ANGULAR ORDERING AND SMALL-\(x\) EVOLUTION\(^*\)

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ABSTRACT

This talk examines the effect of angular ordering on the small-\(x\) evolution of the unintegrated gluon distribution, and discusses the characteristic function for the CCFM equation, as well as some preliminary results on final-state properties.

1. Introduction

For some time now it has been known that angular ordering is an essential element in any description of small-\(x\) final state properties. As a first step of a programme to study the final state in small-\(x\) physics, one should examine the effect of angular ordering on the small-\(x\) evolution of the gluon structure function. Phenomenological studies have already been performed, but this talk will examine the solutions of the CCFM equation from a more theoretical point of view. I will also present some first results on final-state properties.

The main difference between the BFKL and CCFM equations is in the collinear region: in the BFKL case, the \(i\)th emission has a transverse momentum \(q_i > \mu\), with \(\mu\) a cutoff put in by hand and which is taken to zero; this regulates the collinear divergence, which for the gluon structure function cancels out. But in quantities where it doesn’t cancel, such as certain final properties, one obtains the wrong answer. In the CCFM equation, angular ordering of emissions leads to the following condition (see figure 1):

\[ \theta_i > \theta_{i-1}, \quad \Rightarrow \quad q_i > z_{i-1}q_{i-1}, \]

with the corresponding gluon emission distribution being

\[ dP_i = \frac{d^2q_i}{\pi q_i^2} \frac{\bar{\alpha}_S}{z_i} \Delta(z_i, q_i, k_i) \Theta(q_i - z_{i-1}q_{i-1}). \]

The non-Sudakov form factor \(\Delta\), which is analogous to a probability for suppressing any further radiation, is defined by

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The elimination of a large fraction of the small-transverse-momentum emissions means that angular ordering has a big effect on the final state. But in structure function evolution, since collinear singularities cancel, at leading order the BFKL and CCFM structure functions are equivalent.

2. The gluon structure function

As part of a program to carry out a full investigation of the effects of angular ordering at small $x$, this talk examines the component of the next-to-leading order corrections to structure function evolution that arise from angular ordering. Such effects are expected to be part of the full NLL contribution.

Qualitatively, since angular ordering reduces the phase space for evolution the exponent of the small-$x$ growth ought to be reduced. The symmetry, present in the BFKL equation, between large and small scales will be broken, favouring evolution to large momentum scales. Finally diffusion will be reduced because large jumps (down) in scale are suppressed.

There are two limits in which, at small $x$, the effects of angular ordering should disappear: as $\alpha_S \to 0$, because the typical $z_{i-1} \sim \alpha_S$ will be very small (this justifies the assertion that for structure functions the effects of angular ordering are next-to-leading); and in the double-leading-logarithmic limit because the condition $q_i > q_{i-1}$ automatically satisfies the angular ordering condition.

The analytic treatment of the CCFM equation is more complicated than that of the BFKL equation because the gluon density contains one extra parameter, $p$, which defines the maximum angle for the emitted gluons. In DIS it enters through the angle of the quarks produced in the boson-gluon fusion. The equation for the CCFM density, $A(x, k, p)$, of gluons with longitudinal momentum fraction $x$ and transverse momentum $k$ is:
\[
A(x, k, p) = A^{(0)}(x, k, p) + \int \frac{d^2q}{\pi q^2} \frac{dz}{z} \frac{\tilde{\alpha}_S}{z} \Delta(z, q, k) \Theta(p - zq) A(x/z, k', q),
\]

where \( k' = |k + q| \). By analogy to the BFKL equation one can develop some understanding of it by looking for eigensolutions (strictly speaking eigensolutions of the equation without an inhomogeneous term and with no lower limit in the \( z \) integral) of the form

\[
x A(x, k, p) = x^{-\omega} \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right)^{\tilde{\gamma}} G(p/k),
\]

where \( G(p/k) \) parameterises the unknown dependence on \( p \). For \( 0 < \tilde{\gamma} < 1 \), one obtains a coupled pair of equations for \( G \) and \( \omega \):

\[
p \partial_p G(p/k) = \tilde{\alpha}_S \int \frac{d^2q}{\pi q^2} \left( \frac{p}{q} \right)^{\omega} \Delta(p/q, q, k) \left( \frac{q}{k'} \right)^{\tilde{\gamma} - 1} G\left( \frac{q}{k'} \right),
\]

with the initial condition \( G(\infty) = 1 \) and

\[
\omega = \alpha_S \tilde{\chi}(\tilde{\gamma}, \alpha_S) = \alpha_S \int \frac{d^2q}{\pi q^2} \left\{ \left( \frac{k'}{k^2} \right)^{\tilde{\gamma} - 1} G\left( \frac{q}{k'} \right) - \Theta(k - q) G(q/k) \right\}.
\]

In the second of these equations, if \( G = 1 \) one notes that \( \tilde{\chi} \) is just the BFKL characteristic function, \( \chi \). Since \( 1 - G(p/k) \) is formally of order \( \alpha_S \), this demonstrates that angular ordering has a next-to-leading effect on structure function evolution. One can also show that in the limit of \( \gamma \to 0 \) the difference \( \chi(\gamma) - \tilde{\chi}(\gamma, \alpha_S) \) tends to a constant, which implies corrections to the small-\( x \) anomalous dimension of the form \( \alpha^3_S/\omega^2 \).

Though a number of asymptotic properties of \( G(p/k) \) have been determined, it has not so far been possible to obtain its full analytic form. Further understanding requires numerical analysis. This has been carried out and figure \( 2 \) shows the results for \( \tilde{\chi} \) compared to the BFKL characteristic function for three different values of \( \alpha_S \). It illustrates that as \( \alpha_S \to 0 \) the two tend to coincide as happens also in the region \( \gamma \to 0 \) (the DLLA region).

The loss of symmetry under \( \gamma \to 1 - \gamma \) relates to the loss of symmetry between small and large scales. Indeed, in contrast to the BFKL case, there is no longer even a divergence at \( \gamma = 1 \). Correspondingly, the minimum of the characteristic function gets shifted to the right and is lower.

Figure \( 3 \) shows the characteristics of the minimum (or critical point) of \( \tilde{\chi} \) as a function of \( \alpha_S \): the height of the minimum, \( \tilde{\chi}_c \), its position, \( \tilde{\gamma}_c \) and the second derivative of \( \tilde{\chi} \) at the minimum, \( \tilde{\chi}_c'' \). One notes that the dependence of these quantities on
Fig. 2. The BFKL and CCFM characteristic functions as a function of $\gamma$ for different values of $\alpha_S$.

$\alpha_S$ is noticeably non-linear, indicating the presence of substantial corrections beyond next-to-leading logarithms. Indeed, for $\alpha_S = 0.2$, carrying out a simple fit to the linear component of the correction at small $\alpha_S$, one sees that contributions beyond NLL are of the order of about half the total correction.

Angular ordering has a particularly large effect on the second derivative of $\tilde{\chi}$ — for $\alpha_S = 0.2$, it is reduced by a factor of two. The importance of the second derivative is that it is related to the amount of diffusion that is present. For an evolution over a range $x$, if one examines intermediate gluons with longitudinal momentum fraction $x_i$, using the saddle-point approximation one finds that the distribution of their transverse momentum has a width $\Delta \ln k$ which is given by:

$$\Delta \ln k \simeq \sqrt{\frac{\alpha_S \chi_c''}{4}} \frac{\ln x_i \ln x_i/x}{\ln x} \ln x$$

(1)

Figure [4] shows the application of this formula to a particular set of evolution parameters — one sees that angular ordering leads to significant reduction of diffusion into the non-perturbative region. It has been shown by Mueller[8] that diffusion leads to a breakdown in the operator-product expansion (OPE) at a value of $x$ defined by

$$\ln \frac{x_0}{x} \simeq \frac{1}{2 \chi_c''} \ln \frac{Q^2}{\Lambda^2},$$

where $\Lambda$ is the QCD scale, $x_0$ some starting point for the small-$x$ evolution, and $Q^2$ the hard scale of the problem. Accordingly, the effect of angular ordering can be seen
Fig. 3. (a) The value of the minimum of the characteristic function, $\tilde{\chi}_c$, as a function of $\alpha_S$; (b) The position of the minimum of the characteristic function, $\tilde{\gamma}_c$, as a function of $\alpha_S$; (c) The second derivative of the characteristic function, $\tilde{\chi}_c''$, at its minimum, as a function of $\alpha_S$. 
Fig. 4. “Cigars” showing the range of transverse momentum in the evolution as a function of intermediate $x_i$ for the BFKL and CCFM equations. The evolution is to $x = 10^{-4}$, $k = 5$ GeV with $\alpha_S = 0.2$.

as extending the $x$-range over which the OPE remains valid.

3. Associated quantities

The analysis of the gluon distribution is only the first step of a programme which aims to produce a Monte Carlo event generator based on the CCFM equation. An intermediate step is to examine associated final-state properties which are accessible using methods similar to that used for the structure function. Full details, including a description of the methods used, will be given in a forthcoming publication. Here, preliminary results will be presented for two quantities: the probability of having $n$ emissions with a transverse momentum above a certain minimum; and the transverse momentum flow as a function of rapidity. In neither case has the photon-gluon fusion matrix element been included, so the results are not directly comparable with experiment.

The probability of having $n$ primary emitted gluons whose transverse momentum is larger than some minimum $q > \mu$ is an example of a quantity which differs between the BFKL and CCFM equations at leading order: in the BFKL case, for small $\mu$, it varies in proportion to $\log \mu$, whereas in the CCFM case, for small $\mu$ it is independent of $\mu$. Figure 5a shows the probability distribution for $\mu = 1$ GeV, while figure 5b shows it for $\mu = 0.007$ GeV. In the first case the curves are not too different (e.g. the maxima are at the same point), with the main difference being that the BFKL case has a long tail which is suppressed by phase-space constraints when angular
order is introduced. On the other hand, for small $\mu$ the two distributions are quite different, with their maxima at very different positions: in the BFKL case there are many small-momentum emissions which are eliminated by the introduction of angular ordering.

Returning to the phenomenologically more meaningful case of $\mu = 1$ GeV, a simple analytical calculation of the DGLAP result has been also included:

$$P(n) = \frac{\alpha_s \ln k^2 \ln 1/x^n}{(n!)^2}.$$  \hspace{1cm} \text{(2)}

As one would have expected, the CCFM result lies between the BFKL and DGLAP cases.

The other quantity for which we have a preliminary result is the transverse energy flow as a function of rapidity. This is shown in figure 6 where one sees that angular ordering significantly reduces the transverse energy flow.

4. Conclusions and outlook

In this talk, I have presented results on the effect of angular ordering on structure functions and two associated quantities. The main results are to be found in figures 2 and 3, where it is seen that angular ordering reduces the height of the minimum of the small-$x$ characteristic function, shifts its position to larger $\gamma$ and strongly reduces its second derivative (corresponding to a reduction in diffusion).

The technology developed for the study of structure functions is also being applied to the analysis of associated final-state properties and some preliminary results have
been presented here, illustrating that angular ordering tends to reduce the number of emissions, particularly those with low transverse momenta, and that it also reduces the mean transverse energy flow.

To carry out phenomenology, certain extra elements are being implemented, among them the running of $\alpha_S$ and the inclusion of the hard matrix element, which determines the gluon-photon interaction. Also under development is a backward-evolution CCFM Monte Carlo event generator.

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