

Introduction

Vector bundles on complex surfaces have been extensively studied by means of several different methods. See for example the books of Kobayashi [12] and Okonek, Schneider, Spindler [13]. Stable holomorphic bundles on a Kähler surface correspond by a theorem of Donaldson [5] to irreducible anti-self-dual connections on the surface. This result connects the study of holomorphic vector bundles with moduli space of instantons. From the point of view of the study of instantons, vector bundles on the blow up of $\mathbb{P}^2$ appear in Hurtubise’s paper on Instantons and Jumping Lines [10] and in Boyer-Hurtubise-Milgram-Mann [1] in their proof of the Atiyah-Jones conjecture. Subsequently Hurtubise and Milgram [11] proved an extended version of the Atiyah-Jones conjecture for ruled surfaces, by means of studying the structure of holomorphic bundles on ruled surfaces.

To motivate our study of holomorphic vector bundles on blow-ups from a different stand point we mention a fundamental result on the classification of rational surfaces, see Griffiths and Harris [8].

**Theorem**: Every rational surface is obtained by blowing up points on either $\mathbb{P}^2$ or on a rational ruled surface.

The previous theorem suggests that the understanding of vector bundles on rational surfaces depends on the analysis of the behavior of vector bundles under blow-ups. A large amount of work has been done on vector bundles on $\mathbb{P}^2$ (see for example the book by Okonek, Schneider, Spindler [13]). In a sense we can also say that vector bundles on ruled surfaces are well understood (see Brosius [2] [3], Qin [14], Hurtubise and Milgram [11]). Some examples of work on moduli spaces of holomorphic vector bundles on blow-ups are the papers by by Freedman and Morgan [6][7], Brussee [4], and Qin [15].

The blow-up of a point on a surface is a local operation in the sense that one blows-up the point inside one of its coordinate neighborhoods. Roughly speaking we may see the “difference” between moduli spaces of bundles on a rational surface and moduli spaces of bundles on one of its minimal models by studying bundles on the blow up of $\mathbb{C}^2$.

In this work we concentrate on the study of bundles on blow-ups in the local sense, that is in a neighborhood of the exceptional divisor. Our approach is quite concrete, as we give bundles explicitly by their transition matrices and present the moduli spaces as quotients of a vector space $\mathbb{C}^n$ by an equivalence
relation.
In Section 3 we construct a canonical form of transition matrix for rank two bundles on the blow up of \( \mathbb{C}^2 \). Namely, we prove the following:

**Theorem 2.1**: Let \( E \) be a holomorphic rank two vector bundle on \( \tilde{\mathbb{C}}^2 \) with zero first Chern class and let \( j \) be the integer that satisfies \( E_{\ell} \cong \mathcal{O}(j) \oplus \mathcal{O}(-j) \). (Where \( E_{\ell} \) is the restriction of \( E \) to the exceptional divisor.) Then \( E \) has a transition matrix of the form

\[
\begin{pmatrix}
  z^j & p \\
  0   & z^{-j}
\end{pmatrix}
\]

from \( U \) to \( V \), where

\[
p = \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} p_{il} z^l u^i.
\]

In particular \( p \) depends on a finite number of parameters.

We then define the moduli space \( \mathcal{M}_j \) as the space of equivalence classes of such bundles having restriction \( \mathcal{O}(j) \oplus \mathcal{O}(-j) \) to the exceptional divisor, modulo holomorphic equivalence. It follows immediately from our canonical form of a transition matrix that:

**Corollary 2.3**: \( \mathcal{M}_0 \) consists of a single point.

**Corollary 2.5**: \( \mathcal{M}_1 \) consists of a single point.

In Section 4 we continue the study of \( \mathcal{M}_j \) for \( j \geq 2 \). To do this we analyze the problem of when two holomorphic bundles in \( \mathcal{M}_j \) are isomorphic. A simple characterization on the first formal neighborhood of the exceptional divisor.

**Proposition 3.3** On the first formal neighborhood, two holomorphic bundles \( E^{(1)} \) and \( E^{(1)'} \) with transition matrices

\[
\begin{pmatrix}
  z^j & p_1 \\
  0   & z^{-j}
\end{pmatrix}
\]
and

\[
\begin{pmatrix}
  z^j & p'_1 \\
  0 & z^{-j}
\end{pmatrix}
\]

respectively are isomorphic iff \( p'_1 = \lambda p_1 \) for some \( \lambda \in \mathbb{C} - \{0\} \).

Once one passes the first formal neighborhood, the holomorphic equivalences become more intricate. In 5.1 we give a detailed description of \( \mathcal{M}_2 \). Topologically, we have:

**Theorem 4.2**: The moduli space \( \mathcal{M}_2 \) is homeomorphic to the union \( \mathbb{P}^1 \cup \{p, q\} \), of a complex projective plane \( \mathbb{P}^1 \) and two points with a basis of open sets given by

\[
\mathcal{U} \cup \{p, U : U \in \mathcal{U} - \phi\} \cup \{p, q, U : U \in \mathcal{U} - \phi\}
\]

where \( \mathcal{U} \) is a basis for the standard topology on \( \mathbb{P}^1 \).

In 5.2 we describe \( \mathcal{M}_3 \) and in 5.3 we give the generic description of \( \mathcal{M}_j \). Our general results are:

**Theorem 4.4** The generic set of the moduli space \( \mathcal{M}_j \) is a complex projective space of dimension \( 2j - 3 \) minus a closed subvariety of complex codimension bigger than or equal to two.

**Remark 4.6**: The moduli space \( \mathcal{M}_j \) also contains complex projective spaces of every dimension smaller than \( 2j - 3 \), each minus some closed subvariety.

**Remark 4.7**: If we give \( \mathcal{M}_j \) the topology induced from \( \mathbb{C}^N \), then \( \mathcal{M}_j \) is not a Hausdorff space. For example, the direct sum bundle given by

\[
\begin{pmatrix}
  z^j & 0 \\
  0 & z^{-j}
\end{pmatrix}
\]

is arbitrarily close to any other bundle.
**Remark 4.8:** Note that the word generic here is used in the sense that the moduli space $\mathcal{M}_j$ consists of subsets out of which $\mathbb{P}^{2j-3}$ is the subset of highest dimension.

Finally in Section 6 we give some examples of the result of building up bundles on the blow up of a compact surface using our canonical form of a transition matrix for a neighborhood of the exceptional divisor.

Note: This is a quite long file, so I am only sending the ”introduction.” If anyone wants the whole file, be welcome to write to gasparim@ictp.trieste.it