SUBMODELS OF THE SPECIAL COMPRESSIBLE FLUID ON TWO-DIMENSIONAL SUBALGEBRAS

1. Introduction

The differential equation of gas dynamic (EGD) are:

\[ D = \partial_t + u \cdot \nabla \]
\[ Du + \frac{1}{\rho} \nabla p = 0, \]
\[ D\rho + p \text{div} u = 0, \]
\[ DS = 0, \]

(1)

where \( u \) is speed, \( \rho \) is density, \( p \) is pressure, \( S \) is entropy. These variables with the special state equation admit 11 parametrical algebra operators Lee \( L_{11} \). In the Cartesian system of coordinates the \( L_{11} \) bases is the following [1, see also 2]:

\[ X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = t \partial_x + \partial_u, \quad X_5 = t \partial_y + \partial_v, \]
\[ X_6 = t \partial_z + \partial_w, \quad X_7 = y \partial_x - z \partial_y + v \partial_u - w \partial_v, \]
\[ X_8 = z \partial_x - x \partial_y + w \partial_u - u \partial_v, \quad X_9 = x \partial_y - y \partial_x + u \partial_v - v \partial_u, \quad X_{10} = \partial_t, \]
\[ X_{11} = t \partial_t + x \partial_x + y \partial_y + z \partial_z, \]

We considered the special state equation of the kind:

\[ p = \pm \rho^\gamma + F(S), \]

(2)

where \( +\rho^\gamma \) for \( \gamma > 0 \) and \( -\rho^\gamma \) for \( \gamma < 0 \). \( \gamma = 1 \), \( F(S) \) is the function of entropy. It coordinates with the fixed state equation for the fluid at high pressures and high temperatures.

EGD with the equation (2) admit additional operators:

- stretching \( X_{12} = t \partial_t - u \partial_u - v \partial_v - w \partial_w - (\gamma - 2) \rho \partial_p - \gamma \rho \partial_p, \)
- carry \( X_{13} = \partial_p, \)

where \( \gamma = 2\gamma/(\gamma - 1), \gamma \neq 1 \).

Together with \( L_{11} \) L they make up algebra Lee \( L_{13} \).

In cylindrical coordinates \( x = (x, r, \theta), u = (U, V, W), y = r \cos \theta, z = r \sin \theta, u = U, v = V \cos \theta - W \sin \theta, w = V \sin \theta + W \cos \theta \) the basis of algebra \( L_{13} \) is the following:

\[ X_1 = \partial_r, \quad X_2 = \cos \theta \partial_r - \sin \theta r^{-1} (\partial_\theta + W \partial_V - V \partial_W), \]
\[ X_3 = \sin \theta \partial_r + \cos \theta r^{-1} (\partial_\theta + W \partial_V - V \partial_W), \quad X_4 = t \partial_r + \partial_U, \]
\[ X_5 = \cos \theta (t \partial_r - \partial_V) - \sin \theta r^{-1} t (\partial_\theta + W \partial_V - V \partial_W), \quad X_6 = \theta \partial_r + \partial_V + \cos \theta r^{-1} t (\partial_\theta + W \partial_V - V \partial_W), \]
\[ X_7 = \partial_\theta, \quad X_8 = \sin \theta (r \partial_x - x \partial_\theta + V \partial_U - U \partial_V) + \cos \theta (W \partial_U - U \partial_W - x r^{-1} (\partial_\theta + W \partial_V - V \partial_W)), \]
\[ X_9 = - \cos \theta (r \partial_x - x \partial_\theta + V \partial_U - U \partial_V) + \sin \theta (W \partial_U - U \partial_W - x r^{-1} (\partial_\theta + W \partial_V - V \partial_W)), \]
\[ X_{10} = \partial_t, \quad X_{11} = t \partial_t + x \partial_x + r \partial_r, \]
\[ X_{12} = t \partial_t - U \partial_U - V \partial_V - W \partial_W - (\gamma - 2) \rho \partial_p - \gamma \rho \partial_p, \quad X_{13} = \partial_p. \]

For algebra \( L_{13} \) all subalgebras are listed [4]. If

\[ \gamma = -1, 1/3 \] there are more subalgebra than for any \( \gamma \). We shall considers two-dimentional subalgebra from optimum system for \( L_{13} \), apperking only when \( \gamma = -1, 1/3 \). We shall write out not similar two-dimentional subalgebra for this purpose:

2.1. \( X_1 + X_2, aX_4 + X_{13}, a(\gamma - 1) = 0; \)
2.2. \( X_{12}, aX_4 + X_{13}, a \neq 0; \)
2.3. \( X_1 + X_{12}, aX_4 + bX_5 + X_{13}; \)
After twice differenting by $S$, potential component of pressure, It is possible with high pressure (about $10^9\text{kg/cm}^2$) and high temperature (about $10^6\text{K}$). Let us find the meanings $F(S), \Phi(\rho^{-1}), f(\rho^{-1})$.

Comparing $p$ (2), (4) and excluding $T$ with the help of thermodynamics ($\rho, S$ – are independent parameters) we receive the identity:

$$\pm \rho^\gamma + F(S) = \Phi(\rho^{-1}) + (G_S' - F_{SS} \rho^{-1})f(\rho^{-1}), \text{ eqno} (5)$$

where $G(S)$ is determined by the additional experiment.

After twice differenting by $S$:

$$0 = -F_S' - F_{SS}'V f(V) + G_{SS}'f(V), \text{ (6)}$$

where $V = \rho^{-1}$.

Let us $F_{SS} \neq 0$, then:

$$\frac{F_S'}{F_{SS}} = -Vf(V) + \frac{G_{SS}'}{F_{SS}}f(V). \text{ (7)}$$

Once again we differentiate on $S$ and receive: $(\frac{F_S'}{F_{SS}})' = (\frac{G_{SS}'}{F_{SS}})'f(V)$. If $\frac{G_{SS}'}{F_{SS}} \neq 0$, then, dividing by variables we have $f = \text{const} = f_0$ and after the integration we subtitle it in (7). We take the contradiction, that $\rho, S$ are independing parameters.

Means $\frac{G_{SS}''}{F_{SS}} = 0$, i.e.

$$G_{SS}'' = k_0F''_{SS}, F_S' = k_1F''SS, \text{ (8)}$$

and from (7) follows, that $k_1 = -Vf(V) + k_0F(V)$.

The integration of (8) at $k_1 \neq 0$ and the substitution to (5) gives:

$$F(S) = k_1k_2e^{\frac{\rho}{\gamma}} + k_3,$$
\[
\Phi(\rho^{-1}) = \pm \rho^{\gamma} + k_3 - \frac{k_4 k_1}{k_0 - \rho^{-1}}, \tag{9}
\]

\[
f(\rho^{-1}) = \frac{\rho k_1}{\rho k_0 - 1},
\]

\[
G(S) = k_0 k_1 k_2 e^{\frac{S}{k_1}} + k_4 S + k_5,
\]

where \( k_j \) are the constants of integration.

2\textsuperscript{0}. Let us \( F_{SS} = 0 \) (it is equivalent \( k_1 = 0 \)). Then \( F(S) = k_1 S + k_0 \) and from (6) we receive (if \( G''_{SS} \neq 0 \)) \( \frac{k_1}{G'_{SS}} = f(V) = \text{const} = f_0 \). Hence from (5) are follows:

\[
\Phi(\rho^{-1}) = k_1 \rho^{-1} f_0 + k_0 - k_2 f_0 \pm \rho^{\gamma},
\]

\[
f(\rho^{-1}) = f_0,
\]

\[
G(S) = \frac{k_1}{2 f_0} S^2 + G_1(S) + G_0,
\]

where \( G_0, G_1 = \text{const} \).

3\textsuperscript{0}. Let us \( F_{SS} = 0, G_{SS} = 0 \). Then from (7) follows, that \( F_S = 0, F(S) = F_0 \). And from (5) we shall receive:

\[
\Phi(\rho^{-1}) = \pm \rho^{\gamma} + F_0 - G_1 f(\rho^{-1}),
\]

\[
G(S) = G_1(S) + G_0,
\]

where \( F_0, G_0, G_1 \) are constants.

Thus, the state equation (2) coordinates with the equation (4), if the functions \( F(S), f(\rho^{-1}), \Phi(\rho^{-1}) \) are represents in one of kinds: (9), (10), (11).

3. Calculation of invariants

For construction the submodels of the special compressible fluid necessary to calculate invariants of the subalgebras. [2, see also 1].

The algorithm of the calculation invariants consist in the following:

1. We select the system of coordinates, in witch calculates the invariants. If the subalgebra contains the operator \( X_7 \) of the rotation, it is convenient to choose cylindrical coordinates. If the operator of the rotation id not resent are convenient the cartesian coordinates.

2. We write out the operators of subalgebra in convenient system of coordinates from the list (3).

3. We enter the function \( h \), which depends from 9 variables \((t, x, u, \rho, p)\) as required invariants.

4. The function \( h \) is invariants of the subalgebra \( L =< Y_1, Y_2 > \) Only when any operators \( Y \) of subalgebra, working on invariants function, to annul it. Namely, \( Y \cdot h = 0, Y \in L \). We shall work by the operator \( Y_1 \) of the basis subalgebra \( L \) on invariant function. In result we received the linear homogeneous equation with the partial derivatives of the 1-st order. For this equation we write the characteristic equation, the system of the ordinary differential equation [6]. Let us assume, that there is an obviously complete set functionally independent invariant (integrals) \( I^k(t, x, u, \rho, p), k = 1..8 \).
4. We write down the second operator of the basis through received invariants by the rule:

\[ Y_2 = \xi^j \partial_{x^j} = \xi_j \frac{\partial I^k}{\partial x^j} \partial_{x^k} \]  

(12)

5. We shall work by the stayed operator \( Y_2 \) on invariant function \( h(I^k) \). We received the linear homogeneous equation with the partial derivatives of the 1-st order. We write down for it the equation of the characteristic. We find a complete set of invariants.

6. We pass to initial variable.

The received invariants (3) are shown into the table (see the appendix).

Example:

As an example we shall consider the subalgebra \( S_{7'} \) from (3):

\[
\begin{align*}
Y_1 &= X_1 + a X_7 + X_{12} = a \partial \theta + t \partial \theta - U \partial U - V \partial V - W \partial W + \rho \partial \rho - p \partial p, \\
Y_2 &= b X_4 + X_{13} = b t \partial x + b \partial U + \partial p.
\end{align*}
\]

Let us the invariant function \( h(t, x, \rho, p), x = (x, r, \theta), u = (U, V, W) \), satisfying to the equations \( Y_1 \cdot h = 0, Y_2 \cdot h = 0 \).

The second equation looks like \( b t h_x + b h_U + h_p = 0 \).

Let us write down the equation of the characteristics:

\[
\begin{align*}
\frac{dx}{bt} = \frac{dU}{b} = \frac{dp}{1} = \frac{d\theta}{0} = \frac{dV}{0} = \frac{dW}{0} = \frac{dp}{0} = \frac{dt}{0}.
\end{align*}
\]

We find integrals, which form a complete set functionally independent invariants:

\[
t; \rho; W, V; \theta; r; U_1 = U - xt^{-1}; p_1 = p - x(bt)^{-1}.
\]

We shall receive \( h_{1x} = 0 \), when written down the second equation through invariants by a rule (12). Means \( h = h_1(t, r, \theta, V, W, \rho, p_1, U_1) \).

The first equation looks like in new invariants for the known equations variable:

\[
ah_{1r} + h_{1\theta} + (-U+xt^{-1})h_{1U_1} - V h_{1V} - W h_{1W} + p h_{1p} + (-p+x(bt)^{-1})h_{1p_1} = 0;
\]

Having written down the characteristic equation and having calculated integrals, we receive a complete set functionally independent invariants, which in initial variable look like:

\[
r; \theta - aln|t|; Ut - x; Vt; Wt; \rho t^{-1}; pt - xb^{-1}.
\]

(13)

4. Invariant submodels of the second rank

Two-dimensional subalgebra has 5 invariants. The invariant decision is exist if all required functions are defined from expressions for invariants. These invariants are nominated by new functions from others invariants for this purpose. Others invariants necessarily will be the functions of independent variables [2]. All unknown function are defined from the received equality. Thus, the representation of the invariant decision turns out, which is substituted to EGD. The system of the equations turns out which connect only invariants and new invariant functions as a result of the substitution under the theorem of the representation invariant variety [6]. The equations for invariants refers an invariant’s submodel.

We shall write down an invariant submodel for the considered example.

We shall make equality from invariants (13): \( \theta - aln|t| = \theta_1 \).

\[
U t - x = U_1(r, \theta_1), V t = V_1(r, \theta_1), W t = W_1(r, \theta_1), \rho t^{-1} = \rho_1(r, \theta_1), pt - xb^{-1} = p_1(r, \theta_1).
\]
The representation of the invariant decision is defined from these equality: \( U = (U_1 + x)t^{-1}; \ V = V_1t^{-1}; \ W = W_1t^{-1}; \ \rho = \rho_1t; \ p = p_1t^{-1} + x(bt)^{-1} \)

The representation of the invariants decision for \( S \) can be received from the state equation: \( p = \pm\rho^{-1} + S \Rightarrow S = t^{-1}(x(b^{-1}) + S_1) \), where \( S_1 = p_1 \pm p^{-1} \) replaced the state equation in the invariant submodel.

The statement in EGD results to the invariant submodel:

\[
\begin{align*}
D_1 &= (W_1r^{-1} - a)\partial_{\theta_1} + V_1\partial_r, \\
D_1U_1 &= -(\rho_1b)^{-1}, \\
D_1V_1 + p_1\rho_1^{-1} &= W_1^2r^{-1} + V_1, \\
D_1W_1 + p_1(\rho_1r)^{-1} &= W_1 - V_1W_1r^{-1}, \\
D_1\rho_1 + \rho_1(V_1r + r^{-1}W_1\theta_1) &= -\rho_1(2 + V_1r^{-1}) \ \text{(14)} \\
D_1S_1 &= -U_1b^{-1}.
\end{align*}
\]

Any invariant submodel can be resulted to the one of the two initial types by the choice invariants [7]:

- Evolutionary (time-\( t \) is the invariant of the subalgebra)

\[
\begin{align*}
D &= \partial_t + u_2\partial_s, \\
Du_2 + b_1\rho_1^{-1}p_1s &= a_1, \\
Dv_2 &= a_2, \\
Dw_2 &= a_3, \\
D\rho_1 + \rho_1u_2s &= a_4, \\
DS_1 &= a_5, \\
b > 0;
\end{align*}
\]

- Stationary

\[
\begin{align*}
D &= u_2\partial_{x_1} + v_2\partial_{y_1}, \\
Du_2 + b_1\rho_1^{-1}p_{1x_1} &= a_1, \\
Dv_2 + b_2\rho_1^{-1}p_{1y_1} &= a_2, \\
Dw_2 &= a_3, \\
D\rho_1 + \rho_1(u_{2x_1} + v_{2y_1}) &= a_4, \\
DS_1 &= a_5, \\
b_1 > 0, \ b_2 > 0;
\end{align*}
\]

here \( a_i, b, b_i \) is the coefficients of the initial types.

From examining subalgebras (3) is received two submodels of the evolutionary type, and from others subalgebras is received ten submodels of the stationary type.

The canonical types of invariant submodels are tabulated (see appendix), where:

- 1-st column is the number of subalgebra,
2-nd column is the basic system of the coordinates in which considered the
EGD.
3-rd column is the initial type: S is the stationary type, E is the evolutionary
type,
in 4-th column are given invariants,
in 5-th column the factors of the initial type are written down.
The (14) implements to stationary initial type by replacement
\[ r = x_1, \quad \theta = \ln|t|, \quad u_2 = V_1, \quad u_2 = (x_1)^{-1}W_1 + a, \quad u_2 = U_1 \]
with factors: \( a_1 = u_2 + x_1(v_2 + a), \quad a_2 = (v_2 - a)(1 - 2u_2(x_1)^{-1}), \quad a_3 = 1 - (\rho b)^{-1}, a_4 = -\rho_1(u_2x_1^{-1} + 2), \quad a_5 = (1 + \omega_2)b^{-1} + S_1, \quad b_1 = 1, \quad b_2 = x_1^{-2}. \]
The example of reduction subalgebra 2.9’ to the canonical type.
The operators of the subalgebra are those:
\[ Y_1 = aX_7 - 2X_{11} + X_{12}, \quad Y_2 = bx_4 + X_{10} + X_{13}, b(\gamma) = 0, \quad \gamma \neq -1. \]
Invariants from independent variables look like: \( x_1 = (x - b^2 - t^2)r^{-1}, \)
\( y_1 = \theta + a^{-1}ln|\gamma|. \) The representation of the invariant decision enters the name
through the new invariant function: \( V = V_1r^{\frac{1}{2}}, W = W_1r^{\frac{1}{2}}, \rho = \rho_1r^{-\frac{1}{2}}, p = \rho_1r^{\frac{1}{2}} + t, \quad U = U_1r^{\frac{1}{2}} + bt, \) where \( V_1, W_1, \rho, p_1, U_1 \) depends on \( x_1, y_1. \)
The representation for entropy is defined from the state equation \( S = S_1r^{\frac{1}{2}} + t, \)
where \( S_1 = p_1 \pm \rho_1^{-1}. \)
The substitution to the EGD results the following invariant submodel:
\[ D_1 = (U_1 - x_1V_1)\partial_{x_1} + (W_1 + 2a^{-1}V_1)\partial_{y_1}, \]
\[ D_1U_1 + \rho_1^{-1}a - b - 2^{-1}V_1U_1, \]
\[ D_1V_1 + \rho_1^{-1}(p_{1y_1}a - 2^{-1}p_{1x_1}x_1) = W_1^2 - p_1(2\rho_1)^{-1} - 2^{-1}V_1^2, \]
\[ D_1W_1 + \rho_1^{-1}p_{1y_1} = W_1V_1, \]
\[ D_1p_1 + p_{1y_1} + V_{1x_1} + V_{1y_1}a(2r)^{-1} + W_{1y_1} = -3V_1^2 \rho_1^{-1}, \]
\[ D_1S_1 = -1 - V_1S_12^{-1}. \]
The new invariant speeds are entered on expression for \( D_1: \)
\( U_1 - x_1V_1 = u_2, \quad a^{-1}V_1 + W_1 = v_2, \quad W_1 - 2a^{-1}V_1 - x_1U_12a^{-1}, \) with which we
receive replacement: \( x_2 = x_1^2 - av_1, \quad y_2 = y_1 + 2^{-1}a \ln |x_1|, u_3 = 2x_1u_2 - av_2, \quad v_3 = (2x_1)^{-1}au_2 + v_2. \) From which follows the system (16), where:
\[ a_1 = -2x_2b - V_1(x_2U_1 + 2x_2(U_1 - x_1V_1) - aW_1 + V_1x_2(\rho)^{-1}) + (2x_2 - a)(W_1^2 - p_1(2\rho_1)^{-1} - 2^{-1}V_1^2) + 2(U_1 - x_1V_1)^2, \]
\[ a_2 = (a(2x_2)^{-1} + 1)((W_1^2 - p_1(2\rho_1)^{-1} - 2^{-1}V_1^2) - ab(2x_2)^{-1} - aV_1U_1(4x_2)^{-1} + (2x_2)^{-1}a(U_1 - x_1V_1)V_1 - 2^{-1}a^{-2}x_2 + W_1V_1, \]
\[ a_3 = W_1V_1 + 2a^{-1}((W_1^2 - p_1(2\rho_1)^{-1} - 2^{-1}V_1^2) + 2x_2a^{-1}(-b - 2^{-1}V_1U_1) + 2a^{-1}(U_1 - x_1V_1)U_1, \]
\[ a_4 = -\frac{5}{4}\rho_1V_1, \]
\[ a_5 = -1 - 2^{-1}V_1S_1, \]
\[ b_1 = (2x_22^{-1}a^2)^2 + 1 + 4x_2^2, \quad b_2 = \frac{a^2}{4x_2^2}, \quad b_3 = \pm \rho_1^{-1} + S_1. \]
5. Invariant submodel of the third rank
The expressions for invariants are define the speed and the pressure, but it is
impossible to define density (see table, appendix) for subalgebra 2.1” from optimum
system (3) at \( a = 0. \) It is possible to build a regular partially invariant submodel
in this case.
Let is give the definition to the regular partially invariant decisions generally.
Let us for algebra \( H \) are present \( J_1, \ldots, J_k \) - invariants from independent variables
and are present \( J_1, \ldots, J_l \) - invariants from dependent variables. If from invariants
\( J_1, \ldots, J_I \) are defined all dependent variables, it is possible to build an invariant submodel of the rank \( k \), nominating invariants \( J_j \) by functions from \((I_1, \ldots, I_k)\), i.e.

\[
J_j = J_j(I_1, \ldots, I_k), j = 1, \ldots, l.
\] (17)

If it is impossible all dependent variables of invariants \( J_j \), to define, then (17) gives the representation of the regular partially invariant decision of the rank \( k \), and defect \( \sigma \), which is equal to number not determined independent variables, i.e. \( \sigma = m - l \), where \( m \) is the number of dependent variables.

For subalgebra 2.1" the rank equal 3, the defect is equal 1.

Let us consider the subalgebra 2.1" more in detail.

The operators of basis are those:

\[
Y_1 = \partial_t + \partial_p, \\
Y_2 = t\partial_t - u\partial_u - v\partial_v - w\partial_w + 3\rho\partial_\rho + p\partial_p.
\]

Invariants from independent variables are: \( x, y, z \). From others invariants, specified in the table, we receive the representation of the regular partially invariant decision.

\[
u = \rho \frac{4}{3} u_1(x, y, z), : p = t + \rho \frac{4}{3} p_1(x, y, z), : \rho = \rho(t, x, y, z).
\] (18)

The substitution to the EGD, gives:

\[
-\frac{1}{3} u_1(\rho_t + \rho \frac{4}{3} u_1 \nabla \rho) + \rho \frac{4}{3} [(u_1 \nabla) u_1 + \nabla p_1] + \frac{1}{3} \rho \frac{4}{3} p_1 \cdot \nabla \rho = 0,
\] (19)

\[
\rho_t + \frac{2}{3} \rho \frac{4}{3} u_1 \cdot \nabla \rho + \rho \frac{4}{3} \text{div} u_1 = 0.
\] (20)

We shall received the representation of the decision for entropy from the state equation \( S = t + \rho \frac{4}{3} S_1 \), where \( S_1 = p_1 - 1 \).

The substitution to the equation \( DS = 0 \), gives:

\[
\frac{1}{3} S_1 \rho \frac{4}{3} (\rho_t + \rho \frac{4}{3} u_1 \cdot \nabla \rho) + u_1 \cdot \nabla S_1 + 1 = 0.
\] (21)

From (20) and (21) are follows:

\[
\frac{u_1 \cdot \nabla \rho}{\rho} = -9 \frac{(1 + u_1 \cdot \nabla S_1)}{S_1} + 3 \text{div} u_1.
\] (22)

Then it is possible to find \( \rho_t \) from the (21):

\[
\frac{1}{\rho} \rho_t = 3 \rho \frac{4}{3} [-\text{div} u_1 + 2 S_1^{-1} (u_1 \cdot \nabla S_1 + 1)] \equiv \rho \frac{4}{3} B(x).
\] (23)

Replacing \( p_1 \) on \( S_1 + 1 \) and substituting (23), (22) to the (19) we receive:

\[
\frac{1}{\rho} \nabla \rho = \left[ -\frac{1}{S_1} u_1 (1 + u_1 \cdot \nabla S_1) - \nabla S_1 - (u_1 \cdot \nabla) u_1 \right] \frac{3}{S_1 + 1} \equiv A(x).
\] (24)
By the substitution (24) to (22), we exclude $\rho$:

$$\left(\frac{u_1^2}{S_1} + 3\right)(u_1 \cdot \nabla S_1 + 1) + \frac{S_1}{S_1 + 1}(u_1 \cdot \nabla)(S_1 + \frac{1}{2} u_1^2) = 0.$$

Equating the mixed derivative functions $\ln\rho$ from (23), (24), we receive $\nabla B = \frac{1}{3} BA$, $rot A = 0$. From the last equality follows, that $A = \nabla \varphi$ and $\nabla (3\ln B - \varphi) = 0 \Rightarrow 3\ln B - \varphi = 0 \Rightarrow B = e^{3\varphi}$. From (24) follows $\rho = b(t)e^{3\varphi}$. Then from (21) we receive $b' = \frac{b}{3} S$.

The integration gives $b = (\frac{t}{3})^3$, where constant of the integration is made by zero with the help of carries on $t$ and on $p$, admitted by EGD.

So, density as $\rho = t^3 \rho_1(x, y, z)$, i.e. is the representation of the invariant decision for one-dimensional subalgebra $Y_2$.

Thus, the is a reduction of the partially invariant decision to the invariants:

$$(u_1 \cdot \nabla)u_1 + \rho_1^{-1} \cdot \rho_1 = u_1,$$

$$u_1 \cdot \nabla \rho_1 + \rho_1 \text{div} u_1 = -3\rho_1,$$

$$(25)$$

$$u_1 \cdot S_1 = -S_1,$$

where $S_1 = \rho_1 - \rho_1^{\frac{1}{3}}$, $S = t S_1$.

Literature

[1] Ovsyannikov L.V. Group properties of differential equations. - Novosibirsk: SO AN SSSR, 1962. - 240 p.
[2] Ovsyannikov L.V. Group analysis of differential equations. - M.: Nauka, 1978. - 400 p.
[3] Khabirov S.V. The invariant decisions of the rank 1 in gas dynamics // Works of the international conference "Modeling, calculation, designing in conditions of uncertainty". - Ufa: USATU, 2000, P. 104-115.
[4] Khabirov S.V. The optimum systems of the subalgebras, admitted by the equations of gas dynamics. - Ufa: Institute of the mechanics USC of RAS, 1998.- 33.
[5] Stanukovich K.P. The unsteady movements of continous environment. - M.: GITTL, 1955. - 804 p.
[6] Gunter N.M. The integration first order partial differential equations. - L.M.: GTTI, 1934. - 359 p.
[7] Khabirov S.V. The reduction of invariant submodel gas dynamics to canonical form // Mat. Zametki. - 1999. - V. 66. - N. 3. - pp. 439 - 444.

Appendix

2.1’. C.S.: D, Type: S;

Invariants: $y, z, tv, tw, tu - x + \ln | t |, \rho t^{-1}, t S - xa^{-1} + a^{-1} \ln | t |$;

Submodel (16): $a_1 = u_2, a_2 = v_2, a_3 = 1 - (a \rho_1)^{-1}, a_4 = -2 \rho_1, a_5 = S_1 + a^{-1}(1 - w_2)$, $b_1 = b_2 = 1$;

2.2’. C.S.: D, Type: S;

Invariants: $y, z, tv, tw, tu - x, \rho t^{-1}, t S - xa^{-1}$;

Submodel (16): $a_1 = u_2, a_2 = v_2, a_3 = -(a \rho_1)^{-1}, a_4 = -2 \rho_1, a_5 = S_1 + a^{-1}w_2$, $b_1 = b_2 = 1$;

2.3’. C.S.: D, Type: S;
Invariants: \( x - ayb^{-1} - \ln |t|, z, tu - ab^{-1}tv - 1, tw, ab^{-1}(tu - ayb^{-1} - 1) + tv - y, \rho r^{-1}, tS - ya^{-1}; \)

Submodel (16): \( a_1 = u_2 + a(b^2 \rho_1)^{-1} + 1, a_2 = v_2 \rho_1, a_3 = ab^{-1}(u_2 + 1) - (b \rho_1)^{-1}, a_4 = -3 \rho_1, a_5 = S_1 - ba^{-2}w_2 + a^{-1}u_2, b_1 = a^2b^2 + 1, b_2 = 1; \)

2.4'. C.S.: D, Type: E;

Submodel (15): \( a_1 = -2(1 + a^2t^2 + S^2)^{-1}[u_2(a^2t^2 - s tu_2) + w_2(t v_2 - s)] = at(\rho_1)^{-1} - sp_1(\rho_1)^{-1}, a_2 = (1 + a^2t^2 + s^2)^{-1}[2av_2 - u_2a^3t^2 - 2as w_2 + (v_2 + u_2)(2a^2t + w_2) - w_2v_2 + sv_2w_2 + w_2u_2 + s(u_2)^2] + at(\rho_1)^{-1} - sp_1(\rho_1)^{-1}, a_3 = (\rho_1)^{-1}(p_1 + a^2t^2 + at) + (1 + a^2t^2 + s^2)^{-1}[2a^2t^2 w_2 - (w_2 - su_2)^2t - 2sv_2 + a s^2 w_2] + (u_2)^2, a_4 = -2p((w_2 - su_2)(1 + a^2t^2 + s^2)^{-1}, a_5 = [w_2(S_1 - s) + v_2 - u_2(a^2t + S_1 s)](1 + a^2t^2 + s^2)^{-1}, b = (1 + a^2t^2 + s^2), p_1 = \rho_1^{-1} + S_1; \)

2.5'. C.S.: D, Type: E;

Invariants: \( t, \theta + a \ln |t|, r^{-1}(aV + W), r^{-1}U, ar^{-1}(W a - V), pr, r^{-1}(S - x); \)

Submodel (15): \( a_1 = -a(\rho_1)^{-1}(S_1 + (\rho_1)^{-1}) + (a(a^2 + 1))^{-1}(w_2^2 - a^2 u_2^2 + 2u_2w_2, a_2 = -p(\rho_1)^{-1} - v_2(a(a^2 + 1))^{-1}(au_2 - w_2), a_4 = -p(a(a^2 + 1))^{-1}(au_2 - w_2), a_5 = -v_2, b = a^2 + 1; \)