INTEGRABLE WEYL GEOMETRY IN MULTIDIMENSIONAL COSMOLOGY. NUMERICAL INVESTIGATION

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Abstract

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1 Introduction

The multidimensional gravitation theories are very attractive in the context of the unification of fundamental interactions. Moreover, several modern theories require space-time to have more than four dimensions [1-8]. The nonobservability of additional dimensions in such theories needs an explanation. Among different possible ways of such explanation the hypothesis about dynamical contraction of internal manifold during expansion of the universe is very popular. This idea is realized in many exact cosmological solution of multidimensional Einstein’s equations [9-22]. As a rule such models require additional fields and do not avoid initial big bang singularity. The introduction of additional fields in multidimensional gravitation theories destroy their pure geometrical character and require an additional motivation [4]. Such motivation may be done in the framework of some generalizations of Riemannian geometry. In four dimensional case such generalization in several cases leads to removing of cosmological big bang singularity [23-25]. That is why the unification of generalized geometric structures and multidimensional gravity seems to be very attractive. Unfortunately, only in several papers the multidimensional gravitation theory and cosmology are considered in the scope of some generalization of Riemannian geometry [26-28].

One of the simplest generalization of the Riemannian geometry is the integrable Weyl geometry with the connection components

\[ \Gamma_{\beta\gamma}^\alpha = \tilde{\Gamma}_{\beta\gamma}^\alpha - \frac{1}{2} \left( \omega_{\beta}^{\delta\gamma} + \omega_{\gamma}^{\delta\beta} - g_{\beta\gamma} \omega^{\delta} \right), \]

where \( \tilde{\Gamma}_{\beta\gamma}^\alpha \) are the Christoffel symbols, \( \omega_{\alpha} = \omega_{,\alpha} \), \( \omega \) is a scalar field, \( \delta_{\beta}^{\delta} \) are the Kroneker symbols, \( g_{\alpha\beta} \) is a metric tensor; the small Greek indices take values from 0 to \( n-1 \), \( n \) is a dimension of space-time. The Ricci tensor and the curvature scalar of the connection (1) are equal to

\[ R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{n-2}{2} g_{\mu\lambda} \Box \omega + \frac{1}{2} g_{\mu\nu} \Box \omega + \frac{n-2}{4} \left( \omega_{\mu} \omega_{\nu} - g_{\mu\nu} \omega^{\lambda} \omega_{\lambda} \right), \]

\[ R = \tilde{R} + (n-1) \Box \omega - \frac{(n-1)(n-2)}{4} \omega^{\lambda} \omega_{\lambda}, \]

where the tildes denote the quantities calculated in the connection \( \tilde{\Gamma}_{\beta\gamma}^\alpha \), two parallel vertical bars and \( \Box \) denote the covariant derivative and the d’Alembert operator of this connection. It is necessary to note that the integrable Weyl space-time is also conformally-Riemannian, since there is a conformal transformation of metric tensor \( g_{\alpha\beta} \) which maps the Riemannian space-time into integrable Weyl
space-time. As the integrable Weyl space-time is defined by the pair \((g_{\alpha\beta}, \omega)\) the gravitation theory in this space-time does not coincide with Einsteinian general relativity because the field \(\omega\) must be contained in the Lagrangian independently from \(g_{\alpha\beta}\) and cannot be excluded by the conformal transformation.

Some features of the Einsteinian cosmological models with scalar fields were recently considered by several authors [9,11,19,21,28,29-37] both in 4-dimensional and in \((4+d)\)-dimensional space-times. The cosmological models in four-dimensional Weyl-integrable space-time were recently considered by Novello et al. in [23], where the existence of nonsingular open cosmological models was shown. The appearance of Weyl geometry in multidimensional cosmology was discussed also in [27].

In this paper we consider the influence of Weyl geometry on the evolution of Friedman-Robertson-Walker (FRW) cosmological models in multidimensional gravitation theory. As usually the space-time is assumed to have the structure of direct product \(M^4 \times V^d\) of four-dimensional FRW space-time \(M^4\) and \(d\)-dimensional interior space \(V^d\) that is supposed to be \(d\)-sphere \(S^d\) or \(d\)-torus \(T^d\). The metric of space-time is supposed to be block-diagonal

\[
ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - \bar{g}_{ab} du^a du^b, \tag{4}
\]

where \(k = +1, \ 0, \ -1\) for closed, plane and open models, \(d\Omega^2\) is a line element on two-sphere, \(u^a\), \(a = 1, \ldots, d\), and \(\text{wide}\bar{g}_{\alpha\beta}\) are the coordinates and metric tensor of the interior space \(V^d\). Once we consider only spatially homogeneous FRW cosmologies, it is natural to make the Weyl scalar field \(\omega\) to be a function of cosmic time \(t\) only: \(\omega = \omega(t)\). We consider both vacuum case and non vacuum case with the additional scalar field \(\varphi\) with non minimal coupling. The 4-dimensional case will be briefly considered also for completeness. The existence of the conformal map between Riemannian and integrable Weyl space-times may be used for generation of exact solutions from the known solutions of general relativity. Such approach admits obtaining only the particular solutions. Therefore to demonstrate general qualitative behavior of the models we solve the system of cosmological equations numerically with initial values given at \(t = 0\) and satisfying the constraint equation. For that purpose we use adaptive numerical methods with automatic choice of integration step and with the stiffness checking. The geometrical units where \(G = c = 1\) are used in what follows.

2 Integrable Weyl cosmology in vacuum

Following [23] we shall consider the vacuum cosmological models in the gravitation theory with the Lagrangian

\[
L = R + \xi \omega_\alpha \omega^\alpha \tag{5}
\]

where \(R\) is defined by (3) and \(\xi = \text{const.}\). After excluding the total derivatives of the scalar field Lagrangian (5) takes the form

\[
L = \bar{R} - \frac{(n - 1)(n - 2)}{4} - 4\xi \omega^\alpha \omega_\alpha \tag{6}
\]

So, the theory differs from the Einstein theory with the massless scalar field by the coefficient before the square of the scalar field gradient and has different geodesic lines. Note also that due to the definition of the Weyl connection (1) the scalar field \(\omega\) cannot be renormalized and hence the
coefficient $\xi$ before $\omega^\alpha \omega_\alpha$ cannot be put to $\pm 1$ as it may be done in the pure Einstein theory with massless scalar field. Variation of (6) with respect to the pair $(g_{\alpha\beta}, \omega)$ of independent variables yields the equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} - \frac{(n-1)(n-2) - 4\xi}{4} \left( \omega_{\mu} \omega_{\nu} - \frac{1}{2} g_{\mu\nu} \omega^\alpha \omega_\alpha \right) = 0,$$

(7)

and

$$\Box \omega = 0$$

(8)

The equations (7), (8) coincide with the Einstein equations for the massless scalar field, whose solutions for the FRW cosmological models were investigated both in four-dimensional [32] and multidimensional cases [37]. By this reason here we only summarize briefly the main results.

2.1 Four-dimensional case. As the scalar field $\omega$ is a function on $t$ only, equation (8) yields the first integral

$$\dot{\omega} = \frac{\gamma}{a^3}$$

(9)

where overdot denotes time differentiation and $\gamma = \text{const}$ is the integration constant. Due to (9) equations (7) take the form

$$\dot{a}^2 + k - \frac{\lambda \gamma^2}{12a^4} = 0$$

(10)

and

$$2a\ddot{a} + \dot{a}^2 + k + \frac{\lambda \gamma^2}{4a^4} = 0$$

(11)

where $\lambda = (3 - 2\xi)$. As it is easy to see from (10), only singular and static solution of equations (10)-(11) exist if $\lambda > 0$. For negative values of $\lambda$ solution exists only for the open models. In this case $a(t) \geq a_0 = (\xi - 3)\gamma^2/12$ and so the cosmological singularity is absent. The qualitative behavior of scale factor $a(t)$ for negative $\lambda$ is shown at figure 1 and its features are discussed in detail in [23].

Fig. 1.

Qualitative behavior of the scale factor $a(t)$ of the open universe in four-dimensional Weyl-integrable space-time.

2.2. Multidimensional case. In the multidimensional case the behavior of the model depends not only on the parameter $\xi$, as in the previous case, but on the structure of the interior space also. For simplicity only 5- and 6-dimensional models will be considered in the following. We consider these two cases separately. The main qualitative features of models in general $n$-dimensional ($n > 6$) case are the same as in 5- and 6-dimensions.

2.2.1. 5-dimensional models. In 5-dimensions space-time interval (4) reads

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - s^2(t) du^2$$

(12)

where $u \in S^1$ is the interior space coordinate. Assuming, as above, a scalar field $\omega$ to be a function of the cosmological time only, the first integral of equation (8) takes the form

$$\dot{\omega} = \frac{\gamma_1}{a^3(t)s(t)}$$

(13)
where $\gamma_1 = \text{const}$. Due to (12)-(13) equations (7) become after simplification

$$3 \frac{\ddot{a}}{a} \frac{s}{a} + 3 \left( \frac{\dot{a}}{a} \right)^2 s + 3 \frac{k}{a^2} s^2 - \frac{\gamma_2^2 (3 - \xi)}{2a^5 s^2} = 0,$$

(14)

$$\ddot{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{a} \dot{s}}{a s} + \frac{k}{a^2} = 0,$$

(15)

and

$$\frac{\ddot{s}}{s} + 3 \frac{\dot{a} \dot{s}}{a s} = 0$$

(16)

The last equation has the first integral

$$\dot{s} = \frac{\gamma_2}{a^3},$$

(17)

where $\gamma_2 = \text{const}$. It is easy to see that analogous to the four-dimensional case the nonsingular solutions of equations (15), (17) exist only for the open models ($k = -1$). In this case for $t < 0$ the scale factor of 3-space $a(t)$ decreases monotonically from infinity to its minimal value $a_0$ and then grows to infinity at $t > 0$, while the Weyl field $\omega(t)$ and the scale factor of interior space evaluates monotonically from $\omega_- = \lim_{t \to -\infty} \omega(t)$ and $s_- = \lim_{t \to -\infty} s(t)$ to $s_+ = \lim_{t \to \infty} s(t)$, where $a_0$, $\omega_\pm$, and $s_\pm$ are defined by the integration constants and may have arbitrary values. Note that if $\gamma_2 < 0$ than the constants $s_-$ and $s_+$ satisfy the condition $s_- > s_+$ and so the standard dimensional reduction scenario is realized. The typical shape of the functions $a(t)$ and $s(t)$ are shown on the figures (2a,b).

Fig. 2.

The qualitative behavior of the scale factors $a(t)$ (2a) and $s(t)$ (2b) of five-dimensional Weyl integrable cosmological model with open 3-space.

The figure (2a) shows that unlike the 4-dimensional case the evolution of 3-space in 5-dimensional model is time-asymmetric. This asymmetry appears because the equation (15) depends not only on $\dot{s}(t)$ but also on the time-asymmetric interior space scale factor $s(t)$.

2.2.2. 6-dimensional models. In 6-dimensional case we consider two types of topological structures for the interior space: the 2-sphere $S^2$ and 2-dimensional torus $T^2$. Therefore the space-time metric (4) may have one of two forms

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - s^2(t) \left( \frac{du^2}{1 - u^2} + u^2 dv^2 \right)$$

(18)

or

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - s_1^2(t) du^2 - s_2^2(t) dv^2,$$

(19)

where $\{u, v\}$ are the coordinates on $S^2$ or $T^2$ respectively. First integrals of equation (8) take the form

$$\dot{\omega} = \frac{q_1}{a^5 s^2},$$

(20)

for metric (18) and

$$\dot{\omega} = \frac{q_2}{a^3 s_1 s_2},$$

(21)

for metric (18).
Equations (7) for the metric (18) after simplification take the form

\[
\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a} \dot{s}}{a s} + \frac{2k}{a^2} = 0, \tag{22}
\]

\[
\frac{\ddot{s}}{s} + \left( \frac{\dot{s}}{s} \right)^2 + 2 \frac{\dot{a} \dot{s}}{a s} + \frac{1}{s^2} = 0, \tag{23}
\]

and the constraint equation

\[
3 \frac{\ddot{a}}{a} \left( \frac{\dot{a}}{a} + 2 \frac{\dot{s}}{s} \right) + \frac{3k}{a^2} + \frac{1}{s^2} - \frac{q_1(5 - \xi)}{2a^0 s^4} = 0. \tag{24}
\]

The first two equations are dynamical and the last is the constraint.

It is easy to see that only singular solutions of equations (22)-(24) exist: the scale factor \(s(t)\) of the interior space evolves from zero at \(t = t_0\) to its maximal value \(s_{\text{max}}\) and return to zero at \(t = t_1 > t_0\). The behavior of \(a(t)\) depends on the sign of \(k\). Namely, if \(k = +1\) then the qualitative evolution of \(a(t)\) is the same as the evolution of \(s(t)\). If \(k = 0\) than \(a(t)\) increase from zero at \(t = t_0\) to infinity at \(t = t_1\) or decrease from infinity to zero; the unstable solutions with \(a(t) = 0\) are also exist. Finally, if \(k = -1\) then \(a(t)\) evolves from infinity at \(t = t_0\) to its minimum \(a_{\text{min}}\) and then grows to infinity at \(t = t_1\).

For the metric (19) equations (7) after simplification read

\[
\frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \left( \frac{\dot{s}_1}{s_1} + \frac{\dot{s}_2}{s_2} \right) + 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{2k}{a^2} = 0, \tag{25}
\]

\[
\frac{\ddot{s}_1}{s_1} + \frac{3 \ddot{a}}{a} \frac{s_2}{s_1} + \frac{s_1 \dddot{s}_1}{s_1 s_2} + \frac{s_1 \dddot{s}_2}{s_1 s_2} = 0, \tag{26}
\]

\[
\frac{\ddot{s}_2}{s_2} + \frac{3 \ddot{a}}{a} \frac{s_1}{s_2} + \frac{s_2 \dddot{s}_1}{s_1 s_2} + \frac{s_2 \dddot{s}_2}{s_1 s_2} = 0, \tag{27}
\]

and the constraint equation

\[
3 \frac{\ddot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{s}_1}{s_1} + \frac{\dot{s}_2}{s_2} \right) + \frac{s_1 \dddot{s}_1}{s_1 s_2} + \frac{s_2 \dddot{s}_2}{s_1 s_2} + \frac{3k}{a^2} - \frac{q_2(5 - \xi)}{a^0 s_1^2 s_2^2} = 0. \tag{28}
\]

As in 4- and 5-dimensional cases the nonsingular solutions of the equations (25)-(28) exist only for the open models \((k = -1)\). Analogously to 5-dimensional case the scale factor \(3\)-space \(a(t)\) in these models decreases monotonously from infinity to its minimal value \(a_0\) and then grows to infinity at \(t \to +\infty\), while the scale factors \(s_i(t)\), \(i = 1, 2\), of interior space changes monotonously from \(s_{i-} = \lim_{t \to -\infty} s_i(t)\) to \(s_{i+} = \lim_{t \to \infty} s_i(t)\). The necessary condition for the realization of the dimensional reduction scenario in this case are defined by the following inequalities

\[
3 \frac{1}{a_0^2} + \frac{q_2^2(5 - \xi)}{2a_0^6 s_{10} s_{20}} > 0, \tag{29}
\]

and

\[
\dot{s}_1(0) < 0, \dot{s}_2(0) < 0. \tag{30}
\]

It is necessary to note that inequality (29) is the necessary condition for \(\dot{s}_1\) and \(\dot{s}_2\) to be of the same sign. The time behavior of scale factors \(a(t)\), \(s_1(t)\) and \(s_2(t)\) in this case is qualitatively the same as in 5-dimensional case (Figure 2).
3 Integrable Weyl cosmology in theory with non minimal scalar field

In this section we consider cosmological models in gravitational theories with Lagrangian

\[ L = R \left( 1 + \frac{1}{2(n-1)} \varphi^2 \right) + \xi \omega^\alpha \omega_{\alpha} + \eta \varphi^\alpha \varphi_{\alpha}, \]  

(31)

where \( R \) is defined by (3), \( \varphi \) is a real scalar field, \( \eta = \pm 1 \) and \( \xi = \text{const} \) as above. In the limiting case \( \varphi = \text{const} \) Lagrangian (31) coincides with (5) while in another limiting case \( \omega = \text{const} \) it coincides with the Lagrangian for the conformal-invariant scalar field.

The substitution of (3) into (31) gives after simplification

\[ L = \tilde{R} \left( 1 + \frac{\varphi^2}{2(n-1)} \right) - \varphi \varphi^\alpha \omega_{\alpha} - \frac{(n-1)(n-2) - 4\xi}{4} \omega^\alpha \omega_{\alpha} - \frac{(n-2)}{8} \varphi^2 \omega^\alpha \omega_{\alpha} + \eta \varphi^\alpha \varphi_{\alpha}, \]

(32)

where the total derivatives of the scalar fields are omitted.

Variation of (32) with respect to independent variables \( g_{\mu\nu}, \omega \) and \( \varphi \) yields the equations

\[ \left( \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} \right) \left( 1 + \frac{\varphi^2}{2(n-1)} \right) - \frac{(n-1)(n-2) - 4\xi}{4} \left( \omega^\mu \omega^\nu - \frac{1}{2} g_{\mu\nu} \omega^\alpha \omega_{\alpha} \right) - \frac{1}{8} \varphi \left( \omega^\mu_\nu + \varphi^\mu_\nu \omega_{\nu} \right) \frac{n-2}{n-1} \varphi^2 \left( \omega^\mu_\nu - \frac{1}{2} g_{\mu\nu} \omega^\alpha \omega_{\alpha} \right) + \frac{\varphi}{n-1} \left( g_{\mu\nu} \varphi^\mu - \varphi^\mu_\nu \right) + \varphi_{\mu} \varphi_{\nu} \left( \eta - \frac{1}{n-1} \right) + g_{\mu\nu} \varphi^\mu \varphi_{\nu} \left( \frac{1}{n-1} - \frac{\eta}{2} \right) = 0, \]

(33)

and

\[ \eta \Box \varphi - \left( \Box \omega + \frac{1}{n-1} \tilde{R} - \frac{n-2}{4} \omega, \omega, \omega_{\nu} \right) \varphi = 0, \]

(35)

Equation (35) shows that non-Riemannian nature of space-time geometry in the considered model leads to the effective mass generation for the scalar field \( \varphi \).

3.1. Four-dimensional models. In four-dimensional case the equations (33)-(35) consist of the constraint equation

\[ \left( \frac{\dot{a}}{a} + \frac{k}{a^2} \right)^2 \left( 3 + \frac{\varphi^2}{2} \right) + \frac{\dot{a}}{a} \varphi \dot{\varphi} + \frac{\eta}{2} \dot{\varphi}^2 - \frac{1}{2} \eta \dot{\varphi} \dot{\omega} - \frac{1}{8} \varphi^2 \omega^2 - \frac{3 - 2\xi}{4} \omega^2 = 0, \]

(36)

and three dynamical equations

\[ 2 + \frac{\varphi^2}{3} \frac{\ddot{a}}{a} + \frac{1}{3} \varphi \ddot{\varphi} + \frac{2}{3} \left( \frac{\dot{a}}{a} \right)^2 \left( 2 + \frac{\varphi^2}{3} \right) + \frac{5}{3} \frac{\dot{a}}{a} \varphi \dot{\varphi} + \frac{1}{3} \varphi^2 + \left( 2 + \frac{\varphi^2}{3} \right) \frac{2k}{a^2} = 0, \]

(37)

\[ \left( \frac{\varphi^2}{2} - 2\xi + 3 \right) \ddot{\omega} - \varphi \ddot{\varphi} + \left( 9 - 6\xi + \frac{3\varphi^2}{2} \right) \frac{\dot{a}}{a} \ddot{\omega} - 3 \frac{\dot{a}}{a} \varphi \dot{\varphi} - \varphi^2 = 0, \]

(38)

and

\[ \eta \dddot{\varphi} - \varphi \dddot{\omega} + 2 \varphi \frac{\ddot{a}}{a} + 3 \frac{\dot{a}}{a} (\eta \dot{\varphi} - \varphi \dot{\omega}) + \frac{1}{2} \varphi \ddot{\omega}^2 + 2 \varphi \left( \frac{\dot{a}}{a} \right)^2 + \frac{2k}{a^2} = 0. \]

(39)
The coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ in the equations (37)-(39) depend both on the parameters $\xi$, $\eta$ and on the scalar field $\varphi$. The determinant of the matrix of coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ is equal to

$$d = \left(\frac{2}{3} - \frac{3}{4}\right)\varphi^4 + \left(2\eta + \frac{3\xi}{3} - \frac{2\varphi^2}{3} - 4\right)\varphi^2 + 6\eta - 4\eta\xi.$$

The points where $d = 0$ are the singular points of the system (37)-(39). These points are not described by the system (36)-(39) because for fixed $\eta$ and $\xi$ equation $d = \text{const}$ defines not more than four fixed values of $\varphi$ and the system (36)-(39) reduces to the first order system. Therefore the initial value of the field $\varphi$ must be from the open set $d \neq 0$.

For $\eta$ equation $d = 0$ divide the half-plane $(\xi, \varphi^2)$, on three regions that will be denoted as $A$, $B$ and $C$, while for $\eta = -1$ there are only two regions $A$ and $B$ (figure 3a,b). The behavior of the model depends on the region where the point $(\xi, \varphi^2)$ is situated.

**Fig. 3.**
The set $d = 0$ for $\eta = 1$ (3a) and $\eta = -1$ (3b) in 4-dimensions.

Numerical investigation of equations (31)-(33) shows that for the closed ($k = 1$) and flat ($k = 0$) cosmological models only singular solutions exist for any initial conditions. For the open models ($k = -1$) if the pair $(\xi, \varphi^2_0)$ defines the point in the region $B$ (both for $\eta = 1$ and $\eta = -1$) or $C$ (for $\eta = 1$) than only singular solutions of the equations (37)-(39) exist. If the pair $(\xi, \varphi^2_0)$ defines the point in the region $A$ then solutions may be both regular and singular. The numerical investigation does not permit to find the exact conditions of regularity, but it shows that both regular and singular solutions are stable against finite perturbations of the initial conditions. The typical qualitative behavior of the universe scale factor $a(t)$, Weyl field $\omega$ and the matter scalar field $\varphi$ are shown at figure 4a-c.

**Fig. 4.**
The qualitative behavior of the universe scale factor $a(t)$ (4a), Weyl field $\omega$ (4b) and the matter scalar field $\varphi$ (4c) in the open four-dimensional Weyl-integrable space-time model.

The universe scale factor $a(t)$ in the typical nonsingular solution evolves from infinity at $t = -\infty$ to its minimal value $a_0 = a(0)$ and then grows to infinity at $t \to \infty$ (figure 4a). Both scalar fields, the Weyl field $\omega$ and the field $\varphi$ evolves between two limiting values: from $\omega_- = \lim_{t \to -\infty} \omega(t)$ and $\varphi_- = \lim_{t \to -\infty} \varphi(t)$ to $\omega_+ = \lim_{t \to \infty} \omega(t)$ and $\varphi_+ = \lim_{t \to \infty} \varphi(t)$. The difference in the evolution of these fields is that the field $\omega$ evolves monotonously (figure 4b) while the field $\varphi$ near $t = 0$ (i. e. near the minimum of $a(t)$) may have several intermediate extrema with one absolute maximum if $\eta = 1$ (figure 4c) or absolute minimum if $\eta = -1$. As $\varphi(t)$ for big $|t|$ tends asymptotically to constants, the model evolves asymptotically as an empty Weyl cosmological model that is considered in section 2.1. It is necessary to note also that the evolution of the universe scale factor $a(t)$ has a small time-asymmetry in comparison with the case of the empty space. This asymmetry is a result of non symmetrical evolution of the matter field $\varphi$ because the field equations (37)-(39) contain both $\varphi$ and $\ddot{\varphi}$.

**3.2. 5-dimensional models.** In 5-dimensional case equations (33)-(35) after simplification become

$$3\left\{\frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} + \frac{\dot{s}}{s}\right) + \frac{k}{a^2}\right\} \left\{1 + \frac{\varphi^2}{8}\right\} - \frac{\varphi\dot{\varphi}}{4} \left(\frac{3\dot{a}}{a} + \frac{\dot{s}}{s}\right) + \frac{\eta\varphi^2}{2} - \frac{\varphi\ddot{\varphi}}{2} + \frac{\ddot{\varphi}^2}{2} \left(\xi - 3 - \frac{3\varphi^2}{8}\right) = 0, \quad (40)$$
The determinant of the matrix of the coefficients before \( \dot{a}/a, \ddot{a}/a \) in equation (42) is equal to

\[
3 \left\{ 1 + \frac{\varphi^2}{8} \right\} \left\{ \ddot{a}/a + \frac{\ddot{s}}{a} \left( 2 \frac{\dot{a}}{a} + \frac{s}{s} \right) + \frac{2k}{a^2} \right\} + \varphi \ddot{\varphi} + \frac{\varphi}{2} \left( \frac{3}{a} + \frac{s}{s} + \frac{2}{2 \varphi} \right) = 0,
\]

and

\[
\frac{\varphi}{2} \left( \frac{3 \ddot{a}}{a} + \frac{\ddot{s}}{s} \right) + \eta \ddot{\varphi} - \varphi \dddot{\varphi} + \varphi \dddot{\varphi} \left( 3 \ddot{\omega} - 3 \frac{\dot{a}}{a} - \frac{s}{s} \right) + \eta \dddot{\varphi} \left( \frac{3}{a} + \frac{s}{s} \right) + \frac{3 \varphi \ddot{a}}{2a} \left( \frac{\dot{a}}{a} + \frac{s}{s} \right) + \frac{3k \varphi}{2a^2} = 0.
\]

The equation (40) is the constraint that must be satisfied by the initial conditions and the equations (41)-(44) are the dynamical. The determinant of the matrix of the coefficients before \( \dot{a}/a, \ddot{a}/a \) in equation (42) is equal to

\[
d = \left( \frac{9}{256} \eta - \frac{3}{32} \right) \varphi^6 + \left( \frac{27}{32} \eta + \frac{1}{8} \xi - \frac{3}{32} \eta \xi - \frac{3}{2} \right) \varphi^4 + \left( \frac{27}{4} \eta + \xi - \frac{3}{2} \eta \xi - 6 \right) \varphi^2 + 18 \eta - 6 \eta \xi.
\]

The qualitative features of function \( d(\xi, \eta, \varphi) \) are the same as in 4-dimensional case: for \( \eta = 1 \) equation \( d = 0 \) divides the half-plane \( (\xi, \varphi^2 > 0) \) on three regions that are denoted as \( A, B \) and \( C \), while for \( \eta = -1 \) there are only two regions \( A \) and \( B \) (figure 5a,b). The behavior of the model depends on the region where the point \( (\xi, \varphi^2) \) is situated.

Fig. 5.

The sets \( d = 0 \) for \( \eta = 1 \) (5a) and \( \eta = -1 \) (5b) in 5-dimensions.

Numerical investigation of equations (40)-(44) shows that as well as in the previous 4-dimensional case only singular solutions exist at any initial conditions for the closed \( (k = 1) \) and flat \( (k = 0) \) cosmological models. For the open models \( (k = -1) \) if the pair \( (\xi, \varphi^2) \) defines the point in the region \( B \) (both for \( \eta = 1 \) and \( \eta = -1 \)) or \( C \) (for \( \eta = 1 \)) than only singular solutions of the equations (40)-(44) exist, while if the pair \( (\xi, \varphi^2) \) defines the point in the region \( A \) than the solution may be both regular and singular. The regularity of solutions depends on the constants of integration that may be considered as the initial conditions at \( t = 0 \). It was found that the regularity of solutions depends mainly on the signs of \( \dot{s}(0), \ddot{s}(0) \) and \( \varphi(0) \). Their possible combinations that give nonsingular solutions of equations (41)-(44) are represented in table 1. The last column of this table shows the general direction of the interior space evolution by means of the signs of the difference \( \Delta = s_+ - s_-, \) where \( s_{\pm} = \lim_{t \to \pm \infty} s(t) \).

Table 1.

Conditions of the solutions regularity and the direction of \( s(t) \) evolution
The typical behavior of the nonsingular solution of the equations (41)-(44) for $\eta = 1$ is shown at the figures 6a-d for the case $\Delta \leq 0$, i. e. for the contracting interior space.

Fig. 6.

The qualitative behavior of the scale factors $a(t)$, (4a) and $s(t)$ (4b), the matter scalar field $\varphi(t)$ (4c) and Weyl field $\omega(t)$ (6d) in the nonsingular 5-dimensional Weyl-integrable space-time model.

In general nonsingular solution the radius of the universe changes monotonously from infinity at $t = -\infty$ to minimal value $a_0$ and then grows to infinity (figure 6a), while the radius of the internal space starts from $s_0 = \lim_{t \to -\infty} s(t)$, passes through several (one or two) intermediate extrema, that are situated near minimum of $a(t)$ and may be absent in some cases, and then changes to $s_+ = \lim_{t \to \infty} s(t)$ (figure 6b). Note that $s_+$ and $s_-$ may be of the same or different order. The field $\varphi$ evolves analogously to 4-dimensional case (figure 6c). Note that the extremal points of the functions $a(t)$, $s(t)$ and $\varphi(t)$ do not coincide with each other in general case and the function $a(t)$ is time asymmetrical especially near its minimum. Finally the Weyl field $\omega$ changes monotonously between two limiting values: $\omega_- = \lim_{t \to -\infty} \omega(t)$ and $\omega_+ = \lim_{t \to \infty} \omega(t)$ (figure 6d). In the case $\eta = -1$ the model evolves as above but the extremal points of the field $\varphi$ change type: minimum become maximum and vice versa.

| $\text{sign} \dot{s}(0)$ | $\text{sign} \dot{\omega}(0)$ | $\text{sign} \dot{\varphi}(0)$ | $\text{sign}(s_+ - s_-)$ |
|------------------------|------------------------|------------------------|------------------------|
| -1                     | -1                     | 0                      | -1                     |
| -1                     | +1                     | 0                      | -1                     |
| -1                     | -1                     | +1                     | -1                     |
| -1                     | +1                     | -1                     | -1                     |
| 0                      | +1                     | 0                      | -1                     |
| 0                      | -1                     | 0                      | +1                     |
| +1                     | +1                     | -1                     | +1                     |
| +1                     | -1                     | 0                      | +1                     |
| +1                     | +1                     | 0                      | +1                     |

4 Concluding remarks

We have considered the qualitative evolution of multidimensional cosmological models based on the integrable Weyl geometry both in vacuum space-time and in the presence of nonminimal scalar field. The existence of nonsingular solutions of field equations for open cosmological models that realized the dimensional reduction scenario was demonstrated. It was shown that in multidimensional case the evolution of the scale factor of the universe $a(t)$ becomes time-asymmetric unlike the four-dimensional case. We have shown also that all nonsingular cosmological models considered above have some common features. In particular the evolution of the universe scale factor (radius) $a(t)$ for big $|t|$ is asymptotically linear. Further in all nonsingular models Weyl scalar field $\omega(t)$ as well as the matter field $\varphi(t)$ in the models with nonminimal coupling tend asymptotically to constants. So the models tend to the pure Einsteiinian models of the corresponding dimensions and the change of the collapse era into expansion one may be considered as a cosmological phase transition induced by the transition of scalar fields $\omega(t)$ and $\varphi(t)$ from one stationary state $\omega = \omega_-$ and $\varphi = \varphi_-$ into another stationary state $\omega = \omega_+$ and $\varphi = \varphi_+$. At the late stages of the universe evolution the fields $\omega(t)$ and $\varphi(t)$ are unobservable.
There are several qualitative differences between the vacuum models and the models with non-minimal scalar field. First of all in vacuum models the existence of cosmological singularity depends only on the parameters of the theory while in the case of nonminimal scalar field it depends on the initial conditions also. Secondly, in the models with nonminimal scalar field the evolution of the internal space scale factor $s(t)$ may be nonmonotonous. In the typical scenario one of the limiting values of $s(t)$ at $t = \pm \infty$ is much smaller than another but in several models both limiting values of internal radius $s(t)$ may be arbitrary small and it become finite only near minimum of the universe scale factor $a(t)$.

We have discussed here only the models with the one- or two-dimensional interior space because if interior space has dimension $d \geq 3$ and direct product topology of torus on several spheres then the models have the same qualitative features as considered above. In particular, the nonsingular solutions exist only for toroidal interior space topology.

The models considered above show that the real geometrical structure of space-time may have a non-Riemannian nature but the universe may evolve in such a way that its non-Riemaniann nature is essential only near $t = 0$ and become unobservable at late stages of the evolution. Therefore, the consideration of generalized geometrical structures in multidimensional cosmology may be of a considerable interest. In particular, the models considered above may be generalized in the following manner. First of all, both Weyl scalar field $\omega(t)$ and matter field $\varphi(t)$ may be massive and have nonlinear potential. Secondly, the possible influence of the cosmological term $\Lambda$ must be considered also. At last, the term $R\varphi^2/2(n - 1)$ in the action integral (25) may have negative sign. One may suppose that in this case nonsingular solutions of the field equations may be obtained not only for open models, but for closed and flat models also. These possibilities will be considered elsewhere.

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