BOUNDDED $H_\infty$-CALCULUS FOR BOUNDARY VALUE PROBLEMS ON MANIFOLDS WITH CONICAL SINGULARITIES

NIKOLAOS ROIDOS, ELMAR SCHROHE, AND JÖRG SEILER

Abstract. Realizations of differential operators subject to differential boundary conditions on manifolds with conical singularities are shown to have a bounded $H_\infty$-calculus in appropriate $L_p$-Sobolev spaces provided suitable conditions of parameter-ellipticity are satisfied. Applications concern the Dirichlet and Neumann Laplacian and the porous medium equation.

CONTENTS

1. Introduction 1
2. Boundary value problems for cone differential operators 3
3. Domains and realizations 4
4. Parameter-dependent Green operators 10
5. Bounded $H_\infty$-calculus for parameter-elliptic realizations 12
6. The Dirichlet and Neumann Laplacian 18
7. The porous medium equation on conic manifolds with boundary 24
8. Appendix 26
References 30

1. INTRODUCTION

In this article, we study the $H_\infty$-calculus of parameter-elliptic boundary value problems on manifolds with conical singularities on the boundary, following up on earlier work in [2, 8, 25]. Moreover, we present a new way of determining their realizations. As an application we treat the porous medium equation.

A manifold of dimension $n + 1$ with conical singularities on the boundary is a compact topological space $D$, which contains finitely many points $\{d_1, \ldots, d_N\}$ such that

1. $D_{\text{reg}} := D \setminus \{d_1, \ldots, d_N\}$ is an $(n + 1)$-dimensional smooth manifold with boundary,
2. each point $d_j$ has a neighborhood which is homeomorphic to a cone $C_j = [0,1) \times Y_j/\{0\} \times Y_j$ with an $n$-dimensional manifold with boundary $Y_j$.

We replace $C_1 \cup \ldots \cup C_N$ by the cylinder $[0,1) \times Y$, where $Y$ is the disjoint union of the $Y_j$, obtaining a space denoted by $\mathbb{D}$. Note that $D_{\text{reg}}$ can be identified with $\mathbb{D} \setminus \{(0) \times Y\}$. The boundary $\mathbb{B}_{\text{reg}} := \partial D_{\text{reg}}$ is the regular part of an $n$-dimensional conic manifold (without boundary) $\mathbb{B}$ which contains the cylinder.

Key words and phrases. Conic manifolds with boundary, bounded $H_\infty$-calculus, realizations of elliptic boundary value problems.

N. Roidos and E. Schrohe were supported by Deutsche Forschungsgemeinschaft, grant SCHR 319/9-1, in the priority program Geometry at Infinity.
\[0,1) \times \partial Y\]. On the cylindrical parts, we use variables \((x, y)\) with \(x \in [0, 1)\) and \(y \in Y\). For more details we refer the reader to [21 Section 1.1] or [15 Section 3].

Vector bundles over \(D\) are assumed to be smooth over \(D_{\text{reg}}\). On the cylindrical part a vector bundle \(E\) over \(D\) is the pull-back of a smooth vector bundle \(E_0\) over \(Y\) under the canonical projection \((x, y) \mapsto y\); the same applies to vector bundles over \(B\). All bundles are supposed to carry a hermitian structure compatible with the product structure on the cylindrical part.

A cone differential operator on \(D\) acting on sections of a bundle \(E\) (or, more generally, between sections of two possibly different bundles) is a differential operator with smooth coefficients on the regular part, while, on the cylindrical part, it has the form

\[A = x^{-\mu} \sum_{j=0}^{\mu} a_j(x)(-x \partial_x)^{j}, \quad \mu = \text{order of } A,\]

where each \(a_j(x)\) is a family of differential operators of order \(j - \mu\) on \(Y\) acting on sections of \(E_0\), smooth up to \(x = 0\). Initially, we consider \(A\) as a map in \(\mathcal{E}^{\infty,\infty}(D, E)\), the space of smooth sections of \(E\) that vanish to infinite order in the tip \(x = 0\). To give an example, let \(D_{\text{reg}}\) be endowed with a Riemannian metric which, on the cylindrical part, has the form

\[g = dx^2 + x^2 h(x),\]

with a family \(h(x)\) of Riemannian metrics on \(Y\), smooth up to \(x = 0\) (i.e., \(g\) is the metric of a warped cone). Then the Laplacian associated with \(g\) is a second order cone differential operator on \(D\). See Section 6 for further details.

A differential boundary condition for \(A\) as above is a vector

\[T = (T_0, \ldots, T_{\mu-1}), \quad T_j = \gamma_0 \circ B_j : \mathcal{E}^{\infty,\infty}(D, E) \rightarrow \mathcal{E}^{\infty,\infty}(B, F_j),\]

where each \(B_j\) is a cone differential operator of order \(j\) on \(D\) acting from sections of \(E\) to sections of some other bundle \(F_j\) and where \(\gamma_0\) denotes the operator of restriction to the boundary (and \(F_j\) is the restriction of \(F_j\) to the boundary). We allow some of the \(F_j\) to be of dimension 0; in that case the condition \(T_j\) is void. Setting \(F := F_0 \oplus \ldots \oplus F_{\mu-1}\), we will consider \(T\) as a map \(\mathcal{E}^{\infty,\infty}(D, E) \rightarrow \mathcal{E}^{\infty,\infty}(B, F)\).

Given \(A\) and \(T\) we study the operator \(A_T\), acting like \(A\) on the domain

\[\mathcal{D}(A_T) = \mathcal{E}^{\infty,\infty}(D, E)_T := \mathcal{E}^{\infty,\infty}(D, E) \cap \ker T\]

as an unbounded operator in weighted \(L^p\)-Sobolev spaces \(\mathcal{H}^{p,\gamma}_p(D, E)\); here \(s\) measures smoothness, on the cylindrical part with respect to \(x\)-\(\partial_x\)- and \(\partial_y\)-derivatives, while \(\gamma \in \mathbb{R}\) refers to a weight function which coincides with \(x^{\gamma}\) on the cylindrical part; see the appendix for details. The main objective of this article is to establish the existence of a bounded \(H_\infty\)-calculus for closed extensions \(A_T\) of \(A_T\) with domain \(\mathcal{D}(A_T) \subseteq \mathcal{H}^{p,\gamma}_p(D, E) \cap \ker T\); such extensions are also called realizations of \(A\) subject to the condition \(T\).

This problem has already been considered in [6], where it has been shown that a bounded \(H_\infty\)-calculus exists, provided the resolvent of \(A_T\) has a specific pseudodifferential structure. So far, however, only few cases were known where the resolvent is of this kind. Combining the techniques developed in [25] for conic manifolds without boundary with results of Krainer [14], we are now able to treat all realizations \(A_T\) that are parameter-elliptic in the sense of Section 5. The resolvent is then constructed with the help of a pseudodifferential calculus for boundary value problems on manifolds with edges, as presented e.g. in Kapanadze, Schulze [13].

Our methods pertain, in particular, to the Dirichlet and Neumann Laplacian as discussed in Section 6. As an application we show in Section 7 the existence of a
short time solution to the porous medium equation on the conic manifold \( \mathbb{D} \) with Neumann boundary conditions for positive data.

In the appendix of this paper we recall basic definitions of function spaces on manifolds with conic singularities and present key elements of a calculus for pseudodifferential operators on manifolds with conical singularities on the boundary which we will need in the proof of our main theorem.

2. Boundary value problems for cone differential operators

Let \( A \) and \( T \) be as in (1.1) and (1.3), respectively. Consider the boundary value problem

\[
(2.1) \quad A := \begin{pmatrix} A \\ T \end{pmatrix} : \mathcal{C}^{\infty, \infty}(\mathbb{D}, E) \rightarrow \mathcal{C}^{\infty, \infty}(\mathbb{B}, F).
\]

After a normalization of the orders of the boundary operators, \( A \) can be considered as an element of Boutet de Monvel’s algebra for boundary value problems in the sense of Schrohe, Schulze [21,22]. In this class, Shapiro-Lopatinskiĭ ellipticity is characterized by a number of (principal) symbols associated with each element. This leads us to associate with \( A \) the following symbols:

1. The principal symbol \( \sigma_\psi^\mu(A) = \sigma_\psi^\mu(A) \), an endomorphism of \( \pi_\psi^* E_0 \), where \( \pi_\psi: T^* \mathbb{D}_{\text{reg}} \setminus 0 \rightarrow \mathbb{D}_{\text{reg}} \) is the canonical projection.

2. The principal boundary symbol \( \sigma_B^\mu(A) \), a morphism

\[
\mathcal{J}(\mathbb{R}+) \otimes \pi_B^* E_{\bar{\mathbb{B}}} \rightarrow \mathcal{J}(\mathbb{R}+) \otimes \pi_B^* (E_{\bar{\mathbb{B}}} \oplus F),
\]

where \( \pi_B: T^* \mathbb{B}_{\text{reg}} \setminus 0 \rightarrow \mathbb{B}_{\text{reg}} \) is the canonical projection.

3. The conormal symbol \( \sigma_M^\mu(A) \), a polynomial on \( \mathbb{C} \), taking values in boundary problems on \( \partial Y \). Its definition is recalled below.

Both principal symbol and principal boundary symbol degenerate near \( x = 0 \). Using variables \( (x, y, \xi, \eta) \) on the cotangent bundle of the cylindrical part \( (0, 1) \times Y \), the rescaled principal symbol is

\[
(2.2) \quad \tilde{\sigma}_\psi^\mu(A)(y, \eta, \xi) = \lim_{x \to 0^+} x^\mu \sigma_\psi^\mu(A)(x, y, x^{-1} \xi, \eta).
\]

Similarly, with \( (x, y', \xi, \eta') \) in the cotangent bundle of \( (0, 1) \times \partial Y \), the rescaled principal boundary symbol is

\[
(2.3) \quad \tilde{\sigma}_B^\mu(A)(y', \xi, \eta') = \lim_{x \to 0^+} \left( x^\mu \sigma_B^\mu(A)(x, y', x^{-1} \xi, \eta') \right),
\]

where \( S(x) = \text{diag}(1, x, \ldots, x^{\mu-1}) \). The rescaled principal symbol is an endomorphism of \( \pi^* E_0 \), while the rescaled principal boundary symbol is a morphism

\[
\mathcal{J}(\mathbb{R}+) \otimes \pi_B^* E_{\bar{\mathbb{B}}}|_{\partial Y} \rightarrow \mathcal{J}(\mathbb{R}+) \otimes \pi_B^* (E_{\bar{\mathbb{B}}} \oplus F_0).
\]

The conormal symbol of \( A \) is the polynomial

\[
(2.4) \quad \sigma_M^\mu(A)(z) = \sum_{j=0}^{\mu} a_j(0) z^j : \mathbb{C} \rightarrow \text{Diff}^\mu(Y, E_0)
\]

with \( a_j \) as in (1.1), taking values in the differential operators of order \( \mu - j \) on the cross-section \( Y \). For the boundary condition \( T \)

\[
(2.5) \quad \sigma_M^\mu(T)(z) = \left( \gamma_0 \sigma_M^0(B_0)(z), \ldots, \gamma_0 \sigma_M^{\mu-1}(B_{\mu-1})(z) \right),
\]
where $\gamma_0$ denotes the operator of restriction to the boundary of $Y$. In particular, the conormal symbol $\sigma^\mu_M(\mathcal{A}) = (\sigma_{s}^{\mu}(\mathcal{A}))$ furnishes a map

\begin{equation}
\sigma^\mu_M(\mathcal{A})(z) : H^s_p(Y, E_0) \to \bigoplus_{j=0}^{n-1} H^{s-j-\mu/p}(\partial Y, F_0), \quad z \in \mathbb{C},
\end{equation}

for every $1 < p < +\infty$ and $s > \mu - 1 + 1/p$ (with a slight abuse of notation, the trace spaces on the right are the usual Besov spaces with indices $p = q$).

**Definition 2.1.** The boundary value problem $\mathcal{A}$ is called $\mathbb{D}$-elliptic, if each of the four symbols $\sigma^\mu_0(\mathcal{A})$, $\sigma^\mu_0(\mathcal{A})$, $\sigma^\mu_0(\mathcal{A})$, and $\sigma^\mu_0(\mathcal{A})$ is invertible outside the zero-section. It is called elliptic with respect to the weight $\gamma \in \mathbb{R}$, if additionally the conormal symbol $\sigma^\mu_M(\mathcal{A})$ is invertible for all $z \in \mathbb{C}$ with $\text{Re} \, z = \frac{\pi}{2} - \gamma$.

It can be shown, cf. \cite[Theorem 4.1.6]{22}, that the conormal symbol of a $\mathbb{D}$-elliptic boundary value problem is meromorphically invertible in the complex plane, independently of the choice of $s$ and $p$, due to spectral invariance in Boutet de Monvel’s algebra of boundary value problems. Ellipticity with respect to a weight $\gamma$ is thus just the requirement that none of the poles has real part equal to $\frac{\pi}{2} - \gamma$.

**2.1. $\mathbb{D}$-ellipticity with parameter.** Of crucial importance for this paper will be the notion of ellipticity with respect to a parameter in a sector. For $\theta > 0$ define

\begin{equation}
\Lambda = \Lambda_\theta = \{re^{i\varphi} \in \mathbb{C} \mid r > 0, \quad \theta \leq \varphi \leq 2\pi - \theta\}.
\end{equation}

**Definition 2.2.** The boundary value problem $\mathcal{A}$ is called $\mathbb{D}$-elliptic with parameter in $\Lambda$, if $\lambda - \sigma^\mu_0(\mathcal{A})$, $\lambda - \sigma^\mu_0(\mathcal{A})$, $\lambda - \sigma^\mu_0(\mathcal{A})$, and $\lambda - \sigma^\mu_0(\mathcal{A})$ are invertible for every $\lambda \in \Lambda$.

Ellipticity with parameter implies that $T$ is a normal boundary condition in the sense of Grubb \cite[Definition 1.4.3]{11} and \cite[Definition 3.6]{5}; the argument is the same as in the smooth case, see \cite[Lemma 1.5.7]{11}. In particular

\begin{equation}
T : \mathcal{H}^{s,\gamma}_p(\mathbb{D}, E) \to \bigoplus_{j=0}^{n-1} \mathcal{H}^{s-j-\mu/p,\gamma-j-1/2}_p(\mathbb{D}, F_j)
\end{equation}

is surjective for every choice of $\gamma \in \mathbb{R}$, $1 < p < +\infty$ and $s > \mu - 1 + 1/p$. There exists a right-inverse in Boutet de Monvel’s calculus, as constructed for example in Lemma 3.4 and Proposition 3.7 of \cite{5}. The trace spaces on the right are weighted Besov spaces with parameters $p = q$. Though some of the discussion below remains valid for $\mathbb{D}$-elliptic boundary value problems with normal boundary condition, we shall make the following assumption which stands throughout the whole paper:

**Assumption:** The boundary value problem $\mathcal{A} = (\mathcal{A}_p)$ is $\mathbb{D}$-elliptic with parameter in $\Lambda$.

### 3. Domains and realizations

In this section let $\gamma \in \mathbb{R}$ and $1 < p < +\infty$ be fixed.

**3.1. Closed extensions of the full boundary value problem.** Let us now consider $\mathcal{A}$ as an unbounded operator

\begin{equation}
\mathcal{A} : \mathcal{H}^{0,\gamma}_p(\mathbb{D}, E) \to \mathcal{H}^{0,\gamma}_p(\mathbb{D}, E)
\end{equation}

(3.1)
We let \( \mathcal{D}_{\min}(\mathcal{A}) \) denote the domain of the closure, while \( \mathcal{D}_{\max}(\mathcal{A}) \) consists of all elements \( u \in H^p_\gamma(\mathbb{D}, E) \) such that \( Au \) belongs to the space on the right-hand side of \([20,1]\). Note that in the definition of \( \mathcal{D}_{\max}(\mathcal{A}) \) the a-priori regularity \( \mu \) is required. Obviously it would be more precise to use notations like \( \mathcal{D}^{\mu}_{\max}(\mathcal{A}) \) to indicate the dependence on \( \gamma \) and \( p \). However, to keep notation more lean we shall not do so.

The first statement, below, is shown in \([14, \text{Lemma 4.10}]\), the second follows from \([5, \text{Proposition 4.3}]\).

\[ Y \equiv \text{There exists a finite-dimensional space} \]

\[ D \]

function on \( E \)

\[ \gamma \]

dependence on \( \omega \)

\[ A \]

If

\[ Y \equiv \text{Obviously it would be more precise to use notations like} \]

\[ \mathcal{D}^{\mu}_{\max}(\mathcal{A}) \]

of \( (3.1) \). Note that in the definition of \( \mathcal{D}^{\mu}_{\max}(\mathcal{A}) \) the a-priori regularity \( \mu \) is required.

\[ (3.2) \]

\[ (3.3) \]

and define recursively the following functions:

\[ g_0 = 1, \quad g_\ell = -(T^{-\ell}f_0^{-1}) \sum_{j=0}^{\ell-1} (T^{-j}f_{\ell-j})g_j, \quad \ell \in \mathbb{N}, \]

\[ (3.4) \]

\[ (\gamma_0 \sigma_{M}^{\mu-k}(B_0)(z), \ldots, \gamma_0 \sigma_{M}^{\mu-1-k}(B_{\mu-1})(z)) \]

\[ (3.5) \]
where the shift-operator \( T^\rho, \rho \in \mathbb{R}, \) acts on meromorphic functions by \( (T^\rho h)(z) = h(z + \rho). \) Note that the \( g_\ell \) are meromorphic with values in Boutet de Monvel’s algebra on \( Y \) and that the recursion is equivalent to

\[
\sum_{\ell=0}^n (T^{-\ell} f_{\ell-\gamma}) g_\ell = \begin{cases} f_0 & : \ell = 0 \\ 0 & : \ell \geq 1 \end{cases}.
\]

In the following we let

\[
S_\gamma = \{ \sigma \in \mathbb{C} \mid \sigma \text{ is a pole of } f_0^{-1} \text{ and } \frac{n+1}{2} - \gamma - \mu < \text{Re } \sigma < \frac{n+1}{2} - \gamma \}
\]

and we use the Mellin transform

\[
(Mu)(z) = \hat{u}(z) = \int_0^\infty x^z u(x) \frac{dx}{x}.
\]

**Theorem 3.3.** For \( \sigma \in S_\gamma \) and \( \ell \in \mathbb{N}_0 \) define

\[
G_{\sigma}^{(\ell)} : \mathcal{S}^\infty(\partial Y^\wedge, \partial F^\wedge) \oplus \mathcal{S}^\infty(Y^\wedge, E^\wedge) \rightarrow \mathcal{C}^\infty(Y^\wedge, E^\wedge)
\]

by

\[
(G_{\sigma}^{(\ell)} u)(x) = x^\ell \int_{|z| = \varepsilon} x^{-z} g_\ell(z) \Pi_\sigma(f_0^{-1} \hat{u})(z) dz,
\]

where \( \Pi_\sigma h \) denotes the principal part of the Laurent series of a meromorphic function \( h \) in \( \sigma \) \((\Pi_\sigma h = 0, \text{ if } h \text{ is holomorphic in } \sigma), dz = \frac{dz}{2\pi i}, \) and \( \varepsilon > 0 \) is so small that none of the \( z \) with \( 0 < |z| < \varepsilon \) is a pole of the integrand. Moreover, let

\[
G_{\sigma} := \sum_{\ell=0}^{\mu_{\sigma}} G_{\sigma}^{(\ell)}, \quad \mu_{\sigma} := \left\lfloor \text{Re } \sigma + \mu + \gamma - \frac{n+1}{2} \right\rfloor.
\]

where \( \lfloor r \rfloor \) denotes the integer part of \( r \in \mathbb{R}. \) Then

\[
\mathcal{F} = \bigoplus_{\sigma \in S_\gamma} \mathcal{F}_\sigma, \quad \mathcal{F}_\sigma := \text{range } G_{\sigma}.
\]

The proof is a straightforward adaption of the proof of [25, Section 3].

### 3.2. Realizations subject to a boundary condition.

We shall next determine the closed extensions in \( \mathcal{H}_p^0,\gamma(\mathbb{D}) \) of the operator \( A_T \) introduced in (1.4). Let \( \mathcal{D}_{\min}(A_T) \) be the domain of the closure of \( A_T \) and define the maximal extension of \( A_T \) by the action of \( A \) on

\[
\mathcal{D}_{\max}(A_T) = \{ u \in \mathcal{H}_p^0,\gamma(\mathbb{D}, E) \mid Au \in \mathcal{H}_p^0,\gamma(\mathbb{D}, E), \ T u = 0 \}.
\]

There is a natural relation between the closed extensions of \( A \) and those of \( A_T: \)

**Lemma 3.4** (\cite{14} Lemma 4.10). We have

\[
\mathcal{D}_{\max}(A_T) / \mathcal{D}_{\min}(A) \cong \mathcal{D}_{\max}(A_T) / \mathcal{D}_{\min}(A_T)
\]

and the map

\[
V \mapsto V_T := V \cap \ker T
\]

is a bijection between the lattice of intermediate spaces \( \mathcal{D}_{\min}(A) \subseteq V \subseteq \mathcal{D}_{\max}(A) \) and the lattice of intermediate spaces \( \mathcal{D}_{\min}(A_T) \subseteq V_T \subseteq \mathcal{D}_{\max}(A_T). \)
More precisely, let $R$ be a right-inverse of $T$ as in (2.8) with $\gamma$ replaced by $\gamma + \mu$. Then
\begin{equation}
(3.11) \quad u + D_{\min}(\mathcal{A}) \mapsto (u - RTu) + D_{\min}(\mathcal{A}_T)
\end{equation}
provides an isomorphism (3.10). Note that $1 - RT$ is a projection onto the kernel of $T$, since $T(1 - RT) = 0$. In particular, if $\mathcal{F}$ is the space from Theorem 3.2 then
\begin{equation}
(3.12) \quad D_{\max}(\mathcal{A}_T) = D_{\min}(\mathcal{A}_T) \oplus \mathcal{E}, \quad \mathcal{E} := \{\omega u - RT(\omega u) \mid u \in \mathcal{F}\},
\end{equation}
provides a (non-canonical) description of the maximal domain. We obtain:

**Theorem 3.5.** Any extension $\mathcal{A}_T$ of $\mathcal{A}_T$ with $\mathcal{A}_T \subseteq \mathcal{A}_{T,\max}$ corresponds to a choice of a subspace of $\mathcal{E}$, i.e. has a domain of the form
\begin{equation}
(3.13) \quad D(\mathcal{A}_T) = D_{\min}(\mathcal{A}_T) \oplus \mathcal{E}, \quad \mathcal{E} \text{ subspace of } \mathcal{E}.
\end{equation}

By (3.12), $\mathcal{E} \subset H^\infty_p(\gamma, \mu, \gamma + \mu)$ for some $\varepsilon > 0$. Moreover, since $T(\omega u)$ vanishes for small $x$, $RT(\omega u)$ vanishes for small $x$, $RT(\omega u)$ vanishes for small $x$. In particular,
\[ \mathcal{E} + H^\infty_p(\gamma, \mu, \gamma + \mu) = \omega \mathcal{F} + H^\infty_p(\gamma, \mu, \gamma + \mu). \]

If $\mathcal{A}$ is elliptic with respect to the weight $\gamma + \mu$, then, by [5 Proposition 4.3],
\[ D_{\min}(\mathcal{A}_T) = H^\infty_p(\gamma, \mu, \gamma + \mu) | \mathcal{F} = H^\infty_p(\gamma, \mu, \gamma + \mu) \cap \ker T. \]

### 3.3. Operators on the model cone.

If $A$ is as in (3.11) we define the differential operator
\[ \hat{A} = x^{-\mu} \sum_{j=0}^{\mu} a_j(0)(-x\partial_x)^j \]
by freezing the coefficients of $A$ in $x = 0$. Proceeding analogously with the operators $B_k$, we obtain the boundary condition $\hat{T} = (\gamma_0 B_{0}, \ldots, \gamma_0 B_{-1})$. Together they define the so-called model boundary value problem $\hat{\mathcal{A}} = (\hat{T})$ on the infinite cylinder $Y^\wedge = (0, +\infty) \times Y$. Similarly as above in Section 3.1 we shall consider $\hat{\mathcal{A}}$ as an unbounded operator
\begin{equation}
(3.14) \quad \hat{\mathcal{A}} : \mathcal{D}(Y^\wedge, E^\wedge) \subset K^0_p(\gamma, \mu, \gamma + \mu, \gamma + \mu - \varepsilon) \longrightarrow K_{p-1}^{\mu_k-1/p, \gamma + \mu - k - 1/2}(\partial Y^\wedge, F^\wedge) \oplus K^p_{\mu_k} \end{equation}
again, the trace spaces on the right-hand side are Besov spaces with $p = q$.

The discussion of the corresponding closed extensions is parallel to that above. The domain of the closure $\hat{D}_{\min}(\hat{\mathcal{A}})$ and the maximal domain $\hat{D}_{\max}(\hat{\mathcal{A}})$ are contained in $K^0_p(\gamma, \mu, \gamma + \mu, \gamma + \mu - \varepsilon)$ and
\[ \hat{D}_{\min}(\hat{\mathcal{A}}) = \hat{D}_{\max}(\hat{\mathcal{A}}) \cap \bigcap_{\varepsilon > 0} K^p_{\mu_k} \gamma + \mu - \varepsilon(\gamma, \mu, \gamma + \mu, \gamma + \mu - \varepsilon). \]

If $\mathcal{A}$ is elliptic with respect to the weight $\gamma + \mu$, then
\begin{equation}
(3.15) \quad \hat{D}_{\min}(\hat{\mathcal{A}}) = K^\gamma(\gamma, \mu, \gamma + \mu, \gamma + \mu - \varepsilon) \end{equation}

The following theorem combines the results of Krainer in [14 Section 6] with the above representation.

**Theorem 3.6.** Let $S_\gamma$ be as in (3.7) and $G_\gamma^{(f)}$, $G_\sigma$ be as in (3.8) and (3.9), respectively. Then
\[ \hat{D}_{\max}(\hat{\mathcal{A}}) = \hat{D}_{\min}(\hat{\mathcal{A}}) \oplus \omega \hat{\mathcal{F}}, \]
where
\[ \hat{\mathcal{F}} = \bigoplus_{\sigma \in S_\sigma} \hat{\mathcal{F}}_\sigma, \quad \hat{\mathcal{F}}_\sigma := \text{range } G_\sigma^{(f)} \]
The mappings

$$\theta_\sigma : \mathcal{F}_\sigma \to \mathcal{F}_\sigma, \quad \mathcal{G}_\sigma u \mapsto \mathcal{G}_\sigma^{(0)} u$$

are well-defined isomorphisms, hence induce an isomorphism $$\theta : \mathcal{F} \to \mathcal{F}$$. In particular,

$$\mathcal{D}_{\text{max}}(\mathcal{A})/\mathcal{D}_{\text{min}}(\mathcal{A}) \cong \mathcal{D}_{\text{max}}(\mathcal{A})/\mathcal{D}_{\text{min}}(\mathcal{A}).$$

In other words, the map $$\theta$$ gives rise to a bijection $$\Theta$$ between the subspaces of $$\mathcal{F}$$ and those of $$\mathcal{F}$$, i.e. an isomorphism

$$\Theta : \text{Gr}(\mathcal{F}) \to \text{Gr}(\mathcal{F})$$

between the corresponding Grassmannians. This isomorphism has been first described in [9] in the case of manifolds without boundary and in [14] in the case with boundary; the present, equivalent, construction extends that of [25] to the case with boundary. It induces a one-to-one correspondence between the closed extensions of $$\mathcal{A}$$ and those of $$\mathcal{A}$$ by

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}_{\text{min}}(\mathcal{A}) \oplus \omega \mathcal{F} \mapsto \mathcal{D}(\mathcal{A}) := \mathcal{D}_{\text{min}}(\mathcal{A}) \oplus \omega \Theta \mathcal{F}.$$  

**Remark 3.7.** By construction, $$\mathcal{F} \subseteq \ker \hat{\mathcal{A}}$$. In fact,

$$\hat{\mathcal{A}} \mathcal{G}_\sigma^{(0)} u(x) = \hat{A} \int_{|z-\sigma|=\varepsilon} x^{-\gamma} \Pi_\sigma(f_0^{-1} \bar{u})(z) dz = \int_{|z-\sigma|=\varepsilon} x^{-\gamma} f_0(z)(f_0^{-1} \bar{u})(z) dz = 0,$$

since $$\Pi_\sigma$$ can be omitted by the residue theorem. In particular, $$\mathcal{F} \subseteq \ker \hat{T}$$.

Our next topic are the realizations of $$\hat{\mathcal{A}}$$ in $$\mathcal{K}^{0,\gamma}(Y^\wedge, E^\wedge)$$ subject to the boundary condition $$\hat{T}$$, i.e., the extensions of the unbounded operator $$\hat{A}_{\hat{T}}$$ acting like $$\hat{A}$$ on the domain

$$\{u \in \mathcal{S}(Y^\wedge, E^\wedge) \mid \hat{T} u = 0\} \subseteq \mathcal{K}^{0,\gamma}(Y^\wedge, E^\wedge).$$

Let $$\mathcal{D}_{\text{min}}(\hat{A}_{\hat{T}})$$ be the domain of the closure of $$\hat{A}_{\hat{T}}$$, and define the maximal extension $$\hat{A}_{\hat{T},\text{max}}$$ by the action of $$\hat{A}$$ on

$$\mathcal{D}_{\text{max}}(\hat{A}_{\hat{T}}) = \{u \in \mathcal{K}^{0,\gamma}(Y^\wedge, E^\wedge) \mid \hat{T} u = 0\}.$$ 

Proceeding as above, using a right-inverse $$\hat{R}$$ to $$\hat{T}$$ in Boutet de Monvels calculus on the infinite cone, we find a non-canonical decomposition

$$\mathcal{D}_{\text{max}}(\hat{A}_{\hat{T}}) = \mathcal{D}_{\text{min}}(\hat{A}_{\hat{T}}) \oplus \hat{\delta}, \quad \hat{\delta} := \{\omega u - \hat{R}\hat{T}(\omega u) \mid u \in \mathcal{F}\}.$$ 

Since both the differential boundary condition $$\hat{T}$$ and the right-inverse $$\hat{R}$$ preserve rapid decay of functions at infinity, there exists an $$\varepsilon > 0$$ such that

$$\hat{\delta} \subset \mathcal{S}^{\gamma+\varepsilon}(Y^\wedge, E^\wedge).$$

**Remark 3.8.** $$\hat{R}$$ can be chosen such that it commutes with dilations, i.e. given $$u \in \mathcal{K}^{0,\gamma}(\partial Y^\wedge, E^\wedge)$$ and writing $$u_\lambda(x, y) = u(\lambda x, y)$$ for $$\lambda > 0$$ we have $$\hat{R} u_\lambda = (\hat{R} u)_\lambda$$. This follows from the fact that $$\hat{T}$$ commutes with dilations together with the construction in [5, Section 3].

**Remark 3.9.** In general, multiplication by a cut-off function $$\omega$$ will not commute with the trace operator $$\hat{T}$$. If $$\omega$$ and $$\hat{T}$$ commute, $$\hat{T}(\omega u) = 0$$ for $$u \in \mathcal{F}$$, so that $$\hat{\delta} = \omega \mathcal{F}$$. This is the case e.g. for Dirichlet and Neumann boundary conditions.
The isomorphism $\Theta$ provides a one-to-one correspondence between the closed extensions of $\mathcal{A}_T$ and $\hat{\mathcal{A}}_T$, which will be of crucial importance below:

**Definition 3.10.** Let $\hat{\mathcal{A}}_T$ be a closed extension of $\mathcal{A}_T$. According to (3.13) and (3.14) it has domain $\mathcal{D}(\hat{\mathcal{A}}_T) = \mathcal{D}_{\min}(\mathcal{A}_T) \oplus \mathcal{E}$, where

$$\mathcal{E} = \{\omega u - RT(\omega u) \mid u \in \mathcal{F}\}, \quad \mathcal{F} \subset \mathcal{F}.$$ 

Then let $\hat{\mathcal{A}}_T$ be the closed extension of $\mathcal{A}_T$ defined by the domain

$$\mathcal{D}(\hat{\mathcal{A}}_T) = \mathcal{D}_{\min}(\hat{\mathcal{A}}_T) \oplus \mathcal{E}, \quad \mathcal{E} = \{\omega u - RT(\omega u) \mid u \in \Theta \mathcal{F}\}.$$ 

The mapping $\mathcal{D}(\hat{\mathcal{A}}_T) \mapsto \mathcal{D}(\hat{\mathcal{A}}_T)$ does not depend on the choice of the right-inverses $R$ and $\hat{R}$, respectively, but only on the isomorphism $\Theta$ from (3.17).

Finally let us remark that, in case of ellipticity with respect to the weight $\gamma + \mu$,

(3.22) $\mathcal{D}_{\min}(\hat{\mathcal{A}}_T) = K_{p,\gamma+\mu}^{\mu,\gamma}(Y^\wedge, E^\wedge)_{\hat{T}} := K_{p,\gamma+\mu}^{\mu,\gamma}(Y^\wedge, E^\wedge) \cap \ker \hat{T}$.

### 3.4. Invertibility and Fredholm property.

**Proposition 3.11.** Consider the operator

$$\hat{\mathcal{A}}_T : \mathcal{H}^{\mu,\gamma+\mu}_p(D, E)_T \oplus \mathcal{E} \longrightarrow \mathcal{H}^{\mu,\gamma}_p(D, E).$$

If this is a Fredholm operator for some choice $s = s_0$, $p = p_0$ with $\mu \leq s_0 \in \mathbb{N}_0$, $1 < p_0 < +\infty$, then it is a Fredholm operator for all $1 < p < +\infty$, $s \in \mathbb{Z}$, $s > \mu - 1 + 1/p$. Also the index then is independent of $s$ and $p$. Similarly, if it is invertible for $s_0$, $p_0$, then it is invertible for all $1 < p < +\infty$, $s > \mu - 1 + 1/p$.

**Proof.** Since $T$ is surjective and $\mathcal{E}$ is finite-dimensional, [5] Theorem 8.3] implies that $\hat{\mathcal{A}}_T$ is a Fredholm operator if and only if

$$\mathcal{H}^{\mu,\gamma}_p(D, E) \oplus \mathcal{E} \longrightarrow \mathcal{H}^{\mu,\gamma+\mu}_p(D, E)_T$$

is a Fredholm operator; in that case, their indices differ by the dimension of $\mathcal{E}$. By Corollary 50 in [15] the Fredholm property of $\mathcal{A}$ implies that it is after normalization of the orders of the boundary operators — an elliptic cone pseudodifferential operator in the sense of [19]. Hence it has a parametrix and therefore is a Fredholm operator for all other choices of $s$ and $p$. Moreover, the index is independent of $s$ and $p$ by [13] Corollary 50].

Next suppose $\hat{\mathcal{A}}_T$ is invertible for some fixed choice $s_0, p_0$. It follows from the first part of the proof that $\hat{\mathcal{A}}_T$ is a Fredholm operator of index zero for the other values of $s$ and $p$. Hence it will suffice to establish the injectivity of $\hat{\mathcal{A}}_T$. Suppose $\mathcal{A}(u + e) = 0$ for some $u \in \mathcal{H}^{\mu,\gamma+\mu}_p(D)_T$ and $e \in \mathcal{E}$. Then $Au = -Ae \in \mathcal{G}^{\infty,\gamma+c}(\mathbb{D})$ for some $c > 0$. By elliptic regularity in the cone algebra we conclude that $u \in \mathcal{G}^\infty,\mu+\gamma+c(\mathbb{D})$ for some $c' > 0$. This shows that the kernel of $\hat{\mathcal{A}}_T$ does not depend on $s$ and $p$. Hence $\mathcal{A}_T$ is invertible for all $1 < p < +\infty, s > \mu - 1 + 1/p$. \hfill $\square$

In the above proof, our standing assumption of $\mathbb{D}$-ellipticity of $\mathcal{A}$ with respect to $\Lambda$ was not needed. A result similar to Proposition 3.11 holds for model cone operators.

**Proposition 3.12.** Suppose that $0 \neq \lambda \in \Lambda$ and that, in addition to the $\mathbb{D}$-ellipticity of $\mathcal{A}$, the conormal symbol of $\mathcal{A}$ is invertible on the line $\text{Re} \, z = \frac{\text{dim} \mathbb{D}}{2} - \gamma$. If

(3.23) $\lambda - \hat{\mathcal{A}}_T : \mathcal{K}^{\mu,\gamma+\mu}_p(Y^\wedge, E^\wedge)_T \oplus \mathcal{E} \longrightarrow \mathcal{K}^{\mu,\gamma}_p(Y^\wedge, E^\wedge)$

is invertible for some choice of $s$ and $p, s \in \mathbb{Z}, s > \mu - 1 + 1/p, 1 < p < +\infty$, then it is invertible for all other choices.
Proof. The $D$-ellipticity together with the fact that $\lambda \neq 0$ implies that
\[
\left( \lambda - \hat{A}_T \right) : K_p^{s+\mu,\gamma+\mu}(Y^\wedge, E^\wedge) \rightarrow K_p^{s,\gamma}(Y^\wedge, E^\wedge)
\]
is a Fredholm operator for all above choices of $p$ and $s$: This follows from Theorem 6.2.19 in [12], since, after normalization of the orders of the boundary operators, $(\lambda - \hat{A}_T)$ is an elliptic element in the cone algebra on the infinite cone.

Since $\hat{T}$ is surjective, we obtain the Fredholm property for
\[
\lambda - \hat{A}_T : K_p^{s+\mu,\gamma+\mu}(Y^\wedge, E^\wedge) \rightarrow K_p^{s,\gamma}(Y^\wedge, E^\wedge)
\]
and thus for the extension in (4.1). The kernel of this extension is actually independent of $s$ and $p$: Suppose $(\lambda - \hat{A}_T)(u + e) = 0$ for some $u \in K_p^{s+\mu,\gamma+\mu}(Y^\wedge, E^\wedge)_T$ and $e \in \tilde{E}$. Then $(\lambda - \hat{A})u = (\lambda - \hat{A})e \in \mathscr{S}^{\gamma_1+\epsilon}(Y^\wedge, E^\wedge)$ for sufficiently small $\epsilon > 0$, so that, by elliptic regularity, $u \in \mathscr{S}^{\gamma_1+\epsilon}(Y^\wedge, E^\wedge)$ for some $\epsilon > 0$, which is a common subset of all domains, independent of $s$ and $p$.

Furthermore, the invertibility of the conormal symbol on $\text{Re } z = \frac{n+1}{2} - \gamma$ implies that the formal adjoint $\hat{A}_T$ of $\hat{A}$ with the adjoint boundary condition $\hat{T}'$ is also $D$-elliptic according to [5, Corollary 7.3]. An analog of [5, Theorem 4.6] shows that the adjoint of $\lambda - \hat{A}_T$ is $\lambda - \hat{A}'_T$, acting on a domain of the form $K_p^{s+\mu,\gamma+\mu}(Y^\wedge, E^\wedge)_T \oplus \tilde{E}'$ for a suitable $\tilde{E}'$.

Hence the index of $\lambda - \hat{A}_T$ is always zero, as this is the case for the choice of $s$ and $p$, where it is invertible. Moreover, since the kernel dimension is also constant, it must be zero. Therefore $\lambda - \hat{A}_T$ is invertible for all $s$ and $p$. $\square$

4. Parameter-dependent Green operators

Green symbols are parameter-dependent families of integral operators on the model cone with smooth kernels that depend in a specific way on the covariant $\eta$. We will show that they can be characterized by their mapping properties, a result that will be needed in the proof of Theorem 5.13. In the sequel, $[\cdot]$ denotes a smooth function on $\mathbb{R}^n$ with $[\cdot] \equiv 1$ near zero and $[\cdot] = |\cdot|$ for $|\cdot| \geq 1$.

**Definition 4.1.** Let $\nu, \gamma_0, \gamma_1 \in \mathbb{R}$. Then $\mathcal{R}_{\nu}^0(Y^\wedge, \Sigma; \gamma_0, \gamma_1)$ consists of all operator families $a(\eta)$ of the form
\[
(a(\eta)u)(x, y) = [\eta]^{n+1} \int_0^{\infty} \int_Y k_u(\eta, x[y], y, s[\eta], t)u(s, t)s^\nu ds dt
\]
with integral kernel satisfying, for some $\epsilon > 0$,
\[
(4.1) \quad k_u(\eta, x, y, s, t) \in S^\nu(\Sigma, \mathscr{S}^{\gamma_1+\epsilon}(Y^\wedge, E^\wedge) \otimes_{\Sigma} \mathscr{S}^{-\gamma_0-\epsilon}(Y^\wedge, E^\wedge)).
\]

For better readability, we do not mention the vector bundle $E^\wedge$ in the notation. In [11, 15], $\otimes_{\Sigma}$ denotes the completed projective tensor product of Fréchet spaces. For a Fréchet space $F$, $S^\nu(\Sigma, F)$ denotes the space of $F$-valued symbols of order $\nu$ on $\Sigma$, i.e. the smooth functions $a : \Sigma \rightarrow F$ such that, for every multi-index $\alpha$ and every continuous semi-norm $q$ on $F$, there exists a constant $C_{\alpha, q}$ with
\[
q(D^\alpha a(\eta)) \leq C_{\alpha, q}[\eta]^{-|\alpha|}, \quad \eta \in \Sigma.
\]

While $\nu$ in $\mathcal{R}_{\nu}^0$ has the interpretation of the order of symbols, the parameter 0 refers to the class or type of singular Green operators in Boutet de Monvel’s algebra.
Green symbols behave naturally under composition: If $a_j \in R_G^{\nu_j,0}(Y^\land, \Sigma; \gamma_j, \gamma_{j+1})$, $j = 0,1$, then $a_1 a_0 \in R_G^{\nu_0 + \nu_1,0}(Y^\land, \Sigma; \gamma_0, \gamma_2)$.

If $\phi \in C_c^\infty(Y^\land)$ and $a \in R_G^{\nu,0}(Y^\land, \Sigma; \gamma_0, \gamma_1)$, then both $a \phi$ and $\phi a$ belong to $R_G^{\nu,0}(Y^\land, \Sigma; \gamma_0, \gamma_1)$, i.e., are rapidly decreasing in the parameter.

For further details on Green symbols see also Schrohe, Schulze [23].

**Definition 4.2.** The space $C_G^{\nu,0}(\mathbb{D}, \Sigma; \gamma_0, \gamma_1)$ of parameter-dependent Green operators of order $\nu$ and class zero on $\mathbb{D}$ consists of all operator families $g$ of the form

$$g(\eta) = \omega_1 a(\eta) \omega_0 + r(\eta),$$

where $\omega_0, \omega_1$ are cut-off functions, $a \in R_G^{\nu}(\Sigma; \gamma_0, \gamma_1)$, and $r$ has an integral kernel which, for some $\varepsilon = \varepsilon(g) > 0$, belongs to

$$\mathcal{S}(\Sigma, C^{\infty,\gamma_1+\varepsilon}(\mathbb{D}, E) \widehat{*}_\pi C^{\infty,-\gamma_0+\varepsilon}(\mathbb{D}, E)).$$

In the representation of $g$ above, the cut-off functions can be changed at the cost of substituting $r$ by another element of the same structure. Composition behaves as above, i.e., if $g_j \in C_G^{\nu_j,0}(\mathbb{D}, \Sigma; \gamma_j, \gamma_{j+1})$, $j = 0,1$, then $g_j g_0 \in C_G^{\nu_j,0}(\mathbb{D}, \Sigma; \gamma_0, \gamma_2)$.

For later reference let us also note the following:

**Remark 4.3.** If $g \in C_G^{\nu,0}(\mathbb{D}, \Sigma; \gamma_0, \gamma_1)$ and $\omega$ is a cut-off function, then $\omega g \omega \in R_G^{\nu,0}(Y^\land, \Sigma; \gamma_0, \gamma_1)$. Moreover, $(1 - \omega)g$ and $g(1 - \omega)$ belong to $C_G^{\nu,0}(\mathbb{D}, \Sigma; \gamma_0, \gamma_1)$ and therefore have an integral kernel [22].

We now come to the characterization of Green symbols in terms of mapping properties and symbol estimates rather than through the structure of their integral kernels. To this end we briefly recall the concept of operator-valued pseudodifferential symbols in spaces with group action.

A group action on a Banach space $X$ is a strongly continuous map $\kappa : \mathbb{R}_+ \to \mathcal{L}(X)$ with $\kappa_0 \kappa_\lambda = \kappa_{\lambda \rho}$ for every $\lambda, \rho > 0$ and $\kappa_1 = 1$. All function and distribution spaces over $Y^\land$ appearing in this paper will have the same group action, defined by

$$\kappa_\lambda u(x, y) = \lambda^{(n+1)/2} u(\lambda x, y);$$

the factor $\lambda^{(n+1)/2}$ makes this a group of unitary operators in $K_2^{\nu,0}(Y^\land, E^\land)$, which is the $L^2$ space with respect to the cone-degenerate metric $dx^2 + x^2 h(0)$.

Given two Banach spaces $X^0, X^1$ with respective group actions $\kappa^0, \kappa^1$, we denote by $S^\nu(\Sigma; X^0, X^1)$ the space of all smooth $a : \Sigma \to \mathcal{L}(X^0, X^1)$ such that

$$\|\kappa_{i/|\eta|}^0 D_\eta^\alpha a(\eta)\kappa_{|\eta|}^0\|_{\mathcal{L}(X^0, X^1)} \leq C_\alpha[|\eta|^{-\alpha}], \quad \eta \in \Sigma,$$

with some constants $C_\alpha$. If $X^1$ is the projective limit of Banach spaces $X^1_0 \supset X^1_1 \supset X^1_2 \supset \ldots$ and the restriction of the group action in $X^1_0$ yields a group action of the other spaces, we set

$$S^\nu(\Sigma; X^0, X^1) = \bigcap_{j \in \mathbb{N}} S^\nu(\Sigma; X^0, X^1_j).$$

In both cases, $S^\nu(\Sigma; X^0, X^1)$ is a Fréchet space in a natural way.

A smooth function $a : \Sigma \setminus \{0\} \to \mathcal{L}(X^0, X^1)$ is called homogeneous of degree $\nu$, if

$$\kappa_{1/|\eta|}^1 a(\rho|\eta|)\kappa_{|\eta|}^1 = \rho^\nu a(\eta), \quad \eta \neq 0, \quad \rho > 0.$$

Similarly, $a \in C_c^\infty(\Sigma, \mathcal{L}(X^0, X^1))$ is called homogeneous of degree $\nu$ for large $|\eta|$, if there exists an $R > 0$ such that the above relation holds for $|\eta| \geq R$ and $\rho \geq 1$.

In this case $a$ belongs to $S^\nu(\Sigma; X^0, X^1)$. 
Proposition 4.4. The following two properties are equivalent:

1. \( a \in \mathcal{R}_C^{0} (Y^\wedge, \Sigma; \gamma_0, \gamma_1) \)
2. There exists an \( \varepsilon > 0 \) such that \( a \in S^\varepsilon (\Sigma; K_2^{0, \gamma_0} (Y^\wedge, E^\wedge)), \mathcal{F}^{\gamma_1, +\varepsilon} (Y^\wedge, E^\wedge) \) and \( a^* \in S^\varepsilon (\Sigma; K_2^{0, -\gamma_1} (Y^\wedge, E^\wedge)), \mathcal{F}^{-\gamma_0, +\varepsilon} (Y^\wedge, E^\wedge) \).

In (2), the pointwise adjoint refers to the pairings induced by the inner product of \( K_2^{0, 0} (Y^\wedge, E^\wedge) \). The group action is given by \( \mathcal{F}^{\varepsilon} \).

The proof is analogous to that of [25, Proposition 4.6] for the case without boundary.

This class differs from the class \( \mathcal{R}_G^{0, 0} (Y^\wedge, \Sigma; \gamma_0, \gamma_1) \) introduced in Section 8.2 where we require the symbols to be classical. The same applies to \( C_G^{0, 0} (\mathbb{D}, \Sigma; \gamma_0, \gamma_1) \) in Definition 4.2 and \( C_G^{0, 0} (\mathbb{D}, \Sigma; \gamma_0, \gamma_1) \) in Definition 8.6.

5. Bounded \( H_{\infty} \)-calculus for parameter-elliptic realizations

5.1. Assumptions. In the sequel we fix \( \gamma \in \mathbb{R}, 1 < p < +\infty, \) and a function \( \tilde{x} : \mathbb{D} \to \mathbb{R}, \) equal to \( x \) on \([0, 1) \times Y\) and strictly positive otherwise. We assume that \( A = (\frac{\partial}{\partial t}) \) satisfies the following conditions. As before \( \Lambda \) denotes the sector \( (2.7) \).

1. \( A \) is \( \mathbb{D} \)-elliptic with parameter in the sense of Definition 2.2

2. There exists an \( \varepsilon > 0 \) such that the conormal symbol \( \sigma^\varepsilon (\frac{\Lambda}{\Lambda}) \) of \( A \) is invertible whenever \( \mathrm{Re} z = \frac{n+1}{2} - \mu \) or \( \mathrm{Re} z = \frac{n+1}{2} - \gamma + \mu \).

We now consider a closed extension \( \mathcal{A} \mathcal{A}_T \) of \( A \) with domain

\[ \mathcal{D}(\mathcal{A}_T) = \mathcal{H}^{\mu, \gamma + \mu} (\mathbb{D}, E)_T \oplus E. \]

In order to simplify the analysis, we conjugate by \( \tilde{x}^{\gamma} \) and work with the weight \( \gamma = 0 \), i.e. we define the operator \( A_0 = \tilde{x}^{-\gamma} A \tilde{x}^{\gamma} \) and the boundary condition \( T_0 = \tilde{x}^{-\gamma} T \tilde{x}^{\gamma} \) and study the closed extension \( \mathcal{A}_0 \mathcal{A}_T \) of \( A_0 \) with domain

\[ \mathcal{D}(\mathcal{A}_0 T_0) = \mathcal{H}^{\mu, \gamma + \mu} (\mathbb{D}, E)_{T_0} \oplus \tilde{x}^{-\gamma} E. \]

The above conditions (E1) and (E2) then take the form

1. \( A_0 \) is \( \mathbb{D} \)-elliptic with parameter in the sense of Definition 2.2

2. The conormal symbol \( \sigma^\varepsilon (\frac{\Lambda}{\Lambda}) \) of \( A_0 \) is invertible whenever \( \mathrm{Re} z = \frac{n+1}{2} - \mu \) or \( \mathrm{Re} z = \frac{n+1}{2} - \gamma + \mu \).

We shall also consider the corresponding extension \( \mathcal{A}_0 \mathcal{A}_T \) of the model cone operator for \( p = 2 \), i.e.

\[ \mathcal{D}(\mathcal{A}_0 \mathcal{A}_T) = K^{\mu, \gamma + \mu} (Y^\wedge, E)_{\mathcal{T}} \oplus x^{-\gamma} E. \]

We require that

1. There exist \( C, R > 0 \) such that \( (\lambda - \mathcal{A}_0 \mathcal{T}) \) is invertible for \( \lambda \in \Lambda, |\lambda| \geq R, \)

\[ \| \lambda (\lambda - \mathcal{A}_0 \mathcal{T})^{-1} \|_{L(K_2^{0, 0} (Y^\wedge, E^\wedge))} \leq C. \]

We state the condition in (E3) with \( p = 2 \), because we will work with Hilbert space adjoints in the proof. Also the fact that we require the invertibility of the conormal symbol for \( \mathrm{Re} z = \frac{n+1}{2} \) is a consequence of this technique.
5.2. Special boundary conditions. When working with a concrete realization \( A_T \) of a boundary value problem, it can be inconvenient to make the transition to \( A_0, \tau_0 \). In case multiplication with a cut-off function \( \omega \) commutes with the boundary operator \( T \) as it is the case e.g. for Dirichlet and Neumann boundary conditions, this can be avoided. In order show this, we shall use a variant of the \( \mathcal{K}_{p,\gamma} \)-spaces with weights at infinity: For \( \rho \in \mathbb{R} \) let

\[
\mathcal{K}_p^\rho(\mathcal{Y}, \mathcal{E}) = [x]^{-\rho} \mathcal{K}_p^\rho(\mathcal{Y}, \mathcal{E}).
\]

**Proposition 5.1.** Suppose the boundary condition \( \hat{T} \) commutes with cut-off functions \( \omega \). Then

\[
(5.1) \quad \lambda - \hat{A}_T : \mathcal{K}_p^{\mu,\gamma+\mu}(\mathcal{Y}, \mathcal{E})^\rho \oplus \mathbb{C} \longrightarrow \mathcal{K}_p^{\mu,\gamma}(\mathcal{Y}, \mathcal{E})^\rho
\]

is invertible for any weight \( \rho \) and \( \lambda \in \Lambda \), \( |\lambda| \) sufficiently large, if and only it is invertible for \( \rho = 0 \) and \( \lambda \in \Lambda \) sufficiently large. Moreover, if the operator norm is \( O(\lambda^{-1}) \) for some fixed \( \rho \), then this is the case for all \( \rho \).

**Proof.** Suppose the operator \( \lambda - \hat{A}_T \) in (5.1) is invertible. By assumption, the boundary condition \( T \) is normal, hence so is \( \hat{T} \). Since normality is invariant under conjugation by \([x]^{\rho}\) and implies surjectivity, we obtain the surjectivity of

\[
\hat{T} : \mathcal{K}_2^{\mu,\gamma+\mu}(\mathcal{Y}, \mathcal{E})^\rho \longrightarrow \mathcal{K}_2^{\mu,\gamma}(\mathcal{Y}, \mathcal{E})^\rho
\]

The invertibility of \( \lambda - \hat{A}_T \) in (5.1) therefore is equivalent to that of

\[
(5.2) \quad \left( \lambda - \hat{A}_T \right) : \mathcal{K}_2^{\mu,\gamma+\mu}(\mathcal{Y}, \mathcal{E})^\rho \oplus \mathbb{C} \longrightarrow \mathcal{K}_2^{\mu,\gamma}(\mathcal{Y}, \mathcal{E})^\rho
\]

As a consequence,

\[
(5.3) \quad \left( \lambda - \hat{A}_{[\omega]} \right) : \mathcal{K}_{2,\gamma}^{\mu,\gamma+\mu}(\mathcal{Y}, \mathcal{E}) \longrightarrow \mathcal{K}_{2,\gamma}^{\mu,\gamma}(\mathcal{Y}, \mathcal{E})
\]

is a Fredholm operator of index \(-N = -\dim \mathbb{C}\), and the same is true for

\[
(5.3) \quad \left( \lambda - \hat{A}_{[\omega]} \right) : \mathcal{K}_{2,\gamma}^{\mu,\gamma+\mu}(\mathcal{Y}, \mathcal{E}) \longrightarrow \mathcal{K}_{2,\gamma}^{\mu,\gamma}(\mathcal{Y}, \mathcal{E})
\]

where \( \hat{A}_{[\omega]} = [x]^{\rho}\hat{A}[x]^{-\rho} \) and \( \hat{T}_{[\omega]} = [x]^{\rho}\hat{T}[x]^{-\rho} \). Also the operator \( \left( \frac{3}{2} \right) \) acts between the spaces in (5.3). Moreover, the difference \( \left( \frac{3}{2} \right) - \left( \hat{T}_{[\omega]} \right) \) vanishes for small \( x \) and is an operator of order \( \mu - 1 \). Hence, the difference is compact between the spaces in (5.3), and the operator \( \left( \frac{3}{2} \right) \) acting as in (5.3) also has index \(-N\). As a consequence,

\[
(5.3) \quad \left( \lambda - \hat{A}_{[\omega]} \right) : \mathcal{K}_{2,\gamma}^{\mu,\gamma+\mu}(\mathcal{Y}, \mathcal{E}) \oplus \mathbb{C} \longrightarrow \mathcal{K}_{2,\gamma}^{\mu,\gamma}(\mathcal{Y}, \mathcal{E})
\]

has index zero. In fact, it is invertible: In view of the ellipticity of \( \left( \frac{3}{2} \right) \), any function in its kernel is rapidly decreasing as \( x \to \infty \) and therefore also belongs to the kernel.
of \((\hat{A})\) acting as in \([5.22]\), which by assumption is \(\{0\}\). The surjectivity of the boundary operator then implies the invertibility of

\[
\lambda - \hat{A}_\mathcal{F} : \mathcal{K}_2^{0,\gamma+\mu}(\mathcal{Y}, \mathcal{E}^\gamma)_{\mathcal{F}} \oplus \hat{\mathcal{E}} \rightarrow \mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma).
\]

Let us finally check the estimates. If \(\|(\lambda - \hat{A}_\mathcal{F})^{-1}\|_{\mathcal{L}(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)_{\mathcal{F}})} = O(\lambda^{-1})\), then also

\[
(5.4) \quad \|x^\rho(\lambda - \hat{A}_\mathcal{F})^{-1}\|_{\mathcal{L}(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma))} = O(\lambda^{-1}).
\]

We can therefore consider the difference of \((\lambda - \hat{A}_\mathcal{F})^{-1} - x^\rho(\lambda - \hat{A}_\mathcal{F})^{-1}\) and \((\lambda - \hat{A}_\mathcal{F})^{-1}\) as bounded operators in \(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)\). The range of \(x^\rho(\lambda - \hat{A}_\mathcal{F})^{-1}\) is the image of the domain in \([5.1]\) under multiplication by \(x^\rho\), i.e.,

\[
\mathcal{K}_2^{0,\gamma+\mu}(\mathcal{Y}, \mathcal{E}^\gamma)_{\mathcal{F}} \oplus \{x^\rho \hat{\mathcal{E}}\}.
\]

In view of the fact that the boundary condition \(\hat{\mathcal{T}}\) commutes with cut-off functions \(\omega\), this is a subset of the maximal domain of \(\hat{A}\): The projection \(\hat{R}\hat{T}\) in \([3.20]\) is not needed, so that \(\hat{\mathcal{E}} = \omega \hat{\mathcal{E}}\). Since we can take \(\omega\) to have support so close to \(x = 0\) that \(\|x\| \equiv 1\) on its support, \(\|x^\rho \hat{\mathcal{E}}\) = \(\hat{\mathcal{E}}\). Hence \(\hat{A}\) maps the range of \((\lambda - \hat{A}_\mathcal{F})^{-1}\) to \(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)\) and we can write

\[
(\lambda - \hat{A}_\mathcal{F})^{-1} - (\lambda - \hat{A}_\mathcal{F})^{-1} = (\lambda - \hat{A}_\mathcal{F})^{-1}(\hat{A}_\mathcal{F} - \hat{A})(\lambda - \hat{A}_\mathcal{F})^{-1}.
\]

Since \(A\) and \(A_\mathcal{F}\) coincide for small \(x\) and differ by an operator of order \(\mu - 1\),

\[
\hat{A}_\mathcal{F} - \hat{A} : \mathcal{K}_2^{\mu-1,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma) \rightarrow \mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)
\]

is bounded. On the other hand, by interpolation for the \(\mathcal{K}^s_2\)-spaces, see e.g. \([25]\), Lemma 4.1], we conclude from \([3.21]\) that

\[
\|((\lambda - \hat{A}_\mathcal{F})^{-1})\|_{\mathcal{L}(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma), \mathcal{K}_2^{0,\gamma-1}(\mathcal{Y}, \mathcal{E}^\gamma))} = O(\lambda^{-1/\mu}).
\]

This allows us to conclude that, for sufficiently large \(|\lambda|, \lambda \in \Lambda\),

\[
(\lambda - \hat{A}_\mathcal{F})^{-1} = (\lambda - \hat{A}_\mathcal{F})^{-1}\left(1 - (\hat{A}_\mathcal{F} - \hat{A})(\lambda - \hat{A}_\mathcal{F})^{-1}\right)^{-1}.
\]

Since the second factor on the right-hand side is bounded for large \(\lambda \in \Lambda\), we obtain the resolvent estimate for \((\lambda - \hat{A}_\mathcal{F})^{-1}\) on \(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)\).

Similarly, we can derive the estimate for any other \(\rho\) from that for \(\rho = 0\). \(\square\)

**Corollary 5.2.** Suppose that \(\hat{\mathcal{T}}\) commutes with cut-off functions. Then condition \((E3)\) is equivalent to the following: There exists a \(C \geq 0\) such that for all \(\lambda \in \Lambda, |\lambda|\) sufficiently large, the realization \(\lambda - \hat{A}_\mathcal{F}\) is invertible in \(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)\) and

\[
(5.5) \quad \|((\lambda - \hat{A}_\mathcal{F})^{-1})\|_{\mathcal{L}(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma))} = O(\lambda^{-1}).
\]

**Proof.** \((E3)\) is equivalent to the invertibility of \(\lambda - \hat{A}_\mathcal{F} : \mathcal{K}_2^{0,\gamma+\mu}(\mathcal{Y}, \mathcal{E}^\gamma)_{\mathcal{F}} \oplus \hat{\mathcal{E}} \rightarrow \mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)\) with an \(O(\lambda^{-1})\) estimate for \((\lambda - \hat{A}_\mathcal{F})^{-1}\) on \(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)\). By Proposition \([5.1]\) this is equivalent to the invertibility of \(\lambda - \hat{A}_\mathcal{F} : \mathcal{K}_2^{0,\gamma+\mu}(\mathcal{Y}, \mathcal{E}^\gamma)_{\mathcal{F}} \oplus \hat{\mathcal{E}} \rightarrow \mathcal{K}_2^{\mu}(\mathcal{Y}, \mathcal{E}^\gamma)\) with an \(O(\lambda^{-1})\) estimate for the inverse on \(\mathcal{K}_2^{0,\gamma}(\mathcal{Y}, \mathcal{E}^\gamma)\). \(\square\)
5.3. Dilation invariant domains. We call a subspace \( \mathcal{D} \) of \( K_0^0(Y^\wedge, E^\wedge) \) dilation-invariant, if \( u \in \mathcal{D} \) implies that \( \kappa_\rho u \in \mathcal{D} \) for arbitrary \( \rho > 0 \) with \( \kappa \) as in \( [13] \).

Suppose \( \mathcal{D}(\hat{T}) \) is dilation invariant. Then, for all \( \eta \in \Sigma \) and \( \rho > 0 \),

\[
\eta^\mu - \hat{A}_\rho \eta = \rho^\mu \kappa_\rho((\eta/\rho)^\mu - \hat{A}_\rho) : \mathcal{D}(\hat{T}) \to K_0^0(Y^\wedge, E^\wedge).
\]

Choosing \( \rho = |\eta| \), we conclude that \( \eta^\mu - \hat{A}_\rho \eta \) is invertible for large \( |\eta| \), \( \eta \in \Sigma \), if and only if it is invertible for all \( \eta \in \Sigma \setminus \{0\} \), if and only if it is invertible for all \( \eta \in \Sigma \) with \( |\eta| = 1 \).

The dilation invariance of \( \mathcal{D}(\hat{T}) \) in \( K_0^0(Y^\wedge, E^\wedge) \) implies that of \( \mathcal{D}(\hat{T}_0, \overline{\rho}) \) in \( K_0^0(Y^\wedge, E^\wedge) \). The fact that \( \kappa_\rho \) is unitary on \( K_0^0(Y^\wedge, E^\wedge) \) then shows that

\[
\| (\eta^\mu - \hat{A}_0, \overline{\rho})^{-1} \| \cdot (K_0^0(Y^\wedge, E^\wedge)) = \| \kappa_\rho^{-1}(\eta^\mu - \hat{A}_0, \overline{\rho})^{-1} \kappa_\rho \| \cdot (K_0^0(Y^\wedge, E^\wedge)) \\
= |\eta|^{-\mu} \| (\eta/|\eta|)^\mu - \hat{A}_0, \overline{\rho})^{-1} \| \cdot (K_0^0(Y^\wedge, E^\wedge)) \leq C|\eta|^{-\mu}
\]

provided the inverse on the right-hand side exists for \( \eta \in \Sigma, |\eta| = 1 \). We see:

**Proposition 5.3.** For dilation invariant domains, condition (E3) is equivalent to the existence of \( (\lambda - \hat{T})^{-1} \) for \( \lambda \in \Lambda, |\lambda| = 1 \).

5.4. The \( H_\infty \)-calculus. In order to obtain a precise structure of the resolvent, we shall make use of the pseudodifferential calculus for boundary value problems on manifolds with edges as presented in Section 4 of \([13]\) and Section 7 of \([12]\). The basic elements are recalled in Section 5.2 of the appendix.

**Proposition 5.4 (Parametrix).** Assume that (E1), (E2,∗) are fulfilled and that

\[
(5.6) \quad \eta^\mu - \hat{A} : K_2^{\mu, \gamma+\mu}(Y^\wedge, E^\wedge) \to K_2^{\mu-j-1/2, \gamma-j-1/2}(\partial Y^\wedge, F_k^\wedge)
\]

is injective for \( 0 \neq \eta \in \Sigma \). Then there exists a \( B \in C^{-\mu, \Sigma; \gamma, \gamma+\mu} \) with the following properties:

(i) \( B(\eta) \in \mathcal{L}(H_2^{\mu, \gamma}(\mathbb{D}), H_2^{\mu, \gamma+\mu}(\mathbb{D})_T) \) for every \( \eta \in \Sigma, |\eta| \) large.

(ii) \( (\eta^\mu - A)B(\eta) - 1 = G_R(\eta) \) with \( G_R \in C_G^{\mu, \Sigma; \gamma, \gamma+\mu} \).

(iii) \( B(\eta)(\eta^\mu - A) = 1 + G_L(\eta) \) for some \( G_L \in C_G^{\mu, \Sigma; \gamma+\mu} \), and \( B(\eta)(\eta^\mu - A) = 1 \) on \( H_2^{\mu, \gamma+\mu}(\mathbb{D})_T \).

**Proof.** In order to unify the orders in the operator matrix, we replace \( T \) by \( T_1(\eta) = \text{diag}(R^{\mu-1/2}(\eta), R^{\mu-3/2}(\eta), \ldots, R^{1/2}(\eta))T \), where \( R^{\mu-j+1/2}(\eta) \) is a parameter-dependent order reduction on \( \mathbb{B} \), so that

\[
R^{\mu-j-1/2}(\eta) : K_2^{\mu-j-1/2, \gamma-j-1/2}(\mathbb{B}, F_j) \to K_2^{0, \gamma}(\mathbb{B}, F_j)
\]

is an isomorphism. Clearly, \( \ker T(\eta) = \ker T \). Also \( \ker \hat{T}_1(\eta) = \ker \hat{T} \), since the edge symbols of the order reducing operators are invertible.

Now \( \left( \eta^\mu - A \atop T_1(\eta) \right) \) is a symbol of order \( \mu \) and type \( \mu \) in Schulze’s parameter-dependent cone calculus, and its principal edge symbol,

\[
(5.7) \quad \left( \eta^\mu - \hat{A} \atop \hat{T}_1(\eta) \right) : K_2^{\mu, \gamma+\mu}(Y^\wedge, E^\wedge) \to K_2^{0, \gamma}(\partial Y^\wedge, F^\wedge)
\]

is a...
is injective. Following an argument by Krainer, see the proof of Theorem 7.21 in [14], we find an operator family \( \tilde{K}_1(\eta) : C^d \to \mathscr{D}'(\Sigma; 0, \gamma) \), \( \eta \in \Sigma \cap \{ |\eta| = 1 \} \), such that

\[
\begin{pmatrix}
\eta^\mu - \hat{A} & K_1(\eta) \\
T_1(\eta) & 0
\end{pmatrix} : C^d \to \mathcal{K}_2^{(0, \gamma)}(\Sigma; 0, \gamma) \to C^d
\]

is invertible. In fact, \( \tilde{K}_1 \) can be chosen to have an integral kernel in \( \mathcal{D}'(\Sigma \cap \{ |\eta| = 1 \}) \otimes \mathcal{D}'(\Sigma; 0, \gamma) \otimes C^d \); it can be extended \( \mu \)-homogeneously to \( \Sigma \) by

\[
\tilde{K}_1(\eta) = |\eta|^{g_{\gamma}} \tilde{K}_1(\eta/|\eta|).
\]

Let \( K_1(\eta) = \omega \chi(\eta) \tilde{K}_1(\eta) \) for a zero-excision function \( \chi \) and a cut-off function \( \omega \). Then

\[
(5.8)
\begin{pmatrix}
\eta^\mu - A & K_1(\eta) \\
T_1(\eta) & 0
\end{pmatrix} : \mathcal{H}_2^{(0, \gamma + \mu)}(\Sigma, \gamma + \mu) \to C^d
\]

is an elliptic element in Schulze’s parameter-dependent cone calculus. Hence there exists a pararectrix modulo regularizing Green operators.

The operator in (5.8) is invertible for large \( |\eta| \), and we can modify the pararectrix so that it coincides with the inverse for large \( |\eta| \). Denote this pararectrix by

\[
\begin{pmatrix}
B(\eta) & K(\eta) \\
S(\eta) & Q(\eta)
\end{pmatrix}.
\]

Then

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
\eta^\mu - A & K_1(\eta) \\
T_1(\eta) & 0
\end{pmatrix} \begin{pmatrix}
B(\eta) & K(\eta) \\
S(\eta) & Q(\eta)
\end{pmatrix}
\]

shows that \( B(\eta) \) maps into the kernel of \( T \) and that \( (\eta^\mu - A)B(\eta) - 1 = -K_1(\eta)S(\eta) \). This shows (i) and (ii). Interchanging the order of factors, one obtains (iii). \( \square \)

**Theorem 5.5.** Let the conditions (E1)-(E3) be satisfied for \( A_0 \) with boundary condition \( T_0 \). Then \( \hat{A}_T \) has at most finitely many spectral points in \( \Sigma \), and there exists a parameter-dependent operator

\[
(5.9)
C \in C^{-\mu,0}(\Sigma; \gamma, \gamma + \mu) + C^{-\mu,0}_G(\Sigma; \gamma, \gamma)
\]

such that \( (\eta^\mu - \hat{A}_T)^{-1} = C(\eta) \) for sufficiently large \( \eta \in \Sigma \).

In the above decomposition \( \mathfrak{C}_G \), \( C^{-\mu,0}(\Sigma; \gamma, \gamma + \mu) \) is the symbol class introduced in Section 5.2 while \( C^{-\mu,0}_G(\Sigma; \gamma, \gamma) \) is as in Definition 4.2.

**Proof.** By conjugation with \( \tilde{x}_\gamma \) it is sufficient to show the assertion for \( A_{0,T_0} \) or, equivalently, to assume that the weight \( \gamma \) is zero, so that \( A_0 = A \) and \( T_0 = T \).

It follows from [14] Theorem 8.1] that \( (\eta^\mu - \hat{A}_T)^{-1} \) exists for \( \eta \in \Sigma, |\eta| \) sufficiently large, and

\[
|||\eta^\mu - \hat{A}_T|||_{\mathcal{D}'(\Sigma; 0, \mu)} = O(||\eta||^{-\mu}).
\]

Assumption (E3) implies that \( \eta^\mu - \hat{A}_T \) is injective on \( \mathscr{D}(\hat{A}_T, \min) = \mathcal{K}_2^{(0, \gamma)}(Y^\gamma, E^\gamma) \).

Thus also the operator \( (5.9) \) is injective. By Proposition 5.4 we find \( B(\eta) \in C^{-\mu,0}(\Sigma; 0, \mu) \) and \( G_R \in C^{0,0}_G(\Sigma; 0, 0) \) such that, for sufficiently large \(|\eta|\),

\[
(5.10)
B(\eta) = (\eta^\mu - \hat{A}_T)^{-1} + (\eta^\mu - \hat{A}_T)^{-1}G_R(\eta).
\]

Let \( \tilde{T} \) be the adjoint boundary condition for \( A_T \) in the sense of [3] Definition 3.12]. Then \( (\hat{A}_T)^* = \hat{A}_T^\dagger \), i.e. the adjoint of \( \hat{A}_T \) is a suitable realization of the formal
adjoint $A'$ with boundary condition $\tilde{T}$, see [5] Section 4. Let us check that it also satisfies (E1) and (E2).

Clearly, parameter-ellipticity of the principal pseudodifferential symbol holds for $A'$. If we write the boundary symbol of $\left(\begin{smallmatrix} A' & 0 \\ 0 & \partial_r \end{smallmatrix}\right)$ as $\left(\begin{smallmatrix} a^{0,0} & \partial_r \\ \partial_r & 0 \end{smallmatrix}\right)$ and that of $\left(\begin{smallmatrix} A & 0 \\ 0 & \partial_r \end{smallmatrix}\right)$ as $\left(\begin{smallmatrix} a^{0,0} & \partial_r \\ \partial_r & 0 \end{smallmatrix}\right)$, then the boundary symbol realizations in $L^2(\mathbb{R}_+)$ satisfy $(a^{0,0})_\partial = ((a^{0,0})_\rho)^*$, with a corresponding relation for the rescaled symbols, see Grubb [11, Theorem 1.6.9]. Therefore (E1) also holds for the adjoint.

Moreover, [5] Corollary 7.3 applied with $\gamma = 0$ and $\gamma = \mu$ implies that the conormal symbol of $\left(\begin{smallmatrix} A & 0 \\ 0 & \partial_r \end{smallmatrix}\right)$ is invertible for $\Re z = \frac{n+1}{2} - \mu$ and $\Re z = \frac{n+1}{2}$. Hence (E2) holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & \partial_r \end{smallmatrix}\right)$.

It follows from [5] Proposition 7.2 that $\tilde{\hat{T}} = \hat{\tilde{T}}$, since $\hat{T}$ and $\hat{\tilde{T}}$ are differential operators so that the model cone boundary condition is determined by the conormal symbol. Therefore $(\hat{\tilde{A}}_T)^*$ is a realization of $\hat{\tilde{A}}_T$, subject to the boundary condition $\hat{T}$. Since this realization has no spectrum in $\Lambda \cap \{|\lambda| \geq R\}$, the operator $\lambda - \hat{\tilde{A}}_T$ is injective for large $\lambda \in \Lambda$, and thus, by homogeneity, for all $\lambda \in \Lambda \setminus \{0\}$.

So we can apply once more Proposition 5.3 and find $\tilde{B}(\eta) \in C^{-\mu,0}(\mathbb{D}, \Sigma; 0, \mu)$ and $\tilde{G}_R \in C_G^{0,0}(\mathbb{D}, \Sigma; 0, 0)$ such that

$$\tilde{B}(\eta) = (\nabla^\mu - (\hat{\tilde{A}}_T)^*)^{-1} + (\nabla^\mu - (\hat{\tilde{A}}_T)^*)^{-1} \tilde{G}_R(\eta).$$

Taking adjoints in the above equation we obtain

$$\tilde{B}^*(\eta) = (\nabla^\mu - \hat{\tilde{A}}_T)^{-1} + \tilde{G}_R^*(\eta)(\nabla^\mu - \hat{\tilde{A}}_T)^{-1}.\]

Hence

$$(\nabla^\mu - \hat{\tilde{A}}_T)^{-1} = B(\eta) + \tilde{B}^*G_R - \tilde{G}_R^*(\eta)(\nabla^\mu - \hat{\tilde{A}}_T)^{-1}G_R(\eta).$$

By the rules of the calculus, $\tilde{B}^*G_R \in C_G^{-\mu,0}(\mathbb{D}, \Sigma; 0, 0)$; so it remains to show that $\tilde{G}_R^*(\eta)(\nabla^\mu - \hat{\tilde{A}}_T)^{-1}G_R(\eta) \in C_G^{-\mu,0}(\mathbb{D}, \Sigma; 0, 0)$.

From the fact that $D_k^\mu(\eta^\mu - \hat{\tilde{A}}_T)^{-1}$ is a linear combination of terms of the form $p\mu(\eta^\mu - \hat{\tilde{A}}_T)^{-1}$, where $k + l = |\alpha|$ and $p\mu$ is a polynomial of degree at most $(\mu - 1)l - k$, we conclude that, for $\alpha \in \mathbb{N}_2^\nu$, there exists a constant $C_\alpha$ with

$$\|D_k^\mu(\eta^\mu - \hat{\tilde{A}}_T)^{-1}\|_{L^2(\mu^0(\mathbb{D}, \Sigma))} \leq C_\alpha |\eta|^{-\mu - |\alpha|}.$$

Since the group action $\kappa$ is unitary on $K_2^{0,0}(Y^\wedge, E^\wedge)$, we see that, for any cut-off function $\omega$,

$$\omega(\eta^\mu - \hat{\tilde{A}}_T)^{-1}\omega \in S^{-\mu}(\Sigma; K_2^{0,0}(Y^\wedge, E^\wedge), K_2^{0,0}(Y^\wedge, E^\wedge)).$$

Furthermore, $\omega G_R(\eta)\omega, \omega G_R^* \omega \in R_G^{0,0}(Y^\wedge, \Sigma; 0, 0)$, so that

$$R(\eta) := \omega G_R(\eta)\omega^\omega(\eta^\mu - \hat{\tilde{A}}_T)^{-1}\omega^2 G_R(\eta)\omega \in S^{-\mu}(\Sigma; K_2^{0,0}(Y^\wedge, E^\wedge), \mathcal{S}(Y^\wedge, E^\wedge)).$$

By Remark 13 we may omit the two factors $\omega^2$ at the expense of modifying $R$ by an element of $S^{-\infty}(\Sigma; K_2^{0,0}(Y^\wedge, E^\wedge), \mathcal{S}(Y^\wedge, E^\wedge))$. In particular, this preserves the symbol class on the right-hand side. As $R_G^{0,0}(Y^\wedge, \Sigma; 0, 0)$ is invariant under adjoints, the argument applies also to the adjoints. So $R^*$ and its modification belong to $S^{-\mu}(\Sigma; K_2^{0,0}(Y^\wedge, E^\wedge), \mathcal{S}(Y^\wedge, E^\wedge))$, and Proposition 13 implies that

$$\omega G_R(\eta^\mu - \hat{\tilde{A}}_T)^{-1}G_R \omega \in R_G^{-\mu,0}(Y^\wedge, \Sigma; 0, 0).$$

We conclude again from Remark 13 that $\tilde{G}_R^*(\eta^\mu - \hat{\tilde{A}}_T)^{-1}G_R \in C_G^{-\mu,0}(\mathbb{D}, \Sigma; 0, 0)$, and the proof is complete.
The following theorem is a consequence of Theorem 4.3 and [5, Theorem 4.1], noting that the required holomorphicity of the principal interior symbol is immediate from the fact that it arises as the inverse of the principal symbol of \( \eta^2 - A \). This has been shown already in the proof of [6, Theorem 5.4].

**Theorem 5.6.** Under the assumptions (E1)-(E3) there exists a constant \( c > 0 \) such that \( c + \Delta_T \) has a bounded \( H_\infty \)-calculus on \( \mathcal{H}^0_{p,\gamma}(\mathbb{D}, E) \) for all \( 1 < p < +\infty \).

6. The Dirichlet and Neumann Laplacian

Given a metric that coincides with \( dx^2 + x^2h(x) \) on the collar part \((0, 1) \times Y \) of \( \mathbb{D} \), the associated Laplacian is, on the collar part,

\[
\Delta = x^{-2} \left( (x \partial_x)^2 + (n-1) + H(x) \right),
\]

where \( \Delta_Y(x) \) is the Laplacian on \( Y \) induced by \( h(x) \) and \( 2H(x) = x \partial_x (\log \det h(x)) \).

We shall consider realizations subject to the Dirichlet boundary operator \( \gamma_D = \gamma_0 \) and to the Neumann boundary operator \( \gamma_N = \gamma_0 B_1 \), where \( B_1 = x^{-1} \partial_x \) in the sense of (1.1) and \( \nu_x \) is a unit vector field in a collar-neighborhood of \( \partial Y \) that coincides on \( \partial Y \) with the exterior normal with respect to the metric \( h(x) \).

According to Green’s formula, both \( \Delta_D := \Delta_{\gamma_D} \) and \( \Delta_N := \Delta_{\gamma_N} \) are symmetric in \( \mathcal{H}^{0,0}_{0,\gamma}(\mathbb{D}) \). They have been described in [5] in the special case of a metric that is constant in \( x \), i.e., \( h(x) \equiv h(0) \).

We write \( A_D = (\Delta_D) \) and \( A_N = (\Delta_N) \) and obtain the model cone operators

\[
\tilde{\Delta}_D = (\tilde{\Delta}_D) \quad \text{and} \quad \tilde{\Delta}_N = (\tilde{\Delta}_N),
\]

with the Dirichlet and Neumann boundary operators \( \tilde{\gamma}_D \) and \( \tilde{\gamma}_N \), respectively, on \( Y^\gamma \) equipped with the metric \( dx^2 + x^2h(0) \). Both commute with multiplication by cut-off functions, cf. Remark 3.9.

The resulting principal conormal symbols are

\[
f_{0,D/N}(z) = \sigma_{2s}^D(A_{D/N})(z) = \left( z^2 - (n-1)z + \Delta_Y(0) \right).\]

Denoting by \( \Delta_{Y,D/N}(0) \) the Dirichlet respectively Neumann realization of the Laplacian on \( Y \), let us now set, for \( z \in \mathbb{C} \),

\[
f_0(z) = z^2 - (n-1)z + \Delta_Y(0), \quad f_{0,D/N}(z) = z^2 - (n-1)z + \Delta_{Y,D/N}(0).
\]

**Lemma 6.1.** Let \( K_{D/N} \) be a right-inverse of \( \gamma_{D/N} \) in Boutet de Monvel’s algebra for \( Y \). For \( z \in \mathbb{C} \),

\[
f_{0,D}(z) : H_2^{s+2}(Y) \rightarrow H_2^s(Y) \oplus H_2^{s+3/2}(\partial Y), \quad s > -3/2,
\]

and

\[
f_{0,N}(z) : H_2^{s+2}(Y) \rightarrow H_2^s(Y) \oplus H_2^{s+1/2}(\partial Y), \quad s > -1/2,
\]

respectively, are invertible if and only if

\[
f_{0,D/N}(z) : H_2^{s+2}(Y)_{D/N} \rightarrow H_2^s(Y)
\]

are invertible; in this case

\[
f_{0,D/N}(z)^{-1} = \left( f_{0,D/N}(z)^{-1} \left( 1 - f_{0,D/N}(z)^{-1}f_0(z) \right) K_{D/N} \right)
\]

\[
= f_{0,D/N}(z)^{-1} \left( 1 - f_0(z)K_{D/N} \right) + \left( 0 \quad K_{D/N} \right).
\]
Proof. The first fact follows from the surjectivity of the boundary operators, see [5] Corollary 8.2. The formula for the inverse then results from the identity

\[
\left( f_0(z) \right)
\begin{pmatrix}
  f_0(z)K_{D/N} & 1 \\
  0 & 1
\end{pmatrix}
\]

\[
\text{□}
\]

Corollary 6.2. In Theorem 5.8 applied to \( A_D \) or \( A_N \), the operators \( G_{\sigma}^{(f)} \) can be substituted by the operators

\[
G_{\sigma,D/N}^{(f)} : \mathcal{S}(Y) \rightarrow \mathcal{S}(Y)
\]

defined by

\[(G_{\sigma,D/N}^{(f)}v)(x) = x^f \int_{|z-\sigma|=\epsilon} x^{-z} g_{l,D/N}(z) \Pi_{\sigma}(f_{0,D/N}^{-1} \tilde{v})(z) \, dz.\]

Proof. By Lemma 6.1, \( f_{0,D/N}(z)^{-1} \equiv f_{0,D/N}(z)^{-1} \left( 1 - f_0(z)K_{D/N} \right) \) modulo holomorphic functions. Now let \( u = (u_1,u_2) \in \mathcal{S}(Y) \oplus \mathcal{S}(\partial Y) \). Then

\[
\left( 1 - f_0(z)K_{D/N} \right) \left( \begin{array}{c}
\tilde{u}_1(z) \\
\tilde{u}_2(z)
\end{array} \right) = \tilde{u}_1(z) - (f_0(-x\partial_x)K_{D/N}u_2)(z) = \tilde{v}(z)
\]

with \( v(x) := u_1(x) - f_0(-x\partial_x)K_{D/N}u_2(x) \in \mathcal{S}(Y) \). Hence the range of \( G_{\sigma,D/N}^{(f)} \) coincides with that of \( G_{\sigma,D/N}^{(f)} \).

\[
\text{□}
\]

6.1. Extensions on the model cone. Let \( \lambda_{D/N}^j \), \( j \in \mathbb{N}_0 \), be the eigenvalues of the Dirichlet and Neumann Laplacian on \( Y \) with respect to \( h(0) \), respectively. Assuming that \( \mathbb{D} \) is connected and has non-empty boundary, we have \( \lambda_0^D < 0 \) for the Dirichlet case while \( \lambda_0^N = 0 \) for the Neumann case. Then \( f_{0,D/N}(z) \) is invertible for all \( z \) except for the values

\[(6.2) \quad q_{j,D/N}^\pm := \frac{n-1}{2} \pm \sqrt{\left( \frac{n-1}{2} \right)^2 - \lambda_j^{D/N}}, \quad j \in \mathbb{N}_0.
\]

Note the relation \( q_{j,D/N}^- + q_{j,D/N}^+ = n - 1 \).

Let \( E_{j,D/N} \) denote the eigenspace associated with \( \lambda_j^{D/N} \) and \( \pi_{j,D/N} \) the \( L^2 \)-orthogonal projection onto \( E_{j,D/N} \). Then, in case \( n \geq 2 \),

\[
f_{0,D/N}(z) = \sum_{j=0}^{\infty} \frac{1}{q_{j,D/N}^+ - q_{j,D/N}^-} \left( \frac{1}{z - q_{j,D/N}^+} - \frac{1}{z - q_{j,D/N}^-} \right) \pi_{j,D/N},
\]

hence

\[(6.3) \quad \Pi_{q_{j,D/N}^+} f_{0,D/N}(z) = \pm \frac{\pi_{j,D/N}}{q_{j,D/N}^+ - q_{j,D/N}^-} (z - q_{j,D/N}^\pm)^{-1}.
\]

In case \( n = 1 \) this holds also true in the Dirichlet case, and in the Neumann case whenever \( j \geq 1 \). Moreover, in case \( n = 1 \), \( q_{0,N}^+ = q_{0,N}^- = 0 \) is a double pole and

\[
\Pi_{0} f_{0,N}(z)^{-1} = \pi_0 z^{-2}.
\]

Definition 6.3. Let \( E_{j,D/N} \) be the eigenspace associated with \( \lambda_j^{D/N} \). Define

\[
\tilde{E}_{q_{j,D/N}^+} = E_{j,D/N} \oplus x^{-q_{j,D/N}^+} = \left\{ e(y)x^{-q_{j,D/N}^+} \mid e \in E_{j,D/N} \right\}, \quad j \in \mathbb{N}_0,
\]

unless \( n = 1, j = 0 \) and we have Neumann boundary conditions; then we set

\[
\tilde{E}_{0,N} = E_{0,N} \oplus E_{N} \otimes \log x.
\]
For \( \gamma \in \mathbb{R} \) define the set
\[
I_{\gamma,D/N} = \left\{ q_{j,D/N}^\pm \mid j \in \mathbb{N}_0 \right\} \cap \left( \frac{n+1}{2} - \gamma - 2, \frac{n+1}{2} - \gamma \right).
\]
By Theorem 6.7 and straight-forward calculations using the residue theorem we obtain:

**Proposition 6.4.** The maximal extension of \( \tilde{\Delta}_{D/N} \) in \( \mathcal{K}_0^0,\gamma(Y^\gamma) \), \( \gamma \in \mathbb{R} \), subject to Dirichlet/Neumann boundary conditions has the domain
\[
\mathcal{D}_{\max}(\tilde{\Delta}_{D/N}) = \mathcal{D}_{\min}(\tilde{\Delta}_{D/N}) \oplus \omega \hat{\mathcal{E}}_{D/N}^\gamma, \quad \hat{\mathcal{E}}_{D/N}^\gamma = \bigoplus_{q \in I_{\gamma,D/N}} \hat{\mathcal{E}}_{q,D/N}.
\]
In case \( \frac{n+1}{2} - \gamma - 2 \neq q_{j,D/N}^\pm \) for every \( j \), the minimal domain coincides with \( \mathcal{K}_p^{2,\gamma+2}(Y^\gamma)_{D/N} \).

The description of the adjoints of closed extensions makes use of the bilinear form
\[
(u,v)_{D/N} := (\tilde{\Delta}_{D/N}(\omega u),\omega v)_{\mathcal{K}_2^0,\gamma(Y^\gamma)} - (\omega u, \tilde{\Delta}_{D/N}(\omega v))_{\mathcal{K}_2^0,\gamma(Y^\gamma)},
\]
which is non-degenerate as a map
\[
\hat{\mathcal{E}}_{D/N}^\gamma \times \hat{\mathcal{E}}_{D/N}^{-\gamma} \longrightarrow \mathbb{C};
\]
it does not depend on the choice of the cut-off function \( \omega \).

The result below, is an analog of [5, Proposition 6.3]:

**Lemma 6.5.** Let \( \tilde{\Delta}_{D/N} \) be an extension in \( \mathcal{K}_0^0,\gamma(Y^\gamma) \) with domain
\[
\mathcal{D}(\tilde{\Delta}_{D/N}) = \mathcal{D}_{\min}(\tilde{\Delta}_{D/N}) \oplus \omega \hat{\mathcal{E}}_{D/N}^\gamma.
\]
Then its adjoint, considered as an unbounded operator in \( \mathcal{K}_p^{0,-\gamma}(Y^\gamma) \), is the Laplacian acting on the domain
\[
\mathcal{D}(\tilde{\Delta}_{D/N}^\gamma) = \mathcal{D}_{\min}(\tilde{\Delta}_{D/N}) \oplus \omega \hat{\mathcal{E}}_{D/N}^\gamma\#,
\]
where \( \hat{\mathcal{E}}_{D/N}^\gamma\# \) is the orthogonal space to \( \hat{\mathcal{E}}_{D/N}^\gamma \) with respect to the pairing \( \hat{\mathcal{E}}_{D/N}^\gamma \).

6.2. **Extensions with property (E3).** Of the three ellipticity conditions (E1), (E2) and (E3), generally the last one is the most difficult to check. Theorem 6.7 below, gives a simple sufficient condition. We focus on extensions \( \tilde{\Delta}_{D/N} \) of \( \hat{\Delta}_{D/N} \) in \( \mathcal{K}_2^{0,\gamma}(Y^\gamma) \) with domain of the form
\[
\mathcal{D}(\tilde{\Delta}_{D/N}) = \mathcal{D}_{\min}(\tilde{\Delta}_{D/N}) \oplus \omega \hat{\mathcal{E}}_{D/N}^\gamma, \quad \hat{\mathcal{E}}_{D/N}^\gamma = \bigoplus_{q \in I_{\gamma,D/N}} \hat{\mathcal{E}}_{q,D/N},
\]
where \( \hat{\mathcal{E}}_{q,D/N} \) is an arbitrary subspace of \( \hat{\mathcal{E}}_{q,D/N} \), except in the case of the Neumann condition and \( n = 1 \), where for \( q = 0 \) we confine ourselves to the following three choices: \( \hat{\mathcal{E}}_{0,N} = \{0\} \), \( \hat{\mathcal{E}}_{0,N} = E_0^N \otimes 1 \), or \( \hat{\mathcal{E}}_{0,N} = \hat{\mathcal{E}}_{E_0,N} \).

Let \( \hat{\mathcal{E}}_{q,D/N}^\perp = \hat{\mathcal{E}}_{q,D/N} \ominus x^{-q_{j,D/N}} \) with a subspace \( \hat{\mathcal{E}}_{q,D/N}^\perp \) of \( \hat{\mathcal{E}}_{q,D/N}^\perp \), cf. Definition 6.6.

We define
\[
\hat{\mathcal{E}}_{E_{q,D/N}}^\perp := E_{q,D/N}^\perp \ominus x^{-q_{j,D/N}} \subseteq \hat{\mathcal{E}}_{q,E_{q,D/N}}^\perp
\]
(note the sign change), where \( \hat{\mathcal{E}}_{q,D/N}^\perp \) is the orthogonal complement in \( E_{q,D/N} \) with respect to the \( L^2(Y) \) inner product, with the only exception for \( \hat{\mathcal{E}}_{q,N} = E_0^N \otimes 1 \) in case \( n = 1 \), where instead we define \( \hat{\mathcal{E}}_{0,N}^\perp \).

**Lemma 6.6.** If \( \hat{\mathcal{E}}_{D/N}^\gamma \) is as in (6.6) then \( \hat{\mathcal{E}}_{D/N}^{\gamma,\#} = \bigoplus_{q \in I_{\gamma,D/N}} \hat{\mathcal{E}}_{q,D/N}^\perp \).
Proof. The result is based on the description of adjoint operators in [5, Section 6.3]. We shall focus on the Neumann case with \( n = 1 \); this is the most involved case, since then \( q_0^+ = q_0^- = 0 \) is a double pole of the conormal symbol. The other cases are treated analogously.

Let us write \( q_j^+ := q_j^+ \). Since \( n = 1 \), we have \( I_q := I_{q_j^+} = (-1, 1 - \gamma) \cap \{ q_j^+ | j \geq 0 \} \) and \( q_j^- = -q_j^+ \) for all \( j \). By symmetry it is enough to consider the case \( 0 \leq \gamma < 1 \).

**Step 1:** Let \( e_j, f_j \in E_j^N \), \( e_k \in E_k^N \), be arbitrary. A direct calculation yields

\[
[e_j x^{-q_j^+}, e_k x^{-q_k^+}] = -(q_j^+ - q_k^+)(e_j, e_k)_{L^2(Y)} \int_0^{+\infty} \partial_x \omega^2(x) x^{-q_j^+ - q_k^+} dx.
\]

The second factor on the right-hand side equals zero whenever \( j \neq k \). In case \( j = k \),

\[
[e_j x^{-q_j^+}, f_j x^{-q_j^+}] = [e_j x^{-q_j^+}, f_j x^{-q_j^-}] = 0
\]

as well as

\[
[e_j x^{-q_j^+}, f_j x^{-q_j^-}] = -(e_j x^{-q_j^+}, f_j x^{-q_j^-}) = -(q_j^- - q_j^+)(e_j, f_j)_{L^2(Y)} \int_0^{+\infty} \partial_x \omega^2(x) dx = (q_j^- - q_j^+)(e_j, f_j)_{L^2(Y)}.
\]

For \( e, f \in E_j^N \), \( e_j \in E_j^N \), \( j > 0 \), one obtains

\[
[e, f] = [e \log x, f \log x] = [e, e_j x^{-q_j^+}] = [e \log x, e_j x^{-q_j^+}] = 0
\]

and

\[
[e, f \log x] = -(e, f)_{L^2(Y)} \int_0^{+\infty} \partial_x \omega^2(x) dx = (e, f)_{L^2(Y)}.
\]

**Step 2:** Let \( q \in I_q \setminus I_{-\gamma} \). Then \( q = q_j^- \) for some \( j > 0 \). From the above calculations for the pairing it follows that

\[
\hat{\mathcal{E}}_q^\# = \hat{\mathcal{E}}_q^\perp \oplus_{p \in I_{-\gamma}, p \neq -q} \hat{\mathcal{E}}_p^\perp.
\]

Consequently, as the orthogonal complement of a sum of spaces is the intersection of all respective orthogonal complements, we obtain

\[
(\bigoplus_{q \in I_q \setminus I_{-\gamma}} \hat{\mathcal{E}}_q)^\# = \bigoplus_{p \in I_{-\gamma} \cap I_q} \hat{\mathcal{E}}_p^\perp \oplus_{q \in I_q \setminus I_{-\gamma}} \hat{\mathcal{E}}_q^\perp.
\]

Now let \( q \in I_q \cap I_{-\gamma} \). Then also \(-q \in I_q \cap I_{-\gamma} \). If \( q \neq 0 \), the above calculations yield that

\[
(\hat{\mathcal{E}}_q \oplus \hat{\mathcal{E}}_{-q})^\# = \hat{\mathcal{E}}_q^\perp \oplus_{q \in I_q \setminus I_{-\gamma}} \hat{\mathcal{E}}_{-q}^\perp \oplus_{p \in I_{-\gamma}} \hat{\mathcal{E}}_p^\perp,
\]

while for \( q = 0 \), due to our choices of \( \hat{\mathcal{E}}_0^\# \), we find

\[
\hat{\mathcal{E}}_0^\# = \hat{\mathcal{E}}_0^\perp \oplus_{p \in I_{-\gamma}} \hat{\mathcal{E}}_p^\perp.
\]

It follows that

\[
(\bigoplus_{q \in I_q \cap I_{-\gamma}} \hat{\mathcal{E}}_q \oplus \hat{\mathcal{E}}_{-q})^\# = \bigoplus_{q \in I_q \cap I_{-\gamma}, q < 0} \hat{\mathcal{E}}_q^\perp \oplus_{q \in I_q \cap I_{-\gamma}, q < 0} \hat{\mathcal{E}}_{-q}^\perp \oplus_{p \in I_{-\gamma}} \hat{\mathcal{E}}_p^\perp.
\]

Taking the intersection of (6.7) and (6.8) yields the claim. \( \square \)
Theorem 6.7. Let $|\gamma| < (n + 1)/2$, and suppose that the $\text{Re} q_{D/N}^\pm$ are different from both $q_{D/N}^j - \gamma$ and $q_{D/N}^{j+1} - \gamma - 2$ for all $j$. Moreover, let $\hat{\Delta}_{D/N}$ be an extension with domain as in (6.6), where the spaces $\hat{\mathcal{E}}_{q,D/N}$ are chosen such that:

1. $\hat{\mathcal{E}}_{q,D/N} = \hat{\mathcal{E}}_{q-1,D/N}$ for $q \in I_1 \cap I_\gamma$,
2. $\hat{\mathcal{E}}_{q,D/N} = \hat{\mathcal{E}}_{q,D/N}$ for $\gamma \geq 0$ and $q \in I_\gamma \setminus I_\gamma$,
3. $\hat{\mathcal{E}}_{q,D/N} = \{0\}$ for $\gamma \leq 0$ and $q \in I_\gamma \setminus I_\gamma$.

Then $\hat{\Delta}_{D/N}$ satisfies (E3) for every sector $\Lambda \subseteq \mathbb{C} \setminus \mathbb{R}_+$.

Proof. All extensions of the form (6.6) are invariant under dilations in the sense of Section 6.4.1 of [25]. The decay condition in (E3) therefore follows via homogeneity, provided we can establish the invertibility of $\eta^\mu - x^{-\gamma}\hat{\Delta}_{D/N}x^\gamma$ in $K^0_2(Y')$ for $|\eta| = 1$, $\eta \in \mathbb{C} \setminus \mathbb{R}_+$. This in turn is equivalent to the invertibility of $\eta^\mu - \hat{\Delta}_{D/N}$ in $K^0_2(Y')$. Since both the Dirichlet and the Neumann boundary condition commute with cut-off functions, Proposition 5.1 shows that it suffices to establish the invertibility of $\eta^\mu - \hat{\Delta}_{D/N}$ in $K^0_2(Y')$. As observed in the proof of Proposition 5.12, $\eta^\mu - \hat{\Delta}_{D/N}$ is a Fredholm operator. Moreover, we may assume $\gamma \geq 0$ by possibly going over to the adjoint problem, which satisfies the conditions (1) and (2) above by [5, Theorem 6.3]. Then we argue in the same way as for Theorem 5.7 in [24]. \qed

6.3. An extension of the Neumann Laplacian. With a view towards an application discussed below, we will study a particular extension of the Neumann Laplacian. We recall that $q_{1,N}^- < 0$ and, as in [17, 18, 19, 24, 25], we fix $\gamma$ with

(6.9) $\gamma = \frac{n - 3}{2} + \delta$ with some $0 < \delta < \min\{-q_{1,N}^-, 2\}, \quad 2 - \delta \neq q_{j,N}^+$ for all $j$.

Then $I_{\gamma,N}$ contains $q_{0,N}^+$, but none of the $q_{j,N}^-$ for $j > 0$, whereas $I_{-\gamma,N}$ contains $q_{j,N}^-$ for $j > 0$, but none of the $q_{j,N}^+$ for $j > 0$. Moreover, $I_{\gamma,N} \cap I_{-\gamma,N}$ contains at most the poles $q_{0,N}^+ = 0$ and $q_{0,N}^- = n - 1$. In fact, for $n = 1$, $I_{\gamma,N} \cap I_{-\gamma,N}$ contains only $q_{0,N}^- = q_{0,N}^+ = 0$. For $n = 2$, the intersection consists of both $q_{0,N}^- = 0$ and $q_{0,N}^+ = 1$, provided $\delta < 1$, else it is empty. For $n \geq 3$, the intersection is always empty. Let us determine the space $\hat{\mathcal{E}}_{0,N} = \theta_0^{-1}\hat{\mathcal{E}}_{0,N}$. The computation extends that in Section 6.1.1 of [25] to the Neumann Laplacian.

We need the conormal symbol $\sigma^j_M(A_N)$. Recall the unit vector field $\nu_x$ defined at the beginning of this section. For every $y$ in a collar-neighborhood of the boundary $\partial Y$, $x \mapsto \nu_x(y)$ represents a smooth curve in $T_yY$; let

$$\nu_x'(y) = \frac{d}{dx}\nu_x(y) \in T_yY.$$ 

This defines a vector-field $\nu_x'$ for each $x \geq 0$. Then we have

$$\sigma^j_M(A_N)(z) = \left(\frac{\Delta_Y(0) - H'(0)z}{\gamma_{1,N}}\right), \quad \gamma' := \gamma_D\partial\nu_x'.$$

Note that locally constant functions on $Y$ belong both to the kernel of $\Delta_Y(0)$ and the kernel of $\gamma'_{1,N}$, i.e., $\Delta_Y(0)\pi_{0,N} = 0$ and $\gamma'_{1,N}\pi_{0,N} = 0$. \qed
Proof. Clearly, the Neumann Laplacian satisfies condition (E1). By our choice of \( \delta < 1 \), the origin is the only pole of \( f_{0,N}(z)^{-1} \) in \( I_\gamma = (-\delta, 2 - \delta) \) and (6.3) implies that

\[
g_{1,N}(z)\Pi_0(f_{0,N}(z)^{-1}\tilde{v}(z)) = -f_{0,N}(z - 1)^{-1}\left(\Delta_0'(0) - H'(0)\right)\frac{\pi_{0,N}\tilde{v}(0)}{(n - 1)z} = f_{0,N}(z - 1)^{-1}H'(0)\pi_{0,N}\tilde{v}(0)/(n - 1)
\]

is holomorphic in \( z = 0 \). Hence \( G^{(1)}_{0,N} = 0 \) and \( G_{0,N} = G^{(0)}_{0,N} \). We conclude that, for every choice of \( 0 < \delta < \min\{q_1, 2\} \),

\[
\delta_{0,N} = \tilde{\delta}_{0,N} = E_0^N \otimes 1.
\]

The case \( n = 1 \): By direct calculation

\[
(G^{(0)}_{0,N}v)(x) = \int_{|z| < \varepsilon} x^{-z} \pi_{0,N}\left(\frac{\tilde{v}(0)}{z} + \frac{\tilde{v}'(0)}{z}\right) dz = \log x \pi_{0,N}\tilde{v}(0) + \pi_{0,N}\tilde{v}'(0),
\]

showing that

\[
\delta_{0,N} = \{e_0 + e_1 \log x \mid e_0, e_1 \in E_0^N\} = E_0^N \otimes 1 + E_0^N \otimes \log x.
\]

By definition, \( G_{0,N} = G^{(0)}_{0,N} \) for \( \delta < 1 \), while for \( \delta \geq 1 \), similarly as before,

\[
g_{1,N}(z)(\Pi_0 f_{0,N}^{-1}\tilde{v})(z) = f_{0,N}(z - 1)^{-1}H'(0)\pi_{0,N}\left(\frac{\tilde{v}(0)}{z} + \tilde{v}'(0)\right).
\]

Therefore

\[
(G^{(1)}_{0,N}v)(x) = xa_N\pi_{0,N}\tilde{v}(0), \quad a_N := (1 + \Delta_{Y,N}(0))^{-1}H'(0) \in \mathcal{O}^\infty(Y).
\]

We conclude that

\[
\delta_{0,N} = \begin{cases} \tilde{\delta}_{0,N}, & \delta < 1 \\ \{e_0 + e_1(\log x + xa_N) \mid e_0, e_1 \in E_0^N\}, & \delta \geq 1. \end{cases}
\]

The isomorphism \( \theta_0 : \delta_{0,N} \to \tilde{\delta}_{0,N} \) is the identity map in case \( \delta < 1 \), otherwise

\[
\theta_0(e_0 + e_1(\log x + xa_N)) = e_0 + e_1 \log x, \quad e_0, e_1 \in E_0^N.
\]

As a result of this computation we obtain:

Corollary 6.8. If \( \mathcal{E}_{0,N} = E_0^N \otimes 1 \), then \( \mathcal{E}_{0,N} = E_0^N \otimes 1 \).

Theorem 6.9. Let \( \gamma \) be as in (E2) and \( 1 < p < +\infty \). If \( \Delta_N \) denotes the extension of the Neumann Laplacian with domain

\[
\mathcal{D}(\Delta_N) = \mathcal{H}^{2\gamma+2}_{L^p}(\mathbb{D})_N \oplus \omega\mathcal{E}_{0,N},
\]

then \( c - \Delta_N \) has a bounded \( H_\infty \)-calculus in \( \mathcal{H}^{0,\gamma}_{L^p}(\mathbb{D}) \) for sufficiently large \( c > 0 \).

Proof. Clearly, the Neumann Laplacian satisfies condition (E1). By our choice of \( \gamma \), also (E2) holds. Finally, Theorem 6.7 in connection with Corollary 6.8 implies condition (E3). Hence the assertion follows from Theorem 6.6.
7. The porous medium equation on conic manifolds with boundary

Following up on the investigations in [13, 20] and [25] for the case of conic manifolds without boundary, we shall show how the above results can be applied to the porous medium equation

\[ u'(t) - \Delta u^m(t) = f(t, u) \quad \text{in } \mathbb{D} \text{ for } t \in (0, T), \]

\[ \gamma_N u(t) = 0 \quad \text{on } \partial \mathbb{D} \text{ for } t \in (0, T), \]

\[ u(0) = u_0 \quad \text{in } \mathbb{D}, \]

where \( m > 0, T > 0, f \) is a forcing term and \( u_0 \) is some given initial datum.

As long as \( u \) is strictly positive, we can make the transformation \( u = v^m \) and obtain the equivalent system

\[ v'(t) - mv^{(m-1)/m}\Delta v(t) = g(t, v) \quad \text{in } \mathbb{D} \text{ for } t \in (0, T), \]

\[ \gamma_N v(t) = 0 \quad \text{on } \partial \mathbb{D} \text{ for } t \in (0, T), \]

\[ v(0) = v_0 \quad \text{in } \mathbb{D}, \]

with \( g(t, v) = f(t, v^{1/m}) \). In the sequel we will assume that \( g \) is holomorphic in \( v \) and Lipschitz in \( t \).

Equation (7.1) is a quasilinear evolution equation to which we will apply the following theorem of Clément and Li.

**Theorem 7.1.** Consider the quasilinear evolution equation

\[ v'(t) + A(v(t))v(t) = g(t, v), \quad v(0) = v_0. \]

Let \( X_0 \) and \( X_1 \) be Banach spaces and \( V \) an open neighborhood of \( v_0 \) in the real interpolation space \( X_{1-1/q,q} = (X_0, X_1)_{1-1/q,q} \) such that \( A(v_0) : X_1 \to X_0 \) has maximal \( L^q \)-regularity and that, for some \( T_0 > 0, \)

\[ (H1) \ A \in C^1([0, T_0]; X_1, X_0)), \]

\[ (H2) \ f \in C^{1-1/q}([0, T_0] \times X_0). \]

Then there exists a \( T \in (0, T_0] \) and a unique solution \( v \in L_q(0, T; X_1) \cap W_q^{1,0}(0, T; X_0) \) on \( (0, T) \). In particular, \( v \in C([0, T]; X_{1-1/q,q}) \) by [21] Theorem III.4.10.2].

A central property is the maximal \( L^q \)-regularity of the operator \( A(v_0) \). We recall that all the Mellin-Sobolev spaces used here are UMD Banach spaces and therefore the existence of a bounded \( H_\infty \)-calculus implies the \( \mathcal{R} \)-sectoriality for the same sector according to Clément and Prüss, [3] Theorem 4]. Moreover, every operator, which is \( \mathcal{R} \)-sectorial on \( \Lambda(\theta) \) for \( \theta < \pi/2 \), has maximal \( L^q \)-regularity, \( 1 < q < +\infty \), see Weis [20] Theorem 4.2].

For \( \gamma \) and \( \delta \) as in (6.1) we fix \( 1 < p, q < +\infty \) such that

\[ \frac{n+1}{p} + \frac{2}{q} < 1 \quad \text{and} \quad \frac{2}{q} < \delta. \]

We shall apply the theorem of Clément and Li with \( A_c(v) = c - mv^{(m-1)/m}\Delta_N \), where \( \Delta_N \) is the realization of the Neumann Laplacian with the domain in \([6,10]\), and the Banach spaces \( X_0 = H^0(p,\gamma)(\mathbb{D}) \) and \( X_1 = H^0(p,\gamma+\delta)(\mathbb{D}) \oplus \omega\theta_{0,N} = \mathcal{D}(\Delta_N) \).

7.1. Interpolation spaces. The following observation will be useful in the sequel.

**Lemma 7.2.** Let \( X_0, X_1 \) and \( U \) be Banach spaces, all continuously embedded in the same Hausdorff topological vector space. Assume that \( U \) has finite dimension. Then

\[ (X_0 + U, X_1 + U)_{\theta,q} = (X_0, X_1)_{\theta,q} + U \]

for every choice of \( 1 < q < +\infty \) and \( 0 < \theta < 1 \).
Proposition 7.4. Let \( R \) be a topological complement of \( U \cap X_j \) in \( X_j \). Then \( X_j + U = X_j \oplus U \) with equivalent norms. Write \( x = x_0 + x_1 \) with \( x_i \in X_i \) and \( u \in U \). Since the norms are equivalent, there exists a \( C \geq 0 \) such that
\[
\|x_0\|_X + t\|x_1\|_X \leq \|x_0\|_X + t\|x_1\|_X + \|u_0\| + t\|u_1\| \leq C(\|x_0\|_X + \|u_0\| + t\|x_1 + u_1\|_X + \|u_1\|_X)
\]
for every \( t > 0 \), whenever \( x = x_0 + x_1 \) with \( x_i \in X_i \) and \( u = u_0 + u_1 \) with \( u_j \in U \). By passing to the infimum over all such representations we find
\[
K(t, x; X_0 + U, X_1 + U),
\]
where \( K(\cdot) \) is the usual \( K \)-functional in the definition of the real interpolation method. It follows that \( \tilde{x} \in (X_0, X_1)_{\theta,q} \hookrightarrow (X_0, X_1)_{\theta,q} \).

Proof. Let \( s_1, s_0, \gamma_1, \gamma_0 \in \mathbb{R} \), \( 1 < p, q < +\infty \), and \( 0 < \theta < 1 \) be arbitrary. Then
\[
H_{p+r}^{s+c} (\mathbb{D}) \hookrightarrow (H_{p+r}^{s_0, \gamma_0} (\mathbb{D}), H_{p+r}^{s_1, \gamma_1} (\mathbb{D}))_{\theta,q} \hookrightarrow H_{p+r}^{s+c, \gamma} (\mathbb{D})
\]
for all \( \varepsilon > 0 \), where \( s = (1 - \theta)s_0 + \theta s_1 \) and \( \gamma = (1 - \theta)\gamma_0 + \theta \gamma_1 \).

Proof. Let \( 2\mathbb{D} \) be the smooth manifold with boundary obtained by gluing two copies of \( \mathbb{D} \) along \( \{0\} \times Y \). It is then well-known that
\[
H_{p+r}^{s+c} (2\mathbb{D}) \hookrightarrow (H_{p+r}^{s_0} (2\mathbb{D}), H_{p+r}^{s_1} (2\mathbb{D}))_{\theta,p} \hookrightarrow H_{p+r}^{s+c} (2\mathbb{D})
\]
Let \( 2Y \) denote a closed manifold containing \( Y \). Proceeding as in the proof of [4], Lemma 5.4 and using duality, one finds that
\[
H_{p+r}^{s+c, \gamma} (\mathbb{R}^+ \times 2Y) \hookrightarrow (H_{p+r}^{s_0, \gamma_0} (\mathbb{R}^+ \times 2Y), H_{p+r}^{s_1, \gamma_1} (\mathbb{R} \times 2Y))_{\theta,p} \hookrightarrow H_{p+r}^{s+c, \gamma} (\mathbb{R}^+ \times 2Y).
\]
With the help of a continuous extension operator as well as the restriction operator, one finds the latter embeddings also for the spaces on \( \mathbb{R}^+ \times Y \). By a standard partition of unity argument we obtain (7.6) in case \( p = q \). The general case follows from the embedding results for interpolation spaces:
\[
(X_0, X_1)_{\theta,p} \hookrightarrow (X_0, X_1)_{\theta',p} \hookrightarrow (X_0, X_1)_{\theta'',p} \text{, } \theta'' < \theta' < \theta, 1 < p, q < +\infty,
\]
see [1] (2.5.2).
Hence, for \( \theta > \rho \),
\[
(c - \Delta_N)^{\theta} = \frac{\sin \pi \theta}{\pi} \int_0^\infty s^{-\theta} R_N(s) \, ds
\]
with the integral converging in \( L^p(\mathcal{H}^{0,\gamma}_p(D), \mathcal{H}^{2,\gamma}_p(D)) \). For \( 2\rho > 1 + 1/p \) we find that
\[
\gamma_N(c - \Delta_N)^{\theta} = 0,
\]
so that condition (H1) is fulfilled. As (H2) holds by assumption, the theorem of Clément and Li shows the existence of some \( g > 0 \) such that \( \gamma_N v^{(m^{-1})} \Delta_N \) is \( \mathcal{R} \)-sectorial of angle \( \theta \).

In particular, \( \mathcal{A}(v) \) has maximal \( L^2 \)-regularity.

This follows with minor modifications as in case without boundary, see the proof of Theorem 6.1 in [19].

Remark 7.6. Combining the proof of [19] Theorem 6.1 with the method used in the proof of [7] Theorem 5.7, we obtain the above result even for the case of a continuous function \( v \) on \( D \) which is bounded away from zero.

Theorem 7.7. Choose \( \gamma, p \) and \( q \) as in (7.2) and (7.3). Then the porous medium equation (7.1) has a unique short time solution
\[
v \in W^{1, q}(0, T; \mathcal{H}^{0, \gamma}_p(D)) \cap L^q(0, T; \mathcal{H}^{2, \gamma + 2}_p(D)_N \oplus \omega_0(N))
\]
for every strictly positive initial datum \( v_0 \) in \( (X_0, X_1)_{1-1/q, q} \).

In particular, \( v \in C([0, T]; (X_0, X_1)_{1-1/q, q}) \) which, according to Proposition 7.5, embeds into \( C([0, T]; \mathcal{H}^{2, \gamma + 2}_p(D)_N \oplus \omega_0(N)) \) for every \( \varepsilon > 0 \). If \( g \) is independent of \( t \), then we additionally have \( v \in C^\infty((0, T); \mathcal{H}^{2, \gamma + 2}_p(D)_N \oplus \omega_0(N)) \).

Proof. According to Proposition 7.5, the operator \( \mathcal{A}(v_0) \) has maximal regularity. Choose a neighborhood \( V \) of \( v_0 \) in \( (X_0, X_1)_{1-1/q, q} \) such that \( 0 < c_1 \leq \Re v \leq c_2 \) for all \( v \in V \) and positive constants \( c_1, c_2 \). Since the interpolation space embeds into \( C(D) \), the space of continuous functions on the compact space \( D \), the mapping \( v \mapsto mv^{(m^{-1})} \) is a smooth map from \( V \) to \( C(D) \); its range consists of functions with real part bounded and bounded away from zero. In particular,
\[
v \mapsto A(v) \in C^\infty(V; \mathcal{L}(X_1, X_0)),
\]
so that condition (H1) is fulfilled. As (H2) holds by assumption, the theorem of Clément and Li shows the existence of some \( T > 0 \) and a unique
\[
v \in W^{1, q}(0, T; \mathcal{H}^{0, \gamma}_p(D)) \cap L^q(0, T; \mathcal{H}^{2, \gamma + 2}_p(D)_N \oplus \omega_0(N))
\]
solving (7.7) together with (7.6). In particular, \( v \in C([0, T]; (X_0, X_1)_{1-1/q, q}) \). If \( g \) is independent of \( t \), (7.7) together with [16] Theorem 5.2.1 implies that \( v \in C^\infty((0, T); X_1) \). \( \square \)

Remark 7.8. As \( v \) is continuous, bounded and strictly positive, \( u = v^{1/m} \) furnishes a solution \( u \in W^{1, q}(0, T; \mathcal{H}^{0, \gamma}_p(D)) \cap L^q(0, T; \mathcal{H}^{2, \gamma + 2}_p(D)_N \oplus \omega_0(N)) \) to the porous medium equation in the original form, see [20] Remark 2.12.

8. Appendix

By \( \omega, \omega_1, \omega_2 \) we denote cut-off functions near \( x = 0 \), i.e. smooth non-negative functions on \( D \), supported in \([0, 1) \times Y \), equal to 1 for small \( x \). For simplicity of the presentation we shall mostly omit the reference to the vector bundles.
8.1. Function spaces on conic manifolds with boundary. We briefly recall the definition of the function spaces used in this article. More details can be found for example in [14] or [12].

We denote by $2Y$ the double of $Y$; it is a closed manifold. Then

$$
\begin{align*}
H^s_p(\mathbb{R} \times Y) := & H^s_p(\mathbb{R} \times 2Y)|_{\mathbb{R} \times Y}, \\
H^s_0(\mathbb{R} \times Y) := & \left\{ u \in H^s_p(\mathbb{R} \times 2Y) \mid \text{supp } u \subseteq \mathbb{R} \times Y \right\},
\end{align*}
$$

where one uses the product structure on $\mathbb{R} \times 2Y$. The inner product of $L^2(\mathbb{R} \times Y)$ yields an identification of the dual space of $H^s_p(\mathbb{R} \times Y)$ with $\check{H}^{-s}_p(\mathbb{R} \times Y)$, where $1/p + 1/p' = 1$.

We let $\mathcal{C}^\infty(Y)$ be the space of smooth (up to and including the boundary) functions on $Y$, and

$$
\mathcal{C}^\infty_c(\mathbb{R} \times Y) = \mathcal{C}^\infty_c(\mathbb{R}, \mathcal{C}^\infty(Y)), \quad \mathcal{S}(\mathbb{R} \times Y) = \mathcal{S}(\mathbb{R}, \mathcal{C}^\infty(Y)).
$$

We extend the map

$$
S_\gamma : \mathcal{C}^\infty_c(\mathbb{R}_+ \times Y) \rightarrow \mathcal{C}^\infty_c(\mathbb{R} \times Y), \quad (S_\gamma u)(r, y) = e^{(\gamma-1/2)r}u(e^{-r}, y)
$$

to the dual (distribution) spaces. Then we define

$$
H^s_\gamma(\mathbb{R}_+ \times Y) := S_\gamma^{-1}(H^s_p(\mathbb{R} \times Y)), \quad \gamma \in \mathbb{R},
$$

with the canonically induced norm; analogously we define $\check{H}^s_\gamma(\mathbb{R}_+ \times Y)$. Moreover, we let $T^\gamma(\mathbb{R}_+ \times Y)$ be the pre-image of $\mathcal{S}(\mathbb{R} \times Y)$ under $S_\gamma^{-1}$. Note that $H^s_\gamma(\mathbb{R}_+ \times Y)$ for $s \in \mathbb{N}_0$ and $1 \leq p < +\infty$ consists of all functions $u(x, y)$ such that

$$
x^{\frac{\gamma - 1}{2} - j}(x \partial_x)^j \partial_y^ju(x, y) \in L^p(\mathbb{R}_+ \times Y, \frac{dxdy}{x}), \quad j + |\alpha| \leq s.
$$

8.1.1. Function spaces on $\mathbb{D}$ and $\mathbb{B}$.

**Definition 8.1.** $H^s_\gamma(\mathbb{D})$ denotes the space of all $u \in H^s_\gamma(\mathbb{D}_{reg})$ such that $\omega u \in H^s_\gamma(\mathbb{R}_+ \times Y)$, equipped with the norm

$$
\|u\|^2_{H^s_\gamma(\mathbb{D})} = \|\omega u\|^2_{H^s_\gamma(\mathbb{R}_+ \times Y)} + \|(1 - \omega)u\|^2_{H^s_\gamma(2\mathbb{D})},
$$

where $2\mathbb{D}$ is the double of $\mathbb{D}$ obtained by gluing two copies of $\mathbb{D}$ along $\{0\} \times Y$. Analogously one defines the space $\check{H}^s_\gamma(\mathbb{D})$.

The inner product of $H^s_\gamma(\mathbb{D})$ allows for the identification

$$
H^s_\gamma(\mathbb{D})' = \check{H}^{-s}_\gamma(\mathbb{D}).
$$

**Definition 8.2.** Let $\mathcal{C}^{\infty,\gamma}(\mathbb{D})$ denote the space of all $u \in \mathcal{C}^{\infty}(\mathbb{D}_{reg})$ such that $\omega u \in T^\gamma(\mathbb{R}_+ \times Y)$. Moreover, $\mathcal{C}^{\infty,\infty}(\mathbb{D}) := \cap_{\gamma} \mathcal{C}^{\infty,\gamma}(\mathbb{D})$.

Replacing $Y$ by $\partial Y$ and $\mathbb{D}$ by $\mathbb{B}$ one obtains the spaces $H^s_\gamma(\mathbb{B})$, $\mathcal{C}^{\infty,\gamma}(\mathbb{B})$, and $\mathcal{C}^{\infty,\infty}(\mathbb{B})$. Interpolation then furnishes the Besov spaces $H^s_{p,q}(\mathbb{D})$ on $\mathbb{B}$.

8.1.2. Function spaces on model cones. Recall that we write $Y^\wedge = \mathbb{R}_+ \times Y$.

**Definition 8.3.** $T^\gamma(Y^\wedge)$ is the space of all $u \in \mathcal{C}^{\infty}(Y^\wedge)$ such that $\omega u \in T^\gamma(Y^\wedge)$ and $(1 - \omega)u \in T^\gamma(\mathbb{R} \times Y)$. Moreover, $\mathcal{S}^{\infty}(Y^\wedge) := \cap_{\gamma} \mathcal{S}^{\infty}(Y^\wedge)$.

Let $(U_1, \kappa_1, \ldots, U_N, \kappa_N)$ be an atlas of the manifold $2Y$. Using the charts $(\mathbb{R} \times U_j, \kappa_j)$ with $\kappa_j(r, y) = (r, \gamma\kappa_j(y))$, allows to define the Sobolev spaces $H^s_{p,\infty}(\mathbb{R} \times 2Y)$ as the pullback of the standard Sobolev spaces, see [22] Section 4.2 for details.

As in [8.1] one obtains the spaces $H^s_{p,\infty}(\mathbb{R} \times Y)$ and $\check{H}^s_{p,\infty}(\mathbb{R} \times Y)$. 

Definition 8.4. $K_{p}^{\gamma}(Y^{\wedge})$ denotes the space of all $u \in H_{p,loc}^{s}(Y^{\wedge})$ such that $\omega u \in H_{p}^{s,\gamma}(\mathbb{D})$ and $(1 - \omega)u \in H_{p,\infty}^{s}(\mathbb{R} \times Y)$; an analogous construction yields $K_{p}^{\gamma}(Y^{\wedge})$.

Similarly one defines the spaces $\mathcal{S}^{\gamma}(\partial Y^{\wedge})$, $\mathcal{S}^{\infty}(\partial Y^{\wedge})$, and $K_{p}^{\gamma}(\partial Y^{\wedge})$ with $\partial Y^{\wedge} = \mathbb{R}_{+} \times \partial Y$. Interpolation yields the Besov spaces $K_{p}^{\gamma}(\partial Y^{\wedge})$ on $\partial Y^{\wedge}$.

8.2. Elements of a parameter-dependent edge type calculus on manifolds with boundary and conical singularities. We recall a few basic facts concerning a parameter-dependent calculus on conic manifolds with boundary. This is a version of the boundary edge calculus as developed in [12 Section 7.2] and [13 Section 4] for the case where the edge is a single point.

By $B^{0,\beta}(Y; \Sigma)$ we denote the parameter-dependent elements of order $\mu$ and type $d$ in Boutet de Monvel’s calculus with parameter space $\Sigma$. See Section 9 in [5] for a concise presentation. We write $\Gamma_{\beta} = \{ z \in \mathbb{C} \mid \text{Re} z = \beta \}$.

For hermitian vector bundle $E_{0}, E_{1}$ over $Y$ and $F_{0}, F_{1}$ over $\partial Y$, $\gamma_{0}, \gamma_{1} \in \mathbb{R}$ and $\ell_{0}, \ell_{1} \in \mathbb{N}_{0}$ we write $E^{\gamma}_{j}$ and $F^{\gamma}_{j}$ for the pullback of $(E_{j})_{| x = 0}$ and $(F_{j})_{| x = 0}$ to $Y^{\wedge}$ and $\partial Y^{\wedge}$ and define the spaces

\[
\mathcal{K}_{2}^{\gamma_{0}}(Y^{\wedge}; E^{\gamma}_{0}, F^{\gamma}_{0}, C^{\ell}_{0}) = \mathcal{K}_{2}^{\gamma_{1}}(Y^{\wedge}; E^{\gamma}_{1}, F^{\gamma}_{1}, C^{\ell}_{1}),
\]

\[
\mathcal{S}^{\gamma_{0}}(Y^{\wedge}; E^{\gamma}_{0}, F^{\gamma}_{0}, C^{\ell}_{0}) = \mathcal{S}^{\gamma_{1}}(Y^{\wedge}; E^{\gamma}_{1}, F^{\gamma}_{1}, C^{\ell}_{1}).
\]

Definition 8.5. For $\nu \in \mathbb{Z}$, $d \in \mathbb{N}_{0}$, $\gamma_{0}, \gamma_{1} \in \mathbb{R}$ we denote by $R_{G}^{\nu,0}(\mathbb{D}, \Sigma; \gamma_{0}, \gamma_{1})$ the space of all operator families $g(\sigma), \sigma \in \Sigma$ such that, for some $\varepsilon > 0$,

\[
g(\sigma) \in S^{0}_{\delta}(\Sigma; \mathcal{K}_{2}^{\gamma_{0}}(Y^{\wedge}; E^{\gamma}_{0}, F^{\gamma}_{0}, C^{\ell}_{0}), \mathcal{S}^{\gamma_{0}+\varepsilon}(Y^{\wedge}; E^{\gamma}_{0}, F^{\gamma}_{0}, C^{\ell}_{0})),
\]

\[
g^{*}(\sigma) \in S^{0}_{\delta}(\Sigma; \mathcal{K}_{2}^{\gamma_{1}-\varepsilon}(Y^{\wedge}; E^{\gamma}_{1}, C^{\ell}_{1}), \mathcal{S}^{\gamma_{1}+\varepsilon}(Y^{\wedge}; E^{\gamma}_{1}, C^{\ell}_{1})),
\]

where the asterisk denotes the pointwise adjoint with respect to the inner product in $\mathcal{K}_{2}^{\gamma_{0}}(Y^{\wedge}; E^{\gamma}_{0}, F^{\gamma}_{0}, C^{\ell}_{0})$ and $\mathcal{K}_{2}^{\gamma_{1}}(Y^{\wedge}; E^{\gamma}_{1}, C^{\ell}_{1})$.

For $d > 0$, the space $R_{G}^{d,0}(\mathbb{D}, \Sigma; \gamma_{0}, \gamma_{1})$ consists of all operator families of the form

\[
\sum_{j=0}^{d} g_{j}(\sigma) \partial_{\nu}^{j} : \mathcal{K}_{2}^{\gamma_{0}}(Y^{\wedge}; E^{\gamma}_{0}, F^{\gamma}_{0}, C^{\ell}_{0}) \rightarrow \mathcal{S}^{\gamma_{0}+\varepsilon}(Y^{\wedge}; E^{\gamma}_{0}, F^{\gamma}_{0}, C^{\ell}_{0})), \quad s > d - 1/2,
\]

with $g_{j} \in R_{G}^{\nu,0}(\mathbb{D}, \Sigma; \gamma_{0}, \gamma_{1})$ and the normal derivative $\partial_{\nu}$.

In the previous definition, the data $E_{j}$, $F_{j}$ and $C^{\ell}_{j}$ should be part of the notation; we have omitted them here for better legibility.

Similarly to the above notation we let

\[
\mathcal{H}_{2}^{\gamma_{0}}(\mathbb{D}; E_{j}, F_{j}, C^{\ell}_{j}) = \mathcal{H}_{2}^{\gamma_{1}}(\mathbb{D}; E_{j}) \oplus \mathcal{H}_{2}^{\gamma_{1}+\varepsilon}(\mathbb{B}; F_{j}) \oplus C^{\ell}_{j},
\]

\[
\mathcal{G}^{\gamma_{0}+\varepsilon}(\mathbb{D}; E_{j}, F_{j}, C^{\ell}_{j}) = \mathcal{G}^{\gamma_{1}-\varepsilon}(\mathbb{D}; E_{j}) \oplus \mathcal{G}^{\gamma_{1}+\varepsilon}(\mathbb{B}; F_{j}) \oplus C^{\ell}_{j}.
\]

Definition 8.6. The space $C_{G}^{\nu,0}(\mathbb{D}, \Sigma; \gamma_{0}, \gamma_{1})$ consists of all operator-families $g(\sigma)$ of the form

\[
g(\sigma) = \omega_{1} a(\sigma) \omega_{0} + r(\sigma),
\]

where $\omega_{0}, \omega_{1}$ are cut-off functions, $a \in R_{G}^{\nu,0}(\mathbb{D}, \Sigma; \gamma_{0}, \gamma_{1})$, and $r$ has an integral kernel in

\[
\mathcal{S}(\Sigma; \mathcal{G}^{\gamma_{0}+\varepsilon}(\mathbb{D}, E_{1}, F_{1}, C^{\ell}_{1}) \hat{\otimes} \mathcal{G}^{\gamma_{0}+\varepsilon}(\mathbb{D}, E_{0}, F_{0}, C^{\ell}_{0})),
\]

for some $\varepsilon = \varepsilon(g) > 0$. For $d \geq 1$, $C_{G}^{\nu,d}(\mathbb{D}, \Sigma; \gamma_{0}, \gamma_{1})$ is the space of all operator families $\sum_{j=0}^{d} g_{j}(\sigma) \partial_{\nu}^{j}$ with $g_{j} \in C_{G}^{\nu,0}(\mathbb{D}, \Sigma; \gamma_{0}, \gamma_{1})$. 
We define the principal operator-valued symbol $\sigma^\nu_\gamma(g)$ of $g$ to be the principal operator-valued symbol of $a$.

**Definition 8.7.** A holomorphic Mellin symbol of order $\mu$ and type $d$, depending on the parameter $\sigma \in \Sigma$, is a holomorphic function $h : \mathbb{C} \to B^{\mu,d}(Y; \Sigma)$ such that

$$\delta \mapsto h(\delta + it, \sigma) : \mathbb{R} \to B^{\mu,d}(Y; \mathbb{R} \times \Sigma)$$

is continuous. We denote the space of all such symbols by $M^{\mu,d}_\Sigma(Y; \Sigma)$ and set

$$M^{\mu,d}_{\Omega}(\mathbb{R} \times Y; \Sigma) = \mathcal{C}^\infty(\mathbb{R}_+ \times \mathcal{O}^{\mu,d}_\Sigma(Y; \Sigma)).$$

The space $M^{\gamma-n/2,d}_\Sigma(Y)$ of smoothing Mellin symbols of type $d$ consists of all maps $h_0$ which are holomorphic in an $\varepsilon$-strip around the line $\Gamma_{\varepsilon\gamma^{-\infty}}$, $\varepsilon$ arbitrarily small, taking values in $B^{-\infty,d}(Y)$.

For every $\sigma \in \Sigma$, a holomorphic Mellin symbol $h$ and a smoothing Mellin symbol $h_0$ define an operator

$$\text{op}^{\gamma-n/2}_M(h + h_0)(\sigma) : \mathcal{C}^\infty(\mathbb{R}_+ \times Y, E_0^n) \oplus \mathcal{C}^\infty(\mathbb{R}_+ \times \partial Y, F_0^n) \to \mathcal{C}^\infty(\mathbb{R}_+ \times Y, E_1^n) \oplus \mathcal{C}^\infty(\mathbb{R}_+ \times \partial Y, F_1^n)$$

by

$$(8.4) \quad [\text{op}^{\gamma-n/2}_M(h + h_0)(\sigma)u](x) = \int_{\Gamma_{\frac{1}{2}, -\gamma}} x^{-z}(h(x,z,x\sigma) + h_0(z))(\tilde{u})(z) \, dz,$$

where $\tilde{u}$ denotes the Mellin transform.

In the sequel, we will consider $\text{op}^{\gamma-n/2}_M(h + h_0)$ also as an operator

$$\text{op}^{\gamma-n/2}_M(h + h_0) : K^\nu_{\Sigma}(Y; E_0^n, F_0^n, \mathcal{C}^{\nu,0}) \to K^\nu_{\Sigma}(Y; E_1^n, F_1^n, \mathcal{C}^{\nu,1}),$$

by identifying $\text{op}^{\gamma-n/2}_M(h + h_0)$ with the operator-matrix

$$\begin{pmatrix}
\text{op}^{\gamma-n/2}_M(h + h_0) & 0 \\
0 & 0
\end{pmatrix}.$$

**Definition 8.8.** $C^{\mu,d}(\mathbb{D}, \Sigma; \gamma, \gamma - \nu)$ denotes the space of all operator-valued symbols

$$(8.6) \quad b(\sigma) = x^{-\nu}\omega \text{op}^{\gamma-n/2}_M(h)(\sigma)\omega_1 + (1 - \omega)b_{\text{int}}(\sigma)(1 - \omega_2)
+ x^{-\nu}\tilde{\omega}(x[\sigma])\text{op}^{\gamma-n/2}_M(h_0)\tilde{\omega}(x[\sigma]) + g(\sigma),$$

where $h$ and $h_0$ are as above, $b_{\text{int}} \in B^{\nu,d}(2\mathbb{D}; \Sigma)$, $\omega, \omega_1, \omega_2$ and $\tilde{\omega}$ are cut-off functions near $x = 0$, $\omega_1 = \omega$, $\omega_2 = \omega_2$, and $g \in C^{\nu,0}_\Sigma(\mathbb{D}, \Sigma; \gamma, \gamma - \nu)$.

The principal edge symbol associated with the operator-valued symbol $b$ in $(8.6)$ is

$$(8.7) \quad \sigma^\nu_\gamma(b)(\sigma) = x^{-\nu}\text{op}^{\gamma-n/2}_M(h|_{x=0})(\sigma)
+ x^{-\nu}\tilde{\omega}(x[\sigma])\text{op}^{\gamma-n/2}_M(h_0)\tilde{\omega}(x[\sigma]) + \sigma^\nu_\gamma(g)(\sigma),$$

where $\sigma^\nu_\gamma(g)$ is the principal operator-valued symbol of $g$; it is a map

$$K^\nu_{\Sigma}(Y; E_0, F_0, \mathcal{C}^{\nu,0}) \to K^\nu_{\Sigma}(Y; E_1, F_1, \mathcal{C}^{\nu,1}).$$

This definition follows the approach in [13] Section 4.6 and uses the alternative representation of the symbols in [13, Theorem 4.6.29], going back to [10].

Localized to any open $U \subseteq \mathbb{D}_{\text{reg}}$, the operator $b(\sigma)$ is given by a parameter-dependent operator $B_U \in B^{\mu,d}(2\mathbb{D}; \Sigma)$ (here we make the same identification as in $(8.3)$). The principal symbols of these patches to a smooth homogeneous interior principal symbol and a smooth homogeneous boundary symbol on $D^{\nu}_{\text{reg}} \setminus 0$ and $D^{\nu}_{\text{reg}} \setminus 0$, respectively. Similarly as for the symbols introduced after $(2.1)$, these degenerate as $x \to 0$ and can be rescaled as explained in $(2.2)$ and $(2.3)$. 

This is a typical example of how techniques from complex analysis can be applied to problems in partial differential equations, particularly in the study of symbols and their transformations. The Mellin transform, for instance, is a tool that allows for the analysis of operators and their symbols in different domains of the parameter $\sigma$. The definition of the principal operator-valued symbol is crucial for understanding the behavior of these operators near singularities. The approach taken in the document emphasizes the use of holomorphic symbols and Mellin transforms to study operators on conic manifolds with boundary, providing a framework for dealing with the complexities that arise in such geometries.
We call $b$ elliptic, if the interior principal symbol, the rescaled interior principal symbol, the boundary symbol, the rescaled boundary symbol and the operator-valued symbol $\sigma^{\nu}_{\gamma}(\sigma)$ are all invertible. Following [13] Section 4.5 we obtain:

**Theorem 8.9.** Let $b \in C^{\nu,d}(\Sigma; \gamma, \gamma - \nu)$, $d \leq \max\{\nu, 0\}$, be elliptic. Then there exists a parametrix $c \in C^{-\nu,d}(\Sigma; \gamma, \gamma - \nu)$, $d' \leq \max\{-\nu, 0\}$ which inverts $b$ modulo smoothing Green operators, i.e.

$$bc - 1 \in C^{-\infty,d'}(\Sigma; \gamma - \nu, \gamma - \nu) \text{ and } cb - 1 \in C^{-\infty,d}(\Sigma; \gamma, \gamma).$$

### References

1. H. Amann. *Linear and Quasilinear Parabolic Problems.* Monographs in Mathematics **89**, Birkhäuser Verlag, Basel, 1995.

2. O. Bilyj, E. Schrohe and J. Seiler. $H_{\infty}$-calculus for hypoelliptic pseudodifferential operators. Proc. Amer. Math. Soc. **138**, no. 5, 1645-1656 (2010).

3. P. Clément and J. Prüss. An operator-valued transference principle and maximal regularity on vector-valued $L_p$-spaces. In G. Lumer and L. Weis (eds.), *Proc. of the 6th. International Conference on Evolution Equations.* Marcel Dekker, New York, 2001.

4. S. Coriasco, E. Schrohe and J. Seiler. Differential operators on conic manifolds: Maximal regularity and parabolic equations. Bull. Soc. Roy. Sci. Liège **70**, no. 4, 207–229 (2001).

5. S. Coriasco, E. Schrohe and J. Seiler. Realizations of differential operators on conic manifolds with boundary. Ann. Global Anal. Geom. **31**, no. 3, 223–285 (2007).

6. S. Coriasco, E. Schrohe and J. Seiler. Bounded $H_{\infty}$-calculus for differential operators on conic manifolds with boundary. Comm. Partial Differential Equations **32**, no. 2, 229–255 (2007).

7. R. Denk, M. Hieber and J. Prüss. $R$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Amer. Math. Soc. **166**, 2003.

8. R. Denk, J. Saal and J. Seiler. Bounded $H_{\infty}$-calculus for pseudodifferential Douglis-Nirenberg systems of mild regularity. Math. Nachr. **282**, no. 3, 1-22 (2009).

9. J. Gil, T. Krainer and G. Mendoza. Geometry and spectra of closed extensions of elliptic cone operators. Canad. J. Math. **59**, no. 4, 742–794 (2007).

10. J.B. Gil, B.-W. Schulze and J. Seiler. Cone pseudodifferential operators in the edge symbolic calculus. Osaka J. Math. **37**, no. 1, 221–260 (2000).

11. G. Grubb. *Functional Calculus of Pseudodifferential Boundary Problems.* Progress in Mathematics **95**, Birkhäuser Boston, Boston, MA, 1996.

12. G. Harutyunyan and B.-W. Schulze. *Elliptic Mixed, Transmission and Singular Crack Problems,* EMS Tracts in Mathematics **4**, European Mathematical Society (EMS), Zürich, 2008.

13. D. Kapanadze and B.-W. Schulze. *Crack Theory and Edge Singularities.* Mathematics and its applications **561**, Kluwer Academic Publishers, Dordrecht, 2003.

14. T. Krainer. Resolvents of elliptic boundary problems on conic manifolds. Comm. Partial Differential Equations **32**, no. 2, 257–315 (2007).

15. P. Lopes and E. Schrohe. Spectral Invariance of Pseudodifferential Boundary Value Problems on Manifolds with Conical Singularities. J. Fourier Anal. Appl. **25**, no. 3, 1147-1202 (2019).

16. J. Prüss and G. Simonett. *Moving Interfaces and Quasilinear Parabolic Evolution Equations.* Monographs in Mathematics **105**, Birkhäuser/Springer, Cham, 2016.

17. N. Roidos and E. Schrohe. *The Cahn-Hilliard equation and the Allen-Cahn equation on manifolds with conical singularities.* Comm. Partial Differential Equations **38**, no. 5, 925–943 (2013).

18. N. Roidos and E. Schrohe. Bounded imaginary powers of cone differential operators on higher order Mellin-Sobolev spaces and applications to the Cahn-Hilliard equation. J. Differential Equations **257**, no. 3, 611–637 (2014).

19. N. Roidos and E. Schrohe. Existence and maximal $L^p$-regularity of solutions for the porous medium equation on manifolds with conical singularities. Comm. Partial Differential Equations **41**, no. 9, 1441–1471 (2016).

20. N. Roidos and E. Schrohe. Smoothness and long time existence for solutions of the porous medium equation on manifolds with conical singularities. Comm. Partial Differential Equations **43**, no. 10, 1456–1484 (2018).

21. E. Schrohe and B.-W. Schulze. *Boundary value problems in Boutet de Monvel’s algebra for manifolds with conical singularities.* I: Pseudo-differential calculus and mathematical physics, 97–209, Math. Top., **5**, Adv. Partial Differential Equations, Akademie Verlag, Berlin, 1994.
[22] E. Schrohe and B.-W. Schulze. *Boundary value problems in Boutet de Monvel’s algebra for manifolds with conical singularities*, II. Boundary value problems, Schrödinger operators, deformation quantization, 70–205, Math. Top., 8, Adv. Partial Differential Equations, Akademie Verlag, Berlin, 1995.

[23] E. Schrohe and B.-W. Schulze. *Mellin and Green operators for boundary value problems on manifolds with edges*. Integral Equations Operator Theory 34, no. 3, 339–363 (1999).

[24] E. Schrohe and J. Seiler. *The resolvent of closed extensions of cone differential operators*. Can. J. Math. 57, no. 4, 771–811 (2005).

[25] E. Schrohe and J. Seiler. *Bounded $H_\infty$-calculus for cone differential operators*. J. Evolution Equations 18, no. 3, 1395–1425 (2018).

[26] L. Weis. *Operator-valued Fourier multiplier theorems and maximal $L_p$-regularity*. Math. Ann. 319, no. 4, 735-758 (2001).

Department of Mathematics, University of Patras, Rio Patras, Greece
*Email address*: riodos@math.upatras.gr

Institut für Analysis, Leibniz Universität Hannover, Germany
*Email address*: schrohe@math.uni-hannover.de

Dipartimento di Matematica, Università degli Studi di Torino, Turin, Italy
*Email address*: joerg.seiler@unito.it