PATH INTEGRAL TECHNIQUES ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we will prove a finite dimensional approximation scheme for the Wiener measure on closed Riemannian manifolds, establishing a generalization for $L_1$-functionals, of the approach followed by Andersson and Driver on [2]. This scheme is motivated by the measure theoretic techniques of [15]. Moreover, we will embed the concept of stochastic line integral in this scheme. This concept will propitiate some applications of path integration in Riemannian manifolds that provides with an alternative formulation of classical geometric concepts bringing to them an original point of view.

1. Introduction

In 1920, N. Wiener, based on Daniell notion of integral [5, 6, 7], defined in [21] an integral for bounded and continuous functionals $F : C_{x_0}[a, b] \to \mathbb{R}$, where the notation $C_{x_0}[a, b]$ stands for the space of continuous functions $u : [a, b] \to \mathbb{R}$ satisfying $u(a) = x_0$. In later papers [21, 22, 23, 24, 25, 26], he connected this notion to that of Brownian motion and he defined the so-called Wiener Process. In posterior works, he generalized his results for general measurable functionals defining a probability measure $\mu_{x_0}$ on the measurable space $(C_{x_0}[a, b], B_{x_0})$, where $B_{x_0}$ stands for the Borel $\sigma$-algebra of $C_{x_0}[a, b]$ endowed with the uniform convergence topology, satisfying for each finite subset $T = \{t_1, t_2, ..., t_n\}$ of $[a, b]$ and each family $(B_t)_{t \in T} \subset B_{\mathbb{R}}$, the identity

$$
\mu_{x_0}(\pi_T^{-1}(B_t)_{t \in T}) = \int_{B_{t_1}} \cdots \int_{B_{t_n}} \prod_{j=1}^{n} p_{t_{j+1} - t_j}(x_j, x_{j-1}) \prod_{j=1}^{n} dx_j,
$$

where $p_t(x, y)$ is the heat kernel of $\mathbb{R}$, $\pi_T : C_{x_0}[a, b] \to \mathbb{R}^T$ is the projector defined by $\pi_T(u) = (u(t))_{t \in T}$ for each $u \in C_{x_0}[a, b]$ and $x_0 = x_0$, $t_0 = a$. The measure $\mu_{x_0}$ is the well-known Wiener measure.

It seems there is no easy way to compute the integral of an arbitrary measurable functional $F : C_{x_0}[a, b] \to \mathbb{R}$. Nevertheless, Wiener proved in [21] an analogue of Jessen’s formula [13, 15] for the measure $\mu_{x_0}$. More explicitly, he proved that given a bounded and continuous functional $F \in L_1(\mathcal{C}_{x_0}[a, b], \mu_{x_0})$ and a partition $\mathcal{P} = \{(t_n^i)_{i=1}^{n} \}_{n \in \mathbb{N}}$ of $[a, b]$ satisfying the limit condition $\max_{2 \leq i \leq n} |t_n^i - t_n^{i-1}| \to 0$ as $n \to \infty$, then the integral of $F$ can be computed by means of finite dimensional integrals as

$$
\int_{C_{x_0}[a, b]} F(u) \, d\mu_{x_0}(u) = \lim_{n \to \infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} F_n(x_1, x_2, ..., x_n) \prod_{j=1}^{n} p_{t_n^j - t_n^{j-1}}(x_j, x_{j-1}) \prod_{j=1}^{n} dx_j,
$$

with $x_0 = x_0$, $t_n^0 = a$ and $F_n(x_1, x_2, ..., x_n) := F(u(x_1, x_2, ..., x_n))$ where $u(x_1, x_2, ...,)$ denotes the linear interpolation of the points $x_1, x_2, ..., x_n$, for each $n \in \mathbb{N}$. In [15], the author generalizes this formula to every $L_1$ functional proving that for each $F \in L_1(C_{x_0}[a, b], \mu_{x_0})$,
there exists a finite dimensional functional sequence \((F_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_1(\mathbb{R}^n, \mu^n_{\mathbf{x}_0})\) such that

\[
\int_{C_{\mathbf{x}_0}[a,b]} F \, d\mu_{\mathbf{x}_0} = \lim_{n \to \infty} \int_{\mathbb{R}^n} F_n \, d\mu^n_{\mathbf{x}_0},
\]

where

\[
d\mu^n_{\mathbf{x}_0} = \prod_{j=1}^{n} p_{t^n_j - t^n_{j-1}}(x_j, x_{j-1}) \prod_{j=1}^{n} dx_j.
\]

A similar discussion can be done for the category of Riemannian Manifolds. Given a compact connected Riemannian manifold \((M, g)\) (closed Riemannian manifold), we can construct, analogously as it is done for \(\mathbb{R}\), the measure space \((C_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})\), where \(C_{\mathbf{x}_0}(M)\) denotes the space of continuous curves on \(M\) beginning at \(\mathbf{x}_0\) and \(\mu_{\mathbf{x}_0}\) the Wiener measure on \(C_{\mathbf{x}_0}(M)\), i.e., a measure satisfying an analogous of equation (1) for this setting (see Section 3). Similar versions of Jessen type formula have been developed for the category of Riemannian manifolds in [2], in which Andersson and Driver proved that given a bounded and continuous functional \(F : C_{\mathbf{x}_0}(M) \to \mathbb{R}\), the identity

\[
\int_{C_{\mathbf{x}_0}(M)} F \, d\mu_{\mathbf{x}_0} = \lim_{|\mathcal{P}| \to 0} \int_{H_{\mathcal{P}}(M)} F(\sigma) \, d\nu_{\mathcal{P}}(\sigma)
\]

holds, where \((H_{\mathcal{P}}(M), \nu_{\mathcal{P}})\) is a finite dimensional measure space based on the geometrical data of \((M, g)\) and \(\mathcal{P} = \{\{t^n_i\}_{i=1}^{n}\}_{n \in \mathbb{N}}\) is a partition of \([0, 1]\) with norm \(|\mathcal{P}|\).

The first aim of this article is to establish a generalization of equation (3) for every integrable functional \(F \in L_1(C_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})\) not necessarily bounded and continuous in the vein of the analogous result (2) for the classical Wiener measure proved in [15] and to show that it can be proved by means of classical measure theoretic techniques and without the use of the underlying Riemannian structure. Moreover, we prove the existence of certain identification \(\mathfrak{S}\) between \(L_p(C_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})\) for \(1 \leq p < \infty\) and some space consisted of sequences of finite dimensional functions, \(\lim_n L_p(M^n, \nu^n_{\mathbf{x}_0})\) and that this correspondence is, in fact, an isometric isomorphism. Furthermore, we will give the explicit correspondence between this spaces. This identification will simplify all the considerations regarding infinite dimensional integration on Riemannian Manifolds, since instead of working with functionals defined in spaces of infinite number of variables, we can work with sequences of finite dimensional ones. More explicitly, we prove the following result, where the notation involved will be subsequently defined in the next sections.

**Theorem 1.1.** Let \(1 \leq p < \infty\), then the operator

\[
\mathfrak{S} : \lim_n L_p(M^n, \nu^n_{\mathbf{x}_0}) \quad \rightarrow \quad L_p(C_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})
\]

\[
(f_n)_{n \in \mathbb{N}} \quad \mapsto \quad \lim_{L_p(\mu_{\mathbf{x}_0})} \Phi(f_n)
\]

defines an isometric isomorphism. In consequence, given an integrable functional \(F \in L_1(C_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})\), there exists a functional sequence \((f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_1(M^n, \nu^n_{\mathbf{x}_0})\) such that

\[
\int_{C_{\mathbf{x}_0}(M)} F \, d\mu_{\mathbf{x}_0} = \lim_{n \to \infty} \int_{M} \prod_{i=1}^{n} f_n(x_1, x_2, ..., x_n) \prod_{i=1}^{n} p_{t^n_i - t^n_{i-1}}(x_i, x_{i-1}) \prod_{i=1}^{n} d\mu(x_i).
\]

where \(p_t(x, y)\) is the heat kernel of \(M\) and \(\mathcal{P} = \{\{t^n_i\}_{i=1}^{n}\}_{n \in \mathbb{N}}\) is a partition of \([0, 1]\) whose norm \(|\mathcal{P}|\) tends to 0 as \(n \to \infty\).
Exactly the same considerations will be done for the pinned Wiener space \( C^x_\gamma(M) \), i.e., the space of continuous curves \( \gamma : [0,1] \to M \) such that \( \gamma(0) = x, \gamma(1) = y \), and in particular for the loop space \( \mathcal{L}_x(M) = C^x_\gamma(M) \).

The second aim of the article is to embed the notion of the Stratonovich stochastic integral in the space \( L_2(\mathcal{L}_x(M), \mu_{x_0}) \). Once we have this notion defined in \( L_2(\mathcal{L}_x(M), \mu_{x_0}) \), we will be able to integrate a differential form over every loop in \( \mathcal{L}_x(M) \), and not only for smooth ones (as is usually defined), obtaining some interesting results that interconnects Topology with Real Analysis. Among them, we will give a computational method to establish if the fundamental group \( \pi_1(M, x_0) \) of a given Riemannian Manifold \((M, g)\) is nonzero, establishing the existence of a one dimensional hole via path integration. More precisely, we will prove the following.

**Theorem 1.2.** Let \((M, g)\) be a compact connected Riemannian manifold. If there exists a closed form \( \omega \in \Gamma(T^*M) \) such that

\[
\int_{\mathcal{L}_{x_0}(M)} \exp \left\{ - \left\| \int_0^1 \omega \right\| \right\} d\mu_{x_0}(\gamma) \neq 1,
\]

then \( \pi_1(M, x_0) \neq \{0\} \).

The paper is organized as follows. In the first section, we review Banach Inductive Limits, a concept introduced in [15] to prove the finite dimensional decomposition of the classical Wiener measure and that will be also used in this article to prove Theorem [11.1]. It includes a direct proof of the key Theorem 2.2 that do not involves the abstract completion theorem that was used in its proof in [15]. In section three we develop the main theory and we prove Theorem 1.1 for both, the classical space \( C_{x_0}(M) \) and the pinned space \( C^x_\gamma(M) \). In section four, we embed in \( L_2(\mathcal{L}_{x_0}(M), \mu_{x_0}) \) the concept of Stratonovich stochastic integral and we prove Theorem 1.2. The end of the article is dedicated to illustrate some applications of path integration techniques on manifolds. Among them, we will study path integration on the circle \( S^1 \) to give some results concerning Jacobi’s theta function and the evaluation of infinite sums. Finally, we will reformulate the singular cohomology group of the circle \( H^1(S^1, \mathbb{R}) \) in terms of path integration, proving that the obstruction represented by this group can be equivalently expressed as a path integral morphism.

As we have briefly illustrate, the applications of path integration in Riemannian manifolds provides with an alternative formulation of classical geometric concepts bringing to them an original point of view. This geometric reformulation is the essence of this paper.

### 2. Banach Inductive Limits

In this section we will recall some facts about Banach inductive limits, a concept introduced in [15]. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of Banach spaces and \((T_n)_{n \in \mathbb{N}}\) be a family of linear isometries \( T_n : X_n \to X_{n+1} \). We will call each family \((X_n, T_n)_{n \in \mathbb{N}}\) satisfying this property an **inductive chain**. Given two inductive chains \((X_n, T_n)_{n \in \mathbb{N}}, (Y_n, Q_n)_{n \in \mathbb{N}}\), and a sequence \( \mathcal{G} := (G_n)_{n \in \mathbb{N}} \) consisting of linear continuous operators \( G_n : X_n \to Y_n \), we say that \( \mathcal{G} \) is a **chains homomorphism** if the identity \( G_{n+1} \circ T_n = Q_n \circ G_n \) holds for each \( n \in \mathbb{N} \). We will denote a chain homomorphism by \( \mathcal{G} : (X_n, T_n)_{n \in \mathbb{N}} \to (Y_n, Q_n)_{n \in \mathbb{N}} \). We define the **category of inductive chains** as the category \text{Ind} whose objects are the inductive chains and whose morphisms are the corresponding chain homomorphisms. We say that the inductive chains \((X_n, T_n)_{n \in \mathbb{N}}, (Y_n, Q_n)_{n \in \mathbb{N}}\) are **isomorphic** if there exists a morphism \( \mathcal{G} : (X_n, T_n)_{n \in \mathbb{N}} \to (Y_n, Q_n)_{n \in \mathbb{N}} \) with \( \mathcal{G} := (G_n)_{n \in \mathbb{N}} \), such that each operator \( G_n : X_n \to Y_n \) is an isometric isomorphism. If the chains \((X_n, T_n)_{n \in \mathbb{N}}, (Y_n, Q_n)_{n \in \mathbb{N}}\) are isomorphic, we will denote it by \((X_n, T_n)_{n \in \mathbb{N}} \simeq (Y_n, Q_n)_{n \in \mathbb{N}}\).
We say that a chain \((X_n, T_n)_{n \in \mathbb{N}}\) is simple if \(T_n : X_n \hookrightarrow X_{n+1}\) is the inclusion. As we have shown in [5, Section 2], given an inductive chain \((X_n, T_n)_{n \in \mathbb{N}}\), there exists a simple chain \((Y_n, i_n)_{n \in \mathbb{N}}\) such that \((X_n, T_n)_{n \in \mathbb{N}} \simeq (Y_n, i_n)_{n \in \mathbb{N}}\). If \((Y_n, i_n)_{n \in \mathbb{N}}\) is simple, then since \(Y_n \subset Y_{n+1}\) for each \(n \in \mathbb{N}\), the set \(\bigcup_{n \in \mathbb{N}} Y_n\) is a normed space that inherits the norm of each \(Y_n\). We say that a Banach space \(X\) is the completion of a simple chain \((Y_n, i_n)_{n \in \mathbb{N}}\) if \(X\) is the completion of \(\bigcup_{n \in \mathbb{N}} Y_n\). Under these definitions, we can define the concept of Banach inductive limit of a given chain.

**Definition 2.1.** We define the Banach inductive limit of an inductive chain \((X_n, T_n)_{n \in \mathbb{N}}\) as the completion of any simple chain isomorphic to \((X_n, T_n)_{n \in \mathbb{N}}\). That is

\[
\mathcal{IB}an(X_n, T_n)_{n \in \mathbb{N}} := \mathcal{C} \left( \bigcup_{n \in \mathbb{N}} Y_n \right)
\]

where \((Y_n, i_n)_{n \in \mathbb{N}}\) is any simple chain isomorphic to \((X_n, T_n)_{n \in \mathbb{N}}\) and \(\mathcal{C}(X)\) denotes the completion of the normed space \(X\).

We can interpret the Banach Inductive Limit of a chain \((X_n, T_n)_{n \in \mathbb{N}}\) as the minimal Banach space that contains an ordered isometric copy of each \(X_n\). Observe that the Banach Inductive Limit defines a covariant functor \(\mathfrak{F} : \mathfrak{Ind} \rightarrow \mathfrak{Ban}\), from the category of inductive chains to the category of Banach spaces that assigns to every inductive chain \((X_n, T_n)_{n \in \mathbb{N}}\), its Banach Inductive Limit \(\mathcal{IB}an(X_n, T_n)_{n \in \mathbb{N}}\) and to every chain homomorphism \(\mathfrak{G} : (X_n, T_n)_{n \in \mathbb{N}} \rightarrow (Y_n, Q_n)_{n \in \mathbb{N}}\) the linear continuous operator \(\mathfrak{F}(\mathfrak{G}) : \mathcal{IB}an(X_n, T_n)_{n \in \mathbb{N}} \rightarrow \mathcal{IB}an(Y_n, Q_n)_{n \in \mathbb{N}}\) defined as follows: Let \((X'_n, i'_n)_{n \in \mathbb{N}}\) and \((Y'_n, i'_n)_{n \in \mathbb{N}}\) be two simple chains isomorphic to \((X_n, T_n)_{n \in \mathbb{N}}\) and \((Y_n, Q_n)_{n \in \mathbb{N}}\) respectively, then consider the operator \(T : \bigcup_{n \in \mathbb{N}} X'_n \rightarrow \bigcup_{n \in \mathbb{N}} Y'_n\) defined in the last paragraph and define \(\mathfrak{F}(\mathfrak{G}) : \mathcal{C}(\bigcup_{n \in \mathbb{N}} X'_n) \rightarrow \mathcal{C}(\bigcup_{n \in \mathbb{N}} Y'_n)\) to be the unique operator whose restriction to \(\bigcup_{n \in \mathbb{N}} X'_n\) is \(T\).

In the rest of this section, we will present a simple representation of the Banach inductive limit of a given inductive chain. Let \((X_n, T_n)_{n \in \mathbb{N}}\) be an inductive chain and consider the linear space over \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\)

\[
\bigoplus_{n \in \mathbb{N}} X_n := \{ (x_n)_{n \in \mathbb{N}} : x_i \in X_i \text{ for each } i \in \mathbb{N} \}
\]

and the quotient space \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\) defined by

\[
\mathcal{F}(X_n)_{n \in \mathbb{N}} := \left\{ (x_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} X_n : \lim_{n \to \infty} \|x_n\|_X < +\infty \right\} / \sim
\]

where given the sequences \((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} X_n\), we identify \((x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}\) if \(\lim_{n \to \infty} \|x_n - y_n\|_X = 0\). It is easy to see that \(\sim\) is an equivalence relation and therefore the space \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\) is well defined. Roughly speaking, this binary relation identifies sequences with similar “tails”. If we define in \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\) the functional \(\| (x_n)_{n \in \mathbb{N}} \|_\mathcal{F} := \lim_{n \to \infty} \|x_n\|_X\), then the pair \((V, \| \cdot \|_\mathcal{F})\) defines a normed space for each linear space \(V\) contained in \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\), \(V \leq \mathcal{F}(X_n)_{n \in \mathbb{N}}\). It must be observed that \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\) is not a linear space. Since \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\) is the ambient space in which we will work, we would like to embed the spaces \(X_n\) in \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\). For this purpose, we define for each \(N \in \mathbb{N}\) the copy of \(X_N\) in \(\mathcal{F}(X_n)_{n \in \mathbb{N}}\) by

\[
\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}} := \left\{ (x_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} X_n : \exists y_N \in X_N, T_n^N(y_N) = x_n, \forall n \geq N \right\} / \sim
\]
where for every $n, m \in \mathbb{N}$ with $n \geq m$, the notation $T^m_n$ stands for $T_{n-1} \circ T_{n-2} \circ \cdots \circ T_n$. Observe that $\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}} \leq \mathcal{F}(X_n)_{n \in \mathbb{N}}$ and therefore ($\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}, \|\cdot\|$) is a normed space. It is easy to prove [15, Proposition 2.2] that $\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}$ is isometrically isomorphic to $X_N$ via the isometric isomorphism

$$Q_N : \mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}} \rightarrow X_N$$

(\begin{array}{c}
(x_n) \\
y_n
\end{array}) \rightarrow y_N$$

where $y_N \in X_N$ is the element satisfying $T^N_n(y_N) = x_n$ for each $n \geq N$. Moreover, it was proved that the morphism $\mathcal{O} : (\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}, i_N)_{N \in \mathbb{N}} \rightarrow (X_n, T_n)_{n \in \mathbb{N}}, \mathcal{O} := (G_N)_{N \in \mathbb{N}}$ was an isomorphism proving that the inductive chain $(\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}, i_N)_{N \in \mathbb{N}}$ is simple and isomorphic to $(X_n, T_n)_{n \in \mathbb{N}}$. In this way, we have a canonical representation of a simple chain isomorphic to a given inductive chain. Furthermore, if we define the copy of $\bigcup_{N \in \mathbb{N}} X_N$ in $\mathcal{F}(X_n)_{n \in \mathbb{N}}$ by

$$\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}} := \bigcup_{N \in \mathbb{N}} \mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}},$$

the last discussion allows to consider the Banach inductive limit of a given chain $(X_n, T_n)_{n \in \mathbb{N}}$ as

$$\mathcal{\text{Ban}}(X_n, T_n)_{n \in \mathbb{N}} = \mathcal{C}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}}).$$

However, the space $\mathcal{C}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}})$ is difficult to deal with because it consists of double sequences. The main result of this section is to simplify the space $\mathcal{C}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}})$ identifying it with a subspace of $\mathcal{F}(X_n)_{n \in \mathbb{N}}$. The candidate to the simplified space will be the subspace

$$\lim_{(T_n)_{n \in \mathbb{N}}} X_n := \left\{ (x_n)_{n \in \mathbb{N}} \in \mathcal{F}(X_n)_{n \in \mathbb{N}} : \|T^m_n(x_n) - x_m\|_{x_n,m \rightarrow 0} \leq \mathcal{F}(X_n)_{n \in \mathbb{N}} \right\}$$

or equivalently

$$\lim_{(T_n)_{n \in \mathbb{N}}} X_n := \left\{ (x_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} X_n : \|T^m_n(x_n) - x_m\|_{x_n,m \rightarrow 0} \right\} / \sim$$

where $\sim$ is the equivalence relation defined on the space $\mathcal{F}(X_n)_{n \in \mathbb{N}}$. Since $\lim_{(T_n)_{n \in \mathbb{N}}} X_n \leq \mathcal{F}(X_n)_{n \in \mathbb{N}}$, the pair $(\lim_{(T_n)_{n \in \mathbb{N}}} X_n, \|\cdot\|)$ defines a normed space. We will see that $\lim_{(T_n)_{n \in \mathbb{N}}} X_n$ is isometrically isomorphic to $\mathcal{C}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}})$.

In [15, Theorem 2.3] we prove this result by means of the completion theorem. Here we give a new prove that avoid the use of this abstract result and gives the explicit isometric isomorphism. To read the proof of this result is convenient to recall the definition of the completion of a given normed space $X$. The space $\mathcal{C}(X)$ is defined by

$$\mathcal{C}(X) := \left\{ (x_n)_{n \in \mathbb{N}} \subset X : (x_n)_{n \in \mathbb{N}} \text{ is Cauchy on } X \right\} / \sim,$$

where we identify two sequence $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ if $\lim_{n \rightarrow \infty} \|x_n - y_n\|_X = 0$. The norm of $\mathcal{C}(X)$ is given by $\|(x_n)_{n \in \mathbb{N}}\|_\mathcal{C} = \lim_{n \rightarrow \infty} \|x_n\|_X$ for each $(x_n)_{n \in \mathbb{N}} \in \mathcal{C}(X)$. With this definitions, the pair $(\mathcal{C}(X), \|\cdot\|_\mathcal{C})$ defines the unique (up to isometric isomorphism) Banach space containing $X$ as a dense subspace. Observe that $\lim_{(T_n)_{n \in \mathbb{N}}} X_n$ can be viewed as an ordered version of $\mathcal{C}(X)$.

**Theorem 2.2.** The normed spaces $\mathcal{C}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}})$ and $\lim_{(T_n)_{n \in \mathbb{N}}} X_n$ are isometrically isomorphic,

$$\mathcal{C}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}}) \simeq \lim_{(T_n)_{n \in \mathbb{N}}} X_n.$$
In particular, \( \lim_{(T_n)_{n \in \mathbb{N}}} X_n \) is the completion of \( \mathcal{F}(X_n, T_n)_{n \in \mathbb{N}} \) and

\[
\mathbb{Ban}(X_n, T_n)_{n \in \mathbb{N}} \simeq \lim_{(T_n)_{n \in \mathbb{N}}} X_n.
\]

**Proof.** First, we will prove the result for simple inductive chains \((X_n, i_n)_{n \in \mathbb{N}}\), where \(i_n : X_n \hookrightarrow X_{n+1}\) is the inclusion. For the sake of notation we will denote \(\lim_{i_n} X_n\) instead of \(\lim_{(i_n)} X_n\). We define the operator \(\mathfrak{T} : \lim_{i_n} X_n \to (\bigcup_{n \in \mathbb{N}} X_n)\) by

\[
\mathfrak{T} : (\lim_{i_n} X_n, \| \cdot \|_{X}) \quad \mapsto \quad (\bigcup_{n \in \mathbb{N}} X_n, \| \cdot \|_{X})
\]

We will prove there things:

1. \(\mathfrak{T}\) is well defined and linear: We will see that \((x_n)_{n \in \mathbb{N}} \in \lim_{i_n} X_n\), then

\[
\| (x_n)_{n \in \mathbb{N}} \|_{\mathfrak{T}} = \lim_{n \to \infty} \| x_n \|_{X_n} = \lim_{n \to \infty} \| x_n \|_{\bigcup_{\mathbb{N}} X_n} = \| (x_n)_{n \in \mathbb{N}} \|_{\mathfrak{T}}.
\]

2. \(\mathfrak{T}\) is an isometry: Take \((x_n)_{n \in \mathbb{N}} \in \lim_{i_n} X_n\), then

\[
\| x_n - x_m \|_{\bigcup_{\mathbb{N}} X_n} = \| x_n - x_m \|_{n,m \to \infty}.
\]

3. \(\mathfrak{T}\) is onto: Take \((y_n)_{n \in \mathbb{N}} \in \mathcal{C}(\bigcup_{n \in \mathbb{N}} X_n)\) and choose \(\{N_n\}_{n \in \mathbb{N}} \subset \mathbb{N}\) such that \(y_n \in X_{N_n}\) and \(N_1 < N_2 < \cdots < N_n < \cdots\).

Define the sequence \((x_n)_{n \in \mathbb{N}}\) by

\[
x_n := \begin{cases} 
0 & \text{if } n < N_1 \\
y_n & \text{if } N_n \leq n < N_{n+1}.
\end{cases}
\]

We will see that \((x_n)_{n \in \mathbb{N}} \in \lim_{i_n} X_n\). Clearly \((x_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} X_n\) since \(x_n = y_n \in X_{N_n}\) and \(N_n \leq n < N_{n+1}\) implies that \(X_{N_n} \subset X_n\). On the other hand, since \((y_n)_{n \in \mathbb{N}}\) is Cauchy and \(m \to \infty\) implies \(n_i \to \infty\) as \(n \to \infty\),

\[
\| x_n - x_m \|_{X_n} = \| y_n - y_m \|_{X_{N_n}} \to 0.
\]

We have proved that \((x_n)_{n \in \mathbb{N}} \in \lim_{i_n} X_n\). Finally, we will see that \((x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}\) in \(\mathcal{C}(\bigcup_{n \in \mathbb{N}} X_n)\) which implies \(T(x_n)_{n \in \mathbb{N}} = (y_n)_{n \in \mathbb{N}}\). Since \((y_n)_{n \in \mathbb{N}}\) is Cauchy and since \(n_i \to \infty\) as \(n \to \infty\),

\[
\lim_{n \to \infty} \| x_n - y_n \|_{X_n} = \lim_{n \to \infty} \| y_n - y_n \|_{X_n} = 0.
\]

Therefore, \(\mathfrak{T}\) is onto.

In the general case, since \((\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}, i_N)_{N \in \mathbb{N}}\), where \(i_N\) is the inclusion

\[
\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}} \hookrightarrow \mathcal{F}^{N+1}(X_n, T_n)_{n \in \mathbb{N}},
\]

is a simple chain, we have that \(\mathcal{C}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}}) \simeq \lim_{N} \mathcal{F}^{N}(X_n, T_n)_{n \in \mathbb{N}}\). Therefore, it will be enough to prove that \(\lim_{N} \mathcal{F}^{N}(X_n, T_n)_{n \in \mathbb{N}} \simeq \lim_{(T_n)} X_N\). For this, we will see that the operator \(\mathcal{S} : \lim_{N} \mathcal{F}^{N}(X_n, T_n)_{n \in \mathbb{N}} \to \lim_{(T_n)} X_N\) defined by

\[
\mathcal{S} : (\lim_{N} \mathcal{F}^{N}(X_n, T_n)_{n \in \mathbb{N}}, \| \cdot \|_{\mathcal{F}}) \quad \mapsto \quad (\lim_{(T_n)} X_N, \| \cdot \|_{X_N})
\]

is an isometric isomorphism.
(1) $\mathcal{S}$ is well defined and linear: By the definition of the operator $Q_N$, it is clear that $(Q_N(x^N))_{N \in \mathbb{N}} \in \bigoplus_{N \in \mathbb{N}} X_N$. Moreover, since the morphism

$$\mathcal{S} : (\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}, i_N)_{N \in \mathbb{N}} \rightarrow (X_N, T_N)_{N \in \mathbb{N}}$$

defined by $\mathcal{S} := (Q_N)_{N \in \mathbb{N}}$ is a chain morphism, we have that

$$T^M_N \circ Q_N = Q_M \circ i^M_N$$

for each $N < M$

and therefore we infer that

$$\|T^M_N(Q_N(x^N)) - Q_M(x^M)\|_{X_M} = \|Q_M(x^N) - Q_M(x^M)\|_{X_M} = \|x^N - x^M\|_{\mathcal{F}_M} \xrightarrow[N,M \rightarrow \infty]{} 0,$$

where $\| \cdot \|_{\mathcal{F}_M}$ denotes the norm of $\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}$. This implies $(Q_N(x^N))_{N \in \mathbb{N}} \in \lim_{(T_N)_{N \in \mathbb{N}}} X_N$. It is straightforward to verify that $\mathcal{S}$ is a linear map.

(2) $\mathcal{S}$ is an isometry: Let $(x^N)_{N \in \mathbb{N}} \in \lim_{N}(\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}$, then

$$\|(x^N)_{N \in \mathbb{N}}\|_F = \lim_{n \rightarrow \infty} \|x^N\|_{\mathcal{F}_N} = \lim_{n \rightarrow \infty} \|Q_N(x^N)\|_{X_N} = \|\mathcal{S}(x^N)_{N \in \mathbb{N}}\|_{\mathcal{S}_X}.$$

(3) $\mathcal{S}$ is onto: Let $(y_N)_{N \in \mathbb{N}} \in \lim_{(T_N)_{N \in \mathbb{N}}} X_N$ and let us define the vector $(x^N)_{N \in \mathbb{N}}$ where $x^N := (x^N_n)_{n \in \mathbb{N}}$ is given by

$$x^N_n := \begin{cases} 0 & \text{if } n < N \\ T^N_N y_N & \text{if } n \geq N. \end{cases}$$

Observe that $(x^N)_{N \in \mathbb{N}} \in \bigoplus_{N \in \mathbb{N}} \mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}$ and since the norm $\| \cdot \|_{\mathcal{F}_N}$ is the restriction to $\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}$ of the norm $\| \cdot \|_F$ defined in $\mathcal{F}(X_n)_{n \in \mathbb{N}}$, we have that

$$\|x^N - x^M\|_{\mathcal{F}_M} = \lim_{n \rightarrow \infty} \|x^N_n - x^M_n\|_{X_n} = \|T^M_N y_N - y_M\|_{X_M} \xrightarrow[N,M \rightarrow \infty]{} 0.$$  

Hence, we conclude that $(x^N)_{N \in \mathbb{N}} \in \lim_{N}(\mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}}$. Moreover, it is quite evident that $\mathcal{S}(x^N)_{N \in \mathbb{N}} = (y_N)_{N \in \mathbb{N}}$.

Finally, given any inductive chain $(X_n, T_n)_{n \in \mathbb{N}}$, we have proved that

$$\mathcal{S}(\mathcal{F}(X_n, T_n)_{n \in \mathbb{N}}) \simeq \lim_N \mathcal{F}^N(X_n, T_n)_{n \in \mathbb{N}} \simeq \lim_{(T_N)_{N \in \mathbb{N}}} X_N.$$ 

This concludes the proof. $\square$

3. Wiener Measure on Riemannian Manifolds

In this section we will use the Banach inductive limit techniques to simplify the structure of the Banach spaces $L_p(C_{x_0}(M), B_{x_0}, \mu_{x_0})$ for $1 \leq p < \infty$, where $(M, g)$ is a compact connected Riemannian manifold (closed Riemannian manifold for brevity) with a fixed base point $x_0 \in M$, in terms of well known spaces $L_p(M^n, \nu_{x_0}^n)$, where $\nu_{x_0}^n$ is certain borel measure on $M^n$ that will be defined later. The notation $C_{x_0}(M)$ stands for the space of continuous paths $\gamma \in C([0, 1], M)$ such that $\gamma(0) = x_0$, $B_{x_0}$ for the borel $\sigma$-algebra of $C_{x_0}(M)$ with respect to the uniform convergence topology given by the induced metric of $M$ and $\mu_{x_0}$ for the Wiener measure on $M$ with base point $x_0$. 


3.1. Definition of the Wiener Measure. We will start defining this measure space. Consider in \((M, g)\), the measure \(\mu : \mathcal{B}_M \to [0, +\infty]\) induced by the metric \(g\), where \(\mathcal{B}_M\) denotes the Borel \(\sigma\)-algebra of \(M\). This measure is locally given by the expression

\[
d\mu = \sqrt{\det(g_{ij})} \ dx_1 \wedge \cdots \wedge dx_m
\]

where \(m\) is the dimension of \(M\) and \((g_{ij})_{ij}\) is the matrix of \(g\) in a local chart. For each closed Riemannian manifold \((M, g)\), there exists a heat kernel \(p_t(x, y)\), for \(t > 0, x, y \in M\), i.e., the Schwartz kernel of the self-adjoint operator \(e^{t\Delta}\) on \(L^2(M, \mu)\), where \(\Delta\) denotes the Laplace-Beltrami operator on \((M, g)\). The proof of the existence of this map can be found in [3, 10]. It is well known by the Kolmogorov extension Theorem [27, Theorem 6.1], the existence of a probability measure

\[
\mu_{x_0} : \mathcal{P}(M) \to [0, +\infty]
\]

on \((\times_{t \in [0, 1]} M, \mathcal{P}(M))\), where \(\mathcal{P}(M)\) denotes the borel \(\sigma\)-algebra of \(\times_{t \in [0, 1]} M\) with respect to the product topology, satisfying

\[
\mu_{x_0}(\pi^{-1}_T(B_t)_{t \in T}) := \int_{B_{x_0}} \prod_{j=1}^{n} p_{t_j - t_{j-1}}(x_j, x_{j-1}) \prod_{j=1}^{n} d\mu(x_j)
\]

for each finite set \(T = \{t_1, t_2, \ldots, t_n\} \subset [0, 1]\) with \(0 = t_0 < t_1 < \cdots < t_{n-1} < t_n\) and each \((B_t)_{t \in T} \subset \mathcal{B}_M\), where \(x_0 = x_0\) and

\[
\pi_T : \times_{t \in [0, 1]} M \to \times_{t \in T} M
\]

is the projector defined by \(\pi_T(\gamma)_{t \in [0, 1]} = (\gamma(t))_{t \in T}\) for each \((\gamma)_{t \in [0, 1]} \in \times_{t \in [0, 1]} M\). Since \((M, g)\) is compact, is in particular stochastically complete (see for instance [10]), and therefore

\[
\int_M p_t(x, y) d\mu(y) = 1
\]

for each \(t > 0\) and \(x \in M\). This implies that the measure \(\mu_{x_0}\) is of probability. Moreover this measure satisfies

\[
\mu_{x_0}(\mathcal{H}^\theta_{x_0}(M)) = 1\quad \text{for each } \theta \in (0, 1/2)
\]

where \(\mathcal{H}^\theta_{x_0}(M)\) stands for the space of Hölder continuous paths on \(M\) of exponent \(\theta\) satisfying \(\gamma(0) = x_0\). Therefore, since \(C_{x_0}(M) \in \mathcal{P}(M)\) and \(\mathcal{B}_{x_0} = \mathcal{P}(M) \cap C_{x_0}(M)\), we can consider the restricted measure space \((C_{x_0}(M), \mathcal{B}_{x_0}, \mu_{x_0})\). The restricted measure \(\mu_{x_0}\) is called the Wiener measure of \(M\) with base point \(x_0\). The proof of this facts can be found in [3, 10].

3.2. Discretization of the Wiener measure. For the main theorem, we will need a discrete version of the Wiener measure space \((C_{x_0}(M), \mathcal{B}_{x_0}, \mu_{x_0})\). Consider the discrete compact product space

\[
\Omega_M := \times_{t \in \mathbb{Q} \cap [0, 1]} M.
\]

Let us denote \(\mathbb{Q}_* = \mathbb{Q} \cap [0, 1]\) for the sake of notation. We define the \(\sigma\)-algebra \(\bigotimes_{t \in \mathbb{Q}_*} \mathcal{B}_M := \sigma(\mathcal{R})\) where

\[
\mathcal{R} := \{\pi^{-1}_T(B_t)_{t \in T} : B_t \in \mathcal{B}_M \text{ for each } t \in T \text{ and } T \subset \mathbb{Q}_* \text{ finite}\}
\]
and $\pi_T : \Omega_M \to \bigtimes_{t \in T} M$ is the projector defined by $\pi_T(\omega_t)_{t \in Q_*} = (\omega_t)_{t \in T}$ for each $(\omega_t)_{t \in Q_*} \in \Omega_M$. Then the pair $(\Omega_M, \bigotimes_{t \in Q_*} \mathcal{B}_M)$ defines a measurable space. Since

$$\bigotimes_{t \in Q_*} \mathcal{B}_M = \mathcal{P}(M) \cap \Omega_M,$$

we can consider the restricted measure space $(\Omega_M, \bigotimes_{t \in Q_*} \mathcal{B}_M, \nu_{x_0})$ where $\nu_{x_0}$ is the restriction of $\mu_{x_0}$ to $\Omega_M$. Denote by $\mathcal{C}_{x_0}(Q_*, M)$ the space of continuous functions $(\omega_t)_{t \in Q_*} \in \Omega_M$ satisfying $\omega_0 = x_0$ and by $\mathcal{H}_{x_0}^\theta(Q_*, M)$ the space of Hölder continuous paths in $\Omega_M$ of exponent $\theta$ satisfying $\omega_0 = x_0$. Then, this spaces are measurable and it can be shown that the identity

$$\nu_{x_0}(\mathcal{H}_{x_0}^\theta(Q_*, M)) = 1$$

holds for each $\theta \in (0, 1/2)$. Therefore, we can consider the restricted probability space $(\mathcal{H}_{x_0}^\theta(Q_*, M), \mathcal{B}_{x_0}, \nu_{x_0})$, where

$$\mathcal{B}_{x_0} := \bigotimes_{t \in Q_*} \mathcal{B}_M \cap \mathcal{H}_{x_0}^\theta(Q_*, M).$$

We will see that if we restrict to $\mathcal{H}_{x_0}^\theta(M)$ for a given $\theta \in (0, 1/2)$, the continuous and discret models coincide. Let $\Gamma$ be the bijective measurable operator defined by

$$\Gamma : (\mathcal{H}_{x_0}^\theta(Q_*, M), \mathcal{B}_{x_0}) \rightarrow (\mathcal{H}_{x_0}^\theta(M), \mathcal{B}_\gamma)$$

where $\gamma_\omega \in \mathcal{C}_{x_0}(M)$ is the unique continuous function such that $\gamma_\omega(t) = \omega_t$ for each $t \in Q_*$. It must be observed that since $\gamma : Q^* \rightarrow M$ is uniformly continuous and since $M$ is complete as a metric space, by the extension theorem [17, Theorem 3.4.9], there exists a unique continuous extension of $\gamma$ to $[0, 1]$. Moreover this extension is $\theta$-Hölder continuous. This implies that $\Gamma$ is well defined. The map $\Gamma$ is measure preserving as we will see in the next result.

**Lemma 3.1.** The identity $\mu_{x_0}(B) = \nu_{x_0}(\Gamma^{-1}(B))$ holds for each $B \in \mathcal{B}_{x_0}$.

**Proof.** By [12, Proposition 2.2], we have $\mathcal{B}_{x_0} = \sigma(\mathcal{R}') \cap \mathcal{C}_{x_0}(M)$ where

$$\mathcal{R}' := \{\pi^{-1}_T(B_t)_{t \in T} : B_t \in \mathcal{B}_M \text{ for each } t \in T \text{ and } T \subset [0, 1] \text{ finite}\}.$$

Therefore, it is enough to prove the result for $\mathcal{R}' \cap \mathcal{C}_{x_0}(M)$. Since $\mu_{x_0}$ and $\nu_{x_0}$ satisfies equation (4) for rational $t$-values, if $T$ is a finite subset of $Q^*$ and $(B_t)_{t \in T} \subset \mathcal{B}_M$,

$$\mu_{x_0}(\pi^{-1}_T(B_t)_{t \in T} \cap \mathcal{C}_{x_0}(M)) = \mu_{x_0}(\pi^{-1}_T(B_t)_{t \in T})$$

$$= \nu_{x_0}(\pi_T^{-1}(B_t)_{t \in T})$$

$$= \nu_{x_0}(\pi_T^{-1}(B_t)_{t \in T} \cap \mathcal{H}_{x_0}^\theta(Q_*, M))$$

$$= \nu_{x_0}(\Gamma^{-1}(\pi_T^{-1}(B_t)_{t \in T} \cap \mathcal{C}_{x_0}(M)))$$

and this implies that the result is true for $\mathcal{R} \cap \mathcal{C}_{x_0}(M)$. Take $B \in \mathcal{R}' \setminus \mathcal{R}$, then there exists a sequence $(A_n)_{n \in \mathbb{N}} \subset \mathcal{R}$ such that $\chi_{A_n} \rightarrow \chi_B$ pointwise as $n \rightarrow \infty$. Therefore, by the dominated convergence theorem,

$$\mu_{x_0}(B) = \int_{\mathcal{C}_{x_0}(M)} \chi_B \, d\mu_{x_0} = \lim_{n \rightarrow \infty} \int_{\mathcal{C}_{x_0}(M)} \chi_{A_n} \, d\mu_{x_0} = \lim_{n \rightarrow \infty} \mu_{x_0}(A_n).$$

Hence, since the thesis is satisfied for $\mathcal{R} \cap \mathcal{C}_{x_0}(M)$, we get

$$\mu_{x_0}(B) = \lim_{n \rightarrow \infty} \mu_{x_0}(A_n) = \lim_{n \rightarrow \infty} \nu_{x_0}(\Gamma^{-1}(A_n)).$$
On the other hand, since $\Gamma, \nu_{x_0} = \nu_{x_0}(\Gamma^{-1}(\cdot))$ is also a probability measure, by the same argumentation we arrive to

$$\nu_{x_0}(\Gamma^{-1}(B)) = \lim_{n \to \infty} \nu_{x_0}(\Gamma^{-1}(A_n)).$$

(7)

The result is proved joining (6) and (7).

Now, we will prove that the operator $\Gamma$ induces an isometric isomorphism between the $L_p$ spaces of $(\Omega_M, \bigotimes_{t \in Q_\ast} B_M, \nu_{x_0})$ and $(C_{x_0}(M), B_{x_0}, \mu_{x_0})$. This identification allows to work with the continuous space $C_{x_0}(M)$ in a discrete way thorough $\Omega_M$. This will be the essence of the finite dimensional decomposition of $L_p(C_{x_0}(M), \mu_{x_0})$.

**Proposition 3.2.** Let $1 \leq p < \infty$, then the following spaces are isometrically isomorphic

$$L_p(\Omega_M, \nu_{x_0}) \simeq L_p(C_{x_0}(M), \mu_{x_0}).$$

**Proof.** Let us consider the operator $\Phi : L_p(C_{x_0}(Q_\ast, M), \nu_{x_0}) \to L_p(C_{x_0}(M), \mu_{x_0})$ defined by

$$[\Phi(f)](\gamma) = \begin{cases} f(\Gamma^{-1}(\gamma)) & \text{if } \gamma \in H^\theta_{x_0}(M) \\ 0 & \text{if } \gamma \in C_{x_0}(M) \setminus H^\theta_{x_0}(M) \end{cases}$$

for $f \in L_p(C_{x_0}(Q_\ast, M), \nu_{x_0})$. Clearly $\Phi$ is well defined and bijective. By the change of variable formula, we have for every $f \in L_p(C_{x_0}(Q_\ast, M), \nu_{x_0})$

$$\|\Phi(f)\|_{L_p(\mu_{x_0})}^p = \int_{C_{x_0}(M)} |\Phi(f)|^p d\mu_{x_0} = \int_{H^\theta_{x_0}(M)} |f(\Gamma^{-1}(\gamma))|^p d\nu_{x_0}(\gamma)$$

$$= \int_{H^\theta_{x_0}(M)} |f(\Gamma^{-1}(\gamma))|^p d\nu_{x_0}(\Gamma^{-1}(\gamma)) = \int_{H^\theta_{x_0}(Q_\ast, M)} |f|^p d\nu_{x_0}$$

$$= \int_{C_{x_0}(Q_\ast, M)} |f|^p d\nu_{x_0} = \|f\|_{L_p(\nu_{x_0})}^p.$$ 

Therefore, $\Phi$ is an isometric isomorphism. On the other hand, since $\nu_{x_0}(C_{x_0}(Q_\ast, M)) = 1$, we have that the operator

$$\Lambda : L_p(C_{x_0}(Q_\ast, M), \nu_{x_0}) \to L_p(\Omega_M, \nu_{x_0})$$

$$f \mapsto f \chi_{C_{x_0}(Q_\ast, M)}$$

is also an isometric isomorphism. Finally, composing the last operators, we get that $\Phi \circ \Lambda^{-1} : L_p(\Omega_M, \nu_{x_0}) \to L_p(C_{x_0}(M), \mu_{x_0})$ defines also an isometric isomorphism.

**3.3. Finite dimensional Approximation Theorem.** Once we have reduced the problem of the decomposition of $L_p(C_{x_0}(M), \mu_{x_0})$ to the discrete version $L_p(\Omega_M, \nu_{x_0})$, we will use the Banach inductive limit techniques to this last space to reduce it in terms of finite dimensional ones following the philosophy of Theorem 2.2. First of all we have to define the inductive chain we will work with.

**Definition 3.3.** A family $\mathcal{P} = \{\{t^n_i\}_{i=0}^n\}_{n \in \mathbb{N}}$ of subsets of $Q_\ast$ satisfying for each $n \in \mathbb{N}$ the statements

1. $\{t^n_i\}_{i=0}^n \subset \{t^i_{n+1}\}_{i=0}^{n+1}$ for each $n \in \mathbb{N}$,
2. $0 = t^0_n < t^1_n < t^2_n < \cdots < t^n_n$ for each $n \in \mathbb{N}$,
3. $\bigcup_{n \in \mathbb{N}} \{t^n_i\}_{i=0}^n = Q_\ast$,

will be called a Wiener partition of $Q_\ast$. 


Let \( \mathcal{P} = \{ \{ t^n_i \}_{i=0}^n \}_{n \in \mathbb{N}} \) be a fixed Wiener partition. For this partition, consider the probability spaces \( (M^n, \bigotimes_{i=1}^n \mathcal{B}_M, \nu^n_{x_0}) \) with \( \nu^n_{x_0} \) the measure defined by the density

\[
d\nu^n_{x_0} = \prod_{i=1}^n p_{t^n_i-t^n_{i-1}}(x_i, x_{i-1}) \prod_{i=1}^n d\mu(x_i)
\]

with \( x_0 = x_0 \), for each \( n \in \mathbb{N} \). Throughout this section, we will identify the spaces

\[
L_p(M^n, \nu^n_{x_0}) \simeq \{ f(x_{t^n_1}, x_{t^n_2}, \ldots, x_{t^n_n}) : f \in L_p(M^n, \nu^n_{x_0}) \} \subset L_p(\Omega_M, \nu_{x_0}).
\]

Note that this spaces are ordered by inclusion as follows

\[
L_p(M, \nu_{x_0}) \subset L_p(M^2, \nu_{x_0}^2) \subset \cdots \subset L_p(M^n, \nu_{x_0}^n) \subset \cdots \subset L_p(\Omega_M, \nu_{x_0}).
\]

**Proposition 3.4.** Let \( 1 \leq p < \infty \), then the subspace \( \bigcup_{n \in \mathbb{N}} L_p(M^n, \nu^n_{x_0}) \) is dense in \( L_p(\Omega_M, \nu_{x_0}) \).

**Proof.** We have by definition that \( \bigotimes_{t \in \mathbb{Q}} \mathcal{B}_M = \sigma(\mathcal{R}) \), where \( \mathcal{R} \) is defined by \( \Box \), therefore since \( (\Omega_M, \nu_{x_0}) \) has finite measure, by [1, Lemma 3.4.6], \( \text{Span} \{ \chi_R : R \in \mathcal{R} \} \) is dense in \( L_p(\Omega_M, \nu_{x_0}) \). Since

\[
\{ \chi_R : R \in \mathcal{R} \} \subset \bigcup_{n \in \mathbb{N}} L_p(M^n, \nu^n_{x_0}),
\]

the proof is concluded. \( \square \)

Consider the chain \( (L_p(M^n, \nu^n_{x_0}), T_n)_{n \in \mathbb{N}} \) where \( T_n : L_p(M^n, \nu^n_{x_0}) \hookrightarrow L_p(M^{n+1}, \nu^n_{x_0}) \) is the canonical embedding defined by \( T_n(f) = f \) for every \( f \in L_p(M^n, \nu^n_{x_0}) \). Then, it is straightforward to verify that the chain \( (L_p(M^n, \nu^n_{x_0}), T_n)_{n \in \mathbb{N}} \) defines an inductive chain. This is the chain we will work with. The Banach inductive limit associated to this chain is defined through the Banach space

\[
\lim_n L_p(M^n, \nu^n_{x_0}) := \left\{ (f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_p(M^n, \nu^n_{x_0}) : \| f_n - f_m \|_{L_p(\nu^n_{x_0})} \xrightarrow{n,m \to \infty} 0 \right\} / \sim,
\]

where we are identifying \( T_n(f_n) \equiv f \) for each \( n \in \mathbb{N} \). Recall that we relate \( (f_n)_{n \in \mathbb{N}} \sim (g_n)_{n \in \mathbb{N}} \) if and only if \( \lim_{n \to \infty} f_n - g_n \|_{L_p(\nu^n_{x_0})} = 0 \) and we are considering the norm

\[
\| (f_n)_{n \in \mathbb{N}} \|_{\lim_n L_p} = \lim_{n \to \infty} \| f_n \|_{L_p(\nu^n_{x_0})}.
\]

We will identify through an isometric isomorphism the spaces \( \lim_n L_p(M^n, \nu^n_{x_0}) \) and \( L_p(\mathcal{C}_{x_0}(M), \mu_{x_0}) \), simplifying the structure of the last one in terms of finite dimensional spaces. We will follow the philosophy of Theorem \( \Box \) in the proof of this fact. The following is the main result of this section.

**Theorem 3.5.** Let \( 1 \leq p < \infty \), then the operator

\[
\mathcal{F} : \lim_n L_p(M^n, \nu^n_{x_0}) \rightarrow L_p(\mathcal{C}_{x_0}(M), \mu_{x_0})
\]

\[
(f_n)_{n \in \mathbb{N}} \mapsto \lim_{n \to \infty} \mathcal{F}(f_n)
\]

where \( \mathcal{F} : L_p(\mathcal{C}_{x_0}(\mathbb{Q}, \Omega), \nu_{x_0}) \rightarrow L_p(\mathcal{C}_{x_0}(M), \mu_{x_0}) \) is the isometric isomorphism defined by \( \Box \), defines an isometric isomorphism.

**Proof.** Given \( (f_n)_{n \in \mathbb{N}} \in \lim_n L_p(M^n, \nu^n_{x_0}) \), since \( \mathcal{F} \) is a linear isometry, we have that

\[
\| \mathcal{F}(f_n) - \mathcal{F}(f_m) \|_{L_p(\nu_{x_0})} = \| f_n - f_m \|_{L_p(\nu^n_{x_0})} = \| f_n - f_m \|_{L_p(\nu^n_{x_0})} \xrightarrow{n,m \to \infty} 0.
\]
Therefore the sequence \((\Phi(f_n))_{n \in \mathbb{N}}\) is Cauchy in \(L_p(C_{\mathbb{X}_0}(M), \mu_{\mathbb{X}_0})\) and this implies that \(\mathfrak{T}\) is well defined. It is straightforward to verify the linearity of \(\mathfrak{T}\). On the other hand, for each \((f_n)_{n \in \mathbb{N}} \in \lim_n L_p(M^n, \nu_{\mathbb{X}_0}^n)\) we have

\[
\|\mathfrak{T}(f_n)_{n \in \mathbb{N}}\|_{L_p(\mu_{\mathbb{X}_0})} = \left| \lim_{n \to \infty} \frac{\|f_n\|_{L_p(\mu_{\mathbb{X}_0})}}{\|\Phi(f_n)\|_{L_p(\mu_{\mathbb{X}_0})}} \right| = \lim_{n \to \infty} \|f_n\|_{L_p(\nu_{\mathbb{X}_0}^n)} = \|f_n\|_{L_p(M^n, \nu_{\mathbb{X}_0}^n)} \|f_n\|_{L_p(M^n, \nu_{\mathbb{X}_0}^n)}.
\]

Hence \(\mathfrak{T}\) is a linear isometry. Finally, we will prove that this operator is onto. Take \(f \in L_p(C_{\mathbb{X}_0}(M), \mu_{\mathbb{X}_0})\), then \(\Phi^{-1}(f) \in L_p(C_{\mathbb{X}_0}(\mathbb{Q}, M), \nu_{\mathbb{X}_0})\). Since \(\bigcup_{n \in \mathbb{N}} L_p(M^n, \nu_{\mathbb{X}_0}^n)\) is dense in \(L_p(C_{\mathbb{X}_0}(\mathbb{Q}, M), \nu_{\mathbb{X}_0})\), there exists a sequence \((g_n)_{n \in \mathbb{N}} \in \bigcup_{n \in \mathbb{N}} L_p(M^n, \nu_{\mathbb{X}_0}^n)\) such that

\[
g_n \xrightarrow{n \to \infty} \Phi^{-1}(f) \text{ in } L_p(C_{\mathbb{X}_0}(\mathbb{Q}, M), \nu_{\mathbb{X}_0}).
\]

Choose integers \((m_n)_{n \in \mathbb{N}} \subset \mathbb{N}\) such that \(g_n \in L_p(M^{m_n}, \nu_{\mathbb{X}_0}^{m_n})\) and

\[m_1 < m_2 < m_3 < \cdots < m_n < \cdots.\]

Define \((f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_p(M^n, \nu_{\mathbb{X}_0}^n)\) by

\[
f_n := \begin{cases} 0 & \text{if } n < m_1 \\ g_{m_i} & \text{if } m_i \leq n < m_{i+1} \end{cases}.
\]

Since \((g_n)_{n \in \mathbb{N}}\) converges, is a Cauchy sequence and hence \((f_n)_{n \in \mathbb{N}} \in \lim_n L_p(M^n, \nu_{\mathbb{X}_0}^n)\) with

\[
\mathfrak{T}(f_n)_{n \in \mathbb{N}} = \lim_{L_p(\mu_{\mathbb{X}_0})} \Phi(f_n) = \lim_{L_p(\mu_{\mathbb{X}_0})} \Phi(g_n) = (\Phi \circ \Phi^{-1})(f) = f.
\]

This concludes the proof. \(\square\)

As a rather direct application of this result, we have that if \(F \in L_p(C_{\mathbb{X}_0}(M), \mu_{\mathbb{X}_0})\), there exists a sequence \((f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_p(M^n, \nu_{\mathbb{X}_0}^n)\) such that

\[
\int_{C_{\mathbb{X}_0}(M)} |F|^p \ d\mu_{\mathbb{X}_0} = \lim_{n \to \infty} \int_{M^n} |f_n|^p \ d\nu_{\mathbb{X}_0}^n.
\]

Furthermore, if we take into account that given \(F \in L_1(C_{\mathbb{X}_0}(M), \mu_{\mathbb{X}_0})\), then we can write \(F = F^+ - F^-\) with \(F^+, F^-\) positive and \(F^+, F^- \in L_1(C_{\mathbb{X}_0}(M), \mu_{\mathbb{X}_0})\), we get the following result.

**Corollary 3.6.** Let \(F \in L_1(C_{\mathbb{X}_0}(M), \mu_{\mathbb{X}_0})\), then there exists \((f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_1(M^n, \nu_{\mathbb{X}_0}^n)\) such that

\[
\int_{C_{\mathbb{X}_0}(M)} F \ d\mu_{\mathbb{X}_0} = \lim_{n \to \infty} \int_{M^n} f_n \ d\nu_{\mathbb{X}_0}^n.
\]

More precisely, given an integrable functional \(F \in L_1(C_{\mathbb{X}_0}(M), \mu_{\mathbb{X}_0})\), there exists a functional sequence \((f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_1(M^n, \nu_{\mathbb{X}_0}^n)\) such that

\[
\int_{C_{\mathbb{X}_0}(M)} F \ d\mu_{\mathbb{X}_0} = \lim_{n \to \infty} \int_M f_n(x_1, x_2, \ldots, x_n) \prod_{i=1}^n p_{n_i - i_{n-1}}(x_i, x_{i-1}) \ d\mu(x_i).
\]
3.4. **Pinner Wiener Measure.** The same considerations can be done for the pinned space defined for fixed points $x, y \in M$, by

$$C^0_x(M) := \{ \gamma \in C([0,1], M) : \gamma(0) = x, \gamma(1) = y \}.$$

There can be constructed a Wiener measure $\lambda^y_x$ on $C^0_x(M)$, *mutatis mutandis* as we have done for $C^0_x_0(M)$, i.e., a measure $\lambda^y_x : B^y_x \to [0, +\infty]$ on the measurable space $(C^0_x(M), B^y_x)$, where $B^y_x$ denotes the borel $\sigma$-algebra of the measurable space with respect to the topology of uniform convergence, satisfying

$$\lambda^y_x(\pi^{-1}_T(B_t)_{t \in T}) = \int_{B_{t_1}} \cdots \int_{B_{t_n}} \prod_{i=1}^{n+1} p_{t_i-t_{i-1}}(x_i, x_{i-1}) \prod_{i=1}^n d\mu(x_i)$$

for each finite set $T = \{t_1, t_2, ..., t_n\} \subset [0, 1]$ with $0 = t_0 < t_1 < ... < t_n < t_{n+1} = 1$ and each $(B_t)_{t \in T} \subset B_M$, where $x_0 = x, x_{n+1} = y$.

As in the previous discussion, the measure $\lambda^y_x$ is the restriction to $C^0_x(M)$ of a measure satisfying equation (9), defined in the larger measurable space

$$\left( \times_{t \in [0,1]} M, \mathcal{P}(M) \right),$$

whose existence follows also from the Kolmogorov extension theorem. The details can be found in [3]. It must be observed that the measure $\lambda^y_x$ is not necessarily of probability since

$$\lambda^y_x(C^0_x(M)) = p_1(x, y) > 0.$$

Nevertheless, we can rewrite this measure as $\mu^y_x = p_1(x, y)^{-1}\lambda^y_x$ to transform the original measure to a probability one. Hereinafter, we will work with the probability measure $\mu^y_x$.

Observe that the last considerations can be rephrased for this measure and in particular the identity

$$\mu^y_x(\pi^{-1}_T(B_t)_{t \in T}) = \int_{B_{t_1}} \cdots \int_{B_{t_n}} p_1(x, y)^{-1} \prod_{i=1}^{n+1} p_{t_i-t_{i-1}}(x_i, x_{i-1}) \prod_{i=1}^n d\mu(x_i)$$

holds for each finite set $T = \{t_1, t_2, ..., t_n\} \subset [0, 1]$ with $0 = t_0 < t_1 < ... < t_n < t_{n+1} = 1$ and each $(B_t)_{t \in T} \subset B_M$, where $x_0 = x, x_{n+1} = y$. Moreover, this measure $\mu^y_x$ is concentrated on $C^0_{x,y}(M)$, i.e.,

$$\mu^y_x(C^0_{x,y}(M)) = 1,$$

where the notation $C^0_{x,y}(M)$ stands for the space of Hölder continuous paths of exponent $\theta$ satisfying $\gamma(0) = x, \gamma(1) = y$.

Consider the restricted probability space $(\Omega_M, \bigotimes_{t \in \mathbb{Q}^*} B_M, \tau_{x,y})$, where $\tau_{x,y}$ is the restriction of $\mu^y_x$ to $\Omega_M$. Denote by $C^0_x(Q_*, M)$ the space of continuous functions $(\omega_t)_{t \in \mathbb{Q}^*} \in \Omega_M$ satisfying $\omega_0 = x, \omega_1 = y$ and by $C^0_{x,y}(Q_*, M)$ the space of Hölder continuous paths in $\Omega_M$ of exponent $\theta$ satisfying $\omega_0 = x, \omega_1 = y$. Then

$$\tau_{x,y}(C^0_{x,y}(Q_*, M)) = 1$$

for each $\theta \in (0, 1/2)$. Therefore, we can consider the restricted space $(C^0_{x,y}(Q_*, M), B^0_{x,y}, \tau_{x,y})$, where

$$B^0_{x,y} := \bigotimes_{t \in \mathbb{Q}^*} B_M \cap C^0_{x,y}(Q_*, M).$$

Consider the injective measurable operator $\Gamma$ defined by
\[ \Gamma : \left(C_{x,y}^0(Q, M), B_{x,y}^0 \right) \rightarrow \left(C_{x,y}^0(M), B_x^0 \right) \]

where \( \gamma_\omega \in C_x^0(M) \) is the unique continuous function such that \( \gamma_\omega(t) = \omega_t \) for each \( t \in Q \).

It must be observed that also in this case \( \Gamma \) is measure preserving by a similar argument based on the proof of Lemma 3.1. Therefore, we have

\[ \mu_x^\omega(B) = \tau_{x,y}(\Gamma^{-1}(B)) \]

for each \( B \in B_x^0 \), and hence the operator \( \Gamma \) induces an isometric isomorphism between the \( L_p \) spaces of \( (\Omega_M, \tau_{x,y}) \) and \( (C_x^0(M), \mu_x^\omega) \). The explicit isomorphism is given through the operator

\[ \Phi : L_p(C_x^0(Q, M), \tau_{x,y}) \rightarrow L_p(C_x^0(M), \mu_x^\omega) \]

defined by

\[ \Phi(f) = \{ f(\Gamma^{-1}(\gamma)) \text{ if } \gamma \in C_{x,y}^0(M) \} \]

under the philosophy of Proposition 3.2. In this setting we will need a slightly modified version of the Wiener partition.

**Definition 3.7.** A family \( \mathcal{P} = \{ \{ t_n^{i+1} \}_{i=0}^{n} \} \) of subsets of \( Q \) satisfying for each \( n \in \mathbb{N} \) the statements

1. \( \{ t_n^{i+1} \}_{i=0}^{n} \subset \{ t_n^{i+1} \}_{i=0}^{n+2} \) for each \( n \in \mathbb{N} \),
2. \( 0 = t_0^n < t_1^n < \cdots < t_{n+1}^n = 1 \) for each \( n \in \mathbb{N} \),
3. \( \bigcup_{n \in \mathbb{N}} \{ t_n^{i+1} \}_{i=0}^{n} = Q \),

will be called a \( \mathcal{L} \)-Wiener partition of \( Q \).

Fix a \( \mathcal{L} \)-Wiener partition \( \mathcal{P} = \{ \{ t_n^{i+1} \}_{i=0}^{n} \} \). Consider the finite dimensional probability spaces \( (M^n, \bigotimes_{i=1}^n B_M, \tau_{x,y}^n) \) with \( \tau_{x,y}^n \) the measure defined by the density

\[ d\tau_{x,y}^n = p_1(x, y)^{-1} \prod_{i=1}^{n+1} p_{t_i^n - t_{i-1}^n}(x_i, x_{i-1}) \prod_{i=1}^{n} d\mu(x_i), \]

with \( x_0 = x, x_{n+1} = y \). This spaces allow us to consider the Banach space

\[ \lim_n L_p(M^n, \tau_{x,y}^n) := \left\{ (f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_p(M^n, \tau_{x,y}^n) : \| f_n - f_m \|_{L_p(\tau_{x,y}^n)} \xrightarrow{n,m \to \infty} 0 \right\} / \sim. \]

Finally, rephrasing the proof of Theorem 3.5 we obtain the analogous result for the pinned Wiener space.

**Theorem 3.8.** Let \( 1 \leq p < \infty \), then the operator

\[ \Phi : \lim_n L_p(M^n, \tau_{x,y}^n) \rightarrow L_p(C_x^0(M), \mu_x^\omega) \]

where \( \Phi : L_p(C_x^0(Q, M), \tau_{x,y}) \rightarrow L_p(C_x^0(M), \mu_x^\omega) \) is the isometric isomorphism defined by (12), defines an isometric isomorphism. In consequence, given \( F \in L_1(C_x^0(M), \mu_x^\omega) \), there exists a functional sequence \( (f_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_1(M^n, \tau_{x,y}^n) \) such that

\[ \int_{C_x^0(M)} F \, d\mu_x^\omega = \lim_{n \to \infty} \int_{M^n} f_n \, d\tau_{x,y}^n. \]
In particular, if we denote \( L_\alpha(M) = C_\alpha^\infty(M) \) the based loop space, we have that the map \( \mathfrak{T} : \lim_n L_p(M^n, \tau^\infty_{x,M}) \to L_p(L_\alpha(M), \mu^\infty_\alpha) \) is an isomorphism. Hereinafter, we will denote for the sake of notation \( \mu^{x_0} := \mu_{x_0} \) (when there is no confusion with the measure \( \mu_{x_0} \) defined in \( C_{x_0}(M) \)) and \( \tau^\infty_{x,X} := \tau^\infty_x \).

4. Stochastic Line Integrals

In this section, we will embed the concept of Stratonovich stochastic integral in the space \( L_2(L_{x_0}(M), \mu_{x_0}) \). This will be essential for our forthcoming analysis since we would like to work with the concept of line integrals for differential forms not only defined for smooth curves, instead we would like to define this concept for every \( \gamma \in L_{x_0}(M) \).

Let us firstly recall some basic facts about stochastic integration. Let \((X, Y) = (\{X_t\}_{t \in [0,1]}, \{Y_t\}_{t \in [0,1]})\) be a pair of bounded \( \mathbb{R} \)-valued semimartingales defined in the probability space \((\Omega, \mathcal{F}, \mu)\). Then, the Stratonovich integral of \( X \) with respect to \( Y \) is defined by the relation

\[
\int_0^1 X_t \circ dY_t := \lim_{\lambda_\mu \to \infty} \sum_{n=1}^\infty \frac{X_{t_n^\mu} + X_{t_{n-1}^\mu}}{2}(Y_{t_n^\mu} - Y_{t_{n-1}^\mu}) \in L_2(\Omega, \mu)
\]

where \( \mathcal{A} = \{\{t_n^{\mu}\}_{n=0}^\infty\}_{n \in \mathbb{N}} \) is a fixed Wiener (or L-Wiener) partition of \( Q_\ast = Q \cap [0,1] \). It is related to the Itô stochastic integral by

\[
\int_0^1 X_t \circ dY_t = \int_0^1 X_t \ dY_t + [X, Y]_t
\]

where \([X, Y]_t\) denotes the covariation of the processes \((X, Y)\). In the case in which \((X, Y) = (\{X_t\}_{t \in [0,1]}, \{Y_t\}_{t \in [0,1]})\) are bounded \( \mathbb{R}^N \)-valued semimartingales, we define

\[
\int_0^1 X_t \circ dY_t = \sum_{i=1}^N \int_0^1 X^i_t \circ dY^i_t.
\]

It is worth to mention that the usual definition of the Stratonovich integral is under convergence in probability [14, Theorem 26, Chapter V], but since we will deal with semimartingales defined on a compact manifold, we only need the definition for bounded ones, in which the convergence in probability implies the convergence in \( L_2 \) as a consequence of Vitali’s convergence Theorem.

It must be taking into account that if \((M, g)\) is a closed Riemannian manifold embedded in the euclidean space \( \mathbb{R}^N \), by [11] Proposition 3.2.1, the \( M \)-valued stochastic process \( X = \{X_t\}_{t \in [0,1]} \) defined by the coordinate functions of \((C_{x_0}(M), \mu_{x_0})\) defines a \( \mathbb{R}^N \)-valued bounded semimartingale, and this implies by [11] Proposition 1.2.7, (i) that \( \{f(X_t)\}_{t \in [0,1]} \) is a real valued semimartingale for each \( f \in C^\infty(M) \) and therefore the Stratonovich stochastic integral of \( f(X) \) with respect to \( X \), where \( f \in C^\infty(M, \mathbb{R}^N), f = (f_1, f_2, ..., f_N) \), is well defined.

The main result of this section is the following.

**Theorem 4.1.** Let \((M, g)\) be a closed Riemannian manifold embedded in the euclidean space \( \mathbb{R}^N \), \( f \in C^\infty(M, \mathbb{R}^N) \), \( f = (f_1, f_2, ..., f_N) \), and \( \{X_t\}_{t \in [0,1]} \) the \( M \)-valued semimartingale defined by the coordinate functions of \((C_{x_0}(M), \mu_{x_0})\), then

\[
\mathfrak{T} \left( \sum_{j=1}^N \sum_{i=1}^n \frac{f_j(x_i) + f_j(x_{i-1})}{2}(x^j_i - x^j_{i-1}) \right)_{n \in \mathbb{N}} = \int_0^1 f(X_t) \circ dX_t,
\]

where \( x_0 = x_0 \) and \( \mathfrak{T} : \lim_n L_2(M^n, \nu^\infty_{x_0}) \to L_2(C_{x_0}(M), \mu_{x_0}) \) is the isometric isomorphism defined in Theorem 7.3.
Proof. Since $M$ is compact and embedded in $\mathbb{R}^N$, the process $X = \{X_t\}_{t \in [0,1]}$ is, in particular, a bounded $\mathbb{R}^N$-valued semimartingale and its Stratonovich integral is defined in the usual manner by

$$
\int_0^1 f(X_t) \circ dX_t := \sum_{j=1}^N \int_0^1 f_j(X_t) \circ dX_t^j.
$$

$$
= \sum_{j=1}^N \lim_{L_2(\mu_{x_0})} \sum_{i=1}^n \frac{f_j(X_t^i) + f_j(X_t^{i-1})}{2} (X_t^i - X_t^{i-1})
= \lim_{L_2(\mu_{x_0})} \sum_{j=1}^N \sum_{i=1}^n \frac{f_j(X_t^i) + f_j(X_t^{i-1})}{2} (X_t^i - X_t^{i-1}).
$$

On the other hand, by the definition $[\mathcal{R}]$ of $\Phi : L_2(C_{x_0}(\mathbb{Q}_*, M), \nu_{x_0}) \to L_2(C_{x_0}(M), \mu_{x_0})$, if we define $F_n^j : C_{x_0}(\mathbb{Q}_*, M) \to \mathbb{R}$ by

$$
F_n^j(\omega) = \sum_{i=1}^n \frac{f_j(\omega_t^i) + f_j(\omega_t^{i-1})}{2} (\omega_t^i - \omega_t^{i-1}),
$$

for each $j \in \{1, 2, ..., N\}$ and $n \in \mathbb{N}$, then $F_n^j \in L_2(C_{x_0}(\mathbb{Q}_*, M), \nu_{x_0})$ (since is bounded) and

$$
\Phi(F_n^j) = \sum_{i=1}^n \frac{f_j(X_t^i) + f_j(X_t^{i-1})}{2} (X_t^i - X_t^{i-1}).
$$

Therefore, if we prove that $(F_n^j)_{n \in \mathbb{N}} \in \lim_n L_2(M^n, \nu_{x_0}^n)$ for each $j \in \{1, 2, ..., N\}$, by the definition of $\mathcal{K}$ and the linearity of $\Phi$, we will deduce that

$$
\mathcal{K} \left( \sum_{j=1}^N F_n^j \right) = \lim_{L_2(\mu_{x_0})} \sum_{j=1}^N \Phi(F_n^j)
= \lim_{L_2(\mu_{x_0})} \sum_{j=1}^N \sum_{i=1}^n \frac{f_j(X_t^i) + f_j(X_t^{i-1})}{2} (X_t^i - X_t^{i-1})
= \int_0^1 f(X_t) \circ dX_t.
$$

Let us see that $(F_n^j)_{n \in \mathbb{N}} \in \lim_n L_2(M^n, \nu_{x_0}^n)$ for each $j \in \{1, 2, ..., N\}$. Since $\Phi$ is an isometric isomorphism, we have

$$
\|F_n^j - F_m^j\|_{L_2(\nu_{x_0}^n)} = \|F_n^j - F_m^j\|_{L_2(\nu_{x_0}^m)} = \|\Phi(F_n^j) - \Phi(F_m^j)\|_{L_2(\mu_{x_0})} \to 0.
$$

The last step follows from the convergence of $\{\Phi(F_n^j)\}_{n \in \mathbb{N}}$ that is justified by the existence of the integral $\int_0^1 f(X_t) \circ dX_t$. \hfill $\Box$

Exactly the same considerations can be done for the measure space $(\mathcal{L}_{x_0}(M), \mu_{x_0})$. In this case, by [11] Proposition 5.5.6, the $M$-valued stochastic process $X = \{X_t\}_{t \in [0,1]}$ defined by the coordinate functions of $(\mathcal{L}_{x_0}(M), \mu_{x_0})$, defines a $\mathbb{R}^N$-valued bounded semimartingale and mutatis mutandis Theorem 4.4 we deduce the following.
Theorem 4.2. Let \((M, g)\) be a closed Riemannian manifold embedded in the euclidean space \(\mathbb{R}^N\), \(f \in C^\infty(M, \mathbb{R}^N)\), \(f = (f_1, f_2, ..., f_N)\), and \(\{X_i\}_{t \in [0, 1]}\) the \(M\)-valued semimartingale defined by the coordinate functions of \((\mathcal{L}_{x_0}(M), \mu_{x_0})\), then

\[
\mathbb{E} \left( \sum_{j=1}^N \sum_{i=1}^{n+1} f_j(x_i) + f_j(x_{i-1}) \frac{1}{2} (x_i^j - x_{i-1}^j) \right)_{n \in \mathbb{N}} = \int_0^1 f(X_t) \circ dX_t,
\]

where \(x_0 = x_{n+1} = x_0\) and \(\mathbb{E} : \lim_n L^2(M^n, \tau^t_{\mathbb{R}^N}) \rightarrow L^2(\mathcal{L}_{x_0}(M), \mu_{x_0})\) is the isometric isomorphism defined in Theorem 3.3.

4.1. Stochastic Line Integrals. Nash’s embedding theorem asserts that every compact connected Riemannian manifold \((M, g)\) can be isometrically embedded into an euclidean space \(\mathbb{R}^N\). Therefore, if we denote by \(x_1, x_2, ..., x_N\) the coordinate functions of \(\mathbb{R}^N\), there exists a partition of unity \(\{\phi_\alpha\}_\alpha\) of \(M\) and a family \(\{J_\alpha\}_\alpha\) of subsets of cardinality \(\dim M\) of \(\{1, 2, ..., N\}\) such that, for each \(\alpha\), the subfamily \(\{x_i\}_{i \in J_\alpha}\) is a local system of coordinates in a neighbourhood of the support of \(\phi_\alpha\). See [S] pp. 10-11. Let \(\omega \in \Gamma(\mathcal{T}^*M)\). Since \(\phi_\alpha \omega\) vanishes outside the support of \(\phi_\alpha\), we have that \(\phi_\alpha \omega = \sum_{i=1}^N \omega^i_\alpha dx^i\) with coefficients \(\omega^i_\alpha\) vanishing outside the support of \(\phi_\alpha\) and null when \(i \notin J_\alpha\). If we define \(f_i\) by the locally finite sum \(f_i = \sum_{\alpha} \omega^i_\alpha\), then \(\omega = \sum_{i=1}^N f_i dx^i\). Therefore if \((M, g)\) is embedded in \(\mathbb{R}^N\), every \(\omega \in \Gamma(\mathcal{T}^*M)\) can be expressed globally as

\[
\omega = \sum_{i=1}^N f_i dx^i
\]

for some \(f_i \in C^\infty(M)\). See [S] Section 2.17.

Let \((M, g)\) be a closed manifold embedded isometrically on \(\mathbb{R}^N\), \(\omega \in \Gamma(\mathcal{T}^*M)\) and \(X = \{X_i\}_{t \in [0, 1]}\) be a \(M\)-valued semimartingale. We define following [S] Definition 7.3 the line integral of \(\omega\) over \(X\) by

\[
\int_X \omega := \int_0^1 f(X_t) \circ dX_t \in L^2(\Omega, \mu),
\]

where \(\omega = \sum_{i=1}^N f_i dx^i\) and \(f = (f_1, f_2, ..., f_N)\). An important property of this definition is that the line integral has a similar behaviour under gradient fields as the classical concept of line integral over differentiable curves: If \(\omega\) is exact, i.e., there exists \(f \in C^\infty(M)\) such that \(\omega = df\), then

\[
\int_X \omega = f(X_1) - f(X_0) \text{ almost surely.}
\]

Let \(X = \{X_i\}_{t \in [0, 1]}\) be the \(M\)-valued semimartingale defined by the coordinate functionals of \((\mathcal{L}_{x_0}(M), \mu_{x_0})\), then \(\int_X \omega \in L^2(\mathcal{L}_{x_0}(M), \mu_{x_0})\) and we define the line integral of \(\omega\) over a continuous curve \(\gamma \in \mathcal{L}_{x_0}(M)\) by

\[
\int_\gamma \omega := \left[ \int_X \omega \right](\gamma).
\]

In the same manner, let \(X = \{X_i\}_{t \in [0, 1]}\) be the \(M\)-valued semimartingale given by the coordinate functionals of \((\mathcal{L}_{x_0}(M), \mu_{x_0})\), then \(\int_X \omega \in L^2(\mathcal{L}_{x_0}(M), \mu_{x_0})\) and we define

\[
\int_\gamma \omega := \left[ \int_X \omega \right](\gamma) \text{ for each } \gamma \in \mathcal{L}_{x_0}(M).
\]

Thanks to Theorems 4.1 and 4.2, we can easily embed the concept of line integral on \(L^2(\mathcal{L}_{x_0}(M), \mu_{x_0})\) and \(L^2(\mathcal{L}_{x_0}(M), \mu_{x_0})\).
Theorem 4.3. Let $X = \{X_t\}_{t \in [0,1]}$ be the $M$-valued semimartingale defined by the coordinate functionals of $(\mathcal{C}_{x_0}(M), \mu_{x_0})$ and $\omega \in \Gamma(T^*M)$ given by $\omega = \sum_{i=1}^{N} f_i dx_i$, then

$$\mathcal{J} \left( \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{f_j(x_i) + f_j(x_{i-1})}{2} (x_i^j - x_{i-1}^j) \right)_{n \in \mathbb{N}} = \int_{X} \omega,$$

where $x_0 = x_0$ and $\mathcal{J} : \lim_{n} L_2(M^n, v_{x_0}^n) \to L_2(\mathcal{C}_{x_0}(M), \mu_{x_0})$ is the isometric isomorphism defined in Theorem 3.3. Moreover, if $X = \{X_t\}_{t \in [0,1]}$ is the $M$-valued semimartingale defined by the coordinate functionals of $(\mathcal{L}_{x_0}(M), \mu_{x_0})$, then

$$\mathcal{J} \left( \sum_{i=1}^{N} \sum_{j=1}^{n+1} \frac{f_j(x_i) + f_j(x_{i-1})}{2} (x_i^j - x_{i-1}^j) \right)_{n \in \mathbb{N}} = \int_{X} \omega,$$

where $x_0 = x_{n+1} = x_0$ and $\mathcal{J} : \lim_{n} L_2(M^n, \tau_{x_0}^n) \to L_2(\mathcal{L}_{x_0}(M), \mu_{x_0})$ is the isometric isomorphism defined in Theorem 3.3.

4.2. Path Integration and Holes. Thanks to the concept of stochastic line integral that allows to consider line integrals over every continuous curve, we are able to state our first geometric result. It consists in an algorithm to determine if the fundamental dimensional hole in a given Riemannian manifold in terms of a limit of finite dimensional integrals over $M$.

Theorem 4.4. Let $(M, g)$ be a closed Riemannian manifold. If there exists a closed form $\omega \in \Gamma(T^*M)$ such that

$$\int_{\mathcal{L}_{x_0}(M)} \exp \left\{ - \left\| \oint_{\gamma} \omega \right\| \right\} d\mu_{x_0}(\gamma) \neq 1,$$

then $\pi_1(M, x_0) \neq \{0\}$. Moreover, if $M$ is embedded in an euclidean space $\mathbb{R}^N$ and there exists a closed form $\omega \in \Gamma(T^*M)$ defined by $\omega = \sum_{j=1}^{n} f_j dx_j$, such that the limit

$$\lim_{n \to \infty} \int_{M^n} \exp \left\{ - \left| \sum_{j=1}^{n+1} \frac{f_j(x_i) + f_j(x_{i-1})}{2} (x_i^j - x_{i-1}^j) \right| \right\} \prod_{i=1}^{n+1} p_{1_{n-1}}(x_i, x_{i-1}) \prod_{i=1}^{n} d\mu(x_i)$$

differs from $p_1(x_0, x_0)$, then necessarily $\pi_1(M, x_0) \neq \{0\}$.

Proof. Let us start proving that $H_{dR}^1(M) \neq \{0\}$, where $H_{dR}^1(M)$ is the first de Rham cohomology group of $M$. Suppose $H_{dR}^1(M) = \{0\}$. Then each $\omega \in \Gamma(T^*M)$ can be written as $\omega = df$ for some $f \in \mathcal{C}^\infty(M)$. Therefore, for each $\omega \in \Gamma(T^*M)$, we have

$$\oint_{\gamma} \omega = \left( \int_{X} \omega \right)(\gamma) = f(X_1(\gamma)) - f(X_0(\gamma)) = 0$$

for almost every $\gamma \in \mathcal{L}_{x_0}(M)$ and this implies that

$$\int_{\mathcal{L}_{x_0}(M)} \exp \left\{ - \left\| \oint_{\gamma} \omega \right\| \right\} d\mu_{x_0}(\gamma) = 1,$$

a contradiction. Therefore $H_{dR}^1(M) \neq \{0\}$. If $\pi_1(M, x_0) = \{0\}$, then $M$ would be simply connected and hence $H_{dR}^1(M) = \{0\}$, a contradiction. This proves the first statement.
We will prove the second. Let us denote \( \phi(x) = \exp(-|x|/2) \) for \( x \in \mathbb{R} \). Define the functions \( f \in L^2(L_{x_0}(M), \mu_{x_0}) \) and \( f_n \in L^2(M^n, \tau^n_{x_0}) \) by

\[
    f(\gamma) = \int_{\gamma} \omega \quad \text{and} \quad f_n(x_1, ..., x_n) = \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} \frac{f_j(x_i) + f_j(x_{i-1})}{2} (x_i^j - x_{i-1}^j),
\]

respectively. Since \( \phi \in L^\infty(\mathbb{R}) \), we have that \( \phi \circ f \in L^2(L_{x_0}(M), \mu_{x_0}) \) and \( \phi \circ f_n \in L^2(M^n, \tau^n_{x_0}) \) for each \( n \in \mathbb{N} \). Since \( \phi \) is Lipschitz continuous, we have

\[
    \| \phi \circ f_n - \phi \circ f \|^2_{L^2(\tau^n_{x_0})} = \int_{M^n} |\phi \circ f_n - \phi \circ f|^2 \, d\tau^n_{x_0} 
\]

\[
    \leq C \int_{M^n} |f_n - f|^2 \, d\tau^n_{x_0} 
\]

\[
    = C\|f_n - f\|^2_{L^2(\tau^n_{x_0})} \quad \text{as} \quad n \to \infty.
\]

where the last step follows from \( (f_n)_{n \in \mathbb{N}} \in \lim_n L^2(M^n, \tau^n_{x_0}) \). This implies that \( (\phi \circ f_n)_{n \in \mathbb{N}} \in \lim_n L^2(M^n, \tau^n_{x_0}) \). On the other hand, by Theorem 4.3 again, we have \( \Sigma(f_n)_{n \in \mathbb{N}} = f \) and therefore

\[
    \|\Phi(\phi \circ f_n) - \phi \circ f\|^2_{L^2(\mu_{x_0})} = \|\phi \circ \Phi(f_n) - \phi \circ f\|^2_{L^2(\mu_{x_0})}
\]

\[
    = \int_{L_{x_0}(M)} |\phi \circ \Phi(f_n) - \phi \circ f|^2 \, d\mu_{x_0} 
\]

\[
    \leq C \int_{L_{x_0}(M)} |\Phi(f_n) - f|^2 \, d\mu_{x_0} 
\]

\[
    = C\|\Phi(f_n) - f\|^2_{L^2(\mu_{x_0})} \quad \text{as} \quad n \to \infty.
\]

From this we get that

\[
    \Sigma(\phi \circ f_n)_{n \in \mathbb{N}} = \lim_{L^2(\mu_{x_0})} \Phi(\phi \circ f_n) = \phi \circ f,
\]

what is equivalent to

\[
    \Sigma(\phi \circ f_n)_{n \in \mathbb{N}} = \phi \left( \int_{\gamma} \omega \right).
\]

Now, since \( \Sigma \) is an isometry, we have

\[
    \lim_{n \to \infty} \int_{M^n} |\phi \circ f_n|^2 p_1^{-1}(x_0, x_0) \prod_{i=1}^{n+1} p_{\ell_i-1}(x_i, x_{i-1}) \prod_{i=1}^{n} d\mu(x_i) = \int_{L_{x_0}(M)} |\phi \circ f|^2 \, d\mu_{x_0}.
\]

Therefore the limit of the statement of the this theorem equals

\[
    p_1(x_0, x_0) \int_{L_{x_0}(M)} \exp \left\{ - \left| \int_\gamma \omega \right| \right\} \, d\mu_{x_0}(\gamma).
\]

In consequence, the conclusion follows from the first statement. \( \square \)

The same method can be used to determine the nonvanishing property of the singular cohomology group \( H^1(M, \mathbb{R}) \).

**Theorem 4.5.** Let \((M, g)\) be a closed Riemannian manifold. If there exists a closed form \( \omega \in \Gamma(T^*M) \) such that

\[
    \int_{L_{x_0}(M)} \exp \left\{ - \left| \int_\gamma \omega \right| \right\} \, d\mu_{x_0}(\gamma) \neq 1,
\]

then \( H^1(M, \mathbb{R}) \neq \{0\} \).
Proof. Let us suppose that (13) holds, then by the same argument of the proof of Theorem 4.4, we have that \( H^1_{dR}(M) \neq \{0\} \). The de Rham isomorphism \( H^1_{dR}(M) \cong H^1(M, \mathbb{R}) \) concludes the proof. \( \square \)

5. Applications of Path Integration on Manifolds

In this section, we will use path integral methods on manifolds to give some algebro-geometric results. Let us start with a given closed Riemannian manifold \((M, g)\) with a fixed point \(x_0\). We can rewrite \( \mathcal{L}_{x_0}(M) \) as the union of its path-connected components

\[
\mathcal{L}_{x_0}(M) = \bigcup_{\eta \in \pi_1(M, x_0)} [\eta]
\]

where \([\eta]\) denotes the homotopy class of \(\eta\). Let us consider the universal covering space \(\tilde{M}\) of \(M\) with covering projection \(p : \tilde{M} \to M\). It is well known, see for instance [13, Corollary 4, Section 6, Chapter 2], that the fundamental group of \(M\) based on \(x_0\), \(\pi_1(M, x_0)\) is isomorphic to the group of Deck(covering) transformations of the covering \(p : \tilde{M} \to M\), subsequently denoted by \(\text{Aut}_M \tilde{M}\). Choose \(y_0 \in p^{-1}(x_0)\), then the isomorphism is explicitly given by

\[
\Phi : \pi_1(M, x_0) \longrightarrow \text{Aut}_M \tilde{M}
\]

\[
[\eta] \mapsto \varphi_{\eta}
\]

where \(\varphi_{\eta}\) is the unique covering transformation that sends \(y_0\) to \(\tilde{\eta}(1)\), where \(\tilde{\eta}\) is the unique lifting of \(\eta\) with \(\tilde{\eta}(0) = y_0\). In this way, we can rewrite (14) in the form

\[
\mathcal{L}_{x_0}(M) = \bigcup_{\varphi \in \text{Aut}_M \tilde{M}} \mathcal{L}_{x_0}^\varphi(M)
\]

where \(\mathcal{L}_{x_0}^\varphi(M)\) denotes the path component of \(\mathcal{L}_{x_0}(M)\) corresponding to the homotopy class \(\Phi^{-1}(\varphi)\). It is not difficult to prove [3, Theorem 4.3] (see also [20]) that the map

\[
\Lambda : \bigcup \{C_{y_0}^{\varphi(y_0)}(\tilde{M}) : \varphi \in \text{Aut}_M \tilde{M}\} \longrightarrow \mathcal{L}_{x_0}(M), \quad \tilde{\gamma} \mapsto p \circ \tilde{\gamma},
\]

where here we use the notion tilde over \(\eta\) to emphasise that the curve is defined over \(\tilde{M}\), is a homeomorphism with respect to the uniform convergence topology that preserves the Wiener measure, i.e.

\[
\lambda_{x_0}(B) = \sum_{\varphi \in \text{Aut}_M \tilde{M}} \lambda^{\varphi(y_0)}_{y_0}(\Lambda^{-1}(B)), \quad B \in \mathcal{B}_{x_0}.
\]

In particular, since

\[
\Lambda(C_{y_0}^{\varphi(y_0)}(\tilde{M})) = \mathcal{L}_{x_0}^{\varphi(y_0)}(M)
\]

for each \(\varphi \in \text{Aut}_M \tilde{M}\), the restricted map \(\Lambda : C_{y_0}^{\varphi(y_0)}(\tilde{M}) \to \mathcal{L}_{x_0}^{\varphi(y_0)}(M)\) is also a homeomorphism and preserves the Wiener measure, i.e., for each \(B \in \mathcal{B}_{x_0} \cap \mathcal{L}_{x_0}^{\varphi(y_0)}(M)\),

\[
\lambda_{x_0}(B) = \lambda_{y_0}^{\varphi(y_0)}(\Lambda^{-1}(B)).
\]

In particular, from this we infer that each path connected component of \(\mathcal{L}_{x_0}(M)\) has non-vanishing Wiener measure since by equation (15) and (10), we have

\[
\lambda_{x_0}(\mathcal{L}_{x_0}^{\varphi}(M)) = \lambda_{y_0}^{\varphi(y_0)}(C_{y_0}^{\varphi(y_0)}(\tilde{M})) = \tilde{p}_1(y_0, \varphi(y_0)) > 0,
\]

where \(\tilde{p}_1(x, y)\) denotes the heat kernel of \(\tilde{M}\).
Taking into account these observations, if \( \omega \in \Gamma(T^*M) \) is a closed form and \( \phi \in L_\infty(\mathbb{R}) \), then the following equality holds

\[
\int_{\mathcal{L}_{x_0}(M)} \phi \left( \oint_\gamma \omega \right) d\mu_{x_0}(\gamma) = \sum_{\varphi \in \text{Aut}_M} \int_{\mathcal{L}_{x_0}(M)} \phi \left( \oint_\gamma \omega \right) d\mu_{x_0}(\gamma).
\]

Moreover, if the random variable \( \int_X \omega \) is constant in each path connected component of \( \mathcal{L}_{x_0}(M) \), we get the following interesting formula

(16) \[
\int_{\mathcal{L}_{x_0}(M)} \phi \left( \oint_\gamma \omega \right) d\mu_{x_0}(\gamma) = \sum_{\varphi \in \text{Aut}_M} \phi \left( \mathcal{I}(\varphi) \right) \mu_{x_0}(\mathcal{L}_{x_0}^\varphi(M))
\]

where \( \mathcal{I}(\varphi) \) represent the common value of \( \int_X \omega \) on the path connected component \( \mathcal{L}_{x_0}^\varphi(M) \). Equation (16) will be the key to understand the relationship between geometry and path integration. This equation can be seen as a real counterpart of the DeWitt-Laidlaw formula concerning the relationship between Feynman path integral and homotopy classes [9, 16].

Let us denote by \( \mathcal{M} \) the class of closed Riemannian manifolds \((M, g)\) such that \( \int_X \omega \) is constant in each path connected component of \( \mathcal{L}_{x_0}(M) \) for every closed form \( \omega \in \Gamma(T^*M) \). I want to remark that it will be fundamental to obtain a criteria to determine if a given closed Riemannian manifold belongs to \( \mathcal{M} \). There are some approaches to this issue in terms of Malliavin calculus, see [1, Theorem 1.1]. I firmly believe that \( \mathcal{M} \) consists in all closed Riemannian manifolds and I think it can be proven using the techniques of [1]. This will be developed in future works and from now, we left it as a conjecture.

To work with locally constant line integrals without using stochastic integration we can define it directly as follows. Let \((M, g)\) be a closed Riemannian manifold, then it is well known that if \( \gamma_0, \gamma_1 \) are path-homotopic smooth closed curves (they live in the same path connected component of \( \mathcal{L}_{x_0}(M) \)) then the classical line integral over every closed form \( \omega \in \Gamma(T^*M) \) coincides

\[
\oint_{\gamma_0} \omega = \oint_{\gamma_1} \omega.
\]

Then we can define for each closed form \( \omega \in \Gamma(T^*M) \) the step functional \( S(\omega) : \mathcal{L}_{x_0}(M) \rightarrow \mathbb{R} \) by

\[
[S(\omega)](\gamma) := \sum_{\varphi \in \text{Aut}_M} \left( \oint_{\hat{\gamma}} \omega \right) \chi_{\mathcal{L}_{x_0}(M)}(\gamma), \ \gamma \in \mathcal{L}_{x_0}(M),
\]

where \( \hat{\gamma} \) is a smooth representative of the homotopy class of \( \gamma \). It is straightforward to verify that \( S(\omega) \in L_p(\mathcal{L}_{x_0}(M), \mu_{x_0}) \) for \( 1 \leq p \leq \infty \). Hereinafter, we will use the notation

\[
[S(\omega)](\gamma) = \oint_{\gamma} \omega.
\]

It should not be confused with the stochastic line integral defined in the last section. observe that identity (16) still holds for this definition where in this case

\[
\phi(\mathcal{I}(\varphi)) = \phi \left( \oint_{\hat{\gamma}} \omega \right).
\]

Taking this route, that apparently is simpler than the stochastic techniques, makes us pay a price. The stochastic integral definition gives us a finite dimensional approximation formula that allows to compute path integrals in a rather algorithmic way. However, this simpler approach does not provides us with this gift. Nevertheless, we still hope that this definitions coincides.
5.1. The circle $S^1$. Thanks to the last results, we can give a path integral expression for the Jacobi Theta function $\theta_3$ defined by the expression

$$\theta_3(z, \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi in^2 \tau + 2\pi in z), \quad z, \tau \in \mathbb{C}, \Re \tau > 0.$$ 

It has important applications to the field of Number Theory and Abelian Varieties.

**Theorem 5.1.** Let us consider the circle $S^1 \simeq \mathbb{R}/2\sqrt{\pi} \mathbb{Z}$ of radius $1/\sqrt{\pi}$. Fix a point $x_0 \in S^1$, then

$$\theta_3(z, i) = \frac{\sqrt{\pi}}{\Gamma(\frac{i}{2})} \int_{\mathcal{L}_{x_0}(S^1)} \exp \left\{ iz \oint_{\gamma} \omega \right\} d\mu_{x_0}(\gamma), \quad z \in \mathbb{C}$$

where $\omega \in H_{dR}^1(S^1)$ is the restriction to $S^1$ of the form defined on $\mathbb{R}^2 \setminus \{0\}$ by the expression $\omega = (-ydx + xdy)/(x^2 + y^2)$.

**Proof.** It is well known that the universal covering of $M = S^1$ is $\tilde{M} = \mathbb{R}$ and that

$$\text{Aut}_M \tilde{M} = \{ \varphi^n : n \in \mathbb{Z} \}$$

where $\varphi^n(x) = x + 2\sqrt{\pi}n, \quad x \in \mathbb{R}, \text{ for each } n \in \mathbb{N}$. Computing the line integral $\oint_{\gamma} \omega$ for smooth representatives of each path connected component of $\mathcal{L}_{x_0}(S^1)$, we have that

$$\oint_{\gamma} \omega = 2\pi n$$

for almost all $\gamma \in \mathcal{L}_{x_0}(S^1)$.

In this way, by identity (10), we deduce

$$\int_{\mathcal{L}_{x_0}(S^1)} \exp \left\{ iz \oint_{\gamma} \omega \right\} d\mu_{x_0}(\gamma) = \sum_{n=-\infty}^{\infty} e^{2\pi inz} \mu_{x_0}(\mathcal{L}_{x_0}^{\varphi^n}(S^1))$$

$$= \sum_{n=-\infty}^{\infty} e^{2\pi inz} \frac{\lambda_{x_0}(\mathcal{L}_{x_0}^{\varphi^n}(S^1))}{\lambda_{x_0}(\mathcal{L}_{x_0}(S^1))}$$

$$= \sum_{n=-\infty}^{\infty} e^{2\pi inz} \frac{\tilde{p}_1(x_0, \varphi^n(x_0))}{\lambda_{x_0}(\mathcal{L}_{x_0}(S^1))},$$

where $\tilde{p}$ is the heat kernel of the universal covering space $\tilde{M} = \mathbb{R}$. Since the universal covering of $S^1 \simeq \mathbb{R}/2\sqrt{\pi} \mathbb{Z}$ is $\mathbb{R}$ with heat kernel

$$\tilde{p}_t(x, y) = \frac{1}{\sqrt{4\pi t}} \exp \left\{ -\frac{|x - y|^2}{4t} \right\},$$

and $\varphi^n(x) = x + 2\sqrt{\pi}n$, for each $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, we deduce that

$$\tilde{p}_1(x_0, \varphi^n(x_0)) = \frac{1}{\sqrt{4\pi}} e^{-\pi n^2}.$$

On the other hand

$$\lambda_{x_0}(\mathcal{L}_{x_0}(S^1)) = \sum_{n=-\infty}^{\infty} \lambda_{x_0}(\mathcal{L}_{x_0}^{\varphi^n}(S^1)) = \sum_{n=-\infty}^{\infty} \tilde{p}_1(x_0, \varphi^n(x_0)) = \frac{1}{\sqrt{4\pi}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2}.$$

The last sum is expressed in terms of known constants [28] as

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2} = \frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})}.$$
Recovering this expressions and the definition of the Jacobi Theta function, we finally obtain
\[
\int_{L_{x_0}(S^1)} \exp \left\{ i z \oint \omega \right\} \, d\mu_{x_0}(\gamma) = \frac{\Gamma \left( \frac{3}{4} \right)}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2} e^{2\pi i m z} = \frac{\Gamma \left( \frac{3}{4} \right)}{\sqrt{\pi}} \theta_3(z, i).
\]
This concludes the proof. □

As another application, we give a path integral formulation for infinite sums. This formulation yields for every positive infinite sum a path integral, as we state in the next result.

**Theorem 5.2.** Let \( \varphi : \mathbb{Z} \to \mathbb{R} \) such that \( \varphi(n) \geq 0 \) for each \( n \in \mathbb{Z} \). Fix a base point \( x_0 \in S^1 \), where \( S^1 \simeq \mathbb{R}/2\sqrt{\pi} \mathbb{Z} \), then
\[
\sum_{n=-\infty}^{\infty} \varphi(n) = \frac{3 \sqrt{\pi}}{\Gamma \left( \frac{3}{4} \right)} \int_{L_{x_0}(S^1)} \varphi \left( \oint \omega \right) \exp \left\{ \pi \left( \oint \omega \right)^2 \right\} \, d\mu_{x_0}(\gamma),
\]
where \( \omega \in H^1_{dR}(S^1) \) is the restriction to \( S^1 \) of the form defined in \( \mathbb{R}^2 \setminus \{0\} \) by the expression
\[
\omega = \frac{1}{2\pi} (-y dx + x dy)/(x^2 + y^2).
\]

**Proof.** Computing the line integral \( \oint \omega \) for smooth representatives of each path connected component of \( L_{x_0}(S^1) \), we have that
\[
\oint \omega = n \quad \text{for almost all} \quad \gamma \in L_{x_0}(\mathbb{S}^1),
\]
and therefore the measurable function \( F : L_{x_0}(\mathbb{S}^1) \to \mathbb{R} \) defined by
\[
F(\gamma) = \varphi \left( \oint \omega \right) \exp \left\{ \pi \left( \oint \omega \right)^2 \right\}
\]
is expressed as the step function
\[
F(\gamma) = \sum_{n=-\infty}^{\infty} \varphi(n) e^{\pi n^2} \chi_{L_{x_0}(\mathbb{S}^1)}(\gamma).
\]
Define the sequence of measurable functions \( F_m : L_{x_0}(\mathbb{S}^1) \to \mathbb{R} \) by
\[
F_m(\gamma) = \sum_{n=-m}^{m} \varphi(n) e^{\pi n^2} \chi_{L_{x_0}(\mathbb{S}^1)}(\gamma).
\]
Then \( F_m \geq 0 \) for each \( m \in \mathbb{N} \) and
\[
0 \leq F_1 \leq F_2 \leq \cdots \leq F_m \leq \cdots.
\]
By the Monotone Convergence Theorem,
\[
\int_{L_{x_0}(\mathbb{S}^1)} F(\gamma) \, d\mu_{x_0} = \lim_{m \to \infty} \int_{L_{x_0}(\mathbb{S}^1)} F_m(\gamma) \, d\mu_{x_0}.
\]
From the proof of Theorem 5.1, we have
\[
\mu_{x_0}(L_{x_0}^m(\mathbb{S}^1)) = \frac{\Gamma \left( \frac{3}{4} \right)}{\sqrt{\pi}} e^{-\pi n^2},
\]
and therefore
\[
\int_{L_{x_0}(\mathbb{S}^1)} F_m(\gamma) \, d\mu_{x_0}(\gamma) = \sum_{n=-m}^{m} \varphi(n) e^{\pi n^2} \mu_{x_0}(L_{x_0}^m(\mathbb{S}^1)) = \frac{\Gamma \left( \frac{3}{4} \right)}{\sqrt{\pi}} \sum_{n=-m}^{m} \varphi(n).
This implies that
\[ \int_{L_{x_0}(S^1)} F(\gamma) \, d\mu_{x_0} = \lim_{m \to \infty} \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}} \sum_{n=-m}^{m} \varphi(n) = \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \varphi(n) \]
concluding the proof.

We can apply the last result to the particular cases in which \( \varphi(n) = \exp\{-|n|\} \) and \( \varphi(n) = 1/(n^2 + 1) \). In this cases
\[ \sum_{n=-\infty}^{\infty} e^{-|n|} = \frac{e + 1}{e - 1} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} = \pi \coth(\pi) \]
and therefore we get the following.

**Proposition 5.3.** The following identities hold:
\[ \int_{L_{x_0}(S^1)} \exp \left\{ - |\oint_{\gamma} \omega| + \pi \left( \oint_{\gamma} \omega \right)^2 \right\} \, d\mu_{x_0}(\gamma) = \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}} \frac{e + 1}{e - 1}. \]
\[ \int_{L_{x_0}(S^1)} \frac{\exp \left\{ \pi \left( \oint_{\gamma} \omega \right)^2 \right\}}{\left( \oint_{\gamma} \omega \right)^2 + 1} \, d\mu_{x_0}(\gamma) = \Gamma(\frac{3}{4}) \pi^{3/4} \coth \pi. \]

### 5.2. Cohomology of one dimensional manifolds.

In the rest of this section, we will establish a reformulation of the first singular cohomology group \( H^1(S^1, \mathbb{R}) \) in terms of the path integration techniques developed in this article. This approach differs from the one used in Theorem 4.5 since it will rely in exact sequence methods, an algebraic linear point of view in contrast with the nonlinear methods used in Theorem 4.5. We study the one dimensional case since in this case the obstruction represented by the cohomology can be expressed as the obstruction to the exactness of the differential sequence. Let us consider the differential sheaf exact sequence on the circle \( S^1 \),
\[ 0 \to \mathbb{R} \stackrel{i}{\to} \Omega^0 \stackrel{d}{\to} \Omega^1 \to 0 \]
where \( \mathbb{R} \) denotes the constant sheaf of the real numbers, \( \Omega^0 \) the sheaf of differential functions on \( S^1 \) and \( \Omega^1 \) the sheaf of differential 1-form on \( S^1 \). The sheaf morphism \( i \) is the inclusion and \( d \) the differential of 1-form. Looking at the stalks, it is easy to see that this sequence is exact. Therefore, it induces a long exact sequence in cohomology whose first branch is
\[ 0 \to \mathbb{R} \to \Omega^0(S^1) \to \Omega^1(S^1) \to H^1(S^1, \mathbb{R}), \]
where \( H^1(S^1, \mathbb{R}) \) denotes the sheaf cohomology of \( S^1 \) with coefficients in the constant sheaf \( \mathbb{R} \). The cohomology group \( H^1(S^1, \mathbb{R}) \) is known to be isomorphic to the de Rham cohomology group \( H^1_{dR}(S^1) \) that is at the same time isomorphic to the singular cohomology \( H^1(S^1, \mathbb{R}) \) via de Rham isomorphism. Therefore we can see \( H^1(S^1, \mathbb{R}) \) to represent the obstruction to the exactness of the sequence of global sections
\[ 0 \to \mathbb{R} \to \Omega^0(S^1) \to \Omega^1(S^1) \to 0. \]

In our final result, we reinterpret the obstruction to the exactness of this sequence in terms of path integration and therefore it establishes a possible alternative to the sheaf cohomology group \( H^1(S^1, \mathbb{R}) \).

Before stating the result, it is convenient to give some remarks. We will use the isomorphisms
\[ \{ \gamma_0^n : n \in \mathbb{Z} \} = \pi_1(S^1, x_0) \simeq \text{Aut}_{S^1} \mathbb{R} = \{ \varphi^n : n \in \mathbb{Z} \}, \quad \gamma_0^n \mapsto \varphi^n, \]
where $\gamma_0$ is the generator of $\pi_1(S^1, x_0)$ and $\varphi$ the generator of $\text{Aut}_{S^1} \mathbb{R}$. Let us define the positively oriented loop space $L_{x_0}^+(S^1)$ by

$$L_{x_0}^+(S^1) = \bigcup_{n=1}^{\infty} L_{x_0}^n(S^1).$$

**Theorem 5.4.** The sequence

$$0 \rightarrow \mathbb{R} \rightarrow \Omega^0(S^1) \rightarrow \Omega^1(S^1) \xrightarrow{\mathcal{P}} \mathbb{R}$$

is exact, where the morphism $\mathcal{P} : \Omega^1(S^1) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{P}(\omega) = \int_{L_{x_0}^+(S^1)} [\Lambda(\omega)](\gamma) \, d\mu_{x_0}(\gamma), \quad \omega \in \Omega^1(S^1).$$

**Proof.** The sheaf exact sequence on $S^1$,

$$0 \rightarrow \mathbb{R} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \rightarrow 0$$

induce the exact sequence in cohomology

$$0 \rightarrow \mathbb{R} \rightarrow \Omega^0(S^1) \rightarrow \Omega^1(S^1) \xrightarrow{\Lambda} H^1(S^1, \mathbb{R}) \simeq H^1_{dR}(S^1)$$

where the morphism $\Lambda : \Omega^1(S^1) \rightarrow H^1_{dR}(S^1) \simeq \text{Hom}(\pi_1(S^1, x_0), \mathbb{R})$ is defined for each $\omega \in \Omega^1(S^1)$ by

$$(17) \quad [\Lambda(\omega)](\gamma) = \int_{\tilde{\gamma}} \omega, \quad \gamma \in \pi_1(S^1, x_0),$$

where $\tilde{\gamma}$ is a smooth representative of the homotopy class of $\gamma$. Hence

$$[\Lambda(\omega)](\gamma) = [S(\omega)](\gamma) = \int_\gamma \omega.$$

Therefore, we can rewrite the morphism $\mathcal{P}$ as

$$\mathcal{P}(\omega) = \int_{L_{x_0}^+(S^1)} [\Lambda(\omega)](\gamma) \, d\mu_{x_0}(\gamma), \quad \omega \in \Omega^1(S^1).$$

From this expression and the linearity of the integral, we deduce that $\mathcal{P}$ is a group homomorphism. On the other hand, denote by $\gamma_0$ the unique positively oriented generator of $\pi_1(S^1, x_0) \simeq \mathbb{Z}$ and

$$\pi_1^+(S^1, x_0) := \{ \gamma_0^n : n \in \mathbb{N} \}.$$ 

Observe that under this notation, we have

$$L_{x_0}^+(S^1) = \bigcup_{\gamma \in \pi_1^+(S^1, x_0)} [\gamma].$$

Since $\Lambda(\omega) = S(\omega)$ is constant in each path connected component and recalling the proof of Theorem 5.1, we deduce

$$\mathcal{P}(\omega) = \sum_{\gamma \in \pi_1^+(S^1, x_0)} [\Lambda(\omega)](\gamma) \cdot \mu_{x_0}([\gamma]) = \sum_{n=1}^{\infty} [\Lambda(\omega)](\gamma_0^n) \cdot \mu_{x_0}([\gamma_0^n])$$

$$= \sum_{n=1}^{\infty} n \cdot [\Lambda(\omega)](\gamma_0) \cdot \mu_{x_0}([\gamma_0^n]) = [\Lambda(\omega)](\gamma_0) \sum_{n=1}^{\infty} n \cdot \mu_{x_0}([\gamma_0^n])$$

$$= [\Lambda(\omega)](\gamma_0) \frac{\Gamma \left( \frac{1}{2} \right)}{\sqrt{\pi}} \sum_{n=1}^{\infty} ne^{-\pi n^2} = \zeta \cdot [\Lambda(\omega)](\gamma_0),$$
for a given positive constant $\zeta$. Therefore $\ker(P) = \ker(\Lambda)$ and the proof is concluded. \qed

This result establishes that the obstruction to the exactness of the differential sheaf sequence over global sections, given by the sheaf cohomology, can be reinterpreted in terms of path integration. This is illustrated in the following diagram.

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^0(S^1) & \longrightarrow & \Omega^1(S^1) \longrightarrow H^1(S^1, \mathbb{R}) \\
& & & & & & \downarrow \cong \\
& & & & & & \mathbb{R}
\end{array}
$$

Finally, we will see that the path integration morphism gives an explicit isomorphism between the cohomology group $H^1_{dR}(S^1)$ and $\mathbb{R}$.

**Corollary 5.5.** The morphism $\overline{\mathcal{P}} : H^1_{dR}(S^1) \to \mathbb{R}$ given in terms of path integration by

$$
\overline{\mathcal{P}}(\omega) = \int_{\mathcal{L}^+_0(S^1)} \left( \oint_{\gamma} \omega \right) d\mu_{x_0}(\gamma), \quad \omega \in H^1_{dR}(S^1),
$$

is a group isomorphism.

**Proof.** The following isomorphisms hold

$$
H^1_{dR}(S^1) \xrightarrow{\Phi} \text{Hom}(\pi_1(S^1, x_0), \mathbb{R}) \xrightarrow{\Gamma} \mathbb{R},
$$

where $\Phi : H^1_{dR}(S^1) \to \text{Hom}(\pi_1(S^1, x_0), \mathbb{R})$ is defined for each $\omega \in H^1_{dR}(S^1)$ by

$$
[\Phi(\omega)](\gamma) = [\mathcal{S}(\omega)](\gamma) = \oint_{\gamma} \omega, \quad \gamma \in \pi_1(S^1, x_0),
$$

and $\Gamma : \text{Hom}(\pi_1(S^1, x_0), \mathbb{R}) \to \mathbb{R}$ by

$$
\Gamma(f) = f(\gamma_0), \quad f \in \text{Hom}(\pi_1(S^1, x_0), \mathbb{R}),
$$

where $\gamma_0$ is the positively oriented generator of $\pi_1(M, x_0)$. Therefore the map $\Phi \circ \Gamma : H^1_{dR}(S^1) \to \mathbb{R}$ defined by $(\Phi \circ \Gamma)(\omega) = [\Phi(\omega)](\gamma_0)$ is an isomorphism. Applying the same argumentation of the proof of Theorem 5.4, we have

$$
\overline{\mathcal{P}}(\omega) = \zeta \cdot (\Phi \circ \Gamma)(\omega).
$$

Since $\Phi \circ \Gamma$ is an isomorphism, the proof is concluded. \qed

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