On the plus and the minus Selmer groups for elliptic curves at supersingular primes

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Abstract

Let \( p \) be an odd prime number, \( E \) an elliptic curve defined over a number field. Suppose that \( E \) has good reduction at any prime lying above \( p \), and has supersingular reduction at some prime lying above \( p \). In this paper, we construct the plus and the minus Selmer groups of \( E \) over the cyclotomic \( \mathbb{Z}_p \)-extension in a more general setting than that of B.D. Kim, and give a generalization of a result of B.D. Kim on the triviality of finite \( \Lambda \)-submodules of the Pontryagin duals of the plus and the minus Selmer groups, where \( \Lambda \) is the Iwasawa algebra of the Galois group of the \( \mathbb{Z}_p \)-extension.

1 Introduction

Let \( p \) be an odd prime number, \( F_0 \) a finite extension of \( \mathbb{Q} \), \( F_\infty/F_0 \) the cyclotomic \( \mathbb{Z}_p \)-extension and \( F_n \) the \( n \)-th layer. Denote \( \Lambda = \mathbb{Z}_p[[\text{Gal}(F_\infty/F_0)]] \). Let \( E \) be an elliptic curve defined over \( F_0 \).

When \( E \) has good ordinary reduction at any prime of \( F_0 \) lying above \( p \), the Pontryagin dual of the \( p \)-primary Selmer group of \( E \) over \( F_\infty \) is conjectured to be \( \Lambda \)-torsion. This conjecture is proved in several cases now. For example if the \( p \)-primary Selmer group of \( E \) over \( F_0 \) is finite, or if \( E \) is defined over \( \mathbb{Q} \) and \( F_0/\mathbb{Q} \) is abelian, then the conjecture is known to be true (cf. [2] p.55).

On the contrary, when \( E \) has good supersingular reduction at some prime of \( F_0 \) lying above \( p \), the Pontryagin dual of the \( p \)-primary Selmer group of \( E \) over \( F_\infty \) is no longer \( \Lambda \)-torsion. S. Kobayashi [11] defined the plus and the minus Selmer groups \( \text{Sel}^\pm(F_\infty, E[p^\infty]) \) when \( E \) is defined over \( \mathbb{Q} \), \( a_p = 1 + p - \#\tilde{E}(\mathbb{F}_p) = 0 \), and \( F_0 = \mathbb{Q}(\mu_p) \), where \( \tilde{E} \) denotes the reduction of \( E \) at \( p \), and \( \mu_p \) denotes the group of \( p \)-th roots of unity. He proved that the Pontryagin duals \( \text{Sel}^\pm(F_\infty, E[p^\infty])^\vee \) are \( \Lambda \)-torsion. A. Iovita and R. Pollack [6] generalized definitions of the plus and the minus Selmer groups to the case when \( F_0 \) is a
number field in which \( p \) splits completely, \( E \) is defined over \( \mathbb{Q} \) and \( a_p = 0 \). Further B.D. Kim \([7]\) generalized them to the case when \( F_0 \) is a number field in which \( p \) is unramified, \( E \) is defined over \( \mathbb{Q} \) and \( a_p = 0 \).

B.D. Kim proved in \([8]\) the following theorem on the triviality of finite \( \Lambda \)-submodules of \( \text{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee \).

**Theorem 1.1** (\([8]\) Theorem 1.1). Let \( F_0 \) be a finite extension of \( \mathbb{Q} \) in which \( p \) is unramified, \( E \) an elliptic curve defined over \( \mathbb{Q} \), and \( a_p = 0 \).

1. Assume that \( \text{Sel}^-(F_\infty, E[p^\infty])^\vee \) is \( \Lambda \)-torsion. Then \( \text{Sel}^-(F_\infty, E[p^\infty])^\vee \) has no nontrivial finite \( \Lambda \)-submodule.
2. Assume further that \( p \) splits completely in \( F_0 \), and \( \text{Sel}^+(F_\infty, E[p^\infty])^\vee \) is \( \Lambda \)-torsion. Then \( \text{Sel}^+(F_\infty, E[p^\infty])^\vee \) has no nontrivial finite \( \Lambda \)-submodule.

Throughout this paper, we assume that \( \text{Sel}^{\pm}(F_\infty, E[p^\infty])^\vee \) is \( \Lambda \)-torsion as in the above theorem. The following proposition on the local conditions of the plus and the minus Selmer groups was an important ingredient for Theorem 1.1.

**Proposition 1.2** (\([7]\) Proposition 3.13, \([8]\) Proposition 2.2 and Proposition 2.3). Let \( k_0 \) be a finite unramified extension of \( \mathbb{Q}_p \), and \( k_\infty \) the cyclotomic \( \mathbb{Z}_p \)-extension of \( k_0 \). We denote the completed group ring \( \mathbb{Z}_p[[\text{Gal}(k_\infty/k_0)]]) \) by \( \Lambda \).

1. We have
   \[
   (E^-(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \Lambda^@d
   \]
   where \( d = [k_0 : \mathbb{Q}_p] \).
2. If \( k_0 = \mathbb{Q}_p \), we have
   \[
   (E^+(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \Lambda.
   \]

When B.D. Kim studied the plus Selmer group in \([8]\), he restricted the base field \( F_0 \) as in Theorem 1.1 (2) to apply Proposition 1.2, and he got the result only in such a case. Even when he studied the minus Selmer group in \([8]\), the base field \( F_0 \) was a finite extension of \( \mathbb{Q} \) in which \( p \) is unramified. He did not consider the case when \( \mu_p \subset F_0 \).

In this paper, we consider more general fields for \( F_0 \) and \( k_0 \), and a more general elliptic curve \( E \), than those of the above known results. We do not restrict these base fields even when we study the plus Selmer group. We also note that we also consider the case when \( \mu_p \subset F_0 \). We get the following result.

**Main Theorem 1.3** (Theorem 4.8). Let \( F \) be a finite extension of \( \mathbb{Q} \), \( F_0 = F(\mu_p) \), \( F_\infty/F_0 \) the cyclotomic \( \mathbb{Z}_p \)-extension, and \( E \) an elliptic curve defined over a subfield \( F' \) of \( F \). Let \( S_p^\text{ss} \) be the set of all primes of \( F' \) lying above \( p \) where \( E \) has supersingular reduction. Assume the following conditions:

1. \( E \) has good reduction at any prime of \( F' \) lying above \( p \),
2. \( S_p^\text{ss} \) is nonempty,
3. any prime \( w \in S_p^\text{ss} \) is unramified in \( F \),
4. \( F'_w = \mathbb{Q}_p \) for any prime \( w \in S_p^\text{ss} \), where \( F'_w \) is the completion of \( F' \) at the prime \( w \),
(v) $a_w = 1 + p - \#\tilde{E}_w(F_p) = 0$ for any prime $w \in S_p^\infty$, where $\tilde{E}_w$ is the reduction of $E$ at $w$, and

(vi) both $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ are $\Lambda$-torsion.

Then both $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ have no nontrivial finite $\Lambda$-submodule.

Remark 1.4. (1) We assume the condition (i) since we expect the condition (vi) automatically holds true under this condition.

(2) In the case $S_p^\infty = \emptyset$, we have $\text{Sel}^\pm(F_\infty, E[p^\infty]) = \text{Sel}(F_\infty, E[p^\infty])$. Finite $\Lambda$-submodules of $\text{Sel}(F_\infty, E[p^\infty])^\vee$ was studied by Hachimori and Matsuno [3]. Thus we will be interested in the case (ii).

(3) The conditions (iii) and (iv) on the fields $F$ and $F'$ will be used in applying the local result discussed in Section 3.

(4) The condition (v) is crucial in this paper. In our method, it is important to study the local conditions $E^+(k_n)$ and $E^-(k_n)$, where $k_n$ is the $n$-th layer of the cyclotomic $\mathbb{Z}_p$-extension $k_\infty/k_0$ with $k_0$ a finite extension of $\mathbb{Q}_p$. In the case when $a_w \neq 0$ for some $w \in S_p^\infty$, one might need another submodule of $\text{Sel}(F_\infty, E[p^\infty])$ instead of $\text{Sel}^\pm(F_\infty, E[p^\infty])$. F. Sprung [16] defined $\text{Sel}^0(F_\infty, E[p^\infty])$ and $\text{Sel}^0(F_\infty, E[p^\infty])$ instead of the plus and the minus Selmer groups in the case when $F_0 = \mathbb{Q}(\mu_p)$, and $E$ is defined over $\mathbb{Q}$ which has supersingular reduction at $p$ without assuming $a_\mu = 0$. He defined $E^3(k_\infty)$ and $E^3(k_\infty)$ in the case when $k_0 = \mathbb{Q}_p(\mu_p)$, however, did not define $E^3(k_n)$ nor $E^3(k_n)$. We can not yet apply our method in the case when $a_w \neq 0$ for some $w$.

(5) In some cases, $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ is actually known to be $\Lambda$-torsion. For example, let $F_0$ be a finite abelian extension of $\mathbb{Q}$, and $F_\infty/F_0$ the cyclotomic $\mathbb{Z}_p$-extension. Suppose that $E$ is defined over $\mathbb{Q}$ and has supersingular reduction at $p$ with $a_p = 0$. In this case, one can actually show that $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ is $\Lambda$-torsion. Our main theorem implies that $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ has no nontrivial finite $\Lambda$-submodule for any finite abelian field $F_0$. On the other hand, we need a certain assumption on $F_0$ to apply B.D. Kim’s result.

For the proof of Theorem 1.3 we will generalize Proposition 1.2 to the case of our setting. In the study of $(E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$, we find that its $\Lambda$-module structure in our setting is generally different from those in the settings of previous works. We now explain some known results on the $\Lambda$-module structure of $(E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$.

Takeji [17] considered the case when $k_0$ is a quadratic unramified extension of $\mathbb{Q}_p$. He generalized Proposition 1.2 to this case, i.e. he proved that $(E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a free $\Lambda$-module of $\Lambda$-rank 2. Applying this result, he proved that $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ has no nontrivial finite $\Lambda$-submodule in the case when $F_0$ is a quadratic number field in which $p$ inerts, which is a generalization of Theorem 1.1.

M. Kim proved in his dissertation [10] that $E^\pm(k_n)$ are cyclic $\mathbb{Z}_p[\text{Gal}(k_n/\mathbb{Q}_p)]$-modules for all $n$, where $k_n$ is the $n$-th layer of the cyclotomic $\mathbb{Z}_p$-extension $k_\infty/k_0$, in the case when $k$ is a general finite unramified extension of $\mathbb{Q}_p$ and $k_0 = k(\mu_p)$, however, he did not notice that the assumption $[k : \mathbb{Q}_p] \not\equiv 0 \pmod{4}$ is needed. From this cyclicity, one can show that $(E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ is a free $\Lambda$-module of $\Lambda$-rank $[k_0 : \mathbb{Q}_p]$, which is a generalization of Proposition 1.2. B.D. Kim [9] independently generalized Proposition 1.2 to the case when $k_0$ is a finite unramified extension of $\mathbb{Q}_p$ and $[k_0 : \mathbb{Q}_p] \not\equiv 0 \pmod{4}$, i.e. he proved in
this case that \((E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) is a free \(\Lambda\)-module of \(\Lambda\)-rank \([k_0 : \mathbb{Q}_p]\) (cf. [9] Theorem 2.8). Applying this result, we can generalize Theorem 1.2 to the case when \(F_0\) is a finite extension of \(\mathbb{Q}\) in which \(p\) is unramified and \([F_{0,v} : \mathbb{Q}_p] \neq 0 \pmod{4}\) for all primes \(v\) of \(F_0\) lying above \(p\), where \(F_{0,v}\) is the completion of \(F_0\) at the prime \(v\).

We remark that we consider more general settings than those of all the above known results.

Here we prepare some notations of our settings and explain an obstruction for generalizing Proposition 1.5 to the case of our setting, which we have overcome in this paper. Let \(k\) be a finite unramified extension of \(\mathbb{Q}_p\) of degree \(d\), \(k_0 = k(\mu_p)\), \(k_\infty\) the cyclotomic \(\mathbb{Z}_p\)-extension of \(k_0\), \(k_n\) the \(n\)-th layer, \(\Delta = \text{Gal}(k(\mu_p)/k)\), \(\Gamma = \text{Gal}(k_\infty/k_0)\), \(\Gamma_n = \text{Gal}(k_\infty/k_n)\), \(G_n = \text{Gal}(k_n/\mathbb{Q}_p)\) and \(\Lambda = \mathbb{Z}_p[[\Gamma]]\).

An essential property, expected in all previous works [7], [8], [9], [10], and [17] was that \((E^\pm(k_n))\) are cyclic \(\mathbb{Z}_p[G_n]\)-modules for all \(n\). From this expected property, we can show that \((E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) is a free \(\Lambda\)-module of \(\Lambda\)-rank \([k_0 : \mathbb{Q}_p]\). On the contrary to this expectation, we find that \((E^+(k_\infty))\) are not cyclic \(\mathbb{Z}_p[G_n]\)-modules when \(d \equiv 0 \pmod{4}\) (cf. Proposition 3.16 and Remark 3.17). An obstruction is that this non-cyclicity makes \((E^+(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) more complicated. In fact, we find that \((E^+(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) is not a free \(\Lambda\)-module in the case when \(d \equiv 0 \pmod{4}\) (cf. Remark 3.26). Therefore, the same statement with the conclusion of Proposition 1.5 does not hold in the general setting.

A crucial step for the proof of our main theorem is to investigate the \(\Lambda\)-module structure of \((E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) more precisely. We prove the following proposition on the local conditions of the plus and the minus Selmer groups, which is a generalization of Proposition 1.5 and an important ingredient for our main theorem.

**Proposition 1.5** (Proposition 3.28). \((E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) has no nontrivial finite \(\Lambda\)-submodule and its \(\Lambda\)-rank is \([k_0 : \mathbb{Q}_p]\).

We prove this proposition by calculating the \(\Gamma_n\)-coinvariants of the \(\chi\)-component \((E^\pm(k_\infty)^\vee \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) for all \(n\), where \(\chi : \Delta \to \mathbb{Z}_p^\times\) is a character of \(\Delta\). The original idea for such calculations, to calculate the \(\Gamma\)-coinvariants and to apply Nakayama’s lemma, was suggested by M. Kurihara. The authors are very grateful to him.

As a consequence of Proposition 1.5 we prove the following proposition which is also an important ingredient for our main theorem.

**Proposition 1.6** (Proposition 3.32). We have

\[
\left( H^1(k_\infty, E[p^\infty]) \right)^\vee \cong \Lambda^{[k_0 : \mathbb{Q}_p]}.
\]

In comparison with the strategy for the proof of Theorem 1.1 we develop another strategy also for the proof of our main theorem. Our strategy is to prove that the triviality of finite \(\Lambda\)-submodules of \(\text{Sel}(F_\infty, E[p^\infty])^\vee\) is inherited to that of \(\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee\), and use the following theorem.

**Theorem 1.7** (Theorem 4.5). Assume that both \(\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee\) are \(\Lambda\)-torsion. Then \(\text{Sel}(F_\infty, E[p^\infty])^\vee\) has no nontrivial finite \(\Lambda\)-submodule.
The above discussion is enough to prove our main theorem, however, we can determine the explicit structure of the $\Lambda$-module $(E^\pm(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)\vee$, on which we explain here. We prepare some notations concerning the character decomposition. Let $\chi: \Delta \to \mathbb{Z}^\times_\mathbb{p}$ be a character of $\Delta = \text{Gal}(k(\mu_\mathbb{p})/k)$. If $M$ is a $\mathbb{Z}_p[\Delta]$-module, then $M$ is decomposed into

$$M = \bigoplus_\chi \varepsilon_\chi M,$$

where $\varepsilon_\chi = \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \chi(\sigma)\sigma^{-1} \in \mathbb{Z}_p[\Delta]$. We denote by $M^\chi$ the $\chi$-component $\varepsilon_\chi M$. We fix a topological generator $\gamma \in \Gamma$, and identify $\mathbb{Z}_p[[\Gamma]]$ with the ring of formal power series $\mathbb{Z}_p[[X]]$ by identifying $\gamma$ with $1 + X$. We get the following theorem.

**Theorem 1.8 (Theorem 3.33).** Let $\chi: \Delta \to \mathbb{Z}_\mathbb{p}^\times$ be a character. We have

$$(E^+(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \Lambda^{\otimes d} \oplus (\Lambda/X)^{\otimes \delta},$$

$$(E^-(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \Lambda^{\otimes d}$$

where

$$\delta = \begin{cases} 0 & \text{if } d \neq 0 \text{ (mod 4)} \text{ or } \chi \neq 1, \\ 2 & \text{otherwise.} \end{cases}$$

The outline of this paper is as follows. In Section 2, we define the plus and the minus Selmer groups following Kobayashi, and fix a global setting. In Section 3, we study the local conditions in a local setting. Subsection 3.1 is a preparation for the rest of Section 3. In Subsection 3.2, we give a description of $E^\pm(k_\infty)^\chi$ in terms of formal groups and a system of local points. In Subsection 3.3, we study the $\Lambda$-modules $(E^\pm(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ and $(H^1(k_\infty, E[p_\infty])/(E^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p))^\vee$. In Subsection 3.4, we further determine the explicit structure of the $\Lambda$-module $(E^\pm(k_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. In Section 4, we study finite $\Lambda$-submodules of the Pontryagin duals of the Selmer groups $\text{Sel}(F_\infty, E[p_\infty])^\vee$ and $\text{Sel}^\pm(F_\infty, E[p_\infty])^\vee$. In Subsection 4.1, we prove that the usual $p$-primary Selmer group has no nontrivial finite $\Lambda$-submodule under the same assumption with the main theorem. In this step, it is essential to assume that both $\text{Sel}^\pm(F_\infty, E[p_\infty])^\vee$ are $\Lambda$-torsion. In Subsection 4.2, we prove our main theorem.

## 2 The plus and the minus Selmer groups

Let $p$ be a prime number, $F$ a finite extension of $\mathbb{Q}$, and $E$ an elliptic curve defined over $F$. For a finite extension $K/F$, the $p$-primary Selmer group for $E$ over $K$ is defined by

$$\text{Sel}(K, E[p_\infty]) := \text{Ker} \left( H^1(K, E[p_\infty]) \rightarrow \prod_v H^1(K_v, E[p_\infty]) \right),$$

where $v$ runs through all places of $K$, $K_v$ is the completion of $K$ at the place $v$, and $E(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is regarded as a subgroup of $H^1(K_v, E[p_\infty])$ by the
Kummer map. For a number field $K$ that is an infinite extension of $F$, we define the $p$-primary Selmer group for $E$ over $K$ by

$$\text{Sel}(K, E[p^\infty]) := \lim_{\to} \text{Sel}(K', E[p^\infty]),$$

where $K'$ runs through all the subfields of $K$ which are finite extensions of $F$, and transition maps are restriction maps between cohomology groups.

We denote $F_n = F(\mu_{p^n+1})$, $F_{-1} = F$ and $F_{\infty} = \bigcup_n F_n$, where $\mu_{p^n}$ denotes the group of $p^n$-th roots of unity. We fix a generator $(\zeta_{p^n})$ of $\mathbb{Z}_{p^n}(1)$, namely, for each $n \geq 0$, $\zeta_{p^n}$ is a primitive $p^n$-th root of unity such that $\zeta_{p^n+1} = \zeta_{p^n}$.

Then by definition, we have

$$\text{Sel}(F_{\infty}, E[p^\infty]) = \lim_{\to n} \text{Sel}(F_n, E[p^\infty]).$$

Throughout this paper, we fix the following notations:

- $p$ is an odd prime number,
- $F$ is a finite extension of $\mathbb{Q}$,
- $E$ is an elliptic curve defined over a subfield $F'$ of $F$.

Denote $S_p^{ss}$ the set of all primes of $F'$ lying above $p$ where $E$ has supersingular reduction.

Throughout this paper, we assume the following:

- $E$ has good reduction at any prime $w|p$ of $F'$,
- $S_p^{ss}$ is nonempty,
- any prime $w \in S_p^{ss}$ is unramified in $F$,
- $F'_w = \mathbb{Q}_p$ for any prime $w \in S_p^{ss}$, where $F'_w$ is the completion of $F'$ at the prime $w$, and
- $a_w = 1 + p - \#\bar{E}_w(F_p) = 0$ for any prime $w \in S_p^{ss}$, where $\bar{E}_w$ is the reduction of $E$ at $w$.

When $p \geq 5$, the condition $a_w = 0$ is automatically satisfied since we have $p|a_w$ and $|a_w| \leq 2\sqrt{p}$.

Denote $S_p^{ss,F}$ the set of all primes of $F$ lying above $S_p^{ss}$.

Following S. Kobayashi \[11\] we define subgroups $E^+(F_{n,v})$ and $E^-(F_{n,v})$ of $E(F_{n,v})$ for each prime $v \in S_p^{ss,F}$, and define plus and minus Selmer groups $\text{Sel}^+(F_n, E[p^\infty])$, $\text{Sel}^-(F_{\infty}, E[p^\infty])$ as the following (see also \[7\] and \[10\]).

**Definition 2.1.** (1) For a prime $v \in S_p^{ss,F}$ and $n \geq -1$, let $F_{n,v}$ be the completion of $F_n$ at the unique prime of $F_n$ lying above $v$. We define

$$E^+(F_{n,v}) = \{ P \in E(F_{n,v}) | \text{Tr}_{n/m+1} P \in E(F_{m,v}) \text{ for all even } m, -1 \leq m \leq n - 1 \},$$

$$E^-(F_{n,v}) = \{ P \in E(F_{n,v}) | \text{Tr}_{n/m+1} P \in E(F_{m,v}) \text{ for all odd } m, -1 \leq m \leq n - 1 \}$$

where $\text{Tr}_{n/m+1} : E(F_{n,v}) \to E(F_{m+1,v})$ is the trace map.
(2) The plus and the minus Selmer groups are defined by

\[
\text{Sel}^+(F_n, E[p^\infty]) := \text{Ker} \left( \text{Sel}(F_n, E[p^\infty]) \rightarrow \bigoplus_{v \in S_{p,F}} \frac{H^1(F_{n,v}, E[p^\infty])}{E^\infty(F_{n,v}) \otimes \mathbb{Q}_p / \mathbb{Z}_p} \right),
\]

\[
\text{Sel}^+(F_\infty, E[p^\infty]) := \lim_n \text{Sel}^+(F_n, E[p^\infty]).
\]

We denote the Pontryagin dual of a module \( M \) by \( M^\vee \). Let \( G_n = \text{Gal}(F_n/F) \) and \( G_\infty = \text{Gal}(F_\infty/F) \). Then \( \mathbb{Z}_p[G_n] \) acts naturally on \( \text{Sel}^+(F_n, E[p^\infty])^\vee \) and \( \Lambda(G_\infty) := \mathbb{Z}_p[[G_\infty]] \) on \( \text{Sel}^+(F_\infty, E[p^\infty])^\vee \).

The Pontryagin dual of the usual \( p \)-primary Selmer group \( \text{Sel}(F_\infty, E[p^\infty])^\vee \) is not a torsion \( \Lambda(G_\infty) \)-module, however, \( \text{Sel}^+(F_\infty, E[p^\infty])^\vee \) is known to be \( \Lambda(G_\infty) \)-torsion in the case \( F = \mathbb{Q} \) (cf. [11] Theorem 2.2).

3 The formal groups and the norm subgroups

Let \( E/\mathbb{Q}_p \) be an elliptic curve with \( a_p = 0 \) and \( \hat{E} \) the formal group over \( \mathbb{Z}_p \) associated with the minimal model of \( E \) over \( \mathbb{Q}_p \). Let \( k \) be a finite unramified extension of \( \mathbb{Q}_p \) of degree \( d = [k: \mathbb{Q}_p] \) and \( \mathcal{O}_k \) the ring of integers of \( k \). For each \( n \geq -1 \), let \( k_n = k(\mu_{p^n+1}) \) and \( m_n \) be the maximal ideal of \( k_n \). Let \( k_\infty = \bigcup_{n \geq -1} k_n \) and \( m_\infty = \bigcup_{n \geq -1} m_n \). Let \( G_n = \text{Gal}(k_n/\mathbb{Q}_p) \), \( G_\infty = \text{Gal}(k_\infty/\mathbb{Q}_p) \), \( \Gamma_n = \text{Gal}(k_\infty/k_n) \), \( \Gamma = \Gamma_0(= \text{Gal}(k_\infty/k_0)) \) and \( \Delta = \text{Gal}(k(k_\infty)/k) = \text{Gal}(k_0/k_{-1}) \).

Let \( \varphi \) be the Frobenius homomorphism in \( \text{Gal}(k/\mathbb{Q}_p) = G_{-1} \) characterized by \( x^p \equiv x \pmod{p \mathcal{O}_k} \). We denote \( \Lambda = \mathbb{Z}_p[[\Gamma]] \). We fix a topological generator \( \gamma \in \Gamma \). Then we identify \( \mathbb{Z}_p[[\Gamma]] \) with \( \mathbb{Z}_p[[X]] \), and \( \mathbb{Z}_p[[G_\infty]] \) with \( \mathbb{Z}_p[\mathcal{O}_0][[X]] \) by identifying \( \gamma \) with \( 1 + X \).

3.1 The formal groups associated to \( E \)

**Proposition 3.1.** For any \( n \), \( \hat{E}(m_n) \) is \( \mathbb{Z}_p \)-torsion-free.

**Proof.** We can prove this by the same method as the proof of [11] Proposition 8.7. \( \square \)

The above proposition implies that the formal logarithm \( \log_{\hat{E}}(X) \) induces an injective homomorphism \( \log_{\hat{E}} : \hat{E}(m_n) \rightarrow k_n \) for all \( n \), since the kernel of the logarithm of a formal group precisely consists of the elements of finite order.

For such a one-dimensional formal group \( \mathcal{F} \) defined over \( \mathbb{Z}_p \) with height 2, the formal logarithm \( \log_{\mathcal{F}} \) induces isomorphisms as in the following proposition (cf. The proof of Proposition 2.1 in [13], and Lemma 2.4 in [4]).

**Proposition 3.2.** Let \( \mathcal{F} \) be a one-dimensional formal group defined over \( \mathbb{Z}_p \) with height 2. For a finite extension \( K/\mathbb{Q}_p \), denote by \( m_K \) its maximal ideal. Then the logarithm \( \log_{\mathcal{F}} : \mathcal{F}(m_K) \rightarrow K \) induces isomorphisms

\[
\log_{\mathcal{F}} : \mathcal{F}(m_K^j) \rightarrow m_K^j
\]

for all \( j > v_K(p)/(p^2 - 1) \), where \( v_K \) is the normalized valuation of \( K \) so that \( v_K(\pi_K) = 1 \) for a uniformizer \( \pi_K \) of \( K \).
Following [7] and [10] we construct a system of local points \((d_n)_n\).

Fix a generator \(\zeta\) of the group of roots of unity in \(k\). Then \(\zeta\) is a primitive \((p^d - 1)\)-th root of unity, and we have \(k = \mathbb{Q}_p(\zeta)\).

Let \(g(X) = (X + \zeta)^p - \zeta^p \in \mathcal{O}_k[X], g^{(m)}(X) = g^{p^m-1} \circ g^{p^{m-2}} \circ \cdots \circ g(X) = (X + \zeta)^{p^m} - \zeta^{p^m}\) for \(m \geq 1\) and \(g^{(0)}(X) = X\). We define a formal power series \(\log_{\varphi}(X)\) by

\[
\log_{\varphi}(X) = \sum_{m=0}^{\infty} \frac{(-1)^m g^{(2m)}(X)}{p^m}.
\]

We can check that

\[
(\log_{\varphi}^{- (n + 1)}) = (X^{p^2}) + p \log_{\varphi}^{- (n + 1)}(X) \equiv 0 \pmod{p}
\]

and \((\log_{\varphi}^{- (n + 1)})'(X) \in \mathcal{O}_k[[X]]\) for each \(n\). This means that \(\log_{\varphi}^{- (n + 1)}(X)\) is of the Honda type \(t^2 + p\) for each \(n\). Hence by Honda theory (cf. [5] we see that

- there is a formal group \(G_n\) defined over \(\mathcal{O}_k\) whose formal logarithm \(\log_{G_n}\) is given by \(\log_{\varphi}^{- (n + 1)}\) for each \(n\), and
- the power series \(\exp_{\mathcal{O}_k} \circ \log_{G_n}\) is contained in \(\mathcal{O}_k[[X]]\) and gives an isomorphism \(G_n \to \mathcal{E}\) over \(\mathcal{O}_k\) for each \(n\).

We fix a generator \((\zeta_{p^n})\) of \(\mathbb{Z}_p(1)\), namely, for each \(n \geq 0\), \(\zeta_{p^n}\) is a primitive \(p^n\)-th root of unity such that \(\zeta_{p^{n+1}} = \zeta_{p^n}\). Let \(\pi_n = \zeta_{p^{- (n + 1)}}(\zeta_{p^{n+1}} - 1) \in m_n\) for \(n \geq -1\) and \(\pi_n = 0\) for \(n < -1\). For each \(n\), we can easily show that

\[
g^{(m), p^{- (n + 1)}}(\pi_n) = \pi_{n-m} \tag{3.1}
\]

for any \(m \geq 0\) by direct calculation.

Put

\[
\epsilon_n = \zeta_{p^{- (n + 3)}} - \zeta_{p^{- (n + 5)}} p^2 + \zeta_{p^{- (n + 7)}} p^4 - \cdots
\]

\[
= \sum_{i=1}^{\infty} (-1)^{i-1} \zeta_{p^{-(n+2i)}} p^i \in m_k
\]

for \(n \geq -1\). Since \(\log_{G_n} : G_n(m_k) \to m_k\) is an isomorphism for all \(n\) (cf. Proposition 3.2), there is \(\epsilon_n \in G_n(m_k)\) such that \(\log_{G_n}(\epsilon_n) = \epsilon_n\) for \(n \geq -1\).

**Definition 3.3.** We define

\[
d_n = \exp_{\mathcal{O}_k} \circ \log_{G_n}(\epsilon_n[+])_{G_n} \pi_n
\]

for \(n \geq -1\), where \([+]_{G_n}\) is the addition of \(G_n\).

For \(n \geq m\), we denote by \(\text{Tr}_{n/m} : E(m_n) \to E(m_m)\) the trace (norm) with respect to the group-law \(E(X, Y)\).

**Proposition 3.4.** The system of local points \((d_n)_n \in \prod_{n \geq -1} E(m_n)\) satisfies

1. \(\text{Tr}_{n/m}(d_n) = -d_{n-2}\) for each \(n \geq 1\),
2. \(\text{Tr}_{0/-1}(d_0) = -(\varphi + \varphi^{-1})d_{-1}\).
Proof. We prove this by the same method as the proof of Lemma 8.9 in [11]. Since $\log \hat{E}$ is injective (cf. Proposition 3.1) and commute with the action of $G_n$ on $\hat{E}(m_n)$, it is enough to show that the relation holds after applying $\log \hat{E}$ to both sides of the equality.

We have

$$\log \hat{E}(d_n) = \log g_n(\epsilon_n + \pi_n) = \log g_n(\epsilon_n) + \log g_n(\pi_n) = \epsilon_n + \sum_{m=0}^{\infty} (-1)^m \frac{\pi_{n-2m}}{p^m}.$$  

Here the last equality follows from (3.1) and $\pi_n = 0$ for $n \leq -1$.

For $n \geq 1$, we have

$$\text{Tr}_{n/n-1} \log \hat{E}(d_n) = p\epsilon_n - \zeta^\phi^{-m+1} p + \sum_{m=1}^{n-1} (-1)^m \frac{\pi_{n-2m}}{p^{m-1}} = -\epsilon_{n-2} - \sum_{m=0}^{\infty} (-1)^m \frac{\pi_{n-2-2m}}{p^m}$$

For $n = 0$, we have

$$\text{Tr}_{0/-1} \log \hat{E}(d_0) = (p-1)\epsilon_0 - \zeta^\phi^{-1} p = -(\phi + \phi^{-1})\epsilon_0$$

Remark 3.5. As long as we define local points as values of certain power series at certain points, the factor $\phi + \phi^{-1}$ in the condition (2) always appears (cf. [12] Proposition 3.10, (3.3)). Although B.D. Kim did not mention explicitly in [7], [8], this factor $\phi + \phi^{-1}$ was an obstruction. He assumed in [7] and [8] that $k = Q_p$ when he considered the plus Selmer groups in order to make the situation simpler, i.e. $\phi + \phi^{-1} = 2$ in $Z_p[G_{-1}] = Z_p$. In this paper, we consider general unramified extension $k/Q_p$, carefully taking into account this factor $\phi + \phi^{-1}$ in $Z_p[G_{-1}]$.

Lemma 3.6. $\phi + \phi^{-1}$ is a unit in $Z_p[G_{-1}]$ if and only if $d \not\equiv 0$ (mod 4).

Proof. First we note that $\phi + \phi^{-1} \in Z_p[G_{-1}]$ if and only if $1 + \phi^2 \in Z_p[G_{-1}]$. If $d$ is odd, we have

$$(1 + \phi^2)(1 - \phi^2 + \phi^4 - \cdots + (-1)^{d-2} \phi^{2d-2}) = 1 + (-1)^{d-1} = 2.$$  

If $d$ is even, we have

$$(1 + \phi^2)(1 - \phi^2 + \phi^4 - \cdots + (-1)^{d-2} \phi^{d-2}) = 1 + (-1)^{d-2} = \begin{cases} 2 & \text{if } d \not\equiv 0 \pmod{4}, \\ 0 & \text{if } d \equiv 0 \pmod{4}, \end{cases}$$
and $1 - \varphi^2 + \varphi^4 - \cdots + (-1)^{d-2} \varphi^{d-2} \neq 0$ in $\mathbb{Z}_p[G_{-1}]$. Since 2 is invertible in $\mathbb{Z}_p[G_{-1}]$, we get the conclusion of Lemma 3.6. \hfill $\square$

**Remark 3.7.** In the proof of Lemma 3.6, we have proved the following; (1) $\varphi + \varphi^{-1}$ is a unit if $d \not\equiv 0 \pmod{4}$, (2) $\varphi + \varphi^{-1}$ is a zero-divisor if $d \equiv 0 \pmod{4}$.

Moreover, we can easily check the following lemma.

**Lemma 3.8.** We have
$$\text{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1})$$

$$= \left\{ \begin{array}{ll} 0 & \text{if } d \not\equiv 0 \pmod{4}, \\
(1 - \varphi^2 + \varphi^4 - \cdots - \varphi^{d-2} \mathbb{Z}_p[G_{-1}] & \text{if } d \equiv 0 \pmod{4}. 
\end{array} \right.$$  

and $\text{rank}_{\mathbb{Z}_p} \text{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1}) = 2$ if $d \equiv 0 \pmod{4}$.

To describe the quotient modules $\hat{E}(m_n)/\hat{E}(m_{n-1})$ using the local points $(d_n)_n$, we prepare the following lemma.

**Lemma 3.9.** We have $m_n/m_{n-1} = \langle \pi_n \rangle_{\mathbb{Z}_p[G_n]}$ for $n \geq 0$.

**Proof.** It is enough to show that $\zeta(\zeta_{p^{n+1}} - 1)$ generates $m_n/m_{n-1}$ as a $\mathbb{Z}_p[G_n]$-module.

We first observe the ring of integers $O_{k_n}$ of $k_n$. Let $P_m = \{ (\zeta_{p^m} - 1)^\tau | \tau \in \text{Gal}(k_m/k) \}$ for $m \geq 1$ and $P_1 = \{1\}$. Since $O_{k_n} = O_k[\zeta_{p^{n+1}}]$, we have
$$O_{k_n} = (P_0 \cup P_1 \cup \cdots \cup P_{n+1})O_k.$$  

Thus, for $x \in O_{k_n}$, we can write $x = a_0 + \sum_{m=1}^{n+1} \sum_{\tau \in \text{Gal}(k_m/k)} a_{m,\tau}(\zeta_{p^m} - 1)^\tau$ with $a_0, a_{m,\tau} \in O_k$. With this notation, since each $(\zeta_{p^m} - 1)^\tau$ already has positive valuation, we see that $x \in m_n$ if and only if $a_0 \in m_k$.

Take any class in $m_n/m_{n-1}$ with a representative $x \in m_n$. Write $x = a_0 + \sum_{m=1}^{n+1} \sum_{\tau \in \text{Gal}(k_m/k)} a_{m,\tau}(\zeta_{p^m} - 1)^\tau$. In this summation, the summands $a_0$ and $a_{m,\tau}(\zeta_{p^m} - 1)^\tau$ with $1 \leq m \leq n$ are contained in $m_{n-1}$. Since $O_k = \mathbb{Z}_p[\zeta] = (\zeta)_{\mathbb{Z}_p[G_{-1}]}$, each $a_{m+1,\tau} \in O_k$ can be written as $a_{m+1,\tau} = \sum_{i=0}^{d-1} b_{\tau,i} \zeta^{\iota^i}$ with $b_{\tau,i} \in \mathbb{Z}_p$. Therefore we have
$$x \equiv \sum_{\tau \in \text{Gal}(k_{n+1}/k)} a_{n+1,\tau}(\zeta_{p^{n+1}} - 1)^\tau$$
$$\equiv \sum_{\tau \in \text{Gal}(k_{n+1}/k)} \sum_{i=0}^{d-1} b_{\tau,i} \zeta^{\iota^i}(\zeta_{p^{n+1}} - 1)^\tau \pmod{m_{n-1}}.$$  

Here, $(\zeta_{p^{n+1}} - 1)^\tau$ for each $i$ with $0 \leq i \leq d-1$, and each $\tau \in \text{Gal}(k_{n+1}/k)$, is exactly a Galois conjugate of $(\zeta_{p^{n+1}} - 1)$ by $G_n = G_{-1} \times \text{Gal}(k_{n+1}/k)$. This completes the proof. \hfill $\square$

**Proposition 3.10.** For $n \geq 0$, we have $\log_E(\hat{E}(m_n)) \subseteq m_n + k_{n-1}$ and the formal logarithm $\log_E$ induces canonical isomorphisms of $\mathbb{Z}_p[G_n]$-modules,
$$\hat{E}(m_n)/\hat{E}(m_{n-1}) \xrightarrow{\sim} m_n/m_{n-1}.$$
By these isomorphisms, \( d_n \) is sent to \( \pi_n \). In particular, we have

\[
\hat{E}(m_n)/\hat{E}(m_{n-1}) = \langle d_n \rangle_{\mathbb{Z}_p[G_n]}.
\]

**Proof.** We prove this by the same method as the proof of Proposition 8.11 in [11] or Proposition 4.9 in [6].

For the first statement, we only note that by the commutative diagram

\[
\begin{array}{ccc}
\hat{E}(m_n) & \xrightarrow{\log_E} & k_n, \\
\exp_{\mathcal{G}_1} \circ \log_E & \cong & \log_{\mathcal{G}_1}
\end{array}
\]

it is enough to consider \( \log_{\mathcal{G}_1} (= \log_E) \) on \( \mathcal{G}_1(m_n) \) instead of \( \hat{E}(m_n) \).

Then we can show that \( \log_{\mathcal{G}_1}(\mathcal{G}_1(m_n)) \subseteq m_n + k_{n-1} \) as in [11], [6].

Since we have \( \log_E(m_n) \cap k_{n-1} = \log_E(m_n-1) \), the natural map

\[
\hat{E}(m_n)/\hat{E}(m_{n-1}) \to (m_n + k_{n-1})/{k_{n-1}} \cong m_n/m_{n-1}
\]

is injective. By direct calculation, we have \( \log_E(d_n) = \pi_n \) (mod \( k_{n-1} \)). Since \( \pi_n \) generates \( m_n/m_{n-1} \) as a \( \mathbb{Z}_p[G_n] \)-module (cf. Lemma 3.9), the above injection is in fact a bijection.

**Corollary 3.11.** We have

\[
\hat{E}(m_n) = \begin{cases} 
\langle d_{-1} \rangle_{\mathbb{Z}_p[G_{-1}]} & \text{if } n = -1, \\
\langle d_n, d_n-1 \rangle_{\mathbb{Z}_p[G_n]} & \text{if } n \geq 0.
\end{cases}
\]

**Proof.** The case \( n = -1 \) follows from \( \hat{E}(m_{-1}) \cong m_{-1} \) (see Proposition 3.2) and Nakayama’s lemma. The case \( n \geq 0 \) follows easily from Proposition 3.10 and the trace relations satisfied by the \( d_n \) (see Proposition 3.4).

**Remark 3.12.** We defined the system of local points \( (d_n)_n \) following B.D. Kim [7] and M. Kim [10] in the above. We can take another system of local points instead of \( (d_n)_n \). Indeed, what we need for the following discussion is a system of local points \( (d_n)_n \) which satisfies the following three conditions

1. \( \text{Tr}_{n-1}(d_n) = -d_{n-2} \) for each \( n \geq 1 \) (Proposition 3.4 (1)),
2. \( \text{Tr}_{0-1}(d_0) = -(\varphi + \varphi^{-1})d_{-1} \) (Proposition 3.4 (2)),
3. \( \hat{E}(m_n)/\hat{E}(m_{n-1}) = \langle d_n \rangle_{\mathbb{Z}_p[G_n]} \) (Proposition 3.10).

Such a system \( (d_n)_n \) obviously admits at least a difference of multiplication by a unit in \( \mathbb{Z}_p[G_{-1}]^\times \). S. Kobayashi constructed such a system of local points also in [12].

Proof of Proposition 3.12 by using another formal power series \( \ell_{\varepsilon}(X) \) and a system \( (\zeta_{\rho + 1} - 1)_n \) instead of \( \log_{\mathcal{G}_E}(X) \) and a system \( (\zeta_n[+]_{\mathcal{G}_E})_{\pi_n})_n \).

In our setting, the formal power series \( \ell_{\varepsilon}(X) \) is defined for each \( \varepsilon \in \hat{E}(m_k) \) by

\[
\ell_{\varepsilon}(X) = \varepsilon + \sum_{m=0}^{\infty} (-1)^m \frac{\ell(2m)(\varepsilon X)}{p^m} \in k[[X]].
\]
Lemma 3.14. For our purpose, isomorphisms \( n \) are satisfied. We also note that we can take such \( \epsilon \) in \( m_k \), since \( m_k \) is known to be a cyclic \( \mathbb{Z}_p[G_n] \)-module.

3.2 The norm subgroups

Following S. Kobayashi \[11\] (and M. Kim \[10\]), we define the \( n \)-th plus subgroup \( \hat{E}^+(m_n) \), the \( n \)-th minus subgroup \( \hat{E}^-(m_n) \) and the \( n \)-th norm subgroup \( \mathcal{C}(m_n) \) of \( \hat{E}(m_n) \):

**Definition 3.13.** We define

\[
\begin{align*}
\hat{E}^+(m_n) &= \{ P \in \hat{E}(m_n) | \text{Tr}_{n/m+1} P \in \hat{E}(m_n) \text{ for all even } m, -1 \leq m \leq n - 1 \}, \\
\hat{E}^-(m_n) &= \{ P \in \hat{E}(m_n) | \text{Tr}_{n/m+1} P \in \hat{E}(m_n) \text{ for all odd } m, -1 \leq m \leq n - 1 \},
\end{align*}
\]

for \( n \geq 0 \). We denote \( \hat{E}^\pm(m_{\infty}) = \bigcup_n \hat{E}^\pm(m_n) \). We also define

\[
\mathcal{C}(m_n) = \{ P \in \hat{E}(m_n) | \text{Tr}_{n/m+1} P \in \hat{E}(m_n) \text{ for all } m \equiv n \text{ (mod 2)}, -1 \leq m \leq n - 1 \}
\]

for \( n \geq 0 \) and \( \mathcal{C}(m_{-1}) = \hat{E}(m_{-1}) \).

By the following lemma, it is enough to study \( \hat{E}^\pm(m_n) \) instead of \( E^\pm(k_n) \) for our purpose.

**Lemma 3.14.** We can show that the natural maps \( \hat{E}^\pm(m_n) \to E^\pm(k_n) \) induce isomorphisms \( \hat{E}^\pm(m_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong E^\pm(k_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \) for all \( n \), and thus we have

\[
\hat{E}^\pm(m_{\infty}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong E^\pm(k_{\infty}) \otimes \mathbb{Q}_p/\mathbb{Z}_p.
\]

**Proof.** Indeed, we consider the following commutative diagrams

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \hat{E}(m_n) & \longrightarrow & E(k_n) & \longrightarrow & \hat{E}(F_k) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hat{E}^\pm(m_n) & \longrightarrow & E^\pm(k_n) & \longrightarrow & A^\pm_n & \longrightarrow & 0
\end{array}
\]

where \( \hat{E} \) is the reduction of \( E \) modulo \( p \), \( F_k \) is the residue field of \( k \) and \( A^\pm_n \) is the cokernel of \( \hat{E}^\pm(m_n) \to E^\pm(k_n) \). Since \( \hat{E}^\pm(m_n) = \hat{E}(m_n) \cap E^\pm(k_n) \), we see that the right vertical arrows \( \iota^\pm_n \) are injective. Thus \( A^\pm_n \) are finite as \( \hat{E}(F_k) \) is finite. We also note that \( A^\pm_n[p^\infty] = 0 \), since \( E/\mathbb{Q}_p \) has supersingular reduction. From the above, our claim will follow immediately.

By comparing two definitions, we get the following relations between the plus subgroups (the minus subgroups) and the norm subgroups.

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Lemma 3.15. We have

\[
\hat{E}^+(m_n) = \begin{cases} C(m_n) & \text{if } n \text{ is even}, \\ \bC(m_{n-1}) & \text{if } n \text{ is odd}, \end{cases}
\]

\[
\hat{E}^-(m_n) = \begin{cases} \bC(m_n) & \text{if } n \text{ is odd}, \\ \bC(m_{n-1}) & \text{if } n \text{ is even}. \end{cases}
\]

We now describe \(\bC(m_n)\) in terms of the system of local points \((d_n)_n\), and thus we get a description of plus and minus subgroups \(\hat{E}^\pm(m_n)\).

Proposition 3.16. (1) For each \(n \geq -1\), the \(n\)-th norm subgroup is generated by \(d_n\) and \(d_{-1}\) as a \(\bZ_p[\bG_n]\)-module;

\[
\bC(m_n) = \langle d_n, d_{-1} \rangle_{\bZ_p[\bG_n]}.
\]

(2) For each \(n \geq 0\), we have an exact sequence

\[
0 \longrightarrow \hat{E}(m_{-1}) \longrightarrow \bC(m_n) \oplus \bC(m_{n-1}) \longrightarrow \hat{E}(m_n) \longrightarrow 0,
\]

where the first map is diagonal embedding by inclusions, and the second map is \((a, b) \mapsto a - b\).

Proof. We will prove this by the same method as the proof of Proposition 8.12 in [11]. The main difference is the element \(d_{-1}\) in the first statement.

We first show that \(\bC(m_n) \cap \bC(m_{n-1}) = \hat{E}(m_{-1})\) for \(n \geq 0\). It is clear that \(\bC(m_n) \cap \bC(m_{n-1}) \supseteq \hat{E}(m_{-1})\) by definition. Let \(P \in \bC(m_n) \cap \bC(m_{n-1})\). We show that if \(P \in \hat{E}(m_{m_n})\) for some \(m \geq 0\) then \(P \in \hat{E}(m_{m_{n-1}})\). In the case \(m \equiv n \pmod 2\), we have \(p^{m-n}P = \Tr_{m/n}(P) \in \hat{E}(m_{m_{n-1}})\) by the definition of \(\bC(m_{m_n})\). Therefore for \(\sigma \in \Gal(k_m/k_{m-1})\), we have \(p^{m-n}(P^\sigma - P) = 0\). Hence, by Proposition 8.11 \(P \in \hat{E}(m_{m_{n-1}})\). In the case \(m \equiv n - 1 \pmod 2\), our claim is shown similarly by considering \(\bC(m_n)\) instead of \(\bC(m_{m_n})\) in the above argument. The other inclusion is clearly true.

For the moment, let \(\bC'(m_n)\) be the \(\bZ_p[\bG_n]\)-submodule of \(\hat{E}(m_n)\) generated by \(d_n\) and \(d_{-1}\). By the trace relations on \(d_n\), clearly we have \(\bC(m_n) \supseteq \bC'(m_n)\).

We now prove

\[
\bC(m_n) = \bC'(m_n), \quad \bC(m_n) + \bC(m_{n-1}) = \hat{E}(m_n)
\]

for \(n \geq 0\), simultaneously by induction.

In the case \(n = 0\), we have

\[
\bC(m_0) = \hat{E}(m_0) = \langle d_0, d_{-1} \rangle_{\bZ_p[\bG_0]} = \bC'(m_0),
\]

\[
\bC(m_0) + \bC(m_{-1}) = \hat{E}(m_0) + \hat{E}(m_{-1}) = \hat{E}(m_0)
\]

by Corollary 3.11.

In the case \(n \geq 1\), by the induction hypothesis we have

\[
\hat{E}(m_{n-1}) = \bC'(m_{n-1}) + \bC'(m_{n-2}), \quad \bC'(m_n) = \bC'(m_n)
\]

(3.3)
and by the trace relation we have $\mathcal{C}(m_n) \subseteq \mathcal{C}(m_{n-1})$. Therefore, by Proposition 3.10 and 3.3, we have

$$\bar{E}(m_n) = \langle d_n \rangle_{\mathbb{Z}_p[G_n]} + \bar{E}(m_{n-1}) \subseteq \mathcal{C}(m_n) + \mathcal{C}(m_{n-1}).$$

In particular, we have $\mathcal{C}(m_n) \subseteq \mathcal{C}'(m_n) + \mathcal{C}(m_{n-1})$. This implies that $\mathcal{C}(m_n) = \mathcal{C}'(m_n)$. Indeed, if $P \in \mathcal{C}(m_n)$, then there exist $Q \in \mathcal{C}'(m_n)$ and $R \in \mathcal{C}(m_{n-1})$ such that $P = Q + R$. Then we see that $R = P - Q \in \mathcal{C}(m_n) \cap \mathcal{C}(m_{n-1}) = \bar{E}(m_{n-1})$. Note that $\bar{E}(m_{n-1}) \subseteq \mathcal{C}'(m_n)$ since $d_{n-1} \in \mathcal{C}'(m_n)$. So we get $P = Q + R \in \mathcal{C}'(m_n)$ and thus $\mathcal{C}(m_n) = \mathcal{C}'(m_n)$. It is now clear that $\mathcal{C}(m_n) + \mathcal{C}(m_{n-1}) = \mathcal{C}(m_n) + \mathcal{C}(m_{n-1}) = \bar{E}(m_n)$. □

**Remark 3.17.** We check here that the norm subgroup $\mathcal{C}(m_n)$ is not a cyclic $\mathbb{Z}_p[G_n]$-module generated by $d_n$ if and only if $d = [k : \mathbb{Q}_p] \equiv 0 \pmod{4}$ and $n$ is even.

1. When $n$ is odd or $d \not\equiv 0 \pmod{4}$, we see that $d_{n-1}$ is automatically contained in $\langle d_n \rangle_{\mathbb{Z}_p[G_n]}$. Thus in these cases we see that the norm subgroup $\mathcal{C}(m_n)$ is a cyclic $\mathbb{Z}_p[G_n]$-module generated by $d_n$ for each $n$;

$$\mathcal{C}(m_n) = \langle d_n \rangle_{\mathbb{Z}_p[G_n]}.$$

Indeed, when $n$ is odd, we have

$$d_{n-1} = (-1)^{\frac{n-1}{2}} \text{Tr}_{1/0} \cdots \text{Tr}_{n-2/3} \text{Tr}_{n/1} d_n \in \langle d_n \rangle_{\mathbb{Z}_p[G_n]}.$$

When $d \not\equiv 0 \pmod{4}$ and $n$ is even, we have

$$d_{n-1} = (-1)^{\frac{n}{2}} (\varphi + \varphi^{-1})^{-1} \text{Tr}_{0/1} \cdots \text{Tr}_{n-2/3} \text{Tr}_{n/1} d_n \in \langle d_n \rangle_{\mathbb{Z}_p[G_n]}.$$

since $\varphi + \varphi^{-1} \in \mathbb{Z}_p[G_{-1}]^\times$ by lemma 3.6.

2. When $d \equiv 0 \pmod{4}$, $d_{n-1}$ cannot be contained in $\langle d_n \rangle_{\mathbb{Z}_p[G_n]}$ for any even $n$. Thus in this case the norm subgroup $\mathcal{C}(m_n)$ is not a cyclic $\mathbb{Z}_p[G_n]$-module generated by $d_n$ for each even $n$. Indeed, if $\bar{E}(m_n) = \langle d_n \rangle_{\mathbb{Z}_p[G_0]}$, then $\bar{E}(m_{n-1}) = \langle \text{Tr}_{0/1}(d_n) \rangle_{\mathbb{Z}_p[G_{-1}]}$ since $\text{Tr}_{0/1} : \bar{E}(m_0) \to \bar{E}(m_{-1})$ is surjective. Since $\bar{E}(m_{-1}) \cong \mathbb{Z}_p[G_{-1}]$, this means that $\mathbb{Z}_p[G_{-1}] = (\varphi + \varphi^{-1}) \mathbb{Z}_p[G_{-1}]$, which is impossible by lemma 3.6.

**Definition 3.18.** Define $d^+_{n}$ by

$$d^+_{n} = \begin{cases} (-1)^{\frac{n+1}{2}} d_n & \text{if } n \text{ is even}, \\ (-1)^{\frac{n}{2}} d_{n-1} & \text{if } n \text{ is odd}, \end{cases}$$

and $d^-_{n}$ by

$$d^-_{n} = \begin{cases} (-1)^{\frac{n+1}{2}} d_n & \text{if } n \text{ is even}, \\ (-1)^{\frac{n}{2}} d_{n-1} & \text{if } n \text{ is odd}. \end{cases}$$

**Remark 3.19.** By the relation between $\mathcal{C}(m_n)$ and $\bar{E}^\pm(m_n)$ (cf. Lemma 3.15), we can translate Proposition 3.10 in terms of the plus and the minus systems of points $(d^+_{n})_n$ and $(d^-_{n})_n$ such that $\bar{E}^+(m_n) = \langle d^+_{n}, d^-_{0} \rangle_{\mathbb{Z}_p[G_n]}$ and $\bar{E}^-(m_n) = \langle d^+_{n}, d^-_{0} \rangle_{\mathbb{Z}_p[G_n]}$. As in Remark 3.17 we see that the the plus subgroups $\bar{E}^+(m_n)$ are cyclic $\mathbb{Z}_p[G_n]$-modules generated by $d^+_{n}$ for all $n$ if and only if $d \not\equiv 0 \pmod{4}$, on the other hand the minus subgroups $\bar{E}^-(m_n)$ are always cyclic $\mathbb{Z}_p[G_n]$-modules generated by $d^-_{n}$ for all $n$. 


Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character of $\Delta = \text{Gal}(k(\mu_p)/k)$. If $M$ is a $\mathbb{Z}_p[\Delta]$-module, then $M$ is decomposed into

$$M = \bigoplus_{\chi} \varepsilon_\chi M,$$

where $\varepsilon_\chi = \frac{1}{p-1} \sum_{\sigma \in \Delta} \chi(\sigma) \sigma^{-1} \in \mathbb{Z}_p[\Delta]$. We denote by $M^\chi$ the $\chi$-component $\varepsilon_\chi M$.

Since we have $G_n \cong G_{-1} \times \Delta \times \text{Gal}(k_n/k_0)$, we can regard a $\mathbb{Z}_p[G_n]$-module as a $\mathbb{Z}_p[\Delta]$-module.

**Corollary 3.20.** Let $\chi : \Delta \to \mathbb{Z}_p^\times$ be a character and $q_n = \sum_{i=0}^n (-1)^i p^{n-i}$. Then we have

$$\text{rank}_{\mathbb{Z}_p} \mathcal{C}(m_n)^\chi = \begin{cases} d(q_n + 1) & \text{if } n : \text{odd and } \chi = 1, \\ dq_n & \text{otherwise}, \end{cases}$$

for each $n \geq 0$ and

$$\text{rank}_{\mathbb{Z}_p} \mathcal{C}(m_{-1})^\chi = \begin{cases} d & \text{if } \chi = 1, \\ 0 & \text{if } \chi \neq 1. \end{cases}$$

**Proof.** Since $\mathcal{C}(m_{-1}) = \widehat{E}(m_{-1}) \cong \mathbb{Z}_p[G_{-1}]$, we obtain the latter statement. Moreover, from the exact sequence (3.2) we get a recurrence sequence

$$\text{rank}_{\mathbb{Z}_p} \mathcal{C}(m_n)^\chi + \text{rank}_{\mathbb{Z}_p} \mathcal{C}(m_{n-1})^\chi = \text{rank}_{\mathbb{Z}_p} \widehat{E}(m_{-1})^\chi + \text{rank}_{\mathbb{Z}_p} \widehat{E}(m_n)^\chi,$$

since the $\mathbb{Z}_p$-rank of $\widehat{E}(m_n)^\chi$ is $dp^n$ for each $n \geq 0$. The former statement follows from this. \hfill \square

We introduce here some notation that will be used throughout the rest of the paper. Let $\omega_n(X) := (1 + X)^{p^n} - 1$ and $\Phi_n(X) := \sum_{i=0}^{p^n-1} X^i$ be the $p^n$-th cyclotomic polynomial. We define $\widetilde{\omega}_n(X) := 1$ and

$$\begin{align*}
\widetilde{\omega}_n^+(X) &= \prod_{1 \leq m \leq n, m \text{ even}} \Phi_m(1 + X), \\
\widetilde{\omega}_n^-(X) &= \prod_{1 \leq m \leq n, m \text{ odd}} \Phi_m(1 + X),
\end{align*}$$

Note that $\omega_n(X) = \widetilde{\omega}_n^+(X) \omega_n^-(X)$ for all $n \geq 0$. We write $\omega_n(X)$, $\widetilde{\omega}_n^+(X)$ and $\widetilde{\omega}_n^-(X)$ simply by $\omega_n$, $\widetilde{\omega}_n^+$ and $\widetilde{\omega}_n^-$ respectively.

We identify $\mathbb{Z}_p[G_n]$ with $\mathbb{Z}_p[G_0][X]/(\omega_n)_{\mathbb{Z}_p[G_0]}[X]$ by sending $\gamma_n$ to $1 + X$, where $\gamma_n$ is the image of $\gamma$ in $\mathbb{Z}_p[G_n]$.

Set $q_n = \sum_{i=0}^n (-1)^{n-i} p^i$ as in Corollary 3.20 and $q_{-1} := 1$. Put

$$q_n^+ := \begin{cases} q_n & \text{if } n \text{ is even}, \\ q_{n-1} & \text{if } n \text{ is odd}, \end{cases} \quad q_n^- := \begin{cases} q_n & \text{if } n \text{ is odd}, \\ q_{n-1} & \text{if } n \text{ is even}. \end{cases}$$

Note that $q_n^+ + q_n^- = p^n$ for all $n \geq 0$.

For later use, we rephrase Corollary 3.20 in terms of $\widehat{E}^\pm(m_n)^\chi$ and $q_n^\pm$ as in the following corollary.
Corollary 3.21. Let \( \chi : \Delta \to \mathbb{Z}_p^\times \) be a character. Then we have

\[
\text{rank}_{\mathbb{Z}_p}(\hat{E}^+(m_n)^{\chi}) = dq_n^+,
\]

and

\[
\text{rank}_{\mathbb{Z}_p}(\hat{E}^-(m_n)^{\chi}) = \begin{cases} d(q_n^- + 1) & \text{if } \chi = 1, \\ dq_n^- & \text{if } \chi \neq 1. \end{cases}
\]

For a character \( \chi \) of \( \Delta \), we define \( \mathbb{Z}_p[\chi] \) to be the \( \mathbb{Z}_p[\Delta] \)-module which is \( \mathbb{Z}_p \) as a \( \mathbb{Z}_p \)-module, and on which \( \Delta \) acts via \( \chi \), namely \( \sigma \cdot x = \chi(\sigma)x \) for \( \sigma \in \Delta \) and \( x \in \mathbb{Z}_p[\chi] \).

Proposition 3.22. Let \( \chi : \Delta \to \mathbb{Z}_p^\times \) be a character. We have

\[
\hat{E}^+(m_n)^{\chi} \cong \begin{cases} \mathbb{Z}_p[G_0][X] \oplus \mathbb{Z}_p[G_{-1}] & \text{if } \chi = 1, \\ \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_0][X] / \langle \omega_n \rangle \mathbb{Z}_p[G_{-1}][X] & \text{if } \chi \neq 1, \end{cases}
\]

and

\[
\hat{E}^-(m_n)^{\chi} \cong \begin{cases} \mathbb{Z}_p[G_{-1}][X] / \langle \omega_n \rangle \mathbb{Z}_p[G_{-1}][X] & \text{if } \chi = 1, \\ \mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_{-1}][X] / \langle \omega_n \rangle \mathbb{Z}_p[G_{-1}][X] & \text{if } \chi \neq 1, \end{cases}
\]

as \( \mathbb{Z}_p[G_0] \)-modules.

Proof. There is a surjective homomorphism

\[
\psi : \mathbb{Z}_p[G_0][X] \oplus \mathbb{Z}_p[G_{-1}] \longrightarrow \langle d_n^+, d_0^- \rangle_{\mathbb{Z}_p[G_0]} = \hat{E}^+(m_n)
\]

obtained by sending \((1,0)\) to \(d_n^+\) and \((0,1)\) to \(\frac{d_0^-}{p-1}(= \frac{d_n^-}{p-1})\). Since we have that

\[
\omega_n d_n^+ = \omega_n - 2d_n^+ = \cdots = 0, \quad \text{and} \quad \varepsilon_1 \omega_n^+ d_n^+ = \frac{1}{p-1} \text{Tr}_{0 \to -1} \omega_n^+ d_n^+ = (\varphi + \varphi^{-1}) \frac{d_n^-}{p-1},
\]

the map \( \psi \) induces a surjective homomorphism

\[
\bar{\psi} : \mathbb{Z}_p[G_0][X] / \langle \varepsilon_1 \omega_n^+ \rangle \mathbb{Z}_p[G_0][X] \longrightarrow \langle d_n^+, d_0^- \rangle_{\mathbb{Z}_p[G_0]} = \hat{E}^+(m_n).
\]

This map \( \bar{\psi} \) is injective since the source and the target of \( \bar{\psi} \) are free \( \mathbb{Z}_p \)-modules of the same \( \mathbb{Z}_p \)-rank \( d(p-1)q_n^+ \) (cf. Corollary 3.21). Thus we have

\[
\hat{E}^+(m_n) \cong \frac{\mathbb{Z}_p[G_0][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\varepsilon_1 \omega_n^+, - (\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_0][X]}},
\]

and

\[
\hat{E}^-(m_n) \cong \frac{\mathbb{Z}_p[G_{-1}][X] / \langle \omega_n \rangle \mathbb{Z}_p[G_{-1}][X]}{\langle (\varepsilon_1 \omega_n^+, - (\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_0][X]}},
\]

\[
\cong \bigoplus_{\chi} \frac{\varepsilon_\chi \mathbb{Z}_p[G_0][X] \oplus \varepsilon_\chi \mathbb{Z}_p[G_{-1}]}{\langle (\varepsilon_\chi \omega_n^+, 0), (\varepsilon_\chi (\varepsilon_1 \omega_n^+), - (\varphi + \varphi^{-1})) \rangle_{\mathbb{Z}_p[G_0][X]}}.
\]
as \(Z_p[G_n]-\)modules, where the last isomorphism is obtained by the character decomposition. Since we have

\[
\varepsilon \chi Z_p[G_0][X] \oplus \varepsilon \chi Z_p[G_{-1}] \cong \begin{cases} 
Z_p[G_{-1}][X] \oplus Z_p[G_{-1}] & \text{if } \chi = 1, \\
Z_p[G_{-1}][X] & \text{if } \chi \neq 1,
\end{cases}
\]

and

\[
\langle (\varepsilon \chi \omega_n^+, 0), (\varepsilon \chi \phi_1 \omega_n^+, -\varepsilon \chi (\varphi + \varphi^{-1})) \rangle_{Z_p[G_0][X]}
\]

\[
\cong \begin{cases} 
\langle (\omega_n^+, 0), (\phi_1 \omega_n^+, -\varphi + \varphi^{-1}) \rangle_{Z_p[G_{-1}][X]} & \text{if } \chi = 1, \\
\langle (\omega_n^+)Z_p[G_{-1}][X] & \text{if } \chi \neq 1
\end{cases}
\]

as \(Z_p[G_{-1}][X]\)-modules, we have

\[
E^+(m_n)^\chi \cong \frac{\varepsilon \chi Z_p[G_0][X] \oplus \varepsilon \chi Z_p[G_{-1}]}{\langle (\varepsilon \chi \omega_n^+, 0), (\varepsilon \chi \phi_1 \omega_n^+, -\varepsilon \chi (\varphi + \varphi^{-1})) \rangle_{Z_p[G_0][X]}}
\]

\[
\cong \begin{cases} 
\frac{Z_p[G_{-1}][X] \oplus Z_p[G_{-1}]}{\langle (\omega_n^+, 0), (\phi_1 \omega_n^+, -\varphi + \varphi^{-1}) \rangle_{Z_p[G_{-1}][X]}} & \text{if } \chi = 1, \\
\frac{Z_p[G_{-1}][X]}{Z_p[G_{-1}] \oplus Z_p[G_{-1}]} & \text{if } \chi \neq 1
\end{cases}
\]

as \(Z_p[G_n]\)-modules. Since \((\omega_n^+, 0) = X(\phi_1 \omega_n^+, -\varphi + \varphi^{-1})\), we get the conclusion for \(E^+(m_n)^\chi\).

Similarly to the above, we have

\[
E^-(m_n) = (d_n)^\chi Z_p[G_n]
\]

\[
\cong Z_p[G_0][X]/(\omega_n^-, (\sigma - 1)\phi_1 \omega_n^- | \sigma \in \Delta)_{Z_p[G_0][X]}
\]

\[
\cong \frac{Z_p[G_{-1}][X]}{(\omega_n^-)Z_p[G_{-1}][X]} \oplus \bigoplus_{\chi \neq 1} \frac{Z_p[G_{-1}][X]}{(\omega_n^-)Z_p[G_{-1}][X]} 
\]

as \(Z_p[G_n]\)-modules. So we get the conclusion for \(E^-(m_n)^\chi\).

**Remark 3.23.** When \(d \not\equiv 0 \pmod{4}\) and \(\chi = 1\), the description of the Galois module \(E^+(m_n)^\chi\) in Proposition 3.22 can be made more simpler. Explicitly, we claim that the homomorphism

\[
Z_p[G_{-1}][X]/(\omega_n^+) \xrightarrow{\sim} \frac{Z_p[G_{-1}][X] \oplus Z_p[G_{-1}]}{(\omega_n^+, -\varphi + \varphi^{-1})}
\]

given by \(x \mapsto (x, 0)\) is an isomorphism. Indeed, since \(\varphi + \varphi^{-1} \in Z_p[G_{-1}]^\chi\) in this case (see Lemma 3.6), \((x, y) \in Z_p[G_{-1}][X] \oplus Z_p[G_{-1}]\) is equivalent to \((x + y(\varphi + \varphi^{-1})^{-1}\omega_n^+, 0)\) and thus the map is surjective. On the other hand, if \((x, 0) \in (\omega_n^+, -\varphi + \varphi^{-1}))\) for \(x \in Z_p[G_{-1}][X]\), then there exists \(a(X) \in Z_p[G_{-1}][X]\) such that \(x = a(X)\omega_n^+\) and \(0 = -a(0)(\varphi + \varphi^{-1})\). Again by Lemma 3.6 we see that \(a(0) = 0\). So we get \(x = a(X)\omega_n^+ \in (\omega_n^+)\) and thus the map is injective.
In the rest of this paper, we abbreviate $\mathbb{Z}_p[G_{-1}][X]$-modules $\langle S \rangle_{\mathbb{Z}_p[G_{-1}][X]}$ generated by some set $S$ to $\langle S \rangle$ as in the above remark.

### 3.3 The plus and the minus local conditions

In this subsection, we study the $\Lambda$-module $(\hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ and prove Proposition 3.28. We also study the $\Lambda$-module $\left( H_1(k_\infty, E[\mathfrak{m}_\infty]) \chi \otimes \mathbb{Q}_p/\mathbb{Z}_p \right)^\vee$ and prove Proposition 3.32.

We first study $(\hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$.

Since $\hat{E}^\pm(\mathfrak{m}_\infty)$ are $\mathbb{Z}_p$-torsion-free, we have an exact sequence

$$0 \rightarrow \hat{E}^\pm(\mathfrak{m}_n)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0.$$

From this exact sequence, we get the $\Gamma_n$-invariant-coinvariant exact sequence

$$(\hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma_n \rightarrow (\hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p)[p\infty] \rightarrow 0,$$

for each $n \geq 0$. We will compute the rightmost modules $(\hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma_n[p\infty]$ for all $n \geq 0$ to study the $\Lambda$-module $(\hat{E}^\pm(\mathfrak{m}_\infty)^{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$.

Define $\delta$ by

$$\delta = \begin{cases} 0 & \text{if } d \not\equiv 0 \pmod{4} \text{ or } \chi \not= 1, \\ 2 & \text{otherwise.} \end{cases}$$

Proposition 3.24. Let $\chi : \Delta \rightarrow \mathbb{Z}_p^\times$ be a character. Then $\left( (\hat{E}^\pm(\mathfrak{m}_\infty)^{\chi})_{\Gamma_n[p\infty]} \right)^\vee$ are free $\mathbb{Z}_p$-modules for all $n$, and we have

$$\text{rank}_{\mathbb{Z}_p} \left( (\hat{E}^+(\mathfrak{m}_\infty)^{\chi})_{\Gamma_n[p\infty]} \right)^\vee = dq_n^+ + \delta,$$

$$\text{rank}_{\mathbb{Z}_p} \left( (\hat{E}^-(\mathfrak{m}_\infty)^{\chi})_{\Gamma_n[p\infty]} \right)^\vee = \begin{cases} 0 & \text{if } \chi = 1, \\ dq_n^+ - 1 & \text{if } \chi \neq 1. \end{cases}$$
More precisely, we have
\[
\left( \hat{E}^+(m_\infty)^\chi \right)_{\Gamma_n} \left[ p^{\infty} \right] 
\cong \begin{cases} 
\left( \frac{Z_p[G_{-1}][X]}{(\omega_n^+)} \oplus \text{Ann}_{Z_p[G_{-1}]}(\varphi + \varphi^{-1}) \right) \otimes \mathbb{Q}_p / \mathbb{Z}_p & \text{if } \chi = 1, \\
\left( \frac{Z_p[G_{-1}][X]}{(\omega_n^+)} \right) \otimes \mathbb{Q}_p / \mathbb{Z}_p & \text{if } \chi \neq 1,
\end{cases}
\]

\[
\left( \hat{E}^-(m_\infty)^\chi \right)_{\Gamma_n} \left[ p^{\infty} \right] 
\cong \begin{cases} 
\frac{Z_p[G_{-1}][X]}{(\omega_n^+)} \otimes \mathbb{Q}_p / \mathbb{Z}_p & \text{if } \chi = 1, \\
\frac{Z_p[G_{-1}][X]}{(\omega_n^+)} \otimes \mathbb{Q}_p / \mathbb{Z}_p & \text{if } \chi \neq 1
\end{cases}
\] as \( \mathbb{Z}_p \)-modules.

**Proof.** We prove the claim for \( \left( \hat{E}^+(m_\infty)^\chi \right)_{\Gamma_n} \left[ p^{\infty} \right] \) in the case where \( \chi = 1 \). We can prove the rest of the claims similarly.

We have
\[
\left( \hat{E}^+(m_\infty)^\chi \right)_{\Gamma_n} \left[ p^{\infty} \right] = \left( \hat{E}^+(m_\infty)^\chi / \omega_n \hat{E}^+(m_\infty)^\chi \right) \left[ p^{\infty} \right]
\cong \lim_{m \geq n} \left( \hat{E}^+(m_n)^\chi / \omega_n \hat{E}^+(m_n)^\chi \right) \left[ p^{\infty} \right], \tag{3.5}
\]
where transition maps
\[
\left( \hat{E}^+(m_n)^\chi / \omega_n \hat{E}^+(m_n)^\chi \right) \left[ p^{\infty} \right] \longrightarrow \left( \hat{E}^+(m_{n+1})^\chi / \omega_n \hat{E}^+(m_{n+1})^\chi \right) \left[ p^{\infty} \right]
\] are multiplication-by-\( p \) maps when \( m \) is odd and identity maps when \( m \) is even. We will calculate \( \left( \hat{E}^+(m_n)^\chi / \omega_n \hat{E}^+(m_n)^\chi \right) \left[ p^{\infty} \right] \) for each \( n, m \). Since \( \hat{E}^+(m_n) = \hat{E}^+(m_{n-1}) \) if \( m \) is odd, we may assume \( m \) is even.

We consider the case \( n \) is even. By Proposition 3.22 we have
\[
\hat{E}^+(m_n)^\chi / \omega_n \hat{E}^+(m_n)^\chi \cong \frac{Z_p[G_{-1}][X] \oplus Z_p[G_{-1}]}{\langle (\omega_n^+, -(\varphi + \varphi^{-1})), (\omega_n, 0) \rangle}, \tag{3.6}
\]

We can show that
\[
\frac{Z_p[G_{-1}][X] \oplus Z_p[G_{-1}]}{\langle (\omega_n^+, -(\varphi + \varphi^{-1})), (\omega_n, 0) \rangle} \left[ p^{\infty} \right] \cong \frac{Z_p[G_{-1}][X]}{(p^{\frac{n-1}{2}}, \omega_n^+)} \oplus \frac{\text{Ann}_{Z_p[G_{-1}]}(\varphi + \varphi^{-1})}{\langle p^{\frac{n-1}{2}} \rangle}. \tag{3.7}
\]

Indeed, since we have \( \omega_n^+ \equiv p^{\frac{n-1}{2}} \omega_n^+ \pmod{\omega_n} \) and \( \omega_n = \omega_n^+ \omega_n^+ \), there is an
exact sequence

\[
0 \longrightarrow \frac{\mathbb{Z}_p[G_{-1}][X]}{\langle \hat{p} \hat{m} \hat{n}, \tilde{\omega}_n \rangle} \oplus \frac{\text{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1})}{\langle \hat{p} \hat{m} \hat{n} \rangle} \\
\longrightarrow \frac{\mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\tilde{\omega}_n, -((\varphi + \varphi^{-1})), (\omega_n, 0)) \rangle} \\
\longrightarrow \frac{\mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\tilde{\omega}_n, -((\varphi + \varphi^{-1})), (\omega_n, 0), (\alpha \tilde{\omega}_n, 0)) \rangle} \rightarrow 0,
\]

(3.8)

where the first map is \((x, y) \mapsto (x \omega_n^+ + y \tilde{\omega}_n^+, 0)\) and \(\alpha\) is a generator of the \(\mathbb{Z}_p[G_{-1}]\)-module \(\text{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1})\) (cf. Lemma 3.8). There is another exact sequence

\[
0 \longrightarrow \frac{\mathbb{Z}_p[G_{-1}][X]}{(\omega_n, \alpha \omega_n)} \\
\longrightarrow \frac{\mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]}{\langle (\tilde{\omega}_n, -((\varphi + \varphi^{-1})), (\omega_n, 0)), (\alpha \tilde{\omega}_n, 0) \rangle} \\
\longrightarrow \frac{\mathbb{Z}_p[G_{-1}]}{(\varphi + \varphi^{-1})} \rightarrow 0,
\]

(3.9)

whose leftmost and rightmost modules are both \(\mathbb{Z}_p\)-free, where the first map is \(x \mapsto (x, 0)\) and the second map is \((x, y) \mapsto y\). Thus the rightmost module in (3.8), which is the same as the middle module in (3.9), is \(\mathbb{Z}_p\)-free. Our claim (3.7) follows from this.

By (3.5), (3.6), and (3.7), we get

\[
\left(\hat{E}^+(m_\infty)^{\lambda}\right)_{\Gamma_n}[p^\infty] \cong \lim_{m \geq n} \left(\hat{E}^+(m_\infty)^{\lambda}/\omega_n \hat{E}^+(m_\infty)^{\lambda}\right)[p^\infty] \\
\cong \lim_{m \geq n} \left(\mathbb{Z}_p[G_{-1}][X] \oplus \mathbb{Z}_p[G_{-1}]\right)[p^\infty] \\
\cong \lim_{m \geq n} \left(\langle \hat{p} \hat{m} \hat{n}, -((\varphi + \varphi^{-1})), (\omega_n, 0) \rangle \right)[p^\infty] \\
\cong \lim_{m \geq n} \left(\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle \hat{p} \hat{m} \hat{n}, -((\varphi + \varphi^{-1})), (\tilde{\omega}_n, 0) \rangle} \oplus \frac{\text{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1})}{\langle \hat{p} \hat{m} \hat{n} \rangle}\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p
\]

when \(n\) is even. By replacing \(\hat{m} \hat{n}\) with \(\hat{m} \hat{(n-1)}\) in the above discussion, we get the statement also in the case where \(n\) is odd.

From this description, we see that \(\left(\hat{E}^+(m_\infty)^{\lambda}\right)_{\Gamma_n}[p^\infty]\) is \(\mathbb{Z}_p\)-free and

\[
\text{rank}_{\mathbb{Z}_p} \left(\left(\hat{E}^+(m_\infty)^{\lambda}\right)_{\Gamma_n}[p^\infty]\right)^{\vee} \\
= \text{rank}_{\mathbb{Z}_p} \left(\frac{\mathbb{Z}_p[G_{-1}][X]}{\langle \tilde{\omega}_n \rangle}\right) + \text{rank}_{\mathbb{Z}_p} \left(\text{Ann}_{\mathbb{Z}_p[G_{-1}]}(\varphi + \varphi^{-1})\right) \\
= dq_n + \delta.
\]
Corollary 3.25. Let \( \chi: \Delta \to \mathbb{Z}_p^\times \) be a character. Then the \( \Gamma_n \)-coinvariants 
\((\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) are free \( \mathbb{Z}_p \)-modules for all \( n \), and we have

\[
\text{rank}_{\mathbb{Z}_p}\left((\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\right)_{\Gamma_n} = dp^n + \delta, \\
\text{rank}_{\mathbb{Z}_p}\left((\hat{E}^- (m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\right)_{\Gamma_n} = dp^n.
\]

**Proof.** It follows from Corollary 3.20, Proposition 3.24, and the exact sequence of Proposition 3.24.

**Remark 3.26.** From Corollary 3.25 we find that \((\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) is not a free \( \Lambda \)-module in the case when \( \delta = 2 \), i.e. the case when \( d \equiv 0 \) (mod 4) and \( \chi = 1 \). Indeed, the \( \mathbb{Z}_p \)-rank of the \( \Gamma_n \)-coinvariant of a free \( \Lambda \)-module is divisible by \( p^n \) for each \( n \). On the other hand, the \( \mathbb{Z}_p \)-rank of the \( \Gamma_n \)-coinvariant of \((\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) is not divisible by \( p^n \) as in the corollary.

**Proposition 3.27.** Let \( \chi: \Delta \to \mathbb{Z}_p^\times \) be a character. There exist injective homomorphisms of \( \Lambda \)-modules

\[
(\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \to \Lambda^{\oplus d} \oplus (\Lambda/X)^{\oplus \delta}, \\
(\hat{E}^- (m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \to \Lambda^{\oplus d}
\]

with finite cokernels.

**Proof.** We prove the claim for \((\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\). We can prove the rest of the claims similarly.

We first note that \((\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee\) has no nontrivial finite \( \Lambda \)-submodule since its \( \Gamma_n \)-coinvariants are free \( \mathbb{Z}_p \)-modules for all \( n \geq 0 \) (see Corollary 3.25). Thus by the structure theorem for \( \Lambda \)-modules, there exist irreducible distinguished polynomials \( f_j \), nonnegative integers \( r, s, t, m_i, n_j \), and an injective homomorphism

\[
f: (\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \to \Lambda^{\oplus r} \oplus \bigoplus_{i=1}^s \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^t \Lambda/f_j^{n_j} =: \mathcal{E}
\]

with finite cokernel \( Z \).

We show that

\[
\begin{align*}
\{ r = d, \\
s = 0 \text{ (in other words } m_i = 0 \text{ for all } i), \\
t = 0 \text{ if } d \not\equiv 0 \text{ (mod } 4) \text{ or } \chi \neq 1, \\
\text{otherwise, and } \quad & \quad (f^{m_1}_1, \ldots, f^{m_t}_t) = (X, X) \text{ if } t = 2.
\end{align*}
\]

From the exact sequence

\[
0 \to (\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \xrightarrow{f} \mathcal{E} \to Z \to 0,
\]

we get the \( \Gamma_n \)-invariant-coinvariant exact sequences

\[
Z^{\Gamma_n} \to (\hat{E}^+(m_\infty)^{X} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee_{\Gamma_n} \to \mathcal{E}/\omega_n \mathcal{E} \to Z/\omega_n Z \to 0
\]

(3.10)
for all \( n \). Note that, the first maps in (3.10) are 0-maps for all \( n \), since 
\[
(\hat{E}^+(m_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \Gamma_n \text{ is } \mathbb{Z}_p\text{-free. Then we see that } m_i = 0 \text{ and } f_j^n|_{\omega_n}
\] 
(and \( n_j \leq 1 \)) for all sufficiently large \( n \) since \( \mathbb{Z}/\omega_n\mathbb{Z} \) is bounded as \( n \to \infty \). Thus we get \( s = 0 \) here. We now have
\[
dp^n + \delta = \text{rank}_{\mathbb{Z}_p} (\hat{E}^+(m_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \Gamma_n
\]
\[
= \text{rank}_{\mathbb{Z}_p} (E/\omega_n E) = rp^n + \sum_{j=1}^t n_j \deg f_j
\]  
(3.11)
for all sufficiently large \( n \). Thus we get \( r = d \).

In the case when \( d \not\equiv 0 \pmod{4} \) or \( \chi \not\equiv 1 \), we get 0 = \( \sum_{j=1}^t n_j \deg f_j \) from the above discussion. Thus we get \( n_j = 0 \) which is the desired result, i.e. \( t = 0 \).

We finally consider the case when \( d \equiv 0 \pmod{4} \) and \( \chi = 1 \). We may assume \( n_j = 1 \) for all \( j \). In this case, we have
\[
2 = \sum_{j=1}^t \deg f_j
\]  
(3.12)
\[
f_j|_{\omega_n} = (1 + X)^{p^n} - 1
\]  
(3.13)
for all sufficiently large \( n \). We narrow down the possible combinations of \((t, (f_1, \ldots, f_t))\) satisfying these two conditions (3.12) and (3.13). If \( p \geq 5 \), there is a unique combination \((t, (f_1, \ldots, f_t)) = (2, (X, X))\), since \( \deg f_j \leq 2 \). If \( p = 3 \), since \( \omega_1 = X(X^2 + 3X + 3) \), there are two possible combinations \((t, (f_1, \ldots, f_t)) = (2, (X, X)), (1, (X^2 + 3X + 3))\). By showing that the last combination is impossible, we complete the proof. Indeed, we have \( \text{rank}_{\mathbb{Z}_p} E/\omega_0 E = d+1 \) with the combination \((t, (f_1, \ldots, f_t)) = (1, (X^2 + 3X + 3))\). On the other hand, from the exact sequence (3.10) for \( n = 0 \), we must have \( \text{rank}_{\mathbb{Z}_p} E/\omega_0 E = d + 2 \) and thus we get the desired conclusion.

We now get the following proposition which is an important ingredient for the proof of Proposition 3.29.

**Proposition 3.28.** Let \( \chi : \Delta \to \mathbb{Z}_p^\times \) be a character. Then \( \hat{E}^\pm(m_\infty)^\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \) has no nontrivial finite \( \Lambda \)-submodule and its \( \Lambda \)-rank is \( d \).

**Proof.** This follows from Corollary 3.25 and Proposition 3.27.

In the rest of this subsection, we study the \( \Lambda \)-module
\[
\left( \frac{H^1(k_\infty, E[p^\infty])}{\hat{E}^\pm(m_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee.
\]

We consider the following exact sequence;
\[
0 \to \left( \frac{H^1(k_\infty, E[p^\infty])}{\hat{E}^\pm(m_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee \to \left( H^1(k_\infty, E[p^\infty]) \right)^\vee \to 0. \quad (3.14)
\]
We studied the \( \Lambda \)-module structure of the rightmost module. We also know the \( \Lambda \)-module structure of the middle module by the following fact (Proposition 3.29).
Proposition 3.29 (Greenberg [1] §3 Corollary 2). Let $K$ be a finite extension of $\mathbb{Q}_p$ and $K_\infty$ a $\mathbb{Z}_p$-extension of $K$. Put $\Lambda_K = \mathbb{Z}_p[\text{Gal}(K_\infty/K)]$. If $E(K_\infty)[p^\infty] = 0$, then $H^1(K_\infty, E[p^\infty])^\vee$ is a free $\Lambda_K$-module and its $\Lambda_K$-rank is $2[K:\mathbb{Q}_p]$;

$$H^1(K_\infty, E[p^\infty])^\vee \cong \Lambda_K^\oplus[K:\mathbb{Q}_p].$$

We can apply Proposition 3.29 in our setting as $K = k_0$, $K_\infty = k_\infty$. Indeed we see that $E(k_\infty)[p^\infty] = 0$ by Proposition 3.1.

Here we recall the following useful lemma on equivalent conditions on freeness of $\Lambda$-modules and on triviality of finite $\Lambda$-submodules.

Lemma 3.30. Let $M$ be a finitely generated $\Lambda$-module.

(1) $M$ is a free $\Lambda$-module if and only if $M^\Gamma = 0$ and $M^\Gamma$ is a free $\mathbb{Z}_p$-module.

(2) $M$ has no non-trivial finite $\Lambda$-submodule if and only if $M^\Gamma$ is a free $\mathbb{Z}_p$-module.

Proof. See for example [15] Proposition 5.3.19.

Applying the following lemma to the exact sequence (3.14), we can now determine the $\Lambda$-module structure of $(H^1(k_\infty, E[p^\infty])/(\hat{E}^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p))^\vee$.

Lemma 3.31. Let $f : M \to N$ be a surjective homomorphism of $\Lambda$-modules. Suppose that $M$ is a free $\Lambda$-module of $\Lambda$-rank $r$, and that $N$ is a $\Lambda$-module of $\Lambda$-rank $s$ which has no non-trivial finite $\Lambda$-submodule. Then its kernel $\text{Ker} f$ is a free $\Lambda$-module of rank $r - s$.

Proof. We put $M_0 := \text{Ker} f$. Then by taking the invariant-coinvariant exact sequence, we have

$$0 \to M_0^\Gamma \to M^\Gamma \to N^\Gamma \to M_0, \to M^\Gamma.$$

Since $M$ is a free $\Lambda$-module, we have $M^\Gamma = 0$ and $M^\Gamma$ is a free $\mathbb{Z}_p$-module by Lemma 3.30. Since $N$ has no non-trivial finite $\Lambda$-submodule, we see that $N^\Gamma$ is a free $\mathbb{Z}_p$-module by Lemma 3.30. Hence we have $M_0^\Gamma = 0$ and $M_0$ is a free $\mathbb{Z}_p$-module. Thus $M_0$ is a free $\Lambda$-module again by Lemma 3.30. It is easy to see that the $\Lambda$-rank of $M_0$ is $r - s$.

Proposition 3.32. We have

$$\left( \frac{H^1(k_\infty, E[p^\infty])}{\hat{E}^\pm(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right)^\vee \cong \Lambda^\oplus[k_0, \mathbb{Q}_p].$$

Proof. It follows from Proposition 3.28, Proposition 3.29 and Lemma 3.31 for the exact sequence (3.14).

3.4 More on the plus and the minus local conditions

The discussion in the previous subsections is enough to prove our main theorem. In this subsection, we proceed to determine the explicit structure of the $\Lambda$-module $(\hat{E}^\pm(m_\infty)^x \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. For that purpose, we show the following lemma.
Lemma 3.33. Let \( f : M \to N \) be an injective homomorphism of \( \Lambda \)-modules with finite cokernel. Suppose that \( M/\omega_n M \) is \( \mathbb{Z}_p \)-free and \( N_{\Lambda, \text{tors}} = \{ x \in N | \omega_n x = 0 \} \) for all sufficiently large \( n \). Then \( f \) induces an isomorphism
\[
\mathcal{J} : M/M_{\Lambda, \text{tors}} \cong N/N_{\Lambda, \text{tors}}
\]
for all sufficiently large \( n \).

Proof. We regard \( M \) as a \( \Lambda \)-submodule of \( N \) by \( f \). Since \( N/M \) is finite, we see that \( \omega_n N \subset M \) for all sufficiently large \( n \). We thus have
\[
\text{Coker}(\mathcal{J}) = N/(M + N_{\Lambda, \text{tors}}) \\
\xrightarrow{\times \omega_n} \omega_n N/\omega_n M \\
\hookrightarrow M/\omega_n M
\]
for all sufficiently large \( n \). Since \( M/\omega_n M \) is \( \mathbb{Z}_p \)-free for all sufficiently large \( n \), we get \( M/M_{\Lambda, \text{tors}} \cong N/N_{\Lambda, \text{tors}} \) for such \( n \).

Theorem 3.34. Let \( \chi : \Delta \to \mathbb{Z}_p^\times \) be a character. We have
\[
\left( \hat{E}^+ (m_\infty)^X \otimes \mathbb{Q}_p / \mathbb{Z}_p \right)^{\vee} \cong \Lambda^{\oplus d} \oplus (\Lambda / X)^{\oplus \delta}, \\
\left( \hat{E}^- (m_\infty)^X \otimes \mathbb{Q}_p / \mathbb{Z}_p \right)^{\vee} \cong \Lambda^{\oplus d}.
\]

Proof. We prove this theorem for \( (\hat{E}^+(m_\infty)^X \otimes \mathbb{Q}_p / \mathbb{Z}_p)^{\vee} \). We can prove the rest of the claim similarly.

Let \( M = \left( \hat{E}^+(m_\infty)^X \otimes \mathbb{Q}_p / \mathbb{Z}_p \right)^{\vee} \), \( N = \Lambda^{\oplus d} \oplus (\Lambda / X)^{\oplus \delta} \), and \( f \) be the map obtained in Proposition 3.27. We consider the following commutative diagram;
\[
0 \to M_{\Lambda, \text{tors}} \to M \to M/M_{\Lambda, \text{tors}} \to 0 \\
\downarrow f \quad \downarrow \mathcal{J} \quad \downarrow \quad \downarrow \\
0 \to (\Lambda / X)^{\oplus \delta} \to \Lambda^{\oplus d} \oplus (\Lambda / X)^{\oplus \delta} \to \Lambda^{\oplus d} \to 0.
\]

Here we note that \( \mathcal{J} \) is actually injective. Indeed, we see that \( \text{Ker}(\mathcal{J}) \) is a \( \Lambda \)-torsion submodule of a \( \Lambda \)-torsion-free module \( M/M_{\Lambda, \text{tors}} \), since the \( \Lambda \)-rank of \( M/M_{\Lambda, \text{tors}} \) is \( d \) and \( \text{Coker}(\mathcal{J}) \) is finite.

We see that \( M_{\Lambda, \text{tors}} \cong (\Lambda / X)^{\oplus \delta} \), since \( \text{Coker}(f_0) \) is finite. On the other hand, the assumptions in Lemma 3.33 are satisfied (see Proposition 3.27, Corollary 3.25). Thus we have \( M/M_{\Lambda, \text{tors}} \cong \Lambda^{\oplus d} \) by Lemma 3.33. Therefore, the above horizontal exact sequence splits and thus we get
\[
M \cong M/M_{\Lambda, \text{tors}} \oplus M_{\Lambda, \text{tors}} \cong \Lambda^{\oplus d} \oplus (\Lambda / X)^{\oplus \delta}.
\]

4 Finite \( \Lambda \)-submodules

We use the notations and the assumptions introduced in Section 2.

Let \( \Gamma = \text{Gal}(F_\infty / F_0) \) and \( \Lambda = \mathbb{Z}_p[[\Gamma]] \). We fix a topological generator \( \gamma \in \Gamma \). Then we identify the completed group ring \( \mathbb{Z}_p[[\Gamma]] \) with the ring of power series \( \mathbb{Z}_p[[X]] \) by identifying \( \gamma \) with \( 1 + X \).
4.1 Finite $\Lambda$-submodules of $\text{Sel}(F_\infty, E[p^\infty])^\vee$

In this subsection, we study finite $\Lambda$-submodules of the Pontryagin dual of the $p$-primary Selmer group. The aim of this subsection is to prove Theorem 4.3.

The following proposition is due to Matsuno [14, Theorem 2.4] (see also Hachimori–Matsuno [3]).

Proposition 4.1. Let $K$ be a finite extension of $\mathbb{Q}$, $K_\infty/K$ a $\mathbb{Z}_p$-extension, $K_n$ its $n$-th layer, and $E$ an elliptic curve defined over $K$. Put $\Gamma_K = \text{Gal}(K_\infty/K)$, and $\Lambda_K = \mathbb{Z}_p[[\Gamma_K]]$. Let $X_n$ be the kernel of the restriction map

$$\text{Sel}(K_n, E[p^\infty]) \rightarrow \text{Sel}(K_\infty, E[p^\infty])$$

and $X_\infty := \varprojlim X_n$ where the projective limit is taken with respect to the corestriction maps.

Assume that the $\mathbb{Z}_p$-rank of $\text{Sel}(K_n, E[p^\infty])^\vee$ is bounded as $n \to \infty$. Then the maximal finite $\Lambda_K$-submodule of $\text{Sel}(K_\infty, E[p^\infty])^\vee$ is isomorphic to $X_\infty$.

In particular if we further assume that $E(K)[p] = 0$, then $\text{Sel}(K_\infty, E[p^\infty])^\vee$ has no nontrivial finite $\Lambda_K$-submodule.

We check that we can apply the above proposition in our setting $K = F_0$, $K_\infty = F_\infty$.

Lemma 4.2. The morphisms $\text{Sel}^\pm(F_n, E[p^\infty]) \rightarrow \text{Sel}^\pm(F_\infty, E[p^\infty])$ are injective for all $n \geq 0$.

Proof. We can prove this by the same method as the proof of Lemma 9.1 in [11].

We assume from here that both $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ are $\Lambda$-torsion. We denote the Iwasawa $\Lambda$-invariant of $\text{Sel}^\pm(F_\infty, E[p^\infty])^\vee$ by $\lambda^\pm$.

Let

$$\text{Sel}^1(F_n, E[p^\infty]) := \text{Ker} \left( \text{Sel}(F_n, E[p^\infty]) \rightarrow \prod_{v \in S_{p,F}^\infty} \frac{H^1(F_n, E[p^\infty])}{E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right),$$

where $S_{p,F}^\infty$ is the set of all primes of $F$ lying above $p$ where $E$ has supersingular reduction.

By the exact sequence (3.2), we have an exact sequence

$$0 \rightarrow \frac{H^1(F_{n,v}, E[p^\infty])}{E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \rightarrow \frac{H^1(F_{n,v}, E[p^\infty])}{E'(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \oplus \frac{H^1(F_{n,v}, E[p^\infty])}{E^-(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \rightarrow \frac{H^1(F_{n,v}, E[p^\infty])}{E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \rightarrow 0$$

for each $n$ and for each prime $v$ of $F$ lying above $p$. Thus, for each $n$, we get the following exact sequence

$$0 \rightarrow \text{Sel}^1(F_n, E[p^\infty]) \xrightarrow{\iota} \text{Sel}^+(F_n, E[p^\infty]) \oplus \text{Sel}^-(F_n, E[p^\infty]) \xrightarrow{\eta} \text{Sel}(F_n, E[p^\infty])$$

(4.1)

where $\iota$ is the diagonal embedding by inclusions and $\eta$ is $(x, y) \mapsto x - y$. 25
Proposition 4.3. The cokernel of $\eta$ in the exact sequence (4.1) is finite.

Proof. We can prove this by the same method as the proof of Lemma 10.1 in [11]. □

Proposition 4.4. The $\mathbb{Z}_p$-rank of $\text{Sel}(F_n, E[p^{\infty}])^\vee$ is bounded as $n \to \infty$. More precisely, we have

$$\text{rank}_{\mathbb{Z}_p} \text{Sel}(F_n, E[p^{\infty}])^\vee \leq \lambda^+ + \lambda^-$$

for every $n$.

Proof. Since the restriction map $\text{Sel}(F_n, E[p^{\infty}]) \to \text{Sel}^1(F_n, E[p^{\infty}])$ is injective, we have

$$\text{rank}_{\mathbb{Z}_p} \text{Sel}(F_n, E[p^{\infty}])^\vee \leq \text{rank}_{\mathbb{Z}_p} \text{Sel}^1(F_n, E[p^{\infty}])^\vee.$$

Hence by Lemma 4.2 and Proposition 4.3, we get

$$\text{rank}_{\mathbb{Z}_p} \text{Sel}(F_n, E[p^{\infty}])^\vee \leq \text{rank}_{\mathbb{Z}_p} \text{Sel}^1(F_n, E[p^{\infty}])^\vee + \text{rank}_{\mathbb{Z}_p} \text{Sel}^-(F_n, E[p^{\infty}])^\vee \leq \lambda^+ + \lambda^-$$

for every $n$ from (4.1). The boundedness of the $\mathbb{Z}_p$-ranks follows from this immediately. □

From the above argument, we can prove the following theorem.

Theorem 4.5. Assume that both $\text{Sel}^+(F_{\infty}, E[p^{\infty}])^\vee$ and $\text{Sel}^-(F_{\infty}, E[p^{\infty}])^\vee$ are $\Lambda$-torsion. Then $\text{Sel}(F_{\infty}, E[p^{\infty}])^\vee$ has no nontrivial finite $\Lambda$-submodule.

Proof. The $\mathbb{Z}_p$-rank of $\text{Sel}(F_n, E[p^{\infty}])^\vee$ is bounded as $n \to \infty$ (cf. Proposition 4.4). Further, we have $E(F_0)[p] = 0$ by Proposition 3.1. Thus we can apply Proposition 4.1 and get the desired result. □

4.2 Finite $\Lambda$-submodules of $\text{Sel}^\pm(F_{\infty}, E[p^{\infty}])^\vee$

Finally, we study finite $\Lambda$-submodules of the Pontryagin duals of the plus and the minus Selmer groups. The aim of this subsection is to prove our main theorem (Theorem 4.5).

We prove that the triviality of finite $\Lambda$-submodules of $\text{Sel}(F_{\infty}, E[p^{\infty}])^\vee$ is inherited to that of $\text{Sel}^\pm(F_{\infty}, E[p^{\infty}])^\vee$.

Let us consider the following exact sequence of $\Lambda$-modules coming from the definition of the Selmer groups:

$$\bigoplus_{v \in S_{p, F}} \left( \frac{H^1(F_{\infty, v}, E[p^{\infty}])}{E^\pm(F_{\infty, v}) \otimes \mathbb{Q}_p / \mathbb{Z}_p} \right)^\vee \overset{i^\pm}{\longrightarrow} \text{Sel}(F_{\infty}, E[p^{\infty}])^\vee \longrightarrow \text{Sel}^\pm(F_{\infty}, E[p^{\infty}])^\vee \longrightarrow 0. \quad (4.2)$$

Proposition 4.6. Assume that $\text{Sel}^\pm(F_{\infty}, E[p^{\infty}])^\vee$ is $\Lambda$-torsion. If each direct summand $H^1(F_{\infty, v}, E[p^{\infty}])/(E^\pm(F_{\infty, v}) \otimes \mathbb{Q}_p / \mathbb{Z}_p)^\vee$ of the leftmost in the exact sequence (4.2) is a torsion-free $\Lambda$-module of $\Lambda$-rank $[F_0, v : \mathbb{Q}_p]$, then the map $i^\pm$ in (4.2) is injective.
Proof. We have rank$_{\Lambda}(\text{Sel}(F_\infty, E[p^{\infty}]))^\vee \geq \sum_{v \in S^{\infty}_{p,F}}[F_{0,v} : \mathbb{Q}_p]$ (cf. [4] Theorem 1.7). From (4.2) and our assumptions, we see that

$$\sum_{v \in S^{\infty}_{p,F}}[F_{0,v} : \mathbb{Q}_p] = \text{rank}_{\Lambda} \left( \bigoplus_{v \in S^{\infty}_{p,F}} \left( H^1(F_\infty,v,E[p^{\infty}])^\vee \otimes_{E^\pm(F_\infty,v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right) \right) \geq \text{rank}_{\Lambda}(\text{Sel}(F_\infty, E[p^{\infty}]))^\vee.$$ 

Thus we get

$$\text{rank}_{\Lambda} \left( \bigoplus_{v \in S^{\infty}_{p,F}} \left( H^1(F_\infty,v,E[p^{\infty}])^\vee \otimes_{E^\pm(F_\infty,v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right) \right) = \text{rank}_{\Lambda}(\text{Sel}(F_\infty, E[p^{\infty}]))^\vee.$$ 

From this, we see that the kernel Ker $\iota^\pm$ is $\Lambda$-torsion. Therefore we get the conclusion since the leftmost direct sum in the exact sequence (4.2) is a torsion-free $\Lambda$-module.

The following proposition is a key tool for the proof of our main theorem.

**Proposition 4.7** (Greenberg [2] p.104–105). Let $f : M \to N$ be an injective homomorphism of $\Lambda$-modules. Suppose that $N$ is a finitely generated $\Lambda$-module which has no non-trivial finite $\Lambda$-submodule, and that $M$ is a free $\Lambda$-module. Then the cokernel Coker$(f)$ has no nontrivial finite $\Lambda$-submodule.

**Proof.** We put $N_1 := \text{Coker}(f)$. Then by taking the invariant-coinvariant exact sequence of $0 \to M \to N \to N_1 \to 0$, we get an exact sequence

$$M^\Gamma \longrightarrow N^\Gamma \longrightarrow N_1^\Gamma \longrightarrow M_\Gamma. \quad (4.3)$$

Since $M$ is a free $\Lambda$-module, we see that $M^\Gamma = 0$ and $M_\Gamma$ is a free $\mathbb{Z}_p$-module by Lemma 3.30. Since $N$ has no non-trivial finite $\Lambda$-submodule, we see that $N^\Gamma$ is a free $\mathbb{Z}_p$-module by Lemma 3.30. Thus from the exact sequence (4.3), we see that $N_1^\Gamma$ is a free $\mathbb{Z}_p$-module and therefore $N_1$ has no nontrivial finite $\Lambda$-submodule again by Lemma 3.30.

**Theorem 4.8.** Assume that both $\text{Sel}^+(F_\infty, E[p^{\infty}])^\vee$ and $\text{Sel}^-(F_\infty, E[p^{\infty}])^\vee$ are $\Lambda$-torsion. Then both $\text{Sel}^+(F_\infty, E[p^{\infty}])^\vee$ and $\text{Sel}^-(F_\infty, E[p^{\infty}])^\vee$ have no non-trivial finite $\Lambda$-submodule.

**Proof.** By Proposition 3.32, we have

$$\left( \frac{H^1(F_\infty,v,E[p^{\infty}])^\vee}{E^\pm(F_\infty,v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right) \cong \Lambda^\oplus[F_{0,v}, \mathbb{Q}_p]$$

for each prime $v \in S^{\infty}_{p,F}$. Thus we can apply Proposition 4.6 and Proposition 4.7 for $f = \iota^\pm$. Thus, by Theorem 1.5 we get the desired result.

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