A Simplified Mathematical Model for the Formation of Null Singularities Inside Black Holes II

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Abstract

We study a simple system of two hyperbolic semi-linear equations, inspired by the Einstein equations. The system, which was introduced in \cite{1}, is a model for singularity formation inside black holes. We show for a particular case of the equations that the system demonstrates a finite time blowup. The singularity that is formed is a null singularity. Then we show that in this particular case the singularity has features that are analogous to known features of models of black-hole interiors — which describe the inner-horizon instability. Our simple system may provide insight into the formation of null singularities inside spinning or charged black holes.
1 Introduction

This research examines the singularity formation in a system of equations motivated by Einstein’s equations. This study may lead to a better understanding of the internal structure of black holes and more specifically, the singularity created within them. The system of equations studied in this work is motivated by the former studies of the Reissner-Nordström (RN) based models [1]. It is a continuation of the work in [1] where the system of equations and their physical motivation were presented. Here we present a new approach to the study of the model and prove the conjecture presented in [1] for a simplified form of the equations.

The mass inflation model presented in [3] demonstrates the genericity of null singularities inside realistic black holes. In [1] it was conjectured that a simpler dynamical system – an “active ingredient” of the Einstein equations is capable of producing black hole-like configurations of generic null weak singularities subjected to certain characteristic features. It was also shown that the evolution equations in Reissner-Nordström-de Sitter (RNDS) spacetime with two intersecting null fluids can be written in the
form:

\[
R_{,uv} = e^s F(R), \\
s_{,uv} = e^s F'(R).
\]  \hfill (1)

\[
s_{,uv} = e^s F'(R).
\]  \hfill (2)

\(F(R)\) is defined as:

\[
F(R) = -h'(R)
\]  \hfill (3)

where

\[
R \equiv r^2, \quad e^s \equiv rf,
\]  \hfill (4)

for the following metric in double-null coordinates \((u, v)\):

\[
ds^2 = -2f(u, v)du dv + r^2(u, v)d\Omega^2,
\]  \hfill (5)

and

\[
h(R) = 2R_1^2 - 4m + \frac{2Q^2}{R^2} - \frac{2\Lambda}{3} R^2,
\]  \hfill (6)

where \(m, Q\) and \(\Lambda\) are the mass, charge and the cosmological constant in the RNDS solution. The conjecture was formulated in \[1\] for a general function \(h(R)\) which has three roots \(h(R_i) = 0\) (Fig. 1). The roots of \(h(R)\) correspond to the locations of the ”horizons” in a flux-free solution (defined below). \(R = R_1, R_2, R_3\) correspond in the RNDS model to the cosmological, event and inner horizons respectively. The origin of axes is placed at \(R = R_a\).

In addition we define

\[
k_i \equiv \left| h'(R_i) \right| \quad (i = 1, 2, 3),
\]  \hfill (7)

as a generalization of the surface gravity. In this work we study the singularity formation for a simple type of \(h(R)\) - a saw-tooth function. We will show that the formed singularity and its properties depend only on the surface gravities at the horizons and their relative values.

Our semi-linear system is accompanied by two conserved fluxes:

\[
e^s(e^{-s} R_{,u})_u = \Psi(u),
\]  \hfill (8)

\[
e^s(e^{-s} R_{,v})_v = \Phi(v),
\]  \hfill (9)

where \(\Psi(u)\) and \(\Phi(v)\) are arbitrary functions which stand for the perturbations of the flux-free solution \(\Psi(u) = \Phi(v) = 0\). These functions are proportional to the energy density in the outflux and influx respectively. Eqs. (8,9) can be derived from our semi-linear system (see [1]). Too strong
null fluids may ruin the singularity structure. Thus, we consider “small” \( \Psi(u) \) and \( \Phi(v) \). Later we will define exactly what we mean by “small”.

In addition there exists a gauge freedom \((u \to \tilde{u}(u), v \to \tilde{v}(v))\) for which \( s(u,v) \) is changed according to the following rule:

\[
\tilde{s} = s - \ln \left( \frac{d\tilde{v}}{dv} \right) - \ln \left( \frac{d\tilde{u}}{du} \right).
\]

There are two quantities which play an important role in the conjecture and are related to the mass function and the null fluid fluxes:

- \( e^{-s} R_u R_v \) - A gauge invariant quantity.
- \( e^{-s} R_u \) (res. \( e^{-s} R_v \)) - Invariant under gauge transformations in \( u \) (res. \( v \)).

The paper is organized as follows. In section 2 we study the singularity formation in the flux-free case using the method of characteristics. This method will allow us to study the singularity formation in the non-linear case (with two null fluids - perturbations of the flux-free case) as well. For this purpose we introduce in section 3 the general form of characteristic equations. In the discussion (section 4) we review the conjecture about the singularity formation (which was presented in [1]) and indicate the particular case which will be studied in the rest of the paper - \( h(R) \) as a saw-tooth function. In sections 5 and 6 we prove the conjecture for a saw-tooth function under few reasonable assumptions.

## 2 Singularity in the flux-free case and the method of characteristics

Our semi-linear system (12) for the functions \( R(u,v) \) and \( s(u,v) \) is a system of second order hyperbolic PDE. To obtain a unique solution inside the
positive quadrate \((u \geq 0, \ v \geq 0)\) we should have four known functions on the boundary of the region: \(R(u,0), \ R(0,v), \ s(0,v)\) and \(s(u,0)\). This type of initial value problem is usually called a Goursat Problem (see for example [5]). Notice that the initial data is given on the characteristic curves of the equations. The Goursat problem for linear hyperbolic PDE is a well-posed problem and has a global unique solution. Our system of equations is semi-linear, so we have only local existence and uniqueness in general. This local existence and uniqueness can be proved using successive approximations as in the linear PDE case.

In addition to the initial data we have a gauge freedom in the coordinates as our set of equations is invariant under gauge transformations. We can choose any function for \(\frac{\partial u}{\partial \tilde{u}}\) and \(\frac{\partial v}{\partial \tilde{v}}\) (recall the gauge transformation (8)). This freedom enables us to set a gauge in which \(s(0,v) = 0\) and \(s(u,0) = 0\). We will refer to this gauge later as the “standard gauge”. In this gauge we still have to set only two functions: \(R(u,0)\) and \(R(0,v)\). Relating to these two functions, we divide the initial value problem into two cases:

1. The linear case - \(R(u,0)\) and \(R(0,v)\) are linear functions. This is the case when we do not have perturbations (flux-free).

2. The non-linear case - The two functions \(R(u,0)\) and \(R(0,v)\) are non-linear.

In the standard gauge, for the flux-free case, eqs. (6,7) on the boundary \(v = 0\) and \(u = 0\) are

\[
\begin{align*}
R_{,uu} &= 0 \quad (v = 0), \\
R_{,vv} &= 0 \quad (u = 0).
\end{align*}
\]

Therefore, on the \(u\) axis \(R\) is linear with respect to \(u\) and on the \(v\) axis \(R\) is linear with respect to \(v\). It is consistent with eqs. (9,10) to assume that \(R(u,v)\) is monotonous on the axes:

\[
\begin{align*}
R(u,0) &= R_a - bu \\
R(0,v) &= R_a + av,
\end{align*}
\]

where \(R_a\) as noted in Fig. 1 is placed at the origin of axes. Equating the expression \(e^{-s}R_{,u}R_{,u}\) (which is gauge invariant) in the static gauge to the same expression in the standard gauge at the point \(R = R_a\), we obtain:

\[
-h(R_a) = -e^{-s}R_x^2 = e^{-s}R_yR_{,y} = -ab.
\]

5
Eqs. (6,7) in the flux-free case take the following form:

\[ e^s (e^{-s} R_u)_u = 0, \]  
\[ e^s (e^{-s} R_v)_v = 0. \]  
(13)

(14)

Our goal is to derive a first order PDE whose characteristic curves will determine the behavior of the solution of the original semi-linear system. We start by integrating eqs. (13,14):

\[ \begin{cases} R_u = e^s \phi(v) \\ R_v = e^s \psi(u) \end{cases} \]  
(15)

φ(v), ψ(u) are defined below. After differentiation, we have:

\[ \begin{cases} R_{uv} = e^s \phi'(v) + e^s s_v \phi(v) \\ R_{vu} = e^s \psi'(u) + e^s s_u \psi(u). \end{cases} \]  
(16)

Since \( R_{uv} = R_{vu} \) \((R(u, v) \in C^2)\), we obtain a first order PDE:

\[ \phi(v)s_u - \psi(u)s_v + \phi'(v) - \psi'(u) = 0. \]  
(17)

The characteristic curves for eq. (17) are:

\[ \begin{cases} \dot{v} = -\phi(v) \\ \dot{u} = \psi(u) \\ \dot{s} = \phi'(v) - \psi'(u). \end{cases} \]  
(18)

We can find explicitly the functions \( \phi(v) \) and \( \psi(u) \). In the standard gauge \( s(u, 0) = 0 \) so (16, 1) yields

\[ R_{uv} = F(R) = \psi'(u), \]  
(19)

\[ \psi(u) = \int_0^u F(R(u', 0)) du' + \psi(0). \]  
(20)

Eqs. (11) and (15) in the standard gauge yield:

\[ \psi(u) = \int_{R_a}^R F(R) \frac{du'}{dR'} dR' + R_v(0, 0) = - \int_{R_a}^R \frac{F(R')}{b} dR' + a = \frac{h(R(u, 0)) - h(R_a)}{b} + a = h(R_a - bu), \]  
(21)

(Here we use the definition of \( F(R) \) - eq. (3) and the value of \( h(R_a) \) is from eq. (12)).
In the same way, on \( u = 0 \)

\[
\phi(v) = \int_{R_a}^{R} F(R) \frac{dv'}{dR'} dR' + R_u(0,0) = \int_{R_a}^{R} F(R') \frac{dR'}{a} - b = - \frac{h(R(0,v)) - h(R_a)}{a} - b = - \frac{h(R_a + av)}{a}.
\] (22)

Now, we can rewrite the characteristic as

\[
\begin{cases}
\dot{v} = a^{-1} h(R_a + av) \\
\dot{u} = b^{-1} h(R_a - bu) \\
\dot{s} = h'(R_a - bu) - h'(R_a + av).
\end{cases}
\] (23)

Note the behavior of the function \( R(u,v) \) in the region \( (u \geq 0, v \geq 0) \): From eqs. (15) and (22) we obtain an expression for \( R_u \):

\[ R_u = -a^{-1} e^s h(R_a + av). \] (24)

We find out (from the form of the function \( h(R) \)) that \( R_u(v) \) is monotonously decreasing in regions I and II (Fig. 2) on the rays \( v = \text{const} \) in consistency with the known expression for \( v = 0 \) (eq. (11)). In an analogous manner, from eqs. (15) and (21), we obtain an expression for \( R_v \):

\[ R_v = b^{-1} e^s h(R_a - bu). \] (25)

Thus, \( R(u,v) \) is monotonously increasing in region I on the rays \( u = \text{const} \) in consistency with the known expression for \( u = 0 \) (eq. (11)). In region II \( R(u,v) \) is monotonously decreasing on \( u = \text{const} \). We summarize the behavior of \( R(u,v) \) in Fig. 2 below.

Next we study the singularity of \( s \). As we approach the point (2,1) in Fig. 2 we see that since \( h'(R_2) > 0, h'(R_1) < 0 \) and \( h(R) > 0 \) in region I (\( R(u,v) \) is monotonous),

\[ \dot{u}, \dot{v} \to 0^+ \]

(the limit is from the positive side of zero). Thus, \( u \) and \( v \) attain their asymptotic values. Therefore from the last equation in (23) we obtain:

\[ \dot{s} \to h'(R_2) - h'(R_1) = \text{const} > 0 \] (26)

Then, \( s \to \infty \). However, as we approach the point (3,1) we have two generic cases:

1. \( k_3 > k_1 \) - \( \dot{s} = \text{const} < 0 \) while \( \dot{v} \to 0^+ \Rightarrow s \to -\infty \).
Figure 2: The sign of partial derivatives of $R$ in the different regions of the flux-free solution in the standard gauge. The indications $R_1$, $R_2$ and $R_3$ label the corresponding characteristic lines.

2. $k_1 > k_3 - \dot{s} = \text{const} > 0$ while $\dot{v} \to 0^+ \Rightarrow s \to \infty$.

From (23) we may also consider the flow along the characteristics:

$$\frac{du}{dv} = \frac{ah(R_a - bu)}{bh(R_a + av)}$$

(27)

The direction of this flow is represented in Fig 3.

Figure 3: The characteristic curves in the flux-free (linear) case in the standard gauge.
3 The general characteristic equations

In this section we derive the characteristic equations in their most general form, i.e., for any perturbation. Furthermore, we show that the characteristic curves have physical meaning - they are the curves of constant $R$. Then we show that the parameter $t$ of the characteristic curves also has a physical interpretation since it is invariant under gauge transformations.

3.1 Derivation of the characteristic equations

Similarly to (17), we can derive an equation for $s(u, v)$ for any perturbation to the flux-free case. In this case, however, the equations are non-linear (and, also, non local).

Integration of (6, 7) gives:

$$e^{-s}R_{,u} = \int e^{-s(u',v)}\Psi(u')du' + \phi(v),$$

(28)

$$e^{-s}R_{,v} = \int e^{-s(u',v')}\Phi(v')dv' + \psi(u),$$

(29)

The functions $\phi, \psi, \Phi, \Psi$ are determined from $R(0, v)$ and $R(u, 0)$ (in the standard gauge) as shown below. For convenience, we rewrite (28, 29) as

$$R_{,u} = e^{8f_1(u, v)},$$

(30)

$$R_{,v} = e^{8f_2(u, v)}. $$

(31)

Differentiate (30) with respect to $v$, (31) with respect to $u$, and equate $R_{,uv} = R_{,vu}$ to obtain

$$s_vf_1 - s UIF_2 + f_1,v - f_2,u = 0.$$

(32)

The equations of the characteristic curves for this equation are:

$$\begin{cases}
\dot{v} = -f_1(u, v) \\
\dot{u} = f_2(u, v) \\
\dot{s} = f_1,v - f_2,u.
\end{cases}$$

(33)

From (30, 31) and (33), we obtain:

$$\frac{du}{dv} = -\frac{f_2}{f_1} = -\frac{R_v}{R_u}. $$

(34)

It follows
Proposition 3.1 A curve in the domain \( u \geq 0 \) \( v \geq 0 \) is a characteristic curve of (32) if and only if \( R = \text{const} \) on the curve.

As in the linear case, we can obtain from (28-31):

\[
R_{,uv} = e^s s_{,v}(\int_0^u e^{-s(u,v')}\Phi(v')dv' + \psi(u)) - e^s(\int_0^v s_{,u}e^{-s(v')}\Psi(u')du' - \phi'(v)).
\] (35)

\[
R_{,vu} = e^s s_{,u}(\int_0^v e^{-s(u,v')}\Phi(v')dv' + \psi(u)) - e^s(\int_0^u s_{,v}e^{-s(u')}\Psi(u')du' - \phi'(u)).
\] (36)

From (35) on \( u = 0 \) in the standard gauge \((s(0,v) = 0)\) we obtain \( R_{,uv} = \phi'(v) \). From (36) on \( v = 0 \) in the standard gauge \((s(u,0) = 0)\) we obtain \( R_{,vu} = \psi'(u) \). Equating these expressions to eq. (1), we arrive at the same results for the derivatives of \( \psi(u) \) and \( \phi(v) \) as in the linear case:

\[
- h'(R(u,0)) = F(R(u,0)) = \psi'(u)
\] (37)

\[
- h'(R(0,v)) = F(R(0,v)) = \phi'(v).
\] (38)

Note also that in the standard gauge the conserved fluxes \((6, 7)\) takes, on the axis \( u = 0, \) \( v = 0 \), the form:

\[
\Phi(v) = R_{,vv}(0, v).
\] (40)

Combining the above results we can rewrite the characteristic equations (33) explicitly (in terms of the data \( R(u,0) \) and \( R(0,v) \)), as

\[
\begin{cases}
\dot{v} = - \int_0^u e^{-s} R_{,uu}(u',0)du' + \int_0^v h'(R(0,v'))dv' \\
\dot{u} = \int_0^v e^{-s} R_{,vv}(0,v')dv' - \int_0^u h'(R(u',0))du' \\
\dot{s} = h'(R(u,0)) - h'(R(0,v)) + \int_0^v (e^{-s})_{,u} R_{,uu}(u',0)du' - \int_0^v (e^{-s})_{,v} R_{,vv}(0,v')dv'.
\end{cases}
\] (41)

3.2 The physical interpretation of \( t \)

The parameter \( t \) of the characteristic curves is related to the proper time in the same way as the coordinates \( r^* \) or \( t \) are related to it\(^1\). The proper time \( \tau \) is defined by:

\[
d\tau^2 \equiv \frac{2e^s}{R^2}dudv.
\] (42)

\(^1\)Here is the parameter that we use in the usual black-hole metric, \( r^* \) is the *tortoise coordinate*, defined by \( r^* = \int_1^{-\frac{dr}{2mR}} \) for example in the Schwarzschild metric.
From eqs. (33) and (30) we obtain:
\[ \dot{u} = R_v e^{-s}. \] (43)

From (31) and (33) we obtain:
\[ \dot{v} = -e^{-s} R_u. \] (44)

Combining the last three results, we find that the proper time \( \tau \) and the parameter \( t \) are connected through the scalar \( e^{-s} R_u R_v \):
\[ (\dot{\tau})^2 = \frac{2 e^{-s} R_u R_v}{R^2}. \] (45)

Therefore the parameter \( t \) is a geometric quantity and not coordinate dependent. In the linear case we have:
\[ (\dot{\tau})^2 = \frac{2h(R)}{R^2}. \] (46)

It appears to be like the connection between \( \tau \) and \( t \) on \( r = const \), where \( t \) is the usual Schwarzschild coordinate (recall that \( \frac{h(R)}{2R^2} = 2f \) in the static solution), but here the characteristic curves are more general. For example, the relation (46) is valid for curves of the type \( u = const \) and \( v = const \) even though it is not true for the Schwarzschild \( t \) coordinate. Such curves are the characteristics along the “horizons”. Furthermore, on each characteristic curve \( R = const \) and therefore \( (\dot{\tau})^2 = const \). Thus, the proper time is proportional to the parameter \( t \), except for the horizons (roots of \( h(R) \)) along which \( \dot{\tau} = 0 \). To conclude, in the linear case the parameter \( t \) is regarded as an extension of the Schwarzschild coordinate \( t \) to include the horizons as well.

4 Discussion

Let us summarize the features of the singularity obtained in the linear case \( (R_{uv}(u,0) = R_{uv}(0,v) \equiv 0) \). We have a singularity that develops from regular initial conditions. This singularity forms at finite \( u \) and \( v \) - a finite time blowup. Its structure suits the structure of the RNDS spacetime (with an additional “horizon” because of the additional zero in \( h(R) \)). The singularity consists of a “point singularity” where \( s \to \infty \) and a ray of singularity emerging from it where \( s \to \pm \infty \), depending on inequalities between
Figure 4: The general singularity structure. The three horizons are displayed, denoted by $H_i$ ($i = 1, 2, 3$). The singular inner horizon is displayed by a thick line emerging from the point singularity in the middle.

the derivatives of $h(R)$ at the horizons. The structure is illustrated in Fig. 4. These inequalities and the resulting singularities of $s$ correspond to the blue-shift/red-shift at the inner horizon. In addition, we found out that each horizon corresponds to a certain $R$ value (which is constant on the horizon); in the linear case the horizons are characteristic curves of the first order PDE (17) according to Proposition 3.1. In the non-linear case the curves of constant $R$ deviate from the horizons due to the perturbations. Therefore we will use in the non-linear case the notation $H_i$ ($i = 1, 2, 3$) for the horizons. $H_1, H_2, H_3$ correspond in the RNDS model to the cosmological, event and inner horizons respectively.

We reformulate below the main ingredient of the conjecture\(^2\) for the non-linear case (nonzero flux in both directions):

**Conjecture**

For small enough non-linear perturbations the structure of the linear case is preserved. That is, a singularity point is formed from which the inner horizon emerges ($H_3$ in Fig. 4). In addition, we expect the gauge invariant quantities to diverge at the inner horizon. Since they are related to the mass inflation phenomenon mentioned in the introduction, we expect the conditions for their divergence to be the same as in the mass inflation model of Brady and Poisson [4].

At the table below we summarize the divergence conditions of different quantities at the inner horizon based on [1] for the linear and non-linear

\(^2\)The full form of the conjecture can be found in [1]
cases.

| The quantity | Type of divergence | Condition for the divergence |
|--------------|--------------------|------------------------------|
| $s$          | linear             | $k_1 > k_3$                  |
|              | non-linear         | $k_3 > k_1$                  |
| $e^{-s}R_{uv}$| $-\infty$         | $k_3 > 2k_1$                |
| $e^{-s}R_{uv}$| $+\infty$         | $k_3 > 2k_1$                |

Table 1: The divergence of various quantities at the inner horizon $H_3$ expected to be found in the linear and non-linear cases.

$h(R)$ as a saw-tooth function

From now on we write the characteristic equations for a particular case of the function $h(R)$. We choose $h(R)$ to be a saw-tooth function as illustrated in Fig. 5. The origin of the $(u, v)$ axes is positioned on the curve $R = R_a$. The region where $R_a < R(u, v) < R_1$ will be referred as region 1. The region where $R_b < R(u, v) < R_a$ is region 2. These, and regions 3 and 4 are displayed in Fig. 6.

\[ s = s_1(u) + s_2(v), \quad (47) \]

where $s_1(u)$ is an arbitrary function of $u$ and $s_2(v)$ - a function of $v$. 

Figure 5: The function $h(R)$ as a “saw-tooth” function.

For the saw-tooth function the evolution equation for $s$ [2] turns to be in each region $s_{uv} = 0$ whose solution is
Figure 6: The regions 1, 2, 3, 4 in the solution - the forms of these regions are determined by the form of the function \( h(R) \). The general directions of the characteristic curves in each region are indicated by arrows.

5 Reduction to a system of ODE

In this section we reduce our semi-linear system to a system of four first order ODE along a characteristic curve. First we begin the discussion by investigating the characteristic curve \( R(u, v) = R_a \). We choose the parameter \( t \) along this characteristic curve to be the independent variable for the equations. Our semi-linear system (1,2) takes the following form near \( R(u, v) = R_a \):

\[
\begin{align*}
R_{,uv} &= e^s[(k_1 + k_2)\theta(R - R_a) - k_2] \\
\theta_{,uv} &= e^s(k_1 + k_2)\delta(R - R_a),
\end{align*}
\]

where \( \theta(s) \) is the Heaviside step function (\( \theta(s) = 1 \) for \( s \geq 0 \) and \( \theta(s) = 0 \) for \( s < 0 \)). These equations are valid in the neighborhood of \( R = R_a \). In the standard gauge \( s = s_1(u) \) is the solution in region 1 and \( s = s_2(v) \) is the solution in region 2 - see (47). On the curve \( R = R_a \):

\[
s_1(u) = s_2(v).
\]

\(^3\)We assume here that there is no “hiding” of the singularity — namely, the curve \( R_b \) is monotonously decreasing (Fig. 6). This assumption is included in the statement that \( s \) is only a function of \( v \). In the next section we will formulate sufficient conditions for it.
Eq. (48) takes the form $R_{,uv} = k_1 e^{s_1(u)}$ in region 1. We integrate this equation with respect to $u$:

$$R_{,v} = \int u k_1 e^{s_1(u')} du' + R_{,v}(0,v),$$

(50)

and then differentiate with respect to $t$. Using (43,44) we obtain the following equation for $R_{,v}$:

$$\dot{R}_{,v} = k_1 R_{,v} - R_{,vv}(0,v) e^{-s} R_{,u}.$$  

(51)

We obtain an equation for $R_{,u}$ by integrating eq. (48) in region 2 with respect to $v$, and then differentiate with respect to $t$ (using again (43,44)):

$$\dot{R}_{,u} = k_2 R_{,u} + R_{,uu}(u,0) e^{-s} R_{,v}.$$  

(52)

We obtain the third equation by integrating eq. (48) with respect to $u$ in region 2 (in the standard gauge $s_{,v}(0,v) = 0$):

$$s_{,v}'(v) = s_{,v} = (k_1 + k_2) \int_0^u e^{s'(u',v)} \delta(R - R_a) du' = (k_1 + k_2) \frac{e^{s_2(v)}}{|R_{,u}(u_0,v)|},$$

(53)

where $u_0$ is the solution of $R(u_0,v) = R_a$. Then using (44) we have

$$\dot{s} = -(k_1 + k_2) \frac{R_{,u}}{|R_{,u}(R_a)|},$$

(54)

where we denote by $R_{,u}(R_a)$ the function $R_{,u}$ on the curve $R(u,v) = R_a$. Note that the equation for $s$ is written in a form which is valid in all of region 2. When we add (44) to the above three equations we get the complete system of ODE on the curve $R = R_a$:

$$\begin{align*}
\dot{R}_{,v} &= k_1 R_{,v} - R_{,vv}(0,v) e^{-s} R_{,u} \\
\dot{R}_{,u} &= k_2 R_{,u} + R_{,uu}(u,0) e^{-s} R_{,v} \\
\dot{s} &= -(k_1 + k_2) \frac{R_{,u}}{|R_{,u}(R_a)|} \\
\dot{u} &= e^{-s} R_{,v}.
\end{align*}$$

(55)

Our goal now is to extend this set of four equations to any characteristic curve in region 2. It is only the first equation which has to be replaced in region 2. For this we recall the characteristic equations (41) and equate the equation for $u$ along the characteristic with (43). After differentiation with respect to $t$ and using (43,44) we obtain

$$\dot{R}_{,v} = -k_2 R_{,v} - (k_1 + k_2) \frac{R_{,u}}{|R_{,u}(R_a)|} - R_{,vv}(0,v) e^{-s} R_{,u}.$$  

(56)
This set of four equations is accompanied by initial conditions on the $u$ axis for $(R_v, R_u, s, u)$ derived from our original semi-linear system. For a characteristic curve in the standard gauge originated at $(\pi, 0)$ we obtain:

$$
\begin{cases}
R_v(0) = R_v(0, 0) - k_2 \pi \\
R_u(0) = R_u(\pi, 0) \\
s(0) = 0 \\
u(0) = \pi.
\end{cases}
$$

(57)

In regions 3 and 4 we shall only need the equation for the function $s$. By the same argument as in the previous discussions, $s(u, v)$ is only a function of $v$ in region 4 (in the standard gauge). We denote $s$ in region 4 by $s_4(v)$. In region 3 we can write the solution as $s(u, v) = s_4(v) + s_3(u)$, where $s_4(v)$ is the solution in region 4 (by continuity). Now we write our semi-linear system (1, 2) with the relevant terms in the neighborhood of the curve $R = R_b$:

$$
\begin{align*}
R_{uv} &= e^s [-(k_2 + k_3)\theta(R - R_b) + k_3] \\
R_{uv} &= e^s [(k_1 + k_2)\delta(R - R_a) - (k_2 + k_3)\delta(R - R_b)].
\end{align*}
$$

(58)

In regions 3 and 4 we have to integrate the second of (58) with respect to $u$ over the delta functions across $R = R_a$ and $R = R_b$ and use the first equation in (58) to obtain:

$$
s_4'(v) = e^{s_2(v)} \left( \frac{(k_1 + k_2)}{|R_u(R_a)|} - \frac{(k_2 + k_3)}{|R_u(R_b)|} \right),
$$

(59)

where $R_u(R_a)$ is the function $R_u(v)$ on the curve $R = R_a$ and $R_u(R_b)$ is the function $R_u(v)$ on the curve $R = R_b$.

6 The non-linear case

6.1 Singularity formation in the non-linear case

The structure of Fig 6 where $R_a$ is the boundary between regions 1 and 2, implies $R_u(0, 0) < 0$. Therefore there exists $\hat{t} > 0$, so that $R_a < 0$ on $R = R_a$ for $0 \leq t < \hat{t}$. On this segment, (55) holds while the third equation is simplified into $\dot{s} = k_1 + k_2$. Then in the standard gauge we obtain:

$$
s = (k_1 + k_2) t.
$$

(60)

Proposition 6.1 Assume $m_1 < R_v(0, v) < M_1$ and $-M_2 < R_u(u, 0) < -m_2$ where $m_1, M_1, m_2, M_2$ are arbitrary positive constants. Then, there exists singularity $s \to \infty$ on the curve $R = R_a$ at finite values of $u$ and $v$. 

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Proof

Using eqs. (44, 43) and the first two equations in (55) we obtain

\[
\begin{align*}
R_v(t) &= k_1 e^{k_1 t} \int_0^t e^{-k_1 t'} R_v(0, v) dt' + R_v(0, v) \\
R_u(t) &= k_2 e^{k_2 t} \int_0^t e^{-k_2 t'} R_u(u, 0) dt' + R_u(u, 0).
\end{align*}
\] (61)

From this and the assumption of the proposition on \(R_v(0, v)\) and \(R_u(u, 0)\):

\[
\begin{align*}
m_1 e^{k_1 t} < R_v(t) < M_1 e^{k_1 t} \\
-M_2 e^{k_2 t} < R_u(t) < -m_2 e^{k_2 t}.
\end{align*}
\] (62)

Now we substitute (60) into (44, 43) to obtain

\[
\begin{align*}
\dot{u} &= e^{-(k_1+k_2)t} R_v \\
\dot{v} &= -e^{-(k_1+k_2)t} R_u,
\end{align*}
\] (63)

which, together with (62), implies

\[
\begin{align*}
u(t) &< \frac{M_1}{k_2} (1 - e^{-k_2 t}) \\
v(t) &< \frac{M_1}{k_2} (1 - e^{-k_1 t}).
\end{align*}
\] (64)

In conclusion, when \(t \to \infty\), \(s \to \infty\) but \(\lim_{t \to \infty} u(t) < \frac{M_1}{k_2}\) and \(\lim_{t \to \infty} v(t) < \frac{M_1}{k_1}\). Thus, the singularity forms at a finite \(u\) and \(v\). \(\square\)

Under the assumption of Proposition 6.1, we obtain from (62) that \(R_u(t) < 0\) for \(t \in [0, \infty)\) on \(R = R_a\). The same conclusion is valid for any characteristic curve in region 2 since the equation for \(R_u\) takes the same form as on \(R = R_a\).

In Fig. 6 we assume that the curve \(R = R_b\) is monotonously decreasing. This is guaranteed under the following reasonable assumption

**Assumption 6.1** \(R_{vv}(0, v) < 0\) (the weak energy condition)

The characteristic curve which plays an important role in determining whether \(R_b\) is monotonous or not is the curve which starts on the \(u\) axis at the point \(u_h = k_2^{-1} R_v(0, 0)\). We denote the constant value of \(R\) on this curve by \(R_h\). In the next result we show that \(R = R_h\) is monotonously decreasing, and that its \(u_h\) is the minimal value of \(u\) with this property.

**Proposition 6.2** Let Assumption 6.1. The characteristic curve originated at \(u_h = k_2^{-1} R_v(0, 0)\) on the \(u\) axis is monotonously decreasing (\(\dot{u} < 0\)) provided \(u_b > u_h\) (see Fig. 6).
Proof Let us write the characteristic equation for $u(t)$ (41) in region 2:

$$\dot{u} = \int_0^v e^{-s} R_{vv}(0, v')dv' + R_v(0, 0) - k_2 u. $$

Even though we still do not know the shape of the curve $R = R_h$, we know that in a small region near the axis, $s$ is a function only of $v$ (a consequence of the choice in the standard gauge). This region is shown in Figure 7. There exists $0 < t_f$ such that for any $t \in [0, t_f]$, $s = s(v)$. Now we differentiate this equation with respect to $t$ and use (44) to obtain

$$\ddot{u} + k_2 \dot{u} + e^{-2s(v)} R_{vv}(0, v) R_v = 0 \quad (65)$$

By the choice $u_h = k_2^{-1} R_v(0, 0)$ we have $\dot{u}(0) = 0$. Next we show that $\dot{v}(t) < 0$ for any $t \in (0, t_f)$. Substituting $\dot{u}(t) = c(t)e^{-k_2 t}$ into (65) we find

$$\dot{c}(t)e^{-k_2 t} = -e^{-2s} R_{vv}(0, v) R_v. \quad (66)$$

Since $R_{vv}(0, v) < 0$ (assumption 6.1 - the weak energy condition) and $R_v < 0$ in region 2 ($\dot{v} = -e^{-s} R_v > 0$ in region 2 from (41)), we find that $\dot{c}(t) < 0$. In addition, $c(0) = \dot{u}(0) = 0$. Therefore $\dot{u} < 0$ for any $t \in (0, t_f)$. Note that $\dot{v} > 0$ in region 2 where $s = s(v)$, thus $\frac{du}{dv} < 0$ on the characteristic curve $R = R_h$.

![Figure 7](image-url)

Figure 7: The beginning of the curve $R = R_h$ (the curve that starts at the point $u_h = k_2^{-1} R_v(0, 0)$). Here we assume that the curve starts monotonously decreasing. According to Proposition 6.2 this is the correct possibility.

We now argue that $t_f = \infty$. Indeed, if $t_f < \infty$ then, necessarily, $\dot{u}(t_f) = 0$. This, however, is impossible via (66), whose right side is negative in region 2. □
Figure 8: The curve $R = R_h$ and some other characteristic curves. Notice that the curve $R = R_h$ is the first characteristic curve which is monotonously decreasing (first in the sense of minimal $u$ values). The characteristic curves before $R = R_h$ are not monotonous.

The characteristic curve originated at $(u_h, 0)$ plays an important role in defining the boundary of region 2. If this point is part of the boundary of region 2, then the singularity is not hidden. Based on the assumption that $-M_2 < R_u(u, 0) < -m_2$ we obtain:

$$R_a - M_2 u < R(u, 0) < R_a - m_2 u.$$  

Thus we can bound $u_b$ from below:

$$u_b > \frac{R_a - R_b}{M_2}. \quad (67)$$

Therefore, a sufficient condition for $u_b \geq u_h (k_2^{-1}R_v(0,0))$ in the non-linear case is

$$R_a - k_2^{-1}M_2R_v(0,0) > R_b. \quad (68)$$

**Proposition 6.3** All the characteristic curves in region 2 meet at one point where $s \to \infty$.

**Proof**

First we argue that any characteristic curve $R \neq R_a$ cannot intersect with $R_a$ at $t < \infty$. If such an intersection takes place, then the gradient of $R$ will blow up. The latter is impossible for $t < \infty$ according to (62). The same
argument applies for any two characteristic curves in region 2. We conclude that the characteristic curves in region 2 cannot intersect for any \( t < \infty \).

We showed in Proposition 6.1 that the singularity exists on the curve \( R = R_a \). Let us denote the point singularity on the curve \( R = R_a \) by \((u_s, v_s)\). Since in region 2, \( s(u, v) \) is a function only of \( v \), we have the locus of the singularities in region 2:

\[
R^\infty(u_0) \equiv \lim_{t \to \infty} R(t) ; \quad v(0) = s(0) = 0, u(0) = u_0, R(0) = R(u_0, 0),
\]

where \( 0 \leq u_0 \leq u_b \) (see Fig. 9 where the locus and one of the characteristic curves \( R = \tilde{R} \) are displayed). On \( R^\infty(u) \) we have a divergence of \( s \) via (60). Since no two characteristic curves can intersect at \( t < \infty \) in region 2, then \( R^\infty(u_1) \geq R^\infty(u_2) \) if \( 0 \leq u_1 < u_2 \leq u_b \). If, for some \( u_1 < u_2 \), \( R^\infty(u_1) > R^\infty(u_2) \), then

\[
\int_{u_1}^{u_2} R^\infty_u du = R^\infty(u_2) - R^\infty(u_1) < 0,
\]

which implies that \( |R^\infty_{,u}| \to \infty \) for almost any \( u \) in the interval \((u_1, u_2)\). But \( R^\infty_{,u} = \lim_{t \to \infty} R_{,u}(t) = \infty \) (from (62)) where \( R_{,u}(t) \) is given by (61), and we obtain a contradiction. It follows that \( R^\infty \) is a constant in any such interval. \( \square \)

Figure 9: The curve \( R = \tilde{R} \) and the curve \( R = R_a \) hit the ray of singularity (\( R^\infty \)). The situation in this figure is impossible as we show in the proof of Proposition 8.3

In order to proceed to regions 3 and 4, we need another result about region 2 which is formulated in the following Lemma.
Lemma 6.1 In region 2 $\frac{R_u(R')}{R_u(R'')} \to 1$ when $t \to \infty$, for any $R', R'' \in [R_a, R_b]$. 

Proof

From equation (48) we obtain, analogously to (50), 

$$R_u = -k_2 \int v e^{s_2(v')} dv' + R_u(u,0).$$

This equation is valid for any characteristic curve in region 2. So, look at the limit 

$$\lim_{v \to v_s} R_u(R') = \lim_{v \to v_s} -k_2 \int v e^{s_2(v')} dv' + R_u(u',0).$$  (69)

$u'(v)$ and $u''(v)$ describe the curves $R(u,v) = R'$ and $R(u,v) = R''$ respectively. According to (60) the integrals in the denominator and the numerator diverge. Using L’Hospital’s rule we obtain:

$$\lim_{v \to v_s} \frac{R_u(R')}{R_u(R'')} = \lim_{v \to v_s} -k_2 + R_{uu}(u'(v),0)e^{-s_2(v')} \frac{du'}{dv}.$$  (70)

$\lim_{v \to v_s} e^{-s_2(v')} \frac{du'}{dv} = 0$ on any characteristic because $\lim_{v \to v_s} \int v e^{s_2(v')} dv'$ diverges but $\lim_{v \to v_s} \int v \frac{du}{dv} dv' = u(v_s)$ is finite. Therefore $\lim_{v \to v_s} \frac{R_u(R')}{R_u(R'')} = 1$. □

Let us summarize the properties of the inner horizon of our toy-model in the following proposition.

Proposition 6.4 Assume (68). Then, at the inner horizon $H_3$:

a. If $k_3 > k_1$, $s \to \infty$, while if $k_1 < k_3$, $s \to -\infty$.

b. $R_u e^{-s}$ diverges if $k_3 > 2k_1$ and converges if $k_3 < 2k_1$.

c. $e^{-s} R_v R_u$ diverges if $k_3 > 2k_1$ and converges if $k_3 < 2k_1$.

Proof

a. In region 4 we have an expression for $s_4(v)$ as a function of $s_2(v)$ and $R_u$ in region 2 (eq. (59)):

$$s_4(v) = \int_0^v \frac{1}{R_u(R_b)} \left[ k_2 + k_3 - \frac{(k_1 + k_2) R_u(R_b)}{R_u(R_a)} \right] e^{s_2(v')} dv'.$$  (71)

Let us look at the integral when $v \to v_s$. The integrand is written in terms of quantities in region 2. Recall that in region 2 $R_u < 0$. Then since $s_2(v) \to \infty$ when $v \to v_s$ and using Lemma 6.1 we see that if $k_3 > k_1$ then
$s_4(v) \to -\infty$ and if $k_1 > k_3$, $s_4(v) \to \infty$. Note that $s_4(v)$ is regular in all region 4 for $v < v_s$.

In region 3, although $s(u, v) = s_4(v) + s_3(u)$, we have the same type of singularity as in region 4 along the horizon $H_3$ (except for the point singularity). $s_3(u)$ is regular in region 3, except for the point singularity. The reason for this regularity of $s_3(u)$ is the following: If it was not regular in region 3, we would have a singularity in the same $u$ on the curve $R = R_b$ (Fig. 10). This, of course, contradicts our previous results about the regularity of $s$ in region 2. Therefore the only point where $s_3(u)$ might diverge is the point singularity.

Figure 10: A situation where there is a singularity in $s_3(u)$ in $u_{\nu} \neq u_s$. Then we would have a singularity on $R = R_a$ at a regularity point of $s_3$.

b. From the inequality for $R_{u}$ (62), which is valid in all region 2, we obtain the asymptotic behavior $R_{u} \sim e^{k_2 t}$. Substituting the asymptotic expressions of $R_{u}$ and $e^{s_2(v)}$ into eq. (44), we find that $\dot{v} \sim e^{-k_1 t}$. Let us denote by $\Delta v$ the distance to the singularity on the $v$ axes. Then, asymptotically, $\Delta v \sim e^{-k_1 t}$ near the singularity ($t \to \infty$). Substituting the asymptotic expressions of $R_{u}$, $e^{s_2(v)}$ and $\dot{v}$ into eq. (71) we obtain:

$$e^{s_4} \sim e^{(k_3-k_1)t}.$$  \hspace{1cm} (72)

Substituting $\Delta v$, we obtain:

$$e^{s_4} \sim \Delta v \frac{k_3}{k_4 - 1}.$$  \hspace{1cm} (73)
in region 4. Equating the expression for \( \dot{u} \) in (41) with (43) and substituting the asymptotic behavior (73) we obtain

\[
e^{-s} R_v \sim \int^u (\Delta v')^{1-k_3} R_{vu}(0, v') \, dv' - \int^u h'(R(u', 0)) \, du'.
\]  

(74)

Therefore \( e^{-s} R_v \) diverges near the inner horizon \( H_3 \) if \( k_3 > 2k_1 \) and converges if \( k_3 < 2k_1 \).

c. Recall the evolution equation for \( R(u, v) \) in region 4:

\[
R_{uv} = k_3 e^{s_4(v)}.
\]

Integrating this equation with respect to \( v \) we obtain:

\[
R_u = k_3 \int^v e^{s_4(v')} \, dv' + R_u(u, 0).
\]  

(75)

Substituting the asymptotic expression for \( e^{s_4} \) (73) into the expression for \( R_{uv} \), we find that at the inner horizon \( R_u \) converges to a constant different than zero. Therefore the divergences of \( e^{-s} R_v R_u \) and \( e^{-s} R_v \) at the inner horizon are the same. \( \square \)

### 6.2 The No-Hair Phenomenon

Our semi-linear system demonstrate a no-hair phenomenon. Asymptotically near the singularity the contribution of the perturbations disappear. The best way to see it is to look at the equations and the asymptotic form of their components expressed with the parameter \( t \). This parameter is appropriate for this purpose as well since \( t \to \infty \) when we expect the perturbations to vanish. Considering (55) and noting that the equation for \( s \) can be approximated by \( \dot{s} \approx k_1 + k_2 \), we obtain the asymptotic forms of \( e^s \) and \( R_{uv} \) in region 2 as \( \dot{R}_v \sim e^{k_1 t}, e^s \sim e^{(k_1+k_2)t} \). Substituting these asymptotic forms in the perturbative term in the first equation of (55) we obtain

\[
R_{vu}(0, v) e^{-s} R_u \sim R_{vu}(0, v) e^{-k_1 t}.
\]

Thus, when \( t \to \infty \), the contribution of the perturbation decreases exponentially. Thus the asymptotic form of the equation for \( \dot{R}_v \) is \( \dot{R}_v \sim k_1 R_v \). Therefore the asymptotic form of \( R_v \) is \( R_v \sim e^{k_1 t} \). Substituting the latter in the second equation of (55) we find that:

\[
R_{uu}(u, 0) R_v e^{-s} \sim R_{uu}(u, 0) e^{-k_2 t}.
\]

Hence, when \( t \to \infty \), the effect of the perturbations on the equations (and their solution) vanishes. This is the no-hair phenomenon - the characters of the black hole do not depend on the explicit form of the perturbations.
7 Conclusions

The main goal of this research was to provide evidence for the validity of the conjecture, which was described in section 4 (and in [1]). We actually proved the conjecture for the particular case of $h(R)$ as a saw-tooth function under few assumptions. The saw-tooth function is assumed to be generic enough, so any other function $h(R)$ will yield a similar structure of singularity for an open neighborhood of the linear case in the space of initial data.

For $h(R)$ a saw-tooth function our semi-linear system demonstrates a finite time blowup in region 2 under the following assumptions:

- $m_1 < R_v(0,v) < M_1$ and $-M_2 < R_u(u,0) < -m_2$ where $m_1, M_1, m_2, M_2$ are arbitrary positive constants.
- $R_{vv}(0,v) < 0$ (the weak energy condition).

- $R_u - \frac{M_2 R_v(0,0)}{k_2} > R_b$.

In addition our semi-linear system demonstrates a no-hair phenomenon in region 2. We proved that the singularity in region 2 is actually a point singularity (which corresponds to future timelike infinity). The asymptotic expressions near the singular lines in regions 3 and 4 are in full agreement with the conjectured asymptotic behavior (Table 5.1):

- $e^s \sim (\Delta v)^{\frac{k_1}{\kappa_1} - 1}$.
- $e^{-s}R_v \sim (\Delta v)^{2 - \frac{k_2}{\kappa_1}}$.

- $e^{-s}R_vR_u \sim (\Delta v)^{2 - \frac{k_2}{\kappa_1}}$.

The “half gauge invariant scalar” $e^{-s}R_v$ is related to the divergent flux at the inner horizon in various models (see [1]). The gauge invariant scalar $e^{-s}R_vR_u$ is related to the mass function and the scalar curvature singularity which is formed when the model contains two intersecting streams of null fluid.

The curve $R = R_h$ (the curve that starts at the point $u_h = k_2^{-1} R_v(0,0)$ on the $u$ axis) has turned out to be very important in the definition of region 2 since it is the first curve in the region which is monotonically decreasing (see Fig. 8). This curve has an important physical meaning for our model: It is the first characteristic curve ($R = const$) which is completely inside the trapped region. Namely, $R_v, R_u < 0$ along this curve. The apparent horizon is defined as the outer boundary of the trapped region inside the
black hole. In the linear case (which corresponds to the structure of a static black hole) the event horizon and the apparent horizon coincide (see Fig. 2). In the non-linear case the apparent horizon is inside the black hole. In Fig. 8 we can see how the apparent horizon passes if we join the extremal points of the curves behind the horizon $H_2$ by a line - the parts of the curves which are monotonously decreasing are inside the trapped region.

In conclusion, we found that our semi-linear system with $h(R)$ as a saw-tooth function demonstrates many features that have physical meaning and are analogous to the black-hole inner structure (the causal structure and location of the singularity, the infinite blue shift and the inner horizon, mass inflation, the trapped regions etc.). We even found in the model features of the spacetime outside the black-hole (e.g. no-hair principle). Further research is needed to confirm (or contradict) the conjecture under general data. Even more challenging question is the validity of the conjecture for a general $h(R)$.

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