THE FREE PRODUCT OF TOPOLOGICAL GROUPS
BEING HAUSDORFF IS HAUSDORFF – A NEW PROOF

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Abstract. The explicit description of the topology of the free product of topological groups being Hausdorff is given. In particular, it is shown that it coincides with the so-called $X_0$-topology for the corresponding colimit $X$ in the category of topological spaces. Applying this fact, a new short proof of the well-known Graev’s theorem asserting that the free product of topological groups being Hausdorff is Hausdorff is given.

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1. Introduction

In [4] Graev has proved that the free product of topological groups being Hausdorff is Hausdorff. However, Graev’s proof uses the complicated technique of pseudometrics. Moreover, it does not give an explicit description of the topology of the free product. The aim of this paper is to give such a description and then, applying it, to give a new short proof of the Graev’s result.

To give here the found description, we first note that since the forgetful functor

$$\mathcal{F} : \text{Top}(\text{Grp}) \rightarrow \text{Grp},$$

has a right adjoint (sending a discrete group to itself, but equipped with an antidiscrete topology), the free product of topological groups algebraically coincides with their free product in the category of discrete groups. Here $\text{Top}(\text{Grp})$ denotes the category of topological groups, while $\text{Grp}$ denotes the category of discrete groups.

In this paper we prove that the topology of the free product $G$ of Hausdorff topological groups $G_i$ ($i \in I$, $I$ is an arbitrary set) coincides with the $X_0$-topology in the sense of Mal’tsev, where $X$ is the union of sets $G_i$ equipped with the topology, where open subsets are those ones $W$ for which $W \cap G$ is open in $G_i$, for any $i$ from $I$. The proved fact implies the following characterization of the topology of $G$. A subset $O$
of $G$ is open if and only if $O \cap G_i$ is open in $G_i$, for any $i$, and, moreover, for any $n \geq 2$ and $x_1, x_2, \ldots, x_n \in X$ with

$$x_1 x_2 \cdots x_n \in O,$$

there exist neighborhoods $W_1, W_2, \ldots, W_n$ in $X$ of resp. $x_1, x_2, \ldots, x_n$ with

$$W_1 W_2 \cdots W_n \subset O.$$

Finally we point out that, as is well-known, there is a common construction for a free group and the free product of groups, when both are considered as groups defined by generators and relations. In [4] Graev posed the problem of finding the common construction for a free Hausdorff topological group and the free product of Hausdorff topological groups, and by this to provide one more parallel between the theory of groups and the theory of topological groups. Since the free Hausdorff group over a completely regular space $X$ has the $X_0$-topology, as it was proved first by Mal’tsev [7] in the case of a compact $X$ and later by Burgin [2] in the general case of any completely regular space $X$, our construction of the free product of Hausdorff groups gives an answer to the posed problem.

2. THE TOPOLOGY OF THE FREE PRODUCTS OF TOPOLOGICAL GROUPS

In [7] Markov has introduced the notion of the free Hausdorff topological group over a completely regular space and proved its existence. This existence immediately implies that Markov’s definition is equivalent to the following one. For a completely regular topological space $X$, a Hausdorff topological group $F$ equipped with a continuous mapping $\eta : X \to F$ is called a free Hausdorff topological group over $X$ if for any Hausdorff topological group $G$ and any continuous mapping $\theta : X \to G$, there exists a unique continuous homomorphism $\phi : F \to G$ with $\phi \eta = \theta$. One can define a free topological group over a topological space as a topological group satisfying the universality condition obtained from the formulated one by omitting all “Hausdorff” in it. Applying the fact that the partially ordered set of topologies rendering a group to be a topological group is a complete sup-semilattice (Graev [3]), one can easily prove the existence of the free topological group over any topological space and the following

**Lemma 2.1.** As a group, the free topological group over a topological space $X$ is the free group (with the same canonical mapping) over the set $X$. 

The issue whether the similar statement is valid for free Hausdorff topological groups is by no means so simple. The corresponding result is one of the central ones in Markov’s theory. Namely, we have

**Theorem 2.2** (Markov [7]). As a group, the free Hausdorff topological group over a completely regular space $X$ is the free group (with the same canonical mapping) over the set $X$. Moreover, the canonical mapping $\eta$ is a closed homeomorphic embedding.

We obtain

**Proposition 2.3.** For a completely regular space $X$, the free topological group $F$ and the free Hausdorff topological group $F'$ over $X$ are isomorphic as topological groups (and this isomorphism is compatible with the canonical mappings).

**Proof.** We have a continuous homomorphism $h : F \rightarrow F'$ such that $h\eta = \eta'$. From Lemma 2.1 and Theorem 2.2 we conclude that $h$ is an isomorphism of groups, and hence $F$ is Hausdorff. This implies that $h$ is an isomorphism of topological groups. \hfill $\square$

Let $X$ be a topological space, and let $G$ be a topological group containing $X$ as a subspace. One has the so-called $X_0$-topology on $G$ (Mal’tsev [6]). The open sets in this topology are subsets $O$ such that for any group term $t(x_1, x_2, \ldots, x_n)$ on the variables $x_1, x_2, \ldots, x_n$ from $X$ with

$$t(x_1, x_2, \ldots, x_n) \in O,$$

there exist neighborhoods $W_1, W_2, \ldots, W_n$ in $X$ of resp. $x_1, x_2, \ldots, x_n$ such that

$$t(W_1, W_2, \ldots, W_n) \subset O.$$

Below we will use the notion of the $X_0$-topology in the wider context of an arbitrary (discrete or not discrete) group containing a topological space $X$ as a subset. Under this assumption the following statement is valid.

**Lemma 2.4.** Let $G$ be a topological group, and let $X$ be its subspace. Let $N$ be a normal subgroup of $G$, and $X/N$ be the quotient-set determined by the equivalence relation induced by $N$, equipped with the quotient–topology. Let $G$ have the $X_0$-topology. Then the quotient–group $G/N$ has the $(X/N)_0$-topology.

**Proof.** Let $O$ be open in the $(X/N)_0$-topology. Consider the union $O'$ of equivalence classes from $O$ and show that it is open in the $X_0$-topology. To this end, consider $x_1, x_2, \ldots, x_n \in X$ and a term $t$ with

$$t(x_1 x_2 \cdots x_n) \in O'.$$
Then
\[ t([x_1][x_2] \cdots [x_n]) \in O, \]
for the equivalence classes \([x_1], [x_2], \ldots, [x_n]\) containing resp. \(x_1, x_2, \ldots, x_n\). Hence there are open neighborhoods \(W_1, W_2, \ldots, W_n\) in \(X/N\) of resp. \([x_1], [x_2], \ldots, [x_n]\) with
\[ t(W_1W_2 \cdots W_n) \subset O. \]
The unions \(W'_1, W'_2, \ldots, W'_n\) of equivalence classes from resp. \(W_1, W_2, \ldots, W_n\) are open in \(X\) and contain resp. \(x_1, x_2, \ldots, x_n\). Moreover,
\[ t(W'_1W'_2 \cdots W'_n) \subset O'. \]
Thus \(O\) is open in the quotient–topology.
The converse follows from the fact that the operations of \(A/N\) are continuous in the quotient-topology. \qed

One can easily verify the following

**Lemma 2.5.** Let \(G\) be a group and \(X\) be a topological space being a subset of \(G\). If \(X\) is closed under inverse elements and the operation of taking such an element is compatible with respect to the topology of \(X\), then a subset \(O\) of \(G\) is open in the \(X_0\)-topology if and only if the following conditions are satisfied:

(i) \(O \cap X\) is open in \(X\);
(ii) for any \(n \geq 2\) and \(x_1, x_2, \ldots, x_n \in X\) with
\[ x_1x_2 \cdots x_n \in O, \]
there exist neighborhoods \(W_1, W_2, \ldots, W_n\) in \(X\) of resp. \(x_1, x_2, \ldots, x_n\) with
\[ W_1W_2 \cdots W_n \subset O. \]

Note that, in general, the \(X_0\)-topology is not compatible with the algebraic structure of \(G\). However, sometimes this is the case.

**Theorem 2.6.** (Burgin [2]) The free Hausdorff topological group over a completely regular space \(X\) has the \(X_0\)-topology.

**Remark 2.7.** This theorem, in the case of a compact space \(X\), was first proved by Mal’tsev in [6]. However, in that case, the first characterization of the topology of a free Hausdorff group was found earlier by Graev [3], but it was given in different terms. After Graev the issue of characterizing this topology was studied by many authors. We refer the interested reader to the survey paper [1] by Arkhangel’skii and more recent [8] by Sipacheva.
Let $I$ be an arbitrary set and $G_i$ be a topological group, for any $i \in I$. Without loss of generality we will assume that $G_i \cap G_j = \{1\}$, for any $i, j \in I$. Let $G$ be the free product of $(G_i)_{i \in I}$ with the canonical morphisms $\varphi_i : G_i \to X$, and let $X$ be the colimit of the diagram $(\{1\} \to G_i)_{i \in I}$ in the category of topological spaces, with the canonical morphisms $\psi_i : G_i \to G$. Recall that $X$, as a set, is the union of groups $G_i$ ($i \in I$), while a subset $W$ is open in it if and only if, for all $i$, the intersection $W \cap A_i$ is open in $A_i$.

We have a continuous mapping $\omega : X \to G$ such that $\omega \psi_i = \varphi_i$, for all $i \in I$.

**Theorem 2.8.** Let all $G_i$ be Hausdorff. Then $\omega$ is injective and the free product $G$ has the $X_0$-topology. A subset $O$ of $G$ is open in this topology if and only if the conditions (i) and (ii) of Lemma 2.5 are satisfied.

**Proof.** The injectivity of $\omega$ follows from the fact that the functor $\mathcal{F}$ mentioned in the Introduction preserves free products. According to Pontryagin’s theorem, all groups $G_i$ are completely regular. Hence the space $X$ is also of this kind. Consider the free topological group $F$ with the canonical mapping $\eta : X \to F$. By Proposition 2.3 the pair $(F; \eta)$ is the free Hausdorff topological group over $X$. According to Theorem 2.4, the group $F$ has the $X_0$-topology. It is obvious that $G$ is the quotient-algebra of $F$ by some normal subgroup $N$. Obviously $N$ induces the trivial equivalence relation on $X$, and hence $X/N$ is homeomorphic to $X$. Now Lemma 2.4 implies that $G$ has the $X_0$-topology. The rest of the proof follows from Lemma 2.5. □

### 3. A New Proof of Graev’s Theorem

We first give some auxiliary facts on free products of discrete groups $G_i$ ($i \in I$) (in the category of such groups).

Let $t$ be a group term

\[
x_1 x_2 \ldots x_n
\]

with $x_1, x_2, \ldots, x_n \in X$, where $X$ is the union of groups $G_i$. Under a subterm of $t$ we mean any term of the form

\[
x_{k_1} x_{k_2} \ldots x_{k_l},
\]

where $1 \leq k_1 < k_2 < \ldots < k_l \leq n$. A subterm (3.2) is called uniform if all $x_{k_1}, x_{k_2}, \ldots, x_{k_l}$ lie in one and the same $G_i$.

The following two lemmas easily follow from the well-known construction of the free product of groups (see, for instance, [5]).

**Lemma 3.1.** If the value of a term (3.1) in the free product $G$ of groups $G_i$ ($i \in I$) is equal to 1, then, for any $i$, the product of all elements $x_k$ belonging to $G_i$, taken in the same order as in (3.1) is equal to 1.
Lemma 3.2. Let $t$ and $t'$ be terms

$$x_1x_2...x_n$$

and

$$y_1y_2...y_n$$

resp. over $X$, such that the following conditions are satisfied:

(i) if a subterm

$$y_{k_1}y_{k_2}...y_{k_1},$$

of $t'$ is uniform, then the corresponding subterm (3.2) of $t$ is uniform as well;

(ii) if a subterm (3.2) of $t$ is uniform and its value in $G$ differs from 1, then the value of the subterm (3.3) of $t'$ also differs from 1.

Under these assumptions, if the value of $t$ in $G$ differs from 1, then the same is valid for $t'$.

Proof. Suppose the value of $t'$ is equal to 1. This means that there exists a sequence of cancellation transformations reducing $t'$ to 1. In each step the cancellation is performed on some uniform subterm of $t'$. The corresponding subterm of $t$ is uniform, according to the condition (i). But since the value of $t$ differs from 1, some of these subterms also differ from 1. But then, by the condition (ii), the corresponding subterm of $t'$ also is not equal to 1. Contradiction. □

Theorem 3.3. The free product of Hausdorff topological groups in the category of (not-necessarily Hausdorff) topological groups is Hausdorff.

Proof. From Theorem 2.8 we know that the free product $G$ of topological groups $G_i$ has the $X_0$-topology. We will show that the set $G \setminus \{1\}$ is open in this topology. Since all $G_i$ are Hausdorff, the intersection of this set with $X$ is open in $X$. Let us now verify that if the value in $G$ of the term

$$x_1x_2...x_n$$

where $n \geq 1$ and $x_1, x_2, ..., x_n \in X$, differs from 1, then there exist neighbourhoods $W_1, W_2, ..., W_n$ in $X$ of $x_1, x_2, ..., x_n$ with

$$1 \notin W_1W_2...W_n.$$  (3.5)

We first consider the case where (3.4) is a uniform term. Let all $x_1, x_2, ..., x_n$ lie in $G_i$. Since the group $G_i$ is Hausdorff, there exist neighbourhoods $U_1, U_2, ..., U_n$ in $G_i$ of $x_1, x_2, ..., x_n$ with

$$1 \notin U_1U_2...U_n.$$  (3.5)

Let us, for $x_m \neq 1$, take

$$W_m = U_m \cap (G_i \setminus \{1\}),$$  (3.6)
while for $x_m = 1$, take
\[ W_m = U_m \cup \bigcup_{j \in I, j \neq i} V_j \]
where $V_j$ is any neighbourhood of 1 in $G_j$. Lemma 3.2 implies that such system of neighbourhoods is desired.

Suppose now that (3.4) is not necessarily uniform. For any uniform subterm $t' = x_{k_1}x_{k_2}...x_{k_l}$ of (3.4) whose value differs from 1, we, by the above-proved, have neighbourhoods $W'_{k_1}, W'_{k_2}, ..., W'_{k_l}$ in $X$ of resp. $x_{k_1}, x_{k_2}, ..., x_{k_l}$ with
\[ 1 \notin W'_{k_1}W'_{k_2}...W'_{k_l} \]
and such that if $x_{k_r}$ ($1 \leq r \leq l$) is not equal to 1 and lies in $G_i$, then
\[ W'_{k_r} \subset G_i \setminus \{1\}. \] (3.7)

Let us now, for any $x_m$, take the neighbourhood
\[ W_m = \bigcap_{t' \neq 1, \text{ \textit{t'} is uniform}} W'_m. \]

Consider any term
\[ y_1y_2...y_n \] (3.8)
with $y_1 \in W_1, y_2 \in W_2, ..., y_n \in W_n$. If $x_m \neq 1$ ($1 \leq m \leq n$), then obviously $W_m \subset W'_m$, for $t' = x_m$. Then (3.7) implies that if $y_m \in G_i$, then $x_m \in G_i$. Thus (3.8) satisfies the condition (i) of Lemma 3.2. The validity of the condition (ii) is obvious. Hence we have (3.5). \[ \square \]

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