Multipolar Expansions for the Relativistic N-Body Problem in the Rest-Frame Instant Form.

David Alba  
*Dipartimento di Fisica*  
*Università di Firenze*  
*L.go E.Fermi 2 (Arcetri)*  
50125 Firenze, Italy  
*E-mail: ALBA@FI.INFN.IT*

and

Luca Lusanna  
*Sezione INFN di Firenze*  
*L.go E.Fermi 2 (Arcetri)*  
50125 Firenze, Italy  
*E-mail: LUSANNA@FI.INFN.IT*

and

Massimo Pauri  
*Dipartimento di Fisica*  
*Università di Parma*  
*Parco Area Scienze 7/A*  
43100 Parma, Italy  
*E-mail: PAURI@PR.INFN.IT*

Abstract

Dixon’s multipoles for a system of N relativistic positive-energy scalar particles are evaluated in the rest-frame instant form of dynamics. The Wigner hyperplanes (intrinsic rest frame of the isolated system) turn out to be the natural framework for describing multipole kinematics. In particular, concepts like the *barycentric tensor of inertia* can be defined in special relativity only by means of the quadrupole moments of the isolated system.
I. INTRODUCTION.

In a recent paper [1] we have given a complete treatment of the kinematics of the relativistic N-body problem in the rest-frame instant form of dynamics [2–5]. We have shown in particular how to perform the separation of the center-of-mass motion in the relativistic case. This requires the reformulation of the theory of relativistic isolated systems on arbitrary spacelike hypersurfaces [1]. This description is also able to incorporate the coupling to the gravitational field. This is essentially Dirac’s reformulation [6] of classical field theory (suitably extended to particles) on arbitrary spacelike hypersurfaces (equal time surfaces) and provides the classical basis of the Tomonaga-Schwinger formulation of quantum field theory. For each isolated system (containing any combination of particles, strings and fields) one gets a reformulation as a parametrized Minkowski theory [2], with the extra bonus of having the theory already predisposed to the coupling to gravity in its ADM formulation, but with the price that the functions $z^\mu(\tau, \vec{\sigma})$ describing the embedding of the spacelike hypersurface in Minkowski spacetime become configuration variables in the action principle. Since the action is invariant under separate $\tau$-reparametrizations and space-diffeomorphisms, there emerge first class constraints ensuring the independence of the description from the choice of the 3+1 splitting. The embedding configuration variables $z^\mu(\tau, \vec{\sigma})$ are the gauge variables associated with this kind of general covariance.

Let us remark that, since the intersection of a timelike worldline with a spacelike hypersurface corresponding to a value $\tau$ of the time parameter is identified by 3 numbers $\vec{\sigma} = \vec{\eta}(\tau)$ instead of four, in parametrized Minkowski theories each particle must have a well defined sign of the energy: therefore we cannot describe the two topologically disjoint branches of the mass hyperboloid simultaneously like in the standard manifestly Lorentz-covariant theory. As a consequence there are no more mass-shell constraints. Therefore, each particle with a definite sign of the energy is described by the canonical coordinates $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)$ with the derived 4-position of the particles given by $x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$. The derived 4-momenta $p^\mu_i(\tau)$ are $\vec{\kappa}_i$-dependent solutions of $p^2_i - \epsilon m^2_i = 0$ with the chosen sign of the energy.

As said, parametrized Minkowski theories have separate spatial and time reparametrization invariances, which imply the independence of the description from the choice of the 3+1 splitting of Minkowski spacetime. In Minkowski spacetime we can restrict the foliation to have spacelike hyperplanes as leaves. Then, for each configuration of the isolated system with timelike 4-momentum, we can restrict ourselves to the special foliation whose leaves are the hyperplanes orthogonal to the conserved system 4-momentum, which have been named Wigner hyperplanes. This special foliation is intrinsically determined only by the configuration of the isolated system. In this way [2] it is possible to arrive at the definition of the Wigner-covariant rest-frame instant form of dynamics for every isolated system whose configurations have well defined and finite Poincaré generators with timelike total 4-momentum (see Ref. [7] for the traditional forms of dynamics).

This formulation clarifies the non-trivial definition of a relativistic center of mass. As well known, no such definition can enjoy all the properties of the non-relativistic center of

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1Such hypersurfaces are the leaves of a foliation of Minkowski spacetime (namely one among its 3+1 splittings) and are equivalent to a congruence of timelike accelerated observers.
mass. See Refs. [8,9,11–13] for a partial bibliography of all the existing attempts.

As shown in Appendix A of Ref. [1] only four first class constraints survive in the rest-frame instant form on Wigner hyperplanes and the original configuration variables \( z^\mu(\tau, \bar{\sigma}) \), \( \bar{\eta}_i(\tau) \) and their conjugate momenta \( \rho_\mu(\tau, \bar{\sigma}), \bar{\kappa}_i(\tau) \) are reduced to:

i) a decoupled particle \( \tilde{x}^\mu_s(\tau) \), \( p^\mu_s \) (the only remnant of the spacelike hypersurface) with a positive mass \( \epsilon_s = \sqrt{p^2_s} \) determined by the first class constraint \( \epsilon_s - M_{sys} \approx 0 \) and with its rest-frame Lorentz scalar time \( T_s = \frac{\tilde{x}_s \cdot p_s}{\epsilon_s} \) put equal to the mathematical time as the gauge fixing \( T_s - \tau \approx 0 \) to the previous constraint. Here, \( \tilde{x}^\mu_s(\tau) \) is a non-covariant canonical variable for the 4-center of mass. After the elimination of \( T_s \) and \( \epsilon_s \) with the previous pair of second class constraints, one remains with a decoupled free point (point particle clock) of mass \( M_{sys} \) and canonical 3-coordinates \( \bar{z}_s = \epsilon_s(\vec{x}_s - \frac{\vec{p}_s}{\epsilon_s} \tilde{x}^\mu_s) \), \( \bar{k}_s = \frac{\vec{p}_s}{\epsilon_s} \). The non-covariant canonical \( \tilde{x}^\mu_s(\tau) \) must not be confused with the 4-vector \( x^\mu_s(\tau) = z^\mu(\tau, \bar{\sigma} = 0) \) identifying the origin of the 3-coordinates \( \bar{\sigma} \) inside the Wigner hyperplanes. The worldline \( x^\mu_s(\tau) \) is arbitrary because it depends on \( x^\mu_s(0) \) and its 4-velocity \( \dot{x}^\mu_s(\tau) \) depends on the Dirac multipliers associated with the 4 left first class constraints [1], as it will be shown in the next Section. The unit timelike 4-vector \( u^\mu(p_s) = \frac{p^\mu_s}{\epsilon_s} \) is orthogonal to the Wigner hyperplanes and describes their orientation in the chosen inertial frame.

ii) the particle canonical variables \( \bar{\eta}_i(\tau), \bar{\kappa}_i(\tau) \) inside the Wigner hyperplanes. They are restricted by the three first class constraints (the rest-frame conditions) \( \bar{\kappa}_+ = \sum_{i=1}^N \bar{\kappa}_i \approx 0 \). Since the role of the relativistic decoupled 4-center of mass is taken by \( \tilde{x}^\mu_s(\tau) \) (or, after the gauge fixing \( T_s - \tau \approx 0 \), by an external 3-center of mass \( \bar{z}_s \), defined in terms of \( \tilde{x}^\mu_s \) and \( p^\mu_s \)), the rest-frame conditions imply that the internal 3-center of mass \( \bar{q}_+ = \bar{\sigma}_{com} \) is a gauge variable, which can be eliminated with gauge fixings [1].

Therefore, we need a doubling of the concepts:

1) There is the external viewpoint of an arbitrary inertial Lorentz observer, who describes the Wigner hyperplanes, leaves of a foliation of Minkowski spacetime, determined by the timelike configurations of the isolated system. A change of inertial observer by means of a Lorentz transformation rotates the Wigner hyperplanes and induces a Wigner rotation of the 3-vectors inside each Wigner hyperplane. Every such hyperplane inherits an induced internal Euclidean structure while an external realization of the Poincaré group induces the internal Euclidean action. As said above, an arbitrary worldline (centroid) \( x^\mu_s(\tau) \) is chosen

\[ ^2M_{sys} \] being the invariant mass of the isolated system.

\[ ^3\bar{z}_s/\epsilon_s \] is the classical analogue of the Newton-Wigner 3-position operator, only covariant under the Euclidean subgroup of the Poincaré group.

\[ ^4 \]Therefore this arbitrary worldline may be considered as an arbitrary centroid for the isolated system.

\[ ^5 \]They are Wigner spin-1 3-vectors, like the coordinates \( \bar{\sigma} \).

\[ ^6 \]For instance \( \bar{q}_+ \approx 0 \) implies that the internal 3-center of mass is put in the origin \( x^\mu_s(\tau) = z^\mu(\tau, \bar{\sigma} = 0) \).
as origin of the internal 3-coordinates on the Wigner hyperplanes.

Three external concepts of 4-center of mass can be defined by using the external realization of the Poincaré algebra (to each one there corresponds a 3-location inside the Wigner hyperplanes):

a) the external non-covariant canonical 4-center of mass (also named 4-center of spin) \( \bar{x}_s^\mu \) (with 3-location \( \bar{\sigma} \)),

b) the external non-covariant non-canonical Møller 4-center of energy \( R_s^\mu \) (with 3-location \( \bar{\sigma}_R \)),

c) the external covariant non-canonical Fokker-Pryce 4-center of inertia \( Y_s^\mu \) (with 3-location \( \bar{\sigma}_Y \)).

Only the canonical non-covariant center of mass \( \bar{x}_s^\mu(\tau) \) is relevant in the Hamiltonian treatment with Dirac constraints, while only the Fokker-Pryce \( Y_s^\mu \) is a 4-vector by construction. See Ref. [7] for the construction of the 4-centers starting from the corresponding 3-centers (3-center of spin [12], 3-center of energy [10], 3-center of inertia [11,12]).

2) There is the internal viewpoint inside the Wigner hyperplanes associated to an unfaithful internal realization of the Poincaré algebra: the total internal 3-momentum of the isolated system vanishes due to the rest-frame conditions. The internal energy and angular momentum are the invariant mass \( M_{\text{sys}} \) and the spin (the angular momentum with respect to \( \bar{x}_s^\mu(\tau) \)) of the isolated system respectively.

With the internal realization of the Poincaré algebra we can define three internal 3-centers of mass: the internal canonical 3-center of mass, the internal Møller 3-center of energy and the internal Fokker-Pryce 3-center of inertia. But, due to the rest-frame conditions, they coincide. As a natural gauge fixing to the rest-frame conditions we can add the vanishing of the internal Lorentz boosts: it is equivalent to locate the internal canonical 3-center of mass \( \bar{q}_s \) in \( \bar{\sigma} = 0 \), i.e. in the origin \( x_s^\mu(\tau) \approx z^\mu(\tau, 0) \). With these gauge fixings and with \( T_s - \tau \approx 0 \), the worldline \( x_s^\mu(\tau) \) becomes uniquely determined except for the arbitrariness in the choice of \( x_s^\mu(0) \) \([ u^\mu(p_s) = p_s^\mu / \epsilon_s ] \)

\[
x_s^\mu(\tau) = x_s^\mu(0) + u^\mu(p_s)T_s,
\]

and coincides with the external covariant non-canonical Fokker-Pryce 4-center of inertia, \( x_s^\mu(\tau) = x_s^\mu(0) + Y_s^\mu \) [1].

This doubling of concepts with the external non-covariant canonical 4-center of mass \( \bar{x}_s^\mu(\tau) \) (or with the external 3-center of mass \( \bar{z}_s \) when \( T_s - \tau \approx 0 \)) and with the internal canonical 3-center of mass \( \bar{q}_s \approx 0 \) replaces the separation of the non-relativistic 3-center of mass due to the Abelian translation symmetry. The non-relativistic conserved 3-momentum is replaced by the external \( \bar{p}_s = \epsilon_s \bar{k}_s \), while the internal 3-momentum vanishes, \( \bar{k}_+ \approx 0 \), as a definition of rest frame.

In the final gauge we have \( \epsilon_s \equiv M_{\text{sys}} \), \( T_s \equiv \tau \) and the canonical basis \( \bar{z}_s, \bar{k}_s, \bar{\eta}_i, \bar{\kappa}_i \) is restricted by the three pairs of second class constraints \( \bar{\kappa}_+ = \sum_{i=1}^N \bar{\kappa}_i \approx 0, \bar{q}_+ \approx 0 \), so that 6N canonical variables describe the N particles like in the non-relativistic case. We still need a canonical transformation \( \bar{\eta}_i, \bar{\kappa}_i \mapsto \bar{q}_s \approx 0, \bar{\kappa}_+ \approx 0, \bar{p}_a, \bar{\pi}_a \) \([a = 1, \ldots, N - 1] \) identifying a set of relative canonical variables. The final 6N-dimensional canonical basis is \( \bar{z}_s, \bar{k}_s, \bar{p}_a, \bar{\pi}_a \). To get this result we need a highly non-linear canonical transformation [1], which can be obtained by exploiting the Gartenhaus-Schwartz singular transformation [13].

In the end we obtain the Hamiltonian for relative motions as a sum of N square roots, each one containing a squared mass and a quadratic form in the relative momenta, which
goes into the non-relativistic Hamiltonian for relative motions in the limit \( c \to \infty \). This fact has the following implications:

a) if one tries to make the inverse Legendre transformation to find the associated Lagrangian, it turns out that, due to the presence of square roots, the Lagrangian is a hyperelliptic function of \( \tilde{\rho}_a \) already in the free case. A closed form exists only for \( N=2, m_1 = m_2 = m \): 
\[
L = -em\sqrt{4 - \dot{\rho}^2}.
\]
This exceptional case already shows that the existence of the limiting velocity \( c \) (or in other terms the Lorentz signature of spacetime) forbids a linear relation between the spin (center-of-mass angular momentum) and the angular velocity.

b) the \( N \) quadratic forms in the relative momenta appearing in the relative Hamiltonian cannot be simultaneously diagonalized. In any case the Hamiltonian is a sum of square roots, so that concepts like reduced masses, Jacobi normal relative coordinates and tensor of inertia cannot be extended to special relativity. As a consequence, for example, a relativistic static orientation-shape SO(3) principal bundle approach \([15]\)
\footnote{See Ref. \([1]\) for a review of this approach used in molecular physics for the definition and study of the vibrations of molecules.} can be implemented only by using non-Jacobi relative coordinates.

c) the best way of studying rotational kinematics (the non-Abelian rotational symmetry, associated with the conserved internal spin) is based on the canonical spin bases with the associated concepts of spin frames and dynamical body frames introduced in Ref. \([1]\): they can be build in the same way as in the non-relativistic case \([16]\) starting from the canonical basis \( \tilde{\rho}_a, \tilde{\pi}_a \).

Let us clarify this point.

In the non-relativistic N-body problem it is easy to make the separation of the absolute translational motion of the center of mass from the relative motions, due to the Abelian nature of the translation symmetry group. This implies that the associated Noether constants of motion (the conserved total 3-momentum) are in involution, so that the center-of-mass degrees of freedom decouple. Moreover, the fact that the non-relativistic kinetic energy of the relative motions is a quadratic form in the relative velocities allows the introduction of special sets of relative coordinates, the Jacobi normal relative coordinates, which diagonalize the quadratic form and correspond to different patterns of clustering of the centers of mass of the particles. Each set of Jacobi normal relative coordinates organizes the N particles into a hierarchy of clusters, in which each cluster of two or more particles has a mass given by an eigenvalue (reduced masses) of the quadratic form; Jacobi normal coordinates join the centers of mass of pairs of clusters.

However, the non-Abelian nature of the rotation symmetry group, whose associated Noether constants of motion (the conserved total angular momentum) are not in involution, prevents the possibility of a global separation of absolute rotations from the relative motions, so that there is no global definition of absolute vibrations. This has the consequence that an isolated deformable body can undergo rotations by changing its own shape (see the examples of the falling cat and of the diver). It was just to deal with these problems that the theory of the orientation-shape SO(3) principal bundle approach \([15]\) has been developed. Its essential content is that any static (i.e. velocity-independent) definition of body frame for a deformable
body must be interpreted as a gauge fixing in the context of a SO(3) gauge theory. Both the laboratory and the body frame angular velocities as well as the orientational variables of the static body frame become thereby unobservable gauge variables. This approach is associated with a set of point canonical transformations, which allow to define the body frame components of relative motions in a velocity-independent way.

Since in many physical applications (e.g. nuclear physics, rotating stars,...) angular velocities are viewed as measurable quantities, it is desirable to have an alternative formulation complying with this requirement, possibly generalizable to special relativity. This program has been realized in a previous paper [16], of which Ref. [1] is the relativistic extension. Let us summarize the main points of our formulation.

First of all for \( N \geq 3 \) we have constructed in Ref. [16] a class of non-point canonical transformations, which allows to build the canonical spin bases quoted above, which are connected to the patterns of the possible clusterings of the spins associated with relative motions. The definition of these spin bases is independent of Jacobi normal relative coordinates, just as the patterns of spin clustering are independent of the patterns of center-of-mass Jacobi clustering. We have found two basic frames associated to each spin basis: the spin frame and the dynamical body frame. Their construction is guaranteed by the fact that, besides the existence on the relative phase space of a Hamiltonian symmetry left action of SO(3)\(^8\), it is possible to define as many Hamiltonian non-symmetry right actions of SO(3)\(^9\) as the possible patterns of spin clustering. While for \( N=3 \) the unique canonical spin basis coincides with a special class of global cross sections of the trivial orientation-shape SO(3) principal bundle, for \( N \geq 4 \) the existing spin bases and dynamical body frames turn out to be unrelated to the local cross sections of the static non-trivial orientation-shape SO(3) principal bundle, and evolve in a dynamical way dictated by the equations of motion. In this new formulation both the orientation variables and the angular velocities become measurable quantities in each canonical spin basis by construction.

For each \( N \) every allowed spin basis provides a physically well-defined separation between rotational and vibrational degrees of freedom. The non-Abelian nature of the rotational symmetry implies that there is no unique separation of absolute rotations and relative motions. The unique body frame of rigid bodies is replaced by a discrete number of evolving dynamical body frames and of spin canonical bases, both of which are grounded on patterns of spin couplings, direct analogues of the coupling of quantum angular momenta.

In this paper we complete our study of relativistic kinematics for the \( N \)-body system by evaluating its rest-frame Dixon multipoles \([17]\). Let us recall that this method is being used for treating extended systems in astrophysics. The starting point is the definition of

\[8\] We adhere to the definitions used in Ref. [16]; in the mathematical literature our left action is a right action.

\[9\] Their generators are the center-of-mass angular momentum Noether constants of motion.

\[10\] Their generators are not constants of motion.

\[11\] See Ref. [18] for the definition of Dixon’s multipoles in general relativity.
the energy momentum tensor of the N positive-energy particles on the Wigner hyperplane. In this way we see that the Wigner hyperplane is the natural framework for reorganizing a lot of kinematics connected with multipoles. Moreover, only in this way, a concept like the *barycentric tensor of inertia* can be introduced in special relativity by means of the quadrupole moments.

A review of the rest-frame instant form of dynamics for N scalar free positive-energy particles is given in Section II.

In Section III we evaluate the energy momentum tensor on the Wigner hyperplanes. Dixon’s multipoles for the system are defined in Section IV. A special study of monopole, dipole and quadrupole moments is given together with the multipolar expansion.

Other properties of Dixon’s multipoles are reviewed in Section V.

Some comments on open problems are given in the Conclusions.

The non-relativistic N-particle multipolar expansion is given in Appendix A, while in Appendix B there is a review of symmetric trace-free (STF) tensors.
II. REVIEW OF THE REST-FRAME INSTANT FORM.

Let us review the system of N free scalar positive-energy particles in the framework of parametrized Minkowski theory (see Appendices A and B and Section II of Ref. [1]). As said in the Introduction, each particle is described by a configuration 3-vector $\vec{z}(\tau, \vec{s})$. The particle worldline is $x^\mu(\tau) = z^\mu(\tau, \vec{\eta}(\tau))$, where $z^\mu(\tau, \vec{s})$ are the embedding configuration variables describing the spacelike hypersurface $\Sigma$. To say in the Introduction, each particle is described by a configuration 3-vector $\vec{z}(\tau)$. The action is invariant under separate $\tau$- and $\vec{s}$-reparametrizations.

The canonical momenta are

$$\rho_\mu(\tau, \vec{s}) = -\frac{\partial L(\tau, \vec{s})}{\partial \dot{z}_\mu(\tau, \vec{s})} = \sum_{i=1}^{N} \delta^3(\vec{s} - \vec{\eta}_i(\tau))m_i \frac{z_{\tau\mu}(\tau, \vec{s}) + z_{\vec{s}\mu}(\tau, \vec{s})\dot{\eta}_i^\tau(\tau)}{\sqrt{g_{\tau\tau}(\tau, \vec{s}) + 2g_{\tau\vec{s}}(\tau, \vec{s})\dot{\eta}_i^\tau(\tau) + g_{\vec{s}\vec{s}}(\tau, \vec{s})\dot{\eta}_i^{\vec{s}}(\tau)}} = \left[ \rho_{\mu}^\text{b} \right] l_\mu + \left[ \rho_{\mu}^\text{a} \right] \gamma_{\vec{s}}^s z_{\dot{z}\mu}(\tau, \vec{s}),$$

$$\kappa_{ij}(\tau) = -\frac{\partial L(\tau)}{\partial \dot{\eta}_i^\tau(\tau)} = \frac{m_i \sqrt{g_{\tau\tau}(\tau, \vec{s}) + 2g_{\tau\vec{s}}(\tau, \vec{s})\dot{\eta}_i^\tau(\tau) + g_{\vec{s}\vec{s}}(\tau, \vec{s})\dot{\eta}_i^{\vec{s}}(\tau)}}{\sqrt{g_{\tau\tau}(\tau, \vec{s}) + 2g_{\tau\vec{s}}(\tau, \vec{s})\dot{\eta}_i^\tau(\tau) + g_{\vec{s}\vec{s}}(\tau, \vec{s})\dot{\eta}_i^{\vec{s}}(\tau)}}.$$  

\{z^\mu(\tau, \vec{s}), \rho_{\mu}(\tau, \vec{s})\} = -\eta^\mu_\nu \delta^3(\vec{s} - \vec{s}'), \quad \{\dot{\eta}_i^\mu(\tau), \kappa_{ij}(\tau)\} = -\delta_{ij} \delta^\mu_\nu. \quad (2.2)$$

The canonical Hamiltonian $H_c$ is zero, but there are the primary first class constraints

\(^{12}\)The foliation is defined by an embedding $R \times \Sigma \rightarrow M^4$, $(\tau, \vec{s}) \mapsto z^\mu(\tau, \vec{s})$, with $\Sigma$ an abstract 3-surface diffeomorphic to $R^3$. $\Sigma$ is the Cauchy surface of equal time. The metric induced on it is $g_{AB}[z] = z_A^\mu \eta_{\mu\nu} z_B^\nu$ on $\Sigma$, a functional of $z^\mu$, and the embedding coordinates $z^\mu(\tau, \vec{s})$ are considered as independent fields. We use the notation $\sigma^A = (\tau, \vec{s})$ of Refs. [2-4]. The $z_A^\mu(\sigma) = \partial z^\mu(\sigma)/\partial \sigma^A$ are flat cotetrad fields on Minkowski spacetime with the $z_A^\mu$'s tangent to $\Sigma$. While in Ref. [1] we used the metric convention $\eta_{\mu\nu} = \epsilon(+-+-)$ with $\epsilon = \pm$, in this paper we shall use $\epsilon = 1$ like in Ref. [2].
\[ \mathcal{H}_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - l_\mu(\tau, \vec{\sigma}) \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{m_i^2 - \gamma^{\tilde{r}\tilde{s}}(\tau, \vec{\sigma}) \kappa_{ir}(\tau) \kappa_{is}(\tau)} - \\
- z^{\tilde{r}\mu}(\tau, \vec{\sigma}) \gamma^{\tilde{r}\tilde{s}}(\tau, \vec{\sigma}) \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \kappa_{is} \approx 0, \quad (2.3) \]

so that the Dirac Hamiltonian is \( H_D = \int d^3 \sigma \lambda^\mu(\tau, \vec{\sigma}) \mathcal{H}_\mu(\tau, \vec{\sigma}) \), with the \( \lambda^\mu(\tau, \vec{\sigma}) \) Dirac multipliers.

The conserved Poincaré generators are (the suffix “s” denotes the hypersurface \( \Sigma_\tau \))

\[ p_\mu^s = \int d^3 \sigma \rho^\mu(\tau, \vec{\sigma}), \]
\[ J^\mu_{s\nu} = \int d^3 \sigma [z^\mu(\tau, \vec{\sigma}) \rho^\nu(\tau, \vec{\sigma}) - z^\nu(\tau, \vec{\sigma}) \rho^\mu(\tau, \vec{\sigma})]. \quad (2.4) \]

After the restriction to spacelike hyperplanes the Dirac Hamiltonian is reduced to \( H_D = \lambda_\mu(\tau) \tilde{\mathcal{H}}^\mu(\tau) + \lambda_\mu(\tau) \tilde{\mathcal{H}}^\nu(\tau) \) (only ten Dirac multipliers survive) with the remaining ten constraints given by

\[ \tilde{\mathcal{H}}^\mu(\tau) = \int d^3 \sigma \mathcal{H}^\mu(\tau, \vec{\sigma}) = p_\mu^s - l_\mu^s \sum_{i=1}^{N} \sqrt{m_i^2 + \kappa_i^2(\tau)} + b^s_{\nu}(\tau) \sum_{i=1}^{N} \kappa_{ir}(\tau) \approx 0, \]
\[ \tilde{\mathcal{H}}^{\mu\nu}(\tau) = b^s_{\nu}(\tau) \int d^3 \sigma \sigma^\mu \mathcal{H}^{\nu}(\tau, \vec{\sigma}) - b^s_{\mu}(\tau) \int d^3 \sigma \sigma^\nu \mathcal{H}^{\mu}(\tau, \vec{\sigma}) = S^{\mu\nu}_{s}(\tau) - \{b^s_{\mu}(\tau) b^s_{\nu}(\tau) - b^s_{\nu}(\tau) b^s_{\mu}(\tau) \} \sum_{i=1}^{N} \eta_i^s(\tau) \sqrt{m_i^2 + \kappa_i^2(\tau)} - \\
- \{b^s_{\mu}(\tau) b^s_{\nu}(\tau) - b^s_{\nu}(\tau) b^s_{\mu}(\tau) \} \sum_{i=1}^{N} \eta_i^s(\tau) \kappa_i^s(\tau) \approx 0. \quad (2.5) \]

Here \( S^{\mu\nu}_{s} \) is the spin part of the Lorentz generators

\[ J^{\mu\nu}_{s} = x^\mu_{s} p^\nu_{s} - x^\nu_{s} p^\mu_{s} + S^{\mu\nu}_{s}, \]
\[ S^{\mu\nu}_{s} = b^s_{\nu}(\tau) \int d^3 \sigma \sigma^\mu \mathcal{H}^{\nu}(\tau, \vec{\sigma}) - b^s_{\mu}(\tau) \int d^3 \sigma \sigma^\nu \mathcal{H}^{\mu}(\tau, \vec{\sigma}). \quad (2.6) \]

On the Wigner hyperplane\textsuperscript{13} we have the following constraints and Dirac Hamiltonian

\[ \tilde{\mathcal{H}}^\mu(\tau) = p_\mu^s - u^\mu(p_s) \sum_{i=1}^{N} \sqrt{m_i^2 + \kappa_i^2} + \epsilon_i^\mu(u(p_s)) \sum_{i=1}^{N} \kappa_{ir} = \]

\textsuperscript{13}On it we use the notation \( A = (\tau, \vec{r}) \). The 3-vectors \( \vec{B} = \{ b^r \} \) on the Wigner hyperplanes are Wigner spin-1 3-vectors.

\textsuperscript{14}\( \epsilon_i^\mu(u(p_s)) = L^\mu_\tau(p_s, \vec{0}) \) and \( \epsilon_i^\mu(u(p_s)) = u^\mu(p_s) = L^\mu_\nu(p_s, \vec{0}) \) are the columns of the standard Wigner boost for timelike Poincaré orbits. See Appendix B of Ref. \textsuperscript{1}. 

\[ u^\mu(p_s)\left[\epsilon_s - \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{\kappa}_i^2} \right] + e_\nu^\mu(u(p_s)) \sum_{i=1}^{N} \kappa_{ip} \approx 0, \]

or

\[ \epsilon_s - M_{sys} \approx 0, \quad M_{sys} = \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{\kappa}_i^2}, \]

\[ \vec{p}_{sys} = \vec{\kappa}_+ = \sum_{i=1}^{N} \vec{\kappa}_i \approx 0, \]

\[ H_D = \lambda^\mu(\tau)\vec{H}_\mu(\tau) = \lambda(\tau)[\epsilon_s - M_{sys}] - \vec{\lambda}(\tau) \sum_{i=1}^{N} \vec{\kappa}_i, \]

\[ \lambda(\tau) \approx -\dot{x}_{sp}(\tau)u^\mu(p_s), \]

\[ \lambda_r(\tau) \approx -\dot{x}_{sp}(\tau)e^\mu_\nu(u(p_s)), \]

\[ \dot{x}^\mu_s(\tau) = -\lambda(\tau)u^\mu(p_s), \]

\[ \dot{x}^\mu_s(\tau) \equiv \{x^\mu_s(\tau), H_D\} = \dot{\lambda}_\nu(\tau)\{x^\mu_s(\tau), \vec{H}_\nu(\tau)\} \approx \]

\[ -\dot{\lambda}^\mu(\tau) = -\lambda(\tau)u^\mu(p_s) + e_\nu^\mu(u(p_s))\lambda_r(\tau), \]

\[ \dot{x}^2_s(\tau) = \lambda^2(\tau) - \vec{\lambda}^2(\tau) > 0, \quad \dot{x}_s \cdot u(p_s) = -\lambda(\tau), \]

\[ U^\mu_s(\tau) = \frac{\dot{x}^\mu_s(\tau)}{\sqrt{\dot{x}^2_s(\tau)}} = \frac{-\lambda(\tau)u^\mu(p_s) + \lambda_r(\tau)e^\mu_\nu(u(p_s))}{\sqrt{\lambda^2(\tau) - \vec{\lambda}^2(\tau)}}, \]

\[ \Rightarrow \quad x^\mu_s(\tau) = x^\mu_s(0) - u^\mu(p_s) \int_0^\tau d\tau_1 \lambda(\tau_1) + e^\mu_\nu(u(p_s)) \int_0^\tau d\tau_1 \lambda_r(\tau_1). \quad (2.7) \]

While the Dirac multiplier \( \lambda(\tau) \) is determined by the gauge fixing \( T_s - \tau \approx 0 \), the 3 Dirac’s multipliers \( \vec{\lambda}(\tau) \) describe the classical zitterbewegung of the origin \( x^\mu_s(\tau) = z^\mu(\tau, \vec{0}) \) of the coordinates on the Wigner hyperplane. Each gauge-fixing \( \vec{\chi}(\tau) \approx 0 \) to the 3 first class constraints \( \vec{\kappa}_+ \approx 0 \) (defining the internal rest-frame) gives a different determination of the multipliers \( \vec{\lambda}(\tau) \). Therefore it identifies a different worldline for the covariant non-canonical origin \( x^\mu_s(\vec{\chi})(\tau) \) which carries with itself the definition \( \vec{\sigma}_{com} \) of the internal 3-center of mass, conjugate with \( \vec{\kappa}_+ \).\[ \]

Let us remark that the constant \( x^\mu_s(0) \) [and, therefore, also \( \vec{x}^\mu_s(0) \)] is arbitrary, reflecting the arbitrariness in the absolute location of the origin of the internal coordinates on each hyperplane in Minkowski spacetime. The origin \( x^\mu_s(\tau) \) corresponds to the unique special

\[ \text{15} \text{Obviously each choice } \vec{\chi}(\tau) \text{ leads to a different set of conjugate canonical relative variables.} \]
relativistic center-of-mass-like worldline of Refs. \[19\] 16, which unifies previous proposals of Syngue, Møller and Pryce quoted in that paper.

The only remaining canonical variables describing the Wigner hyperplane in the final Dirac brackets are the non-covariant canonical coordinate \( \vec{x}_s^\mu(\tau) \) and \( p_\mu^s \). The point with coordinates \( \vec{x}_s^\mu(\tau) \) is the decoupled canonical external 4-center of mass of the isolated system, which can be interpreted as a decoupled observer with his parametrized clock (point particle clock). Its velocity \( \dot{\vec{x}}_s^\mu(\tau) \) is parallel to \( p_\mu^s \), so that it has no classical zitterbewegung.

The relation between \( x_\mu^s(\tau) \) and \( \vec{x}_s^\mu(\tau) \) (\( \vec{\tau} \) is its 3-location on the Wigner hyperplane) is

\[
\vec{x}_s^\mu(\tau) = (\vec{x}_s^\mu(\tau); \vec{\tau}_s(\tau)) = z^\mu(\tau, \vec{\tau}) = x_\mu^s(\tau) - \frac{1}{\epsilon_s(p_\mu^s + \epsilon_s)} \left[ p_{s\nu} S_\mu^\nu + \epsilon_s (S_o^\mu - S_s^\nu \frac{p_\nu p^\mu_s}{\epsilon_s^2}) \right],
\]

After the separation of the relativistic canonical non-covariant external 4-center of mass \( \vec{x}_s^\mu(\tau) \), on the Wigner hyperplane the N particles are described by the 6N Wigner spin-1 3-vectors \( \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau) \) restricted by the rest-frame condition \( \vec{\kappa}_+ = \sum_{i=1}^N \vec{\kappa}_i \approx 0 \).

Inside the Wigner hyperplane, three degrees of freedom of the isolated system \[19\] become gauge variables. To eliminate the three first class constraints \( \vec{\kappa}_+ \approx 0 \) the natural gauge fixing is \( \vec{\chi}(\tau) = \vec{\sigma}_{\text{com}} = \vec{q}_+ \approx 0 \) such that \( \vec{q}_+ \approx 0 \) implies \( \lambda_s(\tau) = 0 \): in this way the internal 3-center of mass is located at the origin \( x_\mu^s(\tau) = z^\mu(\tau, \vec{\sigma} = 0) \) of the Wigner hyperplane.

The various spin tensors and vectors are \[2\]

\[
J_s^{\mu\nu} = x_\mu^s p_\nu^s - x_\nu^s p_\mu^s + S_s^{\mu\nu} = \vec{x}_s^\mu p_\nu^s - \vec{x}_s^\nu p_\mu^s + \vec{S}_s^{\mu\nu},
\]

\[
S_s^{\mu\nu} = \left[ u(\mu(p_s)) e^\nu(u(p_s)) - u(\nu(p_s)) e^\mu(u(p_s)) \right] S_s^{\mu\nu} + \epsilon_s(u(p_s)) \epsilon^\nu(u(p_s)) S_s^{\mu\nu} \equiv \epsilon^\nu(u(p_s)) \epsilon^\mu(u(p_s)) S_s^{\mu\nu} \equiv \left[ e^\mu(r(p_s)) e^\nu(u(p_s)) - e^\nu(r(p_s)) e^\mu(u(p_s)) \right] \sum_{i=1}^N |\eta_i^r| \nu_{i,1}^2 c^2 + \kappa_i^2 + \left[ e^\mu(r(p_s)) e^\nu(u(p_s)) - e^\nu(r(p_s)) e^\mu(u(p_s)) \right] \sum_{i=1}^N |\eta_i^u| \kappa_i^2,
\]

\[
\vec{S}_s^{AB} = \epsilon^A_r(u(p_s)) \epsilon^B_u(u(p_s)) S_s^{\mu\nu},
\]

\[16\]See Refs. \[20\] [21] for the definition of this concept in general relativity. By using the interpretation of Ref. \[20\], also the special relativistic limit of the general relativistic Dixon centroid of Ref. \[18\] gives the centroid \( x_\mu^s(\tau) \): it coincides with the special relativistic Dixon centroid of Ref. \[17\] defined by using the conserved energy momentum tensor, as we shall see in Section IV.

\[17\]It describes a point living on the Wigner hyperplanes and has the covariance of the little group O(3) of timelike Poincaré orbits, like the Newton-Wigner position operator.

\[18\]As already said, they describe an internal center-of-mass 3-variable \( \vec{\sigma}_{\text{com}} \) defined inside the Wigner hyperplane and conjugate to \( \vec{\kappa}_+ \); when the \( \vec{\sigma}_{\text{com}} \) are canonical variables they are denoted \( \vec{q}_+ \).
\[ S_{rs}^s \equiv \sum_{i=1}^{N} (\eta_i^s \kappa_i^r - \eta_i^r \kappa_i^s), \quad \bar{S}_{rs}^s \equiv -\sum_{i=1}^{N} \eta_i^s \sqrt{m_i^2 c^2 + \kappa_i^s}, \]

\[ \bar{S}_{\mu
u}^s = S_{\mu
u}^s + \frac{1}{\sqrt{\epsilon_p^2 + \epsilon_p^2}} [p_{s\beta}(S_{s\beta}^{\alpha\mu}p_{s}^{\alpha} - S_{s\beta}^{\alpha\nu}p_{s}^{\alpha}) + \sqrt{p_{s}^{2}(S_{s\alpha}^{\mu\nu} - S_{s\alpha}^{\nu\mu})}], \]

\[ S_{ij}^s = \delta_{ir} \delta_{js} \bar{S}_{rs}^s, \quad \bar{S}_{oi}^s = -\delta_{ir} \bar{S}_{rs}^s p_{s}^o, \]

\[ \tilde{S} \equiv \bar{S} = \sum_{i=1}^{N} \tilde{\eta}_i \times \tilde{\kappa}_i \approx \sum_{i=1}^{N} \tilde{\eta}_i \times \tilde{\kappa}_i - \tilde{\eta}_+ \times \tilde{\kappa}_+ = \sum_{a=1}^{N-1} \tilde{\rho}_a \times \tilde{\pi}_a. \tag{2.9} \]

Let us remark that while \( L_{\mu\nu}^s = \bar{x}_{s\mu}^\nu - x_{s\mu}^\nu \) and \( S_{\mu\nu}^s \) are not constants of the motion due to classical zitterbewung, both \( \bar{L}_{\mu\nu}^s = \bar{x}_{s\mu}^\nu - \bar{x}_{s\mu}^\nu \) and \( \bar{S}_{\mu\nu}^s \) are conserved.

The canonical variables \( \bar{x}_{s\mu}^\nu, p_{s}^\mu \) for the external 4-center of mass, can be replaced by the canonical pairs \[ T_s = \frac{p_{s} \cdot \bar{x}_s}{\epsilon_s} = \frac{p_{s} \cdot \bar{x}_s}{\epsilon_s}, \]

\[ \epsilon_s = \pm \sqrt{\epsilon_p^2}, \]

\[ \bar{z}_s = \epsilon_s (\bar{z}_s - \frac{\bar{p}_s}{\epsilon_s} \bar{z}_s), \]

\[ \bar{k}_s = \frac{\bar{p}_s}{\epsilon_s}, \tag{2.10} \]

with the inverse transformation

\[ \bar{x}_s^o = \sqrt{1 + \bar{k}_s^2 (T_s + \frac{\bar{k}_s \cdot \bar{z}_s}{\epsilon_s})}, \]

\[ \bar{x}_s = \frac{\bar{z}_s}{\epsilon_s} + (T_s + \frac{\bar{k}_s \cdot \bar{z}_s}{\epsilon_s}) \bar{k}_s, \]

\[ p_{s}^o = \epsilon_s \sqrt{1 + \bar{k}_s^2}, \]

\[ \bar{p}_s = \epsilon_s \bar{k}_s. \tag{2.11} \]

This non-point canonical transformation can be summarized as \[ [\epsilon_s - M_{sys} \approx 0, \bar{k}_+ = \sum_{i=1}^{N} \bar{k}_i \approx 0] \]

\[ \begin{bmatrix} \bar{x}_s^o & \bar{p}_s^o \\ \bar{z}_s & \bar{k}_s \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \epsilon_s & \bar{z}_s & \bar{k}_s \\ T_s & \bar{k}_s & \bar{z}_s \end{bmatrix} \tag{2.12} \]

The invariant mass \( M_{sys} \) of the system, which is also the internal energy of the isolated system, replaces the non-relativistic Hamiltonian \( H_{rel} \) for the relative degrees of freedom.

\[ ^{19} \text{It makes explicit the interpretation as a point particle clock.} \]
after the addition of the gauge-fixing $T_s - \tau \approx 0$ this reminds of the frozen Hamilton-Jacobi theory, in which the time evolution can be reintroduced by using the energy generator of the Poincaré group as Hamiltonian.

After the gauge fixings $T_s - \tau \approx 0$, the final Hamiltonian and the embedding of the Wigner hyperplane into Minkowski spacetime are

$$H_D = M_{sys} - \lambda(\tau) \cdot \vec{k}_s,$$

$$z^\mu(\tau, \vec{\sigma}) = x_s^\mu(\tau) + \epsilon^\mu_s(u(p_s))\sigma^\nu = x_s^\mu(0) + u^\mu(p_s)\tau + \epsilon^\mu_s(u(p_s))\sigma^\nu,$$

with

$$\dot{x}_s^\mu(\tau) = d x_s^\mu(\tau) \frac{d}{d\tau} + \{x_s^\mu(\tau), H_D\} = u^\mu(p_s) + \epsilon^\mu_s(u(p_s))\lambda_\nu(\tau),$$

where $x_s^\mu(0)$ is an arbitrary point.

The particles’ worldlines in Minkowski spacetime and the associated momenta are

$$x^\mu_s(\tau) = z^\mu(\tau, \vec{\eta}_s(\tau)) = x_s^\mu(\tau) + \epsilon^\mu_s(u(p_s))\eta^\nu_s(\tau),$$

$$p^\mu_s(\tau) = \sqrt{m_s^2 + \vec{\kappa}_s^2(\tau)}u^\mu(p_s) + \epsilon^\mu_s(u(p_s))\kappa^\nu(\tau) \Rightarrow \epsilon^\mu_s = m_s^2.$$

The external rest-frame instant form realization of the Poincaré generators with non-fixed invariants $\epsilon^2_s = M^2_{sys}$, $-\epsilon^2_s \tilde{S}^2_s \approx -\epsilon^2 S^2$, is obtained from Eq.(2.9):

$$p^\mu_s, \quad J^{\mu\nu}_s = \tilde{x}^\mu_p \tilde{x}^\nu - \tilde{x}^\mu p^\nu + \tilde{S}^{\mu\nu},$$

$$p^0_s = \sqrt{\epsilon^2_s + \vec{p}_s^2} = \epsilon_s \sqrt{1 + \vec{k}_s^2} \approx \sqrt{M^2_{sys} + \vec{p}_s^2} = M_{sys} \sqrt{1 + \vec{k}_s^2},$$

$$\vec{p}_s = \epsilon_s \vec{k}_s \approx M_{sys} \vec{k}_s,$$

$$J^i_j = \tilde{x}^i p^j - \tilde{x}^j p^i + \delta^{ij}s \sum_{\alpha=1}^N (\eta^i_s \kappa^\nu_s - \eta^\nu_s \kappa^i_s) \approx \delta^{ij}s \sum_{\alpha=1}^N (\eta^i_s \kappa^\nu_s - \eta^\nu_s \kappa^i_s) = \delta^{ij}s \sum_{\alpha=1}^N \left( \eta^i_s \kappa^\nu_s - \eta^\nu_s \kappa^i_s \right)$$

$$= -\sqrt{1 + \vec{k}_s^2} z^i_s \tilde{x}^j_s + \frac{\delta^{ij}s \epsilon^2_s \tilde{S}^u_s \kappa^\nu_s \kappa^i_s}{1 + \vec{k}_s^2} \approx \tilde{x}^j_s \epsilon^i_s - \tilde{x}^i_s \sqrt{M^2_{sys} + \vec{p}_s^2} - \frac{\delta^{ij}s \epsilon^2_s \tilde{S}^u_s \kappa^\nu_s \kappa^i_s}{M_{sys} + \sqrt{M^2_{sys} + \vec{p}_s^2}}. \quad (2.15)$$

---

20It implies $\lambda(\tau) = -1$ and identifies the time parameter $\tau$ with the Lorentz scalar time of the center of mass in the rest frame, $T_s = p_s \cdot \tilde{x}_s / M_{sys}$; $M_{sys}$ generates the evolution in this time.

21See Refs. [23] for a different derivation of this result.

22As in every instant form of dynamics, there are four independent Hamiltonians $p^0_s$ and $J^{oi}_s$, functions of the invariant mass $M_{sys}$; we give also the expression in the basis $T_s, \epsilon_s, \tilde{x}_s, \vec{k}_s$. 

14
On the other hand the internal realization of the Poincaré algebra is built inside the Wigner hyperplane by using the expression of $\bar{S}^{AB}_s$ given by Eq. (2.9)

\[ M_{sys} = H_M = \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2}, \]

\[ \vec{\kappa}_+ = \sum_{i=1}^{N} \vec{\kappa}_i (\approx 0), \]

\[ \vec{J} = \sum_{i=1}^{N} \vec{\eta}_i \times \vec{\kappa}_i, \quad J^r = \vec{S}^r = \frac{1}{2} \epsilon^{ruvw} \bar{S}_{uw} \equiv \bar{S}_s^r, \]

\[ \vec{K} = -\sum_{i=1}^{N} \sqrt{m_i^2 + \vec{k}_i^2} \vec{\eta}_i = -M_{sys} \vec{R}_+, \quad K^r = J^{or} = \bar{S}_s^{rr}, \]

\[ \Pi = M_{sys}^2 - \vec{\kappa}_+^2 \approx M_{sys}^2 > 0, \]

\[ W^2 = -\epsilon(M_{sys}^2 - \vec{\kappa}_+^2) \bar{S}_s^2 \approx -\epsilon M_{sys}^2 \bar{S}_s^2. \quad (2.16) \]

The constraints $\epsilon_s - M_{sys} \approx 0, \vec{\kappa}_+ \approx 0$ mean: i) the constraint $\epsilon_s - M_{sys} \approx 0$ is the bridge which connects the external and internal realizations; ii) the constraints $\vec{\kappa}_+ \approx 0$, together with $\vec{K} \approx 0$, imply a unfaithful internal realization, in which the only non-zero generators are the conserved energy and spin of an isolated system.

The determination of $\vec{q}_+$ for the N particle system was done with the group theoretical methods of Ref. [24] in Section III of Ref. [1]. Given a realization of the ten Poincaré generators on the phase space, one can build three 3-position variables in terms of them only. For N free scalar relativistic particles on the Wigner hyperplane with $\vec{p}_{sys} = \vec{\kappa}_+ \approx 0$ and by using the internal realization (2.16) they are:

i) a canonical internal center of mass (or center of spin) $\vec{q}_+$;

ii) a non-canonical internal Møller center of energy $\vec{R}_+$;

iii) a non-canonical internal Fokker-Pryce center of inertia $\vec{y}_+$.

It can be shown [1] that, due to $\vec{\kappa}_+ \approx 0$, they coincide: $\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+$.

Therefore the gauge fixings $\vec{x}(\tau) = \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \approx 0$ imply $\lambda(\tau) \approx 0$ and force the three internal collective variables to coincide with the origin of the coordinates, which now becomes

\[ x_s^{(\vec{q}_+)}(T_s) = x_s(0) + u(\vec{p}_s)T_s. \quad (2.17) \]

As shown in Section IV, the addition of the gauge fixings $\vec{x}(\tau) = \vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \approx 0$ implies that the Dixon center of mass of an extended object [18] and the Pirani [25] and Tulczyjew

\[ ^{23} \text{This internal Poincaré algebra realization should not be confused with the previous external one based on } \bar{S}_{s}^{\mu \nu}; \Pi \text{ and } W^2 \text{ are the two non-fixed invariants of this realization.} \]

\[ ^{24} \text{The external spin coincides with the internal angular momentum due to Eqs. (A11) of Ref. [1].} \]

\[ ^{25} \text{As we shall see in the next Section, } \vec{K} \approx 0 \text{ is implied by the natural gauge fixing } \vec{q}_+ \approx 0. \]
centroids all simultaneously coincide with the origin $x^\mu_s(\tau)$.

The external realization (2.15) allows to build the analogous external 3-variables $\vec{q}_s$, $\vec{R}_s$, $\vec{Y}_s$. It is then shown in Ref. [1] how to build the associated external 4-variables and their location on the Wigner hyperplane

$$\tilde{x}^\mu_s = (\tilde{x}^\mu_s; \vec{z}_s) = \frac{1}{\epsilon_s} \left[ z_s + \frac{\vec{S}_s \times \vec{p}_s}{\epsilon_s [1 + u^o(p_s)]} \right] + (T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s}) \vec{k}_s = x^\mu_s + \epsilon^\mu_u(u(p_s)) \tilde{\sigma}^u,$$

$$Y^\mu_s = (\tilde{x}^0_s; \vec{Y}_s) = \left( \tilde{x}^0_s; \vec{z}_s \frac{1}{\epsilon_s} \left[ z_s + \frac{\vec{S}_s \times \vec{p}_s}{\epsilon_s [1 + u^o(p_s)]} \right] + (T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s}) \vec{k}_s \right) = \tilde{x}^\mu_s + \epsilon^\mu_u(u(p_s)) \sigma^\mu_Y,$$

$$R^\mu_s = (\tilde{x}^0_s; \vec{R}_s) = \left( \tilde{x}^0_s; \vec{z}_s \frac{1}{\epsilon_s} \left[ z_s + \frac{\vec{S}_s \times \vec{p}_s}{\epsilon_s [1 + u^o(p_s)]} \right] + (T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s}) \vec{k}_s \right) = \tilde{x}^\mu_s - \epsilon^\mu_u(u(p_s)) \sigma^\mu_R.$$

$$T_s = u(p_s) \cdot x_s = u(p_s) \cdot \tilde{x}_s = u(p_s) \cdot Y_s = u(p_s) \cdot R_s,$$

$$\tilde{\sigma}^r = \epsilon_{r\mu}(u(p_s)) [x^\mu_s - \tilde{x}^\mu_s] = \frac{\epsilon_{r\mu}(u(p_s)) [u_o(p_s) S^o_{s\mu} + S^{o\mu}_{s\nu}]}{[1 + u^o(p_s)]} = \epsilon_{r\mu}(u(p_s)) \frac{S^o_{s\mu}}{[1 + u^o(p_s)]} = \epsilon_{r\mu}(u(p_s)) \frac{S^o_{s\nu} u^o(p_s)}{1 + u^o(p_s)} \approx \epsilon_{r\mu}(u(p_s)) \frac{S^o_{s\nu} u^o(p_s)}{1 + u^o(p_s)},$$

$$\sigma^r_Y = \epsilon_{r\mu}(u(p_s)) [x^\mu_s - Y^\mu_s] = \tilde{\sigma}^r - \epsilon_{r\mu}(u(p_s)) \frac{\vec{S}_s \times \vec{p}_s}{\epsilon_s [1 + u^o(p_s)]} \approx \epsilon_{r\mu}(u(p_s)) \frac{S^o_{s\nu} u^o(p_s)}{1 + u^o(p_s)} \approx 0,$$

$^{26}$See Ref. [28] for the application of these methods to find the center of mass of a configuration of the Klein-Gordon field after the preliminary work of Ref. [29] on the center of phase for a real Klein-Gordon field.
\[ \sigma^r_R = \epsilon_{r\mu}(u(p_s))[x^\mu_s - R^\mu_s] = \bar{\sigma}^r + \epsilon_{ru}(u(p_s)) \frac{(\bar{S}_s \times \bar{p}_s)^u}{\epsilon_s u^o(p_s)[1 + u^o(p_s)]} = \]

\[ = \bar{\sigma}^r - \frac{S^r_s u^s(p_s)}{u^o(p_s)[1 + u^o(p_s)]} = \epsilon_s R^r_s + \frac{[1 - u^o(p_s)]S^r_s u^s(p_s)}{u^o(p_s)[1 + u^o(p_s)]} \approx \]

\[ \approx \frac{[1 - u^o(p_s)]S^r_s u^s(p_s)}{u^o(p_s)[1 + u^o(p_s)]}, \]

\[ \Rightarrow x^{(\bar{q}_s)\mu}(\tau) = Y^\mu_s, \quad (2.18) \]

namely the external Fokker-Pryce non-canonical center of inertia coincides with the origin \( x^{(\bar{q}_s)\mu}(\tau) \) carrying the internal center of mass.
III. THE ENERGY-MOMENTUM TENSOR ON THE WIGNER HYPERPLANE
AND DIXON’S RELATIVISTIC MULTipoles.

A. The Euler-Lagrange Equations and the Energy-Momentum Tensor of
Parametrized Minkowski Theories.

The Euler-Lagrange equations associated with the Lagrangian (3.1) are (the symbol ’\( \overset{o}{=} \)’
means evaluated on the solutions of the equations of motion)

\[
\frac{\partial L}{\partial z^\mu} - \frac{\partial_A}{\partial z^A} \left( \frac{\partial L}{\partial z^A} \right)(\tau, \vec{\sigma}) = \eta_{\mu\nu} \partial_A [\sqrt{g} T^{AB} z_B^\nu](\tau, \vec{\sigma}) \overset{o}{=} 0,
\]
\[
\frac{\partial L}{\partial \vec{\eta}_i} - \partial_{\vec{r}} \frac{\partial L}{\partial \vec{n}_i} = - \left[ \frac{1}{2} \frac{T^{AB}}{\sqrt{g}} \right]_{\vec{\sigma} = \vec{\eta}} \frac{\partial g_{AB}}{\partial \vec{n}_i} - \partial_{\vec{r}} \frac{g_{rr} + g_{rs} \vec{\eta}^s}{\sqrt{g_{rr} + 2g_{rw} \vec{\eta}^r + g_{uv} \vec{\eta}^u \vec{\eta}^v}} |_{\vec{\sigma} = \vec{\eta}} \overset{o}{=} 0,
\]

(3.1)

where we have introduced the energy-momentum tensor (here \( \vec{\eta}_i^s(\tau) = (1; \vec{\eta}_i(\tau)) \))

\[
T^{AB}(\tau, \vec{\sigma}) = - \frac{2}{\sqrt{g}} \delta S \left| \frac{\partial g_{AB}}{\partial \vec{n}_i} \right| (\tau, \vec{\sigma}) = - N \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \frac{m_i \vec{\eta}_i^A(\tau) \vec{\eta}_i^B(\tau)}{\sqrt{g_{rr} + 2g_{rw} \vec{\eta}^r + g_{uv} \vec{\eta}^u \vec{\eta}^v}} (\tau, \vec{\sigma}).
\]

(3.2)

Due to the delta functions the Euler-Lagrange equations for the fields \( z^\mu(\tau, \vec{\sigma}) \) are trivial
\((0 \overset{o}{=} 0)\) everywhere except at the positions of the particles. They may be rewritten in a form
valid for every isolated system

\[
\partial_A T^{AB} z_B^\mu \overset{o}{=} - \frac{1}{\sqrt{g}} \partial_A [\sqrt{g} z_B^\mu] T^{AB}.
\]

(3.3)

When \( \partial_A [\sqrt{g} z_B^\mu] = 0 \) as it happens on the Wigner hyperplanes in the gauge \( \vec{q}_+ \approx 0 \) and
\( T_s - \tau \approx 0 \), we get the conservation of the energy-momentum tensor \( T^{AB} \), i.e. \( \partial_A T^{AB} \overset{o}{=} 0 \).
Otherwise there is a compensation coming from the dynamics of the surface.

On the Wigner hyperplane, where we have

\[
x_s^\mu(\tau) = x_0^\mu(\tau, \vec{\eta}_i(\tau)) = x_s^\mu(\tau) + \epsilon_s^\mu(u(p_s)) \vec{\eta}_i(\tau),
\]
\[
x_s^1(\tau) = x_0^1(\tau, \vec{\eta}_i(\tau)) + x_s^1(\tau, \vec{\eta}_i(\tau)) \vec{\eta}_i^1(\tau) = \dot{x}_s^1(\tau) + \epsilon_s^1(u(p_s)) \vec{\eta}_i^1(\tau),
\]
\[
x_s^2(\tau) = g_{rr}(\tau, \vec{\eta}_i(\tau)) + 2g_{rr}(\tau, \vec{\eta}_i(\tau)) \vec{\eta}_i^2(\tau) + g_{rs}(\tau, \vec{\eta}_i(\tau)) \vec{\eta}_i^s(\tau) \vec{\eta}_i^r(\tau) = \dot{x}_s^2(\tau) + 2\dot{x}_{s\mu}(\tau) \vec{\epsilon}_s^\mu(u(p_s)) \vec{\eta}_i^r(\tau) - \dot{z}_i^2(\tau),
\]
\[
p_s^\mu(\tau) = \sqrt{m_s^2 - \gamma^{rs}(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau) \delta^\mu(\tau, \vec{\eta}_i(\tau)) - \kappa_{ir}(\tau) \gamma^{rs}(\tau, \vec{\eta}_i(\tau)) z_s^{\nu}(\tau, \vec{\eta}_i(\tau))} = \sqrt{m_s^2 + \vec{\kappa}_i^2(\tau) u^\mu(p_s) + \vec{\epsilon}_s^\mu(u(p_s)) \vec{\kappa}_i^r(\tau)} \Rightarrow p_i^2 = m_i^2,
\]
\[
p_s^\mu = \int d^3 \sigma \rho_\mu(\tau, \vec{\sigma}) \approx \sum_{i=1}^{N} p_i^\mu(\tau),
\]

(3.4)

the energy-momentum tensor \( T^{AB}(\tau, \vec{\sigma}) \) has the form
\[ T^{\tau\tau}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \frac{m_i}{\sqrt{\dot{x}_i^2(\tau) + 2\dot{x}_{sp}(\tau)\epsilon^p_\mu(u(p_s)) - \dot{\eta}_i^2(\tau)}} \]
\[ T^{r\tau}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \frac{m_i\dot{n}_i^r(\tau)}{\sqrt{\dot{x}_i^2(\tau) + 2\dot{x}_{sp}(\tau)\epsilon^p_\mu(u(p_s)) - \dot{\eta}_i^2(\tau)}} \]
\[ T^{rs}(\tau, \vec{\sigma}) = -\sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \frac{m_i\dot{n}_i^r(\tau)\dot{\eta}_i^s(\tau)}{\sqrt{\dot{x}_i^2(\tau) + 2\dot{x}_{sp}(\tau)\epsilon^p_\mu(u(p_s)) - \dot{\eta}_i^2(\tau)}}. \] (3.5)

B. The Energy-Momentum Tensor of the Standard Lorentz-Covariant Theory.

The same form is obtained from the restriction to positive energies \(^{27}\) of the energy momentum tensor of the standard manifestly Lorentz covariant theory with Lagrangian

\[ S_S = \int d\tau L_S(\tau) = -\sum_{i=1}^{N} m_i \int d\tau \sqrt{\dot{x}_i^2(\tau)} \]  

On the Wigner hyperplanes with \( T_s - \tau \approx 0 \) we will get

\[ T^{\mu\nu}(\tau, \vec{\sigma}) = -\left( \frac{2}{\sqrt{g}} \frac{\delta S_S}{\delta g_{\mu\nu}} \right)_{\tau = z(\tau, \vec{\sigma})} = \]
\[ = \sum_{i=1}^{N} m_i \int d\tau_1 \frac{\dot{x}_i^\mu(\tau_1)\dot{x}_i^\nu(\tau_1)}{\sqrt{\dot{x}_i^2(\tau_1)}} \delta^4(x_i(\tau_1) - z(\tau, \vec{\sigma})) = \]
\[ = \epsilon_A^\mu(u(p_s))\epsilon_B^\nu(u(p_s))T^{AB}(\tau, \vec{\sigma}). \] (3.6)

On the other hand, from the restriction of the standard theory we get

1) On arbitrary spacelike hypersurfaces

\[ T^{\mu\nu}(z(\tau, \vec{\sigma})) = \sum_{i=1}^{N} m_i \int d\tau_1 \frac{\dot{x}_i^\mu(\tau_1)\dot{x}_i^\nu(\tau_1)}{\sqrt{\dot{x}_i^2(\tau_1)}} \delta^4(x_i(\tau_1) - z(\tau, \vec{\sigma})) = \]
\[ = \sum_{i=1}^{N} m_i \int \frac{d\tau_1}{\sqrt{\dot{x}_i^2(\tau_1)}} \delta^4(z(\tau_1, \vec{\eta}_i(\tau_1)) - z(\tau, \vec{\sigma})) = \]
\[ = \sum_{i=1}^{N} m_i \left[ \dot{x}_i^\mu(\tau_1, \vec{\eta}_i(\tau_1))\dot{x}_i^\nu(\tau_1, \vec{\eta}_i(\tau_1))\dot{\eta}_i^r(\tau_1) \right] \]
\[ + \left[ \dot{z}_i^\mu(\tau_1, \vec{\eta}_i(\tau_1))\dot{z}_i^\nu(\tau_1, \vec{\eta}_i(\tau_1))\dot{\eta}_i^r(\tau_1) \right] + \]
\[ + \left[ \dot{z}_i^\mu(\tau_1, \vec{\eta}_i(\tau_1))\dot{z}_i^\nu(\tau_1, \vec{\eta}_i(\tau_1))\dot{\eta}_i^r(\tau_1) \right] \dot{\eta}_i^r(\tau_1) + \]

\(^{27}\) \( p_i^\mu = m_i \frac{\dot{x}_i^\mu}{\sqrt{\dot{x}_i^2}} > 0, \) \( \epsilon(\sum_{i=1}^{N} p_i^\mu)^2 = \epsilon(\sum_{i=1}^{N} m_i \frac{\dot{x}_i^\mu}{\sqrt{\dot{x}_i^2}})^2 > 0. \)
\[ + z''_\tau (\tau_1, \bar{\eta}_i (\tau_1)) z''_\tau (\tau_1, \bar{\eta}_i (\tau_1)) \bar{\eta}_i^\tau (\tau_1) \bar{\eta}_i^\tau (\tau_1) \] = 
\[ = \sum_{i=1}^{N} \frac{m_i}{\sqrt{\bar{\mu}^\tau_r}} \delta^3 \left( \bar{\sigma} - \bar{\eta}_i (\tau) \right) \left( \text{det} \left| z''_\alpha (\tau, \bar{\sigma}) \right| \right)^{-1} \]
\[ \left[ z''_\tau (\tau, \bar{\eta}_i (\tau_1)) z''_\tau (\tau, \bar{\eta}_i (\tau_1)) + \left( z''_\tau (\tau, \bar{\eta}_i (\tau)) z''_\tau (\tau, \bar{\eta}_i (\tau)) + \right. \]
\[ + z''_\tau (\tau, \bar{\eta}_i (\tau)) z''_\tau (\tau, \bar{\eta}_i (\tau)) \left| \bar{\eta}_i^\tau (\tau) \bar{\eta}_i^\tau (\tau) \right| \right] = 
\[ = \sum_{i=1}^{N} \frac{m_i}{\sqrt{\bar{\mu}^\tau_r}} \delta^3 \left( \bar{\sigma} - \bar{\eta}_i (\tau) \right) \]
\[ \left[ z''_\tau (\tau, \bar{\eta}_i (\tau_1)) z''_\tau (\tau, \bar{\eta}_i (\tau_1)) + \left( z''_\tau (\tau, \bar{\eta}_i (\tau)) z''_\tau (\tau, \bar{\eta}_i (\tau)) + \right. \]
\[ + z''_\tau (\tau, \bar{\eta}_i (\tau)) z''_\tau (\tau, \bar{\eta}_i (\tau)) \left| \bar{\eta}_i^\tau (\tau) \bar{\eta}_i^\tau (\tau) \right| \right], \quad (3.7) \]

since \( \text{det} \left| z''_\alpha \right| = \sqrt{g} = \sqrt{\gamma (g_{rr} - \gamma^{rs} g_{rs})} \), \( \gamma = |\text{det} g_{rs}| \).

2) On arbitrary spacelike hyperplanes, where it holds [4]

\[ z''^\rho (\tau, \bar{\sigma}) = x''^\rho (\tau) + b''_u (\tau) \sigma^u, \quad x''_s (\tau) = x''^\rho (\tau) + b''_u (\tau) \eta''_s (\tau), \]
\[ z''_\tau (\tau, \bar{\sigma}) = b''_v (\tau), \quad z''_\tau (\tau, \bar{\sigma}) = z''^\rho (\tau) + b''_u (\tau) \sigma^u = l'' / \sqrt{g^{rr}} - g_{rr} z''_r, \]
\[ g_{rr} = [x''_s + b''_u \sigma^u]^2, \quad g_{rr} = b_{r\mu} [x''_s + b''_u \sigma^u], \]
\[ g_{rs} = -\delta_{rs}, \quad \gamma^{rs} = -\delta^{rs}, \quad \gamma = 1, \]
\[ g = g_{rr} + \sum_r g_{rr}^2, \]
\[ g_{rr}^r = 1/[l_{\mu} (x''_s + b''_u \sigma^u)]^2, \quad g_{rr} = g_{rr} g_{rr} = b_{r\mu} (x''_s + b''_u \sigma^u) / [l_{\mu} (x''_s + b''_u \sigma^u)]^2, \]
\[ g_{rs} = -\delta^{rs} + g_{rr} g_{rr} g_{rs} = -\delta^{rs} + b_{r\mu} (x''_s + b''_u \sigma^u) b_{r\nu} (x''_s + b''_u \sigma^v) / [l_{\mu} (x''_s + b''_u \sigma^u)]^2, \quad (3.8) \]

we get

\[ T^\mu_\nu [x''_s (\tau) + b''_v (\tau) \sigma^v] = \sum_{i=1}^{N} \frac{m_i \delta^3 (\bar{\sigma} - \bar{\eta}_i (\tau))}{\sqrt{g(\tau, \bar{\sigma})} \sqrt{g_{rr}(\tau, \bar{\sigma}) + 2g_{rr}(\tau, \bar{\sigma}) \bar{\eta}_i (\tau) - \bar{\eta}_i (\tau)}} \]
\[ \left[ (x''_s (\tau) + b''_v (\tau) \eta''_s (\tau)) (x''_s (\tau) + b''_v (\tau) \eta''_s (\tau)) + \right. \]
\[ + \left( (x''_s (\tau) + b''_v (\tau) \eta''_s (\tau)) b''_v (\tau) (x''_s (\tau) + b''_v (\tau) \eta''_s (\tau)) b''_v (\tau) \right) \bar{\eta}_i (\tau) + \]
\[ + b''_v (\tau) b''_v (\tau) \bar{\eta}_i (\tau) \bar{\eta}_i (\tau) \right]. \quad (3.9) \]

3) On Wigner’s hyperplanes, where it holds [2]

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\( z^\mu(\tau, \vec{\sigma}) = x^\mu_0(\tau) + \epsilon^\mu_\nu(u(p_s))\sigma^\nu, \quad x^\mu_0(\tau) = x^\mu_0(\tau) + \epsilon^\mu_\nu(u(p_s))\eta^\nu_0(\tau), \)

\[
z^\mu = \epsilon^\mu_\nu(u(p_s)), \quad l^\mu = u^\mu(p_s), \quad z^\mu = \dot{x}^\mu_s(\tau), \quad g = \{x_s(\tau) \cdot u(p_s)\}^2, \quad g_{\tau\tau} = \dot{x}^2_s, \quad g_{\tau\nu} = \dot{x}_{s\mu}e^\mu_\nu(u(p_s)), \quad g_{rs} = -\delta_{rs},
\]

\[
g^{\tau\tau} = 1/[\dot{x}_{s\mu}u^\mu(p_s)]^2, \quad g^{\tau\nu} = \dot{x}_{s\mu}e^\mu_\nu(u(p_s))/[\dot{x}_{s\mu}u^\mu(p_s)]^2,
\]

\[
g^{rs} = -\delta^{rs} + \dot{x}_{s\mu}e^\mu_\nu(u(p_s))\dot{x}_{s\mu}e^\mu_\nu(u(p_s))/[\dot{x}_{s\mu}u^\mu(p_s)]^2,
\]

\[
ds^2 = \dot{x}^2_s(\tau)d\tau^2 + 2\dot{x}_s(\tau) \cdot \epsilon_\nu(u(p_s))d\tau d\sigma^\nu - d\vec{\sigma}^2,
\]

we get

\[
T^{\mu\nu}[x^\beta_s(\tau) + \epsilon^\beta_\nu(u(p_s))\sigma^\nu] = \frac{1}{\dot{x}_s(\tau) \cdot u(p_s)} \sum_{i=1}^{N} m_i \sqrt{\dot{x}^2_s(\tau) + 2\dot{x}_s(\tau)\dot{x}_s(\tau) e^\mu_\nu(u(p_s))\dot{\eta}_i^\mu(\tau) - \dot{\eta}_i^\mu(\tau)} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))
\]

\[
\frac{\dot{x}_s(\tau) \cdot u(p_s)}{\sqrt{\dot{x}^2_s(\tau) + 2\dot{x}_s(\tau)\dot{x}_s(\tau) e^\mu_\nu(u(p_s))\dot{\eta}_i^\mu(\tau) - \dot{\eta}_i^\mu(\tau)}} \left[ \dot{x}_s^\mu(\tau) \dot{x}_s^\nu(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) + \dot{x}_s^\mu(\tau) \dot{x}_s^\mu(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \dot{\eta}_i^\mu(\tau) + \epsilon^\mu_\nu(u(p_s)) \epsilon^\nu_\mu(u(p_s)) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \dot{\eta}_i^\mu(\tau) \dot{\eta}_i^\mu(\tau) \right] = \frac{1}{\dot{x}_s(\tau) \cdot u(p_s)} \sum_{i=1}^{N} m_i \sqrt{\dot{x}^2_s(\tau) + 2\dot{x}_s(\tau)\dot{x}_s(\tau) e^\mu_\nu(u(p_s))\dot{\eta}_i^\mu(\tau) - \dot{\eta}_i^\mu(\tau)} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \dot{\eta}_i^\mu(\tau) \dot{\eta}_i^\mu(\tau).
\]

Since the volume element on the Wigner hyperplane is \( u^\mu(p_s) d^3\sigma \), we obtain the following total 4-momentum and total mass of the N free particle system (Eqs. (2.17) are used)

\[
P^\mu_T = \int d^3\sigma T^{\mu\nu}[x^\beta_s(\tau) + \epsilon^\beta_\nu(u(p_s))\sigma^\nu]u_\nu(p_s) = -\lambda(\tau) \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \vec{\kappa}^2_i(\tau) u^\mu(p_s) + \kappa^\nu_i(\tau) \epsilon^\mu_\nu(u(p_s))} = \sum_{i=1}^{N} p^\mu_i(\tau) = p^\mu_s,
\]

\[
M_{sys} = P^\mu_T u_\mu(p_s) = -\lambda(\tau) \sum_{i=1}^{N} m_i \sqrt{m_i^2 c^2 + \vec{\kappa}^2_i(\tau)} = \sum_{i=1}^{N} \sqrt{m_i^2 + \vec{\kappa}^2_i(\tau)},
\]

which turn out to be in the correct form only if \( \lambda(\tau) = -1 \). This shows that the agreement with parametrized Minkowski theories on arbitrary spacelike hypersurfaces is obtained only on Wigner hyperplanes in the gauge \( T_s - \tau \approx 0 \), which indeed implies \( \lambda(\tau) = -1 \).
C. The Phase-Space Version of the Standard Energy-Momentum Tensor.

The same result may be obtained by first reformulating the standard energy-momentum in phase space and then by imposing the restriction \( p_i^\mu(\tau) = \sqrt{m_i^2 - \gamma^{rs}(\tau, \bar{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau) l^\mu(\tau, \bar{\eta}_i(\tau)) - \kappa_{ir}(\tau) \gamma^{rs}(\tau, \bar{\eta}_i(\tau)) z_s^\mu(\tau, \bar{\eta}_i(\tau))} \rightarrow \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau) u^\mu(u(p))} + \kappa_i^r e^r_s(u(p)) \) \(^{28}\) [the last equality refers to the Wigner hyperplane, see Eq.(2.14)]:

\[
T^{\mu\nu}(z(\tau, \bar{\sigma})) = \sum_{i=1}^{N} \frac{1}{m_i} \int d\tau_1 \sqrt{\dot{x}_1^2(\tau_1) p_i^\mu(\tau_1) p_i^\nu(\tau_1) \delta^4(x_1(\tau_1) - z(\tau, \bar{\sigma}))} = \\
= \sum_{i=1}^{N} \frac{\sqrt{\dot{x}_1^2(\tau)}}{m_i} p_i^\mu(\tau) p_i^\nu(\tau) \delta^4(\bar{\sigma} - \bar{\eta}_i(\tau)), \\
\Downarrow \text{on Wigner’s hyperplanes} \\
T^{\mu\nu}[x_\beta^s(\tau) + e_\alpha^s(u(p)) \sigma^\alpha] = \sum_{i=1}^{N} \frac{\dot{x}_i^2 + 2 \dot{x}_i \epsilon_i^\alpha(u(p)) \dot{\eta}_i^\alpha - \dot{\bar{\eta}}_i(\tau) p_i^\mu(\tau) p_i^\nu(\tau) \delta^4(\bar{\sigma} - \bar{\eta}_i(\tau))} {m_i \sqrt{\dot{x}_i \cdot u(p)}} \\
= \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \frac{\dot{x}_i^2 + 2 \dot{x}_i \epsilon_i^\alpha(u(p)) \dot{\eta}_i^\alpha - \dot{\bar{\eta}}_i(\tau)} {m_i \sqrt{\dot{x}_i \cdot u(p)}} \\
\left[ (m_i^2 + \bar{\kappa}_i^2(\tau)) u^\mu(p) u^\nu(p) + \right. \\
+ \kappa_i^r(\tau) \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau)} \left( u^\mu(p) e^r_s(u(p)) + u^\nu(p) e^r_s(u(p)) \right) + \\
+ \kappa_i^r(\tau) \kappa_i^s(\tau) e^r_s(u(p)) e^s_s(u(p)) \right] = \\
= \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \frac{\dot{x}_i^2 + 2 \dot{x}_i \epsilon_i^\alpha(u(p)) \dot{\eta}_i^\alpha - \dot{\bar{\eta}}_i(\tau)} {m_i \sqrt{\dot{x}_i \cdot u(p)}} \\
\left[ \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau)} \left( u^\mu(p) u^\nu(p) + \right. \\
+ \kappa_i^r(\tau) \left( u^\mu(p) e^r_s(u(p)) + u^\nu(p) e^r_s(u(p)) \right) + \\
+ \kappa_i^r(\tau) \kappa_i^s(\tau) e^r_s(u(p)) e^s_s(u(p)) \right]. \quad (3.13)
\]

Now the total 4-momentum and total mass are

\[
P_T^\mu = \int d^3\sigma T^{\mu\nu}[x_\beta^s(\tau) + e_\alpha^s(u(p)) \sigma^\alpha] u_\nu(p) = \\
= \sum_{i=1}^{N} \frac{\dot{x}_i^2 + 2 \dot{x}_i \epsilon_i^\alpha(u(p)) \dot{\eta}_i^\alpha - \dot{\bar{\eta}}_i(\tau)} {\dot{x}_i \cdot u(p)} \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau)} \\
\]

\(^{28}\)In this way we are sure to have imposed the restriction to positive energy particles and to have excluded the other \( 2^N - 1 \) branches of the total mass spectrum.
\[
[\sqrt{m_i^2 + \kappa_i^2(\tau)} u^\mu(p_s) + \epsilon_i^\mu(u(p_s))] = \\
\sum_{i=1}^N \frac{\sqrt{\lambda^2(\tau) - [\bar{\eta}_i(\tau) + \bar{\lambda}(\tau)]^2} \sqrt{m_i^2 + \kappa_i^2(\tau)}}{m_i} \lambda^\mu(\tau),
\]

\[
M_{sys} = P_T^\mu u_\mu(p_s) = \sum_{i=1}^N \sqrt{\lambda^2(\tau) - [\bar{\eta}_i(\tau) + \bar{\lambda}(\tau)]^2} \frac{\sqrt{m_i^2 + \kappa_i^2(\tau)}}{m_i} \sqrt{m_i^2 + \kappa_i^2(\tau)}. \tag{3.14}
\]

These equations show that the total 4-momentum evolved from the energy-momentum tensor of the standard theory restricted to positive energy particles is consistent \(^{29}\) with the description on the Wigner hyperplanes with its gauge freedom \(\lambda(\tau), \bar{\lambda}(\tau)\), only by working with the Dirac brackets of the gauge \(T_s \equiv \tau\), where one has \(\lambda(\tau) = -1\) and

\[
\begin{align*}
\dot{x}_s^\mu(\tau) &= u^\mu(p_s) + \epsilon_i^\mu(u(p_s)) \lambda_i(\tau), \\
\dot{x}_s^\mu(\tau) &= x_s^\mu(0) + \tau u^\mu(p_s) + \epsilon_i^\mu(u(p_s)) \int_0^\tau d\tau_1 \lambda_i(\tau_1),
\end{align*}
\]

because \(m_i/\sqrt{1 - [\bar{\eta}_i(\tau) + \bar{\lambda}(\tau)]^2} = \sqrt{m_i^2 c^2 + \kappa_i^2(\tau)}\).

Therefore, for every \(\bar{\lambda}(\tau)\), we get

\[
T^{\mu\nu}[x_s^\beta(T_s) + \epsilon_u^\beta(u(p_s))\sigma^\nu] = \epsilon_A^\mu(u(p_s))\epsilon_B^\nu(u(p_s)) T^{AB}(T_s, \bar{\sigma}) = \\
= \sum_{i=1}^N \bar{\eta}_i(T_s) \left[ \sqrt{m_i^2 + \kappa_i^2(T_s)} u^\mu(p_s) u^\nu(p_s) + k_i^\nu(T_s) (u^\mu(p_s) \epsilon_i^\nu(u(p_s)) + u^\nu(p_s) \epsilon_i^\mu(u(p_s))) + \frac{k_i^\nu(T_s) k_i^\nu(T_s)}{\sqrt{m_i^2 + \kappa_i^2(T_s)}} \epsilon_i^\nu(u(p_s)) \epsilon_i^\mu(u(p_s)) \right],
\]

\[
T^{\tau\tau}(T_s, \bar{\sigma}) = \sum_{i=1}^N \delta^3(\bar{\sigma} - \bar{\eta}_i(T_s)) \left[ \sqrt{m_i^2 + \kappa_i^2(T_s)} \right],
\]

\[
T^{\tau\tau}(T_s, \bar{\sigma}) = \sum_{i=1}^N \delta^3(\bar{\sigma} - \bar{\eta}_i(T_s)) k_i^\tau(T_s),
\]

\[
T^{\tau\rho}(T_s, \bar{\sigma}) = \sum_{i=1}^N \delta^3(\bar{\sigma} - \bar{\eta}_i(T_s)) k_i^\tau(T_s) k_i^\rho(T_s),
\]

\[
P_T^\mu = p_T^\mu = M u^\mu(p_s) + \epsilon_i^\mu(u(p_s)) k_i^\tau \approx M u^\mu(p_s),
\]

\[
M = \sum_{i=1}^N \sqrt{m_i^2 c^2 + \kappa_i^2(T_s)},
\]

\[
T^{\mu\nu}[x_s^\beta(T_s) + \epsilon_u^\beta(u(p_s))\sigma^\nu] u_\nu(p_s) = \epsilon_A^\mu(u(p_s)) T^{A\tau}(T_s, \bar{\sigma}) = \]

\(^{29}\) Namely we get \(P_T^\mu = p_T^\mu\) and \(M_{sys} = \sum_{i=1}^N \sqrt{m_i^2 + \kappa_i^2(\tau)}\).
\[ T^\mu_{\mu} \left[ x^\beta_s(T_s) + \epsilon^\beta_u(p_s) \sigma^u \right] = T^A_A(T_s, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(T_s)) \frac{m_i^2}{\sqrt{m_i^2 + \kappa_i^2(T_s)}}. \]
IV. DIXON’S MULTIPOLES ON THE WIGNER HYPERPLANE.

In this Section we shall define the special relativistic Dixon multipoles on the Wigner hyperplane for the N-body problem\(^{30}\). We list the non-relativistic multipoles for N free particles in Appendix A by comparison.

Let us now consider an arbitrary timelike worldline \(w^\mu(\tau) = z^\mu(\tau, \vec{\eta}(\tau)) = x_s^{(\vec{q} +)}(\tau) + \epsilon^\mu_r(u(p_s))\eta^r(\tau)\) and let us evaluate the Dixon multipoles \([7, 9]\) on the Wigner hyperplanes in the natural gauge with respect to the given worldline. A generic point will be parametrized as

\[
z^\mu(\tau, \vec{\sigma}) = x_s^{(\vec{q} +)}(\tau) + \epsilon^\mu_r(u(p_s))\sigma^r = u^\mu(\tau) + \epsilon^\mu_r(u(p_s))[\sigma^r - \eta^r(\tau)] \equiv u^\mu(\tau) + \delta z^\mu(\tau, \vec{\sigma})
\]

so that \(\delta z^\mu(\tau, \vec{\sigma})u^\mu(p_s) = 0\).

For \(\vec{\eta}(\tau) = 0\) we will get the multipoles with respect to the origin \(x_s^{(\vec{q} +)}(\tau)\).

A. Dixon’s Multipoles.

Lorentz covariant Dixon’s multipoles and their Wigner covariant counterparts on the Wigner hyperplanes are defined as

\[
t_T^{\mu_1...\mu_n\mu}(T_s, \vec{\eta}) = t_T^{(\mu_1...\mu_n)(\mu\nu)}(T_s, \vec{\eta}) = \\
e^{\mu_1}_{\tau_1}(u(p_s))...e^{\mu_n}_{\tau_n}(u(p_s))\epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s))q^{\tau_1...\tau_n AB}(T_s, \vec{\eta}) = \\
= \int d^3\sigma \delta z^{\mu_1}(T_s, \vec{\sigma})...\delta z^{\mu_n}(T_s, \vec{\sigma})T^{\mu\nu}[x_s^{(\vec{q} +)}(T_s) + \epsilon^\beta_u(u(p_s))\sigma^u] = \\
= \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s)) \int d^3\sigma \delta z^{\mu_1}(T_s, \vec{\sigma})...\delta z^{\mu_n}(T_s, \vec{\sigma})T^{AB}(T_s, \vec{\sigma}) = \\
= \epsilon^{\mu_1}_{\tau_1}(u(p_s))...e^{\mu_n}_{\tau_n}(u(p_s)) \\
\left\{[u^\mu(p_s)u^\nu(p_s)\sum_{i=1}^{N}[\eta_i^{\tau_1}(T_s) - \eta^{\tau_1}(T_s)]...[\eta_i^{\tau_n}(T_s) - \eta^{\tau_n}(T_s)]\sqrt{m_i^2 + \kappa_i^2(T_s)} + \\
+ \epsilon^\mu_r(u(p_s))\epsilon^\nu_r(u(p_s)) \\
\sum_{i=1}^{N}[\eta_i^{\tau_1}(T_s) - \eta^{\tau_1}(T_s)]...[\eta_i^{\tau_n}(T_s) - \eta^{\tau_n}(T_s)]\kappa_i^{\tau_1}(T_s)\kappa_i^{\tau_n}(T_s)\right\} + \\
+ [u^\mu(p_s)\epsilon^\nu_r(u(p_s)) + u^\nu(p_s)\epsilon^\mu_r(u(p_s))] + \ldots
\]

\(^{30}\)Previous studies of multipole theories of point particles can be found in Refs. \[26, 27, 31, 33, 35\].

\(^{31}\)See this paper for the previous definitions given by Bielecki, Mathisson and Weyssenhof \[38, 39, 44, 46\].
\[
\sum_{i=1}^{N} [\eta_i^1(T_s) - \eta^1(T_s)] [\eta_i^{\alpha}(T_s) - \eta^{\alpha}(T_s)] \kappa_i^\tau(T_s),
\]

\[
q_T^{r_1...r_n A B}(T_s, \bar{\eta}) = \delta_A^A \delta_B^B \sum_{i=1}^{N} [\eta_i^1(T_s) - \eta^1(T_s)] [\eta_i^{\alpha}(T_s) - \eta^{\alpha}(T_s)] \sqrt{m_i^2 + \kappa_i^2(T_s)} +
\]

\[
+ \delta_A^A \delta_B^B \sum_{i=1}^{N} [\eta_i^1(T_s) - \eta^1(T_s)] [\eta_i^{\alpha}(T_s) - \eta^{\alpha}(T_s)] \frac{\kappa_i^\mu(T_s) \kappa_i^\nu(T_s)}{\sqrt{m_i^2 + \kappa_i^2(T_s)}} +
\]

\[
+ (\delta_A^A \delta_B^B + \delta_A^B \delta_B^A) \sum_{i=1}^{N} [\eta_i^1(T_s) - \eta^1(T_s)] [\eta_i^{\alpha}(T_s) - \eta^{\alpha}(T_s)] \kappa_i^\tau(T_s),
\]

\[
u_{\mu_1}(p_s) t_T^{\mu_1...\mu_n \mu \nu}(T_s, \bar{\eta}) = 0,
\]

\[
t_T^{\mu_1...\mu_n \mu \nu}(T_s, \bar{\eta}) \overset{def}{=} e^{-\tau \eta_{\mu_1}(u(p_s))} ... e^{-\tau \eta_{\mu_n}(u(p_s))} q_T^{r_1...r_n A}(T_s, \bar{\eta}) =
\]

\[
\int d^3 \sigma \delta z^{\alpha_1}(\tau, \bar{\sigma}) ... \delta z^{\alpha_n}(\tau, \bar{\sigma}) T_{\mu \nu} [x^\nu(\bar{\eta}) \eta_{\mu}(T_s) + e^{\mu}(u(p_s)) \sigma^\nu] =
\]

\[
e^{-\tau \eta_{\mu_1}(u(p_s))} ... e^{-\tau \eta_{\mu_n}(u(p_s))}
\]

\[
\sum_{i=1}^{N} [\eta_i^1(T_s) - \eta^1(T_s)] [\eta_i^{\alpha}(T_s) - \eta^{\alpha}(T_s)] \frac{m_i^2}{\sqrt{m_i^2 + \kappa_i^2(T_s)}} =
\]

\[
\tilde{t}_T^{\mu_1...\mu_n \mu \nu}(T_s, \bar{\eta}) = t_T^{\mu_1...\mu_n \mu \nu}(T_s, \bar{\eta}) u_{\mu}(p_s) u_{\nu}(p_s) =
\]

\[
e^{-\tau \eta_{\mu_1}(u(p_s))} ... e^{-\tau \eta_{\mu_n}(u(p_s))} q_T^{r_1...r_n A \tau}(T_s, \bar{\eta}) =
\]

\[
e^{-\tau \eta_{\mu_1}(u(p_s))} ... e^{-\tau \eta_{\mu_n}(u(p_s))}
\]

\[
\sum_{i=1}^{N} [\eta_i^1(T_s) - \eta^1(T_s)] [\eta_i^{\alpha}(T_s) - \eta^{\alpha}(T_s)] \frac{m_i^2}{\sqrt{m_i^2 + \kappa_i^2(T_s)}}.
\]

(4.2)

Related multipoles are

\[
p_T^{\mu_1...\mu_n \mu}(T_s, \bar{\eta}) = t_T^{\mu_1...\mu_n \mu \nu}(T_s, \bar{\eta}) u_{\nu}(p_s) =
\]

\[
e^{-\tau \eta_{\mu_1}(u(p_s))} ... e^{-\tau \eta_{\mu_n}(u(p_s))} e_{\nu}(u(p_s)) q_T^{r_1...r_n A \tau}(T_s, \bar{\eta}) =
\]

\[
e^{-\tau \eta_{\mu_1}(u(p_s))} ... e^{-\tau \eta_{\mu_n}(u(p_s))}
\]

\[
\sum_{i=1}^{N} [\eta_i^1(T_s) - \eta^1(T_s)] [\eta_i^{\alpha}(T_s) - \eta^{\alpha}(T_s)]
\]

\[
\left[ \sqrt{m_i^2 + \kappa_i^2(T_s)} u^{\mu}(p_s) + \kappa_i^\tau(T_s) e_{\tau}(p_s) \right],
\]

\[
u_{\mu_1}(p_s) p_T^{\mu_1...\mu_n \mu}(T_s, \bar{\eta}) = 0,
\]

\[
p_T^{\mu_1...\mu_n \mu}(T_s, \bar{\eta}) u_{\mu}(p_s) = \tilde{t}_T^{\mu_1...\mu_n \mu}(T_s, \bar{\eta}),
\]

\[
n = 0 \Rightarrow p_T^{\mu}(T_s, \bar{\eta}) = e_{\tau}(u(p_s)) q_T^{A \tau}(T_s) = P_T^{\mu} \approx p_s^{\mu}.
\]

(4.3)

The inverse formulas, giving the multipolar expansion, are
\[
T^{\mu\nu}[w^\beta(T_s) + \delta z^\beta(T_s, \vec{\sigma})] = T^{\mu\nu}[x_s^\alpha(T_s) + \epsilon^\beta_r(u(p_s))\sigma^r] = \\
= \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s))T^{AB}(T_s, \vec{\sigma}) = \\
= \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s)) \sum_{n=0}^{\infty} (-1)^n \frac{q^{r_1...r_n}_{T}(T_s, \vec{\eta})}{n!} \\
\partial^n_{\partial\sigma^{r_1}...\partial\sigma^{r_n}} \delta^3(\vec{\sigma} - \vec{\eta}(T_s)) = \\
= \sum_{n=0}^{\infty} (-1)\frac{1}{n!} \epsilon^{\mu_1...\mu_n}_{\eta_1}(T_s, \vec{\eta}) \epsilon_{\sigma_1,\mu_1}(u(p_s))...\epsilon_{\sigma_n,\mu_n}(u(p_s)) \\
\partial^n_{\partial\sigma^{r_1}...\partial\sigma^{r_n}} \delta^3(\vec{\sigma} - \vec{\eta}(T_s)).
\]

The quantities \(q^{r_1...r_n\tau\tau}_{T}(T_s, \vec{\eta})\), \(q^{r_1...r_n\tau\tau}_T(T_s, \vec{\eta}) = q^{r_1...r_n\tau\tau}_T(T_s, \vec{\eta})\), \(q^{r_1...r_n\mu\nu}_T(T_s, \vec{\eta})\) are the mass density, stress tensor and momentum density multipoles with respect to the worldline \(u^\mu(T_s)\) (barycentric for \(\vec{\eta} = 0\)) respectively.

B. Monopoles.

The monopoles correspond to \(n = 0\) \(^{32}\) and have the following expression \(^{33}\).

\[
q^{AB}_T(T_s, \vec{\eta}) = \delta^A_\tau \delta^B_\tau M + \delta^A_\mu \delta^B_\nu \sum_{i=1}^{N} \kappa^u_i \kappa^v_i + (\delta^A_\tau \delta^B_\tau + \delta^A_\mu \delta^B_\nu)\kappa^u_+ \approx \\
\rightarrow_{\alpha \rightarrow \infty} \sum_{i=1}^{N} \frac{1}{m^2_i + \vec{k}^2_i} + \frac{\kappa^u_+ (\infty) \kappa^v_+ (\infty)}{H_i(\infty)} = \\
= \sum_{i=1}^{N} \frac{1}{m^2_i + \vec{k}^2_i} + \sum_{de} \gamma_i \gamma_j \vec{q}_{dq} \cdot \vec{q}_{qe} + \\
\]

\(^{32}\)\(q^{\mu\nu}_T(T_s, \vec{\eta}) = \epsilon^\mu_A(u(p_s))\epsilon^\nu_B(u(p_s))q^{AB}_T(T_s, \vec{\eta}), \; p^{\mu}_T(T_s, \vec{\eta}) = \epsilon^\mu_A(u(p_s))q^{Ar}_T(T_s, \vec{\eta}), \; \vec{\eta}_T(T_s, \vec{\eta}) = q^{\tau\tau}_T(T_s, \vec{\eta}).\)

\(^{33}\)They are \(\vec{\eta}\) independent; see Appendix C of Ref. [1] and Appendix A for the non-relativistic limit.

\(^{34}\)In this Section we use results from Section V of Ref. [1], where the rest-frame (\(\vec{\kappa}_+ = \vec{q}_+ = 0\)) canonical relative variables with respect to the internal 3-center of mass were found by means of a classical Gartenhaus-Schwartz transformation \(^{14}\). This transformation is a sequence of canonical transformations depending on a parameter \(\alpha\). The final rest-frame relative variables are obtained in the limit \(\alpha \rightarrow \infty\). In Eqs.\(^{4.13}\), \(^{4.9}\), \(^{4.13}\), \(^{4.14}\) we use the following symbols from Ref. [1]: \(\vec{\kappa}(\infty) = \sqrt{N} \sum_{a=1}^{N-1} \gamma_\alpha \vec{q}_a, \; \Pi = M_{sys}^2 - \vec{\kappa}_+^2, \; H_i = \sqrt{m^2_i + \vec{\kappa}_i^2}, \; H_i(\infty) = \sqrt{m^2_i + \sum_{ab}^{1,N-1} \gamma_\alpha \gamma_\beta \vec{q}_a \cdot \vec{q}_b}.\) Also the non-relativistic limits, \(c \rightarrow \infty\), are shown.
\[ q^{\tau \tau}_T(T_s, \bar{\eta}) \rightarrow c \rightarrow \infty \sum_{i=1}^{N} m_i c^2 + \frac{1}{2} \sum_{a b} \frac{N \gamma_{ai} \gamma_{bi}}{m_i} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb} + O(1/c) = \]

\[ = \sum_{i=1}^{N} m_i c^2 + H_{rel nr} + O(1/c), \]

\[ q^{\tau}_T(T_s, \bar{\eta}) = \kappa^\tau_+ \approx 0, \text{ rest-frame condition (also at the non-relativistic level)}, \]

\[ q^{u v}_T(T_s, \bar{\eta}) \rightarrow c \rightarrow \infty \sum_{ab} \sum_{i=1}^{N} \frac{N \gamma_{ai} \gamma_{bi}}{m_i} \bar{\pi}_{qa} \bar{\pi}_{qb} + O(1/c) = \]

\[ = \sum_{ab} k_{ab} \bar{\pi}_{qa} \bar{\pi}_{qb} + O(1/c) = \sum_{ab} k_{ab} \rho_a \rho_b + O(1/c), \]

\[ q^{A}_T(T_s, \bar{\eta}) = t^{\mu}_T(T_s, \bar{\eta}) = \sum_{i=1}^{N} \frac{m_i^2}{\sqrt{m_i^2 + k_i^2}} \]

\[ \rightarrow \alpha \rightarrow \infty \sum_{i=1}^{N} \frac{m_i^2}{H_i(\infty)} = \sum_{i=1}^{N} \frac{m_i^2}{ \sqrt{m_i^2 + N \sum_{de} \gamma_{di} \gamma_{ei} \bar{\pi}_{qd} \cdot \bar{\pi}_{qe} }} \]

\[ = \sum_{i=1}^{N} m_i c^2 - \frac{1}{2} \sum_{a b} \sum_{i=1}^{N} \frac{N \gamma_{ai} \gamma_{bi}}{m_i} \bar{\pi}_{qa} \cdot \bar{\pi}_{qb} + O(1/c) = \]

\[ = \sum_{i=1}^{N} m_i c^2 - H_{rel nr} + O(1/c). \quad (4.5) \]

where we used Eqs. (5.10), (5.11) of Ref. [1] to obtain their expression in terms of the internal relative variables.

Therefore, in the rest-frame instant form, the mass monopole is the invariant mass \( M = \sum_{i=1}^{N} \sqrt{m_i^2 + k_i^2} \), while the momentum monopole vanishes.

C. Dipoles.

The dipoles correspond to \( n = 1 \)

\[ q^{r A B}_T(T_s, \bar{\eta}) = \delta^A_r \delta^B_r M[R^+(T_s) - \eta^r(T_s)] + \delta^A_r \delta^B_r \left[ \sum_{i=1}^{N} \eta_i^r \kappa_i^u \kappa^u_i \right] (T_s) - \eta^r(T_s) q^{u v}_{r A B}(T_s, \bar{\eta}) + \]

\[ + (\delta^A_r \delta^B_r + \delta^A_r \delta^B_r) \left[ \sum_{i=1}^{N} \eta_i^r \kappa_i^u \right] (T_s) - \eta^r(T_s) \kappa^u_+ ], \]

\[ \eta^{\mu \nu}_T(T_s, \bar{\eta}) = \frac{\epsilon^{\mu \nu}_r(u(p_s)) \epsilon^{\mu}_A(u(p_s)) \epsilon^{\nu}_A(u(p_s)) q^{r A B}_{T}(T_s, \bar{\eta}), \quad \bar{\mu}^{\mu}_T(T_s, \bar{\eta}) = \frac{\epsilon^{\mu \nu}_r(u(p_s)) q^{r \tau \tau}_{T}(T_s, \bar{\eta}), \quad p^{\mu \nu}_T(T_s, \bar{\eta}) = \epsilon^{\mu}_r(u(p_s)) e^{\mu}_A(u(p_s)) q^{r A \tau r}_{T}(T_s, \bar{\eta}). \]
\[ q^{\tau A}_{T}(T_s, \vec{\eta}) = e_{\mu 1} \left( u(p_s) \right) t^\mu_{\mu} \mu (T_s, \vec{\eta}) = \sum_{i=1}^{N} \frac{[\eta_i^r - \eta_i^\mu] m_i^2}{\sqrt{m_i^2 + \kappa_i^2}} (T_s). \]  

(4.6)

The vanishing of the mass dipole identifies the internal Møller center of energy \( \vec{R}_+ \approx \vec{q}_+ \approx \vec{y}_+ \) and therefore the rest-frame internal center of mass \( \vec{q}_+ \): 

\[ q^{\tau\tau}_{T}(T_s, \vec{\eta}) = e_{\mu 1} \left( u(p_s) \right) t^\mu_{\mu} \mu (T_s, \vec{\eta}) = 0, \quad \Rightarrow \vec{\eta}(T_s) = \vec{R}_+ = 0 \quad \text{for} \quad x^\mu_a = x_a^{(\vec{q}_+)\mu}. \]  

(4.7)

The time derivative of the mass dipole identifies the center-of-mass momentum-velocity relation for the system when \( \vec{\eta} = 0 \)

\[ \frac{dq^{\tau\tau}_{T}(T_s, \vec{\eta})}{dT_s} \approx \vec{\kappa}_+ - M \vec{\eta}^r (T_s) \rightarrow \vec{\eta} \rightarrow 0. \]  

(4.8)

The expression of the dipoles in terms of the internal relative variables when \( \vec{\eta} = \vec{R}_+ = \vec{q}_+ = 0 \) is obtained by using Eqs. (5.10), (5.11), (5.22), (5.25) of Ref.
\[ \rightarrow a \to \infty \sum_{a=1}^{N-1} \left( \frac{1}{\sqrt{N}} \sum_{ij} (\gamma_{ai} - \gamma_{aj}) \frac{\sqrt{m_i^2 + N \sum_{de} \gamma_{dj} \gamma_{ej} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}}}{\sqrt{m_i^2 + N \sum_{de} \gamma_{di} \gamma_{ei} \vec{\pi}_{qd} \cdot \vec{\pi}_{qe}}} \right) \rho_a^r \nonumber \]

\[ \rightarrow c \to \infty \sum_{ij} \sum_{a=1}^{1 \ldots N} \sqrt{N} (\gamma_{ai} - \gamma_{aj}) \rho_a^r \frac{m_i N}{m_i m} \sum_{be} \gamma_{ba} \gamma_{ci} \vec{\pi}_{qb} \cdot \vec{\pi}_{qc} + O(1/c) = \nonumber \]

\[ = \sqrt{N} \sum_{abc} \left[ N \sum_{i=1}^{1 \ldots N-1} \frac{\gamma_{ai} \gamma_{ba} \gamma_{ci}}{m_i} - \frac{\sum_{j=1}^{1 \ldots N-1} m_j \gamma_{aj}}{m} \right] \rho_a^r \vec{\pi}_{qb} \cdot \vec{\pi}_{qc} + O(1/c). \quad (4.9) \]

The antisymmetric part of the related dipole \( p_T^{\mu \nu}(T_s, \vec{\eta}) \) identifies the spin tensor. Indeed, the spin dipole \(^3_{\text{spin}}\) is

\[ S_T^{\mu \nu}(T_s)[\vec{\eta}] = 2p_T^{[\mu \nu]}(T_s, \vec{\eta}) = \nonumber \]

\[ = M[R^s(T_s) - \eta^s(T_s)] \left[ \epsilon^{\mu}_{\nu}(u(p_s)) u^\nu(p_s) - \epsilon^\nu_{\nu}(u(p_s)) u^\mu(p_s) \right] + \nonumber \]

\[ + \sum_{i=1}^N \left( \eta^s_i(T_s) \eta^s(T_s) \right) \left[ \epsilon^{\mu}_{\nu}(u(p_s)) \epsilon^\nu_{\nu}(u(p_s)) - \epsilon^\nu_{\nu}(u(p_s)) \epsilon^\mu_{\nu}(u(p_s)) \right], \nonumber \]

\[ \downarrow \]

\[ S_T^{\mu \nu}(T_s)[\vec{\eta} = 0] = S_T^{\mu \nu} \equiv \nonumber \]

\[ \sum_{i=1}^N \left( \frac{m_i \eta^s_i(T_s)}{1 - \vec{\eta}^2_i(T_s)} \right) \left[ \epsilon^{\mu}_{\nu}(u(p_s)) u^\nu(p_s) - \epsilon^\nu_{\nu}(u(p_s)) u^\mu(p_s) \right] + \nonumber \]

\[ + \sum_{i=1}^N \left( \frac{m_i \eta^s_i(T_s) \eta^s_i(T_s)}{1 - \vec{\eta}^2_i(T_s)} \right) \left[ \epsilon^{\mu}_{\nu}(u(p_s)) \epsilon^\nu_{\nu}(u(p_s)) - \epsilon^\nu_{\nu}(u(p_s)) \epsilon^\mu_{\nu}(u(p_s)) \right], \nonumber \]

\[ m_{u(p_s)}(T_s, \vec{\eta}) = u\mu(p_s) S_T^{\mu \nu}(T_s)[\vec{\eta}] = -\epsilon^{\nu}_{\nu}(u(p_s)) [S_T^{\mu \nu} - M \eta^s(T_s)] = \nonumber \]

\[ = -\epsilon^{\nu}_{\nu}(u(p_s)) M[R^s(T_s) - \eta^s(T_s)] = -\epsilon^{\nu}_{\nu}(u(p_s)) q_T^{s \nu}(T_s, \vec{\eta}), \nonumber \]

\[ \Rightarrow \quad u\mu(p_s) S_T^{\mu \nu}(T_s)[\vec{\eta}] = 0, \quad \Rightarrow \vec{\eta} = 0. \quad (4.10) \]

This explains why \( m_{u(p_s)}(T_s, \vec{\eta}) \) is also called the mass dipole moment.

Therefore, \( x_T^{(q)}(T_s) \) is also simultaneously the Tulczyjew centroid \(^{36}\) and, due to \( j_T^{(q)}(T_s) = u\mu(p_s) \), also the Pirani centroid \(^{37}\). Usually, in absence of a relation

\[ 36\text{See Ref.} \quad [17] \text{for the ambiguous definitions of} \quad p_T^{\mu}, \quad S_T^{\mu \nu} \text{given by Papapetrou, Urich and Papapetrou, and by B.Tulczyjew and W.Tulczyjew} \quad [38,39]. \]

\[ 37\text{Defined by} \quad S_T^{\mu \nu}(T_s)[\vec{\eta}] u\nu(p_s) = 0, \text{namely with} \quad S_T^{\mu \nu} = 0 \text{in the momentum rest frame.} \]

\[ 38\text{Defined by} \quad S_T^{\mu \nu}(T_s)[\vec{\eta}] j_T^{s \nu} = 0, \text{namely with} \quad S_T^{\mu \nu} = 0 \text{in the instantaneous velocity rest frame.} \]
between 4-momentum and 4-velocity they are different centroids \[.\]

Let us remark that non-covariant centroids could also be connected with the non-covariant external center of mass \(\vec{x}_n\) and the non-covariant external Möller center of energy.

### D. Quadrupoles.

The *quadrupoles* correspond to \(n = 2\) \[39\]:

\[
q_{T}^{r_1r_2AB}(T_s, \eta) = \delta^A \delta^B \sum_{i=1}^{N} [\eta_{i}^{r_1}(T_s) - \eta^{r_1}(T_s)] [\eta_{i}^{r_2}(T_s) - \eta^{r_2}(T_s)] \sqrt{m_i^2 + \kappa_i^2(T_s)} +
\]

\[
+ \delta^A \delta^B \sum_{i=1}^{N} [\eta_{i}^{r_1}(T_s) - \eta^{r_1}(T_s)] [\eta_{i}^{r_2}(T_s) - \eta^{r_2}(T_s)] \frac{\kappa_i^u \kappa_i^v}{\sqrt{m_i^2 + \kappa_i^2(T_s)}} +
\]

\[
+ (\delta^A \delta^B + \delta^A \delta^D) \sum_{i=1}^{N} [\eta_{i}^{r_1}(T_s) - \eta^{r_1}(T_s)] [\eta_{i}^{r_2}(T_s) - \eta^{r_2}(T_s)] \kappa_i^n(T_s),
\]

\[
q_{T}^{r_1r_2\tau\tau}(T_s, \vec{R}_+) = \sum_{i=1}^{N} (\eta_{i}^{r_1} - R_{i}^{r_1})(\eta_{i}^{r_2} - R_{i}^{r_2}) H_i =
\]

\[
= \frac{1}{N} \sum_{ijk} \sum_{ab} (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \rho_{a}^{T_1} \rho_{b}^{T_2} \frac{H_i H_j H_k}{H_M^3},
\]

\[
q_{T}^{r_1r_2\nu\nu}(T_s, \vec{R}_+) = \sum_{i=1}^{N} (\eta_{i}^{r_1} - R_{i}^{r_1})(\eta_{i}^{r_2} - R_{i}^{r_2}) \kappa_i^n =
\]

\[
= \frac{1}{N} \sum_{ijk} \sum_{ab} (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \rho_{a}^{T_1} \rho_{b}^{T_2} \frac{\kappa_i^u H_j H_k}{H_M^3},
\]

\[
q_{T}^{r_1r_2\nu\nu}(T_s, \vec{R}_+) = \sum_{i=1}^{N} (\eta_{i}^{r_1} - R_{i}^{r_1})(\eta_{i}^{r_2} - R_{i}^{r_2}) \frac{\kappa_i^u \kappa_i^v}{H_i} =
\]

\[
= \frac{1}{N} \sum_{ijk} \sum_{ab} (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \rho_{a}^{T_1} \rho_{b}^{T_2} \frac{\kappa_i^u \kappa_i^v H_j H_k}{H_i H_M^3}. \quad (4.11)
\]

Dixon’s definition of *barycentric tensor of inertia* follows the non-relativistic pattern starting from the mass quadrupole

\[
q_{T}^{r_1r_2\tau\tau}(T_s, \eta) = \sum_{i=1}^{N} [\eta_{i}^{r_1} \eta_{i}^{r_2} \sqrt{m_i^2 + \kappa_i^2}] (T_s) + M[\eta^{r_1} \eta^{r_2} - \eta^{r_1} R_{i}^{r_2} - \eta^{r_2} R_{i}^{r_1}] (T_s), \quad (4.12)
\]

\[39\]For instance, the so-called *background Corinaldesi-Papapetrou centroid* \[41\] is defined by the condition \(S_{T}^{\mu \nu}(T_s)[\eta_{i}^{r_1} v_{\nu} = 0\), where \(v^\mu\) is a given fixed unit 4-vector.

\[40\]The \(T_{T}^{\mu_1 \mu_2 \nu} = e_{r_1}^{\mu_1}(u(p_s)) e_{r_2}^{\mu_2}(u(p_s)) e_{A}^{\nu}(u(p_s)) q_{T}^{r_1 r_2 A B}, \quad T_{T}^{\mu_1 \mu_2 \nu} = e_{r_1}^{\mu_1}(u(p_s)) e_{r_2}^{\mu_2}(u(p_s)) q_{T}^{r_1 r_2 A B}, \quad T_{T}^{\mu_1 \mu_2 \nu} = e_{r_1}^{\mu_1}(u(p_s)) e_{r_2}^{\mu_2}(u(p_s)) e_{A}^{\nu}(u(p_s)) q_{T}^{r_1 r_2 A B}.

\[41\]For instance, the so-called background Corinaldesi-Papapetrou centroid \[41\] is defined by the condition \(S_{T}^{\mu \nu}(T_s)[\eta_{i}^{r_1} v_{\nu} = 0\), where \(v^\mu\) is a given fixed unit 4-vector.
\[ I_{\text{thorne}}^{r1r2}(T_s) = \delta^{r_1r_2} \sum_u q_T^{uA}_r(T_s, 0) - q_T^{r1r2A}_r(T_s, 0) = \]

\[ = \sum_{i=1}^N [(\delta^{r_1r_2}\eta_i^2 - \eta_i r_1r_2)\sqrt{m_i^2 + k_i^2}](T_s) \]

\[ \rightarrow_{\gamma \rightarrow \infty} \frac{1}{N} \sum_{ijk} \sum_{ab} (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \]

\[ [\bar{\rho}_{qa} \cdot \bar{\rho}_{qb}\delta^{r_1r_2} - \rho_{qa}\rho_{qb}] \frac{H_i(\infty)H_j(\infty)H_k(\infty)}{\Pi} = \]

\[ = \frac{1}{N} \sum_{ijk} \sum_{ab} (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \]

\[ \sqrt{m_i^2 + N \sum_{de} \gamma_{di}\pi_{qd} \cdot \pi_{qe}} \]

\[ (\sum_{h=1}^N \sqrt{m_i^2 + N \sum_{de} \gamma_{dh} \pi_{qd} \cdot \pi_{qe}})^2 \]

\[ \left[ [\bar{\rho}_{qa} \cdot \bar{\rho}_{qb}\delta^{r_1r_2} - \rho_{qa}\rho_{qb}] \right] \]

\[ -\gamma \rightarrow \infty \sum_{ab} \sum_{ijkl} m_m m_j m_k (\gamma_{ai} - \gamma_{aj})(\gamma_{bi} - \gamma_{bk}) \bar{\rho}_{qa} \cdot \bar{\rho}_{qb}\delta^{r_1r_2} - \rho_{qa}\rho_{qb} \] \times

\[ \left[ 1 + \frac{1}{c} \left( \frac{N \sum_{cd}^{1N-1} \gamma_{cd} \pi_{qc} \cdot \pi_{qd}}{2m_i^2} + \frac{N \sum_{cd}^{1N-1} \gamma_{cd} \gamma_{de} \pi_{qc} \cdot \pi_{qd}}{2m_j^2} \right) \right] = \]

\[ = 1.0 \sum_{ab} k_{ab} \bar{\rho}_{qa} \cdot \bar{\rho}_{qb}\delta^{r_1r_2} - \rho_{qa}\rho_{qb} + O(1/c) = \]

\[ I^{r1r2}[q_{rn}] + O(1/c). \quad (4.13) \]

In the non-relativistic limit we recover the tensor of inertia of Eqs. (4.11). On the other hand, Thorne’s definition of barycentric tensor of inertia [12] is
where for the \( \partial \) the \( \partial T^{\mu \nu} \not= 0 \) imply the Papapetrou-Dixon-Souriau equations of motion for the total momentum \( P^{\mu}_{T}(T_{s}) = \epsilon_{A}^{\mu}(u(p_{s}))q^{\mu}e_{T}(T_{s}) \approx p_{s}^{\mu} \) and the spin tensor \( S_{T}^{\mu \nu}(T_{s})[\bar{\eta} = 0] \) restricted to positive energy particles [see Eqs. (5.4) and (5.7)].
\[
\frac{d P^\mu_T(T_s)}{dT_s} = 0, \\
\frac{d S^{\mu
u}_T(T_s)[\vec{\eta} = 0]}{dT_s} = 2 P^\mu_T(T_s) u^\nu(p_s) = 2 \kappa_+ \epsilon_{\mu u}^\nu(u(p_s)) u^\nu(p_s) \approx 0. 
\] (4.16)
V. MORE ON DIXON’S MULTIPOLES.

In this Section we shall consider multipoles with respect to the origin, i.e. with \( \vec{r} = 0 \) [we use the notation \( t_T^{\mu_1...\mu_n\nu}(T_s, 0) = t_T^{\mu_1...\mu_n\nu}(T_s) \)].

As shown in Ref. [17], if a field has a compact support \( W \) on the Wigner hyperplanes \( \Sigma_{W} \) and if \( f(x) \) is a \( C^\infty \) complex-valued scalar function on Minkowski spacetime with compact support \( \Sigma \), we have

\[
<T^{\mu\nu}, f> = \int d^4x T^{\mu\nu}(x) f(x) = \int dT_s \int d^3 \sigma f(x_s + \delta x_s) T^{\mu\nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\phi] = \int dT_s \int d^3 \sigma \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot [x_s(T_s) + \delta x_s(\vec{\sigma})]} T^{\mu\nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\phi] = \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot x_s(T_s)} \int d^3 \sigma T^{\mu\nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\phi] = \int T_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot x_s(T_s)} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{\mu_1}...k_{\mu_n} t_T^{\mu_1...\mu_n\nu}(T_s), \tag{5.1}
\]

and, but only for \( f(x) \) analytic in \( W [17, 17] \), we get

\[
<T^{\mu\nu}, f> = \int dT_s \sum_{n=0}^{\infty} \frac{1}{n!} t_T^{\mu_1...\mu_n\nu}(T_s) \frac{\partial^n f(x)}{\partial x^{\mu_1}...\partial x^{\mu_n} |_{x=x_s(T_s)}}, \tag{5.2}
\]

For a \( N \) particle system this equation may be rewritten as Eq.(4.4).

For non-analytic functions \( f(x) \) we have

\[
<T^{\mu\nu}, f> = \int dT_s \sum_{n=0}^{N} \frac{1}{n!} t_T^{\mu_1...\mu_n\nu}(T_s) \frac{\partial^n f(x)}{\partial x^{\mu_1}...\partial x^{\mu_n} |_{x=x_s(T_s)}} + \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot x_s(T_s)} \sum_{n=N+1}^{\infty} \frac{(-i)^n}{n!} k_{\mu_1}...k_{\mu_n} t_T^{\mu_1...\mu_n\nu}(T_s), \tag{5.3}
\]

and, as shown in Ref. [17], from the knowledge of the moments \( t_T^{\mu_1...\mu_n\nu}(T_s) \) for all \( n > N \) we can get \( T^{\mu\nu}(x) \) and, therefore, all the moments with \( n \leq N \).

\[41\]So that its Fourier transform \( \tilde{f}(k) = \int d^4x f(x)e^{ik\cdot x} \) is a slowly increasing entire analytic function on Minkowski spacetime \(|(x^0 + iy)^n(x^3 + iy^3)^p f(x^\mu + iy^\mu)| < C_{q_0...q_3} a_0 |y|^{q_0} + ... + a_3 |y|^3 | a_\mu > 0, q_\mu \) positive integers for every \( \mu \) and \( C_{q_0...q_3} > 0 \), whose inverse is \( f(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k)e^{-ik\cdot x} \).

\[42\]See this paper for related results of Mathisson and Tulczyjew \([36,31,37,26]\).
The Hamilton equations imply \(^{43}\) imply (we omit the dependence on \(\bar{\eta} = 0\) of the multipoles)

\[
\frac{dp_T^{\mu}(T_s)}{dT_s} = 0, \quad \text{for } n = 0,
\]

\[
\frac{dp_T^{(\mu_1\cdots\mu_n\nu)}(T_s)}{dT_s} = -nu^{(\mu_1}(p_s))p_T^{(\mu_2\cdots\mu_n)\nu}(T_s) + nt^{(\mu_1\cdots\mu_n)\nu}(T_s), \quad n \geq 1. \tag{5.4}
\]

If we define for \(n \geq 1\) \([S_T^\mu = 2p_T^{[\mu\nu]} = 2c_T^\mu]\)

\[
b_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = p_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = \epsilon_T^{(\mu_1}(u(p_s))\cdots\epsilon_T^{\mu_n}(u(p_s))\epsilon_A^\nu(u(p_s))q_T^{(\nu)\cdots\nu}A^\tau(T_s),
\]

\[
\epsilon_T^{\nu_1}(u(p_s))\cdots\epsilon_T^{\nu_n}(u(p_s))b_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = \frac{1}{n + 1}u^{\mu}(p_s)q_T^{(\nu)\cdots\nu\tau\tau}(T_s) + \epsilon_T^{\nu}(u(p_s))q_T^{(\nu)\cdots\nu\tau\tau}(T_s),
\]

\[
e_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = \epsilon_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = p_T^{(\mu_1\cdots\mu_n\nu)}(T_s) - p_T^{(\mu_1\cdots\mu_n\nu)}(T_s) =
\]

\[
[\epsilon_T^{\nu_1}(u(p_s))\cdots\epsilon_T^{\nu_n}(u(p_s))\epsilon_A^\nu(u(p_s)) -
\]

\[
- \epsilon_T^{(\nu_1}(u(p_s))\cdots\epsilon_T^{\nu_n}(u(p_s))\epsilon_A^\nu(u(p_s))\epsilon_A^\nu(u(p_s))]q_T^{(\nu)\cdots\nu\tau\tau}(T_s),
\]

\[
\epsilon_T^{\nu_1}(u(p_s))\cdots\epsilon_T^{\nu_n}(u(p_s))c_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = \frac{n}{n + 1}u^{\mu}(p_s)q_T^{(\nu)\cdots\nu\tau\tau}(T_s) +
\]

\[
+ \epsilon_T^{\nu}(u(p_s))[q_T^{(\nu)\cdots\nu\tau\tau}(T_s) - q_T^{(\nu)\cdots\nu\tau\tau}(T_s)], \tag{5.5}
\]

and then for \(n \geq 2\)

\[
d_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = d_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = t_T^{(\mu_1\cdots\mu_n\nu)}(T_s) -
\]

\[
- \frac{n + 1}{n}t_T^{(\mu_1\cdots\mu_n\nu)}(T_s) + t_T^{(\mu_1\cdots\mu_n\nu)}(T_s) +
\]

\[
+ \frac{n + 2}{n}t_T^{(\mu_1\cdots\mu_n\nu)}(T_s) =
\]

\[
[\epsilon_T^{\nu_1}\epsilon_R^{\nu_n}\epsilon_A^\nu\epsilon_B^\nu - \frac{n + 1}{n}(\epsilon_T^{\nu_1}\epsilon_R^{\nu_n}\epsilon_A^\nu)\epsilon_B^\nu +
\]

\[
+ \epsilon_T^{(\mu_1\cdots\mu_n\nu)}\epsilon_A^\nu\epsilon_B^\nu] + \frac{n + 2}{n}c_T^{(\mu_1\cdots\mu_n\nu)}(T_s),
\]

\[
d_T^{(\mu_1\cdots\mu_n\nu)}(T_s) = 0,
\]

\(^{43}\)In Ref. \([7]\) this is a consequence of \(\partial_\mu T^{\nu\mu} \equiv 0\).
\[ t^{\mu\nu}_1(T_s) = t^{(\mu\nu)}_T(T_s) \triangleq p^\mu_T(T_s)u^\nu(p_s) + \frac{1}{2} \frac{d}{dT_s}(S^{\mu\nu}_T(T_s) + 2b_T^{\mu\nu}(T_s)), \]

\[ \downarrow \]

\[ t^{\mu\nu}_T(T_s) \triangleq p^\mu_T(T_s)u^\nu(p_s) + \frac{d}{dT_s}b_T^{\mu\nu}(T_s) = Mu^\mu(p_s)u^\nu(p_s) + \kappa_+[\epsilon^\mu(u(p_s))\epsilon^\nu(u(p_s))] + \epsilon^\mu(u(p_s))\epsilon^\nu(u(p_s)) \sum_{i=1}^N \frac{n_i^\mu n_i^\nu}{\sqrt{m_i^2 + \kappa_i^2}}, \]

\[ \frac{d}{dT_s}S^{\mu\nu}_T(T_s) \triangleq 2p_T^{(\mu)}u^{(\nu)}(p_s) = 2\kappa^\tau\epsilon^\mu(u(p_s))u^{\nu}(p_s) \approx 0, \]

\[ 2t^{(\mu\nu)}_T(T_s) \triangleq 2u^{(\mu)}(p_s)b_T^{(\nu)}(T_s) + u^{(\rho)}(p_s)S^{(\mu\nu)}_T(T_s) + \frac{d}{dT_s}(t^{(\mu\nu)}_T(T_s) + c^{(\mu\nu)}_T(T_s)), \]

\[ \downarrow \]

\[ t^{(\mu\nu)}_T(T_s) \triangleq u^{(\mu)}(p_s)b_T^{(\nu)}(T_s) + S^{(\mu\nu)}_T(T_s)u^{\nu}(p_s) + \frac{d}{dT_s}(\frac{1}{2}b_T^{(\mu\nu)}(T_s) - c_T^{(\mu\nu)}(T_s)), \]

\[ 3) \quad n \geq 3 \]

\[ t^{\mu_1\cdots\mu_n\nu}(T_s) \triangleq \frac{d^{\mu_1\cdots\mu_n\nu}}{dT_s}(T_s) + u^{(\mu_1)}(p_s)b_T^{(\mu_2\cdots\mu_n)(\nu)}(T_s) + 2u^{(\mu_1)}(p_s)e_T^{(\mu_2\cdots\mu_n)(\nu)}(T_s) + \frac{2}{n}c_T^{(\mu_1\cdots\mu_n)(\nu)}(T_s), \]

This allows to rewrite \( T^{\mu\nu}, f > \) in the following form

\[ < T^{\mu\nu}, f > = \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k)e^{-ik\cdot x_s(T_s)} \left[ u^{(\mu)}(p_s)p_T^{(\nu)}(T_s) - ik_{\rho}S^{(\rho\mu)}_T(T_s)u^{(\nu)}(p_s) + \sum_{n=2}^{\infty} \frac{(-i)^n}{n!} k_{p_1} \cdots k_{p_n} I^{(\mu_1\cdots\mu_n\nu)}_T(T_s) \right], \]
with

\[ I_T^{\mu_1...\mu_n\nu}(T_s) = I_T^{(\mu_1...\mu_n)(\nu)}(T_s) = \frac{1}{2} I_T^{\mu_1...\mu_n\nu}(T_s) - \frac{2}{n-1} u^{(\mu_1} (p_s) e_T^{\mu_2...\mu_n)(\nu)}(T_s) + \]
\[ + \frac{2}{n} e_T^{\mu_1...\mu_n}(\mu(T_s) u^{(\nu)}(p_s) = \]
\[ = \left[ \epsilon_{\mu_1}^{\nu_1} \epsilon_{\mu_2}^{\nu_2} ... \epsilon_{\mu_n}^{\nu_n} \right] - \frac{1}{n} \left[ \epsilon_{\nu_1}^{\nu_1} \epsilon_{\nu_2}^{\nu_2} ... \epsilon_{\nu_n}^{\nu_n} \right] - \frac{2}{n} e_{T}^{\mu_1...\nu_n} u^{(\mu(T_s) u^{(\nu)}(p_s))} \]
\[ q_{T}^{r_1...r_n AB}(T_s) - \]
\[ - \left[ \frac{2}{n-1} u^{(\mu_1} (p_s) \left( \epsilon_{\nu_1}^{\nu_1} \epsilon_{\nu_2}^{\nu_2} ... \epsilon_{\nu_n}^{\nu_n} \right) - \right. \]
\[ - \frac{2}{n} \left( \epsilon_{\nu_1}^{\nu_1} \epsilon_{\nu_2}^{\nu_2} ... \epsilon_{\nu_n}^{\nu_n} \right) \right] (u(p_s)) \]
\[ q_{T}^{r_1...r_n AB}(T_s), \]
\[ I_T^{(\mu_1...\mu_n\nu)}(T_s) = 0, \]
\[ e_{\mu_1}^{\nu_1} (u(p_s)) ... e_{\mu_n}^{\nu_n} (u(p_s)) I_T^{\mu_1...\mu_n\nu}(T_s) = \frac{n+3}{n+1} u^{(\mu_1} (p_s) u^{(\nu)}(p_s) q_{T}^{r_1...r_n ... \nu_1}(T_s) + \]
\[ + \frac{1}{n} [u^{(\mu_1} (p_s) e_{r_1}^{\nu_1}(u(p_s)) + u^{(\nu)}(p_s) e_{r_1}^{\mu_1}(u(p_s))] q_{T}^{r_1...r_n ... \nu_1}(T_s) + \]
\[ + \epsilon_{r_1}^{\nu_1}(u(p_s)) \epsilon_{r_2}^{\nu_2}(u(p_s)) q_{T}^{r_1...r_n ... \nu_1}(T_s) - \]
\[ - \frac{n+1}{n} q_{T}^{r_1...r_n ... \nu_1}(T_s) + q_{T}^{r_1...r_n ... \nu_1}(T_s) + \]
\[ + q_{T}^{r_1...r_n ... \nu_1}(T_s) \right]. \quad (5.9) \]

For a N particle system Eq. (5.8) implies Eq. (4.15). Finally, a set of multipoles equivalent to the \( I_T^{\mu_1...\mu_n\nu} \) is 44

for \( n \geq 0 \)

\[ J_T^{\mu_1...\mu_n\nu\rho\sigma}(T_s) = J_T^{(\mu_1...\mu_n)[\nu\rho\sigma]}(T_s) = I_T^{\mu_1...\mu_n[\nu\rho\sigma]}(T_s) = \]
\[ = \frac{1}{n+1} [u^{(\mu_1} (p_s) e_T^{\nu_1...\nu_n}[\rho\sigma]}(T_s) + \]
\[ + u^{[\rho}(p_s) e_T^{\sigma]}^{[\mu_1...\mu_n]}(T_s)] = \]
\[ \left[ \epsilon_{\mu_1}^{\nu_1} ... \epsilon_{\mu_n}^{\nu_n} \right] e_T^{[\nu \rho \sigma]}(u(p_s)) q_{T}^{r_1...r_n AB}(T_s) - \]

\[ 44 \quad A^{[\mu[\nu\rho\sigma]} \equiv \frac{1}{4} (A^{\mu\nu\rho\sigma} - A^{\nu\mu\rho\sigma} - A^{\mu\sigma\nu\rho} + A^{\nu\sigma\mu\rho}). \]

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We can decompose these Cartesian tensors in their irreducible STF (symmetric trace-free) components:

\[ u_{\mu_1}(p_s)J_{T}^{\mu_1...\mu_n\nu\rho\sigma}(T_s) = J_{T}^{\mu_1...\mu_n-1|\mu_\nu\rho\sigma}(T_s) = 0, \quad \text{for } n \geq 1, \]

\[ I_{T}^{\mu_1...\mu_n\nu}(T_s) = \frac{4(n-1)}{n+1}J_{T}^{\mu_1...\mu_n-1|\mu_\nu}(T_s), \quad \text{for } n \geq 2, \]

\[ \epsilon_{\mu_1}^{\nu_1}(u(p_s))...\epsilon_{\mu_n}^{\nu_n}(u(p_s))J_{T}^{\mu_1...\mu_n\nu\rho\sigma}(T_s) = \left[ \epsilon_{\nu_1}^{\nu_1}\epsilon_{\nu_2}^{\nu_2}|\epsilon_{\nu_3}^{\nu_3}\epsilon_{\nu_4}^{\nu_4}\right] (u(p_s))q_{T}^{\nu_1...r_{n+3}}(T_s) - \]

\[ - \frac{1}{n+1}\left[ u^{\nu_1}(p_s)\epsilon_{\nu_1}^{\nu_1}(u(p_s))\epsilon_{\nu_2}^{\nu_2}(u(p_s))\epsilon_{\nu_3}^{\nu_3}(u(p_s)) + \right. \]

\[ + u^{\nu_1}(p_s)\epsilon_{\nu_1}^{\nu_1}(u(p_s))\epsilon_{\nu_2}^{\nu_2}(u(p_s))\epsilon_{\nu_3}^{\nu_3}(u(p_s)) \right] q_{T}^{\nu_1...r_{n+3}}(T_s). \]

\[ (n + 4)(3n + 5) \quad \text{linearly independent components,} \]

The \( J_{T}^{\mu_1...\mu_n\nu\rho\sigma} \) are the Dixon 2\( ^{n+2} \)-pole inertial moment tensors of the extended system: they (or equivalently the \( I_{T}^{\mu_1...\mu_n\nu\rho\sigma} \)) determine its energy-momentum tensor together with the multipole \( P_{T}^{\mu} \) and the spin dipole \( S_{T}^{\mu\nu} \). The equations \( \partial_{\mu}T^{\mu\nu} = 0 \) are satisfied due to the equations of motion for \( P_{T}^{\mu} \) and \( S_{T}^{\mu\nu} \) without any need of the equations of motion for the \( J_{T}^{\mu_1...\mu_n\nu\rho\sigma} \). When all the multipoles \( J_{T}^{\mu_1...\mu_n\nu\rho\sigma} \) are zero (or negligible) one speaks of a pole-dipole system.

On the Wigner hyperplane the content of these 2\( ^{n+2} \)-pole inertial moment tensors is replaced by the Euclidean Cartesian tensors \( q_{T}^{\nu_1...r_{n+3}}, q_{T}^{\nu_1...r_{n+3}}, q_{T}^{\nu_1...r_{n+3}} \). As shown in Appendix B we can decompose these Cartesian tensors in their irreducible STF (symmetric trace-free) parts (the STF tensors).

The multipolar expansion \( \text{[4.13]} \) may be rewritten as

\[ T^{\mu\nu}[x_s^{(\bar{\sigma}+\beta)}(T_s) + \epsilon_{\mu}^{\nu}(u(p_s))\sigma^\nu] = T^{\mu\nu}[w^{\beta}(T_s) + \epsilon_{\mu}^{\nu}(u(p_s))\sigma^\nu - \eta^\nu(T_s)] = \]

\[ = u^{\nu}(p_s)\epsilon_{A}^{\nu}(u(p_s))\delta_{\nu}^{A}\eta + \delta_{A}^{A}u^{\nu}(p_s)\delta_{\nu}^{A}(p_s) + \]

\[ + \frac{1}{2}S_{T}^{\nu}(T_s)[\bar{\eta}] u^{\nu}(p_s)\epsilon_{\nu}^{\nu}(u(p_s))\frac{\partial}{\partial\sigma^\nu}\delta_{\nu}(\bar{\eta}(T_s)) + \]

\[ + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{n+3}{n+1} u^{\mu}(p_s)u^{\nu}(p_s)q_{T}^{\nu_1...r_{n+3}}(T_s, \bar{\eta}) + \]

\[ + \frac{1}{n} [u^{\mu}(p_s)\epsilon_{\nu}(u(p_s)) + u^{\nu}(p_s)\epsilon_{\nu}(u(p_s))] q_{T}^{\nu_1...r_{n+3}}(T_s, \bar{\eta}) + \]

\[ \text{[45]} \text{The so called Papapetrou-Dixon-Souriau equations given in Eq.\text{[4.16]}.} \]
\[ + \epsilon_{s_1}^\mu (u(p_s)) \epsilon_{s_2}^\nu (u(p_s)) [q_{r_1...r_n s_1 s_2}^s(T_s, \bar{\eta}) -
- \frac{n + 1}{n} (q_{r_1...r_n s_1}^s(T_s, \bar{\eta}) + q_{r_1...r_n s_2}^s (T_s, \bar{\eta})) + q_{r_1...r_n s_1 s_2}^s (T_s, \bar{\eta})] \]

\[ \frac{\partial q}{\partial \sigma^1...\partial \sigma^n} \delta^3 (\bar{\sigma} - \bar{\eta}(T_s)). \] (5.11)
In this paper we have completed our study of the relativistic kinematics of the system of N scalar positive-energy particles in the rest-frame instant form of dynamics on Wigner hyperplanes orthogonal to the system total 4-momentum initiated in Ref. [1].

We have evaluated the energy momentum tensor of the system on the Wigner hyperplane and then determined Dixon’s multipoles for the N-body problem with respect to the internal 3-center of mass located at the origin of the Wigner hyperplane. In the rest-frame instant form of dynamics these multipoles are Cartesian (Wigner-covariant) Euclidean tensors. While the study of the monopole and dipole moments in the rest frame gives informations on the mass, the spin and the internal 3-center of mass, the quadrupole moment give the only (even if not unique) way to introduce the concept of barycentric tensor of inertia for extended systems in special relativity.

Finally, let us observe that, by exploiting the canonical spin bases of Refs. [16,1], after the elimination of the internal 3-center of mass ($\vec{q}_+ = \vec{\kappa}_+ = 0$) the Cartesian multipoles $q^{r_1...r_n}_{AB}$ can be expressed in terms of 6 orientational variables (the spin vector and the three Euler angles identifying the dynamical body frame) and of $6N - 6$ (rotational scalar) shape variables.

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46 So that the origin is also the Fokker-Pryce center of inertia and Pirani and Tulczyjew centroids of the system.
APPENDIX A: NON-RELATIVISTIC MULTIPOLAR EXPANSIONS FOR N FREE PARTICLES.

In the review paper in Ref. [18] there is a study of the Newtonian multipolar expansions for a continuum isentropic distribution of matter characterized by a mass density \( \rho(t, \vec{\sigma}) \), a velocity field \( U^r(t, \vec{\sigma}) \) \(^{47}\) and a stress tensor \( \sigma^{rs}(t, \vec{\sigma}) \). If the system is isolated, the only dynamical equations are the mass conservation and the continuum equations of motion respectively

\[
\frac{\partial \rho(t, \vec{\sigma})}{\partial t} - \frac{\partial \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma})}{\partial \sigma^r} = 0,
\]

\[
\frac{\partial \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma})}{\partial t} - \frac{\partial [\rho U^r U^s - \sigma^{rs}](t, \vec{\sigma})}{\partial \sigma^s} = 0.
\]  

(A1)

We can adapt this description to an isolated system of N particles in the following way. The mass density

\[
\rho(t, \vec{\sigma}) = \sum_{i=1}^{N} m_i \delta^3(\vec{\sigma} - \vec{\eta}_i(t)),
\]  

(A2)

satisfies

\[
\frac{\partial \rho(t, \vec{\sigma})}{\partial t} = - \sum_{i=1}^{N} m_i \dot{\vec{\eta}}_i(t) \cdot \vec{\sigma} \delta^3(\vec{\sigma} - \vec{\eta}_i(t)) \overset{\text{def}}{=} \frac{\partial}{\partial \sigma^r} [\rho U^r](t, \vec{\sigma}),
\]  

(A3)

while the momentum density \(^{48}\)

\[
\rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) = \sum_{i=1}^{N} m_i \dot{\vec{\eta}}_i(t) \delta^3(\vec{\sigma} - \vec{\eta}_i(t)),
\]  

(A4)

The associated constant of motion is the total mass \( m = \sum_{i=1}^{N} \).

If we define a function \( \zeta(\vec{\sigma}, \vec{\eta}_i) \) concentrated in the N points \( \vec{\eta}_i \), \( i=1,..,N \), such that \( \zeta(\vec{\sigma}, \vec{\eta}_i) = 0 \) for \( \vec{\sigma} \neq \vec{\eta}_i \) and \( \zeta(\vec{\eta}_i, \vec{\eta}_j) = \delta_{ij} \) \(^{49}\) then the velocity field associated to N particles becomes

\[
\vec{U}(t, \vec{\sigma}) = \sum_{i=1}^{N} \dot{\vec{\eta}}_i(t) \zeta(\vec{\sigma}, \vec{\eta}_i(t)).
\]  

(A5)

The continuum equations of motion are replaced by

\(^{47}\) \( \rho(t, \vec{\sigma}) \vec{U}(t, \vec{\sigma}) \) is the momentum density.

\(^{48}\) This can be taken as the definitory equation for the velocity field, even if strictly speaking we do not need it in what follows.

\(^{49}\) It is a limiting concept deriving from the characteristic function of a manifold.
\[ \frac{\partial}{\partial t} [\rho(t, \vec{\sigma}) U^r(t, \vec{\sigma})] = \frac{\partial}{\partial \sigma^s} \sum_{i=1}^{N} m_i \dot{\eta}_i^r(t) \eta_i^s(t) \delta^3(\vec{\sigma} - \vec{n}_i(t)) + \sum_{i=1}^{N} m_i \ddot{\eta}_i^r(t) = \frac{\partial}{\partial \sigma^s} [\rho U^r U^s - \sigma^r s](t, \vec{\sigma}). \tag{A6} \]

For a system of free particles we have \( \ddot{\eta}_i(t) = 0 \) so that \( \sigma^r s(t, \vec{\sigma}) = 0 \). If there are inter-particle interactions, they will determine the effective stress tensor.

Let us consider an arbitrary point \( \vec{\eta}(t) \). The multipole moments of the mass density \( \rho \) and momentum density \( \rho \vec{U} \) and of the stress-like density \( \rho U^r U^s \) with respect to the point \( \vec{\eta}(t) \) are defined by setting \( N \geq 0 \)

\[ m^{r_1 \ldots r_n}[\vec{\eta}(t)] = \int d^3[\sigma^{r_1} - \eta^{r_1}(t)] \ldots [\sigma^{r_n} - \eta^{r_n}(t)] \rho(t, \vec{\sigma}) = \]

\[ = \sum_{i=1}^{N} m_i [\eta_i^{r_1}(t) - \eta^{r_1}(t)] \ldots [\eta_i^{r_n}(t) - \eta^{r_n}(t)], \]

\[ \text{with } n = 0 \quad m[\vec{\eta}(t)] = m = \sum_{i=1}^{N} m_i, \]

\[ p^{r_1 \ldots r_n}[\vec{\eta}(t)] = \int d^3[\sigma^{r_1} - \eta^{r_1}(t)] \ldots [\sigma^{r_n} - \eta^{r_n}(t)] \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) = \]

\[ = \sum_{i=1}^{N} m_i \dot{\eta}_i^r(t) [\eta_i^{r_1}(t) - \eta^{r_1}(t)] \ldots [\eta_i^{r_n}(t) - \eta^{r_n}(t)], \]

\[ \text{with } n = 0 \quad p^r[\vec{\eta}(t)] = \sum_{i=1}^{N} m_i \dot{\eta}_i^r(t) = \sum_{i=1}^{N} \dot{\kappa}_i^r = \kappa_+^r \approx 0, \]

\[ p^{r_1 \ldots r_n s}[\vec{\eta}(t)] = \int d^3[\sigma^{r_1} - \eta^{r_1}(t)] \ldots [\sigma^{r_n} - \eta^{r_n}(t)] \rho(t, \vec{\sigma}) U^r(t, \vec{\sigma}) U^s(t, \vec{\sigma}) = \]

\[ = \sum_{i=1}^{N} m_i \dot{\eta}_i^r(t) \dot{\eta}_i^s(t) [\eta_i^{r_1}(t) - \eta^{r_1}(t)] \ldots [\eta_i^{r_n}(t) - \eta^{r_n}(t)]. \tag{A7} \]

The mass monopole is the conserved mass, while the momentum monopole is the total 3-momentum, which vanishes in the rest frame.

The point \( \vec{\eta}(t) \) is the center of mass if the mass dipole vanishes

\[ m^r[\vec{\eta}(t)] = \sum_{i=1}^{N} m_i [\eta_i^r(t) - \eta^r(t)] = 0 \Rightarrow \vec{\eta}(t) = \vec{q}_{nr}. \tag{A8} \]

The time derivative of the mass dipole is

\[ \frac{dm^r[\vec{\eta}(t)]}{dt} = p^r[\vec{\eta}(t)] - m \dot{\eta}^r(t) = \kappa_+^r - m \dot{\eta}^r(t). \tag{A9} \]

When \( \vec{\eta}(t) = \vec{q}_{nr} \), from the vanishing of this time derivative we get the momentum-velocity relation for the center of mass

\[ p^r[\vec{q}_{nr}] = \kappa_+^r = m \dot{\eta}_+^r \ \approx 0 \ \text{in the rest frame}. \tag{A10} \]
so that the barycentric mass quadrupole and tensor of inertia are respectively

\[ m^rs[\vec{q}_\nu] = \sum_{i=1}^{N} m_i \eta^r_i(t) \eta^s_i(t) - m q^r_{nr} q^s_{nr}, \]

\[ I^{rs}[\vec{q}_\nu] = \delta^{rs} \sum_u m^{uu} [\vec{q}_\nu] - m^r[\vec{q}_\nu] = \sum_{i=1}^{N-1} k_{ab} (\bar{\rho}_a \cdot \bar{\rho}_b \delta^{rs} - \rho^r_a \rho^s_b), \]

\[ \Rightarrow m^r[\vec{q}_\nu] = \delta^{rs} \sum_{a,b=1}^{N-1} k_{ab} \bar{\rho}_a \cdot \bar{\rho}_b - I^{rs}[\vec{q}_\nu]. \] (A12)

The antisymmetric part of the barycentric momentum dipole gives rise to the spin vector in the following way

\[ p^r[\vec{q}_\nu] = \sum_{i=1}^{N} m_i \epsilon^{r} \eta^s_i(t) - q^r_{nr} p^s[\vec{q}_\nu] = \sum_{i=1}^{N-1} \eta^r_i(t) \kappa^s_i(t) - q^r_{nr} \kappa^s_{nr}, \]

\[ S^u = \frac{1}{2} \epsilon^{urs} p^r[\vec{q}_\nu] = \sum_{a=1}^{N-1} (\bar{\rho}_a \times \bar{\pi}_a)\nu. \] (A13)

The multipolar expansions of the mass and momentum densities around the point \( \vec{q}(t) \) are

\[ \rho(t, \vec{q}) = \sum_{n=0}^{\infty} \frac{m^{r_1, r_2, \ldots, r_n}[\vec{q}]}{n!} \frac{\partial^n}{\partial \sigma^{r_1} \partial \sigma^{r_2} \ldots \partial \sigma^{r_n}} \delta^3(\vec{q} - \vec{q}(t)), \]

\[ \rho(t, \vec{q})U^r(t, \vec{q}) = \sum_{n=0}^{\infty} \frac{p^{r_1, r_2, \ldots, r_n}[\vec{q}]}{n!} \frac{\partial^n}{\partial \sigma^{r_1} \partial \sigma^{r_2} \ldots \partial \sigma^{r_n}} \delta^3(\vec{q} - \vec{q}(t)). \] (A14)

For the barycentric multipolar expansions we get

\[ \rho(t, \vec{q}) = m \delta^3(\vec{q} - \vec{q}_n) - \frac{1}{2} \left( I^{rs}[\vec{q}_n] - \frac{1}{2} \delta^{rs} \sum_u I^{uu}[\vec{q}_n] \right) \frac{\partial^2}{\partial \sigma^r \partial \sigma^s} \delta^3(\vec{q} - \vec{q}_n) + \]

\[ + \sum_{n=3}^{\infty} \frac{m^{r_1, r_2, \ldots, r_n}[\vec{q}_n]}{n!} \frac{\partial^n}{\partial \sigma^{r_1} \partial \sigma^{r_2} \ldots \partial \sigma^{r_n}} \delta^3(\vec{q} - \vec{q}_n), \]

\[ \rho(t, \vec{q})U^r(t, \vec{q}) = \kappa^r_+ \delta^3(\vec{q} - \vec{q}_n) + \left[ \frac{1}{2} \epsilon^{rsu} S^u + p^{(sr)}[\vec{q}_n] \right] \frac{\partial}{\partial \sigma^s} \delta^3(\vec{q} - \vec{q}_n) + \]

\[ + \sum_{n=2}^{\infty} \frac{p^{r_1, r_2, \ldots, r_n}[\vec{q}_n]}{n!} \frac{\partial^n}{\partial \sigma^{r_1} \partial \sigma^{r_2} \ldots \partial \sigma^{r_n}} \delta^3(\vec{q} - \vec{q}_n). \] (A15)
APPENDIX B: SYMMETRIC TRACE-FREE TENSORS.

In the applications to gravitational radiation [45,46,42,47] one does not use Cartesian tensors but irreducible symmetric trace-free Cartesian tensors (STF tensors). While a Cartesian multipole tensor of rank \( l \) (like the rest-frame Dixon multipoles) on \( \mathbb{R}^3 \) has \( 3^l \) components, of which in general \( \frac{1}{2}(l+1)(l+2) \) are independent, a spherical multipole moment of order \( l \) has only \( 2l+1 \) independent components. Even if spherical multipole moments are preferred in calculations of molecular interactions, spherical harmonics have various disadvantages in numerical calculations: for analytical and numerical calculations the Cartesian moments are often more convenient (see for instance Ref. [48] for the case of the electrostatic potential). Therefore one prefers to use the irreducible Cartesian STF tensors [49,50] (with \( 2^l \) independent components if of rank \( l \)), which are obtained by using Cartesian spherical (or solid) harmonic tensors in place of spherical harmonics.

Given an Euclidean tensor \( A_{k_1\ldots k_l} \) on \( \mathbb{R}^3 \), one defines the completely symmetrized tensor

\[
S_{k_1\ldots k_I} \equiv A_{(k_1\ldots k_I)} = \frac{1}{I!} \sum_{\pi} A_{k_{\pi(1)}\ldots k_{\pi(I)}}.
\]

Then, the associated STF tensor is obtained by removing all traces (\( \lfloor I/2 \rfloor = \text{largest integer} \leq I/2 \)):

\[
A_{k_1\ldots k_I}^{(STF)} = \sum_{n=0}^{\lfloor I/2 \rfloor} a_n \delta(k_1k_2\ldots\delta_2k_{2n-1}k_{2n}S_{k_{2n+1}\ldots k_I})i_1i_2\ldots j_{2n},
\]

\[
a_n \equiv (-1)^n \frac{l!(2l-2n-1)!!}{(l-2n)!(2l-1)!!(2n)!!}.
\]

For instance \( (T_{abc})^{STF} \equiv T_{(abc)} - \frac{1}{5} \left[ \delta_{ab}T_{(iic)} + \delta_{ac}T_{(ibi)} + \delta_{bc}T_{(aii)} \right] \).
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