An eigenvalue localization theorem for stochastic matrices and its application to Randić matrices

Anirban Banerjee\textsuperscript{1,2} and Ranjit Mehatari\textsuperscript{1}

\textsuperscript{1}Department of Mathematics and Statistics
\textsuperscript{2}Department of Biological Sciences
Indian Institute of Science Education and Research Kolkata
Mohanpur-741246, India
\{anirban.banerjee, ranjit1224\}@iiserkol.ac.in

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Abstract

A square matrix is called stochastic (or row-stochastic) if it is non-negative and has each row sum equal to unity. Here, we constitute an eigenvalue localization theorem for a stochastic matrix, by using its principal submatrices. As an application, we provide a suitable bound for the eigenvalues, other than unity, of the Randić matrix of a connected graph.

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1 Introduction

Stochastic matrices occur in many fields of research, such as, computer-aided-geometric designs [17], computational biology [16], Markov chains [20], etc. A stochastic matrix $S$ is irreducible if its underlying directed graph is strongly connected. In this paper, we consider $S$ to be irreducible. Let $e$ be the column vector whose all entries are equal to 1. Clearly, 1 is an eigenvalue of $S$ with the corresponding eigenvector $e$. By Perron-Frobenius theorem (see Theorem 8.4.4 in [8]), the multiplicity of the eigenvalue 1 is one and all other eigenvalues of $S$ lie in the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. The eigenvalue 1 is called the Perron eigenvalue (or Perron root) of the matrix $S$, whereas, the eigenvalues other than 1 are known as non-Perron eigenvalues of $S$. 


Here, we describe a method for localization of the non-Perron eigenvalues of $S$. The eigenvalue localization problem for stochastic matrices is not new. Many researchers gave significant contribution to this context \cite{6, 9, 10, 12, 13}. In this paper, we use Geršgorin disc theorem \cite{7} to localize the non-Perron eigenvalues of $S$. Cvetković et al. \cite{6} and Li et al. \cite{12, 13} derived some useful results, using the fact that any non-Perron eigenvalue of $S$ is also an eigenvalue of the matrix $S-(ee^T)\text{diag}(c_1, c_2, \cdots, c_n)$, where $c_1, c_2, \cdots, c_n \in \mathbb{R}$.

In \cite{6}, Cvetković et al. found a disc which contains all the non-Perron eigenvalues of $S$.

**Theorem 1.1.** \cite{6} Let $S = [s_{ij}]$ be a stochastic matrix, and let $s_i$ be the minimal element among the off-diagonal entries of the $i$-th column of $S$. Taking $\gamma = \max_{i \in \mathbb{N}}(s_{ii} - s_i)$, for any $\lambda \in \sigma(S) \setminus \{1\}$, we have

$$|\lambda - \gamma| \leq 1 - \text{trace}(S) + (n - 1)\gamma.$$ 

Theorem \[1.1\] was further modified by Li and Li \cite{12}. They found another disc with different center and different radius.

**Theorem 1.2.** \cite{12} Let $S = [s_{ij}]$ be a stochastic matrix, and let $S_i = \max_{j \neq i} s_{ji}$. Taking $\gamma' = \max_{i \in \mathbb{N}}(S_i - s_{ii})$, for any $\lambda \in \sigma(S) \setminus \{1\}$, we have

$$|\lambda + \gamma'| \leq \text{trace}(S) + (n - 1)\gamma' - 1.$$ 

In this paper, we show that there exist square matrices of order $n - 1$, whose eigenvalues are the non-Perron eigenvalues of $S$. We apply Geršgorin disc theorem to those matrices in order to obtain our results. We provide an example where our result works better than Theorem 1.1 and Theorem 1.2.

Let $\Gamma = (V, E)$ be a simple, connected, undirected graph on $n$ vertices. Two vertices $i, j \in V$ are called neighbours, written as $i \sim j$, if they are connected by an edge in $E$. For a vertex $i \in V$, let $d_i$ be its degree and $N_i$ be the set neighbours of the vertex $i$. For two vertices $i, j \in V$, let $N(i, j)$ be the number of common neighbours of $i$ and $j$, that is, $N(i, j) = |N_i \cap N_j|$. Let $A$ denote the adjacency matrix \cite{5} of $\Gamma$ and let $D$ be the diagonal matrix of vertex degrees of $\Gamma$. The Randić matrix $R$ of $\Gamma$ is defined by $R = D^{-1/2}AD^{-1/2}$ which is similar to the matrix $R = D^{-1}A$. Thus, the matrices $R$ and $R$ have the same eigenvalues. The matrix $R$ is an irreducible stochastic matrix and its $(i,j)$-th entry is

$$R_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

The name Randić matrix was introduced by Bozkurt et al. \cite{3} because $R$ has a connection with Randić index \cite{14, 18}. In recent days, Randić matrix becomes more popular to researchers. The Randić matrix has a direct connection with normalized Laplacian matrix $L = I_n - R$ studied in \cite{4} and with $\Delta = I_n - R$ studied is \cite{11, 15}. Thus, for any graph $\Gamma$, if $\lambda$ is an eigenvalue of the normalized Laplacian matrix, then $1 - \lambda$ is an eigenvalue of the Randić matrix.
In Section 3, we localize non-Perron eigenvalues of $\mathcal{R}$. We provide an upper bound for the largest non-Perron eigenvalue and a lower bound for the smallest non-Perron eigenvalue of $\mathcal{R}$ in terms of common neighbours of two vertices and their degrees. The eigenvalue bound problem was studied previously in many articles [2, 4, 11, 19], but the lower bound of the smallest eigenvalue of $\mathcal{R}$ given by Rojo and Soto [19] is the only one which involves the same parameters as in our bound. We recall the Rojo-Soto bound for Randić matrix.

**Theorem 1.3.** [19] Let $\Gamma$ be a simple undirected connected graph. If $\rho_n$ is the eigenvalue with the largest modulus among the negative Randić eigenvalues of $\Gamma$, then

$$|\rho_n| \leq 1 - \min_{i \sim j} \left\{ \frac{N(i,j)}{\max\{d_i, d_j\}} \right\},$$

where the minimum is taken over all pairs $(i, j)$, $1 \leq i < j \leq n$, such that the vertices $i$ and $j$ are adjacent.

One of the drawbacks of Theorem 1.3 is that it always produces the trivial lower bound of $\rho_n$, if the graph contains an edge which does not participate in a triangle. Though the bound in Theorem 1.3 and our bound (Theorem 3.1) are incomparable but, in many occasions, our bound works better than Rojo-Soto bound. We illustrate this by a suitable example.

## 2 Localization of the eigenvalues of an irreducible stochastic matrix

Let $e_1, e_2, \ldots, e_n$ be the standard orthonormal basis for $\mathbb{R}^n$ and let $e' = [1 \ -1 \ -1 \ \cdots \ -1]^T$. For $k \geq 1$, let $j_k$ be the $k \times 1$ matrix with each entry equal to 1 and $0_k$ be the $k \times 1$ zero matrix. We define the matrix $P$ as

$$P = [\ e \ e_2 \ e_3 \ \cdots \ e_n \ ].$$

It is easy to verify that the matrix $P$ is nonsingular and its inverse is

$$P^{-1} = [\ e' \ e_2 \ e_3 \ \cdots \ e_n \ ].$$

We use $S(i|i)$ to denote the principal submatrix of $S$ obtained by deleting $i$-th row and the $i$-th column. Now we have the following theorem.

**Theorem 2.1.** Let $S$ be a stochastic matrix of order $n$. Then $S$ is similar to the matrix

$$\begin{bmatrix} 1 & x^T \\ 0_{n-1} & B \end{bmatrix}$$

where $x^T = [s_{12} \ s_{13} \ \cdots \ s_{1n}]$, and $B = S(1|1) - j_{n-1}x^T$. 

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Proof. Let \( y = [ s_{21} \ s_{31} \ \ldots \ s_{n1} ]^T \). Then the matrices \( S, P, P^{-1} \) can be partitionoid as,

\[
S = \begin{bmatrix}
s_{11} & x^T \\
y & S(1|1)
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
1 & 0^T_{n-1} \\
j_{n-1} & I_{n-1}
\end{bmatrix},
\]

\[
P^{-1} = \begin{bmatrix}
1 & 0^T_{n-1} \\
-j_{n-1} & I_{n-1}
\end{bmatrix}.
\]

Now

\[
P^{-1}SP = \begin{bmatrix}
1 & 0^T_{n-1} \\
-j_{n-1} & I_{n-1}
\end{bmatrix} \begin{bmatrix}
1 & 0^T_{n-1} \\
y & S(1|1)
\end{bmatrix} \begin{bmatrix}
1 & 0^T_{n-1} \\
j_{n-1} & I_{n-1}
\end{bmatrix} = \begin{bmatrix}
1 & 0^T_{n-1} \\
j_{n-1} & I_{n-1}
\end{bmatrix}
\]

\[
\sum_{j=1}^{n}s_{1j}x^T - s_{11}j_{n-1} + S(1|1)j_{n-1} - j_{n-1}x^Tj_{n-1} - S(1|1)j_{n-1}x^T.
\]

For \( i = 2, 3, \ldots, n \), we have \((P^{-1}SP)_i = s_{i1} - s_{11} + \sum_{j=2}^{n}s_{ij} - \sum_{j=2}^{n}s_{1j} = 0\) and hence the result follows.

\[\square\]

**Theorem 2.2.** Let \( S = [s_{ij}] \) be a stochastic matrix of order \( n \). Then any eigenvalue other than 1 is also an eigenvalue of the matrix

\[
S(k) = S(k|k) - j_{n-1}s(k)^T, \ k = 1, 2, \ldots, n
\]

where \( s(k)^T = [s_{k1} \ \ldots \ s_{k,k-1} \ s_{k,k+1} \ \ldots \ s_{kn}] \) is the \( k \) deleted row of \( S \).

Proof. If \( k = 1 \) then the proof is straightforward from Theorem 2.1.

For \( k > 1 \), consider the permutation matrix \( P_k = [e_2 \ e_3 \ \ldots \ e_k \ e_1 \ e_1+1 \ \ldots \ e_n] \).

Therefore, the matrix \( S \) is similar to the matrix

\[
P_k^{-1}SP_k = \begin{bmatrix}
s_{kk} & x^T \\
y & S(k|k)
\end{bmatrix},
\]

where \( x = s(k) = [s_{k1} \ \ldots \ s_{k,k-1} \ s_{k,k+1} \ \ldots \ s_{kn}]^T \)

and \( y = [s_{1k} \ \ldots \ s_{k-1,k} \ s_{k+1,k} \ \ldots \ s_{nk}]^T \).

Now, applying Theorem 2.1 to \( P_k^{-1}SP_k \), we get that \( S \) is similar to the matrix

\[
\begin{bmatrix}
1 & s(k) \\
0_{n-1} & S(k|k) - j_{n-1}s(k)^T
\end{bmatrix}.
\]

Thus, any eigenvalue of \( S \), other than 1, is also an eigenvalue of the matrix \( S(k), k = 1, 2, \ldots, n \). \[\square\]
Theorem 2.3. (Geršgorin[7]) Let $A = [a_{ij}]$ be an $n \times n$ complex matrix. Then the eigenvalues of $A$ lie in the region
\[ G_A = \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}. \]

Theorem 2.4. Let $S$ be a stochastic matrix of order $n$. Then the eigenvalues of $S$ lie in the region
\[ \bigcap_{i=1}^{n} \left[ G_{S(i)} \cup \{1\} \right], \]
where $G_{S(i)} = \bigcup_{k \neq i} \{ z \in \mathbb{C} : |z - s_{kk} + s_{ik}| \leq \sum_{j \neq k} |s_{kj} - s_{ij}| \}$.

Proof. By Theorem 2.2, we have, for all $i$,
\[ \sigma(S) = \sigma(S(i)) \cup \{1\}. \]

By Geršgorin disc theorem, $\sigma(S(i)) \subseteq G_{S(i)}$, for $i = 1, 2, \ldots, n$. Therefore,
\[ \sigma(S) \subseteq \bigcap_{i=1}^{n} \left[ G_{S(i)} \cup \{1\} \right]. \]

Again, applying Theorem 2.3 to $G_{S(i)}$, we get
\[ G_{S(i)} = \bigcup_{k=1, k \neq i}^{n} \left\{ z \in \mathbb{C} : |z - S(i)_{kk}| \leq \sum_{j \neq k} |S(i)_{kj}| \right\} \]
\[ = \bigcup_{k=1, k \neq i}^{n} \left\{ z \in \mathbb{C} : |z - s_{kk} + s_{ik}| \leq \sum_{j \neq k} |s_{kj} - s_{ij}| \right\}. \]

Hence, the proof is completed.

Remark. Theorem 2.4 works nicely in some occasions even if Geršgorin disc theorem fails to provide a non-trivial result. For example, let $S$ be an irreducible stochastic matrix with at least one diagonal element zero. Then, by Geršgorin disc theorem, $G_S \supseteq \{ z \in \mathbb{C} : |z| \leq 1 \}$. But, in this case, Theorem 2.4 may provide a non-trivial eigenvalue inclusion set (see Example 2.1 and Example 3.1). Again, Theorem 1.1 and Theorem 1.2 always provide larger single discs, whereas, the eigenvalue inclusion set in Theorem 2.4 is a union of smaller regions. Example 2.1 gives a numerical explanation to this interesting fact.

Example 2.1. Consider the $4 \times 4$ stochastic matrix
\[ S = \begin{bmatrix} 0.25 & 0.25 & 0.3 & 0.2 \\ 0 & 0.5 & 0.33 & 0.17 \\ 0.6 & 0.4 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{bmatrix}. \]
Then we have

\[ S(1) = \begin{bmatrix} 0.25 & 0.03 & -0.03 \\ 0.15 & -0.3 & -0.2 \\ -0.05 & 0 & 0.2 \end{bmatrix}, \]

\[ S(2) = \begin{bmatrix} 0.25 & -0.03 & 0.03 \\ 0.6 & -0.33 & -0.17 \\ 0.1 & -0.03 & 0.23 \end{bmatrix}, \]

\[ S(3) = \begin{bmatrix} -0.35 & -0.15 & 0.2 \\ -0.6 & 0.1 & 0.17 \\ -0.5 & -0.2 & 0.4 \end{bmatrix}, \]

\[ S(4) = \begin{bmatrix} 0.15 & 0.05 & 0 \\ -0.1 & 0.3 & 0.03 \\ 0.5 & 0.2 & -0.3 \end{bmatrix}. \]

The eigenvalues of \( S \) are \(-0.307, 0.174, 0.282, 1\). Figure 1 shows that any eigenvalue other than 1 lies in each \( G_{S(k)} \). Also, from Figure 1, it is clear that \( \sigma(S) \subseteq \cap_{k=1}^{4} \{G_{S(k)} \cup \{1\}\} = G_{S(1)} \cup \{1\} \).

Now, we estimate the eigenvalue inclusion sets in Theorem 1.1 and Theorem 1.2. We have \( s_1 = 0, s_2 = 0.2, s_3 = 0.3, s_4 = 0 \) and \( S_1 = 0.6, S_2 = 0.4, S_3 = 0.33, S_4 = 0.2 \). Therefore,

\[ \gamma = \max\{0.25, 0.3, -0.3, 0.4\} = 0.4 \]

and

\[ \gamma' = \max\{0.35, -0.1, 0.33, -0.2\} = 0.35. \]

By Theorem 1.1, any eigenvalue \( \lambda \neq 1 \) of \( S \) satisfies

\[ |\lambda - 0.4| \leq 1.05. \]

Again, by Theorem 1.2, for any \( \lambda \in \sigma(S) \setminus \{1\} \), we have

\[ |\lambda + 0.35| \leq 1.2. \]

It is easy to verify that \( G_{S(1)} \) is contained in both the discs. Therefore, in this example, Theorem 2.4 works better than the other two.

### 3 Bound for Randić eigenvalues

In this section, we give a nice bound for non-Perron eigenvalues of the Randić matrix of a connected graph \( \Gamma \). Since \( R \) is symmetric, the eigenvalues of \( R \) (or \( R \)) are all real and lie in the closed interval \([-1, 1]\). We arrange the eigenvalues of \( R \) as

\[-1 \leq \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 < \lambda_1 = 1.\]

Now we have the following theorem.
Theorem 3.1. Let $\Gamma$ be a simple connected graph of order $n$. Then

$$-2 + \max_{i \in \Gamma} \min_{k \neq i} \{\alpha_{ik} \} \leq \lambda_n(\mathcal{R}) \leq \lambda_2(\mathcal{R}) \leq 2 - \max_{i \in \Gamma} \min_{k \neq i} \{\beta_{ik} \},$$

where, for $k \neq i$, $\alpha_{ik}$ and $\beta_{ik}$ are given by

$$\alpha_{ik} = \begin{cases} 
\frac{1}{d_k} + \frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \sim i \\
\frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \not\sim i
\end{cases}$$

and

$$\beta_{ik} = \begin{cases} 
\frac{1}{d_k} + \frac{2}{d_i} + \frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \sim i \\
\frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \not\sim i
\end{cases}$$

Proof. Let $\lambda$ be a non-Perron eigenvalue of $\mathcal{R}$. By Theorem 2.2, $\lambda$ is also an eigenvalue of $\mathcal{R}(i) = \mathcal{R}(i|i) - j_{n-1}r(i)^T$, where $r(i)^T$ is the $i$-deleted row of $\mathcal{R}$, for $i = 1, 2, \ldots, n$. So $\lambda$ lies
in the regions $G_{R(i)}$ with

$$G_{R(i)} = \bigcup_{k \neq i} \left\{ z \in \mathbb{C} : |z + r_{ik}| \leq \sum_{j \neq k} |r_{kj} - r_{ij}| \right\} = \bigcup_{k=1}^{n} G_{R(i)(k)},$$

where $G_{R(i)(k)}$ are the Geršgorin discs for $R(i)$. Now, we consider each individual disc of $G_{R(i)}$. For the vertex $k \in \Gamma$, $k \neq i$, we calculate the centre and the radius of $G_{R(i)(k)}$. Here two cases may arise.

**Case I:** Let $k \sim i$. Then $r_{ik} = \frac{1}{d_i}$ and $r_{ki} = \frac{1}{d_k}$. Thus, the disc $G_{R(i)(k)}$ is given by

$$|z + \frac{1}{d_i}| \leq \sum_{j \neq i,k} |r_{kj} - r_{ij}| = \sum_{j \sim i, j \neq k} |r_{kj} - r_{ij}| + \sum_{j \sim i, j = k} |r_{kj} - r_{ij}| + \sum_{j \sim i, j = k} |r_{kj} - r_{ij}| = N(i, k) \left( \frac{1}{d_k} \right) + \frac{d_k - N(i, k) - 1}{d_k} + \frac{d_i - N(i, k) - 1}{d_i} + 0
$$

$$= 2 - \frac{1}{d_k} - \frac{1}{d_i} - \frac{2N(i, k)}{\max\{d_i, d_k\}}.$$

**Case II:** If $k \not\sim i$. Then $r_{ik} = 0$ and $r_{ki} = 0$. Thus, we have the disc

$$|z| \leq \sum_{j \neq i, k} |r_{kj} - r_{ij}| = \sum_{j \sim i} |r_{kj} - r_{ij}| + \sum_{j \sim i, j = k} |r_{kj} - r_{ij}| + \sum_{j \sim i, j = k} |r_{kj} - r_{ij}| = N(i, k) \left( \frac{1}{d_k} \right) + \frac{d_k - N(i, k) - 1}{d_k} + \frac{d_i - N(i, k) - 1}{d_i} + 0
$$

$$= 2 - \frac{2N(i, k)}{\max\{d_i, d_k\}}.$$

Now, we consider the whole region $G_{R(i)}$. Since the eigenvalues of $R$ are real, by combining Case I and Case II, we obtain that any non-Perron eigenvalue $\lambda$ of $R$ must satisfy

$$-2 + \min_{k \neq i} \{ \alpha_{ik} \} \leq \lambda \leq 2 - \min_{k \neq i} \{ \beta_{ik} \},$$

for all $i = 1, 2, \ldots, n$.

Therefore, by Theorem 2.4 we obtain our required result. \qed

**Corollary 3.1.** Let $\Gamma$ be a simple connected graph. If $\rho_2$ and $\rho_n$ are the smallest and the largest nonzero normalized Laplacian eigenvalue of $\Gamma$, then

$$-1 + \max_{i \in \Gamma} \{ \min_{k \neq i} \{ \beta_{ik} \}, 1 \} \leq \rho_2 \leq \rho_n \leq 3 - \max_{i \in \Gamma} \{ \min_{k \neq i} \{ \alpha_{ik} \}, 1 \},$$

where $\alpha_{ik}$, $\beta_{ik}$ are the constants defined as in Theorem 3.1.

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Corollary 3.2. Let $\gamma$ be a connected $r$-regular graph on $n$ vertices. If $\lambda \neq 1$ be any eigenvalue of $R$, then

$$-2 + \frac{1}{r} \max_i \min_{k \neq i} \{\gamma_{ik}\}, 1\} \leq \lambda \leq 2 - \frac{1}{r} \max_i \min_{k \neq i} \{\delta_{ik}\}, 1\},$$

where

$$\gamma_{ik} = \begin{cases} 1 + 2N(i, k), & \text{if } k \sim i \\ 2N(i, k), & \text{if } k \not\sim i \end{cases}$$

and

$$\delta_{ik} = \begin{cases} 3 + 2N(i, k), & \text{if } k \sim i \\ 2N(i, k), & \text{if } k \not\sim i. \end{cases}$$

Figure 2: A graph containing an edge which is not a part of a triangle.

Below we give an example where Theorem 1.3 is improved by Theorem 3.1.

Example 3.1. Let $\Gamma$ be the graph as in Figure 2. The vertex degrees of $\Gamma$ are $d_1 = 4$, $d_2 = 5$, $d_3 = d_4 = d_5 = d_6 = 4$, $d_7 = 3$. The sets of neighbours of each vertex are given by

$$N_1 = \{2, 3, 6, 7\},$$

$$N_2 = \{1, 3, 4, 5, 6\},$$

$$N_3 = \{1, 2, 4, 7\},$$

$$N_4 = \{2, 3, 5, 6\},$$

$$N_5 = \{2, 4, 6, 7\},$$

$$N_6 = \{1, 2, 4, 5\},$$

$$N_7 = \{1, 4, 5, 6\}.$$
Let $\alpha_i = \min_{k \neq i} \alpha_{ik}$ and $\beta_i = \min_{k \neq i} \beta_{ik}$.

The numbers of common neighbours of the vertex $2 \in \Gamma$ with all other vertices are $N(2, 1) = 2$, $N(2, 3) = 2$, $N(2, 4) = 3$, $N(2, 5) = 2$, $N(2, 6) = 3$ and $N(2, 7) = 3$. Also note that the vertex 2 is adjacent to all other vertices other than the vertex 7. Thus we obtain

$$\alpha_2 = \min \left\{ \frac{1}{4} + \frac{1}{4}, \frac{1}{4} + \frac{2}{5}, \frac{2}{5} + \frac{6}{5} \right\} = 1.05$$

and

$$\beta_2 = \min \left\{ \frac{1}{4} + \frac{2}{5} + \frac{1}{4} + \frac{4}{5}, \frac{2}{5} + \frac{6}{5} \right\} = 1.2$$

Similarly, for all other vertices of $\Gamma$ we get, $\alpha_1 = 0.75$, $\beta_1 = 1.25$, $\alpha_3 = 0.75$, $\beta_3 = 1.25$, $\alpha_4 = 0.75$, $\beta_4 = 1$, $\alpha_5 = 0.333$, $\beta_5 = 0.833$, $\alpha_6 = 0.75$, $\beta_6 = 1$, $\alpha_7 = 1$, $\beta_7 = 1$.

Therefore, using Theorem 3.1, we get

$$\lambda_2 \leq 0.75 \text{ and } \lambda_7 \geq -0.95.$$ 

Note that, since $N(5, 7) = 0$, the lower bound for $\lambda_7$ in [4] becomes $-1$.

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References

[1] A. Banerjee, J. Jost, On the spectrum of the normalized graph Laplacian, Linear Algebra Appl., 428 (2008) 3015-3022.

[2] F. Bauer, J. Jost, S. Liu, Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator, Math. Res. Lett. 19 (2012) 1185-1205.

[3] Ş.B. Bozkurt, A.D. Güngör, I. Gutman, A.S. Çevik, Randić matrix and Randić energy, MATCH Commun. Math. Comput. Chem., 64 (2010) 239-250.

[4] F. Chung, Spectral Graph Theory, AMS (1997).
[5] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs - Theory and Application*, Academic Press (1980).

[6] L.J. Cvetković, V. Kostić, J.M. Peña, *Eigenvalue localization refinements for matrices related to positivity*, SIAM J. Matrix Anal. Appl. 32 (2011) 771-784.

[7] S. A. Geršgorin, *Über die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk, 6 (1931) 749-754.

[8] R. A. Horn, C. R. Johnson, *Matrix analysis*, Cambridge University press (2013).

[9] S. Kirkland, *A cycle-based bound for subdominant eigenvalues of stochastic matrices*, Linear Multilinear Algebra, 57 (2009) 247-266.

[10] S. Kirkland, *Subdominant eigenvalues for stochastic matrices with given column sums*, Electron. J. Linear Algebra, 18 (2009) 784-800.

[11] J. Li, J-M. Guo, Y. C. Shiu, *Bounds on normalized Laplacian eigenvalues of graphs*, J. Inequal. Appl., (2014) 1-8.

[12] C. Li, Y. Li, *A modification of eigenvalue localization for stochastic matrices*, Linear Algebra Appl., 460 (2014) 221-231.

[13] C. Li, Q.Liu, Y. Li, *Geršgorin-type and Brauer-type eigenvalue localization sets of stochastic matrices*, Linear Multilinear Algebra, 63 (2014) 2159-2170.

[14] X. Li, Y. Shi, *A survey on the Randić index*, MATCH Commun. Math. Comput. Chem. 59 (2008) 127-156.

[15] R. Mehatari, A. Banerjee, *Effect on normalized graph Laplacian spectrum by motif attachment and duplication*, Applied Math. Comput. 261 (2015) 382-387.

[16] M. Newman, *Networks: an introduction*, Oxford University Press (2010).

[17] J.M. Peña, *Shape Preserving Representations in Computer Aided-Geometric Design*, Nova Science Publishers, Hauppage, NY, 1999.

[18] M. Randić, *On characterization of molecular branching*, J. Am. Chem. Soc. 97 (1975) 6609-6615.

[19] O. Rojo, R. L. Soto, *A New Upper Bound on the Largest Normalized Laplacian Eigenvals*, Oper. Matrices, 7 (2013) 323-332.

[20] E. Seneta, *Non-Negative Matrices and Markov Chains* Springer-Verlag (1981).