NON-UNIQUE WEAK SOLUTIONS IN LERAY-HOPF CLASS FOR THE 3D HALL-MHD SYSTEM

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Abstract. Non-unique weak solutions in Leray-Hopf class are constructed for the three dimensional magneto-hydrodynamics with Hall effect. We adapt the widely appreciated convex integration framework developed in a recent work of Buckmaster and Vicol [10] for the Navier-Stokes equation, and with deep roots in a sequence of breakthrough papers for the Euler equation.

KEY WORDS: Hall-magneto-hydrodynamics; Leray-Hopf solutions; non-uniqueness; convex integration.

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1. INTRODUCTION

To capture the fast process of the magnetic reconnection phenomena in plasma physics, the following model of the incompressible magneto-hydrodynamics (MHD) with Hall effect

\begin{align}
  u_t + u \cdot \nabla u - B \cdot \nabla u + \nabla p &= \Delta u, \\
  B_t + u \cdot \nabla B - B \cdot \nabla u + \zeta \nabla \times ((\nabla \times B) \times B) &= \Delta B, \\
  \nabla \cdot u &= 0, 
\end{align}

was proposed by astrophysicists. In (1.1), \( u, p \) and \( B \) represent the fluid velocity field, the scalar pressure, and the magnetic field, respectively; they are the unknown functions on the spacial-time domain \( \Omega \times (0, \infty) \). In the present paper, we take \( \Omega = \mathbb{T}^3 \). The parameter \( \zeta \) in front of the Hall term indicates the strength of the Hall effect. For mathematical study on this model during the last few decades, we refer to [1, 13, 14, 15, 17, 22, 25, 31] and references therein.

We notice that system (1.1) with \( \zeta = 0 \) is the usual MHD model. In this case, one also observes that the magnetic field equation is essentially linear in \( B \), while the velocity equation is obviously the Navier-Stokes equation (NSE) with a force term. Due to the linear feature of the magnetic field equation, it is expected that the properties of solutions to the MHD system do not seriously deviate from those of the solutions to the NSE. In fact, a vast amount of work for the MHD and the NSE have shown this consistence.

However, for the Hall MHD system (1.1) with \( \zeta > 0 \), the situation is drastically different, comparing to the usual MHD system. On one hand, the equation of \( B \) is nonlinear with a strong nonlinear Hall term which is actually more singular than \( u \cdot \nabla u \) in the NSE; on the other hand, a natural scaling does not exist for the Hall MHD system, while the MHD system shares the same natural scaling as for the NSE. More discussion on the scaling analysis will be provided at a later point. Due to the obvious difference of the two systems, a natural question is that: how does

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the presence of the Hall term change the behavior of solutions? Since the Hall term is more singular than other nonlinear terms in the system, one expectation is that it is probably more approachable to construct wild solutions and to discover severe ill-posedness for the Hall-MHD system. Searching wild solutions and justifying ill-posedness for fluid equations remains mathematically interesting and physically important before one can give an affirmative answer to the global regularity problem of these equations.

As for the 3D NSE, Leray’s conjecture regarding the appearance of singularity at finite time has been a long-standing open problem; the uniqueness of Leray-Hopf weak solutions is not known either. Since the time of these problems raised in 1930s, much effort has been taken to tackle them from the negative side in the means of constructing blow-up solutions, wild solutions, or wild data-to-solution maps. Wild solutions for the Euler equation were first constructed in \[47, 48, 49\]. In \[36\], Jia and Šverák showed non-uniqueness of Leray-Hopf weak solutions in \(L^\infty(L^{3,\infty})\) with the assumption that certain spectral condition holds for a linearized Navier-Stokes operator. For an averaged NSE with modified nonlinear term, Tao constructed a smooth solution which blows up in finite time in \[43\]; moreover, the author proposed a program for adapting the blowup construction to the true NSE. Very recently, in the groundbreaking paper \[10\], Buckmaster and Vicol constructed nontrivial weak solutions for the 3D NSE by developing the convex integration scheme using intermittent Beltrami flows, which leads to non-uniqueness of weak solutions. It is a significant progress towards settling the problem of non-uniqueness of Leray-Hopf weak solutions, although the weak solutions constructed there belong to the space \(C^0(0, T; H^\beta(T^3))\) for a very small number \(\beta > 0\).

The convex integration method developed in \[10\] dates back to \[41\] and a sequence of breakthrough work for the Euler equation in the last decade. It was first introduced by De Lellis and Székelyhidi in \[27, 28, 29\] to study the non-uniqueness of weak solutions and the existence of dissipative continuous solutions for the Euler equation. The framework of convex integration was further developed in \[4, 5, 26, 34\] and eventually leads to a complete resolution of the second half of Onsager’s conjecture \[43\] by Isett \[35\], and Buckmaster, De Lellis, Székelyhidi and V. Vicol \[6\].

Back to the dissipative equations, as mentioned above, the non-uniqueness of Leray-Hopf weak solutions to the 3D NSE is still open. Following the convex integration method in \[10\], one may expect to construct non-trivial solutions in \(C^0(H^\beta)\) for \(\beta < 1/2\) and close enough to \(1/2\); while crossing \(1/2\) spatial regularity would be a major barrier. The reason is that \(H^{1/2}\) is critical for the 3D NSE, in which the regularity implies uniqueness. When the dissipation is weak, as for the hyperviscous Navier-Stokes equation with fractional Laplacian \((-\Delta)^\theta\) with \(\theta \in (0, 1/5)\) in \[21\], Colombo, De Lellis and De Rosa showed the non-uniqueness of Leray weak solutions, that is, solutions with finite energy and in the space \(C^0(H^\theta)\). The result was extended to the case with \(\theta < \frac{1}{6}\) by De Rosa \[30\].

Regarding the hyperviscous NSE with \(\theta < 5/4\), adapting the convex integration techniques of \[10\], Luo and Titi in \[39\] established the non-uniqueness of weak solutions by slightly simplifying the original construction of Buckmaster and Vicol. In another work \[3\], Buckmaster, Colombo, and Vicol constructed wild solutions for the 3D NSE, whose singular set in time has Hausdorff dimension strictly less than 1. Moreover, the result holds for the hyperviscous NSE with \(\theta < 5/4\) as well.
Thus, along with the uniqueness result for \( \theta \geq 5/4 \) by Lions [38], the work of [39] and [3] indicates the well-posedness criticality of the exponent \( \theta = 5/4 \).

Other wild solutions for the Navier-Stokes equations were also constructed in [3, 18, 19, 20]. Nonuniqueness of weak solutions was studied for other fluid equations as well, see [7, 11, 42]. For more detailed background on recent development of convex integration method for fluid equations, the reader can consult the survey papers [8, 9].

The main purpose of this paper is to address the problem of non-uniqueness of weak solutions in Leray-Hopf space for the Hall MHD system (1.1) with \( \zeta > 0 \). A scaling analysis will be helpful to demonstrate why it is approachable to study this problem by adapting the convex integration techniques. We first look at the MHD system, that is (1.1) with \( \zeta = 0 \). If the triplet \((u(x,t), B(x,t), p(x,t))\) solves the MHD system with data \((u_0(x), B_0(x))\), the following scaled functions

\[
\begin{align*}
  u_\lambda &= \lambda u(\lambda x, \lambda^2 t), \\
  B_\lambda &= \lambda B(\lambda x, \lambda^2 t), \\
  p_\lambda &= \lambda^2 p(\lambda x, \lambda^2 t),
\end{align*}
\]

(1.2)
solve the MHD system as well with scaled data \((\lambda u_0(\lambda x), \lambda B_0(\lambda x))\). In the case of vanishing magnetic field \(B\), such scaling holds for the NSE. Under scaling (1.2), the space \( H^{1/2} (\mathbb{T}^3) \times H^{1/2} (\mathbb{T}^3) \) is critical for the 3D MHD system. It is known that regularity and hence uniqueness holds in subcritical spaces \( H^s \) with \( s > 1/2 \). Since the MHD system with zero magnetic field reduces to the NSE, the non-uniqueness result of the 3D NSE in [10] immediately provides a proof of non-uniqueness of weak solutions for the 3D MHD system. Similarly as for the 3D NSE, the uniqueness of Leray-Hopf solutions to the 3D MHD remains an open problem. The attempt to construct non-unique Leray-Hopf solutions via the convex integration method might not succeed since the criticality of \( 1/2 \) spacial regularity would be a crucial obstacle to overcome.

Now we turn to the Hall MHD system (1.1) with \( \zeta > 0 \), a natural scaling no longer holds due to the presence of the Hall term \( \nabla \times ((\nabla \times B) \times B) \). One can see that the Hall term is more singular than other nonlinear terms in the system and the most singular one in the magnetic equation. This motivates us to consider the magnetic equation with vanishing velocity field as the first step. Thus we analyze the so-called Hall equation (also referred as the electron MHD),

\[
B_t + \nabla \times ((\nabla \times B) \times B) = \Delta B
\]

(1.3)
which has the natural scaling

\[
B_\lambda = B(\lambda x, \lambda^2 t).
\]

(1.4)

We observe that if \( \nabla \cdot B(x,0) = 0 \), \( \nabla \cdot B(x,t) = 0 \) holds for all time \( t > 0 \). The basic energy law for the Hall equation (1.3) is

\[
\frac{1}{2} \frac{d}{dt} \|B(t)\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = 0.
\]

(1.5)

A Leray-Hopf type of weak solution to (1.3) is a function \( B \in L^\infty(L^2) \cap L^2(H^1) \) which satisfies (1.3) and (1.5) in the distributional sense. On the other hand, under scaling (1.4), the Sobolev space \( H^{3/2} \) (the same as \( H^{3/2} \) on periodic domains) is critical for (1.3) in three dimensions. One can expect global regularity of solution in \( H^{3/2} \) and uniqueness in the space as a consequence. Since the critical index \( 3/2 \) of regularity is larger than the Leray-Hopf weak solution regularity index 1, non-uniqueness of Leray-Hopf weak solutions in \( C^0(H^1) \) constructed by the convex
integration method would not contradict with anything according to the scaling properties. 

Inspired by the aforementioned analysis, we adapt the convex integration scheme to the Hall equation (1.3) and establish the first main result as follows.

**Theorem 1.1.** For any nonnegative smooth function \( E(t) : [0, T] \rightarrow \mathbb{R}_{\geq 0} \), there exists a weak solution \( B \in L^\infty([0, T]; L^2(\mathbb{T}^3)) \cap C^0([0, T]; H^1(\mathbb{T}^3)) \) to the Hall equation (1.3), such that

\[
\int_{\mathbb{T}^3} |\nabla \times B|^2 \, dx = E(t), \quad t \in [0, T].
\]

The statement implies non-uniqueness of weak solutions to the Hall equation (1.3) in Leray-Hopf class. Indeed, the vorticity of the weak solutions can have any nonnegative energy profiles and thus a constant (in particular, zero) is not the only weak solution.

Concerning the strategy to prove Theorem 1.1, we take the curl of the Hall equation and apply the convex integration method to the resulted equation of the current density \( J = \nabla \times B \). Section 4 will be devoted to this purpose.

Once we have the convex integration scheme for the Hall equation, we turn to the coupled Hall MHD system. At each level of the convex integration which produces \( B_q \), we solve the velocity field equation – the NSE with the Lorentz force \((B_q \cdot \nabla) B_q\). We show that there exists a Leray-Hopf weak solution \( u_q \) to the NSE based on the estimates on \( B_q \). In the end, we illustrate that the sequence \( \{u_q, B_q\} \) converges to a pair of functions \( \{u, B\} \) which is a weak solution of the Hall MHD system (1.1). Therefore, we are able to prove the second main result stated below.

**Theorem 1.2.** For any nonnegative smooth function \( E(t) : [0, T] \rightarrow \mathbb{R}_{\geq 0} \), there exists a weak solution \((u, B)\) to the Hall MHD system (1.1) with \( \zeta > 0 \) on \([0, T]\), such that, \( u, B \in L^\infty([0, T]; L^2(\mathbb{T}^3)) \cap L^2([0, T]; H^1(\mathbb{T}^3)) \) and

\[
\int_{\mathbb{T}^3} |\nabla \times B|^2 \, dx = E(t), \quad t \in [0, T].
\]

Analogously, Theorem 1.2 suggests \((0, 0, p)\) is not the only Leray-Hopf weak solution of (1.1). Thus we provide a construction of non-unique Leray-Hopf weak solutions for the 3D Hall MHD system. The proof of Theorem 1.2 will be laid out in Section 5.

In the regime of applying convex integration techniques to the MHD system, the two recent articles [2] and [32] obtained astonishing results. In [2], Beekie, Buckingham and Vicol constructed finite energy weak solutions to the ideal MHD system whose magnetic helicity is not conserved. Consequently, it shows the existence of finite energy weak solutions to the ideal MHD which cannot be obtained in the zero viscosity-resistivity limit. It indicates that the ideal-MHD version of Taylor’s conjecture is false. In [32], Faraco, Lindlerg and Székelyhidi showed the existence of infinitely many bounded solutions to the 3D ideal MHD. Interestingly, for such solutions, the total energy and cross helicity are not conserved; but the magnetic helicity is preserved. Moreover, the authors showed that the 2D ideal MHD has no nontrivial compactly supported solutions with finite energy.

We conclude this section by a few well-posedness results for the Hall MHD system. In a previous paper [22], the author showed that system (1.1) with \( \zeta > 0 \) is locally well-posed in the Sobolev space \( H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \) with \( s > n/2 \). Eventually in [23], the author established the local well-posedness of the system in
$H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)$ with $s > n/2 - 1$, which appears to be optimal in regards to the scaling of the NSE and scaling \([1,4]\) for the Hall equation.

2. Preliminaries

2.1. Notation. For the sake of brevity, we first fix some notations. We denote by: $A \lesssim B$ an estimate of the form $A \leq CB$ with an absolute constant $C$; $A \sim B$ an estimate of the form $C_1B \leq A \leq C_2B$ with absolute constants $C_1, C_2$.

2.2. The Hall equation. To analyze the effect of the Hall term, we first consider the Hall equation, which is recalled here

$$B_t + \nabla \times ((\nabla \times B) \times B) = \Delta B. \quad (2.1)$$

Note that $\nabla \cdot B(t) = 0$ for all $t \geq 0$ if $\nabla \cdot B(0) = 0$. It is easy to verify that a smooth solution of the Hall equation satisfies the energy identity,

$$\frac{1}{2} \frac{d}{dt} \|B(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2 = 0.$$

**Definition 2.1.** We say that $B$ is a Leray-Hopf weak solution of (2.1), if for any $\varphi \in C_\infty_c([0, T] \times \mathbb{T}^3)$, the following integral equation

$$\int_0^T \int_{\mathbb{T}^3} B \cdot \varphi_t + (B \otimes B) : \nabla \nabla \times \varphi \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} \nabla B : \nabla \varphi \, dx \, dt$$

is satisfied, and $B$ belongs to $L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))$.

**Remark 2.2.** Note that the definition is valid thanks to the vector identity

$$(\nabla \times B) \times B = \nabla \cdot (B \otimes B) - \nabla \frac{|B|^2}{2}$$

which holds for divergence free vector field $B$. Indeed, the integral with the non-linear term $\nabla \times ((\nabla \times B) \times B)$ can be handled as, by using integration by parts

$$\int_0^T \int_{\mathbb{T}^3} \nabla \times ((\nabla \times B) \times B) \cdot \varphi \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{T}^3} \left( \nabla \cdot (B \otimes B) - \nabla \frac{|B|^2}{2} \right) \cdot \nabla \varphi \, dx \, dt$$

$$= - \int_0^T \int_{\mathbb{T}^3} (B \otimes B) : \nabla \nabla \times \varphi \, dx \, dt + \int_0^T \int_{\mathbb{T}^3} \frac{|B|^2}{2} \nabla \cdot (\nabla \times \varphi) \, dx \, dt$$

$$= - \int_0^T \int_{\mathbb{T}^3} (B \otimes B) : \nabla \nabla \times \varphi \, dx \, dt,$$

where the last step follows from the fact that $\nabla \cdot (\nabla \times \varphi) = 0$.

The existence of Leray-Hopf weak solutions to (2.1) is trivial; for instance, it can be established by the standard Galerkin’s approximating method.

Taking curl on the Hall equation leads to

$$(\nabla \times B)_t + \nabla \times \nabla \times ((\nabla \times B) \times B) = \Delta \nabla \times B.$$  

By introducing the vorticity of the magnetic field, current density, $J = \nabla \times B$, we give two formulations of the equation. The first one reads as

$$J_t + \nabla \times \nabla \times (J \times B) = \Delta J. \quad (2.2)$$
By applying a few vector calculus identities, see Section 6, the current density equation can be formulated in a more symmetric way, namely

\[ J_t + \nabla \cdot (B \otimes (\nabla \times J)) - (\nabla \times J) \otimes B - 2\nabla \times (J \nabla B) = \Delta J \quad (2.3) \]

where \( J \nabla B = (J_t \partial_3 B_1, J_t \partial_2 B_1, J_t \partial_3 B_1) \). In the rest of the paper, we will need to refer to both formulations to have a more complete vision of the structure of the Hall equation.

2.3. Leray-Hopf weak solution of the Hall-MHD.

**Definition 2.3.** We say that \((u, p, B)\) is a Leray-Hopf weak solution of \((1.1)\), if for any \(\psi, \varphi \in C_0^\infty([0, T] \times \mathbb{T}^3)\) with \(\nabla \cdot \psi = 0\), the following integral equations

\[
\int_0^T \int_{\mathbb{T}^3} u \cdot \psi_t + (u \otimes u) : \nabla \psi - (B \otimes B) : \nabla \psi \, dx \, dt = \int_0^T \int_{\mathbb{T}^3} \nabla u : \nabla \psi \, dx \, dt,
\]

\[
\int_0^T \int_{\mathbb{T}^3} B \cdot \varphi_t + (u \otimes B) : \nabla \varphi - (B \otimes B) : \nabla \varphi + \varsigma (B \otimes B) : \nabla \nabla \times \varphi \, dx \, dt
\]

are satisfied; and

\[(u, B) \in (L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3)))^2.\]

The existence of Leray-Hopf weak solutions of \((1.1)\) can be found in [12].

2.4. Estimates for periodic functions and anti-derivative operator. The following lemma regards improved Hölder’s inequality for periodic functions.

**Lemma 2.4.** [10] Let \(\lambda \in \mathbb{N}\) and \(f, g : \mathbb{T}^d \to \mathbb{R}\) be smooth functions. Denote \(g_\lambda : \mathbb{T}^d \to \mathbb{R}\) by the \(1/\lambda\) periodic function defined by \(g_\lambda(x) := g(\lambda x)\). Then for every \(p \in [1, \infty]\), we have

\[
\|f g_\lambda\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^p} + C_{\lambda^p} \|f\|_{C^1} \|g\|_{L^p}.
\]

We note that

\[
\|D^k g_\lambda\|_{L^p(\mathbb{T}^d)} = \lambda^k \|D^k g\|_{L^p(\mathbb{T}^d)}, \quad \forall \ k \in \mathbb{N}, \ \forall \ p \in [1, \infty],
\]

and in particular,

\[
\|g_\lambda\|_{L^p(\mathbb{T}^d)} = \|g\|_{L^p(\mathbb{T}^d)}, \quad \forall \ p \in [1, \infty].
\]

A type of commutator estimate for periodic functions is introduced below.

**Lemma 2.5.** [10] Assume \(\kappa \geq 1\), \(p \in (1, 2]\) and \(L \in \mathbb{N}\) is sufficiently large. Let function \(a \in C^L(\mathbb{T}^3)\) be such that there exists \(1 \leq \lambda \leq \kappa\) and \(C_a > 0\) with

\[
\|D^j a\|_{L^\infty} \leq C_a \lambda^j
\]

for all \(0 \leq j \leq L\). Assume in addition that \(\int_{\mathbb{T}^3} a(x) P_{\geq \kappa} f(x) \, dx = 0\). Then the estimate

\[
\|\nabla^{-1}(a P_{\geq \kappa} f)\|_{L^p} \lesssim C_a \left(1 + \frac{\lambda^L}{\kappa^{L-2}}\right) \frac{\|f\|_{L^p}}{\kappa}\]

holds for any \(f \in L^p(\mathbb{T}^3)\), where the implicit constant depends on \(p\) and \(L\).

We also introduce an estimate for the symmetric anti-divergence operator.
Lemma 2.6. \[27\] There exists a linear operator $\mathcal{R}$ of order $-1$, such that

$$
\nabla \cdot \mathcal{R}(u) = u - \int_{\mathbb{T}^3} u \, dx.
$$

It satisfies the Calderon-Zygmund and Schauder estimates, for $1 < p < \infty$,

$$
\|\mathcal{R}\|_{L^p \to W^{1,p}} \lesssim 1, \quad \|\mathcal{R}\|_{C^0 \to C^0} \lesssim 1, \quad \|\mathcal{R} \mathcal{P} \mathcal{P} P \mathcal{P}^{-1} \mathcal{P} \mathcal{P} u\|_{L^p} \lesssim \|\nabla^{-1} \mathcal{P} \mathcal{P} P \mathcal{P}^{-1} \mathcal{P} \mathcal{P} u\|_{L^p}.
$$

3. The Hall equation and intermittent Beltrami flows

In this part, we analyze the structure of the equation of the current density $J = \nabla \times B$ and lay out the intermittent Beltrami flows introduced in \[10\]. The analysis will reveal the fact that the equation of the current density is analogous to the NSE near the intermittent Beltrami flows.

3.1. Analyzing the equation. If we apply the convex integration scheme directly to equation (2.2), we would consider the approximating equation

$$
\partial_t J_q + \nabla \times \nabla \times (J_q \times B_q) = \Delta J_q + \nabla \times \nabla \times M_q,
$$

with $J_q = \nabla \times B_q$, and $M_q$ being certain vector with the property that $M_q \to 0$ in an appropriate sense as $q \to \infty$. The main idea would be to construct building blocks for the increments $v_{q+1} = B_{q+1} - B_q$ and $w_{q+1} = J_{q+1} - J_q$, which give rise to a new pair $(B_{q+1}, J_{q+1})$ and consequently a new vector $M_{q+1}$ according to equation (3.1) at the level of $q+1$. Most importantly, the construction should be designed in such a way that: at level $q + 1$, the major contribution of nonlinear interaction to the new vector $M_{q+1}$ cancels $M_q$; and hence the sequence $\{M_q\}$ converges to zero eventually.

However, we realize that it has certain advantages to apply the convex integration scheme to the slightly more symmetric equation (2.3). In fact, we will work with the approximating form of (2.3)

$$
\partial_t J_q + \nabla \cdot (B_q \otimes (\nabla \times J_q) - (\nabla \times J_q) \otimes B_q) - 2\nabla \times (J_q \nabla B_q) = \Delta J_q + \nabla \cdot R_q \tag{3.2}
$$

where $R_q$ is recognized as an error stress tensor. Note that (3.2) is formally equivalent to

$$
\partial_t J_q + \nabla \times \nabla \times (J_q \times B_q) = \Delta J_q + \nabla \cdot R_q, \tag{3.3}
$$

Both (3.2) and (3.3) will be used in the analysis below. To keep the algebra as simple as possible, we may refer (3.3) rather than (3.2).

The main element is that we need to design building blocks for the increments $v_{q+1}$ and $w_{q+1}$, which in turn yield the triplet $(B_{q+1}, J_{q+1}, R_{q+1})$ with the property: the significantly large part of $R_{q+1}$ from the nonlinear interaction represented by $B \otimes (\nabla \times J) - (\nabla \times J) \otimes B$ cancels the previous level of stress tensor $R_q$. A crucial observation is that:

- if we take $B = W(x)$ as the Beltrami wave defined in Section 3.2 and $J = \nabla \times W(x) = \Lambda W(x)$, then we can verify

$$
\nabla \cdot (B \otimes (\nabla \times J) - (\nabla \times J) \otimes B) - 2\nabla \times (J \nabla B) = 0;
$$

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if we take \( B = \eta(x,t)W(x) \) as the intermittent Beltrami wave and \( J = \nabla \times (\eta(x,t)W(x)) \), with an appropriate choice of \( \eta(t,x) \) defined in Section 3.2, one can make sure the difference

\[
|\nabla \cdot (B \otimes (\nabla \times J) - (\nabla \times J) \otimes B) - 2\nabla \times (J\nabla B) - \nabla \Pi| - \nabla \cdot (J \otimes J)
\]  

is small for some pressure term \( \Pi \).

This indicates that, the stationary Beltrami wave is a solution of the Hall equation; while near certain intermittent Beltrami waves, equation (2.3) is "close" to

\[
J_t + \nabla \cdot (J \otimes J) + \nabla \Pi = \Delta J
\]

which is the NSE. An important motivation we obtain is that an analogous construction scheme by using the convex integration method as for the NSE in [10] would possibly lead to the non-uniqueness of weak solutions of equation (2.3) with divergence free. Of course, in our case, two functions \( J \) involved in the construction; and the relation \( J = \nabla \times B = \nabla \times (\eta W) \) will generate many error terms. On the other hand, it is also crucial to determine how to apply the important geometric lemma in the current context.

We will describe the convex integration scheme in detail for equation (2.3) by considering its approximation sequence (3.2) in the following section.

3.2. Building blocks. We adapt the construction idea of [10] using intermittent Beltrami flows. While we have to keep in mind that, rather than dealing with one function satisfying the NSE, we deal with the pair \( (B, J) \) with \( J = \nabla \times B \) satisfying (2.3) in our context.

We first fix \( \xi, \alpha, B_\xi \), and \( a_\xi \) as defined in [10]:

\[
\xi \in S^2 \cap \mathbb{Q}^3, \quad A_\xi \in S^2 \cap \mathbb{Q}^3, \quad A_\xi \cdot \xi = 0, \quad A_\xi = A_{-\xi},
\]

\[
a_\xi \in \mathbb{C}, \quad \overline{a_\xi} = a_{-\xi}, \quad B_\xi = \frac{1}{\sqrt{2}} (A_\xi + i\xi \times A_\xi).
\]

The stationary Beltrami wave is taken as

\[
W(x) = \sum_{\xi \in \Lambda} a_\xi W_\xi := \sum_{\xi \in \Lambda} a_\xi B_\xi e^{i\lambda \xi \cdot x},
\]

where \( \Lambda \) is a given finite subset of \( S^2 \) such that \( \Lambda = -\Lambda \), and \( \lambda \) is an integer such that \( \lambda \Lambda \subset \mathbb{Z}^3 \). One can verify that \( W(x) \) is real-valued and satisfies

\[
\nabla \cdot W = 0, \quad \nabla \times W = \lambda W, \quad \nabla \cdot (W \otimes W) = \nabla |W|^2, \quad \frac{1}{2} \int_{\mathbb{T}^3} W \otimes W \, dx = \frac{1}{2} \sum_{\xi \in \Lambda} |a_\xi|^2 (\text{Id} - \xi \otimes \xi).
\]

Lemma 3.1. [10] For any \( N \in \mathbb{N} \), we can find \( \varepsilon_\gamma > 0 \) and \( \lambda > 1 \) with the following property. Let \( B_{\varepsilon_\gamma}(\text{Id}) \) be the ball of symmetric 3 \times 3 matrices, centered at \( \text{Id} \) of radius \( \varepsilon_\gamma \). There exists pairwise disjoint subsets

\[
\Lambda_\alpha \subset S^2 \cap \mathbb{Q}^3, \quad \alpha \in \{1, \ldots, N\},
\]

with \( \lambda \Lambda_\alpha \subset \mathbb{Z}^3 \), and smooth positive functions

\[
\gamma_\alpha^\xi \in C^\infty (B_{\varepsilon_\gamma}(\text{Id})), \quad \alpha \in \{1, \ldots, N\}, \quad \xi \in \Lambda_\alpha,
\]
with derivatives that are bounded independently of $\lambda$, such that:

1. $\xi \in \Lambda_\alpha$ implies $-\xi \in \Lambda_\alpha$ and $\gamma_\xi = \gamma_{-\xi}$;
2. For each $R \in B_\varepsilon,(Id)$ we have the identity

$$R = \frac{1}{2} \sum_{\xi \in \Lambda_\alpha} (\gamma_\xi(R))^2 (Id - \xi \otimes \xi).$$

Next we describe the intermittent Beltrami flows by adding oscillations to the Beltrami waves. We start with the Dirichlet kernel $D_n$

$$D_n(x) = \sum_{\xi=-n}^{n} e^{i\xi x} = \frac{\sin((n + \frac{1}{2})x)}{\sin(\frac{x}{2})}$$

which satisfies for $p > 1$

$$\|D_n\|_{L^p} \sim n^{\frac{1}{p} - \frac{1}{2}}.$$

We define the lattice cube

$$\Omega_r := \{\xi = (j,k,l) : j,k,l \in \{-r,...,r\}\}$$

and the 3D normalized Dirichlet kernel

$$D_r(x) := \frac{1}{(2r + 1)^{\frac{1}{2}}} \sum_{\xi \in \Omega_r} e^{i\xi x}$$

satisfying

$$\|D_r\|_{L^2}^2 = (2\pi)^3, \quad \|D_r\|_{L^p} \lesssim r^{\frac{3}{2} - \frac{1}{p}}, \quad p > 1,$$

where the implicit constant depends only on $p$. The parameter $r$ refers to the number of frequencies along edges of $\Omega_r$.

We shall define a directed and rescaled periodic Dirichlet kernel with period $(T/(\lambda \sigma))^3$. The small constant $\sigma$ is chosen such that $\lambda \sigma \in N$ which parameterizes the spacing between frequencies; and $\sigma r \ll 1$. We fix an integer $N_0 \geq 1$ such that

$$\{N_0\xi, N_0 A_\xi, N_0 \xi \times A_\xi \} \subset N_0 S^2 \cap \mathbb{Z}^3$$

for all $\xi \in \Lambda_\alpha$ and $\alpha \in \{1,...,N\}$. We also introduce a parameter $\mu \in (\lambda, \lambda^2)$, which adjusts the temporal oscillation. It is then ready to define the modified Dirichlet kernel

$$\eta_{\xi,\lambda,\sigma,\tau,\mu}(x,t) = D_r(\lambda\sigma N_0(\xi \cdot x + \mu t), \lambda\sigma N_0 A_\xi \cdot x, \lambda\sigma N_0(\xi \times A_\xi) \cdot x)$$

for $\xi \in \Lambda_\alpha^+$; while $\eta_{\xi,\lambda,\sigma,\tau,\mu}(x,t) = \eta_{-\xi,\lambda,\sigma,\tau,\mu}(x,t)$ for $\xi \in \Lambda_\alpha^-$. We take the short notation $\eta_{\xi}(x,t) = \eta_{\xi,\lambda,\sigma,\tau,\mu}(x,t)$. It is important to notice that

$$\mu^{-1} \partial_t \eta_{\xi}(x,t) = \pm (\xi \cdot \nabla) \eta_{\xi}(x,t), \quad \forall \xi \in \Lambda_\alpha^\pm,$$

which is the crucial identity used to design temporal oscillation in the increments later.

One also observe that

$$\int_{\mathbb{T}^3} \eta_{\xi}^2(x,t) \, dx = \int_{\mathbb{T}^3} D_r^2(x) \, dx = 1, \quad \|\eta_{\xi}(\cdot,t)\|_{L^p} = \|D_r\|_{L^p} \lesssim r^{\frac{3}{2} - \frac{1}{p}},$$

for all $1 < p \leq \infty$.

Now we are ready to introduce the intermittent wave $\mathbb{W}_\xi$:

$$\mathbb{W}_\xi(x,t) = \eta_{\xi}(x,t) B_\xi e^{i\lambda \xi \cdot x}.$$

(3.8)
It is worth to point out that $\mathcal{W}_\xi$ is supported on certain frequencies. Indeed, we have
\[
\begin{align*}
P_{\leq 2\lambda^{2}\sigma rN^{2}0} \eta_{\xi} & = \eta_{\xi}, \\
P_{\leq 2\lambda} P_{\geq \lambda/2} \mathcal{W}_\xi & = \mathcal{W}_\xi, \\
P_{\leq 4\lambda} P_{\geq c_0 \lambda} (\mathcal{W}_\xi \otimes \mathcal{W}_{\xi'}) & = \mathcal{W}_\xi \otimes \mathcal{W}_{\xi'},
\end{align*}
\]
where $c_0$ is a small constant and $\xi' \neq -\xi$.

Another important fact regarding $\mathcal{W}_\xi$ is given by
\[
\begin{align*}
\nabla \cdot (\mathcal{W}_\xi \otimes \mathcal{W}_{-\xi} + \mathcal{W}_{-\xi} \otimes \mathcal{W}_\xi) & = \nabla \eta^2_{\xi} - (\xi \cdot \nabla)\eta^2_{\xi} = \nabla \eta^2_{\xi} - \frac{\xi}{\mu} \partial_t \eta^2_{\xi}.
\end{align*}
\]
It is the main motivation that we need to include the temporal oscillation $w_{q+1}$ into the construction later.

Different from the Beltrami wave $W_\xi(x) = B_\xi e^{i\lambda \xi \cdot x}$, the intermittent Beltrami wave $\mathcal{W}_\xi$ is not divergence free or an eigenfunction of curl, i.e.
\[
\nabla \cdot \mathcal{W}_\xi \neq 0, \quad \nabla \times \mathcal{W}_\xi \neq \lambda \mathcal{W}_\xi.
\]
Instead, we have
\[
\begin{align*}
\nabla \cdot \mathcal{W}_\xi & = \nabla \eta \xi \cdot W_\xi, \\
\nabla \times \mathcal{W}_\xi & = \lambda \mathcal{W}_\xi + \nabla \eta \xi \times W_\xi.
\end{align*}
\]
Parameters $\lambda, \sigma, r$, and $\mu$ will be chosen in an appropriate way such that $\nabla \eta \xi \cdot W_\xi$ and $\nabla \eta \xi \times W_\xi$ are sufficiently small.

For such intermittent Beltrami waves $\mathcal{W}_\xi$ and $\Lambda_\alpha, \varepsilon_{\gamma}, \gamma_{\xi}$ as in Lemma 3.1, we have the following geometric lemma, which is a key ingredient in the construction.

Lemma 3.2. [10] Assume $a \xi \in \mathbb{C}$ are constants satisfying $a \xi = a_{-\xi}$. The vector field
\[
\sum_\alpha \sum_{\xi \in \Lambda_\alpha} a \xi \mathcal{W}_\xi(x)
\]
is real valued. Moreover, for each matrix $R \in B_{\varepsilon_{\gamma}} (\text{Id})$ we have
\[
\sum_{\xi \in \Lambda_\alpha} (\gamma_{\xi}(R))^2 \int_{\mathbb{T}^3} \mathcal{W}_\xi \otimes \mathcal{W}_{-\xi} \, dx = \sum_{\xi \in \Lambda_\alpha} (\gamma_{\xi}(R))^2 B_{\xi} \otimes B_{-\xi} = R. \tag{3.9}
\]

3.3. Analogy of equation (2.3) with the NSE near intermittent Beltrami flows. In this part, we further analyze the structure of the nonlinearity of equation (2.3) by comparing it with the NSE near the intermittent Beltrami flows introduced above. We can take the magnetic field $B$ as
\[
\mathcal{W}^B_\xi = \frac{1}{\lambda} \mathcal{W}_\xi = \frac{1}{\lambda} \eta \xi W_\xi.
\]
An important observation is that
\[
\nabla \times \mathcal{W}^B_\xi = \eta \xi W_\xi + \frac{1}{\lambda} \nabla \eta \xi \times W_\xi = \mathcal{W}_\xi + \frac{1}{\lambda} \nabla \eta \xi \times W_\xi
\]
and
\[
\| \frac{1}{\lambda} \nabla \eta \xi \times W_\xi \|_{L^2} \lesssim \sigma r,
\]
where the upper bound \( \sigma r \) can be sufficiently small by choosing the parameters appropriately. We denote \( \mathbb{W}_\xi = \frac{1}{\lambda} \nabla \eta \times W_\xi \) to be the small error term between \( \nabla \times \mathbb{W}_\xi^B \) and \( \mathbb{W}_\xi \). Thus we can naturally adapt \( J = \nabla \times \mathbb{W}_\xi^B \).

Now we show that the difference (3.4) is actually small near the intermittent Beltrami flows. Namely, by taking \( B = \mathbb{W}_\xi^B = \frac{1}{\lambda} \mathbb{W}_\xi \) and \( J = \nabla \times \mathbb{W}_\xi^B = \mathbb{W}_\xi + \mathbb{W}_\varepsilon \), a straightforward computation shows the difference

\[
\nabla \cdot \left( (\nabla \times J) \otimes (\nabla \times J) - (\nabla \times J) \otimes \mathbb{W}_\xi^B - 2\nabla \times (J \nabla \mathbb{W}_\xi^B) \right) - \frac{\nabla |J|^2}{2} - \nabla \cdot (J \otimes J) \\
= \nabla \cdot \left( \frac{1}{\lambda} \mathbb{W}_\xi \otimes (\nabla \mathbb{W}_\xi + \mathbb{W}_\varepsilon) - (\nabla \mathbb{W}_\xi + \mathbb{W}_\varepsilon) \otimes \frac{1}{\lambda} \mathbb{W}_\xi \right) \\
- 2\nabla \times ((\mathbb{W}_\xi + \mathbb{W}_\varepsilon) \nabla \mathbb{W}_\xi) + \nabla \left| \mathbb{W}_\xi + \mathbb{W}_\varepsilon \right|^2 \right) - \nabla \cdot ((\mathbb{W}_\xi + \mathbb{W}_\varepsilon) \otimes (\mathbb{W}_\xi + \mathbb{W}_\varepsilon)) \\
\sim \nabla \cdot \left( \frac{1}{\lambda} \mathbb{W}_\xi \otimes (\nabla \mathbb{W}_\xi) - (\nabla \mathbb{W}_\xi) \otimes \frac{1}{\lambda} \mathbb{W}_\xi \right) - 2\nabla \times (\mathbb{W}_\xi \nabla \mathbb{W}_\xi) \\
+ \frac{\nabla |\mathbb{W}_\xi|^2}{2} - \nabla \cdot (\mathbb{W}_\xi \otimes \mathbb{W}_\xi) \\
\sim 0.
\]

Thus, near the intermittent Beltrami flows \((B, J) = (\mathbb{W}_\xi^B, \nabla \times \mathbb{W}_\xi^B)\), equation (2.3) (the curl of the Hall equation) is indeed “close” to the NSE. Also, an obvious fact is that \( J = \nabla \times B \) scales as the velocity field in the NSE. This is the main motivation to investigate the problem of non-uniqueness of weak solutions for the Hall-MHD system in Leray-Hopf space by adhering to what has been done for the NSE in [10]. Of course, new difficulties arise in the construction. In particular, rather than one function, involved here are a pair of functions \( B \) and \( J \), which are related through \( J = \nabla \times B \). On the other hand, to apply the rigid geometric lemma, one has to regroup the nonlinear interactions in a suitable way such that error terms can be controlled. It is also non-trivial to determine how to introduce the temporal oscillation. In the end, to show non-uniqueness of Leray-Hopf weak solutions for the Hall-MHD system, we need to design a scheme of combining the convex integration method for the magnetic field equation and the classical regularity theory for the NSE. We will address all of these problems in the rest of the article.

4. Convex integration for the Hall equation

In this part, we adapt the convex integration method to construct Leray-Hopf weak solutions of the Hall equation with nonnegative energy profiles for the current density field. The main strategy is to design an iteration scheme for the approximating equation (3.2) illustrated in Proposition 4.1.

We start with fixing several parameters: for large enough constants \( a \gg 1 \) and \( b \gg 1 \), and small enough positive constant \( \beta \ll 1 \), we define:

\[
\lambda_q = a^{q^4}, \quad \delta_q = \lambda_1^{3\beta} \lambda_q^{-2\beta}, \quad r = \lambda_q^{\frac{4}{q+1}}, \quad \sigma = \lambda_q^{-\frac{4}{q+1}}, \quad \mu = \lambda_q^{\frac{4}{q+1}}, \quad \ell = \lambda_q^{-20}.
\]

(4.1) (4.2)

It is easy to see that \( \lambda_{q+1} = \lambda_q^b \).
Proposition 4.1. There exists an absolute constant $C > 0$ and a sufficiently small parameter $\varepsilon > 0$ depending on $b$ and $\beta$ such that the following inductive statement holds. Let $(B_q, J_q, R_q)$ be a solution of the approximating equation (3.2) on $\mathbb{T}^3 \times [0, T]$ satisfying:

\begin{align}
\|B_q\|_{C^1_t} &\leq \lambda_q^3, \\
\|J_q\|_{C^1_t} &\leq \lambda_q^4,
\end{align}

for $0 \leq E(t) - \int_{\mathbb{T}^3} |J_q|^2 \, dx \leq \delta_{q+1},$

and

\[ E(t) - \int_{\mathbb{T}^3} |J_q|^2 \, dx \leq \frac{\delta_{q+1}}{100} \]

implies $J_q(\cdot, t) \equiv 0$ and $R_q(\cdot, t) \equiv 0.$

In addition, we assume

\begin{align}
\nabla \cdot R_q &= \nabla \cdot \tilde{R}_q + \nabla \times \nabla \times \tilde{M}_q + \nabla \cdot \nabla \tilde{Q}_q + \nabla \tilde{p}_{q+1}
\end{align}

for a traceless symmetric tensor $\tilde{R}_q$, vector field $\tilde{M}_q$ and $\tilde{Q}_q$, and a scalar pressure function $\tilde{p}_{q+1},$ which satisfy

\begin{align}
\|\tilde{R}_q\|_{L^\infty(L^1)} + \|\tilde{M}_q\|_{L^\infty(L^1)} + \|\tilde{Q}_q\|_{L^\infty(L^1)} &\leq \lambda_q^{-\varepsilon R} \delta_{q+1}, \\
\|R_q\|_{C^1_t} &\leq \lambda_q^{12}.
\end{align}

Then we can find another solution $(B_{q+1}, J_{q+1}, R_{q+1})$ of (3.2) satisfying (4.3)-(4.9) with $q$ replaced by $q + 1.$ Moreover, the increments $v_{q+1} = B_{q+1} - B_q$ and $w_{q+1} = J_{q+1} - J_q$ satisfy

\begin{align}
\|v_{q+1}\|_{L^2} &\leq C\lambda_{q+1}^{-1/2}\delta_{q+1}, \\
\|w_{q+1}\|_{L^2} &\leq C\delta_{q+1}^{1/2}.
\end{align}

This proposition leads to a proof of Theorem 1.1 immediately.

Proof of Theorem 1.1. At the first step, we take $(B_0, J_0, R_0) = (0, 0, 0)$ which satisfies (4.3)-(4.8), and (4.5)-(4.6) for large enough $a > 0.$ For $q \geq 1,$ we apply Proposition 4.1 to obtain a sequence of approximating solutions $\{(B_q, J_q, R_q)\}$ satisfying (4.3)-(4.9). It follows from (4.10) that

\[ \sum_{q \geq 0} \|J_{q+1} - J_q\|_{L^2} = \sum_{q \geq 0} \|w_{q+1}\|_{L^2} \lesssim \sum_{q \geq 0} \delta_{q+1}^{1/2} < \infty.\]

which implies the strong convergence of $J_q = \nabla \times B_q$ to a function $J$ in $C^0(0, T; L^2),$ and the strong convergence of $B_q$ to a function $B$ in $C^0(0, T; H^1)$ with $J = \nabla \times B$ and $\nabla \cdot B = 0.$

While $\|\tilde{R}_q\|_{L^\infty(0, T; L^1)} \to 0$ and $\|\tilde{M}_q\|_{L^\infty(0, T; L^1)} \to 0$ as $q \to \infty,$ we conclude $J$ is a weak solution of (2.4), and $B$ is a weak solution of (2.1); moreover, it is obvious that $B \in L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3)),$ since $B$ is divergence free.

The proof of Proposition 4.1 will be carried out in Sections 4.1 - 4.5 below.
4.1. Construction of the perturbation \((v_{q+1}, w_{q+1})\). Based on the building blocks introduced in Section 3.2, we proceed to construct the perturbation \(v_{q+1} = B_{q+1} - B_q\),

\[ v_{q+1} := v_{q+1}^p + v_{q+1}^c + v_{q+1}^t \]

where \(v_{q+1}^p\) and \(v_{q+1}^c\) are defined as

\[
v_{q+1}^p = \sum_{\xi \in \Lambda_n} a_\xi \hat{W}_\xi^B = \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda_n} a_\xi \eta_\xi W_\xi, \\
v_{q+1}^c = \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda_n} \nabla (a_\xi \eta_\xi) \times W_\xi,
\]

while \(v_{q+1}^t\) will be defined through \(w_{q+1}^t\) later. One can verify that

\[
\nabla \cdot (v_{q+1}^p + v_{q+1}^c) = \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda_n} \nabla \cdot (\nabla \times (a_\xi \eta_\xi W_\xi)) = 0.
\]

We now define the perturbation \(w_{q+1} = J_{q+1} - J_q\) as

\[ w_{q+1} = w_{q+1}^p + w_{q+1}^c + w_{q+1}^t \]

with

\[
w_{q+1}^p = \nabla \times v_{q+1}^p, \quad w_{q+1}^c = \nabla \times v_{q+1}^c, \\
w_{q+1}^t = \mu^{-1} \sum_\xi P_{H \neq 0} (a_\xi^2 \eta_\xi^2 \xi).
\]

In the end, we define \(v_{q+1}^t\) through \(w_{q+1}^t = \nabla \times v_{q+1}^t\) up to a gradient which we can take as zero. Indeed, for \(v_{q+1}^t \in L^2\), we can decompose \(v_{q+1}^t\) as

\[ v_{q+1}^t = v_{q+1}^{t,0} + \nabla \phi, \quad \text{with} \quad \nabla \cdot v_{q+1}^{t,0} = 0. \]

In our case, we simply take \(v_{q+1}^{t,0}\) to be \(v_{q+1}^t\), since \(\nabla \times \nabla \phi = 0\). Thus, \(\nabla \cdot v_{q+1}^t = 0\) holds. Along with the fact \(\nabla \cdot (v_{q+1}^p + v_{q+1}^c) = 0\), we have

\[ \nabla \cdot v_{q+1} = 0. \]

On the other hand, it is obvious that

\[ \nabla \cdot w_{q+1}^p = \nabla \cdot w_{q+1}^c = \nabla \cdot w_{q+1}^t = \nabla \cdot w_{q+1} = 0 \]

and

\[ w_{q+1} = \nabla \times v_{q+1}. \]

4.2. Estimates of building blocks. The main purpose of adding the oscillation \(\eta_\xi\) to the Beltrami waves is to make sure the \(L^1\) norm of the waves is significantly smaller than the \(L^2\) norm. This can be seen in the following lemma.

**Lemma 4.2.** \[10\] The bounds

\[
\|\nabla^N \partial_t^K \hat{W}_\xi\|_{L^p} \lesssim \lambda^N (\lambda \sigma r \mu)^K r^{\frac{2}{p} - \frac{2}{q}}, \quad (4.11)
\]

\[
\|\nabla^N \partial_t^K \eta_\xi\|_{L^p} \lesssim \lambda^N (\lambda \sigma r)^N (\lambda \sigma r \mu)^K r^{\frac{2}{p} - \frac{2}{q}}, \quad (4.12)
\]

hold for all \(1 < p \leq \infty\).
We point out that, following [10], in order to avoid a loss of derivative, the pair $(v_q, w_q)$ at each level needs to be regularized by using standard Friedrichs mollifiers. Moreover, the corresponding stress tensor $R_q$ is not spatially homogenous. To fix it, cutoff functions that form a partition of unity can be introduced to decompose $R_q$ into slices. The two steps involve delicate computations, which will be omitted in our presentation. Rather, we do adapt the regularization parameter $\ell$ from the first step. We also adapt the partition of unity: let $0 \leq \tilde{\chi}_0, \tilde{\chi} \leq 1$ be smooth functions supported on $[0, 4]$ and $[\frac{1}{4}, 4]$ respectively; and $\tilde{\chi}_i(z) = \tilde{\chi}(4^{-i}z)$ satisfying

$$\tilde{\chi}_0^2(z) + \sum_{i \geq 1} \tilde{\chi}_i(z) \equiv 1, \quad \forall z > 0.$$ 

Then we define the amplitude function $a_\xi$ for the intermittent Beltrami flows as,

$$a_{\xi, i+1} = \rho_i \chi_i \gamma(\xi) \left( \text{Id} - \frac{R_i}{\rho_i} \right) \tag{4.13}$$

where $\rho_i$ and $\chi_i$ are defined as

$$\rho_i = \lambda_q^{-\gamma} \delta_{q+1}^{i+\epsilon_0}, \quad i \geq 1,$$

$$\chi_i(x, t) = \tilde{\chi}_i \left( \frac{R(x, t)}{100 \lambda_q^{-\gamma} \delta_{q+1}} \right).$$

Here we use the notation $\langle A \rangle = (1 + |A|^2)^{1/2}$ with $| \cdot |$ being the Euclidean norm of a matrix. Referring to [10], we have

$$4^{\max(i)} \lesssim \ell^{-1} \tag{4.14}$$

To make sure the inequality (4.15) holds, we need to choose $\rho_0$ as follows,

$$\rho(t) = \frac{1}{3|\mathbb{T}|} \left( \int_{\mathbb{T}} \chi_0^2 \, dx \right)^{-1} \max \left( E(t) - \int_{\mathbb{T}} |J_q|^2 \, dx - \frac{3}{2} \sum_{i \geq 1} \rho_i \int_{\mathbb{T}} \chi_i^2 \, dx - \frac{\delta_{q+1}}{2}, 0 \right)$$

$$\rho_0 = \left( (\rho^{1/2} \ast \varphi) \right)^2,$$

where $\varphi$ is the standard Friedrichs mollifier at time scale $\ell$. It was shown in [10], such defined $\rho_0$ satisfies

$$\| \rho_0 \|_{C^0} \lesssim 2 \delta_{q+1}, \quad \| \rho_0^{1/2} \|_{C^0} \lesssim \delta_{q+1}^{1/2} \ell^{-N},$$

for $N \geq 1$.

Below is a collection of estimates satisfied by the amplitude function $a_\xi$.

**Lemma 4.3.** The following bounds hold

$$\| \chi_i \|_{L^1} \lesssim 4^{-i}, \tag{4.15}$$

$$\| a_\xi \|_{L^2} \lesssim \delta_{q+1}^{1/2}, \tag{4.16}$$

$$\| a_\xi \|_{L^\infty} \lesssim \delta_{q+1}^{1/2} \ell^{-\frac{1}{2}}, \tag{4.17}$$

$$\| a_\xi \|_{L^p} \lesssim \delta_{q+1}^{1/2} \ell^{-\frac{1}{2}(1 - \frac{1}{p})}, \quad \text{for } p \geq 1, \tag{4.18}$$

$$\| a_\xi \|_{C^N_{x,t}} \lesssim \ell^{-N}, \quad \text{for } N \geq 1. \tag{4.19}$$
Proof: We only need to show (4.18), since other ones were shown in [10]. In view of (4.15), we deduce
\[
\|a_\xi\|_{L^1} \lesssim \rho_1^{\frac{1}{p}} \|\chi_i\|_{L^1} \lesssim \lambda_q^{-\varepsilon R/2} \delta_{q+1}^{\frac{1}{p}}.
\]

Thus, by interpolation we obtain
\[
\|a_\xi\|_{L^p} \lesssim \|a_\xi\|_{L^\infty} \|a_\xi\|_{L^1} \lesssim \delta_{q+1}^{\frac{1}{p}(1-\frac{2}{p})} \ell^{-\frac{1-\varepsilon R}{2}(1-\frac{1}{p})} \lambda_q^{-\varepsilon R} \delta_{q+1} \lesssim \delta_{q+1}^{\frac{1}{p}(1-\frac{2}{p})} \ell^{-\frac{1-\varepsilon R}{2}(1-\frac{1}{p})}.
\]

4.3. Estimates of the perturbation.

Lemma 4.4. The increment \(v_{q+1} = B_{q+1} - B_q\) satisfies the following estimates
\[
\text{(4.20)} \quad \|v_{q+1}^p\|_{L^2} \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^{\frac{1}{p}},
\]
\[
\text{(4.21)} \quad \|v_{q+1}^c\|_{L^2} \lesssim \ell^{-1} \mu^{-1} \lambda_{q+1}^{-1} \delta_{q+1}^{\frac{1}{p}},
\]
\[
\text{(4.22)} \quad \|v_{q+1}^t\|_{L^2} \lesssim \ell^{-1} \mu^{-1} (\lambda_{q+1} \sigma)^{-1} \delta_{q+1}^{\frac{1}{p}},
\]
\[
\text{(4.23)} \quad \|v_{q+1}^p\|_{L^p} \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^{\frac{1}{p}} \ell^{-\frac{1}{2}(1-\frac{2}{p})} \lambda_{q+1}^{-\frac{1}{2}}, \quad p \geq 1,
\]
\[
\text{(4.24)} \quad \|v_{q+1}^c\|_{L^p} \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^{\frac{1}{p}} \ell^{-\frac{1}{2}(1-\frac{2}{p})} \sigma \lambda_{q+1}^{-\frac{1}{2}}, \quad p \geq 1,
\]
\[
\text{(4.25)} \quad \|v_{q+1}^t\|_{L^p} \lesssim \ell^{-1} \mu^{-1} (\lambda_{q+1} \sigma)^{-1} \delta_{q+1}^{-\frac{1}{2}}, \quad p \geq 1,
\]
\[
\text{(4.26)} \quad \|v_{q+1}^p\|_{W^{1,p}} + \|v_{q+1}^c\|_{W^{1,p}} \lesssim \ell^{-2} \mu^{-\frac{1}{2}p}, \quad p \geq 1,
\]
\[
\text{(4.27)} \quad \|v_{q+1}^t\|_{W^{1,p}} \lesssim \mu^{-1} \delta_{q+1}^{-1} \ell^{-1} \mu^{-\frac{1}{2}p}, \quad p \geq 1,
\]
\[
\text{(4.28)} \quad \|v_{q+1}^p\|_{C^{\alpha}_{N,x}} + \|v_{q+1}^c\|_{C^{\alpha}_{N,x}} \lesssim \lambda_{q+1}^{1+\alpha},
\]
\[
\text{(4.29)} \quad \|v_{q+1}^t\|_{C^{\alpha}_{N,x}} \lesssim \lambda_{q+1}^{1+\alpha},
\]
\[
\text{(4.30)} \quad \|B_{q+1}\|_{C^{\alpha}_{N,x}} \lesssim \lambda_{q+1}^{1+\alpha}.
\]

Proof: Adhering to Lemma 2.4 (4.11), (4.18) and (4.19), we obtain for 1 \(\leq p \leq \infty\)
\[
\|v_{q+1}^p\|_{L^p} \leq \lambda_{q+1}^{-1} \|a_\xi\|_{L^p} \|W_\xi\|_{L^p}
\]
\[
\lesssim \lambda_{q+1}^{-1} \sum_{\xi \in A_n} \|a_\xi\|_{L^p} \|W_\xi\|_{L^p} + \lambda_{q+1}^{-\frac{1}{p}} \|a_\xi\|_{C^{1}_{N}} \|W_\xi\|_{L^p}
\]
\[
\lesssim \lambda_{q+1}^{-1} \delta_{q+1}^{\frac{1}{p}} \ell^{-\frac{1}{2}(1-\frac{2}{p})} \lambda_{q+1}^{-\frac{1}{2}} \delta_{q+1}^{-\frac{1}{2}}.
\]
In view of Lemma 2.4, (4.18), (4.19), and the choice of parameters (4.28) and (4.29), and the choice of parameters (4.11)-(4.12), we obtain

\[ \|v_{q+1}^c\|_{L^p} \leq \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda_\alpha} \|\nabla (a_\xi \eta_\xi)\|_{L^p} \]

\[ \lesssim \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda_\alpha} \|a_\xi \nabla \eta_\xi\|_{L^p} + \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda_\alpha} \|a_\xi \eta_\xi\|_{L^p} \]

\[ \lesssim \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda_\alpha} \left( \|a_\xi\|_{L^p} \|\nabla \eta_\xi\|_{L^p} + \lambda_{q+1}^{-\frac{p}{2}} \|a_\xi\|_{C^1} \|\nabla \eta_\xi\|_{L^p} \right) \]

\[ + \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda_\alpha} \|\nabla a_\xi\|_{L^\infty} \|\eta_\xi\|_{L^p} \]

\[ \lesssim \lambda_{q+1}^{-2} \lambda_{q+1}^{\frac{1}{2}} \epsilon^{-\frac{1}{2}(1-\frac{1}{p})} \lambda_{q+1} \sigma_1 ^{\frac{1}{2}} + \lambda_{q+1}^{-2} \epsilon^{-1} \sigma _1 ^{\frac{1}{2}} \]

\[ \lesssim \lambda_{q+1}^{-2} \lambda_{q+1}^{\frac{1}{2}} \epsilon^{-\frac{1}{2}(1-\frac{1}{p})} \sigma_1 ^{\frac{1}{2}}. \]

The proof of estimates on other norms of \(v_{q+1}^c\) and \(v_{q+1}^c\) can be found in [10] (by multiplying each estimate the factor \(\lambda_{q+1}^{-1}\)). We only need to show the estimates for \(v_{q+1}^c\). Recall that \(v_{q+1}^c\) satisfies

\[ \nabla \times v_{q+1}^c = w_{q+1}^t = \mu^{-1} \sum_{\xi \in \Lambda} \mathbb{P}_H \mathbb{P}_{\neq 0}(a_\xi^2 \eta_\xi^2 \xi) \]

and \(v_{q+1}^c\) is divergence free. Thus by Lemma 2.4, we deduce

\[ \|v_{q+1}^c\|_{L^2} \leq \mu^{-1} \left\| \sum_{\xi \in \Lambda} \text{curl}^{-1} \mathbb{P}_H \mathbb{P}_{\neq 0}(a_\xi^2 \eta_\xi^2 \xi) \right\|_{L^2} \]

\[ \lesssim \mu^{-1} \left\| \sum_{\xi \in \Lambda} \text{curl}^{-1} \left(a_\xi^2 \mathbb{P}_{\geq \lambda_{q+1} \sigma_2 / 2} (\eta_\xi^2 \xi) \right) \right\|_{L^2} \]

\[ \lesssim \mu^{-1} \sum_{\xi \in \Lambda} \|a_\xi^2\|_{L^\infty} \left( \lambda_{q+1}^{-1 \sigma_2} \left( 1 + \frac{1}{\ell L (\lambda_{q+1} \sigma_2)^{-2}} \right) \right) \|\eta_\xi^2\|_{L^2} \]

\[ \lesssim \mu^{-1} \delta_{q+1} \ell^{-1} (\lambda_{q+1} \sigma_2)^{-1} \epsilon^{\frac{3}{2}} . \]

In an analogous way, we can obtain

\[ \|v_{q+1}^c\|_{L^p} \lesssim \mu^{-1} \delta_{q+1} \ell^{-1} (\lambda_{q+1} \sigma_2)^{-1} \epsilon^{\frac{3}{2}} . \]

\[ \|v_{q+1}^c\|_{W^{1,p}} \lesssim \mu^{-1} \delta_{q+1} \ell^{-1} (\lambda_{q+1} \sigma_2)^{-1} \epsilon^{\frac{3}{2}} . \]

Proof of inequality (4.28) can be referred to [10]; inequality (4.29) follows from (4.28) and (4.29).
Lemma 4.5. The increment \( w_{q+1} = J_{q+1} - J_q \) satisfies the following estimates,
\[
\| w_{q+1}^p \|_{L^2} \lesssim \delta_{q+1}^{\frac{1}{2}}, \\
\| w_{q+1}^c \|_{L^2} + \| w_{q+1}^t \|_{L^2} \lesssim \ell^{-1} \mu^{-1} \delta_{q+1}^{\frac{1}{2}}, \\
\| w_{q+1} \|_{L^p} \lesssim \delta_{q+1}^{\frac{1}{2}} (1 - \frac{1}{p}) \ell^{-\frac{3}{2} - \frac{1}{p}}, \quad p \geq 1,
\]
(4.31)
(4.32)
(4.33)
(4.34)
(4.35)
(4.36)
(4.37)
(4.38)

Proof: Recall that
\[
w_{q+1}^p = \nabla \times v_{q+1}^p = \sum_{\xi \in \Lambda} a_\xi \mathbb{W}_\xi + \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \nabla (a_\xi \eta_\xi) \times W_\xi
\]
(4.39)

with
\[
\mathbb{W}^J = \lambda_{q+1}^1 w_{q+1}^p, \quad \mathbb{W}_{\epsilon_1} = \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \nabla (a_\xi \eta_\xi) \times W_\xi,
\]
and
\[
w_{q+1}^c = \nabla \times v_{q+1}^c = \lambda_{q+1}^{-2} \nabla \times \left( \sum_{\xi \in \Lambda} \nabla (a_\xi \eta_\xi) \times W_\xi \right) = \lambda_{q+1}^{-1} \nabla \times \mathbb{W}_{\epsilon_1}.
\]

Note that \( \mathbb{W}^J \) is the intermittent wave defined for the principle part of the velocity increment \( u_{q+1}^p - u_q^p \) in [10]; while the temporal oscillation part \( w_{q+1}^t \) is defined the same way as in [10]. Thus the estimates on \( \mathbb{W}^J \) and \( w_{q+1}^t \) can be adapted from [10]. Therefore, it is sufficient to estimate \( \mathbb{W}_{\epsilon_1} \) and \( \nabla \times \mathbb{W}_{\epsilon_1} \).

In addition, we notice that \( \mathbb{W}_{\epsilon_1} = \lambda_{q+1}^1 c_{q+1}^c \). It then follows from Lemma 4.3 that
\[
\| \mathbb{W}_{\epsilon_1} \|_{L^2} \lesssim \lambda_{q+1} \| c_{q+1}^c \|_{L^2} \lesssim \delta_{q+1}^{\frac{1}{2}}, \\
\| \mathbb{W}_{\epsilon_1} \|_{L^p} \lesssim \lambda_{q+1} \| c_{q+1}^c \|_{L^p} \lesssim \delta_{q+1}^{\frac{1}{2}} \ell^{-\frac{3}{2} - \frac{1}{p}}, \\
\| \partial_t \mathbb{W}_{\epsilon_1} \|_{L^p} \lesssim \lambda_{q+1} \| \partial_t c_{q+1}^c \|_{L^p} \lesssim \ell^{-2} \lambda_{q+1} \sigma \mu^{\frac{3}{2} - \frac{1}{p}}, \\
\| \mathbb{W}_{\epsilon_1} \|_{C_{N, \mu}^N} \lesssim \lambda_{q+1} \| c_{q+1}^c \|_{C_{N, \mu}^N} \lesssim \lambda_{q+1}^{\frac{3 + 5}{2}}.
\]
(4.40)

The estimates of \( \nabla \times \mathbb{W}_{\epsilon_1} \) and hence \( w_{q+1}^c \) are carried out as follows. First, a direct computation leads to
\[
w_{q+1}^c = \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda} (W_\xi \cdot \nabla \nabla (a_\xi \eta_\xi) - W_\xi \cdot \Delta (a_\xi \eta_\xi) - \nabla (a_\xi \eta_\xi) \cdot \nabla W_\xi)
\]
(4.41)
where we used the fact that $\nabla \cdot W_\xi = 0$. Thus, we have
\[
\|w^c_{q+1}\|_{L^2} \lesssim \lambda_{q+1}^{-2} \left( \|W_\xi \nabla (a_\xi \eta_\xi)\|_{L^2} + \|\nabla (a_\xi \eta_\xi) \nabla W_\xi\|_{L^2} \right) \\
\lesssim \lambda_{q+1}^{-2} \left( \|\nabla (a_\xi \eta_\xi)\|_{L^2} + \lambda_{q+1}\|\nabla (a_\xi \eta_\xi)\|_{L^2} \right) \\
\lesssim \lambda_{q+1}^{-2} \left( \|a_\xi \nabla^2 \eta_\xi\|_{L^2} + \|\nabla a_\xi \nabla \eta_\xi\|_{L^2} + \|\nabla^2 a_\xi \eta_\xi\|_{L^2} \right) \\
+ \lambda_{q+1}^{-1} \left( \|a_\xi \nabla \eta_\xi\|_{L^2} + \|\nabla a_\xi \eta_\xi\|_{L^2} \right).
\]
Following Lemma 2.4 for $L^2$ norm, we obtain
\[
\|a_\xi \nabla^2 \eta_\xi\|_{L^2} \lesssim \|a_\xi\|_{L^2} \|\nabla \eta_\xi\|_{L^2} + \lambda_{q+1}^{\frac{1}{2}} \|a_\xi\|_{C^1} \|\nabla^2 \eta_\xi\|_{L^2} \\
\lesssim \delta_{q+1}^3 (\lambda_{q+1} \sigma) \epsilon^2 + \lambda_{q+1}^{-1} (\lambda_{q+1} \sigma) \epsilon^2 \\
\lesssim \lambda_{q+1}^{\frac{1}{2}} \epsilon^{-1} \mu^{-1} \delta_{q+1}^2 \delta_{q+1}^2 \delta_{q+1}^2.
\]
due to (4.12) and (4.16), and the choice of parameters (4.1) and (4.2). The other terms are treated in an analogous way,
\[
\|\nabla a_\xi \nabla \eta_\xi\|_{L^2} \lesssim \|\nabla a_\xi\|_{L^\infty} \|\nabla \eta_\xi\|_{L^2} \lesssim \epsilon^{-1} \lambda_{q+1} \sigma \epsilon \lesssim \lambda_{q+1}^{\frac{1}{2}} \epsilon^{-1} \mu^{-1} \delta_{q+1}^2 \delta_{q+1}^2;
\]
\[
\|\nabla^2 a_\xi \eta_\xi\|_{L^2} \lesssim \|\nabla^2 a_\xi\|_{L^\infty} \|\eta_\xi\|_{L^2} \lesssim \epsilon^{-2} \lesssim \lambda_{q+1}^{\frac{1}{2}} \epsilon^{-1} \mu^{-1} \delta_{q+1}^2 \delta_{q+1}^2;
\]
\[
\|a_\xi \nabla \eta_\xi\|_{L^2} \lesssim \|a_\xi\|_{L^2} \|\nabla \eta_\xi\|_{L^2} \lesssim \delta_{q+1} \lambda_{q+1} \sigma \epsilon \lesssim \lambda_{q+1} \epsilon^{-1} \mu^{-1} \delta_{q+1}^2 \delta_{q+1}^2;
\]
\[
\|\nabla a_\xi \eta_\xi\|_{L^2} \lesssim \|\nabla a_\xi\|_{L^\infty} \|\eta_\xi\|_{L^2} \lesssim \epsilon^{-1} \lesssim \lambda_{q+1} \epsilon^{-1} \mu^{-1} \delta_{q+1}^2 \delta_{q+1}^2.
\]
Combining the estimates above yields
\[
\|w^c_{q+1}\|_{L^2} \lesssim \epsilon^{-1} \mu^{-1} \delta_{q+1}^2 \delta_{q+1}^2 \delta_{q+1}^2
\]
which concludes the proof of (4.32).

Now we estimate the $L^p$ norm of $w^p_{q+1}$, $w^c_{q+1}$, and $w^t_{q+1}$. Again we recall that $w^p_{q+1} = \lambda_{q+1} v^p_{q+1} + \lambda_{q+1} v^c_{q+1}$. The estimates (4.23) and (4.24) give immediately
\[
\|w^p_{q+1}\|_{L^p} \lesssim \delta_{q+1}^3 \epsilon^{-\frac{1}{2}(1-\frac{1}{p})} \delta_{q+1}^2 \delta_{q+1}^2 + \delta_{q+1} \epsilon^{-\frac{1}{2}(1-\frac{1}{p})} \sigma \delta_{q+1}^2 \delta_{q+1}^2 \\
\lesssim \delta_{q+1}^3 \epsilon^{-\frac{1}{2}(1-\frac{1}{p})} \delta_{q+1}^2 \delta_{q+1}^2 \delta_{q+1}^2.
\]
In an analogous way of estimating $\|w^c_{q+1}\|_{L^2}$, we can obtain
\[
\|w^c_{q+1}\|_{L^p} \lesssim \epsilon^{-1} \mu^{-1} \delta_{q+1}^3 \delta_{q+1}^3 \delta_{q+1}^3.
\]
While we deal with $w^t_{q+1}$ as follows, by using (4.17) and (4.12)
\[
\|w^t_{q+1}\|_{L^p} \lesssim \mu^{-1} \sum_{\xi \in \Lambda_n} \|a_\xi \eta_\xi\|_{L^p} \\
\lesssim \mu^{-1} \sum_{\xi \in \Lambda_n} \|a_\xi\|_{L^\infty} \|\eta_\xi\|_{L^2} \\
\lesssim \mu^{-1} \delta_{q+1} \epsilon^{-1} \mu^{-1} \delta_{q+1}^3 \delta_{q+1}^3 \delta_{q+1}^3.
\]
Combining the last three estimates yields
\[
\|w^t_{q+1}\|_{L^p} \lesssim \delta_{q+1}^3 \epsilon^{-\frac{1}{2}(1-\frac{1}{p})} \delta_{q+1}^2 \delta_{q+1}^2
\]
which proves (4.33).
Next we estimate $\|w_{q+1}^c\|_{W^{1,p}}$. It follows from (4.11), Lemma 4.3, Lemma 4.5 and (4.1)–(4.2) that
\[
\|w_{q+1}^c\|_{W^{1,p}} \lesssim \lambda_{q+1}^{-2} \left( \|W_\xi \nabla^2 (a_\xi \eta_\xi)\|_{W^{1,p}} + \|\nabla W_\xi \nabla (a_\xi \eta_\xi)\|_{W^{1,p}} \right)
\lesssim \lambda_{q+1}^{-2} \left( \|\nabla^3 (a_\xi \eta_\xi)\|_{L^p} + \lambda_{q+1} \|\nabla^2 (a_\xi \eta_\xi)\|_{L^p} \right)
\lesssim \lambda_{q+1}^{-2} \|a_\xi\|_{C^3} \left( \|\nabla^3 \eta_\xi\|_{L^p} + \lambda_{q+1} \|\nabla^2 \eta_\xi\|_{L^p} \right)
\lesssim \lambda_{q+1}^{-2} \ell^{-3} \left( (\lambda_{q+1} \sigma r)^3 + \lambda_{q+1} (\lambda_{q+1} \sigma r)^2 \right) r^{\frac{3}{p} - \frac{3}{2}}
\lesssim \ell^{-2} \lambda_{q+1} r^{\frac{3}{p} - \frac{3}{2}}.
\]
Thus, the proof of (4.33) is also complete.

To prove (4.35), we proceed to estimate $\|\partial_t w_{q+1}^c\|_{L^p}$,
\[
\|\partial_t w_{q+1}^c\|_{L^p} \lesssim \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda} \left( \|\partial_t \nabla^2 (a_\xi \eta_\xi)\|_{L^p} + \lambda_{q+1} \|\partial_t \nabla (a_\xi \eta_\xi)\|_{L^p} \right)
\lesssim \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda} \|a_\xi\|_{C^3} \left( \|\partial_t \nabla^2 \eta_\xi\|_{L^p} + \|\nabla^2 \eta_\xi\|_{L^p} \right)
+ \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \|a_\xi\|_{C^2} \left( \|\partial_t \nabla \eta_\xi\|_{L^p} + \|\nabla \eta_\xi\|_{L^p} \right)
\lesssim \lambda_{q+1}^{-2} \ell^{-3} \left( \lambda_{q+1} \sigma r \mu + 1 \right) (\lambda_{q+1} \sigma r)^2 r^{\frac{3}{p} - \frac{3}{2}}
+ \lambda_{q+1}^{-1} \ell^{-2} \lambda_{q+1} \sigma r \mu + 1 \lambda_{q+1} \sigma r)^2 r^{\frac{3}{p} - \frac{3}{2}}
\lesssim \ell^{-2} \lambda_{q+1} \sigma r \mu r^{\frac{3}{p} - \frac{3}{2}}.
\]
In the end, we estimate $\|w_{q+1}^c\|_{C^{N,1}_{x,t}}$,
\[
\|w_{q+1}^c\|_{C^{N,1}_{x,t}} \lesssim \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda} \left( \|\nabla^2 (a_\xi \eta_\xi)\|_{C^{N,1}_{x,t}} + \lambda_{q+1} \|\nabla (a_\xi \eta_\xi)\|_{C^{N,1}_{x,t}} \right)
\lesssim \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda} \left( \|a_\xi\|_{C^{N,1}_{x,t}} \|\nabla^2 \eta_\xi\|_{C^{N,1}_{x,t}} + \|\nabla a_\xi\|_{C^{N,1}_{x,t}} \|\nabla \eta_\xi\|_{C^{N,1}_{x,t}} + \|\nabla^2 a_\xi\|_{C^{N,1}_{x,t}} \|\eta_\xi\|_{C^{N,1}_{x,t}} \right)
+ \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \left( \|a_\xi\|_{C^{N,1}_{x,t}} \|\nabla \eta_\xi\|_{C^{N,1}_{x,t}} + \|\nabla a_\xi\|_{C^{N,1}_{x,t}} \|\eta_\xi\|_{C^{N,1}_{x,t}} \right)
\lesssim \lambda_{q+1}^{-2} \left( \ell^{-N} (\lambda_{q+1} \sigma r)^2 + \ell^{-N-1} \lambda_{q+1} \sigma r + \ell^{-N-2} \right) (\lambda_{q+1} \sigma r \mu)^N r^{\frac{3}{p}}
+ \lambda_{q+1}^{-1} \left( \ell^{-N} \lambda_{q+1} \sigma r + \ell^{-N-1} \right) (\lambda_{q+1} \sigma r \mu)^N r^{\frac{3}{p}}
\lesssim \lambda_{q+1}^{-2 + \frac{3}{p}}.
\]
It completes the proof of the inequality (4.36).

Inequality (4.37) can be obtained analogously as (4.34), while (4.38) is implied by (4.37) and (4.2).

\[
4.4 \textbf{Stress tensor } R_{q+1} \textbf{ and its estimate.} \text{ Based on the construction of the increments } v_{q+1} = B_{q+1} - B_q \text{ and } w_{q+1} = J_{q+1} - J_q \text{, and hence } B_{q+1} = B_q + v_{q+1} \text{ and } J_{q+1} = J_q + w_{q+1} \text{, we will derive the new stress tensor } R_{q+1} \text{ such that } (B_{q+1}, J_{q+1}, R_{q+1}) \text{ satisfies}
\]
\[
\partial_t J_{q+1} + \nabla \times \nabla \times (J_{q+1} \times B_{q+1}) = \Delta J_{q+1} + \nabla \cdot R_{q+1},
\]
(4.42)
i.e. equation (3.3) (and (3.2) as well) at the level \(q+1\).

Substituting \(B_{q+1} = B_q + v_{q+1}\) and \(J_{q+1} = J_q + w_{q+1}\) in the left hand side of (4.42), we have
\[
\partial_t J_{q+1} + \nabla \times \nabla \times (J_{q+1} \times B_{q+1}) \\
= \partial_t J_q + \partial_t w_{q+1} + \nabla \times \nabla \times ((J_q + w_{q+1}) \times (B_q + v_{q+1})) \\
= \partial_t J_q + \partial_t w_{q+1} + \nabla \times \nabla \times (J_q \times B_q) + \nabla \times \nabla \times (J_q \times v_{q+1}) \\
+ \nabla \times \nabla \times (w_{q+1} \times B_q) + \nabla \times \nabla \times (w_{q+1} \times v_{q+1}).
\]

(4.43)

Recall that \((B_q, J_q, R_q)\) satisfies equation (3.3), i.e.
\[
\partial_t J_q + \nabla \times \nabla \times (J_q \times B_q) = \Delta J_q + \nabla \cdot R_q.
\]

Since \(\Delta J_q = \Delta J_{q+1} - \Delta w_{q+1}\), we have
\[
\partial_t J_q + \nabla \times \nabla \times (J_q \times B_q) = \Delta J_{q+1} - \Delta w_{q+1} + \nabla \cdot R_q.
\]

Therefore, substituting \(\partial_t J_q + \nabla \times \nabla \times (J_q \times B_q)\) by \(\Delta J_{q+1} - \Delta w_{q+1} + \nabla \cdot R_q\) in the right hand side of (4.43) leads to
\[
\partial_t J_{q+1} + \nabla \times \nabla \times (J_{q+1} \times B_{q+1}) \\
= \Delta J_{q+1} + \nabla \cdot R_q + \partial_t w_{q+1} - \Delta w_{q+1} + \nabla \times \nabla \times (J_q \times v_{q+1}) \\
+ \nabla \times \nabla \times (w_{q+1} \times B_q) + \nabla \times \nabla \times (w_{q+1} \times v_{q+1}).
\]

(4.44)

We now observe from (4.44) that, if we choose \(R_{q+1}\) such that
\[
\nabla \cdot R_{q+1} = \nabla \cdot R_q + \partial_t w_{q+1} - \Delta w_{q+1} + \nabla \times \nabla \times (J_q \times v_{q+1}) \\
+ \nabla \times \nabla \times (w_{q+1} \times B_q) + \nabla \times \nabla \times (w_{q+1} \times v_{q+1}),
\]

(4.45)

then \((B_{q+1}, J_{q+1}, R_{q+1})\) satisfies (4.42).

Next, we rewrite the double curl terms with \(\nabla \times \nabla \times\) in (4.45) into divergence form of \(\nabla\). Applying a few vector identities in Section 6 we have
\[
\nabla \times \nabla \times (J_q \times v_{q+1}) \\
= \nabla \cdot [v_{q+1} \otimes (\nabla \times J_q) - (\nabla \times J_q) \otimes v_{q+1}] \\
+ \nabla \cdot [(\nabla \times v_{q+1}) \otimes J_q - J_q \otimes (\nabla \times v_{q+1})] \\
- 2\nabla \times (J_q \nabla v_{q+1}),
\]

\[
\nabla \times \nabla \times (w_{q+1} \times B_q) \\
= \nabla \cdot [B_q \otimes (\nabla \times w_{q+1}) - (\nabla \times w_{q+1}) \otimes B_q] \\
+ \nabla \cdot [(\nabla \times B_q) \otimes w_{q+1} - w_{q+1} \otimes (\nabla \times B_q)] \\
- 2\nabla \times (w_{q+1} \nabla B_q),
\]

\[
\nabla \times \nabla \times (w_{q+1} \times v_{q+1}) \\
= \nabla \cdot [v_{q+1} \otimes (\nabla \times w_{q+1}) - (\nabla \times w_{q+1}) \otimes v_{q+1}] \\
+ \nabla \cdot [(\nabla \times v_{q+1}) \otimes w_{q+1} - w_{q+1} \otimes (\nabla \times v_{q+1})] \\
- 2\nabla \times (w_{q+1} \nabla v_{q+1}).
\]
Combining the last three equations with (4.45), we obtain

\[ \nabla \cdot R_{q+1} = \partial_t w_{q+1} - \Delta w_{q+1} \]
\[ + \nabla \cdot [v_{q+1} \odot (\nabla \times J_q) - (\nabla \times J_q) \odot v_{q+1}] \]
\[ + \nabla \cdot [(\nabla \times v_{q+1}) \odot J_q - J_q \odot (\nabla \times v_{q+1})] \]
\[ + \nabla \cdot [B_q \odot (\nabla \times w_{q+1}) - (\nabla \times w_{q+1}) \odot B_q] \]
\[ + \nabla \cdot [(\nabla \times B_q) \odot w_{q+1} - w_{q+1} \odot (\nabla \times B_q)] \]
\[ + \nabla \cdot [v_{q+1} \odot (\nabla \times w_{q+1}) - (\nabla \times w_{q+1}) \odot v_{q+1}] \]
\[ + \nabla \cdot [(\nabla \times v_{q+1}) \odot w_{q+1} - w_{q+1} \odot (\nabla \times v_{q+1})] \]
\[ - 2\nabla \times (J_q \nabla v_{q+1}) - 2\nabla \times (w_{q+1} \nabla B_q) - 2\nabla \times (w_{q+1} \nabla v_{q+1}) \]
\[ + \nabla \cdot R_q. \tag{4.46} \]

We further reorder the terms and classify them on the right hand side of (4.46) into linear, correction and oscillation terms:

\[ \nabla \cdot R_{q+1} = \{ \nabla \cdot [\mathcal{R}(\partial_t w_{q+1}^p + \partial_t w_{q+1}^c - \Delta w_{q+1})] \]
\[ + \nabla \cdot [v_{q+1} \odot (\nabla \times J_q) - (\nabla \times J_q) \odot v_{q+1} + (\nabla \times v_{q+1}) \odot J_q - J_q \odot (\nabla \times v_{q+1})] \]
\[ + \nabla \cdot [B_q \odot (\nabla \times w_{q+1}) - (\nabla \times w_{q+1}) \odot B_q + (\nabla \times B_q) \odot w_{q+1} - w_{q+1} \odot (\nabla \times B_q)] \]
\[ - \nabla \cdot [2(\nabla \times (J_q \nabla v_{q+1}) + 2\nabla \times (w_{q+1} \nabla B_q))] \]
\[ + \{ \nabla \cdot [(v_{x+1}^p + v_{y+1}^p) \odot (\nabla \times w_{q+1}) + v_{x+1}^p \odot (\nabla \times (w_{q+1}^c + w_{q+1}^p))] \]
\[ - \nabla \cdot [(\nabla \times (w_{q+1}^c + w_{q+1}^p)) \odot v_{q+1} + (\nabla \times w_{q+1}^p) \odot (v_{x+1}^c + v_{x+1}^p)] \]
\[ + \nabla \cdot [(\nabla \times (w_{x+1}^c + w_{y+1}^c)) \odot w_{q+1} + (\nabla \times w_{y+1}^c) \odot (w_{x+1}^c + w_{x+1}^p)] \]
\[ - \nabla \cdot [(w_{x+1}^c + w_{y+1}^c) \odot (\nabla \times v_{q+1}) - w_{y+1}^p \odot (\nabla \times (v_{x+1}^c + v_{x+1}^p))] \]
\[ - \nabla \cdot [2(\nabla \times (w_{q+1} \nabla w_{q+1}^p)) - 2\nabla \times (w_{q+1} \nabla v_{q+1}^p)] \]
\[ - \nabla \cdot [2(w_{x+1}^c + w_{y+1}^c) \nabla v_{q+1}^p)] \]
\[ + \{ \nabla \cdot [(v_{x+1}^p \odot (\nabla \times w_{q+1}^p)) - (\nabla \times w_{q+1}^p) \odot v_{x+1}^p] \]
\[ + \nabla \cdot [(\nabla \times v_{y+1}^p) \odot w_{q+1}^p - w_{y+1}^p \odot (\nabla \times v_{x+1}^p)] \]
\[ - 2\nabla \times (w_{x+1}^p \nabla w_{y+1}^p) + (\nabla \cdot R_q + \partial_t w_{q+1}^c) \} \]
\[ =: \nabla \cdot R_{\text{linear}} + \nabla \cdot R_{\text{corrector}} + \nabla \cdot R_{\text{oscillation}}. \tag{4.47} \]

On the right hand side of the equation (4.47), the first four lines correspond to linear terms, the middle six lines correspond to correction terms, and the last three lines correspond to oscillation terms.

Before diving into the estimates of the new stress tensor, we will first rearrange \( \nabla \cdot R_{\text{corrector}} \) and \( \nabla \cdot R_{\text{oscillation}} \).

**Lemma 4.6.** The term \( \nabla \cdot R_{\text{corrector}} \) can be reformulated as

\[ \nabla \cdot R_{\text{corrector}} = \nabla \cdot (R_{\text{cor},2} + R_{\text{cor},3}) + \nabla \times \nabla \times \tilde{M}_{q+1} + \nabla \tilde{p}_{q+1,1} \tag{4.48} \]
with \( R_{\text{cor,2}} \), \( R_{\text{cor,3}} \) and \( \hat{M}_{q+1} \) defined as follows

\[
R_{\text{cor,2}} = \nabla \cdot [(v_{q+1}^c + v_{q+1}^t) \otimes (\nabla \times w_{q+1}) + v_{q+1}^p \otimes (\nabla \times (w_{q+1}^c + w_{q+1}^t))]
- \nabla \cdot [[(\nabla \times (w_{q+1}^c + w_{q+1}^t)) \otimes v_{q+1} + (\nabla \times v_{q+1}^c) \otimes (\nabla \times v_{q+1}^t)]
+ \nabla \cdot [[(\nabla \times v_{q+1}^c) \otimes w_{q+1} + (\nabla \times v_{q+1}^t) \otimes (w_{q+1}^c + w_{q+1}^t)]
- \nabla \cdot [(w_{q+1}^c + w_{q+1}^t) \otimes (\nabla \times w_{q+1}) - w_{q+1}^p \otimes (\nabla \times (v_{q+1}^c + v_{q+1}^t))]
- \nabla \cdot [R \nabla \times (2w_{q+1}^c \nabla v_{q+1}^c)]
- 2\nabla \times (w_{q+1} \nabla v_{q+1}^t)
- \nabla \cdot [R \nabla \times (2(w_{q+1}^c + w_{q+1}^t) \nabla v_{q+1}^c)]
= \{ \nabla \cdot [v_{q+1}^c \otimes (\nabla \times w_{q+1}) - (\nabla \times w_{q+1}^p) \otimes v_{q+1}^c] + \nabla \cdot [(\nabla \times v_{q+1}^c) \otimes w_{q+1} - w_{q+1} \otimes (\nabla \times v_{q+1}^c)] - \nabla \cdot [R \nabla \times (2w_{q+1} \nabla v_{q+1}^c)] \}
+ \{ \nabla \cdot [(\nabla \times v_{q+1}^c) \otimes w_{q+1} - w_{q+1} \otimes (\nabla \times v_{q+1}^c)] - \nabla \cdot [R \nabla \times (2w_{q+1} \nabla v_{q+1}^c)] \}
+ \{ \nabla \cdot [(w_{q+1}^c + w_{q+1}^t) \otimes (w_{q+1}^c + w_{q+1}^t) - (w_{q+1}^c + w_{q+1}^t) \otimes (w_{q+1}^c + w_{q+1}^t)]
- \nabla \cdot [R \nabla \times (2(w_{q+1}^c + w_{q+1}^t) \nabla v_{q+1}^c)] \}
= : \nabla \cdot R_{\text{cor,1}} + \nabla \cdot R_{\text{cor,2}} + \nabla \cdot R_{\text{cor,3}}.
\]

We notice that only \( R_{\text{cor,1}} \) involves with \( v_{q+1}^t \). We can further rewrite \( \nabla \cdot R_{\text{cor,1}} \) into

\[
\nabla \cdot R_{\text{cor,1}}
= \{ \nabla \cdot [v_{q+1}^c \otimes (\nabla \times w_{q+1}) - (\nabla \times w_{q+1}^p) \otimes v_{q+1}^c]
+ \nabla \cdot [(\nabla \times v_{q+1}^c) \otimes w_{q+1} - w_{q+1} \otimes (\nabla \times v_{q+1}^c)]
- 2\nabla \times (w_{q+1} \nabla v_{q+1}^c)
\}
= \nabla \times (w_{q+1} \nabla v_{q+1}^t).
\]

Denote

\[
\hat{M}_{q+1} = w_{q+1} \times v_{q+1}^t.
\]
It follows that
\[ \nabla \cdot R_{\text{cor},1} = \nabla \times \nabla \times \tilde{M}_{q+1}. \]

It completes the proof of the lemma.

\[ \square \]

**Lemma 4.7.** The oscillation part \( \nabla \cdot R_{\text{oscillation}} \) of the stress tensor can be written as
\[ \nabla \cdot R_{\text{oscillation}} = \nabla \cdot (R_{\text{osc},1} + R_{\text{osc},2}) + \nabla \tilde{p}_{q+1,2} \quad (4.52) \]
with \( R_{\text{osc},1} \) and \( R_{\text{osc},2} \) defined as follows
\[ R_{\text{osc},1} = \lambda_{q+1}(v_{q+1}^p + \lambda_{q+1}v_{q+1}^p - w_{q+1}^p \otimes w_{q+1}^p + R_q + \mathcal{R}\partial_t w_{q+1}^t), \quad (4.53) \]
\[ R_{\text{osc},2} = 2\nabla \cdot (w_{q+1}^p \otimes \mathcal{W}_{e,1}) + \nabla \cdot (w_{q+1}^c \otimes v_{q+1}^p - v_{q+1}^p \otimes w_{q+1}^c) - \nabla \cdot (\mathcal{R}\nabla \times (2\mathcal{W}_{e,1} \nabla t_{q+1}) - \nabla \cdot (\mathcal{W}_{e,1} \otimes \mathcal{W}_{e,1}), \quad (4.54) \]
while \( \mathcal{W}_{e,1} \) is defined in (4.39) and \( \tilde{p}_{q+1,2} \) is a pressure term to make \( R_{\text{osc},1} \) and \( R_{\text{osc},2} \) traceless.

**Proof:** In fact the oscillation terms, see (13.17), can be written as
\[ \nabla \cdot R_{\text{oscillation}} \]
\[ = \nabla \times \nabla \times (w_{q+1}^p \times v_{q+1}^p) + \nabla \cdot R_q + \partial_t w_{q+1}^t \]
\[ = \nabla \times \nabla \times ((\nabla \times v_{q+1}^p) \times v_{q+1}^p) + \nabla \cdot R_q + \partial_t w_{q+1}^t \]
\[ = \nabla \cdot (\lambda_{q+1}v_{q+1}^p + \lambda_{q+1}v_{q+1}^p) - \nabla \cdot (w_{q+1}^p \otimes w_{q+1}^p) + \nabla \cdot (w_{q+1}^p \otimes w_{q+1}^p) \]
\[ + \nabla \times \nabla \times ((\nabla \times v_{q+1}^p) \times v_{q+1}^p) - \nabla \cdot (\lambda_{q+1}v_{q+1}^p + \lambda_{q+1}v_{q+1}^p) \]
\[ + \nabla \cdot (w_{q+1}^p \otimes w_{q+1}^p). \quad (4.55) \]
We denote the first three terms in (4.55) by \( \nabla \cdot R_{\text{osc},1} \), i.e.
\[ R_{\text{osc},1} = \lambda_{q+1}v_{q+1}^p + \lambda_{q+1}v_{q+1}^p - w_{q+1}^p \otimes w_{q+1}^p + R_q + \mathcal{R}\partial_t w_{q+1}^t. \]
Regarding the last three terms of \( \nabla \cdot R_{\text{oscillation}} \) in (4.55), we need more effort to deal with them. We first recall that
\[ w_{q+1}^p = \nabla \times v_{q+1}^p = \lambda_{q+1}v_{q+1}^p + \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \nabla (a_{\xi} \eta_{\xi}) \times W_{\xi} = \lambda_{q+1}v_{q+1}^p + \mathcal{W}_{e,1}. \]

Thus, we have
\[ \lambda_{q+1}v_{q+1}^p = w_{q+1}^p - \mathcal{W}_{e,1}. \]
On the other hand, we notice that
\[ \nabla \times w_{q+1}^p = \lambda_{q+1} \sum_{\xi \in \Lambda} a_{\xi} \mathcal{W}_{\xi} + \sum_{\xi \in \Lambda} \nabla (a_{\xi} \eta_{\xi}) \times W_{\xi} + \nabla \times \mathcal{W}_{e,1} \]
\[ = \lambda_{q+1}(w_{q+1}^p - \mathcal{W}_{e,1}) + \lambda_{q+1} \mathcal{W}_{e,1} + \nabla \times \mathcal{W}_{e,1} \]
\[ = \lambda_{q+1}w_{q+1}^p + \nabla \times \mathcal{W}_{e,1}. \]
Therefore, a straightforward computation leads to
\[
\nabla \times \nabla \times ((\nabla \times v_{q+1}^p) \times v_{q+1}^p) = -\nabla \cdot (\nabla \cdot (\lambda_{q+1} v_{q+1}^p \otimes \lambda_{q+1} v_{q+1}^p)) + \nabla \cdot (u_{q+1}^p \otimes u_{q+1}^p)
\]
\[
= -\nabla \cdot (v_{q+1}^p \otimes (\nabla \times v_{q+1}^p)) - (\nabla \times v_{q+1}^p) \otimes v_{q+1}^p) - 2\nabla \times (u_{q+1}^p \nabla v_{q+1}^p)
\]
\[
\quad + \nabla \cdot (\nabla \cdot (\lambda_{q+1} v_{q+1}^p \otimes \lambda_{q+1} v_{q+1}^p)) - \nabla \cdot (\nabla \cdot (\nabla \times v_{q+1}^p) \otimes v_{q+1}^p) - 2\nabla \times (W_{e,1} \otimes u_{q+1}^p) + \nabla \cdot (W_{e,1} \otimes u_{q+1}^p)
\]
\[
\quad + \nabla \cdot (u_{q+1}^p \otimes \lambda_{q+1} v_{q+1}^p) + \nabla \cdot (W_{e,1} \otimes u_{q+1}^p)
\]
\[
\quad + \nabla \cdot (u_{q+1}^p \otimes \lambda_{q+1} v_{q+1}^p) - \nabla \cdot (W_{e,1} \otimes \lambda_{q+1} v_{q+1}^p)
\]
\[
= -\nabla \cdot ((u_{q+1}^p - W_{e,1} v_{q+1}^p) \otimes u_{q+1}^p - W_{e,1} v_{q+1}^p) - (u_{q+1}^p - W_{e,1} v_{q+1}^p) \otimes (u_{q+1}^p - W_{e,1} v_{q+1}^p))
\]
\[
\quad - 2\nabla \times (W_{e,1} \nabla v_{q+1}^p) + \nabla \cdot (W_{e,1} \otimes u_{q+1}^p)
\]
\[
\quad + \nabla \cdot (W_{e,1} v_{q+1}^p - W_{e,1} v_{q+1}^p) - \nabla \cdot (W_{e,1} \otimes \lambda_{q+1} v_{q+1}^p) - \nabla \cdot (W_{e,1} \otimes \lambda_{q+1} v_{q+1}^p)
\].
\]
In view of the facts that \(\nabla \cdot w_{q+1}^p = 0\) and \(\nabla \times W_{e,1} = w_{q+1}^c\), we obtain that by continuing with the last equation
\[
\nabla \times \nabla \times ((\nabla \times v_{q+1}^p) \times v_{q+1}^p) = -\nabla \cdot (\nabla \cdot (\lambda_{q+1} v_{q+1}^p \otimes \lambda_{q+1} v_{q+1}^p)) + \nabla \cdot (u_{q+1}^p \otimes u_{q+1}^p)
\]
\[
= 2\nabla \cdot (u_{q+1}^p \otimes W_{e,1}) + \nabla \cdot (u_{q+1}^p \otimes v_{q+1}^p - v_{q+1}^p \otimes u_{q+1}^p)
\]
\[
\quad - \nabla \cdot [R \nabla \times (2W_{e,1} \nabla v_{q+1}^p)] - \nabla \cdot (W_{e,1} \otimes W_{e,1}).
\]

Thus, the conclusion of the lemma follows from (4.56) and (4.58).

We are now ready to accomplish the estimate of the new stress tensor \(R_{q+1}\).

**Lemma 4.8.** Consider equation (4.44) with \(R_{q+1}\) defined by (4.47) (and equivalently (4.47)). There exists another traceless symmetric tensor \(R_{q+1}\) and a scalar pressure function \(p_{q+1}\) such that \(\nabla \cdot R_{q+1}\) can be written as
\[
\nabla \cdot R_{q+1} = \nabla \cdot \hat{R}_{q+1} + \nabla \times \nabla \times \hat{M}_{q+1} + \nabla \cdot \hat{p}_{q+1}.
\]
In addition, there exists \(p > 1\) sufficiently close to 1, and a sufficiently small \(\varepsilon_R > 0\) independent of \(q\) such that
\[
||\hat{R}_{q+1}||_{L^p} + ||\hat{M}_{q+1}||_{L^p} \leq \lambda_{q+1}^{-2p} \delta_{q+2}
\]
holds for some implicit constant which depends on \(p\) and \(\varepsilon_R\).

**Proof:** Recall from (4.47) that
\[
\nabla \cdot R_{q+1} = \nabla \cdot R_{\text{linear}} + \nabla \cdot R_{\text{corrector}} + \nabla \cdot R_{\text{oscillation}}.
\]
Denote
\[ \tilde{R}_{q+1} = R_{\text{linear}} + R_{\text{cor.2}} + R_{\text{cor.3}} + R_{\text{osc.1}} + R_{\text{osc.2}}, \]
\[ \tilde{p}_{q+1} = \tilde{p}_{q+1,1} + \tilde{p}_{q+1,2}. \]
Therefore, (4.57) follows from Lemma 4.6, Lemma 4.7, and (4.59).

The estimate of \( \tilde{R}_{q+1} \) will be established in Lemma 4.9, Lemma 4.10, and Lemma 4.11 below, and the estimate of \( \tilde{M}_{q+1} \) will be obtained in Lemma 4.10. Thus, estimate (4.58) follows from these lemmas.

\[ \square \]

4.4.1. Linear terms. The estimates of the linear terms are relatively easy.

**Lemma 4.9.** For \( p > 1 \) sufficiently close to 1, \( R_{\text{linear}} \) satisfies
\[ \|R_{\text{linear}}\|_{L^p} \lesssim \lambda_{q+1}^{-2} \delta_{q+2}. \]

**Proof:** It follows from Lemma 2.6 and (4.34) that,
\[ \|R_{\text{cor.1}}\|_{L^p} \lesssim \|w_{q+1}\|_{W^{1,p}} \lesssim \ell^{-2} \lambda_{q+1}^2 r^{3-\frac{3}{p}}. \]
While Lemma 2.6 and (4.39) together give
\[ \|R(\partial_t(w^0_{q+1} + w^c_{q+1}))\|_{L^p} = \|R(\partial_t \nabla \times (v^0_{q+1} + v^c_{q+1}))\|_{L^p} \]
\[ = \lambda_{q+1}^{-1} \|R\partial_t \nabla \nabla \nabla \times \nabla \times v^0_{q+1}\|_{L^p} \]
\[ = \lambda_{q+1}^{-1} \|R\partial_t \nabla \times u^0_{q+1}\|_{L^p} \]
\[ \lesssim \lambda_{q+1}^{-1} \|\partial_t u^0_{q+1}\|_{L^p} \]
\[ \lesssim \ell^{-2} \sigma_{\mu r}^{1-\frac{3}{p}}. \]

We have, by \( \|w_{q+1}\|_{L^p} \lesssim \|w_{q+1}\|_{W^{1,p}} \),
\[ \|\nabla \times J_q\|_{L^p} \lesssim \|v_{q+1}\|_{L^p} \]
\[ \lesssim \lambda_q^2 \left( \lambda_{q+1}^{-1} \delta_{q+1}^\frac{3}{2} \ell^{-\frac{1}{2}(1-\frac{3}{p})} r^{\frac{3}{2}-\frac{3}{p}} + \ell^{-1} \mu^{-1} (\lambda_{q+1} \sigma)^{-1} \delta_{q+1} r^{3-\frac{3}{p}} \right) \]
\[ \lesssim \lambda_q^2 \ell^{-2} \lambda_{q+1}^{-1} \delta_{q+1}^{-1} r^{3-\frac{3}{p}}; \]
and similarly, by \( \|w_{q+1}\|_{L^p} \lesssim \|w_{q+1}\|_{W^{1,p}} \),
\[ \|B_q\|_{L^p} \lesssim \|B_q\|_{L^\infty} \|w_{q+1}\|_{W^{1,p}} \]
\[ \lesssim \lambda_q^2 \ell^{-2} \lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}}. \]

Combining (4.38), (4.23)-(4.26), (4.27), (4.30), (4.33), and (4.34) yields
\[ \|\nabla (J_q \times v_{q+1}) + \nabla (w_{q+1} + B_q)\|_{L^p} \]
\[ \lesssim \|\nabla J_q\|_{L^\infty} \|v_{q+1}\|_{L^p} + \|J_q\|_{L^\infty} \|v_{q+1}\|_{W^{1,p}} \]
\[ + \|\nabla B_q\|_{L^\infty} \|w_{q+1}\|_{L^p} + \|B_q\|_{L^\infty} \|w_{q+1}\|_{W^{1,p}} \]
\[ \lesssim \lambda_q^3 \ell^{-1} \mu^{-1} (\lambda_{q+1} \sigma)^{-1} r^{\frac{3}{2}-\frac{3}{p}} + \lambda_q^3 \left( \ell^{-2} \sigma_{\mu r}^{1-\frac{3}{p}} + \mu^{-1} \delta_{q+1} \ell^{-1} r^{4-\frac{3}{p}} \right) \]
\[ + \lambda_q^4 \left( \delta_{q+1}^{1-\frac{3}{p}} r^{\frac{3}{2}-\frac{3}{p}} + \ell^{-2} \lambda_{q+1}^{-1} \delta_{q+1}^{1-\frac{3}{p}} \right). \]
Other terms in $R_{\text{linear}}$ can be estimated similarly. Summarizing the estimates above and taking into account the choice of parameters (4.1)-(4.2) concludes the proof.

4.4.2. Correction terms.

**Lemma 4.10.** For $p > 1$ close enough to 1, and a sufficiently small constant $\varepsilon_R > 0$ depending on $p$, the following estimates hold:

$$
\|R_{\{\text{cor,2}\}}\|_{L^p} + \|R_{\{\text{cor,3}\}}\|_{L^p} + \|\dot{M}_{q+1}\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+1}.
$$

**Proof:** Recall from (4.48)

$$
\nabla \cdot R_{\text{corrector}} = \nabla \cdot (R_{\{\text{cor,2}\}} + R_{\{\text{cor,3}\}}) + \nabla \times \nabla \times \dot{M}_{q+1} + \nabla \tilde{p}_{q+1,1}
$$

with

$$
\dot{M}_{q+1} = w_{q+1} \times v_{q+1}^t,
$$

and $R_{\{\text{cor,2}\}}, R_{\{\text{cor,3}\}}$ defined in (4.49), (4.50) respectively. We can estimate $\dot{M}_{q+1}$ as, in view of (4.33), (4.25) and (4.1)- (4.2)

$$
\|\dot{M}_{q+1}\|_{L^p} \leq \|w_{q+1}\|_{L^p}\|v_{q+1}^t\|_{L^p} \leq \delta_{q+1}^{\frac{1}{3}} \varepsilon \left(1 - \frac{1}{p}\right) \mu^{\frac{1}{2}} \varepsilon^{\frac{1}{6}} \lambda_{q+1}^{-1} \delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \lesssim \varepsilon \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+1}.
$$

We turn to the estimates of $R_{\{\text{cor,2}\}}, R_{\{\text{cor,3}\}},$ and $\dot{M}_{q+1,2}$ which are trivial. Following from (4.24) and (4.34), it has

$$
\|R_{\{\text{cor,2}\}}\|_{L^p} \leq \|w_{q+1}\|_{W^{1,2p}}\|v_{q+1}^c\|_{L^p} \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^\frac{1}{3} \varepsilon \left(1 - \frac{1}{p}\right) \sigma r^{\frac{2}{3} - \frac{2}{p}} \lambda_{q+1} r^{\frac{2}{3} - \frac{2}{p}} \lesssim \varepsilon \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+1}.
$$

By (4.34) and (4.23), we have, for $p > 1$ sufficiently close to 1

$$
\|R_{\{\text{cor,3}\}}\|_{L^p} \leq \|w_{q+1}^c + w_{q+1}^c\|_{W^{1,2p}}\|v_{q+1}^p\|_{L^p} \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^\frac{1}{3} \varepsilon \left(1 - \frac{1}{p}\right) \sigma r^{\frac{2}{3} - \frac{2}{p}} \lambda_{q+1} r^{\frac{2}{3} - \frac{2}{p}} \lesssim \varepsilon \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+1},
$$

where the last step follows from the choice of the parameters (4.1)-(4.2).
4.4.3. Oscillation terms.

**Lemma 4.11.** Recall from (4.32)

\[ \nabla \cdot R_{oscillation} = \nabla \cdot (R_{(osc,1)} + R_{(osc,2)}) \]

with \( R_{(osc,1)} \) and \( R_{(osc,2)} \) respectively defined in (4.33) and (4.34). For \( p > 1 \) sufficiently close to 1 and an arbitrarily small constant \( \varepsilon > 0 \), we have

\[ \| R_{(osc,1)} \|_{L^p} + \| R_{(osc,2)} \|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon R} \delta_{q+1}. \]

**Proof:** Notice that \( \nabla \cdot R_{(osc,1)} \) is a similar oscillation term as \( \nabla \cdot \tilde{R}_{oscillation} \) for the NSE in [10], and hence can be estimated in an analogous way. Without showing details, we claim

\[ \| R_{(osc,1)} \|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon R} \delta_{q+1}. \]  \hspace{1cm} (4.60)

We estimate \( R_{(osc,2)} \) as follows. Applying (4.39), (4.40), and (4.1)-(4.2), we deduce that for \( p > 1 \) close enough to 1

\[ \| u_{q+1}^p \otimes W_{c,1} + W_{c,1} \otimes u_{q+1}^p \|_{L^p} \lesssim \| u_{q+1}^p \|_{L^p} \| W_{c,1} \|_{L^\infty} \]

\[ \lesssim \delta_{q+1} \| q_{c,1}^q \|_{L^1} \leq \frac{\ell}{q_{c,1}^q} \sigma_{q} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\lambda_{q+1}^{-1}}{\lambda_{q+1}} \]

\[ \lesssim \lambda_{q+1}^{-2\varepsilon R} \delta_{q+2}. \]

Using (4.33), (4.23) and (4.1)-(4.2) gives us

\[ \| w_{q+1}^c \otimes v_{q+1}^p + v_{q+1}^p \otimes w_{q+1}^c \|_{L^p} \lesssim \| w_{q+1}^c \|_{L^p} \| v_{q+1}^p \|_{L^p} \]

\[ \lesssim \delta_{q+1} \| q_{c}^q \|_{L^1} \leq \frac{\ell}{q_{c}^q} \sigma_{q} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\lambda_{q+1}^{-1}}{\lambda_{q+1}} \]

\[ \lesssim \lambda_{q+1}^{-2\varepsilon R} \delta_{q+2}. \]

for \( p > 1 \) sufficiently close to 1. Applying (4.40) leads to

\[ \| \nabla \otimes W_{c,1} \|_{L^p} \lesssim \| \nabla \otimes W_{c,1} \|_{L^p} \]

\[ \lesssim \left( \delta_{q+1} \| q_{c}^q \|_{L^1} \leq \frac{\ell}{q_{c}^q} \sigma_{q} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\lambda_{q+1}^{-1}}{\lambda_{q+1}} \right)^2 \lesssim \lambda_{q+1}^{-2\varepsilon R} \delta_{q+2} \]

where the last step holds for \( p > 1 \) close enough to 1 in view of (4.41)-(4.42). In the end, employing (4.40), (4.23) and (4.1)-(4.2) leads to

\[ \| R \nabla \times (W_{c,1} \nabla v_{q+1}^p) \|_{L^p} \lesssim \| \nabla \otimes W_{c,1} \|_{L^p} \]

\[ \lesssim \delta_{q+1} \| q_{c}^q \|_{L^1} \leq \frac{\ell}{q_{c}^q} \sigma_{q} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\delta_{q+1}}{\lambda_{q+1}} \frac{\lambda_{q+1}^{-1}}{\lambda_{q+1}} \]

\[ \lesssim \lambda_{q+1}^{-2\varepsilon R} \delta_{q+2} \]

for \( p > 1 \) close enough to 1. The last inequalities together with equation (4.54) yield

\[ \| R_{(osc,2)} \|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon R} \delta_{q+1}. \]  \hspace{1cm} (4.61)

The conclusion of the lemma follows immediately from (4.60) and (4.61). \( \square \)
4.5. The energy iteration.

Lemma 4.12. If $\rho_0(t) \neq 0$, then the energy of the current density $J_{q+1}$ satisfies

$$|E(t) - \int_{T^3} |J_{q+1}(x, t)|^2 \, dx - (\delta_{q+2})^2| \leq \frac{\delta_{q+2}}{4}.$$

Lemma 4.13. If $\rho_0(t) = 0$, then $J_{q+1}(\cdot, t) \equiv 0$, $R_{q+1}(\cdot, t) \equiv 0$ and

$$E(t) - \int_{T^3} |J_{q+1}(x, t)|^2 \, dx \leq \frac{3}{4} \delta_{q+2}.$$

The proof of Lemma 4.12 and Lemma 4.13 follows closely as the proof of Lemma 6.2 and Lemma 6.3 in [10]. The two estimates in Lemma 4.12 and Lemma 4.13 immediately implies (4.5) for $q + 1$. On the other hand, if $E(t) - \int_{T^3} |J_{q+1}(x, t)|^2 \, dx \leq \frac{\delta_{q+2}}{100}$, it follows from Lemma 4.12 that $\rho_0(t) = 0$. Thus, Lemma 4.13 guarantees $J_{q+1}(t) \equiv 0$ and $R_{q+1}(t) \equiv 0$, which shows (4.6) for $q + 1$.

Now we can conclude that the proof of Proposition 4.1 is complete.

5. Non-uniqueness of the Hall MHD system

In this section, we come back to the 3D Hall-MHD system (1.1) with $\zeta = 1$ and demonstrate that non-unique Leray-Hopf weak solutions can be actually constructed for this coupled system of the NSE and the Hall equation. That is, we prove Theorem 1.2.

We consider the approximating system

$$\begin{align*}
\partial_t u_q + (u_q \cdot \nabla) u_q + \nabla p_q &= \Delta u_q + (B_q - \nabla) B_q, \\
\partial_t J_q + \nabla \times \nabla \times (B_q \times u_q) + \nabla \times \nabla \times (J_q \times B_q) &= \Delta J_q + \nabla \cdot R_{q+1}^s, \\
\nabla \cdot u_q &= 0.
\end{align*}$$

(5.1)

The plan is to apply convex integration framework only to the equation of the current density $J_q$ and solve the NSE with force $(B_q \cdot \nabla) B_q$ at every level of the convex integration. The detailed scheme is described below:

(i) Start with the trivial choice of $(u_0, B_0, J_0, R_0^s) = (0, 0, 0, 0)$ which satisfies (5.1) automatically.

(ii) Construct appropriate (small enough) perturbations $v_1^s = B_1 - B_0$ and $w_1^s = J_1 - J_0$ to obtain $J_1 = J_0 + w_1^s$ and $B_1 = B_0 + v_1^s$.

(iii) Solve the NSE with force $(B_1 \cdot \nabla) B_1$,

$$\partial_t u_1 + (u_1 \cdot \nabla) u_1 + \nabla p_1 = \Delta u_1 + (B_1 \cdot \nabla) B_1$$

(5.2)

and obtain $u_1$.

(iv) With $B_1, J_1$, and $u_1$, we derive the new stress tensor $R_1^s$ which satisfies

$$\partial_t J_1 + \nabla \times \nabla \times (B_1 \times u_1) + \nabla \times \nabla \times (J_1 \times B_1) = \Delta J_1 + \nabla \cdot R_1^s.$$
In principle, the increments (stress tensor $R$ (5.1) and the stress tensor $\{v\}$) prove that the sequence satisfies the last equation. Thus, we complete the second iteration and obtain $u_2$.

(vii) With $B_2$, $J_2$, and $u_2$, we derive the stress tensor $R_2$ satisfying
\[
\partial_t J_2 + \nabla \cdot \nabla \times (B_2 \times u_2) + \nabla \cdot \nabla \times (J_2 \times B_2) = \Delta J_2 + \nabla \cdot R_2.
\]
Thus, we have
\[
\nabla \cdot R_2 = \partial_t J_2 - \Delta J_2 + \nabla \cdot \nabla \times (B_2 \times u_2) + \nabla \cdot \nabla \times (J_2 \times B_2)
\]
\[
= \partial_t (J_1 + w_1) - \Delta (J_1 + w_2) + \nabla \cdot \nabla \times (B_2 \times u_2)
\]
\[
+ \nabla \cdot \nabla \times (J_1 + w_2) (B_1 + v_2).
\]
It follows from the equation of $J_1$ in Step (iv) that
\[
\partial_t J_1 - \Delta J_1 + \nabla \cdot \nabla \times (J_1 \times B_1) = \nabla \cdot R_1 - \nabla \cdot \nabla \times (B_1 \times u_1).
\]
Combining the last two equations leads to
\[
\nabla \cdot R_2 = \partial_t w_2 - \Delta w_2 + \nabla \cdot \nabla \times (J_1 \times v_2)
\]
\[
+ \nabla \cdot \nabla \times (w_2 \times B_1) + \nabla \cdot \nabla \times (w_2 \times v_2)
\]
\[
+ \nabla \cdot R_1 + \nabla \cdot \nabla \times (B_2 \times u_2 - B_1 \times u_1).
\]
Thus, we complete the second iteration and obtain $(u_2, B_2, J_2, R_2)$ with $R_2$ satisfying the last equation.

(viii) Repeat Steps (v)-(vii) iteratively to obtain a sequence $\{(u_{q+1}, B_{q+1}, J_{q+1}, R_{q+1}^*)\}$ satisfying (5.1) and the stress tensor $R_{q+1}^*$ satisfies (5.13) in Lemma 5.3 below.

(viii) Prove that the sequence $\{(u_{q+1}, B_{q+1}, J_{q+1}, R_{q+1}^*)\}$ converges to $(u, B, J, 0)$ with functions $u, B, J$ satisfying
\[
J = \nabla \times B, \quad u \in L^\infty(L^2) \cap L^2(H^1), \quad B \in L^\infty(L^2) \cap L^2(H^1),
\]
and $(u, B)$ is a weak solution of the Hall-MHD system (1.1).

Remark 5.1. We notice that at level $q = 1$, the stress tensor $R_1^*$ for the system (6.1) and the stress tensor $R_1$ for the equation (5.3) are different. Indeed, we have
\[
\nabla \cdot R_1 = \partial_t J_1 - \Delta J_1 + \nabla \cdot \nabla \times (B_1 \times u_1) + \nabla \cdot \nabla \times (J_1 \times B_1),
\]
\[
\nabla \cdot R_1 = \partial_t J_1 - \Delta J_1 + \nabla \cdot \nabla \times (J_1 \times B_1).
\]
In principle, the increments $(v^1_2, w^1_2)$ (or $(v_2, w_2)$) are constructed to correct the stress tensor $R_1^*$ (or $R_1$). Thus, due to the difference of $R_1^*$ and $R_1$, we should have
\[(v^*_2, w^*_2) \neq (v_2, w_2).\] However, \((v^*_2, w^*_2)\) can be constructed analogously as \((v_2, w_2)\) by using intermittent Beltrami flows. The difference relies on the coefficients \(a_\xi\) of the Beltrami flows, which depend on the stress tensor, see details of Subsections 4.1 and 4.2.

**Remark 5.2.** The increments \((v^*_{q+1}, w^*_{q+1})\) for \(q \geq 0\) can be constructed similarly as \((v_{q+1}, w_{q+1})\) such that the estimates in Lemma 4.4 and Lemma 4.5 hold for \((v^*_{q+1}, w^*_{q+1})\) as well. Note that the coefficients \(a_\xi\) of the Beltrami flows in the construction of \((v^*_{q+1}, w^*_{q+1})\) depend on the stress tensor \(R^*_q\).

**Remark 5.3.** In the iterating process, the stress tensor \(R^*_q\) for \(q \geq 0\) satisfies equation (5.13) below, i.e.
\[
\nabla \cdot R^*_q = \nabla \cdot S^*_q + \nabla \times \nabla \times M^*_q + 1
\]

with
\[
\nabla \cdot S^*_q = \nabla \cdot R^*_q + \partial_1 w^*_q - \Delta w^*_q + \nabla \times \nabla \times (J_q \times v^*_q) + \nabla \times \nabla \times (w^*_q \times B_q) + \nabla \times \nabla \times (w^*_q \times v^*_q),
\]
\[
M^*_q = B_{q+1} \times u_{q+1} - B_q \times u_q.
\]

We observe that \(\nabla \cdot S^*_q\) has the same structure as \(\nabla \cdot R^*_q\) of (4.45). Hence, \(R^*_q\) can be dealt with and estimated in an analogous way as that of \(R^*_q\). We would like to point out how to deal with \(\nabla \times \nabla \times M^*_q\) in the first iteration as \(q = 0\):
\[
\nabla \cdot R^*_1 = \nabla \cdot S^*_1 + \nabla \times \nabla \times M^*_1,
\]
\[
M^*_1 = B_1 \times u_1 - B_0 \times u_0 = B_1 \times u_1
\]

which is also reflected in Remark 5.2. We notice that it is sufficient to estimate \(M^*_1\) rather than \(\text{div}^{-1} \nabla \times \nabla \times M^*_1\); indeed, in the process of passing limit in
\[
\int_{\mathbb{T}^3} (\nabla \cdot R^*_q) \cdot \varphi \, dx = \int_{\mathbb{T}^3} (\nabla \cdot S^*_q + \nabla \times \nabla \times M^*_q) \cdot \varphi \, dx
\]
as \(q \to \infty\), we can use integration by parts to move derivatives to the test function \(\varphi\). We construct \(v^*_1 = B_1\) such that \(\|B_1\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq c\delta_2\) for a small constant \(c\) and \(B_1 \otimes B_1 \in L^2(0,T;L^2(\mathbb{T}^3))\). Thus, the force in the NSE (5.2) satisfies
\[
(B_1 \cdot \nabla)B_1 = \nabla \cdot (B_1 \otimes B_1) \in L^2(0,T;W^{-1,2}(\mathbb{T}^3));
\]

consequently, the solution \(u_1\) of (5.2) belongs to the space \(L^\infty(0,T;L^2(\mathbb{T}^3))\), see [46]. Therefore, we have
\[
\|M^*_1\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} = \|B_1 \times u_1\|_{L^\infty(0,T;L^1(\mathbb{T}^3))} \leq \|B_1\|_{L^\infty(0,T;L^1(\mathbb{T}^3))}\|u_1\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq c\delta_2.
\]

The scheme described in (i)-(viii) involves two major bulks: solving the NSE of \(u_q\) and applying convex integration on the \(J_q\) equation. Details are demonstrated by proving the following inductive statement.

**Proposition 5.4.** There exists an absolute constant \(C > 0\) and a sufficiently small parameter \(\varepsilon_R\) depending on \(b\) and \(\beta\) such that the following inductive statement
holds. Let \((u_q, p_q, B_q, J_q, R_q^s)\) be a solution of the approximating equation (5.1) on \(T^3 \times [0, T]\) satisfying:

\[
\|B_q\|_{C^{1}_{x,t}} \leq \lambda_q^3,
\]

\[
\|J_q\|_{C^{1}_{x,t}} \leq \lambda_q^4,
\]

\[
0 \leq E(t) - \int_{\mathbb{T}^3} |J_q|^2 \, dx \leq \delta_{q+1},
\]

and

\[
E(t) - \int_{\mathbb{T}^3} |J_q|^2 \, dx \leq \frac{\delta_{q+1}}{100} \text{ implies } J_q(\cdot, t) \equiv 0 \text{ and } R_q^s(\cdot, t) \equiv 0. \tag{5.6}
\]

In addition, we assume

\[
\nabla \cdot R_q^s = \nabla \cdot \tilde{R}_q^s + \nabla \times \nabla \times M_q^s
\]

with \(\tilde{R}_q^s\) being a symmetric traceless stress tensor and \(M_q^s\) being a vector field which satisfy

\[
\|
\tilde{R}_q^s\|_{L^\infty(L^1)} \leq \lambda_q^{-2} \delta_{q+1},
\]

\[
\|M_q^s\|_{L^\infty(L^1)} \leq \lambda_q^{-2} \delta_{q+1} + C\|z_q\|_{L^2}.
\]

Then we can find another solution \((u_{q+1}, p_{q+1}, B_{q+1}, J_{q+1}, R_{q+1}^s)\) of (5.1) satisfying (5.3)-(5.9) with \(q\) replaced by \(q+1\). Moreover, the increments \(v_{q+1}^s = B_{q+1} - B_q, w_{q+1}^s = J_{q+1} - J_q\) and \(z_{q+1}^s = u_{q+1} - u_q\) satisfy

\[
\|v_{q+1}^s\|_{L^2} \leq C\lambda_{q+1}^{-1} \delta_{q+1}^{1/2}, \quad \|u_{q+1}^s\|_{L^2} \leq C\lambda_{q+1}^{1/2},
\]

\[
\lim_{q \to \infty} \|z_{q+1}^s\|_{L^p} = 0, \quad 1 \leq p \leq 2.
\]

In analogy with Proposition 4.1 and Theorem 1.1, a proof of Theorem 1.2 follows immediately from Proposition 5.4; thus the details are omitted.

In order to prove Proposition 5.4, we adapt the same construction of perturbations \(v_{q+1}^s = B_{q+1} - B_q\) and \(w_{q+1}^s = J_{q+1} - J_q\) as that of \(v_{q+1}^s\) and \(w_{q+1}^s\) for the Hall equation in Section 1, however, with different coefficients for the Beltrami flows, as indicated in Remark 4.2. Since \(v_{q+1}^s\) and \(w_{q+1}^s\) also satisfy the estimates in Lemma 4.4 and Lemma 4.5, respectively, we claim that (5.3), (5.4), and (5.10) hold for \(q + 1\).

We continue to complete the proof of Proposition 5.4 in Subsections 5.1 and 5.2 below.

5.1. **Weak solution** \(u_{q+1}\) of the NSE in \(L^\infty(L^2) \cap L^2(H^1)\). We consider the forced NSE

\[
\partial_t u_{q+1} + (u_{q+1} \cdot \nabla) u_{q+1} + \nabla p_{q+1} = \Delta u_{q+1} + \nabla \cdot (B_{q+1} \otimes B_{q+1}).
\]

By construction, we have

\[
B_{q+1} = B_0 + \sum_{j=0}^{j=q} v_{j+1}^s, \quad J_{q+1} = J_0 + \sum_{j=0}^{j=q} w_{j+1}^s
\]

with \(\|v_{q+1}^s\|_{L^2} \leq C\lambda_{q+1}^{-1} \delta_{q+1}^{1/2}\) and \(\|w_{q+1}^s\|_{L^2} \leq C\lambda_{q+1}^{1/2}\). It is then obvious that \(\|B_{q+1}\|_{L^2} \leq C\) and \(\|J_{q+1}\|_{L^2} \leq C\) which implies \(B_{q+1} \in L^\infty(0,T;H^1(T^3))\), since \(B_{q+1}\) is divergence free.

It follows from the Sobolev embedding theorem that \(B_{q+1} \otimes B_{q+1}\) is in the space \(L^2(0,T;L^3(T^3))\), and hence in \(L^2(0,T;L^3(T^3))\) as well. Thus we have \(\nabla \cdot (B_{q+1} \otimes B_{q+1})\)
Therefore, (5.11) is justified.

Moreover, we have obtained in Section 5.1, we proceed to derive the stress tensor of (5.12) with

\[
\partial_t u_{q+1} = \partial_t J_q + \partial_t w_{q+1}^s + \Delta J_q + \Delta w_{q+1}^s,
\]

we have

\[
\Delta J_q + \Delta w_{q+1}^s,
\]

and

\[
\nabla \times \nabla \times (J_{q+1} \times B_{q+1}) = \nabla \times \nabla \times ((J_q + w_{q+1}^s) \times (B_q + v_{q+1}^s))
\]

\[
= \nabla \times \nabla \times (J_q \times B_q + \nabla \times \nabla \times (J_q \times v_{q+1}^s)
\]

\[
+ \nabla \times \nabla \times (w_{q+1}^s \times B_q) + \nabla \times \nabla \times (w_{q+1}^s \times v_{q+1}^s)
\]

5.2. The new stress tensor $R_{q+1}^s$ and its estimate. With $v_{q+1}^s = B_{q+1} - B_q$ and $w_{q+1}^s = J_{q+1} - J_q$ constructed following the same line of Section 4 and $u_{q+1}$ obtained in Section 5.1, we proceed to derive the stress tensor $R_{q+1}^s$ satisfying the equation of $J_{q+1}$ in (5.11) at the level $q + 1$. Compared to the $J_q$ equation in (3.2), there is one extra term $\nabla \times \nabla \times (B_{q+1} \times u_{q+1})$ in the $J_q$ equation of (3.1). Thus, $R_{q+1}^s$ will be different from $R_{q+1}$ mainly due to the interaction of this extra nonlinear term. In the following, we will show that:

**Lemma 5.5.** Let $v_{q+1}^s = B_{q+1} - B_q$ and $w_{q+1}^s = J_{q+1} - J_q$ be the increments appropriately constructed similarly as in Section 4 which depend on $R_{q+1}^s$. Let $u_{q+1}$ be the new velocity obtained in Subsection 5.1. We choose the symmetric traceless stress tensor $R_{q+1}^s$ such that

\[
\nabla \cdot R_{q+1}^s = \partial_t w_{q+1}^s + \nabla \times \nabla \times (J_q \times v_{q+1}^s) + \nabla \times \nabla \times (w_{q+1}^s \times B_q) + \nabla \times \nabla \times (w_{q+1}^s \times v_{q+1}^s)
\]

\[
+ \nabla \cdot R_{q+1}^s + \nabla \times \nabla \times (B_{q+1} \times u_{q+1} - B_q \times u_q)
\]

with

\[
M_{q+1} = B_{q+1} \times u_{q+1} - B_q \times u_q
\]

and $\nabla \cdot \tilde{R}_{q+1}$ being the rest terms of $\nabla \cdot R_{q+1}^s$. Then, the tuplet $(u_{q+1}, B_{q+1}, J_{q+1}, R_{q+1}^s)$ solves system (5.13) with $q$ replaced by $q + 1$. **Proof:** Since $B_{q+1} = B_q + v_{q+1}^s$, $J_{q+1} = J_q + w_{q+1}^s$, we have

\[
\partial_t J_{q+1} = \partial_t J_q + \partial_t w_{q+1}^s + \Delta J_{q+1} = \Delta J_q + \Delta w_{q+1}^s,
\]

and

\[
\nabla \times \nabla \times (J_{q+1} \times B_{q+1})
\]

\[
= \nabla \times \nabla \times ((J_q + w_{q+1}^s) \times (B_q + v_{q+1}^s))
\]

\[
= \nabla \times \nabla \times (J_q \times B_q + \nabla \times \nabla \times (J_q \times v_{q+1}^s)
\]

\[
+ \nabla \times \nabla \times (w_{q+1}^s \times B_q) + \nabla \times \nabla \times (w_{q+1}^s \times v_{q+1}^s).
\]
Therefore, we continue to deduce
\[
\partial_t J_{q+1} + \nabla \times \nabla \times (B_{q+1} \times u_{q+1}) + \nabla \times \nabla \times (J_{q+1} \times B_{q+1})
= \partial_t J_q + \partial_t w_{s_{q+1}}^a + \nabla \times \nabla \times (B_{q+1} \times u_{q+1})
+ \nabla \times \nabla \times (J_q \times B_q) + \nabla \times \nabla \times (J_q \times v_{s_{q+1}}^a)
+ \nabla \times \nabla \times (w_{s_{q+1}}^a \times B_q) + \nabla \times \nabla \times (w_{s_{q+1}}^a \times v_{s_{q+1}}^a).
\] (5.14)

Recall that the tuplet \((u_q, B_q, J_q, R_q^s)\) satisfies equation \((5.1)\). Hence,
\[
\partial_t J_q + \nabla \times \nabla \times (J_q \times B_q)
= \Delta J_q - \nabla \times \nabla \times (B_q \times u_q) + \nabla \cdot R_q^s.
\] (5.15)

Inserting \((5.15)\) into the right hand side of \((5.14)\) yields
\[
\partial_t J_{q+1} + \nabla \times \nabla \times (B_{q+1} \times u_{q+1}) + \nabla \times \nabla \times (J_{q+1} \times B_{q+1})
= \Delta J_q - \nabla \times \nabla \times (B_q \times u_q) + \nabla \cdot R_q^s + \partial_t w_{s_{q+1}}^a
+ \nabla \times \nabla \times (B_{q+1} \times u_{q+1}) + \nabla \times \nabla \times (J_q \times v_{s_{q+1}}^a)
+ \nabla \times \nabla \times (w_{s_{q+1}}^a \times B_q) + \nabla \times \nabla \times (w_{s_{q+1}}^a \times v_{s_{q+1}}^a).
\] (5.16)

Substituting \(\Delta J_q\) by \(\Delta w_{s_{q+1}}^a\) in \((5.16)\) shows that
\[
\partial_t J_{q+1} + \nabla \times \nabla \times (B_{q+1} \times u_{q+1}) + \nabla \times \nabla \times (J_{q+1} \times B_{q+1})
= \Delta J_{q+1} - \Delta w_{s_{q+1}}^a - \nabla \times \nabla \times (B_q \times u_q) + \nabla \cdot R_q^s + \partial_t w_{s_{q+1}}^a
+ \nabla \times \nabla \times (B_{q+1} \times u_{q+1}) + \nabla \times \nabla \times (J_q \times v_{s_{q+1}}^a)
+ \nabla \times \nabla \times (w_{s_{q+1}}^a \times B_q) + \nabla \times \nabla \times (w_{s_{q+1}}^a \times v_{s_{q+1}}^a).
\] (5.17)

Thus we choose \(R_{q+1}^s\) such that
\[
\nabla \cdot R_{q+1}^s = \partial_t w_{s_{q+1}}^a - \Delta w_{s_{q+1}}^a + \nabla \times \nabla \times (J_q \times v_{s_{q+1}}^a)
+ \nabla \times \nabla \times (w_{s_{q+1}}^a \times B_q) + \nabla \times \nabla \times (w_{s_{q+1}}^a \times v_{s_{q+1}}^a)
+ \nabla \cdot R_q^s + \nabla \times \nabla \times (B_{q+1} \times u_{q+1} - B_q \times u_q),
\]
\[
= \partial_t w_{s_{q+1}}^a - \Delta w_{s_{q+1}}^a + \nabla \times \nabla \times (J_q \times v_{s_{q+1}}^a)
+ \nabla \times \nabla \times (w_{s_{q+1}}^a \times B_q) + \nabla \times \nabla \times (w_{s_{q+1}}^a \times v_{s_{q+1}}^a)
+ \nabla \cdot R_q^s + \nabla \times \nabla \times (B_{q+1} \times u_{q+1} - B_q \times u_q).
\]

We denote \(M_{q+1}^s = B_{q+1} \times u_{q+1} - B_q \times u_q\), and choose \(\bar{R}_{q+1}^s\) such that
\[
\nabla \cdot \bar{R}_{q+1}^s = \nabla \cdot R_q^s + \partial_t w_{s_{q+1}}^a - \Delta w_{s_{q+1}}^a + \nabla \times \nabla \times (J_q \times v_{s_{q+1}}^a)
+ \nabla \times \nabla \times (w_{s_{q+1}}^a \times B_q) + \nabla \times \nabla \times (w_{s_{q+1}}^a \times v_{s_{q+1}}^a).
\] (5.18)

Hence the tuplet \((u_{q+1}, B_{q+1}, J_{q+1}, R_{q+1}^s)\) satisfies \((5.1)\) at the level \(q + 1\) thanks to \((5.17)\).

In the following, we estimate \(\bar{R}_{q+1}^s, M_{q+1}^s\) and hence \(R_{q+1}^s\).

**Lemma 5.6.** For \(p > 1\) sufficiently close to 1 and \(\varepsilon_R > 0\) sufficiently small, the stress tensor \(\bar{R}_{q+1}^s\) satisfies
\[
\|\bar{R}_{q+1}^s\|_\infty \leq \lambda_{q+1}^* \delta_{q+2}.
\]

**Proof:** As pointed out in Remark 5.3, \(\bar{R}_{q+1}^s\) of \((5.18)\) has the same structure as the stress tensor \(R_{q+1}^s\). In analogy of handling \(R_{q+1}^s\) of \((4.45)\) in Subsection 4.4, we can rewrite \(\nabla \cdot \bar{R}_{q+1}^s\) into
\[
\nabla \cdot \bar{R}_{q+1}^s = \nabla \cdot R_{\text{linear}}^s + \nabla \cdot R_{\text{corrector}}^s + \nabla \cdot R_{\text{oscillation}}^s.
\]
with

\[ R_{\text{linear}}^s = R_{\text{linear}}(v_{q+1}^s, w_{q+1}^s), \]
\[ R_{\text{corrector}}^s = R_{\text{corrector}}(v_{q+1}^s, w_{q+1}^s), \]
\[ R_{\text{oscillation}}^s = R_{\text{oscillation}}(v_{q+1}^s, w_{q+1}^s, R_q^s), \]

where \( R_{\text{linear}}(v_{q+1}^s, w_{q+1}^s) \) denotes the linear part of the stress tensor as in \( (4.47) \) with \((v_{q+1}, w_{q+1})\) replaced by \((v_{q+1}^s, w_{q+1}^s)\), \( R_{\text{corrector}}(v_{q+1}^s, w_{q+1}^s) \) the corrector part with \((v_{q+1}, w_{q+1})\) replaced by \((v_{q+1}^s, w_{q+1}^s)\), and \( R_{\text{oscillation}}(v_{q+1}^s, w_{q+1}^s, R_q^s) \) the oscillation part with \((v_{q+1}, w_{q+1}, R_q)\) replaced by \((v_{q+1}^s, w_{q+1}^s, R_q^s)\).

On the other hand, as indicated in Remark \( 5.2 \), the increments \((v_{q+1}, w_{q+1})\) satisfy the same estimates of \((v_{q}, w_{q})\) in Lemma \( 4.3 \) and Lemma \( 4.5 \). Therefore, \( R_{\text{linear}}, R_{\text{corrector}} \) and \( R_{\text{oscillation}} \) satisfy the same estimates as \( R_{\text{linear}}, R_{\text{corrector}} \) and \( R_{\text{oscillation}} \), respectively. Hence, in view of Lemma \( 4.8 \) we have

\[ \| R_{q+1}^s \|_{L^p} \lesssim \lambda_{q+1}^{-2} \eta_{q+2} \]

for \( p > 1 \) sufficiently close to 1 and \( \varepsilon_R > 0 \) sufficiently small.

\[ \square \]

**Lemma 5.7.** For \( p > 1 \) sufficiently close to 1 and \( \varepsilon_R > 0 \) sufficiently small, the vector \( M_{q+1}^s \) satisfies

\[ \| M_{q+1}^s \|_{L^p} \leq \lambda_{q+1}^{-2} \eta_{q+2} + C \| z_{q+1} \|_{L^2}, \]

with some absolute constant \( C > 0 \).

**Proof:** An obvious rearrangement yields

\[ M_{q+1}^s = B_{q+1} \times u_{q+1} - B_q \times u_q = v_{q+1}^s \times u_{q+1} + B_q \times z_{q+1}^s. \]

Therefore, we deduce from \( (4.20), (4.24), (4.26) \) and \( (4.1) - (4.2) \), for \( p > 1 \) close enough to 1,

\[ \| M_{q+1}^s \|_{L^p} \leq \| v_{q+1}^s \times u_{q+1} \|_{L^p} + \| B_q \times z_{q+1}^s \|_{L^p} \]
\[ \lesssim \| v_{q+1}^s \|_{L^{2/p}} \| u_{q+1} \|_{L^2} + \| B_q \|_{L^{2/p}} \| z_{q+1}^s \|_{L^2} \]
\[ \lesssim \| v_{q+1}^s \|_{L^{2/p}} \| u_{q+1} \|_{L^2} + \| z_{q+1}^s \|_{L^2} \sum_{j=0}^{j=q-1} \| v_j^s \|_{L^{2/p}} \]
\[ \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^{1/2} \eta_{q+1}^{1/2} \eta_{q+1} + \| z_{q+1}^s \|_{L^2} \sum_{j=0}^{j=q-1} \lambda_{j+1}^{-1} \delta_{j+1}^{1/2} \lambda_{j+1}^{1/2} \lambda_{j+1}^{-1} \delta_{j+1} \]
\[ \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^{1/2} \eta_{q+1} + \| z_{q+1}^s \|_{L^2} \]
\[ \lesssim \lambda_{q+1}^{-2} \eta_{q+2} + \| z_{q+1}^s \|_{L^2}. \]

According to Lemma \( 5.5 \), Lemma \( 5.6 \) and Lemma \( 5.7 \), \( 5.8 \) and \( 5.9 \) in Proposition \( 5.4 \) are proved to hold for \( q+1 \).

\[ \square \]
5.3. Other estimates of Proposition 5.4. Regarding the energy iteration properties (5.5) and (5.6), they can be obtained in a similar way as of (5.5) and (5.6). Indeed, we notice that the $J_q$ equation in (5.14) differs from the $J_q$ equation (5.2) by the nonlinear term $\nabla \times \nabla \times (u_q \times B_q)$, which is smaller than the nonlinear portion of the Hall term $\nabla \times \nabla \times (J_q \times B_q)$ (up to scale $\lambda_q^{-1}$). Therefore, when deriving (5.3) and (5.4) for $q + 1$, the nonlinear term $\nabla \times \nabla \times (u_{q+1} \times B_{q+1})$ can be treated as a small error term and hence absorbed by other terms in the estimates. Thus, slight modification of the proof of energy iteration in [10] will yield (5.5) and (5.6).

We conclude the proof of Proposition 5.4.

5.4. Proof of Theorem 1.2. We are left to show that the sequence $\{(u_q, B_q)\}_{q=1}^{\infty}$ converges to a pair $(u, B) \in \left(L^\infty(0, T; L^2(T^3)) \cap L^2(0, T; H^{1}(T^3))\right)^2$ which solves the Hall MHD (1.1).

For given $(u_0, B_0, J_0, R_0)$, we apply Proposition 5.4 iteratively to obtain a sequence of approximating solutions $\{(u_q, B_q, J_q, R_q)\}$ satisfying (5.3)-(5.9). It follows from (5.10) that

$$\sum_{q \geq 0} \|J_{q+1} - J_q\|_{L^2} = \sum_{q \geq 0} \|u^*_q\|_{L^2} \lesssim \sum_{q \geq 0} \delta_{q+1}^{1/2} < \infty.$$ 

which implies the strong convergence of $J_q = \nabla \times B_q$ to a function $J$ in $C^0(0, T; L^2)$, and the strong convergence of $B_q$ to a function $B$ in $C^0(0, T; H^1)$ with $J = \nabla \times B$ and $\nabla \cdot B = 0$.

According to the analysis above, we have $u_q \in L^\infty(0, T; L^2(T^3)) \cap L^2(0, T; H^{1}(T^3))$ and $B_q \in C^0(0, T; H^1)$ for all $q \geq 1$. We claim that, there exists a subsequence (with the same notation) such that

$$u_q \to u \quad \text{weakly in} \quad L^2(0, T; H^1(T^3)) \quad \text{and strongly in} \quad L^2(0, T; L^2(T^3)). \quad (5.19)$$

The weak convergence in $L^2(0, T; H^1(T^3))$ is automatic. In order to show the strong convergence in $L^2(0, T; L^2(T^3))$, we apply the Aubin-Lions lemma (cf. [15]). Since $u_q \in L^2(0, T; H^1(T^3))$, $H^1$ is compactly embedded in $L^2$ and $L^2$ is continuously embedded in $H^{-1}$, it is sufficient to prove

$$\partial_t u_q \in L^4([0, T]; H^{-1}(T^3)). \quad (5.20)$$

Hence, the Aubin-Lions lemma implies the strong convergence of $u_q$ in $L^2(0, T; L^2(T^3))$.

We prove (5.20) as follows. Let $\varphi \in H^1(T^3)$ with $\nabla \cdot \varphi = 0$. Taking inner product of equation (5.12) with $\varphi$ (to simplify notations, $q + 1$ in (5.12) is replaced by $q$) and applying integration by parts yields

$$\int_{T^3} \partial_t u_q \cdot \varphi \, dx = \int_{T^3} (u_q \cdot \nabla)\varphi \cdot u_q \, dx - \int_{T^3} \nabla u_q : \nabla \varphi \, dx - \int_{T^3} (B_q \cdot \nabla)\varphi \cdot B_q \, dx.$$

Thus, we obtain

$$\left|\int_{T^3} \partial_t u_q \cdot \varphi \, dx\right| \leq C \|\varphi\|_{H^1(T^3)} \left(\|u_q\|^2_{L^4(T^3)} + \|B_q\|^2_{L^4(T^3)} + \|\nabla u_q\|_{L^2(T^3)}\right)$$

for a constant $C > 0$, which implies

$$\|\partial_t u_q\|_{H^{-1}(T^3)} \leq C \left(\|u_q\|^2_{L^4(T^3)} + \|B_q\|^2_{L^4(T^3)} + \|\nabla u_q\|_{L^2(T^3)}\right), \quad (5.21)$$
Thus, taking curl of the Hall equation

Applying the identities of (6.1), one can rewrite

Assume

Let \( u, B \) of system (1.1). As

we conclude that

Applying the fact of

For a constant

In view of Gagliardo-Nirenberg’s inequality, we have

Applying the fact of

On the other hand, the facts

Combining (5.19) and the fact of

\( \int_0^T \| \partial_t u_q \|_{H^{-1}(\Omega)}^2 \) for a constant \( C > 0 \) that is independent of \( q \). As a consequence, we deduce

for a constant \( C(T) \) dependent on \( T \) and independent on \( q \), where we used the fact \( u_q \in L^2(0, T; H^1(\Omega)) \). This completes the proof of (5.20) and hence (5.19).

Combining (5.19) and the fact of \( B_q \) converging to \( B \) strongly in \( C^0(0, T; H^1(\Omega)) \), we conclude that \((u, B)\) solves the NSE part of (1.1) in the weak sense.

On the other hand, the facts \( \| R_q \|_{L^\infty(0, T; L^3(\Omega))} \to 0 \) and \( \| M_q \|_{L^\infty(0, T; L^3(\Omega))} \to 0 \) as \( q \to \infty \) lead to \( \| R_q \|_{L^\infty(0, T; L^3(\Omega))} \to 0 \) as \( q \to \infty \). Thus, \((u, B)\) also solves the second equation of (1.1) in the weak sense. It follows that \((u, B)\) is a weak solution of system (1.1).

6. Appendix: Vector calculus identities

Let \( A \) and \( B \) be vector valued functions. The following identities hold:

\[
\begin{align*}
\nabla \times (A \times B) &= A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B; \\
(A \cdot \nabla)B &= A\nabla B - A \times (\nabla \times B), \quad \text{with} \quad A\nabla B = (A_i \partial_j B_{ij}); \\
\nabla \times (\nabla \times A) &= \nabla (\nabla \cdot A) - \nabla^2 A = \nabla (\nabla \cdot A) - \Delta A.
\end{align*}
\]

(6.1)

Assume \( B \) is divergence free. Let \( J = \nabla \times B \), then \( \nabla \cdot J = 0 \) and \( \nabla \cdot (\nabla \times J) = 0 \). Applying the identities of (6.1), one can rewrite

\[
\begin{align*}
\nabla \times \nabla \times (|\nabla \times B| \times B) &= \nabla \times ((B \cdot \nabla) J - (J \cdot \nabla)B) \\
&= \nabla \times (B \nabla J) - \nabla \times (B \times (\nabla \times J)) - \nabla \times (J \nabla B) + \nabla \times (J \times (\nabla \times B)) \\
&= -2\nabla \times (J \nabla B) + (B \cdot \nabla)(\nabla \times J) - ((\nabla \times J) \cdot \nabla)B.
\end{align*}
\]

One can also derive the identity,

\[
\Delta (\nabla \times B) = \nabla \times (\Delta B).
\]

Thus, taking curl of the Hall equation

\[
B_t + \nabla \times ((\nabla \times B) \times B) = \Delta B,
\]
we obtain the equation of the current density $J = \nabla \times B$,

$$J_t + \nabla \cdot [B \otimes (\nabla \times J) - (\nabla \times J) \otimes B] - 2\nabla \times (J \nabla B) = \Delta J.$$ 

In general, for $A$ and $B$ with $\nabla \cdot A = \nabla \cdot B = 0$, we have

$$\nabla \times \nabla \times (A \times B) = \nabla \cdot [B \otimes (\nabla \times A) - (\nabla \times A) \otimes B] + \nabla \cdot [(\nabla \times B) \otimes A - A \otimes (\nabla \times B)] - 2\nabla \times (A \nabla B).$$

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