How to regularize singular vectors and kill the dynamical Weyl group

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Abstract

Let g be a simple Lie algebra, and let M λ be the Verma module over g with highest weight λ. For a finite-dimensional g-module U we introduce a notion of a regularizing operator, acting in U, which makes the meromorphic family of intertwining operators \( F_\mu: M_{\lambda+\mu} \to M_\lambda \otimes U \) holomorphic, and conjugates the dynamical Weyl group operators \( A_\mu(\lambda) \in \text{End}(U) \) to constant operators. We establish fundamental properties of regularizing operators, including uniqueness, and prove the existence of a regularizing operator in the case \( g = sl_3 \).

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1. Introduction

Let g be a simple Lie algebra. Let \( M_\lambda \) be the Verma module over g with highest weight \( \lambda \), and \( 1_\lambda \in M_\lambda \) the highest weight vector. A fundamental object of the representation theory of g is a family of intertwining operators

\[
\Phi_\mu^\lambda: M_{\lambda+\mu} \to M_\lambda \otimes U, \quad 1_{\lambda+\mu} \mapsto 1_\lambda \otimes u + \text{lower order terms},
\]

where U is a finite-dimensional g-module, and \( u \in U[\mu] \). In this paper we study singularities of \( \Phi_\mu^\lambda \) as a function of \( \lambda \) and \( u \).

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The intertwining operator $\Phi^\mu_\lambda$ is completely determined by the image of the highest weight vector, which we denote by $\text{Sing}(1_j \otimes u)$. For any $u \in U[\mu]$, the singular vector $\text{Sing}(1_j \otimes u)$ can be written in terms of the Shapovalov form on $M_\lambda$, and is unique for generic $\lambda$. It is a rational function of $\lambda$ with possible poles if $\lambda$ belongs to at least one of the Kac-Kazhdan hyperplanes $\langle \alpha, \lambda + \rho \rangle = \frac{1}{2} \langle \alpha, \alpha \rangle$. For those special values of $\lambda$ the operator $\Phi^\mu_\lambda$ may fail to exist.

One of the objectives of this paper is to regularize the family of operators $\Phi^\mu_\lambda$, that is, to construct a family of intertwining operators $\tilde{\Phi}^\mu_\lambda : M_{\lambda + \mu} \rightarrow M_\lambda \otimes U$, holomorphically depending on $\lambda$ and $u$, such that the correspondence $u \mapsto \tilde{\Phi}^\mu_\lambda$ is still injective.

We also study the relation between the operators $\Phi^\mu_\lambda$ for different values of $\lambda$ and $u$. Recall that for a dominant integral weight $v$ and any element $w$ of the Weyl group $W$, the Verma module $M_v$ contains a submodule isomorphic to $M_{wv}$. We fix this identification and regard $M_{wv}$ as a submodule of $M_v$.

If $\lambda + \mu$ is a dominant integral weight, then for any $w \in W$ the restriction of the operator $\Phi^\mu_\lambda$ to the submodule $M_{w(\lambda + \mu)}$ takes values in the submodule $M_{w(\lambda)} \otimes U$. The relation between this restriction and the operator $\Phi^\mu_{w(\lambda)}$ is given by

$$\Phi^\mu_{w(\lambda)} |_{M_{w(\lambda)}} = \Phi^{A_w(\lambda)u}_{w(\lambda)},$$

where $A_w(\lambda)$ is a rational $\text{End}(U)$-valued function of $\lambda$. The operators $A_w(\lambda)$, $w \in W$, are called the dynamical Weyl group operators.

The specialization of the dynamical Weyl group operators at $\lambda = -\rho$ produces constant operators $\tilde{w} = A_w(-\rho)$, and for a simple root reflection $s_i \in W$ we have $\tilde{s}_i = f_i^{b_i}$. According to the classical results of Verma and Lusztig, the operators $\tilde{s}_i$ form a representation of the Weyl group. The dynamical Weyl group operators may be regarded as meromorphic deformations of the operators $\tilde{w}$ in the class of cocycles on the Weyl group, satisfying

$$A_{w_1w_2}(\lambda) = A_{w_1}(w_2 \cdot \lambda)A_{w_2}(\lambda).$$

We introduce a notion of a regularizing operator $N(\lambda)$, acting in $U$ and polynomially depending on $\lambda$. By definition, we require that $N(\lambda)$ has the following properties.

First, the modified intertwining operators $\Phi^N(\lambda)_w : M_{\lambda + \mu} \rightarrow M_\lambda \otimes U$ are holomorphic in $\lambda$ for any $u \in U[\mu]$.

Second, the conjugated dynamical Weyl group operators $N(w \cdot \lambda)^{-1} A_w(\lambda)N(\lambda)$ are constant operators. This implies, in particular, that the cocycle $A_w(\lambda)$ is cohomologically equivalent to a constant function of $\lambda$.

Third, we impose a minimality condition on $N(\lambda)$ by requiring that $\det N(\lambda)$ is equal to a certain explicitly specified polynomial. (The second property of $N(\lambda)$ implies the divisibility of $\det N(\lambda)$ by this polynomial.)

We prove that the regularizing operator $N(\lambda)$, if it exists, is uniquely determined up to the right multiplication by an operator, polynomially depending on $\lambda$, invertible for all values of $\lambda$, and symmetric with respect to the Weyl group action. If
\( g = \mathfrak{sl}_2 \) it is easy to check that the regularizing operator exists for any irreducible finite-dimensional \( g \)-module \( U \); in each weight subspace of \( U \) the operator \( N(\lambda) \) acts as a scalar, determined by the determinant condition. In this paper we prove the existence property in the case \( g = \mathfrak{sl}_3 \). We conjecture that a regularizing operator exists for any simple Lie algebra \( g \) and any finite-dimensional \( g \)-module \( U \).

Our construction of the regularizing operator \( N(\lambda) \) relies on a certain realization of finite-dimensional \( \mathfrak{sl}_3 \)-modules, based on a special case of the \((\mathfrak{gl}_n, \mathfrak{gl}_n)\) duality. We identify weight subspaces \( U[\mu] \) of an \( \mathfrak{sl}_3 \)-module \( U \) with subspaces of singular vectors in tensor products \( V_{(l,0)} \otimes V_{(l,0)} \otimes V_{(l,0)} \) of finite-dimensional \( \mathfrak{gl}_2 \)-modules. In this realization the action of operators \( A_w(\lambda) \) is identified with the action of suitably renormalized standard rational \( R \)-matrices, the spectral parameters in the \( R \)-matrices being determined by \( \lambda \). These ideas are based on the duality between the difference dynamical equations and qKZ difference equations [TV2]. The duality between differential dynamical and KZ equations is discussed in [TL].

We use the functional realization \( \mathscr{H}[l] \) of the tensor product \( V_{(l,0)} \otimes V_{(l,0)} \otimes V_{(l,0)} \), provided by the representation theory of the Yangian \( Y(\mathfrak{gl}_2) \), to define operators \( \mathcal{N}[l; z] : \mathscr{H}[l] \to V_{(l,0)} \otimes V_{(l,0)} \otimes V_{(l,0)} \), which conjugate the renormalized \( R \)-matrices to the identity. We then use results on the reducibility of the tensor product \( V_{(l,0)}(z_1) \otimes V_{(l,0)}(z_2) \otimes V_{(l,0)}(z_3) \) of the evaluation \( Y(\mathfrak{gl}_2) \)-modules, to prove that for any \( u \in \text{Im} \mathcal{N}[l; z] \) the operator \( \Phi^u_2 \) is holomorphic in \( \lambda \).

The final step of the construction is an alternative “constant” identification of the functional space \( \mathscr{H}[l]^{\text{sing}} \) with the subspace of singular vectors \( (V_{(l,0)} \otimes V_{(l,0)} \otimes V_{(l,0)})^{\text{sing}} \), which is provided by a certain combinatorial lemma. The composition of the “constant” identification with \( \mathcal{N}[l; z] \) gives the desired regularizing operator \( N(\lambda) \) in our particular realization.

As an application of our results, we use the regularized family of intertwining operators to introduce an \( \text{End}(U[0]) \)-valued function \( \Psi(\lambda, x) \) of \( \lambda \in \mathfrak{h}^* \), \( x \in \mathfrak{h} \), defined as a certain matrix trace of \( \Phi^u_2 \). This Baker–Akhiezer-type function is holomorphic in \( \lambda \), and satisfies certain algebraic identities (resonance conditions), relating values of \( \Psi(\lambda, x) \) for different values of \( \lambda \). The fact that \( N(\lambda) \) conjugates the dynamical Weyl group operators \( A_w(\lambda) \) to the constant operators \( \tilde{w} \) implies that these identities have a simple form

\[ \Psi(\lambda, x)u = \Psi(s_x \cdot \lambda, x)u \]

for suitable \( \lambda, u \). Special cases of such resonance conditions were used in [ES] to establish the algebraic integrability of the generalized quantum Calogero–Sutherland systems.

This paper is organized as follows. In Section 2 we review facts on \( g \)-modules and set the notation. In Section 3 we recall the construction of intertwining operators \( \Phi : M_{\lambda + \mu} \to M_\lambda \otimes U \) and singular vectors in the tensor product.

In Section 4 we review facts about the Weyl group and the dynamical Weyl group. We introduce a distinguished action of \( W \) in \( U \), and show that the dynamical Weyl
group operators can be viewed as its deformation. We establish Theorems 4 and 5, which improve results of [EV,TV].

In Section 5 we introduce the notion of a regularizing operator $N(\lambda)$, and use the axiomatic description to establish properties of regularizing operators. Theorem 9 gives the uniqueness of a regularizing operator. We formulate Theorem 10 on the existence of regularizing operators for $\mathfrak{g} = \mathfrak{sl}_3$. This is the main technical result of the paper.

Section 6 provides the background necessary to construct regularizing operators for $\mathfrak{sl}_3$. We invoke the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-duality and functional spaces to give a realization of an $\mathfrak{sl}_3$-module $U$, and use the realization to construct the fundamental operator $N[\{l; z\}]$, which is the main ingredient in the construction of $N(\lambda)$.

Section 7 contains the construction of regularizing operators for $\mathfrak{sl}_3$ and proofs of all their properties. The short Section 8 is devoted to resonance conditions for trace functions, renormalized by $N(\lambda)$.

In Appendix A we review the calculus of formal monomials, which is used to prove the properties of the dynamical Weyl group operators. Appendix B is devoted to the representation theory of the Yangian $Y(\mathfrak{gl}_2)$. We use Yangian modules to establish properties of the operator $N[\{l; z\}]$. Appendix C contains a proof of a technical combinatorial Lemma 19, which is used in the main construction.

The results of this paper can be generalized to the case of representations of quantum groups. This will be done in a separate paper.

2. Notation

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}$ and a root system $\Delta$ with a polarization $\Delta = \Delta^+ \cup \Delta^-$. We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the Killing form on $\mathfrak{g}$, and denote the induced invariant bilinear form on $\mathfrak{h}^*$ by $\langle \cdot, \cdot \rangle$.

An element $\lambda \in \mathfrak{h}^*$ is called dominant, if $\langle \alpha, \lambda \rangle \geq 0$ for any $\alpha \in \Delta^+$. We write $\lambda \geq \mu$ for $\lambda, \mu \in \mathfrak{h}^*$, if $\lambda - \mu$ is dominant. This is a partial order on $\mathfrak{h}^*$.

An element $\lambda \in \mathfrak{h}^*$ is called integral, if $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for any $\alpha \in \Delta^+$.

Denote $W$ the Weyl group of $\mathfrak{g}$. We introduce two actions of $W$ in $\mathfrak{h}^*$. The standard action is defined by

$$s_\alpha v = v - \frac{\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

for any root reflection $s_\alpha \in W$, and the “dot” action is defined by

$$s_\alpha \cdot v = s_\alpha(v + \rho) - \rho,$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

Let $\{e_i, h_i, f_i\}$ denote the standard generators of the Lie algebra $\mathfrak{g}$, corresponding to a simple root $\alpha_i$. We denote $\mathfrak{sl}_2(\alpha_i)$ the Lie subalgebra of $\mathfrak{g}$, spanned by $e_i, h_i, f_i$. 
We denote \( n^+, n^- \) the Lie subalgebras, generated respectively by \( \{e_i\} \) and \( \{f_i\} \). We have a vector space decomposition \( g = n^+ \oplus h \oplus n^- \).

The root lattice \( Q \) is defined by \( Q = \sum_{i=1}^{\dim h} \mathbb{Z} \alpha_i \). We also set \( Q^+ = \sum_{i=1}^{\dim h} \mathbb{Z}_{\geq 0} \alpha_i \).

The universal enveloping algebra \( \mathcal{U}(g) \) is a \( Q \)-graded associative algebra, with the grading defined by

\[
\text{wt}(e_i) = \alpha_i, \quad \text{wt}(h_i) = 0, \quad \text{wt}(f_i) = -\alpha_i, \quad i = 1, \ldots, \dim h.
\]

The Poincare–Birkhoff–Witt theorem gives us a decomposition

\[
\mathcal{U}(g) = \mathcal{U}(n^-) \otimes \mathcal{U}(h) \otimes \mathcal{U}(n^+),
\]

and we denote \( \pi_0 : \mathcal{U}(g) \to \mathcal{U}(h) \) the induced projection along the subspace \( (n^- \mathcal{U}(g) + \mathcal{U}(g)n^+) \subset \mathcal{U}(g) \).

Introduce an anti-involution \( \varpi \) of \( g \) by

\[
\varpi(e_i) = f_i, \quad \varpi(f_i) = e_i, \quad \varpi(h_i) = h_i, \quad i = 1, \ldots, \dim h.
\]

Define a bilinear \( \mathcal{U}(h) \)-valued form \( S \) on \( \mathcal{U}(g) \), by

\[
S(x, y) = \pi_0(\varpi(x)y) \in \mathcal{U}(h), \quad x, y \in \mathcal{U}(g).
\]

The form \( S \) is contravariant, i.e. satisfies

\[
S(xy, z) = S(y, \varpi(x)z), \quad x, y, z \in \mathcal{U}(g).
\]

We will identify the commutative algebra \( \mathcal{U}(h) \) with the algebra \( \mathbb{C}[h^+] \) of polynomial functions on \( h^+ \), and for any \( \lambda \in h^+ \) we will denote \( S_\lambda \) the \( \mathbb{C} \)-valued contravariant form on \( \mathcal{U}(g) \), obtained by evaluating \( S \) at \( \lambda \).

We consider \( g \)-modules \( V \) with a weight space decomposition

\[
V = \bigoplus_{\mu \in h^+} V[\mu], \quad V[\mu] = \{v \in V \mid hv = \mu(h)v \text{ for all } h \in h\},
\]

with finite-dimensional weight subspaces \( V[\mu] \). We say that \( v \in V \) is homogeneous of weight \( \mu \) and write \( \text{wt}(v) = \mu \), if \( v \in V[\mu] \). Then for any homogeneous \( x \in \mathcal{U}(g) \), we have \( \text{wt}(xv) = \text{wt}(x) + \text{wt}(v) \).

In this paper \( U \) always denotes a finite-dimensional \( g \)-module.

For any \( \lambda \in h^+ \), we denote \( M_\lambda \) the Verma module for \( g \), generated by the highest weight vector \( 1_\lambda \), with relations

\[
n^+ 1_\lambda = 0, \quad h 1_\lambda = \lambda(h) 1_\lambda \text{ for } h \in h.
\]

We identify any Verma module \( M_\lambda \) with \( \mathcal{U}(n^-) \) by sending \( x \in \mathcal{U}(n^-) \) to \( x 1_\lambda \in M_\lambda \).

The Verma module \( M_\lambda \) carries a bilinear \( g \)-contravariant form \( (\cdot|\cdot) \), defined by

\[
(x 1_\lambda | y 1_\lambda) = S_\lambda(x, y), \quad x, y \in \mathcal{U}(g).
\]
This form is called the Shapovalov form. Different weight subspaces of \( M_{\lambda} \) are orthogonal with respect to this form, and its restriction to a weight subspace \( M_{\lambda}[\mu] \) is nondegenerate unless \( \lambda \) belongs to a hyperplane of the form

\[
\lambda_{x,k}(\lambda) \overset{\text{def}}{=} \frac{2\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \alpha \rangle} - k = 0
\]

for some \( \alpha \in \Delta^+ \), \( k \in \mathbb{Z}_{\geq 0} \).

The Verma module \( M_{\lambda} \) is reducible if and only if \( \lambda \) belongs to at least one of those hyperplanes. The kernel of the Shapovalov form is the maximal proper submodule of \( M_{\lambda} \). Let \( V_{\lambda} \) denote the irreducible quotient of the Verma module \( M_{\lambda} \); it inherits a contravariant form and a weight decomposition from \( M_{\lambda} \); moreover, the contravariant form on \( V_{\lambda} \) is nondegenerate. It is known that the module \( V_{\lambda} \) is finite-dimensional if and only if \( \lambda \) is a dominant integral weight.

3. Singular vectors and intertwining operators

A vector \( v \) in a \( g \)-module \( V \) is called singular, if \( n^+ v = 0 \). The subspace of all singular vectors in \( V \) is denoted \( V^{\text{sing}} \). For any \( \lambda \in \mathfrak{h}^* \), we have an isomorphism

\[
\text{Hom}_g(M_{\lambda}, V) \cong V^{\text{sing}}[\lambda],
\]

constructed by associating with any intertwining operator \( \Phi \in \text{Hom}_g(M_{\lambda}, V) \) the image of the vector \( 1_{\lambda} \).

It is known [BGG] that a Verma module \( M_{\lambda} \) contains a unique up to proportionality singular vector of weight \( v \) if and only if there exists a finite sequence of weights

\[
\lambda^{(0)} = \lambda, \lambda^{(1)}, \ldots, \lambda^{(k)} = v,
\]

such that

\[
\lambda^{(i)} = s_{\beta_i} \cdot \lambda^{(i-1)} = \lambda^{(i-1)} - n_i \beta_i, \quad i = 1, \ldots, k
\]

for some \( \beta_i \in \Delta^+ \), \( n_i \in \mathbb{Z}_{\geq 0} \). We write \( v < \lambda \) if this condition is satisfied.

**Theorem 1** (Malikov et al. [FFM]). Let \( \lambda \in \mathfrak{h}^* \), and let \( w \in W \) be such that \( w \cdot \lambda < \lambda \). Let \( w = s_{i_1} \ldots s_{i_l} \) be the reduced decomposition of \( w \) in terms of simple root reflections.

Then the formal monomial

\[
F_w(\lambda) = f_{i_1}^{\gamma_{i_1}(\lambda)} \ldots f_{i_l}^{\gamma_{i_l}(\lambda)}, \quad \gamma_k(\lambda) = \langle \alpha_{i_k}, (s_{k+1} \ldots s_l) \cdot \lambda \rangle,
\]

makes sense, and \( F_w(\lambda) 1_{\lambda} \) is a singular vector of weight \( w \cdot \lambda \) in the Verma module \( M_{\lambda} \).
If $\lambda$ is a dominant integral weight, then $w \cdot \lambda < \lambda$ for any $w \in W$, and $\gamma_k(\lambda) \in \mathbb{Z}_{\geq 0}$ for all $k = 1, \ldots, l$. Therefore, $F_w(\lambda)$ is a well defined element in $\mathcal{U}(\mathfrak{n}^-)$. The meaning of $F_w(\lambda)$ for other $\lambda \in \mathfrak{h}^*$, when we may have $\gamma_k(\lambda) \notin \mathbb{Z}_{\geq 0}$, is explained in Appendix A.

Introduce notation $v_{w, \lambda} = F_w(\lambda)1_{\lambda}$ for the singular vector, constructed in Theorem 1. We identify the submodule of $M_\lambda$, generated by $v_{w, \lambda}$, with $M_{w, \lambda}$ by mapping $1_{w, \lambda}$ to $v_{w, \lambda}$.

We now study singular vectors in tensor products $M_\lambda \otimes U$, $\lambda \in \mathfrak{h}^*$. All tensor products in this paper are considered over the field $\mathbb{C}$.

Let $\{g_i\}$ be a homogeneous basis in $\mathcal{U}(\mathfrak{n}^-)$, and let $(S^{-1})_{ij}^\lambda$ be the inverse to the matrix of the form $S_\lambda$, restricted to $\mathcal{U}(\mathfrak{n}^-)$, with respect to this basis. The matrix elements $(S^{-1})_{ij}^\lambda$ are rational functions of $\lambda \in \mathfrak{h}^*$, with possible simple poles at the hyperplanes $x_{\alpha, k}(\lambda) = 0$, for some $\alpha \in \Delta^+$ and $k \in \mathbb{Z}_{\geq 0}$.

Introduce the Cartan anti-automorphism $\omega$ of the Lie algebra $\mathfrak{g}$ by

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h_i) = h_i, \quad i = 1, \ldots, \dim \mathfrak{h}.$$ 

Consider a rational $\mathcal{U}(\mathfrak{n}^-) \otimes \text{End}_\mathbb{C}(U)$-valued function of $\lambda \in \mathfrak{h}^*$,

$$\mathcal{E}(\lambda) = \sum_{i,j} (S^{-1})_{ij}^\lambda g_i \otimes \omega(g_j),$$

(3.1)

where we regard $\omega(g_j)$ as operators acting in $U$. Note that the summation over $i,j$ is finite, because $(S^{-1})_{ij}^\lambda = 0$ for sufficiently large $i$ and $\omega(g_j)$ acts as zero for sufficiently large $j$. One can check that $\mathcal{E}(\lambda)$ does not depend on the choice of the basis $\{g_j\}$.

For any $\lambda \in \mathfrak{h}^*$ and $u \in U$ such that $\mathcal{E}(\lambda)(1 \otimes u) \in \mathcal{U}(\mathfrak{n}^-) \otimes U$ is well defined, we denote

$$\text{Sing}(1_\lambda \otimes u) = \mathcal{E}(\lambda)(1_\lambda \otimes u) \in M_\lambda \otimes U.$$ 

For generic $\lambda$, the vector $\text{Sing}(1_\lambda \otimes u)$ is the unique singular vector of the form

$$\text{Sing}(1_\lambda \otimes u) = 1_\lambda \otimes u + \text{lower order terms},$$

where the “lower order terms” is a linear combination of vectors $\nu' \otimes u'$ with $\text{wt}(\nu') < \lambda$, see [ES]. The vector $\text{Sing}(1_\lambda \otimes u)$ is a singular vector for any $\lambda$, when it is well defined.

For any $u \in U[\mu]$ such that $\text{Sing}(1_\lambda \otimes u)$ is well defined we introduce an intertwining operator $\Phi_\lambda^{mu} \in \text{Hom}_\mathfrak{g}(M_{\lambda + \mu}, M_\lambda \otimes U)$, uniquely determined by the condition

$$\Phi_\lambda^{mu}1_{\lambda + \mu} = \text{Sing}(1_\lambda \otimes u).$$
4. Weyl group and dynamical Weyl group

The Weyl group \( W \) is generated by simple reflections \( \{ s_i \} \), \( i = 1, \ldots, \dim h \), subject to the relations \( s_i^2 = \text{id} \) and the braid relations

\[
\underbrace{s_is_js_i \ldots}_{n_i \text{ factors}} = \underbrace{s_js_is_j \ldots}_{n_j \text{ factors}} \quad (4.1)
\]

for \( i \neq j \), where \( n_{ij} = 2, 3, 4, 6 \) if \( \frac{4\langle x_i, x_i \rangle^2}{\langle x_i, x_j \rangle \langle x_j, x_i \rangle} = 0, 1, 2, 3 \) respectively.

The length \( l(w) \) of any \( w \in W \) is defined to be the smallest positive integer \( l \) such that \( w = s_{i_1} \ldots s_{i_l} \) for some \( i_1, \ldots, i_l \). For the identity element \( \text{id} \in W \) we set \( l(\text{id}) = 0 \).

Let \( U \) be a finite-dimensional \( g \)-module. For any \( i = 1, \ldots, \dim h \), introduce an operator \( \bar{s}_i \in \text{End}_C(U) \) by setting

\[
\bar{s}_i f_i^k v = f_i^{n-k} v, \quad k = 0, 1, \ldots, n \quad (4.2)
\]

for any \( v \in U \) such that \( e_i v = 0 \) and \( h_i v = n v \) for some \( n \in \mathbb{Z}_{\geq 0} \). One can think of operators \( \bar{s}_i \) as operators \( f_i^{h_i} \).

The following lemma is well known; see [EV] and references therein.

**Lemma 2.** There exists a linear representation \( w \mapsto \bar{w} \) of the Weyl group \( W \) in \( U \), such that the generators \( s_i \) are mapped to the operators \( \bar{s}_i \).

Obviously, for any \( w \in W \) we have

\[
\bar{w} = \bigoplus \bar{w}_\mu, \quad \bar{w}_\mu \in \text{Hom}_C(U[\mu], U[\mu]).
\]

Note that if \( w \in W \) and \( \mu \in h^* \) are such that \( w\mu = \mu \), then \( \bar{w}_\mu = \text{Id}_{U[\mu]} \). One can show that the equivalence class of the representation \( w \mapsto \bar{w} \) of \( W \) is completely determined by these conditions.

**Remark.** The operators \( \bar{w} \) do not preserve the Shapovalov form on \( U \); one might try to modify formula (4.2), and introduce operators \( \tilde{s}_i \) by

\[
\tilde{s}_i f_i^k v = \frac{k!}{(n-k)!} f_i^{n-k} v, \quad k = 0, 1, \ldots, n \quad (4.3)
\]

for any \( v \in U \) such that \( e_i v = 0 \) and \( h_i v = n v \) for some \( n \in \mathbb{Z}_{\geq 0} \). The new operators \( \tilde{s}_i \) clearly preserve the Shapovalov form.

**Conjecture 3.** There exists a representation of the Weyl group \( W \) in \( U \), such that the generators \( s_i \) are mapped to the operators \( \tilde{s}_i \).
This conjecture can be verified in certain cases, including adjoint representations for arbitrary \( g \); however, no general proof is available.

We now study the dynamical Weyl group, introduced in [EV,TV]. The following Theorems 4 and 5 are refinements of previous results.

Let \( \mathbb{C}[h^*], \mathbb{C}(h^*) \) denote respectively the algebras of polynomial and rational functions on \( h^* \). If \( V \) is a vector space and \( F \in V \otimes \mathbb{C}[h^*] \), we denote \( F(\lambda) \in V \) the value of the \( V \)-valued polynomial \( F \) at point \( \lambda \in h^* \).

Theorem 4. For any \( w \in W \) there exists an operator \( A_w \in \text{End}_\mathbb{C}(U) \otimes \mathbb{C}(h^*) \), satisfying the following property.

Suppose \( \lambda, \mu \in h^* \) are such that \( w \cdot (\lambda + \mu) < \lambda + \mu \), and \( u \in U[\mu] \) is such that \( \text{Sing}(1, \otimes u) \) is well defined. Then \( A_w(\lambda)u \) is well defined, and

1. If \( w \cdot \lambda < \lambda \), then

\[
\Phi^w_{\lambda}v_{w(\lambda+\mu)} = v_{w,\lambda} \otimes A_w(\lambda)u + \text{lower order terms}. \tag{4.4}
\]

2. If the condition \( w \cdot \lambda < \lambda \) fails, then \( A_w(\lambda)u = 0 \).

This theorem is proved in Appendix A.

The collection of operators \( A_w \in \text{End}_\mathbb{C}(U) \otimes \mathbb{C}(h^*), \ w \in W \), is called the dynamical Weyl group. Eq. (4.4) implies that

\[
A_w = \bigoplus_{\mu} A^\mu_w, \quad A^\mu_w \in \text{Hom}_\mathbb{C}(U[\mu], U[w\mu]) \otimes \mathbb{C}(h^*).
\]

Example. Let \( U \) be a finite-dimensional \( \mathfrak{sl}_2 \)-module, generated by a highest weight vector \( v \) of weight \( n \in \mathbb{Z}_{\geq 0} \). Then the dynamical Weyl group consists of two operators: \( A_{id} = \text{Id} \) and \( A_{s_1} \), determined by

\[
A_{s_1}(\lambda)f^kv = (-1)^n \frac{(-t-2)(-t-3)\ldots(-t-n+k-1)}{t(t-1)\ldots(t-k+1)} \\
\times \frac{k!}{(n-k)!}f^{n-k}v, \quad k = 0, 1, \ldots, n, \tag{4.5}
\]

where \( t = (\lambda, z_1) \).

For arbitrary \( g \), the operators \( A_{s_{\lambda}}(\lambda) \) act in any irreducible \( \mathfrak{sl}_2(z_{\lambda}) \) submodule of \( U \) by formula (4.5), with \( t = \frac{2\langle \lambda, z_1 \rangle}{\langle z_1, z_1 \rangle} \).

Theorem 5. The dynamical Weyl group operators \( A_w, \ w \in W \), acting in a finite-dimensional \( g \)-module \( U \), have the following properties.
1. The dynamical Weyl group operators satisfy the cocycle condition,

\[ A_{w_1w_2}(\lambda) = A_{w_1}(w_2 \cdot \lambda)A_{w_2}(\lambda), \quad w_1, w_2 \in W. \] (4.6)

2. The operator \( A_w \), regarded as an \( \text{End}_U(U) \)-valued rational function on \( \mathfrak{h}^* \), has at most simple poles, which may occur only at the hyperplanes

\[ \chi_{\alpha,k}(\lambda) = 0, \quad \alpha \in \Delta^+ \cap w(\Delta^-), \quad k \in \mathbb{Z}_{\geq 0}. \]

3. If we identify \( U[\mu] \) and \( U[w\mu] \) in any way not depending on \( \lambda \), then for generic \( \lambda \in \mathfrak{h}^* \) we have

\[ \det A_{w}(\lambda) = \text{const} \prod_{\alpha \in \Delta^+ \cap w(\Delta^-)} \frac{\prod_{k=0}^{\infty} \chi_{\alpha,k}(w \cdot \lambda) \dim U[w\mu+k\alpha]}{\prod_{k=0}^{\infty} \chi_{\alpha,k}(\lambda) \dim U[\mu+k\alpha]}, \]

where the constant depends on the choice of identification.

Remark. The dynamical Weyl group operators \( A_{w}(\lambda) \) were introduced in [EV,TV], using a different normalization of the singular vectors:

\[ v_{w,\lambda} = \frac{f_{mn}}{m_1! \cdots f_{i_j}^{n_j}} 1_{\lambda}, \quad n_k = \langle \alpha_k, (s_{k+1} \cdots s_{i_j}) \cdot \lambda \rangle. \]

Eq. (4.4) was observed in [EV,TV] only for \( \lambda \gg \mu \). Part 2 of Theorem 4 is new.

In [EV,TV], the cocycle condition was satisfied only for \( w_1, w_2 \in W \) such that \( l(w_1w_2) = l(w_1) + l(w_2) \); equivalently, operators \( A_{w}(\lambda) \) formed a cocycle on the braid group. Our operators satisfy the additional conditions \( A_{s_i}(s_{x \cdot \lambda})A_{s_i}(\lambda) = \text{Id} \), and thus form a cocycle on the Weyl group.

Proof of Theorem 5. It follows directly from the construction (see also [EV,TV]) that the cocycle condition (4.6) is satisfied when \( w_1, w_2 \in W \) are such that \( l(w_1w_2) = l(w_1) + l(w_2) \). Therefore, it suffices to show that

\[ A_{s_i}(s_{i \cdot \lambda})A_{s_i}(\lambda) = \text{Id}, \quad i = 1, \ldots, \dim \mathfrak{h}. \]

This follows from the explicit formula (4.5) for the action of \( A_{s_i}(\lambda) \) in any irreducible \( \mathfrak{sl}_2(\mathfrak{z}_i) \)-submodule of \( U \).

Now, let \( w \in W \) have a reduced decomposition \( w = s_{i_1} \cdots s_{i_j} \). Then the cocycle condition implies

\[ A_{w}(\lambda) = A_{s_{i_j}}(s_{i_{j-1}} \cdots s_{i_1} \cdot \lambda) \cdots A_{s_{i_2}}(s_{i_1} \cdot \lambda), \]
and singularities of \( A_w(\lambda) \) are determined by the singularities of \( A_{s_i l_k} (s_{i_k} \ldots s_{i_l} \cdot \lambda) \) for \( k = 1, \ldots, \dim \mathfrak{h} \). From the explicit formula (4.5) it follows that \( A_{s_i l_k} (s_{i_k} \ldots s_{i_l} \cdot \lambda) \) may only have simple poles at the hyperplanes \( \chi_{2_i l_k} (s_{i_k} \ldots s_{i_l} \cdot \lambda) = 0 \), or equivalently

\[
\chi_{\beta_k n}(\lambda) = 0, \quad \beta_k = s_i \ldots s_{i_k} x_i. \tag{4.7}
\]

It is known that if \( w = s_i \ldots s_{i_l} \) is a reduced expression, then \( \{ \beta_k \} \) form (without repetitions) the set \( \Delta^+ \cap w(\Delta^-) \). Therefore, hyperplanes (4.7) are all distinct, and the second assertion follows.

Finally, from the cocycle property we obtain

\[
\det A_w^\mu(\lambda) = \det A_{s_i l_k} (s_{i_k} \ldots s_{i_l} \cdot \lambda) \ldots \det A_{s_i l_k} (\lambda)
\]

and the proof of the general formula reduces to verification of the formula

\[
\det A_w^\mu(\lambda) = \text{const} \prod_{k=0}^{\infty} \chi_{s_i l_k}(s_{i_k} \cdot \lambda)^{\dim U[s_i l_k]} / \prod_{k=0}^{\infty} \chi_{s_i l_k}(\lambda)^{\dim U[\mu + k \mathfrak{h}]}.
\]

which follows from (4.5).

The cocycle condition (4.6) immediately implies that the operators \( A_w(-\rho), \ w \in W \), form a representation of the Weyl group. One can easily check that in fact \( A_w(-\rho) = \tilde{w} \), where operators \( \tilde{w} \) are defined by (4.2).

5. Regularizing operators and their properties

In this section we define the main object of our study—regularizing operators for finite-dimensional representations of a semisimple Lie algebra \( \mathfrak{g} \), and establish some of their properties.

Let \( \mathfrak{g} \) be a simple Lie algebra, and let \( U \) be a finite-dimensional \( \mathfrak{g} \)-module. An operator \( N \in \text{End}_\mathbb{C}(U) \otimes \mathbb{C}[\mathfrak{h}^*] \) is called a regularizing operator, if it satisfies the following conditions.

1. The operator \( N \) preserves weight subspaces, i.e. it can be decomposed as

\[
N = \bigoplus_{\mu} N_\mu, \quad N_\mu \in \text{End}_\mathbb{C}(U[\mu]) \otimes \mathbb{C}[\mathfrak{h}^*].
\]

2. For any \( \mu \in \mathfrak{h}^* \), there exists a nonzero constant \( c_\mu \) such that

\[
\det N_\mu(\lambda) = c_\mu \prod_{\alpha \in \Delta^+} \prod_{k=0}^{\infty} \chi_{\alpha, k}(\lambda)^{\dim U[\mu + k \mathfrak{h}]}.
\]

(5.1)
3. There exist operators $a_w \in \text{End}_C(U)$, such that for any $w \in W$ and generic $\lambda \in h^*$ we have

$$a_w = N(w \cdot \lambda)^{-1} A_w(\lambda) N(\lambda).$$

(5.2)

4. The operator $\Xi_N(\lambda) : U \rightarrow \mathcal{U}(n^-) \otimes U$, defined by

$$\Xi_N(\lambda)u = \Xi(\lambda)(1 \otimes N(\lambda)u), \quad u \in U,$$

depends polynomially on $\lambda$. Here $\Xi(\lambda)$ is given by (3.1).

**Remark.** The last condition is equivalent to the requirement that for any $u \in U$, the singular vector $\text{Sing}(1 \otimes N(\lambda)u) \in \mathcal{M}(\lambda) \otimes U$ is polynomial in $\lambda$.

The following Lemma is convenient for establishing formula (5.1).

**Lemma 6.** Suppose $N \in \text{End}_C(U) \otimes \mathbb{C}[h^*]$ satisfies conditions (1) and (3) above, and is such that $N^{-1}$ is regular outside the hyperplanes $\chi_{a,k}(\lambda) = 0$ for all $a \in A^+$ and $k \in \mathbb{Z}_{\geq 0}$. Then $N$ also satisfies condition (2).

**Proof.** Formula (5.2) implies that for any $w \in W$ and generic $\lambda \in h^*$, we have

$$\det A_w^\mu(\lambda) \det N_\mu(\lambda) = \text{const} \det N_{w\mu}(w \cdot \lambda).$$

(5.3)

The right-hand side of this equation polynomially depends on $\lambda$, and therefore in the left-hand side $\det N_\mu(\lambda)$ must be divisible by the denominator of $\det A_w^\mu(\lambda)$. In the special case when $w = w_0$ is the longest element of the Weyl group, this denominator is precisely $\prod_{a \in A^+} \prod_{k=0}^{\infty} \chi_{a,k}(\lambda)^{\dim U[\mu+k\alpha]}$. Therefore, we conclude that

$$c_\mu(\lambda) = \frac{\det N_\mu(\lambda)}{\prod_{a \in A^+} \prod_{k=0}^{\infty} \chi_{a,k}(\lambda)^{\dim U[\mu+k\alpha]}}$$

is polynomial in $\lambda$, and (5.3) implies that $c_\mu(\lambda) = \text{const} c_{w_0\mu}(w_0 \cdot \lambda)$.

By the assumption of the lemma, $c_\mu(\lambda)$ may only vanish at hyperplanes $\chi_{a,k}(\lambda) = 0$, and similarly $c_{w_0}(w_0 \cdot \lambda)$ may only vanish at hyperplanes $\chi_{a,k}(w_0 \cdot \lambda) = 0$. Since these two families of hyperplanes are disjoint, we conclude that $c_\mu(\lambda)$ is a polynomial in $\lambda$ which never vanishes, i.e. a constant polynomial. Condition (2) follows.

The argument above also shows that condition (2) can be thought of as a minimality condition for $N$.

Here are some other immediate corollaries of the definition.

**Proposition 7.** Let $U$ be a finite-dimensional $g$-module, and let $N$ be a regularizing operator.

1. The operators $a_w \in \text{End}_C(U)$ form a representation of the Weyl group.
2. The inverse operator $N^{-1}$, regarded as an $\text{End}_C(U)$-valued rational function on $\mathfrak{h}^*$, may have only simple poles. The poles may only occur at hyperplanes $Z_{x,k}(\lambda) = 0$ for $x \in A^+$ and $k \in \mathbb{Z}_{\geq 0}$ such that $U[\mu + kx] \neq 0$.

3. Let $C \in \text{End}_C(U)$ be a weight-preserving operator, i.e.

$$C = \bigoplus_{\mu} C_{\mu}, \quad C_{\mu} \in \text{End}(U[\mu]).$$

Then the operator $\tilde{N} = NC$ is also a regularizing operator.

**Proof.** For any $w_1, w_2 \in W$, we compute

$$a_{w_1w_2} = N(w_1w_2 \cdot \lambda)^{-1} A_{w_1w_2}(\lambda) N(\lambda) = N(w_1w_2 \cdot \lambda)^{-1} A_{w_1}(w_2 \cdot \lambda) A_{w_2}(\lambda) N(\lambda)$$

$$= (N(w_1w_2 \cdot \lambda)^{-1} A_{w_1}(w_2 \cdot \lambda) N(w_2 \cdot \lambda))(N(w_2 \cdot \lambda)^{-1} A_{w_2}(\lambda) N(\lambda)) = a_{w_1}a_{w_2},$$

which shows that the correspondence $w \mapsto a_w$ gives a representation of the Weyl group.

From the formula for $\det N(\lambda)$, we see that $N^{-1}$ may only have poles at hyperplanes $Z_{x,k}(\lambda) = 0$, for some $x \in A^+$ and $k \in \mathbb{Z}_{\geq 0}$ such that $U[\mu + kx] \neq 0$. Fix $x \in A^+$ and $k \in \mathbb{Z}_{\geq 0}$. We have

$$N(\lambda)^{-1} = a_{s_x}^{-1} N(s_x \cdot \lambda)^{-1} A_{s_x}(\lambda)$$

since $N(s_x \cdot \lambda)^{-1}$ does not have a pole at the hyperplane $Z_{x,k}(\lambda) = 0$, the singularity of $N(\lambda)^{-1}$ may come only from the operator $A_{s_x}(\lambda)$, which have at most simple pole there.

The operators $\tilde{N}(\lambda) = N(\lambda)C$ are obviously polynomial in $\lambda$, and the conditions on $C_{\mu}$ ensure weight preserving and determinant properties of $\tilde{N}(\lambda)$. Finally,

$$\tilde{a}_w = \tilde{N}(w \cdot \lambda)^{-1} A_w(\lambda) \tilde{N}(\lambda) = C^{-1} N(w \cdot \lambda)^{-1} A_w(\lambda) N(\lambda) C = C^{-1} a_w C$$

do not depend on $\lambda$, and for any $u \in U$ the vector

$$\Xi_N(\lambda)u = \Xi(\lambda)(1 \otimes \tilde{N}(\lambda)u) = \Xi(1 \otimes N(\lambda)Cu) = \Xi_N(\lambda)(Cu)$$

is polynomial in $\lambda$. □

**Theorem 8.** Let $U$ be a finite-dimensional $g$-module, and let $N \in \text{End}_C(U) \otimes \mathbb{C}[\mathfrak{h}^*]$ be a regularizing operator. Fix any $\lambda_0 \in \mathfrak{h}^*$. Then

1. The linear map

$$\Xi_N(\lambda_0) : U \to (M_{\lambda_0} \otimes U)^{\text{sing}}, \quad u \mapsto \text{Sing}(1_{\lambda_0} \otimes N(\lambda_0)u)$$

is injective.
2. We have the following description of the image of $N_{\mu}(\lambda_0)$:

$$\text{Im } N(\lambda_0) = \{ u \in U \mid \text{Sing}(1_{\lambda_0} \otimes u) \text{ is well defined} \}. $$

**Proof.** Let $\Delta_0$ be the root subsystem of $\Delta$, defined by

$$\Delta_0 = \{ \alpha \in \Delta \mid \langle \alpha, \lambda_0 \rangle \in \mathbb{Z} \}. $$

We set $\Delta_0^+ = \Delta_0 \cap \Delta^+$, and denote by $W_0$ the subgroup of $W$, generated by root reflections $s_\alpha$, $\alpha \in \Delta_0$.

Let $w \in W_0$ be such that $w \cdot \lambda_0$ is antidominant with respect to $\Delta_0^+$. Then, $\det N(w \cdot \lambda_0) \neq 0$, and $N(w \cdot \lambda_0)$ is invertible.

According to Theorem 4,

$$\Phi_{\lambda_0} N(\lambda_0) u_{w \cdot \lambda_0 + \mu} = v_{w \cdot \lambda_0} \otimes A_w(\lambda_0) N(\lambda_0) u + \text{lower order terms}. $$

Suppose for some $u \in U[\mu]$ we have $\text{Sing}(1_{\lambda_0} \otimes N(\lambda_0))u = 0$. Then the intertwining operator $\Phi_{\lambda_0} N(\lambda_0) u$ is identically zero, and in particular

$$N(w \cdot \lambda_0) a_w u = A_w(\lambda_0) N(\lambda_0) u = 0. $$

Since $N(w \cdot \lambda_0)$ and $a_w$ are both invertible, we conclude that $u = 0$, and thus $\Xi_N(\lambda_0)$ is injective.

Now, let $u \in U[\mu]$ be such that $\text{Sing}(1_{\lambda_0} \otimes u)$ is well defined. Then the vector $u' = A_w(\lambda_0) u$ is also well defined, and we have

$$u = A_w(\lambda_0)^{-1} u' = N(\lambda_0) a_w^{-1} N(w \cdot \lambda_0)^{-1} u' \in \text{Im } N(\lambda_0). $$

Conversely, property (4) of $N(\lambda_0)$ guarantees that $\text{Sing}(1_{\lambda_0} \otimes u)$ is well defined for any $u \in \text{Im } N_{\mu}(\lambda_0)$. \qED

Finally, we prove that the regularizing operator is determined uniquely up to right multiplication by a polynomially invertible matrix $C(\lambda)$, satisfying certain $W$-invariance conditions.

**Theorem 9.** Let $N, \tilde{N}$ be regularizing operators, and let $a_w, \tilde{a}_w$ be the associated constant representations of the Weyl group. Then the operator

$$C(\lambda) = N(\lambda)^{-1} \tilde{N}(\lambda) \in \text{End}_C(U) $$

is polynomial in $\lambda$, has a decomposition

$$C(\lambda) = \bigoplus_{\mu} C_{\mu}(\lambda), \quad C_{\mu}(\lambda) \in \text{End}_C(U[\mu]),$$
and satisfies

\[ C(w \cdot \lambda)\tilde{a}_w = a_w C(\lambda). \]

Moreover, the inverse operator \( C(\lambda)^{-1} \) is also polynomial in \( \lambda \), and the Weyl group representations \( a_w, \tilde{a}_w \) are equivalent.

**Proof.** Since regularizing operators preserve weight subspaces, we have

\[ C_\mu(\lambda) = N_\mu(\lambda)^{-1} \tilde{N}_\mu(\lambda) \in \text{End}_C(U[\mu]). \]

The only possible singularities of \( C(\lambda) = N(\lambda)^{-1} \tilde{N}(\lambda) \) may be simple poles at one of the hyperplanes \( \chi_{\alpha, k}(\lambda) = 0 \). Fix \( \alpha \in \Delta^+ \), \( k \in \mathbb{Z}_{\geq 0} \), and write

\[ N(\lambda)^{-1} = \frac{X(\lambda)}{\chi_{\alpha, k}(\lambda)} + Y(\lambda) \]

for some \( X(\lambda), Y(\lambda) \in \text{End}_C(U) \), regular at the hyperplane \( \chi_{\alpha, k}(\lambda) = 0 \).

It follows from the identity \( N(\lambda)^{-1} \tilde{N}(\lambda) = \text{Id} \) that for generic \( \lambda_0 \) from this hyperplane, we have \( X(\lambda_0)N(\lambda_0) = 0 \). Hence,

\[ \ker X(\lambda_0) \supset \text{Im} N(\lambda_0) = \text{Im} \tilde{N}(\lambda_0), \]

which implies that \( C(\lambda) \) is well defined at \( \lambda_0 \). Therefore, \( C(\lambda) \) does not have a pole at the hyperplane \( \chi_{\alpha, k}(\lambda) = 0 \), which was arbitrary, and thus \( C(\lambda) \) is polynomial in \( \lambda \).

In view of (5.1), we conclude that \( \det C_\mu(\lambda) = (\det N_\mu(\lambda))^{-1} \det \tilde{N}_\mu(\lambda) \) is a nonzero constant. Hence, each \( C_\mu(\lambda) \) is polynomially invertible.

Finally, we compute

\[ C(w \cdot \lambda)^{-1} a_w C(\lambda) = \tilde{N}(w \cdot \lambda)^{-1} N(w \cdot \lambda)a_w N(\lambda)^{-1} \tilde{N}(\lambda) \]

\[ = \tilde{N}(w \cdot \lambda)^{-1} A_w(\lambda) \tilde{N}(\lambda) = \tilde{a}_w. \]

The invariance condition follows. Setting \( \lambda = -\rho \), we get

\[ \tilde{a}_w = C(-\rho)^{-1} a_w C(-\rho) \]

and the representations \( a_w, \tilde{a}_w \) of the Weyl group are equivalent. \( \square \)

**Example.** Let \( g = sl_2 \). We identify \( \mathfrak{h}^* \) with \( C \) by associating \( \mathfrak{h}^* \ni \lambda \leftrightarrow \langle \alpha_1, \lambda \rangle \in C \). Under this identification, \( \alpha_1 \equiv 2 \) and \( \rho \equiv 1 \).
Let $U$ be an irreducible finite-dimensional $\mathfrak{sl}_2$-module with highest weight $\lambda \in \mathbb{Z}_{\geq 0}$. For $u \in U[1 - 2k]$, $k \in \mathbb{Z}_{\geq 0}$, we have

$$\text{Sing}(1_x \otimes u) = \sum_{j=0}^{k} \frac{(-1)^j}{j! \prod_{i=0}^{j-1} (\lambda - i)} f^j 1_x \otimes e^j u.$$ 

The weight subspaces of $U$ are one-dimensional, the operators $N_{A-2k}(\lambda) \in \text{End}(U[1 - 2k]) \cong \mathbb{C}$ being scalars, and we take

$$N_{A-2k}(\lambda) = \frac{1}{k!} \prod_{j=0}^{k-1} z_{x,j}(\lambda) = \frac{1}{k!} \prod_{j=0}^{k-1} (\lambda - j).$$

One can check that the operator $a_s$ acts in $U$ by

$$a_s : f^k 1_A \mapsto f^{A-k} 1_A, \quad k = 0, 1, \ldots, A.$$ 

In particular, if $A$ is an even integer and $u = f^{A/2} 1_A \in U[0]$, we have $a_s u = u$.

We now state our main result on the existence of regularizing operators.

**Theorem 10.** Let $\mathfrak{g} = \mathfrak{sl}_3$, and let $U$ be an irreducible finite-dimensional $\mathfrak{g}$-module. Then there exists a regularizing operator $N \in \text{End}_\mathbb{C}(U) \otimes \mathbb{C}(\mathfrak{h}^*)$. Moreover, the associated operators $a_s$, corresponding to simple root reflections in $W$, coincide with the operators $s_i$ given by formulas (4.2).

This theorem will be proved in Section 7.

**Example.** Let $U$ be the adjoint representation of $\mathfrak{sl}_3$. The zero weight subspace $U[0]$ is the Cartan subalgebra $\mathfrak{h}$; the generators $h_1, h_2 \in \mathfrak{sl}_3$ form a basis in $U[0]$.

The action of operators $A^{[0]}(\lambda)$ on $U[0]$ is completely determined by the matrices

$$A^{[0]}_{x_1}(\lambda) \begin{pmatrix} -\frac{\lambda_1 - 2}{\lambda_1} & -\frac{\lambda_1 - 1}{\lambda_1} \\ 0 & 1 \end{pmatrix}, \quad A^{[0]}_{x_2}(\lambda) \begin{pmatrix} 1 & 0 \\ -\frac{\lambda_2 - 1}{\lambda_2} & -\frac{\lambda_2 - 2}{\lambda_2} \end{pmatrix},$$

where $\lambda_1 = \langle x_1, \lambda \rangle$, $\lambda_2 = \langle x_2, \lambda \rangle$. The operator $N_{[0]}(\lambda) : U[0] \rightarrow U[0]$ is given by the matrix

$$N_{[0]}(\lambda) = \frac{1}{3} \begin{pmatrix} 2\lambda_1^2 + 2\lambda_1 \lambda_2 - \lambda_2^2 + 4\lambda_1 - \lambda_2 & 2\lambda_1^2 + 2\lambda_1 \lambda_2 - \lambda_2^2 + 2\lambda_1 - 2\lambda_2 \\ 2\lambda_2^2 + 2\lambda_1 \lambda_2 - \lambda_1^2 - 2\lambda_1 + 2\lambda_2 & 2\lambda_2^2 + 2\lambda_1 \lambda_2 - \lambda_1^2 - \lambda_1 + 4\lambda_2 \end{pmatrix}.$$ 

It is a straightforward computation to show that

$$N_{[0]}(s_1 \cdot \lambda)^{-1} A^{[0]}_{x_1}(\lambda) N_{[0]}(\lambda) \equiv \text{Id}_{U[0]}, \quad N_{[0]}(s_2 \cdot \lambda)^{-1} A^{[0]}_{x_2}(\lambda) N_{[0]}(\lambda) \equiv \text{Id}_{U[0]}.$$
and that

$$\det N_0(\lambda) = \lambda_1\lambda_2(\lambda_1 + \lambda_2 + 1) = \chi_{21,1}(\lambda)\chi_{22,1}(\lambda)\chi_{21+22,1}(\lambda).$$

6. The $$(\mathfrak{gl}_m, \mathfrak{gl}_n)$$ duality, and functional realizations of $$\mathfrak{gl}_n$$-modules

In Section 7 we will construct the regularizing operators for $$\mathfrak{g} = \mathfrak{sl}_3$$ using a concrete realization of irreducible finite-dimensional $$\mathfrak{sl}_3$$-modules. This section is devoted to a review of some underlying results on representations of the reductive Lie algebras $$\mathfrak{gl}_n$$.

The space $$C^{mn} = (\mathbb{C}^m) \otimes \mathbb{C}^n$$ has natural commuting actions of $$\mathfrak{gl}_m$$ and $$\mathfrak{gl}_n$$. The algebra $$P_{m,n} = \mathbb{S}(C^{mn})$$ of polynomial functions on $$C^{mn}$$ becomes a module over $$\mathfrak{gl}_m \otimes \mathfrak{gl}_n$$. It has decompositions

$$P_{m,n} = \bigoplus_{\mu_1, \ldots, \mu_m} P_{m,n}^{\mu_1, \ldots, \mu_m} \otimes V_{\mathfrak{gl}_m}^{\mu_1, \ldots, \mu_m} \otimes V_{\mathfrak{gl}_n}^{\mu_1, \ldots, \mu_m},$$

where the subspaces

$$P_{m,n}^{\mu_1, \ldots, \mu_m} \cong \mathbb{S}^{\mu_1}(\mathbb{C}^n) \otimes \cdots \otimes \mathbb{S}^{\mu_m}(\mathbb{C}^n),$$

$$P_{m,n}^{v_1, \ldots, v_n} \cong \mathbb{S}^{v_1}(\mathbb{C}^m) \otimes \cdots \otimes \mathbb{S}^{v_n}(\mathbb{C}^m),$$

are the weight subspaces with respect to the algebras $$\mathfrak{gl}_m$$ and $$\mathfrak{gl}_n$$ respectively.

In the explicit realization of $$P_{m,n}$$ as the polynomial algebra in $$mn$$ variables $$\mathbb{C}[\{x_{ij}\}]$$, $$1 \leq i \leq m$$, $$1 \leq j \leq n$$, the actions of $$\mathfrak{gl}_m, \mathfrak{gl}_n$$ are given by

$$E_{ab}^{\mathfrak{gl}_m} = \sum_{i=1}^n x_{ai} \frac{\partial}{\partial x_{bi}}, \quad a, b = 1, \ldots, m,$$

$$E_{ij}^{\mathfrak{gl}_n} = \sum_{a=1}^m x_{ai} \frac{\partial}{\partial x_{aj}}, \quad i, j = 1, \ldots, n,$$

where $$E_{ij}^{\mathfrak{gl}_k}$$ are the standard generators of the Lie algebra $$\mathfrak{gl}_k$$, corresponding to the elementary matrices with 1 at the intersection of the $$r$$th row and $$s$$th column, and 0 elsewhere.

We have the following theorem, see [H, Zh].

**Theorem 11.** The $$\mathfrak{gl}_m \otimes \mathfrak{gl}_n$$ module $$P_{m,n}$$ has the decomposition

$$P_{m,n} = \bigoplus_{\Lambda \in \mathcal{P}_{\text{max}(m,n)}} V_A^{\mathfrak{gl}_m} \otimes V_A^{\mathfrak{gl}_n},$$

where $$\mathcal{P}_j$$ denotes the collection of finite sequences $$\Lambda$$ of nonnegative integers
$A_1 \geq \cdots \geq A_j$, and $V_{sl_k}^{j}$ for $k \geq j$ denotes the irreducible $gl_k$-module with highest weight $(A_1, \ldots, A_j, 0, \ldots, 0)$.

In particular, this theorem implies that a $gl_n$-module $V_{sl_n}^{A}$ may be realized as the subspace of $gl_m$-singular vectors of weight $A$ in $P_{m,n}$.

We will use the special case of the $(gl_2, gl_3)$ duality, which gives the following decomposition of the polynomial algebra $P_{2,3}$:

$$P_{2,3} = \bigoplus_{m_1 \geq m_2 \geq 0} V_{(m_1,m_2)}^{gl_1} \otimes V_{(m_1,m_2)}^{gl_2}.$$ 

We identify a module $V_{(m_1,m_2)}^{gl_1}$ with the subspace of $gl_2$-singular vectors in the algebra $P_{2,3}$ of weight $(m_1,m_2)$ with respect to $gl_2$:

$$V_{(m_1,m_2)}^{gl_1} \cong \bigoplus_{l_1+l_2+l_3=m_1+m_2} (V_{(l_1,0)}^{gl_2} \otimes V_{(l_2,0)}^{gl_2} \otimes V_{(l_3,0)}^{gl_2})^{\text{sing}}[\{(m_1,m_2)\}].$$

Such identification gives the correspondence of the weight subspaces,

$$V_{(m_1,m_2)}^{gl_1}[(l_1,l_2,l_3)] \cong (V_{(l_1,0)}^{gl_2} \otimes V_{(l_2,0)}^{gl_2} \otimes V_{(l_3,0)}^{gl_2})^{\text{sing}}[\{(m_1,m_2)\}].$$

Our next goal is to describe a realization of $V_{(l,0)}^{gl_1} \otimes V_{(l,0)}^{gl_2} \otimes V_{(l,0)}^{gl_2}$ as a certain functional space. In the remaining part of this section $l = (l_1,l_2,l_3)$ will denote a triple of nonnegative integers.

To shorten formulas, we will omit superscripts for $gl_2$-modules and operators, and write $V_{(l,0)}$, $E_{ab}$ instead of $V_{(l,0)}^{gl_2}$, $E_{ab}^{gl_2}$, respectively. By a slight abuse of notation, we also write $l$ for the highest weight vector of the $gl_2$-module $V_{(l,0)}$.

Let $C[z], C(z)$ denote respectively the algebras of polynomial and rational functions in complex variables $z = (z_1, z_2, z_3)$.

Let $\mathcal{H}_k$ denote the space of symmetric polynomials in variables $t_1, \ldots, t_k$ of degree at most 2 in each $t_1, \ldots, t_k$, with coefficients in $C[z]$. We also set

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathcal{H}_k.$$ 

For any function $\varphi$ of $t_1, \ldots, t_k$, we denote

$$\text{Sym}(\varphi(t_1, \ldots, t_k)) = \sum_{\sigma \in S_k} \varphi(t_{\sigma(1)}, \ldots, t_{\sigma(k)}).$$

**Proposition 12** (Tarasov and Varchenko [TV3]). The space $\mathcal{H}$ has a structure of a $gl_2$-module, depending on $l$, such that the action of generators $\{E_{ab}\}$ on
a function \( \phi \in \mathcal{H}_k \) is given by

\[
(E_{11} \phi)(t_1, \ldots, t_k) = (l_1 + l_2 + l_3 - k)\phi(t_1, \ldots, t_k),
\]

\[
(E_{22} \phi)(t_1, \ldots, t_k) = k\phi(t_1, \ldots, t_k),
\]

\[
(E_{12} \phi)(t_1, \ldots, t_{k-1}) = \lim_{t_k \to \infty} \frac{\phi(t_1, \ldots, t_k)}{t_k^2},
\]

\[
(E_{21} \phi)(t_1, \ldots, t_{k+1}) = \frac{1}{k!} \text{Sym} \left( \phi(t_1, \ldots, t_k) \left( \prod_{a=1}^3 \frac{t_{k+1} - t_i}{t_{k+1} - t_i} \right) \right).
\]

One can easily check that multiplication by any function of \( z \) commutes with the \( \mathfrak{gl}_2 \) action above; thus, \( \mathcal{H} \) can be regarded as a \( \mathfrak{gl}_2 \otimes \mathbb{C}[z] \)-module.

**Proposition 13** (Tarasov and Varchenko [TV3]). The linear map

\[
\phi[\lambda] : V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} \to \mathcal{H},
\]

\[
\phi[\lambda](E_{21}^{k_1} 1_{l_1} \otimes E_{21}^{k_2} 1_{l_2} \otimes E_{21}^{k_3} 1_{l_3}) = \frac{l_1! l_2! l_3!}{(l_1 - k_1)! (l_2 - k_2)! (l_3 - k_3)!} \cdot \text{Sym} \left( \prod_{a=1}^3 \prod_{i=1}^{k_{i+a-1}} \frac{t_i - z_a}{t_i - z_a} \prod_{j=k_1 + \cdots + k_a + 1} t_i - z_a + l_a \right) \times \prod_{i<j} \frac{t_i - t_j + 1}{t_i - t_j},
\]

is a homomorphism of \( \mathfrak{gl}_2 \)-modules.

Let \( \mathcal{H}_k[\lambda] \) denote the subspace of \( \mathcal{H}_k \), consisting of functions \( \phi \in \mathcal{H}_k \), satisfying additional “admissibility” conditions

\[
\phi(z_i, z_i - 1, \ldots, z_i - l_i, t_{i+2}, \ldots, t_k) \equiv 0,
\]

imposed if \( k > l_i, \ i = 1, 2, 3 \). We also denote

\[
\mathcal{H}[\lambda] = \bigoplus_{k \in \mathbb{Z}_{>0}} \mathcal{H}_k[\lambda].
\]

One can check that \( \mathcal{H}[\lambda] \) is a \( \mathfrak{gl}_2 \otimes \mathbb{C}[z] \)-submodule of \( \mathcal{H} \).
Proposition 14. 1. The map \( \phi[l] \) is injective, and

\[
\text{Im } \phi[l] \otimes \mathbb{C}[z] \subseteq \mathcal{H}[l] \otimes \mathbb{C}[z] \subseteq \mathcal{H}(z).
\]  

2. There exists a homomorphism of \( \mathfrak{gl}_2 \otimes \mathbb{C}[z] \)-modules

\[
\mathcal{S}[l] : \mathcal{H}[l] \to (V_{(l, 0)} \otimes V_{(l, 0)} \otimes V_{(l, 0)}) \otimes \mathbb{C}[z],
\]

such that

\[
(\mathcal{S}[l] \circ \phi[l])v = v \quad \text{for any } v \in V_{(l, 0)} \otimes V_{(l, 0)} \otimes V_{(l, 0)}.
\]

Moreover, for any \( \varphi \in \mathcal{H}[l] \), the vector \( \mathcal{S}[l]\varphi \), rationally depending on \( z \), may have at most simple poles, located at the hyperplanes

\[
z_j - r = z_i - l_i, \quad r = 0, 1, \ldots, \min(l_i, l_j) - 1, \quad 1 \leq i \leq j \leq 3.
\]

The proposition is proved in Appendix B.

Introduce \( \mathcal{X}[l] \in \mathbb{C}[z] \) by

\[
\mathcal{X}[l] = \prod_{r=0}^{l-1} (z_1 - l_1 - z_2 + r) \prod_{r=0}^{l-1} (z_1 - l_1 - z_3 + r) \prod_{r=0}^{l-1} (z_2 - l_2 - z_3 + r),
\]

and set

\[
\mathcal{N}[l] = \mathcal{X}[l] \mathcal{S}[l].
\]

Corollary 15. Formula (6.5) defines a homomorphism of \( \mathfrak{gl}_2 \otimes \mathbb{C}[z] \)-modules

\[
\mathcal{N}[l] : \mathcal{H}[l] \to (V_{(l, 0)} \otimes V_{(l, 0)} \otimes V_{(l, 0)}) \otimes \mathbb{C}[z].
\]

Proof. For any \( \varphi \in \mathcal{H}[l] \), the possible simple poles of \( \mathcal{S}[l]\varphi \) are offset by \( \mathcal{X}[l] \). Therefore, \( \mathcal{N}[l]\varphi \) is polynomial in \( z \). The \( \mathfrak{gl}_2 \)-intertwining property for \( \mathcal{N}[l] \) follows from the fact that \( \mathcal{S}[l] \) is a \( \mathfrak{gl}_2 \)-homomorphism, and the commutativity of multiplication by any element from \( \mathbb{C}[z] \) with the action of \( \mathfrak{gl}_2 \). \( \square \)

Now we let formal variables \( z \) take particular complex values; specifying \( z = z_0 \) for some \( z_0 \in \mathbb{C}^3 \) will be reflected by adding \( z_0 \) to the notation. For example, \( \phi[l; z_0] \) will denote the composition of \( \phi[l] \) with the homomorphism \( \mathbb{C}[z] \to \mathbb{C} \) of evaluation at \( z = z_0 \), and \( \mathcal{N}[l; z_0] \) will consist of symmetric polynomials in \( t_1, \ldots, t_k \) with complex coefficients, obtained from elements of \( \mathcal{H}[l] \) by specializing \( z = z_0 \).

We now establish certain properties of operators \( \phi[l; z_0], \mathcal{N}[l; z_0] \) for special values of \( z_0 \). Recall the following standard decompositions of the tensor product of \( \mathfrak{gl}_2 \)-
modules:
\[ V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} \cong \left( \bigoplus_{i=0}^{\min(l_1,l_2)} V_{(l_1+l_2-i,i)} \right) \otimes V_{(l_3,0)} \]
\[ \cong V_{(l_1,0)} \otimes \left( \bigoplus_{i=0}^{\min(l_1,l_3)} V_{(l_2+l_3-i,i)} \right). \]

**Proposition 16.** Let \( z_0 \in \mathbb{C}^3 \).

1. If \( z_0 \) is generic from the hyperplane \( z_1 - z_2 - l_1 + s = 0, \ 0 \leq s < \min(l_1, l_2) \), then

\[ \ker \phi[l; z_0] = \left( \bigoplus_{i=s+1}^{\min(l_1,l_2)} V_{(l_1+l_2-i,i)} \right) \otimes V_{(l_3,0)}. \]

2. If \( z_0 \) is generic from the hyperplane \( z_2 - z_3 - l_2 + s = 0, \ 0 \leq s < \min(l_2, l_3) \), then

\[ \ker \phi[l; z_0] = V_{(l_1,0)} \otimes \left( \bigoplus_{i=s+1}^{\min(l_2,l_3)} V_{(l_2+l_3-i,i)} \right). \]

This Proposition will be proved in Appendix B. As a consequence, we obtain the following properties of operators \( \mathcal{N}[l; z_0] \).

**Corollary 17.** Let \( z_0 \in \mathbb{C}^3 \).

1. If \( z_0 \) is generic from the hyperplane \( z_1 - l_1 - z_2 + s = 0, \ 0 \leq s < \min(l_1, l_2) \), then

\[ \text{im} \ \mathcal{N}[l; z_0] \subset \left( \bigoplus_{i=s+1}^{\min(l_1,l_2)} V_{(l_1+l_2-i,i)} \right) \otimes V_{(l_3,0)}. \]

2. If \( z_0 \) is generic from the hyperplane \( z_2 - l_2 - z_3 + s = 0, \ 0 \leq s < \min(l_2, l_3) \), then

\[ \text{im} \ \mathcal{N}[l; z_0] \subset V_{(l_1,0)} \otimes \left( \bigoplus_{i=s+1}^{\min(l_2,l_3)} V_{(l_2+l_3-i,i)} \right). \]

**Proof.** Let \( \varphi \in \mathcal{H}[l; z_0] \). Then it follows from the definitions and (6.3) that

\[ \phi[l; z_0] \circ \mathcal{N}[l; z_0] \varphi = \mathcal{X}[l; z_0] \varphi. \]
For any point \( z_0 \) from the hyperplane \( z_1 - l_1 - z_2 + s = 0 \) with \( 0 \leq s < \min(l_1, l_2) \), we have that \( \mathcal{H}[l; z_0] = 0 \), and therefore \( \mathcal{H}[l; z_0]|\phi \subset \ker \phi \). Hence, the first assertion follows from Proposition 16.

The proof of the second assertion is similar. □

Note that both the action of \( \mathfrak{gl}_2 \otimes \mathbb{C}[z] \) and the admissibility conditions (6.2) are invariant under simultaneous permutations of triples \( l \) and \( z \), i.e. we have a canonical isomorphism of modules

\[
\mathcal{H}[l; z] = \mathcal{H}[\sigma(l); \sigma(z)], \quad \sigma \in S_3.
\]

**Proposition 18.** There exists a family of linear operators \( \mathcal{R}_{l_p}(u) : V_{(l,0)} \otimes V_{(l',0)} \rightarrow V_{(l,0)} \otimes V_{(l',0)} \), rationally depending on a complex parameter \( u \), such that

\[
\mathcal{R}_{l_p}(u) \mathbf{1}_l \otimes \mathbf{1}_l = \frac{\prod_{j=1}^{l'} (u - l_j + j)}{\prod_{j=1}^{l} (u - l_j + j)} \mathbf{1}_l \otimes \mathbf{1}_l, \quad (6.7)
\]

and the following diagram is commutative for generic \( z \):

\[
\begin{array}{c}
\mathcal{H}[s_1(1); s_1(z)] \xrightarrow{N[s_1(1); s_1(z)]} V_{(l_2,0)} \otimes V_{(l_1,0)} \otimes V_{(l_3,0)} \\
\mathcal{H}[l; z] \xrightarrow{N[l; z]} V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} \\
\mathcal{H}[s_2(1); s_2(z)] \xrightarrow{N[s_2(1); s_2(z)]} V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)}
\end{array}
\]

Proposition 18 is proved in Appendix B, where it is shown that the operators \( \mathcal{R}_{l_p}(u) \) are the standard rational \( R \)-matrices, renormalized by (6.7).

**7. The main construction**

Let \( U \) be an irreducible finite-dimensional \( \mathfrak{sl}_3 \)-module with highest weight \( \lambda \). Set

\[
m = \langle z_1 + 2z_2, \lambda \rangle, \quad k = \langle z_2, \lambda \rangle.
\]

The \( \mathfrak{gl}_3 \)-module \( V_{(m-k,k,0)}^{\mathfrak{gl}_3} \), regarded by restriction as an \( \mathfrak{sl}_3 \)-module, is isomorphic to \( U \). We fix an identification (which is unique up to a scalar)

\[
\Theta : U \rightarrow \bigoplus_{l \in \mathbb{Z}_{\geq 0}^3} (V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}(m - k, k),
\]
so that \( \Theta \) maps a weight subspace \( U[\mu] \) to \((V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)]\),

where the corresponding \( I = (l_1, l_2, l_3) \in \mathbb{Z}_{\geq 0}^3 \) is determined by

\[
l_1 - l_2 = \langle \alpha_1, \mu \rangle, \quad l_2 - l_3 = \langle \alpha_2, \mu \rangle, \quad l_1 + l_2 + l_3 = m. \tag{7.1}
\]

Under this identification the Weyl group for \( \mathfrak{sl}_3 \) coincides with the symmetric group \( S_3 \) naturally acting on \( \mathbb{Z}_{\geq 0}^3 \).

For any \( I \in \mathbb{Z}_{\geq 0}^3 \) we fix a \( \mathbb{C} \)-basis \( \{ v_i[I] \} \) of the space \((V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)]\), such that \( v_i[I] w(I) = \tilde{w} v_i[I] \), where the operators \( \tilde{w} \) correspond to the action of the Weyl group given by Lemma 2. This can be achieved because the restriction of \( \tilde{w} \) on \((V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)]\) is the identity operator when \( w(I) = I \).

We have the following Lemma, proved in Appendix C.

**Lemma 19.** For any \( I = (l_1, l_2, l_3) \in \mathbb{Z}_{\geq 0}^3 \), and \( k \in \mathbb{Z}_{\geq 0} \), the space \( \mathcal{H}_k[I]^{\text{sing}} \) is a free \( \mathbb{C}[z] \)-module. Moreover, we have

\[
\text{rank } \mathcal{H}_k[I]^{\text{sing}} = \dim (V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)].
\]

Using this Lemma, for any \( I \in \mathbb{Z}_{\geq 0}^3 \) we fix a \( \mathbb{C}[z] \)-basis \( \{ z_i[I] \} \) of the space \( \mathcal{H}_k[I]^{\text{sing}} \), such that in addition \( z_i[I; z] = z_i[I; \sigma(I); \sigma(z)] \) for any \( \sigma \in S_3 \).

Now define the operators \( \Upsilon[I]: (V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)] \to \mathcal{H}_k[I]^{\text{sing}} \) by

\[
\Upsilon[I] v_i[I] = z_i[I].
\]

Two crucial properties of these operators, which follow directly from construction, are the commutativity of the diagram

\[
\begin{array}{ccc}
(V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)] & \xrightarrow{\Upsilon[I] v_i[I]} & \mathcal{H}_k[I; z_i[I; \sigma(z)]^{\text{sing}}] \\
\quad \uparrow{s_1} & & \uparrow{\Theta[I]} \\
(V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)] & \xrightarrow{\Upsilon[I] v_i[I]} & \mathcal{H}_k[I; z_i[I; \sigma(z)]^{\text{sing}}] \\
\quad \uparrow{s_2} & & \uparrow{\Theta[I]} \\
(V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)] & \xrightarrow{\Upsilon[I] v_i[I]} & \mathcal{H}_k[I; z_i[I; \sigma(z)]^{\text{sing}}]
\end{array}
\]

and the fact that

\[
\Upsilon[I]^{-1} \circ \phi[I] \in \text{End}((V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m - k, k)]) \otimes \mathbb{C}[z]. \tag{7.3}
\]

In other words, the composition \( \Upsilon[I]^{-1} \circ \phi[I] \) is polynomial in \( z \).
We now introduce our main object—the operators $N_\mu(\lambda) \in \text{End}_C(U[\mu])$—by setting

$$
N_\mu(\lambda) = \Theta^{-1} \circ \mathcal{N}[l; z] \circ \mathcal{Y}[l; z] \circ \Theta.
$$

(7.4)

Here $\mu$ and $l_1, l_2, l_3$ are as in (7.1), and $z_1, z_2, z_3$ satisfy

$$(z_1 - l_1) - (z_2 - l_2) = \langle \alpha_1, \lambda + \rho \rangle, \quad (z_2 - l_2) - (z_3 - l_3) = \langle \alpha_2, \lambda + \rho \rangle. \quad (7.5)$$

**Theorem 20.** The operator $N(\lambda) = \bigoplus_\mu N_\mu(\lambda)$ is a regularizing operator for the $g$-module $U$. The corresponding Weyl group representation $w \mapsto a_w$ is such that $a_{s_i} = \bar{s}_i$.

**Proof.** Corollary 15 implies that the operator $N(\lambda)$ depends on $\lambda \in \mathfrak{h}^*$ polynomially.

To check (5.2), we use the result of [TV2], where it was shown that the action of the dynamic Weyl group operators $A_w$ in $P_{2,3}$ coincides with the action of operators $\mathcal{R}$. More precisely, we have the following commutative diagram:

$$
\begin{array}{ccc}
U[s_1\mu] & \rightarrow & (V_{(l_2,0)} \otimes V_{(l_1,0)} \otimes V_{(l_1,0)})^{\text{sing}} [(m - k, k)] \\
A_{s_1}(l) \uparrow & & \uparrow \mathcal{R}_{l/2}(z_1 - z_2) \otimes 1 \\
U[\mu] & \rightarrow & (V_{(l_1,0)} \otimes V_{(l_1,0)} \otimes V_{(l_1,0)})^{\text{sing}} [(m - k, k)].
\end{array}

(7.6)

Combining (7.6) with (6.8) and (7.2), we get the commutative diagram

$$
\begin{array}{ccc}
U[s_1\mu] & \xrightarrow{N_{s_1}(s_1' - l)} & U[s_1\mu] \\
\bar{s}_1 \downarrow & & \downarrow A'_{s_1}(l) \\
U[\mu] & \xrightarrow{N_\mu(\lambda)} & U[\mu] \\
\bar{s}_2 \downarrow & & \downarrow A'_{s_2}(l) \\
U[s_2\mu] & \xrightarrow{N_{s_2}(s_2' - l)} & U[s_2\mu]
\end{array}
$$

which implies (5.2).

Next, in view of (7.3), the operator $N(\lambda)^{-1} = \Theta \circ \mathcal{X}[l; z] \circ \mathcal{X}[l; z]^{-1} \phi[l; z] \circ \Theta$ is singular only at the hyperplanes determined by $\mathcal{X}[l; z]^{-1}$:

$$
\begin{align*}
z_1 - l_1 - z_2 + s &= 0, \quad 0 \leq s < l_2, \\
z_2 - l_2 - z_3 + s &= 0, \quad 0 \leq s < l_3, \\
z_1 - l_1 - z_3 + s &= 0, \quad 0 \leq s < l_3,
\end{align*}
$$
which correspond respectively to the hyperplanes
\[ \chi_{x_1,l_2-s}(\lambda) = 0, \quad 0 \leq s < l_2, \]
\[ \chi_{x_2,l_3-s}(\lambda) = 0, \quad 0 \leq s < l_3, \]
\[ \chi_{x_1+x_2,l_3-s}(\lambda) = 0 \quad 0 \leq s < l_3. \]

Now it follows from Lemma 6 that \( N(\lambda) \) satisfies (5.1).

Finally, let us show that for any \( u \in U[\mu] \), the vector \( \Xi_N(\lambda)u \in \mathcal{U}(n^-) \otimes U \) polynomially depends on \( \lambda \). According to (3.1), this is equivalent to regularity of the sum \( \sum_j (S^{-1}_\lambda)^{ij}o(g_j)N(\lambda)u \) for any \( i \) at any hyperplane \( \chi_{x,r}(\lambda) = 0, \quad \alpha \in \Delta^+, \quad r = 1, 2, \ldots \).

Consider the case \( \alpha = \alpha_1 \).

Let \( \{g_j\} \) be the Poincare–Birkhoff–Witt basis of \( \mathcal{U}(n^-) \), corresponding to an ordering of \( \Delta^+ \) with \( z_1 \) being the last root. It is known (see [ES]) that \( (S^{-1}_\lambda)^{ij} \) is regular at the hyperplane \( \chi_{x_1,r}(\lambda) = 0 \) unless \( g_j = \tilde{g}_j f_1^r \) for some \( \tilde{g}_j \in \mathcal{U}(n^-) \). In the last case \( (S^{-1}_\lambda)^{ij} \) may have a simple pole there, and the desired regularity will follow if \( o(\tilde{g}_j)e_1^r N(\lambda)u \) vanishes at the hyperplane \( \chi_{x_1,r}(\lambda) = 0 \).

We prove that \( e_1^r N(\lambda)u = 0 \) at the hyperplane \( \chi_{x_1,r}(\lambda) = 0 \) using the explicit realization \( \Theta \) of the module \( U \in \mathbb{P}_{2,3} \). The operator \( e_1^r : U[\mu] \to U[\mu+r\alpha_1] \) corresponds to the operator
\[
(E_{12}^{a_k})^r : (V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m-k,k)] \\
\to (V_{(l_1+r,0)} \otimes V_{(l_2-r,0)} \otimes V_{(l_3,0)})^{\text{sing}}[(m-k,k)]
\]
if \( r \leq l_2 \), and is equal to zero, if \( r > l_2 \). There is nothing to prove in the latter case. The claim in the former case is implied by the following lemma.

**Lemma 21.** Assume that \( r \leq l_2 \). Let \( z_0 \) be a generic point from the hyperplane \( (z_1-l_1)-z_2+(l_2-r) = 0 \). Then the following composition of operators vanishes:
\[
\mathcal{H}[l; z_0] \rightarrow V_{(l_1,0)} \otimes V_{(l_2,0)} \otimes V_{(l_3,0)} \overset{(E_{12}^{a_k})^r}{\rightarrow} V_{(l_1+r,0)} \otimes V_{(l_2-r,0)} \otimes V_{(l_3,0)}.
\]

**Proof.** Since \( r \leq l_2 \), the product \( \mathcal{H}[l; z] \) contains the factor \( z_1-l_1-z_2+(l_2-r) \), corresponding to \( \chi_{z_1,r}(\lambda) \), and \( \mathcal{H}[l; z_0] = 0 \) if \( l_2-r > \min(l_1,l_2) \), then \( \mathcal{H}[l; \varphi] \) is regular at \( z = z_0 \) for any \( \varphi \in \mathcal{H}[l] \), and the operator \( \mathcal{H}[l; z_0] = \mathcal{H}[l; z_0] \mathcal{H}[l; z_0] \) vanishes.
If \( l_2 - r \leq \min(l_1, l_2) \), then by Proposition 17, the image of \( \mathcal{N}[l; z_0] \) is contained in 
\[
(\bigoplus_{i=l_2-r+1}^{\min(l_1, l_2)} V(l_1+\ell-i, i)) \otimes V(l_0, 0).
\]
Since \( (\mathfrak{gl}_2)^r \) commutes with the \( \mathfrak{gl}_2 \) action and 
\[
V(l_1+r, 0) \otimes V(l_2-r, 0) \otimes V(l_0, 0) \cong \left( \bigoplus_{i=0}^{l_2-r} V(l_1+l_2-i, i) \right) \otimes V(l_0, 0),
\]
the composition (7.7) vanishes. \( \square \)

The case \( \alpha = \alpha_2 \) can be considered similarly to the case \( \alpha = \alpha_1 \).
Let now \( \alpha = \alpha_1 + \alpha_2 \). In this case we have \( \alpha = s_2 \alpha_1 \), and therefore 
\[
\chi_{x, r}(\lambda) = \langle \alpha, \lambda + \rho \rangle - r = \langle s_2 \alpha_1, \lambda + \rho \rangle - r = \langle \alpha_1, s_2 \cdot \lambda + \rho \rangle - r = \chi_{s_2, f}(s_2 \cdot \lambda).
\]

Recall that \( \Xi_N(\lambda)u \) is regular at the hyperplane \( \chi_{x, r}(\lambda) = 0 \) if and only if the operator \( \Phi^N(\lambda)u \) is regular at the same hyperplane.

Consider \( \lambda \) from the hyperplane \( \langle \alpha_2, \lambda + \rho \rangle = -n \), for a large enough nonnegative integer \( n \). Then \( M_\lambda \) is contained as a proper submodule in \( M_{s_2, \lambda} \), and we have 
\[
\Phi^N(\lambda)u = \Phi^{A(s_2 \cdot \lambda)}N(s_2 \cdot \lambda)u |_{M_\lambda}.
\]

Since the operator \( \Phi^{N(s_2 \cdot \lambda)}u \) is regular at the hyperplanes \( \chi_{s_2, f}(s_2 \cdot \lambda) = 0 \), so is the operator \( \Phi^N(\lambda)u \). Hence \( \Xi_N(\lambda)u \) has no pole at the hyperplane \( \chi_{s_2, f}(s_2 \cdot \lambda) = 0 \), provided \( \langle \alpha_2, \lambda + \rho \rangle = -n \).

The rational function \( \chi_{x, r}(\lambda) \Xi_N(\lambda)u \) is regular at the hyperplane \( \chi_{x, r}(\lambda) = 0 \), and vanishes at infinitely many points of the hyperplane. Therefore, the function must vanish at the hyperplane identically, that is \( \Xi_N(\lambda)u \) has no pole at the hyperplane \( \chi_{x, r}(\lambda) = 0 \). \( \square \)

8. Resonance conditions

Let \( \mathfrak{g} \) be a semisimple Lie algebra, and let \( U \) be an irreducible finite-dimensional \( \mathfrak{g} \)-module with nontrivial zero weight subspace \( U[0] \).

Let \( \mathcal{N}(\lambda) \in \text{End}_C(U) \) be a regularizing operator. Introduce an \( \text{End}_C(U[0]) \)-valued function \( \Psi(\lambda, x) \), where \( \lambda \in \mathfrak{h}^* \), \( x \in \mathfrak{h} \), by 
\[
\Psi(\lambda, x)u = \text{Tr} |_{M_\lambda} (\Phi^N(\lambda)u e^x).
\]

Special cases of the function \( \Psi(\lambda, x) \) were used in [ES] to give a representation-theoretic proof of the algebraic integrability of Calogero–Sutherland systems. In [EV] it was shown that the trace functions satisfy remarkable differential and difference equations.
An important property of the function $\Psi(\lambda, x)$ is that it satisfies certain resonance conditions.

**Proposition 22.** Let $N = N(\lambda)$ be a regularizing operator acting in a finite-dimensional $g$-module $U$ with $U[0] \neq 0$, and let $\Psi(\lambda, x)$ be the corresponding trace function.

Let $z \in \Delta^+$ and $k \in \mathbb{Z}_{\geq 0}$ be such that $U[kx] \neq 0$. Then for any $\lambda \in \mathfrak{h}^*$ such that $\chi_{z,k}(\lambda) = 0$, and any $u \in \ker N(\lambda) \subset U[0]$, we have

$$\Psi(\lambda, x)u = \Psi(s_z \cdot \lambda, x)u.$$  

**Proof.** Let $\lambda$ be generic from the hyperplane $\chi_{z,k}(\lambda) = 0$. Then $M_\lambda$ contains a unique proper submodule $M_{s_z \cdot \lambda}$, and $N(\lambda)u = 0$ implies that the image of $\Phi_N(\lambda)u$ is contained in $M_{s_z \cdot \lambda} \otimes U$. Therefore, the trace of the operator $\Phi_N(\lambda)u$ in $M_\lambda$ is equal to the trace of its restriction to $M_{s_z \cdot \lambda}$. Moreover, the restriction of $\Phi_N(\lambda)u$ to the submodule $M_{s_z \cdot \lambda}$ coincides with $\Phi_N(s_z \cdot \lambda)u$, and we have

$$\Psi(\lambda, x)u = \text{Tr}|_{M_\lambda}(\Phi_N(\lambda)u) = \text{Tr}|_{M_{s_z \cdot \lambda}}(\Phi_N(s_z \cdot \lambda)u) = \Psi(s_z \cdot \lambda, x)u. \quad \square$$

**Example.** Let $\mathfrak{g} = \mathfrak{sl}_2$. We identify $\mathfrak{h}^*$ with $\mathbb{C}$ by associating $\mathfrak{h}^* \ni \lambda \leftrightarrow \langle x, \lambda \rangle \in \mathbb{C}$.

Let $U$ be an irreducible finite-dimensional $\mathfrak{sl}_2$-module with even highest weight $\Lambda$. In this case the weight subspace $U[0]$ is one-dimensional, $\text{End}(U[0]) \cong \mathbb{C}$, and the function $\Psi(\lambda, x)$ is scalar-valued. It satisfies the resonance conditions

$$\Psi(\lambda, x) = \Psi(s \cdot \lambda, x), \quad \lambda = 0, 1, 2, \ldots, \Lambda/2 - 1.$$ 

**Appendix A. Formal monomials**

To prove Theorem 4 we use the calculus of formal monomials in $\mathfrak{U}(\mathfrak{n}^-)$, developed in [FFM].

**Lemma A.1.** Let $f \in \mathfrak{n}^-$, $X \in \mathfrak{g}$. For any $n \in \mathbb{Z}_{\geq 0}$ we have the following identity in $U(\mathfrak{g})$,

$$X^{\mathfrak{n}^+} = \sum_{k=0}^{\infty} \binom{n}{k} f^{n-k} \prod_{j=1}^{\infty} [\underbrace{[X, f], \ldots, [X, f]}_{k \text{ times}}].$$  

(A.1)

This lemma is proved by induction. Note that the summation over $k$ is finite.

Introduce an associative algebra $\mathbb{A}$, with unit $1$ and generators $f_i^\gamma$, $i = 1, \ldots, \dim \mathfrak{h}$, $\gamma \in \mathbb{C}$, subject to the relations

$$[f_i, \ldots, [f_i, f_j]] = 0, \quad a_{ij} = \frac{2 \langle \gamma_i, \gamma_j \rangle}{\langle \gamma_i, \gamma_i \rangle},$$  

(A.2)
where the commutator $[\cdot, \cdot]$ is defined as $[X, Y] = XY - YX$.

**Lemma A.2.** There exists an algebra homomorphism $i: \mathcal{U}(n^-) \to \mathcal{A}$ such that $i(f_i) = f_i$ for any $i = 1, \ldots, \dim \mathfrak{h}$. The homomorphism $i$ is injective.

**Proof.** It is known that $U(n^-)$ is the quotient of the free Lie algebra with generators $\{f_i\}$, by the ideal, generated by the Serre relations (A.2). Therefore, $i$ extends from the generators to the entire algebra $\mathcal{U}(n^-)$.

Next, we note that the algebra $\mathcal{A}$ is nonzero, i.e. the ideal, generated by the relations (A.4) is proper. Indeed, a linear functional $\varphi$ on $\mathcal{A}$, defined on monomials by

$$\varphi(f_{i_1}^{a_1} \cdots f_{i_l}^{a_l}) = 1,$$

is well defined, and is nonzero, so $\mathcal{A} \neq 0$.

Let $\lambda \in \mathfrak{h}^*$ be generic. Introduce a $\mathfrak{g}$-module $\tilde{M}_\lambda$, which is equal to $\mathcal{A}$ as a vector space, with $f_i$ acting naturally on the left, $h \in \mathfrak{h}$ acting by

$$hf_{i_1}^{a_1} \cdots f_{i_l}^{a_l} = \langle h, \lambda - \sum_{k=1}^l \gamma_k \mathfrak{h}_k \rangle f_{i_1}^{a_1} \cdots f_{i_l}^{a_l}.$$  

The action of operators $e_i$ is determined by the condition $e_i 1 = 0$, and the commutation formula

$$e_i f_j = \sum_{k=0}^{\infty} \binom{\gamma}{k} f_j^{\gamma - k} \left[ \cdots [e_i, f_j], \ldots, f_j \right], \quad \gamma \in \mathfrak{c}.$$  

The module $\tilde{M}_\lambda$ is an obvious analog of the Verma module $M_\lambda$.

For any $X, Y \in \mathcal{U}(n^-)$, consider the action of $\omega(Y) \in \mathcal{U}(\mathfrak{g})$ on the element $i(X) 1 \in \tilde{M}_\lambda$. We have

$$\omega(Y) i(X) 1 = S_\lambda(X, Y) 1 + \text{lower order terms},$$

and therefore $i(X) = 0$ implies $S_\lambda(X, Y) = 0$ for any $Y \in \mathcal{U}(n^-)$. Since the Shapovalov form $S_\lambda$ is nondegenerate for generic $\lambda$, this means that $X = 0$. Thus, $i$ is an injection. \(\square\)

We identify $\mathcal{U}(n^-)$ with its image $i(\mathcal{U}(n^-)) \subset \mathcal{A}$. We say that an element $X \in \mathcal{A}$ makes sense, if $X \in \mathcal{U}(n^-)$.
Example. Let $g = \mathfrak{sl}_3$, and let $g_1, g_2, g_3 \in \mathbb{C}$. If $\gamma_1 + \gamma_2, \gamma_3 \in \mathbb{Z}_{\geq 0}$, and $\gamma_1 + \gamma_3 \geq \gamma_2$, then the monomial $f_1^{\gamma_1} f_2^{\gamma_2} f_3^{\gamma_3}$ makes sense, and we can write

$$f_1^{\gamma_1} f_2^{\gamma_2} f_3^{\gamma_3} = \sum_{j=0}^{\infty} \binom{\gamma_2}{j} \binom{\gamma_3}{j} f_1^{\gamma_1 + \gamma_3 - j} f_2^{\gamma_2 - j} [f_2, f_1]^j.$$ 

Consider the completion $\mathbb{A} \otimes \mathcal{U}(n^-)$, consisting of possibly infinite sums $\sum_i X_i \otimes Y_i$ with $X_i \in \mathbb{A}$ and homogeneous $Y_i \in \mathcal{U}(n^-)$, such that $\text{wt}(Y_i) \to \infty$ when $i \to \infty$.

We following two lemmas can be easily verified; we give sketches of proof, and leave more technical details to a reader.

Lemma A.3. There exists an algebra homomorphism

$$\Delta : \mathbb{A} \to \mathbb{A} \otimes \mathcal{U}(n^-),$$

such that

$$\Delta(f_i^\gamma) = \sum_{j=0}^{\infty} \binom{\gamma}{j} f_i^{\gamma-j} \otimes f_i^j.$$ 

Proof (Sketch). We have to check that the comultiplication above can be extended to an algebra homomorphism, i.e. that it respects the defining relations (A.4). For each such relation of homogeneous weight $\gamma$, it amounts to verifying a family of identities in weight subspaces $\mathbb{A}[v] \otimes \mathcal{U}(n^-)[v - v]$. These identities are valid for any nonnegative integer $\gamma$, because in this case they are equivalent to the consistency of the comultiplication in $\mathcal{U}(n^-)$. Since these identities are polynomial in $\gamma$, they are satisfied for arbitrary $\gamma$. $\square$

It is easy to see that if $X \in \mathcal{U}(n^-) \subset \mathbb{A}$, then $\Delta(X) \in \mathcal{U}(n^-) \otimes \mathcal{U}(n^-) \subset \mathbb{A} \otimes \mathcal{U}(n^-)$.

Lemma A.4. Let $J \in \mathbb{Z}_{\geq 0}$, $\gamma_1, \ldots, \gamma_l \in \mathbb{C}$. Let $\{g_i\}$ be a homogeneous basis of $\mathcal{U}(n^-)$. Then there exist $J_1, \ldots, J_l \in \mathbb{Z}_{\geq 0}$, such that any monomial $f_1^{\gamma_1-j_1} \ldots f_l^{\gamma_l-j_l}$ with $j_1, \ldots, j_l \leq J$, can be written as

$$f_1^{\gamma_1-j_1} \ldots f_l^{\gamma_l-j_l} = f_1^{\gamma_1-J_1} \ldots f_l^{\gamma_l-J_l} \sum_i q_i(\gamma_1, \ldots, \gamma_l) g_i,$$

for some polynomials $q_i(\gamma_1, \ldots, \gamma_l)$.
Proof (Sketch). The statement is obvious for \( l = 1 \). Using induction on \( l \), we can write

\[
f_{i_1}^{\gamma_1-j_1} \ldots f_{i_{k-1}}^{\gamma_{k-1}-j_{k-1}} f_{i_k}^{\gamma_k} = f_{i_1}^{\gamma_1-j_1} \ldots f_{i_{k-1}}^{\gamma_{k-1}-j_{k-1}} \left( \sum q_i(\gamma_1, \ldots, \gamma_l) g_i \right) f_{i_k}^{\gamma_k-j_k}.
\]

Then we can move all \( f_{i_k} \) to the left by commuting them with \( \{ g_i \} \); nilpotency of the adjoint action of \( f_{i_k} \) will guarantee that all only finite number of \( f_{i_k} \)'s can be absorbed. \( \square \)

We now use these result to give a proof of Theorem 4.

Proof of Theorem 4. We represent the left side of (4.4) as

\[
\Phi^\mu_{\lambda}(v \lambda, \lambda + \mu) = \Phi^\mu_{\lambda}(v \lambda, \lambda + \mu) 1_{\lambda + \mu} = \Delta(F_{\lambda + \mu}(\lambda + \mu)) 1_{\lambda + \mu} \otimes u
\]

\[
= \sum_{j_1, \ldots, j_l \in \mathbb{Z}_{\geq 0}} \sum_{k, j} q_{j_1, \ldots, j_l}(\gamma_1(\lambda + \mu), \ldots, \gamma_l(\lambda + \mu))
\times f_{i_1}^{\gamma_1(\lambda + \mu)-j_1} \ldots f_{i_l}^{\gamma_l(\lambda + \mu)-j_l} g_i 1_{\lambda} \otimes (S_{\lambda}^{-1})_{ij} f_{i_l}^{j_l} \ldots f_{i_1}^{j_1} \omega(g_j) u. \quad (A.5)
\]

Here the coefficients \( q_{j_1, \ldots, j_l}(\gamma_1(\lambda + \mu), \ldots, \gamma_l(\lambda + \mu)) \) are certain polynomials in \( \gamma_1(\lambda + \mu), \ldots, \gamma_l(\lambda + \mu) \). The summation in (A.5) is actually finite, because the element \( f_{i_1}^{j_1} \ldots f_{i_l}^{j_l} \) acts as zero in \( U \) if any of \( j_1, \ldots, j_l \) is big enough.

Next, we have that \( \gamma_k(\lambda + \mu) - j_k = \gamma_k(\lambda) + \langle \beta_k, \mu \rangle \) for some \( \beta_k \in \Delta^+ \). Hence, for all nonzero monomials in the sum, we have

\[
\gamma_k(\lambda + \mu) - j_k = \gamma_k(\lambda) + \langle \beta_k, \mu \rangle - j_k \geq \gamma_k(\lambda) - J, \quad k = 1, \ldots, l,
\]

for some \( J \in \mathbb{Z}_{\geq 0} \). Lemma A.4 implies that there exist \( J_1, \ldots, J_l \in \mathbb{Z}_{\geq 0} \), such that we can write

\[
\Delta(F_{\lambda + \mu}(\lambda + \mu)) \Xi(\lambda) = (f_{i_1}^{\gamma_1(\lambda)} - J_1) \ldots f_{i_l}^{\gamma_l(\lambda)} - J_l \otimes \left( \sum g_i \otimes P_i(\lambda) \right), \quad (A.6)
\]

for a certain finite collection of \( \text{End}_C(U) \)-valued rational functions \( P_i(\lambda) \).

We now make two claims, which imply Theorem 4.

First, in the summation formula (A.6) we have

\[
P_i(\lambda) \equiv 0, \quad \text{if } wt(g_i) \notin wt(f_{i_1}^{J_1} \ldots f_{i_l}^{J_l}) - Q^+.
\]

(A.7)
Second, for some rational $\text{End}_C(U)$-valued functions $A_w(\lambda)$, we have

$$
(f^{\gamma_i(\lambda)}_{i_1} \cdots f^{\gamma_i(\lambda)}_{i_j} - J_i \otimes 1) \left( \sum_i g_i \otimes P_i(\lambda) \right) = F_w(\lambda) \otimes A_w(\lambda),
$$

(A.8)

where $\sum_i'$ denotes the sum of terms with $\text{wt}(g_i) = \text{wt}(f^{J_i}_{i_1} \cdots f^{J_i}_{i_j})$.

To verify these claims, we consider a dominant integral weight $\lambda$ such that $\lambda \gg 0$. One can check that for any $u \in U$ we have

$$
\Delta(F_w(\lambda + \mu) \Xi(\lambda) (1_\lambda \otimes u) = \psi_{\lambda, \lambda} \otimes \tilde{u} + \text{lower order terms},
$$

(A.9)

for some $\tilde{u} \in U$. In particular, the term $g_i \otimes P_i(\lambda)$ is nonzero only if

$$
\text{wt}(f^{\gamma_i(\lambda)}_{i_1} \cdots f^{\gamma_i(\lambda)}_{i_j}) + \text{wt}(g_i) + \lambda \leq w \cdot \lambda.
$$

(A.10)

From the definition of $\gamma_k(\lambda)$ it follows that

$$
\lambda + \text{wt}(f^{\gamma_i(\lambda)}_{i_1} \cdots f^{\gamma_i(\lambda)}_{i_j}) = w \cdot \lambda,
$$

and thus (A.9) implies (A.7) for the given $\lambda$. Since the rational functions $P_i(\lambda)$ vanish for all sufficiently large dominant integral weights, they must vanish for all $\lambda$.

Using Eq. (A.8), we define the dynamical Weyl group operator $A_w(\lambda)$ for integral dominant $\lambda \gg 0$ by the rule $A_w(\lambda) u = \tilde{u}$. We now establish that $A_w(\lambda)$ rationally depends on $\lambda$, and verify our second claim.

By Lemma A.4, we can write

$$
F_w(\lambda) f^{\gamma_i(\lambda)}_{i_1} \cdots f^{\gamma_i(\lambda)}_{i_j} \sum_i Q_i(\lambda) g_i
$$

for some polynomials $Q_i(\lambda)$ in $\lambda \in \mathfrak{h}^*$. Eq. (A.9) then becomes

$$
(f^{\gamma_i(\lambda)}_{i_1} \cdots f^{\gamma_i(\lambda)}_{i_j} - J_i \otimes 1) \left( \sum_i g_i \otimes P_i(\lambda) u \right) = (f^{\gamma_i(\lambda)}_{i_1} \cdots f^{\gamma_i(\lambda)}_{i_j} - J_i \otimes 1) \left( \sum_i Q_i(\lambda) g_i \otimes A_w(\lambda) u \right).
$$

(A.11)

or equivalently

$$
\sum_i' g_i \otimes P_i(\lambda) u = \sum_i' Q_i(\lambda) g_i \otimes A_w(\lambda) u.
$$

It is valid for dominant integral $\lambda \gg 0$, and implies that for all $i$ we have

$$
A_w(\lambda) = \frac{P_i(\lambda)}{Q_i(\lambda)}, \text{ if } \text{wt}(g_i) = \text{wt}(f^{J_i}_{i_1} \cdots f^{J_i}_{i_j}).
$$

(A.12)
Therefore, $A_w(\lambda)$ can be extended to a rational $\text{End}_\mathbb{C}(U)$-valued function, such that (A.12) is satisfied for all $\lambda$. Eqs. (A.11), (A.8), and (4.4) follow for all $\lambda$. □

Appendix B. Yangians, weight functions and $R$-matrices

We recall some facts about the Yangian $Y(\mathfrak{gl}_2)$, and explain how to get weight functions as matrix elements of evaluation Yangian modules. For more details on Yangians, see for example [CP].

The Yangian $Y(\mathfrak{gl}_2)$ is a Hopf algebra with generators $T^{(k)}_{ij}$, $i,j = 1,2$; $k = 1,2,3, \ldots$. The relations are most conveniently written in terms of the generating series

$$ T_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} T^{(k)}_{ij} u^{-k}, $$

and have the form

$$ (u - v)[T_{ij}(u), T_{kl}(v)] = T_{kj}(v)T_{il}(u) - T_{kj}(u)T_{il}(v). $$

The coproduct for $Y(\mathfrak{gl}_2)$ is given by

$$ T_{ij}(u) \mapsto \sum_{k=1}^{2} T_{kj}(u) \otimes T_{ik}(u). $$

The Yangian $Y(\mathfrak{gl}_2)$ contains $\mathcal{U}(\mathfrak{gl}_2)$ as a Hopf subalgebra, the embedding $\mathcal{U}(\mathfrak{gl}_2) \to Y(\mathfrak{gl}_2)$ being given by $E_{ij} \mapsto T^{(1)}_{ji}$. We identify $\mathcal{U}(\mathfrak{gl}_2)$ with its image in $Y(\mathfrak{gl}_2)$ under this embedding.

There is a family of homomorphisms $\varepsilon_z: Y(\mathfrak{gl}_2) \to \mathcal{U}(\mathfrak{gl}_2)$, depending on a complex parameter $z$:

$$ \varepsilon_z: T_{ij}(u) \mapsto \delta_{ij} + \frac{E_{ij}}{u - z}. $$

These homomorphism are identical on the subalgebra $\mathcal{U}(\mathfrak{gl}_2) \subset Y(\mathfrak{gl}_2)$.

For any $\mathfrak{gl}_2$-module $V$, we denote $V(z)$ the pullback of $V$ through the homomorphism $\varepsilon_z$; the $Y(\mathfrak{gl}_2)$-module $V(z)$ is called an evaluation module.

The Yangian action in a tensor product of evaluation modules is the underlying structure for the functional realization $\phi[\mathfrak{I}]$ of the tensor product of the corresponding $\mathfrak{gl}_2$-modules.

As before, $\mathfrak{I} = (l_1, l_2, l_3)$ will always denote a triple of nonnegative integers. For any $k \in \mathbb{Z}_{\geq 0}$ set

$$ \mathcal{I}_k = \{(k_1, k_2, k_3) \in \mathbb{Z}_{\geq 0}^3 \mid k_1 + k_2 + k_3 = k\}, $$

$$ \mathcal{I}_k[\mathfrak{I}] = \{(k_1, k_2, k_3) \in \mathcal{I}_k \mid k_1 \leq l_1, k_2 \leq l_2, k_3 \leq l_3\}. $$
Proposition B.1 (Korepin et al. [KBI], Tarasov and Varchenko [TV3]). Let \( z = (z_1, z_2, z_3) \) be a triple of complex numbers. Consider the \( Y(g|_2) \)-module \( V_{(l_1,0)}(z_1) \otimes V_{(l_2,0)}(z_2) \otimes V_{(l_3,0)}(z_3) \). For any \( k \in \mathbb{Z}_{>0}, \quad q \in \mathcal{A}_k[l] \), we have

\[
\left( \prod_{i=1}^{3} \prod_{a=1}^{k} (t_a - z_i) \right) T_{12}(t_1) \cdots T_{12}(t_k)(1_{l_1} \otimes 1_{l_2} \otimes 1_{l_3}) = \sum_{p \in \mathcal{A}_k[l]} w_p[l; z](t_1, \ldots, t_k) E_{12}^{p_1} 1_{l_1} \otimes E_{12}^{p_2} 1_{l_2} \otimes E_{12}^{p_3} 1_{l_3},
\]

\[
\left( \prod_{i=1}^{3} \prod_{a=1}^{k} (t_a - z_i) \right) T_{21}(t_1) \cdots T_{21}(t_k)(E_{12}^{p_1} 1_{l_1} \otimes E_{12}^{p_2} 1_{l_2} \otimes E_{12}^{p_3} 1_{l_3}) = w'_q[l; z](t_1, \ldots, t_k)(1_{l_1} \otimes 1_{l_2} \otimes 1_{l_3}),
\]

where

\[
w_k[l; z](t_1, \ldots, t_k) = \frac{1}{k_1!k_2!k_3!} \text{Sym} \left( \prod_{i=1}^{3} \prod_{a=1}^{k} (t_a - z_i + l_i) \prod_{b=k_1+\cdots+k_b+1} (t_b - z_i) \right) \prod_{a < b} \frac{t_a - t_b - 1}{t_a - t_b}.
\]

\[
w'_k[l; z](t_1, \ldots, t_k) = \frac{1}{k_1!k_2!k_3!} \text{Sym} \left( \prod_{i=1}^{3} \prod_{a=1}^{k} (t_a - z_i) \prod_{b=k_1+\cdots+k_b+1} (t_b - z_i + l_i) \right) \prod_{a < b} \frac{t_a - t_b + 1}{t_a - t_b}.
\]

The functions \( w_k[l; z] \) and \( w'_k[l; z] \) are called the weight functions and the dual weight functions, respectively. They are polynomials in \( t_1, \ldots, t_k, \) and \( z_1, z_2, z_3. \)

Definition (6.1) of the map \( \phi[l] \) can be written as

\[
\phi[l; z](E_{21}^{k_1} 1_{l_1} \otimes E_{21}^{k_2} 1_{l_2} \otimes E_{21}^{k_3} 1_{l_3}) = \frac{k_1!l_1!}{(l_1 - k_1)!} \frac{k_2!l_2!}{(l_2 - k_2)!} \frac{k_3!l_3!}{(l_3 - k_3)!} w'_k[l; z].
\]
Let \( \mathcal{H}_k \) denote the space of symmetric polynomials in variables \( t_1, \ldots, t_k \) of degree at most 2 in each \( t_1, \ldots, t_k \). Set

\[
\mathcal{H}' = \bigoplus_{k \geq 0} \mathcal{H}_k' .
\]

It is clear that \( \mathcal{H}_k = \mathcal{H}_k' \otimes \mathbb{C}[z] \) and \( \mathcal{H} = \mathcal{H}' \otimes \mathbb{C}[z] \).

**Theorem B.2** (Tarasov and Varchenko [TV3]). The space \( \mathcal{H}' \) has a structure of a \( Y(\mathfrak{gl}_2) \)-module, depending on \( l, z \), with the following properties:

1. the \( \mathfrak{gl}_2 \)-module structure on \( \mathcal{H} \) coincides with that, induced by the \( \mathfrak{gl}_2 \)-module structure on \( \mathcal{H} \), described in Proposition 12;
2. the map \( \phi[l; z] : V(l, 0)(z_1) \otimes V(l, 0)(z_2) \otimes V(l, 0)(z_3) \rightarrow \mathcal{H}' \) is a homomorphism of \( Y(\mathfrak{gl}_2) \)-modules.

We now use the representation theory of the Yangian \( Y(\mathfrak{gl}_2) \) to prove the results, announced in Chapter 6. The following theorems go back to [T].

**Theorem B.3** (Tarasov [T]). 1. Let \( l \in \mathbb{Z}_{\geq 0} \), and let \( U \) be an irreducible finite-dimensional \( Y(\mathfrak{gl}_2) \)-module. Then for generic \( z \in \mathbb{C} \), the \( Y(\mathfrak{gl}_2) \)-modules \( U \otimes V(l, 0)(z) \) and \( V(l, 0)(z) \otimes U \) are irreducible and isomorphic to each other.

2. Let \( l, l' \in \mathbb{Z}_{\geq 0} \), and let \( z, z' \in \mathbb{C} \) be such that \( z - l = z' - r \) for some \( r = 0, 1, \ldots, \min(l, l') - 1 \). Then the \( Y(\mathfrak{gl}_2) \)-module \( V(l, 0)(z) \otimes V(l', 0)(z') \) contains a unique proper submodule \( \tilde{V}_{l, l', p}(z) \). As a \( \mathfrak{gl}_2 \)-module, \( \tilde{V}_{l, l', p}(z) \) is isomorphic to

\[
\tilde{V}_{l, l', p}(z) \cong \bigoplus_{x = r + 1}^{\min(l, l')} V(l + l' - s, x) .
\]

**Theorem B.4** (Tarasov [T]). Let \( l, l' \in \mathbb{Z}_{\geq 0} \). There exists a unique linear operator \( R_{l, l'}(z) \in \text{End}_\mathbb{C}(V(l, 0) \otimes V(l', 0)) \), rationally depending on a complex parameter \( z \), such that for generic \( z, z' \) the operator

\[
\tilde{R}_{l, l'}(z - z') = P \circ R_{l, l'}(z - z') : V_l(z) \otimes V_{l'}(z') \rightarrow V_l(z') \otimes V_{l'}(z)
\]

is an isomorphism of \( Y(\mathfrak{gl}_2) \)-modules, \( P \) being the permutation operator: \( P(u \otimes v) = v \otimes u \), and

\[
R_{l, l'}(z) 1_l \otimes 1_{l'} = 1_l \otimes 1_{l'} .
\]

We call the operator \( R_{l, l'}(z) \) the standard rational \( R \)-matrix.
Proof of Proposition 16. Let \( z_0 = (z_1, z_2, z_3) \) be a generic point from the hyperplane \( z_1 - z_2 + l_2 - r = 0 \), with \( r = 0, 1, \ldots, \min(l_1, l_2) - 1 \). Then it follows from Theorem 29 that the \( Y(\mathfrak{g}l_2) \)-module \( V_{(l_1,0)}(z_1) \otimes V_{(l_2,0)}(z_2) \otimes V_{(l_3,0)}(z_3) \) has a unique proper submodule \( \tilde{V}_{l_1,l_2,r}(z_1) \otimes V_{(l_3,0)}(z_3) \), while Proposition B.1 and formula (B.1) imply that \( \ker \phi[l;z_0] \) contains \( \tilde{V}_{l_1,l_2,r}(z_1) \otimes V_{(l_3,0)}(z_3) \). Since \( \phi[l;z_0] \) is a nontrivial homomorphism of \( Y(\mathfrak{g}l_2) \)-modules, the kernel \( \ker \phi[l;z_0] \) must be a proper \( Y(\mathfrak{g}l_2) \)-submodule, which proves the equality

\[
\ker \phi[l;z_0] = \tilde{V}_{l_1,l_2,r}(z_1) \otimes V_{(l_3,0)}(z_3).
\]

The proof of the second part of Proposition 16 is similar. \( \square \)

Proof of Proposition 18. It follows from Proposition B.1 that to determine the image of a vector \( v \in V_{l_1} \otimes V_{l_2} \otimes V_{l_3} \), we must act on it by the element \( (\prod_{i=1}^{3} \prod_{a=1}^{k} (t_a - z_i)) T_{21}(t_1) \ldots T_{21}(t_k) \), and then use the linear functional \( (1_{l_1} \otimes 1_{l_2} \otimes 1_{l_3})^* \). The \( Y(\mathfrak{g}l_2) \)-intertwining property of the \( R \)-matrix implies that we have the commutative diagram

\[
\begin{array}{ccc}
V_{l_2} \otimes V_{l_1} \otimes V_{l_3} & \xrightarrow{\phi[s_1(l);s_1(z)]} & \mathcal{H}[s_1(l);s_1(z)] \\
\downarrow R_{l_2,l_3}(z_1-z_2) \otimes 1 & & \downarrow \\
V_{l_1} \otimes V_{l_2} \otimes V_{l_3} & \xrightarrow{\phi[l;z]} & \mathcal{H}[l;z]
\end{array}
\]

\[
V_{l_1} \otimes V_{l_3} \otimes V_{l_2} \xrightarrow{\phi[s_2(l);s_2(z)]} \mathcal{H}[s_2(l);s_2(z)].
\]

Now denote

\[
\tilde{\mathcal{R}}_{l,p}(u) = \frac{\prod_{j=1}^{p} (u - l + j)}{\prod_{j=1}^{l} (-u - l + j)} R_{l,p}(u).
\]

The commutative diagram (B.2) and the formulas

\[
\frac{\prod_{j=1}^{l_1} (z_1 - z_2 - l_1 + j)}{\prod_{j=1}^{l_2} (z_2 - z_1 - l_2 + j)} = \frac{\mathcal{A}[l;z]}{\mathcal{A}[s_1(l);s_1(z)]},
\]

\[
\frac{\prod_{j=1}^{l_1} (z_2 - z_3 - l_2 + j)}{\prod_{j=1}^{l_2} (z_3 - z_2 - l_3 + j)} = \frac{\mathcal{A}[l;z]}{\mathcal{A}[s_2(l);s_2(z)]},
\]

imply that the operators \( \tilde{\mathcal{R}}_{l,p}(u) \) satisfy the conditions of Proposition 18. \( \square \)

In the remaining part of the appendix we construct the map \( \mathcal{S}[l] \) explicitly, and describe its singularities.
Let $u = (u_1, u_2, u_3)$ be complex variables, and for any $k \in \mathbb{Z}_{\geq 0}$ set $\mathcal{H}_k = \mathcal{H} \otimes \mathbb{C}[u]$.

Denote also

$$\Omega_k[u; z](t_1, \ldots, t_k) = \prod_{i=1}^{3} \prod_{a=1}^{k} \frac{1}{(t_a - z_i + u_i)(t_a - z_i)} \prod_{a \neq b}^{k} \frac{t_a - t_b}{t_a - t_b - 1}.$$  

For any function $F = F(t_1, \ldots, t_k)$, set

$$\text{Res}_{(t_1, \ldots, t_k)} F = \text{Res}_{t_k = i_6} (\text{Res}_{t_{k-1} = i_5} (\ldots \text{Res}_{t_1 = i_1} F(t_1, \ldots, t_k))) \ldots ,$$

Introduce a symmetric $\mathbb{C}(u; z)$-valued bilinear form $\langle \cdot, \cdot \rangle$ on each $\mathcal{H}_k$, so that for any $\varphi, \psi \in \mathcal{H}_k$ the value $\langle \varphi, \psi \rangle_{u, z}$ of the corresponding rational function at generic $u, z$ is given by

$$\langle \varphi, \psi \rangle_{u, z} = \sum_{k \in \mathcal{X}_k} \text{Res}_{t_k = i_6} (\text{Res}_{t_{k-1} = i_5} (\ldots \text{Res}_{t_1 = i_1} \varphi[u; z] \psi[u; z] \Omega_k[u; z])), \quad (B.3)$$

where the "string" $\tau_k^- [z]$ is defined by

$$\tau_k^- [z] = (z_1, \ldots, z_1 - k_1 + 1, z_2, \ldots, z_2 - k_2 + 1, z_3, \ldots, z_3 - k_3 + 1).$$

**Remark.** The definition above is motivated by the fact (see [TV4]) that the integral over the imaginary hyperplane $\text{Re} t_1 = \cdots = \text{Re} t_k = 0$,

$$\frac{1}{(2\pi i)^k} \int \cdots \int \varphi[u; z](t_1, \ldots, t_k) \psi[u; z](t_1, \ldots, t_k) \Omega_k[u; z](t_1, \ldots, t_k) dt_1 \cdots dt_k,$$

converges when $\text{Re}(u_i) \ll \text{Re}(z_i) < 0, \ i = 1, 2, 3$, and represents the rational function $\langle \varphi, \psi \rangle$.

Evaluating the integral by residues, and using the analytic continuation, we get the more convenient algebraic formula $(B.3)$; one can show that it is equivalent to another iterated residue formula:

$$\langle \varphi, \psi \rangle_{u, z} = (-1)^k \sum_{k \in \mathcal{X}_k} \text{Res}_{t_k = i_6} (\varphi[u; z] \psi[u; z] \Omega_k[u; z]), \quad (B.4)$$

where $\tau_k^+[z - u]$ is defined by

$$\tau_k^+[z - u] = (z_1 - u_1, \ldots, z_1 - u_1 + k_1 - 1, z_2 - u_2, \ldots, z_2 - u_2 + k_2 - 1, z_3 - u_3, \ldots, z_3 - u_3 + k_3 - 1).$$
Lemma B.5. For any \( \varphi, \psi \in \tilde{\mathcal{H}}_k \), the only possible singularities of \( \langle \varphi, \psi \rangle_{u, z} \) are simple poles at the following hyperplanes:

\[
    z_i - u_i = z_j - r, \quad i, j = 1, 2, 3, \quad r = 0, 1, \ldots, k - 1.
\]

(B.5)

Proof. Since \( \varphi, \psi \) are polynomials, possible singularities of \( \langle \varphi, \psi \rangle_{u, z} \) are determined by \( \Omega_k[u; z] \). Using explicit formulas, one can check that \( \text{Res}_{z_i}^u(\Omega_k[u; z]) \) has only simple poles located at the hyperplanes

\[
    z_i - u_i = z_j - r, \quad z_i = z_j - r, \quad i, j = 1, 2, 3, \quad r = 0, 1, \ldots, k - 1.
\]

Similarly, \( \text{Res}_{z_i}^u(\Omega_k[u; z]) \) has only simple poles located at the hyperplanes

\[
    z_i - u_i = z_j - r, \quad z_i = z_j - u_j - r, \quad i, j = 1, 2, 3, \quad r = 0, 1, \ldots, k - 1.
\]

Since \( \langle \varphi, \psi \rangle_{u, z} \) is given either by formula (B.3) or (B.4), the function \( \langle \varphi, \psi \rangle_{u, z} \) can have poles only at the hyperplanes which belong to both of these lists, which gives precisely list (B.5). \( \square \)

Let \( H_k[I], k \in \mathbb{Z}_{\geq 0}, \) denote the subspace of functions \( \varphi \in H_k \), satisfying the following conditions for each \( i = 1, 2, 3 \) such that \( l_i < k \):

\[
    \varphi(z_i, \ldots, z_i - l_i, t_{l_i+2}, \ldots, t_k) = 0 \quad \text{at the hyperplane } u_i = l_i.
\]

We have two important imbeddings of \( H_k[I] \) into \( H_k[I] \), given by \( \varphi \mapsto \varphi^- \),

\[
    \varphi^-[u; z](t_1, \ldots, t_k) = \varphi[z](t_1, \ldots, t_k),
\]

and \( \varphi \mapsto \varphi^+ \),

\[
    \varphi^+[u; z](t_1, \ldots, t_k) = \varphi[z - u + 1](t_1, \ldots, t_k).
\]

For \( p \in D_k \), let \( w_p = w_p[u; z] \) be the weight function defined in Proposition B.1, with integers \( I = (l_1, l_2, l_3) \) replaced by variables \( u = (u_1, u_2, u_3) \). It is obvious that \( w_p \in H_k \); moreover, if \( p \in D_k[I] \), then \( w_p \in H_k[I] \).

Lemma B.6. Let \( p \in D_k[I], \) and \( \varphi \in H_k[I] \).

1. The function \( \langle w_p, \varphi^- \rangle_{u, z} \) may have only simple poles, located at the hyperplanes

\[
    z_i - u_i = z_j - r, \quad 1 \leq i \leq j \leq 3, \quad r = 0, 1, \ldots, l_j - 1.
\]
2. The function \( \langle w_p, \varphi^+ \rangle_{u,z} \) may have only simple poles, located at the hyperplanes

\[
\tau_i - u_i = z_j - r, \quad 1 \leq i \leq 3, \quad r = 0, 1, \ldots, l_i - 1.
\]

**Proof.** Since \( \varphi^- [u; z](\tau_q^+ [z]) = \varphi [z](\tau_q^+ [z]) = 0 \) unless \( q \in \mathcal{D}_k[l] \), we have

\[
\langle w_p, \varphi^- \rangle_{u,z} = \sum_{q \in \mathcal{D}_k[l]} \varphi^- (\tau_q^+ [z]) \text{Res}_{\tau_q^+ [z]} (w_p [u; z] \Omega_k [u; z])
\]

\[
= \sum_{q \in \mathcal{D}_k[l]} \varphi^- (\tau_q^+ [z]) \text{Res}_{\tau_q^+ [z]} (w_p [u; z] \Omega_k [u; z]).
\]

Let \( q \in \mathcal{D}_k[l] \). From the explicit formulas for \( w_p [u; z] \) and \( \Omega_k [u; z] \) it follows that the function \( \text{Res}_{\tau_q^+ [z]} (w_p [u; z] \Omega_k [u; z]) \) has no pole at the hyperplane \( \tau_i - u_i = z_j - r \) provided \( i > j \) or \( r > q_j \). Using the description (B.5) of all possible poles, we see that \( \langle w_p, \varphi^- \rangle_{u,z} \) may have only poles corresponding to \( i \leq j \) and \( r < q_j \leq l_i \), which proves the first assertion.

Similarly, using the fact that \( \varphi^+[u; z](\tau_q^+ [z]) \varphi [z - u](\tau_q^+ [z - u]) = 0 \) unless \( q \in \mathcal{D}_k[l] \), we arrive to the formula

\[
\langle w_p, \varphi^+ \rangle_{u,z} = \sum_{q \in \mathcal{D}_k[l]} \varphi^+ (\tau_q^+ [z - u]) \text{Res}_{\tau_q^+ [z - u]} (w_p [u; z] \Omega_k [u; z]).
\]

For any \( q \in \mathcal{D}_k[l] \) the function \( \text{Res}_{\tau_q^+ [z - u]} (w_p [u; z] \Omega_k [u; z]) \) has no pole at the hyperplane \( \tau_i - u_i = z_j - r \) provided \( i > j \) or \( r > q_j \). Therefore, it follows that \( \langle w_p, \varphi^+ \rangle_{u,z} \) may only have poles corresponding to \( i \leq j \) and \( r < q_j \leq l_i \). \( \square \)

**Lemma B.7.** Let \( \varphi, \psi \in \mathcal{H}_k[l] \). The function \( \langle \varphi, \psi \rangle_{u,z} \) analytically continues to \( u = 1 \), and we have

\[
\langle \varphi, \psi \rangle_{1,z} = \sum_{k \in \mathcal{D}_k[l]} \text{Res}_{\tau_k^+ [z]} (\varphi [l; z] \psi [l; z] \Omega_k [l; z])
\]

\[
= (-1)^k \sum_{k \in \mathcal{D}_k[l]} \text{Res}_{\tau_k^+ [l - 1]} (\varphi [l; z] \psi [l; z] \Omega_k [l; z]). \quad (B.6)
\]

In other words, the analytic continuation to \( u = 1 \) can be computed by applying the residue formula to the specialized functions \( \psi [l; z] \), \( \varphi [l; z] \) and \( \Omega_k [l; z] \), rather than treating \( u \) as variables, and the summation is reduced to \( k \in \mathcal{D}_k[l] \subset \mathcal{D}_k \). Note that in general this is not true for arbitrary \( \varphi, \psi \in \mathcal{H}_k \).

**Proof.** One can check that for any \( q \in \mathcal{D}_k \) the rational functions \( \text{Res}_{\tau_q^+ [z]} (\psi [u; z] \Omega_k [u; z]) \) and \( \text{Res}_{\tau_q^+ [z - u]} (\psi [u; z] \Omega_k [u; z]) \) have no poles at the hyperplanes \( u_i = l_i, \ i = 1, 2, 3 \). Together with Lemma 31, this shows that the analytic continuation to \( u = 1 \) is well defined.
Since for $q \notin \mathcal{L}_k[l]$, the expression $\varphi[u; z](\tau_k^{-}(z))$ vanishes at $u = l$, it follows that the summation in the residue formulas (B.3) reduces to $q \in \mathcal{L}_k[l]$. Similar conclusion holds for the residue formula (B.4).

The statement now follows from the fact that for $q \subseteq k \subseteq l$ we have

$$\text{Res}_{t^{-}[z]}(\psi[u; z] \Omega_k[u; z])|_{u = 1} = \text{Res}_{t^{-}[z]}(\psi[l; z] \Omega_k[l; z]),$$

$$\text{Res}_{t^{-}[x-u]}(\psi[u; z] \Omega_k[u; z])|_{u = 1} = \text{Res}_{t^{-}[x-l]}(\psi[l; z] \Omega_k[l; z]).$$

Introduce a partial order $\leq$ on $\mathcal{L}_k$ by saying that $x \leq y$ if and only if $x_1 \leq y_1$ and $x_3 \geq y_3$.

**Lemma B.8.** For any $k, p, q \in \mathcal{L}_k$ we have

$$w_p[l, z](\tau_k^{-}[z]) = 0 \quad \text{unless} \quad k \leq p,$$

$$w'_q[l, z](\tau_k^{-}[z]) = 0 \quad \text{unless} \quad k \geq q.$$  

Moreover, if $z$ is generic, then $w'_q[l, z](\tau_k^{-}[z]) \neq 0$.

The proof is straightforward.

**Lemma B.9** (Tarasov and Varchenko [TV4]). Let $k \in \mathbb{Z} \geq 0$, and let $p, q \in \mathcal{L}_k[l]$. Then, we have

$$\langle w_p, w'_q \rangle_{L^z} = \delta_{p,q} \frac{p_1! l_1!}{(l_1 - p_1)! (l_2 - p_2)! (l_3 - p_3)!}$$

(B.7)

**Proof.** By Lemma B.8 the summation in the residue formula (B.6) reduces to $k$ such that $q \leq k \leq p$, that is

$$\langle w_p, w'_q \rangle_{L^z} = \sum_{q \leq k \leq p} w'_q[l, z](\tau_k^{-}[z]) w_p[l, z](\tau_k^{-}[z]) \text{Res}_{t^{-}[z]}(\Omega_k[l; z]).$$

It follows immediately that $\langle w_p, w'_q \rangle_{L^z} = 0$ unless $q \leq p$.

Similarly, one can check that

$$w_p(\tau_k^{+}[z - l]) = 0 \quad \text{unless} \quad k \geq p,$$

$$w'_q(\tau_k^{+}[z - l]) = 0 \quad \text{unless} \quad k \leq q,$$
Thus, it follows that
\[
\langle w_p, w'_q \rangle_{L_2} = \sum_{p \leq k \leq q} w_p[l; z](\tau_k^+[z - l])w'_q[l; z](\tau_k^+[z - l])(\text{Res}_{\tau_k^+[z - l]})\Omega_k[l, z].
\]
and therefore, \( \langle w_p, w'_q \rangle_{L_2} = 0 \) unless \( p \leq q \).

Hence, \( \langle w_p[l, z], w'_q[l, z] \rangle_{L_2} \) is equal to zero unless \( p = q \). In the last case there is only one term in the residue summation formula, corresponding to \( k = p = q \), and the statement follows from the direct computation of the only remaining residue at \( \tau_k[z] \).

Finally, we have all ingredients necessary to describe explicitly the map \( \mathcal{I}[l] \) and to prove its properties.

**Proof of Proposition 14.** Lemma B.8 implies that the dual weight functions \( \{w'_k[l; z]\} \) are linearly independent as elements of \( \mathcal{H}_k \), which yields the injectivity of the map \( \phi[l] \). Using the explicit combinatorial formulas for \( w'_k[l; z] \), one can check that the admissibility conditions (6.2) are satisfied provided that \( k \in \mathcal{Z}_k[l] \). Hence, \( \text{Im} \phi[l] \otimes C[z] \subset \mathcal{H}_k[l] \otimes C[z] \). To show that these spaces coincide, it is enough to check that for generic \( z \) one has \( \dim \mathcal{H}_k[l; z] \leq \dim \text{Im} \phi[l; z] = \# \mathcal{Z}_k[l] \).

Consider the maps \( e_p[z] : \mathcal{H}_k \to C \),
\[
e_p[z] : \psi \mapsto \psi(\tau_p[z]).
\]

Lemma B.8 implies that for generic \( z \) these maps are linearly independent. It is clear that for any \( \psi \in \mathcal{H}[l; z], \ p \notin \mathcal{Z}_k[l] \) one has \( e_p[z](\psi) = 0 \), which gives \( \dim \mathcal{H}_k[l; z] \leq \# \mathcal{Z}_k[l] \), since \( \dim \mathcal{H}_k = \# \mathcal{Z}_k \).

Now, consider the map
\[
\mathcal{I}[l; z] : \mathcal{H}[l; z] \to V_{(l,0)} \otimes V_{(l,0)} \otimes V_{(l,0)},
\]
\[
\mathcal{I}[l; z] \phi = \sum_{k \in \mathcal{Z}_k[l]} \sum_{p \in \mathcal{Z}_k[l]} \langle w_p, \varphi \rangle_{L_2} E_{21}^p \mathbf{1}_l \otimes E_{21}^p \mathbf{1}_l \otimes E_{21}^p \mathbf{1}_l.
\]
Formulas (6.1) and (B.7) imply (6.4).

According to Lemma B.7, we have
\[
\langle w_p, \varphi \rangle_{L_2} = (\langle w_p, \varphi^- \rangle_{u,z})|_{u=l} (\langle w_p, \varphi^+ \rangle_{u,z})|_{u=l},
\]
and Lemma B.6 implies that \( \langle w_p, \varphi \rangle_{L_2} \) may have poles only at the hyperplanes
\[
z_i - u_i = z_j - r, \quad 1 \leq i < j \leq 3, \quad r = 0, 1, \ldots, \min(l_i, l_j) - 1.
\]
This proves the last part of Proposition 14. \( \square \)
Appendix C

Proof of Lemma 19. The subspace $\mathcal{H}_k[l]^{\text{sing}}$ consists of polynomials of degree at most 1 in each $t_j$, and therefore any function $\zeta(t_1, \ldots, t_k) \in \mathcal{H}_k[l]^{\text{sing}}$ has the form

$$\zeta(t_1, \ldots, t_k) = \sum_{r=0}^k X_r(z_1, z_2, z_3) \sigma_r(t_1, \ldots, t_k)$$

where $X_0, \ldots X_k$ are some coefficients from $\mathbb{C}[z]$, and

$$\sigma_r(t_1, \ldots, t_k) = \sum_{1 \leq i_1 < \cdots < i_r \leq k} t_{i_1} \cdots t_{i_r}, \quad r = 0, 1, 2, \ldots$$

in particular, $\sigma_0(t_1, \ldots, t_k) \equiv 1$, and $\sigma_r(t_1, \ldots, t_k) \equiv 0$ if $r > k$.

Using the combinatorial identity

$$\sigma_r(z, \ldots, z - l, t_{l+2}, \ldots, t_k) = \sum_{s=0}^r \sigma_{r-s}(z, \ldots, z - l) \sigma_s(t_{l+2}, \ldots, t_k),$$

we see that admissibility conditions (6.2) for the function $\zeta(t_1, \ldots, t_k)$ are equivalent to the following system of equations for the coefficients $X_0, \ldots X_k$:

$$\sum_{r=s}^k X_r \sigma_{r-s}(z_i, \ldots, z_i - l_i) = 0, \quad s = 0, \ldots, k - l_i - 1, \quad i = 1, 2, 3. \quad (C.1)$$

Let $d = \dim (V_{(l_0, 0)} \otimes V_{(l_2, 0)} \otimes V_{(l_3, 0)})^{\text{sing}}[(l_1 + l_2 + l_3 - k, k)]$. We will show that system (C.1) determines $X_0, \ldots, X_{k-d}$ as linear combinations of free variables $X_{k-d+1}, \ldots, X_k$ with coefficients, polynomial in $z_1, z_2, z_3$. Then we construct a $\mathbb{C}[z]$-basis $\zeta_s[l], \ s = 1, \ldots, d$ of $\mathcal{H}_k[l]^{\text{sing}}$, by taking

$$X_{k-s+1} = 1, \quad X_{k-r+1} = 0 \quad \text{for} \ r \neq s, \ 1 \leq r \leq d,$$

and computing $X_0, \ldots, X_{k-d}$ from Eqs. (C.1).

Consider the $(k - d + 1) \times (k + 1)$ matrix of system (C.1). It suffices to prove that its $(k - d + 1) \times (k - d + 1)$ minor, corresponding to the variables $X_0, \ldots, X_{k-d}$, is nonzero, and divides all other $(k - d + 1) \times (k - d + 1)$ minors. Before giving a general proof of this statement, we consider an example.

Example. $l_1 = 1, \quad l_2 = 2, \quad k = 3, \quad l_3 > 3$. Then $d = \dim (V_{(l_1, 0)} \otimes V_{(l_2, 0)} \otimes V_{(l_3, 0)})^{\text{sing}}[(l_3, 3)] = 1$, and the matrix of system of (C.1) is equal to

$$\begin{pmatrix}
1 & z_1 + (z_1 - 1) & z_1(z_1 - 1) & 0 \\
0 & 1 & z_1 + (z_1 - 1) & z_1(z_1 - 1) \\
1 & z_2 + (z_2 - 1) + (z_2 - 2) & z_2(z_2 - 1) + z_2(z_2 - 2) + (z_2 - 1)(z_2 - 2) & z_2(z_2 - 1)(z_2 - 2)
\end{pmatrix}$$
The $3 \times 3$ minors of this matrix are given by

\[ M_{012} = 3(z_1 - z_2)(z_1 - z_2 + 1), \]
\[ M_{013} = (z_1 - z_2)(z_1 - z_2 + 1)(2z_1 + z_2 - 2), \]
\[ M_{023} = (z_1 - z_2)(z_1 - z_2 + 1)(z_1^2 + 2z_1z_2 - 3z_1 - z_2 + 2), \]
\[ M_{123} = (z_1 - z_2)(z_1 - z_2 + 1)(3z_1^2z_2 - 3z_1^2z_2 + 5z_1 + z_2 - 2). \]

The leftmost minor $M_{012}$ is nonzero and divides all other minors. The function $\zeta(t_1, t_2, t_3)$, spanning the space $\mathcal{H}[(1, 2, l_3), (z_1, z_2, z_3)]^{\text{sing}}[(l_3, 3)]$, is given by

\[ \zeta(t_1, t_2, t_3) = t_1t_2t_3 - \frac{1}{M_{012}}(M_{013}(t_1t_2 + t_2t_3 + t_1t_3) - M_{023}(t_1 + t_2 + t_3) + M_{123}). \]

We now return to the case of arbitrary $l$. Without loss of generality, we may assume that $l_1 \leq l_2 \leq l_3$. To shorten notation, we denote

\[ \sigma_r^{(i)} = \sigma_r(z_i, z_i - 1, \ldots, z_i - l_i), \quad r = 0, \ldots, k, \quad i = 1, 2, 3. \]

**Case 1:** $k \leq l_1$. Then, there are no admissibility conditions, $d = k + 1$, and all the variables $X_0, \ldots, X_k$ are free.

**Case 2:** $l_1 < k \leq l_2$. Then $d = l_1 + 1$, and the matrix of system (C.1) is equal to

\[
\begin{pmatrix}
\sigma_0^{(1)} & \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_{k-l_1}^{(1)} & \sigma_{k-l_1-1}^{(1)} & \sigma_{k-l_1}^{(1)} & \cdots & \sigma_k^{(1)} \\
0 & \sigma_0^{(1)} & \sigma_1^{(1)} & \cdots & \sigma_{k-l_2}^{(1)} & \sigma_{k-l_2-1}^{(1)} & \sigma_{k-l_2}^{(1)} & \cdots & \sigma_{k-l_1}^{(1)} \\
0 & 0 & \sigma_0^{(1)} & \cdots & \sigma_{k-l_3}^{(1)} & \sigma_{k-l_3-1}^{(1)} & \sigma_{k-l_3}^{(1)} & \cdots & \sigma_{k-l_2}^{(1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & \sigma_0^{(1)} & \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_{h+1}^{(1)}
\end{pmatrix}
\]

Note that some of the terms in the upper right corner can also vanish, because $\sigma_{l+2}^{(1)} = \cdots = \sigma_k^{(1)} = 0$.

Since $\sigma_0^{(1)} \equiv 1$, the leftmost $(k - d + 1) \times (k - d + 1)$ minor is identically equal to 1. Therefore $X_{k-d+1}, \ldots, X_k$ are free variables, and $X_0, \ldots, X_{k-d}$ are their linear combinations with coefficients, polynomial in $z_1, z_2, z_3$.

**Case 3:** $l_2 < k \leq l_3$. If $k \geq l_1 + l_2$, then $d = 0$, and there is nothing to check. Assume that $k < l_1 + l_2$, which gives $d = l_1 + l_2 - k$. The matrix of system (C.1) is equal to
We make two claims now. First, every \((k-d+1) \times (k-d+1)\) minor of this matrix is a polynomial in \(z_1, z_2\), divisible by

\[
D(z_1, z_2) = \prod_{j_1=1}^{k-l_1} \prod_{j_2=1}^{k-l_2} (z_1 - z_2 + j_1 - j_2).
\]

Second, the leftmost \((k-d+1) \times (k-d+1)\) minor of matrix (C.2) is a nonzero polynomial in \(z_1, z_2\) of homogeneous degree \((k-l_1)(k-l_2)\).

These two claims together imply the desired statement. Indeed, since the homogeneous degree of the polynomial \(D(z_1, z_2)\) is equal to \((k-l_1)(k-l_2)\), the leftmost minor is proportional to \(D(z_1, z_2)\) with a nonzero coefficient. Therefore, every \((k-d+1) \times (k-d+1)\) minor is divisible by the leftmost \((k-d+1) \times (k-d+1)\) minor, which means that \(X_0, \ldots, X_{k-d}\) can be expressed as linear combinations of the free variables \(X_{k-d+1}, \ldots, X_k\) with coefficients, polynomially depending on \(z_1, z_2, z_3\).

We first prove the second claim. Note that \(\sigma_r^{(i)}\) is a polynomial of \(z_i\) with the highest degree term \((l_i+1)z_i^r\). Therefore, the homogeneous degree of the leftmost \((k-d+1) \times (k-d+1)\) minor does not exceed \((k-l_1)(k-l_2)\), while the sum of terms of degree \((k-l_1)(k-l_2)\) is given by the determinant

\[
\begin{pmatrix}
\sigma_0^{(1)} & \sigma_1^{(1)} & \cdots & \cdots & \sigma_{k-l_1-1}^{(1)} & \sigma_k^{(1)} \\
0 & \sigma_0^{(1)} & \cdots & \cdots & \sigma_{k-l_1-2}^{(1)} & \sigma_{k-1}^{(1)} \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & \sigma_0^{(1)} & \cdots & \sigma_{l_1-1}^{(1)} \\
\sigma_0^{(2)} & \sigma_1^{(2)} & \cdots & \cdots & \sigma_{k-l_2-1}^{(2)} & \sigma_k^{(2)} \\
0 & \sigma_0^{(2)} & \cdots & \cdots & \sigma_{k-l_2-2}^{(2)} & \sigma_{k-1}^{(2)} \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & \sigma_0^{(2)} & \cdots & \sigma_{l_2-1}^{(2)}
\end{pmatrix}
\]

\[
\text{det}
\begin{pmatrix}
1 & \binom{l_1+1}{1}z_1 & \cdots & \cdots & \binom{l_1+1}{k-l_1-1}z_1^{k-l_1-1} & \cdots & \binom{l_1+1}{k-l_1-1}z_1^{k-l_1-1} \\
0 & 1 & \cdots & \cdots & \binom{l_1+1}{k-l_1-2}z_1^{k-l_1-2} & \cdots & \binom{l_1+1}{k-l_1-2}z_1^{k-l_1-2} \\
\vdots & \vdots & \ddots & \cdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & \cdots & \binom{l_1+1}{k-l_1-1}z_1^{k-l_1} \\
1 & \binom{l_2+1}{1}z_2 & \cdots & \cdots & \binom{l_2+1}{k-l_2-1}z_2^{k-l_2-1} & \cdots & \binom{l_2+1}{k-l_2-1}z_2^{k-l_2-1} \\
0 & 1 & \cdots & \cdots & \binom{l_2+1}{k-l_2-2}z_2^{k-l_2-2} & \cdots & \binom{l_2+1}{k-l_2-2}z_2^{k-l_2-2} \\
\vdots & \vdots & \ddots & \cdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & \cdots & \binom{l_2+1}{k-l_2}z_2^{k-l_2-1}
\end{pmatrix}
\]
The monomial $z_1^{(k-l_1)(k-l_2)}$ enters this determinant with a nonzero coefficient

$$\det\begin{pmatrix}
\begin{pmatrix}
\frac{l_i+1}{k-l_2} & \frac{l_i+1}{k-l_2} & \cdots & \frac{l_i+1}{k-l_2} \\
\frac{l_i+1}{k-l_2-1} & \frac{l_i+1}{k-l_2-1} & \cdots & \frac{l_i+1}{k-l_2-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{l_i+1}{l_i-l_2+1} & \frac{l_i+1}{l_i-l_2+1} & \cdots & \frac{l_i+1}{l_i-l_2+1}
\end{pmatrix}
\end{pmatrix} = \prod_{i=1}^{k-l_1} \prod_{j=1}^{k-l_2} \prod_{s=1}^{l_i+1-k-l_2} \frac{i+j+s-1}{i+j+s-2}.
$$

The last equality follows from the general formula, established in [K]:

$$\det\begin{pmatrix}
\begin{pmatrix}
\frac{a+b}{a} & \frac{a+b}{a+1} & \cdots & \frac{a+b}{a+c} \\
\frac{a+b}{a-1} & \frac{a+b}{a} & \cdots & \frac{a+b}{a+c-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a+b}{a-c} & \frac{a+b}{a-c+1} & \cdots & \frac{a+b}{a}
\end{pmatrix}
\end{pmatrix} = \prod_{i=1}^{c+1} \prod_{j=1}^{a} \prod_{s=1}^{b} \frac{i+j+s-1}{i+j+s-2}.$$

by taking $a = k - l_2, b = l_1 + l_2 - k + 1, c = k - l_1 - 1$.

Finally, we prove the first claim. Recall that we assume $l_1 \leq l_2 < k < l_1 + l_2$.

Take an integer $j$ such that $-l_2 \leq j \leq l_1$ and let $z_1 - z_2 = j = 0$. Then the sets
\{\{z_1, z_1 - 1, \ldots, z_1 - l_1\} and \{z_2, z_2 - 1, \ldots, z_2 - l_2\} have $p(j) + 1$ common points
\{\{\zeta, \zeta - 1, \ldots, \zeta - p(j)\}, where $\zeta = z_1$ for $j < 0, \zeta = z_2$ for $j \geq 0$, and

$$p(j) = \begin{cases} l_2 + j, & -l_2 \leq j < l_1 - l_2, \\
l_1, & l_1 - l_2 \leq j < 0, \\
l_1 - j, & 0 \leq j \leq l_1,
\end{cases}$$

Consider the system of linear equations on variables $X_0, \ldots, X_k$, corresponding to the condition

$$\zeta(\zeta, \zeta - 1, \ldots, \zeta - p(j), t_{p(j)+2}, \ldots, t_k) \equiv 0. \tag{C.3}$$

Let $L \subset \mathbb{C}^k$ denote the linear span of the rows of the corresponding matrix. Since the rows are linearly independent, we have $\dim L = k - p(j)$.

Obviously, if the function $\zeta(t_1, \ldots, t_k)$ obeys condition (C.3), then

$$\zeta(z_1, z_1 - 1, \ldots, z_1 - l_1, t_{l_1+1}, \ldots, t_k) \equiv 0,$$

$$\zeta(z_2, z_2 - 1, \ldots, z_2 - l_2, t_{l_2+1}, \ldots, t_k) \equiv 0.$$

This means that the linear span of the rows of matrix (C.2) is contained in $L$. In particular, the rank of matrix (C.2) does not exceed $\dim L$. Therefore, if $l_1 - k < j < k - l_2$, then the rank of matrix (C.2) is less than its number of rows $2k - l_1 - l_2$ by at least $k + p(j) - l_1 - l_2$, which is positive.

The above consideration shows that every $(k - d + 1) \times (k - d + 1)$ minor of matrix (C.2) is divisible by the product

$$\prod_{j=l_1-k+1}^{k-l_2-1} (z_1 - z_2 - j)^{k+p(j)-l_1-l_2}.$$

It is easy to show that this product coincides with $D(z_1, z_2)$. 

\[\text{ARTICLE IN PRESS} \]
Case 4: $l_1 + l_2 < k \leq l_3$. In this case $\dim \mathcal{H}_k(1; z)^{\text{sing}} = 0$, and there is nothing to check.

Case 5: $k > l_3$. There are three types of admissibility conditions in this case, related to $(z_1, l_1), (z_2, l_2)$ and $(z_3, l_3)$. The consideration, similar to Case 3, shows that the leftmost $(k - d + 1) \times (k - d + 1)$ minor of the corresponding matrix is nonzero and divides every $(k - d + 1) \times (k - d + 1)$ minor. This implies the required statement.

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