The Construction of Pointlike Localized Charged Fields from Conformal Haag-Kastler Nets *

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Abstract

Starting from a chiral conformal Haag-Kastler net on 2 dimensional Minkowski space we construct associated charged pointlike localized fields which intertwine between arbitrary superselection sectors with finite statistics of the theory. This amounts to a proof of the Spin-Statistics theorem, the PCT theorem and the existence of operator product expansions.

This paper generalizes similar results of a recently published paper by Fredenhagen and the author [FrJ] from the neutral vacuum sector to the the full theory with arbitrary charge and finite statistics.

1 Introduction

The formulation of quantum field theory can be done in terms of Haag Kastler nets of local observable algebras (“local quantum physics” [Haag]) or in the framework of Wightman axioms [StW]. Usually, both concepts are assumed to be physically equivalent. There is, however, still no proof for this assumption.

In a recently published paper [FrJ] Fredenhagen and the author could give a partial answer to the question how both concepts are interrelated in the case of chiral conformal quantum field theory. Starting from a chiral conformal Haag-Kastler net on 2 dimensional Minkowski space we constructed associated neutral pointlike localized fields in the vacuum sector. This amounted to a proof of the existence of operator product expansions for observables. The present paper generalizes these results from the neutral vacuum sector to the the full theory with arbitrary charge and finite statistics.

The existence of operator product expansions in the Wightman framework has been postulated by Wilson [Wil]. Especially in 2d conformal field theory, this assumption has turned out to be very fruitful. The existence of a convergent expansion of the product of two fields on the vacuum could been derived from conformal covariance, but the existence of the associated local fields had to be postulated [Lus, Mac, SSV].

The existence of an operator product expansion in the Haag-Kastler framework might be formulated as the existence of sufficiently many Wightman fields such that their linear span

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applied to the vacuum is dense in the Hilbert space. Actually, we are able to derive a stronger result: We prove an expansion of local operators (arbitrary local elements of the reduced field bundle \([\text{FRS1}, \text{FRS2}]\) associated to the net of observables) into charged fields with local coefficients and show that this expansion converges \(*\)-strongly on a dense domain in the physical Hilbert space.

The construction of Wightman fields out of Haag-Kastler nets should be possible by some scaling limit. This heuristic idea, however, is difficult to formulate in an intrinsic way \([\text{Buc}]\). Buchholz and Fredenhagen \([\text{BuF}]\) performed the construction of a pointlike field in the presence of massless particles in a dilation invariant theory where scaling is well defined.

We study the possibly simplest situation: Haag-Kastler nets in 2 dimensional Minkowski space with trivial translations in one light cone direction (“chirality”) and covariant under the real Möbius group which acts on the other lightlike direction.

We construct pointlike localized fields carrying arbitrary charge with finite statistics and therefore intertwining between the different superselection sectors of the theory. (In Conformal Field Theory these objects are known as “Vertex Operators”.) We obtain the unbounded field operators as limits of elements of the reduced field bundle \([\text{FRS1}, \text{FRS2}]\) associated to the net of observables of the theory. Their smeared linear combinations are affiliated to the reduced field bundle and generate it. At the moment, we do not know whether the constructed fields satisfy all Wightman axioms, since we have not yet found an invariant domain of definition.

Our method consists of an explicit use of the representation theory of the universal covering group of \(SL(2, \mathbb{R})\) combined with a conformal cluster theorem. This conformal cluster theorem was proven in \([\text{FrJ}]\) for the vacuum sector and can be generalized in the present paper to the case of arbitrary charge with finite statistics.

As a consequence of the existence of charged pointlike localized fields we can finally prove the Spin-Statistics theorem\(^1\), the PCT theorem for the full theory, a generalized version of the Bisognano-Wichmann property and additivity of the net of local algebras.

\section{Assumptions and Results}

\subsection{Assumptions}

Let \(\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{K}_0}\) be a family of von Neumann algebras on some separable Hilbert space \(H\). \(\mathcal{K}_0\) denotes the set of nonempty bounded open intervals on \(\mathbb{R}\). \(\mathcal{A}\) is assumed to satisfy the following conditions.

\begin{enumerate}
  \item Isotony:
  \[ \mathcal{A}(I_1) \subset \mathcal{A}(I_2) \quad \text{for} \quad I_1 \subset I_2, \quad I_1, I_2 \in \mathcal{K}_0. \]  \(1\)
  \item Locality:
  \[ \mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \quad \text{for} \quad I_1 \cap I_2 = \emptyset, \quad I_1, I_2 \in \mathcal{K}_0 \]  \(2\)
  \[ (\mathcal{A}(I_2)')' \text{ is the commutant of } \mathcal{A}(I_2). \]
  \item There exists a strongly continuous unitary representation \(U\) of \(G = SL(2, \mathbb{R})\) in \(H\) with \(U(-1) = 1\) and
  \[ U(g) \mathcal{A}(I) U(g)^{-1} = \mathcal{A}(gI), \quad I, gI \in \mathcal{K}_0 \]  \(3\)
\end{enumerate}

\(^1\)After the completion of the present paper we received a preprint by Guido and Longo \([\text{GLo}]\) that gives an independent proof of the conformal Spin-Statistics theorem.
$(SL(2, \mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix})$ acts on $\mathbb{R} \cup \{\infty\}$ by $x \mapsto \frac{ax + b}{cx + d}$ with the appropriate interpretation for $x, gx = \infty$.

iv) The conformal Hamiltonian $H$, which generates the restriction of $U$ to $SO(2)$, has non-negative spectrum.

v) There is a unique (up to a phase) $U$-invariant unit vector $\Omega \in H$.

vi) $H$ is the smallest closed subspace containing the vacuum $\Omega$ which is invariant under $U(g), g \in SL(2, \mathbb{R}),$ and $A \in \mathcal{A}(I), I \in \mathcal{K}_0$ (“cyclicity”).

2.2 Known Results

Several results on the structure of the vacuum sector of an conformally invariant Haag-Kastler theory in 1+1 dimensions are well known. With the use of a theorem of Borchers [Bo], it was possible [FröG, BGL] to identify the modular structure with geometrical objects in accordance with the ideas of Bisognano and Wichmann [BiW]. As a consequence, Haag duality and additivity of the net of local algebras and a PCT theorem could be proven. In [FrJ], we recently constructed pointlike localizable conformal field operators in the vacuum sector. These unbounded operators of neutral fields could be shown to be closable, their closures are affiliated with the associated algebra of local observables and generate the local net of observables.

In the same paper, a conformal cluster theorem could be proven and an expansion of local observables into neutral fields with local coefficients was given. 

2.3 New Results

In order to construct charged fields intertwining between the superselection sectors with finite statistics of the theory, we consider the reduced field bundle $\mathcal{F}_{\text{red}} = (\mathcal{F}_{\text{red}}(I))_{I \in \mathcal{K}_0}$ associated to the net of observables $\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{K}_0}$, introduced in [FRS1, FRS2].

The reduced field bundle $\mathcal{F}_{\text{red}}$ is an algebra densely spanned by operators $F = F(e, A)$, linear in the local degree of freedom $A \in \mathcal{A}$ and with a multi-index $e$. This multi-index refers to the charge carried by $F$ as well as to the source sector and the range sector between which $F$ interpolates according to the “fusion rules”. If $\rho, \alpha, \beta$ are charge, source and range of the corresponding “fusion channel” the multi-index $e$ is said to be of type $(\alpha, \rho, \beta)$. The elements of the reduced field bundle act on $H_{\text{red}}$, a realization of the physical Hilbert space. $H_{\text{red}}$ is the direct sum of copies of the vacuum Hilbert space, one for each superselection sector with finite statistics. The direct sum of the representations of the universal covering group of the conformal group for each superselection sector with finite statistics will be called $U(\tilde{G})$.

We are able to prove a generalization of the conformal cluster theorem to arbitrary charged sectors with finite statistics. This theorem specifies the decrease properties of conformal two-point-functions in the algebraic framework to be exactly those known from theories with point-like localization.

**Theorem:** Let $(\mathcal{A}(I))_{I \in \mathcal{K}_0}$ be a conformally covariant local net on $\mathbb{R}$. Let $a, b, c, d \in \mathbb{R}$ and $a < b < c < d$. Let $F \in \mathcal{F}_{\text{red}}((a, b))$, $G \in \mathcal{F}_{\text{red}}((c, d))$, $n \in \mathbb{R}$ and $P_k F \Omega = P_k \tilde{F} \Omega = 0, k < n$. $P_k$ here denotes the projection on the subrepresentation of $U(\tilde{g})$ with conformal dimension $k$. We then have

$$
|\langle \Omega, GF\Omega \rangle| \leq \left(\frac{(b-a)(d-c)}{(c-a)(d-b)}\right)^n \|F\| \|G\|.
$$

(4)
Due to the positivity condition, the representation $U(\hat{G})$ is completely reducible into irreducible subrepresentations and the irreducible components $\tau$ are up to equivalence uniquely characterized by the conformal dimension $n_\tau \in \mathbb{R}_+$ ($n_\tau$ is the lower bound of the spectrum of the conformal Hamiltonian $H$ in the representation $\tau$).

Associated with each irreducible subrepresentation $\tau$ of $U$ we find for each $I \in \mathcal{K}_0$ a densely defined operator valued distribution $\varphi^I_\tau$ on the space $\mathcal{D}(I)$ of Schwartz functions with support in $I$ such that the following statements hold for all $f \in \mathcal{D}(I)$.

i) The domain of definition of $\varphi^I_\tau(f)$ is given by $\mathcal{A}(I')\Omega$.

$$\varphi^I_\tau(f)\Omega \in P_\tau H_{\text{red}}$$

with $P_\tau$ denoting the projector on the module of $\tau$.

ii) $U(\tilde{g}) \varphi^I_\tau(x) U(\tilde{g})^{-1} = (cx + d)^{-2n_\tau} \varphi^{gI}_\tau(\tilde{g}x)$

with the covering projection $\tilde{g} \mapsto g$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, $I, gI \in \mathcal{K}_0$.

iv) $\varphi^I_\tau(f)$ is closable.

v) The exchange algebra of the reduced field bundle \cite{FRS2} and the existence of the closed field operators $\varphi^I_\tau(f)$, mapping a dense set of the vacuum Hilbert space into some charged sector with finite statistics, suffice to construct closed field operators $\varphi^{I,\alpha}_\tau(f)$, mapping a dense set of an arbitrary charged sector $\alpha$ with finite statistics into some (other) charged sector with finite statistics. Here, the irreducible module $\tau$ of $U(\hat{G})$ labels orthogonal irreducible fields defined in the same sector $\alpha$.

vi) The closure of any $\varphi^{I,\alpha}_\tau(f)$ is affiliated to $\mathcal{F}_{\text{red}}(I)$.

vii) $\mathcal{F}_{\text{red}}(I)$ is the smallest von Neumann algebra to which all operators $\varphi^{I,\alpha}_\tau(f)$ are affiliated.

With the existence of pointlike localized fields we are able to give a proof of a generalized Bisognano-Wichmann property. We can identify the conformal group and the reflections as generalized modular structures in the reduced field bundle. Especially, we obtain a PCT operator on $H_{\text{red}}$ proving the PCT theorem for the full theory.

Moreover, the existence of pointlike localized fields gives a proof of the hitherto unproven Spin-Statistics theorem for conformal Haag-Kastler nets in 1+1 dimensions.

It's also possible to generalize the operator product expansions of \cite{FrJ} to charged sectors with finite statistics:

**Theorem:** For each $I \in \mathcal{K}_0$ and each $F \in \mathcal{F}_{\text{red}}(I)$ there is a local expansion

$$F = \sum_{\tau, \alpha} \varphi^{I,\alpha}_\tau(f_{\tau,\alpha,F})$$

into a sum over all sectors $\alpha$ with finite statistics and all irreducible modules $\tau$ of $U(\hat{G})$ with

$$\text{supp} f_{\tau,\alpha,F} \subset I,$$

which converges on $\mathcal{F}_{\text{red}}(I')\Omega$ *-strongly (cf. the definition in \cite{BrR}). Here, $I'$ denotes the complement of $I$ in $\mathcal{K}_0$. 

4
3 Construction of Pointlike Localized Charged Fields

The following idea for the definition of charged conformal fields is the generalization of the idea for the case of neutral fields: Let \( A \) be a local observable, \( A \in \mathcal{A}(I_0) \), \( I_0 \in \mathcal{K}_0 \), let \( \rho \) be a localized and transportable endomorphism of \( \mathcal{A} \) inducing a charged sector with finite statistics, let \( e \) be a field bundle multi-index of type \((0, \rho, \rho)\). Then \( F = F(e, A) \) is a local element of the reduced field bundle. Let now \( n \in \mathbb{R}_+ \) and \( P_n \) the projection on the subspace of conformal dimension \( n \) in \( H_{\text{red}} \). We can think of \( P_n F\Omega \) as a vector of the form \( \varphi_n(h)\Omega \) where \( \varphi_n \) is a conformal field of dimension \( n \) and \( h \) is an appropriate function on \( \mathbb{R} \). The exact relation between \( F \) and \( h \), however, is unknown at the moment. All we have are the transformation properties under \( G \):

\[
U(\hat{g}) P_n F\Omega = \varphi_n(h^{(n)}_\hat{g})\Omega
\]

with \( h^{(n)}_\hat{g}(x) = (cx - a)^2 h(\frac{dx - b}{cx + a}) \) and the covering projection \( \hat{g} \mapsto g \) for \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \).

Scaling the vector \( P_n F\Omega \) by dilations \( D(\lambda) = U \begin{pmatrix} \frac{\lambda}{\lambda^2} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \) we find

\[
D(\lambda) P_n F\Omega = \lambda^n \varphi_n(h_\lambda)\Omega
\]

with \( h_\lambda(x) = \lambda^{-1} h(\frac{x}{\lambda}) \). Hence, we obtain formally for \( \lambda \downarrow 0 \)

\[
\lambda^{-n} D(\lambda) P_n F\Omega \longrightarrow \int dx h(x) \varphi_n(0)\Omega.
\]

We smear over the group of translations \( T(a) = U \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \) with some test function \( f \) and obtain in the formal limit a Hilbert space vector:

\[
\lim_{\lambda \downarrow 0} \lambda^{-n} \int da f(a) T(a) D(\lambda) P_n F\Omega = \int dx h(x) \varphi_n(f)\Omega.
\]

Now, the left hand side can be interpreted as a definition of a conformal field \( \varphi_n \) on the vacuum vector \( \Omega \). Writing down

\[
\varphi_n^I(f) A'\Omega = \pi_\rho(A')\varphi_n^I(f)\Omega, \quad f \in \mathcal{D}(I), \ A' \in \mathcal{A}(I'), \ I \in \mathcal{K}_0,
\]

we obtain operators with a domain of definition that is dense in the vacuum Hilbert space. In this section, \( I' \) always denotes the complement of \( I \) in \( \mathcal{K}_0 \).

In order to make this formal construction meaningful there are two problems to overcome.

The first one is the fact that the limit on the left hand side of (12) does not exist in general if \( F\Omega \) is replaced by an arbitrary vector in \( H_{\text{red}} \). This corresponds to the possibility that the function \( h \) on the right hand side might not be integrable. After smearing the operator \( F \) with a smooth function on \( \hat{G} \) the limit is well defined. Such operators will be called regularized.

The second problem is to show that the smeared field operators \( \varphi^I_n(f) \) are closable, in spite of the nonlocal nature of the projections \( P_n \). This problem will be solved by an argument based on the generalization of the conformal cluster theorem [FrJ] to the charged case.

First, we generalize the conformal cluster theorem to charged sectors with finite statistics.

**Theorem:** Let \((\mathcal{A}(I))_{I \in \mathcal{K}_0}\) be a conformally covariant local net on \( \mathbb{R} \). Let \( a, b, c, d \in \mathbb{R} \) and \( a < b < c < d \). Let \( F \in \mathcal{F}_{\text{red}}((a, b)) \), \( G \in \mathcal{F}_{\text{red}}((c, d)) \), \( n \in \mathbb{R} \) and \( P_k F\Omega = P_k \tilde{F}\Omega = 0 \), \( k < n \).
\( P_k \) here denotes the projection on the subrepresentation of \( U(\tilde{g}) \) with conformal dimension \( k \). We then have
\[
|\langle \Omega, GF\Omega \rangle| \leq \left( \frac{(b-a)(d-c)}{(c-a)(d-b)} \right)^n \|F\| \|G\|. \tag{14}
\]

**Proof:** (Cf. the proof of the conformal cluster theorem in [FrJ] and the idea in [Frc])

Choose \( R > 0 \). We consider the following 1-parameter subgroup of \( G = SL(2, \mathbb{R}) \)
\[
g_t : x \mapsto \frac{x \cos \frac{t}{2} + R \sin \frac{t}{2}}{-x \frac{R}{\sin \frac{t}{2} + \cos \frac{t}{2}}}. \tag{15}
\]

Its generator \( H_R \) is within each subrepresentation of \( U(\tilde{g}) \) unitarily equivalent to the conformal Hamiltonian \( H \). Therefore, the spectrum of \( F\Omega \) and \( F\bar{\Omega} \) w.r.t. \( H_R \) is bounded below by \( n \). Let \(-\pi < t_0 < t_1 < \pi \) such that \( g_{t_0}(b) = c \) and \( g_{t_1}(a) = d \). Because of the conformal covariance of the reduced field bundle, the function
\[
M(z) := \langle \Omega, G \alpha_{g_t}(F)\Omega \rangle, \quad z = e^{it}, \quad -\pi < t < \pi, \quad t \notin [t_0, t_1],
\]
is well-defined in its domain of definition. We consider the analytical properties of
\[
N(z) := (z - z_0)^n (z^{-1} - z_0^{-1})^n M(z), \quad z_0 := e^{\frac{i}{2}(t_0 + t_1)}. \tag{17}
\]

Using the condition of positive energy and weak locality \( N(z) \) can easily be continued analytically (see [FrJ]). We find singularities at \( z = 0, \ z = \infty \) and on (the copies of) the interval \( [e^{it_0}, e^{it_1}] \) and branch-cut (with arbitrary position) have to be introduced which connect the singularities. Hence, as maximal domain of analyticity we obtain a Riemannian surface. To apply the maximum principle of complex analysis we consider the continuation of \( N(\cdot) \) to the Alexandroff-compactification of this Riemannian surface at \( z = 0 \) and \( z = \infty \) (see [SG]). In vicinities of \( z = 0 \) and \( z = \infty \) the function \( N(\cdot) \) is bounded because of the bound on the spectrum of \( H_R \) and can therefore be continued analytically to the compactification [For]. As an analytic function on the compactified Riemannian surface it reaches its maximum on the boundary of its domain of definition, i.e. on (the copies of) \( [e^{it_0}, e^{it_1}] \). Therefore, we obtain the bound:
\[
\sup |N(\cdot)| \leq \|F\| \|G\| |e^{it_0} - e^{\frac{i}{2}(t_0 + t_1)}|^{2n} = \|F\| \|G\| (2 \sin \frac{t_0 - t_1}{4})^{2n}. \tag{18}
\]

This leads, as in [FrJ], to
\[
|\langle \Omega, GF\Omega \rangle| = |M(1)| = |N(1)| |1 - e^{\frac{i}{2}(t_0 + t_1)}|^{-2n} = |N(1)| (2 \sin \frac{t_0 + t_1}{4})^{-2n} \leq \sup |N(2 \sin \frac{t_0 + t_1}{4})^{-2n} \leq \|F\| \|G\| \left( \frac{\sin \frac{t_0 - t_1}{4}}{\sin \frac{t_0 + t_1}{4}} \right)^{2n} \tag{19}
\]

Finally, \( t_0 \) and \( t_1 \) remain to be determined. We obtain
\[
\lim_{R \to \infty} R t_0 = 2(c - b) \quad \text{and} \quad \lim_{R \to \infty} R t_1 = 2(d - a). \tag{20}
\]

We assume \( a - b = c - d \) and find \( \left( \frac{t_0 - t_1}{t_0 + t_1} \right)^2 = \frac{(a - b)(c - d)}{(a - c)(b - d)} =: x \). The bound on \( |\langle \Omega, GF\Omega \rangle| \) can only depend on the conformal cross ratio \( x \). Hence, we can drop the assumption and the theorem is proven.
3.1 Existence of the Pointlike Field Limit in Charged Sectors

$P_n H$ can be identified with copies of $L^2(\mathbb{R}_+, p^{2n-1} dp)$, where $G$ acts according to

$$
U_n \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \Phi(p) = \lim_{\varepsilon, \delta \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}_+} dq \ e^{-ip(x+ix)+iqg^{-1}(x+i\varepsilon)}(a - c(x+i\varepsilon))^{2n-2} \Phi(q). \tag{21}
$$

This fact can be used to investigate the limit in [L2]: Let $\Phi \in P_n H$ now be smeared out with a test function on $G$ such that $\Phi$ is $C^\infty$, i.e. $\tilde{g} \mapsto U_n(\tilde{g}) \Phi$ is $C^\infty$. In the appendix of [FrJ], we have proven in the vacuum case that smeared out functions $\Phi(\cdot)$ are continuous and bounded in $p$. The argument in [FrJ] uses an expansion into normalized associated Laguerre polynomials and can be fully transferred to charged sectors. Having proven continuity and boundedness, straightforward calculation leads to

$$
\left( \int da \ f(a) T(a) D(\lambda) \lambda^{-n} \Phi \right)(p) = \tilde{f}(p) \Phi(\lambda p) \tag{22}
$$

and

$$
\int dp \ p^{2n-1} |\tilde{f}(p)|^2 |\Phi(\lambda p) - \Phi(0)|^2 \longrightarrow 0 \tag{23}
$$

for $\lambda \downarrow 0$, showing the convergence of [L2].

We thus obtained for each $n$ and each $\Phi \in P_n H \cap C^\infty$ with the complex number $\Phi(0) \neq 0$ a multiple of a unitary map $V_{n,\Phi} : L^2(\mathbb{R}_+, p^{2n-1} dp) \to P_n H$ which is defined on the dense set

$$
\{\tilde{f}|_{\mathbb{R}_+} \mid f \in D(\mathbb{R})\}
$$

by

$$
V_{n,\Phi} : \tilde{f}|_{\mathbb{R}_+} \mapsto \Phi(0) |(\tilde{f}|_{\mathbb{R}_+}) > := \lim_{\lambda \downarrow 0} \lambda^{-n} \int da \ f(a) T(a) D(\lambda) \Phi \tag{24}
$$

and intertwines the irreducible representations of $G$.

3.2 Definition of Pointlike Localized Charged Field Operators

Now, we come to the definition of pointlike localized fields. Take a local observable $A \in \mathcal{A}(I_0)$, $I_0 \in \mathcal{K}_0$. Let $\rho$ be a localized and transportable endomorphism of $\mathcal{A}$ inducing a charged sector with finite statistics, let $e$ be a field bundle multi-index of type $(0, \rho, \rho)$. Then $F = F(e, A)$ is a local element of the reduced field bundle. We want $F$ to be regularized, i.e. we choose $F$ such that $\tilde{g} \mapsto \alpha_g(F)$ is $C^\infty$ in the strong operator topology. Let now $n \in \mathbb{R}_+$ and $P_n$ the projection on the subspace of conformal dimension $n$ in $H_{\text{red}}$.

Then the vector $P_n F\Omega$ is $C^\infty$. Hence, we may define operator valued distributions $\varphi^I_{n,F}$ on $\mathcal{D}(I)$, $I \in \mathcal{K}_0$, with a domain of definition dense in the vacuum Hilbert space by

$$
\varphi^I_{n,F}(f) \ B^\prime \Omega := \pi_\rho(B^\prime) V_{n,P_n F\Omega} \tilde{f}|_{\mathbb{R}_+}, \ f \in \mathcal{D}(I), \ B^\prime \in \mathcal{A}(I^\prime). \tag{25}
$$

Let now $e$ be a field bundle multi-index of arbitrary type $(\alpha, \rho, \beta)$. Then $F = F(e, A)$ is a local element of the reduced field bundle mapping $H_\alpha$, the copy of $H$ associated with the superselection sector $\alpha$, onto $H_\beta$, the copy of $H$ associated with the superselection sector $\beta$. In the reduced field bundle one can find [FRS2] the following relations of an exchange algebra with structure constants $R$:

$$
F(e_2, A_2) F(e_1, A_1) = \sum_{f_1 \circ f_2} R_{f_1 \circ f_2}^{\alpha \alpha_1} (+/-) F(f_1, A_1) F(f_2, A_2) \tag{26}
$$
whenever $F_1$ is localized in the right/left complement of the localization domain of $F_2$. Hence, we can define operator valued distributions $\varphi^I_{n,F}$ on $\mathcal{D}(I)$, $I \in \mathcal{K}_0$, with a domain of definition dense in the Hilbert space $H_\alpha$ by

$$\varphi^I_{n,F(e,A)}(f) \mathbf{F}(e',B') \Omega := \sum_{g' \circ g} R^{e \circ e'}_{g' \circ g}(+) F(g', B') \varphi^I_{n,F(g,A)}(f) \Omega$$

for $f \in \mathcal{D}(I)$ and $F(e', B') \in \mathcal{F}_{\text{red}}(I')$ whenever it is localized in the right/left complement of $I$ with respect to $\mathcal{K}_0$.

### 3.3 Properties of the Charged Field Operators

First, we prove the closability of the operators $\varphi^I_{n,F}(f)$. We start with the case of field bundle multi-index $e$ of type $(0, \rho, \rho)$.

**Theorem:** Let $n \in \mathbb{N}$, $I \in \mathcal{K}_0$, $f \in \mathcal{D}(I)$, $B' \in \mathcal{A}(I')$, $G' = F(e, C') \in \mathcal{F}_{\text{red}}(I')$ and let $F = F(e, A)$ be a regularized local element of the reduced field bundle. With the linear operator reversal $F \mapsto \hat{F}$ and the antilinear charge conjugation operation $F \mapsto \bar{F}$ both defined in [FRS2] and with the statistical phase $k_{\rho}$ we then have

$$\langle G' \Omega, \varphi^I_{n,F}(f) B' \Omega \rangle = \left( \frac{1}{k_{\rho}} \right)^{(+/-)1} \langle \hat{G}', \varphi^I_{n,F}(\hat{f}) \Omega, B' \Omega \rangle$$

$$\varphi^I_{n,F}(f)^* = \varphi^I_{n,F}(f)^*|_{\mathcal{F}_{\text{red}}(I')} \Omega = \varphi^I_{n,F}(\bar{f})$$

$\varphi^I_{n,F}(f)$ is closable because $\varphi^I_{n,F}(f)^*$ has a dense domain.

**Remark:** Here and in following proofs we heavily use the property of “weak locality”:

Let $F, G$ be two local elements of the reduced field bundle, $F$ leading from the vacuum sector to a charged sector $\rho$, $G$ leading back from $\rho$ to the vacuum sector. In [FRS2] has been proven that

$$GF = \left( \frac{1}{k_{\rho}} \right)^{(+/-)1} FG$$

whenever $F$ is localized in the left/right complement of $G$. “Weak locality” is the reminiscent of the Haag-Kastler axiom “locality” in the exchange algebra of the reduced field bundle in low-dimensional quantum field theory.

**Proof:** The Casimir operator associated to the representation $U(\cdot)$ of the universal covering of the Lie group $SL(2, \mathbb{R})/\mathbb{Z}_2$ has the following spectral decomposition [Lang]:

$$C_{\tilde{G}} = \sum_{i \in \mathbb{R}} i(i-1) P_i .$$

$C_{\tilde{G}}$ is a second order differential operator in $\tilde{G}$. Hence, it is a local operator in contrast to the global projector $P_n$.

Some algebra leads to

$$\langle G' \Omega, \varphi^I_{n,F}(f) B' \Omega \rangle = \lim_{\lambda \downarrow 0} \int_{\mathbb{R}} dx f(x) (\pi_{\rho}(B')^* G' \Omega, U(x) D(\lambda) \lambda^{-n} P_n F \Omega)$$

1
\[
\lim_{\lambda \downarrow 0} \int_{\mathbb{R}} dx f(x) \left( \pi_\rho (B')^* G' \Omega, U(x) D(\lambda) \lambda^{-n} \left( \prod_{0 < i < n} C_{\tilde{G}} - i(i - 1) \right) P_n F\Omega, \right)
\]

\[
\lim_{\lambda \downarrow 0} \int_{\mathbb{R}} dx f(x) \left( \pi_\rho (B')^* G' \Omega, U(x) D(\lambda) \lambda^{-n} \left( \prod_{0 < i < n} C_{\tilde{G}} - i(i - 1) \right) (1 - \sum_{0 < i < n} P_i) F\Omega, \right)
\]

(Because of equation (32) the polynomial in $C_{\tilde{G}}$ has the property to act as the identity operator on $P_n$ and as the zero operator on all $P_i, i < n$. As a consequence of the conformal cluster theorem, the contribution of conformal energies \(\geq n + 1\) vanishes uniformly in the limit \(\lambda \rightarrow 0\). Since the conformal rotation by \(2\pi\) leaves the observable algebra invariant, the conformal energy in a superselection sector takes values in \(\mathbb{Z} + n\) with a fixed conformal dimension \(n\).)

\[
\lim_{\lambda \downarrow 0} \int_{\mathbb{R}} dx f(x) \left( \pi_\rho (B')^* G' \Omega, U(x) D(\lambda) \lambda^{-n} \left( \prod_{0 < i < n} C_{\tilde{G}} - i(i - 1) \right) F\Omega, \right)
\]

we use weak locality.

\[
\lim_{\lambda \downarrow 0} \int_{\mathbb{R}} dx f(x) \left( \frac{1}{k_\rho} (1^{+/\pm 1})(U(x) D(\lambda) \lambda^{-n} \left( \prod_{0 < i < n} C_{\tilde{G}} - i(i - 1) \right) F\Omega, \right)
\]

\[
\left( \frac{1}{k_\rho} (1^{+/\pm 1})(G' \varphi_{n,F}(\tilde{f}) \Omega, B' \Omega) \right)
\]

\[
\left( \varphi_{n,F}(\tilde{f}) \right) G' \Omega, B' \Omega
\]

(33)

with the definition of $\varphi_{n,F}^I$.

Next, we argue that the closability theorem can be generalized to fields with field bundle multi-index $e$ of arbitrary type $(\alpha, \rho, \beta)$: Since the reduced field bundle only considers superselection sectors with finite statistics [FRS2], the definition (27) of charged fields of arbitrary type only contains a finite sum. It is therefore a straightforward calculation to show the closability of charged field operators of arbitrary type. Yet, we were not able to give an explicit expression for the adjoint operator on a dense domain of definition, because in the general case of operators with multi-index of arbitrary type we have to use the full exchange algebra instead of weak locality in the proof of the above theorem.

Moreover, the closures of the charged field operators are affiliated to the associated local von Neumann algebras of the reduced field bundle: The commutant of the von Neumann algebra of the reduced field bundle localized in $I \in \mathcal{K}_0$ is given by the algebra of observables in $I'$ represented on the Hilbert space of the full theory $H_{\text{red}}$. Therefore, the proof of Prop. 2.5.9 in [BrR] and more detailed in [Jor1] for neutral fields can be transferred to the case of charged fields. For closed field operators $\varphi^I$ localized in $I \in \mathcal{K}_0$ we obtain the affiliation relation

\[
\varphi^I \mathcal{F}_{\text{red}} (I)' \subseteq \mathcal{F}_{\text{red}} (I)' \varphi^I.
\]

(34)

The existence of sufficiently many charged field operators such that their linear span applied to the vacuum vector is dense in the Hilbert space $H_{\text{red}}$ can be proven exactly the same way as done in [Fr] for the vacuum case. For each $F_n \neq 0$ a non-zero field can be constructed. In a similar way elements $F_i$ of the reduced field bundle with a field bundle multi-index $e$ of arbitrary type $(\alpha, \rho, \beta)$ can be chosen such that the fields $\varphi_{n,F_i}$ are non-zero and orthogonal.

Finally, one can easily see that the charged fields transform covariantly:

\[
U(\tilde{g}) \varphi_{n,F}^I (\tilde{f}) U(\tilde{g})^{-1} = \varphi_{n,F}^I (\tilde{g}^n)
\]

(35)
with \( n \in \mathbb{R} \), \( F \) localized in \( I_0 \in \mathcal{K}_0 \) and regularized, \( \tilde{g} \in \tilde{G} \), with the covering projection \( \tilde{g} \mapsto g \), \( I, gI \in \mathcal{K}_0 \) and \( f \in D(I) \).

4 Consequences of the Existence of Pointlike Localized Charged Fields

The coexistence of the formulation of quantum field theory in terms of Haag-Kastler nets of von Neumann algebras and in terms of unbounded field operators with pointlike localization can be used to derive important structural results of the theory. In the present paper, we derive for all charged sectors with finite statistics of a conformally invariant theory in 1+1 dimensions what generalized Bisognano-Wichmann property, the PCT theorem, the Spin-Statistics theorem, an equivalence theorem for both formulations, additivity and an operator product expansion.

In [FrJ], we had the coexistence of both formulations in the vacuum sector and in the vacuum sector the Bisognano-Wichmann property, Haag duality, PCT covariance, equivalence of both formulations, additivity and an operator product expansion could be derived.

4.1 PCT, Spin & Statistics, and All That

We start with a proof of the Spin-Statistics theorem for conformally invariant quantum field theory in 1+1 dimensions. We use the argument of [FRS2].

**Spin-Statistics Theorem:** The statistical phase \( k_{\rho} \) and the chiral scaling dimension \( n_{\rho} \) of a conformal field \( \phi \) with arbitrary charge \( \rho \) with finite statistics fulfill the following relation

\[
e^{2\pi in_{\rho}} = e^{2\pi in_{\bar{\rho}}} = k_{\rho}.
\]

**Proof:** Let \( F, G \) be regularized elements of the reduced field bundle localized in disjoint intervals and \( n \in \mathbb{R} \) such that \( P_n F \Omega \neq 0 \). With weak locality we obtain

\[
(\varphi_{n,F}(x) \Omega, \varphi_{n,G}(y) \Omega) = k_{\rho}^{\text{sign}(y-x)} (\varphi_{n,G}(y) \Omega, \varphi_{n,F}(x) \Omega).
\]

The explicit knowledge of the transformation properties of the charged conformal two-point-function leads to

\[
(\varphi_{n,F}(x) \Omega, \varphi_{n,G}(y) \Omega) = e^{2\pi in_{\text{sign}(y-x)}} (\varphi_{n,G}(y) \Omega, \varphi_{n,F}(x) \Omega).
\]

Since it has been shown in the last section that the charged fields generate a dense set in \( H_{\text{red}} \) from the vacuum sector \( \Omega \), a comparison of the phases yields the theorem.

Next, we derive a generalization of the Bisognano-Wichmann result to the reduced field bundle formalism. The Tomita-Takesaki theory [1ak, BrR] assigns to every pair of a von Neumann algebra \( \mathcal{A} \) and a cyclic and separating vector \( \Psi \) a closable, antilinear operator:

\[
S_0 : A\Psi \mapsto A^*\Psi \quad \text{for all } A \in \mathcal{A}.
\]

\( S_0 \), the closure of \( S_0 \), has a polar decomposition \( S = J\Delta^{\frac{1}{2}} \) and its components fulfill a couple of relations:

\[
J = J^*, \quad J^2 = 1, \quad \Delta^{-\frac{1}{2}} = J\Delta^{\frac{1}{2}} J, \quad J\mathcal{A}J = \mathcal{A}'.
\]
The set of operators $\Delta^{it}$, $t \in \mathbb{R}$, generate the group of modular automorphisms:
\[ \Delta^{it} \mathcal{A} \Delta^{-it} = \mathcal{A}. \quad (41) \]

If $\Psi$ is chosen to be the vacuum vector $\Omega$ and $\mathcal{A}$ an algebra of local observables in a conformally covariant Haag-Kastler net in 1+1 dimensions, [FröG, BG1] have identified $J$ as the geometric reflection of the localization domain onto its complement on the circle and the modular automorphism group as the supgroup of conformal transformations which leave the localization domain invariant.

On the Hilbert space $H_{\text{red}}$ of all charged sectors with finite statistics we consider a generalized modular structure based on the charge conjugation operation $F \mapsto \bar{F}$ instead of the operator adjoint $F \mapsto F^\ast$. In the following, $S_I = J_I \Delta^{i\frac{\pi}{2}}_I$ is defined as the closure of the operator defined by the mapping $F\Omega \mapsto \bar{F}\Omega$ with $F \in \mathcal{F}_{\text{red}}(I)$ for $I \in K_0$.

In order to find a PCT-operator on $H_{\text{red}}$ we define the antilinear Operator $\Theta$:
\[ \Theta \varphi_{n,F}(x) \Omega := (-1)^n \varphi_{n,\bar{F}}(-x) \Omega \quad (42) \]
for $n,x \in \mathbb{R}$ and $F \in \mathcal{F}_{\text{red}}$ regularized. It can be easily seen that $\Theta$ is antiunitary and commutes PCT-covariantly with the representation $U(\tilde{G})$. To prove that $\Theta$ acts geometrically on the reduced field bundle and on the charged field operators we first derive the following generalization of the Bisognano-Wichmann result to charged sectors with finite statistics.

**Bisognano-Wichmann Theorem:** Let $k^{\frac{1}{2}}$ be the operator defined by its eigenvalues $k^{\frac{1}{2}}\rho$ on the Hilbert spaces $H_\rho$ associated with the different superselection sectors. Let $V(\cdot)$ be the dilation subrepresentation of $U(\cdot)$ and let $\Theta$ and $S_I$ be defined as above. We get as a generalization of the result of Bisognano and Wichmann
\[ S_{\mathbb{R}_+} = k^{\frac{1}{2}} \Theta V(i\pi), \quad S_{\mathbb{R}_-} = k^{-\frac{1}{2}} \Theta V(-i\pi). \quad (43) \]

**Proof:** Take the proof in [Jör2] and use weak locality instead of locality.

**Remark:** With this result, we see a posteriori that we could have simplified the construction of charged fields. Instead of using reducible modules $P_n H_{\text{red}}$ we could have started with irreducible modules $P\tau H_{\text{red}}$. As shown in [Fr4] for the vacuum sector, the Bisognano-Wichmann result suffices to prove the closability of field operators constructed with projectors $P\tau$ on irreducible modules.

As a consequence of the identification of a generalized modular structure with objects with well known geometrical meaning in the Bisognano-Wichmann theorem above, we are able to derive PCT covariance in the full theory

**PCT Theorem:** $\Theta$ acts geometrically on the reduced field bundle
\[ \Theta \mathcal{F}_{\text{red}}(\mathbb{R}_+) \Theta = \mathcal{F}_{\text{red}}(\mathbb{R}_-) \quad (44) \]
and on the charged field operators
\[ \Theta \varphi_{n,F}(x) \Theta = \varphi_{n,\Theta F\Theta}(-x) \quad (45) \]
with $n, x \in \mathbb{R}$ and $F \in \mathcal{F}_{\text{red}}$ regularized.

**Proof:** The generalized modular structure based on the antilinear change conjugation operation $F \mapsto \bar{F}$ can be identified with the relative modular structure introduced by Araki (see
Then, Araki’s results on relative modular operators imply the geometrical action of $k^{-\frac{1}{2}}J_{R_+} = k^{\frac{1}{2}}J_{R_-} = \Theta$ on the reduced field bundle. The geometrical action on the charged field operators then follows by straightforward calculation.

**Remark:** The PCT theorem can be obtained more directly. Another PCT operator $\tilde{\Theta}$ can be constructed on the physical Hilbert space $H_{\text{red}}$ in a natural manner: We already have a PCT operator on the vacuum Hilbert space and we know how the PCT operation should intertwine between the different copies of the vacuum Hilbert space. $\tilde{\Theta}$, too, can be shown to act geometrically on the reduced field bundle and on the charged field operators.

### 4.2 Equivalence between the “algebra picture” and the “distribution picture”

The algebra generated by polar and spectral decomposition of all $\varphi_{I,\tau,e}(f)^{**}$, $\tau$ irreducible, $e$ of arbitrary type and $f \in \mathcal{D}(I)$, is invariant under the generalized modular automorphisms $\text{Ad} \Delta^I_\tau$ introduced in subsection 4.1 and has the vacuum $\Omega$ as a cyclic vector. Hence, it coincides with $F_{\text{red}}(I)$ [Tak].

Thereby, we proved the equivalence of the formulation in terms of nets of von Neumann algebras and in terms of unbounded field operators with pointlike localization. Without any loss of information one can change between the “algebraic picture” and the “distribution picture”.

We now prove the additivity of the local von Neumann algebras of the reduced field bundle: If $I = \bigcup \alpha I_\alpha$ with $I, I_\alpha \in K_0$, then

$$F_{\text{red}}(I) = \bigvee \alpha F_{\text{red}}(I_\alpha)$$

where $\bigvee$ denotes the generated von Neumann algebra.

This follows from the fact that the local von Neumann algebras can be constructed by linear combinations of the decomposition parts of the unbounded field operators without building products of field operators.

Additivity can also be proven directly. Using again the identification of the generalized modular automorphism group with the subgroup of dilations, one can transfer the proof in [FrJ] from the vacuum sector to all charged sectors with finite statistics.

### 4.3 Operator Product Expansion in Charged Sectors

In the Haag-Kastler framework, the existence of an operator product expansion might be formulated as the existence of sufficiently many field operators such that their linear span applied to the vacuum vector is dense in the Hilbert space. Actually, in [FrJ] a stronger result with local coefficients and covariance w.r.t. the modular *-operation $S$ has been derived in the vacuum sector. Here, we prove an operator product expansion for arbitrary charged sectors with finite statistics.

**Theorem:** Let $I \in K_0$ and $F \in F_{\text{red}}(I)$. We then obtain a local expansion

$$F = \sum_{\tau, \alpha} \varphi_{I,\tau,\alpha}(f_{\tau,\alpha,F})$$

into a sum over all sectors $\alpha$ with finite statistics and all irreducible subrepresentations $\tau$ of $U(\tilde{G})$ with

$$\text{supp} f_{\tau,\alpha,F} \subset I,$$
which converges on $F_{\text{red}}(I')\Omega$ $\ast$-strongly (cf. the definition in [BrR]).

For a $F$ with a field bundle multi-index $e$ of type $(0, \rho, \rho)$ and for an irreducible $\tau$ the two simultaneous conditions

$$P_{\tau}F\Omega = \varphi_{\tau}(f_{\tau,F})\Omega \quad \text{and} \quad P_{\tau}F\Omega = \overline{\varphi_{\tau}(f_{\tau,F})}\Omega$$

(49)

together fully determine the testfunction $f_{\tau,F}$:

$$f_{\tau,F}(x) = -i^{2n-1} \int_{-\infty}^{x} dy_1 \int_{-\infty}^{y_1} dy_2 \cdots \int_{-\infty}^{y_{2n-2}} dy_{2n-1} (\Omega, [\varphi_{\tau}(y_{2n-1})^*, F]^\wedge \Omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ipx} \int_{-\infty}^{\infty} dy e^{-ipy} (\Omega, [\varphi_{\tau}(y)^*, F]^\wedge \Omega).$$

(50)

We used the abbreviation $[F,G]^\wedge := FG - GF$ with the operator reversal $F \mapsto \bar{F}$.

**Proof:** The formula for $f_{\tau,F}$ in (50) follows from straightforward calculation since the two conditions (49) determine the positive and negative energy content of $f_{\tau,F}$ respectively. It remains to be shown that $f_{\tau,F}$ has support in $I$.

Because of the nontrivial phases occurring in the relation of weak locality, the commutator function in the reduced field bundle has no compact support. Instead, we show that the convolution of the commutator function with the Fourier transform of $p^{-(2n-\lfloor 2n \rfloor)}$

$$(\Omega, [\varphi_{\tau}(x)^*, F]^\wedge \Omega) \otimes x^{2n-\lfloor 2n \rfloor-1} := \int_{-\infty}^{\infty} dy (\Omega, [\varphi_{\tau}(y-x)^*, F]^\wedge \Omega) y^{2n-\lfloor 2n \rfloor-1}$$

(51)

has support in $I$.

Let $x \in I'$. Without restriction of generality we assume $x$ to be on the right side of $I$. After a transformation of variables $y \mapsto -y$ in the second term of the sum of the convoluted commutator, we obtain with the Spin-Statistics theorem the phase factor that occurs in the relation of weak locality. Now, we can interpret the convolution as a line integral over the boundary of the upper complex half plane $C_+$ with an integrand analytic in $C_+$. The contribution of the infinite half circle to the line integral vanishes because of the conformal cluster theorem. Now, again without restriction of generality, we assume $F$ to be regularized. Applying weak locality, one can then see that the integrand is continuous at $y = 0$ and on the whole boundary of $C_+$. Hence, with the theorem of Cauchy from complex analysis the integral vanishes and we have proven that the convoluted commutator function has support in $I$.

This support property and conformal cluster theorem specifies the Fourier transform $G(p)$ of the commutator function $(\Omega, [\varphi_{\tau}(x)^*, F]^\wedge \Omega)$ to be of the form $p^{2n-1}H(p)$, with an appropriate analytic function $H(p)$. Therefore, using the Paley-Wiener theorem ([Bro], theorem 29.2) we see that the support of $f_{\tau,F}(x) = \bar{H}(x)$ is included in the support of the convoluted commutator function. Hence, it is included in $I$.

The local expansion (17) then follows directly from the result above and the rules for charged field operators defined on arbitrary sectors $\alpha$ with finite statistics.

**Remark:** As a by-product of the proof above we obtain a result on the charged two-point-function in a conformally covariant Haag-Kastler net in 1+1 dimensions. Let $F, G$ be any two local elements of the reduced field bundle with disjoint domains of localization. Let $P_k F\Omega = P_k F\Omega = 0, k < n$. Then the conformal cluster theorem implies that the two-point-function $(\Omega, GU(x) F\Omega)$ decreases as $x^{-2n}$. With the argument of the proof above its Fourier transform can be written as $\Theta(p) p^{2n-1} H(p)$ with an appropriate analytic function $H(p)$. 

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References

[Ara] H. Araki: Positive cones for von Neumann algebras (1980) report in the proceedings of the Kingston conference 1980

[BGL] R. Brunetti, D. Guido, R. Longo: Comm. Math. Phys. 156 (1993) 201 Modular structure and duality in conformal quantum field theory

[BiW] J. J. Bisognano, E. H. Wichmann: J. Math. Phys. 16 (1975) 985-1007 On the duality condition for a hermitian scalar field; J. Math. Phys. 17 (1976) 303-321 On the duality condition for quantum fields

[Bor] H. J. Borchers: Comm. Math. Phys. 134 (1992) 315 The CPT-Theorem in Two-dimensional Theories of Local Observables

[BrR] O. Bratelli, D. W. Robinson: Operator algebras and quantum statistical mechanics I (1979) Springer

[Buc] D. Buchholz: On the Manifestation of Particles (1993) DESY-preprint 93-155 and report in the proceedings of the Beer Sheva conference 1993

[BuF] D. Buchholz, K. Fredenhagen: J. Math. Phys. 18 (1977) 1107-1111 Dilations and interactions

[For] O. Forster: Riemannsche Flächen (1977) Springer

[Fre] K. Fredenhagen: Comm. Math. Phys. 97 (1985) 461 A Remark on the Cluster Theorem

[FrJ] K. Fredenhagen, M. Jörß: Conformal Haag-Kastler Nets, Pointlike Localized Fields and the Existence of Operator Product Expansions (to appear in Comm. Math. Phys.)

[FRS1] K. Fredenhagen, K.-H. Rehren, B. Schroer: Comm. Math. Phys. 125 (1989) 201 Superselection sectors with braid group statistics and exchange algebra I

[FRS2] K. Fredenhagen, K.-H. Rehren, B. Schroer: Rev. Math. Phys., special issue (1992) 113 Superselection sectors with braid group statistics and exchange algebra II

[FröG] J. Fröhlich, F. Gabbiani: Comm. Math. Phys. 155 (1993) 569 Operator algebras and conformal field theory

[GLo] D. Guido, R. Longo: The Conformal Spin and Statistics Theorem \[\text{hep-th/9505059}\]

[Haag] R. Haag: Local quantum physics (1992) Springer

[Iso] T. Isola: Modular structure for the crossed product by a compact group dual (to appear in J. Op. Th.)

[Jör1] M. Jörß: Lokale Netze auf dem eindimensionalen Lichtkegel (1991) diploma thesis, FU Berlin

[Jör2] M. Jörß: On the Existence of Pointlike Localized Fields in Conformally Invariant Quantum Physics (1992) DESY-preprint 92-156 and report in the proceedings of the Cambridge conference 1992

[Lang] S. Lang: \(SL_2(\mathbb{R})\) (1975) Springer

[Lüs] M. Lüscher: Comm. Math. Phys. 50 (1976) 23-52 Operator product expansions on the vacuum in conformal quantum field theory in two space-time dimensions
[Mac] G. Mack: Comm. Math. Phys. 53 (1976) 155 Convergence of Operator Product Expansions on the vacuum in Conformally Invariant Quantum Field Theory

[SSV] B. Schroer, J. A. Swieca, A. H. Völkel: Phys. Rev. D 11, 6 (1974) 1509 Global operator expansions in conformally invariant relativistic quantum field theory

[Str] K. Strebel: Vorlesungen über Riemannsche Flächen (1980) Vandenhoeck und Ruprecht

[StW] R. Streater, A. S. Wightman: PCT, Spin & Statistics, and All That (1964) Benjamin

[Tak] M. Takesaki: Tomita’s theory of modular Hilbert algebras and its applications (1970) Springer

[Tre] F. Treves: Topological Vector Spaces, Distributions and Kernels (1967) Academic Press

[Wil] R. Wilson: Phys. Rev. 179, 5 (1969) 1499 Non-Lagrangian Models of Current Algebras