base can be propagated to further conditional assertions, obtaining in all cases the tightest probability bounds. Beside allowing a more flexible and realistic approach to probabilistic default reasoning, this provides an exact illustration of the degradation of System P rules when interpreted in probabilistic terms. The probabilistic approach based on coherence has been adopted in many recent papers, see, e. g., (Biazzo, and Gilio 1999), (Capotorti, and Vantaggi 1999), (Coletti 1995), (Coletti, and Scozzafava 1996), (Coletti, and Scozzafava 1999), (Gilio 1995a), (Gilio 1995b), (Gilio, and Ingassina 1998), (Gilio, and Scozzafava 1994), (Lad 1999), (Lad, Dickey, and Rahman 1990), (Scozzafava 1994). The algorithms described in (Biazzo, and Gilio 1999), based on the linear programming technique, have been implemented with Maple V. The paper is organized as follows. In Section 2 we give some preliminary concepts on coherence and probability logic. In Section 3 we consider the definitions of probabilistic consistency and entailment given by Adams; then we recall the main inference rules of his probabilistic default reasoning. In Section 4 we consider the propagation of lower and upper probability bounds in System P. We also examine the disjunctive Weak Rational Monotony of System P. In Section 5 we examine the propagation of lower bounds with real \( \epsilon \)-values. Then, in Section 6 we apply the results to (a slightly modified version of) an example given in (Bourne, and Parsons 1998). Finally, in Section 7 we give some conclusions.

Some preliminaries

We recall some preliminary concepts on coherence of imprecise probability assessments and on probability logic. Given a family \( \mathcal{F}_n = \{E_1, H_1, \ldots, E_n, H_n\} \) and a vector \( \mathcal{A}_n = (\alpha_1, \ldots, \alpha_n) \) of lower bounds \( P(E_i|H_i) \geq \alpha_i \), with \( i \in J_n = \{1, \ldots, n\} \), we consider the following definition of generalized coherence (g-coherence), given in (Biazzo, and Gilio 1999), which essentially coincides with a previous one introduced in (Gilio 1995a).

**Definition 1** The vector of lower bounds \( \mathcal{A}_n \) on \( \mathcal{F}_n \) is said g-coherent if and only if there exists a precise coherent assessment \( \mathcal{P}_n = (p_1, \ldots, p_n) \) on \( \mathcal{F}_n \), with \( p_i = P(E_i|H_i) \), which is consistent with \( \mathcal{A}_n \), that is such that \( p_i \geq \alpha_i \) for each \( i \in J_n \).

The Definition 1 can be also applied to imprecise assessments like

\[ \alpha_i \leq P(E_i|H_i) \leq \beta_i \quad i \in J_n, \]

since each inequality \( P(E_i|H_i) \leq \beta_i \) amounts to the inequality \( P(E_i^c|H_i) \geq 1 - \beta_i \).

Then, given an imprecise assessment \( \mathcal{A}_n = (\alpha_1, \ldots, \alpha_n) \) on \( \mathcal{F}_n \), a suitable procedure, given in (Gilio 1995b), can be used to check the g-coherence of \( \mathcal{A}_n \). The g-coherent extension of \( \mathcal{A}_n \) to a further conditional event \( E_{n+1}|H_{n+1} \) has been studied in (Biazzo, and Gilio 1999) where, defining a suitable interval \( [p_0, p^0] \subseteq [0, 1] \), the following result has been obtained.

**Theorem 1** Given a g-coherent imprecise assessment \( \mathcal{A}_n \) on the family \( \mathcal{F}_n = \{E_i|H_i, i \in J_n\} \), the extension \( [\alpha_{n+1}, \beta_{n+1}] \) of \( \mathcal{A}_n \) to a further conditional event \( E_{n+1}|H_{n+1} \) is g-coherent if and only if the following condition is satisfied

\[ [\alpha_{n+1}, \beta_{n+1}] \cap [p_0, p^0] \neq \emptyset. \quad (1) \]

In the quoted paper an algorithm has been proposed to determine \( [p_0, p^0] \). Moreover, starting with a g-coherent assessment \( \mathcal{A}_n \), by the same algorithm it is possible to determine the corresponding assessment \( \mathcal{A}_n \) coherent wrt definition given in (Walley 1991).

We can frame our approach to the problem of propagating imprecise conditional probability assessments (probabilistic deduction) from the probability logic point of view, see, e. g., (Frisch, and Haddawy 1994), (Łukasiewicz 1998), (Nilsson 1986). We associate to each conditional assertion \( H \models E \) in the knowledge base \( \mathcal{K} \) a probability interval \( [\alpha, \beta] \). In particular, given a family \( \mathcal{F}_n \) of \( n \) conditional assertions, we consider an interval-valued probability assessment \( \mathcal{A}_n = ([\alpha_i, \beta_i], i = 1, \ldots, n) \). Then, we can look at the pair \( (\mathcal{F}_n, \mathcal{A}_n) \) as a probabilistic knowledge base, where each imprecise assessment \( \alpha_i \leq P(E_i|H_i) \leq \beta_i \) is a probabilistic formula denoted by \( (E_i|H_i) [\alpha_i, \beta_i] \). In our approach a probabilistic interpretation is just a coherent precise conditional probability assessment \( \mathcal{P}_n \) on \( \mathcal{F}_n \). A probabilistic interpretation \( \mathcal{P}_n = (p_1, \ldots, p_n) \) is a model of a probabilistic formula \( (E_i|H_i) [\alpha_i, \beta_i] \) iff \( \mathcal{P}_n \models (E_i|H_i) [\alpha_i, \beta_i] \) that is \( \alpha_i \leq p_i \leq \beta_i \). \( \mathcal{P}_n \) is a model of the probabilistic knowledge base \( \mathcal{K} = (\mathcal{F}_n, \mathcal{A}_n) \), denoted \( \mathcal{P}_n \models \mathcal{K} \), if \( \mathcal{P}_n \models (E|H) [\alpha, \beta] \) for every \( (E|H) [\alpha, \beta] \). Therefore, \( \mathcal{P}_n \) is a model of \( \mathcal{K} = (\mathcal{F}_n, \mathcal{A}_n) \), if \( \mathcal{P}_n \) is consistent with \( \mathcal{A}_n \). A set of probabilistic formulas \( \mathcal{K} \) is satisfiable iff a model of \( \mathcal{K} \) exists, therefore the concept of satisfiability of \( \mathcal{K} = (\mathcal{F}_n, \mathcal{A}_n) \) coincides with that of g-coherence of \( \mathcal{A}_n \) on \( \mathcal{F}_n \). A probabilistic formula \( (E_{n+1}|H_{n+1}) [\alpha_{n+1}, \beta_{n+1}] \) is a logical consequence of \( \mathcal{K} = (\mathcal{F}_n, \mathcal{A}_n) \), denoted \( \mathcal{K} \models (E_{n+1}|H_{n+1}) [\alpha_{n+1}, \beta_{n+1}] \), iff

\[ \alpha_{n+1} \leq \inf \mathcal{I}, \quad \beta_{n+1} \geq sup \mathcal{I}, \]

where \( \mathcal{I} \) is the set of the real values \( p \) such that there exists a model of \( \mathcal{K} \cup \{(E_{n+1}|H_{n+1})[p, p]\} \). As shown by the condition \( \emptyset \), in our approach this amounts to

\[ [p_0, p^0] \subseteq [\alpha_{n+1}, \beta_{n+1}]. \]
A probabilistic formula \((E_{n+1}|H_{n+1})|\alpha_{n+1},\beta_{n+1}\) is a tight logical consequence of \(\mathcal{K} = (\mathcal{F}_n, A_n)\), denoted \(\mathcal{K} \models_{tight} (E_{n+1}|H_{n+1})|\alpha_{n+1},\beta_{n+1}\), iff

\[
\alpha_{n+1} = \inf \mathcal{I}, \quad \beta_{n+1} = \sup \mathcal{I},
\]

that is

\[
\alpha_{n+1} = p_0, \quad \beta_{n+1} = p^*.
\]

Considering a probabilistic query \((E_{n+1}|H_{n+1})|\alpha,\beta\), where \(\alpha\) and \(\beta\) are two different variables, to a probabilistic knowledge base \(\mathcal{K} = (\mathcal{F}_n, A_n)\) a correct answer is any \([\alpha,\beta] = [\alpha_{n+1},\beta_{n+1}] \supseteq [p_0,p^*]\), that is such that \(\mathcal{K} \models (E_{n+1}|H_{n+1})|\alpha_{n+1},\beta_{n+1}\). The tight answer is \([\alpha,\beta] = [p_0,p^*]\).

**Probabilistic consistency and entailment**

We recall that in (Adams 1975) the conditional assertion “if \(A\) then \(B\)” is looked at as \(P(B|A) \geq 1 - \epsilon\) (\(\forall \epsilon > 0\)). Adopting a more realistic point of view we may look at the same conditional assertion as the probabilistic formula \((B|A)|\alpha,\beta\), with \(0 \leq \alpha \leq \beta \leq 1\), where usually \(\beta = 1\). Then, a (probabilistic) knowledge base might be defined as a family of probabilistic formulas \(\mathcal{K} = \{(E|H)|\alpha,\beta\}\). In (Adams 1975) the following definition has been given.

**Definition 2** The knowledge base \(\mathcal{K}\) is probabilistically consistent (p-consistent) if, for every \(\epsilon > 0\), there exists a probability \(P\) on \(\mathcal{A}\), proper for \(\mathcal{K}\), such that \(P(E|H) \geq 1 - \epsilon\) for every \(E|H \in \mathcal{K}\).

In our framework the concept of probabilistic consistency can be defined in the following way.

**Definition 3** The knowledge base \(\mathcal{K}\) is probabilistically consistent (p-consistent) if, for every set of lower bounds \(\mathcal{A} = \{E|H, E|H \in \mathcal{K}\}\) on \(\mathcal{K}\), there exists a precise coherent probability assessment \(P = \{p_{E|H}, E|H \in \mathcal{K}\}\) on \(\mathcal{K}\), with \(p_{E|H} = P(E|H)\), such that, for each \(E|H \in \mathcal{K}\), \(p_{E|H} \geq \alpha_{E|H}\).

We recall the concept of probabilistic entailment as defined in (Adams 1975).

**Definition 4** A p-consistent knowledge base \(\mathcal{K}\) probabilistically entails (p-entails) the conditional \(B|A\) if, for every \(\epsilon > 0\), there exists \(\delta > 0\) such that for all probabilities \(P\), proper for \(\mathcal{K} \cup \{B|A\}\), if \(P(E|H) \geq 1 - \delta\) for each \(E|H \in \mathcal{K}\), then \(P(B|A) \geq 1 - \epsilon\).

In our framework the concept of probabilistic entailment can be defined in the following way.

**Definition 5** A p-consistent knowledge base \(\mathcal{K}\) probabilistically entails the conditional \(B|A\) if there exists a subfamily \(\mathcal{F} \subseteq \mathcal{K}\) such that, for every \(\alpha_{B|A}\), there exists a set of lower bounds \(\mathcal{A} = \{\alpha_{E|H}, E|H \in \mathcal{F}\}\) on \(\mathcal{F}\) such that for all precise coherent probability assessment \(P = \{p_{B|A}, p_{E|H}, E|H \in \mathcal{F}\}\) on \(\mathcal{F} \cup \{B|A\}\), with \(p_{B|A} = P(B|A), p_{E|H} = P(E|H)\), if \(p_{E|H} \geq \alpha_{E|H}\) for each \(E|H \in \mathcal{F}\), then \(p_{B|A} \geq \alpha_{B|A}\).

The probabilistic entailment of \(B|A\) by \(\mathcal{K}\) is denoted by the symbol \(\mathcal{K} \Rightarrow B|A\). In (Adams 1975) a suitable set \(\mathcal{R}\) of seven inference rules has been introduced, by means of which an inferential system can be developed to deduce in an automatic way all the plausible conclusions of a knowledge base \(\mathcal{K}\). See also (Pearl 1988). The fundamental rules of the set \(\mathcal{R}\) are the following ones:

- **R1.** \(A \triangleright \sim C, A \triangleright B \Rightarrow AB \triangleright \sim C\) (Triangulary)
- **R2.** \(AB \triangleright \sim C, A \triangleright B \Rightarrow A \triangleright \sim C\) (Bayes)
- **R3.** \(A \triangleright \sim C, B \triangleright \sim C \Rightarrow A \lor B \triangleright \sim C\) (Disjunction)

The previous rules, among others, have been used (with different names and in the framework of symbolic approaches too) by many authors, with the aim of developing some nonmonotonic logics to formalize the plausible reasoning (see e.g. (Kraus, Lehmann, and Magidor 1990), (Lehmann, and Magidor 1992), (Dubois, and Prade 1994)). See also the survey given in (Benferhat, Dubois, and Prade 1997).

We recall that in (Gilio 1997) the following concept of strict probabilistic consistency has been introduced.

**Definition 6** The knowledge base \(\mathcal{K}\) is strictly p-consistent if the probability assessment \(P\) on \(\mathcal{K}\), such that \(P(E|H) = 1\) for each \(E|H \in \mathcal{K}\), is coherent.

Then the following result, which has some relations with the Theorem 3 given in (Schurz 1998), has been given (without proof).

**Theorem 2** \(\mathcal{K}\) is p-consistent if and only if \(\mathcal{K}\) is strictly p-consistent.

The proof of Theorem 2 is based on the following three assertions:

- a. If \(\mathcal{K}\) is strictly p-consistent, then \(\mathcal{K}\) is p-consistent;
- b. If \(\mathcal{K}\) is p-consistent, then \(\mathcal{K}\) is consistent;
- c. If \(\mathcal{K}\) is consistent, then \(\mathcal{K}\) is strictly p-consistent.

Hence, the property of p-consistency can be simply defined on the basis of the property of strict p-consistency. Moreover, the following well known result

**Theorem 3** If \(\mathcal{K}\) is consistent, then \(\mathcal{K} \Rightarrow B|A\) if and only if \(\mathcal{K} \cup \{B^+|A\}\) is inconsistent.

Can be reformulated in the following way:

**Theorem 4** Given a consistent knowledge base \(\mathcal{K}\) and a conditional \(B|A\), \(\mathcal{K}\) p-entails \(B|A\) if and only if the probability assessment \(P\) on \(\mathcal{K} \cup \{B^+|A\}\), with \(P(B^+|A) = P(E|H) = 1\) for each \(E|H \in \mathcal{K}\), is not coherent.
Exact propagation of probability bounds in System P

We recall that the inference rules in System P are the following ones:

1. \( A \vdash A \) (Reflexivity)
2. \( \vdash A \leftrightarrow B, A \vdash C \Rightarrow B \vdash C \) (Left Logical Equivalence)
3. \( \vdash B \rightarrow C, A \vdash B \Rightarrow A \vdash C \) (Right Weakening)
4. \( A \vdash B, A \vdash C \Rightarrow A \vdash BC \) (And)
5. \( A \vdash C, A \vdash B \Rightarrow AB \vdash C \) (Cautious Monotonicity)
6. \( A \vdash C, B \vdash C \Rightarrow A \lor B \vdash C \) (Or)

Two derived rules in System P are

7. \( AB \vdash C, A \vdash B \Rightarrow A \vdash C \) (Cut)
8. \( AB \vdash C \Rightarrow A \vdash B \rightarrow C \) (S)

As we can see, the rules R1, R2, R3 coincide respectively with the rules Cautious Monotonicity, Cut, Or in System P. In (Adam98) an extended probability logic (System P+) was developed by allowing the disjunction of conditionals in the conclusion of inferences and by adding the following dWRM rule.

9. \( A \vdash C \Rightarrow A \vdash B^c \lor AB \vdash C \) (disjunctive Weak Rational Monotony)

Now we will show how the probability intervals associated with the antecedents in each rule of System P propagate in a precise way to the consequent of the given rule. We will also examine the rules Cut, S and dWRM. These bounds will provide an exact illustration of the degradation of System P rules when interpreted in probabilistic terms. Perhaps, as a by-product, also a better appreciation of the results given in (Hawthorne 1996) on the interpretation of nonmonotonic conditionals in terms of probabilistic support levels could be obtained. We assume that the basic events \( A, B, C \) are logically independent.

1. Reflexivity rule. As for every assertion \( A \) the assessment \( P(A|A) = p \) is coherent if and only if \( p = 1 \), to the conditional assertion \( A \vdash A \) we associate the interval \( [\alpha, \beta] = [1,1] \).

2. Left Logical Equivalence rule. If two assertions \( A, B \) are equivalent, then for every assertion \( C \) the assessment \( (x, y) \) on the pair of conditional events \( C|A, C|B \) is coherent if and only \( x = y \). Therefore we associate the same probability interval to the conditional assertions \( C|A, C|B \). In other words, the assessment \( [\alpha, \beta] \) on \( C|A \) propagates to the same interval on \( C|B \).

3. Right Weakening rule. If \( B \subseteq C \) then, defining the inclusion among conditional events as in (Goodman and Nguyen 1988), it is \( B|A \subseteq C|A \) and then the assessment \( (x, y) \) on the pair of conditional events \( B|A, C|A \) is coherent if and only \( x \leq y \), see (Gilio 1993). Therefore, the assessment \( [\alpha, \beta] \) on \( B|A \) propagates to the interval \( [\alpha,1] \) on \( C|A \).

4. And rule. Given the assessment \( (x, y) \) on the pair of conditional events \( B|A, C|A \), as well known the extension \( P(BC|A) = z \) is coherent if and only if

\[
\text{Max} \{0, x+y-1\} = z' \leq z \leq z'' = \text{Min} \{x, y\}.
\]

Therefore, the probability intervals \([\alpha_1, \beta_1], [\alpha_2, \beta_2]\) on the antecedents \( B|A, C|A \) of the rule propagate to the exact interval \([\alpha_3, \beta_3]\), with

\[
\alpha_3 = \text{Min} \ z' = \text{Max} \ \{0, \alpha_1 + \alpha_2 - 1\} ,
\]

\[
\beta_3 = \text{Max} \ z'' = \text{Min} \ \{\beta_1, \beta_2\} ,
\]

on the consequent \( BC|A \).

5. Cautious Monotonicity rule. Given the assessment \( (x, y) \) on the pair of conditional events \( C|A, B|A \), as proved in (Gilio 1995b), the extension \( P(C|AB) = z \) is coherent if and only if \( z \in [z', z''] \), with

\[
z' = \begin{cases} \frac{x+y-1}{y} & \text{if } x+y > 1 \\ 0 & \text{if } x+y \leq 1 \end{cases},
\]

\[
z'' = \begin{cases} \frac{z}{y} & \text{if } x < y \\ 1 & \text{if } x \geq y \end{cases}.
\]

We observe that, for \((x, y) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\) the function \( f(x, y) \) attains its minimum value at the point \((\alpha_1, \alpha_2)\). Then, it follows

\[
\alpha_3 = \begin{cases} \frac{\alpha_1 + \alpha_2 - 1}{2} & \text{if } \alpha_1 + \alpha_2 > 1 \\ 0 & \text{if } \alpha_1 + \alpha_2 \leq 1 \end{cases},
\]

\[
\beta_3 = \begin{cases} \frac{\alpha_2}{2} & \text{if } \beta_1 < \alpha_2 \\ 1 & \text{if } \beta_1 \geq \alpha_2 \end{cases}.
\]

Moreover, the function \( g(x, y) \) attains its maximum value at the point \((\beta_1, \alpha_2)\). Then it follows

\[
\beta_3 = \begin{cases} \frac{\alpha_2}{2} & \text{if } \beta_1 < \alpha_2 \\ 1 & \text{if } \beta_1 \geq \alpha_2 \end{cases}.
\]

Then, in order to determine the interval \([\alpha_3, \beta_3]\) we have to consider the position of the vertices \((\alpha_1, \alpha_2), (\beta_1, \alpha_2)\) wrt diagonals of the unitary square \([0,1]^2\).

6. Or rule. Given the assessment \( (x, y) \) on the pair of conditional events \( C|A, C|B \), it can be proved that the extension \( P(C|A \lor B) = z \) is coherent if and only if \( z \in [z', z''] \), with

\[
z' = \begin{cases} \frac{xy}{x+y-xy} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases},
\]

\[
z'' = \begin{cases} \frac{xy-2xy}{1-xy} & \text{if } (x, y) \neq (1,1) \\ 1 & \text{if } (x, y) = (1,1) \end{cases}.
\]
Moreover, we observe that both \( z' \) and \( z'' \) increase as either \( x \) or \( y \) increase. Therefore, the probability intervals \([\alpha_1, \beta_1], [\alpha_2, \beta_2]\) on the antecedents \(C\vert A, C\vert B\) of the rule propagate, under the condition \((\alpha_1, \alpha_2) \neq (0, 0), (\beta_1, \beta_2) \neq (1, 1)\), to \([\alpha_3, \beta_3]\), with
\[
\begin{align*}
\alpha_3 &= \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2 - \alpha_1 \alpha_2}, \\
\beta_3 &= \frac{\beta_1 + \beta_2 - 2\beta_1 \beta_2}{1 - \beta_1 \beta_2},
\end{align*}
\]
on the consequent \(C\vert (A \lor B)\).

Concerning the rules Cut and S we have the following results.

(e) Cut rule. Given the assessment \((x, y)\) on the pair of conditional events \(C\vert AB, B\vert A\), it can be proved that the extension \(P(C\vert A) = z\) is coherent if and only if
\[xy \leq z \leq xy + 1 - y.\]

Therefore, the probability intervals \([\alpha_1, \beta_1], [\alpha_2, \beta_2]\) on the antecedents \(C\vert AB, B\vert A\) of the rule propagate to \([\alpha_3, \beta_3]\), with
\[
\begin{align*}
\alpha_3 &= \alpha_1 \alpha_2, \\
\beta_3 &= \beta_1 \alpha_2 + 1 - \alpha_2,
\end{align*}
\]
on the consequent \(C\vert A\).

(f) S rule. As \(C\vert AB \subseteq (B^c \lor C)\vert A\), then the assessment \((x, y)\) on the conditional events \(C\vert AB, (B^c \lor C)\vert A\) is coherent if and only if \(x \leq y\). Therefore, the probability interval \([\alpha_1, \beta_1]\) on the antecedent \(C\vert AB\) of the rule propagates to \([\alpha_2, \beta_2]\), with
\[
\begin{align*}
\alpha_2 &= \alpha_1, \\
\beta_2 &= 1,
\end{align*}
\]
on the consequent \((B^c \lor C)\vert A\).

(g) dWRM rule. Let \(P = (x, y, z)\) a probability assessment on the family \(\{C\vert A, B^c\vert A, C\vert AB\}\). For this family the constituents (possible worlds) are
\[C_0 = A^c, \ C_1 = ABC, \ C_2 = ABC^c, \ C_3 = AB^cC, \ C_4 = AB^cC^c.\]

To the constituents \(C_1, \ldots, C_4\) we associate the points
\[Q_1 = (1, 0, 0), \ Q_2 = (0, 0, 0), \ Q_3 = (1, 1, z), \ Q_4 = (0, 1, z).\]

Then, based on the method given in (Gilio 1995b) and denoting by \(I\) the convex hull of the points \(Q_1, \ldots, Q_4\), it can be proved that the coherence of an assessment \(P \in I\) amounts to the condition \(P \in I\). Notice that in general this condition is necessary but not sufficient for the coherence of an assessment \(P = (p_1, \ldots, p_n)\) on a family \(F = \{E_1\vert H_1, \ldots, E_n\vert H_n\}\). The study of the condition \(P \in I\) requires considering the equations of the four planes determined respectively by the terms of points
\[
\begin{align*}
\{Q_1, Q_2, Q_3\}, & \quad \{Q_2, Q_3, Q_4\}, \\
\{Q_1, Q_2, Q_4\}, & \quad \{Q_1, Q_3, Q_4\}.
\end{align*}
\]

Denoting by \(X, Y, Z\) the axes’ coordinates, the equations are given respectively by
\[
\begin{align*}
Z &= X + (z - 1)Y, & Z &= zY, \\
Z &= X + zY, & Z &= (z - 1)Y + 1.
\end{align*}
\]

Then, given the values \(x, y\), it is
\[P \in I \iff z' \leq z \leq z'',\]
where
\[
\begin{align*}
z' &= \begin{cases} \frac{x - y}{1 - y}, & \text{if } x > y \\ 0, & \text{if } x \leq y \end{cases}, \\
z'' &= \begin{cases} \frac{x - y}{1 - y}, & \text{if } x + y < 1 \\ 1, & \text{if } x + y \geq 1 \end{cases}.
\end{align*}
\]

In order to examine the probabilistic interpretation of the rule we introduce a partition \(\{R_1, R_2, R_3, R_4\}\) of the unitary square \([0, 1]^2\), with
\[
\begin{align*}
R_1 &= \{(x, y) : x + y < 1, x \geq y\}, \\
R_2 &= \{(x, y) : x + y < 1, x < y\}, \\
R_3 &= \{(x, y) : x + y \geq 1, x < y\}, \\
R_4 &= \{(x, y) : x + y \geq 1, x \geq y\}.
\end{align*}
\]

We have to examine the case in which \(x\) is "high", therefore \(R_3\) is not of interest. In \(R_3\), since \(x < y\) if \(x\) is "high" then \(y\) is "high" too. In \(R_1\) and \(R_4\) it is \(z \geq z' = \frac{x - y}{1 - y}\) so that, if \(x\) is "high" and \(y\) is "not high", then \(z\) is "high".

Concerning propagation of probability intervals, if we consider the assessments \([\alpha_1, \beta_1], [\alpha_2, \beta_2]\) on the conditional events \(C\vert AB, B^c\vert A\), then for the interval \([\alpha_3, \beta_3]\) associated with \(C\vert AB\) we first observe that the quantity \(\frac{x - y}{1 - y}\) attains its maximum value at the point \((\beta_1, \alpha_2)\), while the quantity \(\frac{x - y}{1 - y}\) attains its minimum value at the point \((\alpha_1, \beta_2)\). Then, we have:
\[
\alpha_3 = \begin{cases} \frac{\alpha_1 + \beta_2}{1 - \alpha_2}, & \text{if } \alpha_1 \geq \beta_2 \\ 0, & \text{if } \alpha_1 < \beta_2 \end{cases} \quad (8)
\]
\[
\beta_3 = \begin{cases} \frac{\beta_1}{1 - \alpha_2}, & \text{if } \beta_1 + \alpha_2 < 1 \\ 1, & \text{if } \beta_1 + \alpha_2 \geq 1 \end{cases} \quad (9)
\]

**Remark 1** Using the lower bounds computed previously, we can verify the probabilistic entailment in each inference rule on the basis of Definition 3. We have

- **And rule.** For each given value \(\alpha_3\), from (3) we have that, for every \((\alpha_1, \alpha_2) \in [\alpha_3, 1] \times [\alpha_3, 1]\) such that \(\alpha_1 + \alpha_2 = 1 + \alpha_3\), if \(P(B\vert A) \geq \alpha_1, P(C\vert A) \geq \alpha_2\) then \(P(BC\vert A) \geq \alpha_3\).

- **Cautious Monotonicity rule.** For each given value \(\alpha_3\), from (3) we have that, for every \((\alpha_1, \alpha_2) \in [\alpha_3, 1] \times [0, 1]\) such that \(\alpha_1 + (1 - \alpha_3)\alpha_2 = 1\), if \(P(C\vert A) \geq \alpha_1, P(B\vert A) \geq \alpha_2\) then \(P(C\vert AB) \geq \alpha_3\).
• **Or rule.** For each given value \( \alpha_3 \), from (3) we have that, for every \(( \alpha_1, \alpha_2 ) \in [ \alpha_3, 1] \times [ \alpha_3, 1] \) such that \( \alpha_2 = \frac{\alpha_1 \alpha_3}{\alpha_3 (1+\alpha_3) - \alpha_1} \), if \( P(C|A) \geq \alpha_1, P(C|B) \geq \alpha_2 \) then \( P(C|A \lor B) \geq \alpha_3 \).

• **Cut rule.** For each given value \( \alpha_3 \), from (7) we have that for every \(( \alpha_1, \alpha_2 ) \in [ \alpha_3, 1] \times [ \alpha_3, 1] \) such that \( \alpha_2 = \frac{\alpha_1 \alpha_3}{\alpha_3 (1+\alpha_3) - \alpha_1} \), if \( P(C|AB) \geq \alpha_1, P(C|A) \geq \alpha_2 \) then \( P(C|A) \geq \alpha_3 \).

**Propagation with \( \epsilon \)-values**

The results of the previous section can be examined in the particular case in which for \( i = 1, 2 \) it is \([ \alpha_i, \beta_i ] = [1 - \epsilon_i, 1] \). As it can be verified, the \( \epsilon \)-values propagate in the following way.

• **And rule.** From (2), the probability bounds \([1 - \epsilon_1, 1], [1 - \epsilon_2, 1] \) on the antecedents \( B|A, C|A \) of the rule propagate, on the consequent \( BC|A \), to the exact bounds \([1 - \epsilon_3, 1] \), with

\[ \epsilon_3 = \epsilon_1 + \epsilon_2 \quad (10) \]

• **Cautious Monotonicity rule.** From (3), the probability intervals \([1 - \epsilon_1, 1], [1 - \epsilon_2, 1] \) on the antecedents \( C|A, B|A \) of the rule propagate, on the consequent \( C|AB \), to \([1 - \epsilon_3, 1] \), with

\[ \epsilon_3 = \frac{\epsilon_1 - \epsilon_2}{1 - \epsilon_1 \epsilon_2} \quad (11) \]

• **Or rule.** From (4), the probability intervals \([1 - \epsilon_1, 1], [1 - \epsilon_2, 1] \) on the antecedents \( C|A, C|B \) of the rule propagate, on the consequent \( C|A \lor B \), to \([1 - \epsilon_3, 1] \), with

\[ \epsilon_3 = \frac{\epsilon_1 + \epsilon_2 - 2\epsilon_1 \epsilon_2}{1 - \epsilon_1 \epsilon_2} \quad (12) \]

• **Cut rule.** From (5), the probability intervals \([1 - \epsilon_1, 1], [1 - \epsilon_2, 1] \) on the antecedents \( C|AB, B|A \) of the rule propagate, on the consequent \( C|A \), to \([1 - \epsilon_3, 1] \), with

\[ \epsilon_3 = \frac{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2}{1 - \epsilon_1 \epsilon_2} \quad (13) \]

**Remark 2** Our results concerning the value of \( \epsilon_3 \) coincide with that ones obtained in (Bourne, and Parsons 1998) for the rules And and Cautious Monotonicity and are better for the rules Or and Cut, as from (10) and (11) one has respectively

\[ \epsilon_3 = \frac{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2}{1 - \epsilon_1 \epsilon_2} < \epsilon_1 + \epsilon_2 \quad (\epsilon_1 < 1, \epsilon_2 < 1), \]

\[ \epsilon_3 = \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2 < \epsilon_1 + \epsilon_2 \quad (\epsilon_1 > 0, \epsilon_2 > 0). \]

The use of the precise bounds may have some relevance when the inference rules are applied with real \( \epsilon \)-values.

An application

We will now examine an example to give an idea, on one hand, of how much the conclusions may be sensible to the use of methods of exact propagation of probability bounds and, on another hand, of the related phenomenon of degradation of inference rules when interpreted in probabilistic terms. The example is a modified version of an application considered in (Bourne, and Parsons 1998) which was inspired by examples given in (Kraus, Lehmann, and Magidor 1990). We consider a probabilistic knowledge base consisting of some conditional assertions, which concern the fact that a given party has various attributes (the party is great, noisy, Linda and Steve are present, and so on). By the symbol \( A \sim B \) we denote the assessment \( P(B|A) \geq 1 - \epsilon \).

We start with a knowledge base which has the following rules and \( \epsilon \)-values:

1. Linda \( \sim_{0.05} \) great
2. Linda \( \sim_{0.2} \) Steve
3. Linda \& Steve \( \sim_{0.1} \sim \) noisy
4. Steve \( \sim_{0.05} \) Linda
5. \( \sim \) noisy \( \sim_{0.2} \) great

Notice that the conditional "Linda \( \sim_{0.05} \) great" means that the probability of the conditional event "(The party will be great | Linda goes to the party)" is greater than or equal to 1 - 0.05, and so on. We are interested in propagating the previous bounds to find the \( \epsilon \)-values of the following conditionals:

(a) Linda \( \sim_{\epsilon} \sim \) noisy
(b) \( \top \sim_{\epsilon} \sim \) noisy
(c) Linda \& Steve \( \sim_{\epsilon} \sim \) Linda
(d) Steve \( \sim_{\epsilon} \sim \) great \& noisy
(e) Linda \lor Steve \( \sim_{\epsilon} \sim \) noisy

By the symbol \( \top \) we denote (any tautology representing) the certain event.

Applying the Cut rule to the conditionals

Linda \( \sim_{0.2} \) Steve, Linda \& Steve \( \sim_{0.1} \sim \) noisy ,
we obtain the conditional

Linda \( \sim_{0.28} \sim \) noisy.

Then, applying the And rule to the conditionals

Linda \( \sim_{0.28} \sim \) noisy, Linda \( \sim_{0.05} \) great,
we obtain the conditional

Linda \( \sim_{0.33} \) great \& noisy.
Applying the S rule to

\[ \text{Linda} \models_{0.33} \text{great} \land \neg \text{noisy} \]

we obtain

\[ \top \models_{0.33} \neg \text{Linda} \lor \text{great} \land \neg \text{noisy} . \]

Applying the S rule to

\[ \neg \text{noisy} \models_{0.2} \neg \text{great} \]

we obtain

\[ \top \models_{0.2} \text{noisy} \lor \neg \text{great} . \]

Finally, applying the And rule to the conditionals

\[ \top \models_{0.33} \neg \text{Linda} \lor \text{great} \land \neg \text{noisy}, \top \models_{0.2} \text{noisy} \lor \neg \text{great}, \]

we obtain the conditional

\[ \top \models_{0.53} \neg \text{Linda} \land (\text{noisy} \lor \neg \text{great}) . \]

Then, by the Right Weakening rule we have

\[ \top \models_{0.53} \neg \text{Linda} \land (\text{noisy} \lor \neg \text{great}) \Rightarrow \top \models_{0.53} \neg \text{Linda} . \]

Concerning the conditional \((c)\), applying the Cautious Monotonicity rule to

\[ \text{Linda} \models_{0.05} \text{great} , \ \text{Linda} \models_{0.2} \text{Steve} , \]

we obtain

\[ \text{Linda} \land \text{Steve} \models_{0.0625} \text{great} . \]

Then, applying the And rule to the conditionals

\[ \text{Linda} \land \text{Steve} \models_{0.0625} \text{great} , \ \text{Linda} \land \text{Steve} \models_{0.1} \neg \text{noisy} , \]

we obtain the conditional

\[ \text{Linda} \land \text{Steve} \models_{0.0725} \text{great} \land \neg \text{noisy} . \]

Concerning the conditional \((d)\), applying the Cut rule to the conditionals

\[ \text{Linda} \land \text{Steve} \models_{0.1} \neg \text{noisy} , \ \text{Steve} \models_{0.05} \text{Linda} , \]

we obtain the conditional

\[ \text{Steve} \models_{0.145} \neg \text{noisy} . \quad (14) \]

Then, applying the Or rule to the conditionals

\[ \text{Steve} \models_{0.145} \neg \text{noisy} , \ \text{Linda} \models_{0.28} \neg \text{noisy} , \]

we obtain the conditional

\[ \text{Linda} \lor \text{Steve} \models_{0.358} \neg \text{noisy} . \quad (15) \]

We observe that, propagating the bounds with \( \epsilon_3 = \epsilon_1 + \epsilon_2 \), instead of the conditionals \((14)\) and \((15)\) we would obtain respectively

\[ \text{Steve} \models_{0.15} \neg \text{noisy} , \]

and

\[ \text{Linda} \lor \text{Steve} \models_{0.425} \neg \text{noisy} . \]

Conclusions

In this paper the inference rules of System P have been considered in the framework of coherence. We have also examined the disjunctive Weak Rational Monotony proposed by Adams in his extended probability logic, corresponding to System P+. Differently from the probabilistic approaches generally given in the literature, see, in particular, (Hawthorne 1996) and (Schurz 1998), within our framework we can directly manage conditional probability assessments, even if some (or possibly all the) conditioning events have zero probability. We think that this opportunity is important specially in the field of nonmonotonic reasoning where infinitesimal probabilities play a significant role. Moreover, exploiting our algorithms, the lower and upper probability bounds associated with the conditional assertions of a given knowledge base can be propagated to further conditional assertions, obtaining in all cases the precise probability bounds. In particular, beside allowing a more flexible and realistic approach to probabilistic default reasoning, this provides an exact illustration of the degradation of System P rules when interpreted in probabilistic terms.

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