Some Results from the Upwinding Compact Scheme on Continuous and Non-continuous Functions

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Abstract. In this paper, an 8th order Upwinding Compact Scheme is derived by using the idea of the nth order polynomial. Also, the dissipation and dispersion analysis of the scheme is obtained by Fourier analysis after getting the wave number of the scheme. When applying the proposed scheme on continuous functions, high order of accuracy is achieved with high resolution due to the properties of compact schemes. The idea of WENO smoothness indicator is used to detect discontinuity locations. To overcome the global dependency issue, which causes problems near discontinuities in compact schemes, the technic of the decoupling system is used to switch into a local dependency in non-smooth regions.

In this work, the promise and success of the proposed scheme is verified by gaining high order, high resolution, and non-oscillation from various numerical examples. Additionally, there is a plan to confirm the achievement of the present scheme when applying it to multi-dimensional flows with shock-turbulence interaction problems from Computational Fluid Dynamics (CFD).

Keywords: Upwinding Compact Scheme, High order, High resolution, Dispersion errors, Dissipation errors, Discontinuities.

1. Introduction

In standard finite difference schemes, the highest order of accuracy that can be reached should be one less the number of available points. Unless increasing the width of the stencil, finding a high order scheme with stable boundary closure was an essential problem. To overcome this issue, centered compact schemes were proposed by Lele [1]. The compact schemes are implicit, and these schemes not only provide a higher order of accuracy without increasing the number of grid point, but also achieve high resolution properties. Although the advantages of compact schemes are great, they cannot be directly used in problems involved discontinuities. One of the good approaches of schemes, which deal with discontinuities, is the concept of ENO (essentially non-oscillatory) schemes, which were first introduced by Harten et al. [2], and they were improved by Shu et al. [3]. Based on ENO schemes, Liu et al. [4] suggested the concept of WENO (weighted essentially non-oscillatory) schemes by assigning a specific weight for each stencil and combining them instead of choosing the smoothest scheme as used in ENO schemes. This concept was improved by Jiang and Shu [5] to gain higher order of accuracy in smooth areas with better convergence property. Despite the success of WENO schemes, they cannot achieve an optimal order of accuracy near discontinuities. To overcome this drawback, many schemes have been proposed by Liu et al. [6,7] by mixing the concepts of
WENO schemes and compact schemes. Also, we have followed this combination between compact schemes and WENO schemes and established new schemes, see [8,9].

In this paper, we constructed an 8th order upwinding compact scheme by using the idea of the nth order polynomials. The scheme achieves the prospective order of accuracy in smooth reigns without oscillation, and it maintains more accurate solution near shock waves with high resolution by using the idea of decoupling the matrix of the derivatives in the system.

2. The derivation of the scheme

To get a conservative scheme, the discretization of the domain is reconstructed such that the grid and the cell center and size are defined at mid points where the numerical flux and the primitive function are involved [9]. An 8th order Upwinding Compact Scheme (UCS8) can be derived where both function and its first derivatives at five grid points are involved using the formula below:

\[
\frac{a_1 f_{i-2} + a_2 f_{i-1} + f_i + a_3 f_{i+1} + a_4 f_{i+2}}{h} = \frac{1}{h} [a_5 (f_{i-1} - f_{i+1}) + a_6 (f_{i+1} - f_{i+2}) + a_7 (f_{i+2} - f_{i+3})]
\]

Plugging the 8th order polynomial, \(F_8(x) = \sum_{k=0}^{8} c_k x^k\) in both sides of the above formula, leads to determine the values of the coefficients \(a_1, a_2, a_3, \ldots, a_8\) as follows:

\[
a_1 = \frac{1}{36}, a_2 = \frac{4}{9}, a_3 = \frac{4}{9}, a_4 = \frac{1}{36}, a_5 = \frac{25}{216}, a_6 = \frac{185}{216}, a_7 = \frac{185}{216}, \text{and } a_8 = \frac{25}{216}
\]

3. Analysis and numerical results

3.1 Fourier analysis of errors

The Fourier analysis has been used as an efficient tool to measure the resolution of numerical approximations by quantifying the dispersion and dissipation errors [10]. The wave numbers for the UCS8 can be computed as follows:

\[
W = \frac{5}{6} \left[ \frac{32 \sin[k] + 5 \sin[2k]}{18 + 16 \cos[k] + \cos[2k]} \right]
\]

In this analysis, there is a wave number \(W\) for each finite difference scheme. The imaginary part of \(W\) quantifies the dispersion error of the finite difference scheme while the dissipation error is obtained from the real part of \(W\).

The dispersion and dissipation errors of UCS8 are the best compared to WENO [5], WCS [6], and WUCS [9] as illustrated in figures 1 and 2 below.
3.2 Numerical results for the 1-D wave equation

For time integration, Runge-Kutta method [3] is used, and UCS8 is applied to obtain the derivative in the x direction for the one-dimensional wave equation

\[ u_t + u_x = 0 \]

with initial function \( u(x, 0) = u_0(x) , 0 < x < \pi \).

To test the resolution and the discontinuity capturing of the UCS8, two kinds of initial functions are used, continuous and non-continuous

3.2.1 Wave equation with continuous initial function

UCS8 is applied to solve the wave equation with the continuous initial function \( u_0(x) = \sin(20x) \) as in figures 3.

Meanwhile, this continuous initial function is a good high frequency example, which is used to verify the scheme has an 2\(^{nd}\) order of accuracy with high resolution as in figure 4 and Tables 1-3.
Figure 4. The enlarged portion of the selected part that indicated in Figure 3.

Table 1. Errors and order in L_1 Norm from the numerical solution of UCS8 when $u_0(x) = s^2 (20x)$ at $t_k=1$

| N  | $L_1$ ERROR | H/2H   | $L_1$ ORDER |
|----|-------------|--------|-------------|
| 32 | 2.07E-02    |        |             |
| 64 | 4.60E-05    | 4.49E+02| 8.81        |
| 128| 1.58E-07    | 2.91E+02| 8.18        |
| 256| 6.00E-10    | 2.63E+02| 8.04        |
| 512| 2.33E-12    | 2.57E+02| 8.01        |

Table 2. Errors and order in L_2 Norm from the numerical solution of UCS8 when $u_0(x) = s^2 (20x)$ at $t_k=1$

| N  | $L_2$ ERROR | H/2H   | $L_2$ ORDER |
|----|-------------|--------|-------------|
| 32 | 1.34E-02    |        |             |
| 64 | 2.92E-05    | 4.58E+02| 8.84        |
| 128| 9.96E-08    | 2.93E+02| 8.19        |
| 256| 3.77E-10    | 2.64E+02| 8.05        |
| 512| 1.46E-12    | 2.58E+02| 8.01        |
Table 3. Errors and order in $L_\infty$ Norm from the numerical solution of UCS8 when $u_0(x) = s \cdot (20x)$ at $t = 1$

| N  | $L_\infty$ ERROR | H/2H        | $L_\infty$ ORDER |
|----|------------------|-------------|------------------|
| 32 | 1.10E-02         |             |                  |
| 64 | 2.37E-05         | 4.66E+02    | 8.86             |
| 128| 8.01E-08         | 2.95E+02    | 8.21             |
| 256| 3.02E-10         | 2.65E+02    | 8.05             |
| 512| 1.23E-12         | 2.47E+02    | 7.95             |

3.2.2 Wave equation with non-continues initial function

For the initial function with jump discontinuity, the system is divided into subsystems according to the discontinuity locations, and these subsystems are solved using UCS8 as shown in figure 5.

Figure 5. The exact solution compared with the numerical solutions using UCS8, WUCS and WENO at time step $t = 0.2$ when

$$u_0(x) = \begin{cases} 
1, & 0 \leq x \leq 1 \\
-1, & 1 < x \leq 2 \\
0, & 2 < x \leq 3 
\end{cases}$$
3.2.3 shock/entropy Wave Interaction

The resolution and shock-capturing capacity of UCS8 are also verified by applying the scheme to the problem of shock-entropy wave interaction. The initial conditions:

\[(\rho, u, p) = \begin{cases} 
(3.857143, 2.629369, 10.33333), & x < -4 \\
(1 + 0.2 \sin(5x), 0, 1), & x \geq -4 
\end{cases}\]

are used to solve the Euler equations. The discontinuity in above problem moves into a density fluctuation field, and it is going to interact with the fluctuation. The results from idea of the decoupling system of the proposed scheme are computed for density and velocity as in figure 7.

Figures (8 and 9) below illustrate the ability of UCS8 to resolve the fine structure without oscillation in both density and velocity respectively.
4. Conclusion

In this paper, the Upwinding Compact Scheme of order 8 is constructed by solving system of 8 coefficients. From the technique of Fourier analysis, the dispersion and dissipation of the scheme are quantified from the wave number. Compared to other schemes, the present scheme has a better dispersion and dissipation. The numerical results in smooth areas clearly show the high order property of the scheme, and they present the ability of the scheme to maintain problems with high frequency. For non-smooth regions, discontinuities are detected by using the bias weight of WENO scheme, and they are efficiently captured when using the current scheme. Additionally, no serious oscillations have been noticed when applying the proposed scheme, which leads to accurate results compared with some previous schemes. For 2D problems of Navier-Stokes equations, the scheme might be applied with the aid of an appropriate shock detector.

5. Acknowledgment

The author would like to thank Mustansiriyah university -www.uomustansiriyah.edu.iq - (Baghdad – Iraq) for its support in the present work.
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