Recursive utility maximization under partial information

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Abstract. This paper concerns the recursive utility maximization problem under partial information. We first transform our problem under partial information into the one under full information. When the generator of the recursive utility is concave, we adopt the variational formulation of the recursive utility which leads to a stochastic game problem and a characterization of the saddle point of the game is obtained. Then, we study the K-ignorance case and explicit saddle points of several examples are obtained. At last, when the generator of the recursive utility is smooth, we employ the terminal perturbation method to characterize the optimal terminal wealth.

Key words. recursive utility, partial information, dual method, saddle point

Mathematics Subject Classifications. 93E20, 91A30, 90C46

1 Introduction

In this paper, we study the problem of an agent who invests in a financial market so as to maximize the recursive utility of his terminal wealth $X(T)$ on finite time interval $[0, T]$, while the recursive utility is characterized by the initial value $Y(0)$ of the following Backward Stochastic Differential Equation (BSDE for short)

$$Y(t) = u(X(T)) + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)d\hat{W}(s).$$

(1.1)

The market consists of a riskless asset and $d$ risky assets, the latter being driven by a $d$-dimensional Brownian motion. And the investor has access only to the history of interest rates and prices of risky assets, while the appreciation rate and the driving Brownian motion are not directly observed. That is, the filtration generated by the Brownian motion could not be used when the investor chooses his portfolios. This is quite practical in a real financial market. So we are interested in this so called recursive utility maximization problem under partial information.

In the full information case, the problem of maximizing the expected utility of terminal wealth is well understood in a complete or constrained financial market [3], [16]. In an incomplete multiple-priors model, Quenez [23] studied the problem of maximization of utility of terminal wealth in which the asset prices are semimartingales. Schied [24] studied the robust utility maximization problem in a complete market under
the existence of a “least favorable measure”. As for the recursive utility optimization, El Karoui et al. [6] studied the optimization of recursive utilities when the generator of BSDE is smooth. Epstein and Ji [9], [10] formulated a model of recursive utility that captures the decision-maker’s concern with ambiguity about both the drift and ambiguity and studied the recursive utility optimization under G-framework. But all the above works do not accommodate partial information.

In the partial information case, Lakner [17] generalized the martingale method to expected utility maximization problem, see also Pham [21]. Cvitanic et al [2] maximized the recursive utility under partial information. But the generator $f$ in Cvitanic et al [2] doesn’t depend on $z$. Miao [18] studied a special case of recursive multiple-priors utility maximization problem under partial information in which the appreciation rate is assumed to be an $F_0$-measurable, unobserved random variable with known distribution. Actually, they studied the problem under Bayesian framework and did not give the explicit solutions.

In this paper, we first transform our portfolio selection problem under partial information into a one under full information in which the unknown appreciation rate is replaced by its filter estimate and the Brownian motion is replaced by the innovation process. Then, a backward formulation of the problem under full information is built in which instead of the portfolio process, the terminal wealth is regarded as the control variable. This backward formulation is based on the existence and uniqueness theorem of BSDE and was introduced in [6] and [13].

When the generator $f$ of (1.1) is concave, we adopt the variational formulation of the recursive utility which leads to a stochastic game problem. Inspired by the convexity duality method developed in Cvitanic and Karatzas [4], we turn the primal “sup-inf” problem to a dual minimization problem over a set of discounting factors and equivalent probability measures. A characterization of the saddle point of this game is obtained in this paper. Furthermore, the explicit saddle points for several classical examples are worked out.

When the generator $f$ of the BSDE is smooth, we apply the terminal perturbation method developed in Ji and Zhou [12] and Ji and Peng [11] to characterize the optimal terminal wealth of the investor. Once the optimal terminal wealth is obtained, the determination of the optimal portfolio process is a martingale representation problem which we do not involve in this paper.

The rest of this paper is organized as follows. In section 2, we formulate the recursive utility maximization problem under partial information, reduce the original problem to a problem under full information and give the backward formulation. The case of non-smooth generator is tackled in section 3. In section 4, we specialize in K-ignorance model and give explicit saddle points of several examples. In section 5, we characterize the optimal wealth when the generator $f$ is smooth.

2 The problem of recursive utility maximization under partial observation

2.1 Classical formulation of the problem

We consider a financial market consisting of a riskless asset whose price process is assumed for simplicity to be equal to one, and $d$ risky securities (the stocks) whose prices are stochastic processes $S_i(t), i = 0, 1, ..., d$
governed by the following SDEs:

\[ dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right), \quad i = 1, \ldots, d, \]  

(2.1)

where \( W(\cdot) = (W_1(\cdot), \ldots, W_d(\cdot))' \) is a standard \( d \)-dimensional Brownian motion defined on a filtered complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\). \( \mu' = \{\mu'(t) = (\mu_1(t), \ldots, \mu_d(t)), t \in [0, T]\} \) is the appreciation rate of the stocks which is \( \mathcal{F}_t \)-adapted, bounded, and the \( d \times d \) matrix \( \sigma(t) = (\sigma_{ij}(t))_{1 \leq i, j \leq d} \) is the disperse rate of the stocks. Here and throughout the paper 't denotes the transpose operator.

The asset prices are assumed to be continuously observed by the investors in this market, in other words, the information available to the investors is represented by \( \mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0} \), which is the P-augmentation of the filtration generated by the price processes \( \sigma(S(u); 0 \leq u \leq t) \). The matrix disperse coefficient \( \sigma(t) \) is assumed invertible, bounded uniformly and \( \exists \varepsilon > 0, \mu'(t)\sigma'(t)\rho \geq \varepsilon ||\rho||^2, \forall \rho \in \mathbb{R}^d, t \in [0, T], a.s. \). In fact, \( \sigma(t) \) can be obtained from the quadratic variation of the price process. So we assume w.l.o.g. that \( \sigma(t) \) is \( \mathcal{G}_t \)-adapted. However, the appreciation rate \( \mu'(t) := (\mu_1(t), \ldots, \mu_d(t)) \) is not observable for the investors.

A small investor whose actions cannot affect the market prices can decide at time \( t \in [0, T] \) what amount \( \pi_i(t) \) of his wealth to invest in the \( i \)th stock, \( i = 1, \ldots, d \). Of course, his decision can only be based on the available information \( \{\mathcal{G}_t\}_{t=0}^{T} \), i.e., the processes \( \pi'(t) = (\pi_1(t), \ldots, \pi_d(t)) : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) are \( \{\mathcal{G}_t\}_{t=0}^{T} \) progressively measurable and satisfy \( \mathbb{E} \int_0^T ||\pi(t)||^2 dt < \infty \).

Then the wealth process \( X(t) \equiv X^{x, \pi}(t) \) of a self-financing investor who is endowed with initial wealth \( x > 0 \) satisfies the following stochastic differential equation:

\[ dX(t) = \sum_{i=1}^{d} \pi_i(t) \frac{dS_i(t)}{S_i(t)} = \pi'(t)\mu(t)dt + \pi'(t)\sigma(t)dW(t). \]  

(2.2)

Because the only information available to the investor is \( \mathcal{G} \), we could not use the Brownian motion \( W \) to define the recursive utility. As we will show in the following, there exists a Brownian motion \( \hat{W} \) under \( P \) in the filtered measurable space \((\Omega, \mathcal{G})\) which is often referred to as an innovation process. The recursive utility process \( Y(t) \equiv Y^{x, \pi}(t) \) of the investor is defined by the following backward stochastic differential equation:

\[ Y(t) = u(X(T)) + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)d\hat{W}(s), \]  

(2.3)

where \( f \) and \( u \) are functions satisfying the following assumptions.

**Assumption 2.1**

(\( \text{A1} \)) \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) is \( \mathcal{G} \)-progressively measurable for any \( (y, z) \in \mathbb{R} \times \mathbb{R}^d \).

(\( \text{A2} \)) There exists a constant \( C \geq 0 \) such that

\[ |f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|), \forall (t, \omega, y_1, y_2, z_1, z_2) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d. \]

(\( \text{A3} \)) \( f(t, \cdot, \cdot) \) is continuous about \( t \) and \( \mathbb{E} \int_0^T f^2(t, 0, 0)dt < +\infty. \)
**Assumption 2.2** \( u : \mathbb{R}^+ \to \mathbb{R} \) is continuously differentiable and satisfies linear growth condition.

**Remark 2.3** Equation (2.3) is not a standard BSDE because in general \( G \) is strictly larger than the augmented filtration of the \((P, G)\)-Brownian motion \( \tilde{W} \).

We introduce the following spaces:

\[
L^2(\Omega, G_T; P; \mathbb{R}) := \left\{ \xi : \Omega \to \mathbb{R} \mid \xi \text{ is } G_T\text{-measurable, and } E|\xi|^2 < \infty \right\},
\]

\[
M^2_G(0, T; \mathbb{R}^d) := \left\{ \phi : [0, T] \times \Omega \to \mathbb{R}^d \mid (\phi_t)_{0 \leq t \leq T} \text{ is } G\text{-progressively measurable process,}
\right. \quad \text{and} \quad ||\phi|| = \sqrt{E \int_0^T |\phi(t)|^2 dt < \infty},
\]

\[
S^2_G(0, T; \mathbb{R}) := \left\{ \phi : [0, T] \times \Omega \to \mathbb{R} \mid (\phi_t)_{0 \leq t \leq T} \text{ is } G\text{-progressively measurable process,}
\right. \quad \text{and} \quad ||\phi||^2 := \sqrt{E \sup_{0 \leq t \leq T} |\phi(t)|^2} < \infty \right\}.
\]

For notational simplicity, we will often write \( L^2_G, M^2_G \) and \( S^2_G \) instead of \( L^2(\Omega, G_T; P; \mathbb{R}), M^2_G(0, T; \mathbb{R}^d) \) and \( S^2_G(0, T; \mathbb{R}) \) respectively.

We will show in the next subsection that under Assumption 2.1, for any \( \xi \in L^2_G \), the BSDE (2.3) has a unique solution \((Y(\cdot), Z(\cdot)) \in S^2_G \times M^2_G \). Then for each \( \pi \in M^2_G \), \((X(\cdot), \pi(\cdot))\) satisfies Eq. (2.2),\( Y(\cdot), Z(\cdot) \) satisfies Eq. (2.3),\( \pi(\cdot) \in M^2_G \), \( X(\cdot) \) satisfies Eq. (2.4), and Assumption 2.2 ensures that the variable \( u(X(T)) \in L^2_G \). Thus, under Assumptions 2.1 and 2.2 the recursive utility process associated with this terminal value is well defined.

Given an utility function satisfying Assumption 2.2 and initial endowment \( x \), the recursive utility maximization problem with bankruptcy prohibition is formulated as: the investor chooses a portfolio strategy so as to

\[
\text{Maximize } Y^{x, \pi}(0), \text{ s.t. } \begin{cases} 
X(t) \geq 0, & t \in [0, T], \text{ a.s.}, \\
\pi(\cdot) \in M^2_G, \\
(X(\cdot), \pi(\cdot)) \text{ satisfies Eq. (2.2),} \\
(Y(\cdot), Z(\cdot)) \text{ satisfies Eq. (2.3)},
\end{cases}
\]

where \( X(t) \geq 0 \) means that no-bankruptcy is required.

**Definition 2.4** A portfolio \( \pi(\cdot) \) is said to be admissible if \( \pi(\cdot) \in M^2_G \) and the corresponding wealth processes \( X(t) \geq 0, \ t \in [0, T], \text{ a.s.} \).

Given initial wealth \( x > 0 \), denote by \( \mathcal{A}(x) \) the set of investor’s feasible portfolio strategies, that is

\[
\mathcal{A}(x) = \left\{ \pi : \pi \in M^2_G, \ X^{x, \pi}(t) \geq 0, \ dP \otimes dt \text{ a.s.} \right\}.
\]

### 2.2 Reduction to a problem under full information

Define the risk premium process \( \eta(t) = \sigma(t)^{-1} \mu(t) \). Because we have assumed the process \( \mu(\cdot), \sigma(\cdot) \) are uniformly bounded, the process

\[
L(t) := \exp(-\int_0^t \eta'(s)dW(s) - \frac{1}{2} \int_0^t |\eta(s)|^2 ds)
\]
is a \((P, \mathcal{F})\) martingale. So a probability measure \(\tilde{P}\) is defined by
\[
\tilde{P}(A) = E[L(T) I_A], \quad \forall A \in \mathcal{F}_T, \quad \text{where } \frac{d\tilde{P}}{dP} = L(T).
\]

\(\tilde{P}\) is usually called risk neutral probability in the financial market. The process
\[
\tilde{W}(t) := W(t) + \int_0^t \eta(s) ds, \quad 0 \leq t \leq T
\]
is a Brownian motion under \(\tilde{P}\) by Girsanov’s theorem.

Then we can rewrite the stock price processes (2.1) as
\[
dS_i(t) = S_i(t) \left( \sum_{j=1}^d \sigma_{ij}(t) d\tilde{W}_j(t) \right), \quad i = 1, ..., d.
\]

Note that \(\sigma(t)\) is assumed to be bounded, invertible and \(\mathcal{G}\)-adapted. So the filtration \(\mathcal{G}\) coincides with the augmented natural filtration of \(\tilde{W}\) by Theorem V.3.7 in [22].

Let \(\hat{\eta}(t) := E[\eta(t)|\mathcal{G}_t]\) be a measurable version of the conditional expectation of \(\eta\) w.r.t. the filtration \(\mathcal{G}\).

Set \(\hat{\mu}(t) = E[\mu(t)|\mathcal{G}_t]\). Then \(\hat{\mu}(t) = \sigma(t) \hat{\eta}(t)\), since \(\sigma\) is \(\mathcal{G}\)-adapted.

We introduce the process
\[
\tilde{W}(t) := \tilde{W}(t) - \int_0^t \hat{\eta}(s) ds = W(t) + \int_0^t (\eta(s) - \hat{\eta}(s)) ds, \quad t \geq 0. \tag{2.5}
\]

By Theorem 8.1.3 and Remark 8.1.1 in Kallianpur [14], \((\tilde{W}(t), t \geq 0)\) is a \((\mathcal{G}, P)\)-Brownian motion which is the so-called innovations process. Then, we could describe the dynamics of stock price processes and the wealth process within a full observation model:
\[
dS_i(t) = S_i(t) \left( \hat{\mu}_i(t) dt + \sum_{j=1}^d \sigma_{ij}(t) d\tilde{W}_j(t) \right), \quad i = 1, ..., d,
\]
\[
dX(t) = \pi'(t) \hat{\mu}(t) dt + \pi'(t) \sigma(t) d\tilde{W}(t).
\]

Now all the coefficients in our model is observable. So we are in a full observation model and our problem (2.4) can be reformulated as
\[
\begin{aligned}
\text{Maximize } & \quad Y^{x_0, \pi}(0), \quad \text{s.t. } \quad X(t) \geq 0 \quad \forall t \in [0, T] \quad \text{a.s.}, \\
& \quad \pi(\cdot) \in M^2_\mathcal{G}, \\
& \quad (X(\cdot), \pi(\cdot)) \text{ satisfies Eq.}(2.2), \\
& \quad (Y(\cdot), Z(\cdot)) \text{ satisfies Eq.}(2.3).
\end{aligned} \tag{2.6}
\]

### 2.3 Backward formulation of the problem

In this subsection, we first show BSDE (2.3) has a unique solution under some mild conditions and then give an equivalent backward formulation of problem (2.6).
Lemma 2.5 Under Assumption (2.4), for \( \forall \xi \in L^2_G \), there exists a unique solution \((Y,Z) \in S^2_G \times M^2_G\) to the BSDE (2.3).

Since \( G \) is strictly larger than the augmented filtration of the \((P,G)\)-Brownian motion \( \tilde{W} \) in general, equation (2.3) is not a standard BSDE. Fortunately, by Theorem 8.1.1 in [14], every square integrable \( \mathcal{G}_t \)-martingale \( M(t) \) can be represented as

\[
M(t) = M(0) + \int_0^t Z(s)d\tilde{W}(s),
\]

where \( Z(\cdot) \in M^2_G \). Thus, applying similar analysis as in [19], it is easy to prove this lemma.

Let \( q(\cdot) := \sigma(\cdot)'\pi(\cdot) \). Since \( \sigma(\cdot) \) is invertible, \( q(\cdot) \) can be regarded as the control variable instead of \( \pi(\cdot) \). By the existence and uniqueness result of BSDE (2.3), selecting \( q(\cdot) \) is equivalent to selecting the terminal wealth \( X(T) \). If we take the terminal wealth as control variable, the wealth equation and recursive utility process can be written as:

\[
\begin{aligned}
-dX(t) &= -q^{-1}(t)\hat{\mu}(t)dt - q'(t)d\tilde{W}(t), \\
X(T) &= \xi,
\end{aligned}
\]

\[
\begin{aligned}
-dY(t) &= f(t,Y(t),Z(t))dt - Z'(t)d\tilde{W}(t), \\
Y(T) &= u(\xi),
\end{aligned}
\]

where the “control” is the terminal wealth \( \xi \) to be chosen from the following set

\[
U := \{ \xi | \xi \in L^2_G, \xi \geq 0 \}.
\]

From now on, we denote the solution of (2.7) by \((X^\xi(\cdot),q^\xi(\cdot),Y^\xi(\cdot),Z^\xi(\cdot))\). We also denote \( X^\xi(0) \) and \( Y^\xi(0) \) by \( X^\xi_0 \) and \( Y^\xi_0 \) respectively.

As implied by the comparison theorem for BSDE (2.3), the nonnegative terminal wealth, \( \xi = X(T) \geq 0 \) keeps the wealth process nonnegative all the time. This gives rise to the following optimization problem:

Maximize \[ J(\xi) := Y^\xi_0, \text{s.t.} \begin{cases} \xi \in U, \\ X^\xi_0 = x, \\ (X^\xi(\cdot), q^\xi(\cdot), Y^\xi(\cdot), Z^\xi(\cdot)) \text{ satisfies Eq. (2.7)}. \end{cases} \] (2.8)

Definition 2.6 A random variable \( \xi \in U \) is called feasible for the initial wealth \( x \) if and only if \( X^\xi(0) = x \). We will denote the set of all feasible \( \xi \) for the initial wealth \( x \) by \( A(x) \).

It is clear that original problems (2.4) and (2.6) are equivalent to the auxiliary one (2.8). Hence, hereafter we focus ourselves on solving (2.8). Note that \( \xi \) becomes the control variable. The advantage of this approach is that the state constraint in (2.4) becomes a control constraint in (2.8), whereas it is well known in control theory that a control constraint is much easier to deal with than a state constraint. The cost of this approach is the original initial condition \( X^\xi(0) = x \) now becomes a constraint.

A feasible \( \xi^* \in A(x) \) is called optimal if it attains the maximum of \( J(\xi) \) over \( A(x) \). Once \( \xi^* \) is determined, the optimal portfolio is obtained by solving the first equation in (2.7) with \( X^{\xi^*}(T) = \xi^* \).
3 Dual method for recursive utility maximization

In this section, we impose the following concavity condition:

**Assumption 3.1** The function \((y, z) \mapsto f(\omega, t, y, z)\) is concave for all \((\omega, t) \in \Omega \times [0, T]\).

We also need the following assumption on \(u\):

**Assumption 3.2** \(u : (0, \infty) \to \mathbb{R}\) is strictly increasing, strictly concave, continuously differentiable, and satisfies

\[
\lim_{x \to 0^+} u'(x) = \infty, \quad u'(\infty) = 0.
\]

Under Assumption 3.2, Assumption 2.2 seems too restrictive and it precludes some interesting examples. So in the following two sections, for any given utility function \(u\) satisfying Assumption 3.2, we set

\[
U = \{\xi | \xi \in L^2_{G_T}, u(\xi) \in L^2_{G_T} \text{ and } \xi \geq 0\}.
\]

In this section, we assume \(\sigma \equiv \text{Id}\), the d-dimensional identity matrix. Let \(F(t, \beta, \gamma)\) be the Fenchel-Legendre transform of \(f\):

\[
F(\omega, t, \beta, \gamma) := \sup_{(y, z) \in \mathbb{R} \times \mathbb{R}^d} [f(\omega, t, y, z) - y\beta - z'\gamma], \quad (\beta, \gamma) \in \mathbb{R} \times \mathbb{R}^d.
\]

Let the effective domain of \(F\) be \(D_F := \{(\omega, t, \beta, \gamma) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d | F(\omega, t, \beta, \gamma) < +\infty\}\). As was shown in [7], the \((\omega, t)\)-section of \(D_F\), denoted by \(D_F(\omega, t)\), is included in the bounded domain \(B = [-C, C]^{d+1} \subset \mathbb{R} \times \mathbb{R}^d\), where \(C\) is the Lipschitz constant of \(f\).

We have the duality relation by the concavity of \(f\),

\[
f(\omega, t, y, z) = \inf_{(\beta, \gamma) \in D_F(\omega, t)} [F(\omega, t, \beta, \gamma) + y\beta + z'\gamma].
\]

For every \((\omega, t, y, z)\) the infimum is achieved in this relation by a pair \((\beta, \gamma)\) which depends on \((\omega, t)\).

Set

\[
\mathcal{B} = \{(\beta, \gamma) | (\beta, \gamma) \text{ is } G\text{-progressively measurable and } B\text{-valued and } E \int_0^T F(t, \beta_t, \gamma_t)^2 dt < +\infty\}.
\]

Then \(\mathcal{B}\) is a convex set. For any \((\beta, \gamma) \in \mathcal{B}\), let

\[
f^{\beta, \gamma}(t, y, z) = F(t, \beta_t, \gamma_t) + y\beta_t + z'\gamma_t,
\]

and denote by \((Y^{\beta, \gamma}, Z^{\beta, \gamma})\) the unique solution to the linear BSDE with \(f^{\beta, \gamma}\).

By similar analysis as Proposition 3.4 in [7], we have the following variational formulation of \(X^\xi(t)\) and \(Y^\xi(t)\).

**Lemma 3.3** Under Assumption 2.1 and 3.1, for any \(\xi \in U\), the solutions \((X^\xi(\cdot), q^\xi(\cdot)), (Y^\xi(\cdot), Z^\xi(\cdot))\) of Eq. (2.7) can be represented as

\[
X^\xi(t) = \mathcal{L}^{-1}(t)E\mathcal{L}(T)\xi|\mathcal{G}_t],
\]

\[
Y^\xi(t) = \text{ess } \inf_{\beta, \gamma \in \mathcal{B}} Y^{\beta, \gamma}, \quad t \in [0, T], \text{a.s.,}
\]
where

$$\hat{L}(t) := e^{-\int_0^t \hat{\mu}(s) d\hat{W}(s)} - \frac{1}{\mu} \int_0^t |\hat{\mu}(s)|^2 ds,$$

$$Y^\beta,\gamma = E\left[ \int_t^T \Gamma^\beta,\gamma_{t,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{t,T} u(\xi) | \mathcal{G}_t \right],$$

$$\Gamma^\beta,\gamma_{t,s} = e^{\int_t^s (\beta_r - \frac{1}{2} |\gamma_r|^2) dr + \int_t^s \gamma_r d\hat{W}(r)}.$$

Especially, we have $Y^\xi(0) = \inf_{(\beta, \gamma) \in B} E\left[ \int_0^T \Gamma^\beta,\gamma_{0,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{0,T} u(\xi) \right].$

**Remark 3.4** By Theorem 3.1 in [17], we have $\hat{L}(t) = E[L(t)|\mathcal{G}_t]$, $t \in [0, T]$, a.s.

By Lemma 3.3, $A(x) = \{\xi \in U | E[L(T)|\xi] = x\}$. Thus, our problem is equivalent to the following problem:

Maximize $J(\xi) = \inf_{\beta, \gamma \in B} E\left[ \int_0^T \Gamma^\beta,\gamma_{0,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{0,T} u(\xi) \right]$

subject to $\xi \in A(x)$. (3.4)

The maximum recursive utility that the investor can achieve is

$$V(x) := \sup_{\xi \in A(x)} \inf_{\beta, \gamma \in B} E\left[ \int_0^T \Gamma^\beta,\gamma_{0,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{0,T} u(\xi) \right].$$

It is dominated by its “min-max” counterpart

$$\bar{V}(x) := \inf_{(\beta, \gamma) \in B} \sup_{\xi \in A(x)} E\left[ \int_0^T \Gamma^\beta,\gamma_{0,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{0,T} u(\xi) \right].$$

If we can find $(\hat{\beta}, \hat{\gamma}, \hat{\xi}) \in B \times A(x)$ such that

$$V(x) = E\left[ \int_0^T \Gamma^\hat{\beta},\hat{\gamma}_{0,s} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma^\hat{\beta},\hat{\gamma}_{0,T} u(\hat{\xi}) \right] = \bar{V}(x),$$

then the optimal solution of problem (3.4) is $(\hat{\beta}, \hat{\gamma}, \hat{\xi})$.

Let us introduce the monotone decreasing function $I(\cdot)$ as the inverse of the marginal utility function $u'(\cdot)$, and the convex dual

$$\tilde{u}(\zeta) := \max_{x \geq 0} [u(x) - \zeta x] = u(I(\zeta)) - \zeta I(\zeta), \quad \zeta > 0.$$ (3.8)

Then, $\forall \xi \in A(x)$, $\forall (\beta, \gamma) \in B$, $\forall \zeta > 0$,

$$E\left[ \int_0^T \Gamma^\beta,\gamma_{0,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{0,T} u(\xi) \right]$$

$$\leq E\left[ \int_0^T \Gamma^\beta,\gamma_{0,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{0,T} \tilde{u}(\frac{\hat{L}(T)}{\Gamma^\beta,\gamma_{0,T}}) + \zeta \hat{L}(T) \right]$$

$$= E\left[ \int_0^T \Gamma^\beta,\gamma_{0,s} F(s, \beta_s, \gamma_s) ds + \Gamma^\beta,\gamma_{0,T} \tilde{u}(\frac{\hat{L}(T)}{\Gamma^\beta,\gamma_{0,T}}) \right] + \zeta x.$$ (3.9)
Furthermore, we have equality in the above formula for some $\hat{\zeta} \in \mathcal{A}(x)$, $(\hat{\beta}, \hat{\gamma}) \in \mathcal{B}$, $\hat{\zeta} > 0$ if and only if the conditions

$$E[\hat{\zeta} \hat{L}(T)] = x, \quad (3.10)$$

$$\hat{\zeta} = I(\hat{\zeta} \hat{L}(T)_{\frac{\Gamma}{0,T}}), \text{ a.s.} \quad (3.11)$$

are satisfied simultaneously. And in this case, we have

$$E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \hat{\beta}, \hat{\gamma}) ds + \Gamma_{0,s}^0 u(\hat{\zeta}) \right] = E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \hat{\beta}, \hat{\gamma}) ds + \Gamma_{0,s}^0 u(\hat{\zeta}) \right] + \hat{\zeta} x. \quad (3.12)$$

**Lemma 3.5** Under Assumption \ref{assumption:2.1} \ref{assumption:3.1} and \ref{assumption:3.2}, suppose that there exists a quadruple $(\hat{\zeta}, \hat{\beta}, \hat{\gamma}, \hat{\xi}) \in (\mathcal{A}(x) \times \mathcal{B} \times (0, \infty))$ which satisfies \ref{constraint:3.10}, \ref{constraint:3.11} and

$$E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \hat{\beta}, \hat{\gamma}) ds + \Gamma_{0,s}^0 u(\hat{\xi}) \right] \leq E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \beta, \gamma) ds + \Gamma_{0,s}^0 u(\hat{\xi}) \right], \forall(\beta, \gamma) \in \mathcal{B}. \quad (3.13)$$

Then we have $\forall \xi \in \mathcal{A}(x)$, $\forall(\beta, \gamma) \in \mathcal{B}$,

$$E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \hat{\beta}, \hat{\gamma}) ds + \Gamma_{0,s}^0 u(\xi) \right] \leq E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \hat{\beta}, \hat{\gamma}) ds + \Gamma_{0,s}^0 u(\xi) \right] \leq E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \beta, \gamma) ds + \Gamma_{0,s}^0 u(\xi) \right]. \quad (3.14)$$

**Proof:** We only prove the first relationship in \ref{constraint:3.14}. Let $(\beta, \gamma) = (\hat{\beta}, \hat{\gamma})$ and $\zeta = \hat{\zeta}$ in \ref{constraint:3.14}. We get

$$E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \beta, \gamma) ds + \Gamma_{0,s}^0 u(\zeta) \right]$$

$$\leq E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \beta, \gamma) ds + \Gamma_{0,s}^0 u(\zeta) \right] + \hat{\zeta} x$$

$$= E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \beta, \gamma) ds + \Gamma_{0,s}^0 u(\zeta) \right], \forall \xi \in \mathcal{A}(x),$$

by \ref{constraint:3.12}. This completes the proof. $\square$

Let us introduce the value function

$$\tilde{V}(\zeta) \equiv \tilde{V}(\zeta; x) := \inf_{(\beta, \gamma) \in \mathcal{B}} E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \beta, \gamma) ds + \Gamma_{0,s}^0 u(\zeta) \right], 0 < \zeta < \infty. \quad (3.15)$$

By \ref{constraint:3.14}, we have

$$\tilde{V}(x) \leq V_*(x), \quad (3.16)$$

where

$$V_*(x) := \inf_{\zeta > 0, (\beta, \gamma) \in \mathcal{B}} E\left[ \int_0^T \Gamma_{0,s}^0 \beta^s F(s, \beta, \gamma) ds + \Gamma_{0,s}^0 u(\zeta) \right] + \zeta = \inf_{\zeta > 0} (\tilde{V}(\zeta) + \zeta). \quad (3.17)$$
Lemma 3.6 Under the assumptions of lemma 3.5, the followings hold:

(i) \( \hat{\beta}, \hat{\gamma} \) attains the infimum in (3.15) with \( \zeta = \hat{\zeta} \).
(ii) The triple \( (\hat{\zeta}, \hat{\beta}, \hat{\gamma}) \) attains the first infimum in (3.17).
(iii) The number \( \hat{\zeta} \in (0, \infty) \) attains the second infimum in (3.17).
(iv) There is no “duality gap” in (3.11); that is,

\[
V_*(x) = \tilde{V}(x) = \underline{V}(x) = E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right].
\]

Proof: (i) By (3.12) and (3.13),

\[
E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right] + \hat{\zeta} x
\]

\[
= E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right] - \hat{\zeta} x
\]

\[
\leq E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right] - \hat{\zeta} x
\]

\[
\leq E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right], \forall (\beta, \gamma) \in B,
\]

where the last inequality is due to (3.9).

(ii) By (3.12) and (3.13), we have

\[
E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right] + \hat{\zeta} x
\]

\[
= E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right]
\]

\[
\leq E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right], \forall (\beta, \gamma) \in B, \forall \zeta \in (0, \infty)
\]

where the last inequality is an application of (3.9) to \( \xi = \hat{\xi} \).

(iii) By (i), (3.12) and (3.13),

\[
\tilde{V}(\hat{\zeta}) + \hat{\zeta} x
\]

\[
= E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right] + \hat{\zeta} x
\]

\[
= E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right]
\]

\[
\leq E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \hat{\beta}_s, \hat{\gamma}_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right] + \hat{\zeta} x, \forall (\beta, \gamma) \in B, \forall \zeta \in (0, \infty)
\]

So we get

\[
\tilde{V}(\hat{\zeta}) + \hat{\zeta} x \leq \inf_{(\beta, \gamma) \in B} E\left[ \int_0^T \Gamma_{0,s}^\alpha F(s, \beta_s, \gamma_s)ds + \Gamma_{0,T}^\alpha \tilde{u}(\hat{\tilde{L}}(T)) \right] + \hat{\zeta} x = \tilde{V}(\hat{\zeta}) + \hat{\zeta} x, \forall \zeta \in (0, \infty).
\]
(iv) By (ii) and (3.12),
\[
V_*(x) = E\left[\int_0^T \Gamma_0^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_0^{\beta, \gamma} \tilde{u}(\frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}})\right] + \zeta x
\]
\[
= E\left[\int_0^T \Gamma_0^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_0^{\beta, \gamma} u(\hat{\zeta})\right]
\]
\[
= \tilde{V}(x) = V(x).
\]
This completes the proof. \(\square\)

In the following, we prove the existence of the quadruple \((\hat{\zeta}, \hat{\beta}, \hat{\gamma}, \hat{\zeta})\) which is postulated in Lemma 3.5.

Notice that the function \(x \mapsto x\tilde{u}(\frac{1}{x})\) is convex. By similar analysis as in Appendix B of [3], the following lemma holds.

**Lemma 3.7** Under Assumption 2.1, 3.1 and 3.2, for any given \(\zeta > 0\), there exists a pair \((\hat{\beta}, \hat{\gamma}) = (\hat{\beta}_\zeta, \hat{\gamma}_\zeta) \in B\) which attains the infimum in (3.19).

**Lemma 3.8** Under Assumption 2.1, 3.1 and 3.2 and suppose
\[
E\left[\int_0^T \Gamma_0^{\beta, \gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_0^{\beta, \gamma} u(\frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}})\right] < \infty, \quad \forall \zeta > 0, \quad \forall (\beta, \gamma) \in B.
\]
Then for any given \(x > 0\), there exists a number \(\hat{\zeta} = \hat{\zeta}_x \in (0, \infty)\) which attains \(V_*(x) = \inf_{\zeta > 0} [\tilde{V}(\zeta) + \zeta x]\).

**Proof:** **Step 1:** By the convexity of \(\tilde{u}\) and Lemma 3.7, \(\tilde{V}(\cdot)\) is convex. Fix \(\zeta > 0\), denote \((\hat{\beta}, \hat{\gamma}) = (\hat{\beta}_\zeta, \hat{\gamma}_\zeta)\) as in Lemma 3.7. For any \(\delta > 0\), we have
\[
\frac{\tilde{V}(\zeta + \delta) - \tilde{V}(\zeta)}{\delta} \leq \frac{E\left[\Gamma_0^{\beta, \gamma} \tilde{u}(\zeta + \delta + \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}}) - \Gamma_0^{\beta, \gamma} \tilde{u}(\zeta + \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}})\right]}{\delta}
\]
\[
\leq E\left[\hat{L}(T)\tilde{u}'(\zeta + \delta + \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}})\right] = -E\left[\hat{L}(T)I(\zeta + \delta + \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}})\right].
\]
Then, by Levi’s lemma,
\[
\lim_{\delta \to 0^+} \frac{\tilde{V}(\zeta + \delta) - \tilde{V}(\zeta)}{\delta} \leq -E\left[\hat{L}(T)I\left(\zeta + \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}}\right)\right] \tag{3.18}
\]
and
\[
\lim_{\delta \to 0^+} \frac{\tilde{V}(\zeta) - \tilde{V}(\zeta - \delta)}{\delta} \geq -E\left[\hat{L}(T)I\left(\zeta - \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}}\right)\right]. \tag{3.19}
\]
Since \(\tilde{V}(\cdot)\) is convex, we obtain that \(\tilde{V}(\cdot)\) is differentiable on \((0, \infty)\) and \(\tilde{V}'(\zeta) = -E\left[\hat{L}(T)I\left(\zeta + \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}}\right)\right] \).

**Step 2:** Because \(\mu(\cdot)\) is bounded, we have that for any \(\zeta \in (0, \infty)\), \(\frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}} < +\infty, a.s.\). Then,
\[
\tilde{V}'(\infty) := \lim_{\zeta \to +\infty} \tilde{V}'(\zeta) = -\lim_{\zeta \to -\infty} E\left[\hat{L}(T)I\left(\zeta + \frac{\hat{L}(T)}{\Gamma_0^{\beta, \gamma}}\right)\right] = 0,
\]
\[ V'(0) := \lim_{\zeta \to 0^+} V'(\zeta) = -\lim_{\zeta \to 0^+} E\left[ \frac{\tilde{L}(T)I(\zeta \tilde{L}(T))}{\Gamma_{0,T}^{(\hat{\beta}, \hat{\gamma})}} \right] = -\infty. \]

Thus, there exists a number \( \hat{\zeta} \) which attains \( V_*(x) \) and \( V'_{\hat{\zeta}}(\zeta) = -x \in (-\infty, 0) \). This completes the proof.

\[ \square \]

**Lemma 3.9** Under Assumption 3.1, 3.2 and 3.3, \( V_*(x) = E\left[ \int_0^T \Gamma_{0,s}^{(\hat{\beta}, \hat{\gamma})} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds \right] + \hat{\zeta} x \) with \( \hat{\zeta} = \hat{\zeta}_x \) as in lemma 3.8 and \( (\hat{\beta}, \hat{\gamma}) = (\hat{\beta}\hat{\xi}, \hat{\gamma}\hat{\xi}) \) as in lemma 3.7.

**Proof:** We have

\[ E\left[ \int_0^T \Gamma_{0,s}^{(\hat{\beta}, \hat{\gamma})} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds \right] + \Gamma_{0,T}^{(\hat{\beta}, \hat{\gamma})} \tilde{u}(\hat{\zeta}) \]

\[ = \hat{V}(\hat{\zeta}) + \hat{\zeta} x \]

\[ \leq \hat{V}(\hat{\zeta}) + \zeta x \]

\[ \leq E\left[ \int_0^T \Gamma_{0,s}^{(\beta, \gamma)} F(s, \beta_s, \gamma_s) ds \right] + \Gamma_{0,T}^{(\beta, \gamma)} \tilde{u}(\zeta) \], \( \forall (\beta, \gamma) \in \mathcal{B}, \forall \zeta \in (0, \infty), \forall x > 0. \)

This completes the proof. \( \square \)

Our main result is the following theorem.

**Theorem 3.10** Under Assumption 3.1, 3.2 and 3.3, let \( (\hat{\beta}, \hat{\gamma}) \) as in lemma 3.8 and define \( \hat{\xi} = I(\hat{\zeta} \frac{\tilde{L}(T)}{\Gamma_{0,T}^{(\hat{\beta}, \hat{\gamma})}}) \) a.s.. If \( \hat{\xi} \in U \), then \( (\hat{\beta}, \hat{\gamma}, \hat{\xi}) \) satisfies all the conditions in lemma 3.8, that is 3.10, 3.11 and 3.13.

**Proof:** Notice that

\[ \hat{V}(\hat{\zeta}) = \inf_{(\beta, \gamma) \in \mathcal{B}} E\left[ \int_0^T \Gamma_{0,s}^{(\beta, \gamma)} F(s, \beta_s, \gamma_s) ds \right] + \Gamma_{0,T}^{(\beta, \gamma)} \tilde{u}(\hat{\zeta}) \]

Applying the maximum principle in Peng [20], we obtain a necessary condition for \( (\hat{\beta}, \hat{\gamma}) \):

\[ F(t, \hat{\beta}_t, \hat{\gamma}_t) + p_t \hat{\beta}_t + q_t \hat{\gamma}_t \geq F(t, \hat{\beta}_t, \hat{\gamma}_t) + p_t \hat{\beta}_t + q_t \hat{\gamma}_t, \forall (\beta, \gamma) \in \mathcal{B}, \]

where \( (p_t, q_t) \) is the solution of the adjoint equation

\[ \begin{cases} -dp_t = (F(t, \hat{\beta}_t, \hat{\gamma}_t) + p_t \hat{\beta}_t + q_t \hat{\gamma}_t) dt - q_t d\tilde{W}(t), \\ p_T = u(I(\hat{\zeta} \frac{\tilde{L}(T)}{\Gamma_{0,T}^{(\hat{\beta}, \hat{\gamma})}})). \end{cases} \]

\( \forall (\beta, \gamma) \in \mathcal{B}, \) let \( (y_t, z_t) \) and \( (\tilde{y}_t, \tilde{z}_t) \) be the unique solutions of the following two linear BSDEs, respectively,

\[ y_t = u(\hat{\zeta}) + \int_t^T \left( y_s \hat{\beta}_s + z_s \hat{\gamma}_s + F(s, \hat{\beta}_s, \hat{\gamma}_s) \right) ds - \int_t^T z_s d\tilde{W}(s), \]

\[ \tilde{y}_t = u(\hat{\zeta}) + \int_t^T \left( \tilde{y}_s \hat{\beta}_s + \tilde{z}_s \hat{\gamma}_s + F(s, \beta_s, \gamma_s) \right) ds - \int_t^T \tilde{z}_s d\tilde{W}(s). \]

By 3.20 and the comparison theorem of BSDE, we have \( y_t \leq \tilde{y}_t, t \in [0, T], \) a.s., especially \( y_0 \leq \tilde{y}_0. \)
Solving the above linear BSDEs gives
\[ y_0 = E\left[ \int_0^T \Gamma_{0,s}^{\hat{\beta},\hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta},\hat{\gamma}} u(\hat{\xi}) \right] \]
and
\[ \tilde{y}_0 = E\left[ \int_0^T \Gamma_{0,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta,\gamma} u(\hat{\xi}) \right]. \]
So
\[ E\left[ \int_0^T \Gamma_{0,s}^{\hat{\beta},\hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta},\hat{\gamma}} u(\hat{\xi}) \right] \leq E\left[ \int_0^T \Gamma_{0,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{0,T}^{\beta,\gamma} u(\hat{\xi}) \right], \forall (\beta, \gamma) \in \mathcal{B}, \]
which exactly is Eq. (3.13).

By Lemma 3.8, \( \tilde{V}'(\hat{\zeta}) = -x \). By Lemma 3.7,
\[ \tilde{V}(\hat{\zeta}) = E\left[ \int_0^T \Gamma_{0,s}^{\hat{\beta},\hat{\gamma}} F(s, \hat{\beta}_s, \hat{\gamma}_s) ds + \Gamma_{0,T}^{\hat{\beta},\hat{\gamma}} u(\hat{\xi}) \right]. \tag{3.24} \]
Differentiating both sides of (3.24) as functions of \( \hat{\zeta} \), we get
\[ E[I(\hat{\zeta}, \tilde{L}(T)) \tilde{L}(T)] = x. \tag{3.25} \]
This completes the proof. \( \square \)

Remark 3.11 It is worth to pointing out that the adjoint process \( p_t \) in the proof of the above theorem coincides with the optimal utility process \( y_t \) in Eq. (3.22).

4 K-ignorance

In this section, we study a special case which is called K-ignorance by Chen and Epstein [1]. In this case, the generator \( f \) is specified as
\[ f(t, y, z) = -K|z|, \quad K \geq 0. \]
Chen and Epstein interpreted the term \( K|z| \) as modeling ambiguity aversion rather than risk aversion. \( f(z) = -K|z| \) is not differentiable. But it is concave and \( f(z) = \inf_{|\gamma| \leq K} (\gamma z) \). Then, our results in the above section are still applicable.

In this section, we assume \( d = 1, \sigma \equiv 1 \). The wealth equation and recursive utility become
\[
\begin{align*}
-dX(t) &= -q'(t)\tilde{\mu}(t)dt - q'(t)d\tilde{W}(t), \\
X(T) &= \xi, \\
-dY(t) &= -K|Z(t)|dt - Z'(t)d\tilde{W}(t), \\
Y(T) &= u(\xi).
\end{align*}
\]
Our problem is formulated as:

\[
\begin{align*}
\text{Maximize} & \quad J(\xi) := Y_0^\xi, \\
\text{s.t.} & \quad X(0) = x, \\
& \quad (X(\cdot), q(\cdot)), (Y(\cdot), Z(\cdot)) \text{ satisfies Eq. (4.1).}
\end{align*}
\]

(4.2)

Now Lemma 3.3 can be simplified to the following lemma.

**Lemma 4.1** For \( \xi \in U \), the solutions \((X(\cdot), q(\cdot))\) and \((Y(\cdot), Z(\cdot))\) of Eq. (4.1) can be represented as

\[
X(t) = \tilde{L}^{-1}(t)E[\tilde{L}(T)|\mathcal{G}_t], \\
Y(t) = \text{ess inf}_{\gamma \in B} \Gamma_{0,T}^{\gamma}[u(\xi)|\mathcal{G}_t],
\]

where

\[
\tilde{L}(t) = e^{-\int_0^t \hat{\mu}(s)d\tilde{W}(s)} - \frac{1}{2} \int_0^t |\hat{\mu}(s)|^2 ds, \\
\Gamma_{0,T}^{\gamma} = e^{\int_0^T \gamma_t d\tilde{W}(s)} - \frac{1}{2} \int_0^T |\gamma_t|^2 ds, \\
B = \{ \gamma = \{ \gamma_t \}_{t \geq 0} | \gamma_t \text{ is } \mathcal{G}-\text{progressively measurable, } |\gamma_t| \leq K, t \in [0,T], a.s. \}. 
\]

For any \( \gamma \in B \), \( \Gamma_{0,T}^{\gamma} \) is \((\mathcal{G}, P)\)-martingale. Then, a new probability measure \( P_\gamma \) is defined on \( \mathcal{G}_T \) by

\[
\frac{dP_\gamma}{dP} = \Gamma_{0,T}^{\gamma}
\]

and \( \tilde{W}_\gamma(t) = \tilde{W}(t) - \int_0^t \gamma_s ds \) is a Brownian motion under \( P_\gamma \). Thus, \( Y(0) = \text{inf}_{\gamma \in B} E_\gamma [u(\xi)] \) where \( E_\gamma [\cdot] \) is the expectation operator with respect to \( P_\gamma \).

Our problem (4.2) is equivalent to the following problem:

\[
\begin{align*}
\text{Maximize} & \quad J(\xi) = \text{inf}_{\gamma \in B} E_\gamma u(\xi) \\
\text{s.t.} & \quad \xi \in \mathcal{A}(x).
\end{align*}
\]

(4.3)

The auxiliary dual problem in (3.15) becomes

\[
\bar{V}(\zeta) \equiv \bar{V}(\zeta; x) := \text{inf}_{\gamma \in B} E_\gamma [\bar{u}(\zeta Z(\cdot))], \quad 0 < \zeta < \infty,
\]

(4.4)

where \( Z(\cdot) := \frac{\tilde{L}(\cdot)}{\Gamma_{0,T}^{\gamma}}, \quad t \in [0,T], a.s. \) and

\[
V_\gamma(x) := \text{inf}_{\zeta > 0} [E_\gamma [\bar{u}(\zeta Z(\cdot))] + \zeta x] = \text{inf}_{\zeta > 0} [\bar{V}(\zeta) + \zeta x].
\]

(4.5)

Applying the procedure in the previous section, we can find the saddle point. So we list the results without proof except Lemma 4.2 in which a new proof is given.

**Lemma 4.2** Under Assumption 3.2, for any given \( \zeta > 0 \), there exists a unique \( \hat{\gamma} = \hat{\gamma}_\zeta \in B \) which attains the infimum in (4.4).
Assumption 4.7 satisfies the following assumption:

wealth of the investor may be negative. This utility function if \( \hat{u} \) is the solution to problem (4.3) is

\[
\text{Constant absolute risk aversion). Suppose that}
\]

Lemma 4.3 Under Assumption 3.2, if \( g \) exists a number \( \hat{\zeta} \), then for any given \( x > 0 \), there exists a sequence \( \tilde{M}_n(T) \in \text{conv}(M_n(T), M_{n+1}(T), \ldots) \), i.e. \( M_n(T) = \sum_{k=n}^{\infty} \lambda_k M_{k}(T) \), \( \lambda_k \in [0,1] \) and \( \sum_{k=n}^{\infty} \lambda_k = 1 \), such that the sequence \( \{M_n(T)\}_{n \geq 1} \) converges a.s. to a random variable \( M \). By the a.s. closedness of \( M \), we have \( M \in M \), that is \( \mathbb{E} \in B \), s.t. \( M \in M(T) \).

Consider a minimizing sequence \( \{M_n(T)\}_{n \geq 1} \) for (4.4), that is

\[
\lim_{n \to \infty} \mathbb{E}[g(M_n(T))] = \tilde{V}^{\prime} - \mathbb{E} \sum_{k=n}^{\infty} \lambda_k M_k(T) \gamma_k = \hat{\zeta} \in (0, \infty) \] which attains the infimum of \( V_{\hat{\zeta}}(x) = \mathbb{E} \sum_{k=n}^{\infty} \lambda_k M_k(T) \gamma_k \) is closed under a.s. convergence because \( B \) is uniformly integrable. As a consequence, \( M \) is closed under a.s. convergence.

By Komlos’ theorem, there exists a sequence \( \tilde{M}_n(T) \) such that \( \sum_{k=n}^{\infty} \lambda_k M_k(T) \), \( \lambda_k \in [0,1] \) and \( \sum_{k=n}^{\infty} \lambda_k = 1 \), such that the sequence \( \{M_n(T)\}_{n \geq 1} \) converges a.s. to a random variable \( M \). By the a.s. closedness of \( M \), we have \( M \in M \), that is \( \exists \tilde{\gamma} \in B \), s.t. \( M = M_{\tilde{\gamma}}(T) \).

Note that \( g \) is a strictly convex function, hence we have

\[
\mathbb{E}[g(M)] = \mathbb{E}[\lim_{n \to \infty} g(M_{\tilde{\gamma}}(T))] \leq \lim_{n \to \infty} \mathbb{E}[g(M_{\tilde{\gamma}}(T))] \leq \mathbb{E}[\lim_{n \to \infty} g(M_{\tilde{\gamma}}(T))]
\]

\[
\leq \lim_{n \to \infty} \mathbb{E}[g(M_{\tilde{\gamma}}(T))] = \mathbb{E}[\lim_{n \to \infty} g(M_{\tilde{\gamma}}(T))] = \tilde{V}^{\prime} - \mathbb{E} \sum_{k=n}^{\infty} \lambda_k M_k(T) \gamma_k = \hat{\zeta} \in (0, \infty)
\]

The uniqueness follows from the strictly convexity of \( g \). This completes the proof. \( \square \)

Lemma 4.3 Under Assumption 3.2 if \( \mathbb{E}[I(\zeta Z_{\tilde{\gamma}}(T))] < \infty \), \( \forall \zeta > 0 \), \( \forall \gamma \in B \), then for any given \( x > 0 \), there exists a number \( \hat{\zeta} = \hat{\zeta}_x \in (0, \infty) \) which attains the infimum of \( \tilde{V}_{\hat{\zeta}}(x) = \mathbb{E} \sum_{k=n}^{\infty} \lambda_k M_k(T) \gamma_k \) as in lemma 4.3 and \( \hat{\gamma} = \hat{\gamma}_{\hat{\zeta}} \) as in lemma 4.4.

Lemma 4.4 Under Assumption 3.2 \( V_{\hat{\zeta}}(x) = \mathbb{E} \sum_{k=n}^{\infty} \lambda_k M_k(T) \gamma_k \) is the same as in lemma 4.3 and \( \hat{\gamma} = \hat{\gamma}_{\hat{\zeta}} \) as in lemma 4.4.

Theorem 4.5 Under Assumption 3.2 let \( (\hat{\zeta}, \hat{\gamma}) \) is the same as in lemma 4.5, then the optimal terminal wealth of problem (4.3) is

\[
\hat{\xi} = I(\hat{\zeta} Z_{\hat{\gamma}}(T)), \text{ a.s.}
\]

if \( \hat{\xi} \) belongs to \( U \).

In the following, we give some examples to illustrate our above analysis.

Example 4.6 (Constant absolute risk aversion). Suppose that \( u(x) = 1 - e^{-\alpha x}, x \in \mathbb{R}, \alpha > 0 \), and the wealth of the investor may be negative. This utility function \( u \) does not satisfies Assumption 3.2. But it satisfies the following assumption:

Assumption 4.7 \( u \) is strictly increasing, strictly concave, continuously differentiable, and

\[
u'(-\infty) := \lim_{x \to -\infty} u'(x) = \infty, \quad u'(\infty) := \lim_{x \to \infty} u'(x) = 0.
\]
Note that under Assumption 4.7, the results in this section still hold.

For this example, \( I(\zeta) = -\frac{1}{\alpha} \ln x, \zeta > 0, \) and \( \tilde{u}(\zeta) = 1 - \frac{1}{\alpha} + \frac{\zeta}{\alpha}, \zeta > 0. \) Then the value function of the auxiliary dual problem (4.4) is

\[
E_\gamma \tilde{u}(\zeta Z_\gamma(T)) = 1 - \frac{\zeta}{\alpha} + \frac{\zeta}{\alpha} \ln \frac{\zeta}{\alpha} + \frac{\zeta}{2\alpha} \tilde{E}[\ln Z_\gamma(T)] = 1 - \frac{\zeta}{\alpha} + \frac{\zeta}{\alpha} \ln \frac{\zeta}{\alpha} + \frac{\zeta}{2\alpha} \tilde{E} \int_0^T (\hat{\mu}(t) + \gamma_t)^2 dt, \zeta > 0.
\]

Apparently, \( \gamma_t \) (the optimal \( \gamma_t \)) which attains the infimum of Problem (4.4) is independent of \( \zeta. \) It is easy to see that

\[
\gamma_t = (-K) \vee (-\tilde{\mu}(t)) \wedge K, t \in [0, T], \text{a.s.}
\]

The optimal value of Problem (4.4) is

\[
\hat{V}(\zeta) = 1 - \frac{\zeta}{\alpha} + \frac{\zeta}{\alpha} \ln \frac{\zeta}{\alpha} + \frac{\zeta}{2\alpha} \tilde{E} \int_0^T (\hat{\mu}(t) + \gamma_t)^2 dt,
\]

and the Lagrange multiplier in Lemma 4.3 is

\[
\hat{\zeta} = \alpha e^{-\frac{1}{2} \tilde{E} \int_0^T (\hat{\mu}(t) + \gamma_t)^2 dt - \alpha x} = \arg \min \limits_{\zeta > 0} [\hat{V}(\zeta) + \zeta x].
\]

Thus, the optimal terminal wealth in Theorem 4.5 is

\[
\hat{x} = \frac{1}{\alpha} \ln \frac{\hat{\zeta} Z_\gamma(T)}{\alpha}.
\]

Moreover, it is easy to check that \( (Y(t), Z(t)) := (1 - \frac{\zeta}{\alpha} Z_\gamma(t), \frac{\zeta}{\alpha} (\hat{\mu}(t) + \gamma_t) Z_\gamma(t)), t \in [0, T] \) uniquely solves the utility equation in Eq. (4.4) when \( \xi = \hat{x}. \)

**Example 4.8 (Logarithmic utility function)** Suppose \( u(x) = \ln x, \, x > 0. \) In this case,

\[
I(\zeta) = \frac{1}{\zeta}, \text{ and } \tilde{u}(\zeta) = -\ln \zeta - 1, \zeta > 0.
\]

Then the value function of the auxiliary dual problem (4.4) is

\[
E_\gamma \tilde{u}(\zeta Z_\gamma(T)) = E_\gamma [\ln Z_\gamma(T)] - 1
\]

\[
= E_\gamma [\ln Z_\gamma(T)] - 1
\]

\[
= \frac{1}{2} E_\gamma \int_0^T (\hat{\mu}(t) + \gamma_t)^2 dt - \ln \zeta - 1, \zeta > 0.
\]

So the optimal \( \gamma_t \) is independent of \( \zeta. \) Consider the following BSDE

\[
y_\gamma(t) = E_\gamma \left[ \int_t^T (\hat{\mu}(s) + \gamma_s)^2 ds | G_t \right] = \int_t^T ([\hat{\mu}(s) + \gamma_s]^2 + \gamma_s z_\gamma(s)] ds - \int_t^T z_\gamma(s) d\tilde{W}(s).
\]

Set

\[
f(t, z_t) = \inf \limits_{\gamma_t \in \mathcal{B}} ([\hat{\mu}(t) + \gamma_t]^2 + \gamma_t z_t]
\]

\[
= \begin{cases} 
K^2 - 2K \hat{\mu}(t) - K z_t + \hat{\mu}(t)^2, & \text{if } -2\hat{\mu}(t) + 2K < z_t; \\
\frac{1}{2} z_t^2 - \hat{\mu}(t) z_t, & \text{if } -2\hat{\mu}(t) - 2K \leq z_t \leq -2\hat{\mu}(t) + 2K; \\
K^2 + 2K \hat{\mu}(t) + K z_t + \hat{\mu}(t)^2, & \text{if } z_t < -2\hat{\mu}(t) - 2K, \, t \in [0, T], \text{a.s.}
\end{cases}
\]
It is easy to show that \( f(t, z_t) \) is uniformly Lipschitz, so the following BSDE has a unique solution which we still denote by \((y_t, z_t)\).

\[
y_t = \int_t^T f(s, z_s)ds - \int_t^T z_s d\tilde{W}(s). \tag{4.8}
\]

Then the infimum in problem \( \{4.4\} \) is attained at

\[
\hat{y}_t = \arg\inf_{\gamma \in \mathcal{B}} \left[ (\hat{\mu}(t) + \gamma_t)^2 + \gamma_t z_t \right]
\]

\[
= -KI(-2\hat{\mu}(t)+2K < z_t) + (\hat{\mu}(t) - \frac{z_t}{2})I(-2\hat{\mu}(t)-2K \leq z_t \leq -2\hat{\mu}(t)+2K) + KI(z_t < -2\hat{\mu}(t)-2K), \quad t \in [0, T], \text{a.s}.
\]

The Lagrange multiplier in Lemma \( \{4.3\} \) is

\[
\hat{\xi} \equiv \hat{\xi}_x = \frac{1}{x} = \arg\min_{\zeta > 0} [\hat{V}(\zeta) + \zeta x].
\]

The optimal terminal wealth in Theorem \( \{4.5\} \) is

\[
\hat{\xi} = \frac{\hat{x}}{\hat{Z}_\gamma(T)}.
\]

**Example 4.9** Suppose that the appreciation rate \( \mu(t) \) is a bounded deterministic function of \( t \). In this case, \( \mathcal{G}_t = \mathcal{F}_t, \ t \geq 0 \), and we claim that

\[
\hat{y}_t = (-K) \vee (-\mu(t)) \cap K, \quad t \in [0, T]. \tag{4.9}
\]

**Proof:** We show that \( \hat{y} \) defined above attains the infimum of \( \{4.4\} \). Denote

\[
v(t, x) \equiv v(t, x; \xi) := \mathbb{E}[g(x M_{\xi}(T))].
\]

Then \( v(t, x) \) is the solution of the partial differential equation \( \frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} x^2 (\mu(t) + \gamma_t)^2 = 0 \).

\[
\forall \gamma \in \mathcal{B}, \text{applying Itô’s formula to } v(t, M_{\gamma}(t)), \text{we have}
\]

\[
dv(t, M_{\gamma}(t)) = \left[ \frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2}(M_{\gamma}(t)^2 (\mu(t) + \gamma_t)^2) dt + \frac{\partial v}{\partial x} M_{\gamma}(t) (\mu(t) + \gamma_t) d\tilde{W}(t) \right.
\]

\[
= \frac{1}{2} \frac{\partial^2 v}{\partial x^2} (M_{\gamma}(t)^2 (\mu(t) + \gamma_t)^2 - (\mu(t) + \gamma_t)^2) dt + \frac{\partial v}{\partial x} M_{\gamma}(t) (\mu(t) + \gamma_t) d\tilde{W}(t).
\]

By the definition of \( \hat{y}_t \) \( \{4.9\} \), we have \( (\mu(t) + \gamma_t)^2 - (\mu(t) + \hat{\gamma}_t)^2 \geq 0, \ t \in [0, T]. \) The convexity of \( v(t, \cdot) \) guarantees that \( v(t, M_{\gamma}(t)) \) is a submartingale. Thus, \( \forall \gamma \in \mathcal{B}, \)

\[
E_{\gamma}[\hat{z}(\zeta Z_\gamma(T))] = \mathbb{E}[g(M_{\gamma}(T))] = \hat{E}v(T, M_{\gamma}(T)) \geq \hat{E}v(0, M_{\gamma}(0)) = \mathbb{E}[g(M_{\gamma}(T))] = E_{\gamma}[\hat{u}(\zeta Z_\gamma(T))].
\]

This completes the proof. \( \Box \)

**Example 4.10** Suppose that \( |\mu(\cdot)| \leq K, \) a.e., a.s. Then we have

\[
\hat{y}_t = -\hat{\mu}(t), \quad t \in [0, T], \text{a.s.} \tag{4.10}
\]

Note that \( \mu \) belongs to \( \mathcal{B} \) when \( |\mu(\cdot)| \leq K, \) a.e., a.s. Due to the convexity of \( g \), we have that \( \forall \gamma \in \mathcal{B}, \)

\[
\hat{E}[g(M_{\gamma}(t))] \geq g(\hat{E}(M_{\gamma}(T))) = g(1) \equiv g(M_{\gamma}(T)) \equiv \hat{E}[g(M_{\gamma}(T))].
\]

In this case, \( P_{\gamma} \) coincides with the risk neutral probability \( \hat{P} \) on \( \mathcal{G}_T \) which leads to the optimal terminal wealth \( \hat{\xi} = x \). This means that the investor will not invest on the risk assets at all.

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5 Terminal perturbation method

When the generator of the recursive utility \([2.3]\) is non-concave, the dual method is not applicable. In this case, we apply the terminal perturbation method to obtain a characterization of the optimal terminal wealth. We need the following smooth assumption:

**Assumption 5.1** \(f\) is continuously differentiable in \((y, z)\).

Let \(\xi^*\) be an optimal terminal wealth for \([2.8]\), i.e.

\[ Y^{\xi^*}(0) = \sup_{\xi \in A(x)} Y^\xi(0), \]

and \((X^*(\cdot), q^*(\cdot), Y^*(\cdot), Z^*(\cdot))\) be the corresponding state processes of \([2.7]\).

Set

\[ \bar{\Omega} := \{ \omega \in \Omega | \xi^*(\omega) = 0 \}. \]

By the terminal perturbation method in \([11]\) and \([12]\), we have the following stochastic maximum principle.

**Theorem 5.2** Under assumptions \([2.1]\), \([2.2]\) and \([5.1]\), if \(\xi^*\) is the optimal wealth of problem \([2.8]\) then there exists \(h_0 \in \mathbb{R}, h_1 \geq 0\) and \(|h_0|^2 + h_1^2 = 1\) such that

\[ h_0 m(T) + h_1 u'(\xi^*) n(T) \geq 0, \text{ a.s. on } \bar{\Omega}; \]

\[ h_0 m(T) + h_1 u'(\xi^*) n(T) = 0, \text{ a.s. on } \bar{\Omega}^c, \]

where

\[
\begin{cases}
    dm(t) = -\dot{\mu}'(t)\sigma^{-1}(t)m(t)d\overline{W}(t), & m(0) = 1; \\
    dn(t) = f^*_X(t)n(t)dt + f^*_Z(t)n(t)d\overline{W}(t), & n(0) = 1,
\end{cases}
\]

and \(f^*_X(t) := f_Y(t, Y^*(t), Z^*(t)), f^*_Z(t) := f_Z(t, Y^*(t), Z^*(t))\).

**Remark 5.3** Note that we do not need the concavity property of \(u\) in the above theorem.

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