Coupled local/nonlocal models in thin domains

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Abstract

In this paper, we analyze a model composed by coupled local and nonlocal diffusion equations acting in different subdomains. We consider the limit case when one of the subdomains is thin in one direction (it is concentrated to a domain of smaller dimension) and as a limit problem we obtain coupling between local and nonlocal equations acting in domains of different dimension. We find existence and uniqueness of solutions and we prove several qualitative properties (like conservation of mass and convergence to the mean value of the initial condition as time goes to infinity).

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1 Introduction and main results

In this paper we combine a local diffusion equation, the classical heat equation,

\[ \frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) \]  

(1.1)

in a higher dimensional domain \( \Omega \subset \mathbb{R}^N \), with a nonlocal diffusion equation, given by an integrable kernel

\[ \frac{\partial u}{\partial t}(x,t) = \int_{R} J(x-y)(u(y,t) - u(x,t))dy \]  

(1.2)

in \( R \) a different subset of \( \mathbb{R}^N \). Associated with these two domains, \( \Omega \) and \( R \), in [23] and [29] the following kind of energy functional was introduced

\[ E(u,v) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4} \int_{R} \int_{R} J(x-y)(v(y-t) - v(x-t))^2 dydx + \frac{1}{2} \int_{R} \int_{A} G(x-y)(v(x) - u(y))^2 dydx. \]  

(1.3)

Here the set \( A \subset \Omega \) is the whole \( \Omega \) (and we will refer to the resulting model as having a coupling in the source terms, see the next subsection) or a part of the boundary \( A = \Gamma \subset \partial \Omega \) (we refer to this case as coupling at the boundary).

Observe that, the kernels \( J \) and \( G \) do not need to be equal. We will assume that \( J \) and also \( G \) satisfy the following hypotheses that will be assumed along the whole paper without further mention,

\[ J \in C(\mathbb{R}^N, \mathbb{R}) \text{ is nonnegative, with } J(0) > 0, J(-x) = J(x) \text{ for every } x \in \mathbb{R}^N, \text{ and integrable,} \]

\[ G \in C(\mathbb{R}^N, \mathbb{R}) \text{ is nonnegative, nontrivial and integrable.} \]

Remark 1. We can also consider kernels that are not in convolution form, that is, \( J(x,y) \) and \( G(x,y) \) with \( J \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}) \) nonnegative, with \( J(x,x) > 0 \), symmetric \( J(x,y) = J(y,x) \) and integrable, and \( G \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}) \) nonnegative, nontrivial and integrable. To simplify the presentation we will deal with convolution type kernels in the proofs.
Observe that it is common to assume that the integral of $J$ and $G$ is equal to one. This assumption is related to the probabilistic interpretation of the model given in [23] and [29]. For example, in this interpretation, $G((x_1, x_2), y)$ is the probability of a particle (or an individual of a biological species) that is at $(x_1, x_2)$ jumps to $y$ in a time step. So, in this case, we have

$$\int_{R} G(x_1, x_2, y) dy = 1.$$ 

To obtain our results we only need the integrability of the kernels, hence we do not assume that they are normalized to have integral equal to one.

Associated with the energy (1.3) we have the evolution problem given by its gradient flow (with respect to $L^2(\Omega \cup R)$). This gives rise to an diffusion problem. Take $(u, v)$ as the solution of the abstract ODE problem

$$(u, v)'(t) = -\partial E \ [[(u, v)(t)], \quad t \geq 0,$$

with $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$ and $\partial E \ [(u, v)]$ the subdifferential of $E$. Then, it turns out (see [23] and [29]) that $(u, v)$ solves a system composed by a heat equation (local diffusion) of the form (1.1) in $\Omega$ and a nonlocal diffusion equation in $R$, (1.2), coupled via source terms in the equations (when $A = \Omega$ in (1.2)) or via a boundary flux on $\Gamma \subset \partial \Omega$ (when $A = \Gamma$ in (1.2)). See Sections 1.1 and 1.2 below.

Also from [23] and [29] we know that the associated evolution problem is well-posed in the sense that there are existence and uniqueness of solutions. There are two alternative proofs of this fact. The first one uses a fixed point argument while the second relies on semigroup theory. Besides, a comparison principle holds. Also, the total mass of the initial condition is preserved along the evolution and the solutions converge exponentially fast to the mean value of the initial condition. Notice that, according to [23] and [29], we do not impose any continuity of the densities throughout the interface between the local and nonlocal domain, but we can guarantee continuity of the densities $u$ and $v$ inside the local and nonlocal subdomains $\Omega$ and $R$, respectively, by assuming continuity of the initial conditions. Also there is a probabilistic interpretation of this model (we refer one more time to [23] and [29]). In this interpretation individuals cannot diffuse neither jump from the exterior $\mathbb{R}^N \setminus \Omega$ into $\Omega$ or the other way around (the integrals accounting for jumps do not consider the complement of $\Omega$). There is no interchange of mass between $\Omega \cup R$ and its complement. Therefore, the total mass is preserved and we can call our problem as being of Neumann type.

The study of nonlocal problems with smooth kernels has been widely considered recently, see [6, 7, 8, 9, 11, 14, 20, 21, 22] and the book [1]. This kind of equation is getting attention due to its potential applications in ecology, physics, and engineering, and to its flexibility to accurately capture effects that are not easily obtained from classical local models. Biological mobility models of animals and plants are examples of how distinct patterns of mobility can affect the success of invasions [7, 34]. In epidemiology, the effects of long-range interactions are responsible for the spreading of diseases around the world [36]. Nonlocal patterns also play an important role in molecular interactions in dissimilar interfaces, continuum mechanics, [24, 30], and peridynamics (a model of elasticity and mechanics), [31, 32].

There are different strategies for couplings between local and nonlocal models. Let us briefly summarize previous results in [15, 18, 22, 23, 24, 29], see also the review [17]. In [15], local and nonlocal problems are coupled through a prescribed solid region in which both kinds of equations overlap (the value of the solution in the nonlocal part of the domain is used as a Dirichlet boundary condition for the local part and vice-versa). This kind of coupling gives continuity of the solution in the overlapping region but does not preserve the total mass. Here we follow [23] and [29] (see also [22, 26]). In probabilistic terms, in the model described in [23], particles may jump across the interface between the two regions but can not pass coming from the local side unless they jump. Finally, in [29], the authors studied local and nonlocal diffusion models in different zones coupled via the fluxes across the surface that separates the two regions.

Here, we take as the nonlocal region a thin domain, that is, we consider $R_\varepsilon \subset \mathbb{R}^N$ ($R_\varepsilon$ is assumed to be open and bounded), depending on a small parameter $\varepsilon \in (0, 1]$ that will go to zero and that measures the thickness of the domain. Therefore, in our model problem we have two full dimensional domains, the local domain $\Omega \subset \mathbb{R}^N$ (that is fixed) and the nonlocal domain $R_\varepsilon \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. We denote $x = (x_1, x_2)$ a point in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. The domain $R_\varepsilon$ is assumed to be a general thin domain defined as

$$R_\varepsilon = \{(x_1, \varepsilon x_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : (x_1, x_2) \in R\}.$$
with $R \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. Notice that $R_\varepsilon$ is a domain that is thin in the $x_2$-variable. See Figure 1.

Our main goal here is to pass to the limit as $\varepsilon \to 0$ in the previous setting and obtain a nontrivial diffusion model in which we couple local and nonlocal diffusion equations, (1.1) and (1.2) that take place in domains of different dimension (we deal here with local diffusion in the full-dimensional domain and nonlocal diffusion in the lower-dimensional one).

For simplicity, we will concentrate in the product case and take $R_\varepsilon$ as

$$R_\varepsilon = R_1 \times \varepsilon R_2 = \{(x_1, \varepsilon x_2) : x_1 \in R_1, x_2 \in R_2\}.$$  

Our results are valid in a more general setting (see Remark 2 below) but we prefer to avoid extra notations and simplify the changes of variables that are needed in the proofs. The typical configuration under study is depicted in Figure 1.

**Remark 2.** Instead of a thin domain like $R_\varepsilon = \{(x_1, \varepsilon x_2) : x_1 \in R_1, x_2 \in R_2\}$, we could have a more complex domain, which could be described by some function $g$ related to the geometry of the channel $R_\varepsilon$, more exactly, on the way the channel $R_\varepsilon$ collapses to a general manifold $R_1$. If we want to construct a more general geometry of the channel we could, for instance, in two dimensions, consider the channel $R_\varepsilon = \{(x, y) : 0 < x_1 < 1, 0 < x_2 < \varepsilon g(x_1)\}$, although more general and complicated geometries are allowed, see [2].

**Figure 1:** Perturbed domain.

**Figure 2:** Limit domain.

**Main goal.** Let $\Omega_\varepsilon = \Omega \cup R_\varepsilon \subset \mathbb{R}^N$ and consider a local/nonlocal coupling in this domain (see subsections 1.1 and 1.2 for a precise statement of the involved equations and the obtained results). As we have mentioned, our main goal is to study the limit as the nonlocal region, $R_\varepsilon$, gets thinner, that is, to study the limit as $\varepsilon \to 0$. When passing the limit as $\varepsilon \to 0$, the "limit" domain, $\Omega_0$ (see figure 2) will be the union of $\Omega$ and the lower dimensional domain $R_1$. In the limit of the solutions to our coupled models we will obtain solutions to a local equation in the domain $\Omega$ (with a nonlocal source) and a nonlocal equation in a domain of smaller dimension, $R_1$. After obtaining the limit equations, we will also prove some qualitative properties of this limit problem (like conservation of the total mass and study the asymptotic behaviour of the solutions).

Concerning references for equations in thin domains we refer to [2, 3, 1, 27, 5, 33] that develop some techniques and methods to understand the effects of the geometry of the thin domain on the solutions of elliptic and parabolic singular problems. We can find some applications in elastic beam theories (as torsion and warping functions) [28], lubrication [12], fluid flows as ocean dynamics, geophysical fluid dynamics, and fluid flows in cell membranes, see for instance [25].
Our results can be viewed as an extension of [2] and [27]. In [2], the authors investigate the dynamics of a local reaction-diffusion equation with homogeneous boundary condition in a dumbbell domain. The dumbbell domain is composed by two disconnected regions joined by a thin channel, that depends on a thickness parameter $\varepsilon$ and degenerates to a line segment as the parameter $\varepsilon \to 0$. As part of a series of articles (see [3, 4]) the authors also prove some properties about the continuity of the set of equilibria. On the other hand, in [27] the authors deal with nonlocal evolution problems with non singular kernels in thin domains obtaining a limit problem when the thickness of the domain goes to zero, but without considering any coupling with a local part of the problem. Passing to the limit in these coupling terms is the main contribution of this work.

1.1 Coupling using source terms

We need to compare the solutions of the problem posed in the perturbed domain $\Omega_\varepsilon = \Omega \cup R_\varepsilon \subset \mathbb{R}^N$ and the solutions to the limit problem in the limit domain $\Omega_0$. Since the solutions live in different spaces, to obtain convergence we need some care, not only in the choice of the functional space, but also with the metric chosen in this space. Decomposing a function $w \in L^2(\Omega_\varepsilon)$ as $w = u + v$, with $u = w\chi_\Omega$ and $v = w\chi_{R_\varepsilon}$, we define the metric in $L^2(\Omega_\varepsilon)$ as

$$
||w||^2_{L^2(\Omega_\varepsilon)} = \int_\Omega |u|^2 + \frac{1}{\varepsilon N_2} \int_{R_\varepsilon} |v|^2.
$$

(1.4)

Remark that we multiply the norm of the involved functions in the thin part of the domain $R_\varepsilon$ by a factor $\varepsilon^{-N_2}$.

Now, we can define the energy functional

$$
E_\varepsilon(u, v) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon N_2} \int_{R_\varepsilon} \int_{R_\varepsilon} J(x-y)|v(y)-v(x)|^2 dy dx + \frac{1}{\varepsilon N_2} \int_{R_\varepsilon} \int_\Omega G(x-y)|v(x)-u(y)|^2 dy dx,
$$

(1.5)

which is finite in

$$
\mathcal{B} := \{(u, v) \in L^2(\Omega_\varepsilon) : u \in H^1(\Omega), v \in L^2(R_\varepsilon)\}.
$$

Notice that in this energy functional we have two terms,

$$
\frac{1}{2} \int_\Omega |\nabla u|^2 dx \quad \text{and} \quad \frac{1}{\varepsilon N_2} \int_{R_\varepsilon} \int_{R_\varepsilon} J(x-y)(v(y)-v(x))^2 dy dx,
$$

that are naturally associated with the equations (1.1) and (1.2) plus a coupling term given by

$$
\frac{1}{\varepsilon N_2} \int_{R_\varepsilon} \int_\Omega G(x-y)(v(x)-u(y))^2 dy dx.
$$

Now, let us consider the evolution problem obtained as the gradient flow associated with this energy with respect to the norm previously defined in (1.3), that is, $(u(t), v(t))$ will be the solution of the abstract $ODE$ problem

$$(u, v)'(t) = -\partial E_\varepsilon[(u, v)(t)], \quad t \geq 0,$$

with initial data $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$. Here $\partial E[(u, v)]$ denotes the subdifferential of $E$ at the point $(u, v)$. To see what kind of equations we are solving here, let us compute the derivative of $E$ at $(u, v)$ in the direction of $\varphi \in C_0^\infty(\Omega_\varepsilon)$,

$$
\partial_\varphi E_\varepsilon(u, v) = \lim_{h \to 0} \frac{E_\varepsilon(u + h\varphi, v + h\varphi) - E_\varepsilon(u, v)}{h}
$$

$$
= \int_\Omega \nabla u \nabla \varphi dx + \frac{1}{\varepsilon N_2} \int_{R_\varepsilon} \frac{1}{2\varepsilon N_2} \int_{R_\varepsilon} J(x-y)(v(y)-v(x))(\varphi(y) - \varphi(x)) dy dx
$$

$$
+ \frac{1}{\varepsilon N_2} \int_{R_\varepsilon} \int_\Omega G(x-y)(v-v(u))(\varphi) dx dy - \int_\Omega \frac{1}{\varepsilon N_2} \int_{R_\varepsilon} G(x-y)(v(y)-u(x)) \varphi dx dy.
$$
Since \( \langle \partial E[u, v], \varphi \rangle = \partial_x E(u, v) \), we can derive the local/nonlocal problem associated to this gradient flow that is given by the following system of equations:

\[
\begin{aligned}
&\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + \frac{1}{\varepsilon^2} \int_{R_\varepsilon} G(x - y)(v(y, t) - u(x, t))dy, \quad (x, t) \in \Omega \times (0, +\infty), \\
&\frac{\partial u}{\partial \eta}(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, +\infty), \\
&\frac{\partial v}{\partial t}(x, t) = \frac{1}{\varepsilon} \int_{R_\varepsilon} J(x - y) (v(y, t) - v(x, t)) dy - \int_{\Omega} G(x - y)(v(x, t) - u(y, t))dy, \quad (x, t) \in R_\varepsilon \times (0, +\infty), \\
&u(x, 0) = u_0(x), \quad x \in \Omega, \\
&v(x, 0) = v_0(x), \quad x \in R_\varepsilon.
\end{aligned}
\]

As we have mentioned previously, our aim is to pass to the limit as \( \varepsilon \rightarrow 0 \) in this evolution problem \((1.6)\). To introduce a candidate to be a limit problem for \((1.6)\), defined in the domain \( \Omega_0 \) (see Figure 2) we will perform a change of variables (as described in [27]) in the thin domain, \( R_\varepsilon \), in order to fix it. The change of variables is given by

\[ R = R_1 \times R_2 \ni (x_1, x_2) \longmapsto (x_1, \varepsilon x_2) \in R_1 \times \varepsilon R_2 = R_\varepsilon. \]

That is, we take \( \tilde{x}_2 = \frac{x_2}{\varepsilon} \) and \( \tilde{y}_2 = \frac{y_2}{\varepsilon} \). With these variables we can fix the domain which allows us to analyze the asymptotic behavior as \( \varepsilon \rightarrow 0 \) in a fixed space of functions. To fix the initial condition for \( v \) after the change of variables, we take \( v(x, 0) = v_0^\varepsilon(x_1, x_2) = v_0(x_1, \tilde{x}_2) \) for some fixed function \( v_0 \). The problem \((1.6)\) becomes after this change of variables the following equations in the fixed domain \( \tilde{\Omega} = \Omega \cup R \):

\[
\begin{aligned}
&\frac{\partial u^\varepsilon}{\partial t}(x, t) = \Delta u^\varepsilon(x, t) + \int_{R} G_\varepsilon(x - y)(v^\varepsilon(y, t) - u(x, t))dy, \quad (x, t) \in \Omega \times (0, +\infty), \\
&\frac{\partial u^\varepsilon}{\partial \eta}(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, +\infty), \\
&\frac{\partial v^\varepsilon}{\partial t}(\tilde{x}, t) = \int_{R} J_\varepsilon(x - y) (v^\varepsilon(y, t) - v^\varepsilon(\tilde{x}, t)) dy - \int_{\tilde{\Omega}} G_\varepsilon(x - y)(v^\varepsilon(\tilde{x}, t) - u^\varepsilon(y, t))dy, \quad (\tilde{x}, t) \in R \times (0, +\infty), \\
&u^\varepsilon(x, 0) = u_0(x), \quad x \in \Omega, \\
&v^\varepsilon(\tilde{x}, 0) = v_0(\tilde{x}), \quad \tilde{x} \in R,
\end{aligned}
\]

where

\[ J_\varepsilon(x - y) = J(x_1 - y_1, \varepsilon(\tilde{x}_2 - \tilde{y}_2)), \quad G_\varepsilon(x - y) = G(x_1 - y_1, \varepsilon \tilde{x}_2 - \tilde{y}_2) \]

and \( v^\varepsilon(x_1, \tilde{x}_2, t) = v(x_1, \varepsilon \tilde{x}_2, t) \).

Notice that the problem \((1.7)\) is similar to the ones obtained previously in thin domains (see for instance [27, 2]).

Now, we are ready to state our main result for this coupling.

**Theorem 1.1.** Let \( \{ (u^\varepsilon, v^\varepsilon) \}_{\varepsilon > 0} \) be a family of solutions of \((1.7)\). Then, there exist \( (u^*, V^*) \), \( u^* \in C([0, T], H^1(\Omega)) \) and \( V^* \in C([0, T], L^2(R_1)) \), such that

\[
\begin{aligned}
&u^\varepsilon \rightarrow u^* \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)), \\
v^\varepsilon \rightarrow v^* \quad \text{in} \quad L^\infty(0, T; L^2(R_1)) \quad \text{and}, \\
V^*(\cdot) = \int_{R_2} v^*(\cdot, \varepsilon \tilde{x}_2, t)d\tilde{x}_2 \rightarrow V^*(\cdot) = \int_{R_2} v^*(\cdot, 0, t)d\tilde{x}_2 \quad \text{in} \quad L^\infty(0, T; L^2(R_1)).
\end{aligned}
\]
The pair \( \{u^*, V^*\} \) satisfies the following limit problem in \( \Omega_0 = \Omega \cup R_1 \),

\[
\begin{align*}
\frac{\partial u^*}{\partial t}(x,t) &= \Delta u^*(x,t) + \int_{R_1} G^*(x-y)(V^*(y,t) - |R_2|u^*(x,t))dy, \quad (x,t) \in \Omega \times (0, +\infty), \\
\frac{\partial u^*}{\partial n}(x,t) &= 0, \quad (x,t) \in \partial \Omega \times (0, +\infty), \\
\frac{\partial V^*}{\partial t}(x,t) &= |R_2| \int_{R_1} J^*(x-y) (V^*(y,t) - V^*(x,t)) dy - \int_{\Omega} G^*(x-y)(V^*(x,t) - |R_2|u^*(y,t))dy, \\
\quad (x,t) \in R_1 \times (0, +\infty), \\
u^*(x,0) &= u_0(x), \quad x \in \Omega, \\
V^*(x,0) &= V_0^* = \int_{R_2} v_0(x,0) dx_2, \quad x \in R_1,
\end{align*}
\]

where the limit kernels \( J^* \) and \( G^* \) are given by

\[
J^*(x-y) = J(x_1 - y_1,0), \quad G^*(x-y) = G(x_1 - y_1,0 - y_2).
\]

We also include here some properties of the limit problem (1.8). The problem is well posed, the total mass remains constant in time, that is,

\[
\int_{\Omega} u^*(x,t) dx + \int_{R_1} V^*(x,t) dx = \int_{\Omega} u_0^* dx + \int_{R_1} V_0^* dx, \quad \forall t \geq 0,
\]

and solutions converge exponentially to the mean value of the initial condition as \( t \to \infty \), i.e.,

\[
\left\| (u^*, V^*)(\cdot,t) - \int (u_0^*, V_0^*) \right\|_{L^2(\Omega_0)} \leq C e^{-\lambda_1 t}.
\]

for some \( C > 0 \) and \( \lambda_1 > 0 \) (we also obtain that \( \lambda_1 \) can be chosen independent of the initial data).

### 1.2 Coupling at the boundary

Now we want to impose that an individual to pass from the nonlocal domain to the local domain, it necessarily needs to cross the boundary to then get in the local domain. As we did before, first we will define the problem in the perturbed domain (see Figure 1.2) and then derive the limit problem defined in the limit domain (see Figure 2).

Let us, as before, consider the domain \( \Omega_\varepsilon = \Omega \cup R_\varepsilon \subset \mathbb{R}^N \), with \( \Omega \subset \mathbb{R}^N \) and \( R_\varepsilon = R_1 \times \varepsilon R_2 \subset \mathbb{R}^N \), with a small parameter \( \varepsilon \).

Let us consider the metric (1.4) and derive the evolution problem as the flux associated with the energy

\[
E^\varepsilon(u,v) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\varepsilon^2 N_2} \int_{R_1} \int_{R_\varepsilon} J(x-y) (v(y) - v(x))^2 dxdy + \frac{1}{2\varepsilon N_2} \int_{R_\varepsilon} \int_{\Gamma} G(x-y) (v(x) - u(y))^2 d\sigma(y)dx,
\]

with \( \Gamma \) a fixed part of the boundary of \( \Omega \) with \( |\Gamma|_{N-1} > 0 \) (then we have a well defined trace operator from \( H^1(\Omega) \) into \( L^2(\Gamma) \)). Notice that the coupling term

\[
\frac{1}{2\varepsilon N_2} \int_{R_\varepsilon} \int_{\Gamma} G(x-y) (v(x) - u(y))^2 d\sigma(y)dx
\]

involve the values of \( u \) on \( \Gamma \subset \partial \Omega \) instead of the values of \( u \) inside \( \Omega \) (compare with the previous functional \( E_\varepsilon(u,v) \)).
Now, the evolution problem associated to the energy functional $E^\varepsilon(u,v)$ is given by the following system:

\[
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \Delta u(x,t), \quad (x,t) \in \Omega \times (0, +\infty), \\
\frac{\partial u}{\partial \eta}(x,t) &= 0, \quad (x,t) \in \partial \Omega \setminus \Gamma \times (0, +\infty), \\
\frac{\partial u}{\partial \eta}(x,t) &= \frac{1}{\varepsilon^N} \int_{\Gamma} G(x-y)(v(y,t) - u(x,t))d\sigma(y), \quad (x,t) \in \Gamma \times (0, +\infty), \\
\frac{\partial v}{\partial t}(x,t) &= \frac{1}{\varepsilon^N} \int_{\Omega} J(x-y) (v(x,t) - v(y,t)) dy - \int_{\Gamma} G(x-y)(v(y,t) - u(y,t))d\sigma(y), \quad (x,t) \in R \times (0, +\infty) \\
u(x,0) &= u_0(x), \quad x \in \Omega, \\
v(x,0) &= v_0(x), \quad x \in R.
\end{aligned}
\] (1.10)

Notice that the nonlocal part contributes with the normal derivative of $u$ on $\Gamma$ and the local part of the problem appears as before in the source term of the equation for the nonlocal part. The coupling is balanced in such a way that the problem preserves the total mass, see [29].

After the same change of variables that we used before, $\tilde{x}_2 = \frac{x_2}{\varepsilon}$ and $\tilde{y}_2 = \frac{y_2}{\varepsilon}$, we fix the domain and then pass to the limit and obtain the limit problem. Again here we take $v(x,0) = v_0^\varepsilon(x_1, x_2) = v_0(x_1, \tilde{x}_2)$ for some fixed function $v_0$ as the initial condition. Notice that, as we did in the previous subsection, there exists an equivalence between the coupled local/nonlocal problem (1.10) with the following coupled local/nonlocal thin domain problem defined in $\hat{\Omega} = \Omega \cup R$, with $\Gamma$ a fixed part of the boundary of $\Omega$,

\[
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial t}(x,t) &= \Delta u^\varepsilon(x,t), \quad (x,t) \in \hat{\Omega} \times (0, +\infty), \\
\frac{\partial u^\varepsilon}{\partial \eta}(x,t) &= 0, \quad (x,t) \in \partial \hat{\Omega} \setminus \Gamma \times (0, +\infty), \\
\frac{\partial u^\varepsilon}{\partial \eta}(x,t) &= \int_{R} G^\varepsilon(x-y)(v^\varepsilon(y,t) - u^\varepsilon(x,t))dy, \quad (x,t) \in \Gamma \times (0, +\infty), \\
\frac{\partial v^\varepsilon}{\partial t}(\tilde{x},t) &= \int_{R} J^\varepsilon(x-y) (v^\varepsilon(y,t) - v^\varepsilon(\tilde{x},t)) dy - \int_{\Gamma} G^\varepsilon(x-y)(v^\varepsilon(\tilde{x},t) - u^\varepsilon(y,t))d\sigma(y), \quad (\tilde{x},t) \in R \times (0, +\infty) \\
u^\varepsilon(x,0) &= u_0(\tilde{x}), \quad x \in \Omega, \\
v^\varepsilon(\tilde{x},0) &= v_0(\tilde{x}), \quad \tilde{x} \in R,
\end{aligned}
\] (1.11)

with

$$J^\varepsilon(x-y) = J(x_1 - y_1, \varepsilon \tilde{x}_2 - \varepsilon \tilde{y}_2) \quad G^\varepsilon(x-y) = G(x_1 - y_1, x_2 - \varepsilon \tilde{y}_2) \quad \text{and} \quad v^\varepsilon(\tilde{x},t) = v(x_1, \varepsilon \tilde{x}_2, t).$$

Now we can enunciate a convergence result analogous to Theorem 1.2. It says that there is a limit as $\varepsilon \to 0$ of the solutions to the problem (1.11) in the limit domain $\Omega_0 = \Omega \cup R_1$ (see Figure 2).

**Theorem 1.2.** Let $\{v^\varepsilon, v^\varepsilon\}_{\varepsilon > 0}$ be a family of solutions for the problem (1.11). Then, there exists a solution $(u^*, V^*)$, $u^* \in C([0,T], H^1(\Omega))$ and $V^* \in C([0,T], L^2(R_1))$, such that

\[
\begin{aligned}
u^\varepsilon \rightharpoonup u^* \quad &\text{in} \quad L^\infty(0,T; L^2(\Omega)), \\
v^\varepsilon \rightharpoonup v^* \quad &\text{in} \quad L^\infty(0,T; L^2(R)) \quad \text{and,} \\
V^\varepsilon(\cdot) = \int_{R_2} v^\varepsilon(\cdot, \varepsilon \tilde{x}_2, t)d\tilde{x}_2 \rightharpoonup V^*(\cdot) = \int_{R_2} v^*(\cdot, 0, t)d\tilde{x}_2 \quad &\text{in} \quad L^\infty(0,T; L^2(R_1)).
\end{aligned}
\]
The pair \( \{ u^*, V^* \} \) satisfies the following limit problem in \( \Omega_0 = \Omega \times R_1 \),

\[
\begin{aligned}
\frac{\partial u^*}{\partial t}(x,t) &= \Delta u^*(x,t), \quad (x,t) \in \Omega \times (0, +\infty), \\
\frac{\partial u^*}{\partial \eta}(x,t) &= 0, \quad (x,t) \in \partial \Omega \setminus \Gamma \times (0, +\infty), \\
\frac{\partial u^*}{\partial \eta}(x,t) &= \int_{R_1} G^*(x-y)(V^*(y,t)-|R_2|u^*(x,t))dy, \quad (x,t) \in \Gamma \times (0, +\infty), \\
\frac{\partial V^*}{\partial t}(x,t) &= |R_2|\int_{R_1} J^*(x-y) (V^*(y,t)-V^*(x,t))dy - \int_{\Gamma} G^*(x-y)(V^*(x,t)-|R_2|u^*(y,t))d\sigma(y), \\
u^*(x,0) &= u_0^*(x), \quad x \in \Omega, \\
V^*(x,0) &= V_0^*(x) = \int_{R_2} v_0(x,0)dx_2, \quad x \in R_1, 
\end{aligned}
\]

where the limit kernels \( J^* \) and \( G^* \) are given by

\[
J^*(x-y) = J(x_1-y_1,0), \quad G^*(x-y) = G(x_1-y_1,x_2-0).
\]

For this limit problem we also have that it is well posed, the total mass remains constant in time and solutions converge exponentially to the mean value of the initial condition as \( t \to \infty \).

### 1.3 The local part in a thin domain

We can also consider the case in which the local part of the problem takes place in a thin domain (fixing the nonlocal domain). That is, we consider \( \Omega \subset \mathbb{R}^{N_1}, \Omega_2 \subset \mathbb{R}^{N_2} \), and can take \( \Omega_\varepsilon = \Omega_1 \times \varepsilon \Omega_2 \cup R_2 \) as our reference domain. In this case the associated energy takes the form

\[
E(u, v) := \frac{1}{2\varepsilon} \int_{\Omega_1 \times \varepsilon \Omega_2} |\nabla u|^2 dx + \frac{1}{4} \int_R \int_{\Omega_1 \times \varepsilon \Omega_2} J(x-y) (v(y) - v(x))^2 dy dx \\
+ \frac{1}{2\varepsilon} \int_R \int_{\Omega_1 \times \varepsilon \Omega_2} G(x-y) (v(x) - u(y))^2 dy dx.
\]

We can also consider the limit as \( \varepsilon \to 0 \) of solutions to the associated gradient flow in this case. In this case we obtain a limit problem in which the equation for \( u \) involves only the Laplacian in the first \( N_1 \)-variables and the coupling kernel is given by

\[
G^*(x-y) = G(x_1-y_1,0-y_2)
\]

the kernel \( J \) remains unchanged since we are fixing the nonlocal domain \( R_2 \). The proof of this limit can be obtained following \( [2, 3, 4] \) (notice that here we are taking the limit in the local part of the problem) and hence we don’t include the details in this paper.

The paper is organized as follows: in Section 2 we deal with the problem with coupling via source terms and we prove Theorem 1.1; in Section 3 we consider the coupling on the boundary and prove of Theorem 1.2; finally, in Section 4 we include some numerical experiments (based on a discretization of our models) that illustrate the behaviour of the solutions to our limit equations.

## 2 Coupling via source terms. Proof of Theorem 1.1

First, we introduce a result that will be important to study the large time behavior and the limit problems described in the previous section. We state the lemma for the first problem (coupling via source terms) but the same proof can be adapted for the other evolution problem (coupling on the boundary).
Let us denote by \( \hat{\Omega} = \Omega \cup R \) the fixed domain after the change of variables and by \( E \) the functional \( 1.5 \) after the change of variables, that is,

\[
E(u, v) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{4} \int_R \int_R J_\epsilon(x - y) |v(y) - v(x)|^2 dydx + \frac{1}{2} \int_R \int_\Omega G_\epsilon(x - y) |v(x) - u(y)|^2 dydx,
\]

with

\[
J_\epsilon(x - y) = J(x_1 - y_1, \epsilon(\tilde{x}_2 - \tilde{y}_2)), \quad G_\epsilon(x - y) = G(x_1 - y_1, \epsilon \tilde{x}_2 - \tilde{y}_2).
\]

**Lemma 2.1.** Let \( \{\lambda^\epsilon_n\}_{\epsilon > 0} \) be a family of first nontrivial eigenvalues of our evolution problem that are given by

\[
\lambda^\epsilon_n = \inf_{(u, v) : \int_\Omega u^\epsilon dx + \int_R v^\epsilon = 0} \frac{E(u^\epsilon, v^\epsilon)}{\int_\Omega (u^\epsilon)^2 + \int_R (v^\epsilon)^2}.
\]

Then, there exists a constant \( C > 0 \), that does not depends on \( \epsilon \) such that

\[
\lambda^\epsilon_n \geq C > 0,
\]

and hence we have,

\[
E(u^\epsilon, v^\epsilon) \geq C \left( \int_\Omega (u^\epsilon)^2 + \int_R (v^\epsilon)^2 \right),
\]

for every \( (u^\epsilon, v^\epsilon) \) solution to \( 1.7 \), such that \( \int_\Omega u^\epsilon + \int_R v^\epsilon = 0 \).

**Proof.** Let us argue by contradiction. Suppose that \( 2.1 \) is not hold, that means that, for every \( n \in \mathbb{N} \) there exists a subsequence \( \{\epsilon_n\} \to 0 \) and \( \{u^{\epsilon_n}\} = \{(u^\epsilon, v^\epsilon)\} \in L^2(\Omega) \cap H^1(\Omega) \) such that

\[
\int_\Omega u^{\epsilon_n} + \int_R v^{\epsilon_n} = 0,
\]

\[
\int_\Omega (u^{\epsilon_n})^2 + \int_R (v^{\epsilon_n})^2 = 1,
\]

and

\[
\frac{1}{2} \int_\Omega |\nabla u^{\epsilon_n}|^2 dx + \frac{1}{4} \int_R \int_R J_\epsilon(x - y)(v^{\epsilon_n}(y) - v^{\epsilon_n}(x))^2 dydx + \frac{1}{2} \int_R \int_\Omega G_\epsilon(x - y)(v^{\epsilon_n}(x) - u^{\epsilon_n}(y))^2 dx dy \leq \frac{1}{n}.
\]

Taking the limit as \( n \to \infty \) we obtain

\[
\lim_{n \to \infty} \frac{1}{2} \int_\Omega |\nabla u^{\epsilon_n}|^2 dx = 0,
\]

\[
\lim_{n \to \infty} \left( \frac{1}{4} \int_R \int_R J_\epsilon(x - y)(v^{\epsilon_n}(y) - v^{\epsilon_n}(x))^2 dydx \right) = 0,
\]

and

\[
\lim_{n \to \infty} \left( \frac{1}{2} \int_R \int_\Omega G_\epsilon(x - y)(v^{\epsilon_n}(x) - u^{\epsilon_n}(y))^2 dydx \right) = 0.
\]

We have that \( \int_\Omega (u^{\epsilon_n})^2 dx \leq 1 \), that is, \( \{u^{\epsilon_n}\} \) is bounded in \( L^2(\Omega) \). Moreover, we get that \( \{u^{\epsilon_n}\} \) is bounded in \( H^1(\Omega) \). Taking a subsequence, also denoted by \( \{u^{\epsilon_n}\} \), such that \( \epsilon_n \to 0 \) we have

\[
u^{\epsilon_n} \rightharpoonup u^* \quad \text{in} \quad H^1(\Omega)
\]

\[
u^{\epsilon_n} \to u^* \quad \text{in} \quad L^2(\Omega).
\]

Thanks to the Fatou’s lemma we know that

\[
\frac{1}{2} \int_\Omega |\nabla u^*|^2 dx \leq \liminf_{\epsilon_n} \frac{1}{2} \int_\Omega |\nabla u^{\epsilon_n}|^2 dx = 0.
\]
Hence, the limit $u^*$ is constant in $\Omega$.

Also $\{v^{\varepsilon_n}\}$ is bounded in $L^2(R)$. Define $k^{\varepsilon_n} = \int_R v^{\varepsilon_n}$. From the bound in $L^2(R)$ of $v^{\varepsilon_n}$ we obtain that there exists a constant $C$ such that $|k^{\varepsilon_n}| \leq C$ and, moreover, we can take a subsequence $\{v^{\varepsilon_n}\}$ which weakly converges in $L^2(R)$ to some limit $v^*$ as $\varepsilon_n \to 0$ and such that $\{k^{\varepsilon_n}\}$ also converges to a limit that we call $k^*$. Consider $z^{\varepsilon_n} = v^{\varepsilon_n} - k^{\varepsilon_n}$. We have that $\int_R z^{\varepsilon_n} = 0$, therefore, see [10] and [1], there exists a constant $C > 0$ independent of $\varepsilon$ such that
\[
\int_R \int_R J(x_1 - y_1, \varepsilon_n, (x_2 - y_2)) (z^{\varepsilon_n}(y) - z^{\varepsilon_n}(x))^2 dy dx \geq C \int_R (z^{\varepsilon_n}(x))^2 dx.
\]
(2.2)

In fact, since $J$ is continuous, from our hypothesis on $J$, we get that there exists constants $M, \delta > 0$ such that
\[
J(x_1 - y_1, x_2 - y_2) \geq M, \quad \text{whenever} \quad |(x_1 - y_1, x_2 - y_2)| < \delta.
\]
Then, it follows that
\[
J(x_1 - y_1, \varepsilon(x_2 - y_2)) \geq \frac{M}{2}, \quad \text{whenever} \quad |x_1 - y_1| < \frac{\delta}{2}, \quad \varepsilon|x_2 - y_2| < \frac{\delta}{2},
\]
for every $\varepsilon$ small enough. Hence, the inequality (2.2) follows from Lemma 3.1 in [10] and the constant $C$ only depends on $M, \delta$ and $R$ but not on $\varepsilon$.

Note that we have
\[
\lim_{n \to \infty} \left( \frac{1}{4} \int_R \int_R J_{\varepsilon_n}(x - y) (z^{\varepsilon_n}(y) - z^{\varepsilon_n}(x))^2 dy dx \right) = \lim_{n \to \infty} \left( \frac{1}{4} \int_R \int_R J_{\varepsilon_n}(x - y) (v^{\varepsilon_n}(y) - v^{\varepsilon_n}(x))^2 dy dx \right) = 0,
\]
as $\varepsilon_n \to 0$, which yields
\[
0 \geq \lim_{n \to \infty} C \int_R (z^{\varepsilon_n}(x))^2 dx.
\]
From here we conclude that $z^{\varepsilon_n} \to 0$ in $L^2(R)$, which leads to $v^{\varepsilon_n} \to k^*$ strongly in $L^2(R)$. Finally, as $u^{\varepsilon_n} \to u^*$ in $L^2(\Omega)$ and $v^{\varepsilon_n} \to k^*$ in $L^2(R)$, we can take the limit as $\varepsilon_n \to 0$ and obtain
\[
0 = \lim_{n \to \infty} \left( \frac{1}{2} \int_R \int_\Omega G_{\varepsilon_n}(x - y)(v^{\varepsilon_n}(x) - u^{\varepsilon_n}(y))^2 dy dx \right) = \frac{1}{2} \int_R \int_\Omega G^*(x - y)(v^*(x) - u^*(y))^2 dy dx.
\]
From where it follows that $k^* - u^* = 0$, that is, $k^* = u^*$. From
\[
\int_\Omega u^{\varepsilon_n} + \int_R v^{\varepsilon_n} = 0,
\]
it follows that
\[
\int_\Omega u^* + \int_R k^* = 0,
\]
and since we have $k^* = u^*$ we get
\[
k^* = u^* = 0.
\]
Now, from
\[
\int_\Omega (u^{\varepsilon_n})^2 + \int_R (v^{\varepsilon_n})^2 = 1,
\]
and the strong convergence in $L^2$ we obtain
\[
\int_\Omega (u^*)^2 + \int_R (k^*)^2 = 1,
\]
which yields a contradiction. The proof is complete. \[\Box\]
With this lemma, following [23] (see also [29]), we can provide an estimate for the asymptotic behavior of the solutions of the problem [1.7], that is, the solutions \( \{u^\epsilon, v^\epsilon\}_{\epsilon>0} \) converges to the mean value of the initial condition

\[
\left\| (u^\epsilon, v^\epsilon)(\cdot, t) - \int (u_0, v_0) \right\|_{L^2(\Omega)} \leq C_1 e^{-C_2 t},
\]

(2.3)

with \( C_1, C_2 \) finite positive constants, independent of \( \epsilon \) and also, \( C_2 \) independent of the initial condition. Hence, we have that the \( L^2 \)-norm of \( \{u^\epsilon, v^\epsilon\}_{\epsilon>0} \) is bounded (independently of \( \epsilon \)). Here

\[
\int (u_0, v_0) = \int_{\Omega} u_0 + \int_{R} v_0.
\]

Now we are ready to proceed with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** First, we observe that, since \( J \) and \( G \) are continuous functions, we have

\[
J_{\epsilon}(x - y) = J(x_1 - y_1, \epsilon(x_2 - y_2)) \rightarrow J^*(x - y) = J(x_1 - y_1, 0), \quad \text{and}
\]

\[
G_{\epsilon}(x - y) = G(x_1 - y_1, \epsilon(x_2 - y_2)) \rightarrow G^*(x - y) = G(x_1 - y_1, 0 - y_2),
\]

as \( \epsilon \to 0 \), uniformly in \( x, y \).

From Lemma 2.1 since \( \{v^\epsilon\} \) is bounded in \( L^\infty(0, T; L^2(R)) \) we can take a subsequence, also denoted by \( \{v^\epsilon\} \), such that

\[
v^\epsilon \rightarrow v^* \quad \text{weakly in} \quad L^\infty(0, T; L^2(R)) \quad \text{as} \quad \epsilon \to 0.
\]

On the other hand, we have that

\[
\int_{\Omega} |u^\epsilon(x, t)|^2 dx \quad \text{and} \quad \int_{\Omega} |\nabla u^\epsilon(x, t)|^2 dx,
\]

are also bounded in \( L^2(\Omega) \) (uniformly in \( t \in [0, T] \)). Hence, along a subsequence if necessary,

\[
u^\epsilon \rightarrow u^* \quad \text{weakly in} \quad L^\infty(0, T; H^1(\Omega)) \quad \text{as} \quad \epsilon \to 0,
\]

\[
u^\epsilon \rightarrow u^* \quad \text{strongly in} \quad L^\infty(0, T; L^2(\Omega)) \quad \text{as} \quad \epsilon \to 0.
\]

Now we consider the weak form of (1.7), that is, using the symmetry of the kernel \( J \) we have the following identities,

\[
\int_{\Omega} u^*(x, T) \varphi(x, T) dx - \int_0^T \int_{\Omega} u^*(x, t) \frac{\partial \varphi}{\partial t}(x, t) dxdt = \int_{\Omega} u_0(x) \varphi(x, 0) dx - \int_0^T \int_{\Omega} \nabla u^*(x, t) \nabla \varphi(x, t) dxdt
\]

\[
+ \int_0^T \int_R \int_R G_{\epsilon}(x - y)(v^\epsilon(y, t) - u^\epsilon(x, t)) \varphi(x, t) dydxdt,
\]

\[
\int_R v^\epsilon(x, T) \varphi(x, T) dx - \int_0^T \int_R v^\epsilon(x, t) \frac{\partial \varphi}{\partial t}(x, t) dxdt = \int_R v_0(x) \varphi(x, 0) dx
\]

\[
- \frac{1}{2} \int_0^T \int_R J_{\epsilon}(x - y) (v^\epsilon(y, t) - u^\epsilon(x, t)) (\varphi(y, t) - \varphi(x, t)) dydxdt
\]

\[
- \int_0^T \int_R \int_R G_{\epsilon}(x - y) (v^\epsilon(\tilde{x}, t) - u^\epsilon(y, t)) \varphi(x, t) dydxdt,
\]

for every \( \varphi \in C^1(H^1(\Omega) \cup L^2(R)) \).

Now, let us take a test function that depends only on the first variable, for \( x \in R \), that is, \( \varphi = \varphi(x_1) \) and we analyze the limit as \( \epsilon \to 0 \) of each term in the previous equations. We have

\[
\lim_{\epsilon \to 0} \left( \int_0^T \int_{\Omega} \nabla u^\epsilon \nabla \varphi dxdt \right) = \left( \int_0^T \int_{\Omega} \nabla u^* \nabla \varphi dxdt \right).
\]
Now, note that
\[ \int_0^T \int_\Omega \int_{R_1} \int_{R_2} G(x_1 - y_1, x_2 - \varepsilon \tilde{y}_2)(\varphi^\varepsilon(y_1, \tilde{y}_2, t) - u^\varepsilon(x_1, x_2, t)) \varphi(x_1, x_2, t) dy_2 dy_1 dx_2 dx_1 dt \]
\[ = \int_0^T \int_\Omega \int_{R_1} \int_{R_2} \left[ G(x_1 - y_1, x_2 - \varepsilon \tilde{y}_2) - G(x_1 - y_1, x_2) \right] \]
\[ \times (\varphi^\varepsilon(y_1, \tilde{y}_2, t) - u^\varepsilon(x_1, x_2, t)) \varphi(x_1, x_2, t) dy_2 dy_1 dx_2 dx_1 dt \]
\[ + \int_0^T \int_\Omega \int_{R_1} \int_{R_2} G(x_1 - y_1, x_2)(\varphi^\varepsilon(y_1, \tilde{y}_2, t) - u^\varepsilon(x_1, x_2, t)) \varphi(x_1, x_2, t) dy_2 dy_1 dx_2 dx_1 dt. \]

Notice that the measure in \( \Omega \) is the product measure and hence when we integrate we have \( dx = dx_1 dx_2 \).

Since
\[ \left[ G(x_1 - y_1, x_2 - \varepsilon \tilde{y}_2) - G(x_1 - y_1, x_2) \right] \]
goes to zero uniformly and \( u^\varepsilon \) and \( \varphi^\varepsilon \) are bounded in \( L^2 \), the first term goes to zero as \( \varepsilon \to 0 \) and therefore we concentrate in the second. To analyze the limit of the second term, we observe that \( G(x_1 - y_1, x_2) \) does not depend on \( y_2 \) and hence we can rewrite this term as follows,
\[ \int_0^T \int_\Omega \int_{R_1} \int_{R_2} G(x_1 - y_1, x_2)(\varphi^\varepsilon(y_1, \tilde{y}_2, t) - u^\varepsilon(x_1, x_2, t)) \varphi(x_1, x_2, t) dy_2 dy_1 dx_2 dx_1 dt \]
\[ = \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2) \left[ \int_{R_2} \varphi^\varepsilon(y_1, \tilde{y}_2, t) - u^\varepsilon(x_1, x_2, t) \right] dy_2 dy_1 dx_2 dx_1 dt. \]

Let
\[ V^\varepsilon(y_1, t) = \int_{R_2} \varphi^\varepsilon(y_1, \tilde{y}_2, t) dy_2. \] (2.7)

Observe that, since \( \varphi^\varepsilon \) is bounded in \( L^\infty(0, T; L^2(R)) \), then \( V^\varepsilon \) is also bounded in \( L^\infty(0, T; L^2(R)) \) so, taking a subsequence if necessary
\[ V^\varepsilon \to V^* \quad \text{weakly in} \quad L^\infty(0, T; L^2(R_1)). \]

Using (2.7) we obtain
\[ \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2) |V^\varepsilon(y_1) - |R_2|u^\varepsilon(x_1, x_2, t)| dy_1 dx_2 dx_1 dt. \]

Therefore, we can take the limit as \( \varepsilon \to 0 \) and obtain
\[ \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2 - \varepsilon \tilde{y}_2) |V^\varepsilon(y_1) - |R_2|u^\varepsilon(x_1, x_2, t)| dy_1 dx_2 dx_1 dt \]
\[ = \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2) \lim_{\varepsilon \to 0} |V^\varepsilon(y_1)| dy_1 dx_2 dx_1 dt \]
\[ - \int_0^T |R_2| \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2) \lim_{\varepsilon \to 0} |u^\varepsilon(x_1, x_2, t)| dy_1 dx_2 dx_1 dt \]
\[ = \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2 - 0)V^*(y_1) dy_1 dx_2 dx_1 dt \]
\[ - \int_0^T |R_2| \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2 - 0)u^*(x_1, x_2, t) dy_1 dx_2 dx_1 dt. \]

The same idea can be applied for the second integral in the weak form of the problem using the properties of the kernel \( G \) and Fubini’s theorem, which leads to
\[ \int_0^T \int_{R_1} \varphi(x_1, t) \int_\Omega G(x_1 - y_1, 0 - y_2) (V^*(y_1) - |R_2|u^*(x_1, x_2, t)) dy_1 dx_2 dx_1 dt. \]
Concerning the terms that involve time derivatives, from the $L^\infty - L^2$ convergence we obtain
\[
\lim_{\varepsilon \to 0} - \int_0^T \int_\Omega u^\varepsilon(x,t) \frac{\partial \varphi}{\partial t}(x,t) \, dx \, dt = - \int_0^T \int_\Omega u^\varepsilon(x,t) \frac{\partial \varphi}{\partial t}(x,t) \, dx \, dt
\]
and
\[
\lim_{\varepsilon \to 0} - \int_0^T \int_R v^\varepsilon(x,t) \frac{\partial \varphi}{\partial t}(x,t) \, dx \, dt = - \int_0^T \int_R V^\varepsilon(x,t) \frac{\partial \varphi}{\partial t}(x,t) \, dx \, dt
\]
Finally, we will deal with the pure nonlocal integral. By Fubini’s theorem and (2.7) we get
\[
\int_0^T \int_{R_1} \int_{R_2} J^\varepsilon(x-y) (v^\varepsilon(y_1, \tilde{y}_2,t) - v^\varepsilon(x_1, \tilde{x}_2,t)) \varphi(x_1,t) \, dy \, d\tilde{y}_2 \, d\tilde{x}_2 \, dx \, dt
\]
\[
= \int_0^T \int_{R_1} \int_{R_2} J(x_1-y_1, \epsilon(\tilde{x}_2, \tilde{y}_2)) \left[ \int_{R_2} (v^\varepsilon(y_1, \tilde{y}_2,t) - v^\varepsilon(x_1, \tilde{x}_2,t)) \, dy_2 \, d\tilde{x}_2 \right] \, dx \, dt
\]
\[
= \int_0^T \int_{R_1} \varphi(x_1,t) \int_{R_2} J(x_1-y_1, \epsilon(\tilde{x}_2, \tilde{y}_2)) (|R_2|^\varepsilon(y_1) - |R_2| |V^\varepsilon(x_1)|) \, dy_1 \, dx_1 \, dt.
\]
Now, we can take the limit as $\varepsilon \to 0$, it follows that
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{R_1} \varphi(x_1,t) \int_{R_2} J(x_1-y_1, \epsilon(\tilde{x}_2, \tilde{y}_2)) (|R_2|^\varepsilon(y_1) - |R_2| |V^\varepsilon(x_1)|) \, dy_1 \, dx_1 \, dt
\]
\[
= \int_0^T |R_2| \int_{R_1} \varphi(x_1,t) \int_{\Omega_0} J^\varepsilon(x_1-y_1, \epsilon(\tilde{x}_2, \tilde{y}_2)) \lim_{\varepsilon \to 0} (|R_2|^\varepsilon(y_1) - |R_2| |V^\varepsilon(x_1)|) \, dy_1 \, dx_1 \, dt
\]
\[
- \int_0^T |R_2| \int_{R_1} \varphi(x_1,t) \int_{\Omega_0} J^\varepsilon(x_1-y_1, \epsilon(\tilde{x}_2, \tilde{y}_2)) \lim_{\varepsilon \to 0} (|R_2|^\varepsilon(y_1) - |R_2| |V^\varepsilon(x_1)|) \, dy_1 \, dx_1 \, dt
\]
\[
= \int_0^T |R_2| \int_{R_1} \varphi(x_1,t) \int_{R_1} J(x_1-y_1,0)(V^\varepsilon(y_1) - V^\varepsilon(x_1)) \, dy_1 \, dx_1 \, dt.
\]
Hence, since this procedure can be carry over for every $T > 0$, the limit equation, defined in the domain $\Omega_0 = \Omega \cup R_1$, (see Figure 2) is given by the system (1.8).

\[
\begin{cases}
\frac{\partial u^*}{\partial t}(x,t) = \Delta u^*(x,t) + \int_{R_1} G^*(x-y)(V^*(y,t) - |R_2|u^*(x,t)) \, dy, & (x,t) \in \Omega \times (0,\infty), \\
\frac{\partial u^*}{\partial \eta}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\
\frac{\partial V^*}{\partial t}(x,t) = |R_2| \int_{R_1} J^*(x-y)(V^*(y,t) - V^*(x,t)) \, dy \\
- \int_{\Omega} G^*(x-y)(V^*(x,t) - |R_2|u^*(y,t)) \, dy, & (x,t) \in R_1 \times (0,\infty), \\
u^*(x,0) = u_0^*(x), & x \in \Omega, \\
V^*(x,0) = V_0^*(x), & x \in R_1,
\end{cases}
\]
where $J^*(x-y) = J(x_1-y_1,0)$ and $G^*(x-y) = G(x_1-y_1, x_2-y_2)$.  

To finish the proof we show existence and uniqueness of a solution of the solution to the limit problem (1.8) (notice that up to this point we have convergence along subsequences $\varepsilon_j \to 0$, proving uniqueness of the limit we obtain the existence of the full limit as $\varepsilon \to 0$).

Thanks to the limit along subsequences we ensure the existence of a solution $(u^*, V^*)$ for the limit problem. To show the uniqueness let us suppose that there exists two solutions $(u_1^*, V_1^*)$ and $(u_2^*, V_2^*)$ of (1.8). Define $w^* = u_1^* - u_2^*$
and $z^* = V_1^* - V_2^*$. The pair of $(w^*, z^*)$ satisfies the following equations

$$
\begin{align*}
\frac{\partial w^*}{\partial t}(x,t) &= \Delta w^*(x,t) + \int_{R_1} G^*(x-y)(z^*(y,t) - |R_2|w^*(x,t))dy, \\
\frac{\partial w^*}{\partial \eta}(x,t) &= 0, \quad (x,t) \in \partial \Omega \times (0,\infty), \\
\frac{\partial z^*}{\partial t}(x,t) &= |R_2| \int_{R_1} J^*(x-y) \left(z^*(y,t) - z^*(x,t)\right)dy - \int_\Omega G^*(x-y) \left(z^*(x,t) - |R_2|w^*(x,t)\right)dy, \\
\left.\left(w^*(x,0) = 0, \quad x \in \Omega, \right.\right. \\
\left.\left.\left(z^*(x,0) = 0, \quad x \in R_1.\right)\right.\right.
\end{align*}
$$

(2.8)

Multiplying the first equation of the problem (2.8) by $w^*$ and integrating over $\Omega$ and, the second equation by $\frac{\partial z^*}{\partial t}$ and integrating over $R_1$, we get

$$
|R_2| \int_\Omega \frac{\partial w^*}{\partial t} w^* dx + \int_{R_1} \frac{\partial z^*}{\partial t} z^* dx = -|R_2| \int_\Omega \nabla w^* dx - \frac{|R_2|}{2} \int_{R_1} \int_{R_1} J^*(x-y)(z^*(y,t) - z^*(x,t))^2 dy dx
$$

$$
- \int_{R_1} \int_\Omega G^*(x-y) (z^*(y,t) - |R_2|w^*(x,t))^2 dy dx
$$

$$
= -2E(w^*, z^*) \leq 0.
$$

Hence, if we let

$$
f'(t) = |R_2| \int_\Omega \frac{\partial w^*}{\partial t} w^* dx + \int_{R_1} \frac{\partial z^*}{\partial t} z^* dx,
$$

we have

$$
f(t) = \frac{|R_2|}{2} \int_\Omega (w^*)^2 dx + \int_{R_1} (z^*)^2 dx.
$$

Now, from Lemma 2.1 we obtain

$$
2E(w^*, z^*) \geq 2\lambda_1 \left(\frac{|R_2|}{2} \int_\Omega (w^*)^2 dx + \int_{R_1} (z^*)^2 dx\right) = 2\lambda_1 f(t),
$$

which implies $-2E(w^*, z^*) \leq -2\lambda_1 f(t)$ and then we get

$$
f'(t) \leq -2\lambda_1 f(t).
$$

Hence, Gronwall’s inequality gives that

$$
f(t) \leq e^{-2\lambda_1 t} f(0),
$$

where $f(0) = \frac{|R_2|}{2} \int_\Omega (w^*)^2(x,0) dx + \int_{R_1} (z^*)^2(x,0) dx$. Since $f(t) \geq 0$ and $f(0) = 0$ we have that

$$
0 \leq f(t) \leq 0,
$$

that is

$$
f(t) \equiv 0
$$

and hence

$$
w^* = 0 \quad \text{and} \quad z^* = 0,
$$

which means $u^*_1 = u^*_2$ and $V_1^* = V_2^*$. This guarantees the uniqueness of the solution for the problem (1.8) as we wanted to show.

Now, we include several remarks.
Remark 3. From our previous arguments, we also conclude that the limit problem \([1.8]\) is well-posed in \(L^2(\Omega_0)\) (we have existence, uniqueness and continuous dependence with respect to the initial data of the solutions).

Remark 4. We only prove weak convergence of the solution of the problem \([1.6]\) to the solution of the problem \([1.8]\) (we do not prove strong convergence in the \(L^2\)-norm). Moreover, we only guarantee the uniqueness of \(V^*\) and this is not enough to ensure the uniqueness of \(v^*\).

Remark 5. Observe that, instead of the usual metric in \(L^2(\Omega \cup R)\) we choose to work with the metric \([1.4]\). This choice was made to obtain a nontrivial limit. In fact, using this metric we can observe the coupling of the local part of the problem in the domain \(\Omega\) with the nonlocal part in the lower dimensional domain \(R_1\).

Now, if we consider the usual metric in \(L^2\) and the energy functional
\[
E(u, v) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{4\varepsilon N_2} \int_{R_1} \int_{\Omega_1} J(x - y) (v(y) - v(x))^2 \, dy \, dx + \frac{1}{2} \int_{R_1} \int_\Omega G(x - y) (v(x) - u(y))^2 \, dy \, dx,
\]
the associated evolution problem (after the change of variables) is given by
\[
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial t}(x, t) &= \triangle u^\varepsilon(x, t) + \varepsilon \int_{R_1} G_\varepsilon(x - y)(v^\varepsilon(y, t) - u(x, t)) \, dy, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u^\varepsilon}{\partial \eta}(x, t) &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
\frac{\partial v^\varepsilon}{\partial t}(\tilde{x}, t) &= \int_{R_1} J_\varepsilon(x - y) (v^\varepsilon(y, t) - v^\varepsilon(\tilde{x}, t)) \, dy - \int_\Omega G_\varepsilon(x - y)(v^\varepsilon(x, t) - u^\varepsilon(x, t)) \, dy, \quad x \in R, \quad t > 0, \\
u(x, 0) &= u_0(x), \quad x \in \Omega, \\
v(\tilde{x}, 0) &= v_0(\tilde{x}), \quad \tilde{x} \in R,
\end{aligned}
\]
where \(J_\varepsilon(x - y) = J(x_1 - y_1, \varepsilon(x_2 - y_2))\), \(G_\varepsilon(x - y) = G(x_1 - y_1, \varepsilon(x_2 - y_2))\) and \(v^\varepsilon(x_1, \tilde{x}_2, t) = v(x_1, \tilde{x}_2, t)\). Observe that taking the limit as \(\varepsilon \to 0\) the nonlocal term that appears in the equation for \(u^\varepsilon\) goes to zero and hence we will lose the coupling term in the limit (the equation for \(u^*\) will be independent of \(V^*\)). Also in this case, the limit problem will be well-posed, in the sense that we can ensure existence and uniqueness of the solution, but it is less interesting.

As we expected, the limit problem \([1.8]\) preserves the total mass of the solution. This follows from the limit procedure and the fact that the problem \([1.7]\) preserves the total mass for every \(\varepsilon > 0\). We include below a direct proof of this fact for completeness.

**Theorem 2.2.** The solution \((u^*, V^*)\) of the problem \([1.8]\), with initial data \(u_0^* \in H^1(\Omega)\) and \(V_0^* \in L^2(R_1)\) satisfies
\[
\int_\Omega u^*(x, t) \, dx + \int_{R_1} V^*(x, t) \, dx = \int_\Omega u_0^* \, dx + \int_{R_1} V_0^* \, dx, \quad \forall t \geq 0. \tag{2.9}
\]

**Proof.** Differentiating \([2.9]\) with respect to \(t\) we obtain
\[
\begin{aligned}
\int_{\Omega} \frac{\partial u^*}{\partial t} \, dx + \int_{R_1} \frac{\partial V^*}{\partial t} \, dx &= \int_\Omega \triangle u^*(x, t) \, dx + \int_\Omega \int_{R_1} G^*(x - y)(V^*(y, t) - |R_2|u^*(x, t)) \, dy \, dx \\
&\quad + \int_{R_1} |R_2| \int_{R_1} J^*(x - y)(V^*(y, t) - V^*(x, t)) \, dy \, dx \\
&\quad - \int_{R_1} \int_\Omega G^*(x - y)(V^*(x, t) - |R_2|u^*(y, t)) \, dy \, dx \\
&= 0.
\end{aligned}
\]
Indeed, after a change of variables, due to the symmetry of \(G\) and Fubini’s theorem, the second and the fourth integral cancel each other. Also, by the symmetry of \(J\) and Fubini’s theorem, the second integral is zero. Finally, the first integral is zero since we have a Neumann type boundary condition for the local part.

This ends the proof. \(\square\)
Finally, we include the study of the asymptotic behavior of the solutions for the limit problem (1.8).

Notice that from the fact that the constants in (2.3) do not depend on $\varepsilon$ we obtain that the solutions for the limit problem (1.8) converge exponentially to the mean value of the initial condition. We have that

$$\left\|(u^*, V^*)(\cdot, t) - \int\left(u_0, v_0\right)\right\|_{L^2(\Omega_0)} \leq C_1 e^{-C_2 t}.$$ 

However, we can obtain a better control of the constant $C_2$ and obtain an exponential decay in terms of the first nontrivial eigenvalue associated to the limit problem. To this end, we use the $L^2$-norm

$$\|(u^*, V^*)\|_{L^2(\Omega_0)} = |R_2| \int_{\Omega} |u^*|^2 dx + \int_{R_1} |V^*|^2 dx.$$

We can define the energy functional associated to the limit problem (1.8) by

$$E(u^*, V^*) = \frac{|R_2|}{2} \int_{\Omega} |\nabla u^*|^2 dx + \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^*(x-y)(V^*(y) - V^*(x))^2 dy dx$$

$$+ \frac{1}{2} \int_{R_1} \int_\Omega G^*(x-y)(V^*(x) - |R_2|u^*(y))^2 dy dx. $$

Indeed, the gradient flow associated with (2.10), is given by

$$\partial_\varepsilon E(u^*, V^*) = \lim_{h\to 0} \frac{E(u^* + h\varepsilon, V^* + h\varepsilon) - E(u^*, V^*)}{h}$$

$$= |R_2| \int_{\Omega} \nabla u^* \nabla \varepsilon dx + \frac{|R_2|}{2} \int_{R_1} \int_{R_1} J^*(x-y)(V^*(y) - V^*(x))(\varphi(y) - \varphi(x)) dy dx$$

$$+ |R_2| \int_{R_1} \int_\Omega G^*(x-y)(V^*(x) - |R_2|u^*(y))(\varphi(x) dx - |R_2|\varphi(y)) dy dx.$$ 

Hence, using that

$$|R_2| \int_{\Omega} \frac{\partial u^*}{\partial \varepsilon} \varphi(x) dx + \int_{R_1} \frac{\partial V^*}{\partial \varepsilon} \varphi(x) dx = -\partial_\varepsilon E[(u^*, V^*)](t),$$ 

we obtain the limit problem (1.8).

With this energy at hand we can obtain the first nontrivial eigenvalue for our limit problem. Let us take $\lambda_1$ as

$$0 < \lambda_1 = \inf_{u^*, V^* \in W_0} \frac{E(u^*, V^*)}{|R_2| \int_{\Omega} |u^*|^2 + \int_{R_1} |V^*|^2},$$

where $E(u^*, V^*)$ is given by (2.10) and

$$W_0 = \left\{ u^* \in H^1(\Omega), V^* \in L^2(R_1) : |R_2| \int_{\Omega} u^* + \int_{R_1} V^* = 0 \right\}.$$ 

**Lemma 2.3.** Let $\lambda_1$ given by (2.11), then $\lambda_1 > 0$ and therefore,

$$E(u^*, V^*) \geq \lambda_1 \left( |R_2| \int_{\Omega} |u^*|^2 + \int_{R_1} |V^*|^2 \right),$$

for every $u^*, V^*$ solution of (1.8), such that $|R_2| \int_{\Omega} u^* + \int_{R_1} V^* = 0$.

**Proof.** The proof is similar to the one of Lemma 2.1 but we include the details for completeness. Let us suppose that $\lambda_1 = 0$. This implies that there exists a subsequence $\{u^*_n\} \in H^1(\Omega)$ and $\{v^*_n\} \in L^2(R_1)$ such that

$$|R_2| \int_{\Omega} u^*_n + \int_{R_1} v^*_n = 0,$$
\[ |R_2| \int \Omega (u_n^*)^2 + \int_R (V_n^*)^2 = 1, \]
and
\[ \frac{|R_2|}{2} \int \Omega |\nabla u_n^*|^2 dx + \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^* (x-y)(V_n^*(y) - V_n^*(x))^2 dy dx + \frac{1}{2} |R_2| \int \Omega J(x-y)(V_n^*(x) - |R_2| u_n^*(y))^2 dy dx \leq \frac{1}{n}. \]

Taking the limit as \( n \to \infty \) we obtain
\[
\lim_{n \to \infty} \left( \frac{|R_2|}{2} \int \Omega |\nabla u_n^*|^2 dx \right) = 0,
\]
and
\[
\lim_{n \to \infty} \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^* (x-y)(V_n^*(y) - V_n^*(x))^2 dy dx = 0,
\]
and
\[
\lim_{n \to \infty} \left( \frac{1}{2} |R_2| \int \Omega J^*(x-y)(V_n^*(x) - |R_2| u_n^*(y))^2 dy dx \right) = 0.
\]

Recalling that we have
\[ J^*(x-y) = J(x_1 - y_1, 0), \quad \text{and} \quad G^*(x-y) = G(x_1 - y_1, 0 - y_2), \]
it follows that \( |R_2| \int \Omega (u_n^*)^2 dx \leq 1 \), that is, \( \{u_n^*\} \) is bounded in \( L^2(\Omega) \). Moreover, \( \{u_n^*\} \) is also bounded in \( H^1(\Omega) \).

Then, we can extract a subsequence \( \{u_n^*\} \in H^1(\Omega) \) which weakly converges to a limit \( \hat{u} \in H^1(\Omega) \). From the weak convergence in \( H^1(\Omega) \) we obtain strong convergence in \( L^2(\Omega) \). Then, we have that
\[
\frac{1}{2} |R_2| \int \Omega |\nabla \hat{u}|^2 dx \leq \lim \inf \frac{1}{2} |R_2| \int \Omega |\nabla u_n^*|^2 dx = 0.
\]
Hence, the limit \( \hat{u} \) is constant in \( \Omega \).

Also, it follows that \( \{V_n^*\} \) is bounded in \( L^2(R_1) \). Since
\[
\int_{R_1} |V_n^*|^2 dx \leq C \left( \int \Omega (V_n^*)^2 dx \right)^\frac{1}{2} \leq C,
\]
we let \( k_n = \int_{R_1} V_n^* \), and obtain that \( |k_n| \leq C \). Then, we can take a subsequence \( \{V_n^*\} \) which converges in \( L^2(R_1) \), to some limit \( \hat{V} \) as \( n \to \infty \). Consider \( z_n = V_n^* - k_n \), this function is such that \( \int_{R_1} k_n = 0 \). By Lemma 3.1, in [10], there exists a constant \( c_1 > 0 \) such that
\[
\int_{R_1} \int_{R_1} J(x_1 - y_1, 0)(z_n(y) - z_n(x))^2 dy dx \geq c_1 \int_{R_1} (z_n(x))^2 dx.
\]
From this inequality we have
\[
\lim_{n \to \infty} \left( \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^* (x-y)(z_n(y) - z_n(x))^2 dy dx \right) = \lim_{n \to \infty} \left( \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^* (x-y)(V_n^*(y) - V_n^*(x))^2 dy dx \right) \to 0,
\]
which yields
\[
0 \geq c_1 \lim_{n \to \infty} \int_{R_1} \int_{R_1} (z_n(x))^2 dx.
\]
We conclude that \( z_n \to 0 \) strongly in \( L^2(R_1) \), which leads to \( V_n^* \to \hat{V} \) strongly in \( L^2(R_1) \). Finally, as \( u_n^* \to \hat{u} \) in \( L^2(\Omega) \) and \( V_n^* \to \hat{V} \) in \( L^2(R_1) \), we can take the limit as \( n \to \infty \) and obtain
\[
\lim_{n \to \infty} \left( \frac{1}{2} \int_{R_1} \int_{R_1} G^*(x-y)(V_n^*(x) - u_n^*(y))^2 dy dx \right) = \frac{1}{2} \int_{R_1} \int_{\Omega} G^*(x-y)(\hat{V} - |R_2| \hat{u})^2 dy dx \to 0.
\]
Then, we have that \( \hat{V} - |R_2| \hat{u} = 0 \), that is \( \hat{V} = |R_2| \hat{u} \). Hence, it follows that \( \hat{V} = \hat{u} = 0 \), but this is a contradiction with the fact that
\[
|R_2| \int \Omega (u_n^*)^2 + \int_R (V_n^*)^2 = 1
\]
since we have strong convergence in \( L^2 \). The proof is complete. \( \square \)
Thanks to Lemma 2.3 we can show that solutions to the limit problem converge exponentially fast to the mean
value of their initial condition.

**Theorem 2.4.** Given \( u_0^* \in H^1(\Omega) \) and \( V_0^* \in L^2(R_1) \), the solution to (1.8), with initial data \( u_0^*, V_0^* \), converges to its mean value as \( t \to \infty \), with an exponential rate \( \lambda_1 \) (given by (2.11)),

\[
\left\| (u^*, V^*)(\cdot, t) - \int \left( u_0^*, V_0^* \right) \right\|_{L^2(\Omega_0)} \leq C \left( \left\| (u_0^*, V_0^*) \right\|_{L^2(\Omega_0)} \right) e^{-\lambda_1 t}.
\]

**Proof.** We know that \( V^* = |R_2|u^* \), with \( k \) constant, is also a solution of the problem (1.8). Hence, the pair

\[
(h(x, t), z(x, t)) = (R_2)u^*(x, t) - k, z(x, t) = V^*(x, t) - k
\]
is also a solution of (1.8). If we choose

\[
k = |R_2| \int_{\Omega} u_0^* + \int_{R_1} V_0^*
\]
then, using that the mass is preserved in time, we get that \( h \) and \( z \) satisfy

\[
\int_{\Omega} h(x, t)dx + \int_{R_1} z(x, t)dx = 0.
\]

Let

\[
f(t) = \frac{|R_2|}{2} \int_{\Omega} h(x, t)^2dx + \frac{1}{2} \int_{R_1} z(x, t)^2dx.
\]

Differentiating \( f \) with respect to \( t \) we obtain

\[
f'(t) = |R_2| \int_{\Omega} \frac{\partial h}{\partial t}(x, t)h(x, t)dx + \int_{R_1} \frac{\partial z}{\partial t}(x, t)z(x, t)dx
\]

\[
= |R_2| \int_{\Omega} \frac{\partial h}{\partial t}(x, t)h(x, t)dx - |R_2| \int_{R_1} |\nabla h(x, t)|^2dx - \frac{|R_2|}{2} \int_{R_1} \int_{R_1} J^*(x - y)(z(y, t) - z(x, t))^2dydx
\]

\[
- \int_{R_1} \int_{\Omega} G^*(x - y)(z(x, t) - |R_2|h(y, t))z(x, t)dydx + \int_{R_1} \int_{\Omega} G^*(x - y)(z(x, t) - |R_2|h(y, t))|R_2|h(x, t)dydx
\]

\[
= |R_2| \int_{\Omega} |\nabla h(x, t)|^2dx - \frac{|R_2|}{2} \int_{R_1} \int_{R_1} J^*(x - y)(z(y, t) - z(x, t))^2dydx
\]

\[
- \int_{R_1} \int_{\Omega} G^*(x - y)(z(x, t) - |R_2|h(y, t))|^2dydx
\]

\[
= -2E(h, z).
\]

From Lemma 2.3 we get

\[
E(h, z) \geq \lambda_1 \left( |R_2| \int_{\Omega} h^2 + \int_{R_1} z^2 \right).
\]

Hence, we obtain

\[
f'(t) \leq -2\lambda_1 f(t)
\]
so, by Gronwall’s lemma we have that

\[
f(t) \leq e^{-2\lambda_1 t} f(0),
\]

with \( f(0) = \frac{1}{2} \left( |R_2| \int_{\Omega} h_0^2 + \int_{R_1} z_0^2 \right) \). From this it follows that

\[
|R_2| \int_{\Omega} ||u^*(x, t) - k||^2dx + \int_{R_1} |V^*(x, t) - k|^2dx \leq C \left( \left\| (u_0^*, V_0^*) \right\|_{L^2(\Omega_0)} \right) e^{-2\lambda_1 t} \to 0,
\]
as \( t \to \infty \). In particular, it means that \( |R_2|u^* \to k \) in \( L^2(\Omega) \) and \( V^* \to k \) in \( L^2(R_1) \), with \( k \) given by the mean value of the initial condition.

\( \square \)
3 Coupling on the boundary. Proof of Theorem 1.2

Let us first note that the existence and uniqueness of the solutions \( \{u^\varepsilon, v^\varepsilon\} \), of the problem (1.11), for each \( \varepsilon > 0 \), was obtained in [29]. The arguments used to prove the conservation of mass and comparison principle also apply for the problem (1.11) following the ideas presented in [29].

Notice that we have an energy functional for the problem (1.11) given by (1.9),

\[
E_b^\varepsilon(u, v) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon^2 N_2} \int_{R^*} \int_{R^*} J(x-y) (v(y) - v(x))^2 dydx + \frac{1}{2\varepsilon N_2} \int_{R^*} \int_{\Gamma} G(x-y) dy (v(x) - u(y))^2 d\sigma(y) dx.
\]

If we change variables as before we get

\[
E_b(u, v) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{4} \int_{R^*} \int_{R^*} J_{c}(x-y)|v(y) - v(x)|^2 dydx + \frac{1}{2} \int_{R^*} \int_{\Gamma} G_{c}(x-y)|v(x) - u(y)|^2 dydx,
\]

with, as before,

\[
J_{c}(x-y) = J(x_1 - y_1, \varepsilon(x_2 - \tilde{y}_2)), \quad G_{c}(x-y) = G(x_1 - y_1, \varepsilon(x_2 - \tilde{y}_2)).
\]

Now, we just observe that Lemma 2.1 also works here. One can define what is the analogous to the first non-zero eigenvalue for the problem (1.11) as follows:

\[
\alpha_1^\varepsilon := \inf_{u^\varepsilon, v^\varepsilon \in \mathcal{A}} \frac{E_b^\varepsilon(u^\varepsilon, v^\varepsilon)}{\int_\Omega (u^\varepsilon)^2 dx + \int_{R^*} (v^\varepsilon)^2 dx},
\]

with

\[
\mathcal{A} = \{ u^\varepsilon \in H^1(\Omega), v^\varepsilon \in L^2(R) : \int_\Omega u^\varepsilon dx + \int_{R^*} v^\varepsilon dx = 0 \}.
\]

For the positivity of \( \alpha_1^\varepsilon \) we refer to [1]. A uniform lower bound independent of \( \varepsilon \) can be proved as in Lemma 2.1.

The large time behavior for the solutions of (1.11) can be obtained following the ideas developed in [29]. As we find in [29], the solutions of (1.11) converge exponentially to the mean value of the initial data as \( t \) goes to \( \infty \), for each \( \varepsilon \).

Now, we prove Theorem 1.2 taking the limit as \( \varepsilon \) goes to zero in the weak form of (1.11).

Proof of the Theorem 1.2. We proceed as in the proof of Theorem 1.1. First, we obtain convergence along sub-sequences. From Lemma 2.1, since

\[
\int_{R^*} (v^\varepsilon(x))^2 dx
\]

is bounded in \( L^2(R) \) we can take a subsequence, also denoted by \( \{v^\varepsilon\} \), such that

\[
v^\varepsilon \rightharpoonup v^* \quad \text{weakly in} \quad L^\infty(0,T;L^2(R)) \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

On the other hand, we have that

\[
\int_\Omega (u^\varepsilon(x))^2 dx \quad \text{and} \quad \int_\Omega |\nabla u^\varepsilon(x)|^2 dx,
\]

are also bounded, and hence \( u^\varepsilon \) is bounded in \( H^1(\Omega) \). Hence, along a subsequence if necessary,

\[
u^\varepsilon \rightharpoonup u^* \quad \text{weakly in} \quad L^\infty(0,T;H^1(\Omega)) \quad \text{as} \quad \varepsilon \rightarrow 0,
\]

\[
u^\varepsilon \rightarrow u^* \quad \text{strongly in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{as} \quad \varepsilon \rightarrow 0.
\]
Let us consider the weak form of the problem \((\ref{1.11})\) using for the equation for the variable \(v^\varepsilon\) (the second equation of \((\ref{1.11})\)) a test function that depends only on the first variable, that is \(\phi = \varphi(x_1)\). We have,

\[
\int \int_\Omega u^\varepsilon(x_1, x_2, T) \frac{\partial \varphi}{\partial t}(x_1, x_2, T) dx_2 dx_1 - \int_0^T \int \int_\Omega u^\varepsilon(x_1, x_2, t) \frac{\partial \varphi}{\partial t}(x_1, x_2, t) dx_2 dx_1 dt \\
= \int \int_\Omega u^\varepsilon_0(x_1, x_2) \varphi(x_1, x_2, 0) dx_2 dx_1 - \int_0^T \int \int_\Omega \nabla u^\varepsilon \nabla \varphi(x_1, x_2, t) dx_2 dx_1 dt \\
+ \int_0^T \int_\Gamma \int_{R_1}^{R_2} G_\varepsilon(x - y)(v^\varepsilon(y_1, y_2, t) - u^\varepsilon(x_2, t)) \varphi(x_1, x_2, t)d\tilde{y}_2 dy_1 d\sigma(x_2) dt,
\]

\[
\int \int_R R v^\varepsilon(x_1, x_2, T) \frac{\partial \varphi}{\partial t}(x_1, x_2, T) dx_2 dx_1 - \int_0^T \int \int_R R v^\varepsilon(x_1, x_2, t) \frac{\partial \varphi}{\partial t}(x_1, x_2, t) dx_2 dx_1 dt \\
= \int \int_R R v^\varepsilon_0(x_1, x_2) \varphi(x_1, x_2, 0) dx_2 dx_1 \\
+ \int_0^T \int_\Gamma \int_{R_1}^{R_2} J_\varepsilon(x - y)(v^\varepsilon(y_1, y_2, t) - v^\varepsilon(x_1, x_2, t)) \varphi(x_1, x_2, t)d\tilde{y}_2 dy_1 d\tilde{x}_2 dx_1 dt \\
- \int_0^T \int_\Gamma \int_{R_1}^{R_2} G_\varepsilon(x - y)(v^\varepsilon(x_1, x_2, t) - u^\varepsilon(y_2, t)) \varphi(x_1, x_2, t)d\sigma(y_2)d\tilde{x}_2 d\tilde{x}_1 dt.
\]

Now we can take the limit for \(\varepsilon \to 0\) in each integral on the right side of the previous equations as we did in Theorem 1.1. The only difference appears when we analyze the term

\[-\int_0^T \int_{R_1}^{R_2} \int_\Gamma G_\varepsilon(x - y)(v^\varepsilon(x_1, x_2, t) - u^\varepsilon(y_2, t)) \varphi(x_1, x_2, t)d\sigma(y_2)d\tilde{x}_2 d\tilde{x}_1 dt.
\]

In this case, we need the fact that we have a well defined and compact trace operator \(Tr : H^1(\Omega) \to L^2(\Gamma)\), see [19], therefore from the weak convergence

\[u^\varepsilon \rightharpoonup u^* \text{ weakly in } L^\infty(0, T; H^1(\Omega)) \text{ as } \varepsilon \to 0,
\]

we obtain that, along a subsequence,

\[u^\varepsilon \to u^* \text{ strongly in } L^\infty(0, T; L^2(\Gamma)) \text{ as } \varepsilon \to 0.
\]

As before, since \(v^\varepsilon\) is bounded in \(L^\infty(0, T; L^2(R))\), then \(V^\varepsilon\) is also bounded in \(L^\infty(0, T; L^2(R_1))\) so, taking a subsequence if necessary

\[V^\varepsilon \rightharpoonup V^* \text{ weakly in } L^\infty(0, T; L^2(R_1)).
\]

Using \((\ref{2.7})\) we obtain

\[-\int_0^T \int_{R_1}^{R_2} \int_\Gamma G_\varepsilon(x - y)(V^\varepsilon(x_1, x_2, t) - |R_2|u^\varepsilon(y_2, t)) \varphi(x_1, t)d\sigma(y_2)d\tilde{x}_2 dt.
\]

Now we can take the limit as \(\varepsilon \to 0\) to obtain

\[
\lim_{\varepsilon \to 0} -\int_0^T \int_{R_1}^{R_2} \int_\Gamma G_\varepsilon(x - y)(V^\varepsilon(x_1, x_2, t) - |R_2|u^\varepsilon(y_2, t)) \varphi(x_1, t)d\sigma(y_2)d\tilde{x}_2 dt \\
= -\int_0^T \int_{R_1}^{R_2} \int_\Gamma G_\varepsilon(x - y)(V^*(x_1, x_2, t) - |R_2|u^*(y_2, t)) \varphi(x_1, t)d\sigma(y_2)d\tilde{x}_2 dt.
\]

The rest of the terms can be handled as in Theorem 1.1 to obtain the weak form of the equations of the limit problem \((\ref{1.12})\).
Uniqueness of solutions to the limit problem can be obtained as in Theorem 1.1 using the energy

\[ E(u^*, V^*) = \frac{|R_2|}{2} \int_{\Omega} |\nabla u^*|^2 dx + \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^*(x - y) (V^*(y) + V^*(x))^2 dy dx 
+ \frac{1}{2} \int_{R_1} \int_{\Gamma} G^*(x - y)(V^*(x) - |R_2|u^*(y))^2 d\sigma(y) dx \]  

(3.5)

This completes the proof.

Now, we gather some properties of the limit problem (1.12).

The existence and uniqueness of the limit problem (1.12) can be obtained using a fixed point argument as it was done in [29].

The mass conservation in time follows as in the previous section (see also [29]).

To deal with the large time behavior we can be proceed as we did before, since the solutions of the problem (1.12) also converges to the mean value of its initial condition as \( t \) goes to zero. Notice that from (1.12) we can define the associated eigenvalue problem. Let us consider \( \alpha_1^* \) given by

\[ \alpha_1^* = \inf_{u^*, V^* \in A_0} \frac{E(u^*, V^*)}{|R_2| \int_{\Omega} |u^*|^2 dx + \int_{R_1} |V^*|^2 dx}, \]

where \( E \) is given by (3.5) and

\[ A_0 = \left\{ u^* \in H^1(\Omega), V^* \in L^2(R_1) : |R_2| \int_{\Omega} u^* dx + \int_{R_1} V^* dx = 0 \right\}. \]

One can show that \( \alpha_1^* \) is strictly positive and then, the computations of the previous section can be adapted to prove that the solution of the limit problem (1.12) converges exponentially fast for the mean value of the initial datum as \( t \) goes to \( \infty \).

4 Numerical experiments

In this section we propose a discrete numerical scheme for the two models (1.8) and (1.12) described in this paper.

To obtain a fully discretization of the equations in space and time we will use classical methods, centered finite differences for the interior points of the local part, forward and backward differences for the boundary points; while for the nonlocal region and the coupling terms we just approximate the involved integrals by Riemann sums. We use an explicit Euler discretization for the time variable.

As we mentioned in the Introduction, the continuous problems (1.8) and (1.12) have some properties: well-posedness, comparison principle, conservation of mass and convergence to the mean value of the initial datum. In this section we will perform numerical simulations that illustrate these properties.

We will assume that \( \Omega \) is a bidimensional rectangle \( \Omega = \Omega_1 \times \Omega_2 = [a, b] \times [c, d] \) and we take the mesh parameter as \( h_1 = \frac{b-a}{M-1} = \frac{d-c}{N-1} \). Let \( h_1 \), be same in the two directions. For the nonlocal part, the domain \( R_1 \) will be the segment \( R_1 = [b, f] \), with \( h_2 = \frac{f-b}{M-1} \). The time step \( \Delta_t \) is give by the difference between the final time, \( t_f \), with the initial time, \( t_0 \).

We approximate the continuous solution \( u(x, y, t) \), for \( (x, y, t) \in \Omega \times \mathbb{R} \) and \( V(x, t) \), for \( (x, t) \in R_1 \times \mathbb{R} \), by discrete values \( u_{i,j,k}^t \approx u(x_i, y_j, t_k) \) and \( V_{i,j}^k \approx V(z_k, t_k) \), respectively, with \( i, k = 1, \cdots, M, j = 1, \cdots, N \). For simplicity, let us consider a uniform mesh for the local and nonlocal part. The local domain \( \Omega \) was discretized by the mesh \( (x_i, y_j) \), with \( i = 1, \cdots, M, j = 1, \cdots, N \) while, the nonlocal domain, \( R_1 \) is discretized by the points \( z_k, k = 1, \cdots, M \).

We will consider \( h = h_1 = h_2 \) (for simplicity). With this in mind, we call \( x_1 = a, x_i = x_{i-1} + h \) and \( x_M = x_{M-1} + h, y_1 = c, x_i = y_{j-1} + h \) and \( y_N = y_{N-1} + h, z_1 = b, z_k = z_{k-1} + h \) and \( z_M = z_{M-1} + h \).
Then, the numerical approximation of the problem (1.8), is given by the following system of equations: for the
local part we have,

\[
\begin{align*}
    u_{l,i,j}^{t+1} &= u_{l,i,j}^t + \frac{\Delta t}{h^2} \left( u_{l+1,i,j}^t + u_{l,i-1,j}^t + u_{l,i,j+1}^t + u_{l,i,j-1}^t - 4u_{l,i,j}^t \right) \\
    &\quad + \Delta \tau h \sum_{k=1}^{M} G(x_i - z_k, y_j) \left( V_k^l - |R_2|u_{l,i,j}^t \right), \quad i = 2, \ldots, M-1, \quad j = 2, \ldots, N-1 \\
    u_{l,1,j}^{t+1} &= u_{l,1,j}^t + \frac{\Delta t}{h^2} \left( u_{l,2,j}^t + u_{l,2,j}^t - 2u_{l,1,j}^t \right) \\
    u_{l,M,j}^{t+1} &= u_{l,M,j}^t + \frac{\Delta t}{h^2} \left( u_{l,M-1,j}^t + u_{l,M-1,j}^t - 2u_{l,M,j}^t \right) \\
    u_{l,1,1}^{t+1} &= u_{l,1,1}^t + \frac{\Delta t}{h^2} \left( u_{l,2,1}^t + u_{l,1,2}^t - 2u_{l,1,1}^t \right) \\
    u_{l,M,1}^{t+1} &= u_{l,M,1}^t + \frac{\Delta t}{h^2} \left( u_{l,M-1,1}^t + u_{l,M-1,1}^t - 2u_{l,M,1}^t \right) \\
    u_{l,N,j}^{t+1} &= u_{l,N,j}^t + \frac{\Delta t}{h^2} \left( u_{l,N-1,j}^t + u_{l,N-1,j}^t - 2u_{l,N,j}^t \right), \quad i = 2, \ldots, M-1 \\
    u_{l,1,N}^{t+1} &= u_{l,1,N}^t + \frac{\Delta t}{h^2} \left( u_{l,2,N}^t + u_{l,1,N-1}^t - 2u_{l,1,N}^t \right) \\
    u_{l,M,N}^{t+1} &= u_{l,M,N}^t + \frac{\Delta t}{h^2} \left( u_{l,M-1,N}^t + u_{l,M-1,N}^t - 2u_{l,M,N}^t \right) \\
    u_{l,1,1}^{t+1} &= u_{l,1,1}^t + \frac{\Delta t}{h^2} \left( u_{l,2,1}^t + u_{l,1,2}^t - 2u_{l,1,1}^t \right), \quad j = 2, \ldots, N-1 \\
    u_{l,M,1}^{t+1} &= u_{l,M,1}^t + \frac{\Delta t}{h^2} \left( u_{l,M-1,1}^t + u_{l,M-1,1}^t - 2u_{l,M,1}^t \right) \\
    u_{l,1,N}^{t+1} &= u_{l,1,N}^t + \frac{\Delta t}{h^2} \left( u_{l,2,N}^t + u_{l,1,N-1}^t - 2u_{l,1,N}^t \right)\end{align*}
\]

(4.1)

for \( l > 0 \) and, for the nonlocal part,

\[
\begin{align*}
    V_{k}^{t+1} &= V_{k}^{t} + \Delta \tau R_2 \sum_{p=1}^{N} J(z_k - z_p)(V_{p}^{t} - V_{k}^{t}) - \Delta \tau h^2 \sum_{i=2}^{M-1} \sum_{j=2}^{N-1} G(x_i - z_k, y_j)(V_{i}^{t} - |R_2|u_{l,i,j}^t), \quad k = 1, \ldots, M \\
    V_{k}^{0} &= V_{k0}, \quad k = 1, \ldots, M, \quad \text{for } l > 0.
\end{align*}
\]

(4.2)
Similarly, the full discretization for the problem \([1,12]\) is given by: for the local part

\[
\begin{align*}
\begin{cases}
    u_{i,j}^{l+1} = u_{i,j}^{l} + \frac{\Delta t}{h^2} \left( u_{i+1,j}^{l} + u_{i-1,j}^{l} + u_{i,j+1}^{l} + u_{i,j-1}^{l} - 4u_{i,j}^{l} \right) \\
    u_{i,1}^{l+1} = u_{i,1}^{l} + \frac{\Delta t}{h^2} \left( u_{i,2}^{l} + 2u_{i,1}^{l} - 2u_{i,0}^{l} \right) \\
    u_{M,1}^{l+1} = u_{M,1}^{l} + \frac{\Delta t}{h^2} \left( u_{M-1,1}^{l} + u_{M,2}^{l} - 2u_{M,1}^{l} \right) \\
    u_{1,M}^{l+1} = u_{1,M}^{l} + \frac{\Delta t}{h^2} \left( u_{1,M-1}^{l} + u_{1,M}^{l} - 2u_{1,M-1}^{l} \right) \\
    u_{i,n}^{l+1} = u_{i,n}^{l} + \frac{\Delta t}{h^2} \left( u_{i+1,n}^{l} + u_{i-1,n}^{l} + u_{i,n+1}^{l} + u_{i,n-1}^{l} - 3u_{i,n}^{l} \right) \\
    i = 1, \ldots, M - 1, \\
    j = 1, \ldots, N,
\end{cases}
\end{align*}
\]

for \(l > 0\) and, for the nonlocal part

\[
\begin{align*}
\begin{cases}
    V_{k}^{l+1} = V_{k}^{l} + \Delta t R_{2} h \sum_{p=1}^{N} J(z_{k} - z_{p}) (V_{p}^{l} - V_{k}^{l}) - \Delta t h \sum_{i=1}^{M} G(x_{i} - z_{k}, y_{N})(V_{i}^{l} - |R_{2}|u_{i,N}^{l}) \\
    V_{k}^{0} = V_{k0}, \quad k = 1, \ldots, M,
\end{cases}
\end{align*}
\]

for \(l > 0\).

Notice that the main difference between the two discretizations occurs at the coupling terms, that in one case are given by

\[
\sum_{k=1}^{M} G(x_{i} - z_{k}, y_{j})(V_{k}^{l} - |R_{2}|u_{i,j}^{l}) \quad \text{and} \quad \sum_{i=2}^{M-1} \sum_{j=2}^{N-1} G(x_{i} - z_{k}, y_{j})(V_{k}^{l} - |R_{2}|u_{i,j}^{l})
\]

(these terms appear in the discretization of the model coupled via source terms, the double sums corresponds to discretizations of double integrals) and in the second discretization by

\[
\sum_{k=1}^{M} G(x_{i} - z_{k}, y_{N})(V_{k}^{l} - |R_{2}|u_{i,N}^{l}) \quad \text{and} \quad \sum_{i=1}^{M} G(x_{i} - z_{k}, y_{N})(V_{k}^{l} - |R_{2}|u_{i,N}^{l})
\]

(this corresponds to coupling on the boundary, remark that the sums here are discretizations of one dimensional integrals).

For the experiments we will consider the domain \(\Omega = [-1,1] \times [-1,1], \) \(R_{1} = [1,3], \) \(R_{2} = [0,1]\) and a time step which satisfies \(\Delta t \leq \frac{h^2}{4}\) (this comes from stability considerations).

At the simulations we will use the kernel \(J,\) given by the following probability density:

\[
J(x) = \begin{cases}
\frac{1}{2} \cos(x), & \text{if } |x| \leq \frac{\pi}{2}, \\
0, & \text{otherwise}.
\end{cases}
\]

This particular kernel \(J\) satisfies the hypothesis described before, \(J\) is a nonnegative continuous function, symmetric, with \(J(0) > 0\) and integrable.

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4.1 Numerical experiments for coupling via source terms.

Now, we will include some numerical experiments considering the fully discrete scheme for the problem (4.1)–(4.2).

In this case, concerning the kernel \( G \), as the problem (1.8) allows that particles can jump directly inside the interior of \( \Omega \), we will consider \( G \) as a function given by

\[
G(x, y) = \begin{cases} 
\frac{1}{4} \cos(x) \cos(y), & \text{if } |x, y| \leq \frac{\pi}{2} \\
0, & \text{otherwise.}
\end{cases}
\]

(4.6)

The kernel \( G \) satisfies the hypothesis defined in the Introduction.

**Numerical experiment 1.** For this simulation we consider \( M = N = 11, h = 0.2, \Delta t = 0.005 \), as initial conditions, we used \( u_0(x, y) = 0, V_0(x) = 1 \). The mean value of the initial condition is \( \approx 0.083 \).

In Figure 3 we plot the evolution of the local and the nonlocal parts of the solution (for the local part we have depicted the solution \( u(x, y, t) \) at three different time steps, as the same for the nonlocal part of the solution, \( V(x, t) \), we can observe its evolution in four time steps). Both local and nonlocal parts of the solution converge towards the mean value of the numerical initial condition as \( t \) increases.

![Figure 3: The local part (left) and the nonlocal part (right) with two constants as initial conditions.](image)

**Numerical experiment 2.** For this simulation we consider \( M = N = 11, h = 0.2, \Delta t = 0.005, \) as initial conditions, we used \( u_0(x, y) = \cos\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi y}{2}\right), V_0(x) = 1 \). Now, the mean value of the initial condition \( \approx 0.38 \).

Figure 4 contains the plot of the local and the nonlocal parts of the solution. One can see that even with a not constant initial condition for the local part, we observe its fast convergence towards the mean value of the initial condition as \( t \) increases.
Numerical experiment 3. For this simulation we consider $M = N = 11$, $h = 0.2$, $\Delta t = 0.005$, as initial conditions, we used $u_0(x, y) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right)$, $V_0(x) = 9 - x^2$. The mean value of the initial condition is $\approx 0.68$.

In Figure 5 both local and nonlocal initial conditions are non constants and they also verify the convergence to the mean of the initial condition as $t$ increases. Note that, even for $t = 1$ the solution of the local part is closer to the mean value of the initial condition. For the nonlocal part, as $t = 10$ the solution is very close to the mean of the initial condition that is subscribed by the last iteration.
4.2 Numerical experiments for coupling via boundary terms.

Now, we will include some numerical experiments considering the fully discrete scheme for the problem \([1.12]\) given by \((4.3)-(4.4)\).

At the simulations we will use the same kernel \(J\), as we define in \((4.5)\) and the kernel \(G\) as we define in \((4.6)\).

For simplicity, we have considered the local domain \(\Omega\) as a square \(\Omega = [-1,1] \times [-1,1]\), then for the coupling we will consider \(\Gamma\) as a whole side of the domain \(\Omega\), \(\Gamma = \{1\} \times [-1,1]\).

**Numerical experiment 4.** For this simulation we consider \(M = N = 11\), \(h = 0,2\), \(\Delta t = 0,005\), as initial conditions, we used \(u_0(x,y) = 0\), \(V_0(x) = 1\). Mean value of the initial condition \(\approx 0,31\).

In Figure 6 we plot the evolution of the local and the nonlocal parts of the solution. Both local and nonlocal parts of the solution converge towards the mean value of the numerical initial condition as \(t\) increases.

**Numerical experiment 5.** For this simulation we consider \(M = N = 11\), \(h = 0,2\), \(\Delta t = 0,005\), as initial conditions, we used \(u_0(x,y) = x^2 + y^2\), \(V_0(x) = 9 - x^2\). Mean value of the initial condition \(\approx 1,99\).

In Figure 7, we observe that also when we take two non-constants initial conditions, the solution converges towards the mean value of the numerical initial condition as \(t\) increases. We plot the solutions for specific time steps to follow the evolution.
Numerical experiment 6. For this simulation we consider $M = N = 11$, $h = 0.1$, $\Delta t = 0.005$, as initial conditions, we used $u_0(x, y) = x^2 + y^2$, $V_0(x) = x^2$. Mean value of the initial condition $\approx 1.92$. In Figure 8, we define the same initial condition for the local and nonlocal part. Note that we obtain the same behavior along the time, both local and nonlocal solution converge to the mean value of the initial condition.
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References

[1] Andreu-Vaillo, F.; Toledo-Melero, J.; Mazon, J. M.; Rossi, J. D. Nonlocal diffusion problems. Number 165. American Mathematical Soc., 2010.

[2] Arrieta, J. M.; Carvalho, A. N.; Lozada-Cruz, G. Dynamics in dumbbell domains I. Continuity of the set of equilibria. Journal of Differential Equations, 231(2), 551-597, 2006.

[3] Arrieta, J. M.; Carvalho, A. N.; Lozada-Cruz, G. Dynamics in dumbbell domains II. The limiting problem. Journal of Differential Equations, 247(1), 174-202, 2009.

[4] Arrieta, J. M.; Carvalho, A. N.; Lozada-Cruz, G. Dynamics in dumbbell domains III. Continuity of attractors. Journal of Differential Equations, 247(1), 225-259, 2009.

[5] Arrieta, J. M.; Pereira, M. C. Homogenization in a thin domain with an oscillatory boundary. Jour. Math Pures Appl., 96(1), (2011), 29–57.

[6] Bates, P.; Chmaj, A. An integrodifferential model for phase transitions: stationary solutions in higher dimensions. J. Statist. Phys. 95 (1999), no. 5–6, 1119–1139.

[7] Berestycki, H., Coulon, A.-Ch.; Roquejoffre, J-M.; Rossi, L. The effect of a line with nonlocal diffusion on Fisher-KPP propagation. Math. Models Meth. Appl. Sciences, 25.13, (2015), 2519–2562.

[8] Carrillo, C.; Fife, P. Spatial effects in discrete generation population models. J. Math. Biol. 50 (2005), no. 2, 161–188.

[9] Chasseigne, E.; Chaves, M.; Rossi, J. D. Asymptotic behavior for nonlocal diffusion equations. J. Math. Pures Appl. 96(1), (2011), 29–57.

[10] Cortázar, C.; M. Elgueta, M.; Rossi, J. D.; Wolanski, N. Boundary fluxes for non-local diffusion. J. Differential Equations 234 (2007), no. 2, 360–390.

[11] Cortázar, C.; M. Elgueta, M.; Rossi, J. D.; Wolanski, N. How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems. Arch. Ration. Mech. Anal. 187 (2008), no. 1, 137–156.

[12] Bayada, G.; Chupin, L.; Martin, S. Viscoelastic fluids in a thin domain. Quart. Appl. Math., 65(4), (2007), 625-651.

[13] D’Elia, M.; Bochev, P. Formulation, analysis and computation of an optimization-based local-to-nonlocal coupling method. arXiv preprint arXiv:1910.11214, 2019.

[14] D’Elia, M.; Du, Q.; Gunzburger M.; Lehoucq, R. Nonlocal convection-diffusion problems on bounded domains and finite-range jump processes. Comput. Methods Appl. Math. 17 (2017), no. 4, 707–722.

[15] D’Elia, M.; Perego, M.; Bochev, P.; Littlewood, D. A coupling strategy for nonlocal and local diffusion models with mixed volume constraints and boundary conditions. Comput. Math. Appl. 71 (2016), no. 11, 2218–2230.

[16] D’Elia, M.; Ridzal, D.; Peterson, K. J.; Bochev, P.; Shashkov, M. Optimization-based mesh correction with volume and convexity constraints. J. Comput. Phys. 313 (2016), 455–477.

[17] D’Elia, M.; Li, X.; Seleson, P.; Tian, X.; Yu, Y. A review of Local-to-Nonlocal coupling methods in nonlocal diffusion and nonlocal mechanics. to appear in Jour. Peridynamics Nonlocal Modeling, 2020.
[18] Du, Q.; Li, X.H.; Lu, J.; Tian, X. A quasi-nonlocal coupling method for nonlocal and local diffusion models. SIAM J. Numer. Anal. 56 (2018), no. 3, 1386–1404.

[19] Evans, L.C. Partial Differential Equations. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[20] Fife, P. Some nonclassical trends in parabolic and parabolic-like evolutions. In “Trends in nonlinear analysis”, 153–191, Springer, Berlin, 2003.

[21] Fife, P.; Wang, X. A convolution model for interfacial motion: the generation and propagation of internal layers in higher space dimensions. Adv. Differential Equations 3 (1998), no. 1, 85–110.

[22] Gal, C. G.; Warma, M. Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces. Communications in Partial Differential Equations, 42(4) (2017), 579–625.

[23] Gárriz, A.; Quirós, F.; Rossi, J. D. Coupling local and nonlocal evolution equations. Calc. Var. PDE, 59(4), art. 112, (2020).

[24] Han, F., Gilles L.; Coupling of nonlocal and local continuum models by the Arlequin approach. Inter. Journal Numerical Meth. Engineering 89.6 (2012): 671–685.

[25] Iftimie, D.; Raugel, G.; Sell, G. R. Navier-Stokes equations in thin 3D domains with Navier boundary conditions. Indiana Univ. Math. Jour., (2007), 1083–1156.

[26] Kriventsov, D. Regularity for a local-nonlocal transmission problem. Arch. Ration. Mech. Anal. 217 (2015), 1103–1195.

[27] Pereira, M. C.; Rossi, J. D. Nonlocal evolution problems in thin domains. Appl. Anal., 97(12), (2018), 2059–2070.

[28] Rodríguez, J. M.; Viaño, J. M. Asymptotic analysis of Poisson’s equation in a thin domain and its application to thin-walled elastic beams and tubes. Mathematical methods in the applied sciences, 21(3), (1998), 187–226.

[29] dos Santos, B. C.; Oliva, S. M.; Rossi, J. D. A local/nonlocal diffusion model. To appear in Applicable Analysis. arXiv preprint: 2003.02015, 2020.

[30] Seleson, P., Samir B., Serge P.; A force-based coupling scheme for peridynamics and classical elasticity. Computational Materials Science 66 (2013), 34–49.

[31] Silling, S. A.: Reformulation of elasticity theory for discontinuities and long-range forces. Jour. Mech. Physics Solids, 48(1), 2000, 175—209.

[32] Silling, S. A.; Lehoucq, R. B.: Peridynamic theory of solid mechanics. In Advances in applied mechanics (Vol. 44, pp. 73-168). Elsevier, 2010.

[33] Shiuichi, J.; Yoshihisa, M. Remarks on the behavior of certain eigenvalues on a singularly perturbed domain with several thin channels. Comm. Partial Differential Equations, (1992), 17(3-4), 189–226.

[34] Strickland, C.; Gerhard D.; Patrick D. S.; Modeling the presence probability of invasive plant species with nonlocal dispersal. Jour. Math. Biology 69.2 (2014), 267–294.

[35] Wang, X. Metastability and stability of patterns in a convolution model for phase transitions. J. Differential Equations 183 (2002), no. 2, 434–461.

[36] Wang, W.; Xiao-Qiang Z.; A nonlocal and time-delayed reaction-diffusion model of dengue transmission. SIAM Jour. Appl. Math. 71.1 (2011), 147–168.

[37] Zhang, L. Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks. J. Differential Equations 197 (2004), no. 1, 162–196.
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