CORRELATION OF MULTIPLICATIVE FUNCTIONS OVER FUNCTION FIELDS

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ABSTRACT. In this article, we study function field analogs of a result of Kátai [13] on asymptotic behaviour of correlation of multiplicative functions. More precisely, if we set $M_n$ and $P_n$ be the set of all monic polynomials and monic irreducible polynomials of degree $n$ over $\mathbb{F}_q$ respectively then for multiplicative functions $\psi_1, \psi_2 : \mathbb{F}_q[x] \to U$, and $A_1, A_2 \in \mathbb{F}_q[x] \setminus \{0\}$, and $h_1, h_2 \in \mathbb{F}_q[x]$, we obtain asymptotic formula for the following correlation functions

$$\sum_{f \in M_n} \psi_1(A_1 f + h_1)\psi_2(A_2 f + h_2), \quad \text{and} \quad \sum_{P \in P_n} \psi_1(P + h_1)\psi_2(P + h_2)$$

for fixed $q$ and sufficiently large $n$. We also find an asymptotic formula of the first correlation function when $\psi_1$ and $\psi_2$ are so called “Hayes pretentious” multiplicative functions which lead us to deduce a generalized Kátai’s conjecture over function field. We give a new proof Kátai’s conjecture over function fields for a multiplicative function (see Klurman et al. [14] for different proof). We also prove Kátai’s conjecture for pair and triplet of multiplicative functions whose values lies on the unit circle. As a consequence towards probabilistic interpretation, we derive the behaviour of the distribution of the sum of additive functions.

1. Introduction

Consider the polynomial ring $\mathbb{F}_q[x]$ over a field $\mathbb{F}_q$ with $q$ elements. One of the fruitful analogies in number theory is between the integers $\mathbb{Z}$ and the polynomial ring $\mathbb{F}_q[x]$. We will introduce correlation of multiplicative functions over $\mathbb{F}_q[x]$ after highlighting few well known results of correlation of multiplicative functions over the integers. In subsequent sections of the introduction, we will discuss new results in this paper.

1.1. Correlation of multiplicative functions over integers. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Many problem from number theory are connected with asymptotic of the mean

$$M_f(x) := \frac{1}{x} \sum_{n \leq x} f(n).$$

The Distance function. In [11], Granville and Soundararajan defined the “distance” between two multiplicative functions $f, g : \mathbb{N} \to U$ as

$$\mathbb{D}(f, g; y; x) := \left( \sum_{y < p \leq x} \frac{1 - \Re(f(p)g(p))}{p} \right)^{1/2}$$

where $U = \{z \in \mathbb{C} : |z| \leq 1\}$ and in particular $\mathbb{D}(f, g; x) := \mathbb{D}(f, g; 1; x)$. In several instances $\mathbb{D}(f, g; \infty)$ is infinite (for example, $\mathbb{D}(1, \mu; \infty)$ is infinite). However, if $\mathbb{D}(f, g; \infty) < \infty$ then $f$ is said to be $g$-pretentious and the case $\mathbb{D}(f, g; \infty) = \infty$ is known
as \( g \) non-pretentious. The most important property of this distance functions is that it satisfied the following triangle inequality:

\[
D(f, g; x) \leq D(f, h; x) + D(h, g; x)
\]

for any functions \( f, g, h : \mathbb{N} \to \mathbb{U} \). The theory of multiplicative functions get new direction and have been subsequently developed using this new approach called “pretentious approach” in last two decades.

The following theorem of Halász is one of the important theorems related to the asymptotic behaviour of \( M_f(x) \) in terms of distance function as \( x \to \infty \).

**Theorem A** (Halász, 1971) Let \( f : \mathbb{N} \to \mathbb{U} \) be multiplicative. Then

\[
M_f(x) = o(1)
\]

unless there exist \( t \in \mathbb{R} \) such that \( D(f, n^it, \infty) < \infty \) in which case, as \( x \to \infty \) we have

\[
M_f(x) \sim \frac{x^it}{1 + it} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(\sum_{k \geq 0} \frac{f(p^k)p^{-kit}}{p^k}\right) + o(1).
\]

Quantitative improvements of Halász’s theorem have been obtained by several authors (for example [9], [10], [11]). As a natural generalization of Halász’s theorem one would like to find asymptotic behaviour of the following \( k \)-point correlation function

\[
M_x(g_1, \ldots, g_k) := \frac{1}{x} \sum_{n \leq x} g_1(F_1(x)) \cdots g_k(F_k(x)), \quad k \geq 2
\]

where \( g_j \)'s are multiplicative functions with modulus less than or equal to 1 and \( F_j(x) \)'s are polynomials with integer coefficients.

1.1.1. **Non-pretentious world.** If \( g_j = \lambda \), Liouville’s function and \( F_j(x) = x + h_j, j = 1, 2, \ldots, k \) for distinct natural numbers \( h_j \)'s then a famous conjecture of Chowla asserts that \( M_k(x) = o(1) \) as \( x \to \infty \). Chowla’s conjecture remains open for any \( h_1, \ldots, h_k \) with \( k \geq 2 \). On the basis of the breakthrough work of Matomäki and Radziwill [21], Tao [27] proved the following two-point \( (k = 2) \) logarithmic averaged Chowla and Elliott conjecture:

\[
\sum_{x/w(x) < n \leq x} \frac{\lambda(a_1n + b_1)\lambda(a_2n + b_2)}{n} = o(\log w(x)).
\]

In general if \( g_1 \) is “non-pretentious” in the sense that

\[
\inf_{|t| \leq x} D(g_1(n), \chi(n)n^it; x) \to \infty, \quad x \to \infty
\]

for all Dirichlet characters \( \chi \), then

\[
\sum_{x/w(x) < n \leq x} \frac{g_1(a_1n + b_1)g_2(a_2n + b_2)}{n} = o(\log w(x)),
\]

where \( a_1, a_2 \) are natural numbers and \( b_1, b_2 \) are distinct non-negative integers such that \( a_1b_2 - a_2b_1 \neq 0 \), and \( 1 \leq w(x) \leq x \) is an arbitrary function of \( x \) that goes to infinity as \( x \to \infty \).

In recent years, progresses have been made on various averaged forms of Chowla’s conjecture. For instance, Matomäki, Radziwill and Tao [22] established a version of Chowla’s
conjecture where one performs some averaging in the parameters \(h_1, \ldots, h_k\). Also Tao and Teräväinen [29] proved a structure theorem for the logarithmically averaged correlations of multiplicative functions which leads to obtain several new cases of logarithmically averaged Chowla and Elliott conjecture for higher \(k\)-point correlations and in [30], they extend this result to some cases of the unweighted Elliott conjecture at almost all scales.

1.1.2. Pretentious world. Kátaï [13] first studied the asymptotic behaviour of the sum (1) with some assumptions on \(g_j\)’s and \(F_j(x)\)’s are special polynomials but did not provide any error term. Stepanauskas [26] studied the asymptotic formula for sum (1) with explicit error term when \(F_j(x)\)’s are linear polynomials and \(g_j\) are “close” to 1 (which is much stronger condition than “pretend” to 1). In [5], the first author studied the asymptotic behaviour of the sum (1) with explicit error term when \(F_j\)’s are polynomial of degree \(\geq 2\) and \(g_j\)’s are close to 1.

In a fine work [15], Klurman provided an asymptotic formula for the sum (1) for two multiplicative functions (also for the same multiplicative function) \(f, g : \mathbb{N} \to \mathbb{U}\) with \(\mathbb{D}(f(n), n^{it_1}\chi(n), \infty) < \infty\) and \(\mathbb{D}(g(n), n^{it_2}\psi(n), \infty) < \infty\) for some primitive Dirichlet characters \(\chi, \psi\). As an application of this result together with Tao’s theorem [2], Klurman [15] proved Kátaï conjecture that if \(f : \mathbb{N} \to S^1\) is completely multiplicative and the consecutive values of \(f\) are close to each other in the sense that

\[
\sum_{n \leq x} |f(n + 1) - f(n)| = o(x)
\]

then \(f(n) = n^{it}\) for some real number \(t\). In the same article Klurman obtained Erdős-Tao discrepancy problem among several other results.

1.2. Correlation of multiplicative functions over \(\mathbb{F}_q[x]\). Consider the polynomial ring \(\mathbb{F}_q[x]\) over a field with \(q\) elements. Let \(\mathcal{M}_n\) be the set of all monic polynomials of degree \(n\) over \(\mathbb{F}_q\), so that \(|\mathcal{M}_n| = q^n\). Let \(\mathcal{P}_n\) be the set of all monic irreducible polynomials of degree \(n\) over \(\mathbb{F}_q\).

Let \(\psi : \mathcal{M} \to \mathbb{C}\) be a multiplicative function. A central theme is the asymptotic behaviour of the mean

\[
\sigma(n, q; \psi) := \frac{1}{q^n} \sum_{f \in \mathcal{M}_n} \psi(f), \quad \text{as } q^n \to \infty.
\]

There are following three way to study the asymptotic behaviour of sum (3).

a) When \(n \to \infty\) and \(q\) is fixed, which is called “large degree limit”,

b) When \(q \to \infty\) and \(n\) is fixed, which is called “large finite field limit”,

c) When both \(n, q \to \infty\).

In [8], Granville et al. initiated the study of mean values of multiplicative functions over \(\mathbb{F}_q[x]\) by proving the following quantitative analog of Theorem 1 in the large degree limit aspect.

**Theorem B (Granville et al.)** Let \(\psi\) be multiplicative functions on \(\mathcal{M}\) with modulus less than or equal to 1 and \(\sigma(n, q; \psi)\) be as defined (3). Then for all integers \(n \geq 2\), we have

\[
|\sigma(n, q; \psi)| \leq 2(2 + M)e^{-M},
\]
where \( \max_{|z|=1} |\Psi^\perp(z)| := 2ne^{-M} \) and \( \Psi^\perp \) is corresponding power series which has truncated euler product defined in [8] with respect to \( n \).

In the large degree limit, Klurman [14] derived analogs of Wirsing, Halász and Hall’s theorem on \( \mathbb{F}_q[x] \). Motivated by the study of correlation of multiplicative functions over integers one would also like to find the asymptotic behaviour of the following sums:

\[
S_k(n, q) := \sum_{f \in \mathcal{M}_n} \psi_1(A_1f + h_1) \cdots \psi_k(A_k f + h_k)
\]

and

\[
R_k(n, q) := \sum_{P \in \mathcal{P}_n} \psi_1(P + h_1) \cdots \psi_k(P + h_k),
\]

where \( \psi_1, \ldots, \psi_k \) are multiplicative functions on \( \mathcal{M} \) and \( A_j \in \mathbb{F}_q[x] \setminus \{0\} \) and \( h_j \in \mathbb{F}_q[x] \) are fixed polynomials for all \( j = 1, \ldots, k \).

1.2.1. Large finite field aspect. In the large finite field limit, one can obtain much better results what can be done in the case of integers. For example, Carmon and Rudnick [4] proved function field analog of Chowla’s conjecture in the large finite field limit. Also, Bary-Soroker [24] proved the function field analog of the Hardy-Littlewood conjecture over large finite fields.

1.2.2. Large degree aspect. A recent groundbreaking result of Sawin and Shusterman [25] established the Chowla conjecture in function fields in the form

\[
\frac{1}{q^n} \sum_{f \in \mathcal{M}_\leq n} \mu(f + B_1) \cdots \mu(f + B_k) = o(1)
\]

for any \( k \geq 1 \) and any distinct \( B_1, \ldots, B_k \in \mathbb{F}_q[x] \) in the large field case \( q > p^2k^2e^2 \), where \( p = \text{char}(\mathbb{F}_q) \). They used geometric methods to improve on the function field version of the Burgess bound, and showed that, when restricted to certain special subspaces, the Möbius function over \( \mathbb{F}_q[x] \) can be mimicked by Dirichlet characters.

Non-pretentious world in large degree aspect. See section 2.2.4 for the notion of “distance” function over function fields. In contrast with the integer case, a new notion so called “short interval character” (see section 2.2.2) plays a crucial role to obtain the correlation of multiplicative function over function fields which helps to classify wider class of multiplicative functions.

Recently, Klurman et al. [16] proved the following two-point logarithmic averaged Chowla and Elliott conjecture over function fields: Assume that \( \psi_1 \) satisfies the non-pretentious assumption

\[
\min_{M \in \mathcal{M}_\leq W} \min_{\chi \pmod{M}} \min_{\xi \text{ short}} \min_{\theta \in [0,1]} \mathbb{D}^2(\psi_1(P), \chi(P)\xi(P)e^{2\pi i \theta \deg(P)}; N) \to \infty,
\]

as \( N \to \infty \) for every fixed \( W \geq 1 \). Then for any fixed \( B \in \mathbb{F}_q[x] \setminus \{0\} \),

\[
\frac{1}{N} \sum_{f \in \mathcal{M}_\leq N} \frac{\psi_1(f)\psi_2(f + B)}{q^{\deg(f)}} = o(1), \quad \text{as } N \to \infty,
\]

where \( \mathbb{D}(\psi_1, \psi_2; N) \) is defined by (10).
In particular, if \( \psi_1 = \psi_2 = \mu \), where \( \mu : \mathbb{F}_q[t] \to \{-1, 0, +1\} \) is the Möbius function, this result is viewed as two-point logarithmically averaged Chowla’s conjecture in function fields. In the same article, Klurman et al. proved Kátai conjecture over function fields using \( (\mathbb{R}) \) and new argument which is different from the proof in the integer setting. Also, very recently, Klurman et al. \([17]\) studied the Erdős discrepancy problem over function fields.

### 1.3. Pretentious world and main results in large degree aspect

#### 1.3.1. Correlation with constant function

Let \( \psi_j : \mathcal{M} \to \mathbb{U} \) and \( \alpha_j : \mathcal{M} \to \mathbb{C} \) be multiplicative functions such that \( \alpha_j = \mu * \psi_j \) for all \( j = 1, 2 \). For fixed polynomials \( A_j \in \mathbb{F}_q[x] \setminus \{0\} \) and \( h_j \in \mathbb{F}_q[x] \) for all \( j = 1, 2 \) and \( n \geq r \), we define

\[
Q(n) := \prod_{\deg P \leq n} \nu_P \quad \text{and} \quad Q(r, n) = \prod_{r < \deg P \leq n} \nu_P, \tag{7}
\]

\[
Q'(n) := \prod_{\deg P \leq n} \nu'_P \quad \text{and} \quad Q'(r, n) = \prod_{r < \deg P \leq n} \nu'_P, \tag{8}
\]

where

\[
\nu_P := \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1}) \alpha_2(P^{m_2})}{q^{\deg[(P^{m_1}, P^{m_2})]/(A_1 h_2 - A_2 h_1)}} \quad \text{and} \quad \nu'_P := \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1}) \alpha_2(P^{m_2})}{\Phi(P^{m_1}, P^{m_2})}
\]

and the Euler phi function over function field is defined by

\[
\Phi(f) = |f| \prod_{P|f} \left( 1 - \frac{1}{|P|} \right).
\]

The following theorem gives the asymptotic behaviour of \( S_2(n, q) \) with explicit error term in large degree limit in most general situation.

**Theorem 1.** Let \( \psi_1 \) and \( \psi_2 \) be multiplicative functions on \( \mathcal{M} \) with modulus less than or equal to 1. Let \( A_1, A_2 \in \mathbb{F}_q[x] \setminus \{0\} \) and \( h_1, h_2 \in \mathbb{F}_q[x] \) with \( \deg(h_j) < \deg(A_j) \) such that \( (A_1, h_1) = (A_2, h_2) = 1 \) and \( \Delta = A_1 h_2 - A_2 h_1 \neq 0 \). Suppose that \( \gamma := \deg(\Delta) \) and \( A := \max\{d(A_1), d(A_2)\} \). Then there exists a positive absolute constant \( C \) such that for all \( n \geq r \geq \gamma \) and for all \( \frac{1}{2} < \alpha < 1 \), we have

\[
\frac{S_2(n, q)}{q^n} - Q(n) \ll q^{d(A)} (\mathbb{D}(\psi_1, 1; r, n + d(A_1)) + \mathbb{D}(\psi_2, 1; r, n + d(A_2)))
\]

\[
+ q^{(1-2\alpha)n+2(1-\alpha)A} \exp \left( \frac{cq^n r^\gamma}{r} \right) + (rq^n)^{-\frac{1}{2}},
\]

where \( Q(n) \) and \( \mathbb{D}(\psi_j, 1; r, n + d(A_j)) \) are defined as in \( (7) \) and \( (10) \) respectively.

The next theorem gives the asymptotic behaviour of \( R_2(n, q) \) with explicit error term in large degree aspect.

**Theorem 2.** Let \( \psi_1 \) and \( \psi_2 \) be multiplicative functions on \( \mathcal{M} \) with modulus less than or equal to 1. Let \( h_1, h_2 \in \mathbb{F}_q[x] \) such that \( \Delta = h_2 - h_1 \neq 0 \). Suppose that \( \gamma := \deg(\Delta) \).
Then there exists a positive absolute constant $c$ such that for all $n \geq r \geq \gamma$ and for all $
frac{1}{2} < \alpha < 1$, we have

$$
\frac{R_2(n, q)}{|P_n|} - Q'(n) \ll \mathbb{D}(\psi_1, 1; r, n) + \mathbb{D}(\psi_2, 1; r, n) + n^{-A} \exp \left( \frac{c\delta^{2r}}{r} \right) + (rq)^{-\frac{1}{4}}
$$

where $A > 0$ is arbitrary constant and $Q'(n)$ and $\mathbb{D}(\psi_j, 1; r, n)$ are defined as in (8) and (10) respectively.

**Remark 1.** Notice that

$$
v_p = \begin{cases} 
1 - \frac{1}{|P|} \left( \sum_{m=0}^{\infty} \frac{\psi_1(P^m) + \psi_2(P^m)}{|P|^m} \right) - 1 & \text{if } P \nmid \Delta AC \\
1 + \sum_{m=1}^{\infty} \frac{\psi_1(P^m) - \psi_1(P^{m-1})}{|P|^m} & \text{if } P \nmid \Delta, P \nmid A, P|C \\
1 + \sum_{m=1}^{\infty} \frac{\psi_2(P^m) - \psi_2(P^{m-1})}{|P|^m} & \text{if } P \nmid \Delta, P|A, P \nmid C \\
1 & \text{if } P \mid \Delta \text{ but } P \nmid A \text{ and } P \nmid C
\end{cases}
$$

and if $P|\Delta$ but $P \nmid A$ and $P \nmid C$ then

$$
v_p = 1 + \sum_{P \nmid A} \frac{\alpha_1(P)\alpha_2(P)}{|P|^i} + \delta_A \sum_{j>i} \frac{\alpha_1(P^i)\alpha_2(P^j)}{|P|^j} + \delta_C \sum_{j>i} \frac{\alpha_1(P^i)\alpha_2(P^j)}{|P|^j},
$$

where $\delta_f = 0$ when $P|f$ and $\delta_f = 1$ otherwise, and $\Delta_P$ is an integer such that $P^{\Delta_P} \mid \Delta$.

Similar expression can be deduce for $v_p'$.

**Remark 2.** Theorem 1 and Theorem 2 can be extended for $S_k(n, q)$ and $R_k(n, q), k \geq 3$.

1.3.2. Correlation with Dirichlet character.

**Theorem 3.** Let $\psi_1, \psi_2 : \mathcal{M} \to \mathbb{U}$ be multiplicative functions such that $\mathbb{D}(\psi_j, \chi_j e^{\theta_j}; \infty) < \infty$ for some primitive characters $\chi_j$ of conductor $Q_j$ and an angle $\theta_j \in [0, 1]$. Assume that $h_1$ and $h_2$ are fixed polynomials. Then as $n \to \infty$,

$$
\frac{1}{|\mathcal{M}_n|} \sum_{f \in \mathcal{M}_n} \psi_1(f + h_1)\psi_2(f + h_2) = e^{2\pi i (\theta_1 + \theta_2)m} \frac{1}{[[Q_1, Q_2]]} \sum_{f, Q_j : \text{rad}(f) \mid Q_j} \psi_1(f_1)\psi_2(f_2) |f_1, f_2| e^{\theta_1} \left( \frac{f_2}{f_1} \right) e^{\theta_2} \left( \frac{f_1}{f_1, f_2} \right) e_{-(\theta_1 + \theta_2)}([f_1, f_2])
$$

$$
\times \sum_{h([Q_1, Q_2])} \chi_1 \left( h \frac{f_2}{(f_1, f_2)} + D_1 \right) \chi_2 \left( h \frac{f_1}{(f_1, f_2)} + D_2 \right) \prod_{p \in \mathcal{P}} v_p + o(1), \text{ if rad}(Q_1) = \text{rad}(Q_2),
$$

otherwise, the sum in left hand side vanishes, where $v_p$ are defined as in (7) with $\alpha_j = \psi_j \chi_j e^{\theta_j}$, $\Delta = (h_1 - h_2)$, $D_1, D_2$ are polynomials such that $D_1f_1 - D_2f_2 = \Delta$, and the sum runs over all $f_1, f_2$ such that $(f_1, f_2) \mid \Delta$, $\frac{Q_2}{(Q_1, Q_2)} \mid \frac{f_1}{(f_1, f_2)}$ and $\frac{Q_1}{(Q_1, Q_2)} \mid \frac{f_2}{(f_1, f_2)}$.

**Corollary 1.1.** Let $\psi_1, \psi_2 : \mathcal{M} \to \mathbb{U}$ be multiplicative functions such that $\mathbb{D}(\psi_j, \chi_j e^{\theta_j}; \infty) < \infty$ for some primitive characters $\chi_j$ of conductor $Q_j$ and an angle $\theta_j \in [0, 1]$. Let
\[ U = \frac{Q_2}{(q_1,q_2)} \text{ and } V = \frac{Q_1}{(q_1,q_2)}. \] Then as \( n \to \infty \),

\[
\frac{1}{|M_n|} \sum_{f \in M_n} \psi(f)\psi(f+1) = e^{2\pi i(\theta_1+\theta_2)}e^{2\pi i(V\theta_1+U\theta_2+(\theta_1+\theta_2)UV)} \frac{1}{|[Q_1,Q_2]|UV} \psi(U)\psi(V) \times \sum_{h([Q_1,Q_2])} \chi_1(hV+D_1)\chi_2(hU+D_2) \prod_{p|\{Q_1,Q_2\}} v_p + O(1), \]

if \( \text{rad}(Q_1) = \text{rad}(Q_2) \), otherwise, the sum in left hand side vanishes, where

\[
v_p = \left( 1 - \frac{1}{|P|} \right) \left( \sum_{m=0}^{\infty} \frac{\psi_1\chi_1 e^{-\theta_1}}{|P|^m} + \sum_{m=0}^{\infty} \frac{\psi_2\chi_2 e^{-\theta_2}}{|P|^m} \right) - 1,
\]

and \( D_1, D_2 \) are polynomials such that \( UD_1 - VD_2 = 1 \).

**Theorem 4.** Let \( \psi : M \to \mathbb{U} \) be multiplicative functions such that

\[ \mathbb{D}(\psi, \chi e^\theta; \infty) < \infty \]

for some primitive character \( \chi \) of conductor \( Q \) and an angle \( \theta \in [0,1] \). Then as \( n \to \infty \),

\[
\frac{1}{|M_n|} \sum_{f \in M_n} \psi(f + h_1)\psi(f + h_2) = \frac{1}{|Q|} \sum_{f|\Delta} \frac{\psi(f)^2}{|f|} \sum_{h|Q} \chi(h + D_1)\chi(h + D_2) \times \prod_{P \in \mathcal{P}} v_p + O(1),
\]

where \( v_p \) are defined as in (7) with \( \alpha_1 = \alpha_2 = \psi^* e_{-\theta} \), \( \Delta = (h_1 - h_2) \), and \( D_1, D_2 \) are polynomials such that \( (D_1 - D_2)f = \Delta \).

**1.3.3. Correlation with Hayes character.**

**Theorem 5.** Assume that \( 1 \leq l \leq \left\lceil \frac{n-1}{2} \right\rceil \). Let \( \psi : M \to \mathbb{U} \) be multiplicative functions such that

\[ \mathbb{D}(\psi, \chi e^\theta; \infty) < \infty, \]

where \( \chi \) is a primitive Dirichlet character of conductor \( Q \), \( \xi \) is a short interval character of length \( l \), and an angle \( \theta \in [0,1] \). Then for fixed polynomials \( h_1, h_2 \in \mathbb{F}_q[x] \) with \( \deg(h_j) \leq l \), as \( n \to \infty \),

\[
\frac{1}{|M_n|} \sum_{f \in M_n} \psi(f + h_1)\psi(f + h_2) = \frac{1}{|Q|} \sum_{f|\Delta} \frac{\psi(f)^2}{|f|} \sum_{h|Q} \chi(h + D_1)\chi(h + D_2) \times \prod_{P \in \mathcal{P}} v_p + O(1),
\]

where \( v_p \) are defined as in (7) with \( \alpha_1 = \alpha_2 = \psi^* \xi e_{-\theta} \), \( \Delta = (h_1 - h_2) \) and \( D_1, D_2 \) are polynomials such that \( (D_1 - D_2)f = \Delta \).

**1.4. Applications.**
1.4.1. Outcomes of Theorem 1 and Theorem 2. The following corollary is a direct application of Theorem 1 and Theorem 2.

**Corollary 1.2.** Assume the hypothesis of Theorem 1 and Theorem 2. Suppose that $\psi_1$ and $\psi_2$ are pretend to 1. Then as $n \to \infty$,

$$\frac{S_2(n, q)}{q^n} \to \prod_{P \in \mathcal{P}} v_P,$$

and

$$\frac{R_2(n, q)}{|P_n|} \to \prod_{P \in \mathcal{P}} v'_P,$$

where $v_P$ and $v'_P$ are defined as in (7) and (8) respectively.

We define the truncated Liouville function over function field by

$$\lambda_y(P^t) = \begin{cases} (-1)^t \ (= \lambda(P^t)) & \text{if } \deg P \leq y \\ 1 & \text{if } \deg P > y. \end{cases}$$

It is very interesting to establish

$$\sum_{f \in \mathcal{M}_n} \lambda_y(f)\lambda_y(f + h) = o(q^n), \quad \text{as } n \to \infty.$$ 

Note that, if $y = n$ then it is Chowla’s conjecture over function fields in large degree limit. For very small choice of $y$ the following theorem gives a truncated variant of Chowla’s conjecture in large degree limit which is an application of Theorem 1.

**Corollary 1.3.** There is a positive absolute constant $C$ such that if $n \geq 2$, $2 \leq y \leq \log n$ and fixed $h \in \mathbb{F}_q[x]$ with $\deg h \leq y$, then

$$\left| \sum_{f \in \mathcal{M}_n} \lambda_y(f)\lambda_y(f + h) \right| < C \frac{\log^4 y}{y^4} q^n.$$ 

**Remark 3.** Mangerel [20] proved a number field analog of the above theorem with wide range. Following this it may be possible to extend the range of $y$ satisfying $\frac{\log y}{y} \to \infty$.

Let $\mathbb{F}_q^*$ be the group of units in $A := \mathbb{F}_q[x]$. Let $\mathcal{F}_k$ be the set of monic polynomials in $\mathbb{F}_q[x]$ which are $k$-th power free. As a direct application of Theorem 1 we get an asymptotic formula for two simultaneously $k$-free monic polynomials.

**Corollary 1.4.** Let $a \in \mathbb{F}_q^*$. Then we have

$$\frac{1}{q^n} \sum_{f,f+a \in \mathcal{F}_k} 1 = \prod_{P} \left( 1 - \frac{2}{q^{\deg P}} \right) + O \left( \frac{1}{n^B} \right)$$

for any $0 < B < 1$.

We can also apply Theorem 1 and Theorem 2 to $\Phi(f)/|f|$.

**Corollary 1.5.** For a fixed $a \in \mathbb{F}_q^*$ we have

$$\frac{1}{q^n} \sum_{f \in \mathcal{M}_n} \frac{\Phi(f)\Phi(f + a)}{|f||f + a|} = \prod_{P} \left( 1 - \frac{2}{q^{2\deg P}} \right) + O \left( \frac{1}{n^B} \right)$$

and

$$\frac{1}{|P_n|} \sum_{P \in \mathcal{P}_n} \Phi(P)\Phi(P + a) = \prod_{P} \left( 1 - \frac{2}{q^{\deg P(q^{\deg P} - 1)}} \right) + O \left( \frac{1}{(\log n)^B} \right).$$
1.4.2. Outcomes of Theorem 5

Corollary 1.6. Let $\psi : \mathcal{M} \to \mathbb{U}$ be multiplicative functions such that

$$D(\psi, \chi \xi e^\theta; \infty) < \infty,$$

where $\chi$ is a primitive Dirichlet character of conductor $Q$, $\xi$ is a short interval character of length $l$, and an angle $\theta \in [0, 1]$. Then as $n \to \infty$,

$$\frac{1}{|\mathcal{M}_n|} \sum_{f \in \mathcal{M}_n} \psi(f) \overline{\psi(f+1)} = \frac{\mu(Q)}{|Q|} \prod_{P \mid Q} \left( 2 \left( 1 - \frac{1}{|P|} \right) \left( \sum_{m=0}^{\infty} \frac{\Re \left( (\psi \chi \xi e^{-\theta})(P^m) \right)}{|P|^m} \right) - 1 \right) + o(1).$$

Theorem 6 (Kátai Conjecture). Let $\psi : \mathcal{M} \to \mathbb{S}^1$ be a completely multiplicative function. Suppose that as $n \to \infty$,

$$\sum_{f \in \mathcal{M}_n} |\psi(f+1) - \psi(f)| = o(q^n).$$

Then there exists an angle $\theta \in [0, 1)$ and a short interval character $\xi : \mathcal{M} \to \mathbb{S}^1$ such that $\psi(f) = \xi(f)e^{2\pi i \theta \deg(f)}$.

Corollary 1.7 (Kátai conjecture for pairs). Let $\psi, \eta : \mathcal{M} \to \mathbb{S}^1$ be a completely multiplicative function. Suppose that as $n \to \infty$,

$$\sum_{f \in \mathcal{M}_n} |\psi(f+1) - \eta(f)| = o(q^n).$$

Then there exists an angle $\theta \in [0, 1)$ and a short interval character $\xi : \mathcal{M} \to \mathbb{S}^1$ such that $\psi(f) = \eta(f) = \xi(f)e^{2\pi i \theta \deg(f)}$.

As a direct application of Corollary 1.7 we have the following.

Corollary 1.8 (Kátai conjecture for triplets). Let $\psi, \eta, \kappa : \mathcal{M} \to \mathbb{S}^1$ be a completely multiplicative function. Suppose that as $n \to \infty$,

$$\sum_{f \in \mathcal{M}_n} |\psi(f+2) - 2\eta(f+1) - \kappa(f)| = o(q^n).$$

Then there exists an angle $\theta \in [0, 1)$ and a short interval character $\xi : \mathcal{M} \to \mathbb{S}^1$ such that $\psi(f) = \eta(f) = \kappa(f) = \xi(f)e^{2\pi i \theta \deg(f)}$.

1.4.3. Probabilistic viewpoint over $\mathbb{F}_q[x]$. The asymptotic behaviour of the $k$-point correlation functions $S_k(n, q)$ and $R_k(n, q)$ over $\mathbb{F}_q[x]$ are used to get the behaviour of the distribution of the sum

$$\eta_1(A_1f + h_1) + \ldots + \eta_k(A_2f + h_2),$$

and

$$\eta(P + h_1) + \ldots + \eta(P + h_k)$$

where $\eta_1, \ldots, \eta_k$ are real-valued additive functions, $A_j \in \mathbb{F}_q[x] \setminus \{0\}$ and $h_j \in \mathbb{F}_q[x]$.
Therorem 7. Let \( t, x \in \mathbb{R} \). Let \( A_1, A_2 \in \mathbb{F}_q[x] \setminus \{0\} \) and \( h_1, h_2 \in \mathbb{F}_q[x] \) such that \( (A_1, h_1) = (A_2, h_2) = 1 \) and \( \Delta = A_1 h_2 - A_2 h_1 \neq 0 \). Assume that \( \eta_1 \) and \( \eta_2 \) be real-valued additive functions on \( \mathcal{M} \) such that following series converge:

\[
\sum_{|\eta_i(P)| \leq 1} \eta_i(P) q^{-\deg P}, \quad \sum_{|\eta_i(P)| \leq 1} \eta_i(P) q^{-\deg P}, \quad \sum_{|\eta_i(P)| > 1} q^{-\deg P} \quad \forall i = 1, 2.
\]

Then the distribution functions

\[
\frac{1}{|\mathcal{M}_n|} \left\{ f \in \mathcal{M}_n : \eta_1(A_1 f + h_1) + \eta_2(A_2 f + h_2) \leq x \right\}
\]

and

\[
\frac{1}{|\mathcal{P}_n|} \left\{ P \in \mathcal{P}_n : \eta_1(P + h_1) + \eta_2(P + h_2) \leq x \right\}
\]

converges weakly towards a limit distribution whose characteristic functions are equal to \( \prod_{P \in \mathcal{P}} \psi_P \) and \( \prod_{P \in \mathcal{P}} \psi'_P \) where \( \psi_P \) and \( \psi'_P \) are defined as in (11) and (12) respectively with \( \psi_j \) replaced by \( \exp(it\eta_{ij}), \forall j = 1, 2 \).

As a direct consequence of Theorem 7, we get the following corollary.

Corollary 1.9. Let \( z, t \in \mathbb{R} \) and \( a \in \mathbb{F}_q^* \). The distribution functions

\[
\frac{1}{|\mathcal{M}_n|} \left\{ f \in \mathcal{M}_n : \frac{\Phi(f) \Phi(f + a)}{|f||f + a|} \leq e^z \right\}
\]

and

\[
\frac{1}{|\mathcal{P}_n|} \left\{ P \in \mathcal{P}_n : \frac{\Phi(P) \Phi(P + a)}{|P||P + a|} \leq e^z \right\}
\]

converge weakly towards limit distributions. The characteristic functions of these limit distributions are

\[
\prod_{\deg P} \left( 1 + \frac{2}{q^{\deg P}} (1 - q^{-\deg P})^it - 1 \right) \quad \text{and} \quad \prod_{\deg P} \left( 1 + \frac{2}{q^{\deg P}} (1 - q^{-\deg P})^it - 1 \right)
\]

respectively.

2. Preliminaries

2.1. Notation. We start by fixing a finite field \( \mathbb{F}_q \) of odd cardinality \( q = p^r, r \geq 1 \) with a prime \( p \). We denote by \( \mathbb{A} = \mathbb{F}_q[x] \) the polynomial ring over \( \mathbb{F}_q \). For a polynomial \( f \) in \( \mathbb{F}_q[x] \), its degree will be denoted by either \( \deg(f) \) or \( d(f) \).

The set of all monic polynomials and monic irreducible polynomials of degree \( n \) are denoted by \( \mathcal{M}_{n,q} \) (or simply \( \mathcal{M}_n \) as we fix \( q \)) and \( \mathcal{P}_{n,q} \) (or simply \( \mathcal{P}_n \)) respectively. Let \( \mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n \) and \( \mathcal{P} = \bigcup_{n \geq 1} \mathcal{P}_n \). we also denote the set of all monic polynomials and monic irreducible polynomials of degree less or equal to \( n \) by \( \mathcal{M}_{\leq n,q} \) (or simply \( \mathcal{M}_{\leq n} \)) and \( \mathcal{P}_{\leq n,q} \) (or simply \( \mathcal{P}_{\leq n} \)) respectively. Let \( \mathcal{H}_n \) denotes the set of monic square-free polynomials of degree \( n \). Observe that for \( n \geq 1, |\mathcal{M}_n| = q^n \). If \( f \) is a non-zero polynomial \( \mathbb{F}_q[t] \), we define the norm of \( f \) to be \( |f| = q^{d(f)} \). If \( f = 0 \), we set \( |f| = 0 \).

Given polynomials \( f, g \in \mathbb{F}_q[x] \setminus \{0\} \), their greatest common divisor is denoted by \( (f, g) \) and least common multiple is denoted by \( [f, g] \) and defined by \( [f, g] = \frac{fg}{(f, g)} \). For a polynomial
\begin{align*}
f \in \mathbb{F}_q[x], \text{ we write } e_\theta(f) := e(\theta \deg(f)) = e^{2\pi i \theta \deg(f)}. \text{ Further, we use } U := \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } S^1 := \{z \in U : |z| = 1\}.
\end{align*}

\subsection{Background on Function fields}
We say that \( \psi : M \rightarrow \mathbb{C} \) is multiplicative if \( \psi(fg) = \psi(f)\psi(g) \) whenever \( (f, g) = 1 \) and additive if \( \psi(fg) = \psi(f) + \psi(g) \) whenever \( (f, g) = 1 \).

\subsubsection{Short intervals over function fields}
Let \( B \in \mathbb{F}_q[x] \). For \( l \geq 1 \), define
\[ I(B; l) := \{ f \in M : \deg(f - B) < l \}. \]
In other words,
\[ I(B; l) = B + \tilde{P}_{\leq l-1}, \]
where \( \tilde{P}_{\leq l} = \{ g \in \mathbb{F}_q[x] : d(g) \leq l \} \). Hence \( \#I(B; l) = q^l \).

Note that for \( B \) monic, the interval \( I(B; l) \) consists of only monic polynomials. Also, all monic polynomials of degree \( n \) are contained in one of the intervals \( I(B; l) \) with \( B \) monic of degree \( n \). Moreover, for \( B_1, B_2 \in M_n \) and \( l < n \),
\[ I(B_1; l) \cap I(B_2; l) \neq \emptyset \iff d(B_1 - B_2) < l \iff I(B_1; l) = I(B_2; l). \]
Therefore, we get a partition of \( M_n \) into disjoint intervals parameterized by \( B \in M_n \):
\begin{align*}
M_n &= \bigsqcup_{B \in B} I(B; l),
\end{align*}
where \( B = \{ B = t^n + b_{n-1}t^{n-1} + \ldots + b_1t : b_j \in \mathbb{F}_q \} \).

\subsubsection{Hayes characters}
Let \( l \geq 1 \) and \( Q \in M \). Define a relation \( R_{Q,l} \) on \( M \) as follows: if \( A, B \in M \) then
\[ A \equiv B \pmod{R_{Q,l}} \text{ if and only if } A \equiv B \pmod{Q} \text{ and } \text{the leading } l + 1 \text{ coefficients of } A \text{ and } B \text{ are the same.} \]

If \( A, B \in M_n \) then the later condition is equivalent to \( \deg(A - B) < N - l \).

An element of \( M \) is invertible \( \pmod{R_{Q,l}} \) if and only if it is co-prime to \( Q \). The units of \( M/R_{Q,l} \) form an abelian group, denoted by \( (M/R_{Q,l})^\times \). Thus the characters of \( (M/R_{Q,l})^\times \) can be extended to \( M \) by defining them to be zero on non-unit elements. These extentions are Hayes characters (for more details, see \cite{16}). Define,
\[ G(R_{Q,l}) = \left\{ \tilde{\chi} : \tilde{\chi} \in (M/R_{Q,l})^\times \right\}. \]

Any Hayes character \( \tilde{\chi} \in G(R_{Q,l}) \) can be decomposed as a product \( \chi_Q \xi_l \), where \( \chi_Q \) is a Dirichlet character modulo \( Q \), and \( \xi_l \) is a short interval character of length \( l \) (where \( l = \max \{ v : \deg(A - B) < N - v \} \) ). Notice that the group has size \( \phi(Q)q^l \). Note that taking \( Q = 1 \), for any \( f \in I(B; l) \) and any short interval character \( \xi \) of length \( l \), we have \( \xi(f) = \xi(B) \). Also \( \xi \in G(R_{1,l}) \) means a Hayes character with trivial Dirichlet part.

A short interval character \( \xi_l \) is called primitive if it is not equal to a short interval character of length strictly larger than \( l \). A Hayes character \( \tilde{\chi} \in G(R_{Q,l}) \) is called primitive if both \( \chi_Q \) and \( \xi_l \) are primitive. Also a Hayes character is called non-principal if either is non-principal in the Dirichlet character or if the length of its short interval character is non-zero. The Hayes conductor of \( \tilde{\chi} \in G(R_{Q,l}) \) is defined by \( \text{Cond}(\tilde{\chi}) := \text{Cond}(\chi_Q) + \text{len}(\xi_l) = \deg(Q) + l \).
2.2.3. Orthogonality of Hayes characters. The orthogonality relation are given by (see [7])

\[
\frac{1}{\Phi(Q)q^l} \sum_{A \in \mathbb{R}_{Q,l}} \overline{\tilde{\chi}_1(A)} \tilde{\chi}_2(A) = \mathbbm{1}_{\tilde{\chi}_1 = \tilde{\chi}_2}
\]

and

\[
\frac{1}{\Phi(Q)q^l} \sum_{\tilde{\chi} \in G(\mathbb{R}_{Q,l})} \overline{\tilde{\chi}(A)} \tilde{\chi}(B) = \mathbbm{1}_{A \equiv B(\mathbb{R}_{Q,l})}.
\]

Let \(\xi_1\) and \(\xi_2\) be short interval characters of length \(l\). The orthogonality ([16], equation 20) relation with \(Q = 1\) implies that

\[
\frac{1}{q^l} \sum_{A \in \mathbb{R}_{1,l}} \xi_1(A) \overline{\xi_2(A)} = \mathbbm{1}_{\xi_1 = \xi_2}.
\]

If \(A_1, A_2 \in \mathcal{M}_n\) then \(A_1 \equiv A_2\) (mod \(\mathbb{R}_{1,l}\)) if and only if \(\deg(A_1 - A_2) < n - l\) (first \((l + 1)\) coefficients of \(A_1\) and \(A_2\) coincide).

Note that polynomials of the form

\[
t^n + a_{n-1}t^{n-1} + \ldots + a_{n-l}t^{n-l}
\]

represent classes modulo \(\mathbb{R}_{1,l}\) and so \(B\) defined earlier, comprises of exactly these polynomials. Each class modulo \(\mathbb{R}_{1,l}\) can be written as

\[
I(B; n - l) = B + \vec{P}_{\leq n - l - 1}
\]

with \(B \in \mathcal{B}\).

2.2.4. Pretentiousness in function fields. Following Klurman [14], we define the “distance” between two multiplicative functions \(\psi_1, \psi_2 : \mathcal{M} \to \mathbb{U}\) by

\[
\mathbb{D}(\psi_1, \psi_2; m, n) = \left( \sum_{\substack{P \in \mathcal{P} \\text{deg} P \leq n}} \frac{1 - \mathbb{R}(\psi_1(P)\psi_2(P))}{q^{\deg P}} \right)^{1/2}
\]

and \(\mathbb{D}(\psi_1, \psi_2; n) := \mathbb{D}(\psi_1, \psi_2; 1, n)\). If \(\mathbb{D}(\psi_1, \psi_2; \infty) < \infty\) then \(\psi_1\) is said to be \(\psi_2\)-pretentious, otherwise it is called non-pretentious. In this ways, if for some \(\theta \in [0, 1]\), \(\mathbb{D}(\psi, \chi_Q \xi_l e_\theta; \infty) < \infty\) then \(\psi\) is called Hayes pretentious, where \(\chi_Q\) is a Dirichlet character modulo \(Q\), and \(\xi_l\) is a short interval character of length \(l\) and \(e_\theta(P) = e^{2\pi i \theta d(P)}\). More precisely,

\[
\mathbb{D}(f, \chi_Q \xi_l e_\theta; \infty) = \left( \sum_{\substack{P \in \mathcal{P} \\text{deg} P \leq n}} \frac{(1 - \mathbb{R}(f(P)\chi_Q(P)\overline{\xi_l(P)}e^{-2\pi i \theta d(P)})}{q^{d(P)}} \right)^{1/2}
\]

is finite.

Observe that for any Hayes character \(\chi_Q \xi_l \in G(\mathbb{R}_{Q,l})\), and for some \(\theta \in [0, 1]\),

\[
\mathbb{D}(f; \chi_Q \xi_l e_\theta; \infty) < \infty \quad \text{if and only if} \quad \mathbb{D}(f \chi_Q \xi_l e_{-\theta}, 1; \infty) < \infty.
\]
2.3. Basic lemmas. The following lemma follows from Chinese remainder theorem over function fields.

**Lemma 1.** Let $A_1, A_2, g_1, g_2 \in \mathbb{F}_q[x] \setminus \{0\}$ and $h_1, h_2 \in \mathbb{F}_q[x]$ such that $(A_1, h_1) = (A_2, h_2) = 1$. The congruence system

$$A_jf + h_j \equiv 0 \pmod{g_j} \quad j = 1, 2$$

has a solution if and only if $(g_1, g_2) | (A_1h_2 - A_2h_1)$. If the solution exists, it is unique modulo $[g_1, g_2]$.

We now present the analogue of classical prime number theorem for polynomials over finite fields (see [23], Theorem 2.2).

**Lemma 2 (Prime Polynomial Theorem).** Let $\mathcal{P}_n$ denote the number of monic irreducible polynomials in $A$ of degree $n$. Then we have

$$|\mathcal{P}_n| = \frac{q^n}{n} + O\left(\frac{q^n}{n} \right).$$

The following lemma collects some useful estimates over function field.

**Lemma 3.**

a) Let, $q > 1$ and $\gamma > 0$. Then

$$\sum_{m \leq n} q^m m^{-\gamma} = O(q^n n^{-\gamma}).$$

b) We have

$$\sum_{\deg P \leq n} q^{-\deg P} = \log n + c_1 + O(1/n)$$

where $c_1$ is an absolute constant.

c) Also we have

$$\sum_{m \deg P \leq n/2, m \geq 1} q^m \deg P = O\left(\frac{q^n}{n}\right) \quad \text{and} \quad \sum_{m \deg P \leq n, m \geq 1} q^{-(m+1)\deg P} = O(1).$$

**Proof.** The estimates are collected from Section 3.3 of [19].

**Lemma 4** ([2], Lemma 2.2). Let $f \in \mathbb{F}_q[x]$. Then

$$\prod_{P | f} \left(1 + \frac{1}{|P|}\right) = O(\log(\deg(f))).$$

**Lemma 5.** Let $\Delta$ be a polynomial in $\mathbb{F}_q[x]$. Then

$$\sum_{M \in M_{\leq n}, (M, \Delta) = 1} \frac{\mu^2(M)3^\omega(M)}{|M|} \geq cn^3 \prod_{P | \Delta} \left(1 + \frac{3}{|P|}\right)^{-1},$$

where $c$ is an absolute constant.
Proof. Let us consider
\[ F(s) = \sum_{M} \frac{\mu^2(M)3^{\omega(M)}}{|M|^{s+1}}, \quad \Re(s) > 0. \]

We can write
\[ F(s) = \sum_{n=1}^{\infty} \frac{H(n)}{q^{ns}}, \quad \text{where} \quad H(n) = \sum_{M \in \mathcal{M}_n} \frac{\mu^2(M)3^{\omega(M)}}{|M|}. \]

Substituting \( u = q^{-s} \), we define
\[ \tilde{F}(u) = \sum_{n=1}^{\infty} H(n)u^n, \quad |u| < 1. \]

On the other hand, from Euler product we get
\[ \tilde{F}(u) = \frac{\tilde{G}(u)}{(1-u)^3}, \quad \text{where} \quad \tilde{G}(u) = G(s) = \prod_{P} \left(1 + \frac{3}{|P|^{s+1}}\right) \left(1 - \frac{1}{|P|^{s+1}}\right)^3. \]

It is easy to see that \( G(u) \) converges for \( |u| < 1 \) and therefore bounded. Comparing the coefficient of \( \tilde{F}(u) \) there exist constants \( c_1, c_2 > 0 \) such that
\[ c_1n^2 \leq H(n) \leq c_2n^2. \]

Using this, we conclude that
\[
\prod_{P \mid \Delta} \left(1 + \frac{3}{|P|}\right) \sum_{\substack{M \in \mathcal{M} \leq n \\text{(mod } \Delta)}} \frac{\mu^2(M)3^{\omega(M)}}{|M|} \geq \sum_{\substack{M \in \mathcal{M} \leq n \\text{(mod } \Delta)}} \frac{\mu^2(M)3^{\omega(M)}}{|M|} = \sum_{m \leq n} H(m) \geq cn^3,
\]
which completes the proof of the lemma. \qed

2.4. Lemmas on Dirichlet characters. We start with the following standard property of a primitive Dirichlet character over function fields.

**Lemma 6.** Let \( \chi \) be a primitive Dirichlet character of modulus \( Q \) on \( \mathbb{F}_q[x] \). Then for any non-constant polynomial \( D \) dividing \( Q \) satisfying \( \deg(D) < \deg(Q) \), there exists \( C \equiv 1 \mod D, \ (C, Q) = 1 \) such that \( \chi(C) \neq 1 \).

**Lemma 7.** Let \( \chi \) be a primitive Dirichlet character modulo \( P^k \) on \( \mathbb{F}_q[x] \), where \( P \) is an irreducible polynomial. For any \( 1 \leq i < k \), we have
\[
\sum_{L \pmod{P^i}} \chi(A + P^{k-i}L) = 0.
\]

**Proof.** Let
\[
S = \sum_{L \pmod{P^i}} \chi(A + P^{k-i}L).
\]
Using Lemma 6, we get $C \equiv 1 \pmod{P^k - i}$ such that $\chi(C) \neq 1$. Write $C = 1 + BP^{k-i}$ and consequently

$$\chi(C)S = \chi(1 + BP^{k-i}) \sum_{L(P^i)} \chi(A + P^{k-i}L)$$

$$= \sum_{L(P^i)} \chi(A + (B + L)P^{k-i} + BLP^{2(k-i)})$$

If $k \geq 2i$ then $2(k-i) \geq k$, therefore

$$\chi(C)S = \sum_{L(P^i)} \chi(A + (B + L)P^{k-i}) = S,$$

which implies $S = 0$ as $\chi(C) \neq 1$. The other possibility is $k < 2i$ and in that case, by Lemma 6 we can find $C \equiv 1 \pmod{P^i}$ such that $\chi(C) \neq 1$. Writing $C = 1 + DP^i$ and multiplying $\chi(C)$ with $S$, we obtain

$$\chi(C)S = \sum_{L(P^i)} \chi(A + P^{k-i}L + ADP^i)$$

$$= \sum_{L(P^i)} \chi(A + P^{k-i}(L + ADP^{2i-k}))$$

$$= \sum_{L(P^i)} \chi(A + P^{k-i}L) = S$$

since $2i - k > 0$. But this is not possible unless $S = 0$, which completes the proof.

**Lemma 8.** Let $\chi$ be a primitive Dirichlet character of conductor $P^m$, where $P$ is an irreducible polynomial. Then for any $F \in \mathbb{F}_q[x] \setminus \{0\}$ and $D \in \mathbb{F}_q[x]$, we get

$$\sum_{h(P^m)} \chi(hF + D) = \begin{cases} |P|^m \chi(D) & \text{if } P^m | F, \\ 0 & \text{else}. \end{cases}$$

**Proof.** If $P$ does not divide then we have a full character sum which vanishes. Therefore we suppose that $P^c || F$ and write $F = P^c L$ with $L$ coprime to $P$. The polynomial $L$ being invertible, after a change of variable, the left hand side becomes

$$\sum_{h(P^m)} \chi(hP^c + D).$$

Note that it is enough to prove that the sum is nonzero only if $c \geq m$. Suppose $c < m$. We can write the variable $h$ modulo $P^c$ as $h = x + P^{m-c}y$ where $x$ varies over residues
modulo $P^{m-c}$ and $y$ over residues modulo $P^c$. Thus
\[
\sum_{h(P^m)} \chi(hP^c + D) = \sum_{x(P^{m-c}) \atop y(P^c)} \chi((x + P^{m-c}y)P^c + D)
\]
\[
= \sum_{x(P^{m-c}) \atop y(P^c)} \chi(xP^c + D)
\]
\[
= |P|^c \sum_{x(P^{m-c})} \chi(xP^c + D)
\]
\[
= 0,
\]
where we use Lemma 7 in the last step.

**Lemma 9.** Let $P$ be an irreducible polynomial, $\chi_1$ and $\chi_2$ be primitive Dirichlet characters modulo $P^a$ and $P^b$ respectively. For any polynomials $F_1, F_2 \in \mathbb{F}_q[x] \setminus \{0\}$ and $D_1, D_2 \in \mathbb{F}_q[x]$, let
\[
I(\chi_1, \chi_2) = \sum_{h(P^{\max(a,b)})} \chi_1(hF_1 + D_1)\chi_2(hF_2 + D_2).
\]
If $a < b$ then $I(\chi_1, \chi_2) \neq 0$ only if $P^{b-a}|F_2$.

**Proof.** Suppose that $a < b$ and write the variable $h$ modulo $P^b$ as $h = x + P^ay$ with $x$ varying modulo $P^a$ and $y$ modulo $P^{b-a}$. The sum on the left hand side becomes
\[
\sum_{x(P^a)} \chi_1(xF_1 + D_1) \sum_{y(P^{b-a})} \chi_2(xF_2 + D_2 + P^ayF_2).
\]
Putting $A = xF_2 + D_2$, the inner sums
\[
S = \sum_{y(P^{b-a})} \chi_2(A + P^ayF_2).
\]
By Lemma 7, $S$ vanishes if $P$ does not divide $F_2$. Let $P^l||F_2$ with $l \geq 1$. If $l \geq b - a$, then
\[
S = |P|^{b-a} \chi_2(xF_2 + D_2).
\]
Next suppose $1 \leq l < b - a$ and write $F_2 = P^lt$ with $t$ coprime to $P$. Therefore
\[
S = \sum_{y(P^{b-a})} \chi_2(A + P^ct),
\]
where $c = l + a$. Clearly $a < c < b$, so we can write the variable $y$ modulo $P^{b-a}$ as $y = u + P^{b-c}v$ with $u$ varying modulo $P^{b-c}$ and $v$ modulo $P^{c-a}$. This gives
\[
S = |P|^{c-a} \sum_{u(P^{b-c})} \chi_2(A + P^cu) = 0
\]
from Lemma 7. \qed
2.5. **Brun-Titchmarsh inequality over function fields.** Given a non-constant polynomial \( M \in \mathbb{F}_q[x] \) and a polynomial \( B \) coprime to \( M \), let \( \pi_A(n; M, B) \) denote the number of primes \( P \in \mathcal{P}_n \) such that \( P \equiv B \pmod{M} \). The prime polynomial theorem for arithmetic progression ([23], Theorem 4.8) says that

\[
\pi_A(n; M, B) = \frac{q^n}{n\Phi(M)} + O\left(\frac{q^n}{n}\right).
\]

As in classical case, we want to allow \( \deg(M) \) to grow with \( n \). The interesting range of parameter is \( \deg(M) < n \) because if \( \deg(M) \geq n \) there is at most one prime polynomial of degree \( n \) in arithmetic progression \( h \equiv B \pmod{M} \). From (12), we see that if \( \frac{n}{2} \leq \deg(M) < n \) then error term becomes larger than main term. Therefore, we must assume that \( \deg(M) < \frac{n}{2} \).

The following lemma, Brun-Titchmarsh inequality over function field which is a special case of a theorem of Chin-Nung Hsu [12] gives an upper bound when \( \deg(M) < n \).

**Lemma 10** ([12], Theorem 4.3). Let \( \pi_A(n; M, B) \) be defined as above and \( \Phi(M) \) denotes the number of coprime residues modulo \( M \). Then for \( \deg(M) < n \), we have

\[
\pi_A(n; M, B) \leq \frac{2q^n}{\Phi(M)(n - \deg(M) + 1)}.
\]

2.6. **Application of Selberg sieve over \( \mathbb{F}_q[x] \).** The following lemma is an application of Selberg sieve method for polynomials over finite field to estimate \( \pi_A(n, M, B) \) on an average when \( \frac{n}{2} < \deg(M) < n \).

**Lemma 11.** Using the above notations, we have

\[
\Theta(n) := \sum_{\frac{n}{2} < \deg Q \leq n} \Phi(Q)\pi_A^2(n; Q, -h) \ll |\mathcal{P}_n|^2
\]

where the summation varies over all monic irreducible polynomial \( Q \) and \( h \) is a fixed polynomial with \( \deg h < n \).

**Proof.** Expanding square, we obtain

\[
\Theta(n) = \sum_{\frac{n}{2} < \deg Q \leq n} \Phi(Q)\left(\sum_{\substack{P \in \mathcal{P}_n \quad \text{deg} P = \deg h(Q) \quad \text{deg} A = \deg B}} 1\right)\left(\sum_{\substack{P' \in \mathcal{P}_n \quad \text{deg} P' = \deg h(Q) \quad \text{deg} A = \deg B}} 1\right) = \sum_{A, B \in \mathcal{M} \leq \frac{n}{2}} S(A, B),
\]

where

\[
S(A, B) := \sum_{\frac{n}{2} < \deg Q \leq n} \Phi(Q).
\]

Now we have to find upper bound of the set \( S(A, B) \). We define the following sets.

\[
A := \left\{ a_M := M(AM + h)(BM + h) : \deg(M) = n - \deg(A) \right\},
\]

\[
\mathcal{P}_\Delta := \left\{ P \in \mathcal{P} : \deg(P) < [n/2], P \nmid \Delta \right\},
\]

where \( \Delta = AB(Ah - Bh) \). For a monic polynomial \( D \in \mathbb{F}_q \), let us define

\[
\varrho(D) = \#\left\{ M \pmod{D} : a_M \equiv 0 \pmod{D} \right\}.
\]
Also let
\[ \tilde{Q} = \prod_{P \in P_\Delta, \deg P \leq \frac{n}{2}} P \quad \text{and} \quad \mathcal{D} = \left\{ D \in \mathcal{M} : \tilde{Q}, \deg(D) < \frac{n}{5} \right\}. \]

Observe that
\[
S(A, B) \leq \sum_{M \in \mathcal{M}_{n - \deg(A)}} |M| = q^{n - \deg(A)} \sum_{\substack{M \in \mathcal{M}_{n - \deg(A)} \backslash \mathcal{A} \quad a_M \notin A \quad (a_M, Q) = 1}} 1.
\]

Let \( X_D \) be real numbers corresponding to each \( D \) with \( D \in \mathcal{D} \) and \( X_1 = 1 \). We use Theorem 1 of Webb [31] to obtain
\[
S(A, B) \leq q^{2n - 2 \deg(A)} \sum_{D_1, D_2 \in \mathcal{D}} |X_{D_1} \cdot X_{D_2} [\tilde{Q}[D_1, D_2]]^n\),
\]
where
\[
Q = \sum_{D \in \mathcal{D}} \frac{1}{g(D)}, \quad g(D) = f(D) \prod_{P \mid D} \left( 1 - \frac{1}{f(P)} \right)
\]
with \( f(D) = \frac{|D|}{g(D)} \). Since \( g(D) \) is a multiplicative function on the divisors of \( \tilde{Q} \), then we have
\[
\sum_{M \mid D} \frac{1}{g(M)} = \frac{f(D)}{g(D)}, \quad D \in \mathcal{D}.
\]

We see that \( X_D \ll 1 \) for all \( D \in \mathcal{D} \). The above \( O \)-term is bounded above by
\[
\ll q^{n - \deg(A)} \sum_{D_1, D_2 \in \mathcal{D}} 3^{D_1, D_2} \ll q^{2n - \deg(A)} \prod_{\deg(P) \leq \frac{n}{5}} \left( 1 - \frac{3}{|P|} \right)^{-2} \ll n^6 q^{\frac{7n}{5} - \deg(A)}.
\]

Therefore, contribution of \( O \)-term to \( \Theta(n) \) is bounded above by
\[
\ll n^6 q^{\frac{7n}{5}} \sum_{\substack{A, B \in \mathcal{M} \leq \frac{n}{5} \quad \deg A = \deg B}} \frac{1}{|A|} \ll n^6 q^{\frac{19n}{10}},
\]
which is quite small. Using Lemma 5, we have
\[
Q = \sum_{M \in \mathcal{M}} \frac{1}{g(M)} \geq \sum_{M \in \mathcal{M}} \frac{1}{f(D)} = \sum_{\substack{M \in \mathcal{M} \leq \frac{n}{5} \quad (M, \Delta) = 1}} \frac{\mu^2(M) 3^{\omega(M)}}{|M|} \geq cn^3 \prod_{P \mid \Delta} \left( 1 + \frac{3}{|P|} \right)^{-1}
\]
where \( c > 0 \) is an absolute constant. Combining above results we get
\[
S(A, B) \ll q^{2n - 2 \deg(A)} \prod_{P \mid \Delta} \left( 1 + \frac{3}{|P|} \right).
\]

Therefore,
\[
\Theta(n) \ll q^{2n} \sum_{\substack{A, B \in \mathcal{M} \leq \frac{n}{5} \quad \deg A = \deg B}} q^{-2 \deg A} \prod_{P \mid AB(A-B)} \left( 1 + \frac{3}{|P|} \right).
\]
We write

\[
\sum_{A,B \in M \leq \frac{n}{2}, \deg A = \deg B} q^{-2 \deg A} \prod_{P | AB(A-B)h} \left(1 + \frac{3}{|P|}\right)
\]

\[
= \sum_{A,B \in M \leq \frac{n}{2}, \deg A = \deg B} q^{-2 \deg A} \prod_{P | A} \left(1 + \frac{3}{|P|}\right) \prod_{P | B(A-B)} \left(1 + \frac{3}{|P|}\right) \prod_{P | h} \left(1 + \frac{3}{|P|}\right).
\]

We use that \(\prod_{P | h} \left(1 + \frac{3}{|P|}\right) \ll 1\) with constant depending on \(q\) and \(h\). Also we find that

\[
\sum_{A,B \in M \leq \frac{n}{2}, \deg A = \deg B} q^{-2 \deg A} \prod_{P | A} \left(1 + \frac{3}{|P|}\right) \prod_{P | B(A-B)} \left(1 + \frac{3}{|P|}\right)
\]

\[
= \sum_{A \in M \leq \frac{n}{2}} q^{-2 \deg A} \prod_{P | A} \left(1 + \frac{3}{|P|}\right) \sum_{B \deg B = \deg A} \prod_{P | B(A-B)} \left(1 + \frac{3}{|P|}\right).
\]

The inner sum becomes

\[
\sum_{B \deg B = \deg A} \prod_{P | B(A-B)} \left(1 + \frac{3}{|P|}\right) \leq \sum_{B \deg B = \deg A} \prod_{P | B} \left(1 + \frac{3}{|P|}\right) \prod_{P | (A-B)} \left(1 + \frac{3}{|P|}\right)
\]

\[
= \sum_{B \deg B = \deg A} \sum_{D_1 | B} \mu^2(D_1) 3^{\omega(D_1)} \sum_{D_2 | A-B} \mu^2(D_2) 3^{\omega(D_2)} \frac{|D_1|}{|D_2|}
\]

\[
= \sum_{D_1, D_2} \frac{\mu^2(D_1) \mu^2(D_2) 3^{\omega(D_1)} 3^{\omega(D_2)}}{|D_1 D_2|} \sum_{B \in M \leq \frac{n}{2}, B \equiv 0(D_1)} \sum_{B \equiv A(D_2)} 1.
\]

We observe that \((D_1, D_2)|A\). Let \(D = (D_1, D_2)\) and writing \(D_i = DF_i\) we have \((F_i, D) = 1, (F_1, F_2) = 1\) and \(\omega(D_i) = \omega(D) + \omega(F_i)\) for all \(i = 1, 2\).
Using (11), we have

\[
\sum_{D|A} \mu^2(D)3^{2\omega(D)} \frac{1}{|D|^2} \sum_{F_1 \in \mathcal{M}_{\deg A - \deg D}} \mu^2(F_1) \mu^2(F_2)3^{\omega(F_1) + \omega(F_2)} \frac{1}{|F_1 F_2|} \sum_{B' \in \mathcal{M}_{\deg A - \deg D}} \frac{1}{|F_1 F_2|} + O(1)
\]

\[
= q^{\deg A} \sum_{D|A} \mu^2(D)3^{2\omega(D)} \frac{1}{|D|^3} \sum_{F_1 \in \mathcal{M}_{\deg A - \deg D}} \mu^2(F_1) \mu^2(F_2)3^{\omega(F_1) + \omega(F_2)} \frac{1}{|F_1 F_2|^2} \sum_{F_1 \in \mathcal{M}_{\deg A - \deg D}} \frac{1}{|F_1 F_2|} + O\left(\sum_{D|A} \mu^2(D)3^{2\omega(D)} \frac{1}{|D|^2} \sum_{F_1 \in \mathcal{M}_{\deg A - \deg D}} \mu^2(F_1) \mu^2(F_2)3^{\omega(F_1) + \omega(F_2)} \frac{1}{|F_1 F_2|}\right)
\]

\[
\ll q^{\deg A}.
\]

Hence, we conclude that

\[
\Theta(n) \ll \frac{q^{2n}}{n^3} \sum_{A \in \mathcal{M} \leq \frac{n}{2}} q^{-\deg A} \prod_{P|A} \left(1 + \frac{3}{|P|}\right)
\]

\[
= \frac{q^{2n}}{n^3} \sum_{A \in \mathcal{M} \leq \frac{n}{2}} \sum_{D|A} \mu^2(D)3^{\omega(D)} \frac{1}{|D|^2} \ll \frac{q^{2n}}{n^2}
\]

which completes proof of the lemma.

\[
\square
\]

2.7. Certain estimates for large prime polynomials. We introduce some sets which will be used to prove Theorem 1 and Theorem 2. Let \( h_1, h_2 \in \mathbb{F}_q[x] \) and \( A_1, A_2 \in \mathbb{F}_q[x] \setminus \{0\} \) be fixed such that \( \deg(h_k) < \deg(A_k) \). For any \( f \in \mathcal{M} \) and \( k = 1, 2 \), we define

\[
\mathcal{P}_f(k) := \left\{ P \in \mathcal{P} : P^m \parallel A_k f + h_k \text{ and } |1 - \psi_k(P^m)| > \frac{1}{2} \right\}.
\]

For \( r < n \), we consider the following sets:

\[
\mathcal{N}_r = \left\{ f \in \mathcal{M}_n : \exists k \in \{1, 2\} \text{ and } \exists P \in \mathcal{P}_f(k) \text{ with } \deg P > r \right\}
\]

and taking \( A_k = 1 \),

\[
\mathcal{Q}_r = \left\{ P \in \mathcal{P}_n : \exists k \in \{1, 2\} \text{ and } \exists Q \in \mathcal{P}_f(k) \text{ with } \deg Q > r \right\}.
\]
Lemma 12. With notations as above, upper bound for cardinalities of the sets \( N_r \) and \( Q_r \) are as follows.

\[
|N_r| \ll q^n \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; r, n + d(A_j)) + \frac{q^{n-r}}{r}
\]

and

\[
|Q_r| \ll |P_n| \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; r, n) + \frac{|P_n|}{rq^r} + \frac{|P_n|}{q^r}.
\]

Proof. Observe that

\[
|N_r| \ll \sum_{j=1}^{2} \sum_{f \in M_n \atop P^m \parallel A_j, f + h_j \leq \deg P > r} \frac{1}{q^m \deg P} \ll q^n \sum_{j=1}^{2} \sum_{\deg P \leq n + d(A_j) \atop 1 - \psi_j(P^m) > 1/2} \frac{1}{q^\deg P} + q^n \sum_{\deg P > r} q^{-2 \deg P}
\]

\[
\ll q^n \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; r, n + d(A_j)) + \frac{q^{n-r}}{r}.
\]

Interchanging summation we get

\[
|Q_r| \ll \sum_{j=1}^{2} \sum_{\deg Q \leq n} \pi_A(n; Q, -h_j) = \sum_{j=1}^{2} \sum_{\deg Q \leq n} \pi_A(n; Q, -h_j)
\]

\[
+ \sum_{j=1}^{2} \sum_{\deg Q \leq n} \pi_A(n; Q^k, -h_j) =: M_1 + M_2,
\]

where \( \sum^* \) denotes sum over \( Q \in \mathcal{P} \) satisfying \( \deg(Q) > r \) and \( |1 - \psi_j(Q^{k})| > \frac{1}{2} \).

Using Lemma 10 and Cauchy-Schwarz inequality, we have

\[
M_1 \ll \frac{q^n}{n} \sum_{j=1}^{2} \sum_{\deg Q \leq \frac{n}{2}} \frac{1}{\Phi(Q)} + \sum_{j=1}^{2} \left( \sum_{\deg Q \leq n} \frac{1}{\Phi(Q)} \right)^{\frac{1}{2}} \left( \sum_{\deg Q \leq n} \Phi(Q)^2 \pi_A^2(n; Q, -h_j) \right)^{\frac{1}{2}}
\]

\[
\ll \frac{q^n}{n} \sum_{j=1}^{2} \sum_{\deg Q \leq \frac{n}{2}} \frac{|1 - \psi_j(Q)|^2}{\Phi(Q)} + \sum_{j=1}^{2} \left( \sum_{\deg Q \leq n} \frac{|1 - \psi_j(Q)|^2}{\Phi(Q)} \right)^{\frac{1}{2}} (\Theta(n)^2)
\]

\[
\ll |P_n| \sum_{j=1}^{2} \mathbb{D}^2 \left( \psi_j, 1; r, \frac{n}{2} \right) + |P_n| \sum_{j=1}^{2} \mathbb{D} \left( \psi_j, 1; \frac{n}{2} \right),
\]

where we used Lemma 11 in the second term. Also

\[
M_2 \ll \frac{q^n}{n} \sum_{k \deg Q \leq \frac{n}{2} \atop \deg Q > r; k \geq 2} \frac{1}{\Phi(Q^k)} + q^n \sum_{\frac{n}{2} < k \deg Q \leq n} \frac{1}{\Phi(Q^k)} \ll \frac{|P_n|}{rq^r} + \frac{|P_n|}{q^r}.
\]
Combining these estimates, we have
\[ |Q_r| \ll |P_n| \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; r, n) + \frac{|P_n|}{rq^r} + \frac{|P_n|}{q^2}. \]

\[ \square \]

2.8. Variants of Turán-Kubilius inequality over function field. The following lemma is a shifted version of Turán-Kubilius inequality over function field in large degree limit.

**Lemma 13.** For a sequence of complex numbers \( \{\psi(P^m), P \in \mathcal{P}, m \geq 1\} \), we have
\[ \tilde{S} := \sum_{f \in \mathcal{M}_n} \left| \sum_{P^m \parallel f + h} \psi(P^m) - \sum_{m \deg P \leq n} \frac{\psi(P^m)}{q^{m \deg P}} (1 - q^{-\deg P}) \right|^2 \ll q^n \sum_{m \deg P \leq n} |\psi(P^m)|^2 \]
where \( h \) is some fixed polynomial with \( \deg h < n \).

**Proof.** First we assume that \( \psi(P^m) = 0 \) for all irreducible polynomials \( P \) with \( m \deg(P) > \frac{n}{2} \). By opening square of modulus on the left hand side, the coefficient of \( \psi(P^m)\psi(Q^r) \) for distinct irreducible polynomials \( P \) and \( Q \), is
\[ \sum_{f \in \mathcal{M}_n} \left| \sum_{P^m \parallel f + h} \psi(P^m) - \sum_{m \deg P \leq n} \frac{\psi(P^m)}{q^{m \deg P}} (1 - q^{-\deg P}) \right|^2 \ll q^n \sum_{m \deg P \leq n} |\psi(P^m)|^2 \]
where \( h \) is some fixed polynomial with \( \deg h < n \).

Observe that
\[ \sum_{f \in \mathcal{M}_n} \sum_{P^m \parallel f + h} 1 = q^n \left(1 - q^{-\deg P}\right) \left(1 - q^{-\deg Q}\right). \]

By treating three other sums analogously, we find that the coefficient of \( \psi(P^m)\psi(Q^r) \) is zero. Therefore only diagonal terms have non-zero coefficients. It is easy to see that coefficient of \( |\psi(P^m)|^2 \) is \( \leq q^{n - m \deg(P)} \). Thus the diagonal term is bounded above by
\[ \leq q^n \sum_{m \deg P \leq n} |\psi(P^m)|^2 \]

If we assume that \( \psi(P^m) = 0 \) for all monic irreducible polynomials \( P \) with \( m \deg P \leq \frac{n}{2} \). Therefore, if \( f \in \mathcal{M}_n \), there exist at most one prime polynomial power \( P^m \parallel f + h \) such that \( \psi(P^m) \neq 0 \). So we have
\[ \tilde{S} \ll q^n \sum_{m \deg P \leq n} |\psi(P^m)|^2 \]
Finally, we write a general \( \psi \) as \( \psi_1 + \psi_2 \), where \( \psi_1(P^m) = 0 \) for all monic irreducible polynomials with \( m \deg P > \frac{n}{2} \) and \( \psi_2(P^m) = 0 \) with \( m \deg P \leq \frac{n}{2} \) and combining above calculation we get the required result. \( \square \)

As a direct consequence of Lemma 13, using Lemma 3 and Cauchy-Schwarz inequality twice we get the following version of Turán-Kubilius inequality over function field.
Lemma 14. For a sequences of complex numbers \( \{a(P^m), P \in \mathcal{P}, m \geq 1\} \), we have
\[
\sum_{f \in \mathcal{M}_n} \left| \sum_{P^m \mid f+h} a(P^m) - \sum_{m \operatorname{deg} P \leq n} \frac{a(P^m)}{q^m \operatorname{deg} P} \right| \ll q^n \left( \sum_{m \operatorname{deg} P \leq n} \frac{|a(P^m)|^2}{q^m \operatorname{deg} P} \right)^{1/2}
\]
where \( h \) is some fixed polynomial of \( \operatorname{deg} h < n \).

The following lemma is an analog of Lemma 14 for irreducible polynomials.

Lemma 15. Let \( h \) be a fixed polynomial of \( \operatorname{deg} h < n \). For a sequences of complex numbers \( \{a(P^m), P \in \mathcal{P}, m \geq 1\} \), we have
\[
\sum_{P \in \mathcal{P}_n} \left| \sum_{Q^k \mid P+h} a(Q^k) - A(n) \right| \ll |\mathcal{P}_n|B(n)
\]
where
\[
A(n) := \sum_{Q \in \mathcal{P}} \frac{a(Q^k)}{|Q^k|} \quad \text{and} \quad B^2(n) := \sum_{Q \in \mathcal{P}} \frac{|a(Q^k)|^2}{\Phi(Q^k)}.
\]

Proof. For \( m < n \), using triangle inequality we have
\[
(14) \quad \sum_{P \in \mathcal{P}_n} \left| \sum_{Q^k \mid P+h} \psi(Q^k) - A(n) \right| \leq \sum_{P \in \mathcal{P}_n} \left| \sum_{Q^k \mid P+h} \psi(Q^k) - A(m) \right|
\]
\[
+ \sum_{P \in \mathcal{P}_n} \left| \sum_{Q^k \mid P+h, \operatorname{deg}(Q) \leq m} \psi(Q^k) \right| + \sum_{P \in \mathcal{P}_n} |A(n) - A(m)| =: L_1 + L_2 + L_3,
\]

Using Cauchy-Schwarz inequality and Lemma 2 we get
\[
L_1 \leq \left( \sum_{P \in \mathcal{P}_n} 1 \right)^{1/2} \left( \sum_{P \in \mathcal{P}_n} \left| \sum_{Q^k \mid P+h} \psi(Q^k) - A(m) \right|^2 \right)^{1/2} \leq \frac{q^{3\frac{2}{15}}}{n^2} L_4^{1/2},
\]
where
\[
L_4 := \sum_{P \in \mathcal{P}_n} \left| \sum_{Q^k \mid P+h} \psi(Q^k) - A(m) \right|^2.
\]

Note that
\[
\sum_{P \in \mathcal{P}_n \atop Q^k \mid P+h} 1 = \pi_A(n, Q^k, -h) - \pi_A(n, Q^{k+1}, -h)
\]
and
\[
\sum_{P \in \mathcal{P}_n \atop Q^k \mid P+h} 1 = \pi_A(n, Q_1^{k_1} Q_2^{k_2}, -h) - \pi_A(n, Q_1^{k_1+1} Q_2^{K_2}, -h)
\]
\[
- \pi_A(n, Q_1^{k_1} Q_2^{k_2+1}, -h) + \pi_A(n, Q_1^{k_1+1} Q_2^{K_2+1}, -h).
\]
Choosing $m = \frac{n}{4}$, we use (12) and by simplifying square of modulus of $L_4$, we observe that

$$L_4 = \frac{q^n}{n} \sum_{k \text{ deg } Q \leq m} \frac{|\psi(Q^k)|^2}{\Phi(Q^k)} \left(1 - \frac{1}{q^{\text{deg } Q}} \right) \left(1 - \frac{1}{q^{k \text{ deg } Q}} \right) + O\left(\frac{q^{\frac{n}{2} + 2m}}{nm} \sum_{k \text{ deg } Q \leq m} \frac{|\psi(Q^k)|^2}{\Phi(Q^k)} \right).$$

Thus, we have

$$L_4 \ll \frac{q^n}{n} B^2(n).$$

The next term of (14) gives

$$L_2 \leq \sum_{n/4 < k \text{ deg } Q \leq n} |a(Q^k)| \pi(n, Q^k, -h)$$

$$= \sum_{n/4 < k \text{ deg } Q \leq n/2} |a(Q^k)| \pi(n, Q^k, -h) + \sum_{n/2 < k \text{ deg } Q \leq n} |a(Q^k)| \pi(n, Q^k, -h)$$

$$:= L_5 + L_6$$

It is easy to show using Cauchy-Schwarz inequality that

$$L_5 \ll \frac{q^n}{n} B(n).$$

Using Lemma 10, Lemma 11 and Cauchy-Schwarz inequality, we have

$$L_6 = \sum_{P \in \mathcal{P}_n} \left| \sum_{Q^k \parallel P + h} a(Q^k) \right| \leq \sum_{\frac{n}{2} < k \text{ deg } Q \leq n} |a(Q^k)| \pi_A(n; Q^k, -h)$$

$$\ll \left( \sum_{\frac{n}{2} < k \text{ deg } Q \leq n} \frac{|a(Q^k)|^2}{\Phi(Q^k)} \right)^{1/2} \left( \sum_{\frac{n}{2} < k \text{ deg } Q \leq n} \Phi(Q^k) \pi_A^2(n; Q^k, -h) \right)^{1/2}$$

$$\ll B(n) \Theta(n)^{1/2} + B(n) q^n \left( \sum_{\frac{n}{2} < k \text{ deg } Q \leq n} \frac{1}{\Phi(Q^k)} \right)^{1/2}$$

$$\ll B(n) \frac{q^n}{n} + B(n) q^n \ll \frac{q^n}{n} B(n).$$

It is easy to show using Cauchy-Schwarz inequality that

$$|A(m) - A(n)| \leq \left( \sum_{m < k \text{ deg } Q \leq n} \frac{|a(Q^k)|^2}{\phi(Q^k)} \right)^{1/2} \left( \sum_{m < k \leq n} \frac{\phi(Q^k)}{|Q^k|^2} \right)^{1/2} \ll B(n)$$

for any $m < n$ and thus the last term of (14) becomes

$$L_3 \ll \frac{q^n}{n} B(n).$$

This completes the proof of lemma. \qed
2.9. **Probabilistic set-up over** $\mathbb{F}_q[x]$. Let $\psi : \mathcal{M} \to \mathbb{R}$ be a real valued additive function. Define $\Omega := \mathcal{M}_n$, which is a finite set of $q^n$ elements and $\psi_n$ to be the restriction of $\psi$ to $\mathcal{M}_n$. Let $\psi(\Omega) = \{x_1, \ldots, x_l\}$ be an enumeration. The subsets $A_i := \{f \in \Omega : \psi_n(f) = x_i\}$, $i = 1, \ldots, l$, of $\Omega$ are pairwise disjoint and form a partition of $\Omega$. The $\sigma$-algebra $\mathcal{F}$ generated by this partition consists of union of a finite number of subsets $A_i$. For $A \in \mathcal{F}$, let $\nu(A) = \frac{|A|}{q^n}$, where $|A|$ is the cardinality of $A$. Then $\nu$ is a probability measure on $\mathcal{F}$ and $(\Omega, \mathcal{F}, \nu)$ is a finite probability space. Now $\psi_n$ is a random variable on $(\Omega, \mathcal{F}, \nu)$. The distribution function of $\psi_n$ is

$$\nu_n(\psi, x) = \frac{1}{q^n}\left|\{f \in \mathcal{M}_n : \psi_n(f) \leq x\}\right|.$$ 

**Definition 1.** If there exists a distribution function $\Psi$ such that $\frac{1}{q^n}\nu_n(\psi, x)$ converges point-wise to $\Psi(x)$ as $n \to \infty$, then we say that $\psi$ has the limit distribution function $\Psi$.

Associated with a distribution function $F(x)$, the characteristic function is defined by

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x).$$

This characteristic function is defined for all real values of $t$. It is uniformly continuous for $-\infty < t < \infty$ and satisfies $\phi(0) = 1, |\phi(t)| \leq 1$.

**Lemma 16 ([28], Theorem 3).** Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of distribution functions and $\{\phi_n\}_{n=1}^{\infty}$ be the corresponding sequence of characteristic functions. Then $F_n$ converges weakly to the distribution function $F$ if and only if $\phi_n$ converges pointwise on $\mathbb{R}$ to a function $\phi$ which is continuous at $0$. Moreover, $\phi$ is the characteristic function of $F$ and the convergence of $\phi_n$ to $\phi$ is uniform on any compact subset.

### 3. Proof of Theorem [1]

We begin by splitting $\psi_1, \psi_2$ into parts, one is trivial on large primes and other on small primes. For $r \geq 1$ and $j = 1, 2$, we define multiplicative functions $\psi_{jr}$ and $\psi_{jr}^*$, by

$$\psi_{jr}(P^m) = \begin{cases} \psi_j(P^m) & \text{if } \deg P \leq r, \\ 1 & \text{if } \deg P > r, \end{cases} \quad \text{and} \quad \psi_{jr}^*(P^m) = \begin{cases} 1 & \text{if } \deg P \leq r, \\ \psi_j(P^m) & \text{if } \deg P > r. \end{cases}$$

We use Möbius inversion to define

$$\alpha_{jr}(P^m) = \begin{cases} \psi_j(P^m) - \psi_j(P^{m-1}) & \text{if } \deg P \leq r, \\ 0 & \text{if } \deg P > r, \end{cases}$$

so that $\psi_{jr} = 1 * \alpha_{jr}, j = 1, 2$.

**Lemma 17.** For each $j = 1, 2$ and for any $\beta \in (0, 1)$, we have

$$\sum_{g \in \mathcal{M}} \frac{|\alpha_{jr}(g)|}{|g|^{\beta}} \ll \exp\left(c q^{(1-\beta)r}\right)$$

for some absolute constant $c > 0$. 

Proof. Since $\alpha_{jr}$ is multiplicative, we can write

$$\sum_{g \in M} \frac{|\alpha_{jr}(g)|}{|g|^\beta} \leq \prod_{P \in \mathcal{P}, \deg(P) \leq r} \left(1 + \sum_{m=1}^{\infty} \frac{|\alpha_{jr}(P^m)|}{|P^m|^\beta}\right).$$

Recall that $\alpha_{jr}(P^m) = \psi_{jr}(P^m) - \psi_{jr}(P^{m-1})$ and hence $|\alpha_{jr}(P^m)| \leq 2$. Therefore the last product is bounded above by

$$\prod_{P \in \mathcal{P}, \deg(P) \leq r} \left(1 + \sum_{m=1}^{\infty} \frac{2}{q^{3m\deg(P)}}\right) \leq \exp \left(2 \sum_{P \in \mathcal{P}, \deg(P) \leq r} \sum_{m=1}^{\infty} \frac{1}{q^{3m\deg(P)}}\right) \leq \exp \left(c \sum_{P \in \mathcal{P}, \deg(P) \leq r} \frac{1}{q^{\beta\deg(P)}}\right)$$

for a suitable $c > 0$. This completes the proof using $|\mathcal{P}_m| \ll \frac{m}{m}$ for any $m$. \(\square\)

We write

$$\frac{S_2(n, q)}{q^n} - Q(n) = Q(r, n) \left(\frac{1}{q^n} \sum_{f \in M_n} \psi_{1r}(A_1f + h_1)\psi_{2r}(A_2f + h_2) - Q(r)\right) + \frac{1}{q^n} \sum_{f \in M_n} \psi_{1r}(A_1f + h_1)\psi_{2r}(A_2f + h_2) \left(\psi_{1r}^*(A_1f + h_1)\psi_{2r}^*(A_2f + h_2) - Q(r, n)\right).$$

Observe that $Q(r, n) \ll 1$. Therefore, we have

$$\frac{S_2(n, q)}{q^n} - Q(n) \ll \left|\frac{1}{q^n} \sum_{f \in M_n} \psi_{1r}(A_1f + h_1)\psi_{2r}(A_2f + h_2) - Q(r)\right| + \frac{1}{q^n} \sum_{f \in M_n} \left|\psi_{1r}^*(A_1f + h_1)\psi_{2r}^*(A_2f + h_2) - Q(r, n)\right|.$$

Now we see that

$$\sum_{f \in M_n} \psi_{1r}(A_1f + h_1)\psi_{2r}(A_2f + h_2) = \sum_{f \in M_n} \sum_{g_1 | A_1f + h_1} \alpha_{1r}(g_1) \sum_{g_2 | A_2f + h_2} \alpha_{2r}(g_2)$$

$$= \sum_{g_1 \in \mathcal{F}_n \setminus \{0\}} \alpha_{1r}(g_1) \sum_{g_2 : A_2f + h_2} \alpha_{2r}(g_2) \sum_{f : A_1f + h_1} 1.$$ 

By using Lemma 3 we have

$$\left|\left\{f \in M_n : A_1f + h_1 \equiv 0 \pmod{g_1}, A_2f + h_2 \equiv 0 \pmod{g_2}\right\}\right| = \frac{q^n}{[g_1, g_2]} + O(1),$$
whenever \((g_1, g_2)|(A_1 h_2 - A_2 h_1)\). Therefore we obtain
\[
\sum_{f \in M_n} \psi_{1r}(A_1 f + h_1)\psi_{2r}(A_2 f + h_2) = q^n \sum_{g_j \in \mathbb{F}_q[x]\{0\}} \frac{\alpha_{1r}(g_1)\alpha_{2r}(g_2)}{|[g_1, g_2]|} + \]
\[
+ O\left( \sum_{\substack{d(g_j) \leq n + d(A_j) \\
j \neq 1, 2}} |\alpha_{1r}(g_1)\alpha_{2r}(g_2)| \right) =: M_1 + E_1.
\]
So, we have
\[
M_1 = q^n \sum_{g_j \in \mathbb{F}_q[x]\{0\}} \frac{\alpha_{1r}(g_1)\alpha_{2r}(g_2)}{|[g_1, g_2]|} + O\left( q^n \sum_{\deg(g_1) > n + d(A_1)} \sum_{g_2 \in \mathbb{F}_q[x]\{0\}} \frac{|\alpha_{1r}(g_1)\alpha_{2r}(g_2)|}{|[g_1, g_2]|} \right)
\]
\[
= q^n Q(r) + E_2.
\]
Since \((g_1, g_2)|\Delta\) and \(\Delta\) is a fixed polynomial we have \(|(g_1, g_2)| \ll 1\) with constant depending on \(q, A_j\) and \(h_j\). By writing \([g_1, g_2] = \frac{g_{12}}{(g_1, g_2)}\) we get
\[
E_2 \ll q^n \sum_{\deg(g_1) > n + d(A_1)} \frac{|\alpha_{1r}(g_1)|}{|g_1|} \sum_{g_2 \in \mathbb{F}_q[x]\{0\}} \frac{|\alpha_{2r}(g_2)|}{|g_2|}.
\]
Using Lemma 3(b), we observe that
\[
\sum_{g \in \mathbb{F}_q[x]\{0\}} \frac{|\alpha_{jr}(g)|}{|g|} \leq \prod_{\deg P \leq r} \left( 1 + \sum_{k=1}^{\infty} \frac{|\alpha_{jr}(P^k)|}{q^k \deg P} \right) \ll \prod_{\deg P \leq r} \left( 1 + \frac{2}{q^{\deg P} - 1} \right)
\]
\[
\ll \exp \left( c \sum_{\deg P \leq r} q^{-\deg P} \right) \ll r^{c_1}
\]
for some constant \(c, c_1 > 0\). For \(0 < \alpha < 1\), using Lemma 3(a), we have
\[
\sum_{\deg(g) > n + d(A_j)} \frac{|\alpha_{jr}(g)|}{|g|} \ll \frac{1}{q^{(n+d(A_j))\alpha}} \sum_{g \in \mathbb{F}_q[x]\{0\}} \frac{|\alpha_{jr}(g)|}{q^{(1-\alpha)\deg(g)}}
\]
\[
\ll \frac{1}{q^{(n+d(A_j))\alpha}} \exp \left( c_2 \sum_{\deg P \leq r} \frac{1}{q^{(1-\alpha)\deg P}} \right)
\]
\[
\ll \frac{1}{q^{(n+d(A_j))\alpha}} \exp \left( c_3 \sum_{m \leq r} \frac{q^{m\alpha}}{m} \right) \ll \frac{1}{q^{(n+d(A_j))\alpha}} \exp \left( c_4 \frac{r^{\alpha}}{r} \right)
\]
for constants \(c_3 > 0\) and \(c_4 > 0\). Using these estimates, we get
\[
E_1 \ll q^{(2-2\alpha)(n+\alpha)} \exp \left( c \frac{q^{r\alpha}}{r} \right) \quad \text{and} \quad E_2 \ll q^{(1-\alpha)n-\alpha d(A_1)} \exp \left( c \frac{q^{r\alpha}}{r} \right),
\]
where \(A = \max\{d(A_1), d(A_2)\}\). Finally, we have to calculate the following sum
\[
E_3 := \sum_{f \in M_n} \psi_{1r}(A_1 f + h_1)\psi_{2r}^*(A_2 f + h_2) - Q(r, n).
\]
We decompose $E_3$ as

$$E_3 = \sum_{f \in \mathcal{N}_r} \left| \psi_{1r}^*(A_1 f + h_1) \psi_{2r}^*(A_2 f + h_2) - Q(r, n) \right| + \sum_{f \notin \mathcal{N}_r} \left| \psi_{1r}^*(A_1 f + h_1) \psi_{2r}^*(A_2 f + h_2) - Q(r, n) \right| =: E_4 + E_5.$$

Using Lemma 12, we get

$$E_4 \ll |\mathcal{N}_r| \ll q^n \sum_{j=1}^{2} D(\psi_j, 1; r, n + d(A_j)) + \frac{q^{n-r}}{r}.$$ 

We recall that if $\Re(u) \leq 0, \Re(v) \leq 0$, then

(16) \quad |\exp(u) - \exp(v)| \leq |u - v| \quad \text{and} \\
(17) \quad \log(1+z) = z + O(|z|^2), \quad \text{if } |z| \leq 1, |\arg(z)| \leq \frac{\pi}{2}.

Note that

$$\log Q(r, n) = \sum_{r < \deg P \leq n} \log \left(1 + \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{q^{m \deg P}}\right).$$

Using (17),

$$\log \psi_{jr}^*(A_j f + h_j) = \sum_{P^m || A_j f + h_j \atop \deg P > r} (\psi_j(P^m) - 1) + O \left(\sum_{P^m || A_j f + h_j \atop \deg P > r} |\psi_j(P^m) - 1|^2\right).$$

Using (16) and (17), we have

$$E_5 \ll \sum_{j=1}^{2} \sum_{f \in \mathcal{M}_n} \left| \sum_{P^m || A_j f + h_j \atop \deg P > r} (\psi_j(P^m) - 1) - \sum_{m \deg P \leq n \atop \deg P > r} \frac{\psi_j(P^m) - 1}{q^{m \deg P}} \right|$$

$$+ \sum_{f \in \mathcal{M}_n} \left| \sum_{j=1}^{2} \sum_{m \deg P \leq n \atop \deg P > r} \frac{\psi_j(P^m) - 1}{q^{m \deg P}} - \log Q(r, n) \right| + O \left(\sum_{f \in \mathcal{M}_n} \sum_{m \deg P \leq n \atop \deg P > r} |\psi_j(P^m) - 1|^2\right)$$

$$=: E_6 + E_7 + E_8.$$
We obtain

\[ E_8 \ll q^n \sum_{j=1}^{2} \sum_{\substack{m \deg P \leq n+d(A_j) \\ m \geq 1, \deg P > r}} \frac{|\psi_j(P^m) - 1|^2}{q^{m \deg P}} \ll q^n \sum_{j=1}^{2} \sum_{r < \deg P \leq n+d(A_j)} \frac{|\psi_j(P) - 1|^2}{q^{\deg P}} + \frac{q^{n-r}}{r} \]

\[ \ll q^n \left( D^2(\psi_1, 1; r, n + d(A_1)) + D^2(\psi_2, 1; r, n + d(A_2)) \right) + \frac{q^{n-r}}{r}, \]

\[ E_7 = q^n \sum_{j=1}^{2} \sum_{r < \deg P \leq n} \frac{\psi_j(P) - 1}{q^{\deg P}} + O\left( \sum_{\deg P > r} q^{-2 \deg P} \right) - q^n \sum_{r < \deg P \leq n} \frac{\psi_j(P) - 1}{q^{\deg P}} \]

\[ \ll q^n \sum_{\deg P > r} q^{-\deg P} \ll \frac{q^{n-r}}{r}. \]

Following lines of proof of Lemma 14 for the shifts \( A_jf + h_j, \) we have

\[ E_6 \ll q^{n+d(A)} \left( \sum_{j=1}^{2} \sum_{\substack{m \deg P \leq n+d(A_j) \\ m \geq 1, \deg P > r}} \frac{|\psi_j(P^m) - 1|^2}{q^{m \deg P}} \right)^{1/2} \]

\[ \ll q^{n+d(A)} \sum_{j=1}^{2} D(\psi_j, 1; r, n + d(A_j)) + \frac{q^n}{(rq^n)^{1/2}}. \]

Combining the above estimates, we get the theorem.

4. PROOF OF THEOREM 2

We use the functions \( \psi_j^*(j = 1, 2) \) defined in Section 3. Writing analogously, we get

\[ \frac{R_2(n, q)}{|P_n|} - Q'(n) = Q'(r, n) \left( \frac{1}{|P_n|} \sum_{P \in P_n} \psi_{1r}(P + h_1)\psi_{2r}(P + h_2) - Q'(r) \right) \]

\[ + \frac{1}{|P_n|} \sum_{P \in P_n} \psi_{1r}(P + h_1)\psi_{2r}(P + h_2)\left( \psi_{1r}^*(P + h_1)\psi_{2r}^*(P + h_2) - Q'(r, n) \right). \]

Observe that \( Q'(r, n) \ll 1. \) Therefore we have

\[ \frac{R_2(n, q)}{|P_n|} - Q'(n) \ll \left| \frac{1}{|P_n|} \sum_{P \in P_n} \psi_{1r}(P + h_1)\psi_{2r}(P + h_2) - Q'(r) \right| \]

\[ + \frac{1}{|P_n|} \sum_{P \in P_n} \left| \psi_{1r}^*(P + h_1)\psi_{2r}^*(P + h_2) - Q'(r, n) \right| =: E_9 + E_{10} \]
Using Lemma 1 we have
\[
\sum_{P \in \mathcal{P}_n} \psi_1(P + h_1) \psi_2(P + h_2) = \sum_{g_1, g_2 \in M \leq n} \alpha_1(g_1) \alpha_2(g_2)
\]

\[
= \sum_{g_1, g_2 \in M \leq n} \alpha_1(g_1) \alpha_2(g_2) \pi_A(n; [g_1, g_2], M)
\]

\[
= \sum_{g_1, g_2 \in M \leq z} \alpha_1(g_1) \alpha_2(g_2) \left( \pi_A(n; [g_1, g_2], M) - \frac{q^n}{n \Phi([g_1, g_2])} \right) + \frac{q^n}{n} \sum_{g_1, g_2 \in M \leq z} \alpha_1(g_1) \alpha_2(g_2) \Phi([g_1, g_2])
\]

\[
+ O \left( \sum_{g_1, g_2 \in M \leq z} |\alpha_1(g_1) \alpha_2(g_2)| \pi_A(n; [g_1, g_2], M) \right)
\]

where \( M \) is the monic polynomial satisfying \( M \equiv -h_j \pmod{g_j}, j = 1, 2 \) and \( 0 \leq \deg(M) < \deg([g_1, g_2]) \), and \( \sum' \) denotes summation over \( g_1, g_2 \) satisfying \( (g_1, g_2)|(h_2-h_1) \), \( \alpha_{jr}, j = 1, 2 \) are as defined in section 5 and \( r \leq z \leq n/4 \) is to be chosen later.

Therefore we have

\[
E_9 \leq \frac{1}{|\mathcal{P}_n|} \sum_{j=1}^{z} |\alpha_1(g_1) \alpha_2(g_2)| \left| \pi_A(n; [g_1, g_2], M) - \frac{q^n}{n \Phi([g_1, g_2])} \right|
\]

\[
+ O \left( \sum_{g_1, g_2 \in \mathcal{P}_n \setminus \{0\}} \frac{|\alpha_1(g_1) \alpha_2(g_2)|}{\Phi([g_1, g_2])} \right) + O \left( \frac{1}{|\mathcal{P}_n|} \sum_{g_1, g_2 \in M \leq n} |\alpha_1(g_1) \alpha_2(g_2)| \pi_A(n; [g_1, g_2], M) \right)
\]

\[
=: E_{11} + E_{12} + E_{13}.
\]

If \( \deg(g_1) \) and \( \deg(g_2) \) are both \( \leq z \leq n/4 \) then \( \deg([g_1, g_2]) \leq n/2 \) and hence we can apply (122) to get

\[
\pi(n, [g_1, g_2], M) = \frac{q^n}{n \Phi([g_1, g_2])} + O \left( \frac{q^{\frac{n}{2}}}{n} \right).
\]

Using these estimates we obtain

\[
E_{11} \leq \frac{q^{n/2}}{n|\mathcal{P}_n|} \sum_{g_1, g_2 \in M \leq n} |\alpha_1(g_1) \alpha_2(g_2)|
\]

\[
\ll \frac{q^{n/2+2\alpha}}{n|\mathcal{P}_n|} \sum_{g_1, g_2 \in M} \frac{|\alpha_1(g_1) \alpha_2(g_2)|}{|g_1|^\alpha |g_2|^\alpha}
\]

\[
\ll \frac{q^{n/2+2\alpha}}{n|\mathcal{P}_n|} \prod_{1 \leq j \leq 2} \left( \sum_{g_j \in M} \frac{|\alpha_{jr}(g_j)|}{|g_j|^\alpha} \right) \ll \frac{q^{n/2+2\alpha}}{n|\mathcal{P}_n|} \exp \left( 2c_3 \frac{q^{r(1-\alpha)}}{r} \right).
\]
Note that, trivially $\pi(n, [g_1, g_2], M) \ll \left( \frac{q^n}{\|g_1, g_2\|} + 1 \right)$. Therefore

$$E_{13} \ll \frac{1}{|P_n|} \sum_{g_1 \in M \leq n \atop z < \deg(g_2) \leq n} |\alpha_{1r}(g_1)\alpha_{2r}(g_2)| \left( \frac{q^n}{\|g_1, g_2\|} + 1 \right)$$

$$\leq \frac{q^{n-\alpha}}{|P_n|} r^{c_1} \exp \left( c_4 \frac{q^{ar}}{r} \right) + \frac{q^{2\alpha}}{|P_n|} \exp \left( c_5 \frac{q^{r(1-\alpha)}}{r} \right)$$

Similarly we have

$$E_{12} \ll q^{-\alpha z} r^{c_1} \exp \left( c_6 \frac{q^{ar}}{r} \right) \ll q^{-\alpha z} \exp \left( c_7 \frac{q^{ar}}{r} \right).$$

Therefore we write

$$E_{10} = \frac{1}{|P_n|} \sum_{P \in Q_r} |\psi_{1r}^*(P + h_1)\psi_{2r}^*(P + h_2) - Q'(r, n)|$$

$$+ \frac{1}{|P_n|} \sum_{P \notin Q_r} |\psi_{1r}^*(P + h_1)\psi_{2r}^*(P + h_2) - Q'(r, n)| =: \frac{1}{|P_n|}(E_{14} + E_{15}).$$

Using Lemma 12 we get

$$E_{14} \ll |Q_r| \ll |P_n| \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; r, n) + \frac{|P_n|}{rq} + \frac{|P_n|}{q^2}.$$ 

Using (16) and (17), we have

$$|\psi_{1r}^*(P + h_1)\psi_{2r}^*(P + h_2) - Q'(r, n)| \leq \left| \sum_{j=1}^{2} \sum_{Q^k \parallel P + h_j \atop \deg Q > r} (\psi_j(Q^k) - 1) - \log Q'(r, n) \right|$$

$$+ O \left( \sum_{j=1}^{2} \sum_{Q^k \parallel P + h_j \atop \deg Q > r} |\psi_j(Q^k) - 1|^2 \right).$$

Therefore we get

$$E_{15} \leq \sum_{P \notin Q_r} \left| \sum_{j=1}^{2} \sum_{Q^k \parallel P + h_j \atop \deg Q > r} (\psi_j(Q^k) - 1) - \log Q'(r, n) \right| + O \left( \sum_{P \notin Q_r} \sum_{j=1}^{2} \sum_{Q^k \parallel P + h_j \atop \deg Q > r} |\psi_j(Q^k) - 1|^2 \right)$$

$$\leq \sum_{P \notin Q_r} \left| \sum_{j=1}^{2} \sum_{Q^k \parallel P + h_j \atop \deg Q > r} (\psi_j(Q^k) - 1) - \sum_{j=1}^{2} \sum_{k \geq 1 \atop r < \deg(Q) \leq n/k} \frac{\psi_j(Q^k) - 1}{|Q^k|} \right|$$

$$+ \sum_{P \notin Q_r} \left| \sum_{j=1}^{2} \sum_{k \geq 1 \atop r < \deg(Q) \leq n/k} \frac{\psi_j(Q^k) - 1}{|Q^k|} - \log Q'(r, n) \right| + O \left( \sum_{P \notin Q_r} \sum_{j=1}^{2} \sum_{Q^k \parallel P + h_j \atop \deg Q > r} |\psi_j(Q^k) - 1|^2 \right)$$

$$= E_{18} + E_{19} + E_{20}. $$
Applying Lemma 15 with \(a(Q^k) = \psi_j(Q^k) - 1\) for \(Q \in \mathcal{P}\) and \(\deg(Q) > r\), we get
\[
E_{18} \ll |\mathcal{P}_n| \left( \sum_{k \deg Q \leq n, k \geq 1: \deg Q > r} |\psi_j(Q^k) - 1|^2 \right)^{\frac{1}{2}} \ll |\mathcal{P}_n| \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; r, n) + \frac{|\mathcal{P}_n|}{(rq^n)^{\frac{1}{2}}}.
\]

Observe that
\[
Q'(r, n) = \prod_{r < \deg P \leq n} \left( 1 - \frac{2}{\Phi(P)} + \sum_{k=1}^{\infty} \frac{\psi_1(P^k) + \psi_2(P^k)}{q^k \deg P} \right).
\]
It is easy to see that
\[
\log Q'(r, n) = \sum_{Q \in \mathcal{P}} \sum_{r < \deg(Q) \leq n} \left( -\frac{2}{\phi(Q)} + \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{\psi_j(Q^m)}{|Q^m|} \right) + O \left( \frac{1}{rq^n} \right)
\]
and consequently
\[
E_{19} \ll |\mathcal{P}_n|^{\frac{1}{2}}.
\]
In order to estimate \(E_{20}\), we first consider the sum corresponding to \(k = 1\) and a fixed \(j \in \{1, 2\}\) which is as follows.
\[
\sum_{P \not\in \mathcal{Q}_r} \sum_{Q \mid P + h_j, \deg(Q) > r} |1 - \psi_j(Q)|^2
\]
\[
\leq \sum_{Q \in \mathcal{P}} \sum_{r < \deg(Q) \leq n} |1 - \psi_j(Q)|^2 \pi(n, Q, -h_j)
\]
\[
\leq \sum_{Q \in \mathcal{P}} \sum_{r < \deg(Q) \leq n/2} |1 - \psi_j(Q)|^2 \pi(n, Q, -h_j) + \sum_{Q \in \mathcal{P}} \sum_{n/2 < \deg(Q) \leq n} \frac{|1 - \psi_j(Q)|}{\phi(Q)^{1/2}} \phi(Q)^{1/2} \pi(n, Q, -h_j)
\]
\[
\ll \frac{q^n}{n} \sum_{Q \in \mathcal{P}} \sum_{r < \deg(Q) \leq n/2} \frac{|1 - \psi_j(Q)|}{\phi(Q)^{1/2}} + \left( \sum_{Q \in \mathcal{P}} \sum_{n/2 < \deg(Q) \leq n} \frac{|1 - \psi_j(Q)|}{\phi(Q)} \right)^{1/2} \Theta(n)^{1/2}
\]

Similar to estimation of \(E_{14}\), we get
\[
E_{20} \ll \frac{|\mathcal{P}_n|}{rq^n} + \frac{|\mathcal{P}_n|}{q^n}.
\]
Choosing \(z = A \log \log q, A > 0\) and combining all these estimates, we get the required theorem.

5. Proof of Theorem [3] and Corollary [1,1]

Proof of Theorem [3] Given \(f \in \mathcal{M}_n\), define \(f_1, f_2 \in \mathbb{F}_q[x] \setminus \{0\}\) such that
\[
f_1 | (f + h_1) \text{ and } \text{rad}(f_1)|Q_1, \left( \frac{f + h_1}{f_1}, Q_1 \right) = 1
\]
and
\[ f_2|\left( f + h_2 \right) \quad \text{and} \quad \text{rad}(f_2)|Q_2, \quad \left( \frac{f + h_2}{f_2}, Q_2 \right) = 1. \]

Note that \((f_1, f_2)|(h_1 - h_2)\). So, we can write
\[ f + h_1 = G \frac{f_1 f_2}{(f_1, f_2)} + D_1 f_1 \quad \text{and} \quad f + h_2 = G \frac{f_1 f_2}{(f_1, f_2)} + D_2 f_2 \]
such that \(D_1 f_1 - D_2 f_2 = h_1 - h_2\), where \(D_1\) and \(D_2\) are polynomials that depends on \(f_1\) and \(f_2\).

Therefore, using multiplicativity of \(\psi_j\)'s, we obtain
\[
\sum_{f \in \mathcal{M}_n} \psi_1(f + h_1) \psi_2(f + h_2) \quad = \quad \sum_{\text{rad}(f_1)|Q_1 \atop \text{rad}(f_2)|Q_2} \psi_1(f_1) \psi_2(f_2) \sum^*_{d(G) = n - d([f_1, f_2])} \psi_1 \left( G \frac{f_2}{(f_1, f_2)} + D_1 \right) \psi_2 \left( G \frac{f_1}{(f_1, f_2)} + D_2 \right),
\]
where the sum \(\sum^*\) varies over the polynomials \(G\) such that
\[
\left( G \frac{f_2}{(f_1, f_2)} + D_1, Q_1 \right) = 1, \quad \left( G \frac{f_1}{(f_1, f_2)} + D_2, Q_2 \right) = 1.
\]

(20)

Define multiplicative functions \(\tilde{\psi}_j, j = 1, 2\) by
\[
\tilde{\psi}_j(P^k) = \begin{cases} \psi_j(P^k) & \text{if } P \nmid Q_j, \\ 0 & \text{otherwise}. \end{cases}
\]

This gives us
\[
\sum^*_{d(G) = n - d([f_1, f_2])} \psi_1 \left( G \frac{f_2}{(f_1, f_2)} + D_1 \right) \tilde{\psi}_1 \left( G \frac{f_1}{(f_1, f_2)} + D_2 \right) = \sum_{d(G) = n - d([f_1, f_2])} \tilde{\psi}_1 \left( G \frac{f_2}{(f_1, f_2)} + D_1 \right) \tilde{\psi}_2 \left( G \frac{f_1}{(f_1, f_2)} + D_2 \right).
\]

Now we write \(\tilde{\psi}_j\) as
\[
\tilde{\psi}_j(f) = \chi_j(f) \psi'_j(f), \quad j = 1, 2,
\]
where \(\chi_j\) are the Dirichlet characters in the hypothesis. Then \(\chi_1 \chi_2\) is a Dirichlet character modulo \([Q_1, Q_2]\).

In the above sum we write
\[
G = g[Q_1, Q_2] + h,
\]
where \( h \) runs over residue classes modulo \([Q_1, Q_2]\). From the Hypothesis, we have that 
\( d(f+h) = d(f) \) for sufficiently large degree of \( f \). Therefore the above sum becomes

\[
e^{2\pi i(\theta_1 + \theta_2)n} e_{\theta_1} \left( \frac{f_2}{(f_1, f_2)} \right) e_{\theta_2} \left( \frac{f_1}{(f_1, f_2)} \right) e_{-(\theta_1 + \theta_2)}([f_1, f_2])
\]

\[
\times \sum_{h([Q_1, Q_2])} \chi_1 \left( \frac{h f_2}{(f_1, f_2)} + D_1 \right) \chi_2 \left( \frac{h f_1}{(f_1, f_2)} + D_2 \right)
\]

\[
\times \sum_{d(g)=n-d([f_1, f_2])} (\psi'_1 e_{-\theta_1}) \left( \frac{g f_2[Q_1, Q_2]}{(f_1, f_2)} + \frac{h f_2}{(f_1, f_2)} + D_1 \right) (\psi'_2 e_{-\theta_2}) \left( \frac{g f_1[Q_1, Q_2]}{(f_1, f_2)} + \frac{h f_1}{(f_1, f_2)} + D_2 \right)
\]

We apply Theorem 11 to the innermost sum with the condition that

(21)

\[
\Delta = \frac{[Q_1, Q_2]}{(f_1, f_2)} (h_1 - h_2).
\]

Since the inner sum does not depend on the residue classes modulo \([Q_1, Q_2]\), so up to a small error of \(o(1)\), it is equal to

\[
\frac{1}{|[Q_1, Q_2]|} \sum_{d(G)=n-d([f_1, f_2])} (\psi'_1 e_{-\theta_1}) \left( G \frac{f_2}{(f_1, f_2)} + D_1 \right) (\psi'_2 e_{-\theta_2}) \left( G \frac{f_1}{(f_1, f_2)} + D_2 \right).
\]

Gathering these estimates, we conclude that

\[
\sum_{f \in M_n} \psi_1(f + h_1) \psi_2(f + h_2) = e^{2\pi i(\theta_1 + \theta_2)n} \frac{1}{|[Q_1, Q_2]|}
\]

\[
\times \sum_{rad([f_1]) | Q_1 \atop rad([f_2]) | Q_2} \psi_1(f_1) \psi_2(f_2) e_{\theta_1} \left( \frac{f_2}{(f_1, f_2)} \right) e_{\theta_2} \left( \frac{f_1}{(f_1, f_2)} \right) e_{-(\theta_1 + \theta_2)}([f_1, f_2])
\]

\[
\times \sum_{h([Q_1, Q_2])} \chi_1 \left( \frac{h f_2}{(f_1, f_2)} + D_1 \right) \chi_2 \left( \frac{h f_1}{(f_1, f_2)} + D_2 \right)
\]

\[
\times \sum_{d(G)=n-d([f_1, f_2])} (\psi'_1 e_{-\theta_1}) \left( G \frac{f_2}{(f_1, f_2)} + D_1 \right) (\psi'_2 e_{-\theta_2}) \left( G \frac{f_1}{(f_1, f_2)} + D_2 \right).
\]

Using Lemma 8 and Lemma 9 the character sum

(22)

\[
\sum_{h([Q_1, Q_2])} \chi_1 \left( \frac{h f_2}{(f_1, f_2)} + D_1 \right) \chi_2 \left( \frac{h f_1}{(f_1, f_2)} + D_2 \right)
\]

vanishes unless \( \frac{Q_2}{(Q_1, Q_2)} \mid \frac{h}{(f_1, f_2)} \) and \( \frac{Q_1}{(Q_1, Q_2)} \mid \frac{f_2}{(f_1, f_2)} \).

Observe that the hypothesis \( \mathbb{D}(\psi_j, \chi_j e_{\theta_j}; \infty) < \infty \) implies that

\( \mathbb{D}(\psi'_j e_{-\theta_j}, 1; \infty) < \infty. \)

We use Theorem 11 to the above innermost sum to conclude the proof.
**Proof of Corollary 1.4.** In this case, we have \((f_1, f_2) = 1\). From (22),

\[
\sum_{h(Q_1 Q_2)} \chi_1 (hf_2 + D_1) \chi_2 (hf_1 + D_2)
\]

vanishes unless

\[
U = f_1 = \frac{Q_2}{Q_1, Q_2} \quad \text{and} \quad V = f_2 = \frac{Q_1}{Q_1, Q_2}.
\]

6. **Proof of Theorem 5**

Recall that

\[
\mathcal{M}_n = \bigcup_{B \in \mathcal{B}} I(B; n - l),
\]

where

\[
\mathcal{B} = \{ B = t^n + b_{n-1}t^{n-1} + \ldots + b_{n-l}t^{n-l} : b_j \in \mathbb{F}_q \}.
\]

So, we have

\[
\sum_{f \in \mathcal{M}_n} \psi(f + h_1)\overline{\psi(f + h_2)} = \sum_{B \in \mathcal{B}} \sum_{f \in I(B; n - l)} \psi(f + h_1)\overline{\psi(f + h_2)}.
\]

We see that if \(f \in I(B; n - l)\) then we have \(f + h_j \in I(B; n - l)\), since \(\deg(h_j) \leq l\) for all \(j = 1, 2\). Therefore

\[
f + h_j \in I(B; n - l) \implies \xi(f + h_j) = \xi(B).
\]

This gives us

\[
\sum_{B \in \mathcal{B}} \sum_{f \in I(B; n - l)} \psi(f + h_1)\overline{\psi(f + h_2)} = \sum_{B \in \mathcal{B}} \xi(B)\overline{\xi(B)} \sum_{f \in I(B; n - l)} (\overline{\psi \xi})(f + h_1)(\overline{\psi \xi})(f + h_2)
\]

\[
= \sum_{B \in \mathcal{B}} \sum_{f \in I(B; n - l)} (\overline{\psi \xi})(f + h_1)(\overline{\psi \xi})(f + h_2)
\]

\[
= \sum_{f \in \mathcal{M}_n} (\overline{\psi \xi})(f + h_1)(\overline{\psi \xi})(f + h_2).
\]

The hypothesis \(\mathcal{D}(\psi, \chi \epsilon_q; \infty) < \infty\), implies that \(\mathcal{D}(\overline{\psi \xi}, \chi \epsilon_q; \infty) < \infty\). Hence we can apply Theorem 4 to conclude the proof.

7. **Proof of Theorem 4 and Corollary 1.6**

**Proof of Theorem 4**. We follow the arguments of proof of the Theorem 3 to obtain

\[
\sum_{f \in \mathcal{M}_n} \psi(f + h_1)\overline{\psi(f + h_2)} = \frac{1}{|Q|} \sum_{\text{rad}(f_1/Q)} \psi(f_1)\overline{\psi(f_2)} \epsilon_{-\theta} \left( \frac{f_2}{(f_1, f_2)} \right) \epsilon_{\theta} \left( \frac{f_1}{(f_1, f_2)} \right)
\]

\[
\times \sum_{h(Q)} \chi \left( \frac{h}{(f_1, f_2)} + D_1 \right) \chi \left( \frac{h}{(f_1, f_2)} + D_2 \right)
\]

\[
\times \sum_{d(G) = n - d(f_1, f_2)} (\overline{\psi \epsilon_{-\theta}}) \left( G \frac{f_2}{(f_1, f_2)} + D_1 \right) (\overline{\psi \epsilon_{\theta}}) \left( G \frac{f_1}{(f_1, f_2)} + D_2 \right),
\]
where $D_1, D_2$ are polynomials depending on $f_1$ and $f_2$ such that $D_2 f_2 - D_1 f_1 = \Delta$. Now we have to estimate

$$T(Q) := \sum_{h(Q)} \chi \left( h \frac{f_2}{(f_1, f_2)} + D_1 \right) \chi \left( h \frac{f_1}{(f_1, f_2)} + D_2 \right).$$

By Chinese remainder theorem on $\mathbb{F}_q[x]$, we have

$$T(Q) = \prod_{P^k || Q} \sum_{h(p^k)} \chi_{p^k}(h f_2 + D_1) \chi_{p^k}(h f_1 + D_2),$$

where $\chi_{p^k}$ is a primitive Dirichlet character of conductor $P^k$. We claim that $T(Q)$ vanishes when $f_1 \neq f_2$. In this case, there exists an irreducible polynomial say $P \in \mathbb{F}_q[x]$ such that $P^i | f_1$ and $P^j | f_2$ with $j > i$. Then $(\frac{f_1}{(f_1, f_2)}, P) = 1$. Therefore using the change of variable

$$h \mapsto h \frac{f_1}{(f_1, f_2)} \pmod{P^k}$$

the inner sum of $T(Q)$ becomes

$$S(P) = \sum_{h(p^k)} \chi_{p^k}(h P^{j-i} + D'_1) \chi_{p^k}(h + D'_2),$$

where $(t, P) = 1$ and for polynomials $D'_j \in \mathbb{F}_q[x], j = 1, 2$. If $j - i \geq k$ then first term of the sum $S(P)$ is fixed and the second term runs over all residue classes modulo $P^k$, which leads the sum to be zero.

Let us assume that $j - i < k$. We write

$$h = H + P^{k-(j-i)} F,$$

where $H$ runs over all residue classes modulo $P^{k-(j-i)}$ and $L$ over residue classes modulo $P^{j-i}$. Thus, we obtain

$$S(P) = \sum_{H(p^{k-(j-i)})} \chi_{p^k}(H P^{j-i} + D'_1) \sum_{L(p^{j-i})} \chi_{p^k}(H + P^{k-(j-i)} F + D'_2).$$

Applying Lemma 7 we say that the inner sum of $S(P)$ vanishes. Hence, $f_1 = f_2 = f$ (say). Therefore, we conclude that

$$\sum_{f \in M_n} \psi(f + h_1) \overline{\psi(f + h_2)} = \frac{1}{|Q|} \sum_{f \mid \Delta} \frac{1}{\chi(f)} \frac{|\psi(f)|^2}{\chi_\Delta(f)} \times \sum_{h(Q)} \chi(h + D_1) \overline{\chi(h + D_2)} \times \sum_{d(G) = n - d(f)} (\overline{\psi} \chi e_{\theta}) (G + D_1) (\overline{\psi} \chi e_{\theta}) (G + D_2),$$

where $D_1, D_2$ are the polynomials depend on $f_1$ and $f_2$ such that $D_2 f_2 - D_1 f_1 = \Delta$. We use Theorem 4 to the innermost sum to conclude the proof.
Proof of Corollary 1.6. We apply \( h_1 = 0 \) and \( h_2 = 1 \) to the Theorem 5. The conditions \( f \mid \Delta \) and \( (D_1 - D_2)f = \Delta \) implies that \( f = 1 \) and \( D_2 - D_1 = 1 \). Also \( \deg(D_j) \leq 0 \) forces \( D_2 = 1 \) and \( D_1 = 0 \). Therefore,

\[
S(P) = \sum_{h(P^k)} \chi_{P^k}(h)\overline{\chi_P(h+1)} = \begin{cases} -1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2. \end{cases}
\]

This yields \( T(Q) = \mu(Q) \), which concludes the proof.

8. Proof of Theorem 6 and Corollary 1.7

Proof of Theorem 6. Let us consider

\[
\Delta \psi(f) = \psi(f + 1) - \psi(f).
\]

Using Hypothesis we observe that

\[
\sum_{f \in \mathcal{M} \leq N} \frac{|\Delta \psi(f)|^2}{|f|} \leq \sum_{f \in \mathcal{M} \leq N} \frac{2|\Delta \psi(f)|}{|f|} = 2 \sum_{n \leq N} \frac{1}{q^n} \sum_{f \in \mathcal{M}_n} |\Delta \psi(f)| = o(N).
\]

Step 1: We first show that if for some \( 0 < \epsilon < 1 \),

\[
\sum_{f \in \mathcal{M} \leq N} \frac{|\Delta \psi(f)|^2}{|f|} \leq 2(1 - \epsilon)N
\]

holds then there exists a primitive character \( \chi \), a short interval character \( \xi \) and an angle \( \theta \in [0, 1) \) such that

\[
\mathbb{D}(\psi(f), \chi(f)\xi(f)e^{\theta(f)}; \infty) < \infty.
\]

To prove the claim we start by writing

\[
\Re(\psi(f)\overline{\psi(f+1)}) = 1 - \frac{|\Delta \psi(f)|^2}{2}
\]

so that \(23\) gives

\[
\sum_{f \in \mathcal{M} \leq N} \frac{\Re(\psi(f)\overline{\psi(f+1)})}{|f|} \geq \epsilon N.
\]

We can apply the Theorem 1.5 of \[16\] to deduce that for every sufficiently large \( N \), there exist a Dirichlet character \( \chi_N \) of bounded modulus, a short interval character \( \xi_N \) of bounded length, and an angle \( \theta_N \in [0, 1) \) such that

\[
\mathbb{D}(\psi, \chi_N \xi_N e^{\theta_N}; N) \ll 1.
\]

Following the argument in page 54 of \[16\], we can conclude that uniformly in \( N \),

\[
\mathbb{D}(\psi, \chi \xi e^{\theta}; N) \ll 1,
\]

which establishes the claim.

Step 2: We now show that if \( \mathbb{D}(\psi(f), \chi(f)\xi(f)e^{\theta(f)}; \infty) < \infty \) for a primitive Dirichlet character \( \chi \) of modulus \( Q \in \mathbb{F}_q[x] \), a short interval character of length \( l \geq 1 \) and an angle \( \theta \in [0, 1) \) then

\[
\sum_{f \in \mathcal{M} \leq N} \frac{|\Delta \psi(f)|^2}{|f|} = 2 \left( 1 - \mathbb{E}(\psi) + o(1) \right) N,
\]
where

\[ E(\psi) = \frac{\mu(Q)}{|Q|} \prod_{P \mid Q} \left( 2 \left( 1 - \frac{1}{|P|} \right) \left( \sum_{m=0}^{\infty} \frac{\Re\left( (\psi \bar{\psi} e^{\theta})(P^m) \right)}{|P|^m} \right) - 1 \right). \]

This step easily follows from Corollary 1.6 together with the estimate that

\[ \sum_{f \in \mathcal{M} \leq N} \frac{\Delta \psi(f)^2}{|f|} = 2 \left( \sum_{f \in \mathcal{M} \leq N} \frac{1 - \Re(\psi(f)\psi(f + 1))}{|f|} \right). \]

**Step 3**: Combining Step 1, Step 2, and observation (23), we must have that \( E(\psi) = 1 \). From this condition we have to find out the desired form of the function \( \psi \).

Observe that the euler factor

\[ 2 \left( 1 - \frac{1}{|P|} \right) \left( \sum_{m=0}^{\infty} \frac{\Re\left( (\psi \bar{\psi} e^{\theta})(P^m) \right)}{|P|^m} \right) - 1 \geq \frac{|P| - 4}{|P|} \geq -1, \]

where the equality holds only when \(|P| = 2 \left( \deg(P) = \left\lfloor \log_{2} \log q \right\rfloor \right) \) and also

\[ 2 \left( 1 - \frac{1}{|P|} \right) \left( \sum_{m=0}^{\infty} \frac{\Re\left( (\psi \bar{\psi} e^{\theta})(P^m) \right)}{|P|^m} \right) - 1 \leq 2 \left( 1 - \frac{1}{|P|} \right) \left( \sum_{m=0}^{\infty} \frac{1}{|P|^m} \right) - 1 \leq 1. \]

Therefore we must have \( Q \in \mathbb{F}_q^* \) and for all \( P \in \mathcal{P} \),

\[ 2 \left( 1 - \frac{1}{|P|} \right) \left( \sum_{m=0}^{\infty} \frac{\Re\left( (\psi \bar{\psi} e^{\theta})(P^m) \right)}{|P|^m} \right) - 1 = 1, \]

which is possible if and only if \( \psi(P^m) = \xi(P^m)e^{\theta(P^m)} \) for all \( m \geq 1 \).

**Proof of Corollary 1.7** We can write the hypothesis as

\[ \sum_{f \in \mathcal{M}_n} \frac{|\psi(f + 1) - \eta(f)|}{|f|} \to 0, \quad \text{as } n \to \infty. \]  \hspace{1cm} (25)

For any \( A \in \mathcal{M} \), we consider \( h = A(f + 1) - 1 \). So (25) implies that

\[ \sum_{h \in \mathcal{M}_{n+\deg(A)}} \frac{|\psi(h + 1) - \eta(h)|}{|h|} \to 0, \quad \text{as } n \to \infty. \]  \hspace{1cm} (26)

On the other hand, for \( A \in \mathcal{M} \), we also have

\[ \sum_{f \in \mathcal{M}_n} \frac{|\psi(A(f+1)) - \psi(A)f - \eta(f)|}{|f|} \to 0, \quad \text{as } n \to \infty, \]

which turns into

\[ \sum_{f \in \mathcal{M}_n} \frac{|\psi(A(f+1)) - \psi(A)\eta(f)|}{|f|} \to 0, \quad \text{as } n \to \infty. \]  \hspace{1cm} (27)

Also we write (26) as

\[ \sum_{f \in \mathcal{M}_n} \frac{|\psi(A(f+1)) - \eta(A)(f + 1)|}{|Af|} \to 0, \quad \text{as } n \to \infty. \]  \hspace{1cm} (28)
From (27) and (28), we obtain
\[
\sum_{f \in \mathcal{M}} \frac{|\psi(A(f) - \eta(Af + 1))|}{|f|} \to 0, \quad \text{as } n \to \infty.
\]

We use change of variable \(f = (A - 1)Bg\) for some \(B \in \mathcal{M}\) so that \(\deg(g) = \deg(f) - \deg(A) - \deg(B)\). Let \(k = n - \deg(A) - \deg(B)\). Therefore, (29) becomes
\[
\sum_{f \in \mathcal{M}_k} \frac{|\psi(A - 1)\eta(B)\eta(g) - \eta(A - 1)\eta(ABg + 1)|}{|g|} \to 0, \quad \text{as } k \to \infty,
\]
which implies that
\[
\sum_{f \in \mathcal{M}_k} \frac{|\psi(A)\eta(B)\eta(g) - \eta(ABg + 1)|}{|g|} \to 0, \quad \text{as } k \to \infty.
\]

From the symmetry in the \(A\) and \(B\), we also have
\[
\sum_{f \in \mathcal{M}_k} \frac{|\psi(B)\eta(A)\eta(g) - \eta(ABg + 1)|}{|g|} \to 0, \quad \text{as } k \to \infty.
\]

Therefore, (30) and (31) give us
\[
\psi(A)\eta(B) = \psi(B)\eta(A).
\]

The function \(H : \mathcal{M} \to S^1\) defined by
\[
H = \frac{\psi}{\eta}
\]
such that \(H(A) = H(B)\) for all \(A, B \in \mathcal{M}\). This leads us to conclude that \(H\) is constant on \(\mathcal{M}\). Then the complete multiplicativity of \(H\) implies that \(H = 1\), which gives \(\psi = \eta\). The rest the proof follows from Theorem 6.

9. Proof of Corollary 1.3

We choose \(r = y\) and \(\psi_j = \lambda_y\), \(j = 1, 2\). Let \(\alpha_j = \mu * \lambda_y\), \(j = 1, 2\). On the basis of this choice, we find that \(\mathbb{D}(\lambda_y, 1; r, n) = 0\) and
\[
\alpha_j(P^t) = \begin{cases} 2(-1)^t & \text{if } \deg P \leq y \\ 0 & \text{if } \deg P > y. \end{cases}
\]

We use Theorem 11 to obtain
\[
\frac{1}{q^n} \sum_{f \in \mathcal{M}_n} \lambda_y(f)\lambda_y(f + h) \leq |Q(n)| + O((yq^y)^{-\frac{1}{2}} + q^{(1-2\alpha)n} \exp \left(\frac{cq^\alpha y}{y}\right)),
\]
where \(Q(n)\) is defined by (7). Since \(\deg(h) \leq y\) then we have
\[
Q(n) = \prod_{\deg P \leq y} \sum_{m_1 = 0}^{\infty} \sum_{m_2 = 0}^{\infty} \frac{\alpha_1(P^{m_1})\alpha_2(P^{m_2})}{q^{[m_1, m_2]_h} \deg P} = \prod_{\deg P \leq y} v_P.
\]
Note that $\alpha_1 = \alpha_2 = \alpha_3$ (say). We define the non-negative integer $k(P)$ such that $P^{k(P)} \| h$. For deg $P \leq y$, we get

$$v_P = \sum_{m=0}^{k(P)} \frac{\alpha_3(P^m)^2}{q^m \deg P} + 2 \sum_{m=0}^{k(P)} \alpha_3(P^m) \sum_{l=m+1}^{\infty} \frac{\alpha_3(P^l)}{q^{l \deg P}} \left( 1 + 4 \sum_{m=0}^{k(P)} \frac{1}{q^m \deg P} \right) + 4 \sum_{l=1}^{\infty} \frac{(-1)^l}{q^l \deg P}$$

$$+ 8 \sum_{m=1}^{k(P)} \frac{(-1)^{2m+1}}{q^{(m+1) \deg P}} \sum_{j=0}^{\infty} \frac{(-1)^j}{q^j \deg P} = 1 - 4 \frac{q^{k(P) \deg P}}{q^{\deg P} + 1}.$$ 

Finally, using Lemma 3 and Lemma 4 and the hypothesis that deg$(h) \leq y$, we have

$$Q(n) = \prod_{\deg P \leq y} \left( 1 - \frac{4}{q^{\deg P} + 1} \right) \prod_{\deg P \leq y} \left( 1 - 4 \frac{q^{k(P) \deg P}}{q^{\deg P} + 1} \right) \left( 1 - \frac{4}{q^{\deg P} + 1} \right)^{-1}$$

$$\leq C_1 \exp \left( -4 \sum_{\deg P \leq y} q^{-\deg P} + 4 \sum_{\deg P \leq y} q^{-\deg P} \right) \leq C_1 \frac{\log y}{y^4}.$$ 

Using the Hypothesis that $2 \leq y \leq \log n$, we have

$$q^{(1-2\alpha)n} \exp \left( \frac{Cq^ny}{y} \right) \ll (y \log y)^{-1}.$$ 

Combining the above estimates we conclude the proof.

10. Proof of Theorem 7
In the case of monic polynomials, the distribution function is

$$F_n(x) := \frac{1}{q^n} \nu_n \{ f; \psi_1(f + h_1) + \psi_2(f + h_2) \leq x \}$$

and the corresponding characteristic function is

$$\phi_n(t) = \frac{1}{q^n} \sum_{f \in M_n} \exp (it (\psi_1(f + h_1) + \psi_2(f + h_2))).$$

We observe that

$$\sum_P \exp(it \psi_j(P)) - 1 = t \sum_{|\psi_j(P)| \leq 1} \psi_j(P) + O \left( t^2 \sum_{|\psi_j(P)| \leq 1} \frac{\psi_j^2(P)}{q^{\deg P}} + \sum_{|\psi_j(P)| > 1} q^{-\deg P} \right).$$

Therefore from the hypothesis of the theorem, we can say that $\phi(t)$ is convergent for every real $t$. Further, the infinite product $\phi(t)$ is continuous at $t = 0$ because it converges uniformly for $|t| \leq T$ where $T > 0$ is arbitrary.

Also, for $j = 1, 2$, we have

$$\mathbb{D}(\psi_j(P), 1; \infty) \ll t^2 \sum_{|\psi_j(P)| \leq 1} \frac{\psi_j^2(P)}{q^{\deg P}} + \sum_{|\psi_j(P)| > 1} q^{-\deg P}.$$ 

So, using the hypothesis of the theorem we see that $\psi_j$ is close to 1 and choosing $r = \log n$ in Theorem 1, we get that the remainder term disappears when $n \to \infty$. 

Thus the characteristic function $\phi_n(t)$ has the limit $\phi(t)$ for every real $t$ and this limit is continuous at $t = 0$. Therefore by Lemma [16], we get the required Theorem [17].

In the case of monic irreducible polynomials, the distribution function is

$$\tilde{F}_n(x) := \frac{1}{|P_n|} \nu_n \{ P; \psi_1(P + h_1) + \psi_2(P + h_2) \leq x \}$$

and the corresponding characteristic function is

$$\tilde{\phi}_n(t) = \frac{1}{|P_n|} \sum_{P \in P_n} \exp (it(\psi_1(P + h_1) + \psi_2(P + h_2))).$$

Following a similar argument as above, we complete the proof.

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