SIGN RETRIEVAL IN SHIFT-IN Variant SPACES WITH TOTALLY POSITIVE GENERATOR

JOSÉ LUIS ROMERO

Abstract. We show that a real-valued function $f$ in the shift-invariant space generated by a totally positive function of Gaussian type is uniquely determined, up to a sign, by its absolute values $\{|f(\lambda)| : \lambda \in \Lambda\}$ on any set $\Lambda \subseteq \mathbb{R}$ with lower Beurling density $D^-(\Lambda) > 2$.

We consider a totally positive function of Gaussian type, i.e., a function $g \in L^2(\mathbb{R})$ whose Fourier transform factors as

$$
\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} dx = C_0 e^{-\gamma \xi^2} \prod_{\nu=1}^{m} (1 + 2\pi i \delta_{\nu} \xi)^{-1}, \quad \xi \in \mathbb{R},
$$

with $\delta_1, \ldots, \delta_m \in \mathbb{R}, C_0, \gamma > 0, m \in \mathbb{N} \cup \{0\}$, and the shift-invariant space

$$
V^\infty(g) = \left\{ f = \sum_{k \in \mathbb{Z}} c_k g(\cdot - k) : c \in \ell^\infty(\mathbb{Z}) \right\},
$$
generated by its integer shifts within $L^\infty(\mathbb{R})$. As a consequence of (1), each $f \in V^\infty(g)$ is continuous, the defining series converges unconditionally in the weak* topology of $L^\infty$, and the coefficients $c_k$ are unique.

The shift-invariant space $V^\infty(g)$ enjoys the following sampling property [4]: every separated set $\Lambda \subseteq \mathbb{R}$ with lower Beurling density

$$
D^-(\Lambda) := \liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\# \Lambda \cap [x-r,x+r]}{2r}
$$
strictly larger than 1 provides the norm equivalence

$$
||f||_{L^\infty(\mathbb{R})} \simeq ||f||_{\ell^\infty(\Lambda)}, \quad f \in V^\infty(g).
$$
(A similar property holds for all $L^p$ norms, $1 \leq p \leq \infty$.)

In this article, we show the following uniqueness property for the absolute values of real-valued functions in $V^\infty(g)$.

**Theorem 1** (Sign retrieval). Let $g$ be a totally positive function of Gaussian type, as in (1), and $\Lambda \subseteq \mathbb{R}$ with lower Beurling density

$$
D^-(\Lambda) > 2.
$$
Assume that $f_1, f_2 \in V^\infty(g)$ are real-valued and $|f_1| \equiv |f_2|$ on $\Lambda$. Then either $f_1 \equiv f_2$ or $f_1 \equiv -f_2$.

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See [2] for motivation for sign retrieval in shift-invariant spaces. When \( g \) is a Gaussian function - corresponding to \( m = 0 \) in (1) - Theorem 1 was recently obtained in [3, Theorem 1]. Here, that result is extended to all totally positive functions of Gaussian type.

The intuition behind Theorem 1 is as follows. Suppose that \( |f_1| \equiv |f_2| \) on \( \Lambda \), and split \( \Lambda \) into

\[
\begin{align*}
\Lambda_1 &= \{ \lambda \in \Lambda : f_1(\lambda) = f_2(\lambda) \}, \\
\Lambda_2 &= \{ \lambda \in \Lambda : f_1(\lambda) = -f_2(\lambda) \}.
\end{align*}
\]

Then \( f_1 - f_2 \) vanishes on \( \Lambda_1 \) while \( f_1 + f_2 \) vanishes on \( \Lambda_2 \). Under (3), one may expect one of two subsets \( \Lambda_j \) to have lower Beurling density larger than 1. The sampling inequalities [2] would then imply that either \( f_1 - f_2 \) or \( f_1 + f_2 \) are identically zero. This argument breaks down, however, because Beurling’s lower density is not subadditive. For example, \( \Lambda_1 := \Lambda \cap (-\infty, 0] \) and \( \Lambda_2 := \Lambda \cap (0, \infty) \) have always zero lower Beurling density. The proof of the sign retrieval theorem for Gaussian generators in [3] resorts instead to a special property of the Gaussian function, namely that \( V^\infty(g) \cdot V^\infty(g) \) is contained in a dilation of \( V^\infty(\tilde{g}) \) by a factor of 2, where \( \tilde{g} \) is another Gaussian function. Thus, in the Gaussian case, the sampling theorem can be applied after rescaling to the set \( \Lambda \) to conclude that \( (f_1 - f_2) \cdot (f_1 + f_2) \equiv 0 \), and, by analyticity, that either \( f_1 \equiv f_2 \) or \( f_1 \equiv -f_2 \). A similar argument applies to Paley-Wiener spaces [1, Theorem 2.5]. We are unaware of an analogous dilation property for general totally positive generators.

To prove Theorem 1 for all totally positive generators of Gaussian type we take a different route. We define the upper average circular density of a set \( \Lambda \subseteq \mathbb{R} \) as

\[
D^+_{\text{circ}}(\Lambda) = \limsup_{r \to \infty} \frac{4}{\pi r^2} \sum_{\lambda \in \Lambda \cap [-t,t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t},
\]

with the convention that \( D^+_{\text{circ}}(\Lambda) = \infty \), if \( \Lambda \) is uncountably infinite. The density is named circular because, in [5], each point \( \lambda \in [-t,t] \) is weighted with the measure of the largest vertical segment \( \{ \lambda \} \times (-a, a) \) contained in the two-dimensional open disk \( B_t(0) \subseteq \mathbb{R}^2 \).

The upper average circular density can be alternatively described as follows: for any lattice \( \alpha \mathbb{Z}, \alpha > 0 \),

\[
D^+_{\text{circ}}(\Lambda) = \limsup_{r \to \infty} \frac{2\alpha}{\pi r^2} \int_0^r \#(\Lambda \setminus \{0\} \times \alpha \mathbb{Z}) \cap B_t(0) \frac{dt}{t}.
\]

From here, it follows easily that \( D^+_{\text{circ}} \) dominates Beurling’s lower density:

\[
D^+_{\text{circ}}(\Lambda) \geq D^-(\Lambda);
\]

see Lemma 1 below. We call \( D^+_{\text{circ}} \) an upper density because, due to the sublinearity of \( \limsup \),

\[
D^+_{\text{circ}}(\Lambda_1 \cup \Lambda_2) \leq D^+_{\text{circ}}(\Lambda_1) + D^+_{\text{circ}}(\Lambda_2),
\]

for any two sets \( \Lambda_1, \Lambda_2 \subseteq \mathbb{R} \).

Below we prove the following uniqueness result formulated in terms of the upper average circular density of the zero set \( \{ f = 0 \} \) of a function \( f \) (counted without multiplicities).

**Theorem 2.** [Uniqueness theorem] Let \( g \) be a totally positive function of Gaussian type, as in (1). Let \( f \in V^\infty(g) \) be non-zero. Then \( D^+_{\text{circ}}(\{ f = 0 \}) \leq 1 \).
The uniqueness theorem allows one to carry out the natural proof the sign retrieval theorem.

**Proof of Theorem 1 assuming Theorem 2.** Assume that $D^- (\Lambda) > 2$ and write $\Lambda = \Lambda_1 \cup \Lambda_2$ as in (4). Then, by (7) and (8), either $D^+_{\text{circ}} (\Lambda_1) > 1$ or $D^+_{\text{circ}} (\Lambda_2) > 1$. In the first case, Theorem 2 shows that $f_1 \equiv f_2$, while in the second, $f_1 \equiv -f_2$. □

As the proof shows, Theorem 1 remains valid if (3) is relaxed to $D^+_{\text{circ}} (\Lambda) > 2$. Sign retrieval also holds under the same density condition for the shift-invariant spaces $V^p (g)$ defined with respect to $L^p$ norms, $1 \leq p \leq \infty$, as these are contained in $V^{\infty} (g)$.

Towards the proof of Theorem 2, we first prove (6) and (7).

**Lemma 1.** Let $\Lambda \subseteq \mathbb{R}$ and $\alpha > 0$. Then (6) and (7) hold true.

**Proof.** We assume that $\Lambda$ has no accumulation points, since, otherwise, both sides of (6) are infinite. Denote provisionally the right hand side of (6) by $\tilde{D}^+_{\text{circ}} (\Lambda)$, and set $\Lambda' := \Lambda \setminus \{0\}$.

**Step 1.** Let $\varepsilon \in (0, 1)$, and $C = C_{\alpha, \varepsilon} > 0$ a constant to be specified. We claim:

\[
D^+_{\text{circ}} (\Lambda) = \limsup_{r \to \infty} \frac{4}{\pi r^2} \int_C^r \sum_{\lambda \in \Lambda' \cap [-t, t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t},
\]

and does not affect the limit on $r$. Similarly, since $\sqrt{t^2 - \lambda^2} \leq t$,

\[
\frac{4}{\pi r^2} \int_0^C \sum_{\lambda \in \Lambda' \cap [-t, t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t} \leq \#(\Lambda' \cap [-C, C]) \cdot \frac{4C}{\pi r^2}.
\]

The last quantity is finite because $\Lambda$ has no accumulation points. This proves (9), because it shows that limiting the integral to $[C, r]$ does not affect the limit on $r$. Second, a change of variables shows that (10) holds with $C$ replaced by 0. In addition, since $\Lambda$ has no accumulation points, there exists $\eta > 0$ such that $\Lambda' \cap (-\eta, \eta) = \emptyset$. Therefore, we can estimate the part of the integral excluded in (10) as

\[
\frac{2\alpha}{\pi r^2} \int_0^C \#(\Lambda' \times \alpha \mathbb{Z}) \cap B(1+\delta) (0) \frac{dt}{t},
\]

which is finite because $\eta > 0$. This proves (10).

**Step 2.** For $t > 0$, we note that

\[
\#(\Lambda' \times \alpha \mathbb{Z}) \cap B_t (0) = \sum_{\lambda \in \Lambda' \cap [-t, t]} \# \{ k \in \mathbb{Z} : \lambda^2 + \alpha^2 k^2 < t^2 \}
\]

\[
= \sum_{\lambda \in \Lambda' \cap [-t, t]} \# \{ k \in \mathbb{Z} : |k| < \alpha^{-1} \sqrt{t^2 - \lambda^2} \}. \]
Next, we choose the constant $C_{\alpha, \varepsilon}$ so that the following estimates hold for $t \geq C_{\alpha, \varepsilon}$:

$$
\#(\Lambda' \times \alpha \mathbb{Z}) \cap B_{(1+\varepsilon)t}(0) \geq \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{(1+\varepsilon)^2t^2 - \lambda^2} - 1
$$

$$
= \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{(1+\varepsilon)^2t^2 - \lambda^2} - \alpha \sqrt{(1+\varepsilon)^2t^2 - \lambda^2 + \alpha^2/4}
$$

$$
\geq \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{(1+\varepsilon)^2t^2 - \lambda^2} - \alpha(1+\varepsilon)t + \alpha^2/4
$$

where the last estimate requires choosing $C_{\alpha, \varepsilon}$ large. Similarly,

$$
\#(\Lambda' \times \alpha \mathbb{Z}) \cap B_{(1-\varepsilon)t}(0) \leq \sum_{\lambda \in \Lambda' \cap [(1-\varepsilon)t,(1+\varepsilon)t]} \frac{2}{\alpha} \sqrt{(1-\varepsilon)^2t^2 - \lambda^2} + 1
$$

$$
= \sum_{\lambda \in \Lambda' \cap [(1-\varepsilon)t,(1+\varepsilon)t]} \frac{2}{\alpha} \sqrt{(1-\varepsilon)^2t^2 - \lambda^2} + \alpha \sqrt{(1-\varepsilon)^2t^2 - \lambda^2 + \alpha^2/4}
$$

$$
\leq \sum_{\lambda \in \Lambda' \cap [(1-\varepsilon)t,(1+\varepsilon)t]} \frac{2}{\alpha} \sqrt{(1-\varepsilon)^2t^2 - \lambda^2} + \alpha(1-\varepsilon)t + \alpha^2/4
$$

$$
\leq \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{t^2 - \lambda^2},
$$

where the last estimate requires choosing $C_{\alpha, \varepsilon}$ large. Combining the last two estimates with (9) and (10), and letting $\varepsilon \to 0$, we deduce (6).

**Step 3.** To prove (7), let $r > s > 0$ and use (6) with $\alpha = 1$ to estimate

$$
D^+_{\text{circ}}(\Lambda) \geq \liminf_{r \to \infty} \frac{2}{\pi r^2} \int_s^r \inf_{z \in \mathbb{R}^2} \#(\Lambda' \times \mathbb{Z}) \cap B_t(z) \frac{dt}{t}
$$

$$
\geq \inf_{t \geq s} \left[ \frac{1}{B_t(0)} \inf_{z \in \mathbb{R}^2} \#(\Lambda' \times \mathbb{Z}) \cap B_t(z) \right] \liminf_{r \to \infty} \frac{2}{\pi r^2} \int_s^r |B_t(0)| \frac{dt}{t}
$$

$$
= \inf_{t \geq s} \left[ \inf_{z \in \mathbb{R}^2} \frac{1}{|B_t(z)|} \#(\Lambda' \times \mathbb{Z}) \cap B_t(z) \right].
$$

Letting $s \to \infty$, we see that

$$
D^+_{\text{circ}}(\Lambda) \geq \liminf_{t \to \infty} \inf_{z \in \mathbb{R}^2} \frac{1}{|B_t(z)|} \#(\Lambda' \times \mathbb{Z}) \cap B_t(z) = D^-(\Lambda'),
$$

where we used that the two-dimensional lower Beurling density of $\Lambda' \times \mathbb{Z}$ is $D^-(\Lambda)$.

**Proof of Theorem** We proceed by induction on $m$ in (11).
In the case \( m = 0 \), the function \( g \) is a Gaussian \( g(x) = C_0 e^{-ax^2} \), with \( a = \frac{\pi^2}{7} > 0 \). Let \( f = \sum_k c_k g(\cdot - k) \in V^\infty(g) \) with \( c \) bounded be non-zero, and denote by \( \Lambda \subseteq \mathbb{R} \) its zero set.

As shown in [5] Lemma 4.1, \( f \) possesses an extension to an entire function on \( \mathbb{C} \), satisfying the growth estimate \( |f(x + iy)| \leq e^{ay^2} \) for \( x, y \in \mathbb{R} \), where the implied constant depends on \( f \). Let \( n \geq 0 \) be the order of \( f \) at \( z = 0 \), and consider the analytic function

\[
F(z) := C_1 z^{-n} f(z) e^{\frac{a}{2} z^2},
\]

where \( C_1 \in \mathbb{C} \) is chosen so that \( F(0) = 1 \). Then \( F \) satisfies

\[
|F(x + iy)| \leq C e^{ay^2} e^{\frac{a}{2} (x^2 - y^2)} = Ce^{\frac{a}{2} (x^2 + y^2)}, \quad x, y \in \mathbb{R},
\]

for some constant \( C > 0 \). Moreover, the zero set of \( f \) is invariant under addition of \( \frac{\pi}{a} \mathbb{Z} \) [5 Lemma 4.2]. Therefore, the set of complex zeros of \( F \) contains \( (\Lambda \setminus \{0\}) + \frac{\pi}{a} \mathbb{Z} \):

\[
F(\lambda + \frac{\pi}{a} k) = 0, \quad \lambda \in \Lambda \setminus \{0\}, k \in \mathbb{Z}.
\]

By (6) with \( \alpha = \frac{\pi}{a} \), the zero-counting function of \( F \),

\[
n_F(t) := \# \{ z \in \mathbb{C} : F(z) = 0, |z| \leq t \},
\]

satisfies

\[
\limsup_{r \to \infty} \frac{1}{r^2} \int_0^r \frac{n_F(t)}{t} dt \geq \frac{a}{2} D^+_{\text{circ}}(\Lambda).
\]

On the other hand, by Jensen’s formula combined with (11), for all \( r > 0 \),

\[
\frac{1}{r^2} \int_0^r \frac{n_F(t)}{t} dt = \frac{1}{2\pi r^2} \int_0^{2\pi} \log |F(re^{i\theta})| d\theta \leq \frac{\log(C)}{r^2} + \frac{a}{2}.
\]

By combining (12) and (13) we conclude that \( D^+_{\text{circ}}(\Lambda) \leq 1 \), as claimed. Note, in particular, that \( \Lambda \) has no accumulation points.

**Inductive step.** Let \( g \) be as in (11) and \( f = \sum_k c_k g(\cdot - k) \in V^\infty(g) \) non-zero. Since \( g \) is real-valued, by replacing \( c_k \) with \( \Re(c_k) \) or \( \Im(c_k) \) if necessary, we may assume that \( f \) is real-valued. Let \( g_1 \) be given by (11), except that the product runs only up to \( m - 1 \). Then

\[
f_1 := f + \delta_m f' = \sum_{k \in \mathbb{Z}} c_k g_1(\cdot - k)
\]

belongs to \( V^\infty(g_1) \) and is real-valued and non-zero, because the coefficients \( c_k \) are real and at least one of them is non-zero. We assume by inductive hypothesis that \( D^+_{\text{circ}}(\{f_1 = 0\}) \leq 1 \).

Let \( \Lambda = \{f = 0\} \) and \( \Gamma = \{f_1 = 0\} \). Suppose that \( \Lambda \) has no accumulation points. Assume for the moment that, in addition, \( f \) has a non-negative zero, and order the set of non-negative zeros of \( f \) increasingly:

\[
\Lambda \cap [0, \infty) = \{ \lambda_k : k = 0, \ldots, N \},
\]

where \( N \in \mathbb{N} \cup \{\infty\} \), and we do not count multiplicities. Hence, \( \lambda_k > 0 \), if \( k > 0 \).
By Rolle’s Theorem applied to the differential operator \( I + \delta_m \partial_x \), there is a sequence of zeros of \( f_1 \) that interlaces \( \Lambda \cap [0, \infty) \) (see [4, Lemma 5.1] or [5, Lemma 4.8] for details). We parameterize that sequence as \( \Gamma^+ = \{ \gamma_k : k = 1, \ldots, N \} \), with

\[
\lambda_{k-1} < \gamma_k < \lambda_k, \quad k > 0.
\]

The set \( \Gamma^+ \) is empty if \( N = 0 \). For \( t > 0 \), this ordering associates with each \( \lambda_k \in (0, t], k \neq 0 \), a distinct zero of \( f_1 \), \( \gamma_k \in (0, t] \), such that the vertical segment through \( \gamma_k \) contained in the disk \( B_t(0) \) is larger than the corresponding segment through \( \lambda_k \):

\[
\sum_{k>0, \lambda_k \in (0,t]} \sqrt{t^2 - \lambda_k^2} \leq \sum_{k>0, \gamma_k \in (0,t]} \sqrt{t^2 - \gamma_k^2} \leq \sum_{\gamma \in \Gamma \cap (0,t]} \sqrt{t^2 - \gamma^2};
\]

see Figure II. We now add the term corresponding to \( \lambda_0 \), and bound \( \sqrt{t^2 - \lambda_0^2} \leq t \) to obtain

\[
\sum_{\lambda \in \Lambda \cap [0,t]} \sqrt{t^2 - \lambda^2} \leq t + \sum_{\gamma \in \Gamma \cap (0,t]} \sqrt{t^2 - \gamma^2}.
\]

The previous estimate is also trivially true is \( f \) has no non-negative zeros. Arguing similarly with the non-positive zeros of \( f \) we conclude that

\[
\sum_{\lambda \in \Lambda \cap [-t,0]} \sqrt{t^2 - \lambda^2} \leq 2t + \sum_{\gamma \in \Gamma \cap [-t,0]} \sqrt{t^2 - \gamma^2}.
\]

This shows that \( D_{\text{circ}}^+ (\{ f = 0 \}) \leq D_{\text{circ}}^+ (\{ f_1 = 0 \}) \). The latter quantity is bounded by 1 by inductive hypothesis. Finally, if \( \Lambda = \{ f = 0 \} \) has an accumulation point, Rolle’s theorem applied as before shows that so does \( \Gamma = \{ f_1 = 0 \} \), and, therefore, \( D_{\text{circ}}^+ (\{ f_1 = 0 \}) = \infty \) contradicting the inductive hypothesis. \( \square \)
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Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria, and Acoustics Research Institute, Austrian Academy of Sciences, Wohllebengasse 12-14, Vienna, 1040, Austria

E-mail address: jose.luis.romero@univie.ac.at, jlromero@kfs.oeaw.ac.at