On the Invertibility of Motives of Affine Quadrics

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Abstract

We show that the reduced motive of a smooth affine quadric is invertible as an object of the triangulated category of motives $DM(k, \mathbb{Z}[1/e])$ (where $k$ is a perfect field of exponential characteristic $e$). We also establish a motivic version of the conjectures of Po Hu on products of certain affine Pfister quadrics. Both of these results are obtained by studying a novel conservative functor on (a subcategory of) $DM(k, \mathbb{Z}[1/e])$, the construction of which constitutes the main part of this work.

1 Introduction

Voevodsky has constructed a triangulated category $DM(k)$ parametrised on a perfect field $k$ containing the classical category $Chow(k)$ of Chow motives. Like $Chow(k)$, $DM(k)$ is a tensor category. We denote the tensor product by $\otimes = \otimes_{DM(k)}$ and the unit by $\mathbb{1} = \mathbb{1}_{DM(k)}$. As in any tensor category, we have the notion of invertible objects: an object $E \in DM(k)$ is called invertible if there exists an object $F \in DM(k)$ and an isomorphism $E \otimes F \approx \mathbb{1}$. The set of isomorphism classes of invertible objects forms an abelian group under $\otimes$ and is called the Picard group. We denote it $Pic(DM(k))$.

To the best of the author’s knowledge, not much is known about the structure of $Pic(DM(k))$. The shifted tensor unit $\mathbb{1}[1]$ and the twisted tensor unit $\mathbb{1}\{1\}$ (the reduced motive of $\mathbb{P}^1$) generate a summand in $Pic(DM(k))$ which is free of rank two. Beyond this, the study of $Pic(DM(k))$ seems so far to be confined to constructing elements of this group (i.e. invertible motives). In this direction we prove the following result.

Theorem. Let $k$ be a perfect field of exponential characteristic $e$ not two, $\phi(t_1, \ldots, t_n)$ a non-degenerate quadratic form over $k$ and $a \in k^\times$. Write $X^a_\phi$ for the affine quadric defined by the equation $\phi(t_1, \ldots, t_n) = a$.

Then the reduced motive $\mathbb{M}(X^a_\phi) \in DM(k, \mathbb{Z}[1/e])$ is invertible.

This result has a number of predecessors. Work of Voevodsky [17, particularly Lemma 4.8] can be used to show that reduced versions of the Rost motives [16] are invertible. As observed by Hu-Kriz [11, Proposition 5.5], the reduced Rost motives are reduced motives of affine Pfister quadrics. They go further
and explore analogies with the Hopf invariant one problem. In \cite{10} this culminates in certain conjectures about wedge products of affine Pfister quadrics implying their invertibility. Moreover the conjectures are proven in low dimensions. Further evidence that all affine quadrics may be invertible was supplied by Asok-Doran-Fasel in \cite{2} where they show that affine quadrics of special form have invertible stable homotopy types.

The best method the author knows of attacking the study of Picard groups of tensor categories (to the extend that it even deserves the name “method”) is to construct “realisation functors” \( F : \text{DM}(k) \to \mathcal{C} \). If \( F \) is a tensor functor, it induces a homomorphism \( \text{Pic}(\text{DM}(k)) \to \text{Pic}(\mathcal{C}) \). If \( F \) is sufficiently nice, and \( \mathcal{C} \) sufficiently simple, one may hope to compute \( \text{Pic}(\mathcal{C}) \) and relate it to \( \text{Pic}(\text{DM}(k)) \). We mention in passing that a good test for the “niceness” of \( F \) seems to be conservativity (i.e. the property that \( F \) detects isomorphisms). This will be illustrated later.

There are well known realisation functors out of \( \text{DM}(k) \), but none of them seem helpful to our problem. If \( k \subset \mathbb{C} \) there is the Hodge realisation, but this factors through the natural functor \( \text{DM}(k) \to \text{DM}(\mathbb{C}) \) and hence provides no interesting information about quadrics (which over \( \mathbb{C} \) are distinguished by only their dimension). There is also étale realisation, but this factors through \( \text{DM}(k) \to \text{DM}_{et}(k) \). In \( \text{DM}_{et}(k) \) our problem turns out to be very simple and not indicative of the complexity encountered in \( \text{DM}(k) \) (i.e. in the Nisnevich topology). What we propose in this work is to construct purpose-built realisation functors \( \text{DM}(k) \to \mathcal{C} \) into big but easy to understand categories. (Actually we do not quite achieve this; limitations will be explained later.) To motivate our constructions, we explain two analogous but simpler problems obtained by replacing \( \text{DM}(k) \) by another category.

First let \( G \) be a finite group. There exists the stable \( G \)-equivariant homotopy category \( \text{SH}(G) \). Its objects (called genuine \( G \)-spectra) are roughly pointed \( G \)-spaces, where maps inducing weak equivalences on all fixed point sets have been turned into isomorphisms, and all representation spheres are invertible objects. If \( H \leq G \) is a subgroup, the set of cosets \( G/H \) can naturally be turned into a pointed \( G \)-space (adding a base point \( * \) with trivial action). We denote the associated spectrum by \( \Sigma^\infty G/H_+ \). The objects \( \Sigma^\infty G/H_+ \) generate \( \text{SH}(G) \). There is a functor, called geometric fixed points functor, and denoted \( \Phi = \Phi^G : \text{SH}(G) \to \text{SH} \) (where \( \text{SH} = \text{SH}(\{e\}) \) is the classical stable homotopy category) which turns out to be very useful. It is a tensor functor with the property that \( \Phi^G(\Sigma^\infty G/G_+) = S \) (the sphere spectrum), whereas \( \Phi^G(\Sigma^\infty G/H_+) = 0 \) for any proper subgroup \( H < G \). There are also natural functors \( \text{SH}(G) \to \text{SH}(H) \) (treating \( G \)-spaces as \( H \)-spaces) allowing us to construct the more general geometric fixed points functors \( \Phi^H : \text{SH}(G) \to \text{SH}(H) \to \text{SH} \). As it turns out the collection \( \{\Phi^H\}_H \) (with \( H \) ranging over all subgroups of \( G \)) is as nice as one may ask (in particular conservative). Consequently these functors were used in \cite{8} to study \( \text{Pic}(\text{SH}(G)) \).

We now come to a second, more algebraic, example. Let \( R \) be a (commutative unital) ring. Suppose we want to study \( \text{Pic}(\text{D}(R)) \), the Picard group of the
derived category of \( R \)-modules. Let \( m \) be a maximal ideal of \( R \). Recalling that \( D(R) \) can be identified with a subcategory of \( K(P(R)) \), the homotopy category of chain complexes of projective \( R \)-modules, it is easy to construct a functor \( \Phi^m : D(R) \to D(R/m) \) with the property that \( \Phi^m(R[0]) = R/m[0] \). (This is just \( \otimes R/m \).) It turns out that the collection \( \{ \Phi^m \}_m \) (where \( m \) ranges over all maximal ideals) is as nice as one needs (at least when restricted to subcategories of sufficiently small objects in \( D(R) \)). Moreover the categories \( D(R/m) \) are easy to understand. Consequently, these functors have implicitly been used by Fausk in his study of the Picard group of derived categories \([7]\).

Our construction for \( \text{DM}^{gm}(k) \) uses a conglomerate of these ideas. The technical notion of weight structures is the glue that holds our constructions together. We proceed roughly as follows. Recall that \( \text{DM}^{gm}(k) \) is generated as a triangulated category by the Chow motives. Let \( S \) be the triangulated subcategory generated by those Chow motives not affording a (non-vanishing) Tate summand. The basic idea is to consider the (Verdier Quotient) functor \( \varphi_k : \text{DM}^{gm}(k) \to \text{DM}^{gm}(k)/S \). The right hand side does not seem initially easier to understand, but it is at least clear that it is generated by the images of Tate motives. Using weight structure theory one obtains a functor \( t : \text{DM}^{gm}(k)/S \to K^b(Tate) \), where \( Tate \) is the category of (pure) Tate motives, and \( K^b \) means bounded chain homotopy category. Combined with base change to arbitrary fields, we thus obtain a collection of functors \( \Phi^l : \text{DM}^{gm}(k) \to \text{DM}^{gm}(l) \to \text{DM}^{gm}(l)/S \to K^b(Tate) \). We not that if \( T \in Tate \) is a Tate motive then \( \Phi^k(T) = T \). If instead \( M \in \text{Chow} \) affords no (non-zero) Tate summands, then \( \Phi^k(M) = 0 \). This is rather similar to the geometric fixed points functor \( \Phi^G \) from stable equivariant homotopy theory. Since the general \( \Phi^l \) are obtained from \( \Phi^k \) by base change, just as \( \Phi^H \) is obtained from the \( \Phi^G \) construction by base change (restriction to a subgroup), we will call the functors \( \Phi^l \) “generalized geometric fixed points functors.”

A natural question is when these functors have good properties. For our purposes we definitely need tensor functors, which is to say we need \( S \) to be a tensor ideal. This is just not true in general. However, if instead of looking at the full \( \text{DM}^{gm}(k) \) we look at the subcategory \( \text{DQM}^{gm}(k) \) generated by the (products of) smooth projective quadrics, and use coefficients modulo two, then we can show that \( S \) is a tensor ideal. Moreover, using more properties of weight structures, we prove the collection of generalized fixed points functors to be conservative and Pic-injective (i.e. inducing an injection on Picard groups):

**Theorem.** Let \( k \) be a perfect field of exponential characteristic \( e \) not two, and \( \Phi^l : \text{DQM}^{gm}(k, \mathbb{F}_2) \to Tate(\mathbb{F}_2) \) the functors constructed above.

Then \( \{ \Phi^l \}_l \), as \( l \) ranges over finitely generated extensions of \( k \), forms a conservative, Pic-injective family of tensor triangulated functors.

It is then not hard to use general properties of base change and change of coefficients for \( \text{DM} \) to build a conservative and Pic-injective family for

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1Actually for this to be true we need to consider \( \text{DM}^{gm}(k, \mathbb{F}) \) where \( \mathbb{F} \) is a field. This is not really a problem.
DQM\textsuperscript{gm}(k, \mathbb{Z}[1/e]). It turns out that one additional functor \( \Psi : \text{DQM}(k, \mathbb{Z}[1/e]) \to \text{Tate}(\mathbb{Z}[1/e]) \) suffices. (It is related to geometric base change.)

In more detail, the paper is organised as follows. In section 2 we introduce our notations regarding Chow motives and collect some results. The main idea is to use the absence of degree one zero-cycles in a variety to conclude that it is free of Tate summands in a strong sense. This observation is what will allow us in a later section to establish that our “geometric fixed points functors” \( \Phi^l \) are tensor.

In section 3 we review in some detail the categories \( \text{DM}(k, A) \) (triangulated motives over the perfect field \( k \) with coefficients in the commutative ring \( A \)) and their behaviour under change of coefficients and base. All the material is well known, but sometimes hard to source. We then construct a convenient conservative and Pic-injective collection out of \( \text{DM}(k, A) \). The targets are always \( \text{DM}(k', A') \) with either \( k \) simplified (e.g. \( k' \) separably closed) or \( A \) simplified (e.g. \( A' \) a field).

Section 4 constitutes the technical heart of our work. We first rapidly review Bondarko’s theory of weight structures. After that we carry out the programme outlined above of constructing a conservative and Pic-injective family of functors \( \Phi^l : \text{DQM}^\text{gm}(k, \mathbb{F}_2) \to K^B(\text{Tate}(\mathbb{F}_2)) \).

The remaining sections contain applications. In section 5 we prove that all affine quadrics have invertible motives. This is rather satisfying, since affine quadrics are fairly natural “generalised spheres.” Also the result has been known in the étale topology for a long time. Compare the beginning of this introduction for a history of this problem.

Section 6 contains the second set of applications. In [10, Conjecture 1.4] Po Hu has stated certain conjectures about the motivic spectra of affine Pfister quadrics, namely certain formulas they should satisfy under wedge product. We establish the analogues (or “images”) of these formulas in \( \text{DM}(k) \) by an easy computation involving our fixed points functors.

The list of applications of our methods does not end here, but the amount of material we want to stuff into one article does. As directions of future work, let us mention the following possibilities. The structure of \( \text{Pic}(\text{DQM}(k)) \) can be investigated. One may replace the set of projective quadrics by projective homogeneous varieties for a fixed group \( G \). Also using (almost) the same methods, it is possible to study \( \text{DATM}(k) \), the subcategory of \( \text{DM}(k) \) generated by \( M(\text{Spec}(l))\{i\} \) for \( l/k \) finite separable and \( i \in \mathbb{Z} \), i.e. Artin-Tate motives. This will be treated in forthcoming work.

Whenever we talk about quadrics or quadratic forms, we shall assume that the base field has characteristic different from two. This will be restated with the most important theorems.

Our results are stated over perfect base fields, because this is when \( \text{DM}(k) \) is best understood. However actually everything goes through over arbitrary base fields, using [5]. We have elected not to explicitly treat the imperfect case to make the paper more accessible. We do have to work with \( \text{DM}(l) \) for imperfect \( l \) at intermediate steps. This is confined entirely to section 4.

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vestigation and for providing many helpful insights, and Mikhail Bondarko for comments on a draft of this paper.

2 Some Results about Chow Motives

We begin with some notation. We take for granted the notion of an additive category. An additive category $\mathcal{C}$ is called Karoubi-closed if every idempotent endomorphism of an object of $\mathcal{C}$ corresponds to a direct sum decomposition. By a tensor category we mean an additive category provided with a suitably compatible symmetric monoidal structure [6, Section 1]. In particular this means that the monoidal operation is bi-additive. We shall always denote the monoidal operation by $\otimes = \otimes_\mathcal{C}$ and call it tensor product. The tensor unit is generically denoted $1 = 1_\mathcal{C}$.

Now our conventions regarding Chow motives. By $\text{SmProj}(k)$ we denote the category of smooth projective varieties over the field $k$. It is a symmetric monoidal category using cartesian product as monoidal product. We shall assume understood the existence and functoriality properties of the Chow monoidal category using cartesian product as monoidal product. We shall always denote the category of smooth projective varieties over the field $k$. It is a symmetric monoidal category using cartesian product as monoidal product. We shall always denote the category of smooth projective varieties over the field $k$. It is a symmetric monoidal category using cartesian product as monoidal product.

In particular we have $\text{Hom}(\text{SmProj}(k)) = \text{SmProj}(k)$ together with a covariant symmetric monoidal functor $M = M_f : \text{SmProj}(k) \to \text{Chow}(k)$ which has the following properties. The unit object is $1_\text{Chow}(k) = 1 = M(\text{Spec}(k))$. There exists an object $1 \{1\}$ such that $M(F^1) \cong 1 \oplus 1 \{1\}$. We call $1 \{1\}$ the Lefschetz motive. It is invertible. For any $n \in \mathbb{Z}$ and $M \in \text{Chow}(k)$ we write $M \{n\} := M \otimes 1 \{1\}^\otimes n$. For any $X, Y \in \text{SmProj}(k)$ and $i, j \in \mathbb{Z}$ we have

$$\text{Hom}_{\text{Chow}(k)}(M(X)\{i\}, M(Y)\{j\}) = A_{\dim X+i-j}(X \times Y).$$

In particular we have $\text{Hom}(MX, 1\{i\}) = A_i(X, F)$ and $\text{Hom}(1\{i\}, MX) = A_i(X, F)$. Composition is by the usual push-pull convolution.

In the remainder of this section we collect some results about Chow motives which we will need throughout the article. None of them are hard so probably most of this is well known.

Recall first that if $l/k$ is a field extension then $\text{SmProj}(k) \to \text{SmProj}(l)$, $X \to X_l$ induces a functor $\text{Chow}(k, F) \to \text{Chow}(l, F)$ called base change and denoted $M \mapsto M_l$. We need to know something about this in the inseparable case.

**Lemma 1.** Let $l/k$ be a purely inseparable extension of fields of characteristic $p$ and $F$ a coefficient ring on which $p$ is invertible. Then the base change $\text{Chow}(k, F) \to \text{Chow}(l, F)$ is fully faithful.

**Proof.** It suffices to prove that for $X \in \text{SmProj}(k)$ we have $A_*(X, F) = A_*(X_l, F)$. By the definition of rational equivalence as in [6, Section 1.6] it is enough to show that $Z_*(X, F) \to Z_*(X_l, F)$ is an isomorphism for all $X$.

Let $Z \subset X$ be a reduced closed subscheme and $|Z_l|$ the reduced closed subscheme underlying $Z_l$. Then the image of $|Z|$ under $Z_*(X, F) \to Z_*(X_l, F)$
is \( n | Z_i | \), where \( n \) is the multiplicity of \( Z_i \). This is easily seen to be a power of \( p \), whence \( Z_*(X, F) \to Z_*(X_i, F) \) is injective. It is also surjective since \( X_i \to X \) is a homeomorphism on underlying spaces. This concludes the proof.

We now investigate “Tate summands”. Denote by \( Tate(k, F) \subset Chow(k, F) \) the smallest full Karoubi-closed additive subcategory containing \( \mathbb{1}{i} \) for all \( i \). This is independent up to equivalence of \( k \) and we will just write \( Tate(F) \) if no confusion can arise. (It is a tensor category.)

We say \( M \in Chow(k, F) \) is Tate-free if whenever \( M \approx T \oplus M' \) with \( T \in Tate(k, F) \) then \( T \approx 0 \). The next proposition holds in much greater generality, but this version is all we need.

**Proposition 2.** Let \( F \) be a finite ring and \( M \in Chow(k, F) \). Then there exist \( T \in Tate(F) \) and \( M' \in Chow(k, F) \) with \( M' \) Tate-free and \( M \approx T \oplus M' \).

**Proof.** Splitting off Tate summands inductively, the only problem which could occur is that \( M \) might afford arbitrarily large Tate summands. The impossibility of this follows (for example) from the finiteness of étale cohomology of complete varieties [13 Corollary VI.2.8].

**Lemma 3.** Let \( F \) be a field. Then if \( M, N \in Chow(k, F) \) are Tate-free so is \( M \oplus N \).

**Proof.** A motive with \( F \)-coefficients is Tate-free if and only if it affords no summand of the form \( \mathbb{1}\{n\} \) for any \( n \).

Let \( i : \mathbb{1}\{n\} \to M \oplus N \) and \( p : M \oplus N \to \mathbb{1}\{n\} \) be inclusion of and projection to a summand, for \( M, N \) arbitrary. Write \( i = (i_M, i_N)^T \) and \( p = (p_M, p_N) \). Then \( id = pi = p_M i_M + p_N i_N \). Since \( \text{Hom}(\mathbb{1}\{n\}, \mathbb{1}\{n\}) = \mathbb{F} \neq 0 \) we must have \( p_M i_M \neq 0 \) or \( p_N i_N \neq 0 \). Suppose the former holds. Then since \( F \) is a field we may replace \( i_M \) by a multiple \( ci_M \) such that \( p_M(ci_M) = 1 \). Thus \( \mathbb{1}\{n\} \) is a summand of \( M \). Similarly in the other case. This establishes the contrapositive of the lemma.

**Lemma 4.** Let \( F \) be a field. Then any morphism in \( Tate(k, F) \) factoring through a Tate-free object is zero.

**Proof.** Since \( F \) is a field any Tate motive is a sum of \( \mathbb{1}\{n\} \) for various \( n \), so it suffices to consider a morphism \( \mathbb{1}\{n\} \to \mathbb{1}\{m\} \) factoring through a Tate-free object. Since \( \text{Hom}(\mathbb{1}\{n\}, \mathbb{1}\{m\}) = 0 \) for \( n \neq m \) we may assume \( n = m \). Consider \( a \in \text{Hom}(\mathbb{1}\{n\}, M) \) and \( b \in \text{Hom}(M, \mathbb{1}\{n\}) \). If \( ba \neq 0 \) then there exists \( c \in F \) such that \( (cb)a = id \). It follows that \( (cb) \), a present \( \mathbb{1}\{n\} \) as a summand of \( M \). This establishes the contrapositive.

We need tools to recognise Tate-free motives. To do so, we introduce some more notation. For \( X \in SmProj(k) \) there exists the degree map \( \text{deg} : A_0(X, F) \to F \) (corresponding to pushforward along the structure map \( \text{Hom}(\mathbb{1}, MX) \to \text{Hom}(\mathbb{1}, \mathbb{1}) \)). Write \( I_F(X) = \text{deg}(A_0(X, F)) \) for the image of the degree map. This is the ideal inside \( F \) generated by the degrees of closed points. The utility of this notion is as follows.
Lemma 5. Let $F$ be a field and suppose $I_F(X) \neq F$. Then $MX$ is Tate-free.

Proof. As before $MX$ is Tate-free if and only if it affords no summand $1 \{N\}$ for any $N$. Given $i \in \text{Hom}(1 \{N\}, MX) = A_N(X, F)$ and $p \in \text{Hom}(MX, 1 \{N\}) = A_N(X, F)$, the composite $pi \in \text{Hom}(1 \{N\}, 1 \{N\}) = F$ is obtained by push-pull convolution. In this case it is just $\deg(p \cap i)$ and so is contained in $I_F(X)$. Thus $pi \neq 1$ and $(p, i)$ is not a presentation of $1 \{N\}$ as a summand of $X$. \hfill \Box

Lemma 6. Let $X, Y \in \text{SmProj}(k)$. Then $I_F(X \times Y) \subset I_F(X) \cap I_F(Y)$.

Proof. We recall that $I_F(X \times Y)$ is just the ideal generated by degrees of closed points. So let $z \in X \times Y$ be a closed point. Then $z \to X \times Y$ corresponds to morphisms $z \to X$ and $z \to Y$. It follows that $\deg(z) \in I_F(X)$ and similarly $\deg(z) \in I_F(Y)$. This implies the result. \hfill \Box

Suppose $S \subset \text{SmProj}(k)$ is a set of smooth projective varieties. We write $\langle S \rangle_{\text{Chow}(k, F)}^T$ for the smallest additive, Karoubi-closed, tensor subcategory of $\text{Chow}(k, F)$ containing all Tate motives and also $MX$ for each $X \in S$. Assuming $F$ is a field, this means that a general object of $\langle S \rangle_{\text{Chow}(k, F)}^T$ is isomorphic to

$$T \oplus M(X_1^{(1)} \otimes \cdots \otimes X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \otimes \cdots \otimes X_{n_m}^{(m)})\{i_m\},$$

with $T \in \text{Tate}(F)$ and $X_i^{(j)} \in S, i_r \in \mathbb{Z}$.

The following proposition (or rather its failure to generalise) is the basic reason why in the construction of fixed point functors we will need to restrict to subcategories.

Proposition 7. Let $F$ be a finite field and $S \subset \text{SmProj}(k)$ be such that $I_F(X) = 0$ for all $X \in S$ (i.e. such that all closed points of $X$ have degree divisible by the characteristic of $F$). Then any object $M \in \langle S \rangle_{\text{Chow}(k, F)}^T$ can be written as $T \oplus M'$, where $T \in \text{Tate}(F)$ and $M'$ is a summand of

$$M(X_1^{(1)} \otimes \cdots \otimes X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \otimes \cdots \otimes X_{n_m}^{(m)})\{i_m\},$$

for some $X_i^{(j)} \in S, i_r \in \mathbb{Z}$. Moreover any such $M'$ is Tate-free.

Proof. By Lemma 6 we know that $I_F(X_1^{(j)} \times \cdots X_{n_l}^{(j)}) = 0$ and thus by Lemmas 5 and 8 we conclude that any $M'$ as displayed is indeed Tate-free. So it suffices to establish the first part.

By definition we may write

$$M \oplus M'' \approx T \oplus M(X_1^{(1)} \otimes \cdots \otimes X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \otimes \cdots \otimes X_{n_m}^{(m)})\{i_m\},$$

with $T \in \text{Tate}(F)$ and $X_i^{(j)} \in S$. Using Proposition 8 we write $M \oplus M'' \approx M' \oplus M'' \oplus T'$, where $M'$, $M''$ are maximal Tate-free summand in $M, M''$ respectively and $T'$ is Tate. Writing out the inverse isomorphisms $M' \oplus M'' \oplus T' \approx T \oplus M(X_1^{(1)} \cdots) \oplus \ldots$ in matrix form and using Lemma 8 we conclude that $T' \approx T$ via the induced map. The Lemma below yields that $M' \oplus M'' \approx M(X_1^{(1)} \cdots) \oplus \ldots$. This finishes the proof. \hfill \Box
Lemma 8. Let $\mathcal{C}$ be an additive category and let $U, T, X, T' \in \mathcal{C}$ be four objects. Suppose we are given an isomorphism $\phi : U \oplus T \to X \oplus T'$ such that the component $T \to T'$ is also an isomorphism. Then there is an isomorphism $\tilde{\phi} : U \to X$.

Proof. Let us write

$$\phi = \begin{pmatrix} \alpha & a \\ b & f \end{pmatrix} \quad \psi = \begin{pmatrix} \beta & a' \\ b' & g \end{pmatrix},$$

where $\psi$ is the inverse of $\phi$. By assumption $f$ is an isomorphism. Writing out $\phi \psi = \text{id}_{X \oplus T}$ and $\psi \phi = \text{id}_{U \oplus T'}$ one obtains

$$b \beta = -f b' \quad \beta a = -a' f \quad \alpha \beta + ab' = \text{id}_U \quad \beta a + a'b = \text{id}_X.$$

Put $\tilde{\phi} = \alpha - af^{-1}b$. Then the above relations imply that $\tilde{\phi}$ is an isomorphism with inverse $\beta$. \qed

3 Review of Voevodsky Motives

In this section we collect some facts about $\text{DM}(k, A)$ that we will need for the applications. Most of this is well-known, but in some cases we were unable to locate adequate references. For simplicity we only consider $\text{DM}^- (k, A)$ (i.e. “connective motives”). This is not really essential and could be avoided at the cost of using slightly more elaborate techniques. We will assume throughout that $k$ is perfect, and re-state this assumption with each theorem.

Fix a ring $A$. First recall the construction of $\text{DM}^- (k, A)$ [12]. Write $Sm(k)$ for the symmetric monoidal category of smooth schemes over $k$ (monoidal operation being cartesian product) and $Cor(k)$ for the symmetric monoidal category with same objects as $Sm(k)$ but morphisms given by finite correspondences. There is a natural monoidal functor $Sm(k) \to Cor(k)$. Write $\text{Sh}^{tr}(k, A)$ for the abelian category of Nisnevich sheaves of $A$-modules on $Cor(k)$, i.e. presheaves $Cor(k) \to A\text{-Mod}$ such that the restriction $Sm(k) \to Cor(k) \to A\text{-Mod}$ is a sheaf in the Nisnevich topology. There is a functor $A_{tr, \bullet} : Sm(k) \to \text{Sh}^{tr}(k, A)$ sending $X \in Sm(k)$ to the presheaf with transfers it represents (which turns out to be a sheaf). The category $\text{Sh}^{tr}(k, A)$ carries a right exact tensor structure making $A_{tr}$ a monoidal functor.

The category $\text{Sh}^{tr}(k, A)$ has enough injectives, so one may form the right-bounded derived category $D^- (\text{Sh}^{tr}(k, A))$. Let $\mathcal{C} = \mathcal{C}_{A, k}$ be the localising subcategory generated by cones on $A_{tr}(X \otimes \mathbb{A}^1) \to A_{tr}X$ for all $X \in Sm(k)$. The localisation $D^- (\text{Sh}^{tr}(k, A))/\mathcal{C}$ is denoted by $\text{DM}^{\text{eff}, -}(k, A)$. By general theory, the functor $D^- (\text{Sh}^{tr}(k, A)) \to \text{DM}^{\text{eff}, -}(k, A)$ has a fully faithful adjoint. We identify $\text{DM}^{\text{eff}, -}(k, A)$ with its image in $D^- (\text{Sh}^{tr}(k, A))$ and write $L_{A_{tr}}$ for the functor $D^- (\text{Sh}^{tr}(k, A)) \to \text{DM}^{\text{eff}, -}(k, A) \subset D^- (\text{Sh}^{tr}(k, A))$. The composite $Sm(k) \xrightarrow{A_{tr}} \text{Sh}^{tr}(k, A) \subset D^- (\text{Sh}^{tr}(k, A)) \xrightarrow{L_{A_{tr}}} \text{DM}^{\text{eff}, -}(k, A)$ is denoted $M^{\text{eff}}$.

If $k$ is perfect then a model for $L_{A_{tr}}$ is given by the $\mathbb{A}^1$-chain complex $C_*$, and $\text{DM}^{\text{eff}, -}(k, A) \subset D^- (\text{Sh}^{tr}(k, A))$ can be characterised explicitly. For $C^\bullet \in$
$D^-(\text{Shv}^{tr}(k, A))$ denote by $\mathbf{L}^i(C) \in \text{Shv}^{tr}(k, A)$ the cohomology objects. A sheaf $F \in \text{Shv}^{tr}(k, A)$ is called homotopy invariant if for each $X \in \text{Sm}(k)$ the natural map $F(X) \to F(X \otimes \mathbb{A}^1)$ is an isomorphism. Write $\text{HI}^n(k, A)$ for the full subcategory of $\text{Shv}^{tr}(k, A)$ consisting of homotopy invariant sheaves. Then $\text{DM}^{\text{eff},-}(k, A)$ is the subcategory of all objects $C^\bullet \in D^-(\text{Shv}^{tr}(k, A))$ such that $\mathbf{L}^i(C) \in \text{HI}^i(k, A)$ for all $i$.

The triangulated category $\text{DM}^{\text{eff},-}(k, A)$ carries a compatible tensor structure, making the functor $M^{\text{eff}}$ a tensor functor. We denote the unit object variously by $\mathbb{1} = \mathbb{1}_A = \mathbb{1}_k = \mathbb{1}_{k,A}$, as convenient. Still assuming $k$ to be perfect, we have the important cancellation theorem: for $C, D \in \text{DM}^{\text{eff},-}(k, A)$ one has $\text{Hom}(C, D) = \text{Hom}(C \otimes M^{\text{eff}} \mathbb{G}_m, D \otimes M^{\text{eff}} \mathbb{G}_m)$. Recall that $M^{\text{eff}} \mathbb{G}_m = M^{\text{eff}}(\mathbb{A}^1 \setminus \{0\}, 1)$ where for $X \in \text{Sm}(k)$ with rational point $x \in X$ one puts $M^{\text{eff}}(X, x) = \text{cone}(M^{\text{eff}} X \to M^{\text{eff}}_X)[-1]$. It follows (roughly) that we can form a category $\text{DM}^-(k, A)$ by inverting $M^{\text{eff}} \mathbb{G}_m$: there is a fully faithful functor $i : \text{DM}^{\text{eff},-}(k, A) \to \text{DM}^-(k, A)$, the image $iM^{\text{eff}} \mathbb{G}_m$ is invertible, and every object of $\text{DM}^-(k, A)$ is isomorphic to one of the form $iC \otimes (iM^{\text{eff}} \mathbb{G}_m)^{\otimes n}$ for some $C \in \text{DM}^{\text{eff},-}(k, A)$ and $n \in \mathbb{Z}$.

The composite $iM^{\text{eff}}$ is just denoted $M = M_{k, A}$. The triangulated category $\text{DM}^-(k, A)$ is generated (as a localising subcategory) by the compact objects $M(X) \otimes M(\mathbb{G}_m)^{\otimes n}$ for $X \in \text{Sm}(k)$ and $n \in \mathbb{Z}$. The subcategory of compact objects is denoted $\text{DM}^{tr}(k, A)$. By general results [13, Lemma 2.2] it coincides with the thick triangulated category generated by the $M(X) \otimes M(\mathbb{G}_m)^{\otimes n}$.

This concludes our review of the construction of $\text{DM}$. Next we review base change. For this let $f : \text{Spec}(l) \to \text{Spec}(k)$ be a (not necessarily finite) separable extension. There is a natural functor $f_\ast : \text{Shv}^{tr}(l, A) \to \text{Shv}^{tr}(k, A)$ with $f_\ast F(X) = F(X \otimes_k l)$. It affords a right adjoint $f^\ast : \text{Shv}^{tr}(k, A) \to \text{Shv}^{tr}(l, A)$. If $l/k$ is finite, then one has $f^\ast F(X) = F(X)$, where on the right hand side $X$ is viewed as an element of $\text{Sm}(k)$ via $X \to \text{Spec}(l) \to \text{Spec}(k)$. If $l/k$ is algebraic and separable one has

$$f^\ast F(X) = \text{colim}_{X \to X'} F(X'),$$

where the colimit is over all morphisms $X \to X'$ with $X' \in \text{Sm}(k)$. For example if $X = Y \otimes_k l$ for some $Y \in \text{Sm}(k)$ then

$$f^\ast F(X) = \text{colim}_{l'/k} F(Y \otimes_k l'),$$

where the colimit is over subextensions $k \subset l' \subset l$ such that $l'/k$ is finite. For these two results, see e.g. [13, Proposition II.2.2 and Lemma II.3.3].

The functor $f^\ast$ is exact and one has $f^\ast(\text{HI}^n(k, A)) \subset \text{HI}^n(l, A)$. One also has $f^\ast A_{tr, k} X = A_{tr, l} X$. It follows that there is a well-defined induced tensor triangulated functor $f^\ast : \text{DM}^-(k, A) \to \text{DM}^-(l, A)$.

The following result is surely well-known, but we could not find a reference, so include the easy proof.

**Proposition 9.** Let $f : \text{Spec}(l) \to \text{Spec}(k)$ be a (separable) extension of the perfect field $k$, and $A$ a ring such that for each finite subextension $l/l'/k$, the (image of the) integer $[l'/k]$ is a unit in $A$. 

Then \( f^* : \text{DM}^-(k,A) \to \text{DM}^-(l,A) \) is conservative.

Proof. Conservative means detecting zero objects. This can be checked after arbitrary tensor product with invertible objects (e.g. \( M\mathbb{G}_m \)), so we reduce to \( \text{DM}^\text{eff,-} \). Considering cohomology objects, it suffices to show: if \( F \in \text{Shv}^{tr}(k,A) \) and \( f^* F = 0 \), then \( F = 0 \). Let \( X \in \text{Sm}(k), x \in F(X) \). It suffices to show that \( x = 0 \). By example (1) we have \( 0 = f^* F(X \otimes_k l) = \text{colim}_{l'/k} F(X \otimes_k l') \). Thus there exists a finite subextension \( l'/k \) with \( (l'/k)^*(x) = 0 \). But then by a transfer argument one finds that \( [l' : k]x = 0 \), whence \( x = 0 \) since \( [l' : k] \) is a unit in \( A \) by assumption.

Next we consider change of coefficients. The construction and basic properties must be well known, but again we could not find convenient references. Let \( \alpha : A \to B \) be a ring homomorphism. There is a natural adjoint functor pair

\[
\alpha : \text{Shv}^{tr}(k,A) \rightleftarrows \text{Shv}^{tr}(l,A) : \alpha^*.
\]

Here \( \alpha^* \) is the sheaf associated to \( X \mapsto F(X) \otimes_A B \) and \( \alpha^* F(X) = F(X) \), viewed as an \( A \)-module. The functor \( \alpha^* \) is exact and so immediately descends to \( D^-(\text{Shv}^{tr}(k,B)) \). We also have \( \alpha^*(\text{HI}^{\text{eff}}(k,B)) \subset \text{HI}^{\text{eff}}(k,A) \) and so \( \alpha^* \) defines \( R\alpha^* : \text{DM}^\text{eff,-}(k,B) \to \text{DM}^\text{eff,-}(k,A) \).

The situation with \( \alpha \) is slightly more delicate. There is \( L\alpha : D^-(\text{Shv}^{tr}(k,A)) \to D^-(\text{Shv}^{tr}(k,B)) \). This is essentially just derived tensor product. The value \( L\alpha(C^\bullet) \) is computed in the usual way: resolve \( C^\bullet \) as a complex of presheaves by representable sheaves (this is always possible), then apply \( \alpha \) termwise. It follows that \( L\alpha \) is actually a tensor functor, and that \( L\alpha(C_k,A) \subset C_k,B \). Thus there is a well-defined induced functor \( L\alpha : \text{DM}^\text{eff,-}(k,A) \approx D^-(\ldots)/C_k,A \to D^-(\ldots)/C_k,B \approx \text{DM}^\text{eff,-}(k,B) \). Since \( L\alpha \) is a tensor functor, it extends immediately to \( \text{DM}^-(k,A) \).

Resolving \( B \) projectively as an \( A \)-module, it is not difficult to show that for \( E \in \text{DM}^\text{eff,-}(k,B) \) one has \( R\alpha^*(E \otimes M\mathbb{G}_m) \subset \text{DM}^\text{eff,-}(k,A) \otimes \mathbb{G}_m \). An easy computation using adjunction of \( L\alpha \) and \( R\alpha^* \) then shows that \( R\alpha^*(E \otimes M\mathbb{G}_m) \approx R\alpha^*(E) \otimes M\mathbb{G}_m \). Thus \( R\alpha^* \) also extends from \( \text{DM}^\text{eff,-}(k,A) \) to \( \text{DM}^-(k,A) \).

We point out that as usual, all parallel versions of \( L\alpha \) and \( R\alpha^* \) are adjoint. Also any \( f^* \) “commutes” with any \( L\alpha, R\alpha^* \) (whenever the statement makes sense).

Here are some of the basic properties of the change of coefficient functors.

**Proposition 10.** Let \( k \) be perfect, \( \alpha : A \to B \) be flat, \( E \in \text{DM}^{gm}(k,A) \) and \( F \in \text{DM}^-(k,A) \). Then

\[
\text{Hom}(E,F) \otimes_A B \approx \text{Hom}(L\alpha E, L\alpha F).
\]

Proof. By the cancellation theorem, we may assume that \( E,F \) are effective. There is certainly a natural map from the left side to the right. Using the 5-lemma and the fact that \( \text{DM}^{\text{eff, gm}}(k,A) \) is generated by \( MX \) for \( X \in \text{Sm}(k) \), we
may reduce to $E = MX[i]$. In this case $\text{Hom}(MX[i], F)$ is given by the hypercohomology $H^{-i}(X, F^*)$. Since $\otimes_A B$ is exact it commutes with hypercohomology and preserves sheaves, so we have $H^{-i}(X, F^*) \otimes_A B = H^{-i}(X, F^* \otimes_A B) = H^{-i}(X, (\alpha#(F^*)^*)$.

**Proposition 11.** Let $k$ be perfect, $A$ a ring, $a \in A$ a non zero divisor and $\alpha : A \to A/(a)$ the natural map. Then for $E \in \text{DM}^{-}(k, A)$ there is a natural distinguished triangle

$$E \xrightarrow{\alpha} E \to \text{Ra}^* L\alpha#E.$$

This triangle yields the typical Bockstein sequences one expects for reduction of coefficients.

**Proof.** Since $\text{Ra}^*$ and $L\alpha#$ commute with $\otimes MGE$, we may assume that $E$ is effective. Then $\text{Ra}^* L\alpha#E$ is computed by resolving $E$ by a complex of representable sheaves $\mathcal{C}^*$ and then $\mathcal{C}^*/(a)$ is a model for $\text{Ra}^* L\alpha#E$. (Note that since $\mathcal{C}^*$ has homotopy invariant cohomology, so does $\alpha#(\mathcal{C}^*/(a)$, by considering the Bockstein sequence. Hence we may apply $\alpha^*$ immediately to $\alpha#(\mathcal{C}^*$ instead of having to $\mathbb{A}^1$-localise first.) Since a is not a zero divisor the sequence $0 \to \mathcal{C}^* \to \mathcal{C}^*/(a) \to 0$ is exact and yields the desired triangle. □

With this preparation out of the way, we can prove our conservativity and Pic-injectivity theorem. Recall that $\text{Hom}_{\text{DM}(k, A)}(1, 1[i]) = A$ if $i = 0$ and = 0 else.

**Theorem 12.** Let $k$ be a perfect field and $A$ a PID of characteristic zero. Let $f : \text{Spec}(k^*) \to \text{Spec}(k)$ be a separable closure.

The collection of functors $\{f^*\} \cup \{L\alpha_{\pi#}\}$ is conservative. If $A$ has primes of arbitrary large characteristic, the collection is also Pic-injective (both on $\text{DM}^{-}(k, A)$). Here $\alpha_{\pi} : A \to A/[\pi]$ runs through the primes of $A$.

We could prove essentially the same theorem with $A$ replaced by a Dedekind domain (of characteristic zero) with only slightly more work.

**Proof.** We first show conservativity. As usual we may reduce to $\text{DM}^{\text{eff,-}}$. So let $E \in \text{DM}^{\text{eff,-}}(k, A)$ with $L\alpha_{\pi#}E = 0$ for all $\pi$ and $f^*E = 0$. We must show that $E = 0$. Let $T \in \text{DM}^{\text{eff, gm}}(k, A)$. It suffices to prove that $\text{Hom}(T, E) = 0$. Now by proposition 11 we have the triangle $E \xrightarrow{\pi} E \to \text{Ra}_{\pi#}(\alpha#E) = 0$. Thus multiplication by $\pi$ is an isomorphism on $\text{Hom}(T, E)$. Let $K = \text{Frac}(A)$. Since $\pi$ was arbitrary it follows that $\text{Hom}(T, E)$ is a $K$-vector space. Since $K \otimes_A K \neq 0$ we conclude that $\text{Hom}(T, E) = 0$ provided that $\text{Hom}(T, E) \otimes_A K = 0$. Let $\alpha_0 : A \to K$ be the (flat) localisation. By proposition 10 we know that $\text{Hom}(T, E) \otimes_A K = \text{Hom}(L\alpha_{0#}T, L\alpha_{0#}E)$, so it suffices to show that $L\alpha_{0#}E = 0$. But $K$ is of characteristic zero, so by proposition 8 it is enough to show that $f^*L\alpha_{0#}E = 0$. Since $L\alpha_{0#}$ and $f^*$ “commute”, this follows from the assumption that $f^*E = 0$. □
Now we prove Pic-injectivity. Let $E \in \text{DM}^\sim(k, A)$ be such that $f^*E \cong \mathbb{I}_{k^*}$ and $\text{Lo}_\pi^\# E \cong \mathbb{I}_{A/(\pi)}$. As a first step, I claim that there exists a finite extension $k \subset l \subset k^*$ such that $g^*E \cong \mathbb{I}_{l}$, where $g : \text{Spec}(l) \to \text{Spec}(k)$. Indeed it follows from the cancellation theorem that $\text{Hom}(\mathbb{I}_{k^*}, f^*E) = \colim_{k^* \subset l \subset k^*} \text{Hom}(\mathbb{I}_{l}, (l/k)^*E)$, where the colimit is over finite subextensions. Hence there exist $l$ and an element $t \in \text{Hom}(\mathbb{I}_{l}, g^*E)$ such that $(k^*/l)^*(t)$ is an isomorphism. The commutative diagram

\[
\begin{array}{cccc}
\text{Hom}(\mathbb{I}_{l}, g^*E) & \longrightarrow & \text{Hom}(\mathbb{I}_{k^*}, f^*E) \cong A \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{I}_{l,A/(\pi)}, \text{Lo}_\pi^\# g^*E) & \cong & \text{Hom}(\mathbb{I}_{k^*,A/(\pi)}, \text{Lo}_\pi^\# f^*E) \cong A/(\pi)
\end{array}
\]

shows that $\text{Lo}_\pi^\#(t)$ is an isomorphism. Thus by the first part (conservativity), $t$ is an isomorphism.

Now we consider $\text{Hom}(\mathbb{I}_{k}, E)$. From the Bockstein triangles and the assumption $\text{Lo}_\pi^\# \cong \mathbb{I}_{A/(\pi)}$ we get the exact sequences

\[
\text{Hom}(\mathbb{I}_{A/(\pi)}, \text{Lo}_\pi^\# E[-1]) = 0 \to \text{Hom}(\mathbb{I}_{k}, E) \xrightarrow{\partial} \text{Hom}(\mathbb{I}_{k}, E) \\
\to \text{Hom}(\mathbb{I}_{A/(\pi)}, \text{Lo}_\pi^\# E) \cong A/(\pi) \to \text{Hom}(\mathbb{I}_{k}, E[1])
\]

It follows that $\text{Hom}(\mathbb{I}_{k}, E)$ is a torsion-free $A$-module (hence abelian group). Thus by transfer it follows that $\text{Hom}(\mathbb{I}_{k}, E) \to \text{Hom}(\mathbb{I}_{l}, g^*E) \cong A$ is injective. Let us denote the image by $I \subset A$. This is a free $A$-module (of rank zero or one).

Since $\text{Hom}(\mathbb{I}_{k}, g^*(E)[1]) = 0$ it follows by transfer that $\text{Hom}(\mathbb{I}_{k}, E[1])$ is $|l : k|$-torsion. Choosing $\pi$ of sufficiently large characteristic, we find that $A/(\pi) \to \text{Hom}(\mathbb{I}_{k}, E[1])$ is the zero map. Thus $I = \text{Hom}(\mathbb{I}_{k}, E) \neq 0$, i.e. $I \cong A$. It follows that $\text{Hom}(\mathbb{I}_{k}, E) \to \text{Hom}(\mathbb{I}_{A/(\pi)}, \text{Lo}_\pi^\# E) \cong A/(\pi)$ is surjective for each $\pi$.

Consider the commutative diagram

\[
\begin{array}{cccc}
\text{Hom}(\mathbb{I}_{k}, E) & \longrightarrow & \text{Hom}(\mathbb{I}_{l}, g^*E) \cong A \\
\downarrow (\ast) & & \downarrow (\ast\ast) \\
\text{Hom}(\mathbb{I}_{A/(\pi)}, \text{Lo}_\pi^\# E) & \cong & \text{Hom}(\mathbb{I}_{l,A/(\pi)}, \text{Lo}_\pi^\# g^*E) \cong A/(\pi)
\end{array}
\]

The map (\ast\ast) is the natural surjection and (\ast) is surjective as we just proved. It follows that $I + (\pi) = A$ for each $\pi$ and so $I = A$. Thus there exists $t' \in \text{Hom}(\mathbb{I}_{k}, E)$ with $g^*(t') = t$ an isomorphism. Considering the diagram again one finds that $\text{Lo}_\pi^\#(t')$ is also an isomorphism. Thus $t'$ is an isomorphism and we are done.

We need two more auxiliary results. For the first, let $f : \text{Spec}(l) \to \text{Spec}(k)$ be a Galois extension with group $G$. If $M \in \text{DM}^\text{gm}_A(k, A)$ then the $A$-module $\text{Hom}(\mathbb{I}, f^*M) \cong \text{Hom}(\mathbb{I}, M(\text{Spec}(l)), M)$ has a natural action by $G$ (coming from automorphisms of $\text{Spec}(l)$). We denote this action by $\kappa_M : G \to Aut(\text{Hom}(\mathbb{I}, f^*M))$. 

Proposition 13. Let \( f : \text{Spec}(l) \to \text{Spec}(k) \) be (finite) Galois and \([l : k]\) invertible on \( A \). Then the above construction yields an injective homomorphism
\[
\kappa : \text{Ker}(f^* : \text{Pic}(\text{DM}^{gm}(k, A)) \to \text{Pic}(\text{DM}^{gm}(l, A))) \to A^\times.
\]

Proof. Suppose that \( M \in \text{DM}^{gm}(k, A) \), \( f^*M \simeq 1 \) and let us show that \( M \simeq 1 \) if and only if the action is trivial. Necessity is clear, we show sufficiency.

Independent of the assumptions on \([l : k]\) and \( M \) I claim we have the following: if \( t : 1_L \to f^*M \) is any morphism, then \( f^*(\text{tr}(t)) : 1_L \to f^*M \) is the sum of the conjugates under the \( G \)-action. Indeed the action on \( \text{Hom}_L(1_L, f^*M) \approx \text{Hom}_k(1_L, M) \) comes from premultiplication by elements of \( \text{Hom}_k(1_L, 1_L) \), whereas transfer comes from premultiplication with the adjunction morphism. Thus prove the claim we may assume that \( M = 1 \) and \( t = \text{id} \), in which case the result follows from [12 Exercise 1.11].

Thus reinstating our assumptions, let \( t : 1_L \to f^*M \) be an isomorphism and assume that the \( G \)-action is trivial. Then \( \text{tr}(t/[l : k]) : 1 \to M \) is an isomorphism since \( f^*(\text{tr}(t/[l : k])) = t \) is, by Proposition 13.

Finally we have to prove that \( \kappa \) is a homomorphism, i.e. that \( \kappa_{M \otimes N} = \kappa_M \kappa_N \). For this let us denote the adjunction isomorphism \( \text{Hom}_{\text{DM}(k,A)}(M(l), T) \to \text{Hom}_{\text{DM}(l,A)}(1, f^*T) \) generically by \( \text{ad} \). One checks that given \( f \in \text{Hom}(M(l), M), g \in \text{Hom}(M(l), N) \) then \( \text{ad}(f) \otimes \text{ad}(g) = \text{ad}((f \otimes g) \circ \alpha) \), where \( \alpha : M(l) \to M(l) \otimes M(l) \) is the map corresponding to \( l \otimes l \to l, a \otimes b \to ab \). Next observe that \( \alpha \) is \( G \)-equivariant if \( G \) acts diagonally on \( M(l) \otimes M(l) \). The result follows.

For the statement of the next result, we need \( \text{DM}(l, A) \) even if \( l \) is not perfect. It is explained in the next section what we mean by that. Under our assumptions on \( A \), it is equivalent to \( \text{DM}(\overline{l}, A) \), where \( \overline{l} \) is the perfect closure of \( l \).

Lemma 14. Let \( k \) be a perfect field, \( X/k \) a smooth variety, \( A \) a ring in which the exponential characteristic of \( k \) is invertible, and \( M \in \text{DM}(k, A) \).

If for all \( n \in \mathbb{Z} \) and all \( x \in X \) (not necessarily closed) we have that \( \text{Hom}_{\text{DM}(x,A)}(\overline{1} \{n\}, M_x) = 0 \), then also for all \( n \in \mathbb{Z} \) we have \( \text{Hom}_{\text{DM}(k,A)}(MX \{n\}, M) = 0 \).

Proof. We will prove the result by induction on \( \dim X \). Thus in order to prove it for \( X \) we may assume it proved for every smooth, locally closed \( X' \subset X \) with \( \dim X' < \dim X \) (because the residue fields of \( X' \) form a subset of those of \( X \)). If \( \dim X = 0 \) then \( X \) is a disjoint union of spectra of fields, and the result is clear.

To prove the general case, we may assume that \( X \) is connected. Let \( n \in \mathbb{Z} \) and \( \alpha \in \text{Hom}(MX \{n\}, M) \). It suffices to show that \( \alpha = 0 \). By considering the generic point and using continuity [5 Example 2.6(2)] we conclude that there exists a non-empty open subvariety \( U \subset X \) such that \( \alpha|_U = 0 \). Let \( Z = X \setminus U \).

If \( Z \) is empty there is nothing to do. Otherwise there exists a non-empty, smooth, connected open subvariety \( U_1 \subset Z \), since \( k \) is perfect.

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Let $Z' = Z \setminus U_1$, $U' = U \cup U_1 = X \setminus Z'$. Then $U'$ is smooth open in $X$ and we have $X \setminus U' = Z'$, which is strictly smaller than $Z$. We shall prove that $\alpha|_{U'} = 0$. By repeating this argument with $U$ replaced by $U'$ (i.e. Noetherian induction on $Z$) it will follow that $\alpha = 0$.

Note that $U_1 = U' \setminus U$ is closed in $U'$, say of codimension $c$. Thus we get the exact Gysin triangle

$$MU\{n\} \to MU'\{n\} \to MU_1\{n-c\}.$$ 

Now $\text{Hom}(MU_1\{n-c\}, M) = 0$ by the induction on dimension. Thus $\text{Hom}(MU'\{n\}, M) \to \text{Hom}(MU\{n\}, M)$ is injective. But $(\alpha|_{U'})|_{U} = \alpha|_{U} = 0$ by assumption, so $\alpha|_{U'} = 0$. \hfill \Box 

4 Weight Structures and the Geometric Fixed Points Functors

In this section, we will use Bondarko’s theory of weight structures to construct “generalised geometric fixed points functors” and prove that they have good properties. We shall fix a coefficient ring $\mathbb{F}$ on which an integer $e$ is invertible, and only work with fields of exponential characteristic $e$.

We shall have to deal with $\text{DM}(k, \mathbb{F})$ for $k$ an imperfect field. There is now a fairly complete theory of $\text{DM}(X, \mathbb{F})$ for Noetherian schemes over a field of exponential characteristic $e$ (assumed invertible in $\mathbb{F}$) [5]. It satisfies the six functors formalism, in particular continuity. We recall that if $k$ is an imperfect field with perfect closure $k^p$, then the pull back $\text{DM}(k, \mathbb{F}) \to \text{DM}(k^p, \mathbb{F})$ is an equivalence of categories [5, Proposition 8.1 (d)]. This means that essentially all properties known over perfect fields hold over imperfect fields as well. We also mention that all of the categories $\text{DM}(X, \mathbb{F})$ afford DG-enhancements. (This is well known if $k$ is a perfect field and hence holds for $k$ any field by the previous remark, and this is all we need. But it is actually clear that the constructions in [5] all yield DG categories.)

We shall work extensively in this section with weight structures [3], which we now review rapidly.

**Definition.** Let $\mathcal{C}$ be a triangulated category and $\mathcal{C}^{w \geq 0}, \mathcal{C}^{w \leq 0} \subset \mathcal{C}$ two classes of objects. We call this a weight structure if the following hold:

(i) $\mathcal{C}^{w \geq 0}, \mathcal{C}^{w \leq 0}$ are additive and Karoubi-closed in $\mathcal{C}$.

(ii) $\mathcal{C}^{w \geq 0} \subset \mathcal{C}^{w \geq 0}[1], \mathcal{C}^{w \leq 0}[1] \subset \mathcal{C}^{w \leq 0}$

(iii) For $X \in \mathcal{C}^{w \geq 0}, Y \in \mathcal{C}^{w \leq 0}$ we have $\text{Hom}(X, Y[1]) = 0$.

(iv) For each $X \in \mathcal{C}$ there is a distinguished triangle

$$B[-1] \to X \to A$$

with $B \in \mathcal{C}^{w \geq 0}$ and $A \in \mathcal{C}^{w \leq 0}$. \hfill \Box 


These axioms look quite similar to those of a $t$-structure, but in practice weight structures behave rather differently. We call a decomposition as in (iv) a weight decomposition. It is usually far from unique. We put $C^{w \geq n} = C^{w \geq 0}[-n]$ and $C^{w \leq n} = C^{w \leq 0}[-n]$. We also write $C^{w > n} = C^{w \geq n+1}$ etc. The intersection $C^{w=0} := C^{w \geq 0} \cap C^{w \leq 0}$ is called the heart of the weight structure.

A weight structure is called non-degenerate if $\cap_n C^{w \geq n} = 0 = \cap_n C^{w \leq n}$. It is called bounded if $\cup_n C^{w> n} = C = \cup_n C^{w \leq n}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between categories with weight structures is called $w$-exact if $F(C^{w \leq 0}) \subset D^{w \leq 0}$ and $F(C^{w \geq 0}) \subset D^{w \geq 0}$. It is called $w$-conservative if given $X \in \mathcal{C}$ with $F(X) \in D^{w \leq 0}$ we have $X \in C^{w \leq 0}$, and similarly for $w \geq 0$.

Note that a $w$-conservative functor on a non-degenerate weight structure is conservative.

In the following proposition we summarise properties of weight structures we use.

**Proposition 15.** (1) $C^{w \leq 0}$ and $C^{w \geq 0}$ are extension-stable: if $A \to B \to C$ is a distinguished triangle and $A, C \in C^{w \leq 0}$ (respectively $A, C \in C^{w \geq 0}$) then $B \in C^{w \leq 0}$ (respectively $B \in C^{w \geq 0}$).

Moreover $X \in C^{w \geq 0}$ if and only if $\text{Hom}(X,Y) = 0$ for all $Y \in C^{w < 0}$, and similarly $X \in C^{w \leq 0}$ if and only if $\text{Hom}(Y,X) = 0$ for all $Y \in C^{w > 0}$.

(2) Bounded weight structures are non-degenerate.

(3) If $\mathcal{C}$ admits a DG-enhancement and the weight structure is bounded, then there exists a $w$-exact, $w$-conservative triangulated functor

$$t : \mathcal{C} \to K^b(C^{w=0})$$

called the weight complex. Its restriction to $C^{w=0}$ is the natural inclusion.

(4) If the weight structure is bounded and $C^{w=0}$ is Karoubi-closed then so is $\mathcal{C}$.

(5) If $H \subset \mathcal{C}$ is a negative subcategory of a triangulated category (i.e. for $X,Y \in H$ we have $\text{Hom}(X,Y[n]) = 0$ for $n > 0$) generating it as a thick subcategory, then there exists a unique weight structure on $\mathcal{C}$ with $H \subset C^{w=0}$. Moreover $C^{w \leq 0}$ is the smallest extension-stable Karoubi-closed subcategory of $\mathcal{C}$ containing $\cup_n H[n]$, and similarly for $C^{w \geq 0}$. The weight structure is bounded and $C^{w=0}$ is the Karoubi-closure of $H$ in $\mathcal{C}$.

(6) If $\mathcal{D} \subset \mathcal{C}$ is a triangulated subcategory such that $D^{w \leq 0} := \mathcal{D} \cap C^{w \leq 0}$ and $D^{w \geq 0} := \mathcal{D} \cap C^{w \geq 0}$ define a weight structure on $\mathcal{D}$ (we say the weight structure restricts to $\mathcal{D}$) then the Verdier quotient $\mathcal{C}/\mathcal{D}$ affords a weight structure with $(\mathcal{C}/\mathcal{D})^{w \leq 0}$ the Karoubi-closure of the image of $C^{w \leq 0}$ in $\mathcal{C}/\mathcal{D}$, and similarly for $(\mathcal{C}/\mathcal{D})^{w \geq 0}$, $(\mathcal{C}/\mathcal{D})^{w=0}$.

The natural “quotient” functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ is $w$-exact. If $X,Y \in C^{w=0}$ then

$$\text{Hom}(QX,QY) = \text{Hom}(X,Y)/\Sigma_{Z \in \mathcal{D}^{w=0}} \text{Hom}(Z,Y) \circ \text{Hom}(X,Z).$$

The weight structure on $\mathcal{C}/\mathcal{D}$ is bounded if the one on $\mathcal{C}$ is.
Proof. (1) 3 Proposition 1.3.3 (1-3)]. (2) 3 Proposition 1.3.6 (3) and comment after proof. (3) 3 Proposition 3.3.1 (I), (IV) and Section 6.3]. (4) 3 Lemma 5.2.1]. (5) 3 Theorem 4.3.2 (II) and its proof. (6) 3 Proposition 8.1.1).

Weight exactness holds by definition of the weight structure on $C/\mathcal{D}$.

We shall call a triangulated category with a fixed weight structure a $w$-category.

Weight structures mostly come from “stupid truncation” of (generalised) complexes, and this intuition allows us to formulate many true results about weight structures. We point out some examples right away.

Lemma 16. Let $C$ be a $w$-category with heart $H$, and $H' \subset H$ an additive subcategory. Let $C'$ be the triangulated category generated by $H'$ inside $C$.

Then the weight structure of $C$ restricts to $C'$. In particular, if $X \in C'$ then we may choose a weight decomposition $A \to X \to X'$ (i.e. $A \in C_{w \geq 0}$ and $X' \in C_{w < 0}$) with $A, X' \in C'$. 

Proof. This is just Proposition 15 (5) which says that $C'$, being negatively generated by $H'$, carries a natural unique weight structure. By the description provided we find $C_{w \leq 0} \subset C_{w \leq 0}$, $C_{w \geq 0} \subset C_{w \geq 0}$. Hence a weight decomposition in $C'$ is also a weight decomposition in $C$. The rest follows from the definitions.

(It follows from the orthogonality characterisation that $C_{w \leq 0} = C_{w \leq 0} \cap C'$, but we do not need this.)

Lemma 17. Let $F : C \to D$ be a triangulated functor of $w$-categories, and assume that the weight structure on $C$ is bounded. Then $F$ is $w$-exact if and only if $F(C_{w=0}) \subset D_{w=0}$.

Proof. Necessity is clear, we show sufficiency. We find by induction that the subcategory of $C$ generated by $C_{w=0}$ contains $C_{w \leq n} \cap C_{w \geq -n}$ for all $n$, and hence all of $C$ by boundedness. It follows that the weight structure on $C$ is the one described in Proposition 15 (5), i.e. $C_{w \geq 0}, C_{w \leq 0}$ are obtained as extension closures of $\bigcup_{n \geq 0} C_{w=n}, \bigcup_{n \leq 0} C_{w=n}$. The result follows since $D_{w \geq 0}, D_{w \leq 0}$ are extension-stable.

Lemma 18. Let $C$ be a $w$-category which is also a tensor category. Assume that $1 \in C_{w=0}$ and that tensoring is weight-bi-exact, i.e. that $C_{w \leq 0} \otimes C_{w \leq 0} \subset C_{w \leq 0}$ and similarly for $C_{w \geq 0}$.

Then the weight complex functor is tensor whenever $C$ affords a tensor DG-enhancement and Pic-injective whenever additionally the weight structure is bounded.

If moreover $C$ is rigid then the dualisation $D : C^{op} \to C$ is $w$-exact (i.e. $D(C_{w \geq 0}) \subset C_{w \leq 0}$ and vice versa).

Proof. If $D$ is a negative DG tensor category, then $H^0(D)$ is tensor in a natural way and the weight complex functor $t$ manifestly respects the tensor structure. If $C$ is a tensor DG category with the property that $H^n(\text{Hom}(X, Y)) = 0$ for all
Proposition 19. The category \( D(S)^{\text{TM}}(l, \mathbb{F})/\langle S \rangle^{\text{tri}} \) carries natural weight and tensor structures, and \( \varphi_0^l \) is a \( w \)-exact tensor functor. The composite

\[
Tate(\mathbb{F}) \to D(S)^{\text{TM}}(l, \mathbb{F}) \to D(S)^{\text{TM}}(l, \mathbb{F})/\langle S \rangle^{\text{tri}}
\]

is a full embedding with essential image \( D(S)^{\text{TM}}(l, \mathbb{F})/\langle S \rangle^{\text{tri}} \cong D(S)^{\text{TM}}(l, \mathbb{F})/\langle S \rangle^{\text{tri}}^{w=0} \).
Proof. The existence of the weight structure and weight exactness is Proposition \ref{prop:weight_structure} (6). This also says that \((\mathcal{D}(S)\text{TM}(l, F)/\langle S \rangle^{trri})^{w=0}\) is generated as a Karoubi-closed category by \(\varphi^0_0(\mathcal{D}(S)\text{TM}(l, F)^{w=0})\). If \(M \in \mathcal{D}(S)\text{TM}(l, F)^{w=0} = \langle S \rangle^{\otimes T}_{\text{Chow}(l, F)}\) then we may write \(M \approx M' \oplus T\) with \(T\) a Tate and \(M' \in \langle S \rangle^{\otimes T}_{\text{Chow}(l, F)}\), by Proposition \ref{prop:weight_description}. Thus \(\varphi^0_0(M) \approx \varphi^0_0(T)\) and so \(\varphi^0_0 : \text{Tate}(F) \to (\mathcal{D}(S)\text{TM}(l, F)/\langle S \rangle^{trri})^{w=0}\) is essentially surjective up to Karoubi-completing. We shall show it is fully faithful whence its essential image is Karoubi-closed and so \(\varphi^0_0 : \text{Tate}(F) \to (\mathcal{D}(S)\text{TM}(l, F)/\langle S \rangle^{trri})^{w=0}\) will be an equivalence. But by the description in Proposition \ref{prop:weight_description} (6) it suffices to prove that any morphism between Tate objects factoring through \(\langle S \rangle^{\otimes}_{\text{Chow}(l, F)}\) is zero. This follows from Lemma \ref{lem:weight_zero}.

For the existence of the tensor structure we need \(\langle S \rangle^{trri} \otimes \mathcal{D}(S)\text{TM}(l, F) \subset \langle S \rangle^{trri}\); then \(\varphi^0_0\) is automatically tensor. Considering generators, it suffices to show that \(\langle S \rangle^\otimes_{\text{Chow}(l, F)} \otimes \langle S \rangle^\otimes_{\text{Chow}(l, F)} \subset \langle S \rangle^\otimes_{\text{Chow}(l, F)}\). This follows from Proposition \ref{prop:weight_description}.

Let \(l/k\) be any extension. We write \(\Phi^l : \mathcal{D}(S)\text{TM}(k, F) \to K^b(\text{Tate}(F))\) for the composite

\[
\Phi^l : \mathcal{D}(S)\text{TM}(k, F) \to \mathcal{D}(S)\text{TM}(l, F) \to \mathcal{D}(S)\text{TM}(l, F)/\langle S \rangle^{trri} \xrightarrow{t} K^b((\mathcal{D}(S)\text{TM}(l, F)/\langle S \rangle^{trri})^{w=0}) \approx K^b(\text{Tate}(F))
\]

of base change, the Verdier quotient functor \(\varphi^0_0\), and the weight complex \(t\). It is a \(w\)-exact triangulated tensor functor. We can now state the main theorem of this section.

**Theorem 20.** Let \(k\) be a ground field of exponential characteristic \(e\), \(F\) a finite field of characteristic \(p \neq e\). Suppose given for each field extension \(l/k\) a set \(S_l \subset \text{SmProj}(l)\) and a function \(ex = ex_l : S_l \to \mathbb{N}\). Assume that the following hold (for all fields \(l/k\)):

1. For \(x \in X \in S_l\) closed, \(p|\text{deg}(x)\).

2. If \(l'/l\) is a field extension and \(X \in S_l\) has no rational point over \(l'\), then \(X_{l'}\) is isomorphic to an object of \(S_{l'}\) and \(ex(X_{l'}) \leq ex(X)\).

3. If \(l'/l\) is a field extension and \(X \in S_l\) has a rational point over \(l'\), then \(MX_{l'}\) is a summand of a motive of the form

\[
T \oplus M(X^{(1)}_1 \otimes \cdots \otimes X^{(j)}_{n_1})\{i_1\} \oplus \cdots \oplus M(X^{(m)}_1 \otimes \cdots \otimes X^{(m)}_{n_m})\{i_m\},
\]

with \(T \in \text{Tate}(F), X^{(j)}_i \in S_{l'}\) and \(ex(X^{(j)}_i) < ex(X)\) for all \(i, j\).

Then the family \(\{\Phi^l\}_l\), as \(l\) runs through finitely generated extensions of \(k\) is \(w\)-conservative (so in particular conservative) and \(\text{Pic-injective}\). 

\[18\]
We note that (2) and (3) imply that $\langle S_l \rangle \otimes, T_{\text{Chow}}(l, F)$ are stable by base change, i.e. we are in the situation we have been discussing. Also (1) implies that none of the $X \in S_l$ have rational points over $l$. The somewhat obscure functions $e x_i$ are necessary to make an induction step in the proof work. We will always use $e x = \dim$ in applications.

Before proving the result we explain how to compute $\Phi$ in the case that $k$ is perfect (but $l$ need not be).

**Proposition 21.** Assume in addition that $k$ is perfect. Let $l/k$ be a field extension. There exists an essentially unique additive functor $\Phi_0 : \langle S_l \rangle \otimes, T_{\text{Chow}}(k, F) \to \text{Tate}(F)$ such that $\Phi_0|_{\text{Tate}(l, F)} = \text{id}$ and $\Phi_0(M) = 0$ if $M$ is Tate-free. It is tensor and the following diagram commutes (up to natural isomorphism; the lower horizontal arrow is base change of Chow motives):

\[
\begin{array}{ccc}
D\langle S \rangle \text{TM}(k, F) & \xrightarrow{t} & K^b(\text{Tate}(F)) \\
\downarrow & & \downarrow \\
K^b\left(\langle S_k \rangle \otimes, T_{\text{Chow}(k, F)}\right) & \xrightarrow{\phi_0} & K^b\left(\langle S_l \rangle \otimes, T_{\text{Chow}(l, F)}\right) \\
\end{array}
\]

**Proof.** Certainly $\Phi_0$ is essentially unique, using e.g. Proposition 7. The functor $t \circ \phi_0$ satisfies the required properties, so $\Phi_0$ exists. It is tensor by construction.

To establish the commutativity claim, consider the diagram

\[
\begin{array}{ccc}
D\langle S \rangle \text{TM}(k, F) & \xrightarrow{t} & K^b\left(\langle S_k \rangle \otimes, T_{\text{Chow}(k, F)}\right) \\
\downarrow & & \downarrow \\
D\langle S \rangle \text{TM}(l, F) & \xrightarrow{t} & K^b\left(\langle S_l \rangle \otimes, T_{\text{Chow}(l, F)}\right) \\
\phi'_0 \downarrow & & \phi'_0 \downarrow \\
D\langle S \rangle \text{TM}(l, F)/\langle S_l \rangle_{\text{tri}} & \xrightarrow{t} & K^b(\text{Tate}(F)). \\
\end{array}
\]

It suffices to prove that the two squares commute (up to natural isomorphism). This is most readily seen using DG-enhancements: let $D(r)$ be a functorial negative DG-enhancement of $\langle S_r \rangle \otimes, T_{\text{Chow}(r, F)} \subset D\langle S \rangle \text{TM}(r, F)$, for fields $r/k$. Then it suffices to establish strict commutativity of the diagram

\[
\begin{array}{ccc}
D(k) & \xrightarrow{t} & D(k)_0 \\
\downarrow & & \downarrow \\
D(l) & \xrightarrow{t} & D(l)_0 \approx \langle S_l \rangle \otimes, T_{\text{Chow}(l, F)} \\
\phi'_0 \downarrow & & \phi'_0 \downarrow \\
D(l)/\langle S_l \rangle_{\text{tri}} & \xrightarrow{t} & (D(l)/\langle S_l \rangle_{\text{tri}})_0 \approx \text{Tate}(F),
\end{array}
\]
where $D_0$ for a negative DG-category means zero-truncation. (Indeed the previous diagram is obtained by passing to $Ho(\text{Pre-Tr}(\bullet))$.) The upper square commutes by definition and the lower square commutes if and only if it commutes on zero morphisms, which is true essentially by definition of $\Phi^l_0$.

We establish Theorem 20 through a series of lemmas.

**Lemma 22.** Let $C$ be a $w$-category, $X \in C_{w \leq 0}$. Suppose given weight decompositions $A \rightarrow X \rightarrow X'$ (i.e. $A, B \in C_{w \geq 0}$, $X' \in C_{w < 0}$ and $X'' \in C_{w < -1}$).

Then $A, B \in C_{w = 0}$ and for $T \in C_{w = 0}$ there is a natural exact sequence

$$\text{Hom}(T, B) \rightarrow \text{Hom}(T, A) \rightarrow \text{Hom}(T, X) \rightarrow 0.$$  

**Proof.** We have $A, B \in C_{w = 0}$ by (the dual of) [3, Proposition 1.3.3 (6)]. There is an exact sequence

$$\text{Hom}(T, X'[-1]) \rightarrow \text{Hom}(T, A) \rightarrow \text{Hom}(T, X) \rightarrow \text{Hom}(T, X') = 0$$

where the last term is zero because $T \in C_{w \geq 0}, X' \in C_{w < 0}$. In particular $\text{Hom}(T, A) \rightarrow \text{Hom}(T, X)$ is surjective. Applying the same reasoning to $\text{Hom}(T, X'[-1])$ we find that $\text{Hom}(T, B) \rightarrow \text{Hom}(T, X'[-1])$ is surjective and hence

$$\text{Hom}(T, B) \rightarrow \text{Hom}(T, A) \rightarrow \text{Hom}(T, X) \rightarrow 0$$

is exact. This concludes the proof.

**Corollary 23.** Let $X \in \mathcal{D}(S)\mathcal{T}(k, F)_{w \leq 0}$ have a weight decomposition $T \rightarrow X \rightarrow X'$ with $T \in \text{Tate}(F)$ (and $X' \in \mathcal{D}(S)\mathcal{T}(k, F)_{w < 0}$). Suppose that $\varphi^k(X) \in \left(\mathcal{D}(S)\mathcal{T}(k, F)/(S_k)^{\text{tri}}\right)_{w < 0}$.

Then for $T' \in \text{Tate}(F)$ we have $\text{Hom}(T', X) = 0$.

**Proof.** Let $B[1] \rightarrow X' \rightarrow X''$ be a further weight decomposition. Naturality in the above lemma yields the following commutative diagram with exact rows

$$\begin{array}{ccc}
\text{Hom}(T', B) & \rightarrow & \text{Hom}(T', T) & \rightarrow & \text{Hom}(T', X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Hom}(\varphi^k(T'), \varphi^k(B)) & \rightarrow & \text{Hom}(\varphi^k(T'), \varphi^k(T)) & \rightarrow & \text{Hom}(\varphi^k(T'), \varphi^k(X)) & \rightarrow & 0.
\end{array}$$

Since $\varphi^k$ is weight exact we have $\text{Hom}(\varphi^k(T'), \varphi^k(X)) = 0$ and so $\delta$ is surjective. The construction of $\varphi^k$ (in particular Proposition 19) implies that $\alpha$ is surjective and $\beta$ is an isomorphism. It follows that $\gamma$ is surjective, whence $\text{Hom}(T', X) = 0$. This concludes the proof.

The main work in proving our theorem is the following lemma. We let $\varphi^l : \mathcal{D}(S)\mathcal{T}(k, F) \rightarrow \mathcal{D}(S)\mathcal{T}(l, F)/(S_l)^{\text{tri}}$ be the composite of $\varphi^l_0$ and base change.
Lemma 24. Let $X \in D(S)\text{TM}(k, \mathbb{F})^{w \leq 0}$ and suppose that for all $l/k$ finitely generated, $\varphi^l(X) \in (D(S)\text{TM}(l, \mathbb{F})/\langle S_{1, \text{tri}} \rangle)^{w < 0}$. Then $X \in D(S)\text{TM}(k, \mathbb{F})^{w < 0}$.

Proof. We begin by pointing out that Lemma 13 also applies if $k$ is not perfect. Indeed if $k^p/k$ is the perfect closure then $X_{k^p}$ is homeomorphic to $X$, so has the same set of points, and the residue field extensions of $X_{k^p} \to X$ are purely inseparable, so induce equivalences on $\text{DM}(?, \mathbb{F})$. Thus the Lemma holds over $k$ if and only if it holds over $k^p$.

Let $\mathcal{R}$ be the set of finite multi-subsets of $\mathbb{N}$ (i.e. the set of finite non-increasing sequences in $\mathbb{N}$). It is well-ordered lexicographically and so can be used for induction. We extend $ex$ to a function $ex: D(S)\text{TM}(l, \mathbb{F}) \to \mathcal{R}$. First, for $X_1, \ldots, X_n \in S_l$ put $ex(X_1, \ldots, X_n) = \{ex(X_1), \ldots, ex(X_n)\}$. Next, if $Y \in D(S)\text{TM}(l, \mathbb{F})$ then there exist $X_1, \ldots, X_n \in S_l$ such that $Y \in \langle Tate(\mathbb{F}), X_1, \ldots, X_n \rangle^{\text{tri}}$, i.e. $Y$ is in the thick tensor triangulated subcategory generated by the $M_X$ and the Tate motives. We let $ex(Y)$ be the minimum of $ex(X_1, \ldots, X_n)$ such that this holds. We shall abuse notation and write $ex(Y) = ex(X_1, \ldots, X_n)$ to additionally mean that $Y \in \langle Tate(\mathbb{F}), X_1, \ldots, X_n \rangle^{\text{tri}}$.

Let us observe that if $ex(Y) = ex(X_1, \ldots, X_n)$ and $l'/l$ is an extension in which one of the $X_i$ acquires a rational point, then $ex(Y_{l'}) < ex(Y)$, using assumptions (2) and (3).

We shall prove the result by induction on $ex(X)$. Note that is suffices to prove that there is a weight decomposition $A \xrightarrow{\alpha} X \to X'$ (i.e. $A \in D(S)\text{TM}(k, \mathbb{F})^{w = 0}$ and $X' \in D(S)\text{TM}(k, \mathbb{F})^{w < 0}$) with $\alpha = 0$ (because then $X' \approx X \oplus A[1]$ and so $X \in D(S)\text{TM}(k, \mathbb{F})^{w < 0}$, the latter being Karoubi-closed by definition).

If $ex(X) = \emptyset$ then $X$ must must be Tate. By Lemma 10 we may choose a weight decompostion $T \xrightarrow{\alpha} X \to X'$ with $T \in Tate(\mathbb{F})$. By the corollary above (applied to $T' = T$) we find that $\alpha = 0$. This finishes the base case of our induction.

Suppose now $ex(X) = ex(X_1, \ldots, X_n) > \emptyset$. If $l/k$ is any extension such that one of the $X_1, \ldots, X_n$ acquires a rational point over $l$, then we may assume the lemma proved over $l$ by induction, so $X_l \in D(S)\text{TM}(l, \mathbb{F})^{w < 0}$. Let $A \xrightarrow{\alpha} X \to X'$ be a weight decomposition; as before way may choose $A \in \{(X_1, \ldots, X_n)\}^{\otimes T}_{\text{Chow}(k, \mathbb{F})}$. Write $A \approx T \oplus A'$ as in Proposition 7. I claim that $\alpha|_{A'} = 0$. It is enough to show that if $Y$ is a product of the $X_i$ then $\text{Hom}(MY(n), X) = 0$ for all $n$. By Lemma 13 it is enough to show that for all $n \in \mathbb{Z}$ and $p \in Y$ we have that $\text{Hom}_{D(S)\text{TM}(p, \mathbb{F})}(1\{n\}, X_p) = 0$. But every variety has a rational point after base change to any one of its points, so $X_p \in D(S)\text{TM}(p, \mathbb{F})^{w < 0}$ by induction. This proves the claim.

We thus have a weight decomposition $T \oplus A' \xrightarrow{(\alpha, 0)} X \to X'$. Let $Y$ be a cone on $\alpha: T \to X$. We find that $X' \approx Y \oplus A'[1]$ and hence $Y \in D(S)\text{TM}(k, \mathbb{F})^{w < 0}$. Thus $T \xrightarrow{\alpha} X \to Y$ is a weight decomposition. Using the corollary again we get $\text{Hom}(T, X) = 0$ and so $\alpha = 0$. This finishes the induction step. \qed
The rest of Theorem 20 is relatively easy to establish now. We begin with the following.

**Lemma 25.** Let $\mathcal{C}$, $\mathcal{D}$ be $w$-categories with bi-$w$-exact tensor structures. Suppose that $\mathcal{C}$ is rigid and its weight structure is bounded.

Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a $w$-exact tensor functor such that whenever $X \in \mathcal{C}^{w \leq 0}$ and $\Phi(X) \in \mathcal{D}^{w < 0}$ then $X \in \mathcal{C}^{w < 0}$.

Then $\Phi$ is $w$-conservative.

**Proof.** Let $X \in \mathcal{C}$. If $\Phi(X) \in \mathcal{D}^{w \leq 0}$ then also $X \in \mathcal{C}^{w \leq 0}$. Indeed since the weight structure on $\mathcal{C}$ is bounded we have $X \in \mathcal{C}^{w \leq N}$ for some $N$. If $N > 0$ then the assumptions imply that $X \in \mathcal{C}^{w \leq N-1}$, and so on.

Suppose now instead that $\Phi(X) \in \mathcal{D}^{w \geq 0}$. We need to show that $X \in \mathcal{C}^{w \geq 0}$. But $X \in \mathcal{C}^{w \geq 0}$ if and only if $DX \in \mathcal{C}^{w \leq 0}$ by Lemma 18 (use that $X \approx D(DX)$), and $\Phi$ commutes with taking duals (since $\mathcal{C}$ is rigid). Thus $\Phi(DX) = D\Phi(X) \in \mathcal{D}^{w \leq 0}$, so $DX \in \mathcal{C}^{w \leq 0}$ and we are done. □

It follows that $\{ \varphi^l \}$ is a $w$-conservative family. But all our weight structures are bounded so the weight complex functors are $w$-conservative, and thus $\{ \Phi^l \}$ is also a $w$-conservative family.

Finally for Pic-injectivity, let $X \in \mathcal{D}(S)\mathcal{TM}(k, \mathbb{F})$ be invertible with $\Phi^l(X) \approx 1$ for all $l$. Since $1 \in K^b(Tate(\mathbb{F}))^{w=0}$, $w$-conservativity implies that $X \in \mathcal{D}(S)\mathcal{TM}(k, \mathbb{F})^{w=0} = \langle S_k \rangle_{\text{Chow}(k, \mathbb{F})}$. Write $X \approx T \oplus X'$, with $T$ Tate and $X'$ Tate-free. Then $1 \approx \Phi^k(X) = T$ and so $T \approx 1$. It follows that $\Phi^l(X) = 1 \oplus \Phi^l(X')$. For this to be invertible we need $\Phi^l(X') = 0$. Since this is true for all $l$, conservativity implies that $X' = 0$. This finishes the proof of Theorem 20.

## 5 Application 1: Invertibility of Affine Quadrics

We now begin to reap in the benefits of the work of the previous sections. First we construct the conservative and Pic-injective collection of functors we shall use in the remainder of this work. After that we study invertibility of affine quadrics.

We will be dealing with quadratic forms. If $l$ is a field and $\phi$ is a non-degenerate quadratic form over $l$, we write $Y_{\phi} = \text{Proj}(\phi = 0)$ for the projective quadric. This does not really make sense if $\dim\phi = 1$ in which case we put $Y_0 = \emptyset$ by convention. Given $a \in l^\times$ we put $Y_{a\phi} = \text{Proj}(\phi = aZ^2)$ and $X_{a\phi} = \text{Spec}(\phi = a)$. All of these varieties are smooth.

Fix a perfect field $k$ of exponential characteristic $e \neq 2$ and coefficient ring $A$ containing $1/e$. We denote by $\text{QM}(k, A)$ the Karoubi-closed tensor subcategory of $\text{Chow}(k, A)$ generated by the smooth projective quadrics over $k$. Note that $1\{1\} \in \text{QM}(k, A)$.

By [12 Property (14.5.6)] the category $\text{Chow}(k, A)$ embeds into $\text{DM}^{gm}(k, A)$. We write $\text{DQM}^{gm}(k, A)$ for the thick triangulated subcategory of $\text{DM}^{gm}(k, A)$ generated by $\text{QM}(k, A)$. This is a tensor category.
We write $\text{QM}(k) = \text{QM}(k, \mathbb{Z}[1/e])$ and $\text{DQM}^{gm}(k) = \text{DQM}^{gm}(k, \mathbb{Z}[1/e])$. As promised, these categories contain all (smooth) affine quadrics.

**Lemma 26.** If $\phi$ is a non-degenerate quadratic form over the perfect field $k$ of characteristic not two, and $a \in k^\times$, then the affine quadric $X^a_\phi$ satisfies $M(X^a_\phi) \in \text{DQM}^{gm}(k, A)$.

*Proof.* We have $X^a_\phi = Y^a_\phi \setminus Y_\phi$ and $M(Y^a_\phi), M(Y^a_\psi), \mathbb{I}\{1\} \in \text{DQM}^{gm}(k, A)$, so the result follows from the Gysin triangle. \qed

We recall the following result.

**Lemma 27 (Rost).** Let $\phi$ be an isotropic non-degenerate quadratic form. Then there exists a non-degenerate form $\psi$ such that

$$M(Y^a_\phi) \approx \mathbb{I} \oplus M(Y^a_\psi)\{1\} \oplus \mathbb{I}\{\dim Y_\phi\}. $$

Moreover for $a \in k^\times$ the natural inclusion $M(Y_\phi) \to M(Y^a_\phi)$ is given by

$$
\begin{pmatrix} 
\text{id} & 0 & 0 \\
0 & 0 & 0 \\
0 & s\{1\} & i\{1\}
\end{pmatrix} : \mathbb{I} \oplus \mathbb{I}\{\dim Y_\phi\} \oplus M(Y_\psi)\{1\} \to \mathbb{I} \oplus \mathbb{I}\{\dim Y_\phi + 1\} \oplus M(Y^a_\psi)\{1\},
$$

where $i : M(Y_\phi) \to M(Y^a_\phi)$ is the natural inclusion and $s : \mathbb{I}\{\dim Y^a_\phi\} \to M(Y^a_\psi)$ is the fundamental class (dual of the structure map).

*Proof.* This is basically [16 Proposition 2]. Rost starts with $\phi = \mathbb{H} \perp \psi$, but this is equivalent to $\phi$ having a rational point.

For the explicit form of the inclusion, note first that all matrix entries shown as zero have to be so for dimensional reasons. The entries “id” and “$i\{1\}$” follow from naturality of Rost’s construction. For the final entry, we can argue as follows. Note that $\mathbb{Z} = CH^0(Y^a_\phi) = \text{Hom}(\mathbb{I}\{\dim Y^a_\phi + 1\}, MY^a_\psi)\{1\}) \approx \text{Hom}(\mathbb{I}\{\dim Y^a_\phi + 1\}, MY_\phi) = CH^1(Y_\phi).$ The induced map we are interested in corresponds under this identification to the cycle class of the closed subvariety $Y_\phi \subset Y^a_\phi$. So up to verifying a sign, it is enough to show that this class is a generator, which one sees for example by considering the embedding into ambient projective space. \qed

**Lemma 28.** For a field extension $l/k$ let $S_l$ be the set of anisotropic projective smooth quadrics over $l$, and let $ex : S_l \to \mathbb{N}$ be the dimension function $ex(X) = \dim X$. Then Theorem 20 applies, with $F = \mathbb{F}_2$.

*Proof.* Points on an anisotropic quadric have degree divisible by two by Springer’s theorem [11 Chapter 7, Theorem 2.3], hence condition (1) holds. Condition (2) is satisfied essentially by definition. Finally condition (3) follows from Lemma 27. \qed

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It follows from Lemma \[27\] that motives of quadrics are geometrically Tate. Let \( f : \text{Spec}(k^s) \to \text{Spec}(k) \) be a separable closure. It follows that the weight complex functor \( t : \text{DQM}^{\text{gm}}(k^s) \to K^b(\text{Chow}(k^s, \mathbb{Z}[1/e])) \) takes values in \( K^b(\text{Tate}(\mathbb{Z}[1/e])) \). We write \( \Psi \) for the composite

\[
\Psi : \text{DQM}^{\text{gm}}(k) \xrightarrow{\Phi} \text{DQM}^{\text{gm}}(k^s) \xrightarrow{t} K^b(\text{Tate}(\mathbb{Z}[1/e])).
\]

Let \( g : \text{Spec}(l) \to \text{Spec}(k) \) be any field extension and \( \alpha : \mathbb{Z}[1/e] \to \mathbb{F}_2 \) be the natural surjection. Via Lemma \[28\] and Theorem \[20\] we obtain functors \( \Phi^l : \text{DQM}^{\text{gm}}(k, \mathbb{F}_2) \to K^b(\text{Tate}(\mathbb{F}_2)) \). We abuse notation and denote the composite with change of coefficients \( \text{DQM}^{\text{gm}}(k) \xrightarrow{L_\alpha} \text{DQM}^{\text{gm}}(k, \mathbb{F}_2) \to K^b(\text{Tate}(\mathbb{F}_2)) \) also by \( \Phi^l \).

**Theorem 29.** The functors \( \Psi, \Phi^l \) are tensor triangulated. Together (as \( l \) ranges over all finitely generated extension of \( k \)) they are conservative and Pic-injective.

**Proof.** The functors are composites of tensor triangulated functors, so are tensor triangulated.

By Theorem \[12\] the collection \( f^*, \{L_\alpha\}_p \) (where \( p \) ranges over all primes) is conservative and Pic-injective. Since all weight complex functors are conservative and Pic-injective by Lemma \[18\] the collection \( tf^*, \{tL_\alpha\}_p \) is conservative and Pic-injective. We have \( tf^* = \Psi \). By Theorem \[20\] we may replace \( tL_\alpha \) in our collection by \( \{\Phi^l\}_l \).

It remains to deal with \( L_\alpha \) at odd \( p \). Let \( M \in \text{DQM}^{\text{gm}}(k, \mathbb{Z}[1/e]) \). By repeated application of Lemma \[27\] we can find an extension \( L/k \) (which we may assume Galois) of degree a power of 2, such that \( M_L \) is in the triangulated subcategory generated by the Tate motives. In particular \( t(L_\alpha\# M_L) \approx L_\alpha\# \Psi(M) \) (as complexes of Tate motives). Since \([L : k] \) is a power of two, base change along \( L/k \) is conservative in odd characteristic by Proposition \[9\] Thus if \( \Psi(M) \simeq 0 \) then also \( L_\alpha\# M \simeq 0 \) and our collection is conservative.

We need to work a bit harder for Pic-injectivity. Let \( M \in \text{DQM}^{\text{gm}}(k, \mathbb{Z}[1/e]) \) be invertible with \( \Phi^l(M) \simeq 1[0] \) for all \( l/k \) and \( \Psi(M) \simeq 1[0] \). Then we know that \( L_\alpha\# (M) \simeq 1 \) by Theorem \[20\]. We also have \( t(M_L) = \Psi(M) \simeq 1 \), so \( M_L \simeq 1 \) by Lemma \[18\] Consider the mod 2 Bockstein sequence

\[
\text{Hom}(1, L_\alpha\# M[-1]) = 0 \to \text{Hom}(1, M) \xrightarrow{2} \text{Hom}(1, M) \to \text{Hom}(1, L_\alpha\# M) \to \text{Hom}(1, M[1]) \xrightarrow{2} \text{Hom}(1, M[1]) \to \text{Hom}(1, L_\alpha\# M[1]) = 0.
\]

The extremal terms are zero because \( L_\alpha\# M \simeq 1 \), and for the same reason we have that \( \text{Hom}(1, L_\alpha\# M) \simeq \mathbb{F}_2 \). Thus \( \text{Hom}(1, M) \) has no 2-torsion, whereas \( \text{Hom}(1, M[1]) \) has no 2-cotorsion. The composite \( M \to M_L \to M \) of base change and transfer is multiplication by \([L : k] = 2^N\). We conclude that \( \text{Hom}(1, M) \) injects into \( \text{Hom}_L(1_L, M_L) \simeq \mathbb{Z}[1/e] \) and that the kernel of \( \text{Hom}(1, M[1]) \to \text{Hom}_L(1_L, M_L[1]) = 0 \) (i.e. the whole group) is contained in the \( 2^N \)-torsion. But multiplication by 2 is surjective on \( \text{Hom}(1, M[1]) \), whence
that $A$ applied to $L_\beta$ has a kernel index 2, i.e. corresponds to a quadratic subextension.

Consequently we have $\text{Hom}(\mathbb{1}, M) \approx \mathbb{Z}[1/e]$ (since it is an ideal of $\mathbb{Z}[1/e]$ with a non-vanishing quotient, i.e. $\mathbb{F}_2$).

We shall now apply Proposition 13. As we have seen $M_L \cong \mathbb{1}$, so we obtain a $G = \text{Gal}(L/k)$-action on $\text{Hom}(\mathbb{1}, M_L) \approx \mathbb{Z}[1/e]$, i.e. a group homomorphism $\kappa_M : G \to \mathbb{Z}[1/e]^\times$. Since $e$ is prime we have $\mathbb{Z}[1/e]^\times = \{\pm 1\} \times \{e^k| k \in \mathbb{Z}\}$ and since $G$ is finite the image of $\kappa_M$ must be contained in $\{\pm 1\}$. Note that if $\kappa_M = 1$ then $M \cong \mathbb{1}$ and we are done. Indeed it suffices by Theorem 22 to show that $L\alpha_{p^\#}M \cong \mathbb{1}$ for odd $p$. Since $(L\alpha_{p^\#}M)_L \cong \mathbb{1}$, by Proposition 13 this happens if and only if an appropriate Galois action is trivial, but this action is just the reduction $G \overset{\kappa_M}{\to} \mathbb{Z}[1/e]^\times \to (\mathbb{Z}/p)^\times$. So assume now that $\kappa_M$ is non-trivial.

Let $\beta : \mathbb{Z}[1/e] \to \mathbb{Z}[1/(2e)]$ be the natural map. Note that $\kappa_M : G \to \{\pm 1\}$ has a kernel index 2, i.e. corresponds to a quadratic subextension $k \subset k_2 \subset L$. I claim that $L\beta_{\#}M \cong L\beta_{\#}MSpec(k_2)$. Indeed this follows from Proposition 13 applied to $A = \mathbb{Z}[1/(2e)]$, where $f^*$ becomes conservative, and the observation that $\kappa_{MSpec(k_2)} = \kappa_M$.

In particular we must have $\text{Hom}(\mathbb{1}, M_{\beta_{\#}MSpec(k_2)}) \approx \text{Hom}(\mathbb{1}, M) \otimes_{\mathbb{Z}[1/e]} \mathbb{Z}[1/(2e)] = \mathbb{Z}[1/(2e)]$, by Proposition 10 and our previous computation. But one may compute easily that $\text{Hom}(\mathbb{1}, M_{\beta_{\#}MSpec(k_2)}) = 0$. This contradiction concludes the proof.

We can now prove that affine quadrics are invertible. Recall the reduced motive $\tilde{M}(X) = \text{cone}(M(X) \to M(\text{Spec}(k)))[-1]$.

**Theorem 30.** Let $k$ be a perfect field of characteristic not two, $\phi$ a non-degenerate quadratic form over $k$ and $a \in k^\times$. Then $\tilde{M}(X^a_\phi)$ is invertible in $\text{DM}^{gm}(k)$.

**Proof.** We have $\tilde{M}X^a_\phi \in \text{DQM}^{gm}(k)$ by Lemma 26 and we can use Theorem 29. Since the category $\text{DQM}^{gm}(k)$ is generated by rigid objects (Chow motives) it is rigid and so conservative tensor functors detect invertibility, by standard arguments. We thus need to show that $\Psi(\tilde{M}X^a_\phi)$ is invertible and that for each $l/k$, $\Phi^l(\tilde{M}X^a_\phi)$ is invertible. The functor $\Psi$ is computed by first applying geometric base change, so $\phi$ becomes split and the invertibility follows as before from [2, Theorem 2].

Dealing with $\Phi^l$ is a bit harder. Let $d + 2 = \dim \phi$. Let us put $V^a_\phi = D(MX^a_\phi)(d + 1)$ and $\tilde{V}^a_\phi = D(\tilde{M}X^a_\phi)$. From the closed inclusion $i : Y_\phi \to Y^a_\phi$, with complement $X^a_\phi$ we get the dual Gysin triangle

$MY^a_\phi \overset{i}{\to} MY^a_\phi \to V^a_\phi$.

It follows that $t(V^a_\phi) = [MY^a_\phi \overset{i}{\to} \tilde{M}Y^a_\phi]$. Here the dot is used to indicate the term of degree zero in the chain complex. Dualising the defining triangle of $\tilde{M}X^a_\phi$ we obtain

$\text{cone}(d + 1) \overset{a}{\to} V^a_\phi \to \tilde{V}^a_\phi$.

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where $s$ is the fundamental class (dual of the structure map). Hence we finally obtain
\[
\varphi = [MY_\phi \oplus \mathbb{1}\{d+1\}] \xrightarrow{(i,s)} MY_\psi^0 =: C(\phi).
\]

The expression $C(\phi) \in K^b(\text{QM}(k))$ makes sense even if $k$ is not perfect. Using Proposition 21 it suffices to prove: if $l/k$ is any field extension, then $\Phi^l_0 C(\phi_l)$ is invertible. We drop the subscript zero from now on. We may as well prove: if $k$ is any field and $\phi$ is any non-degenerate quadratic form over $k$, then $\Phi^k(C(\phi))$ is invertible. By Lemma 31 below, if $\phi \approx \psi \perp \mathbb{H}$ then $C(\phi) \simeq C(\psi)\{1\}$. We may thus assume that either $\phi$ is anisotropic, or $\phi = \mathbb{H}$, or $\phi$ is of dimension one.

If $\phi = \mathbb{H}$ then $Y_\phi \approx \text{Spec}(k) \bigsqcup \text{Spec}(k)$, $Y_\phi^0 \approx \mathbb{P}^1$ and the result follows easily. If $\phi$ is of dimension one then $MY_\phi = 0$ and either $MY_\phi^0 = \mathbb{1} \oplus \mathbb{1}$ or $MY_\phi^0 = M(k')$, where $k'/k$ is a quadratic extension. Again the result follows easily.

So we may assume that $\phi$ is anisotropic. There are three cases. If $\phi \perp \langle -a \rangle$ is also anisotropic, then none of $MY_\phi, MY_\phi^0$ afford Tate summands, by Proposition 7. Thus $\Phi^k(C(\phi)) = \mathbb{1}\{d+1\}[1]$ is invertible.

If $\phi \perp \langle -a \rangle$ is isotropic, then $\phi \perp \langle -a \rangle = \psi \perp \mathbb{H}$. Suppose that $\psi$ has dimension greater than one. Then by (the contrapositive of) Lemma 32 below, $\psi$ is anisotropic. It follows that $MY_\phi = \mathbb{1} \oplus \mathbb{1}\{d+1\} \oplus MY_\psi^0$ and $\Phi^k(C(\phi)) = \mathbb{1}\{d+1\} \to \mathbb{1} \oplus \mathbb{1}\{d+1\}$. The component $\mathbb{1}\{d+1\} \to \mathbb{1}\{d+1\}$ comes from the fundamental class of $MY_\psi^0$ and so is an isomorphism. Thus $\Phi^k(C(\phi)) \simeq \mathbb{1}$ is invertible.

Finally it might be that $\psi$ has dimension one. Then $Y_\phi^0 \approx \mathbb{P}^1$ whereas $MY_\phi$ affords no Tate summands, and the result follows as in the case of dimension greater than one. This concludes the proof.

\begin{lemma}
Notation as in the theorem. If $\phi \perp \mathbb{H}$ then $C(\phi) \simeq C(\psi)\{1\}$.
\end{lemma}

\begin{proof}
Using the explicit form for the inclusion $MY_\phi \to MY_\phi^0$ from Lemma 27 we find that
\[
C(\phi) = [(\mathbb{1} \oplus \mathbb{1}\{d\} \oplus MY_\psi\{1\}) \oplus \mathbb{1}\{d+1\}] \xrightarrow{\alpha} \mathbb{1} \oplus \mathbb{1}\{d+1\} \oplus MY_\psi^0\{1\},
\]
where $\alpha$ is given by the matrix
\[
\begin{pmatrix}
id & 0 & 0 & 0 \\
0 & 0 & 0 & f \\
0 & s\{1\} & i\{1\} & 0
\end{pmatrix}.
\]
Here $f$ comes from the fundamental class and so is an isomorphism. It follows that $C(\phi) \simeq C(\psi)\{1\} \oplus \text{cone}(\text{id}\{1\})[-1] \oplus \text{cone}(\text{id}\{d+1\})[-1] \simeq C(\psi)\{1\}$. This is the desired result.
\end{proof}

\begin{lemma}
If $\phi \perp \langle a \rangle \approx \psi \perp \mathbb{H} \perp \mathbb{H}$, then $\phi$ is isotropic.
\end{lemma}
Proof. Let $X = Y_{\phi}\perp \phi$. Then $Y_{\phi} = X \cap \{X_0 = 0\}$. Since $\langle a \rangle \perp \phi \approx \psi \perp H \perp H$, we find that $Y_{H \perp H} \subset X$. Then $Y_{\phi} \cap Y_{H \perp H} = Y_{H \perp H} \cap \{X_0 = 0\}$ (intersecting inside $X$). Now we know that after a linear change of coordinates $(X_0 : \cdots : X_r) \mapsto (T_0 : \cdots : T_r)$ the subvariety $Y_{H \perp H}$ of $X$ is given by the equations $T_0T_1 + T_2T_3 = 0, T_i = 0$ for $i > 3$. Thus $Y_{\phi} \cap Y_{H \perp H}$ is obtained by adding a further linear constraint in the $T_0, T_1, T_2, T_3$. It is easy to see that there must be a rational, non-zero solution, so $Y_{\phi}$ has a rational point. This was to be shown.

6 Application 2: Po Hu’s Conjectures for Motives

In this final section we prove a version for motives of Po Hu’s conjectures [10–12, Conjecture 1.4]. We retain notation from the previous section.

For $\underline{a} = (a_1, \ldots, a_n) \in (k^\times)^n$, $b \in k^\times$ let us put

$$U^b_{\underline{a}} = X^b_{\langle (a_1, \ldots, a_n) \rangle},$$

where $\langle (a_1, \ldots, a_n) \rangle$ is the $n$-fold Pfister quadric associated with the symbol $\underline{a}$. We use notation such as $\underline{a}, a' = (a_1, \ldots, a_n, a') \in (k^\times)^{n+1}$ for concatenation of tuples.

**Theorem 33.** Let $k$ be a perfect field of characteristic not two, and $\underline{a} \in (k^\times)^n, b \in k^\times$.

In $DM^{2m}(k)$ there is an isomorphism

$$\hat{M}(U^1_{\underline{a}, b}) \otimes \hat{M}(U^b_{\underline{a}})[1] \approx \hat{M}(U^1_{\underline{a}})(2^n).$$

To prove this, we have to recall some facts about Rost motives. If $\underline{a} \in (k^\times)^n$, then there is the associated Rost motive $R_{\underline{a}} \in \QM(k)$. Recall that one has $H^1_{et}(k, E_2) = k^\times/2$, and hence cup product yields a natural map $\partial = \partial^b : (k^\times)^n \to H^2_{et}(k, \Z/2)$. The Rost motives have the remarkable property that $R_{\underline{a}}$ is irreducible if and only if $\partial(\underline{a}) \neq 0$. In fact there are canonical maps

$$\mathbb{1}(2^n - 1) \to R_{\underline{a}} \to \mathbb{1}$$

(which we call structure maps) and if $\partial(\underline{a}) = 0$ then this is a splitting distinguished triangle. The same statements hold true with $E_2$ coefficients. These results follow from the work of a number of people, see [13] for an overview.

The relationship between Rost motives and $U^b_{\underline{a}}$ is encapsulated in the following proposition.

**Proposition 34.** For $\underline{a} \in (k^\times)^n, b \in k^\times$ there is a distinguished triangle

$$\hat{M}(U^b_{\underline{a}}) \to R_{\underline{a}, b} \to R_{\underline{a}}(2^n - 1) \oplus 1.$$ 

Here $R_{\underline{a}, b} \to \mathbb{1}$ is the structure map, and the composite

$$\mathbb{1}(2^n - 1) \to R_{\underline{a}, b} \to R_{\underline{a}}(2^n - 1)$$

is the $(2^n - 1)$ twist of the structure map $\mathbb{1}(2^n - 1) \to R_{\underline{a}}$. 

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Proof. This is essentially [11, proof of Proposition 5.5].

We know that $U := U^b_a$ is the complement of $X := Y_b(a)$ in $Y := Y^b_{a(b)}$. By the work of Rost [16, Theorem 17 and Proposition 19], if we put $R_n := R_{a,b}$ and $R_{n-1} = R_\omega$, then

$$M(Y) = R_n \oplus \bigoplus_{k=1}^{2^{n-1}-1} R_{n-1}\{k\} = R_n \oplus R', \quad M(X) = \bigoplus_{k=0}^{2^{n-1}-1} R_{n-1}\{k\} = R_{n-1} \oplus R'$$

and the natural map $M(X) \to M(Y)$ is the identity on $R'$. The localisation triangle $M^c(X) = M(X) \to M^c(Y) = M(Y) \to M^c(U)$ fits into the following commutative diagram of (distinguished) triangles:

$$
\begin{array}{ccc}
R' & \longrightarrow & R' \\
\downarrow & & \downarrow \\
M(X) & \longrightarrow & M(Y) \longrightarrow M^c(U) \\
\downarrow & & \downarrow \\
R_{n-1} & & R_n
\end{array}
$$

An application of the octahedral axiom yields a distinguished triangle $R_{n-1} \to R_n \to M^c(U)$. Noting that $DM^c(U) = M(U)\{-(2^n - 1)\}$, $DR_n = R_n\{-(2^n - 1)\}$ and $DR_{n-1} = R_{n-1}\{-(2^{n-1} - 1)\}$, by dualising and twisting the triangle, we find a distinguished triangle $M(U) \to R_n \to R_{n-1}\{2^{n-1}\}$. Adding in the copy of $\mathbb{1}$ implied in $M(U)$, we get the claimed triangle with the correct map $R_n \to \mathbb{1}$.

To see the second claim about the differential, the important point is that in the triangle $R_{n-1} \to R_n \to M^c(U)$ the map $R_{n-1} \to R_n$ is induced from the inclusion $M(X) \to M(Y)$ by passing to the appropriate summands. It follows that $R_{n-1} \to R_n \to \mathbb{1}$ is the structure map of $R_{n-1} \to \mathbb{1}$. The desired result now follows by dualising. \qed

Proof of Theorem 33. By Lemma 26, we have $\tilde{M} (U^b_a) \in DQM_{gm}(k)$, etc. We also know by Theorem 30 that both sides of equation (2) are invertible. Hence if $F : DQM_{gm}(k) \to C$ is a Pic-injective functor, it suffices to prove that $F(LHS) \approx F(RHS)$.

Of course we use the Pic-injective collection from Theorem 30.

From Proposition 34 we know that $t(\tilde{M} (U^b_a) = [\hat{R}_{a,b} \to R_\omega\{2^n\} \oplus \mathbb{1}],$

and we also know certain things about the differential. To compute $\Psi$, we have to consider geometric base change, where the triangle (3) is splitting distinguished. One obtains

$$
\Psi(\tilde{M} (U^b_a)) = [\mathbb{1} \oplus \mathbb{1}\{2^n - 1\} \to \mathbb{1}\{2^{n-1}\} \oplus \mathbb{1}\{2^n - 1\} \oplus \mathbb{1}]$

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\[
\partial_l(a, b) \neq 0 \quad \text{or} \quad \partial_l(a, b) \neq 0 \quad \text{but} \quad \partial_l(a) \neq 0
\]

Table 1: Terms needed to compute \(\Phi^l\).

| \(\Phi^l(U_{a,b}^1)\) | \(\mathbb{I} \{2^{n+1} - 1\} \to \mathbb{I}\) | \(\mathbb{I} \{2^n\} \oplus \mathbb{I} \{2^{n+1} - 1\} \oplus \mathbb{I}\) |
| \(\Phi^l(U_{a,b}^2)\) | \(\mathbb{I} \to \mathbb{I}\) | \(\mathbb{I} \{2^n\} \to \mathbb{I}\) |
| \(\Phi^l(U_{a}^1)\) | \(\mathbb{I} \{2^n - 1\} \to \mathbb{I}\) | \(\mathbb{I} \{2^n - 1\} \to \mathbb{I}\) |

Table 2: Terms needed to compute \(\Phi^l\), simplified form.

| \(\Phi^l(U_{a,b}^1)\) | \(\mathbb{I} \{2^{n+1} - 1\} \to \mathbb{I}\) | \(\mathbb{I} \{2^n\} \to \mathbb{I}\) |
| \(\Phi^l(U_{a,b}^2)\) | \(\mathbb{I} \{2^n\} \to \mathbb{I}\) | \(\mathbb{I} \{2^n - 1\} \to \mathbb{I}\) |

and from the information about the differential given in proposition 34 we deduce that \(\Psi(U_{a,b}^1) \simeq \mathbb{I} \{2^n\} \to \mathbb{I}\). Thus \(\Psi(LHS) \approx \Psi(RHS)\) reads

\[
\mathbb{I} \{2^n\} \to \mathbb{I} \{2^n\} \otimes \mathbb{I} \{2^n\} \to \mathbb{I} \{2^n\} \to \mathbb{I} \{2^n\},
\]

which is certainly true.

Now let \(l/k\) be an arbitrary field extension. We need to prove \(\Phi^l(LHS) \approx \Phi^l(RHS)\). This involves \(\mathbb{R}_{a}, \mathbb{R}_{a,b}, \mathbb{R}_{a,1}\), and \(\mathbb{R}_{a,b,1}\). Depending on \(l\) these may or may not split into Tate motives, so may or may not survive \(\Phi\). We see that \(\mathbb{R}_{a,1}\) and \(\mathbb{R}_{a,b,1}\) always split (because \(\partial^l(1) = 0\)), and that \(\mathbb{R}_{a,b}\) splits whenever \(\mathbb{R}_{a}\) splits (because \(\partial(a,b) = \partial(a) \cup \partial(b)\)).

If \(\mathbb{R}_{a}\) splits then everything is split and \(\Phi^l\) is just mod two reduction of \(\Psi\), so we know the equation is satisfied. Thus there are just two cases and three things in each to compute, which we gather in Table 1.

The differentials can again be figured out using Proposition 34. Using these one can simplify the expressions. We have gathered the results in Table 2.

To complete the proof, we check that \(\Phi^l(LHS) \approx \Phi^l(RHS)\) in both cases.

This is easy.

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