Effective dimension, level statistics, and integrability of Sachdev-Ye-Kitaev-like models

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The Sachdev-Ye-Kitaev (SYK) model attracts attention in the context of information scrambling, which represents delocalization of quantum information and is quantified by the out-of-time-ordered correlators (OTOC). The SYK model contains \( N \) fermions with disordered and four-body interactions. Here, we introduce a variant of the SYK model, which we refer to as the Wishart SYK model. We investigate the Wishart SYK model for complex fermions and that for hard-core bosons. We show that the ground state of the Wishart SYK model is massively degenerate and that the residual entropy is extensive, and that the Wishart SYK model for complex fermions is integrable. In addition, we numerically investigate the OTOC and level statistics of the SYK models. At late times, the OTOC of the fermionic Wishart SYK model exhibits large temporal fluctuations, in contrast with smooth scrambling in the original SYK model. We argue that the large temporal fluctuations of the OTOC are a consequence of a small effective dimension of the initial state. We also show that the level statistics of the fermionic Wishart SYK model is in agreement with the Poisson distribution, while the bosonic Wishart SYK model obeys the GUE or the GOE distribution.

I. INTRODUCTION

Scrambling of quantum information in quantum many-body systems attracts attention in a wide range of fields including high energy physics and condensed matter physics. The Sachdev-Ye-Kitaev (SYK) model exhibits a fascinating feature of scrambling, which is a quantum model of fermions with disordered, all-to-all, and four-body interactions \([1, 2, 8–16]\). Recently, Kitaev proposed the SYK model to address the black hole information paradox \([1, 2]\). In this context, it was conjectured that black holes are the fastest scramblers of quantum information \([5–9]\), where scrambling behavior has been investigated not only in high energy physics, but also in condensed matter physics \([15, 22]\) because of its relevance to non-Fermi liquid, the quantum critical phenomena, and the effect of disorder in strongly correlated systems. For example, there are many extensions of the SYK model: that with \( q \)-point interactions \([22]\), that with hard-core bosons \([23]\), SUSY extensions \([30–35]\), disorder-free tensor models \([36, 37]\), the SYK model with a lattice structure \([23, 24, 27, 29]\), and a kind of coupled or perturbed system \([21, 22, 23, 27, 28]\).

In this paper, we investigate a variant of the SYK model, which we refer to as the Wishart SYK model. This model reduces to a clean SYK model without quenched disorder as a special case. In a previous work \([1]\), two of the authors found that the clean SYK model exhibits large temporal fluctuations in contrast to the original SYK model. Here we will investigate the origin of such large fluctuations from a more general perspective based on the Wishart SYK model.

We find that the ground state of the Wishart SYK model is very degenerate, and the residual entropy is extensive. The degeneracy makes the effective dimension of the initial state smaller, and the small effective dimension leads to large temporal fluctuations. On the other hand, the original SYK model shows a large effective dimension and small temporal fluctuations.

We also numerically investigate the level statistics of the original and the Wishart SYK models. We show that the level statistics of the fermionic Wishart SYK model is in good agreement with the Poisson distribution. Correspondingly, we prove that the fermionic Wishart SYK model is integrable by mapping it onto a particular case of the Richardson-Gaudin model \([25, 26]\), which is known to be Bethe-ansatz solvable.

The SYK model was originally introduced with Majorana fermions \([1, 2]\); the SYK model with complex fermions \([18, 19]\) can also be defined as

\[
H_{\text{SYK}} := \frac{1}{N^{3/2}} \sum_{1 \leq i < k < l \leq N} J_{i,j,k,l} c_i^\dagger c_j c_k c_l,
\]

where \( N \) is the number of sites, and \( c_i (c_i^\dagger) \) is the annihilation (creation) operator of a complex fermion at site \( i \), satisfying the anti-commutation relation \( \{ c_i, c_j^\dagger \} = \delta_{i,j} \) and \( \{ c_i, c_j \} = \{ c_i^\dagger, c_j^\dagger \} = 0 \). The coupling constant \( J_{i,j,k,l} \) is sampled from the complex Gaussian distribution with variance \( J^2 \), satisfying \( J_{i,j,k,l} = J_{l,k,i,j}^\dagger \). Besides, other variants of the SYK model have been investigated not only in high energy physics, but also in condensed matter physics \([15, 22]\) because of its relevance to non-Fermi liquid, the quantum critical phenomena, and the effect of disorder in strongly correlated systems. For example, there are many extensions of the SYK model: that with \( q \)-point interactions \([22]\), that with hard-core bosons \([23]\), SUSY extensions \([30–35]\), disorder-free tensor models \([36, 37]\), the SYK model with a lattice structure \([23, 24, 27, 29]\), and a kind of coupled or perturbed system \([21, 22, 23, 27, 28]\).

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The rest of this paper is organized as follows. In Sec. II, we introduce the Wishart SYK model and investigate its basic properties. In Sec. III, we show the numerical results of the energy spectrum of the original and the Wishart SYK models. In Sec. IV, we numerically show dynamics of the OTOC and the effective dimension. In Sec. V, we investigate the level statistics of the energy level spacings. In Sec. VI, we show the integrability of the fermionic Wishart SYK model. In Appendix A, we show the algebraic independence of the mutually commuting operators of the fermionic SYK model. We should note that the residual entropy in the original SYK model does not represent huge degeneracy indeed hold for the Hamiltonian (2), where the counting theorem, which is given by

\[ \dim[\text{Im}(Q)] + \dim[\text{Ker}(Q)] = \dim[\text{Domain}(Q)], \]

where \( \text{Im} \), \( \text{Ker} \), and \( \text{Domain} \) respectively represent the image, the kernel, and the domain of an operator. In the sector of the fermion number \( N_p \), \( \dim[\text{Ker}(Q)] = Z_{N,N_p}, \dim[\text{Domain}(Q)] = C(N,N_p) \), and \( \dim[\text{Im}(Q)] \leq C(N,N_p - 2) \) hold, where the equality in the last inequality is achieved if \( Q \) is surjective. Thus, a lower bound of \( Z_{N,N_p} \) with \( N_p \leq \left\lfloor \frac{N+1}{2} \right\rfloor \) is given by

\[ Z_{N,N_p} \geq C(N,N_p) - C(N,N_p - 2). \]

Defining \( Z_N := \sum_{N_p=0}^{N} Z_{N,N_p} \) and using inequality (1), we obtain a lower bound of \( Z_N \) as

\[ Z_N \geq \sum_{N_p=0}^{\left\lfloor \frac{N+1}{2} \right\rfloor} Z_{N,N_p} + \sum_{N_p=1, N_p \text{ odd}}^{N} Z_{N,N_p} + \sum_{N_p=2, N_p \text{ even}}^{\left\lfloor \frac{N+1}{2} \right\rfloor} Z_{N,N_p} \]

\[ \geq C(N, \left\lfloor \frac{N+1}{2} \right\rfloor) + C(N, \left\lfloor \frac{N+1}{2} \right\rfloor - 1) \]

\[ = C(N, 1, \left\lfloor \frac{N+1}{2} \right\rfloor). \]

In Appendix A, we show that the equalities of (1) and (2) hold for the Hamiltonian (2), where the counting of the zero-energy states arrives at that of the lowest weight states of the total angular momentum.

Since \( Z_N \) increases exponentially with \( N \), the Wishart SYK model has an extensive residual entropy. This fact reminds us of the residual entropy of the original SYK model. We should note that the residual entropy in the original SYK model does not represent huge degeneracy in the ground state but many low-energy excited states near the ground state in the large-\( N \) limit.

We also consider the SYK model and the Wishart SYK model with hard-core bosons. They are defined by replacing the annihilation (creation) operator for fermions with an annihilation (creation) operator for bosons.
is because the operator we name the model (2) after the Wishart SYK model Wishart-Laguerre ensembles \[34, 35\]. We note that why of the SUSY SYK model is generically described by the characterised by the Gaussian ensembles \[44, 45\], that theory. While the spectral density of the SYK model is.

The SYK model is smooth, that of the fermionic Wishart SYK model. We note that the di-similarity between the Hamiltonian of the Wishart SYK model (2) and the Hamiltonian of \(N = 2\) SUSY SYK model \([30–34]\) defined as \(H_{\text{SUSY}, N=2} := Q^\dagger Q + QQ^\dagger\),

\[Q := \frac{i}{N} \sum_{1 \leq i < j < k \leq N} J_{i,j,k} c_i^\dagger c_j c_k,\]

where \(J_{i,j,k}\) are independent complex Gaussian variables with variance \(J^2\), and the supercharge \(Q\) is nilpotent: \(Q^2 = 0\). It is shown that there are zero-energy ground states in the \(N = 2\) SUSY SYK model because there is nonzero subspace spanned by the vectors with \(Q |\psi\rangle = Q^\dagger |\psi\rangle = 0\). When we assume that \(N\) is even and \(N_P = N/2\) for simplicity, the number of the zero-energy ground states of the \(N = 2\) SUSY SYK model is given by \(2 \cdot 3^{N/2-1}\) as shown in Ref. [2], which is much smaller than \(Z_{N,N/2}\) for the Wishart SYK model. Such a SUSY extension can be defined when the supercharge is a product of \(q\) annihilation operators with odd \(q\). In the case of the Wishart SYK model, \(Q\) is defined with two annihilation operators. The difference between them is remarkable when we consider the fermionic parity \((-1)^{N_P}\). While the supercharge \(Q\) anti-commutes with the fermionic parity \(\{Q, (-1)^{N_P}\} = 0\) for the SUSY case, the operator \(Q^\dagger\) in the fermionic Wishart SYK model commutes with the fermionic parity \(\{Q^\dagger, (-1)^{N_P}\} = 0\).

The SYK model and the SUSY SYK model have been investigated from the viewpoint of the random matrix theory. While the spectral density of the SYK model is characterized by the Gaussian ensembles \([44, 45]\), that of the SUSY SYK model is generically described by the Wishart-Laguerre ensembles \([44, 45]\). We note that why we name the model \([2]\) after the Wishart SYK model is because the operator \(Q\) is represented as a rectangular matrix, which directly leads to the huge ground-state degeneracy of the Wishart SYK model.

III. ENERGY SPECTRUM

In this section, we show the results of numerically exact diagonalization of the Hamiltonian of the SYK models. We set \(J_{\text{mean}} = 0\) in the following sections. In this section, the numerical results are obtained from a single disorder realization.

Figure (a) shows eigenenergies of the SYK model with \(N = 16\) and \(N_P = N/2\). Figures (a) and (b) show that the structures of eigenenergies are very similar between the fermionic and bosonic SYK models. We note that all the eigenenergies are non-degenerate in the sector of \(N_P = N/2\). When we look at the sector of \(N_P = 0\) and 1 (not shown), there are degenerate energy eigenstates at \(E = 0\) whose degeneracy is \(N + 1\). This degeneracy at \(E = 0\) comes from the fact that the sectors of \(N_P = 0\) and 1 are in the kernel of the Hamiltonian of the SYK model.

Figures (c) and (d) show eigenenergies of the fermionic/bosonic Wishart SYK models. All the eigenenergies are non-negative because the Hamiltonian is positive-semidefinite. A huge number of the ground states are observed as discussed in Sec. (a). We have checked that the ground-state degeneracy equals the right-hand side of Eq. (1). For example, the degeneracy with \(N = 16\) and \(N_P = 8\) is \(Z_{N,N_P} = 4862\). Concerning the excited states, the structure of the energy spectrum differs between the fermionic and the bosonic Wishart models. While the spectrum of the bosonic Wishart model is smooth, that of the fermionic Wishart model is rough.

Figure (a) shows the degeneracy of the eigenenergies of the Wishart SYK models. As shown in Fig. (b), there is no degeneracy in the excited states of the bosonic Wishart SYK model. On the other hand, Fig. (a) shows that there are many degenerate excited states in...
the fermionic Wishart SYK model. The degeneracy is given by $2^{l}$, where $l = 0, 2, 4,$ and 6. We also note that the degeneracy tends to decrease as the energy increases. In general, the same degeneracy is also seen in other disordered realizations. We will examine this degeneracy in the excited states in Sec. IV.

IV. DYNAMICS OF OTOC AND EFFECTIVE DIMENSION

In this section, we numerically investigate the effect of the huge ground-state degeneracy on dynamics of the Wishart SYK model. We focus on the out-of-time-ordered correlator (OTOC), which is an indicator of scrambling. An OTOC for operators $A$ and $B$, with an initial state $|\Psi\rangle$, and at time $t$ is defined as

$$C_{AB}(t) := \langle \Psi | A^\dagger(t) B^\dagger(0) A(t) B(0) | \Psi \rangle. \quad (10)$$

We also define the long time average of the OTOC and the temporal fluctuations of its real part.

$$\overline{C_{AB}(t)} := \frac{1}{T} \int_0^T dt C_{AB}(t), \quad (11)$$

$$(\Delta C_{AB})^2 := (\text{Re} C_{AB}(t) - \text{Re} \overline{C_{AB}})^2. \quad (12)$$

Figure 3 shows time dependence of OTOC for the original SYK model and the Wishart SYK model. We set $A = c_1 (b_1)$ and $B = c_1^\dagger (b_1^\dagger)$ for the fermionic (bosonic) case. Figure 3 (a) shows the case of the fermionic models. While the OTOC for the original SYK model shows quick relaxation, the OTOC for the Wishart SYK model shows slower relaxation with larger temporal fluctuations at late times. We note that the temporal fluctuations of the tripartite mutual information of the clean SYK model are also larger than that of the disordered model [11]. On the other hand, Fig. 3 (b) shows that the temporal fluctuations of the OTOCs of the bosonic models are smaller than that of the fermionic Wishart SYK model at late times.

To investigate the dynamics of the OTOCs more systematically, we consider the effective dimension of the initial state. We write the initial state as

$$|\Psi\rangle = \sum_{i} \sum_{\alpha} d_i c_i^\dagger |E_i^\alpha\rangle,$$

where $|E_i^\alpha\rangle$ is an eigenstate with eigenenergy $E_i$ with $\alpha$ being a label of degeneracies, and $d_i$ represents the degeneracy of $E_i$. The effective dimension of $|\Psi\rangle$ is defined as

$$D_{\text{eff}}(|\Psi\rangle) := \left( \sum_{i} p_i \right)^{-1}, \quad (14)$$

where $p_i := \sum_{\alpha=1}^{d_i} |c_i^\alpha|^2$. The effective dimension has been investigated in the context of relaxation of the expectation value of an observable at late times [16, 17]. The temporal fluctuations of an observable $O$ around its long time average, written as $\Delta O$, are bounded as $(\Delta O)^2 \leq C D_G/D_{\text{eff}}$, where $C$ is a constant independent of the system size and $D_G$ is the maximum degeneracy of energy gaps. Thus, a large effective dimension implies relaxation of the expectation value. Although the OTOC cannot be written as the expectation value of any single observable, we expect that a similar bound holds for the temporal fluctuations of the OTOC.
be well investigated to diagnose the conventional quantum chaos in quantum many-body systems \cite{23,49}. It is known that the distribution of energy level spacings follows the Poisson distribution if the system is integrable and the Wigner-Dyson distribution if the system is non-integrable. The Wigner-Dyson distribution is classified into three classes GOE, GUE, and GSE corresponding to symmetries of the Hamiltonian. The level statistics of the (SUSY) SYK model is investigated in Refs. \cite{24,47}.

We adopt the ratio of consecutive level spacings \cite{25} to examine the level statistics of the fermionic/bosonic (Wishart) SYK models. We assume that $E_i < E_j$ ($i < j$). We define the nearest-neighbor spacing as $S_i := E_{i+1} - E_i$. The ratio of consecutive level spacings is then defined as

$$\tilde{r}_i := \frac{\min(S_i, S_{i-1})}{\max(S_i, S_{i-1})} = \min(r_i, 1/r_i), \quad (15)$$

$$r_i := S_i/S_{i-1}. \quad (16)$$

By definition, $\tilde{r}_i$ takes a value within $0 \leq \tilde{r}_i \leq 1$.

The level statistics of the energy level spacings is described by the Wigner-Dyson (Poisson) distribution for the non-integrable (integrable) systems, respectively. As shown in Ref. \cite{50}, the corresponding forms of $P(\tilde{r})$ are given by

$$P_{\text{Poisson}}(\tilde{r}) := 2/(1 + \tilde{r})^2, \quad (17)$$

$$P_{\text{Wigner}}(\tilde{r}) := \frac{2}{Z_\beta} \frac{(\tilde{r} + \tilde{r}^2)^\beta}{(1 + \tilde{r} + \tilde{r}^2)^{1+(3/2)\beta}}, \quad (18)$$

where $\beta = 1$ (GOE), 2 (GUE), 4 (GSE), $Z_1 = 8/27$, $Z_2 = (4/81)(\pi/\sqrt{3})$, and $Z_4 = (4/729)(\pi/\sqrt{3})$.

Figure 5 shows the distribution of the ratio of consecutive level spacings $P(\tilde{r})$ of the SYK models. The average is taken over 24 samples of disorder, and the error bars represent the standard deviation. As shown in Fig. 5, the distributions of the fermionic/bosonic SYK models are in good agreement with the GUE prediction. For the fermionic Wishart SYK model, the distribution follows that of the GUE, except for the case of $N_P = N/2 + 1$ where it follows the GOE distribution (see Fig. 5(c)). As will be shown in Appendix B, we can understand the origin of the GUE distribution for this special case by constructing an anti-unitary operator which commutes with the Hamiltonian. Such an operator can be constructed only when $N_P = N/2 + 1$ (see Appendix B).

It is noteworthy that the distribution of the fermionic Wishart SYK model matches the Poisson distribution, implying two possibilities. One is that the fermionic Wishart SYK model is integrable, and the other is that we missed another symmetry of the Hamiltonian (though we have already considered the $U(1)$ symmetry corresponding to the particle number conservation). In the next section, we will show that the fermionic Wishart SYK model is, in fact, integrable.

We also show the level statistics of the bosonic SYK model with 2-body interactions, whose Hamiltonian is

V. LEVEL STATISTICS

In this section, we consider the level statistics of the Hamiltonian of the SYK models. The level statistics has
VI. INTEGRABILITY OF THE FERMIONIC WISHART SYK MODEL

In this section, we show the integrability of the fermionic Wishart SYK model by mapping it to the Richardson-Gaudin model, which is known to be integrable by the algebraic Bethe ansatz. We also explicitly construct the mutually commuting conserved quantities. We examine the degenerate structure in the excited states shown in Fig. 3(a).

A. Mapping to the Richardson-Gaudin model

For simplicity, we first assume that \( N \) is even with \( N = 2M \) \((M \in \mathbb{N})\) and the coupling strengths \( J_{k,l} \) are real. It is well known that any real skew-symmetric matrix \( J \) can be brought into the following canonical form:

\[
O^T J O = \begin{pmatrix}
0 & 
\lambda_1 & 
0 & 
\cdots & 
0 \\
-\lambda_1 & 0 & 
\lambda_2 & 
\cdots & 
0 \\
0 & 
-\lambda_2 & 0 & 
\cdots & 
0 \\
\vdots & 
\vdots & 
\vdots & 
\ddots & 
\vdots \\
0 & 
0 & 
0 & 
\cdots & 
-\lambda_M \\
-\lambda_M & 0 & 
0 & 
\cdots & 
0
\end{pmatrix},
\]

where the matrix \( O \) is orthogonal, \( \lambda_j \in \mathbb{R} \) \((j = 1, 2, \cdots, M)\) and zero elements of \( J \) are left empty. In a generic case, \( \lambda_j > 0 \) is expected. With the matrix \( O \), we introduce a new set of fermionic operators

\[
(f_{1,\uparrow}, f_{1,\downarrow}, \cdots, f_{M,\uparrow}, f_{M,\downarrow}) := (c_1, c_2, \cdots, c_{2M-1}, c_{2M}) O,
\]

which satisfy the anti-commutation relations:

\[
\{f_{j,\sigma}, f_{k,\tau}\} = \delta_{j,k} \delta_{\sigma,\tau},
\]

\[
\{f_{j,\sigma}, f_{k,\tau}\} = \{f_{j,\sigma}^\dagger, f_{k,\tau}^\dagger\} = 0.
\]

We also define the corresponding number operators: \( n_{j,\sigma} := f_{j,\sigma}^\dagger f_{j,\sigma} \) and \( n_j := n_{j,\uparrow} + n_{j,\downarrow} \). In terms of the new fermion operators, the operator \( Q \) and the Hamilto-
nian are written as
\[ Q = \sum_{j=1}^{M} Q_j, \]  
\[ Q_j := \lambda_j f_{j,\uparrow} f_{j,\downarrow}, \]  
\[ H = \left( \sum_{j=1}^{M} \lambda_j f_{j,\uparrow} f_{j,\uparrow}^{\dagger} \right) \left( \sum_{k=1}^{M} \lambda_k f_{k,\downarrow} f_{k,\downarrow}^{\dagger} \right). \]  
This is nothing but a particular case of the Richardson-Gaudin model [12, 13, 54, 55] which is known to be integrable. We note that the above mapping is possible in spite of the disorder and the distribution of the disordered coupling strength is not important for the mapping.

In the following, using the new set of fermionic operators [24], we explicitly show the integrability of the fermionic Wishart SYK model and the degeneracies in the excited states shown in Fig. 3(a).

**B. Integrability of the fermionic Wishart SYK model**

We now show that the fermionic Wishart SYK model is integrable in the sense of Ref. [58] by explicitly constructing the conserved quantities. The Hamiltonian (24) can be written as a sum of the mutually commuting operators \( \{ h_j \} \) in accordance with Ref. [58]:

\[ H = \sum_{j=1}^{M} \lambda_j^2 h_j, \]  
\[ h_j := \eta_j^+ \eta_j^- - 2 \sum_{k(\neq j)}^{\lambda_k^2} \frac{\lambda_k^2}{\lambda_j^2 - \lambda_j^2} \eta_j^+ \eta_k^-, \]  
\[ - \sum_{k(\neq j)}^{\lambda_k^2} \frac{\lambda_k^2}{\lambda_j^2 - \lambda_j^2} \left( \eta_j^+ \eta_k^- + \eta_k^+ \eta_j^- \right), \]  
where we defined \( \eta \) operators as

\[ \eta_j^+ := f_{j,\uparrow} f_{j,\uparrow}^{\dagger}, \quad \eta_j^- := f_{j,\uparrow}^{\dagger} f_{j,\downarrow}, \]  
\[ \eta_j^\pm := \frac{1}{2} (f_{j,\uparrow} f_{j,\downarrow}^{\dagger} + f_{j,\downarrow}^{\dagger} f_{j,\uparrow} - 1). \]  
One can verify that \( h_j \)'s mutually commute: \( [h_j, h_k] = 0 \). In addition, if these commuting operators are algebraically independent, the Hamiltonian is integrable. We show their linear independence in Appendix B and expect that their algebraic independence also holds. From the above properties of the Hamiltonian, we expect that the integrability of the fermionic Wishart SYK model can be shown by the algebraic Bethe ansatz (ABA). In Appendices B and E, we review the previous results of the ABA in our context. In Appendix C, we define the generators of SU(2) and another algebra, and in Appendix E we write down the ansatz states in the ABA.

**C. Degeneracy of energy eigenstates**

We next investigate the degeneracy of the ground states and the excited states. We can easily construct some of the ground states, which take the form of the ground states, which take the form of the Hamiltonian (26) can be shown by the algebraic Bethe ansatz (ABA). In Appendices D and E, we review the previous results of the ABA in our context. In Appendix E, we write down the ansatz states in the ABA.

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fold degeneracy is easily obtained as $C(M, 2)$.

We can also explain $2^M - 4$-fold degeneracy by considering the Hamiltonian in the sector of $n_i = 1 \ (i \neq i_1, i_2, i_3, i_4)$ and $n_i \neq 1 \ (i = i_1, i_2, i_3, i_4)$. The restricted Hamiltonian is defined as

$$H_1 := \left( \sum_{j \in I} \lambda_j f_j^\dagger f_j^\dagger \right) \left( \sum_{k \in I} \lambda_k f_k f_k \right),$$

where $I := \{ i_1, i_2, i_3, i_4 \}$. This Hamiltonian is represented by a $6 \times 6$ matrix ($C(m, m/2) = 6$ with $m = |I| = 4$, where $| \cdot |$ is the number of elements). There are two eigenstates with zero eigenenergies, which is understood similarly as the discussion about the number of zero-energy eigenstates in Sec. III. In this case, the number of zero-energy eigenstates of $H_1$ is counted as $C(4, 2) - C(4, 1) = 2$. Thus, the number of excited eigenenergies with $2^M - 4$-fold degeneracy is calculated as $C(M, 4)\{ C(4, 2) - C(4, 2) - C(4, 1) \} = C(M, 4)C(4, 1)$. Similarly, we find that the number of excited eigenenergies with $2^M - 2^l$-fold degeneracy is $C(M, 2l)/C(2l, l - 1)$.

We note that the maximum degeneracy is brought by the smallest restricted Hamiltonians $H_{i,j}$. Naturally, we expect that the spectrum is broader if $|I|$ is larger. Thus, the above result explains that the degeneracy becomes smaller in the higher energy region in Fig. 2 (a).

For general $N$ and $N_P$, the above discussion applies in the same manner. We briefly comment on the case of odd $N$. When $N$ is odd, the canonical form of the coupling strengths becomes

$$O^T JO = \begin{pmatrix}
0 & \lambda_1 & 0 & \cdots & 0 & \lambda_M \\
-\lambda_1 & 0 & \cdots & \lambda_1 & \lambda_2 & 0 \\
& & \ddots & & \ddots & \\
& & \lambda_2 & \cdots & 0 & \lambda_M \\
& & & \lambda_1 & -\lambda_2 & 0 \\
& & & & \lambda_M & 0 \\
& & & & & 0
\end{pmatrix},$$

where the last block is $1 \times 1$ and its element is 0. With this structure, the Hamiltonian splits into two parts corresponding to the fermion number of the last block. When the number of the fermion in the last block is 1 (0), the structure of degenerate excited states is the same as one with $N - 1$ sites and $N_P - 1$ particles ($N - 1$ sites and $N_P$ particles).

We also note that the excited states can be generated algebraically for each restricted Hamiltonian. We define lowering operators as

$$S_m^+ := \sum_{j=1}^{M} (\lambda_j)^{2m+1} S_j^+,$$
$$S_m^- := f_j^\dagger f_j^\dagger,$$

where $I$ is the complement of $I$ and $I_I$ is the identity operator defined on $I$. One of the energy eigenstates of the Hamiltonian can be written by $|\Phi, I \rangle \otimes |\Psi, \overline{I} \rangle$, where $|\Phi, I \rangle$ is an excited eigenstate of $H_1$ and $|\Psi, \overline{I} \rangle$ is the “ferromagnetic” state defined by

$$|\Psi, \overline{I} \rangle := \left( \prod_{j \in \overline{I}} f_j^\dagger \right) |\text{vac}_\overline{I} \rangle,$$

where $|\text{vac}_\overline{I} \rangle$ is the vacuum state of $\overline{I}$. Acting with the lowering operators on $|\Phi, I \rangle \otimes |\Psi, \overline{I} \rangle$ repeatedly, we obtain the degenerate excited energy eigenstates.

VII. CONCLUSION

In this paper, we have introduced a variant of the SYK model, which is referred to as the Wishart SYK model. We have numerically investigated the energy spectrum of the original and the Wishart SYK models for fermions/bosons. We have shown that there is a huge number of degeneracy in the ground state of the Wishart SYK models, and the degeneracy is given by Eq. (6). Then we have shown that the OTOC of the fermionic Wishart SYK model exhibits large temporal fluctuations at late times, i.e., $|J_{\text{mean}}|T \gg 1$ or $|J|T \gg 1$. The large fluctuations are explained by the small effective dimension brought by the huge degeneracy. We have also numerically investigated the level statistics and found that the level statistics of the fermionic Wishart SYK model follows the Poisson distribution. Correspondingly, we have shown that the fermionic Wishart SYK model is integrable by mapping it onto a particular case of the Richardson-Gaudin model and by writing the Hamiltonian as a sum of mutually commuting operators.

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Appendix A: Condition of the equalities in (6) and (7)

In this appendix, we show that the equalities in (6) and (7) indeed hold for the fermionic Wishart SYK model. The estimation of the zero-energy states (6) and (7) is based on the rank-nullity theorem (5). If the operator $Q$ is surjective, the equalities in (6) and (7) are achieved.

We transform the operator $Q$ by the conjugation argument in Refs. [55, 56]. Let us consider an invertible
transformation $V$ and introduce the conjugated operator $\tilde{Q}$ as
\begin{align}
\tilde{Q} := VQV^{-1}, \\
V := \prod_{j,\sigma} V_{j,\sigma}, \\
V_{j,\sigma} := 1 + (\sqrt{A_j} - 1)\eta_{j,\sigma}.
\end{align}
We can easily check that the inverse operator of $V$ is given by
\begin{align}
V^{-1} := \prod_{j,\sigma} V_{j,\sigma}^{-1}, \\
V_{j,\sigma}^{-1} := 1 + \{(\sqrt{A_j})^{-1} - 1\}\eta_{j,\sigma}.
\end{align}
With the conjugated operator, we define the corresponding Hamiltonian as
\begin{equation}
\tilde{H}_{wSYK} := \tilde{Q}^\dagger \tilde{Q},
\end{equation}
which has the same number of zero-energy eigenstates as $H_{wSYK}$. We note that the conjugated Hamiltonian $\tilde{H}_{wSYK}$ corresponds to the quasispin limit discussed in Ref. [54].

By the above transformation, we find
\begin{equation}
\tilde{Q} = \sum_{j=1}^{M} \eta_{j}^-.
\end{equation}
We can identify $\eta_{j}^-$ and $\eta_{j}^+$ with angular momentum operators of an SU(2) spin. Then, the conjugated operator $\tilde{Q}$ is the lowering operator of the total angular momentum of $M$ spins. We define
\begin{equation}
\tilde{Q}^\dagger = \sum_{j=1}^{M} \eta_{j}^+,
\end{equation}
whose eigenvalues are written as $k/2$ with $k = -M, -M + 1, \cdots, M - 1, M$. From the theory of angular momentum coupling, $\tilde{Q}$ is surjective in the subspace spanned by the eigenstates with non-positive $k$, if $N_{p} \leq \left\lfloor \frac{N+1}{2} \right\rfloor$. Thus, the equalities in (B1) and (B2) are achieved for the fermionic Wishart SYK model.

Appendix B: Construction of an anti-unitary operator

In Sec. [6] we showed that the level statistics of the bosonic Wishart SYK model is GUE with $N_{p} \neq N/2 + 1$ and GOE with $N_{p} = N/2 + 1$. This result suggests that there exists an anti-unitary operator commuting with the Hamiltonian, which can be defined only for $N_{p} = N/2 + 1$. In this appendix, we explicitly construct such an anti-unitary operator.

We first recall that from the random matrix theory, the level statistics is GOE (GSE) if $P^2 = +I$ ($-I$) is satisfied, where $P$ is an anti-unitary operator commuting with the Hamiltonian.

We consider the Hamiltonian of the Wishart SYK model in the form of Eq. (3). Let $P$ be a particle-hole operator defined as
\begin{equation}
P := K \prod_{i=1}^{N} (a_i + a_i^\dagger),
\end{equation}
where $K$ is an operator of complex conjugation and $a_i = c_i (b_i)$ for the fermionic (bosonic) model. We note that $P$ is anti-unitary. For hard-core bosons, $P$ satisfies $P^2 = +I$. For fermions, $P$ satisfies $P^2 = +I$ if $N = 0$ (mod 4) and $P^2 = -I$ if $N = 2$ (mod 4). We find that $PQP = Q^\dagger$, $PQ^\dagger P = Q$ and $PHP = QQ^\dagger$ by simple calculations.

We define
\begin{equation}
\tilde{P} := PQ|Q|^{-1},
\end{equation}
where $|Q| := \sqrt{Q^\dagger Q}$. We should note that $\tilde{P}$ can be defined only for $N_{p} = N/2 + 1$, because $|Q|^{-1}$ cannot be defined in the other cases. By simple calculations, we find that $\tilde{P}$ is anti-unitary and satisfies $P^2 = \pm I$, and $[H, \tilde{P}] = [Q^\dagger Q, \tilde{P}] = 0$. Thus, the existence of the operator $\tilde{P}$ explains why the level statistics of the bosonic Wishart SYK model obeys GOE only when $N_{p} = N/2 + 1$ as shown in Fig. [6].

Although this discussion holds both for the fermionic and bosonic Wishart SYK models, the level statistics of the fermionic Wishart SYK model is not GOE nor GSE, because it is integrable as discussed in Sec. [6].

As a side remark, let us also consider a variant of the fermionic Wishart SYK model
\begin{equation}
H_{wSYK, s} := Q_4^\dagger Q_4, \\
Q_4 := \sum_{1 \leq k, l, m, n \leq N} J_{k, l, m, n} c_k c_l c_m c_n.
\end{equation}
With this model, the above discussion of the construction of $\tilde{P}$ applies for $N_{p} = N/2 + 2$, and we have numerically confirmed that the level statistics of the model is GSE for $N = 14$ and GOE for $N = 16$ (data not shown). Thus, the model is unlikely to be integrable.

Appendix C: Linear independence of the conserved charges

We show the linear independence of the mutually commuting operators $\{h_j\}$ defined in (28) of Sec. [3]. We first define the projection operator $P_j$ as
\begin{equation}
P_j := \eta_j^+ \eta_j^-.
\end{equation}
We also define the inner product of operators $A, B$ using the trace as
\[(A, B) := \text{tr} [A^\dagger B] / 2^M.\] (C2)

The inner product of $P_j$ and $h_k$ is calculated as
\[(P_j, h_k) = \begin{cases} 1/4 & (j = k), \\ 1/16 & (j \neq k). \end{cases} \] (C3)

Let us consider the following equation:
\[\sum_{i=1}^{M} a_i h_i = 0, \] (C4)
where $a_i \in \mathbb{C}$. Taking the inner product of $P_j$ and the left-hand side of Eq. (C4), we obtain
\[L_{jk} = \begin{cases} 1/4 & (j = k), \\ 1/16 & (j \neq k). \end{cases} \] (C5)
where $L$ is a $M \times M$ matrix, whose elements are given by
\[L_{jk} = \begin{cases} 1/4 & (j = k), \\ 1/16 & (j \neq k). \end{cases} \] (C6)

Since $\det L = (M + 3)3^{M-1}/16^M \neq 0$, Eq. (C5) only has a solution $a_1 = \cdots = a_M = 0$, which implies the linear independence of $\{h_j\}$.

Appendix D: Generator of SU(2) and another algebra

In this appendix, we define the generator of SU(2) and another algebra. Let us first introduce the generators of SU(2) symmetries:
\[S^+_j := f^\dagger_{j,\uparrow} f_{j,\downarrow}, \quad S^-_j := f^\dagger_{j,\downarrow} f_{j,\uparrow}, \] (D1)
\[S^z_j := \frac{1}{2}(f^\dagger_{j,\uparrow} f_{j,\downarrow} - f^\dagger_{j,\downarrow} f_{j,\uparrow}), \] (D2)
\[\eta^+_j := f^\dagger_{j,\uparrow} f_{j,\uparrow}, \quad \eta^-_j := f_{j,\downarrow} f^\dagger_{j,\downarrow}, \] (D3)
\[\eta^z_j := \frac{1}{2}(f^\dagger_{j,\uparrow} f_{j,\downarrow} + f^\dagger_{j,\downarrow} f_{j,\uparrow} - 1). \] (D4)

From the commutation relations among $f^\dagger_{j,\sigma}$ and $f_{j,\sigma}$, we can verify the following relations:
\[S^+_j, S^-_j = 2\delta_{j,k} S^z_j, \quad [S^+_j, S^z_j] = [S^-_j, S^z_j] = 0, \] (D5)
\[[\eta^+_j, \eta^-_k] = 2\delta_{j,k} \eta^z_j, \quad [\eta^+_j, \eta^z_k] = [\eta^-_j, \eta^z_k] = 0, \] (D6)
\[S^z_j, S^z_k = 2\delta_{j,k} \eta^z_j, \quad S^+_j, S^-_j = 0, \] (D7)
\[(\alpha, \beta = z, +, -). \] (D8)
\[(\alpha, \beta = z, +, -). \] (D9)

We note that the SU(2) symmetry generated by $\eta^z$-pairing operators has been discussed in the context of the Hubbard model \[11, 12\]. According to Refs. [12, 13], we also introduce another algebra generated by
\[S^+_m := \sum_{j=1}^{M} (\lambda_j)^{2m+1} S^z_j, \quad S^-_m := \sum_{j=1}^{M} (\lambda_j)^{2m} S^z_j, \] (D10)
\[T^+_m := \sum_{j=1}^{M} (\lambda_j)^{2m} T^z_j, \quad T^-_m := \sum_{j=1}^{M} (\lambda_j)^{2m+1} T^z_j. \] (D11)

From the commutation relations from (D5) to (D7), we can verify the following relations:
\[S^z_m, S^-_n = 2S^-_m, S^z_n = 2S^z_m, S^-_n = \pm S^z_m, S^-_n = \pm S^z_m, \] (D12)
\[(\alpha, \beta = z, +, -). \] (D13)
\[(\alpha, \beta = z, +, -). \] (D14)

Appendix E: Fermionic Wishart SYK model with the algebraic Bethe ansatz

We show the integrability of the fermionic Wishart SYK model with the algebraic Bethe ansatz \[12, 14\]. With $f_{j,\sigma}$ and $f^\dagger_{j,\sigma}$, we introduce $S^\alpha_m$ and $\eta^\alpha_j (\alpha = +, -, z)$ and $S^\beta_m$ and $T^\beta_m (\alpha = +, -, z)$ as in Appendix D. Using $S^\alpha_m$ and $T^\alpha_m$, we write the Hamiltonian of the fermionic Wishart SYK model as
\[H = T^z_m T^z_m. \] (E1)
One can show that the Hamiltonian commutes with all of $S^\alpha_m$. Thus, we can find many conserved charges, for example, $C := i(S^+_0 - S^-_0)$.

The Hamiltonian (E1) can be diagonalized by using the algebraic Bethe ansatz (ABA). To see this, let us introduce the following operators
\[T^\pm(x) = \sum_{j=1}^{M} \frac{\lambda_j}{1 - (\lambda_j)^2 x} \eta^\pm_j, \] (E2)
\[T^z(x) = \sum_{j=1}^{M} \frac{(\lambda_j)^2}{1 - (\lambda_j)^2 x} \eta^z_j. \] (E3)
where $T^\pm(0) = T_0^\pm$. The key relations for the ABA are

$$[T^+(x), T^-(0)] = [T^-(0), T^-(x)] = 2T^z(x), \quad (E4)$$

$$[T^z(x), T^z(y)] = \pm \frac{T^\pm(x) - T^\pm(y)}{x-y}. \quad (E5)$$

$$T^-(x)|\text{vac}\rangle = 0, \quad (E6)$$

$$T^z(x)|\text{vac}\rangle = -\frac{1}{2} \left( \sum_{j=1}^{M} \frac{(\lambda_j)^2}{1 - (\lambda_j)^2} \right) |\text{vac}\rangle. \quad (E7)$$

The eigenstates of $H$ can be constructed by acting with $T^+(z_j)$ ($j = 1, 2, \ldots, n$) on the vacuum. The ansatz state reads

$$|\Psi(z_1, z_2, \ldots, z_n)\rangle := T^+(z_1)T^+(z_2)\cdots T^+(z_n)|\text{vac}\rangle. \quad (E8)$$

Here we assume that $z_j$'s are distinct. By acting with $H$ on this state, we obtain

$$H|\Psi(z_1, z_2, \ldots, z_n)\rangle = \sum_{j=1}^{n} \left( \frac{\lambda_j^2}{1 - (\lambda_j)^2} + 2 \sum_{k \neq j}^{n} \frac{1}{z_j - z_k} \right) |\Psi(0, z_1, \ldots, z_j-1, z_{j+1}, \ldots, z_n)\rangle. \quad (E9)$$

Suppose that none of $z_j$ ($j = 1, 2, \ldots, n$) is 0. Then, the ansatz state (E8) is an eigenstate of $H$ with eigenvalue 0, if $z_j$'s satisfy the following Bethe equations:

$$\sum_{i=1}^{M} \frac{(\lambda_i)^2}{1 - (\lambda_i)^2} + 2 \sum_{k \neq j}^{n} \frac{1}{z_j - z_k} = 0 \quad (E10)$$

for all $j = 1, 2, \ldots, n$. The situation is different when one of $z_j$'s is 0. In this case, for example, $|\Psi(0, z_2, \ldots, z_n)\rangle$ is an eigenstate of $H$ with eigenvalue

$$E = \sum_{i=1}^{M} (\lambda_i)^2 - 2 \sum_{k=2}^{n} \frac{2}{z_k} \quad (E11)$$

demanding that the following equations hold for all $j = 2, 3, \ldots, n$:

$$\sum_{i=1}^{M} \frac{(\lambda_i)^2}{1 - (\lambda_i)^2} + \frac{2}{z_j} + 2 \sum_{k \neq j}^{n} \frac{1}{z_j - z_k} = 0. \quad (E12)$$

This reproduces the previous results in [54].

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