Three-Dimensional Projective Geometry with Geometric Algebra

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Abstract. The line geometric model of 3-D projective geometry has the nice property that the Lie algebra $sl(4)$ of 3-D projective transformations is isomorphic to the bivector algebra of $Cl(3,3)$, and line geometry is closely related to the classical screw theory for 3-D rigid-body motions. The canonical homomorphism from $SL(4)$ to $Spin(3,3)$ is not satisfying because it is not surjective, and the projective transformations of negative determinant do not induce orthogonal transformations in the Plücker coordinate space of lines.

This paper presents our contributions in developing a rigorous and convenient algebraic framework for the study of 3-D projective geometry with Clifford algebra. To overcome the unsatisfying defects of the Plücker correspondence, we propose a group $Pin^+(3,3)$ with $Pin(3,3)$ as its normal subgroup, to quadruple-cover the group of 3-D projective transformations and polarities. We construct spinors in factored form that generate 3-D reflections and rigid-body motions, and extend screw algebra from the Lie algebra of rigid-body motions to other 6-D Lie subalgebras of $sl(4)$, and construct the corresponding cross products and virtual works.

Key words: Projective Geometry; Line Geometry; Screw Theory; Plücker model; Geometric Algebra.

1 Introduction

The study of the geometry of lines in space was invented by Plücker with his introduction of the now so-called Plücker coordinates of lines. It became an active research topic with the establishment of screw theory by Balls [1], where the 6-D Plücker coordinates of a line are decomposed into a pair of 3-D vectors, called the screw form of the line, and the inner product and cross product of vector algebra are extended to screw forms.

A pair of force and torque, called a wrench, are naturally represented by a pair of 3-D vectors, and are geometrically interpreted as a line in space along which the force acts, together with a line at infinity about which the torque acts. On the other hand, a pair of infinitesimal rotation and translation, called an infinitesimal screw motion or rigid-body motion or twist, are represented by
the rotation axis and the translation vector, and are again naturally represented by a pair of 3-D vectors. Geometrically the translation is a special “rotation” about an axis that is at infinity, so the translation vector represents a line at infinity. For a wrench \((f, q)\) and an infinitesimal screw motion \((v, t)\), where \(f\) is the force direction multiplied with the magnitude of force, \(q\) is the composed torque, \(v\) is the rotation axis direction multiplied with the angle of rotation, and \(t\) is the moment of the screw motion, the virtual work of the wrench along the infinitesimal screw motion is the “crossed” inner product

\[
\begin{pmatrix} f \\ q \end{pmatrix} \cdot \begin{pmatrix} v \\ t \end{pmatrix} := f \cdot t + q \cdot v. \tag{1.1}
\]

The inner product (1.1) gives the 6-D space of wrenches a signature \(\mathbb{R}^{3,3}\), where a pure force has zero inner product with itself, called a null vector. A positive (or negative) vector of \(\mathbb{R}^{3,3}\) is one having positive (or negative) inner product with itself. A positive vector is interpreted as a pure force together with an extraneous torque so that the pair follow the right-hand rule, while for the negative vector, the force and torque follow the left hand rule. The group \(SL(4)\) which acts in the 4-space of homogeneous coordinates of points, can be lifted to a group action in the 6-D space of wrenches by acting upon the Plücker coordinates of the lines representing the wrenches. The image of the lift is \(SO_0(3,3)\), the connected component of \(SO(3,3)\) containing the identity [3]. As \(PSL(4) = SL(4)/\mathbb{Z}_2\) is the group of orientation-preserving projective transformations, the crossed inner product provides a 6-D orthogonal geometric model of wrenches to study 3-D projective geometry of points.

The same inner product (1.1) also gives the 6-D space of twists the same signature \(\mathbb{R}^{3,3}\). The interpretation of a null vector of \(\mathbb{R}^{3,3}\) in the setting of twists, is that it represents an infinitesimal pure rotation or pure translation. A positive vector represents an infinitesimal screw motion where the translation along the screw axis follow the right-hand rule with the orientation of the rotation, while a negative vector represents a left-handed infinitesimal screw motion. The lift of group \(SL(4)\) to \(SO_0(3,3)\) then makes the space of infinitesimal rigid-body motions a 6-D orthogonal geometric model to study the orientation-preserving projective geometry of points.

The bold-faced words clearly reveal a conflict. The group of rigid-body motions is 6-D, while the group \(SL(4)\) is 15-D; the former is much smaller. What sense does it make to investigate projective transformations via rigid-body motions? Furthermore, the inner product (1.1) is between the space of wrenches and the space of twists, indicating that the two spaces need to be identified, yet they have to be different spaces by nature. Understanding (1.1) as a pairing between a linear space and its dual space does not make much difference, as the same inner product exists in either space.

Line geometry and screw theory are closely related to each other. In history, the screw forms were first used by Clifford in the name of biquaternions, also known as dual quaternions, in describing 3-D Euclidean transformations. Later on, Balls [1], Study [35], Blaschke [4] established screw theory and developed dual
vector algebra out of Clifford’s dual quaternions, also known as screw algebra. Nowadays line geometry together with screw theory have important applications in mechanism analysis, robotics, computer vision and computational geometry [6], [2], [33], [10], [29], [32], [9].

For two vectors \((x_1, y_1)^T \in \mathbb{R}^{3,3}\) and \((x_2, y_2)^T \in \mathbb{R}^{3,3}\), where \(x_i, y_j \in \mathbb{R}^3\), their cross product, also called dual vector product, is defined as follows:

\[
\begin{pmatrix}
  x_1 \\
  y_1
\end{pmatrix} \times \begin{pmatrix}
  x_2 \\
  y_2
\end{pmatrix} := \begin{pmatrix}
  x_1 \times x_2 \\
  x_1 \times y_2 + y_1 \times x_2
\end{pmatrix}. \tag{1.2}
\]

This product is covariant under the subgroup of \(SO_0(3,3)\) that is the lift of the group of rigid-body motions \(SE(3)\), but not so under the whole group \(SO_0(3,3)\). In other words, it is not a valid operator in 3-D projective geometry; it is valid only for Euclidean geometry.

In dual vector algebra, the dual inner product of two vectors of \(\mathbb{R}^3\) is defined to be a dual number. A dual number is of the form \(\lambda + \epsilon \mu\) where \(\lambda, \mu \in \mathbb{R}\), \(\epsilon^2 = 0\) and \(\epsilon\) commutes with everything. This numbers system is a ring instead of a field, and the corresponding polynomials and modules are drastically different from the usual ones. The dual inner product is invariant under the lift of \(SE(3)\) to \(SO_0(3,3)\), so it is suitable for Euclidean geometry only.

In [25], it was pointed out that dual vector algebra and dual quaternions can be realized in the conformal geometric algebra \(\mathcal{C}l(4,1)\), and can be extended to arbitrary dimensions. The Euclidean geometric parts of the wrench model and the twist model have no conflict, and their identification is natural. The twist model should not have anything beyond Euclidean geometry, otherwise it would be absurd. The wrench model, or more generally the model of lines in space, deserves further attention.

For invariant computing in projective geometry, the traditional algebraic tool is Grassmann-Cayley algebra and bracket algebra [26]. The study of projective geometry by Clifford algebra was initiated by Hestenes and Ziegler [18], and Stolfi [34]. The representation of projective transformations by spinors was initiated by Doran et al. [11], where a homomorphism of the Lie algebra \(gl(n)\) into \(so(n, n)\) was proposed, making it possible to construct projective transformations by elements of \(Pin(n, n)\). Following this line, Goldman and Mann [15] discovered for many 3-D projective transformations their bivector generators in \(\mathcal{C}l(4,4)\), Considering that the dimension of \(so(4,4)\) is \(C_8^2 = 28\), while the dimension of \(sl(4)\) is 15, the embedding space of \(sl(4)\) seems too high [13].

A classical result [3] states that the group \(SL(4)\), which acts upon the 4-space of homogeneous coordinates of points, is in fact isomorphic to the group \(Spin_0(3,3)\), the connect component of \(Spin(3,3)\) containing the identity, and the isomorphism is realized via the Plücker coordinates of lines and the adjoint action of \(Spin(3,3)\) upon \(\mathbb{R}^{3,3}\). This canonical isomorphism indicates the possibility of using the wrench model, the model of spatial lines, to study projective geometry with \(\mathcal{C}l(3,3)\).

In AGACSE 2009, Li and Zhang [27] proposed a new model of 3-D projective geometry by taking the null vectors of \(\mathbb{R}^{3,3}\) as algebraic generators, and defining
points and planes as the two connected components of the set of null 3-spaces of \( \mathbb{R}^{3,3} \) respectively. Whenever an element of \( \text{Spin}_0(3,3) \) acts upon \( \mathbb{R}^{3,3} \), it induces a projective transformation via the outermorphism of the action upon the null 3-vectors representing 3-D points and planes. This approach was later followed by Klawitter [23], who proposed an explicit expression of the spinor inducing a projective transformation in \( 4 \times 4 \) matrix form, and recently by Dorst [13], who constructed bivector generators for many 3-D projective transformations.

When viewed from the homogeneous coordinates model \( \mathbb{R}^4 \) of 3-D projective geometry, the \( \mathbb{R}^{3,3} \) model seems to have too many defects. The map from \( SL(4) \) to \( SO(3,3) \) is not surjective, nor injective. The projective transformations of negative determinant cannot be lifted to \( O(3,3) \), and conversely, the elements of \( O(3,3) \) with negative determinant do not correspond to any projective transformation, but represent projective polarities where points are all mapped to planes. In the \( \mathbb{R}^{3,3} \) model, while lines are represented by vectors, the 3-D points and planes are represented by null 3-vectors, whose embedding vector space has dimension \( C_6^3 = 20 \). To make things worse, the mapping from \( \mathbb{R}^4 \) to the null 3-vectors is quadratic, and defining the subset of null 3-vectors in the 20-D vector space they span is difficult.

From the mathematical viewpoint, establishing the space \( \mathbb{R}^4 \) of homogeneous coordinates from the 6-space \( \mathbb{R}^{3,3} \) spanned by lines requires rigorous mathematical argument. It is the converse procedure of Plücker’s construction of line coordinates from point coordinates. The well-definedness of the points and planes, and the covariance of the construction under suitable transformations of \( \mathbb{R}^{3,3} \) need to be established. The benefits of using the null 3-vectors instead of the linear space \( \mathbb{R}^4 \) to represent points need to be discovered. The groups \( SO_0(3,3) \) and \( \text{Spin}_0(3,3) \) are too small to cover the whole group of all 3-D projective transformations and polarities, and finding suitable covering groups to provide spin representations for all 3-D projective transformations and polarities is indispensable.

So compared with other models of Geometric Algebra for classical geometries [17], [19], [24], [12], [26], [20], [14], [21], the line geometric model of 3-D projective geometry is much less developed. When every problem raised above is solved, then for the group of 3-D Euclidean transformations, a highly mature subject of study in Geometric Algebra, a one-to-one correspondence among the representations in the line geometric model and in other models need to be set up.

As mentioned before, the screw algebra is valid only for Euclidean motions, and the corresponding group \( SE(3) \) is only a subgroup of \( SL(4) \). When \( SE(3) \) is replaced by another 6-D Lie subgroup of \( SL(4) \), then the Lie algebra \( se(3) \) of Euclidean motions is replaced by another 6-D Lie subalgebra of \( sl(4) \). Correspondingly, we can introduce new screw forms for the 6-D Lie subalgebra, together with the new “virtual work” of a wrench, which is still a vector of \( \mathbb{R}^{3,3} \), along an infinitesimal “projective motion” represented by a screw form of the Lie subalgebra. The 6-D Lie subalgebra has its own Lie bracket, so the corresponding screw forms should have a different cross product. The new “virtual work”
should be related to the new cross product, or even be completely determined by it.

We can go one step further by decomposing the 15-D algebra $\text{sl}(4)$ into the direct sum of five 3-D vector spaces, so that instead of using only pairs of 3-D vectors as in classical screw theory for the screw forms of $\text{se}(3)$, we can use 5-tuples of 3-D vectors to represent screw forms of $\text{sl}(4)$, and develop a “super-screw theory”, equipped with “super-cross product” and “super-virtual work”.

For the purpose of developing a mathematically rigorous model out of the peculiar, unfamiliar and seemingly ineffective line geometric model, for more effectively describing and manipulating 3-D projective transformations with Geometric Algebra, and with the ambition to further extend screw theory to projective geometry, we picked up the research subject again in 2014, and after one-year’s hard work, we are confident to announce that from the algebraic viewpoint, this model is sufficiently mature now. The main contributions are summarized as follows:

1. Rigorous establishment of the $\mathbb{R}^{3,3}$ model for 3-D projective geometry.

While in the classical model of projective line geometry only $\text{SL}(4)$ has spin representation, and all the spinors are in $\text{Spin}_0(3,3)$, a rather unsatisfying limitation, the new model completely overcomes the limitation by providing pin group representations for all 3-D projective transformations and polarities, thus enlarging the transformation group four times.

The group of linear regularities of $\mathbb{R}^{3,3}$ is defined by

$$RL(3,3) := \{ B \in GL(3,3) \mid B^TJB = \pm J \},$$

where $J$ is the matrix form of the metric of $\mathbb{R}^{3,3}$. Only when we computed the group acting upon the null 3-vectors induced by $RL(3,3)$ did we find the complete version of the line geometric model. The group $RL(3,3)$ double covers the whole group of 3-D projective transformations and polarities in this manner, and the group $\text{Pin}^{\text{op}}(3,3)$ quadruple-covers the latter, hence it can be used to construct versors for all kinds of 3-D projective transformations and polarities.

The well-definedness of points and planes in the $\mathbb{R}^{3,3}$ model, and the covariance of the representations are established. Some nice properties of reflections in $\mathbb{R}^{3,3}$ are found, together with the classification of 3-D projective transformations induced by two reflections in $\mathbb{R}^{3,3}$.

2. Construction of spinors in factored form inducing 3-D reflections and rigid-body motions, and discovery of the relation between the cross product of the screw forms of $\text{se}(3)$ and the virtual work.

For 3-D reflections and rigid-body motions, the spinors inducing them in $\text{Pin}^{\text{op}}(3,3)$ in factored form are discovered. Since the bivector Lie algebra of $\text{Cl}(3,3)$ is isomorphic to $\text{sl}(4)$, any element of $\text{se}(3)$ has a bivector form, and the cross product of the bivectors equals the cross product of their screw forms as vectors of $\mathbb{R}^{3,3}$. On the other hand, a wrench is only a vector of $\mathbb{R}^{3,3}$. To make pairing with a bivector, a vector needs to be first upgraded to a bivector of $\Lambda^2(\mathbb{R}^{3,3})$ by making inner product (tensor contraction) with a trivector. We show that this trivector is exactly the one complementary to the trivector defining the
cross product of the screw forms of $se(3)$, and the latter trivector is exactly the lift of the quadratic form of $\mathbb{R}^{3,0,1}$. This correspondence shows the intrinsic the connection between the virtual work and the cross product of $se(3)$. The connection between the wrench interpretation and the twist interpretation of the line geometric model is now clarified.

3. Extension of the cross product and virtual work of the screw forms of $se(3)$ to other 6-D Lie subalgebras of $sl(4)$.

For many 6-D Lie subalgebras of $sl(4)$, we have developed the corresponding screw forms together with the cross product and virtual work that are completely determined by the Lie bracket of the subalgebra. In particular, for $so(K)$ where $K$ is a quadratic form of $\mathbb{R}^4$ with rank $\geq 3$, we have established the corresponding screw forms, cross products and virtual works, and discovered a striking fact: the trivectors for constructing new cross products and new virtual works are exactly the lifts of the quadratic form $K$ by the Plücker correspondence and the dual Plücker correspondence to the trivector space. This result demonstrates that there is no intrinsic connection between the $se(3)$-interpretation and the wrench interpretation of line geometry.

This paper is organized as follows. Section 2 is on the Plücker model of 3-D projective geometry, the invariant group $PR(3)$ and its quadruple-covering group $Pin^p(3,3)$. Section 3 is on the covariance of the Plücker transform and the dual Plücker transform, and the explicit expression of the induced group element of $PR(3)$ by an element of $RL(3,3)$. Section 4 is on properties of projective transformations induced by one or two reflections of $\mathbb{R}^{3,3}$, and the construction of spinors in factored form inducing rigid-body motions. Section 5 is on the bivector representation of $se(3)$, the screw forms and the cross products of screw forms. For reflections in space, the generating spinors are also constructed. Section 6 is on extension of classical screw theory to 6-D Lie subalgebras of $sl(4)$ other than $se(3)$. A framework of super-screw theory is also presented.

2 The Plücker model of 3-D projective geometry

Definition 1. A real $n$-space is said to have signature $\mathbb{R}^{p,q,r}$, if it has a basis with respect to which the metric of the $n$-space is diagonal, where the diagonal elements are composed of 1 of multiplicity $p$, and $-1$ of multiplicity $q$, and 0 of multiplicity $r=n-p-q$. $\mathbb{R}^{p,q}$ stands for $\mathbb{R}^{p,q,0}$, called a non-degenerate inner-product space; $\mathbb{R}^p$ stands for $\mathbb{R}^{p,0,0}$, called a Euclidean inner-product space; $\mathbb{R}^{0,0,r}$ is called a null inner-product space, and $\mathbb{R}^{p,1}$ is called a Minkowski inner-product space.

Definition 2. A Witt decomposition of $\mathbb{R}^{3,3}$ refers to a decomposition of $\mathbb{R}^{3,3}$ into the direct sum of two null 3-spaces, say $I_3, J_3$. A Witt basis of the decomposition is composed of a basis of $I_3$, say $E_1, E_2, E_3$, and the corresponding Witt-dual basis of $J_3$, denoted by $E'_1, E'_2, E'_3$, such that $E_i \cdot E'_j = \delta_{ij}$.

Lemma 1. [8] $\mathbb{R}^{3,3}$ has infinitely many Witt decompositions. Let $\mathbb{R}^{3,3} = I_3 \oplus J_3$ be a fixed Witt decomposition of $\mathbb{R}^{3,3}$. Then for any basis $E_1, E_2, E_3$ of the 3-space $I_3$, the corresponding Witt-dual basis $E'_1, E'_2, E'_3$ of the 3-space $J_3$ is unique.
Lemma 2. Fix a Witt decomposition $\mathbb{R}^{3,3} = I_3 \oplus J_3$. For any null 3-space $S_3$ of $\mathbb{R}^{3,3}$, if the dimension $n$ of $S_3 \cap I_3$ is even (or odd), then the dimension $m$ of $S_3 \cap J_3$ is odd (or even).

Proof. When $n = 0$, then $\mathbb{R}^{3,3} = I_3 \oplus S_3$. Let $E_1, E_2, E_3$ be a basis of $I_3$, and let the corresponding Witt-pairing bases in $J_3$ and $S_3$ be respectively $E'_1, E'_2, E'_3$ and $s_1, s_2, s_3$. Then $s_i = E'_i + \sum_j s_{ij} E_j$ where $s_{ij} = -s_{ji}$. Let

$$S = \begin{pmatrix} 0 & s_{12} & s_{13} \\ -s_{12} & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{pmatrix}.$$ 

If $S = 0$ then $S_3 = J_3$ and $m = 3$. If $S \neq 0$, then its rank is 2, so its kernel has dimension 1. Let $X = x_1s_1 + x_2s_2 + x_3s_3 \in S_3 \cap J_3$, then $0 = \sum_i x_i(s_i - E'_i) = \sum_i x_is_iE_j$, so $(x_1, x_2, x_3)^T$ is in the kernel of $ST$. This proves $m = 1$.

When $n = 2$, let $I_1$ be spanned by $E_1, E_2, E_3$, and let $S_3$ be spanned by $E_1, E_2, s$. Then $s = s_3E_3 + \sum_i s_i'E_i$. Since $S_3$ is a null 3-space, by $s \cdot E_1 = s \cdot E_2 = s^2 = 0$, we get $s'_1 = s'_2 = s_3s'_3 = 0$. If $s'_3 = 0$ then $S_3 = I_3$, violating the assumption that $n = 2$. So $s'_3 \neq 0$ and $s_3 = 0$. We get $s = s_3'E_3$ and the 1-space spanned by $E'_3$ is $S_3 \cap J_3$. This proves $m = 1$.

When $n = 1$, let $S_3$ be spanned by $E_1, s, t$, such that $s = s_2E_2 + s_3E_3 + s'_2E'_2 + s'_3E'_3$, and $t = t_2E_2 + t_3E_3 + t'_2E'_2 + t'_3E'_3$. Then $s^2 = t^2 = s \cdot t = 0$, we get

$$s_2s'_2 + s_3s'_3 = t_2t'_2 + t_3t'_3 = s_2t'_2 + s_3t'_3 + t_2s'_2 + t_3s'_3 = 0. \quad (2.1)$$

Since the 1-space spanned by $E_1$ is $I_3 \cap S_3$, $s'_2 = t'_2 \neq 0$. In the 2-space spanned by vectors $(s'_2, s'_3)^T$ and $(s'_2, s'_3)^T$, by (2.1), we have $(s_2, s_3)^T = \lambda(s'_3, -s'_2)^T$, and $(t_2, t_3)^T = \mu(t'_3, -t'_2)^T$, and $\lambda = \mu$. If $\lambda = 0$ then $S_3$ is spanned by $e_1, e_2, e'_3$, so $m = 2$. If $\lambda \neq 0$, then $s'_2 = s'_3 = 0$, so $S_3 \cap J_3 = \{0\}$, and $m = 0$.

When $n = 3$, the conclusion $m = 0$ is trivial. \qed

Corollary 1. For a Witt decomposition $\mathbb{R}^{3,3} = I_3 \oplus J_3$ with Witt basis $E_1, E_2, E_3$ and $E'_1, E'_2, E'_3$, if $S_3 \cap I_3$ is the 1-space spanned by $E_1$, then $S_3$ is spanned by $E_1, s, t$, where

$$s = \lambda E_2 + \mu E'_3, \quad t = \lambda E_3 - \mu E'_2. \quad (2.2)$$

for some $\mu \neq 0$ and $\lambda$. If $S_3 \cap I_3$ is the 2-space spanned by $E_1, E_2$, then $S_3$ is spanned by $E_1, E_2, E_3$.

Lemma 3. Let the intersection of null 3-spaces $P_3, Q_3$ be a 1-space, then for any null 3-space $S_3$ of $\mathbb{R}^{3,3}$, the dimension $n$ of $S_3 \cap P_3$ and the dimension $m$ of $S_3 \cap Q_3$ have the same parity.

Proof. Let $P_3 \cap Q_3$ be the 1-space spanned by vector $E_1$. Assume $S_3 \neq P_3$ and $S_3 \neq Q_3$. \qed
When $n = 0$, for the Witt decomposition $\mathbb{R}^{3,3} = P_3 \oplus S_3$, by Lemma 2, $m = 0$ or 2.

When $n = 1$ and $P_3 \cap Q_3 = P_3 \cap S_3$, let $E_1, E_2, E_3$ be a basis of $P_3$, and let $E_1', E_2', E_3'$ be the Witt-pairing basis of $P_3'$ for a Witt decomposition $\mathbb{R}^{3,3} = P_3' \oplus P_3''$. By Corollary 1, $Q_3$ is spanned by $E_1, \lambda_1 E_2 + \mu_1 E_3'$, $\lambda_2 E_3 - \mu_2 E_2'$, where $\mu_1 \neq 0$, while $S_3$ is spanned by $E_1, \lambda_1 E_2 + \mu_1 E_3', \lambda_1' E_3 - \mu_1 E_2'$, where $\mu_1' \neq 0$. Since $S_3 \neq Q_3$, $\lambda_2 : \mu_2 \neq \lambda_1 : \mu_1$, so the 1-space $E_1$ is the only intersection of $S_3$ and $Q_3$. This proves $m = 1$.

When $n = 1$ but $P_3 \cap Q_3 \neq P_3 \cap S_3$, let $E_1, E_2, E_3$ be a basis of $P_3$ such that $P_3 \cap S_3$ is the 1-space spanned by $E_3$. Let $E_1', E_2', E_3'$ be the Witt-pairing basis of $P_3'$ for a Witt decomposition $\mathbb{R}^{3,3} = P_3' \oplus P_3''$. By Corollary 1, $Q_3$ is spanned by $E_1, \lambda_1 E_2 + \mu_1 E_3$, $\lambda_2 E_3 - \mu_2 E_2'$, where $\mu_1 \neq 0$, while $S_3$ is spanned by $E_1, \lambda_1 E_2 + \mu_1 E_3', \lambda_2 E_3 - \mu_2 E_1'$, where $\mu_2 \neq 0$. Then $S_3 \cap Q_3$ is the 1-space spanned by $\lambda_1 \mu_1 E_1 + \lambda_2 \mu_1 E_2 + \mu_1 \mu_2 E_3'$ and $\mu_1' \neq 0$. Again $m = 1$.

When $n = 2$ and $P_3 \cap Q_3 \subset P_3 \cap S_3$, let $E_1, E_2, E_3$ be a basis of $P_3$ such that $P_3 \cap S_3$ is spanned by $E_1, E_2$. Let $E_1', E_2', E_3'$ be the Witt-pairing basis of $P_3'$ for a Witt decomposition $\mathbb{R}^{3,3} = P_3' \oplus P_3''$, then $S_3$ is spanned by $E_1, E_2, E_3$, and $Q_3$ is spanned by $E_1, \lambda E_2 + \mu E_3', \lambda E_3 - \mu E_1'$, where $\mu \neq 0$. Then $Q_3 \cap S_3$ is the 1-space spanned by $E_1, \lambda E_2 + \mu E_3'$, and $m = 2$.

When $n = 2$ but $P_3 \cap Q_3$ is not in $P_3 \cap S_3$, let $E_1, E_2, E_3$ be a basis of $P_3$ such that $P_3 \cap S_3$ is spanned by $E_1, E_2$, and $P_3 \cap Q_3$ is spanned by $E_3$. Let $E_1', E_2', E_3'$ be the Witt-pairing basis of $P_3'$ for a Witt decomposition $\mathbb{R}^{3,3} = P_3' \oplus P_3''$. Then $S_3$ is spanned by $E_1, E_2, E_3'$, and $Q_3$ is spanned by $E_3, \lambda E_1 + \mu E_2, \lambda E_2 - \mu E_1'$ where $\mu \neq 0$. Obviously $Q_3 \cap S_3 = \{0\}$, and $m = 0$. □

Proposition 1. The set of null 3-spaces can be decomposed into two subsets (connected components): in each subset the dimension of the intersection subspace of any two diferent elements is 1, and between the two subsets, the dimension of the intersection subspace of any two elements, one from each subset, is 0 or 2.

As a corollary, the following concepts of points and planes in the set of null 3-spaces of $\mathbb{R}^{3,3}$ are well defined, and any pair of non-incident point and plane form a Witt decomposition of $\mathbb{R}^{3,3}$.

Definition 3. For fixed Witt decomposition $\mathbb{R}^{3,3} = I_3 \oplus J_3$, if we call $I_3$ a point (the origin), and call $J_3$ a plane (the plane at infinity), then for any null 3-space $S_3$ of $\mathbb{R}^{3,3}$, let $n$ be the dimension of the vector space $S_3 \cap I_3$, if $n$ is even, then $S_3$ is called a plane, and if $n$ is odd, then $S_3$ is called a point.

The above concepts of points and planes have the following background. The classical Plücker map changes a pair of points of Euclidean affine space $E^3$ in their homogeneous coordinates form $X = (x_0, x_1, x_2, x_3)^T$ and $Y = (y_0, y_1, y_2, y_3)^T$ with respect to the basis $e_0, e_1, e_2, e_3$ of $\mathbb{R}^4$, to a vector $XY := X \wedge Y$ of $\mathbb{R}^{3,3}$ where the induced basis is $e_{ij} = e_i e_j$ for $0 \leq i < j \leq 3$, and the image of the two points is represented by its Plücker coordinates with respect to the induced basis.
Henceforth we always denote the outer product of the Grassmann algebra \( A(\mathbb{R}^4) \) generated by \( \mathbb{R}^4 \) by the juxtaposition of participating elements, while denoting the the outer product of the Grassmann algebra \( A(\mathbb{R}^{3,3}) \) by the wedge symbol.

Let \( e_0, e_1, e_2, e_3 \) be an orthonormal basis of \( \mathbb{R}^4 \). Let

\[
\begin{align*}
E_1 &= e_{01}, \quad E_2 = e_{02}, \quad E_3 = e_{03}; \\
E_1' &= e_{23}, \quad E_2' = e_{31}, \quad E_3' = e_{12}.
\end{align*}
\] (2.3)

Then

\[
I_3 := E_1 \wedge E_2 \wedge E_3, \quad J_3 := E_1' \wedge E_2' \wedge E_3'
\] (2.4)

are two null 3-spaces of \( \mathbb{R}^{3,3} \) forming a Witt decomposition.

Let \( e_0 \) represent the origin of the Euclidean affine 3-space \( \mathcal{E}^3 \), and let \( \mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle \) represent the plane at infinity. The basis \( e_0, e_1, e_2, e_3 \) induces a Witt basis (2.3). The two 3-vectors \( I_3, J_3 \) are invariant under any special linear transformation of \( \mathbb{R}^3 \).

For \( X, Y \in \mathbb{R}^4 \), the vector \( XY \in \mathbb{R}^{3,3} \), if not zero, is a null vector. Conversely, any null vector of \( \mathbb{R}^{3,3} \) is the image of either an affine line or a line at infinity of \( \mathcal{E}^3 \) under the Plücker map. A 2-space \( S_2 \) of \( \mathbb{R}^{3,3} \) spanned by null vectors, when interpreted geometrically so that its null 1-spaces are lines in space, has two kinds: (1) a pair of non-intersection lines in space, when the signature of \( S_2 \) is \( \mathbb{R}^{1,1} \); (2) a pair of incident point and plane, \( i.e., \) a pencil of lines incident at a fixed point and at the same time lying on a fixed plane, when \( S_2 \) is null. A null 3-space \( S_3 \) of \( \mathbb{R}^{3,3} \) when interpreted in line geometry, represents either a point or a plane, \( i.e., \) either all lines incident at a fixed point, or all lines lying on a fixed plane.

Definition 3 is based on a fixed basis \( e_0, e_1, e_2, e_3 \) of \( \mathbb{R}^4 \) and the induced Witt decomposition. Now Proposition 1 tells us that no matter what the underlying 4-space \( \mathbb{R}^4 \) is and what the induced Witt decomposition of \( \mathbb{R}^{3,3} \) could be, as long as the fixed null 3-space \( E_1 \wedge E_2 \wedge E_3 \) of \( \mathbb{R}^{3,3} \) is classified as a “point”, then in a new line geometry whose abstract “lines” are the null 1-spaces of \( \mathbb{R}^3 \), any “point” defined with respect to the basis \( e_0, e_1, e_2, e_3 \) of the original \( \mathbb{R}^4 \) is always classified as a point in the new line geometry.

Fix the underlying space \( \mathbb{R}^4 \) of the homogeneous coordinates of Euclidean affine space \( \mathcal{E}^3 \), and fix a basis \( e_0, e_1, e_2, e_3 \) of it. The \textit{projective transformation group} of \( \mathcal{E}^3 \) is the union \( SL(4) \cup SL^-(4) \), where \( SL(4) \) is the linear transformations of determinant 1, while \( SL^-(4) \) is the linear transformations of determinant \(-1\). In the dual space \( (\mathbb{R}^4)^* \) of \( \mathbb{R}^4 \) equipped with the corresponding dual basis \( e_0^*, e_1^*, e_2^*, e_3^* \) such that the pairing between \( e_j \) and \( e_j^* \) is \( \delta_{ij} \), the corresponding linear transformation of \( A \in GL(4) \) is \( A^{-T} \). The pair \( (A, A^{-T}) \) acts upon \( \mathbb{R}^4 \times (\mathbb{R}^4)^* \), and is still called a general linear transformation.

Any non-singular linear mapping from \( \mathbb{R}^4 \) to \( (\mathbb{R}^4)^* \) is called a \textit{projective polarity}. The set of all projective polarities is denoted by \( GP(4) \). Such a mapping is called a \textit{special polarity} if its matrix form \( A \) has determinant 1. The matrix \( A^{-T} \) represents the corresponding linear mapping from \( (\mathbb{R}^4)^* \) to \( \mathbb{R}^4 \). The pair \( (A, A^{-T}) \) acts upon \( \mathbb{R}^4 \times (\mathbb{R}^4)^* \), and is still called a projective polarity. The set
of special polarities is denoted by $SP(4)$, and the set of projective polarities with determinant $-1$ is denoted by $SP^-(4)$.

Obviously, the set
\[ UR(4) := SL(4) \cup SL^- (4) \cup SP(4) \cup SP^- (4) \] (2.5)
is a group, and $SL(4), SL(4)\cup SL^- (4), SL(4)\cup SP(4)$ are three subgroups. $UR(4)$ is called the group of unitary regularities.

On the other hand, consider some general linear transformations of $\mathbb{R}^{3,3}$. Fix a Witt decomposition $I_3 \oplus J_3$ and the corresponding Witt basis $E_1, E_2, E_3, E'_1, E'_2, E'_3$ of $\mathbb{R}^{3,3}$. Let $J$ be the linear transformation in $\mathbb{R}^{3,3}$ interchanging $E_i$ and $E'_i$ for $i = 1, 2, 3$, i.e., its matrix form is
\[ J := \begin{pmatrix} 0 & I_{3\times 3} \\ I_{3\times 3} & 0 \end{pmatrix}. \] (2.6)

Let $T$ be the linear transformation in $\mathbb{R}^{3,3}$ changing $E_i$ to $-E_i$ while preserving $E'_i$ for $i = 1, 2, 3$, i.e., its matrix form is
\[ T := \begin{pmatrix} -I_3 & 0 \\ 0 & I_3 \end{pmatrix}. \] (2.7)

Notice that $T$ interchanges positive vectors and negative vectors of $\mathbb{R}^{3,3}$. The following is obvious:
\[ JT = -TJ = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}. \] (2.8)

The group of non-singular linear transformations $B$ in $\mathbb{R}^{3,3}$ satisfying
\[ B^T J B = \pm J \] (2.9)
is called the group of linear regularities in $\mathbb{R}^{3,3}$, denoted by $RL(3,3)$. When the sign is positive, the corresponding subset forms the group of orthogonal transformations $O(3,3)$; when the sign is negative, the corresponding subset is called the anti-orthogonal transformations, denoted by $AO(3,3)$.

The subgroup of special orthogonal transformations $SO(3,3)$ has two connected components: the component containing the identity transformation $I_{6\times 6}$ is denoted by $SO_0(3,3)$, while the component containing $-I_{6\times 6}$ is denoted by $SO_1(3,3)$.

$J$ is an orthogonal transformations of determinant $-1$. The subset of orthogonal transformations of determinant $-1$ is denoted by $SO^-(3,3)$. It also has two connected components: the component containing matrix $J$ is denoted by $SO^+_0(3,3)$, while the component containing $-J$ is denoted by $SO^-_1(3,3)$.

$T$ is an orthogonal transformations of determinant $-1$. The set of anti-orthogonal transformations of determinant $-1$ is denoted by $SAO^- (3,3)$. It has two connected components: the component containing matrix $T$ is denoted by $SAO^-_0(3,3)$, while the component containing $-T$ is denoted by $SAO^-_1(3,3)$.\[ \]
$J^T$ is an anti-orthogonal transformations of determinant 1. The set of anti-orthogonal transformations of determinant 1 is denoted by $SAO(3, 3)$. It has two connected components: the component containing matrix $J^T$ is denoted by $SAO_0(3, 3)$, while the component containing $-J^T$ is denoted by $SAO_1(3, 3)$. We have

$$RL(3, 3) = SO_0(3, 3) \cup SO_1(3, 3) \cup SO_0^{-1}(3, 3) \cup SO_1^{-1}(3, 3).$$

(2.10)

The two groups $UR(4)$ and $RL(3, 3)$ are related by the Plücker transform and dual Plücker transform defined as follows. Let $e_0, e_1, e_2, e_3$ be a fixed basis of $\mathbb{R}^4$. Let $\Lambda^3(\mathbb{R}^4)$ be the realization space of $(\mathbb{R}^4)^*$, whose basis $e_0^*, e_1^*, e_2^*, e_3^*$ satisfy for all positive permutations $ijk$ of 123, the following:

$$e_0 \vee e_0^* = e_i \vee e_i^* = 1, \quad e_0^* \vee e_i^* = e_{jk} = E'_i,$$

$$e_i^* \vee e_j^* = e_{0k} = E_k.$$  

(2.12)

Here "$\vee$" is the meet product in the Grassmann-Cayley algebra generated by $\mathbb{R}^4$. The pairing between $e_i$ and $e_j^*$ is defined by $e_p \vee e_q^* = \delta_{pq}$. Let (2.3) be the induced Witt basis of $\mathbb{R}^3, 3$.

**Definition 4.** The Plücker transform from $GL(4)$ to $GL(3, 3)$ is defined by $A \in GL(4) \mapsto \wedge^2 A \in GL(3, 3)$, where

$$(\wedge^2 A)e_{ij} = (Ae_i) \wedge (Ae_j) \in \mathbb{R}^{3, 3}.\tag{2.13}$$

The dual Plücker transform from the set of projective polarities $GD(4)$ to $GL(3, 3)$ is defined for any projective polarity $D$ as

$$(\lor^2 D)e_{ij} = (De_i) \lor (De_j) \in \mathbb{R}^{3, 3}.\tag{2.14}$$

For example, for the affine transformation

$$A : \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^4 \mapsto \begin{pmatrix} 1 & 0 \\ t & L \end{pmatrix} \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^4,$$

where $t \in \mathbb{R}^3$ and $L \in GL(3)$, the matrix form of $\wedge^2 A$ with respect to the basis $E_1, E_2, E_3, E'_1, E'_2, E'_3$ is

$$\begin{pmatrix} L & 0 \\ t \times L & L^{-T} \end{pmatrix}. \tag{2.15}$$

The following result is direct.

**Lemma 4.** For any $A \in GL(4)$,

$$\det(\wedge^2 A) = (\det(A))^3.\tag{2.16}$$
For any \( D \in GD(4) \),
\[
\det(\nabla^2 D) = - (\det(D))^3.
\]
(2.17)

Furthermore, let \( A_1, A_2 \in GL(4) \) and \( D_1, D_2 \in GD(4) \), and let "\( \circ \)" denote the composition of mappings, then
\[
\begin{align*}
\wedge^2(A_1 \circ A_2) &= (\wedge^2 A_1) \circ (\wedge^2 A_2), \\
\nabla^2(D_1 \circ D_2) &= (\nabla^2 D_1) \circ (\nabla^2 D_2), \\
\end{align*}
\]
(2.18)

The Plücker transform maps \( SL(4) \) onto \( SO_0(3, 3) \), and the kernel is \( \pm I_{4 \times 4} \). It also maps \( SL^-(4) \) onto \( SAO_0^-(3, 3) \), such that \( \pm A \in SL^-(4) \) are mapped to the same image of \( SAO_0^-(3, 3) \). For example, the matrix \( \text{diag}(-1, 1, 1, 1) \) is mapped to \( T \in SAO_0^-(3, 3) \), and the pre-images of \( T \) are \( \pm \text{diag}(-1, 1, 1, 1) \). The branches \( SO_1(3, 3) \) and \( SAO^-_1(3, 3) \) have no pre-image in \( SL(4) \cup SL^-(4) \).

Similarly, the dual Plücker transform maps \( SP^-(4) \) onto \( SO_0^-(3, 3) \), such that any \( B \in SO_0^-(3, 3) \) has two pre-images \( \pm A \in SP^-(4) \). For example, the mapping \( D : e_i \rightarrow e_i \) for \( i = 0, 1, 2, 3 \) is mapped to \( J \), and the pre-images of \( J \) are \( \pm D \). The dual Plücker transform also maps \( SP(4) \) onto \( SAO_0(3, 3) \), and the pre-images of any \( B \in SAO(3, 3) \) are of the form \( \pm A \in SP(4) \). The branches \( SO_1(3, 3) \) and \( SAO^-_1(3, 3) \) have no pre-image in \( SP(4) \cup SP^-(4) \).

**Proposition 2.** The Plücker transform is a double-covering homomorphism from \( SL(4) \) to \( SO_0(3, 3) \). The Plücker transform and the dual Plücker transform provide a double-covering homomorphism in the sense of (2.18), from \( UR(4) \) to the following subgroup of \( RL(3, 3) \):
\[
RL_0(3, 3) := SO_0(3, 3) \cup SO_0^-(3, 3) \cup SAO_0(3, 3) \cup SAO^-_0(3, 3).
\]
(2.19)

In the setting of Clifford algebra \( Cl(3, 3) \), the Clifford product is always denoted by juxtaposition of participating elements. Any element of the Pin group \( Pin(3, 3) \) is generated by invertible vectors of unit magnitude of \( \mathbb{R}^{3,3} \) under the Clifford product. Any element of the subgroup \( Spin(3, 3) \) is the Clifford product of even number of unit vectors. \( Spin^-(3, 3) \) is the subset of elements that are the Clifford product of odd number of unit vectors. \( Spin(3, 3) \) has two connected components, and the component contain the identity element is denoted by \( Spin_0(3, 3) \), the other component is denoted by \( Spin_1(3, 3) \).

**Proposition 3.** Any element of \( Spin_0(3, 3) \) is the Clifford product of even number of negative vectors and even number of positive vectors; any element of \( Spin_1(3, 3) \) is the Clifford product of odd number of negative vectors and odd number of positive vectors. In particular, \( \pm 1 \) are in \( Spin_0(3, 3) \), while \( \pm I_{3,3} \) are in \( Spin_1(3, 3) \), where
\[
I_{3,3} := E_1 \wedge E_2 \wedge E_3 \wedge E'_4 \wedge E'_2 \wedge E'_3 = E_{1231^*2^*3^*},
\]
(2.20)
is a pseudoscalar of \( A(\mathbb{R}^{3,3}) \) satisfying \( I_{3,3}^2 = 1 \).
Proof. Denote the set of positive vectors by $P(3,3)$, and denote the set of negative vectors by $N(3,3)$. First we prove that each of $P(3,3)$, $N(3,3)$ is connected. By symmetry, we only consider $P(3,3)$. Let $v_1, v_2$ be two positive vectors, then $v_1 \wedge v_2$ is one of $\mathbb{R}^{2,0,0}, \mathbb{R}^{1,0,1}, \mathbb{R}^{1,1,0}$. Hence there exists a positive vector $v_3$ that is orthogonal to both $v_1, v_2$. On Euclidean plane $v_1 \wedge v_3$, $v_1$ and $v_3$ are connected by positive vectors; on Euclidean plane $v_2 \wedge v_3$, $v_2$ and $v_3$ are connected by positive vectors. So $v_1, v_2$ are connected.

Denote the set of elements that are the Clifford product of either two positive vectors or two negative vectors by $P_2(3,3)$, and denote the set of elements that are the Clifford product of either one negative vector and one positive vector, or one positive vector and one negative vector, by $N_2(3,3)$. Next we prove that each of $P_2(3,3), N_2(3,3)$ is connected. Let $E_+, E_-$ be a pair of unit positive vector and unit negative vector in $\mathbb{R}^{3,3}$. Let

$$v_+ = E_+ - \lambda E_-, \quad v_- = E_+ + \lambda E_+,$$

where $0 < \lambda < 1$. They are respectively a positive vector and a negative vector. By the continuity of the Clifford multiplication, we only need prove that $E_+ v_-$ equals the Clifford product of two positive vectors, and $E_+ E_+$ equals the Clifford product of a positive vector and a negative vector. Both are true because

$$E_+ v_- = -\{E_+(E_+ E_+)\}(E_+ E_-)(E_+ + \lambda E_+) = (-E_+)(E_+ - \lambda E_-),$$
$$E_- E_+ = (-E_+)(E_-).$$

Obviously $\pm 1 \in P_2(3,3)$. By $(E_+ E_-)(E_+ E_-) = 1$ and the continuity of the Clifford multiplication, we get that the Clifford product of four invertible vectors, where the number of negative vectors is even, must be in the same connected component with $P_2(3,3)$. By $(E_+ E_-)(E_+ E_-) = E_+ E_-$ and the continuity of the Clifford multiplication, we get that the Clifford product of four invertible vectors, where the number of negative vectors is odd, must be in the same connected component with $N_2(3,3)$.

Denote by $Spin_0(3,3)$ the connected component of $Spin(3,3)$ containing $P_2(3,3)$, and denote by $Spin_1(3,3)$ the connected component containing $N_2(3,3)$. By induction on the number of invertible vector factors in the factorization of an element of $Spin(3,3)$, we get that $Spin_0(3,3)$ contains all elements that are the Clifford product of even number of negative vectors and even number of positive vectors, while $Spin_1(3,3)$ contains all elements that are the Clifford product of odd number of negative vectors and odd number of positive vectors. Since $Spin(3,3) = Spin_0(3,3) \cup Spin_1(3,3)$ and $Spin(3,3)$ is known to have two connected components, $Spin_0(3,3)$ and $Spin_1(3,3)$ are not the same connected component. \qed

Lemma 5. $\mathcal{J} \in SO^-(3,3)$ is double covered by $\pm(E_1-E'_1)(E_1-E'_1)(E_1-E'_1) \in Pin^-(3,3)$, and $-\mathcal{J}$ is double covered by $\pm(E_1+E'_1)(E_1+E'_1)(E_1+E'_1) \in Pin^-(3,3)$. 

Proposition 4. The set \( \text{Spin}^-(3,3) \) has two connected components. The component double-covering \( \mathcal{J} \) is denoted by \( \text{Spin}_0^-(3,3) \), whose elements are each the Clifford product of odd number of negative vectors and even number of positive vectors. The component double-covering \( -\mathcal{J} \) is denoted by \( \text{Spin}_1^-(3,3) \), whose elements are each the Clifford product of even number of negative vectors and odd number of positive vectors. In particular, all positive vectors are in \( \text{Spin}_1^-(3,3) \), while all negative vectors are in \( \text{Spin}_0^-(3,3) \).

The group \( O(3,3) \) is double-covered by \( \text{Pin}(3,3) \), and the covering homomorphism is given as following: for any \( \pm U \in \text{Pin}(3,3) \),

\[
\text{Ad}_U^* X := \epsilon UXU^{-1}, \quad \text{for} \ X \in \mathbb{R}^{3,3}
\]

is a transformation belonging to \( O(3,3) \), where \( \epsilon = 1 \) if \( U \in \text{Spin}(3,3) \), and \( \epsilon = -1 \) if \( U \in \text{Spin}^-(3,3) \).

Denote

\[
\text{Pin}_0(3,3) = \text{Spin}_0(3,3) \cup \text{Spin}_{0}^-(3,3), \quad \text{Pin}_1(3,3) = \text{Spin}_1(3,3) \cup \text{Spin}_{1}^-(3,3).
\]

\( \text{Pin}_0(3,3) \) is a subgroup of \( \text{Pin}(3,3) \), and double covers \( \text{SO}_0(3,3) \cup \text{SO}_{0}^-(3,3) \). In particular, \( \text{SO}_0(3,3) \) is double covered by \( \text{Spin}_0(3,3) \). Now that the Plücker transform provides another double-covering homomorphism of \( \text{SO}_0(3,3) \) by \( \text{SL}(4) \), we get the classical result that the two groups \( \text{SL}(4) \) and \( \text{Spin}_0(3,3) \) are isomorphic.

Below we extend the above double-covering map to \( \text{SAO}(3,3) \cup \text{SAO}^-(3,3) \). We have seen that \( \mathcal{T} \in \text{SAO}^-(3,3) \). For any \( B \in O(3,3) \), obviously \( \mathcal{T}B\mathcal{T} \in O(3,3) \), so any element of \( \text{AO}(3,3) \) must have a unique matrix form \( \mathcal{T}B \) for some \( B \in O(3,3) \). In fact, we have the following:

Definition 5. Define the following isomorphism in group \( \text{Pin}(3,3) \): For any \( U = Y_1 Y_2 \cdots Y_r \in \text{Pin}(3,3) \) where \( Y_i \in \mathbb{R}^{3,3} \),

\[
U^T := (\mathcal{T}Y_1)(\mathcal{T}Y_2)\cdots(\mathcal{T}Y_r).
\]

Lemma 6. For any \( U \in \text{Pin}(3,3) \) and any \( X \in \mathbb{R}^{3,3} \),

\[
\mathcal{T}(\text{Ad}_U^* X) = \text{Ad}_{U^T}^* (\mathcal{T}X).
\]

Proof. If (2.26) is true for \( U = Y_1 \), by induction it is true for any other element of \( \text{Pin}(3,3) \). It holds for \( U = Y_1 \) by direct verification. \( \square \)

Definition 6. By defining a formal associative product between \( \mathcal{T} \) and \( \text{Pin}(3,3) \) satisfying the following commutativity, a new group is generated, denoted by \( \text{Pin}^p(3,3) \): for any \( U \in \text{Pin}(3,3) \),

\[
\mathcal{T} \circ U = U^T \circ \mathcal{T}.
\]

Let \( \mathcal{T}\text{Pin}(3,3) \) be the coset of \( \text{Pin}(3,3) \) with respect to \( \mathcal{T} \), then

\[
\text{Pin}^p(3,3) = \text{Pin}(3,3) \cup \mathcal{T}\text{Pin}(3,3).
\]
The adjoint action of $\mathcal{T}Pin(3,3)$ upon $\mathbb{R}^{3,3}$ is defined as follows: for any $U \circ \mathcal{T} \circ V \in \mathcal{T}Pin(3,3)$, where $U, V \in Pin(3,3)$, for any $X \in \mathbb{R}^{3,3}$,

$$Ad_{U \circ \mathcal{T} \circ V}X := Ad_U(\mathcal{T}(Ad_VX)).$$

(2.29)

$Pin^{sp}(3,3)$ double covers $O(3,3) \cup AO(3,3)$ by the adjoint action. For $i = 0, 1$, let $\mathcal{T}Pin_i(3,3)$ be the coset of $Pin_i(3,3)$ with respect to $\mathcal{T}$ for $i = 0, 1$. Then

$$Pin^{sp}_0(3,3) = Pin_0(3,3) \cup \mathcal{T}Pin_0(3,3)$$

(2.30)

double covers $RL_0(3,3)$. Since $UR(4)$ also double covers $RL_0(3,3)$ by the Plücker transform and dual Plücker transform, $UR(4)$ and $Pin^{sp}_0(3,3)$ are isomorphic.

Below we realize $\mathcal{T}$ in $GL(3,3)$. Let

$$K_2 = E_{111} + E_{222} + E_{333} := e_{01} \wedge e_{23} + e_{02} \wedge e_{31} + e_{03} \wedge e_{12}.$$  

(2.31)

It is called the symplectic form of $\mathbb{R}^{3,3}$ with respect to the Witt decomposition $\mathbb{R}^{3,3} = I_3 \oplus J_3$.

**Lemma 7.** $K_2$ is invariant under any general linear transformation $C$ in the 3-space $I_3$ and the associated linear transformation $C^{-T}$ in the 3-space $J_3$. In other words, it is independent of the the choice of Witt basis of the fixed Witt decomposition.

**Proof.** Let $e_1, e_2, e_3, e'_1, e'_2, e'_3$ be a Witt basis of $I_3 \oplus J_3$, and let $A \in GL(I_3)$ such that $Ae_1 = a_1$. Let $a_i = (a_{i1}, a_{i2}, a_{i3})^T$, and let $a'_i = (a'_{i1}, a'_{i2}, a'_{i3})^T$, where $a_{ij}$ is the minor of $A$ by removing the $i$-th row and $j$-th column. Then $A^{-T}e'_i = a'_i/\det(A)$, and $a'^T a'_j = \delta_{ij} \det(A)$. We have

$$a_1 \wedge a'_1 + a_1 \wedge a'_1 + a_1 \wedge a'_1 = (a_{11}a_{11}' + a_{21}a_{12}' + a_{31}a_{13}')e_1 \wedge e'_1 + (a_{11}a_{12}' + a_{21}a_{22}' + a_{31}a_{23}')e_1 \wedge e'_2 + (a_{11}a_{13}' + a_{21}a_{23}' + a_{31}a_{33}')e_1 \wedge e'_3 + (a_{12}a_{12} + a_{22}a_{22}' + a_{32}a_{32}')e_2 \wedge e'_1 + (a_{12}a_{13}' + a_{22}a_{23}' + a_{32}a_{33}')e_2 \wedge e'_2 + (a_{13}a_{13}' + a_{23}a_{23}' + a_{33}a_{33}')e_3 \wedge e'_1 + (a_{13}a_{12}' + a_{23}a_{22}' + a_{33}a_{32}')e_3 \wedge e'_2 + (a_{13}a_{13}' + a_{23}a_{23}' + a_{33}a_{33}')e_3 \wedge e'_3 = \det(A)(e_1 \wedge e'_1 + e_2 \wedge e'_2 + e_3 \wedge e'_3).$$

Some simple facts about $K_2$:

- $K_2^2 = 3 - 2K_2I_{3,3}$.
- $K_2$ is invertible: $K_2^{-1} = (K_2 + 2I_{3,3})/3$.
- For any $X \in \mathbb{R}^{3,3}$, $K_2XK_2 = (1 - 2I_{3,3})X$.
- For any $X, Y \in \mathbb{R}^{3,3}$, if $X \wedge K_2 = Y \wedge K_2$, then $X = Y$.

**Proposition 5.** For any $X \in \mathbb{R}^{3,3}$,

$$\mathcal{T}X = X \cdot K_2.$$  

(2.32)
We have seen that $UR(4)$ is isomorphic to only half of the group $Pin^{\text{op}}(3,3)$. To study 3-D projective transformations and polarities, the homogeneous model $\mathbb{R}^4$ of 3-D projective geometry does not allow for the whole group $Pin^{\text{op}}(3,3)$ to be used. To overcome this drawback, we need to proceed to use the null 3-space representation of projective points and planes, instead of returning to $\mathbb{R}^4$. This ideal leads to the following new model of 3-D projective geometry.

**Definition 7.** The Plücker model of 3-D projective geometry refers to the study of 3-D projective transformations and polarities by using the adjoint action of the whole group $P$ in $\text{sp}(3,3)$ upon the null 3-spaces of $\mathbb{R}^3$, for any $U \in Pin^{\text{op}}(3,3)$, the adjoint action is the following:

$$(\wedge^3 Ad_U)C_3 := (Ad_U X_1) \wedge (Ad_U X_2) \wedge (Ad_U X_3).$$

(2.33)

**Definition 8.** The 3-D projective regularity group $PR(3)$ is defined as the quotient of $UR(4)$ modulo the equivalence relation “$*$” in which two elements are equivalent if and only if they differ by scale. Usually we write

$$PR(3) := UR(4)/\ast.$$  

(2.34)

In the next section, it will be shown that via the Plücker model, group $Pin^{\text{op}}(3,3)$ quadruple-covers group $PR(3)$, and the kernel of the covering homomorphism is $\{\pm 1, \pm I_{3,3}\}$.

3 Some properties of the Plücker transform

In this section we investigate two topics: null 3-vector representations of 3-D points and planes and their covariance under the group action of $GL(4)$ upon $\mathbb{R}^4$, and the adjoint action of $Pin^{\text{op}}(3,3)$ upon the null 3-vector representations.

**Notations of algebra.** An orthonormal basis $e_0, e_1, e_2, e_3$ is fixed in $\mathbb{R}^4$, which induces a fixed Witt basis $E_i = e_{0i}$ and $E'_i = e_{ijk}$ for positive permutation $ijk$ of 123, where $i = 1, 2, 3$. Denote

$$E_{i_1 \ldots i_r} = E_{i_1} \wedge \cdots \wedge E_{i_r},$$

(3.1)

for $i_j \in \{1, 2, 3, 1', 2', 3'\}$. Denote $e_{123} = e_{1}e_{2}e_{3}$, and for any $x \in \mathbb{R}^3$, denote

$$x^\perp := x \cdot e_{123}.$$  

(3.2)

For example, $e_i^\perp = e_{ijk}$ for positive permutation $ijk$ of 123.

For two 3-vectors $P_3, Q_3 \in \Lambda(\mathbb{R}^4)$, denote by $P_3 \vee Q_3$ the meet product $P_3 \wedge Q_3$. This notation does not conflict with the same usage of the juxtaposition for representing the outer product of two vectors of $\mathbb{R}^4$, as the outer product of two 3-vectors in $\Lambda(\mathbb{R}^4)$ is always zero, so is the meet product of two vectors.

**Notations of geometry.** Let $x \in \mathbb{R}^3$, then vector $(x_0, x) = x_0 e_0 + x \in \mathbb{R}^4$ represents an affine point if and only if $x_0 \neq 0$. A plane is determined by the
equation \( x_0(-d) + n \cdot x = 0 \) for point \( (x_0, x) \) on it, where \( n \in \mathbb{R}^3 \) and \( d \in \mathbb{R} \), and at least one of them is nonzero. When \( d \mathbf{n} \neq 0 \), the equation represents the affine plane normal to vector \( n \) and with signed distance \( d/|n| \) from the origin along direction \( n \). When \( d = 0 \), the plane passes through the origin; when \( n = 0 \), the plane is the plane at infinity. So the pair \( (n, -d) \in \mathbb{R}^3 \times \) represents the plane, and the representation is unique up to scale.

**Proposition 6.** Point \((x_0, x) \in \mathbb{R}^3\) has the following null 3-vector form in \( \Lambda^3(\mathbb{R}^3, \mathbb{R}) \):

\[
(e_0 x) \wedge ((e_0 x) \cdot J_3) - x_0(e_0 x) \wedge K_2 + x_0^2 I_3, \tag{3.3}
\]

and plane \((n, -d)\) has 3-vector form

\[
n^\perp \wedge (n^\perp \cdot I_3) - d n^\perp \wedge K_2 + d^2 J_3, \tag{3.4}
\]

**Proof.** When \( x_0 \mathbf{e}_0 + x \) is an affine point, where \( x = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \), then \( x_0 \neq 0 \), and the following three lines pass through the point, and form a basis of the 3-space of lines through the point: \((x_0 \mathbf{e}_0 + x) \mathbf{e}_1, (x_0 \mathbf{e}_0 + x) \mathbf{e}_2, (x_0 \mathbf{e}_0 + x) \mathbf{e}_3\).
We have

\[
((x_0 \mathbf{e}_0 + x) \mathbf{e}_1) \wedge ((x_0 \mathbf{e}_0 + x) \mathbf{e}_2) \wedge ((x_0 \mathbf{e}_0 + x) \mathbf{e}_3)
= (x_0 \mathbf{e}_{01} - x_2 \mathbf{e}_{12} + x_3 \mathbf{e}_{31}) \wedge (x_0 \mathbf{e}_{02} + x_1 \mathbf{e}_{12} - x_3 \mathbf{e}_{23}) \wedge (x_0 \mathbf{e}_{03} - x_1 \mathbf{e}_{31} + x_2 \mathbf{e}_{23})
= x_0^3 I_3 - x_0^2 \{x_1 \mathbf{e}_{01} \wedge (\mathbf{e}_{02} \wedge \mathbf{e}_{31} + \mathbf{e}_{03} \wedge \mathbf{e}_{12}) + x_2 \mathbf{e}_{02} \wedge (\mathbf{e}_{01} \wedge \mathbf{e}_{23} + \mathbf{e}_{03} \wedge \mathbf{e}_{12})
+ x_3 \mathbf{e}_{03} \wedge (\mathbf{e}_{01} \wedge \mathbf{e}_{23} + \mathbf{e}_{02} \wedge \mathbf{e}_{31})\}
+ x_0 (x_1 \mathbf{e}_{01} + x_2 \mathbf{e}_{02} + x_3 \mathbf{e}_{03}) \wedge \{(-x_1 \mathbf{e}_{12} \wedge \mathbf{e}_{31} + x_2 \mathbf{e}_{12} \wedge \mathbf{e}_{23} - x_3 \mathbf{e}_{31} \wedge \mathbf{e}_{23})
= x_0^3 I_3 - x_0^2 (e_0 x) \wedge K_2 + x_0 (e_0 x) \wedge \{(e_0 x) \cdot (e_{23} \wedge e_{31} \wedge e_{12})\}.
\]

Removing the common factor \( x_0 \), we get (3.3). When \( x_0 = 0 \), then \( e_0 x \) is the line through \( x \) and the origin, and \((e_0 x) \cdot J_3\) are spanned by two lines at infinity that meet line \( e_0 x \), i.e., through the point at infinity \( x \). So \((e_0 x) \wedge ((e_0 x) \cdot J_3)\) is the 3-vector representing the point at infinity \( x \).

For an affine plane \((n, -d)\) where \( n^2 = 1 \), let \( a, b, n \) be an orthonormal frame of \( \mathbb{R}^3 \). Then \( n^\perp = ab \), and

\[
I_3 = (e_0 n) \wedge (e_0 a) \wedge (e_0 b),
J_3 = (n a) \wedge (n b) \wedge (a b),
K_2 = (e_0 n) \wedge (a b) - (e_0 a) \wedge (n b) + (e_0 b) \wedge (n a).
\]

The following three lines span the 3-space of lines on the plane: \((e_0 + d n) a, (e_0 + d n) b, a b\). We have

\[
((e_0 + d n) a) \wedge ((e_0 + d n) b) \wedge (a b)
= (e_0 a) \wedge (e_0 b) \wedge (a b) + d^2 (n a) \wedge (n b) \wedge (a b)
+ d (a b) \wedge \{(e_0 a) \wedge (n b) - (e_0 b) \wedge (n a)\}
= (a b) \wedge ((a b) \cdot I_3) - d (a b) \wedge K_2 + d^2 J_3.
\]

Thus we get (3.4) for affine plane. When \( n = 0 \), then \( J_3 \) is the plane at infinity. \(\square\)
**Definition 9.** The map

\[ F : x_0 e_0 + x \mapsto (e_0 x) \wedge ((e_0 x) \cdot J_3) - x_0 (e_0 x) \wedge K_2 + x_0^2 J_3 \tag{3.5} \]

is called the Plücker representation of the point. The plane \((n, d)\) can be represented by vector \(de_0 + n\); for the plane, the map

\[ F' : de_0 + n \mapsto n^\perp \wedge (n^\perp \cdot I_3) + dn^\perp \wedge K_2 + d^2 J_3 \tag{3.6} \]

is called the Plücker representation of the plane.

**Remark.** The Plücker representations of points and points at infinity \(e_0, e_1, e_2, e_3\) are respectively \(E_{123}, E_{123}'\), \(E_{231}'\), \(E_{312}'\). Furthermore,

\[ F(x_0 e_0 + x) = F(x_0 e_0) + F(x) - ((x_0 e_0) x) \wedge K_2, \]
\[ F'(de_0 + n) = F'(de_0) + F'(n) + (dn^\perp) \wedge K_2. \tag{3.7} \]

Definition 9 relies upon the specific Witt decomposition \(I_3 \oplus J_3\). The following property gives an alternative definition of \(F\) that is independent of the Witt decomposition.

**Proposition 7.** Let \(X, a_1, a_2, a_3\) be a basis of \(R^3\). Then

\[ F(X) = \frac{(X a_1) \wedge (X a_2) \wedge (X a_3)}{[X a_1 a_2 a_3]} \tag{3.8} \]

In particular, \(F(X)\) is independent of the choice of \(a_1, a_2, a_3\) as long as the latter forms a basis of the 4-space together with \(X\). Furthermore,

\[ F' = (\wedge^3 F) \circ F. \tag{3.9} \]

**Proof.** (3.9) is obvious by (3.5) and (3.6). We only need to prove (3.8). When \(X\) is an affine point, by definition,

\[ F(X) = \frac{(X e_1) \wedge (X e_2) \wedge (X e_3)}{[X e_1 e_2 e_3]} \]

For \(i = 1, 2, 3\), let \(a_i = d_i X + c_{i1} e_1 + c_{i2} e_2 + c_{i3} e_3\). Then \(\det(c_{ij})_{i,j=1..3} \neq 0\), as

\[ (X a_1 a_2 a_3) = (X e_1 e_2 e_3) \begin{pmatrix} 1 & d^T \\ 0 & C \end{pmatrix}, \]

where \(d^T = (d_1, d_2, d_3)\), and \(C = (c_{ij})_{i,j=1..3}\). By direct verification,

\[ \frac{(X a_1) \wedge (X a_2) \wedge (X a_3)}{[X a_1 a_2 a_3]} = \frac{\det(C)(X e_1) \wedge (X e_2) \wedge (X e_3)}{\det(C)[X e_1 e_2 e_3]} \]

When \(X\) is a point at infinity, let \(X = \lambda x\), where \(x\) is a unit vector, and let \(x, a, b\) be an orthonormal basis of the 3-space spanned by \(e_1, e_2, e_3\), then \([X e_0 a b] = -\lambda\). By definition,

\[ F(X) = \lambda^2 (e_0 x) \wedge ((e_0 x) \cdot (a b \wedge b x \wedge xa)) = \frac{(X e_0) \wedge (X b) \wedge (X a)}{[X e_0 a b]} \]

The following proposition answers the following question: Given \(F(X)\) for \(X = x_0 e_0 + x\), how to recover \(X\)?
Proposition 8. For any $X = x_0 e_0 + x \in \mathbb{R}^4$, and any $II = d e_0^* + n^* \in (\mathbb{R}^4)^*$,
\[
\mathcal{F}(X) \cdot K_2 = -2x_0(e_0 x),
\]
\[
\mathcal{F}''(II) \cdot K_2 = -2d(e_0^* n^*).
\]  
(3.10)

Proof. Let $x = x_1 e_1 + x_2 e_2 + x_3 e_3$, then
\[
\mathcal{F}(X) \cdot K_2 = \sum_{i=1}^{3} x_i (E_i \wedge (e_0 x)) \cdot J_3 - 3x_0(e_0 x) + x_0 \sum_{i=1}^{3} x_i E_i = -2x_0(e_0 x).
\]

The second result can be proved similarly. □

The following is on the covariance of $\mathcal{F}(X)$: if $X \in \mathbb{R}^4$ undergoes a general linear transformation, how does $\mathcal{F}(X)$ change in $A^3(\mathbb{R}^{3,3})$ accordingly?

Proposition 9. Let $A$ be a non-singular linear transformation in the 4-space spanned by $e_0, e_1, e_2, e_3$. Then for any vector $X = x_0 e_0 + x$ of the 4-space,
\[
\det(A) \mathcal{F}(AX) = \wedge^3(\wedge^2 A) \mathcal{F}(X),
\]  
(3.11)

and
\[
\det(A) \mathcal{F}'(A^{-T} X) = \wedge^3(\mathcal{J} \circ (\wedge^2 A^{-T}) \circ \mathcal{J}) \mathcal{F}'(X).
\]  
(3.12)

Proof. Without loss of generality, let $X$ represent an affine point in space, then
\[
\wedge^3(\wedge^2 A) \mathcal{F}(X) = \frac{(\wedge^2 A)(X e_1) \wedge (\wedge^2 A)(X e_2) \wedge (\wedge^2 A)(X e_3)}{[X e_1 e_2 e_3]}
\]
\[
= \det(A) \frac{(AX)(A e_1) \wedge (AX)(A e_2) \wedge (AX)(A e_3)}{[AX(A e_1)(A e_2)(A e_3)]}
\]
\[
= \det(A) \mathcal{F}(AX).
\]

(3.12) is obvious from (3.11), where $A^{-T}$ is the induced linear transformation in the vector space of planes. □

Alternatively, a plane $(n, d)$ can be represented by a 3-vector in $A(\mathbb{R}^4)$. Let $n = a e_1 + be_2 + ce_3$, then $II = d e_0^* + ae_1^* + be_2^* + ce_3^*$ represents the plane. A point $X = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$ is on the plane if and only if $X \cup II = 0$, i.e., $x_0 d + x_1 a + x_2 b + x_3 c = 0$.

Proposition 10. Let $A \in GL(4)$ act upon $\mathbb{R}^4$. In $A^3(\mathbb{R}^4)$ and $A^2(\mathbb{R}^{3,3})$, the two matrices $\wedge^3 A = \det(A) A^{-T}$ and $B = \sqrt{2}(\wedge^3 A)$ satisfy
\[
\det(B) = (\det(\det(A) A^{-T}))^3 = \det(A)^9.
\]  
(3.13)

Let $n^* = ae_1^* + be_2^* + ce_3^*$. When $II = d e_0^* + n^*$ is an affine plane not through the origin, then $d n^* \neq 0$, and the plane meets the following three planes passing
through the origin: \( e_1^*, e_2^*, e_3^* \), so lines \( IIe_i^* \) for \( i = 1, 2, 3 \) form a basis of the 3-space of lines on plane \( II \). By (2.12),

\[
(IIe_1^*) \wedge (IIe_2^*) \wedge (IIe_3^*)
\]

\[
= (de_{23} - be_{03} + ce_{02}) \wedge (de_{31} + ae_{03} - ce_{01}) \wedge (de_{12} - ae_{02} + be_{01})
\]

\[
= d^3J_3 + d^2 \{ ae_{31} \wedge (e_02 \wedge e_{03} + e_12) + be_{31} \wedge (e_01 \wedge e_{23} + e_{03} \wedge e_{12}) + ce_{12} \wedge (e_01 \wedge e_{23} + e_{03} \wedge e_{02}) \}.
\]

Removing the common factor \( d \), we get the following alternative form of (3.4):

\[
(e_0^* II) \wedge ((e_0^*) \cdot I_3) + d(e_0^* II) \wedge K_2 + d^2 J_3. \tag{3.14}
\]

When \( II \) passes through the origin, then \( d = 0 \), and (3.14) still represents the plane. When \( II \) is the plane at infinity, then \( II = de_0^* \), so \( e_0^* II = 0 \), and (3.14) once again represents the plane.

**Definition 10.** When a plane \( (n, d) \) is represented by a 3-vector \( II = de_0^* + n^* \), the map

\[
\mathcal{F}': de_0^* + n^* \mapsto (e_0^* II) \wedge ((e_0^*) \cdot I_3) + d(e_0^* II) \wedge K_2 + d^2 J_3 \tag{3.15}
\]

is called the **dual Plücker representation** of the plane.

**Remark.**

\[
\mathcal{F}''(de_0^* + n^*) = \mathcal{F}'(de_0^*) + \mathcal{F}'((n^*) + ((de_0^*) n^*) \wedge K_2. \tag{3.16}
\]

**Lemma 8.** For the plane \( (n, d) \) represented by 3-vector \( II = de_0^* + n^* \),

\[
d = IIe_1^* e_2^* e_3^* = II \vee e_1^* \vee e_2^* \vee e_3^* \tag{3.17}
\]

**Lemma 9.** Let \( A \in GL(4) \) act upon \( \mathbb{R}^4 \). Then in \( \mathbb{R}^{3,3} \),

\[
\wedge^2 A = \det(A) (\vee^2 A^{-T}). \tag{3.18}
\]

**Proof.** Let \( A = (a_0 \ a_1 \ a_2 \ a_3) \). On one hand, \( \wedge^2 A(e_{0i}) = A(e_0) \wedge A(e_i) = a_0 \wedge a_i \), and \( \wedge^2 A(e_{ij}) = a_i \wedge a_j \). On the other hand, for positive permutation \( ijk \) of 123,

\[
\vee^2 A^{-T}(e_{0i}) = \vee^2(\det(A)^{-1} \wedge^3 A)(e_j^* \vee e_k^*) = \det(A)^{-2}(a_0 \wedge a_k \wedge a_j) \vee (a_0 \wedge a_i \wedge a_j) = \det(A)^{-1}a_0 \wedge a_i;
\]

\[
\vee^2 A^{-T}(e_{ij}) = \vee^2(\det(A)^{-1} \wedge^3 A)(e_0^* \vee e_k^*) = -\det(A)^{-2}(a_i \wedge a_j \wedge a_k) \vee (a_0 \wedge a_i \wedge a_j) = \det(A)^{-1}a_i \wedge a_j.
\]

\( \square \)
Proposition 11. For \( X \in \mathbb{R}^4 \) and \( II \in \Lambda^3(\mathbb{R}^4) \),
\[
F(X) \cdot F''(II) = -[XII]^2. \tag{3.19}
\]

Proof. Let \( X = x_0e_0 + x \) and \( II = de_0^* + n^* \), then
\[
F(X) \cdot F''(II) = (F(X) + x_0^2F(e_0) - x_0(e_0x) \wedge K_2) \\
\cdot (F''(n^*) + d^2F''(e_0^*) + d(e_0^*n^*) \wedge K_2) \\
= - (x \cdot n)^2 - x_0^2d^2 - 2x_0d (x \cdot n) \\
= - [XII]^2. \]

The proof of the following proposition is similar to those of Proposition 7 and Proposition 9, and is omitted.

Proposition 12. Let \( II, a_1^*, a_2^*, a_3^* \) be a basis of \((\mathbb{R}^4)^*\). Then
\[
F''(II) = \frac{IIa_1^* \wedge (IIa_2^*) \wedge (IIa_3^*)}{IIa_1^*a_2^*a_3^*}. \tag{3.20}
\]
In particular, \( F''(II) \) is independent of the choice of \( a_1^*, a_2^*, a_3^* \) as long as the latter forms a basis of the 4-space together with \( II \).

Let \( A \) be a non-singular linear transformation in \( \mathbb{R}^4 \). It induces a linear transformation \( A^{-T} = \det(A)^{-1} \wedge^3A \) in \((\mathbb{R}^4)^*\). For any \( II = de_0^* + n^* \in (\mathbb{R}^4)^* \),
\[
\det(A^{-T})F''(A^{-T}II) = \wedge^3(\vee^2A^{-T})F''(II). \tag{3.21}
\]

Below we extend the previous results from \( GL(4) \) to \( GD(4) \). The following is a simple fact from linear algebra on the influence of coordinate transformations in \( \mathbb{R}^4 \) upon the matrix form of \( A : \mathbb{R}^4 \rightarrow (\mathbb{R}^4)^* \).

Proposition 13. Let \( C = (c_0 \ c_1 \ c_2 \ c_3) \in GL(4) \) be a coordinate transformation in \( \mathbb{R}^4 \), and let \( A \) be an invertible linear mapping from \( \mathbb{R}^4 \) to \((\mathbb{R}^4)^*\), whose matrix form with respect to the basis \( e_i \)'s of \( \mathbb{R}^4 \) and \( e_i^* \)'s of \((\mathbb{R}^4)^*\) is \( A_{4 \times 4} \). Then with respect to the basis \( c_i \)'s of \( \mathbb{R}^4 \) and its dual basis \( c_i^* \)'s of \((\mathbb{R}^4)^*\), where \( c_i^* = (\wedge^3C)c_i^*/\det(C) \), the matrix form of \( A \) is \( C^TA_{4 \times 4}C \).

Proof. The coordinate transformation \( C \) in \( \mathbb{R}^4 \) induces the following coordinate transformation in \( \Lambda^3(\mathbb{R}^4) \):
\[
(\wedge^3C)e_i^* = \sum_{j=0}^3 \det(C)(C^{-T})_{ji}e_j^*. \tag{3.22}
\]

Let \( Ae_i = \sum_{j=0}^3 a_{ij}e_j^* \), and let \( c_i = (c_{0i}, c_{1i}, c_{2i}, c_{3i})^T \). Then
\[
Ac_i = \sum_{k=0}^3 c_{ki}Ae_k = \sum_{j,k=0}^3 c_{ki}a_{jk}(C^T)_{lj}(\wedge^3C)e_l^* \\
= \sum_{l=0}^3 (C^TAC)_{li}c_l^*.
\]
Notation. For any $A_{4\times 4} = (a_{ij})_{i,j = 0, \ldots, 3}$, denote its $2 \times 2$ minor composed of the $u, v$ rows and $p, q$ columns by $A_{pq}^{uv}$, i.e.,

$$A_{pq}^{uv} := \begin{vmatrix} a_{up} & a_{uq} \\ a_{vp} & a_{vq} \end{vmatrix}. \quad (3.23)$$

Let $A \in GD(4)$, i.e., a non-singular linear map from $\mathbb{R}^4$ to $(\mathbb{R}^4)^*$. Let the matrix form of $A$ with respect to the basis $e_0, e_1, e_2, e_3$ of $\mathbb{R}^4$ and the dual basis $e_0^*, e_1^*, e_2^*, e_3^*$ of $(\mathbb{R}^4)^*$ be $A_{4\times 4} = (a_0, a_1, a_2, a_3)$, where $a_i = (a_{0i}, a_{1i}, a_{2i}, a_{3i})^T$. Then $A e_i = a_i^* := \sum_{j=0}^3 a_{ji} e_j^*$, and

$$(\sqrt{2} A)(e_{ij}) = a_i^* a_j^* = \sum_{0 \leq p < q \leq 3} A_{ij}^{pq} e_{pq}, \quad (3.24)$$

where $pqrs$ is a positive permutation of $0123$.

**Lemma 10.** For the $4 \times 4$ identity matrix $I_{4\times 4}$ representing an element of $GD(4)$, $\sqrt{2} I_{4\times 4} = J$.

**Proposition 14.** Let $A \in GD(4)$ have matrix form $A_{4\times 4}$, and let $B \in GL(3,3)$ be the matrix form of $\sqrt{2} A$. When matrix $A_{4\times 4}$ is taken as a linear transformation of $\mathbb{R}^4$, then $\wedge^2 A_{4\times 4}$ has the matrix form $J B \in GL(3,3)$.

**Proof.** Let $\tilde{A}$ be the non-singular linear transformation in $\mathbb{R}^4$ having the same matrix form $A_{4\times 4} = (a_0, a_1, a_2, a_3)$ with $A$. Then the matrix form $B$ of $\wedge^2 \tilde{A}$ is given by

$$(\wedge^2 \tilde{A})(e_{ij}) = \sum_{0 \leq p < q \leq 3} A_{ij}^{pq} e_{pq}. \quad (3.25)$$

Obviously $B = J B$. \hfill $\square$

**Proposition 15.** Let $A \in GD(4)$. Then for any $X \in \mathbb{R}^4$,

$$\det(A) F''(AX) = \wedge^3 (\sqrt{2} A) F(X). \quad (3.26)$$

**Proof.** We have

$$\wedge^3 (\sqrt{2} A) F(X) = \frac{(AX)(Ae_1) \wedge (AX)(Ae_2) \wedge (AX)(Ae_3)}{[X e_1 e_2 e_3]}.$$ 

Since $(A e_0)(A e_1)(A e_2)(A e_3) = a_0^* a_1^* a_2^* a_3^* = \det(A) e_0^* e_1^* e_2^* e_3^* = \det(A)$, we have $\det(A) [X e_1 e_2 e_3] = (AX)(Ae_1)(Ae_2)(Ae_3)$. On the other hand,

$$F''(AX) = \frac{(AX)(Ae_1) \wedge (AX)(Ae_2) \wedge (AX)(Ae_3)}{(AX)(Ae_1)(Ae_2)(Ae_3)}.$$ 

For any $U \in Pin^p(3,3)$, $\wedge^3 A d_U^p$ maps a null vector to a null vector. If $U \in Spin(3,3) \cup TSpin(3,3)$, then $\wedge^3 A d_U^p$ maps the set of null 3-spaces representing points to the same set; if $U \in Spin^-(3,3) \cup TSpin^-(3,3)$, then $\wedge^3 A d_U^p$ interchanges the set of null 3-spaces representing points and the set of null 3-spaces representing planes.
Lemma 11. For any $X \in \mathbb{R}^4$ and $\Pi \in (\mathbb{R}^4)^*$,\[ F(X) \cdot I_{3,3} = F(X), \quad F''(\Pi) \cdot I_{3,3} = -F''(\Pi). \] (3.27)

Furthermore, for any $P \in \mathbb{R}^{3,3}$, $Ad_{I_{3,3}}^* P = -P$.

Given a null 3-vector, (3.27) can be used to determine whether it represents a point or plane of $\mathcal{E}^3$.

**Notation.** For two algebraic elements $a, b$, if they differ by scale, we write
\[ a \backsimeq b. \] (3.28)

In $A^3(\mathbb{R}^{3,3})$, the representation of a point or plane by a null 3-vector is unique up to scale. So $Ad_{I_{3,3}}^*$ leaves each point and plane of $\mathcal{E}^3$ invariant, and induces the projective transformation represented by the identity transformation $I_{4 \times 4}$ of $\mathbb{R}^4$. As a consequence, the adjoint action of $Pin^{sp}(3, 3)$ upon the null 3-vectors of $A^3(\mathbb{R}^{3,3})$ induces a quadruple-covering homomorphism upon the group $PR(3)$, and the kernel is composed of 4 elements: $\pm 1$, $\pm I_{3,3}$.

**Definition 11.** For $U \in Pin^{sp}(3, 3)$, The 4 × 4 matrix representation of $Ad_{U}^*$ refers to a matrix form of the linear map induced by $Ad_{U}^*$ via the null 3-vector representation of points and planes, from $\mathbb{R}^4$ to $\mathbb{R}^4$ or $(\mathbb{R}^4)^*$. The matrix form is unique up to scale.

As $Pin^{sp}(3, 3)$ double covers $RL(3, 3)$, any element $B \in RL(3, 3)$ also has 4 × 4 matrix representation. By the covariance of $F$ and $F''$, the 4 × 4 matrix representation after rescaling is mapped to one of $\pm B$ by the Plücker transform or dual Plücker transform. We study how to compute the 4 × 4 matrix form $A$ of $B$. Without loss of generality, let $B \in SO_0(3, 3)$. Then $B$ is the image of $A \in SL(4)$ under the Plücker transform.

**Notations used only in this section.** Denote
\[ P(0) := 123, \quad P(1) := 12'3', \quad P(2) := 23'1', \quad P(3) := 31'2'. \] (3.29)

Let $ijk$ be a positive permutation of 123, denote
\[ Q(i) := j'k', \quad Q(ij) := k'. \] (3.30)

**Proposition 16.** Let $A = (a_{i,j,k}) = (a_{p,q})_{p,q=0,3} \in SL(4)$, and let $B = (b_{\alpha,\beta})_{\alpha,\beta=1,2,3,1',2',3'}$ be the matrix of $\wedge^2 A$ with respect to the basis $E_1, E_2, E_3, E_{1'}, E_{2'}, E_{3'}$, i.e., for $1 \leq i, j \leq 3$, let $ipq$ and $juv$ be both positive permutations of 123, then
\[ b_{ij} = A_{0ij}^{0ij}, \quad b_{ij'} = A_{0ij}^{pq}, \quad b_{ij'} = A_{0ij}^{0ij}, \quad b_{ij'} = A_{0ij}^{0ij}. \] (3.31)

Denote by $B_{pqr}^{uvw}$ the 3 × 3 minor formed by the $u, v, w$ rows and $p, q, r$ columns of $B$. Then for any $0 \leq i, j \leq 3$,
\[ \det(A)a_{ij}^2 = B_{p(i)}^{(j)}. \] (3.32)
Proof. Without loss of generality, consider $j = 1$. The following diagram commutes:

$$
\begin{array}{c}
\textbf{e}_1 \xrightarrow{\mathcal{F}} \mathbf{E}_{123'}' \\
\downarrow \quad \downarrow \\
\Lambda^3 \mathbf{B} \quad C_3
\end{array}
$$

where

$$
C_3 = \det(\mathbf{A}) \{a_{01}^2 \mathbf{I}_3 - a_{01} (\mathbf{e}_0 \mathbf{a}_1) \wedge \mathbf{K}_2 + (\mathbf{e}_0 \mathbf{a}_1) \wedge (\mathbf{e}_0 \mathbf{a}_1) \cdot \mathbf{J}_3\}
$$

$$
= \det(\mathbf{A}) \{a_{01}^2 \mathbf{E}_{123} + a_{01} a_{11} (\mathbf{E}_{313'} - \mathbf{E}_{122'}) + a_{01} a_{21} (\mathbf{E}_{121'} - \mathbf{E}_{233'}) + a_{11} a_{31} (\mathbf{E}_{131'} - \mathbf{E}_{222'}) + a_{21} a_{31} (\mathbf{E}_{112'} + \mathbf{E}_{333'})\}. \tag{3.34}
$$

Since $\Lambda^3 \mathbf{B}(\mathbf{E}_{123'}) = \sum_{ijk} \mathbf{B}^{ijk}_{123'} \mathbf{E}_{ijk}$, by comparing the coefficients of $\mathbf{E}_{123}$, $\mathbf{E}_{123'}$, $\mathbf{E}_{233'}$, $\mathbf{E}_{312'}$, we get

$$
\mathbf{B}^{123}_{123'} = \det(\mathbf{A}) a_{01}^2, \quad \mathbf{B}^{123'}_{123'} = \det(\mathbf{A}) a_{11}^2, \quad \mathbf{B}^{123}_{123} = \det(\mathbf{A}) a_{21}^2, \quad \mathbf{B}^{123'}_{123} = \det(\mathbf{A}) a_{31}^2. \quad \square
$$

Remark. By (3.32), a matrix $\mathbf{B} \in SO(3,3)$ is in $SO_0(3,3)$ if and only if $\mathbf{B}^{P(i)}_{P(j)} \geq 0$ for all $0 \leq i, j \leq 3$.

Proposition 17. Let $ijk$ be a positive permutation of 123, and let $l$ be one of 0,1,2,3, then

$$
\begin{align*}
\det(\mathbf{A}) a_{0l} a_{il} &= -\mathbf{B}^{ij}_{P(i)} = -\mathbf{B}^{ik}_{P(i)} = -\mathbf{B}^{jk'}_{P(i)}, \\
\det(\mathbf{A}) a_{il} a_{ij} &= \mathbf{B}^{ij}_{P(i)} = \mathbf{B}^{ij'}_{P(i)} = \mathbf{B}^{ik'}_{P(i)}, \\
\det(\mathbf{A}) a_{0l} a_{il} &= -\mathbf{B}^{ij}_{P(i)} = -\mathbf{B}^{ij'}_{P(i)} = -\mathbf{B}^{ik'}_{P(i)}, \\
\det(\mathbf{A}) a_{il} a_{ij} &= \mathbf{B}^{ij}_{P(i)} = \mathbf{B}^{ij'}_{P(i)} = \mathbf{B}^{ik'}_{P(i)}, \tag{3.35}
\end{align*}
$$

Proof. Without loss of generality, consider $l = 1$. In the diagram (3.33), by comparing the coefficients of $\mathbf{E}_{123'}$, $\mathbf{E}_{313'}$, $\mathbf{E}_{122'}$, $\mathbf{E}_{233'}$, $\mathbf{E}_{312'}$, $\mathbf{E}_{233'}$ in $\sum \mathbf{B}^{ijk}_{123'} \mathbf{E}_{ijk}$ and $C_3$ of (3.34), we get the first line of (3.35) for $l = 1$. By comparing the coefficients of $\mathbf{E}_{131'}$, $\mathbf{E}_{222'}$, $\mathbf{E}_{112'}$, $\mathbf{E}_{333'}$, $\mathbf{E}_{212'}$, $\mathbf{E}_{333'}$, we get the second line of (3.35) for $l = 1$. When replacing $\mathbf{A}$ with $\mathbf{A}^T$, by the trivial fact

$$
\wedge^2 (\mathbf{A}^T) = (\wedge^2 \mathbf{A})^T, \tag{3.36}
$$

we get the last two lines of (3.35). \square

Proposition 18. Let $ijk$ and $pq$ be both positive permutations of 123, then

$$
\begin{align*}
\det(\mathbf{A}) (a_{0p} a_{iq} + a_{0q} a_{ip}) &= -\mathbf{B}^{ij}_{P(q)} - \mathbf{B}^{ij'}_{P(q)} = -\mathbf{B}^{ik'}_{P(q)} - \mathbf{B}^{ik'}_{P(q)}, \\
\det(\mathbf{A}) (a_{ip} a_{jq} + a_{iq} a_{jp}) &= \mathbf{B}^{ij}_{P(q)} + \mathbf{B}^{ij}_{P(q)} = \mathbf{B}^{ij}_{P(q)} + \mathbf{B}^{ij}_{P(q)}. \tag{3.37}
\end{align*}
$$
Proof. Without loss of generality, consider \( i = 1 \) and \( j = 2 \). The following diagram commutes:

\[
\begin{array}{ccc}
e_1 + e_2 & \xrightarrow{f} & E_{123}' + E_{231}' + E_{132}' + E_{223}' \\
A & \downarrow & \wedge^3 B \\
a_1 + a_2 & \xrightarrow{\det(A) \cdot f} & D_3
\end{array}
\]

where for \( C_3 \) of (3.34),

\[
D_3 = C_3 + \det(A) \{ 2a_1a_2E_{123} + (a_0a_1 + a_0a_1)(E_{413'} - E_{122'}) \\
+ (a_0a_2 + a_0a_3)(E_{121'} - E_{233'}) + (a_0a_3 + a_0a_3)(E_{232'} - E_{311'}) \\
+ 2a_1a_2E_{123'} + 2a_2a_3E_{231'} + 2a_3a_3E_{312'} \\
+ (a_1a_2 + a_1a_2)(E_{132'} + E_{223'}) + (a_1a_2 + a_1a_2)(E_{112'} + E_{322'}) \\
+ (a_2a_3 + a_2a_3)(E_{212'} + E_{331'}) \}\}
\]

Since \( \wedge^3 B(E_{132'} + E_{223'}) = \sum_{ijk}(B_{132'}^{ijk} + B_{223'}^{ijk})E_{ijk} \), by comparing the coefficients of \( E_{313'}, E_{122'}, E_{121'}, E_{233'}, E_{232'}, E_{311'}, E_{112'}, E_{223'}, E_{132'}, E_{131'}, E_{212'}, E_{331'} \) in this expression and in \( D_3 - C_3 \), we get for positive permutation \( pqr \) of 123 the following:

\[
\det(A)(a_0a_2 + a_0a_2) = -B_{132'}^{ppq'} - B_{223'}^{ppq'} = -B_{132'}^{ppq'} - B_{223'}^{ppq'},
\]

\[
\det(A)(a_1a_2 + a_1a_2) = B_{132'}^{pqr'} + B_{223'}^{pqr'} = B_{132'}^{pqr'} + B_{223'}^{pqr'},
\]

\[
\square
\]

The 4 \( \times \) 4 matrix representation \( A_{4 \times 4} = (a_{pq})_{p,q=0,\ldots,3} \) of \( B \in SO(3,3) \) is given explicitly by (3.32), (3.35), and (3.37) as following: one entry of the first row of \( A \) must be nonzero, say \( a_{00} \neq 0 \) for some 0 \( \leq q_0 \leq 3 \). From (3.32) we get \( a_{00}^2 \); from \( a_{0q} \) and (3.35) we get \( a_{iq} \) for all 1 \( \leq i \leq 3 \), and \( a_{0q} \) for all \( q \neq q_0 \). From \( a_{0q}a_{iq} + a_{iq}a_{0q} \) of (3.37) we get the rest components \( a_{iq} \) of \( A \). The last step can be made more explicit as follows.

By

\[
\begin{align*}
a_{00}a_{iq} - a_{0q}a_{ip} &= A_{pq}^{01} = b_{ir'}, \\
a_{ip}a_{jq} - a_{iq}a_{jp} &= A_{pq}^{1j} = b_{ik'},
\end{align*}
\]

we get
Proposition 19.

\[ 2a_0a_{iq} = -B^{ij}_{pq} - B^{ij}_{pq} + a_{i(pq)} \det(A) \]
\[ = -B^{ij}_{pq} - B^{ij}_{pq} + a_{i(pq)} \det(A), \]
\[ 2a_0a_{ip} = -B^{ij}_{pq} - B^{ij}_{pq} - a_{i(pq)} \det(A) \]
\[ = -B^{ij}_{pq} - B^{ij}_{pq} - a_{i(pq)} \det(A), \]
\[ 2a_0a_{jq} = B^{ij}_{pq} + B^{ij}_{pq} + b_{i(pq)} \det(A) \]
\[ = B^{ij}_{pq} + B^{ij}_{pq} + b_{i(pq)} \det(A), \]
\[ 2a_0a_{jp} = B^{ij}_{pq} + B^{ij}_{pq} - b_{i(pq)} \det(A) \]
\[ = B^{ij}_{pq} + B^{ij}_{pq} - b_{i(pq)} \det(A). \]

(3.40)

4 From reflections in \( \mathbb{R}^{3,3} \) to rigid-body motions in \( \mathcal{E}^3 \)

Any non-null vector \( \mathbf{X} \in \mathbb{R}^{3,3} \) generate a reflection of \( \mathbb{R}^{3,3} \) by its adjoint action. The induced action in the null 3-spaces interchanges points and planes, so it is a 3-D projective polarity from \( \mathbb{R}^4 \) to \( (\mathbb{R}^3)^* \).

**Notations.** For any \( \mathbf{x} = (x, y, z) \in \mathbb{R}^3 \), denote

\[ \mathbf{E}(\mathbf{x}) := x \mathbf{E}_1 + y \mathbf{E}_2 + z \mathbf{E}_3, \]
\[ \mathbf{E}'(\mathbf{x}) := x \mathbf{E}'_1 + y \mathbf{E}'_2 + z \mathbf{E}'_3. \]

(4.1)

**Definition 12.** Any vector \( \mathbf{X} \in \mathbb{R}^{3,3} \) must be of the form \( \mathbf{E}(\mathbf{x}) + \mathbf{E}'(\mathbf{y}) \) for \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \). The vector

\[ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = (\mathbf{x}, \mathbf{y})^T \in \mathbb{R}^3 \times \mathbb{R}^3 \]

(4.2)

is called the **screw form** of vector \( \mathbf{X} \).

One advantage of the screw form is its capability of representing two different inner products simultaneously by the same dot symbol with no ambiguity. On one hand, \( \mathbb{R}^3 \) is naturally equipped with the standard Euclidean inner product. On the other hand, the inner product of two vectors \( \mathbf{X}_1 = \mathbf{E}(\mathbf{x}_1) + \mathbf{E}'(\mathbf{y}_1) \) and \( \mathbf{X}_2 = \mathbf{E}(\mathbf{x}_2) + \mathbf{E}'(\mathbf{y}_2) \) of \( \mathbb{R}^{3,3} \) is denoted by the same dot symbol. The two inner products are distinguished by the participating vectors, with the lower-case letters denoting vectors of \( \mathbb{R}^3 \), while the capitals denoting vectors of \( \mathbb{R}^{3,3} \).

We have

\[ \mathbf{X}_1 \cdot \mathbf{X}_2 = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_2 \\ \mathbf{y}_2 \end{pmatrix} = \mathbf{x}_1 \cdot \mathbf{y}_2 + \mathbf{y}_1 \cdot \mathbf{x}_2. \]

(4.3)

Vector \( \mathbf{X}_1 \) is invertible if and only if \( \mathbf{x}_1 \cdot \mathbf{y}_1 \neq 0 \). In \( \mathcal{A}(\mathbb{R}^4) \), \( \mathbf{E}'(\mathbf{x}) \) is the dual of \( \mathbf{E}(\mathbf{x}) \). In \( \mathbb{R}^{3,3} \),

\[ \mathbf{E}(\mathbf{x}) \cdot \mathbf{E}'(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}. \]

(4.4)
Notations. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ are the usual Euclidean inner product and vector cross product in vector algebra. For a $3 \times 3$ matrix $\mathbf{M} = (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ where each $\mathbf{m}_i \in \mathbb{R}^3$, denote

$$\mathbf{x} \times \mathbf{M} := (\mathbf{x} \times \mathbf{m}_1, \mathbf{x} \times \mathbf{m}_2, \mathbf{x} \times \mathbf{m}_3). \quad (4.5)$$

Alternatively, matrix $\mathbf{x} \times \mathbf{M}$ is defined by

$$(\mathbf{x} \times \mathbf{M}) \mathbf{y} = \mathbf{x} \times (\mathbf{M} \mathbf{y}), \quad \text{for } \mathbf{y} \in \mathbb{R}^3. \quad (4.6)$$

For example, when $\mathbf{x} = (x, y, z)^T$,

$$\mathbf{x} \times \mathbf{I}_{3 \times 3} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}. \quad (4.7)$$

Let $\mathbf{U} = (\mathbf{x}, \mathbf{y})^T \in \mathbb{R}^{3,3}$ be invertible, then for any $\mathbf{V} = (\mathbf{p}, \mathbf{q})^T \in \mathbb{R}^{3,3}$,

$$\text{Ad}_{\mathbf{U}}^*(\mathbf{V}) = \mathbf{V} - 2 \frac{\mathbf{V} \cdot \mathbf{U}}{\mathbf{U} \cdot \mathbf{U}} \mathbf{U} = \begin{pmatrix} \mathbf{p} - \frac{\mathbf{x} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{y}}{\mathbf{x} \cdot \mathbf{y}} & \mathbf{x} \cdot \mathbf{y} \\ -\mathbf{y} & 0 \end{pmatrix}. \quad (4.8)$$

By direct computation, we get

**Proposition 20.** For invertible $\mathbf{U} = (\mathbf{x}, \mathbf{y})^T \in \mathbb{R}^{3,3}$,

$$(\wedge^3 \text{Ad}_{\mathbf{U}}^*) \mathbf{J}_3 = -\mathbf{F}(\mathbf{x}) \mathbf{x} \cdot \mathbf{y}, \quad (\wedge^3 \text{Ad}_{\mathbf{U}}^*) \mathbf{F}(\mathbf{x}) = - (\mathbf{x} \cdot \mathbf{y}) \mathbf{J}_3. \quad (4.9)$$

In other words, the plane at infinity is mapped to the point at infinity $\mathbf{x}$ by the adjoint action of $\mathbf{U}$. Furthermore, with respect to the basis $\mathbf{e}_i$ of $\mathbb{R}^4$ and $\mathbf{e}_j^*$ of $(\mathbb{R}^4)^*$, the $4 \times 4$ matrix form of $\text{Ad}_{\mathbf{U}}^*$ of unit determinant is

$$\pm \frac{1}{\sqrt{|\mathbf{x} \cdot \mathbf{y}|}} \begin{pmatrix} 0 & -\mathbf{y}^T \\ \mathbf{y} & \mathbf{x} \times \mathbf{I}_{3 \times 3} \end{pmatrix}. \quad (4.10)$$

The matrix is skew-symmetric. Its inverse-transpose is

$$\pm \frac{1}{\sqrt{|\mathbf{x} \cdot \mathbf{y}|}} \begin{pmatrix} 0 & -\mathbf{x}^T \\ \mathbf{x} & \mathbf{y} \times \mathbf{I}_{3 \times 3} \end{pmatrix}. \quad (4.11)$$

$\text{Ad}_{\mathbf{U}}^*$ induces an affine transformation of $\mathcal{E}^3$ if and only if it preserves the plane at infinity. In $\Lambda(\mathbb{R}^{3,3})$, the plane at infinity is represented by $\mathbf{J}_3 = \mathbf{E}_{1\,2\,3}$. The following is a direct corollary of the above proposition.

**Corollary 2.** Let $\mathbf{U}_i = (\mathbf{x}_i, \mathbf{y}_i)^T \in \mathbb{R}^{3,3}$ for $i = 1, 2$ be invertible. Then $\text{Ad}_{\mathbf{U}_1, \mathbf{U}_2}^*$ induces an affine transformation of $\mathcal{E}^3$ if and only if $\mathbf{x}_1 \times \mathbf{x}_2 = 0$. If $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$, then the affine transformation is volume-preserving if and only if $\mathbf{x} \cdot \mathbf{y}_1 = \mathbf{x} \cdot \mathbf{y}_2$. 

Definition 13. Any element \( M \in \Lambda^2(\mathbb{R}^4) = \mathbb{R}^{3,3} \) induces a linear transformation of \( \mathbb{R}^4 \) as following:

\[
\mathcal{L}(M) : X \in \mathbb{R}^4 \mapsto M \cdot X \in \mathbb{R}^4, \tag{4.12}
\]

where the inner product is in \( \mathcal{C}(\mathbb{R}^4) \). It is called the **bilinear form** of vector \( M \in \mathbb{R}^{3,3} \).

The matrix form of the transformation \( \mathcal{L}(M) \) is obviously skew-symmetric. The following lemma is by direct verification:

Lemma 12. For any \( M = (x, y)^T \in \Lambda^2(\mathbb{R}^4) \),

\[
\mathcal{L}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = -\begin{pmatrix} 0 & -x^T \\ x & y \times I_{3 \times 3} \end{pmatrix}. \tag{4.13}
\]

When \( M \in \mathbb{R}^{3,3} \) is invertible, the above matrix is exactly the \( 4 \times 4 \) matrix of \( \text{Ad}^*_M \) from the space of planes \( \mathbb{R}^4 \) to the space of points \( \mathbb{R}^4 \).

The inner product in the space of \( 4 \times 4 \) matrices is that of the embedded space \( \mathbb{R}^{16} \) of components. For any \( M_1, M_2 \in \mathbb{R}^{3,3} \),

\[
\mathcal{L}(M_1) \cdot \mathcal{L}(JM_2) := \text{tr} \left((\mathcal{L}(M_1))^T(\mathcal{L}(JM_2))\right) = 2(M_1 \cdot M_2). \tag{4.14}
\]

Corollary 3. Let \( M = (x, y)^T \in \mathbb{R}^{3,3} \). When \( x \cdot y \neq 0 \), then \( \ker(\mathcal{L}(M)) = 0 \); when \( x \cdot y = 0 \) but \( x, y \) are not both zero, then \( \ker(\mathcal{L}(M)) \) is the 2-space of \( \mathbb{R}^4 \) containing all the projective points of the line with Plücker coordinates \( (x, y)^T \).

An element of \( \text{Spin}(3, 3) \) in factored form with respect to the Clifford product has the following nice property:

Proposition 21. Let \( U = V_1 V_2 \cdots V_r \in \text{Pin}(3, 3) \), where each \( V_i \in \mathbb{R}^{3,3} \) is invertible. Let \( \tilde{U} \) be obtained from \( U \) by replacing the odd-positioned vector factors \( V_r, V_{r-2}, \ldots \) counted from the right in the product with \( JV_r, JV_{r-2}, \ldots \). Let the result be \( \tilde{U} = W_1 W_2 \cdots W_r \). Then the \( 4 \times 4 \) matrix form of \( \text{Ad}^*_U \) is up to scale that of the following transformation:

\[
X \in \mathbb{R}^4 \mapsto W_1 \cdot (W_2 \cdot (\cdots \cdot (W_r \cdot X))) \in \mathbb{R}^4, \tag{4.15}
\]

where the inner product is in \( \mathcal{C}(\mathbb{R}^4) \).

Definition 14. Let \( U = X_1 X_2 \cdots X_r \in \text{Pin}(3, 3) \). The **inverse-transpose** of \( U \) is the following element of \( \text{Pin}(3, 3) \):

\[
U^T := (JX_1)(JX_2) \cdots (JX_r). \tag{4.16}
\]

Lemma 13. Let \( U \in \text{Pin}(3, 3) \). Then as matrices in \( O(3, 3) \),

\[
(\text{Ad}^*_U)^{-T} = \text{Ad}^*_{U^T}. \tag{4.17}
\]
Proposition 22. Let \( x \in \mathbb{R}^3 \) be nonzero, and \( y \in \mathbb{R}^3 \) satisfies \( x \cdot y = 0 \). The rotation of angle \( \pi \) about the axis passing through point \( y \) in direction \( x \) is generated by two reflections induced by the following vectors sequentially (from right to left):

\[
\begin{pmatrix} x \\ \lambda x + x \times y \end{pmatrix}, \quad \begin{pmatrix} x \\ -\lambda x + x \times y \end{pmatrix},
\]

(4.22)
where $\lambda \neq 0$ and is arbitrary. Except for such rotations in $\mathbb{E}^3$, no other rigid-body motion is generated by two reflections in $\mathbb{R}^{3,3}$.

There are five kinds of 2-planes in $\mathbb{R}^{3,3}$ that are spanned by invertible vectors, classified by the signature of the 2-plane: $\mathbb{R}^{2,0,0}, \mathbb{R}^{0,2,0}, \mathbb{R}^{1,0,1}, \mathbb{R}^{0,1,1}, \mathbb{R}^{1,1,0}$. We use them to give the normal form of any element of $SL(4)$ that is the product of two invertible skew-symmetric matrices, under the matrix similarity transformations of $SL(4)$.

**Definition 15.** Two matrices $M, N \in SL(4)$ are said to be conjugate in $SL(4)$, if there exists a matrix $C \in SL(4)$ such that $M = CNC^{-1}$.

**Lemma 14.** Let $U, U_0, V, V_0 \in \mathbb{R}^{3,3}$ be invertible, such that $U^2 = U_0^2$, and $U \wedge V$ has the same signature with $U_0 \wedge V_0$. Then there exists $g \in SO_0(3,3)$ such that $g(U) = U_0$, and $g(V) \in U_0 \wedge V_0$.

**Proof.** Obviously there are two 4-spaces $\mathcal{V}^4$ and $\mathcal{V}_0^4$ of $\mathbb{R}^{3,3}$, each having signature $2,2$, such that $U, V \in \mathcal{V}^4$ and $U_0, V_0 \in \mathcal{V}_0^4$, and the orthogonal map $h$ defined on $U \wedge V$ by $h(U) = U_0$ and $h(V) \in U_0 \wedge V_0$ can be extended to an orthogonal map from $\mathcal{V}^4$ to $\mathcal{V}_0^4$, still denoted by $h$.

Let $\mathcal{V}^2$ be the orthogonal complement of $\mathcal{V}^4$ in $\mathbb{R}^{3,3}$, and let $\mathcal{V}_0^2$ be the orthogonal complement of $\mathcal{V}_0^4$ in $\mathbb{R}^{3,3}$. Let $v_+, v_-$ be an orthogonal basis of $\mathcal{V}^2$, such that $v_+^2 = 1$ and $v_-^2 = -1$, and let $v_{0+}, v_{0-}$ be an orthogonal basis of $\mathcal{V}_0^2$ such that $v_{0+}^2 = 1$ and $v_{0-}^2 = -1$. Extend $h$ to an orthogonal transformation of $\mathbb{R}^{3,3}$ by setting $h(v_+) = v_{0+}$ and $h(v_-) = v_{0-}$. Set $g = h$ on $\mathcal{V}^4$, and

- if $h$ is the composition of the reflections with respect to even number of positive vectors and even number of negative vectors, set $g(v_+) = v_{0+}$ and $g(v_-) = v_{0-}$;
- if $h$ is the composition of the reflections with respect to odd number of positive vectors and odd number of negative vectors, set $g(v_+) = -v_{0+}$ and $g(v_-) = -v_{0-}$;
- if $h$ is the composition of the reflections with respect to even number of positive vectors and odd number of negative vectors, set $g(v_+) = v_{0+}$ and $g(v_-) = -v_{0-}$;
- if $h$ is the composition of the reflections with respect to odd number of positive vectors and even number of negative vectors, set $g(v_+) = -v_{0+}$ and $g(v_-) = v_{0-}$.

Then $g$ is the composition of the reflections with respect to even number of positive vectors and even number of negative vectors, so $g \in SO_0(3,3)$.

**Lemma 15.** Let $U = X_1X_2 \cdots X_{2k} \in \text{Spin}(3,3)$, where each $X_i \in \mathbb{R}^{3,3}$ is invertible. Let $g \in SO_0(3,3)$ be the Plücker transform of $G \in SL(4)$. Denote $g(U) := (gX_1)(gX_2) \cdots (gX_{2k})$. Let the $4 \times 4$ matrix forms of $\text{Ad}_U^g$ and $\text{Ad}_U^{g(U)}$ be $A, C \in SL(4)$ respectively. Then

\[ A = GC G^{-1}. \] (4.23)
Proof. Denote \( gX_i = Y_i \) for \( 1 \leq i \leq 2k \). Let \( g = Ad^*_W \) for some \( W \in Spin(3,3) \), then \( g^{-1} = Ad^*_{W^{-1}} \). We have

\[
Ad_U(X) = (g^{-1}(Y_1Y_2\cdots Y_{2k}))X(g^{-1}(Y_1Y_2\cdots Y_{2k}))^{-1}
\]
\[
= W(Y_1Y_2\cdots Y_{2k})W^{-1}XW(Y_1Y_2\cdots Y_{2k})^{-1}W^{-1}
\]
\[
= (Ad^*_W \circ Ad^*_{g(U)} \circ Ad^*_{W^{-1}})X.
\]

□

**Proposition 23.** Let \( A \) be the product of two skew-symmetric matrices of \( SL(4) \). Then \( A \) is conjugate in \( SL(4) \) to one and only one of the following block diagonal matrices up to scale:

1. 
\[
\begin{pmatrix}
  c & -s & 0 & 0 \\
  s & c & 0 & 0 \\
  0 & 0 & c & -s \\
  0 & 0 & s & c
\end{pmatrix}, \quad \text{where } c = \cos \theta, \text{ and } s = \sin \theta \neq 0.
\]
\[ (4.24) \]

2. 
\[
\begin{pmatrix}
  c & s & 0 & 0 \\
 -s & c & 0 & 0 \\
  0 & 0 & c & -s \\
  0 & 0 & s & c
\end{pmatrix}, \quad \text{where } c = \cos \theta, \text{ and } s = \sin \theta \neq 0.
\]
\[ (4.25) \]

3. 
\[
\begin{pmatrix}
  1 & -\lambda & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & \lambda & 1
\end{pmatrix}, \quad \text{where } \lambda \neq 0.
\]
\[ (4.26) \]

4. 
\[
\begin{pmatrix}
  1 & \lambda & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & \lambda & 1
\end{pmatrix}, \quad \text{where } \lambda \neq 0.
\]
\[ (4.27) \]

5. 
\[
\text{diag}(\lambda^{-1}, \lambda^{-1}, \lambda, \lambda), \quad \text{where } \lambda > 0.
\]
\[ (4.28) \]

6. 
\[
\text{diag}(\lambda^{-1}, \lambda^{-1}, -\lambda, -\lambda), \quad \text{where } \lambda > 0.
\]
\[ (4.29) \]

Proof. When \( A = \pm I_{4\times 4} \), then it belongs to the class \((4.28)\). Below we assume \( A \neq \pm I_{4\times 4} \). Then \( A \) must be the \( 4 \times 4 \) matrix form of \( Ad^*_{UV} \) for some unit vectors \( U, V \in \mathbb{R}^{3,3} \), where \( U \wedge V \neq 0 \).

Case 1. If \( U \wedge V \) has signature \( \mathbb{R}^{2,0,0} \), choose

\[
V_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} e_2 \\ e_2 \end{pmatrix}, \quad U_0 = \frac{c}{\sqrt{2}} \begin{pmatrix} e_2 \\ e_2 \end{pmatrix} + \frac{s}{\sqrt{2}} \begin{pmatrix} e_3 \\ e_3 \end{pmatrix},
\]
\[ (4.30) \]
where \( c = \cos \theta \), and \( s = \sin \theta \neq 0 \). They are both unit vectors, and \( U_0 \wedge V_0 \) has signature \( \mathbb{R}^{2,0,0} \). By Lemma 14, there exists an element \( g \in SO_0(3,3) \) such that \( g(V) = V_0 \), and \( g(U) \in U_0 \wedge \begin{pmatrix} e_3 \\ e_3 \end{pmatrix} \). Obviously there exists a parameter \( \theta \) such that \( g(U) = U_0 \). So the \( 4 \times 4 \) matrix form of \( Ad_{UV}^* \) is conjugate in \( SL(4) \) to that of \( Ad_{U_0V_0}^* \). By (4.18), the \( 4 \times 4 \) matrix form of \( Ad_{U_0V_0}^* \) is (4.24).

Case 2. If \( U \wedge V \) has signature \( \mathbb{R}^{0,2,0} \), then choose unit vectors

\[
V_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix}, \quad U_0 = \frac{c}{\sqrt{2}} \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix} + \frac{s}{\sqrt{2}} \begin{pmatrix} e_3 \\ -e_3 \end{pmatrix},
\]

where \( c = \cos \theta \), and \( s = \sin \theta \neq 0 \). Obviously \( U_0 \wedge V_0 \) has signature \( \mathbb{R}^{0,2,0} \), so an element \( g \in SO_0(3,3) \) maps \( V \) to \( V_0 \), and maps \( U \) to \( U_0 \) for some parameter \( \theta \). The \( 4 \times 4 \) matrix form of \( Ad_{U_0V_0}^* \) is (4.25).

Case 3. If \( U \wedge V \) has signature \( \mathbb{R}^{1,0,1} \), then choose unit vectors

\[
V_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix}, \quad U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix} + \frac{\lambda}{\sqrt{2}} \begin{pmatrix} e_3 \\ 0 \end{pmatrix},
\]

where \( \lambda \neq 0 \). Since \( U_0 \wedge V_0 \) has signature \( \mathbb{R}^{1,0,1} \), an element \( g \in SO_0(3,3) \) maps \( V \) to \( V_0 \), and maps \( U \) to \( U_0 \) for some parameter \( \lambda \). The \( 4 \times 4 \) matrix form of \( Ad_{U_0V_0}^* \) is (4.26).

Case 4. If \( U \wedge V \) has signature \( \mathbb{R}^{0,1,1} \), then choose unit vectors

\[
V_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix}, \quad U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix} + \frac{\lambda}{\sqrt{2}} \begin{pmatrix} e_3 \\ 0 \end{pmatrix},
\]

where \( \lambda \neq 0 \). The \( 4 \times 4 \) matrix form of \( Ad_{U_0V_0}^* \) is (4.27).

Case 5. If \( U \wedge V \) has signature \( \mathbb{R}^{1,1,0} \), there are four subcases: (1) \( U, V \) are both positive, (2) \( U \) is negative while \( V \) is positive, (3) \( U \) is positive while \( V \) is negative, (4) \( U, V \) are both negative.

Let \( V' \in U \wedge V \) be a unit vector orthogonal to \( V \). Then \( V, V' \) have opposite signatures, and \( U \cdot (V \wedge V') \) is a unit vector of \( U \wedge V \) having opposite signature to \( U \). Since

\[
UV = U(V \wedge V')(V \wedge V')V = -V^2 \{ U \cdot (V \wedge V') \} V',
\]

we have \( Ad_{UV}^* = Ad_{\{U \cdot (V \wedge V')\}V'}^* \). So we only need to consider the first two subcases by assuming \( V^2 = 1 \). Then \( (V')^2 = -1 \). Let \( U = aV + bV' \) where \( b \neq 0 \). Then \(|a^2 - b^2| = 1\), so \( a \neq \pm b \).

Choose unit vectors

\[
V_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \quad V'_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -e_1 \end{pmatrix}.
\]

An element \( g \in SO_0(3,3) \) maps \( V \) to \( V_0 \), and maps \( V' \) to \( V'_0 \), so it maps \( UV \) up to scale to

\[
\begin{pmatrix} (a + b)e_1 \\ (a - b)e_1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_1 \end{pmatrix}.
\]
The 4 × 4 matrix form of the adjoint action of (4.35) is
\[ \text{diag}(a + b, a + b, a - b, a - b). \]  (4.36)

Let \( \lambda = \sqrt{\frac{a+b}{a+b}} \). Then \( \lambda > 0 \).

Subcase 5.1. When \( U = 1 \), then \( a^2 - b^2 = 1 \). The 4 × 4 matrix form of the adjoint action of (4.35) is \( \text{diag}(\lambda^{-1}, \lambda^{-1}, \lambda, \lambda) \).

Subcase 5.2. When \( U = -1 \), then \( a^2 - b^2 = -1 \). The 4 × 4 matrix form of the adjoint action of (4.35) is \( \text{diag}(\lambda^{-1}, \lambda^{-1}, -\lambda, -\lambda) \).

\[ \text{Remark.} \] When \( \lambda \neq 0 \), then
\[ \begin{pmatrix} x \\ \lambda x \end{pmatrix} \begin{pmatrix} x \\ \mu x \end{pmatrix} = (\mu + \lambda) + (\mu - \lambda)E(x) \wedge E'(x). \]  (4.37)

For different parameters \( \lambda, \mu \), as long as the ratio \( \lambda : \mu \) is constant, then (4.37) gives the same spinor up to scale. In particular, as long as \( \lambda \neq 0 \), then
\[ \begin{pmatrix} x \\ \lambda x \end{pmatrix} \begin{pmatrix} x \\ -\lambda x \end{pmatrix} \] induces the same rotation of angle \( \pi \) about the axis in direction \( x \) at the origin.

The following is a general criterion on the “compressibility” of a spinor in factored form.

**Proposition 24.** Let \( U \) be the Clifford product of 4 invertible vectors of \( \mathbb{R}^{3,3} \). Then \( U \) can be written as the Clifford product of 2 invertible vectors of \( \mathbb{R}^{3,3} \) if and only if \( \langle U \rangle_4 = 0 \) and \( \langle U \rangle_2 \) has an invertible vector factor.

**Proof.** Let \( U = X_1 X_2 X_3 X_4 \) where each vector \( X_i \) is invertible. When \( U = Y_1 Y_2 \) for some other two invertible vectors \( Y_1, Y_2 \), the two conditions of the proposition are obviously true. Conversely, when both conditions hold, then the \( X_i \) are in some 3-space of \( \mathbb{R}^{3,3} \), so \( \langle U \rangle_2 \) must be decomposable into the outer product of two vectors. Let \( a, b \) be two invertible vector of \( \langle U \rangle_2 \) such that \( \langle U \rangle_2 = a \wedge b \). We claim that there are two invertible vectors \( Y_i = \lambda_i a + \mu_i b \) for \( i = 1, 2 \) such that \( U = Y_1 Y_2 \).

Expanding \( U = \langle U \rangle_0 + a \wedge b = (\lambda_1 a + \mu_1 b)(\lambda_2 a + \mu_2 b) \), we get
\[ \begin{align*}
\lambda_1 \mu_2 - \lambda_2 \mu_1 &= 0, \\
\lambda_1 \lambda_2 a^2 + \mu_1 \mu_2 b^2 + a \cdot b(\lambda_1 \mu_2 + \lambda_2 \mu_1) &= \langle U \rangle_0.
\end{align*} \]  (4.38)

When setting \( \lambda_2 = 0 \), then \( \mu_2 = \lambda_1^{-1} \) and \( \mu_1 = (\langle U \rangle_0 - a \cdot b)\lambda_1 b^{-2} \), so \( U = (a + (\langle U \rangle_0 - a \cdot b)b^{-1})b \). Since \( b \) and \( U \) are both invertible, so is \( Ub^{-1} \).

In the rest of this section, we construct 4-tuples of reflections in \( \mathbb{R}^{3,3} \) inducing rigid-body motions in \( \mathcal{E}^3 \). Affine transformation (4.19) is translation-free if and only if \( y_1 \times y_2 = 0 \). Consider the composition of two such affine transformations. Let
\[ \begin{align*}
U_1 &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, & V_1 &= \begin{pmatrix} x_1 \\ z_1 \end{pmatrix}, & U_2 &= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, & V_2 &= \begin{pmatrix} x_2 \\ z_2 \end{pmatrix},
\end{align*} \]  (4.39)
where \( y_i = \lambda_i z_i \) for \( i = 1, 2 \) such that \( \lambda_i \neq 1 \). For simplicity, assume \( x_i \cdot z_i = 1 \) for \( i = 1, 2 \).

The \( 4 \times 4 \) matrix form of \( Ad^*_U \cdot T_V U_2^* V_2 \) is up to scale the following:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\lambda_1 & 1 & 0 & 0 \\
0 & (1 - \lambda_1) & 0 & 0 \\
0 & 0 & (1 - \lambda_2) & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\lambda_2 & 1 & 0 & 0 \\
0 & (1 - \lambda_2) & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

(4.40)

where

\[
M = \lambda_1 \lambda_2 (1 - \lambda_1) x_2 z_2^T + \lambda_2 (1 - \lambda_2) x_1 z_1^T + (1 - \lambda_1) (1 - \lambda_2) (x_2 \cdot z_1) x_1 z_2^T.
\]

(4.41)

When \( M \) is the rotation matrix

\[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(4.42)

there is more than one way of constructing the four reflection vectors (4.39).

Option 1. Choose \( x_1 = e_1 \) and \( x_2 = e_2 \). Let \( z_1 = e_1 + a_2 e_2 + a_3 e_3 \) and \( z_2 = b_1 e_1 + e_2 + b_3 e_3 \). Comparing (4.41) and (4.42), we get

\[
\lambda_1 = \cos \theta, \quad \lambda_2 = \sec \theta, \quad a_3 = b_3 = 0, \quad a_2 = b_1 = -\cot \frac{\theta}{2}.
\]

(4.43)

So

\[
U_1 = \begin{pmatrix} e_1 \\ e_1 \cos \theta - e_2 \cos \theta \cot \frac{\theta}{2} \end{pmatrix},
\]

\[
V_1 = \begin{pmatrix} e_1 \\ e_1 \cos \frac{\theta}{2} - e_2 \cot \frac{\theta}{2} \end{pmatrix},
\]

\[
U_2 = \begin{pmatrix} e_2 \\ e_2 \sec \theta - e_1 \sec \theta \cot \frac{\theta}{2} \end{pmatrix},
\]

\[
V_2 = \begin{pmatrix} e_2 \\ e_2 \cot \frac{\theta}{2} - e_1 \cot \frac{\theta}{2} \end{pmatrix}.
\]

(4.44)

This 4-tuple of reflection vectors has the strong defect that the following is required: \( \theta \notin \{0, \pm \pi/2\} \).

Option 2. Set \( x_1 = e_1 \) and \( x_2 = c'e_1 + s'e_2 \), where \( c'^2 + s'^2 = 1 \). Set \( z_1 = e_1 + ae_2 \) and \( z_2 = pe_1 + qe_2 \) such that \( x_2 \cdot z_2 = cp + sq = 1 \). Set \( y_1 = \lambda z_1 \) and \( y_2 = \lambda^{-1} z_2 \) for some \( \lambda \neq 1 \). Again by comparing (4.41) and (4.42) and solving the equations, we get

\[
\lambda = -1, \quad a = 0, \quad s' = q = -\sin \frac{\theta}{2}, \quad c' = p = \cos \frac{\theta}{2}.
\]

(4.45)
So

\[
U_1 = \begin{pmatrix} e_1 \\ -e_1 \end{pmatrix},
\]

\[
V_1 = \begin{pmatrix} e_1 \\ e_1 \end{pmatrix},
\]

\[
U_2 = \begin{pmatrix} e_1 \cos \frac{\theta}{2} - e_2 \sin \frac{\theta}{2} \\ -e_1 \cos \frac{\theta}{2} + e_2 \sin \frac{\theta}{2} \end{pmatrix},
\]

\[
V_2 = \begin{pmatrix} e_1 \cos \frac{\theta}{2} - e_2 \sin \frac{\theta}{2} \\ e_1 \cos \frac{\theta}{2} - e_2 \sin \frac{\theta}{2} \end{pmatrix}.
\]

This 4-tuple of reflection vectors is perfect. It can be reformulated as follows:

**Proposition 25.** Let \( x_1, x_2 \) be unit vectors of \( \mathbb{R}^3 \). The rotation of \( \mathbb{R}^3 \) in the \( x_1x_2 \) plane with angle \( 2\angle(x_1, x_2) \) is generated by four reflections in \( \mathbb{R}^3 \) with respect to the following invertible vectors sequentially (from right to left):

\[
\begin{pmatrix} x_2 \\ -x_2 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}.
\]

(4.47)

**Remark.** The adjoint action of \( \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \) induces the rotation of angle \( \pi \) in the plane \( x_1^\perp \). It is a geometric fact that this rotation followed by the rotation in the plane \( x_2^\perp \) leads to the rotation of angle \( 2\angle(x_1, x_2) \) in the \( x_1x_2 \) plane.

Next consider the translation by vector \( t \). In the 4 vectors of (4.39), denote \( w_i = y_i - z_i \) for \( i = 1, 2 \), and assume \( x_i \cdot z_i = 1 \), then the matrix form of \( Ad_{U_1^T, V_1, U_2^T, V_2} \) is up to scale the following:

\[
\begin{pmatrix} 1 \\ -w_1 \times z_1 \end{pmatrix} (x_1 \cdot w_1 + 1) I_{3 \times 3} - x_1 w_1^T \begin{pmatrix} 1 \\ -w_2 \times z_2 \end{pmatrix} \begin{pmatrix} x_2 \cdot w_2 + 1 \end{pmatrix} I_{3 \times 3} - x_2 w_2^T \]

\[
= \begin{pmatrix} 1 & f_1 \\ f_2 & -1 \end{pmatrix},
\]

(4.48)

where

\[
f_1 = -w_1 \times z_1 - (x_1 \cdot w_1) w_2 \times z_2 - w_2 \times z_2 + x_1 |w_1 w_2 z_2|,
\]

\[
f_2 = (x_1 \cdot w_1 + 1) (x_2 \cdot w_2 + 1) I_{3 \times 3} - (x_1 \cdot w_1 + 1) x_2 w_2^T - (x_2 \cdot w_2 + 1) x_1 w_1^T + (x_2 \cdot w_1) x_1 w_2^T.
\]

(4.49)

Notice that in (4.49), vectors \( z_1, z_2 \) occur only in the translation part \( f_1 \). That (4.48) is a translation matrix if and only if \( f_2 = I_{3 \times 3} \). For simplicity, choose \( x_1 = x_2 = x \), and set \( x = e_1 \).

Option 1. To simplify the expression of \( f_2 \), let \( x \cdot w_i = 0 \) for \( i = 1, 2 \). Then equation \( f_2 = I_{3 \times 3} \) becomes

\[
-x (w_2 + (x \cdot w_1) w_2 - w_1)^T = 0,
\]

(4.50)

hence \( w_1, w_2 \) are linearly dependent, and we get \( w_2 = -w_1 \). Choose \( w_1 = e_2 \).
The translation vector is \( t = f_1 = -w_1 \times (z_1 - z_2) \). We can choose \( t \) to be in the direction of \( e_1 \), so that \( t = -\lambda e_1 \) for some scalar \( \lambda \). Choose \( z_2 = e_1 \), then \( z_1 = e_1 + \lambda e_3 \). The 4 reflection vectors inducing the translation by vector \(-\lambda e_1\) are the following from right to left:

\[
\begin{pmatrix}
  e_1 \\
  e_1 + \lambda e_3 + e_2 \\
  e_1 + \lambda e_3 \\
  e_1 - e_2 \\
  e_1 \\
\end{pmatrix}, \quad \begin{pmatrix}
  e_1 \\
  e_1 + \lambda e_3 \\
  e_1 - e_2 \\
  e_1 \\
\end{pmatrix}. \quad (4.51)
\]

This 4-tuple can be formulated in a coordinate-free form as following:

**Proposition 26.** Let \( x, y \) be nonzero vectors of \( \mathbb{R}^3 \) satisfying \( x \cdot y = 0 \). The translation of \( \mathbb{R}^3 \) by vector \( x \) is generated by four reflections induced by the following invertible vectors in \( \mathbb{R}^{3,3} \) sequentially (from right to left):

\[
\begin{pmatrix}
  x \\
  x - y + \frac{x \times y}{y^2} \\
  x + \frac{x \times y}{y^2} \\
  x + y \\
  x \\
\end{pmatrix}, \quad \begin{pmatrix}
  x \\
  x \times y \\
  x \times y \\
  x \times y \\
  x \\
\end{pmatrix}. \quad (4.52)
\]

**Remark.** The first two reflections generate the following affine transformation:

\[
\begin{pmatrix}
  1 \\
  -x \times y + x \\
  x \times y \\
  x \times y \\
  0 \\
\end{pmatrix} \begin{pmatrix}
  I_{3 \times 3} + xy^T \\
\end{pmatrix}; \quad (4.53)
\]

the last two reflections generate the following affine transformation:

\[
\begin{pmatrix}
  1 \\
  x \times y \\
  I_{3 \times 3} - xy^T \\
\end{pmatrix}. \quad (4.54)
\]

The two matrices are always commutative.

Geometrically, when setting \( |y| = 1 \), then (4.54) represents a planar shear transformation followed by a translation by vector \( x \times y \), where the shearing occurs in plane \((x \times y)^\perp\), with shear direction \( x \) and shear factor \(-|x|\); for any \( z \in \mathbb{R}^3 \), \( z \mapsto z - (z \cdot y)x + x \times y \). Similarly, (4.53) represents a planar shear transformation followed by a translation by vector \( x - x \times y \), where the shearing occurs in direction \( x \) of plane \((x \times y)^\perp\), while the shear factor is \( |x| \).

Option 2. Choosing \( x \cdot w_i = -2 \) for \( i = 1, 2 \) simplifies the expression of \( f_2 \) just the same. Then equation \( f_2 = I_{3 \times 3} \) becomes

\[
-x(w_2 + (x \cdot w_1)w_2 + w_1)^T = 0, \quad (4.55)
\]

hence \( w_1, w_2 \) are linearly dependent, and we get \( w_2 = w_1 \). Choose \( w_1 = w_2 = -2e_1 \).

The translation vector is \( t = f_1 = -w_1 \times (z_1 - z_2) \). We can choose \( t \) to be in the direction of \( e_3 \), so that \( t = -\lambda e_3 \) for some scalar \( \lambda \). Choose \( z_2 = e_1 \), then \( z_1 = e_1 - \lambda e_2/2 \). The 4 reflection vectors inducing the translation by vector \(-\lambda e_3\) are the following from right to left:

\[
\begin{pmatrix}
  e_1 \\
  -e_1 - \frac{\lambda}{2} e_2 \\
  e_1 - \frac{\lambda}{2} e_2 \\
  e_1 \\
\end{pmatrix}, \quad \begin{pmatrix}
  e_1 \\
  e_1 + \frac{\lambda}{2} e_2 \\
  e_1 - \frac{\lambda}{2} e_2 \\
  -e_1 \\
\end{pmatrix}. \quad (4.56)
\]

This 4-tuple is just as perfect as the previous option. Its coordinate-free formulation is the following:
Proposition 27. Let \( \mathbf{x}, \mathbf{y} \) be nonzero vectors of \( \mathbb{R}^3 \) satisfying \( \mathbf{x} \cdot \mathbf{y} = 0 \). The translation of \( \mathbb{R}^3 \) by vector \( \mathbf{x} \) is generated by four reflections induced by the following invertible vectors in \( \mathbb{R}^3 \), sequentially (from right to left):

\[
\begin{pmatrix}
\mathbf{y} \\
-\mathbf{y} + \frac{\mathbf{x} \times \mathbf{y}}{2}
\end{pmatrix}, \quad
\begin{pmatrix}
\mathbf{y} \\
\mathbf{y} + \frac{\mathbf{x} \times \mathbf{y}}{2}
\end{pmatrix}, \quad
\begin{pmatrix}
\mathbf{y} \\
-\mathbf{y}
\end{pmatrix}, \quad
\begin{pmatrix}
\mathbf{y}
\end{pmatrix}.
\]

Remark. The first two reflections generate the following affine transformation:

\[
\begin{pmatrix}
1 & 0 \\
\mathbf{x} & -I_{3 \times 3} + 2 \frac{\mathbf{y} \mathbf{y}^T}{\mathbf{y}^2}
\end{pmatrix};
\]

it is the rotation of angle \( \pi \) about the axis in direction \( \mathbf{y} \) at point \( \mathbf{x}/2 \in \mathbb{R}^3 \). The last two reflections generate the following affine transformation:

\[
\begin{pmatrix}
1 & 0 \\
0 & -I_{3 \times 3} + 2 \frac{\mathbf{y} \mathbf{y}^T}{\mathbf{y}^2}
\end{pmatrix}.
\]

It is the rotation of angle \( \pi \) about the axis in direction \( \mathbf{y} \) at the origin. The two matrices are not commutative.

We work out the spinor representation in factored form of a general rigid-body motion \((\mathbf{R}, \mathbf{t}) : \mathbf{x} \in \mathbb{R}^3 \mapsto \mathbf{R} \mathbf{x} + \mathbf{t} \), where \( \mathbf{R} \) is a rotation matrix, and \( \mathbf{t} \in \mathbb{R}^3 \). It is a classical result that a rigid-body motion in space is either a pure translation or a screw motion: a rotation about an affine line called the screw axis, and then a translation along the screw axis by a screw driving distance.

Lemma 16. Let \( \mathbf{R} \) be a rotation of angle \( \theta \) about the axis in unit direction \( \mathbf{v} \) at the origin, and let \( \mathbf{t} \cdot \mathbf{v} = 0 \). Then the rigid-body motion \( \mathbf{x} \mapsto \mathbf{R} \mathbf{x} + \mathbf{t} \) of \( \mathbb{R}^3 \) is a pure rotation, and the axis of rotation is the line in direction \( \mathbf{v} \) and passing through point

\[
\mathbf{c} = \frac{\mathbf{R} \mathbf{v} - \mathbf{t}}{2 \sin \frac{\theta}{2}},
\]

where \( \mathbf{R}_\alpha \) denotes the rotation of angle \( \alpha \) about unit direction \( \mathbf{v} \) at the origin.

Proof. We only need to prove that \( \mathbf{c} \) is a fixed point. Set

\[
\mathbf{v} = \mathbf{e}_3, \quad \mathbf{t} = t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2, \quad \mathbf{c} = c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2, \quad \mathbf{R} = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Substituting them into \( \mathbf{c} - \mathbf{R} \mathbf{c} = \mathbf{t} \), we get

\[
2s \begin{pmatrix}
\mathbf{s} & \mathbf{c} \\
-\mathbf{c} & \mathbf{s}
\end{pmatrix}
\begin{pmatrix}
\mathbf{c}_1 \\
\mathbf{c}_2
\end{pmatrix}
= \begin{pmatrix}
t_1 \\
t_2
\end{pmatrix},
\]

where \( \mathbf{c} = \cos(\theta/2) \) and \( \mathbf{s} = \sin(\theta/2) \). The solution is

\[
\begin{pmatrix}
\mathbf{c}_1 \\
\mathbf{c}_2
\end{pmatrix}
= \frac{1}{2s}
\begin{pmatrix}
\mathbf{s} & -\mathbf{c} \\
\mathbf{c} & \mathbf{s}
\end{pmatrix}
\begin{pmatrix}
t_1 \\
t_2
\end{pmatrix}.
\]
Corollary 4. Let $R$ be a rotation of angle $\theta$ about the axis in unit direction $v$ at the origin. Then the rigid-body motion $x \mapsto Rx + t$ of $\mathbb{R}^3$ is a screw motion, whose screw axis is in direction $v$ and passing through point $t$

$$c = \frac{R_{\alpha_2} (t - (t \cdot v)v)}{2 \sin \frac{\theta}{2}}. \quad (4.64)$$

In fact, point $c$ is the foot drawn from the origin to the screw axis, called the original center of the screw motion. The screw driving distance is $d = t \cdot v$.

Proof. Since $t = (t - (t \cdot v)v) + (t \cdot v)v$, point $c$ of (4.64) is first mapped to $Rc + (t - (t \cdot v)v) = c$ by (4.60), then mapped to $c + (t \cdot v)v$, so it is on the screw axis.

Consider a rigid-body motion $(R, t)$ where the rotation axis of $R$ is $e_3$. Let $e_1$ be rotated to $e_1 \cos \theta + e_2 \sin \theta$, and let $t = t_1 e_1 + t_2 e_2 + t_3 e_3$ be the translation vector. Motivated by the 4-reflection generators of pure rotation and pure translation, consider the 4 reflections generated by the following vectors:

$$U_1 = \begin{pmatrix} e_1 \\ y_1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} e_1 \\ y_2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} ce_1 - se_2 \\ q_1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} ce_1 - se_2 \\ q_2 \end{pmatrix},$$

where

$$y_1 = -e_1 + \alpha_3 e_2 - \alpha_2 e_3, \quad y_2 = e_1 + \alpha_3 e_2 - \alpha_2 e_3, \quad q_1 = -(ce_1 - se_2) + \beta_3 (se_1 + ce_2) - \beta_2 e_3, \quad q_2 = (ce_1 - se_2) + \beta_3 (se_1 + ce_2) - \beta_2 e_3. \quad (4.66)$$

The $4 \times 4$ matrix of the affine transformation induced by the above 4 reflections is

$$\begin{pmatrix} 1 & 0 \\ -2(\alpha_3 e_3 + \alpha_2 e_2) & \text{I}_{3 \times 3} + 2e_1 e_1^T \\ -2(\beta_3 e_3 + \beta_2 (se_1 + ce_2)) & \text{I}_{3 \times 3} + 2(ce_1 - se_2)(ce_1 - se_2)^T \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ t' & \text{R}' \end{pmatrix}, \quad (4.67)$$

where

$$t' = e_1(-2s\beta_2) + e_2(-2\alpha_2 + 2c\beta_2) + e_3(-2\alpha_3 + 2\beta_3), \quad R' = \text{I}_{3 \times 3} - 2e_1 e_1^T + 2(ce_1 + se_2)(ce_1 - se_2)^T = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.68)$$

By $t' = t$, we get

$$\alpha_2 = -\frac{ct_1 + st_2}{2s}, \quad \beta_2 = -\frac{t_1}{2s}, \quad -\alpha_3 + \beta_3 = \frac{t_3}{2}. \quad (4.69)$$
When \( s \neq 0 \), i.e., \( \theta \neq 0 \), we can set
\[
\beta_3 = 0, \quad \alpha_3 = -\frac{t_3}{2}. \tag{4.70}
\]
Then by (4.64) and \( d = t_3 \),
\[
\alpha_3 e_2 - \alpha_2 e_3 = e_1 \times (c + \frac{d}{2} e_1),
\beta_3 (se_1 + ce_2) - \beta_2 e_3 = (ce_1 - se_2) \times c. \tag{4.71}
\]
Alternatively, we can set
\[
\alpha_3 = 0, \quad \beta_3 = \frac{t_3}{2}. \tag{4.72}
\]
Then
\[
\alpha_3 e_2 - \alpha_2 e_3 = e_1 \times c,
\beta_3 (se_1 + ce_2) - \beta_2 e_3 = (ce_1 - se_2) \times (c - \frac{d}{2} e_3). \tag{4.73}
\]
When \( s = 0 \), the rigid-body motion is a translation, and there is no rotation plane \( e_1 e_2 \). Still as long as \( t_1 = 0 \), we get from (4.69) and (4.72) \( \beta_2 = \alpha_3 = 0 \), and \( \alpha_2 = -\frac{t_2}{2} \), and \( \beta_3 = \frac{t_3}{2} \). The 4 reflection vectors generating the translation are (from right to left):
\[
\left( e_1 - \frac{t_2}{2} e_3 \right), \quad \left( e_1 + \frac{t_2}{2} e_3 \right), \quad \left( -e_1 + \frac{t_3}{2} e_2 \right), \quad \left( e_1 + \frac{t_3}{2} e_2 \right). \tag{4.74}
\]
It gives the third 4-tuple of reflection vectors generating pure translation by \( t = t_2 e_2 + t_3 e_3 \) other than (4.51) and (4.56).

**Proposition 28.** Any rigid-body motion is induced by two pairs of reflections in \( \mathbb{R}^{3,3} \) such that each pair induces an affine transformation. For a rotation in a plane spanned by two orthonormal directions \( v_1, v_2 \) with angle \( \theta \neq 0 \), followed by a translation along vector \( t \), the corresponding 4 reflections are the following (from right to left):
\[
\left( cv_1 + sv_2 \right), \quad \left( cv_1 + sv_2 + m_1 \right), \quad \left( v_1 \right), \quad \left( v_1 + m_2 \right),
\tag{4.75}
\]
where \( c = \cos(\theta/2) \) and \( s = \sin(\theta/2) \), and we can choose either
\[
m_1 = (cv_1 + sv_2) \times (c + \frac{d}{2} n), \quad m_2 = v_1 \times c, \tag{4.76}
\]
or
\[
m_1 = (cv_1 + sv_2) \times c, \quad m_2 = v_1 \times (c - \frac{d}{2} n). \tag{4.77}
\]
Here \( c \) is the original center (4.60) of the screw motion, and \( d \) is the screw driving distance.
By (4.76), the first two reflections generate the rotation of angle \( \pi \) about the axis in direction \( c v_1 + s v_2 \) at point \( c + d n/2 \), and the last two reflections generate the rotation of angle \( \pi \) about the axis in direction \( v_1 \) at point \( c - d n/2 \). When \( \theta = 0 \), for the choice of \( v_1 \) such that \( t \cdot v_1 = 0 \), (4.75) is still valid by taking the limit of \( \theta \to 0 \).

**Corollary 5.** Any rigid-body motion is the composition of two 3-D rotations of angle \( \pi \).

Given a 3-D projective transformation or polarity, we see that the elements of \( Pin^p(3, 3) \) generating it in factored form is far from being unique. On the other hand, the expanded forms of the elements differ at most by a factor in \( \{ \pm 1, \pm I_{3,3} \} \). For a rigid-body motion, the exponential of a bivector of \( A^2(\mathbb{R}^{3,3}) \) provides a spinor that when expanded is equal to the spinors in different factored forms that we find in this section. This is the topic of the bivector representation of \( se(3) \), the Lie algebra of rigid-body motions.

### 5 Lie subalgebra of rigid-body motions and classical screw theory

First we recall some basic facts of the Lie algebra \( se(3) \). By direct verification, we know that

**Proposition 29.** Let \( R \) be the rotation matrix of angle \( \theta \) about the axis in unit direction \( v \) at the origin. Then

\[
\begin{pmatrix}
1 & 0 \\
\mathbf{t} & \mathbf{R}
\end{pmatrix} = \exp\begin{pmatrix}
0 & 0 \\
\mathbf{u} & \theta \mathbf{v} \times I_{3 \times 3}
\end{pmatrix},
\]

where

\[
\mathbf{u} = (\mathbf{t} \cdot \mathbf{v})\mathbf{v} + \frac{\theta}{2 \sin \frac{\theta}{2}} \mathbf{R}_{-\frac{\theta}{2}}(\mathbf{t} - (\mathbf{t} \cdot \mathbf{v})\mathbf{v}) = \theta \mathbf{c} \times \mathbf{v} + d \mathbf{v},
\]

and the original center \( c \) is given by (4.64), and \( d = \mathbf{t} \cdot \mathbf{v} \).

The Lie algebra \( se(3) \) is composed of \( 4 \times 4 \) matrices of the form

\[
\begin{pmatrix}
0 & 0 \\
\mathbf{t} & \mathbf{v} \times I_{3 \times 3}
\end{pmatrix},
\]

where \( \mathbf{t}, \mathbf{v} \in \mathbb{R}^3 \). Hence we can use a pair of vectors \( \mathbf{t}, \mathbf{v} \) to represent the matrix.

**Definition 16.** The map

\[
\begin{pmatrix}
0 & 0 \\
\mathbf{t} & \mathbf{v} \times I_{3 \times 3}
\end{pmatrix} \in se(3) \mapsto \begin{pmatrix}
\mathbf{v} \\
\mathbf{t}
\end{pmatrix} \in \mathbb{R}^{3,3}
\]

(5.3)

is called the **screw representation** of \( se(3) \). The Lie bracket of \( se(3) \) agrees with the following **cross product** of screw forms:

\[
\begin{pmatrix}
\mathbf{v}_1 \\
\mathbf{t}_1
\end{pmatrix} \times \begin{pmatrix}
\mathbf{v}_2 \\
\mathbf{t}_2
\end{pmatrix} := \begin{pmatrix}
\mathbf{v}_1 \times \mathbf{v}_2 \\
\mathbf{v}_1 \times \mathbf{t}_2 + \mathbf{v}_2 \times \mathbf{t}_1
\end{pmatrix}.
\]

(5.4)
By (5.1), the screw form of the element of $se(3)$ generating the motion $x \in \mathbb{R}^3 \mapsto Rx + t$ where $R \neq I_{3\times3}$, is
\[
\theta \left( v \times v + \lambda v \right),
\]
where $\lambda = d/\theta = (t \cdot v)/\theta$ is called the screw ratio. It is the ratio of the screw driving distance $d$ by the angle of rotation $\theta$. The screw motion is positive if $\lambda > 0$, in which case the screw driving direction and the rotation orientation follow the right-hand rule. The screw motion is negative if $\lambda < 0$, in which case the screw driving direction and the rotation orientation follow the left-hand rule. The screw motion is a pure rotation if $\lambda = 0$.

Any nonzero vector $X \in \mathbb{R}^{3,3}$ is up to scale either of the form
\[
\begin{pmatrix}
0 \\
t
\end{pmatrix} = E'(t),
\]
or of the form
\[
X = \begin{pmatrix}
1 \\
 c \times l + \lambda l
\end{pmatrix},
\]
where $l$ is a unit vector of $\mathbb{R}^3$, $c \in \mathbb{R}^3$ and $\mu \in \mathbb{R}$. In terms of infinitesimal generators of screw motions, (5.6) generates a pure rotation in direction $t$. As to (5.7), since $X^2 = 2\lambda$, the vector is null if $\lambda = 0$, and represents a line in direction $l$ and passing through point $c$, at the same time it is the generator of a pure rotation about the axis represented by the 6-D vector itself. The vector is positive if $\lambda > 0$, and generates a positive screw motion about the axis $(l, c \times l)^T$; the vector is negative if $\lambda > 0$, and represents a negative screw motion about the same axis. So the screw representation of $se(3)$ provides a motion-generator interpretation to all vectors of $\mathbb{R}^{3,3}$. In contrast, in line geometry only null vectors of $\mathbb{R}^{3,3}$ have geometric interpretation.

Traditionally, a “screw” refers to the screw form of a pure rotation or pure translation, i.e., a null vector of $\mathbb{R}^{3,3}$, while a general vector of $\mathbb{R}^{3,3}$ is called a twist, as it generates a screw motion. In this paper, we call the vectors unanimously a screw form.

The following proposition states that the cross product of the screw forms of two screw motions is the screw form of a third screw motion, the latter being along the common perpendicular of the two screw axes of the former two motions.

**Proposition 30.** For $i = 1, 2$, let
\[
X_i = \begin{pmatrix}
l_i \\
c_i \times l_i + \lambda_i l_i
\end{pmatrix} \in \mathbb{R}^{3,3},
\]
such that (1) each $l_i$ is a unit vector, (2) $c_i \cdot l_i = 0$, and (3) $l_1 \times l_2 \neq 0$. Let $y \in \mathbb{R}^3$ be an arbitrary point on the common perpendicular of the two lines $(l_1, c_i \times l_i)$, and let $d$ be the signed distance from the first line to the second along
The result is invariant under a shift of the subscripts 1, 2, 3. In particular, when
\[ \mathbf{q}_i = c_i \times \mathbf{l}_i + \lambda_i \mathbf{l}_i, \] the result is
\[ \mathbf{l}_1 \cdot (c_3 - c_2)(l_2 \cdot l_3) + \mathbf{l}_2 \cdot (c_1 - c_3)(l_3 \cdot l_1) + \mathbf{l}_3 \cdot (c_2 - c_1)(l_1 \cdot l_2) + (\lambda_1 + \lambda_2 + \lambda_3)[l_1 l_2 l_3]. \]
Lemma 17 (Decomposition of vectors of $\mathbb{R}^3$ with respect to the cross product). For any $x, y \in \mathbb{R}^3$, let $z \in \mathbb{R}^3$ be nonzero such that $z \cdot x = 0$ and $z \cdot y = 0$. Then
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \frac{1}{z^2} \begin{pmatrix} z \times x \\
z \times y \\
0
\end{pmatrix}.
\] (5.17)

Proposition 32 (Orthogonal decomposition of infinitesimal screw motions of $\mathbb{R}^4$). Let $v_1, v_2, v_3$ be an orthonormal basis of $\mathbb{R}^3$. Then
\[
\begin{pmatrix} v_1 \\
\lambda v_3 \times v_1 + \mu v_1
\end{pmatrix} = \begin{pmatrix} v_2 \\
\lambda v_3 \times v_2 + \mu v_3
\end{pmatrix}.
\] (5.18)

It states that an infinitesimal screw motion about the axis in direction $v_1$ and through point $\lambda v_3$, is the cross product of an infinitesimal rotation about the axis in direction $v_3 \times v_1$ and through point $\lambda v_3$, with an infinitesimal screw motion about the axis in direction $v_3$ and through the origin.

We come back to the definition of the cross product in $\mathbb{R}^{3,3}$. It has some limited covariance in $SO(3, 3)$. Consider $B \in SO(3, 3).$ Let its matrix form with respect to Witt decomposition $\mathbb{R}^{3,3} = I_3 \oplus J_3$ be $\begin{pmatrix} B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}$. Matrix $B$ is called affine if its induced transformation in $\mathbb{R}^4$ preserves the plane at infinity of $\mathcal{E}^3$, i.e., $B_{12} = 0$. When $B$ is affine, by $B^T J B = J$, we have
\[
B_{22} = B_{11}^{-T}, \quad B_{21}^T B_{11} = -B_{11}^T B_{21}.
\] (5.19)

When $B$ is affine, it is Euclidean if its induced transformation in $\mathbb{R}^4$ preserves the inner product of $\mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle$, i.e., matrix $B_{11}$ is orthogonal, and $B_{22} = B_{11}$. If $B$ induces a translation, then $B_{11} = B_{22} = \pm I_{3 \times 3}$; if it induces a rotation about the origin, then $B_{21} = 0$.

Proposition 33. The cross product is covariant under the Euclidean subgroup of $SO(3, 3)$ inducing rigid-body motions of $\mathcal{E}^3$, i.e., for any
\[
M = \begin{pmatrix} R & 0 \\
C & R
\end{pmatrix} \in SO(3, 3),
\] (5.20)

where $R$ is a rotation matrix, and $C = -AC^T A$.

\[
M \begin{pmatrix} v_1 \\
t_1
\end{pmatrix} \times M \begin{pmatrix} v_2 \\
t_2
\end{pmatrix} = M \begin{pmatrix} v_1 \\
t_1
\end{pmatrix} \times M \begin{pmatrix} v_2 \\
t_2
\end{pmatrix}.
\] (5.21)

Proof. By direct computation, we get that the cross product is covariant under the subgroup of $SO(3, 3)$ inducing translations, and is also covariant under the subgroup of $SO(3, 3)$ inducing rotations at the origin. So it is covariant under the Euclidean subgroup.

In classical screw theory, a force is represented by a line in space, and a torque is represented by a line at infinity. For example, the force along line $(e_0 + x)l$
with magnitude \( f \) is represented by \( f(e_01 + x1) \in A^2(\mathbb{R}^4) \), the torque in the plane normal to unit vector \( n \) with scale \( t \) is represented by \( tn^\perp \in A^2(\mathbb{R}^4) \). The linear space spanned by forces and torques is the space of wrenches. A general wrench is represented by a vector of \( \mathbb{R}^{3,3} \), while a pure force or torque is represented by a null vector of \( \mathbb{R}^{3,3} \).

When an infinitesimal rigid-body motion is represented in screw form, i.e., as a vector of \( \mathbb{R}^{3,3} \), then the virtual work of a wrench and a screw form is defined as the inner product of the two corresponding vectors in \( \mathbb{R}^{3,3} \). Let \((f, q)^T\) be a wrench, where \( p \) is the force part and \( q \) is the torque part, and let \((v, u)^T\) be the screw form of a screw motion with axis direction \( v \). Then the virtual work between them is

\[
(f) \cdot (v) + q \cdot v = f \cdot u + q \cdot v.
\]  

(5.22)

We try to understand the virtual work in the \( 4 \times 4 \) matrix form of \( se(3) \). By \((4.12)\), any vector \( X \in \mathbb{R}^{3,3} \) corresponds to a bilinear form \( \mathcal{L}(X) \in sl(4) \). When \( X = (f, q)^T \) represents a wrench, then the inner product of \( \mathcal{L}(X) \) with the infinitesimal motion matrix \( (0, v \times I_{3\times3}) \) is

\[
- \begin{pmatrix} 0 & -f^T \\ f & q \times I_{3\times3} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ u & v \times I_{3\times3} \end{pmatrix} = -f \cdot u - 2(q \cdot v). \]  

(5.23)

It is almost the virtual work except for the ratio of the two part of works by the force and the torque respectively.

Since \( Spin(3,3) \) covers \( SL(4) \), the Lie algebra \( A^2(\mathbb{R}^{3,3}) \) of \( Spin(3,3) \) is isomorphic to \( sl(4) \). In particular, any element of \( se(3) \) has a bivector form in \( A^2(\mathbb{R}^{3,3}) \). On the other hand, a wrench is only a vector of \( \mathbb{R}^{3,3} \). How is the virtual work represented by some pairing between a bivector and a vector of \( A(\mathbb{R}^{3,3}) \)? Before embarking on the investigation of this problem, we make an analysis of the bivector representation of \( se(3) \) by computing the expanded form of the spinors \((4.46), (4.51), (4.75)\) inducing rigid-body motions and the corresponding exponential form.

**Proposition 34.** Let \( x = e_1 \) and \( y = c e_1 + s e_2 \), where \( c = \cos(\theta/2) \) and \( s = \sin(\theta/2) \), then

\[
\frac{1}{4} \begin{pmatrix} y & -y \\ y & -y \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = \exp\left(\frac{\theta}{2}(E_{21}' - E_{12}')\right).
\]

(5.24)

**Proof.** On one hand,

\[
\frac{1}{4} \begin{pmatrix} y & -y \\ y & -y \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = \left( (cE_1 + sE_2) \wedge (cE_1' + sE_2') \right) (E_1 \wedge E_1')
\]

\[
= c^2 + cs(E_{21}' - E_{12}') - s^2E_{121}'E_{12}'.
\]

(5.25)
On the other hand, let \( B = (E_{21'} - E_{12'})/2 \), let \( I = E_{12'21'} \), and let \( L = (1+I)/2 \). Then
\[
\begin{align*}
I^2 &= 1, \\
BI &= IB = B, \\
B^2 &= -L, \\
L^2 &= L.
\end{align*}
\]

So
\[
\exp(\theta B) = 1 + B\theta + \frac{-L\theta^2}{2!} + \frac{-B\theta^3}{3!} + \frac{L\theta^4}{4!} + \frac{B\theta^5}{5!} + \ldots
\]
\[
= B(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots) + 1 - L + L(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots)
\]
\[
= B\sin\theta + \frac{1 - I}{2} + \frac{1 + I}{2}\cos\theta
\]
\[
= \cos^2\frac{\theta}{2} - I\sin^2\frac{\theta}{2} + 2B\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}.
\]

**Proposition 35.** The translation along vector \( x \in \mathbb{R}^3 \) is induced by the following spinor of \( Cl(3,3) \):
\[
1 - \frac{E(x) \cdot J_3}{2} = \exp(-\frac{1}{2}E(x) \cdot J_3). \tag{5.27}
\]

**Proof.** Expanding the Clifford product of the four vectors in (4.51) from left to right, we get \( 4 + 2\lambda E_{21'}/2 \). The exponential form of spinor \( 1 + \lambda E_{21'}/2 \) is obviously \( \exp(\lambda E_{21'}/2) \). \( \square \)

**Definition 17.** Let \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Define the following linear maps from \( \mathbb{R}^3 \) to \( A^2(\mathbb{R}^{3,3}) \):
\[
\begin{align*}
EE : \quad x &= x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto x_1 E_{23} + x_2 E_{31} + x_3 E_{12}; \\
E'E' : \quad x &= x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto x_1 E_{2'3'} + x_2 E_{3'1'} + x_3 E_{1'2'}; \\
E_2 E' : \quad x &= x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto x_1 E_{23'} + x_2 E_{31'} + x_3 E_{12'}; \\
E_3 E' : \quad x &= x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto x_1 E_{32'} + x_2 E_{13'} + x_3 E_{21'}.
\end{align*}
\]

Obviously, \( EE(x) = E'(x) \cdot I_4 \), and \( E'E'(x) = E(x) \cdot J_3 \). Under a special orthogonal transformation of \( \mathbb{R}^3 \), the basis \( e_1, e_2, e_3 \) is changed into another basis, and the basis vectors \( E_4, E'_4 \) of \( \mathbb{R}^{3,3} \) change accordingly. With respect to the new basis, four new maps can be defined as above. The following invariance can be proved easily:

**Proposition 36.** The maps \( EE, E'E' \) and \( E_2 E' - E_3 E' \) are all invariant under \( SO(3) \). Furthermore, for any \( x_1, x_2 \in \mathbb{R}^3 \),
\[
(E_2 E' - E_3 E')(x_1) \cdot (E_2 E' - E_3 E')(x_2) = -2(x_1 \cdot x_2). \tag{5.29}
\]
The elements of $E'E'(x)$ for $x \in \mathbb{R}^3$ form a Lie subalgebra of $sl(4)$; it is the Lie subalgebra $\mathbb{R}^3$ of translations. The elements of $(E_xE' - E_xE')(x)$ for $x \in \mathbb{R}^3$ form the Lie subalgebra $so(3)$. It is easy to prove that any nonzero element of $so(3)$ in bivector form is not the outer product of two vectors of $\mathbb{R}^3$.

Consider the expanded form of the Clifford product of the 4 vectors in (4.75):

$$
4 \left( E_{11'} - \frac{t_1}{2} E_{1'2'} + \frac{c t_1 - s t_2}{2 s'} E_{3'1'} \right) \left( c'^2 E_{11'} + s^2 E_{22'} + c' s' (E_{12'} + E_{21'}) + \frac{c t_1}{2 s'} E_{3'1'} - \frac{t_1}{2} E_{2'3'} \right) \right.
= 4 \left( c'^2 - s^2 E_{12'21'} - \frac{s}{2} (st_1 + ct_2) E_{11'2'3'} - \frac{s}{2} (st_2 - ct_1) E_{21'2'3'} \right.
+ c s (E_{12'} - E_{21'}) - \frac{t_3}{2} E_{1'2'} + \frac{c}{2} (st_2 - ct_1) E_{2'3'} - \frac{c}{2} (st_1 + ct_2) E_{3'1'}\right). \tag{5.30}
$$

After some computation, we get that (5.30) equals $\exp(B_2)$, where if $\theta \neq 0$, then

$$
B_2 = \frac{\theta}{2} (E_{21'} - E_{12'}) - \frac{t_3}{2} E_{1'2'} - \frac{\theta}{4 \sin \frac{\theta}{2}} \left( (t_1 \cos \frac{\theta}{2} + t_2 \sin \frac{\theta}{2}) E_{2'3'} \right. + \left. (-t_1 \sin \frac{\theta}{2} + t_2 \cos \frac{\theta}{2}) E_{3'1'} \right). \tag{5.31}
$$

If $\theta = 0$, the above expression is still valid by (5.27), as the limit of the right side when $\theta \to 0$ is $-(t_1 E_{2'3'} + t_2 E_{3'1'})/2$, and $e_1, e_2$ are selected so that the translation vector $t \in$ the $e_1 e_2$ plane.

**Proposition 37.** The rigid-body motion $x \mapsto Rx + t$, where $R$ is the rotation matrix of angle $\theta$ about the axis in unit direction $v$ at the origin, and $t \in \mathbb{R}^3$, corresponds to the following spinor of $\mathbb{R}^{3,3}$, where $c$ is the original center defined by (4.64), and $d = t \cdot v$ is the screw driving distance:

$$
\exp \frac{1}{2} \{(E_xE' - E_xE')(\theta v) - E'E'(\theta c \times v + dv)\}. \tag{5.32}
$$

The Lie algebra se(3) in bivector form is spanned by $E'E'(x)$ and $(E_xE' - E_xE')(y)$ for $x, y \in \mathbb{R}^3$.

**Proposition 38.** The cross product of $\mathbb{R}^{3,3}$ is induced by the following trivector of $A^3(\mathbb{R}^{3,3})$:

$$
C_3 = \mathcal{F}(e_1) + \mathcal{F}(e_2) + \mathcal{F}(e_3) = E_{12'3'} + E_{23'1'} + E_{31'2'}, \tag{5.33}
$$

such that for any $X, Y \in \mathbb{R}^{3,3}$,

$$
X \times Y = -(X \wedge Y) \cdot C_3. \tag{5.34}
$$

Furthermore, for $X = (x, y)^T$,

$$
X \cdot C_3 = E'E'(y) - (E_xE' - E_xE')(x) \in se(3). \tag{5.35}
$$
Let \( L_2 = (E_2 E' - E_2 E')(x) - E' E'(y) \) be an element of \( se(3) \). The map

\[
L_2 \mapsto \begin{pmatrix} x \\ y \end{pmatrix}
\]

(5.36)
gives the screw form of bivector \( L_2 \in se(3) \). It is also called the \( se(3) \)-lift of vector \( x, y \).  

The transformation \( X \mapsto L_2 \cdot X \) for \( X \in \mathbb{R}^3 \) has the following \( 6 \times 6 \) matrix form:

\[
M := \begin{pmatrix} x \times I_{3 \times 3} & 0 \\ y \times I_{3 \times 3} & x \times I_{3 \times 3} \end{pmatrix}
\]

(5.37)

For \( X = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^{3,3} \),

\[
L_2 \cdot X = MX = \begin{pmatrix} x \times p \\ x \times q + y \times p \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \times \begin{pmatrix} p \\ q \end{pmatrix}.
\]

(5.38)

So the cross product of two vectors of \( \mathbb{R}^{3,3} \) equals the inner product of the \( se(3) \)-lift of the first vector with the second vector. When both vectors are lifted to bivectors by the \( se(3) \)-lift, then the cross product of the two bivectors of \( se(3) \) equals the cross product of the two original vectors of \( \mathbb{R}^{3,3} \).

**Proposition 39.** By the following image of \( C_3 \) under \( \wedge^3 \mathcal{J} \):

\[
D_3 := \mathcal{F}''(e_1^*) + \mathcal{F}''(e_2^*) + \mathcal{F}''(e_3^*) = E_{4'23} + E_{2'31} + E_{3'12},
\]

(5.39)

we have

\[
X \cdot D_3 = EE(x) - (E_2 E' - E_2 E')(y) \in so(3, 0, 1).
\]

(5.40)

Furthermore,

\[
\left( \begin{pmatrix} v \\ u \end{pmatrix} \cdot C_3 \right) \cdot \left( \begin{pmatrix} f \\ q \end{pmatrix} \cdot D_3 \right) = -f \cdot u - 2(q \cdot v).
\]

(5.41)

**Proof.** That \( X \cdot D_3 \in so(3, 0, 1) \), the Lie algebra of the special orthogonal group of \( \mathbb{R}^{3,0,1} \), will be made clear in the next section. All others are by simple computation. \( \Box \)

It is easy to prove that both \( C_3 \) and \( D_3 \) are invariant under any transformation of \( SO(I_3) \) and the induced transformation of \( SO(J_3) \) with the same matrix form. In this sense, the two trivectors are basis-independent.

By (5.23) and (5.41), for two vectors \( X, Y \in \mathbb{R}^{3,3} \), if \( X \) is lifted by \( C_3 \) to a bivector \( A_2 \) of \( se(3) \), and the other is lifted by \( D_3 \) to a bivector \( B_2 \) of \( so(3, 0, 1) \), then their inner product in \( A^2(\mathbb{R}^{3,3}) \) is the inner product of the \( 4 \times 4 \) matrix form of \( A_2 \in sl(4) \) with the bilinear form \( \mathcal{L}(Y) \). The lift by \( D_3 \) realizes a pairing between a vector and a bivector of \( se(3) \), and the result is almost the virtual work between a wrench and an infinitesimal screw motion.

We have the following more general result on the pairing:
Proposition 40. For \( C_3, D_3 \) given by (5.33) and (5.39),
\[
\left( \begin{array}{c} x_1 \\ y_1 \\
\end{array} \right) \cdot (C_3 + \lambda I_3) \cdot \left( \begin{array}{c} x_2 \\ y_2 \\
\end{array} \right) \cdot (D_3 + \lambda J_3) = -2x_1 \cdot y_2 - (1 + \lambda^2)y_1 \cdot x_2.
\]
(5.42)

Having talked enough of \( se(3) \), we consider the whole group of Euclidean transformations, i.e., affine transformations of the form \( x \in \mathbb{R}^3 \mapsto Gx + t \), where \( G \in O(3, 3) \). For matrix
\[
T := \text{diag}(-1, 1, 1),
\]
(5.43)
we have
\[
T \begin{pmatrix} 1 & 0 \\ t & G \\
\end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ -t & -G \\
\end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & G \\
\end{pmatrix} T = -\begin{pmatrix} 1 & 0 \\ t & -G \\
\end{pmatrix}.
\]
(5.44)

Both results are in \( SO(3, 3) \), so we can simply compute the spinor generator of each matrix, say \( U_l, U_r \) respectively, and then get the generator of \( x \mapsto Gx + t \) in \( T Spin(3, 3) \):
\[
T \begin{pmatrix} U_l \\ U_r \\
\end{pmatrix} = \begin{pmatrix} U_l \\ U_r \\
\end{pmatrix}
\]
Lemma 18. The screw motion of rotation angle \( \pi \) and screw driving distance \( d \) about the axis in normal direction \( n \) and passing though point \( c \in \mathbb{R}^3 \), where \( c \cdot n = 0 \), has the following \( 4 \times 4 \) matrix form:
\[
\begin{pmatrix} 1 & 0 \\ 2c + dn & 2mn^T - I_{3 \times 3} \\
\end{pmatrix}.
\]
(5.46)

Proof. By (4.64), the translation vector is \( t = 2c + dn \). The rotation matrix of angle \( \pi \) about the axis in direction \( n \) at the origin is \( 2mn^T - I_{3 \times 3} \). □

By (5.44), (5.45) and (5.46), we get

Corollary 6. The reflection with respect to plane \( (n, d) \) is the composition of \( T \) with the screw motion of angle \( \pi \) and screw driving distance \( 2d \) about the axis in direction \( n \) at the origin.

Proposition 41. The screw form of \( se(3) \) realizing the reflection with respect to plane \( (n, d) \) is \( (\pi n, 2dn)^T \in \mathbb{R}^{3 \times 3} \). The corresponding element of \( T Spin(3, 3) \) is the following:
\[
T \begin{pmatrix} v_2 \\ v_2 - dv_1 \\
\end{pmatrix} \begin{pmatrix} v_2 \\ v_2 - dv_1 \\
\end{pmatrix} \begin{pmatrix} v_1 \\ -v_1 \\
\end{pmatrix} \begin{pmatrix} v_1 \\ v_1 \\
\end{pmatrix},
\]
(5.47)
where \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{n} \) form an orthonormal basis of \( \mathbb{R}^3 \). When \( d = 0 \), the above element can be simplified to

\[
T \left( \begin{array}{c} \mathbf{n} \\ -\mathbf{n} \end{array} \right) \mathbf{n}.
\]

(5.48)

6 Other Lie subalgebras and corresponding screw forms

The 6-D Lie algebra \( \text{se}(3) \) of rigid-body motions provides an interpretation of vectors of \( \mathbb{R}^3 \) as screw forms of infinitesimal rigid-body motions. The inner product of \( \mathbb{R}^3 \) can be transferred to \( \text{se}(3) \), although incompatible with the inner product of the bivector representation of \( \text{se}(3) \); conversely, the cross product of \( \text{se}(3) \) in bivector form can be transferred to \( \mathbb{R}^3 \), and the result is the cross product in classical screw theory for line geometry. The transformation-generator interpretation of vectors of \( \mathbb{R}^3 \) is not necessarily restricted to the specific Lie subalgebra \( \text{se}(3) \). As long as there is a 6-D Lie subalgebra of \( \text{sl}(4) \), a screw representation may be assigned to the Lie subalgebra, and the Lie bracket of the Lie subalgebra may be translated to a new cross product of the screw forms.

First we investigate several typical 3-D Lie algebras of \( \text{sl}(4) \).

1. Perspectivity group:

A perspectivity \([13]\) has the following matrix form:

\[
\begin{pmatrix}
1 & \mathbf{x}^T \\
0 & \mathbf{I}_{3 \times 3}
\end{pmatrix}.
\]

(6.1)

It induces the following rational linear transformation on the line through the origin in unit direction \( \mathbf{l} \):

\[
\lambda \mathbf{l} \in \mathbb{R}^3 \mapsto \frac{\lambda}{1 + \lambda \mathbf{x} \cdot \mathbf{l}} \mathbf{l} \in \mathbb{R}^3.
\]

(6.2)

When \( \lambda \to \infty \), the point at infinity of the line is mapped to point \( 1/(\mathbf{x} \cdot \mathbf{l}) \). When \( \mathbf{x} \cdot \mathbf{l} = 0 \), every point on the line is invariant. The origin is fixed, so is every line through the origin.

Every point on the plane normal to \( \mathbf{x} \) and through the origin is fixed. Furthermore, draw a family of parallel planes normal to \( \mathbf{x} \), and let \( \lambda \) be the signed distance of the plane from the origin along \( \mathbf{x} \). Then the plane with signed distance \( \lambda \) is mapped to the plane with signed distance \( \lambda/(1 + \lambda d) \). In particular, the plane with signed distance \( -1/d \) is mapped to the plane at infinity, while the plane at infinity is mapped to the plane with signed distance \( 1/d \).

**Proposition 42.** The bivector generator of (6.1) in \( \Lambda^2(\mathbb{R}^3) \) is \( \mathbf{E}' \mathbf{E}'(\mathbf{x})/2 \). The Lie algebra of the group is isomorphic to \( \mathbb{R}^3 \).

2. Anisotropic dilation group:

The transformation \((1, t_1, t_2, t_3) \in \text{GL}(4)\) where \( t_1 t_2 t_3 > 0 \), is called an anisotropic dilation (or non-uniform scaling). When all the \( t_i > 0 \), the dilation is said to be positive. All anisotropic dilations of \( \mathcal{E}^3 \) form an Lie group,
denoted by \( AD(3); \) all positive anisotropic dilations of \( \mathcal{E}^3 \) form a Lie subgroup, denoted by \( AD^+(3). \) We have

\[
AD(3) = AD^+(3) \cup \text{diag}(1,1,-1,-1)AD^+(3) \\
\cup \text{diag}(1,-1,1,-1)AD^+(3) \cup \text{diag}(1,-1,1)AD^+(3). \tag{6.3}
\]

Lemma 19. Bivector \(-\lambda E_{11}/2\) induces the following transformation of \( \mathcal{E}^3: \)

\[
\text{diag}(1,1,e^{\lambda},e^{-\lambda}). \tag{6.4}
\]

The corresponding element of \( sl(3) \) is \( \text{diag}(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}). \)

Proof. Let \( U = \exp(\frac{\lambda}{2} E_{11}^r) = \cosh \frac{\lambda}{2} + E_{11}^r \sinh \frac{\lambda}{2}. \) Then \( \text{Ad}_U^* E_1 = e^{\lambda} E_1 \) and \( \text{Ad}_U^* E_{11} = e^{-\lambda} E_{11}, \) while all other basis vectors are fixed. By direct computing, we get that \( \text{Ad}_U^* \) has the following 4 \( \times \) 4 matrix form: \( \text{diag}(e^{\frac{\lambda}{2}}, e^{\frac{-\lambda}{2}}, e^{\frac{\lambda}{2}}, e^{\frac{-\lambda}{2}}). \)

Definition 18. Denote

\[
F_1 = \frac{E_{22} + E_{33} - E_{11}}{2}, \quad F_2 = \frac{E_{33} + E_{11} - E_{22}}{2}, \quad F_3 = \frac{E_{11} + E_{22} - E_{33}}{2}. \tag{6.5}
\]

The following are linear maps from \( \mathbb{R}^3 \) to \( \Lambda^2(\mathbb{R}^3)_3: \)

\[
\begin{align*}
E_1' & : \ (x_1, x_2, x_3)^T \mapsto x_1 (E_{22} + E_{33}) + x_2 (E_{33} + E_{11}) + x_3 (E_{11} + E_{22}); \\
E_2' & : \ (x_1, x_2, x_3)^T \mapsto x_1 F_1 + x_2 F_2 + x_3 F_3.
\end{align*}
\tag{6.6}
\]

Proposition 43. Any element of the Lie algebra of \( AD^+(3) \) is of the form \( \text{diag}(-(t_1 + t_2 + t_3), t_1, t_2, t_3), \) and the corresponding bivector of \( \Lambda^2(\mathbb{R}^3)_3 \) is

\[
-\frac{1}{2} \{ t_1 (E_{22} + E_{33}) + t_2 (E_{33} + E_{11}) + t_3 (E_{11} + E_{22}) \}. \tag{6.7}
\]

For example, the isotropic positive dilation \( \text{diag}(1,e^t,e^t,e^t) \) is generated by \(-t (E_{11} + E_{22} + E_{33}). \)

3. Upper triangular shear transformation group:

An upper triangular shear transformation of \( \mathcal{E}^3 \) by vector \( t = (t_1, t_2, t_3)^T \) has the following matrix form:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & t_3 & t_2 \\
0 & 0 & 1 & t_1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\tag{6.8}
\]

Proposition 44. \( (6.8) \) is generated by the following element of \( \Lambda^2(\mathbb{R}^3)_3: \)

\[
\frac{1}{2} (t_3 E_{12} + t_1 E_{23} + (t_2 - \frac{t_1 t_3}{2}) E_{13}). \tag{6.9}
\]
The corresponding element of $sl(4)$ is
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & t_3 & t_2 - \frac{t_1^2}{2} \\
0 & 0 & 0 & t_1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\tag{6.10}
\]

**Proof.** Let $H = x_3E_{12'} + x_1E_{23'} + x_2E_{13'}$. By $\exp(H) = 1 + H - x_1x_3E_{122'}$, we get the $6 \times 6$ matrix form of $\text{Ad}_{\exp(H)}^*$ as following:
\[
\begin{pmatrix}
1 & 2x_3 & 2(x_2 + x_1x_3) & 0 & 0 & 0 \\
0 & 1 & 2x_1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2x_3 & 1 & 0 \\
0 & 0 & 0 & -2(x_2 - x_1x_3) & -2x_1 & 1 \\
\end{pmatrix}.
\tag{6.11}
\]

The corresponding $4 \times 4$ matrix of $SL(4)$ is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 2x_3 & 2(x_2 + x_1x_3) \\
0 & 0 & 1 & 2x_1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

4. Other triangular shear transformation groups:

In the upper triangular shear transformation (6.8), basis vector $e_1$ is fixed, $e_2$ is sheared towards $e_1$, and $e_3$ is sheared towards plane $e_1e_2$. The roles of $e_1, e_2, e_3$ can be interchanged, resulting in five other branches of shear transformations. The six different branches of shear transformations are generated by the following six 3-D Lie subalgebras, each being represented by its basis:
\[
E_{12'}, E_{13'}, E_{23'}; \quad E_{12'}, E_{13'}, E_{32'}; \\
E_{21'}, E_{23'}, E_{31'}; \quad E_{21'}, E_{23'}, E_{13'}; \\
E_{31'}, E_{32'}, E_{12'}; \quad E_{31'}, E_{32'}, E_{21'}.
\tag{6.12}
\]

For example, the lower triangular shear transformation \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & t_3 & 1 & 0 \\
0 & t_2 & t_1 & 1 \\
\end{pmatrix}
\]
generated by the following element of $A^2(\mathbb{R}^{3,3})$:
\[
\frac{1}{2}(t_1E_{32'} + t_3E_{21'} + (t_2 - \frac{t_1t_3}{2})E_{31'}). 
\tag{6.13}
\]

5. 3-D Lie subalgebra $sl(2)$:

In $\mathbb{E}^3$ let $X \in \mathbb{R}^4$ represent a point, and let $\Pi \in (\mathbb{R}^4)^*$ represent a plane not incident to the point. The subgroup of $SL(4) \cup SL^{-}(4)$ fixing point $X$ and plane $\Pi$ is the group of 2-D projective transformations of the plane with respect to the point, and is isomorphic to $SL(3) \cup SL^{-}(3)$. 
This embedding of \( SL(3) \cup SL^{-}(3) \) into \( SL(4) \) may be suitable for modeling a pin-hole camera. Let \( \mathbf{C}, \mathbf{X} \in \mathbb{R}^4 \), and let \( \Pi \in A^4(\mathbb{R}^{3,3}) \) represent a plane, then the intersection of line \( \mathbf{CX} \) with the plane is represented by the bivector \( (\mathbf{CX}) \cdot \Pi \in A^2(\mathbb{R}^{3,3}) \). A null 2-space of \( \Pi \) represents a unique point on the plane. It can be taken as the local representation of the spatial point on the plane. The global representation of the point is \( (\mathbf{CX}) \land \langle (\mathbf{CX}) \cdot \Pi \rangle \in A^4(\mathbb{R}^{3,3}) \).

In the special case where the point is the origin, and the plane is the one at infinity, The 8-D Lie subalgebra \( sl(3) \) is spanned by the following elements of \( A^2(\mathbb{R}^{3,3}) \): \( \mathbf{E}(x) \land \mathbf{E}'(y) \) for \( x, y \in \mathbb{R}^3 \).

When vectors \( e_0, e_3 \) are fixed, and the 2-space \( \langle e_1, e_2 \rangle \) is also fixed, the corresponding Lie subgroup of \( SL(3) \) is \( SL(2) \). The Lie subalgebra \( sl(2) \) has the following bivector basis: \( \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{11}' - \mathbf{E}_{22}' \).

**Lemma 20.** The Plücker transform induces the following Lie algebraic isomorphism from \( sl(4) \) to \( so(3,3) \):

\[
M = \begin{pmatrix}
-\text{tr}(N) & n_0^T \\
m_0 & N
\end{pmatrix} \in sl(4) \mapsto \begin{pmatrix}
-\text{tr}(N)I_{3\times3} + N & -n_0 \times I_{3\times3} \\
m_0 \times I_{3\times3} & \text{tr}(N)I_{3\times3} - N^T
\end{pmatrix},
\]

where \( N = (m_1 \ m_2 \ m_3) \), and \( m_i, n_0 \in \mathbb{R}^3 \) for \( 0 \leq i \leq 3 \).

**Proof.** As \( M \) is the derivative of \( \exp(tM) \) at \( t = 0 \), for \( X, Y \in \mathbb{R}^4 \),

\[
(\exp(tM)X) \land (\exp(tM)Y) = (X + tMX + o(t)) \land (X + tMY + o(t)) = X \land Y + t(M \otimes I_{4\times4} + I_{4\times4} \otimes M)(X \land Y) + o(t),
\]

where

\[
M \otimes I_{4\times4}(X \land Y) = \frac{1}{2}M \otimes I_{4\times4}(X \otimes Y - Y \otimes X) = \frac{1}{2}(MX \otimes Y - MY \otimes X).
\]

So the lift of \( M \in sl(4) \) to \( so(3,3) \) is the \( 6 \times 6 \) matrix representation of the action of \( M \otimes I_{4\times4} + I_{4\times4} \otimes M \) upon \( A^2(\mathbb{R}^4) = \mathbb{R}^{3,3} \). For any positive permutation \( ijk \) of 123, by

\[
(M \otimes I_{4\times4} + I_{4\times4} \otimes M)e_{0i} = -\text{tr}(N)e_{0i} + m_0e_i + e_0m_i,
\]

\[
(M \otimes I_{4\times4} + I_{4\times4} \otimes M)e_{jk} = -n_0e_i + e_jm_k - e_km_j,
\]

we get (6.14).
Definition 19. The following are notations of some linear maps from $\mathbb{R}^3$ to $3 \times 3$ matrices:

$$\text{diag} : \quad x = x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto \text{diag}(x_1, x_2, x_3);$$

$$\text{skew} : \quad x = x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto \begin{pmatrix} 0 & x_3 & 0 \\ 0 & 0 & x_1 \\ x_2 & 0 & 0 \end{pmatrix};$$

$$(6.17)$$

$$\text{skew}^T : \quad x = x_1 e_1 + x_2 e_2 + x_3 e_3 \mapsto \begin{pmatrix} 0 & 0 & x_2 \\ x_3 & 0 & 0 \\ 0 & x_1 & 0 \end{pmatrix} = (\text{skew}(x))^T.$$  

Proposition 45. The Plücker transform and the adjoint action of $\text{Spin}(3, 3)$ induce the following The Lie algebraic isomorphism from $sl(4)$ to $A^2(\mathbb{R}^{3,3})$: let

$$N = \text{diag}(n_1) + \text{skew}(n_2) + \text{skew}^T(n_3),$$

and let $m_0, n_i \in \mathbb{R}^3$ for $0 \leq i \leq 3$, then

$$M = \begin{pmatrix} -\text{tr}(N) & n_0^T \\ m_0 & N \end{pmatrix} \in sl(4) \mapsto \frac{1}{2} \left\{ -E'E'(m_0) + EE(n_0) - E_-E'(n_1) + E<E'(n_2) + E>E'(n_3) \right\}.$$  

$$(6.19)$$

Proof. The right side of (6.19), denoted by $B_2 \in A^2(\mathbb{R}^{3,3})$, induces a linear transformation $X \in \mathbb{R}^{3,3} \mapsto B_2 \cdot X \in \mathbb{R}^{3,3}$. It has the same matrix form (6.14).

Definition 20. Given an $n \times n$ symmetric real matrix $K$, the set of matrices $M \in SL(n)$ satisfying $M^T KM = K$ is denoted by $SO(K)$, called the special orthogonal group with respect to $K$. Its Lie algebra is denoted by $so(K)$.

Proposition 46. Let $K$ be a $4 \times 4$ symmetric real matrix of rank $\geq 3$. Then $so(K)$ is isomorphic to one of $so(4), so(3, 1), so(3, 0, 1), so(2, 2), so(2, 1, 1)$.

Proof. Let $R \in O(4)$ be a matrix to diagonalize $K$ by similarity transformation: $R^T KR = L$, where $L$ is a real diagonal matrix whose nonzero entries are the eigenvalues of $K$. Let $T$ be a real diagonal matrix to change each nonzero entry of $L$ to $\pm 1$ by matrix congruent transformation: $T^T LT = N$, where $N$ is a real diagonal matrix whose nonzero entries are $\pm 1$. Up to sign, $N$ is one of
diag$(0, 1, 1, 1), \ \text{diag}(0, -1, 1, 1), \ \mathbf{I}_{4 \times 4}, \ \text{diag}(-1, 1, 1, 1), \ \text{diag}(-1, -1, 1, 1),$ and

$$(RT)^T K(RT) = N.$$  

$$(6.20)$$

Let $M \in so(K)$. Then $\exp(M^T)K\exp(M) = K$. It can be written in the following form by using (6.20):

$$\{ (RT)^{-1} \exp(M)(RT) \}^T N \{ (RT)^{-1} \exp(M)(RT) \} = N.$$  

$$(6.21)$$
So \((RT)^{-1} \exp(M)(RT) \in SO(N)\), and \(so(K)\) is isomorphic to \(so(N)\).

We start to discuss 6-D Lie subalgebras of type \(so(K)\).

6. \(so(4)\):
   It is composed of skew-symmetric matrices, \(i.e.,\) matrices of the form
   \[
   \begin{pmatrix}
   0 & x^T \\
   -x & y \times I_{3 \times 3}
   \end{pmatrix},
   \]
   (6.22)
   for \(x, y \in \mathbb{R}^3\). The following is a bivector basis of \(so(4)\):
   \[
   E_{2:3'} + E_{23}, \quad E_{3'1'} + E_{31}, \quad E_{1'2'} + E_{12}, \\
   E_{32'} - E_{23'}, \quad E_{13'} - E_{31'}, \quad E_{21'} - E_{12'}. 
   \]
   (6.23)
   For \(x = (x_1, x_2, x_3)^T\) and \(y = (y_1, y_2, y_3)^T\) of \(\mathbb{R}^3\), let
   \[
   Q(x) = x_1(E_{32'} - E_{23'}) + x_2(E_{13'} - E_{31'}) + x_3(E_{21'} - E_{12'}),
   \]
   \[
   P(y) = y_1(E_{23'} + E_{23}) + y_2(E_{31'} + E_{31}) + y_3(E_{12'} + E_{12}). 
   \]
   (6.24)
   If we denote by \(\begin{pmatrix} x \\ y \end{pmatrix}\) the bivector \(Q(x) + P(y)\), then
   \[
   \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \times_{so(4)} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \times x_2 + y_1 \times y_2 \\ x_1 \times y_2 + y_1 \times x_2 \end{pmatrix},
   \]
   (6.25)
   where \(\times_{so(4)}\) denotes the Lie bracket of \(so(4)\). So \(so(4)\) defines its own cross product of screw forms.

   If we set \(\begin{pmatrix} x \\ y \end{pmatrix} = E(x) + E'(y)\), then the corresponding trivector of \(A^3(\mathbb{R}^{3,3})\) realizing the cross product of \(so(4)\) is
   \[
   E_{123} + E_{12'3'} + E_{23'1'} + E_{31'2'} = F(e_0) + F(e_1) + F(e_2) + F(e_3). 
   \]
   (6.26)
   It corresponds to the matrix \(\text{diag}(1, 1, 1, 1)\) of the quadratic form preserved by group \(SO(4)\).

7. \(so(3, 1)\):
   It is composed of matrices of the form
   \[
   \begin{pmatrix}
   0 & x^T \\
   x & y \times I_{3 \times 3}
   \end{pmatrix},
   \]
   (6.27)
   for \(x, y \in \mathbb{R}^3\). It has the following bivector basis in \(A^3(\mathbb{R}^{3,3})\):
   \[
   E_{2:3'} - E_{23}, \quad E_{3'1'} - E_{31}, \quad E_{1'2'} - E_{12}, \\
   E_{32'} - E_{23'}, \quad E_{13'} - E_{31'}, \quad E_{21'} - E_{12'}. 
   \]
   (6.28)
   Let
   \[
   Q(x) = (E'E' - EE)(x), \quad P(y) = (E'E' - E'')E(y). 
   \]
   (6.29)
If we denote by \( \begin{pmatrix} x \\ y \end{pmatrix} \) the bivector \( Q(x) + P(y) \), then

\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \times_{so(3,1)} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \times x_2 - y_1 \times y_2 \\ x_1 \times y_2 + y_1 \times x_2 \end{pmatrix},
\]

(6.30)

where “\( \times_{so(3,1)} \)” denotes the Lie bracket of \( so(3,1) \). So \( so(3,1) \) defines another cross product of screw forms.

If we set \( \begin{pmatrix} x \\ y \end{pmatrix} = E(x) + E'(y) \), then the corresponding trivector realizing the cross product of \( so(3,1) \) is

\[
-E_{123} + E_{12'3'} + E_{23'1'} + E_{31'2'} = -\mathcal{J}(e_0) + \mathcal{J}(e_1) + \mathcal{J}(e_2) + \mathcal{J}(e_3).
\]

(6.31)

It corresponds to the matrix diag\((-1, 1, 1, 1)\) of the quadratic form preserved by group \( SO(3,1) \).

8. \( so(3,0,1) \):

It is composed of matrices of the form

\[
\begin{pmatrix} 0 & x^T \\ 0 & y \times I_{3\times3} \end{pmatrix},
\]

(6.32)

for \( x, y \in \mathbb{R}^3 \). It has the following bivector basis in \( \Lambda^3(\mathbb{R}^3,3) \):

\[
E_{23}, \quad E_{31}, \quad E_{12}, \quad E_{42'} - E_{23'}, \quad E_{13'} - E_{31'}, \quad E_{21'} - E_{12'}.
\]

(6.33)

Let

\[
Q(x) = EE(x), \quad P(y) = (E_{>E'} - E_{<E'})'(y).
\]

(6.34)

If we denote by \( \begin{pmatrix} x \\ y \end{pmatrix} \) the bivector \( Q(x) + P(y) \), then

\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \times_{so(3,0,1)} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \times x_2 \\ x_1 \times y_2 + y_1 \times x_2 \end{pmatrix},
\]

(6.35)

where “\( \times_{so(3,0,1)} \)” denotes the Lie bracket of \( so(3,0,1) \). So \( so(3,0,1) \) defines the same cross product of screw forms with that of \( se(3) \), and the two algebras are isomorphic under the transpose of \( 4 \times 4 \) matrices.

The trivector realizing the cross product of \( so(3,0,1) \) is

\[
E_{12'3'} + E_{23'1'} + E_{41'2'} = \mathcal{J}(e_1) + \mathcal{J}(e_2) + \mathcal{J}(e_3).
\]

(6.36)

It corresponds to the matrix diag\((0, 1, 1, 1)\) of the quadratic form preserved by group \( SO(3,0,1) \). The transpose of matrices from \( se(3) \) to \( so(3,0,1) \) is realized in \( \Lambda^2(\mathbb{R}^{3,3}) \) by \( \wedge^2 \mathcal{J} \).

9. \( so(2,2) \):
It is composed of matrices of the form
\[
\begin{pmatrix}
0 & x_1 & x_2 & x_3 \\
-x_1 & 0 & y_3 & -y_2 \\
x_2 & y_3 & 0 & y_1 \\
x_3 & -y_2 & -y_1 & 0
\end{pmatrix}.
\] (6.37)

It has the following bivector basis in \( \Lambda^2(\mathbb{R}^{3,3}) \):
\[
E_{2'3'} + E_{23}, \quad E_{3'1'} - E_{31}, \quad E_{1'2'} - E_{12}, \\
E_{32'} - E_{23'}, \quad E_{13'} + E_{31'}, \quad E_{21'} + E_{12'}.
\] (6.38)

**Definition 21.** For \( \mathbf{x} = (x_1, x_2, x_3)^T \) and \( \mathbf{y} = (y_1, y_2, y_3)^T \) of \( \mathbb{R}^3 \),
\[
\mathbf{x} \times_1 \mathbf{y} := \begin{pmatrix} x_2y_3 - x_3y_2 \\ -(x_3y_1 - x_1y_3) \\ -(x_1y_2 - x_2y_1) \end{pmatrix}, \quad \mathbf{x} \times_2 \mathbf{y} := \begin{pmatrix} -(x_2y_3 - x_3y_2) \\ -(x_3y_1 + x_1y_3) \\ x_1y_2 + x_2y_1 \end{pmatrix}.
\] (6.39)

In \( so(2,2) \), for \( \mathbf{x} = (x_1, x_2, x_3)^T \) and \( \mathbf{y} = (y_1, y_2, y_3)^T \) of \( \mathbb{R}^3 \), let
\[
Q(\mathbf{x}) = x_1(-E_{32'} + E_{23'}) + x_2(-E_{13'} - E_{31'}) + x_3(E_{21'} + E_{12'}), \\
P(\mathbf{y}) = y_1(E_{2'3'} + E_{23}) + y_2(E_{3'1'} - E_{31}) + y_3(E_{1'2'} - E_{12}).
\] (6.40)

If we denote by \( \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \) the bivector \( Q(\mathbf{x}) + P(\mathbf{y}) \), then
\[
\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \end{pmatrix} \times_{so(2,2)} \begin{pmatrix} \mathbf{x}_2 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \times_1 \mathbf{x}_2 + \mathbf{y}_1 \times \mathbf{y}_2 \\ \mathbf{y}_1 \times_2 \mathbf{x}_2 - \mathbf{y}_2 \times_2 \mathbf{x}_1 \end{pmatrix},
\] (6.41)

where “\( \times_{so(2,2)} \)” denotes the Lie bracket of \( so(2,2) \). So \( so(2,2) \) defines its own cross product of screw forms.

If we set \( \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = E(\mathbf{x}) + E'(\mathbf{y}) \), then the corresponding trivector realizing the cross product of \( so(2,2) \) is
\[
-E_{123} - E_{12'3'} + E_{23'1'} + E_{31'2'} = -\mathcal{F}(\mathbf{e}_0) - \mathcal{F}(\mathbf{e}_1) + \mathcal{F}(\mathbf{e}_2) + \mathcal{F}(\mathbf{e}_3).
\] (6.42)

It corresponds to the matrix \( \text{diag}(-1, -1, 1, 1) \) of the quadratic form preserved by group \( SO(2,2) \).

10. \( so(2,1,1) \):

It is composed of matrices of the form
\[
\begin{pmatrix}
0 & x_1 & x_2 & x_3 \\
0 & 0 & y_3 & -y_2 \\
0 & y_3 & 0 & y_1 \\
0 & -y_2 & -y_1 & 0
\end{pmatrix}.
\] (6.43)

It has the following bivector basis in \( \Lambda^2(\mathbb{R}^{3,3}) \):
\[
E_{23}, \quad -E_{31}, \quad E_{12}, \\
E_{32'} - E_{23'}, \quad E_{13'} + E_{31'}, \quad E_{21'} + E_{12'}.
\] (6.44)
In \( so(2,1,1) \), for \( x = (x_1, x_2, x_3)^T \) and \( y = (y_1, y_2, y_3)^T \) of \( \mathbb{R}^3 \), let
\[
Q(x) = x_1(E_{32'} - E_{23'}) + x_2(E_{13'} + E_{31'}) + x_3(E_{21'} + E_{12'}),
\]
\[
P(y) = y_1E_{23} - y_2E_{31} + y_3E_{12},
\]
(6.45)

If we denote by \( \binom{x}{y} \) the bivector \( Q(x) + P(y) \), then
\[
\binom{x_1}{y_1} \times_{so(2,1,1)} \binom{x_2}{y_2} = \binom{x_1 \times_1 x_2}{y_1 \times_2 x_2 - y_2 \times_2 x_1},
\]
(6.46)

where \( \times_{so(2,1,1)} \) denotes the Lie bracket of \( so(2,1,1) \). So \( so(2,1,1) \) defines its own cross product of screw forms.

If we set \( \binom{x}{y} = E(x) + E'(y) \), then the corresponding trivector realizing the cross product of \( so(2,1,1) \) is
\[
-E_{12'3'} + E_{23'1'} + E_{34'1'} = -F(e_1) + F(e_2) + F(e_3).
\]
(6.47)

It corresponds to the matrix \( \text{diag}(0, -1, 1, 1) \) of the quadratic form preserved by group \( SO(2,1,1) \).

**Definition 22.** The *Hadamard product* of vectors in \( \mathbb{R}^3 \) is the following multi-linear, associative and commutative product:
\[
\binom{x_1}{y_1} \odot \binom{x_2}{y_2} := \binom{x_1 x_2}{y_1 y_2}{z_1 z_2}.
\]
(6.48)

The \( i \)-th *Hadamard power* of a vector \( x \in \mathbb{R}^3 \) is
\[
\odot^i x := x \odot \cdots \odot x \quad (i \text{ times}).
\]
(6.49)

The *exponential function* of \( x \) in the Hadamard product is denoted by \( e^\odot x \). In particular,
\[
\odot^0 x = e^\odot 0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]
(6.50)

11. 6-D Lie subalgebra of general anisotropic dilation group:

The general anisotropic dilation group is composed of matrices of the form
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\lambda_1 & 0 & 0 & 0 \\
\lambda_2 & 0 & 0 & 0 \\
\lambda_3 & 0 & 0 & 0
\end{pmatrix}.
\]
(6.51)

The matrix represents either an anisotropic dilation centering at an affine point, or a pure translation. The corresponding Lie subalgebra is denoted by \( gad(3) \),
and is composed of matrices of the form
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
u_1 & \mu_1 & 0 & 0 \\
u_2 & 0 & \mu_2 & 0 \\
u_3 & 0 & 0 & \mu_3
\end{pmatrix}.
\] (6.52)

Its bivector basis is spanned by elements of the form \(E'E'(x), \ E_xE'(y)\) for \(x, y \in \mathbb{R}^3\).

Let
\[
Q(x) = x_1E_{2'3'} + x_2E_{3'1'} + x_3E_{1'2'},
\]

\[
P(y) = y_1F_1 + y_2F_2 + y_3F_3,
\] (6.53)

where the \(F_i\)'s are defined by (6.5). If we denote by \(\begin{pmatrix} x \\ y \end{pmatrix}\) the bivector \(Q(x) + P(y)\), then
\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \times_{gad} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \circ y_2 - y_1 \circ x_2 \\ 0 \end{pmatrix},
\] (6.54)

where "\(\times_{gad}\)" denotes the Lie bracket of this Lie algebra.

12. 7-D invariant group of \(\mathbb{R}^{2,2}\) and its 6-D subgroups:

Let \(L_4\) be a fixed 4-space of signature \(\mathbb{R}^{2,2}\), and let its orthogonal complement in \(\mathbb{R}^{3,3}\) be \(L'_4\). Then \(L'_4\) has signature \(\mathbb{R}^{1,1}\), so its null 1-spaces represent a pair of non-intersecting lines in space, say \(l_1\) and \(l_2\). The null 1-spaces of \(L_4\) is denoted by \(N(L_4)\); it represents all the lines in space incident to both \(l_1\) and \(l_2\).

**Proposition 47.** Any projective point on line \(l_1\) or \(l_2\) is on infinitely many lines of \(N(L_4)\), while any other projective point is on one and only one line of \(N(L_4)\). Similarly, any plane passing through \(l_1\) or \(l_2\) also passes through infinitely many lines of \(N(L_4)\), while any other plane passes through one and only one line of \(N(L_4)\).

**Proof.** Let \(X_3\) be a null 3-space of \(A(\mathbb{R}^{3,3})\). When \(X_3\) represents a point of \(l_1\), then it is incident to all the lines connecting \(X_3\) and \(l_2\); when \(X_3\) represents a plane through \(l_1\), then it meets line \(l_2\) at a point \(Y_3\), and so is incident to all the lines connecting \(Y_3\) and \(l_1\).

When \(X_3\) represents a point on neither \(l_1\) nor \(l_2\), then in \(\mathbb{R}^{3,3}\), the intersection of the 3-space \(X_3\) with the 4-space \(L_4\) is a null 1-space or 2-space. If point \(X_3\) is on more than one line of \(N(L_4)\), then \(l_1, l_2\) must be coplanar, violating the requirement that the two lines do not intersect. So point \(X_3\) is on on and only one line of \(N(L_4)\). Similarly, when \(X_3\) represents a plane supporting neither \(l_1\) nor \(l_2\), it must be incident to one and only one line of \(N(L_4)\). \(\square\)

Consider the subgroup of \(SO(3,3)\) that leaves the 4-space \(L_4\) invariant. Denote the subgroup by \(Inv(L_4)\). Then
\[
Inv(L_4) = (SO(1,1) \oplus SO(2,2)) \cup (SO^-(1,1) \oplus SO^-(2,2)).
\] (6.55)

It is a 7D Lie subgroup.
A typical example is that \(\mathbf{l}_1\) is an affine line, say \(\mathbf{l}_1 = \mathbf{e}_{01}\), the line through the origin and in direction \(\mathbf{e}_1\), and \(\mathbf{l}_2\) is a line at infinity, say \(\mathbf{l}_2 = \mathbf{e}_{23}\), the line at infinity normal to \(\mathbf{e}_1\). Then \(N(\mathbf{L}_4)\) is composed of all lines meeting \(\mathbf{l}_1\) and perpendicular to it. Then \(\mathbf{L}_4 = \mathbf{E}_{2233}\), and denote

\[
\text{Inv}(2, 2) := \text{Inv}(\mathbf{E}_{2233}).
\]

(6.56)

The Lie algebra \(\text{inv}(2, 2)\) of \(\text{Inv}(2, 2)\) is spanned by \(\mathbf{E}_{11}, \mathbf{E}_{22}, \mathbf{E}_{33}, \mathbf{E}_{23}, \mathbf{E}_{23}', \mathbf{E}_{23'}, \mathbf{E}_{32}\). Its matrix form in \(\text{sl}(4)\) is

\[
\begin{pmatrix}
\mathbf{A} & 0 \\
0 & \mathbf{D}
\end{pmatrix},
\]

(6.57)

where \(\mathbf{A}, \mathbf{D}\) are two \(2 \times 2\) matrices such that \(\text{tr}(\mathbf{A}) + \text{tr}(\mathbf{D}) = 0\). The group \(\text{Inv}(2, 2)\) is composed of matrices of the form

\[
\begin{pmatrix}
\mathbf{A} & 0 \\
0 & \mathbf{D}
\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}
0 & \mathbf{B} \\
\mathbf{C} & 0
\end{pmatrix},
\]

(6.58)

where \(\det(\mathbf{AD}) = 1\), and \(\det(\mathbf{BC}) = -1\).

For a general 4-space \(\mathbf{L}_4\) of signature \(\mathbb{R}^{2,2}\), \(\text{Inv}(\mathbf{L}_4)\) is isomorphic to \((\text{GL}(2) \times \text{GL}(2))/\ast\), where two elements are equivalent if and only if they differ by scale. The space \(\mathbf{L}_4\) is called the space of coupled projective screws with respect to the pair of axes \(\mathbf{l}_1, \mathbf{l}_2\). The projective screw ratio refers to \(\det(\mathbf{A})/\det(\mathbf{D})\) in the first case of (6.58), and \(\det(\mathbf{B})/\det(\mathbf{C})\) in the second case.

A typical 6-D Lie subalgebra of \(\text{inv}(2, 2)\) is spanned by \(\mathbf{E}_{22}, \mathbf{E}_{33}, \mathbf{E}_{23}, \mathbf{E}_{23}', \mathbf{E}_{23'}, \mathbf{E}_{32}\). For \(\mathbf{x} = (x_1, x_2, x_3)^T\) and \(\mathbf{y} = (y_1, y_2, y_3)^T\), let

\[
\begin{align*}
P(\mathbf{x}) &= x_1(\mathbf{E}_{33} + \mathbf{E}_{22}) + x_2\mathbf{E}_{23} + x_3\mathbf{E}_{23'}, \\
Q(\mathbf{y}) &= y_1(\mathbf{E}_{33'} - \mathbf{E}_{22'}) + y_2\mathbf{E}_{32'} + y_3\mathbf{E}_{23'}.
\end{align*}
\]

(6.59)

Then

\[
\begin{align*}
P(\mathbf{x}) \times Q(\mathbf{y}) &= 0, \\
P(\mathbf{x}) \times P(\mathbf{y}) &= -(x_2y_3 - x_3y_2)(\mathbf{E}_{33} + \mathbf{E}_{22}) + 2(x_1y_2 - x_2y_1)\mathbf{E}_{23} \\
&\quad + 2(x_3y_1 - x_1y_3)\mathbf{E}_{23'}, \\
Q(\mathbf{x}) \times Q(\mathbf{y}) &= (x_2y_3 - x_3y_2)(\mathbf{E}_{33'} - \mathbf{E}_{22'}) + 2(x_1y_2 - x_2y_1)\mathbf{E}_{32'} \\
&\quad + 2(x_3y_1 - x_1y_3)\mathbf{E}_{23'}.
\end{align*}
\]

(6.60)

For the whole Lie algebra \(\text{sl}(4)\), by (6.17), any element of it is naturally decomposed into the direct sum of five 3-D vectors:

\[
\mathbf{M} = \begin{pmatrix}
-\text{tr(diag}(\mathbf{c})) \\
\mathbf{m} & \mathbf{n}^T \\
\text{diag}(\mathbf{c}) + \text{skew}(\mathbf{u}) + \text{skew}^T(\mathbf{d})
\end{pmatrix},
\]

(6.61)

where \(\mathbf{m}, \mathbf{n}, \mathbf{c}, \mathbf{u}, \mathbf{d} \in \mathbb{R}^3\). They allow for defining a matrix of \(\text{sl}(4)\) as a “superscrew” in \(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3\), and defining the Lie bracket as a “supercross product”. Such a representation of \(\text{sl}(4)\) is called the superscrew representation. For symmetry consideration, instead of choosing the 3-D Lie subalgebra decomposition (6.61), we make the following decomposition of \(\text{sl}(4)\) into 3-spaces:
Definition 23. When $\text{sl}(4)$ is represented by bivectors of $\Lambda^2(\mathbb{R}^{3,3})$, a superscrew of 3-D projective geometry is defined as follows: for any $x_i \in \mathbb{R}^3$ where $1 \leq i \leq 5$, \[
abla \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} := \mathcal{E}(x_1) + \mathcal{E}^\prime(x_2) + \mathcal{E}_0(x_3) + (\mathcal{E}_\geq \mathcal{E}^\prime + \mathcal{E}_\leq \mathcal{E}^\prime)(x_4) + (\mathcal{E}_\geq \mathcal{E}^\prime - \mathcal{E}_\leq \mathcal{E}^\prime)(x_5). \] (6.62)

Definition 24. The symmetric cross product of two vectors of $\mathbb{R}^3$ is defined as follows: for any $x, y \in \mathbb{R}^3$ \[
abla \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \times_{\text{sl}(4)} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}. \] (6.63)

By direct computation, we get \[
abla \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \times_{\text{sl}(4)} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}, \] (6.64)

where \[
z_1 = -x_1 \otimes y_3 + x_3 \otimes y_1 + x_1 \star y_4 - x_4 \star y_1 + x_1 \times y_5 + x_5 \times y_1, \\
z_2 = -x_2 \star y_4 + x_4 \star y_2 + x_2 \otimes y_3 - x_3 \otimes y_2 + x_2 \times y_5 + x_5 \times y_2, \\
z_3 = (-x_1 \cdot y_2 + x_2 \cdot y_1)\mathcal{E}^{\otimes 0} - x_1 \otimes y_2 + x_2 \otimes y_1 + 4(x_4 \otimes y_5 - x_5 \otimes y_4) \times \mathcal{E}^{\otimes 0}, \\
z_4 = \frac{1}{2}(x_1 \star y_2 - x_2 \star y_1) - (x_3 \times \mathcal{E}^{\otimes 0}) \otimes y_5 + x_5 \otimes (y_3 \times \mathcal{E}^{\otimes 0}) - x_4 \times y_5 - x_5 \times y_4, \\
z_5 = \frac{1}{2}(x_1 \times y_2 + x_2 \times y_1) + (x_3 \times \mathcal{E}^{\otimes 0}) \otimes y_4 - x_4 \otimes (y_3 \times \mathcal{E}^{\otimes 0}) + x_4 \times y_4 + x_5 \times y_5. \] (6.65)

7 Conclusion

In this paper, we establish a rigorous mathematical foundation for the line geometric model of 3-D projective geometry. We also extend screw theory from rigid-body motions to projective transformations, in hope that non-Newtonian mechanics may find an algebraic language for the development of virtual work of projective motions.
The connection between the $\mathcal{Cl}(3,3)$ model of 3-D projective geometry and the $\mathcal{Cl}(4,1)$ model of 3-D conformal geometry is an interesting topic, and will be investigated in another paper. This connection makes it possible to investigate 3-D non-Euclidean geometry together with its various realizations in Euclidean space via pin-hole cameras of $\mathbb{R}^{4,1}$.

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