Integer-valued fixed point index for compositions of acyclic maps

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Dedicated to the memory of Jean Leray

Abstract

An integer-valued fixed point index for compositions of acyclic multivalued maps is constructed. This integer-valued fixed point index has the properties: additivity, homotopy invariance, normalization, commutativity, multiplicativity. The acyclicity is with respect to the Čech cohomology with integer coefficients. The technique of chain approximation is used.

Keywords: Fixed point index, multivalued maps, acyclic maps

The fixed point index (f. p. i.) for compositions of $\mathbb{F}$-acyclic multivalued maps of locally compact ANR spaces was constructed in [18, 22, 23, 24, 6, 12]. This f. p. i. has values in the field $\mathbb{F}$ and has the properties: additivity, homotopy invariance, normalization, commutativity, multiplicativity and mod $p$. The $\mathbb{F}$-acyclicity of the multivalued maps is with respect to the Čech homology, i.e., the reduced Čech homology with coefficients in $\mathbb{F}$ of the images of all points is equal to zero. The construction used the chain approximation technique.

An integer-valued f. p. i. for $\mathbb{Z}$-acyclic multivalued maps of ENR spaces, i.e., finite dimensional ANR spaces, with all properties mentioned above, was constructed in [5], see also [13], p. 547-550. The $\mathbb{Z}$-acyclicity is with respect to the Čech cohomology. The construction of this integer-valued f. p. i. is based on some homotopic considerations. Both constructions are presented in [12], p. 163-173 and p. 251-276, where all necessary definitions and results used in this note are given, see also [13].

The question was raised about the existence of integer valued f. p. i. with all properties for $\mathbb{Z}$-acyclic maps, with respect to Čech (co)homology, of compact ANR spaces, see [5], [12], ch. IV, Section 34, p. 173. The difficulty is that the chain approximation technique, developed in [18, 22, 23, 24] and based on [11, 26], does not work in the case of $\mathbb{Z}$-acyclic maps with respect to the Čech homology, see the example in [5]. Moreover, the technique of [5] is based on some results and arguments from geometric topology for which finite dimensionality of the spaces is essential. A particular solution, based on the homotopy method, was proposed in [15].

In this note we shall give a construction of integer-valued f. p. i. with all properties for compositions of $\mathbb{Z}$-acyclic maps of compact ANR spaces. The acyclicity is with respect to the Čech cohomology with integer coefficients. The chain approximation technique is applied. For this a relation between Čech cohomology theory with integer coefficients and the Steenrod-Sitnikov homology with integer coefficients and some results about the last homologies are used.

1 Preliminaries

Here we shall define some notions and fix some notations. As usual, we denote by $\mathbb{N}$ the natural numbers and by $\mathbb{Z}$ the integers.

All topological spaces are assumed to be compact metric spaces.
For a given space $X$ we denote by $\text{Cov}(X)$ the set of all its finite open coverings. For $\alpha, \beta \in \text{Cov}(X)$ we say that $\beta$ is a refinement of $\alpha$, denoted as $\beta \triangleright \alpha$, if for every element $U \in \beta$ there is an element $V \in \alpha$ such that $U \subset V$.

Let $\alpha = \{U_0,\ldots,U_n\} \in \text{Cov}(X)$ and let $A$ be a closed subset of $X$. By $\alpha | A$ we denote the open covering of the set $A$ whose elements are all nonempty sets $U_0 \cap A, \ldots, U_n \cap A$.

If $A$ is a subset of $X$ and $\alpha \in \text{Cov}(X)$, we denote by $\text{St}(A, \alpha)$ the union of all elements of the covering $\alpha$ which meet $A$.

We shall use the standard definitions for chain complexes, chain maps, chain homotopies, cohomologies, see e.g. [3, 25]. For compact metric spaces we use Čech cohomology and Čech homology with coefficients in given Abelian groups, [3, 25, 20]. We use also the Steenrod-Sitnikov homology theory with integer coefficients for compact metric spaces, [19, 20].

For a given space $X$ and $\alpha \in \text{Cov}(X)$ we denote by $N(\alpha)$ the nerve of the covering $\alpha$. The vertices of this (abstract) simplicial complex are the elements of the covering $\alpha$. Furthermore, the $k+1$ vertices $U_0,\ldots,U_k$ are vertices of a $k$-simplex $\sigma^k = [U_0,\ldots,U_k]$ in $N(\alpha)$ if and only if

$U_0 \cap \ldots \cap U_k$ is not empty. The support $\text{supp}(\sigma^k)$ of the simplex $\sigma^k$ is the set $U_0 \cup \ldots \cup U_k$. Let $c = \sum_{i=1}^n g_i \sigma^{k_i} \in C_k(N(\alpha); G)$, $g_i \neq 0$, be a chain of the simplicial complex $N(\alpha)$ with coefficients in an Abelian group $G$. Then the support $\text{supp}(c)$ of $c$ is the set $\text{supp}(\sigma^{k_1}) \cup \ldots \cup \text{supp}(\sigma^{k_n})$.

By $N(\alpha)^{(n)}$ we denote the $n$-th skeleton of the simplicial complex $N(\alpha)$.

Denote by $\pi(\beta, \alpha) : N(\beta) \rightarrow N(\alpha)$ an induced simplicial map for given coverings $\alpha, \beta \in \text{Cov}(X)$ with $\beta \triangleright \alpha$. For a given closed subset $A$ in $X$ we denote also by $\pi(\beta, \alpha) : N(\beta | A) \rightarrow N(\alpha | A)$ the restriction of the map $\pi(\beta, \alpha)$ on the simplicial complex $N(\beta | A)$.

A multivalued map $F : X \rightarrow Y$ is a map which assigns to every point $x \in X$ a nonempty compact set $F(x)$ in $Y$. The graph $\Gamma(F)$ of the map $F$ is the set $\Gamma = \{(x,y) \in X \times Y : y \in F(x)\}$. The map $F$ is called upper semi-continuous (u.s.c.) if the graph $\Gamma(F)$ is a closed subset in the space $X \times Y$. We have two projections $p : \Gamma(F) \rightarrow X$, defined by $p(x,y) = x$ and $q : \Gamma(F) \rightarrow Y$, defined by $q(x,y) = y$ for $(x,y) \in \Gamma(F)$. Then $F(x) = q(p^{-1}(x))$ for $x \in X$. For u.s.c. maps and their general properties see [11, 12].

Let $\tilde{H}$ be a reduced homology or cohomology theory with coefficients in an Abelian group $G$. A compact space $Z$ is called $G$-acyclic with respect to (w.r.t.) $\tilde{H}$ if $\tilde{H}(Z; G) = 0$. An u.s.c. map $F : X \rightarrow Y$ is called $G$-acyclic w.r.t. to $\tilde{H}$ if the spaces $F(x)$ are $G$-acyclic w.r.t. $\tilde{H}$ for all $x \in X$.

A single-valued continuous map $f : X \rightarrow Y$ is called $G$-acyclic w.r.t. $\tilde{H}$ if the inverse map $f^{-1} : Y \rightarrow X$ of $f$ is u.s.c. multivalued $G$-acyclic w.r.t. $\tilde{H}$.

Let $\Gamma(F)$ be the graph of the $G$-acyclic w.r.t. $\tilde{H}$ multivalued map $F$ and $p : \Gamma(F) \rightarrow X$, $q : \Gamma \rightarrow Y$ the projections defined above. Then the single-valued map $p : \Gamma(F) \rightarrow X$ is $G$-acyclic w.r.t. $\tilde{H}$. The map $p : \Gamma(F) \rightarrow X$ induces a homomorphism in the (co)homologies $\tilde{H}$. This homomorphism is an isomorphism in the cases:

(a) $\tilde{H}^*$ are the reduced Čech cohomologies with coefficients in an arbitrary Abelian group $G$, [25], Theorem 15, p. 344.

(b) $\tilde{H}_*$ are the reduced Čech homologies with coefficients in an arbitrary field $G$, [1], Theorem 2, p. 538.

(c) $\tilde{H}_*$ are the reduced Steenrod-Sitnikov homology with integer coefficients [14], Theorem 3.2, p. 57.

In the cases (b) and (c) we did not state the most general results, for them see [20], Ch. 8, 6.2.

In the case (a) we define the homomorphism $F^* : H^*(Y; G) \rightarrow H^*(X; G)$ by $F^* = (p^*)^{-1}q^*$, where $p^* : H^*(X; G) \rightarrow H^*(\Gamma(F); G)$ and $q^* : H^*(Y; G) \rightarrow H^*(\Gamma(F); G)$ are the homomorphisms induced by the projections $p$ and $q$, respectively.

Similarly, in the cases (b) and (c) we define $F_* : \tilde{H}_*(Y; G) \rightarrow \tilde{H}_*(X; G)$ by $F_* = q_*(p_*)^{-1}$,
where \( p_* : \tilde{H}_*(\Gamma(F); G) \to \tilde{H}_*(X; G) \) and \( q_* : \tilde{H}^*(\Gamma(F); G) \to \tilde{H}^*(Y; G) \) are the homomorphisms induced by the maps \( p \) and \( q \), respectively.

## 2 Maps of order \( n \) with respect to \( G \)

Here we shall give, in an appropriate form, some definitions and results of E. Begle, \cite{1}.

**Definition 1** Let \( X \) be a compact metric space and let \( G \) be an Abelian group. Let \( n \) be a natural number. The compact \( X \) is called \((n, G)\)-compact if for every covering \( \alpha \in \text{Cov}(X) \) there is a covering \( \mu \in \text{Cov}(X) \) such that \( \mu > \alpha \) and the homomorphism

\[
\pi(\mu, \alpha)_k : \tilde{H}_k(N(\mu); G) \to \tilde{H}_k(N(\alpha); G)
\]

is the zero homomorphism for \( 0 \leq k \leq n \).

**Remark 1** Every \((n, G)\)-compact space is \((n, G)\)-acyclic with respect to the Čech homology, i.e., the reduced homology \( \tilde{H}_k(X; G) = 0 \) for \( k = 0, \ldots, n \).

The converse is not true, e.g., the \( 2 \)-adic solenoid is \((1, \mathbb{Z})\)-acyclic with respect to the Čech homology with integer coefficients, but is not \((1, \mathbb{Z})\)-compact space, see \cite{8}, Ch. X, Exercise F. Both properties are equivalent when \( G \) is a field.

**Definition 2** (cf. \cite{1}, Section 3) Let \( F : X \to Y \) be an u.s.c. multivalued map of the compact metric space \( X \) in the compact metric space \( Y \). Let \( G \) be an Abelian group. The map \( F \) is called map of order \( n \) w.r.t. the group \( G \), written \((n, G)\)-map, if \( F(x) \) is \((n, G)\)-compact for each point \( x \in X \).

In \cite{1} this definition is given for maps \( F = f^{-1} \), where \( f : Y \to X \) is a single-valued continuous onto map. E. Begle called \( f \) Vietoris map of order \( n \). We say that \( f \) is a single-valued \((n, G)\)-map.

Recall that E. Begle proved the Vietoris theorem for single-valued onto \((n, G)\)-maps \( f : Y \to X \). For such maps the induced homomorphism \( f_k : \tilde{H}_k(Y; G) \to \tilde{H}_k(X; G) \) of the reduced Čech homologies is an isomorphism for \( 0 \leq k \leq n \), \cite{1}, Section 3, Theorem 1. This theorem is not true in the case when the map \( f \) is \((n, \mathbb{Z})\)-acyclic with respect to the Čech homology, \cite{3}.

E. Begle derived the Vietoris theorem from the two lemmas below, in the case of the map \( F = f^{-1} \).

**Lemma 1** (cf. \cite{1}, Section 4, Lemma 2) If \( F : X \to Y \) is an u.s.c. multivalued \((n, G)\)-map, then for each covering \( \alpha \in \text{Cov}(X) \) and each covering \( \beta \in \text{Cov}(Y) \) there is a covering \( \nu = \nu(\alpha, \beta) \in \text{Cov}(X) \), with \( \nu > \alpha \), and a chain map \( T(\nu, \alpha) = \{ T(\nu, \beta)_k \}, k = 0, \ldots, n + 1 \)

\[
T(\nu, \alpha)_k : C_k(N(\nu)^{(n+1)}; G) \to C_k(N(\beta)^{(n+1)}; G)
\]

such that

1. for any \( k \)-simplex \( \sigma \in N(\nu) \) there is a point \( x(\sigma) \in X \) with
   (a) \( \text{supp}(\sigma) \subset \text{St}(x(\sigma), \alpha) \),
   (b) \( \text{supp}(T(\nu, \beta)_k(\sigma)) \subset \text{St}(F(x(\sigma)), \beta) \),
2. \( fT(\nu, \alpha)_k(\sigma) \) is chain homotopic to \( \sigma \) on \( N(\alpha) \).

3
For the next Lemma we use the notations of Lemma 1.

**Lemma 2** (cf. [1], Section 4, Lemma 3) Let \( F : X \rightarrow Y \) be an u.s.c. multivalued \((n,G)\)-map. Let \( \alpha \) and \( \alpha_1 \) be coverings of \( X \) with \( \alpha_1 > \alpha \). Let \( \beta \) and \( \beta_1 \) be coverings of \( Y \) with \( \beta_1 > \beta \). Let \( \nu = \nu(\alpha,\beta) \) and \( \nu_1 = \nu(\alpha_1,\beta_1) \). Let \( T(\nu,\beta) \) and \( T_1(\nu_1,\beta_1) \) be the chain maps from Lemma 1. Then there is a common refinement \( \gamma \) of the coverings \( \nu \) and \( \nu_1 \) such that the chain map \( T(\nu,\beta)\pi(\gamma,\nu) \) is chain homotopic with \( \pi(\beta_1,\beta)T_1(\nu_1,\beta_1)\pi(\gamma,\nu_1) \) to a chain homotopy \( D \) with the property: for every simplex \( \sigma \in N(\gamma) \) there is a point \( c(\sigma) \in X \) with:

1. \( \text{supp}(\sigma) \subset \text{St}(c(\sigma),\alpha) \),

2. \( \text{supp}(D(\sigma)) \subset \text{St}(F(c(\sigma)),\beta) \).

We say that the homotopy \( D \) is \((F,\alpha,\beta)\)-small.

**Remark 2** 1. Lemma 2 and Lemma 3 in [1] are stated for the multivalued map \( F = f^{-1} \), i.e., for the inverse of the single-valued map \( f : Y \rightarrow X \) in an equivalent form for Vietoris chains. The more general form given above follows easily from Lemma 2 and Lemma 3 in [1].

2. The chain maps \( \{T(\nu,\beta),\alpha \in \text{Cov}(X),\beta \in \text{Cov}(Y),\nu = \nu(\alpha,\beta) \in \text{Cov}(X)\} \) induce the homomorphisms \( F_k : \tilde{H}_k(X;G) \rightarrow \tilde{H}_k(Y;G) \) for \( 0 \leq k \leq n \).

3. The properties 1a),b) of Lemma 1 and 1, 2 of Lemma 2 are not stated in Lemma 2, Lemma 3 in [1], in the case \( F = f^{-1} \), but are explicit in the proofs of these lemmas given there.

### 3 Chain approximations and approximation systems for u.s.c. maps

Chain approximations for u.s.c. multivalued maps were used by L. Vietoris (see [20]), S. Eilenberg and D. Montgomery ([2]), E. Begle ([1, 2]) and B. O’Neil ([16]). The explicit definition is given in [18]. They are developed further in [2, 23, 24, 6, 12], Ch. 4, p. 251-276. In all these papers the authors, except E. Begle in [1], work with \( G \)-acyclic maps w.r.t. the Čech homology with coefficients in a field \( G \). The most general case is considered by E. Begle in [1] for Čech homology with coefficients in an Abelian group \( G \) for single-valued \((n,G)\)-maps. As mentioned before every \((n,G)\)-map is \((n,G)\)-acyclic but not vice versa.

Here we shall give the definitions of chain approximations and approximation systems for \((n,G)\)-maps. They are the same as the definitions given in [6, 12], Ch. 4, p. 251-276, in the case where \( G \) is a field.

The chain approximations and the approximation systems for \((n,Z)\)-maps are defined in the same way and have the same properties as the chain approximations and the approximation systems for \( F \)-acyclic maps for \( F \) a field. Moreover, the proofs of the corresponding properties for \((n,G)\)-maps are the same as for \( F \)-acyclic maps. For this reason we skip these proofs, but give exact references for the corresponding proofs in [12].

**Definition 3** (cf. [12], Definition 50.29, p. 255) Let \( F : X \rightarrow Y \) be an u.s.c. multivalued map and let \( G \) be an Abelian group. Let \( \alpha, \overline{\alpha} \in \text{Cov}(X), \overline{\alpha} > \alpha \), and let \( \beta \in \text{Cov}(Y) \). An augmentation preserving chain map \( \varphi : C_\ast(N(\overline{\alpha})^{(n+1)};G) \rightarrow C_\ast(N(\beta)^{(n+1)};G) \) is called \((n,\alpha,\beta)\)-approximation of the map \( F \) provided for each simplex \( \sigma \in N(\overline{\alpha})^{(n+1)} \) there is a point \( x(\sigma) \in X \) such that

1. \( \text{supp}(\sigma) \subset \text{St}(x(\sigma),\alpha) \),
Remark 3 Let $F : X \rightarrow Y$ be an u.s.c. multivalued $(n,G)$-map. The chain map $T(\nu, \beta)$ from Lemma 1 is an $(n, \alpha, \beta)$-approximation of the map $F$.

Definition 4 Let $F : X \rightarrow Y$ be an u.s.c. multivalued map. An $(n,G)$-approximation system, written $(n,G,A)$-system, of the map $F$ is a collection of chain maps $\{\varphi(\nu, \beta) : \alpha, \nu = \nu(\alpha, \beta) \in Cov(X), \beta \in Cov(Y)\}$, where

- the chain map $\varphi(\nu, \beta) : C_*(N(\nu)^{(n+1)}; G) \rightarrow C_*(N(\beta)^{(n+1)}; G)$ is an $(n, \alpha, \beta)$-approximation of $F$.

- Let $\nu = \nu(\alpha, \beta) \in Cov(X)$ correspond to a given $\alpha \in Cov(X)$ and $\beta \in Cov(Y)$. Let $\nu_1 = \nu(\alpha_1, \beta_1) \in Cov(X)$ correspond to a given $\alpha_1 > \alpha$ and $\beta_1 > \beta$. Then it follows: there exists $\gamma \in Cov(X)$ wit $\gamma > \nu$ and $\gamma > \nu_1$ such that the chain maps $\varphi(\nu, \beta)\pi(\gamma, \nu)$ and $\pi(\beta_1, \beta)\varphi(\nu_1, \beta_1)\pi(\gamma, \beta_1)$ are homotopic with a chain homotopy which is $(F, \alpha, \beta)$-small.

Remark 4 1. Let $F : X \rightarrow Y$ be an u.s.c. multivalued $(n,G)$-map. Lemma 1 and Lemma 2 imply that the collection of chain maps $A(F) = \{T(\nu, \beta) : \alpha \in Cov(X), \nu = \nu(\alpha, \beta) \in Cov(X), \beta \in Cov(Y)\}$ is an $(n,G,A)$-system for the map $F$. We call $A(F)$ the induced $(n,G,A)$-system of the map $F$.

2. For the definition of an $(n,G,A)$-system it is not necessary to consider all coverings of the spaces $X$ and $Y$. It is enough to consider only a fundamental sequence of coverings $\{\alpha_k\} \in Cov(X)$ and $\{\beta_k\} \in Cov(Y)$. Then an $(n,G,A)$-system is a collection of augmentation preserving chain maps $\{\varphi(l,k) = \varphi(\nu_l, \beta_k) : k,l \in \mathbb{N}\}$, which satisfy the conditions of the previous definition with $\nu_l = \nu(\alpha_k, \beta_k)$. In the case where $X$ and $Y$ are finite polyhedra with given triangulations $\tau$, $\mu$, respectively, we take the covering $\alpha_k$ to consist of all open stars of the vertices of the $k$-th barycentric subdivision $\tau^k$ of the triangulation $\tau$ w.r.t. $\tau^k$. Similarly for $\beta_k$. Compare with [12], Definitions 50.17, 50.18, p. 152.

Now we shall consider a composition of u.s.c. multivalued maps and shall define a composition of $(n,G,A)$-systems. From [12], p. 259-262 follows:

Lemma 3 (cf. [12], Proposition (50.37), p. 260) Let $F_i : X_i \rightarrow X_{i+1}, i = 1,2$, be u.s.c. multivalued maps. Let $\alpha \in Cov(X_1), \gamma \in Cov(X_2)$. There exists $\beta \in Cov(X_2)$ such that if $\varphi(\nu_2, \gamma) : C_*(N(\nu_2)^{(n+1)}; G) \rightarrow C_*(N(\gamma)^{(n+1)}; G)$ is an $(n, \beta, \gamma)$-approximation of $F_2$ for $\nu_2 = \nu(\beta, \gamma) > \beta$, and $\varphi(\nu_1, \nu_2) : C_*(N(\nu_1)^{(n+1)}; G) \rightarrow C_*(N(\nu_2)^{(n+1)}; G)$ is an $(n, \alpha, \nu_2)$-approximation of $F_2$ for $\nu_1 = \nu(\alpha, \nu_2) > \alpha$, then $\varphi(\nu_2, \gamma)\varphi(\nu_1, \nu_2) : C_*(N(\nu_1)^{(n+1)}; G) \rightarrow C_*(N(\gamma)^{(n+1)}; G)$ is an $(n, \alpha, \gamma)$-approximation of the map $F_2F_1$.

Lemma 4 (cf. [12], Proposition (50.39), p. 261) Let $F_i : X_i \rightarrow X_{i+1}, i = 1,2$, be u.s.c. multivalved maps. Assume that $A_1 = \{\varphi(\nu_1, \beta) : \alpha \in Cov(X_1), \nu_1 = \nu(\alpha, \beta) \in Cov(X_1), \beta \in Cov(X_2)\}$ is an $(n,G,A)$-system of the map $F_1$ and $A_2 = \{\varphi(\nu_2, \gamma) : \beta \in Cov(X_2), \nu_2 = \nu(\beta, \gamma) \in Cov(X_2), \gamma \in Cov(X_3)\}$ is an $(n,G,A)$-system of the map $F_2$. Then $\{\varphi(\nu_2, \gamma)\varphi(\nu_1, \nu_2) : \alpha \in Cov(X_1), \gamma \in Cov(X_3)\}$ is an $(n,G,A)$-system of the map $F_2F_1$.

We call this $(n,G,A)$-system for the map $F_2F_1$ the composition of the $A$-systems $A_1$ and $A_2$ and denote it by $A_2 \circ A_1$.

From this lemmas and Remark 3, 4 follows
Corollary 1 Let \( F_i : X_i \to X_{i+1} \), \( i = 1, \ldots, k - 1 \) be \((n,G,A)\)-maps. Let \( \mathcal{A}(F_i) \) be the induced \((n,G,A)\)-system of the maps \( F_i \), defined in Remark 4.1. Then the composition \( \mathcal{A}(F_{k-1}) \circ \cdots \circ \mathcal{A}(F_1) \) is an \((n,G,A)\)-system of the map \( F = F_{k-1} \cdots F_1 : X_1 \to X_k \).

We denote this \((n,G,A)\)-system of \( F = F_{k-1} \cdots F_1 \) by \( \mathcal{A}(F) \).

Definition 5 (cf. [12], Definition 50.19, p. 252) Let \( F_1, F_2 : X \to Y \) be u.s.c. multivalued maps. Let \( H : X \times I \to Y \) be an u.s.c. multivalued homotopy joining \( F_1 \) and \( F_2 \). Let \( \mathcal{A}(F_i) \) be an \((n,G,A)\)-system of \( F_i, i = 1, 2 \). The approximation systems \( \mathcal{A}(F_1) \) and \( \mathcal{A}(F_2) \) are called \( H \)-homotopic if for all sufficiently fine coverings \( \alpha \in \text{Cov}(X) \), \( \beta \in \text{Cov}(Y) \) and \( \varphi_1(\nu_1, \beta) \in \mathcal{A}(F_1), \varphi_2(\nu_2, \beta) \in \mathcal{A}(F_2) \) the chain maps \( \varphi_1(\nu_1, \beta)_* \pi(\gamma, \nu_1) \) and \( \varphi_2(\nu_2, \beta)_* \pi(\gamma, \nu_2) \) are chain homotopic with a chain homotopy \( D \) for all \( \gamma > \nu_1, \nu_2 \). Moreover, we assume that the chain homotopy \( D \) satisfies the following condition: for every simplex \( \sigma \in N(\gamma) \) there is a point \( d(\sigma) \in X \) such that

- \( \text{supp}(\sigma) \subset \text{St}(d(\sigma), \alpha) \),
- \( \text{supp}(D\sigma) \subset \text{St}(H(d(\sigma) \times I), \beta) \).

Lemma 5 (cf. [12], Lemma 51.8, p. 265) Let \( F_1, F_2 : X \to Y \) be \((n,G)\)-maps. Let \( H : X \times I \to Y \) be an \((n,G)\)-map, which is a homotopy joining \( F_1 \) and \( F_2 \). Let \( \mathcal{A}(F_i) \) be the induced \((n,G,A)\)-system of \( F_i, i = 1, 2 \). Then the approximation systems \( \mathcal{A}(F_1) \) and \( \mathcal{A}(F_2) \) are \( H \)-homotopic.

Definition 6 Assume that \( F = F_{k-1} \cdots F_1 \) and \( \Phi = \Phi_{k-1} \cdots \Phi_1 : X_1 \to X_k \) are compositions of the maps \( F_i, \Phi_i : X_i \to X_{i+1}, i = 1, \ldots, k - 1 \). Assume that \( F_i \) and \( \Phi_i \) are homotopic with a homotopy \( H_i : X_i \times I \to X_{i+1}, i = 1, \ldots, k - 1 \). Then we say that the maps \( F \) and \( \Phi \) are composition-homotopic with a homotopy \( H = H_{k-1}(H_{k-2} \circ \text{id}) \cdots (H_1 \circ \text{id}) \), where \( \text{id} : I \to I \) is the identity.

Corollary 2 (cf. [12], Proposition 51.9, p. 265) Assume that \( F = F_{k-1} \cdots F_1, \Phi = \Phi_{k-1} \cdots \Phi_1 : X_1 \to X_k \) are compositions of the \((n,G)\)-maps \( F_i, \Phi_i : X_i \to X_{i+1}, i = 1, \ldots, k - 1 \). Assume that \( F \) and \( \Phi \) are composition-homotopic with a homotopy \( H \) Then the induced \((n,G,A)\)-systems \( \mathcal{A}(F) = \mathcal{A}(F_{k-1}) \circ \cdots \circ \mathcal{A}(F_1) \) and \( \mathcal{A}(\Phi) = \mathcal{A}(\Phi_{k-1}) \circ \cdots \circ \mathcal{A}(\Phi_1) \) are \( H \)-homotopic.

4 Fixed point index of an approximation system

Let \((K, \tau)\) be a finite simplicial complex with triangulation \( \tau \). By \( \tau^k \) we denote the \( k \)-th barycentric subdivision of the triangulation \( \tau \). Let \( C_*(\tau^k) = C_*(K, \tau^k; \mathbb{Z}) \) be the chain complex with integer coefficients of the triangulation \( \tau^k \). By \( b(k, l) : C_*(\tau^k) \to C_*(\tau^l) \) we denote the barycentric subdivision operation, \( l > k \). We consider the fundamental sequence of open coverings \( \{\alpha_k\} \) of the space \( K \), where \( \alpha_k \) is the covering of \( K \) by the open stars of the vertices of \( \tau^k \), w.r.t. the triangulation \( \tau^k \).

Let \( U \) be an open set in \( K \) such that its closure \( \overline{U} \) is a subcomplex in \((K, \tau)\). By \( p(U, k) = \{p(U, k)_i\} : C_*(\tau^k) \to C_*(\tau^k | \overline{U}) \) we denote the projection homomorphism. Here \( \tau^k | \overline{U} \) is the restriction of the triangulation \( \tau^k \) on \( \overline{U} \).

Let \( F : U \to K \) be an u.s.c. multivalued map without fixed points on the boundary \( \partial U \) of the set \( U \), i.e., \( x \notin F(x) \) for \( x \in \partial U \), or the same, \( \text{Fix}(F) = \{x : x \in F(x)\} \subset U \). In this case we call the triple \((K, F, U)\) admissible.
Definition 7 We call a quadruple \((K, F, U; A)\) admissible if \((K, F, U)\) is an admissible triple and \(A\) is an \((n, \mathbb{Z}, A)\)-system of the map \(F\), with \(n = \dim K\). Denote by \(K_A\) the set of all admissible quadruples.

Definition 8 (Fixed point index on \(K_A\), cf. [12], Ch. 4, Definition 50.21) The fixed point index \(I_A: K_A \to \mathbb{Z}\) is defined as follows. Let \((K, F, U; A) \in K_A\) and let \(A = \{\varphi(l, k), \nu_l = \nu(\alpha_k, \alpha_l)\}\) be an \((n, \mathbb{Z}, A)\)-system of the map \(F: \overline{U} \to K\), see Remark 4.2. Let \(\psi\) be the graded homomorphism \(\psi = \{\psi_i\} = \{p(U, k)\varphi(l, k)b(k, l) : C_*(\tau^k | \overline{U}) \to C_*(\tau^k | \overline{U})\}\) and let \(k\) be a sufficiently large natural number. Then \(I_A(K, F, U)\) is defined by the Lefschetz number of the graded homomorphism \(\psi\), i.e., \(I_A(K, F, U) = \lambda(\psi) = \sum (-1)^i \text{tr}(\psi_i)\), where \(\text{tr}(\psi_i)\) is the trace of the homomorphism \(\psi_i\).

Remark 5 The definition of the fixed point index \(I_A(K, F, U)\) is correct, i.e., it does not depend on the number \(k\). This follows as in Lemma and Definition 1.2, [13].

Lemma 6 The fixed point index \(I_A(K, F, U)\) has the following properties:

1. **Additivity**
   
   Let \(U_1, U_2\) be open, disjoint and polyhedral subsets of \(U\) and \(\text{Fix}(F) = \{x : x \in F(x)\} \subseteq U_1 \cup U_2\). Then \(I_A(K, F, U) = I_A(K, F, U_1) + I_A(K, F, U_2)\).

2. **Homotopy invariance**
   
   Let \(H = H_t : \overline{U} \times I \to K\) be an u.s.c homotopy such that \((K, H_t, U)\) is an admissible triple for all \(t \in I\). Let \(A_0, A_1\) be \(H\)-homotopic \((n, \mathbb{Z}, A)\)-systems for the maps \(H_0, H_1\), respectively. Then \(I_{A_0}(K, F, U) = I_{A_1}(K, F, U)\).

3. **Commutativity**
   
   Let \(K, L\) be finite simplicial complexes. Let \(W \subset K\) be an open subset and let \(F_1 : K \to L\), \(F_2 : L \to K\) be u.s.c. multivalued maps. Assume that \(x \notin F_2F_1(x)\) for \(x \in \partial W\) and \(y \notin F_1F_2(y)\) for \(y \in \partial F_2^{-1}(W)\). Assume further that
   
   \[y \in \text{Fix}(F_1F_2) \setminus F_2^{-1}(W) \implies F_2(y) \cap \text{Fix}(F_2F_1 | \overline{W}) = \emptyset.\]
   
   Then for all \((n, \mathbb{Z}, A)\)-systems \(A_1, A_2\) of \(F_1, F_2\), respectively, follows
   
   \[I_{A_1 \circ A_2}(L, F_1F_2, F_2^{-1}(W)) = I_{A_2 \circ A_1}(K, F_2F_1, W).\]
   
   Here \(n \geq \max\{\dim K, \dim L\}\).

4. **Normalization**
   
   \[I_A(K, F, K) = \lambda(\psi_*),\]
   
   where \(\psi_* : H_*(K; \mathbb{Z}) \to H_*(K; \mathbb{Z})\) is the homomorphism induced by the chain map \(\varphi(l, k)b(k, l)\), where \(\varphi(k, l) \in A\) and \(k\) is sufficiently large.

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7
Now we define the fixed point index $I_A(K,F,V)$ for all open sets $V$ and maps $F : \overline{V} \to K$ such that $Fix(F) \subset V$. We call such a triple admissible.

Take an open polyhedral set $U$, such that $\overline{U} \subset V$ and $Fix(F) \cap (V \setminus U) = \emptyset$. Then define $I_A(K,F,V) = I_A(K,F,U)$.

The proofs of Propositions 1.5, 1.6, [18] imply the Homotopy invariance and the Commutativity property.

The Normalization follows from [11].

**Definition 9** Let $\mathcal{K}(n) = \{(K,F,U)\}$ where

- $K$ is a finite simplicial complex, and the triple $(K,F,U)$ is admissible;
- the map $F = F_{k-1} \ldots F_1$ with $F_i : X_i \to X_{i+1}$, $X_1 = U$, $X_k = K$;
- the maps $F_i$ are $(n,\mathbb{Z})$-maps, $n = \dim K$, see Definition 2.

**Definition 10** Let $(K,F,U) \in \mathcal{K}(n)$. Let $A(F) = A(F_{k-1}) \circ \ldots \circ A(F_1)$ be the induced $(n,G,A)$-system of the map $F$. Then $(K,F,U;A(F)) \in \mathcal{K}_A$, see Definition 7. The fixed point index $I : \mathcal{K}(n) \to \mathbb{Z}$ is defined as $I(K,F,U) = I_{A(F)}(K,F,U)$.

From Lemma 6 follows

**Corollary 3** The fixed point index $I : \mathcal{K}(n) \to \mathbb{Z}$ has the properties Additivity, Homotopy invariance, Commutativity and Normalization.

**Remark 6** For the definition of the fixed point index $I : \mathcal{K}(n) \to \mathbb{Z}$ a block structure of the simplicial complex could be used. Then one obtains also the Multiplicativity of $I(K,F,U)$, as in [24].

## 5 \(\mathbb{Z}\)-acyclic and \((n,G)\)-maps

Here we shall describe relations between the $\mathbb{Z}$-acyclicity w.r.t. the Čech cohomology, w.r.t. the Steenrod-Sitnikov homology of compact metric spaces and $(n,\mathbb{Z})$-compact spaces, see Definition 1.

**Lemma 7** A compact metric space is $\mathbb{Z}$-acyclic w.r.t. the Čech cohomology if and only if it is $\mathbb{Z}$-acyclic w.r.t. the Steenrod-Sitnikov homology.

Proof. Assume that the compact metric space is $\mathbb{Z}$-acyclic w.r.t. the Čech cohomology. Then the exact sequence

$$0 \to Ext(H^{n+1}(X;\mathbb{Z});\mathbb{Z}) \to H_n(X;\mathbb{Z}) \to Hom(H^n(X;\mathbb{Z}),\mathbb{Z}) \to 0,$$

see [20], (14), p. 221, implies that $X$ is $\mathbb{Z}$-acyclic w.r.t. the Steenrod-Sitnikov homology.

For the inverse assertion assume that the compact space is $\mathbb{Z}$-acyclic w.r.t. the Steenrod-Sitnikov homology. The above exact sequence gives that $Ext(H^{n+1}(X;\mathbb{Z});\mathbb{Z}) = 0$ and $Hom(H^n(X;\mathbb{Z}),\mathbb{Z}) = 0$. Since $X$ is a compact metric space the groups $H^n(X;\mathbb{Z})$ are countable. Then the theorem of Stein-Serre implies that the groups $H^n(X;\mathbb{Z})$ are free, see [21], Proposition 1.2, Proposition 1.3 and Remark on p. 374. Then $Hom(H^n(X;\mathbb{Z}),\mathbb{Z}) = 0$ gives that the space $X$ is $\mathbb{Z}$-acyclic w.r.t. the Čech cohomology.
Lemma 8 Let $F : X \rightarrow Y$ be an u.s.c. multivalued map. Assume that $F$ is $G$-acyclic w.r.t. the Steenrod-Sitnikov homology with coefficients in a countable Abelian group $G$. Then $F$ is an $(n,G)$-map for every natural number $n$.

Proof. Consider the representation $F = qp^{-1}$ with the projections $p : \Gamma(F) \rightarrow X$ and $q : \Gamma(F) \rightarrow Y$. The Proposition 1.5. from [14] gives that $p^{-1}(x)$ are $(n,G)$-compact spaces for $x \in X$ and every $n$. Then by Definition 2 the map $F$ is an $(n,G)$-map for every $n$.

Remark 7 In the proof of Proposition 1.5. from [14] is used that the Mittag-Leffler property of projective systems $\{A_i\}$ of Abelian groups is equivalent to the vanishing of the first derived functor $\lim^1 \{A_i\}$ of the projective system $\{A_i\}$. This is true for countable projective systems of countable Abelian groups, see [21], Proposition 1.2, p. 371. For this reason we assume that the spaces are compact and metric.

6 Main Theorem

Definition 11 Let $X$ be a compact ANR and let $U$ be an open set in $X$. The triple $(X,F,U)$ is called acyclic admissible if

- $F : \overline{U} \rightarrow X$ has no fixed points on the boundary $\partial U$ of the set $U$, i.e., $\text{Fix}(F) \subset U$;
- There is a natural number $k$ such that $F = F_{k-1} \ldots F_1 : \overline{U} \rightarrow X$ and $F_i : X_i \rightarrow X_{i+1}$, $i = 1, \ldots, k-1$, $X_1 = \overline{U}, X_k = X$;
- the maps $F_i$, $i = 1, \ldots, k-1$ are $\mathbb{Z}$-acyclic w.r.t. Čech cohomology with integer coefficients.

Denote by $\mathcal{K}$ the set of all acyclic admissible triples.

For $(X,F,U) \in \mathcal{K}$ we have a homomorphism $F^* = F_1^* \ldots F_{k-1}^* : H^*(X;\mathbb{Z}) \rightarrow H^*(\overline{U};\mathbb{Z})$, where $F_i^* : H^*(X_{i+1};\mathbb{Z}) \rightarrow H^*(X_i;\mathbb{Z})$ is defined as follows. Consider the representation $F_i = q_i(p_i)^{-1}$ of the map with the projections $p_i : \Gamma(F_i) \rightarrow X_i$ and $q_i : \Gamma(F_i) \rightarrow X_{i+1}$. The Vietoris theorem implies that the homomorphism $p_i^* : H^*(X_i;\mathbb{Z}) \rightarrow H^*(\Gamma(F_i);\mathbb{Z})$ is an isomorphism, see [25], Theorem 15, p. 344. Then $F_i^* = (p_i^*)^{-1}q_i^* : H^*(X_{i+1}) \rightarrow H^*(X_i)$, and $F^* = F_1^* \ldots F_{k-1}^* : H^*(X;\mathbb{Z}) \rightarrow H^*(\overline{U};\mathbb{Z})$.

Consider the case $X = U$. Since $X$ is compact ANR then $H^*(X;\mathbb{Z})$ is a finetely generated Abelian group. Then the Lefschetz number $\lambda(F^*) = \sum_k (-1)^k tr(F^k)$ is defined. Here $F^* = \{F^k\}$ and $F^k : H^k(X;\mathbb{Z}) \rightarrow H^k(X;\mathbb{Z})$ is the homomorphism induced by the map $F$.

Let $(X,F,U),(X,G,U) \in \mathcal{K}$. We call the triples $(X,F,U)$ and $(X,G,U)$ admissible homotopic if the maps $F$ and $G$ are composition-homotopic with a homotopy $H$ such that $x \not\in H(x,t)$ for all $x \in \partial U$ and $t \in I = [0,1]$, see Definition 6.

Theorem 1 (integer fixed point index on $K$) There is a function $i : \mathcal{K} \rightarrow \mathbb{Z}$ defined on the set of all acyclic admissible triples $\mathcal{K}$ with integer values with the following properties:

1. Additivity

Let $(X,F,U) \in \mathcal{K}$ and let $U_1, U_2$ be open subsets of $U$ with $\text{Fix}(F) \subset U_1 \cup U_2$, then

$$i(X,F,U) = i(X,F,U_1) + i(X,F,U_2);$$
2. Homotopy invariance

Let \((X,F,U),(X,G,U) \in \mathcal{K}\) be admissible homotopic triples, then

\[ i(X,F,U) = i(X,G,U); \]

3. Commutativity

Let \(F : X \to Y, G : Y \to X\) be compositions of acyclic maps. Let \(U\) be an open set in \(X\). Assume that \((X,GF,U), (Y,FG,G^{-1}(U)) \in \mathcal{K}\) and \(G(Fix(FG) \setminus G^{-1}(U)) \cap Fix(GF \mid U) = \emptyset\), then

\[ i(X,GF,U) = i(Y,FG,G^{-1}(U)); \]

4. Normalization

\[ i(X,F,X) = \lambda(F^*). \]

Proof.

Let \(\mathcal{K}'\) be the subset of \(\mathcal{K}\) consisting of all acyclic admissible triples \((K,F,U) \in \mathcal{K}\) with \(K\) a finite simplicial complex.

Lemma 8 implies that if \((K,F,U) \in \mathcal{K}'\) then \((K,F,U) \in \mathcal{K}(n)\) with \(n = \text{dim} K\), see Definition 9.

Consider \(I : \mathcal{K}(n) \to \mathbb{Z}\), see Definition 10. Define \(i(K,F,U) = I(K,F,U)\). Lemma 6 implies that the fixed point index \(i : \mathcal{K}' \to \mathbb{Z}\) has Additivity, Homotopy invariance and Commutativity properties.

The Normalization property in Lemma 6 is stated as follows: \(i(K,F,K) = \lambda(F_k)\), where \(F_k = (F_{k-1})_* \ldots (F_1)_*\) is a homomorphism in the Čech homology with integer coefficients. It follows that \(\lambda(F_k) = \lambda(F^*)\), i.e., the Normalization property from Theorem 1 also follows for the index \(i : \mathcal{K}' \to \mathbb{Z}\).

Since the fixed point index \(i : \mathcal{K}' \to \mathbb{Z}\) has the Commutativity property we can apply the extension procedure from [12], Ch. 4, Section 53. As a result we obtain a fixed point index \(i : \mathcal{K} \to \mathbb{Z}\) with Additivity, Homotopy invariance, Commutativity and Normalization properties.

Remark 8 1. The fixed point index \(i(X,F,U)\) coincides with the fixed point index defined in [3] for \(\text{dim} X < \infty\) and \(F\) a \(\mathbb{Z}\)-acyclic map.

2. Using the technique from [23] one can prove that the fixed point index \(i : \mathcal{K} \to \mathbb{Z}\) has also the Multiplicativity property, i.e., for \((X_1,F_1,U_1),(X_2,F_2,U_2) \in \mathcal{K}\) follows \((X_1 \times X_2,F_1 \times F_2,U_1 \times U_2) \in \mathcal{K}\) and

\[ i(X_1 \times X_2,F_1 \times F_2,U_1 \times U_2) = i(X_1,F_1,U_1)i(X_2,F_2,U_2) \]

3. With an adaptation of the technique of [9,10] one can obtain an integer fixed point index for compositions of multivalued, \(\mathbb{Z}\)-acyclic, weighted maps with an integer multiplicity function. For these maps see [17].

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