Exact Scale Invariance in Mixing of Binary Candidates in Voting Model

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We introduce a voting model and discuss the scale invariance in the mixing of candidates. The Candidates are classified into two categories \( \mu \in \{0, 1\} \) and are called as ‘binary’ candidates. There are in total \( N = N_0 + N_1 \) candidates, and voters vote for them one by one. The probability that a candidate gets a vote is proportional to the number of votes. The initial number of votes (‘seed’) of a candidate \( \mu \) is set to be \( s_\mu \). After infinite counts of voting, the probability function of the share of votes of the candidate \( \mu \) obeys gamma distributions with the shape exponent \( s_\mu \) in the thermodynamic limit \( Z_0 = N_1 s_1 + N_0 s_0 \rightarrow \infty \). Between the cumulative functions \( \{x_\mu\} \) of binary candidates, the power-law relation \( 1 - x_1 \sim (1 - x_0)^\alpha \) with the critical exponent \( \alpha = s_1/s_0 \) holds in the region \( 1 - x_0, 1 - x_1 \ll 1 \). In the double scaling limit \( (s_1, s_0) \rightarrow (0, 0) \) and \( Z_0 \rightarrow \infty \) with \( s_1/s_0 = \alpha \) fixed, the relation \( 1 - x_1 = (1 - x_0)^\alpha \) holds exactly over the entire range \( 0 \leq x_0, x_1 \leq 1 \). We study the data on horse races obtained from the Japan Racing Association for the period 1986 to 2006 and confirm scale invariance.

KEYWORDS: scale invariance, voting model, branching process, gamma distribution, ROC, accuracy ratio

1. Introduction

Scale-invariant behaviour has attracted considerable attention on account of its ubiquity in natural and man-made phenomena.1) Many possible candidate mechanisms that gives rise to power-law distributions have been proposed thus far. The Yule process is a widely applicable mechanism for generating power-law distributions.2) Originally, it has been proposed to explain why the distribution of the number of species in a genus, a family, or any other taxonomic group follows a power law.3) Now, it has found wide applications in other areas.1, 4)

Consider the distribution of the number of species in a genus. Suppose first that new species appear but they never die; species are only ever added to genera and never removed. Species are added to genera by speciation, the splitting of one species into two. If we assume that this happens at some stochastically constant rate, then it follows that a genus with \( k \) species will gain new species at a rate proportional to \( k \), since each of the \( k \) species has the same chance per unit time of dividing into two. In addition, suppose that a new species that...
belongs to a new genus is added once every \( m \) speciation events. So \( m + 1 \) new species appear for each new genus and there are \( m + 1 \) species per genus. Thus the number of genera goes up steadily as does the number of species within each genus. We denote the fraction of genera that has \( k \) species by \( p_{k,n} \), where \( n \) denotes the total number of genera and \( n \) measures the passage of time in the model. At each time-step one new species founds a new genus, thereby increasing \( n \) by 1, and \( m \) other species are are added to various pre-existing genera which are selected in proportion to the number of species they already have. By solving the master equation for \( p_{k,n} \) in the limit \( n \to \infty \), \( p_k \equiv \lim_{n \to \infty} p_{k,n} \) behaves as \( p_k \sim k^{2+\frac{1}{m}} \). The Yule process has been adopted and generalized to explain power laws in many other systems. An important feature of this process is that the probability that a genus with \( k \) species will gain new species is proportional to \( k \). This ‘rich-get-richer’ process is the most important factor in exhibiting power-law behaviour. The feature that \( n \) increases infinitely is also important in generating power-law behaviour.

In this study, we introduce a voting model, a multivariate Polya-Eggenberger model,\(^5\) and discuss the scale invariance in the mixing of candidates. The candidates are classified into two categories \( \mu \in \{0, 1\} \) and are called as ‘binary’ candidates. The probability that a candidate get a vote is proportional to the number of votes, which is the same as the relation in the Yule process. The main difference between the voting model and the Yule process is that the number of candidates is fixed in our model. In the Yule process, \( n \) increases and in the limit \( n \to \infty \), power-law behaviour is observed. In our model, the distribution of the number of votes does not show power-law behaviour. However, our model exhibits scale-invariant behaviour. This behaviour is observed in the mixing of the binary candidates. Furthermore, the power law holds over the entire range in a double scaling limit.

This kind of voting model has been introduced in the literatures of social-choice problems on preference formation in a voting population.\(^6\) The voting paradox, the possibility of individual preference patterns leading to in-transitivity, ask about the likelihood that certain kinds of cycles occurs, given that people can choose at random among all possible profiles, rankings of choices. In order that majority rule does work in decision making process, or to fix the Condorcet’s winner, there must exist a transitive ordering among profiles. The voting model is a simple Polya-variety urn model. A homogeneity parameter relates to measures of similarity among voters. The model is a rough model for contagion diseases, such that each occurrence increases the chance of further occurrences. We can interpret the homogeneity parameter as the contagion parameter or as the amount of similarity-homogeneity among voters, the extent to which voters influence one another. It was concluded that as the preference similarity among voters increases, or stronger mutual influence among voters, there is a lesser chance for the paradox of occurring. Our conclusion is that in the ranking of the horses, the mutual influence among voters induces the scale invariance in the mixing.
The organization of this paper is as follows. In §2, we introduce the voting model. We select a candidate (initial number of votes $s_{\mu}$) and show that the probability density function of the share of votes $u_{\mu}$ of the candidate obeys a gamma distribution function with the shape exponent $s_{\mu}$ in the thermodynamic limit $Z_0 = N_1s_1 + N_0s_0 \to \infty$. We also show that the joint probability density function of $u$ for any $k$ candidates is given by the direct product of the gamma distributions in the same limit. We discuss the scale invariance in the mixing of the binary candidates in §3. The cumulative function $1 - x_{\mu}$ of candidates $\mu$ is given by the incomplete gamma function. The power-law relation $1 - x_1 \sim (1 - x_0)^{\alpha}$ with the exponent $\alpha = s_1/s_0$ holds in the region $1 - x_0, 1 - x_1 << 1$. Furthermore, in the double scaling limit $\{s_{\mu}\} \to 0$ and $Z_0 \to \infty$ with $\alpha = s_1/s_0$ fixed, the relation $1 - x_1 = (1 - x_0)^{\alpha}$ holds exactly over the entire range $0 \leq x_0, x_1 \leq 1$. Using the data on horse races, we verify these results in §4. We show that scale invariance holds over the wide range of cumulative functions. In addition, we show that the probability distribution functions of $u$ are well described by gamma distributions. Section 5 is dedicated to the summary and concluding remarks. Appendix A is devoted to the derivation of the joint probability distribution function of $u$ for any $k$ candidates. In Appendix B, we map the voting model to a branching process and easily derive the gamma distribution function.

2. Voting Model for Binary Candidates

Consider a voting model for $N$ candidates. Voters vote for them one by one, and the result of each voting is announced promptly. The time variable $t \in \{0, 1, 2 \cdots, T\}$ counts the number of the votes. The candidates are classified into two categories $\mu \in \{0, 1\}$ and are called as binary candidates. There are $N_{\mu}$ candidates in each category and $N_0 + N_1 = N$. The main result of this section is that the scaled share of votes $u^\mu_i$ of a candidate $\mu$ obeys a gamma distribution with the shape exponent $s_{\mu}$ in the thermodynamic limit $N_0, N_1 \to \infty$.

We denote the number of votes of $i$th candidate $\mu \in \{0, 1\}$ at time $t$ as $\{X^\mu_{i,t}\}_{i \in \{1, \cdots, N_{\mu}\}}$. At $t = 0$, $X^\mu_{i,t}$ takes the initial value $X^\mu_{i,0} = s_{\mu} > 0$. If the $i$th candidate $\mu$ gets a vote at $t$, $X^\mu_{i,t}$ increases by one unit.

$$X^\mu_{i,t+1} = X^\mu_{i,t} + 1.$$  

A voter casts a vote for the total $N$ candidates at a rate proportional to $X^\mu_{i,t}$. The probability $P^\mu_{i,t}$ that the $i$th candidate $\mu$ gets a vote at $t$ is

$$P^\mu_{i,t} = \frac{X^\mu_{i,t}}{Z_t}$$  (1)

$$Z_t = \sum_{\mu=0}^{1} \sum_{i=1}^{N_{\mu}} X^\mu_{i,t} = N_1s_1 + N_0s_0 + t.$$  (2)

The problem of determining the probability of the $i$th candidate $\mu$ getting $n$ votes up to
\( T \) is equivalent to the famous Pólya’s urn problem.\(^{5,6,8,9}\) If the change in \( X^\mu_{i,t} \) is given by
\[
\Delta X^\mu_{i,t} \equiv X^\mu_{i,t} - X^\mu_{i,t-1},
\]
the sequence \((\Delta X^\mu_{i,1}, \cdots, \Delta X^\mu_{i,T})\) is called Pólya’s urn sequence. This sequence is an exchangeable stochastic process, and the joint distribution of \((X^\mu_{i,1}, \cdots, X^\mu_{i,T})\) is given by
\[
\text{Prob}(\Delta X^\mu_{i,1} = x_1, \cdots, \Delta X^\mu_{i,T} = x_T) = \frac{(s_\mu)_{k}(Z_0 - s_\mu)_{T-k}}{(Z_0)_{T}}.
\]
Here, \( k = \sum_{t=1}^{T} x_t \) and \((a)_n \equiv a \cdot (a+1) \cdot (a+2) \cdots (a+n-1)\) is the rising factorial. This distribution depends only on \( k \), and not on the particular order of \((x_1, \cdots, x_T)\). This distribution is invariant under the permutations of the entries and, hence, it is called exchangeable.

Furthermore, the expectation value of \( \Delta X^\mu_{i,t} \), denoted by \( p_\mu \), does not depend on \( t \).
\[
\langle \Delta X^\mu_{i,t} \rangle = \frac{s_\mu}{Z_0}.
\]

The correlation function \( \rho_\mu \) between \( \Delta X^\mu_{i,t} \) and \( \Delta X^\mu_{i,t'} \) \((t' \neq t)\) is also constant\(^9\) as \( \rho_\mu \).
\[
\rho_\mu \equiv \text{Corr}(\Delta X^\mu_{i,t}, \Delta X^\mu_{i,t'}) \equiv \frac{\langle \Delta X^\mu_{i,t} \Delta X^\mu_{i,t'} \rangle - p^2}{p(1-p)} = \frac{1}{Z_0 + 1}, \quad t \neq t'.
\]

The probability that the \( i \)th candidate \( \mu \) gets \( n \) votes up to \( T \) is given by the beta binomial distribution
\[
\text{Prob}(X^\mu_{i,T} - s_\mu = n) = T C_n \cdot \frac{(s_\mu)_n(Z_0 - s_\mu)_{T-n}}{(Z_0)_{T}}.
\]

\((a)_n\) is written as \((a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\) and this relation can also be written as
\[
\text{Prob}(X^\mu_{i,T} - s_\mu = n) = T C_n \cdot \frac{\Gamma(s_\mu + n) \Gamma(Z_0 - s_\mu + T - n) \Gamma(Z_0)}{\Gamma(s_\mu) \Gamma(Z_0 - s_\mu) \Gamma(Z_0 + T)}.
\]

Using a definition of beta function \( B(a,b) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \), we can rewrite the expression as
\[
\text{Prob}(X^\mu_{i,T} - s_\mu = n) = T C_n \cdot \frac{B(s_\mu + n, Z_0 - s_\mu + T - n)}{B(s_\mu, Z_0 - s_\mu)}.
\]

\(B(a,b)\) is also written as \( B(a,b) = \int_0^1 p^{a-1}(1-p)^{b-1} dp\), we get the next expression
\[
\text{Prob}(X^\mu_{i,T} - s_\mu = n) = T C_n \cdot \int_0^1 p^n(1-p)^{T-n} B(s_\mu-1)p^Z_0-s_\mu B(s_\mu, Z_0 - s_\mu) dp.
\]

After infinite counts of voting, i.e. \( T \to \infty \), the share of votes \( x^\mu_i \equiv \lim_{T \to \infty} X^\mu_{i,T} - s_\mu \) becomes the beta distributed random variable \( B(s_\mu, Z_0 - s_\mu) \) on \([0,1]\).
\[
p(x) \equiv \lim_{T \to \infty} \text{Prob}(X^\mu_{i,T} - s_\mu = T x) = \frac{x^{s_\mu-1}(1-x)^{Z_0-s_\mu-1}}{B(s_\mu, Z_0 - s_\mu)}.
\]

Here, we use the identity \( \lim_{T \to \infty} T C_n T x p^{T x}(1-p)^{T(1-x)} = \delta(x-p) \). This result has been derived by Pólya.\(^5\)

Next, we focus on the thermodynamic limit \( N_0, N_1 \to \infty \) and \( Z_0 = N_0 s_0 + N_1 s_1 \to \infty \). The expectation value of \( x^\mu_i \) is \( \langle x_i^\mu \rangle = p_\mu = \frac{s_\mu}{Z_0} \). We introduce a variable \( u^\mu_i \equiv (Z_0 - s_\mu - 1)x^\mu_i \).
The distribution function \( p_{s\mu}(u) \) in the thermodynamic limit is given as
\[
 p_{s\mu}(u) \equiv \lim_{Z_0 \to \infty} p(x_i^\mu = u \frac{Z_0 - s_\mu - 1}{Z_0 - s_\mu - 1}) = \frac{1}{\Gamma(s_\mu)} e^{-u^{s_\mu} - 1}. \tag{10}
\]
The share of votes, \( u \), of a candidate \( \mu \) obeys a gamma distribution function with \( s_\mu \).

In general, the joint probability distribution function of the scaled share of votes of \( k \) different candidates becomes the direct product of \( k \) gamma distribution functions in the limit \( Z_0 \to \infty \). We denote the \( k \) candidates as \( \{\mu_j, i_j\}_{j=1, \ldots, k} \) and denote the scaled share of votes as \( \{u_j\}_{j=1, \ldots, k} \). The joint probability distribution function is given as
\[
 p(u_1, \cdots, u_k) = \prod_{j=1}^k p_{s\mu_j}(u_j). \tag{11}
\]
The derivation of the result is given in Appendix A. It should be noted that in the thermodynamic limit, the correlation among \( \{u_j\}_{j=1, \ldots, k} \) vanishes. Hence, by mapping the voting problem to a continuous time branching process, we can derive the gamma distribution function \( p_{s\mu}(u) \) easily (refer Appendix B). In the branching process, the stochastic processes of the increase in \( \{X_{i,t}^\mu\} \) are independent of each other.

3. Scale Invariance in Mixing of Binary Candidates

In this section, we discuss the mixing of the binary candidates. After many counts of voting \( T \to \infty \), the binary candidates are distributed in the space of \( u \) according to the gamma distribution in the thermodynamic limit \( Z_0 \to \infty \). If \( s_1 > s_0 \), a candidate belonging to category \( \mu = 1 \) has a higher probability of getting many votes than a candidate belonging to category \( \mu = 0 \). Even the latter can obtain many votes. It is also possible that the former may get few votes. Thus, there is a mixing of the binary candidates. We see a scale invariant behaviour appears in the mixing. Between the cumulative functions of the binary candidates \( 1 - x_\mu \), the power-law relation \( 1 - x_1 \sim (1 - x_0)^\alpha \) with the exponent \( \alpha = s_1/s_0 \) holds.

In order to study the mixing configuration, we arrange the \( N \) candidates according to the size of \( u_i^\mu \) as
\[
 u_{i_1}^{\mu_1} > u_{i_2}^{\mu_2} > \cdots > u_{i_N}^{\mu_N}, \quad \mu_k \in \{0, 1\}. \tag{12}
\]
Using the ranking information \( \{\mu_k\}_{k=1, \ldots, N} \), we draw a path \( \{(x_{0,k}, x_{1,k})\}_{k=0, \ldots, N} \) in two-dimensional space \((x_0, x_1)\) from \((x_{0,0}, x_{1,0}) = (0, 0)\) to \((x_{0,N}, x_{1,N}) = (1, 1)\) as
\[
 x_{\mu,k} = \frac{1}{N_\mu} \sum_{j=1}^k \delta_{\mu_j, \mu}. \tag{13}
\]
See Fig. 1. If \( \mu_k = \mu \), the path extends in \( x_\mu \) direction. The pictorial representation of the mixing of binary objects is known as a receiver operating characteristic (ROC) curve.\(^{10}\) If \( s_1 \gg s_0 \), the binary candidates are well separated on the axis of \( u \), and the first \( N_1 \) candidates belong to category \( \mu = 1 \) and the last \( N_0 \) candidates belong to category \( \mu = 0 \). The path goes
straight from (0, 0) to (0, 1) and then turns right to the end point (1, 1). If \( s_1 = s_0 \), the path almost runs diagonally to the end point. If \( s_1 > s_0 \) holds, the path resembles a upward convex curve from (0, 0) to (1, 1).

\[ x \]

\[ \bigcirc \times \bigcirc \times \bigcirc \times \times \times \]

\[ x_1 \]

\[ \frac{3}{N_1} \]

\[ \frac{2}{N_1} \]

\[ \frac{1}{N_1} \]

\[ \frac{1}{N_0} \]

\[ \frac{2}{N_0} \]

\[ \frac{3}{N_0} \]

\[ \frac{4}{N_0} \]

\[ \frac{5}{N_0} \]

\[ x_0 \]

\[ \bigcirc \ : \mu = 1 \]

\[ \times \ : \mu = 0 \]

Fig. 1. ROC curve of mixing configuration. \( \bigcirc \) represents candidate belonging to category \( \mu = 1 \). \( \times \) represents candidate belonging to category \( \mu = 0 \). At the top of the figure, the order of three candidates from category \( \mu = 1 \) and five candidates from category \( \mu = 0 \) is shown.

The distribution function of the candidate \( \mu \) on the axis of \( u \) is given by the gamma distribution with the shape exponent \( s_\mu \). The ROC curve \((x_0(t), x_1(t))\) of the parameter \( t \in [0, \infty] \) is given by its cumulative function as

\[
x_\mu(t) = \int_t^\infty p_{s_\mu}(u)du.
\]  

(14)

Using the incomplete gamma function of the first kind \( \gamma(s, t) \equiv \int_0^t e^{-u} \cdot u^{s-1} du \), the ROC curve is given as

\[
1 - x_\mu(t) = \frac{1}{\Gamma(s_\mu)} \cdot \gamma(s_\mu, t).
\]  

(15)

Near the end point, \((x_0, x_1) \simeq (1, 1)\), in other words, in the small \( u \) region \((t \simeq 0)\), the incomplete gamma function \( \gamma(s_\mu, t) \) behaves as

\[
\gamma(s_\mu, t) \sim t^{s_\mu}.
\]  

(16)

As \( 1 - x_{s_\mu}(t) \propto t^{s_\mu} \), the following relation holds:

\[
1 - x_1 \sim (1 - x_0)^\alpha \quad \text{with} \quad \alpha = \frac{s_1}{s_0}.
\]  

(17)
The density of good candidates, $\rho_1$, in terms of the cumulative function of bad candidates, $1 - x_0$, is given as

$$\rho_1 = \frac{d(1 - x_1)}{d(1 - x_0)} \propto (1 - x_0)^{\alpha - 1}.$$  \hspace{1cm} (18)$$

$\rho_1$ obeys the power law with the exponent $\alpha - 1$.

Furthermore, in the limit $(s_1, s_0) \to (0, 0)$ with $\alpha = s_1/s_0$ fixed, the relation $1 - x_1 = (1 - x_0)^\alpha$ holds. The proof is given as follows. $\gamma(s, t)$ is expressed using Kummer’s confluent hypergeometric function $M(a, b, t)$ as

$$\gamma(s, t) = \frac{1}{s} t^s \cdot M(s, s + 1, -t).$$  \hspace{1cm} (19)$$

The cumulative function $1 - x_\mu(t)$ is then given as

$$1 - x_\mu(t) = \frac{t^s}{\Gamma(s + 1)} \cdot M(s_\mu, s_\mu + 1, -t).$$  \hspace{1cm} (20)$$

Thus, we obtain

$$\left(1 - x_0\right)^\alpha = \left(1 - x_1\right) \frac{\Gamma(s_1 + 1)}{\Gamma(s_1, s_1 + 1, -t)} \left(\frac{M(s_0, s_0 + 1, -t)}{\Gamma(s_0 + 1)}\right)^\alpha.$$  \hspace{1cm} (21)$$

In the limit $s_\mu \to 0$, both $\Gamma(s_\mu + 1)$ and $M(s_\mu, s_\mu + 1, -t)$ become equal to 1 and the following relation holds.

$$1 - x_1 = \left(1 - x_0\right)^\alpha, \quad 0 \leq x_0, x_1 \leq 1.$$  \hspace{1cm} (22)$$

Thus, the scale-invariant relation holds over the entire range $0 \leq x_0, x_1 \leq 1$. The feature is remarkable from the viewpoint of statistical physics. Usually, the power-law relation does hold only in the tail.

The relative probability that a candidate gets the first vote ($t = 0$) is given by $s_\mu$. If the candidate gets the first vote, his/her score increases by 1 and the relative probability becomes $s_\mu + 1$. In the limit $s_\mu \to 0$, the additional score +1 or the weight of a single vote becomes crucially important. The probability that the candidate gets the next vote becomes equal to 1, which is exemplified by the behaviour of $\rho_\mu$, given by eq.(4).

$$\rho_\mu = \frac{1}{Z_0 + 1} = \frac{1}{N_0 s_0 + N_1 s_1 + 1} \to 1 \text{ if } \{s_\mu\} \to 0.$$  \hspace{1cm} (23)$$

After infinite counts of voting, the candidate occupies the first position in the order of candidates according to the number of votes. Then, we neglect this candidates in the voting problem and consider the remaining $N - 1$ candidates. Similarly, if a candidate is selected randomly with the relative probability $s_\mu$, he/she occupies the second position. Thus, the voting problem reduces to a random choice problem with the relative probability $s_\mu$ in the limit $\{s_\mu\} \to 0$. At $(x_0, x_1)$ on the ROC curve, the probability that the next candidate belongs category $\mu$ is proportional to $(1 - x_\mu)s_\mu$. The coordinates of the ROC curve $(x_0, x_1)$ grow according to the following relation:

$$dx_\mu \propto (1 - x_\mu) \cdot s_\mu.$$
Solving this relation, we get eq.(22).

Finally, we discuss the limit in the derivation of the exact scale invariance. In the derivation of the gamma distribution, we take the thermodynamic limit $Z_0 = N_1 s_1 + N_0 s_0 \to \infty$. With the gamma distribution, eq.(22) holds in the limit $\{s_\mu\} \to 0$. In order that eq.(22) holds, these two limits, $Z_0 \to \infty$ and $\{s_\mu\} \to 0$, should go together. $\{s_\mu\}$ approaches zero more slowly than $\{N_\mu\}$ approaches infinity. In other words, in the double scaling limit $Z_0 \to \infty$ and $\{s_\mu\} \to 0$ with $\alpha = s_1/s_0$ fixed, eq.(22) holds. So the above intuitive explanation of the exact scale invariance may be too naive. If we take the limit $\{s_\mu\} \to 0$ without the limit $Z_0 \to \infty$, $\rho_\mu$ becomes 1. The firstly chosen candidate get all the remaining votes and there does not occur the mixing of the binary candidates. The double scaling limit is crucial in the emergence of the exact scale invariance.

4. Data Analysis of Horse Races

We verify the results of the voting model, particularly the scale invariance in the mixing of binary candidates. We study all the data on horse race betting obtained from the Japan Racing Association (JRA) for the period 1986 to 2006. There have been 71549 races and in which a total of 901366 horses have participated. We select the winning horses as candidate belonging to category $\mu = 1$. For candidate belonging to category $\mu = 0$, we consider two cases; losing horses and horses finishing second. In a race, no one knows which horse will win. Betters only have partial information on the horses, which is embedded in the initial values $\{s_\mu\}$. The results of betting are announced at short intervals. Betters usually presume that the horses which get many votes are strong. They come to know which horses are considered to be strong by other betters. These features are incorporated in the voting model. Betters do not always bet to strong horses. Some betters may prefer betting to a horse that can coin more money even if it is considered to be ‘weaker’ than a horse that can coin less money. However, in the bet to win, only the better who bets to the winning horse coin the bet. Hence, the assumption is not so unrealistic. We also note the reason why we can treat multiple categories, 2nd finishing horse and losing horse, as the category $\mu = 0$. For the betters, the only difference between the losing horses and finishing second ones is their confidence. By tuning parameter $s_0$, we can treat the two categories on the same footing.

Next, we explain the meaning of the initial values $\{s_\mu\}$. The probability that a candidate $\mu$ is selected is proportional to $s_\mu$ as $\langle \Delta X_{i,t}^{\mu} \rangle = s_\mu/Z_0$. The ratio $s_1/s_0$ is a measure of the accuracy of the knowledge of betters. On the other hand, $\rho_\mu$ is given by eq.(4). If the scale of $\{s_\mu\}$ is small, the decisions of betters are crucially affected by the choices of other betters. In the limit $\{s_\mu\} \to \infty$, their decisions are not affected by the choices of other betters. The scale of $\{s_\mu\}$ is a measure of the degree of similarity (‘copycat’) of the choices of betters.

In the early stage of voting, $\{s_\mu\}$ is the only available information. Voters decide on horses on the basis of $\{s_\mu\}$ and they are ‘intelligent’, because their decisions are not affected by the
choices of other betters. As the voting process proceeds, the importance of the cumulative number of votes exceeds that of the initial scores, and voters become ‘copycat’. If one control the scale of \( \{ s_\nu \} \) (or the weight of a single vote), the passage timing from the initial ‘intelligent’ stage to the late ‘copycat’ stage should change.

Table 1. Data on horse race betting obtained from the Japan Racing Association (JRA) for the period 1986 to 2006. There are 71549 races and 71650 winning (finishing first) horses. 71590 horses are finishing second. The difference between \( N_{\text{Win}} \) and \( N_{\text{2nd}} \) indicates that there occurs a tie in the race. In the third column, we show the average value of the share of the votes in each category. The fourth column shows the values \( v^\nu/c \), here \( c \) is the estimated value of the scale parameter in (24). About the estimation of \( c \), please see the main text and Figure 2.

| Category \( \nu \) | \( N^\nu \) | \( v^\nu [%] \) | \( v^\nu/c \) |
|----------------------|------------|---------------|--------------|
| Win                  | 71650      | 21.23         | 1.769        |
| 2nd                  | 71590      | 15.40         | 1.283        |
| Lose                 | 829716     | 6.80          | 0.567        |

We denote the three categories of horses as \( \nu \in \{ \text{Win}, \text{2nd}, \text{Lose} \} \) and the number of horses in each category as \( N^\nu \). \( v^\nu_i \) denotes the share of votes of the \( i \)th horse in the category \( \nu \), and \( v^\nu \) denotes the average value of \( v^\nu_i \). In Table I, we summarize the data on horse races. A difference between \( N_{\text{Win}} \) and \( N_{\text{2nd}} \) indicates that there is a tie in the race.

We have shown that the share of votes, \( u \), obeys a gamma distribution function with \( s_\mu \). In order to check whether \( v^\nu_i \) obeys a gamma distribution function, we have to set the scale \( c \) between \( v^\nu_i \) and \( u \) as follows:

\[
v^\nu_i = c \cdot u. \tag{24}
\]

The same \( c \) should be used for all categories. Assuming that \( u \) obeys the gamma probability distribution with \( s_\nu \), \( v^\nu_i \) obeys the following probability distribution function:

\[
p(v^\nu_i = v) = p_{s_\nu}(v) = \frac{1}{c \cdot \Gamma(s_\nu)} \left( \frac{v}{c} \right)^{s_\nu-1} \exp\left( -\frac{v}{c} \right).
\]

The expectation value of \( v^\nu_i \) is

\[
< v^\nu_i >_\mu = \int_0^\infty p_{s_\nu}(v)vdv = c \cdot s_\nu.
\]

If we set \( c \), it is possible to estimate \( s_\nu \) of the horses in category \( \nu \) as \( s_\nu = v^\nu/c \).

Figure 2 shows the probability distribution functions \( p(v) \) of \( v^\mu \). In the same figure, we show the result of fitting with the gamma probability functions. Using the least square method in the range \( v \in [0.01, 1.0] \), we set \( c = 0.12 \) and \( s_{\text{Win}} = 1.659, s_{\text{2nd}} = 1.258 \) and \( s_{\text{Lose}} = 0.529 \). Comparing with the values in the fourth column in Table I, it is observed the values of \( s_\nu \) and \( v^\nu/c \) are close to each other in all categories, implying that the bulk shapes of the
Fig. 2. Logarithmic plot of probability distribution functions $p(v)$ of shares of votes. The curves from the top to bottom indicate the data for $\nu = \text{Win}$ (solid), 2nd (dashed) and Lose (dotted). The gamma distribution functions with $c = 0.12$ and $s_\nu$ are also plotted (chain lines).

probability functions of $v'\nu$ are well described by the gamma distributions. We also notice clear discrepancies in the figure. $p(v)$ does not obey the gamma distribution for the larger shares. The bulk shape of $p(v)$ is not crucial in our argument, because we are interested in the critical properties, or small win bet fraction regime. We think the discrepancies come from that the voters' confidence $s_\mu$ has some variance.

We study the cumulative functions $1 - x_\mu$ in the small share region, $v \to 0$. Figure 3 shows the cumulative functions $D(v)$ of $v'_\mu$, which is defined as

$$D(v) \equiv \int_0^v p(v)dv. \quad (25)$$

We are interested in the power law behaviour of $D(v) \propto v^{s_\nu}$ and the figure shows the double logarithmic plot. We see that they do not obey the power law, as have been predicted in eq.(20). In the figure, we show the result of fitting result with $(v - v_c)^{s_\nu}$. We set $v_c = 0.0014$ and the figure shows that the winning and finishing second horses's $D(v)$ obey the power law with cut-off $v_c$. On the other hand, about the losing horses, the fitting only works for the region $D(v) \geq D_c \equiv 0.003$ and $v \geq v_c$. The reason why $D(v)$ does not obey the power law is not clear. We think that there are some voters who want to vote to the horses with remarkable small shares. The odds are very large and for the voters, the horses look very attractive. If so, we can understand the existence of the cut-off $v_c$.

We study the mixing properties of the binary horses by employing the method explained
Fig. 3. Double-logarithmic plot of cumulative distribution functions $D(v)$ of shares of votes. The curves from the top to bottom indicate the data for $\nu = \text{Lose}, \text{2nd}$ and $\text{Win}$. The fitted functions $(v - v_c)^{\alpha}$ are also plotted. We set $v_c = 0.0014$.

Fig. 4. Double-logarithmic plot of ROC curves $(1 - x_0, 1 - x_1)$. The curves of the Win-Lose pair (solid line) and the Win-2nd pair (dashed line) are plotted. The fitting curves given by $1 - x_1 = a \cdot (1 - x_0)^\alpha$ (dash-dotted line) are also plotted.
Table II. The initial value $s_\mu, s'_\mu$ in each category and the critical exponent $\alpha$ are plotted. In the last two column, we show the predicted values of $\alpha$ by the voting model.

| Pair          | $s_1$ | $s_0$ | $s'_1$ | $s'_0$ | $\alpha$ | $s_1/s_0$ | $s'_1/s'_0$ |
|--------------|-------|-------|--------|--------|----------|-----------|------------|
| Win vs Lose  | 1.659 | 0.529 | 2.03   | 1.15   | 1.81     | 3.134     | 1.765      |
| Win vs 2nd   | 1.659 | 1.258 | 2.03   | 1.86   | 1.12     | 1.318     | 1.091      |

in the text. We adopt the Win-Lose pair and Win-2nd pair as the binary pairs. Figure 4 shows the double-logarithmic plot of the ROC curve $(1 - x_0, 1 - x_1)$ for the two pairs. The plots show scale-invariant behaviour over the wide range of $1 - x_1$. In the case of the Win-2nd pair, scale invariance holds over the range $10^{-5} < 1 - x_1 < 10^{-1}$, which can be anticipated from the bahaviour of $D(v)$. About the Win-Lose pair, the range is restricted for $1 - x_0 \geq D_c = 0.003$. In order to see the scale invariance for the region $1 - x_0 \leq D_c$, many more results of the races ($N^{Win} \sim 10^6$) are necessary. Using the least square method in the range $0 \leq 1 - x_0 \leq 0.1$, we estimate the critical exponent $\alpha$. The values of the parameters and other data are summarized in Table II. The estimated values of $\alpha$ are considerably different from those predicted by the model; i.e. $s_1/s_0$. In the table, we also show the values $s'_\mu$ estimated by fitting $D(v)$ with $(v - v_c)^{s'_\mu}$. The estimated values of $\alpha$ are closer than the values from the bulk values $s_\mu$.

5. Concluding Remarks

In this study, we have introduced a simple voting model in order to discuss the mixing of binary candidates with initial number of votes $s_0$ and $s_1$. As the voting process proceeds, the candidates are mixed in the space of the share of votes, $u$. We have shown that the probability distribution of $u$ of a candidate $\mu$ obeys a gamma distribution function with the shape exponent $s_\mu$ in the thermodynamic limit $Z_0 \to 0$. The joint probability distribution of $k$ different candidates is given as the direct product of the gamma distributions. The mixing configuration of the binary candidates exhibits scale invariance in the small $u$ region. In particular, in the double scaling limit $Z_0 \to \infty$ and $\{s_\mu\} \to 0$ with $\alpha = s_1/s_0$ fixed, the scale invariance holds over the entire range. The cumulative function of the binary candidates obeys $1 - x_1 = (1 - x_0)^{\alpha}$ for $0 \leq x_0, x_1 \leq 1$.

The data on horse races obtained from JRA also show that scale invariance holds over the wide range of cumulative functions. The distribution functions of the share of votes, $u$, are to some extent described by the gamma distribution functions, implying that the behaviour of betters is described by the voting model. However a clear discrepancy is observed in the critical behaviour. The bulk properties of the probability function $p(v)$ and the critical properties of the cumulative functions $D(v)$ should be discussed separately. Although our voting model describes the mechanism of scale invariance in the mixing of binary candidates, it may be too simple to describe the behaviour of betters in real cases. Thus far, dividends have been
reported to exhibit power-law behaviour. Another betting model has been proposed in\textsuperscript{12,13) A detailed study of real data, in particular the time series of the number of votes, should clarify the mechanism of scale invariance in betting systems.\textsuperscript{14) We also note that our model is related to the random Young diagram problem.\textsuperscript{15) This problem pertains to the probabilistic growth of a Young diagram. A parabolic shape\textsuperscript{16) and a quadrant shape\textsuperscript{17) have been obtained for the asymptotic shape. The complementary part of the ROC curve, which is embedded in the fourth quadrant, corresponds the Young diagram. In our model, the ROC curve \((x_0(t), x_1(t))\) given by (15) describes the asymptotic shape of the Young diagram. In particular, it is described by the relation \(1 - x_1 = (1 - x_0)\alpha\) in the double scaling limit. Figure 5 shows the correspondence between the voting model and the random Young diagram problem. As the voting process proceeds, the order of the binary candidates and the Young diagram change.

It is also possible to study the voting model with many categories of candidates with the usage of many different initial values \(\{s_\mu\}\). \(u\) of the candidates in each category becomes a gamma distributed random variable. Scale invariance does hold between any pair of categories. Figure 6 shows the triple-logarithmic plot of the cumulative functions of the winning \((x_{1st})\), finishing second \((x_{2nd})\) and finishing third \((x_{3rd})\) horses. In the linear part of the curve, scale invariance holds between any pair of categories.

Acknowledgment

We thank Dr. Emmanuel Guitter for his useful discussions on the branching process. We also thank Takumi Nakaso for helping us with the data analysis of horse races. This work was supported by Grant-in-Aid for Challenging Exploratory Research 21654054.
Fig. 6. Triple-logarithmic plot of ROC curve \( (1 - x_{1st}, 1 - x_{2nd}, 1 - x_{3rd}) \). \( x_\nu \) denotes the cumulative function of the horses finishing 1st, 2nd and 3rd.
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Appendix A: Joint probability distribution function

We start from the expression of the joint probability function given by

\[ \text{Prob}(\{X_{ij,T}^{\mu_j} - s_{\mu_j} = n_j\}_{j=1,\ldots,k}) = \frac{T!}{(Z_0)_T} \prod_{j=1}^{k+1} \left[ \frac{(s_{\mu_j})_{n_j}}{n_j!} \right]. \]  \hspace{1cm} (A-1)\]

Here, \( s_{\mu_{k+1}} = Z_0 - \sum_{j=1}^{k} s_{\mu_j} \) and \( n_{k+1} = T - \sum_{j=1}^{k} n_j \). Using the Dirichlet distribution function, we can rewrite the expression as

\[ \text{Prob}(\{X_{ij,T}^{\mu_j} - s_{\mu_j} = n_j\}_{j=1,\ldots,k}) = \frac{T!}{\prod_{i=1}^{k+1} n_i} \prod_{i=1}^{k} \left[ \int_0^{1-\sum_{j=1}^{i-1} p_j} dp_i \right] \prod_{i=1}^{k+1} \left[ \frac{p_{n_i} + s_{\mu_i} - 1}{\Gamma(s_{\mu_i})} \right] \Gamma(Z_0). \]  \hspace{1cm} (A-2)\]

The expectation value of \( \Delta X_{ij,t}^{\mu_j} = X_{ij,t+1}^{\mu_j} - X_{ij,t}^{\mu_j} \) is given by

\[ p_{\mu_j} = \langle \Delta X_{ij,t}^{\mu_j} \rangle = \frac{s_{\mu_j}}{Z_0}. \]  \hspace{1cm} (A-3)\]
The correlation between $\Delta X_{i,j,t}$ and $\Delta X_{i,k,t}$ ($k \neq j$) is given as
\[ \rho_{\mu_j,\mu_k} = -\sqrt{\frac{s_{\mu_j} s_{\mu_k}}{(1 - \frac{s_{\mu_j}}{Z_0})(1 - \frac{s_{\mu_k}}{Z_0}) Z_0 (1 + Z_0)}}. \quad (A-4) \]

By changing the integral variables from $\{p_i\}_{i=1,\ldots,k}$ to $\{h_i\}_{i=1,\ldots,k}$ as $p_i = (1 - \sum_{j=1}^{i-1} p_j) h_i = \prod_{j=1}^{i-1} (1 - h_j) h_i$, we obtain
\[
\text{Prob}(\{X_{i,j,T}^\mu - s_{\mu_j} = n_j\}_{j=1,\ldots,k}) = \frac{\Gamma(Z_0)}{\Gamma(s_{\mu_{k+1}})} \prod_{i=1}^{k} \left[ \frac{1}{\Gamma(s_{\mu_i})} \left( T - \sum_{j=1}^{i-1} n_j \right) C_{n_i} \right] \int_0^1 dh_i h_i^{n_i + s_{\mu_i} - 1} (1 - h_i)^{T - \sum_{j=1}^{i-1} n_j + Z_0 - \sum_{j=1}^{i-1} s_{\mu_j} - 1} \right]. \quad (A-5)
\]

We focus on the share of votes of candidates in the limit $T \to \infty$. We introduce $y_i$ as $n_i = (T - \sum_{j=1}^{i-1} n_j) y_i = T \prod_{j=1}^{i-1} (1 - y_j) y_i$ and define the joint distribution function as
\[
P(\{y_j\}_{j=1,\ldots,k}) \equiv \lim_{T \to \infty} \text{Prob}(\{X_{i,j,T}^\mu - s_{\mu_j} = n_j\}_{j=1,\ldots,k}) = T \prod_{i=1}^{k} (1 - y_i) y_i \prod_{i=1}^{k} (T - \sum_{j=1}^{i-1} n_j). \quad (A-6)
\]

The joint function $P(\{y_j\}_{j=1,\ldots,k})$ is given by
\[
P(\{y_j\}_{j=1,\ldots,k}) = \frac{\Gamma(Z_0)}{\Gamma(s_{\mu_{k+1}})} \prod_{i=1}^{k} \left[ \frac{1}{\Gamma(s_{\mu_i})} y_i^{s_{\mu_i} - 1} (1 - y_i)^{Z_0 - \sum_{j=1}^{i-1} s_{\mu_j} - 1} \right]. \quad (A-7)
\]

We introduce the variable $x_i$ as $x_i = (1 - \sum_{j=1}^{i-1} x_j) y_i$, which is related to $n_i$ as $n_i = T \cdot x_i$. The joint probability function $P(\{x_j\}_{j=1,\ldots,k})$ is then given as
\[
P(\{x_j\}_{j=1,\ldots,k}) = \frac{\Gamma(Z_0)}{\Gamma(s_{\mu_{k+1}})} \prod_{i=1}^{k} \left[ \frac{1}{\Gamma(s_{\mu_i})} x_i^{s_{\mu_i} - 1} \right] (1 - \sum_{j=1}^{k} x_j)^{s_{\mu_{k+1}} - 1}. \quad (A-8)
\]

Finally, we introduce the variable $\{u_i\}$ as $u_i \equiv (s_{\mu_{k+1}} - 1) x_i$. In the thermodynamic limit $Z_0, s_{\mu_{k+1}} \to \infty$, we obtain
\[
P(\{u_j\}_{j=1,\ldots,k}) = \prod_{j=1}^{k} \left[ \frac{e^{-u_j}}{\Gamma(s_{\mu_j})} u_j^{s_{\mu_j} - 1} \right] = \prod_{j=1}^{k} \rho_{\mu_j}(u_j). \quad (A-9)
\]

**Appendix B: Continuous time branching process**

We translate the discrete time voting problem $\{X_{i,t}^\mu\}_{i=1,\ldots,N_t}$ to a continuous time branching process $\{X_{i,t}^\mu(t)\}_{i=1,\ldots,N_t}$ because the latter is more tractable than the former.\(^9\) Figure B-1 shows the mapping process. Let $X_{i,t}^\mu(t)$ denote the number of offspring of $s_{\mu}$ individuals. Each individual is substituted by two offspring at its death (branching) and the probability that an individual dies during time $dt$ is given by $dt$. The number of offspring of each individual is denoted as $\{x_{i,k}^\mu(t)\}_{k=1,\ldots,s_{\mu}}$.
\[
X_{i,t}^\mu(t) = \sum_{k=1}^{s_{\mu}} x_{i,k}^\mu(t), \quad x_{i,k}^\mu(0) = 1. \quad (B-1)
\]
The substitution of the individuals by two offspring corresponds to the process of getting a vote. The frequency of deaths or the probability of getting another vote is proportional to $X_i^\mu(t)$. This relation is the same as that in the discrete time voting model. The counts of voting, $t$, corresponds to the counts of branchings. If branching takes place $t$ times up to $t'$, the following relation holds.

$$X_i^\mu(t') = X_i^\mu_{i,t}$$

Fig. B-1. Mapping voting model to branching process. The left-hand side figure shows a voting process with $N_1 = N_0 = 2$. ○ represents candidate belonging to category $\mu = 1$, $s_1 = 2$ and × represents candidate belonging to category $\mu = 0$, $s_0 = 1$. The right-hand side figure shows the corresponding branching process. ● represents the initial individual and offspring. Candidate belonging to category $\mu = 1$ (0) is composed of two individuals (one individual).

The expectation values $\langle x_{i,k}^\mu(t) \rangle$ and $\langle X_i^\mu(t) \rangle$ increase with $e^t$. Next, we introduce the scaled variables $U_i^\mu(t)$ and $u_{i,k}^\mu(t)$ as

$$U_i^\mu(t) \equiv e^{-t}X_i^\mu(t) \quad \text{and} \quad u_{i,k}^\mu(t) \equiv e^{-t}x_{i,k}^\mu(t). \quad \text{(B-2)}$$

We focus on the following probability distributions:

$$p_{s_i}(u)du \equiv \lim_{t \to \infty} \text{Prob}(u \leq U_i^\mu(t) \leq u + du) \quad \text{(B-3)}$$

$$p(u)du \equiv \lim_{t \to \infty} \text{Prob}(u \leq u_{i,k}^\mu(t) \leq u + du). \quad \text{(B-4)}$$

In order to obtain $p(u)$, we consider the situation in which an individual splits at $t = \tau$ for the first time. The resulting two offspring continue the branching process. The scaled number of offspring of the individual is denoted as $u$. Those of the two offspring are denoted as $u_1$ and $u_2$. Figure B-2 gives a pictorial representation of the relation among $u$, $u_1$ and $u_2$. We observe
Fig. B·2. Pictorial representation of self-consistent relation among \(u, u_1\) and \(u_2\). An individual splits at \(t = \tau\) for the first time producing two offspring appears. Because of the time lag \(\tau\), the relation \(u = (u_1 + u_2)e^{-\tau}\) holds.

that these variables satisfy the following relation:

\[ u = (u_1 + u_2)e^{-\tau}. \]

Furthermore, \(u_1\) and \(u_2\) obey the same probability distribution as that obeyed by \(u\), and the probability that an individual splits for the first time during \(\tau \leq t \leq \tau + d\tau\) is \(e^{-\tau}d\tau\). Thus, we obtain

\[ p(u) = \int_0^\infty e^{-\tau}d\tau \int_0^\infty du_1 \int_0^\infty du_2 p(u_1)p(u_2)\delta(u - (u_1 + u_2)e^{-\tau}). \]  
(B·5)

Introducing \(X = e^{-\tau}\), the relation is rewritten as

\[ p(u) = \int_0^1 dX \int_0^\infty du_1 \int_0^\infty du_2 p(u_1)p(u_2)\delta(u - (u_1 + u_2)X). \]  
(B·6)

Using the Laplace transform of \(p(u)\), \(\hat{p}(s) \equiv \int_0^\infty p(u)e^{-su}du\), it can be shown that \(\hat{p}(s)\) satisfies the following integral equation:

\[ \hat{p}(s) = \frac{1}{s} \int_0^s \hat{p}(v)dv. \]  
(B·7)

Differentiating (B·7) with respect to \(s\), we obtain the following differential equation.

\[ s \frac{d\hat{p}(s)}{ds} = \hat{p}^2(s) - \hat{p}(s). \]  
(B·8)

(B·8) can be solved easily to obtain

\[ \hat{p}(s) = \frac{1}{1 + as}. \]  
(B·9)
Using the normalization condition $<u> = 1$ and the inverse Laplace transform, we get

$$p(u) = e^{-u}. \quad (B-10)$$

We obtain $p_{s_{\mu}}(u)$ by convolution as

$$p_{s_{\mu}} = \prod_{i=1}^{s_{\mu}} \left[ \int_{0}^{\infty} du_i p(u_i) \right] \delta(u - \sum_{i=1}^{s_{\mu}} u_i)$$

$$= \frac{1}{\Gamma(s_{\mu})} u^{s_{\mu}-1} e^{-u}. \quad (B-11)$$

$U_{i}^{\mu}$ obeys a gamma distribution with the shape exponent $s_{\mu}$ given by (10). We note that the result (10) is derived in the thermodynamic limit, where the correlation among $\{u_j\}_{j=1,\ldots,k}$ vanishes. On the other hand, in the continuous time branching process, the splitting processes of each individual and offspring are independent of each other. As a result, we obtain the gamma distribution which appears in the voting model in the thermodynamic limit.