Thermal conductivity of color-flavor locked quark matter

Matt Braby, Jingyi Chao and Thomas Schäfer

Physics Department
North Carolina State University
Raleigh, NC 27695, USA

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Abstract

We compute the thermal conductivity of color-flavor locked (CFL) quark matter. At temperatures below the scale set by the gap in the quark spectrum, transport properties are determined by collective modes. In this work we focus on the contribution from the lightest modes, the superfluid phonon and the massive neutral kaon. The calculation is done in the framework of kinetic theory, using variational solutions of the linearized Boltzmann equation. We find that the thermal conductivity due to phonons is \( \kappa^{(P)} \sim 1.04 \times 10^{26} \, \mu_{500}^8 \Delta_{50}^{-6} \) erg cm\(^{-1}\) s\(^{-1}\) K\(^{-1}\) and the contribution of kaons is \( \kappa^{(K)} \sim 2.81 \times 10^{21} f_{\pi,100}^4 T_{\text{MeV}}^{1/2} m_{10}^{-5/2} \) erg cm\(^{-1}\) s\(^{-1}\) K\(^{-1}\). These values are smaller than previous estimates, but still much larger than (in the case of phonons) or similar to (for kaons) the corresponding values in nuclear matter. From the phonon thermal conductivity we estimate that a CFL quark matter core of a compact star becomes isothermal on a timescale of a few seconds.
I. INTRODUCTION

In this paper, we explore the thermal conductivity of quark matter in the regime of large baryon density and low temperature. It is expected that cold dense quark matter is a color superconductor, and that at asymptotically high density the ground state of three-flavor quark matter is the color-flavor locked (CFL) phase [1]. The work described in this paper is part of an ongoing research effort with the goal of determining the transport properties of not only the CFL phase, but also of other, less dense, phases of quark matter. Previous work has focused on the shear viscosity [2], bulk viscosity [3, 4], and neutrino emissivity of the CFL phase [5, 6]. A general discussion of the kinetics of a CFL superfluid is given in [7]. There are also calculations of the transport properties of unpaired quark matter [8–10], and some results for the transport properties of a kaon condensed CFL phase [11]. The long term goal is to connect these calculations of transport properties to possible observational signatures of high density quark matter phases in the core of compact stars.

The thermal conductivity of dense matter plays an important part in the cooling of compact stars [12–14]. The cooling history of a star depends on the rate of energy loss by neutrino emission from the bulk and, at very late times, photon emission from the surface, on the specific heat, and on the thermal conductivity. Thermal conductivity determines how quickly different layers of the star become isothermal. It is generally assumed that the core of the star becomes isothermal very quickly, but that the outer crust is a poor conductor of heat and may take several hundreds of years to reach the temperature of the inner core [15–17]. In order to verify that this is true for CFL quark matter core we have to compute the thermal conductivity of CFL matter. There is an earlier calculation of the thermal conductivity of the CFL phase [18]. This calculation was based on a simple mean free path estimate. In the present paper we will perform a more definitive calculation based on the linearized Boltzmann equation. We will show that while the mean free path estimate is not reliable, the main conclusion of [18] is valid – a CFL quark matter core becomes isothermal on a very short time scale.

The thermal conductivity, like other transport properties, depends on the properties of quasiparticle excitations. In the CFL phase up, down, and strange quarks are all gapped, with the gaps typically much larger than the appropriate temperature of the compact stars. Hence, the quarks are unlikely to have any significant contribution to transport properties. Below the gap the excitation spectrum consists of collective modes associated with the spontaneously broken symmetries of the CFL phase, see [19–22]. There is a phonon mode related to superfluidity, which we can view as
the Goldstone boson associated with the breaking of the $U(1)$ of baryon number, and there is a meson octet coming from the breaking of the $SU(3)_c \times SU(3)_L \times SU(3)_R$ color and chiral flavor symmetry to the diagonal subgroup $SU(3)_{C+F}$. This diagonal subgroup is the residual symmetry of the color-flavor locked diquark condensate. The quantum numbers of the meson octet coincide with those arising from the breaking of chiral symmetry in the QCD vacuum, and we will refer to these modes as pions, kaons, etc. However, the mass hierarchy of the meson octet in CFL is different, [20], such that the kaons are the lightest excitations. In this paper, we will calculate the thermal conductivity due to the lightest excitations in the CFL phase, phonons and kaons.

This paper is organized as follows. In Section II and III we will outline the calculation of the thermal conductivity in kinetic theory, in Section IV we will discuss the interaction among the light excitations and derive the associated collision terms, and in Section V we will present numerical results. We end with some conclusions in Section VI.

II. THERMAL CONDUCTIVITY AND TRANSPORT THEORY

The thermal conductivity is defined in hydrodynamics as

$$q = -\kappa \nabla T$$  \hspace{1cm} (2.1)

where $q$ is the heat flow and $\kappa$ is the thermal conductivity. In kinetic theory, the heat flux can be written in terms of the quasi-particle distribution function as

$$q = \int \frac{d^3p}{(2\pi)^3} v_p E_p \delta f_p$$  \hspace{1cm} (2.2)

where $v_p = \partial E_p / \partial p$ is the particle velocity and $\delta f_p = f_p - f_p^0$ is the deviation of the distribution function from the form in local thermal equilibrium, given by

$$f_p^0 = \frac{1}{e^{(p_\mu u^\mu - \mu)/T} - 1}. \hspace{1cm} (2.3)$$

Here, $u_\mu$ is the local fluid velocity and $\mu$ is the chemical potential (which is zero in the case of phonons and equal to the hypercharge chemical potential in the case of the kaons). Since we are only interested in contributions to the stress-energy tensor arising from thermal gradients we can write $\delta f_p$ as

$$\delta f_p = -f_p^0 (1 + f_p^0) T \frac{g(p)}{T^3} p \cdot \nabla T,$$  \hspace{1cm} (2.4)
where \( g(p) \) is a dimensionless function of the magnitude of the momentum. Inserting this into Eq. (2.2), we can then read off the heat flux as

\[
q = -\frac{\nabla T}{3T^3} \int \frac{d^3p}{(2\pi)^3} f_p^0 (1 + f_p^0) v_p E_p p g(p)
\]

and the thermal conductivity as

\[
\kappa = \frac{1}{3T^3} \int \frac{d^3p}{(2\pi)^3} f_p^0 (1 + f_p^0) g(p) v_p E_p p
\]

Kinetic theory also gives some constrains on the form that \( \delta f_p \) can take. Conservation of particle number, energy and momentum imply [25]

\[
\int d\Gamma \delta f_p = \int d\Gamma E_p \delta f_p = \int d\Gamma p \delta f_p = 0,
\]

where \( d\Gamma = \frac{d^3p}{(2\pi)^3} \). Note that the number of phonons is not explicitly conserved. However, we will only consider \( 2 \leftrightarrow 2 \) processes and the particle number constraint is expected to hold. We can show that for \( \delta f_p \) of the form given in Eq. (2.4) the first and second constraints (energy and momentum conservation) are automatically satisfied, but the third one (particle number conservation) is non-trivial. We get

\[
0 = \int d\Gamma p \delta f_p = \frac{\nabla T}{3T^3} \int d\Gamma f_p^0 (1 + f_p^0) g(p) p^2.
\]

Note that if \( v_p \) is independent of \( p \), then the contribution to the thermal conductivity vanishes due to the constraint. This implies that phonons with an exactly linear dispersion relation do not contribute to the thermal conductivity. This result was first derived in connection with the transport properties of superfluid helium [26]. The thermal conductivity of superfluid helium is dominated by rotons and phonon-roton scattering. Note that the phonon and roton are part of the same excitation curve. The name roton refers to the part of the excitation curve in which the dispersion relation is very non-linear and \( E(p) \) develops a second minimum. In the following we will compute the thermal conductivity of the CFL phase due to non-linearities in the phonon dispersion relation, and due to massive kaons.

To solve for the thermal conductivity, we need to determine the form of \( g(p) \). The quasi-particle distribution functions satisfies the Boltzmann equation,

\[
\frac{df_p}{dt} = \frac{\partial f_p}{\partial t} + \mathbf{v}_p \cdot \frac{\partial f_p}{\partial \mathbf{x}} + \mathbf{F}_{ext} \cdot \frac{\partial f_p}{\partial \mathbf{p}} = C[f],
\]

where \( \mathbf{F}_{ext} \) is an external force and \( C[f] \) is the collision term. In the case of binary scattering the collision integral is given by

\[
C[f_p] = \frac{1}{2E_p} \int \frac{(2\pi)^4}{k,k',p'} |\mathcal{M}|^2 D_{2-2};
\]
where
\[ \int_q = \int \frac{d^3q}{(2\pi)^3 2E_q} \] (2.11)
is the one-particle phase space integral, and \( D_{2\rightarrow 2} = f_pf_k(1 + f_{p'})(1 + f_{k'}) - (1 + f_p)(1 + f_k)f_{p'}f_{k'} \) is the term involving the distribution functions for \( 2 \leftrightarrow 2 \) scattering. Linearizing in \( \delta f \) we can write
\[ \delta D = - \frac{f_pf_k(1 + f_{p})(1 + f_{k})}{T^3} \Delta_g \cdot \nabla T, \] (2.12)
and
\[ \Delta_g = g(p)p + g(k)k - g(k')k' - g(p)p'. \] (2.13)
Note that the collision integral will vanish for \( D = D_0 \). This is a consequence of detailed balance, which implies that \( |M|^2 \) is invariant under \( (p, k) \leftrightarrow (p', k') \) and that
\[ f_p^0 f_k^0 (1 + f_{p'}^0)(1 + f_{k'}) = (1 + f_p^0)(1 + f_k^0)f_{p'}^0 f_{k'}^0. \]
We can then write the collision integral as
\[ C[f_p] \equiv - \frac{F \cdot \nabla T}{T^2}, \] (2.14)
where
\[ F = \frac{1}{2E_p} \int d\Gamma_{k,k',p'} \frac{f_p^0 f_k^0 (1 + f_{p'}^0)(1 + f_{k'})}{T} \Delta_g \] (2.15)
and
\[ \int d\Gamma_{k,k',p'} = \int \frac{(2\pi)^4 \delta(P + K - P' - K') |M|^2}{(P + K - P' - K')}. \] (2.16)
We can simplify the left-hand side of the Boltzmann equation using our definition of the distribution function in Eq. (2.3). In general, we can write
\[ \frac{df}{dt} = -\alpha_p \frac{f_p^0 (1 + f_p^0)}{T^2} p \cdot \nabla T, \] (2.17)
where the form of \( \alpha_p \) depends on the properties of the quasi-particles we are studying. In the following we will consider two cases. The first is phonons with a non-linear dispersion relation
\[ E_p^{(P)} = vp \left( 1 + \gamma p^2 \right) \equiv vp \left( 1 + \epsilon \frac{v^2 p^2}{T^2} \right), \] (2.18)
where \( v = 1/\sqrt{3} \) is the speed of sound, \( \gamma \) controls the curvature of the dispersion relation, and \( \epsilon \equiv \gamma T^2/v^2 \) is a dimensionless measure of the non-linearity. The parameter \( \gamma \) was computed by Zarembo [27], who finds \( \gamma = -11/(540 \Delta^2) \), where \( \Delta \) is the gap in the fermion spectrum. The second case is massive kaons with
\[ E_p^{(K)} = \sqrt{v_k^2 p^2 + m_K^2} \cong \frac{v_k^2 p^2}{2m_K} + m_K, \] (2.19)
where $v_K = v$. We will specify the kaon mass $m_K$ in Section IV B. The quasi-particle velocities, $v_p = \frac{\partial E_p}{\partial p}$, are given by
\[
v_{p}^{(P)} = v \left(1 + 3\epsilon v_p^2 \frac{T^2}{2}ight), \quad v_{p}^{(K)} = \frac{v^2 p}{m_K}.
\]
(2.20)

The explicit form of $\alpha_p$ is derived in Appendix A. We find
\[
a_p^{(P)} = 4\epsilon v^2 \left(\frac{v^2 p^2}{T^2} - \frac{20\pi^2}{7}\right), \quad a_p^{(K)} = \frac{v^4 p^2}{2m_K^2} - \frac{5v^2 T}{2m_K}.
\]
(2.21)

We have now expressed the LHS of the Boltzmann equation, Eq. (2.17), in terms of the function $\alpha_p$. The RHS of the Boltzmann equation, Eq. (2.14), is given in terms of the unknown function $g(p)$. Once $g(p)$ is determined the thermal conductivity is given by Eq. (2.6). In practice we will solve the Boltzmann equation using a variational procedure. This procedure is based on an expression for the thermal conductivity in terms of a suitable integral over the collision term, which we will now derive. Consider the following integral over the LHS of the Boltzmann equation,
\[
\frac{c}{T} \int d\Gamma \frac{df}{dt}g(p)p = -\frac{c\nabla T}{3T^3} \int d\Gamma f_p^0(1 + f_p^0)p^2\alpha_p g(p),
\]
(2.22)

where $c$ is a constant. We observe that the RHS of this equation has the same structure as Eq. (2.5) for the heat flux. For the two expressions to be equal, we need
\[
c \int d\Gamma f_p^0(1 + f_p^0)p^2\alpha_p g(p) = \int d\Gamma f_p^0(1 + f_p^0)p^2\frac{v_p E_p}{p} g(p).
\]
(2.23)

The two sides of this equation can be matched by using the constraint Eq. (2.8). The constraint implies that constant terms in $v_p E_p/p$ and $\alpha_p$ do not contribute to Eq. (2.23). We can then determine $c$ by matching the $p^2$ terms in $v_p E_p/p$ and $\alpha_p$. We find $c = 1$ for both phonons and kaons. Using this fact together with the Boltzmann equation we can express the heat current in terms of the collision integral
\[
q = -\frac{\nabla T}{3T^3} \int d\Gamma g(p) p \cdot F.
\]
(2.24)

The corresponding expression for the thermal conductivity is
\[
\kappa = \frac{1}{12T^4} \int d\Gamma_{p,k,k',g'} \Delta_g^2,
\]
(2.25)

where
\[
\int d\Gamma_{p,k,k',g'} = \int_{p_k p'_k} (2\pi)^4 \delta(P + K - K' - P') |\mathcal{M}|^2 f_p^0 f_{p'}^0 (1 + f_k^0) (1 + f_{k'}^0)
\]
(2.26)

and we have used Eq. (2.15) and the symmetries of the matrix elements to derive Eq. (2.25). We now have two expressions for the thermal conductivity, Eq. (2.6) and Eq. (2.25). Both equations
depend on the unknown function \(g(p)\), but the equivalence of the two equations rests on the fact that \(g(p)\) satisfies the Boltzmann equation. As we will now show, this fact can be used to determine \(g(p)\).

III. VARIATIONAL SOLUTION TO THE BOLTZMANN EQUATION

We will solve the Boltzmann equation by expanding \(g(p)\) in a basis of orthogonal polynomials. The procedure is variational in the sense that the result for \(\kappa\) obtained in a truncated basis provides an upper bound on \(\kappa\) for the exact solution. The expansion has the form

\[
g(p) = \sum_s b_s B_s(p^2),
\]

(3.1)

where \(B_s(p^2)\) is a polynomial in \(p^2\) of order \(s\). The coefficient of the highest power is set to one. This means that \(B_0 = 1, B_1 = p^2 + c_{10},\) etc. The polynomials \(B_s(p^2)\) are orthogonal with regard to the inner product

\[
\int d\Gamma f_0 p (1 + f_0 p) p^2 B_s(p^2) B_t(p^2) \equiv A_s \delta_{st}. \tag{3.2}
\]

The functions \(B_s(p^2)\) are a generalization of the Laguerre polynomials used in solutions of the linearized Boltzmann equation in classical physics [25]. Starting with \(B_0 = 1\), we can solve for all higher polynomials and their normalizations, \(A_s\). This is laid out in more detail in Appendix B.

Inserting the expansion Eq. (3.1) into the constraint Eq. (2.8) we get

\[
0 = \int d\Gamma f_p^0 (1 + f_p^0) p^2 B_s(p^2) B_0(p^2), \tag{3.3}
\]

where we have used \(B_0 = 1\). We conclude that the constraint is satisfied if \(b_0 = 0\). Using the polynomial expansion in the first expression for \(\kappa\), Eq. (2.6), gives

\[
\kappa = \frac{1}{3T^3} \sum_{s \neq 0} b_s \int d\Gamma f_p^0 (1 + f_p^0) p^2 B_s(p^2) (a_0 B_0 + a_1 B_1) = \frac{a_1}{3T^3} b_1 A_1, \tag{3.4}
\]

where we have written \(v_p E_p/p = a_0 B_0 + a_1 B_1\). The coefficients \(a_0\) and \(a_1\) are determined by Eq. (2.20). We get \(a_1 = 4 \epsilon v^4/T^2\) for phonons and \(a_1 = v^4/2m_K^2\) for kaons. Substituting the polynomial expansion into the second expression for the thermal conductivity, Eq. (2.25), gives

\[
\kappa = \frac{1}{12T^4} \sum_{s, t \neq 0} b_s b_t M_{st}, \tag{3.5}
\]
where

\[ M_{st} = \int d\Gamma_{p,k,k',p'} Q_s \cdot Q_t, \quad Q_s = B_s(p^2) p + B_s(k^2) k - B_s(k'^2) k' - B_s(p'^2) p'. \] (3.6)

Requiring Eq. (3.4) and Eq. (3.5) to be equal gives an equation for \( b_s \)

\[ \frac{a_1}{3T^3} \sum_{s \neq 0} b_s A_s \delta_{s1} = \frac{1}{12T^4} \sum_{s,t \neq 0} b_s b_t M_{st}, \] (3.7)

which is equivalent to the linear equation

\[ \sum_{t \neq 0} M_{st} b_t = \frac{4a_1 T}{3} A_1 \delta_{s1}. \] (3.8)

This equation can be solved by inverting matrix \( M \). We get

\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots
\end{pmatrix} = \frac{4a_1 T}{3} A_1 M^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}
\] (3.9)

This equation determines \( b_1 \) and, using (3.4), the thermal conductivity. We find

\[ \kappa = \left( \frac{4a_1^2}{9T^2} \right) A_1^2 M_{11}^{-1}, \] (3.10)

where \( M_{11}^{-1} \) is the (1,1)-element of the matrix \( M \), the constants \( a_1 \) are given below Eq. (3.4) and \( A_1 \) is given in Appendix A. \( M \) is an infinite matrix, and in practice we solve for \( \kappa \) by restricting the dimension of the matrix to a finite number \( N \). This procedure can be shown to be variational in the sense that [28]

\[ \kappa \geq \left( \frac{4a_1^2}{9T^2} \right) \frac{(b_1 A_1)^2}{\sum_{s,t \neq 0} b_s b_t M_{st}} \] (3.11)

for any value of \( N \). This right hand side can be written as

\[ \kappa \geq \left( \frac{4a_1^2}{9cT^2} \right) A_1^2 M_{11}^{-1}, \] (3.12)

where \( M^{-1} \) is the the inverse of the truncated \( N \times N \) matrix. The bound is saturated as \( N \rightarrow \infty \).

IV. COLLISION TERMS

The matrix \( M \) depends on the 2 \( \leftrightarrow \) 2 scattering amplitudes. In this Section we will compute the phonon and kaon scattering amplitude using a low energy effective lagrangian for the CFL phase.
The effective lagrangian for the phonon field $\phi$ is given by [29]

$$\mathcal{L} = \frac{1}{2} (\partial_0 \phi)^2 - \frac{1}{2} v^2 (\partial_i \phi)^2 - \frac{\pi}{9\mu_q^2} \partial_0 \phi (\partial_\mu \phi \partial^\mu \phi) + \frac{\pi^2}{108\mu_q^4} (\partial_\mu \phi \partial^\mu \phi)^2 + \ldots,$$

(4.1)

where $v = 1/\sqrt{3}$ is the speed of sound and $\mu_q$ is the quark chemical potential. We have displayed the leading three and four-phonon vertices. Higher order terms include higher powers of $\phi$ or additional derivatives. These terms are suppressed by powers of the typical momentum over the quark chemical potential. The speed of sound as well as the coefficients of the three and four-phonon vertices are given to leading order in the strong coupling constant. The lowest order diagrams that contribute to phonon-phonon scattering are shown in Figure 1.

The corresponding matrix elements were previously computed in connection with the phonon contribution to the shear viscosity of the CFL phase [2]. The matrix element $\mathcal{M}(P, K; P'K')$ where $P, K$ and $P', K'$ are the in and out-going four momenta can be written as the sum of the contact term $\mathcal{M}_c$ and the $s, t, u$-channel phonon exchange diagrams $\mathcal{M}_{s,t,u}$. The individual terms are

$$i\mathcal{M}_c = \lambda [(P \cdot K)(P' \cdot K) + (P \cdot K')(P' \cdot K) + (P \cdot P')(K \cdot K')]$$

$$i\mathcal{M}_s = g^2 \left[2(p_0 + k_0)P \cdot K + p_0K^2 + k_0P^2\right] \left[2(p'_0 + k'_0)P' \cdot K' + p'_0K'^2 + k'_0P'^2\right] G(P + K)$$

$$i\mathcal{M}_t = i\mathcal{M}_s(P \leftrightarrow K')$$

$$i\mathcal{M}_u = i\mathcal{M}_t(P \leftrightarrow K),$$

(4.2)

where $G(Q)$ is the phonon propagator and the coupling constants $\lambda$ and $g$ are given by

$$\lambda = \frac{4\pi^2}{108\mu_q^4}, \quad g = \frac{2\pi}{9\mu_q^2}.$$

(4.3)

At leading order the phonon propagator is given by $G(Q) = (q_0^2 - E_q^2)^{-1}$ with $E_q = vq$. The thermal conductivity is sensitive to non-linearities in the dispersion relation and we will use $E_q$ as given in
Eq. (2.18). We note that non-linear effects are generated by higher derivative terms in the effective lagrangian. Except for the non-linearities in the dispersion relation the structure of these terms has not been determined. We have verified that the matrix elements $M_{st}$ are not sensitive to higher order corrections in the vertices. We also note that a non-linear dispersion (with $\epsilon < 0$) regulates a possible on-shell divergence of the matrix element, and as consequence there is no need to include self-energy corrections in the fermion propagator as in [2, 30].

We can now insert the scattering amplitude into Eq. (3.6) and compute the matrix elements $M_{st}$. Using the energy and momentum conserving delta functions as well as spherical symmetry the 12-dimensional phase space integral in Eq. (3.6) can be reduced to a 5-dimensional integral. First, we use the momentum conserving delta function to integrate over $p'$. We can choose $p$ to lie along the $z$-direction and perform the corresponding angular integrals. The energy conserving part of the delta function can be used to fix the magnitude of $k'$. Finally, we note that the integrand only depends on the relative azimuthal angle of $k$ and $k'$ and we can trivially integrate over the other azimuthal angle. The remaining 5-dimensional integral is computed numerically using standard methods, such as the VEGAS routine [31].

B. Massive Kaon

The effective lagrangian for the meson octet is given by [19–21]

$$
\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr} \left[ \nabla_0 \Sigma \nabla_0 \Sigma^\dagger - \nu^2 \nabla \Sigma \nabla \Sigma^\dagger \right] + \frac{af_\pi^2}{2} \text{Tr} \left[ M^{-1} |M| (\Sigma + \Sigma^\dagger) \right]
$$

(4.4)

where $\Sigma = \exp(\frac{i\theta^a t^a}{f_\pi})$ is the chiral field, $t^a$ with $a = 1, \ldots, 8$ are the $SU(3)$ Gell-Mann matrices, $f_\pi$ is the pion decay constant, $M = \text{diag}(m_u, m_d, m_s)$ is the quark mass matrix, and $|M|$ is the determinant of the mass matrix. The covariant derivative of the chiral field is given by $\nabla_0 \Sigma = \partial_0 \Sigma - i[A, \Sigma]$ where $A$ is the effective chemical potential $A = -M^2/(2\mu_q)$. At asymptotically high density the constants $f_\pi$, $\nu$, and $a$ can be determined by matching the effective theory to perturbative QCD [20, 21, 32]

$$
f_\pi^2 = \frac{21 - 8 \ln 2}{18} \frac{\mu^2}{2\pi^2}, \quad \nu = \frac{1}{\sqrt{3}}, \quad a = \frac{3\Delta^2}{\pi^2 f_\pi^2},
$$

(4.5)

where $\Delta$ is the fermionic energy gap at zero temperature. For $m_s \gg m_u, m_d$ the lightest excitations in the theory are kaons. The leading terms in the effective theory involving kaons are $\mathcal{L}_0 + \mathcal{L}_{int}$
Energy

$K^-$

$\uparrow \delta m$

$\bar{K}^0$

$\downarrow m^2_s/2\mu$

$K^+$

$\uparrow \delta m$

$K^0$

$FIG. 2$: Spectrum of energies for the charged and neutral kaons. The mass splitting between $K^0$ and $K^+$ is given by $\delta m$ and the larger splitting between the kaons containing $\bar{s}$ and those containing $s$ is given by $m^2_s/2\mu$.

with

$$L_0 = \frac{1}{2}(\partial_\mu K^0)(\partial^\mu K^0) + \frac{1}{2}(\partial_\mu K^+)(\partial^\mu K^-) - \frac{1}{2}m^2_{K^0}K^0K^0 - \frac{1}{2}m^2_{K^+}K^+K^-,$$

$$L_{int} = -\frac{1}{24}\left[\lambda_0 (K^0\bar{K}^0)^2 + \lambda_+ (K^+K^-)^2 + (\lambda_0 + \lambda_+) (K^0\bar{K}^0)(K^+K^-)\right], \quad (4.6)$$

where $\lambda_{0,+} = m^2_{K^0,+}/f^2_\pi$ and

$$m^2_{K^0} = am_s(m_d + m_s), \quad m^2_{K^+} = am_d(m_u + m_s). \quad (4.7)$$

We have not included kaon interactions that involve derivatives of the kaon field. These terms arise from expanding the first term in (4.4), see [33]. For energies and momenta that are small compared to the kaon mass these terms are suppressed as compared to the terms in Eq. (4.6).

The spectrum of kaons is shown schematically in Figure 2. The mass difference between the isospin doublets $(K^0, K^+)$ and $(\bar{K}^0, K^-)$ is proportional to $m^2_s/(2\mu_q)$, while the mass splitting with the doublets is $\delta m^2 = am_s(m_d - m_u)$. In the following we will only include the lightest kaon, the $K^0$. The relevant interaction term is

$$L_{int} = -\frac{m^2_{K^0}}{24f^2_\pi}(K^0\bar{K}^0)^2, \quad (4.8)$$

and the corresponding scattering amplitude is

$$|\mathcal{M}|^2 = \frac{m^4_{K^0}}{f^2_\pi} \quad (4.9)$$
This expression can be inserted into Eq. (3.6) in order to compute the matrix $M_{st}$. As in the case of phonon scattering, the phase space integral can be reduced to a 5-dimensional integral over the magnitudes of $p$ and $k$, the polar angles $\cos(\theta_k)$ and $\cos(\theta_{k'})$ describing the angles between $p$ and the respective momenta, and one azimuthal angle.

V. RESULTS

Before we show numerical results for the thermal conductivity, we would like to discuss the dependence of $\kappa$ on the dimensionful parameters that appear in the problem, the temperature $T$, the quark chemical potential $\mu_q$, the fermion gap $\Delta$, and the mass of the kaon. We begin by studying the scaling of the matrix elements $M_{st}$. In the phonon case, we can rescale all momenta by the temperature and write $M_{st}$ as a product of the dimensionful parameters and a dimensionless phase space integral. A subtlety arise due to the possible role of on-shell divergences. There is no on-shell sensitivity in the phonon contribution to the shear viscosity and the corresponding matrix elements exhibit the naive scaling behavior, but the phonon mean free path is sensitive to the phonon self energy near the on-shell point [2, 30]. We find that the thermal conductivity also depends on the behavior of the phonon propagator near the on-shell point. In the present calculation this behavior is governed by non-linearities in the dispersion relation. Consider the contribution to $M_{st}$ from nearly on-shell phonons in the $s, t$ or $u$-channel. We find

$$M_{st}^{(P)} \sim \frac{T^{2s+2t+14}}{\mu^8} \int \frac{dq^2 d\Gamma'}{(q^2 + \epsilon f(q, \Gamma'))^2} \sim \frac{T^{2s+2t+14}}{\mu^8} \frac{1}{|\epsilon|} \sim \frac{T^{2s+2t+12} \Delta^2}{\mu^8}$$

(5.1)

where $q$ represent the momentum transfer in the relevant channel, $d\Gamma'$ is a phase space integral over the remaining momenta, and $f(q, \Gamma')$ is a function of these variables.

In the kaon case we can assume that the temperature is smaller than all the other parameters and write the dispersion as $E_q = m_K + q^2/(2m_K)$. We rescale the momenta by $\sqrt{m_K T}$ and extract all dimensionful parameters from the phase space integral. We get

$$M_{st}^{(K)} \sim \frac{\lambda_0^2}{m_K^3} (m_K T)^{s+t+9/2} e^{-2\delta m/T} \sim \frac{m_K}{f_{\pi}^4} (m_K T)^{s+t+9/2} e^{-2\delta m/T}$$

(5.2)

Using Eq. (5.1) and Eq. (5.2), as well as the scaling of the normalization factor $A_1$ derived in Appendix B, we find

$$\kappa \sim \begin{cases} \frac{\mu^8}{\Delta^6} & \text{phonons,} \\ \frac{f_{\pi}^4}{m_K} \sqrt{\frac{T}{m_K}} & \text{kaons.} \end{cases}$$

(5.3)
We note that the contribution to the thermal conductivity from phonons is independent of temperature. The naive power counting gives $\kappa \sim \mu^8/T^6$, but the contribution from nearly on-shell phonons modifies the naive scaling behavior by a factor $\epsilon^3$. Two powers of $\epsilon$ come from the $a_1^2$ term and one power of $\epsilon$ comes from the nonlinear dispersion cutting off the collinear regimes in the propagators. Using $\epsilon \sim T^2$ then leads to a thermal conductivity that is independent of temperature.

The thermal conductivity due to kaons is not exponentially suppressed by $\exp(-m_K/T)$ as one might naively expect, because the exponential factors in $A_1^2$ and $M_{11}$ cancel at leading order. This result is analogous to the well known fact that the transport properties of a dilute gas are independent of the density of the gas. We should note, however, that in practice the range of validity of this result is limited to temperatures that are not too small. At very small temperature the kaon mean free path is bigger than the system size, and heat transport is no longer a diffusive process.

Numerical results for the thermal conductivity are shown in Figs. 3 and 4. In Fig. 3 we show the dependence of $\kappa$ on the number of basis functions. We observe that the convergence in the case
FIG. 4: Numerical results for the contribution of phonons and kaons to the thermal conductivity of the CFL as a function of temperature. The results shown in this figure were obtained using the variational method with $N = 5$. We have used $\mu = 400$ MeV, $\Delta = 100$ MeV for the phonons and $m_K = 10$ MeV, $\mu_K = 5$ MeV and $f_\pi = 100$ MeV for the kaons.

of the kaon contribution is excellent – using $N = 1, 2$ gives results within a few percent of the final result. Also, the rate of convergence is independent of the temperature. In the case of the phonon contribution the rate of convergence is slower, but going up to $N = 5$ appears to give converged results.

Fig. 4 shows the thermal conductivity as a function of temperature. We have used the results for $N = 5$. We observe that thermal conductivity due to kaons follows the square root dependence predicted by Eq. (5.3). Also, the thermal conductivity due to phonons is approximately temperature independent, in agreement with Eq. (5.3). The numerical results are well represented by

$$\kappa^{(P)} \gtrsim 4.01 \times 10^{-2} \frac{\mu^8}{\Delta^6} \text{MeV}^2,$$

$$\kappa^{(K)} \simeq 0.86 \frac{f_\pi^4}{m_K^2} \sqrt{\frac{T}{m_K}} \text{MeV}^2.$$  (5.4)

We observe that for $m_K$ in the few MeV range the phonon contribution is significantly larger than the kaon contribution.
VI. CONCLUSIONS

We can convert these result into CGS units. We get

\[ \kappa^{(P)} \gtrsim 1.04 \times 10^{26} \frac{\mu_{500}^8}{\Delta_{50}^9} \frac{\text{erg}}{\text{cm s K}} , \quad \kappa^{(K)} \simeq 2.81 \times 10^{21} f_{\pi,100}^4 T_{\text{MeV}}^{1/2} m_{10}^{-5/2} \frac{\text{erg}}{\text{cm s K}} , \]  

(6.1)

where \( \mu_{500} \) is the quark chemical potential in units of 500 MeV, \( \Delta_{50} \) is the gap in units of 50 MeV, \( f_{\pi,100} \) is the pion decay constant in units of 100 MeV, \( T_{\text{MeV}} \) is the temperatures in units of MeV and \( m_{10} \) is the kaon mass in units of 10 MeV.

These numbers can be compared to a number of results in the literature. The thermal conductivity of the CFL phase was previously studied by Shovkovy and Ellis [18]. Based on a mean free path estimate they find that the contribution due to phonons is \( \kappa^{(P)} \simeq 0.7 \times 10^{32} T_{\text{MeV}}^3 R_{0,\text{km}} \frac{\text{erg}}{\text{cm s K}} \), where \( R_{0,\text{km}} \) is the phonon mean free path in units of 1 km. This particular number was chosen because the authors argued that the mean free path is so long that it is effectively cut off by the size of the quark matter core. In the present work we have demonstrated that in the framework of kinetic theory there is no simple connection between the phonon mean free path and the thermal conductivity. We note, however, that if one replaces the mean free path in the Shovkovy-Ellis estimate with a more accurate result from kinetic theory, which is on the order of millimeters rather than kilometers [2], one obtains a thermal conductivity which is close to our result. Shovkovy and Ellis also argued that there is a large contribution to \( \kappa \) from photons. The photon mean free path is indeed on the order of the size of the star [5]. This implies, however, that photons are not thermally coupled to the a CFL quark matter core.

The thermal conductivity of unpaired quark matter at low temperature is \( \kappa = 0.5 m_D^2/\alpha_s^2 \) [9], where \( m_D \) is the gluon screening mass and \( \alpha_s \) is the strong coupling constant. Using \( m_D \simeq 500 \) MeV and \( \alpha_s = 0.3 \) this gives \( \kappa \simeq 10^4 \) MeV\(^2\), about 5 orders of magnitude larger than the thermal conductivity of the CFL phase. The thermal conductivity of nuclear matter was studied in [34–38]. A typical value at nuclear matter saturation density and \( T \simeq 1 \) MeV is \( \kappa \simeq (10^{20} - 10^{21}) \frac{\text{erg}}{\text{cm s K}} \), which is about 5 orders of magnitude smaller than the thermal conductivity of the CFL phase.

We can convert the thermal conductivity to a timescale for thermal diffusion. The thermal diffusion constant is \( \chi = \kappa/c_V \), where \( c_V \) is the specific heat. The time scale for thermal diffusion over a distance \( R \) is

\[ \tau \simeq \frac{c_V R^2}{\kappa} . \]  

(6.2)

Using \( c_V = 2\pi^2 T^3/(15v^3) \) for phonons, as well as \( T = 1 \) MeV, \( \mu = 500 \) MeV, \( \Delta = 50 \) MeV, and
$R = 1 \text{ km}$ we get $\tau \simeq 10 \text{ s}$, indicating that a CFL quark core will become isothermal rapidly.

In this paper we have studied the thermal conductivity of the CFL phase using a linearized Boltzmann equation including a collision term that involves $2 \leftrightarrow 2$ scattering between phonons or kaons only. We have not studied a coupled transport equation for phonon-kaon scattering. Given that the relaxation times of phonons and kaons are quite different, it should be possible to find approximate solutions to the coupled problem. We have only provided a simple estimate for the relaxation time of temperature gradients. More accurate estimates will require a detailed model of the initial temperature profile, and a full solution of the dissipative two-fluid hydrodynamic equations for a CFL superfluid.

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APPENDIX A: STREAMING TERMS IN THE BOLTZMANN EQUATION

1. Superfluid Phonon

In this appendix we compute the quantity $\alpha_p$, which is related to the streaming terms (the left-hand side) of the Boltzmann equation. We will use the dispersion relation for the phonon given by

$$E_p = v_p \left(1 + \gamma p^2\right) \equiv v_p \left(1 + \epsilon \frac{v^2 p^2}{T^2}\right). \quad (A.1)$$

The left-hand side of the Boltzmann equation can be written as

$$\frac{df_p}{dt} = \frac{\partial f_p}{\partial t} + v_p \cdot \nabla f_p,$$

where $v_p$ is the particle velocity. In local thermal equilibrium the time dependence of $f_p$ arises from the time dependence of the local temperature and fluid velocity. We have

$$\frac{\partial f_p}{\partial t} = -\frac{f_p^0 (1 + f_p^0)}{T} \left(\frac{E_p}{T} \frac{\partial T}{\partial t} + p \cdot \frac{\partial u}{\partial t}\right), \quad (A.3)$$

where we have used the fact that by going to the local rest frame we can set the local fluid velocity (but not its derivatives to zero). We have also used that the number of phonons is not conserved, and the phonon chemical potential is zero. The spatial derivatives can be written as

$$v_p \cdot \nabla f_p = -\frac{f_p^0 (1 + f_p^0)}{T} \left(\frac{E_p}{T} v_p \cdot \nabla T + v_p \cdot \nabla (p \cdot u)\right). \quad (A.4)$$

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We can simplify these expressions using the Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho} \nabla P = -\frac{1}{\rho \, dT} \nabla T$$  \hspace{1cm} (A.5)$$

where $P$ is the pressure and $\rho$ is the mass density. The mass density is defined by

$$\pi = \rho \mathbf{u} ,$$  \hspace{1cm} (A.6)$$

where $\pi$ is the momentum density of the fluid. We can now collect all terms that contain a gradients of $T$. We find

$$\frac{df_p}{dt} = -\frac{f_p^0(1 + f_p^0)}{T} \left( \frac{E_p}{T} \mathbf{v}_p \cdot \nabla T - \frac{1}{\rho \, dT} \mathbf{p} \cdot \nabla T \right) .$$  \hspace{1cm} (A.7)$$

Noting that $\mathbf{v}_p = (v_p/p) \mathbf{p}$ and using our definition of $\alpha_p$ in Eq. (2.17), we can write

$$\alpha_p = \frac{T}{\rho \, dT} \frac{v_p}{p} - E_p \frac{v_p}{p} .$$  \hspace{1cm} (A.8)$$

The pressure of the phonon gas is given by

$$P = -T \int d\Gamma \ln \left( 1 - e^{-E_p/T} \right) ,$$  \hspace{1cm} (A.9)$$

leading to

$$\frac{dP}{dT} = \frac{1}{3T} \int d\Gamma (3E_p + v_p \mathbf{p}) f_p^0 \simeq \frac{2\pi^2 T^3}{45v^4} \left( 1 + \frac{60\epsilon \pi^2}{7} \right) ,$$  \hspace{1cm} (A.10)$$

where we have kept terms up to linear order in $\epsilon$. The mass density is given by

$$\rho = \frac{1}{3T} \int d\Gamma \mathbf{p}^2 f_p^0(1 + f_p^0) \simeq \frac{2\pi^2 T^4}{45v^5} \left( 1 + 20\epsilon \pi^2 \right) .$$  \hspace{1cm} (A.11)$$

Inserting these results into Eq. (A.8) we get

$$\alpha_{p}^{(P)} = 4\epsilon v^2 \left[ \frac{v^2 p^2}{T^2} - \frac{20\pi^2}{7} \right] .$$  \hspace{1cm} (A.12)$$

We note that $\alpha_p = 0$ for $\epsilon = 0$. This result is consistent with Eq. (2.8). In the case of phonons with an exactly linear dispersion relation, the LHS side (the streaming term) of the Boltzmann equation is zero. The RHS (the collision term) is zero if $\delta f_p$ is a zero mode of the linearized collision operator. The zero mode associated with momentum conservation satisfies the constraints in Eq. (2.7) and, because of Eq. (2.8), gives a vanishing thermal conductivity. For $\epsilon \neq 0$ both the LHS and the RHS are not zero, and the corresponding solution of the linearized Boltzmann equation leads to a finite thermal conductivity.
2. Massive Kaon

In this section we compute the streaming term for kaons. There are two important differences as compared to the phonon case: kaons are massive, and they carry conserved charges (hypercharge and isospin) which couple to chemical potentials. For simplicity we consider only the neutral kaon, which couples to a single chemical potential (a linear combination of isospin and hypercharge). We will also consider the limit of low temperature, such that $T \ll m, \mu, \delta m \equiv m - \mu$, where $m \equiv m_K$ and $\mu \equiv \mu_K$. The left-hand side of the Boltzmann equation can be written as

$$\frac{df_p}{dt} = \frac{\partial f_p}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f_p}{\partial T} \frac{dT}{dt} + \frac{\partial f_p}{\partial \mu} \frac{d\mu}{dt} + \mathbf{v}_p \cdot \text{gradient terms}$$ (A.13)

The time derivatives of the temperature and the chemical potential can be converted to spatial derivatives using the continuity equation. These terms do not contribute to the thermal conductivity. The derivative of the velocity field can be rewritten using the Euler equation

$$\frac{du}{dt} = -\frac{1}{\rho} \nabla P.$$ (A.14)

Spatial derivatives of the velocity field only contribute to the shear viscosity. The spatial derivative of the chemical potential can be rewritten using the Gibbs-Duhem relation $dP = nd\mu + sdT$, where $n$ is the density and $s$ is the entropy density. We get

$$\nabla \mu = \frac{1}{n} \nabla P - \frac{s}{n} \nabla T.$$ (A.15)

Collecting gradients of the temperature and pressure we find

$$\frac{df_p}{dt} = \left( \frac{\partial f_p}{\partial T} - \frac{s}{n} \frac{\partial f_p}{\partial \mu} \right) \mathbf{v}_p \cdot \nabla T + \left( \frac{\mathbf{v}_p \partial f_p}{n \partial \mu} - \frac{1}{\rho} \frac{\partial f_p}{\partial u} \right) \cdot \nabla P.$$ (A.16)

At this point we need to specify explicit expressions for the thermodynamic quantities. The pressure of an ideal gas of massive bosons is given by

$$P = -T \int d\Gamma \ln \left( 1 - e^{-\left(E_p - \mu\right)/T} \right).$$ (A.17)

The entropy density, particle density, and mass density are

$$s = \frac{1}{3T} \int d\Gamma \, f_p^0 \left( 3 \left( E_p - \mu \right) + v_p p \right) \simeq \left( \frac{\delta m}{T} + \frac{5}{2} \right) n,$$

$$n = \int d\Gamma \, f_p^0 \simeq \left( \frac{mT}{2\pi v^2} \right)^{3/2} e^{-\delta m/T},$$

$$\rho = \frac{1}{3T} \int d\Gamma \, p^2 f_p^0 (1 + f_p^0) \simeq \frac{m}{v^2} n,$$ (A.18)
where we have given analytical results in the low temperature limit. We also need the derivatives of the distribution function with respect to the thermodynamic quantities,
\[
\frac{\partial f}{\partial T} = \frac{f_p^0(1 + f_p^0)}{T} E_p - \mu
\]
\[
\frac{\partial f}{\partial \mu} = \frac{f_p^0(1 + f_p^0)}{T}
\]
\[
\frac{\partial f}{\partial u} = \frac{f_p^0(1 + f_p^0)}{T} p.
\] (A.19)

The coefficient of the $\nabla P$ term in Eq. (A.16) is given by
\[
\frac{v_p}{n} \frac{\partial f}{\partial \mu} - \frac{1}{\rho} \frac{\partial f}{\partial u} = \frac{f_p^0(1 + f_p^0)}{T} \left( \frac{v_p}{n} - \frac{m}{\rho} \right).
\] (A.20)

Using $v_p = v^2 p/E_p \sim v^2 p/m$ we can see that $v_p/p \sim v^2/m$. We can also show that $n/\rho = v^2/m$.

This implies that the coefficient of $\nabla P$ is zero. The $\nabla T$ term is
\[
\frac{df}{dt} = \left( \frac{\partial f}{\partial T} - \frac{s}{n} \frac{\partial f}{\partial \mu} \right) v_p \cdot \nabla T = \frac{f_p^0(1 + f_p^0)}{T} \left( E_p - \mu - \frac{s}{n} \right) v_p \cdot \nabla T.
\] (A.21)

We can then read off the coefficient $\alpha_p$. We find
\[
\alpha_p^{(K)} = \frac{v^2}{m} \left[ E_p - \mu - \frac{T s}{n} \right] \sim \frac{v^4 p^2}{2m^2} - \frac{5v^2 T}{2m}.
\] (A.22)

**APPENDIX B: ORTHOGONAL POLYNOMIALS**

In this appendix we collect some explicit expressions for the orthogonal polynomials $B_s(p^2)$ introduced in section III. The starting point is Eq. (3.2)
\[
\int d\Gamma f_p^0(1 + f_p^0)p^2 B_s(p^2)B_t(p^2) \equiv A_s \delta_{st},
\] (B.1)

together with $B_0 = 1$. We write
\[
B_0 = 1
\]
\[
B_1 = p^2 + c_{10}
\]
\[
B_2 = p^4 + c_{22} p^2 + c_{20}
\]
\[
B_3 = p^6 + c_{34} p^4 + c_{32} p^2 + c_{30}
\]
\[
\ldots
\] (B.2)

and iteratively determine the coefficients $c_{st}$. For example, we find
\[
c_{10} = \begin{cases} -\frac{20\pi^2 T^2}{v^2} & \text{phonons,} \\ -\frac{5mT}{v^2} & \text{kaons.} \end{cases}
\] (B.3)
which implies that $B_1 \propto \alpha_p$. This relation plays a role in ensuring the consistency of the linearized Boltzmann equation. The variational function $g(p) = B_0$ is a zero mode of the linearized collision operator. We already showed that a term $g(p) \propto B_0$ is eliminated by the constraints, Eq. (2.8), and that it does not contribute to the thermal conductivity. The collision operator acting on any function $g(p)$ is orthogonal to $B_0$, because $B_0$ is a zero mode, and the collision operator is hermitean. This implies that the streaming term must be orthogonal to $B_0$ also. The streaming term is proportional to $\alpha_p$, and orthogonality to $B_0$ follows from the relation $\alpha_p \propto B_1$.

The final result for the thermal conductivity requires the normalization constant $A_1$. Using the result for $B_1$ we get

$$A_1 = \int d\Gamma f_p^0(1 + f_p^0)p^2(p^2 + c_{10})^2 = \begin{cases} \frac{256\pi^6 T^9}{245v^2} & \text{phonons}, \\ \frac{15}{32\pi^{3/2}} \left(\frac{2mT}{v^2}\right)^{9/2} e^{-\delta m/T} & \text{kaons}. \end{cases}$$

(B.4)

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