The influence of the irregular forces on the motions in the three-body problem

Solovaya N. A., Sternberg Astronomical Institute,
Lomonosov Moscow State University,
University Prospect 13, 119 899 Moscow, solov@sai.msu.ru

Abstract

The influence of the irregular forces on the evolution of a triple hierarchical stellar system moving in the field of stars have been studied. Triple hierarchical stellar systems are stable in contrary to the stellar systems with comparable distances between all components. We considered the motion in the frame of the general three-body problem using differential equations of the motion with the Hamiltonian without short-periodic terms.

For isolated triple stellar systems, where we took into account the perturbations until the third order, we obtained the solution in which the mean motions of both components have the secular accelerations. Under the influence of perturbations of the distant component the mean motion in the near pair is slowed and vice versa. The mean motion of the distant star is constantly increasing. These changes are small, but on the cosmological time interval the hierarchical systems will convert into stellar systems, in which all components have comparable distances between each other. Such systems become unstable.

In a general case, if we take into account the irregular forces, the angular momentum of this system and its summary energy might be either loss or gain. These changes may influence on the dynamical evolution and stability of the stellar system.

Introduction

We have investigated the dynamical evolution of the triple stellar system using the analytical theory of the particular case of the nonrestricted three-body problem (Orlov and Solovaya, 1988). Masses of components are comparable and the ratio of the semi-major axes of their orbits is the small parameter. The value of the eccentricities and the inclinations of both orbits are without any limitations. Unlike multiples systems with the comparable distances between components, these systems are thought to be stable.

The dynamical stability is understood as the conservation of the configuration of the system over the astronomical long time interval – the eccentricities of the orbits remain less 1, the mutual inclinations change in small limits, there are no close approaches among the bodies, and the triple system is located inside of a stellar cluster. The study of the isolated triple stellar system, where we took into account the perturbations until the third order in the Hamiltonian (Solovaya, 1977), showed that the mean motions of both components have the secular accelerations (Solovaya, 2010).

Due to the perturbations of the distant component the mean motion of the close pair slows, while for the distant component increases. These changes are small, but on the cosmological time interval hierarchical systems will converted into stellar systems, in which all components have comparable distances between each other.

If the triple stellar system is located inside of a cluster, it must be subjected to perturbations of the surrounding stars. We have done attempt to estimate the effect of their influence on the dynamical evolution and stability of the triple stellar system, moving through the gravitational field of surrounding stars. In general case it may be either loss or gain of the kinetic energy. We used the Harrington $S$ criterion of stability (Harrington, 1972). For direct motion, when the mutual inclination of the two orbital planes is less than 90°, $S = a_2 (1 - e_2) / a_1 > 3.5$. For the retrograde motion $S > 2.75$. Here $a_1$ and $a_2$ are the semi-major axes of the orbits and $e_2$ is the eccentricity of the distant component’s orbit.
The gravitational perturbations

The motion of the isolated triple stellar system was studied using the analytical theory of the general three-body problem (Orlov and Solovaya, 1988). We have used the Hamiltonian without short-periodic terms. As the intermediate orbits we used non-keplerian ellipses. For the computation was used the Jacobian coordinate system and the canonical Delaunay elements – $L_i, G_i, l_i, g_i$ ($i=1,2$) taking the invariable plane as a reference plane. Expanded in terms of the Legendre polynomials and truncated after the terms of the third order, the Hamiltonian has the form:

$$F = \frac{\gamma_1}{2L_1^2} + \frac{\gamma_2}{2L_2^2} - \frac{1}{16}\gamma_3 \frac{L_1^4}{L_2^2 G_2^3} \left[1 - 3 q^2 \right] \left( 5 - 3 \eta^2 \right) - 15 \left(1 - q^2 \right) \left(1 - \eta^2 \right) \cos 2g_1 + R, \quad (1)$$

where $R$ are the perturbations of the third order, the coefficients $\gamma_1$, $\gamma_2$, and $\gamma_3$ depend on masses, and

$$q = \frac{c^2 - G_1^2 - G_2^2}{2G_1 G_2}, \quad \eta = \sqrt{1 - e_1^2}. \quad (2)$$

c is the constant of the angular momentum, $g_1$ is the argument of the periastron of the close pair in the invariable plane, and $q$ is the cosine of the mutual inclination of the orbits. The solution in hyperelliptic integrals $I_i$ ($i = 1, 2, 3$) with the Hamiltonian until the terms of the second order was obtained by the method of Hamilton-Jacobi (Orlov and Solovaya, 1988).

Denoting the mean motions of stars $\nu_i$, which equals $2 \pi P_i^{-1}$, where $P_i$ are the periods of revolution, we found that the mean anomalies $l_1$ and $l_2$ are expressed as:

$$l_1 = B_1 + \nu_1 (t - t_0) + \text{periodic terms},$$
$$l_2 = B_2 + \nu_2 (t - t_0) + \text{periodic terms}. \quad (3)$$

a) The mean motion of the star $S_1$

$$\nu_1 = \left\{ 1 + \frac{1}{16} \frac{\gamma m^2}{(1 - e_1^2)^2} \left[4 A_3 + \frac{6 \delta}{\Sigma_1 G_2^2} (\Sigma_1 Q_1 + \Sigma_2 Q_2 + \Sigma_3 Q_3) \right] \right\} n_1. \quad (4)$$

b) The mean motion of the star $S_2$

$$\nu_2 = \left[ m - \frac{1}{16} \frac{\gamma^2}{(1 - e_2^2)^2} \frac{3 A_3 \sqrt{1 - e_2^2}}{G_2} \right] n_1, \quad (5)$$

where $\Sigma_i$ ($i = 1, 2, 3$) are the secular parts of the hyperelliptic integrals.

For stars moving along unperturbed orbits their mean motions may be expressed according to the third Keplerian law by formulae:

$$n_1 = k \sqrt{\frac{m_0 + m_1}{a_1^3}}, \quad n_2 = k \sqrt{\frac{m_0 + m_1 + m_2}{a_2^3}}, \quad (6)$$

where $a_1$ and $a_2$ are the semi-major axes of their orbits. Due to the mutual perturbations the value of the mean motion of the close pair slows by $\Delta n_1 = \nu_1 - n_1$, while that of the distant component up by $\Delta n_2 = \nu_2 - n_2$.

To the illustration of the theory the stellar system $\varepsilon$ Hydrae (ADS6993) was selected. The three brighter components of the system are moving on the orbits with the semi-major axes $a_1 = 3.967$ AU.
and $a_2 = 75.76$ AU. The eccentricities of their orbits have values $e_1 = 0.67$ and $e_2 = 0.29$. The masses of the components in the solar mass are $m_0 = 1.50$, $m_1 = 1.30$, and $m_2 = 2.74$ (Heintz, 1963).

The results showed that for the close pair

$$\nu_1 - n_1 = -0.019 \text{ year}^{-1},$$

and for the distant component

$$\nu_2 - n_2 = +0.002 \text{ year}^{-1}.$$  

Adding the perturbations of the third order $R$ in the Hamiltonian the mean motions $\nu_1$ and $\nu_2$ have the secular accelerations. The accelerations are small but on the cosmological time scale may change the configuration of the system.

The perturbation function has the following form:

$$R = \frac{15}{512} \gamma_4^4 \frac{L_1^6}{L_2^8} \frac{e_1 e_2 \sqrt{1 - e_2^2}}{(1 - e_2)^3(1 + e_2)^3} \times$$

$$\times \left[ (4 + 3 e_1^2) (-1 - 11 q + 5 q^2 + 15 q^3) \cos (g_1 - g_2) \right]$$

(Solovaya, 1997), where

$$\gamma_4 = k^2 \mu_1 \frac{m_0 - m_1}{m_0 + m_1} \frac{\beta_1^6}{\beta_2^8}.$$  

The mean motion $\nu_1$ of the close pair changes with time according to the formula

$$\nu_1 = n_{10} - \Delta n_1 - \Delta n_1^{(3)} - \sigma_1 t,$$  

while the semi-major axis $a_1$ of the close pair increases. Mean motion $\nu_2$ of the distant star increased with time too, according to the formula

$$\nu_2 = n_{20} + \Delta n_2 + \Delta n_2^{(3)} + \sigma_2 t,$$  

where $\sigma_1$ and $\sigma_2$ are the secular accelerations. Finally, the semi-major axis $a_2$ of the distant component decreases.

The evolution of the semi-major axes $a_1$ and $a_2$ within the interval of $4.3 \times 10^5$ years under the influence of the gravitational perturbation is shown in Figure 1 and 2. It is seen that under the influence of gravitational perturbations the triple hierarchical stellar system converts into the stellar system in which the components have comparable distance between each other. The coefficient $S$ is less than 3.5. Such system can be unstable.

**Effect of the dynamical friction**

Let the triple stellar system moves through the gravitational field of the surrounding stars in a stellar cluster. The triple system must be subjected to action of the irregular forces, due to random star approaches. One of them is the dynamical friction. In general case, the triple stellar system may loss or gain the kinetic energy.

Stars in the cluster are considered as point masses interacting with each other according to the Newton’s law. Each approach of the test star with the field star is based on the two-body approximation. Mass of the test star is bigger than mass of the field star. We have no information about the motion of the stars in the system. We attempt roughly to estimate the influence of dynamical friction on the motion of the stars in the triple system located in an isolated globular or open cluster. Such clusters contains several thousand stars. We shall assume that the velocities of the stars in the triple...
system not differ too much from the mean velocity of the field stars. Then the effect of the dynamical friction may be expressed by the following simplified equation:

\[ \frac{\Delta v}{\Delta t} = -\eta v, \quad (13) \]

where \( \eta \) is the coefficient of dynamical friction, \( \Delta v \) is the mean change in the velocity of the test star during the time interval \( \Delta t \).

Chandrasekhar (1943) concluded that the coefficient of dynamical friction must be of the order of the reciprocal of the time of relaxation of the system. Consequently the effect of the dynamical friction can be apparent in systems with relatively short time of relaxation. We do not know the velocity of every star. But we can use the virial theorem, which confirms that for the stable state in the gravitating spherical distribution of equal mass objects, the average potential energy must be equal the average kinetic energy, within a fractal two.

From the virial theorem

\[ 2 < K_{\text{tot}} > = < U >, \quad < V^2 > = G \frac{n m_f}{R_c}, \quad (14) \]

where \( < K_{\text{tot}} > \) is the average value of the kinetic energy and \( < U > \) is the average value of the potential energy. \( < V > \) is the average value of the velocity of a field star, \( R_c \) is the radius of the cluster, \( n \) is number of stars in the cluster, and \( m_f \) is mass of the field star.

Introduce the dynamical friction relaxation time \( T_{fr} \), defined by the equation

\[ T_{fr} = \frac{2 m_f}{m_f + m_t} T_d, \quad (15) \]

(Bertin, 2000), where \( T_d \) is the relaxation time relative to the deflection, defined as

\[ T_d = \frac{N}{\ln(N)} \tau_d, \quad (16) \]

and \( \tau_d \) is the natural crossing time

\[ \tau_d = \frac{R_c}{< V >}. \quad (17) \]

Then the dynamical friction equation is

\[ \frac{\Delta v}{\Delta t} = -\frac{v_t}{T_{fr}} = -\frac{4 \pi G^2 m_t \rho_f F(v) \ln \Lambda}{< v_t^2 >} v_t, \quad (18) \]

(Bertin, 2000), where \( m_t \) is the mass of the test star, \( m_f \) is the mass of the field star, \( F(v) \) is the relation of the velocity \( v_f \) of the field star to the velocity \( v_t \) of the test star, \( \rho_f = m_f n \) is the mass density of the cluster.

The Coulomb logarithm is defined as

\[ \ln \Lambda = \ln \frac{b_2}{b_1}, \quad (19) \]

where \( b_1 \) and \( b_2 \) are the minimum and maximum values of the impact parameters.

We used formulae of the two-body approximation. If the velocity of a star increases (decreases) on the \( \Delta v \) in its apocenter (pericenter), then the distance of the pericenter (apocenter) grows (falls) to the value

\[ 4 \Delta v \left[ \frac{a^3 (1 - e)}{\mu (1 + e)} \right]^{\frac{1}{2}}, \quad (20) \]

where \( \mu = k^2 (m_0 + m_1) \).
Results

We have used Equation (18) for the estimation of the change of velocities of components in the triple stellar system in two cases, when this system is located in the globular cluster and in the open cluster.

It is known that the average linear diameter $R_c$ of the globular cluster range from 20 pc to 2000 pc. For our calculations we adopted values $n = 10^6$, $R_c = 35$ pc, and $G = 4.3 \times 10^{-3}$ pc M$^{-1}$ km$^2$ s$^{-2}$. For the globular cluster with the equal star masses $m_f = 1$ M$_\odot$ we see result in Figures 1 and 2. In this case the dynamical friction have no the influence on the evolution of the semi-major axis of the stars in the triple system.

For the next example we took the open cluster similar to the Pleiades, with the number of members $n = 10^3$, the mass $M = 800$ M$_\odot$, and the radius of the cluster $R_c = 2$ pc. But the impact parameter $b_1$ in Coulomb logarithm was taken as a free parameter. The values of the impact parameters were selected $b_1 = 20$ AU and $b_2 = 2$ pc. The gravitational constant $G = 4.3 \times 10^{-3}$ pc M$^{-1}$ km$^2$ s$^{-2}$.

The evolution of the semi-major axes are presented in Figures 3 and 4. It is seen that within the interval $t = 5.6 \times 10^5$ years the coefficient $S$ is greater than 3.5. If close approaches are possible, the dynamic friction has the influence on the evolution of the semi-major axis of the stars in the triple system.
Figure 3. The evolution of the semi-major axis of the triple stellar system in the open cluster with $n = 3 \times 10^3$ members during the interval $t = 5.6 \times 10^5$ years under the influence of the dynamical friction for the growing values of $\Delta v$.

Figure 4. The evolution of the semi-major axis of the triple stellar system in the open cluster with $n = 3 \times 10^3$ members during the interval $t = 5.6 \times 10^5$ years under the influence of the dynamical friction for the slowing down values of $\Delta v$.

Conclusions

The mean motions of both components in the triple stellar system under the influence of the gravitational perturbations have the secular accelerations. The mean motion of the close pair is slowed and vice versa. The mean motion of the distant star is constantly increasing. On the cosmological time scale the configuration of the triple stellar system converts from the hierarchical system into the stellar system, in which all components have comparable distances between each other.

The influence of the dynamical friction on the change of the semi-major axes of the stars in the triple stellar system is more less in the comparison with the gravitational perturbations of third order. In the case of a close approach of the field star to the triple stellar system the strong perturbations are possible. In this case the perturbations from the dynamical friction are greater than the gravitational perturbations of the third order. The influence of the dynamical friction on the evolution of the semi-major axes of the both orbits in the triple stellar system is evidently and the system can stay hierarchical.

References

Bertin, G.: 2000. *Dynamics of galaxies*. Cambridge Univ. Press, 74.
Chandrasekhar, S.: 1943. *Astrophys. J.*, 97, 255.
Evans, D.: 1968. *Q. H. R. Astron. Soc.*, 9, 388.
Harrington, R.: 1972. *Celestial Mech.*, 9, 322.
Heintz, W. D.: 1963. *Zeitschrift für Astrophysics*, 57, 3.

Orlov, A. A. and Solovaya, N. A.: 1988. In *Few Body Problem*. Ed. N. Valtonen Kluwer Acad. Publish., Dordrecht, 243.

Solovaya, N. A.: 1977. *Vestn. Mosk. Univ. Fiz. Astron.*, 4, 47.

Solovaya, N. A.: 2010. *Vestn. Mosk. Univ. Fiz. Astron.*, 4, 322.