Brownian excursions and critical quantum random graphs

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Abstract

Let us consider a continuum analog of Erdős-Rényi random graphs, called the quantum random graphs. In these graphs, the vertices are replaced by circles punctured at Poisson points of arrivals, and connections are derived through another Poisson process on the circle. The quantum random graph is an instance of an inhomogeneous random graph, but it does not belong to the class of rank-1 models. In this paper, we first prove that at criticality the largest connected component of such graph scales like $N^{2/3}$. Then, we prove that the scaled and ordered sizes of connected components converge in distribution to the ordered lengths of excursions above zero of a reflected Brownian motion with a drift. In the process, we also prove that the surplus edges converge to a point process whose intensity is given by the same reflected diffusion.

1 Introduction

The Erdős-Rényi random graph [8] is the simplest but probably the most important example of a random graph. In this graph, denoted by $G(N, p)$ each pair of the $N$ vertices is connected with probability $p$ independently of other pairs. The phase transition phenomenon it exhibits has been greatly studied in the past decades. It has been proved [8, 5, 14] that for $p = c N$ with $c > 1$, the largest component has $\Theta(N)$ vertices and the second largest $O(\ln N)$ vertices (as $N \to \infty$ with probability 1); for $p = \frac{c}{N}$ with $c < 1$ the largest component has $O(\ln N)$ vertices; when $p = \frac{1}{N}$, the largest component of $G(N, p)$ has $\Theta(N^{2/3})$ vertices.

In 1997, Aldous considered the Erdős-Rényi random graph inside the “scaling window” of the phase transition. He proved in [2] that the ordered set of the component sizes rescaled by $N^{-2/3}$ converge to an ordered set of excursion lengths of reflected inhomogeneous Brownian

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motion with a certain drift. This work received a lot of attention since its publication. A vast variety of random graph models studied in the past decades have been shown to exhibit a phase transition phenomenon similar to the Erdős-Rényi random graph. Many of these models have been proved to show same type of behaviour in the near critical regime and some of them have been shown to fall into the different universality class.

One of the models above is a random multigraph with $N$ vertices constructed using the configuration model with its vertex degrees being independent identically distributed random variables with the distribution $\nu$. Joseph in 2011 [10] showed that the limit of the ordered set of the critical random multigraph component sizes rescaled appropriately depends on the distribution $\nu$ in the following way. If $\nu$ has finite third moment then the right scaling is $N^{-2/3}$ and the model falls into the same universality class as the Erdős-Rényi random graph. However, if $\nu_k \sim c k^{-\tau}$ as $k \to \infty$, $c > 0$, $\tau \in (3, 4)$ then the scaling is $N^{-(\tau-2)/(\tau-1)}$ and the limit is an ordered set of the excursion lengths of a certain drifted process with independent increments above past minima. Riordan [17] proved the same result as finite third moment but extended it to the scaling window.

Nachmias and Peres in 2007 [16] among other results proved that random graph obtained by performing percolation on a random $d$-regular ($d \geq 3$) graph on $N$ vertices with percolation probability $p = \frac{1+a N^{-1/3}}{d-1}$, $a \in \mathbb{R}$ falls into the same universality class as the Erdős-Rényi random graph.

The near critical behavior of the Rank-1 model, which is a special case of the general inhomogeneous random graph studied in [6], received much attention in the last years. This model can be seen as a random graph with vertices having a weight ($x_i$ of vertex $i$) associated to them, and the weight of a vertex moderates its degree in a sense that the probability of connection between two vertices $i$ and $j$ is $\min\{c x_i x_j N^2, 1\}$, where $c$ is a positive constant possibly depending on $N$. We refer the reader to [18] for further details. In the Rank-1 model in the “critical window”, the weights $x_1, x_2, ..., x_N$ are independent identically distributed random variables with finite third moment and the probability of connection is $p_{i,j} = \min\{\frac{x_i x_j N}{2}, 1 + a N^{1/3}\}$. This model, formulated, however, differently was first proved by Aldous in [2] to belong to the same universality class as the Erdős-Rényi random graph. Later, in [19] and in [3] independently, similar result was reproved.

The Rank-1 model with power - law degrees was considered in [4]. It has been proved that for exponent $\tau \in (3, 4)$, the sizes of the components rescaled by $N^{-(\tau-2)/(\tau-1)}$ converge to hitting times of certain Lévy process. Note that for $\tau \in (3, 4)$ the degrees have finite variance but infinite third moment.

The purpose of this paper is to study the near critical behavior of the so-called quantum random graph. As it is explained in [12] both the motivation and the choice of terminology come from the stochastic geometric (Fortuin–Kasteleyn type) representation of the quantum Curie-Weiss model in transverse magnetic field, which was originally developed in the general ferromagnetic context in [1, 7]. We refer the reader to [11] where the stochastic geometric representations of different types are explained in more detail.

The quantum random graph, as is explained below, was first introduced in [12]. It is a quantum version of the classical Erdős-Rényi random graph. More precisely:
Let $S_\beta$ denote the circle of length $\beta$. The model is built on the space $G_N^\beta = G_N \times S_\beta$, that is to each site $i \in G_N$ we attach a copy $S_{\beta}^i$ of $S_{\beta}$, where we use $S_{\beta}^i = i \times S_{\beta}$. We shall call the second coordinate of a point in this space - the time coordinate. In each $S_{\beta}^i$, make finite random number of holes using the Poisson point process of holes $H_i$ on $S_{\beta}^i$ with intensity $\lambda_i$. $H_i$-s are assumed to be independent for different $i$-s. The punched circle $S_{\beta}^i \setminus H_i$ consists of $k$ disjoint connected intervals,

$$S_{\beta}^i \setminus H_i = I^1_i \cup I^2_i \cup ... \cup I^k_i. \quad (1.1)$$

Of course $k = 1$ whenever the cardinality $\#H_i = 0, 1$. Next, draw finite random number of links between various points of $G_N^\beta$ with the same time coordinates, that is between points of the type $(i, t)$ and $(j, t)$ where $i \neq j$ and $t \in S_{\beta}$ in the following way. With each (unordered) pair of vertices $i, j \in G_N$ we associate a copy $S_{\beta}^i \cup S_{\beta}^j$ of $S_{\beta}$ and a Poisson point process of links $\mathcal{L}_{i,j}^\beta$ on $S_{\beta}^i \cup S_{\beta}^j$ with intensity $\frac{1}{N}$. Processes $\mathcal{L}_{i,j}^\beta \subset \mathcal{L}_{j,i}^\beta$ are assumed to be independent for different $(i, j)$ and also independent of the processes of holes $H_i$. Two intervals $I^k_i$ and $I^j_i$ in the decomposition $(1.1)$ are said to be connected if there exists $t \in S_{\beta}^i \cup S_{\beta}^j$ such that $t \in \mathcal{L}_{i,j}^\beta$, whereas $(i, t) \in I^k_i$ and $(j, t) \in I^j_i$. The decomposition

$$G_N^\beta \setminus H = C_1 \lor ... \lor C_k. \quad (1.2)$$

of $G_N^\beta \setminus H$ into maximal connected components is, thereby, well defined. We used $H$ above to denote the total collection of all the holes. An example for $N = 4$ is given in Figure 1. For each fixed $x \in G_N^\beta$, the probability $P_{N}^{\beta,\lambda}(x \in H) = 0$. Thus, for given $x \in G_N^\beta$, the notion $C(x)$ of the connected component containing $x$ in the above decomposition is well defined.

**Remark:** If $\lambda = 0$ then there are no holes and $S_{\beta}^i \setminus H_i$ always contains only one connected component, which, of course, equals to $S_{\beta}^i$ itself. In the latter case, the probability that $S_{\beta}^i$ and $S_{\beta}^j$ are connected equals $(1 - e^{-\beta})$ and we are back to a setup similar to that of Erdős-Rényi.

We wish to emphasise here that all the random graph models mentioned in this section mimic the idea of rank-1 random graphs in the sense that the probability of connecting any two vertices depends on the *weights* associated with the vertices. It has been noticed in [9] that our random graph model can be treated as an instance of the general inhomogeneous random graph, if each interval is treated as a vertex. However, in our random graph model the connection between any two vertices depends on the size of overlap of the two intervals, and not only on the individual lengths of the intervals (which can be considered as weights). This property not only separates our model from the class of rank-1 models, but also marks it as the first model which is inherently different from those random graph models which appear to mimic the idea of rank-1 random graphs.

The critical curve of the quantum random graph in the $(\beta, \lambda)$-coordinate quarter plane is
already known (cf. [12]), and is given by:

\[ F(\beta, \lambda) = \frac{2}{\lambda} \left( 1 - e^{-\lambda \beta} \right) - \beta e^{-\lambda \beta} = 1 \]  

which follows from comparisons with critical branching process with the offspring distribution given by the cut-gamma distribution \( \Gamma_\beta(2, \lambda) \).

It has been proved that if \((\beta, \lambda) \in \mathbb{R}_+\) such that \(F(\beta, \lambda) > 1\) an emergence of an \(\Theta(N)\)-giant connected component happens in the disjoint decomposition \((1.2)\) and if \((\beta, \lambda) \in \mathbb{R}_+\) such that \(F(\beta, \lambda) < 1\) typical connected component of any point \((i, t) \in G_N^\beta\) is of order \(O(\ln N)\). This is a quantum version of Erdős-Rényi phase transition phenomenon.

The objective of this paper is primarily two fold: first to analyze the component sizes at criticality and then to analyze this model for values of the parameters lying in a small window around the criticality. Before stating the results we remark here that our definition of size of a connected component in the quantum random graph model is the number of intervals in the component.

We note here that unlike the cases dealt with in [2, 3, 10, 15, 16], the exploration process (or the breadth first walk) corresponding to our model can visit an already visited vertex (circle). In general, it can do so as many times as the number of intervals sharing the same vertex, which is a finite but unbounded random variable. This difference makes our analysis quite challenging.
Although, in proving results related to the sizes of component we have managed to surpass the above technical difficulty by proving that these multiple returns take place with very small probability. However, these probabilities are not small enough to be ignored when we consider the scaling limits of ordered set of component sizes in our model. In fact, only the first return plays a crucial role, and the subsequent returns indeed carry such small probabilities that they do not play any major role in our results. The difference between the limiting process in [2] and in our work is the coefficient $\beta$. As it becomes clear from the proof, it represents the already explored (during the first visit) portion of a vertex (circle). Finally, we note here that questions in the spirit of convergence of random graphs as metric spaces are still open for our model primarily because of this precise problem of multiple visits to the same vertex.

We see through our proofs that our model belongs to the Erdős-Rényi universality class primarily due to two reasons: first is the very small probability of multiple returns to the same vertex, and second is the fact that the exploration process in our model seems to forget it’s past, and behave almost like a Markov process. This feature of our model can be used to check many other inhomogeneous random graph models if they belong to the Erdős-Rényi universality class. We also note here that the first reason that of small probability of multiple returns can further be relaxed but the second reason that of approximate Markovian behaviour is crucial.

1.1 Main results

The first result of this paper complements [12], as it answers questions similar to those in [12], but at criticality.

**Theorem 1.1** Let $C_{\text{max}}$ be the largest component in the critical quantum random graph. Then,

(a) there exist constants $N_0$ and $A_0$, such that for all $N > N_0$ and $A > A_0$, there exists a constant $c_*$ such that

$$\mathbb{P}\left(|C_{\text{max}}| > AN^{2/3}\right) \leq \frac{c_*}{A^{3/2}}.$$

(b) there exist $\delta_0$ and $N_1$ such that for all $\delta < \delta_0$ and $N > N_1$, we have

$$\mathbb{P}\left(|C_{\text{max}}| < \lfloor \delta N^{2/3} \rfloor \right) \leq 15\delta^{3/5}.$$

Consequently, one can conclude that the size of a typical connected component is $O(N^{2/3})$, which matches well with the corresponding results for the Erdős-Rényi random graph. This naturally leads one to believe that the quantum random graph may indeed fall in the same universality class as the Erdős-Rényi random graph, which is precisely the second result of this paper.

However, in order to state the result, we first need to define some related quantities. To begin with, as in [2], the surplus edges in a component are, intuitively, those edges, which
when removed, make the component become a tree. The precise definition will be given in step (iv) of exploration process in section 2.1, and in that notation
\[
\text{number of surplus links} = \text{number of links} - (\text{number of intervals} - 1) \geq 0.
\]
Next, starting with standard Brownian motion \(\{W(s), 0 \leq s < \infty\}\), we define a new process
\[
W^{a,\beta}(s) = W(s) + \frac{1}{2\beta^2}s^2,
\]
where \(a \in \mathbb{R}\). Further, we define the following process
\[
B^{a,\beta}(s) = W^{a,\beta}(s) - \min_{0 \leq s' \leq s} W^{a,\beta}(s'),
\]
and a corresponding marks process \(N^{a,\beta}(\cdot)\), which can be characterized as the counting process for which
\[
N^{a,\beta}(s) - \int_0^s B^{a,\beta}(u)\,du \quad \text{is a martingale.}
\]
Then,

**Theorem 1.2** Let \(C_{1,N}, C_{2,N}, \ldots\) denote the ordered sizes of the components in the quantum random graph with parameters \(\{(\lambda, \beta) : E(|I|) = 1 + \frac{a}{N^{1/3}}\}\) for some \(a \in \mathbb{R}\); and let \(\sigma^{a,N}_j\) be the surplus of the \(j\)-th component. Then as \(N \to \infty\)
\[
(N^{-2/3}C_{1,N}, \sigma^{a,N}_1), (N^{-2/3}C_{2,N}, \sigma^{a,N}_2), \ldots \to^d (\gamma_1, \mu(\gamma_1)), (\gamma_2, \mu(\gamma_2), \ldots)
\]
where \(\gamma_1, \gamma_2, \ldots\) denote the ordered lengths of the excursions of \(B^{a,\beta}\) above zero, and \(\mu(\gamma_1), \mu(\gamma_2), \ldots\) denote the mark counts corresponding of the process \(N^{a,\beta}\). Moreover, the convergence of the rescaled component sizes holds with respect to the \(l^2\), topology defined in [2].

Our proofs evolve around an exploration process which is a tool to estimate the size of a connected component in random graphs.

Rest of this paper is partitioned into two parts. In Section 2, we shall prove Theorem 1.1 by adapting the ideas set forth in [15] to our model. The primary idea is to cleverly construct two counting processes which stochastically dominate (from above and below) the original counting process which determines the size of the component. This way we avoid the process of precise counting, and the difficulties associated with it.

However, the same cannot be done in proving Theorem 1.2, since it requires precise estimates, rather than crude upper and lower bounds. In Section 3, we shall put together the proof of Theorem 1.2, which is presented in multiple steps, leading to the main result. Two of the main results, which set our analysis apart are Propositions 3.1 and 3.2, which are statements about the number of vertices which have already been visited once, twice, or more number of times. These crucial estimates brings our analysis one step closer to that of [2], and allows us to follow the program of [2].

In Sections 3.1 and 3.2, we prove the convergence of breadth first walk and the process of marking surplus edges to appropriate continuous time analogs, respectively. Finally, in Section 3.3, we collate all the results, and prove Theorem 1.2.
2 Proof of Theorem 1.1

As mentioned in the previous section, the problem of revisiting the same vertex multiple number of times makes it difficult for us to get working estimates for the distribution of the exploration process. We get over this problem by introducing two processes, which stochastically dominate the exploration process from above and below. This enables us to get appropriate upper and lower bounds for the probabilities appearing in Theorem 1.1.

In the following section we shall first describe the exploration process. Thereafter in Sections 2.2 and 2.3, we shall present the proofs of the two parts of Theorem 1.1.

2.1 Exploration process

Let us start with defining the exploration process corresponding to our model, originally defined in [12] for one cluster. Using this algorithm we shall construct/sample the quantum random graph, interval by interval.

First, order the vertices (circles). At time $k$ let $A_k$ be the number of active intervals and the number of explored intervals will be $k$.

Initial stage: Take the first vertex (say $v$) and sample an interval $I(v,0)$ around $(v,0)$ in the following way, which is governed by the process of holes. Let $U,V$ be two independent $\text{Exp}(\lambda)$ random variables. If $U + V \geq \beta$ then $I(v,0) = S_\beta$. Otherwise, $I(v,0) = (-V, U) \subset S_\beta$.

In the sequel we shall refer to the distribution of the length of $I(v,0)$ as to $\Gamma_\beta(2,\lambda)$. At time $k = 0$ the interval $I(v,0)$ is the only active interval. Analogous to the exploration process of the Erdős-Rényi random graph we call all the other parts of the space neutral. Since the intervals are not sampled yet it makes more sense to call the whole circles neutral (or free) rather than intervals. Finally, set $A_0 = 1$.

At time $k$:

(a) If $A_{k-1} > 0$, let $I_k$ be the active interval belonging to the vertex with the smallest ordering number.

(b) If $A_{k-1} = 0$ and there exists at least one free vertex (whole neutral circle), then sample the interval including $(w,0)$ with $w$ being the first free vertex in the same way as described for $I(v,0)$ above. Let $I_k$ be this interval.

(c) If $A_{k-1} = 0$ and there is no free vertex, then sample the neutral interval with the smallest beginning point and let $I_k$ be this interval.

Connections of $I_k$: Denote by $w$ the vertex $I_k$ belongs to. For each vertex $i$ run $L_{w,i}$ on the $S^w_\beta \cap I_k$. For each link check:

(i) If the resulting link is connected to the already explored interval, simply ignore (erase) this link.
(ii) If the resulting link is connected to a free vertex, sample an interval $I$ around the point of connection $(i, s)$ in the following way ruled by the process of holes. Let $U, V$ be two independent $\text{Exp}(\lambda)$ random variables. If $U + V \geq \beta$ then $I = \mathbb{S}_\beta$. Otherwise, $I = (s - V, s + U) \subset \mathbb{S}_\beta$. Namely, $|I| \sim \Gamma_\beta(2, \lambda)$ defined above.

(iii) If the resulting link is connected to the neutral part $(t_1, t_2)$ of a vertex which already has explored interval(s), sample an interval $I$ around the point of connection $(i, s)$ in the following way ruled by the process of holes. Let $U, V$ be two independent $\text{Exp}(\lambda)$ random variables, then $I = (s - \min\{U, |s - t_1|\}, s + \min\{V, |t_2 - s|\})$.

(iv) If the resulting link is connected to an active interval, keep this link. These type of links play the role of surplus edges. Namely, if one erases these type of links, the resulting cluster will be a tree.

Denote by $\eta_k$ the number of neutral intervals connected to $I_k$ and change their status to active. Then set $I_k$ itself explored. Note that in order to be able to construct the quantum random graph we need to keep track of all the active and explored intervals (beginning and ending points). By the definition of $A_k$ we have the following recursion:

\[
A_k = \begin{cases} 
A_{k-1} + \eta_k - 1, & \text{if } A_{k-1} > 0 \\
\eta_k, & \text{if } A_{k-1} = 0.
\end{cases} \tag{2.1}
\]

Although we do not define the distribution of $\eta_k$ explicitly, one can get it from “Connections of $I_k$” part of the algorithm. However, it is obvious that the conditional distribution of $\eta_k$ given $|I|$ is stochastically dominated by $\text{Bin}(N, 1 - e^{-|I|/N})$, with $|I| \sim \Gamma_\beta(2, \lambda)$. Note that the first time when the process $A_k = 0$, is when the exploration of the cluster which includes the origin of the process $A_k$, ends. In other words, let us define $\tau = \min\{k \geq 1 : A_k = 0\}$, then by $\tau$ the number of explored intervals equals $|C(v, 0)|$, the size of the cluster which includes the vertex $v$.

Due to the intricate nature of our model, it is rather difficult to analyze this exploration process without resorting to approximations. Thus, we shall define two more counting processes which will dominate the aforementioned exploration process from above and below. These processes will be far more amenable to analyze, hence paving the way for easier computations.

### 2.2 Overcounting

Let us begin with a collection of independent pairs of random variables $\{(|I_i|, \xi_i)\}_{i \geq 1}$ such that $|I_i|$ are all i.i.d. with $|I_i| \sim \Gamma_\beta(2, \lambda)$, and given $|I_i|$, the random variable $\xi_i$ is distributed as $\text{Bin}(N, 1 - e^{-|I_i|/N})$. Then, define the following process

\[
S_k = 1 + \sum_{i=1}^{k} (\xi_i - 1).
\]
Clearly, since the random variables $\eta_ks$ is dominated by $\xi_ks$, therefore the process $A_k$ is consequently dominated by $S_k$. However note that this is not a martingale, and define

$$S_k^* = 1 + \sum_{i=1}^{k} (\xi_i - E\xi_i),$$

as the martingale extracted out of the process $S_k$. Also, note that $S_k^* > S_k$ for all $k$, since $(1 - E\xi_1) > 0$, which in turn implies that $S_k^*$ also dominates $A_k$. Moreover, we can couple $S_k^*$ to $A_k$ in such a way that $S_k^* > A_k$ for all $k$.

Note that for a fixed $H$, the set $\{|C(v,0)| > H^2\}$ can be rewritten as $\{\tau > H^2\}$. Next let us define

$$\gamma = \min\{k : S_k^* = 0, \text{ or } S_k^* > H\}.$$

Since $S_k^* > A_k$, $\tau > H^2$ implies that either $\gamma > H^2$ or $S_\gamma^* > H$. Therefore, we have

$$P(\tau > H^2) \leq P(S_\gamma^* > H) + P(\gamma > H^2). \quad (2.2)$$

Thus, in order to get an upper bound for $P(\tau > H^2)$, we need to estimate the two probabilities appearing on the right hand side of the above equation.

Applying optional sampling theorem, together with Markov’s inequality we get

$$P(S_\gamma^* > H) \leq \frac{E(S_\gamma^*)}{H} = \frac{1}{H}. \quad (2.3)$$

Thereafter, let $\xi$ be another copy of $\xi_1$, then we have the following which is akin to Lemma 5 of [15].

$$P(S_\gamma^* - H > m|S_\gamma^* > H) = \sum_k P(\xi_k - E\xi_k + S_{k-1}^* - H > m|S_\gamma^* > H, \gamma = k) P(\gamma = k|S_\gamma^* > H) \leq \sum_k P(\xi - E\xi > m) P(\gamma = k|S_\gamma^* > H)$$

which is a consequence of the same set of arguments as set forth in Lemma 5 of [15]. Thus, we have

$$P(S_\gamma^* - H > m|S_\gamma^* > H) \leq \sum_k P(\xi > m)P(\gamma = k|S_\gamma^* > H) = P(\xi > m),$$

which implies that the overshoot $(S_\gamma^* - H)$ conditioned on the event $\{S_\gamma^* > H\}$ is stochastically bounded by the random variable $\xi$. Therefore, using Corollary 6 of [15] with $f(x) = (2Hx + x^2)$ for the process $S_k^*$, we get,
\[ \mathbb{E} \left[ (S_\gamma^*)^2 | S_\gamma^* \geq H \right] \leq H^2 + 3H, \quad \text{for large enough } H, \quad (2.4) \]

Also note that \( \{(S_k^*)^2 - k \text{Var}(\xi_1)\} \) is a martingale, which can be used to conclude that
\[ 1 + \mathbb{E} \left( N e^{-|I|/N} (1 - e^{-|I|/N}) \right) \mathbb{E} \gamma = \mathbb{E} \left( (S_\gamma^*)^2 \right) = \mathbb{P}(S_\gamma^* \geq H) \mathbb{E} \left[ (S_\gamma^*)^2 | S_\gamma^* \geq H \right] \quad (2.5) \]

Subsequently, taking \( H \) so that \( H \leq N - 3 \), and using equations (2.3) & (2.4), we get
\[ \mathbb{E} \gamma \leq \frac{H + 2}{h_l}, \]
where \( 0 < h_l = e^{-\beta/N} \mathbb{E} \left( N(1 - e^{-|I|/N}) \right) \leq \mathbb{E} \left( N e^{-|I|/N} (1 - e^{-|I|/N}) \right) \). Thus, for large enough \( H \) we can conclude that there exists a \( c > 0 \) such that
\[ \mathbb{P}(\gamma \geq H^2) \leq \frac{H + 2}{H^2 h_l} \leq \frac{c}{H}. \]

Subsequently, writing \( \gamma^* = \gamma \wedge H^2 \), we have
\[ \mathbb{P} \left( S_{\gamma^*}^* > 0 \right) \leq \mathbb{P} \left( S_{\gamma^*}^* \geq H \right) + \mathbb{P} \left( \gamma \geq H^2 \right) \leq \frac{c_*}{H}. \]

Let \( T = H^2 \), then \( |C(v,0)| > H^2 \) implies \( S_{\gamma^*}^* > 0 \), which in turn implies
\[ \mathbb{P}(|C(v,0)| > T) \leq \frac{c_*}{\sqrt{T}}. \]

Now, we shall use this bound on typical component size to obtain a similar bound for the largest component \( C_{\text{max}} \), for which let us introduce \( N_T \) as the cumulative size of components, which have at least \( T \) vertices. Clearly, \( N_T \) is either zero, or is larger than \( T \), therefore, \( C_{\text{max}} > T \) implies that there must be at least one component whose size is larger than \( T \), thus resulting in the following set of inequalities,
\[ \mathbb{P}(|C_{\text{max}}| > T) \leq \mathbb{P}(|N_T| > T) \leq \frac{\mathbb{E} N_T}{T} \leq \frac{N \mathbb{P}(|C(v,0)| > T)}{T}, \]
for \( T \in (c_*^2 \vee (1 + h_{\lambda,\beta}), (N - 3)^2) \cap \mathbb{N} \). Next, for any \( A > 1 \) choosing \( T = \left( \sqrt{A N^2/3} \right)^2 \) the above inequality simplifies to
\[ \mathbb{P} \left( |C_{\text{max}}| > AN^{2/3} \right) \leq \mathbb{P} \left( |C_1| > T \right) \leq \frac{2c_*}{A^{3/2}}, \quad (2.6) \]
which proves the first part of Theorem 1.1.

Now that we have an upper bound, we shall try to get a lower bound, which in turn will shed some light on the order of the size of the largest component.
2.3 Undercounting

The principal idea to obtain the lower bound is to consider a similar process as above, but one which counts only the free vertices. As it has been mentioned earlier, in our model, each $S_\beta$ can be revisited many times, and by counting only the free vertices we overrule the possibility of revisiting the same vertex, thus achieving the goal of undercounting. Namely, in this process, each component contain at most the same amount of intervals as the original exploration process. This undercounting process is similar to the exploration process in [15]. However, for the sake of clarity we shall define this process. First order the vertices (circles).

At time $k \in \{1, ..., N\}$, let $A^f_k$ be the number of active intervals and the number of explored intervals will be $k$.

**Initial stage:** Take the first vertex (say $v$) and sample an interval $I^f_{(v,0)}$ around $(v,0)$ in the same way as described in the original exploration process algorithm in the beginning of this Section. Namely $I^f_{(v,0)} \sim \Gamma_\beta(2,\lambda)$. At time $k = 0$ the interval $I^f_{(v,0)}$ is the only active interval and all other vertices are neutral (free). Finally, set $A^f_0 = 1$.

**At time $k$:**

(a) If $A^f_{k-1} > 0$, let $I^f_k$ be the active interval belonging to the vertex with the smallest ordering number.

(b) If $A^f_{k-1} = 0$ and there exists at least one free vertex (whole neutral circle), then sample the interval including $(w,0)$ with $w$ being the first free vertex in the same way as described for $I^f_{(v,0)}$ above. Let $I^f_k$ be this interval.

(c) If $A^f_{k-1} = 0$ and there is no free vertex, then the process is over.

**Connections of $I^f_k$:** Denote by $w$ the vertex $I^f_k$ belongs to. For each vertex $i$ run $\mathcal{L}_{w,i}$ on the $S^i_{\beta,w} \cap I^f_k$. For each link check:

(i) If the resulting link is connected to the already explored interval, simply ignore (erase) this link.

(ii) If the resulting link is connected to a free vertex, sample an interval $I^f$ around the point of connection $(i,s)$ in the same way as described in the original exploration process algorithm in the beginning of this Section. Namely, $|I^f| \sim \Gamma_\beta(2,\lambda)$.

(iii) If the resulting link is connected to the neutral part of a vertex which already has an explored interval/s, simply ignore (erase) this link.

(iv) If the resulting link is connected to an active interval, keep this link.

Denote by $\eta^f_k$ the number of neutral intervals connected to $I^f_k$ and change their status to active. Then set $I^f_k$ itself explored. By the definition of $A^f_k$ we have the following recursion:

$$A^f_k = \begin{cases} 
A^f_{k-1} + \eta^f_k - 1, & \text{if } A^f_{k-1} > 0 \\
\eta^f_k, & \text{if } A^f_{k-1} = 0.
\end{cases}$$
Define $N_k^f = N - A_k^f - k - 1_{A_k^f = 0}$, then the conditional distribution of $\eta_k^f$ given $A_k^f$ and $|I_k^f|$ is Bin($N_k^f, 1 - e^{-|I_k^f|/N}$), where $|I_k^f| \sim \Gamma(2, \lambda)$.

Let us begin with $h$, $T_1$ and $T_2$ positive integers, then define

$$\tau_h = \left( \min \{k \leq T_1 : A_k^f \geq h \} \right) \wedge T_1,$$

with the usual convention that minimum of an empty set is taken as $+\infty$. If $A_{k-1}^f > 0$, then

$$(A_k^f)^2 - (A_{k-1}^f)^2 = (\eta_k^f - 1)^2 + 2(\eta_k^f - 1)A_{k-1}^f.$$ 

Now equipped with the facts that $\mathbb{E}(|I_k^f|) = 1$ at criticality and $|I_k^f| \leq \beta$, if we assume $0 < A_{k-1}^f \leq h$, then for large enough $N$ we have

$$\mathbb{E} \left[ (A_k^f)^2 - (A_{k-1}^f)^2 | A_{k-1}^f \right] \geq \frac{(N - h - k)}{N} \mathbb{E} |I_k^f| \cdot \left( 1 - \frac{|I_k^f|}{N} \right) - 2 \frac{h}{N}(h + k) + o \left( \frac{1}{N} \right)$$

(2.7)

If we further assume that $h < \sqrt{N}/4$ and $k \leq T_1 = \left[ \frac{N}{8k} \right]$, then the above reduces to

$$\mathbb{E} \left[ (A_k^f)^2 - (A_{k-1}^f)^2 | A_{k-1}^f \right] > \frac{1}{2}$$

(2.8)

We can redo these calculations and prove (2.8) when $A_{k-1}^f = 0$, in which case $A_k^f = \eta_k^f$. Thus implying that $\left( (A_k^f)^2 - (\tau_h)^2 / 2 \right)$ is a submartingale. Subsequently, using Corollary 6 of [15], with $f(x) = (2Hx + x^2)$ for the process $A_k^f$, we get

$$\mathbb{E}(A_{\tau_h}^f)^2 \leq h^2 + 3h \leq 2h^2$$

for $h \geq 3$. Then using similar set of inequalities as in (2.5), we get

$$2h^2 \geq \mathbb{E}(A_{\tau_h}^f)^2 \geq \frac{1}{2} \mathbb{E} A_{\tau_h}^f \geq \frac{1}{2} T_1 \mathbb{P}(\tau_h = T_1),$$

which in turn implies that

$$\mathbb{P}(\tau_h = T_1) \leq \frac{4h^2}{T_1} \leq \frac{32h^3}{N},$$

which is the same as in [15].

Through above estimates what we have learnt is that $\tau_h$ is of the order of $N$ with a very small probability, which means that there exists at least one fairly large cluster. Next we need to ensure that this fairly large cluster is not very large.

For this, let us define $\tau_0 = \min \{ s : A_s^f = 0 \}$, if this set is nonempty, and $\tau_0 = T_2$ otherwise. Let $M_s = h - \min \{ h, A_{\tau_0+s}^f \}$, and assume $0 < M_{s-1} < h$, then it implies that $0 < A_{\tau_0+s-1}^f < h$. Moreover we have

$$M_s = h - \min \{ h, A_{\tau_0+s-1}^f + \eta_{\tau_0+s}^f - 1 \} \leq h - \min \{ h, A_{\tau_0+s-1}^f \} + (1 - \eta_{\tau_0+s}^f).$$
Also, note that
\[
M_s^2 - M_{s-1}^2 \leq \left( \eta_{rh+s}^f - 1 \right)^2 + 2 \left( 1 - \eta_{rh+s}^f \right) M_{s-1}.
\]

Further observing that at criticality \( E(|I_k^f|) = 1 \), we get
\[
E \left[ M_s^2 - M_{s-1}^2 | A_{rh+s-1}^f, \tau_h \right] 
\leq 1 + \left( 2M_{s-1} - 1 \right) \frac{A_{rh+s-1}^f + (\tau_h + s - 1)}{N} + \frac{E \left( |I_k^f| - 1 \right)^2}{N} + 2 \frac{A_{rh+s-1}^f + (\tau_h + s - 1)}{N} E \left( |I_k^f|^2 - |I_k^f| \right)
\]
\[ + \left( \frac{A_{rh+s-1}^f + (\tau_h + s - 1)}{N} \right)^2 E \left( |I_k^f|^2 \right) + o \left( \frac{1}{N} \right)
\]

Next assuming that \( h < \sqrt{N}/4 \) and \( s \leq T_2 \leq N/8h \), and noting that we had begun with assuming that \( M_{s-1} \) and \( A_{rh+s-1}^f \) were bounded from below and above by 0 and \( h \), respectively, we get
\[
E \left[ M_s^2 - M_{s-1}^2 | A_{rh+s-1}^f, \tau_h \right] \leq 2.
\]

The above inequality can also be checked even when \( A_{rh+s-1}^f \geq h \), thus making \( \{ M_{s-1}^2 - 2(s \wedge \tau_0) \}^{T_2}_{s=0} \) a super martingale. From here on, we can follow the arguments set forth in the proof of Theorem 2 of [15] verbatim to conclude that:
\[
P(\tau_0 < T_2) \leq \frac{32h^3}{N^2} + \frac{2T_2}{h^2}.
\]

Finally, as in [15], choosing \( T_2 = \lfloor \delta N^{2/3} \rfloor \) and appropriate \( h \) so as to minimize the right hand side of the above inequality, we get \( h = \lfloor \frac{\delta^{1/5} N^{1/3}}{(24)^{1/5} s^{1/5}} \rfloor \), which also satisfies \( T_2 \leq N/8h \), and makes the RHS of the above inequality less than \( 15\delta^{3/5} \).

Writing \( |C_{\max}^f| \) as the size of the cluster obtained via the exploration process on the free vertices only, we observe that \( |C_{\max}^f| < T_2 \) implies \( \tau_0 < T_2 \), which in turn implies that \( P(|C_{\max}^f| < T_2) \leq P(\tau_0 < T_2) \). However, note that \( |C_{\max}^f| < |C_{\max}| \), thus,
\[
P(|C_{\max}| < T_2) \leq P(|C_{\max}^f| < T_2) \leq P(\tau_0 < T_2),
\]
thus completing the proof of second part of Theorem 1.1.

Now putting together the two parts of Theorem 1.1, we conclude that the largest component in the critical quantum random graph is indeed \( \Theta(N^{2/3}) \), which was the goal of this section.
3 Proof of Theorem 1.2

3.1 Brownian excursions and critical quantum random graph

In order to describe our model, one can equivalently consider the length of circle as \( \lambda \beta = c \) (say), and run a unit intensity Poisson process for creating holes, and another set of i.i.d. Poisson processes with intensity \( 1/(\lambda N) \) for all the pairs of circles for creating links between them. For technical reasons, we shall adopt this interpretation of our model throughout the rest of this work.

The distribution of the length of sampled interval \( I \), is characterised as

\[
|I| = \begin{cases} 
  c, & \text{if } U + V \geq c \\
  U + V, & \text{o.w.}
\end{cases}
\]

where \( U \) and \( V \) are two independent, unit intensity Exponential random variables. In the notation used in the previous section, we shall write \(|I| \sim \Gamma_c(2,1)\). Thus,

\[
E(|I|) = 2 \left( 1 - e^{-c} \right) - ce^{-c},
\]

Then, under this notation, using [12] the critical curve can be expressed as the set of all those \( \lambda \) and \( c \) such that

\[
2 \left( 1 - e^{-c} \right) - ce^{-c} = \lambda.
\]

Subsequently, for any fixed \( a \in \mathbb{R} \), let us define the critical window as

\[
\{(\lambda, c) \in \mathbb{R}^2_+ : E(|I|) = \lambda \left( 1 + \frac{a}{N^{1/3}} \right) \}.
\]

Note that we can rewrite \( W^{a,\beta} \) (defined in (1.4)) as follows:

\[
W^{a,\lambda,c}(s) = W(s) + as - \frac{\lambda}{2c}s^2.
\]

Let \( (Z^a_{N,\lambda,c}(k), 0 \leq k \leq \text{total number of intervals}) \) be the breadth-first walk associated with the quantum random graph with parameters in the critical window. Formally, \( Z^a_{N,\lambda,c}(0) = 0 \), \( Z^a_{N,\lambda,c}(k) = Z^a_{N,\lambda,c}(k-1) + (\eta_k - 1) \), where \( \eta_k \) are the number of neutral intervals connected to the interval which is screened at time \( k \), which were first introduced in the algorithm defining the exploration process in (2.1).

**Theorem 3.1** Rescale \( Z^a_{N,\lambda,c}(\cdot) \) by defining

\[
\tilde{Z}^a_{N,\lambda,c}(s) = N^{-1/3}Z^a_{N,\lambda,c}(\lfloor N^{2/3}s \rfloor).
\]

Then \( \tilde{Z}^a_{N,\lambda,c} \to W^{a,\lambda,c} \) in distribution as \( N \to \infty \).
Proof. From the general theory we know that we can decompose $Z^{a,\lambda,c}_N$ as follows

$$Z^{a,\lambda,c}_N = M^{a,\lambda,c}_N + B^{a,\lambda,c}_N,$$  \hspace{1cm} (3.2)

where $M^{a,\lambda,c}_N$ is a martingale and $B^{a,\lambda,c}_N$ is a predictable, bounded variation process. Then

$$\left(M^{a,\lambda,c}_N\right)^2 = Q^{a,\lambda,c}_N + D^{a,\lambda,c}_N,$$  \hspace{1cm} (3.3)

where $Q^{a,\lambda,c}_N$ is a martingale and $D^{a,\lambda,c}_N$ is a predictable, monotone increasing process.

Rescaling as in (3.1) to define $\bar{B}^{a,\lambda,c}_N$, $\bar{D}^{a,\lambda,c}_N$, $\bar{M}^{a,\lambda,c}_N$, we shall show that for a fixed $s_0$, as $N \to \infty$

$$\sup_{s \leq s_0} |\bar{B}^{a,\lambda,c}_N(s) - \rho^{\lambda,c} (s)| \to_p 0, \text{ where } \rho^{\lambda,c} (s) = as - \frac{\lambda}{2e} s^2;$$  \hspace{1cm} (3.4)

$$\bar{D}^{a,\lambda,c}_N (s_0) \to_p s_0,$$  \hspace{1cm} (3.5)

$$\mathbb{E} \sup_{s \leq s_0} \left| \bar{M}^{a,\lambda,c}_N(s) - \bar{M}^{a,\lambda,c}_N(s-)^2 \right| \to 0.$$  \hspace{1cm} (3.6)

This as a consequence of functional central limit theorem for continuous time martingales, will prove the main result of this theorem. \hfill \Box

From here onwards, we shall drop the indices $(a,\lambda,c)$ from $Z^{a,\lambda,c}_N$, and all related random variables.

**Proposition 3.1** Let $\nu^i_m$ denote the number of vertices visited $m$ times by time $(i + 1)$. Then, for $m \geq 2$, and $i \leq N^{2/3} s_0$,

$$\nu^i_m = O(N^{1/3}).$$

**Proof:** Note that by definition,

$$\nu^i_2 = \sum_{k=1}^{\nu^i_1} 1_{(k-th \text{ vertex is visited once more)}}.$$

Then, applying Markov inequality for $\nu^i_2$ we get

$$\mathbb{P} \left( \nu^i_2 > N^{1/3} A \right) \leq \left( N^{1/3} A \right)^{-1} \mathbb{E} \left( \sum_{k=1}^{\nu^i_1} 1_{(k-th \text{ vertex is visited once more)}} \right)$$  \hspace{1cm} (3.7)

Observing that $\nu^i_1 \leq N^{2/3} s_0$ and the probability of connection to the visited site is less or equal than to the free one, therefore, the probability that at step $i$ there will appear a connection to the visited site is of order $O(N^{-1/3})$, which implies

$$\mathbb{P} \left( \nu^i_2 > N^{1/3} A \right) = \frac{s_0}{A},$$  \hspace{1cm} (3.8)
which proves the lemma for \( m = 2 \). Similar set of arguments can be used to prove the result for \( m > 2 \). We also note here that for \( m > 2 \), the order is much smaller. \( \square \)

Let us define \( \mathcal{F}_k \) as the sigma algebra containing all the information up to time \( k \) of the exploration process defined in Section 2.1. Then by standard theory we know that

\[
B_N(k) = \sum_{i=1}^{k} \mathbb{E}(Z_i - Z_{i-1}|\mathcal{F}_{i-1}) = \sum_{i=1}^{k} \mathbb{E}(\eta_i - 1|\mathcal{F}_{i-1}),
\]  

(3.9)

which can further be simplified as follows:

**Lemma 3.1**

\[
B_N(k) = \sum_{i=1}^{k} \left[ -1 + \frac{1}{N\lambda}(N - \nu^i)\mathbb{E}|I| + \sum_{j=1}^{\nu^i} \frac{1}{N\lambda}\mathbb{E}(\mathbb{E}|I_i \cap I_j^c| |\mathcal{F}_{i-1}) + O(N^{-2/3}) \right],
\]

where \( \nu^i \), as defined earlier, is the number of vertices visited once until \( (i+1) \).

**Proof:** Observe that

\[
\mathbb{E}(\eta_i|\mathcal{F}_{i-1}) = \mathbb{E}\left( \sum_{I_j \in \{\text{all intervals}\}} 1_{\{I_i \text{ and } I_j \text{ are linked}\}} |\mathcal{F}_{i-1}) \right)
\]

(3.10)

The above sum can be broken down into three categories: intervals that belong to free vertices; intervals which belong to vertices which have been visited once; and intervals belonging to vertices which have been visited more than once. Using this partition, and Proposition 3.1, we get

\[
\mathbb{E}(\eta_i|\mathcal{F}_{i-1}) = \frac{1}{N\lambda}(N - \nu^i)\mathbb{E}|I| + \sum_{j=1}^{\nu^i} \frac{1}{N\lambda}\mathbb{E}(\mathbb{E}|I_i \cap I_j^c| |\mathcal{F}_{i-1}) + O(N^{-2/3}),
\]

(3.11)

where we have also used first order approximation of \( (1 - e^{-|I_i|/N\lambda}) \) and \( (1 - e^{-|I_i \cap I_j|/N\lambda}) \).

Subsequently, recalling the arguments used to prove Proposition 3.1, we can show that for \( m \geq 3 \), \( \nu^i_m = O(N^{-1/3}) \), which in turn implies that the contribution from vertices visited more than twice is \( O(N^{-4/3}) \), which proves the lemma. \( \square \)

**Remark:** We note here that \( I_i \) can be an interval which is part of a free vertex, in which case all \( |I_i| \)'s have the same distribution. However, in the case when \( I_i \) is part of a vertex which has already been visited (at least once), then the distribution of various such \( |I_i| \)'s depends on \( i \). Also, typically for \( i \leq s_0N^{2/3} \), the interval \( I_i \) is part of a free vertex with probability \( (1 - O(N^{-1/3})) \).

From here on we consider only those \( I_i \)'s which are part of free vertices (which happens with probability \( (1 - O(N^{-1/3})) \)).
Proposition 3.2 In the notation of Lemma 3.1, for \( i \leq s_0N^{2/3} \), we have
\[
\nu_i = i + O(N^{1/3}), \\
\nu_0 = N - i + O(N^{1/3}).
\]

Proof. Writing \( \iota(i) \) for the label of the component to which the \( i \)-th interval belongs, we have
\[
\nu_i = \left( i - \sum_{j=2}^{i} (j-1)\nu_j^i \right) + A_i + (\iota(i+1) - \iota(i)),
\]
where \( \nu_j^i \) is as defined earlier.

Consequently by Proposition 3.1, we can conclude that
\[
\nu_i = i + A_i + (\iota(i+1) - \iota(i)) + O(N^{1/3}).
\]

Therefore, it suffices to prove that
\[
A_i + (\iota(i+1) - \iota(i)) = O(N^{1/3}). \tag{3.12}
\]

However, recall that
\[
A_i + (\iota(i+1) - \iota(i)) = Z_N(i) + \iota(i + 1).
\]

Thus, enough to show that
\[
Z_N(i) + \iota(i + 1) = O(N^{1/3}), \tag{3.13}
\]

which will, in turn, prove the lemma.

Corresponding to the breadth-first walk on intervals, it can be shown that: \( \iota(i) = 1 - \min_{j \leq (i-1)} Z_N(j) \) (refer equations (5)-(7) of [2]). Therefore, we get
\[
|Z_N(i) + \iota(i + 1)| \leq |Z_N(i)| + 1 + \min_{k \leq i} Z_N(k) \leq 1 + 2 \max_{k \leq i} |Z_N(k)|. \tag{3.14}
\]

Hence proving (3.13) is equivalent to proving that
\[
N^{-1/3} \max_{i \leq s_0N^{2/3}} |Z_N(i)| \text{ is stochastically bounded,} \tag{3.15}
\]

for which we shall adapt Aldous’ method to our case. Define
\[
T_N = \min \left( s_0N^{2/3}, \min \{ i : |Z_N(i)| > KN^{1/3} \} \right).
\]

Then, note that for all \( i < T_N \), equation (3.15) is satisfied (by the definition of \( T_N \)), which in turn implies equation (3.13). Thus, for \( i < T_N \), we have proven the main result of this lemma.

However, for general \( i \), we shall proceed as follows: we shall begin with collecting all bits of results we need to prove (3.15).
First, we shall begin with recalling $D_N$ from equation (3.3), which can further be expressed as

$$D_N(k) = \sum_{i \leq k} \left[ \mathbb{E}(\eta_i^2 | F_{i-1}) - (\mathbb{E}(\eta_i | F_{i-1}))^2 \right].$$

(3.16)

Then using optional sampling theorem, we note that

$$\mathbb{E}(M^2_N(T_N)) = \mathbb{E}(D_N(T_N)) \quad (3.17)$$

However, since $D_N$ is a monotone non-decreasing process, and $T_N \leq s_0 N^{2/3}$, therefore,

$$\mathbb{E}(M^2_N(T_N)) \leq \mathbb{E}(D_N(s_0 N^{2/3})) \quad (3.18)$$

Next, using the definition of $D_N$ in (3.16), and the fact that $(\mathbb{E}(\eta_i^2 | F_{i-1}))^2 \geq 0$, we get

$$\mathbb{E}(M^2_N(T_N)) \leq \sum_{i \leq s_0 N^{2/3}} \mathbb{E}(\eta_i^2)$$

Recall that $\eta_i$ is stochastically bounded by $\xi \sim \text{Bin}(N, 1 - e^{\frac{|I|}{N^2s}})$ where $|I| \sim \Gamma_c(2, 1)$, which implies that $\mathbb{E}(\eta_i^2) \leq \mathbb{E}(\xi^2)$. However, we also note that $\mathbb{E}(\xi^2) = O(1)$ for large $N$, therefore we get

$$\mathbb{E}(M^2(T_N)) \leq c_0 N^{2/3},$$

which in turn implies that

$$\mathbb{E}(|M(T_N)|) \leq c_0^{1/2} N^{1/3}.$$

(3.19)

Next, to analyze $B_N(l)$, recall its expression from Lemma 3.1, and note that $|I_i \cap I_j| \leq |I_i|$ for any collection of intervals $I_i$ and $I_j$. Therefore, we get

$$B_N(l) \leq \sum_{i=1}^l \left[ -1 + \frac{1}{\lambda} \mathbb{E}(|I|) + O(N^{-2/3}) \right].$$

However, we also note that $\mathbb{E}(|I|) = \lambda(1 + aN^{-1/3})$, which then implies that

$$B_N(l) \leq a l N^{-1/3} + O(l N^{-2/3}).$$

Similarly, a simple lower bound for $B_N(l)$ can be obtained by dropping the sum over intervals, which have been visited once.

$$B_N(l) \geq \sum_{i=1}^l \left[ aN^{-1/3} - \frac{\nu_i^l}{N} \left( 1 + aN^{-1/3} \right) + O(N^{-2/3}) \right]$$

Now recall the set of equations (3.14), and that $\nu_i^l = i + Z_N(i) + i(i+1)$+$O(N^{1/3})$. Therefore,

$$|B_N(T_N)| \leq \max \left[ \left( a T_N N^{-1/3} + T_N O(N^{-2/3}) \right), \right.$$  

$$\left( \sum_{i=1}^{T_N} \left\{ aN^{-1/3} + \frac{\nu_i^l}{N} \left( 1 + aN^{-1/3} \right) + O(N^{-2/3}) \right\} \right)]$$

(3.20)
Let us analyze the second term in the above expression

\[
\sum_{i=1}^{T_N} \left\{ aN^{-1/3} + \frac{\nu_i}{N} \left( 1 + aN^{-1/3} \right) + O(N^{-2/3}) \right\}
\]

\[
= aT_N N^{-1/3} + \sum_{i=1}^{T_N-1} \left\{ \frac{\nu_i}{N} \left( 1 + aN^{-1/3} \right) + O(N^{-2/3}) \right\}
\]

\[
+ \frac{\nu_1 T_N}{N} \left( 1 + aN^{-1/3} \right) + O(N^{-2/3})
\]

(3.21)

For the second part of the above expression, recall that \( \nu_i = i + O(N^{1/3}) \) for \( i < T_N \). Then using \( T_N = O(N^{2/3}) \), we can rewrite the above expression as

\[
= aT_N N^{-1/3} + \sum_{i=1}^{T_N-1} \left\{ i + O(N^{1/3}) \left( 1 + aN^{-1/3} \right) + O(N^{-2/3}) \right\}
\]

\[
+ \frac{\nu_1 T_N}{N} \left( 1 + aN^{-1/3} \right) + O(N^{-2/3})
\]

(3.22)

Next, by observing that \( \mathbb{E}[\nu_1 T_N] = O(N) \), we conclude that \( \mathbb{E}\left[ \frac{\nu_1 T_N}{N} \left( 1 + aN^{-1/3} \right) \right] \) is at worst \( O(1) \). Subsequently, combining equations (3.20) and (3.22), we can claim that

\[
\mathbb{E}|B_N(T_N)| = O(N^{1/3}).
\]

(3.23)

Now that we have all the estimates we need to prove (3.15). Note that

\[
\mathbb{E}|Z_N(T_N)| \leq \mathbb{E}|M_N(T_N)| + \mathbb{E}|B_N(T_N)|
\]

Applying equations (3.19) and (3.23), we get

\[
\mathbb{E}|Z_N(T_N)| \leq O(N^{1/3})
\]

(3.24)

Thus,

\[
\mathbb{P}\left( \sup_{i \leq s_0 N^{2/3}} |Z_N(i)| > KN^{1/3} \right) = \mathbb{P}\left( |Z_N(T_N)| > KN^{1/3} \right) \leq O(1),
\]

which proves that \( N^{-1/3} \sup_{i \leq s_0 N^{2/3}} |Z_N(i)| \) is stochastically bounded as \( N \to \infty \). Thus proving the claim in equation (3.13), and hence the proposition.

Equipped with the above proposition, we are now in a position to prove equation (3.5).

**Lemma 3.2**

\[
N^{-2/3} D_N(N^{2/3}s) \to s, \quad \text{in } \mathbb{P},
\]

(3.25)

which is the assertion of equation (3.5).
**Proof.** Recall that $D_N$ can be expressed as

$$D_N(k) = \sum_{i \leq k} \left[ E(\eta_i^2 | F_{i-1}) - (E(\eta_i | F_{i-1}))^2 \right] = \sum_{i \leq k} \text{var}(\eta_i | F_{i-1}).$$

Next, recall the random variables $\eta_i^f$ and $\xi_i$ defined in Section 2, and notice that conditioned on $F_{i-1}$, the random variables $\eta_i$, $\eta_i^f$ and $\xi_i$, are sums of independent Bernoulli random variables. Therefore, we have

$$\sum_{i \leq k} \text{var}(\eta_i^f | F_{i-1}) \leq \sum_{i \leq k} \text{var}(\eta_i | F_{i-1}) \leq \sum_{i \leq k} \text{var}(\xi_i | F_{i-1}), \quad (3.26)$$

Note that $\xi_i$ is independent of $F_{i-1}$, thus we have

$$\text{var}(\xi_i) = E(N e^{-|I|/N\lambda}(1 - e^{-|I|/N\lambda})).$$

Therefore,

$$N^{-2/3} \sum_{i \leq \lfloor N^{2/3} s \rfloor} \text{var}(\xi_i) = s E(|I|) + o(1), \quad \text{for large enough } N$$

Next, using Proposition 3.2, we get

$$\text{var}(\eta_i^f | F_{i-1}) = E\left(N - i + O(N^{1/3})e^{-|I|/N\lambda}(1 - e^{-|I|/N\lambda})\right),$$

which implies that

$$N^{-2/3} \sum_{i \leq \lfloor N^{2/3} s \rfloor} \text{var}(\eta_i^f | F_{i-1}) = s E(|I|) + o(1), \quad \text{for large enough } N,$

proving the statement of this lemma.

Next, we shall prove equation (3.6), which will bring us one step closer to proving Theorem 3.1.

**Lemma 3.3** Equation (3.6) is true.

**Proof.** Recall that from the definition of $B_N(k)$, the martingale $M_N$ can be written as

$$M_N(k) = \sum_{i=1}^{k} (\eta_i - E(\eta_i | F_{i-1})).$$

In view of this expression, it suffices to prove the following

$$N^{-2/3} E \max_{i \leq s_0 N^{2/3}} (\eta_i - E(\eta_i | F_{i-1}))^2 \to 0.$$
Now by the same arguments as used in the proof of Lemma 3.2, we get that $E(\eta_i | F_{i-1}) = 1 + O(N^{-1/3})$. Therefore,

$$N^{-2/3} \max_{i \leq s_0 N^{2/3}} (\eta_i - E(\eta_i | F_{i-1}))^2 \leq N^{-2/3} \left( 1 + E \max_{i \leq s_0 N^{2/3}} \eta_i^2 \right).$$

However recall that,

$$\max_{i \leq s_0 N^{2/3}} \eta_i^2 \leq \max_{i \leq s_0 N^{2/3}} \xi_i^2$$

Then, using the relationship between expectation of a non-negative random variable, and the sum of its tail probabilities, and using Cauchy-Schwarz for $(\xi^2_i)$ we get

$$E \max_{i \leq s_0 N^{2/3}} \eta_i^2 \leq \sum_{k=0}^N \left[ 1 - \left( 1 - \frac{E(\xi_i)^4}{k^2} \right) s_0 N^{2/3} \right].$$

(3.27)

Next, using the standard formula for the fourth moment of Binomial distribution, and that $|I| \leq c$, we obtain

$$E(\xi_i)^4 = m + O(N^{-1/3}),$$

(3.28)

where $m = \left[ 1 + 7c \frac{1}{4} + 6c^2 \frac{1}{12} + c^3 \frac{1}{3} \right]$. Thus, putting together equations (3.27) and (3.28), and writing $c_N = m + O(N^{-1/3})$ we get

$$E \max_{i \leq s_0 N^{2/3}} \eta_i^2 \leq \sum_{k=0}^N \left[ 1 - \left( 1 - \frac{c_N}{k^2} \right) s_0 N^{2/3} \right].$$

(3.29)

Subsequently, for a fixed $0 < \epsilon < 1$, we can break the sum on the right hand side into two parts: $0 \leq k \leq N^{2/3-\epsilon}$ and $N^{2/3-\epsilon} < k \leq N$. Clearly, the first part can be bounded above by $N^{2/3-\epsilon}$, whereas for the second part, we note that it’s a sum of decreasing sequence of numbers, implying that the sum can be bounded above by the number of terms in the sum times the largest of the summands. Therefore, we get

$$E \max_{i \leq s_0 N^{2/3}} \eta_i^2 \leq O \left( N^{2/3-\epsilon} \right) + N \left[ 1 - \left( 1 - \frac{c_N}{N^{2/3-2\epsilon}} \right) \right].$$

(3.30)

which after some simple analysis boils down to

$$E \max_{i \leq s_0 N^{2/3}} \eta_i^2 \leq O \left( N^{2/3-\epsilon} \right) + O \left( N^{4/3+2\epsilon} \right),$$

(3.31)

which, in turn, implies that $N^{-2/3} \max_{i \leq s_0 N^{2/3}} \eta_i^2 = o(1)$, thereby proving equation (3.6). \(\square\)
Now we are left with proving equation (3.4) to finish the proof of Theorem 3.1. Note that using Lemma 3.1 and Proposition 3.2, we can write $\bar{B}_N(l)$ as

$$
\bar{B}_N(l) = la - \frac{l^2}{2} + O(N^{-1/3}) + N^{-1/3} \frac{1}{N\lambda} \sum_{i=1}^{N^{2/3}} \sum_{j=1}^{N^{2/3}} \mathbb{E}(|I_i \cap I_j^c||\mathcal{F}_{i-1}).
$$

(3.32)

Let us introduce $\mathcal{I}$ and $\mathcal{I}'$ as two independent random intervals on two distinct free vertices, such that $|\mathcal{I}|, |\mathcal{I}'| \sim \Gamma_c(2, 1)$, and $\mathbb{E}|\mathcal{I}| = \lambda$. Notice that,

$$1 - \frac{\mathbb{E}(|I \cap I^c|)}{\mathbb{E}(|I|)} = \frac{\mathbb{E}(|I \cap I^c|)}{\mathbb{E}(|I|)},$$

where $I^c$ is the complement of $I$ on $S_c$. Also, observe that at criticality,

$$\frac{\mathbb{E}(|I \cap I^c|)}{\mathbb{E}(|I|)} = \frac{\lambda}{c}.$$

Thus, in view of the above observations, to show that equation (3.4) is indeed true, it suffices to prove that

$$
\lim_{N \to \infty} \bar{B}_N(l) = la - \frac{l^2}{2} + \frac{\mathbb{E}(|I \cap I^c|)}{\mathbb{E}(|I|)} \frac{l^2}{2},
$$

(3.33)

which, in turn, can be derived as a direct consequence of the following lemma.

**Lemma 3.4**

$$
\lim_{N \to \infty} N^{-4/3} \lambda^{-1} \sum_{1 \leq j < i \leq N^{2/3}} \mathbb{E}(|I_i \cap I_j^c||\mathcal{F}_{i-1}) = \frac{\mathbb{E}(|I \cap I^c|)}{\mathbb{E}(|I|)} \frac{l^2}{2}.
$$

(3.34)

In order to prove this lemma, we first need to introduce one more process related to our model, for which we shall recall the exploration process defined in Section 2.1. Consider the collection of trees generated by the exploration process by ignoring the surplus edges. We refer to Fig.1 of [2] for an illustration of this tree. Let us denote the various levels of the tree as *generations*. Now corresponding to each branch (a path from the origin to a leaf) define a process of the points $X_i$ around which intervals are constructed as described in the exploration process. Each such process is clearly a Markov chain conditioned on the event that each vertex in the cluster is visited only once.

**Lemma 3.5** Consider any process defined as above. Let $X_i|_{I_j}$ for some fixed $j < i$ be $X_i$ conditioned on interval $I_j$, which was built around $X_j$. Then, there exists $q \in (0, 1)$ such that writing $\pi$ for normalized Lebesgue measure on $S_c$, and $\pi_{i,j}$ for the measure induced by the random variable $X_i|_{I_j}$ on $S_c$ conditioned on the event that all the vertices appearing in tree are visited only once, we have

$$d_{TV}(\pi_{N,i,j}, \pi) < q^{i-j}.$$
**Proof:** Consider the process \( (X_t | I_j, \{ \text{each vertex in the cluster is visited only once} \}) \). Note that following the same arguments as in the proof of Proposition 3.1, the event that each vertex in the cluster is visited only once has probability \( (1 - O(N^{-1/3})) \). Thus \( \{X_t | I_j \} \) is a Markov chain on a set of probability \( (1 - O(N^{-1/3})) \). Let us call this set \( \Omega_1 \).

Next, writing \( \pi_{N,i,j} \) as the measure on \( S_c \), which is induced by the random variable \( (X_t | I_j, \{ \text{each vertex in the cluster is visited only once} \}) \), and \( \pi \) for the uniform measure on \( S_c \), by Proposition 4.1, we have the desired result. \( \Box \)

With these results, we shall now embark upon proving Lemma 3.4.

**Proof of Lemma 3.4:** In view of the fact that, \( \mathbb{E}(|\mathcal{I}|) = \lambda \), we can rewrite (3.34) as follows

\[
\lim_{N \to \infty} N^{-4/3} \sum_{j < i \leq N^{2/3}} \mathbb{E}(|I_i \cap I_j^c| | \mathcal{F}_{i-1}) = \frac{1}{2} \mathbb{E}(|\mathcal{I} \cap \mathcal{I}^c|) \tag{3.35}
\]

Next, note that although the conditioning with respect to \( \mathcal{F}_{i-1} \) adds no information to the length of the interval \( I_i \), however the conditioning does tell us the point \( X_t \) around which the interval \( I_i \) is constructed. Therefore, there exists a measurable and bounded \( F \), such that \( \mathbb{E}(|I_i \cap I_j^c| | \mathcal{F}_{i-1}) = F(X_t, I_j) \). Note that the sub-index \( i \) in this case comes from the exploration process and differs from Lemma 3.5, where it corresponds to the order on a branch. Using this notation, we can introduce \( F(U, \mathcal{I}) \), where \( U \) is a uniform random variable on \( S_c \), then naturally, \( F(U, \mathcal{I}) \Delta \equiv \mathbb{E}(|\mathcal{I} \cap \mathcal{I}^c|) \). With this notation, it suffices to prove the following:

\[
\frac{1}{i^2} \mathbb{E} \left( \sum_{j \leq i} F(X_t, I_j) - F(U, \mathcal{I}) \right)^2 \to 0. \tag{3.36}
\]

By Lemma 3.5, for any fixed \( j \) and \( j' \), we have

\[
d_{TV}(\pi_{N,i,j}, \pi) \to 0, \quad \text{as } d_t(i, j) \to \infty \tag{3.37}
\]

and writing \( \pi_{N,i,j,j'} \) as the measure induced by \( X_t | I_j, I_j' \), conditioned on the event \( \Omega_1 \), we have

\[
d_{TV}(\pi_{N,i,j,j'}, \pi) \to 0, \quad \text{as } \min\{d_t(i, j), d_t(i, j')\} \to \infty. \tag{3.38}
\]

where \( d_t(i, j) \) is a distance on tree. Indeed, if \( i \) and \( j \) in (3.37) and \( i, j \) and \( j' \) in (3.38) belong to the same branch the above equations are direct consequences of the Lemma 3.5. If \( i \) and \( j \) in (3.37) belong to different branches but \( d_t(i, j) \to \infty \) then there exists a common ancestor \( k \) of \( i \) and \( j \) such that \( d_t(i, k) \to \infty \) or \( d_t(j, k) \to \infty \) implying by Lemma 3.5 the desired result. The same type of arguments imply (3.38) when \( i, j \) and \( j' \) do not belong to the same branch.

Let us consider the LHS of equation (3.36), since \( \mathbb{E}(F(X_t, I_j) - F(U, \mathcal{I}))^2 \) is bounded, therefore \( i^{-2} \sum_{j \leq i} \mathbb{E}(F(X_t, I_j) - F(U, \mathcal{I}))^2 \to 0 \) as \( i \to \infty \). Thus, only the crossproduct terms may have some nontrivial contribution to the LHS of (3.36). However, we shall show that the crossproduct terms also vanish when divided by \( i^2 \). Clearly,

\[
|\mathbb{E} \left( (F(X_t, I_j)(F(X_t, I_j')) - \mathbb{E} \left[ (F(U, I))^2 \right] \right) |
\leq |\mathbb{E} \left( (F(X_t, I_j)(F(X_t, I_j')) - \mathbb{E} \left[ (F(U, I))^2 \right] \right) |
+ |\mathbb{E} \left[ (F(U, I)F(U, I')) \right] - \mathbb{E} \left[ (F(U, I))^2 \right] |; \tag{3.36}
\]
where $F(U, I_j) = \mathbb{E}(|I \cap I_j|)$. Notice that when restricted to the set $\Omega_1$, the expected value of $((F(X_i, I_j))F(X_i, I_j'))$ converges to the expected value of $((F(U, I))F(U, I'))$ by equation (3.38). However, we also note that the measure of $\Omega_1$ converges to unity, which implies that the first part of the RHS of the above equation converges to 0. For the second part of the RHS, we notice that as $|j - j'| \to \infty$, $F(U, I_j)$ and $F(U, I'_j)$ are asymptotically independent\(^1\), therefore,

$$
\mathbb{E}[(F(U, I_j))(F(U, I'_j))] \to \mathbb{E}[(F(U, I))^2].
$$

which proves equation (3.35), thus proving the result. \(
\square
\)

Finally, this proves the claims mentioned in equations (3.4), (3.5) and (3.6), which completes the proof of Theorem 3.1.

### 3.2 Surplus edges

Another interesting process which appears naturally as an offshoot of the exploration process described in Section 2.1 is the counting process corresponding to the surplus edges. Recall from step (iv) of the regular exploration process from Section 2.1, that surplus edges are generated through connections to active intervals. Thus at time $i$, the probability of generating a surplus edge can be expressed as:

$$
\sum_{j=1}^{A_i} \left( \frac{\mathbb{E}(|I_i \cap I_j| \mathcal{F}_{i-1})}{N\lambda} + O \left( \frac{1}{N^2} \right) \right)
$$

The primary objective of this section is to prove that, under appropriate scaling, the aforementioned process converges to a counting process whose instantaneous intensity is given by $(W^{a, \lambda, c}(s) - \min_{0 \leq s' \leq s} W^{a, \lambda, c}(s'))$.

Note that in order to prove the above convergence, it suffices to prove that the respective rates converge to the desired quantity.

**Theorem 3.2**

$$
N^{2/3} \sum_{j=1}^{A_j N^{2/3}} \frac{\mathbb{E}(|I_{jN^{2/3}} \cap I_j| \mathcal{F}_{tN^{2/3}-1})}{N\lambda} \to \left( W^{a, \lambda, c}(s) - \min_{0 \leq u \leq s} W^{a, \lambda, c}(u) \right) \frac{\lambda}{c},
$$

as $N \to \infty$.

**Remark:** Note that for counting processes, only the time need to be scaled, which explains the multiplicative factor $N^{2/3}$ appearing in the statement of the above theorem.

\(^1\)Since $X_i|I_j \to U$ in total variation, and that the limit does not depend on $j$, we can also interpret this as for a fixed $i$, as we go back far into the past, then the past is independent of the present (by a simple application of Baye’s theorem). Using this kind of argument, it follows that $I_j$ and $I'_j$ are asymptotically independent as $|j - j'| \to \infty$. 

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Proof. Recall that \( A_i = Z_N(i) + \iota(i) \). Next divide collection of active intervals into two classes: those belonging to vertices which have been visited once, and the rest. However, by using an argument similar to the one put forth in Proposition 3.1, we can conclude that the number of intervals in the second group, which corresponds to the number of active intervals belonging to vertices which have been visited more than twice is \( O(N^{-1/3}) \). Therefore, the probability of getting a surplus edge from connections to this class of intervals is \( O(N^{-4/3}) \). Thus the contribution to equation (3.39) is \( O(N^{-5/3}) \).

Simplifying the expression under consideration

\[
N^{2/3} \sum_{j=1}^{A_1 \times N^{2/3}} \frac{\mathbb{E} \left( |I_{[sN^{2/3}]} \cap I_j| \right) \mathcal{F}_{[sN^{2/3}]} - 1}{N^2} = N^{2/3} \frac{1}{N} \left( 1 + \frac{a}{N^{1/3}} \right) \mathbb{E} \left( |I_{[sN^{2/3}]} \cap I_j| \right) \mathcal{F}_{[sN^{2/3}]} - 1
\]

Noting that \( \iota(i) = 1 - \min_{j \leq (i-1)} Z_N(j) \), we get

\[
\left( \tilde{Z}_N(s) - \min_{u \leq s} \tilde{Z}_N(u) \right) \left( 1 + \frac{a}{N^{1/3}} \right) \mathbb{E} \left( |I_{[sN^{2/3}]} \cap I_j| \right) \mathcal{F}_{[sN^{2/3}]} - 1
\]

where recall that \( \lambda_c = \frac{\mathbb{E}(Z \cap Z')}{\lambda} \).

The first part of the limit is primarily a consequence of Theorem 3.1, whereas the second part can be obtained by repeating the arguments used in Lemma 3.4.

As a consequence of the above theorem one can prove the joint convergence of the breadth first walk and the counting process to \((W_{a,\lambda,c}(s) - \min_{0 \leq s' \leq s} W_{a,\lambda,c}(s'))\) and a counting process whose intensity in given in the above theorem, respectively.

3.3 Joint convergence of component sizes

Recall the statement of Theorem 1.2. In this section, we shall wrap up the proof of this theorem by putting together all the results from previous sections.

As pointed out in [2], Theorem 1.2 primarily has two parts:
1. First is to prove that the excursions of the limit process are matched by the excursions of the breadth first random walk.

2. Second is to arrange these excursions in the decreasing order. This can be achieved if one can ascertain that there exists a random point after which one is sure (with high probability) not to see large excursions.

In order to settle the first issue, we shall invoke Lemmas 7 and 8 of [2], which can be applied verbatim to our case, together with Theorem 3.1 proved in previous subsection.

Thus, we need only be concerned about the second issue, for which we shall need to prove an appropriate version of Lemma 9 of [2] suited to our case.

Let us define

\[ T(y) = \min\{s : W^a_{\lambda,c}(s) = -y\} \]
\[ T_N(y) = \min\{i : Z_N(i) = -\lfloor yn^{1/3} \rfloor\} \]

As a consequence of Theorem 3.1

\[ N^{-2/3}T_N(y) \to_d T(y). \]

**Lemma 3.6** Let us denote by \( p(N, y, \delta) \) the probability that the quantum random graph with the parameters \( (\lambda, c) \in \{(\lambda, c) : \mathbb{E}(|I|) = \lambda(1 + aN^{-1/3})\} \), contains a component (collection of connected intervals) of size greater than or equal to \( \delta N^{2/3} \) which does not contain any vertex \( i \) with \( 1 \leq i \leq yN^{1/3} \). Then,

\[ \lim_{y \to \infty} \limsup_N p(N, y, \delta) = 0 \quad \text{for all } \delta > 0 \]

**Proof:** Fix \( \delta > 0 \). Let \( v_{C_i} \) be the minimal vertex of the cluster \( C_i \), then for an interval \( I \), define

\[ q(N, I) = \mathbb{E}\left( \sum_{C_i \text{ clusters}} 1(|C_i| \geq \delta N^{2/3}; v_{C_i} \in N^{1/3} I) \right). \] (3.40)

Then conditioned on component sizes, the labels of the vertices are going to be in random order (because we are going to order the components in decreasing order). For a component \( C_i \) of size \( bN^{2/3} \), writing \( v_{C_i} \) as the label of the minimal vertex of \( C_i \), define

\[ \chi_N(C_i) = N^{-1/3}v_{C_i}, \]

and

\[ U = N^{-2/3} \text{(number of vertices in the component)}. \]

Then note that

\[ \mathbb{P}(v_{C_i} > N^{1/3}x \mid U) = \left( 1 - \frac{U N^{2/3}}{N} \right)^{N^{1/3}x}, \]

which implies that the conditional distribution of \( \chi_N \) given \( U \), converges to \( \text{Exp}(U) \) as \( N \to \infty \). The conditional exponential distribution implies

\[ \mathbb{P}(\chi_N > y \mid U) \sim \frac{e^{-Uy}}{1 - e^{-U}} \mathbb{P}(\chi_N \leq 1 \mid U). \] (3.41)
Conditional on cluster sizes being \( |C_i| = bN^{2/3} \),

\[
\mathbb{P}\left( v_{C_i} \in [yN^{1/3}, \infty) \right) = \mathbb{E}\left( \mathbb{P}\left( v_{C_i} \in [yN^{1/3}, \infty) \mid U \right) \right). \tag{3.42}
\]

At this point we note that using the arguments put forth in Proposition 3.1, we can conclude that \( b/3 < U < b \) with probability \( (1 - O(N^{-4/3})) \). Then conditioned on cluster sizes being \( |C_i| = bN^{2/3} \), we get

\[
\mathbb{E}\left( \mathbb{P}\left( v_{C_i} \in [yN^{1/3}, \infty) \mid U \right) \right) \leq \frac{e^{-by/3}}{1 - e^{-b/3}} \mathbb{P}\left( v_{C_i} \in [0, N^{1/3}] \right) + O(N^{-4/3}).
\]

Recall the definition of \( q(N, [0, \infty)) \) from (3.40), and let us denote by \( M_b \) the number of clusters of the size \( bN^{2/3} \), then

\[
q(N, [y, \infty)) = \mathbb{E}\left( \sum_{b=\delta}^{\infty} \mathbb{E}\left( \sum_{C_i:|C_i|=bN^{2/3}} 1(v_{C_i} \in [yN^{1/3}, \infty)) \mid M_b \right) \right).
\]

Clearly, given the size of the component, the minimal vertex of various different components of the same size, are identically distributed, therefore,

\[
q(N, [y, \infty)) = \mathbb{E}\left( \sum_{b=\delta}^{\infty} M_b \mathbb{P}\left( v_{C_i} \in [yN^{1/3}, \infty) \mid |C_i| = bN^{2/3} \right) \right)
\]

This, together with (3.41) implies that

\[
q(N, [y, \infty)) \leq \frac{e^{-\delta y/3}}{1 - e^{-\delta/3}} q(N, [0, 1]) + O(N^{-1/3}).
\]

Now observe that \( p(N, y, \delta) \leq q(N, [y, \infty)) \), then in view of the above set of inequalities, in order to prove the theorem, it suffices to prove that

\[
\sup_N q(N, [0, 1]) < \infty. \tag{3.43}
\]

However, notice that the expected number of components of size larger than \( \delta N^{2/3} \), with minimal vertex smaller than \( N^{1/3} \) is smaller than \( N^{1/3} \mathbb{P}(|C_0| \geq \delta N^{2/3}) \). Moreover, if we define \( T \) as the size of the over counting tree, then \( \mathbb{P}(|C_0| \geq \delta N^{2/3}) \leq \mathbb{P}(T \geq \delta N^{2/3}) \). Thus, taking these observations into account, proving (3.43) is equivalent to proving the following

\[
\mathbb{P}(T \geq \delta N^{2/3}) \leq N^{-1/3}\text{const.} \tag{3.44}
\]

We note here that when \( a < 0 \), then proving (3.44) for this case is rather easy.

**Case I: \( a < 0 \)**

Note that \( \mathbb{E}(1 - e^{-\theta TN^{-2/3}}) < \theta \mathbb{E}(TN^{-2/3}) \).
Also, \( T - 1 = \sum_{i=1}^{M_N} T_i \), where \( M_N \) is the number of subtrees originating from the root, and \( T_i \) is the size of the \( i \)-th subtree. Clearly, \( ET = 1 + E(M_N)E(T) \), implying \( ET = \frac{1}{1 - E(M_N)} \). However note that \( E(M_N) = E \left( N(1 - e^{-1/(N\lambda)}) \right) \leq 1 + \frac{\theta}{N^{1/3}} = Q \) (say). Therefore, \( ET < \frac{N^{1/3}}{\theta} \), which in turn implies that

\[
N^{-2/3}ET < \text{const} \ N^{-1/3},
\]

thus proving equation (3.44).

**Case II: \( a \geq 0 \)**

However, for the case when \( a > 0 \), we shall need to adopt a more stringent approach. Let us begin with

\[
\mathbb{P}(T \geq \delta N^{2/3}) = \mathbb{P}(1 - e^{-\theta T N^{-2/3}} \geq 1 - e^{-\delta}) \leq \frac{E(1 - e^{-\theta T N^{-2/3}})}{1 - e^{-\delta}}.
\]

Therefore, it suffices to prove that

\[
E(1 - e^{-\theta T N^{-2/3}}) \leq N^{-1/3}c(\theta).
\]

(3.45)

Define \( \psi(\theta) = E(1 - e^{-\theta T}) \), \( \hat{\psi}(\theta) = E(e^{-\theta T}) \), and \( \varepsilon = N^{-1/3} \).

In this notation, need to prove:

\[
\psi(\theta \varepsilon^2) \leq \varepsilon c(\theta).
\]

(3.46)

Recall

\[ T - 1 = \sum_{i=1}^{M_N} T_i, \]

By conditioning on \( M_N \),

\[
\psi(\theta \varepsilon^2) = E(1 - e^{-\theta T \varepsilon^2}) = E(1 - [1 - \psi(\theta \varepsilon^2)]^{M_N} e^{-\theta \varepsilon^2})
\]

Consequently,

\[
\hat{\psi}(\theta \varepsilon^2) = e^{-\theta \varepsilon^2} E \left[ \hat{\psi}(\theta \varepsilon^2)^{M_N} \right],
\]

which can be rewritten as

\[
\theta = \frac{1}{\varepsilon^2} \log E \left[ \hat{\psi}(\theta \varepsilon^2)^{M_N-1} \right], \quad \forall \theta
\]

But recall from equation (3.46) that we need to prove \( \hat{\psi}(\theta \varepsilon^2) \geq 1 - \varepsilon c(\theta) \), which boils down to comparing the above with \( f(x) = \frac{1}{x^2} \log E \left( [1 - x \lambda]^{M_N-1} \right) \).

Therefore, if we prove that for all \( \varepsilon \leq \varepsilon_0 \),

\[
\frac{1}{\varepsilon^2} \log E \left( [1 - c(\theta) \varepsilon]^{M_N-1} \right) > \theta,
\]

(3.47)
this will imply that
\[ \frac{1}{\varepsilon^2} \log E \left[ (1 - c(\theta)\varepsilon)^{MN-1} \right] > \frac{1}{\varepsilon^2} \log E \left[ \psi(\theta \varepsilon^2)^{MN-1} \right], \]
then by monotonicity of the above function we can conclude that \( \psi(\theta \varepsilon^2) \geq 1 - \varepsilon c(\theta) \)
Therefore, enough to prove equation (3.47) which can be restated as
\[ E \left[ (1 - c(\theta)\varepsilon)^{MN-1} \right] > e^{\theta \varepsilon^2}. \]
Since \( a > 0 \) the above inequality is equivalent to
\[ E \left[ \frac{(1 - c(\theta)\varepsilon)^{MN-1} - 1}{a \varepsilon^2} \right] > \frac{e^{\theta \varepsilon^2} - 1}{a \varepsilon^2}. \]
Let us take \( \varepsilon_0(\theta) \) such that for all \( \varepsilon \leq \varepsilon_0 \)
\[ \frac{2\theta}{a} \geq \frac{e^{\theta \varepsilon^2} - 1}{a \varepsilon^2}. \]
Thus, we need to prove that for all \( \varepsilon \leq \varepsilon_0 \)
\[ E \left[ \frac{(1 - c(\theta)\varepsilon)^{MN} - (1 - c(\theta)\varepsilon)}{a \varepsilon^2} \right] > [1 - c(\theta)\varepsilon] \frac{2\theta}{a}. \]
equivalently,
\[ E \left[ \frac{(1 - c(\theta)\varepsilon)^{MN} - (1 - c(\theta)\varepsilon)}{a \varepsilon^2} \right] > [1 - c(\theta)\varepsilon] \frac{2\theta}{a}. \]
Therefore, it is enough to prove
\[ E \left[ \frac{(1 - c(\theta)\varepsilon)^{MN} - (1 - c(\theta)\varepsilon)}{a \varepsilon^2} \right] > \frac{2\theta}{a}. \]
However,
\[ E \left[ (1 - \varepsilon c(\theta))^{MN} \right] = E \left( E \left[ (1 - \varepsilon c(\theta))^{MN} \mid I \right] \right) \]
\[ = E \left[ \left( 1 - \varepsilon c(\theta) + \varepsilon c(\theta)e^{-|I|/\lambda N} \right)^{MN} \right] \]
\[ = g(\varepsilon, \theta, |I|, N) \quad \text{(say)}. \]
Then it suffices to prove that
\[ E \left( \frac{g(\varepsilon, \theta, |I|, N) - (1 - \varepsilon c(\theta)(1 + a \varepsilon))}{a \varepsilon^2} \right) - c(\theta) > \frac{2\theta}{a}. \]
Note that \( E(|I|/\lambda) = 1 + a \varepsilon \). Therefore, the above expression boils down to
\[ \frac{1}{a \varepsilon^2} E \left( \frac{g(\varepsilon, \theta, |I|, N) - (1 - \varepsilon c(\theta) |I|/\lambda)}{\lambda} \right) - c(\theta) > \frac{2\theta}{a}. \] (3.48)
Now note that using the generalized mean value theorem, we have for $0 < \rho < \varepsilon$

$$g(\varepsilon, \theta, |I|, N) \geq \mathbb{E} \left[ 1 - \varepsilon c(\theta) \frac{|I|}{\lambda} + \frac{1}{2} \varepsilon^2 c(\theta) \frac{|I|^2}{\lambda^2} \left( 1 - \varepsilon c(\theta) \left( 1 - e^{-|I|/\lambda N} \right) \right) N^{-2} + O \left( \frac{1}{N} \right) \right]$$

Thus implying that

$$\frac{1}{a\varepsilon^2} \mathbb{E} \left( g(\varepsilon, \theta, |I|, N) - \left[ 1 - \varepsilon c(\theta) \frac{|I|}{\lambda} \right] \right) - c(\theta) \geq \frac{1}{2a} \mathbb{E} \left( c^2(\theta) \frac{|I|^2}{\lambda^2} \left( 1 - \varepsilon c(\theta) \left( 1 - e^{-|I|/\lambda N} \right) \right) N^{-2} + O \left( \frac{1}{N} \right) \right) - c(\theta)$$

In view of the above and equation (3.48), it suffices to prove that

$$\frac{1}{2a} \mathbb{E} \left( c^2(\theta) \frac{|I|^2}{\lambda^2} \left( 1 - \varepsilon c(\theta) \left( 1 - e^{-|I|/\lambda N} \right) \right) N^{-2} + O \left( \frac{1}{N} \right) \right) - c(\theta) > \frac{2\theta}{a}. \quad (3.49)$$

Equivalently,

$$\frac{c^2(\theta)}{2a} \mathbb{E} \left( \frac{|I|^2}{\lambda^2} \left( 1 - \varepsilon c(\theta) \left( 1 - e^{-|I|/\lambda N} \right) \right) \right) N^{-2} + O \left( \frac{1}{N} \right) - c(\theta) > \frac{2\theta}{a}. \quad (3.50)$$

Note that the above looks almost like a quadratic equation in $c(\theta)$, with the coefficient of the second order term being positive, and for large enough $N$ we can have

$$\mathbb{E} \left( \frac{|I|^2}{\lambda^2} \left( 1 - \varepsilon c(\theta) \left( 1 - e^{-|I|/\lambda N} \right) \right) \right) N^{-2} + O \left( \frac{1}{N} \right) > \frac{1}{2},$$

which means that it suffices to prove that

$$\frac{1}{4a} (c(\theta))^2 - c(\theta) - \frac{2\theta}{a} > 0. \quad (3.51)$$

However, note that there exists a positive root, say $\alpha(\theta)$, of this quadratic equation, such that for all $c(\theta) > \alpha(\theta)$, equation (3.45) is satisfied, which proves the Lemma. \(\square\)

This lemma also concludes the proof of Theorem 1.2.

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4 Appendix

Consider discrete time Markov chain \((X_n)_{n \geq 0}\) on circle of finite length, such that the transition probability density is given by \(p(\cdot)\), and that \(p(x) \geq q > 0\).

**Proposition 4.1** This MC has a unique stationary distribution, and it is given by uniform on the circle. Moreover,
\[
d_{TV}(\pi_n, \pi_\infty) \leq q^n.
\]

**Proof:** Define a new coupled process \((Y_n)_{n \geq 0}\) given by, such that \(Y_0\) is uniformly distributed on the circle, and that \(X_0 = x\), a fixed point on the circle. The joint process \((X_n, Y_n)_{n \geq 0}\) is made to evolve like a MC. Define \(p_1(x)\) such that
\[
p(x) = q \pi_\infty(x) + (1 - q) p_1(x),
\]
which gives
\[
p_1(x) = \frac{1}{1 - q} (p(x) - q \pi_\infty(x))
\]
Next take a sequence of random variables \((\eta_i)_{i \geq 0}\) which are i.i.d. Bernoulli\((q)\). Then the relationship between the two sequences is given as follows:

- if \(\eta_i = 1\), then let \(X_{i+1} = Y_{i+1}\), where \(Y_{i+1}\) is a freshly generated uniform random variable, and then onwards both the chains are equal, and the transition probability density is given by \(p(x)\) defined earlier.
- if \(\eta_i = 0\), then generate \(X_{i+1}\) according to \(\mathbb{P}(X_{i+1} \in y + dy|X_i = x) = p_1(y - x) dy\), and generate \(Y_{i+1}\) according to \(\mathbb{P}(Y_{i+1} \in y + dy|Y_i = x) = p_1(y - x) dy\).

Clearly, \(p\) is the transition density for both the processes, and
\[
p(X_n \neq Y_n) \leq P(\eta_0 = 1, \ldots, \eta_{n-1} = 1) = q^n.
\]
Now using Doeblin Theorem to get \(d_{TV}(\pi_n, \pi_\infty)\).