Nonlinear Electron Oscillations in a Viscous and Resistive Plasma

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(Dated: May 13, 2010)

New non-linear, spatially periodic, long wavelength electrostatic modes of an electron fluid oscillating against a motionless ion fluid (Langmuir waves) are given, with viscous and resistive effects included. The cold plasma approximation is adopted, which requires the wavelength to be sufficiently large. The pertinent requirement valid for large amplitude waves is determined. The general non-linear solution of the continuity and momentum transfer equations for the electron fluid along with Poisson’s equation is obtained in simple parametric form. It is shown that in all typical hydrogen plasmas, the influence of plasma resistivity on the modes in question is negligible. Within the limitations of the solution found, the non-linear time evolution of any (periodic) initial electron number density profile \(n_e(x,t=0)\) can be determined (examples). For the modes in question, an idealized model of a strictly cold and collisionless plasma is shown to be applicable to any real plasma, provided that the wavelength \(\lambda \gg \lambda_{min}(n_0, T_e)\), where \(n_0 = \text{const}\) and \(T_e\) are the equilibrium values of the electron number density and electron temperature. Within this idealized model, the minimum of the initial electron density \(n_e(x_{min}, t=0)\) must be larger than half its equilibrium value, \(n_0/2\). Otherwise, the corresponding maximum \(n_e(x_{max}, t = \tau_p/2)\), obtained after half a period of the plasma oscillation blows up. Relaxation of this restriction on \(n_e(x, t = 0)\) as one decreases \(\lambda\), due to the increase of the electron viscosity effects, is examined in detail. Strong plasma viscosity is shown to change considerably the density profile during the time evolution, e.g., by splitting the largest maximum in two.

I. INTRODUCTION

In this paper, which is an extension and generalisation of an earlier Letter [1], we present new features of an important mode in a plasma, i.e., electron plasma waves, also called Langmuir oscillations or space charge waves. The non-linear (fluid) features of these waves will be examined in full detail, along with approximate inclusion of the dissipative effects due to electron viscosity and plasma resistivity.

Electron plasma waves were first examined, both experimentally and theoretically, in 1929 by Tonks and Langmuir. A simple theory was given [2], treating the plasma electrons as non-interacting particles oscillating against the motionless ions. This simple picture, neglecting random thermal motions, led to the equation of motion for each electron to be that for a harmonic oscillator with angular frequency

\[\omega^2 = \omega_{pe}^2 = 4\pi n_0 e^2/m_e,\]

where \(e\) and \(m_e\) are the electron charge and mass, and \(n_0 = \text{const}\) is the equilibrium value of the electron number density. A similar equation was obtained for the electric field \(E\) under the assumption that \(\nabla \times E = 0\), i.e., that there is no magnetic field associated with the wave. This means that the electric current due to electron oscillations is fully compensated by the displacement current \(\dot{E}/(4\pi)\). This fact and formula (1) defining the frequency of electron oscillations are the most essential features of Langmuir oscillations in the “cold plasma approximation”, neglecting thermal motions.

The influence of random thermal motions on small oscillations of electrons can be examined within the kinetic theory based on the linearised Vlasov equation. This was done correctly for the first time in 1946 by Landau [3] who arrived at the dispersion relation in the form

\[\omega^2 = \omega_{pe}^2 + k^2(3T_e/m_e) \equiv \omega_{pe}^2[1 + 3(k\lambda_D)^2],\]

valid in the long wavelength limit, \((k\lambda_D)^2 \ll 1\), where \(k = 2\pi/\lambda\) is the wavenumber, \(\lambda\) is the wavelength, \(\lambda_D = v_{th}/\omega_{pe}\) is the Debye length, \(v_{th} = \sqrt{T_e/m_e}\) is the electron thermal speed, and \(T_e\) is the equilibrium electron temperature.

PACS numbers: 52.30.-q,51.20.+d

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in energy units. In this long wavelength limit, the phase velocity of the wave, \( v_{ph} = \omega/k \approx \omega_{pe}/k \) is much larger than \( v_{th} \). Landau's analysis also indicated that the electron plasma waves in this limit undergo a small collisionless damping with decrement given by

\[
\omega_{pe} \sqrt{\frac{\pi}{8} \frac{1}{(k \lambda D)^3}} \exp \left[ -\frac{1}{2(k \lambda D)^2} \right].
\]

These results, along with the contour integrals used in the derivations, turned out to be very important for further progress in the kinetic theory of various waves in plasmas, see e.g., [4]. The decrement given by (3) is nowadays referred to as Landau damping. The dispersion relation (2) is commonly associated with the names of Bohm and Gross who derived it independently three years later [5]. However, their derivation, not based on either Vlasov theory or any other systematic method, had no chance of generalisations or finding damping.

The physics behind Landau damping is nowadays well understood. It appears for any particle distribution function which decreases as one increases the particle velocity in some vicinity of the phase velocity of the wave. In that case there are more particles that are a bit slower than the wave and are accelerated on account of the wave energy than those decelerated. Large Landau damping is obtained if the phase velocity of the wave is close to the particle thermal speed where the slope of the distribution function is large.

The simplest model that can be used to examine waves in a plasma is the fluid model in which electrons and each kind of ions are treated as fluids. In the small amplitude limit which allows for linearisation of the macroscopic speed where the slope of the distribution function is large.

\[
\frac{\nu}{\nu_0} = \frac{\nu_{th}}{\nu_0} = \frac{\rho_i}{\rho_e}, \frac{T_i}{T_e}, \frac{v_{th,i}}{v_{th,e}} = \frac{n_i}{n_e} = \frac{T_i}{T_e}.
\]

In the linearised theory, they will also be solutions, oscillating with periodic profiles with the same \( \lambda \). The real part of such a solution is one-dimensional, periodic in \( x \) with wavelength \( \lambda = 2\pi/k \). Its profile is purely cosinusoidal. More complicated periodic profiles with the same \( \lambda \) can be obtained by adding higher harmonics, with \( k \) replaced by \( nk \), \( n = 1, 2, \ldots \). In the linearised theory, it will also be solutions, oscillating with \( \omega = \omega_{pe} \). The aim of this paper is to describe the behaviour of these 1D solutions as one increases the amplitudes so that the non-linear terms cannot be neglected, and also with dissipative effects included.

For a purely cosinusoidal initial profile of \( n_e - n_0 \) and no dissipation, this problem was solved formally by Dawson in 1959 by introducing a Lagrangian coordinate and in more detail by Davidson and Schram in 1968 [7]. The solution had a somewhat unexpected initial amplitude limitation. This will follow from our results as a very special case.

In the linearised theory one can superpose plane plasma waves propagating in various directions. Such higher dimensional solutions are beyond the scope of our fully non-linear 1D analysis. They are not so important as the 1D Langmuir waves that are important recently revisited in connection with laser-driven plasma-based electron accelerators. Such accelerators are capable of supporting fields even in excess of 100 GV/m [8].

While in the kinetic theory of Langmuir waves as a rule one assumes that ions are motionless, in the fluid model one can easily include motions of all plasma species, to check this assumption quantitatively. To close the set of fluid equations involving the continuity and momentum transfer equations for all species, one has to postulate a pressure–density relation. If the polytropic one is assumed (\( p_\alpha/n_\alpha^{\alpha} = const. \)) the relevant equations for the longitudinal waves propagating along the \( x \) axis as follows [6] (\( \alpha = e \) for electrons and 1, 2, \ldots, \( N_i \) for ions).

Dispersion relation:

\[
1 - \sum_\alpha \frac{\left(\frac{\omega_{pe}}{\omega}\right)^2}{1 - \gamma_\alpha (v_{th,\alpha}/v_{ph})^2} = 0, \quad \omega_{ph}^2 = \frac{4\pi n_0 q_e^2}{m_\alpha \rho_\alpha}. \tag{4}
\]

Complex amplitudes of the macroscopic velocity \( v_\alpha \) versus electric field \( E \):

\[
v_\alpha = i \frac{q_e}{\omega m_\alpha} \frac{E}{1 - \gamma_\alpha (v_{th,\alpha}/v_{ph})^2}, \tag{5}
\]

where \( q_e = -e, q_i = Z_i e, v_{th,\alpha} = \sqrt{T_\alpha/m_\alpha}, v_{ph} = \omega/k, n_{\alpha,0} \) and \( T_\alpha \) are the equilibrium values of the number density and temperature in energy units for the \( \alpha \) species \( (n_{e,0} \equiv n_0) \). The thermal corrections (subtracted from 1 in the
denominators) must be small as compared to unity, to avoid strong Landau damping. Neglecting them we obtain
\[ \omega^2 = \omega_p^2 \equiv \sum_{\alpha} \omega_{p\alpha}^2 = \omega_{pe}^2 \left[ 1 + \sum_{i=1}^{N_i} O(m_e/m_i) \right], \]  
(6)

\[ v_i = -Z_i \frac{m_e}{m_i} v_e. \]  
(7)

Eqs. (6) and (7) justify the assumption of motionless ions \((m_i \to \infty)\). Under this assumption, Eq. (4) simplifies to
\[ \omega^2 = \omega_{pe}^2 + k^2 \left( \frac{\gamma_e T_e}{m_e} \right). \]  
(8)

This dispersion relation following from the fluid model will coincide with that from the kinetic theory, Eq. (2), if we choose \(\gamma_e = 3\). This value is thus appropriate for the fluid description of Langmuir waves, which can be interpreted as a one-dimensional adiabatic compression of the electron gas, associated with these waves.

II. BASIC EQUATIONS

For \(x\) dependent electron plasma waves and a motionless ion fluid, the model equations describing the electron fluid are: the continuity and momentum transfer equations along with the adiabatic pressure–density relation with \(\gamma_e = 3\), and Poisson’s equation, see e.g., [9] (Gaussian units),
\[ \frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x} \left( n_e v_e \right) = 0, \]  
(9)
\[ m_e n_e \left( \frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} \right) = \frac{\partial p_e}{\partial x} + e n_e E \left( \frac{4}{3} n_e \frac{\partial v_e}{\partial x} \right) - \eta e^2 n_0 n_e v_e, \]  
(10)
\[ \frac{p_e}{n_e^2} = \frac{p_0}{n_0^2} = \frac{T_e}{n_0^2}, \]  
(11)
\[ \frac{\partial E}{\partial x} = 4\pi e (n_0 - n_e), \]  
(12)

where the equilibrium electron temperature \(T_e\) is in energy units, \(v_e\) is the electron viscosity coefficient, \(\eta\) is the plasma resistivity, and an ideal gas equation of state for the electron gas in equilibrium is assumed, \(p_0 = n_0 T_e\).

We assume that the first term on the right hand side in Eq. (10) is negligible as compared to the second one, i.e., that the wave is driven by the electric field rather than by electron pressure (cold plasma approximation). For small amplitudes that allow for linearisation, that will be the case if the thermal correction in the dispersion relation (2) is negligible, i.e.,
\[ 3 \left( k \lambda_D \right)^2 \equiv \left( \frac{3\pi}{e^2} \right) \frac{T_e}{n_0 \lambda^2} \ll 1, \]  
where
\[ k \lambda_D = 2\pi \frac{\lambda}{\lambda_D} \approx \frac{k v_{th}}{\omega} \approx \frac{v_{th} \omega}{v_{ph}} \equiv \frac{\tau_p v_{th}}{\lambda}, \]  
(13a)

and \(\tau_p = 2\pi/\omega\) is the period of plasma oscillations. This condition is fulfilled in any plasma, provided that the wavelength \(\lambda\) is sufficiently large as compared to the Debye length. In that case, \(\lambda\) is also large as compared to the distance travelled by an electron moving with its thermal speed during one period of plasma oscillation, and phase velocity of the wave is large as compared to the electron thermal speed. The non-linear counterpart of condition (13) will be derived in Sec. VI.

Analytical formulas for the electron viscosity coefficient \(\nu_e\) and resistivity \(\eta\) involve several approximations, see Sec. VI. Using the latest values of the physical constants [10], the relevant formulas given in the first reference of [9] take the form (two component plasma of electrons and one kind of \(Z = 1\) ions, i.e., protons, deuterons, or tritons):
\[ \nu_e = 0.73 \frac{3}{{4\sqrt{2\pi}} \frac{T_e}{\lambda C^2}} = 4.01318 \times 10^{-8} \left( \frac{T_e [eV]}{\lambda C/10} \right)^{5/2}, \]  
(14a)
\[ \eta = 0.51 \frac{4\sqrt{2\pi m_e \lambda_C e^2}}{3T_e^{3/2}} = 5.86059 \times 10^{-14} \frac{\lambda_C/10}{(T_e [eV])^{3/2}} \]  

(14b)

where \( \lambda_C \) is a slowly varying Coulomb logarithm which will be treated as a constant:

\[
\lambda_C = \begin{cases} 
23.4 - \ln(n_0^{1/2} T_e^{-3/2}) & \text{if } T_e \leq 50 \text{ eV}, \\
25.356 - \ln(n_0^{1/2} T_e^{-1}) & \text{if } T_e \geq 50 \text{ eV}.
\end{cases}
\]  

(15)

(Typically \( \lambda_C \approx 10^{-20} \)).

Neglecting the electron pressure term, we can write Eq. (10) in the form

\[ \frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} = -\frac{e}{m_e} E + \frac{\nu}{m_e n_e} \frac{\partial^2 v_e}{\partial x^2} - \eta \frac{e^2 n_0}{m_e} v_e. \]  

(16)

where \( \nu = \frac{4}{3} \nu_e \).

### III. NEW VARIABLES

The nonlinearity on the left hand side of Eq. (16) can be eliminated by introducing Lagrangian coordinates: \( x_0(x, t) \), the initial position (at \( t = 0 \)) of an electron fluid element which at time \( t \) was at \( x \), and time in the electron fluid rest frame, \( \tau (= t) \). The basic transformation equations between Eulerian and Lagrangian coordinates are (see [11] for more detail)

\[ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + v_e \frac{\partial}{\partial x}, \quad x = x_0 + \int_0^\tau v_e(x_0, \tau') d\tau'. \]  

(17)

The continuity equation (9) in the electron fluid rest frame is now simply

\[ n_e \frac{\partial x}{\partial x_0} = n_{e0}(x_0) \equiv n_e(x_0, t = 0). \]  

(18)

This indicates that the nonlinear operator \( n_e^{-1} \partial / \partial x \) (= \( n_e^{-1} \partial / \partial x_0 \)) in Eq. (16) gets a bit simpler in the Lagrangian coordinates. However, to make it linear, a new space variable \( s \) has to be introduced satisfying:

\[ \frac{1}{n_e(x, t)} \frac{\partial}{\partial x} = \frac{1}{n_{e0}(x_0)} \frac{\partial}{\partial x_0} = \frac{1}{n_0} \frac{\partial}{\partial s}. \]  

(19)

The second equality here implies the definition of the auxiliary variable \( s \):

\[ s = n_0^{-1} \int n_{e0}(x_0) dx_0, \]  

(20)

which turns out to be proportional to the integral of the initial electron density profile.

Using Eqs. (17) and (19), Eq. (16) takes the form

\[ \frac{\partial v_e}{\partial \tau} + \frac{e}{m_e} E - \nu \frac{\partial}{\partial s} \left( \frac{n_e}{n_0} \frac{\partial v_e}{\partial s} \right) + \eta \frac{e^2 n_0}{m_e} v_e = 0. \]  

(21)

An important point is that \( E \) can be linearly expressed in terms of \( v_e \). Indeed, using Eqs. (12) and (9) it follows that

\[ \frac{\partial E}{\partial t} = 4\pi e n_e v_e, \]  

(22)

which expresses the fact that the electric current, \(-en_e v_e\), is compensated by the displacement current \((4\pi)^{-1} \partial E / \partial t\). Adding this equation to \( v_e \) times Eq. (12) we end up with

\[ \frac{\partial E}{\partial \tau} = 4\pi e n_0 v_e, \]  

(23)
where the right hand side is linear in $v_c$ as promised. Therefore, if we linearise the viscous term in Eq. (21) by replacing $n_c$ by $n_0$, differentiate this equation $\partial / \partial \tau$ and use (23), we obtain

$$\frac{\partial^2 v_c}{\partial \tau^2} - \frac{\nu}{m_e n_0} \frac{\partial^2 v_c}{\partial s^2} + \eta \frac{\omega_{pe}^2}{4\pi} \frac{\partial v_c}{\partial \tau} + \omega_{pe}^2 v_c = 0,$$

(24)

where $\omega_{pe}^2$ is defined by Eq. (1). Linearisation of the viscous term is a simplification justified by the fact that this term only corrects the motion, whereas the driving electric force and the convective non-linearities in the fluid equations are taken into account exactly. Furthermore, the formula for the viscosity coefficient, Eq. (14a), is approximate, up to a factor of two or three.

Eq. (24) is a linear partial differential equation (PDE) for $v_c(s, \tau)$ with constant coefficients. Solutions of such PDEs are any superpositions of the normal modes $\exp[i(ks - \Omega \tau)]$, for which Eq. (24) leads to the dispersion relation

$$\Omega = \pm \omega(k) - i\alpha(k),$$

(25a)

$$\alpha(k) = \frac{\nu k^2}{2m_e n_0} + \frac{\eta e^2 n_0}{2m_e},$$

(25b)

$$\omega(k) = \sqrt{\omega_{pe}^2 - \alpha^2(k)}.$$  

(25c)

Assuming that $k$ is real and superposing the normal modes corresponding to the plus and minus signs in $\Omega$ given by Eq. (25) we obtain four real solutions:

$$v_c = e^{-\alpha(k)\tau} f(\omega(k)\tau) g(ks),$$

(26)

where $f, g = \sin$ or $\cos$.

For each $\tau$, $v_c$ given by Eq. (26) is a periodic function of $s$ with wavelength $\lambda = 2\pi/k$. By adding higher harmonics, obtained from Eq. (26) on replacing $k \rightarrow nk$, $n = 1, 2, \ldots$, and multiplying by an amplitude, any solution periodic in $s$ with wavelength $\lambda$ can be obtained. Replacing $a \sin(\omega\tau) + b \cos(\omega\tau) \equiv A \cos(\omega\tau + \varphi)$, the solution in question can be written as

$$v_c = \sum_{n=1}^{\infty} e^{-\alpha(nk)\tau} \left\{ A_{1n} \cos[\omega(nk)\tau + \varphi_{1n}] \sin(nks) + A_{2n} \cos[\omega(nk)\tau + \varphi_{2n}] \cos(nks) \right\},$$

(27)

where $A_{1n}, \varphi_{1n}, A_{2n},$ and $\varphi_{2n}$ are arbitrary constants.

### IV. GENERAL SOLUTION IN PARAMETRIC FORM

Our equations and final results take a simple and universal form if we introduce dimensionless quantities:

$$\tilde{x} = kx, \quad \tilde{s} = ks, \quad \tilde{t} = \omega_{pe} t, \quad \tilde{\tau} = \omega_{pe} \tau,$$

(28a)

$$\tilde{\omega}_n = \frac{\omega(nk)}{\omega_{pe}} = \sqrt{1 - \alpha_n^2}, \quad \tilde{\alpha}_n = n^2 \nu + \tilde{\eta},$$

(28b)

$$\tilde{\eta} = \frac{\eta n_0 e^2}{2m_e \omega_{pe}} = 1.31642 \times 10^{-10} \frac{\lambda_c}{10} \frac{n_0^{1/2}}{(T_e [eV])^{3/2}},$$

(28c)

$$\tilde{\nu} = \frac{2\nu_e k^2}{3m_e n_0 \omega_{pe}} = 2.70376 \times 10^6 \frac{T_e [eV]}{\tilde{\eta} n_0 \lambda^2},$$

(28d)

$$\tilde{v}_e = \frac{v_e}{v_{ph}}, \quad v_{ph} = \frac{\omega_{pe}}{k},$$

(28e)

$$\tilde{A}_{1n} = \frac{A_{1n}}{v_{ph}}, \quad \tilde{A}_{2n} = \frac{A_{2n}}{v_{ph}},$$

(28f)

$$\tilde{n}_e = \frac{n_e}{n_0}, \quad \tilde{E} = \frac{E e k}{m_e \omega_{pe}^2}.$$  

(28g)
Thus, Eqs. (12), (19) and (23) are now:

$$\frac{\partial E}{\partial x} = 1 - n_e, \quad \frac{\partial s}{\partial x} = \bar{n}_e, \quad \frac{\partial E}{\partial \tau} = \bar{v}_e. \quad (29)$$

Integrating the first two over $dx$ and the last one over $d\tau$ and using (27), we end up with equations which define all relevant quantities: $x, E, \bar{n}_e$ and $\bar{v}_e$ in terms of $s$ and $\tau$ ($= t$). Dropping bars for simplicity, the final results for the dimensionless quantities become:

$$x(s, t) = s + E, \quad (30a)$$

$$E(s, t) = -\sum_{n=1}^{\infty} e^{-\alpha_n t} \left[ A_{1n} g_{1n}(t) \sin(ns) + A_{2n} g_{2n}(t) \cos(ns) \right], \quad (30b)$$

$$n_e^{-1}(s, t) = 1 - \sum_{n=1}^{\infty} e^{-\alpha_n t} \left[ A_{1n} g_{1n}(t) \cos(ns) - A_{2n} g_{2n}(t) \sin(ns) \right], \quad (30c)$$

$$v_e(s, t) = \sum_{n=1}^{\infty} e^{-\alpha_n t} \left[ A_{1n} f_{1n}(t) \sin(ns) + A_{2n} f_{2n}(t) \cos(ns) \right], \quad (30d)$$

$$f_{jn}(t) = \cos(\omega_n t + \varphi_{jn}), \quad j = 1, 2,$$

$$g_{jn}(t) = \alpha_n f_{jn}(t) - \omega_n \sin(\omega_n t + \varphi_{jn}) \equiv \cos(\omega_n t + \varphi_{jn} + \varphi_{0n}), \quad (30e)$$

$$\varphi_{0n} = \arctan(\omega_n/\alpha_n). \quad (30f)$$

The identity in Eq. (30e) is due to the fact that $\alpha_n^2 + \omega_n^2 = 1$, and therefore one can always find such $\varphi_{0n}$ that cos($\varphi_{0n}$) = $\alpha_n$, and sin($\varphi_{0n}$) = $\omega_n$.

The independent parameters $A_{1n}$, $\varphi_{1n}$, $A_{2n}$, and $\varphi_{2n}$ must be chosen so as to ensure reality of all dependent parameters defined by Eqs. (30). $x$, $E$, $n_e$, and $v_e$.

For modes with $n$ not too large, such that $\alpha_n \equiv n^2 \nu + \eta < 1$, $\omega_n$ and $\varphi_{0n}$ are real and positive, and $f_{jn}(t)$ and $g_{jn}(t)$ are purely oscillating cosine functions, with period $2\pi/\omega_n$ and arguments shifted by $\varphi_{0n}$.

For higher modes, either $\omega_n = \varphi_{0n} = 0$, if $\alpha_n = 1$, leading to $f_{jn}(t) = g_{jn}(t) = \cos(\varphi_{jn}) = \text{const}$, or $\omega_n$ and $\varphi_{0n}$ become purely imaginary ($= i[\omega_n]$ and $i[\varphi_{0n}]$), if $\alpha_n > 1$. Only the latter case requires further attention, whereas in the remaining cases ($\alpha_n \leq 1$) we can choose $A_{jn}$ to be real, $j = 1, 2$.

Thus if $\alpha_n > 1$, real values of the phase shifts $\varphi_{jn}$ in general lead to $f_{jn}(t)$ and $g_{jn}(t)$ being complex and having time dependent phases. Therefore in that case one cannot produce real results by an appropriate choice of complex coefficients $A_{1n}$ and $A_{2n}$. The only exceptions are $\varphi_{jn} = 0$ or $\varphi_{jn} = (\varphi_{0n} \pm \pi)/2$.

If we choose $\varphi_{jn} = 0$, the resulting $f_{jn}(t) = \cos(\omega_n t)$ and $g_{jn}(t) = \cos(\omega_n t + |\varphi_{0n}|)$ will be real. A similar situation will arise if we choose $\varphi_{jn} = -\varphi_{0n}$, for which $f_{jn}(t) = \cos(\omega_n |t - |\varphi_{0n}|)$, $g_{jn}(t) = \cos(\omega_n |t|)$. In all these cases we can choose $A_{jn}$ to be real, $j = 1, 2$.

If we choose $\varphi_{jn} = \pm \pi/2$ (and also for $\varphi_{jn} = -\varphi_{0n} \pm \pi/2$), it is convenient to replace the amplitudes $A_{jn}$ by $A_{jn}/\omega_n$. With this choice Eq. (30) will hold if new definitions of the functions $f_{jn}(t)$ and $g_{jn}(t)$ are adopted, i.e.,

$$f_{jn}(t) = \pm \frac{\sin(\omega_n t)}{\omega_n}, \quad (31a)$$

$$g_{jn}(t) = \pm \left[ \frac{\alpha_n \sin(\omega_n t)}{\omega_n} + \cos(\omega_n t) \right], \quad (31b)$$

or

$$f_{jn}(t) = \pm \left[ \frac{\alpha_n \sin(\omega_n t)}{\omega_n} - \cos(\omega_n t) \right], \quad (32a)$$

$$g_{jn}(t) = \pm \frac{\sin(\omega_n t)}{\omega_n}, \quad (32b)$$

where $\frac{\sin(\omega_n t)}{\omega_n}$ must be replaced by $t$ if $\omega_n = 0$. It can be seen that the right hand sides of Eqs. (31) and (32) are well defined and real for any $n$ (if $\alpha_n > 1$, $\cos(\omega_n t) \equiv \cos(\omega_n t)$ and $\sin(\omega_n t)/\omega_n \equiv \sin(\omega_n t)/|\omega_n|$). Their values at $t = 0$ are either zero or $\pm 1$. Thus, if Eq. (31) holds for both $j = 1$ and $j = 2$, this corresponds to the unperturbed value of the initial electron velocity ($v_e(x, t = 0) = 0$), and maximally perturbed initial values of the electric field $E$ and $n_e$. If in that case ($\varphi_{jn} = \mp \pi/2$) we choose $A_{2n} = 0$ and $A_{1n} = A_n$, we will reproduce our earlier results given in [1]. Similarly, Eq. (32) corresponds to the unperturbed initial values of the electric field and the electron density ($E(x, t = 0) \equiv 0$ and $n_e(x, t = 0) = 1$), and maximally perturbed initial value of the electron velocity.
Real values of $f_{jn}(t)$ and $g_{jn}(t)$ for $\alpha_n > 1$ will also be obtained if we choose $\varphi_{jn}$ to be pure imaginary. Eq. (30c) is only meaningful if the sum is smaller than unity, which imposes a limitation on the amplitudes $A_{1n}$ and $A_{2n}$.

If $\varphi_{1n} = \varphi_{2n}$, the mode in question is a product of a function of $t$ and a function of $s$, i.e., represents a standing wave.

Note that all modes, both with periodic $f_{jn}(t)$ and $g_{jn}(t)$ (for $\alpha_n < 1$) and aperiodic ones (for $\alpha_n \geq 1$) are damped exponentially as $t \to \infty$ (with decrement $\alpha_n$ if $\alpha_n \leq 1$ or $\alpha_n - \sqrt{\alpha_n^2 - 1} \equiv [\sqrt{\alpha_n^2 - 1} + \alpha_n]^{-1}$ if $\alpha_n \geq 1$).

V. SOLUTION GIVEN IN TERMS OF PHYSICAL VARIABLES

One can eliminate the parameter $s$ from Eqs. (30) thereby making $x$ and $t$ the independent variables. This parameter has to be determined in terms of $x$ and $t$ from Eq. (30a) and used in the remaining Eqs. (30). While numerically this is a simple task, analytical formulas are complicated, see Eqs. (33)–(37). At the same time, the parametric form (30), which involves simple elementary functions, can also be used to plot $E$, $n_e$, and $v_x$ as functions of the physical variables $x$ and $t$ (see Figs. 1–5 and 8–11 obtained by using “ParametricPlot3D” of Mathematica 7).

The electron density $n_e(x, t)$ is a periodic function of $x$ with wavelength $2\pi$. It can be expanded in a Fourier series in $x$ with time dependent Fourier coefficients:

$$n_e(x, t) = 1 + \sum_{m=1}^{\infty} \left[ B_{1m}(t) \cos(mx) + B_{2m}(t) \sin(mx) \right],$$

where $(n_e \, dx = ds)$

$$B_{1m}(t) \equiv \frac{1}{\pi} \int_0^{2\pi} n_e(x, t) \cos(mx) \, dx = \frac{1}{\pi} \int_0^{2\pi} \cos \left\{ m \left[ s + E(s, t) \right] \right\} ds,$$

and similarly,

$$B_{2m}(t) = \frac{1}{\pi} \int_0^{2\pi} \sin \left\{ m \left[ s + E(s, t) \right] \right\} ds,$$

with $E(s, t)$ given by Eq. (30b).

Equation (33) along with the first equation in (29) integrated over $dx$ leads to the Fourier expansion of $E(x, t)$:

$$E(x, t) = \sum_{m=1}^{\infty} \frac{1}{m} \left[ -B_{1m}(t) \sin(mx) + B_{2m}(t) \cos(mx) \right].$$

And finally, Eqs. (22) and (28) lead to

$$n_e(x, t) v_x(x, t) = \frac{\partial E(x, t)}{\partial t}.$$

This equation along with (36) and (33) defines $v_x(x, t)$.

The freedom in the choice of the constants $A_{1n}$, $\varphi_{1n}$, $A_{2n}$, and $\varphi_{2n}$ can be reduced from four to two if we prescribe the initial electron density profile $n_{e0}(x) \equiv n_e(x, t = 0)$. Putting $t = 0$ in Eq. (30c) and using standard formulas for Fourier coefficients we arrive at $(n_e^{-1} \, ds = dx)$

$$A_{1n} g_{1n}(0) = -\frac{1}{n\pi} \int_0^{2\pi} \cos \left\{ n \left[ x - E(x, 0) \right] \right\} \, dx$$

$$= -\frac{1}{n\pi} \int_0^{2\pi} \cos \left\{ n \int_0^{x} n_{e0}(x') \, dx' \right\} \, dx,$$

where $E(x, 0)$ is given by Eq. (36) and we have chosen $s(0, 0) = 0$. An analogous formula for $A_{2n} g_{2n}(0)$ can be obtained from Eq. (38) if we drop the minus signs in front of the integrals $dx$ and replace the cosine functions by sines.

In general the integrals in Eqs. (34), (35) and (38) (first line) are not expressible in terms of elementary functions, but if the sum over $m$ or $n$ is truncated at some $M$ or $N$, one can calculate as many integrals as needed (numerically).
If either \( M = 1 \) or \( N = 1 \) (first harmonics only), and additionally \( B_{21}(0) = 0 \) or \( A_{21} = 0 \) (i.e., \( n_{e0}(x) \) or \( n_{e1}^{-1}(s, t) \) as a function of \( s \) are even functions) the relevant coefficients are expressible in terms of Bessel functions. Thus using the identity [12] (p. 423)

\[
\int_0^\pi \cos(ax - z \sin x) \, dx = \pi J_n(z),
\]

we obtain

\[
A_{1n}g_{1n}(0) = \frac{2}{n} (-1)^{n+1} J_n(nB_{11}(0)),
\]

and \( A_{2n} \equiv 0 \), if \( B_{1m}(0) = 0 \) for \( m > 1 \) and \( B_{2m} \equiv 0 \).

Similarly,

\[
B_{1m}(t) = 2J_m[mA_{11}e^{-\alpha t} g_{11}(t)],
\]

and \( B_{2m}(t) \equiv 0 \), if \( A_{1n} = 0 \) for \( n > 1 \) and \( A_{2n} \equiv 0 \).

VI. APPLICABILITY CONDITIONS

As for applicability of Eqs.(14a), (14b), one assumes that the plasma is quasineutral \((n_e \approx n_i)\) and the distribution functions are not far from local Maxwellsians. Other approximations come from standard assumptions of the Chapman–Enskog method. Therefore, uncertainty factors of two or three cannot be excluded [9].

If the requirement that the electron pressure term in Eq. (10) is negligible as compared to the electric force is fulfilled for reasonable values of \( \lambda \) the last column of Table I gives the values of \( \eta \). In Table I we present \( n_{e0}^\frac{3\pi}{e^2} T_e^{-\frac{2}{3}} \) as a function of \( n_0 \) and \( T_e \). As oscillating results are obtained only for \( \nu < 1 \), values of the wavelength \( \lambda \) must be greater than \( \lambda_{\text{min}} \equiv \lambda(\nu = 1) \) given in the last column of Table I. Values of \( \lambda \) in \( \text{cm} \) can be obtained by multiplying the last column of Table I by \( \sqrt{\nu^{-1}} \). For example, for \( \nu = 10^{-10}, \sqrt{\nu^{-1}} = 10^5 \) and the last column of Table I gives the values of \( \lambda \) in \( \text{km} \). Similarly for \( \nu = 10^{-4}, \) it gives \( \lambda \) in \( \text{m} \), etc.

Table I indicates that for realistic plasmas, \( \eta \) is always much smaller than one. Therefore the requirement (43) is nearly always fulfilled. Thus, if \( \nu < 1 \), the viscous term in Eq. (21) is significant, both quasi-neutrality and linearisation of Eq. (21) require that the dimensionless \( n_e(x, t) \) cannot be significantly different from unity \((n_{e, \text{max}}^4 \approx 1)\). In the opposite limit of \( \nu \ll 1 \), the viscous term in Eq. (21) is insignificant and there is no need to worry about its linearisation or quasi-neutrality. In that case \( n_{e, \text{max}} \) can be large, but condition Eq. (43) will be fulfilled for reasonable values of \( n_{e, \text{max}} \) (if \( n_{e, \text{max}}^4 \nu < 1 \)).
can in the initial distribution as to introduce a noticeable damping, growth of the maximum in question is reduced, and the corresponding minimum an exploding maximum after a half period of plasma oscillation. However, when the plasma viscosity
\( \eta \) and blows up if
\( \nu \) is increased so
\( \nu < \nu_{\text{max}} \). This will be illustrated in Section VII.

As \( E(s,t) \) in Eqs. (44a), (44b) is a periodic function of \( t \), the Fourier coefficients \( B_{1m}(t) \) and \( B_{2m}(t) \) given by (34) and (35) are also periodic, with period \( \pi \). This in turn implies the same periodicity of all plasma parameters. They will all oscillate, with frequency (in real time) equal to the plasma frequency \( \omega_p \). The fact that this frequency is independent of the amplitude of nonlinear oscillations is a characteristic feature of an ideal plasma, i.e., of any real plasma for which \( \nu, \eta \ll 1 \). This fact is not new but is demonstrated here for the general solution (44).

Another characteristic feature of the general solution (44) is that the dimensionless \( n_e(x,t) \) is always greater than \( \frac{1}{2} \) and blows up if \( n_e(x,t) \to \frac{1}{2} \) somewhere. One obtains \( n_e \to \infty \) when the sum in Eq. (44c) tends to \( -1 \) (its unreachable minimum) while its value at time shifted by \( \pi \) (the half period) tends to \( 1 \) (its unreachable maximum). The fact that in a cold and collisionless plasma \( n_e \) must be greater than \( \frac{1}{2} \) must be kept in mind when prescribing the initial electron density profile \( n_{e0}(x) \equiv n_e(x,t=0) \). Any minimum in the initial density distribution tending to \( \frac{1}{2} \) produces an exploding maximum after a half period of plasma oscillation. However, when the plasma viscosity \( \nu \) is increased so as to introduce a noticeable damping, growth of the maximum in question is reduced, and the corresponding minimum in the initial distribution can fall below \( \frac{1}{2} \). This will be illustrated in Section VII.

Table I indicates that for a typical thermonuclear plasma with \( T_e = 10 \) keV, \( \eta \) is much smaller than one and so condition (43) for the plasma being “cold” is fulfilled if \( n_e^{\text{max}} \nu < 1 \). This would be so even if one could increase the number density \( n_0 \) of such a plasma to the value reached in a laser plasma, \( n_0 = 10^{20} \) cm\(^{-3} \), which would result in \( \eta = 1.52 \times 10^{-6} \).

### VII. EXAMPLES

In view of the fact that for realistic plasmas \( \eta \) is always much smaller than one, the actual value of \( \eta \) will have no visible effect on the plots of \( n_e(x,t) \), etc. where it will be taken equal to zero. The plots will depend on whether or not \( \nu \) is much smaller than one. In any case, values of \( \lambda \) in cm can be obtained by multiplying the last column of Table I by \( \sqrt{\nu^{-1}} \).
FIG. 1: (color online). Plot of $n_e(x,t)$ when only $A_{11} = 0.65$ is nonzero, and $\nu = 0$. This describes all situations with $\nu \ll 1$. The corresponding $\lambda$, for given $n_0$ and $T_e$, can be found from Eqs. (28c), (28d) or Table I.

In Figs. 1 and 2 we present typical examples of the electron density evolution for a small number of modes included in Eqs. (30). Various spatially periodic structures can be produced but the initial density distribution follows.

We can prescribe the initial density distribution $n_e(x, t = 0)$ either directly, or by defining initial values of its Fourier coefficients $B_{1m}(0)$, $B_{2m}(0)$. In the latter case periodicity is guaranteed and in any case, the Fourier coefficients $A_{1n}$ and $A_{2n}$ can be calculated by using Eq. (38). In the simplest case of only the first harmonic present, $A_{1n}$ are expressible in terms of Bessel functions, see Eq. (40). Otherwise numerical integration is necessary which, however, is a simple numerical task. Examples are shown in Figs. 3–5.

The particular solution (30) in which $A_{1n}$ is given by (40) with $\varphi_{1n} = -\pi/2$, reduces to that of [7] if $\nu = \eta = 0$, though in a different notation. Denoting, as in [1], the amplitude $B_{11}(0)$ of the initial density deviation from equilibrium by $B_1(0)$, the known condition $B_1(0) \leq \frac{1}{2}$ follows from the identity $\sum_{n=1}^{\infty} J_n(nz) = \frac{z}{2(1-z)}$ [12] (p. 935).

FIG. 2: (color online). Plot of $n_e(x,t)$ when only $A_{11} = 0.4$ and $A_{12} = 0.2$ are nonzero, and $\nu = 0.02$. ($\sqrt{\nu^{-1}} = 7.07$, see Table I.) The viscous damping is evident after a single period.
FIG. 3: (color online). Plot of $n_e(x,t)$ when only $B_1(0) \equiv B_{11}(0) = 0.55$ is nonzero, $\nu = 0.015$ and $N = 40$. ($\sqrt{\nu - 1} = 8.16$, see Table I.) Note that in the presence of even weak viscosity $B_1(0)$ can exceed $1/2$, see Fig. 6. For $B_1(0) = \frac{1}{2}$, $n_e(x,t = 0)$ reaches its minimum equal $\frac{1}{2}$ at $x = \pi$ and $n_e^{-1}$ becomes zero at $s = t = \pi$ ($n_e \to \infty$). This illustrates the general property of an ideal plasma formulated at the end of Sec. VI.

FIG. 4: (color online). Plot of $n_e(x,t)$ when only $B_1(0) \equiv B_{11}(0) = 0.55$ is nonzero, $\nu = 0.1$ and $N = 40$. ($\sqrt{\nu - 1} = 3.16$, see Table I.) The observed bifurcation of the maximum under strong viscosity is a new nonlinear effect.

The behavior of the solution in question for $\nu > 0$ is shown in Figs. 3 and 4. In a viscous plasma $B_1(0)$ can exceed $1/2$, see Fig. 6. This figure assumes $\eta \ll 1$ and gives the minimal admissible value of $\nu$ for given $B_1(0)$. For $\nu = \nu_{\text{min}}[B_1(0)]$, the smallest minimum of $n_e^{-1}(x = \pi,t)$ at $t = t_{\text{min}}$ becomes zero. Note also that if $\nu$ is sufficiently large, see Fig. 4, where $\nu = 0.1$, a new nonlinear effect can be noticed, i.e., the largest density maximum splits in two, with a saddle point between the peaks. This effect is due to the presence of the integer $n^2$ in front of $\nu$ in Eq. (28b).

The integral $dx$ in Eq. (38) is elementary if $n_{e0}(x)$ is a sequence of step functions. An example of this is shown
FIG. 5: (color online). Plot of $n_e(x,t)$ when only $B_{11}(0) = 0.4$ and $B_{12}(0) = 0.3$ are nonzero, $\nu = 0.015$ and $N = 40$. ($\sqrt{\nu^{-1}} = 8.16$, see Table I.)

FIG. 6: Minimal admissible value of $\nu$ as a function of $B_1(0)$.

in Fig. 7, where $n_{e0}(x) - 1$ is given in the form of two pulses, $p_1$ and $p_2$, emerging from a reference level $r$. The parameters $p_1$, $p_2$, and $r$ can be positive, negative or zero but none of them can be smaller than $-1$, as that would

FIG. 7: An example of possible initial electron density deviation from equilibrium, $n_e(x) - 1$. Here: $x_1 = 1$, $x_2 = 2$, $x_3 = 3.5$, $x_4 = 5$, $p_1 = 0.5$, $p_2 = 0.3$, and $r = -0.251$ as calculated from (46).
FIG. 8: (color online). Plot of $n_e(x,t)$ for a two pulse profile, $p_1 = p_2 = 0.7$, $\nu = 0$ and $N = 40$. See caption to Fig. 1 for the consequences of $\nu = 0$.

FIG. 9: (color online). Plot of $n_e(x,t)$ for a two pulse profile, $p_1 = 0.5$, $p_2 = -0.4$, $\nu = 0$ and $N = 100$. 
FIG. 10: (color online). Plot of $n_e(x,t)$ for a saw-tooth initial density profile, $b = 0.63$, $\nu = 0.015$ and $N = 40$. ($\sqrt{\nu^{-1}} = 8.16$, see Table 1.) Note that in the presence of even weak viscosity $b$ can exceed $\frac{1}{2}$, i.e., the minimum of $n_e(x,t = 0)$ can fall below $\frac{1}{2}$.

make $n_{e,0} < 0$. Using Eq. (38) (the second line) we easily find:

$$A_{1n}g_{1n}(0) = -\frac{1}{\pi n^2} \left\{ \frac{1}{1 + r} \left[ \sin(ns_1) + \sin(ns_3) - \sin(ns_2) - \sin(ns_4) \right] + \frac{1}{1 + p_1} \left[ \sin(ns_2) - \sin(ns_1) \right] + \frac{1}{1 + p_2} \left[ \sin(ns_4) - \sin(ns_3) \right] \right\},$$

where

$$s_1 = (1 + r)x_1,$$
$$s_2 = s_1 + (1 + p_1)(x_2 - x_1),$$
$$s_3 = s_2 + (1 + r)(x_3 - x_2),$$
$$s_4 = s_3 + (1 + p_2)(x_4 - x_3).$$

$A_{2n}g_{2n}(0)$ will be given by the right hand side of Eq. (45) if we replace sin by cos.

The integral $\int_0^{2\pi} (n_{e,0} - 1) \, dx$ must be zero, leading to a constraint on the independent parameters. Solving it for $r$ we obtain

$$r = -\frac{p_1(x_2 - x_1) + p_2(x_4 - x_3)}{2\pi - (x_2 - x_1) - (x_4 - x_3)}. \quad (46)$$

Examples are shown in Figs. 8 and 9.
If the integral $dx'$ in Eq. (38) is expressible analytically, the integral $dx$ in this equation can easily be found numerically. For example, for the saw-tooth initial density profile we obtain ($\varphi_{1n} = -\pi/2$):

$$A_{1n} = -\frac{2}{n\pi} \int_0^\pi \cos\left\{ nx[(1 + b - bx/\pi)] \right\} dx, \quad A_{2n} \equiv 0,$$

(47)

where $b$ is the “amplitude” of the initial density perturbation ($n_e(x = 0) = 1 + b$). The results are shown in Figs. 10 and 11.

Note that if we linearise Eq. (16), by neglecting $v_e \partial v_e/\partial x$ on the left hand side and replacing $n_e$ by $n_0$ on the right hand side, the resulting equation will be identical with the linearised form of Eq. (21) if we replace $s$ by $x$ and $\tau$ by $t$. This means that $v_e(x, t)$ for the linearised problem will be given by Eq. (27), or its dimensionless equivalent Eq. (30d), if we replace $s$ by $x$ and $\tau$ by $t$. Then, using the third equation in (29) (with $\tau \rightarrow t$) and the first equation we can find $E(x, t)$ and $n_e(x, t)$. Finally, $E(x, t)$ will be given by Eq. (30b) with $s$ replaced by $x$, and for $n_e$ we will get

$$n_e(x, t) = 1 + \sum_{n=1}^\infty e^{-\alpha_n t} \left[ A_{1n} g_1(t) \cos(nx) - A_{2n} g_2(t) \sin(nx) \right].$$

(48)

Putting here and in Eqs. (30b) and (30d) $A_{1n} = 0$ for $n > 1$, $A_{2n} \equiv 0$, and $\varphi_{11} = -\pi/2$ we obtain

$$E(x, t) = -A_{11} G_1(t) \sin x,$$

$$n_e(x, t) = 1 + A_{11} G_1(t) \cos x,$$

$$v_e(x, t) = A_{11} e^{-\alpha_1 t} \frac{\sin(\omega_1 t)}{\omega_1} \sin x,$$

$$G_1(t) = e^{-\alpha_1 t} \left[ \frac{\alpha_1 \sin(\omega_1 t)}{\omega_1} + \cos(\omega_1 t) \right],$$

(49a-49d)

where $\omega_1 = \sqrt{1 - \alpha_1^2}$, $\alpha_1 = \nu + \eta$. This result becomes a linear counterpart of the nonlinear solution discussed earlier (only $B_{11}(0)$ nonzero) if we put $A_{11} = B_{11}(0)$. The amplitude $A_{11}$ can take any value between $-1$ and $1$, but there is no maximum amplification shown in Fig. 3. There is also no maximum splitting for large enough viscosity shown in Fig. 4. Both are clearly nonlinear effects, not present in the linearised theory.

FIG. 11: (color online). Plot of $n_e(x, t)$ for a saw-tooth initial density profile, $b = 0.55$, $\nu = 0.1$ and $N = 40$. ($\sqrt{\nu^{-1}} = 3.16$, see Table I.) Again the bifurcation of the maximum under strong viscosity can be seen.
VIII. CONCLUSIONS AND FINAL REMARKS

The 1D Langmuir waves are importantly recently revisited in connection with laser-driven plasma-based electron accelerators. A full non-linear fluid description of these waves, with the dissipative effects included, is given by Eqs. (30). They define the dynamics of the macroscopic parameters of the wave in a simple parametric form, only containing trigonometric and exponential functions. This simple form can be used to represent graphically the time evolution of the wave, \(n_e(x,t)\), etc., while direct analytical formulas for these quantities, Eqs. (33)–(37), are much more complicated.

Our analysis indicates that in real plasmas, both space and laboratory, the influence of plasma resistivity and electron pressure forces on the waves in question is negligible.

The role of electron viscosity can be noticeable as one decreases the wavelength \(\lambda\) of the Langmuir wave excited in a plasma. However, for \(\lambda\) sufficiently large as compared to \(\lambda_{\text{min}}\) given in Table I (that is the dimensionless viscosity coefficient \(\mu\) sufficiently small compared to unity), the effect of viscosity is also negligible. In that case, the waves are well described by an idealized model of a plasma being strictly cold and collisionless \((T_e = \nu = \eta = 0)\). Within this model, all plasma parameters oscillate in time with the plasma frequency \(\omega_{pe}\) (1), independently of the initial density shape or its amplitude. Furthermore, the electron number density \(n_e\) is always greater than half its equilibrium value \(\frac{1}{2}n_0\). Any minimum in the initial electron density distribution \(n_e(x,t=0)\) tending to \(\frac{1}{2}n_0\) would produce an exploding maximum after half a period of the plasma oscillation. These facts were known for a purely cosinusoidal deviation of the initial electron number density from its equilibrium value \(n_0\), but are demonstrated here for arbitrary deviation.

When increasing the wavelength \(\lambda\) (i.e., for \(\nu\) small but finite, e.g., \(\nu = 0.015\)), growth of the maximum is reduced, and the minimum can fall below \(\frac{1}{2}n_0\), see Figs. 3 and 10. And for \(\nu\) relatively large, e.g., \(\nu = 0.1\), the maximum bifurcates as shown in Figs. 4 and 11.

Our requirement that \(n_e/n_0\) should not be considerably larger than unity concerns the accuracy of the viscous force (for validity of linearisation and quasi-neutrality). However, this term only corrects the motion, while the driving electric force and the convective non-linearities are taken into account exactly. In fact, only for relatively large \(\nu\), must one stipulate that \(n_e\) should not exceed \(n_0\) too much.

[1] E. Infeld, G. Rowlands, and A. A. Skorupski, Phys. Rev. Lett. 102, 145005 (2009).
[2] L. Tonks, and I. Langmuire, Phys. Rev. 33, 195 (1929).
[3] L. D. Landau, JETF 16, 574 (1946); J. Phys. USSR 10, 25 (1946).
[4] N. A. Krall and A. W. Trivelpiece, Principles of Plasma Physics, (McGraw–Hill, New York, 1973) Chap. 8.
[5] D. Bohm, and E. P. Gross, Phys. Rev. 75, 1851 (1949).
[6] A. A. Skorupski, Linear waves and instabilities in fluid plasmas. Modern plasma physics (IAEA, Vienna, 1981).
[7] J. M. Dawson, Phys. Rev. 113, 383 (1959); R. C. Davidson and P. P. Schram, Nuclear Fusion 8, 183 (1968); R. C. Davidson, Methods in Nonlinear Plasma Theory, (Academic, N. Y., 1972) Chap. 3.
[8] E. Esarey, C. B. Schroeder, and W. P. Leemans, Rev. Mod. Phys. 81, 1229 (2009).
[9] S. I. Braginski, in Voprosy Teorii Plazmy, ed. by M. A. Leontovitch, vol. 1 (Gosatomizdat, Moscow, 1963); P. C. Clemmow, J. P. Dougherty, Electrodynamics of Particles and Fields, (Addison–Wesley, N. Y., 1990) Chap. 11.
[10] C. Amsler et al., Phys. Lett. B 667, 1 (2008).
[11] E. Infeld and G. Rowlands, Nonlinear Waves, Solitons and Chaos, 2nd ed. (Cambridge University Press, Cambridge, England, 2000) Chap. 6; J. Tech. Phys. 28, 607 (1997) and 29, 3 (1998).
[12] I. S. Gradshteyn, I. M. Rizhik, Table of Integrals, Series, and Products, 7th ed. (Academic, Amsterdam, 2007).