A PROOF OF A MULTIVARIABLE ELLIPTIC SUMMATION FORMULA CONJECTURED BY Warnaar

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Abstract. We prove a multivariable elliptic analogue of Jackson’s $8W_7$ summation formula, which was recently conjectured by S. O. Warnaar.

1. Introduction

Elliptic hypergeometric series form a natural generalization of hypergeometric and basic hypergeometric (or $q$-) series. It is surprising that they were introduced only very recently, by Frenkel and Turaev [FT], who expressed the $6j$-symbols corresponding to certain elliptic solutions of the Yang–Baxter equation, cf. [DJ], in terms of the $10\omega$-sums defined below. It is expected that elliptic hypergeometric series play a fundamental role in the representation theory of elliptic quantum groups, though so far there has been little work in this direction.

Recall that a series $\sum_n a_n$ is called hypergeometric if $f(n) = a_{n+1}/a_n$ is a rational function of $n$ and basic hypergeometric if $f$ is a rational function of $q^n$ for some $q$. This can be compared with Weierstrass’ theorem, stating that a meromorphic function of $z$ which satisfies an algebraic addition theorem is either a rational function, a rational function of $q^z$, or, in the most general case, an elliptic function. This suggests that an elliptic hypergeometric series should be a series $\sum_n a_n$ with $a_{n+1}/a_n$ an elliptic function of $n$. Actually, the series introduced by Frenkel and Turaev only fit this description if one interprets the term “elliptic” somewhat loosely. Nevertheless, their properties stem from addition theorems for elliptic functions (it is worth noting that the Yang–Baxter equation is an algebraic addition theorem for matrix-valued functions).

Let us write $[x]$ for the “elliptic number” (a Jacobi theta function, normalized so that $[1] = 1$)

$$[x] = \frac{q^{-\frac{x}{2}} \prod_{j=0}^{\infty} (1 - q^{-x}p^j)(1 - q^{-x}p^{j+1})}{q^{-\frac{x}{2}} \prod_{j=0}^{\infty} (1 - qp^j)(1 - q^{-1}p^{j+1})},$$

where $p$ and $q$ are fixed parameters with $|p| < 1$. When $p = 0$, $q = e^{2ih}$, we have the trigonometric number

$$[x] = \frac{q^x - q^{-x}}{q^\frac{x}{2} - q^{-\frac{x}{2}}} = \frac{\sin(hx)}{\sin(h)},$$

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which tends to the rational number \([x] = x\) as \(q\) tends to 1. Returning to the general case, we write
\[
[x]_n = [x][x + 1] \cdots [x + n - 1]
\]
for the elliptic Pochhammer symbols. The elliptic, or modular, hypergeometric series occurring in [FT] are finite sums of the form
\[
_{r+1}\omega_r(a; -n, b_1, \ldots, b_{r-3})
\]
where
\[
(r - 3)(a + 1) = 2(1 - n + \sum_i b_i).
\]
When \(p = 0\), this is a terminating very-well-poised balanced basic hypergeometric series [GH], which tends to the corresponding hypergeometric series as \(q\) tends to 1. As was pointed out in [FT], the series \(_{r+1}\omega_r\) has remarkable invariance properties under the standard action of \(SL(2, \mathbb{Z})\) on \(p\) and \(q\).

Most (or possibly all) known identities involving terminating \(q\)-series may be proved by induction, using the trigonometric addition formula
\[
[x + z]_{p=0}[x - z]_{p=0} = [x + y]_{p=0}[x - y]_{p=0} + [y + z]_{p=0}[y - z]_{p=0}.
\]
However, only a tiny subset of these identities may be obtained from the elliptic addition formula
\[
[x + z][x - z][y + w][y - w] = [x + y][x - y][z + w][z - w] + [x + w][x - w][y + z][y - z]
\]
satisfied by the elliptic numbers. At least as a rule of thumb, these are the identities involving series which are both well-poised and balanced, and thus only these admit elliptic analogues. In particular, Frenkel and Turaev obtained the elliptic Jackson–Dougall summation formula
\[
_{8}\omega_7(a; -n, b, c, d, e) = \frac{[a + 1]_n[a + 1 - b - c]_n[a + 1 - b - d]_n[a + 1 - c - d]_n}{[a + 1 - b]_n[a + 1 - c]_n[a + 1 - d]_n[a + 1 - b - c - d]_n}
\]
and (more generally) the elliptic Bailey transformation formula
\[
_{10}\omega_9(a; -n, b, c, d, e, f, g) = \frac{[a + 1]_n[a + 1 - e - f]_n[\lambda + 1 - e]_n[\lambda + 1 - f]_n}{[a + 1 - e]_n[a + 1 - f]_n[\lambda + 1 - e - f]_n[\lambda + 1]_n}
\times \frac{1}{10}\omega_9(\lambda; -n, \lambda + b - a, \lambda + c - a, \lambda + d - a, e, f, g),
\]
where \(\lambda = 2a + 1 - b - c - d\); note that the balanced condition \(\mathbb{I}\) is assumed.

If one wants to further develop the theory of elliptic hypergeometric series, there are two natural directions: quadratic (or higher) transformation formulas and multivariable series. In [W], Warnaar initiated the investigation of both topics. We will be concerned with the multivariable theory. As Warnaar pointed out, progress in this direction requires essentially new ideas, since the known proofs in the trigonometric and rational case usually depend on “lower level” identities, corresponding to the degenerate addition theorem \(\mathbb{I}\).
The purpose of this paper is to prove an identity conjectured by Warnaar in [W], cf. Theorem 2.1, which is a generalization of (3) connected with the root system $C_n$. Our main tool will be a different generalization of (3), obtained by Warnaar [W] from a determinant evaluation.

We mention that one degenerate case of Theorem 2.1 is the terminating case of a multivariable $q\psi_6$ sum due to van Diejen [D]. It generalizes various Macdonald–Morris-type identities for root systems, cf. [D] for a detailed discussion. Moreover, van Diejen’s sum gives the norm evaluation for the multivariable $q$-Racah polynomials studied by van Diejen and Stokman [DS].

When [W] was published, Theorem 2.1 was new even in the trigonometric case ($p = 0$). This case of the conjecture was settled by van Diejen and Spiridonov [DS], who deduced it from a certain multiple integral due to Gustafson [G], which reduces to the Nassrallah–Rahman integral [NR] in the one-variable case. The multiple $q$-series in question appears as a sum of residues of the integrand. Moreover, it was demonstrated that both sides of the equality in Theorem 2.1 are invariant under the action of $\text{SL}(2, \mathbb{Z})$. Using the theory of modular forms, it was then proved that for $q = e^{2\pi i h}$, the two sides are equal at least up to order $h^{10}$ around $h = 0$; a strong indication that Warnaar’s conjecture is true. Finally, van Diejen and Spiridonov conjectured an elliptic generalization of Gustafson’s integral, involving the elliptic gamma function introduced by Ruijsenaars [R]. A proof of this identity would yield another proof of Theorem 2.1, completely different from the one given here. The one-variable case of the integral is treated in [Sp].

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2. Notation and statement of results

In the rest of the paper we will use the “multiplicative” notation of [W] rather than the “additive” notation of [FT] used in the introduction. Since the elliptic modulus $p$ is fixed we suppress it from the notation. Thus we write

$$E(x) = \prod_{j=0}^{\infty} \left(1 - xp^j\right) \left(1 - p^{j+1}/x\right),$$

$$E(x_1, \ldots, x_m) = E(x_1) \cdots E(x_m),$$

$$(a; q)_k = \prod_{j=0}^{k-1} E(aq^j), \quad k \in \mathbb{Z}_{\geq 0},$$

$$(a; q)^{-k} = \frac{1}{(aq^{-k}; q)_k}, \quad k \in \mathbb{Z}_{< 0},$$

$$(a_1, \ldots, a_m; q)_k = (a_1; q)_k \cdots (a_m; q)_k,$$

$$(a; q, x)_\lambda = \prod_{j=1}^{n} (ax^{1-j}; q)_{\lambda_j}, \quad \lambda \in \mathbb{Z}^n,$$

$$(a_1, \ldots, a_m; q, x)_\lambda = (a_1; q, x)_\lambda \cdots (a_m; q, x)_\lambda.$$
We will use without comment standard identities such as \((a; q)_n (aq^n; q)_k = (a; q)_{n+k}\).

We also mention the easily verified identity

\[
(aq; q)_n \prod_{1 \leq i < j \leq n} E(aq^{i+j}) = (aq; q^2)_n \prod_{1 \leq i < j \leq n} E(aq^{i+j-1}).
\]

We can now state the main result of the paper, conjectured by Warnaar [W].

**Theorem 2.1.** In the notation above,

\[
\sum_{\lambda} \prod_{i=1}^{n} \left( \frac{E(ax^{2(1-i)} q^{2\lambda_i})}{E(ax^{2(1-i)})} q^{\lambda_i} x^{2(i-1)\lambda_i} \right) \prod_{1 \leq i < j \leq n} \left( \frac{E(x^{j-i} q^{\lambda_i - \lambda_j})}{E(x^{j-i})} \frac{E(ax^{2-i-j} q^{\lambda_i + \lambda_j})}{E(ax^{2-i-j})} \right) \times \left( \frac{(aq x^{1-i-j}; q)_{\lambda_i + \lambda_j}}{(aq x^{1-i-j}; q)_{\lambda_i + \lambda_j}} \frac{(aq x^{j-1-i}; q)_{\lambda_i - \lambda_j}}{(aq x^{j-1-i}; q)_{\lambda_i - \lambda_j}} \right) \frac{N_{ax}}{(aq/b, aq/c, aq/d, aq/e, aq^{N+1}; q, x)_\lambda} = \frac{(aq, aq/bc, aq/bd, aq/cd; q N)^n}{(aq/b, aq/c, aq/d, aq/bed; q, x)_N},
\]

where the sum is over the partitions \(\lambda \in \Lambda_{nN} = \{\lambda \in \mathbb{Z}^n; N \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}\) and where \(bcdex^{n-1} = a^2 q^{N+1}\).

Here \(N^n\) denotes the partition with \(\lambda_i = N, i = 1, \ldots, n\). Our main tool will be the following identity, again due to Warnaar.

**Lemma 2.2.** In the notation above,

\[
\sum_{k_1, \ldots, k_n = 0}^{1} \prod_{i=1}^{n} \frac{(bx_i, cx_i, dx_i, ex_i; q)_{k_i}}{(aq x_i/b, aq x_i/c, aq x_i/d, aq x_i/e; q)_{k_i}} (-1)^{k_i} q^{(i-1)k_i} \prod_{1 \leq i < j \leq n} \frac{E(q^{k_i-k_j} x_i/x_j)}{E(x_i/x_j)} \frac{E(ax_i x_j q^{k_i+k_j})}{E(ax_i x_j q)}
\]

\[
= (aq/bc, aq/bd, aq/cd; q^{-1})_n \prod_{i=1}^{n} \frac{E(ax_i^2)}{E(ax_i^2 - bcd x_i, aqx_i/b, aqx_i/c, aqx_i/d)},
\]

where \(a^2 q^{3-n} = bcde\).

In fact, Warnaar proved the more general identity [W], Theorem 5.1]

\[
\sum_{k_1, \ldots, k_n = 0}^{N} \prod_{i=1}^{n} \frac{E(ax_i^2 q^{2k_i})}{E(ax_i^2)} \frac{(ax_i^2, bx_i, cx_i, dx_i, ex_i, x_i, q^{-N}; q)_{k_i}}{(aq x_i/b, aq x_i/c, aq x_i/d, aq x_i/e, aq^{N+1} x_i^2; q)_{k_i}} q^{k_i} \times \prod_{1 \leq i < j \leq n} \frac{E(q^{k_i-k_j} x_i/x_j)}{E(x_i/x_j)} \frac{E(ax_i x_j q^{k_i+k_j})}{E(ax_i x_j q N)}
\]

\[
= \prod_{i=1}^{n} \frac{(aq x_i^2, aq^{2-i}/bc, aq^{2-i}/bd, aq^{2-i}/cd; q)_N}{(aq^{2-n}/bcd x_i, aqx_i/b, aqx_i/c, aqx_i/d; q)_N},
\]
where $a^2q^{N+2-n} = bcde$. For $n = 1$, this is equivalent to (3) and for $N = 1$ it reduces to (3). The case $p = 0$ is due to Schlosser [S].

We will prove Theorem 2.1 by induction on the “terminator” $N$. However, because of a duality property for the sums in question, cf. Proposition 4.1, we can alternatively formulate the proof as an induction on the number $n$ of variables. In this context we remark that, for $p = 0$, (6) is a special case not only of Schlosser’s identity but also of yet another multivariable Jackson–Dougall formula due to Denis and Gustafson [DG] and Milne and Lilly [ML]. The degeneration of the latter to the $_6\psi_6$-level, together with induction on the number of variables, was used by van Diejen [D] to prove the trigonometric $6\psi_6$-version of Theorem 2.1. Nevertheless, our proof is essentially different from the one in [D], since we only need a very special case of the (as yet unproved) elliptic Denis–Gustafson–Milne–Lilly identity.

3. Proof of Theorem 2.1

We will prove Theorem 2.1 by induction on $N$. The argument allows us to deduce the case $N = 1$, equivalent to (3), from the trivial case $N = 0$, so we obtain in particular a direct proof of the one-variable elliptic Jackson–Dougall formula.

Assume that Theorem 2.1 holds for a fixed value of $N$. Let us fix parameters with

$$a^2q^{N+2-n} = bcde.$$

We write the right-hand side of (5) with $N$ replaced by $N + 1$ as

$$R = \frac{(aq, aq/bc, aq/bd, aq/cd; q, x)_n}{(aq/b, aq/c, aq/d, aq/bcd; q, x)_n} \sum_\lambda \prod_{i=1}^n \left( \frac{E(ax^{2(i-1)}q^{2\lambda_i+1})}{E(ax^{2(i-1)}q)} \right)$$

$$\times \prod_{1 \leq i < j \leq n} \left( \frac{E(x^{j-i}q^{\lambda_i-\lambda_j})}{E(x^{j-i})} \right)$$

$$\times \frac{(aqx^{-1}, b, c, d, eq, q^{-N}; q, x)_\lambda}{(qx^{-1}, aq^2/b, aq^2/c, aq^2/d, aq/e, aq^{N+2}; q, x)_\lambda}.$$
which allows us to write

\[
(aq/bc, aq/bd, aq/cd; x^{-1})_n \prod_{i=1}^{n} E(ax^{2(i-1)}q^{2\lambda_i+1})
\]
\[
= \prod_{i=1}^{n} E(ax^{-i}q^{1-\lambda_i}/bcd, ax^{-i}q^{\lambda_i+1}/b, ax^{-i}q^{\lambda_i+1}/c, ax^{-i}q^{\lambda_i+1}/d)
\]
\[
\times \sum_{k_1, \ldots, k_n=0}^{1} \prod_{i=1}^{n} \frac{(bx^{-i}q^{\lambda_i}, cx^{-i}q^{\lambda_i}, dx^{-i}q^{\lambda_i}, ex^{-i}q^{\lambda_i-N}; x)_{k_i}(-1)^{k_i}x^{(i-1)k_i}}{ax^{-i}q^{\lambda_i+1}/b, ax^{-i}q^{\lambda_i+1}/c, ax^{-i}q^{\lambda_i+1}/d, ax^{-i}q^{\lambda_i+N+1}/e; x)_{k_i}}
\]
\[
\times \prod_{1 \leq i < j \leq n} \frac{E(x^{-i-j}q^{\lambda_i+\lambda_j}; q)_{\lambda_i+\lambda_j-k_i-k_j} (x^{-i-j+1}; q)_{\lambda_i+\lambda_j-k_i-k_j}}{E(x^{-i-j}q^{\lambda_i+\lambda_j}; q)_{\lambda_i+\lambda_j-k_i-k_j} (x^{-i-j+1}; q)_{\lambda_i+\lambda_j-k_i-k_j}}
\]
\[
\times \frac{aqx^{3-i-j}; q)_{\lambda_i+\lambda_j-k_i-k_j} (x^{j-i+1}; q)_{\lambda_i+\lambda_j-k_i-k_j}}{aqx^{1-n}; b, c, d, eq, q^{-N}; q, x)_{\lambda-k}}
\]
\[
\times \frac{aqx^{1-n}; b, c, d, eq, q^{-N}; q, x)_{\lambda-k}}{(q^{-n-1}, aq^{2}/b, aq^{2}/c, aq^{2}/d, aq/e, aq^{N+2}; q, x)_{\lambda-k}}.
\]

Plugging this into the previous identity and then replacing \( \lambda \) by \( \lambda - k \) in the summation yields

\[
R = \frac{(aq; x^{-1})_n}{(aq/b, aq/c, aq/d, aq/bcd; x^{-1})_n}
\]
\[
\times \sum_{\lambda, k} \prod_{i=1}^{n} \left( \frac{E(ax^{-i}q^{\lambda_i+1-k_i+1}/bcd, ax^{-i}q^{\lambda_i+1-k_i+1}/b, ax^{-i}q^{\lambda_i+1-k_i+1}/c, ax^{-i}q^{\lambda_i+1-k_i+1}/d)}{E(ax^{2(i-1)})} \right)
\]
\[
\times \frac{Q_{\lambda_i+\lambda_j-k_i-k_j}(x^{j-i+1}; q)_{\lambda_i+\lambda_j-k_i-k_j}}{Q_{\lambda_i+\lambda_j-k_i-k_j}(x^{j-i+1}; q)_{\lambda_i+\lambda_j-k_i-k_j}}
\]
\[
\times \frac{aqx^{1-n}; b, c, d, eq, q^{-N}; q, x)_{\lambda-k}}{(q^{-n-1}, aq^{2}/b, aq^{2}/c, aq^{2}/d, aq/e, aq^{N+2}; q, x)_{\lambda-k}}.
\]

We will identify the sum with respect to \( k \) as a case of (6) with \( q \) replaced by \( x^{-1} \). Since \( k_i \in \{0, 1\} \), we can write

\[
(ex^{-i}q^{\lambda_i-k_i-N}; x)_{k_i} = (ex^{-i}q^{\lambda_i-1-N}; x^{-1})_{k_i},
\]

\[
\frac{(aq^{1-n}; eq, q^{-N}; q, x)_{\lambda-k}}{(qx^{-n-1}, aq/e, aq^{N+2}; q, x)_{\lambda-k}} = \prod_{i=1}^{n} \frac{(x^{-n-i}q^{\lambda_i}, ax^{-i}q^{\lambda_i}; e, ax^{-i}q^{\lambda_i+N+1}; x^{-1})_{k_i}}{(ax^{2-n-i}q^{\lambda_i}, ex^{-i}q^{\lambda_i}, x^{-i}q^{\lambda_i-N}; x^{-1})_{k_i}},
\]
\[
(b; q, x)_{\lambda-k} \prod_{i=1}^{n} (bx^{-i}q^{\lambda_i-k_i}; x)_{k_i} = (b; q, x)_{\lambda}.
\]
and similarly with $b$ replaced by $c$ and $d$. Using the reflection formula $E(x) = -xE(1/x)$ and recalling (7), we have

$$\frac{E(ax^{i-n}q^{k_i-k_j+1}/bcd)}{(ax^{i-1}q^{\lambda_i-k_j+1}/e; x)_{k_i}} = q^{k_i} \frac{E(ex^{i-1}q^{\lambda_i-N-1}/a)}{(ax^{i-1}q^{\lambda_i+N+1}/e; x^{-1})_{k_i}}.$$

Considering the four cases $k_i, k_j = 0, 1$ separately, we find that the factor in curly brackets may be written as

$$\frac{E(x^{j-i+k_j-k_i}q^{\lambda_i-\lambda_j}, ax^{3-i-j-k_i-k_j}q^{\lambda_i+\lambda_j}, ax^{1-i-j})}{E(x^{j-i}, ax^{3-i-j}, ax^{2-i-j})} \frac{(ax^{3-i-j}; q)_{\lambda_i+\lambda_j}(x^{j-i+1}; q)_{\lambda_i-\lambda_j}}{(ax^{1-i-j}; q)_{\lambda_i+\lambda_j}(qx^{j-i-1}; q)_{\lambda_i-\lambda_j}}.$$

Finally, by (4), we have

$$(aq; x^{-1}) \prod_{i=1}^{n} \frac{1}{E(ax^{2(1-i)})} \prod_{1 \leq i < j \leq n} \frac{E(ax^{1-i-j})}{E(ax^{2-i-j})} = 1.$$

These simplifications lead to

$$R = \frac{1}{(aq/bcd; x^{-1})_n} \sum_{\lambda} \prod_{i=1}^{n} \left( E(ex^{i-1}q^{\lambda_i-N-1}/a)q^{\lambda_i}x^{2(1-i)\lambda_i} \right)$$

$$\times \prod_{1 \leq i < j \leq n} \left( \frac{E(x^{j-i}q^{\lambda_i-\lambda_j}, ax^{2-i-j}q^{\lambda_i+\lambda_j})}{E(x^{j-i}, ax^{3-i-j})} \frac{(ax^{3-i-j}; q)_{\lambda_i+\lambda_j}(x^{j-i+1}; q)_{\lambda_i-\lambda_j}}{(ax^{1-i-j}; q)_{\lambda_i+\lambda_j}(qx^{j-i-1}; q)_{\lambda_i-\lambda_j}} \right)$$

$$\times \frac{(b, c, d, ax^{1-n}, eq, q^{-N}; q, x)_{\lambda}}{(qx^{n-1}, aq/b, eq/c, aq/d, eq/e, eq^{N+2}; q, x)_{\lambda}}$$

$$\times \sum_{k} \prod_{1 \leq i < j \leq n} \frac{E(x^{j-i+k_j-k_i}q^{\lambda_i-\lambda_k})}{E(x^{j-i}q^{\lambda_i-\lambda_j})} \frac{E(ax^{3-i-j-k_k}q^{\lambda_i+\lambda_j})}{E(ax^{2-i-j}q^{\lambda_i+\lambda_j})}$$

$$\times \prod_{i=1}^{n} \frac{E(x^{n-i}q^{\lambda_i}, ax^{1-i}q^{\lambda_i}/e, ax^{1-i}q^{\lambda_i+N+1}, \lambda_i^{-1}q^{\lambda_i-N-1}; x^{-1})_{k_i}(-1)^{k_i}x^{-(i-1)k_i}}{(ax^{2-n-i}q^{\lambda_i}, ex^{1-i}q^{\lambda_i}, x^{1-i}q^{\lambda_i-N-1}, ax^{1-i}q^{\lambda_i+N+1}/e; x^{-1})_{k_i}}.$$

The sum in $k$ is the left-hand side of (8) with

$$(a, b, c, d, e, x_i, q) \mapsto (ax, x^{n-1}, a/e, eq^{N+1}, eq^{-N-1}, x^{1-i}q^{\lambda_i}, x^{-1}),$$

and thus equals

$$\prod_{i=1}^{n} \frac{E(ax^{2(1-i)}q^{2\lambda_i}, ex^{i-n}, x^{1-i}q^{N+1}, ex^{i-1}q^{N-1}/a)}{E(ax^{2-i-n}q^{\lambda_i}, ex^{1-i}q^{\lambda_i}, x^{1-i}q^{\lambda_i-N-1}, ex^{i-1}q^{-\lambda_i-N-1}/a)}$$

$$= \frac{(aq/bcd; x^{-1})_n(e, q^{-N-1}; q, x)_{\lambda}}{(eq^{-N}; q, x)_{\lambda}} \prod_{i=1}^{n} \frac{E(ax^{2(1-i)}q^{2\lambda_i})}{E(ax^{2-i-n}q^{\lambda_i}, ex^{1-i}q^{-\lambda_i-N-1}/a)}.$$
Finally, we use (4) to write

$$(aqx^{1-n}; q, x) \lambda \prod_{i=1}^{n} \frac{1}{E(ax^{2-i-n} q^{\lambda_i})} \prod_{1 \leq i < j \leq n} \frac{1}{E(ax^{2-i-j})}$$

$$= (ax^{1-n}; q, x) \lambda \prod_{i=1}^{n} \frac{1}{E(ax^{2(1-i)})} \prod_{1 \leq i < j \leq n} \frac{1}{E(ax^{2-i-j})}.$$ 

Putting all this together we find that $R$ equals

$$\sum_{\lambda} \prod_{i=1}^{n} \left( \frac{E(ax^{2(i-1)} q^{2\lambda_i})}{E(ax^{2(i-1)})} q^{\lambda_i} x^{2(i-1)\lambda_i} \right) \prod_{1 \leq i < j \leq n} \left( \frac{E(x^{j-i} q^{\lambda_i-\lambda_j})}{E(x^{j-i})} \frac{E(ax^{2-i-j} q^{\lambda_i+\lambda_j})}{E(ax^{2-i-j})} \frac{(ax^{3-i-j}; q)_{\lambda_i+\lambda_j} (x^{3-i-j}; q)_{\lambda_i-\lambda_j}}{(aqx^{1-i-j}; q)_{\lambda_i+\lambda_j} (qx^{j-i-1}; q)_{\lambda_i-\lambda_j}} \right),$$

which is indeed the left-hand side of (3) with $N$ replaced by $N + 1$. This completes the proof of Theorem 2.1.

4. Duality

In this section we prove a duality property for sums of the type occurring in Theorem 2.1. To state the result, we use the notation

$$r+1 \Omega_r^{(n)}(a; b_1, \ldots, b_{r-3}, q^{-N}; q, x)$$

$$= \sum_{\lambda \in \Lambda_{nN}} \prod_{i=1}^{n} \left( \frac{E(ax^{2(i-1)} q^{2\lambda_i})}{E(ax^{2(i-1)})} q^{\lambda_i} x^{2(i-1)\lambda_i} \right) \prod_{1 \leq i < j \leq n} \left( \frac{E(x^{j-i} q^{\lambda_i-\lambda_j})}{E(x^{j-i})} \frac{E(ax^{2-i-j} q^{\lambda_i+\lambda_j})}{E(ax^{2-i-j})} \frac{(ax^{3-i-j}; q)_{\lambda_i+\lambda_j} (x^{3-i-j}; q)_{\lambda_i-\lambda_j}}{(aqx^{1-i-j}; q)_{\lambda_i+\lambda_j} (qx^{j-i-1}; q)_{\lambda_i-\lambda_j}} \right) \left( \frac{ax^{1-n}; b_1, \ldots, b_{r-3}, q^{-N}; q, x}{aq/b_1, \ldots, aq/b_{r-3}, aq^{N+1}; q, x} \right)$$

It is natural to assume the balanced condition $(aq)^{r-3} = (x^{n-1} q^{1-N} \prod_i b_i)^2$, though we do not need it to prove the following proposition.

**Proposition 4.1.** One has

$$r+1 \Omega_r^{(n)}(a; b_1, \ldots, b_{r-3}, q^{-N}; q, x) = r+1 \Omega_r^{(N)}(aqx; b_1, \ldots, b_{r-3}, x^n; x^{-1}, q^{-1}).$$

In fact, this holds to a termwise summation symmetry between the two sums, the change of summation variable $\Lambda_{nN} \rightarrow \Lambda_{Nn}$ being conjugation of partitions. Let us write $\lambda'$ for the conjugate of a partition $\lambda$. Note that, since we consider partitions into non-negative parts, $\lambda'$ depends not only on the Young diagram of $\lambda$ but also on the choice of $n$ and $N$. For instance, $(3, 2, 0) \in \Lambda_{33}$ and $(3, 2, 0) \in \Lambda_{34}$ has conjugate $(2, 2, 1)$ and $(2, 2, 1, 0)$, respectively.

To prove Proposition 4.1, we observe that since clearly $(b; q, x)_{\lambda} = (b; x^{-1}, q^{-1})_{\lambda'}$, it is enough to show that, for $\lambda \in \Lambda_{nN}$, the two quantities

$$A_{\lambda} = \prod_{1 \leq i \leq j \leq n} \frac{E(ax^{2-i-j} q^{\lambda_i+\lambda_j})}{E(ax^{2-i-j})} \prod_{1 \leq i \leq j \leq n} \frac{(ax^{3-i-j}; q)_{\lambda_i+\lambda_j} (ax^{1-n}; q, x)_{\lambda}}{(aqx^{1-i-j}; q)_{\lambda_i+\lambda_j} (aq^{N+1}; q, x)_{\lambda}}$$
and
\[ B_\lambda = \prod_{i=1}^n q^{\lambda_i} x^{2(i-1) \lambda_i} \prod_{1 \leq i < j \leq n} E(x^{j-i} q^{\lambda_i-\lambda_j}) \frac{(x^{j-i+1}; q)_{\lambda_i-\lambda_j} (q^{-N}; q, x)^{\lambda}}{E(x^{j-i}) (q x^{j-i-1}; q)_{\lambda_i-\lambda_j} (q x^{n-1}; q, x)^{\lambda}} \]
are invariant under the transformation \((a, q, x, n, N, \lambda) \mapsto (aq x, x^{-1}, q^{-1}, N, n, \lambda')\).

We prove the invariance of \(A_\lambda\), the case of \(B_\lambda\) being similar. We fix \(n\) and \(N\) and proceed by induction on the number of boxes in the Young diagram of \(\lambda\), starting from the trivial case of zero boxes. Suppose that the invariance holds for a fixed partition \(\lambda\). We will show that it also holds for any partition \(\lambda^+\) obtained by adding a box to the Young diagram of \(\lambda\). There exist \(k\) and \(l\) with \(1 \leq k \leq n\), \(1 \leq l \leq N\) such that \(\lambda_j^+ = \lambda_j\) for \(j \neq k\), \(\lambda_k = l - 1\) and \(\lambda_k^+ = l\). After straightforward simplifications, we may write
\[ \frac{A_{\lambda^+}}{A_\lambda} = \frac{E(ax^{2-2k} q^{2l}, ax^{1-2k} q^{2l-1}, ax^{2-n-k} q^{l-1}) \prod_{i=1}^n E(ax^{2-i-k} q^{l+1}, ax^{3-i-k} q^{l+i-1})}{E(ax^{2-2k} q^{2l-1}, ax^{3-2k} q^{2l-2}, ax^{1-k} q^{l+n}) \prod_{i=1}^n E(ax^{2-i-k} q^{l+1-1}, ax^{1-i-k} q^{l+i})} \]
Next we observe that
\[ \prod_{i=1}^n \frac{E(b x^{1-i} q^{\lambda_i})}{E(b x^{-1} q^{\lambda_i})} = \prod_{i=1}^n \frac{E(b x^{1-i} q^{N})}{E(b x^{-i} q^{N})} \prod_{i=\lambda_i+1}^{N-1} \frac{E(b x^{1-i} q^{N-1})}{E(b x^{-1} q^{N-1})} \times \cdots \times \prod_{i=\lambda_i+1}^n \frac{E(b x^{1-i})}{E(b x^{-1})} \]
which gives
\[ \frac{A_{\lambda^+}}{A_\lambda} = \frac{E(ax^{2-2k} q^{2l}, ax^{1-2k} q^{2l-1}, ax^{2-n-k} q^{l+1-1}) \prod_{i=1}^n E(ax^{1-k} q^{l+1-i}, ax^{2-k} q^{l+i-2})}{E(ax^{2-2k} q^{2l-1}, ax^{3-2k} q^{2l-2}, ax^{1-k-n} q^{l}) \prod_{i=1}^n E(ax^{1-k} q^{l+i}, ax^{2-k} q^{l+i-1})}. \]
This agrees with the expression obtained from the previous one by substituting \((a, q, x, n, N, \lambda, k, l) \mapsto (aq x, x^{-1}, q^{-1}, N, n, \lambda', l, k)\). Thus the invariance of \(A_\lambda\) implies that of \(A_{\lambda^+}\).

\begin{thebibliography}{99}

[DJ] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, \textit{Exactly solvable SOS models: local height probabilities and theta function identities}, Nuclear Phys. B 290 (1987), 231–273.

[DG] R. Y. Denis and R. A. Gustafson, \textit{An SU(n) q-beta integral transformation and multiple hypergeometric series identities}, SIAM J. Math. Anal. 23 (1992), 552–561.

[D] J. F. van Diejen, \textit{On certain multiple Bailey, Rogers and Dougall type summation formulas}, Publ. Res. Inst. Math. Sci. 33 (1997), 483–508.

[DS] J. F. van Diejen and V. P. Spiridonov, \textit{An elliptic Macdonald-Morris conjecture and modular hypergeometric sums}, Math. Res. Lett., to appear.

[DSvD] J. F. van Diejen and J. V. Stokman, \textit{Multivariable q-Racah polynomials}, Duke Math. J. 91 (1998), 89–136.

[FT] I. B. Frenkel and V. G. Turaev, \textit{Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions}, The Arnold–Gelfand mathematical seminars, 171–204, Birkhäuser, Boston, MA, 1997.

[GR] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[G] R. A. Gustafson, \textit{Some q-beta integrals on SU(n) and Sp(n) that generalize the Askey-Wilson and Nasrallah-Rahman integrals}, SIAM J. Math. Anal. 25 (1994), 441–449.

[ML] S. C. Milne and G. M. Lilly, \textit{Consequences of the Ai and Ci Bailey transform and Bailey lemma}, Discrete Math. 139 (1995), 319–346.

\end{thebibliography}
[NR] B. Nassrallah and M. Rahman, Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials, SIAM J. Math. Anal. 16 (1985), 186–197.

[R] S. M. Ruijsenaars, First order analytic difference equations and integrable quantum systems, J. Math. Phys. 38 (1997), 1069–1146.

[S] M. Schlosser, Summation theorems for multidimensional basic hypergeometric series by determinant evaluations, Discrete Math. 210 (2000), 151–169.

[Sp] V. P. Spiridonov, On an elliptic beta function, Russian Math. Surveys, to appear.

[W] S. O. Warnaar, Summation and transformation formulas for elliptic hypergeometric series, \protect\url{http://arxiv.org/abs/math/0001006} (math.QA/0001006).

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