Characters of admissible representations of the affine superalgebra $\hat{sl}(2|1; \mathbb{C})_k$

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Abstract

We calculate characters and supercharacters for irreducible, admissible representations of the affine superalgebra $\hat{sl}(2|1; \mathbb{C})_k$ in both the Ramond and Neveu–Schwarz sectors and discuss their modular properties in the special case of level $k = -\frac{1}{2}$. We also show that the non-degenerate integrable $\hat{sl}(2|1; \mathbb{C})_k$ characters coincide with some $N = 4$ superconformal characters.

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1 Introduction.

The classical simple complex Lie superalgebra $A(1, 0)$ has a realisation provided by $\mathfrak{sl}(2|1; \mathbb{C})$, the set of $3 \times 3$ complex matrices whose supertrace is zero. The affinisation of one of its real forms, $\mathfrak{sl}(2|1; \mathbb{R})$, plays a central role in the construction of the topological gauged $SL(2|1; \mathbb{R})/SL(2|1; \mathbb{R})$ Wess-Zumino-Novikov-Witten (WZNW) model, which is believed to be intimately related to the non-critical $N = 2$ string theory. In order to substantiate the evidence, based on similar observations made in \cite{1, 2, 16} for the non-critical bosonic and $N = 1$ strings, one should calculate the space of physical states in both theories. Our paper is a contribution towards this task. Indeed, the partition function of the $SL(2|1; \mathbb{R})/SL(2|1; \mathbb{R})$ theory splits in three sectors: a level $k$ and a level $-(k + 2)$ WZNW models based on $SL(2|1; \mathbb{R})$, as well as a system of four fermionic and four bosonic ghosts corresponding to the four even and four odd generators of $SL(2|1; \mathbb{R})$ (see e.g.\cite{24, 1}). The physical states of the theory should be obtained as elements of the cohomology of the BRST charge. A crucial step in this approach \cite{4} is to pass from the cohomology on the Fock space to the irreducible representations of $\hat{\mathfrak{sl}}(2|1; \mathbb{C})_k$ at level $k$. When the matter coupled to supergravity in the $N = 2$ non-critical string is minimal, i.e. taken in an $N = 2$ super Coulomb gas representation with central charge

$$c_{\text{matter}} = 3(1 - \frac{2p}{u})$$

\[p, u \in \mathbb{N}, \gcd(p, u) = 1, \]  

(1.1)

the level of the ‘matter’ affine superalgebra $\hat{\mathfrak{sl}}(2|1; \mathbb{C})$ appearing in the $SL(2|1; \mathbb{R})/SL(2|1; \mathbb{R})$ model is of the form

$$k = \frac{p}{u} - 1,$$

(1.2)

i.e. the level precisely takes the values for which admissible representations of $\hat{\mathfrak{sl}}(2|1; \mathbb{C})_k$ exist \cite{22}. It is therefore interesting to study the representation theory of this affine superalgebra in the context of the $N = 2$ non-critical string, but also in connection with a large class of applications, from conformal field theory to integrable systems.

The Kac-Kazhdan determinant formula for $\mathfrak{sl}(2|1; \mathbb{C})_k$ \cite{11, 23} encodes crucial information on the singular vectors of the theory. In \cite{3}, four classes of representations and the corresponding embedding diagrams were given, each of them having infinitely many singular vectors in the highest weight Verma module. In this paper, we present character formulas for a subset of the above classes, which correspond to integrable and admissible representations. It turns out that for integer values of the level $k$, the character formulae coincide with those of the superconformal algebra $\mathcal{N} = 4$, calculated in \cite{13, 14}. We offer some comments on this rather surprising fact later in the paper.

When the level is fractional and of the form (1.2), admissible representations do occur for the embedding diagrams of class IV (in the notations of \cite{3}), which are identical in structure to the embedding diagrams for the minimal representations of the $N = 2$ superconformal algebra as found in \cite{12} or in \cite{14}. This analogy between embedding diagrams is reminiscent of the analogy between $\hat{\mathfrak{sl}}(2, \mathbb{C})$ (resp. $\hat{\mathfrak{o}}\mathfrak{s}\mathfrak{p}(1|2; \mathbb{C})$) and Virasoro (resp. $N = 1$) diagrams,
and is correlated to the relation between $\hat{sl}(2|1; \mathbb{C})_k$ and $N = 2$ theories through hamiltonian reduction [2, 30].

The paper is organised as follows. The character formulae corresponding to class IV embedding diagrams are given in section 2 for arbitrary level $k$. It is also explained there that these characters only form a finite representation of the modular group when class V characters, corresponding to representations whose highest weight state quantum numbers take values at the edge of the class IV domain, are included. In section 3, the characters of integrable representations stemming from class I embedding diagrams are presented and compared with massive $N = 4$ superconformal characters, while the unique class IV integrable character coincides with the massless vacuum $N = 4$ character. Analogies between the two algebras are stressed there, and the corresponding Kac-Kazhdan determinant formulas are compared in the appendix. Section 4 analyzes the structure of simple pole singularities in non-integrable characters, when $\sigma$, the angular variable keeping track of the isospin of the states within an irreducible representation, tends to zero. It is found that, upon multiplying by a given modular form, the residues at the poles are minimal $N = 2$ superconformal characters, in complete parallel with a similar study in [29] for $\hat{sl}(2, \mathbb{C})$ characters, where the Virasoro minimal characters emerge as residues at the poles. In section 5, the branchings of $\hat{sl}(2|1; \mathbb{C})$ admissible characters at level $k = -\frac{1}{2}$ into $\hat{sl}(2, \mathbb{C})$ characters at the same level are calculated, and the characters are shown to provide a finite representation of the modular group.

2 The characters and supercharacters of $\hat{sl}(2|1; \mathbb{C})_k$.

The Lie superalgebra $A(1, 0) \equiv sl(2|1; \mathbb{C})$ has rank two and its simple roots can be chosen to be fermionic. The root diagram (Fig.1) can be represented in a 2-dimensional Minkowski space with the fermionic roots $\pm \alpha_1$ and $\pm \alpha_2$ in the lightlike directions (see § for notations),
The currents of the untwisted affine superalgebra $A(1,0)^{(1)} \equiv \hat{sl}(2|1;\mathbb{C})$ have the following Laurent expansions,

\[ J(e_{\pm(\alpha_1+\alpha_2)})(z) = \sum_n J_n^\pm z^{-n-1} \]

\[ J(h_+)(z) = 2 \sum_n J_n^3 z^{-n-1} \]

\[ J(h_-)(z) = 2 \sum_n U_n z^{-n-1} \]

\[ J(e_{\pm\alpha_1})(z) = \sum_n j_n^\pm z^{-n-1} \]

\[ J(e_{\pm\alpha_2})(z) = \sum_n j_n^\pm z^{-n-1}. \quad (2.1) \]

In terms of these Laurent modes, the commutation relations for $A(1,0)^{(1)}$ at arbitrary level $k$ are,

\[ [J_m^+, J_n^-] = 2J_{m+n}^3 + km\delta_{m+n,0} \]

\[ [J_m^3, J_n^3] = \pm J_m^\pm J_n^\mp \]

\[ [J_m^3, j_n^\pm] = \pm j_{m+n}^\pm \]

\[ [2J_m^3, j_n^\pm] = \pm j_{m+n}^\pm \]

\[ [U_m, U_n] = -\frac{k}{2} m\delta_{m+n,0} \]

\[ [U_m, j_n^\pm] = \mp j_{m+n}^\pm \]

\[ \{j_m^+, j_n^\pm\} = (U_{m+n} - J_{m+n}^3) - km\delta_{m+n,0} \]

\[ \{j_m^+, j_n^-\} = (U_{m+n} + J_{m+n}^3) + km\delta_{m+n,0} \]

\[ \{j_m^-, j_n^\pm\} = J_{m+n}^\pm. \quad (2.2) \]

Also, the Sugawara energy-momentum tensor is given by,

\[ T(z) = \frac{1}{2(k+1)} \{2(J^3)^2(z) - 2U^2(z) + J^+ J^-(z) + J^- J^+(z) \]

\[ + j^+ j^-(z) - j^- j^+(z) - j^+ j^- (z) + j^- j^+(z) \}. \quad (2.3) \]

Until otherwise stated, we shall work in the Ramond sector, where the suffix of all generators is an integer.

As first discussed in [22], the level $k$ of an affine Lie (super)algebra must be of the form

\[ k + h^\vee = \frac{p}{u}, \quad p, u \in \mathbb{N} \quad \text{gcd}(p, u) = 1, \quad (2.4) \]

for admissible representations to exist. In the above expression, $h^\vee$ is the dual Coxeter number of the Lie (super)algebra and $h^\vee = 1$ for $sl(2|1;\mathbb{C})$. Setting $u = 1$, the level is an integer ($k \in \mathbb{Z}_+$ and $k \in \mathbb{N} \cup \{0\}$ when $h^\vee = 1$), which is a necessary condition for integrable representations.

In this paper, we identify a family of irreducible highest weight state (hws) representations which are non-integrable but nevertheless whose characters provide a finite representation of the modular group. This indicates that one can build rational conformal field theories based on $sl(2|1;\mathbb{C})_k$ at fractional level.
The superalgebra $A(1,0)$ possesses two sets of Weyl inequivalent simple roots, for instance, \{\alpha_1, \alpha_2\} and \{\alpha_1 + \alpha_2, -\alpha_2\}. Clearly, the definition of hws depends crucially on the choice of positive roots. In this paper, we have chosen $\alpha_1, \alpha_2$ and $\alpha_1 + \alpha_2$ as positive roots, and $|\Lambda\rangle$ is a hws when

$$j_0^+ |\Lambda\rangle = j_0^- |\Lambda\rangle = J^-_0 |\Lambda\rangle = 0.$$  \hspace{1cm} (2.5)

If instead, the positive roots are chosen to be $\alpha_1 + \alpha_2, -\alpha_2$ and $\alpha_1$, the hws $|\Lambda'\rangle$ is defined by,

$$J_0^+ |\Lambda'\rangle = j_0^- |\Lambda'\rangle = j_0^+ |\Lambda'\rangle = 0,$$  \hspace{1cm} (2.6)

but the qualitative analysis of characters is unchanged.

An $\hat{sl}(2|1; \mathbb{C})_k$ hws $|\Lambda\rangle$ is further characterized by its isospin $\frac{1}{2} h_-$ and its hypercharge $\frac{1}{2} h_+$, \textit{i.e.},

$$J_0^0 |\Lambda\rangle = \frac{1}{2} h_- |\Lambda\rangle, \hspace{0.5cm} U_0 |\Lambda\rangle = \frac{1}{2} h_+ |\Lambda\rangle,$$  \hspace{1cm} (2.7)

while its conformal weight, calculated from the Sugawara tensor \((2.3)\), is given by,

$$h = \frac{1}{4(k+1)}(h_-^2 - h_+^2).$$  \hspace{1cm} (2.8)

For some specific values of $h_-$ and $h_+$, dictated by the Kac-Kazhdan determinant formula, the Verma module built on such a hws contains singular vectors. The identification of their quantum numbers and of their embedding patterns within the Verma module is of crucial importance in constructing irreducible characters. A classification of embedding diagrams for $\hat{sl}(2|1; \mathbb{C})_k$ singular vectors appearing in Verma modules built on hws $|\Lambda\rangle$ whose isospin quantum number $h_-$ obeys the constraint

$$h_- + \frac{P}{u} m - n = 0,$$

$$0 \leq m \leq u - 1, \hspace{0.5cm} 0 \leq n \leq p - 1,$$  \hspace{1cm} (2.9)

was provided in [5].

Of particular interest here is the class where the isospin $\frac{1}{2} h_-$ obeys the condition (2.9) with $n = 0$, but also where the hypercharge $\frac{1}{2} h_+$ is constrained by

$$h_- - h_+ = -2 \frac{P}{u} m', \hspace{0.5cm} m' \in \mathbb{Z}_+, \hspace{0.5cm} m' - m \leq 0.$$  \hspace{1cm} (2.10)

These two conditions on the hws quantum numbers $\frac{1}{2} h_-$ and $\frac{1}{2} h_+$ can be reformulated as

$$h_- + h_+ + 2(k+1)(m - m') = 0,$$

$$h_- - h_+ + 2m'(k+1) = 0,$$  \hspace{1cm} (2.11)

where $m, m' \in \mathbb{Z}_+, 0 \leq m' \leq m \leq u - 1$ and $k + 1 = \frac{P}{u}$. They correspond to class IV in [5].
However, in order to obtain a rational conformal field theory, one is led to consider, together with this class IV, a new class of representations for which the hws has isospin given by (2.9) with \( n = p \), and hypercharge \( \frac{1}{2} h_+ \) given by,

\[
  h_- - h_+ = 2\frac{p}{u}(m' + 1), \quad m' \in \mathbb{Z}_+, \quad m + m' \leq u - 2.
\]  \tag{2.12}

These latter two conditions are equivalent to the following constraints on \( h_- \) and \( h_+ \),

\[
  h_- + h_+ - 2(k + 1)(u - m - m' - 1) = 0,
  h_- - h_+ - 2(k + 1)(m' + 1) = 0,
\]  \tag{2.13}

where \( m, m' \in \mathbb{Z}_+, 0 \leq m + m' \leq u - 2 \) and \( k + 1 = \frac{p}{u} \). This new class is labelled class V.

The embedding diagrams for classes IV and V have the same pattern, given in Fig. 2. However, the quantum numbers of singular vectors in the two classes are different. We reproduce the class IV data and give the new, class V data in the following tables (where \( a \in \mathbb{Z}_+ \)).

In the notation of \( \mathcal{H} \), \( Z_0' \) is the bosonic hws. \( Z_{a+1}' \) and \( T_{a+1}' \) are bosonic singular vectors while \( T_a^+, Z_{a+1}^+, T_a^- \) and \( Z_{a+1}^- \) are fermionic singular vectors.
If the hws $Z'_0$ has conformal weight $h$, isospin $\frac{1}{2}h_-$ and hypercharge $\frac{1}{2}h_+$ satisfying the criteria in (2.11), then the quantum numbers of the singular vectors are given in Table 1 and Table 2.

| Family      | Conformal weight                                                                 |
|-------------|----------------------------------------------------------------------------------|
| $Z'_a$      | $h + a^2pu - apm$                                                                |
| $T'_{a+1}$  | $h + (a + 1)^2pu + (a + 1)pm$                                                   |
| $Z^{-}_{a+1}$ | $h + m' - m + (a + 1)^2pu + (a + 1)(u - pm)$                                   |
| $T^{-}_a$   | $h + m' + a^2pu + a(u + pm)$                                                    |
| $Z^+_{a+1}$ | $h - m' + (a + 1)^2pu + (a + 1)(u - pm)$                                       |
| $T^+_a$     | $h + m - m' + a^2pu + a(u + pm)$                                                |

Table 1: Conformal weights of class IV Ramond singular vectors.

| Family      | $2 \times$ Isospin | $2 \times$ Charge |
|-------------|---------------------|-------------------|
| $Z'_a$      | $2ap - m(k + 1)$    | $h_+$             |
| $T'_{a+1}$  | $-2(a + 1)p - m(k + 1)$ | $h_+$     |
| $Z^{-}_{a+1}$ | $1 + 2(a + 1)p - m(k + 1)$ | $h_+ - 1$ |
| $T^{-}_a$   | $-1 - 2ap - m(k + 1)$ | $h_+ - 1$       |
| $Z^+_{a+1}$ | $1 + 2(a + 1)p - m(k + 1)$ | $h_+ + 1$ |
| $T^+_a$     | $-1 - 2ap - m(k + 1)$ | $h_+ + 1$       |

Table 2: Isospin and Charge of class IV Ramond singular vectors.

If instead, the hws $Z'_0$ has conformal weight $h$, isospin $\frac{1}{2}h_-$ and hypercharge $\frac{1}{2}h_+$ satisfying the criteria in (2.13), then the quantum numbers of the singular vectors are given in Table 3 and Table 4, with $M = u - 2 - m - m'$ and $M' = m'$. 

6
Table 3: Conformal weights of class V Ramond singular vectors.

| Family   | Conformal weight                                                                 |
|----------|----------------------------------------------------------------------------------|
| $Z'_a$   | $h + a^2pu - ap(M + M' + 2)$                                                      |
| $T'_{a+1}$ | $h + (a + 1)^2pu + (a + 1)p(M + M' + 2)$                                        |
| $Z_{a+1}^-$ | $h - M' - 1 + (a + 1)^2pu + (a + 1)(u - p(M + M' + 2))$                  |
| $T_a^-$   | $h + M + 1 + a^2pu + a(u + p(M + M' + 2))$                                     |
| $Z_{a+1}^+$ | $h - M - 1 + (a + 1)^2pu + (a + 1)(u - p(M + M' + 2))$                  |
| $T_a^+$   | $h + M' + 1 + a^2pu + a(u + p(M + M' + 2))$                                     |

Table 4: Isospin and Charge of class V Ramond singular vectors.

| Family   | $2 \times \text{Isospin}$ | $2 \times \text{Charge}$ |
|----------|----------------------------|--------------------------|
| $Z'_a$   | $-2ap + (M + M' + 2)(k + 1)$ | $h_+$                    |
| $T'_{a+1}$ | $2(a + 1)p + (M + M' + 2)(k + 1)$ | $h_+$                   |
| $Z_{a+1}^-$ | $-2(a + 1)p - 1 + (M + M' + 2)(k + 1)$ | $h_+ - 1$            |
| $T_a^-$   | $2ap + 1 + (M + M' + 2)(k + 1)$ | $h_+ - 1$               |
| $Z_{a+1}^+$ | $-2(a + 1)p - 1 + (M + M' + 2)(k + 1)$ | $h_+ + 1$             |
| $T_a^+$   | $2ap + 1 + (M + M' + 2)(k + 1)$ | $h_+ + 1$               |

The information above is sufficient to derive the Ramond sector $\hat{sl}(2|1; \mathbb{C})_k$ characters for the representations in classes IV and V, when the level is given by $k = -1 + \frac{p}{u}, p, u \in \mathbb{N}$ and gcd$(p, u) = 1$.

In the Ramond sector, the Kac-Weyl denominator for $\hat{sl}(2|1; \mathbb{C})_k$ corresponds to the free action of all negatively moded generators of the algebra, together with $J_0^-, J_0^-$ and $J_0^-'$. It reads,

$$\left(F^R(\sigma, \nu, \tau)\right)^{-1} = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2(1 - zq^n)(1 - z^{-1}q^{n-1})}{(1 + z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^n)(1 + z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}}q^n)(1 + z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{-n})(1 + z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-n-1})}, \quad (2.14)$$
where,

\[ q^d = \exp(2\pi i\tau), \quad \tau \in \mathbb{C} \quad \text{Im}(\tau) > 0 \Rightarrow |q| < 1, \]
\[ z^d = \exp(2\pi i\sigma), \quad \sigma \in \mathbb{C}, \]
\[ \zeta^d = \exp(2\pi i\nu), \quad \nu \in \mathbb{C}. \]

(2.15)

The complex variables \( q, z, \zeta \) respectively keep track of the conformal weight, isospin and charge of the states in the Verma module. Using the standard technique of subtracting off the submodules generated from the singular vectors, whose quantum numbers are given in Tables 1 - 4, one obtains the Ramond characters for irreducible, admissible representations of class IV as,

\[
\chi_{h_{+, R}^-, h_{+, R}^+}^{\text{R, IV}, \hat{sl}(2|1; \mathbb{C})}_k (\sigma, \nu, \tau) = q^{h_{+, R}^-} z^{\frac{1}{2} h_{+, R}^+} \zeta^{\frac{1}{2} h_{+, R}^+} F^R(\sigma, \nu, \tau) \times \\
\sum_{a \in \mathbb{Z}} q^{a^2 p_a + a p_m} z^{-a p} \frac{1 - q^{2 a + m} z^{-1}}{(1 + q^{a + m} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{a + m} z^{-\frac{1}{2}} \zeta^{1/2})},
\]

(2.16)

where \( m, m' \in \mathbb{Z}_+, 0 \leq m' \leq m \leq u - 1 \), and

\[ h_{-, R}^- = -m(k + 1), \]
\[ h_{+, R}^+ = (2m' - m)(k + 1). \]

(2.17)

On the other hand, the Ramond characters for irreducible, admissible representations of class V are,

\[
\chi_{h_{+, R}^-, h_{+, R}^+}^{\text{R, V}, \hat{sl}(2|1; \mathbb{C})}_k (\sigma, \nu, \tau) = q^{h_{+, R}^-} z^{\frac{1}{2} h_{+, R}^+} \zeta^{\frac{1}{2} h_{+, R}^+} F^R(\sigma, \nu, \tau) \times \\
\sum_{a \in \mathbb{Z}} q^{a^2 p_u + a p_{M+M'} + 2} z^{-a p} \frac{1 - q^{2 a + M + M' + 2} z}{(1 + q^{a + M + M' + 1} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^{a + M + M' + 1} z^{-\frac{1}{2}} \zeta^{1/2})},
\]

(2.18)

where \( M, M' \in \mathbb{Z}_+, 0 \leq M + M' \leq u - 2 \), and

\[ h_{-, R}^- = (M + M' + 2)(k + 1), \]
\[ h_{+, R}^+ = (M - M')(k + 1). \]

(2.19)

The Neveu–Schwarz sector of the theory may be obtained from the Ramond sector using a variety of spectral flows, corresponding to several isomorphisms of the algebra \( A(1, 0)^{(1)} \).

For instance, the following ‘tilde’ generators,

\[
\tilde{J}_{n+2\beta}^+ = J_n^+, \\
\tilde{J}_{n+\beta+\alpha}^\pm = j_n^\pm, \\
\tilde{J}_{n+\beta\pm\alpha}^\pm = j_n^\pm, \\
\tilde{J}_n^3 = J_n^3 + k\beta\delta_{n,0}, \\
\tilde{U}_n = U_n - k\alpha\delta_{n,0}, \\
\tilde{L}_n = L_n + 2\beta J_n^3 - 2\alpha U_n - k(\alpha^2 - \beta^2)\delta_{n,0},
\]

(2.20)
obey the same algebra as the original Ramond generators for any real values of the parameters \(\alpha\) and \(\beta\). In particular, if the pair \((\alpha, \beta) = (0, \pm \frac{1}{2})\) or \((\pm \frac{1}{2}, 0)\), one relates the Ramond sector to the Neveu-Schwarz sector of the theory. The first two values of \((\alpha, \beta)\) correspond to isospin spectral flows, while the latter two correspond to hypercharge spectral flows. Although these flows may be used to derive the Neveu-Schwarz characters of \(\hat{\mathfrak{sl}}(2|1; \mathbb{C})\) from the Ramond characters given above, it should be emphasized that the isospin flows do not always map highest weight state representations to highest (or lowest) weight representations. This is due to the fact that some admissible Ramond representations at fractional level have an infinite number of states at grade zero. There exists, however, an inner automorphism of the algebra \(\mathfrak{sl}(2|1; \mathbb{C})\) which changes the sign of all nonzero roots in the root diagram and which corresponds to the following isomorphism of the affine version of the algebra,

\[
\begin{align*}
J_{n}^{3, \text{NS}} &= -J_{n}^{3, R} + \frac{1}{2} k \delta_{n,0}, \\
L_{n}^{\text{NS}} &= L_{n}^{R} - J_{n}^{3, R} + \frac{1}{4} k \delta_{n,0}, \\
J_{n \pm 1}^{\mp, \text{NS}} &= J_{n \pm 1}^{\pm, R} + \frac{1}{4} k \delta_{n,0}, \\
J_{n \pm 1}^{\mp, \text{NS}} &= J_{n \pm 1}^{\pm, R} + \frac{1}{4} k \delta_{n,0}.
\end{align*}
\]

(2.21)

The central generator \(k\) remains unchanged. The above isomorphism maps a Ramond hw to a Neveu–Schwarz hw, whose quantum numbers are given in terms of the quantum numbers of the Ramond hw by,

\[
\begin{align*}
h_{n}^{\text{NS}} &= h_{n}^{R} - \frac{1}{2} h_{n}^{R} + \frac{1}{4} k, \\
\frac{1}{2} h_{n}^{\text{NS}} &= -\frac{1}{2} h_{n}^{R} + \frac{1}{4} k, \\
\frac{1}{2} h_{n}^{\text{NS}} &= \frac{1}{2} h_{n}^{R} + \frac{1}{4} k.
\end{align*}
\]

(2.22)

The Neveu–Schwarz characters may then be obtained from the Ramond characters by using the spectral flow (2.21) as follows,

\[
\chi_{h_{n}^{\text{NS}}, h_{n}^{\text{NS}}}^{\text{NS,}\hat{\mathfrak{sl}}(2|1; \mathbb{C})_{k}}(\sigma, \nu, \tau) \overset{d}{=} \text{tr} \exp(2\pi i(\tau L_{0}^{\text{NS}} + \sigma J_{0}^{3, \text{NS}} + \nu U_{0}^{\text{NS}})) \\
= \text{tr} \exp(2\pi i(\tau (L_{0}^{R} - J_{0}^{3, R} + \frac{1}{4} k) + \sigma (-J_{0}^{3, R} + \frac{1}{2} k)) + \nu U_{0}^{R}) \\
= q^{\frac{1}{4} k} z^{\frac{1}{2} k} \text{tr} \exp(2\pi i(\tau L_{0}^{R} - (\sigma + \tau) J_{0}^{3, R} + \nu U_{0}^{R}) \\
= q^{\frac{1}{4} k} z^{\frac{1}{2} k} \chi_{h_{n}^{R}, h_{n}^{R}}^{\text{NS,}\hat{\mathfrak{sl}}(2|1; \mathbb{C})_{k}}(-\sigma - \tau, \nu, \tau).
\]

(2.23)

Defining the infinite product,

\[
F_{\text{NS}}^{\text{NS}}(\sigma, \nu, \tau) = \\
\prod_{n=1}^{\infty} \frac{(1 + z^{\frac{1}{2}} \zeta^{\frac{1}{2}} q^{-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{-\frac{1}{2}})(1 + z^{\frac{1}{2}} \zeta^{-\frac{1}{2}} q^{-\frac{1}{2}})(1 + z^{-\frac{1}{2}} \zeta^{\frac{1}{2}} q^{-\frac{1}{2}})}{(1 - q^{n})^{2}(1 - z q^{n})(1 - z^{2} q^{n-1})},
\]

(2.24)

the Neveu–Schwarz characters for class IV irreducible, admissible representations of \(\hat{\mathfrak{sl}}(2|1; \mathbb{C})_{k}\)
therefore read,
\[
X_{h_{-}^{NS},h_{+}^{NS}}^{NS,IV,\hat{sl}(2|1;\C)_{k}}(\sigma, \nu, \tau) = q^{h_{NS}}z^{h_{NS}}\zeta^{1/2}F_{NS}(\sigma, \nu, \tau) \times \\
\sum_{a \in \mathbb{Z}} q^{a^{2}p_{u}+a_{p}(1+m)} z^{-ap} \frac{1 - q^{2a_{u}+1+m}z}{(1 + q^{a_{u}+m'+\frac{1}{2}}z^{\frac{1}{2}}\zeta^{\frac{1}{2}})(1 + q^{a_{u}-m'+\frac{1}{2}}z^{\frac{1}{2}}\zeta^{\frac{1}{2}})}, \tag{2.25}
\]
where \(m, m' \in \mathbb{Z}_{+}\) and \(0 \leq m' \leq m \leq u - 1\). Also, the Neveu–Schwarz characters for class V irreducible, admissible representations are,
\[
X_{h_{-}^{NS},h_{+}^{NS}}^{NS,IV,\hat{sl}(2|1;\C)_{k}}(\sigma, \nu, \tau) = q^{h_{NS}}z^{h_{NS}}\zeta^{1/2}F_{NS}(\sigma, \nu, \tau) \times \\
\sum_{a \in \mathbb{Z}} q^{a^{2}p_{u}+ap(M+M'+1)} z^{-ap} \frac{1 - q^{2a_{u}+M+M'+1}z^{-1}}{(1 + q^{a_{u}+M+M'+\frac{1}{2}}z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}})(1 + q^{a_{u}+M'\frac{1}{2}}z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}})}, \tag{2.26}
\]
where \(M, M' \in \mathbb{Z}_{+}\) and \(0 \leq M + M' \leq u - 2\).

It is now straightforward to derive the supercharacters by shifting \(\sigma \longrightarrow \sigma + 1\) in the expressions for the characters (2.16),(2.18),(2.25),(2.26) and dividing by \(e^{2\pi i h_{R}^{1/2} NS}\). This amounts to the same as inserting the operator \((-1)^{F}\) in the formal definition of a character because the isospin of a fermionic singular vector is offset from that of a bosonic singular vector by \(\frac{1}{2}\). In the Ramond sector, one obtains the class IV supercharacters as,
\[
S_{X_{h_{-}^{R},h_{+}^{R}}^{R,IV,\hat{sl}(2|1;\C)_{k}}}^{R}(\sigma, \nu, \tau) = q^{h_{R}}z^{h_{R}}\frac{1}{2} h_{R}^{1/2} F_{R}(\sigma + 1, \nu, \tau) \times \\
\sum_{a \in \mathbb{Z}} q^{a^{2}p_{u}+apm} z^{-ap} \frac{1 - q^{2a_{u}+m}z^{-1}}{(1 - q^{a_{u}+m'}z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}})(1 - q^{a_{u}-m}z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}})}, \tag{2.27}
\]
where \(m, m' \in \mathbb{Z}_{+}\) and \(0 \leq m' \leq m \leq u - 1\), while the class V Ramond supercharacters are,
\[
S_{X_{h_{-}^{R},h_{+}^{R}}^{R,IV,\hat{sl}(2|1;\C)_{k}}}^{R}(\sigma, \nu, \tau) = q^{h_{R}}z^{h_{R}}\frac{1}{2} h_{+}^{1/2} F_{R}(\sigma + 1, \nu, \tau) \times \\
\sum_{a \in \mathbb{Z}} q^{a^{2}p_{u}+ap(M+M'+1)} z^{-ap} \frac{1 - q^{2a_{u}+M+M'+2}z}{(1 - q^{a_{u}+M+1}z^{\frac{1}{2}}\zeta^{\frac{1}{2}})(1 - q^{a_{u}+M'+1}z^{\frac{1}{2}}\zeta^{\frac{1}{2}})}, \tag{2.28}
\]
where \(M, M' \in \mathbb{Z}_{+}\) and \(0 \leq M + M' \leq u - 2\). Upon putting \(\sigma = \nu = 0\) into the nonsingular (see section 4) supercharacters we discover that all such supercharacters reduce to unity (class IV) or zero (class V). These numbers may be interpreted as Witten indices. In the case of class V, although the Witten index vanishes, the supersymmetry is unbroken.

In the Neveu–Schwarz sector, the class IV supercharacters are,
\[
S_{X_{h_{-}^{NS},h_{+}^{NS}}^{NS,IV,\hat{sl}(2|1;\C)_{k}}}^{NS}(\sigma, \nu, \tau) = q^{h_{NS}}z^{h_{NS}}\zeta^{1/2}F_{NS}(\sigma + 1, \nu, \tau) \times \\
\sum_{a \in \mathbb{Z}} q^{a^{2}p_{u}+ap(m+1)} z^{-ap} \frac{1 - q^{2a_{u}+m+1}z}{(1 - q^{a_{u}+m'+\frac{1}{2}}z^{\frac{1}{2}}\zeta^{\frac{1}{2}})(1 - q^{a_{u}+m-\frac{1}{2}}z^{\frac{1}{2}}\zeta^{\frac{1}{2}})}, \tag{2.29}
\]
where $m, m' \in \mathbb{Z}_+$ and $0 \leq m' \leq m \leq u-1$, while the class V Neveu-Schwarz supercharacters are,

$$S_{\chi_{h_{NS}^+ h_{NS}^+}^V}^{NS,k}(\sigma, \nu, \tau) = q^{h_{NS}^+} z^{\frac{1}{2} h_{NS}^+} \zeta^{\frac{1}{2} h_{NS}^+} F^{NS}(\sigma + 1, \nu, \tau) \times$$

$$\sum_{a \in \mathbb{Z}} q^{a^2 su^a + ap(M + M' + 1)} z^{- ap} \frac{1 - q^{a u + M + M' + 1} z^{-1}}{(1 - q^{a u + M + M' + 1} z^{-1} \zeta^{- \frac{1}{2}})(1 - q^{a u + M' + 1} z^{-1} \zeta^{\frac{1}{2}})},$$

where $M, M' \in \mathbb{Z}_+$ and $0 \leq M + M' \leq u - 2$.

We have thus obtained the Ramond sector $\hat{sl}(2|1; \mathbb{C})$ characters at fractional level $k$, when the hws quantum numbers are given by (2.11) and (2.13). The Neveu-Schwarz characters were obtained by spectral flow. Should one insist on an integer level $k$, the only possible hws quantum numbers are $h_+ = h_- = 0$, stemming from (2.11) with $u = 1$, and the corresponding representation in the Ramond sector is the vacuum representation. Remarkably, this integrable vacuum representation and other integrable, non-vacuum representations arising from class I (in the classification of [5]) have characters identical to those of the $N = 4$ superconformal algebra at the same level $k$. We shall discuss this relationship now.

## 3 Integrable representations and their characters.

Let $\hat{L}(\Lambda)$ be a highest weight module over $A(1,0)^{(1)}$. If $\alpha_1$ and $\alpha_2$ are the simple roots of $A(1,0)$, we can parametrize $\Lambda$ by,

$$\Lambda = (\frac{1}{2}(h_- + h_+)\alpha_1 + \frac{1}{2}(h_- - h_+)\alpha_2, k, 0).$$

This module is integrable if and only if one of the following conditions is satisfied: $h_- \in \mathbb{N}$ and $h_+$ is unconstrained, or $h_- = h_+ = 0$. We also require the level $k \in \mathbb{Z}_+$ with $k \geq h_- [21]$. Since $k$ is to be an integer, one should specialize to the value $u = 1$ in (2.9). A straightforward analysis of the hws quantum numbers in classes I, II, III, IV and V reveals that integrable representations occur in all classes but III and V. It is a remarkable fact that the integrable characters of class I are the same as the massive $N = 4$ characters discussed in [13], while the unique integrable representation in class IV (which is the vacuum representation) is the same as the massless $N = 4$ representation at isospin $\ell = 0$ (see [13]). Before we establish these relations with the characters of the $N = 4$ superconformal algebra, let us stress that the $\hat{sl}(2|1; \mathbb{C})_k$ integrable characters just mentioned obey the “non-degeneracy” conditions,

$$\frac{1}{2}(h_- - h_+) \neq 0 \Rightarrow \frac{1}{2}(h_- - h_+) \text{ is not divisible by } k + 1,$$

$$\frac{1}{2}(h_- + h_+) \neq 0 \Rightarrow \frac{1}{2}(h_- + h_+) \text{ is not divisible by } k + 1,$$

and that the class IV integrable vacuum character agrees with an explicit formula obtained by Kac and Wakimoto in [21].
We will now indicate how to relate integrable characters of classes I and IV to massive and massless $N = 4$ characters respectively. The integrable characters in class I correspond to representations whose highest weight $\Lambda$ has arbitrary charge \( \frac{1}{2} h_+ \), isospin,

\[
\frac{1}{2} h_+ = \frac{1}{2} n, \quad 1 \leq n \leq k = p - 1, \quad n \in \mathbb{N}, \tag{3.3}
\]

and conformal weight

\[
h^R_{\widehat{sl}(2|1;\mathbb{C})_k} = \frac{1}{4(k + 1)}((h_-^R)^2 - (h_+^R)^2). \tag{3.4}
\]

Using the embedding diagram and quantum numbers of class I singular vectors given in [5], one easily derives the following $\widehat{sl}(2|1;\mathbb{C})_k$ Ramond sector, integrable characters,

\[
\chi_{h^R_+}^{R.I. \widehat{sl}(2|1;\mathbb{C})_k}(\sigma, \nu, \tau) = q^{h^R_+} z^{h^R_+} \zeta^{h^R_+} F^R(\sigma, \nu, \tau) \times \sum_{a \in \mathbb{Z}} q^{a^2(k+1)+ah^R_+}(z^{a(k+1)} - z^{-a(k+1)-h^-_+}). \tag{3.5}
\]

Direct comparison with the massive $N = 4$ characters given in [13] shows that, once the latter are multiplied by the “Casimir” factor $q^{-\frac{1}{2}k} = q^{-\frac{k}{4}}$, once the conformal weights of the hw are related by $h^R_{N=4} = h^R_{\widehat{sl}(2|1;\mathbb{C})_k} + \frac{k}{4}$, and once one identifies

\[
h^-_+ = 2\ell, \quad 2\pi\sigma = \theta \quad \text{and} \quad 2\pi\nu = \varphi, \tag{3.6}
\]

the two expressions are identical up to a factor $\zeta^{\frac{1}{2}h_+}$. The reason for this is that in the $N = 4$ characters, the variable $y$ keeps track of the $U(1)$ charge of the supersymmetry generators, and it was assumed that the hw had charge zero. Setting $h_+ = 0$, one then may write the Ramond sector, integrable characters of class I as,

\[
\chi_{h^R_+}^{R.I. \widehat{sl}(2|1;\mathbb{C})_k}(\sigma, \nu, \tau) = \frac{1}{\eta(\tau)} \left( \chi_0^{\widehat{sl}(2|1;\mathbb{C})_2}(\sigma, \tau) \chi_{\frac{1}{2}}^{\widehat{sl}(2|1;\mathbb{C})_2}(\sigma, \tau) + \chi_0^{\widehat{sl}(2|1;\mathbb{C})_2}(\sigma, \tau) \chi_{\frac{1}{2}}^{\widehat{sl}(2|1;\mathbb{C})_2}(\sigma, \tau) \right) \chi_{\frac{1}{2}(h^R_+ - 1)}(\sigma, \tau). \tag{3.7}
\]

In the above, the integrable $\hat{sl}(2;\mathbb{C})_k$ characters are given by,

\[
\chi_{\ell}^{\hat{sl}(2;\mathbb{C})_k}(\sigma, \tau) = \frac{\vartheta_{2\ell+1,k+2}(\sigma, \tau) - \vartheta_{-2\ell-1,k+2}(\sigma, \tau)}{\vartheta_{1,2}(\sigma, \tau) - \vartheta_{-1,2}(\sigma, \tau)}, \quad 0 \leq \ell \leq \frac{k}{2}, \quad 2\ell \in \mathbb{Z}, \tag{3.8}
\]

where the generalised theta functions (see e.g. [20])

\[
\vartheta_{m,k}(\sigma, \tau, w) \overset{d}{=} e^{2\pi i k w} \sum_{a \in \mathbb{Z}} e^{2\pi i a(k+a \frac{m}{2k})} e^{2\pi i a(k+a \frac{m}{2k})} = e^{2\pi i k w} \vartheta_{m,k}(\sigma, \tau), \tag{3.9}
\]
are evaluated at $w = 0$, and where $\eta(\tau)$ is Dedekind’s function,

$$\eta(\tau) \overset{d}{=} q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \vartheta_{1,6}(0, \tau) - \vartheta_{5,6}(0, \tau). \quad (3.10)$$

The Neveu-Schwarz characters are obtained by spectral flow and read,

$$\chi_{k_{NS},0}^{NS,I,\hat{sl}(2|1;\mathbb{C})_k}(\sigma, \nu, \tau) = \frac{1}{\eta(\tau)} \left( \chi_{\frac{1}{2}}(\sigma, \tau) \chi_{\frac{1}{2}}(\nu, \tau) + \chi_{0}^{\hat{sl}(2|1;\mathbb{C})_1}(\sigma, \tau) \chi_{0}^{\hat{sl}(2|1;\mathbb{C})_1}(\nu, \tau) \right) \chi_{\frac{1}{2}k_{NS}}^{\hat{sl}(2|1;\mathbb{C})_{k-1}}(\sigma, \tau). \quad (3.11)$$

These characters have the following S-transform,

$$\chi_{k_{NS},0}^{NS,I,\hat{sl}(2|1;\mathbb{C})_k}(\sigma, \nu, \tau, \frac{-1}{\tau}) = (-i\tau)^{-1/2} e^{-\frac{i2k\pi}{12}} e^{-\frac{i\pi}{12}} \sum_{h'_{NS}} S_{h'_{NS},h_{NS}}^{NS,I,\hat{sl}(2|1;\mathbb{C})_k}(\sigma, \nu, \tau), \quad (3.12)$$

where

$$S_{h'_{NS},h_{NS}} = \sqrt{\frac{2}{k+1}} \sin\left(\pi \frac{(h_{NS} + 1)(h'_{NS} + 1)}{k + 1}\right). \quad (3.13)$$

We now show that the massless, Ramond sector character of the vacuum representation of the $N = 4$ superconformal algebra studied in [13] is identical to the class IV Ramond sector vacuum character of $\hat{sl}(2|1;\mathbb{C})_k$ when $k$ is an integer. The latter is the expression (2.16) when $u = 1$ and $m = m' = 0$, i.e.,

$$\chi_{0,0}^{R,IV,\hat{sl}(2|1;\mathbb{C})_k}(\sigma, \nu, \tau) = F^R(\sigma, \nu, \tau) \times \sum_{a \in \mathbb{Z}} q^{a^2p} z^{-ap} \frac{1 - q^{2a}z^{-1}}{(1 + q^a z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^a z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})}. \quad (3.14)$$

From [13], we have the following expression for the massless $N = 4$ unitary characters at integer level $k$ and isospin $\ell = 0, \frac{1}{2}, \ldots, \frac{k}{2}$,

$$c^R_o(k, \ell; \sigma, \nu, \tau) = q^{\frac{k}{2}} F^R(\sigma, \nu, \tau) \times \sum_{m \in \mathbb{Z}} \left( z^{(k+1)m+\ell} q^{(k+1)m^2+2\ell m} \frac{1 - q^{2a}z^{-1}}{(1 + q^a z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^a z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})} - z^{-(k+1)m-\ell} q^{(k+1)m^2+2\ell m} \frac{1 - q^{2a}z^{-1}}{(1 + q^a z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})(1 + q^a z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}})} \right), \quad (3.15)$$

where we used again the identifications (3.6).
Multiplying the numerator and denominator of the second term of the sum in (3.15) by $q^{2m}z^{-1}$, while relabeling $m \to -m$ in the first term and setting $\ell = 0$, we get,

$$\sum_{a \in \mathbb{Z}} q^{(k+1)a^2} z^{-(k+1)a} \frac{1 - q^{2a}z^{-1}}{(1 + q^{a}z^{-\frac{1}{2}}\zeta^{\frac{1}{2}})(1 + q^{a}z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}})}. \quad (3.16)$$

With $k + 1 = p$ this is identical to the sum in (3.14). Thus the class IV vacuum character at integer level $k$ is identical to the massless $N = 4$ vacuum character at the same level, i.e.,

$$\chi_{k,0}^{NS,IV,\hat{sl}(2;\mathbb{C})} (\sigma, \nu, \tau) = q^{\frac{k}{2} - \frac{1}{2}} F^{NS}(\sigma, \nu, \tau) \left( 1 - q^{2a+1}z \right) \frac{1 - q^{2a+1}z}{(1 + q^{a+\frac{1}{2}}z^{\frac{1}{2}}\zeta^{\frac{1}{2}})(1 + q^{a+\frac{1}{2}}z^{\frac{1}{2}}\zeta^{-\frac{1}{2}})}, \quad (3.18)$$

and one easily shows that,

$$\chi_{k,0}^{NS,IV,\hat{sl}(2;\mathbb{C})} (\sigma, \nu, \tau) = q^{\frac{k}{2} - \frac{1}{2}} \chi_{0}^{NS}(k, \frac{1}{2}; \nu, \tau) \quad (3.19)$$

in total agreement with the spectral flow.

We now recall the behaviour of the character in (3.19) under the modular transformation $S : \tau \to -\frac{1}{\tau}$. In [14], the decomposition of this character at level $k = 1$ into $\hat{sl}(2;\mathbb{C})_1$ characters was given as,

$$\chi_{k=1,0}^{NS,IV,\hat{sl}(2;\mathbb{C})} (\sigma, \nu = 0, \tau) =$$

$$\left( \frac{-\chi_{\frac{1}{2}}^{\hat{sl}(2;\mathbb{C})_1} (\tau)}{(\chi_{0}^{\hat{sl}(2;\mathbb{C})_1} (\tau))^2 + (\chi_{\frac{1}{2}}^{\hat{sl}(2;\mathbb{C})_1} (\tau))^2} + h_3(\tau) \chi_0^{\hat{sl}(2;\mathbb{C})_1} (\sigma, \tau) \right) \chi_0^{\hat{sl}(2;\mathbb{C})_1} (\sigma, \tau)$$

$$\quad \quad \quad + \left( \frac{\chi_{0}^{\hat{sl}(2;\mathbb{C})_1} (\tau)}{(\chi_{0}^{\hat{sl}(2;\mathbb{C})_1} (\tau))^2 + (\chi_{\frac{1}{2}}^{\hat{sl}(2;\mathbb{C})_1} (\tau))^2} + h_3(\tau) \chi_{\frac{1}{2}}^{\hat{sl}(2;\mathbb{C})_1} (\sigma, \tau) \right) \chi_{\frac{1}{2}}^{\hat{sl}(2;\mathbb{C})_1} (\sigma, \tau) \quad (3.20)$$

where,

$$h_3(\tau) = \frac{1}{\eta(\tau) \theta_3(\tau)} \sum_a q^{\alpha^2/2 - 1/8} \quad (3.21)$$

and

$$h_3(\tau) + h_3(-1/\tau) = \frac{1}{\eta(\tau)} \int_{-\infty}^{\infty} d\alpha \frac{q^{\alpha^2}}{2 \cosh \pi \alpha}. \quad (3.22)$$
is the Mordell integral \[28\]. It is a straightforward exercise to show that

\[
\chi_{k=1,0}^{NS,IV,\hat{sl}(2|1;\mathbb{C})_k}(\sigma = 0, \nu = 0, \frac{-1}{\tau}) = -\chi_{k=1,0}^{NS,IV,\hat{sl}(2|1;\mathbb{C})_k}(\sigma = 0, \nu = 0, \tau)
\]

\[
+ \int_{-\infty}^{\infty} da \frac{q^{\frac{2}{\tau}}}{2 \cosh \pi \alpha} \times \chi_{k=1,0}^{NS,I,\hat{sl}(2|1;\mathbb{C})_k}(\sigma = 0, \nu = 0, \tau)
\]

(3.23)

where

\[
\chi_{k=1,0}^{NS,I,\hat{sl}(2|1;\mathbb{C})_k}(\sigma = 0, \nu = 0, \tau) = \frac{1}{\eta(\tau)}((\chi_0^{\hat{sl}(2|1;\mathbb{C})_1}(\tau))^2 + (\chi_{\frac{1}{2}}^{\hat{sl}(2|1;\mathbb{C})_1}(\tau))^2),
\]

(3.24)

according to (3.11).

We end this section by noting, interestingly, that the striking identity between integrable \(\hat{sl}(2|1;\mathbb{C})_k\) and \(N = 4\) characters is deeply rooted in the similar nature of singular vectors in the two theories. A close inspection of the generalised Malikov-Feigin-Fuchs (MFF) construction for singular vectors given in \[5\] for \(\hat{sl}(2|1;\mathbb{C})_k\) in the Ramond sector and partially in \[13\] for \(N = 4\) reveals that there is a one-to-one correspondence between the set of generators \(\{j_0^-, j_0^-, J_0^-, J_1^+\}\) in \(\hat{sl}(2|1;\mathbb{C})_k\) and the set \(\{G_0^2, G_0^2, T_0^-, T_0^+\}\) in \(N = 4\). In particular, it is obvious that the singular vectors in massive \(N = 4\) Verma modules based on a hws \(|\Lambda'\rangle\) defined by

\[
G_0^2|\Lambda'\rangle = G_0^1|\Lambda'\rangle = T_1^-|\Lambda'\rangle = T_0^+|\Lambda'\rangle = 0,
\]

(3.25)

and the singular vectors in class I \(\hat{sl}(2|1;\mathbb{C})_k\) Verma modules based on a hws defined by \(\Box 5\), have the same analytic expressions once the above correspondence is implemented. One can also establish a similar dictionary between the singular vectors of the Ramond massless vacuum module and those of the unique class IV integrable \(\hat{sl}(2|1;\mathbb{C})_k\) module. For instance, the state \(G_0^2|\Lambda'\rangle\) (resp. \(G_0^2|\Lambda'\rangle\)) becomes singular in the \(N = 4\) massless case and corresponds to the state \(j_0^-|\Lambda\rangle\) (resp. \(j_0^-|\Lambda\rangle\)) in class IV \(\hat{sl}(2|1;\mathbb{C})_k\). Although the currents in the two theories have non-matching conformal spin, they have the same isospin and \(U(1)\)-charge quantum numbers, which implies that the singular vectors have the same quantum numbers, a fact that can indeed be checked by direct inspection of the Kac-Kazhdan determinant formulas of the two algebras (see appendix).

Whether the relation between \(N = 4\) and \(\hat{sl}(2|1;\mathbb{C})_k\) goes beyond coincidence of character formulas is an open question. One should analyze the correlation functions in the two theories, a task beyond the scope of this paper, but certainly facilitated by the knowledge of the MFF representation of singular vectors.

4 Character singularities.

We now return to the class IV and class V \(\hat{sl}(2|1;\mathbb{C})_k\) characters presented in section 2, and analyse their behaviour in the limit \(\sigma \to 0\). It turns out, as we will see below, that some
characters develop a simple pole in this limit, while others are regular at $\sigma = 0$. We find that upon multiplying by a certain modular form, the residues at the pole are (non)unitary, minimal $N = 2$ superconformal characters $[10]$. We can use these residues to identify the branching functions of $\hat{sl}(2|1; \mathbb{C})_k$ characters into $\hat{sl}(2; \mathbb{C})_k$ characters, which makes the discussion of modular properties of characters in the next section much easier. In [29], Mukhi and Panda had already remarked that some $\hat{sl}(2; \mathbb{C})$ characters for admissible representations (which were worked out in [22]) developed a simple pole in a certain limit. Upon multiplication by $\eta^2$, the residues at these poles were found to be the Virasoro characters in the unitary and non–unitary minimal series. The results in [29] and our analysis are especially interesting in the light of the relationship between $\hat{sl}(2; \mathbb{C})_k$ and the Virasoro algebra and between $\hat{sl}(2|1; \mathbb{C})_k$ and the $N = 2$ superconformal algebra through quantum hamiltonian reduction.

Simple poles arise in the factor $F^R(\tau, \sigma, \nu)$ (see (2.14)) or $F^{NS}(\tau, \sigma, \nu)$ (see (2.24)) of the $\hat{sl}(2|1; \mathbb{C})_k$ characters at $\sigma = 0$. The following lemmas identify which characters are singular at $\sigma = 0$. We shall see that this simple pole is cancelled by the vanishing of the sums in the characters only for special values of $m$ in the case of class IV and special values of $M + M' + 1$ in the case of class V.

**Lemma 1** The Ramond characters of class IV (resp. class V) are nonsingular at $\sigma = 0$ iff $m = 0$ (resp. $M + M' + 1 = u - 1$).

**Proof.** We prove the lemma only for class IV but the proof for class V is identical. The sum from (2.16) is,

$$\sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} z^{-ap} \frac{1 - q^{2au+m} z^{-1}}{(1 + q^{a+m'} z^{1-\frac{i}{2}\zeta^{-\frac{1}{2}}})(1 + q^{au+u-m'} z^{-\frac{i}{2}\zeta^{-\frac{1}{2}}})},$$

where $z \overset{d}{=} \exp(2\pi i \sigma)$. Then when $\sigma$ is small the sum becomes,

$$\sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} \frac{1 - q^{2au+m} - 2\pi i \sigma (ap + q^{2au+m}(1 + ap)) + O(\sigma^2)}{f(\sigma; a)},$$

where the denominator $f(\sigma; a)$ is defined as,

$$f(\sigma; a) \overset{d}{=} (1 + q^{au+u-m'} \zeta^{\frac{1}{2}})(1 + q^{au+u-m'} \zeta^{-\frac{1}{2}}) - i\pi \sigma (q^{au+u-m'} \zeta^{\frac{1}{2}} + q^{au+u-m'} \zeta^{-\frac{1}{2}} + 2q^{2au+m}) + O(\sigma^2).$$

We can pair up each term in the sum with the one that has the equal but opposite value of $a$. Then the sum becomes,

$$\frac{1}{2} \left\{ \sum_{a \in \mathbb{Z}} q^{a^2 pu + apm} \frac{1 - q^{2au+m} - 2\pi i \sigma (ap + q^{2au+m}(1 + ap)) + O(\sigma^2)}{f(\sigma; a)} + (a \to -a) \right\}$$

16
\[
\sum_{a \in \mathbb{Z}} q^{a^2p}u^{-q^{2ua} - 2\pi i a} \frac{O(a^2)}{f(a;\sigma)} + \sum_{a \in \mathbb{Z}} q^{a^2p}u^{-q^{2ua} - 2\pi i a} \frac{O(a^2)}{f(a;\sigma)}
\]

iff \(m = 0\). Then multiplying numerator and denominator of the second term by \(q^{2ua}\), we see that the last line vanishes at \(\sigma = 0\).

\textbf{Corollary 1} Integrable characters are nonsingular at \(\sigma = 0\).

\textbf{Proof.} We set \(u = 1\) to obtain integrable characters in class IV. Since \(0 \leq m \leq u - 1\), \(u = 1 \Rightarrow m = 0\).

Similar arguments show that class IV Neveu–Schwarz characters are nonsingular only when \(m = u - 1\). However, the type of argument above does not prove that the class V Neveu–Schwarz characters are ever nonsingular. In the example of the next section we shall see that the class V Neveu–Schwarz character that appears is singular. We conjecture that the class V Neveu–Schwarz characters are all singular.

The finite dimensional algebra \(sl(2|1)\) is a subalgebra of the Ramond sector \(\hat{sl}(2|1; \mathbb{C})\). The next lemma, when combined with Lemma 1, shows that the class IV Ramond sector characters of the affine algebra are nonsingular iff the highest weight module \(L(\Lambda_h)\) of the finite dimensional algebra is finite dimensional.

\textbf{Lemma 2} Let \(\Lambda_h = \left(\frac{1}{2}h_-, \frac{1}{2}h_+\right)\) be the Ramond highest weight of \(sl(2|1)\) where,

\[
\frac{1}{2}h_- = -\frac{mp}{2u} \quad \text{and} \quad \frac{1}{2}h_+ = \frac{p}{2u}(2m' - m).
\]

Then the highest weight module \(L(\Lambda_h)\) is finite dimensional iff \(m = 0\).

\textbf{Proof.} Let \(\Pi = \{\alpha_1 + \alpha_2, -\alpha_1\}\) be a basis of simple roots for \(sl(2|1)\). \(\alpha_1 + \alpha_2\) is an even root. \(\alpha_1\) and \(\alpha_2\) are isotropic odd roots. \(\alpha_1 \cdot \alpha_2 = 1\) so, \((\alpha_1 + \alpha_2)^2 = 2\). The scalar product on the roots is computed with a Minkowski signature metric. Between them, Kac [19] and Cornwell [3] prove \(\) that the graded representation defined by \(\Lambda_h\) is finite dimensional iff the numerical mark \(n_1\) (corresponding to \(\alpha_1 + \alpha_2\)) is a non–negative integer, i.e., iff,

\[
\Lambda_h \cdot (\alpha_1 + \alpha_2) \in \mathbb{Z}_+.
\]

Now, if we take \(\alpha_1 = \frac{1}{2}(1, 1)\) and \(\alpha_2 = \frac{1}{2}(1, -1)\) then

\[
\Lambda_h = -\frac{mp}{u} \alpha_1 + \frac{m'p}{u} (\alpha_1 - \alpha_2)
\]

and,

\[
\Lambda_h \cdot (\alpha_1 + \alpha_2) = -\frac{mp}{u} \in \mathbb{Z}_+ \iff m = 0. \square
\]

Although \(\Lambda_h\) is different for class V, it is easy to see that the lemma holds there too for \(M + M' + 1 = u - 1\).

\(^1\)Kac proves sufficiency, Cornwell, necessity.
We end this section by giving the residues of class IV and class V Neveu–Schwarz and Ramond $sl(2|1; \mathbb{C})_k$ characters at the pole $\sigma = 0$.

One has,

$$\lim_{\sigma \to 0} 2\pi i \sigma \chi_{h_{NS}^-, h_{R+}^R}^{NS, \hat{s}l(2|1; \mathbb{C})_k} (\sigma, \nu, \tau) = \frac{\vartheta_{0,2}(\frac{i}{2} \nu, \tau) + \vartheta_{2,2}(\frac{i}{2} \nu, \tau)}{\eta^3(\tau)} \chi_{r,s}^{R+, N=2} (\zeta^\frac{1}{2}, q)$$

where the $N = 2$ superconformal characters are given by,

$$\chi_{r,s}^{NS, N=2} (\zeta^\frac{1}{2}, q) = \zeta^{(r-s)p} q^{\frac{(r+s)}{2u}} \prod_{n=1}^{\infty} \frac{1 + \zeta^\frac{1}{2} q^n - \frac{1}{2}}{1 - q} \times$$

$$\sum_{a \in \mathbb{Z}} q^{a^2 up + ap(r+s)} \frac{1 - q^{2ua+r+s}}{(1 + q^{ua+s} \zeta^\frac{1}{2}) (1 + q^{ua+r} \zeta^{-\frac{1}{2}})}.$$

with central charge,

$$c = 3(1 - \frac{2p}{u}), \quad u \in \mathbb{N} \setminus \{1\}, \quad p \in \mathbb{N}, \quad \gcd(u, p) = 1.$$

In the above expression, we have changed notation as follows. For class IV, where $h_{NS}^- = m(k + 1) + k$ and $h_{NS}^- = (2m' - m)(k + 1)$, one has, $m = r + s - 1, m' = r - \frac{1}{2}$. For class V, where $h_{NS}^- = -(M + M' + 2)(k + 1) + k$ and $h_{NS}^+ = (M - M')(k + 1)$, one has, $M = r - \frac{1}{2}, M' = s - \frac{1}{2}$. From the ranges of $m, m', M$ and $M'$, we can deduce ranges for $r$ and $s$ for each class and then comparing with Boucher, Friedan and Kent [3] we see that these $N = 2$ characters as residues are unitary when $p = 1$ and nonunitary when $p > 1$.

In the Ramond sector, the residues of class IV and class V $sl(2|1; \mathbb{C})_k$ characters at the pole $\sigma = 0$ are,

$$\lim_{\sigma \to 0} 2\pi i \sigma \chi_{h_{NS}^+, h_{R}^R}^{R, \hat{s}l(2|1; \mathbb{C})_k} (\sigma, \nu, \tau) = \frac{\vartheta_{1,2}(\frac{i}{2} \nu, \tau) + \vartheta_{-1,2}(\frac{i}{2} \nu, \tau)}{\eta^3(\tau)} \chi_{r,s}^{R+, N=2} (\zeta^\frac{1}{2}, q)$$

where,

$$\chi_{r,s}^{R+, N=2} (\zeta^\frac{1}{2}, q) = \zeta^{(-r-s)p} q^{\frac{(r+s)}{2u}} \prod_{n=1}^{\infty} \frac{1 + \zeta^\frac{1}{2} q^n - \frac{1}{2}}{1 - q} \times$$

$$\sum_{a \in \mathbb{Z}} q^{a^2 up + ap(r+s)} \frac{1 - q^{2ua+r+s}}{(1 + q^{ua+s} \zeta^\frac{1}{2}) (1 + q^{ua+r} \zeta^{-\frac{1}{2}})}.$$

For class IV, where $h_{R}^R$ and $h_{+}^R$ are given by (2.17), $m = r + s$ and $m' = r$, but for class V, with $h_{NS}^+$ and $h_{R+}^R$ given by (2.19), $r = M + 1$ and $s = M' + 1$. The $N = 2$ characters derived from class V are unitary or not according to whether $p = 1$ or $p > 1$. The $N = 2$ characters derived from class IV are unitary when $m' > 0$ and $p = 1$, and are nonunitary otherwise.
5 \( \hat{sl}(2|1; \mathbb{C})_k \) admissible characters at level \( k = -\frac{1}{2} \).

In this section we show how some low level admissible \( \hat{sl}(2|1; \mathbb{C})_k \) characters may be decomposed in a basis of admissible \( \hat{sl}(2; \mathbb{C})_k \) characters, as \( \hat{sl}(2; \mathbb{C})_k \) is a subalgebra of \( \hat{sl}(2|1; \mathbb{C})_k \).

From [22] or from [29], we have the following expression for the \( \hat{sl}(2; \mathbb{C})_k \) admissible characters,

\[
\chi_{n,n'}^{\hat{sl}(2;\mathbb{C})_k}(\sigma, \tau) = \frac{\vartheta_{b_+ a}^{\frac{\sigma}{u}}(\tau) - \vartheta_{b_- a}^{\frac{\sigma}{u}}(\tau)}{\vartheta_{1,2}^{\sigma}(\tau) - \vartheta_{-1,2}^{\sigma}(\tau)},
\]

(5.1)

where the level is parametrized as,

\[
k = \frac{t}{u}, \quad (t, u) = 1, \quad u \in \mathbb{N}, \quad t \in \mathbb{Z},
\]

(5.2)

where \( 0 \leq n \leq 2u + t - 2 \) and \( 0 \leq n' \leq u - 1 \) and,

\[
b_{\pm} \equiv u(\pm(n + 1) - n'(k + 2)) \quad a \equiv u^2(k + 2).
\]

(5.3)

Note that integer level requires \( u = 1 \), which implies \( n' = 0 \), and (5.1) is then the well-known expression (3.8), which is regular when \( \sigma \to 0 \). For \( u \neq 1 \) however, it is shown in [24] that, at fixed level \( k \), some \( \hat{sl}(2; \mathbb{C})_k \) characters develop a pole at \( \sigma = 0 \), and that the residues at the pole are the minimal Virasoro characters characterized by the pair \((t + 2u, u)\), multiplied by \( \frac{1}{\eta^2} \). On the other hand, we showed in the previous section that the residue at the pole \( \sigma = 0 \) in Ramond \( \hat{sl}(2|1; \mathbb{C})_k \) characters is the product of the modular function \( \frac{1}{\eta^2} \) by an \( N = 2 \) minimal character and a linear combination of \( A_2 \) characters, where \( A_2 \) is the rational torus algebra with one extra spin 2 generator \( \mathfrak{sl}_3 \). One can therefore speculate that the branching functions of \( \hat{sl}(2|1; \mathbb{C})_k \) characters into \( sl(2; \mathbb{C})_k \) characters involve, for \( k + 1 = \frac{2}{u} \), the \( A_2 \) characters \( \frac{\vartheta_m \vartheta^{\nu}(\tau)}{\eta(\tau)} \), \( m = 0, 1, 2, 3 \), multiplied by the ratio of \( N = 2 \) and Virasoro minimal characters. From a ‘coset’ construction point of view, the central charge \( c_{sl(2|1)} \) associated with \( \hat{sl}(2|1; \mathbb{C})_k \) through the Sugawara energy-momentum tensor is zero for any level \( k \), while the corresponding central charge for \( \hat{sl}(2; \mathbb{C})_k \) is \( c_{sl(2)} = \frac{3k}{k + 2} \). One therefore expects the coset to have central charge \( c_{\text{coset}} = c_{sl(2|1)} - c_{sl(2)} = -\frac{3k}{k + 2} \). For \( k + 1 = \frac{2}{u} \), one can rewrite,

\[
c_{\text{coset}} = 1 + 3(1 - \frac{2p}{u}) - (1 - \frac{6p^2}{u(p + u)}),
\]

(5.4)

where one can read off, on the right hand side, the central charges of the torus algebra \( A_2 \), of minimal \( N = 2 \) and of minimal Virasoro for the pair \((p + u, u)\).

When \( p = 1 \), \( u = 2 \), \( c_{\text{coset}} = 1 \) is the \( A_2 \) central charge. The following detailed analysis of \( sl(2|1; \mathbb{C}) \) at level \( k = -\frac{1}{2} \) certainly confirms the above heuristic argument. When \( k = -\frac{1}{2} \), i.e. when \( t = -1 \) and \( u = 2 \), there are four admissible \( \hat{sl}(2; \mathbb{C})_{-\frac{1}{2}} \) characters \( \text{viz.} \chi_{0,0}, \chi_{1,0}, \chi_{0,1}, \chi_{1,1} \). The first pair are regular at \( \sigma = 0 \) and the latter pair are singular there [29]. On the other hand, there are three \( \hat{sl}(2|1; \mathbb{C})_{-\frac{1}{2}} \) characters in class IV, with
\((h^R, h^R) = (0, 0), (-\frac{1}{2}, \pm \frac{1}{2})\), while there is only one in class V, with \((h^R, h^R) = (1, 0)\).

As was mentioned above in section 4, not all the \(\hat{sl}(2|1; \mathbb{C})\) characters develop a pole at \(\sigma = 0\). For the case at hand, the class IV Ramond sector characters developing a pole have \((h^R, h^R) = (-\frac{1}{2}, \pm \frac{1}{2})\). By contrast the only singular class IV Neveu--Schwarz character has \((h^{-NS}, h^{+NS}) = (-\frac{1}{2}, 0)\). In class V, the single Ramond character is non-singular and the Neveu--Schwarz character is singular. Combined use of residues at the poles and spectral flow arguments allows for an analytic derivation of the branching functions when decomposing \(\hat{sl}(2|1; \mathbb{C})\) characters at level \(k = -\frac{1}{2}\) into \(\hat{sl}(2; \mathbb{C})\) characters at the same level. The \(N = 2\) and Virasoro characters--as--residues are evaluated at \(c = 0\) and so they appear as factors of unity on the RHS. Letting \(\mu\) be 0 or 1, we find in the Ramond sector,

\[
\chi_{\mu, 0}^{R, \hat{sl}(2|1; C), -\frac{1}{2}}(\sigma, \nu, \tau) = \frac{1}{\eta(\tau)} \sum_{\rho = 0}^{1} \vartheta_{2\mu + 2\rho, 2}(\frac{1}{2}\nu, \tau) \chi_{\rho, 0}^{\hat{sl}(2; C), -\frac{1}{2}}(\sigma, \tau),
\]

and for the Neveu--Schwarz sector,

\[
\chi_{\mu, 0}^{NS, \hat{sl}(2|1; C), -\frac{1}{2}}(\sigma, \nu, \tau) = \frac{1}{\eta(\tau)} \sum_{\rho = 0}^{1} \vartheta_{2\mu + 2\rho, 2}(\frac{1}{2}\nu, \tau) \chi_{\rho, 1}^{\hat{sl}(2; C), -\frac{1}{2}}(\sigma, \tau),
\]

\[
\chi_{0, \mu, -\frac{1}{2}}^{NS, \hat{sl}(2|1; C), -\frac{1}{2}}(\sigma, \nu, \tau) = \frac{1}{\eta(\tau)} \sum_{\rho = 0}^{1} \vartheta_{2\mu + 2\rho - 1, 2}(\frac{1}{2}\nu, \tau) \chi_{\rho, 0}^{\hat{sl}(2; C), -\frac{1}{2}}(\sigma, \tau).
\]

The singular (resp. non-singular) \(\hat{sl}(2|1; \mathbb{C}), -\frac{1}{2}\) characters are expanded into the singular (resp. non-singular) \(\hat{sl}(2; \mathbb{C}), -\frac{1}{2}\) characters. For the modular transformations of the Ramond and Neveu--Schwarz characters, it is useful to have the Neveu--Schwarz supercharacters branched into \(\hat{sl}(2; \mathbb{C})\) characters too. This is easily obtained from the branchings of the Neveu--Schwarz characters in (5.6) above upon shifting \(\sigma \to \sigma + 1\) and dividing by \(e^{2\pi i \frac{1}{2} h^{NS}}\). We obtain,

\[
S_{\chi_{\mu, 0}^{NS, \hat{sl}(2|1; C), -\frac{1}{2}}}(\sigma, \nu, \tau) = \frac{1}{\eta(\tau)} \sum_{\rho = 0}^{1} (-1)^{\mu + \rho} \vartheta_{2\mu + 2\rho, 2}(\frac{1}{2}\nu, \tau) \chi_{\rho, 1}^{\hat{sl}(2; C), -\frac{1}{2}}(\sigma, \tau),
\]

\[
S_{\chi_{0, \mu, -\frac{1}{2}}^{NS, \hat{sl}(2|1; C), -\frac{1}{2}}}(\sigma, \nu, \tau) = \frac{1}{\eta(\tau)} \sum_{\rho = 0}^{1} (-1)^{\rho} \vartheta_{2\mu + 2\rho - 1, 2}(\frac{1}{2}\nu, \tau) \chi_{\rho, 0}^{\hat{sl}(2; C), -\frac{1}{2}}(\sigma, \tau).
\]

It is now straightforward to deduce the behaviour of the Ramond and Neveu--Schwarz \(\hat{sl}(2|1; \mathbb{C}), -\frac{1}{2}\) characters under the modular group \(PSL(2, \mathbb{Z})\). First of all, let us recall the S modular transform of the generalised theta functions (3.9) (see [20]),

\[
\vartheta_{m,k}(\tau, \frac{\nu}{\tau}, \frac{1}{2}, v + \frac{\nu^2}{2}) = \sqrt{-\frac{i\pi}{2k}} \sum_{r=0}^{2k-1} e^{-ir\tau} \vartheta_{r,k}(\nu, \tau, v). \tag{5.8}
\]
From there, and using (3.10), one also has,

\[ \eta(-\frac{1}{\tau}) = \sqrt{-i\tau} \eta(\tau). \] (5.9)

Furthermore, in view of the definition (3.1), the S transform of the admissible \( \hat{sl}(2; \mathbb{C})_k \) characters are,

\[ \chi_{\hat{sl}(2; \mathbb{C})_k}(\sigma, \nu, \tau) = e^{-\frac{i\pi k\sigma}{2\tau}} \sum_{\nu=0}^{2u-t-2} \sum_{\nu'=0}^{u-1} S_{\nu\nu'}^{\hat{sl}(2; \mathbb{C})_k}(\sigma, \nu, \tau), \] (5.10)

with

\[ S_{\nu\nu'} = \sqrt{\frac{2}{u^2(k+2)}} (-1)^{\nu'(n+1)+(n+1)\nu'} e^{-i\pi(k+2)\nu'} \sin \left[ \frac{\pi(n+1)(\nu+1)}{k+2} \right]. \] (5.11)

So, under the transformation \( S : (\sigma, \nu, \tau) \rightarrow (\frac{\sigma}{\tau}, \frac{\nu}{\tau}, -\frac{1}{\tau}) \), the Ramond and Neveu-Schwarz \( \hat{sl}(2|1; \mathbb{C})_{-\frac{1}{2}} \) characters respectively transform as follows,

\[ \chi_{R, \hat{sl}(2|1; \mathbb{C})_{-\frac{1}{2}}}(\sigma, \nu, \tau) = -\frac{1}{2} e^{\frac{i\pi(e^2-\nu^2)}{2\tau}} \left[ \sum_{\rho=0}^{\frac{1}{2}} e^{2i\pi(\mu-\frac{1}{4})_{\rho}} S_{\rho, \nu}^{NS, \hat{sl}(2|1; \mathbb{C})_{-\frac{1}{2}}}(\sigma, \nu, \tau) - \sum_{\rho=0}^{\frac{1}{2}} e^{2i\pi(\mu-\frac{1}{4})_{\rho}} S_{\rho, 0}^{NS, \hat{sl}(2|1; \mathbb{C})_{-\frac{1}{2}}}(\sigma, \nu, \tau) \right], \] (5.12)

and

\[ \chi_{-\frac{1}{2}, -\frac{1}{2}}(\sigma, \nu, \tau) = \frac{1}{2} e^{\frac{i\pi(e^2-\nu^2)}{2\tau}} \left[ i \sum_{\rho=0}^{\frac{1}{2}} e^{2i\pi(\mu-\rho)} S_{\rho, -\frac{3}{2}, 0}^{NS, \hat{sl}(2|1; \mathbb{C})_{-\frac{1}{2}}}(\sigma, \nu, \tau) - \sum_{\rho=0}^{\frac{1}{2}} e^{2i\pi(\mu-\rho)} S_{\rho, 0, -\frac{1}{2}}^{NS, \hat{sl}(2|1; \mathbb{C})_{-\frac{1}{2}}}(\sigma, \nu, \tau) \right], \]
\[
\chi_{\mu,\nu,\tau}^{NS,\hat{sl}(2|1;C),-\frac{1}{2}}(\frac{\sigma}{\tau}, \frac{\nu}{\tau}, \frac{1}{\tau}) = \frac{1}{2} e^{i \pi \left(\frac{(\sigma - \nu)^2}{2 \tau} + \frac{\mu}{2} \right)} \\
\left[ 1 + \sum_{\rho=0}^{1} e^{2i \pi \rho} \chi_{\rho,\mu,\nu,\tau}^{NS,\hat{sl}(2|1;C),-\frac{1}{2}}(\frac{\sigma}{\tau}, \frac{\nu}{\tau}, \frac{1}{\tau}) \right],
\]
(5.13)

As can be checked easily, when evaluated at \( \sigma = \nu = 0 \), the matrices \( S^R \) and \( S^{NS} \) of \( S \) transform are unitary and \( (S^{NS})^4 = 1 \). Thus we see that the Ramond characters transform into the super Neveu–Schwarz characters and the Neveu–Schwarz characters transform into themselves under \( S \), as expected.

Finally, let us mention how these admissible characters transform under \( T : (\sigma, \nu, \tau) \to (\sigma, \nu, \tau + 1) \). It is straightforward to see that,
\[
\chi^{R,\hat{sl}(2|1;C)k}_{h^R,h^R_+}(\sigma, \nu, \tau + 1) = e^{2i \pi h^R} \chi^{R,\hat{sl}(2|1;C)k}_{h^R,h^R_+}(\sigma, \nu, \tau) \\
\chi^{NS,\hat{sl}(2|1;C)k}_{h^-_{NS},h^+_NS}(\sigma, \nu, \tau + 1) = e^{2i \pi h^{NS}} S \chi^{NS,\hat{sl}(2|1;C)k}_{h^-_{NS},h^+_NS}(\sigma, \nu, \tau).
\]
(5.14)

for class IV and class V separately at any level \( k \).

We have therefore verified that the affine superalgebra \( \hat{sl}(2|1;C) \) at fractional level \( k = -\frac{1}{2} \) allows for four irreducible admissible representations whose characters form a finite representation of the modular group. We expect to obtain such admissible representations in classes IV and V for any fractional level of the form \( k + 1 = \frac{p}{u}, p, u \in \mathbb{N}, \gcd(p, u) = 1 \). However, the corresponding character formulas in the form given in section 2 are not directly suited for the analysis of modular transformations, and work is in progress to rewrite them in terms of modular forms and functions.

6 Conclusion

In this paper, we provide character formulas for integrable and admissible, irreducible representations of the affine superalgebra \( \hat{sl}(2|1;C)_k \). We explicitly show how the characters of the torus algebra \( A_2 \) arise as branching functions of admissible \( \hat{sl}(2|1;C) \) characters at level \( k = -\frac{1}{2} \) into \( \hat{sl}(2;C) \) at the same level. This rewriting of admissible characters enables us to easily derive their modular transformations. It is also argued that branching functions for other fractional values of the level \( [1,2] \) should involve the product of \( A_2 \) characters with a ratio of minimal \( N = 2 \) and Virasoro characters.
The most surprising result however is that one can identify the integrable \( \hat{sl}(2|1; \mathbb{C})_k \) characters with those of the superconformal \( N = 4 \) algebra. Striking similarities between the zero mode generators of the two algebras are responsible for the existence of singular vectors with identical embedding structures and identical quantum numbers in both theories, leading to identical character formulas. These observations are reflected in the presence of common factors in the Kac-Kazhdan determinant formulas, as explained in the appendix. Whether or not the relation goes beyond matching character formulas is not clear, and we wish to conclude our work with further remarks and speculations. For instance, it is possible, using the Malikov-Feigin-Fuchs construction for singular vectors, to derive differential equations obeyed by the correlators of free fields in both theories. These equations should encode the conformal spins of the generators, which obviously differ in \( \hat{sl}(2|1; \mathbb{C})_k \) and \( N = 4 \). However, another intriguing coincidence is that the Coulomb gas representation of the \( N = 4 \) SCA at level \( k \) [26, 27], which can be obtained by hamiltonian reduction from the affine \( A(1/1) \) superalgebra [18] but also [31, 7, 6] from a representation of the doubly extended \( N = 4 \) SCA, uses three \( SU(2) \) currents at level \( k - 1 \), four free real fermions and a \( U(1) \) current. This is very similar to the current content of \( sl(2|1; \mathbb{C})_k \). The mismatch in conformal spins could possibly be resolved by the appropriate twisting of one algebra. The fact that \( A(1,1) \) contains \( A(1,0) \) as a subalgebra, that its affinisation provides the \( N = 4 \) superconformal algebra through hamiltonian reduction while \( A(1,0)^{(1)} \) reduces to the \( N = 2 \) superconformal algebra may ultimately shed a new light on the relation between the ubiquitous \( N = 2 \) string theory and its more complicated \( N = 4 \) counterpart.

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Appendix

We show how to identify common factors in the Kac-Kazhdan determinant formulas of \( N = 4 \) and \( sl(2|1; \mathbb{C}) \) at level \( k \). The former was conjectured in [25] and proven in [27] using a Coulomb gas representation of \( N = 4 \) and screening charges. In the Ramond sector (setting \( \rho = \eta = 0 \) in formula (9) of [25]), it reads,

\[
\det M_{\nu,s,c}(h, k, t) = A \prod_{m,n>0} f_{m,n}(h, k, t)^{P^4(\nu-mn,s,c)} \prod_{m \in \mathbb{Z}, n>0} g_{m,n}(h, k, t)^{P^4(\nu-|m|n,s+\text{sgn}(m)n,c)} \\
\times \prod_{m \in \mathbb{Z}, \epsilon = \pm 1} h_{m,\epsilon}(h, k, t)^{P^4_{\nu,c}(\nu-|m|,s+\frac{1}{2}\text{sgn}(m),c-\epsilon)}, \tag{A.1}
\]
with \( A \) a non-zero constant,

\[
\begin{align*}
    f_{m,n}(h,k,t) &= 4t^2 - (k + 1)(4h - k) - ((k + 1)m + n)^2, \\
    g_{m,n}(h,k,t) &= 2t \sgn(m) + |m|(k + 1) - n, \\
    h_{m,\epsilon}(h,k,t) &= 4h - k + 4m(2t + m(k + 1)),
\end{align*}
\]  

(A.2)

and \( \sgn(0) = 1 \). The partition functions \( P^4 \) and \( P^4_{m,\epsilon} \) are given in [25], but we shall only use the result that \( P^4(0,0,0) = P^4_{m,\epsilon}(0,0,0) = 1 \). Also, in the above, \( k \) is the level of the \( N = 4 \) SCA in the notations of [13], and the quantum numbers \( h \) and \( t \) are the conformal weight and the isospin of the singular vector on which a reducible Verma module is built. The Kac-Kazhdan determinant formula enables one to identify the quantum numbers of singular vectors in this reducible Verma module, given some specific values of \( h \) and \( t \) obtained whenever (at least) one factor of the determinant vanishes.

Suppose now that \( h \) and \( t \) are the quantum numbers of an \( N = 4 \) hws in a unitary representation, i.e.,

\[
\begin{align*}
    h &\geq \frac{k}{4} \quad \text{and} \quad \frac{k}{2} \geq t \geq 0, \quad 2t \in \mathbb{Z},
\end{align*}
\]  

(A.3)

and let \( c \) be its \( U(1) \) charge. The \( g \)-product in (A.1) corresponds, in the \( \hat{sl}(2|1; \mathbb{C})_k \) Kac-Kazhdan determinant formula [23, 11], to the factors,

\[
\begin{align*}
    \tilde{\phi}_n^{(0)}((\tilde{\alpha}_1 + \tilde{\alpha}_2),0,m), \\
    \tilde{\phi}_n^{(0)}(-(\tilde{\alpha}_1 + \tilde{\alpha}_2),0,1+m), \quad m \in \mathbb{Z}_+, \quad n \in \mathbb{N},
\end{align*}
\]  

(A.4)

in the notations of [3], when the \( \hat{sl}(2|1; \mathbb{C})_k \) hws has isospin \( \frac{1}{2}h_- \) and hypercharge \( \frac{1}{2}h_+ \). So, whenever a factor of the \( g \)-product vanishes, i.e. whenever the isospin quantum number \( t \) of the hws obeys,

\[
2t \sgn(m) + (k + 1)|m| - n = 0, \quad \text{for} \quad m \in \mathbb{Z} \quad \text{and} \quad n \in \mathbb{N},
\]  

(A.5)

there exists a singular vector in the reducible hws Verma module with conformal weight \( h + mn \), isospin \( t - \sgn(m)n \) and charge \( c \). Using the formula iteratively provides the quantum numbers of singular vectors appearing in the Verma module built on a massive \( N = 4 \) hws. These quantum numbers, as well as the corresponding embedding diagrams, are identical to those in the \( \hat{sl}(2|1; \mathbb{C})_k \) theory, therefore showing how the massive \( N = 4 \) characters at level \( k \) coincide with the class I integrable \( \hat{sl}(2|1; \mathbb{C})_k \) characters.

Now consider the \( h \)-factors in (A.1) when the conformal weight of the \( N = 4 \) hws is \( h = \frac{k}{4} \),

\[
4m(2t + m(k + 1)), \quad m \in \mathbb{Z}.
\]  

(A.6)

The second factor corresponds to the factors,

\[
\begin{align*}
    \tilde{\phi}_n^{(1)}((\tilde{\alpha}_i,0,m), \\
    \tilde{\phi}_n^{(1)}(-(\tilde{\alpha}_i,0,1+m)), \quad m \in \mathbb{Z}_+, \quad i = 1, 2,
\end{align*}
\]  

(A.7)
in [5], when the $\hat{sl}(2|1;\mathbb{C})_k$ hws has hypercharge $\frac{1}{2}h_+ = 0$. So, whenever the $N = 4$ hws quantum number $t$ obeys,

$$2t + (k + 1)m = 0, \quad m \in \mathbb{Z}, \quad (A.8)$$

there exists a singular vector in the hws Verma module with conformal weight $\frac{k}{4} + |m|$, isospin $t - \frac{1}{2}\text{sgn}(m)$ and hypercharge $c \pm 1$. In the unitary domain however, the only possible isospin value for the hws is $t = 0$. So, provided that $c = 0$, the vacuum massless $N = 4$ character at level $k$ coincides with the vacuum integrable class IV $\hat{sl}(2|1;\mathbb{C})_k$ character.

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