Robust Output Feedback Stabilization of MIMO Invertible Nonlinear Systems with Output-Dependent Multipliers  
(extended version)  
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Abstract—This note studies the robust output feedback stabilization problem of multi-input multi-output invertible nonlinear systems with output-dependent multipliers. An “ideal” state feedback is first designed under certain mild assumptions. Then, a set of extended low-power high-gain observers is systematically designed, providing a complete estimation of the “ideal” feedback law. This yields a robust output feedback stabilizer such that the origin of the closed-loop system is semiglobally asymptotically stable, while improving the numerical implementation with the power of high-gain parameters up to 2.

Index Terms—Multi-input multi-output; Extended high-gain observer; Output feedback; Invertibility

I. INTRODUCTION

The problem of output feedback stabilization to the zero equilibrium for nonlinear systems is a fundamental problem in the field of systems and control. Several methodologies have been developed such as high-gain observer-based, backstepping-based, and passivity-based control [1], [2], that differ in the kind of system structure (normal form or lower triangular form), and in assumptions on the internal stability (input-output stability or output-to-state stability).

For single-input single-output (SISO) nonlinear systems, particular attention has been devoted to systems having a normal form, for which a high-gain observer (HGO) can be employed to derive an output feedback stabilizer, providing asymptotic/practical stability in a semiglobal sense. Along this line, an extended high-gain observer (EHGO)-based approach is developed in [3], where an “ideal” state feedback, consisting of a linear stabilizing term and a term to cancel the undesired terms (referred to as a perturbation term uniformly), is first designed. A robust output feedback stabilizer can then be designed by using an extended high-gain observer (EHGO) to not only estimate states in the stabilizing term, but also provide a partial estimation to the perturbation term. As a result, one can recover an “ideal” system performance obtained by the ideal state feedback. The EHGO technique has been extended to multi-input multi-output (MIMO) nonlinear systems with a well-defined vector relative degree [4], [5], for which a static state feedback law for feedback linearization can be designed to decouple all input-output channels. However, the class of systems considered in [4], [5] is a very particular one, while the stabilization of more general classes of MIMO nonlinear systems is generally nontrivial and cannot be achieved via direct extensions of SISO results.

Recently, several authors have studied a general class of MIMO nonlinear systems [6], [7], [8], [9], [10], referred to as invertible MIMO nonlinear systems [11], [12], for which a vector relative degree is not necessary. A significant feature of these systems is that the input-output behavior cannot be fully decoupled by a static feedback linearization law [2], due to the presence of nonzero “multipliers” when running the Structure Algorithm [12], [9] to derive a multivariable normal form. In [6], with an input-output linearizable assumption (i.e., having constant multipliers), by defining a “virtual” output as a linear combination of actual outputs and their derivatives, the invertible systems can be transformed to an “intermediate” form with a unitary vector relative degree, for which the feedback stabilization problem can be solved. This linearizable assumption is later relaxed in [8] by permitting state-dependent “multipliers” in a special structure such that a dynamical feedback linearization can be used, but at the price of requiring a trivial zero dynamics [1].

To apply the EHGO technique to robustify the stabilizer while recovering the dynamical feedback linearizing performance, [10] has further proposed a recursive design method for the same class of invertible MIMO nonlinear systems with a lower-triangular high-frequency gain matrix. In spite of these impressive results, we note that the output feedback stabilization of invertible MIMO nonlinear systems with non-constant multipliers and nontrivial zero dynamics is still an open problem.

On the other hand, in [5], [4], [10] the maximum power of the high-gain parameter increases as the number of states increases, which in practice may create numerical implementation problems when the dimension of the system to be estimated is very large. To solve this problem, the low-power technique in [13], [14] can be employed. However, the combination with the low-power technique is nontrivial, particularly for invertible MIMO nonlinear systems.

Motivated by the previous context, this technical note studies the problem of robust output feedback stabilization for MIMO invertible nonlinear systems with output-dependent multipliers and nontrivial zero dynamics. Compared to the relevant works [8], [10], a nontrivial zero dynamics is permitted and a lower triangular high-frequency gain matrix is

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1 The zero dynamics is said to be trivial if the constraint that the outputs are zero implies that the states are zero. Otherwise, we say that it is nontrivial.
not needed, though this note requires a stronger condition on multipliers. To the best knowledge of authors, currently available approaches cannot be used to solve the considered problem.

In this paper, assuming that all states are accessible, an ideal state feedback law is first designed, rendering an asymptotically stable closed-loop system under a strongly minimum-phase condition. Taking advantage of both EHGO and low-power techniques, a set of extended low-power high-gain observers (ELPHGOs) is systematically designed, providing a complete estimation of the ideal state feedback law. This in turn yields a robust output feedback stabilizer such that the origin of the resulting closed-loop system is semiglobally asymptotically stable. Meanwhile, each EHGO has the power of its high-gain parameter only up to 2, which to some extent solves the numerical implementation problem. It is worth noting that our ELPHGOs are designed with an estimation of the entire perturbation term to achieve a complete estimation of the ideal state feedback law, which is different from the partial estimation in [3, 4, 10]. As further discussed in Remark [4] this complete estimation in turn adds extra difficulties and challenges to the stability analysis.

**Notations:** $| \cdot |$ denotes the standard Euclidean norm. $\otimes$ denotes the Kronecker product. A continuous function $\alpha : \mathbb{R}_+ := [0, \infty) \rightarrow \mathbb{R}_+$ is said to be of class $K$ if $\alpha$ is strictly increasing and $\alpha(0) = 0$, and of class $K_\infty$ if it is also unbounded. For any positive integer $d$, $A_d$ denotes a $d \times 1$ vector, whose entries are all zero, and $(A_d, B_d, C_d)$ is used to denote the matrix triplet in the prime form. Namely, $A_d$ denotes a shift matrix of dimension $d \times d$, $B_d = (0 \cdots 0 1)^T \in \mathbb{R}^d$, and $C_d = (1 0 \cdots 0) \in \mathbb{R}^{1 \times d}$. For any $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, we denote col $(x_1, \ldots, x_n)$ as a vector in $\mathbb{R}^{\sum_{i=1}^d}$, by concatenating all $x_i$’s in order. We denote $satv_i(s)$ as a smooth vector-valued saturation function with saturation level $l$: $satv_i(s) = s$ if $|s| \leq l$; $0 < \frac{dsatv_i(s)}{ds} < 1$ for all $|s| > l$; and $\lim_{s \to \infty} satv_i(s) = l + \epsilon_0$ with $0 < \epsilon_0 \ll 1$. For convenience, $\nabla satv_i$ denotes the Jacobian matrix of $satv_i(\cdot)$.

**II. PRELIMINARIES**

**A. Problem Formulation**

Consider invertible MIMO nonlinear systems of the form

\[
\begin{align*}
\dot{x}_0 &= f_0(x_0, \xi, u) \\
\dot{\xi}_i &= a_{i1}(x) + b_{i1}(x)u, \quad 1 \leq i \leq r_1 - 1 \\
\dot{\xi}_1 &= a_{11}(x) + b_{11}(x)u \\
\dot{\xi}_{k,i} &= a_{k,i}(x) + b_{k,i}(x)u, \quad 1 \leq i \leq r_k - 1, \quad 2 \leq k \leq m \quad (1)
\end{align*}
\]

where internal states $x_0 \in \mathbb{R}^{n_0}$ and partial states $\xi \in \mathbb{R}^n$ with $\xi = \text{col} \{\xi_1, \ldots, \xi_m\} \in \mathbb{R}^n$ and $r = \sum_{i=1}^m r_i$, output $y = \text{col} \{y_1, \ldots, y_m\}$, and control input $u \in \mathbb{R}^n$. For convenience, we let

\[
x = \text{col} \{x_0, \xi\} \in \mathbb{R}^n, \quad \text{with } n = n_0 + r. \quad (2)
\]

Let $a(x) \in \mathbb{R}^m$ be a vector with the $i$-th entry $a_i(x)$, and $b(x) \in \mathbb{R}^{m \times m}$ be a matrix with the $i$-th row $b_i(x)$. Throughout this paper, we suppose all mappings $f_0, a_i, b_i, \delta_{k,i+1}^s$ in (1) are sufficiently smooth, and $a(0) = 0$ and $b(x)$ is invertible for all $x \in \mathbb{R}^n$. In this setting, this paper is interested in the problem of semiglobal asymptotic stabilization of system (1) via output feedback. As in [9], we assume system (1) is strongly—and also locally exponentially—minimum-phase.

**Assumption 1:** There exists an ISS Lyapunov function $V_0(x_0)$ for the $x_0$-subsystem with $\xi$ as an input, uniformly in $u$, such that along the $x_0$-subsystem in (1), for every $(x_0(0), \xi(0)) \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

\[
V_0(x_0) \leq -\beta_1(|x_0|) + \beta_2(|\xi|) \quad (3)
\]

holds for some $\beta_1, \beta_2 \in K_\infty$. The origin of system $\dot{x}_0 = f_0(x_0, 0, u)$ is locally exponentially stable, uniformly in $u$.

If $x$ is available for feedback and if the functions $a(\cdot)$ and $b(\cdot)$ are known, we can design an “ideal” control law

\[
u^* = b^{-1}(x)[-a(x) + v] \quad (4)
\]

with the residual control $v = \text{col} \{v_1, \ldots, v_m\}$.

This ideal control reduces the input-output model of (1) to

\[
\begin{align*}
\dot{\xi}_1 &= \dot{\xi}_1 \\
\dot{\xi}_k &= \dot{\xi}_k + \sum_{i=1}^{k-1} M_k(y) v_i, \quad 2 \leq k \leq m
\end{align*}
\]

where $M_k : \mathbb{R}^m \rightarrow \mathbb{R}^{r_k}$ are defined by

\[
M_k(y) = \left( \frac{0_{r_{k-1}}^T}{1} \cdot \delta_{k,r_1+1}^s(y) \cdots \delta_{k,r_k}^s(y) \right)^T.
\]

Regarding the design of $v$ stabilizing [5], several approaches can be applied, such as high-gain method [11] for a semiglobal stabilizer. In this paper, since our focus is not on the design of $v$ for [5], and to ease the subsequent analysis, we additionally make the following assumption.

**Assumption 2:** All multipliers $\delta_{k,i+1}^s(y)$ in (1) are bounded for all $y \in \mathbb{R}^m$.

This then motivates us to select $K_k \in \mathbb{R}^{r_k \times r_k}$ such that $A_{rk} - B_{rk} K_k$ is Hurwitz, and design $v$ as

\[
u_k = -K_k \xi_k, \quad 1 \leq k \leq m. \quad (6)
\]

Thus, by Assumption 2 it can be easily seen that the origin of the $\xi_1$-subsystem in (5) is globally exponentially stable (GES), and the $\xi_2$-subsystem is input-to-state stable (ISS) with respect to input $\xi_1$ with a linear gain. Hence, the origin of the $(\xi_1, \xi_2)$-subsystems turns out GES by the small-gain theorem [21]. Similarly, we can further consider the $\xi_k$-subsystem recursively for $k = 3, \ldots, m$ and eventually obtain that the origin of (5) is GES with a quadratic Lyapunov function. This, with (3), implies that the origin of system (1) with (4) is globally asymptotically stable using the standard small-gain theorem [20, 21], permitting a constructive Lyapunov function $V_x(x)$ for (1) and an $\alpha_x \in K_\infty$ such that

\[
\dot{V}_x \leq -\alpha_x(|x|). \quad (7)
\]

We note that the feedback law [4] is not implementable due to inaccessibility of the full state $x$. Motivated by this, this note develops a new set of high-gain observers driven only by
the output $y$, which provides an estimate of the controller $\delta_1$, stabilizing the origin of system (1) in a semiglobal sense.

**Remark 1:** The considered class of systems (1) can be regarded as a particular case of the multivariable normal form (1) with output-dependent multipliers $\delta_{k,i+1}$, and can be easily verified to be invertible in the sense of [12]. The derivations of such form (1) can follow the Structure Algorithm [9]. We note that the stabilization problem of invertible MIMO system (1), to the best knowledge of authors, has not been studied yet and cannot be solved by the existing approaches [6, 7, 8, 10].

Compared to the invertible MIMO nonlinear systems having constant multipliers in [6, 7], the multipliers $\delta_{k,i+1}(y)$ in (1) are output-dependent, for which the vector relative degree \{1, 1, \ldots, 1\} cannot be obtained from “virtual” outputs that are defined by a linear function of actual outputs and their derivatives. As a result, the recursive observer design approach in [6, 7] cannot be applied. In contrast with [8, 10], our model (1) permits the existence of the zero dynamics (or $x_0$-dynamics in (1)) and the high-frequency gain matrix $b(x)$ to be in a general, instead of lower-triangular, structure.

**Remark 2:** In Assumption 1 [6] indeed characterizes that the system (1) is weakly uniformly 0-degradable in the sense of [15]. We note that in the present semiglobal setting, Assumption 2 is not necessary and (6) can be replaced by such as a high-gain feedback law (1).

### III. OBSERVER AND CONTROL DESIGN

Let $C_x \subset \mathbb{R}^n$ be any compact set, and $c > 0$ be such that

$$C_x \subset \Omega_c := \{x \in \mathbb{R}^n : V_x(x) \leq c\}$$

where $V_x(x)$ is defined in (7). As in [4], we assume that the high-frequency gain matrix $b(x)$ satisfies the property below.

**Assumption 3:** There exist a constant nonsingular matrix $\hat{B} \in \mathbb{R}^{m \times m}$ and a number $0 < \mu_0 < 1$ such that

$$|\Delta_b(x)| \leq \mu_0$$

for all $x \in \Omega_{c+1}$. Define the perturbation term

$$\sigma(x,u) := a(x) + [b(x) - \hat{B}]u,$$

which indicates $a(x) + b(x)u = \hat{B}u + \sigma$. If

$$u = -\hat{B}^{-1}(\sigma + Kx)$$

with $K = \text{blkdiag}(K_1, \ldots, K_m)$, then $a(x) + b(x)u = -K\xi$, rendering the $\xi$-subsystem in (1) to the ideal form (10).

In view of this, if there is a desired observer that can provide estimates for both the partial states $\xi$ and the perturbations $\sigma$, then a complete estimate of the ideal feedback control can be obtained. However, it is noted that the perturbation $\sigma$ defined in (9) is in fact a function of the control input $u$, and appears not only in the bottom equation of each set of (1), but also in the middle equations of the $k$-th set, $k = 2, \ldots, m$. This makes the observer design and the stability analysis challenging.

Bearing in mind the previous analysis, we denote $M_{k,j}(y)$ as the $j$-th entry of vector $M_k(y)$ and note that $M_{k,j}(y) = \delta_{k,j+1}(y)$ for $j = r_1, r_1 + 1, \ldots, r_k - 1$. We propose a set of high-gain observers of the form

$$
\begin{align*}
\eta_{k,1} &= \eta_{k,1}^2 + \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,1}^2)
\eta_{k,2} &= \eta_{k,2}^2 + \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,i} &= \eta_{k,i+1}^2 + \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,i}^2), \quad 1 \leq i \leq r_1 - 2
\eta_{k,i} &= \eta_{k,r_1}^2 + \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,i}^2), \quad 1 \leq i \leq r_1 - 2
\eta_{k,r_1} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,r_1}^2)
\eta_{k,2r_1} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2r_1}^2)
\eta_{k,2r_1-1} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2r_1-1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,1} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,i+1} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,i+1}^2)
\eta_{k,i} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,i}^2), \quad 1 \leq i \leq r_k - 2
\eta_{k,2r_k-1} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2r_k-1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,1} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_1)\gamma_{k,1}(y_k - \eta_{k,2}^2)
n\end{align*}
$$

and for $k = 2, \ldots, m$,

$$
\begin{align*}
\eta_{k,1} &= \eta_{k,1}^2 + \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,1}^2)
\eta_{k,2} &= \eta_{k,2}^2 + \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,i} &= \eta_{k,i+1}^2 + \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,i}^2), \quad 1 \leq i \leq r_k - 2
\eta_{k,r_k} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,r_k}^2)
\eta_{k,2r_k-1} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,2r_k-1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,1} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,i+1} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,i+1}^2)
\eta_{k,i} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,i}^2), \quad 1 \leq i \leq r_k - 2
\eta_{k,2r_k-1} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,2r_k-1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,2}^2)
\eta_{k,1} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,1}^2)
\eta_{k,2} &= \frac{1}{2}(\ell_k)\gamma_{k,1}(y_k - \eta_{k,2}^2)
n\end{align*}
$$

where $\gamma_{k,i} = \text{col}(\gamma_{k,j}, \gamma_{k,j}^2)$ with $k = 1, \ldots, m$. Moreover, the expressions of which are given by

$$\dot{\hat{x}}_k = (I_{r_k} \otimes C_2)\eta_{k,y}, \quad \ddot{\hat{x}}_k = \gamma_{k,r_k}^2, \quad k = 1, \ldots, m. \quad (13)$$

3For readability, the arguments $(x,u)$ of $\sigma$ will be omitted occasionally.
Letting $\eta = \col{\eta_1, \ldots, \eta_m}$, $\xi = \col{\xi_1, \ldots, \xi_m}$ and $\hat{\sigma} = \col{\hat{\sigma}_1, \ldots, \hat{\sigma}_m}$, instead of the ideal feedback control (9)-(10), we propose an implementable feedback law as
\begin{equation}
    u = -\hat{B}^{-1} \satv_l \left( \hat{\sigma} + K \hat{\xi} \right)
\end{equation}
where $\satv_l(\cdot)$ is a smooth vector-valued saturation function with saturation level $l$ designed as
\begin{equation}
    l \geq \sup_{x \in \Omega_{c+1}} \frac{1}{1 - \mu_0} |ax| + K\xi + 1.
\end{equation}

Remark 3: The observer (11)-(12) is comprised of $m$ high-gain observers, the $k$-th of which is used to estimate not only the partial state $\xi_k$, but also the perturbation term $\sigma_k$ (i.e., the $k$-th entry of $\sigma$). In this respect, the observer (11)-(12) is a kind of extended high-gain observer [3], [4]. On the other hand, the design of (11)-(12) also utilizes the low-power technique developed in (13) for the purpose of solving the numerical implementation problem when $r_k$ is very large. As one can see, the high-gain parameter $\ell_k$ of each observer is powered up to only 2, rather than $r_k + 1$ as in [3], [4], although the dimension of the observer increases to $2r_k$.

IV. Stability Analysis

A. Change of Coordinates

The aim of this subsection is to derive the estimation error dynamics, whose stability will be analyzed later.

For $1 \leq k \leq m$, define the scaled estimation errors as
\begin{equation}
\begin{aligned}
    \hat{\eta}_{k,1} &= (\ell_k) y_k - \eta_{k,1} \\
    \hat{\eta}_{k,2} &= (\ell_k) y_{k-1} - \eta_{k,2} \\
    \hat{\eta}_{k,i} &= (\ell_k) y_{k-i+1} - \eta_{k,i}, \quad 1 \leq i \leq r_k - 1 \\
    \hat{\eta}_{k,r_k} &= \ell_k (\xi_{k,r_k} - \eta_{k,r_k}) \\
    \hat{\eta}_{k,r_k+1} &= \hat{\sigma} - \eta_{k,r_k}
\end{aligned}
\end{equation}
with $\sigma_k$ being the $k$-th element of vector $\sigma$ defined in (9).

Let $\hat{\eta}_{k,j} = \col{\hat{\eta}_{k,j,1}, \hat{\eta}_{k,j,2}, \ldots, \hat{\eta}_{k,j,m,r_m}}$, for $1 \leq k \leq m$ and $1 \leq j \leq r_k$, and $\hat{\eta} = \col{\hat{\eta}_{1}, \ldots, \hat{\eta}_{m}}$. Let
\begin{equation}
    \hat{\sigma} = \col{\hat{\eta}_{1, r_1,1}, \hat{\eta}_{2, r_2,2}, \ldots, \hat{\eta}_{m, r_m,m}}.
\end{equation}

Remark 4: From the bottom equation of (16), it can be seen that $\hat{\eta}_{k,r_k+1}$ is used to estimate the entire perturbation $\sigma_k$, which is motivated by (17), (18) and different from [3], [4], where the extra state provided by the extended-state observer is used to partially estimate the perturbations, i.e., to estimate $\hat{\sigma} = a(x) - b(x) - B\hat{B}^{-1} \satv_l (\hat{\sigma} + K\hat{\xi})$. The gap between $\hat{\sigma}$ and the perturbation $\sigma$ eventually prevents analyzing the closed-loop stability in our setting. In this paper, this gap vanishes by using the complete estimation. As will be shown later, the effect of the perturbations in the estimation error dynamics can be fully dominated by adjusting the high-gain parameters. This enables us to analyze the closed-loop stability by appropriately designing the high-gain parameters. Since the perturbations $\sigma$ to be estimated depend on the control input $u$, the corresponding stability analysis will be more complicated than that of [3], [4].

Now we proceed to rewrite the closed-loop system (11)-(12)-(46) in coordinates $(x, \hat{\eta})$. With (16), we rewrite (46) as
\begin{equation}
    u = -\hat{B}^{-1} \satv_l (\sigma(x,u) + K\xi - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})
\end{equation}
where $\Lambda_\ell = \bldiag(\Lambda_{\ell_1}, \ldots, \Lambda_{\ell_m})$ with $\Lambda_{\ell_k} = \diag(\ell_{k,1}, \ldots, \ell_{k,k})$, and $\sigma$, as defined in (9), depends on $x, u$. The following lemma shows that the equation (18) has the unique solution $u$.

Lemma 1: Set $\psi(u) = u + \hat{B}^{-1} \satv_l (\sigma(x,u) + K\xi - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})$. Then, with Assumption 3, there exists a unique solution of the equation $\psi(u) = 0$ for all $x \in \Omega_{c+1}$.

With Assumption 3, some simple calculations show that the Jacobian $\frac{\partial \psi(u)}{\partial u}$ is uniformly nonsingular, which in turn proves Lemma 1. We omit the corresponding details. Then, recalling (2) and (17), the unique solution $u$ of (18) is a function of $(x, \hat{\eta})$, and we denote it as $u = \pi(x, \hat{\eta})$.

Thus, we can rewrite (11) with (46) in coordinates $(x, \hat{\eta})$ as
\begin{equation}
\begin{aligned}
    \dot{x}_0 &= f_0(x_0, \xi, \pi(x, \hat{\eta})) \\
    \dot{\xi} &= (A - B(y)K)\xi + B(y)\phi(x, \hat{\eta}) - \hat{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}
\end{aligned}
\end{equation}
in which $A = \bldiag(A_{r_1}, \ldots, A_{r_m})$, and
\begin{equation}
    B(y) = \begin{pmatrix}
        B_{r_1} \\
        M_1^{(y)} \\
        \vdots \\
        M_m^{(y)} \\
        M_{r_m}^{(y)} \\
        B_{r_m}
    \end{pmatrix}
\end{equation}
\begin{equation}
    \phi(x, \hat{\eta}) = \begin{pmatrix}
        0 \\
        0 \\
        \vdots \\
        0 \\
        \pi(x, \hat{\eta})
    \end{pmatrix}
\end{equation}
Taking the derivative of the estimation errors in (16) yields
\begin{equation}
\begin{aligned}
    \dot{\eta}_{k,1} &= \ell_k (-\gamma_{k,1} \eta_{k,1} + \gamma_{k,1}) \\
    \dot{\eta}_{k,2} &= \ell_k (-\gamma_{k,2} \eta_{k,2} + \gamma_{k,2}) \\
    \dot{\eta}_{k,i} &= \ell_k (\gamma_{k,i} \eta_{k,i-1} - \gamma_{k,i} \eta_{k,i} + \gamma_{k,i}), \quad 1 \leq i \leq r_k - 1 \\
    \dot{\eta}_{k,r_k} &= \ell_k \gamma_{k,r_k} \eta_{k,r_k} - \gamma_{k,r_k} \eta_{k,r_k} \\
    \dot{\eta}_{k,r_k+1} &= \dot{\sigma} - \gamma_{k,r_k} \eta_{k,r_k}
\end{aligned}
\end{equation}
and for $k = 2, \ldots, m$,
\begin{equation}
\begin{aligned}
    \dot{\eta}_{k,1} &= \ell_k (-\gamma_{k,1} \eta_{k,1} + \gamma_{k,1}) + (\ell_k)^{r_k-1} \sum_{j=1}^{k-1} M_{j,1}^{(y)}(\eta_{j,1})^2 \\
    \dot{\eta}_{k,2} &= \ell_k (-\gamma_{k,2} \eta_{k,2} + \gamma_{k,2}) + (\ell_k)^{r_k-1} \sum_{j=1}^{k-1} M_{j,2}^{(y)}(\eta_{j,2})^2 \\
    \dot{\eta}_{k,i} &= \ell_k (\gamma_{k,i} \eta_{k,i-1} - \gamma_{k,i} \eta_{k,i} + \gamma_{k,i}) \\
    &\quad + (\ell_k)^{r_k-1} \sum_{j=1}^{k-1} M_{j,i}^{(y)}(\eta_{j,i})^2, \quad 2 \leq i \leq r_k - 1 \\
    \dot{\eta}_{k,r_k} &= \ell_k (\gamma_{k,r_k} \eta_{k,r_k} - \gamma_{k,r_k} \eta_{k,r_k} + \gamma_{k,r_k}) \\
    \dot{\eta}_{k,r_k+1} &= \dot{\sigma} - \gamma_{k,r_k} \eta_{k,r_k}
\end{aligned}
\end{equation}
Putting all bottom equations of (21) and (22) together, and recalling (17), we have
\begin{equation}
\hat{\sigma} = \bld{H} L \ell \hat{\eta} + \hat{\sigma}
\end{equation}
where $L_\ell = \diag(\ell_{1,1}, \ldots, \ell_{m,r_m})$, and
\begin{equation}
\bld{H} = \bldiag(H_{1,1}, \ldots, H_{m,m})
\end{equation}
\begin{equation}
H_k = (0 \cdots 0 \gamma_{k,r_k}^2 - \gamma_{k,r_k}^2 0) \in \mathbb{R}^{2r_k}.\end{equation}
Recalling (9) and (46), we observe that
\[
\sigma = a(x) - \Delta_b(x) \text{sat}_v(\sigma + K\xi - \tilde{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\tilde{\eta})
\]
whose derivative, by setting \(\Delta_0 := \Delta_b(x)\nabla \text{sat}_v\), is given by
\[
\dot{\sigma} = a(x) - \dot{b}(x) \tilde{\eta} - \Delta_0[\dot{\sigma} + K\xi - \tilde{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\tilde{\eta}].
\]
By adding \(\Delta_0\dot{\sigma}\) on both sides of equation (25), and setting
\[
\Delta_1 = a(x) - \dot{b}(x) \tilde{\eta} - \Delta_0[\dot{\sigma} + K\xi - \tilde{\sigma} - K(\Lambda_\ell^{-1} \otimes C_2)\tilde{\eta}],
\]
the equation (25) can be rewritten as
\[
(I_m + \Delta_0)\dot{\sigma} = \Delta_1 + \Delta_0(\hat{\sigma} + K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}).
\]
We then observe that \(\Delta_0\) and \(\Delta_1\) have the following properties.

**Lemma 2:** With Assumption 3 for all \(x \in \Omega_{c+1}\),
(i) \(|\Delta_0| \leq \mu_0 < 1\), and \(I_m + \Delta_0\) is invertible,
(ii) there exists a constant \(\delta_1 > 0\), independent of \(\ell = \text{col}\{\ell_1, \ldots, \ell_k\}\), such that \(|\Delta_1| \leq \delta_1\) holds for all \(\tilde{\eta} \in \mathbb{R}^{2r}\).

The proof of Lemma 2(i) is straightforward using Assumption 3 and the fact that \(\nabla \text{sat}_v\) is a diagonal matrix whose entries are less than one, while the proof of (ii) can be easily concluded by deriving the explicit expression of \(\Delta_1\) and is also omitted. Using the first part of Lemma 2 implies
\[
\dot{\sigma} = (I_m + \Delta_0)^{-1}[\Delta_1 + \Delta_0(\hat{\sigma} + K(\Lambda_\ell^{-1} \otimes C_2)\hat{\eta})].
\]
It can be verified that \((\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}\) is independent of \(\hat{\eta}_{rk}\), \(k = 1, \ldots, m\), and thus \((\Lambda_\ell^{-1} \otimes C_2)\hat{\eta}\) can be expressed as a linear function of \(\tilde{\eta}\) from (21)-(22). Namely,
\[
(\Lambda_\ell^{-1} \otimes C_2)\tilde{\eta} = J(\ell)\tilde{\eta}
\]
where \(J(\ell)\) is a matrix dependent of \(\ell\). To be precise, bearing in mind the definition of \(\Lambda_\ell\) given after (18), \(J(\ell)\) has the property that for any \(\ell_i \geq 1, i = 1, \ldots, m\), there exists \(\delta_2 > 0\), independent of \(\ell_i\)’s, such that
\[
|J(\ell)| \leq \delta_2.
\]
Substituting (28) and (29) into (23), we obtain
\[
[I_m - (I_m + \Delta_0)^{-1}\Delta_0]\dot{\sigma} = HL_{\ell_1}\tilde{\eta} + (I_m + \Delta_0)^{-1}[\Delta_1 + \Delta_0KJ(\ell)\tilde{\eta}].
\]
By observing that \([I_m - (I_m + \Delta_0)^{-1}\Delta_0] = (I_m + \Delta_0)^{-1}\), we further obtain
\[
\dot{\sigma} = (I_m + \Delta_0)HL_{\ell_1}\tilde{\eta} + \Delta_1 + \Delta_0KJ(\ell)\tilde{\eta}.
\]
Thus, the equations of the re-scaled estimation errors (21) and (22) can be compactly described by
\[
\hat{\eta} = [F + G\Delta_0H + G\Delta_0KJ(\ell)L_{\ell_1}^{-1}]L_{\ell_1}\tilde{\eta} + G\Delta_1
\]
where
\[
F = \begin{pmatrix}
\frac{1}{\ell_1}L_{21}(\ell_2)B_{21}^\top & 0 & \cdots & 0 \\
\cdots & \ddots & \cdots & \cdots \\
\frac{1}{\ell_m}L_{m1}(\ell_m)B_{m1}^\top & \frac{1}{\ell_m}L_{m2}(\ell_m)B_{m2}^\top & \cdots & F_m
\end{pmatrix},
\]
\[
G = \text{blkdiag} (B_{2r1}, \ldots, B_{2rm})
\]
\[
L_{ij}(\ell_i, y) = \begin{pmatrix}
(\ell_i)r_i - r_{i-1} + \frac{\beta_i}{m} & \cdots & 0 \cr 
\cdots & \cdots & \cdots \cr 
(\ell_i)^2 \sigma_i & \cdots & 0
\end{pmatrix},
\]
\(1 \leq j < i \leq m\).

By Assumption 2, it is clear that given \(\ell_i \geq 1\), there exists \(\nu_{ij} > 0\), independent of \(\ell_i\) such that
\[
|L_{ij}(\ell_i, y)| \leq \nu_{ij}\ell_i^{2(r_i - r_{i-1})}.
\]

**B. Stability Analysis of the Estimation Error Dynamics (32)**

Before presenting the main result of this subsection, a fundamental lemma, proven in Appendix A is given below.

**Lemma 3:** Suppose Assumption 3 holds. There exist symmetric positive definite matrices \(P_i\) and positive constants \(\lambda_i > 0, i = 1, \ldots, m\) such that
\[
\sum_{i=1}^m \tilde{\eta}_i^T (P_iF_i + F_i^\top P_i)\tilde{\eta}_i + 2\tilde{\eta}^T PG\Delta_0H\tilde{\eta} \leq - \sum_{i=1}^m \lambda_i|\tilde{\eta}_i|^2
\]
with \(P = \text{blkdiag} (P_1, \ldots, P_m)\), holds for all \(x \in \Omega_{c+1}\). The stability property of (32) is formulated as below.

**Proposition 1:** Given any \(\tau_{\text{max}} > 0\) and \(R > 0\), suppose \(x \in \Omega_{c+1}\) for all \(t \in [0, \tau_{\text{max}}]\), and the initial conditions \(|\tilde{\eta}(0)| \leq R\). Let \(\Gamma_{ij}\) be chosen as in Lemma 3 so that (34) is satisfied, and choose the design parameters as
\[
\ell_m = \frac{g_mK}{\kappa^*},
\]
\[
\ell_i = \frac{g_i}{g_{i+1}}(\ell_{i+1})^{r_{i+1} - r_i + 1}, \quad \text{for } 1 \leq i < m - 1.
\]
Then for every \(\tau_2 < \tau_{\text{max}}\) and every \(\epsilon > 0\), there exist \(g_i > 0, i = 1, \ldots, m\), independent of \(\kappa\), and \(\kappa^* \geq 1\) such that for all \(\kappa \geq \kappa^*\),
\[
|\tilde{\eta}(t)| \leq 2\epsilon, \quad \text{for all } t \in [\tau_2, \tau_{\text{max}}].
\]

**Proof.** Let \(V_c(\tilde{\eta}) = \tilde{\eta}^\top L_tP\tilde{\eta}\), and \(\alpha_1 = \min\{\text{eig}(P)\}\) and \(\alpha_2 = \max\{\text{eig}(P)\}\) with \(\text{eig}(P)\) denoting the set of all eigenvalues of matrix \(P\). It is clear that
\[
V_c(\tilde{\eta}) \geq \alpha_1 \sum_{i=1}^m \ell_i|\tilde{\eta}_i|^2 \geq \alpha_1 \ell_{\text{min}}|\tilde{\eta}|^2
\]
\[
V_c(\tilde{\eta}) \leq \alpha_2 \sum_{i=1}^m \ell_i|\tilde{\eta}_i|^2 \leq \alpha_2 \ell_{\text{max}}|\tilde{\eta}|^2
\]
where \(\ell_{\text{max}}, \ell_{\text{min}}\) denote the maximum and minimum of \(\ell_1, \ldots, \ell_m\), respectively.
We compute the derivative of $V_c$ along system (32) as

$$
\dot{V}_c = 2\eta^\top L_\ell P [F + G_\Delta \mathbf{H} + G_\Delta K J(\ell) L_\ell^{-1}] L_\ell \eta
+ 2\eta^\top L_\ell P G_\Delta \eta
+ m \sum_{i=1}^{m-1} \ell_i \eta_i^\top (P_i F_i + F_i^\top P_i) \ell_i \eta_i + 2\eta^\top L_\ell P G_\Delta \mathbf{H} L_\ell \eta
$$

where the inequality is obtained by using Lemma 3 and (33).

Letting $\lambda_{\min} = \min \{ \lambda_1, \ldots, \lambda_m \}$, and using Young’s Inequality (30) and Lemma 2, we have

$$
2\eta^\top L_\ell P G_\Delta \mathbf{H} L_\ell \eta \leq \frac{\lambda_{\min}}{8} \sum_{k=1}^{m} \ell_k^2 |\eta_k|^2 + \frac{8}{\lambda_{\min}} \sum_{k=1}^{m} |P_j^2 \mu_0^2 K_j^2 \delta_t^2 |\eta_k|^2
$$

The first of the above inequalities further indicates that

$$
2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \iota_{ij} |P_j| \ell_i^{1-r_j + r_j+2} |\eta_i| \cdot |\eta_j| \leq \frac{\lambda_{\min}}{8} \sum_{k=1}^{m} \ell_k^2 |\eta_k|^2 + \frac{8}{\lambda_{\min}} \sum_{k=1}^{m} |P_j^2 \mu_0^2 K_j^2 \delta_t^2 |\eta_k|^2
$$

Therefore, we have

$$
\dot{V}_c \leq -\frac{\lambda_{\min}}{2} \ell_m^2 - \eta_0 |\eta_m|^2 + \eta_1
$$

where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

(38)

with $i = 1, \ldots, m - 1$.

Therefore, we have

$$
\dot{V}_c \leq -\frac{\lambda_{\min}}{2} \ell_m^2 - \eta_0 |\eta_m|^2 + \eta_1
$$

where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

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with $i = 1, \ldots, m - 1$.

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$$

where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

(38)

with $i = 1, \ldots, m - 1$.

Therefore, we have

$$
\dot{V}_c \leq -\frac{\lambda_{\min}}{2} \ell_m^2 - \eta_0 |\eta_m|^2 + \eta_1
$$

where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

(38)

with $i = 1, \ldots, m - 1$.

Therefore, we have

$$
\dot{V}_c \leq -\frac{\lambda_{\min}}{2} \ell_m^2 - \eta_0 |\eta_m|^2 + \eta_1
$$

where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

(38)

with $i = 1, \ldots, m - 1$.

Therefore, we have

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where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

(38)

with $i = 1, \ldots, m - 1$.

Therefore, we have

$$
\dot{V}_c \leq -\frac{\lambda_{\min}}{2} \ell_m^2 - \eta_0 |\eta_m|^2 + \eta_1
$$

where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

(38)

with $i = 1, \ldots, m - 1$.

Therefore, we have

$$
\dot{V}_c \leq -\frac{\lambda_{\min}}{2} \ell_m^2 - \eta_0 |\eta_m|^2 + \eta_1
$$

where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

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with $i = 1, \ldots, m - 1$.

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Therefore, we have

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where $\eta_0 = \sum_{i=1}^{m} \frac{4}{\lambda_i} \ell_i^2 + \sum_{j=i+1}^{m} \frac{2(j-1) \ell_j^2 |P_j|^2}{\lambda_j} \ell_j^{2(r_j-r_j+1)} - \eta_0 |\eta_i|^2$

(38)

with $i = 1, \ldots, m - 1$.
Now, let us consider the closed-loop system (19)-(32) as
\[
\dot{x}_0 = f_0(x_0, \xi, u)
\]
\[
\dot{\xi} = (A - B(y)K)\xi + B(y)\phi(x, \bar{\eta}) - \text{satv}_1(\phi(x, \bar{\eta}) - \bar{\sigma} - K(\Lambda_2^1 \otimes C_2)\bar{\eta})
\]
\[
\bar{\eta} = [F + G_{D_0}(x(t))L_{\eta}^1\bar{\eta} + G\Delta_1)
\]
According to the Mean Value Theorem, we know from (18) that there exists \(u' \in \mathbb{R}^m\) such that
\[
\pi(x, \bar{\eta}) = -\tilde{B}^{-1}\nabla \text{satv}_1(u')(\sigma(x, \pi) + K\xi - \bar{\sigma} - K(\Lambda_2^1 \otimes C_2)\bar{\eta})
\]
which, for \(x \in \Omega_{c+1}\), yields
\[
\pi(x, \bar{\eta}) = -\tilde{B}^{-1}[I + \nabla \text{satv}_1(u')\Delta(x)]^{-1}\nabla \text{satv}_1(u')
\]
Recalling the definition of \(\phi(x, \bar{\eta})\) in (20), we have
\[
\phi(x, \bar{\eta}) = (I - \Delta_3)[K\xi + a(x)] + \Delta_3[\bar{\sigma} + K(\Lambda_2^1 \otimes C_2)\bar{\eta}]
\]
with \(\Delta_3 := \Delta_3(I + \nabla \text{satv}_1(u')\Delta(x))^{-1}(\nabla \text{satv}_1(u')\Delta(x)]^{-1}\nabla \text{satv}_1(u')\). For all \(x \in \Omega_{c+1}\), by \(\nabla \text{satv}(u') \leq 1 \) and (8), we have
\[
|\Delta_3| \leq |\Delta_3(x)| \cdot |(I + \nabla \text{satv}_1(u')\Delta(x)]^{-1}| \leq \frac{\rho_0}{1 - \rho_0}
\]
which yields
\[
|\phi(x, \bar{\eta}) - \bar{\sigma} - K(\Lambda_2^1 \otimes C_2)\bar{\eta})| \leq \frac{1}{1 - \rho_0}|K\xi + a(x)| + \frac{1}{1 - \rho_0}|\bar{\sigma} + K(\Lambda_2^1 \otimes C_2)\bar{\eta})|
\]
where the last inequality is obtained by setting \(\rho := \frac{|K| + 1}{1 - \rho_0}\), using the definition of the saturation level \(l_i\) in (15) and \(\xi_i \geq 1\), \(i = 1, \ldots, m\).
In view of the above analysis, given any \(\tau_2 < \tau_1\), according to Proposition 1 for any sufficiently small \(c > 0\) there exists a sufficiently large \(\kappa\) such that \(\rho_0(\bar{\eta}_i(t)) \leq 2\rho_0 < 1\) for all \(t \in [\tau_2, \tau_1]\). This implies that
\[
\text{satv}_1(\phi(x, \bar{\eta}) - \bar{\sigma} - K(\Lambda_2^1 \otimes C_2))\bar{\eta}) = \phi(x, \bar{\eta}) - \bar{\sigma} - K(\Lambda_2^1 \otimes C_2)\bar{\eta}) \leq 2\rho_0 \leq 2\rho_0
\]
for \(t \in \Omega_{c+1}\). Thus, the \(\xi\)-subsystem in (42) reduces to
\[
\dot{\xi} = (A - B(y)K)\xi + B(y)\bar{\sigma} + K(\Lambda_2^1 \otimes C_2)\bar{\eta})
\]
where \(\bar{\sigma} + K(\Lambda_2^1 \otimes C_2)\bar{\eta}) \leq 2\rho_0 \leq 2\rho_0\).
Towards this end, pick any number \(0 < c' < c\) and consider the “annular” compact set \(S_{c'} = \{x : c' \leq V(x) \leq c + 1\}\). Let \(\nu_{c'} = \nu_{\text{min}}(\sigma_k)\) be such that \(2\rho_0 \sup_{x \in S_{c'}^1} |\frac{\partial \phi}{\partial x}B(y)| \leq \frac{1}{2}\nu_{c'}\). It then follows that \(V_{c}^*(x) \leq -\frac{1}{2}\nu_{c'} \leq \frac{1}{2}\nu_{c'}\) so long as \(x \in S_{c'}^1\).
In this setting, the ideal feedback control can be designed as
\[
u^* = -(b(x)^{-1})a_1(x) + K\xi) = -(\frac{1}{2} - \frac{\sin(\xi_2)}{3}) (x_0\xi_2 + K_1\xi_1)
\]
with \(K_1 = (\frac{1}{3} 1)\) and \(K_2 = (\frac{1}{3} 1)\). This in turn yields a \(\xi\)-dynamics of the form
\[
\begin{cases}
\xi_{11} = \xi_{12} \\
\xi_{12} = -K_1\xi_1 \\
\xi_{21} = \xi_{22} \\
\xi_{22} = \xi_{23} - \delta_2^1(y)K_1\xi_1 \\
\xi_{23} = -K_2\xi_2
\end{cases}
\]
which is clearly globally exponentially stable at the origin with a Lyapunov function \(V_\xi(\xi) = \xi_1^TP_1\xi_1 + \xi_2^TP_2\xi_2\) with
\[
P_{\xi_1} = 100\begin{pmatrix} 2.625 & 2 \\ 2 & 2.5 \end{pmatrix}, \quad P_{\xi_2} = \begin{pmatrix} 4.2266 & 6.8594 & 4.0000 \\ 6.8594 & 15.8750 & 9.8125 \\ 4.0000 & 9.8125 & 6.8750 \end{pmatrix}.
\]
Computing the time derivative of $V_{\xi}(\xi)$ yields

$$\dot{V}_{\xi} \leq -100|\xi_1|^2 - |\xi_2|^2 + 16|\xi_1||\xi_2| \leq -2|\xi_1|^2 - 0.3|\xi_2|^2.$$ 

Towards this end, the ideal feedback control law (44) globally exponentially stabilizes system (43) with a Lyapunov function $V_{x}(x) = V_0(x_0) + V_{\xi}(\xi)$, satisfying

$$\dot{V}_{x}(x) \leq -\frac{1}{2}|x_0|^2 - |\xi_1|^2 - 0.3|\xi_2|^2.$$ 

With the above ideal control law (44) and the Lyapunov function $V_{x}(x)$, we take $c = 2$ and $\Omega_\gamma = \{x \in \mathbb{R}^7 : V_{x}(x) \leq c\}$. Then, it can be verified that $	ext{sup}_{x \in \Omega_\gamma} \frac{1}{\gamma}|a(x) + K\xi| \leq 24$, and we thus design the saturation level $I = 25$. The desired output-feedback control law is given by

$$u = -\hat{B}^{-1}\text{sat}_I\left(\hat{\sigma} + K\beta\right) \quad (46)$$

where $\hat{\sigma} = \text{col}(\eta_{12}, \eta_{23})$ and $\hat{\beta} = \text{col}(\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22}, \eta_{23})$ are provided by the following extended low-power high-gain observer

$$\begin{align*}
\eta_{11} &= \eta_{11}^2 + \ell_1 \gamma_{11}^1(y_1 - \eta_{11}) \\
\eta_{12} &= \eta_{12}^2 + \tilde{B}_1 u + (\ell_1)^2 \gamma_{12}^2(y_1 - \eta_{12}) \\
\eta_{13} &= \eta_{13}^2 + \ell_1 \gamma_{13}^1(y_1 - \eta_{13}) \\
\eta_{21} &= \eta_{21}^2 + \ell_2 \gamma_{21}^2(y_2 - \eta_{21}) \\
\eta_{22} &= \eta_{22}^2 + \ell_2 \gamma_{22}^2(y_2 - \eta_{22}) \\
\eta_{23} &= \eta_{23}^2 + \ell_2 \gamma_{23}^2(y_2 - \eta_{23})
\end{align*}$$

in which $\gamma_{1i}^1 = \frac{2.5}{4.6}$, $\gamma_{1i}^2 = \frac{2.5}{1.533}$ for $i = 1, 2$, and $\gamma_{2i}^3 = \frac{2.5}{0.511}$ such that matrices

$$F_1 = \begin{pmatrix} F_{11} & D_2 \\ \Gamma_{21} B_2^T & F_{12} \end{pmatrix}, \quad F_2 = \begin{pmatrix} F_{21} & D_2 \\ \Gamma_{22} B_2^T & F_{22} \end{pmatrix}$$

are Hurwitz, with $F_{ki} = A_2 - \Gamma_{ki} C_2$, and

$$D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_{ki} = \begin{pmatrix} \gamma_{ki}^1 \\ \gamma_{ki}^2 \end{pmatrix}, \quad i = 1, \ldots, r_k, \quad k = 1, 2.$$ 

The simulations are finally performed with high-gain parameters $\ell_1 = 5 \times 10^5$ and $\ell_2 = 200$. The resulting evolutions of $|x(t)|$ are given in Figure 2 from which it can be seen that $|x(t)|$ is lower than $6 \times 10^{-4}$ at about $t = 30s$. We are also interested in the robustness of the proposed output-feedback stabilizer. As a comparison, we add a sinusoidal perturbation $0.1 \sin t$ to the function $q_2(x)$, and the resulting evolutions of $|x(t)|$ are given in Figure 2 from which $|x(t)|$ is lower bounded by about $3 \times 10^{-3}$ at about $t = 30s$.

VI. CONCLUSIONS

This note studied the robust output feedback stabilization problem of multivariable invertible nonlinear systems with output-dependent multipliers. We first assumed that all states were accessible and proposed an “ideal” state feedback law. By systematically designing a set of extended low-power high-gain observers, we showed that this “ideal” law can be approximately estimated, providing a robust output feedback stabilizer such that the origin of the resulting closed-loop system is semiglobally asymptotically stable. Moreover, each EHGO has the power of its high-gain parameter up to 2, which to some extent solves the numerical implementation problem.

APPENDIX

A. Proof of Lemma 3

It is noted that $\gamma_{1i}^1 > 0$ and $\gamma_{1i}^2 > 0$ are selected such that matrix $F_{i}$ is Hurwitz [16]. With these choices of $(\gamma_{1i}^1, \gamma_{1i}^2)$, we then consider the system

$$\begin{align*}
\hat{z}_i &= F_{i1} \hat{\eta}_i + G_i u_i, \quad i = 1, \ldots, m \\
u &= \Delta_0 y, \quad y_i = H_i \hat{\eta}_i, \quad i = 1, \ldots, m
\end{align*}$$

(48)
where state $\bar{\eta} = (\bar{\eta}_1, \ldots, \bar{\eta}_m)$ with $\bar{\eta}_i = \text{col}(\bar{\eta}_{1,i}, \ldots, \bar{\eta}_{r,i})$ and $\bar{\eta}_{i,j} = \text{col}(\bar{\eta}_{1,j} \bar{\eta}_{i,j}^2)$, output $y := \text{col}(y_1, \ldots, y_m)$ and input $u := \text{col}(u_1, \ldots, u_m)$. By taking the change of variables

$$x_{i,k} = \left( \Pi_{j=1}^{i-1} \chi_{i,j} \right) \left( \bar{\eta}_{1,k} \right) \chi_{k+1}, \quad 1 \leq k \leq r_i$$

$$x_{i,r_i,k} = \left( \Pi_{j=1}^{r_i-1} \chi_{i,j} \right) \chi_{i,r_i,k+1}, \quad 1 \leq k \leq r_i$$

$$x_{i,2r_i} = \left( \Pi_{j=1}^{r_i-1} \chi_{i,j} \right) \chi_{i,2r_i}$$

$$u = \Delta_0 y, \quad y_i = \chi_{i,1}$$

in which we have defined $y_i = \chi_{i,1}$, system (43) is transformed into the lower triangular form

$$x_{i,k} = \left( \Pi_{j=1}^{i-1} \chi_{i,j} \right) \left( \bar{\eta}_{1,k} \chi_{k+1} + \chi_{k+1} \right), \quad 1 \leq k \leq r_i$$

$$x_{i,r_i,k} = \left( \Pi_{j=1}^{r_i-1} \chi_{i,j} \right) \chi_{i,r_i,k+1} + \chi_{i,r_i,k+1}, \quad 1 \leq k \leq r_i$$

$$x_{i,2r_i} = \left( \Pi_{j=1}^{r_i-1} \chi_{i,j} \right) \chi_{i,2r_i}$$

(49)

Since $\chi_{i,j}$ and $\bar{\eta}_{i,j}$ are nonzero constants, the above change of variables defines a nonsingular matrix $T_i \in \mathbb{R}^{2r_i \times 2r_i}$ such that $x_i = T_i \bar{\eta}_i$ with $x_i = \text{col}(x_1, \ldots, x_{2r_i})$.

For compactness, system (49) can be rewritten as

$$x_i = F_i x_i + G_i u_i, \quad u = \Delta_0 y, \quad y_i = H_i x_i$$

(50)

in which $F_i = T_i F_i T_i^{-1}$, $G_i = T_i G_i$ and $H_i = H_i T_i^{-1}$. From (49), it can be easily seen that the triplet $(F_i, G_i, H_i)$ is controllable and observable. Denote the minimal polynomial of Hurwitz $F_i$ as $P_i(s) = p_{i,0} + p_{i,1} s + \ldots + p_{i,r_i} s^{r_i-1} + s^{r_i}$. By some straightforward but lengthy calculations, we can deduce that $p_{i,0} = \Pi_{j=1}^{r_i} \chi_{i,j}^2$. With this being the case, let $G(s)$ denote the state transfer function of system (50), given by $G(s) = \text{diag}(G_i(s), \ldots, G_m(s))$ where

$$G_i(s) = \left( \frac{\chi_{i,1}}{P_i(s)} \right), \quad i = 1, \ldots, m.$$ 

It is clear that $|G_i(\infty)| = 1 < \frac{1}{\mu_0}$.

By the Bounded Real Lemma [9] Theorem 3.1, there is a symmetric positive definite matrix $P_i$ and a $\lambda_i$ such that

$$2 \chi_i^T \bar{F}_i \chi_i + G_i u_i \leq -\lambda_i |\chi_i|^2 + \frac{1}{\mu_0^2} |u_i|^2 - |y_i|^2$$

for $i = 1, \ldots, m$. This then suggests

$$\sum_{i=1}^m 2 \chi_i^T \bar{F}_i \chi_i + G_i u_i \leq - \sum_{i=1}^m \lambda_i |\chi_i|^2 + \frac{1}{\mu_0^2} |u_i|^2 - |y_i|^2.$$ 

Since $|u| = |\Delta_0 y| \leq |\Delta_0| |y| \leq \mu_0 |y|$, we have

$$\sum_{i=1}^m 2 \chi_i^T \bar{F}_i \chi_i + G_i u_i \leq - \sum_{i=1}^m \lambda_i |\chi_i|^2.$$ 

Thus, letting $P_i = T_i P_i T_i$ and $\lambda_i \leq \lambda_i |T_i|^2$ for $i = 1, \ldots, m$, the inequality (43) can be obtained, which completes the proof.

### B. Proof of Lemma 2

Let

$$\Psi_m = \frac{\lambda_m}{4} t_m^2 - \varrho_0 - \kappa \ell_m$$

$$\Psi_i = \frac{\lambda_i}{4} t_i^2 - \sum_{j=i+1}^m \frac{2(\lambda_j - 1) t_j^2}{\lambda_j} \ell_j (\ell_j - \ell_i) - \varrho_0 - \kappa \ell_i$$

with $1 \leq i \leq m - 1$, which indicates that the proof is completed if it is shown that $\Psi_i \geq 0$ for all $1 \leq i \leq m$. We proceed to show this by a recursive method.

**Step 1:** Let us consider the case that $i = m$. With the choice of $\ell_m$ given in (53), choosing $g_m > \frac{\sqrt{\mu_0^2 - \lambda_m}}{4 \lambda_m}$ and letting $\mu_m = \frac{\lambda_m}{4} g_m^2 - g_m$, we observe that $\mu_m > 0$. Thus, it can be seen that $\Psi_m \geq 0$ for all $\kappa \geq \theta_m$ with $\theta_m = \max\{1, \sqrt{\mu_0^2 - \lambda_m} \}$.

**Step 2:** With the choice of $\ell_{m-1}$ in (53), $\Psi_{m-1}$ reads as

$$\Psi_{m-1} = \frac{\lambda_m - 4}{4} g_m^2 - \frac{2(\lambda_m - 1) g_m^2}{\lambda_m} \ell_m (\ell_m - \ell_m) - \varrho_0 - \kappa \ell_m$$

$$\geq \mu_{m-1} - \kappa \ell_{m-1}$$

where the inequality is obtained using $\kappa \geq 1$ and defining

$$\mu_{m-1} := \frac{\lambda_m - 4}{4} g_m^2 - \frac{2(\lambda_m - 1) g_m^2}{\lambda_m} g_m (\ell_{m-1} - \ell_{m-1}) - \varrho_0$$

Given any fixed $g_m$, it is clear that there exists a positive constant $g_m > 0$, independent on $\kappa$ such that $\mu_{m-1} > 0$ for all $g_m > g_m$. This further indicates $\Psi_{m-1} \geq 0$ for all $\kappa \geq \theta_{m-1}$ with

$$\theta_{m-1} = \max\{1, \sqrt{\mu_0^2 - \lambda_m} \}. \quad (51)$$

**Step m+i:** Following the previous design, we now proceed to the $m+i$-th step, $i = 1, \ldots, m$, and have fixed $g_j$ and $\theta_j$ for $j = i+1, \ldots, m$. With (53), we observe that

$$\Psi_i(\kappa) = \omega_{i,i} \kappa^{2(\ell_{i+k} - \ell_{i+k-1} + 1)}$$

$$- \sum_{j=i+1}^m \omega_{i,j} \kappa^2 (\ell_{j} - \ell_{j+1} + 1)$$

$$- \omega_{i,0} \kappa$$

(51)

where

$$\omega_{i,j} = \frac{\lambda_j}{4} \Pi_{k=1}^{j-1} (g_k)^2$$

$$\omega_{i,0} = \frac{1}{4} \Pi_{j=1}^{m-i+1} (g_{j+1}) (\ell_{j+1} - \ell_{j+1})$$

(52)

In this way, the function $\Psi_i$ in (51) is expressed as a polynomial of $\kappa$. Moreover, it is noted that $\ell_i \leq \ell_{i+1} \leq \cdots \leq \ell_m$, and the inequality

$$\Pi_{k=1}^{m-i} (\ell_{i+k} - \ell_{i+k-1} + 1)$$

$$\geq (\ell_{j} - \ell_{j+1} + 1) \Pi_{k=1}^{j-1} (\ell_{j+k} - \ell_{j+k-1} + 1) + 1 \geq 0$$

holds for all $j = i+1, \ldots, m$. Thus, given $\kappa \geq 1$ we have

$$\Psi_i \geq \mu_i \kappa^{2(\ell_{i+k} - \ell_{i+k-1} + 1)} - \varrho_0$$

with $\mu_i := \omega_{i,i} - \sum_{j=i+1}^m \omega_{i,j} - \omega_{i,0}$. Recalling (52), it can be seen that given any fixed $g_j$, $j = i+1, \ldots, m$, there always exists $g_j > 0$, independent on $\kappa$ such that $\mu_i > 0$ for all $g_j > g_j^*$. With the above choice of $g_i$ being the case, it then can be easily shown that there exists a $\theta_i > 0$ such that for all $\kappa \geq \theta_i$, the polynomial function $\Psi_i$ is positive.
Finally, following the previous recursive design, at the step $m$ we can fix $g_1$ and $\theta_1$. Therefore, choosing $\theta^* = \max\{\theta_1, \ldots, \theta_m\}$, we can conclude that for any $\kappa \geq \theta^*$, $\Psi_i \geq 0$ hold for all $i = 1, \ldots, m$, which completes the proof.

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