The orbit type stratification of the moduli space of Higgs bundles

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Abstract
The moduli space of Higgs bundles can be constructed as a quotient of an infinite-dimensional space and hence admits an orbit type decomposition. In this paper, we show that the orbit type decomposition is a complex Whitney stratification such that each stratum is a complex symplectic submanifold and hence admits a complex Poisson bracket. Moreover, these Poisson brackets glue to a Poisson bracket on the structure sheaf of the moduli space so that the moduli space is a stratified complex symplectic space.

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1 Introduction

Let \( C \) be a hyperKähler manifold, and \( G \) a Lie group acting on \( C \) and preserving the hyperKähler structure. We also assume that \( G \) admits a complexification \( G^C \) such that the \( G \)-action on \( C \) can be extended to a holomorphic \( G^C \)-action with respect to some complex structure \( I \) on \( C \). Suppose there is a hyperKähler moment map \( m = (\mu, \mu_C) \) such that \( \mu \) is a moment map for the \( G \)-action with respect to the Kähler form induced by \( I \), and \( \mu_C \) is a complex moment map for the \( G^C \)-action with respect to the complex symplectic form induced by the other complex structures \( J \) and \( K \). Then, we may consider the hyperKähler quotient \( m^{-1}(0)/G \). Although the quotient \( m^{-1}(0)/G \), in general, may be highly singular, Mayrand showed in [9] that if \( C \) is finite-dimensional, and \( G \) is compact, then the hyperKähler quotient \( m^{-1}(0)/G \) is a complex space and can be decomposed into smooth hyperKähler manifolds by orbit types such that the decomposition is a complex Whitney stratification. The hyperKähler structure on each stratum comes from \( C \) and is compatible with the complex space structure. Moreover, the complex symplectic structure on each stratum induces a complex Poisson bracket such that these Poisson brackets glue to a complex Poisson bracket on the structure sheaf of the complex space \( m^{-1}(0)/G \). Finally, the complex space \( m^{-1}(0)/G \) is locally biholomorphic to an affine complex symplectic quotient such that the biholomorphism is a Poisson map and preserves the orbit type decompositions. Note that Mayrand’s result is a natural generalization of Sjamaar–Lerman [14] from symplectic quotients to hyperKähler quotients.

The purpose of this paper is to show that Mayrand’s results can be extended to the moduli space of Higgs bundles, which is promised in [3]. More precisely, let \( X \) be a closed Riemann surface with genus \( \geq 2 \). To parametrize Higgs bundles, we fix a smooth Hermitian vector bundle \( E \) over \( X \), and let \( g_E \) be the bundle of skew-Hermitian endomorphisms of \( E \). For convenience, we may assume that the degree of \( E \) is 0. Then, by the Chern correspondence and the fact that \( \dim_X = 1 \), the space of holomorphic structures on \( E \) can be identified with the space \( A \) of unitary connections on \( E \). Set \( C = A \times T^1,0(g_E) \), and the configuration space \( B \) of the Higgs bundles with underlying smooth bundle \( E \) is defined as

\[
B = \{(A, \Phi) \in C : \partial A \Phi = 0\}.
\]  

(1.1)

Note that the complex gauge group \( G^C = \text{Aut}(E) \) acts on \( B \) and preserves the subspaces \( B^{ss} \) and \( B^{ps} \) consisting of semistable and polystable Higgs bundles, respectively (see [17] for more details). The moduli space of Higgs bundles is defined as the quotient \( M = B^{ps}/G^C \) equipped with the \( C^\infty \)-topology. To see how hyper-Kähler geometry comes into the picture, let us recall that the moduli space \( M \) can be realized as a singular hyperKähler quotient in the following way. Note that \( C \) is an infinite-dimensional affine hyperKähler manifold modeled on \( T^1,0(g_E) \oplus T^1,0(g_C) \) (see [7, §6]). The complex gauge group \( G^C \) acts on \( C \) holomorphically with respect to the complex structure \( I \) that is given by the multiplication by \( \sqrt{-1} \). (In this paper, we routinely identify \( T^1,0(g_E) \) with \( T^0,1(g_E) \).) The subgroup \( G \) of \( G^C \) consisting of unitary gauge transformations preserves the hyperKähler structure. The \( G \)-action also
admits a hyperKähler moment map as follows. Hitchin’s equation

\[ \mu(A, \Phi) = FA + [\Phi, \Phi^*] \]  

(1.2)
can be regarded as a real moment map for the \( G \)-action with respect to the Kähler form induced by \( I \). Moreover, the holomorphicity condition \( \mu_C(A, \Phi) = \tilde{\partial}_A \Phi \) can be regarded as a complex moment map for the \( G^C \)-action with respect to the complex symplectic form induced by the other two complex structures \( J \) and \( K \). Then, the Hitchin–Kobayashi correspondence (see [7,18]) states that the inclusion \( m^{-1}(0) \hookrightarrow \mathcal{B}^{ps} \) induces a homeomorphism

\[ i : m^{-1}(0)/\mathcal{G} \sim \mathcal{B}^{ps}/G^C = \mathcal{M}, \]  

(1.3)

where \( m = (\mu, \mu_C) \), and the quotient \( m^{-1}(0)/\mathcal{G} \) is equipped with the \( C^\infty \)-topology. Therefore, we are now in Mayrand’s setting except that \( C^\infty \), \( G \) and \( G^C \) are infinite-dimensional. We now define the orbit type decompositions of \( m^{-1}(0)/\mathcal{G} \) and \( \mathcal{M} \). Let \( H \) be a \( \mathcal{G} \)-stabilizer at some Higgs bundle in \( m^{-1}(0) \) and \((H)\) the conjugacy class of \( H \) in \( G \). Consider the subspace

\[ m^{-1}(0)_{(H)} = \{ (A, \Phi) \in m^{-1}(0) : (H)_{(A, \Phi)} \in (H) \}. \]  

(1.4)

It is \( \mathcal{G} \)-invariant, and the orbit type decomposition of the singular hyperKähler quotient \( m^{-1}(0)/\mathcal{G} \) is defined as

\[ m^{-1}(0)/\mathcal{G} = \coprod_{(H)} \text{components of } m^{-1}(0)_{(H)}/\mathcal{G}. \]  

(1.5)

By abusing the notation, we generally use \( \pi \) to denote the quotient map \( \mathcal{B}^{ps} \to \mathcal{M} \) or \( m^{-1}(0) \to m^{-1}(0)/\mathcal{G} \). Then, we will prove the following. It is a slight generalization of Hitchin’s construction of the moduli space of stable Higgs bundles in [7, §5 and §6] (cf. [15, Proposition 2.21]).

**Theorem A** Every stratum \( Q \) in the orbit type decomposition of the hyperKähler quotient \( m^{-1}(0)/\mathcal{G} \) is a locally closed smooth manifold, and \( \pi^{-1}(Q) \) is a smooth submanifold of \( \mathcal{C} \) such that the restriction \( \pi : \pi^{-1}(Q) \to Q \) is a smooth submersion. Moreover, the restriction of the hyperKähler structure from \( \mathcal{C} \) to \( \pi^{-1}(Q) \) descends to \( Q \).

Similarly, if \( L \) is a \( \mathcal{G}^C \)-stabilizer at some Higgs bundle in \( \mathcal{B}^{ps} \), and \((L)\) denotes the conjugacy class of \( L \) in \( \mathcal{G}^C \), then we consider the subspace

\[ \mathcal{B}^{ps}_{(L)} = \{ (A, \Phi) \in \mathcal{B}^{ps} : (\mathcal{G}^C)_{(A, \Phi)} \in (L) \}. \]  

(1.6)

It is \( \mathcal{G}^C \)-invariant, and the orbit type decomposition of the moduli space \( \mathcal{M} \) is defined as

\[ \mathcal{M} = \coprod_{(L)} \text{components of } \mathcal{B}^{ps}_{(L)}/\mathcal{G}^C. \]  

(1.7)
In [3], it is shown that $\mathcal{M}$ is a normal complex space. Then, we will prove the following.

**Theorem B** Every stratum $Q$ in the orbit type decomposition of the moduli space $\mathcal{M}$ is a locally closed complex submanifold of $\mathcal{M}$, and $\pi^{-1}(Q)$ is a complex submanifold of $\mathcal{C}$ with respect to the complex structure $I$ such that the restriction $\pi : \pi^{-1}(Q) \to Q$ is a holomorphic submersion. This decomposition is a complex Whitney stratification.

Here, by complex Whitney stratification, we mean that the orbit type decomposition of $\mathcal{M}$ is a disjoint union of locally closed complex submanifolds such that if $\mathcal{Q}_1 \cap \mathcal{Q}_2 \neq \emptyset$ then $\mathcal{Q}_1 \subset \mathcal{Q}_2$ for any strata $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in the decomposition. This is called the frontier condition. Moreover, this decomposition is required to satisfy Whitney conditions $A$ and $B$. Although Whitney conditions $A$ and $B$ are conditions for submanifolds in Euclidean space, they make sense for complex spaces, since they are local conditions and invariant under diffeomorphisms (see [9, Definition 2.2, 2.5, 2.7] for more details).

Moreover, the Hitchin–Kobayashi correspondence $i$ preserves the orbit type decompositions in the following way.

**Theorem C** If $Q$ is a stratum in the orbit type decomposition of $m^{-1}(0)/\mathcal{G}$, then $i(Q)$ is a stratum in the orbit type decomposition of $\mathcal{M}$, and the restriction $i : Q \to i(Q)$ is a biholomorphism with respect to the complex structure $I_Q$ on $Q$ coming from $\mathcal{C}$ and the natural complex structure on $i(Q)$.

Therefore, each stratum $Q$ in the orbit type decomposition of $\mathcal{M}$ acquires a complex symplectic structure from the corresponding stratum in the orbit type decomposition of $\mathcal{m}^{-1}(0)/\mathcal{G}$. As a consequence, each $Q$ admits a complex Poisson bracket. We will show that these Poisson brackets glue to a complex Poisson bracket on the structure sheaf of $\mathcal{M}$. To state the result more precisely, we recall that any Higgs bundle $(A, \Phi) \in \mathcal{m}^{-1}(0)$ defines a deformation complex $C_{\mathcal{M}C}$, which is an elliptic complex (see Sect. 2).

Let $H^1$ denote the harmonic space $H^1(C_{\mathcal{M}C})$. In [3], it is shown that $H^1$ is a complex symplectic vector space, and the $\mathcal{G}C$-stabilizer $H^C$ acts linearly on it and preserves the complex symplectic structure, where $H$ is the $\mathcal{G}$-stabilizer at $(A, \Phi)$. Let $v_{0,C}$ be the canonical complex moment map for the $H^C$-action on $H^1$. By [3], around $(A, \Phi)$, the moduli space $\mathcal{M}$ is a biholomorphism to an open neighborhood of $[0]$ in the complex symplectic quotient $v_{0,C}^{-1}(0) \to H^C$, which is an affine geometric invariant theory (GIT) quotient. Note that $v_{0,C}^{-1}(0) \to H^C$ also has an orbit type decomposition, since every point in $v_{0,C}^{-1}(0) \to H^C$ has a unique closed orbit, and this orbit has an orbit type (see Sect. 5). By Mayrand [9], the orbit type decomposition of $v_{0,C}^{-1}(0) \to H^C$ is a Whitney stratification, and each stratum is a complex symplectic submanifold and hence admits a complex Poisson bracket. Moreover, these Poisson brackets glue to a Poisson bracket on the structure sheaf such that the inclusion from each stratum to $v_{0,C}^{-1}(0) \to H^C$ is a Poisson map. Then, we will prove the following.

**Theorem D** There is a unique complex Poisson bracket on the structure sheaf of $\mathcal{M}$ such that the inclusion $Q \hookrightarrow \mathcal{M}$ is a Poisson map for each stratum $Q$ in $\mathcal{M}$. Moreover, we have the following.

1. The local biholomorphism between $\mathcal{M}$ and $v_{0,C}^{-1}(0) \to H^C$ preserves the orbit type stratifications and is a Poisson map.
(2) Its restriction to each stratum $Q$ in $\mathcal{M}$ is a complex symplectomorphism, and hence serves as complex Darboux coordinates on $Q$.

Following Mayrand [9] and Sjamaar-Lerman [14], a complex space is called a stratified complex symplectic space if it admits a complex Whitney stratification, a complex symplectic structure on each stratum, and a complex Poisson bracket on the structure sheaf such that the inclusion from each stratum to the complex space is a holomorphic Poisson map. As a consequence of the main theorems proved in this paper, we conclude the following.

**Corollary** The moduli space $\mathcal{M}$ of Higgs bundles is a stratified complex symplectic space with the orbit type decomposition as the complex Whitney stratification.

To prove Theorem A and the first part of Theorem B, the basic tools are local slice theorems for the $G$-action and the $G^C$-action. Since the $G$-action is proper, its local slice theorem is available. To obtain a local slice theorem for the $G^C$-action around Higgs bundles satisfying Hitchin’s equation, we adapt Buchdahl and Schumacher’s argument in [1, Proposition 4.5]. To prove the second part of Theorem B, we simply follow Mayrand’s arguments in [9, §4.6, §4.7]. The idea is that the Whitney conditions and the frontier condition are local conditions and therefore can be checked on an open neighborhood of $[0]$ in $v_{0,C}^{-1}(0) \parallel H^C$, provided that the biholomorphism between $\mathcal{M}$ and a local model $v_{0,C}^{-1}(0) \parallel H^C$ preserves the orbit type decompositions. We will prove that this is the case. These results will be proved in Sects. 4 and 5. Moreover, in Sect. 2, we will review results in [3].

To prove Theorem C, the major obstacle is to show that the Hitchin–Kobayashi correspondence preserves orbit types. We will follow Sjamaar’s argument in [13, Theorem 2.10]. However, this argument crucially relies on Mostow’s decomposition for complex reductive Lie groups. Since $G^C$ is infinite-dimensional, we need to extend Mostow’s decomposition to $G^C$ in the following way.

**Theorem E [Mostow’s decomposition]** Let $H$ be a compact subgroup of $G$, $\mathfrak{h}$ its Lie algebra and $\mathfrak{h}^\perp$ the $L^2$-orthogonal complement of $\mathfrak{h}$ in the Lie algebra $\Omega^0(\mathfrak{g}_E)$ of $G$. Let $\mathfrak{h}^\perp \times_H G$ be the quotient of $\mathfrak{h}^\perp \times G$ by $H$, where the action of $H$ is given by

$$h_0 \cdot (s, u) = (h_0 sh_0^{-1}, h_0 u).$$

Then, the map

$$\mathfrak{h}^\perp \times_H G \rightarrow G^C/H^C, \quad [s, u] \mapsto H^C \exp(is)u,$$  

is a $G$-equivariant bijection, where $G$ acts on both sides by right multiplication.

It is likely that the map mentioned in Theorem E is not only a bijection but also a diffeomorphism. That said, for the purpose of this paper, a bijection is all we need. Once Mostow’s decomposition for $G^C$ is established, the rest of the proof follows easily. To prove Theorem E, we will instead prove that the map $H^C \times_H (\mathfrak{h}^\perp \times G) \rightarrow G^C$ is a bijection (see Theorem 3.1 for a more precise statement). To this end, following the Heinzner and Schwarz’s idea in [6, §9], we will realize $\mathfrak{h}^\perp \times G$ as a zero set of some
moment map on $\mathcal{G}^C$. Therefore, we need to show that $\mathcal{G}^C$ is a weak Kähler manifold and that the left $H$-action on $\mathcal{G}^C$ is Hamiltonian with a suitable moment map. In [8], Huebschmann and Leicht provided a framework to deal with this problem. Although their results are in finite-dimensional settings, they can be carried out for $\mathcal{G}^C$ without any problems. For the sake of completeness, we provide the details in the Appendix, and the proofs are taken or adapted from [8]. Then, it will be shown that every $H^C$-orbit in $\mathcal{G}^C$ intersects $h^1 \times \mathcal{G}$, and the intersection is a single $H$-orbit. Here, we will use the framework laid out in Mundet i Riera’s paper [10]. All these results will be proved in Sect. 3.

To prove Theorem D, we need to define a complex Poisson bracket on the structure sheaf of $\mathcal{M}$. Since every stratum in the orbit type decomposition has a complex Poisson bracket, and $\mathcal{M}$ is a disjoint union of these strata, we may pointwise define the complex Poisson bracket of any two holomorphic functions on $\mathcal{M}$. Therefore, the real question is to answer whether the resulting function is still holomorphic. We will show that the local biholomorphism between $\mathcal{M}$ and a local model $\nu^{-1}(0)/\mathcal{G}^C$ is a Poisson map. Then, Theorem D follows from this. Now the key observation to see that the local biholomorphism is a Poisson map is that the Kuranishi map $\theta$ (see [3] for the construction of Kuranishi maps and Kuranishi local models) induces the local biholomorphism and preserves the complex symplectic structures on $H^1$ and $\mathcal{G}$. Moreover, all the complex symplectic structures on the strata in the orbit type decompositions of $\mathcal{M}$ and $\nu^{-1}(0)/\mathcal{G}^C$ come from those on $\mathcal{G}$ and $H^1$.

Finally, we want to say a few words on the topologies we will be using on various spaces throughout this paper. By definition, the moduli space $\mathcal{M}$ and the hyperKähler quotient $\mathcal{m}^{-1}(0)/\mathcal{G}$ are equipped with the $C^\infty$-topology. That said, in order to use the implicit function theorem, in most of the proofs, we need to complete the spaces $\mathcal{G}^C$, $\mathcal{G}$ and $\mathcal{C}$ with respect to the Sobolev $L^2_{k+1}$-norm and $L^2_k$-norm. Here, $k > 1$. Then, the resulting spaces $\mathcal{G}_{k+1}$ and $\mathcal{G}^C_{k+1}$ are Banach Lie groups acting smoothly on the Banach affine manifold $\mathcal{C}_k$. Moreover, we also need to extend the moment map $\mathcal{m}$ to a moment map $\mathcal{m}_k$ on $\mathcal{G}_k$. By the Sobolev multiplication theorem, this is well-defined. On the other hand, by the regularity results in [15, Theorem 3.17] and [3, Lemma 3.11, 3.12 and Corollary 3.13], the natural maps $\mathcal{m}^{-1}(0)/\mathcal{G} \to \mathcal{m}_k^{-1}(0)/\mathcal{G}_{k+1}$ and $\mathcal{M} \to \mathcal{B}_k^\nu/\mathcal{G}^C_{k+1}$ are homeomorphisms. Moreover, it will be clear in the proofs of Theorems A and B that they preserve orbit type decompositions. As a result, for notational convenience, we will drop these subscripts that indicate the Sobolev completions and work with these Sobolev completions in the proofs whenever necessary. This should not cause any confusion. Finally, it should be noted that Theorem E, strictly speaking, should be a result for the Banach Lie groups $\mathcal{G}_{k+1}$ and $\mathcal{G}^C_{k+1}$.

2 Preliminaries

In this section, we review some useful results in [3]. We start with deformation complexes. Every Higgs bundle $(A, \Phi) \in \mathcal{m}^{-1}(0)$ defines a deformation complex

$$C_{\mu_C} : \quad \Omega^0(\mathcal{G}_E) \xrightarrow{D''} \Omega^{0,1}(\mathcal{G}_E) \oplus \Omega^{1,0}(\mathcal{G}_E) \xrightarrow{D''} \Omega^{1,1}(\mathcal{G}_E),$$

(2.1)
where $D'' = \tilde{\partial}_A + \Phi$.  

**Proposition 2.1** ([11, §1] and [12, §10]) $C_{\mu_C}$ is an elliptic complex and a differential graded Lie algebra. Moreover, the Kähler identities,

$$ (D'')^* = -i[* , D'] , \quad (D')^* = +i[* , D''] , \quad (2.2) $$

hold, where $D' = \partial_A + \Phi^*$ and $\ast$ is the Hodge star.

Another elliptic complex (see [7, p. 85]) associated with $(A, \Phi)$ is the following:

$$ C_{Hit}: \Omega^0(g_E) \xrightarrow{d_1} \Omega^1(g_E) \oplus \Omega^{1,0}(g_E^c) \xrightarrow{d_2 \oplus D''} \Omega^2(g_E) \oplus \Omega^{1,1}(g_E^c). \quad (2.3) $$

Here, $d_1(u) = (d_A u, [\Phi, u])$ and $d_2$ is the derivative of $\mu$ from (1.2) at $(A, \Phi)$. By direct computation, we have the following. As a consequence, throughout this paper, we will use $H^1$ to denote either $H^1(C_{\mu_C})$ or $H^1(C_{Hit})$.

**Proposition 2.2** The map $\Omega^1(g_E) \rightarrow \Omega^{0,1}(g_E)$ given by $\alpha \mapsto \alpha''$ induces an isomorphism $H^1(C_{Hit}) \simeq H^1(C_{\mu_C})$, where $\alpha''$ denotes the $(0, 1)$ component of $\alpha$.

Now, we review the Kuranishi local models used to construct the moduli space $\mathcal{M}$ as a normal complex space (for more details, see [3]). Fix $(A, \Phi) \in m^{-1}(0)$ with $\mathcal{B}$-stabilizer $H$ and consider the subspace $\mathcal{B} = (1 - \text{Har})\mu_C$, where $\text{Har}$ is the harmonic projection from $\Omega^{1,1}(g_E^c)$ onto $H^2(C_{\mu_C})$. Then, locally around $(A, \Phi)$, $\mathcal{B}$ is a complex submanifold of $\mathcal{C}$. Moreover, the holomorphic map

$$ F : \Omega^{0,1}(g_E^c) \oplus \Omega^{1,0}(g_E^c) \rightarrow \Omega^{0,1}(g_E^c) \oplus \Omega^{1,0}(g_E^c), \quad F(\alpha, \eta) = (\alpha, \eta) + (D'')^*G[\alpha'', \eta], \quad (2.4) $$

is $H^C_{\mu}$-equivariant and restricts to

$$ F : \mathcal{B} \cap ((A, \Phi) + \ker(D''))^* \rightarrow H^1. \quad (2.5) $$

In fact, $F$ maps an open neighborhood of $(A, \Phi)$ in $\mathcal{B} \cap ((A, \Phi) + \ker(D''))^*$ homeomorphically onto an open ball (in $L^2$-norm) $B \subset H_1$ around 0. Its inverse, viewed as a map $\theta : B \rightarrow \mathcal{C}$, is called a Kuranishi map. It can be shown that $\theta$ can be extended to a holomorphic map $BH^C \rightarrow \mathcal{C}$ that is $H^C_{\mu}$-equivariant, where $BH^C = \bigcup_{h \in H_C} Bh$. The hyperKähler structure on $\mathcal{C}$ restricts to $H^1$, and hence $H^1$ has a linear complex symplectic structure $\omega_C$. Since $H^C$ acts linearly on $H^1$ and preserves $\omega_C$, there is a standard complex moment map $\nu_{0,C} : H^1$ such that $\nu_{0,C}(0) = 0$. Let $Z = B \cap \nu_{0,C}^{-1}(0)$. It is proved that $\theta$ maps $\mathcal{Z}$ homeomorphically onto an open neighborhood of $(A, \Phi)$ in $\mathcal{B}^{ss} \cap ((A, \Phi) + \ker(D''))^*$. Moreover, $x \in \mathcal{Z}$ has a closed $K^C$-orbit in $H^1$ if and only if $\theta(x)$ is a polystable Higgs bundle, provided that $B$ is sufficiently small. Furthermore, $B$ can be arranged so that $\mathcal{Z} H^C / H^C$ is an open neighborhood of $[0]$ in $\nu_{0,C}^{-1}(0) / H^C$ and that $\theta$ induces a biholomorphism $\varphi : \mathcal{Z} H^C / H^C \rightarrow \mathcal{M}$ onto an open neighborhood of $[A, \Phi]$ in $\mathcal{M}$. We will also call $\varphi$ a Kuranishi map. More precisely, $\varphi[x] = [r\theta(x)]$ for any $x \in \mathcal{Z}$, where $r : \mathcal{B}^{ss} \rightarrow \mu^{-1}(0)$ is the retraction.
defined by the Yang-Mills-Higgs flow. The local inverse of $\phi$ is given by $[B, \Psi] \mapsto \{x\}$, where $(B, \Psi)$ can be chosen such that there are unique $g \in G^C$ and $x \in Z$ such that $(B, \Psi) = \theta(x)g$. Moreover, $g$ and $x$ depend on $(B, \Psi)$ holomorphically.

3 Mostow’s decomposition

In this section, we will prove Mostow’s decomposition for $G^C$, Theorem E. In fact, we will prove the following Theorem 3.1, and Theorem E follows as a corollary.

Let $H$ be a compact subgroup of $G$ and $\mathfrak{h}$ its Lie algebra. The compactness of $H$ implies that $\mathfrak{h}$ is a finite-dimensional subspace of $\Omega^0(\mathfrak{g}_E)$ and hence closed. Therefore, $\mathfrak{h}$ has a $L^2$-orthogonal complement $\mathfrak{h}^\perp$ in $\Omega^0(\mathfrak{g}_E)$ so that $\Omega^0(\mathfrak{g}_E) = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Moreover, $g$ and $x$ depend on $\mathfrak{h}$ holomorphically.

Theorem 3.1 (cf. [6, Corollary 9.5]) The map

$$H^C \times_H (\mathfrak{h}^\perp \times \mathscr{G}) \to \mathscr{G}^C, \quad [h, s, u] \mapsto h \exp(is)u,$$

(3.1)
is a bijection, where $H$ acts on $H^C \times (\mathfrak{h}^\perp \times \mathscr{G})$ by

$$h_0 \cdot (h, s, u) = (hh_0^{-1}, h_0sh_0^{-1}, h_0u).$$

(3.2)

To prove Theorem 3.1, we adapt the proof of [6, Corollary 9.5]. Recall that the polar decomposition $u(n) \times U(n) \to GL_n(\mathbb{C})$ induces a polar decomposition

$$\Omega^0(\mathfrak{g}_E) \times \mathscr{G} \to \mathscr{G}^C, \quad (s, u) \mapsto \exp(is)u.$$

(3.3)

Via the polar decomposition, the left multiplication of $H^C$ on $\mathscr{G}^C$ induces a left $H^C$-action on $\Omega^0(\mathfrak{g}_E) \times \mathscr{G}$. In particular, $H$ acts on $\Omega^0(\mathfrak{g}_E) \times \mathscr{G}$ by $h_0 \cdot (s, u) = (hh_0^{-1}, h_0sh_0^{-1}, h_0u)$. In the Appendix, we will show that both $\Omega^0(\mathfrak{g}_E) \times \mathscr{G}$ and $\mathscr{G}^C$ are weak Kähler manifolds such that the polar decomposition is an isomorphism of Kähler manifolds. Moreover, the left $H$-action is Hamiltonian with a moment map given by

$$\kappa: \Omega^0(\mathfrak{g}_E) \times \mathscr{G} \to \mathfrak{h}, \quad (s, u) \mapsto Ps,$$

(3.4)

where $P: \Omega^0(\mathfrak{g}_E) \to \mathfrak{h}$ is the projection. Now, we routinely identify $\Omega^0(\mathfrak{g}_E) \times \mathscr{G}$ with $\mathscr{G}^C$ using the polar decomposition. Then, Theorem 3.1 follows from the following.

Lemma 3.2

1. Every $H^C$-orbit in $\mathscr{G}^C$ intersect $\kappa^{-1}(0)$.
2. $\kappa^{-1}(0) \cap H^Cg = Hg$ for every $g \in \mathscr{G}^C$.

Proof Since $H$ is a compact Lie group (hence finite-dimensional), [10, Lemma 5.2 and Theorem 5.4] apply. Therefore, it suffices to show that

$$\lim_{t \to \infty} (\kappa(\exp(it)s)g, s)_{L^2} > 0$$

(3.5)
for any \( s \in \mathfrak{h} \) and \( g \in G^C \). Using the polar decomposition, we may write

\[
\exp(its)g = \exp(it\eta(t))u(t) \tag{3.6}
\]

for some \( \eta(t) \in \Omega^0(g_E) \) and \( u(t) \in \mathcal{G} \). Hence,

\[
(\kappa(\exp(its)g), s)_{L^2} = (P\eta(t), s)_{L^2} = (\eta(t), s)_{L^2}. \tag{3.7}
\]

Since \( H^C \) acts on \( \mathcal{G}^C \) freely, by [10, Lemma 2.2], \( (\eta(t), s)_{L^2} \) is a strictly increasing function of \( t \). Therefore, it suffices to prove that if \( t \gg 0, (\eta(t), s)_{L^2} \geq 0 \). Hence, we may assume that \( \eta(t) \neq 0 \) for any \( t \). By the proof of [16, Theorem 5.12], we see that

\[
\lim_{t \to \infty} \frac{\eta(t)}{\|\eta(t)\|_{L^2}} = \frac{s}{\|s\|_{L^2}} \tag{3.8}
\]

in \( L^2 \)-norm so that

\[
\lim_{t \to \infty} \left( \frac{\eta(t)}{\|\eta(t)\|_{L^2}}, \frac{s}{\|s\|_{L^2}} \right)_{L^2} = 1. \tag{3.9}
\]

Therefore, if \( t \gg 0, (\eta(t), s)_{L^2} > 0 \).

**Proof of Theorem 3.1** Consider the map

\[
H^C \times \mathcal{H} \kappa^{-1}(0) \to \mathcal{G}^C, \quad [h, s, u] \mapsto h \exp(isu). \tag{3.10}
\]

The surjectivity and the injectivity follow from (1) and (2) in Lemma 3.2, respectively. Moreover, \( \kappa^{-1}(0) = \mathfrak{h}^\perp \times \mathcal{G} \).

As a corollary of Mostow’s decomposition, Theorem E, we obtain the following that will be used often in this paper.

**Corollary 3.3** Let \( H \) and \( K \) be compact subgroups of \( \mathcal{G} \). Then, \( H^C \) and \( K^C \) are conjugate in \( \mathcal{G}^C \) if and only if \( H \) and \( K \) are conjugate in \( \mathcal{G} \).

**Proof** This follows from Mostow’s decomposition (Theorem E) and the first paragraph in the proof of [13, Theorem 2.10]. Note that all we need is the fact that the map in Theorem E is a \( \mathcal{G} \)-equivariant bijection.

**4 The orbit type decompositions**

**4.1 Orbit types in the hyperKähler quotient**

In this section, we will prove Theorem A.
**Proof of Theorem A.** Fix \([A, \Phi] \in Q\) such that \((A, \Phi)\) is of class \(C^\infty\). Hence, gauge transformations in its \(G\)-stabilizer \(H\) are of class \(C^\infty\). By definition, \(Q\) is a component of \(m^{-1}(0)_{(H)}/G\). Since the \(G\)-action is proper, a standard argument (e.g. [4, Proposition 4.4.5]) shows that there is an \(H\)-invariant open neighborhood \(S\) of \((A, \Phi)\) in \((A, \Phi) + \ker d_1^\Psi\) such that the natural map \(f : S \times_H G \to G\) is a \(G\)-equivariant diffeomorphism onto an open neighborhood of \((A, \Phi)\), where \(d_1\) is defined in the complex \(C_H\) (see Sect. 2). Therefore, the restriction

\[
f : ((m^{-1}(0) \cap S) \times_H G)_{(H)} \to m^{-1}(0)_{(H)}
\]  

is a \(G\)-equivariant homeomorphism onto an open neighborhood of \((A, \Phi)\) in \(m^{-1}(0)_{(H)}\). Since \(Q\) is open in \(m^{-1}(0)_{(H)}/G\), \(\pi^{-1}(Q)\) is open in \(m^{-1}(0)_{(H)}\) and contains \((A, \Phi)\). By shrinking \(S\), we may further assume that \(f\) takes values in \(\pi^{-1}(Q)\). We claim that

\[
((m^{-1}(0) \cap S) \times_H G)_{(H)} = (m^{-1}(0) \cap S^H) \times_H G = (m^{-1}(0) \cap S^H) \times (G/H).
\]  

(4.1)

where \(S^H\) consists of elements in \(S\) that are fixed by \(H\). The second equality is obvious. To show the first one, let \([B, \Psi, g]\) be a point in \((m^{-1}(0) \cap S) \times_H G\) with \(G\)-stabilizer conjugate to \(H\) in \(G\). As a consequence,

\[
G_{[B, \Psi, 1]} = gG_{[B, \Psi, g]}g^{-1} \in (H).
\]  

(4.3)

Since \(S\) is a local slice for the \(G\)-action on \(G\), \(G_{[B, \Psi, 1]} \subset H\). Since \(G_{[B, \Psi, 1]}\) and \(H\) have the same dimension and the same number of components, we see that \(G_{[B, \Psi, 1]} = H\). Hence, \(H\) fixes \((B, \Psi)\), and the claim follows. Therefore, the map \(\pi^{-1}(Q) \to Q\) can be locally identified with the projection

\[
(m^{-1}(0) \cap S^H) \times (G/H) \to m^{-1}(0) \cap S^H.
\]  

(4.4)

Moreover, since \(H\) is compact, the quotient map \(m^{-1}(0) \cap S \to (m^{-1}(0) \cap S)/H\) is closed. Since \(S^H\) is closed in \(S\), we conclude that \((m^{-1}(0) \cap S^H)/H\) is closed in \((m^{-1}(0) \cap S)/H\). Since \(m^{-1}(0) \cap S^H = (m^{-1}(0) \cap S)/H\) is homeomorphic to an open neighborhood of \([A, \Phi]\) in \(Q\), \(Q\) is a locally closed subset of \((m^{-1}(0))/G\). Then, we prove that \(m^{-1}(0) \cap S^H\) is a submanifold of \(S^H\). As a consequence, \(\pi^{-1}(Q)\) is a submanifold of \(G\), \(Q\) is a smooth manifold, and \(\pi : \pi^{-1}(Q) \to Q\) is a smooth submersion.

To show that \(m^{-1}(0) \cap S^H\) is a submanifold of \(S^H\), we adapt the proof of [15, Theorem 2.24]. Let \(\mu_i\) be a component of the hyperKähler moment map \(m\). We first show that the restriction \(\mu_i|_{S^H}\) has a constant finite corank so that \(S^H \cap \mu_i^{-1}(0)\) is a submanifold of \(S^H\). After that, we show that \(S^H \cap m^{-1}(0) = \cap_{i=1}^3 S^H \cap \mu_i^{-1}(0)\) is a submanifold of \(S^H\). Fix \((B, \Psi) \in S^H\). Note that \(T_{(B, \Psi)} S^H = (\ker d_1^\Psi)^H\). Consider the sequence

\[
\text{ Springer}
\]
\[
\Omega^0(\mathcal{G}_E) \xrightarrow{d_1} T_{(B, \Psi)} \mathcal{C} \xrightarrow{d_{\mu_i}} \Omega^2(\mathcal{G}_E), \tag{4.5}
\]

where \(d_{\mu_i}\) is the derivative of \(\mu_i\) at \((B, \Psi)\). Note that \(H\) acts on each term by conjugation, and both \(d_1\) and \(d_{\mu_i}\) are \(H\)-equivariant. Since the complex \(C_{Hit}\) is elliptic, the symbol of \(d_{\mu_i}\) is surjective so that the Hodge decomposition

\[
\Omega^2(\mathcal{G}_E) = \ker(d_{\mu_i})^* \oplus \text{im} d_{\mu_i} \tag{4.6}
\]

holds, where \((d_{\mu_i})^*\) is the \(L^2\)-formal adjoint of \(d_{\mu_i}\). We claim that

\[
d_{\mu_i} \left( (T_{(B, \Psi)} \mathcal{C})^H \right) = (\text{im} d_{\mu_i})^H. \tag{4.7}
\]

Since \(d_{\mu_i}\) is \(H\)-equivariant, the inclusion “\(\subset\)” is obvious. Conversely, suppose \(y = d_{\mu_i}(x)\) is fixed by \(H\) for some \(x \in T_{(B, \Psi)} \mathcal{C}\). Since \(H\) is compact, \(\int_H (x \cdot h) dh\) is well-defined and fixed by \(H\). Therefore, the inclusion “\(\supset\)” follows from

\[
y = \int_H (y \cdot h) dh = \int_H d_{\mu_i}(x \cdot h) dh = d_{\mu_i} \left( \int_H (x \cdot h) dh \right). \tag{4.8}
\]

Then, the \(H\)-equivariance of \(d_{\mu_i}\) implies that

\[
\Omega^2(\mathcal{G}_E)^H = (\ker(d_{\mu_i})^*)^H \oplus (\text{im} d_{\mu_i})^H = (\ker(d_{\mu_i})^*)^H \oplus d_{\mu_i} \left( (T_{(B, \Psi)} \mathcal{C})^H \right). \tag{4.9}
\]

Moreover, the formula \((d_{\mu_i})^* = -I_i d_1^*\) implies that

\[
\dim \ker(d_{\mu_i})^* = \dim \ker d_1 = \dim \mathcal{G}_{(B, \Psi)}. \tag{4.10}
\]

Since \(S\) is a local slice for the \(\mathcal{G}\)-action on \(\mathcal{C}\), \(\mathcal{G}_{(B, \Psi)} = H\). Finally, since

\[
(T_{(B, \Psi)} \mathcal{C})^H = (\text{im} d_1)^H \oplus (\ker d_1^*)^H = (\text{im} d_1)^H \oplus (T_{(B, \Psi)} S^H) \tag{4.11}
\]

and \(d_{\mu_i} d_1 = 0\), we conclude that \(\mu_i |_{S^H} : S^H \rightarrow \Omega^2(\mathcal{G}_E)^H\) has a constant finite corank so that \(\mu_i^{-1}(0) \cap S^H\) is a smooth submanifold of \(S^H\).

Now, we show that \(\cap_{i=1}^3 S^H \cap \mu_i^{-1}(0)\) is a submanifold of \(S^H\). Let \(\Delta : S^H \rightarrow (S^H)^3\) be the diagonal map. If we can show that \(\Delta\) is transversal to

\[
W = (\mu_1^{-1}(0) \cap S^H) \times (\mu_2^{-1}(0) \cap S^H) \times (\mu_3^{-1}(0) \cap S^H) \subset (S^H)^3, \tag{4.12}
\]

then \(\Delta^{-1}(W) = \cap_{i=1}^3 S^H \cap \mu_i^{-1}(0)\) is a smooth submanifold of \(S^H\). So, we fix \((B, \Psi) \in \Delta^{-1}(W)\). Then, we have

\[
T_{\Delta(B, \Psi)} W = \bigoplus_{i=1}^3 \left( \ker d_{\mu_i} \cap (\ker d_1^*)^H \right) \tag{4.13}
\]
and
\[ \Delta_* T_{(B, \psi)} S^H = \{(v, v, v) : v \in (\ker d_1^*)^H \}. \]  
(4.14)

Note that \( \ker d_1 \mu_i = (I_i \text{ im } d_1) \perp \). Therefore, if \( u_i \in (\ker d_1^*)^H (i = 1, 2, 3) \), then we may write \( u_i = u_i' + u_i'' \) for \( u_i' \in I_i \text{ im } d_1 \) and \( u_i'' \in (I_i \text{ im } d_1) \perp \). Since \( \mu_i(B, \Psi) = 0 \), it is not hard to check that \( u_i'' \in (\ker d_1^*)^H \). Moreover, since \( I_i \text{ im } d_1 \) are orthogonal to each other, we may further write
\[
   u_i = \left( u_i' - \sum_{j \neq i} u_j'' \right) + u_1'' + u_2'' + u_3''.
\]  
(4.15)

Therefore, \( T_{\Delta(B, \psi)} W \) and \( \Delta_* T_{(B, \psi)} S^H \) generate \( (T(B, \psi) S^H)^3 \), and we are done.

By construction, we see that \( m^{-1}(0) \cap S^H \) is a smooth manifold with tangent space
\[
   \ker d m \cap (\ker d_1^*)^H = (\ker d_2 \cap \ker d_3^*)^H = H^1(C_{H}it)^H = (H^1)^H
\]  
(4.16)
at \([A, \Phi]\). Since the \( L^2 \)-metric on \( C \) is preserved by the \( G \)-action, it descends to a metric on \( Q \). By the proof of [7, Theorem 6.7], we see that \( I, J \) and \( K \) restrict to \( H^1 \). Since they are preserved by the \( H \)-action on \( H^1 \), they further restrict to \( (H^1)^H \). Moreover, since they are preserved by the \( G \)-action, they, together with the \( L^2 \)-metric on \( (H^1)^H \), define an almost hyperKähler structure on \( Q \).

Let \( \Omega_i, \Omega_J \) and \( \Omega_K \) be the Kähler forms on \( C \) associated with complex structures \( I, J \) and \( K \), respectively. By the proof of [7, Theorem 6.7], we see that if \( v \in \text{ im } d_1 \oplus H^1 \) then \( \Omega_i(v, \xi) = 0 \) for any \( i \in \{I, J, K\} \) and any vector \( \xi \) tangent to the \( G \)-orbit. This also holds for \( v \in \text{ im } d_1 \oplus (H^1)^H = T_{(A, \Phi)} \pi^{-1}(Q) \). Therefore, there are unique Kähler forms \( \omega_i, i \in \{I, J, K\} \), on \( Q \) such that \( \pi^* \omega_i = \Omega_{i, \pi^{-1}(Q)} \) for \( i \in \{I, J, K\} \). Therefore, each \( \omega_i \) is closed. Then, the integrability of complex structures on \( Q \) follows from [7, Lemma 6.8].

Finally, by the elliptic regularity, it is easy to see that elements in \( \ker d_1^* \cap m^{-1}(0) \) are of class \( C^\infty \). Since \( S^H \cap m^{-1}(0) \subset \ker d_1^* \cap m^{-1}(0) \), our heuristic use of infinite-dimensional manifolds can be justified by working with Sobolev completions.

### 4.2 A local slice theorem

Now we study the strata in the orbit type decomposition of \( M \). Therefore, we need a local slice theorem for the \( G \)-action on \( C \).

**Theorem 4.1** Let \( (A, \Phi) \) be a Higgs bundle in \( m^{-1}(0) \) with \( G \)-stabilizer \( H \). Then, there exists an open neighborhood \( O \) of \( (A, \Phi) \) in \( (A, \Phi) + \ker(D'')^* \) such that the natural map \( O H^C \times_{H^C} G^C \to C \) is a biholomorphism onto an open neighborhood of \( (A, \Phi) \).
**Proof** Note that the \( G^\mathbb{C} \)-stabilizer of \((A, \Phi)\) is \( H^\mathbb{C} \) and acts on \((D'')^*\). Consider the natural map

\[
f : ((A, \Phi) + \ker(D'')^*) \times_{H^\mathbb{C}} G^\mathbb{C} \to \mathcal{C}.
\]

Its derivative at \([A, \Phi, 1]\) is given by

\[
\ker(D'')^* \oplus H^0(C_{\mu^C})^\perp \to \Omega^{0,1}(g^\mathbb{C}_E) \oplus \Omega^{1,0}(g^\mathbb{C}_E) \quad (x, u) \mapsto x + D''u,
\]

and hence an isomorphism, since \( T_{(A, \Phi)}\mathcal{C} = \ker(D'')^* \oplus \text{im } D''. \) Therefore, there are open neighborhoods \( O \times N \) of \((A, \Phi, 1)\) and \( W \) of \((A, \Phi)\) such that \( f : \pi(O \times N) \to W \) is a biholomorphism, where \( \pi \) is the quotient map. Then, we consider the restriction

\[
f : O H^\mathbb{C} \times_{H^\mathbb{C}} G^\mathbb{C} \to \mathcal{C}.
\]

Since \( O H^\mathbb{C} G^\mathbb{C} = W G^\mathbb{C} \), its image is \( \pi(O) G^\mathbb{C} \), which is an open neighborhood of \((A, \Phi)\) in \( \mathcal{C} \). Since \( G^\mathbb{C} = \bigcup_{g \in G^\mathbb{C}} Ng \) and \( f \) is \( G^\mathbb{C} \)-equivariant, \( f \) is a local biholomorphism. Therefore, it remains to show that \( f \) is injective provided that \( O \) is small enough. We will follow the proof of [1, Proposition 4.5]. Suppose that

\[
((A, \Phi) + (\alpha_1, \eta_1))g = (A, \Phi) + (\alpha_2, \eta_2)
\]

for some \( g \in G^\mathbb{C} \) and \((D'')^*(\alpha_i, \eta_i) = 0\). Equivalently,

\[
D''g + (\alpha''_1 g - g \alpha''_2, \eta_1 g - g \eta_2) = 0,
\]

where \( \alpha''_i \) is the \((0, 1)\)-component of \( \alpha_i \). We show that if each \( \|\alpha_i, \eta_i\|_{L^2_k} \) is small enough, then \( g \in H^\mathbb{C} \). Write \( g = g_0 + g_1 \) for some \( g_0 \in H^0(C_{\mu^C}) \) and \( g_1 \in H^0(C_{\mu^C})^\perp \). The idea is to show that \( D''g_1 = 0 \) so that \( g = g_0 \in H^\mathbb{C} \). Applying \((D'')^*\), we obtain

\[
(D'')^*D''g + (D'')^*(\alpha''_1 g - g \alpha''_2, \eta_1 g - g \eta_2) = 0.
\]

We first claim that \( D'g_0 = 0 \), where \( D' = \partial_A + \Phi^* \). In fact, using the Kähler identity, \( D^* = +i[^*, D''] \), we have

\[
\|D'g_0\|^2_{L^2} = (D'^*D'g_0, g_0)_{L^2} = i(D''D'g_0, g_0)_{L^2} = -i(D'D''g_0, g_0)_{L^2} = 0.
\]

Here, we have used the fact that \( D''D' + D'D'' = 0 \), since \( \mu(A, \Phi) = F_A + [\Phi, \Phi^*] = 0 \). Then, using the Kähler’s identity, \((D'')^* = -i[^*, D']\), we see that \( D'(\alpha_i, \eta_i) = 0 \) for each \( i \), and

\[
(D'')^*(\alpha''_1 g - g \alpha''_2, \eta_1 g - g \eta_2) = (D'')^*(\alpha''_1 g_1 - g_1 \alpha''_2, \eta_1 g_1 - g_1 \eta_2).
\]
As a consequence,
\[\|D''g_1\|_{L^2}^2 = - \left( (\alpha''^\prime g_1 - g_1\alpha''^\prime, \eta_1 g_1 - g_1\eta_2), D''g_1 \right)_{L^2}\]
\[\leq \|(\alpha''^\prime g_1 - g_1\alpha''^\prime, \eta_1 g_1 - g_1\eta_2)\|_{L^2} \|D''g_1\|_{L^2}\]
\[\leq (\|\alpha''^\prime\|_{C^0} + \|\alpha''^\prime\|_{C^0} + \|\eta_1\|_{C^0} + \|\eta_2\|_{C^0})\|g_1\|_{L^2} \|D''g_1\|_{L^2}\]
\[\leq C(\|\alpha''^\prime\|_{L^2} + \|\alpha''^\prime\|_{L^2} + \|\eta_1\|_{L^2} + \|\eta_2\|_{L^2})\|g_1\|_{L^2} \|D''g_1\|_{L^2},\]  \hspace{1cm} (4.25)

where we have used the Sobolev embedding \(L^2 \hookrightarrow C^0\). Moreover, since \(g_1 \in H^0(C_{\mu})\),\(\frac{\partial}{\partial x}\),
\[\|g_1\|_{L^2} \leq \|g_1\|_{L^1} = \|D''g_1\|_{L^1} \leq C \|D''g_1\|_{L^2}.\]  \hspace{1cm} (4.26)

Therefore,
\[\|D''g_1\|_{L^2}^2 \leq C(\|\alpha''^\prime\|_{L^2} + \|\alpha''^\prime\|_{L^2} + \|\eta_1\|_{L^2} + \|\eta_2\|_{L^2})\|D''g_1\|_{L^2}.\]  \hspace{1cm} (4.27)

Since the isomorphism \(\Omega^1(g_{\Phi}) \to \Omega^{1,0}(g_{\Phi}^C)\) given by \(\alpha \mapsto \alpha''\) is a homeomorphism in the \(L^2\)-topology, we conclude that that if \(\|(\alpha_i, \eta_i)\|_{L^2}^2\) is small enough for every \(i\), then \(D''g_1 = 0\). \(\square\)

As a corollary of Proposition 4.1, every Kuranishi map \(\theta: B \to \mathcal{C}\) (see Sect. 2) preserves stabilizers in the following sense.

**Proposition 4.2** If \(B\) is sufficiently small, then \((H^C)_x = (\mathcal{G}^C)_{\theta(x)}\) for every \(x \in \mathscr{X}\).

**Proof** By [3, Proposition 3.4], if \(x_1 = x_2g\) for some \(x_1, x_2 \in B\) and \(g \in H^C\) then \(\theta(x_1) = \theta(x_2)g\). This proves the inclusion “\(\subset\)”. To prove the inclusion “\(\supset\)”, we shrink \(B\) so that \(\theta(\mathscr{X}) \subseteq O\) where \(O\) is obtained in Proposition 4.1. As a consequence, if \(\theta(x)g = \theta(x)\) for some \(g \in \mathcal{G}\) then \(g \in H^C\). Since \(F\) is \(H^C\)-equivariant and \(F(\theta(x)) = x\) for every \(x \in B\), we conclude that \(xg = x\). \(\square\)

### 4.3 Orbit types in the moduli space

Now we are able to prove Theorem B. Before giving the proof, we first show that there is a one-to-one correspondence between the conjugacy classes appearing in the orbit type decompositions of \(\mathcal{M}\) and \(m^{-1}(0)/\mathcal{G}\).

**Proposition 4.3** Every conjugacy class \((L)\) appearing in the orbit type decomposition of \(\mathcal{M}\) is equal to a conjugacy class \((H^C)\) for some \(\mathcal{G}\)-stabilizer \(H\) at some Higgs bundle in \(m^{-1}(0)\).

**Proof** Let \((A, \Phi)\) be a polystable Higgs bundle whose \(\mathcal{G}^C\)-stabilizer is conjugate to \(L\) in \(\mathcal{G}^C\). By the Hitchin–Kobayashi correspondence, \(\mu((A, \Phi)g) = 0\) for some \(g \in \mathcal{G}^C\). Therefore,
\[\left(\mathcal{G}(A, \Phi)g\right)^C = \left(\mathcal{G}^C\right)_{(A, \Phi)g} = g^{-1}\left(\mathcal{G}^C\right)_{(A, \Phi)g},\]  \hspace{1cm} (4.28)

\(\square\)
Let $H = \mathcal{G}_{(A, \Phi)}g$ and the $\mathcal{G}^C$-stabilizer of $(A, \Phi)$ is conjugate to $H^C$ in $\mathcal{G}^C$. □

**Proof of Theorem B** Fix $[A, \Phi] \in Q$ such that $(A, \Phi) \in \mathfrak{m}^{-1}(0)$ is of class $C^\infty$. Therefore, gauge transformations in the $\mathcal{G}$-stabilizer $H$ at $(A, \Phi)$ are of class $C^\infty$. Since $H^C$ is the $\mathcal{G}^C$-stabilizer at $(A, \Phi)$, $Q$ is a component of $\mathcal{B}^{ps}_{(H^C)}/\mathcal{G}^C$. By Theorem 4.1, there is an open neighborhood $O$ of $(A, \Phi) + \ker(D')^*$ such that the natural map $OH^C \times H^C \mathcal{G}^C \rightarrow \mathcal{C}$ is a diffeomorphism onto an open neighborhood of $(A, \Phi)$. The $\mathcal{G}^C$-equivariance implies that

$$
\left( OH^C \times H^C \mathcal{G}^C \right)_{(H^C)} \rightarrow \mathcal{C}(H^C),
$$

(4.29)

Since $H^C$ has finitely many components and is finite-dimensional, following the proof of Theorem A, we see that

$$
\left( OH^C \times H^C \mathcal{G}^C \right)_{(H^C)} = OH^C \times (\mathcal{G}^C/H^C).
$$

(4.30)

As a consequence, the natural map

$$
\left( OH^C \cap \mathcal{B}^{ss} \right) \times \left( \mathcal{G}^C/H^C \right) \rightarrow \mathcal{B}^{ss}_{(H^C)}
$$

(4.31)

is a diffeomorphism onto an open image. On the other hand, the Kuranishi map $\theta : \mathcal{Z} \rightarrow O \cap \mathcal{B}^{ss}$ is a homeomorphism if $B$ is sufficiently small. Since $\theta$ preserves stabilizers (Proposition 4.2), we see that $\theta : \mathcal{Z}^H \rightarrow OH^C \cap \mathcal{B}^{ss}$ is a homeomorphism. Since the $H^C$-action on $H^1$ is holomorphic with respect to $I$, $(H^1)^{H^C}$ is a complex symplectic subspace of $H^1$ so that $H^1 = F \oplus (H^1)^{H^C}$ where $F$ is the $\omega^C$-complement of $(H^1)^{H^C}$. As a consequence,

$$
v_0^{-1}(0, \omega) = (v_0, \omega|_F)^{-1}(0) \times (H^1)^{H^C}
$$

(4.32)

so that $\mathcal{Z}^{H^C}$ is an open subset of $v_0^{-1}(0)^{H^C} = (H^1)^{H^C}$. Since every point in $(H^1)^{H^C}$ has closed $H^C$-orbits, $\theta(\mathcal{Z}^{H^C}) \subset OH^C \cap \mathcal{B}^{ps}$. Moreover, since $Q$ is open in $\mathcal{B}^{ps}_{(H^C)}/\mathcal{G}^C$, $\pi^{-1}(Q)$ is open in $\mathcal{B}^{ps}_{(H^C)}$. Therefore, if $O$ and $B$ are sufficiently small, we obtain a well-defined map

$$
f : \mathcal{Z}^{H^C} \times (\mathcal{G}^C/H^C) \rightarrow \pi^{-1}(Q), \quad (x, [g]) \mapsto \theta(x)g,
$$

(4.33)

which is a homeomorphism onto its open image. This already shows that $\pi^{-1}(Q)$ is a complex submanifold of $\mathcal{C}$. Moreover, $f$ induces a well-defined map $\mathcal{Z}^{H^C} \rightarrow Q$ given by $[x] \mapsto [\theta(x)]$. It is exactly the restriction of the local chart (see Sect. 2) $\varphi : \mathcal{Z}^{H^C} \parallel H^C \rightarrow \mathcal{M}$, since

$$
v_0^{-1}(0) \parallel H^C = (v_0, \omega|_F)^{-1}(0) \parallel H^C \times (H^1)^{H^C},
$$

(4.34)
and $\theta(x)$ is polystable for every $x \in \mathcal{Z}_{\mathbb{C}}^c$. Therefore, we see that $Q$ is a complex submanifold of $\mathcal{M}$. Moreover, the quotient map $\pi: \pi^{-1}(Q) \to Q$ can be locally identified with the projection
\[ \mathcal{Z}_{\mathbb{C}}^c \times (\mathcal{G}^c / H^c) \to \mathcal{Z}_{\mathbb{C}}^c, \tag{4.35} \]
which is clearly a holomorphic submersion. Finally, by elliptic regularity, elements in $H^1$ are of class $C^\infty$. Since $\mathcal{Z}_{\mathbb{C}}^c \subset H^1$, our heuristic use of infinite-dimensional manifolds can be justified by working with Sobolev completions.

## 5 Complex Whitney stratification

To show that the orbit type decomposition of $\mathcal{M}$ is a complex Whitney stratification, we follow Mayrand’s arguments in [9, §4.6 and §4.7]. The idea is that the problem can be reduced to a local model $v_{0,\mathbb{C}}^{-1}(0) \sslash H^c$ near $[0]$, once we show that the Kuranishi map $\varphi: \tilde{U} \to U$ preserves the orbit type decompositions, where $\tilde{U} = \mathcal{Z}_{\mathbb{C}}^c \sslash H^c$ and $U = \varphi(\tilde{U})$ (see Sect. 2). To clarify different possible partitions on $v_{0,\mathbb{C}}^{-1}(0) \sslash H^c$, we adopt the following notation. By [5], the natural map $v_{0,\mathbb{C}}^{-1}(0)^{ps} \hookrightarrow v_{0,\mathbb{C}}^{-1}(0)$ induces a bijection
\[ v_{0,\mathbb{C}}^{-1}(0)^{ps} / H^c \sim v_{0,\mathbb{C}}^{-1}(0) \sslash H^c, \tag{5.1} \]
where $v_{0,\mathbb{C}}^{-1}(0)^{ps}$ is the subspace of $v_{0,\mathbb{C}}^{-1}(0)$ consisting of polystable points, or equivalently points whose $H^c$-stabilizers are closed in $H^1$. Let $L$ be a $H^c$-stabilizer at some point in $v_{0,\mathbb{C}}^{-1}(0)$ and $(L)_{H^c}$ the conjugacy class of $L$ in $H^c$. Then, we may define
\[ v_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{H^c}} = \left\{ x \in v_{0,\mathbb{C}}^{-1}(0)^{ps} : (H^c)_x \in (L)_{H^c} \right\}. \tag{5.2} \]
As a consequence, $v_{0,\mathbb{C}}^{-1}(0) \sslash H^c$ has a partition
\[ \mathcal{P}^c_h = \left\{ v_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{H^c}} / H^c : L = (H^c)_x \text{ for some } x \in v_{0,\mathbb{C}}^{-1}(0) \right\}. \tag{5.3} \]
Here, we have identified $v_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{H^c}}$ with its image in $v_{0,\mathbb{C}}^{-1}(0) \sslash H^c$. The orbit type decomposition of $v_{0,\mathbb{C}}^{-1}(0) \sslash H^c$ is defined as the refinement $\mathcal{P}^c_\circ_h$ of $\mathcal{P}^c_h$ into connected components. If $(L)_{\mathcal{G}^c}$ is the conjugacy class of $L \subset H^c$ in $\mathcal{G}^c$, then we may similarly define the partition
\[ \mathcal{P}^c_{\mathcal{G}^c} = \left\{ v_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{\mathcal{G}^c}} / H^c : L = (H^c)_x \text{ for some } x \in v_{0,\mathbb{C}}^{-1}(0) \right\}. \tag{5.4} \]
Finally, note that $v_{0,\mathbb{C}}^{-1}(0) \sslash H^c$ can be realized as a hyperKähler quotient as follows. Since $H$ acts linearly on $H^1$ and preserves the Kähler form $\omega_I$, there is a moment map
\(v_0\) associated with the Kähler form \(\omega_f\) such that \(v_0(0) = 0\). Then, \(n_0 = (v_0, v_0, C)\) is a hyperKähler moment map for the \(H\)-action. By [5], the inclusion \(n_0^{-1}(0) \hookrightarrow v_0^{-1}(0)\) induces a homeomorphism

\[
n_0^{-1}(0)/H \sim v_0^{-1}(0) \parallel H^C.
\] (5.5)

Then, the same proof of Proposition 4.3 shows that every conjugacy class \((L)\) appearing in \(\tilde{\mathcal{P}}_H^c\) or \(\tilde{\mathcal{P}}_g^c\) is equal to a conjugacy class \((H^C_1)\) for some \(H\)-stabilizer \(H_1\) at some point in \(n_0^{-1}(0)\). Then, the relation between \(\tilde{\mathcal{P}}_H^c\) and \(\tilde{\mathcal{P}}_g^c\) is stated below.

**Lemma 5.1** [9, Lemma 4.2] \(\tilde{\mathcal{P}}_H^c = \tilde{\mathcal{P}}_g^c\).

**Proof** This follows from the same proof of [9, Lemma 4.2]. The only difference is that Mowstow’s decomposition in that proof must be replaced by Theorem E. \(\square\)

Now, let \(\mathcal{P}\) be the partition of \(\mathcal{M}\) defined as

\[
\mathcal{P} = \left\{ \mathcal{B}^{\text{ps}}_{(L)g^c}/G^c : L = (G^c)_{(A, \Phi)} \text{ for some } (A, \Phi) \in \mathcal{B}^{\text{ps}} \right\}.
\] (5.6)

Note that the orbit type decomposition of \(\mathcal{M}\) is simply \(\mathcal{P}^o\). Then, we have the following.

**Proposition 5.2** The Kuranishi map \(\varphi : (\tilde{U}, (\tilde{\mathcal{P}}_H^c|\tilde{U})^o) \to (U, (\mathcal{P}^o|U)^o)\) is an isomorphism of partitioned spaces.

**Proof** We first record a simple fact without a proof. \(\square\)

**Lemma 5.3** Let \(X\) be a space and \(\mathcal{P}\) a partition of \(X\). If \(U\) is an open subset of \(X\), then \((\mathcal{P}^o|U)^o = (\mathcal{P}|U)^o\).

Then, by Lemmas 5.1 and 5.3,

\[
(\tilde{\mathcal{P}}_H^c|\tilde{U})^o = (\tilde{\mathcal{P}}_g^c|\tilde{U})^o = (\tilde{\mathcal{P}}_g|\tilde{U})^o
\] (5.7)

and \((\mathcal{P}^o|U)^o = (\mathcal{P}|U)^o\). Then, it suffices to show that

\[
\varphi : (\tilde{U}, \tilde{\mathcal{P}}_g|\tilde{U}) \to (U, \mathcal{P}|U)
\] (5.8)

is an isomorphism of partitioned spaces. If \([x] \in \tilde{U}\) is such that the \(H^C\)-orbit of \(x\) is closed in \(H^1\), then \(\varphi[x] = [\theta(x)]\). The rest follows from \((G^c)_{\theta(x)} = (H^C)_x\) (Proposition 4.2).

**Theorem 5.4** The orbit type decomposition of \(\mathcal{M}\) is a complex Whitney stratification.

**Proof** We first show that \(\mathcal{P}^o\) satisfies the frontier condition. In other words, we need to show that if \(Q_1, Q_2 \in \mathcal{P}^o\) and \(Q_1 \cap Q_2 \neq \emptyset\), then \(Q_1 \subset \tilde{Q}_2\). Fix \([A, \Phi] \in \mathcal{M}\) such that \(\mu(A, \Phi) = 0\) and \(\mathcal{P}_{(A, \Phi)} = H\). Let \(\varphi : \tilde{U} \to U\) be the Kuranishi map. Let \(Q\) be the component of \(\mathcal{B}^{\text{ps}}_{(H^C)}/G^c\) containing \([A, \Phi]\). If \([x] \in \varphi^{-1}(Q \cap U)\) such
that its $H^C$-orbit is closed in $H^1$. By Proposition 4.2, $(G^C)_{\theta(x)} = (H^C)_x$. Since $[\theta(x)] \in Q$, we conclude that $(H^C)_x = H^C$. From the proof of Theorem B in Sect. 4.3, we see that
\[
\varphi^{-1}(Q \cap U) = 2^pH^C = B \cap (H^1)^H^C. \tag{5.9}
\]
This shows that $Q \cap U$ is connected so that $Q \cap U \in (\mathcal{P}^0|_U)^\circ$. By [9, Lemma 4.7], it suffices to show that $(\mathcal{P}^0|_U)^\circ$ is conical at $Q \cap U$ (see [9, p. 18] for the definition). Since $B \cap (H^1)^H^C \in (\tilde{\mathcal{P}}^0_{(H^C)})^\circ$, Proposition 5.2 implies that it suffices to show that $(\tilde{\mathcal{P}}^0_{(H^C)})^\circ \cap (\tilde{\mathcal{P}}^0_{(H^C)})^\circ$ is conical at $B \cap (H^1)^H^C$. Moreover, by Lemma 5.3, it suffices to show that $(\tilde{\mathcal{P}}^0_{(H^C)})^\circ$ is conical at $B \cap (H^1)^H^C$. This follows from the proof of [9, Proposition 4.8].

Now we show that $\mathcal{P}$ satisfies the Whitney conditions at every point of $\mathcal{M}$. Fix $[A, \Phi] \in \mathcal{M}$ and let $\varphi: \tilde{U} \to U$ be a Kuranishi map such that $[A, \Phi] \in U$. Then, it suffices to check that $(\mathcal{P}^0|_U)^\circ$ satisfies the Whitney conditions at $[A, \Phi]$. By Proposition 5.2, it suffices to check that $(\tilde{\mathcal{P}}^0_{(H^C)})^\circ \cap (\tilde{\mathcal{P}}^0_{(H^C)})^\circ$ satisfies the Whitney conditions at $[0]$. This follows from [9, Proposition 4.12]. \hfill \Box

6 The Hitchin–Kobayashi correspondence

Finally, we show that the Hitchin–Kobayashi correspondence preserves the orbit type decompositions, Theorem C (cf. [9, Proposition 4.6]).

**Proof of Theorem C** Suppose $Q$ is a component of $m^{-1}(0)/(H)/\mathcal{G}$ for some $\mathcal{G}$-stabilizer at a Higgs bundle in $m^{-1}(0)$. We first show that the restriction
\[
i: m^{-1}(0)/(H)/\mathcal{G} \to \tilde{\mathcal{P}}_{(H^C)}^p/\mathcal{G}^C \tag{6.1}
\]
is a bijection and hence a homeomorphism. As a consequence, $i(Q)$ is a stratum in the orbit type decomposition of $\mathcal{M}$. The injectivity is obvious. To show the surjectivity, let $[A, \Phi] \in \tilde{\mathcal{P}}_{(H^C)}^p/\mathcal{G}^C$. We may further assume that $(\mathcal{G}^C)_{(A, \Phi)} = H^C$. The Hitchin–Kobayashi correspondence provides some $g \in \mathcal{G}^C$ such that $(A, \Phi)g \in m^{-1}(0)$. Hence,
\[
(\mathcal{G}^C)_{(A, \Phi)}g = g^{-1}(\mathcal{G}^C)_{(A, \Phi)}g = g^{-1}H^Cg. \tag{6.2}
\]
Then, Corollary 3.3 implies that $\mathcal{G}^C_{(A, \Phi)}g$ is conjugate to $H$ in $\mathcal{G}$.

Then, we show that the restriction $i: Q \to i(Q)$ is holomorphic. Consequencely, since $Q$ and $i(Q)$ are smooth, $i|_Q$ is a biholomorphism. By the proofs of Theorems A and B, we see that $i|_Q$ can be locally identified with a map $m^{-1}(0) \cap S^H \to (H^1)^H^C$. More precisely, this map is given by $(B, \Psi) \mapsto x$ where $x$ is determined by the equation $(B, \Psi) = \theta(x)g$ for a unique $g \in \mathcal{G}^C$. It is holomorphic, since $x$ and $g$ depend on $(B, \Psi)$ holomorphically, which can be seen by Proposition 4.1 and the holomorphicity of $\theta: B \to \mathcal{G}$.
7 Poisson structure

Let \( Q \) be a stratum in the orbit type stratification of \( \mathcal{M} \). As a consequence, \( i^{-1}(Q) \) is a stratum in the orbit type stratification of \( \mathfrak{m}^{-1}(0)/\mathcal{G} \), where \( i \) is the Hitchin–Kobayashi correspondence. We have shown in Theorem A that \( i^{-1}(Q) \) is a hyperKähler manifold and hence has a complex symplectic form \( \omega_C = \omega_J + \sqrt{-1} \omega_K \). Using the Hitchin–Kobayashi correspondence \( i \), we may transport \( \omega_C \) to \( Q \) so that \( Q \) is also a complex symplectic manifold. We will still use \( \omega_C \) to denote the resulting complex symplectic form on \( Q \). Alternatively, \( \omega_C \) can be defined as follows. Let \( \pi : \mathcal{B}^{ps} \rightarrow \mathcal{M} \) be the quotient map. It is shown in Theorem B that \( \pi^{-1}(Q) \) is a complex submanifold of \( \mathcal{C} \), and \( \pi : \pi^{-1}(Q) \rightarrow Q \) is a holomorphic submersion. Then, it follows that \( \pi^* \omega_C = \Omega_C|_{\pi^{-1}(Q)} \), where \( \Omega_C = \Omega_J + \sqrt{-1} \Omega_K \) is the complex symplectic form on \( \mathcal{C} \). This can be seen by Theorem A and the definition of \( i \). Then, a complex Poisson bracket can be defined on the sheaf of \( \mathcal{M} \) as follows. Let \( U \) be an open subset of \( \mathcal{M} \) and \( f, g : U \rightarrow \mathbb{C} \) holomorphic functions. Let \( Q \) be a stratum in the orbit type stratification of \( \mathcal{M} \). Therefore, the restrictions \( f|_{U \cap Q} \) and \( g|_{U \cap Q} \) are holomorphic so that the Poisson bracket \( \{f|_{U \cap Q}, g|_{U \cap Q}\}_{Q} \) is well-defined using the complex symplectic form \( \omega_C \) on \( Q \). Consequently, there is a unique function \( \{f, g\} : U \rightarrow \mathbb{C} \) such that

\[
\{f, g\}|_{U \cap Q} = \{f|_{U \cap Q}, g|_{U \cap Q}\}_{Q} \tag{7.1}
\]

for every stratum \( Q \). Then, it remains to show that \( \{f, g\} : U \rightarrow \mathbb{C} \) is holomorphic. Since the complex space \( \mathcal{M} \) is constructed by gluing Kuranishi local models, \( \{f, g\} \) is holomorphic if and only if its pullback along any Kuranishi map is holomorphic. On the other hand, from Sect. 5, we see that \( v_{0,\mathcal{C}}^{-1}(0) \parallel H^\mathcal{C} \) can be realized as a singular hyperKähler quotient. Hence, the structure sheaf of \( v_{0,\mathcal{C}}^{-1}(0) \parallel H^\mathcal{C} \) has a Poisson structure by [9, Theorem 1.4]. Then, the holomorphicity of the Poisson bracket on \( \mathcal{M} \) follows from the following result (cf. [9, Proposition 4.18]).

**Theorem 7.1** (=Theorem D) Every Kuranishi map \( \varphi : \tilde{U} \rightarrow U \) is a Poisson map. In other words, if \( f, g : U \rightarrow \mathbb{C} \) are holomorphic functions, then \( \varphi^*\{f, g\} = \{\varphi^*f, \varphi^*g\} \).

Before giving the proof, we need the following lemmas.

**Lemma 7.2** The Kuranishi map \( \theta : B \rightarrow \mathcal{C} \) preserves the complex symplectic forms.

**Proof** By construction of \( \theta \) (see Sect. 4.2), it suffices to show that the map

\[
F : \tilde{B} \cap ((A, \Phi) + \ker(\mathcal{D}''^*)) \rightarrow H^1,
\]

\[
F(\alpha'', \eta) = (\alpha'', \eta) + \frac{1}{2} (\mathcal{D}''^*)^* G[\alpha'', \eta; \alpha'', \eta], \tag{7.2}
\]

preserves the restrictions of Kähler forms \( \Omega_J \) and \( \Omega_K \). For notational convenience, let \( M = \tilde{B} \cap ((A, \Phi) + \ker(\mathcal{D}''^*)) \). If \( (B, \Psi) = (A, \Phi) + (\alpha_0'', \eta_0) \in M \), then

\[
d_{(B, \Psi)} F(\alpha, \eta) = (\alpha, \eta) + (\mathcal{D}''^*)^* G[\alpha_0'', \eta_0; \alpha, \eta], \quad (\alpha, \eta) \in T_{(B, \Psi)} M. \tag{7.3}
\]
Then, we need to show that
\[
\Omega_i(d((B, \Psi) F(\alpha''_1; \eta_1), d((B, \Psi) F(\alpha''_2; \eta_2)) = \Omega_i(\alpha''_1; \eta_1; \alpha''_2, \eta_2) \quad (7.4)
\]
for any \( i \in \{J, K\}, (B, \Psi) \in M \), and \( (\alpha''_j, \eta_j) \in T_{(B, \Psi)} M \). This amounts to show that
\[
(1) \ \Omega_i(\alpha''_1; \eta_1; (D'')^* G[\alpha''_0; \eta_0; \alpha''_2, \eta_2]) = 0, \quad \text{and} \quad \Omega_i((D'')^* G[\alpha_0, \eta_0, \alpha''_1; \eta_1]; (D'')^* G[\alpha''_0; \eta_0; \alpha''_2, \eta_2]) = 0
\] for any \( i \in \{J, K\} \) and \( (\alpha''_j, \eta_j) \in \ker(D'')^* \) for \( j = 0, 1, 2 \). To show (1), we compute
\[
\Omega_J(\alpha''_1, \eta_1; (D'')^* G[\alpha''_0; \eta_0; \alpha''_2, \eta_2]) = g(D'' J(\alpha''_1; \eta_1), G[\alpha''_0, \eta_0; \alpha''_2, \eta_2]), \quad (7.5)
\]
where \( g \) is the \( L^2 \)-metric. Moreover,
\[
D'' J(\alpha''_1; \eta_1) = D''(i \eta^*, -i \alpha''^*) = -i \tilde{\partial}_A \alpha''^* + i [\Phi, \eta^*] = (i D'(\alpha''_1; \eta_1))^* = 0 \quad (7.6)
\]
where the last equality follows from the Kähler’s identity, \( (D'')^* = -i[*; D'] \) and the assumption that \( (D'')^*(\alpha''_1; \eta_1) = 0 \). Similarly,
\[
\Omega_K(\alpha''_1, \eta_1; (D'')^* G[\alpha''_0; \eta_0; \alpha''_2, \eta_2]) = g(D'' K(\alpha''_1; \eta_1), G[\alpha''_0, \eta_0; \alpha''_2, \eta_2]) \quad (7.7)
\]
and
\[
D'' K(\alpha''_1, \eta_1) = D''(-\eta^*, \alpha''^*) = \partial_A \alpha''^* - [\Phi, \eta^*] = D'(\alpha''_1; \eta_1)^* = 0 \quad (7.8)
\]
Finally, the same argument shows (2). \hfill \Box

Now, let \( \tilde{C} \) be a connected component of \( \tilde{Q} \cap \tilde{U} \), where \( \tilde{Q} \) is a stratum in the orbit type stratification of \( \nu_{0, C}(0) \parallel H^C \). In other words, \( \tilde{C} \in (\mathcal{P}^o_{H^C})(\tilde{U})^o \). By Proposition 5.2, there is some connected component \( C \) of \( Q \cap U \) for some stratum \( Q \) in the orbit type stratification of \( \mathcal{M} \) such that the restriction \( \phi: \tilde{C} \to C \) is a biholomorphism. Let \( \pi \) denote the projections \( \mathcal{B}^\iota \to \mathcal{M} \) and \( \nu_{0,C}(0) \to v_{0,C}(0) \parallel H^C \). By Theorem B and [9, Lemma 4.14], \( \pi^{-1}(C) \) and \( \pi^{-1}(\tilde{C}) \) are complex submanifolds of \( C \) and \( H^1 \), respectively. Moreover, the following diagram commutes
\[
\pi^{-1}(\tilde{C}) \cap B \xrightarrow{\theta} \pi^{-1}(C) \quad (7.9)
\]
\[
\pi \downarrow \quad \pi \downarrow
\]
\[
\tilde{C} \xrightarrow{\phi} C
\]
Now, by Lemma 7.2, \( \theta \) preserves the restrictions of the complex symplectic forms \( \Omega_C \) on \( C \) and \( \omega_C \) on \( H^1 \). By Theorem B and [9, Lemma 4.14] again, we see that these restrictions of complex symplectic forms descend to \( \tilde{C} \) and \( C \). As a consequence, we obtain the following.
Lemma 7.3 The Kuranishi map \( \varphi: \tilde{\mathcal{C}} \to \mathcal{C} \) is a complex symplectomorphism. In particular, it preserves the complex Poisson brackets.

Proof of Theorem 7.1 Let \( f, g: U \to \mathbb{C} \) be holomorphic functions. Then, we compute
\[
(\varphi^*\{f, g\}|_{\tilde{\mathcal{C}}}) = (\varphi|_{\tilde{\mathcal{C}}})^*(\{f, g\}|_\mathcal{C})
= (\varphi|_{\tilde{\mathcal{C}}})^*(\{f|_\mathcal{C}, g|_\mathcal{C}\}|_{\tilde{\mathcal{C}}})
= \{(\varphi|_{\tilde{\mathcal{C}}})^*(f|_\mathcal{C}), (\varphi|_{\tilde{\mathcal{C}}})^*(g|_\mathcal{C})\}|_{\tilde{\mathcal{C}}}
= \{(\varphi^* f)|_{\tilde{\mathcal{C}}}, (\varphi^* g)|_{\tilde{\mathcal{C}}}\}|_{\tilde{\mathcal{C}}}
= \{\varphi^* f, \varphi^* g\}|_{\tilde{\mathcal{C}}}
\]
for any connected component \( \tilde{\mathcal{C}} \) of \( \tilde{\mathcal{Q}} \cap \tilde{\mathcal{U}} \) for some stratum \( \tilde{\mathcal{Q}} \) in the orbit type stratification of \( v_{0, \mathcal{C}}^{-1}(0) \parallel H^\mathcal{C} \). This completes the proof.

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Appendix

In this section, we will prove that \( \mathcal{G}^\mathcal{C} \) is a weak Kähler manifold such that the left \( \mathcal{G} \)-action on \( \mathcal{G}^\mathcal{C} \) is Hamiltonian with a moment map \( \kappa: \mathcal{G}^\mathcal{C} \to \Omega^0(\mathfrak{g}_E) \) given by \( \kappa(\exp(\imath s)u) = s \). Most of the proofs are taken or adapted from [8].

We first describe a weak symplectic form on \( \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \). Define the 1-form \( \tau \) on \( \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \) by
\[
\tau_{(s, u)}(z, w) = (s, wu^{-1})_{L^2}, \quad (z, w) \in \Omega^0(\mathfrak{g}_E) \oplus T_u \mathcal{G},
\]
where \( \cdot, \cdot \)_{L^2} is the \( L^2 \)-metric on \( \Omega^0(\mathfrak{g}_E) \). Note that if \( \mathcal{G} \) were finite-dimensional, then \( \tau \) would be exactly the tautological 1-form on the cotangent bundle \( T^* \mathcal{G} = \text{Lie}(\mathcal{G}) \times \mathcal{G} \). There are left and right actions of \( \mathcal{G} \) on \( \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \) given by
\[
u_0 \cdot (s, u) = (u_0 s u_0^{-1}, u_0 u) \quad \text{and} \quad (s, u) \cdot u_0 = (s, uu_0).
\]
By direct computation, we see that both the left and right \( \mathcal{G} \)-actions preserve \( \tau \) and hence the 2-form \( \omega := -d\tau \). In this section, a map is said to be left (resp. right) \( \mathcal{G} \)-equivariant if it is \( \mathcal{G} \)-equivariant with respect to the left (resp. right) \( \mathcal{G} \)-action, and \( \mathcal{G} \)-equivariant if it is \( \mathcal{G} \)-equivariant with respect to both the left and right \( \mathcal{G} \)-actions.

Proposition 7.4 If the 2-form \( \omega \) is non-degenerate, then the projection \( \kappa: \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \to \Omega^0(\mathfrak{g}_E) \) onto the first factor is a moment map for the left \( \mathcal{G} \)-action on \( \Omega^0(\mathfrak{g}_E) \times \mathcal{G}, \omega \).

Proof To verify that \( \kappa \) is a moment map for the left \( \mathcal{G} \)-action on \( \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \), fix \( \xi \in \Omega^0(\mathfrak{g}_E) \) and let \( \xi^* \) denote the vector field generated by the left \( \mathcal{G} \)-action on
Since the left \( G \)-action preserves \( \omega \), the Lie derivative of \( \omega \) along \( \xi^* \) vanishes. Therefore, \(-i\xi^*d\tau = di\xi^*\tau\). Moreover,

\[
\xi^*_{(s, u)} = \left. \frac{d}{dt} \right|_{t=0} (\text{Ad}(e^{i\xi})s, e^{i\xi}u) = ([\xi, s], \xi u),
\]

and hence \( \tau(\xi^*)(s, u) = (s, \xi)_{L^2} \). Finally, it is easy to verify that \( \kappa \) is left \( G \)-equivariant.

Now, we describe a complex structure \( J \) on \( \Omega^0(g_E) \times G \) and later verify that \( \omega \) is compatible with \( J \) and positive so that \( \omega \) is a Kähler form on \( \Omega^0(g_E) \times G \). Let \( \psi: \Omega^0(g_E) \times G \to G^C \) be the polar decomposition given by \( \psi(s, u) = \exp(is)u \). It is clear that \( \psi \) is \( G \)-equivariant. Then, there is a unique complex structure \( J \) on \( \Omega^0(g_E) \times G \) such that \( \psi \) is a biholomorphism. To see the relation between the symplectic form \( \omega \) and the complex structure \( J \), we also view \( P = \Omega^0(g_E) \times G \) as a principal \( G \)-bundle over \( \Omega^0(g_E) \), and show that \( P \) has a connection induced by the complex structure \( J \).

**Proposition 7.5** Every tangent vector of \( P \) at a point \((s, u) \in P\) can be uniquely written as \( \xi^#_{(s, u)} + J\eta^#_{(s, u)} \) for some \( \xi, \eta \in \Omega^0(g_E) \), where \( \xi^# \) is the tangent vector of \( P \) at \((s, u) \) generated by the right \( G \)-action.

**Proof** Note that any tangent vector of \( G^C \) at \( \psi(s, u) \) can be uniquely written as \( Z^#_{\psi(s, u)} \) for some \( Z \in \Omega^0(g_E^C) \), where \( Z^#_{\psi(s, u)} \) is the tangent vector on \( G^C \) generated by the right translations. Then, write \( Z = \xi + i\eta \) for some \( \xi, \eta \in \Omega^0(g_E) \). Since the right \( G \)-action and \( \psi \) are holomorphic, and \( \psi \) is \( G \)-equivariant, we obtain

\[
Z^#_{\psi(s, u)} = (\xi + i\eta)^#_{\psi(s, u)} = \xi^#_{\psi(s, u)} + i\eta^#_{\psi(s, u)} = d(s, u)\psi(\xi^#_{(s, u)} + J\eta^#_{(s, u)}),
\]

where \( i \) in the second equality also denotes the complex structure on \( G^C \). The rest follows from the fact that the derivative \( d(s, u)\psi \) is an isomorphism.

By Proposition 7.5, we are able to define a \( \Omega^0(g_E) \)-valued 1-form \( \gamma \) by

\[
\gamma(\xi^#_{(s, u)} + J\eta^#_{(s, u)}) = \xi.
\]

Another \( \Omega^0(g_E) \)-valued 1-form \( \chi \) on \( P \) is given by

\[
\chi(\xi^#_{(s, u)} + J\eta^#_{(s, u)}) = \eta.
\]

It is clear that \( \chi = -\gamma J \). Since \( \psi \) is \( G \)-equivariant, the right \( G \)-action on \( P \) is holomorphic. Therefore, it is easy to verify that both \( \gamma \) and \( \chi \) are \( G \)-equivariant in the sense that \( R(u_0)^*\gamma u_0 = \text{Ad}(u_0^{-1})\gamma \) and \( R(u_0)^*\chi = \text{Ad}(u_0^{-1})\chi \), where \( R(u_0) \) is the right \( G \)-action on \( P \) given by \( u_0 \). The following are some useful formulas.
Proposition 7.6 For any \((s, e) \in P\) and \((z, w) \in T_{(s, e)} P = \Omega^0(\mathfrak{g}_E) \oplus \Omega^0(\mathfrak{g}_E)\), the formulas for \(\chi\) and \(\gamma\) are given by

\[
\begin{align*}
\gamma_{(s, e)}(z, w) &= \frac{1 - \cos \text{ad } s}{\text{ad } s} z + w, \\
\chi_{(s, e)}(z, w) &= \frac{\sin \text{ad } s}{\text{ad } s} z.
\end{align*}
\] (7.17)

Proof The derivative \(d_{(s, e)} \psi\) is given by

\[
d_{(s, e)} \psi(z, w) = \left. \frac{d}{dt} \right|_{t=0} \exp(is + itz) \exp(tw) \\
= \left. \frac{d}{dt} \right|_{t=0} \exp(is + itz) + \left. \frac{d}{dt} \right|_{t=0} \exp(is) \exp(tw).
\] (7.18)

Moreover, by the formula for the derivative of the exponential map (e.g. [2, Theorem 1.5.3]),

\[
\exp(is) \left( -1 \frac{d}{dt} \right|_{t=0} \exp(is + itz) = \frac{1 - \exp(-\text{ad}(is))}{\text{ad}(is)} (iz) \\
= \left( \frac{1 - \cos \text{ad } s}{\text{ad } s} + i \frac{\sin \text{ad } s}{\text{ad } s} \right) z.
\] (7.19)

As a consequence,

\[
\psi(s, e) \left( -1 \frac{d}{dt} \right|_{t=0} \psi(z, w) = \frac{1 - \cos \text{ad } s}{\text{ad } s} z + w + i \frac{\sin \text{ad } s}{\text{ad } s} z.
\] (7.20)

The rest follows from the proof of Proposition 7.5. \(\square\)

Consider a right \(\mathcal{G}\)-equivariant map \(\bar{\kappa}\) given by \(\bar{\kappa}(s, e) = \kappa(s, e)\). In other words, \(\bar{\kappa}(s, u) = u^{-1} s u\).

Proposition 7.7

1. \(\tau = (\bar{\kappa}, \gamma)_{L^2}\).
2. If \(f : P \to \mathbb{R}\) is a function given by \(f(s, u) = \frac{1}{2} \|s\|_{L^2}\), then \((\bar{\kappa}, \chi)_{L^2} = df\).
3. There is a unique right \(\mathcal{G}\)-equivariant \(\text{Hom}(\Omega^0(\mathfrak{g}_E), \Omega^0(\mathfrak{g}_E))\)-valued 1-form \(\Psi\) on \(P\) such that \(d_Y \bar{\kappa}(s, u) = \Psi(s, u) \chi(s, u)\) for any \((s, u) \in P\), where \(d_Y \bar{\kappa}\) is the covariant derivative of \(\bar{\kappa}\). More explicitly,

\[
d_Y \bar{\kappa}_{(s, e)}(z, w) = \cos \text{ad } s(z), \\
\Psi_{(s, e)} = \cos \text{ad } s \frac{\text{ad } s}{\sin \text{ad } s},
\] (7.21)

for any \((z, w) \in T_{(s, e)} P\).
Before giving the proof, we claim that $\sin \text{ad} \frac{s}{\text{ad} s}$ is invertible so that $\frac{\text{ad} s}{\sin \text{ad} s}$ simply means its inverse. In fact, if $\xi \in \Omega^0(g_E)$, then, by definition, $\chi(s,e)(J\xi^#(s,e)) = \xi$. Moreover, if $J\xi^#(s,e) = (z, w)$ for some $(z, w) \in T_{(s,e)}P$, then

$$\xi = \chi(s,e)(J\xi^#(s,e)) = \chi(s,e)(z, w) = \chi(s,e)(z, 0).$$

(7.22)

Therefore, the map

$$\Omega^0(g_E) \to \Omega^0(g_E), \quad z \mapsto \chi(s,e)(z, 0) = \frac{\sin \text{ad} s}{\text{ad} s} z,$$

(7.23)

is invertible.

**Proof of Proposition 7.7** If $(z, w) \in T_{(s,e)}P$, then Proposition 7.6 implies that

$$(\bar{\kappa}, \gamma)_{L^2}(z, w) = (s, \frac{1 - \cos \text{ad} s}{\text{ad} s} z + w)_{L^2}. $$

(7.24)

Since $(s, [s, z])_{L^2} = 0$,

$$(s, \frac{1 - \cos \text{ad} s}{\text{ad} s} z)_{L^2} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j)!} \left(s, (\text{ad} s)^{2j-1} z \right)_{L^2} = 0.$$  

(7.25)

Similarly,

$$(\bar{\kappa}, \chi)_{L^2}(z, w) = \left(s, \frac{\sin \text{ad} s}{\text{ad} s} z \right)_{L^2} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j + 1)!} \left(s, (\text{ad} s)^{2j} z \right)_{L^2} = (s, z)_{L^2}. $$

(7.26)

Therefore, the identities (1) and (2) hold at $(s, e)$. By the right $G$-equivariance, they hold everywhere.

Then, we prove the formula for the covariant derivative $d_\gamma \bar{\kappa}(s,e)$. Note that $d_\gamma \bar{\kappa} = d\bar{\kappa} + [\gamma, \bar{\kappa}]$. Therefore, if $(z, w) \in T_{(s,e)}P$, we have

$$d_\gamma \bar{\kappa}(s,e)(z, w) = d\bar{\kappa}(s,e)(z, w) + \left[\frac{1 - \cos \text{ad} s}{\text{ad} s} z + w, s \right] \text{ad} s(w) + z + \text{ad} s \left(\frac{\cos \text{ad} s - 1}{\text{ad} s} z - w \right)$$

(7.27)

$$= \cos \text{ad} s(z).$$
To define $\Psi$, it is enough to define $\Psi_{(s,e)}$ which needs to satisfy
\[
\cos \text{ad } s(z) = \Psi_{(s,e)} \frac{\sin \text{ad } s(z)}{\text{ad } s(z)}.
\] (7.28)

As a consequence,
\[
\Psi_{(s,e)} = \cos \text{ad } s \frac{\text{ad } s}{\sin \text{ad } s}. \tag{7.29}
\]

The following results verify that $\omega$ is compatible with $J$. Then, we will verify that $\omega$ is positive so that $\omega$ is a Kähler form on $\Omega^0(g_E) \times \mathcal{G}$.

**Proposition 7.8** The following hold for any $\xi, \eta \in \Omega^0(g_E)$:

1. $\omega(\xi^#, J \eta^#) = (\xi, \Psi(\eta))_{L^2}$.
2. $\omega(J\xi^#, J \eta^#) = \omega(\xi^#, \eta^#)$.
3. $(\xi, \Psi(\eta))_{L^2} = (\eta, \Psi(\xi))_{L^2}$.

As a consequence, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$.

**Proof** We compute
\[
\omega(\xi^#, J \eta^#) = -d(\bar{\kappa}, \lambda)_{L^2}(\xi^#, J \eta^#)
= -\xi^#(\bar{\kappa}, \lambda(J \eta^#))_{L^2} + J \eta^#(\bar{\kappa}, \lambda(\xi^#)) + (\bar{\kappa}, \lambda([\xi^#, J \eta^#]))_{L^2}
= (\bar{\kappa}(J \eta^#), \xi)_{L^2} + (\bar{\kappa}, \lambda(J[\xi^#, \eta^#]))_{L^2}
= (d\gamma \bar{\kappa}(J \eta^#), \xi)_{L^2}
= (\Psi(\eta), \xi)_{L^2}.
\] (7.30)

Here, we have used the formula that $[\xi^#, J \eta^#] = J[\xi^#, \eta^#]$, since $J$ commutes with the right $\mathcal{G}$-action. Moreover, $d\bar{k}(J \eta^#) = d\gamma \bar{k}(J \eta^#)$, since $J \eta^#$ is horizontal. This proves (1). To prove (2), we compute
\[
\omega(J\xi^#, J \eta^#) = (\bar{\kappa}, \gamma)_{L^2}([J\xi^#, J \eta^#])_{L^2}
= (\bar{\kappa}, \gamma)_{L^2}(-[\xi^#, \eta^#])_{L^2}
= (\bar{\kappa}, \gamma)_{L^2}(-[\xi, \eta])_{L^2}
= -(\bar{\kappa}, [\xi, \eta])_{L^2}.
\] (7.31)

On the other hand,
\[
\omega(\xi^#, \eta^#) = -\xi^#(\bar{\kappa}, \gamma(\eta^#))_{L^2} + \eta^#(\bar{\kappa}, \gamma(\xi^#))_{L^2} + (\bar{\kappa}, \gamma([\xi^#, \eta^#]))
= -[[\bar{\kappa}, \xi], \eta]_{L^2} + ([\bar{\kappa}, \eta], \xi)_{L^2} + ([\bar{\kappa}, \xi], \eta)_{L^2}
= (\bar{\kappa}, [\eta, \xi])_{L^2}.
\] (7.32)
Finally, to prove (3), we first compute
\[
d(\bar{k}, \chi)_{L^2}(J\xi^#, J\eta^#) = J\xi^#(\bar{k}, \chi(J\eta^#))_{L^2} - J\eta^#(\bar{k}, \chi(J\xi^#))_{L^2}
- (\bar{k}, \chi([J\xi^#, \eta^#])_{L^2}
= (d\bar{k}(J\xi^#), \eta)_{L^2} - (d\bar{k}(J\eta^#), \xi)_{L^2}
= (d\gamma\bar{k}(J\xi^#), \eta)_{L^2} - (d\gamma\bar{k}(J\eta^#), \xi)_{L^2}
= (\Psi\chi(J\xi^#), \eta)_{L^2} - (\Psi\chi(J\eta^#), \xi)_{L^2}
= (\Psi(\xi), \eta)_{L^2} - (\Psi(\eta), \xi)_{L^2},
\]
and then note that \(d(\bar{k}, \chi)_{L^2} = 0\) by Proposition 7.7.

**Proposition 7.9** The metric \(\omega(\cdot, J\cdot)\) is positive-definite. In particular, \(\omega\) is non-degenerate.

**Proof** If \(\xi, \eta \in \Omega^0(\mathfrak{g}_E)\), then
\[
g(\xi^# + J\eta^#, \xi^# + J\eta^#) = \omega(\xi^# + J\eta^#, J\xi^# - \eta^#)
= \omega(\xi^#, J\xi^#) - \omega(\xi^#, \eta^#) + \omega(J\eta^#, J\xi^#) - \omega(J\eta^#, \eta^#)
= (\xi, \Psi(\xi))_{L^2} + (\eta, \Psi(\eta))_{L^2} - 2\langle s, [\xi, \eta]\rangle_{L^2}.
\]
(7.34)

Since \(g\) is right \(\mathcal{G}\)-invariant, it is enough to show the positive-definiteness at \((s, e)\). Hence, we need to show that
\[
(\xi, \Psi(s,e)(\xi))_{L^2} + (\eta, \Psi(s,e)(\eta))_{L^2} - 2\langle s, [\xi, \eta]\rangle_{L^2}
= \int_X (\xi, \Psi(s,e)(\xi))_{L^2} + (\eta, \Psi(s,e)(\eta))_{L^2} - 2\langle s, [\xi, \eta]\rangle_{L^2} > 0
\]
(7.35)
for every \(s \in \Omega^0(\mathfrak{g}_E)\) and nonzero \(\xi + i\eta \in \Omega^0(\mathfrak{g}_E^\mathbb{C})\). Note that the integrand is positive pointwise. In fact, after fixing a base point \(x \in X\), the polar decomposition \(\psi\) becomes the usual polar decomposition \(\psi = \bar{\psi}\cdot \lambda\). The unique complex structure on \(u(n) \times U(n)\) making \(\psi\) a biholomorphism is compatible with the tautological 1-form on \(u \times U(n) = T^*U(n)\) so that the tautological 1-form is a Kähler form (see [8, Theorem 5.1 and Remark 5.2]). The resulting Kähler metric on \(u(n) \times U(n)\) evaluated at \((\xi^# + J\eta^#)(x)\) is exactly the integrand and hence positive. Here, \((\xi^# + J\eta^#)(x)\) is the value of the section \(\xi^# + J\eta^# \in \Omega^0(\mathfrak{g}_E) \oplus \Omega^0(\mathfrak{g}_E)\) at \(x\).

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