Zeta Functional Analysis
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Abstract
We intimate deeper connections between the Riemann zeta and gamma functions than often reported and further derive a new formula for expressing the value of \(\zeta(2n+1)\) in terms of zeta at other fractional points. This paper also establishes and presents new expository notes and perspectives on zeta function theory and functional analysis. In addition, a new fundamental result, in form of a new function called omega \(\Omega(s)\), is introduced to analytic number theory for the first time. This new function together with some of its most fundamental properties and other related identities are here disclosed and presented as a new approach to the analysis of sums of generalised harmonic series, related alternating series and polygamma functions associated with Riemann zeta function.

1 Introduction
The Riemann zeta function is defined by the generalised harmonic series
\[
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s},
\]
where \(s = \sigma + it\) and \(\sigma > 1\). In his 1859 paper [1], Riemann introduced the functional equation
\[
\frac{\zeta(s)\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = \frac{\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{1-s}{2}}}
\]
which suggests replacing the value of \(s\) by \(1-s\) without changing the result of the outcome after substitution. Our imagination is tickled by this property to investigate
\[
\log\left(\frac{\zeta(s)\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}}\right) = \log\left(\frac{\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{1-s}{2}}}\right)
\]
hoping that we might find an elementary expression for the value of \(\Gamma(1-s)\). Observe
\[
\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = \log\left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right) + (s - \frac{1}{2})\log(\pi) = \log\left(\frac{(2\pi)^s}{\pi^{s/2}}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\right),
\]
which implies
\[
\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = \log\left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right) + (s - \frac{1}{2})\log(\pi) = s\log(2\pi) - \log(\pi) + \log\left(\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\right).
\]
Without hesitation, we are induced to introduce \(1-s\) in place of \(s\) into the last equation to produce:
\[
\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = \log\left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right) + (1-s)\log(\pi) = (1-s)\log(2\pi) - \log(\pi) + \log\left(\cos\left(\frac{\pi s}{2}\right)\right) + \log(\Gamma(s)),
\]
similar to Riemann’s idea. We digress from the original motivation of finding an expression for \(\Gamma(1-s)\) to envision many great uses for these simple formulae created in 5 and 5 taking on great feats such as the challenge of finding a new formula for \(\zeta(2n+1)\) resulting in a new development and progress towards tackling this long standing open problem in number theory.
2 Polygamma function and Riemann zeta at odd integers

Lemma 2.1. Assuming \( n \geq 1 \) is an integer number,

\[
\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))).
\] (7)

Proof. According to [6]

\[
\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) + \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})).
\] (8)

Lemma 2.2. Assuming \( n \geq 1 \) is an integer number,

\[
\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{1-s}{2}))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})).
\] (9)

Proof. Again, this results as a direct consequence of [6]

Lemma 2.3. The following identities are valid:

\[
\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{1-s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))); \tag{10}
\]

and

\[
\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(1-s))) = -\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) + \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(\frac{1-s}{2}))) - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\sin(\frac{\pi s}{2}))). \tag{11}
\]

Proof. Again, this results as a direct consequence of [6]

Theorem 2.4. Assuming \( n \) is an integer number, the following is a formula expressing the value of \( \zeta(2n+1) \).

\[
(-1)^{2n+1}(2^{2n+1}-1)\zeta(2n+1)\Gamma(2n+1) = \psi^{(2n)}(\frac{1}{2}) = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})) \bigg|_{s \to \frac{1}{2}} - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))) \bigg|_{s \to \frac{1}{2}} \tag{12}
\]

Proof.

\[
\frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\Gamma(s))) \bigg|_{s \to \frac{1}{2}} = \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\frac{\zeta(1-s)}{\zeta(s)})) \bigg|_{s \to \frac{1}{2}} - \frac{d^{(2n+1)}}{ds^{(2n+1)}}(\log(\cos(\frac{\pi s}{2}))) \bigg|_{s \to \frac{1}{2}} \tag{13}
\]

According to K.S. Köhlig [2], special values of polygamma function [3][4] defined as \( \psi^{(s-1)}(x) = \frac{d^{s-1}}{dx^{s}} \psi(x) = \frac{d}{dx} \ln \Gamma(x) \) may be combined to compute (or compose) the value of \( \zeta(2n+1) \), e.g. \( \psi^{(k)}(\frac{1}{2}) + \psi^{(k)}(\frac{3}{2}) \) as in the case of:

\[
\zeta(n) = (-1)^{n} \frac{\psi^{(n-1)}(\frac{1}{2}) + \psi^{(n-1)}(\frac{3}{2})}{2^{n} (2^{n} - 1) \Gamma(n)} \tag{14}
\].
Theorem 2.5. Let \( n \) be an integer number, then

\[
- \zeta(2n + 1) = \left( \psi^{(2n)} \left( \frac{1}{4} \right) + \psi^{(2n)} \left( \frac{3}{4} \right) \right) \left( \frac{1}{2^{2n+1} \Gamma(2n+1)} \right) = \frac{1}{2} \left( \frac{d^{(2n+1)}(\log(\zeta(1-s)))}{ds^{(2n+1)}} \right) \bigg|_{s \to \frac{1}{2}} + \frac{d^{(2n+1)}(\log(\pi s/2))}{ds^{(2n+1)}} \bigg|_{s \to \frac{1}{2}}.
\]

Proof. Notice that

\[
\psi^{(2n)} \left( \frac{1}{4} \right) = \frac{d^{(2n+1)}(\log(\Gamma(s)))}{ds^{(2n+1)}} \bigg|_{s \to \frac{1}{2}}, \quad \psi^{(2n)} \left( \frac{3}{4} \right) = \frac{d^{(2n+1)}(\log(\Gamma(s)))}{ds^{(2n+1)}} \bigg|_{s \to \frac{3}{4}},
\]

Therefore

\[
\psi^{(2n)} \left( \frac{1}{4} \right) + \psi^{(2n)} \left( \frac{3}{4} \right) = 2 \left( \frac{d^{(2n+1)}(\log(\zeta(s)))}{ds^{(2n+1)}} \right) \bigg|_{s \to \frac{1}{2}} + \frac{d^{(2n+1)}(\log(\pi s/2))}{ds^{(2n+1)}} \bigg|_{s \to \frac{1}{2}}.
\]

Clearly the following theorems are valid and do not require explicit proofs.

Theorem 2.6. In general,

\[
\psi^{(2n)}(s) + \psi^{(2n)}(1-s) = 2 \left( \frac{d^{(2n+1)}(\log(\zeta(s)))}{ds^{(2n+1)}} \right) + \frac{d^{(2n+1)}(\log(\tan(\frac{\pi s}{2}))}{ds^{(2n+1)}}
\]

where \( s \notin \{0, 1\} \).

Theorem 2.7. In general,

\[
\psi^{(2n)}(s) - \psi^{(2n)}(1-s) = \frac{d^{(2n+1)}(\log(\cos(\frac{\pi s}{2}) \sin(\frac{\pi s}{2}))}{ds^{(2n+1)}}
\]

where \( s \notin \{0, 1\} \).

3 Further analysis of zeta functional equations

Theorem 3.1. Given that \( s \) is any (real of complex) number, except 0 and 1, then

\[
\log \left( \frac{\Gamma \left( \frac{1-s}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} \right) = \log \left( \frac{\zeta \left( \frac{1-s}{2} \right)}{\zeta(1-(\frac{1-s}{2}))} \right) - (s - \frac{1}{2}) \log(2\pi) - \log \left( \frac{\sin \left( \frac{\pi (1-s)}{2} \right)}{\sin \left( \frac{\pi s}{2} \right)} \right);
\]

\[
\log \left( \frac{\zeta(s)}{\zeta(1-s)} \right) = \log \left( \frac{\zeta \left( \frac{s+1}{2} \right)}{\zeta(1-(\frac{s+1}{2}))} \right) - (s - \frac{1}{2}) \log(2) - \log \left( \frac{\sin \left( \frac{\pi (1-s)}{2} \right)}{\sin \left( \frac{\pi (1-s)}{2} \right)} \right).
\]
Many important identities are derivable for this last set of equations.

**Proof.** Observe \( \log \left( \frac{\zeta(s)}{\zeta(1-s)} \right) - s \log(2\pi) + \log(\pi) - \log(\sin(\pi s/2)) = \log(\Gamma(1-s)) \) which implies \( \log \left( \frac{\zeta(1-s)}{\zeta(s)} \right) - (1-s) \log(2\pi) + \log(\pi) - \log(\cos(\pi s/2)) = \log(\Gamma(s)) \). Since \( \frac{1}{2} = 1 - \left( \frac{s}{2} \right)^2 = 1 - (s/2)^2 \), the following identities are valid:

\[
\log(\Gamma(1-s)) = \log(\Gamma(1-(s+1)/2))) = \log(\frac{\zeta(s+1/2)}{\zeta(1-(s+1)/2)}(1-1/2)) = \log(\Gamma(1-s/2)) (22) \\
\log(\Gamma(s)) = \log(\Gamma(1-(s/2))) = \log(\frac{\zeta(s+1/2)}{\zeta(1-(s+1)/2)}(1-1/2)) = \log(\Gamma(s/2)) (23) \\
\]

Subtracting eq. (23) from (22) gives:

\[
\log(\frac{\Gamma(1-s)}{\Gamma(s/2)}) = \log(\frac{\zeta(s+1/2)}{\zeta(1-(s+1)/2)}(1-1/2)) - (s-1/2) \log(2\pi) - \log(\sin(\pi(s+1)/2)); \\
\log(\frac{\zeta(s)}{\zeta(1-s)}) = \log(\frac{\zeta(s+1/2)}{\zeta(1-(s+1)/2)}(1-1/2)) - (s-1/2) \log(2\pi) - \log(\sin(\pi(s+1)/2)); \\
\]

since \( \log(\frac{\Gamma(1-s)}{\Gamma(s/2)}) = \log(\frac{\zeta(s)}{\zeta(1-s)}) - (s - 1/2) \log \pi. \)

The following identities demonstrate a combinatorial perspective on \( \log \left( \frac{\zeta(s)}{\zeta(1-s)} \right) \) decomposition:

**Lemma 3.2.**

\[
\log(\zeta(s+1)/\zeta(-s)) = \log(\frac{\zeta(s+1/2)}{\zeta(1-(s+1)/2)}(1-1/2)) - (s+1/2) \log(2\pi) - \log(\sin(\pi(s+1)/2)); \\
\log(\frac{\zeta(s)}{\zeta(1-s)}) = \log(\frac{\zeta(s+1/2)}{\zeta(1-(s+1)/2)}(1-1/2)) - (s-1/2) \log(2\pi) - \log(\sin(\pi(s+1)/2)); \\
\log(\frac{\zeta(s-1)}{\zeta(2-s)}) = \log(\frac{\zeta(s-1/2)}{\zeta(1-(s-1/2))}(1-1/2)) - (s-3/2) \log(2\pi) - \log(\sin(\pi(s-1)/2)); \\
\log(\frac{\zeta(s-2)}{\zeta(3-s)}) = \log(\frac{\zeta(s-2/2)}{\zeta(1-(s-2/2))}(1-1/2)) - (s-5/2) \log(2\pi) - \log(\sin(\pi(s-2)/2)); \\
\]

Many important identities are derivable for this last set of equations.

**Theorem 3.3.**

\[
\log\left( \frac{\zeta(s+1/2)}{\zeta(1-s)} \right) - \log\left( \frac{\sin(\pi s/2)}{\sin(\pi(2-s)/2)} \right) = \log\left( \frac{\zeta(s-1)}{\zeta(2-s)} \right) - \log\left( \frac{\sin(\pi(s-1)/2)}{\sin(\pi(2-s)/2)} \right) - \frac{(s-1)(s-1/2)}{(2\pi)^2} - \frac{\sin(\pi(s-1/2))}{\sin(\pi(2-s)/2)} \\
\]

**Proof.** Note

\[
\log(\frac{\zeta(s-2)}{\zeta(3-s)}) = \log(\frac{\zeta(s-2/2)}{\zeta(1-(s-2/2))}(1-1/2)) - (s-5/2) \log(2\pi) - \log(\frac{\sin(\pi(s-1)/2)}{\sin(\pi(2-s)/2)}), \\
\]
This suggests

\[
\log\left(\frac{\zeta(s + \frac{1}{2})}{\zeta(\frac{1}{2} - s)}\right) = \log\left(\frac{\zeta(s - \frac{1}{2})}{\zeta(\frac{1}{2} - s)}\right) - \log\left(\frac{\sin\left(\frac{\pi(s + \frac{1}{2})}{2}\right)}{\sin\left(\frac{\pi(\frac{1}{2} - s)}{2}\right)}\right)
\]

from which the identity

\[
\log\left(\frac{\zeta(s + \frac{1}{2})}{\zeta(\frac{1}{2} - s)}\right) - \log\left(\frac{\sin\left(\frac{\pi(s + \frac{1}{2})}{2}\right)}{\sin\left(\frac{\pi(\frac{1}{2} - s)}{2}\right)}\right) = \log\left(\frac{\zeta(s - \frac{1}{2})}{\zeta(\frac{1}{2} - s)}\right) - \log\left(\frac{\sin\left(\frac{\pi(s - \frac{1}{2})}{2}\right)}{\sin\left(\frac{\pi(\frac{1}{2} - s)}{2}\right)}\right)
\]

is derived.

### 3.1 Zeta at 1+s, -s, 1-s and s

**Theorem 3.4.**

\[
\log\left(\frac{\zeta^2(1+s)}{\zeta^2(-s)}\right) = -\log\left(\frac{\zeta^2(1-s)}{\zeta^2(s)}\right) - \log\left(\frac{s^2}{(2\pi)^2}\right) + \log\left(\frac{\sin\left(\frac{\pi(s+1)}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(s-1)}{2}\right)}{\sin\left(\frac{\pi(-s)}{2}\right)}\right)
\]

**Proof.** According to lemma 3.2 and \(\frac{\zeta(s-2)}{\zeta(s-1)}\) - an independent discovery by the author similar in comparison to one of Henrik Stenlund’s results, the identity

\[
\log\left(\frac{\zeta(s+1)}{\zeta(-s)}\right) - \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(s-1)}{\zeta(2-s)}\right) - \log\left(\frac{\zeta(s-2)}{\zeta(3-s)}\right) - \left(\log\left(\frac{\zeta\left(\frac{s+2}{2}\right)}{\zeta\left(\frac{1}{2} - s\right)}\right) - \log\left(\frac{\zeta\left(\frac{s-2}{2}\right)}{\zeta\left(\frac{3}{2} - s\right)}\right)\right)
\]

\[
= -2\log 2 - \log\left(\frac{\sin\left(\frac{\pi(s+1)}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(s-1)}{2}\right)}{\sin\left(\frac{\pi(-s)}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{\pi(s-2)}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(s-3)}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)}\right)
\]

\[
\log\left(\frac{\zeta(s+2)}{\zeta(s-2)}\right) = \log\left(\frac{\zeta(s+1)}{\zeta(s-1)}\right) - \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) - \log\left(\frac{\zeta(s-1)}{\zeta(2-s)}\right) + \log\left(\frac{\zeta(s-2)}{\zeta(3-s)}\right) \quad (29)
\]
The recursion relations between the first six terms may be combined to produce:

\[
\log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(s + 1) \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(s) - \log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(-s)
\]

\[
= -2 \log \left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{(s - 2)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) \log\left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{(s - 2)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right)
\]

(30)

or alternatively,

\[
\log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(s + 1) \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(s) - \log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(-s)
\]

\[
= -2 \log \left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{(s - 2)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) \log\left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{(s - 2)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right)
\]

(31)

As a result of this aggregation

\[
\log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(s + 1) \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(s) - \log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 1}{\pi^2}\right) \zeta^2(-s) - \log\left(\frac{s - 2}{\pi^2}\right) \zeta^2(-s)
\]

\[
= -2 \log \left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{(s - 2)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) \log\left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{(s - 2)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{(s - 1)\pi}{2}\right)}{\sin\left(\frac{(1 - s)\pi}{2}\right)}\right)
\]

(32)

is produced. Further aggregation results in

\[
\log\left(\frac{s - 1}{s - 2}\right) \zeta^2(1 + s) \zeta^2(1 - s) = - \log\left(\frac{s}{s - 2}\right) - \log\left(\frac{s - 1}{s - 2}\right) - 2 \log 2
\]

(33)

\[
\Rightarrow
\log\left(\frac{s - 1}{s - 2}\right) \zeta^2(1 + s) \zeta^2(1 - s) = - \log\left(\frac{s - 1}{s - 2}\right) - \log\left(\frac{s}{s - 2}\right) - 2 \log 2
\]

(34)

\[
\Rightarrow
\log\left(\frac{s - 1}{s - 2}\right) \zeta^2(1 + s) \zeta^2(1 - s) = - \log\left(\frac{s - 1}{s - 2}\right) - \log\left(\frac{s}{s - 2}\right) - 2 \log 2
\]

(35)
and finally
\[
\log(\frac{\zeta^2(1+s)}{\zeta^2(-s)}) = -\log(\frac{\zeta^2(1-s)}{\zeta^2(s)}) - \log(\frac{s^2}{(2\pi i)^2}) + \\
- \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) + \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) + \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s)}{2})}).
\]

\[\Box\]

### 3.2 On the decomposition of the derivatives of logarithm of zeta

Here we would like to prove the following

**Theorem 3.5.**

\[
\log(\frac{(s-2)(s-1)}{(2\pi i)^2}) = \log(\frac{\zeta^2(s)}{\zeta^2(1-s)})
\]

\[
= \log(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})}) + \log(\frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})}) + \log(\frac{\zeta(\frac{s+1}{2})}{\zeta(1-\frac{s+1}{2})}) + \log(\frac{\zeta(\frac{s-1}{2})}{\zeta(1-\frac{s-1}{2})}) - (2s-3) \log 2 + \\
- \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})})
\]

\[
= \log(\frac{\zeta(s+1)}{\zeta(s-1)}) + \log(\frac{\zeta(s-1)}{\zeta(2-s)}) + \log(\frac{\zeta(s-2)}{\zeta(3-s)}) - \log(\frac{\zeta(s+2)}{\zeta(1-s)}) + \log(\frac{\zeta(s-2)}{\zeta(1-s)})
\]

\[
= -(4s-4) \log 2 - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\pi(1-s))}) - \log(\frac{\sin(\pi(1-s))}{\sin(\frac{\pi(1-s)}{2})})
\]

\[
\Rightarrow
\]

\[
\log(\frac{(s-1)s\zeta^2(s+1)}{(2\pi i)^2}) + \log(\frac{(s-2)(s-1)}{(2\pi i)^2}) = \log(\frac{\zeta^2(s)}{\zeta^2(1-s)}) - \log(\frac{\zeta(s+2)}{\zeta(1-\frac{s+2}{2})}) + \log(\frac{\zeta(s-2)}{\zeta(1-\frac{s-2}{2})})
\]

\[
= -(4s-4) \log 2 - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})})
\]

\[
= -(4s-4) \log 2 - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s)}{2})}) - \log(\frac{\sin(\pi(1-s))}{\sin(\frac{\pi(1-s)}{2})}).
\]
From this last equation we subtract eq. 32

\[ \log \left( \frac{(s - 1)s \zeta^2(s + 1)}{(2\pi i)^2} \right) - \log \left( \frac{(s - 2)(s - 1) \zeta^2(s)}{(2\pi i)^2} \right) + \log \left( \frac{\zeta(s - 2)}{(2\pi i)^2} \right) \]

\[ = -2 \log 2 - \log \left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) - \log \left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) + \log \left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) \]

to yield

\[ 2 \log \left( \frac{(s - 2)(s - 1) \zeta^2(s)}{(2\pi i)^2} \right) = \]

\[ \left( \log \left( \frac{\zeta\left(\frac{s}{2}\right)}{\zeta(1 - \frac{s}{2})} \right) + \log \left( \frac{\zeta\left(\frac{s}{2}\right)}{\zeta(1 - \frac{s}{2})} \right) \right) + \left( \log \left( \frac{\zeta^2\left(\frac{s}{2}\right)}{\zeta^2(1 - \frac{s}{2})} \right) + \log \left( \frac{\zeta^2\left(\frac{s}{2}\right)}{\zeta^2(1 - \frac{s}{2})} \right) \right) \]

\[ + \log\left( \frac{\zeta(s - 1)\zeta(s)}{(2\pi i)^2} \right) - \log \left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) - \log\left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) \]

which shows the decomposition relation of \( \log\left( \frac{\zeta^2(s)}{\zeta^2(1 - s)} \right) \) as

\[ 2 \log \left( \frac{(s - 2)(s - 1) \zeta^2(s)}{(2\pi i)^2} \right) = \]

\[ \left( \log \left( \frac{\zeta\left(\frac{s}{2}\right)}{\zeta(1 - \frac{s}{2})} \right) + \log \left( \frac{\zeta\left(\frac{s}{2}\right)}{\zeta(1 - \frac{s}{2})} \right) \right) + \left( \log \left( \frac{\zeta^2\left(\frac{s}{2}\right)}{\zeta^2(1 - \frac{s}{2})} \right) + \log \left( \frac{\zeta^2\left(\frac{s}{2}\right)}{\zeta^2(1 - \frac{s}{2})} \right) \right) \]

\[ + \log\left( \frac{\zeta(s - 1)\zeta(s)}{(2\pi i)^2} \right) - \log \left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) - \log\left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) \]

simplified to

\[ \log \left( \frac{(s - 2)(s - 1) \zeta^2(s)}{(2\pi i)^2} \right) = \log \left( \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} \right) + \log \left( \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} \right) + \log \left( \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} \right) \]

\[ + \log\left( \frac{\zeta(s - 1)\zeta(s)}{(2\pi i)^2} \right) - \log \left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) - \log\left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) \]

or

\[ \log \left( \frac{(s - 2)(s - 1) \zeta^2(s)}{(2\pi i)^2} \right) = \log \left( \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} \right) + \log \left( \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} \right) + \log \left( \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} \right) \]

\[ - \log \left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) - \log\left( \frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(1 - s)}{2}\right)} \right) \]

depending on the identity

\[ \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} = \log \left( \frac{\zeta\left(\frac{s - 1}{2}\right)}{(2\pi i)^2} \right) + \log \left( \frac{\zeta\left(\frac{s + 1}{2}\right)}{\zeta(1 - \frac{s + 1}{2})} \right) \]

\[ \square \]
Lemma 3.6. The following identity is true.

\[ \log(\frac{\sin(\frac{\pi}{2} (\frac{s}{2} + \frac{1}{2}))}{\cos(\frac{\pi}{2} (\frac{s}{2} + \frac{1}{2}))}) = \log(\frac{\sin(\frac{\pi}{2} (\frac{s-4}{2} + \frac{1}{2}))}{\cos(\frac{\pi}{2} (\frac{s-4}{2} + \frac{1}{2}))}). \] (44)

Proof.

\[ \log(\frac{\zeta^2(s)}{\zeta^2(1-s)}) = \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-\frac{s}{2})}) + \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-\frac{s}{2})}) \]

\[ \zeta(1-\frac{s}{2}) \zeta(1+(\frac{s}{2}+1)) \]

\[ \zeta(1-\frac{s}{2}) \zeta(1-(\frac{s}{2}+1)) \]

\[ \zeta(1-\frac{s}{2}) \zeta(1+(\frac{s}{2}+1)) \]

\[ \zeta(1-(\frac{s}{2}+1)) \zeta(1+(\frac{s}{2}+1)) \]

Recall the identity

\[ \log(\frac{\zeta(s)}{\zeta(1-s)}) = \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-\frac{s}{2})}) -(s+\frac{1}{2})\log(2) - \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}), \]

therefore

\[ \log(\frac{\zeta^2(s)}{\zeta^2(1-s)}) = \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-\frac{s}{2})}) -(s+\frac{1}{2})\log(2) - 2\log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}) = \]

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-\frac{s}{2})}) \]

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) \]

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) \]

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) \]

implying

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) -(s+\frac{1}{2})\log(2) - 2\log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}) = \]

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) -(s+\frac{1}{2})\log(2) - 2\log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}) = \]

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) -(s+\frac{1}{2})\log(2) - 2\log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}) = \]

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) -(s+\frac{1}{2})\log(2) - 2\log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}) = \]

Changing s such that \( s \to s - 2 \) leads to:

\[ \log(\frac{\zeta(s+\frac{1}{2})}{\zeta(1-s/2)}) = \log(\frac{\zeta(s-\frac{3}{2})}{\zeta(1-s/2)}) \]

\[ + (2) \log(2) - \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}) = \]

\[ + (2) \log(2) - \log(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s/2)}{2})}) = \]

9
Substituting this last result into eq. \ref{eq:48} produces

\[
\log(\frac{\zeta(s+1)}{\zeta(1-s+1)} \frac{\zeta(\frac{s}{2})}{\zeta(1-\frac{s}{2})}) = \left( \log(\frac{\zeta(s-3)}{\zeta(1-s+3)} \frac{\zeta(s-4)}{\zeta(1-s+4)}) +
\right. \\
\left. + (2) \log 2 - \log\left(\frac{s-4}{(2\pi i)^2}\right) + \log\left(\frac{\sin(\frac{\pi(s-1)}{2})}{\sin(\frac{\pi(1-s+1)}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(s-3)}{2})}{\sin(\frac{\pi(1-s+3)}{2})}\right)\right)
\]

(50)

and

\[
\log(\frac{\zeta(s-3)}{\zeta(1-s+3)} \frac{\zeta(s-4)}{\zeta(1-s+4)}) = \left( \log(\frac{\zeta(s+3)}{\zeta(1-s+3)} \frac{\zeta(s+4)}{\zeta(1-s+4)}) +
\right. \\
\left. + (4) \log 2 - \log\left(\frac{(s-1)(s-2)(s-3)(s-4)}{(2\pi i)^4}\right) + \log\left(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s+1)}{2})}\right) - \log\left(\frac{\sin(\frac{\pi(s+3)}{2})}{\sin(\frac{\pi(1-s+3)}{2})}\right)\right).
\]

(51)

Taking an analytical look at the right hand term

\[
\log(\frac{\zeta(s-3)}{\zeta(1-s+3)} \frac{\zeta(s-4)}{\zeta(1-s+4)}) = \log(\frac{s-3}{(2\pi i)^2} \frac{s-1}{(2\pi i)^2} \frac{s+1}{(2\pi i)^2} \frac{s+2}{(2\pi i)^2} \frac{\zeta(s+3)}{\zeta(s+4)})
\]

implies

\[
\log(\frac{\zeta(s+1)}{\zeta(1-s+1)} \frac{\zeta(s+2)}{\zeta(1-s+2)}) = \log(\frac{s+3}{(2\pi i)^2} \frac{s+1}{(2\pi i)^2} \frac{s+2}{(2\pi i)^2} \frac{\zeta(s+3)}{\zeta(s+4)}) +
\]

(53)

and as a result of further simplification,

\[
\log\left(\frac{\sin(\frac{\pi(s+1)}{2})}{\sin(\frac{\pi(1-s+2)}{2})}\right) = \log\left(\frac{\sin(\frac{\pi(s+1+4)}{2})}{\sin(\frac{\pi(1-s+2+4)}{2})}\right);
\]

(54)

confirming

\[
\log\left(\frac{\sin(\frac{s+1}{2})}{\cos(\frac{\pi(s+1)}{2})}\right) = \log\left(\frac{\sin(\frac{s+4}{2})}{\cos(\frac{\pi(s+4)}{2})}\right).
\]

(55)

4 Further analysis involving the gamma function

Here, we first prove the following theorem before presenting the related identity in terms of zeta function.
Theorem 4.1.
\[
\log\left(\frac{\Gamma(2 + \frac{s}{2})}{\Gamma\left(2 - \frac{s}{2}\right)}\right) - \log\left(\frac{\Gamma(2 - \frac{s}{2})}{\Gamma\left(2 + \frac{s}{2}\right)}\right) = \log\left(\frac{\Gamma\left(\frac{s}{2} - 1\right)}{\Gamma\left(\frac{s}{2} + 1\right)}\right) - \log\left(\frac{\Gamma\left(\frac{s}{2} + 1\right)}{\Gamma\left(\frac{s}{2} - 1\right)}\right) + \log\left(\frac{s}{s + 4}\right)
\]

(56)

Proof. Due to theorem 3.3

\[
\log\left(\frac{\Gamma^2\left(-\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2}\right)}\right) + \log(\pi^{2(1+s-\frac{1}{2})}) = -\log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2} + 1\right)}\right) - \log(\pi^{2(1-s-\frac{1}{2})}) - \log\left(\frac{2s^2}{(s - 2)}\right) + \frac{(s - 1)}{(2\pi i)^2} + \\
- \log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right).
\]

(57)

Also

\[
\log\left(\frac{\Gamma^2\left(-\frac{s-4}{2}\right)}{\Gamma^2\left(\frac{s-4}{2}\right)}\right) + \log(\pi^{2(1+s-\frac{1}{2})})
\]

\[
= -\log\left(\frac{\Gamma^2\left(\frac{s-4}{2}\right)}{\Gamma^2\left(\frac{s-4}{2} + 1\right)}\right) - \log(\pi^{2(1-s-\frac{1}{2})}) - \log\left(\frac{2(s - 4)}{(s - 2)}\right) + \frac{(s - 1)}{(2\pi i)^2} + \\
- \log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right),
\]

(58)

because of the following identities which are derived in relation to the identity 54

\[
\log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right) = \log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right);
\]

\[
\log\left(\frac{\sin\left(\frac{\pi(s + 1)}{2}\right)}{\sin\left(\frac{\pi(s + 1)}{2}\right)}\right) = \log\left(\frac{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{s}{2}\right)}\right);
\]

\[
\vdots
\]

and so on. Subtracting eq. 58 from 57

\[
\log\left(\frac{\Gamma^2\left(-\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2}\right)}\right) + \log(\pi^{2(1+s-\frac{1}{2})}) - \left(\log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2} + 1\right)}\right) + \log(\pi^{2(1-s-\frac{1}{2})})\right) =
\]

\[
- \log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2} + 1\right)}\right) - \log(\pi^{2(1-s-\frac{1}{2})}) - \log\left(\frac{2s^2}{(s - 2)}\right) + \frac{(s - 1)}{(2\pi i)^2} + \\
- \left(\log\left(\frac{\Gamma^2\left(\frac{s}{2} - 1\right)}{\Gamma^2\left(\frac{s}{2} + 1\right)}\right) - \log(\pi^{2(1-s-\frac{1}{2})}) - \log\left(\frac{2(s - 4)}{(s - 2)}\right) + \frac{(s - 1)}{(2\pi i)^2}\right)
\]

\[
\Rightarrow
\]

\[
\log\left(\frac{\Gamma^2\left(-\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2}\right)}\right) - \log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2} + 1\right)}\right) = -\log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{s}{2} + 1\right)}\right) - \log\left(\frac{s^2}{(s - 2)}\right) + \frac{s - 1}{(2\pi i)^2} + \\
+ \log\left(\frac{\Gamma^2\left(\frac{s}{2} - 1\right)}{\Gamma^2\left(\frac{s}{2} + 1\right)}\right) + \log\left(\frac{(s - 4)^2}{(s - 6)}\right)
\]

(60)
\[ \log\left(\frac{\Gamma^2\left(-\frac{s}{2}\right)}{\Gamma^2\left(-\frac{s}{2} + 1\right)}\right) + \log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(-\frac{s}{2} + 1\right)}\right) + \log\left(\frac{s^2}{(s-2)(s-1)}\right) \]

\[ = \log\left(\frac{\Gamma^2\left(2 - \frac{s}{2}\right)}{\Gamma^2\left(2 - \frac{s}{2} - 2\right)}\right) + \log\left(\frac{\Gamma^2\left(\frac{s}{2} - 2\right)}{\Gamma^2\left(2 + \frac{s}{2}\right)}\right) + \log\left(\frac{(s-4)^2}{(s-2)(s-3)}\right) \]

\[ \Rightarrow \]

\[ \log\left(\frac{\Gamma^2\left(-\frac{s}{2}\right)}{\Gamma^2\left(-\frac{s}{2} + 1\right)}\right) + \log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(-\frac{s}{2} + 1\right)}\right) + \log\left(\frac{s^2}{2}\right) = \log\left(\frac{\Gamma^2\left(2 - \frac{s}{2}\right)}{\Gamma^2\left(2 - \frac{s}{2} - 2\right)}\right) + \log\left(\frac{\Gamma^2\left(\frac{s}{2} - 2\right)}{\Gamma^2\left(2 + \frac{s}{2}\right)}\right) + \log\left(\frac{(s-4)^2}{2}\right) \]

\[ \Rightarrow \]

\[ \log\left(\frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}\right) + \log(s) = \log\left(\frac{\Gamma\left(2 - \frac{s}{2}\right)}{\Gamma\left(1/2 - 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(\frac{s}{2} - 2\right)}{\Gamma\left(1/2 + 2 - \frac{s}{2}\right)}\right) + \log(s-4) \]

\[ s \to -s:\]

\[ \log\left(\frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}\right) + \log(-s) = \log\left(\frac{\Gamma\left(2 + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(-\frac{s}{2} - 2\right)}{\Gamma\left(\frac{1}{2} + 2 + \frac{s}{2}\right)}\right) + \log(-s-4). \]

Subtracting eq. 65 from 64

\[ \log\left(\frac{\Gamma\left(2 + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(-\frac{s}{2} - 2\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) = \log\left(\frac{\Gamma\left(2 - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(\frac{s}{2} - 2\right)}{\Gamma\left(\frac{1}{2} + 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{s-4}{s+4}\right) \]

Therefore

\[ \log\left(\frac{\Gamma\left(2 + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) - \log\left(\frac{\Gamma\left(2 - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) = \log\left(\frac{\Gamma\left(\frac{s}{2} - 2\right)}{\Gamma\left(\frac{1}{2} + 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{s-4}{s+4}\right) \]

or

\[ -\log\left(\frac{\Gamma\left(2 + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(2 - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - 2 + \frac{s}{2}\right)}\right) = -\log\left(\frac{\Gamma\left(\frac{s}{2} - 2\right)}{\Gamma\left(\frac{1}{2} + 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{\Gamma\left(-\frac{s}{2} - 2\right)}{\Gamma\left(\frac{1}{2} + 2 + \frac{s}{2}\right)}\right) + \log\left(\frac{s+4}{s-4}\right) \]

as a result of \( s \to -s \) substitution.

\[ \Box \]

4.1 Logarithm of zeta at \( \pm s \pm 4 \) and \( 1 - (\pm s \pm 4) \)

The related zeta version of the last equation may be presented in form of

\[ -\log\left(\frac{\zeta\left(s + 4\right)}{\zeta\left(-3 - s\right)}\right) + \log\left(\frac{\zeta\left(-s + 4\right)}{\zeta\left(-3 + s\right)}\right) \]

\[ = -\log\left(\frac{\zeta\left(s - 4\right)}{\zeta\left(5 - s\right)}\right) + \log\left(\frac{\zeta\left(-s - 4\right)}{\zeta\left(-5 + s\right)}\right) + \log\left(\frac{s - 4}{s + 4}\right) \]
\[ \Rightarrow \]

\[ -\log\left( \frac{\zeta(s+4)}{\zeta(-3-s)} \right) + \log\left( \frac{\zeta(-s+4)}{\zeta(-3+s)} \right) + \log\left( \frac{\pi^{\frac{1}{2}}}{\pi^{\frac{1}{2}}-s+4} \right) = -\log\left( \frac{\zeta(s-4)}{\zeta(5-s)} \right) + \log\left( \frac{\zeta(-s-4)}{\zeta(-3+s)} \right) + \log\left( \frac{s-4}{s+4} \right) \]

(68)

\[ \Rightarrow \]

\[-\log\left( \frac{\zeta(s+4)}{\zeta(-3-s)} \right) + \log\left( \frac{\zeta(-s+4)}{\zeta(-3+s)} \right) = -\log\left( \frac{\zeta(s-4)}{\zeta(5-s)} \right) + \log\left( \frac{\zeta(-s-4)}{\zeta(-3+s)} \right) + \log\left( \frac{s-4}{s+4} \right) \]

(69)

\[-\log\left( \frac{\zeta(s+4)}{\zeta(-3-s)} \right) + \log\left( \frac{\zeta(-s+4)}{\zeta(-3+s)} \right) = -\log\left( \frac{\zeta(s-4)}{\zeta(5-s)} \right) + \log\left( \frac{\zeta(-s-4)}{\zeta(-3+s)} \right) + \log\left( \frac{s-4}{s+4} \right). \]

(70)

This last statement is not difficult to justify. To this aim, we use the recursion relation substitutes as demonstrated:

\[ -\log\left( \frac{\zeta(s+4)}{\zeta(-3-s)} \right) + \log\left( \frac{\zeta(-s+4)}{\zeta(-3+s)} \right) = -\log\left( \frac{\zeta(s-4)}{\zeta(5-s)} \right) + \log\left( \frac{\zeta(-s-4)}{\zeta(-3+s)} \right) + \log\left( \frac{s-4}{s+4} \right) \]

(71)

which then confirms the obvious (but almost unreported) identity

\[ \log\left( \frac{(s-4)(s-3)(s-2)(s-1)(s+1)(s+2)(s+3)}{(2\pi i)^4} \right) - \log\left( \frac{(-s-4)(-s-3)(-s-2)(-s-1)(-s+1)(-s+2)(-s+3)}{(2\pi i)^4} \right) = \log\left( \frac{s-4}{s+4} \right). \]

(72)

presented here for the first time ever.

4.2 Logarithm of zeta at \( \pm s \) and \( 1 \mp s \)

Another important discovery involving eq. 70 is on the proof of the next theorem.
Theorem 4.2.

\[
\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) = \log\left(\frac{\zeta(-s+8)}{\zeta(7+s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) - \log\left(\frac{s-8}{s}\right)
\]

(73)

Proof. The fact that

\[
-\log\left(\frac{\zeta(s+4)}{\zeta(-3-s)}\right) + \log\left(\frac{\zeta(-s+4)}{\zeta(-3+s)}\right) = -\log\left(\frac{\zeta(s-4)}{\zeta(5-s)}\right) + \log\left(\frac{\zeta(-s-4)}{\zeta(5+s)}\right) + \log\left(\frac{s-4}{s+4}\right)
\]

suggests

\[
-\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s+8)}{\zeta(7-s)}\right) = -\log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) + \log\left(\frac{s-8}{s}\right)
\]

(74)

\[
-\log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) + \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) = -\log\left(\frac{\zeta(-s+8)}{\zeta(7-s)}\right) - \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) + \log\left(\frac{s-8}{s}\right)
\]

(75)

This implies

\[
\log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s)}{\zeta(1+s)}\right) = \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) + \log\left(\frac{\zeta(-s+8)}{\zeta(7-s)}\right) - \log\left(\frac{s-8}{s}\right)
\]

(76)

Substituting \(s \to -s+8\) into (77)

\[
\log\left(\frac{\zeta(s+8)}{\zeta(7+s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9-s)}\right) = \log\left(\frac{\zeta(s)}{\zeta(1-s)}\right) + \log\left(\frac{\zeta(-s+16)}{\zeta(-15+s)}\right) - \log\left(\frac{s-8}{s+8}\right)
\]

(78)

Substituting \(s \to s+8\) into (77)

\[
\log\left(\frac{\zeta(s+8)}{\zeta(7-s)}\right) + \log\left(\frac{\zeta(s-8)}{\zeta(9+s)}\right) = \log\left(\frac{\zeta(s)}{\zeta(1+s)}\right) + \log\left(\frac{\zeta(-s+16)}{\zeta(-15-s)}\right) - \log\left(\frac{s-8}{s+8}\right)
\]

(79)
In other words,
\[
\log\left(\frac{\zeta(s)}{(1-s)}\right) + \log\left(\frac{\zeta(-s)}{(1+s)}\right) = \\
= \log\left(\frac{\zeta(-s + 8)}{\zeta(-7 + s)}\right) + \log\left(\frac{\zeta(s - 8)}{\zeta(9 - s)}\right) - \log\left(\frac{s - 8}{s + 8}\right) \\
= \log\left(\frac{\zeta(-s + 16)}{\zeta(-15 + s)}\right) + \log\left(\frac{\zeta(s - 16)}{\zeta(17 - s)}\right) - \log\left(\frac{s - 16}{s + 16}\right) + \log\left(\frac{s}{s + 8}\right) \\
= \log\left(\frac{\zeta(s + 16)}{\zeta(-15 - s)}\right) + \log\left(\frac{\zeta(-s - 16)}{\zeta(17 + s)}\right) - \log\left(\frac{s + 16}{s - 16}\right) + \log\left(\frac{s}{s - 8}\right)
\]

5 An introduction to the new omega function $\Omega(s)$

Here, we wish to present and prove

**Theorem 5.1.**

\[
\Omega(s) = \psi'(2n)\left(\frac{s}{2}\right) + \psi'(2n)\left(\frac{1 - s}{2}\right) - \psi'(2n)\left(1 - \frac{s}{2}\right) - \psi'(2n)\left(1 + \frac{s}{2}\right) = \\
\left(\psi'(2n)\left(\frac{s}{2}\right) - \psi'(2n)\left(\frac{1 + s}{2}\right)\right) = \\
(-2)^{2n+1}\Gamma(2n + 1)\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k + s)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + 1 - s)^{2n+1}}\right) = (80)
\]

**Proof.** According to theorem 3.4

\[
\log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{1 + s}{2}\right)}\right) + \log(\pi^{2(1 + s - \frac{s}{2})}) = - \log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{1 - s}{2}\right)}\right) - \log(\pi^{2(1 - s - \frac{s}{2})}) - \log(2s^2)\left(\frac{s - 1}{(s - 2)(2\pi)^2}\right) + \\
- \log\left(\frac{\sin\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi (1 - s)}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi (1 - s)}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{\pi (1 - s)}{2}\right)}{\sin\left(\frac{\pi (1 - (1 - s))}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi (1 - s)}{2}\right)}\right)
\]

\[
\Rightarrow \\
\frac{d^{2n+1}}{ds^{2n+1}}\log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{1 - s}{2}\right)}\right) = \frac{d^{2n+1}}{ds^{2n+1}}\log\left(\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{1 - s}{2}\right)}\right) - \frac{d^{2n+1}}{ds^{2n+1}}\log\left(\frac{2s^2}{(s - 2)(2\pi)^2}\right) + \\
\frac{d^{2n+1}}{ds^{2n+1}}\left(\frac{\sin\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi (1 - s)}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi (1 - s)}{2}\right)}\right) - \log\left(\frac{\sin\left(\frac{\pi (1 - s)}{2}\right)}{\sin\left(\frac{\pi (1 - (1 - s))}{2}\right)}\right) + \log\left(\frac{\sin\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi (1 - s)}{2}\right)}\right)
\]

\[
\Rightarrow
\]
\[-\frac{\psi'(2)(-\frac{s}{2}) + \psi'(2)(\frac{1+s}{2})}{2^{2n}} = -\frac{\psi'(2)(\frac{s}{2}) + \psi'(2)(\frac{1-s}{2})}{2^{2n}} - \frac{\Gamma(2n+1)}{2^{2n}(-\frac{s}{2})^{2n+1}} - \frac{d^{2n+1}}{ds^{2n+1}}(\log(s^2)) + \frac{d^{2n+1}}{ds^{2n+1}}\left( - \log\left( \frac{\sin\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)} \right) + \log\left( \frac{\sin\left(\frac{\pi(\frac{s}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s}{2})}{2}\right)} \right) \right) \quad (82)\]

Note the following identities:

\[\psi'(2)(\frac{a}{b}) = \frac{(-1)^{2n+1} \Gamma(2n+1)}{\left(\frac{a}{b}\right)^{2n+1}} + \psi'(2)(1 + \frac{a}{b});\]

\[\psi'(2)(1 + \frac{a}{b}) = \frac{(-1)^{2n+1} \Gamma(2n+1)}{\left(\frac{b}{a}\right)^{2n+1}} + \psi'(2)(\frac{a}{b});\]

i.e.

\[\psi'(2)(-\frac{s}{2}) = -\frac{\Gamma(2n+1)}{\left(-\frac{s}{2}\right)^{2n+1}} + \psi'(2)(1 - \frac{s}{2}).\]

Therefore

\[-\psi'(2)(1 - \frac{s}{2}) + \psi'(2)(\frac{1+s}{2}) = \frac{\psi'(2)(\frac{s}{2}) + \psi'(2)(\frac{1-s}{2})}{2^{2n}} - \frac{\Gamma(2n+1)}{2^{2n}(-\frac{s}{2})^{2n+1}} - \frac{d^{2n+1}}{ds^{2n+1}}(\log(s^2)) + \frac{d^{2n+1}}{ds^{2n+1}}\left( - \log\left( \frac{\sin\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)} \right) + \log\left( \frac{\sin\left(\frac{\pi(\frac{s}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s}{2})}{2}\right)} \right) \right) \quad (83)\]

\[\Rightarrow\]

\[-\psi'(2)(1 - \frac{s}{2}) + \psi'(2)(\frac{1+s}{2}) = \frac{\psi'(2)(\frac{s}{2}) + \psi'(2)(\frac{1-s}{2})}{2^{2n}} + \frac{2\Gamma(2n+1)}{s^{2n+1}} - \frac{d^{2n+1}}{ds^{2n+1}}(\log(s^2)) + \frac{d^{2n+1}}{ds^{2n+1}}\left( - \log\left( \frac{\sin\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)} \right) + \log\left( \frac{\sin\left(\frac{\pi(\frac{s}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s}{2})}{2}\right)} \right) \right) \quad (84)\]

\[\Rightarrow\]

\[-\psi'(2)(1 - \frac{s}{2}) + \psi'(2)(\frac{1+s}{2}) = \frac{\psi'(2)(\frac{s}{2}) + \psi'(2)(\frac{1-s}{2})}{2^{2n}} \quad (85)\]

\[\frac{d^{2n+1}}{ds^{2n+1}}\left( - \log\left( \frac{\sin\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)} \right) + \log\left( \frac{\sin\left(\frac{\pi(\frac{s}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s}{2})}{2}\right)} \right) \right) = \psi'(2)(\frac{s}{2}) + \psi'(2)(\frac{1-s}{2}) - \psi'(2)(\frac{1}{2}) + \frac{d^{2n+1}}{ds^{2n+1}}\left( - \log\left( \frac{\sin\left(\frac{\pi(\frac{s+1}{2})}{2}\right)}{\sin\left(\frac{\pi(1-s)}{2}\right)} \right) + \log\left( \frac{\sin\left(\frac{\pi(\frac{s}{2})}{2}\right)}{\sin\left(\frac{\pi(1-\frac{s}{2})}{2}\right)} \right) \right) \quad (86)\]

\[-\frac{\Gamma(2n+1)}{2^{2n}} \sum_{k=0}^{\infty} \frac{1}{(k + \frac{s}{2})^{2n+1}} + \frac{\Gamma(2n+1)}{2^{2n}} \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1-s}{2})^{2n+1}} - \frac{\Gamma(2n+1)}{2^{2n}} \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2})^{2n+1}} \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1-s}{2})^{2n+1}} - \frac{\Gamma(2n+1)}{2^{2n}} \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2})^{2n+1}} \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1-s}{2})^{2n+1}} \quad (86)\]
s=0 and s=1, the omega function, i.e. Ω(0) or Ω(1), requires the estimation of

\[ \sum_{j \in \mathbb{Z}} \text{ where } j \text{ is any integer number. This omega function has poles at integer points. For instance, at the point \( \phi \). As a result the following fundamental properties emerge in general:

5.1 The omega function of golden ratio

Let \( \phi \) represent the golden ratio constant \( \frac{1 + \sqrt{5}}{2} \). The implication of the omega functional equation is that \( \Omega(\phi) = \Omega(1 - \phi) = \Omega(1 + \frac{1}{\phi}) = \Omega(-\frac{1}{\phi}) \) because \( \phi = 1 + (\phi - 1) = 1 + \frac{1}{\phi} \). This suggests that all the series sums below will produce the same result.
\[ s \to \phi: \]
\[
\frac{\Omega(\phi)}{-\Gamma(2n+1)} = \sum_{k=0}^{\infty} \frac{1}{(k + \frac{\phi}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{\phi}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{\phi}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 - \frac{\phi}{2})^{2n+1}}. \tag{92}
\]

\[ s \to 1 - \phi: \]
\[
\frac{\Omega(1 - \phi)}{-\Gamma(2n+1)} = \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{\phi}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{\phi}{2})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 + \frac{\phi}{2})^{2n+1}}. \tag{93}
\]

\[ s \to 1 + \frac{1}{\phi}: \]
\[
\frac{\Omega(1 + \frac{1}{\phi})}{-\Gamma(2n+1)} = \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{1}{2\phi})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 + \frac{1}{2\phi})^{2n+1}}. \tag{94}
\]

\[ s \to -\frac{1}{\phi}: \]
\[
\frac{\Omega(-\frac{1}{\phi})}{-\Gamma(2n+1)} = \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} - \frac{1}{2\phi})^{2n+1}} + \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2} + \frac{1}{2\phi})^{2n+1}} - \sum_{k=0}^{\infty} \frac{1}{(k + 1 + \frac{1}{2\phi})^{2n+1}}. \tag{95}
\]

6 Discussion

In this paper we derived a new formula for representing the actual value of \( \psi^{2n}(s) \) from which we further derived another formula for expressing the value of \( \zeta(2n + 1) \) employing a method dependent on \( \psi^{2n}(s) + \psi^{2n}(1 - s) \) calculation. For instance, we uncovered the new formula for the first time

\[
\zeta(2n + 1) = \frac{\psi^{(2n)}(\frac{1}{2}) + \psi^{(2n)}(\frac{3}{2})}{2^{2n+1} \cdot (2^{2n+1} - 1)} \cdot \frac{1}{\Gamma(2n + 1)} = \frac{2^d(2n+1)}{\Gamma(2n+1) \cdot (\log(\zeta(1-s)))} \bigg|_{s \to \frac{1}{2}} + \frac{(2n+1)^d}{\Gamma(2n+1) \cdot (\log(\tan(\frac{\pi}{2})))} \bigg|_{s \to \frac{1}{2}}.
\]

We then presented new strategies for calculating the value of the logarithm of \( \frac{\psi(x)}{\zeta(1-s)} \) based on fractions of other zeta functions at smaller points of arguments. New identities relating the logarithm of Riemann zeta function to that of gamma function were also presented and all proved primarily using elementary functions only. We considered presenting new combinatorial results and perspectives on contemporary zeta functional analysis to establish deeper connections with several related zeta functions for the first time. We then used one of our main results about a recent discovery (also presented here) to uncover and present the relation.
\[ \Omega(s) = \psi^{(2n)}(\frac{s}{2}) + \psi^{(2n)}(\frac{1-s}{2}) - \psi^{(2n)}(1) - \psi^{(2n)}(\frac{s}{2}) - \psi^{(2n)}(\frac{1+s}{2}) = \]
\[ (\psi^{(2n)}(\frac{s}{2}) - \psi^{(2n)}(\frac{1}{2}) + \psi^{(2n)}(\frac{1}{2} - s)) + (\psi^{(2n)}(\frac{1-s}{2}) - \psi^{(2n)}(\frac{1}{2} + s)) = \]
\[ (-2)^{2n+1} \Gamma(2n+1) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+s)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1-s)^{2n+1}} \right) = \]
\[ 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} ( \log \left( \frac{\sin(\frac{s+1}{2})}{\cos(\frac{s}{2})} \right) + \log \left( \frac{\sin(\frac{s+1}{2})}{\cos(\frac{s}{2})} \right) ) - \log \left( \frac{\sin(\frac{1}{2})}{\cos(\frac{s}{2})} \right) + \log \left( \frac{\sin(\frac{1}{2})}{\cos(\frac{s}{2})} \right) \right) \]
which may provide new optimised and efficient strategies for analysing and evaluating sums of alternating series related to zeta, Dirichlet beta, and important nonelementary functions. For example,
\[ -2(2^{2n+1})2^{2n+1} \beta(2n+1) = \Omega\left(\frac{1}{4}\right) = 2\psi^{(2n)}(\frac{1}{4}) - 2\psi^{(2n)}(\frac{3}{4}) = \]
\[ (-2)^{2n+1} \Gamma(2n+1) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})^{2n+1}} \right) = \]
\[ 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} \left( \log \left( \frac{\sin(\frac{s+1}{2})}{\cos(\frac{s}{2})} \right) + \log \left( \frac{\sin(\frac{s+1}{2})}{\cos(\frac{s}{2})} \right) \right) - \log \left( \frac{\sin(\frac{1}{2})}{\cos(\frac{s}{2})} \right) + \log \left( \frac{\sin(\frac{1}{2})}{\cos(\frac{s}{2})} \right) \right) \]

It is a delight knowing that the same technique could be used to analyse slightly more complicated results such as
\[ \Omega(i) = \psi^{(2n)}(\frac{i}{2}) + \psi^{(2n)}(\frac{1-i}{2}) - \psi^{(2n)}(1) - \psi^{(2n)}(\frac{1+i}{2}) = \]
\[ (-2)^{2n+1} \Gamma(2n+1) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+i)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1-i)^{2n+1}} \right) = \]
\[ (-2)^{2n+1} \Gamma(2n+1) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+i)^{2n+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k-1+i)^{2n+1}} \right) = \]
\[ (-2)^{2n+1} \Gamma(2n+1) \left( \frac{1}{(i)^{2n+1}} + \sum_{k=1}^{\infty} \frac{(-1)^k((k-i)^{2n+1}-(k+i)^{2n+1})}{(k^2+1)^{2n+1}} \right) = \]
\[ 2^{2n} \frac{d^{2n+1}}{ds^{2n+1}} \left( \log \left( \frac{\sin(\frac{s+1}{2})}{\cos(\frac{s}{2})} \right) + \log \left( \frac{\sin(\frac{s+1}{2})}{\cos(\frac{s}{2})} \right) \right) - \log \left( \frac{\sin(\frac{1}{2})}{\cos(\frac{s}{2})} \right) + \log \left( \frac{\sin(\frac{1}{2})}{\cos(\frac{s}{2})} \right) \right) \]
which has instantaneously provided a stable mechanism of evaluating the series sum in terms of derivatives of the logarithm of elementary trigonometric functions; as a result, avoiding actual computations of \( \psi^{(2n)}(\frac{1}{2}) \), \( \psi^{(2n)}(\frac{1-i}{2}) \), \( \psi^{(2n)}(1 - \frac{i}{2}) \) and \( \psi^{(2n)}(\frac{1+i}{2}) \) without requiring factorising \( (k-i)^{2n+1} - (k+i)^{2n+1} \).
We then ask: how important is the result

\[
\frac{1}{i^{2n+1}} + \sum_{k=1}^{\infty} \frac{(-1)^k((k-i)^{2n+1} - (k+i)^{2n+1})}{(k^2 + 1)^{2n+1}} \quad (98)
\]

which was derived after just a few additional steps.

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