The Automorphism Conjecture for Ordered Sets of Width $\leq 11$

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Abstract

We introduce a recursive method to deconstruct the automorphism group of an ordered set. By connecting this method with deep results for permutation groups, we prove the Automorphism Conjecture for ordered sets of width less than or equal to 11. Subsequent investigations show that the method presented here could lead to a resolution of the Automorphism Conjecture.

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1 Introduction

An ordered set consists of an underlying set $P$ equipped with a reflexive, antisymmetric and transitive relation $\leq$, the order relation. An order-preserving self-map, or, an endomorphism, of an ordered set $P$ is a self map $f : P \to P$ such that $p \leq q$ implies $f(p) \leq f(q)$. Consistent with standard terminology, endomorphisms with an inverse that is an endomorphism, too, are called automorphisms. The set of endomorphisms is
denoted \( \text{End}(P) \) and the set of automorphisms is denoted \( \text{Aut}(P) \). Rival and Rutkowski’s Automorphism Problem (see [12], Problem 3) asks the following.

**Open Question 1.1 (Automorphism Problem.)** Is it true that

\[
\lim_{n \to \infty} \max_{|P|=n} \frac{|\text{Aut}(P)|}{|\text{End}(P)|} = 0?
\]

The **Automorphism Conjecture** states that the Automorphism Problem has an affirmative answer. In light of the facts that, for almost every ordered set, the identity is the only automorphism (see [11], Corollary 2.3a), and that every ordered set has at least \( 2^{\frac{n}{2}} \) endomorphisms (see [5], Theorem 1), this conjecture is quite natural. Indeed, if, for ordered sets with “many” automorphisms, we could show that there are “enough” endomorphisms to guarantee the ratio’s convergence to zero (for examples of this technique, see [7, 8], or Proposition 7.7 here), the conjecture would be confirmed. However, the Automorphism Conjecture has been remarkably resilient against attempts to prove it in general.

Recall that an **antichain** is an ordered set in which no two elements are comparable and that the **width** \( w(P) \) of an ordered set \( P \) is the size of the largest antichain contained in \( P \). The Automorphism Conjecture for ordered sets of small width has recently gathered attention in [2]. It is easy to slightly improve Theorem 1 in [5] to, for ordered sets of bounded width, provide at least \( 2^{(1-\varepsilon)n} \) endomorphisms, see Lemma 7.3. With such lower bounds available, it is natural to also consider upper bounds on the number of automorphisms. We will see here that the search for upper bounds on the number of automorphisms is linked with numerous insights on the connection between the combinatorial structure of an ordered set and the structure of its automorphism group.

We start our investigation with ordered sets that have a lot of local symmetry in Section 2. Proposition 2.8 essentially shows that, if too much local symmetry is allowed, then, for any automorphism, the remainder of the ordered set is locked into following the automorphism’s action on a small subset. Section 3 provides an overall framework to investigate this “transmission of local actions of automorphisms.” Proposition 3.10 shows that the framework decomposes into subsets, called interdependent orbit unions, on which the actions of automorphisms are independent of each other. Section 4 focuses on the automorphism groups of such interdependent orbit unions. Theorem
4.13 is the key to splitting the automorphism group into two parts, which can then be analyzed separately. Via the deconstruction of automorphisms in Section 5, we can leverage the substantial body of work on permutation groups. In Section 6, Theorem 6.17 then uses bounds for permutation groups induced on individual orbits to obtain a bound for $|\text{Aut}(P)|$ for many types of ordered sets. Section 7 shows the utility of this approach by, in Theorem 7.11, confirming the Automorphism Conjecture for ordered sets of width $\leq 11$. Section 8 improves the lower bound for the number of endomorphisms from [5] in Theorem 8.3 to $2^{0.7924n}$ for large $n$. Finally, Section 9 gives further upper bounds on the number of automorphisms for some classes of ordered sets. The overall total of these advances suggests that a resolution of the Automorphism Conjecture may well be within our reach.

2 Max-locked Ordered Sets

An ordered set with “many” automorphisms must have a high degree of symmetry. In terms of counting techniques, this means that, even when the values of an automorphism are known for a “large” number of points, the automorphism would still not be uniquely determined by these values. It is thus natural to try to identify smaller subsets $S \subset P$ such that every automorphism of the ordered set $P$ is uniquely determined by its values on $S$.

For sets $B, T \subseteq P$, we will write $B < T$ iff every $b \in B$ is strictly below every $t \in T$. For singleton sets, we will omit the set braces. Recall that a nonempty subset $A \subseteq P$ is called order-autonomous iff, for all $z \in P \setminus A$, we have that existence of an $a \in A$ with $z < a$ implies $z < A$, and, existence of an $a \in A$ with $z > a$ implies $z > A$. An order-autonomous subset $A \subseteq P$ will be called nontrivial iff $|A| \notin \{1, |P|\}$.

Lemma 2.1 below shows the simple idea that we will frequently use: When there are no nontrivial order-autonomous antichains, then an automorphism’s values on an antichain are determined by the automorphism’s values away from the antichain. Recall that $\uparrow x = \{p \in P : p \geq x\}$ and $\downarrow x = \{p \in P : p \leq x\}$.

**Lemma 2.1** Let $P$ be an ordered set with at least 3 points, let $A \subseteq P$ be an antichain that does not contain any nontrivial order-autonomous antichains, and let $\Phi, \Psi \in \text{Aut}(P)$ be so that both $\Phi$ and $\Psi$ map $A$ to itself and $\Phi|_{P \setminus A} = \Psi|_{P \setminus A}$. Then $\Phi = \Psi$. 

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Proof. Let \( \Phi, \Psi \in \text{Aut}(P) \) be so that both \( \Phi \) and \( \Psi \) map \( A \) to itself and \( \Phi|_{P\setminus A} = \Psi|_{P\setminus A} \), and let \( x \in A \). Then \( \uparrow \Phi(x) \setminus \{ \Phi(x) \} = \Phi[\uparrow x \setminus \{x\}] = \Psi[\uparrow x \setminus \{x\}] = \Psi(x) \setminus \{ \Psi(x) \} \), and similarly, \( \downarrow \Phi(x) \setminus \{ \Phi(x) \} = \downarrow \Psi(x) \setminus \{ \Psi(x) \} \). Hence, \( \{ \Phi(x), \Psi(x) \} \subseteq A \) is an order-autonomous antichain. By hypothesis, \( \{ \Phi(x), \Psi(x) \} \subseteq A \) is not a nontrivial order-autonomous antichain. Because there is at least one more point, we have \( \{ \Phi(x), \Psi(x) \} \neq P \) and hence \( \Phi(x) = \Psi(x) \).

Recall that an element \( x \) of a finite ordered set \( P \) is said to be minimal or of rank 0, and we set \( \text{rank}(x) := 0 \), iff there is no \( z \in P \) such that \( z < x \). Recursively, the element \( x \) is said to be of rank \( k \), and we set \( \text{rank}(x) := k \), iff \( x \) is minimal in \( P \setminus \{ z \in P : \text{rank}(z) \leq k - 1 \} \). It is easy to see that the rank of a point is preserved by automorphisms.

**Definition 2.2** Let \( P \) be an ordered set. For every nonnegative integer \( j \), we define \( R_j \) to be the set of elements of rank \( j \). The largest number \( h \) such that \( R_h \neq \emptyset \) is called the height of \( P \).

Clearly, the symmetric group \( S_n \) and the alternating group \( A_n \) are the largest permutation groups on \( n \) elements. In fact, the very deep Corollary 1.4 in [10] (see Theorem 6.5 here) shows that these two are by far the largest permutation groups on \( n \) elements. Hence we start our analysis with ordered sets on which a set \( R_k \) carries a maximum-sized alternating group. We will return to a more detailed investigation of permutation groups on antichains in Section 6.

**Definition 2.3** For any set \( S \), we define \( A_{|S|}(S) \) to be the alternating group on \( S \).

**Lemma 2.4** (Folklore.) Let \( n \geq 6 \) be a composite number, let \( k \) be a nontrivial divisor of \( n \) and let \( \ell \leq k \) be the smallest nontrivial divisor of \( n \). Then \( k! \left( \left( \frac{n}{k} \right)! \right)^k \leq \ell! \left( \left( \frac{n}{\ell} \right)! \right)^\ell < (n - 1)! \).

**Proof.** For the first inequality, every factor of \( k! \left( \left( \frac{n}{k} \right)! \right)^k \) that is greater than 1 can be matched with a corresponding equal or larger factor of \( \ell! \left( \left( \frac{n}{\ell} \right)! \right)^\ell \).

For the second inequality, because \( \ell \) is the greatest nontrivial divisor of \( n \), two factors 2 of \( \ell! \left( \left( \frac{n}{\ell} \right)! \right)^\ell \) can be matched with a factor greater than or equal to 4 of \( (n - 1)! \) and each of the remaining factors greater than 1 can be matched with a corresponding equal or larger factor of \( (n - 1)! \).
Lemma 2.5 Let $P$ be an ordered set of height 1 such that the following hold.

1. $P$ has $w \geq 3$ minimal elements.

2. There is an $\ell \in \{1, \ldots, w-1\}$ such that every maximal element is above exactly $\ell$ minimal elements.

3. $P$ does not contain any nontrivial order-autonomous antichains.

4. $\{\Phi|_{R_0} : \Phi \in \text{Aut}(P)\}$ contains $A_w(R_0)$.

Then $R_1$ has exactly $\binom{w}{\ell}$ elements. For $\ell \geq 2$, $P$ is isomorphic to the ordered set that consists of the singleton subsets and the $\ell$-element subsets of $R_0$ ordered by inclusion. For $\ell = 1$, $P$ is isomorphic to the pairwise disjoint union $wC_2$ (see Figure 1) of $w$ chains with 2 elements each.

Proof. Let $x \in R_1$. Let $A := \downarrow x \cap R_0$ and let $B$ be any $\ell$-element subset of $R_0$.

We first claim that there is a $\Phi_\beta \in \text{Aut}(P)$ such that $\Phi_\beta[A] = B$. First consider the case $\ell \geq 2$. In this case, there are $|A \setminus B|$ pairwise disjoint transpositions whose composition $\gamma$ maps $A \setminus B$ to $B \setminus A$ and which leaves all other points fixed. Hence $\gamma$ maps $A$ to $B$. If $|A \setminus B|$ is even, then $\gamma \in A_w(R_0)$ and we set $\beta := \gamma$. If $|A \setminus B|$ is odd, let $\tau$ be a transposition that interchanges two elements of $B$. Then $\beta := \tau \circ \gamma$ maps $A$ to $B$ and $\beta \in A_w(R_0)$. By assumption, there is a $\Phi_\beta \in \text{Aut}(P)$ such that $\Phi_\beta|_{R_0} = \beta$, which proves the claim for $\ell \geq 2$.

For $\ell = 1$, let $A = \{a\}$, let $B = \{b\}$, let $\gamma := (ab)$ and let $\tau$ be a transposition that interchanges two elements of $R_0 \setminus B$. Then $\beta := \tau \circ \gamma$ is even and maps $A$ to $B$. By assumption, there is a $\Phi_\beta \in \text{Aut}(P)$ such that $\Phi_\beta|_{R_0} = \beta$, which proves the claim for $\ell = 1$.

Because $\Phi_\beta \in \text{Aut}(P)$ and $\Phi_\beta[A] = B$, we conclude that $\downarrow \Phi_\beta(x) \cap R_0 = B$. Because $B$ was arbitrary, for every $\ell$-element subset $L$ of $R_0$, there is an $x_L \in R_1$ such that $\downarrow x_L \cap R_0 = L$. Because $P$ does not contain any nontrivial order-autonomous antichains, this element $x_L$ is unique. Because every element of $R_1$ is above exactly $\ell$ elements of $R_0$, there are no further elements in $R_1$.

For $\ell = 1$, the claimed isomorphism is clear. For $\ell \geq 2$, the claimed isomorphism maps every $z \in R_0$ to $\{z\}$ and every $x \in R_1$ to $\downarrow x \cap R_0$. The claim about the number of elements in $R_1$ follows easily. ■
Definition 2.6 (See Figure 1.) Let $w \geq 3$. We define the ordered set $S_w$ to be the ordered set consisting of the 1-element subsets and the $w - 1$-element subsets of $\{1, \ldots, w\}$ ordered by inclusion.

Recall that an ordered set $P$ is coconnected iff there are no two nonempty subsets $A$ and $B$ such that $A < B$ and $P = A \cup B$.

Lemma 2.7 Let $P$ be a coconnected ordered set of height 1 and width $w \geq 3$ such that $\{\Phi|_{R_0} : \Phi \in \text{Aut}(P)\}$ contains $A_w(R_0)$. Then $P$ is isomorphic to an ordered set $S_w$ or an ordered set $wC_2$. In particular $|\text{Aut}(P)| = w!$.

Proof. Recall that automorphisms map order-autonomous antichains to order-autonomous antichains. If $R_0$ were to contain a nontrivial order autonomous antichain, then, by Lemma 2.4, we conclude that, for $w \geq 6$, we have $|\{\Phi|_{R_0} : \Phi \in \text{Aut}(P)\}| < (w - 1)! < \frac{1}{2} w!$. For $w = 4$, the same argument leads to $|\{\Phi|_{R_0} : \Phi \in \text{Aut}(P)\}| \leq 8 < \frac{1}{2} 4!$. Because $\{\Phi|_{R_0} : \Phi \in \text{Aut}(P)\}$ contains $A_w(R_0)$, which has $\frac{1}{2} w!$ elements, we conclude that $R_0$ cannot contain any order-autonomous antichains. Moreover, because $\{\Phi|_{R_0} : \Phi \in \text{Aut}(P)\}$ contains $A_w(R_0)$ and $P$ has width $w$, we conclude that $|R_0| = w$.

Because $P$ is coconnected, there are an $\ell \in \{1, \ldots, w - 1\}$ and an $x \in R_1$ such that $|\downarrow x \cap R_0| = \ell$. Obtain $P^*$ from $P$ by selecting exactly one element from every order-autonomous antichain that has either 0 or exactly $\ell$ strict lower bounds. Then $P^*$ is coconnected, has height 1, width $w$, no nontrivial order-autonomous antichains, $\{\Phi|_{R_0} : \Phi \in \text{Aut}(P^*)\}$ contains $A_w(R_0)$, and every non-minimal element has exactly $\ell$ strict lower bounds. By Lemma 2.5, because $P^*$ has width $w$, we obtain that $\ell \in \{1, w - 1\}$ and that $P$ is isomorphic to an ordered set $S_w$ or an ordered set $wC_2$. A fortiori, because $P$ has width $w$, we obtain that $P$ contains no nontrivial order-autonomous antichains. Hence $P^* = P$ which establishes the claimed isomorphism as well as $|\text{Aut}(P)| = w!$.  

\[\text{Figure 1: The standard example } S_5 \text{ and the disjoint union } 5C_2 \text{ of five 2-chains.}\]
Proposition 2.8 Let $P$ be a coconnected ordered set width $w \geq 3$ such that there is a $k$ such that \( \{ \Phi_{|R_k} : \Phi \in \text{Aut}(P) \} \) contains $A_w(R_k)$. Then, for every $j$ such that $R_{j+1} \neq \emptyset$, we have that $R_j \cup R_{j+1}$ is isomorphic to $S_w$ or $wC_2$. In particular, $|\text{Aut}(P)| = w!$.

Proof. First consider the case that there are elements of rank greater than $k$. Then $|R_k| = w$ and, because $P$ is coconnected, $R_k \cup R_{k+1}$ is a coconnected ordered set of height 1 and width $w$ such that \( \{ \Phi_{|R_k} : \Phi \in \text{Aut}(P) \} \) contains $A_w(R_0)$. By Lemma 2.7, we have that $R_k \cup R_{k+1}$ is isomorphic to $S_w$ or $wC_2$.

By Lemma 2.1 applied to $R_k \cup R_{k+1}$, we obtain that every $\Phi_{|R_{k+1}}$ is uniquely determined by $\Phi_{|R_k}$. Because \( \{ \Phi_{|R_k} : \Phi \in \text{Aut}(P) \} \) contains $A_w(R_k)$, \( \{ \Phi_{|R_{k+1}} : \Phi \in \text{Aut}(P) \} \) contains $A_w(R_{k+1})$. Inductively, with $h$ denoting the height of $P$, we infer that, for every $j \in \{k, \ldots, h-1\}$, the ordered set $R_j \cup R_{j+1}$ is isomorphic to $S_w$ or $wC_2$.

Now, independent of whether $k = h$ or $k < h$, the elements of rank $k$ are the elements of dual rank $h-k$. The dual of the preceding argument finishes the proof.

Definition 2.9 An ordered set $P$ of width $w \geq 2$ such that, for every $j$ such that $R_{j+1} \neq \emptyset$, we have that $R_j \cup R_{j+1}$ is isomorphic to $S_w$ or $wC_2$ will be called max-locked.

Remark 2.10 Note that max-locked ordered sets such that, for every $j$ such that $R_{j+1} \neq \emptyset$, we have that $R_j \cup R_{j+1}$ is isomorphic to $wC_2$ need not be unions of pairwise disjoint chains: There can be $k \geq 0$ and $\ell \geq 2$ such that $R_k < R_{k+\ell}$.

3 Interdependent Orbit Unions

The proof of Proposition 2.8 shows how the action of the automorphism group on a single set of elements of rank $k$ can, in natural fashion, “transmit vertically” though the whole ordered set. In the situation of Proposition 2.8, the sets $R_k$ happen to be orbits (see Definition 3.1 below) of $P$. In general, the “transmission” must focus on the orbits, not the sets $R_k$, and it can “transmit” the action of the automorphism group on one orbit $O$ to orbits that have no points that are comparable to any element of $O$. It should be noted that interdependent orbit unions (see Definition 3.5 below) in graphs
have also proven useful in the set reconstruction of certain graphs, see [14]. Although the presentation up to Proposition 3.10 translates directly to and from the corresponding results in [14], all proofs are included to keep this presentation self-contained.

**Definition 3.1** (Compare with Definition 8.1 in [14].) Let $P$ be an ordered set, let $G$ be a subgroup of $\text{Aut}(P)$ and let $x \in P$. Then the set $G \cdot x := \{\Phi(x) : \Phi \in G\}$ is called the **orbit of $x$ under the action of $G$ or the $G$-orbit of $x$**. Explicit mention of $G$ or $x$ can be dropped when there is only one group under consideration or when specific knowledge of $x$ is not needed. When no group $G$ is explicitly mentioned at all, we assume by default that $G = \text{Aut}(P)$. The group generated by a single automorphism is denoted $\langle \Phi \rangle$.

Note that, if a strict subset $Q \subset P$ was obtained by removing a union of $\text{Aut}(P)$-orbits, then $\text{Aut}(P)$-orbits that are contained in $Q$ can be strictly contained in $\text{Aut}(Q)$-orbits: The $\text{Aut}(P)$-orbits of the ordered set $P$ in Figure 2 are marked by ovals. We can see that, for $X \in \{A, B, C, D\}$, the $\text{Aut}(P)$-orbits $X$ and $\tilde{X}$ are strictly contained in the $\text{Aut}(P \setminus M)$-orbit $X \cup \tilde{X}$. For this reason (further elaborated later in Remark 3.11), dictated orbit structures/orbit frames will be useful for the representation of $\text{Aut}(P)$ in Proposition 3.10 and they are vital for the remaining investigation.

**Definition 3.2** (Compare with Definition 8.2 in [14].) Let $P$ be an ordered set and let $\mathcal{D}$ be a partition of $P$ into antichains. Then $\text{Aut}_{\mathcal{D}}(P)$ is the set of automorphisms $\Phi : P \to P$ such that, for every $\Phi$-orbit $O := \langle \Phi \rangle \cdot x$ of $\Phi$, there is a $D \in \mathcal{D}$ such that $O \subseteq D$. In this context, the partition $\mathcal{D}$ is called a **dictated orbit structure** ([14]) or an **orbit frame**\(^1\) for $P$, and the pair $(P, \mathcal{D})$ is called a **structured ordered set**. $\text{Aut}_{\mathcal{D}}(P)$-orbits will, more briefly, be called $\mathcal{D}$-orbits. Sets $D \in \mathcal{D}$ will be called **frames**.

The partition of $P$ into its $\text{Aut}(P)$-orbits is called the **natural orbit structure/frame** of $P$, which will typically be denoted $\mathcal{N}$. When working with the natural orbit structure/frame, explicit indications of the automorphism set, usually via subscripts or prefixes $\mathcal{D}$, will often be omitted.

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\(^1\)Although this is a revision of the author’s own terminology, the investigation in this paper and the fact that the elements of $\mathcal{D}$ need not be orbits make it clear that the terminology “orbit frame” is more appropriate. The author thanks Frank a Campo for this suggestion.
Clearly, $\text{Aut}_D(P)$ is a subgroup of the automorphism group $\text{Aut}(P)$. Moreover, $\text{Aut}_D(P) \neq \text{Aut}(P)$ iff there are an orbit $O$ of $P$ and a $D \in \mathcal{D}$ such that $O \cap D \neq \emptyset$ and $O \not\subseteq D$.

The proof of Proposition 2.8 has already shown that, for an automorphism $\Phi$, the values on a single orbit of $\Phi$ can completely determine $\Phi$. The relation of direct interdependence in Definition 3.3 below provides a more detailed view of this observation as well as of the simple observation in Lemma 2.1. Recall that $x \sim y$ denotes the fact that $x \leq y$ or $x \geq y$.

**Definition 3.3** *(Compare with Definition 8.3 in [14].)* Let $(P, \mathcal{D})$ be a structured ordered set and let $C, D$ be two $\mathcal{D}$-orbits of $P$ such that there are a $c_1 \in C$ and a $d_1 \in D$ such that $c_1 < d_1$. We will write $C \lhd \mathcal{D} D$ and $D \lhd \mathcal{D} C$ iff there are $c_2 \in C$ and a $d_2 \in D$ such that $c_2 \not\sim d_2$. In case $C \lhd \mathcal{D} D$ or $C \lhd \mathcal{D} D$, we write $C \lhd \lhd \mathcal{D} D$ and say that $C$ and $D$ are **directly interdependent**.

Figure 2 shows how (connection through) direct interdependence can allow one orbit to determine the values of automorphisms on many other orbits: In the ordered set in Figure 2, we have $A \lhd \mathcal{N} B \lhd \mathcal{N} M \lhd \mathcal{N} \tilde{B} \lhd \mathcal{N} \tilde{A}$ and the values of any automorphism on $A \cup B \cup M \cup \tilde{B} \cup \tilde{A}$ are determined by automorphism’s values on $A$. Note, however, that even direct interdependence of two orbits $C \lhd \mathcal{N} D$ does not mean that automorphisms are determined by the values on either orbit $C$ or $D$: Consider a 6-crown in which every element is replaced with an order-autonomous 2-antichain: The minimal elements form an orbit, as do the maximal elements, the two orbits are directly interdependent, but no automorphism is completely determined solely by its values on the minimal (or the maximal) elements.
We now turn to interdependent orbit unions, which are the unions of the connected components of the orbit graph defined in Definition 3.4 below. Orbit graphs themselves will take center stage starting in Section 4.

**Definition 3.4** Let \((P, D)\) be a structured ordered set. The **orbit graph** \(O(P, D)\) of \(P\) is defined to be the graph whose vertex set is the set of all \(D\)-orbits such that two \(D\)-orbits \(D_1\) and \(D_2\) are adjacent iff \(D_1 \upharpoonright D \subseteq D_2\).

Although our main focus will ultimately be on orbit graphs for structured ordered sets in which every \(D \in D\) is a \(D\)-orbit, also see part 3 of Definition 3.13 below, until that time, we must keep in mind that the \(D \in D\) are frames which need not be orbits themselves.

**Definition 3.5** (Compare with Definition 8.5 in [14].) Let \((P, D)\) be a structured ordered set and let \(O(P, D)\) be its orbit graph. If \(E\) is a connected component of \(O(P, D)\), then we will call the set \(\bigcup E \subseteq P\) an **interdependent** \(D\)-orbit union.

**Definition 3.6** (Compare with Definition 8.6 in [14].) Let \((P, D)\) be a structured ordered set and let \(Q \subseteq P\) such that, for all \(D\)-orbits \(D\), we have \(D \subseteq Q\) or \(D \cap Q = \emptyset\). The **orbit frame** for \(Q\) **induced by** \(D\), denoted \(D|Q\), is defined to be the set of all \(D\)-orbits that are contained in \(Q\).

For the natural orbit frame \(N\) for \(P\), the orbit frame for \(Q\) induced by \(N\) will be called the **naturally required** orbit frame \(N|Q\).

Note that, for induced orbit frames \(D|Q\), every \(D \in D|Q\) is an orbit. Proposition 3.7 below now shows how interdependent \(D\)-orbit unions reside in an ordered set. It also lays the groundwork for representing automorphisms through certain automorphisms on the non-singleton interdependent orbit unions in Proposition 3.10.

**Proposition 3.7** (Compare with Proposition 8.7 in [14].) Let \((P, D)\) be a structured ordered set and let \(U\) be an interdependent \(D\)-orbit union. Then the following hold.

1. For all \(x \in P \setminus U\) and all \(C \in D|U\), the following hold.

   (a) If there is a \(c \in C\) such that \(c < x\), then \(C < x\).

   (b) If there is a \(c \in C\) such that \(c > x\), then \(C > x\).
2. For every $\Phi \in \text{Aut}_D(P)$, we have that $\Phi|_U \in \text{Aut}_{D|U}(U)$. In particular, this means that the $D|U$-orbits are just the sets in $D|U$.

3. For every $\Psi \in \text{Aut}_{D|U}(U)$, the function $\Psi^P(x) := \begin{cases} 
\Psi(x); & \text{if } x \in U, \\
\cdot & \text{if } x \in P \setminus U,
\end{cases}$ is an automorphism of $P$.

**Proof.** To prove part 1a, let $x \in P \setminus U$ and $C \in D|U$ be so that there is a $c \in C$ such that $c < x$. Let $X$ be the $D$-orbit of $x$. Because $x \notin U$, we must have $X \notHING \subset D|U$. Because $C \ni c < x \in X$, we must have that $C < X$ and hence $C < x$.

Part 1b is proved dually.

Part 2 follows directly from the definitions.

To prove part 3, let $\Psi \in \text{Aut}_{D|U}(U)$. Clearly, $\Psi^P$ is bijective. To prove that $\Psi^P$ is order-preserving, let $x < y$. If $x, y$ are both in $U$ or both in $P \setminus U$, we obtain $\Psi^P(x) < \Psi^P(y)$. In case $x \in P \setminus U$ and $y \in U$, let $Y \in D|U$ be so that $y \in Y$. Then $\Psi(y) \in Y$. By part 1b, we have that $x < Y$ and hence $\Psi^P(x) = x < Y \ni \Psi^P(y)$. The case in which $y \in P \setminus U$ and $x \in U$ is handled dually.

**Definition 3.8** (Compare with Definition 8.8 in [14].) Let $(P, D)$ be a structured ordered set and let $U$ be an interdependent $D$-orbit union. We define $\text{Aut}^P_{D|U}(U)$ to be the set of automorphisms $\Psi^P \in \text{Aut}(P)$ as in part 3 of Proposition 3.7.

Let $P$ be an ordered set, let $D$ be an orbit frame for $P$ and let $U, U'$ be disjoint interdependent $D$-orbit unions. Then, clearly, for $\Psi^P \in \text{Aut}^P_{D|U}(U)$ and $\Phi^P \in \text{Aut}^P_{D|U'}(U')$, we have $\Psi^P \circ \Phi^P = \Phi^P \circ \Psi^P$. Hence the following definition is sensible.

**Definition 3.9** (Compare with Definition 8.9 in [14].) Let $P$ be an ordered set and let $A_1, \ldots, A_z \subseteq \text{Aut}(P)$ be sets of automorphisms such that, for all pairs of distinct $i, j \in \{1, \ldots, z\}$, all $\Phi_i \in A_i$ and all $\Phi_j \in A_j$, we have $\Phi_i \circ \Phi_j = \Phi_j \circ \Phi_i$. We define $\bigcirc_{j=1}^z A_j$ to be the set of compositions $\Psi_1 \circ \cdots \circ \Psi_z$ such that, for $j = 1, \ldots, z$, we have $\Psi_j \in A_j$.

**Proposition 3.10** (Compare with Proposition 8.10 in [14].) Let $P$ be an ordered set with natural orbit frame $N$ and let $U_1, \ldots, U_z$ be the non-singleton interdependent orbit unions of $P$. Then $\text{Aut}(P) = \bigcirc_{j=1}^z \text{Aut}_{N|U_j}(U_j)$, and consequently $|\text{Aut}(P)| = \prod_{j=1}^z |\text{Aut}_{N|U_j}(U_j)|$. 

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Proof. The containment $\text{Aut}(P) \supseteq \bigodot_{j=1}^z \text{Aut}^P_{\mathcal{N}[U_j]}(U_j)$ follows from part 3 of Lemma 3.7.

By part 2 of Lemma 3.7, for every $\Phi \in \text{Aut}(P)$ and every $j \in \{1, \ldots, z\}$, we have that $\Phi|_{U_j} \in \text{Aut}_{\mathcal{N}[U_j]}(U_j)$. Because $\Phi$ fixes all points in $P \setminus \bigcup_{j=1}^z U_j$, we have $\Phi = \Phi|_{U_1} \circ \cdots \circ \Phi|_{U_z}$. Hence $\text{Aut}(P) \subseteq \bigodot_{j=1}^z \text{Aut}^P_{\mathcal{N}[U_j]}(U_j)$. □

Remark 3.11 (Compare with Remark 8.11 in [14].) Although the naturally required orbit frame $\mathcal{N}|U$ may look more technical than natural, it is indispensable for the representation in Proposition 3.10. Consider the ordered set in Figure 2. The natural interdependent orbit unions in this ordered set are $U_1 := A \cup B \cup M \cup \tilde{B} \cup \tilde{A}$ and $U_2 := C \cup D \cup \tilde{D} \cup \tilde{C}$. However, when considering $U_2$ as an ordered set by itself, the Aut($U_2$)-orbits are $C \cup \tilde{C}$ and $D \cup \tilde{D}$, whereas the Aut($P$)-orbits in $U_2$ are $C, D, \tilde{C}$ and $\tilde{D}$. Hence, we cannot use the automorphism groups Aut($U_j$) in place of their subgroups Aut$_{\mathcal{N}[U_j]}(U_j)$ in Proposition 3.10.

The same effect was observed in the introduction in the case that $M$ is removed from the ordered set. This is why orbit frames are also crucial for the proof of Theorem 6.17. □

When bounding the number of automorphisms, Proposition 3.10 allows us to focus our efforts on ordered sets that consist of a single interdependent orbit union $(U, \mathcal{D})$. Even when not stated explicitly later, any upper bound $|\text{Aut}_D(U)| \leq 2^{|U|}$ with $c \leq 0.66$ (or, in light of Theorem 8.3, $c \leq 0.79$) gives another class of ordered sets for which the Automorphism Conjecture is confirmed.

We conclude this section by introducing ideas and terminology that will be fundamental for the remainder of this paper.

Definition 3.12 We will say that the structured ordered set $(P, \mathcal{D}_P)$ is (dually) isomorphic to the structured ordered set $(Q, \mathcal{D}_Q)$ iff there is a (dual) isomorphism $\Phi : P \rightarrow Q$ such that, for all $D \in \mathcal{D}_P$, we have $\Phi[D] \in \mathcal{D}_Q$.

Definition 3.13 Let $(P, \mathcal{D})$ be a structured ordered set.

1. We call $(P, \mathcal{D})$ an interdependent orbit union iff $\bigcup \mathcal{D}$ is an interdependent $\mathcal{D}$-orbit union.

2. We say $(P, \mathcal{D})$ is without slack iff none of the sets in $\mathcal{D}$ contains a nontrivial order-autonomous antichain.
3. We call \((P, \mathcal{D})\) **tight** iff it is without slack and each \(D \in \mathcal{D}\) is a \(\mathcal{D}\)-orbit, that is, it is without slack and \(\text{Aut}_\mathcal{D}(P)\) acts transitively on every \(D \in \mathcal{D}\).

**Definition 3.14** Let \((P, \mathcal{D})\) be a structured ordered set and let \(S \subseteq P\). We define \(\Lambda_\mathcal{D}(S) := \{ \Phi|_S : \Phi \in \text{Aut}_\mathcal{D}(P) \}\).

Our main focus will be on interdependent orbit unions \((U, \mathcal{D})\) such that no \(\Lambda_\mathcal{D}(D)\) contains \(A_{|D|}(D)\). Because we will first focus on indecomposable ordered sets, we will assume that \((U, \mathcal{D})\) is without slack. Because we can always refine the orbit frame, we are free to assume that \((U, \mathcal{D})\) is tight.

### 4 The Orbit Graph of an Interdependent Orbit Union

To bound the number of automorphisms on a tight interdependent orbit union \((U, \mathcal{D})\), we will ultimately perform an induction on the number of frames. Because \((U, \mathcal{D})\) is tight, every \(D \in \mathcal{D}\) is an orbit. Hence the vertex set of \(\mathcal{O}(U, \mathcal{D})\) is \(\mathcal{D}\). When a frame \(D_n \in \mathcal{D}\) is removed, the orbit graph can become disconnected and the resulting structured ordered set may contain nontrivial order-autonomous antichains. We start with some notation that assures that the orbits with indices smaller than \(n\) form a component of the resulting graph \(\mathcal{O}(U, \mathcal{D}) - D_n\), and which allows easy reference to orbits with certain properties. This notation will be used throughout the remaining sections.

**Notation 4.1** Throughout, \((U, \mathcal{D})\) will be a tight interdependent orbit union with \(|\mathcal{D}| \geq 3\), with the labeling of the elements of \(\mathcal{D} = \{D_1, \ldots, D_m\}\) and \(n \geq 3\) chosen so that the following hold.

1. \(\{D_1, \ldots, D_{n-1}\}\) is a connected component of the graph \(\mathcal{O}(U, \mathcal{D}) - D_n\).
2. The orbits that are directly interdependent with \(D_n\) are \(D_s, \ldots, D_{n-1}\) and \(D_r, \ldots, D_m\).
3. The orbits that contain nontrivial \(U \setminus \bigcup_{i=n}^{m} D_i\)-order-autonomous antichains are \(D_t, \ldots, D_{n-1}\).
Note that Notation 4.1 allows for \( n = m \), that is, for \( D_n \) to not be a cutvertex of \( O(U,D) \). Moreover, because \((U,D)\) is an interdependent orbit union, \( O(U,D) \) is connected, so \( s \leq n - 1 \). For the other parameters, \( r > m \) or \( t > n - 1 \) shall indicate that there are no orbits as described via \( r \) or \( t \).

To easily refer to the (possibly trivial) \( U \setminus \bigcup_{i=n}^{m} D_i \)-order-autonomous antichains in the orbits \( D_s, \ldots, D_{n-1} \), we introduce the following notation.

**Notation 4.2** For every \( j \in \{s, \ldots, n-1\} \), we let \( A^j_1, \ldots, A^j_{\ell_j} \) be the maximal \( U \setminus \bigcup_{i=n}^{m} D_i \)-order-autonomous antichains that partition \( D_j \). We set \( \mathcal{A}^j := \{A^j_1, \ldots, A^j_{\ell_j}\} \). Moreover, for every \( i \in \{1, \ldots, \ell_j\} \), we choose a fixed element \( a^j_i \in A^j_i \). Note that, for \( s \leq j \leq t - 1 \), the sets \( A^j_i \) are singletons.

The fact that automorphisms map maximal order-autonomous antichains to maximal order-autonomous antichains motivates the definition below, which will frequently be used and which leads to our first insights.

**Definition 4.3** Let \( S \) be a set, let \( \mathcal{A} = \{A_1, \ldots, A_{\ell}\} \) be a partition of \( S \) and let \( \Phi : S \to S \) be a permutation of \( S \). We say that \( \Phi \) **respects the partition** \( \mathcal{A} \) iff, for every \( i \in \{1, \ldots, \ell\} \), there is a \( j \in \{1, \ldots, \ell\} \) such that \( \Phi[A_i] = A_j \).

**Lemma 4.4** For every \( j \in \{s, \ldots n-1\} \) we have that \( \ell_j > 1 \) and every \( \Phi \in \text{Aut}_D(U) \) respects the partition \( \mathcal{A}^j \) of \( D_j \). Moreover, for all \( i, k \in \{1, \ldots, \ell_j\} \), we have \( |A^j_i| = |A^j_k| \).

**Proof.** Let \( j \in \{s, \ldots, n-1\} \). Because \( n \geq 3 \) and \( \{D_1, \ldots, D_{n-1}\} \) is a connected component of the graph \( O(U,D) - D_n \), we have that \( D_j \) is directly interdependent with another \( D_{j'} \) with \( j' \in \{1, \ldots, n-1\} \setminus \{j\} \). Consequently, \( D_j \) itself is not a \( U \setminus \bigcup_{i=n}^{m} D_i \)-order-autonomous antichain. Hence \( \ell_j > 1 \).

Let \( \Phi \in \text{Aut}_D(U) \). Because \( \Phi|_{U \setminus \bigcup_{i=n}^{m} D_i} \in \text{Aut}_{D \setminus \{D_n, \ldots, D_m\}}(U \setminus \bigcup_{i=n}^{m} D_i) \), we have \( \Phi[D_j] = D_j \), and because automorphisms map maximal order-autonomous antichains to maximal order-autonomous antichains, \( \Phi \) must map every \( A^j_i \) to another \( A^j_k \), that is, \( \Phi \) respects \( \mathcal{A}^j \).

Because \( (U,D) \) is tight, for any \( i, k \in \{1, \ldots, \ell_j\} \), there is a \( \Phi \in \text{Aut}_D(U) \) such that \( \Phi(A^j_i) = A^j_k \). Consequently, because \( \Phi \) respects \( \mathcal{A}^j \), we have \( \Phi[A^j_i] = A^j_k \), and hence \( |A^j_i| = |A^j_k| \). \( \blacksquare \)

**Lemma 4.5** For all \( j \in \{s, \ldots, n-1\} \), \( i \in \{1, \ldots, \ell_j\} \) and distinct \( x, y \in A^j_i \), there is a \( d \in D_n \) such that \( d \) is comparable to one of \( x \) and \( y \), but not the other.
Proof. Let $j \in \{1, \ldots , n - 1\}$ and $i \in \{1, \ldots , \ell_j\}$. Because $D_j$ is not directly interdependent with any $D_k$ with $k > n$, we have that $A^j_i$ is order-autonomous in $U \setminus D_n$. Because, for any distinct $x, y \in A^j_i$, the set $\{x, y\}$ is not order-autonomous in $U$, there must be a $d \in D_n$ such that $d$ is comparable to one of $x$ and $y$, but not the other. \hfill \blacksquare

In our analysis, the order-autonomous antichains $A^j_i$ will be collapsed into singletons. Hence we introduce the following and again establish some natural properties.

**Definition 4.6** We define the $D_n$-pruned and compacted ordered set $U_n := \bigcup_{i=1}^{s-1} D_i \cup \bigcup_{j=1}^{n-1} \{a^j_i : i \in \{1, \ldots , \ell_j\}\}$ and we define

$$D_n := \{D_j \cap U_n : j \in \{1, \ldots , n - 1\}\}$$

$$= \{D_j : j \in \{1, \ldots , s - 1\}\} \cup \big\{\big\{a^j_i : i \in \{s, \ldots , n - 1\}\big\} : j \in \{s, \ldots , n - 1\}\big\}.$$  

For every $\Phi \in \text{Aut}_D(U)$, we define the function $\Phi_n : U_n \to U_n$ by $\Phi_n|_{\bigcup_{j=1}^{n-1} D_j} := \Phi|_{\bigcup_{j=1}^{n-1} D_j}$, and by, for any $j \in \{s, \ldots , n - 1\}$ and $i \in \{1, \ldots , \ell_j\}$, setting $\Phi_n(a^j_i)$ to be the unique element of $\Phi [A^j_i] \cap U_n$.

**Lemma 4.7** For every $\Phi \in \text{Aut}_D(U)$, we have that $\Phi_n \in \text{Aut}_{D_n}(U_n)$. Moreover, $(U_n, D_n)$ is a tight interdependent orbit union.

Proof. Clearly, $D_n$ is an orbit frame for $U_n$.

Because $\{D_1, \ldots , D_{n-1}\}$ is a connected component of the graph $\mathcal{O}(U, D) - D_n$, $(U_n, D_n)$ is an interdependent orbit union.

Because $(U, D)$ is without slack and because we choose exactly one element from each maximal $U \setminus \bigcup_{i=1}^{m} D_i$-order-autonomous antichain to be in $U_n$, we obtain that $(U_n, D_n)$ is an interdependent orbit union without slack.

For any $\Phi \in \text{Aut}_D(U)$, it follows from the definitions that $\Phi_n \in \text{Aut}_{D_n}(U_n)$. Because, for any $j \in \{s, \ldots , n - 1\}$ and $x, y \in \{1, \ldots , \ell_j\}$, there is a $\Phi \in \text{Aut}_D(U)$ with $\Phi [A^j_x] = A^j_y$, we conclude that $(U_n, D_n)$ is tight. \hfill \blacksquare

With the “early orbits” $D_1, \ldots , D_{n-1}$ thus discussed, we now turn to the remaining orbits $D_n, \ldots , D_m$ as well as the connection between $D_n$ and the sets $A^j_i$.

**Definition 4.8** Let $Q := \bigcup_{j=s}^{m} D_j$, let $\mathcal{E}_Q := \{A^j_i : j = s, \ldots , n - 1; i = 1, \ldots , \ell_j\} \cup \{D_j : j = n, \ldots , m\}$ and let $\mathcal{D}_Q$ be the set of all $\mathcal{E}_Q$-orbits.
Lemma 4.9 \((Q, \mathcal{D}_Q)\) is a tight structured ordered set.

Proof. First note that, for \(j \geq n\), all \(D_j\) that are directly interdependent with \(D_j\) are contained in \(\bigcup_{k=n}^{m} D_k = Q\). Therefore, any \(Q\)-order-autonomous antichain in a set \(D_j\) with \(j \geq n\) would be \(U\)-order-autonomous. Hence for \(j \geq n\), no \(D_j\) contains a nontrivial \(Q\)-order-autonomous antichain.

By Lemma 4.5, for \(j \in \{s, \ldots, n-1\}\) and \(i \in \{1, \ldots, \ell_j\}\), no \(A_i^j\) contains a nontrivial \(Q\)-order-autonomous antichain. We conclude that \((Q, \mathcal{E}_Q)\) is a structured ordered set without slack. Because \(\mathcal{D}_Q\) is the set of all \(\mathcal{E}_Q\)-orbits, we conclude that \((Q, \mathcal{D}_Q)\) is a tight structured ordered set. \(\blacksquare\)

Lemma 4.9 may feel a little unsatisfying compared to Lemma 4.7 in that \((Q, \mathcal{D}_Q)\) need not be an interdependent orbit union. However, Figure 3 gives examples that this really need not be the case: Any of the structured ordered sets given there could be a structured ordered set \((Q, \mathcal{D}_Q)\) in the case in which \(m = n\), \(D_n\) is a pendant vertex, and \(|U_n \cap D_{n-1}| = 2\).

Definition 4.10 With \(\Phi_n\) as in Definition 4.6, we define \(\text{Aut}_U^U(Q) := \{ \Psi \in \text{Aut}_D(U) : \Psi_n = \text{id}_{U_n} \}\). For every \(\Delta \in \text{Aut}_{\mathcal{D}_Q}(Q)\), we define \(\Delta^U\) by

\[
\Delta^U(x) := \begin{cases} 
\Delta(x); & \text{if } x \in Q, \\
x; & \text{if } x \in U \setminus Q.
\end{cases}
\]

Lemma 4.11 \(\text{Aut}_{\mathcal{D}_Q}^U(Q) = \{ \Psi \in \text{Aut}_D(U) : \Psi_n = \text{id}_{U_n} \} = \{ \Delta^U : \Delta \in \text{Aut}_{\mathcal{D}_Q}(Q) \}\).
Proof. First note that, for every $\Delta \in \text{Aut}_{\mathcal{D}_{Q}}(Q)$, the fact that $\Delta$ fixes all sets $A_i^j$ implies that $\Delta^U \in \text{Aut}_{\mathcal{D}}(U)$ and that $\Delta^U_n = \text{id}_{U_n}$.

Conversely, if $\Psi \in \text{Aut}_{\mathcal{D}}(U)$ satisfies $\Psi_n = \text{id}_{U_n}$, then $\Psi$ maps every $A_i^j$ to itself. Hence $\Psi|_Q \in \text{Aut}_{\mathcal{D}_{Q}}(Q)$ and $\Psi = (\Psi|_Q)^U$.

So far, we have investigated the structured ordered sets $(U, \mathcal{D})$, $(U_n, \mathcal{D}_n)$, and $(Q, \mathcal{D}_Q)$, and groups directly associated with them. Unfortunately, $\text{Aut}_{\mathcal{D}_n}(U_n)$ can be a proper subset of $\text{Aut}_{\mathcal{D}}(U)$. Therefore, to fully utilize the power of the ideas presented so far for a recursive deconstruction of $\text{Aut}_{\mathcal{D}}(U)$, we must focus on subgroups $G^*$ of $\text{Aut}_{\mathcal{D}}(U)$.

Definition 4.12 Let $G^*$ be a subgroup of $\text{Aut}_{\mathcal{D}}(U)$. We define $\text{Aut}^*_n(U_n) := \{\Phi_n : \Phi \in G^*\}$, $\text{Aut}^*_{\mathcal{D}_{Q}}(Q) := \{\Psi \in G^* : \Psi_n = \text{id}_{U_n}\}$ and $\text{Aut}^*_{\mathcal{D}_{Q}}(Q) := \{\Psi|_Q : \Psi \in \text{Aut}^*_{\mathcal{D}_{Q}}(Q)\}$.

Theorem 4.13 Let $G^*$ be a subgroup of $\text{Aut}_{\mathcal{D}}(U)$. The set $\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ is a normal subgroup of $G^*$ and the factor group $G^*/\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ is isomorphic to $\text{Aut}^*_n(U_n)$. Consequently $|G^*| = |\text{Aut}^*_n(U_n)| |\text{Aut}^*_{\mathcal{D}_{Q}}(Q)|$.

Proof. First note that, because $\Phi_n \Psi_n = (\Phi \Psi)_n$ and $(\Phi^{-1})_n = (\Phi_n)^{-1}$, we have that $\text{Aut}^*_n(U_n)$ is a subgroup of $G^*$ and that $\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ is a subgroup of $G^*$.

Let $\Delta^U \in \text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ and let $\Phi \in G^*$. Then $(\Phi^{-1} \Delta^U \Phi)_n = \Phi^{-1}_n \Delta^U_n \Phi_n = \Phi^{-1}_n \Phi_n = \text{id}_{U_n}$. Hence $\Phi^{-1} \text{Aut}^*_{\mathcal{D}_{Q}}(Q) \Phi = \text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ and therefore $\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ is a normal subgroup of $G^*$.

Moreover, for all $\Phi, \Psi \in \text{Aut}_{\mathcal{D}}(U)$, we have $\Phi \text{Aut}^*_{\mathcal{D}_{Q}}(Q) = \Psi \text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ iff $\Phi_n = \Psi_n$. Hence the factor group $G^*/\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ is, via $\Phi \text{Aut}^*_{\mathcal{D}_{Q}}(Q) \mapsto \Phi_n$, isomorphic to $\text{Aut}^*_n(U_n)$. The equation now follows because $\Psi \mapsto \Psi|_Q$ is an isomorphism from $\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$ to $\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$.

The notations carry over to subgroups $G^*$.

Definition 4.14 Let $G^*$ be a subgroup of $\text{Aut}_{\mathcal{D}}(U)$. We define $\mathcal{D}^*_n$ to be the set of $\text{Aut}^*_n(U_n)$-orbits, and we define $\mathcal{D}^*_{\mathcal{D}_{Q}}(Q)$ to be the set of $\text{Aut}^*_{\mathcal{D}_{Q}}(Q)$-orbits.

Definition 4.15 Let $(P, \mathcal{D})$ be a structured ordered set and let $G^*$ be a subgroup of $\text{Aut}_{\mathcal{D}}(P)$. For all $S \subseteq P$, we define $\Lambda^*_P(S) := \{\Phi|_S : \Phi \in G^*\}$.
Because all we did is “re-tighten” the interdependent orbit unions (if this is even necessary), we have $\text{Aut}^*_D(U_n) = \text{Aut}^*_A(U_n)$ and $\text{Aut}^*_D(Q) = \text{Aut}^*_Q(Q)$.

**Definition 4.16** Let $G^*$ be a subgroup of $\text{Aut}_D(U)$. The **separation partition** $S^*(D_n)$ of $D_n$ is the partition of $D_n$ that is contained in $D^*_Q$.

**Lemma 4.17** Let $G^*$ be a subgroup of $\text{Aut}_D(U)$. Every $\Phi \in G^*$ respects $D^*_Q$. Every nontrivial $D^*_Q$-orbit $D$ that is contained in a set $A^*_i$ is directly interdependent with a $D^*_Q$-orbit $S \in S^*(D_n)$, and all $D^*_Q$-orbits that are directly interdependent with $D$ are contained in $D_n$.

**Proof.** Suppose, for a contradiction, there are a $\Phi \in G^*$ and an $S \in D^*_Q$ such that $\Phi[S] \not\in D^*_Q$. Then $\Phi[S]$ intersects two distinct sets $B, C \in D^*_Q$ or $\Phi[S]$ is strictly contained in a set $D \in D^*_Q$. Because we are free to work with the inverse, it suffices to consider the case in which $\Phi[S]$ intersects two distinct sets $B, C \in D^*_Q$. Because $S \in D^*_Q$ is an $\text{Aut}^*_D(Q)$-orbit, there is a $\Delta^U \in \text{Aut}^*_D(Q)$ that maps a $b \in \Phi^{-1}[B] \cap S$ to a $c \in \Phi^{-1}[C] \cap S$. Now $\Phi(b) \in B$, $\Phi\Delta^U\Phi^{-1}(\Phi(b)) = \Phi(c) \in C$ and $\Phi\Delta^U\Phi^{-1} \mid_Q \in \text{Aut}^*_D(Q)$, a contradiction to $B, C \in D^*_Q$. We thus conclude that every $\Phi \in G^*$ respects $D^*_Q$.

Finally, by Lemma 4.5, any nontrivial $D^*_Q$-orbit that is contained in an $A^*_i$ is directly interdependent with an $S \in S^*(D_n)$. Because no two distinct $A^*_i$ are directly interdependent and no $A^*_i$ is directly interdependent with a $D_k$ with $k > n$, all orbits that are directly interdependent with an orbit in $A^*_i$ must be contained in $D_n$.

### 5 Deconstructing Automorphisms

Lemma 5.2 below is the motivation for the recursive deconstruction of $\text{Aut}_D(U)$ given in Definition 5.3. At every step, we will remove a noncutvertex of the orbit graph. At the end of the deconstruction, we will be left with an interdependent orbit union with 2 orbits. Because the removed vertices of the orbit graph are not cutvertices, the residual structured ordered sets $(Q, D_Q)$ have bipartite orbit graphs. Hence we start with bipartite orbit graphs.
Lemma 5.1 Let $(P, D)$ be a tight structured ordered set such that every $D \in \mathcal{D}$ is directly interdependent with another $E \in \mathcal{D}$, and such that $P$ can be partitioned into sets $B$ and $T$ such that, if $D, E \in \mathcal{D}$ are directly interdependent, then $D \subseteq B$ and $E \subseteq T$ or $D \subseteq T$ and $E \subseteq D$. Let $G^*$ be a subgroup of $\text{Aut}_\mathcal{D}(P)$. Then $|G^*| = |\Lambda^*_\mathcal{D}(B)| = |\Lambda^*_\mathcal{D}(T)|$.

Proof. For every $p \in P$, let $D_p$ be the unique orbit in $\mathcal{D}$ that contains $p$. If $p \in B$, then, because $p \in D_p \cap B$, by hypothesis, we have $D_p \subseteq B$. Similarly, if $p \in T$, then $D_p \subseteq T$. We define a new order $\sqsubseteq$ on $P$ by, for $b \in B$ and $t \in T$, setting $b \sqsubseteq t$ iff $D_b \sqsubseteq_D D_t$ and $b \sim t$.

Let $b_1, b_2 \in B$ be two distinct elements such that $D_{b_1} = D_{b_2}$. To prove that $\{b_1, b_2\}$ is not a $\sqsubseteq$-order-autonomous antichain, we argue as follows. Because $(P, \mathcal{D})$ is tight, and hence without slack, without loss of generality, there is an $a \in P \setminus \{b_1, b_2\}$ such that $a \sim b_1$ and $a \not\sim b_2$. In particular, this means that $D_a \neq D_{b_1} = D_{b_2}$ satisfies $D_a \sqsubseteq_D D_{b_1} = D_{b_2}$, and hence $a \sqsupseteq b_1$ and $a \not\sqsubseteq b_2$. Thus $\{b_1, b_2\}$ is not a $\sqsubseteq$-order-autonomous antichain. Similarly, for any two distinct elements $t_1, t_2 \in T$ such that $D_{t_1} = D_{t_2}$, we have that $\{t_1, t_2\}$ is not a $\sqsubseteq$-order-autonomous antichain.

Now let $\Phi \in \text{Aut}_\mathcal{D}(P)$ and let $b \sqsubseteq t$. Then $b \in D_b \sqsubseteq_D D_t \ni t$ and $b \sim t$. Because $\Phi \in \text{Aut}_\mathcal{D}(P)$, we have $\Phi(b) \in D_b \sqsubseteq_D D_t \ni \Phi(t)$ and $\Phi(b) \sim \Phi(t)$. Hence $\Phi(b) \sqsubseteq \Phi(t)$. Because $\Phi$ is bijective, $\Phi$ is a $\sqsubseteq$-automorphism. In particular, every function in $G^*$ is a $\sqsubseteq$-automorphism.

Finally let $\Phi \in G^*$. For any $t \in T$, we apply Lemma 2.1 to $B \cup D_t$ and $\Phi|_{B \cup D_t}$ with $A = D_t$ to conclude that $\Phi|_{D_t}$ is determined by $\Phi|_B$. Thus every $\Phi \in G^*$ is uniquely determined by $\Phi|_B$. Similarly, every $\Phi \in G^*$ is uniquely determined by $\Phi|_T$. We conclude that $|G^*| = |\Lambda^*_\mathcal{D}(B)| = |\Lambda^*_\mathcal{D}(T)|$. 

Lemma 5.2 Let $(U, \mathcal{D})$ be a tight interdependent orbit union, let $G^*$ be a subgroup of $\text{Aut}_\mathcal{D}(U)$, and let $D_m$ not be a cutvertex of $\mathcal{O}(U, \mathcal{D})$. Let $b, c > 0$ be so that the following hold.

1. $|\text{Aut}_\mathcal{D}^*(U_m)| \leq 2^{\frac{b}{2^{b^c}}\ell[U_m]}$.

2. For all $j \in \{t, \ldots, m\}$, we have $|\Lambda^*_\mathcal{D}^Q(D_j)| \leq 2^{c[D_j]}$.

3. For all $j \in \{t, \ldots, m - 1\}$ and $i \in \{1, \ldots, \ell_j\}$, we have $|A^j_i| \geq b$.

Then $|G^*| \leq 2^{\frac{b}{2^{b^c}}\ell[U]}$. If, in addition, we have $c \geq 1$ and $|\text{Aut}_\mathcal{D}^*(U_m)| \leq 2^{c[U_m]}$, then $|G^*| \leq 2^{\min\{\frac{b}{2^{b^c}}\ell[U] \geq 2^{c[U_m]}\}} = 2^{\min\{\frac{b}{2^{b^c}}\ell[U], c[U_m]\}}$. 

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Proof. In case $|\text{Aut}^*_Q(Q)| = 1$, the conclusions trivially follow from Theorem 4.13. For the remainder, we can assume that $|\text{Aut}^*_Q(Q)| > 1$. In particular, this means that $t < m$.

By Lemma 4.17, no two distinct $\mathcal{D}_Q$-orbits in $\bigcup_{j=t}^{m-1} D_j$ are directly interdependent and every $\mathcal{D}_Q$-orbit in $\bigcup_{j=t}^{m-1} D_j$ is directly interdependent with a $\mathcal{D}_Q$-orbit in $D_m$. Let $D \subseteq D_m$ be a $\mathcal{D}_Q$-orbit. Clearly, $D$ is not directly interdependent with any $\mathcal{D}_Q$-orbits contained in $D_m$. Hence there is an $a \in \bigcup_{j=t}^{m-1} D_j$ that is comparable to some, but not all, elements of $D$, and $D$ is directly interdependent with a $\mathcal{D}_Q$-orbit in $\bigcup_{j=t}^{m-1} D_j$. Therefore, we can apply Lemma 5.1 to $(Q, \mathcal{D}_Q)$ with $B := D_m$ and $T := \bigcup_{j=t}^{m-1} D_j$. We obtain

$$|\text{Aut}^*_Q(Q)| = |\Lambda^*_Q(B)| = |\Lambda^*_Q(T)| = |\Lambda^*_Q(D_m)| = |\Lambda^*_Q(\bigcup_{j=t}^{m-1} D_j)|.$$

We first prove that $|G^*| \leq 2^{\frac{b}{b-1}|U|}$.

By hypothesis 2, with $n_1 := |D_m|$, we have $|\Lambda^*_Q(D_m)| \leq 2^{cn_1}$, and with $n_2 := \sum_{j=t}^{m-1} |D_j|$, we have $|\Lambda^*_Q(\bigcup_{j=t}^{m-1} D_j)| \leq \prod_{j=t}^{m-1} |\Lambda^*_Q(D_j)| \leq \prod_{j=t}^{m-1} 2^{d|D_j|} \leq 2^{cn_2}$. Thus $|\text{Aut}^*_Q(Q)| \leq 2^{c\min\{n_1, n_2\}}$.

By hypothesis 3, for $j = t, \ldots, m-1$ and $i = 1, \ldots, \ell_j$, we have $|A_{ij}^2| \geq b$. Hence $|U_m| \leq |U| - n_1 - n_2 + \frac{1}{b} \max\{n_1, n_2\}$. Now we obtain the following via Theorem 4.13 and hypothesis 1. Because all computations are in the exponents, we consider the base 2 logarithm.

$$\log(|G^*|) = \log\left(|\text{Aut}^*_{D_m}(U_m)| |\text{Aut}^*_Q(Q)|\right)$$

$$= \log\left(|\text{Aut}^*_{D_m}(U_m)|\right) + \log\left(|\text{Aut}^*_Q(Q)|\right)$$

$$\leq \frac{b}{2b-1} c|U_m| + c \min\{n_1, n_2\}$$

$$\leq \frac{b}{2b-1} c\left(|U| - n_1 - n_2 + \frac{1}{b} \max\{n_1, n_2\}\right) + c(n_1 + n_2 - \max\{n_1, n_2\})$$

$$= \frac{b}{2b-1} c|U| + c\left(\frac{b-1}{2b-1}(n_1 + n_2) - \frac{2b-2}{2b-1} \max\{n_1, n_2\}\right)$$

$$\leq \frac{b}{2b-1} c|U|$$

It remains to be proved that, in case $c \geq 1$ and $|\text{Aut}^*_{D_m}(U_m)| \leq 2^{\frac{c}{c+1}|U_m|}$, we have $|G^*| \leq 2^{\min\{\frac{c}{c+1}, \frac{b}{2b-1}\} c|U|}$. Clearly, in this case, we can assume that
\( b \in \{2, 3\} \). Let \( Z \) be the number of elements in 2-antichains \( A_i^2 \), let \( T \) be the number of elements in 3-antichains \( A_i^3 \), let \( n_1 := |D_m| \), and let \( n_2 := \sum_{j=t}^{m-1} |D_j| \). Because every point in \( \bigcup_{j=t}^{m-1} D_j \setminus (Z \cup T) \) is in a set \( A_i^j \) with at least 4 elements, we obtain

\[
|U_m| \leq |U| - n_1 - n_2 + \frac{1}{2}Z + \frac{1}{3}T + \frac{1}{4}(n_2 - Z - T)
\]

\[
= |U| - n_1 - \frac{3}{4}n_2 + \frac{1}{4}Z + \frac{1}{12}T.
\]

**Case 1:** \( n_2 \leq n_1 + \frac{6c - \frac{7}{2}}{4c}Z + \frac{20c - \frac{56}{3}}{12c}T \). Let \( D_{Q}^2 \) and \( D_{Q}^3 \) be comprised of the sets \( D_j \) that are partitioned into sets \( A_i^j \) that contain 2 or 3 elements, respectively. Let \( D_{Q}^4 \) be comprised of the sets \( D_j \) that are partitioned into sets \( A_i^j \) that contain 4 or more elements. We obtain the following via Lemmas 4.17 and 5.1.

\[
\left| \text{Aut}^*_Q(Q) \right| \leq \left| \Lambda^*_{D_{Q}} \left( \bigcup_{j=t}^{m-1} D_j \right) \right|
\]

\[
\leq \prod_{D \in D_{Q}^2} \left| \Lambda^*_{D_{Q}}(D) \right| \prod_{D \in D_{Q}^3} \left| \Lambda^*_{D_{Q}}(D) \right| \prod_{D \in D_{Q}^4} \left| \Lambda^*_{D_{Q}}(D) \right|
\]

\[
\leq 2^\frac{1}{4}Z6^\frac{7}{3}T^{c(n_2-Z-T)}
\]

\[
\leq 2^\frac{1}{4}Z9^\frac{5}{3}T^{c(n_2-Z-T)}
\]

\[
= 2^{c(n_2 + (\frac{1}{2} - c)Z + (\frac{8}{9} - c)T)}.
\]

The above leads to the following estimate for \( |G^*| \). Because all computations are in the exponent, we work with the base 2 logarithm.

\[
\lg (|G^*|) = \lg \left( \left| \text{Aut}^*_m(U_m) \right| \left| \text{Aut}^*_Q(Q) \right| \right)
\]

\[
= \lg \left( \left| \text{Aut}^*_m(U_m) \right| \right) + \lg \left( \left| \text{Aut}^*_Q(Q) \right| \right)
\]

\[
\leq \frac{4}{7}c|U_m| + \left( cn_2 + \left( \frac{1}{2} - c \right)Z + \left( \frac{8}{9} - c \right)T \right)
\]

\[
\leq \frac{4}{7}c \left( |U| - n_1 - \frac{3}{4}n_2 + \frac{1}{4}Z + \frac{1}{12}T \right) + \frac{4}{7}c \left( \frac{7}{4}n_2 + \frac{7}{4}c \right)Z + \frac{7}{4}c \left( \frac{7}{4} - c \right)T
\]

\[
= \frac{4}{7}c \left( |U| - n_1 + n_2 + \frac{1}{4}Z + \frac{7}{4}c \right)Z + \frac{1}{12}T + \frac{7}{4}c \left( \frac{7}{4} - c \right)T
\]

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\[ = \frac{4}{7} c \left( |U| - n_1 + n_2 + \frac{1}{4c} cZ + \frac{7}{2} - 7c \right) + \frac{1}{12c} cT + \frac{56}{3} - 21c \]
\[ = \frac{4}{7} c \left( |U| - n_1 + n_2 + \frac{7}{2} - 6c \right) + \frac{56}{3} - 20c \]
\[ = \frac{4}{7} c \left( |U| + n_2 - \left( n_1 - \frac{7}{2} - 6c \right) \right) + \frac{56}{3} - 20c \]
\[ = \frac{4}{7} c \left( |U| + n_2 - \left( n_1 + 6c - \frac{7}{2} \right) \right) \]
\[ \leq \frac{4}{7} c|U|, \]
where the last step follows from the hypothesis for this case.

\textbf{Case 2: } \( n_2 > n_1 + \frac{6c - \frac{7}{2}}{4c} Z + \frac{20c - \frac{56}{3}}{12c} T \). In this case, by Lemma 5.1, we have \( |\text{Aut}^*_D(Q)| \leq 2^{cn_1} \), by assumption we have \( n_1 - n_2 < \frac{7}{2} - 6c + \frac{56}{3} - 20c \), and we obtain the following.

\[ \lg \left(|G^*|\right) = \lg \left(|\text{Aut}^*_D(U_m)|\right) + \lg \left(|\text{Aut}^*_D(Q)|\right) \]
\[ \leq \frac{4}{7} c|U_m| + cn_1 \]
\[ \leq \frac{4}{7} c \left( |U| - n_1 - \frac{3}{4} n_2 + \frac{3}{4} Z + \frac{1}{12} T \right) + \frac{4}{7} c \left( \frac{7}{4} n_1 \right) \]
\[ = \frac{4}{7} c \left( |U| + \frac{3}{4} n_1 - \frac{3}{4} n_2 + \frac{1}{4} Z + \frac{1}{12} T \right) \]
\[ = \frac{4}{7} c \left( |U| + \frac{3}{4} (n_1 - n_2) + \frac{1}{4} Z + \frac{1}{12} T \right) \]
\[ \leq \frac{4}{7} c \left( |U| + \frac{3}{4} \left( \frac{7}{2} - 6c \right) Z + \frac{56}{3} - 20c \right) \]
\[ = \frac{4}{7} c \left( |U| + \frac{21}{16} Z + \frac{56}{48c} T \right) \]
\[ = \frac{4}{7} c \left( |U| + \frac{21}{16} Z + \frac{56}{48c} T \right) \] (recall \( c \geq 1 \))
\[ \leq \frac{4}{7} c|U|. \]
We conclude this section with some terminology to facilitate the recursive application of Lemma 5.2.

**Definition 5.3** Let \((U, D)\) be a tight interdependent orbit union. We define \((U^{m+1}, D^{m+1}) := (U, D)\). Recursively, for \(j = m, \ldots, 3\), we can choose \(D^j\) to be a noncutvertex of \((U^{j+1}, D^{j+1})\). We can define \((U^j, D^j)\) to be the pruned and compacted ordered set from applying Definition 4.6 to \((U^{j+1}, D^{j+1})\) and \(D^j\). We can define \((Q^j, D_{Q^j})\) to be the set \((Q, D_Q)\) obtained by applying Definition 4.8 to \((U^{j+1}, D^{j+1})\) and \(D^j\). A sequence \(\{(U^j, D^j)\}_{j=m+1}^3\) obtained in this fashion is called a **deconstruction sequence** of \((U, D)\). The corresponding sequence \(\{(Q^j, D_{Q^j})\}_{j=m}^3\) obtained in this fashion is called the **sequence of residuals** for \(\{(U^j, D^j)\}_{j=m}^3\). Finally, we also call \((U^3, D^3)\) the final residual.

**Definition 5.4** Let \((U, D)\) be a tight interdependent orbit union. For \(b > 0\), a deconstruction sequence \(\{(U^j, D^j)\}_{j=m+1}^3\) of \((U, D)\) is called a **\(b\)-deconstruction sequence** of \((U, D)\) iff, for all \(j = m+1, \ldots, 3\), all nontrivial order-autonomous antichains of \(U^{j+1} \setminus D^j\) (also see Notation 4.2) have at least \(b\) elements.

Clearly, any tight interdependent orbit union has a \(2\)-deconstruction sequence.

**Remark 5.5** It should be noted that we have not used many details on the the combinatorial structure of the residuals \((Q^j, D_{Q^j})\). An earlier draft of this paper, which is now completely superseded by the application of results on permutation groups in the next section, extensively analyzed the structure of residuals \((Q, D_Q)\) to arrive at a, rather technical, proof of Lemma 7.2 below. Similar combinatorial work should allow for further advances on the Automorphism Conjecture in the future.

It is easy to see that, when the removed orbit \(D_m\) is a noncutvertex in \(\mathcal{O}(U, D)\), any two interdependent orbit unions in \((Q, D_Q)\) are isomorphic. Via Lemma 5.1 the analysis of interdependent orbit unions in \((Q, D_Q)\) can therefore be reduced to analyzing ordered sets of height 1 such that, without loss of generality, for any two orbits \(D_1\) and \(D_2\) that consist of minimal elements, the groups \(\Lambda_D(D_1)\) and \(\Lambda_D(D_2)\) are permutation equivalent, that is, there is an isomorphism that is induced by a bijection between the points of the respective base sets \(D_1\) and \(D_2\).
6 Primitive Nestings

We can now discuss some deep results from the theory of permutation groups that should lead to significant advances on the Automorphism Problem. In the case of Theorem 6.6, we present a simple improvement that can be obtained with a rather trivial computation. Theorem 6.5 rests on the shoulders of the giant which is the Classification of Finite Simple Groups. Consequently, the subsequent results on the Automorphism Problem are rather deep mathematics. Thanks to the accessible writing in [3] and [10], this depth is within reach of the author’s humble combinatorial means.

We start by reviewing a few key concepts from the theory of permutation groups.

**Definition 6.1** Let $G$ be a permutation group on the set $X$. We call $G$ transitive iff for all $x, u \in X$, there is a $\sigma \in G$ such that $\sigma(x) = u$.

**Definition 6.2** Let $G$ be a permutation group on the set $X$. We call $|X|$ the degree of $G$.

Clearly, for a tight interdependent orbit union $(U, D)$ and $D \in D$, we have that $\Lambda_D(D)$ is a transitive permutation group of degree $|D|$.

**Definition 6.3** Let $G$ be a permutation group on the set $X$. A subset $B \subseteq X$ is called a $(G)$-block iff, for all $\sigma \in G$, we have that $\sigma[B] = B$ or $\sigma[B] \cap B = \emptyset$. A block is called nontrivial iff it is not a singleton and not equal to $X$.

By Lemma 4.4, for every $j \in \{s, \ldots, n-1\}$, the partition $\mathcal{A}^j$ is a partition of $D_j$ into blocks of $\Lambda_D(D_j)$. Let $G^*$ be a subgroup of $\text{Aut}_D(U)$. By Lemma 4.17, for every $j \in \{t, \ldots, m\}$, the partition of $D_j$ induced by $D^* D^*$ is a partition of $D_j$ into blocks of $\Lambda^*_D(D_j)$.

**Definition 6.4** A permutation group $G$ is called primitive iff it has no nontrivial $G$-blocks. A permutation group that is not primitive is called imprimitive.

**Theorem 6.5** (See Corollary 1.4 in [10].) Let $G$ be a primitive permutation group of degree $n$ not containing $A_n$. If $|G| > 2^{n-1}$, then $G$ has degree at most 24, and is permutation isomorphic to one of the 24 groups in Table 1 with their natural permutation representation if not indicated otherwise in the table.
| Degree $n$ | Group $G$          | Order $|G|$ | $\log(|G|)/n \leq$ | transitivity |
|----------|------------------|--------|--------------------|--------------|
| 5        | $\text{AGL}(1,5)$ | 20     | 0.8644             | s2           |
| 6        | $\text{PSL}(2,5)$ | 60     | 0.9845             | 2p           |
| 6        | $\text{PGL}(2,5)$ | 120    | 1.1512             | s3           |
| 7        | $\text{PSL}(3,2)$ | 168    | 1.0561             | 2            |
| 8        | $\text{ATL}(1,8)$ | 168    | 0.9241             | 2p           |
| 8        | $\text{PSL}(2,7)$ | 168    | 0.9241             | 2p           |
| 8        | $\text{PGL}(2,7)$ | 336    | 1.0491             | s3           |
| 8        | $\text{ASL}(3,2) = \text{AGL}(3,2)$ | 1344    | 1.2991             | 3            |
| 9        | $\text{AGL}(2,3)$ | 432    | 0.9728             | 2            |
| 9        | $\text{PSL}(2,8)$ | 504    | 0.9975             | s3p          |
| 9        | $\text{PGL}(2,8)$ | 1512   | 1.1736             | 3p           |
| 10       | $\text{PGL}(2,9)$ | 720    | 0.9492             | s3           |
| 10       | $M_{10}$         | 720    | 0.9492             | s3           |
| 10       | $S_6$ primitive on 10 elt. | 720    | 0.9492             | 2p           |
| 10       | $\text{PGL}(2,9)$ | 1440   | 1.0492             | 3            |
| 11       | $M_{11}$         | 7920   | 1.1774             | s4           |
| 12       | $M_{11}$ on 12 elements | 7920    | 1.0793             | 3p           |
| 12       | $M_{12}$         | 95040  | 1.3781             | s5           |
| 13       | $\text{PSL}(3,3)$ | 5616   | 0.9582             | 2            |
| 15       | $\text{PSL}(4,2)$ | 20160  | 0.9533             | 2            |
| 16       | $2^4 : A_7$      | 40320  | 0.9563             | 3            |
| 16       | $2^4 : L_4(2) = \text{AGL}(4,2)$ | 322560 | 1.1438             | 3            |
| 23       | $M_{23}$         | 10200960 | 1.0123             | 4            |
| 24       | $M_{24}$         | 244823040 | 1.1612             | 5            |

Table 1: The permutation groups given in [10] which have degree $n$ and order $\geq 2^{n-1}$. Data from [3]. In case of mismatched names, both names are given. The upper bound for $\log(|G|)/n$ is obtained by rounding up to the fourth digit. The number $k$ in the transitivity column gives the $k$-transitivity of the group, “s” stands for “sharply,” “p” stands for “primitive,” and both are more stringent than $k$-transitivity.
Theorem 6.6 (Compare with Corollary 1.2 in [10].) If \( G \) is a primitive subgroup of \( S_n \) not containing \( A_n \), then \( |G| < 2^{1.38n} \). Moreover, if \( n > 24 \), then \( |G| < 2^n \).

**Proof.** The only difference to [10], Corollary 1.2 is that [10], Corollary 1.2 guarantees, for a primitive subgroup \( G \) of \( S_n \) not containing \( A_n \), that \( |G| < 3^n \). By Theorem 6.5, we have that \( |G| \leq 2^{n-1} \) unless \( G \) is permutation isomorphic to one of the 24 groups in Table 1 with their natural permutation representation if not indicated otherwise in the table.

For every group \( G \) in Table 1, the degree \( n \) and an upper bound \( c_G \) for \( \log(|G|) \) are given. Clearly, \( |G| \leq 2^{cn} \), and all values \( c_G \) are smaller than the factor 1.38 used in the statement above.

For the following, the author apologizes for possibly (even likely) restating some items from the theory of permutation groups. However, standard works on permutation groups, such as [4], or references, such as [10], formulate results primarily in the language of groups, which does not seem to have a simple connection to interdependent orbit unions \((U_m, D_m)\), structured ordered sets \((Q, D_Q)\), or Lemma 5.2. The following will keep the presentation self-contained, and, in Lemmas 6.14 and 6.16, make the requisite connections.

**Definition 6.7** Let \( G \) be a permutation group on the set \( X \) and let \( C \subseteq X \). We define \( G \cdot C := \{ \sigma[C] : \sigma \in G \} \) and \( G|_C := \{ \sigma|_C : \sigma \in G, \sigma[C] \subseteq C \} \).

**Proposition 6.8** (Folklore.) Let \( G \) be a transitive permutation group on the set \( X \) and let \( B \) be a \( G \)-block. Then \( G \cdot B \) is a partition of \( X \), every \( \sigma \in G \) respects \( G \cdot B \), and, for any two \( B_1, B_2 \in G \cdot B \), we have that \( G|_{B_1} \) is isomorphic to \( G|_{B_2} \). Finally, if \( A \) is a block that contains \( B \), then every block in \( G \cdot B \) that intersects \( A \) is contained in \( A \).

**Proof.** Let \( \sigma, \tau \in G \) such that \( \sigma[B] \cap \tau[B] \neq \emptyset \). Then \( \tau^{-1}\sigma[B] \cap B \neq \emptyset \). Hence, because \( B \) is a block, we infer \( \tau^{-1}\sigma[B] \cap B = B \), and then \( \sigma[B] = \tau[B] \). This proves that \( G \cdot B \) is a partition of \( X \) and that every \( \sigma \in G \) respects \( G \cdot B \).

Now let \( B_1, B_2 \in G \cdot B \) and let \( \mu \in G \) be so that \( \mu[B_1] = B_2 \). The function \( \Phi(\sigma) := \mu\sigma\mu^{-1}|_{B_2} \) is a homomorphism from the subgroup \( \{ \sigma \in G : \sigma[B_1] = B_1 \} \) onto the group \( G|_{B_2} \). Its kernel is the set \( \ker(\Phi) = \{ \sigma \in G : \sigma|_{B_1} = \text{id}_{B_1} \} \) and the Isomorphism Theorem now provides the claimed isomorphism.

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Finally, let $A$ be a block that contains $B$ and let $C := \sigma[B]$ intersect $A$. Then $\sigma[A] \cap A \neq \emptyset$, and therefore $\sigma[A] = A$. Consequently, $C = \sigma[B] \subseteq \sigma[A] = A$.  

**Definition 6.9** Let $G$ be a permutation group on the set $X$ and let $B \subseteq X$ be a block. For $\sigma \in G$, we define $\sigma \uparrow G \cdot B$ by, for all $C \in G \cdot B$, setting $\sigma \uparrow G \cdot B(C) := \sigma[C]$. Let $A$ be a block that contains $B$. We define $A[G \cdot B] := \{C \in G \cdot B : C \subseteq A\}$ and $G \uparrow_A B := \{\sigma \uparrow G \cdot B |_{A[G \cdot B]} : \sigma \in G, \sigma[A] \subseteq A\}$.  

**Definition 6.10** Let $G$ be a transitive permutation group on the set $X$. A sequence of blocks $\{x\} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_m = X$ such that, for all $j \in \{0, \ldots, m-1\}$ we have that $G \uparrow_{B_j} B_{j+1}$ is primitive is called a primitive nesting.

By starting with a singleton $B_0 = \{x\}$ and successively choosing a smallest size block that contains, but is not equal to, the block currently under consideration, we see that every finite permutation group has indeed many primitive nestings. Theorem 6.6 shows that, except for alternating or symmetric groups $G \uparrow_{B_{j+1}}$ and $G|_{B_1}$, the sizes of these groups are bounded by $2^{1.38n}$. Similar to some results in [10], it makes sense to exclude groups $G \uparrow_{B_{j+1}}$ that contain the alternating group on their respective domains.

**Definition 6.11** Let $G$ be a transitive permutation group on the set $X$ and let $\{x\} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_m = X$ be a primitive nesting for $G$. If, for $j = 0, \ldots, m-1$, no $G \uparrow_{B_{j+1}}$ contains a copy of $A|_{B_{j+1}}(B_{j+1}[G \cdot B_j])$ with $|\frac{B_{j+1}}{B_j}| \geq 6$, then the primitive nesting is called proper. Otherwise, the primitive nesting is called improper and every index $j$ for which $G \uparrow_{B_{j+1}}$ contains a copy of $A|_{B_{j+1}}(B_{j+1}[G \cdot B_j])$ is called an index of a factorial factor.

Clearly, for a primitive permutation group $G$ on the set $X$, the only primitive nestings are of the form $\{x\} \subsetneq X$, and they are proper iff $G$ does not contain the alternating group $A|_X$. For imprimitive groups with a proper primitive nesting, we record the following.

\[2\] Unless the author is mistaken, the $G \uparrow_{A} B$ are sections of $G$. However, the definition of sections might include other structures and, to the author’s knowledge, does not explicitly refer to the block structures we define here to make the connection to Lemma 5.2.
Lemma 6.12  Let $G$ be a transitive permutation group on the set $X$ such that there is a proper primitive nesting $\{x\} = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_m = X$. Then $|G| \leq 2^{1.7376 |X|}$.

Proof. Because the nesting is proper, for every $j \in \{0, \ldots, m-1\}$, we have $|G^{G:B_j}| \leq 2^{d_j \frac{|B_{j+1}|}{|B_j|}}$, where $d_j$ is the maximum factor for primitive groups of degree $\frac{|B_{j+1}|}{|B_j|} \in \{6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 23, 24\}$, which can be found in Table 1 and is is $\leq 1.38$, or, for $\frac{|B_{j+1}|}{|B_j|} \leq 5$, $d_j := \log \left( \frac{|B_{j+1}|}{|B_j|} \right)$, or, see Theorem 6.5, for degrees $\geq 14$ not covered so far, $d_j := 1$. Thus, for all $j$, we have $d_j \leq 1.3814$.

By Proposition 6.8, for any block $B$ and any two $C_1, C_2 \in G \cdot B$, we have that $G|C_1$ is isomorphic to $G|C_2$. Hence

$$|G| \leq \prod_{j=0}^{m-1} \left|G^{G:B_j}\right|^{\frac{|X|}{|B_{j+1}|}} \leq \prod_{j=0}^{m-1} \left( 2^{d_j \frac{|B_{j+1}|}{|B_j|}} \right)^{\frac{|X|}{|B_{j+1}|}} = \prod_{j=0}^{m-1} 2^{d_j \frac{|X|}{|B_j|}} = 2^{|X| \left( \sum_{j=0}^{m-1} \frac{d_j}{B_j} \right)} = 2^{|X| \left( \sum_{j=0}^{s} \frac{d_j}{|B_j|} \right) 2^{|X| \left( \sum_{j=s+1}^{m-1} \frac{1.38}{2} \right)}} \leq 2^{|X| \left( \sum_{j=0}^{s} \frac{d_j}{|B_j|} \right) 2^{|X| 1.38 \left( \sum_{j=s+1}^{m-1} \frac{1}{2} \right)}} = 2^{|X| \left( \sum_{j=0}^{s} d_j \prod_{i=0}^{j-1} \frac{|B_i|}{|B_{i+1}|} \right) 2^{|X| 1.38 \frac{1}{2}}}
$$

It is simple to write the nested loops that compute an upper bound for $\sum_{j=0}^{s} d_j \prod_{i=0}^{j-1} \frac{|B_i|}{|B_{i+1}|}$ for $s = 7$: For any degree $\frac{|B_{j+1}|}{|B_j|}$ that is $\leq 5$ or tabulated in Table 1, we use $d_j$ as indicated above and we use $\frac{|B_j|}{|B_{j+1}|}$ as a factor in the product. For any degree $\frac{|B_{j+1}|}{|B_j|} > 5$ that is not tabulated in Table 1, we use $d_j = 1$ and we use $\frac{1}{14}$ as an upper bound for the factor $\frac{|B_j|}{|B_{j+1}|}$ in the product.
We obtain $\sum_{j=0}^{7} d_j \prod_{i=0}^{j-1} \frac{|B_i|}{|B_{i+1}|} \leq 1.7268$. Together with $\frac{138}{26} \leq 0.0108$, we obtain the claimed estimate.

Let $(U, D)$ be an interdependent orbit union. As was mentioned after Definition 6.3, by Lemma 4.4, for every $j \in \{s, \ldots, n-1\}$, the partition $\mathcal{A}_j$ is a partition of $D_j$ into blocks of $\Lambda_\mathcal{D}(D_j)$. In a deconstruction sequence, the same orbit $D \in \mathcal{D}$ can be split multiple times in this fashion, and there may be further ways to split $D$ into blocks of $\Lambda_\mathcal{D}(D_j)$ which may or may not arise in deconstruction sequences, and which might also not be nested with blocks that arise as sets $A^j_i$. (Also see Remark 6.19 below on this subject.)

To connect Lemma 6.12 with Lemma 5.2, we must assure that primitive nestings split into primitive nestings and that no new primitive nestings could arise when a $\Lambda_\mathcal{D}(D)$ is split by a partition into sets $A^j_i$.

**Lemma 6.13** Let $G$ be a transitive permutation group on the set $X$, let $\{x\} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_m = X$ be a primitive nesting for $G$ and let $k \in \{1, \ldots, m-1\}$. For $j = k, \ldots, m$, define $\hat{B}_j := \{C \in G \cdot B_k : C \subseteq B_j\}$. Then $\{B_k\} = \hat{B}_k \subsetneq \hat{B}_{k+1} \subsetneq \cdots \subsetneq \hat{B}_m = G \cdot B_k$ is a primitive nesting for $G^\ast X$ and all primitive nestings of $G^\ast X$ are of this form.

**Proof.** The fact that $\{B_k\} = \hat{B}_k \subsetneq \hat{B}_{k+1} \subsetneq \cdots \subsetneq \hat{B}_m = G \cdot B_k$ is a primitive nesting for $G^\ast X$ follows straight from the definitions, as the only adjustment is that $G^\ast X$ acts on the blocks in $G \cdot B_k$, whereas $G$ acts on the elements of $X$.

Conversely, let $\{D_k\} = C_k \subseteq C_{k+1} \subseteq \cdots \subseteq C_n = G \cdot B_k$ be a primitive nesting for $G^\ast X$. For $j \in \{k+1, \ldots, n\}$, let $D_j := \bigcup C_j$. We can now find a primitive nesting $\{x\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_q = D_k \subseteq D_{k+1} \subseteq \cdots \subseteq D_n$ for $G$. (Because different block structures could be chosen, we need not have $q = k$, but, as before, the rest follows straight from the definitions.) This is the requisite primitive nesting for $G$ that induces $\{D_k\} = C_k \subsetneq C_{k+1} \subsetneq \cdots \subseteq C_n = G \cdot B_k$ as indicated.

**Lemma 6.14** Let $(U, D = \{D_1, \ldots, D_m\})$ be a tight interdependent orbit union such that $D_m$ is not a cutvertex of the orbit graph $\mathcal{O}(U, D)$. Let $G^\ast$ be a subgroup of $\text{Aut}_\mathcal{D}(U)$ such that, for all $D \in \mathcal{D}$, $\Lambda_\mathcal{D}(D)$ is transitive and every primitive nesting of $\Lambda_\mathcal{D}(D)$ is proper. Then, $\mathcal{D}_m = \mathcal{D}_m$ and, for all $D \in \mathcal{D}_m$, $\Lambda_\mathcal{D_m}(D)$ is transitive and every primitive nesting for $\Lambda_\mathcal{D_m}(D)$ is proper.
Proof. We can assume that \( \mathcal{D} = \{ D_1, \ldots, D_m \} \) is labeled as in Notation 4.1. We will use the notation from Section 4. For \( j \in \{ 1, \ldots, m - 1 \} \), transitivity of the \( \Lambda^*_{\mathcal{D}}(D_j) \) implies that \( \mathcal{D}^*_m = \mathcal{D}_m \). Hence, for all \( D_j \cap U_m \in \mathcal{D}^*_m \), we have that \( \Lambda^*_{\mathcal{D}^*_m}(D_j \cap U_m) \) is transitive.

For \( j < t \), we have that \( D_j = D_j \cap U_m \) and \( \Lambda^*_{\mathcal{D}^*_m}(D_j \cap U_m) = \Lambda^*_{\mathcal{D}^*_m}(D_j) \), which establishes that, for \( j < t \), all primitive nestings of \( \Lambda^*_{\mathcal{D}^*_m}(D_j) \) are proper.

Let \( j \in \{ t, \ldots, m - 1 \} \) and let \( \{ b \} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z = D_j \cap U_m \) be a primitive nesting for \( \Lambda^*_{\mathcal{D}^*_m}(D_j \cap U_m) \). For each \( B_k \) define \( \hat{B}_k := \{ A^j_i : a^j_i \in B_k \} \).

Let \( \hat{B}_0 \) be the unique element of \( \mathcal{B}_{\hat{D}} \). Note that the only change in going from \( \{ b \} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z = D_j \cap U_m \) to \( \{ \hat{B}_0 \} = \hat{B}_0 \subsetneq \hat{B}_1 \subsetneq \cdots \subsetneq \hat{B}_z = \mathcal{A}^i = \Lambda^*_{\mathcal{D}^*_m}(D_j) \cdot \hat{B}_0 \) is that permutations in \( \Lambda^*_{\mathcal{D}^*_m}(D_j \cap U_m) \) are traded for permutations of blocks in \( \Lambda^*_{\mathcal{D}^*_m}(D_j) \cap \Lambda^*_{\mathcal{D}^*_m}(D_j) \cdot \hat{B}_0 \). Hence \( \{ \hat{B}_0 \} = \hat{B}_0 \subsetneq \hat{B}_1 \subsetneq \cdots \subsetneq \hat{B}_z = \mathcal{A}^i = \Lambda^*_{\mathcal{D}^*_m}(D_j) \cdot \hat{B}_0 \) is a primitive nesting for \( \Lambda^*_{\mathcal{D}^*_m}(D_j) \cdot \hat{B}_0 \).

By Lemma 6.13, this primitive nesting is isomorphic to the upper part of a primitive nesting for \( \Lambda^*_{\mathcal{D}^*_m}(D_j) \), and hence it is proper. Consequently \( \{ b \} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z = D_j \cap U_m \) is proper, too. \( \blacksquare \)

Lemma 6.15 Let \( G \) be a transitive permutation group on the set \( X \), let \( B \) be a block of \( G \) and let \( \{ x \} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z = B \) be a primitive nesting for \( G|_B \). Then there is a primitive nesting for \( G \) that contains the above primitive nesting. Moreover, if the containing primitive nesting is proper, then so is \( \{ x \} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z = B \).

Proof. Any block of \( G|_B \) is a block of \( G \), too. Thus \( B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z \) are blocks of \( G \). Conversely, any block of \( G \) that is contained in \( B \) is a block of \( G|_B \). Therefore, there is no \( j \in \{ 1, \ldots, z \} \) such that there is a block \( C \) of \( G \) such that \( B_{j-1} \subsetneq C \subsetneq B_j \). Now \( B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z \) can be extended to a primitive nesting for \( G \). Clearly, if this primitive nesting is proper, then so is \( \{ x \} = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_z = B \). \( \blacksquare \)

Lemma 6.16 Let \( (U, \mathcal{D}) \) be a tight interdependent orbit union such that \( D_m \) is not a cutvertex of the orbit graph \( \mathcal{O}(U, \mathcal{D}) \). Let \( G^* \) be a subgroup of \( \text{Aut}_D(U) \) such that, for all \( D \in \mathcal{D} \), \( \Lambda^*_{\mathcal{D}}(D) \) is transitive and every primitive nesting of \( \Lambda^*_{\mathcal{D}}(D) \) is proper. Then, for all \( D \in \mathcal{D}^*_Q \), \( \Lambda^*_{\mathcal{D}^*_Q}(D) \) is transitive and every primitive nesting for \( \Lambda^*_{\mathcal{D}^*_Q}(D) \) is proper. In particular, for all \( j \in \{ t, \ldots, m - 1 \} \), we have \( |\Lambda^*_{\mathcal{D}^*_Q}(D_j)| \leq 2^{1.7376|D_j|} \).
**Proof.** Let $D \in \mathcal{D}_Q^*$. By definition, $D$ is an orbit of $\text{Aut}_{\mathcal{D}_Q^*}(Q)$, and hence $\Lambda_{\mathcal{D}_Q^*}^*(D)$ acts transitively on $D$. By Lemma 4.17, every $\Phi \in G^*$ respects $\mathcal{D}_Q^*$, which implies that $D$ is a block of $\Lambda_{\mathcal{D}_Q^*}^*(D_j)$.

By Lemma 6.15, any primitive nesting for $\Lambda_{\mathcal{D}_Q^*}^*(D)$ is contained in a, necessarily proper, primitive nesting for $\Lambda_{\mathcal{D}_Q^*}^*(D)$, and hence it is proper. Consequently, by Lemma 6.12, for every $D \in \mathcal{D}_Q^*$, we have $|\Lambda_{\mathcal{D}_Q^*}^*(D)| \leq 2^{1.7376|D|}$. Now, for all $j \in \{t, \ldots, m-1\}$, we have $|\Lambda_{\mathcal{D}_Q^*}^*(D_j)| \leq \prod_{D \in \mathcal{D}_Q^*, D \subseteq D_j} |\Lambda_{\mathcal{D}_Q^*}^*(D)| \leq \prod_{D \in \mathcal{D}_Q^*, D \subseteq D_j} 2^{1.7376|D|} = 2^{1.7376|D_j|}$.

**Theorem 6.17** Let $(U, \mathcal{D})$ be a tight interdependent orbit union such that, for all $D \in \mathcal{D}$, every primitive nesting for $\Lambda_{\mathcal{D}}^*(D)$ is proper. If $(U, \mathcal{D})$ has a $b$-deconstruction sequence, then $|\text{Aut}_{\mathcal{D}}(U)| \leq 2^{\min\{\frac{4}{7}, \frac{b}{26-7}\}1.7376|U|}$. In case $|\mathcal{D}| = 2$, we even have $|\text{Aut}_{\mathcal{D}}(U)| \leq 2^{\frac{1}{2}1.7376|U|}$.

**Proof.** We will prove by induction on $m = |\mathcal{D}|$ that, for every tight interdependent orbit union $(U, \mathcal{D})$ with a $b$-deconstruction sequence, and for every subgroup $G^*$ of $\text{Aut}_{\mathcal{D}}(U)$ such that, for all $D \in \mathcal{D}$, $\Lambda_{\mathcal{D}}^*(D)$ is transitive and every primitive nesting of $\Lambda_{\mathcal{D}}^*(D)$ is proper, we have $|G^*| \leq 2^{\min\{\frac{4}{7}, \frac{b}{26-7}\}1.7376|U|}$, and that, in case $m = 2$, we even have $|G^*| \leq 2^{\frac{1}{2}1.7376|U|}$.

**Base step** $m = 2$: By Lemma 6.12, for $i = 1, 2$, we obtain $|\Lambda_{\mathcal{D}}^*(D_i)| \leq 2^{1.7376|D_i|}$. Via Lemma 5.1, with $B = D_1$, $T = D_2$ and $c = 1.7376$, we obtain $|G^*| = |\Lambda_{\mathcal{D}}^*(D_1)| = |\Lambda_{\mathcal{D}}^*(D_2)| \leq 2^{1.7376 \min\{|D_1|, |D_2|\}} \leq 2^{1.7376\frac{3}{2}t}$.

**Induction step** $(m-1) \rightarrow m$: Let $\mathcal{D} = \{D_1, \ldots, D_m\}$, labeled such that $D_m$ is the orbit that is removed in the first step of a $b$-deconstruction sequence. Then clearly, $(U_m, \mathcal{D}_m)$ has a $b$-deconstruction sequence. By Lemma 6.14, $\mathcal{D}_m^* = \mathcal{D}_m$ and, for all $D \in \mathcal{D}_m^*$, $\Lambda_{\mathcal{D}_m^*}^*(D)$ is transitive and every primitive nesting for $\Lambda_{\mathcal{D}_m^*}^*(D)$ is proper. Thus, by induction hypothesis, $|\text{Aut}_{\mathcal{D}_m^*}(U_m)| \leq 2^{\min\{\frac{4}{7}, \frac{b}{26-7}\}1.7376|U_m|}$.

By Lemma 6.16, the claim follows from Lemma 5.2 with $c = 1.7376$. 

It is not surprising that bounds for the automorphism group of the ordered set will be a bit tighter than the bounds for the permutation groups induced on the individual orbits: The direct interdependences of an orbit $D$ with other orbits bring more points into play, and that is all that is used in Theorem 6.17.
Remark 6.18 Because every deconstruction sequence is a 2-deconstruction sequence, Theorem 6.17 guarantees $|\text{Aut}_D(U)| \leq 2^{\frac{3}{4}1.747[U]} \leq 2^{0.999[U]}$ when all primitive nestings of all $\Lambda_D(D)$ are proper.

Example (3) in [5] shows that there is a family of ordered sets with $|\text{End}(P)| = (\sqrt{4 + \sqrt{11}})^{|P|} \leq 2^{1.4356|P|}$. A lower bound on the number of endomorphisms of the form $2^{|P|}$ would be very helpful indeed, and Problem (2) in [5] asks if there is an infinite family of ordered sets with fewer than $2^{1.2716|P|}$ endomorphisms. However, until such bounds are established, we cannot exclude that there may be families of ordered sets with fewer endomorphisms.

Hence, although the “exponential bound $2^{0.999|P|}$ unless there are blocks with 6 or more elements which carry their alternating group” from Theorem 6.17 is a significant (and, given the nature of the results needed to prove it, deep) step forward for the theory of automorphisms of ordered sets, further analysis is needed to resolve the Automorphism Conjecture. The challenge for better bounds for the number of automorphisms lies in factorial factors and, possibly, in certain small primitive factors that could be excluded to improve the estimate in Lemma 6.12: If we, in addition to $A_n$ and $S_n$ for $n \geq 6$, also exclude $A_4$, $S_4$, $A_5$, $S_5$, $PGL(2,5)$, $AGL(3,2)$, and $M_{12}$ from all primitive nestings, the same argument we gave here would lead to a bound of $2^{0.76[U]}$. This is less than the new lower bound for the number of endomorphisms that will be established in Theorem 8.3 below.

Remark 6.19 The hypothesis in Theorem 6.17 that, for all $D \in \mathcal{D}$, every primitive nesting for $\Lambda_D(D)$ is proper is rather strong, but, via Lemmas 6.14 and 6.16, it facilitates the recursive application of Lemma 5.2. The problem we face in the use of Lemma 5.2 is that a block in a $\Lambda_D(D)$ may be split by $\mathcal{A}^j$ in a way that leaves too large a factor for $\text{Aut}_{D_m}(U_m)$.

For example, consider the set $X := \{(i, \{j, k\}) : i, j, k \in \{1, \ldots, n\}, j \neq k\}$ with the permutation group $G$ that consists of all permutations $\bar{\sigma}(i, j) := (\sigma(i), \{\sigma(j), \sigma(k)\})$, where $\sigma \in S_n$. Clearly, all rows $R_i := \{(i, \{j, k\}) : j, k \in \{1, \ldots, n\}, j \neq k\}$ and columns $C_{\{j, k\}} := \{(i, \{j, k\}) : i \in \{1, \ldots, n\}\}$ are blocks of $G$. Moreover, $|G| = n!$, $|X| = n \binom{n}{2} = \frac{1}{2}(n^3 - n^2)$ and, as $n$ grows, $2^{\frac{1}{2}(n^3 - n^2)} \gg n!$. However, if $G$ were to be split by a partition $\mathcal{A}^j$ that partitions into rows, then we would have $n!$ permutations on the $n$ elements in the corresponding orbit of the set $(U_m, \mathcal{D}_m)$. A split into the columns would
still give $n!$ permutations on the $n$ elements in the corresponding orbit, but we would have $\frac{1}{2}(n^2 - n)$ elements to facilitate an adequate estimate. In either case, the corresponding groups on the residuals consist of the identity.

It should be possible to address some of these issues with a more detailed refinement of Lemma 5.2 and of the definitions for deconstruction sequences given here, but this should be addressed in future work.

7 The Automorphism Conjecture for Ordered Sets of Width 11

The results presented so far provide substantial new insights into the structure of automorphism groups of ordered sets. Almost as a “proof of concept,” we now turn to confirming the Automorphism Conjecture for ordered sets of width up to 11.

We will call an interdependent union $(U, D)$ max-locked iff $U$ is a max-locked ordered set.

Lemma 7.1 Let $(U, D)$ be an interdependent orbit union of width $\leq 11$ such that there is a $D \in D$ with $|D| \geq 6$ such that $\Lambda_D(D)$ contains $A_{|D|}(D)$. Then $(U, D)$ is max-locked.

**Proof.** Let $D \in D$ with $|D| \geq 6$ such that $\Lambda_D(D)$ contains $A_{|D|}(D)$ and let $E \parallel_D D$ be directly interdependent with $D$. By Lemma 2.5, because $|E| \leq 11 < 15 = \binom{6}{2}$, we conclude, with $w = |D|$ that $D \cup E$ is isomorphic to $wC_2$ or $S_w$. In particular, $|E| = |D|$ and $\Lambda_D(E)$ contains $A_{|E|}(E)$.

Because the above applies to any $D \in D$ with $|D| \geq 6$ such that $\Lambda_D(D)$ contains $A_{|D|}(D)$ and because $O(U, D)$ is connected, we conclude that, for any two $D, E \in D$ with $D \parallel_D E$, we have that $D \cup E$ is isomorphic to $wC_2$ or $S_w$. Finally, because any two orbits of size at least 6 must have comparable elements, we conclude that $(U, D)$ is max-locked.

Lemma 7.2 Let $(U, D)$ be a tight interdependent orbit union such that, for all $D \in D$, we have $|D| \leq 11$, and all orbits $|D|$ such that $\Lambda_D(D)$ contains $A_{|D|}(D)$ satisfy $|D| \leq 5$. Then $|\text{Aut}_D(U)| \leq 2^{0.993|U|}$.

**Proof.** Let $D \in D$ such that $\Lambda_D(D)$ contains a factorial factor on $n$ elements. Because $|D| \leq 11$, we have that $n \geq 6$ would imply that $\Lambda_D(D)$
contains \( A_{\rightharpoonup D}(D) \), which was excluded. Therefore \( n \leq 5 \). Hence every primitive nesting for \( \Lambda_D(D) \) is proper.

Because \( D \in \mathcal{D} \) was arbitrary, we can apply Theorem 6.17 with \( b = 2 \) and we obtain \(|\text{Aut}_D(U)| \leq 2^{4^{1.7376|U|}} \leq 2^{0.993|U|} \).\n
To prove the Automorphism Conjecture for ordered sets of width up to 11 in Theorem 7.11 below, we first establish a lower bound on the number of endomorphisms in Lemma 7.3. Then, from Proposition 7.5 through Proposition 7.7, we show that a relative abundance of max-locked interdependent orbit unions of height 1 guarantees that the Automorphism Conjecture holds. Lemma 7.8 combines the work so far into an upper bound for the number of automorphisms when there are no nontrivial order-autonomous antichains, and Proposition 7.10 shows that a single execution of the lexicographic sum construction, as occurs for example when nontrivial order-autonomous antichains are inserted, does not affect the status of the Automorphism Conjecture.

**Lemma 7.3** Let \( w \in \mathbb{N} \) and \( \varepsilon > 0 \). There is an \( N \in \mathbb{N} \) such that every ordered set \( P \) of width \( \leq w \) with \( n := |P| \geq N \) elements has at least \( 2^{(1-\varepsilon)n} \) endomorphisms.

**Proof.** Let \( P \) be an ordered set of height \( h \) with \( n \) elements. The proof of Theorem 1 in [5] (on page 20 of [5]) shows that \( P \) has at least \( 2^{h/n} \) endomorphisms that are surjective onto a chain of length \( h \).

Let \( N \in \mathbb{N} \) be so that \( \frac{(N/w)^{w-1}}{[(N/w)^{w-1}]+1} > 1 - \varepsilon \) and let \( P \) be an ordered set of width \( w \) with \( n \geq N \) elements. Then the height \( h \) of \( P \) satisfies \( h \geq \frac{n}{w} - 1 \geq \frac{N}{w} - 1 \). Hence \( P \) has at least \( 2^{h/n} \geq 2^{(1-\varepsilon)n} \) endomorphisms.\n
Similar to \( \text{Aut}_D(P) \), we define \( \text{End}_D(P) \).

**Definition 7.4** Let \((P, \mathcal{D})\) be a structured ordered set. We define \( \text{End}_D(P) \) to be the set of order-preserving maps \( f : P \to P \) such that, for all \( D \in \mathcal{D} \), we have \( f[D] \subseteq D \).

**Proposition 7.5** Let \( P \) be an ordered set and let \( \mathcal{U} \) denote the set of nontrivial natural interdependent orbit unions of \( P \). Then

\[
\frac{|\text{Aut}(P)|}{|\text{End}(P)|} = \frac{|\text{Aut}_{\mathcal{U}}(P)|}{|\text{End}_{\mathcal{U}}(P)|} \leq \prod_{U \in \mathcal{U}} \frac{|\text{Aut}_{\mathcal{U}[U]}(U)|}{|\text{End}_{\mathcal{U}[U]}(U)|}.
\]
Proof. The functions in each set \( \text{End}_{\mathcal{N}|U}(U) \) can be combined into endomorphisms of \( P \) in the same way as the functions in \( \text{Aut}_{\mathcal{N}|U}(U) \) are combined in Proposition 3.10.

Lemma 7.6 \( |\text{End}_{\mathcal{N}}(wC_2)| \geq w^w \) and \( |\text{End}_{\mathcal{N}}(S_w)| \geq (w-1)^w \).

Proof. The result about \( |\text{End}_{\mathcal{N}}(wC_2)| \) follows from the fact that every self map of the minimal elements can be extended to a function in \( \text{End}_{\mathcal{N}}(wC_2) \). For the result about \( |\text{End}_{\mathcal{N}}(S_w)| \), let \( t \in S_w \) be maximal in \( S_w \). Now we can map the maximal elements of \( S_w \) to \( t \), we can map every one of the \( w \) minimal elements of \( S_w \) to any of the \( w-1 \) minimal elements in \( \downarrow t \), and, through each such choice, we obtain a function in \( \text{End}_{\mathcal{N}}(S_w) \).

Proposition 7.7 The Automorphism Conjecture is true for the class of ordered sets \( P \) such that at least \( \lg(|P|) \) elements of \( P \) are contained in max-locked interdependent orbit unions of height 1.

Proof. Let \( P \) be an ordered set with \( |P| = n \) elements such that at least \( \lg(n) \) elements of \( P \) are contained in max-locked interdependent orbit unions of height 1.

In case \( \geq \sqrt{\lg(n)} \) elements of \( P \) are contained in one max-locked interdependent orbit union \( U \) of height 1 and width \( w \), we have \( w \geq \frac{1}{2} \sqrt{\lg(n)} \). By Lemma 7.6, we obtain \( |\text{End}_{\mathcal{N}|U}(U)| \geq (w-1)^w \). Moreover, \( U \) has exactly \( w! \) automorphisms. By Proposition 7.5, we obtain the following.

\[
\frac{|\text{Aut}(P)|}{|\text{End}(P)|} \leq \frac{w!}{(w-1)^w} \leq \left( \frac{\frac{1}{2} \sqrt{\lg(n)}}{\frac{1}{2} \sqrt{\lg(n)} - 1} \right)^{\frac{1}{2} \sqrt{\lg(n)}},
\]

and the right hand side goes to zero as \( n \to \infty \).

In case no max-locked interdependent orbit union \( U \subseteq P \) of height 1 contains \( \geq \sqrt{\lg(n)} \) elements, we have that \( P \) contains \( \geq \sqrt{\lg(n)} \) max-locked interdependent orbit unions \( (U, \mathcal{N}|U) \) of height 1. By Lemma 7.6, we have \( \frac{|\text{Aut}_{\mathcal{N}|U}(U)|}{|\text{End}_{\mathcal{N}|U}(U)|} \leq \frac{w!}{(w-1)^w} \), which, for \( w \geq 3 \), is \( \leq \frac{3}{4} \). For width \( w = 2 \), we have \( \frac{|\text{Aut}_{\mathcal{N}|U}(U)|}{|\text{End}_{\mathcal{N}|U}(U)|} \leq \frac{2}{4} < \frac{3}{4} \). By Proposition 7.5, we obtain

\[
\frac{|\text{Aut}(P)|}{|\text{End}(P)|} \leq \left( \frac{3}{4} \right)^{\sqrt{\lg(n)}},
\]

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and the right hand side goes to zero as $n \to \infty$.

Therefore, the Automorphism Conjecture is true for this class of ordered sets.

**Lemma 7.8** There is an $N \in \mathbb{N}$ such that, for every ordered set $P$ of width $w \leq 11$ with $n \geq N$ elements, no nontrivial order-autonomous antichains, and at most $\lg(n)$ elements contained in max-locked interdependent orbit unions of height 1, we have that $\frac{|\text{Aut}(P)|}{|\text{End}(P)|} \leq 2^{-0.005n}$.

**Proof.** By Lemma 7.2, for all tight interdependent orbit unions $(U, D)$ such that, for all $D \in D$ we have $|D| \leq 11$, and all orbits $|D|$ such that $\Lambda_D(D)$ contains $A_{|\Lambda_D(D)}(D)$ satisfy $|D| \leq 5$, we have that $|\text{Aut}_D(U)| \leq 2^{0.993|U|}$.

Let $(U, D)$ be a tight interdependent orbit union such that there is an orbit $D$ with more than 5 elements such that $|\Lambda_D(D)|$ contains $A_{|\Lambda_D(D)}(D)$. By Lemma 7.1, $(U, D)$ is max-locked of width $v \leq w \leq 11$. Now either $U$ has height 1, or, a quick computation with factorials shows that $|\text{Aut}_D(U)| = v! \leq 2^{2.4v} \leq 2^{0.8|U|}$.

Let $Z \subseteq P$ be the set of all points in $P$ that are contained in (possibly trivial) interdependent orbit unions that are not max-locked of height 1. Let $K := |Z|$. By assumption $n - K \leq \lg(n)$. By the above, the number of restrictions of automorphisms of $P$ to $Z$ is bounded by $2^{0.993|Z|}$. The number of restrictions of automorphisms of $P$ to $P \setminus Z$ is bounded by $|P \setminus Z|! = (n - K)! \leq \lg(n)! \leq \lg(n)^{\lg(n)} = 2^{\lg(n) \lg(\lg(n))}$.

Hence, the number of automorphisms of $P$ is bounded by $2^{0.993n - \lg(n) \lg(\lg(n))}$, which, for large enough $n$, is smaller than $2^{0.994n}$. The claim now follows from Lemma 7.3 with $\varepsilon := 0.001$.

We are left with the task to consider ordered sets with nontrivial order-autonomous antichains. Proposition 7.10 below focuses on the more general lexicographic sum construction.

**Definition 7.9** (See, for example, [6, 9].) Let $T$ be a nonempty ordered set considered as an index set. Let $\{P_t \mid t \in T\}$ be a family of pairwise disjoint nonempty ordered sets that are all disjoint from $T$. We define the **lexicographic sum** $L\{P_t \mid t \in T\}$ (of the $P_t$ over $T$) to be the union $\bigcup_{t \in T} P_t$ ordered by letting $p_1 \preceq p_2$ iff either

1. There are distinct $t_1, t_2 \in T$ with $t_1 < t_2$, such that $p_i \in P_{t_i}$, or
2. There is a $t \in T$ such that $p_1, p_2 \in P_t$ and $p_1 \preceq p_2$ in $P_t$. 

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The ordered sets $P_t$ are the pieces of the lexicographic sum and $T$ is the index set.

**Proposition 7.10** Let $\mathcal{C}$ be a class of finite ordered sets for which the Automorphism Conjecture holds and let $\mathcal{L}_\mathcal{C}$ be the class of lexicographic sums $L\{P_t|t \in T\}$ such that all $P_t \in \mathcal{C}$ and $T \in \mathcal{C}$ is indecomposable or a chain, or, all $P_t$ are antichains and $T \in \mathcal{C}$ does not contain any nontrivial order-autonomous antichains. Then the Automorphism Conjecture holds for $\mathcal{L}_\mathcal{C}$.

**Proof.** For every $n \in \mathbb{N}$, let

$$h(n) := \max_{P \in \mathcal{C}, |P| \geq n} \frac{|\text{Aut}(P)|}{|\text{End}(P)|}.$$ 

Because $\lim_{n \to \infty} h(n) = 0$, for $n \geq 2$, we have that $h(n) < 1$.

For every $L\{P_t|t \in T\} \in \mathcal{L}_\mathcal{C}$, let $A(P_t : t \in T) := \sum_{|P_t| > 1} |P_t|$.

Let $\varepsilon > 0$. Fix $K \in \mathbb{N}$ such that, for all $k \geq K$, we have that $(h(2))^k < \varepsilon$ and $h(k) < \varepsilon$. Choose $N \in \mathbb{N}$ such that $N \geq K^2$ and such that, for all $n \geq N$, we have $h(n - K^2) \leq \varepsilon^2$ and $(K^2)! \leq \frac{1}{\sqrt{h(n - K^2)}}$. Then, for every $L\{P_t|t \in T\} \in \mathcal{L}_\mathcal{C}$ with $n \geq N$ elements, we obtain the following.

Recall that automorphisms map order-autonomous sets to order-autonomous sets. Therefore, for every automorphism $\Phi \in \text{Aut}(P)$, all endomorphisms of all sets $P_s$ translate into pairwise distinct order-preserving maps from $P_s$ to $\Phi[P_s]$. In case $A(P_t : t \in T) > K^2$, this provides the following estimates.

In case $A(P_t : t \in T) > K^2$ and there is an $s \in T$ with $|P_s| > K$, we have

$$\frac{|\text{Aut}(L\{P_t|t \in T\})|}{|\text{End}(L\{P_t|t \in T\})|} \leq \frac{|\text{Aut}(P_s)|}{|\text{End}(P_s)|} \leq h(|P_s|) < \varepsilon$$

In case $A(P_t : t \in T) > K^2$ and, for all $t \in T$ we have $|P_t| \leq K$, let $R$ be the set of indices $r \in T$ such that $|P_r| > 1$. Then $|R| \geq K$ and we have

$$\frac{|\text{Aut}(L\{P_t|t \in T\})|}{|\text{End}(L\{P_t|t \in T\})|} \leq \prod_{r \in R} \frac{|\text{Aut}(P_r)|}{|\text{End}(P_r)|} \leq (h(2))^K < \varepsilon$$

This leaves the case $A(P_t : t \in T) \leq K^2$. Because $T \in \mathcal{C}$ is indecomposable or a chain or $T \in \mathcal{C}$ does not contain any nontrivial order-autonomous antichains and all $P_t$ are antichains, automorphisms map sets $P_t$ to sets $P_t$ and
every automorphism induces a corresponding automorphism on \( T \). Hence we obtain the following.

\[
\frac{|\text{Aut}(L\{P_t|t \in T\})|}{|\text{End}(L\{P_t|t \in T\})|} \leq \frac{|\text{Aut}(T)| \cdot A(P_t : t \in T)!}{|\text{End}(T)|} \\
\leq h(|T|) A(P_t : t \in T)! \\
\leq h(|T|) (K^2)! \\
\leq \sqrt{h(n - K^2)} < \varepsilon
\]

\[\blacksquare\]

**Theorem 7.11** The Automorphism Conjecture is true for ordered sets of width \( w \leq 11 \).

**Proof.** By Proposition 7.7 and Lemma 7.8, the Automorphism Conjecture is true for ordered sets of width \( w \leq 11 \) without nontrivial order-autonomous antichains. Now apply Proposition 7.10, using the ordered sets of width \( \leq 11 \) without order-autonomous antichains and the antichains as the base class \( C \). This proves the Automorphism Conjecture for a class that contains all ordered sets of width \( w \leq 11 \).

\[\blacksquare\]

**Remark 7.12** By Table 1, the only primitive group of degree \( |X| = 13 \) satisfies \( |G| \leq 2^{0.9582|X|} \). Thus orbits of size 13 could be added to the key Lemma 7.2. However, for the imprimitive groups of degree \( |X| = 12 \), the order \((6!)^22!\) exceeds \(2^{1.5|X|}\). It should not be too hard to obtain an analogue of Lemma 7.1 to manage this permutation group and to make sure that there are no other problematic cases. However, to establish the Automorphism Conjecture for all ordered sets of width \( \leq 13 \), we would also need to treat the situation of orbits as in Example 9.7 below (with \( M = 6 \)). Hence such an attempt at an extension will encounter additional challenges.

Although this possible extension feels very feasible (also see Remark 7.13 below), and the author is not superstitious, it stands to reason that energy expended on small orbits should be expended as indicated in Remark 5.5: Strengthenings of Lemma 5.2 are likely to have more far-reaching consequences, as can be seen from Section 6.

**Remark 7.13** An ordered set \( P \) is called **graded** iff there is a function \( g : P \to \mathbb{N} \) such that, if \( y \) is an upper cover of \( x \), then \( g(y) - g(x) = 1 \). For
any automorphism $\Phi : P \to P$ of a graded ordered set $P$, and for all $x \in P$, we have that $g(\Phi(x)) = g(x)$. Hence, for a graded ordered set $P$, Lemma 5.1 can be applied with $B := \{x \in P : g(x) = 2k, k \in \mathbb{N}\}$ and $T := \{x \in P : g(x) = 2k - 1, k \in \mathbb{N}\}$. Moreover, the computation in Lemma 6.12 can be done for primitive nestings in which no factor with $\frac{|B_{j+1}|}{|B_j|} \geq 7$ contains an alternating group with $\frac{|B_{j+1}|}{|B_j|}$ elements to obtain a bound of $|G| \leq 2^{1.95|X|}$. Hence, for a graded tight interdependent orbit union $(U, D)$, we can obtain a version of Theorem 6.17 that guarantees $|Aut_D(U)| \leq 2^{\frac{1}{7}1.95|U|}$ under the condition that no factor with $\frac{|B_{j+1}|}{|B_j|} \geq 7$ in a primitive nesting of a $\Lambda_D(D)$ contains an alternating group with $\frac{|B_{j+1}|}{|B_j|}$ elements. Now the same argument as given in this section proves the Automorphism Conjecture for graded ordered sets of width at most 13.

8 More Endomorphisms for Heights 2 and 3

Theorem 7.11 clearly demonstrates the utility of the trivial realization that having more endomorphisms is better when considering the Automorphism Conjecture: The guarantee of at least $2^{(1-\varepsilon)n}$ endomorphisms in Lemma 7.3 allows for slightly less stringent estimates for the number of automorphisms to carry the day. Remark 6.18 indicates that exclusion of a few more factors in primitive nestings even guarantees $|Aut_D(U)| \leq 2^{0.76|U|}$. In Corollary 9.2 below, we will encounter a natural situation in which $|Aut_D(U)| \leq 2^{0.69|U|}$.

Theorem 1 of [5] guarantees that every ordered set $P$ of height $h$ has at least $2^n$ endomorphisms. Clearly, this is a “near miss” when $|Aut_D(U)| \leq 2^{0.69|U|}$. Theorem 1 of [5] is proved by showing that every ordered set $P$ of height 1 has at least $2^n$ endomorphisms (see (C) in Section 2 of [5]), and that every ordered set of height $h$ has at least $2^{\frac{h}{h+1}n}$ endomorphisms (see (D) in Section 2 of [5]). Hence we can improve the lower bound on the number of endomorphisms by improving the lower bounds for heights 2 and 3. The improvement for height 2 suffices for the situation in Corollary 9.2. Since the factor in the improvement exceeds the factor $\frac{3}{4}$ for height 3, it is natural to go one step further to height 3.

**Lemma 8.1** Let $P$ be an ordered set of height 2 with $n$ elements. Then we have $\frac{|Aut(P)|}{|End(P)|} \leq 2^{\sqrt{2}\log(n)}$, or, for $n \geq 22$, we have $|End(P)| \geq 2^{\frac{|k(3)|}{2}n} > 2^{0.7924n}$. 

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Proof. Let \( n_j \) denote the number of elements of rank \( j \). If \( P \) has more than \( \lg(n) \) irreducible points, then, by Theorem 1 in [8], we have \( \frac{|\text{Aut}(P)|}{|\text{End}(P)|} \leq 2\sqrt{\frac{\lg(n)}{2}} \). We are left with the case that \( P \) has fewer than \( \lg(n) \) irreducible points.

If all non-maximal elements of rank 1 are irreducible, then \( n_1 < \lg(n) \) and \( P \setminus R_1 \) is an ordered set of height 1 with more than \( n - \lg(n) \) elements and hence more than \( 2^{n-\lg(n)} \) endomorphisms. For \( n \geq 22 \), we have \( 2^{n-\lg(n)} > 2^{\frac{\lg(3)}{2}n} \). We are left with the case that there is a non-irreducible non-maximal element \( c \) of rank 1.

Let \( b_1, b_2 < c \) be two lower covers of \( c \) and let \( t_1, t_2 > c \) be two upper covers of \( c \). Mapping all elements of \( R_1 \) to \( c \), mapping the elements of \( R_0 \) in any fashion to \( \{b_1, b_2, c\} \), and mapping the elements of \( R_2 \) in any fashion to \( \{c, t_1, t_2\} \) generates \( 3^{n_0+n_1} = 2^{\frac{\lg(3)}{2}(n_0+n_2)} \) endomorphisms. Hence, in case \( n_0 + n_2 \geq \frac{n}{2} \), we have at least \( 2^{\frac{\lg(3)}{2}n} \) endomorphisms. We are left with the case that \( n_0 + n_2 < \frac{n}{2} \).

In this case, we have \( n_1 \geq \frac{n}{2} \). Mapping \( R_0 \) to \( b_1, R_2 \) to \( t_1 \) and mapping the elements of \( R_2 \) to \( \{b_1, c, t_1\} \) in any fashion generates \( 3^{n_1} \geq 2^{\frac{\lg(3)}{2}n} \) endomorphisms.

Lemma 8.2 Let \( P \) be an ordered set of height 3 with \( n \) elements. Then we have \( \frac{|\text{Aut}(P)|}{|\text{End}(P)|} \leq 2\sqrt{\frac{\lg(n)}{2}} \), or, for \( n \geq 27 \), we have \( |\text{End}(P)| \geq 2^{\frac{\lg(3)}{2}(n-\lg(n))} \), which, for \( n \geq 180,000 \), exceeds \( 2^{0.7924n} \).

Proof. Let \( n_j \) denote the number of elements of rank \( j \). If \( P \) has more than \( \lg(n) \) irreducible points, then, by Theorem 1 in [8], we have \( \frac{|\text{Aut}(P)|}{|\text{End}(P)|} \leq 2\sqrt{\frac{\lg(n)}{2}} \). We are left with the case that \( P \) has fewer than \( \lg(n) \) irreducible points.

Consider the case that all non-maximal elements of rank 2 are irreducible. In this case, \( n_2 < \lg(n) \) and \( P \setminus R_2 \) is an ordered set of height 2 with more than \( n - \lg(n) \geq 22 \) elements. The image of an irreducible point under an automorphism is determined by its unique upper or lower cover. Therefore \( |\text{Aut}(P)| = |\text{Aut}(P \setminus R_2)| \). Now, via Lemma 8.1, we obtain \( \frac{|\text{Aut}(P)|}{|\text{End}(P)|} \leq \frac{|\text{Aut}(P \setminus R_2)|}{|\text{End}(P \setminus R_2)|} \leq 2\sqrt{\frac{\lg(|P \setminus R_2|)}{2}} \leq 2\sqrt{\frac{\lg(n)}{2}} \), or, for \( n \geq 27 \), \( |\text{End}(P)| \geq |\text{End}(P \setminus R_2)| \geq 2^{\frac{\lg(3)}{2}(n-\lg(n))} \). Similarly, if all non-minimal elements of dual rank 2 are irreducible, then there are fewer than \( \lg(n) \) elements of dual rank 2, \( P \setminus \{x \in P \setminus \text{min}(P) : \text{dual rank}(x) = 2\} \) is
an ordered set of height 2 with more than \( n - \lg(n) \) elements, and, again, we obtain the requisite bounds via Lemma 8.1. We are left with the case that there is a non-irreducible non-maximal element \( c \) of rank 2 and that there is a non-irreducible non-minimal element \( d \) of dual rank 2.

Let \( t_1, t_2 > c \) be two upper covers of \( c \) and let \( a < b < c \).

If \( n_2 + n_3 \geq \frac{n}{2} \), mapping \( R_0 \cup R_1 \) to \( a \), mapping \( R_2 \) to \( \{a, b, c\} \), and mapping \( R_3 \) to \( \{c, t_1, t_2\} \) generates \( 3^{n_2 + n_3} \geq 2^{\frac{\lg(3)}{2} - n} \) endomorphisms. If \( n_0 + n_3 \geq \frac{n}{2} \), mapping \( R_0 \) to \( a \), mapping \( R_2 \) to \( c \), mapping \( R_1 \) to \( \{a, b, c\} \), and mapping \( R_3 \) to \( \{c, t_1, t_2\} \) generates \( 3^{n_0 + n_3} \geq 2^{\frac{\lg(3)}{2} - n} \) endomorphisms.

If there is an \( i \in \{0, 1, 2\} \) such that \( n_i = \frac{n}{2} \), then, dual to the preceding argument, we can use \( d \), two of its lower covers, and a chain \( d < b < a \), to construct \( \geq 2^{\frac{\lg(3)}{2} - n} \) endomorphisms.

Using the larger of \( n_0 \) and \( n_3 \) and the larger of \( n_1 \) and \( n_2 \), we conclude that, for some \( j \in \{0, 1, 2\} \), we must have \( n_j + n_3 \geq \frac{n}{2} \), or, for some \( i \in \{0, 1, 2\} \), we must have \( n_0 + n_i \geq \frac{n}{2} \). This, and a quick computation to verify the final claim, concludes the proof.

9 Orbits Carrying Primitive Permutation Groups

Section 6 shows the significant utility of the ideas presented here beyond the proof of concept in Section 7. The only drawback to Section 6 is that there is no easily stated class of ordered sets for which the Automorphism Conjecture is confirmed. In this section we will focus on tight interdependent orbit unions such that all orbits carry primitive permutation groups. We will see that the
Automorphism Conjecture can be confirmed for ordered sets whose natural interdependent orbit unions are so that all orbits carry primitive permutation groups, and which do not exhibit the problem indicated in Example 9.7. In particular, this gives a first insight how to possibly handle factorial factors.

**Proposition 9.1** Let \((U, D)\) be a tight interdependent orbit union such that all \(\Lambda_D(D)\) are primitive. Then, for any \(D \in \mathcal{D}\), we have \(|\text{Aut}_D(U)| = |\Lambda_D(D)|\).

**Proof.** We will prove by induction, that, for any subgroup \(G^*\) of \(|\text{Aut}_D(U)|\) such that all \(\Lambda^*_D(D)\) are primitive, for any \(D \in \mathcal{D}\), we have that, \(|G^*| = |\Lambda^*(D)|\).

The proof is an induction on the number \(m = |\mathcal{D}|\) of orbits. For the base case \(m = 2\), first note that, because order-autonomous antichains in \(D\) constitute blocks of \(\Lambda^*_D(D)\), no orbit \(D \in \mathcal{D}\) can contain a nontrivial order-autonomous antichain. In the absence of nontrivial order-autonomous antichains within either of \(D_1\) and \(D_2\), by Lemma 2.1, every \(\Phi \in \text{Aut}_D(D)\) is completely determined by \(\Phi|_{D_1}\) or by \(\Phi|_{D_2}\), which leads to \(|G^*| = |\Lambda^*(D_i)|\) for either \(i \in \{1, 2\}\).

For the induction step \(\{1, \ldots, m - 1\} \to m\), let \(\mathcal{D} \in \mathcal{D}\). We first note that \(\mathcal{O}(U, \mathcal{D})\) has at least two noncutvertices. Therefore, we can relabel the orbits in \(\mathcal{D}\) such that \(D_m\) is a noncutvertex that is not equal to \(D\).

By Theorem 4.13, we have \(|G^*| = |\text{Aut}^*_{D_m}(U_m)| \cdot |\text{Aut}^*_{DQ}(Q)|\). Because all \(\Lambda^*_D(D)\) are primitive, all partitions \(\mathcal{A}^j\) from Notation 4.2 must be partitions into singletons. Therefore, the only element of \(\text{Aut}^*_{DQ}(Q)\) is the identity. Moreover, for all \(j \in \{1, \ldots, m - 1\}\), we have that the permutation group induced by \(\text{Aut}^*_{D_m}(U_m)\) on \(D_j\) is \(\Lambda^*_D(D_j)\) and hence it is primitive. Therefore we can apply the induction hypothesis to \(\text{Aut}^*_{DQ}(Q)\).

We conclude that \(|G^*| = |\text{Aut}^*_{D_m}(U_m)| = |\Lambda^*(D)|\), which concludes the induction. The result now follows by using \(G^* = \text{Aut}_D(U)\) in the claim we just proved.

**Corollary 9.2** Let \((U, \mathcal{D})\) be a tight interdependent orbit union such that all \(\Lambda_D(D)\) are primitive and don’t contain \(A_{|D|}\). Then \(|\text{Aut}_D(U)| \leq 2^{1.38 \min\{|D|: D \in \mathcal{D}\}} \leq 2^{0.69|U|}\).

**Proof.** By Theorem 6.6, we obtain that, for all \(D \in \mathcal{D}\), we have \(|\Lambda_D(D)| < 2^{1.38|D|}\). By Proposition 9.1, we obtain that, for any \(D \in \mathcal{D}\),
we have $|\text{Aut}_D(U)| = |\Lambda_D(D)| < 2^{1.38|D|}$. Because there is a $D \in \mathcal{D}$ with $|D| \leq \frac{1}{2}|U|$, we obtain the conclusion. ■

**Proposition 9.3** Let $(U, \mathcal{D})$ be a tight interdependent orbit union such that all $\Lambda_D(D)$ are primitive, such that there is a $D \in \mathcal{D}$ such that $\Lambda_D(D)$ contains a copy of $A_{|D|}(D)$ and such that not all orbits $C \in \mathcal{D}$ have the same size. Then $|U| \geq \frac{1}{2}|D|(|D| + 1)$ and hence $|\text{Aut}_D(U)| \leq 2^{|U|}$.

**Proof.** Let $D$ be an orbit such that $\Lambda_D(D)$ contains a copy of $A_{|D|}(D)$. By Lemma 2.5 (and trivially for $|D| = 2$), if $D \parallel \mathcal{D} E$ and $|D| = |E| =: w$, then $D \cup E$ is isomorphic to an ordered set $wC_2$ or $S_w$ and hence $\Lambda_D(E)$ contains a copy of $A_{|E|}(E)$. Because not all frames in $\mathcal{D}$ are of the same size, we can assume, without loss of generality, that there is an $E \in \mathcal{D}$ such that $D \parallel \mathcal{D} E$ and $|D| \neq |E|$. Because $\Lambda_D(E)$ is primitive, we obtain $|D| \geq 3$. By Lemma 2.5, we have that there is a $k \in \{2, \ldots, |D| - 2\}$ such that $|E| = \left(\frac{|D|}{k}\right) \geq \frac{1}{2}|D|(|D| - 1)$.

Hence $|U| \geq |D| + \frac{1}{2}|D|(|D| - 1) = \frac{1}{2}|D|(|D| + 1)$. By Proposition 9.1, we have $|\text{Aut}_D(U)| \leq |\Lambda_D(D)| \leq |D|! \leq 2^{0.4605 \frac{1}{2}|D|(|D| + 1)} \leq 2^{|U|}$. ■

**Remark 9.4** Theorem 8.3, Corollary 9.2 and Proposition 9.3 show that, when considering the Automorphism Conjecture, for the case of $(U, \mathcal{D})$ being a tight interdependent orbit union such that all $\Lambda_D(D)$ are primitive, the only case that remains is the case that there is a $D$ such that $\Lambda_D(D)$ contains a copy of $A_{|D|}(D)$ and such that all orbits $C \in \mathcal{D}$ have the same size.

More progress is possible with the ideas from Lemma 7.6 and Proposition 7.7: When the orbit graph is a tree, it can be proved that $|\text{End}_D(U)| \geq (w - 1)^w$, where $w$ is the width of an orbit. However, although Lemma 9.6 below shows that the structure must be very specific when the orbit graph contains cycles, Example 9.7 shows that $\mathcal{D}$-endomorphisms will not provide the complete answer here. Although this may be a bit anticlimactic, Example 9.7 is representative for the only remaining problem for these interdependent orbit unions, which should be resolvable in the future.

**Definition 9.5** Let $(U, \mathcal{D})$ be an interdependent orbit union and let $\mathcal{C} = \{D_1 \parallel \mathcal{D} D_2 \parallel \mathcal{D} \cdots \parallel \mathcal{D} D_N \parallel \mathcal{D} D_1\}$ be a cycle in $\mathcal{O}(U, \mathcal{D})$. Then $\mathcal{C}$ is called a **lock cycle** iff there is an $M > 1$ such that, for all $i \in \{1, \ldots, N\}$ and with index arithmetic modulo $N$, we have that $D_i \cup D_{i+1}$ is isomorphic to $S_M$ or to
For $x, y \in D_1$, we write $x \circ y$ and say $y$ is **cycle locked** to $x$ iff there is a sequence $x = x_1, x_2, \ldots, x_N, x_{N+1} = y$ such that, for all $i \in \{1, \ldots, N\}$, we have that $x_i \in D_i$, and, if $D_i \cup D_{i+1}$ is isomorphic to $S_M$, then $x_i \not\sim x_{i+1}$, and, if $D_i \cup D_{i+1}$ is isomorphic to $MC_2$, then $x_i \sim x_{i+1}$.

**Lemma 9.6** Let $(U, D)$ be an interdependent orbit union. Let $D_1 \upharpoonright_D D_2 \upharpoonright_D \cdots \upharpoonright_D D_N \upharpoonright_D D_1$ be a lock cycle in $O(U, D)$ such that there are distinct $x, y \in D_1$ such that $x \circ y$. Then $|\Lambda_D(D_1)| \leq \frac{1}{M-1}M!$. In particular, $\Lambda_D(D_1)$ does not contain a copy of $A_{|D_1|}(D_1)$.

**Proof.** Every $\Phi \in \text{Aut}_D(U)$ must preserve the $\circ$-relation and every $x \in D_1$ is $\circ$-related to exactly one element in $D_1$. Therefore, if $x \circ y$ and $x \neq y$, then, for every $\Phi \in \text{Aut}_D(U)$, $\Phi(x)$ uniquely determines $\Phi(y)$ as the unique element that is $\circ$-related to $\Phi(x)$. Hence $|\Lambda_D(D_1)| \leq \frac{1}{M-1}M!$. 

Lemma 9.6 gives further information about the structure of interdependent orbit unions with induced permutation groups $\Lambda_D(D)$ that contain the alternating group on $D$ and such that all orbits are of the same size: In every lock cycle, all elements are cycle locked to themselves. Unfortunately, we cannot easily dispense with this remaining case in the same way we did when the width was bounded by 11 because of the following situation (and similar ones).

**Example 9.7** Let $M \in \mathbb{N} \setminus \{1, 2\}$ and consider the interdependent orbit union $(U, D)$ with $D = \{D_1, D_2, D_3, D_4\}$, $D_i = \{x_1^i, \ldots, x_M^i\}$ such that $x_j^1 < x_j^k$ iff $j \neq k$, $x_j^2 < x_j^k$ iff $j = k$, $x_j^3 < x_j^k$ iff $j > k$, $x_j^4 < x_j^k$ iff $j = k$, and no further comparabilities. Then every $f \in \text{End}_D(U)$ is an automorphism.

Let $f \in \text{End}_D(U)$. Without loss of generality, we can assume that $f$ is a retraction onto its range. Hence $f|_{D_1 \cup D_2}$ is a retraction that preserves minimal and maximal elements. Let $x_j^1 \in f[D_1]$. Then no element of $D_2 \setminus \{x_j^2\}$ is mapped to $x_j^2$. Similarly, if $x_j^2 \in f[D_2]$, then no element of $D_1 \setminus \{x_j^1\}$ is mapped to $x_j^1$.

Let $x_j^1 \in f[D_1]$. Then $x_j^4 \in f[D_4]$, $x_j^3 \in f[D_3]$ and $x_j^2 \in f[D_2]$. Similarly, if $x_j^2 \in f[D_2]$, then $x_j^1 \in f[D_1]$. Hence $x_j^1 \in f[D_1]$ iff $x_j^2 \in f[D_2]$. Via the preceding paragraph, we conclude that no element in $D_1 \setminus f[D_1]$ can be mapped to any element of $f[D_1]$. Hence $f[D_1] = D_1$ and $f$ must be an automorphism.
10 Conclusion

The insights presented in Sections 3 and 4 are a significant addition to our understanding of the automorphic structure of ordered sets. The connection to permutation groups on orbits in Section 5 is a significant advance towards confirming the Automorphism Conjecture. The remarks throughout this presentation indicate a wide variety of fruitful paths forward. Remark 5.5 indicates that deeper use of the order-theoretical properties of the residual sets \((Q, \mathcal{D}_Q)\) should allow a further tightening of the bounds so far. Remark 6.18 articulates that the main remaining challenges will be with factor groups of the \(\Lambda_\mathcal{D}(D)\) which contain a subgroup \(A_{|D|}(D)\) and possibly with some factor groups from among the small primitive groups from Table 1. Remark 6.19 indicates that further refinements of the key Lemma 5.2 are a worthy target for future investigations. The work in Section 9 shows that further progress can be made when the problematic factors in \(\Lambda_\mathcal{D}(D)\) necessitate that certain orbits \(E \upharpoonright \mathcal{D} D\) must be large. Although Section 9 focuses on primitive groups \(\Lambda_\mathcal{D}(D)\), a group \(G \triangleleft^G_B A\), where \(A, B\) are blocks of \(\Lambda_\mathcal{D}(D)\), that contains an alternating group on the blocks in \(A[G \cdot B]\) will, through a variation on Lemma 2.5, for every \(E \upharpoonright \mathcal{D} D\), require that \(D \cup E\) contains two levels of a Boolean algebra or a set \(wC_2\), but these levels will be made up of blocks rather than points, and there may be multiple parallel copies. Unfortunately, this does not automatically create the size imbalance that was used in Proposition 9.3, because the blocks in \(E\) might contain structures that, in turn, require larger blocks in another level of the primitive nesting of \(D\). This and the fact that, aside from the alternating and symmetric groups, all remaining problematic groups are small, and can indeed be explicitly identified in a short list, makes the author very hopeful for the future of this work.

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11 Declarations

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