Multi-query quantum sums

David A. Meyer\textsuperscript{1} and James Pommersheim\textsuperscript{1,2}

\textsuperscript{1}Department of Mathematics
University of California/San Diego, La Jolla, CA 92093-0112
\textsuperscript{2}Department of Mathematics
Reed College, Portland, OR 97203
dmeyer@math.ucsd.edu, jamie@reed.edu

Abstract. \textsc{Parity} is the problem of determining the parity of a string $f$ of $n$ bits given access to an oracle that responds to a query $x \in \{0, 1, \ldots, n-1\}$ with the $x^{th}$ bit of the string, $f(x)$. Classically, $n$ queries are required to succeed with probability greater than $1/2$ (assuming equal prior probabilities for all length $n$ bitstrings), but only $\lceil n/2 \rceil$ quantum queries suffice to determine the parity with probability 1. We consider a generalization to strings $f$ of $n$ elements of $\mathbb{Z}_k$ and the problem of determining $\sum f(x)$. By constructing an explicit algorithm, we show that $n - r$ ($n \geq r \in \mathbb{N}$) entangled quantum queries suffice to compute the sum correctly with worst case probability $\min\{\lfloor n/r \rfloor / k, 1\}$. This quantum algorithm utilizes the $n - r$ queries sequentially and adaptively, like Grover’s algorithm, but in a different way that is not amplitude amplification.

1. Introduction

\textsc{Parity} is the oracle (or black-box) problem of determining the parity of an $n$-bit string by querying positions in the string. Since even a single unqueried bit can change the parity, $n$ classical queries are required to solve this problem with probability 1, assuming all $n$-bit strings are possible.

When $n = 2$, this is Deutsch’s problem [1], for which a single quantum query, used properly, suffices [2]. Beals \textit{et al.} show that in general $\lceil n/2 \rceil$ quantum queries suffice by applying the solution to Deutsch’s problem to the bits in pairs [3]. In their algorithm the quantum queries are \textit{independent} of one another—they can be asked in parallel since none depends on the responses of the oracle to the others—and they are also \textit{incoherent}—after each query is processed, the state is measured and the resulting information (the parity of a pair of the bits) is combined classically at the end of the algorithm.

This same independence of multiple queries is a feature of existing multi-query quantum algorithms for abelian [4] and non-abelian (\textit{e.g.}, [5,6,7]) hidden subgroup problems, which range from incoherent [4] through partially [5,6] to completely [7] coherent. Grover’s quantum search algorithm [8], and quantum (random walk) search algorithms on graphs [9,10] more generally, however, utilize coherent sequences of \textit{adapted} queries—the quantum state is modified by each oracle response before it is returned to the oracle for the next query, so the queries are not independent. These algorithms all use \textit{amplitude amplification} [11] to adapt their sequential queries.
But amplitude amplification, which identifies an element in the preimage of 1 for some bit-valued function, does not apply to \textsc{Parity}, nor to its generalization:

\textsc{Sum}. Let \( f : \mathbb{Z}_n \to \mathbb{Z}_k \), where \( f \) is accessed via an oracle that responds with \( f(x) \) when queried about \( x \in \mathbb{Z}_n \). Find \( \sum_{x \in \mathbb{Z}_n} f(x) \pmod{k} \).

As they are for \textsc{Parity}, \( n - 1 \) classical queries are useless for \textsc{Sum} when \( f \) is chosen uniformly at random, \textit{i.e.}, the \( 1/k \) prior probability of each possible sum is unchanged after the oracle responds to the queries [12]. Our Uselessness Theorem: if \( 2q \) classical queries are useless, then \( q \) quantum queries are useless [12], implies that \( \lfloor (n - 1)/2 \rfloor \) quantum queries are therefore useless for \textsc{Sum}. This raises the question of how well we can do using more than a useless number of queries; to answer it we construct an \( n - r \) quantum query algorithm that computes the sum correctly with worst case probability \( \min\{\lfloor n/r \rfloor/k, 1\} \), for each \( 1 \leq r \in \mathbb{N} \), and that returns a result that is within \( |kr/2n| \) of the sum with probability at least \( 4/\pi^2 \). This quantum algorithm utilizes the \( n - r \) queries sequentially and adaptively, like quantum search algorithms, but in a different way that is not amplitude amplification.

We motivate the development of our algorithm in the next section by considering the simplest new instances of \textsc{Sum}, computing the sum of 2 or 3 trits. In §3 we state and prove two basic lemmas and combine them to construct the general algorithm in §4. We conclude in §5 by recalling the result of van Dam that strings of \( n \) bits can be identified with high probability using \( n/2 + O(\sqrt{n}) \) queries, and hence any function of them can be computed with at least the same probability [13]. We generalize this result to \( k > 2 \) and show that, unsurprisingly—since it is designed to do more than just compute the sum of the string values, it gives success probabilities less than those of our algorithm.

2. Sums of trits

The simplest generalization of Deutsch’s problem is to add two trits rather than two bits, \textit{i.e.}, the \( n = 2 \) and \( k = 3 \) version of \textsc{Sum}. As with Deutsch’s problem, if all possible functions \( f : \mathbb{Z}_2 \to \mathbb{Z}_3 \) are equally likely, a single classical query is useless—the prior probabilities of \( 1/3 \) for each value of \( \sum f(x) \) are unchanged after a single query—while two classical queries suffice to determine the sum with probability 1. Thus the goal of a quantum algorithm for this problem should be to determine the sum with a single quantum query with probability greater than \( 1/3 \).

PROPOSITION 1. \textit{Using a single quantum query the sum of two trits can be determined with worst case probability 2/3}.

Before giving the proof we recall some standard notation: We will work in the Hilbert space \( \mathbb{C}^n \otimes \mathbb{C}^k \), with computational basis \( \{|x\rangle|y\rangle \mid x \in \mathbb{Z}_n, y \in \mathbb{Z}_k \} \). The shift operator acts by \( X : |z\rangle \mapsto |z+1\rangle \) and the oracle acts by \( O_f : |x\rangle|y\rangle \mapsto |x\rangle|y + f(x)\rangle = |x\rangle X f(x)|y\rangle \).
Finally, \( \omega = e^{2\pi i/k} \), and the Fourier transform on \( \mathbb{C}^k \) acts by

\[
\mathcal{F} : |y\rangle \mapsto \frac{1}{\sqrt{k}} \sum_{\ell=0}^{k-1} \omega^{\ell y} |\ell\rangle =: |\omega^{-y}\rangle,
\]

since \( X|\omega^{-y}\rangle = \omega^{-y}|\omega^{-y}\rangle \). These “Fourier” (or “character”) basis states will be used to implement the widely useful generalization to dimensions greater than 2 [14,15,16] of the “phase kickback trick” [2].

To use these states in a quantum algorithm for the two trit problem we might expect simply to query the oracle with a state of the form

\[
\frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right) \otimes |\phi\rangle,
\]

where \( \phi = \omega^{-y} \) for some \( y \in \{1, 2\} \subset \mathbb{Z}_3 \), as if we were solving Deutsch’s problem. Notice, however, that the relative phase of the two components in the state returned by the oracle would be \( \phi^f(0) - f(1) \), so it would not encode the sum \( f(0) + f(1) \), unlike the two bit case in which \( -f(1) \equiv f(1) \pmod{2} \). A different query state is required:

**Proof** (of Proposition 1). It suffices to exhibit a single query algorithm that succeeds with probability 2/3.

0. Initialize to the state

\[
\frac{1}{\sqrt{2}} \left( |1\rangle |\omega^1\rangle + |0\rangle |\omega^{-1}\rangle \right).
\]

1. Call the oracle \( \mathcal{O}_f \) to obtain the state

\[
\frac{1}{\sqrt{2}} \left( \omega^{f(1)} |1\rangle |\omega^1\rangle + \omega^{-f(0)} |0\rangle |\omega^{-1}\rangle \right).
\]

Notice that the relative phase of the two terms is \( \omega^{f(0)} + f(1) \). We could argue at this point that there is a POVM that identifies which of the three possible states we have with probability 2/3 [17], but as a simple sequence of unitary transformations avoids the necessity for anything beyond a complete von Neumann measurement in the computational basis, we describe it explicitly in the following steps.

2. Act by \( X \otimes I \) to obtain the state

\[
\frac{1}{\sqrt{2}} \left( \omega^{f(1)} |0\rangle |\omega^1\rangle + \omega^{-f(0)} |1\rangle |\omega^{-1}\rangle \right).
\]

3. Act by \( K \) to obtain the state

\[
\frac{1}{\sqrt{2}} \left( |0\rangle \left( \omega^{f(1)} |\omega^1\rangle + \omega^{-f(0)} |\omega^0\rangle \right) \right),
\]
where \( K \) acts on \( \mathbb{C}^n \otimes \mathbb{C}^k \) by

\[
K : |x\rangle|y\rangle = \begin{cases} 
|0\rangle|\omega^0\rangle & \text{if } x = n - 1 \text{ and } y = k - 1; \\
(n - 1)|\omega^{k-1}\rangle & \text{if } x = 0 \text{ and } y = 0; \\
|x\rangle|\omega^y\rangle & \text{otherwise.}
\end{cases}
\]

Note that while \( K \) is a complicated unitary operation, it is independent of \( f \), i.e., it does not call the oracle.

The \( C^3 \) tensor factor in the final state (3) can be rewritten as:

\[
\frac{1}{\sqrt{2}} \left( \omega^{f(1)}|1\rangle + \omega^{-f(0)}|0\rangle \right)
\]

\[
= \omega^{-f(0)} \frac{1}{\sqrt{2}} \left( \omega^{\Sigma f} |1\rangle + |0\rangle \right)
\]

\[
= \omega^{-f(0)} \frac{1}{\sqrt{6}} \left( (1 + \omega^{\Sigma f}) |0\rangle + (1 + \omega^{\Sigma f - 1}) |1\rangle + (1 + \omega^{\Sigma f - 2}) |2\rangle \right),
\]

using the definition (1), so now measurement of the \( C^3 \) tensor factor will return \( \sum f(x) \) with probability \( 2/3 \).

To obtain this probability our initial query (2) was an entangled state, rather than the usual tensor product state; this is the first innovation in the algorithm up to which we are building. The next step is to consider adding \( n = 3 \) trits. In this case two classical queries are useless, so one quantum query is useless [12], and we must consider algorithms with two coherent quantum queries.

**Proposition 2.** Two quantum queries suffice to solve \( \text{SUM} \) with probability 1 when \( n = k = 3 \).

**Proof.** It suffices to exhibit a two query algorithm that succeeds with probability 1.

0. Initialize to the entangled state

\[
\frac{1}{\sqrt{3}} \left( |1\rangle|\omega^1\rangle + |0\rangle|\omega^{-1}\rangle + |0\rangle|\omega^{-2}\rangle \right).
\]

1. Call the oracle \( O_f \) to obtain the state

\[
\frac{1}{\sqrt{3}} \left( \omega^{f(1)}|1\rangle|\omega^1\rangle + \omega^{-f(0)}|0\rangle|\omega^{-1}\rangle + \omega^{-2f(0)}|0\rangle|\omega^{-2}\rangle \right).
\]

2. Act by \( X \otimes I \) to obtain the state

\[
\frac{1}{\sqrt{3}} \left( \omega^{f(1)}|2\rangle|\omega^1\rangle + \omega^{-f(0)}|1\rangle|\omega^{-1}\rangle + \omega^{-2f(0)}|1\rangle|\omega^{-2}\rangle \right).
\]
3. Act by \( J_1 \) to obtain the state

\[
\frac{1}{\sqrt{3}} (\omega^{f(1)}|2\rangle|\omega^2\rangle + \omega^{-f(0)+f(2)}|2\rangle|\omega^1\rangle + \omega^{-2f(0)-f(1)}|1\rangle|\omega^{-1}\rangle),
\]

where \( J_r \) acts on \( \mathbb{C}^n \otimes \mathbb{C}^k \) by

\[
J_r : |x\rangle|\omega^y\rangle = \begin{cases} 
|x\rangle|\omega^0\rangle & \text{if } y = 0; \\
|x+r\rangle|\omega^1\rangle & \text{if } y = -1; \\
|x\rangle|\omega^{y+1}\rangle & \text{otherwise.}
\end{cases}
\]

Note that like \( K \), while \( J_r \) is a complicated unitary operation, it is independent of \( f \), i.e., it does not call the oracle.

4. Call the oracle \( O_f \) a second time to obtain the state

\[
\frac{1}{\sqrt{3}} (\omega^{f(1)+2f(2)}|2\rangle|\omega^2\rangle + \omega^{-f(0)+f(2)}|2\rangle|\omega^1\rangle + \omega^{-2f(0)-f(1)}|1\rangle|\omega^{-1}\rangle).
\]

5. Act by \( X \otimes I \) again to obtain the state

\[
\frac{1}{\sqrt{3}} (\omega^{f(1)+2f(2)}|0\rangle|\omega^2\rangle + \omega^{-f(0)+f(2)}|0\rangle|\omega^1\rangle + \omega^{-2f(0)-f(1)}|2\rangle|\omega^{-1}\rangle).
\]

6. Act by \( K \) to obtain the state

\[
\frac{1}{\sqrt{3}} |0\rangle (\omega^{f(1)+2f(2)}|\omega^2\rangle + \omega^{-f(0)+f(2)}|\omega^1\rangle + \omega^{-2f(0)-f(1)}|\omega^0\rangle).
\]

The \( \mathbb{C}^3 \) tensor factor in the final state (6) can be rewritten as:

\[
\frac{1}{\sqrt{3}} (\omega^{f(1)+2f(2)}|\omega^2\rangle + \omega^{-f(0)+f(2)}|\omega^1\rangle + \omega^{-2f(0)-f(1)}|\omega^0\rangle) \\
= \frac{1}{\sqrt{3}} \omega^{f(0)+2f(1)} (\omega^{-\Sigma f}|\omega^2\rangle + \omega^{-2\Sigma f}|\omega^1\rangle + \omega^{-3\Sigma f}|\omega^0\rangle) \\
= \omega^{f(0)+2f(1)}|\Sigma f\rangle,
\]

using the definition (1), so now measurement of this tensor factor will return \( \sum f(x) \) with probability 1.

The key piece of algebra is that the phases of the terms in (6), each a linear combination of two values of \( f \), are also linear combinations of all three values of \( f \), with a coefficient of 0 in front of the third value: \((0,1,2) \cdot f, (−1,0,1) \cdot f = (2,0,1) \cdot f, \text{ and } (−2,−1,0) \cdot f = (1,2,0) \cdot f, \) where \( f = (f(0), f(1), f(2)) \). Written this way it is clear that the coefficient vectors are
successive cyclic shifts $\sigma$ of $(0, 1, 2)$, so if we factor out the last phase factor the other two become:

$$(0, 1, 2) \cdot f - \sigma^2(0, 1, 2) \cdot f = (2, 2, 2) \cdot f = -\sum f$$

$$\sigma(0, 1, 2) \cdot f - \sigma^2(0, 1, 2) \cdot f = \sigma(1, 1, 1) \cdot f = -2\sum f,$$

the phases of the first two terms in (7).

This algorithm is optimal since it uses only one more than the useless number of quantum queries. Notice that its two coherent quantum queries are sequential rather than parallel, and that the second query is adapted in the sense that the state (5) that is passed to the oracle as the second query depends on the response of the oracle to the first query, unitarily transformed by $J(X \otimes I)$. This adaptation differs from amplitude amplification [11] and is the second innovation in our quantum summation algorithm.

### 3. Two basic lemmas

To generalize the quantum algorithms given in the previous section for summing trits, it is convenient first to state two basic lemmas.

**LEMMA 3.** For $A \in \mathbb{Z}_k$ and $k \geq s \in \mathbb{N}$, let

$$|A_s\rangle = \frac{1}{\sqrt{s}} \sum_{\ell=1}^{s} \omega^{-\ell A} |\omega^{s-\ell}\rangle \in \mathbb{C}^k. \tag{8}$$

Measurement of $|A_s\rangle$ in the computational basis returns $|y\rangle$, $y \in \mathbb{Z}_k$, with probability

$$|\langle y|A_s\rangle|^2 = \frac{1}{sk} \left( \frac{\sin \pi s(y - A)/k}{\sin \pi (y - A)/k} \right)^2,$$  \tag{9}$$
defined to be a continuous function of $y - A$. The probability $|\langle y|A_s\rangle|^2$ takes its maximum value, $s/k$, at $y = A$, and the probability that the measurement is within $\pm \lfloor k/2s \rfloor$ of $A$ is at least $4/\pi^2$.

**Proof.** This is an elementary (and familiar from phase estimation; see, e.g., [2]) calculation using the definition (1):

$$\langle y|A_s\rangle = \frac{1}{\sqrt{sk}} \sum_{\ell=1}^{s} \omega^{-\ell A - (s-\ell)y} = \frac{1}{\sqrt{sk}} \omega^{-sy} \sum_{\ell=1}^{s} \omega^{\ell(y-A)} = \frac{1}{\sqrt{sk}} \omega^{(1-s)y-A} \frac{1 - \omega^{s(y-A)}}{1 - \omega^{y-A}}.$$

Taking the norm squared of this expression gives (9), which by continuity takes the value $s/k$ when $y = A$. That this is the maximum follows from the fact that in this case all the terms in the sum above are 1.
Writing $d = y - A$, $|d| \leq k/2s$ implies $|\sin \pi sd/k| \geq |\pi sd/k|/|\pi/2| = 2s|d|/k$, since the argument of $\sin$ has absolute value no more than $\pi/2$. Also, $|\sin \pi d/k| \leq |\pi d/k|$. Using these bounds in (9) gives
\[ |\langle A + d|A_s\rangle|^2 \geq \frac{1}{sk} \left( \frac{2s|d|/k}{\pi d/k} \right)^2 = \frac{4}{\pi^2} \frac{s}{k}, \]
so
\[ \sum_{|d| \leq k/2s} \frac{4}{\pi^2} \frac{s}{k} \geq \left\lfloor \frac{k}{s} \right\rfloor \cdot \frac{4}{\pi^2} \frac{s}{k} \geq \frac{4}{\pi^2}. \]

When $k = 3 = s$ and $A = \sum f(x)$, the state (8) is equal to the $C^3$ tensor factor in the final state of the algorithm in Proposition 2, up to an overall phase. Similarly, in the algorithm of Proposition 1, if rather than factoring out the phase $\omega^{-f(0)}$ in (4), we factor out $\omega^{f(0)+2f(1)}$, we obtain (8) with $k = 3$, $s = 2$, and $A = \sum f(x)$. The success probabilities of 1 and $2/3$ in these two algorithms are the values $s/k$ given by Lemma 3.

When $A = \sum_{x \in \mathbb{Z}_n} f(x)$, each component of the state (8) depends on all of the $n$ values of $f$. The next lemma says that, up to an overall phase, this state is equivalent to one in which each component depends on fewer than $n$ values of $f$.

**Lemma 4.** Let $1 \leq r \in \mathbb{N}$, let $r|n$, and let $s = n/r$. Then
\[
\omega^{\sum_{m=1}^{s} m[f((m-1)r)+\cdots+f(mr-1)]} \frac{1}{\sqrt{s}} \sum_{\ell=1}^{s} \omega^{-\ell s} f|\omega^{s-\ell}| = \frac{1}{\sqrt{s}} \sum_{\ell=1}^{s} \omega^{\sum_{m=1}^{s} (m-\ell)[f((m-1)r)+\cdots+f(mr-1)]}|\omega^{s-\ell}|.
\]

For each value of $\ell$, namely for each component, in the sum on the right hand side of this equation, there is a term in the sum in the exponent which vanishes because $m = \ell$. Since each of these terms depends on $r$ values of $f$, each component depends on $sr - r = n - r$ values of $f$. In the algorithm of Proposition 1, $n = 2$ and $r = 1$ so $s = n/r = 2$, and each component of the final state (3) depends on $n - r = 2 - 1 = 1$ value of $f$. In the algorithm of Proposition 2, $n = 3$ and $r = 1$ so $s = n/r = 3$, and each component of the final state (6) depends on $n - r = 3 - 1 = 2$ values of $f$.

**4. The general SUM problem**

Summing two and three trits are special cases of the general SUM problem that motivate the two innovations in our general algorithm. Propositions 1 and 2 are special cases of the following theorem.

**Theorem 5.** Let $f : \mathbb{Z}_n \to \mathbb{Z}_k$. Using $n - r$ quantum queries the sum $\sum_{x \in \mathbb{Z}_k} f(x)$ can be computed correctly with worst case probability $\min\{\lfloor n/r \rfloor/k, 1\}$, for each $n \geq r \in \mathbb{N}$.
Furthermore, the same algorithm outputs a result within \(\lfloor kr/2n\rfloor\) of the correct sum with probability at least \(4/\pi^2\).

**Proof.** First consider the case \(r|n\) and let \(s = n/r \in \mathbb{N}\). If \(s \leq k\) and we can construct the state

\[
\frac{1}{\sqrt{s}} \sum_{\ell=1}^{s} \omega^{-\ell \sum_{m=1}^{s} m} |\omega^{s-\ell}\rangle,
\]

then by Lemma 3 we can find \(\sum f(x)\) with probability \(s/k = n/(rk)\), and even when the output is wrong, it is likely to be close—within \(\lfloor rk/2n\rfloor\) with probability at least \(4/\pi^2\). By Lemma 4 we need only construct the state

\[
\frac{1}{\sqrt{s}} \sum_{\ell=1}^{s} \omega^{\sum_{m=1}^{s} (m-\ell) |f((m-1)r)+\cdots+f(mr-1)|} |\omega^{s-\ell}\rangle,
\]

in which each component depends on \(n-r\) values of \(f\). The following algorithm does so, using \(n-r\) quantum queries:

0. Initialize to the entangled state

\[
\frac{1}{\sqrt{s}} (|r\rangle |\omega^1\rangle + |0\rangle |\omega^{-1}\rangle + \cdots + |0\rangle |\omega^{-(s-1)}\rangle).
\]

1. Apply \(K(((X \otimes I)\mathcal{O}_f)^r)(J_r((X \otimes I)\mathcal{O}_f)^r)^{s-2}\) to obtain the state

\[
\frac{1}{\sqrt{s}} |0\rangle \sum_{\ell=1}^{s} \omega^{\sum_{m=1}^{s} (m-\ell) |f((m-1)r)+\cdots+f(mr-1)|} |\omega^{s-\ell}\rangle.
\]

2. Measure the \(\mathcal{O}^k\) tensor factor in the computational basis.

Notice that when \(n = k\) and \(r = 1\), i.e., using \(k-1\) quantum queries, this algorithm returns \(\sum f(x)\) with probability 1.

If \(s > k\), or equivalently, if \(r < n/k\), then \(n = uk + v\) with \(u \geq r\) and \(0 \leq v < k\), so we can use \(k-1\) queries in this algorithm applied to each block of length \(k\), using a total of \(uk - u = n - v - u\) queries, leaving \(v + u - r \geq v\) queries to identify the last \(v\) values of \(f\). Thus when \(s > k\), we can find \(\sum f(x)\) with probability 1.

Second, and similarly, if \(r\not|n\), let \(s = \lfloor n/r \rfloor\). Then \(n = rs + w\) with \(0 < w < r\). Using the algorithm applied to the first \(n-w\) values of \(f\), we can compute

\[
\sum_{x=0}^{rs-1} f(x), \text{ with probability } \min\{1, \lfloor n/r \rfloor/k\},
\]

using \(n-w-r\) queries, leaving \(w\) queries to identify the last \(w\) values of \(f\).
Thus in all cases, this algorithm uses $n - r$ quantum queries to return $\sum f(x)$ with probability $\min\{1, [n/r]/k\}$, and a value within $[kr/2n]$ of the sum with probability at least $4/\pi^2$.

We believe this algorithm is optimal, but we have only proved it to be so for $r = n - 1$, i.e., a single query [18].

5. Conclusion

Since the number of queries $n - r \geq 0$, the success probability of our algorithm is always at least $1/k$, as it should be. Furthermore, since $[n/r] = 1$ until $r \leq n/2$, fewer than $n/2$ quantum queries in this algorithm are useless, as they must be according to the Uselessness Theorem [12]. When $k = 2$, Theorem 5 says that for $r \leq n/2$ the success probability is 1, as we know from the solution to PARITY [3].

For $k > 2$ we know of no algorithms to which to compare ours. Van Dam’s quantum algorithm for obtaining all the information about a function $\mathbb{Z}_n \to \mathbb{Z}_2$ with high probability using $n/2 + O(\sqrt{n})$ queries [13], however, can be generalized to functions $f : \mathbb{Z}_n \to \mathbb{Z}_k$:

**Theorem 6.** Let $f : \mathbb{Z}_n \to \mathbb{Z}_k$. There is a quantum algorithm using $q$ queries that correctly identifies the function with worst case probability

$$p_q = \frac{1}{kn} \sum_{j=0}^{q} \binom{n}{j}(k-1)^j. \quad (10)$$

The cumulative distribution function (10) for this binomial probability distribution is greater than 0.95 (almost 0.98) provided $q > n(k-1)/k + 2\sqrt{n(k-1)/k}$, namely the mean plus two standard deviations. Thus with this many queries we can determine the oracle correctly with probability more than 0.95, and thus compute the sum of its values correctly. More precisely, using this algorithm with $q$ queries, we can compute $\sum f(x)$ with probability less than $p_q + (1 - p_q)/k$ (obtained by bounding the probability of computing the sum correctly by $1/k$ when the algorithm fails to output the correct $f$). Figure 1 plots this upper bound on the success probability as a function of the number of queries, along with the success probability of the algorithm of Theorem 5.

The success probability of the algorithm of Theorem 5 is greater than or equal to that of the generalized van Dam algorithm of Theorem 6, for any number of queries, an unsurprising result since the latter is using those queries to try to determine the whole function, not just its sum. To succeed with probability greater than a constant, the former requires a fraction of $n$ approaching 1 like $1/k$ quantum queries, while the latter requires this many plus $O(\sqrt{n})$. 

9
Fig. 1. Success probabilities of the algorithms from Theorems 5 (steps) and 6 (smooth).

Finally, to succeed only approximately (i.e., within $\epsilon k$) with probability greater than a constant ($4/\pi^2$; the sum of probabilities calculated for the central peak of (9) in Lemma 3), the algorithm of Theorem 5 requires $n(1 - \epsilon)$ quantum queries, independent of $k$.

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