Distributed Task Management in the Heterogeneous Fog: A Socially Concave Bandit Game

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Abstract

Fog computing has emerged as a potential solution to the explosive computational demand of mobile users. This potential mainly stems from the capacity of task offloading and allocation at the network edge, which reduces the delay and improves the quality of service. Despite the significant potential, optimizing the performance of a fog network is often challenging. In the fog architecture, the computing nodes are heterogeneous smart devices with distinct abilities and capacities, thereby, preferences. Besides, in an ultra-dense fog network with random task arrival, centralized control results in excessive overhead, and therefore, it is not feasible. We study a distributed task allocation problem in a heterogeneous fog computing network under uncertainty. We formulate the problem as a social-concave game, where the players attempt to minimize their regret on the path to Nash equilibrium. To solve the formulated problem, we develop two no-regret decision-making strategies. One strategy, namely bandit gradient ascent with momentum, is an online convex optimization algorithm with bandit feedback. The other strategy, Lipschitz Bandit with Initialization, is an EXP3 multi-armed bandit algorithm. We establish a regret bound for both strategies and analyze their convergence characteristics. Moreover, we compare the proposed strategies with a centralized allocation strategy named Learning with Linear Rewards. Theoretical and numerical analysis shows the superior performance of the proposed strategies for efficient task allocation compared to the state-of-the-art methods.

Index Terms

Distributed task management, equilibrium, fog computing, multi-armed bandit.

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I. INTRODUCTION

In the past years, the world has witnessed the explosive demand for excessively resource-consuming wireless services and applications such as online gaming and on-demand streaming. Besides, the ubiquitous connectivity allows countless devices to form networks for different purposes instead of functioning individually. The data dynamism and heterogeneity resulting from this expected explosive expansion of connected devices together with demanded services require significant processing and knowledge extraction capabilities [1]. Therefore, effective computing platforms that handle extortionate heterogeneous data and devices are essential parts of future network architectures.

Cloud computing is a paradigm that allows accessing a set of shared and configurable computing resources (e.g., networks, servers, storage facilities, applications) [2]. As such, it virtually enables an unlimited storage capacity and also processing power [3]. The concept, which first emerged in [4] in 2007, has attracted immense attention since then, becoming an inseparable component of the Internet of Things [5] due to its dynamic characteristics such as elasticity, reduced management efforts, and flexible pricing model (pay-per-use) [6]. At the same time, cloud computing suffers from some shortcomings, the most fundamental ones being delay and excessive bandwidth consumption, both caused by the long (physical or logical) distance between the centralized cloud servers and devices [7].

To address the challenges, a core idea is to utilize the continuously-growing computational ability of handheld devices such as smartphones. The fog computing paradigm realizes this idea by extending the cloud architecture towards the network edge. Indeed, it brings the computation in the proximity of end-users by introducing a hierarchy of computing capacities such as fog nodes and cloudlets between edge and cloud. The architecture exploits the idle communication and computational resources at the demand side to provide low-latency computing, storage, and network services.

In the future ultra-dense networks with ubiquitous connectivity, the end-users offload a wide range of tasks, including transmission, sensing, and signal processing. Besides, a wide range of powerful end-user, edge, and access devices act as fog nodes. Such nodes are distinct concerning software and hardware; thus, they have different characteristics and capabilities and perform offloaded tasks at various quality and efficiency levels. In realizing the true potential of fog computing, efficient allocation of the offloaded tasks to the fog nodes plays a crucial role. The challenge becomes aggravated in the absence of prior knowledge and in a distributed structure, where avoiding information exchange is essential to retain low signaling and feedback overhead. Besides, the fog nodes might be reluctant to share their resources for extra tasks to avoid expensive computation and energy loss unless they receive appropriate compensation [8]. Hence,
it is crucial to consider the trade-off between individual interest and social welfare. Based on the discussion above, it is imperative to develop task allocation policies that optimize the consumption of computational resources while guaranteeing an acceptable quality of service to the end-users. Such methods should be amenable to distributed implementation and robust to information shortage and uncertainty.

A. Related Work

Xiao et al. present a system that allocates the cloud computing resources dynamically based on application demands [9]. The authors introduce the concept of “skewness” to quantify the multidimensional resource utilization of a server. By minimizing the skewness, the method combines different types of workloads, thereby improving the overall utilization of server resources. Reference [10] develops a systematic framework to solve a workload allocation problem in cloud-fog computing. The authors solve the allocation problem by decomposing it into several subproblems: (i) workload allocation in fog and cloud servers based on the power consumption and time delay, and (ii) minimizing communication delay in dispatch between fog and cloud servers. In [11], the authors propose an efficient approximate solution for offloading and allocation decisions between users and the cloud. To that end, they use separable semidefinite relaxation, taking the computation- and communication energy, also the delay, into account. Then a computing access point, which serves as the network gateway and a computation service provider, is connected with the extended mobile cloud computing [12]. Indeed, the connection replaces direct interaction between the cloud and users.

The cutting-edge research described above solves the task sharing and resource pooling problems using centralized optimization methods, often given precise information about the crucial variables. Nevertheless, such methods often suffer from excessive complexity and heavy signaling- and feedback overhead. Besides, centralized methods are blind to potentially conflicting incentives and benefits of fog nodes. One potential solution to these challenges is to delegate the decision-making about sharing resources and performing tasks to the fog nodes. In such a case, the fog nodes decide based on the potential reward to compensate for their consumption of resources and computation [8], [13]. Therefore, individual incentives play a remarkable role. Besides, each fog node selects tasks given only local information, thereby decreasing the communication overhead significantly. Finally, the fog nodes divide the burden of computation rather than barging on a central controller. In the past few years, a large body of research has resorted to methods based on game theory and reinforcement learning to solve the problem of distributed allocation of generalized resources, including radio- and computational resources, in computing platforms. Kaewpuang et al. propose
a decision-making framework in a mobile cloud computing environment. The problem constitutes resource allocation and revenue management via cooperation formation among mobile service providers \cite{14}. They model the cooperation formation among providers as a distributed game and use the best-response dynamics as a solution. A similar algorithm concerning cooperation under uncertainty appears in \cite{15}. The method is generic and applies to a wide range of application, including fog computing. In \cite{16}, the authors formulate the task offloading problem as a matching game with externalities. They propose a strategy based on the deferred acceptance algorithm that enables efficient allocation in a distributed mode and ensures stability over the matching outcome. References \cite{17} and \cite{18} consider the task offloading problem from a game-theoretic perspective. They model the problem as a potential game that admits a pure strategy Nash equilibrium. The former uses the finite improvement property to achieve Nash equilibrium, whereas the latter proposes a best-response adaption algorithm to achieve Nash equilibrium. Besides, \cite{18} extends their framework and proposes a near-optimal solution for resource allocation to address the time complexity of determining the equilibrium. In \cite{19}, the authors model the task offloading problem as a minority game and obtain the Nash equilibrium. Reference \cite{20} uses a reverse auction model for resource allocation and task management in a computation offloading platform as a part of a software-defined network.

While the research described above uses conventional models from game theory, a few papers allow for uncertainty in the game model or irrational players’ behavior. For example, in \cite{21}, the authors study a task allocation problem among heterogeneous cyber-physical systems under state uncertainty, where task preferences of each cyber-physical system depend on the state. They use the concept of deterministic equivalence and sequential core to solve the problem and use the Walrasian auction process to implement the core. As another example, in \cite{22}, the authors propose a supply-demand market model for task allocation, where the fog nodes and offloading user, respectively, represent the sellers that set the service prices and the consumer that uses the services at the given prices. Unlike previous research, this work takes the potential irrational decision-makers into account.

Alongside game theory, reinforcement learning is another widely-used mathematical tool to efficiently share or allocate resources under uncertainty. In \cite{23}, the authors propose a multi-armed bandit setting with linear rewards to solve the combinatorial user-channel allocation problem. Reference \cite{24} develops a decision-making method based on multi-armed bandit theory to select the most suitable server in a dynamic time-variant network. In \cite{25}, the authors attempt to minimize the computation latency by proposing an online learning algorithm. Reference \cite{26} develops a two-stage task offloading approach. In the first stage, the algorithm designs a contract that specifies the contribution and associated reward to encourage fog servers to share resources.
The second stage consists of the upper confidence bound (UCB) method to connect the user and fog server. In [8], a similar two-stage resource sharing and task offloading approach is developed, which extends [26] by integrating contract theory with computational intelligence.

B. Motivation and Contributions

The cutting-edge research studies the task allocation and resource pooling problems under a variety of objectives; nevertheless, it neglects the following aspects to a great extent: i) Selfishness of the fog servers; (ii) Heterogeneity of the fog nodes in terms of task preferences and abilities; iii) Divisibility of tasks; iv) Partial feedback and limited information.

In this paper, we investigate the distributive task allocation problem in a distributed fog computing architecture. Against the cutting-edge research, our model and solution offers the following novelty and contribution:

- Our system model is generic as we allow for arbitrary heterogeneous fog nodes in terms of capacity and capability. We quantify their distinct characteristics and preferences using a well-designed utility function.
- We consider a realistic scenario, where the fog nodes do not have any prior information about each others’ types, the tasks’ utility, and cost.
- Taking the selfishness and rationality of the intelligent fog nodes into account, we model the task allocation problem as a sequential decision-making game. Each fog node makes decisions based on the learned task preference and average computational cost.
- We prove that the task allocation game is a social-concave game in the bandit setting, which converges to the Nash equilibrium when every player uses a no-regret learning strategy to select tasks. Besides, we prove that the Nash equilibrium in the modeled task allocation game is unique. Based on [27] and [28], we propose two decision-making policies and prove their performance characteristics such as regret bound and convergence.
- We compare the proposed strategies to an existing centralized method from different perspectives and prove that the proposed strategies are more efficient in solving the task allocation problem of fog computing.
- Through intensive numerical analysis, we prove the uniqueness of Nash equilibrium in task allocation game and evaluate the performance of our scheme in comparison to several methods based on different principles.

The rest of paper is organized as follows. Section II presents the system model and basic assumptions. In section III we formulate the problem of task sharing among heterogeneous entities under uncertainly. In Section IV we model the formulate task allocation problem as a game and analyze the existence and uniqueness of Nash equilibrium. In section V we develop
and analyze two no-regret decision-making strategies that converge to Nash equilibrium: The first one is based on bandit gradient descent (section V-A), while the second one is based on Lipschitz Bandit (section V-B). Section VI describe a centralized strategy that we use as the benchmark for performance evaluation and comparison. Section VII includes numerical analysis and the subsequent discussions. Section VIII concludes the paper and suggests some directions for future research.

II. System Model

We consider a fog computing system consisting of $K$ fog nodes gathered in the set $\mathcal{K} = \{1, 2, \ldots, K\}$. Besides, there is a task pool, where the end-user devices offload a set of $M$ tasks denoted by $\mathcal{M} = \{1, 2, \ldots, M\}$. Fig. 1 shows an instant of such fog computing system.

Each fog node decides to which task(s) and at which level it contributes. The vector $x_k = (x_{k,1}, \ldots, x_{k,M})$, $0 \leq x_{k,m} \leq 1$, gathers the fraction of each task $m \in \mathcal{M}$ that shall be performed by the fog node $k \in \mathcal{K}$. For every task $m \in \mathcal{M}$, we assume that $\exists k \in \mathcal{K}$ so that $x_{k,m} > 0$; that is, at least one fog node is interested in performing some fraction of each task. As mentioned before, the fog nodes have different types concerning ability and capacity; therefore, they have different task preferences. We quantify such a type heterogeneity using a performance index vector for every fog node $k \in \mathcal{K}$ as $\rho_k = (\rho_{k,1}, \ldots, \rho_{k,M})$. Besides, for each fog node $k \in \mathcal{K}$, we characterize the cost, e.g., the memory cost or delay, using a cost index vector $\kappa_k = (\kappa_{k,1}, \ldots, \kappa_{k,M})$.

After each fog node $k \in \mathcal{K}$ announces its preferred task share $x_k$, the tasks are allocated among all fog nodes to satisfy their request as far as possible. For the fog node $k$, we denote the allocation vector by $a_k = (a_{k,1}, \ldots, a_{k,M})$, where

$$a_{k,m} = \frac{x_{k,m}}{\sum_{i \in \mathcal{K}} x_{i,m}},$$

Fig. 1: Distributed task allocation among the fog nodes.

(1)
where $\sum_{i \in K} x_{i,m} > 0$ by assumption. The mechanism described by (1) corresponds to a proportional allocation mechanism, where every fog node receives a fraction of the task equivalent to its requested proportion divided by the sum of all fog nodes requested proportion.

For every fog node $k \in K$ that performs $a_{k,m}$ fraction of task $m \in M$, the type-dependent reward is given by (2)

$$\varphi_{k,m}(a_{k,m}) = \rho_{k,m}(1 - e^{-a_{k,m}}).$$

By (2), for a specific fraction of task $m$, a higher performance index (more suitable type) results in more utility. In addition, after submitting its preferred task share, each fog node will reserve some resources to perform the task. For a fog node $k \in K$, the cost of resource reservation is proportional to the requested task fraction. Formally,

$$c_{k,m}(x_{k,m}) = \kappa_{k,m} x_{k,m}.$$ 

Therefore, the total utility of the fog node $k \in K$ yields

$$u_k(x_k, X_{-k}) = \sum_{m \in M} \rho_{k,m}(1 - e^{-a_{k,m}}) - \kappa_{k,m} x_{k,m},$$

where $X_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K)$ denotes the joint decision of all fog nodes excluding $k$.

In the following, we state two propositions concerning the utility function defined in (3).

**Proposition 1.** For every $k \in K$, the utility function defined in (3) satisfies the Lipschitz condition

$$|u_k(x) - u_k(y)| \leq L \cdot |x - y|, \text{ for any } x, y \in [0, 1]^M.$$  

*Proof. See Appendix IX-B*

**Proposition 2.** For every $k \in K$, the utility function defined in (3) is concave in the proposal of fog node $k$, $x_k$, and convex in the actions of other fog nodes, $X_{-k}$.

*Proof. See Appendix IX-C*

### III. Problem Formulation

The task management procedure follows in successive rounds, $t = 1, 2, \ldots, T$. To introduce the decision-making rounds to the notation, we add the superscript $t$. As the task arrival is random, the task share, thereby the utility and the cost of each fog node is random. Formally,

1 Alternatively, one can use the *barrier to entry* concept, i.e., add a small constant to the denominator. [29].
at each round \( t \), by proposing \( x^t_k \), the total utility of fog node \( k \in K \) yields \( u^t_k(x^t_k, X^t_{-k}) + N_0 \), where \( N_0 \) is a zero-average Gaussian noise; that is, the fog node observes a noisy version of the average utility resulted from \((x^t_k, X^t_{-k})\), given by \((3)\).

Primarily, each selfish fog node \( k \in K \) opts to maximize its accumulated utility. Formally, the optimization problem to solve is given by

\[
\text{maximize } \sum_{t=1}^{T} \sum_{m=1}^{M} \rho_{k,m} (1 - e^{-\frac{s^t_{k,m}}{\sum_{i \in K} x^t_{i,m}}}) - \kappa_{k,m} x^t_{k,m}.
\]  

Solving optimization problem \((5)\) is not feasible as (i) the fog nodes do not have any prior information about the task arrival and the utility functions, (ii) the utility of each fog node depends on the action of all other nodes, and (iii) after each round of decision-making, each fog node only observes the utility of the performed action and receives no other feedback.

The scenario described above corresponds to a multi-armed bandit setting \([30]\), in which the performance measure follows the conventional concept of expected total regret. We define the cumulative regret of fog node \( k \) up to time \( T \) as

\[
R^T_k(x_k) = \max_{x_k} \sum_{t=1}^{T} u^t_k(x_k, X^t_{-k}) - \sum_{t=1}^{T} u^t_k(x^t_k, X^t_{-k}).
\]  

Then, instead of solving \((5)\), each fog node minimizes its accumulated regret. Regret minimization procedures correspond to decision-making with bounded rationality due to limited information availability. In the multi-agent setting, they enable every agent to have a guarantee on its utility regardless of the actions of others \([31]\). Formally,

\[
\text{minimize } R^T_k(x_k).
\]  

Besides maximizing individual utility, from a distributed system perspective, the fog nodes must achieve a steady-state. Reaching such a steady-state in the absence of a centralized controller is not trivial. In Section \( IV \) and Section \( V \) we solve this problem using a social-concave bandit game model and no-regret learning strategies.

### IV. Task Allocation Game

We define the described task allocation game \( \Gamma \) by a tuple \( \prec K, \{S_k\}_{k \in K}, \{x_k\}_{k \in K}, \{\varphi_k\}_{k \in K}, \{u_k\}_{k \in K} \succ \), where: (i) \( K \) is the set of players, implying that each fog node is a player; (ii) \( S_k \in [0,1]^M \) is player \( k \)’s strategy space; (iii) \( x_k \in [0,1]^M \) is player \( k \)’s action; (iv) \( \varphi_k : [0,1]^M \to \mathbb{R} \) is the value function of player \( k \) assigning a value/reward function to allocation; (v) \( u_k(x_k, X_{-k}) : \bigotimes_{k \in K} S_k \to \mathbb{R} \) is the utility function of player \( k \), defined as \( u_k(x_k, X_{-k}) = \varphi_k(a_k(x_k, X_{-k})) - \kappa_k x_k \).
Before proceeding to analyze the task allocation game, we present some concepts.

**Definition 1** (Nash Equilibrium [31]). A strategy profile \(X^* = (x^*_1, \ldots, x^*_k, \ldots, x^*_K)\) is called Nash equilibrium if for all \(k \in K\) and all actions \(x'_k\), strategy profile \(X = (x^*_1, \ldots, x'_k, \ldots, x^*_K)\) yields

\[
u_k(x^*_k, X_{-k}) \geq u_k(x'_k, X_{-k}).\]

A strategy profile \(X^*\) is called \(\epsilon\)-Nash equilibrium if for all \(k \in K\) and all actions \(x'_k\), strategy profile \(X^*\) yields

\[
u_k(x^*_k, X_{-k}) \geq u_k(x'_k, X_{-k}) - \epsilon.\]

**Definition 2** (Concave Game [32]). A game \(\Gamma\) is concave if for all \(k \in K\), \(u_k(x_k)\) is a concave function in \(x_k\) for every fixed \(X_{-k} \in S_{-k}\).

**Definition 3** (Socially Concave Game [31]). A game \(\Gamma\) is socially concave if

(i) \(\exists \{\lambda_k\}_{k \in K}, \lambda_k > 0, \sum_{k \in K} \lambda_k = 1\), such that the weighted sum of utility functions \(\sigma(X, \lambda) = \sum_{k=1}^{K} \lambda_k u_k(x_k)\) is a concave function in every \(x_k\), where \(X = (x_1, \ldots, x_k, \ldots, x_K)\) is the joint strategy profile.

(ii) The utility function of each player \(k \in K\) is convex in the actions of the other players, i.e., for every \(x_k \in S_k\), the function \(u_k(x_k, X_{-k})\) is convex in \(X_{-k} \in S_{-k}\).

The following propositions describe some properties of the task allocation game \(\Gamma\) concerning equilibrium.

**Proposition 3.** The task allocation game \(\Gamma\) by a tuple \(<K, \{S_k\}_{k \in K}, \{x_k\}_{k \in K}, \{\varphi_k\}_{k \in K}, \{u_k\}_{k \in K}>\) is a socially concave game and a concave game.

*Proof.* See Appendix IX-D.

**Proposition 4.** If the task allocation game \(\Gamma\) converges to a Nash equilibrium, then that equilibrium is unique.

*Proof.* See Appendix IX-E.

V. **E**FFICIENT **E**QUILIBRIUM IMPLEMENTATION VIA **N**O-**R**EGRET **B**ANDIT **S**TRATEGY

As formulated by (7), in the task allocation procedure, each fog node \(k \in K\) aims at minimizing its regret. Furthermore, as discussed in Section IV from a system perspective, the fog nodes’ interactions must converge to a steady-state or equilibrium. In this section, we model the task
allocation game as a bandit game. To solve the formulated game, we propose two bandit learning strategies based on the bandit gradient descent (BGD) algorithm [27] and Lipschitz Bandit (LB) algorithm [28]. In the following, we describe the developed decision-making policies and establish a regret bound. More precisely, we prove that both proposed strategies are no-regret, meaning that they guarantee sub-linear regret growth. Table I summarizes the frequently-used notations of this section.

Before proceeding, we note the following: By (3), the utility of every fog node $k \in K$ is additive over all tasks $m \in M$. Besides, the fog node selects the fraction of tasks to perform independently of each other. Therefore, the multi-task allocation problem boils down to $M$ independent single-task allocation problems with the same set of participants.
A. Bandit Gradient Ascent with Momentum

The BGD strategy calculates the one-point approximation of the gradient. It then uses that estimation to the online convex optimization algorithm proposed by Zinkevich [33]. Our proposed method differs from the BGD strategy in two crucial aspects: (i) we use adaptive learning rate instead of the fixed learning rate, and (ii) we add momentum in the update rule to accelerate the convergence.

The BGD policy only applies on a symmetric action space; therefore, we shift the action space $[0, 1]$ to $[-\xi, \xi]$, with $\xi$ being a positive constant. Formally, let $y^t$ be the running parameter and $z^t$ denote the shifted action so that

$$z^t = y^t + \sigma c^t.$$  

with $c^t$ being a unit vector selected uniformly at random. As $z^t$ is the shifted action, we have

$$x^t_{k,m} = z^t + \xi.$$  

Besides, we define the function $u_t(z^t)$, which is the utility function with the shifted action space; that is, $u^t(z^t) = u^t_{k,m}(z^t + \xi) = u^t_{k,m}(x^t_{k,m})$. Let [27]

$$\hat{u}^t(y^t) = \mathbb{E}_{c^t \in \mathbb{B}}[u^t(y^t + \sigma c^t)] = \mathbb{E}_{c^t \in \mathbb{B}}[u^t(z^t)],$$  

where $\sigma$ is a positive constant, and $\mathbb{B} = \{x \in \mathcal{R} ||x| \leq 1\}$ is a unit ball centered around the origin in action space. Thus $y^t$ can be interpreted as the approximation of the shifted action.

Lemma 1 ([27]). Fix $\sigma > 0$, over the random unit vectors $c$,

$$\mathbb{E}[u(y + \sigma c)c] = \sigma \nabla \hat{u}(y).$$  

According to Lemma 1 in the algorithm BGAM, we can use

$$\sigma \nabla \hat{u}^t(y^t) = u^t(z^t)c^t = u^t_{k,m}(x^t_{k,m})c^t$$  

to estimate the gradient.

As mentioned previously, we further enhance the algorithm with momentum, which is a trick in gradient-based optimization algorithms. It allows gradient-based optimizers to speed up along low curvature directions. Here, we apply the momentum with the one-point gradient estimation that provably achieves the same enhancement as in gradient algorithms. Hence the update rule
follows as
\[ v^t = \beta v^{t-1} + \sigma \nabla \hat{u}^t(y^t), \quad (15) \]
\[ y^{t+1} = P_{(1-\alpha)S}(y^t + v^t v^t), \quad (16) \]
where \( \alpha \) and \( \beta \) are constant numbers. Besides, \( P(\cdot) \) is the projection function \( P_S(y) = \min_{x \in S} \|x - y\| \).

The reason to obtain \( y^{t+1} \) in \((1-\alpha)S\) is to avoid the value of \( z^t \) out of the defined action range \[27\]. Algorithm 1 summarizes the proposed BGAM decision-making strategy.

**Algorithm 1** Bandit Gradient Ascent with Momentum

for fog node \( k \in K \) and task \( m \in M \)

**Parameters:** \( \xi = 0.5 \)

**Initialization:** \( y^1 = 0, v^1 = 0 \).

1: for \( t = 1, 2, 3, \ldots, T \) do
2: Select unit vector \( c_t \) uniformly at random;
3: Calculate \( z^t \) using (10);
4: Take action \( x^t_{k,m} \) using (11);
5: Estimate the gradient using (14);
6: Update \( v^t \) using (15);
7: Obtain \( y^{t+1} \) using (16);
8: end for

1) **Complexity:** BGAM algorithm only stores the updated parameters. As such, the space requirement is \( O(1) \). The total runtime at \( T \) is \( O(T) \).

2) **Regret Analysis:** In the following, we establish a regret-bound for the BGAM algorithm. Before proceeding to the theory, it is essential to mention that introducing a momentum and adaptive learning rate in BGAM do not influence its convergence. Rather, they accelerate the convergence rate.

**Proposition 5.** Let \( u^1, u^2, \ldots, u^T : S \to \mathbb{R} \) be a sequence of concave and differentiable functions. Besides, \( g^1, g^2, \ldots, g^T \) are the single-point estimation of gradient with \( g^t = \nabla \hat{u}^t(y^t) \) and \( \|g^t\| \leq G \). Let \( L \) be the Lipschitz constant of utility function. Assume that each \( y^t \) generated by Algorithm 1 satisfies \( \|y^i - y^j\|_2 \leq D \), for all \( i, j \in \{1, 2, \ldots, T\} \) and \( D \) being a positive constant. Select \( v^t = \frac{\nu}{\sqrt{t}}, \sigma = T^{-0.25} \sqrt{\frac{RUr}{3(Lr+U)}}, \text{ and } \alpha = \frac{2}{r} \), where \( U \) is the maximum value of utility function. Also, \( r \) and \( R \) satisfy the condition that shifted strategy space \( S \) contains the ball of radius \( r \) centered at the origin and is contained in the ball of radius \( R \) \[27\]. Here, we select...
\( r = R = \xi \). Then the decision-making policy BGAM guarantees the following regret bound:

\[
E[R(T)] \leq \tilde{O}(T^{\frac{3}{4}})
\]  

(17)

**Proof.** See Appendix IX-F.

\[ \square \]

**B. Lipschitz Bandit with Initialization**

In the seminal MAB problem, the set of arms is finite. In many applications, that setting only serves as an imprecise model of the situation, which potentially forces the learner to pull only suboptimal arms, thereby causing a linear regret \([34]\). In contrast, the Continuum-Armed Bandit (CAB) defines the set of arms over a continuous space. Such a setting has attracted intensive attention in the past few years due to its ability to model general situations. In particular, it fits the problems where the expected reward is a Lipschitz function of the arm, known as \textit{Lipschitz Bandits}.

**Proposition 6.** For fog node \( k \), the task allocation game is a Lipschitz multi-armed bandit problem.

**Proof.** See Appendix IX-G.

\[ \square \]

A simple approach to solving the MAB problems with continuous arm space is discretizing the action space. However, determining the optimal quantization intervals is challenging and has a remarkable impact on the regret bound. Reference \([28]\) proposes a decision-making policy based on discretization, which consists of two phases: In the first phase, the method explores uniformly. It thus finds a rather crude estimate of the Lipschitz constant and determines the optimal number of intervals for discretization. In the second phase, the method finds the optimal quantization interval using a standard exploration-exploitation strategy.

The BGAM algorithm proposed in Section V-A only requires the bandit feedback; nevertheless, the information about the Lipschitz constant is necessary to optimize the hyperparameter according to Proposition 5 that achieves the lower regret bound of expected regret is achieved. However, the Lipschitz bandit algorithm proposed in \([28]\) does not impose such a limitation, as the available approximation of the Lipschitz constant suffices for optimization. Besides, it guarantees a sublinear regret growth.

1) \textit{Lipschitz Bandit with Initialization:} Based on the properties of the task allocation game, here we adopt the seminal \textit{Exponential-weight algorithm for Exploration and Exploitation} (EXP3), whose core idea is to assign some selection probability to each discretized action at every trial, which is proportional to the exponentially weighted accumulated reward of that action so far.
Reference [28] shows that the optimal number of discretization intervals directly depends on the number of rounds $T$; i.e., a longer game requires more discretization intervals. Nevertheless, too many periods expand the exploration rate dramatically, hence reducing the efficiency and wasting computational resources. Therefore, we modify the first phase of the seminal Lipschitz bandit strategy to estimate the Lipschitz constant.

Besides, we enhance that policy by adding several steps between the two phases. More precisely, we implement an initialization process for the second phase, which uses the collected information during the first phase. Despite adding some memory cost this method accelerates the convergence.

In the first phase of the LBWI strategy, the agent quantizes the action space into $N$ intervals (arms), where $N$ is a random positive integer. Afterward, it performs pure exploration by pulling each arm $A$ times. We use $\lambda^{t}_{k,m}$ to denote the selected arm at time $t$, where $\lambda^{t}_{k,m} \in \{0, 1, \ldots, N-1\}$ is an integer to indicate the selected discretized interval. To model a general situation with continuous action space, the real action $x^{t}_{k,m}$ is sampled from the corresponding interval as

$$x^{t}_{k,m} = \text{uniform}_\text{sample}\left(\frac{\lambda^{t}_{k,m}}{N}, \frac{\lambda^{t}_{k,m} + 1}{N}\right).$$

Let $\hat{\mu}_{k,m} \in \mathbb{R}^{N}$ be a vector to memorize the average utility for specific arm and $A^{t}_{k,m} \in \mathbb{R}^{N}$ to memorize the number that specific arm has been selected,

$$\hat{\mu}^{t}_{k,m}[n] = \begin{cases} \frac{\hat{\mu}^{t-1}_{k,m}[n] \cdot A^{t-1}_{k,m}[n] + u^{t}_{k,m}}{A^{t-1}_{k,m}[n] + 1}, & \text{if } \lambda^{t}_{k,m} = n, \\ \hat{\mu}^{t-1}_{k,m}[n], & \text{otherwise,} \end{cases}$$

$$A^{t}_{k,m}[n] = \begin{cases} 1 + A^{t-1}_{k,m}[n], & \text{if } \lambda^{t}_{k,m} = n, \\ A^{t-1}_{k,m}[n], & \text{otherwise}. \end{cases}$$

At the same time, in the first phase a weighted vector $\Omega_{k,m} \in \mathbb{R}^{N}$ and a probability vector $p_{k,m} \in \mathbb{R}^{N}$ related to the utility are recorded. The update rule of the weighted matrix here is the same as that in EXP3 strategy. The first phase provides an approximation of the Lipschitz constant as

$$\hat{L} = N \max_{n \in \{0, 1, \ldots, N-1\}} \max_{i \in \{-1, 1\}} |\hat{\mu}^{T_1}_{k,m}[n] - \hat{\mu}^{T_1}_{k,m}[n + i]|,$$

$$\tilde{L} = \hat{L} + N \sqrt{\frac{2}{A} \ln(2NT)}.$$  

Algorithm [2] summarizes the proposed LBWI decision-making strategy in the first phase.
Algorithm 2 Lipschitz Bandit with Initialization Strategy—Pure Exploration Phase

**Parameters:**
- \( N \): number of intervals for discretization in Phase I;
- \( \gamma \): Real constant number \( \gamma \in (0, 1] \);

**Initialization:**
- \( \Omega \): Weight matrix with \( \Omega_{k,m}^{1}[n] = 1, n = 0, \ldots, N - 1 \);

1: for \( t = 1, 2, 3, \ldots, T_1 \) do
2: Select the arm \( \lambda_{k,m}^{t} \) randomly;
3: Sample action \( x_{k,m}^{t} \) randomly according to (18) and get the utility \( u_{k,m}^{t} \);
4: Update the average utility vector \( \hat{\mu}_{k,m} \) according to (19) and action selection matrix \( A_{k,m} \) according to (20);
5: Set \( p_{k,m}^{t}[n] = (1 - \gamma) \frac{\Omega_{k,m}^{t}[n]}{\sum_{i=1}^{N} \Omega_{k,m}^{t}[i]} + \frac{\gamma}{N}, n = 0, 2, \ldots, N - 1 \);
6: Set \( \Omega_{k,m}^{t+1}[n] = \begin{cases} \Omega_{k,m}^{t}[n] \exp \left\{ \frac{\gamma u_{k,m}^{t} N p_{k,m}^{t}[n]}{\Omega_{k,m}^{t}[n]} \right\}, & \text{if } \lambda_{k,m}^{t} = n \\ \Omega_{k,m}^{t}[n], & \text{otherwise} \end{cases} \)
7: end for
8: Obtain \( \hat{L} \) using (21) and \( \tilde{L} \) using (22).

The number of intervals in Phase II is related to the approximated Lipschitz constant as

\[
\tilde{N} = N \left\lceil \frac{\tilde{L}^{\frac{2}{3}} T_1^{\frac{1}{3}}}{N} \right\rceil.
\]

As described before, besides estimating the Lipschitz constant and the optimal number of discretization intervals, we use the weight matrix \( \Omega_{k,m}^{T_1} \) to offer some prior information to Phase II. Nonetheless, as the number of intervals changes in Phase II, it is essential to redistribute the weights \( \Omega_{k,m}^{T_1} \) in Phase I to initialize the weights \( \omega_{k,m}^{T_1+1} \) in Phase II. Algorithm 3 summarizes the redistribution process and the EXP3 decision-making strategy in second phase.

2) **Complexity:** The complexity of the two-phase Lipschitz bandit algorithm is more than that of BGAM. In the exploration phase, the runtime is \( O(T_1) \), \( T_1 << T \), whereas the runtime of the EXP3 phase is \( O(T^{\frac{1}{3}}) \) because of the round-related discretization number \( \tilde{N} \). Therefore, its total runtime yields \( O(T^{\frac{1}{3}}) \). The total space requirement is \( O(T^{\frac{1}{3}}) \).

3) **Regret Analysis:** In the following, we establish a regret-bound for the LBWI algorithm.

**Proposition 7.** Let \( L \) be the Lipschitz constant of utility function and let \( H \) be the uniform bound of utility function’s Hessians. Select \( N \) for division in phase I satisfying \( N \geq \frac{8H}{L} \). With
Algorithm 3 Lipschitz Bandit with Initialization Strategy—EXP3 Phase

Initialization:
\[ \omega_{k,m}^{T_1+1} \in \mathbb{R}^{\tilde{N}}; \text{ Weight matrix, } n = 0, 1, 2, \ldots, \tilde{N} - 1; \]

for \( n = 0, 1, \ldots, \tilde{N} - 1 \) do
\[ i = \lceil \frac{n}{\tilde{N}} \rceil, \omega_{k,m}^{T_1+1}[n] = \frac{N}{\tilde{N}} \Omega_{k,m}[i]; \]
end for

1: **EXP3 Strategy:**
2: for \( t = T_1 + 1, \ldots, T \) do
3: Set \( p_{k,m}^t[n] = (1 - \gamma) \frac{\omega_{k,m}^t[n]}{\sum_{i=1}^{\tilde{N}} \omega_{k,m}^t[i]} + \frac{\gamma}{\tilde{N}}, n = 0, 1, \ldots, \tilde{N} - 1; \)
4: Select arm \( \lambda_{k,m}^t \) according to probability matrix \( p_{k,m}^t; \)
5: According to selected arm sample the corresponding proportion \( x_{k,m}^t \) with
\[ x_{k,m}^t = \text{uniform_sample}(\frac{\lambda_{k,m}^t}{\tilde{N}}, \frac{\lambda_{k,m}^t + 1}{\tilde{N}}); \] (24)
6: Receive the reward/utility \( u_{k,m}^t \) and update the weight matrix as
\[ \omega_{k,m}^{t+1}[n] = \begin{cases} \omega_{k,m}^t[n] \exp \left\{ \frac{\gamma u_{k,m}^t}{N p_{k,m}^t[n]} \right\}, & \text{if } \lambda_{k,m}^t = n \\ \omega_{k,m}^t[n], & \text{otherwise} \end{cases} \]
7: end for

Proof. See Appendix IX-H

Proposition 8. Consider the task allocation game formulated in Section IV. If all of the fog nodes implement BGAM or LBWI decision-making strategies, then the game converges to the unique Nash equilibrium.

Proof. See Appendix IX-I

VI. LEARNING WITH LINEAR REWARDS STRATEGY

Learning with linear rewards (LLR) strategy [23] is a decision-making policy for the stochastic combinatorial multi-armed bandit problem with linear rewards, i.e., when the total reward is a linearly-weighted combination of the selected random variables. The policy combines the upper confidence bound (UCB) and combinatorial optimization strategies such as maximum weighted matching. We briefly review this strategy and use it later as a benchmark for numerical
Different with proposed strategies, in LLR strategy the tasks are not divisible and every node selects one task at most, meaning \( x_{k,m} \in \{0, 1\} \) and \( \sum_{m \in \mathcal{M}} x_{k,m} \leq 1 \). Therefore, if \( M > K \), some tasks remain unassigned. To store the information after taking an action at a time slot, the policy uses two metrics, namely sample mean matrix \( \hat{\theta}_{K \times M} \) and observed times matrix \( C \), defined as

\[
\hat{\theta}_{k,m}^t = \begin{cases} 
\hat{\theta}_{k,m}^{t-1} - \frac{u_{k,m}^t}{c_{k,m}^{t-1} + 1}, & \text{if } x_{k,m}^t = 1, \\
\hat{\theta}_{k,m}^{t-1}, & \text{otherwise},
\end{cases}
\]

(26)

\[
C_{k,m}^t = \begin{cases} 
1 + C_{k,m}^{t-1}, & \text{if } x_{k,m}^t = 1, \\
C_{k,m}^{t-1}, & \text{otherwise}.
\end{cases}
\]

(27)

For initialization, the LLR policy runs a random play to ensure that every action is played at least once \( (C_{k,m} > 0) \). Afterward, unlike distributed strategies, it uses the maximum weighted matching to allocate tasks in a centralized way. Given graph \( G = (V, E) \), and \( V = \{\mathcal{K}, \mathcal{M}\} \) is the vertex set includes the fog nodes and tasks. Upper confidence bound is utilized to model the weights associated with each edge \( (k, m) \) as the following equation

\[
\mathcal{W}_{k,m}^t = \hat{\theta}_{k,m}^t + \sqrt{\frac{(\Xi + 1) \ln t}{C_{k,m}^t}},
\]

(28)

where \( \Xi = \min\{K, M\} \). Algorithm 4 summarizes the LLR strategy.

Algorithm 4 Learning with Linear Rewards

1: for \( t = 1, 2, 3, \ldots, T_1 \) do
2: Select the arm \( x_{k,m}^t \) randomly;
3: Update the average utility vector \( \hat{\theta}_{k,m} \) according to (26) and action selection matrix \( C_{k,m} \) according to (27);
4: end for
5: for \( t = T_1 + 1, \ldots, T \) do
6: Play an action which solves the maximum weight matching problem with weights matrix \( W \) according to (28);
7: Update the average utility vector \( \hat{\theta}_{k,m} \) according to (26) and action selection matrix \( C_{k,m} \) according to (27).
8: end for

I) Analysis: Unlike BGAM and LBWI, LLR is a centralized policy. It requires all fog nodes to share their information with a central coordinator, thus increasing the overhead and computational cost. Besides, it allocates each task only as a whole and does not divide any task. That aspect, despite reducing the computational cost slightly, substantially accelerates the regret growth. More
precisely, the regret increases linearly with time. Also, concerning the privacy aspects and the selfishness of the fog nodes, BGAM and LBWI are more efficient than LLR. We establish this claim also through numerical analysis in the next section.

VII. NUMERICAL ANALYSIS

We divide the numerical analysis into two parts. In Section VII-A, we consider a toy scenario with two fog nodes and two tasks. The goal is to clarify the workflow of the developed task allocation schemes. Then, in Section VII-B, we expand the scenario significantly by increasing the number of tasks and fog nodes. The goal is to analyze the performance of the proposed strategies compared with the state-of-art solutions.

A. Game I

1) Model Parameter: In the two-server two-tasks allocation game, we show the two fog nodes’ action profiles as $[x_{11}, x_{12}]$. Table II shows the performance index $\rho$ and the basic energy consumption $\kappa$, where $\rho_{k,m}, \kappa_{k,m}, k, m \in \{1, 2\}$ respectively correspond to the performance index and basic energy consumption of node $k$ in task $m$. Fig. 2 shows the 3D plots of each node’s utility function. It indicates that the utility functions are Lipschitz constant. Also, if Node 2 refuses to participate in Task 1, the utility of both fog nodes becomes maximum. Similarly, if Node 1 does not contribute to Task 2, both fog nodes’ utility is maximum, which is consistent with the mathematical proofs as well as the intuition of decision-making: If the cost consumption index is higher than the performance index for some specific task, then the fog server prefers to avoid performing that task. For example, for $\rho_{1,2} \leq \kappa_{1,2}$, then Node 1 hesitates to contribute to Task 2.

2) Results and Discussions: In this section, we compare the performance of BGAM and LBWI policies with that of BGD and LB, respectively. Note that BGD and the proposed BGAM algorithm belong to the class of bandit gradient optimization algorithms, while the Lipschitz

| Paramter: | $\rho$ | $\kappa$ |
|-----------|--------|----------|
| FN Task   | $T_1$  | $T_2$    | $T_1$  | $T_2$  |
| FN1       | 0.9    | 0.1      | 0.1    | 0.4    |
| FN2       | 0.2    | 0.8      | 0.45   | 0.05   |
**Fig. 2:** 3D plots of the utility of two fog nodes.

EXP3 bandit, also with initialization, is a type of Lipschitz bandit strategy. **Fig. 3** shows the evolution of the average actions of each fog node in bandit gradient optimization algorithms. The results demonstrate that the average actions converge. Besides, the convergence point is consistent with the conclusion that we drew from the utility plots. Both $x_{1,2}$ (fraction of Task 2 for Node 1) and $x_{2,1}$ (fraction of Task 1 for Node 2) converge towards zero while the action of Node 1 in Task 1 and Node 2 in Task 2 keep searching in a specific action range. The reason is that, compared to choosing not to contribute at all, it is much more difficult to find the optimal action value. Finally, BGAM converges faster than BGD. Besides, we compare the evolution of actions of LBWI and LB. As described before, the first phase of the Lipschitz bandit strategies involves only uniform random selection; as such, we omit the selection frequency histograms for the initial phase. **Fig. 4** demonstrates the selection frequency of different action fractions between $[0, 1]$ by the LBWI and LB strategies after round $t = 45000$. We observe the same convergence tendency as **Fig. 3**. After 90% of rounds, the average for actions with LBWI strategy is $\bar{x}_{1,1} \approx 0.6$, $\bar{x}_{1,2} \approx 0.15$, $\bar{x}_{2,1} \approx 0.15$, and $\bar{x}_{2,2} \approx 0.6$. These results approve that LBWI is an effective strategy in the task allocation game. Compared to LB, LBWI performs better in identifying the suitable action fraction range, which means that initialization after the first phase enhances the performance of the Lipschitz bandit strategy. Nevertheless, compared to BGAM, the average action is not ideal, especially concerning $x_{1,2}$ and $x_{2,1}$, which shall be zero according to the analysis. The reason is that the Lipschitz bandit strategies sample the action fraction from the interval; thus, it is hard to achieve the boundary values such as zero or one.
Fig. 3: Actions of two fog nodes that use the bandit gradient algorithms.

Besides, in a lengthy game, the interval is divided very often; that increases the computational cost and expands the exploration period. Consequently, the performance of BGAM is superior to that of LBWI.

Fig. 4: Histogram of actions’ frequency of two fog nodes in the multi-armed bandit algorithms after round $t = 45000$.

To validate the uniqueness of Nash equilibrium and to compare the performance of the proposed strategies, in Fig. 5 we show the average actions within a small number of rounds near the end. We do the performance evaluation in different contexts, namely, online learning and game theory:

- **Greedy Projection (GP)** [33]: Select an arbitrary initial value $x_{k,m}^1$ and a sequence of learning rate $\eta^1, \eta^2, \ldots \in \mathbb{R}^+$. At time step $t$, select the next vector $x_{k,m}^{t+1}$ as

$$ x_{k,m}^{t+1} = P(x_{k,m}^t + \eta^t \nabla u_{k,m}(x_{k,m}^t)). $$
Best Response (BR) \cite{36}: Given $X_{-k}$ as the strategies of all players excluding player $k$, then the player $k$’s best response strategy is

$$x_k^* = \arg\max_{x_k} u_k(x_k, X_{-k}).$$

Fig. 5: Average action in the final 1000 rounds.

Both GP and BR strategies require full feedback. Given that, they converge swiftly. Besides, their convergence points are almost the same, i.e., they converge to the same Nash equilibrium. Finally, among four bandit-feedback strategies, the average actions of BGAM have the minimum distance to the optimal actions at Nash equilibrium. The average action profile of LBWI is comparable to that of BGD despite more deviation. Fig. 6 shows the average utility of each node generated by different strategies. This figure confirms the results that appear in Fig. 5. BGAM offers the highest utilities compared to other strategies with bandit feedback. Besides, both BGAM and LBWI perform better than their former version, BGD, and LB, respectively. In other words, our proposed modifications are remarkably effective, especially to solve the task allocation problem. The utility of LBWI is slightly less than BGD, which is due to its deviation. The utility of both fog nodes increases over time; However, unlike bandit gradient strategies, the Lipschitz bandit strategies suffer a low, non-increasing utility at the beginning, which arises due to the random selection in the first phase, and vanishes later. Also, in the second phase, the Lipschitz bandit strategies execute exploration in a vast space $O(T^{\frac{1}{2}})$, which escalates the computational cost. Therefore, especially in a long-run intensive sequential experiment, Lipschitz
bandit strategies spend plenty of time and computational resources in exploration, which degrades their performance compared to the bandit gradient algorithms.

![Utilities of FN 1 and FN 2](image)

**Fig. 6**: Utilities of two fog nodes achieved by different strategies.

**B. Game II**

1) **Model Parameters**: In this section, for rigorous performance evaluation, we consider a larger network with five fog nodes \((K = 5)\) that perform seven tasks \((M = 7)\) cooperatively. The matrix below gathers the performance- and cost indices of fog nodes.

\[
\rho_{k,m} = \begin{bmatrix}
0.95 & 0.32 & 0.87 & 0.75 & 0.21 & 0.13 & 0.98 \\
0.71 & 0.93 & 0.52 & 0.31 & 0.09 & 0.88 & 0.21 \\
0.11 & 0.53 & 0.82 & 0.17 & 0.93 & 0.21 & 0.89 \\
0.23 & 0.79 & 0.12 & 0.29 & 0.33 & 0.71 & 0.34 \\
0.45 & 0.16 & 0.32 & 0.83 & 0.67 & 0.31 & 0.10 
\end{bmatrix},
\]

\[
\kappa_{k,m} = \begin{bmatrix}
0.05 & 0.45 & 0.27 & 0.15 & 0.46 & 0.33 & 0.11 \\
0.33 & 0.09 & 0.41 & 0.32 & 0.30 & 0.11 & 0.44 \\
0.42 & 0.35 & 0.20 & 0.36 & 0.23 & 0.41 & 0.25 \\
0.37 & 0.17 & 0.32 & 0.11 & 0.47 & 0.08 & 0.07 \\
0.36 & 0.28 & 0.19 & 0.22 & 0.10 & 0.29 & 0.34 
\end{bmatrix}.
\]
Note that in such a setting, there are \((2^7)^5 = 34359738368\) combinations for only binary selection, and with the continuous action set \(s\), the set of possible allocation profiles becomes infinite.

As discussed in Section VII-A, Greedy Projection and Best response dynamics reach the Nash equilibrium easily with the help of full-information feedback; As such, they count as optimal strategies. Consequently, we use their obtained utility values at the convergence point as the maximum achievable utility to measure the regret of proposed policies. Besides the LLR strategy described in Section VI, we use the uniformly random action selection as a benchmark.

Fig. 7 and Fig. 8 respectively show the cumulative regret and the expected average regret based on ten-time average to eliminate outliers. Based on this figure, we conclude the followings:

- BGAM has the best performance with the least regret and the swiftest convergence. Compared with BGD, momentum and adaptive learning rate play a significant role in that achievement.
- The overlap part of LBWI, LB, and RS at the beginning is due to the pure random exploration in the first phase of algorithms LBWI and LB.
- The essential information required by the decision-making policies decreases in the following order: LLR, BGD(BGAM), LBWI, and LB; Therefore, their regret increases in the reverse order. More precisely, compared to the BGD and LB, LLR needs more information among the fog nodes at each iterative time round; hence the regret decreases faster than BGD and LB at the beginning. However, unlike other policies, the LLR’s regret growth is linear with time. The reasons are its indivisible allocation and the one-agent-one-task matching rule. Therefore, in the long run, LLR incurs a higher regret whose average does not tend to zero. On the horizon of the figure, BGD’s regret is comparable to that of LLR; nevertheless, its tendency to decrease is observable, whereas LLR’s average regret remains almost fixed. Finally, because of adaptive learning rate and the momentum, BGAM converges faster than BGD, and achieves much better performance than LLR, even with less information.
- The cumulative regrets of LBWI and LB are more than others due to the following reasons. Both methods have an initial phase of pure exploration in the action space, and for the LB, the number of intervals concluded from the first phase is always large (> 50), which extends exploration. Because of our modifications, the initialization with information collected in the first phase is assistive in LBWI; hence, it incurs a lower cumulative regret than the LB strategy.

Remark 1. According to Fig. 8 only after \(t = 25000\) the average regret decreases to less than a reasonable threshold (< 10%). That seems to be implausible in several applications; Nevertheless, as stated by Theorem 3 one can suffice to a near-optimal solution, namely, \(\epsilon\)-Nash
equilibrium, instead of achieving the unique Nash equilibrium. That shortens the convergence time significantly with negligible performance degradation.

**Fig. 7:** Total cumulative regret of different algorithms.

**Fig. 8:** Expected dynamic regret of different algorithms.

**VIII. CONCLUSION**

We studied the task pooling problem in a fog computing infrastructure. Every fog node has some preference over tasks, which is initially unknown. The self-interested fog nodes aim to divide the tasks in a distributed manner so that the outcome of consecutive interactions is an efficient equilibrium. We formulate the scenario as an online convex optimization problem. We developed two decision-making policies for the formulated problem, namely BGAM and LBWI. Theoretically, we established that both methods guarantee a sublinear upper bound for
the regret growth, and they converge to the unique Nash equilibrium. Numerical results showed
that the proposed BGAD algorithm achieves a close-to-optimal utility performance superior to
the existing algorithms.

IX. APPENDIX

A. Some Auxiliary Results

Here, we state some auxiliary results and materials from game theory and MABs that are
necessary for the proofs.

Theorem 1 ([37], [38]). Let \( f : U \rightarrow \mathbb{R} \) be a twice continuously differentiable function defined
on \( U \subset \mathbb{R}^n \), i.e. \( f \in C^2(U) \). Then

(i) \( f \) is convex if and only if the Hessian matrix \( D^2 f(x) \) is positive semidefinite for all \( x \in U \),
i.e.,

\[
\langle D^2 f(x) h, h \rangle \geq 0, \text{ for any } h \in \mathbb{R}^n .
\]  

(29)

If \( \langle D^2 f(x) h, h \rangle > 0 \), for any \( h \in \mathbb{R}^n \setminus \{0\} \), then \( f \) is strictly convex.

(ii) \( f \) is concave if and only if the Hessian matrix \( D^2 f(x) \) is negative semidefinite for all \( x \in U \),
i.e.,

\[
\langle D^2 f(x) h, h \rangle \leq 0, \text{ for any } h \in \mathbb{R}^n .
\]  

(30)

If \( \langle D^2 f(x) h, h \rangle < 0 \), for any \( h \in \mathbb{R}^n \setminus \{0\} \), then \( f \) is strictly concave.

Lemma 2 ([31]). Consider a socially concave game \( \Gamma \). If for every player \( k \in K \) the utility
function \( u_k(x_k, X_{-k}) \) is twice differentiable in every \( x_k \in S_k \), then \( \Gamma \) is a concave game.

Definition 4 (Diagonally Strictly Concave [39]). Define a weighted sum of utility functions
\( \sigma(X, r) = \sum_{k=1}^{K} r_k u_k(X) \), where \( r = \{r_1, \ldots, r_K\}, r_k \geq 0, \forall k \). The pseudogradient of \( \sigma(X, r) \) for any nonnegative \( r \) is defined as

\[
\nabla \sigma(X, r) = \begin{bmatrix}
    r_1 \nabla_1 u_1(X) \\
    \vdots \\
    r_K \nabla_K u_K(X)
\end{bmatrix},
\]  

(31)

where \( \nabla_k \) refers to the gradient with respect to \( x_k \).
The function \( \sigma(X, r) \) is called diagonally strictly concave (DSC) for a given non-negative \( r \) if
for every distinct pair $X^0, X^1 \in S$, we have
\[
(X^1 - X^0)^T(\nabla \sigma(X^1, r) - \nabla \sigma(X^0, r)) < 0.
\] (32)

**Theorem 2** ([39]). Consider a game for which the constraint set is orthogonal and the function $\sigma(X, r)$ is diagonally strictly concave with positive definite $r$. Then, if a Nash equilibrium exists, it is unique.

**Theorem 3** ([31]). Consider a socially concave game $\Gamma$ with $K$ players. If every player $k$ plays according to a procedure with external regret bound $R_k(t)$, then at time $t$, the followings hold:

(i) The average strategy vector $\hat{X}^t$ is an $\epsilon$-Nash equilibrium, where $\epsilon_t = \frac{1}{\lambda_{\min}} \sum_{i \in K} \frac{\lambda_i R_i}{t}$ and $\lambda_{\min} = \min_{i \in K} \lambda_i$.

(ii) The average utility of each player $k$ is close to her utility at $\hat{X}^t$, the average vector of strategies. Formally,
\[
|u^t_k - u_k(\hat{X}^t)| \leq \frac{1}{\lambda_k} \sum_{i \in K} \frac{\lambda_i R_i}{t}.
\] (33)

**Lemma 3** ([31]). Consider a socially concave game $\Gamma$ with $K$ players. If every player $k$ plays according to a procedure with non-external regret so that $\lim_{T \to \infty} \frac{1}{T} R_T = 0$, then the players’ joint action profile converges to a Nash equilibrium.

**Lemma 4** ([40]). Let $u^1, u^2, \cdots, u^T : S \to \mathbb{R}$ be a sequence of concave and differentiable functions. Besides, $g^1, g^2, \cdots, g^T$ are the single-point estimation of gradient with $g^t = \nabla \hat{u}^t(y^t)$ and $\|g^t\| \leq G$. Besides, $g_{1:T} = [g^1, \ldots, g^T]$, and $\beta$ is a constant. Select $\nu^t = \frac{\nu}{\sqrt{t}}$. The following holds
\[
T \sum_{t=1}^T \frac{\nu^t (v^t)^2}{\sqrt{t}} \leq \|g_{1:T}\|_4^2 \sqrt{1 + \log T} \frac{\nu}{(1-\beta)^2}.
\] (34)

**Definition 5.** $H$ is a uniform bound for the Hessians of utility function $u$ if
\[
|\langle D^2 u_k(x_k) y, y \rangle| \leq H \|y\|_\infty^2,
\] (35)
for any $x_k \in [0, 1]^M$, $y \in [0, 1]^M$.

**Lemma 5** ([28]). Let $N$ be the number of discretization intervals in the first phase of LBWI strategy. For $N \geq 3$, with probability at least $1 - \frac{1}{T}$,
\[
L - \frac{7H}{N} \leq \tilde{L} \leq L + 2N \sqrt{\frac{2}{A} \ln(2NT)},
\] (36)
where $H$ is defined in Definition 5, $L$ is the Lipschitz constant, and $\tilde{L}$ is given by (21). Besides, $A$ is the number of times each arm is pulled independently.

B. Proof of Proposition 1

The gradient of utility function defined in (3) is

$$
\nabla u_k = \begin{bmatrix}
\nabla u_{k,1} \\
\vdots \\
\nabla u_{k,M}
\end{bmatrix},
$$

(37)

where $\nabla u_{k,m} = \frac{\rho_{k,m}x_{k,m}}{(x_{k,m}+X_{-k,m})^2} \exp\left(-\frac{x_{k,m}}{(x_{k,m}+X_{-k,m})}\right) - \kappa_{k,m}$ and $X_{-k,m} = \sum_{i \in K \setminus k} x_{i,m}$.

Because $\exp\left(-\frac{x_{k,m}}{(x_{k,m}+X_{-k,m})}\right) \leq 1$ and $\frac{x_{k,m}}{(x_{k,m}+X_{-k,m})^2} \leq \frac{1}{X_{-k,m}}$, there exits an $L$, such that $|\nabla u_{k,m}| < L$, $\forall x_{k,m} \in [0, 1], k \in K, m \in M$, that completes the proof.

C. Proof of Proposition 2

According to Theorem 1 at first we need to calculate the Hessian matrix. The Hessian matrix of $u_k$ in $x_k$ is

$$
D^2 u_k(x_k) = \begin{bmatrix}
\frac{\partial^2 u_k}{\partial x_{k,1}^2} & 0 & \ldots & 0 \\
0 & \frac{\partial^2 u_k}{\partial x_{k,2}^2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{\partial^2 u_k}{\partial x_{k,M}^2}
\end{bmatrix}
$$

(38)

where $\frac{\partial^2 u_k}{\partial x_{k,m}^2} = -\rho_{k,m}e^{\frac{x_{k,m}}{x_{k,m}+X_{-k,m}}} \left(\frac{2X_{-k,m}}{(x_{k,m}+X_{-k,m})^3} + \frac{x_{k,m}^2}{(x_{k,m}+X_{-k,m})^4}\right)$ is non-positive for every $m \in M$. Thus for any $h \in \mathbb{R}^M$, it satisfies

$$
\langle D^2 u_k(x_k), h \rangle \leq 0,
$$

meaning that the utility function $u_k(x_k, X_{-k})$ is concave in $x_k$.

Similarly, the Hessian matrix of $u_k$ in $X_{-k}$ is

$$
D^2 u_k(X_{-k}) = \begin{bmatrix}
\frac{\partial^2 u_k}{\partial X_{-k,1}^2} & 0 & \ldots & 0 \\
0 & \frac{\partial^2 u_k}{\partial X_{-k,2}^2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{\partial^2 u_k}{\partial X_{-k,M}^2}
\end{bmatrix}
$$

(39)

The barrier to entry in denominator of $a_{k,m}$ will avoid the extreme value of $L$. 
where \( \frac{\partial^2 \varphi}{\partial x_{k,m}^2} = \rho_{k,m} e^{-\frac{x_{k,m}}{x_{k,m} + x_{-k,m}}} \frac{2x_{k,m}x_{-k,m} + x_{k,m}^2}{(x_{k,m} + x_{-k,m})^2} \) is non-negative for every \( m \in M \). Thus for any \( h \in \mathbb{R}^M \), it satisfies

\[
\langle D^2 u_k(X_{-k}), h \rangle \geq 0,
\]

meaning that \( u_k \) is convex in \( X_{-k} \). That completes the proof.

D. Proof of Proposition 3

According to Proposition 2, the utility function \( u_k \) is concave in \( x_k \), the proposal of fog node \( k \), and convex in the actions of other fog nodes \( X_{-k} \), which satisfies the two conditions in Definition 3. Therefore, the task allocation game \( \Gamma \) is a socially concave game. Besides, the utility function \( u_k \) is twice differentiable and therefore, by Lemma 2, \( \Gamma \) is also a concave game (Definition 2). This completes the proof.

E. Proof of Proposition 4

Define a weighted sum of the utility functions in task allocation game \( \Gamma \) as

\[
\sigma(X, r) = \sum_{k=1}^{K} r_k u_k(X) = \sum_{k=1}^{K} r_k u_k(x_k, X_{-k}),
\]

where \( \forall k, r_k > 0 \). The pseudogradient of \( \sigma(X, r) \) is defined by (31), where \( \nabla_k u_k(X) = \nabla u_k(x_k) \), which is given by (37) with

\[
\nabla u_{k,m} = \frac{\rho_{k,m} x_{-k,m}}{(x_{k,m} + x_{-k,m})^2} \exp \left( -\frac{x_{k,m}}{x_{k,m} + x_{-k,m}} \right) - \kappa_{k,m}.
\]

To prove the uniqueness of Nash equilibrium, we need to prove the inequality (32), which is equivalent to the following inequality in task allocation game \( \Gamma \):

\[
\sum_k \sum_m r_k (x^1_{k,m} - x^0_{k,m}) (\nabla u_{k,m}(x^1_{k,m}) - u_{k,m}(x^0_{k,m})) < 0.
\]

Because \( \frac{\partial^2 u_{k,m}}{\partial x_{k,m}^2} = -\rho_{k,m} e^{-\frac{x_{k,m}}{x_{k,m} + x_{-k,m}}} \frac{2x_{k,m} + x_{-k,m}^2}{(x_{k,m} + x_{-k,m})^3} \) \( \frac{2x_{k,m} + x_{-k,m}^2}{(x_{k,m} + x_{-k,m})^3} \) \( \frac{2x_{k,m} + x_{-k,m}^2}{(x_{k,m} + x_{-k,m})^3} \) < 0, \( \forall x_{k,m} \in [0, 1] \). Therefore, according to the mean value theorem, there exists \( x'_{k,m} \in [x^0_{k,m}, x^1_{k,m}] \) for any \( x^0_{k,m} \neq x^1_{k,m} \) so that

\[
\frac{\nabla u_{k,m}(x^1_{k,m}) - \nabla u_{k,m}(x^0_{k,m})}{x^1_{k,m} - x^0_{k,m}} = \frac{\partial^2 u_{k,m}}{\partial x_{k,m}^2} < 0.
\]

That is then equivalent to

\[
r_k (x^1_{k,m} - x^0_{k,m}) (\nabla u_{k,m}(x^1_{k,m}) - \nabla u_{k,m}(x^0_{k,m})) < 0,
\]
for all $r_k > 0$. Summing up with $k \in K$, $m \in M$, we conclude
\[
\sum_k \sum_m r_k(x^1_{k,m} - x^0_{k,m})(\nabla u_{k,m}(x^1_{k,m}) - \nabla u_{k,m}(x^0_{k,m})) < 0.
\]
According to Definition 4, $\sigma(x, r)$ is diagonally strictly concave. Therefore, by Theorem 2, Nash equilibrium of task allocation game $\Gamma$ is unique. That completes the proof.

F. Proof of Proposition 5
Because $u^t$ is the same as the utility function with the shifted action space, the regret
\[
R^T_{k,m}(x_{k,m}) = \max_{x_{k,m}} \sum_{t=1}^T u^t_{k,m}(x_{k,m}) - \sum_{t=1}^T u^t_{k,m}(x_{k,m})
\]
equals to
\[
R^T(z) = \max_z \sum_{t=1}^T u^t(z) - \sum_{t=1}^T u^t(z').
\] (42)
Thus, in the following, we use the regret $R^T(z)$ in (42). By [12] we have [27]
\[
\hat{u}^t(y) = \mathbb{E}[u^t(z^t)],
\]
which results
\[
\max_z \mathbb{E}\left[\sum_{t=1}^T u^t(z)\right] = \max_{(y + \sigma c)} \mathbb{E}\left[\sum_{t=1}^T u^t(y + \sigma c)\right] = \max_y \sum_{t=1}^T \hat{u}^t(y)
\]
Then, the expectation of regret $R^T(z)$ will be
\[
\mathbb{E}[R^T(z)] = \max_y \sum_{t=1}^T \hat{u}^t(y) - \sum_{t=1}^T \hat{u}^t(y').
\] (43)
Define $y^*$ as the optimal $y$ that maximize $\hat{u}^t(y)$. We can bound the expected difference between $\hat{u}^t(y^*)$ and $\hat{u}^t(y^t)$ in terms of gradient with [27]
\[
\mathbb{E}[\hat{u}^t(y^*) - \hat{u}^t(y^t)] \leq \mathbb{E}[g^t(y^* - y^t)].
\] (44)
From the update rules (15) and (16), we can conclude that
\[
y^{t+1} = P_{(1-\alpha)}S(y^t + \nu^t(\beta v^{t-1} + g^t)).
\] (45)
Following Zinkevich and Flaxman’s analysis \cite{27, 33}, we have \(|y^* - P_s(y)| \leq |y^* - y|\). Therefore,
\[
\|y^* - y^{t+1}\|^2 \leq \|y^* - y^t\|^2 - 2\nu^t(\beta\nu^{t-1} + g^t)(y^* - y^t) + (\nu^t\nu^t)^2. \tag{46}
\]
Rearranging the expression, we get
\[
g^t(y^* - y^t) \leq \frac{1}{2\nu^t}(\|y^* - y^t\|^2 - \|y^* - y^{t+1}\|^2) - \beta\nu^{t-1}(y^* - y^t) + \frac{\nu^t}{2}(\nu^t)^2,
\]
\[
\leq \frac{1}{2\nu^t}(\|y^* - y^t\|^2 - \|y^* - y^{t+1}\|^2) + \frac{\nu^t}{2}(\nu^t)^2. \tag{47}
\]
By summing inequality (47) and combining it with the inequality (44), we arrive at
\[
\sum_{t=1}^T \mathbb{E}[\hat{u}^t(y^*) - \hat{u}^t(y^t)] \leq \sum_{t=1}^T \mathbb{E}[g^t(y^* - y^t)]
\leq \frac{1}{2\nu^t}(y^* - y^1)^2 + \frac{1}{2}\sum_{t=1}^T [(y^* - y^t)^2(\frac{1}{\nu^t} - \frac{1}{\nu^{t-1}}) + \nu^t(\nu^t)^2]
\leq \frac{D^2}{2\nu^t} + \frac{D^2}{2}\left(\frac{1}{\nu^t} - \frac{1}{\nu}\right) + \sum_{t=1}^T \frac{\nu^t}{2}(\nu^t)^2
\leq \frac{D^2(1 + \sqrt{T})}{2\nu} + \sum_{t=1}^T \frac{\nu^t}{2}(\nu^t)^2. \tag{48}
\]
The second and third inequality follow from the bounding assumptions \(|y^i - y^j|^2 \leq D\). According to Lemma 4, we can conclude
\[
\sum_{t=1}^T \mathbb{E}[\hat{u}^t(y^*) - \hat{u}^t(y^t)] \leq \frac{D^2(1 + \sqrt{T})}{2\nu} + \frac{\nu\sqrt{1 + \log T}}{2}\|g_{1:T}\|_4^2 \frac{1}{(1 - \beta)^2}. \tag{49}
\]
From (14), it follows that
\[
\|g^t\| = \left\|\frac{1}{\sigma}u^t(y^t + \sigma c^t)c^t\right\| \leq \frac{U}{\sigma}, \tag{50}
\]
where \(U\) is the maximum value of utility function. According to references \cite{40, 41}, \(\|g_{1:T}\|_4^2 \leq \frac{U}{\sigma}\sqrt{T}\). Therefore, we have
\[
\max_{y \in (1-\alpha)S} \sum_{t=1}^T \hat{u}^t(y) - \mathbb{E}\left[\sum_{t=1}^T \hat{u}^t(y^t)\right] \leq \frac{D^2(1 + \sqrt{T})}{2\nu} + \frac{\nu U\sqrt{1 + \log T}}{2\sigma}\frac{\sqrt{T}}{(1 - \beta)^2}. \tag{51}
\]
In Proposition [1] the utility function has a Lipschitz constant $L$. Hence

$$|u^t(y^t) - u^t(y')| \leq \sigma L$$

$$|u^t(y^t) - u^t(z')| \leq 2\sigma L.$$ 

These imply

$$\max_{z \in (1-\alpha)S} \sum_{t=1}^{T} (u^t(z) - \sigma L) - \mathbb{E} \left[ \sum_{t=1}^{T} (u^t(z') + 2\sigma L) \right] \leq \frac{D^2(1 + \sqrt{T})}{2\nu} + \frac{\nu U \sqrt{1 + \log T}}{2\sigma} \frac{\sqrt{T}}{(1 - \beta)^2}.$$ 

$$\max_{z \in (1-\alpha)S} \sum_{t=1}^{T} u^t(z) - \mathbb{E} \left[ \sum_{t=1}^{T} u^t(z') \right] \leq 3\sigma LT + \frac{D^2(1 + \sqrt{T})}{2\nu} + \frac{\nu U \sqrt{1 + \log T}}{2\sigma} \frac{\sqrt{T}}{(1 - \beta)^2}. \quad (52)$$

Because $u^t$ is concave in $z^t$ and $0 \in S$,

$$\max_{z \in (1-\alpha)S} u^t(z) = \max_{z \in S} u^t((1-\alpha)z) \geq \max_{z \in S} (\alpha u^t(0) + (1 - \alpha)u^t(z)) \geq \max_{z \in S} (\alpha (u^t(0) - u^t(z)) + u^t(z)) \geq \max_{z \in S} (-2\alpha U + u^t(z)). \quad (53)$$

$$\max_{z \in S} \sum_{t=1}^{T} u^t(z) - \mathbb{E} \left[ \sum_{t=1}^{T} u^t(z') \right] \leq 3\sigma LT + 2\alpha UT + \frac{D^2(1 + \sqrt{T})}{2\nu} + \frac{\nu U \sqrt{1 + \log T}}{2\sigma} \frac{\sqrt{T}}{(1 - \beta)^2}. \quad (54)$$

With the above inequality (52) we have

$$\max_{z \in S} \sum_{t=1}^{T} u^t(z) - \mathbb{E} \left[ \sum_{t=1}^{T} u^t(z') \right] \leq 3\sigma LT + 2\alpha UT + \frac{D^2(1 + \sqrt{T})}{2\nu} + \frac{\nu U \sqrt{1 + \log T}}{2\sigma} \frac{\sqrt{T}}{(1 - \beta)^2}. \quad (55)$$

Select $\sigma = T^{-0.25} \sqrt{\frac{RU}{3(Lr + U)}}, \alpha = \frac{\sigma}{r}$, the expected regret is bounded by

$$\mathbb{E}[R(T)] \leq \frac{D^2(1 + \sqrt{T})}{2\nu} + \frac{\nu U T^{\frac{3}{4}} \sqrt{1 + \log T}}{2 \sqrt{\frac{RU}{3(Lr + U)}}} \frac{1}{(1 - \beta)^2} + \frac{2U}{r} T^{\frac{3}{4}} \sqrt{\frac{RU}{3(Lr + U)}} \quad (56)$$

That completes the proof.

G. Proof of Proposition [6]

According to Proposition [1] the utility function of fog node $k$’s action satisfies the Lipschitz condition. Therefore, fog node $k$ with action space $S_k \in [0, 1]^M$ and utility function $u_k$ in task allocation game $\Gamma$ can be modelled as a Lipschitz bandit. That completes the proof.
H. Proof of Proposition 7

Considering the average utility in \(N\) discretized bins, the average value of \(u_{k,m}\) indexed by \(n = 0, 1, \ldots, N - 1\) is

\[
\bar{u}_{k,m}[n] = \int_{\frac{n}{N}}^{\frac{n+1}{N}} u_{k,m}(x) \, dx.
\]

Because the utility function is L-Lipschitz, we have

\[
\max_{x \in [0,1]} u_{k,m}(x) - \max_{n} \bar{u}_{k,m}(n) \leq \frac{L}{N}.
\]

Therefore, the expected regret bound of exploration-exploitation strategy with \(N\)-dimensional discrete action space yields

\[
\mathbb{E}[R(T)] \leq T \frac{L}{N} + \mathcal{R}(T, N),
\]

where \(\mathcal{R}(T, N)\) is a function that depends on round number \(T\) and Lipschitz constant \(L\) to represent the regret bound under discrete condition. By substitute the parameters as given in our algorithm we conclude,

\[
\mathbb{E}[R(T)] \leq T_{1} + \mathbb{E}\left[\frac{LT}{N} + \mathcal{R}(T - T_{1}, \tilde{N})\right] = \mathcal{A} N + \mathbb{E}\left[\frac{LT}{N} + \mathcal{R}(T - \mathcal{A} N, \tilde{N})\right],
\]

where \(\mathcal{A} N\) is the regret caused by random selection in phase I and \(\mathcal{R}(T - \mathcal{A} N, \tilde{N})\) is the regret generated by the EXP3 strategy in phase II. The initialization of the weight matrix in EXP3 does not influence its regret bound and \(\mathcal{R}(T', N') = 2.63\sqrt{T N' \ln N'}\) according to [35]. With inequality

\[
\tilde{N} = N \left[ \frac{\tilde{L}^\frac{2}{3} T^\frac{1}{3}}{N} \right] \leq N \left[ \frac{\tilde{L}^\frac{2}{3} T^\frac{1}{3}}{N} (1 + \frac{N}{\tilde{L}^\frac{2}{3} T^\frac{1}{3}}) \right] = \tilde{L}^\frac{2}{3} T^\frac{1}{3} (1 + \frac{N}{\tilde{L}^\frac{2}{3} T^\frac{1}{3}}),
\]

and by substituting the parameters in the regret bound \(\mathcal{R}(T', N') = 2.63\sqrt{T N' \ln N'}\) we arrive at

\[
\mathcal{R}(T, \tilde{N}) \leq r \sqrt{\tilde{L}^\frac{2}{3} T^\frac{1}{3} (1 + \frac{N}{\tilde{L}^\frac{2}{3} T^\frac{1}{3}}) \ln \left( \tilde{L}^\frac{2}{3} T^\frac{1}{3} (1 + \frac{N}{\tilde{L}^\frac{2}{3} T^\frac{1}{3}}) \right)}
\]

\[
\leq r \sqrt{\tilde{L}^\frac{2}{3} T^\frac{1}{3} e \ln \left( e \tilde{L}^\frac{2}{3} T^\frac{1}{3} \right)}
\]

\[
\leq r \sqrt{e \tilde{L}^\frac{2}{3} T^\frac{1}{3} \tilde{L}^\frac{2}{3} T^\frac{1}{3}} = r' \tilde{L}^\frac{2}{3} T^\frac{5}{6}.
\]
The first inequality follows from \((1 + \frac{N}{L_2 T^3}) < e\) whenever \(N < (e - 1)L_2^\frac{z}{2}T^\frac{3}{4}\). The second inequality is concluded from the inequality \(\ln(e x) = 1 + \ln x \leq x\) whenever \(x > 0\). Besides, we have \(r = 2.63\) and \(r' = 2.63\sqrt{e} \approx 4.34\). According to Lemma 5 and inequality \((x_1 + \cdots + x_p)^r \leq x_1^r + \cdots + x_p^r\) [28], we can conclude the following: With probability at least \(1 - \frac{1}{T}\), it holds
\[
\mathcal{R}(T, \tilde{N}) \leq r' L_2^\frac{z}{2}T^\frac{5}{6}
\leq r'[L + 2N\sqrt{\frac{2}{A} \ln(2NT)})]^\frac{3}{2}T^\frac{5}{6}
\leq r'T^\frac{5}{6}(L_2^\frac{z}{2} + (2N\sqrt{\frac{2}{A} \ln(2NT)})^\frac{3}{2}).
\] (63)

Moreover, with probability at least \(1 - \frac{1}{T}\), we have
\[
\tilde{L} \geq L - \frac{7H}{N} \geq \frac{L}{8},
\] (64)
if \(N \geq \frac{8H}{L}\), where \(H\) is defined in Definition 5. Therefore,
\[
\frac{LT}{N} \leq \frac{LT}{L_2^\frac{z}{2}T^\frac{3}{4}} \leq \frac{LT^\frac{z}{2}}{L^\frac{2}{3}} = 4L^\frac{1}{2}T^\frac{5}{6}.
\] (65)

Putting things together, we get
\[
\mathbb{E}[R(T)] \leq T_1 + \mathbb{E}[\frac{LT}{N} + \mathcal{A}(T - T_1, \tilde{N})]
\leq \mathcal{A}N + 4L^\frac{1}{2}T^\frac{5}{6} + r'T^\frac{5}{6}(L_2^\frac{z}{2} + (2N\sqrt{\frac{2}{A} \ln(2NT)})^\frac{3}{2}),
\] (66)
which can be concluded as the following: With probability at least \(1 - \frac{1}{T}\),
\[
\mathbb{E}[R(T)] \leq \tilde{O}(T^\frac{5}{6}).
\] (67)

That completes the proof.

I. Proof of Proposition 8

By Proposition 3, the task allocation game \(\Gamma\) is a socially concave game. Besides, by Proposition 4 if the game converges to a Nash equilibrium, then that equilibrium is the unique equilibrium of the game. Therefore, by Theorem 3 to achieve equilibrium while also ensuring (7) for every fog node \(k \in \mathcal{K}\), it suffices that each fog node \(k\) plays according to a no-regret decision-making strategy. By Proposition 5 and Proposition 7, BGAM and LBWI are no-regret policies. That completes the proof.
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