Higher Order Corrections to Holographic Black Hole Chemistry

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We investigate the holographic Smarr relation beyond the large $N$ limit. By making use of the holographic dictionary, we find that the bulk correlates of sub-leading $1/N$ corrections to this relation are related to the couplings in Lovelock gravity theories. We likewise obtain a holographic equation of state, and check its validity for a variety of interesting and non-trivial black holes, including rotating planar black holes in Gauss-Bonnet-Born-Infeld gravity, and non-extremal rotating black holes in minimal 5$d$ gauged supergravity. We provide an explanation of the $N$-dependence of the holographic Smarr relation in terms of contributions due to planar and non-planar diagrams in the dual theory.

I. INTRODUCTION

For nearly two decades the AdS/CFT correspondence [1] has been the subject of intense research, motivated by the fact that it posits a connection between an anti de Sitter (AdS) black hole and a conformal field theory (CFT) defined on its boundary. In the context of black hole thermodynamics, an understanding of the physics of the AdS black hole can be reinterpreted in terms of a thermal system on the boundary field theory and vice-versa.

The general assumption underlying nearly all investigations of the AdS/CFT correspondence is that the cosmological constant is a fixed parameter. However increasing interest has focussed on regarding the cosmological constant as a thermodynamic variable via the relation

$$P = -\frac{\Lambda}{8\pi G_d} = \frac{(d-1)(d-2)}{16\pi l^2 G_d}$$

where $P$ is the pressure of the black hole system [2]. With this comes the associated concept of volume $V$ for a black hole, which is the thermodynamic conjugate of pressure. The extension of thermodynamic phase space to include these two variables has led to the realization that black holes can exhibit enormously rich and diverse phase behaviour, including Van der Waals phase transitions for charged black holes [3, 4], triple points analogous to water [5], and even superfluid phase transitions analogous to those seen in gels and polymers [6], and even superfluid phase transitions analogous to those in superfluid helium [7]. This burgeoning subfield is now referred to as black hole chemistry [8].

It is therefore of interest to ask what black hole chemistry implies for the variables in the boundary field theory. What do the first law of thermodynamics, the Smarr relation, and so on look like on the CFT side? The quantity $l$ in (1) is the AdS radius and is related to the number of colours $N$ in the dual gauge theory via a holographic relation of the form [12]

$$\frac{l^{d-2}}{G_d} \sim N^2$$

where the $d$-dimensional gravitational constant $G_d$ has a length dimension of $d – 2$. This kind of relation was first introduced in the AdS/CFT correspondence from string theory [1], in which an $\text{AdS}_5 \times S^5$ spacetime appears to be the near horizon geometry of $N$ coincident $D_3$ branes in type IIB supergravity. The correspondence between an $\text{AdS}_5 \times S^5$ spacetime and a $\mathcal{N} = 4$ SU$(N)$ Yang-Mills theory on its boundary was expressed as follows

$$l^4 = \frac{\sqrt{2}\ell_{Pl}^3}{\pi^2} N,$$

where $\ell_{Pl}$ is the 10-dimensional Planck length. From the two preceding relations we can remark that the variation of the AdS radius $l$ amounts to the variation of the color number $N$ in the boundary Yang-Mills theory.

An interesting subject to think about is on the nature of the connection between the bulk and the CFT on its boundary as well as its implications when we are in presence of another theory of gravity.

The suggestion that varying the pressure, or $\Lambda$, is equivalent to varying the number of colors, $N$, in the boundary Yang–Mills theory has been proposed by a few authors [9–11], with $V$ being interpreted in the boundary field theory as an associated chemical potential $\mu$ for colour. This has the consequence that variation of $\Lambda$ in the bulk moves one around the space of field theories in the boundary. Alternatively, one could keep $N$ fixed, so that field theory remains the same, in which case varying $\Lambda$ in the bulk corresponds to varying the curvature radius governing the space on which the field theory is defined [12].

From this latter perspective, a generalized Smarr relation can be derived by considering the thermodynamics...
of the dual field theory \[12\]. Noting that the free energy of the field theory scales simply as \(N^2\), we have

\[\Omega(N, \mu, T, l) = N^2\Omega_0(\mu, T, l)\]

in the limit of large \(N\). For a conformal field theory the equation of state reads

\[E = (d - 2)pV\]

and together with (22) can be used to obtain the standard Smarr relation \((d - 3)M = (d - 2)TS - 2PV\) for an uncharged AdS black hole. In this sense equation (22) can be regarded as a ‘holographic Smarr relation’. Since \(N^2 \approx \frac{d+2}{d-2}\), varying \(\Lambda\) is equivalent to varying the AdS length \(l\), and since \(N\) is fixed, \(G_d\) must also be varied.

The purpose of this paper is to investigate the holographic Smarr relation (22) beyond the large \(N\) limit, including sub-leading corrections to this relation. We will see that relation (22) can be generalized to a form that includes subleading \(1/N\) corrections whose bulk correlates are related to the couplings in Lovelock gravity theories. Lovelock theories are higher curvature or derivative generalizations of Einstein’s theory, and in the context of string theory are understood as quantum corrections to Einstein gravity. We shall show that the Lovelock couplings are related to a function of \(N\), with variations of the Lovelock couplings in the bulk dictating the behaviour of the corresponding CFT via the variation of these functions.

Inquiries on how far this generalization extends (or what are the limits thereof) for a given black hole are also of considerable interest. The bulk Smarr relation and the corresponding CFT equation of state are both expected to be satisfied at the lowest order (Einstein-Hilbert action). We shall look at what happens at higher order, considering especially the CFT equation of state to see whether or not it breaks down.

In section 2, we review some important notions and relations for Lovelock black holes, particularly the first law of thermodynamics and the Smarr relation. In section 3 we investigate how these important relations in the bulk theory are viewed in the CFT, particularly with regards to the derivation of the equation of state in the boundary field theory. Section 4 is devoted to the holographic derivation of the Smarr relation where we mostly make use of the equation (36) of the holographic dictionary and by regarding the grand canonical function \(\Omega\) as a homogeneous function of functions of \(N\). In section 5 we check the validity of the equation of state, introduced earlier in section 3, for some particular cases of black holes, including spherically symmetric AdSlovellock black holes, rotating planar black holes in Gauss-Bonnet-Born-Infeld gravity, and non-extremal rotating black holes in minimal 5d gauged supergravity. An explanation of the dependence of the function of \(N\) is in section 6, and in the last section we make some concluding remarks.

II. A REVIEW OF LOVELock BLACK HOLES

In this current section we review the derivation of thermodynamic quantities associated with Lovelock black holes and some relations implied by these quantities.

Lovelock gravity is a generalization of Einstein’s theory whose action and field equations are nonlinear in the curvature whilst always maintaining second-order differential equations for the metric. Its Lagrangian has the form \[14\]

\[L = \frac{1}{16\pi G_d} \sum_{k=0}^{d-1} \dot{\alpha}(k)L^{(k)}\]

with \(d\) the spacetime dimension, \(\dot{\alpha}(k)\) the Lovelock coupling constants for the \(k\)-th power of curvature, and \(L^{(k)}\) the Euler density of dimension \(2k\). These Euler densities are expressed as

\[L^{(k)} = \frac{1}{2k(2k-1)} \delta_{a_1b_1...a_kb_k} R^{a_1d_1...a_kd_k} R_{a_1b_1...a_kb_k} (6)\]

where the \(\delta_{a_1b_1...a_kb_k}\) are the totally antisymmetric in both set of indices of the Kronecker delta functions and \(R^{a_1d_1...a_kd_k}\) the Riemann tensors.

From the Lagrangian (6, 7), the variational principle yields the vacuum equations of motion for Lovelock gravity, which are

\[\nabla_a G^{(k)}_{ab} = 0\]

with \(G^{(k)}_{ab}\) the Einstein-like tensors, which read as

\[G^{(k)}_{ab} = \frac{1}{2k(2k-1)} \delta_{bc_1d_1...c_kd_k} R^{a_1d_1...a_kd_k} R_{a_1b_1...a_kb_k} (9)\]

and each of them satisfy independently the conservation law \(\nabla_a G^{(k)}_{ab} = 0\).

If we minimally couple to a Maxwell field \(F_{ab}\) the action is

\[S = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} \left[ \sum_{k=0}^{d-1} \dot{\alpha}(k)L^{(k)} - 4\pi G_d F_{ab} F^{ab} \right] (10)\]

and yields

\[\sum_{k=0}^{d-1} \dot{\alpha}(k)G^{(k)}_{ab} = 8\pi G_d [F_{ac} F^c_b - \frac{1}{4} g_{ab} F_{cd} F^{cd}] (11)\]

for the equations of motion. Without solving these equations, it can be shown for solutions of asymptotic constant curvature that the first law of thermodynamics and
the Smarr relation respectively are

\[ \delta M = T \delta S + \mu \delta Q - \frac{1}{16\pi G_d} \sum_{k=0}^{d-1} \Psi^{(k)} \delta \hat{\alpha}^{(k)} \]
\[ (d-3)M = (d-2)TS + (d-3)\mu Q \]
\[ + \sum_{k=0}^{d-1} 2(k-1) \frac{16\pi G_d}{(d-3)} \Psi^{(k)} \hat{\alpha}^{(k)} \]

(12)

where the solution is characterized by a mass \( M \), a charge \( Q \), Lorentz coupling constants \( \hat{\alpha}^{(k)} \) each having thermodynamic conjugate \( \Psi^{(k)} \) and (if it is a black hole) a temperature \( T \), and an entropy \( S \).

Restricting attention to spherically symmetric metrics

\[ ds^2 = -f(r)dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2 \]
\[ F = \frac{Q}{r} dt \wedge dr \]

(13)

where \( d\Omega^2 \) is the line element of a compact space of dimension \( (d-2) \) with constant curvature \( (d-2)(d-3)\kappa \) \((\kappa = -1, 0, 1)\), the equations of motion \((11)\) for charged spherically symmetric black holes of mass \( M \) read [21–24, 26–28]

\[ \frac{1}{2^k} \left( \frac{\kappa - f}{r^2} \right)^k = \frac{16\pi G_d M}{(d-2)\omega_{d-2}^n r^{d-1}} - \frac{8\pi G_d Q^2}{(d-2)(d-3)r^2(d-2)} \]

where the charge is given by

\[ Q = \frac{1}{2\omega_{d-2}^n} \int *F \]

(15)

and where \( \omega_{d-2}^{(1)} = 2\omega_{d-2}^{(d-1)/2} \) and

\[ \alpha_0 = \frac{\hat{\alpha}^{(0)}}{(d-1)(d-2)}, \quad \alpha_1 = \hat{\alpha}^{(1)} \]
\[ \alpha_k = \hat{\alpha}^{(k)} \prod_{n=3}^{2k} (d-n) \quad \text{for } k \geq 2 \]

(16)

is a simple and useful rescaling of the Lovelock couplings.

Computing the temperature \( T = \frac{\omega_{d-2}^{(1)}}{4\pi r_+} \) via standard Wick rotation arguments, we need not explicitly know \( f(r) \) in order to determine the mass \( M \), temperature \( T \), entropy \( S \) and electric potential \( \mu \) of the black holes. These thermodynamic quantities are [24, 25]

\[ M = \frac{\omega_{d-2}^{(k)}(d-2)}{16\pi G_d} \sum_{k=0}^{d-3} \alpha_k \kappa r_+^{d-1-2k} + \frac{\omega_{d-2}^{(0)}Q^2}{2(d-3)r_+^{d-3}} \]
\[ T = \frac{1}{4\pi r_+ D(r_+)} \left[ \sum_{k=0}^{d-3} \kappa \alpha_k (d-2k-1)(\frac{\kappa}{r_+^{d-3}})^k - \frac{8\pi G_d Q^2}{(d-2)r_+^{2(d-3)}} \right] \]
\[ S = \frac{\omega_{d-2}^{(0)}(d-2)}{4G_d} \sum_{k=0}^{d-3} \kappa r_+^{d-2k} - \kappa r_+^{d-2k} - \mu = \frac{\omega_{d-2}^{(0)}Q}{(d-3)r_+^{d-3}} \]

with \( D(r_+) = \sum_{k=0}^{d-3} k\alpha_k (\kappa r_+^{d-2})^{k-1} \) and \( r_+ \) the horizon radius. From the extended first law (12) and Smarr relation (12) we can obtain the thermodynamic conjugate quantities \( \Psi^{(k)} \) [16, 17]

\[ \Psi^{(k)} = \frac{\omega_{d-2}^{(k)}(d-2)}{16\pi G_d} \frac{1}{r_+ d-2k} \left( \frac{\kappa}{r_+^{d-2k}} - \frac{4\pi kT}{d-2k} \right), \quad k \geq 0 \]

(18)

in terms of the rescaled coupling constants. The above quantities satisfy the Smarr relation (12). From these quantities it follows that the thermodynamic pressure and volume are given by

\[ P = -\frac{\Lambda}{8\pi G_d} = \frac{(d-1)(d-2)}{16\pi G_d} \alpha_0 \quad V = \omega_n^{(c)} \frac{r_+^{n+1}}{d-1} \]

(19)

where \( n = d-2 \).

Before proceeding we pause to comment on the relationship between these quantities and the more standard notions of thermodynamics bulk pressure and volume in black hole thermodynamics, which are [15]

\[ P_b = -\frac{\Lambda}{8\pi G_d} \quad \text{and} \quad V_b = \frac{\partial M}{\partial P_b} \mid_{S, Q_0, n \geq 1}. \]

(20)

The first relation is the standard identification of \( \alpha_0 \) with the cosmological constant \[16]. Hence

\[ P_b V_b = \alpha_0 \Psi^{(0)}. \]

(21)

The CFT pressure and volume can be defined as \( p = \omega_n^{(c)} R^c \). The pressure \( p \) has a length dimension of \( -(n+1) \) and \( R \) is the radius of the sphere on which the CFT is defined.

III. EQUATION OF STATE

In this section we derive the equation of state by looking at how transformations of parameters on the field theory lead to transformations in the bulk or vice versa. A example of this is the proposed correspondence between varying the cosmological constant \( \Lambda \sim \alpha_0 \) in the bulk and variations in the number of colors \( N \) in the field theory [9, 11, 18–20].

The Smarr relation (12) has been posited [12] to be derivable from the scaling properties of the free energy of the dual field theory in the limit of a large number of colors \( N \). The free energy \( \Omega(N, \mu, T) \) of the field theory dual to Einstein-AdS gravity scales as

\[ \Omega(N, \mu, T) = N^2 \Omega_0(\mu, T) \]

(22)

where \( N^2 \) is the central charge. Extending \( \Omega(N, \mu, T, l) \) to Lovelock gravity theory we posit

\[ \Omega(N, \mu, T, \alpha_j, R) = \sum_{k=0}^{n} g_k(N) \Omega^k(\mu, T, \alpha_j, R) \]

(23)

where the \( g_k(N) \) are assumed to be polynomial functions (as suggested in [12]) of \( N \). We will see in the next section
that this form is of great interest in the derivation of the holographic Smarr relation for Lovelock gravity.

Noting that the thermal properties of AdS black holes can be reinterpreted as those of a CFT at the same finite temperature [13], the grand canonical free energy and its density are expressible in terms of the on-shell action the (Euclidean) bulk solution as [12]

\[ \Omega = M - TS - \mu Q \quad \ddot{\Omega} = \dot{M} - T \dot{S} - \mu \dot{Q} \quad (24) \]

where the quantities $\dot{M}$, $\dot{S}$ and $\dot{Q}$ are the respective mass, entropy and charge per unit volume of the CFT. Note that these thermodynamic quantities are defined on the boundary and have the following form:

\[ Q \sim Q_i l, \quad \mu \sim \mu_0/l, \quad \alpha_k^F = \alpha_k l^{2(1-k)} \quad \text{and} \quad \Psi^{(k)} = \Psi^{(k)} l^{2(k-1)} \]

while others are kept unchanged.

Let us consider conformal field theories, whose equations of states is obtained by taking into account the behaviour of the thermodynamic quantities under an infinitesimal scale transformation

\[ dS = 0, \quad dQ = 0, \quad d\alpha_k^F = 0 \quad (k \geq 1), \quad dM = Md\lambda \quad dp = (n + 1)p d\lambda \quad dv = -nvd\lambda \quad (25) \]

$\lambda$ is the parameter associated to the scale transformation. From these relations and the extended first law of thermodynamics

\[ dM = TdS + \mu dQ + \sum_k \Psi^{(k)} d\alpha_k^F \quad (26) \]

we are led to the equation of state

\[ \dot{M} = (n + 1)p. \quad (27) \]

Also the extended first law is reduced to

\[ dM = vd\rho \quad (28) \]

and (under the constraints of (25)) knowing that

\[ v = \frac{\partial \Omega}{\partial p} = (\dot{\Omega} \partial R v + v \partial R \dot{\Omega}) \frac{\partial R}{\partial p} \quad (29) \]

we get

\[ \partial_R p = \frac{n}{R} \dot{\Omega} + \partial_R \dot{\Omega} \rightarrow n \dot{\Omega} + R \partial_R \dot{\Omega} = -(n + 1) \rho \quad (30) \]

where $R \partial_R p = -(n + 1) \rho$ because of the length dimension of $p$. Inserting the equation of state (27) into (30) yields

\[ \dot{M} = -(n \dot{\Omega} + R \partial_R \dot{\Omega}) \quad (31) \]

or alternatively, using (24),

\[ (n + 1) \dot{M} = n(T \dot{S} + \mu \dot{Q}) - R \partial_R \dot{\Omega} \quad (32) \]

which is the holographic equation of state.

For rotating black holes (32) has to be slightly modified: we have to add one more condition to (25)

\[ dJ_i = J_i d\lambda \quad (33) \]

with $J_i$ the angular momentum associated to the $i$-th angular variable. The additional condition takes (32) to the new expression

\[ (n + 1) \dot{M} = n(T \dot{S} + \mu \dot{Q}) + (n + 1) \sum_i \omega_i \dot{J}_i - R \partial_R \dot{\Omega} \quad (34) \]

where $\dot{\Omega}_i$ is the angular velocity associated with the $i$-th angular variable and $\Omega = M - TS - \mu Q - \sum_i \omega_i \dot{J}_i$.

### IV. HOLOGRAPHIC SMARR RELATION

The grand canonical free energy (23) introduced in the previous section is a polynomial on the variable $N^2$. In Einstein gravity [12] only the first term of $\Omega$ is taken into account. This can be justified by the fact that the dual field theories to the black holes are considered to be in the large $N$ limit.

For a Lovelock black hole nonzero additional terms appear due to contributions from the higher curvature terms. Without knowing explicitly their form, the dimensionality

\[ [\alpha_k] = 2(k - 1) \quad \text{or} \quad \alpha_k \sim l^{2(k - 1)} \quad (35) \]

of the Lovelock couplings implies that

\[ \beta_k(\alpha_k)^{\delta - 2} = g_k(N) \quad (36) \]

for which the $k = 0$ term is

\[ \beta_0 l^{\delta - 2} = N^2 \quad (37) \]

recovering the relationship (2) obtained previously [12]. Here $\beta_0 = \frac{k}{\omega_0 \alpha_k}$, with $\delta$ an arbitrary dimensionless constant.

Equation (36) implies

\[ \alpha_k \frac{\partial X}{\partial \alpha_k} = \frac{d - 2}{2(k - 1)} g_k(N) \frac{\partial X}{\partial g_k} \quad (38) \]

for any arbitrary function $X$ of the parameters $\alpha_k$. Setting $X = \Omega$, equation (38) then becomes

\[ \alpha_k \frac{\partial \Omega}{\partial \alpha_k} = \frac{d - 2}{2(k - 1)} g_k(N) \frac{\partial \Omega}{\partial g_k} \quad (39) \]

After multiplying both sides by $2(k - 1)$ and summing over $k$ we have

\[ \sum_{k=0}^d 2(k - 1) \alpha_k \Psi^{(k)} = (d - 2) \sum_{k=0}^d g_k \frac{\partial \Omega}{\partial g_k} \quad (40) \]

where $\Psi^{(k)} = \frac{\partial \Omega}{\partial \alpha_k}$. We thus have the general relation

\[ \frac{d \partial \Omega}{\partial l} + \sum_{k=1}^d 2(k - 1) \alpha_k \frac{\partial \Omega}{\partial \alpha_k} = (d - 2) \sum_{k=0}^d g_k \frac{\partial \Omega}{\partial g_k} \quad (41) \]
noting that $-2\alpha_0 \partial_{\alpha_0} = l \partial_l$. Recalling that the Euler scaling relation $f(tx_1, \ldots, tx_m) = t^n f(x_1, \ldots, x_m)$ implies

$$nf(x_1, \ldots, x_m) = \sum_j x_j \frac{\partial f}{\partial x_j}$$

(42)

for a homogeneous function of order $n$, it is straightforward to see that equation (40) can be written as

$$\sum_{k=0}^{2} (k-1) \alpha_k \Psi^k = (d-2) \Omega$$

(43)

using

$$\Omega = \sum_{k=0}^{2} g_k \frac{\partial \Omega}{\partial g_k}$$

(44)

which holds since $\Omega$ is an homogeneous function of the $g_k$ of degree 1.

More generally $\Omega$ is a function of $(g_k, R, Q)$ and not just the $g_k$. For any function $f(l, Z)$, its derivative with respect to $l$ will be

$$\partial_l f(l, Z) |_{Z_0} = \partial_l f|_Z + \frac{Z}{l} \partial_Z f |_l.$$  

(45)

if the quantity $Z$ has scaling behaviour $Z = Z_0 l^p$ for some constant $Z_0$. For charged black holes,

$$A_b = lA, \quad \mu_b = l \mu, \quad Q_b = Q/l$$

(46)

after converting to a canonical normalized field strength of dimension 2, and the radius $R = R_0 l$ for the boundary CFT since

$$ds^2_{\text{boundary}} = -dt^2 + l^2 d\Omega^2_{d-2}$$

(47)

is the boundary metric [12]. Hence we obtain

$$l \frac{\partial}{\partial t} + \sum_{k=1}^{2} (k-1) \alpha_k \frac{\partial}{\partial \alpha_k} = (d-2) \sum_{k=0}^{2} g_k \frac{\partial}{\partial g_k} + R \frac{\partial}{\partial R} + \frac{Q}{\partial Q}$$

(48)

and so (40) now reads as

$$\sum_{k=0}^{2} (k-1) \alpha_k \Psi^k = (d-2) \sum_{k=0}^{2} g_k \frac{\partial}{\partial g_k} \Omega_{\mu, T} + R \partial_R \Omega_{\mu, T} \alpha_{k \geq 1} + Q \partial_Q \Omega_{\mu, T} \alpha_{k}$$

$$= (d-2) \Omega - M - \mu Q$$

$$= (d-3) M - (d-2) T S - (d-3) \mu Q$$

(49)

upon using (24), which implies

$$d\Omega = -SdT - Qd\mu + vdp + \sum_{k \geq 1} \Psi^k d\alpha_k$$

(50)

so that $\partial_\mu \Omega = -\mu$ and $\partial_R \Omega = v \partial_R p = -M$ from (27) and (30). We see that (49) is the Smarr relation (12).

V. SOME CASES

The main purpose of this section is to check the validity of the holographic equation of state (34) for a variety of special cases.

We shall consider spherically symmetric AdS Lovelock black holes whose metrics are given in (13) and (14). The thermodynamic quantities for these black holes are given by

$$\tilde{M} = \frac{d-2}{16\pi G_d} \frac{1}{R^{d-2}} \sum_{k=0}^{2} \alpha_k \kappa^k r_+^{d-2k} + \frac{Q^2}{2(d-3) r_+^{d-3} R^{d-2}}$$

$$T S = \frac{d-2}{4G_d} \frac{T}{R^{d-2}} \sum_{k=1}^{2} kr_+^{k-1} \alpha_k r_+^{d-2k}$$

$$\mu Q = \frac{Q^2}{(d-3) r_+^{d-3} R^{d-2}}$$

(51)

and

$$\tilde{\Psi}^{(k)} = \frac{d-2}{16\pi G_d} \frac{kr_+^{k-1} r_+^{d-2k}}{R^{d-2}} \left[ \frac{\kappa}{r_+} + \frac{4\pi k T}{d-2k} \right]$$

$$V = \omega_{d-2}^{(k)} R^{d-2}, \quad R = l.$$  

(52)

From these equations the free energy density looks like

$$\tilde{\Omega} = \frac{d-2}{16\pi G_d} \frac{1}{R^{d-2}} \sum_{k=0}^{2} \alpha_k \kappa^k r_+^{d-2k-1} \left[ \kappa - \frac{4\pi k r_+ T}{d-2k} \right]$$

$$= \frac{Q^2}{2(d-3) r_+^{d-3} l^{d-2}}$$

(53)

To compute $R \partial_R \tilde{\Omega}_{1, \mu, T, \alpha_k}$ we have determine how the other quantities scale in term of $l$. It is easy to notice that $l^{d-2} / G_d \sim l^{d-2}$, $r_+ \sim l^2 T$, $Q \sim l^{d-2-l} d^{d-2}$, $\alpha_k \sim \alpha_k^{(k+1)}$. $Q = l Q_b$ and so

$$R \partial_R \tilde{\Omega}_{1, \mu, T, \alpha_k} = l \partial_l \tilde{\Omega} |_{d-2 \mathcal{F} G_d \mu, T, \alpha_k}$$

$$= \frac{d-2}{16\pi G_d l^{d-2}} \sum_{k=0}^{2} kr_+^{k-1} \alpha_k r_+^{d-2k-1} \frac{2k - 8\pi (k-1) r_+ T}{d-2k}$$

(54)

and using (17) and (51) it is also easy to check that

$$(d-1) \tilde{M} + R \partial_R \tilde{\Omega}$$

$$= (d-2) \left[ (d-2) T \right] \frac{1}{4G_d l^{d-2}} \sum_{k=0}^{2} kr_+^{k-1} \alpha_k r_+^{d-2k} + \frac{Q^2}{(d-3) r_+^{d-3} l^{d-2}}$$

$$= (d-2)(T S + \mu Q)$$

(55)

recovering the equation of state (32) with $n = d-2$.

We next consider rotating planar Lovelock black holes in Gauss-Bonnet-Born-Infeld Gravity. The action in $d$ dimensions is given by [31]

$$I_G = -\frac{1}{16\pi G_d} \int_{\mathcal{M}} d^d x \sqrt{-g} \left[ R - 2\Lambda + \alpha(R_{\mu \nu \gamma \delta} R^{\mu \nu \gamma \delta} - 4R_{\mu \nu} R^{\mu \nu} + R^2) + L(F) \right]$$

$$- \frac{1}{8\pi G_d} \int_{\partial \mathcal{M}} \sqrt{-\gamma} \left[ \Theta + 2\alpha(J - 2\hat{G}_{ab} \Theta^{ab}) \right]$$

(56)
where $\Lambda = -(d-2)(d-1)/2l^2$ is the cosmological constant, $\alpha$ the Gauss-Bonnet coefficient and $L(F)$ the Born-Infeld Lagrangian

$$L(F) = 4\beta^2 \left(1 - \sqrt{1 + \frac{F^2}{2\beta^2}}\right)$$  \hspace{1cm} (57)$$

where $\beta$ is the Born-Infeld parameter which has a dimension of mass, $F^2 = F^{\mu\nu} F_{\mu\nu}$ with $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$.

Regarding the boundary term, $\Theta$ is the trace of extrinsic curvature $\Theta^{ab}$ of the boundary, $G^{ab}(\gamma)$ is the Einstein tensor on the boundary and $J$ the trace of

$$J_{ab} = \frac{1}{3} (\Theta_{cd} \Theta^{cd} \Theta_{ab} + 2 \Theta_{ac} \Theta_{b}^c - 2 \Theta_{ac} \Theta^{cd} \Theta_{db} - \Theta^2 \Theta_{ab})$$  \hspace{1cm} (58)$$

In order to solve the equations of motion derived from the action (56) we consider a $d$ dimensional asymptotically AdS spacetime with $k$ rotation parameters, whose metric reads $[32, 33]$

$$ds^2 = -f(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\phi_i \right)^2 + \frac{r^2}{l^2} \sum_{i=1}^{k} (a_i dt - \Xi l^2 d\phi_i)^2 + \frac{dr^2}{f(r)} - \frac{r^2}{l^2} \sum_{i<j} (a_i d\phi_j - a_j d\phi_i)^2 + r^2 dX^2$$  \hspace{1cm} (59)$$

where $\Xi = \sqrt{1 + \sum_{i=1}^{k} a_i^2/l^2}$ and $dX^2$ a $(d - 2 - k)$ dimensional Euclidean metric. Using the ansatz

$$A_\mu = h(r) (\Xi \delta_\mu^0 - \delta_\mu^i a_i)$$  \hspace{1cm} (60)$$

the equations of motion for the vector potential yield

$$h(r) = -\sqrt{\frac{d-2}{2(d-3)}} \frac{\eta}{r^{d-3} dF_1 \left(\frac{1}{2} \frac{d-3}{2(d-2)}; \frac{3d-7}{2(d-2)}; -\eta\right)}$$  \hspace{1cm} (61)$$

where $2F_1(a; b; c; z)$ is a hypergeometric function and

$$\eta = \frac{(d-3)(d-2)a^2}{2\beta^2 r^{2(d-2)}}$$  \hspace{1cm} (62)$$

Inserting (59) into the gravitational field equations gives

$$f(r) = \frac{r^2}{2(d-4)(d-3)\alpha} (1 - \sqrt{g(r)})$$  \hspace{1cm} (63)$$

where

$$g(r) = 1 - 10 \frac{(d-4)\alpha \beta^2 \eta}{d-1} 2F_1 \left(\frac{1}{2} \frac{d-3}{2(d-2)}; \frac{3d-7}{2(d-2)}; -\eta\right)$$
$$+ 4 \frac{(d-4)(d-3)\alpha}{(d-2)(d-1)r d-1} (2\Lambda r^{d-1} + (d-2)(d-1)m - 4\beta^2 r^{d-1}(1 - \sqrt{1+\eta}))$$  \hspace{1cm} (64)$$

Setting $f(r_+) = 0$, from the above expression it follows that $m$ is given by

$$m = -\frac{2\Lambda r^{d-1}}{(d-2)(d-1)} + \frac{4\beta^2 r^{d-1}}{(d-2)(d-1)} (1 - \sqrt{1+\eta_+}) + 2(d-2)^2 \frac{\eta^2}{r^{d-3}} 2F_1 \left(\frac{1}{2} \frac{d-3}{2(d-2)}; \frac{3d-7}{2(d-2)}; -\eta_+\right)$$  \hspace{1cm} (65)$$

with $r_+$ the horizon radius.

The thermodynamic quantities associated with these black holes are
For the $d$ dimensional black holes whose thermodynamic quantities given above we can see that

$$(d - 1)\bar{M} = \frac{1}{16\pi G_d} b (d - 1)^2 - 2 + (d - 1)^2 m + \sum_i a_i^2 l^2$$

which is (34) upon setting $n = d - 2$, provided $R\partial R\bar{\Omega}$ vanishes.

To compute $R\partial R\bar{\Omega}$, we have to keep in mind that the bulk quantities $Q_b, \mu_b, J_i^b, \omega_i^b$ are redefined in the CFT as

$$Q = Q_b l, \mu = \mu_b / l, \ J_i = J_i^b / l, \ \omega_i = \omega_i^b;$$

we also have $l^{d-2} / G_d \sim l^b, \ r_+ \sim l^2 T, \ q \sim \mu l r^{d-3}$ and $a_i \sim \omega_i$ and $R = l$. Hence a direct computation of (24) yields

$$\bar{\Omega} = 0$$

as expected.

Finally we consider non-extremal rotating black holes in minimal 5d gauged supergravity. This provides an interesting non-trivial example with both charge and angular momentum. The metric in the Boyer- Lindquist coordinates $x^\mu = (t, r, \phi, \psi)$ reads [34]

$$ds^2 = \frac{\Delta_\theta}{\Xi_a \Xi_b} [(1 + g_1^2) \rho^2 dt + 2 q \rho^2 d\psi] + \frac{\Delta_\rho}{\Xi_a \Xi_b} (dt - \zeta)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2$$

\[+ \frac{r^2 + a^2}{\Xi_a} \sin^2 \theta d\phi^2 + \frac{r^2 + b^2}{\Xi_b} \cos^2 \theta d\psi^2 \]

\[A = \sqrt{3} q \frac{\Delta_\theta}{\Xi_a \Xi_b} (dt - \zeta) \]

where

$$\nu = b \sin^2 \theta d\phi + a \cos^2 \theta d\psi \quad \zeta = a \sin^2 \theta d\psi \Xi_a + b \cos^2 \theta d\psi \Xi_b$$

$$\Delta_\theta = 1 - a^2 g_1^2 \cos^2 \theta - b^2 g_1^2 \sin^2 \theta \quad \Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 + g_1^2 r^2) + q^2 + 2 ab q}{r} - 2 m$$

\[\rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta \quad \Xi_a = 1 - a^2 g_1^2 \quad \Xi_b = 1 - b^2 g_1^2 \quad f = 2 m \rho^2 - q^2 + 2 ab q g_1^2 \rho^2 \]

with $a, b$ the rotation parameters associated to the coordinates $\phi, \psi$ respectively and $g$ is a constant with dimension of length.
The associated thermodynamic quantities are

\[
M = \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2\pi q ag^2(\Xi_a + \Xi_b)}{4\Xi_a^2 \Xi_b G_5}
\]

\[
\omega_a = \frac{a(r_+^2 + b^2)(1 + g^2 r_+^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}
\]

\[
\omega_b = \frac{b(r_+^2 + a^2)(1 + g^2 r_+^2) + ag}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}
\]

\[
T = \frac{r_+^4 [1 + g^2(2r_+^2 + a^2 + b^2)] - (q + ab)^2}{2\pi r_+ [(r_+^2 + a^2)(r_+^2 + b^2) + abq]}
\]

\[
S = \frac{\pi^2[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{2\Xi_a \Xi_b r_+ G_5}
\]

\[
J_a = \frac{\pi[2am + qb(1 + a^2 g^2)]}{4\Xi_a^2 \Xi_b G_5}
\]

\[
J_b = \frac{\pi[2bm + qa(1 + b^2 g^2)]}{4\Xi_a \Xi_b^2 G_5}
\]

\[
Q = \frac{\sqrt{3\pi}}{4\Xi_a \Xi_b G_5}
\]

\[
\mu = \frac{\sqrt{3q a g^2}}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}
\]

where \(r_+\) is the horizon radius and

\[
m = \frac{(r_+^2 + a^2)(r_+^2 + b^2)(1 + g^2 r_+^2) + q^2 + 2abq}{2r_+^2}
\]

which means that for higher order curvature theories \((k \geq 2)\) the functions \(\phi_k(N)\) are highly suppressed in the large \(N\) field limit. We shall here restrict our attention to the leading terms of each higher curvature contribution. When we look at a higher curvature theory of gravity, the additional contributions are seen as correction terms to the Einstein-Hilbert action.

To better illustrate what is meant, we can consider a pure Yang-Mills theory, or a field theory coupled to a Yang-Mills gauge theory, whose Lagrangian \(L^{(1)}\) gives rise to planar and non planar diagrams. We can also consider a Kaluza-Klein-like model which couples gravity to a Yang-Mills gauge theory so that the general Lagrangian is similar to that of a pure gravity theory. In such theories gauge field self-interaction terms are part of the Lagrangian \(L^{(1)}\), whereas higher order diagrams come explicitly from higher curvature terms \((R_{d+1} = R_d + gF^2 + ...)\).

Considering only non-planar diagrams with 4 vertices, as shown in (1), we know that these kinds of diagrams bring a contribution to the scattering amplitude proportional to \(N^2 g_{YM}^2 \lambda^2\), where \(g_{YM}\) and \(\lambda\) are the Yang-Mills and 't Hooft couplings respectively. It clearly appears that these diagrams lead to contributions of the order of \(N^6\) in the computation of the scattering amplitude. This reasoning generalizes to higher curvature
and corresponds to $L^{(1)}$ in the Lovelock theory. As we know that every closed loop comes with a factor $N$.

(b) The non planar diagram, which is of great interest here, contributes to the scattering amplitude with a term proportional to $N^2(g_{YM})^2 = N^2\lambda^2$ and is linked to $L^{(1)}$. The Yang-Mills coupling $g_{YM}$ appears at each vertex of the diagram.

The free energy density reads as

$$N^2(g_{YM})^{2k} = N^{2(1-k)}\lambda^{2k}.$$ This suggests the correspondence (74). These results are applicable to any higher-curvature theory of gravity.

VII. CONCLUSION

By considering the grand canonical free energy $\Omega$ in both the bulk and the field theory on its boundary, we have derived a holographic Smarr relation valid beyond the large $N$ limit, with subleading terms of the from $g_k(N)$ arising from higher-curvature corrections of the type found in $k$-th order Lovelock gravity. By assuming that $\Omega$ in the CFT is a homogeneous function of degree one of the $g_k(N)$ functions we were able to obtain the holographic equation of state (34). We illustrated its validity for several non-trivial cases in Lovelock gravity and in minimal gauged supergravity in 5 dimensions.

We expect that asymptotically AdS black holes will in general satisfy the relations we have derived, testifying to the robustness of the correspondence between the bulk relations such as the Smarr relation and the equation of state in the CFT. It was shown [12] that Einstein-gravity black holes whose dual field theories are the large $N$ gauge theories with hyperscaling violation satisfy a modified equation of state in the large $N$ limit. We therefore expect that many other non trivial examples of black holes exist where the equation of state beyond this limit has a slightly modified form from the one we obtained. Some of these black holes are the black $Dn$ branes, which are dual to maximally supersymmetric gauge theories in $n+1$ dimensions. Higher-curvature theories that are dual to gauge theories with hyperscaling violation should be good examples of such theories. An interesting project for future work would be to investigate for these black holes the form of the equation of state at higher order.

APPENDIX

We consider here a computation of the free energy density of rotating black holes in minimal 5d gauged supergravity. From the thermodynamic quantities given in (71) we see that

$$4\hat{M} - 3(T\hat{S} + \mu\hat{Q}) - 4\omega_a\hat{J}_a - 4\omega_b\hat{J}_b = \frac{\pi}{\Xi_2\Xi_b} m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)\frac{1}{G_5l^5} + \frac{2\pi}{\Xi_2\Xi_b} qabg^2(\Xi_a + \Xi_b)\frac{1}{G_5l^5} - 3\frac{1}{G_5l^5} TS - 3\frac{1}{G_5l^5}\mu Q - \frac{\pi}{\Xi_2\Xi_b} [2am + q(1 + a^2g^2)]\frac{\omega_a}{G_5l^5} - \frac{\pi}{\Xi_2\Xi_b} [2bm + qa(1 + b^2g^2)]\frac{\omega_b}{G_5l^5}.$$

The free energy density reads as
\[\Omega = \frac{\pi m}{4 \Xi_a \Xi_b G_5 L^3} (2 \Xi_a + 2 \Xi_b - \Xi_a \Xi_b) + \frac{\pi q ab g^2 (\Xi_a + \Xi_b)}{2 \Xi_a \Xi_b G_5 L^3} - \frac{\pi^2}{2 \Xi_a \Xi_b G_5 L^3} T (r_+^2 + a^2) (r_+^2 + b^2) - \frac{\pi^2}{2 \Xi_a \Xi_b G_5 L^3} T \]

\[\mu \left[ 2 \Xi_a \Xi_b G_5 L^3 r_+ \right] - \frac{\pi \alpha}{2 \Xi_a \Xi_b G_5 L^3} \left[ \frac{\omega_a}{l} - \frac{\pi b (1 + a^2 g^2) \omega_a}{2 \Xi_a \Xi_b G_5 L^3} \right] - \frac{\pi a (1 + b^2 g^2) \omega_b}{2 \Xi_a \Xi_b G_5 L^3} \]

Therefore, we have

\[l \partial \Omega |_{\gamma = G_5 L^3} = \frac{\pi}{2 \Xi_a \Xi_b G_5 L^3} \left\{ 0 - 4 m + \frac{4 \Xi_a \Xi_b G_5 L^3}{l} + \frac{3(a^2 - b^2) r_+^2}{r_+^2} - 2 \frac{\pi}{G_5 L^3} \Omega_a \right\} + \frac{9 \alpha}{2 \Xi_a \Xi_b G_5 L^3} \left\{ \frac{4 \pi b (1 + a^2 g^2) \omega_a}{G_5 L^3} + \frac{3 \pi a (1 + b^2 g^2) \omega_b}{G_5 L^3} \right\} - \frac{\pi}{G_5 L^3} \omega_a \]

and which can finally be written as

\[l \partial \Omega |_{\gamma = G_5 L^3} = -4 M + 3 T S + 4 \alpha_0 \tilde{J}_a + 4 \alpha_0 \tilde{J}_b + \frac{9 \pi}{G_5 L^3} \frac{q r_+^2}{l} \left\{ \alpha_0 \tilde{Q} + 4 \alpha_0 \tilde{J}_a + 4 \alpha_0 \tilde{J}_b \right\} \]

using (71).

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