GEOMETRY AND ALGEBRA OF PRIME FANO 3-FOLDS OF
GENUS 12

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Abstract. The connection between these Fano 3-folds and plane quartic curves is explained.

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1. Introduction

According to Mukai [Muk92] every prime Fano 3-fold of genus 12 has geometric realizations in three different ways:

(a) Let \( V \) be a 7-dimensional vectorspace and let \( \eta : \Lambda^2 V \rightarrow N \) be a net of alternating forms on \( V \), i.e. \( N \) is 3-dimensional. Denote by \( \mathbb{G}(3, V, \eta) = \{ E \in \mathbb{G}(3, V) | \Lambda^2 E \subset \ker(\eta : \Lambda^2 V \rightarrow N) \} \) the Grassmannian of isotropic 3-spaces of \( V \).

(b) Let \( f \in S_4(U) \) be the equation of a plane quartic \( F \subset \mathbb{P}(U) = \mathbb{P}^2 \). A polar hexagon \( \Gamma \) of \( F \) is the union of six lines \( \Gamma = \{ l_1 \cdot \ldots \cdot l_6 = 0 \} \) such that \( f = l_1^4 + \ldots + l_6^4 \). We can also identify \( \Gamma \) with a sixtuple of points in \( \mathbb{P}^2 = \mathbb{P}(U^*) \), i.e. a point of \( \text{Hilb}_6(\mathbb{P}^2) \). Then the variety of sums of powers presenting \( f \) is

\[ VSP(F, 6) = \{ \Gamma \in \text{Hilb}_6(\mathbb{P}^2) | \Gamma \text{ is polar to } F \}, \]

the closure of the set of polar hexagon of \( F \).

(c) Let \( W \) be a 4-dimensional vector space, and let \( q: U^* \hookrightarrow S_2 W^* \) be a net of quadrics in \( \mathbb{P}^3 = \mathbb{P}(W^*) \). Consider the Hilbert scheme \( \text{Hilb}_{3t+1}(\mathbb{P}^3) = H_1 \cup H_2 \) and

\[ Date: 8.11.1999. \]
\[ 1991 \textit{Mathematics Subject Classification.} 14J45, 14H45, 13D02. \]
\[ Key words and phrases. Fano 3-folds, plane quartics, theta characteristics, instable planes, syzygies, polar hexagons, sums of powers, twisted cubics, net of quadrics. \]
the component $H_1$ containing the twisted cubic curves $C \subset \mathbb{P}(W) = \mathbb{P}^3$. Let 

$$H \subset G(3, S_2 W)$$

be the image of $H_1$ under the map

$$\text{Hilb}_{3+1}(\mathbb{P}^3) \to G(3, H^0(\mathbb{P}^3, O(2))) = G(3, S_2 W),$$

which sends

$$C \mapsto H^0(\mathbb{P}^3, I_C(2)).$$

Denote by

$$H(q) = H \cap G(3, \ker(S_2 W \to U)) \subset G(3, S_2 W)$$

the variety of twisted cubics, whose quadratic equations are annihilated by $U^* \subset S_2 W^*$.

**Theorem 1.1.** (Mukai [Muk92, Muk95]) Let $X$ be a prime Fano 3-fold of genus 12 over an algebraically closed field of characteristic 0. Then there exist (a) a net of alternating forms $\eta: \Lambda^2 V \to N$, (b) a plane quartic $F$, (c) a net of quadrics $q: U^* \rightarrow S_2(W)^*$ such that

$$X \cong G(3, V, \eta) \cong VSP(F, 6) \cong H(q).$$

Conversely a general net of alternating forms, a general quartic or a general net of quadrics gives a smooth prime Fano 3-fold of genus 12.

**Corollary 1.2.** The moduli spaces $M_{\text{Fano}}$ of prime Fano 3-folds of genus 12, $M_3$ of curves of genus 3, $M_q$ of nets of quadrics, and $M_3, \vartheta_{\text{ev}}$ of curves of genus 3 together with a non-vanishing theta characteristic are birational to each other.

$M_{3, \vartheta_{\text{ev}}}$ occurs, since a net of quadrics in $\mathbb{P}^3$ is determined by its discriminant, a plane quartic, together with the associated non-vanishing theta characteristic. The connection between (a) and (b) is sketched in [Muk92]. The surprising fact that $M_3$ and $M_{3, \vartheta_{\text{ev}}}$ are birational, is actually an old result due to Scorza [Sc1899a] recently reconsider by Dolgachev and Kaney [DK93]. The purpose of this paper is to give a detailed description, how the three realizations are related to each other. In particular it is proved that Mukai’s and Scorza’s constructions give the same birational transformation. For non-algebraically closed ground fields our investigation gives that the models of type (a) and (b) exists over the field of definition of the $V_{22}$, while the space curve model (c) is in general only defined after a field extension.

**Acknowledgement** I am grateful to David Eisenbud, Geir Ellingsrud, Toni Iarrobino, János Kollár, Shigeru Mukai, Kristian Ranestad and Stein-Arild Strømme for fruitful discussions on the material of this paper.

2. **Apolarity and sums of powers**

Let $k$ be a field of characteristic zero. Consider $S = k[x_0, \ldots, x_r]$ and $T = k[\partial_0, \ldots, \partial_r]$. $T$ acts on $S$ by differentiation:

$$\partial^{\alpha}(x^{\beta}) = \alpha! \frac{\beta}{\alpha} x^{\beta - \alpha}$$

if $\beta \geq \alpha$ and 0 otherwise. Here $\alpha$ and $\beta$ are multi-indices, $(\frac{\beta}{\alpha}) = \prod (\frac{\beta_i}{\alpha_i})$ and so on. In particular we have a perfect pairing, apolarity, between forms of degree $n$ and homogenous differential operators of order $n$. 
Note that the polar of a form \( f \in S \) in a point \( a \in \mathbb{P}^r \) is given by \( P_a(f) \) for \( a = (a_0, \ldots, a_r) \) and \( P_a = \sum a_i \partial_i \in T \).

One can interchange the role of \( S \) and \( T \) by defining \( x^\alpha(\partial^\beta) = \alpha! \binom{\alpha}{\beta} \partial^{\alpha-\beta} \).

With this notation we have for forms of degree \( n \)

\[ P^n_a(f) = f(P^n_a) = n! f(a). \]

Moreover

\[ f(P^m_a) = 0 \iff f(a) = 0 \quad (2.1) \]

if \( m \geq n \).

A polarity allows to define artinian Gorenstein graded quotient rings of \( T \) via forms: For \( f \) a homogenous form of degree \( n \) define

\[ f^\perp = \{ D \in T | D(f) = 0 \} \]

and

\[ A^f = T/f^\perp. \]

The socle of \( A^f \) is in degree \( n \). Indeed \( P_a(D(f)) = 0 \forall P_a \in T_1 \iff D(f) = 0 \) or \( D \in T_n \). In particular the socle of \( A^f \) is 1-dimensional, and \( A^f \) is indeed Gorenstein.

Conversely for a graded Gorenstein ring \( A = T/I \) with socle in degree \( n \) multiplication in \( A \) induces a linear form \( f: S^n(T_1) \to k \) which can be identified with a homogenous polynomial \( f \in S \) of degree \( n \). This proves:

**Theorem 2.1.** (Macaulay, [Mac1916]) The map \( F \mapsto A^F \) gives a bijection between hypersurfaces \( F = \{ f = 0 \} \subset \mathbb{P}^r \) of degree \( n \) and Artinian graded Gorenstein quotient rings \( A = T/I \) of \( T \) with socle in degree \( n \).

Note, that

\[ (f^\perp : D) = D^\perp(f) \quad (2.2) \]

for any homogenous \( D \in T \).

**Definition 2.2.** (cf. [Jar84, Syl1886]) For forms \( f \) of even degree \( 2n \) the matrix

\[ \text{Cat}(f) = (D_iD_j(f))_{1 \leq i,j \leq (n+r)} \]

with \( D_1, \ldots, D_{(n+r)} \in T_n \) a basis, is called catalectican matrix of \( f \). \( f \) is called non-degenerate, if \( \text{Cat}(f) \) has maximal rank.

The Hilbert function and syzygies of \( A^F \) depends on subtle properties of \( F \), cf. [Jar84, IK96]. For example for plane quartics, we have (cf. Clebsch [Cle1861]):

**Theorem 2.3.** Let \( F = \{ f = 0 \} \) be a plane quartic. The following integers are equal:

(1) \( \dim_k A^F_2 \),
(2) \( \text{rank } \text{Cat}(f) \),
(3) the minimal \( s \) such that over the algebraic closure \( \overline{k} \) of \( k \) \( f \) lies in the closure of forms \( l_1^4 + \ldots + l_s^4 \).
Proof. \( \dim_k A^F_2 = \rank \Cat(f) \) holds because multiplication in \( A^F \) gives perfect pairings \( A^F_2 \times A^F_2 \to A^F_4 \cong k \).

Suppose \( f = l_1^4 + \ldots + l_s^4 \). The corresponding lines \( L_1, \ldots, L_s \in \PP^2 \), viewed as points in the dual space, impose at most \( s \) conditions on quadrics \( \{ D = 0 \} \subset \PP^2 \).

Hence \( \dim_k f_2^+ \geq 6 - s \) and \( \dim_k A^F_2 \leq s \). Since the Hilbert function of \( A^F \) varies semi-continuously with \( F \), \( \dim_k A^F_2 \leq s \) holds for all forms in the closure of the set of forms of the sums of \( s \) powers.

Conversely, suppose \( \dim_k A^F_2 = s \). \( A^F = T/I \) is Gorenstein of codimension 3. Hence the structure theorem of Buchsbaum-Eisenbud \[BE77\] applies: \( A^F \) has syzygies

\[
0 \leftarrow A^F \leftarrow T \leftarrow F_1 \xrightarrow{\phi} F_2 \leftarrow T(-7) \leftarrow 0,
\]

with \( F_1 = \bigoplus_{i=1}^{2r+1} T(-a_i), F_2 \cong F_1(7), \phi \) skew-symmetric, and \( I \) generated by the \( 2r \times 2r \) pfaffians of \( \phi \). Conversely, any sufficiently general skew-symmetric homomorphism \( \phi \in \Hom_T(F_2, F_1) \) defines via its pfaffians a graded Artinian Gorenstein ring \( A = T/I \) with the same Hilbert function as \( A^F \). Therefor it suffices to establish the sum presentation \( f = l_1^4 + \ldots + l_s^4 \) for an \( f \) corresponding to an \( A = A^F \) with sufficiently general syzygy matrix \( \phi \) for each possible numerical type of syzygies.

There are only a few number of numerical cases:

| Hilbert function | \( (a_1, \ldots, a_{2r+1}) \) | \( m \) | \( p \) | \( (b_{r+1}, \ldots, b_{2r+1}) \) |
|------------------|-----------------|-----|-----|------------------|
| \( (1,3,6,3,1) \) | \( (3,3,3,3,3,3,3) \) | 3   | 0   | \( (4,4,4) \) |
| \( (1,3,5,3,1) \) | \( (2,3,3,3,3) \)  | 2   | 0   | \( (4,4) \)   |
| \( (1,3,4,3,1) \) | \( (2,2,3) \)     | 0   | 1   | \( (4) \)     |
| \( (1,3,4,3,1) \) | \( (2,2,3,3,4) \) | 1   | 1   | \( (3,4) \)   |
| \( (1,3,3,3,1) \) | \( (2,2,2,4,4) \) | 0   | 2   | \( (3,3) \)   |
| \( (1,2,2,2,1) \) | \( (1,3,3) \)     | 1   | 0   | \( (4) \)     |
| \( (1,2,2,2,1) \) | \( (1,2,4) \)     | 0   | 1   | \( (3) \)     |
| \( (1,1,1,1,1) \) | \( (1,1,5) \)     | 0   | 1   | \( (2) \)     |

We argue in each of the cases separately but similarly: Let \( n \) be the number of cubic generators of \( I = f^\perp \) and \( m = \left\lfloor \frac{n}{2} \right\rfloor \). Consider the \( n \times n \) submatrix \( \tilde{\phi} \) of \( \phi \) corresponding to the linear coefficients of the quartic syzygies.

Suppose there is a \( m \times m \) (skew) symmetric submatrix of zeroes in \( \tilde{\phi} \). The corresponding quartic syzygies and all the syzygies which involve only equations of degree \( \leq 2 \) give \( r = m + p \) syzygies of degrees \( (b_{r+1}, \ldots, b_{2r+1}) \) as indicated above between equations of degrees \( (a_1, \ldots, a_{r+1}) \). Here \( p \) is the number of syzgies which involve only equations of degree \( \leq 2 \), and \( m + p = r \). Thus we obtain a block decomposition

\[
\phi = \left( \begin{array} {ccc} 0 & * & \psi \\ -* & 0 & \psi \\ -\psi^T & 0 & \end{array} \right) \left( \begin{array} {cc} r+1 \\ r \end{array} \right)
\]

with a \( (r+1) \times r \) matrix \( \psi \). The \( r+1 \) minors of \( \psi \) are among the pfaffians of \( \phi \); they are precisely the generators involved in our \( r \) syzygies. By Hilbert-Burch \[Eis93\, Thm 20.15\] these minors generate the homogenous ideal \( J_\Gamma \) of a set \( \Gamma \subset \PP^2 \) of distinct points with syzygies

\[
0 \leftarrow J_\Gamma \leftarrow \bigoplus_{i=1}^{r+1} T(-a_i) \xrightarrow{\psi} \bigoplus_{j=r+1}^{2r+1} T(-b_j) \leftarrow 0,
\]
if $\psi$ is sufficiently general. $J_\Gamma$ is 2-regular, since $b_j \leq 4$ for $j \geq r + 1$. Hence the Hilbert function of $R = R_\Gamma = T/J_\Gamma$ takes the values

$$h_{R}(t) = \dim_k(R_\Gamma)t = \deg \Gamma$$

for $t \geq 2$. On the other hand

$$\dim_k(R_\Gamma)_{2} = \dim_k A^f_2 = s$$

as $(J_\Gamma)_{\leq 2} = I_{\leq 2}$ by construction. Thus for sufficiently general $\psi$ $\Gamma$ consist of $s$ points $L_1 = \{ t_1 = 0 \}, \ldots, L_s = \{ t_s = 0 \}$. To prove that there exists a sum presentation $f = \lambda_1 t_1^4 + \cdots + \lambda_s t_s^4$ we consider $T \to R \to A$ and the induces inclusions

$$\Hom(A_4, k) \subset \Hom(R_4, k) \subset \Hom(T_4, k).$$

The linear forms

$$\{D \mapsto D(t_i^j)\}$$

are contained in $\Hom(R_4, k)$. Moreover since $\Gamma$ imposes $s$ independent conditions on quartics, these linear forms span the image. In particular

$$\{D \mapsto D( f )\} \in \Hom(A_4, k)$$

is contained in this space, i.e. $D( f ) = D(\lambda_1 t_1^4 + \cdots + \lambda_s t_s^4)$ for all $D \in T_4$ for suitable $\lambda_1, \ldots, \lambda_s \in k$. Hence $f = \lambda_1 t_1^4 + \cdots + \lambda_s t_s^4$ as desired. Taking roots of the $\lambda_i$’s we can put them into the equations $l_i$.

It remains to prove the existence of a $m \times m$ block of zeroes in $\hat{\phi}$, possibly after row and column operations. Let $V_f := \text{Tor}^2_T(A^f, k)$. Then $\hat{\phi}$ corresponds to a net of alternating forms $\Lambda^2 V_f \to T_1$ and we are looking for a subspace $E \in \mathbb{G}(m, n) = \mathbb{G}(m, V_f)$ such that $\Lambda^2 E \subset \ker(\Lambda^2 V_f \to T_1)$. If $m = 1$ there is nothing to prove. If $m \geq 2$ then $E$ exists, because for $j = \binom{n}{2}$ and $c_j = c_j(\Lambda^2 \mathcal{E}^*)$ the $j^{th}$ Chern class, where $\mathcal{E}$ denotes the universal subbundle on $\mathbb{G}(m, n)$, we have $c_j \neq 0$. \hfill $\square$

Notice that we expect a 3-dimensional family in case $s = 6$, a 1-dimensional family of sum presentations for $f$ with $s = 5$, or $s = 4$ and $r = 3$, or $s = 3$ and $h_A(1) = 2$, and a unique presentation otherwise.

**Remarks 2.4.** 1) The fact that, despite the dimension count, a general plane quartic is not the sum of 5 powers, goes back to Clebsch [Cle1861].

2) It is not true, that every $f$ with $\dim_k A^f_2 = s$ is a sum of $s$ powers. For $s$ with an unique presentation examples are rather obvious. But even in case $s = 5$ this occurs: Eg.

$$f = (1 - \frac{1}{t^2})x_1^4 + x_1^3(x_0 - \frac{4}{t}x_2) + \frac{1}{t^2}(x_1 + tx_2)^4 + x_2^3(x_0 - 4tx_1) + (1 - t^2)x_2^3$$

is not the sum of $5$ powers. The reason is that the quadric in $I = f^{\perp}$ is a double line. Hence distinct points in a $\Gamma$ would give a linear form in $I$, a contradiction. For $f$ the 1-dimensional family of sum decompositions degenerates to the family parametrized by $t$ of decompositions into five summands as above.

**Definition 2.5.** For $F = \{ f = 0 \} \subset \mathbb{P}^n$ a hypersurface we call a scheme $X \subset \mathbb{P}^n$ apolar to $F$ if $I_X \subset F^{\perp}$. The family of zero-dimensional apolar subschemes of degree $s$ of $F$ is denoted by $\text{VPS}(F, s)$. 
Note that with this definition
\[ VSP(F, s) \subset VPS(F, s) \]
is an open subscheme and equality holds if \( VPS(F, s) \) is irreducible and \( VSP(F, s) \) non-empty.

**Theorem 2.6.** Let \( F = \{ f = 0 \} \subset \mathbb{P}(U) \) be a non-degenerate plane quartic. Then
\[ VPS(F, 6) \cong \mathbb{G}(3, V_f, \eta_f), \]
where \( V_f = (f^{-1})^* N_f = U^* \) and \( \eta : \Lambda^2 V_f \rightarrow N_f \) the skew-symmetric syzygy matrix of \( N_f \). Conversely, for a net \( \eta : \Lambda^2 V \rightarrow N \) of skew-forms on a 7-dimensional vector space, whose pfaffians define a ideal \( I \) of codimension 3 in \( S(N) \), the dual socle quartic \( F = F(V, \eta) \subset \mathbb{P}(N^*) \) is a non-degenerate quartic, and \( \operatorname{VPS}(F(V, \eta), 6) \cong \mathbb{G}(3, V, \eta) \).

**Proof.** By 2.1, the structure theorem of Buchsbaum-Eisenbud [BE77], and 2.3
\[ F \mapsto (\Lambda^2 V_f \rightarrow \eta_f) \]
and
\[ (\eta : \Lambda^2 V \rightarrow N) \mapsto F(V, \eta) \]
give bijections between
\[ \{ F | \det(\text{Cat}(f)) \neq 0 \} \leftrightarrow \{ \eta : \Lambda^2 V \rightarrow N | \text{codim} \ I = 3 \}. \]
Moreover points \( p \in \mathbb{G}(3, V, \eta) \) correspond to block decompositions
\[ \phi = \begin{pmatrix} 0 & * & \psi \\ -* & 0 & 0 \\ -\psi^t & 0 & 0 \end{pmatrix} \]
of the syzygy matrix \( \phi \). We claim that for every \( \psi \) the ideal of minors \( I(\psi) \) has codimension 2, hence defines of a subscheme of length 6 in \( \mathbb{P}^2 \).

Assume that, \( I(\psi) \) has not depth 2. Then by Hilbert-Burch, the corresponding minors have a common factor. Since the minors are minimal generators of \( I_{pf} \), the factor has to be a linear form \( t \in T_1 \). So \( \psi \) is a matrix of syzygies among 4 quadrics without a common factor. The quadrics generate an ideal \( I \) of codimension \( \geq 2 \). Let \( B = T/J \). \( B \) has Hilbert function \((1, 3, 2, 1, \ldots) \). If \( \dim B = 0 \) then 3 general quadrics in \( J \) form a regular sequence, whose quotient has Hilbert function \((1, 3, 3, 1, 0) \) and the fourth quadric cuts down to a ring with Hilbert function \((1, 3, 2, 0) \). This is not the case. So \( \dim B = 1 \) and \( B \) has Hilbert function \((1, 3, 2, 1, 1, \ldots) \). Such quotients \( B \) exist: \( B \) is defined by 4 quadrics in the homogeneous ideal of a point \( p \in \mathbb{P}^2 \). However such a \( \psi \) does not occur as part of a skew-symmetric matrix \( \phi \), whose pfaffians have codimension 3. The syzygies of \( B \) start
\[ 0 \leftarrow B \leftarrow T \leftarrow 4T(-2) \leftarrow 3T(-3) \oplus T(-4) \oplus \ldots \ldots \]
Since \( tJ \subset I_{pf} \) the syzygy \( 4T(-2) \leftarrow T(-4) \) gives a relation among the pfaffians. But this relation is not in the space generated by the columns of \( \phi \), since the sequence
\[ 3T(-3) \xrightarrow{-\psi^t} 4T(-4) \leftarrow T(-6) \leftarrow 0 \]
is exact. This contradicts the exactness of the pfaffian complex.

Thus we have a well-defined morphism \( \alpha : \mathbb{G}(3, V, \eta) \rightarrow \text{Hilb}_6(\mathbb{P}^2) \). To prove that \( \alpha \) is an isomorphism onto its image, consider the open part \( \text{Hilb}_6(\mathbb{P}^2)^{\circ} \) of the Hilbert
scheme of length 6 subschemes, which impose independent conditions on quadrics, and the embedding \( \text{Hilb}_6(\mathbb{P}^2) \hookrightarrow \mathcal{G}(4,T_3) \). The diagram

\[
\begin{array}{ccc}
\text{Hilb}_6(\mathbb{P}^2) & \hookrightarrow & \mathcal{G}(4,T_3) \\
\alpha \uparrow & & \uparrow \\
\mathcal{G}(3,V,\eta) & \hookrightarrow & \mathcal{G}(4,V^*)
\end{array}
\]

commutes, where \( V^* = (f^\perp)_3 \subset T_3 \).

Finally, note that the image of \( \alpha \) contains all polar hexagons of \( F \). Indeed, if \( f = l_1^4 + \ldots + l_6^4 \) for distinct lines \( \Gamma = \{L_1, \ldots, L_6\} \subset \mathbb{P}^2 \), then \( \Gamma \) imposes independent conditions on quadrics by Thm 2.3 Hence syzygies of \( \Gamma \) are of type:

\[
0 \leftarrow R_\Gamma \leftarrow T \leftarrow 4T(-3) \leftarrow 3T(-4) \leftarrow 0.
\]

By (2.1) the ideal \( J_\Gamma \subset f^\perp \). Hence we have a sequence

\[
0 \leftarrow A^I \leftarrow R_\Gamma \leftarrow I_{A/R} \leftarrow 0.
\]

Since \( A^I \) and \( R = R_\Gamma \) have Hilbert functions \((1,3,6,3,1)\) and \((1,3,6,6,6,\ldots)\) respectively, \( I_{A/R} \) has 3 cubic generators with 4 linear relations:

\[
0 \leftarrow I_{A/R} \leftarrow 3T(-3) \leftarrow 4T(-4).
\]

The minors of the presentation matrix are contained in the annihilator, which is \( J_\Gamma \). Hence, this matrix is \( \psi^\perp \) again, and \( I_{A/R} \cong \omega_R(-4) \). A mapping cone between the complex (2.3) and its dual over the sequence (2.4), gives syzygies of \( A^I \) with the desired block structure.

We do not claim at this point, that every non-degenerate \( f \) has a non-degenerate polar hexagon. However, if there is one, then the points in the image of \( \alpha \) corresponding to them, form an open subset. \( \square \)

**Corollary 2.7.** For a general plane quartic \( F \) the variety of polar hexagons \( VSP(F,6) \) is a smooth Fano 3-fold of genus 12.

**Proof.** Since for the tautological subbundle \( \mathcal{E} \) on \( \mathcal{G}(3,V) \) the sheaf \( \Lambda^3 \mathcal{E}^* \) is globally generated by \( \Lambda^2 V^* \), a general net \( N \) of skew-forms defines a smooth subscheme \( \mathcal{G}(3,V,\eta) \) of codimension 9. Since \( \omega_{\mathcal{G}(3,V)} \cong \mathcal{O}_{\mathcal{G}(3,V)}(-7) \) and \( \Lambda^3(3\Lambda^2 \mathcal{E}^*) \cong \mathcal{O}_{\mathcal{G}(3,V)}(-6) \) one has \( \omega_{\mathcal{G}(3,V,\eta)} \cong \mathcal{O}_{\mathcal{G}(3,V,\eta)}(-1) \). By degree reasoning \( \mathcal{G}(3,V,\eta) \) is irreducible and hence it is a Fano 3-fold. \( \square \)

### 3. The Scorza map

In this section we recall some results of Scorza from [DK93].

A plane cubic \( C \) is called anharmonic, if \( C \) lies in the \( \text{PGL}(3) \)-orbit closure of \( \{x_0^3 + x_1^3 + x_2^3 = 0\} \). The reason is that the cross-ratio of the Fermat cubic is anharmonic. Let \( A \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(3))^*) \cong \mathbb{P}^9 \) denote the variety of anharmonic cubics. The \( \text{PGL} \)-orbit closure of a general cubics is a hypersurface of degree 12. Due to the additional automorphism of the Fermat cubic, \( A \) is hypersurface of degree 4. In terms of coordinates \((a,\ldots,j)\) of \( \mathbb{P}^9 \), \( ax_0^3 + bx_1^3 + cx_2^2 + 3dx_0^2x_1 + \)
Consider Lemma 3.1. Let $S$ be a smooth anharmonic cubic all three quadrics in $I$ intersect in 3 points. Then $S$ is a smooth Fermat cubic, and then $S$ is not a complete intersection of three quadrics, or $A^g$ has syzygies

$$0 \leftarrow A^g \leftarrow T \leftarrow \bigoplus_{i=1}^{5} T(-a_i) \leftarrow \bigoplus_{j=1}^{5} T(-b_j) \leftarrow T(-6) \leftarrow 0,$$

with $(a_1, \ldots, a_5) = (2, 2, 2, 3, 3)$ and $b_i = 6 - a_i$. In the second case the three quadrics of $I = f^2$ intersect in 3 points $\{L_1, L_2, L_3\} \in \mathbb{P}^2$ (possibly infinitesimal near), and as in section 2 we obtain $g = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^2$. Conversely, if $g$ is a smooth anharmonic cubic all three quadrics of $I$ vanish in $\{L_1, L_2, L_3\} \in \mathbb{P}^2$, and $A^g$ is not a complete intersection.

Since all cones are anharmonic cubics, this proves the lemma. \qed

Let $F = \{f = 0\} \subset \mathbb{P}^2$ be a non-degenerate plane quartic. Consider

$$S_F = \{a \in \mathbb{P}^2 | P_a(F) \in A\}.$$ 

Then either $S_F = \mathbb{P}^2$ or $S_F$ is a plane quartic. The first case does not occur for non-degenerate quartics. \cite[6.6.3]{DK93}. We call $S_F$ the covariant quartic of $F$. Consider

$$T_F = \{(a, b) \in \mathbb{P}^2 \times \mathbb{P}^2 | \text{rank} \ P_{a,b}(F) \leq 1\}.$$ 

Lemma 3.2. \cite[6.8.1]{DK93} Let $F \subset \mathbb{P}^2$ be a general quartic. Then $S_F$ is a smooth quartic and $T_F$ is a smooth symmetric correspondence of type (3,3) on $S_F \times S_F$ without united points.

Proof. For a complete proof see \cite{DK93}. The reason, why $T_F$ is such a correspondence on $S_F \times S_F$ is the following:

Suppose $(a, b) \in T_F$, say $P_{a,b}(f) = h^2$. Set $t_0 = P_b \in T_1$ and $(t_1, t_2) = (h^\perp)_1 \subset T_1$. Then $t_0 t_1, t_0 t_2 \in (P_a(f)^\perp)_2$ and $(P_a(f)^\perp)$ is not a complete intersection of quadrics. By Lemma 3.1, $P_a(F)$ is an anharmonic cubic. So $a \in S_F$.

For general $F$ and general $a \in S_F$, $P_a(F)$ is a smooth Fermat cubic, and then the points $b \in \mathbb{P}^2$ such that rank $P_{a,b}(F) \leq 1$ are the 3 vertices of the Hessian triangle of $P_a(F)$. So $T_F$ is symmetric of type (3,3). Since

$$2 \frac{\partial^2 f}{\partial x_i \partial x_j} (a) = \frac{\partial^2}{\partial x_i \partial x_j} P_{a,a}(f)$$

by (2.1), $T_F$ has no united points, if rank $(\frac{\partial^2 f}{\partial x_i \partial x_j}) (a) \geq 2$ for all $a \in S_F$. This is the case for general $F$, because then the Hessian $H e(F) = \{\text{det}(\frac{\partial^2 f}{\partial x_i \partial x_j}) = 0\}$ is smooth. \qed
Let \( S \) be a smooth plane quartic and \( \vartheta \) an even theta characteristic on \( S \). Consider the \( \vartheta \)-correspondence
\[
\vartheta \in \{ (a, b) \in S \times S | h^0(S, \vartheta(a - b)) \geq 1 \}.
\]
\( \vartheta \) is not effective. \( \deg \vartheta(a) = 3 \), and \( h^0(S, \vartheta(a - b)) = h^1(S, \vartheta(a - b)) = h^0(S, \vartheta(b - a)) \). So \( T_\vartheta \) is a symmetric correspondence of type \((3, 3)\) without united points.

**Theorem 3.3.** ([DK93, 7.6] ) Let \( k \) be a algebraically closed field. If \( F \) is a quartic such that \( S_F \) is a smooth quartic, then there exists a unique theta characteristic \( \vartheta = \vartheta_F \) on \( S_F \) such that
\[
T_F = T_\vartheta.
\]

**Proof.** For a general \( F \) and a general \( (a, b) \in T_F \) consider the polar Hessian triangles \( T_F(a) = b + b_1 + b_2 \) and \( T_F(b) = a + a_1 + a_2 \), ie. the vertices of the Hessian of \( P_a(F) \) and \( P_b(F) \) respectively. All 6 points are different. If \( P_{a,b}(f) = h^2 \) then both \( b_1, b_2 \) and \( a_1, a_2 \) span the line \( \{ h = 0 \} \). So \( S_F \cap \{ h = 0 \} = \{ a_1, a_2, b_1, b_2 \} \) and
\[
T_F(a) - a + T_F(b) - b = b_1 + b_2 + a_1 + a_2 \tag{3.1}
\]
is a canonical divisor on \( S_F \). Moreover
\[
T_F(a) - a \equiv T_F(b) - b \tag{3.2}
\]
for any 2 points \( a, b \in S_F \). To establish this consider the map
\[
S_F \rightarrow \text{Pic}^2(S_F), a \mapsto \mathcal{O}(T_F(a) - a).
\]
Since even for degenerate polar Hessians the three (possibly infinitesimal near) points \( T_F(a) = b_1 + b_2 + b_3 \) are not colinear, \( h^0(S_F, \mathcal{O}(T_F(a))) = 1 \) for all \( a \in S_F \). So
\[
h^0(S_F, \mathcal{O}(T_F(a) - a)) = 0,
\]
as \( T_F \) has no united points. It follows, that the image of \( S_F \) does not intersect the \( \Theta \)-divisor of \( \text{Pic}^2(S_F) \). Since \( \Theta \) is ample, \( S_F \) maps to a point. By (3.1),(3.2)
\[
\vartheta = \mathcal{O}(T_F(a) - a))
\]
is an non-vanishing theta characteristic. \( \Box \)

**Remark 3.4.** Although the \([\vartheta] \in \text{Pic} C\) is a point defined over the ground field \( k \), the line bundle \( \vartheta \) may not be defined over the ground field. For an example see 3.7 below.

**Theorem 3.5.** (Scorza, [DK93, 7.8,7.11] ) Let \( k \) be algebraically closed. The rational map induced by
\[
s: F \mapsto (S_F, \vartheta_F)
\]
from the moduli spaces of curves of genus 3
\[
s: \mathcal{M}_3 \rightarrow \mathcal{M}_{3, \vartheta_F}
\]
to the moduli of curves of genus 3 together with a even theta characteristic is birational.
Remarks 3.6. (1) The projection $\mathcal{M}_{3,θ\varpi} \to \mathcal{M}_3$ is a finite cover of degree 36:1, since a curve of genus 3 has precisely 36 even theta characteristics.

(2) For a pair $(S, θ)$ of a plane quartic together with a non-vanishing theta characteristic, the quartic $F = s^{-1}(S, θ)$ is called Scorza quartic of $(S, θ)$. Dolgachev and Kanev give two description of $s^{-1}$. In section 5 we will give another one.

(3) More general, for a canonical curve $S \subset \mathbb{P}^{q-1}$ of genus $g$, and a non-vanishing theta characteristic $θ$ with some plausible, but yet unproven hypothesis Scorza constructs a quartic hypersurface $F \subset \mathbb{P}^{q-1}$. See [Sco1899] and [DK93].

4. Rank 2 vector bundles on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 3$

Let $W$ be a 4-dimensional vector space, $\mathbb{P}^3 = \mathbb{P}(W^*)$ and $q: U^* \hookrightarrow S_2 W^*$ a net of quadrics in $\mathbb{P}^3$, whose general element is smooth. Let

$$W \to U \otimes W^*$$

the associated symmetric matrix with entries in $U$, and

$$b_q: W \otimes \mathcal{O}_{P^2}(-2) \to W^* \otimes \mathcal{O}_{P^2}(-1)$$

the associated map of sheaves on $P^2 = \mathbb{P}(U)$ twisted. Let $S = S_q = \{\det(b_q) = 0\} \subset P^2$ denote the discriminant of the net, and $\vartheta = \vartheta_q = \operatorname{coker}(b_q)$. Since $b_q$ is symmetric,

$$\vartheta \cong \operatorname{Ext}^1_{\mathbb{P}^2}(\vartheta, \mathcal{O}_{P^2}) \cong \operatorname{Hom}_{\mathcal{O}_{P^2}}(\vartheta, \omega_S).$$

If $S$ is smooth, then $\vartheta$ is an invertible $\mathcal{O}_S$-module, hence $\vartheta$ is a non-vanishing theta characteristic on $S$.

Conversely, given a plane quartic $S$ and a torsion free rank 1 $\mathcal{O}_S$-module $\vartheta$ which satisfies (4.1), we denote by $W^* = H^0(S, \vartheta(1))$ and

$$q = q(S, \vartheta): U^* \hookrightarrow S_2 W^*$$

the corresponding net of quadrics in $\mathbb{P}(W^*) = \mathbb{P}^3$.

If $S$ is smooth, then

$$\phi_{\vartheta(1)}: S \hookrightarrow \mathbb{P}^3$$

is an embedding, and the image $\tilde{S} = \phi_{\vartheta(1)}(S)$ is the variety of vertices of the singular quadrics in the net. Equation of $\tilde{S} \subset \mathbb{P}^3 = \mathbb{P}(W^*)$ are given by the $3 \times 3$ minors of

$$\tilde{b}_q: W \otimes \mathcal{O}_{P^3}(-1) \to U \otimes \mathcal{O}_{P^3}.$$
Proof. The number of syzygy \( \dim \text{Tor}_j^R(A^q, k)_j \) in the above sequence take the minimal possible values for an Artinian ring \( A \) with Hilbert function \((1, 4, 3, 0, \ldots)\). Thus by semi-continuity it suffices to establish the existence of one example \( q \) with such syzygies. The Kleinian net

\[
g_{\text{Klein}} = (\frac{1}{2}z_1^2 - z_0z_2, \frac{1}{2}z_2^2 - z_0z_3, \frac{1}{2}z_3^2 - z_0z_1)
\]

where \( T = k[z_0, z_1, z_2, z_3] \), has this property. \( \square \)

Consider the map \( \phi_1 \) in the complex above, and its kernel sheafivied and twisted by \( \otimes \mathcal{O}_{\mathbb{P}^3}(5) \):

\[
\mathcal{E} = \mathcal{E}_q = \ker(7\mathcal{O}_{\mathbb{P}^3}(3) \xrightarrow{\phi_1} 8\mathcal{O}_{\mathbb{P}^3}(2)).
\]

**Proposition 4.2.** Let \( q \) be a general net of quadrics. Then \( \mathcal{E}_q \) is a stable rank 2 vectorbundle with Chern classes \( c_1 = 0, c_2 = 3 \) and syzygies

\[
0 \leftarrow \mathcal{E}_q \leftarrow 8\mathcal{O}_{\mathbb{P}^3}(-2) \leftarrow 7\mathcal{O}_{\mathbb{P}^3}(-3) \leftarrow \mathcal{O}_{\mathbb{P}^3}(-5) \leftarrow 0. \tag{4.2}
\]

Its \( H^2 \)-cohomology module is

\[
A^q(5) = \bigoplus_n H^2(\mathbb{P}^3, \mathcal{E}_q(n)).
\]

Proof. Since \( A^q \) Artinian the first syzygy module sheafivied is a rank 6 vectorbundle,

\[
0 \leftarrow \mathcal{O}_{\mathbb{P}^3}(5) \leftarrow 7\mathcal{O}_{\mathbb{P}^3}(3) \leftarrow \mathcal{F} \leftarrow 0. \tag{4.3}
\]

\( \mathcal{E}_q \) is the kernel of by \( \phi_1 \),

\[
(0 \leftarrow \mathcal{F} \leftarrow 8\mathcal{O}_{\mathbb{P}^3}(2) \leftarrow \mathcal{E}_q \leftarrow 0. \tag{4.4}
\]

One expects that \( \mathcal{F} \leftarrow 8\mathcal{O}_{\mathbb{P}^3}(2) \) is surjective outside a set of codimension \( 8-6+1 = 3 \), hence that \( \mathcal{E}_q \) is a vector bundle of rank 2 outside a finite set of points. The expected number of these points is 0 by Porteous formula. Thus either \( \phi_1 \) has rank \( \leq 5 \) along at least a curve, or \( \mathcal{E}_q \) is a rank 2 vector bundle with Chern polynomial

\[
c_t(\mathcal{E}_q) = \frac{(1 + 5t)(1 + 2t)^7}{(1 + 3t)^8} \equiv 1 + 3t^2 \mod t^4.
\]

For a general \( q \) the second alternative takes place, as one can check by considering an example, eg. the Kleinian net.

Since \( \mathcal{E}_q \) has rank 2 and \( c_1 = 0 \), wedge product

\[
\mathcal{E}_q \otimes \mathcal{E}_q \rightarrow \Lambda^2 \mathcal{E}_q \cong \mathcal{O}_{\mathbb{P}^3}
\]
gives \( \mathcal{E}_q^* \cong \mathcal{E}_q \). Thus the dual of the sequences (4.3) and (4.4) give the exact sequence (4.2). Since this complex is short enough to stay exact on global sections for arbitrary twists, this is the minimal resolution. The last statement follows from the cohomology sequence of (4.2.2) and (4.2.3). \( \mathcal{E} \) is stable, because \( H^0(\mathbb{P}^3, \mathcal{E}) = 0 \), cf. \cite[Lemma II 1.2.5]{OSS80}.

\( \square \)

**Corollary 4.3.** If \( q \) is general, then there are natural isomorphism

\[
U \cong \text{Tor}_4^R(A^q, k)_6 \cong (\text{Tor}_2^R(A^q, k)_4)^* \text{ and a skew-symmetrical self-duality}
\]

\[
\text{Tor}_3^R(A^q, k)_5 \cong (\text{Tor}_1^R(A^q, k)_3)^*.
\]

The maps \( 3R(-4) \leftarrow 8R(-5) \) and \( 8R(-5) \leftarrow 3R(-6) \) in the resolution (4.2) are dual to each other under these isomorphisms.
Proof. The matrices yield minimal presentations

\[ 0 \leftarrow \bigoplus_n H^1(\mathbb{P}^3, \mathcal{E}(n)) \leftarrow 3R(1) \leftarrow 8R \]

and

\[ 0 \leftarrow Ext^1_R(A^q, R) \leftarrow 3R(6) \leftarrow 8R(5) \]

respectively. By Serre duality and \(A^q(5) = \bigoplus_n H^2(\mathbb{P}^3, \mathcal{E}(n))\),

\[ Ext^1_R(A^q(5), R) = \bigoplus_n H^1(\mathbb{P}^3, \mathcal{E}^*(n)) \]

Since \(\mathcal{E} \cong \mathcal{E}^*\), the modules are isomorphic, hence the desired isomorphisms follow by comparison of the presentations. The self-duality on \(\text{Tor}^R_3(A^q, k)_5\) is skew, since the isomorphism \(\mathcal{E} \cong \mathcal{E}^*\) has this property. Finally, we note

\[ U \cong A^q \cong H^2(\mathbb{P}^3, \mathcal{E}(-3)) \cong H^1(\mathbb{P}^3, \mathcal{E}^*(-1))^* \]

\[ \cong H^1(\mathbb{P}^3, \mathcal{E}(-1))^* \cong \text{Tor}^R_2(A^q, k)_5 \cong \text{Tor}^R_3(A^q, k)_6. \]

\[ \square \]

**Corollary 4.4.** \(\tilde{S} = \tilde{S}_q \subset \mathbb{P}^3\) is the variety of unstable planes of \(\mathcal{E}_q\). \(\tilde{S}\) determines \(\mathcal{E}_q\) up to isomorphism. The moduli space \(\mathcal{M}_{\mathbb{P}^3}(2; 0, 3)\) of rank 2 vector bundles on \(\mathbb{P}^3\) with \(c_1 = 0\) and \(c_2 = 3\) has a component birational to \(G(3, S_2W^*)\).

Proof. A plane \(H = \{h = 0\} \subset \mathbb{P}^3\) is unstable for \(\mathcal{E}\), iff \(H^0(H, \mathcal{E} \mid_H) \neq 0\), equivalently, if multiplication with \(h\) is not injective on

\[ H^1(\mathbb{P}^3, \mathcal{E}(-1)) \xrightarrow{h} H^1(\mathbb{P}^3, \mathcal{E}). \]

\[ H^1(\mathbb{P}^3, \mathcal{E}(-1)) \cong U^* \text{ and } H^1(\mathbb{P}^3, \mathcal{E}) \cong W^*. \] A quadric \(q_1 \in U^* \subset S_2W^*\) is annihilated by \(h \in W\), iff \(Q_1 = \{q_1 = 0\} \subset \mathbb{P}^3\) is a cone with vertex \(H \in \mathbb{P}^3\). So the variety of unstable planes coincides with the variety \(\tilde{S} \subset \mathbb{P}^3\) of vertices of the cones. \(\tilde{S}\) determines \(q\), which in turn determines \(A^q\) and \(\mathcal{E}_q\). The vector bundles obtained from points \(q \in G(3, S_2W^*)\) form an open part of the moduli scheme \(\mathcal{M}_{\mathbb{P}^3}(2; 0, 3)\), since by semi-continuity and minmality, the cohomology modules \(\bigoplus_n H^2(\mathbb{P}^3, \mathcal{E}(n))\) have the same numerical type of syzygies for an open part of \(\mathcal{M}_{\mathbb{P}^3}(2; 0, 3)\). \[ \square \]

5. Twisted Cubics Annihilated by a Net of Quadrics

Let \(q\): \(U^* \hookrightarrow S_2W^*\) be a net of quadrics as before. Let \(H(q)\) denote the variety of twisted cubics \(C \subset \mathbb{P}^3\), whose equations \(H^0(\mathbb{P}^3, I_C(2)) \subset S_2W\) are annihilated by \(q\). Let \(V_q = (q^+)\_2 \subset S_2W\). Since a twisted cubic is defined by its quadrics and \(h^0(\mathbb{P}^3, I_C(2)) = 3\), \(H(q)\) is a subset of \(G(3, V_q)\) in a natural way. We are looking for a net of alternating forms on \(V_q\), which defines \(H(q) \subset G(3, V_q)\).

For a description of \(\text{Hilb}_{3d+1}(\mathbb{P}^3)\) and the map

\[ \text{Hilb}_{3d+1}(\mathbb{P}^3) \to G(3, S_2W^*) \]

see [EPS87], [PS88].

Consider the syzygies of \(A^q\). By definition of \(A^q\) we have

\[ \text{Tor}^R_4(A^q, k)_2 \cong V_q. \]

Define

\[ N_q = \text{Tor}^R_2(A^q, k)_4 \]
and consider
\[ \eta_q : \Lambda^2 V_q \to N_q \]
given by multiplication in the algebra \( \text{Tor}^R_{-1}(A^q, k) \), cf. [Eis95] Exercise A3.20
\[ \eta(p_1 \wedge p_2) = 0 \]
for \( p_1, p_2 \in V_q \), iff the Koszul syzygy
\[ p_1 \otimes p_2 - p_2 \otimes p_1 \in \ker(R \leftarrow 7R(-2)) \]
lies in \( \text{Im}(7R(-2) \leftarrow 8R(-3)) \).

**Theorem 5.1.**
\[ H(q) \cong \mathbb{G}(3, V_q, \eta_q) \]
for a general net \( q \).

**Proof.** Let \( C \subset \mathbb{P}^3 \) be a rational normal curve whose ideal \( I_C = (p_1, p_2, p_3) \) is generated by three quadrics \( p_1, p_2, p_3 \in V \). \( C \) has syzygies
\[ 0 \leftarrow \mathcal{O}_C \leftarrow 3\mathcal{O}_{V_3}(-2) \leftarrow 2\mathcal{O}_{V_3}(-3) \leftarrow 0. \quad (5.1) \]
Hence all syzygies among \( p_1, p_2, p_3 \) are generated by linear relations, and \( E = (I_C)_2 = (p_1, p_2, p_3)_2 \subset \mathbb{G}(3, V, \eta) \).

Conversely, suppose that \( E = (p_1, p_2, p_3) \subset \mathbb{G}(3, V, \eta) \). We will prove that \( p_1, p_2, p_3 \) generate the homogenous ideal of a curve \( C \) of degree 3 and arithmetic genus 0. Choose \( p_4, \ldots, p_7 \in V \), such that \( p_1, \ldots, p_7 \) form a basis. By definition of \( \eta \) there is a matrix \( 8R(-3) \xleftarrow{\psi} 3R(-4) \), such that
\[
\phi_1 \cdot \psi = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
gives the matrix of Koszul relations, where \( \phi_1 \) is the matrix of linear syzygies in Lemma 4.1. With
\[ \tau = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \]
we have
\[ \phi_1 \cdot \psi \cdot \tau = 0. \]
But for \( \ker(7\mathcal{O}_{V_3}(-2) \xleftarrow{\psi_1} 8\mathcal{O}_{V_3}(-3)) = \mathcal{E}(1) \) we have \( H^0(\mathbb{P}^3, \mathcal{E}(1)) = 0 \) by Prop. 4.2. So
\[ \psi \cdot \tau = 0, \]
ie. \( p_1, p_2, p_3 \) are three quadrics with some linear relations
\[ r_1 \cdot p_1 + r_2 \cdot p_2 + r_3 \cdot p_3 = 0. \quad (5.2) \]
The coefficients \( r_1, r_2, r_3 \in R_1 \) are linearly independent for a general linear combination of rows of \( \psi \). Because otherwise any two elements of \( E \) would have a common linear factor, which implies, that all three elements \( p_1, p_2, p_3 \) have a common factor, and \( \text{Tor}^3_1(A^q, k)_4 \neq 0 \). But this group is zero by Lemma 4.1. Thus
\[ (p_1, p_2, p_3) = \Lambda^2 \tau_2 \]
for $2 \times 3$ matrix
\[ \tau_2 = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \end{pmatrix} \]
of linear forms $r_i \in R_1$, (cf. [Sch91], Lemma 4.3). Since the minors $p_1, p_2, p_3$ have no common factor, the Hilbert-Burch complex of $\tau_2$ is exact, and $p_1, p_2, p_3$ generate the ideal of a curve $C \subset P^3$ of degree 3 and arithmetic genus 0. $C \subset \text{Hilb}_{h+1}(P^3)$ lies in the component $H_1$, which contains the twisted cubics, (cf. [PS85, EPS87]).

Note, that boundary points corresponding to plane nodal cubics with an embed-
ded point at the node, do not occur, since all $C$ are arithmetically Cohen-Macaulay. □

**Proposition 5.2.** For a general net $q: U^* \hookrightarrow S_2 W^*$ of quadrics the pfaffians of the net of alternating forms $\eta_q: \Lambda^2 V_q \to N_q$ defines an Artinian Gorenstein ring with Hilbert function $(1, 3, 6, 3, 1)$ with a smooth dual socle quartic $F_q = F(V_q, \eta_q)$.

**Proof.** Since the desired property is an open condition on nets $q$ of quadrics, it suffices to exhibit an example. For the Kleimian net $q_{\text{Klein}}$ we obtain as dual socle quartic $F_{\text{Klein}} = \{x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0\}$. $F_{\text{Klein}}$ is smooth for $\text{char}(k) \neq 7$. For $\text{char}(k) = 7$ one can take some other example. □

6. The Hilbert schemes of lines on $X$

Let $F = \{f = 0\} \subset P^2 = P(U)$ be a non-degenerate plane quartic. In this section we prove that the circle of constructions

\[
F \mapsto (S_F, \vartheta_F), \quad (S, \vartheta) \mapsto q_{S, \vartheta},
\]

$q \mapsto A^q,$

$A^q \mapsto (\eta_q: \Lambda^2 V_q \to N_q),$  \quad \text{Scorza,}

$(\eta: \Lambda^2 V \to N) \mapsto A_{V, \eta},$  \quad \text{net corresponding to $\vartheta$,}

$A \mapsto F_A,$  \quad \text{apolarity,}$

$(\eta_q: \Lambda^2 V_q \to N_q)$ \quad \text{Tor multiplication,}

$A_{V, \eta},$  \quad \text{pfaffians,}$

$F_A$ \quad \text{dual socle quartic,}$

gives the identity transformation on an open set of quadrics. Note that, since $N_q \cong U^*$ by Cor. 4.3, $F_A$ is again a quartic in $P(U)$.

For $X = G(3, V, \eta)$ we denote by $H_X$ the Hilbert scheme of lines in $X$ with respect to the Plücker embedding $X \subset G(3, V) \hookrightarrow P(\Lambda^2 V^*)$.

**Theorem 6.1.** Let $q$ be a general net of quadrics in $\mathbb{P}^3$. Let $E_q$ be the corresponding vector bundle on $\mathbb{P}^3$, $X = H(q) = G(3, V_q, \eta_q)$ and $F = F_q$ the dual socle quartic of $A_{V_q, \eta_q}$. The following curves are isomorphic:

(a) the discriminant $S_q$ of $q$,

(b) the variety $\tilde{S}_q$ of unstable planes of $E_q$,

(c) the Hilbert scheme $H_X$ of lines on $X$,

(d) the covariant quartic $S_F$ of $F$.

**Proof.** (a) $\leftrightarrow$ (b) $: S_q \cong \tilde{S}_q$ holds by section 4. (b) $\leftrightarrow$ (c) : Let $H = \{r = 0\} \subset P^3$ be an unstable plane. Then

\[ U^* = H^1(\mathbb{P}^3, E_q(-1)) \hookrightarrow W^* = H^1(\mathbb{P}^3, E_q) \]
is not injective, equivalently, \[ \mu_r: W \hookrightarrow U \]
not surjective. So \( \ker(\mu_r) \) is at least 2 dimensional, i.e. there are 2 elements \( r_1, r_2 \in R_1 \) such that \( p_1 = r \cdot r_1, p_2 = r \cdot r_2 \in V_q \).

\((r_1, r_2) \cap V_q \subset S_2W\) is at least \(7 - 3 = 4\) dimensional. So there are further 2 quadrics

\[ p_3 = a_1 r_1 + a_2 r_2, p_4 = b_1 r_1 + b_2 r_2 \in (r_1, r_2) \cap V_q. \]

Let \( C_{(\alpha; \beta)} \subset \mathbb{P}^3 \) be the curve (!) defined by

\[ (p_1, p_2, \alpha p_3 + \beta p_4) = \Lambda^2 \begin{pmatrix} r_2 & -r_1 & 0 \\ \alpha a_1 + \beta b_1 & \alpha a_2 + \beta b_2 & -r \end{pmatrix}. \]

By Thm. 5.1,

\[ p_1 \wedge p_2 \wedge (\alpha p_3 + \beta p_4) \in \mathbb{G}(3, V_q, \eta_q) \subset \mathbb{P}(\Lambda^3V_q^*), \quad (\alpha : \beta) \in \mathbb{P}^1, \quad (6.1) \]

gives a point in \( \mathcal{H}_X \).

Conversely, every line in \( H(q) \) is of type (6.1) for some \( p_1, p_2, p_3, p_4 \in V_q \). \( p_1 \) and \( p_2 \) have a common factor \( r \), since

\[ \bigcup_{(\alpha; \beta)} C_{(\alpha; \beta)} \subset \{ p_1 = p_2 = 0 \} \subset \mathbb{P}^3, \]

and \( H = \{ r = 0 \} \) is an unstable plane.

\((c) \rightarrow (d) : F = \{ f = 0 \} \) is non-degenerate by Cor. 5.2 and Thm. 2.3. Moreover

\[ \mathbb{G}(3, V_q, \eta_q) = \mathbb{G}(3, V_f, \eta_f) \cong VSP(F, 6). \]

From this point of view a line

\[ p_1 \wedge p_2 \wedge (\alpha p_3 + \beta p_4) \in \mathbb{G}(3, V_f, \eta_f) \]

corresponds to a syzygy matrix

\[ \phi = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{15} & a_{16} & a_{17} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} & a_{27} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} & a_{37} \\ 0 & 0 & -a_{34} & 0 & a_{45} & a_{46} & a_{47} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & a_{57} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & a_{67} \\ -a_{17} & -a_{27} & -a_{37} & -a_{47} & -a_{57} & -a_{67} & 0 \end{pmatrix}, \quad (6.2) \]

and the family of submatrices

\[ \psi_{(\alpha; \beta)} = \begin{pmatrix} 0 & a_{15} & a_{16} & a_{17} \\ 0 & a_{25} & a_{26} & a_{27} \\ a_{34} & \alpha a_{15} + \beta a_{45} & \alpha a_{36} + \beta a_{46} & \alpha a_{37} + \beta a_{47} \end{pmatrix} \]

corresponds to 1-parameter family of polar hexagons with 3 fixed lines \( \in \mathbb{P}^2 \) defined by

\[ \begin{pmatrix} a_{15} & a_{16} & a_{17} \\ a_{25} & a_{26} & a_{27} \end{pmatrix} \]

and three moving lines through the common point \( a_{34} \in \mathbb{P}^2 \). Hence the polar \( P_{a_{34}}(F) \in A, \) i.e. \( a_{34} \in S_F \subset \mathbb{P}^2 \).

\((c) \leftarrow (d) : \) Conversely given \( a \in S_F \). Since \( F \) is non-degenerate \( g = P_a(f) \) is not a cone. By Lemma 6.1 \( A^g \) is not a complete intersection. Hence there are three quadrics \( b_1, b_2, b_3 \in \{ g^\perp \} \) with precisely 2 linear syzygies. Then \( ab_1, ab_2, ab_3 \in (f^\perp)_3 \), and the 2 linear syzygies give 2 of the columns of \( \phi \) with many zeroes. Thus this gives a decomposition of \( \phi \) of shape (6.2). We only have to check that none of
the 6 possibly non-zero entries of the 2 columns can lie on the diagonal. Suppose
1 or 2 entries lie on the diagonal. Then, since \(b_1, b_2, b_3\) are the minors of the \(3 \times 2\)
matrix, either one quadrics is zero, or they have a common factor and a further
syzygy. Both cases are impossible. Thus every point \(a \in S_F\) gives a uniquely
determined decomposition of \(\phi\) of shape (6.2). Hence a well-defined point in \(\mathcal{H}_X\).
Since \(a = a_{34}\) in this correspondence this is the inverse of \((c) \rightarrow (d)\).

Notice, that under the isomorphisms of section 5 and 4.3,
\[
\eta(\rho_1 \wedge \rho_4) = a = a_{34} \in N \cong U^* \cong H^1(\mathbb{P}^3, \mathcal{E}(-1)).
\]
Moreover \((\rho_1, \rho_2, \rho_3, \rho_4)\) have the relations
\[
\tilde{\phi}_1 = \begin{pmatrix}
-r_2 & a_1 & b_1 & 0 \\
   r_1 & a_2 & b_2 & 0 \\
   0   & -r & 0   & -p_4 \\
   0   & 0   & -r & +p_3
\end{pmatrix}.
\]
\[
\rho = \begin{pmatrix}
(a_1 b_2 - b_1 a_2) \\
p_4 \\
-p_3 \\
r
\end{pmatrix}.
\]
Thus, if we choose a basis for \(8R(3) \oplus 3R(-4)\) in the complex of Cor. 4.3 with
these four relation corresponding basis elements, \(\rho\) gives one column of the matrix
\[
8R(3) \oplus 3R(-4) \leftarrow 8R(-5).
\]
Since
\[
0 \leftarrow \bigoplus_n H^1(\mathbb{P}^3, \mathcal{E}(n)) \leftarrow (3R(-4) \leftarrow 8R(-5)) \otimes R(5)
\]
is a presentation, we obtain, that \(\eta(\rho_3 \wedge \rho_4) = a \in H^1(\mathbb{P}^3, \mathcal{E}(-1))\) is annihilated
by \(r\). Since \(h^0(H, \mathcal{E}|_H) = 1\) for \(H = \{r = 0\}\) any unstable plane, there is only
one quadric cone \(Q \in U^*\) with vertex \(H \in \mathbb{P}^3\) up to scalars. Thus under the
isomorphisms of Cor. 4.3, section 5 and \((a) \leftrightarrow (d)\) the curves
\(S_q, S_F \in \mathbb{P}(U)\)
are actually equal.

Let \(X = X_q = \mathbb{G}(3, V_q, N_q)\). Denote by
\[
T_H = \{(L_1, L_2) \in \mathcal{H}_X \times \mathcal{H}_X | L_1 \cap L_2 \neq \emptyset, L_1 \neq L_2\}
\]
the correspondence of intersecting lines in \(X\).

**Corollary 6.2.** Let \(q\) be a general net of quadrics. Then \(B_X\) consists of all singlar
twisted cubics in \(H(q)\), and \(S_F \subset \mathbb{P}(U)\) is the set of the tripel points of polar
hexagons to \(F = F_q\). The correspondences
\[
\begin{align*}
(a) & \quad T_{\tilde{\phi}_q} \text{ on } S_q, \\
(b) & \quad T_H \text{ on } \mathcal{H}_X, \\
(c) & \quad T_F \text{ on } F_q
\end{align*}
\]
are isomorphic.

**Proof.** From the proof of the theorem we see, that the curves \(C_{(\alpha; \beta)} \in H(q)\) on
a line
\[
\{p_1 \wedge p_2 \wedge (\alpha p_3 + \beta p_4)\}_{(\alpha; \beta) \in \mathbb{P}^1} \in \mathcal{H}
\] (6.1)
are all singular, and that they have the component \( \{r_1 = r_2 = 0\} \subset \mathbb{P}^3 \) in common. Conversely, if a curve \( C \in H(q) \) is singular, it is reducible and one of its components is a line \( \{r_1 = r_2 = 0\} \) in the intersection of two reducible quadrics \( p_1 = r_1 \cdot r, p_2 = r_2 \cdot r \in H^0(\mathbb{P}^3, I_C(2)) \subset V_q \), \( r \) defines a unstable plane of \( \mathcal{E}_q \) and gives a line (6.1).

Now take the point of view from polar hexagons. If a point \( \Gamma = \{L_1, \ldots, L_6\} \) lies on a line in \( V.SP(F, 6) \), then three lines of the hexagon pass through a common point \( a_{34} \in \mathbb{P}^2 \). Conversely, if \( \{L_1, L_2, L_3\} \subset \Gamma \) pass through a point \( a \in \mathbb{P}^2 \), then \( P_a(F) \) is anharmonic, ie. \( a \in S_F \) and \( f = f - (\lambda_4 l_4 + \lambda_5 l_5 + \lambda_6 l_6) \) is a quartic with \( h_{A/1}(1) = 2 \) and \( h_{A/1}(2) = 3 \), since \( f \) is not a sum of five powers. So by Thm 2.3 there is a pencil of 3-tuples of lines presenting \( f \), and this gives the family \( \Gamma(\alpha: \beta) \) defined by (6.1). For three values of \( (\alpha : \beta) \in \mathbb{P}^1 \) one of the moving lines passes through an intersection point \( b \) of a pair of the fixed lines. This corresponds to an intersection of two different lines in \( \mathcal{H} \), ie. a point in \( T_H \), and also to the point \( (a, b) \in T_F \).

Finally to prove \( T_F \cong T_{g_\vartheta} \), note that we already know \( S_F = S_q \). Thus \( T_F \) and \( T_{g_\vartheta} \) correspond both to one of the 36 even theta characteristics on \( S_F \). Since \( M_3 \) and \( M_{3, g^{ev}} \) have the same dimension, this implies, that the circle of constructions induces a covering transformation of

\[
M_{3, g^{ev}} \rightarrow M_3
\]

over an open set corresponding to general nets \( q \). Since \( M_q \) hence also \( M_{3, g^{ev}} \) is irreducible, it suffices to varify \( T_F = T_{g_\vartheta} \) in one example, where all steps are defined. For example one can check this for the Kleinian net \( q_{Klein} \).

Note, that \( F_{q_{Klein}} = S_{q_{Klein}} \) is the Klein curve. Thus, \( g_{Klein} \) is the unique theta characteristic on the Klein curve invariant under the whole automorphism group \( G_{168} \), cf. [Bur1911, §232].

**Corollary 6.3.** Over an algebraically closed field the circle of constructions \( F \mapsto (S_{F}, g_{F}), \quad (S, \vartheta) \mapsto q_{S, g_{\vartheta}}, \quad q \mapsto \Lambda^q, \quad \Lambda^q \mapsto (\eta_q; \Lambda^2 V_q \rightarrow N_q), \quad (\eta; \Lambda^2 V \rightarrow N) \mapsto \Lambda V_{g_{\vartheta}}, \quad A \mapsto F_A \) gives the identity transformation on an open set of quartics.

**Corollary 6.4.** Over an algebraically closed field the circle of constructions define birational transformations of the moduli spaces \( M_{Fano} \) of nets of alternating forms (equivalently of prime Fano 3-folds of genus 12 by Mukai’s Theorem), \( M_3 \) of curves of genus 3, \( M_{3, g^{ev}} \) of curves of genus 3 together with a non-vanishing theta characteristic, and \( M_q \) of nets of quadrics.

**Corollary 6.5.** Let \( K \) be an arbitrary field of characteristic 0 and \( X \) a smooth prime Fano 3-fold of genus 12. Then the Grassmannian model \( \mathbb{G}(3, V, \eta) \) and the plane model \( V.SP(F, 6) \) are defined over \( K \).

Proof. By Mukai’s Theorems [Muk92, Muk95] all 3 models are defined over the algebraic closure of \( K \). However the Hilbert scheme \( H_X \) of lines on \( X \) is defined over \( K \), and so is the correspondence \( T_H \) of intersecting lines. Identifying \( T_H = T_F \) and \( S_F = H \) in its canonical embedding, we obtain a quadratic system of equations for the coefficients of the defining equation \( F = \{f = 0\} \) with coefficients in \( K \). So the by Mukai’s result unique solution with a non-degenerate quartic is defined over \( K \). The equivalence of the Grassmannian model and space model is defined over \( K \). So also \( \eta \) is defined over \( K \).
The space model \( H(q) \) is in general not defined over the ground field, due to the descent from \( T_\vartheta \) to \( \vartheta \). For the real numbers we note:

**Remark 6.6.** 1) Let \( X \cong VSP(F,6) \) be a prime Fano 3-fold of genus 12 defined over \( k \) with smooth covariant quartic \( S_F \). If the covariant quartic \( S_F \) contains a \( k \)-rational point, then the space model \( H(q) \) is defined over \( k \). Indeed if \( a \in S_F \) is defined over \( k \), then the fiber \( T_F(a) \) and line bundle \( \vartheta = O(T_F(a) - a) \) are defined over \( k \).

2) The quadrics

\[
q_0 = w_0^2 + w_1^2 - w_2^2 - w_3^2, \quad q_1 = w_0w_2 + w_1w_3,
\]

\[
q_2 = (w_1 + w_2 + w_3)^2 + (w_0 + w_1 - w_3)^2 - (w_0 + w_1 + w_3)^2 - (w_0 + w_1 - w_2)^2
\]

span a net \( q \), whose discriminant \( S_q \) contains no real point. Indeed the catalecticant \( \text{Cat}(S_q) \) is positive definite. So the existence of a point is sufficient but not necessary for the existence of the space model, even for the ground field \( \mathbb{R} \).

**Example 6.7.** For the Mukai-Umemura quartic cf. [MU83], there are 2 different plane models over \( \mathbb{R} \),

\[
F_{MU} = \{(x^2 + y^2 + z^2)^2 = 0\} \quad \text{and} \quad F_{MU}' = \{(x^2 + y^2 - z^2)^2 = 0\}.
\]

Their covariant quartics are equal to themselves.

For the space model \( H(q_{MU}) \) there is only one version. The net of quadrics \( q_{MU} \) is the ideal of the twisted cubic, which, if defined over \( \mathbb{R} \), is isomorphic to \( \mathbb{P}^1_\mathbb{R} \). Its discriminant is the indefinite \( F_{MU}' \). Thus for \( VSP(F_{MU},6) \) there is no \( \mathbb{R} \)-isomorphic space model.

Since the image of \( \{ q = (q_0, q_1, q_2) \} \) over \( \mathbb{R} \) \( \rightarrow \{ (\eta; \Lambda^2 V \rightarrow \mathbb{R}^3) \} \) over \( \mathbb{R} \) \( \rightarrow \{ F \subset \mathbb{P}^2 \text{ over } \mathbb{R} \} \) is closed (in a neighborhood of \( F_{MU} \)) and \( F_{MU} \) is not in the image, we obtain that for any quartic \( F \) over \( \mathbb{R} \) nearby \( F_{MU} \), the Fano 3-fold \( VSP(F,6) \) has no \( \mathbb{R} \)-isomorphic space model.

**References**

[ACGHR85] Arbarello, E., Cornalba M., Griffiths P. A., Harris J., *Geometry of algebraic curves, Vol. I*, Grundlehren der Mathematischen Wissenschaften, 267, Springer-Verlag, New York-Berlin 1985

[BE77] Buchsbaum, D., Eisenbud, D., *Algebraic structures for finite free resolutions and some structure theorems for ideals of codimension 3*, Amer. J. Math 99 (1977), 447-485

[Bur1911] Burnside, W., *Theory of Groups of Finite Order*, Cambridge Univ. Press, Cambridge 1911; reprint by Dover New York

[Cle1861] Clebsch, A., *¨Uber Curven vierter Ordnung*, J. Reine Angew. Math. 59 (1861), 125-145

[DK93] Dolgachev, I, Kaney V., *Polar covariants of plane cubics and quartics*, Adv. Math. 98 (1993), 216-301

[Eis95] Eisenbud, D., *Commutative algebra with a view toward algebraic geometry*, GTM 150, Springer-Verlag, New York, 1995

[EPS87] Ellingsrud, G., Piene, R., Stromme, S.-A., *On the variety of nets of quadrics defining twisted cubics*, in *Space Curves*, Proceedings, Rocca di Papa 1985, SLN 1266 (1987), 84-96

[Fan1937] Fano, G., *Sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche*, Mem. R. Acc. d’Italia 8 (1937) 23-64

[Har77] Harthshorne, R., *Algebraic Geometry*, GTM 52, Springer-Verlag, New York, Heidelberg, Berlin, 1977

[Iar84] Iarrobino, A., *Compressed algebras: Artin algebras having socle degree and maximal length*, Trans. Amer. Math. Soc. 285 (1984), 337-378

[Iar94] Iarrobino, A., *Associated Graded Algebra of a Gorenstein Artin Algebra*, Memoirs of AMS, 514, (1994)
[Iar95] Iarrobino, A., Inverse systems of a symbolic power. II: The Waring problem for forms, J. of Algebra, 174 (1995), 1091-1110

[IK96] Iarrobino, A., Kanev, V., The length of a homogeneous form, determinental loci of catalecticants and Gorenstein algebras, manuscript May 1996

[Mac1916] Macaulay, F.S., Algebraic theory of modular systems, Cambridge University Press, London, (1916)

[MAC] Bayer, D., Stillman, M., MACAULAY: A system for computation in algebraic geometry and commutative algebra, Source and object code available for Unix and Macintosh computers. Contact the authors, or download from zariski.harvard.edu via anonymous ftp.

[Muk92] Mukai, S., Fano 3-folds, Complex Projective Geometry, London Math. Soc. L. N. S. 179, Cambridge University Press (1992), 255-263

[Muk95] Mukai, S., Curves and Symmetric Spaces, I, Americal J. Math. 117,(1995) 1627-1644

[MU83] Mukai, S., Umemura, H., Minimal rational 3-folds, in Algebraic Geometry, Proceedings Tokyo, Kyoto 1982, SLN 1016 (1983), 490-518

[OSS80] Okonek, C., Schneider, M., Spindler, H., Vector Bundles on Complex Projective Spaces, Birhäuser Boston 1980

[PS85] Piene, R., Schlessinger, M., On the Hilbert scheme compactification of the space of twisted cubics, Amer. J. Math. 107 (1985), 761-774

[Ros1873] Rosanes, J., Über ein Prinzip der Zuordnung algebraischer Formen, J. über r. u. angew. Math. 76 (1873), 312-330

[Sal1885] Salmon, G., Modern Higher Algebra, 4. Edition. Hodges, Figgis, and Co., Dublin (1885)

[Sch91] Schreyer, F.-O., A standard basis approach to syzygies of canonical curves, J. reine angew. Math. 421, (1991), 81-123

[Sco1899a] Scorza, G., Sopra la teoria delle figure polari delle curve piane del 4. ordine, Ann. di Mat. (3) 2 (1899), 155-202

[Sco1899b] Scorza, G., Un nuovo teorema sopra le quartiche piane generali, Math., Ann. 52 (1899), 457-461

[Syl1851] Sylvester, J.J., Sketch of a memoir on elimination, transformation and canonical forms, Collected Works, Vol. I, Cambridge University Press, (1904), 184-197

[Syl1851] Sylvester, J.J., An essay on canonical forms, supplemented by a sketch of a memoir on elimination, transformation and canonical forms, Collected Works, Vol. I, Cambridge University Press, (1904), 203-216

[Syl1886] Sylvester, J.J., Sur une extension d’un théorème de Clebsch relatif aux courbes de quatrième degré, Compte Rendus de l’Acad. de Science 102 (1886), 1532-1534 (Collected Math. Works IV, 527-530)

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