Quantum implications of a scale invariant regularisation

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Abstract

We study scale invariance at the quantum level in a perturbative approach. For a scale-invariant classical theory the scalar potential is computed at three-loop level while keeping manifest this symmetry. Spontaneous scale symmetry breaking is transmitted at quantum level to the visible sector (of $\phi$) by the associated Goldstone mode (dilaton $\sigma$) which enables a scale-invariant regularisation and whose vev $\langle \sigma \rangle$ generates the subtraction scale ($\mu$). While the hidden ($\sigma$) and visible sector ($\phi$) are classically decoupled in $d = 4$ due to an enhanced Poincaré symmetry, they interact through (a series of) evanescent couplings $\propto \epsilon^k$, ($k \geq 1$), dictated by the scale invariance of the action in $d = 4 - 2\epsilon$. At the quantum level these couplings generate new corrections to the potential, such as scale-invariant non-polynomial effective operators $\phi^{2n+4}/\sigma^{2n}$ and also log-like terms ($\propto \ln^k \sigma$) restoring the scale-invariance of known quantum corrections. The former are comparable in size to “standard” loop corrections and important for values of $\phi$ close to $\langle \sigma \rangle$. For $n = 1, 2$ the beta functions of their coefficient are computed at three-loops. In the infrared (IR) limit the dilaton fluctuations decouple, the effective operators are suppressed by large $\langle \sigma \rangle$ and the effective potential becomes that of a renormalizable theory with explicit scale symmetry breaking by the “usual” DR scheme (of $\mu =$constant).

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1 Introduction

It is a common view that Standard Model (SM) is only a low-energy effective theory and “new physics” could arise at some scale below $M_{\text{Planck}}$. The scale of “new physics” can be the vacuum expectation value (vev) of a scalar field $\sigma$ present beyond the SM spectrum. It is then natural to ask how the higgs mass is protected from large quantum corrections associated with $\langle \sigma \rangle$. One long-held answer is TeV-supersymmetry. Scale invariance may also protect the higgs mass against large quantum corrections. This starts from the observation that for a vanishing higgs mass parameter, SM has an increased symmetry: it is scale invariant. This means that the classical action is invariant under a transformation: $x \rightarrow \rho^{-1} x$, $\phi \rightarrow \rho^{d_\phi} \phi$ ($d_\phi$ is the mass dimension of $\phi$). Scale symmetry was noticed to play a role in protecting the electroweak scale [3–6] with classically scale-invariant extensions of the SM considered in [7–26]. But to address the mass hierarchy problem one must go beyond the classical scale symmetry, since the counterterms are actually dictated by the quantum symmetry. This could protect naturally [27] the higgs mass from large quantum corrections [28] associated with a high scale $\langle \sigma \rangle$ of “new physics”. For studies of quantum scale invariance (broken spontaneously) and applications to SM see [29–38].

Our goal is to study further the models in which the classical scale symmetry is extended at the quantum level and is broken only spontaneously. In such theory all scales are generated by fields’ vev’s. Such theory can predict ratios of scales (vev’s) only, in terms of ratios of dimensionless couplings. A hierarchy of physical mass scales can then be generated by a hierarchy of such couplings. The latter is easier to protect by a symmetry (e.g. an enhanced Poincaré symmetry [39]) and is more fundamental than a hierarchy of dimensionful physical scales. Indeed, in a fundamental theory, any physical scale should ultimately be determined in terms of dimensionless couplings and fields vev’s.

Since scale symmetry is broken in the real world, we assume it is broken spontaneously. A flat direction exists and the spectrum contains the associated Goldstone boson (dilaton, hereafter $\sigma$) beyond the spectrum of the initial model. The subtraction scale $\mu$ (used in loop calculations) that would break quantum scale symmetry explicitly, is replaced by the field $\sigma$ which thus maintains scale symmetry at quantum level and after spontaneous breaking generates $\mu \sim \langle \sigma \rangle$, see Englert et al. [29]. This gives a scale-invariant regularisation (SR).

In this paper we discuss further consequences of the original idea of Englert et al. [29]. The SR scheme can be applied to any gauge theory, although we restrict our study to a scalar theory. We study more quantum effects in this scheme and stress the role of symmetries. In $d=4$ the hidden sector (of the dilaton $\sigma$) is classically decoupled from the visible sector (of higgs-like $\phi$), by invoking an enhanced Poincaré symmetry $P_v \times P_h$ of these two sectors [39].

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1 In the absence of “new physics” below $M_{\text{Planck}}$ and ignoring gravity, SM has no hierarchy problem.

2 TeV-SUSY models have large fine-tuning [1] which cannot coexist with a good data fit $\chi^2$/dof $\approx 1$ [2].

3 Classical scale symmetry is often broken by quantum calculations since the UV regulator breaks it explicitly. The classical flat direction is then lifted and a light pseudo-Goldstone boson exists. See for example [5]. We do not follow this approach and implement instead a quantum scale symmetry.

4 Versions of this scheme were used in [30–37] (in some cases classical higgs-dilaton mixing was present which concealed the enhanced Poincaré symmetry and the effects discussed below).
At the quantum level, the manifest scale symmetry of the action in $d = 4 - 2\epsilon$ introduces evanescent couplings $\propto \epsilon \tilde{\sigma}/\langle \sigma \rangle$ of the hidden to the visible sector ($\tilde{\sigma}$: dilaton fluctuations). The SR scheme is thus reformulated as an “ordinary” DR scheme of $\mu=$constant ($\propto \langle \sigma \rangle$) plus an additional field $\langle \sigma \rangle$ with an infinite series of evanescent couplings to the visible sector.

At the quantum level, such evanescent couplings have physical effects. When these couplings multiply poles of momentum integrals, they generate new (finite or infinite) counterterms, all scale invariant. For example one finds non-polynomial operators generated radiatively, such as $\phi^{2n+4}/\sigma^{2n}$, $n \geq 1$ (but also higher derivative operators suppressed by $\sigma$). They can transmit scale symmetry breaking to the visible sector. Such operators can be understood via their Taylor expansion about $\sigma = \langle \sigma \rangle + \tilde{\sigma}$, when they become polynomial. Scale symmetry acts at the quantum level as an organizing principle that re-sums the polynomial ones. We shall study closer these operators, since they are important at large $\phi$. Because of their presence, the quantum scale invariant theory is non-renormalizable.

We compute in a manifest scale invariant way the quantum corrections to the scalar potential in two-loop order (diagrammatically) and three-loop (via Callan-Symanzik equation), for a scale-invariant classical theory. The two-loop (three-loop) potential contains effective operators as finite (infinite) counterterms, respectively. In the infrared (IR) decoupling limit of the dilaton (large $\langle \sigma \rangle$) effective operators vanish; one then recovers the effective potential and trace anomaly of a renormalizable theory (if classical theory was so) with only classical scale invariance and explicit scale symmetry breaking (SSB) by the “usual” DR scheme of $\mu=$constant (no dilaton). The combined role of quantum scale invariance and enhanced Poincaré symmetry in protecting the scalar mass at large $\langle \sigma \rangle$ is also reviewed.

Since $M_{\text{Planck}}$ breaks scale symmetry, this analysis is valid for field values well below this scale. One should extend this study to a Brans-Dicke-Jordan theory of gravity with non-minimal coupling where the dilaton vev $\langle \sigma \rangle$ fixes spontaneously $M_{\text{Planck}}$. We restrict the analysis to a perturbative (quantum) scale symmetry. At very high momentum scales some couplings (e.g. hypercharge) may become non-perturbative, but such scale is above $M_{\text{Planck}}$, where flat space-time description used here fails anyway.

2 From classical to quantum scale invariance

2.1 Implementing quantum scale invariance

Consider a classical scale invariant action, e.g. a toy model or the SM with vanishing higgs mass parameter, extended by the dilaton $\sigma$. We assume that there is no classical interaction between the visible sector (of fields $\phi_j$) and the hidden sector (of dilaton $\sigma$). Then

$$S = \int d^4x L_v(\phi_j, \partial \phi_j) + \int d^4y L_h(\sigma, \partial \sigma)$$  (1)

\footnote{By evanescent coupling we understand a coupling that is non-zero in $d = 4 - 2\epsilon$ and is vanishing in $d = 4$.}
The action in $d = 4$ has an enhanced Poincaré symmetry $(P_v \times P_h)$ associated with both sectors, which forbids a classical coupling $\lambda \phi^2 \sigma^2$. Such coupling can be naturally set to $\lambda_m = 0$ and remains so at the quantum level 

Below we work with the canonical dilaton $\sigma$ related to the actual Goldstone by $\sigma = \langle \sigma \rangle e^\tau$, so that it transforms in a “standard” way under scaling while $\tau$ transforms with a shift

$$x \to \rho^{-1} x, \quad \sigma \to \rho \sigma, \quad \tau \to \tau + \ln \rho$$

The most general potential for $\sigma$ allowed by scale invariance in $d = 4$ is then $\kappa_0 e^{4\tau} \sim \lambda \sigma \sigma^4$. But Poincaré symmetry in the dilaton sector demands a flat potential, so $\lambda \sigma = 0$.

Demanding spontaneous scale symmetry breaking $\langle \sigma \rangle \neq 0$ means “we live” along a flat direction. This is in the end a tuning of the cosmological constant and is present anyway in e.g. TeV supersymmetry. The details of how $\sigma$ acquires a vev are not relevant below.

At the quantum level it is natural to use the dilaton to generate dynamically the subtraction scale $\propto \langle \sigma \rangle$ in order to preserve scale symmetry during quantum calculations. We use DR in $d = 4 - 2\epsilon$, then the only possibility dictated by dimensional arguments is

$$\mu = z \sigma^{2/(d-2)}$$

with $z$ is an arbitrary dimensionless parameter (scaling factor); it keeps track of the vev of $\sigma$ after SSB. The $d = 4$ potential $V(\phi_j)$ of the visible sector is then analytically continued to $d = 4 - 2\epsilon$, into $\mu^2 V(\phi_j)$. Therefore the potential in $d = 4 - 2\epsilon$ is actually

$$\tilde{V}(\phi_j, \sigma) = \left[ z \sigma^{2/(d-2)} \right]^{4-d} V(\phi_j),$$

and becomes a function of $\sigma$! This ensures the $d = 4$ couplings remain dimensionless in $d = 4 - 2\epsilon$ and can be used for perturbative calculations. Therefore, the visible ($\phi_j$) and hidden ($\sigma$) sectors have evanescent couplings dictated by the scale symmetry alone of the (regularized) action in $d = 4 - 2\epsilon$. To see these couplings expand in powers of $\epsilon$ (loops) and then in terms of fluctuations $\tilde{\sigma}$ about the vev $\langle \sigma \rangle$ of $\sigma$:

$$\tilde{V}(\phi_j, \sigma) = \mu_0^3 \left[ 1 + 2\epsilon \left( \frac{\eta - \frac{1}{2} \eta^2 + \frac{1}{3} \eta^3 + \mathcal{O}(\eta^4)}{} \right) + \epsilon^2 \left( 2\eta + \eta^2 - \frac{4}{3} \eta^3 + \mathcal{O}(\eta^4) \right) + \mathcal{O}(\epsilon^3) \right] V(\phi_j)$$

where

$$\mu_0 = z \langle \sigma \rangle^{1-\epsilon}, \quad \sigma = \langle \sigma \rangle + \tilde{\sigma}, \quad \eta = \frac{\tilde{\sigma}}{\langle \sigma \rangle}.$$

$\text{Technically } \beta_{\lambda_m} \propto \lambda_m \text{ at two-loop}$.

$\mu$ has mass dimension one, while $\sigma$ and $\langle \sigma \rangle$ have dimension $(d - 2)/2$. 

3
A scale invariant regularization is then re-expressed as an ordinary DR scheme with $\mu = \mu_0$ plus an extra field ($\sigma$) with (infinitely many) evanescent couplings eq.(5). Since the lhs is scale invariant, so is the rhs if one does not truncate the expansion in field fluctuations. In practice one can still use a truncated expansion (see below). From eq.(4) one can read the new vertices of evanescent interactions $\propto \epsilon^n (n \geq 1)$, between $\tilde{\sigma}$ and $\phi_j$ and the Feynman rules of the scale invariant quantum theory. While these interactions vanish in $d=4$ or in the dilaton decoupling limit ($\eta \to 0$), at the loop level have physical effects.

At quantum level, a coupling proportional to $\epsilon^n, (n \geq 1)$ in an amplitude can bring new corrections to it when multiplying the poles $1/\epsilon^k$ of the integrals over loop momenta. One generates finite quantum corrections (if $n = k$) or new poles/counterterms ($n < k$) beyond those of the theory with $\mu =$constant. If $n = k$, a scattering amplitude that involves the dilaton depends only on the couplings of initial $d=4$ theory, without new parameters needed (counterterm couplings). This can be used to set lower bounds on the scale $\langle \sigma \rangle$.

Since the new couplings are suppressed, $\eta \sim 1/\langle \sigma \rangle$, the counterterms are higher dimensional. They must however respect the scale symmetry of the lhs of eq.(5); one can then “restore” this symmetry “broken” by working with the truncation of the rhs expression, by simply replacing $1/\langle \sigma \rangle \to 1/\sigma$ in their expression. Therefore, the new counterterms of the theory are suppressed by powers of $\sigma$ and are non-polynomial in fields; log-terms in $\sigma$ are also possible, however (see later).

For example, for $V(\phi) = \lambda \phi^4/4!$, a first counterterm is found by inserting a single internal line of $\tilde{\sigma}$ in an amplitude, which brings a factor $(\epsilon/\langle \sigma \rangle)^2$; if this multiplies a $1/\epsilon^3$ pole from a three-loop momentum integral it generates a $1/\epsilon$ pole and a corresponding counterterm $\sigma^6/\sigma^2$ for the 6-point amplitude $(\phi^6)$ [37]. By the same argument, finite quantum corrections appear at two-loops (if due to dynamics of $\sigma$) or even one-loop (due to scale symmetry alone).

Since the theory is scale invariant and so it has no dimensionful couplings, diagrams that would otherwise be proportional to masses automatically vanish. Then the only possibility to construct scale invariant $d = 4$ counterterms that are suppressed by powers of $\sigma$ is to involve appropriate powers $\phi^n, n > 4$ and higher derivatives of $\phi$ and $\tilde{\sigma}$. Therefore the new counterterms are found on dimensional grounds as

$$\sum_{n,m \geq 0} a_{mn} \frac{\partial^{2n} \alpha^{m+4}}{\sigma^{2n+m}}, \quad \alpha = \phi, \sigma.$$  

where the derivatives act in all possible ways in the numerator. This includes the dilaton-dilaton scattering ($\partial^2 \sigma^4/\sigma^4$ (see a-theorem [11]) which emerges at three-loops.

We see that quantum scale-invariant theories are non-renormalizable [37], unlike their counterpart with $\mu =$constant which is not quantum scale invariant but is renormalizable (if initial $d = 4$ action was so). The latter case is recovered in the limit of large $\langle \sigma \rangle$, when fluctuations $\tilde{\sigma}$ decouple, see eq.(5). This picture also applies to gauge theories.
2.2 One-loop potential

Let us first review the quantum corrections to the potential in a scale invariant toy model at one-loop, before going to higher loops. Consider $L$ below in $d = 4$ for a scalar $\phi$

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - V(\phi), \quad V(\phi) = \frac{\lambda}{4!} \phi^4. \quad (8)$$

In $d = 4 - 2\epsilon$ the potential becomes $\tilde{V}(\phi, \sigma)$ of eq.(4) with $V$ as above, so $\phi$ and $\sigma$ do interact as dictated by scale symmetry of analytically continued $L$. The one-loop potential is then

$$V_1 = \tilde{V} - \frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \text{Tr} \ln \left(p^2 - \tilde{M}_\alpha^2 + i\epsilon \right)$$

$$= \tilde{V} + \frac{1}{4\kappa} \sum_{s=\phi,\sigma} \tilde{M}_s^4 \left(\frac{-1}{\epsilon} + \ln \frac{\tilde{M}_s^2}{c_0}\right), \quad \kappa = (4\pi)^2. \quad (9)$$

where $c_0 = 4\pi e^{\frac{3}{2} - \gamma_E}$. $\tilde{M}_s^2$ are field-dependent (masses)$^2$, eigenvalues of the second derivatives matrix $\tilde{V}_{\alpha\beta}$, with $\alpha, \beta = \phi, \sigma$. One eigenvalue $\tilde{M}_s^4 \propto \epsilon^2$, thus it cannot generate counterterms at one-loop. Then

$$V_1 = \tilde{V} + \mu^2 \frac{V_{\phi\phi}}{4\kappa} \left\{ -\frac{1}{\epsilon} + \left(\ln V_{\phi\phi}(z\sigma)^2 - \frac{1}{2}\right) \right\}, \quad \text{with} \quad V_{\phi\phi} = \frac{1}{2} \lambda \phi^2. \quad (10)$$

with $\ln A \equiv \ln A/(4\pi e^{1-\gamma_E})$. It is important to note that the factor $\mu^2\epsilon$ is a function of $\sigma$ (see eq.(3)) and maintains scale invariance$^9$ in $d = 4 - 2\epsilon$. Here we work in the minimal subtraction scheme (MS). Thus the (scale-invariant) counterterm is

$$\delta L_1 = -\mu^2 \frac{1}{4!} \delta^{(1)}_{\lambda} \lambda \phi^4 \quad \text{with} \quad \delta^{(1)}_{\lambda} \equiv Z_\lambda - 1 = \frac{3\lambda}{2\kappa\epsilon}. \quad (11)$$

Then the one-loop potential in $d = 4$ is

$$U = V(\phi) + \frac{1}{4\kappa} V_{\phi\phi}^2 \left[ \ln V_{\phi\phi}(z\sigma)^2 - \frac{1}{2} \right]; \quad (12)$$

where we took the limit $\epsilon \to 0$. Note that $U$ has acquired a dependence on $\sigma$ at quantum level (under the log) and for this reason its expression is now scale invariant (in $d = 4$).

Since dimensionless $z$ keeps track of the presence of $\langle \sigma \rangle$, the one-loop beta function $\beta^{(1)}_\lambda$ is found by demanding the bare coupling $\lambda^B = \mu(\sigma)^{2\epsilon} \lambda Z_\lambda Z^{-2}$ be independent of scaling parameter $z$:

$$\frac{d \lambda^B}{d \ln z} = 0 \quad \Rightarrow \quad \beta^{(1)}_\lambda = \frac{d \lambda}{d \ln z} = \frac{3}{\kappa} \lambda^2 \quad (13)$$

which is identical to the result for the case $\mu =$constant$^{10}$. The Callan-Symanzik (CS)

$^9$This can also be relevant if one wanted to define and use instead a non-minimal subtraction scheme.

$^{10}$Unlike in theories with no dilaton (with explicit SSB by quantum corrections), $\beta_\lambda = 0$ is not a necessary
equation for a scale-invariant theory [33] is easily verified:

\[
\frac{dU}{d \ln z} = \left( \frac{\partial}{\partial \ln z} + \beta(1) \frac{\partial}{\partial \lambda} \right) U = O(\lambda^3).
\]  (14)

Consider now the limit when the dilaton decouples. For this Taylor expand the potential for \( \sigma = \langle \sigma \rangle + \tilde{\sigma} \) where \( \tilde{\sigma} \) are field fluctuations. The result is

\[
U = V(\phi) + \frac{1}{4\kappa} V_{\phi\phi}^2 \left[ \ln \left( \frac{V_{\phi\phi}}{(z(\sigma))^2} \right) - \frac{1}{2} \right] + \Delta U
\]  (15)

with

\[
\Delta U = \frac{1}{4\kappa} V_{\phi\phi}^2 \left( -\frac{\tilde{\sigma}}{\langle \sigma \rangle} + \frac{1}{2} \frac{\tilde{\sigma}^2}{\langle \sigma \rangle^2} + \cdots \right)
\]  (16)

For \( \tilde{\sigma} \ll \langle \sigma \rangle \), \( \Delta U = 0 \) and we recover the Coleman-Weinberg result in a \( d = 4 \) renormalizable theory obtained in the usual DR scheme of \( \mu = \text{constant} = z(\sigma) \) with explicit SSB (no dilaton). Obviously, the CS equation is still respected. One then proceeds to impose boundary conditions to define the quartic self-coupling at \( \phi = \langle \sigma \rangle \): \( \lambda(\sigma) = \partial^4 U / \partial \phi^4 |_{\phi = \langle \sigma \rangle} \), as usual.

The analysis is very similar if more fields \( \phi_j \) are present, of potential \( V(\phi_j) \). The result is found from eqs. (15), (16) by replacing \( V_{\phi\phi} \) by the eigenvalues of matrix \( V_{ij} = \partial^2 V / \partial \phi_i \partial \phi_j \) and summing over them. Again the dilaton does not contribute counterterms at one-loop, but enforces the scale invariance of \( U \) (via \( \ln \sigma \)). The second term in the CS equation in (14) is now a sum over all quartic couplings in \( V \). Including fermions and gauge bosons is immediate by extending the sum over field dependent masses, with appropriate factors.

### 2.3 Two-loop potential

The two-loop correction to the potential of \( \phi \) can be written as

\[
V_2 = V_2^a + V_2^b + V_2^c
\]  (17)

with the diagrams below computed from the background field method\[11\]

\[
V_2^a = \frac{i}{12}; \quad V_2^b = \frac{i}{8}; \quad V_2^c = \frac{i}{2}
\]  (18)

The vertices and propagators in these diagrams receive evanescent corrections from the dilaton field, as seen from the background field expansion. We Taylor expand

\[
\tilde{V}(\phi + \delta \phi, \sigma + \delta \sigma) = \tilde{V}(\phi, \sigma) + \tilde{V}_\alpha s_\alpha + \frac{1}{2} \tilde{V}_a s_a s_\beta + \frac{1}{3!} \tilde{V}_{\alpha\beta\gamma} s_\alpha s_\beta s_\gamma + \frac{1}{4!} \tilde{V}_{\alpha\beta\gamma\rho} s_\alpha s_\beta s_\gamma s_\rho + \cdots
\]  (19)

condition for having scale symmetry in our case here [32,33] since the spectrum is extended to include a dilaton (spontaneous SSB); thus a non-zero \( \beta_\lambda \) does not mean the theory cannot be scale invariant.

\[11\] We use the approach of [30] but without a classical coupling \( \lambda_m \phi^2 \sigma^2 \).
where \( s_\alpha = \delta \phi, \delta \sigma \) are the actual field fluctuations. The vertices \( \bar{V}_{\alpha \beta \ldots} = \partial^2 \bar{V} / \partial \alpha \partial \beta \ldots \) 
(\( \alpha, \beta, \ldots = \phi, \sigma \)) contain terms proportional to powers of \( \epsilon \), e.g. \( \bar{V}_{\phi \sigma} = \lambda (\phi^2 / \sigma) \epsilon + O(\epsilon^2) \). The propagators, obtained from the inverse of the matrix \( (\mu^2 \delta_{\alpha \beta} - \bar{V}_{\alpha \beta}) \), also acquire \( \epsilon \)-dependent shifts. We retain all these corrections up to and including \( O(\epsilon^2) \); these can multiply the poles of the loop integrals \( 1 / \epsilon^2 \) or \( 1 / \epsilon \) to generate finite quantum corrections\(^{12}\), as discussed in Section 2.1. Here we shall identify these corrections. One finds

\[
V_2 = \mu^{2 \epsilon} \lambda^3 \frac{\phi^4}{32 \kappa^2} \left\{ - \frac{3}{\epsilon^2} + \frac{2}{\epsilon} + O(\epsilon^0) \right\}.
\]

with \( \mu^{2 \epsilon} \) a function of \( \sigma \) which maintains the scale invariance in \( d = 4 - 2 \epsilon \), see eq. (3). The counterterm is scale invariant and in the MS scheme is given by

\[
\delta L_2 = \frac{1}{2} \left( \partial_{\mu} \phi \right)^2 \delta \phi - \mu^{2 \epsilon} \frac{1}{4!} \lambda \phi^4 \delta \lambda
\]

and

\[
\delta \phi^2 = \frac{\lambda^2}{\kappa^2} \left( \frac{9}{4} \epsilon^2 - \frac{3}{2} \epsilon \right), \quad \delta \phi^4 = \frac{-\lambda^2}{24 \kappa^2} \epsilon.
\]

From these and with the coefficients \( Z_\lambda = 1 + \delta \lambda, Z_\phi = 1 + \delta \phi \) and since \( \lambda^B = \mu^{2 \epsilon} \lambda \), \( d\lambda^B / d(ln z) = 0 \), one obtains the two-loop corrected beta function

\[
\beta_\lambda = \frac{3}{\kappa} \lambda^2 - \frac{17}{3 \kappa^2} \lambda^3
\]

\( \beta_\lambda \) is identical to that of the \( \phi^4 \) theory with \( \mu = \text{constant} \) (no dilaton) \(^{44, 46}\). No new poles (i.e. counterterms) are generated at two-loop beyond those of the theory with \( \mu = \text{constant} \).

The two-loop potential we find is

\[
U = \frac{\lambda}{4!} \phi^4 \left\{ 1 + \frac{3 \lambda}{2 \kappa} \left( \ln \frac{V_{\phi \phi}}{(z \sigma)^2} - \frac{1}{2} \right) + \frac{3 \lambda^2}{4 \kappa^2} \left( \frac{4}{\ln} \frac{V_{\phi \phi}}{(z \sigma)^2} + 3 \ln \frac{V_{\phi \phi}}{(z \sigma)^2} \right) + \frac{5 \lambda^2}{\kappa^2} \phi^2 + \frac{7 \lambda^2}{24 \kappa^2} \phi^4 \right\},
\]

where\(^{13}\) \( A_0 = -(8/3) \sqrt{3} \text{Cl}_2(\pi/3) \approx -4.688 \cdots \).

Eq. (24) is an interesting result. First, \( U \) is scale invariant. The last two terms in \( U \) are new, finite two-loop corrections in the form of non-polynomial operators \( (\phi^6 / \sigma^2, \phi^8 / \sigma^4, \ldots) \) and cannot be removed by a different subtraction scheme. These terms are independent of the dimensionless subtraction parameter \( z \) and bring corrections beyond those obtained for \( \mu = \text{constant} \) (of explicit SSB). Their presence is easily understood in the light of the

\(^{12}\) New 1/\( \epsilon \) poles from \( (\epsilon - \text{shifts}) \times 1/\epsilon^2 \) do not emerge here, unless a classical mixing \( \phi - \sigma \) exists.

\(^{13}\) The Clausen function \( \text{Cl}_2 \) is defined as \( \text{Cl}_2(x) = - \int_0^x d\theta \ln |2 \sin \theta/2| \).
The field-dependent masses entering the loop calculation, as eigenvalues of the second derivative matrix $\tilde{\mathcal{V}}_{\alpha\beta}$, contain terms suppressed by $\mu^2 \sim \sigma^2$, since the sole dependence on $\sigma$ is $\tilde{\mathcal{V}} \sim \sigma^4$. This explains the presence of positive powers of $\sigma$ only in the denominators of the non-polynomial terms. Even the simplest quantum scale invariant theory is then non-renormalizable (unlike the case with $\mu=$constant which is renormalizable but not quantum scale invariant).

The one-loop terms which are $O(\lambda/\kappa)$ (for $\log \sim 1$) dominate the new two-loop non-polynomial terms if

$$\frac{\lambda \phi^n}{\kappa \sigma^n} < 1, \quad n = 2, 4. \quad (25)$$

The non-polynomial terms can be larger than the “standard” two-loop correction; they are comparable in size for $\phi \sim \sigma$. Higher loops are expected to generate more such operators of larger powers and with new couplings (if they are counterterms\textsuperscript{14}). They are relevant if one is interested in the stability of the potential at large field values $\phi \sim \langle \sigma \rangle$. The non-polynomial terms vanish in the limit $\phi \ll \sigma$.

The result in eq.(24) can be Taylor expanded about the vev of $\sigma$ using $\sigma = \langle \sigma \rangle + \tilde{\sigma}$. Retaining only the leading term corresponds to decoupling the dilaton. Then

$$U = \frac{\lambda}{4!} \phi^4 \left\{ 1 + \frac{3\lambda}{2\kappa} \left[ \ln \frac{V_{\phi\phi}}{\langle \sigma \rangle^2} - \frac{1}{2} \right] + \frac{3\lambda^2}{4\kappa^2} \left( 4 + A_0 - 4 \ln \frac{V_{\phi\phi}}{\langle \sigma \rangle^2} + 3 \ln \frac{V_{\phi\phi}}{\langle \sigma \rangle^2} \right) \right\} + O\left( \frac{1}{\langle \sigma \rangle} \right) \quad (26)$$

Ignoring $O(1/\langle \sigma \rangle)$ terms, eq.(26) is the “standard” two-loop result obtained for $\mu=$constant (no dilaton, explicit SSB) in MS scheme \textsuperscript{42}, more exactly for $\mu = z\langle \sigma \rangle$. The difference between eq.(24) and eq.(26) is made of higher dimensional operators suppressed by large $\langle \sigma \rangle$; these suppressed terms are responsible for maintaining manifest scale invariance of (24).

The generic form of the Callan-Symanzik equation is \textsuperscript{33}

$$\left( \frac{\partial}{\partial \ln z} + \beta_{\lambda_j} \frac{\partial}{\partial \lambda_j} + \phi \gamma_\phi \frac{\partial}{\partial \phi} + \sigma \gamma_\sigma \frac{\partial}{\partial \sigma} \right) U(\phi_j, \sigma, \lambda_j, z) = 0, \quad (27)$$

and we use it to check the result of (24). Here\textsuperscript{15}

$$\gamma_\phi = \frac{d \ln \phi}{d \ln z} = \frac{-1}{2} \frac{d \ln Z_{\phi}}{d \ln z}. \quad (28)$$

To check eq.(27), first use eq.(24) to introduce a decomposition $U = V + V^{(1)} + V^{(2)} + V^{(2,n)}$ to denote the tree-level ($V$), one-loop ($V^{(1)}$), the “usual” two-loop correction with $\mu \rightarrow z\sigma$.

\textsuperscript{14}This is discussed in the next section.

\textsuperscript{15}At two-loop $\gamma_\phi$ is $\gamma_\phi^{(2)} = -\lambda^2/(12\kappa^2)$, from eq.(22).
(\(V^{(2)}\)) and finally, the new finite two-loop correction (\(V^{(2,n)}\)) of the non-polynomial operators (the sum of the last two terms in \(24\)). Eq. \(27\) is decomposed into 3 equations:

\[
\frac{\partial V^{(1)}}{\partial \ln z} + \beta^{(1)}_\lambda \frac{\partial V}{\partial \lambda} = O(\lambda^3)
\]

\[
\frac{\partial V^{(2)}}{\partial^4 \ln z} + \left( \beta^{(2)}_\lambda \frac{\partial}{\partial \lambda} + \beta^{(2)}_\phi \frac{\partial}{\partial \phi} + \beta^{(2)}_\sigma \frac{\partial}{\partial \sigma} \right) V + \beta^{(1)}_\lambda \frac{\partial V^{(1)}}{\partial \lambda} = O(\lambda^4)
\]

\[
\frac{\partial V^{(2,n)}}{\partial \ln z} = O(\lambda^4),
\]

where \(\beta^{(k)}_\lambda, k = 1, 2, \ldots\) denote the \(k\)-loop correction to the beta function of \(\lambda\) (similar for \(\gamma^{(2)}_{\phi,\sigma}\)). We verified that eqs. \(29\)-\(31\) are respected. This is a consistency check of eq. \(24\).

The Callan-Symanzik equation is also respected in the non-scale-invariant case, eq. \(26\), where \(\mu=\)constant (\(\mu = z(\sigma), \)explicit SSB). This is obvious from the above check because \(z\) is tracking exactly this scale and the non-polynomial terms in \(24\) are \(z\)-independent.\(^\text{16}\)

### 2.4 Three-loop potential

In this section we use the three-loop Callan-Symanzik equation for the scalar potential to identify the three-loop correction to the potential without doing the diagrammatic calculation. As in the two-loop case, this correction is a sum of two terms \(V^{(3)}+V^{(3,n)}\). \(V^{(3)}\) is the “usual” three-loop correction obtained with \(\mu=\)constant (no dilaton) \(^{42, 43}\), but with the formal replacement \(\mu \to z(\sigma)\); \(V^{(3,n)}\) is a new correction that contains non-polynomial terms. To find these we use the three-loop counterterms for this theory nicely computed in \(^{37}\)

\[
\delta L_3 = \frac{1}{2} \frac{\delta^{(3)}_\phi}{(\partial_\mu \phi)^2} - \mu 2^{\epsilon} \left( \frac{1}{4!} \delta^{(3)}_\lambda \lambda \phi^4 + \frac{1}{6} \delta^{(3)}_{\lambda_6} \lambda_6 \frac{\phi^6}{\sigma^2} + \frac{1}{8} \delta^{(3)}_{\lambda_8} \lambda_8 \frac{\phi^8}{\sigma^4} \right)
\]

\(\delta L_3\) is scale-invariant in \(d = 4 - 2\epsilon\) (as it should) because \(\mu\) depends on \(\sigma\), eq. \(8\). The terms \(\phi^6/\sigma^2\) and \(\phi^8/\sigma^4\) are expected since they were present as finite operators at two-loop; also

\[
\delta^{(3)}_\phi = -\frac{\lambda^3}{4\kappa^3} \left( \frac{1}{6\epsilon^2} - \frac{1}{12\epsilon} \right)
\]

in the MS scheme, giving \(\gamma^{(3)}_\phi = \lambda^3/(16\kappa^3)\) and

\[
\delta^{(3)}_{\lambda_6} = \frac{3}{2} \frac{\lambda^4}{\lambda_6 \kappa^3 \epsilon}, \quad \delta^{(3)}_{\lambda_8} = \frac{275}{864} \frac{\lambda^4}{\lambda_8 \kappa^3 \epsilon}.
\]

\(^{16}\)This changes at 3-loops, see \(V^{(3,n)}\) in the next section.
With $\lambda_6^B = \mu^{2\varepsilon}(\sigma)\lambda_6 Z_{\lambda_6} Z^{-3}_\sigma Z\sigma$, etc., and with $(d/d\ln z) \lambda_6^B = 0$, we find

$$
\beta_{\lambda_6} = \frac{\lambda^2 \lambda_6}{2\kappa^2} + \frac{\lambda^3}{\kappa^3} \left(9\lambda - \frac{3}{8}\lambda_6\right)
$$

$$
\beta_{\lambda_8} = \frac{2\lambda^2 \lambda_8}{3\kappa^2} + \frac{\lambda^3}{4\kappa^3} \left(\frac{275}{36} \lambda - 2\lambda_8\right)
$$

Both beta functions have a two-loop part (hereafter denoted $\beta_{\lambda_{6,8}}^{(2)} \sim 1/\kappa^2$) that is absent if $\lambda_{6,8} = 0$ in the classical Lagrangian, which is our case here\footnote{Otherwise the terms $\phi^6/\sigma^2$ and $\phi^8/\sigma^4$ would have been counterterms already at two-loop, in eq. (24).}. Then the three-loop part (hereafter $\beta_{\lambda_{6,8}}^{(3)} \sim 1/\kappa^3$) is induced by $\lambda$ alone. These beta functions enter in the CS equations in the presence of $\lambda_6$ and $\lambda_8$, due to their associated counterterms. In their presence, eq. (29) is unaffected, but eq. (30) is modified such as $V$ is now replaced by

$$
V \rightarrow V + \Delta V, \quad \Delta V = \frac{\lambda_6 \phi^6}{6} + \frac{\lambda_8 \phi^8}{8} \sigma^2.
$$

(36)

and $\beta_{\lambda_{6,8}}^{(2)}$ are also included in the first term under the big bracket of (30). Using these and “new” $V$ above, one immediately sees that (30) is verified for non-zero $\lambda_{6,8}$.

Further, there is a CS equation at order $\lambda^4$ for $(V^{(3)} + V^{(3,n)})$, which we divide into two CS equations, eqs. (37) and (40) below. One equation is for the “usual” correction $V^{(3)}$ and is identical to that obtained for $\mu = \text{constant}$ ($= z(\sigma)$)

$$
\frac{\partial V^{(3)}}{\partial \ln z} + \beta_\lambda^{(1)} \frac{\partial V^{(2)}}{\partial \lambda} + \beta_\lambda^{(2)} \frac{\partial V^{(1)}}{\partial \lambda} + \gamma_\phi \frac{\partial V}{\partial \ln \phi} + \gamma_\phi^{(3)} \frac{\partial V}{\partial \ln \phi} = \mathcal{O}(\lambda^3_j). \quad (37)
$$

We integrate (37) to find $V^{(3)}$ up to an unknown “constant” of integration term $\propto Q$

$$
V^{(3)} = \frac{\lambda^4}{\kappa^3} \left[ Q + \left(\frac{97}{128} + \frac{27}{64} A_0 + \frac{\zeta[3]}{4}\right) \ln \frac{V_{\phi\phi}}{(z\sigma)^2} - \frac{31}{96} \ln \frac{V_{\phi\phi}}{(z\sigma)^2} + \frac{9}{64} \ln \frac{V_{\phi\phi}}{(z\sigma)^2} \right]. \quad (38)
$$

$Q$ can be read from the “usual” three-loop computation at $\mu = \text{constant}$\footnote{\cite{12}} in MS scheme:

$$
Q = \frac{1}{288} \left\{ -1673 - \frac{34\pi^4}{15} + 8\pi^2 \ln^2 2 - 8 \ln 2 \ln^2 2 - 192 \text{Li}_4 \left[ \frac{1}{2} \right] + 72\zeta[3] \right\}. \quad (39)
$$

$A_0$ is defined after eq. (23), $\text{Li}_4[x]$ is the polylogarithm and $\zeta[x]$ is the Riemann Zeta function.

Finally, there is one last three-loop CS equation, similar to (51), that involves $V^{(3,n)}$

$$
\frac{\partial V^{(3,n)}}{\partial \ln z} + \beta_{\lambda_j}^{(1)} \frac{\partial V^{(2,n)}}{\partial \lambda_j} + \beta_{\lambda_j}^{(3,n)} \frac{\partial V}{\partial \lambda_j} = \mathcal{O}(\lambda^5_j), \quad \lambda_j = \lambda, \lambda_6, \lambda_8. \quad (40)
$$

where $\beta_{\lambda_j}^{(3,n)}$ denotes possible three-loop corrections beyond $\beta_{\lambda_j}^{(3)}$. Eq. (40) is actually a field-dependent condition. As usual $V^{(3,n)}$ only involves new field operators beyond $V^{(3)}$, sup-
pressed by $\sigma$, e.g. $\phi^6/\sigma^2$, etc. The last term in the lhs with $\lambda_j \rightarrow \lambda$ would bring a term $\propto \phi^4$ which cannot be cancelled, being the only one of this structure. Then the only way to respect the above field-dependent condition is that $\beta^{(3,n)}_\lambda = 0$. This is also seen from (10) in the decoupling limit of large $\langle \sigma \rangle$. Therefore, the three-loop beta function in the quantum scale invariant effective theory is just that of the theory with $\mu =$-constant\[18,19\]. We then integrate eq.(10) using the replacement $\ln z \rightarrow (-1/2) \ln(V_{\phi\phi}/(z \sigma)^2)$ which fixes the “constant” of integration in a scale invariant way. We find

\[ V^{(3,n)} = \frac{\lambda^3}{2\kappa^3} \phi^4 \left\{ \left( 27\lambda - \frac{\lambda_6}{2} \right) \phi^2 \frac{\sigma^2}{8 \sigma^2} + \left( \frac{401\lambda}{72} - \lambda_8 \right) \frac{\phi^4}{16 \sigma^4} \right\} \ln \frac{V_{\phi\phi}}{(z \sigma)^2} \]  

(41)

$V^{(3,n)}$ is correct up to a possible additional presence of a scale invariant $z$-independent three-loop finite (non-polynomial) term $(\lambda^4/\kappa^3) \phi^{10}/\sigma^6$ that cannot be captured by the CS differential equation but only in the diagrammatic approach. In the limit of large field $\sigma$ and similar to $V^{(2,n)}$ at two-loop, $V^{(3,n)} \rightarrow 0$, leaving “usual” $V^{(3)}$ as the sole three-loop correction to the potential, with only a log-dependence on $\sigma$.

To conclude, quantum scale invariance demands the presence of non-polynomial operators. This symmetry arranges them in a series expansion in powers of $\phi/\sigma$ that contributes to the scalar potential. Each of these operators is actually an infinite sum of polynomial operators (in fields), after a Taylor expansion about $\sigma = \langle \sigma \rangle + \tilde{\sigma}$. $V^{(2,n)}$, $V^{(3,n)}$, $\Delta V$ are relevant for the behaviour of the potential at large $\phi \sim \sigma$ and are suppressed at $\phi \ll \sigma$.

2.5 More operators

Having seen the scale invariant non-polynomial operators generated at loop level, it is of interest to see their role if they are included in the action already at classical level, as in

\[ V = \frac{\lambda}{4!} \phi^4 + \frac{\lambda_6}{6} \phi^6 \quad \ldots \]  

(42)

where we ignore similar higher order terms. The last term breaks the enhanced Poincaré symmetry $(P_v \times P_h)$ only mildly, since this symmetry is restored at large $\sigma$. In a consistent setup like Brans-Dicke-Jordan theory of gravity, this operator suppressed by $\langle \sigma \rangle \sim M_{\text{Planck}}$ could mediate gravitational interactions of the two sectors. Such operator is also generated when going from Jordan to Einstein frame, after a conformal transformation\[21\].

The one-loop computation of the potential proceeds as before and has three contributions, all scale invariant. First, there is a one-loop contribution similar to that in eq.(12) with $V_{\phi\phi}$ replaced by the (two) field-dependent (masses)\[2\] which are eigenvalues of the matrix of second derivatives of $V$ above wrt $\phi$ and $\sigma$, then sum over these.

\[ ^{18}\text{Therefore we have } \beta^{(3)}_\lambda = \frac{\lambda^4}{\kappa^3} \left( 145/8 + 12 \zeta(3) \right) \]  

\[ ^{19}\text{This is also consistent with } Z_\sigma = 1 \text{ at three-loops. A three-loop wavefunction correction to } \sigma \text{ generated by a coupling } \sigma \phi^4 \text{ would then be proportional to } \propto e^2 \times (1/e^2) \text{, so no new poles emerge in this order.} \]  

\[ ^{20}\text{“Constants” of integration } \phi^6/\sigma^2, \phi^8/\sigma^4, \phi^4 \text{ are not allowed, being “fixed” in } [59], (38), [59] \text{ for } \phi^4. \]  

\[ ^{21}\text{We ignore here the effect of } \phi \text{ on the vev of } \sigma. \]
A second contribution to the potential exists. The two field-dependent masses derived from $\tilde{V}$ of eq. (11) with $V$ as above have a correction $O(\epsilon)$ induced by $\lambda_6$; when this multiplies $1/\epsilon$ of eq. (9), it generates a finite correction $V^{(1,n)} \propto \lambda_6$ already at one-loop

$$V^{(1,n)} = \frac{\lambda_6}{6\kappa} \phi^4 \left( 4\lambda \frac{\phi^4}{\sigma^4} + 24\lambda_6 \frac{\phi^6}{\sigma^6} + 5\lambda_6 \frac{\phi^8}{\sigma^8} \right).$$

(43)

Finally, there are also one-loop counterterms, of the form $(Z_{\lambda_p} - 1) \lambda_p \phi^p / (p \sigma^{p-4})$, where $p = 6, 8, 10, 12$ and where $Z_{\lambda_p} = 1 + \gamma_{\lambda_p}/(\kappa \epsilon)$ and $\gamma_{\lambda_6} = 9\lambda$, $\gamma_{\lambda_8} = 56\lambda_6/\lambda_8$, $\gamma_{\lambda_{10}} = 20\lambda_6^2/\lambda_{10}$, $\gamma_{\lambda_{12}} = 3\lambda_6^2/\lambda_{12}$. Therefore the potential has a third contribution

$$\Delta V = \sum_p \frac{\lambda_p}{p} \frac{\phi^p}{\sigma^{p-4}}, \quad p = 6, 8, 10, 12.$$  

(44)

$V^{(1,n)}$ and $\Delta V$ are similar to $V^{(2,n)}$, $V^{(3,n)}$, $\Delta V$ found in the previous section, except that they are generated at one-loop, due to non-zero $\lambda_6$. The one-loop beta functions of $\lambda_p$ are

$$\beta^{(1)}_{\lambda_p} = \frac{2}{\kappa} \lambda_p \gamma_{\lambda_p},$$

(45)

with $p$ as above and they vanish if $\lambda_6 = 0$. We checked that the one-loop CS equation is again verified in the presence of these operators. For large $\langle \sigma \rangle$, dilaton fluctuations are suppressed and the above corrections to the potential vanish, to leave the “usual” result (first contribution above), obtained in the renormalizable theory with $\mu$ constant ($= z\langle \sigma \rangle$).

The generalisation to more operators in the classical action is immediate.

### 2.6 Symmetries, regularisations and mass hierarchy

From the above examples, we see that a combination of quantum scale invariance and enhanced Poincaré symmetry \[^{39}\] of the two sectors can ensure a protection of the mass corrections to $\phi$ against a quadratic dependence on the scale of symmetry breaking $\langle \sigma \rangle$ (the only UV physical scale here). No term such as $\lambda \phi^2 \sigma^2 = \lambda \langle \sigma \rangle^2 \phi^2 + \cdots$ was generated at the quantum level in the potential, with $\lambda$ the higgs self-coupling \[^{22}\]; if present this would have required the usual SM-like fine-tuning of $\lambda$. Further, if one introduces a classical “mixing” coupling $\lambda_m$, with a tree-level term $\lambda_m \phi^2 \sigma^2$ which would break the enhanced $P_v \times P_h$ symmetry, this would require a tuning of $\lambda_m$ (rather than $\lambda$) upon replacing $\sigma \rightarrow \langle \sigma \rangle + \tilde{\sigma}$, in order to keep the correction to the mass of $\phi$ under control. But such tuning of $\lambda_m$ is natural and needs to be done only once at the classical level, since the beta function $\beta_{\lambda_m} \sim \lambda_m$ at one-loop \[^{22,30,35,39}\] and two-loops \[^{26}\]. Further, for large $\langle \sigma \rangle$ the non-polynomial operators that broke the $P_v \times P_h$ symmetry vanish and this symmetry and its “protective” role (on $\lambda_m$) are restored. Therefore, this protection remains true in the presence of non-polynomial operators e.g. $\lambda_6 \neq 0$.\[^{23}\]

---

\[^{22}\] This term is forbidden for large $\langle \sigma \rangle$ by the enhanced Poincaré symmetry (restored in this limit).

\[^{23}\] Since we are using spontaneous SSB and a SR scheme, the conclusions of \[^{47}\] do not apply here.
The SR scheme used here is based on the DR scheme which may be considered unsuitable to capture the quadratic UV-scale dependence of the scalar (mass)\(^2\). It is important to note, however, that in our approach any scale is generated by fields vevs after spontaneous SSB. The field-dependence (e.g. counterterms, etc) of the quantum corrected action is not affected by the regularisation and is actually dictated by the symmetries of the theory (including scale invariance), which our SR scheme respects (unlike DR). Therefore, the dependence of the quantum action on the mass scales (generated by these fields vev’s) cannot be affected. The UV behaviour of the mass of \(\phi\) i.e. its dependence on \(\langle \sigma \rangle\) (our physical UV scale), is thus not affected by a regularisation that respected all symmetries of the theory.\(^{24}\)

3 Conclusion

Following the original idea of Englert et al and using a perturbative approach, we examined the quantum implications of a regularization scheme that preserves the scale invariance of the classical theory. To this purpose, we demanded that the analytical continuation of the theory to \(d = 4 - 2\epsilon\) preserves the scale symmetry of the \(d = 4\) action. This is possible under the additional presence of a dilaton field (\(\sigma\)), the Goldstone mode of scale symmetry breaking. This field is classically decoupled from the visible sector, following an enhanced Poincaré symmetry of the two sectors, but there are nevertheless quantum effects.

The scale invariance in \(d = 4 - 2\epsilon\) and the dilaton it demands have two main effects:

a) introduce new “evanescent” interactions \((\propto \epsilon)\) which have quantum consequences;

b) generate the subtraction scale \(\mu \sim \langle \sigma \rangle\) after spontaneous scale symmetry breaking.

As a result, a scale invariant regularisation is re-formulated into an ordinary DR scheme of \(\mu = \text{constant} \ (\propto \langle \sigma \rangle)\) plus an additional field (dilaton) with an infinite series of evanescent couplings to the visible sector. When evanescent interactions multiply the poles of loop integrals, new quantum corrections (finite or infinite counterterms) are generated, not present in the quantum version of the same theory regularized with \(\mu = \text{constant}\) (i.e. no dilaton, explicit breaking). These corrections, which also include log-like terms in the potential (such as \(\ln \sigma\) already at one-loop!), are scale-invariant. They have effects such as transmission of scale symmetry breaking after its spontaneous breaking in the dilaton (hidden) sector, or dilaton-dilaton scattering.

The scalar potential was computed at two-loops by direct calculation and at three loops by integrating its Callan-Symanzik equation. The result is scale invariant. It contains new log-like corrections (in the dilaton \(\sigma\)) similar to those obtained by naively replacing \(\mu \to \sigma\) in the result obtained in the “usual” DR scheme with \(\mu = \text{constant}\). In addition, depending on the details of the classical theory, scale invariant non-polynomial effective operators are also generated from one- or two-loops onwards, in a series of the form \(\phi^4 \times (\phi/\sigma)^{2n}\). These

\(^{24}\)To appreciate the role of \(d = 4\) enhanced Poincaré symmetry, consider a different scale-invariant regularization that violates the \(P_v \times P_h\) symmetry. For \(V = \lambda \phi^4/4!\), use a momentum “cutoff” regularisation: \(k^2 \leq \sigma^2\); \(\sigma\) is a hidden sector field with \(\langle \sigma \rangle\) the scale of new physics. At one-loop \(\Delta V \propto \int_0^{\sigma^2/24} k \ln (1 + \lambda \phi^2 / (2k^2)) = \lambda \phi^2 \sigma^2 + ...\). This term (absent in our case) requires the “usual” order-by-order fine-tuning of self-coupling \(\lambda\).
operators are important for large field values $\phi \sim \sigma$ and can be comparable to “standard” log terms of the loop corrections; the beta functions of their couplings were also computed.

These operators are a generic presence and can be understood via their Taylor series expansion about the scale $\langle \sigma \rangle \neq 0$ of spontaneous SSB, when they become polynomial. Scale symmetry acts at the quantum level as an organising principle that re-sums the polynomial ones. Therefore, maintaining at the quantum level the scale symmetry of the classical action makes the theory non-renormalizable. In the decoupling limit of the dilaton these operators vanish and one recovers the quantum result of a renormalizable theory with explicit SSB (if classical theory was renormalizable).

The role of the quantum scale symmetry and enhanced Poincaré symmetry in protecting a mass hierarchy $m_\phi^2 \ll \langle \sigma \rangle^2$ was reviewed. This protection cannot be affected by working in a regularisation ultimately based on a DR scheme, because all scales and thus hierarchy thereof are generated by vev’s of the fields present in the quantum corrected action (after spontaneous SSB); its counterterms i.e. fields dependence are dictated by the symmetries of the theory (including scale symmetry), that our regularisation respects (unlike DR), hence the aforementioned protection. This remains true in the presence of the non-polynomial terms (i.e. despite non-renormalizability) since at large $\langle \sigma \rangle$ the enhanced Poincaré symmetry is restored. The study can be extended to gauge theories.

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