Strong in-domatic number in digraphs.

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Abstract

Let $D = (V, A)$ be a digraph and $\mathcal{S}$ a partition of $V(D)$. We say that $\mathcal{S}$ is a strong in-domatic partition if every $S$ in $\mathcal{S}$ holds that every vertex not in $S$ has at least one out-neighbor in $S$, that is $S$ is an in-dominating set, and $D(S)$ is strongly connected. The maximum number of elements in a strong in-domatic partition is called the strong in-domatic number of $D$ and it is denoted by $d^-_s(D)$.

In this paper we introduce those concepts and determine the value of $d^-_s(D)$ for semicomplete digraphs and planar digraphs. We show some structural properties of digraphs which have a strong in-domatic partition and we see some bounds for $d^-_s(D)$. Then we study this concept in the Cartesian product, composition, line digraph and other associated digraphs.

In addition, we characterize strong in-domatic critical digraphs and we give two families strong in-domatic critical digraphs which hold some properties, where a strong in-domatic critical digraph $D$ holds that $d^-_s(D - e) = d^-_s(D) - 1$ for every $e$ in $A(D)$.

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1 Introduction

Let $G$ be a graph, a domatic partition of $V(G)$, say $\mathcal{S}$, is a partition of $V(G)$ such that for every $S$ in $\mathcal{S}$, $S$ is a dominating set. The maximum number of elements in such partition is called the domatic number of $G$, denoted by $d(G)$. This concept was introduced by Cockayne and Hedetniemi in [8]. In [10], Garey and Johnson showed that, for every $k \geq 3$, determine whether or not the domatic number of a given graph is $k$ is NP-complete. In [17], Poon, Yen and Ung proved that finding a domatic partition into 3 dominating sets is NP-complete on planar bipartite graphs, and finding a domatic partition with $d(G)$ elements in co-bipartite graphs is NP-complete. In [18], Riege, Rothe, Spakowski and Yamamoto showed that, given an arbitrary graph $G$, it is possible to determine if $V(G)$ can be partitioned into 3 disjoint dominating sets with a deterministic algorithm in time $2.695^n$ (up to polynomial factors) and in polynomial space.

Domatic partitions in graphs have been studied for some researches due its applications and theoretical results (see [9], [16], [12] [16] [17], [18]). Due a large amount of kinds of dominating sets (see [13] and [14]), several authors defined variants on the domatic number in graphs, for instance, total domatic number (Cockayne, Hedetniemi and Dawes [7]), idomatic number (Cockayne and Hedetniemi [8]), $k$-domatic number (Zelinka [20]) and tree domatic number (Chen [4]). In the same spirit, Laskar and Hedetniemi [15] introduced the connected domatic number as follows: for a digraph $G$ a connected domatic partition of $V(G)$ is a domatic partition such that every element in such partition induces a connected graph in $G$. The maximum number of elements in a connected domatic partition of $V(G)$ is the domatic connected number of $G$, denoted by $d^c(G)$. Whenever a graph $G$ holds that $d^c(G-a) < d^c(G)$ for every edge $a$ of $G$, it is said that $G$ is a connectively domatically critical graph. Such graphs were introduced and characterized by Zelinka in [19]. Hartnell and Rall studied the connected domatic number in planar graphs [11], in particular, they showed the following results.

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Theorem 1.1. [11] Let \( G \) be a planar graph. The connected domatic number of \( G \) is at most 4, and \( K_4 \) is the only planar graph achieving this bound.

Theorem 1.2. [11] Let \( G \) be a graph such that \( d_s(G) = 3 \) and let \( \{V_1, V_2, V_3\} \) be any connected domatic partition of \( V(G) \). If \( G \) is planar, then \( \langle V_1 \rangle, \langle V_2 \rangle \) and \( \langle V_3 \rangle \) are paths.

In [21], Zelinka extended the concept of domatic number to digraphs as follows: for a digraph \( D \), an in-domatic partition of \( V(D) \) is a partition of \( V(D) \) into in-dominating sets. The maximum number of classes in an in-domatic partition is the \textit{in-domatic number of the digraph} \( D \), denoted by \( d^-(D) \). Benítez-Bobadilla and Pastrana-Ramírez [5] studied this parameter in the Cartesian product of digraphs and some associated digraphs, as the line digraph.

In this paper we extend the concept of connected domatic number to digraphs as follows: for a digraph \( D \) a strong in-domatic partition of \( V(D) \) is a partition of \( V(D) \) into strong in-dominating sets. The maximum number of classes in a strong in-domatic partition is the \textit{strong in-domatic number of \( D \)}, denoted by \( d_s^-(D) \). A strong in-domatic partition of \( D \) with \( d_s^-(D) \) classes is called a \( d_s^-(D) \)-partition of \( V(D) \). We say that a digraph \( D \) is a strong in-domatic critical digraph if for every arc \( a \) of \( D \), \( D - a \) is strong and \( d_s^-(D - a) = d_s^-(D) - 1 \).

In this paper we show some properties of strong in-domatic partitions in digraphs and some bounds for the strong in-domatic number.

This paper follows the next order: in section 3, some bounds for the strong in-domatic number are given. We prove a characterization of strong in-domatic critical digraphs, and an infinite family of such digraphs will be shown. Also, we show an extension of Theorem 1.1 and Theorem 1.2 for planar digraphs.

In section 4, we study the strong in-domatic number in the Cartesian product and composition of digraphs. As a consequence of the result for composition of digraphs, we will show that, given two natural numbers, say \( p \) and \( m \), there exists a digraph of order \( p \) with strong in-domatic number \( m \). Also, we will show that, given a natural number \( p \geq 2 \) and a divisor of \( p \), say \( n \), there exists a strong in-domatic critical digraph of order \( p \) with strong in-domatic number \( n \). In section 5, we work these new concepts in certain associated digraphs, as line digraph. We finish the paper with a brief note about the strong out-domatic number.

## 2 Terminology and notation

For general concepts we refer the reader to [3] and [14]. Let \( G = (V(G), E(G)) \) be a simple undirected graph. An isolated vertex of \( G \) is a vertex whose degree is zero. For a nonempty subset of \( V(G) \), say \( S \), the subgraph induced by \( S \) is denoted by \( \langle S \rangle \). If \( S \) is such that \( \langle S \rangle \) is complete, then we say that \( S \) is a clique of \( G \). We say that \( S \) is a dominating set if for every \( x \) in \( V(G) \) \( \setminus S \) there exists \( z \) in \( S \) such that \( xz \in E(G) \). We say that a set \( S \) of vertices of \( G \) is a dominating clique if \( S \) is a dominating set and \( S \) is a clique. The minimum cardinality among all dominating cliques of \( G \), denoted by \( \gamma_d(G) \), is called the clique domination number of \( G \). For a connected graph \( G \), a vertex-cut is a set \( S \) of vertices of \( G \) such that \( G - S \) is disconnected. A vertex cut of minimum cardinality is called minimum vertex-cut and this cardinality is denoted by \( \kappa(G) \). A vertex-cut with \( \kappa(G) \) vertices is a \( \k \)-set of \( G \).

Throughout the paper, \( D = (V(D), A(D)) \) denotes a loopless digraph with vertex set \( V(D) \) and arc set \( A(D) \). For an arc \((u,v)\), \( u \) and \( v \) are its end-vertices; we say that the end-vertices are adjacent, we also say that \( u \) out-dominates \( v \) and \( v \) in-dominates \( u \). We say that the arc \((u,v)\) is symmetric if \((v,u) \in A(D) \). Let \( S \) be a subset of \( V(D) \) and \( x \) in \( V(D) \), we say that \( x \) is in-dominated by \( S \) (\( x \) is out-dominated by \( S \)) if \( z \) in-dominates \( x \) for some \( z \) in \( S \) (\( z \) out-dominates \( x \) for some \( z \) in \( S \)). We say that \( S \) is an in-dominating set (out-dominating set) if every vertex in \( V(D) \) \( S \) is in-dominated by \( S \) (out-dominated by \( S \)). If \( x \) is a vertex of \( D \), then the \textit{ex-neighborhood of} \( x \) is the set \( \{ z \in V(D) : (x,z) \in A(D) \} \), denoted by \( N^+(x) \), the \textit{in-neighborhood of} \( x \) is the set \( \{ z \in V(D) : (z,x) \in A(D) \} \), denoted by \( N^-(x) \). The \textit{neighborhood of} \( x \) is the set \( N^+(x) \cup N^-(x) \) and it is denoted by \( N(x) \). The \textit{out-degree} \( \delta^+_D(x) \) of a vertex \( x \) is the number \(|N^+(x)|\), the \textit{in-degree} \( \delta^-_D(x) \) of a vertex \( x \) is the number \(|N^-(x)|\).
A vertex \( v \) is called isolated if \( \delta_D^+(v) = 0 = \delta_D^-(v) \). For a subset \( S \) of \( V(D) \), the subdigraph of \( D \) induced by \( S \), denoted by \( D(S) \), has \( V(D(S)) = S \) and \( A(D(S)) = \{ (u,v) : \{u,v\} \subseteq S \} \). For a subset \( E \) of \( A(D) \), the subdigraph of \( D \) induced by the arc set \( E \), denoted by \( D[E] \), has \( V(D[E]) = \{ x \in V(D) : x \text{ is an end-vertex of some } e \in E \} \) and \( A(D[E]) = E \).

A pair of digraphs \( D \) and \( H \) are isomorphic, denoted by \( D \cong H \), if there exists a bijection \( f : V(D) \rightarrow V(H) \) such that \( (u,v) \in A(D) \) if and only if \( (f(u), f(v)) \in A(H) \). Let \( S_1 \) and \( S_2 \) be subsets of \( V(D) \), an arc \((u,v)\) of \( D \) will be called an \( S_1 \)-arc or \( S_2 \)-arc whenever \( u \in S_1 \) and \( v \in S_2 \). If \( S_1 = \{ x \} \) or \( S_2 = \{ x \} \), then we will write \( x \)-arc or \( x \)-arc, respectively.

A directed walk \( W \) in \( D \) is a sequence of vertices \( (x_0, x_1, \ldots, x_n) \) such that \( (x_i, x_{i+1}) \in A(D) \) for every \( i \in \{0, 1, \ldots, n-1\} \). We say that \( W \) is an \( x_0x_n \)-walk. The length of \( W \) is the number \( n \). If \( x_i \neq x_j \) for all \( i \) and \( j \) such that \( \{i, j\} \subseteq \{0, \ldots, n\} \) and \( i \neq j \), then \( W \) is called a directed path \((x_0x_n)\)-path. Let \( \{x_i, x_j\} \) be a subset of \( V(W) \), with \( i \leq j \), the \( x_ix_j \)-walk \((x_i, x_{i+1}, \ldots, x_{j-1}, x_j)\) contained in \( W \) will be denoted by \((x_i, W, x_j)\). If \( T = (x_1, \ldots, x_n) \) and \( T' = (z_1, \ldots, z_m) \) are directed walks in \( D \) and \( x_n = z_1 \), we denote by \( T \cup T' \) the directed walk \((x_1, \ldots, x_n = z_1, z_2, \ldots, z_m)\). A directed cycle is a directed walk \((v_1, v_2, \ldots, v_n, v_1)\) such that \( v_i \neq v_j \) for all \( i \) and \( j \) such that \( \{i, j\} \subseteq \{1, \ldots, n\} \) and \( i \neq j \). In what follows, we write walk, path and cycle instead of directed walk, directed path and directed cycle, respectively.

We say that \( D \) is strong if, for every pair of vertices \( u \) and \( v \) in \( D \), there exists a \( uv \)-walk and there exists a \( vu \)-walk in \( D \). If \( D \) is a digraph and \( S \) is a subset of \( V(D) \) we say that \( S \) is a strong in-dominating set if \( S \) is in-dominating set and \( D(S) \) is a strong digraph. If \( D \) is a strong digraph, a nonempty subset of \( A(D) \), say \( E \), is a strong cover of \( D \) if \( D[E] \) is a spanning strongly connected subdigraph of \( D \). Set \( \mathcal{E} = \{ |E| : E \text{ is a partition of } A(D) \text{ into strong covers} \} \), and let \( \Lambda(D) \) be the maximum of \( \mathcal{E} \). A partition of \( A(D) \) into strong covers with \( \Lambda(D) \) elements is called a \( \Lambda \)-partition of \( A(D) \).

A digraph is semicomplete if for every \( u \) and \( v \) in \( V(D) \) we have that \( \{(u,v), (v,u)\} \cap A(D) \neq \emptyset \). The line digraph of \( D \), denoted by \( L(D) \), is the digraph such that \( V(L(D)) = A(D) \), and \( ((u,v), (w,z)) \in A(L(D)) \) if and only if \( u = w \).

Let \( D \) be a digraph. The subdivision digraph of \( D \), denoted by \( S(D) \), the root digraph of \( D \), denoted by \( R(D) \), the middle digraph of \( D \), denoted by \( Q(D) \), and the total digraph of \( D \), denoted by \( T(D) \), are defined as follows.

\[
V(S(D)) = V(R(D)) = V(Q(D)) = V(T(D)) = V(D) \cup A(D).
\]

And for every vertex \( x \) in \( V(D) \cup A(D) \),

\[
N^+_S(D)(x) = \begin{cases} 
\{(u,v) \in A(D) : u = x\} & \text{if } x \in V(D) \\
\{v\} & \text{if } x = (u,v) \text{ for some } (u,v) \in A(D) 
\end{cases}
\]

\[
N^+_R(D)(x) = \begin{cases} 
\{(u,v) \in A(D) : u = x\} \cup N^+_R(x) & \text{if } x \in V(D) \\
\{v\} & \text{if } x = (u,v) \text{ for some } (u,v) \in A(D) 
\end{cases}
\]

\[
N^+_Q(D)(x) = \begin{cases} 
\{(u,v) \in A(D) : u = x\} & \text{if } x \in V(D) \\
\{v\} \cup \{(v,y) \in A(D) : y \in V(D)\} & \text{if } x = (u,v) \text{ for some } (u,v) \in A(D) 
\end{cases}
\]

\[
N^+_T(D)(x) = \begin{cases} 
\{(u,v) \in A(D) : u = x\} \cup N^+_T(x) & \text{if } x \in V(D) \\
\{v\} \cup \{(v,y) \in A(D) : y \in V(D)\} & \text{if } x = (u,v) \text{ for some } (u,v) \in A(D) 
\end{cases}
\]

Notice that \( Q(D) \) is a spanning subdigraph of \( T(D) \) and \( T(D) \cap A(D) \) is the line digraph of \( D \). It is straightforward to see that \( S(D) \) and \( R(D) \) have no in-dominating vertex if \( D \) has at least one arc.
The *Cartesian product* of two digraphs $D$ and $H$, denoted by $D \Box H$, is the digraph whose vertex set is $V(D) \times V(H)$ and $((x, z), (u, v))$ is an arc of $D \Box H$ if and only if either $x = u$ and $(z, v) \in A(H)$ or $z = v$ and $(x, u) \in A(D)$. The horizontal level of the vertex $x_0$ in $D \Box H$ is $H_{x_0} = \{(x_0, y) \in V(D \Box H) : y \in V(H)\}$ and the vertical level of the vertex $y_0$ is $D_{y_0} = \{(x, y_0) \in V(D \Box H) : x \in V(D)\}$. Notice that $(D \Box H)\langle H_{x_0} \rangle \cong H$ and $(D \Box H)\langle D_{y_0} \rangle \cong D$ for every $x_0$ in $V(D)$ and every $y_0$ in $V(H)$, respectively.

Let $D$ a digraph and $\alpha = (D_v)_{v \in V(D)}$ be a sequence of digraphs which are pairwise vertex disjoint. The *composition* of $D$ respect to $\alpha$, denoted by $D[\alpha]$, is the digraph obtained from $D$ replacing every vertex $v$ of $D$ by the digraph $D_v$ and joining every vertex from $V(D_v)$ to every vertex in $V(D_u)$ whenever $(v, u) \in A(D)$.

**Remark 2.1.** For every $v$ in $V(D)$, $D[\alpha]\langle V(D_v) \rangle = D_v$. If $x_v$ is an arbitrary vertex in $D_v$, then $D[\alpha]\langle \{x_v : v \in V(D)\} \rangle \cong D$.

For a digraph $D$, the underlying graph of $D$, denoted by $UG(D)$, is the graph such that $V(UG(D)) = V(D)$ and $uv$ is an edge in $UG(D)$ if either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. A digraph is planar if $UG(D)$ is a planar graph.

The following results will be useful.

**Lemma 2.1.** [6] A digraph is strong if and only if it has a closed spanning walk.

**Lemma 2.2.** [1] Let $D$ be a nontrivial digraph without isolated vertex, with at least one arc, and $L(D)$ its line digraph. $L(D)$ is strong if and only if $D$ is strong.

**Lemma 2.3.** [2] If $D$ and $H$ are vertex disjoint digraphs, then $D \Box H$ is strong if and only if $D$ and $H$ are strong.

It is straightforward to see the following.

**Lemma 2.4.** Let $D$ be a strong digraph and $(u, v)$ in $A(D)$. If there exists a $uv$-walk in $D$ which does not contain $(u, v)$, then $D - (u, v)$ is strong.

**Lemma 2.5.** Let $D$ be a digraph with at least one arc and $L(D)$ its line digraph. If $E \subseteq A(D)$ is nonempty, then $L(D)\langle E \rangle = L(D[E])$.

## 3 First results

In this section we prove some properties digraphs with at least one strong in-domatic partition.

**Theorem 3.1.** Let $D$ be a digraph. $D$ has a strong in-domatic partition if and only if $D$ is strong.

**Proof.** Notice that if $D$ is strong, then $\{V(D)\}$ is a strong in-domatic partition of $V(D)$.

For the converse, let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a strong in-domatic partition of $D$ and $\{u, v\}$ a subset of $V(D)$. If $\{u, v\} \subseteq S_i$ for some $i$ in $\{1, \ldots, k\}$, then there exists a $uv$-walk contained in $\langle S_i \rangle$. If $u \in S_j$ and $v \in S_i$ with $i \neq j$, since $S_i$ is an in-dominating set, there exists $x$ in $S_i$ such that $(u, x) \in A(D)$. On the other hand, since $\langle S_i \rangle$ is strong, there exists a $uv$-walk contained in $\langle S_i \rangle$, say $C'$, so $C = (u, x) \cup C'$ is a $uv$-walk in $D$, concluding that $D$ is strong. \[\Box\]

It follows from Theorem 3.1 that we will consider only strongly connected digraphs. On the other hand, despite determinate whether or not a graph has a domatic partition into $k$ disjoint dominating sets ($k \geq 3$) is a NP-complete problem, the following result shows that, given a strong in-domatic partition of the vertices of a digraph $D$ into $k$ disjoint sets ($1 \leq k \leq d^-(D)$), it is straightforward to show a strong in-domatic partition into $n$ disjoint sets for every $n \in \{1, \ldots, k\}$.
Proposition 3.1. If $D$ is a digraph and $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ is a strong in-domatic partition of $D$, then the following holds.

a) $\cup_{i \in I} S_i$ is a strong in-dominating set of $D$ for every nonempty subset $I$ of $\{1, \ldots, k\}$.

b) If $I$ is a nonempty subset of $\{1, \ldots, k\}$, then $\mathcal{S}_I = \{S_i | i \notin I\} \cup \cup_{i \in I} S_i$ is a strong in-domatic partition of $V(D)$.

c) For every $n$ in $\{1, \ldots, d^-_s(D)\}$ there exists a strong in-domatic partition of $V(D)$, say $\mathcal{S}$, such that $|\mathcal{S}| = n$.

Proof. a) Let $I$ be a nonempty subset of $\{1, \ldots, k\}$. Suppose that $I = \{\alpha_1, \ldots, \alpha_r\}$ and $S = \cup_{i \in I} S_i$. Notice that $\{S_{\alpha_1}, \ldots, S_{\alpha_r}\}$ is a in-domatic partition of $V(\langle S \rangle)$ and by Theorem 3.1, $\langle S \rangle$ is strong. On the other hand, since $S_{\alpha_1}$ is an in-dominating set of $D$ and $S_{\alpha_1} \subseteq S$, then $S$ is also an in-dominating set of $D$, concluding that $S$ is a strong in-dominating set.

b) Since $\mathcal{S}$ is a partition of $V(D)$, then $\mathcal{S}_I$ is also a partition of $V(D)$. According to (a), every $S \in \mathcal{S}_I$ is a strong in-domatic set of $D$.

c) Consider $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ a $d^-_s(D)$-partition of $D$. If $I = \{n, n + 1, \ldots, k\}$, according to (b), then $\mathcal{S}_I = \{S_i | i \notin I\} \cup \cup_{i \in I} S_i$ is a strong in-domatic partition of $D$ such that $|\mathcal{S}| = n$.

3.1 Some bounds

Zelinka [19] showed that the vertex connectivity number of a graph is an upper bound for the connected domatic number. In the same spirit, we have the following upper bound.

Proposition 3.2. If $D$ is a non-semicOMPlete strong digraph, then $d^-_s(D) \leq \kappa(UG(D))$.

Proof. Let $U$ be a vertex-cut of $UG(D)$ and $S$ a strong in-dominating set in $D$, we will show that $S \cap U \neq \emptyset$. Suppose that $S \cap U = \emptyset$. Since $\langle S \rangle$ is strong and $UG(D) - U$ is disconnected, then $S \subseteq V(H)$ for some connected component $H$ of $UG(D) - U$. It follows that no vertex in $V(UG(D) - U) \setminus V(H)$ is in-dominated by $S$ in $D$, which is no possible. Hence $S \cap U \neq \emptyset$.

Let $U$ be a $\kappa$-set in $UG(D)$ and $\mathcal{S} = \{S_1, \ldots, S_t\}$ a $d^-_s(D)$-partition of $V(D)$. It follows that $S_i \cap U \neq \emptyset$ for every $i$ in $\{1, \ldots, t\}$. Hence, $d^-_s(D) \leq \kappa(UG(D))$.

Zelinka [11] proved that every digraph $D$ holds $d^-(D) \leq \delta^+(D) + 1$. Corollary 3.1 is a consequence of this last result.

Corollary 3.1. If $D$ is a nontrivial strong digraph, then $d^-_s(D) \leq \delta^+(D) + 1$.

Proof. Let $\mathcal{S}$ be a $d^-_s(D)$-partition of $V(D)$. Since $\mathcal{S}$ is an in-domatic partition of $V(D)$, then $d^-_s(D) \leq d^-(D)$. Hence, $d^-_s(D) \leq \delta^+(D) + 1$.

Remark 3.1. For a complete digraph, it is straightforward to see that $d^-_s(D) = \delta^+(D) + 1 = |V(D)|$.

The upper bound showed in Corollary 3.1 can be improved in digraphs with no isolated vertices.

Proposition 3.3. If $D$ is a strong digraph without in-dominating vertex, then $d^-_s(D) \leq \delta^+(D)$.

Proof. Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a $d^-_s(D)$-partition of $V(D)$ and $x$ in $V(D)$ such that $\delta^+(x) = \delta^+(D)$. For $i$ in $\{1, \ldots, k\}$, if $x \in S_i$ then $N^+(x) \cap S_j \neq \emptyset$ for every $j$ in $\{1, 2, \ldots, k\}$ with $j \neq i$. On the other hand, since $x$ is not an in-dominating vertex, then $S_i$ is a nontrivial strong set, which implies that $N^+(x) \cap S_i \neq \emptyset$. Hence, $d^-_s(D) = |\mathcal{S}| \leq \delta^+(x) = \delta^+(D)$.

Proposition 3.4. If $D$ is a strong digraph such that $d^-_s(D) = \delta^+(D) + 1$, then $D$ has an in-dominating vertex. Moreover, every vertex of minimum out-degree is in-dominating.
Proof. Since \( d^-_s(D) > \delta^+(D) \), it follows from Proposition 3.3 that \( D \) has an in-dominating vertex. On the other hand, consider a \( d^-_s \)-partition of \( V(D) \), say \( \mathcal{G} = \{ S_1, \ldots, S_k \} \), and \( x \) in \( V(D) \) such that \( x \) has minimum out-degree. Let \( j \) in \( \{ 1, \ldots, k \} \) such that \( x \in S_j \). Notice that \( |S_j| = 1 \), otherwise \( S_j \) is a nontrivial strong set, so \( x \) has at least one out-neighbor in \( S_j \). Since every \( S_i \) is an in-dominating set, we conclude that \( x \) has an out-neighbor in \( S_i \) for every \( i \) in \( \{ 1, \ldots, k \} \) and then \( \delta^+(x) \geq |\mathcal{G}| \), but this is no possible, because \( \delta^+(x) = \delta^+(D) \) and \( |\mathcal{G}| = \delta^+(D) + 1 \). Therefore, \( S_j = \{ x \} \). Hence, \( x \) is an in-dominating vertex. \( \square \)

**Corollary 3.2.** Let \( D \) be a strong digraph such that \( d^-_s(D) = \delta^+(D) + 1 \) and \( N_0 \) the set of vertices of minimum out-degree. The following holds:

a) \( N_0 \) is an in-dominating set and \( D\langle N_0 \rangle \) is a complete digraph.

b) \( \gamma_{cl}(UG(D)) \leq |N_0| \).

**Proof.** It follows from Proposition 3.4 that \( N_0 \) is an in-dominating set and \( \langle N_0 \rangle \) is a complete digraph. Hence, \( N_0 \) is a dominating clique in \( UG(D) \), which implies that \( \gamma_{cl}(UG(D)) \leq |N_0| \). \( \square \)

**Remark 3.2.** If \( D \) is a digraph and \( H \) is a spanning strong subdigraph of \( D \), then every strong in-domatic partition of \( H \) is also a strong in-domatic partition of \( D \). In particular, \( d^-_s(H) \leq d^-_s(D) \).

The following proposition shows an upper and a lower bound for a particular case of spanning subdigraphs. It is worth mentioning that Proposition 3.5 will be useful in order to define strong in-domatic critical digraphs, which will be characterized in Section 3.2.

**Proposition 3.5.** Let \( D \) be a digraph such that \( d^-_s(D) \geq 2 \) and \( a \) an arc in \( D \). If \( D - a \) is strong, then \( d^-_s(D) - 1 \leq d^-_s(D - a) \leq d^-_s(D) \).

**Proof.** Suppose that \( a = (u, v) \). Since \( D - a \) is a spanning subdigraph of \( D \), by Remark 3.2, \( d^-_s(D - a) \leq d^-_s(D) \). On the other hand, consider \( D' = D - a \). Notice that if \( d^-_s(D) = 2 \), then \( d^-_s(D) - 1 \leq d^-_s(D - a) \) and the first inequality holds. Suppose that \( d^-_s(D) \geq 3 \). Let \( \mathcal{G} = \{ S_1, S_2, \ldots, S_k \} \) be a \( d^-_s \)-partition of \( D \).

Consider two cases on \( \{ u, v \} \).

**Case 1.** \( u \in S_i \) and \( v \in S_j \) for some \( \{ i, j \} \) subset of \( \{ 1, \ldots, k \} \), with \( i \neq j \).

Since \( r \geq 3 \), we can choose an index \( r \) in \( \{ 1, \ldots, k \} \setminus \{ i, j \} \) and set \( S_0 = S_j \cup S_r \). Consider

\[ \mathcal{G}' = \{ S_l \in \mathcal{G} | l \notin \{ j, r \} \} \cup \{ S_0 \}. \]

We claim that \( \mathcal{G}' \) is a strong in-domatic partition of \( V(D') \). Let \( S \) be an arbitrary element in \( \mathcal{G}' \). If \( S \neq S_0 \), then \( S \) is a strong in-dominating set in \( D' \). If \( S = S_0 \), since \( S_r \) is a in-dominating set in \( D' \), then \( S_0 \) is also a in-dominating set in \( D' \). Moreover, given that \( \{ S_j, S_r \} \) is a strong in-domatic partition of \( D'(S_0) \), by Proposition 3.1, \( D'(S_0) \) is strong, concluding that \( \mathcal{G}' \) is a strong in-domatic partition of \( D' \).

Therefore \( d^-_s(D) - 1 \leq d^-_s(D - a) \).

**Case 2.** \( \{ u, v \} \subseteq S_j \) for some \( j \) in \( \{ 1, \ldots, k \} \).

Let \( x_0 \) be a vertex in \( V(D) \setminus S_j \), notice that \( x_0 \notin \{ u, v \} \) because \( k \geq 3 \). Let \( T = (x_0, \ldots, x_n = v) \) be an \( xv \)-walk in \( D' \) and consider \( \alpha = \max \{ i \in \{ 0, \ldots, n - 1 \} | x_i \notin S_j \} \). Let \( S_r \in \mathcal{G} \) such that \( x_\alpha \in S_r \) and set \( S_0 = S_j \cup S_r \). Consider \( \mathcal{G}' = \{ S_l \in \mathcal{G} | l \notin \{ j, r \} \} \cup \{ S_0 \} \). We claim that \( \mathcal{G}' \) is a strong in-domatic partition of \( V(D') \). Let \( S \) be an arbitrary element in \( \mathcal{G}' \). If \( S \neq S_0 \), then \( S \) is a strong in-dominating set of \( D' \). If \( S = S_0 \), since \( S_r \) is a in-dominating set of \( D' \), then \( S_0 \) is also a in-dominating set in \( D' \). On the other hand, since \( S_r \) is a strong in-dominating set in \( D' \), there exists \( x \) in \( S_r \) such that \( (u, x) \in A(D') \) and there exists a \( xx_\alpha \)-walk in \( D'(S_0) \), say \( T' \), concluding that \( (u, x) \cup T' \cup (x_\alpha, T, v) \) is a uv-walk contained in \( D(S_0) - a \). Therefore, it follows from Lemma 3.1 (a) and Lemma 2.4 that \( D'(S_0) = D(S_0) - a \) is strong. Thus, \( \mathcal{G}' \) is a strong in-domatic partition of \( V(D') \) and \( d^-_s(D) - 1 \leq d^-_s(D - a) \). \( \square \)
3.2 Strong in-domatic critical digraphs

In [19], Zelinka defined a conectively comatically critical graph as a graph \( G \) such that \( d_{e}(G-e) < d_{e}(G) \) for every edge \( e \) of \( G \), and the author showed a characterization of such graphs. In the same spirit, we say that a digraph \( D \) is a strong in-domatic critical digraph if for every arc \( a \) of \( D \), \( D - a \) is strong and \( d_{s}^{-}(D-a) = d_{s}^{-}(D) - 1 \). We will show a characterization of such digraphs (Theorem 3.2) and we will give an infinite family of strong in-domatic critical digraphs (Corollary 3.3).

**Theorem 3.2.** Let \( D \) be a digraph such that \( d_{s}^{-}(D) \geq 2 \) and \( D - a \) is strong for every \( a \) in \( A(D) \). The following are equivalent:

a) \( D \) is a strong in-domatic critical digraph.

b) If \( \mathcal{S} = \{S_1, \ldots, S_k\} \) is a \( d_{s}^{-} \)-partition of \( D \), then \( \mathcal{S} \) holds:

\[ b.1 \quad D(S_i) - a \text{ is not strong for every } i \in \{1, \ldots, k\} \text{ and every } a \text{ in } A(D(S_i)). \]

\[ b.2 \quad |S_i \cap N^{+}(u)| = 1 \text{ for every } i \in \{1, \ldots, k\} \text{ and every } u \text{ in } V(D) \setminus S_i. \]

**Proof.** Suppose that \( D \) is a strong in-domatic critical digraph and let \( \mathcal{S} = \{S_1, \ldots, S_k\} \) be a \( d_{s}^{-} \)-partition of \( V(D) \), we claim that \( \mathcal{S} \) holds (b.1) and (b.2). Proceeding by contradiction, suppose that \( \mathcal{S} \) does not fulfill either (b.1) or (b.2). If \( \mathcal{S} \) does not hold (b.1), then there exist \( i \) in \( \{1, \ldots, k\} \) and \( a \) in \( A(D(S_i)) \) such that \( D(S_i) - a \) is strong, which implies that \( \mathcal{S} \) is a strong in-domatic partition of \( D - a \). Hence, \( d_{s}^{-}(D-a) \geq d_{s}^{-}(D) \), which is a contradiction since \( D \) is a strong in-domatic critical digraph. In the same way, if \( \mathcal{S} \) does not hold (b.2), then there exist \( i \) in \( \{1, \ldots, k\} \), a vertex \( u \) in \( V(D) \setminus S_i \) and \( \{x, z\} \) a subset of \( S_i \) such that \( \{(u, x), (u, z)\} \subseteq A(D) \). In that case, \( \mathcal{S} \) is a strong in-domatic partition of \( D - (u, x) \). Hence, \( d_{s}^{-}(D - (u, x)) \geq d_{s}^{-}(D) \), a contradiction. Therefore, every \( d_{s}^{-} \)-partition of \( D \) holds (b.1) and (b.2).

Suppose that every \( d_{s}^{-} \)-partition of \( D \) holds (b.1) and (b.2), but \( D \) is not a strong in-domatic critical digraph. Let \( (x, z) \in A(D) \) such that \( d_{s}^{-}(D - (x, z)) = d_{s}^{-}(D) \), and consider a \( d_{s}^{-} \)-partition of \( D - (x, z) \), say \( \mathcal{S} = \{S_1, S_2, \ldots, S_k\} \). Since \( d_{s}^{-}(D - (x, z)) = d_{s}^{-}(D) \), we have that \( \mathcal{S} \) is also a \( d_{s}^{-} \)-partition of \( D \). If \( \{x, z\} \) is a subset of \( S_i \) for some \( i \in \{1, \ldots, k\} \), then \( D(S_i) - (x, z) = (D - (x, z))(S_i) \). On the other hand, given that \( \mathcal{S} \) is a strong in-domatic partition of \( D - (x, z) \), we have that \( (D - (x, z))(S_i) \) is strong, which is no possible by (b.1). In the case \( x \in S_j \) and \( z \in S_i \) for some subset \( \{i, j\} \) of \( \{1, \ldots, k\} \) with \( i \neq j \), given that \( S_i \) is an in-dominating set in \( D - (x, z) \), we have that there exists a vertex \( w \) in \( S_i \setminus \{z\} \) such that \( (x, w) \in A(D - (x, z)) \), a contradiction with (b.2). Therefore, \( D \) is a strong in-domatic critical digraph. \( \square \)

The following corollary shows an infinite family of strong in-domatic critical digraphs.

**Corollary 3.3.** For \( n \) in \( \mathbb{N} \), with \( n \geq 3 \), there exists a strong in-domatic critical digraph \( D \) with order \( 2n \) and \( d_{s}^{-}(D) = n \).

**Proof.** Let \( n \) in \( \mathbb{N} \) with \( n \geq 3 \), \( U = \{u_1, u_2, \ldots, u_n\} \) and \( V = \{v_1, v_2, \ldots, v_n\} \) disjoint sets. Consider the digraph \( D \) with vertex set \( V \cup U \) and the arc set given by:

- \( (u_i, u_j) \in A(D) \) if and only if \( i < j \).
- \( (v_i, v_j) \in A(D) \) if and only if \( i < j \).
- \( (u_i, u_j) \in A(D) \) if and only if \( i \geq j \).
- \( (u_i, v_j) \in A(D) \) if and only if \( i \geq j \).

We will prove that \( D - a \) is strong for every \( a \in A(D) \). Consider the following paths:

\[ T_1 = (u_1, u_2, \ldots, u_n). \]

\[ T_2 = (v_1, v_2, \ldots, v_n). \]

\[ T_3 = (v_n, u_n, v_{n-1}, u_{n-1}, \ldots, u_2, v_1, u_1). \]
Given that $C_1 = T_1 \cup (u_n, v_1) \cup T_2 \cup (v_n, u_1)$ and $C_2 = (v_1, v_n) \cup T_3 \cup (u_1, u_n) \cup T_4$ are spanning closed walks in $D$ that are arc disjoint, it follows that $D - a$ is strong for every $a \in A(D)$.

On the other hand, since the partition $\mathcal{S}_0 = \{\{u_i, v_i\} : i \in \{1, \ldots, n\}\}$ is a strong in-domatic partition of $V(D)$, we have that $n \leq d_s^-(D)$. Moreover, since $D$ has no dominating vertex, it follows from Proposition 3.3 that $d_s^-(D) \leq \delta^+(D)$, concluding that $d_s^-(D) = n$ (because $\delta^+(u_1) = n$).

Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be a $d_s^+$-partition of $V(D)$. Given that $D$ has no in-dominating vertex, we conclude that for every $i$ in $\{1, \ldots, n\}$, $|S_i| \geq 2$. On the other hand, since $D$ has order $2n$ and $\mathcal{S}$ is a partition of $V(D)$ with $n$ elements, we get that for every $i$ in $\{1, \ldots, n\}$, $|S_i| = 2$. Therefore, for every $i$ in $\{1, \ldots, n\}$ we have proved that $S_i = \{u_j, v_j\}$ for some $j$ in $\{1, \ldots, n\}$ and so $\mathcal{S} = \mathcal{S}_0$.

Since $\mathcal{S}_0$ holds the conditions (b.1) and (b.2) of Proposition 3.3, we have that $D$ is a strong in-domatic critical digraph. \hfill \Box

### 3.3 Strong in-Domatic number in planar digraphs

In this section, we show the version for digraphs of Theorem 1.1 and Theorem 1.2, namely Theorem 3.3 and Theorem 3.4. The following result will be useful in order to show an upper bound for the strong in-domatic number in planar digraphs.

**Lemma 3.1.** If $D$ is a digraph with at least one strong in-domatic partition, then $UG(D)$ has a connected domestic partition and $d_s^-(D) \leq d_c(UG(D))$.

**Proof.** We claim that if $\mathcal{S}$ is a strong in-domatic partition of $V(D)$, then $\mathcal{S}$ is a connected domestic partition of $V(UG(D))$. Since $D(S)$ is a strong digraph for every $S$ in $\mathcal{S}$ then $UG(D)(S)$ is a connected graph. On the other hand, since $S$ is an in-dominator of $D$ for every $S$ in $\mathcal{S}$, then $S$ is a dominating set in $UG(D)$. Therefore $\mathcal{S}$ is a connected domestic partition of $V(UG(D))$. In a particular case, if $\mathcal{S}$ is a $d_s^+$-partition of $v(D)$, then $|\mathcal{S}| \leq d_c(UG(D))$ and it follows that $d_s^-(D) \leq d_c(UG(D))$. \hfill \Box

**Theorem 3.3.** If $D$ is a strong planar digraph, then $d_s^-(D) \leq 4$. Moreover, $d_s^-(D) = 4$ if and only if $D$ is a complete digraph with order 4.

**Proof.** Since $D$ is planar, then $UG(D)$ is a planar graph, that implies that $d_c(UG(D)) \leq 4$. Thus, by Proposition 3.1, we get that $d_s^-(D) \leq 4$.

On the other hand, suppose that $D$ is a strong planar digraph such that $d_s^-(D) = 4$. It follows from Proposition 3.1 and Theorem 1.1 that $d_c(UG(D)) = 4$ and so by Theorem 1.1 we get that $UG(D)$ is $K_4$, which implies that $D$ is a semicomplete digraph of order 4. Since $d_s^-(D) = 4$ it follows from Proposition 3.1 that $\delta^+(D) = 3$, concluding that $D$ is a complete digraph.

Suppose that $D$ is a complete digraph with order 4. It follows from remark 3.1 that $d_s^-(D) = 4$. \hfill \Box

**Theorem 3.4.** Let $D$ be a planar strong digraph such that $d_s^-(D) = 3$. If $\mathcal{S} = \{S_1, S_2, S_3\}$ is a $d_s^+$-partition of $V(D)$, then $\langle S_i \rangle$ is a symmetric path in $D$ for every $i$ in $\{1, 2, 3\}$.

**Proof.** Consider the following cases on the order of $D$.

- **Case 1.** $D$ has order at least 5.

  In this case, we have from Proposition 1.1 that $d_c(UG(D)) \leq 3$. Hence, by assumption and Proposition 3.1, we conclude that $d_c(UD(D)) = 3$. It is straightforward to see that every $d_s$-partition of $D$, say $\{V_1, V_2, V_3\}$, is a connected domatic partition in $UG(D)$ and by Theorem 1.2 we conclude that $UG(D) \langle V_1 \rangle$, $UG(D) \langle V_2 \rangle$ and $UG(D) \langle V_3 \rangle$ are paths in $UG(D)$. In that case, since $V_i$ is a strong set in $D$ for every $i \in \{1, 2, 3\}$, then $D(V_1)$, $D(V_2)$ and $D(V_3)$ are symmetric paths in $D$.

- **Case 2.** $D$ has order at most 4.

  Since $D$ has order at most 4 and $\{V_1, V_2, V_3\}$ is a $d_s$-partition of $D$, then every set $V_i$ has at most two vertices. It follows that $D(V_1)$, $D(V_2)$ and $D(V_3)$ are symmetric paths in $D$. \hfill \Box
4 Strong in-domatic number in Cartesian Product and composition of digraphs

First, we will show a lower bound of the strong in-domatic number in the Cartesian product.

**Theorem 4.1.** If \( D \) and \( H \) are vertex disjoint strong digraphs, then

\[
d^*_s(D \square H) \geq \max\{d^*_s(D), d^*_s(H)\}.
\]

**Proof.** Suppose without loss of generality that \( d^*_s(H) \leq d^*_s(D) \) and consider a \( d^*_s \)-partition of \( D \), say \( \mathcal{S} = \{S_1, \ldots, S_k\} \). Define \( V_i = \{(x, y) \in V(D \square H) : x \in S_i\} \) and \( \mathcal{V} = \{V_1, V_2, \ldots, V_k\} \). We claim that \( \mathcal{V} \) is a partition of \( V(D \square H) \) into strong in-dominating sets.

1. \( \mathcal{V} \) is a partition of \( V(D \square H) \).

   It follows from the fact that \( \mathcal{S} \) is a partition of \( V(D) \).

2. For every \( i \in \{1, \ldots, k\} \), \( V_i \) is an in-dominating set.

   In order to show that \( V_i \) is an in-dominating set in \( D \square H \), consider \( (x, v) \in V(D \square H) \setminus V_i \), and we will show that there exists \( (u, z) \in V_i \) such that \( ((x, v), (u, z)) \in A(D \square H) \). Since \( (x, v) \in V(D \square H) \setminus V_i \), we get that \( x \notin S_i \). Because of \( S_i \) is an in-dominating set in \( D \), there exists \( y \in S_i \) such that \( (x, y) \in A(D) \). On the other hand, by definition of \( V_i \) it follows that \( (y, v) \in V_i \), and by definition of \( D \square H \), we have that \( ((x, v), (y, v)) \in A(D \square H) \). Therefore \( V_i \) is an in-dominating set for every \( i \) in \( \{1, \ldots, k\} \).

3. For every \( i \in \{1, \ldots, k\} \), \( (D \square H)/V_i \) is strong.

   Since \( D\langle S_i \rangle \) and \( H \) are strong digraphs, it follows from Lemma 2.3 that \( D\langle S_i \rangle \square H \) is strong. On the other hand, it is straightforward to see that \( D\langle S_i \rangle \square H = (D \square H)/S_i \times V(H) \), and \( (D \square H)/S_i \times V(H) = (D \square H)/V_i \), concluding that \( D \square H/V_i \) is strong.

By the above, \( \mathcal{V} \) is a strong in-domatic partition of \( V(D \square H) \). In particular, \( |\mathcal{V}| \leq d^*_s(D \square H) \) and by supposition, \( d^*_s(D \square H) \geq \max\{d^*_s(D), d^*_s(H)\} \).

The following theorem shows a lower bound for the strong in-domatic number in the composition of digraphs.

**Theorem 4.2.** Let \( D \) be a nontrivial strong digraph and \( \alpha \) a sequence of pairwise vertex disjoint digraphs, say \( \alpha = (D_v)_{v \in V(D)} \). The composition of \( D \) respect to \( \alpha \) holds that \( d^*_s(D[\alpha]) \geq \min\{|V(D_v)| : v \in V(D)\} \).

**Proof.** Suppose that \( V(D) = \{v_1, \ldots, v_p\} \), and for every \( i \in \{1, \ldots, p\} \), let \( \{x^i_1, \ldots, x^i_n\} \) be the vertex set of \( D_{v_i} \). Consider \( n = \min\{n_i : i \in \{1, \ldots, p\}\} \), and for every \( k \in \{1, \ldots, n - 1\} \), we define \( S_k = \{x^i_k : i \in \{1, \ldots, p\}\} \) and \( S_n = V(D[\alpha]) \setminus \bigcup_{i=1}^{n-1} S_i \).

We denote by \( \mathcal{F} \) the set \( \{S_k : k \in \{1, \ldots, n\}\} \), and we will show that \( \mathcal{F} \) is a strong in-domatic partition of \( D[\alpha] \). Clearly \( \mathcal{F} \) is a partition of \( V(D[\alpha]) \). It remains to show that for every \( i \in \{1, \ldots, n\} \), \( S_i \) is a strong in-dominating set of \( D[\alpha] \).

**Claim 1.** For every \( k \in \{1, \ldots, n\} \), \( S_k \) is an in-dominating set in \( D[\alpha] \).

Consider \( x^i_k \) in \( V(D[\alpha]) \setminus S_k \). Since \( v_j \) has at least one out-neighbor in \( D \), say \( v_t \) (because \( D \) is a nontrivial strong digraph), it follows that for \( x^i_k \) there exists \( x^j_k \in S_k \) such that \( (x^j_k, x^i_k) \in A(D) \), concluding that \( S_k \) is an in-dominating set in \( D[\alpha] \) for every \( k \in \{1, \ldots, n\} \).

**Claim 2.** For every \( k \in \{1, \ldots, n\} \), \( D[\alpha]\langle S_k \rangle \) is strong.

If \( k \in \{1, \ldots, n - 1\} \) it follows from remark 2.1 that \( D[\alpha]\langle S_k \rangle \cong D \), concluding that \( D[\alpha]\langle S_k \rangle \) is a strong digraph for every \( k \in \{1, \ldots, n - 1\} \).
It remains to show that $D[\alpha] \langle S_n \rangle$ is a strong digraph. Let $x_i^j$ and $x_s^t$ be two vertices in $S_n$. We will denote by $W$ the set $\{ x_i^j : i \in \{1, \ldots, p\} \}$ and notice that $W \subseteq S_n$. Consider the following cases.

**Case 1.** $\{ x_i^j, x_s^t \} \subseteq W$.

Since $D[\alpha] \langle W \rangle \cong D$ and $D$ is a strong digraph, it follows that there exists an $x_i^j x_s^t$-walk in $D[\alpha] \langle S_n \rangle$.

**Case 2.** $x_i^j \notin W$ and $x_s^t \in W$.

Consider an out-neighbor of $v_j$ in $D$, say $v_r$. By definition of $D[\alpha]$ it follows that $x_r^s$ is a vertex in $W$ such that $(x_i^j, x_r^s)$ is an arc in $D[\alpha]$. By case 1, there exists an $x_r^s x_i^j$-walk in $D[\alpha] \langle W \rangle$, say $T$. Hence $(x_i^j, x_r^s) \cup T$ is an $x_i^j x_s^t$-walk in $D[\alpha] \langle S_n \rangle$.

Now consider an in-neighbor of $v_j$ in $D$, say $v_t$. In the same way, $x_t^l$ is a vertex in $W$ such that $(x_n^t, x_l^t)$ is an arc of $D[\alpha]$. By case 1, there exists an $x_n^t x_l^t$-walk in $D[\alpha] \langle W \rangle$, say $T'$. Hence $T' \cup (x_t^l, x_i^j)$ is an $x_i^j x_s^t$-walk in $D[\alpha] \langle S_n \rangle$.

**Case 3.** $\{ x_i^j, x_s^t \} \cap W = \emptyset$.

Consider an out-neighbor of $v_j$ in $D$, say $v_r$. It follows from definition of $D[\alpha]$ that $x_r^s$ is a vertex in $W$ and $(x_i^j, x_r^s)$ is an arc of $D[\alpha]$. By case 2, there exists an $x_r^s x_i^j$-walk in $D[\alpha] \langle W \rangle$, say $T$. Hence, $(x_i^j, x_r^s) \cup T$ is an $x_i^j x_s^t$-walk in $D[\alpha] \langle S_n \rangle$.

It follows from the preceding cases that $D[\alpha] \langle S_n \rangle$ is a strong digraph.

By Claim 1 and Claim 2 we have that $\mathcal{D}$ is a strong in-domatic partition of $V(D[\alpha])$. Therefore, $d_s^\alpha(D[\alpha]) \geq \min \{|V(D_v)| : v \in V(D)\}$.

As a consequence of the previous results, we have the following corollaries.

**Corollary 4.1.** Let $m$ and $p$ be two natural numbers with $0 < m \leq \frac{p}{2}$ and $p \geq 3$. Then there exists a digraph $D$ of order $p$ such that $d_s^\alpha(D) = m$.

**Proof.** Let $q$ and $r$ be two natural numbers such that $m > r \geq 0$ and $p = mq + r$. Notice that $m \leq \frac{p}{2}$ implies that $q \geq 2$. Consider a cycle $H$ of order $q$, say $(v_1, \ldots, v_q, v_1)$, and $\alpha$ a sequence of pairwise vertex disjoint digraphs, say $(D_{v_1}, \ldots, D_{v_q})$, such that $A(D_{v_i}) = \emptyset$ for every $i$ in $\{1, \ldots, q\}$, $|V(D_{v_i})| = m + r$ and $|V(D_{v_i})| = m$ for every $i$ in $\{1, \ldots, q - 1\}$.

We claim that the digraph $D$ defined by $H[\alpha]$ is the desired digraph. It is straightforward to see that $D$ has order $p$. In order to prove that $d_s^\alpha(D) \leq m$, we will show that $D$ holds the hypothesis of Proposition 3.3, that is, $D$ has no in-dominating vertex. Consider the following cases.

**Case 1.** $q \geq 3$.

In this case we have that $H$ has at least three vertices, so $D$ has no in-dominating vertex.

**Case 2.** $q = 2$.

Since $p = mq + r$ and $m \leq \frac{p}{2}$, we have that $r = 0$, which implies that $p = mq$. Hence $m \geq 2$, because $p \geq 3$. Therefore $D$ is a bipartite digraph without in-dominating vertex.

By the above, $D$ has no in-dominating vertex and we get from Proposition 3.3 that $d_s^\alpha(D) \leq \delta^+(D)$.

On the other hand, notice that for every vertex $x$ in $V(D) \setminus V(D_{v_{q-1}})$ we have that $\delta_D^+(x) = m$, and for every $x$ in $V(D_{v_{q-1}})$ we have that $\delta_D^+(x) = m + r$, which implies that $\delta^+(D) = m$. Hence, $d_s^\alpha(D) \leq m$.

Finally, by Proposition 4.2 we have that $m \leq d_s^\alpha(D)$, which implies that $d_s^\alpha(D) = m$. Therefore, $D$ is the desired digraph.

**Corollary 4.2.** Let $p$ and $n$ be two natural numbers such that $p \geq 2$ and $n \geq 2$. If $n$ divides $p$, then there exists a strong in-domatic critical digraph of order $p$ such that $d_s^\alpha(D) = n$.

**Proof.** Notice that if $p = n$, then the complete digraph of order $p$ is the desired digraph, by Remark 3.1. So we can assume that $p \neq n$. Let $t$ be in $\mathbb{N}$ such that $p = nt$, $H$ a cycle of order $t$, say $(w_1, \ldots, w_t, w_1)$, and $\alpha$ a sequence of $t$ pairwise vertex disjoint digraphs, say $\alpha = (D_{w_1}, \ldots, D_{w_t})$, such that for every $i$ in $\{1, \ldots, t\}$ we have that $A(D_{w_i}) = \emptyset$ and $|V(D_{w_i})| = n$. We claim that the digraph $D$ defined by $H[\alpha]$ is the desired digraph.

**Claim 1.** $D$ has order $p$ and $d_s^\alpha(D) = n$. 

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An analogous proof as in Corollary 4.2 will show that $D$ has order $p$ and $d^+_s(D) = n$.

On the other hand, in order to show that $D$ is a strong in-domatic critical digraph, we will show that $D$ holds the hypotheses of Theorem 3.2.

Claim 2. $D - (u, v)$ is strong for every arc $(u, v)$ in $A(D)$.

Let $(u, v)$ be an arc in $D$. We prove that there exists a $uv$-walk on $D$ which does not contain the arc $(u, v)$ and then, in order to conclude, we use Lemma 2.4. By construction of $D$, there exists $k$ in \{1, ..., $t$\} such that $u \in V(D_{w_k})$ and $v \in V(D_{w_{k+1}})$ (indices modulo $t$). On the other hand, for every $i$ in \{1, ..., $t$\} \ \{k, k - 1\} consider a vertex $x_i$ in $D_{w_i}$. Since $n \geq 2$, we can choose $x_k$ in $V(D_{w_k}) \ \{u\}$ and $x_{k+1}$ in $V(D_{w_{k+1}}) \ \{v\}$. Notice that $C = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_1)$ is a cycle in $D - (u, v)$. So, $(u, x_{k+1}) \cup (x_{k+1}, C, x_1) \cup (x_1, C, x_k) \cup (x_k, v)$ is a $uv$-walk that does not contain the arc $(u, v)$. Therefore $D - (u, v)$ is strong.

Claim 3. If $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a $d^-_s$-partition of $D$ then $|S_i \cap V(D_{w_j})| = 1$ for every $i$ in \{1, ..., $n$\} and every $j$ in \{1, ..., $t$\}.

Notice that, by definition of $D$, we have that $t = \min\{|l(C) : C$ is a cycle in $D\}$. From Lemma 2.1 we have that $D(S_i)$ has a spanning closed walk for every $i$ in \{1, ..., $n$\} which implies that $D(S_i)$ has a cycle, concluding that $|V(D(S_i))| \geq t$ for every $i$ in \{1, ..., $n$\}. If there exists $k$ in \{1, ..., $n$\} such that $|S_k| > t$, then

$$|V(D)| = \sum_{i=1}^{n} |S_i| > \sum_{i=1}^{n} t = nt = |V(D)|,$$

which is no possible. So, every element of $\mathcal{S}$ has $t$ vertices of $D$.

Since $D(S_i)$ has a spanning closed walk (by Lemma 2.1), then $D(S_i)$ contains a cycle $C$ which has length at least $t$. Because of $|S_i| = t$ we conclude that $l(C) = t$; that is $D(S_i) = C$.

Hence $|S_i \cap V(D_{w_j})| = 1$ for every $i$ in \{1, ..., $n$\} and every $j$ in \{1, ..., $t$\}.

Claim 4. $|S_i \cap N^+(x)| = 1$ for every $i$ in \{1, ..., $n$\} and every $x$ in $V(D) \setminus S_i$.

Let $x$ be in $V(D) \setminus S_i$ and suppose that $x \in V(D_{w_{k-1}})$ for some $k$ in \{2, ..., $t + 1$\}. Since $S_i$ is an in-dominating set in $D$, then there exists a vertex $z$ in $S_i$ such that $(x, z) \in A(D)$. Notice that $z \in V(D_{w_k})$. Since $|S_i \cap V(D_{w_k})| = 1$ and $N^+(x) \subseteq V(D_{w_k})$, then $S_i \cap N^+(x) = \{z\}$, that is $|S_i \cap N^+(x)| = 1$.

Claim 5. $D(S_i) - a$ is not strong for every $i$ in \{1, ..., $k$\} and every $a$ in $A(D(S_i))$.

Since $D(S_i)$ is a cycle, we have that $D(S_i) - a$ is not strong for every arc $a$ in $D(S_i)$ and every $i$ in \{1, ..., $n$\}.

Therefore, it follows from Claims 1, 2, 4 and 5, and by Theorem 3.2, that $D$ is a strong in-domatic critical digraph. 

\[\square\]

5 Strong in-domatic number in line digraph and other associated digraphs

Proposition 5.1. Let $D$ be a nontrivial strong digraph, $L(D)$ its line digraph and $E$ a nonempty subset of $A(D)$. $E$ is a strong cover of $D$ if and only if $E$ is a strong in-dominating set of $L(D)$.

Proof. For the sufficiency, consider a strong cover of $D$, say $E$. Since $D[E]$ is strong, it follows that $L(D[E])$ is strong (by Lemma 2.2), which implies that $L(D)[E]$ is strong (by Lemma 2.5).

In order to prove that $E$ is an in-dominating set in $L(D)$, consider a vertex in $V(L(D)) \setminus E$, say $(x, z)$. Since $E$ is a strong cover, we get that $D[E]$ is a spanning subdigraph of $D$, which implies that $z \in V(D[E])$. Because of $D[E]$ is a non trivial strong subdigraph, there exists $w$ in $N^+_D(D[E])$. Hence $(x, z)$ is in-dominated by $(z, w)$ in $L(D)$. So, $E$ is a strong in-dominating set in $L(D)$.

For the necessary condition of Proposition 5.1, suppose that $E$ is a strong in-dominating set in $L(D)$. In order to prove that $D[E]$ is spanning subdigraph of $D$, consider a vertex $x$ in $V(D)$. Since $D$ is a nontrivial strong digraph, then there exists $u$ in $N_D(x)$. If $(u, x) \in E$, then we have that $x \in V(D[E])$. If $(u, x) \in V(L(D)) \setminus E$, since $E$ is an in-dominating set in $L(D)$, then there exists $b$ in $E$ such that
that \((u, x), b \in A(L(D))\). Hence, it follows from definition of \(L(D)\) that \(b = (x, z)\) for some \(z \in V(D)\), concluding that \(x \in V(D[E])\). Therefore, \(D[E]\) is a spanning subdigraph of \(D\).

We will prove that \(D[E]\) is strong. Since \(L(D)[E]\) is strong, we have that \(L(D[E])\) is strong (by Lemma 2.5). So, \(D[E]\) is strong (by Lemma 2.2).

Therefore, \(E\) is a strong cover of \(D\).

**Theorem 5.1.** If \(D\) is a strong digraph of order at least three, then \(d^-_s(L(D)) = \Lambda(D)\).

**Proof.** Consider a \(\Lambda\)-partition of \(A(D)\), say \(\mathcal{F}\). According to Proposition 5.1 we have that \(F\) is an in-dominating set in \(L(D)\) for every \(F \in \mathcal{F}\), which implies that \(\mathcal{F}\) is a strong in-domatic partition of \(V(L(D))\), and so \(\Lambda(D) \leq d^-_s(L(D))\). In the same way, if \(\mathcal{F}\) is a \(d^-_s\)-partition of \(V(L(D))\), it follows from Proposition 5.1 that \(S\) is a strong cover of \(D\) for every \(S \in \mathcal{F}\), concluding that \(\mathcal{F}\) is a partition of \(A(D)\) into strong covers, so \(\Lambda(D) \leq d^-_s(L(D))\). Therefore, \(\Lambda(D) = d^-_s(L(D))\).

**Lemma 5.1.** If \(D\) is a strong digraph, then \(d^-_s(S(D)) = d^-_s(R(D)) = 1\).

**Proof.** If \(D\) has order 1, Lemma 5.1 holds. If \(|V(D)| \geq 2\), then for every arc \(a \in A(D)\) we have that \(|N^-_{S(D)}(a)| = 1\). It follows from definition of \(S(D)\) that it has no in-dominating vertex, hence \(d^-_s(S(D)) = 1\) (by Proposition 3.3). A similar proof shows that \(d^-_s(R(D)) = 1\).

**Proposition 5.2.** If \(D\) is a nontrivial strong digraph, then \(d^-_s(L(D)) \leq d^-_s(Q(D))\).

**Proof.** Let \(\mathcal{U}\) be a \(d^-_s\)-partition of \(V(L(D))\), where \(\mathcal{U} = \{S_1, \ldots, S_k\}\). Consider the set \(\mathcal{F} = \{S_1 \cup V(D), S_2, \ldots, S_k\}\). Since \(V(Q(D)) = V(D) \cup A(D)\) we have that \(\mathcal{F}\) is a partition of \(V(Q(D))\). We will show that every set in \(\mathcal{F}\) is a strong in-dominating set in \(Q(D)\).

**Claim 1.** Every set in \(\mathcal{F}\) is an in-dominating set in \(Q(D)\).

First, we will show that Claim 1 holds for the set \(S_1 \cup V(D)\). Let \(x\) be a vertex in \(V(Q(D)) \setminus (S_1 \cup V(D))\). Since \(x \in A(D)\) and \(S_1\) is an in-dominating set in \(L(D)\), then there exists \(b \in S_1\) such that \((x, b) \in A(L(D))\); since \(A(L(D)) \subseteq A(Q(D))\), then \((x, b) \in A(Q(D))\). Therefore, \(S_1 \cup V(D)\) is an in-dominating set in \(Q(D)\).

Now we will show that Claim 1 holds for \(S_i\), where \(i \in \{2, \ldots, k\}\). Let \(x\) be a vertex in \(V(Q(D)) \setminus S_i\). If \(x \in A(D)\), since \(S_i\) is an in-dominating set, then there exists \(b \in S_i\) such that \((x, b) \in A(L(D))\); because of \(A(L(D)) \subseteq A(Q(D))\), we get that \((x, b) \in A(Q(D))\).

Suppose that \(x \in V(D)\). Since \(S_i\) is a strong in-dominating set in \(L(D)\) it follows from Proposition 5.1 that \(S_i\) is a strong cover of \(D\). So, there exists an arc \(a \in S_i\) such that \(a = (x, z)\) for some \(z \in V(D)\), which implies that \((x, a) \in A(Q(D))\).

Therefore, every set in \(\mathcal{F}\) is an in-dominating set in \(Q(D)\).

**Claim 2.** For every set \(W\) in \(\mathcal{F}\), \(Q(D)[W]\) is strong.

Since \(L(D)\) is an induced subdigraph of \(Q(D)\) and \(L(D)[S_i]\) is a strong digraph for every \(i \in \{1, \ldots, k\}\), then we have that \(Q(D)[S_i]\) is strong for every \(i \in \{1, \ldots, k\}\). It remains to prove that \(Q(D)[S_1 \cup V(D)]\) is a strong.

Let \(x\) and \(z\) be two vertices in \(S_1 \cup V(D)\). Consider the following three cases.

- **Case 1.** \(\{x, z\} \subseteq S_1\).

Since \(L(D)[S_1]\) is strong, then there exists an \(xz\)-walk contained in \(L(D)[S_1]\). Since \(L(D)\) is a subdigraph of \(Q(D)\), we have that there exists an \(xz\)-walk contained in \(Q(D)[S_1]\).

- **Case 2.** \(x \in V(D)\) and \(z \in S_1\).

Since \(S_1\) is a strong cover of \(D\) (by Proposition 5.1), we get that there exist arcs \(a\) and \(b\) in \(S_1\) such that \(a = (v, x)\) and \(b = (x, u)\) for some \(v\) and \(u\) in \(V(D)\). It follows from Case 1 that there exists an \(za\)-walk contained in \(Q(D)[S_1]\), say \(C_1\). Therefore, \(C_1 \cup (a, x)\) is a \(zx\)-walk in \(Q(D)[S_1 \cup V(D)]\). In the same way we can prove that there exists a \(bz\)-walk in \(Q(D)[S_1]\), say \(C_2\), which implies that \((x, b) \cup C_2\) is an \(xz\)-walk in \(Q(D)[S_1 \cup V(D)]\).
• Case 3. \( \{x, z\} \subseteq V(D) \).

Consider \( a \) in \( S_1 \). By Case 2 we have that there exists an \( xa \)-walk contained in \( Q(D)\langle S_1 \cup V(D) \rangle \), say \( C_1 \), and there exists an \( az \)-walk contained in \( Q(D)\langle S_1 \cup V(D) \rangle \), say \( C_2 \). It follows that \( C_1 \cup C_2 \) is an \( xa \)-walk contained in \( Q(D)\langle S_1 \cup V(D) \rangle \).

Therefore, \( \mathcal{I} \) is a strong in-domatic partition of \( V(Q(D)) \). In particular, \( |\mathcal{I}| \leq d_s^-(Q(D)) \). Thus, \( d_s^-(L(D)) \leq d_s^-(Q(D)) \).

Proposition 5.3. If \( D \) is a strong digraph of order at least three, then \( d_s^-(L(D)) + 1 \leq d_s^-(T(D)) \).

Proof. Let \( \mathcal{I}' \) be a \( d_s^- \)-partition of \( V(L(D)) \), say \( \mathcal{I}' = \{S_1, \ldots, S_k\} \), and consider \( \mathcal{I} = \{V(D), S_1, \ldots, S_k\} \). We claim that \( \mathcal{I} \) is a strong in-domatic partition of \( V(T(D)) \). It follows from definition of \( T(D) \) that \( \mathcal{I} \) is a partition of \( V(T(D)) \). It only remains to show that every element in \( \mathcal{I} \) is a strong in-dominating set in \( T(D) \).

Claim 1. For every set \( W \) in \( \mathcal{I} \), \( T(D)(W) \) is strong.

Since \( L(D)\langle S_i \rangle \) is strong for every \( i \) in \( \{1, \ldots, k\} \), and \( L(D)\langle S_i \rangle \) is a spanning subdigraph of \( T(D)(S_i) \), then \( T(D)(S_i) \) is a strong digraph. On the other hand, since \( D \) is strong, then \( T(D)(V(D)) \) is strong. Therefore, for every set \( W \) in \( \mathcal{I} \), \( T(D)(W) \) is strong.

Claim 2. Every element in \( \mathcal{I} \) is an in-dominating set in \( T(D) \).

We will prove that \( V(D) \) is an in-dominating set in \( T(D) \). Consider a vertex \( x \) in \( V(T(D)) \setminus V(D) \), it follows that \( x = (u, v) \) for some \( u \) and \( v \) in \( V(D) \) and, by definition of \( T(D) \), we conclude that \( (x, v) \in A(T(D)) \). Therefore, \( V(D) \) is an in-dominating set in \( T(D) \).

On the other hand, we will prove that \( S_j \) is an in-dominating set for every \( i \) in \( \{1, \ldots, k\} \). Let \( S_i \) in \( \mathcal{I} \) and \( x \) in \( V(T(D)) \setminus S_i \) for some \( i \) in \( \{1, \ldots, k\} \). If \( x \notin V(D) \), since \( S_i \) is an in-dominating set in \( L(D) \), then \( x \) is in-dominated by \( S_i \). If \( x \in V(D) \), since \( S_i \) is a strong cover of \( D \) (by Proposition 5.1), it follows that there exists an \( a \)-arc \( a \) in \( S_i \) such that \( a = (x, v) \) for some \( v \) in \( V(D) \), which implies that \( x \) is in-dominated by \( S_i \). Therefore, every element in \( \mathcal{I} \) is an in-dominating set in \( D \).

Since \( \mathcal{I} \) is a strong in-dominating partition of \( V(T(D)) \) we have that \( |\mathcal{I}| \leq d_s^-(T(D)) \), which implies that \( d_s^-(L(D)) + 1 \leq d_s^-(T(D)) \). \( \Box \)

5.1 A note on strong out-domatic number

Let \( D \) be a digraph, the converse of \( D \), denoted by \( \overrightarrow{D} \), is the digraph such that \( V(\overrightarrow{D}) = V(D) \) and \( (u, v) \in A(\overrightarrow{D}) \) if and only if \( (v, u) \in A(D) \). Notice that if \( S \) is an in-dominating set in \( V(D) \), then for every vertex \( x \) in \( V(\overrightarrow{D}) \setminus S \) there exists \( w \) in \( S \) such that \( (w, x) \in A(\overrightarrow{D}) \). Therefore, we can consider the following definition; an out-domatic partition of \( V(D) \) is a partition of \( V(D) \), say \( \mathcal{G} \), such that for every \( S \) in \( \mathcal{G} \), \( D(S) \) is a strong digraph and every vertex not in \( S \) has at least one in-neighbor in \( S \). Notice that \( \mathcal{G} \) is a strong in-domatic partition of \( V(D) \) if and only if \( \mathcal{G} \) is a strong out-domatic partition of \( V(\overrightarrow{D}) \). The maximum number of elements in an out-domatic partition is called the strong out-domatic number of \( D \) and it is denoted by \( d_s^+(D) \). It is straightforward to see that \( d_s^-(D) = d_s^+(\overrightarrow{D}) \).

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