ON EMBEDDABILITY OF THE UNION OF THREE CONES

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ABSTRACT. Theorem. (S. Parsa) Let \( K \) be a \( d \)-dimensional simplicial complex and \( K \ast [3] \) the union of three cones over \( K \) along their common bases. If \( 2m \geq 3d + 3 \) and \( K \ast [3] \) embeds into \( \mathbb{R}^{m+2} \), then \( K \) embeds into \( \mathbb{R}^m \). The proof is based on the Haefliger-Weber ‘configuration spaces’ embeddability criterion, equivariant suspension isomorphism theorem and simple properties of joins and cones.

This note is an invitation to add details into the following sketch of proof, and to publish the resulting complete proof. See also Remark 3.

Theorem 1 (S. Parsa). Let \( K \) be a \( d \)-dimensional simplicial complex and \( K \ast [3] \) the union of three cones over \( K \) along their common bases. If \( 2m \geq 3d + 3 \) and \( K \ast [3] \) embeds into \( \mathbb{R}^{m+2} \), then \( K \) embeds into \( \mathbb{R}^m \).

Sketch of a proof. Denote by \( \pi_{\mathbb{Z}_2}^m(X) \) the set of \( \mathbb{Z}_2 \)-equivariant maps from a \( \mathbb{Z}_2 \)-complex \( X \) to the \( m \)-sphere. Denote by \( Y \times ^2 \Delta \) and \( Y \ast ^2 \Delta \) the deleted product and the deleted join of \( Y \). The theorem follows because for any \( d \)-complex \( K \)

- if \( K \) embeds into \( \mathbb{R}^m \), then \( \pi_{\mathbb{Z}_2}^{m-1}(K \times ^2 \Delta) \neq \emptyset \);
- if \( \pi_{\mathbb{Z}_2}^{m-1}(K \times ^2 \Delta) \neq \emptyset \) and \( 2m \geq 3d + 3 \), then \( K \) embeds into \( \mathbb{R}^m \) [We67];
- there is a 1–1 correspondence

\[
\pi_{\mathbb{Z}_2}^{m-1}(K \times ^2 \Delta) \to \pi_{\mathbb{Z}_2}^m(K \ast [3] \times ^2 \Delta)
\]

which is a composition

\[
\pi_{\mathbb{Z}_2}^{m-1}(K \times ^2 \Delta) \xrightarrow{\Sigma} \pi_{\mathbb{Z}_2}^m((\Sigma K) \times ^2 \Delta) \to \pi_{\mathbb{Z}_2}^m((\Sigma K) \times ^2 \Delta) \to \pi_{\mathbb{Z}_2}^m(K \ast [3] \times ^2 \Delta)
\]

of 1–1 correspondences from [CF60, Theorem 2.5], [Sk02, Theorem 2.5 and the Cone Lemma 4.2.1], [Ma03, Exercise 4 to §5.5];

- \((K \ast [3]) \times ^2 \Delta \cong_{\mathbb{Z}_2} K \times ^2 \Delta \ast [3] \times ^2 \Delta \cong_{\mathbb{Z}_2} (\Sigma^2 K \times ^2 \Delta)\), so that there is a 1–1 correspondence

\[
\Sigma^2 : \pi_{\mathbb{Z}_2}^m((K \ast [3]) \times ^2 \Delta) \to \pi_{\mathbb{Z}_2}^{m+2}(K \ast [3] \times ^2 \Delta)
\]

\( \square \)

Remark 2 (historical). Theorem 1 appears in [Pa20a, Pa20b]. The paper [Pa20a] contains many unclear or even wrong sentences, starting with wrong definition of the main notion ‘\( K \) embeds into \( \mathbb{R}^n \)’ in the first paragraph. When I refereed an earlier version of [Pa20a] in the summer of 2019, besides writing the previous sentence I suggested the above generalization to \( 2m \geq 3d + 3 \) with the above sketch of proof. In

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1The part (4.2.1) of [Sk02, Cone Lemma 4.2] is easy and could have been known in folklore before [Sk02].
the hope that my suggestion would be elaborated in the direction of a clear publishable paper, I wrote ‘I would be glad to recommend a clear two-page paper by S. Parsa stating and proving these results.’

However, the paper [Pa20b] elaborates my suggestion in the opposite direction. That paper contains many unclear or even wrong sentences. E.g.,

- in §2, the last paragraph of the proof of theorem 1, \( [3]^{2*} \cong_{Z_2} S^1 \) is wrong because \( [3]^{2*} \cong K_{3,3} \).
- in §2, the last paragraph of the proof of theorem 1, I could not find in [Pa20b, reference [3]] proof of the fact stated before the sentence ‘For a proof see [3]’.
- the paper uses many times the undefined object \( \pi^k(\cdot) \) (the usual understanding of this notation, i.e. \( \pi^k(\cdot) = [\cdot, S^k] \), makes most sentences involving this notation incorrect).

These mistakes are easy to correct for a qualified person. However, it is easier for him/her to elaborate the above sketch than to find and correct mistakes in [Pa20b].

Thus I invite people to add details into the above sketch and to publish the resulting complete proof (of course properly mentioning S. Parsa’s contribution).

Despite the criticism, I attribute Theorem 1 to S. Parsa, in order to concentrate not on priority question but on mathematics and on important steps to prepare a quality journal submission, see Remark 3.

Remark 3 (important steps to prepare a quality journal submission). 2 (a) It is advisable before putting a paper to arxiv to discuss it among specialists in its area. Such a pre-submission discussion usually allows the author to check whether his/her results are clearly stated, new, and completely proved. This allows to improve quality of the paper. A dissatisfaction which might appear during such work is a natural part of improving quality of the paper and qualifications of the author. Such work is interesting if authors (=developers) recognize the importance of learning and fulfilling the wishes of their readers (=users). Such work is annoying only if authors write under the assumption (however unconscious) that their work need not be useful. Improved quality of the paper publicly available on arxiv improves the author’s reputation, while low quality damages it.

It is advisable to put a paper on arxiv before submitting it to a journal. Without this simple procedure there remains a possibility that the results of a paper are already (partly) known. There are many mathematicians, so what is unknown to one group can be known to another. Putting a paper on arxiv allows including into pre-submission

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2This remark is based on my experience as an author, as well as a referee for ‘Advances in Math.’, ‘Algebraic and Geometric Topology’, ‘Archiv der Math.’, ‘Arnold Math. J.’, ‘Ars Combinatoria’, ‘Commentarii Math. Helvetici’, ‘Contemporary Math.’, AMS book series, ‘Discrete and Computational Geometry’, ‘European J. of Math.’, ‘Izvestiya Ross. Akad. Nauk’, ‘J. of Dynamics and Differential Equations’, ‘J. of Graph Theory’, ‘J. of Math. Physics’, ‘J. of Math. Analysis and Applications’, ‘Math. Proceedings, Cambridge Philosophical Society’ ‘Mat. Sbornik’, ‘Mat. Zametki’, ‘Proceedings of the American Math. Society’, ‘Revista Mat. Iberoamericana’, ‘SIAM J. on Discrete Math.’, ‘St Petersburg Math. J.’ ‘Topology and Its Application’, ‘Transformation Groups’.

I omit ‘in my opinion’ for brevity.

“A genius makes his own rules, but a ‘how to’ article is written by one ordinary mortal for the benefit of another... Authors of articles such as this one know that, but in the first approximation they must ignore it, or nothing would ever get done.” [Ha74]
discussion (see the previous paragraph) people who work in related areas but are not in contact with the author. Improved quality of a paper published in a journal improves the author’s reputation, while low quality damages it even more significantly.

In a less formal pre-submission discussion it is easier to help the author, to share ideas with him/her, and to minimize the critical part of such help. Publication of a paper on arxiv without prior discussion with a colleague means that the author expects a public, not a private approval or criticism of this colleague. The colleague might still prefer private criticism, but could be compelled to criticize publicly if the arxiv text contain flaws which obstruct progress of mathematics.

Journal publications practically rule mathematical world. So writing a referee report on a paper is a responsible task involving double-checking. In this time-consuming form it is much harder to help the author than via informal discussions, see e.g. Remark 2.

The above steps do not absolutely protect against significant flaws, see e.g. \cite{Sk08p}. However, with the above steps done, the responsibility is shared with math community.

(b) During pre-submission discussion specialists in the area might send their specific suggestions/criticism which they consider important (below this is shortened to just ‘suggestions’). Then it is advisable to put on arxiv (or submit to a journal) a revised version approved by specialists. Of course the authors can disagree with some suggestions (and cannot be sure that they would not receive another stupid or essential suggestions the day after they finalized their work, based on previous suggestions). Hence the authors can decide to submit their paper to arxiv/journal even if some suggestions have not been taken into account or if a specialist was not given a chance to see whether his/her suggestions are properly incorporated. Then it is fair to mention this in the text. E.g.\footnote{When I saw the arxiv papers \cite{MW16} and \cite{FK17}, I informed the authors of my opinion expressed in the bullet points and suggested to update the arxiv papers. When the corresponding updates will appear, I would be glad to remove these examples.}

• because of the way M. Skopenkov’s work is mentioned in \cite{FK17}, I find it misleading that \cite{FK17} does not mention that pre-arxiv versions of \cite{FK17} were sent by the authors to M. Skopenkov, who liked the idea of proof and sent the authors important specific criticism on its realization, but \cite{FK17} was put to arxiv without the authors sending arxiv versions to M. Skopenkov (to check that he approves the way his critical remarks were taken into account, if they have been).

• because of the way my name and work is mentioned in \cite{MW16}, I find it misleading that \cite{MW16} does not mention that a pre-arxiv version of \cite{MW16} was sent by the authors to me, I liked the idea of proof and had important specific criticism on its realization, but \cite{MW16} was put to arxiv without the authors sending it to me; cf. \cite{Sk17o}.

• see a forthcoming arxiv note on \cite{Cu20}.

On the other hand, mathematicians would not find it misleading that \cite{Ad18} does not mention that I had some specific criticism on the argument (their opinion might or might not change if they have learned of the criticism); it is sufficient that Karim removed upon my request my name from acknowledgements in version 3 or later.
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