AVERAGE DIMENSION OF FIXED POINT SPACES WITH APPLICATIONS

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Abstract. Let $G$ be a finite group, $F$ a field, and $V$ a finite dimensional $FG$-module such that $G$ has no trivial composition factor on $V$. Then the arithmetic average dimension of the fixed point spaces of elements of $G$ on $V$ is at most $(1/p) \dim V$ where $p$ is the smallest prime divisor of the order of $G$. This answers and generalizes a 1966 conjecture of Neumann which also appeared in a paper of Neumann and Vaughan-Lee and also as a problem in The Kourovka Notebook posted by Vaughan-Lee. Our result also generalizes a recent theorem of Isaacs, Keller, Meierfrankenfeld, and Moretó. Various applications are given. For example, another conjecture of Neumann and Vaughan-Lee is proven and some results of Segal and Shalev are improved and/or generalized concerning BFC groups.

Dedicated to Peter M. Neumann on the occasion of his 70th birthday.

1. Introduction

Let $G$ be a finite group, $F$ a field, and $V$ a finite dimensional $FG$-module. For a non-empty subset $S$ of $G$ we define

$$\text{avgdim}(S, V) = \frac{1}{|S|} \sum_{s \in S} \dim C_V(s)$$

to be the arithmetic average dimension of the fixed point spaces of all elements of $S$ on $V$. In his 1966 DPhil thesis Neumann \[12\] conjectured that if $V$ is an irreducible $FG$-module then $\text{avgdim}(G, V) \leq (1/2) \dim V$. This problem was restated in 1977 by Neumann and Vaughan-Lee \[13\] and was posted in 1982 by Vaughan-Lee in The Kourovka Notebook \[9\] as Problem 8.5. The conjecture was proved by Neumann and Vaughan-Lee \[13\] for solvable groups $G$ and also in the case when $|G|$ is invertible in $F$. Later Segal and Shalev \[17\] showed that $\text{avgdim}(G, V) \leq (3/4) \dim V$ for an arbitrary finite group $G$. This was improved by Isaacs, Keller, Meierfrankenfeld, and Moretó \[8\] to $\text{avgdim}(G, V) \leq ((p+1)/2p) \dim V$ where $p$ is the smallest prime factor of $|G|$. Our first main theorem is

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Theorem 1.1. Let $G$ be a finite group, $F$ a field, and $V$ a finite dimensional $FG$-module. Let $N$ be a normal subgroup of $G$ that has no trivial composition factor on $V$. Then $\text{avgdim}(Ng, V) \leq (1/p) \dim V$ for every $g \in G$ where $p$ is the smallest prime factor of the order of $G$.

Theorem 1.1 not only solves the above-mentioned conjecture of Neumann and Vaughan-Lee but it also generalizes and improves the problem in many ways. First of all, $G$ need not be irreducible on $V$; the only restriction we impose is that $G$ has no trivial composition factor on $V$. Secondly, we prove the bound $(1/2) \dim V$ not just for $\text{avgdim}(G, V)$ but for $\text{avgdim}(S, V)$ where $S$ is an arbitrary coset of a normal subgroup of $G$ with a certain property. Thirdly, Theorem 1.1 involves a better general bound, namely $(1/p) \dim V$ where $p$ is the smallest prime divisor of $|G|$. Note that the example [8, Page 3129] of a completely reducible $FG$-module $V$ for an elementary abelian $p$-group $G$ shows that $\text{avgdim}(G, V) = (1/p) \dim V$ can occur in Theorem 1.1. There are examples for equality in Theorem 1.1 even when $V$ is an irreducible module. Just consider a non-trivial irreducible representation of $G = C_p$ for $p$ an arbitrary prime. Our other example works for arbitrarily large dimensions. Let $p$ be an arbitrary odd prime, let $G$ be the extraspecial $p$-group of order $p^{1+2a}$ (for a positive integer $a$) of exponent $p$, let $N = Z(G)$, let $F$ be an algebraically closed field of characteristic different from $p$, and let $V$ be an irreducible $FG$-module of dimension $p^a$. Then for every element $x \in G \setminus N$ we have $\dim C_V(x) = (1/p) \dim V$ and so $\text{avgdim}(Ng, V) = (1/p) \dim V$ for every $g \in G$. In particular we have $\text{avgdim}(H, V) = (1/p) \dim V$ for every subgroup $H$ of $G$ containing $N$.

In his DPhil thesis [12], Neumann showed that if $V$ is a non-trivial irreducible $FG$-module for a field $F$ and a finite solvable group $G$ then there exists an element of $G$ with small fixed point space. Specifically, he showed that there exists $g \in G$ with $\dim C_V(g) \leq (7/18) \dim V$. Neumann conjectured that in fact, there should exist $g \in G$ such that $\dim C_V(g) \leq (1/3) \dim V$. Segal and Shalev [17] proved, for an arbitrary finite group $G$, that there exists an element $g \in G$ with $\dim C_V(g) \leq (1/2) \dim V$. Later, under milder conditions ($V$ is a completely reducible $FG$-module with $C_V(G) = 0$), Isaacs, Keller, Meierfrankenfeld, and Mórtens [8] showed that there exists an element $g \in G$ with $\dim C_V(g) \leq (1/p) \dim V$ where $p$ is the smallest prime divisor of $|G|$. Under even weaker conditions we improve this latter result.

Corollary 1.2. Let $G$ be a finite group, $F$ a field, and $V$ a finite dimensional $FG$-module. Let $N$ be a normal subgroup of $G$ that has no trivial composition factor on $V$. Let $x$ be an element of $G$ and let $p$ be the smallest prime factor of the order of $G$. Then there exists an element $g \in Nx$ with $\dim C_V(g) \leq (1/p) \dim V$ and there exists an element $g \in N$ with $\dim C_V(g) < (1/p) \dim V$.

Note that Corollary 1.2 follows directly from Theorem 1.1 just by noticing that $\dim C_V(1) = \dim V$. Note also that if $V$ is irreducible and faithful in Corollary 1.2 then no non-trivial normal subgroup of $G$ has a non-zero fixed point on $V$ and so the $N$ above can be any non-trivial normal subgroup of $G$. During the last stage of the writing of this paper Neumann’s above-mentioned conjecture was proved in [6]; if $V$ is a non-trivial irreducible $FG$-module for a finite group $G$ then there exists an element $g \in G$ such that $\dim C_V(g) \leq (1/3) \dim V$. 

Let \( \text{cl}_G(g) \) denote the conjugacy class of an element \( g \) in a finite group \( G \), and for a positive integer \( n \) and a prime \( p \) let \( n_p \) denote the \( p \)-part of \( n \). In [8] Isacs, Keller, Meierfrankenfeld, and Moretó conjecture that for any primitive complex irreducible character \( \chi \) of a finite group \( G \) the degree of \( \chi \) divides \( |\text{cl}_G(g)| \) for some element \( g \) of \( G \). Using their result mentioned before the statement of Corollary 1.2 they showed that if \( \chi \) is an arbitrary primitive complex irreducible character of a finite solvable group \( G \) and \( p \) is a prime divisor of \( |G| \) then \( \chi(1)_p \) divides \( (|\text{cl}_G(g)|)^3 \) for some \( g \in G \). Using Theorem 1.4 we may prove more than this.

**Corollary 1.3.** Let \( \chi \) be an arbitrary primitive complex irreducible character of a finite solvable group \( G \) and let \( p \) be a prime divisor of \( |G| \). Then the number of \( g \in G \) for which \( \chi(1)_p \) divides \( (|\text{cl}_G(g)|)^3 \) is at least \( (2|G|)/(1 + k) \) where \( k = \log_p |G|_p \). Furthermore if \( \chi(1)_p > 1 \) then there exists a \( p' \)-element \( g \in G \) for which \( p^3 \cdot \chi(1)_p \) divides \( (|\text{cl}_G(g)|)^3 \).

Recall that a chief factor of a finite group is a section \( X/Y \) of \( G \) with \( Y < X \) both normal in \( X \) and \( Y \). Note that \( X/Y \) is a direct product of isomorphic simple groups. If \( X/Y \) is abelian, then it is an irreducible \( G \)-module. If \( X/Y \) is non-abelian, then \( G \) permutes the chief factors transitively. A chief factor is called central if \( G \) acts trivially on \( X/Y \) and non-central otherwise. Let \( G \) be a finite group acting on another finite group \( Z \) by conjugation. For a non-empty subset \( S \) of \( G \) define

\[
\text{geom}(S, Z) = \left( \prod_{s \in S} |C_Z(s)| \right)^{1/|S|}
\]

to be the geometric mean of the sizes of the centralizers of elements of \( S \) acting on \( Z \). Similarly, for a non-empty subset \( S \) of \( G \) define

\[
\text{avg}(S, Z) = \frac{1}{|S|} \sum_{s \in S} |C_Z(s)|
\]

to be the arithmetic mean of the sizes of the centralizers of elements of \( S \) acting on \( Z \). Our next result is a non-abelian version of Theorem 1.1 proved using some recent work of Fulman and the first author [5].

**Theorem 1.4.** Let \( G \) be a finite group with \( X/Y = M \) a non-abelian chief factor of \( G \) with \( X \) and \( Y \) normal subgroups in \( G \). Then, for any \( g \in G \), we find that \( \text{geom}(Xg, M) \leq \text{avg}(Xg, M) \leq |M|^{1/2} \).

In fact, one can do slightly better than \( 1/2 \) in the exponent of the statement of Theorem 1.3 but it is a bit easier to write down the proof of this result. It is easy to see that one cannot do better than \( 1/3 \) (consider \( G = \text{SL}(2, q) \) with \( q = 2^e > 2 \) – then all non-trivial elements have centralizers of orders \( q - 1, q \), or \( q + 1 \) which are approximately \( |G|^{1/3} \)).

Let \( \text{ccf}(G) \) and \( \text{ncl}(G) \) be the product of the orders of all central and non-central chief factors (respectively) of a finite group \( G \). (In case these are not defined put them equal to 1.) These invariants are independent of the choice of the chief series of \( G \). Let \( F(G) \) denote the Fitting subgroup of \( G \). Note that \( F(G) \) acts trivially on every chief factor of \( G \). Using Theorems 1.3 and 1.4 we prove
Theorem 1.5. Let $G$ be a finite group. Then $\text{geom}(G, G) \leq \text{ccf}(G) \cdot (\text{ncf}(G))^{1/p}$ where $p$ is the smallest prime factor of the order of $G/F(G)$.

By taking the reciprocals of both sides of the inequality of Theorem 1.5 and multiplying by $|G|$, we obtain the following result.

Corollary 1.6. Let $G$ be a finite group. Then $\text{ncf}(G) \leq \left( \prod_{g \in G} |\text{cl}_G(g)| \right)^{p/(p-1)|G|}$ where $p$ is the smallest prime factor of the order of $G/F(G)$.

A group is said to be a BFC group if its conjugacy classes are finite and of bounded size. A group $G$ is called an $n$-BFC group if it is a BFC group and the least upper bound for the sizes of the conjugacy classes of $G$ is $n$. One of B. H. Neumann’s discoveries was that in a BFC group the commutator subgroup $G'$ is finite [11]. One of the purposes of this paper is to give an upper bound for $|G'|$ in terms of $n$ for an $n$-BFC group $G$. Note that $C_G(G')$ is a finite index nilpotent subgroup. Thus, $F(G)$ is well defined for BFC groups.

If $G$ is a BFC group, then there is a finitely generated subgroup $H$ with $H' = G'$ and $G = H C_G(G') = H F(G)$. Then $H$ has a finite index central torsionfree subgroup $N$. Set $J = H/N$. So $J'$ and $G'$ are $G$-isomorphic. In particular, $\text{ncf}(J) = \text{ncf}(G)$. Clearly, $G/F(G) \cong J/F(J)$. Thus, for the next result, it suffices to consider finite groups. Our first main theorem on BFC groups follows from Corollary 1.6 (by noticing that $|\text{cl}_G(1)| = 1$ and that in that result, we may always assume the action is faithful).

Theorem 1.7. Let $G$ be an $n$-BFC group with $n > 1$. Then $\text{ncf}(G) < n^{\log n} \leq n^2$, where $p$ is the smallest prime factor of the order of $G/F(G)$.

Theorem 1.7 solves [13, Conjecture A].

Not long after B. H. Neumann’s proof that the commutator subgroup $G'$ of a BFC group is finite, Wiegold [20] produced a bound for $|G'|$ for an $n$-BFC group $G$ in terms of $n$ and conjectured that $|G'| \leq n^{(1/2)(1+\log n)}$ where the logarithm is to base 2. Later Macdonald [10] showed that $|G'| \leq n^{6n(\log n)^3}$ and Vaughan-Lee [19] proved Wiegold’s conjecture for nilpotent groups. For solvable groups the best bound to date is $|G'| \leq n^{(1/2)(5+\log n)}$ obtained by Neumann and Vaughan-Lee [13]. In the same paper they showed that for an arbitrary $n$-BFC group $G$ we have $|G'| \leq n^{(1/2)(3+5\log n)}$. Using the Classification of Finite Simple Groups (CFSG) Cartwright [2] improved this bound to $|G'| \leq n^{(1/2)(41+\log n)}$ which was later further sharpened by Segal and Shalev [17] who obtained $|G'| \leq n^{(1/2)(13+\log n)}$. Applying Theorem 1.7 at the bottom of [17, Page 511] we arrive at a further improvement of the general bound on the order of the derived subgroup of an $n$-BFC group.

Theorem 1.8. Let $G$ be an $n$-BFC group with $n > 1$. Then $|G'| < n^{(1/2)(7+\log n)}$.

A word $\omega$ is an element of a free group of finite rank. If the expression for $\omega$ involves $k$ different indeterminates, then for every group $G$, we obtain a function from $\mathbb{Z}^k$ to $G$ by substituting group elements for the indeterminates. Thus we can consider the set $G_\omega$ of all values taken by this function. The subgroup generated by $G_\omega$ is called the verbal subgroup of $\omega$ in $G$ and is denoted by $\omega(G)$. An outer
commutator word is a word obtained by nesting commutators but using always
different indeterminates. In [4] Fernández-Alcober and Morigi proved that if \( \omega \) is an
outer commutator word and \( G \) is any group with \( |G_\omega| = m \) for some positive integer
\( m \) then \( |\omega(G)| \leq (m - 1)^{m-1} \). They suspect that this bound can be improved to a
bound close to one obtainable for the commutator word \( \omega = [x_1, x_2] \). By noticing
that every conjugacy class of a group \( G \) has size at most the number of commutators
of \( G \) we see that Theorem 1.8 yields

**Corollary 1.9.** Let \( G \) be a group with \( m \) commutators for some positive integer \( m \)
at least 2. Then \( |G'| < m^{(1/2)(7+\log m)} \).

Segal and Shalev [17] showed that if \( G \) is an \( n \)-BFC group with no non-trivial
abelian normal subgroup then \( |G| < n^4 \). We improve and generalize this result
in Theorem 1.10. For a finite group \( X \) let \( k(X) \) denote the number of conjugacy
classes of \( X \).

**Theorem 1.10.** Let \( G \) be an \( n \)-BFC group with \( n > 1 \). If the Fitting subgroup
\( F(G) \) of \( G \) is finite, then \( |G| < n^2 k(F(G)) \). In particular, if \( G \) has no non-trivial
abelian normal subgroup then \( |G| < n^2 \).

Since \( F(G) \) has finite index in \( G \), the hypotheses of Theorem 1.10 imply that
\( G \) is finite. Note that even more is true than Theorem 1.10 if \( G \) is a finite group then
\( |G| \leq a^2 k(F(G)) \) where \( a = |G|/k(G) \) is the (arithmetic) average size of a
conjugacy class in \( G \) (this is [17 Theorem 10 (i)]). If \( b \) denotes the maximal size of a
set of pairwise non-commuting elements in \( G \) then, by Turán’s theorem [18] applied
to the complement of the commuting graph of \( G \), we have \( a < b + 1 \). Thus if \( G \)
is a finite group with no non-trivial abelian normal subgroup then \( |G| < (b + 1)^2 \).
This should be compared with the bound \( |G| < c^{(\log b)^3} \) holding for some universal
constant \( c \) with \( c \geq 2^{20} \) which implicitly follows from [15, Lemma 3.3 (ii)] and
should also be compared with the remark in [15, Page 294] that for a non-abelian
finite simple group \( G \) we have \( |G| \leq 27 \cdot b^3 \).

The final main result concerns \( n \)-BFC groups with a given number of generators.
Segal and Shalev [17] proved that in such groups the order of the commutator
subgroup is bounded by a polynomial function of \( n \). In particular they obtained the bound
\( |G'| \leq n^{5d+4} \) for an arbitrary \( n \)-BFC group \( G \) that can be generated by \( d \) elements. By applying Theorem 1.7 to [17, Page 515] we may improve this result.

**Corollary 1.11.** Let \( G \) be an \( n \)-BFC group that can be generated by \( d \) elements.
Then \( |G'| \leq n^{3d+2} \).

Finally, the following immediate consequence of Corollary 1.11 sharpens [17,
Corollary 1.5].

**Corollary 1.12.** Let \( G \) be a \( d \)-generator group. Then
\[
|\{[x, y] : x, y \in G\}| \geq |G'|^{1/(3d+2)}.
\]

The example \( T_m(p) \) [13, Page 213] shows that Theorem 1.8, Corollary 1.9, Corol-
lary 1.11 and Corollary 1.12 are close to best possible.
We point out that Theorem 1.1 for \( p \) odd requires only the Feit-Thompson Odd Order Theorem \cite{FeitThompson}. However, most of the results in this paper depend on CFSG as do the results in \cite{Isaacs} and \cite{Burnside} (for groups of even order).

2. Proof of Theorem 1.1

Our first lemma sharpens and generalizes \cite{Isaacs} Theorem 6.1.

**Lemma 2.1.** Let \( G \) be a finite group, \( F \) a field, and \( V \) a finite dimensional \( FG \)-module. Let \( N \) be an elementary abelian normal subgroup of \( G \) such that \( C_V(N) = 0 \). Then \( \text{avgdim}(Ng,V) \leq (1/p) \text{dim}V \) for every \( g \in G \) where \( p \) is the smallest prime factor of the order of \( G \).

*Proof.* Let us consider a counterexample with \( |G| \) and \( \text{dim}V \) minimal. It clearly suffices to assume that \( G = \langle g,N \rangle \). We may assume that \( V \) is irreducible (since if we have the inequality on each composition factor of \( V \) we have it on \( V \)). We may also assume that \( V \) is absolutely irreducible. Finally, we may assume that \( N \) acts faithfully on \( V \). If \( N \) does not act homogeneously on \( V \), then \( g \) transitively permutes the components in an orbit of size \( t \geq p \) and so every element in \( Ng \) has a fixed point space of dimension at most \( (1/t) \text{dim}V \leq (1/p) \text{dim}V \). So we may assume that the elementary abelian group \( N \) acts homogeneously on \( V \). This means that it acts as scalars on \( V \). Thus \( N \leq Z(G) \) and \( G/Z(G) \) is cyclic. It follows that \( G \) is abelian and so \( \text{dim}V = 1 \). At most 1 element in the coset \( Ng \) is the identity and so \( \text{avgdim}(Ng,V) \leq (1/|N|) \text{dim}V \leq (1/p) \text{dim}V \). The result follows. \( \square \)

We first need a result about generation of finite groups. This is an easy consequence of the proof of the main results of \cite{Aschbacher}.

**Theorem 2.2.** Let \( G \) be a finite group with a minimal normal subgroup \( N = L_1 \times \ldots \times L_t \) for some positive integer \( t \) with \( L_i \cong L \) for all \( i \) with \( 1 \leq i \leq t \) for a non-abelian simple group \( L \). Assume that \( G/N = \langle xN \rangle \) for some \( x \in G \). Then there exists an element \( s \in L_1 \leq N \) such that \( |\{ g \in Nx : G = \langle g,s \rangle \}| > (1/2)|N| \).

*Proof.* First suppose that \( t = 1 \). This is an immediate consequence of \cite{Aschbacher} Theorem 1.4] unless \( G \) is one of \( Sp(2n,2), n > 2 \), \( S_{2m+1} \) or \( L = \Omega^+(8,2) \) or \( A_6 \).

If \( G = Sp(2n,2), n > 2 \), then the result follows by \cite{Aschbacher} Proposition 5.8. If \( G = S_{2m+1} \), then apply \cite{Aschbacher} Proposition 6.8.

Suppose that \( L = A_6 \). Note that the overgroups of \( s \) of order 5 in \( A_6 \) are two subgroups isomorphic to \( A_5 \) (of different conjugacy classes) and the normalizer of the Sylow 5-subgroup generated by \( s \). The result follows trivially from this observation.

Finally consider \( L = \Omega^+(8,2) \). We take \( s \) to be an element of order 15. It follows by the discussion in \cite{Aschbacher} Section 4.1] that given \( G \), there is an element of order 15 satisfying the result (although it is possible that the choice of \( s \) depends on which \( G \) occurs).

Now assume that \( t > 1 \). Write \( x = (u_1, \ldots, u_t) \sigma \) where \( \sigma \) just cyclically permutes the coordinates of \( N \) (sending \( L_i \) to \( L_{i+1} \) for \( i < t \)) and \( u_i \in \text{Aut}(L_i) \). By conjugating by an element of the group \( \text{Aut}(L_1) \times \ldots \times \text{Aut}(L_t) \) we may assume that
Let \( f : Nx \to \text{Aut}(L_1) \) be the map sending \( wx \) to the projection of \((wx)^t\) in \( \text{Aut}(L_1) \). Write \( w = (w_1, \ldots, w_t) \) with \( w_i \in L_i \). Then \( f(wx) = w_tw_{t-1} \cdots w_1u_1 \) is in \( L_1u_1 \). Moreover, we see that every fiber of \( f \) has the same size. By the case \( t = 1 \), we know that the probability that \( (f(wx), s) = \langle L_1, u_1 \rangle \) is greater than \( 1/2 \).

We claim that if \( L_1 \leq \langle (f(wx), s) \rangle \), then \( G = \langle wx, s \rangle \). The claim then implies the result. So assume that \( L_1 \leq \langle (f(wx), s) \rangle \) and set \( H = \langle wx, s \rangle \). Let \( M \leq N \) be the normal closure of \( s \) in \( J := \langle (wx)^t, s \rangle \). This projects onto \( L_1 \) by assumption, but is also contained in \( L_1 \), whence \( M = L_1 \). So \( L_1 \leq H \). Since any element of \( Nx \) acts transitively on the \( L_i \), it follows that \( N \leq H \) and so \( G = H \).

The next result we need is Scott’s Lemma [16]. See [14] for a slightly easier proof of the result which depends only on the rank plus nullity theorem in linear algebra.

**Lemma 2.3** (Scott’s Lemma). Let \( G \) be a subgroup of \( \text{GL}(V) \) with \( V \) a finite dimensional vector space. Suppose that \( G = \langle g_1, \ldots, g_r \rangle \) with \( g_1 \cdots g_r = 1 \).

\[
\sum_{i=1}^r \dim[g_i, V] \geq \dim V + \dim[G, V] - \dim C_V(G).
\]

We will apply this in the case \( r = 3 \). Noting that \( \dim V = \dim[x, V] + \dim C_V(x) \) for any \( x \), we can restate this as:

\[
\sum_{i=1}^3 \dim C_V(g_i) \leq \dim V + \dim C_V(G) + \dim V/[G, V].
\]

**Theorem 2.4.** Let \( G \) be a finite group. Assume that \( G \) has a normal subgroup \( E \) that is a central product of quasisimple groups. Let \( V \) be a finite dimensional \( FG \)-module for some field \( F \) such that \( E \) has no trivial composition factor on \( V \). If \( g \in G \), then \( \text{avgdim}(gE, V) \leq (1/2) \dim V \).

**Proof.** Let us consider a counterexample with \( |G| \) and \( \dim V \) minimal. There is no loss of generality in assuming that \( F \) is algebraically closed, \( G = \langle E, g \rangle \), and then assuming that \( V \) is an irreducible (hence absolutely irreducible) and faithful \( FG \)-module. If \( Z(E) \neq 1 \), the result follows by Lemma 2.1 (by taking \( N = Z(E) \) and noting that \( Z(E) \) is completely reducible on \( V \) with \( C_V(Z(E)) = 0 \) (since \( V \) is a faithful \( FG \)-module)). So we may assume that \( E \) is a direct product of non-abelian simple groups. If \( V \) is not a homogeneous \( FE \)-module, then \( g \) transitively permutes the homogeneous components and so any element in \( gE \) has fixed point space of dimension at most \((1/2) \dim V \). So we may assume that \( V \) is a homogeneous \( FE \)-module. Thus \( E = L_1 \times \cdots \times L_m \) with the \( L_i \)’s non-abelian simple groups. So \( V \) is a direct sum of say \( t \) copies of \( V_1 \otimes \cdots \otimes V_m \) where \( V_i \) is an irreducible nontrivial \( FL_i \)-module. (Since \( G/E \) is cyclic and \( V \) is irreducible, it follows that \( t = 1 \) (by Clifford theory) but we will not use this fact.) We may replace \( E \) by a minimal normal subgroup of \( G \) contained in \( E \) (the hypothesis on the minimal normal subgroup will hold by Clifford’s theorem) and so assume that \( g \) transitively permutes the isomorphic subgroups \( L_1, \ldots, L_m \).
Let \( s \in L_1 \leq E \) be chosen so that \( Y := \{ y \in gE : \langle y, s \rangle = G \} \) has size larger than \((1/2)|E|\). Such an element exists by Theorem 2.2. Set \( c = \dim C_V(s) \). If \( y \in Y \) then, by Lemma 2.3 (applied to the triple \((y, s, (ys)^{-1})\)), we have
\[
c + \dim C_V(y) + \dim C_V(ys) \leq \dim V.
\]
For any \( y \in Y' := gE \setminus Y \), we at least have
\[
\dim C_V(y) + \dim C_V(ys) \leq \dim V + c.
\]
Thus,
\[
2 \sum_{y \in gE} \dim C_V(y) = \sum_{y \in gE} \left( \dim C_V(y) + \dim C_V(ys) \right)
\]
is at most
\[
|Y| (\dim V - c) + |Y'| (\dim V + c) < |E| \dim V.
\]
This gives \( \text{avgdim}(gE) \leq (1/2) \dim V \) as required. \( \square \)

We now prove Theorem 1.1. As usual, we may assume that \( F \) is algebraically closed, \( V \) is an irreducible \( FG \)-module, and \( N \) acts faithfully on \( V \). Let \( A \) be a minimal normal subgroup of \( G \) contained in \( N \). Since \( V \) is a faithful completely reducible \( FN \)-module, \( A \) has no trivial composition factor on \( V \). Now apply Lemma 2.1 and Theorem 2.4 to conclude that \( \text{avgdim}(Ag, V) \leq (1/p) \dim V \) where \( p \) is the smallest prime divisor of \( |G| \). Since \( Ng \) is the union of cosets of \( A \), the result follows.

3. Proof of Corollary 1.3

Let us first prove the first statement of Corollary 1.3. By making the assumptions of the proof of [8, Corollary D], it is sufficient to show that the number of \( g \in G \) such that \( \dim C_V(g) \leq (1/2) \dim V \) is at least
\[
\frac{2|G|}{1 + \log_p |G|} \leq \frac{2|G|}{2 + \dim V}.
\]
But this is clear by Theorem 1.1 noting that \( \dim V \) is even.

Let us prove the second statement of Corollary 1.3. Use the notations and assumptions of the last part of the proof of [8, Corollary D]. Let \( H \) be a Hall \( p' \)-subgroup of \( G \). Since \( V \) is a completely reducible \( G \)-module with \( C_V(G) = 0 \), the vector space \( V \) is also a completely reducible \( H \)-module with \( C_V(H) = 0 \). Hence applying Corollary 1.2 to the \( H \)-module \( V \) we get that there exists \( g \in H \) with \( \dim C_V(g) < (1/2) \dim V \). So the last displayed inequality of the proof of [8, Corollary D] becomes
\[
\frac{[\text{cl}_G(g)]_p}{p} \geq (1)^{1/3}
\]
since \( \dim V \) is even. From this we get that \( p^3 \chi(1) \leq [\text{cl}_G(g)]_p^3 \).
4. Proof of Theorem 1.4

Note that $Y$ centralizes $M$ and so there is no loss in working in $G/Y$ and assuming that $X = M$ is a minimal normal subgroup of $G$. We can replace $G$ by $\langle M, g \rangle$ and so assume that $g$ acts transitively on the direct factors of $M$.

What we need to show is that the arithmetic mean of the positive integers $|C_M(x)|$ for $x \in gX$ is at most $|M|^{1/2}$. If there are $t > 1$ direct factors in $M$, then every element in $gM$ has centralizer at most $|M|^{1/t} \leq |M|^{1/2}$ and the result follows. So assume that $t = 1$ and $M$ is simple.

We compute the arithmetic mean of the positive integers $|C_M(x)|$ for $x \in gM$. All elements in a given $M$-conjugacy class in $gM$ have the same centralizer size. If $h \in gM$, then the $M$-conjugacy class of $h$ has $|M : C_M(h)|$ elements. Thus, we see that the arithmetic mean is precisely the number of conjugacy classes in $gM$. By [5, Lemma 2.1], this is at most $k(M)$, the number of conjugacy classes in $M$, and again by [5, Proposition 5.3], this is at most $|M|^{41} < |M|^{1/2}$, whence the result.

5. Proof of Theorem 1.5

Let us fix a chief series for a finite group $G$. Let $N$ be the set of non-central chief factors of this series. Let $p$ be the smallest prime factor of the order of $G/F(G)$. If $N \in \mathcal{N}$ is abelian then, by Theorem 1.4 (noting that $F(G)$ acts trivially on $N$), we have $\text{geom}(G, N) \leq |N|^{1/p}$. If $N \in \mathcal{N}$ is non-abelian then, by Theorem 1.4 and the Feit-Thompson Odd Order Theorem [3], we again have $\text{geom}(G, N) \leq |N|^{1/p}$. Notice also that for any $g \in G$ we have the inequality $|C_G(g)| \leq \text{ccf}(G) \prod_{N \in \mathcal{N}} |C_N(g)|$. From these observations Theorem 1.5 already follows since

$$\text{geom}(G, G) = \left( \prod_{g \in G} |C_G(g)| \right)^{1/|G|} \leq \text{ccf}(G) \left( \prod_{g \in G} \prod_{N \in \mathcal{N}} |C_N(g)| \right)^{1/|G|} = \text{ccf}(G) \left( \prod_{N \in \mathcal{N}} \text{geom}(G, N) \right) \leq \text{ccf}(G) \cdot \left( \prod_{N \in \mathcal{N}} |N|^{1/p} \right) = \text{ccf}(G) \cdot (\text{ncf}(G))^{1/p}.$$ 

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