Methods of modeling and probabilistic analysis of large deviations of dynamic systems

A A Kabanov¹, S A Dubovik¹

¹ Sevastopol State University, 33, Universitetskaya str., Sevastopol, 299053, Russia

E-mail: KabanovAleksey@gmail.com

Abstract. The paper presents a new method of analyzing large deviations for nonlinear systems defined through matrices with state-dependent coefficients. Large deviations of the controlled process from some standard state is the basis to forecast any critical situation. The task of forecasting is limited to the optimal control problem of Lagrange-Pontryagin optimal control. Two effective methods – State-Dependent Riccati Equations (SDRE) and approximated sequence of Riccati equations (ASRE) – are used. The presented approach to the Lagrange-Pontryagin problem differs from the approach previously used for linear cases by the fact that it uses control in the form of a feedback instead of software open-loop control. This eliminates the need to calculate the end-time boundary value for a conjugate variable, which is the most time-consuming task in nonlinear cases.

1. Introduction

The paper deals with the development of methods of mathematical modeling and probabilistic analysis of large deviations of dynamic systems. These tools can be used to forecast the critical states and synthesis of supervisor control systems based on the principles of analysis of large deviations (analysis of critical situations) and development of anti-crisis controls aimed at overcoming and escape from critical (abnormal) situations of the controlled process.

The concept of action functional is used as the main tool of large deviation analysis [1]. The additional advantage of the action functional is that it allows setting and solving the problems of evaluating large deviations as the optimal control problems [2]. The relations between the perturbed dynamic system and the corresponding optimal control problem for the so-called path system (see [3]) means that the paths of this path system with the optimal counter-control are promising as large deviations of the considered perturbed dynamic system. So, the counter-control path system corresponding to the optimal law makes it possible to assess the asymptotics of large deviations thus determining the development profile of a critical situation (A-profile [3]) and the probability of its occurrence.

Due to their inherent complexity the optimal control problems are often solved within simplified schemes where some form of approximation is performed. The approximate methods make it possible to escape the original non-linear Euler-Lagrange equations and, hence, do not require to search for initial Lagrange multiplier values. However, the price of such simplifications is a loss of optimality, since as a result we get non-optimal solutions.

One group of such approximate methods is introduced using State-Dependent Coefficients (SDC) [4-9]. State-Dependent Riccati Equations (SDRE) are probably the best-known example of the approximate method because of their simplicity and efficiency in many applications [4-6]. This method considers the initial non-linear optimal control problem pointwise as a linear-quadratic regulator (LQR)
problem. As a result, the set of LQR problems is solved sequentially at each time point, to which the entire time domain is sampled. This is done through factorization of the system of equations in the form of state-dependent matrices, which are determined pointwise at each time step. SDRE makes it possible to form the feedback control, the control law being a function of the current state. The nonstationarity of control for problems at a finite time interval leads to the following issue: the exact solution of the problem is impossible without the knowledge of future states of the system, which are currently unknown. The [6] proposes an approximate algorithm using the hypothesis of weak change of the state vector (hypothesis of “frozen” coefficients).

Another approach for synthesizing non-linear optimal control over a finite time interval is presented by the method of approximated sequence of Riccati equations (ASRE) [7-9]. This method also requires qualitative representation of dynamics and criteria in the appropriate form with matrices, which coefficients depend on the state, and leads to iterative solution of a number of LQR problems. For each current LQR, the SDC matrix problems are evaluated using the solution on the previous iteration.

The differences between SDRE and ASRE methods are as follows. SDRE method solves the sequence of pointwise LQR problems at specific time intervals \( t_k, t_{k+1}, \ldots \). ASRE method performs several iterations of solutions defined throughout the time domain \([t_0, t_f]\). The efficiency of both methods is high. But in different problems, one or the other method has its advantages. Thus, in [4], the Van der Pol oscillator shows that for a problem at an infinite time interval, the SDRE method has the advantage when increasing the initial conditions of the system. It provides optimal state feedback for the considered problem. However, for example, in [7], for a problem at a finite time interval, ASRE is more advantageous than SDRE in terms of solution optimality. These results confirm the efficiency of both methods and the relevance to study their development and practical application.

In this work, the approximate SDC methods of solving the corresponding optimal control problems are used for modeling and analysis of large deviations of perturbed non-linear dynamic systems. The feature of setting the problem of analyzing large deviations as a problem of accurate terminal control at a final time interval, in the absence of a penalty for the state of the system in action functional, allows obtaining the analytical solution of the problem, which is especially relevant for implementation in real time.

The rest of the paper is as follows. Section 2 shows the problem statement. The SDRE and ASRE methods solve the problem in sections 3 and 4. The results are discussed and the conclusions are given in sections 5, 6.

2. Problem statement
A closed nonlinear dynamic system is considered:

\[
\frac{d}{dt} \mathbf{x} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0,
\]

(1)
disturbed by small noise

\[
\frac{d}{dt} \mathbf{x} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{\sigma}(\mathbf{x})\mathbf{w}, \quad \mathbf{x}(t_0) = \mathbf{x}_0,
\]

(2)

where, \( \mathbf{x} \in \mathbb{R}^n \), \( \mathbf{w} \) – r-vector of “white noise”, \( \varepsilon > 0 \) – small parameter, \( \mathbf{f}(\mathbf{x}), \mathbf{\sigma}(\mathbf{x}) \) – smooth matrix functions, and \( \mathbf{f}(0) = 0, \forall t \in \mathbb{R} \).

At \( \varepsilon \to 0 \) with a probability tending to 1, the paths of the perturbed system approach the paths of the undisturbed system at any finite time interval. The paths of the system (2), which are far from the paths (1), are of interest. These are events, which probabilities are close to zero, but which can be distinguished by those that are overwhelmingly more likely than others. The estimation of probabilistic quantities associated with these rare events, which are called large deviations, includes a variational formulation closely related to the dynamics of the perturbed system.

The complexity of movement along a given path \( \mathbf{\Phi} \) is measured by the cost of control actions
\( v \in \mathbb{R}^r \) (counter-controls) required to control the state of the system when moving along that path \( \bar{\varphi} : \)

\[
\frac{d}{dt} \varphi = f(\varphi) + \sigma(\varphi)v, \quad \varphi(t_0) = x_0, \tag{3}
\]

where counter-control costs \( v \) are measured by the functional:

\[
S_{t_0}^{t_f}(v) = \frac{1}{2} \int_{t_0}^{t_f} v^T v \, dt. \tag{4}
\]

This link between the controlled system of paths (3), (4) and the perturbed system (2) means that the available paths (3), (4) are promising as large deviations of the perturbed system (2) (i.e. those, to which the system (3) can be directed using counter-control \( v \) corresponding to constraints (4)).

Thus, the global properties of the system (2) are described by means of a system of paths (3), taking into account, first of all, the possibility of deviations of the state of the system (2) from zero in the direction of the boundary \( \partial D \) of the operational area \( D \subset O_x \) (\( D \) – open set, \( O_x \) – domain of attraction, \( X \) – state of stable equilibrium (attractor) of the undisturbed system (1)).

For set \( D \) and system (3) the following equality [2] is true:

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \ln P\{x \in \mathbb{R}^n / D\} = - \min_{\varphi \in D} S_{t_0}^{t_f}(\varphi, v), \tag{5}
\]

where the functional \( S_{t_0}^{t_f}(\varphi, v) \) is determined in accordance with (4) for the solutions of the controlled system (3), for which another boundary condition of reaching the critical state by the moment of time \( t_f \) is indicated:

\[
y(t_f) = C \varphi(t_f) \in \partial X, \tag{6}
\]

where \( C \) – full rank matrix.

The probability in (5) can be estimated by solving the optimal control problem (Lagrange-Pontryagin problem): for solutions of the system of paths (3) to minimize the action functional (4) under the boundary condition (6).

The task is to solve this Lagrange-Pontryagin problem using approximate numerical-analytical methods – SDRE and ASRE.
3. Solution of Lagrange-Pontryagin problem by SDRE method

Let us present the path system (3) in SDC form:

\[
\frac{d}{dt} \phi = A(\phi)\phi + B(\phi)v, \quad \phi(t_0) = \phi_o, \quad A(\phi)\phi = f(\phi), \quad B(\phi) = \sigma(\phi),
\]

and formulate the problem of minimizing the criterion (4) within constraints (6).

The solution of problem (7), (4), (6) in the form of feedback obtained using SDRE method and “freezing” coefficients looks as follows [10]:

\[
v = -B^T(\phi)W^T C^T M^{-1}(CW\phi - y(t_f)),
\]

where

\[
\frac{d}{dt} W = -WA(\phi), \quad W(t_f) = I,
\]

\[
\frac{d}{dt} M = -CB(\phi)B^T(\phi)W^T C^T, \quad M(t_f) = 0.
\]

From where matrices \( W(t) \) and \( M(t) \) we get

\[
W(t) = -\int_{t_f}^t WA(\phi)d\tau, \quad W(t_f) = I, \quad M(t) = -C\int_{t_f}^t WB(\phi)B^T(\phi)W d\tau C^T, \quad M(t_f) = 0. \tag{9}
\]

Following the hypothesis of “frozen” coefficients [6], for each \( \phi(t) \) we denote \( A_s \equiv A(\phi), B_s \equiv B(\phi) \), then from (9) we get

\[
W(t) = e^{A_s(t_f-t)}, \quad M(t) = C(W(t))D(W^T(t) - D)C^T,
\]

where the matrix \( D \) is the solution of the algebraic Lyapunov equation (shall be solved pointwise \( \forall \phi(t) \) at a considered time interval)

\[
A_s D + DA_s^T - B_s B_s^T = 0. \tag{10}
\]

The integral for \( M(t) \) can be presented as (9) based on the following equality

\[
-\int_{t_f}^t WB(\phi)B^T(\phi)W d\tau = \int_{t_f}^t \frac{d}{d\tau} WD W^T d\tau = W(t)D(W^T(t) - D).
\]

The last expression can be checked by taking the derivative

\[
\frac{d}{dt} WD W^T = \frac{d}{dt} W \cdot D W^T + WD \cdot \frac{d}{dt} W^T = W(-WA(\phi) - A^T(\phi)W^T)W^T.
\]

The minimum value of functional (4) – normalized action functional:

\[
S_{\phi(t_f, t)}(\phi, v) = \frac{1}{2} \int_{t_0}^{t_f} (CW\phi - y(t_f))^T \left( \frac{d}{dt} M^{-1} \right) (CW\phi - y(t_f)) dt = \left. (CW\phi - y(t_f))^T M^{-1}(CW\phi - y(t_f)) \right|_{t=t_0}.
\]

Solving this particular problem by the SDRE method gives analytical expressions for countercontrol, for the critical situation the development profile (A-profile [3]) and for the minimum value of the quality criterion, and therefore for probability estimation. Besides, the present solution has the form
of feedback. The condition for this solution is the pointwise controllability of the pair \((A(\phi), B(\phi))\) and the Hurwitz matrix \(A(\phi) \forall \phi(t), t \in [t_0, t_f]\).

4. Solution of Lagrange-Pontryagin problem by ASRE method

The solution of problem (7), (4), (6) by ASRE method includes several stages. The initial step is to solve the problem 0 defined as:

\[
\frac{d\phi^{(0)}}{dt} = A^{(0)}\phi^{(0)} + B^{(0)}v^{(0)}, \quad y(t_f) = C\phi^{(0)}(t_f) \in \partial K,
\]

\[
S_{t_0 t_f}(v) = \frac{1}{2} \int_{t_0}^{t_f} (v^{(0)})^Tv^{(0)} dt \rightarrow \min_v.
\]

where \(A^{(0)} = A(\phi_0), B^{(0)} = B(\phi_0)\).

The problem (12) is a standard LQR problem since all matrix arguments are given and are constant.

The solution to this problem defines \(\phi^{(0)}(t)\) and \(v^{(0)}(t), t \in [t_0, t_f]\):

\[
\frac{d\phi^{(0)}}{dt} = A^{(0)}_{cl}\phi^{(0)} + G^{(0)}y(t_f), \quad \phi^{(0)}(t_0) = \phi_0,
\]

\[
A^{(0)}_{cl} = A^{(0)} - G^{(0)}C^T M^{(0)-1} CW^{(0)}, \quad G^{(0)} = B^{(0)}B^{(0)T} W^{(0)} C^T M^{(0)-1}.
\]

\[
v^{(0)} = -B^{(0)}T W^{(0)} C^T M^{(0)-1} (CW^{(0)}\phi^{(0)} - y(t_f)).
\]

The solution to equation (13) is as follows

\[
\phi^{(0)}(t) = e^{A^{(0)}_{cl} t} \phi^{(0)} + \int_{t_0}^{t} e^{A^{(0)}_{cl} (t-t')} G^{(0)} y(t_f) dt'.
\]

In general, for some iteration \(k\), the problem is formulated as follows

\[
\frac{d\phi^{(k)}}{dt} = A^{(k)}(t)\phi^{(k)} + B^{(k)}(t)v^{(k)}, \quad S_{t_0 t_f}(v) = \frac{1}{2} \int_{t_0}^{t_f} (v^{(k)})^Tv^{(k)} dt,
\]

where \(A^{(k)}(t) = A(\phi^{(k-1)}(t)), B^{(k)}(t) = B(\phi^{(k-1)}(t))\).

In fact, (14) is a non-stationary LQR problem (note that \(\phi^{(k-1)}(t)\) and \(v^{(k-1)}(t)\) are solutions to the problem at step \(k - 1\)). Its solution has the form (8), (9). In solving the problem \(k\), we will get \(\phi^{(k)}(t)\) and \(v^{(k)}(t), t \in [t_0, t_f]\).

Iterations continue until the convergence condition is met. Convergence is achieved by setting the required error rate between the sequence of solutions

\[
\|\phi^{(k)}(t) - \phi^{(k-1)}(t)\| \leq \mu,
\]

where \(\mu > 0\) – some constant. In [7] it is proved that the sequence of solutions \(\phi^{(k)}(t), v^{(k)}(t)\) converges to the solution of the original problem (7), (4), (6) provided that \(A(\phi)\) and \(B(\phi)\) are continuous in terms of their arguments according to Lipschitz.

5. Discussion and future research

An important aspect of the SDC methods is the need to provide a certain representation for dynamics in
the form of a system with state-dependent matrices. This task is referred to as parameterization or factorization of the system (see, for example, [8, 9]). Indeed, although different SDC matrices could be expected to produce the same result, this is not the case in practice. Besides, the use of different representations for SDC matrices gives different results in terms of system path and quality functional.

The property that SDRE and ASRE methods require is the controllability of the pair \((A(\phi), B(\phi))\) in (7). Hence, SDC matrices shall be selected so that the system is controlled for all possible paths. It is desirable to choose factorization, which provides maximum controllability, because the performance of the solution is strictly related to the controllability of the system. Increased controllability results in reduced control effort and probably lower costs. Considering the non-linear dynamics, the choice of a state-dependent matrix \(A(x)\) is a degree of freedom that can be used to provide the best characteristics in terms of algorithm convergence and control force. The effect of factorization methods on the results of large deviation analysis is the subject of further research.

6. Conclusion
The paper proposes the method of analysis of large deviations based on Lagrange-Pontryagin optimal control problem for non-linear systems based on the SDC approach. Two of the most common and effective methods – SDRE and ASRE – are used. The proposed method of analysis of large deviations enables to obtain a solution in the form of feedback on the state, which simplifies the solution of Lagrange-Pontryagin problem, since it eliminates the need to calculate the boundary values for conjugated variables. It is assumed that this feature can be used in solving the problems of long-term forecast of critical situations based on the analysis of large deviations.

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