Random analysis of coupled vehicle–bridge systems with local nonlinearities based on explicit time-domain method

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Received: 30 March 2021 / Accepted: 28 December 2021 / Published online: 18 January 2022
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Abstract Based on the explicit time-domain method, an efficient analysis algorithm is developed for the random vibration analysis of the coupled vehicle–bridge system with local nonlinear components. In this work, the coupled vehicle–bridge system is divided into two subsystems, including the linear bridge subsystem and the nonlinear vehicle subsystem. Firstly, using the equivalent linearization technique, the equivalent linearized subsystem is constructed for the vehicle subsystem with the hysteretic suspension spring at a given time instant. Then, the explicit expressions of the responses of the linearized vehicle subsystem and the linear bridge subsystem are constructed corresponding to contact forces, respectively. Further, the explicit expression of the contact forces is derived in consideration of the compatibility condition. Lastly, the dimension-reduction vibration analysis for the equivalent linearized coupled vehicle–bridge system can be carried out based on the explicit time-domain method. The numerical example about a coupled vehicle–bridge system under the random irregular excitation is investigated, and the results indicate that the proposed approach is of feasibility.

Keywords Vehicle–bridge system · Hysteretic suspension · Equivalent linearization technique · Explicit time-domain method

1 Introduction

During the last two decades, much research was carried out about the vehicle–bridge interaction problem and many algorithms have been developed for the deterministic dynamic analysis of the coupled vehicle–bridge system [1–4]. For most practical engineering problems, the excitations, which the coupled vehicle–bridge systems are subjected to, are not always deterministic excitations. Some of them are random in nature, e.g., wind and earthquake. In recent years, the random vibration problem of the coupled vehicle–bridge system has aroused the concern and interest of researchers [5–10] with the increase in speed of passenger trains. Besides wind load and seismic load which cause random vibration of the coupled vehicle–bridge system, the random irregular excitation from a bridge deck or a rail is another kind of significant excitation that can not be overlooked. Due to the time-varying characteristic of the coupled vehicle–bridge system, the dynamic responses of the coupled vehicle–bridge system show typical
nonstationary random processes, even if the random irregular excitation is a stationary random process. Therefore, the ride comfort and the random vibration analyses of the coupled vehicle–bridge system need to be taken into account. In order to improve the ride comfort, the nonlinear suspension spring with hysteretic properties is used in the suspension system design of the vehicle subsystem. Therefore, it is of great significance to investigate the random vibration of the coupled vehicle–bridge system that has local nonlinearities.

Thus far, different methods have been developed to evaluate the random responses of nonlinear systems, such as the Fokker–Planck–Kolmogorov equation method [11], the stochastic average method [12], the moment equation method [13], the equivalent linearization method [14], the Monte Carlo simulation method (MCS) [15], and so on. Most of them are limited to nonlinear systems with a limited number of degrees of freedom under stationary random excitations. The equivalent linearization method and the MCS are the methods suitable for the random vibration analyses of large-scale nonlinear systems. Although the MCS is a useful method for evaluating the random responses of the nonlinear systems, it is unpopular due to its high computational cost in interpreting the results. Even though, the MCS is often used to evaluate the other methods. In consideration of the shortcomings of the methods mentioned above, the equivalent linearization method is widely accepted as an effective analytical tool for its high computational efficiency.

As is known, the basic idea of the equivalent linearization method is to substitute the original nonlinear system with a linear system according to a certain criterion. For the random vibration analysis of the coupled vehicle–bridge with nonlinearities, Zhang and Knothe [16] extended the equivalent linearization method to the nonlinear contact geometry and nonlinear contact mechanics of the curved track under random track irregularities and solved the linearized equation of motion by the power spectral density method (PSDM). Recently, Jin et al. [17] studied the nonlinear wheel-rail contact of the coupled vehicle–bridge system under the rail irregularity using the equivalent linearization method. Further, they obtained the statistical moments of the equivalent linear system based on the pseudo-excitation method (PEM), which is a modified PSDM. The above research focused on the nonlinear wheel-rail contact geometry with the bridge subsystem remaining linear, which is in fact a local nonlinear problem. As the responses of the coupled vehicle–bridge system present nonstationary characteristic, it needs to transfer the random analysis of the equivalent linearized system into a series of deterministic dynamic analysis at discretized frequency points [18] for the PSDM and the PEM. In another word, hundreds of dynamic time-history analyses are needed for these methods in order to obtain the equivalent linear system for each time instant, which are still extremely time-consuming.

In order to improve the computational efficiency for calculating the nonstationary responses of the equivalent linearized system, Su and his students [19–21] proposed an explicit time domain method (ETDM), which is devoted to solving the nonstationary responses of large-scale linear and nonlinear systems with high efficiency. The ETDM is a pure time-domain analysis method, where the uncoupled treatment of physical and probabilistic evolution mechanisms for the random vibration analysis of linear systems and equivalent linearized systems [22] are carried out, leading to the dimension-reduction analysis for the nonstationary responses of the linear systems and the equivalent linearized systems. As for the linear systems, only one dynamic time-history analysis is needed for the ETDM to obtain the explicit expressions of responses. When the ETDM is used to settle the equivalent linearized problems, only several dynamic time-history analyses are needed to obtain the equivalent linear system for each time instant. Obviously, the times of dynamic time-history analyses needed for the ETDM is far less compared with both the PSDM and the PEM.

The purpose of this study is to extend the ETDM in conjunction with the equivalent linearization technique to the random vibration analysis of the coupled vehicle–bridge system with local nonlinear components. In this paper, the hysteretic nonlinear spring is used in the suspension system of the vehicle subsystem with the bridge subsystem remaining a linear system. Firstly, the linearized vehicle subsystem for a given time instant is constructed using the equivalent linearization technique. Then, the explicit expressions
of the responses of the linearized vehicle subsystem and the linear bridge subsystem are constructed about contact forces, respectively. Further, the explicit expression of the contact forces, which occur between the bridge subsystem and the vehicle subsystem, is derived based on the compatibility condition. Lastly, the dimension-reduction vibration analysis for the equivalent linearized coupled vehicle–bridge system can be carried out based on the ETDM.

2 Equivalent linearization for the nonlinear coupled vehicle–bridge system

2.1 Restoring force model for hysteretic spring

Based on the Bouc–Wen model [23], the restoring force of a hysteretic spring can be expressed as

\[
\begin{align*}
\dot{y} &= A\dot{z} - \varphi|\dot{y}|\dot{z}|^{\theta-1} - \psi|\dot{y}|^{\theta}
\end{align*}
\]

where \( y \) and \( z \) are the real displacement and the hysteretic displacement, respectively; \( \alpha \) denotes the ratio of post-yield to pre-yield stiffness and \( k \) is the initial stiffness. As can be seen from Eq. (1), the hysteretic restoring force consists of two parts, the elastic force \( \alpha k y \) and the hysteretic force \( (1 - \alpha)kz \). In Eq. (1), \( \varphi, \psi, A \) and \( \theta \) are the four parameters; \( \varphi \) and \( \psi \) are used to determine the hysteresis shape; \( A \) represents the amplitude of the hysteretic force; \( \theta \) refers to the smoothness from the elastic zone to plastic zone. By adjusting these parameters, one may obtain softening or hardening hysteretic restoring force models with different capacities of energy dissipation.

The equivalent linear equation for Eq. (1) can be rewritten as [24]

\[
\begin{align*}
f(y, z) &= \alpha k y + (1 - \alpha)k z \\
\dot{z} &= p y + h z
\end{align*}
\]

where the two parameters \( p \) and \( h \) are equivalent coefficients determined by minimizing the mean square of the difference between Eqs. (1) and (2). In this study, the value of \( \theta \) is selected as 1. When the system is subjected to Gaussian process, the responses of the equivalent linear system will also follow Gaussian distributions. Under this condition, when \( \theta = 1 \), the detailed expressions of \( p \) and \( h \) are expressed as [24]

\[
\begin{align*}
p &= A - \sqrt{\frac{2}{\pi}} \left( \varphi \frac{E(\dot{y}z)}{\sigma_y} + \psi \sigma_z \right) \\
h &= -\sqrt{\frac{2}{\pi}} \left( \varphi \sigma_y + \psi \frac{E(\dot{y}z)}{\sigma_z} \right)
\end{align*}
\]

where \( E(\cdot) \) denotes the mathematical expectation.

In the following section, the hysteretic suspension spring, which is modelled by the Bouc–Wen model [23], is used in the simple vehicle subsystem of a coupled vehicle–bridge system.

2.2 Equivalent linear subsystem for the nonlinear vehicle subsystem

Considering a simple mechanical vehicle–bridge model as shown in Fig. 1, the vehicle–bridge system is taken as a 2-degree-of-freedom (DOF) mass-spring model, where the Bouc–Wen model [23] is used to model the suspension spring. Meanwhile, the bridge model is taken as a simply supported uniform beam. In Fig. 1, \( m_1 \) and \( m_2 \) represent the mass of the vehicle body and the wheel, respectively, \( c \) is the damping of the suspension part, \( k_1 \) is the stiffness of the suspension spring and \( k_2 \) is the stiffness of the wheel. The vertical displacements of the vehicle body and the wheel are denoted \( y_1 \) and \( y_2 \), respectively, which are related to the equilibrium position under the force of the gravity of the bridge subsystem. \( y_b \) is the displacement of the wheel-bridge contact point, which is related to the equilibrium position of the bridge subsystem. \( v \) denotes the speed of the vehicle and \( x \) represents the distance of the vehicle travelling along the bridge.

By introducing the vehicle–bridge contact force \( F(t) \) as shown in Fig. 1, the equation of motion for the nonlinear vehicle subsystem at a specified time instant \( \tau \) can be expressed as

\[
\begin{align*}
m_1 \ddot{y}_1 + C(y_1 - \dot{y}_2) + \alpha k_1 (y_1 - y_2) + (1 - \alpha)k_1 z = 0 \\
\dot{z} &= A\dot{y}_{1-2} - \varphi|\dot{y}_{1-2}|z - \psi|\dot{y}_{1-2}|z \\
m_2 \ddot{y}_2 - C(y_1 - \dot{y}_2) - \alpha k_1 (y_1 - y_2) - (1 - \alpha)k_1 z \\
&= -F(t)
\end{align*}
\]

where \( t \in [0, \tau]; y_1, \dot{y}_1 \) and \( \ddot{y}_1 \) represent the vertical displacement, the vertical velocity and the vertical acceleration of the vehicle body, respectively; \( y_2, \dot{y}_2 \) and \( \ddot{y}_2 \) represent the vertical displacement, the vertical velocity and the vertical acceleration of the wheel,
respectively; and $y_{1-2} = y_1 - y_2$. The vehicle–bridge contact force $F(t)$ is determined by the following compatibility equation

$$y_2(t) - y_b(t) - r(t) = -\frac{F(t)}{k_2}$$

(6)

where $r(t)$ is the time-domain roughness height of the rail, which represents the random rail irregularity, and it is supposed to be a homogeneous Gaussian random process.

As stated in Sect. 2.1, Eq. (4) can be replaced by the following equivalent linear equation of motion

$$
\begin{bmatrix}
  m_1\ddot{y}_1 + c\dot{y}_1 - c\dot{y}_2 + zk_1(y_1 - y_2) + (1 - \alpha)k_1z = 0 \\
  \ddot{z} = p(\tau)\dot{y}_{1-2} + h(\tau)z
\end{bmatrix}
$$

(7)

where $p(\tau)$ and $h(\tau)$ have the same form as Eq. (3). It is noted that $p(\tau)$ and $h(\tau)$ only depend on the response for the specified time instant $\tau$, namely

$$
\begin{cases}
  p(\tau) = A - \frac{\sqrt{2}}{\tau} \left( \frac{\Phi \psi \sigma_{y_{1-2}}(\tau)z(\tau)}{\sigma_{y_{1-2}}(\tau)} \right) \\
  h(\tau) = -\frac{\sqrt{2}}{\tau} \left( \frac{\Phi \psi \sigma_{y_{1-2}}(\tau)z(\tau)}{\sigma_{y_{1-2}}(\tau)} \right)
\end{cases}
$$

(8)

Let $Y_v = [y_1, y_2, z]^T$ with the superscript “$T$” for matrix transposition, Eqs. (5) and (7) can be written as a matrix form and the following equation can be derived by combining them

$$M_v \ddot{Y}_v + C_v^e(\tau)\dot{Y}_v + K_v^e(\tau)Y_v = \Phi F(t)$$

(9)

where $M_v$ is the extended mass matrix; $C_v^e(\tau)$ and $K_v^e(\tau)$ are the extended equivalent damping matrix and the extended equivalent stiffness matrix of the vehicle subsystem, which rely on the statistical responses for the specified time instant $\tau$; $M_v$, $C_v^e(\tau)$ and $K_v^e(\tau)$ are expressed as

$$M_v = \begin{bmatrix} m_2 & m_2 \\ m_2 & 0 \end{bmatrix},$$

$$C_v^e(\tau) = \begin{bmatrix} c & -c & 0 \\ -c & c & 0 \\ -p(\tau) & p(\tau) & 1 \end{bmatrix},$$

$$K_v^e(\tau) = \begin{bmatrix} zk_1 & -zk_1 & 0 \\ -zk_1 & zk_1 & 0 \\ 0 & 0 & -h(\tau) \end{bmatrix}$$

(10)

Equation (9) is the equation of motion for the equivalent linear system, where $\Phi = [0, -1, 0]^T$ is a position vector.

2.3 Equation of motion for the bridge subsystem

The numerical model of the bridge subsystem as shown in Fig. 1 can be construct by using the finite element model (FEM) with the beam element. Then, the equation of motion for the linear bridge subsystem can be expressed in the following general form

$$M_b\ddot{Y}_b + C_b\dot{Y}_b + K_bY_b = L_b(x)[F(t) - G_v]$$

(11)

where $M_b$, $C_b$ and $K_b$ are the global mass matrix, the damping matrix and the stiffness matrix of the bridge subsystem, respectively, which are dependent of each other and remain unchanged throughout the whole dynamic time-history analysis; $Y_b$, $\dot{Y}_b$ and $\ddot{Y}_b$ denote the displacement vector, the velocity vector and the acceleration vector of the bridge subsystem, respectively; $F(t)$ is the wheel-bridge contact force; $L_b(x)$ is the position vector used to locate the position of $F(t)$, which is moving along the bridge subsystem; $G_v = (m_1 + m_2)g$.  

![Fig. 1 Mechanical model of a coupled vehicle–bridge system](Image)
The following assumptions are used as follows [25]

\[ Y_{v,i} = Y_{v,i-1} + [(1 - \gamma)\ddot{Y}_{v,i-1} + \gamma \dot{Y}_{v,i}] \Delta t \quad (i = 1, 2, \ldots, n) \quad (12) \]

\[ Y_{v,i} = Y_{v,i-1} + \dot{Y}_{v,i-1} \Delta t + \frac{1}{2} [(1 - 2\beta)\ddot{Y}_{v,i-1} + 2\beta \dot{Y}_{v,i}] \Delta t^2 \quad (i = 1, 2, \ldots, n) \quad (13) \]

where \( \gamma \) and \( \beta \) are two parameters used to control the Newmark-\( \beta \) integration stability; \( n = \frac{\tau}{\Delta t} \) with \( \Delta t \) being the time step; The subscripts “\( i - 1 \)” and “\( i \)” denote \( t_{i-1} = (i-1)\Delta t \) and \( t_i = i\Delta t \), respectively. In this study, \( \gamma = 0.5 \) and \( \beta = 0.25 \) are used and the Newmark-\( \beta \) method will be unconditionally stable. Based on Eqs. (12) and (13), one can obtain the acceleration and the velocity at time instant \( t_i \) and they can be expressed as

\[ \ddot{Y}_{v,i} = a_0 (Y_{v,i} - Y_{v,i-1}) - a_1 \dot{Y}_{v,i-1} - a_2 \ddot{Y}_{v,i-1} \quad (i = 1, 2, \ldots, n) \quad (14) \]

\[ \dot{Y}_{v,i} = a_3 (Y_{v,i} - Y_{v,i-1}) - a_4 \dot{Y}_{v,i-1} - a_5 \ddot{Y}_{v,i-1} \quad (i = 1, 2, \ldots, n) \quad (15) \]

where

\[
\begin{align*}
\frac{1}{\beta \Delta t^2}, \quad \frac{1}{\beta \Delta t}, \quad \frac{1}{2\beta} - 1, \\
\frac{\gamma}{\beta \Delta t}, \quad \frac{\gamma}{\beta} - 1, \quad \frac{\Delta t}{2} (\frac{\gamma}{\beta} - 2)
\end{align*}
\]

The equation of motion for the equivalent linear vehicle subsystem at time instant \( t_i \) can be written as

\[ M_Y \ddot{Y}_{v,i} + C_Y^e (\tau) \dot{Y}_{v,i} + K_Y^e (\tau) Y_{v,i} = \Phi F_i \quad (17) \]

By substituting Eqs. (14) and (15) into Eq. (17), it yields

\[ Y_{v,i} = \Phi^{-1} F_i \quad (18) \]

where

\[ \Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix} \quad (19) \]

\[ \dot{Y}_{v,i} = \Phi F_i + M_Y (a_0 Y_{v,i-1} + a_1 \dot{Y}_{v,i-1} + a_2 \ddot{Y}_{v,i-1}) + C_Y^e (\tau) (a_3 Y_{v,i-1} + a_4 \dot{Y}_{v,i-1} + a_5 \ddot{Y}_{v,i-1}) \quad (20) \]

In addition, based on Eq. (17), the acceleration vector at time instant \( t_i \) can also be expressed as

\[ \ddot{Y}_{v,i} = M_Y^{-1} [\Phi F_i - C_Y^e (\tau) \dot{Y}_{v,i} - K_Y^e (\tau) Y_{v,i}] \quad (21) \]

Analogously, one has

\[ \ddot{Y}_{v,i-1} = M_Y^{-1} [\Phi F_{i-1} - C_Y^e (\tau) \dot{Y}_{v,i-1} - K_Y^e (\tau) Y_{v,i-1}] \quad (22) \]

By substituting Eq. (22) into Eq. (20), it yields

\[ Y_{v,i} = H_{11} Y_{v,i-1} + H_{12} \ddot{Y}_{v,i-1} + R_1 \Phi F_{i-1} + R_2 \Phi F_i \quad (23) \]

where

\[
\begin{align*}
H_{11} &= K_Y^{-1} [S_1 - S_2 M_Y^{-1} K_Y^e (\tau)], \\
H_{12} &= K_Y^{-1} \left[ S_2 - S_3 M_Y^{-1} C_Y^e (\tau) \right], \\
R_1 &= K_Y^{-1} S_2 M_Y^{-1}, \quad R_2 = K_Y^{-1} S_1 - R_1 \Phi C_Y^e (\tau), \quad S_1 = a_0 M_Y + a_2 C_Y^e (\tau), \quad S_2 = a_2 M_Y + a_4 C_Y^e (\tau) \quad (24) \]
\]

By substituting Eq. (23) into (15) and considering Eq. (22), it yields

\[ Y_{v,i} = H_{21} Y_{v,i-1} + H_{22} \ddot{Y}_{v,i-1} + R_3 \Phi F_{i-1} + R_4 \Phi F_i \quad (25) \]

where

\[
\begin{align*}
H_{21} &= a_0 (H_{11} - 1) + a_4 M_Y^{-1} K_Y^e (\tau), \quad H_{22} = a_4 M_Y^{-1} - a_1 + a_4 M_Y^{-1} C_Y^e (\tau), \\
R_3 &= a_3 R_1 - a_2 M_Y^{-1}, \quad R_4 = a_4 R_2 \quad (26) \]
\]

Based on Eqs. (23) and (25), one can derive the following recursion formula
\[ V_{v,i} = T_v V_{v,i-1} + Q_v F_{i-1} + Q_v^2 F_i \]
\[ (i = 1, 2, \ldots, n) \]

where

\[ V_v = \begin{bmatrix} Y_v \\ Y_v \end{bmatrix} \]
\[ T_v, Q_v^1 \text{ and } Q_v^2 \text{ are given as follows} \]
\[ T_v = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad Q_v^1 = \begin{bmatrix} R_1 \\ R_3 \end{bmatrix} \Phi, \quad Q_v^2 = \begin{bmatrix} R_2 \\ R_4 \end{bmatrix} \Phi \]

Without loss of generality, \( V_{v,0} = V_v(0) = 0 \) and \( \dot{y}_1(0) = \ddot{y}_2(0) = 0 \). Then, based on Eq. (27), the explicit expression for the state vector \( V_v(\tau) \) at \( t = \tau \) can be written as

\[ V_v(\tau) = A_{v,1}^v F_1 + A_{v,2}^v F_2 + \cdots + A_{v,n}^v F_n \]

where \( A_{v,j}^v (j = 1, 2, \ldots, n) \) are the coefficient vectors and they are expressed in closed forms as

\[ A_{n,n}^v = Q_v^2 A_{n,n-1}^v = T_v Q_v^2 + Q_v^1, \quad A_{n,n-j}^v = T_v A_{n,n-j+1}^v \]
\[ (j = 2, 3, \ldots, n - 1) \]

Note that, from the physical evolutionary point of view, the coefficient vectors \( A_{n,n}^v, A_{n,n-1}^v, \ldots, A_{n,2}^v \) and \( A_{n,1}^v \) exactly represent the state vectors at \( t_1, t_2, \ldots, t_{n-1}, t_n \), respectively, induced by an unit impulse of the wheel-bridge contact force \( F(t) \) applied at \( t = t_1 \) [19]. Equation (30) can be further expressed in a compact form as

\[ V_v(\tau) = A_{v,[n]}^v F_{[n]} \]

where \( A_{v,[n]}^v = [A_{n,n}^v A_{n,n-1}^v \cdots A_{n,1}^v] \) denotes the state matrix and \( F_{[n]} = [F_1 F_2 \cdots F_n]^T \) denotes the load vector.

3.2 Explicit expressions of dynamic responses for the bridge subsystem

For the equation of motion of the bridge subsystem corresponding to the concerned time instant \( \tau \), as shown in Eq. (11), the following state vector can be defined as

\[ V_b = \begin{bmatrix} Y_b \\ Y_b \end{bmatrix} \]

Using a similar derivation to the dynamic responses for the above equivalent linear vehicle subsystem, the explicit expression for the state vector \( V_b \) can also be written as

\[ V_b(\tau) = A_{b,[n]}^b (F_1 - G_v) + A_{b,2}^b (F_2 - G_v) + \cdots + A_{b,n}^b (F_n - G_v) \]

Note that, as \( F(t) \) changes with time, \( A_{n,i}^b \) \((i = 1, 2, \ldots, n)\) represents the state vector at \( t = t_i \) exerted by an unit impulse of the wheel-bridge contact force \( F(t) \) applied at \( t = t_i \). In other words, if there are \( n \) time instants, there will be \( n \) wheel-bridge contact points. Therefore, in order to obtain \( A_{n,i}^b \) \((i = 1, 2, \ldots, n)\), it needs \( n \)-time determinate dynamic history analyses for the bridge subsystem.

For the sake of simplicity, Eq. (34) can be also written as

\[ V_b(\tau) = A_{b,[n]}^b (F_{[n]} - G_{[n]}) \]

where \( A_{b,[n]}^v = [A_{b,1}^b A_{b,2}^b \cdots A_{b,n}^b] \) is the state matrix; \( F_{[n]} = [F_1 F_2 \cdots F_n]^T \) is the load vector and \( G_{[n]} = [G_v G_v \cdots G_v]^T \) is a \( n \times 1 \) vector.

In fact, the wheel-bridge contact point will not always fall onto the node of the finite element. In the circumstances that the contact point falls onto somewhere between the element nodes, the interpolation function for displacements of the beam elements and the equivalent node force need to be taken into account for the derivation of \( A_{b,i}^b \) \((i = 1, 2, \ldots, n)\). For example, given that an unit impulse of the wheel-bridge contact force \( F(t) \) at time \( t = t_i \) \((i = 1, 2, \ldots, n)\) falls onto somewhere between the element nodes \( d \) and \( e \), as shown in Fig. 2, the unit impulse can be denoted as \( \tilde{F}(t_i) = 1 \). Using the interpolation function for displacements of the beam elements, \( \tilde{F}(t_i) \) can be converted into the equivalent node forces, namely \( p_d \) and \( r_d \) applied at node \( d \), \( p_e \) and \( r_e \) applied at node \( e \). Then, one can obtain \( A_{n,i}^b \) through the state vector at \( t = t_i \) exerted by the equivalent node forces \( p_d, r_d, p_e \) and \( r_e \).
3.3 Explicit expressions of contact forces

It can be seen from Sects. 3.1 and 3.2 that the responses of the vehicle subsystem and the bridge subsystem depend on the contact forces, which occur at the current time instant \( \tau \) and all time instants before \( \tau \). Thus, the key point is to find these contact forces. In view of the following compatibility condition at the current time instant \( \tau \) and all time instants before the current time instant \( \tau \) between the vehicle subsystem and the bridge subsystem as shown by Eq. (6), one can obtain the following expression

\[
y_{2|n} - y_{b|n} - r_{|n} = - \frac{F_{|n}}{k_2}
\]

where \( r_{|n} = [r(x_1), r(x_2), \ldots, r(x_n)]^T \) with \( x_i \) \( (i = 1, 2, \ldots, n) \) representing the distance of the vehicle moving along the bridge; \( y_{2|n} \) is expressed as

\[
y_{2|n} = A_{v2}F_{|n}
\]

in which \( y_{2|n} = [y_2(t_1), y_2(t_2), \ldots, y_2(t_n)]^T \) denotes the displacement vector of the wheel corresponding to time instants \( t_i \) \( (i = 1, 2, \ldots, n) \); \( A_{v2} \) is a matrix constructed from the row vectors of \( A_{b|i} \) \( (1 \leq j \leq n) \) accordingly and can be expressed as

\[
A_{v2} = \begin{bmatrix}
a_{1,1} & 0 & \cdots & 0 \\
a_{2,1} & a_{2,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix}
\]

where \( a_{i,j}^{v2} \) \( (1 \leq j \leq n, 1 \leq i \leq j) \) are the elements of the row vector in \( A_{b|i} \) \( (1 \leq j \leq n) \) corresponding to \( y_j(t_j) \) \( (1 \leq j \leq n) \).

In Eq. (36), \( y_{b|n} \) can be expressed as

\[
y_{b|n} = [y_b(x_1, t_1), y_b(x_2, t_2), \ldots, y_b(x_n, t_n)]^T
\]

where \( y_b(x_j, t_j) \) \( (j = 1, 2, \ldots, n) \) denote the displacements of the bridge deck, which are related to the vehicle–bridge contact points corresponding to time instants \( t_i \) \( (i = 1, 2, \ldots, n) \); and \( y_{b|n} \) can be written as

\[
y_{b|n} = A_b(F_{|n} - G_{|n})
\]

in which \( A_b \) is a matrix constructed from the row vectors of \( A_{b|i} \) \( (1 \leq j \leq n) \) accordingly and can be given as

\[
A_b = \begin{bmatrix}
a_{1,1}^b & 0 & \cdots & 0 \\
a_{2,1}^b & a_{2,2}^b & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1}^b & a_{n,2}^b & \cdots & a_{n,n}^b
\end{bmatrix}
\]

where \( a_{i,j}^b \) \( (1 \leq j \leq n, 1 \leq i \leq j) \) are the elements of the row vector in \( A_{b|i} \) \( (1 \leq j \leq n) \) corresponding to \( y_b(x_j, t_j) \) \( (1 \leq j \leq n) \).

By substituting Eqs. (37) and (40) into Eq. (36), one can easily obtain

\[
F_{|n} = Ar_{|n} + B
\]

where

\[
A = \left(A_{v2} - A_b + \frac{I}{k_2}\right)^{-1} \\
B = -AA_bG_{|n}
\]

Thus far, one has obtained the explicit expression of the contact forces \( F_{|n} \) as shown by Eq. (42). In fact, Eq. (42) reflects the physical evolution mechanism of the equivalent linear vehicle–bridge system. From Eq. (42), one can easily obtain a contact force vector \( F_{|n} \) when a sample of the random rail irregularity is substituted into Eq. (42). In other words, the statistical moments of the contact forces \( F_{|n} \) can be obtained based on Eq. (42) using the MCS, which can be used in the random analysis of the equivalent linear coupled vehicle–bridge. As a matter of fact, the statistical moments of the contact forces \( F_{|n} \) can be obtained based on Eq. (42) in another direct way, which will be discussed in the following section.
3.4 Random analysis

The explicit expressions of dynamic responses for the equivalent linear vehicle subsystem and the linear bridge subsystem are obtained in Sects. 3.1–3.2. The statistical moments of dynamic responses will be discussed in detail in this section.

The mean vector of the contact forces $F_{[n]}$ can be obtained on the basis of Eq. (42) as

$$E(F_{[n]}) = AE(r_{[n]}) + B$$

(44)

where $E(F_{[n]})$ is the mean vector of $F_{[n]}$; $E(r_{[n]})$ denotes the mean vector of the time-domain roughness height of the rail, which can be expressed as follows

$$E(r_{[n]}) = [\mu_r(x_1) \mu_r(x_2) \cdots \mu_r(x_n)]^T$$

(45)

where $\mu_r(x_i) = E[r(x_i)](1 \leq i \leq n)$ represents the mean function of the random irregularity excitation.

Similarly, the correlation matrix of the contact forces $F_{[n]}$ can be given as follows

$$\text{cov}(F_{[n]}, F_{[n]}) = \text{A} \text{cov}(r_{[n]}, r_{[n]}) \text{A}^T$$

(46)

where $\text{cov}(r_{[n]}, r_{[n]})$ is the correlation matrix of the random irregularity excitation expressed by

$$\text{cov}(r_{[n]}, r_{[n]}) =
\begin{bmatrix}
R_r(x_1, x_1) - \mu^2_r(x_1) & R_r(x_1, x_2) - \mu_r(x_1)\mu_r(x_2) & \cdots & R_r(x_1, x_n) - \mu_r(x_1)\mu_r(x_n) \\
R_r(x_2, x_1) - \mu_r(x_2)\mu_r(x_1) & R_r(x_2, x_2) - \mu^2_r(x_2) & \cdots & R_r(x_2, x_n) - \mu_r(x_2)\mu_r(x_n) \\
\vdots & \vdots & \ddots & \vdots \\
R_r(x_n, x_1) - \mu_r(x_n)\mu_r(x_1) & R_r(x_n, x_2) - \mu_r(x_n)\mu_r(x_2) & \cdots & R_r(x_n, x_n) - \mu^2_r(x_n)
\end{bmatrix}$$

(47)

where $R_r(x_i, x_j)(1 \leq i \leq n, 1 \leq j \leq n)$ refers to the correlation function of the random irregularity excitation.

For the case when local nonlinearities exist in the vehicle subsystem, only a small number of statistical moments, e.g., $E[\dot{y}_{1-2}(\tau)z(\tau)]$, $\sigma_{\dot{y}_{1-2}}(\tau)$ and $\sigma_z(\tau)$, need to be calculated in Eq. (8) to obtain the equivalent linear vehicle–bridge coupled system. Based on Eq. (32), one can pick out $\dot{y}_{1}(\tau)$, $\dot{y}_{2}(\tau)$ and $z(\tau)$ from Eq. (32), namely

$$\dot{y}_{1}(\tau) = A_{\dot{y}_{1}[n]}F_{[n]}$$

(48)

$$\dot{y}_{2}(\tau) = A_{\dot{y}_{2}[n]}F_{[n]}$$

(49)

$$z(\tau) = A_{z[n]}F_{[n]}$$

(50)

where $A_{\dot{y}_{1}[n]}$, $A_{\dot{y}_{2}[n]}$ and $A_{z[n]}$ are the row vectors extracted from $A_{\dot{y}_{1}[n]}^T$, respectively. From Eqs. (48) and (49), the relative velocity $\dot{y}_{1-2}$ can be easily obtain as

$$\dot{y}_{1-2}(\tau) = A_{\dot{y}_{1-2}[n]}F_{[n]}$$

(51)

with $A_{\dot{y}_{1-2}[n]} = A_{\dot{y}_{1}[n]} - A_{\dot{y}_{2}[n]}$.

By using Eqs. (50) and (51), one can obtain the mean of $\dot{y}_{1-2}$ and $z(\tau)$, respectively

$$\mu_{\dot{y}_{1-2}}(\tau) = A_{\dot{y}_{1-2}[n]}E[F_{[n]}]$$

(52)

$$\mu_z(\tau) = A_{z[n]}E[F_{[n]}]$$

(53)

Then, $E[\dot{y}_{1-2}(\tau)z(\tau)]$, $\sigma_{\dot{y}_{1-2}}(\tau)$ and $\sigma_z(\tau)$ can be written as

$$E[\dot{y}_{1-2}(\tau)z(\tau)] = A_{\dot{y}_{1-2}[n]}\text{cov}(F_{[n]}, F_{[n]})A_{z[n]}^T$$

(54)

$$\sigma_{\dot{y}_{1-2}}(\tau) = \sqrt{A_{\dot{y}_{1-2}[n]}\text{cov}(F_{[n]}, F_{[n]})A_{\dot{y}_{1-2}[n]}^T - \mu^2_{\dot{y}_{1-2}}(\tau)}$$

(55)

$$\sigma_z(\tau) = \sqrt{A_{z[n]}\text{cov}(F_{[n]}, F_{[n]})A_{z[n]}^T - \mu^2_z(\tau)}$$

(56)

Once $E[\dot{y}_{1-2}(\tau)z(\tau)]$, $\sigma_{\dot{y}_{1-2}}(\tau)$ and $\sigma_z(\tau)$ are obtained, the equivalent coefficients $p(\tau)$ and $h(\tau)$ can be updated through Eq. (8), and the equivalent linear equation of motion shown in Eq. (9) can also be
updated accordingly. Repeat this calculation until the statistical moments of $E[y_2(t)z(t)]$, $\sigma_{y_1}(t)$ and $\sigma_z(t)$ converge. Then, one can calculate the statistical moments of the other required responses of the bridge subsystem. Since one has obtained the explicit expression of dynamic responses of the bridge subsystem shown by Eq. (35), any other dynamic response can be extracted from Eq. (35). For instance, one may concern a dynamic response of the bridge, the required response can be given as follows

$$v_b(t) = a_{b[n]}(F_{[n]} - G_{[n]})$$

(57)

where $v_b(t)$ represents the concerned response extracted from $V_b(t)$ and $a_{b[n]}$ refers to the corresponding row vector extracted from $A^b$. Then, one can obtain the first- and the second-order statistical moments of $v_b(t)$ as

$$\mu_{v_b}(t) = E[v_b(t)] = a_{b[n]}(E[F_{[n]}] - G_{[n]})$$

(58)

$$E[v^2_b(t)] = a_{b[n]} \text{cov}(F_{[n]}, F_{[n]}) a_{b[n]}^T$$

(59)

The variance of $v_b(t)$ can be expressed as

$$\sigma^2_{v_b}(t) = E[v^2_b(t)] - E^2[v_b(t)]$$

(60)

Once the required responses are obtained by using Eqs. (58)–(60), one can move on to the next time instant. Repeating the above process, the responses at all time instants concerned can be obtained.

4 Solution procedure for the linearized vehicle–bridge system

To illustrate the above proposed method for the random vibration analysis of the coupled vehicle–bridge system with local nonlinearities under the random irregularity excitation, an iterative solution procedure is given as follows:

(1) Calculate the explicit expression of the instantaneous dynamic responses for the linear bridge subsystem shown by Eq. (35), which can be implemented based on Eq. (34).

(2) Assign initial values of $E[y_{1-2}(t)z(t)]$, $\sigma_{y_{1-2}}(t)$ and $\sigma_z(t)$ to equivalent coefficients $p(t)$ and $h(t)$ in Eq. (8) (the converged results for the previous time instance may be used).

(3) Solve the equivalent linearized vehicle subsystem with Eqs. (30) and (31), and obtain the explicit expression of dynamic responses for the equivalent linearized vehicle subsystem shown by Eq. (32).

(4) Calculate the explicit expression of the contact forces expressed by Eq. (42), which can be implemented based on Eq. (43).

(5) Obtain the mean vector and the correlation matrix of the contact forces $F_{[n]}$ by using Eqs. (44) and (46).

(6) Calculate $E[y_{1-2}(t)z(t)]$, $\sigma_{y_{1-2}}(t)$ and $\sigma_z(t)$ based on Eqs. (54)–(56).

(7) Substitute the new values of $E[y_{1-2}(t)z(t)]$, $\sigma_{y_{1-2}}(t)$ and $\sigma_z(t)$ into $p(t)$ and $h(t)$, and update the equivalent coefficients.

(8) Repeat steps (2)–(7) until the statistical moments are convergent.

(9) Calculate the statistical moments of any concerned response by using Eqs. (58)–(60).

(10) Move on to the next time instant. Repeat (2)–(9) until the required statistical moments of responses at all time instants concerned are obtained.

For clarity, the above solution procedure of the proposed method can be vividly depicted by the following flowchart as shown in Fig. 3.

5 Numerical examples

In this example, the parameters of the coupled vehicle–bridge system model are set to be the same as those in Refs. [1, 26]. Data for the vehicle–bridge model shown in Fig. 1 are given as follows: $EI = 2,658,069 \text{ kN} \cdot \text{m}^2$, $\rho A = 6067 \text{ kN} \cdot \text{m}^2$, $m_1 = 12,000 \text{ kg}$, $m_2 = 500 \text{ kg}$, $k_1 = 280,000 \text{ N/m}$ and $k_2 = 156,000 \text{ N/m}$. The stiffness reduction ratio for the suspension spring $k_1$ is set to be $\varepsilon = 0.5$. The parameters of the hysteretic displacement are $A_1 = 1$, $\varphi_1 = 100 \text{ m}^{-1}$, $\psi_1 = 100 \text{ m}^{-1}$ and $\theta_1 = 1$. The bridge length is set to be $l = 40 \text{ m}$. The vehicle is travelling at speed $v = 20 \text{ m/s}$. The random rail irregularity excitation $r(t)$ is supposed to be a homogeneous Gaussian random process with a zero mean value and its power spectral density is given by
$$S(\omega) = \frac{1}{\pi} \frac{4\gamma \beta \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}$$

where $\omega_0^2 = \beta^2 + \chi^2$ with $\beta = 0.1$, $\chi = 0.3$ and $\gamma = 1$ cm$^2$. Since $S(\omega)$ decreases rapidly under the condition of $\omega \geq \omega_0$, the frequency range of integration is set to be $\omega \in (-10, 10)$ rad/s, which is good enough for this example.

For random vibration analysis of the nonlinear coupled vehicle–bridge system subjected to random rail irregularities, the proposed method and the MCS are used. The bridge subsystem is modeled by using 20 plane beam elements, and this meshing scheme is sufficient to obtain accurate solution. The duration of the vehicle for crossing the bridge is $T = (40/20)$ s $= 2$ s. The time step used for the proposed method is set to be $\Delta t = 0.02$ s. For the MCS, the total number of samples is set to $N = 1000$, and the time step is set to be $\Delta t = 0.002$ s for the nonlinear vehicle subsystem and $\Delta t = 0.02$ s for the linear bridge subsystem, respectively. It is noteworthy that the sizes of time step for the nonlinear vehicle subsystem and the linear bridge subsystem are different. That’s because the smaller time step is required to
improve the solution accuracy for the nonlinear vehicle subsystem, compared with that for the linear vehicle subsystem under the same condition. Therefore, $\Delta t = 0.002$ s is selected for the nonlinear vehicle subsystem so that the solution accuracy can be guaranteed when using the MCS.

The mean time history and the standard deviation time history of the mid-span displacement of the bridge subsystem are shown in Figs. 4 and 5, respectively. From Figs. 4 and 5, it can be seen that the results obtained by using the proposed method and the MCS are in good agreement. In addition, the mean time history and the standard deviation time history of the mid-span displacement given in Ref. [26] are also plotted in Figs. 4 and 5. As can be seen from Figs. 4 and 5, even though the difference for the mean time history between the nonlinear system and the linear system is small, the difference for the standard deviation between the nonlinear system and the linear system cannot be neglected. It means that the hysteretic suspension spring significantly influences the responses of the coupled vehicle–bridge system.

Further, the acceleration of the vehicle body is investigated, because it plays an important role in the ride comfort. Figures 6 and 7 give the mean value time history and the standard deviation time history of the vertical acceleration $a$ of the vehicle body $m_1$. The results from Ref. [26] are also demonstrated in Figs. 6 and 7. It shows that the vertical acceleration of the
vehicle body decreases when the suspension system exhibits hysteretic properties.

In addition, the number of iterations required by the proposed method is investigated. The tolerance of convergence for the concerned statistical moments is set to be $\varepsilon = 1 \times 10^{-8}$, $1 \times 10^{-10}$ and $1 \times 10^{-20}$, respectively. The numbers of iterations for each time instant are shown in Fig. 8, and the standard deviation time histories of the vertical acceleration $a$ for the vehicle body $m_1$ are shown in Fig. 9. As can be seen from Figs. 8 and 9, the proposed method can reach satisfactory accuracy when $\varepsilon = 1 \times 10^{-8}$ and the number of iterations needed for each time instant is not more than four times. With the tolerance of convergence decreasing, the accuracy is not improved much while the number of iterations needed for each time instant increases rapidly, which leads to the increase in calculation. Therefore, $\varepsilon = 1 \times 10^{-8}$ is suitable for this example, and the proposed method shows good convergency rate.

Furthermore, three cases for the velocity, that is $v = 16$ m/s, 20 m/s and 40 m/s, are taken into account to investigate the influence of the moving velocity of the vehicle, respectively. The mean time histories and the standard deviation time histories of the mid-span displacement of the bridge subsystem are shown in Figs. 10 and 11 under different settings of velocities. At the same time, the mean value time histories and the standard deviation time histories of the vertical acceleration $a$ are presented in Figs. 12 and 13.
can be seen from Figs. 10, 11, 12, and 13 that good agreement is also achieved for the statistical moments obtained by the proposed method and the MCS.

Lastly, in order to investigate the influence of the hysteretic suspension spring on the responses of the coupled vehicle–bridge system, the stiffness reduction ratio for the suspension spring $k_1$ is set to be $\alpha = 0.8$, 0.5 and 0.2, respectively. The time histories of the concerned statistical moments for the bridge subsystem and the vehicle subsystem are shown in Figs. 14, 15, 16, and 17, respectively. Obviously, Figs. 14, 15, 16, and 17 show that the relative error of the proposed method increases apparently with the increase in nonlinear levels of the hysteretic suspension spring. This mainly attributes to the Gaussian assumption of responses adopted in the equivalent linearization technique. When the stiffness reduction is severe, e.g., $\alpha = 0.2$, the responses may depart severely from the Gaussian distribution, resulting in bad calculating accuracy.

6 Conclusions

An efficient analysis algorithm for the random vibration analysis of the coupled vehicle–bridge system with the local nonlinearities is developed based on the framework of the explicit time-domain method. In this work, the coupled vehicle–bridge system is divided into the linear bridge subsystem and the nonlinear
vehicle subsystem. Firstly, using the equivalent linearization technique, the hysteretic vehicle subsystem is replaced by an equivalent linear vehicle subsystem. Then, the explicit expressions of the responses of the linearized vehicle subsystem and the linear bridge subsystem are constructed corresponding to contact forces, respectively. Further, in consideration of the compatibility condition of the two subsystems, the specific explicit expression of contact forces is derived. Lastly, the statistical moments of the contact forces and the responses concerned can be carried out efficiently based on the explicit time-domain method. A numerical example is investigated, and the results indicate that the proposed approach can achieve satisfactory accuracy and high efficiency for the random vibration analysis of the coupled vehicle–bridge system with local nonlinearities.

Based on the efficient approach proposed in this paper, it is of great significance to further construct a series of high-performance algorithms for sensitivity analysis and design optimization of coupled vehicle–bridge systems with local nonlinearities. It is noted that, the proposed approach cannot consider the influence of nonlinear Hertz contact in the coupled vehicle–bridge system so far, which is an important problem worth studying, and a series of corresponding work will be done in the future.

Acknowledgements The project is funded by the Natural Science Foundation of Guangdong Province, China (Grant No. 2020A1515010611, Grant No. 2021A1515012280).

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

Additional information The datasets generated during the current study are available from the corresponding author on reasonable request.

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