On the thermodynamics-based equilibrium beach profile derived by Jenkins and Inman (2006)

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August 22, 2019

Abstract

Based on the second law of thermodynamics, Jenkins and Inman (2006 J. Geophys. Res., 111, C02003) proposed that an equilibrium beach profile described by an elliptic cycloid maximises the rate of wave energy dissipation. However, here we i) highlight that the solution proposed by Jenkins and Inman (the elliptic cycloid) is difficult to recover due to important information being absent; and ii) show that, in fact, other curves can be proposed (e.g. a line) that yield larger rates of energy dissipation as formulated by the aforementioned authors, thus invalidating their hypothesis. Combined, these two crucial aspects associated with the reproducibility and validity of the research invite further scrutiny of the work and conclusions reached by Jenkins and Inman (2006). This paper also serves as an appendix to Maldonado (under review).

1 Context

Energy-dissipation-based theories have been employed in the past to derive analytically curves describing equilibrium beach profiles (see e.g. [2, 3]). However, to the best of the authors’ knowledge, Jenkins and Inman [1] were the first to hypothesise that equilibrium beach profiles may adopt shapes that maximise the rate of energy dissipation of both breaking and non-breaking waves (interestingly, this hypothesis diametrically opposes that by [3] for the latter case). Our focus here is on profiles under non-breaking waves because of their relevance in [4], for which this paper serves as an appendix. Jenkins and Inman [1] arrive at this hypothesis via the maximum entropy production formulation of the second law of thermodynamics, supplemented by certain assumptions pertaining to the shorezone system (e.g. that the system is isothermal). By means of linear wave theory, [1] formulate an integral associated with the dissipation of wave energy in profiles under non-breaking waves (referred to as ‘shorerise profiles’ in [1]), for which a maximum is sought; namely (eq. 19 in their manuscript):

\[ \int h^{-3(n+1)/4} \sqrt{1 + x^2} dh, \]  

(1)
where $h$ is the local water depth, which varies with cross-shore distance, $x$ (note that $h = h(x)$ defines the equilibrium beach profile); $x' \equiv dx/dh$ is the reciprocal of the local bed slope, $dh/dx$; and $n$ is some positive constant that characterises the variation of the bed shear stress magnitude, $\tau_o$, with the flow velocity amplitude at the bed, $u_m$, according to $\tau_o \propto u_m^n$.

Thus, [1] reduce the problem to that of finding an equilibrium beach profile, given by the function $h(x)$, that maximises the above integral, where the limits of integration are the boundaries of the shoreline profile. Then, [1] proceed to find a solution to the problem, using calculus of variations, and claim that an elliptic cycloid (i.e. the curve traced by the trajectory of a point on the perimeter of a rolling ellipse) represents the function $h(x)$ that maximises [1]. Therefore, the hypothesis of [1] that we aim to scrutinise here is the following: an equilibrium beach profile described by an elliptic cycloid maximises the rate of energy dissipation of non-breaking waves, in turn associated with the integral [1].

2 Critique

Despite its novelty and promising results (calibrated elliptic cycloids do indeed compare well against the measured profiles considered), the work by [1] invites scrutiny and revision of several aspects, from the assumptions adopted to the mathematical derivations. However, we focus here on two specific points that, in our view, refute the hypothesis discussed above posed by Jenkins and Inman; namely:

1. The proposed solution cannot be readily verified
2. The proposed solution does not maximise the integral [1]

The first point relates to the reproducibility of the research under consideration, while the second point is concerned with its validity.

2.1 On the reproducibility of the research

An objective of [1] is to find a function $x(h)$ –the inverse of $h(x)$– that maximises the integral [1]. Therefore, the functional to be maximised is (the limits of integration are discussed in §2.2):

$$J[x(h)] = \int_{h_1}^{h_2} L(h, x, x') dh = \int_{h_1}^{h_2} h^{-3(n+1)/4} \sqrt{1 + x'^2} dh.$$  \hspace{1cm} (2)

For eq. (2) to attain a stationary value at $x(h)$, presumed here to be a maximum, the Euler-Lagrange equation must be satisfied. This eventually reduces (see [1] and [4]) to solving the following integral:

$$\int \sqrt{\frac{\Omega h^\alpha}{1 - \Omega h^\alpha}} dh \hspace{1cm} (3)$$

(which is the dimensional version of eq. 21 in [1]), where $\Omega$ is an integration constant and $\alpha = 3(n + 1)/2$. Jenkins and Inman then ‘rationalize the integrand (...) using two separate Euler substitutions (....)’, but, crucially, do not mention what these substitutions are. Moreover, the general solution provided, which has two roots, takes the following form (see eq. 22 in [1]):

$$x = \frac{\Omega(\alpha - 1)/\alpha}{\epsilon \sqrt{R}} \left[ -\sqrt{\frac{h^\alpha}{\Omega} - h^{2\alpha}} + \frac{1}{2\Omega} \arccos \left( 1 - 2\Omega h^\alpha \right) \right],$$  \hspace{1cm} (4)
where $\epsilon$ is, for our purposes, a constant. The first root of the solution is then given as (see eq. 22a in [I]):

$$R = \left( \frac{\pi}{2I_e^{(2)}} \right)^2 \left[ 4\Omega h^\alpha - 4\Omega^2 h^{2\alpha} + \frac{2}{1 + \alpha} (1 - 4\Omega h^\alpha + 4\Omega^2 h^{2\alpha}) \right],$$

with the second root being similar in form but dependent on $I_e^{(1)}$, where [sic] ‘$I_e^{(1)}$ and $I_e^{(2)}$ are elliptic integrals of the first and second kind, respectively’. However, Jenkins and Inman do not mention whether they refer to incomplete or complete elliptic integrals and, more importantly, do not give the argument(s) of said functions (see Appendix A), which precludes us from recovering their solution to the variational problem or verifying, analytically, that it is correct.

The authors of this paper do not wish to assert that the above solution to the variational problem put forward by Jenkins and Inman is incorrect, but do wish to highlight that, given that said solution is arguably a central contribution of [I], the fact that arriving at it is made difficult (we have been unable to recover it) by omitting the details discussed above invites further scrutiny and fails to promote the reproducibility of their derivations, which should be a main objective of any scientific publication.

### 2.2 On the validity of the solution

Jenkins and Inman [I] rewrite their solution (eq. 22 in their manuscript) in the form of a curve describing an elliptic cycloid. The main calibration parameter is then the eccentricity of the ellipse, $e$, in turn related to $n$ in $\tau_0 \propto u_m^n$ via (see eq. 28 in [I]):

$$e = \left[ 1 - \frac{4}{3n + 5} \right]^{1/2}.$$

To test whether the solution by Jenkins and Inman maximises (2), we simply compare the value of the functional (2) yielded by Jenkins and Inman’s solution against that obtained from some arbitrarily neighbouring curves. We do so for the seaward or shorerise part of the profile solely, as discussed previously. The arbitrary curves to be tested are as follows.

Curve A – a linear profile. Reason for selection: simplicity,

$$x(h) = ah + b.$$

Curve B – from [II]. Reason for selection: to test another curve that also depends on $n$,

$$x(h) = ah^{(3n+7)/4} + b.$$

Curve C – particular case of Curve B, when $n = 2$. Reason for selection: to test some arbitrary non-linear profile,

$$x(h) = ah^{13/4} + b.$$

Values of the constants $a$ and $b$ in the above expressions are given by the boundary conditions $x(h = h_1) = x_1$ and $x(h = h_2) = x_2$, in turn obtained from Jenkins and Inman’s solutions, as shown in fig. 1 below.

For comparison, we use the six profiles (a, b, ... , f) shown in fig. 8 of Jenkins and Inman [II]. Table 1 shows the values of $e$ reported for the shorerise profile by [II], and corresponding $n$ according to eq. (6). These are the values of $n$ that we use in (2) and in Curve B for comparison against Jenkins and Inman’s solution.

Table 1, which gives the ratio of $J[x(h)]$ (eq. 2) yielded by Curves A, B and C to that obtained from Jenkins and Inman’s solution, illustrates the following points:
Figure 1: Comparison of Jenkins and Inman’s solution (an elliptic cycloid) against the three arbitrary curves proposed here. Solely the shoaling part of the beach profile (or shorerise profile) is considered. The measured profile is that labelled ‘a) Survey Range PN 1180 March 1981’ in fig. 8 of [1].

| Jenkins & Inman (2006) profiles a | b | c | d | e | f |
|-----------------------------------|---|---|---|---|---|
| value of e                        | 0.70 | 0.66 | 0.74 | 0.77 | 0.70 | 0.76 |
| corresponding n                   | 0.96 | 0.67 | 1.31 | 1.56 | 0.95 | 1.48 |

Table 1: Value of the calibration parameter e reported by Jenkins and Inman [1] for the shoaling part of each of the six profiles considered (see fig. 8 in [1]), and corresponding n according to (6).

- The solution by Jenkins and Inman does not maximise the functional [2]. Other curves yield greater values of $J[x(h)]$; most notably, the linear profile (Curve A).
- The solution by Jenkins and Inman does not minimise the functional [2] either, and so it does not even represent an extremum (see values $< 1$ for Curves B and C).

The claim by [1] that an equilibrium beach profile described by an elliptic cycloid maximises the waves’ rate of energy dissipation, in turn related to eq. (1), is therefore incorrect.

3 Conclusions

Jenkins and Inman [1] posed the hypothesis that an equilibrium beach profile described by an elliptic cycloid maximises the rate of energy dissipation of both breaking and non-breaking waves. However, focusing on the latter only, we have shown here that i) other curves (e.g. a line) yield larger rates of energy dissipation as formulated by Jenkins and Inman themselves; and ii) the
Table 2: Ratio of the functional \( J[x(h)] \) (eq. 2) obtained by the curves proposed here to that yielded by Jenkins and Inman’s solution; i.e. ratio of \( J[x(h)] = \text{curve shown in left column} \) to \( J[x(h)] = \text{Jenkins and Inman’s solution} \). Shorerise profiles are those shown in fig. 8 of [1].

| Curve | a  | b  | c  | d  | e  | f  |
|-------|----|----|----|----|----|----|
| Curve A | 1.30 | 1.21 | 1.29 | 1.38 | 1.23 | 1.38 |
| Curve B | 1.06 | 1.08 | 0.84 | 0.83 | 1.07 | 0.94 |
| Curve C | 0.97 | 1.00 | 0.94 | 0.78 | 1.00 | 0.88 |

solution proposed by Jenkins and Inman [1] to the associated variational problem invites further scrutiny given the missing information which is crucial to recover it (at least, readily).

Codes and data employed in this paper can be found at [https://github.com/sergio-maldonado/on-JI2006-solution](https://github.com/sergio-maldonado/on-JI2006-solution).

Acknowledgements: MU wishes to acknowledge the support received via the scholarship Erasmus+ France (2018), which allowed her to complete an internship at the University of Southampton, where much of this work was developed.

A Appendix

Incomplete elliptic integral of the first kind:

\[
F(\varphi, k) = \int_{0}^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.
\] (10)

Incomplete elliptic integral of the second kind:

\[
E(\varphi, k) = \int_{0}^{\varphi} \sqrt{1 - k^2 \sin^2 \theta} d\theta.
\] (11)

Complete elliptic integral of the first kind:

\[
K(k) = \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.
\] (12)

Complete elliptic integral of the second kind:

\[
E(k) = \int_{0}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta.
\] (13)

References

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