CONVOLUTIONS OF SETS WITH BOUNDED VC-DIMENSION ARE UNIFORMLY CONTINUOUS

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Abstract. We introduce a notion of VC-dimension for subsets of groups, defining this for a set $A$ to be the VC-dimension of the family $\{A \cap (xA) : x \in A \cdot A^{-1}\}$. We show that if a finite subset $A$ of an abelian group has bounded VC-dimension, then the convolution $1_A * 1_{-A}$ is Bohr uniformly continuous, in a quantitatively strong sense. This generalises and strengthens a version of the stable arithmetic regularity lemma of Terry and Wolf [20] in various ways. In particular, it directly implies that the Polynomial Bogolyubov–Ruzsa Conjecture — a strong version of the Polynomial Freiman–Ruzsa Conjecture — holds for sets with bounded VC-dimension. We also prove some results in the non-abelian setting.

1. Introduction

There are many results in additive combinatorics that attempt to describe the algebraic structure of a set satisfying some combinatorial hypotheses. A popular and general type of hypothesis for a finite subset $A$ of an abelian group is that of small-doubling, taking the form $|A + A| \leq K|A|$, where $A + A := \{a + b : a, b \in A\}$ is the sumset of $A$ with itself. Powerful conclusions are known about such sets: it is known, for example, that in this case the four-fold sumset $4A = A + A + A + A$ must contain a coset of a large subgroup if the ambient group has bounded exponent, say — such results are associated with the names of Bogolyubov and Ruzsa in the literature. Motivated by model-theoretic considerations, Terry and Wolf [20] recently considered a different type of hypothesis, which they called $k$-stability. We will define this in Section 5, but for now we note their main theorem, termed an arithmetic regularity lemma due to its relation to a a more general statement of this name due to Green [9].

Theorem 1.1 (Terry–Wolf stable arithmetic regularity). Let $\epsilon \in (0, 1)$, let $k \geq 2$ and let $p$ be a prime. Suppose $G = \mathbb{F}_p^n$ with $n \geq n_0(\epsilon, k, p)$, and that $A \subseteq G$ is $k$-stable. Then there is a subspace $H \leq G$ of codimension at most $O_k(\epsilon^{-O_k(1)})$ such that for each $x \in G$, either $|A \cap (x + H)| \leq \epsilon|H|$ or $|A \cap (x + H)| \geq (1 - \epsilon)|H|$. In other words, each coset of $H$ is either almost disjoint from $A$ or almost contained in $A$. Moreover, there is a union $W$ of cosets of $H$ such that $|A \triangle W| \leq \epsilon|G|$.

The point here is that the quantitative aspects of the conclusion are far stronger than what one obtains for similar statements without the $k$-stability condition, these being necessarily of tower-type with the height of the tower increasing with $1/\epsilon$, as proved by Green [9], akin to the bound proved by Gowers [8] in the graph regularity context; we refer the reader to [20] for more background. On the other hand, it turns out that $k$-stability is a very strong hypothesis. We shall show that a similarly strong quantitative conclusion holds if $A$ satisfies a much weaker hypothesis, and, indeed, we shall show a
more general uniform continuity result. To state these, we introduce our key definition:

**Definition 1.2.** Let $G$ be a group and let $A, B \subseteq G$. We define the VC-dimension of $B$ relative to $A$ to be the VC-dimension of the family $\{A \cap (xB) : x \in A \cdot B^{-1}\}$, and denote this by $\dim_{\text{VC}}(A, B)$. If $B = A$ we write just $\dim_{\text{VC}}(A)$ and call this the VC-dimension of $A$.

Here $A \cdot B^{-1} := \{ab^{-1} : a \in A, b \in B\}$, and the VC-dimension $\dim_{\text{VC}}(F)$ of a family $F$ of sets is the largest size of a set $X$ such that $F$ shatters $X$, i.e., for which every subset $Y \subseteq X$ can be written in the form $Y = X \cap F$ for some $F \in F$.

We shall look at this concept of VC-dimension in more depth in Section 4. For now, let us just note that it has some nice properties, such as being invariant under Freiman isomorphism, and that the commonly considered ‘nice’ sets in additive combinatorics, such as generalised arithmetic progressions and subgroups, have small VC-dimension.

As we show in Section 6, having bounded VC-dimension is a strictly weaker hypothesis than that of stability: if $A$ is $k$-stable in the sense of Terry–Wolf then $\dim_{\text{VC}}(A) \leq k - 1$, but there are sets of bounded VC-dimension that are not boundedly stable. While VC-dimension is certainly of interest in model theory, our motivation for this definition came instead from probability — it is a natural assumption when studying the suprema of empirical processes — and was indeed arrived at prior to [20]. Although not our main result, let us for comparison state a version of Theorem 1.1 valid for sets with bounded VC-dimension.\(^\dagger\)

**Theorem 1.3 (VC-bounded arithmetic regularity).** Let $\epsilon \in (0, 1)$, and let $G = \mathbb{F}_q^n$. If $A \subseteq G$ has size least $\alpha|G|$ and $\dim_{\text{VC}}(A) \leq d$, then there is subspace $H \subseteq A - A$ of codimension at most $Cde^{-C}\log(2/\alpha)$ and a union $W$ of cosets of $H$, contained in $A + H$, such that $|A \Delta W| \leq \epsilon|A|$.

We note that a density assumption like ours is implicit in Theorem 1.1: the main claim is trivial if $\epsilon \geq |A|/|G|$, just by taking $H = G$. Let us also remark that the exponent of $\epsilon^{-1}$ is very reasonable here: certainly one can take $C = 4 + o(1)$.

A group-theoretic definition of VC-dimension very similar to ours was used in a paper of P. Simon [13] and was also arrived at recently from a graph-theoretic perspective by Alon, Fox and Zhao [1], who independently from this work proved a result along the lines of the above theorem using a different argument. We make some remarks on this in Section 7.

In fact, we shall prove a more general result, valid for arbitrary finite abelian groups. For the definitions surrounding Bohr sets, see Section 3 and for a technically more complete statement, see Section 5.

**Theorem 1.4 (Arithmetic regularity, simplified Bohr set version).** Let $\epsilon \in (0, 1)$, and let $G$ be a finite abelian group. If $A \subseteq G$ has size least $\alpha|G|$ and $\dim_{\text{VC}}(A) \leq d$, then there is a Bohr set $H \subseteq A - A$ of rank $m \leq Cde^{-C}\log(2/\alpha)$ and radius at least $c\epsilon^2/m^2$, and a subset $A' \subseteq A$, such that $|A \Delta (A' + H)| \leq \epsilon|A|$.

\(^\dagger\)See the end of this section for a word on notation, in particular on the use of the letter $C$. 

From our point of view, the arithmetic regularity lemmas stated above are not the natural type of statement that our method leads to. Indeed, our definition of VC-dimension arose from its suitability in establishing continuity results for convolutions; Theorems 1.3 and 1.4 are quick corollaries. To state these continuity results properly, let us introduce some notation.

For a group $G$ and functions $f, g : G \to \mathbb{C}$, we define their convolution to be

$$f \ast g(x) = \sum_{y \in G} f(y)g(y^{-1}x),$$

provided this is well-defined, as well as the related skew convolution operation

$$f \circ g = f \ast \tilde{g},$$

where $\tilde{g}(x) = g(x^{-1})^2$. Denoting the indicator function of a set $X$ by $1_X$, a close connection between our definition of VC-dimension and convolution comes from the fact that $1_A \circ 1_B(x) = |A \cap (xB)|$. Note in particular that $A \cdot B^{-1} = \text{supp}(1_A \circ 1_B)$. It will be convenient to normalise certain sums, and for this purpose we write $\mu_A = 1_A/|A|$ for finite sets $A$. We also extend this to define measures: $\mu_A(X) := |A \cap X|/|X|$. We furthermore define the translation operator $\tau_t$ for $t \in G$ by $\tau_t f(x) = f(tx)$.

It is well-known that convolutions of indicator functions (and more general functions) are somewhat smooth, particularly if the sets satisfy some combinatorial condition like small doubling; see for example [4, 7, 6], where a notion of $L^p$-smoothness is proved for $p \geq 2$, or [17], where $L^\infty$-smoothness is proved for convolutions of three sets. Our main results say that if a set has bounded VC-dimension, then one in fact has $L^\infty$-smoothness, i.e. uniform continuity — or uniform almost-periodicity — even for a convolution of two sets. Our first result is valid for arbitrary groups:

**Theorem 1.5.** Let $\epsilon \in (0, 1]$ and $d, k \in \mathbb{N}$. Let $G$ be a group and let $A, B \subseteq G$ be finite subsets with $\dim_{VC}(A, B) \leq d$. If $|S \cdot A| \leq K|A|$ for some set $S \subseteq G$, then there is a set $T \subseteq S$ of size at least $0.99K^{-Cd^k/2^d}|S|$ such that, for each $t \in (T^{-1}T)^k$,

$$\|\tau_t(\mu_A \circ 1_B) - \mu_A \circ 1_B\|_\infty \leq \epsilon.$$

Here $X^k$ denotes $X \cdot X \cdots X$ with $k$ copies of $X$. We note that if $G$ is finite and $A$ has size at least $\alpha|G|$, then one can take $S = G$ and $K = 1/\alpha$. Similarly, if $A \subseteq [N] := \{1, 2, \ldots, N\} \subseteq \mathbb{Z}$ has $|A| \geq \alpha N$, then one can take $S = [N]$ and $K = 2/\alpha$. This small sumset condition is thus a generalisation of a density condition.

For abelian groups, we can impose some structure on the set of almost-periods, ensuring uniform continuity in the so-called Bohr topology (in a quantitatively strong sense); we again refer to Section 3 for the definitions of the terms involved.

**Theorem 1.6.** Let $\epsilon \in (0, 1]$ and $d \in \mathbb{N}$. Let $G$ be a finite abelian group and let $A, B \subseteq G$ be subsets with $\dim_{VC}(A, B) \leq d$ and $\eta := |A|/|B| \leq 1$. If $|A + S| \leq K|A|$ for some set $S \subseteq G$, then there is a regular Bohr set $T$ of rank $m \leq Cd^{-2}(\log K)(\log 2/\epsilon \eta)^2 + C\log(2/\mu_G(S))$ and radius at least $c\eta^{1/2}/m$ such that, for each $t \in T$,

$$\|\mu_A \circ 1_B(\cdot + t) - \mu_A \circ 1_B\|_\infty \leq \epsilon.$$

\[^2\]This will hopefully not cause confusion with the standard notation for composition, which we will not be using.
For a better \( \epsilon \)-dependence in the above theorems when \( B = A \), in light of a lemma from [1] discovered after the completion of this work, see Section 7.

With \( B = A \) and \( \epsilon = \frac{1}{q} \), since \( 1_A \circ \mu_A(0) = 1 \) we obtain as an essentially immediate consequence the following Bogolyubov–Ruzsa-type corollary, which we restrict to the finite field setup for clarity of exposition. We employ here some non-standard terminology:

**Definition 1.7.** We say that a set \( H \subseteq \mathbb{F}_q^n \) has \( A \)-codimension at most \( d \) if \( |H| \geq q^{-d}|A| \).

**Corollary 1.8.** Let \( A \subseteq \mathbb{F}_q^n \) be a set with \( \dim_{VC}(A) \leq d \) and \( |A - A| \leq K|A| \). Then \( A - A \) contains a subspace of \( (A - A) \)-codimension at most \( Cd \log K + C \log q \).

Thus the so-called Polynomial Bogolyubov–Ruzsa Conjecture holds for sets of bounded VC-dimension in the finite field setting, even in the strong sense of working with only the difference set \( A - A \) instead of something like \( 2A - 2A \). We note that this conjecture is in general stronger than the well-known Polynomial Freiman–Ruzsa Conjecture [11].

In the dense regime where \( |A| \geq \alpha|G| \), we have the following version.

**Corollary 1.9.** Let \( A \subseteq \mathbb{F}_q^n \) be a set with \( \dim_{VC}(A) \leq d \) and \( |A| \geq \alpha|G| \). Then \( A - A \) contains a subspace of codimension at most \( Cd \log(1/\alpha) \).

For general sets \( A \) of density at least \( \alpha \), the best known results [15, 6, 2] on subspaces in \( A - A \) say roughly that \( A - A \) contains a subspace of dimension at least \( \alpha n \), whereas in the bounded VC-dimension-case we instead get a strong upper bound on codimension. The so-called niveau set example of Ruzsa [14] (see [10, Theorem 9.4] in the finite field context) implies that one could not hope for more than codimension \( c \sqrt{n} \) in general, even for \( \alpha \) close to \( 1/2 \).

In the non-abelian context, Conant, Pillay and Terry [5] recently generalised Theorem 1.1 to arbitrary finite groups. We make some remarks on this in Section 6.

**Paper layout.** In Section 2 we prove our main general theorem, Theorem 1.5, followed in Section 3 by a review of Bohr sets and a proof of the abelian version, Theorem 1.6. In Section 4 we establish some basic properties of VC-dimension and look at some examples. Section 5 contains our applications: the proofs of the arithmetic regularity lemmas and Corollaries 1.8 and 1.9. In Section 6 we note some relationships between \( k \)-stability and VC-dimension. Finally, we end with some remarks in Section 7.

**Some notation.** Throughout the paper, we employ the very convenient ‘constantly changing constant’ device, meaning that, unless otherwise specified, the letters \( c \) and \( C \) denote positive absolute constants that can vary from occurrence to occurrence, and can be picked to make the statements true and the arguments work. In the statements of results, we always assume that the input-sets are non-empty. In the context of non-abelian groups, we write \( X^{\otimes k} \) for the Cartesian product \( X \times X \times \cdots X \), to distinguish it from the iterated product set \( X^k \). Whenever we speak of the finite field \( \mathbb{F}_q \), we allow \( q \) to be an arbitrary prime power. Finally, for two sets \( A \) and \( B \), we denote their symmetric difference by \( A \triangle B = (A \setminus B) \cup (B \setminus A) \).
2. Proof of $L^\infty$-almost-periodicity for arbitrary groups

Here we prove Theorem 1.5. Our argument largely follows the probabilistic method employed in [7], incorporating a Glivenko–Cantelli-type uniform law of large numbers valid in the bounded VC-dimension setup (and even somewhat more generally). Theorem 1.5 follows immediately from the following version with $k = 1$, by the triangle inequality.

**Theorem 2.1.** Let $\epsilon \in (0, 1]$ and $d \in \mathbb{N}$. Let $G$ be a group and let $A, B \subseteq G$ be finite subsets with $\dim_{\text{VC}}(A, B) \leq d$. If $|S : A| \leq K |A|$ for some set $S \subseteq G$, then there is a set $T \subseteq S$ of size at least $0.99 K^{-Cd/\epsilon^2} |S|$ such that, for each $t \in T^{-1}T$,

$$\|\tau_t(\mu_A \circ 1_B) - \mu_A \circ 1_B\|_\infty \leq \epsilon.$$  

Proof. Let $n$ be a parameter to be specified later, and let $\vec{a} \in A^{\otimes n}$ be sampled uniformly at random. We write

$$\mu_{\vec{a}} \circ 1_B(x) = \mathbb{E}_{j \in [n]} 1_B(x^{-1} a_j),$$

a random variable indexed by $x$, with expectation $\mu_A \circ 1_B(x)$. Our first goal is to show that with high probability,

$$\|\mu_{\vec{a}} \circ 1_B - \mu_A \circ 1_B\|_\infty \leq \frac{1}{2} \epsilon. \quad (2.1)$$

This in fact follows directly from empirical process theory, using techniques pioneered by Koltchinskii and Pollard coupled with results of Dudley and Haussler. Indeed, we apply [3, Theorem 13.7], with the class $\mathcal{A} = \{ A \cap (xB) : x \in A \cdot B^{-1} \}$, the probability measure $\mu_A(X) = |A \cap X|/|A|$ on $G$, the random variables $X_t = a_t$ and $n = Cd/\epsilon^2$.

(Note that $\mu_A(xB) = \mu_A \circ 1_B(x)$. ) Since $\mathcal{A}$ has VC-dimension at most $d$ by assumption, the theorem says that

$$\mathbb{E}\|\mu_{\vec{a}} \circ 1_B - \mu_A \circ 1_B\|_{L^\infty(A \cdot B^{-1})} \leq c \epsilon.$$  

Since the convolutions are supported on $A \cdot B^{-1}$, we may extend the norm to the whole of $G$. Thus, by Markov’s inequality, the claimed inequality holds with probability at least 0.99.

With this probability estimate in place, one may follow the averaging argument of [7] to obtain the set $T$; we include the details for completeness. Call the tuples $\vec{a} \in G^{\otimes n}$ satisfying (2.1) good, so that $\mathbb{P}_{\vec{a} \in A^{\otimes n}} (\vec{a} \text{ is good}) \geq 0.99$. Define

$$T_{\vec{a}} = \{ t \in S : t^{-1} \vec{a} \text{ is good} \} \subseteq S.$$  

Then

$$\mathbb{E}_{\vec{a} \in (S \cdot A)^{\otimes n}} |T_{\vec{a}}| = \sum_{t \in S} \mathbb{P}_{\vec{a} \in (S \cdot A)^{\otimes n}} (t^{-1} \vec{a} \text{ is good}) \geq \frac{|A|^n}{|S \cdot A|^n} \mathbb{P}_{\vec{a} \in A^{\otimes n}} (\vec{a} \text{ is good}) |S| \geq 0.99 K^{-n} |S|.$$  

We now fix some $\vec{a}$ such that $T_{\vec{a}}$ has at least this size, and let this set be our $T$.

We next show that every $t \in T^{-1}T$ is an almost-period, completing the proof. Indeed, if $t = r^{-1}s$ with $r, s \in T$, then, since $r^{-1} \vec{a}$ is good,

$$\|\tau_r(\mu_{\vec{a}} \circ 1_B) - \mu_A \circ 1_B\|_\infty \leq \frac{1}{2} \epsilon,$$
and similarly for $s$. Thus
\[
\|\tau_t(\mu_A \circ 1_B) - \mu_A \circ 1_B\|_\infty = \|\tau_s\tau_r^{-1}(\mu_A \circ 1_B) - \mu_A \circ 1_B\|_\infty \\
\leq \|\tau_s\tau_r^{-1}(\mu_A \circ 1_B) - \tau_s(\mu_\tilde{a} \circ 1_B)\|_\infty + \|\tau_s(\mu_\tilde{a} \circ 1_B) - \mu_A \circ 1_B\|_\infty \\
= \|\mu_A \circ 1_B - \tau_t(\mu_\tilde{a} \circ 1_B)\|_\infty + \|\tau_s(\mu_\tilde{a} \circ 1_B) - \mu_A \circ 1_B\|_\infty \\
\leq \epsilon. \quad \square
\]

Remark 2.2. We could also have used the theory surrounding randomly sampled so-called $\epsilon$-approximations here.

As noted earlier, Theorem 1.5 follows immediately by the triangle inequality.

3. Bohr sets and the abelian case

We now bootstrap Theorem 1.5 to show Theorem 1.6, employing a simple Fourier-analytic argument. Let us first introduce some terminology and notation.

Definition 3.1. For an abelian group $G$, write $\hat{G} = \{\gamma : G \to S^1 : \gamma \text{ a homomorphism}\}$ for the dual group of $G$, consisting of homomorphisms from $G$ to the unit circle in $\mathbb{C}$, endowed with pointwise multiplication of functions as the group law. For a subset $\Gamma \subseteq \hat{G}$ and a real $\rho \geq 0$, we define the Bohr set on these data by
\[
\text{Bohr}(\Gamma, \rho) = \{x \in G : |\gamma(x) - 1| \leq \rho \text{ for all } \gamma \in \Gamma\},
\]
and call $|\Gamma|$ the rank and $\rho$ the radius of the Bohr set. For $B = \text{Bohr}(\Gamma, \rho)$ and $\tau \geq 0$, we write $B_\tau = \text{Bohr}(\Gamma, \tau \rho)$ for the Bohr set with its radius dilated by $\tau$.

For the basics surrounding Bohr sets, the reader may consult [19, Chapter 4]. We note the following standard results, meant to illustrate that Bohr sets are large and structured.

Lemma 3.2. Let $G$ be a finite abelian group, and let $B \subseteq G$ be a Bohr set of rank $d$ and radius $\rho \leq 1$. Then $|B| \geq \left(\frac{2}{\pi} \rho\right)^d |G|$.

We include the proof of the following result as an illustrative example.

Lemma 3.3. Let $G$ be an abelian group, and let $B \subseteq G$ be a Bohr set of rank $d$. If $G$ has exponent $q$, then $B$ contains a subgroup of $G$ of index at most $q^d$. In particular, if $G = \mathbb{F}_q^n$ is a vector space over the finite field $\mathbb{F}_q$, then $B$ contains a subspace of codimension at most $d$.

Proof. For every $\gamma \in \hat{G}$ we have $\gamma(x)^q = \gamma(qx) = \gamma(0) = 1$, whence every $\gamma$ takes values in the $q$-th roots of unity $U_q \subseteq \mathbb{C}^\times$. The function $\varphi : G \to U_q^\Gamma$ defined by $x \mapsto (\gamma(x))_{\gamma \in \Gamma}$ is a homomorphism whose kernel is $\Gamma^\perp = \text{Bohr}(\Gamma, 0)$, and $G/\ker(\varphi) \cong \text{Im}(\varphi) \subseteq U_q^\Gamma$. \quad \square

For groups without a nice subgroup structure, for example the groups $\mathbb{Z}/N\mathbb{Z}$ of prime order, Bohr sets still contain very naturally structured sets, such as long arithmetic progressions and dense generalised arithmetic progressions (see Definition 4.10). The general case is a mix of these two: Bohr sets contain large so-called coset progressions.
We will not elaborate on this here; see [19] Chapter 4 for further details. We shall, however, require one technical aspect of Bohr sets for our proof of the Bohr-set arithmetic regularity lemma; again the details are in [19].

**Definition 3.4 (Regular Bohr sets).** We say that a Bohr set $B$ of rank $d$ is regular if

$$1 - 12d|\tau| \leq |B_{1+\tau}| \leq 1 + 12d|\tau|$$

whenever $|\tau| \leq 1/12d$.

We shall apply this through the following direct consequence:

**Lemma 3.5.** If $B$ is a regular Bohr set of rank $d$ and $t \in B_\delta$ with $\delta \leq \epsilon/24d$, then

$$\sum_{x \in G} |\mu_B(x + t) - \mu_B(x)| \leq \epsilon.$$

Not all Bohr sets are regular in this way, but regular dilates are easy to find:

**Lemma 3.6.** If $B$ is a Bohr set, then there is a $\tau \in [1/2, 1]$ for which $B_\tau$ is regular.

For our proof of Theorem 1.6, we shall use the Fourier transform:

**Definition 3.7.** Let $G$ be a finite abelian group. We define the Fourier transform of a function $f : G \to \mathbb{C}$ to be the function $\hat{f} : \hat{G} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \sum_{x \in G} f(x) \overline{\gamma(x)}.$$

With this normalisation, the Fourier inversion formula, convolution identity and Parseval’s identity take the form

$$f = \mathbb{E}_\gamma \hat{f}(\gamma) \gamma, \quad \hat{f} \ast \hat{g} = \hat{f} \cdot \hat{g}, \quad \mathbb{E}_\gamma |\hat{f}(\gamma)|^2 = ||f||_2^2,$$

where $\mathbb{E}_\gamma = \frac{1}{|G|} \sum_{\gamma \in \hat{G}}$ and $||f||_2^2 = \sum_{x \in G} |f(x)|^2$.

We now perform the bootstrapping. The idea is that we convolve $\mu_A \circ 1_B$ with an iterated convolution of the set of almost-periods from Theorem 1.5 using Chang’s lemma to control the dimension. Similar arguments are used in [16,6,17].

**Proof of Theorem 1.6.** We apply Theorem 1.5 with parameters $\epsilon/3$ and $k = \lceil C \log(2/\epsilon\eta^{1/2}) \rceil$, giving us a set $T \subseteq S$ of size at least $0.99 K^{-Cdk^2/\eta^2} |S|$ such that, for each $t \in kT - kT$,

$$||\mu_A \circ 1_B(-t) - \mu_A \circ 1_B(x)||_{\infty} \leq \frac{1}{3}\epsilon.$$

In particular, by averaging,

$$||\mu_A \circ 1_B \ast \mu - \mu_A \circ 1_B||_{\infty} \leq \frac{1}{3}\epsilon$$

where $\mu = \mu_T^{(k)} \circ \mu_T^{(k)}$ and $\mu_T^{(k)}$ denotes the $k$-fold convolution of $\mu_T$ with itself. Thus, for any $t \in G$ we have by the triangle inequality that

$$||\mu_A \circ 1_B\cdot + t) - \mu_A \circ 1_B||_{\infty} \leq 2 ||\mu_A \circ 1_B - \mu_A \circ 1_B \ast \mu||_{\infty}$$

$$+ \||\mu_A \circ 1_B \ast \mu(-t) - \mu_A \circ 1_B - \mu||_{\infty} \leq \frac{2}{3}\epsilon + \||\mu_A \circ 1_B \ast \mu(-t) - \mu_A \circ 1_B - \mu||_{\infty}. \quad (3.1)$$
Let us write $\Gamma = \{ \gamma \in \hat{G} : |\hat{\mu}_T(\gamma)| \geq \frac{1}{2} \}$ for the large spectrum of $\mu_T$. Then for any $t \in \text{Bohr} (\Gamma, \frac{1}{3} \epsilon n^{1/2})$ and any $x \in G$ we have
\[
|\mu_A \circ 1_B * \mu(x + t) - \mu_A \circ 1_B * \mu(x)| \leq E_\gamma |\hat{\mu}_A| |\hat{1}_B|^2 |\hat{\mu}_T|^{2k} |\gamma(t) - 1| \\
\leq \frac{1}{3} \epsilon n^{1/2} E_\gamma |\hat{\mu}_A| |\hat{1}_B|^2 |\hat{1}_G|^{n/2} |\hat{1}_\Gamma| \\
\leq \frac{1}{3} \epsilon n^{1/2} \|\mu_A\|_2 \|1_B\|_2 = \frac{1}{6} \epsilon,
\]
where we have used the Fourier inversion formula, the convolution identity, the triangle inequality, the Cauchy–Schwarz inequality and Parseval’s identity. Combining this with \cite{11}, we are almost done: we need to bound the rank of the Bohr set. By Chang’s lemma \cite{19}, $\Gamma$ is contained in the $\{0, \pm 1\}$-span of a set of $m \leq C \log (2/\mu(T))$ characters $\Lambda$. We thus take the Bohr set of the conclusion to be $\text{Bohr} (\Lambda, \frac{c}{m} \epsilon n^{1/2}) \subseteq \text{Bohr} (\Gamma, \frac{1}{3} \epsilon n^{1/2})$, the constant $c \in [1/6, 1/3]$ being picked for regularity. \hfill \Box

4. Examples and basic properties of group VC-dimension

In this section we note some basic properties of our notion of group-theoretic VC-dimension, and give some examples of types of set with small VC-dimension.

Basic properties of group VC-dimension. Our notion of $\dim_{\text{VC}}(A, B)$ is of course not monotone with respect to $B$, but it is with respect to the ground set $A$:

**Proposition 4.1** (Monotonicity). Let $A, B$ be subsets of a group $G$.

(i) If $A \subseteq A'$, then $\dim_{\text{VC}}(A, B) \leq \dim_{\text{VC}}(A', B)$.

(ii) Writing $d = \dim_{\text{VC}} \{A \cap (xB) : x \in G\}$, we have $d - 1 \leq \dim_{\text{VC}}(A, B) \leq d$.

(iii) $\dim_{\text{VC}}(G, A) - 1 \leq \dim_{\text{VC}}(A) \leq \dim_{\text{VC}}(G, A)$.

**Proof.** The first item is immediate, as if $X \subseteq A$ is shattered by $\{A \cap (xB) : x \in A \cdot B^{-1}\}$ then it is also shattered by $\{A' \cap (xB) : x \in A' \cdot B^{-1}\}$. The second item follows from the fact that the only potential difference between the corresponding set systems is the empty set. The upper bound in the third item follows from \cite{11}. For the lower bound, suppose $\{xA : x \in G\}$ shatters a set $X$. Then $X \subseteq tA$ for some $t \in G$. The family $\{A \cap (xA) : x \in G\}$ then shatters $t^{-1}X \subseteq A$, and so the bound follows from the lower bound in \cite{11}. \hfill \Box

As one would expect, VC-dimension is translation-invariant:

**Lemma 4.2** ($\dim_{\text{VC}}$ under translation). Let $A, B \subseteq G$. Then $\dim_{\text{VC}}(At, Bt) = \dim_{\text{VC}}(A, B) = \dim_{\text{VC}}(tA, tB)$ for every $t \in G$. In particular, $\dim_{\text{VC}}(At) = \dim_{\text{VC}}(A) = \dim_{\text{VC}}(tA)$.

**Proof.** We have
\[
\dim_{\text{VC}}(At, Bt) = \dim_{\text{VC}} \{At \cap xBt : x \in A \cdot B^{-1}\},
\]
and it is apparent that the family $\{A \cap xB : x \in A \cdot B^{-1}\}$ shatters $X$ iff the above family shatters $Xt$. For left-translation we are similarly done, as
\[
\dim_{\text{VC}}(tA, tB) = \dim_{\text{VC}} \{tA \cap tyB : y \in A \cdot B^{-1}\}. \hfill \Box
Definition 4.3. Let $A, B$ be subsets of a group $G$, and $C, D$ subsets of a group $H$. A pair of maps $\varphi_A : A \to C$, $\varphi_B : B \to D$ is called a (Freiman) 2-isomorphism if each map is a bijection and it holds for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ that
\[
a_1 b_1^{-1} = a_2 b_2^{-1} \text{ iff } \varphi_A(a_1) \varphi_B(b_1)^{-1} = \varphi_A(a_2) \varphi_B(b_2)^{-1}.
\]
The pairs $(A, B)$ and $(C, D)$ are said to be (Freiman) 2-isomorphic if such a pair of maps exists. If $\varphi_A = \varphi_B = \varphi$, so in particular $A = B$ and $C = D$, then we say that $\varphi$ is a (Freiman) 2-isomorphism, and that $A$ and $C$ are (Freiman) 2-isomorphic.

Note in particular that a pair of maps as above induces a well-defined bijection $\varphi : A \cdot B^{-1} \to C \cdot D^{-1}$, given by $\varphi(ab^{-1}) = \varphi_A(a) \varphi_B(b)^{-1}$.

Some typical examples of Freiman isomorphisms are translations, dilations in certain contexts, embeddings of subsets of infinite groups into finite ones — for example embedding a subset of $\mathbb{Z}$ into $\mathbb{Z}/N\mathbb{Z}$. A particularly useful application concerns embedding a subset $A$ of an abelian group satisfying $|A + A| \leq K|A|$ into a finite abelian group $G$ where $|A| \geq |G|/C(K)$, that is where the isomorphic copy is dense — this is known in the literature as modelling. We include such a lemma in Section 5 and otherwise refer the reader to [19] for more information on Freiman isomorphisms.

Lemma 4.4 (dim_{VC} under Freiman isomorphism). Let $A, B$ be subsets of a group $G$, and $C, D$ subsets of a group $H$. If $(A, B)$ and $(C, D)$ are 2-isomorphic, then $\dim_{VC}(A, B) = \dim_{VC}(C, D)$. In particular, if $A$ and $C$ are 2-isomorphic, then $\dim_{VC}(A) = \dim_{VC}(C)$.

**Proof.** Take a pair of maps $\varphi_A, \varphi_B$ giving a 2-isomorphism. We claim that
\[
\{C \cap yD : y \in C \cdot D^{-1}\} = \{\varphi_A(A \cap xB) : x \in A \cdot B^{-1}\},
\]
and since the latter family is clearly isomorphic to $\{A \cap xB : x \in A \cdot B^{-1}\}$, this proves the lemma. To prove the claim, let us denote the image of $x \in A \cdot B^{-1}$ under the induced bijection $\varphi$ described above by $y_x$, so that we can rewrite the left-hand side of the claim as $\{\varphi_A(A) \cap y_x \varphi_B(B) : x \in A \cdot B^{-1}\}$. We will thus be done upon showing that, for each $x = a_x b^{-1}_x$,
\[
\varphi_A(A) \cap y_x \varphi_B(B) = \varphi_A(A \cap xB).
\]
To see this, note that $\varphi_A(a) = y_x \varphi_B(b) = \varphi_A(a_x) \varphi_B(b_x)^{-1} \varphi_B(b)$ holds iff $a = a_x b_x^{-1} b = x b$, by the Freiman isomorphism property, whence the claim follows.

Note that the conclusion that $\dim_{VC}(A) = \dim_{VC}(C)$ in fact holds under the weaker assumption that the pairs $(A, A)$ and $(C, C)$ are 2-isomorphic, allowing one to pick different maps for the two copies of $A$.

There was an arbitrary choice in our definition of VC-dimension, corresponding to the same choice in our definition of convolution: we mostly consider multiplication acting on the group on the left; thus our definition of $\dim_{VC}(A, B)$ might more appropriately be termed left VC-dimension. There is similarly a notion of right VC-dimension:

Definition 4.5. For $A, B \subseteq G$, we write $\dim_{VC}(A, B) = \dim_{VC}\{A \cap (Bx) : x \in B^{-1} \cdot A\}$, and just $\dim_{VC}(A)$ if $B = A$, and call these right VC-dimensions.
Clearly $\dim_{\text{VC}}$ and $\dim_{\text{rVC}}$ coincide for abelian groups. We have the following relationship.

**Lemma 4.6** (dim$_{\text{VC}}$ under inverses). Let $A, B \subseteq G$. Then $\dim_{\text{VC}}(A^{-1}, B^{-1}) = \dim_{\text{rVC}}(A, B)$. In particular, if $G$ is abelian then $\dim_{\text{VC}}(-A) = \dim_{\text{VC}}(A)$.

**Proof.** One of the families shatters a set $X$ iff the other shatters $X^{-1}$. □

(Note that inversion is in general not a Freiman isomorphism, though it is for abelian groups.)

**Examples of sets with small VC-dimension.** Sets with VC-dimension 0 have a straightforward description:

**Proposition 4.7** (Subgroups). Let $A \subseteq G$ be non-empty. Then $\dim_{\text{VC}}(A) = 0$ if and only if $A$ is a (left or right) coset of a subgroup.

**Proof.** A set system $\mathcal{A}$ has VC-dimension 0 iff $|\mathcal{A}| = 1$. Thus $\dim_{\text{VC}}(A) = 0$ iff
\[
\{A \cap (xA) : x \in A \cdot A^{-1}\} = \{A\},
\] since $A \cdot A^{-1}$ contains the identity. This holds iff $A \subseteq xA$ for all $x \in A \cdot A^{-1}$, which by translation and symmetry of $A \cdot A^{-1}$ holds iff $A = xA$ for all $x \in A \cdot A^{-1}$. This is equivalent to $A = \langle A \cdot A^{-1} \rangle \cdot A$, which, fixing any $t \in A$, holds iff $A = \langle A \cdot A^{-1} \rangle \cdot t$. □

Groups like $\mathbb{Z}$ do not contain non-trivial finite subgroups, but they do contain plenty of interesting alternatives, like arithmetic progressions. In the below, we use interval notation to denote the corresponding real intervals intersected with $\mathbb{Z}$.

**Proposition 4.8** (Arithmetic progressions). Let $A \subseteq \mathbb{Z}$ be an arithmetic progression, with $|A| \geq 3$. Then $\dim_{\text{VC}}(A) = 2 = \dim_{\text{VC}}(\mathbb{Z}, A)$.

**Proof.** For the first claim, we may translate and dilate the arithmetic progression to assume that $A = [0, N]$ with $N \geq 2$. The relevant set system is then
\[
\{[0, 0], [0, 1], [0, 2], \ldots, [0, N], [1, N], [2, N], \ldots, [N, N]\}.
\]
This shatters the set $\{0, 1\}$, and so $\dim_{\text{VC}}(A) \geq 2$. On the other hand, any collection of intervals cannot shatter a set of three elements, since if $a < b < c$, then any interval containing $a$ and $c$ automatically contains $b$. Thus $\dim_{\text{VC}}(\mathbb{Z}, A) \leq 2$, and so the result follows by monotonicity. □

**Proposition 4.9** (Boxes). Let $A = [0, N_1] \times \cdots \times [0, N_d] \subseteq \mathbb{Z}^d$. Then $\dim_{\text{VC}}(A) \leq \dim_{\text{VC}}(\mathbb{Z}^d, A) \leq 2d$.

This can be sharpened, but we give the above bound as it is simple to prove, and we include the (standard) proof to give an idea of the style of argument.

**Proof.** We prove the stronger, and well-known, claim that the family $\mathcal{B}$ of axis-aligned boxes in $\mathbb{R}^d$ has VC-dimension at most $2d$. Define the box-span of a set $X \subseteq \mathbb{R}^d$ to be
\[
\text{boxspan}(X) = [\min_{x \in X} x_1, \max_{x \in X} x_1] \times \cdots \times [\min_{x \in X} x_d, \max_{x \in X} x_d];
\]
this is the smallest axis-aligned box containing $X$. If $B$ shatters a non-empty set $X$, let $Y = \{p_1, \ldots, p_{2d}\} \subseteq X$ be such that $\text{boxspan}(X) = \text{boxspan}(Y)$ — such points exist since at most $2d$ numbers define the box-span. Then $B$ cannot distinguish between $X$ and its subset $Y$, since any axis-aligned box containing $Y$ contains $X$. Thus $Y = X$ and $|X| \leq 2d$.

This immediately implies a bound for the natural analogue in $\mathbb{Z}$, namely generalised arithmetic progressions:

**Definition 4.10.** Let $G$ be an abelian group, and let $x_1, \ldots, x_d \in G$. A generalised arithmetic progression of rank $d$ in $G$ is a set $A \subseteq G$ of the form

$$A = a + \{\lambda_1 x_1 + \cdots + \lambda_d x_d : \lambda_i \in [0, N_i]\}.$$  

If $|A| = N_1 \cdots N_d$, that is if all the sums are distinct, then $P$ is called proper.

**Proposition 4.11** (Generalised arithmetic progressions). Let $A \subseteq \mathbb{Z}$ be a proper generalised arithmetic progression of rank $d$, with $A - A$ also proper. Then $\dim(VC(A)) \leq 2d$.

**Proof.** This follows from Lemma 4.4 — the invariance of $\dim(VC)$ under Freiman isomorphism — and Proposition 4.10 as the set $A$ as in Definition 4.10 is the image of $[0, N_1] \times \cdots \times [0, N_d]$ under the obvious map, which is a Freiman isomorphism by properness, and so $\dim(VC(A)) = \dim(VC([0, N_1] \times \cdots \times [0, N_d])) \leq 2d$. 

We again refer the reader to [19] for more information on generalised arithmetic progressions.

### 5. Arithmetic regularity and structures in difference sets

In this section we apply the uniform continuity result to prove Theorems 1.3 and 1.4 — our VC-dimension versions of the arithmetic regularity lemma of Terry and Wolf — as well as prove the corollaries of Bogolyubov–Ruzsa type.

In fact, we shall prove the following more general result.

**Theorem 5.1** (Arithmetic regularity, Bohr set version). Let $\epsilon \in (0, 1)$, $\nu \in [0, 1]$, and let $G$ be a finite abelian group. If $A \subseteq G$ has size least $\alpha |G|$ and $\dim_{VC}(A) \leq d$, then there is a regular Bohr set $H \subseteq A - A$ of rank $m \leq Cd^{-C} \log(2/\alpha)$ and radius $\nu e^2/m^2$, and a subset $A' \subseteq A$, such that $|A \triangle (A' + H)| \leq \epsilon |A|$. Moreover, we may take $|A'| \geq (1 - \epsilon)|A|$, and there is some radius-dilate $D = Cm/\epsilon$ such that $|A \cap (x + H_D)| \geq (1 - \epsilon)|H_D|$ for all $x \in A' + H$.

Theorem 1.3 is an immediate consequence of this, taking $\nu = 0$, so that $H$ is a subspace of the required codimension by Lemma 3.3.

We remark that if one assumes the more general condition $|A + S| \leq K|A|$ instead of the density condition, then one gets the same conclusion but with the bound $m \leq C^{-C} \log K + C \log(2/\mu_G(S))$ on the rank instead. 

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3Viewed as a GAP with the same basis elements $x_1, \ldots, x_d$ in the natural way.
Proof of Theorem 5.1. We apply Theorem 1.6 with $B = A$ and some parameter $\delta = \delta(\epsilon) \leq 1/10$ to be specified later in place of $\epsilon$, taking $S = G$ and $K = 1/\alpha$. This gives us a regular Bohr set $T$ of rank $m \leq C \delta^{-2}(\log 2/\delta)^2 \log(2/\alpha)$ and radius $c\epsilon/m$ such that

$$|\mu_A \circ 1_A(x + t) - \mu_A \circ 1_A(x)| \leq \delta$$

for all $x \in G$ and $t \in T$. Since $\mu_A \circ 1_A(0) = \mathbb{E}_{a \in A} 1_A(a) = 1$, looking at just $x = 0$ tells us that $\mu_A \circ 1_A(t) \geq 1 - \delta$ for all $t \in T$. Note in particular that $T \subseteq A - A$. Averaging this over all $t \in T$, we have

$$\mu_A \circ 1_A \ast \mu_T(0) \geq 1 - \delta,$$

and so

$$\mathbb{E}_{a \in A} 1_A \ast \mu_T(a) \geq 1 - \delta.$$ 

Thus, on average over $a \in A$, $1_A \ast \mu_T(a) = |A \cap (a + T)|/|T| \geq 1 - \delta$. Let

$$A' = \{a \in A : 1_A \ast \mu_T(a) \geq 1 - \delta^{1/2}\} \subseteq A.$$

By Markov’s inequality, $\mu_A(A') \geq 1 - \delta^{1/2}$, since $1_A \ast \mu_T(a) \leq 1$ for all $a$. Let $H = B_T$, where $\tau = c\nu \delta^{1/2}/m$ is picked so that $H$ is regular and so that Lemma 3.5 yields

$$|1_A \ast \mu_T(a + t) - 1_A \ast \mu_T(a)| \leq \sum_{x \in G} |\mu_T(x + t) - \mu_T(x)| \leq \delta^{1/2}$$

for every $t \in H$ and every $a \in G$. Then

$$1_A \ast \mu_T(x) \geq 1 - 2\delta^{1/2}$$

for all $x \in A' + H$.

Equivalently, $(1 - 2\delta^{1/2}) 1_{A' + H} \leq 1_A \ast \mu_T$, and summing this over the whole group yields that

$$(1 - 2\delta^{1/2}) |A' + H| \leq |A|.$$ 

Since $|A \cap (A' + H)| \geq |A'| \geq (1 - \delta^{1/2}) |A|$, we thus have

$$|A \triangle (A' + H)| = |A| + |A' + H| - 2|A \cap (A' + H)| \leq 10\delta^{1/2}|A|.$$ 

Taking $\delta = \epsilon^2/100$ thus yields the result. \qed

Remark 5.2. Note that, in the bounded exponent context where one may take $H = T$ to be a subgroup, one actually has that $|A \cap (x + H)| \geq (1 - \epsilon)|H|$ for all $x \in A' + H$.

Remark 5.3. One could, using known ideas, work with dense subsets of Bohr sets here instead of assuming density in the overall group, but we do not have any applications in mind.

Let us also record the following small doubling version valid over finite fields.

Theorem 5.4 (Arithmetic regularity, small doubling version). Let $\epsilon \in (0, 1)$, and let $G = \mathbb{F}_q^n$. If $A \subseteq G$ has $|A - A| \leq K|A|$ and $\dim_{VC}(A) \leq d$, then there is subspace $H \subseteq A - A$ of $(A - A)$-codimension at most $C d e^{-C} \log K + C \log q$ and a union $W$ of cosets of $H$, contained in $A + H$, such that $|A \triangle W| \leq \epsilon|A|$.

There are several routes to get this kind of result; this specific version follows using a so-called modelling lemma, allowing one to Freiman-isomorphically embed a set $A \subseteq \mathbb{F}_q^n$ with $|A - A| \leq K|A|$ into a group whose size is at most $C(K)|A|$, at which point the tools of the density world apply. We do not detail this here, but we shall use a similar argument shortly.
Structures in difference sets. We turn now to Corollaries \ref{cor:ab} and \ref{cor:ab2}. Again we prove a generalisation in terms of Bohr sets. Throughout this section, \( G \) represents a finite abelian group.

**Theorem 5.5.** If \( A \subseteq G \) has size at least \( \alpha |G| \) and \( \dim_{VC}(A) \leq d \), then \( A - A \) contains a Bohr set of rank at most \( m \leq Cd \log(2/\alpha) \) and radius at least \( c/m \).

We remark again that if one assumes \( |A + S| \leq K |A| \) instead of the density condition, then one gets the same conclusion but with rank \( m \leq C d \log K + C \log(2/\mu_G(S)) \). Note also that the result almost follows directly from Theorem \ref{thm:main} taking \( \epsilon = 1/2 \), but with slightly worse radius. The proof is, however, much simpler.

**Proof.** Applying our main theorem, Theorem \ref{thm:main} with \( \epsilon = 1/2 \), \( B = A \), \( S = G \) and \( K = 1/\alpha \), we get a Bohr set \( T \) of the required rank and radius such that, for each \( t \in T \) and \( x \in G \),

\[
|\mu_A \circ 1_A(x + t) - \mu_A \circ 1_A(x)| \leq 1/2.
\]

Taking \( x = 0 \) and using that \( \mu_A \circ 1_A(0) = 1 \), we see that \( \mu_A \circ 1_A(t) \geq 1/2 \) for all \( t \in T \). In particular, \( t \in A - A \) for all \( t \in T \), and so we are done. \( \square \)

The almost-periodicity result of course says more: it says that not only is \( T \) contained in \( A - A \), but every element of \( T \) is well-represented as a difference of elements in \( A \).

Corollary \ref{cor:ab2} follows immediately from the above theorem and Lemma \ref{lem:free}. To prove the small doubling variant, Corollary \ref{cor:ab} we shall use the following lemma, which is well-known in the additive combinatorics community. It is a variant of \cite{F2} Proposition 6.1], but we include a proof as we have not been able to locate a citable reference.

**Lemma 5.6** (Freiman modelling over finite fields). Let \( s \geq 2 \). Suppose \( A \subseteq \mathbb{F}_q^m \) has \( |A - A| \leq K |A| \) or \( |A + A| \leq K |A| \). Then \( A \) is Freiman \( s \)-isomorphic to a subset of \( G = \mathbb{F}_q^n \), where \( |G| \leq q \cdot K^{2s} |A| \).

The definition of Freiman \( s \)-isomorphism is similar to that of 2-isomorphism given in Definition \ref{def:2-isom} a bijection \( \varphi : A \rightarrow B \) is said to be a Freiman \( s \)-isomorphism if, for all \( a_i \in A \),

\[
a_1 + \cdots + a_s = a_{s+1} + \cdots + a_{2s} \iff \varphi(a_1) + \cdots + \varphi(a_s) = \varphi(a_{s+1}) + \cdots + \varphi(a_{2s}).
\]

We remark again that an \( s \)-isomorphism \( A \rightarrow B \) extends naturally to an \( s/k \)-isomorphism from \( A \pm A \pm \cdots \pm A \rightarrow B \pm B \pm \cdots B \) whenever \( k \) divides \( s \), where there are \( k \) copies of the sets on each side and the same number of plusses and minuses on both sides.

**Proof of Lemma 5.6.** Let \( 1 \leq m \leq n \) be a parameter, and let \( V \subseteq \mathbb{F}_q^n \) be a subspace of codimension \( m \), picked uniformly at random. The quotient map \( \varphi : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n/V \cong \mathbb{F}_q^m \) will then be a Freiman \( s \)-isomorphism when restricted to \( A \) provided \( (sA - sA) \cap V = \{0\} \).

Now, a standard basis counting argument gives that the probability that \( V \) contains a given non-zero vector \( x \) is

\[
\mathbb{P}(x \in V) = \frac{q^{n-m} - 1}{q^n - 1} < q^{-m}.
\]
Thus the probability that $V$ contains at least one of the non-zero elements of $sA - sA$ is less than $q^{-m}|sA - sA|$. Thus, provided $m$ is picked so that $|sA - sA| \leq q^m$, the earlier probability will be less than 1. In particular, taking the smallest such $m$, there exists some subspace $V$ of codimension $m$ such that the corresponding quotient map is a Freiman $s$-isomorphism from $A$ to its image, which in turn is isomorphic to a subset of $\mathbb{F}_q^n$. Finally, we have $q^m < q|sA - sA| \leq q \cdot K^2|A|$ by the Plünnecke–Ruzsa–Petridis inequality [18, Corollary 6.29] (see [13] for an elegant proof).

We can now prove Corollary 1.8 which we restate for convenience.

**Corollary 1.8.** Let $A \subseteq \mathbb{F}_q^n$ be a set with $\operatorname{dim}_{VC}(A) \leq d$ and $|A - A| \leq K|A|$. Then $A - A$ contains a subspace of $(A - A)$-codimension at most $Cd \log K + C' \log q$.

**Proof.** By translating if necessary, we assume without loss of generality that $A$ contains 0. By the modelling lemma, we may embed $A$ into $G = \mathbb{F}_q^m$, where $|G| \leq qK^8|A|$, by a 4-isomorphism $\varphi$ taking 0 to 0. Being a 4-isomorphism, $\varphi$ extends to a 2-isomorphism $A - A \rightarrow \varphi(A) - \varphi(A)$, whence $|A - A| = |\varphi(A) - \varphi(A)|$. Applying Theorem 5.5 to $\varphi(A)$ in $G$, taking $S = -\varphi(A)$ in the bounds in the remark immediately following the theorem, we get that $\varphi(A) - \varphi(A)$ contains a subspace $V$ of $(A - A)$-codimension at most $Cd \log K + C' \log q$. Hence $A - A$ contains $H := \varphi^{-1}(V)$. Since $\varphi^{-1}$ is a 2-isomorphism $V \rightarrow H$ taking 0 to 0, $H$ is also a subspace, and we are done.

We remark that the log $q$ term is of course somewhat artificial, coming from how the size of $4A - 4A$ relates to a power of $q$ in the modelling lemma.

6. VC-DIMENSION AND K-STABILITY

We make here some brief remarks about the relationship between our notion of VC-dimension and the notion of $k$-stability used by Terry and Wolf in [20], this being defined as follows.\footnote{\cite{TerryWolf}

**Definition 6.1.** Let $A$ be a subset a group $G$. Then $A$ is said to have the $k$-order property if there exist $a_1, \ldots, a_k, b_1, \ldots, b_k \in G$ such that $a_ib_j \in A$ if and only if $i \leq j$. If $A$ does not have the $k$-order property, it is called $k$-stable.

We first show that $k$-stable sets have small VC-dimension.

**Lemma 6.2.** Let $A \subseteq G$ be $k$-stable. Then $\operatorname{dim}_{VC}(A) \leq \operatorname{dim}_{VC}(G, A) \leq k - 1$.

**Proof.** We show that if $\operatorname{dim}_{VC}(G, A) \geq k$, then $A$ has the $k$-order property. Let $X \subseteq G$ be a set of size $k$ shattered by the family $\{xA : x \in G\}$. Writing $b_1, \ldots, b_k$ for the elements of $X$, let $a_1, \ldots, a_k \in G$ be elements such that

$$\left(a_i^{-1}A\right) \cap X = \{b_i, \ldots, b_k\};$$

such elements exist by shattering. Then $a_ib_j \in A$ iff $b_j \in \left(a_i^{-1}A\right) \cap X$, which is true iff $i \leq j$ by definition of $a_i$. Thus the $k$-order property holds.

In the other direction, no meaningful bound exists in general:
Lemma 6.3. For every $k \geq 2$, there is a set $A$ with $\text{dim}_{\text{VC}}(A) \leq 2$ that is not $k$-stable.

Proof. The arithmetic progression $A := [0, k) \subseteq \mathbb{Z}$ has the $k$-order property, taking $a_i = i - 1$ and $b_j = k - j$ for $i, j = 1, \ldots, k$. On the other hand, by Proposition 4.8, $\text{dim}_{\text{VC}}(A) = 2$ for $k \geq 3$, and equals 1 if $k = 2$. □

In fact, Conant, Pillay and Terry [5] recently proved the following quantitatively ineffective but structurally strong result.

Theorem 6.4 (Conant–Pillay–Terry). For any $k \geq 1$ and $\varepsilon > 0$, there are $n = n(k, \varepsilon)$ and $N = N(k, \varepsilon)$ such that the following holds. Suppose $G$ is a finite group of size at least $N$, and $A \subseteq G$ is $k$-stable. Then there is a normal subgroup $H \leq G$, of index at most $n$, such that for each coset $C$ of $H$ either $|A \cap C| \leq \varepsilon|H|$ or $|A \cap C| \geq (1 - \varepsilon)|H|$. Moreover, there is a union $W$ of cosets of $H$ such that $|A \triangle W| \leq \varepsilon|H|$.

This is another indication of a fundamental difference between $k$-stability and bounded VC-dimension: the property of having bounded VC-dimension is a useful concept even in groups of prime order, in which the only bounded index subgroup is the group itself. The theorem of Conant–Pillay–Terry shows that $k$-stable sets in such groups are necessarily either very small or very large. On the other hand, these groups contain sets of bounded VC-dimension of any size.

Remark 6.5. In the case of stability, both the results of Terry–Wolf and Conant–Pillay–Terry show that for each $t \in A + H$, $|A \cap (t + H)|$ is either very large or very small. (For $t \notin A + H$, the intersection is empty.) In the VC-bounded case, we obtain a Bohr set $H$ and an almost-full subset $A' \subseteq A$ for which these intersections are large for all $t \in A' + H$. For $t \in (A + H) \setminus (A' + H)$, however, we say nothing; the case of arithmetic progressions indicates that some of the intersections can be of medium size — thus one cannot in general hope to obtain the same kind of dichotomy.

7. Aftermath

As noted in the introduction, independently of our work, Alon, Fox and Zhao [1] recently proved an arithmetic regularity lemma for sets $A$ where the family $\{t + A : t \in G\}$ has small VC-dimension. In our notation, this notion of VC-dimension corresponds precisely to $\text{dim}_{\text{VC}}(G, A)$ (in the abelian setting). It turns out that this notion was also looked at earlier by P. Simon [18] in the context of locally compact groups, with motivation coming from model theory (as with Terry–Wolf). (We note that on the face of it, however, the main result of [18] is phrased topologically and has no content for discrete groups.) An example of where this notion is different to our definition of $\text{dim}_{\text{VC}}(A)$ is for $A$ being a coset of a proper subgroup of $G$: then $\text{dim}_{\text{VC}}(G, A) = 1$, whereas $\text{dim}_{\text{VC}}(A) = 0$, by Proposition 4.7. In fact, differing by 1 is as bad as the difference can get, by Proposition 4.1. We remark that in general $\text{dim}_{\text{VC}}(B, A)$ can be very different to $\text{dim}_{\text{VC}}(A)$, however.

It turns out that a key lemma [1 Lemma 2.2] of Alon–Fox–Zhao, an additive consequence of Haussler’s discrete sphere packing lemma, can be used in combination with our methods to improve the $\varepsilon$-dependence in Theorem 1.6 in the case $B = A$, and hence...
in all the abelian results in this paper where \( B = A \). The result one obtains is the following:

**Theorem 7.1.** Let \( \epsilon \in (0, 1] \). Let \( G \) be a finite abelian group and suppose \( A \subseteq G \) is a subset with \( |A| \geq \alpha|G| \) and \( \dim_{VC}(A) \leq d \). Then there is a regular Bohr set \( T \) of rank \( m \leq Cd \log(2/\epsilon \alpha) \) and radius at least \( c \epsilon/m \) such that, for each \( t \in T \),

\[
\| \mu_A \circ 1_A (\cdot + t) - \mu_A \circ 1_A \|_\infty \leq \epsilon.
\]

In particular, this means that the codimension and rank bounds in Theorems 1.3, 1.4, and 5.1 can all be improved to \( Cd \log(2/\epsilon \alpha) \), and in Theorem 5.4 the codimension can be taken to be \( Cd \log(2K/\epsilon) + C \log q \) (by a slight generalisation of the above).

We remark that one can improve this even further in the case of groups of bounded exponent.

**Other notions of dimension.** Most of the results of this paper would be valid under more general assumptions than VC-dimension being small; one could for example look at primal shatter dimension, geometric notions, or Rademacher complexity. VC-dimension seems to be the most widely studied notion, however, so we have chosen to phrase the results in terms of this. Furthermore, it would be natural to consider functions more general than the indicators \( 1_B \) in the continuity results; we have not aimed for the most general statements.

**The locally compact setting.** Theorem 1.5 should generalise readily to the setting of second countable locally compact groups. The only required changes are to replace each reference to cardinality by (left) Haar measure instead, and to add measurability conditions.

**Questions.** There are a number of natural questions. For example, it is not clear what the appropriate generalisation of the regularity lemmas to non-abelian groups is. The result of Conant, Pillay and Terry (Theorem 6.4) suggests a potential form for such a statement, but as noted in Section 6 one cannot hope for as straightforward a conclusion for VC-dimension, as bounded-index proper normal subgroups need not exist in groups that contain medium-sized sets with bounded VC-dimension.

On the additive side, the following seems a natural question. If \( A \subseteq \mathbb{F}_2^n \) has density at least 0.49, must there be a dense subset \( B \subseteq A \) with \( \dim_{VC}(B) \leq C \sqrt{n} \)? Or just \( o(n) \)?

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**References**

[1] N. Alon, J. Fox and Y. Zhao, *Efficient arithmetic regularity and removal lemmas for induced bipartite patterns*, preprint available at [https://arxiv.org/abs/1801.04675](https://arxiv.org/abs/1801.04675).

[2] T. F. Bloom, *A quantitative improvement for Roth’s theorem on arithmetic progressions*, J. Lond. Math. Soc. (2) 93 (2016), no. 3, 643–663. [http://arxiv.org/abs/1405.5800](http://arxiv.org/abs/1405.5800).

[3] S. Boucheron, G. Lugosi and P. Massart, *Concentration inequalities*, OUP, 2013.

[4] J. Bourgain, *On arithmetic progressions in sums of sets of integers*, A tribute to Paul Erdős, 105–109 (CUP, 1990).
[5] G. Conant, A. Pillay and C. Terry, A group version of stable regularity, preprint available at https://arxiv.org/abs/1710.06309

[6] E. Croot, I. Laba and O. Sisask, Arithmetic progressions in sumsets and $L^p$-almost-periodicity, Combin. Probab. Comput. 22 (2013), no. 3, 351–365. https://arxiv.org/abs/1103.6000

[7] E. Croot and O. Sisask, A probabilistic technique for finding almost-periods of convolutions, Geom. Funct. Anal. 20 (2010), no. 6, 1367–1396. https://arxiv.org/abs/1003.2978

[8] W. T. Gowers, Lower bounds of tower type for Szemerédi’s uniformity lemma, Geom. Funct. Anal. 7 (1997), no. 2, 322–337.

[9] B. Green, A Szemerédi-type regularity lemma in abelian groups, with applications, Geom. Funct. Anal. 15 (2005), no. 2, 340–376. https://arxiv.org/abs/math/0310476

[10] B. Green, Finite field models in additive combinatorics, Surveys in combinatorics 2005, 1–27, London Math. Soc. Lecture Note Ser. 327, CUP, 2005. https://arxiv.org/abs/math.NT/0409420

[11] B. Green, Notes on the Polynomial Freiman–Ruzsa conjecture, available at http://people.maths.ox.ac.uk/greenbj/papers/PFR.pdf

[12] B. Green and I. Z. Ruzsa, Freiman’s theorem in an arbitrary abelian group, J. Lond. Math. Soc. (2) 75 (2007), no. 1, 163–175. https://arxiv.org/abs/math/0505198

[13] G. Petridis, New proofs of Plünnecke-type estimates for product sets in groups, Combinatorica 32 (2012), no. 6, 721–733.

[14] I. Z. Ruzsa, Arithmetic progressions in sumsets, Acta Arith. 60 (1991), no. 2, 191–202.

[15] T. Sanders, Green’s sumset problem at density one half, Acta Arith. 146 (2011), no. 1, 91–101. https://arxiv.org/abs/1103.5649

[16] T. Sanders, On the Bogolyubov–Ruzsa lemma, Anal. PDE 5 (2012), no. 3, 627–655. https://arxiv.org/abs/1011.0107

[17] T. Schoen and O. Sisask, Roth’s theorem for four variables and additive structures in sums of sparse sets, Forum of Mathematics, Sigma 4 (2016), e5 (28 pages). https://doi.org/10.1017/fms.2016.2

[18] P. Simon, VC-sets and generic compact domination, Israel J. Math. 218 (2017), no. 1, 27–41. https://arxiv.org/pdf/1502.04513.pdf

[19] T. Tao and V. H. Vu, Additive Combinatorics, CUP, 2006.

[20] C. Terry and J. Wolf, Stable arithmetic regularity in the finite-field model, preprint available at https://arxiv.org/abs/1710.02021

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