Coweight lattice $A_n^*$ and lattice simplices

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Abstract

There exist as many index-$k$ sublattices of the hexagonal lattice up to isometry as there exist lattice triangles with normalized volume $k$ up to unimodular equivalence, which can be explained using orbifolds. In dimension 3, it was noted that the number of sublattices of the fcc and the bcc lattices and the number of lattice tetrahedra all seem to be the same. We provide a bijection between the sublattices of the coweight lattice $A_n^*$ and the $n$-dimensional lattice simplices. It explains, proves, and generalizes the observed coincidences to arbitrary dimension.

Sections 1 and 2 are introductory: the needed concepts known from literature and our notation for them are introduced. Some coincidences between the number of sublattices and the number of lattice simplices are noted. In Section 3, the result is stated and proven: using only elementary methods, we provide a bijection between the sublattices of the $A_n^*$ lattice and the $n$-dimensional lattice simplices that preserves certain equivalence relations, which ultimately explains the said coincidences.

1 Sublattices

The full-rank sublattices of the $n$-dimensional root lattice $A_n$ and its dual, the coweight lattice $A_n^*$, have been studied and counted in different contexts. Bernstein et al. (1997) and Rutherford (2009) have counted the sublattices of the hexagonal lattice $A_2$ (or $A_2^*$) of any given index. Davey et al. (2010), Hanany et al. (2010), Hanany and Seong (2011) have drawn connections between abelian orbifolds, simplical toric diagrams, brane tilings, and the sublattices of certain lattices (in particular, noting that the sublattices of the hexagonal lattice up to isometry and the lattice triangles up to unimodular equivalence both correspond to abelian orbifolds of $\mathbb{C}^3$). Hart and Forcade (2008) have counted the sublattices of the fcc lattice $A_3$ and the bcc lattice $A_3^*$, obtaining matching numbers. Amini and Manjunath (2010) have related the sublattices of $A_n$ to the discrete Laplacians of finite graphs. Also, Montagard and Ressayre (2009) studied the connection between regular lattice simplices and the root system $A_n$. An introduction to the lattices $A_n$ and $A_n^*$ is given, e. g., in Martinet (2003), Sec. 4.2.

A sublattice can be defined by its basis. Columns of coefficients of the basis vectors with respect to the parent $n$-dimensional lattice can be assembled into an integer $n \times n$ matrix $B$. We denote the sublattice (of some lattice, depending on the context) generated by the basis $B$ as $L(B)$. We allow negative orientation of the basis, so the index of the sublattice is given by $|\det B|$. The choice of the basis of a sublattice is apparently ambiguous. The unique representative of the class of basis matrices generating a particular sublattice having a certain canonical form is known as the Hermite normal form (we will not need it though).

Most of the above-mentioned papers deal with the following equivalence relation for sublattices.
Definition 1. We say that two sublattices $\mathcal{L}(B_1)$ and $\mathcal{L}(B_2)$ of some parent lattice $\mathcal{L}$ are isometric, denoted $\mathcal{L}(B_1) \sim \mathcal{L}(B_2)$, if one transforms into another via an isometric automorphism of $\mathcal{L}$.

For example, the two sublattices of $A_2$ shown in Fig. 1 are isometric because they are related via a $\pi/3$ rotation (or via a certain reflection which is also an isometry of $A_2$).

We denote the number of equivalence classes of index-$k$ sublattices of $A_n$ with respect to isometricity by $\beta_{n,k}$. The sequence $\{\beta_{2,k}\}_k$ is OEIS A003051; that entry provides several explicit formulas for this sequence based on the above-referenced papers. The sequence $\{\beta_{3,k}\}_k$ is OEIS A159842; the explicit formula for this sequence had been derived by Hanany et al. (2010) using Polya’s enumeration theorem (it is given for “the tetrahedral lattice”, which is more commonly known as the diamond crystal structure, consisting of two interpenetrating copies of the $A_3$ lattice; only one copy can contain a sublattice).

There is another commonly used equivalence relation for sublattices.

Definition 2. We say that two sublattices $\mathcal{L}(B_1)$ and $\mathcal{L}(B_2)$ of some parent lattice $\mathcal{L}$ are properly isometric, denoted $\mathcal{L}(B_1) \sim^+ \mathcal{L}(B_2)$, if one transforms into another via an orientation-preserving isometric automorphism of $\mathcal{L}$.

That is, no reflections are allowed in this case. The two sublattices in Fig. 1 are properly isometric, too. We denote the number of index-$k$ sublattices of $A_n$ inequivalent with respect to proper isometricity by $\beta^+_{n,k}$. The sequence $\{\beta^+_{2,k}\}_k$ is OEIS A145394.

If we consider affine lattices, their affine sublattices, and their isometric affine automorphisms (i.e., including translations) instead, the equivalence classes of sublattices defined by properly adjusted definitions 1 and 2 remain the same. It is possible to consider an equivalence relation that identifies the sublattices related via an arbitrary isometry of the ambient vector space, that is, via non-crystallographic rotations (and maybe reflections), but the problem of counting such equivalence classes is more difficult. It is related to the study of coincidence site lattices, which were counted in $A_4$ by Baake and Zeiner (2008), Heuer and Zeiner (2010).

2 Lattice simplices

The studies of the lattice polytopes is an established field [Haase et al. (2012)]. Lattice polytopes are commonly studied up to the unimodular equivalence: two lattice polytopes $S_1$ and $S_2$ (or any other geometric entities defined on a lattice) are unimodularly equivalent, denoted $S_1 \cong S_2$, if they are related via an affine transformation of the ambient space preserving the parent lattice. Lattice polytopes are closely related to toric varieties [Cox et al. (2011)].

We will only consider $n$-dimensional lattice simplices, which can be identified with unordered collections of $n + 1$ vertices. The particular type of the parent lattice does not matter because no isometries are involved. Without loss of generality, we will always assume that one of the vertices is at the origin. The columns of the coordinates of the other $n$ vertices with respect to the basis of the parent lattice form the integer non-degenerate matrix $T$. We denote the ordered lattice simplex (i.e., the $(n+1)$-tuple of vertices), which is the convex hull of the vectors given by
Figure 2: Two unimodularly equivalent unordered, unoriented lattice triangles with lattice volume 2.

T together with the origin, by $S(T)$, the corresponding oriented lattice simplex (i.e., the lattice simplex ordered modulo even permutations) as $|S(T)|$, and the (unordered, unoriented) lattice simplex formed by these vectors by $|S(T)|$. The volume of that simplex (induced by the parent lattice) is $\frac{1}{n!}|\det T|$; in the terms of [Haase et al. (2012)], it has normalized volume (which is the volume induced by the parent lattice times $n!$) $\det T$. The unimodular equivalence preserves the volume.

The notion of the unimodular equivalence is applicable to ordered and oriented simplices as well as the unordered, unoriented ones. $S(T_1) \sim S(T_2) \Rightarrow \lfloor S(T_1) \rfloor \sim \lfloor S(T_2) \rfloor \Rightarrow |S(T_1)| \sim |S(T_2)|$ for all $T_{1,2}$, but the converse is not true. This motivates the following equivalence relations.

**Definition 3.** We say that two ordered lattice simplices, $S(T_1)$ and $S(T_2)$, are **unimodularly equivalent as unordered, unoriented lattice simplices**, denoted $S(T_1) \sim S(T_2)$, if $|S(T_1)| \sim |S(T_2)|$.

For example, the two lattice triangles shown in Fig. 2 are unimodularly equivalent as unordered, unoriented lattice simplices (in fact, the order of their vertices or their orientation is not even chosen and shown).

This equivalence relation respects the volume. We denote the number of equivalence classes of $n$-dimensional ordered lattice simplices with normalized volume $k$ with respect to this equivalence relation by $\tau_{n,k}$.

This equivalence relation is the same as the one introduced in [Davey et al. (2010)] for toric diagrams based on the barycentric coordinates and the “topological character”. The sequence $\{\tau_{3,k}\}_k$, which counts lattice tetrahedra, was first computed by J.-O. Moussafr, and [Hart and Forcade (2008)] noted the coincidence with $\{\beta_{3,k}\}_k$. The sequences $\{\tau_{n,k}\}_k$ for $n = 4, 5, 6$ are OEIS A173824, A173877, A173878; they were studied in [Hanany and Seong (2011)] and also computed by [Balletti (2020)].

Karpenkov (2013) studies the lattice geometry, lattice trigonometry, and their relation to the continued fractions; toric geometry is also known to have connections with continued fractions and Hirzebruch–Jung continued fractions [Cox et al. (2011), §10.2]. In Sec. 6.5 of Karpenkov (2013), the oriented lattice triangles are counted up to the unimodular equivalence (also called integer congruence). This inspires the following equivalence relation.

**Definition 4.** We say that two ordered lattice simplices, $S(T_1)$ and $S(T_2)$, are **unimodularly equivalent as oriented lattice simplices**, denoted $S(T_1) \sim^+ S(T_2)$, if $|S(T_1)| \sim |S(T_2)|$.

We denote the number of equivalence classes of $n$-dimensional ordered lattice simplices with normalized volume $k$ with respect to unimodularly equivalence as oriented lattice simplices by $\tau^+_{n,k}$. Karpenkov (2013) gives $\tau^+_{2,k}$ for $k \in \{1, \ldots, 20\}$. One can see that they coincide with the corresponding values of $\beta^+_{2,k}$.

3 Bijection

The above-reviewed connections and observations lead to the following conjecture.
Proposition 1. \( \beta_{n,k} = \tau_{n,k} \) and \( \beta_{n,k}^+ = \tau_{n,k}^+ \) for all \( n,k \).

We will prove it using only elementary methods.

We need the following theorem.

**Theorem 1.** There is a bijection between the ordered lattice simplices of lattice volume \( k \) on an \( n \)-dimensional lattice and the bases of index-\( k \) sublattices of the coweight lattice \( \Lambda_n^* \). That bijection maps

- bases generating the same sublattices to unimodularly equivalent ordered lattice simplices and vice versa,
- bases generating properly isometric sublattices to lattice simplices unimodularly equivalent as oriented lattice simplices and vice versa,
- bases generating isometric sublattices to lattice simplices unimodularly equivalent as unordered, unoriented lattice simplices and vice versa.

**Proof.** It is known that \( \mathcal{L}(B_1) = \mathcal{L}(B_2) \Leftrightarrow \exists L \in \text{GL}_n(\mathbb{Z}) : B_1 = B_2 L \), i.e., a basis change is represented in coordinates by the right multiplication by a unimodular integer matrix.

The isometries of the parent lattice are represented by the left multiplication of the sublattice basis matrix by certain matrices from \( \text{GL}_n(\mathbb{Z}) \). The group of isometric automorphisms of the \( \Lambda_n^* \) coweight lattice is isomorphic to \( \mathfrak{S}_{n+1} \times \mathbb{Z}_2 \) where \( \mathbb{Z}_2 \) corresponds to the space inversion (which is already taken into account since we consider the basis change simultaneously with the isometries) and the full symmetric group \( \mathfrak{S}_n \) corresponds to the reflections and rotations permuting the set of \( n \) certain appropriately chosen minimal lattice vectors \( e_1, \ldots, e_n \), together with the vertex \( e_0 = -(e_1 + \cdots + e_n) \), also minimal \([\text{Martinet} (2003)]\). In the basis of these minimal vectors, these transformations are represented by the set \( \mathfrak{P}_n \) of matrices generated by the set of all \( n \times n \) permutation matrices, \( \Pi_n \), and the matrices \( P_{n,i} \) for \( i \in \{1, \ldots, n\} \), where their elements are \( P_{n,i;j,i} = -1 \) and \( P_{n,i;j,\ell} = \delta_{j,\ell} \) for \( j \in \{1, \ldots, n\} \), \( \ell \in \{1, \ldots, n\} \setminus \{i\} \). So \( \mathcal{L}(B_1) \sim \mathcal{L}(B_2) \Leftrightarrow \exists L \in \text{GL}_n(\mathbb{Z}), R \in \mathfrak{P}_n : B_1 = RB_2 L ; \mathcal{L}(\cdot) \) hereafter denotes a sublattice of the \( \Lambda_n^* \) lattice in the basis of the appropriate minimal vectors \( e_1, \ldots, e_n \).

The orientation-preserving isometric automorphisms of \( \Lambda_n^* \) are represented by the matrices from \( \mathfrak{P}_n \) with determinant 1. Consequently, \( \mathcal{L}(B_1) \sim^+ \mathcal{L}(B_2) \Leftrightarrow \exists L \in \text{GL}_n(\mathbb{Z}), R \in \mathfrak{P}_n \cap \text{SL}_n(\mathbb{Z}) : B_1 = RB_2 L \).

On the other hand, the unimodular equivalence of the ordered lattice simplices is equivalent to the existence of a unimodular matrix relating their coordinates: \( S(T_1) \cong S(T_2) \Leftrightarrow \exists L \in \text{GL}_n(\mathbb{Z}) : T_1 = LT_2 \).

Reordering of the vertices of the simplex \( |S(T)| \) includes the permutations of the \( n \) non-origin vertices represented by the right multiplication of the matrix \( T \) by the permutation matrices from \( \Pi_n \) and translating the simplex so that one of its vertices, \( t_i \), moves to the origin and the former origin vertex takes its place in the tuple of vertices as the new vertex \( -t_i \). These translations are represented by the matrices \( P_{n,i}^T \) where \( P_{n,i} \) is defined as above. The matrix group generated by \( \Pi_n \) and \( \{P_{n,i}^T\}_i \) is \( \mathfrak{P}_n^T \). So \( |S(T_1)| = |S(T_2)| \Leftrightarrow \exists R \in \mathfrak{P}_n^T : T_1 = T_2 R \), and \( S(T_1) \sim S(T_2) \Leftrightarrow \exists L \in \text{GL}_n(\mathbb{Z}), R \in \mathfrak{P}_n^T : T_1 = LT_2 R \).

Similarly, since \( \mathfrak{P}_n^T \cap \text{SL}_n(\mathbb{Z}) = (\mathfrak{P}_n \cap \text{SL}_n(\mathbb{Z}))^T \), for the unimodular equivalence as oriented lattice simplices we have: \( S(T_1) \sim^+ S(T_2) \Leftrightarrow \exists L \in \text{GL}_n(\mathbb{Z}), R \in (\mathfrak{P}_n \cap \text{SL}_n(\mathbb{Z}))^T : T_1 = LT_2 R \).

From the above-given coordinate representations of the equivalence relations of the sublattices and the lattice simplices, it follows that there is a bijection between the bases of the sublattices of \( \Lambda_n^* \) and the ordered lattice polytopes preserving the equivalence relations defined on them. That bijection is given by the matrix transposition:

\[
\mathcal{L}(B_1) = \mathcal{L}(B_2) \Leftrightarrow S(B_1^T) \cong S(B_2^T) \\
\mathcal{L}(B_1) \sim \mathcal{L}(B_2) \Leftrightarrow S(B_1^T) \sim S(B_2^T) \\
\mathcal{L}(B_1) \sim^+ \mathcal{L}(B_2) \Leftrightarrow S(B_1^T) \sim^+ S(B_2^T)
\]
The bijection of Theorem 1 in the case \( n = 2, k = 6 \) is illustrated in Fig. 3.

To get Proposition 1, we now only need to prove the following proposition and apply it to the lattice \( A_n \).

**Proposition 2.** The number of mutually non-isometric or non-properly isometric index-\( k \) sublattices is the same for a lattice \( L \) and its dual \( L^* \).

There exists a canonical bijection between the sublattices of a lattice \( L \) and the superlattices of its dual \( L^* \), but it is of no use here. The automorphism groups of \( L \) and \( L^* \) are the same. So if we invoke Burnside’s lemma to count the non-isometric sublattices (as is done by Rutherford (2009) and Hanany et al. (2010)), to prove Proposition 2 we only need to show that the number of sublattices of the given index fixed by any particular isometry is the same for \( L \) and \( L^* \). It would follow from the following proposition.

**Proposition 3.** For any isometric automorphism of a lattice \( L \), there exist bases in \( L \) and \( L^* \) such that they are unimodularly transformed by that isometric automorphism in the same way.

**Proof.** Let \( L = \mathcal{L}(B) \) be an \( n \)-dimensional lattice generated by some basis vectors (written in the orthonormal basis of the ambient space \( \mathbb{R}^n \)) forming the matrix \( B \in \text{GL}_n(\mathbb{R}) \) (we can assume that the lattice is unimodular without loss of generality). Then \( \mathcal{L}(B^*) = \mathcal{L}(B^{-T}) \) where \( B^{-T} \equiv (B^T)^{-1} \). An isometric automorphism of \( \mathcal{L}(B) \) is an orthogonal transformation of the coordinates \( R_o \in \text{O}_n(\mathbb{R}) \) such that \( \mathcal{L}(R_oB) = \mathcal{L}(B) \), which means that \( R_oB = BR \) for some \( R \in \text{GL}_n(\mathbb{Z}) \), and also \( R_oB^{-T} = B^{-T}R' \), \( R' \in \text{GL}_n(\mathbb{Z}) \). The statement of the proposition means that there is a basis \( B' \) for \( L^* \) such that \( R_oB' = B'R \). It holds if we take \( B' = B^{-T}L \) where \( L = (R')^{-1}R \in \text{GL}_n(\mathbb{Z}) \). \( \square \)
Proposition 2 follows. Recalling Theorem 1, Proposition 1 follows.

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