The similarity degree of an operator algebra, II

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Abstract. For every integer $d \geq 1$, there is a unital closed subalgebra $A_d \subset B(H)$ with similarity degree equal precisely to $d$, in the sense of our previous paper. This means that for any unital homomorphism $u: A_d \to B(H)$ we have $\|u\|_{cb} \leq K\|u\|^d$ with $K > 0$ independent of $u$, and the exponent $d$ in this estimate cannot be improved. The proof that the degree is larger than $d - 1$ crucially uses an upper bound for the norms of certain Gaussian random matrices due to Haagerup and Thorbjørnsen. We also include several complements to our previous publications on the same subject.

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§0. Introduction

This article is a continuation of our earlier papers [P1, P2]. We denote by \( B(H) \) the algebra of all bounded operators on a Hilbert space \( H \). Let \( A \) be a unital operator algebra i.e. a closed unital subalgebra of \( B(H) \). Assume that every bounded morphism (= unital homomorphism) \( u: A \to B(H) \) is automatically completely bounded (c.b. in short). Then (cf. [P1]) there is an integer \( d \) and a constant \( K \) such that any such \( u \) satisfies
\[
\|u\|_{cb} \leq K\|u\|^d.
\]
The smallest \( d \) for which this holds is called the similarity degree of \( A \) and is denoted by \( d(A) \). By convention, we set \( d(A) = \infty \) if there is a bounded morphism \( u: A \to B(H) \) which is not c.b. The main result proved in section 2 below is

**Theorem 0.1.** For any \( d \geq 1 \), there is a (nonself-adjoint) unital operator algebra \( A_d \) such that \( d(A_d) = d \).

Note that the existence of unital (nonself-adjoint) operator algebras \( A \) with \( d(A) = \infty \) is well known. In view of this, the preceding result is not too surprising. However its verification has proved to be much more difficult than expected, although the algebras \( A_d \) themselves are rather canonical and easy to define.

In [P1], we gave examples of \( C^* \)-algebras with degree equal to 1, 2 and 3 but we could not construct any examples (self-adjoint or not) with finite degree \( > 3 \). The preceding result fills this gap in the nonself-adjoint case, but the case of \( C^* \)-algebras remains open. Note that a well known conjecture of Kadison [Ka] implies, modulo [P1], that there is a universal bound for the similarity degree of \( C^* \)-algebras, but we are convinced that the opposite is what happens.

The proof of Theorem 0.1 uses "maximal operator spaces" in the sense of [BP]. A typical example is the space \( \ell_1 \) (or its \( m \)-dimensional version \( \ell_1^m \)) equipped with the dual operator space structure to the commutative \( C^* \)-algebra \( c_0 \) (or \( \ell_\infty^m \)). The next result, perhaps of independent interest, is the main key tool which we use to prove Theorem 0.1.

**Theorem 0.2.** Let \( m \geq 1 \) and \( d \geq 1 \) be integers. We denote simply \( \ell_1^m \) for \( \max(\ell_1^m) \). Let
\[
E_m^d = \ell_1^m \otimes_h \cdots \otimes_h \ell_1^m \ (d \text{ times})
\]
and let
\[
J_{m,d} : E_m^d \to \max(\ell_2^m \otimes_2 \cdots \otimes_2 \ell_2^m)
\]
be the identity map. Then

\[ 2^{-2(d-1)} m^{\frac{d-1}{2}} \leq \| J_{m,d} \|_{cb} \leq m^{\frac{d-1}{2}}. \]

Moreover, the identity map \( I_d \) on \( E^d_m \) satisfies

\[ 2^{-2(d-1)} m^{\frac{d-1}{2}} \leq \| I_d : E^d_m \rightarrow \max(E^d_m) \|_{cb} \leq m^{\frac{d-1}{2}}. \]

We can restate the preceding result in non-technical terms. In order to do this, we will now give a more explicit description of the constants estimated in (0.1) and (0.2) above.

Consider a family of scalars \( \{ \lambda_i | i = (i_1, \ldots, i_d) \in [1, \ldots, m]^d \} \) and let

\[ P = \sum \lambda_i X_{i_1}^1 X_{i_2}^2 \cdots X_{i_d}^d \]

be the associated “polynomial” (homogeneous of degree \( d \)) in the non-commutative variables \( \{ X_j^k \mid 1 \leq k \leq d, 1 \leq j \leq m \} \). We introduce the norm

\[ |||P||| = \sup \left\| \sum \lambda_i x_{i_1}^1 x_{i_2}^2 \cdots x_{i_d}^d \right\|_{B(\ell_2)} \]

where the supremum runs over all families \( \{ x_j^k \mid 1 \leq k \leq d, 1 \leq j \leq m \} \) in the unit ball of \( B(\ell_2) \) (actually, the supremum is the same if we restrict to the case when \( x_j^1 = x_j^2 = \ldots = x_j^d \) for all \( 1 \leq j \leq m \)).

We denote by \( C(m, d) \) the smallest constant \( C \geq 0 \) with the following property: for any family \( \{ x_i \mid i \in [1, \ldots, m]^d \} \) in \( B(\ell_2) \) such that

\[ \forall i \in [1, \ldots, m]^d \quad \| \sum \lambda_i x_i \| \leq \| \sum \lambda_i X_{i_1}^1 X_{i_2}^2 \cdots X_{i_d}^d \| \]

we can find operators \( \{ x_p^k \mid 1 \leq k \leq d, 1 \leq p \leq m \} \) in the unit ball of \( B(\ell_2) \) such that

\[ \forall i \in [1, \ldots, m]^d \quad x_i = C x_{i_1}^1 x_{i_2}^2 \cdots x_{i_d}^d. \]

(Note: Let \( (e_i) \) be the natural basis of \( E^d_m \). If we fix the family \( (x_i) \) then the smallest constant \( C \geq 0 \) for which (0.4) holds is the \( cb \)-norm of the map from \( E^d_m \) to \( B(\ell_2) \) taking \( e_i \) to \( x_i \) and (0.3) means that this map is assumed of norm \( \leq 1 \).) It is easy to check that

\[ \left( \sum |\lambda_i|^2 \right)^{1/2} \leq \| \sum \lambda_i X_{i_1}^1 \cdots X_{i_d}^d \|. \]
so that for (0.3) to hold it suffices to have

\[ \forall \lambda_i \in C \ (i \in [1, \ldots, m]^d) \quad \left\| \sum \lambda_i x_i \right\| \leq \left( \sum |\lambda_i|^2 \right)^{1/2}. \]

We denote by \( C'(m, d) \) the smallest constant \( C \) such that any family \((x_i)\) in \( B(\ell_2) \) satisfying the estimate (0.5) admits a factorization of the form (0.4).

Then by known results (see [CS2]), we have \( C'(m, d) = \|J_{m,d}\|_{cb} \) and \( C(m, d) = \|Id: E_m^d \to \max(E_m^d)\|_{cb} \), whence the following reformulation of Theorem 0.2.

**Theorem 0.2 bis.** We have for all \( m, d \geq 1 \)

\[ 2^{-2(d-1)} m^{d-1} \leq C'(m, d) \leq C(m, d) \leq m^{d-1}. \]

Note that when the “degree” \( d \) is fixed and \( m \to \infty \) our estimates give the sharp order of magnitude (i.e. the exponent \((d - 1)/2\) is sharp). The delicate point is the lower bound in (0.1), (0.2) or (0.6). Our proofs of this uses random matrices. In particular, we make crucial use of a remarkable upper bound for the norm of Gaussian random matrices with matrix coefficients due to Haagerup and Thorbjørnsen [HT] which we state below as Theorem 1.1. Indeed, the operators \( x_i \) which achieve the lower bound in (0.6) are actually obtained by a random selection of matrices of suitably large size which we can always view as elements of \( B(\ell_2) \) by adding coefficients equal to zero. However, the precise form of our random matrices is rather complicated (see (2.4) below).

As obvious idea which comes to mind is to let \( \{x_i \mid i \in [1, \ldots, m]^d\} \) be an independent collection of \( N \times N \) random matrices with independent entries, each one being Gaussian with mean zero and \( L_2 \)-norm equal to \( N^{-1/2} \). It is well known that, up to a numerical factor, when \( N \to \infty \) (and \( m, d \) remain fixed), these will satisfy (0.5) with large probability (say \( > 1/2 \)). We are thus reduced to estimate the best possible \( C \) for which (0.4) holds for these. However, although this works when \( d = 2 \), this choice of \((x_i)\) is definitely not the right one when \( d \) is larger. Indeed, we show at the end of §2 that this way of choosing \( x_i \) leads to weaker lower bounds in (0.6) when \( d > 2 \) (and even that the resulting estimate is not sensitive enough to “distinguish” between the cases \( d = 2k \) and \( d = 2k + 1! \)). This explains why we use a more complicated definition of our random choice of the matrices \((x_i)\), in which the above mentioned result from [HT] is crucial to show that (0.5) still holds.
The algebras $A_d$ appearing in Theorem 0.1 can be described as follows. Let $E$ be an operator space and let $OA(E)$ be the universal unital operator algebra generated by $E$. This means that any complete contraction $v: E \to B(H)$ uniquely extends to a completely contractive morphism $\hat{v}: OA(E) \to B(H)$. We construct $OA(E)$ as a suitable completion of the tensor algebra of $E$. Let $I_d(E) \subset OA(E)$ be the ideal generated by $E \otimes \cdots \otimes E (d + 1) - \text{times})$. Then we define $A_d(E) = OA(E)/I_d(E)$, and we let $A_d = A_d(\ell_1)$. By [BRS], we know that these quotients are (completely isometrically) operator algebras. The estimates in Theorem 0.2 will allow us to prove:

**Theorem 0.3.** Let $E$ be any infinite dimensional maximal operator space. Then

$$d(A_d(E)) = d \quad \forall d \geq 1.$$  

**Remark.** If $E$ is not completely isomorphic to a maximal operator space, then $d(A_d(E)) = \infty$. Indeed, this follows from [Pa3], where the case $d = 1$ of the preceding result is proved. More precisely, note that if $k < d$, then $A_k(E)$ is a quotient of $A_d(E)$, so that $d(A_k(E)) \leq d(A_d(E))$. Taking $k = 1$, we find that if the degree of $A_d(E)$ is finite, then necessarily $d(A_1(E)) < \infty$ which implies (see [Pa3]) that $E$ is completely isomorphic to a maximal operator space since any bounded map $v: E \to B(H)$ must be c.b.

**Remark.** By the non-commutative version of von Neumann’s inequality proved in [Bo], it is easy to see that $OA(\ell_1)$ can be identified (completely isometrically) with the unital closed subalgebra of $C^*(F_\infty)$ generated by the generators only (and not their inverses); here $F_\infty$ denotes the free group with countably infinitely many generators. Moreover, $OA(E)$ coincides with the unital closed subalgebra generated by $E$ in the $C^*$—algebra of $E$ in the sense of [Pes].

After some background in §1, we prove these results at the end of §2. Then in §3 we prove several complements. We return to the general framework adopted in [P1] of a similarity setting, i.e. an operator space generating an operator algebra. In particular, we will prove:

**Theorem 0.4.** Let $A$ be a unital algebra. Assume that, for some $\varepsilon > 0$, any morphism $u: A \to B(H)$ with $\|u\| \leq 1 + \varepsilon$ is completely bounded. Then necessarily the same property holds for all $\varepsilon > 0$ and hence $d(A) < \infty$.  

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More generally, for any $c \geq 1$, let us denote by $C_c$ the class of all morphisms $u: A \to B(H)$ such that $\|u\| \leq c$.

In addition, we will say that two morphisms $u_1: A \to B(H_1)$ and $u_2: A \to B(H_2)$ are similar if there exists an isomorphism $\xi: H_1 \to H_2$ such that

$$\forall a \in A \quad u_2(a) = \xi u_1(a) \xi^{-1}.$$  

Then we can state one more result to be proved in §3.

**Theorem 0.5.** Let $A$ be a unital operator algebra and let $1 \leq \theta < c < \infty$ be fixed. Then the following are equivalent:

(i) Every morphism in $C_c$ is similar to a morphism in $C_\theta$.

(ii) Every bounded morphism $u: A \to B(H)$ is similar to one in $C_\theta$.

**Remark 0.6.** When (i) and (ii) above hold, the results of [P1] can be applied and yield that there are $\alpha > 0$ and $K$ such that for any bounded morphism $u: A \to B(H)$ there is an invertible operator $\xi: H \to H$ with $\|\xi\| \|\xi^{-1}\| \leq K \|u\|^{\alpha}$ such that $\|u\xi\|_{cb} \leq \theta$, where we have set $u\xi(\cdot) = \xi^{-1}u(\cdot)\xi$. Moreover, the smallest such $\alpha$ is an integer. This is nothing but the similarity degree of a certain “enveloping operator algebra” which is denoted $\tilde{A}_\theta$ in [P1].

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§1. Background

We recall that an “operator space” is a closed subspace $E \subset B(H)$ of the $C^*$-algebra of all bounded operators on a Hilbert space $H$. When $H = \ell_2$, we will denote by $K$ the subalgebra of all compact operators on $\ell_2$. Let $E_1, E_2$ be operator spaces. We denote by $E_1 \otimes E_2$ their algebraic tensor product (as vector spaces). Assume $E_i \subset B(H_i)$ ($i = 1, 2$). Then $E_1 \otimes E_2$ can be identified with a linear subspace of $B(H_1 \otimes_2 H_2)$. The completion of $E_1 \otimes E_2$ for the induced norm is called the minimal (= spatial) tensor product and is denoted $E_1 \otimes_{\text{min}} E_2$. Obviously the resulting embedding $E_1 \otimes_{\text{min}} E_2 \subset B(H_1 \otimes_2 H_2)$ allows us to view $E_1 \otimes_{\text{min}} E_2$ as an operator space. We will denote its norm by $\| \|_{E_1 \otimes_{\text{min}} E_2}$, or simply by $\| \|_{\text{min}}$ when there is no ambiguity.

If $\dim(H) = n$, we identify $B(H)$ with the space $M_n$ of all $n \times n$ matrices with complex entries equipped with the usual operator norm.

Then, if $E$ is an operator space, $M_n \otimes_{\text{min}} E$ can be identified with the space $M_n(E)$ of all $n \times n$ matrices with entries in $E$. In particular, if $E = M_p$ for some integer $p \geq 1$, $M_n \otimes_{\text{min}} M_p$ can be identified with $M_{np}$. Let $I_X$ denote the identity on a space $X$. A linear mapping $u: E_1 \to E_2$ is called completely bounded (in short c.b.) if $I_K \otimes u$ defines a bounded linear map from $K \otimes_{\text{min}} E_1$, to $K \otimes_{\text{min}} E_2$, and we set

$$\| u \|_{\text{cb}} = \| I_K \otimes u: K \otimes_{\text{min}} E_1 \to K \otimes_{\text{min}} E_2 \|.$$ 

For short we will often write $I$ instead of $I_K$. We refer the reader to [Pa1] and [P3] for more information on c.b. maps and to [BP] and [ER1–ER3] for more on “Operator Space Theory”.

A mapping $u$ with $\| u \|_{\text{cb}} \leq 1$ is called “completely contractive” or a “complete contraction”, which we both abbreviate by c.c. We will also use the abbreviations o.s. and o.s.s. for “operator space” and “operator space structure”.

By the term “morphism” we always mean a unital homomorphism between two unital algebras.

We will need the notion of “sum” of operator spaces, in the sense of [P5]. This is defined as follows. Let $\{ E_\pi \mid \pi \in I \}$ be a finite family of operator spaces indexed by some (finite) index set $I$. 

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We assume that this family is “compatible” (this is the term used in interpolation theory, cf. [BL]), i.e. we assume given a specific family $J_\pi: E_\pi \to X$ of continuous linear injective maps into a common Banach space $X$. Thus we may think of the spaces $E_\pi$ as “included” in $X$. This allows us to “compare” an element $x$ in $E_\pi$ with one $x'$ in $E_{\pi'}$, ($\pi \neq \pi'$) by considering their images in $X$. Thus the Banach space $\sum_{\pi \in I} E_\pi$ is defined as the subspace of $X$ formed of all elements $x$ in $X$ of the form $x = \sum_{\pi \in I} J_\pi(x_\pi)$ with $x_\pi \in E_\pi$ for all $\pi$. Equipped with the norm $\|x\| = \inf \sum \|x_\pi\|$ (with the infimum running over all possible representations of $x$), this space becomes a Banach space. The latter space can be identified with the quotient $\left( \bigoplus_{\pi \in I} E_\pi \right) / \Delta$

where $\left( \bigoplus_{\pi \in I} E_\pi \right)_{1}$ denotes the $\ell_1$-direct sum of the family and where

$$\Delta = \{(x_\pi)_{\pi \in I} \mid \sum_{\pi \in I} x_\pi = 0\}.$$

By classical results from operator space theory (based on Ruan’s Theorem), the notions of $\ell_1$-direct sum and the notion of quotient have been extended from the Banach to the operator space category. Therefore the same is true of course for the above “sum” $\sum_{\pi \in I} E_\pi$. For details, see [P5, §2] or [P6, p.35]. Note that we only use the case when the index set $I$ is finite and in that case the space $\bigotimes_{\min} \left( \sum_{\pi \in I} E_\pi \right)$ can be identified with equivalent norms with the space $\sum_{\pi \in I} \bigotimes_{\min} E_\pi$, but for $I$ infinite this does not remain true.

We will use repeatedly the notion of “maximal” operator space introduced in [BP], and further studied in [Pa2]. Let us recall its definition: let $E$ be any normed space. Let $I$ be the class of all maps $u: B \to B(H_u)$ with $\|u\| \leq 1$ (and say $\dim H_u \leq \text{card}(E)$). We let $J: E \to \bigoplus_{u \in I} B(H_u)$ be the isometric embedding defined by $J(x) = \bigoplus_{u \in I} u(x)$. Then, $\text{max}(E)$ is defined as the operator space $J(E) \subset B \left( \bigoplus_{u \in I} H_u \right)$, and any operator space which is of this form (up to complete isometry) is called “maximal”.

The “maximal” operator spaces are characterized by the property that, for any linear map $u: E \to B(H)$ we have $\|u\|_{cb} = \|u\|$. The following slightly more explicit description of their operator space structure from [Pa2] is often useful: for any $n$ and any $x$ in
\[ M_n(\max(E)) \text{ we have } \|x\| < 1 \text{ iff, for some integer } N, \text{ there is a diagonal matrix } D \in M_N(E) \text{ and scalar matrices } \beta \in M_{n,N} \text{ and } \gamma \in M_{N,n} \text{ such that} \]

\[ x = \beta D \gamma \quad \text{and} \quad \|\beta\| \|D\| \|\gamma\| < 1. \]

We refer the reader to [Pa2] for more information on this.

By a “complex Gaussian” random variable, we mean a \( \mathbb{C} \)-valued random variable \( g \) with mean zero such that its real and imaginary parts are independent Gaussian variables with equal variance, so that the covariance matrix of \( g \) viewed as \( \mathbb{R}^2 \)-valued is a multiple of the identity.

We will make crucial use of the following remarkable result of Haagerup and Thorbjørnsen.

**Theorem 1.1.** ([HT]) Let \( r \geq 1 \) and let \( g_1, \ldots, g_r \) be a collection of independent random \( N \times N \) matrices such that the entries \((g_k)_{ij}\) are independent (mean zero) Gaussian complex valued random variables with \( \mathbb{E}|(g_k)_{ij}|^2 = 1/N \), whenever \( 1 \leq k \leq r, 1 \leq i, j \leq N \). Then for any \( p \) and any \( a_1, \ldots, a_r \) in \( M_p \) we have almost surely

\[
\limsup_{N \to \infty} \left\| \sum_{j=1}^{r} a_j \otimes g_j \right\|_{M_p(M_N)} \leq \left\| \sum a_j^* a_j \right\|^{1/2} + \left\| \sum a_j a_j^* \right\|^{1/2},
\]

hence a fortiori \( \leq 2 \left( \sum \|a_j\|^2 \right)^{1/2} \).

**Remark.** For example, let us consider a family \( \{g_{ij}\} \) as in Theorem 1.1 but indexed by \( r = m^2 \) this time so that \( g_{ij} \) is indexed by a pair \( i, j \) with \( 1 \leq i, j \leq m \). We denote as usual by \((e_{ij})\) the canonical basis of \( M_m \) and we introduce the random matrix (of size \( mN \times mN \))

\[ Y_{m,N} = \sum_{ij=1}^{m} e_{ij} \otimes g_{ij}. \]

Observe then that

\[ \left\| \sum_{ij} e_{ij} e_{ij}^* \right\| = \left\| \sum_{ij} e_{ij}^* e_{ij} \right\| = m. \]

Hence Theorem 1.1 implies in particular that

\[
(1.1) \quad \limsup_{N \to \infty} \|Y_{m,N}\| \leq 2m^{1/2}.
\]
Note that actually, as observed in [HT], inequalities such as (1.1) are known to probabilists (cf. [Ge]) and can be obtained by a much more direct proof (not using [HT]) but since the remaining part of our argument depends crucially on [HT], for brevity we content ourselves with the preceding derivation of (1.1) from Theorem 1.1.

**Corollary 1.2.** Fix an integer $d$. Let $(g^{(1)}_1, \ldots, g^{(r)}_1)$ be random matrices as in Theorem 1.1 but with size $N_1 \times N_1$. Then let $(g^{(2)}_1, \ldots, g^{(r)}_2)$ be an analogous $r$-tuple but with size $N_2 \times N_2$ and independent of the preceding collection, and so on until we reach $(g^{(d)}_1, \ldots, g^{(d)}_r)$. Now fix an integer $p$ and let $a_{j_1 \ldots j_d}$ be a collection in $M_p$ indexed by $(j_1, \ldots, j_d) \in [1, \ldots, r]^d$. We have then almost surely

$$\limsup_{N_1 \to \infty} \left( \limsup_{N_2 \to \infty} \ldots \left( \limsup_{N_d \to \infty} \left| \sum a_{j_1 j_2 \ldots j_d} \otimes g^{(1)}_{j_1} \otimes g^{(2)}_{j_2} \otimes g^{(d)}_{j_d} \right| \right) \right) \leq 2^d \left( \sum \|a_{j_1 j_2 \ldots j_d}\|^2 \right)^{1/2}$$

where the summations run over all indices $(j_1, \ldots, j_d)$ with $1 \leq j_k \leq r$.

**Proof.** This is an immediate consequence of Theorem 1.1, by iterated applications. 

We now wish to estimate the following random variable

$$Z(r, p; N_1, \ldots, N_d) = \sup \left\{ \left| \sum a_{j_1 \ldots j_d} \otimes g^{(1)}_{j_1} \otimes \cdots \otimes g^{(d)}_{j_d} (\omega) \right| \right\}$$

where the sup runs over all families $\{a_{j_1 \ldots j_p} \mid 1 \leq j_k \leq r\}$ in $M_p$ such that $\sum \|a_{j_1 \ldots j_p}\|^2 \leq 1$.

We will prove

**Corollary 1.3.** For all $p$ and $r$ we have almost surely

$$\limsup_{N_1 \to \infty} \limsup_{N_2 \to \infty} \ldots \limsup_{N_d \to \infty} Z(r, p; N_1, \ldots, N_d)) \leq 2^d.$$

**Proof.** Let $X$ be the finite dimensional Banach space of all families $a = (a_{j_1 \ldots j_d})$ with $a_{j_1 \ldots j_d} \in M_p$ equipped with the norm $\|a\| = \left( \sum \|a_{j_1 \ldots j_d}\|^2 \right)^{1/2}$. Let $B$ denote the (compact) unit ball of $X$. Fix $0 < \varepsilon < 1$ and let $\Lambda$ be a finite $\varepsilon$-net in $B$. By a classical elementary argument (cf. e.g. [P4, p. 49]) we have

$$Z(r, p; N_1, \ldots, N_d) \leq (1 - \varepsilon)^{-1} \sup_{a \in \Lambda} \left| \sum a_{j_1 \ldots j_d} \otimes g^{(1)}_{j_1} \otimes \cdots \otimes g^{(d)}_{j_d} (\omega) \right|.$$
Now, since $\Lambda$ is finite, Corollary 1.2 implies that we have almost surely
\[ \limsup_{N_1 \to \infty} \left( \ldots \left( \limsup_{N_d \to \infty} \max_{a \in \Lambda} \left\| \sum a_{j_1 \ldots j_d} \otimes g_{j_1}^1 \otimes \ldots \otimes g_{j_d}^d \right\| \right) \right) \leq 2^d. \]
Then Corollary 1.3 follows immediately since $\varepsilon$ can be chosen arbitrarily small.

We will also need the following elementary fact.

**Lemma 1.4.** With the same notation as above, let us denote for each $j = (j_1, \ldots, j_d)$ in $[1, \ldots, m]^d$
\[ U_j = g_{j_1}^1 \otimes \cdots \otimes g_{j_d}^d. \]
We have then almost surely
\[ \liminf_{N_1 \to \infty} \left( \ldots \left( \liminf_{N_d \to \infty} \left\| \sum U_j \otimes U_j^\dagger \right\| \right) \ldots \right) \geq m^d. \]

**Proof.** Let $N = N_1 \times \cdots \times N_d$. It is well known that
\[ \left\| \sum U_j \otimes U_j^\dagger \right\| = \sup \left\{ \left| \text{tr} \left( \sum U_j x U_j^* y \right) \right| \right\} \]
where the supremum runs over $x, y$ in the unit ball of the $N \times N$ Hilbert-Schmidt matrices.
Taking $x$ and $y$ both equal to $N^{-1/2}$-times the identity we obtain
\begin{equation}
\left\| \sum U_j \otimes U_j^\dagger \right\| \geq N^{-1} \text{tr} \left( \sum U_j U_j^* \right). \tag{1.2} \end{equation}
But
\[ N^{-1} \text{tr}(U_j U_j^*) = \prod_{k=1}^d \left[ N_k^{-1} \text{tr}(g_{j_k}^k g_{j_k}^{k*}) \right] \]
and, for each $k$, by the law of large numbers (and the concentration of the $\chi^2$-distribution around its mean) we know that almost surely
\[ \lim_{N_k \to \infty} \frac{1}{N_k} \text{tr}(g_{j_k}^k g_{j_k}^{k*}) = 1. \]
Hence, if $\min(N_1, \ldots, N_d) \to \infty$ we have for each $j$
\[ N^{-1} \text{tr}(U_j \otimes U_j^\dagger) \to 1 \]
whence
\[ N^{-1} \text{tr} \left( \sum U_j \otimes U_j^\dagger \right) \to m^d, \]
from which Lemma 1.4 follows immediately by (1.2).
§2. The examples

The algebras $A_d$ are somewhat canonical. To describe them we start with a universal algebra $OA(E)$ which can be defined as follows.

Let $E$ be an operator space. Let $\mathcal{T}(E)$ be the tensor algebra of $E$, i.e. $\mathcal{T}(E) = \mathbb{C} \oplus E \oplus E^\otimes 2 \oplus \cdots$. Any element of $\mathcal{T}(E)$ can be written as a finite sum $x = x_0 + x_1 + \cdots$ with $x_d \in E^\otimes d$ for all $d \geq 1$ and $x_0 \in \mathbb{C}$. We will denote by $P_j: \mathcal{T}(E) \to E^\otimes j$ the mapping defined by $P_j x = x_j$. Any linear map $v: E \to B(H)$ admits a unique canonical extension $\hat{v}: \mathcal{T}(E) \to B(H)$, as a morphism on the unital algebra $\mathcal{T}(E)$. Let $I$ be the collection of all linear maps $v: E \to B(H_v)$ with $\|v\|_{cb} \leq 1$ (and with, say, $\text{card}(H_v) \leq \text{card}(E)$), and let $J: \mathcal{T}(E) \to \bigoplus_{v \in I} B(H_v)$ be the mapping defined by

$$J(x) = \bigoplus_{v \in I} \hat{v}(x).$$

Using this embedding, we equip $\mathcal{T}(E)$ with a unital operator algebra structure, and we denote by $OA(E)$ the completion of the latter. We have clearly a canonical completely isometric embedding $E \subset OA(E)$. More generally, it is known (see [P1, Prop. 1.10] for details) that the closed subspace of $OA(E)$ generated by $E^\otimes d$ can be identified with the Haagerup tensor product $E \otimes_h \cdots \otimes_h E$ ($d$ times). Let us denote this subspace by $E_d$. For any $d \geq 1$, we denote by $I_d$ the closed ideal generated in $OA(E)$ by $E_{d+1}$. Equivalently $I_d$ can be described as the closed span of $\{E_m \mid m > d\}$. We can then form the quotient algebra

$$A_d(E) = OA(E)/I_d.$$

In the particular case $E = \ell_1$, we simply denote it by

$$A_d = A_d(\ell_1) = OA(\ell_1)/I_d.$$

We will need to describe a bit more the structure of the space $A_d(E)$, as follows.

Proposition 2.1.

(i) $P_j: \mathcal{T}(E) \to E^\otimes j$ defines a completely contractive projection on $OA(E)$.

(ii) Let $q: OA(E) \to A_d(E)$ be the canonical quotient map, and let $\widetilde{E}_j = q(E_j)$ ($0 \leq j \leq d$). Then $\widetilde{E}_j$ is closed and

$$A_d(E) \simeq \widetilde{E}_0 \oplus \cdots \oplus \widetilde{E}_d.$$
More precisely, any $x$ in $A_d(E)$ can be uniquely written as a sum $x = \sum_0^d x_j$ with $x_j \in \tilde{E}_j$, and the projection $x \mapsto x_j$ is a complete contraction from $A_d(E)$ onto $\tilde{E}_j$.

(iii) Finally, for any $0 \leq j \leq d$, the restriction $q|_{E_j}$ is a complete isometry from $E_j$ to $\tilde{E}_j$.

**Proof.** For any $z \in \mathbb{C}$ with $|z| \leq 1$, let $T(z): OA(E) \to OA(E)$ be the completely contractive morphism associated to the linear map $v: E \to OA(E)$ equal to $z$ times the embedding of $E$ into $OA(E)$. For any $y$ in $E^\otimes d$, we have $T(z)y = z^d y$. In particular $T(z)(I_d) \subset I_d$, so that $T(z)$ canonically induces a completely contractive morphism $\theta(z): A_d(E) \to A_d(E)$. Consider $y$ in $OA(E)$ of the form $y = \sum_{j \geq 0} y_j$ (finite sum) with $y_j \in E_j$. We have $T(z)y = \sum_{j \geq 0} z^j y_j$ hence $y_j = \int zT(z) y dm(z)$, which implies $\|y_j\| \leq \|y\|$. Thus, we have a contractive projection $y \to y_j$ from $OA(E)$ onto $E_j$. This averaging argument actually shows that it is a complete contraction, whence (i).

Applying the quotient map $q$, we obtain similarly

$$\theta(z)q(y) = \sum_{j \geq 0} z^j q(y_j)$$

whence again

$$\|q(y_j)\| \leq \|q(y)\| = \left\| \sum q(y_j) \right\|$$

which shows that the mapping $q(y) \mapsto q(y_j)$ defines a completely contractive projection from $A_d(E)$ onto $\tilde{E}_j = q(E_j)$. Also note that $\tilde{E}_j$ is closed. Of course we have $q(E_j) = 0$ $\forall j > d$, whence the decomposition (2.1), completing the proof of (ii).

Note that $q$ restricted to $E_0 + \ldots + E_d$ is clearly injective. To show the last point (iii), consider $y$ in $E_j$ ($0 \leq j \leq d$) with $\|q(y)\| < 1$. Then there is $y'$ in $I_d$ with $\|y + y'\| < 1$.

But since $P_j(y + y') = y$ we have $\|y\| < 1$. Thus by homogeneity we have $\|y\| \leq \|q(y)\|$, which shows that $q|_{E_j}$ is an isometry. The proof that it is a complete isometry is similar and left to the reader. 

**Lemma 2.2.** Let $E$ be a maximal operator space. Then any bounded morphism $u: A_d(E) \to B(H)$ is c.b. and satisfies

$$\|u\|_{cb} \leq \sum_{j=0}^d \|u\|^j \leq (d + 1)\|u\|^d,$$
in particular $d(A_d(E)) \leq d$.

**Proof.** Let $\mathcal{K}_0 = \bigcup_n \mathcal{M}_n$. Consider an element $y$ in $\mathcal{K}_0 \otimes A_d(E)$ with $\|y\| < 1$. We can write $y = q(x) = q(x_0 + \cdots + x_d)$ with $\|x\| < 1$ and $x_j \in \mathcal{K}_0 \otimes E$ where $q: \mathcal{K}_0 \otimes OA(E) \to \mathcal{K}_0 \otimes A_d(E)$ is the quotient map. Therefore if we let $v = uq|_E$ (we identify $E$ with $E_1$) we have

$$(I \otimes u)(y) = (I \otimes \hat{v})[x_0 + \cdots + x_d] = \sum_0^d (I \otimes v^{\otimes j})(x_j)$$

hence

$$\|(I \otimes u)(y)\| \leq \sum_0^d \|v^{\otimes j}: E_j \to B(H)\|_{cb} \|x_j\|$$

and by (i) in Proposition 2.1 we have $\|x_j\| \leq \|x\| < 1$, whence the conclusion since our assumption that $E$ is “maximal” ensures that $\|v\|_{cb} \leq \|v\| \leq \|u\|$.

Let $E$ be an operator space. Let $E_m$ be as above ($m \geq 0$) with the convention that $E_0 = \mathbb{C}$. For any partition $\pi = (m_1, \ldots, m_K)$ with $m_i \geq 0$ such that $\sum m_i = d$ we have a natural product map

$$E_{m_1} \otimes \cdots \otimes E_{m_K} \to E_d$$

which is a complete contraction from $E_{m_1} \otimes_h \cdots \otimes_h E_{m_K}$ to $E_d$. A fortiori, we have a complete contraction from

$$\max(E_{m_1}) \otimes_h \cdots \otimes_h \max(E_{m_K}) \to E_d.$$ 

Let us denote whenever $\pi = (m_1, \ldots, m_K)$

$$E(\pi) = \max(E_{m_1}) \otimes_h \cdots \otimes_h \max(E_{m_K}).$$

If $m_i > 0$ for each $i$, it can be shown that the latter map is always injective (we skip the details, see e.g. [BP, Th. 3.11] which implies that the Haagerup tensor product of two injective maps is injective). This is obvious if $E$ is finite dimensional, or say if $E = \ell_1$. Using these natural continuous injections $E(\pi) \subset E_d$, we define

$$\mathcal{E}_k = \sum_\pi E(\pi)$$
where the sum (this is a sum of operator spaces as defined above, in §1) is relative to all partitions $\pi = (m_1, \ldots, m_K)$ of $d$ into $K$ disjoint nonempty blocks with $K \leq k$ and with $\sum m_i = d$. For example, $\mathcal{E}_1 = \max(E_d)$, $\mathcal{E}_2 = \max(E_d) + \sum_{p=1}^{d-1} \max(E_p) \otimes_h \max(E_{d-p})$, and so on.

Thus, by what precedes, we have a complete contraction

$$\Phi_k: \mathcal{E}_k \rightarrow E_d.$$  

**Lemma 2.3.** If $d(A_d(E)) \leq k$, then necessarily $\Phi_k$ is a complete isomorphism, i.e. $(\Phi_k)^{-1}$ is a c.b. map from $E_d$ to $\mathcal{E}_k = \sum_\pi E(\pi)$.

**Proof.** It is proved in [P1, Th. 4.2] that if $d(A_d(E)) \leq k$, then the product map defines a complete surjection from

$$\max(A_d(E)) \otimes_h \cdots \otimes_h \max(A_d(E)) \text{ (k times)}$$

onto $A_d(E)$.

More precisely, let

$$X_k = \max(A_d(E)) \otimes_h \cdots \otimes_h \max(A_d(E)) \text{ (k times)}$$

and let us denote by $p_k: X_k \rightarrow A_d(E)$ this product map. Then there is a constant $C$ such that for any $x$ in $\mathcal{K} \otimes_{\min} A_d(E)$ there is an element $y$ in $\mathcal{K} \otimes_{\min} X_k$ with $\|y\| \leq C\|x\|$ such that $(I \otimes p_k)(y) = x$. By Proposition 2.1, we have seen that

$$A_d(E) \simeq E_0 \oplus \cdots \oplus E_d$$

hence

$$\max(A_d(E)) \simeq \max(E_0) \oplus \cdots \oplus \max(E_d)$$

and therefore

$$X_k \simeq \bigoplus [\max(E_{m_1}) \otimes_h \cdots \otimes_h \max(E_{m_k})]$$

where the direct sum runs over all $\pi = (m_1, \ldots, m_k)$ with $0 \leq m_i \leq d$.  

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Moreover the projections onto the coordinates of this direct sum are complete contractions. Therefore $y$ can be written as

$$y = \sum_\pi y_\pi \quad \text{with} \quad y_\pi \in \mathcal{K} \otimes E(\pi).$$

If we apply $T(z) \otimes \cdots \otimes T(z)$ ($k$ times) to this equality we obtain

$$(I \otimes T(z)^{\otimes k})y = \sum_\pi z^{\mid \pi \mid}y_\pi$$

where $\mid \pi \mid = m_1 + \cdots + m_k$.

Thus if we actually apply all this to an element $x$ in $\mathcal{K} \otimes \tilde{E}_d$, we have $x = (I \otimes p_k)(y)$ hence

$$z^d x = (I \otimes \theta(z))(x) = (I \otimes \theta(z)p_k)(y) = (I \otimes p_k T(z)^{\otimes k})(y) = \sum_\pi z^{\mid \pi \mid}(1 \otimes p_k)(y_\pi).$$

Therefore we must have

$$x = \sum_{\pi : \mid \pi \mid = d} (1 \otimes p_k)(y_\pi).$$

Moreover since $y \to y_\pi$ is a (completely) contractive projection, we have

$$\sum_{\pi : \mid \pi \mid = d} \|y_\pi\| \leq C'\|y\| \leq C'C\|x\|$$

where $C' = \operatorname{card}\{\pi \mid \mid \pi \mid = d\}$. This is what we want, except that we have allowed $m_i = 0$ in $\pi = (m_1, \ldots, m_k)$. Here is how this point can be fixed: by deleting the zero blocks (but maintaining the order) each such $\pi$ defines a partition $\tilde{\pi} = (m'_1, \ldots, m'_K)$ with $m'_i > 0$ and $K \leq k$. Then we can write

$$\sum_{\pi : \mid \pi \mid = d} y_\pi = \sum_\omega \sum_{\pi : \tilde{\pi} = \omega} y_\pi$$

where the sum $\sum_\omega$ is restricted to partitions of $d$ of the form $\omega = (m'_1, \ldots, m'_K)$ with $m'_i > 0$ as appearing in the definition of $\mathcal{E}_k$. 

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Modulo repeated identifications of \( \mathbb{C} \otimes E \) with \( E \), we have

\[
x = (I \otimes \Phi_k)(\sum_\omega y'_\omega) \quad \text{where} \quad y'_\omega = \sum_{\pi: \pi = \omega} y_\pi \in \mathcal{K} \otimes E(\omega).
\]

Thus we obtain that

\[
\|[I \otimes \Phi_k^{-1}](x)\| \leq C'C\|x\|
\]

or equivalently \( \|\Phi_k^{-1}\|_{cb} \leq C'C \).

\[\hfill \blacksquare\]

**Lemma 2.4.** When \( k = d - 1 \) and \( E \) is a maximal operator space, the space \( \mathcal{E}_k \) actually reduces (completely isometrically) to

\[
(2.3) \quad \sum_{\substack{p+q+r=d \\ \text{for} \ q \geq 1}} E_p \otimes_h \max(E_q) \otimes_h E_r,
\]

with the convention that \( E_p \otimes_h \max(E_q) \otimes_h E_r \) should be replaced by \( E_p \otimes_h \max(E_q) \) if \( r = 0 \) and \( p > 0 \), by \( \max(E_q) \otimes_h E_r \) if \( p = 0 \) and \( r > 0 \), and finally by \( \max(E_q) \) if \( p = r = 0 \).

**Proof.** Indeed, let \( \sigma = (m'_i) \) with \( m'_1 = \cdots = m'_{p-1+1} = 1 \), \( m'_p+1 = q > 1 \) and \( m'_{p+i+1} = 1 \) for \( i = 1, \ldots, r \). Then \( E(\sigma) = E_p \otimes_h \max(E_q) \otimes_h E_r \) where of course \( |\sigma| = \sum m'_i = d \) and \( \sigma \) has at most \( d - 1 \) disjoint blocks since \( q > 1 \). Thus when \( k = d - 1 \), the sum appearing in (2.3) is canonically included in \( \mathcal{E}_k \). Now take any \( \pi = (m_i)_{i \leq K} \) where \( K < d \). Since we have less than \( d \) blocks we have \( m_i > 1 \) for some \( i \). Clearly if \( p = m_1 + \cdots + m_{i-1} \), if \( q = m_i \) and if \( r = m_{i+1} + \cdots + m_K \) we have then

\[
\max(E_{m_1}) \otimes_h \cdots \otimes_h \max(E_{m_K}) \subset E_p \otimes_h \max(E_q) \otimes_h E_r
\]

whence the conclusion of Lemma 2.4.

\[\hfill \blacksquare\]

**Lemma 2.5.** For any \( m \geq 1 \), we consider the operator space \( \max(\ell^m_1) \). We will denote

\[
E^d_m = \max(\ell^m_1) \otimes_h \cdots \otimes_h \max(\ell^m_1) \ (d \ \text{times}),
\]

and also

\[
\mathcal{E}^{d-1}_m = \sum_{\substack{p+q+r=d \\ \text{for} \ q \geq 1}} E^p_m \otimes_h \max(E^q_m) \otimes_h E^r_m
\]

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with the convention $E_m^0 = C$. Then there is a constant $\delta > 0$ (independent of $m$) such that for all $m \geq 1$ we have
\[
\|i: E_m^d \to E_m^{d-1}\|_{cb} \geq \delta \sqrt{m}
\]
where $i$ denotes the identity map.

For the proof we need separate estimates as follows.

**Sublemma 2.6.** Let $X$ be an arbitrary operator space and let $\otimes^\wedge$ denote the operator space projective tensor product. Then the identity map from $\ell_1^m \otimes_h X$ (resp. $X \otimes \ell_1^m$) into $\ell_1^m \otimes^\wedge X$ (resp. $X \otimes^\wedge \ell_1^m$) has c.b. norm $\leq \sqrt{m}$.

**Proof.** The identity of $\ell_1^m$ factorizes through the “row” space $R_m$ as $\ell_1^m \xrightarrow{a} R_m \xrightarrow{a^{-1}} \ell_1^m$ with $\|a\|_{cb} \leq 1$, $\|a^{-1}\|_{cb} \leq \sqrt{m}$. On the other hand it is well known (cf. [ER2, Th. 4.3] and [B, Prop. 2.3 (ii)] ) that $R_m \otimes_h X$ can be identified with $R_m \otimes^\wedge X$ completely isometrically. Thus we obtain a factorization
\[
\ell_1^m \otimes_h X \xrightarrow{a \otimes I} R_m \otimes_h X = R_m \otimes^\wedge X \xrightarrow{a^{-1} \otimes I} \ell_1^m \otimes^\wedge X
\]
from which the announced result follows immediately. The proof of the transposed statement is analogous (with $C_m$ instead of $R_m$).

**Remark.** More generally, let $Y$ be an $m$-dimensional operator space and let $a: R_m \to Y$ be a complete isomorphism. Then the preceding argument shows that
\[
\|Y \otimes_h X \to Y \otimes^\wedge X\|_{cb} \leq \|a\|_{cb}\|a^{-1}\|_{cb}.
\]

**Sublemma 2.7.** The identity map $i$ satisfies
\[
\|i: E_m^{d-1} \to \max(E_m^d)\|_{cb} \leq m^{\frac{d-2}{2}}.
\]

**Proof.** By the canonical property of the “sum”, it suffices to show that if $p + q + r = d$ with $q > 1$ we have
\[
\|i: E_m^p \otimes_h \max(E_m^q) \otimes_h E_m^r \to \max(E_m^d)\|_{cb} \leq m^{\frac{d-2}{2}}.
\]
Let $X = \max(E_m^q)$. By iterated applications of Sublemma 2.6 (using the associativity of $\otimes_h$ and $\otimes^\wedge$, cf. [BP, ER1-4]) we have

$$\|i: E_m^p \otimes_h X \otimes_h E_m^r \to \ell_1^m \otimes^\wedge \ell_1^m \otimes^\wedge X \otimes^\wedge \ell_1^m \otimes^\wedge \cdots \otimes^\wedge \ell_1^m\|_{cb} \leq m^{\frac{p+r}{m}}.$$  

Then we note that by the projectivity of $\otimes^\wedge$ the space $\ell_1^m \otimes^\wedge \cdots \otimes^\wedge \ell_1^m \otimes^\wedge X \otimes^\wedge \ell_1^m \otimes^\wedge \cdots \otimes^\wedge \ell_1^m$ is clearly a maximal operator space (completely) contractively included in $E_m^d$. (Indeed, by [BP, p. 289] the class of maximal operator spaces is stable under the o.s. projective tensor product.) Since it is maximal, it is completely contractively included in $\max(E_m^d)$. Thus we obtain the announced result by noting simply that $q > 1$ ensures $p + r \leq d - 2$.

The next estimate is the key point of this paper.

**Sublemma 2.8.** The identity map $i$ satisfies

$$\|i: E_m^d \to \max(E_m^d)\|_{cb} \geq \delta m^{\frac{d-1}{m}}$$

where $\delta > 0$ is a constant independent of $m$. (We will obtain $\delta = 2^{-2(d-1)}$.)

**Proof.** Fix $m \geq 1$. We denote $[m] = [1, 2, \ldots, m]$. For any $i = (i_1, \ldots, i_d)$ in $[m]^d$, we denote by

$$e_i = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$$

the canonical basis vectors in the space $E_m^d$. Now we fix a number $0 < \varepsilon < 1$ throughout the proof.

We will show that we can find matrices $\{U_i \mid i = (i_1, \ldots, i_d) \in [m]^d\}$ of size $N \times N$ (for a suitable $N = N(m, d, \varepsilon)$ such that:

(i) $\|\sum U_i \otimes \overline{U_i}\| \geq (1 - \varepsilon)m^d$.

(ii) $\forall \lambda_i \in \mathbb{C}$ $(i \in [m]^d)$ \quad $\|\sum \lambda_i U_i\| \leq (1 + \varepsilon)2^{d-1}(\sum |\lambda_i|^2)^{1/2}$.

(iii) The linear map $v: E_m^d \to M_N$ defined by $v(e_i) = U_i$ satisfies $\|v\|_{cb} \geq (1 + \varepsilon)^{-1}2^{-d+1}m^{\frac{d-1}{m}}$.

Let us first check that (i), (ii) and (iii) imply Sublemma 2.8. To do this, first observe that (ii) implies $\|v\| \leq (1 + \varepsilon)2^{d-1}$. This follows form the fact that if $C_m$ denotes the column Hilbert space in dimension $m$, then we have trivially a complete contraction $\ell_1^m \to C_m$, hence a complete contraction $E_m^d \to C_m \otimes_h \cdots \otimes_h C_m$ (d times) but $C_m \otimes_h \cdots \otimes_h C_m$ is
(completely) isometric to the $m^d$-dimensional column Hilbert space (cf. [ER2, p. 272]). Hence (ii) implies $\|v\| \leq (1 + \varepsilon)2^{d-1}$. Therefore, we have

$$\|i: E_m^d \to \max(E_m^d)\|_{cb} \geq \frac{\|v\|_{cb}}{\|v\|} \geq (1 + \varepsilon)^{-2}2^{-2(d-1)}m^{d-1},$$

and Sublemma 2.8 follows. Thus it suffices to produce $(U_i)$ satisfying (i), (ii), (iii).

The matrices $U_i$ will be defined as products of the following form

$$U_{i_1 i_2 \ldots i_d} = U_{i_1 i_2}^1 U_{i_2 i_3}^2 \ldots U_{i_{d-1} i_d}^{d-1},$$

where $U_{ij}^k$ are matrices in $M_N$ for all $1 \leq i, j \leq m$. We will make sure that these matrices satisfy

(iv) For each $k = 1, 2, \ldots, d - 1$,

$$\left\| \sum_{ij \leq m} e_{ij} \otimes U_{ij}^k \right\|_{M_m(M_N)} \leq 2(1 + \varepsilon)m^{1/2}.$$

Let us now verify that (i) and (iv) together imply (iii). Let $\xi_i = \xi_{i_1} \otimes \cdots \otimes \xi_{i_d}$ the canonical basis vectors of $(E_m^d)^* \cong \ell_\infty^m \otimes_h \cdots \otimes_h \ell_\infty^m$. Let $U^k = \sum_{p, q=1}^m e_{pq} \otimes U_{pq}^k$ and let $\hat{e}_{pq} = e_{pq} \otimes I$. Then we have

$$e_{11} \otimes U_{i_1 i_2 \ldots i_d} = \hat{e}_{1i_1} U^1 \hat{e}_{i_2 i_2} U^2 \ldots \hat{e}_{i_{d-1} i_d} U^{d-1} \hat{e}_{i_d 1}.$$

Let $V: (E_m^d)^* \to M_N$ be the linear map associated to the tensor $\sum U_i \otimes e_i \in M_N \otimes E_m^d$ so that $V(\xi_i) = U_i$. Then we have $\|\sum U_i \otimes e_i\|_{M_N(E_m^d)} = \|V\|_{cb}$, but the last identity implies that

$$V(\xi_{i_1} \otimes \cdots \otimes \xi_{i_d}) = v_1(\xi_{i_1})v_2(\xi_{i_2}) \ldots v_d(\xi_{i_d})$$

where the linear maps $v_k: \ell_\infty^m \to M_m(M_N)$ are defined as follows, for all $1 \leq j \leq m$

$$v_1(\xi_j) = \hat{e}_{1j} U^1, \quad v_2(\xi_j) = \hat{e}_{jj} U^2, \ldots, \quad v_{d-1}(\xi_j) = \hat{e}_{jj} U^{d-1} \quad \text{and finally} \quad v_d(\xi_j) = \hat{e}_{j1}.$$

Then we have obviously for all $2 \leq k \leq d - 1$

$$\|v_k\|_{cb} \leq \|U^k\|$$

hence using (iv) $\|v_k\|_{cb} \leq 2(1 + \varepsilon)m^{1/2}$. Similarly, we have (by a well known fact)

$$\|v_1\|_{cb} \leq m^{1/2}\|U^1\| \leq 2(1 + \varepsilon)m$$

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and 
\[ \|v_d\|_{cb} \leq m^{1/2}. \]

Thus we conclude by the classical results on multilinear c.b. maps ([CS1-2, PS]) that we have
\[ \|V\|_{cb} \leq \prod_{i=1}^{d} \|v_k\|_{cb} \leq (2(1 + \varepsilon))^{d-1} m^{\frac{d+1}{2}}. \]

Equivalently, this means that (cf. [BP, ER1])
\[ \left\| \sum U_i \otimes e_i \right\|_{M_N(E_m^d)} = \|V\|_{cb} \leq (2(1 + \varepsilon))^{d-1} m^{\frac{d+1}{2}}. \]

The same proof actually shows that
\[ \left\| \sum \overline{U}_i \otimes e_i \right\|_{M_N(E_m^d)} \leq (2(1 + \varepsilon))^{d-1} m^{\frac{d+1}{2}}. \]

But now by (i) and by the definition of the c.b. norm (since \( v(e_i) = U_i \)) we have
\[
(1 - \varepsilon)m^d \leq \left\| \sum U_i \otimes \overline{U}_i \right\| = \left\| \sum U_i \otimes U_i \right\| \leq \|v\|_{cb(E_m^d, M_N)} \left\| \sum \overline{U}_i \otimes e_i \right\|_{M_N(E_m^d)}
\]
whence we find
\[ \|v\|_{cb(E_m^d, M_N)} \geq (1 - \varepsilon)(2(1 + \varepsilon))^{-(d-1)} m^{\frac{d+1}{2}}, \]
which concludes the proof that (i) and (iv) imply (iii).

We now come to the main point: the construction of matrices \( U_i \) satisfying (i), (ii) and (iv). Actually the product appearing in the definition of \( U_i \) will be a tensor product, i.e. we will have

\begin{equation}
U_{i_1i_2...i_d} = Y_{i_1i_2}^1 \otimes Y_{i_2i_3}^2 \otimes \cdots \otimes Y_{i_{d-1}i_d}^{d-1}
\end{equation}

where \( Y_{ij} \) are matrices of sizes \( N_k \times N_k \).

In other words, we will set
\[ U_{pq}^k = 1 \otimes \cdots \otimes 1 \otimes Y_{pq}^k \otimes 1 \otimes \cdots \otimes 1 \]
and \( N = N_1 \times \cdots \times N_{d-1} \). The matrices \( Y_{ij}^k \) will be chosen at random independently according to a Gaussian distribution. More precisely, the family \( \{ Y_{ij}^k \mid 1 \leq i, j \leq m, 1 \leq \)}
\( k \leq d - 1 \) is taken to be an independent collection of random variables, and for each \( i, j \) and \( k \) (\( 1 \leq i, j \leq m, 1 \leq k \leq d - 1 \)), \( Y_{ij}^k \) is a random \( N_k \times N_k \) matrix, the entries of which are independent complex valued Gaussian variables with mean zero and such that \( \mathbb{E} |(Y_{ij}^k)_{pq}|^2 = (N_k)^{-1} \), exactly as in Theorem 1.1 above. Then, for clarity of notation we introduce the random variables \( Z_2 \) and \( Z_4 \) defined by

\[
Z_2 = \sup_{\sum |\lambda_i|^2 \leq 1} \left\| \sum \lambda_i Y_{ii_1i_2}^1 \otimes \cdots \otimes Y_{ii_{d-1}i_d}^{d-1} \right\|_{MN}
\]
and

\[
Z_4 = m^{-1/2} \sup_{1 \leq k \leq d-1} \left\| \sum_{p,q=1}^m e_{pq} \otimes Y_{pq}^k \right\|_{M_m(M_{N_k})}.
\]

Then by Corollary 1.3 we know that

\[
\limsup_{N_1 \to \infty} \ldots \limsup_{N_{d-1} \to \infty} Z_2 \leq 2^{d-1}
\]
and by Theorem 1.1 (and the remark following it) we have

\[
\limsup_{N_1 \to \infty} \ldots \limsup_{N_{d-1} \to \infty} Z_4 \leq 2.
\]

Since we also have almost surely

\[
\liminf_{N_1 \to \infty} \ldots \liminf_{N_d \to \infty} \left\| \sum U_j \otimes \overline{U_j} \right\| \geq m^d,
\]
it is now clear that the event corresponding to (i), (ii) and (iv) occurs with positive probability (actually = 1) if \( N_1, N_2, ..., N_d \) are suitably large, thus establishing the existence of matrices satisfying (i)–(iv) which completes the proof of Sublemma 2.8.

**Remark.** It is also possible to produce unitary matrices \( (U_i) \) satisfying essentially the properties (ii), (iii), (iv) but with an additional numerical factor in front of the constants involved. (Hint: Use the concentration of measure phenomenon (see e.g. \[P4, p. 44\]) and a comparison principle such as the one appearing in \[MP, p. 84\].) The inequality (i) is then automatically true (and actually becomes an equality).

**Proof of Lemma 2.5.** We have obviously

\[
\| i: E_m^d \to \max(E_m^d) \|_{cb} \leq \| i: E_m^d \to \mathcal{E}_m^{d-1} \|_{cb} \cdot \| i: \mathcal{E}_m^{d-1} \to \max(E_m^d) \|_{cb}.
\]
Hence the result immediately follows from the preceding two sublemmas.

**Proof of Theorem 0.1.** We let $E = \ell_1$ and $A_d = A_d(\ell_1)$. By Lemma 2.2, it suffices to prove that $d(A_d) > d - 1$. Assume to the contrary that $d(A_d) \leq d - 1$. Then, by Lemma 2.3, $(\Phi_{d-1})^{-1}$ is a c.b. map from $E_d = \ell_1 \otimes_h \cdots \otimes_h \ell_1$ (d-times) to $E_{d-1}$. By Lemma 2.4, this means that we have a c.b. mapping

$$E_d \rightarrow \sum_{\substack{p+q+r=d \ q>1}} E_p \otimes_h \max(E_q) \otimes_h E_r$$

which reduces to the identity on $E \otimes \cdots \otimes E$ (d times). But this clearly contradicts Lemma 2.5.

**Proof of Theorem 0.2.** By iterated applications of Sublemma 2.6, we find

$$\|Id: E_m^d \rightarrow \ell_1^m \otimes^\wedge \cdots \otimes^\wedge \ell_1^m\|_{cb} \leq m^{\frac{d-1}{2}},$$

and (as already noted for Sublemma 2.7) we know that $\ell_1^m \otimes^\wedge \cdots \otimes^\wedge \ell_1^m$ (d-times) is a maximal operator space (which can be identified with $\ell_1^{md}$), completely contractively included in $E_m^d$. Therefore, we have a fortiori

$$\|Id: E_m^d \rightarrow \max(\ell_m^d)\|_{cb} \leq m^{\frac{d-1}{2}},$$

and (as already observed in the proof of Sublemma 2.8) we have $\|J_{m,d}\| \leq 1$ whence

$$\|Id: \max(E_m^d) \rightarrow \max(\ell_2^m \otimes_2 \cdots \otimes_2 \ell_2^m)\|_{cb} \leq 1$$

hence the last estimate yields

$$\|J_{m,d}\|_{cb} \leq m^{\frac{d-1}{2}}.$$  

Thus we have proved the right side of both (0.1) and (0.2). The left side of (0.2) is but Sublemma 2.8. Now a close look at the proof of Sublemma 2.8 shows that the mapping $v$ appearing there actually satisfies

$$\|v: \max(\ell_2^m \otimes_2 \cdots \otimes_2 \ell_2^m) \rightarrow M_N\|_{cb} = \|v: \ell_2^m \otimes_2 \cdots \otimes_2 \ell_2^m \rightarrow M_N\| \leq (1 + \varepsilon)2^{d-1}.$$
and
\[ \|vJ_{m,d}\|_{cb} \geq (1 + \varepsilon)^{-1}2^{-d+1}m^{d-1}. \]
Thus we must have \[ \|J_{m,d}\|_{cb} \geq (1 + \varepsilon)^{-2}2^{-2(\varepsilon-1)}m^{d-1} \] from which the left side of (0.1) follows.

**Proof of Theorem 0.3.** Let \( E \) be a maximal operator space such that for each \( m \geq 1 \) the natural inclusion \( \ell^m \to \ell^m_1 \) admits a factorization \( \ell^m_1 \xrightarrow{a_m} E \xrightarrow{b_m} \ell^m_2 \) with

\[ C = \sup_m \|a_m\|: \max(\ell^m_1) \to E \|b_m\|: E \to \max(\ell^m_2)\|_{cb} < \infty. \]

We then claim that \( d(A_d(E)) = d. \)

Assume to the contrary that \( d(A_d(E)) < d. \) Then as explained above, we must have a c.b. map from \( E_d \) to \( \sum_{p+q+r=d} \sum_{q \geq 2} E_p \otimes_h \max(E_q) \otimes_h E_r. \) Let \( C' \) be the cb-norm of this map. For clarity let us denote again

\[ E_{d-1}(E) = \sum_{p+q+r=d} \sum_{q \geq 2} E_p \otimes_h \max(E_q) \otimes_h E_r. \]

Then, after composing with \( a_m \) and \( b_m \), we obtain that the identity map from \( E^d_m = \max(\ell^m_1) \) to \( E_{d-1}(\max(\ell^m_2)) \) has \( cb \)-norm \( \leq C d C' \).

But by the remark following Sublemma 2.6 (noting \( d_{cb}(\max(\ell^m_2), R_m) = d_{cb}(\max(\ell^m_2), C_m) \leq \sqrt{m} \)), we have

\[ \|E_{d-1}(\max(\ell^m_2)) \to \max(\max(\ell^m_2)\|_{cb} \leq m^{d-2} \]

hence a fortiori

\[ \|E_{d-1}(\max(\ell^m_2)) \to \max(\ell^m_2 \otimes_2 \cdots \otimes_2 \ell^m_2)\|_{cb} \leq m^{d-2}. \]

Hence we obtain

\[ \|J_{m,d}\|_{cb} \leq C d C' m^{d-2} \]
which contradicts Theorem 0.2.

Thus, assuming \( C < \infty, \) we have proved that \( d(A_d(E)) = d. \) This implies Theorem 0.3.
Indeed, for any infinite dimensional Banach space \( B, \) there is, for any \( \varepsilon > 0, \) a factorization
\(\ell_1^m \xrightarrow{a_m} B \xleftarrow{b_m} \ell_2^m\) of the canonical inclusion \(\ell_1^m \to \ell_2^m\) with sup \(\|a_m\| \|b_m\| \leq 1 + \varepsilon\). This follows from Dvoretzky’s Theorem (cf. e.g. [P4, p. 41]) applied to the dual \(B^*\). Therefore, if \(E = \max(B)\), we have the factorization considered in the first part of this proof with \(C = 1 + \varepsilon\).

As already explained in the introduction, when \(d > 2\) we cannot prove Sublemma 2.8 using an independent collection of Gaussian random matrices indexed by \([1, \ldots, m]^d\) as a substitute for the collection \((U_i)\). To convince the reader of this impossibility we will now give the estimates resulting from this choice. Although we include them for the record, they may be of independent interest.

Let \(Y\) be an \(N \times N\) random matrix for which the entries \(\{Y(i, j) \mid 1 \leq i, j \leq N\}\) are independent complex Gaussian random variables with \(\mathbb{E}|Y(i, j)|^2 = N^{-1}\) for all \(i, j\). To abbreviate, we will say below that such a random matrix is “standard of size \(N \times N\)”. By a well known result (which follows from (1.1)), there is an absolute constant \(K'\) such that for all \(N\)

\[(\mathbb{E}|Y|^2_{M_N})^{1/2} \leq K'.\]

More generally, by the concentration of measure phenomenon (see e.g. [P4, p. 44]), whenever \((Y_1, \ldots, Y_R)\) is an independent collection of copies of \(Y\), and \(N \geq \log R\), we have

\[(\mathbb{E} \sup_{1 \leq r \leq R} \|Y_r\|^2)^{1/2} \leq K''\]

where \(K''\) is another absolute constant. Now let \(\{Y_i \mid i \in [1, \ldots, m]^d\}\) be an independent collection of copies of the variable \(Y\). For each \(i = (i_1, \ldots, i_d)\) in \([1, \ldots, m]^d\), we let \(e_i = e_{i_1} \otimes \cdots \otimes e_{i_d}\) as before and we consider the random variable

\[y = \sum_{i \in [1, \ldots, m]^d} Y_i \otimes e_i.\]

By an abuse of notation, we will consider \(e_i\) as the natural basis either of \(E^d_m\) or of its dual \(E^*_m\). Thus we may view \(y\) as a random element either of \(M_N(E^d_m)\) or of \(M_N(E^*_m)\). Recall that if \(v: E^d_m \to M_N\) is the linear mapping which takes \(e_i\) to \(Y_i\), then

\[\|v\|_{\text{cb}(E^d_m, M_N)} = \|y\|_{M_N(E^*_m)}.\]
Then, we can state

**Theorem 2.9.** There are absolute constants $K$ and $\delta > 0$ such that: if $d$ is even we have

\[
\delta m^{d/4} \leq (\mathbb{E}\|y\|_{M_N(E_m^d)^*}^2)^{1/2} \leq Km^{d/4}
\]

(2.8)

\[
\delta m^{3d/4} \leq (\mathbb{E}\|y\|_{M_N(E_m^d)}^2)^{1/2} \leq Km^{3d/4}
\]

(2.9)

while if $d \geq 3$ is odd we have

\[
\delta m^{d-1/4} \leq (\mathbb{E}\|y\|_{M_N(E_m^d)}^2)^{1/2} \leq Km^{d-1/4}
\]

(2.10)

\[
\delta m^{3d+1/4} \leq (\mathbb{E}\|y\|_{M_N(E_m^d)}^2)^{1/2} \leq Km^{3d+1/4}
\]

(2.11)

**Proof.** For simplicity of notation we set

\[
a = (\mathbb{E}\|y\|_{M_N(E_m^d)^*}^2)^{1/2} \quad \text{and} \quad b = (\mathbb{E}\|y\|_{M_N(E_m^d)}^2)^{1/2}.
\]

We first claim that $ab \geq m^d$. Indeed, using (2.7) it is easy to check (as in the proof of Lemma 1.4) that

\[
ab \geq \mathbb{E}\left\| \sum Y_i \otimes Y_i^\ast \right\| \geq \mathbb{E}N^{-1} \left( \text{tr} \sum Y_iY_i^\ast \right) = m^d.
\]

From this claim it follows that it is enough to prove both upper bounds in (2.8) and (2.9) (or in (2.10) and (2.11)), the lower bounds then follow automatically.

Now to prove these upper bounds, we will use the associativity of $\otimes_h$ and the (completely) isometric identities

\[
M_n(E) \simeq C_n \otimes_h E \otimes_h R_n
\]

(2.12)

and

\[
C_m \otimes_h C_n = C_{mn} \quad \text{and} \quad R_m \otimes_h R_n = R_{mn}
\]

proved in [BP] and [ER2] for any operator space $E$, where $C_n$ (resp. $R_n$) denotes the column (resp. row) $n$-dimensional Hilbert space. We have completely contractive (in short c.c.) inclusions $C_m \rightarrow \ell^m_{\infty}$ and $R_m \rightarrow \ell^m_{\infty}$ (induced by the identity). Moreover $E_m^d = \ell^m_{\infty} \otimes_h \cdots \otimes_h \ell^m_{\infty}$. Therefore, taking first $E$ one dimensional in (2.12), we have c.c. inclusions

\[
M_{m^{d/2}} \simeq C_{m^{d/2}} \otimes_h R_{m^{d/2}} \rightarrow E_m^d \quad (d \text{ even})
\]
and taking now $E = \ell_m^\infty$ in (2.12), we have

$$M_m \overset{d-1}{\to} (\ell_m^\infty) \simeq C_m \overset{d-1}{\to} \ell_m^\infty \otimes \ell_m^\infty \otimes H_m \overset{d-1}{\to} E_m^d \quad (d \text{ odd}).$$

Hence, if $m$ is even we find

$$a \leq (\|E\|_2^2 M_N(M_m/d^2))^{1/2}.$$  

But now the matrix $m^{-d/4} y$ is “standard” of size $Nm^{d/2} \times Nm^{d/2}$ (in the above sense), hence by (2.5) $a \leq K'm^{d/4}$, whence the right side of (2.8). Similarly, if $d$ is odd and $d \geq 3$ we find using (2.6) instead

$$a \leq K'_{m^{d-1}}$$

whence the right side of (2.10).

We now turn to the upper bounds for $b$. Since the identity maps $C_m \to \ell_1^m$, $R_m \to \ell_m^m$ and $\ell_m^\infty \to \ell_1^m$ have cb-norms respectively equal to $m^{1/2}$, $m^{1/2}$ and $m$ we have if $m$ is even:

$$\|M_m/d^2 \to E_m^d\|_{cb} \leq m^{d/2} \quad \text{and if } m \text{ is odd: } \|M_m \overset{d-1}{\to} (\ell_m^\infty) \overset{d-1}{\to} E_m^d\|_{cb} \leq m^{d+1/2}. $$

Hence repeating the preceding argument we find $b \leq K'm^{3d/4}$ if $d$ is even and $b \leq K''m^{4d+1/4}$ if $d$ is odd. We thus obtain the right sides of (2.10) and (2.12), which concludes the proof of Theorem 2.9 by our original claim that $ab \geq m^d$.

§3. Complements

In this section, we wish to develop several points which have been overlooked in [P1]. For the sake of generality, we return to the framework of “similarity settings”. A similarity setting is a triple $(i, E, A)$ where $A$ is a unital algebra, $E$ is an operator space and $i: E \to A$ is a linear embedding. We will always assume (to avoid degeneracy) that there is at least one injective morphism $u_0: A \to B(H)$ with $\|u_0i\|_{cb} \leq 1$. We will also assume that $A$ is generated by $i(E)$ and the unit. For any $c \geq 1$, we denote by $C_c$ the class of all morphisms $u: A \to B(H)$ with $\|ui\|_{cb} \leq c$ (and, say, $\text{card}(H) \leq \text{card}(A)$). We then define an operator space $\tilde{A}_c$ as follows: we introduce an embedding

$$J_c: A \to \bigoplus_{u \in C_c} B(H_u)$$

by setting

$$\forall x \in A \quad J_c(x) = \bigoplus_{u \in C_c} u(x).$$
This embedding provides us with a norm on $A$. We denote by $\tilde{A}_c$ the completion of $A$ for the corresponding norm. Clearly $\tilde{A}_c$ is actually an operator algebra and (by construction) $J_c$ extends to an isometric morphism from $\tilde{A}_c$ into $B(H)$ with $H = \bigoplus_{u \in C_c} H_u$.

Let $OA(E)$ be the universal operator algebra of $E$ as defined in §2. Since $i_1: E \to \tilde{A}_1$ is completely contractive it extends to a c.c. morphism $\pi_1: OA(E) \to \tilde{A}_1$ which is a complete metric surjection.

The next result is a reformulation of Theorem 1.7 in [P1] (the latter was inspired by Peller’s results in [Pe]).

**Theorem 3.1.** Let $c \geq 1$. Consider $f$ in $K \otimes A$ with $\|f\|_{K \otimes_{\min} \tilde{A}_c} < 1$. We denote by $E^{(j)}$ the space $E \otimes \cdots \otimes E$ ($j$-times) viewed as a subspace of $T(E) \subset OA(E)$. Then for some $N \geq 1$ there are elements $F_j$ in $K \otimes E^{(j)}$ ($0 \leq j \leq N$) satisfying

$$\sup_{j \geq 0} c^j \|F_j\|_{K \otimes_{\min} OA(E)} \leq \left\| \sum_{j=0}^{N} c^j F_j \right\|_{K \otimes_{\min} OA(E)} < 1$$

such that

$$(Id_K \otimes \pi_1) \left( \sum_{j=0}^{N} F_j \right) = f.$$
by Paulsen’s results (see [P1, Prop. 1.8] for details), (i) holds iff the canonical morphism $\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}_\theta$ is a complete isomorphism, i.e. there is a constant $K > 0$ such that for any $f$ in $\mathcal{K} \otimes \mathcal{A}$ we have

$$\|f\|_{\mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_c} \leq K \|f\|_{\mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_\theta}. \quad (3.2)$$

Assume that this holds. Then select the smallest integer $d$ such that $\sum_{j > d} (\frac{2}{c})^j \leq 1/2K$. We will show that (iii) follows for some $C$. Let $T_d$ be the closed subspace of $OA(E)$ generated by $C \oplus E \oplus E^\otimes 2 \oplus \cdots \oplus E^\otimes d$. Let $u: \mathcal{A} \to B(H)$ be a morphism and let $b = \|u_i\|_{cb}$.

Note that by [P1, Prop. 1.10] and by Proposition 2.1 (i) above, we have

$$\|u\pi_{1|T_d}\|_{cb} \leq \sum_{j=0}^{d} b^j. \quad (3.3)$$

Now consider $f$ in $\mathcal{K} \otimes \mathcal{A}$ with $\|f\|_{\mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_\theta} < 1$ and hence by (3.2) $\|f\|_{\mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_c} < K$.

We claim that $f$ can be decomposed in $\mathcal{K} \otimes \mathcal{A}$ as $f = (I_K \otimes \pi_1)(x_0) + f'$ with $x_0 \in \mathcal{K} \otimes T_d$, $f' \in \mathcal{K} \otimes \mathcal{A}$ satisfying

$$\|x_0\|_{\mathcal{K} \otimes_{\min} T_d} \leq C' \quad \text{and} \quad \|f'\|_{\mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_\theta} < \frac{1}{2}. \quad \text{where } C' = K \sum_{j=0}^{d} c^{-j}.$$ 

From this claim (iii) follows immediately. Indeed, iterating the claim, we find a sequence $(x_n)_{n \geq 0}$ in $\mathcal{K} \otimes T_d$ such that $\|x_n\|_{\mathcal{K} \otimes_{\min} T_d} \leq C'2^{-n}$ and

$$\left\| f - (I_K \otimes \pi_1) \left( \sum_{j=0}^{n} x_j \right) \right\|_{\mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_\theta} < 2^{-n-1}.$$  

Let $y$ be the sum of the series $\sum_{j=0}^{\infty} x_j$ which converges in $\mathcal{K} \otimes_{\min} T_d$. Note that $\|y\|_{\mathcal{K} \otimes_{\min} T_d} \leq 2C'$. Since $\pi_{1|T_d}: T_d \to \tilde{A}_1$ actually is continuous into $\tilde{\mathcal{A}}_\theta$, we can write $f = (I_K \otimes \pi_1)(y)$ in $\mathcal{K} \otimes_{\min} \tilde{\mathcal{A}}_\theta$. Therefore

$$\|(I_K \otimes u)(f)\| \leq \|u\pi_{1|T_d}\|_{cb} \|y\|_{\mathcal{K} \otimes_{\min} T_d} \leq \left( \sum_{j=0}^{d} b^j \right) 2C',$$

which implies that (iii) holds with $C = 2C'$. Thus, to complete the proof, it suffices to prove the above claimed decomposition for $f$. By Theorem 3.1, we can write

$$f = \sum_{j=0}^{N} (I_K \otimes \pi_1)(F_j)$$
with \( \sup_{j \geq 0} c^j \| F_j \|_{\mathcal{K} \otimes_{\min} O_A(E)} < K \). We can always assume (adding zero terms if necessary) that \( N \geq d \). Then we have

\[
f = (I_{\mathcal{K}} \otimes \pi_1)(x_0) + f'
\]

where

\[
x_0 = \sum_{0}^{d} F_j
\]

and

\[
f' = \sum_{j > d} (I_{\mathcal{K}} \otimes \pi_1)(F_j).
\]

Then by (3.1) and our original choice of \( d \), we have

\[
\| f' \|_{\mathcal{K} \otimes_{\min} \tilde{A}_\theta} \leq \sum_{j > d} \theta^j \| F_j \|_{\mathcal{K} \otimes_{\min} O_A(E)} < \sum_{j > d} (\theta/c)^j K \leq 1/2.
\]

On the other hand, by (3.1) again

\[
\| x_0 \|_{\mathcal{K} \otimes_{\min} T_d} \leq \sum_{0}^{d} \| F_j \|_{\mathcal{K} \otimes_{\min} O_A(E)} \leq C',
\]

which establishes the above claim.

**Remark 3.4.** Just like in Theorem 2.6 in [P1] the preceding proof works just as well if we replace \( \mathcal{K} \) throughout the proof by a subspace \( X \subset \mathcal{K} \) for which there is a projection \( P: \mathcal{K} \to X \) with \( \| P \|_{cb} = 1 \). If \( X \otimes_{\min} \tilde{A}_\theta \) is isomorphic to \( X \otimes_{\min} \tilde{A}_c \), then \( X \otimes_{\min} \tilde{A}_\theta = X \otimes_{\min} \tilde{A}_b \) for all \( b \geq \theta \). In particular this applies when \( X \) is 1-dimensional. In this case, we find that if \( \tilde{A}_\theta \) and \( \tilde{A}_c \) have equivalent norms, then \( \tilde{A}_\theta \) and \( \tilde{A}_b \) have equivalent norms for all \( b \geq \theta \).

**Proof of Theorems 0.4 and 0.5.** These statements are nothing but Theorem 3.2 in the particular case \( E = \max(A) \) with \( i \) equal to the identity on \( A \).
Remark. Theorem 3.1 can be applied in the situation considered in [P2]. Let $A$ be a
unital operator algebra and let $A_1, A_2$ be unital (closed) subalgebras, let $\mathcal{A}$ be the algebra
generated by $A_1 \cup A_2$. We assume $\mathcal{A}$ dense in $A$. The associated similarity setting is:
$E = A_1 \oplus_1 A_2$ (operator space $\ell_1$-direct sum) with $i: E \to A$ defined by $i((x_1, x_2)) = x_1 + x_2$.
Let us denote here for simplicity

$$\mathcal{K}(A) = \mathcal{K} \otimes_{\min} A.$$  

Clearly $\mathcal{K}(A)$ is an operator algebra which we may view as formed of bi-infinite matrices
with entries in $A$.

We will say that $(A_1, A_2)$ generate $A$ with length $\leq d$ if any $x$ in $\mathcal{K}(A)$ can be written
as

$$(3.4) \quad x = x_1 x_2 \ldots x_d + y_1 y_2 \ldots y_d$$

with $x_i \in \mathcal{K}(A_1) \cup \mathcal{K}(A_2)$, $y_i \in \mathcal{K}(A_1) \cup \mathcal{K}(A_2)$ ($1 \leq i \leq d$) and also $x_1 \in \mathcal{K}(A_1)$ and
$y_1 \in \mathcal{K}(A_2)$. This implies that the natural product map is a surjection from $[\mathcal{K}(A_1) \hat{\otimes} \cdots] \oplus_1
[\mathcal{K}(A_2) \hat{\otimes} \cdots]$ onto $\mathcal{K}(A)$. Hence by the open mapping theorem (and by a well known
“matrix trick”), there is a constant $K$ such that we can always find $x_i, y_i$ as above satisfying
moreover

$$\prod_{i=1}^d \|x_i\| + \prod_{i=1}^d \|y_i\| \leq K\|x\|.$$  

We denote by $\ell(A_1, A_2)$ the smallest $d$ such that $\ell(A_1, A_2) \leq d$.

Note that this definition is equivalent to [P2, Definition 5]: indeed an elementary
argument allows to pass from the approximate version of (3.4) given in [P2] to equality in
(3.4). In this case, $\ell(A_1, A_2)$ is equal to the degree of the setting $(i, E, \mathcal{A})$, and there is a
one to one correspondence between morphisms $u: \mathcal{A} \to B(H)$ with $\|ui\|_{cb} \leq c$ and pairs
of morphisms $u_i: A_i \to B(H)$ with $\max_{i=1,2} \{\|u_i\|_{cb}\} \leq c$. We refer the reader to [P2] for more
variations on this theme.
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