Dynamic characteristics of liquid sloshing in cylindrical tanks filled with porous media

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Abstract. The dynamic characteristics of liquid sloshing in cylindrical tanks filled with porous media are analyzed in this paper. As the tank is subjected to the small external excitations, the flow is assumed to be in Darcy's flow regime, and then the porous medium can provide damping effect linearly proportional to the fluid velocity. Based on the pore flow theory, the linear equivalent mechanical model with the damping effect due to porous medium is established. This equivalent model consists of a rigidly fixed mass and infinite sets of mass-damping-spring systems, which can exert the same total dynamic force as the linearized actual fluid field. The equivalent system can explain the dynamic characteristics of each modal response of sloshing in a cylindrical porous-medium tank. Therefore it is more convenient to analyze the sloshing behavior from the structural dynamic analysis point of view. By this model, the natural frequency and damping ratio of the cylindrical porous-medium tank are revealed. Both parameters are related to the tank dimensions, porosity and permeability of the porous medium.

1. Introduction
The sloshing behavior has long been studied in civil and ocean engineering. Sloshing is known as the dynamic behavior of the free surface inside a container partially filled with liquid. Critical issues include stabilization of satellites [1], unexpected damages of liquid storage tanks under seismic excitations [2], and structural vibration control by tuned liquid damper (TLD) [3]. It is essential to be analyzed by simple and reliable methods. Earlier researchers started from the linear wave theory [4]. Afterwards, the nonlinear sloshing has become more and more important when the sloshing oscillation occurs at or near the resonance. These are mainly solved by various kinds of numerical methods [5,6]. Overall, they all achieved great results. A comprehensive survey about sloshing phenomena and its applications has been monographed [7]. A new issue, that is the liquid sloshing behavior in a tank filled with porous media, has been investigated recently [8,9]. Installing porous medium inside an impervious tank has two practical objectives. One is to suppress the fluid oscillation as the tank is under external excitations [8]. It has been proven a simple and effective way to rapidly reduce the wave responses. The other is to enhance the damping effect of sloshing liquid for better performance as a dynamic vibration absorber [9]. Compared to the slat screens often used in recent years [10], it could be more effective and applicable.

This article demonstrates the analytical method for solving the sloshing problem in an impervious
cylindrical tank filled with porous media. As the tank is subjected to the small external excitations, the flow is assumed to be in Darcy’s flow regime. Therefore the pressure loss due to porous medium can be linearly proportional to the fluid velocity. Based on linear wave theory and pore-flow fluid dynamic, the equivalent mechanical model of a cylindrical porous-medium tank is established. In this equivalent model it comprises a fixed mass and infinite sets of mass-damping-spring systems, which account for the rigid-body motion and the liquid convection, respectively. Therefore it could be dynamically equivalent to the actual flow field. Therefore it is more convenient to analyze the sloshing behavior in porous-medium tank from the structural analysis point of view by this model. In contrast to the other analytic model for inviscid flow, the damping effect can be taken into account. Both natural frequency and damping ratio of the cylindrical porous-medium tank are revealed. The derivation and discussion are made in sections 2 and 3, respectively.

2. Mathematical formulae

2.1. Governing equation and boundary conditions

The sloshing phenomenon in an isotropic porous medium inside an impervious cylindrical tank under excitation is studied. In Figure 1, the radius of the cylindrical tank is \( R \) and the water depths is \( h \). Assume the porous matrix is rigid and the liquid is incompressible; the continuity and momentum equations of liquid within the tank are [11];

\[
\nabla \cdot \mathbf{u} = 0
\]

\[
\left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla P + \nabla P_d
\]

where \( \mathbf{u} \) is the intrinsic fluid velocity, \( P \) the pressure, \( P_d \) is the pressure drop due to porous medium, and \( \rho \) the fluid density. In this paper, the tank is only considered to subject to the small external excitations, therefore the flow can be assumed to be in Darcy’s flow regime, and the gradient of the pressure drop can be expressed as:

\[
\nabla P_d = -\frac{\mu \gamma}{\kappa} (\mathbf{u} - \mathbf{v})
\]

where \( \mathbf{v} \) the velocity of tank, \( \mu \) the dynamic viscosity, \( \gamma \) the porosity, \( \kappa \) the permeability. Equation (3) shows the linear relation between damping and fluid velocity. Assume the flow field is irrotational, the velocity potential of fluid \( \phi \) and tank \( \psi \) can be defined as:

\[
\mathbf{u} = \nabla \phi
\]

\[
\mathbf{v} = \nabla \psi
\]

Due to the continuity of fluid and rigid-body motion of tank, both velocity potentials satisfy the Laplace equation as:

\[
\nabla^2 \phi = 0
\]

\[
\nabla^2 \psi = 0
\]

There are two kinds of boundary conditions: the Dirichlet boundary condition given on the free surface and the Neumann boundary condition given on walls and bottom. On the free surface, the kinematic and dynamic boundary conditions are expressed as equations (8) and (9), respectively [11].

\[
\frac{\partial R}{\partial t} = \frac{\mathbf{u}}{\gamma}
\]
\[ \frac{\partial \phi}{\partial t} + \nabla \phi \cdot \nabla \phi + \gamma \alpha (\phi - \psi) + g(z - h) = 0 \] (9)

where \( R \) is the location of particle on the free surface, \( \eta \) is the vertical deviation of water surface from the mean free surface, \( \alpha = \frac{\mu}{\kappa \rho} \), and \( g \) the gravitational acceleration. On the impervious bottom and walls, the impermeable condition can be expressed as:

\[ \frac{\partial \phi}{\partial n} = v_n \] (10)

where \( n \) is the unit outward vector, and \( v_n \) the normal velocity of the tank boundary.

2.2. Transient and steady-state solutions of linear waves in cylindrical porous-medium tank

The linearized kinematic boundary condition on the free surface in porous media is given as:

\[ \frac{\partial \eta}{\partial t} = \frac{1}{\gamma} \frac{\partial \phi}{\partial z} \] (11)

Substituting equation (11) into the linearized form of equation (9), the velocity potentials at \( z = h \) satisfy:

\[ \frac{\partial^2 \phi}{\partial t^2} + \gamma \alpha \left( \frac{\partial \phi}{\partial t} - \frac{\partial \psi}{\partial t} \right) + \frac{\gamma}{\gamma} \frac{\partial \phi}{\partial z} = 0 \] (12)

To solve the general solution, the velocity of tank can be given as \( \frac{\partial \psi}{\partial t} = 0 \), which means the tank is stationary, therefore the fluid will oscillate freely inside the tank. For simplicity, only the unidirectional sloshing motion is considered so that the even-mode fluid motions will vanish due to the anti-symmetrical fluid motion. Hence the transient solutions of fluid motions from the Laplace equation can be expressed as:

\[ \phi(r, \theta, z, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{mn} \cos \theta J_m(k_{mn}r) \cosh k_{mn}z \ e^{s_n t} \] (13)

where \( D_{mn} \) is a constant determined by the initial condition, \( J_m \) is the Bessel’s function of the first kind, and \( k_{mn} \) is a constant satisfies \( J_m'(k_{mn}R) = 0 \). Substituting equation (13) into equation (12) gives:

\[ s_n^2 \cosh k_{mn}h + s_n \gamma \alpha \cosh k_{mn}h + \frac{\gamma}{\gamma} k_{mn} \sinh k_{mn}h = 0 \] (14)

Similarly, \( s_n \) should be a complex number for an underdamped system such as:

\[ s_n = \frac{-\gamma \alpha}{2} \pm i \left( \frac{\gamma}{\gamma} k_{mn} \tanh k_{mn}h - \frac{\gamma^2 a^2}{4} \right)^{1/2} \] (15)

Therefore the modal sloshing frequency \( \omega_{dmn} \) of the cylindrical tank is solved as:

\[ \omega_{dmn} = \left( \frac{\gamma}{\gamma} k_{mn} \tanh k_{mn}h - \frac{\gamma^2 a^2}{4} \right)^{1/2} \] (16)

Consider the cylindrical tank is subjected to a harmonic excitation along x-direction as:

\[ x_e(t) = A \sin \omega t \] (17)

where \( A \) is the amplitude of displacement, and \( \omega \) is the forcing frequency. The velocity potential of the tank becomes:

\[ \psi = A \omega r \cos \theta \cos \omega t \] (18)

Therefore the velocity potential of fluid can be expressed in the cylindrical coordinate as:

\[ \phi(r, \theta, z, t) = A \omega r \cos \theta \cos \omega t + A \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( C_{mn} \cos \omega t + D_{mn} \sin \omega t \right) \cos \theta J_m(k_{mn}R) \cosh k_{mn}z \] (19)
Substituting equation (19) into equation (12) gives the following identity:

\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( -\omega^2 C_{mn} + \gamma \omega D_{mn} + \frac{g y k_m \tanh k_m h D_{mn}}{D_{mn}} \cos \omega t + \left( -\omega^2 D_{mn} + \gamma \omega C_{mn} + \frac{g y k_m \tanh k_m h D_{mn}}{D_{mn}} \sin \omega t \right) \cos \theta \right) \int_f (k_{mn} R) \cosh \left( \frac{\lambda h}{2} \right) \]

Since \( r \cdot \cos \theta \) can be replaced by Fourier-Bessel series as:

\[ r \cdot \cos \theta = \sum_{n=0}^{\infty} \frac{4(-1)^n}{a_n^2 L} \sin a_n x \]  

The coefficients \( C_{1n} \) and \( D_{1n} \) can be solved as:

\[ C_{1n} = \frac{p_n}{p_n^2 + q^2 \cosh k_{1n} h} \frac{\omega^3}{2R} \frac{2R}{J_1(k_{1n} R)} \]  

\[ D_{1n} = \frac{q}{p_n^2 + q^2 \cosh k_{1n} h} \frac{\omega^3}{2R} \frac{2R}{J_1(k_{1n} R)} \]

The horizontal resultant force \( F \) applied to the cylindrical tank induced by sloshing can be obtained as:

\[ F = \gamma \rho \int_0^h \int_0 r 2 \alpha \frac{\partial}{\partial t} (R, \theta, z, t) \cos \theta \, R \, d\theta \, dz \]

\[ = A \gamma \rho \pi R^2 \left[ -\frac{\omega^2}{2} \sin \omega t + \sum_{n=0}^{\infty} \frac{2 \omega^2 \tanh \left( \frac{\lambda_n h}{2} \right)}{k_{1n} (k_{1n}^2 R^2 - 1)} \right. \frac{\sin \omega t + q}{p_n^2 + q^2} \]

2.3. Equivalent mechanical model of cylindrical porous-medium tank

The equivalent mechanical system shown in Figure 2 consists of a fixed mass \( m_f \) and infinite sets of moving masses \( m_n \) located at the heights \( Z_f \) and \( Z_n \), respectively. The moving masses have their equivalent damping coefficients \( c_n \) and equivalent stiffness \( k_n \). The fixed mass explains the rigid-body motion of liquid, while the moving mass-damping-spring systems account for the oscillation motion of the tank. Under the same excitation as equation (19), the total horizontal force \( \bar{F} \) act on the ground can be obtained as follows:

\[ \bar{F} = (m_f + \sum_{n=0}^{\infty} m_n) \ddot{x}_g + \sum_{n=0}^{\infty} m_n \frac{A \omega^4}{p_n^2 + (2 \omega \omega_n^0)^2} (-p_n \sin \omega t + 2 \xi_n \omega_n \cos \omega t) \]

where \( \omega_n = (k_n/m_n)^{1/2} \), and \( \xi_n = c_n/2m_n \omega_n \).

Since the equivalent model is dynamically equivalent to the actual fluid, they must exert the same total horizontal force. The equivalent moving masses, stiffness, damping ratio and fixed mass can be determined by comparing equations (24) and (25) as follows:

\[ m_n = \frac{2 \gamma \rho R^3}{\lambda_n (\lambda_n^2 - 1)} \tanh \left( \frac{\lambda_n h}{R} \right) \]

\[ k_n = m_n \omega_n^2 = \frac{2 \gamma \rho R^2}{\lambda_n (\lambda_n^2 - 1)} \tanh^2 \left( \frac{\lambda_n h}{R} \right) \]

\[ \xi_n = \frac{\gamma \omega_n}{2 \lambda_n} = \frac{\gamma \omega_n}{2 \lambda_n} \left[ \frac{\gamma R}{\lambda_n \omega_n \mu} \coth \left( \frac{\lambda_n h}{R} \right) \right]^{1/2} \]

\[ m_f = m_w - \sum_{n=0}^{\infty} m_n = \gamma \rho R^2 \left[ 1 - \sum_{n=0}^{\infty} \frac{2 R}{\lambda_n (\lambda_n^2 - 1)} \tanh \left( \frac{\lambda_n h}{R} \right) \right] \] (29)
example $\lambda_0=1.8412$, $\lambda_1=5.3314$ and so on, $m_w=\gamma\rho\pi R^2$ denotes the total water mass of a cylindrical porous-medium tank. The wave elevation on the side wall at $x=R$ can be expressed in terms of the equivalent displacement $x_n$ as:

$$\eta(x=R) = \sum_{n=0}^{\infty} \frac{2\lambda_n}{\gamma(\lambda_n^2-1)} \tanh \left( \frac{\lambda_n h}{R} \right) \cdot x_n$$  \hspace{1cm} (30)

Figure 1. Coordinate system and dimensions of a cylindrical porous-medium tank.  

Figure 2. The equivalent mechanical system of sloshing in a cylindrical porous-medium tank.  

3. Example: Dynamic characteristics of cylindrical porous-medium tank  
Consider the porous media is fully fill in an impervious cylindrical tank, which is partially filled with water. The dynamic viscosity of water $\mu=10^{-3}$ Pa·s and its density $\rho=1000$ kg/m$^3$. The first-mode normalized natural frequencies of a cylindrical porous-medium tank are shown in Figure 3. As porosity decrease, the natural frequency becomes lower regardless of water depth. For a certain value of porosity, the natural frequencies dramatically increase as the water depth increases. After $h/R>0.5$, it will finally converge to a constant value. The normalized permeability of the porous medium in cuboid tank at the case of $\gamma=0.8$ are shown in Figure 4. The lower permeability results in higher damping ratio. In general, the optimal damping ratio would be 4%–8% for the vibrational damper. Therefore the permeability needs to be within the area between two dashed lines shown in Figure 4. For the shallow-water tank, this area becomes smaller.

Figure 3. The first-mode normalized natural frequencies of a cylindrical porous-medium tank for different values of $h/R$.  

Figure 4. The normalized permeability of the cylindrical porous-medium tank for different values of $h/R$ when $\gamma=0.8$.  

\[ \text{Example: } \lambda_0=1.8412, \lambda_1=5.3314 \text{ and so on, } m_w=\gamma\rho\pi R^2 \text{ denotes the total water mass of a cylindrical porous-medium tank. The wave elevation on the side wall at } x=R \text{ can be expressed in terms of the equivalent displacement } x_n \text{ as:} \]

\[ \eta(x=R) = \sum_{n=0}^{\infty} \frac{2\lambda_n}{\gamma(\lambda_n^2-1)} \tanh \left( \frac{\lambda_n h}{R} \right) \cdot x_n \]  \hspace{1cm} (30)
According to the equivalent model, the resultant force act on the ground shown in Equation (25) is related to the fixed and moving masses. Figure 5 shows the ratios between the summation of the first-N-mode equivalent masses and the total water mass. As the water depth increases to $h/R \geq 1$ (deep-water tank), the fixed mass becomes over 50% of the total water mass. Therefore the horizontal sloshing force will be more in phase with the external excitations. The first-mode moving mass is significant to the entire water mass, while the fixed mass and the first-five-mode moving masses will be over 98% of the total water mass. Figure 6 shows the ratios between the $2^{nd}$-$5^{th}$ modal damping ratios and the first modal damping ratio. The higher mode always has the lower damping effect. For the shallow-water tank, the fifth-mode damping ratio is less than 30% of the first-mode one. From Figures 5 and 6, the equivalent masses and damping ratios of any mode higher than the fifth mode are small enough to be ignored. It would be reasonable to simply take the first five mode into account in the analysis for convenience.

![Figure 5](image1.png)  ![Figure 6](image2.png)

**Figure 5.** The ratios between the summation of the first-N-mode equivalent masses and the total water mass of a cylindrical porous-medium tank.

**Figure 6.** The ratios between the modal damping ratios and the first-mode damping ratio of a cylindrical porous-medium tank.

### 4. Conclusions

The linear sloshing behavior of Darcy’s flow regime in an impervious cylindrical tank filled with porous media can be expressed by the equivalent mechanical system derived in this paper. The equivalent masses, stiffness’s and damping ratios are revealed so that the dynamic analysis can be easily carried out without using any empirical formula. Therefore it is valuable for engineers since it is simple to analyze the sloshing behavior from the structural dynamic analysis point of view.

Both the natural frequency and damping ratio are related to the tank dimensions, porosity and permeability of the porous medium. For deep-water tanks, the horizontal sloshing force will be more in phase with the external excitations. Since the equivalent mass and damping ratio of higher modes are very small compared to that of the first mode, the summation of the first five modal responses would be accurate enough for the sloshing analysis.

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