Theoretical studies of atmospheric perturbations caused by the gravity field inhomogeneities

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Abstract. In solving atmospheric dynamics problems, the influence of the gravity field inhomogeneities (GFI) is neglected. The traditional basis for this is that the amplitude of gravity variations even in highly anomalous regions do not exceed $10^{-3}$ ms$^{-2}$ in order of magnitude, i.e. 4 orders of magnitude less than average gravity. But in the presence of gravity anomalies, inhomogeneous forces of the same order act in the directions tangent to the mean Earth ellipsoid. The atmosphere dynamics is very sensitive to the forces of such directions. In highly anomalous regions, they are comparable with the main horizontal forces acting in the atmosphere, in particular with the forces of the pressure gradient and Coriolis forces. Therefore, it seems appropriate to analyze the possible effect of spatial variations of gravity on the dynamics of the atmosphere, primarily for mesoscale atmospheric disturbances. In a number of the authors' previous works, theoretical studies of the possible effect of GFI on the atmosphere dynamics are carried out. This report contains some new relevant findings. A three-dimensional analytical model of geostrophic wind disturbances under the influence of GFI is developed. An analogy of atmospheric disturbances caused by thermal inhomogeneities of the underlying surface and GFI is shown. Attention is drawn to the possibility of "accumulation" of atmospheric effects associated with the gravitational field inhomogeneities.

1. Introduction

In a number of the authors' previous works, theoretical studies of the possible effect of the gravity field inhomogeneities (GFI) on the atmosphere dynamics are carried out [1-3]. The present report contains some new relevant findings. A three-dimensional analytical model of geostrophic wind disturbances under the influence of GFI is developed. An analogy of atmospheric disturbances caused by thermal inhomogeneities of the underlying surface and GFI is shown. Attention is drawn to the possibility of "accumulation" of atmospheric effects associated with the gravitational field inhomogeneities.

2. Three-dimensional analytical model of geostrophic wind disturbances influenced by the gravity field inhomogeneities

The results of two-dimensional model of the authors [1] are generalized to the case when the GFI depends on all three coordinates. In the absence of GFI, a uniform geostrophic flow is specified along one of the horizontal axes $x$:
\[ U = -\frac{1}{f\rho} \frac{\partial \tilde{p}}{\partial y}. \]  

(1)

Here \( y \) is the second horizontal coordinate (in direction transverse to the flow), \( f \) is the Coriolis parameter (\( f \)-plane approximation is used), \( \rho \) is the pressure. It is assumed that the background density and pressure distributions (indicated by a bar) depend not only on the height \( z \), but also on one of the horizontal coordinates \( y \). For example, for analysis it is convenient to use model

\[
\bar{\rho} = \rho_0 \exp\left[-\left(\frac{h + y}{L_\rho}\right)\right], \quad \bar{p} = \rho_0 p_0 H \exp\left[-\left(\frac{h + y}{L_\rho}\right)\right], \quad U = \frac{gH}{fL_\rho} = \text{const},
\]

(2)

where axis \( z \) is directed vertically upwards; the meaning of constants \( \rho_0, H, L_\rho \) is fairly obvious. This specifying the background state in the simplest cases allows to reduce the problem to a system of equations with constant coefficients.

Disturbances introduced in this flow by gravity field inhomogeneities are studied in the linear approximation. The horizontal and vertical components of these "additional" accelerations are described respectively by the values \( g_x(x, y, z) \), \( g_y(x, y, z) \) and \( g_z(x, y, z) \). The total gravity is the vector sum of these perturbations and the average gravity, that below is indicated by a constant \( g \), as usual. If the potential of gravity is denoted by \( \Phi \), then the relations are

\[
\frac{\partial \Phi}{\partial x} = -g_x, \quad \frac{\partial \Phi}{\partial z} = g - g_z, \quad \frac{\partial g_x}{\partial z} = \frac{\partial g_z}{\partial x} \quad \text{etc.}
\]

(3)

The linearized system of equations for the stationary disturbances of velocity, pressure and density in the ideal incompressible medium [1, 3] for a three-dimensional problem has a form:

\[
\begin{aligned}
U \frac{\partial \tilde{u}'}{\partial x} &= -\frac{\partial \tilde{p}'}{\partial x} + f\tilde{v}' + \tilde{p}_g, \\
U \frac{\partial \tilde{v}'}{\partial y} &= -\frac{\partial \tilde{p}'}{\partial y} - f\tilde{u}' + \tilde{p}_g, \\
U \frac{\partial \tilde{w}'}{\partial z} &= -\frac{\partial \tilde{p}'}{\partial z} - g\rho' + \tilde{p}_g, \\
\frac{\partial \tilde{u}'}{\partial x} + \frac{\partial \tilde{v}'}{\partial y} + \frac{\partial \tilde{w}'}{\partial z} &= 0, \\
U \frac{\partial \tilde{p}'}{\partial x} + v \frac{\partial \tilde{p}'}{\partial y} + w \frac{\partial \tilde{p}'}{\partial z} &= 0.
\end{aligned}
\]

(4)

Here \( v, w \) are the disturbances of the horizontal and vertical components of velocity \( y \) and \( z \) respectively; the disturbances of other quantities are indicated by a prime.

There is given a non-permeability condition at the lower boundary of the medium. At a solid horizontal surface it has a form \( w_{|z=0} = 0 \). When \( z \to \infty \), the damping of disturbances is assumed.

This formulation of the problem generalizes the two-dimensional problem considered in the works of the authors [1, 3].

By eliminating part of the unknowns from set (4), it is not difficult to derive a system of two equations with constant coefficients

\[
\begin{aligned}
\frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{\partial^2 \tilde{w}}{\partial z^2} - \frac{1}{H} \frac{\partial \tilde{w}}{\partial z} + \frac{g}{H^2 U} \tilde{w} &= -\frac{\partial^2 \tilde{v}}{\partial y \partial z} + \frac{1}{H} \frac{\partial \tilde{v}}{\partial z} - f \frac{\partial \tilde{v}}{\partial y} + \frac{1}{H} \frac{\partial \tilde{g}_x}{\partial z}, \\
\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2} - \frac{1}{L_\rho} \frac{\partial \tilde{v}}{\partial y} - f \frac{\partial \tilde{v}}{\partial x} + f \left[ \frac{1}{gH} \frac{\partial \tilde{w}}{\partial z} - \frac{1}{UH} \frac{\partial \tilde{w}}{\partial y} + \frac{w}{gH} \frac{\partial \tilde{g}_x}{\partial z} \right].
\end{aligned}
\]

(5)
Quantity $L_p$ entering the denominator of one term is very large in comparison with other spatial scales in the given problem (if $f = 10^{-4}$ s$^{-1}$, $H = 10^4$ m, $U = 10$ ms$^{-1}$, then $L_p = 10^8$ m). Therefore, the above-mentioned term can be confidently neglected and the second equation (5) can be written in the form

$$\Delta_h v \approx -\frac{\partial^2 w'}{\partial y \partial z} + \frac{f}{U} \frac{\partial w'}{\partial z} - \frac{f}{UH} w' + \frac{f}{gH} g_x,$$

$$\Delta_h \approx \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$  \hspace{1cm} (6)

(also, the smallness of Froude number $F = U/\sqrt{gH}$ is taken into account). Applying operator $\Delta_h$ to the first equation (5), we obtain

$$\Delta_h \Delta_h w - \frac{1}{H} \frac{\partial}{\partial z} \Delta_h w' + \frac{g}{HU^2} \Delta_h w' = \left( - \frac{\partial^2}{\partial y \partial z} + \frac{1}{H} \frac{\partial}{\partial y} - \frac{f}{U} \frac{\partial}{\partial z} \right) \Delta_h v + \frac{1}{HU} \Delta_h g_x,$$

$$\Delta_h \approx \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$  \hspace{1cm} (7)

Substitution (6) in the latest equation reduces the system to a single equation for $w$. In general view, it is cumbersome, but scale analysis makes it possible to significantly simplify the problem.

We denote the characteristic horizontal scales of inhomogeneities of the field of gravity by $L_x$ and $L_y$. The characteristic vertical scale will then be of the order of the smaller among them (this can be seen easily from a scale analysis of the Laplace equation for the gravitational potential). It would be advisable to dwell on situations where the horizontal scales $L_x$, $L_y$ are of the same order of magnitude (the case $L_y \gg L_x$ is close to the previously considered two-dimensional problem, and the reverse inequality corresponds to an even less interesting case of flow directed along inhomogeneities). Let us introduce the dimensionless variables $X = x/L$, $Y = y/L$, $Z = z/L$, where $L$ – is the characteristic spatial scale of gravity field inhomogeneities. The equation (7) can be written as

$$\xi^2 \Delta_H \Delta_v - \frac{\xi F}{\partial Z} \frac{\partial}{\partial Z} \Delta_H w + \Delta_H w' - \xi^2 \xi^2 \frac{\partial^4 w}{\partial Y^2 \partial Z^2} + \frac{\xi F}{\partial Y \partial Z} - \frac{2 \xi F}{\partial Y \partial Z} \frac{\partial^2 w}{\partial Z} +$$

$$\frac{\xi^2 B}{\partial Z} \frac{\partial^2 w}{\partial Z^2} + \frac{F}{\partial Y} \frac{\partial w}{\partial Y} = \frac{U}{g} \left( \Delta_H g_x - \frac{\xi F}{\partial Y} \frac{\partial^2 g_x}{\partial Z} + \frac{F}{\partial Y} \frac{\partial g_x}{\partial Y} - \frac{\xi B}{\partial Z} \frac{\partial^2 g_x}{\partial Z} \right),$$

$$\Delta_v \approx \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}, \hspace{1cm} \Delta_H \approx \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}.$$  \hspace{1cm} (8)

The designations for dimensionless parameters are introduced here: the inverse Burger number $\xi^2 = B = f^2 L_0^2 / gH$, the Rossby number $Ro = U/L$, geometric factor $\xi = H/L$. We consider the situations when spatial scales of disturbances $L$ are larger or about 100 km. With such characteristic spatial scales and for the above-mentioned values of the remaining parameters, all dimensionless coefficients in the equation (8) are small. By neglecting the terms with small coefficients, we obtain:

$$\Delta_H w \approx \frac{U}{g} \Delta_H g_x, \hspace{1cm} w \approx \frac{U g x}{g}.$$  \hspace{1cm} (9)

Its meaning is completely clear: it describes flowing of surfaces with equal gravitational potential. But this solution does not meet the boundary condition $w|_{Z=0} = 0$. The latter is also understandable: if the underlying surface does not coincide with any equipotential surface, then the non-permeability
condition on the underlying surface is obviously incompatible with the ideal flow around the curved isosurfaces of the potential. Therefore, a relatively thin boundary layer [1] is formed near the underlying surface; in which the solution (9) is substantially violated. In deriving (9), the terms with small coefficients for the highest derivatives by \( z \) in the dimensionless equations were not taken into account. For a correct description of the boundary layer, it is necessary to take them into account.

Let us dwell on the important limiting case for the small Rossby numbers: \( L_{x,y} \gg U/f \) (in addition to the above assumptions, in particular, \( L_{x,y} \gg H \)). In this case, the trapped disturbances are generated, and a scale analysis results in a significant simplification of the system (5):

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} - \frac{1}{H} \frac{\partial w}{\partial z} + \frac{g}{HU^2} w \approx \frac{1}{H} \frac{\partial v}{\partial y} - \frac{f}{U} \frac{\partial v}{\partial z} + \frac{1}{HU} g_x,
\]

\[\Delta_h v \approx \frac{f}{U} \frac{\partial w}{\partial z} - \frac{f}{UH} w + \frac{f}{gH} g_x.\]

Let us estimate the ratio of the two terms in the right part of the first of these equations:

\[\left(\frac{1}{H} \frac{\partial v}{\partial y} / \frac{f}{U} \frac{\partial v}{\partial z}\right) \approx \frac{U \Delta z}{fH}.\]

Here \( \Delta z \) is the characteristic vertical scale on which the velocity components (the thickness of the above-mentioned boundary layer) vary significantly. If this scale is less or on the order of the characteristic thickness of atmosphere \( H \), then ratio (11) is much less than unity, and the first term on the right-hand side of the first equation (10) can be neglected (the smallness of this term can be verified further a posteriori). Then system (10) is reduced to the single equation

\[\Delta_h \Delta_v w - \frac{1}{H} \frac{\partial \Delta_v w}{\partial z} + \frac{g}{HU^2} \Delta_h w + \left(\frac{f}{U}\right)^2 \frac{\partial^2 w}{\partial z^2} - \frac{1}{H} \left(\frac{f}{U}\right)^2 \frac{\partial w}{\partial z} \approx \frac{1}{HU} \Delta_h g_x - \frac{f^2}{gHU} \frac{\partial g_x}{\partial z}.\]

In order to obtain a solution in an explicit analytical form, we consider the simplest model with periodic on \( x \) and \( y \) inhomogeneity of the gravity field:

\[g_x = G \exp(-kz) \cos(k_x x) \cos(k_y y), \quad k_{x,y} = \frac{2\pi}{L_{x,y}},\]

\[g_y = -G \frac{k_y}{k_x} \exp(-kz) \sin(k_x x) \sin(k_y y), \quad k = \left(k_x^2 + k_y^2\right)^{1/2},\]

\[g_z = -G \frac{k}{k_x} \exp(-kz) \sin(k_x x) \cos(k_y y),\]

where \( G \) is GFI amplitude. The solution for disturbances is also sought in the form of a horizontal harmonic; in particular,

\[w(x,z) = W(z) \cos(k_x x) \cos(k_y y).\]

For the amplitude function \( W \) the approximate equation is derived

\[\frac{d^2 W}{\partial z^2} - \frac{1}{H} \frac{\partial W}{\partial z} - \frac{gk^2}{Hf^2} W \approx -\frac{GUk^2}{Hf^2} \exp(-kz)\]

(the smallness of the dimensionless parameter \( f^2 / kg \) is taken into account). The general solution of the last equation is sought in the standard way in the sum of the general solution of the homogeneous
equation and the particular solution of the inhomogeneous $W_n$. The last, taking into account the smallness of the dimensionless parameters $f^2 / kg$, $Hf^2 / g$, has the form

$$W_n \approx U \frac{G}{g} \exp(-kz). \quad (14)$$

This is a special case of the approximate solution (9). To meet the boundary condition on the surface, it is necessary to add the corresponding solution of the homogeneous equation. Its characteristic equation has the form

$$\sigma^2 - \sigma / H - gk^2 / Hf^2 = 0. \quad (15)$$

Let, for example, $k = 3 \cdot 10^{-6}$ m$^{-1}$, that corresponds to a minimum half-wave length of an inhomogeneity of the order of 1000 km. In this case, for the values of the other parameters indicated above, the inverse Burger number $B = f^2 / gHk^2$ is of the order of $10^{-2}$, so we limit ourselves to the case of small values $B$, when

$$\sigma \approx \pm \frac{k}{f} \left( \frac{g}{H} \right)^{1/2}. \quad (16)$$

According to the boundary conditions, we choose a negative value $\sigma = -kS$, where $S = (1 / f)(g / H)^{1/2}$ is the dimensionless number, so that the solution for the vertical velocity has the form

$$w \approx U \frac{G}{g} \left[ \exp(-kz) - \exp(-Skz) \right] \cos(k_x x) \cos(k_y y). \quad (17)$$

For the components of horizontal velocity an approximate solution is obtained:

$$u' \approx \frac{k_y}{k_x} \frac{f}{k_y} \frac{g}{G} \left[ \exp(-kz) - S \exp(-Skz) \right] \sin(k_x x) \sin(k_y y),$$

$$v \approx \frac{f}{k} \frac{G}{g} \left[ \exp(-kz) - S \exp(-Skz) \right] \cos(k_x x) \cos(k_y y).$$

Qualitatively, the solution has similarities to the previously considered two-dimensional problem [1]. For the parameter values under consideration, the dimensionless parameter $S \approx 300$. This means that a boundary layer of thickness on the order $(kS)^{-1}$ that ensures the fulfillment of the boundary condition (5) is added to slowly dampening expressions of the type (9), (14). In the numerical example considered, the thickness of the boundary layer is of the order of 1 km.

In the three-dimensional problem, inhomogeneities of equipotential surfaces can be flowed not only from above, but also in horizontal direction. Therefore in continuity equation the terms $\partial u^\prime / \partial x$, $\partial v / \partial y$ can be main ones, but not $\partial u^\prime / \partial x$, $\partial w / \partial z$, as in the two-dimensional problem.

A boundary layer appears on the underlying surface, due to the fact that the flow around curved equipotential surfaces is incompatible with the non-permeability condition (absence of vertical velocity components) at the lower horizontal boundary. In this boundary layer, noticeable perturbations of buoyancy, pressure, and horizontal velocity can occur. The characteristic amplitudes of the velocity deviations due to local anomalies of gravity are obtained on the order of the product of the deviations of the geoid and the Brunt-Väisälä frequency. For highly anomalous regions, these amplitudes can reach several tenths of meters per second.
An analogy of atmospheric disturbances caused by thermal inhomogeneities of the underlying surface and inhomogeneities of the gravitational field

A comparative analysis of linear stationary models of mesoscale flows associated with thermal inhomogeneities of the underlying surface and with inhomogeneities of the gravitational field is performed. For both types of problems, there are considered the linear perturbations in a stably stratified semi-bounded volume of a medium rotating around a vertical axis. The Boussinesq approximation is used; it is assumed that the density of the medium linearly depends on the perturbations of the potential temperature.

For the problem of perturbations introduced by thermal inhomogeneities of the underlying surface, the two-dimensional stationary system of equations for linear perturbations has the form

$$\begin{align*}
0 &= -\frac{\partial P}{\partial x} + \nu \Delta u + f v, \\
0 &= \nu \Delta v - f u, \\
0 &= -\frac{\partial P}{\partial z} + \nu \Delta w + \alpha g T,
\end{align*}
$$

(18)

$$\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\
\gamma w &= \kappa T.
\end{align*}
$$

(19)

Here $P$ is the ratio of the pressure perturbation to the average background density of the medium $\rho$; $T$ is temperature deviation; $\alpha$ is the thermal coefficient of the medium expansion; $\nu, \kappa$ are the exchange coefficients; $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial z^2$ is two-dimensional Laplace operator. Temperature stratification is assumed to be stable, so there is a background gradient of potential temperature. $\gamma > 0$.

On the lower boundary, stationary two-dimensional periodical horizontally thermal inhomogeneities and conditions of adhesion and impermeability are defined:

$$c \rho \kappa \frac{\partial T}{\partial z} = -Q \cos kx, \quad w = u = v = 0 \quad \text{при} \quad z = 0.
$$

(20)

Here $c$ is medium heat capacity, the meaning of parameters $Q$ и $k$ is obvious.

We seek the periodical horizontally solutions in the form

$$\begin{align*}
u(x,z) &= U(z) \sin kx, \\
v(x,z) &= V(z) \sin kx, \\
w(x,z) &= W(z) \cos kx,
\end{align*}
$$

(21)

The system of equations for amplitudes has the form

$$\begin{align*}
-k \Phi &= \nu \left( \frac{d^2 U}{dz^2} - k^2 U \right) + fV, \\
0 &= \nu \left( \frac{d^2 V}{dz^2} - k^2 V \right) + fU,
\end{align*}
$$

(22)

$$\begin{align*}
d\Phi/dz &= \nu \left( \frac{d^2 W}{dz^2} - k^2 W \right) + \alpha g \theta, \\
kU + \frac{dW}{dz} &= 0, \\
\gamma W &= \kappa \left( \frac{d^2 \theta}{dz^2} - k^2 \theta \right)
\end{align*}
$$

(23)

Excluding from the last system all unknowns except $W$, we obtain the equation

$$\left( \frac{d^2}{dz^2} - 1 \right)^3 W = -\operatorname{Ta} \frac{d^2 W}{dz^2} + RW,
$$

(24)

$$R = \frac{N^2}{\kappa \nu k^4}, \quad \operatorname{Ta} = \frac{f^2}{\nu k^4} = \left( \frac{4}{kh_E} \right)^4, \quad N = (\alpha g T)^{1/2}, \quad h_E = \left( \frac{2\nu}{f} \right)^{1/2}.
$$

The buoyancy frequency $N$, dimensionless variable $Z = kZ$ are introduced here; the control dimensionless parameters $R > 0, \operatorname{Ta} > 0$ are some analogues of Rayleigh and Taylor numbers.
The solution of equation (24) is standardly sought in the form of a linear combination of exponentials of type \( \exp(\sigma Z) \). The characteristic equation has the form

\[
(\sigma^2 - 1) = -Ta\sigma^2 + R.
\]  

(25)

Given the attenuation at \( z \to \infty \), the vertical velocity solution is a linear combination of three exponentials

\[
w(x, z) = \sum_{j=1}^{3} C_j \exp(k\sigma_j z)\cos kx.
\]  

(26)

From here, the remaining unknowns are easy to express.

It makes sense to limit ourselves to the atmospheric values of the parameters under consideration, for which the inequalities are fulfilled

\[1 << R^{2/3} << Ta << R.\]  

(27)

In this case, three roots of the characteristic equation (25) are expressed as follows:

\[
\sigma_1 \approx -\left(\frac{R}{Ta}\right)^{1/2} = -\left(\frac{\nu}{\kappa}\right)^{1/2} \frac{N}{f}, \quad \sigma_{2,3} \approx -(1 \pm i)\left(\frac{Ta}{4}\right)^{1/4},
\]  

(28)

at that

\[|\sigma_{2,3}| = Ta^{1/4} >> |\sigma_1| >> 1.\]  

(29)

The approximate solution has the form:

\[
u \approx \frac{\nu}{\kappa} \left(\frac{N}{f}\right)^2 \frac{kQ}{\rho_0\gamma} \left[ \exp(-z/h_B) - \frac{2}{\delta} \exp(-z/h_E)\sin(z/h_E) \right] \sin kx,
\]  

\[
v \approx -\frac{fQ}{\rho_0\kappa kN} \left[ \exp(-kz) - \exp(-z/h_B) \right] \sin kx,
\]  

\[
w \approx \left(\frac{\nu}{\kappa}\right)^{1/2} \frac{N}{f} \frac{kQ}{\rho_0\gamma} \left[ \exp(-z/h_B) - \exp(-z/h_E) \left(\sin(z/h_E) + \cos(z/h_E)\right) \right] \cos kx,
\]  

\[
T \approx -\frac{fQ}{(\nu\kappa)^{1/2} \rho_0 kN} \left[ \exp(-z/h_B) - \left(\frac{\kappa}{\nu}\right)^{1/2} \frac{f}{N} \exp(-kz) \right] \cos kx.
\]  

(30)

Here the small parameter \( \delta = (2R)^{1/2} / Ta^{3/4} \) and spatial scales are introduced.

\[
h_B = (k\sigma_1)^{-1} = \frac{1}{k} \left(\frac{R}{Ta}\right)^{1/2} \left(\frac{\kappa}{\nu}\right)^{1/2} \frac{f}{kN}, \quad h_E = \frac{1}{k} \left(\frac{Ta}{4}\right)^{1/4} = \left(\frac{2\nu}{f}\right)^{1/2}.
\]

In the problem on disturbances caused by inhomogeneities of gravity field, there is supposed investigation of linear two-dimensional disturbances related to the presence of gravity inhomogeneities when its acceleration is \( g = (g_x(x, z), 0, -g + g_z(x, z)) \), where \( g_x, g_z \) are small deviations, connected by the ratios (3). In this case, the first and third dynamic equations, compared to (18), are modified as follows:

\[
0 = -\frac{\partial P}{\partial x} + \nu \Delta u + f v + g_x, \quad 0 = -\frac{\partial P}{\partial z} + \nu \Delta w + ag T + g_z.
\]  

(31)

At the lower boundary, instead of the first condition (20), it is assumed that there are no temperature deviations:

\[T = 0 \quad \text{at} \quad z = 0.\]  

(32)
Far from the lower boundary, it is assumed that a static regime exists in the absence of heat transfer, i.e., without taking into account the influence of the underlying surface (horizontal heat transfer in the geometry of the problem is insignificant). The latter means that isobars, isopycnics, and isolines far from the lower boundary coincide with equipotential surfaces, and velocity perturbations decay. We denote by $\Phi$ and $\eta$, respectively, the deviations of the potential of gravity and the vertical deviations of equipotential surfaces associated with GFI. By definition, $\eta = -\Phi / g = \int g_x dx / g$, where the lower limit of integration is the “reference” point at which the mentioned deviations are absent. Accordingly, the upper boundary condition for the temperature disturbance has the form:

$$\theta \to -\gamma \eta = -\gamma \int g_x dx / g \quad \text{at} \quad z \to \infty$$

As in the previous problem, we consider one horizontal harmonic of the disturbing force:

$$g_x = G \exp(-kz) \sin kx, \quad g_z = G \exp(-kz) \cos kx,$$

where $G$ is the amplitude, $k^{-1}$ is the spatial scale of gravity force inhomogeneity correspondingly; we are looking for harmonic solution, similar to (21).

Despite the modification of the original equations, as can be easily verified, Eq. (24) and the roots of the characteristic equation remain unchanged, and the solution turns out to be close to (30).

This looks nontrivial, since in one case the disturbances are related to the inhomogeneity of the boundary conditions (heat flows at the lower boundary), and in the other, with the inhomogeneous volume gravity forces under homogeneous boundary conditions of the first kind. The relationship between the amplitudes of such flows has been found. Gravity field inhomogeneities with amplitude $G$ (m/s²) lead to the appearance of flows with approximately the same amplitude and structure as horizontal inhomogeneities of a heat flow at the underlying surface (W/m²) with amplitude

$$Q_{eff} \approx \left(\kappa \nu\right)^{1/2} c\bar{\gamma}NG / fg. \quad (33)$$

4. On the possibility of "accumulation" of atmospheric effects associated with gravity field inhomogeneities

The effect of GFI on atmospheric dynamics in some respects is similar to the effects of weak slopes of the underlying surface — in both cases, the direction of gravity is somewhat deviated from the normal to the surface. Therefore, for the problems under consideration, many years of experience in studying slope flows in the atmosphere can be useful.

Let, for example, a liquid (gaseous) medium in a gravitational field be bounded below by a cooling inclined surface. Cooled at the surface and, therefore, a denser layer of the medium will flow down under its own weight. In an inhomogeneous gravitational field, when there is a component tangential to the horizontal surface, the situation is similar. The well-known Prandtl solution describes a stationary flow in a stably stratified medium over an infinite cooled surface:

$$u = \theta_0 \sqrt{\frac{ag}{\gamma}} \exp\left(-\frac{\xi}{\eta}\right) \sin \xi, \quad \theta = \theta_0 \exp\left(-\frac{\xi}{\eta}\right) \cos \xi,$$

$$\xi = \frac{n}{h}, \quad h = \left(\frac{4\kappa \nu}{ag \gamma \sin^2 \varphi}\right)^{1/4} = \left(\frac{2\sqrt{\kappa \nu}}{N \sin \varphi}\right)^{1/2}. \quad (34)$$

Here $\varphi$ is the angle of inclination of the underlying surface to the horizon, $u$ is the component of velocity along this surface, axis $n$ is perpendicular to that, $\theta$ is a temperature deviation from background (for the air in atmosphere – deviation of the potential temperature), $\theta_0$ is the given value of this deviation at the surface.
The physical meaning of this solution seems quite clear. When \( \theta_0 < 0 \), along the slope a cooled layer flows down with a thickness of the order of \( h \), increasing with diminishing angle \( \varphi \). Warmer volumes of the medium come from above, they are cooled from the surface, etc.

According to the Prandtl' solution, the maximal absolute value of velocity \( u_{\text{max}} \) is achieved at a height of the order of \( h \) and is about

\[
0.3 \left| \theta_0 \right| \sqrt{\frac{\alpha g}{\gamma}} = 0.3 \left| \theta_0 \right| \frac{\alpha g}{N}.
\]

The absolute value flux of mass along slope

\[
M \sim u_{\text{max}} h \sim \left| \theta_0 \right| \sqrt{\frac{4\alpha g \kappa v}{\gamma^3 \sin^2 \varphi}}.
\]

At first glance, it seems paradoxical that the value \( u_{\text{max}} \) does not depend on the slope angle \( \varphi \), i.e. there is no passage to the limit \( \varphi \to 0 \) toward the case of a horizontal surface, when there are no physical reasons for the occurrence of any flows. Even more surprising seems the fact that the mass flow along the slope at \( \varphi \to 0 \) tends to infinity. We also note that when stratification tends to be neutral \( (\gamma, N \to 0) \), the velocity, as well as thickness \( h \) and flux of mass \( M \) tend to infinity.

The solution seems even more paradoxical if at the lower boundary \( n = 0 \) there is given the boundary condition of the second kind (the disturbance of heat flux \( Q \)) – a condition that is often more justified for geophysical problems:

\[
c_p \rho \kappa \frac{d \theta}{dn} = -Q \quad \text{при} \quad n = 0.
\]

In this case, the solution can be written formally in the same form as (4), where now instead \( \theta_0 \) the quantity is

\[
\Theta_0 = \frac{Q h}{c_p \rho \kappa} = \frac{Q}{c_p \rho} \left( \frac{2}{N \sin \varphi} \sqrt{\frac{v}{\kappa^3}} \right)^{1/2},
\]

that also has meaning of temperature at the lower boundary \( n = 0 \). Since the denominator of the last expression contains \( \sin \varphi \), at \( \varphi \to 0 \) the amplitude of the velocity disturbance \( u_{\text{max}} \) not only does not vanish, but, on the contrary, tends to infinity.

These paradoxical properties of the slope flows (intense response to small surface slopes) were analyzed in detail in [4]. The fact is that with a decrease in the tilt angles, the time for establishing stationary solutions increases. At small time intervals, the effects of weak slopes are negligible, but they accumulate at sufficiently large times. In particular, with stable stratification of the medium, the above mentioned stationary solutions are established in a time of the order of \( \tau = \left( \frac{N \sin \varphi}{s^{-1}} \right)^{1/2} \). If \( N = 10^{-2} \) s\(^{-1} \), \( \varphi = 10^{-4} \) (tilt angle of equipotential surfaces characteristic of intense gravity anomalies), then \( \tau \) is of the order of 10 days. Thus, the effects of small surface tilts (weak GFI), in principle, can accumulate over time.

5. Conclusion
The performed analysis shows that the influence of GFI leads to perturbations of the horizontal velocity of relatively small amplitude – up to several tenths of ms\(^{-1}\) even in highly anomalous regions. But it must be borne in mind that the solutions obtained imply the existence of stable, albeit slow, but ordered flows over large territories. Consequently, the effect of GFI has a quasi-systematic character,
manifested primarily in the systematic errors of mathematical modeling of the atmosphere. The obtained results do not pretend to the conclusion that the effects under consideration are significant, but it seems that in any case they indicate, which disturbances can be induced by the considered mechanism, and allow reasonably decide to whether or not to take into account the influence of GFI.

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