On Central Extensions of Associative Dialgebras

Isamiddin S. Rakhimov
Dept. of Math., Faculty of Science, and Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, Malaysia.
E-mail: risamiddin@gmail.com

Abstract. The concept of central extensions plays an important in constructing extensions of algebras. This technique has been successfully used in the classification problem of certain classes of algebras. In 1978 Skjelbred and Sund reduced the classification of nilpotent Lie algebras in a given dimension to the study of orbits under the action of automorphism group on the space of second degree cohomology of a smaller Lie algebra with coefficients in a trivial module. Then W. de Graaf applied the Skjelbred and Sund method to the classification problem of low-dimensional nilpotent Lie and associative algebras over some fields. The main purpose of this note is to establish elementary properties of central extensions of associative dialgebras and apply the above mentioned method to the classification of low dimensional nilpotent associative dialgebras.

1. Introduction
A dissociative algebra is a vector space with two bilinear binary associative operations \(\ll, \rr\), satisfying certain conditions. Associative algebras are particular case of the diassociative algebras where the two operations coincide. The class of the associative dialgebras has been introduced by Loday in 1990 (see [6] and references therein). The main motivation of Loday to introduce this class of algebras was the search of an “obstruction” to the periodicity in algebraic \(K\)-theory. Later some their important relations with classical and non-commutative geometry, and physics have been discovered.

Definition 1.1. Associative dialgebra (the term “dissociative algebra” also is used) \(D\) over a field \(\mathbb{F}\) is an algebra equipped with two bilinear binary associative operations \(\ll, \rr\), called left and right products, respectively, satisfying the following axioms:

\[
(x \ll y) \ll z = x \ll (y \rr z),
(x \rr y) \ll z = x \rr (y \ll z),
(x \ll y) \rr z = x \rr (y \rr z),
\]

for all \(x, y, z \in D\).

A homomorphism from associative dialgebra \((D_1, \ll_1, \rr_1)\) to associative dialgebra \((D_2, \ll_2, \rr_2)\) is a linear map \(f : D_1 \rightarrow D_2\) satisfying

\[f(x \ll_1 y) = f(x) \ll_2 f(y)\text{ and }f(x \rr_1 y) = f(x) \rr_2 f(y),\text{ for all }x, y \in D_1.\]
The kernel and the image of a homomorphism is defined naturally. A bijective homomorphism is called isomorphism. Main problem in structural theory of algebras is the problem of classification. The classification means the description of the orbits under base change linear transformations and list representatives of the orbits.

The center of $D$ is defined by 

$$Z(D) = \{x \in D \mid x \ast y = y \ast x = 0 \text{ for all } y \in D, \text{ where } \ast = \dashv \text{ and } \vdash\}.$$ 

The natural extensions of the concepts of nilpotency and solvability of algebras to diassociative algebras case have been given in [1]. The automorphism group of $D$ is denoted by $AutD$.

There is the following naive approach for classifying of algebra structures on a vector space which often is being used up to now. It runs as follows. Let us fix a basis of the underlying vector space, then according to the identities which the algebra satisfies we get a system of equations with respect to the structure constants of the algebra on this basis. Solving this system of equations we get a redundant, in general, list of algebras via the tables of multiplications. The second step is to make the obtained list irredundant. The irredundancy can be achieved by identifying those algebras which are obtained from others by a base change. This approach has been applied to get classifications of two and three-dimensional diassociative algebras over $\mathbb{C}$. The classification results can be found in [2], [3], [7] in low-dimensional cases and four-dimensional nilpotent case has been obtained in [1]. In this paper we propose a different approach to the classification problem of associative dialgebras based on central extensions and action of group automorphisms on the Grassmanian of subspaces of the second cohomological groups of algebra with a smaller dimension.

So far central extensions of groups and algebras mostly are extensively studied in Lie algebras case and the results successfully applied in various branches of physics. In the theory of Lie groups, Lie algebras and their representation theory, a Lie algebra extension $M$ is an enlargement of a given Lie algebra $L$ by another Lie algebra $K$. Extensions arise in several ways. There is the trivial extension obtained by taking a direct sum of two Lie algebras. Other types are the split extension and the central extension. Extensions may arise naturally, for instance, when forming a Lie algebra from projective group representations. It is proven that a finite-dimensional simple Lie algebra has only trivial central extensions. Central extensions are needed in physics, because the symmetry group of a quantized system usually is a central extension of the classical symmetry group, and in the same way the corresponding symmetry Lie algebra of the quantum system is, in general, a central extension of the classical symmetry algebra. Kac-Moody algebras have been conjectured to be a symmetry groups of a unified superstring theory. The centrally extended Lie algebras play a dominant role in quantum field theory, particularly in conformal field theory, string theory and in $M$-theory.

The central extensions also are applied in the process of pre-quantization, namely in the construction of prequantum bundles in geometric quantization.

Starting with a polynomial loop algebra over finite-dimensional simple Lie algebra and performing two extensions, a central extension and an extension by a derivation, one obtains a Lie algebra which is isomorphic with an untwisted affine Kac-Moody algebra. Using the centrally extended loop algebra one may construct a current algebra in two spacetime dimensions. The Virasoro algebra is the universal central extension of the Witt algebra.

Due to the Lie correspondence, the theory, and consequently the history of Lie algebra extensions, is tightly linked to the theory and history of group extensions. A systematic study of group extensions has been given by the Austrian mathematician O. Schreier in 1923. Lie algebra extensions are most interesting and useful for infinite-dimensional Lie algebras. In 1967, Victor Kac and Robert Moody independently generalized the notion of classical Lie algebras, resulting in a new theory of infinite-dimensional Lie algebras, now called Kac-Moody algebras. They
generalize the finite-dimensional simple Lie algebras and can often concretely be constructed as extensions.

The organization of the paper is as follows. Section 1 is an introduction followed by Section 2 where we introduce the concept of central extension of associative dialgebras. Then we define 2-cocycles and 2-boundaries of associative dialgebras with coefficients in a trivial module in Section 3. In Section 4 we establish a close relationship between the central extensions and cocycles and in Section 5 we suggest an algorithm on application of the central extension approach to the classification problem of associative dialgebras.

2. Central Extensions of Associative dialgebras

In this section we introduce the concept of central extension of associative dialgebras.

Definition 2.1. Let $D_1, D_2$ and $D_3$ be associative dialgebras over a field $F$. The associative dialgebra $D_2$ is called the extension of $D_3$ by $D_1$ if there are associative dialgebra homomorphisms $\alpha : D_1 \rightarrow D_2$ and $\beta : D_2 \rightarrow D_3$ such that the following sequence

$$0 \rightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \rightarrow 0$$

is exact.

Definition 2.2. An extension is called trivial if there exists an ideal $I$ of $D_2$ complementary to $\ker \beta$, i.e.,

$$D_2 = \ker \beta \oplus I \quad (\text{the direct sum of algebras}).$$

It may happen that there exist several extensions of $D_3$ by $D_1$. To classify extensions the notion of equivalent extensions is defined.

Definition 2.3. Two sequences

$$0 \rightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \rightarrow 0$$

and

$$0 \rightarrow D_1 \xrightarrow{\alpha'} D_2' \xrightarrow{\beta'} D_3 \rightarrow 0$$

are called equivalent extensions if there exists a associative dialgebra isomorphism $f : D_2 \rightarrow D_2'$ such that $f \circ \alpha = \alpha'$ and $\beta' \circ f = \beta$.

The equivalence of extensions is an equivalent relation.

Definition 2.4. An extension

$$0 \rightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \rightarrow 0$$

is called central if the kernel of $\beta$ is contained in the center $Z(D_2)$ of $D_2$, i.e., $\ker \beta \subset Z(D)$.

From the definition above it is easy to see that in a central extension the algebra $D_1$ must be abelian. A central extension of an associative dialgebra $D_2$ by an abelian associative dialgebra $D_1$ can be obtained with the help of 2-cocycles on $D_3$.

3. Cocycles on Diassociative algebras

This section introduces the concept of 2-cocycle for diassociative algebras and gives some of simple but important properties of the 2-cocycles.
Definition 3.1. Let $D$ be an associative dialgebra over a field $\mathbb{F}$ and $V$ be a vector space over the same field. A pair $\Theta = (\theta_1, \theta_2)$ of bilinear maps $\theta_1: D \times D \rightarrow V$ and $\theta_2: D \times D \rightarrow V$ is called a 2-cocycle on $D$ with values in $V$ if $\theta_1$ and $\theta_2$ satisfy the conditions

$$
\begin{align*}
\theta_1(x \triangleleft y, z) &= \theta_1(x, y \triangleleft z) = \theta_1(x, y \triangleright z), \\
\theta_2(x \triangleright y, z) &= \theta_2(x, y \triangleright z) = \theta_2(x \triangleright y, z), \\
\theta_1(x \triangleright y, z) &= \theta_2(x, y \triangleright z),
\end{align*}
$$

(3.1)

for all $x, y, z \in D$.

The set of all 2-cocycles on $D$ with values in $V$ is denoted by $Z^2(D, V)$. One easily sees that $Z^2(D, V)$ is a vector space if one defines the vector space operations as follows:

$$(\Theta_1 \oplus \Theta_2)(x, y) := \Theta_1(x, y) + \Theta_2(x, y) \quad \text{and} \quad (\lambda \circ \Theta)(x) := \lambda \Theta(x).$$

It is easy to see that a linear combination of 2-cocycles again is a 2-cocycle. A special types of 2-cocycles given by the following lemma are called 2-coboundaries.

Lemma 3.1. Let $\nu: D \rightarrow V$ be a linear map, and define $\varphi_1(x, y) = \nu(x \triangleleft y)$ and $\varphi_2(x, y) = \nu(x \triangleright y)$. Then $\Phi = (\varphi_1, \varphi_2)$ is a cocycle.

Proof. Let us check the axioms (3.1) one by one.

$$
\begin{align*}
\varphi_1(x \triangleright y, z) &= \nu((x \triangleright y) \triangleleft z) \\
&= \nu(x \triangleleft (y \triangleright z)) = \varphi_1(x, y \triangleright z), \\
\varphi_1(x, y \triangleright z) &= \nu(x \triangleleft (y \triangleright z)) = \varphi_1(x, y \triangleright z), \\
\varphi_2(x \triangleright y, z) &= \nu((x \triangleright y) \triangleright z) \\
&= \nu(x \triangleright (y \triangleright z)) = \varphi_2(x, y \triangleright z), \\
\varphi_2(x, y \triangleright z) &= \nu(x \triangleright (y \triangleright z)) \\
&= \nu((x \triangleright y) \triangleright z) = \varphi_2(x \triangleright y, z), \\
\varphi_1(x \triangleright y, z) &= \nu((x \triangleright y) \triangleleft z) \\
&= \nu(x \triangleright (y \triangleleft z)) = \varphi_2(x, y \triangleleft z).
\end{align*}
$$

The set of all coboundaries is denoted by $B^2(D, V)$. Clearly, $B^2(D, V)$ is a subgroup of $Z^2(D, V)$. The group $H^2(D, V) = Z^2(D, V)/B^2(D, V)$ is said to be a second cohomology group of $D$ with values in $V$. Two cocycles $\Theta_1$ and $\Theta_2$ are said to be cohomologous cocycles if $\Theta_1 - \Theta_2$ is a coboundary. If we view $V$ as a trivial $D$-bimodule, then $H^2(D, V)$ is an analogue of the second Hochschild-cohomology space.
Theorem 3.1. Let $D$ be an associative dialgebra, $V$ a vector space,

$$\theta_1 : D \times D \rightarrow V \text{ and } \theta_2 : D \times D \rightarrow V$$

be bilinear maps. Set $D_\Theta = D \oplus V$, where $\Theta = (\theta_1, \theta_2)$. For $x, y \in D$, $v, w \in V$ we define

$$(x + v) \leftarrow (y + w) = x \leftarrow y + \theta_1(x, y) \text{ and } (x + v) \rightarrow (y + w) = x \rightarrow y + \theta_2(x, y).$$

Then $(D_\Theta, \leftarrow, \rightarrow)$ is an associative dialgebra if and only if $\Theta = (\theta_1, \theta_2)$ is a 2-cocycle.

Proof. Let $D_\Theta$ be the diassociative defined above. We show that the functions $\theta_1, \theta_2$ are 2-cocycles. Indeed, according to axioms (1.1) we have

$$(x + v) \leftarrow ((y + w) \leftarrow (z + t)) = (x + v) \leftarrow (y + (w \leftarrow (z + t))) = (x + v) \leftarrow (y + (w \leftarrow (z + t))).$$

Since

$$(x + v) \leftarrow (y + w) \leftarrow (z + t) = ((x \leftarrow y) + \theta_1(x, y)) \leftarrow (z + t) \quad (3.2)$$

$$= (x \leftarrow y) \leftarrow z + \theta_1(x \leftarrow y, z).$$

$$(x + v) \leftarrow ((y + w) \leftarrow (z + t)) = (x + v) \leftarrow (y + z + \theta_1(y, z)) \quad (3.3)$$

$$= (x \leftarrow y) \leftarrow z + \theta_1(x, y \leftarrow z).$$

$$(x + v) \leftarrow ((y + w) \leftarrow (z + t)) = (x + v) \leftarrow (y + z + \theta_2(y, z)) \quad (3.4)$$

$$= (x \leftarrow y) \leftarrow z + \theta_2(x, y \leftarrow z).$$

$$(x + v) \leftarrow (y + w) \leftarrow (z + t) = ((x \leftarrow y) + \theta_2(x, y)) \leftarrow (z + t) \quad (3.5)$$

$$= (x \leftarrow y) \leftarrow z + \theta_2(x \leftarrow y, z).$$

$$(x + v) \leftarrow ((y + w) \leftarrow (z + t)) = (x + v) \leftarrow (y + z + \theta_2(y, z)) \quad (3.6)$$

$$= (x \leftarrow y) \leftarrow z + \theta_2(x, y \leftarrow z).$$

$$(x + v) \leftarrow (y + w) \leftarrow (z + t) = (x \leftarrow y + \theta_1(x, y)) \leftarrow (z + t) \quad (3.7)$$

$$= (x \leftarrow y) \leftarrow z + \theta_1(x \leftarrow y, z).$$

$$(x + v) \leftarrow (y + w) \leftarrow (z + t) = (x \leftarrow y + \theta_2(x, y)) \leftarrow (z + t) \quad (3.8)$$

$$= (x \leftarrow y) \leftarrow z + \theta_2(x \leftarrow y, z).$$

$$(x + v) \leftarrow ((y + w) \leftarrow (z + t)) = (x + v) \leftarrow (y + z + \theta_2(y, z)) \quad (3.9)$$

$$= (x \leftarrow y) \leftarrow z + \theta_2(x, y \leftarrow z).$$
Comparing (3.2), (3.3) and (3.4) we get
\[ \theta_1(x \dashv y, z) = \theta_1(x, y \dashv z) = \theta_1(x, y \vdash z). \]
Comparing (3.5), (3.6) and (3.7) we obtain
\[ \theta_2(x \vdash y, z) = \theta_2(x, y \vdash z) = \theta_2(x \dashv y, z). \]
Comparing (3.8) and (3.9) we derive
\[ \theta_1(x \vdash y, z) = \theta_2(x, y \dashv z). \]

Lemma 3.2. Let \( \Theta \) be a cocycle and \( \Phi \) a coboundary. Then \( D_\Theta \cong D_{\Theta + \Phi} \).

Proof. The isomorphism
\[ f : D_\Theta \rightarrow D_{\Theta + \Phi}, \]

is given by \( f(x + v) = x + \nu(x) + v \). First of all we note that \( f \) is a bijective linear transformation. The linear transformation \( f \) obeys the algebraic operations \( \dashv \) and \( \vdash \) as well. Indeed,
\[
\begin{align*}
f((x + v) \dashv (y + w)) &= f(x \dashv y + \theta_1(x, y)) \\
&= f(x \vdash y) + f(\theta_1(x, y)) \\
&= x \vdash y + \nu(x \vdash y) + \theta_1(x, y) \\
&= x \vdash y + \varphi_1(x, y) + \theta_1(x, y) \\
&= x \vdash y + (\varphi_1 + \theta_1)(x, y) \\
&= ((x + (\varphi_1 + \theta_1)(x) + v)) \dashv (y + (\varphi_1 + \theta_1)(y) + w) \\
&= f(x + v) \dashv f(y + w).
\end{align*}
\]

For \( \vdash \) the proof is carried out similarly.

The proof of the following corollary is straightforward.

Corollary 3.1. Let \( \Theta_1, \Theta_2 \) be cohomologous 2-cocycles on a diassociative algebra \( D \) and \( D_1, D_2 \) be the central extensions constructed with these 2-cocycles, respectively. Then the central extensions \( D_1 \) and \( D_2 \) are equivalent extensions. Particularly, a central extension defined by a coboundary is equivalent with a trivial central extension.

Let \( V \) be a \( k \)-dimensional vector space with a basis \( e_1, e_2, \ldots, e_k \) and \( \Theta = (\theta_1, \theta_2) \in Z^2(D, V) \). Then we have \( Z^2(D, V) \cong Z^2(D, \mathbb{F})^k \), \( B^2(D, V) \cong B^2(D, \mathbb{F})^k \) and
\[ \Theta(x, y) = (\theta_1(x, y), \theta_2(x, y)) = \left( \sum_{i=1}^{k} \theta_1^i(x, y)e_i, \sum_{i=1}^{k} \theta_2^i(x, y)e_i \right). \]

Lemma 3.3. \( \Theta = (\theta_1, \theta_2) \in Z^2(D, V) \) if and only if \( \Theta^i = (\theta_1^i, \theta_2^i) \in Z^2(D, \mathbb{F}) \).

Proof. The proof is the direct verification of axioms (3.1).
4. Central Extensions and Cocycles.

There is a close relationship between the central extensions and cocycles which is established in this section.

**Theorem 4.1.** There exists one to one correspondence between elements of $H^2(D,V)$ and nonequivalents central extensions of the diassociative algebra $D$ by $V$.

**Proof.** The fact that for a 2-cocycle $\Theta$ the algebra $D_\Theta$ is a central extension of $D$ has been proven early.

Let us prove the converse, i.e., suppose that $D_\Theta$ is a central extension of $D$:

$$0 \longrightarrow V \overset{\alpha}{\longrightarrow} D_\Theta \overset{\beta}{\longrightarrow} D \longrightarrow 0.$$ We construct a 2-cocycle which generates this central extension. Considering the extension with the property $\text{im} \alpha = \ker \beta \subset Z(D)$ we can find bilinear maps $\theta_1 : D_2 \times D_2 \longrightarrow D_1$ and $\theta_2 : D_2 \times D_2 \longrightarrow D_1$ which satisfy (3.1) for all $x,y,z \in D$ ($V = D_1$).

In case $D_1$ is a one-dimensional associative dialgebra it can be identified with $\mathbb{F}$. To obtain maps satisfying equations (3.1) we consider a linear map $s : D_3 \longrightarrow D_2$ satisfying $\beta \circ s = id_{D_2}$. A map with this property is called a section $\theta$. We have

$$\theta(x,y) := s(x \upharpoonright y) - s(x) \upharpoonright s(y), \quad (4.1)$$

Notice that $\theta_i$, $i = 1,2$ is identically zero if $s$ is an associative dialgebra homomorphism. Moreover, since $\beta$ is an associative dialgebra homomorphism one sees that $\beta \circ \theta_i = 0$, $i = 1,2$. Hence, for all $x,y \in D_3$ we have

$$\theta_i(x,y) \in \ker \beta \subset Z(D_2). \quad (4.3)$$

Using this property and the associative dialgebra axioms one readily verifies that $\Omega = (\theta_1, \theta_2)$ satisfies

$$\begin{align*}
\theta_1(x \upharpoonright y) &= \theta_1(x, y \upharpoonright z) = \theta_1(x, y \upharpoonright y) = \theta_1(x, y \upharpoonright z), \\
\theta_2(x \upharpoonright y) &= \theta_2(x, y \upharpoonright z) = \theta_2(x, y \upharpoonright y) = \theta_2(x, y \upharpoonright z).
\end{align*} \quad (4.4)$$

In the final step to obtain the 2-cocycle $\Theta = (\theta_1, \theta_2)$ we use the injectivity of the map $\alpha : D_1 \longrightarrow D_2$. The map $\Theta = (\theta_1, \theta_2)$, where $\theta_i : D_3 \times D_3 \longrightarrow D_1$, $i = 1,2$ are defined by $\theta_i := \alpha^{-1} \circ \theta_i$, $i = 1,2$, is the required cocycle. \qed

The algebra $D_\Theta$ is a $(\dim V$-dimensional) central extension of $D$ by $V$ since we have the exact sequence of associative dialgebras

$$0 \longrightarrow V \longrightarrow D_\Theta \longrightarrow D \longrightarrow 0.$$ 

Let us now transfer the description of the orbits with respect to the “transport of structure” action of $\text{GL}(D_\Theta)$ ($\dim D_\Theta = n$) on the variety of dialgebras $\text{Dias}_n$ to the description of orbits of action of the $\text{Aut}D$ on the Grassmanian of $H^2(D,\mathbb{F})$. The action of $\sigma \in \text{Aut}D$ on $\Theta \in Z^2(D,V)$ is defined as follows:

$$\sigma(\Theta(x,y)) = (\sigma(\theta_1(x,y)), \sigma(\theta_2(x,y))), \text{ where } \sigma(\theta_i(x,y)) = \theta_i(\sigma(x), \sigma(y)), \ i = 1,2,$$

for $x, y \in D$. Thus $\text{Aut}D$ operates on $Z^2(D,V)$. It is easy to see that $B^2(D,V)$ is stabilized by this action, so that there is an induced action of $\text{Aut}D$ on $H^2(D,V)$.
Theorem 4.2. Let

$$\Theta(x, y) = (\theta_1(x, y), \theta_2(x, y)) = \left( \sum_{i=1}^{k} \theta_1^{i}(x, y)e_i, \sum_{i=1}^{k} \theta_2^{i}(x, y)e_i \right)$$

and

$$\Omega(x, y) = (\omega_1(x, y), \omega_2(x, y)) = \left( \sum_{i=1}^{k} \omega_1^{i}(x, y)e_i, \sum_{i=1}^{k} \omega_2^{i}(x, y)e_i \right).$$

Then $D_\Theta \cong D_\Omega$ if and only if there exists $\varphi \in \text{Aut}(D)$ such that $\varphi(\omega_j^i)$ span the same subspace of $H^2(D, \mathbb{F})$ as the $\theta_j^i$, $j = 1, 2$ and $i = 1, 2, \ldots, k$.

Proof. As vector spaces $D_\Theta = D \oplus V$ and $D_\Omega = D \oplus V$. Let $\sigma : D_\Theta \longrightarrow D_\Omega$ be an isomorphism. Since $V$ is the center of both dialgebras, we have $\sigma(V) = V$. So $\sigma$ induces an isomorphism of $D_\Theta/V = D$ to $D_\Omega/V = D$, i.e., it generates an automorphism $\varphi$ of $D$. Let $D = \text{Span}\{x_1, \ldots, x_n\}$.

Then we write $\sigma(x_i) = \varphi(x_i) + v_i$, where $v_i \in V$, and $\sigma(e_i) = \sum_{j=1}^{s} a_{ji}e_j$. Also write

$$x_i + x_j = \sum_{k=1}^{n} \gamma_{ij}^k x_k, \quad x_i \vdash x_j = \sum_{k=1}^{n} \delta_{ij}^k x_k,$$

and $v_i = \sum_{l=1}^{s} \beta_{ij}^l e_l$. Then the relations

$$\sigma(x_i \vdash \theta) = \sigma(x_i) \vdash \sigma(x_j)$$

amount to

$$\omega_j^l(\varphi(x_i), \varphi(x_j)) = \sum_{k=1}^{s} a_{kl} \theta_j^k(x_i, x_j) + \sum_{k=1}^{s} \gamma_{ij}^k \beta_{kl}, \text{ for } 1 \leq l \leq s \text{ and } r = 1, 2. \quad (4.5)$$

$$\omega_j^l(\varphi(x_i), \varphi(x_j)) = \sum_{k=1}^{s} a_{kl} \theta_j^k(x_i, x_j) + \sum_{k=1}^{s} \delta_{ij}^k \beta_{kl}, \text{ for } 1 \leq l \leq s \text{ and } r = 1, 2. \quad (4.6)$$

Now define the linear function $f_l : D \longrightarrow \mathbb{F}$ by $f_l(x_k) = \beta_{kl}$. Then

$$f_1^l(x_i \vdash x_j) = \sum_{k=1}^{n} \gamma_{ij}^k \beta_{kl} \text{ and } f_2^l(x_i \vdash x_j) = \sum_{k=1}^{n} \delta_{ij}^k \beta_{kl}.$$

From (4.5) and (4.6) it is obvious that modulo $B^2(D, \mathbb{F})$, $\varphi(\omega_j^i)$ and $\theta_j^i$ span the same space.

Let now suppose that there exists an automorphism $\varphi \in D$ such that the cocycles $\varphi(\omega_j^i)$ and $\theta_j^i$, $j = 1, 2$ and $i = 1, 2, \ldots, k$ span the same space in $Z^2(D, \mathbb{F})$, modulo $B^2(D, \mathbb{F})$. Then there are linear functions $f_l : D \longrightarrow \mathbb{F}$ and $a_{kl} \in \mathbb{F}$ so that

$$\omega_j^l(\varphi(x_i), \varphi(x_j)) = \sum_{k=1}^{s} a_{kl} \theta_j^k(x_i, x_j) + f_l(x_i \vdash x_j) \text{ for } 1 \leq l \leq s \text{ and } r = 1, 2. \quad (4.7)$$

$$\omega_j^l(\varphi(x_i), \varphi(x_j)) = \sum_{k=1}^{s} a_{kl} \theta_j^k(x_i, x_j) + f_l(x_i \vdash x_j) \text{ for } 1 \leq l \leq s \text{ and } r = 1, 2. \quad (4.8)$$

If we take $\beta_{kl} = f(x_k)$ then (4.5) and (4.6) hold. Now define $\sigma : D_\Theta \longrightarrow D_\Omega$ as follows

$$\sigma(x_i) = \varphi(x_i) + \sum_{i=1}^{s} \beta_{il}^i e_i, \quad \sigma(e_i) = \sum_{j=1}^{s} a_{ji} e_j.$$

Then the $\sigma$ is the required isomorphism.
5. Application
Let $D'$ be an associative dialgebra and $Z(D')$ be its center which we suppose to be nonzero. Set $V = Z(D')$ and $D = D'/Z(D')$. Then there is a $\Theta \in H^2(D, V)$ such that $D' = D\Theta$. We conclude that any associative dialgebra with a nontrivial center can be obtained as a central extension of an associative dialgebra of smaller dimension. So in particular, all nilpotent dialgebras can be constructed by this way.

Procedure: Let $D$ be an associative dialgebra of dimension $n - s$. The procedure outputs all nilpotent algebras $D'$ of dimension $n$ such that $D'/Z(D') = D$. It runs as follows:

1. Determine $Z^2(D, F)$, $B^2(D, F)$ and $H^2(D, F)$.
2. Determine the orbits of $\text{Aut}(D)$ on $s$-dimensional subspaces of $H^2(D, F)$.
3. For each of the orbits let $\Theta$ be the cocycle corresponding to a representative of it, and construct $D\Theta$.

6. Acknowledgements
The research was supported by Grant 01-02-14-1591FR MOHE, Malaysia.

References
[1] Basri W, Rakhimov I S and Rikhsiboev I M 2015 J. Generalized Lie Theor. Appl. 9
[2] Basri W, Rakhimov I S and Rikhsiboev I M 2007 Proc. 3rd Conf. Research and Educ. in Math. (ICREM3), UPM, Malaysia 164–170.
[3] González CM 2013 Commun. in Algebra, 41 1903–1912.
[4] de Graaf W A 2007 J. Algebra 309 640–653.
[5] de Graaf W A 2010 arXiv: 1009.5339
[6] Loday JL, Frabetti A, Chapoton F and Goichot F 2001 Lecture Notes in Math. 1763 (Springer- Berlin)
[7] Rikhsiboev I M, Rakhimov I S and Basri W 2010 Malays. J. Math. Sci. 4 (2) 241–254.
[8] Skjelbred T and Sund T 1978 C. R. Acad. Sci. Paris Ser. A-B, 286 (5)