On Rainbow-$k$-Connectivity of Random Graphs

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Abstract

A path in an edge-colored graph is called a rainbow path if the edges on it have distinct colors. For $k \geq 1$, the rainbow-$k$-connectivity of a graph $G$, denoted $rc_k(G)$, is the minimum number of colors required to color the edges of $G$ in such a way that every two distinct vertices are connected by at least $k$ internally vertex-disjoint rainbow paths. In this paper, we study rainbow-$k$-connectivity in the setting of random graphs. We show that for every fixed integer $d \geq 2$ and every $k \leq O(\log n)$, $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n, p)) \leq d$. This substantially generalizes a result in [Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster, On rainbow connection, Electron. J. Comb., 15, 2008], stating that $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_1(G(n, p)) \leq 2$. As a by-product, we obtain a polynomial-time algorithm that makes $G(n, p)$ rainbow-$k$-connected using at most one more than the optimal number of colors with probability $1 - o(1)$, for all $k \leq O(\log n)$ and $p = n^{-\epsilon(1 \pm o(1))}$ for any constant $\epsilon \in [0, 1)$.

1 Introduction

All graphs considered in this paper are finite, simple, undirected and contain at least 2 vertices. We follow the notation and terminology of [3]. The following notion of rainbow-$k$-connectivity was proposed by Chartrand et al. [8, 9] as a strengthening of the canonical connectivity concept in graphs. Given an edge-colored graph $G$, a path in $G$ is called a rainbow path if its edges have distinct colors. For an integer $k \geq 1$, an edge-colored graph is called rainbow-$k$-connected if any two different vertices of $G$ are connected by at least $k$ internally vertex-disjoint rainbow paths. The rainbow-$k$-connectivity of $G$, denoted by $rc_k(G)$, is the minimum number of colors required to color the edges of $G$ to make it rainbow-$k$-connected. Note that such coloring does not exist if $G$ is not $k$-vertex-connected, in which case we simply let $rc_k(G) = \infty$. When $k = 1$ it is alternatively called rainbow-connectivity or rainbow connection number in literature, and is conventionally written as $rc(G)$ with the subscript $k$ dropped.

Besides its theoretical interest as being a natural combinatorial concept, rainbow connectivity also finds applications in networking and secure message transmitting [6, 11, 15]. The following motivation is given in [6]: Suppose we want to route messages in a cellular network such that each link on the route between two vertices is assigned with a distinct channel. Then the minimum number of used channels is exactly the rainbow-connectivity of the underlying graph.

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Some easy observations regarding rainbow-$k$-connectivity include that $rc_k(G) = 1$ if and only if $k = 1$ and $G$ is a clique, that $rc(G) \leq n-1$ for all connected $G$, and that $rc(G) = n-1$ if and only if $G$ is a tree, where $n$ is the number of vertices in $G$. Chartrand et al. [8] determined the rainbow-connectivity of several special classes of graphs, including complete multipartite graphs. In [9] they investigated rainbow-$k$-connectivity in complete graphs and regular complete bipartite graphs. The extremal graph-theoretic aspect of rainbow-connectivity was studied by Caro et al. [5], who proved that $rc(G) = O(d(n \log d/d))$ with $d$ being the minimum degree of $G$. This tradeoff was later improved to $rc(G) < 20n/d$ by Krivelevich and Yuster [13], and was recently shown to be $rc(G) \leq 3n/(d+1) + 3$ by Chandran et al. [7] which is essentially tight. Chakraborty et al. [6] studied the computational complexity perspective of this notion, proving among other results that given a graph $G$ deciding whether $rc(G) = 2$ is NP-complete.

Another important setting that has been extensively explored for studying various graph concepts is the Erdős-Rényi random graph model $G(n, p)$ [10], in which each of the $\binom{n}{2}$ pairs of vertices appears as an edge with probability $p$ independently from other pairs. we say an event $E$ happens almost surely if the probability that it happens approaches 1 as $n \to \infty$, i.e., $\Pr[E] = 1 - o(n)/(1)$. We will always assume that $n$ is the variable that tends to infinity, and thus omit the subscript $n$ from the asymptotic notations. For a graph property $P$, a function $p(n)$ is called a threshold function of $P$ if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ almost surely satisfies $P$; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ almost surely does not satisfy $P$.

Furthermore, $p(n)$ is called a sharp threshold function of $P$ if there exist two positive constants $c$ and $C$ such that:

- for every $r(n) \geq C \cdot p(n)$, $G(n, r(n))$ almost surely satisfies $P$; and
- for every $r'(n) \leq c \cdot p(n)$, $G(n, r'(n))$ almost surely does not satisfy $P$.

Clearly a sharp threshold function of a graph property is also a threshold function of it; yet the converse may not hold, e.g., the property of containing a triangle [2].

It is known that every non-trivial monotone graph property possesses a threshold function [4][12]. Obviously for every $k, d$, the property $rc_k(G) \leq d$ is monotone, and thus has a threshold. Caro et al. [5] proved that $p = \sqrt{\log n}/n$ is a sharp threshold function for the property $rc_1(G(n, p)) \leq 2$. In this paper, we significantly extend their result by establishing sharp thresholds for the property $rc_k(G(n, p)) \leq d$ for all constants $d$ and logarithmically increasing $k$. Our main theorem is as follows.

**Theorem 1.** Let $d \geq 2$ be a fixed integer and $k = k(n) \leq O(\log n)$. Then $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n, p)) \leq d$.

We also investigate rainbow-$k$-connectivity from the algorithmic point of view. The NP-hardness of determining $rc(G)$ is shown by Chakraborty et al. [6]. We show that the problem (even the search version) becomes easy in random graphs, by designing an algorithm for coloring random graphs to make it rainbow-$k$-connected with near-optimal number of colors.

**Theorem 2.** For any constant $\epsilon \in [0, 1)$, $p = n^{-\epsilon(1+o(1))}$ and $k \leq O(\log n)$, there is a randomized polynomial-time algorithm that, with probability $1 - o(1)$, makes $G(n, p)$ rainbow-$k$-connected using at most one more than the optimal number of colors, where the probability is taken over both the randomness of $G(n, p)$ and that of the algorithm.
Our result is quite strong, since almost all natural edge probability functions \( p \) encountered in various scenarios satisfy \( p = n^{-\epsilon(1+o(1))} \) for some \( \epsilon > 0 \). Note that \( G(n, n^{-\epsilon}) \) is almost surely disconnected when \( \epsilon > 1 \) \cite{10}, which makes the problem become trivial. We therefore ignore these cases.

In Section 2 we present the proof of Theorem 1 and in Section 3 we show the correctness of Theorem 2.

## 2 Threshold of Rainbow-\( k \)-Connectivity

This section is devoted to proving Theorem 1. Throughout the paper “\( \ln \)” denotes the natural logarithm, and “\( \log \)” denotes the logarithm to the base 2. Hereafter we assume \( d \geq 2 \) is a fixed integer, \( c_0 \geq 1 \) a positive constant, and \( k = k(n) \leq c_0 \log n \) for all sufficiently large \( n \). To establish a sharp threshold function for a graph property the proof should be two-fold. We first show the easy direction.

**Theorem 3.** \( rc_k(G(n, (\ln^{1/d} n) / n^{(d-1)/d})) \geq d + 1 \) almost surely holds.

We need the following fact proved by Bollobás \cite{1}.

**Lemma 1** (Restatement of part of Theorem 6 in \cite{1}). Let \( c \) be a positive constant and \( d \geq 2 \) a fixed integer. Let \( p' = (\ln((n^2 / c)^{1/d}) / n^{(d-1)/d}) \). Then,

\[
\lim_{n \to \infty} \Pr[G(n, p') \text{ has diameter at most } d] = e^{-c/2}.
\]

of Theorem 3. Fix an arbitrary \( \epsilon > 0 \) and choose a constant \( c > 0 \) so that \( e^{-c/2} < \epsilon / 2 \). Let \( p' = (\ln((n^2 / c)^{1/d}) / n^{(d-1)/d}) \) and \( p = (\ln(n)^{1/d}) / n^{(d-1)/d} \). Clearly \( p \leq p' \) for all \( n > c \).

By Lemma 1 and the definition of limits, there exists an \( N_1 > 0 \) such that for all \( n > N_1 \),

\[
\Pr[G(n, p') \text{ has diameter at most } d] < e^{-c/2} + \epsilon / 2 < \epsilon, \quad \text{by our choice of } c.
\]

Thus, for every \( n > \max\{c, N_1\} \),

\[
\Pr[G(n, p) \text{ has diameter at most } d] \leq \Pr[G(n, p') \text{ has diameter at most } d] < \epsilon.
\]

Due to the arbitrariness of \( \epsilon \), this implies that the probability of \( G(n, p) \) having diameter at most \( d \) is \( o(1) \). This completes the proof of Theorem 3 since the rainbow-\( k \)-connectivity of a graph is at least as large as its diameter.

We are left with the other direction stated below. Fix \( C = 2^{20} \cdot c_0 \).

**Theorem 4.** \( rc_k(G(n, C(\log n)^{1/d} / n^{(d-1)/d})) \leq d \) almost surely holds.

The key component of our proof of Theorem 4 is the following theorem.

**Theorem 5.** With probability at least \( 1 - n^{-\Omega(1)} \), every two different vertices of \( G(n, C(\log n)^{1/d} / n^{(d-1)/d}) \) are connected by at least \( 2^{10d} c_0 \log n \) internally vertex-disjoint paths of length exactly \( d \).

Before demonstrating Theorem 5 we show how Theorem 4 follows from it.
of Theorem 4. Let \( G \) be an instance of \( G(n, C(\log n)^{1/d}/n^{(d-1)/d}) \) for which the condition in Theorem 5 holds; that is, every two different vertices of \( G \) have at least \( c_1 \log n \) internally vertex-disjoint paths of length \( d \) connecting them, where \( c_1 := 2^{10d}c_0 \). To establish Theorem 4 it suffices to prove that \( rc_k(G) \leq d \) for every such \( G \), since by Theorem 5 the condition holds with probability at least \( 1 - n^{-\Omega(1)} = 1 - o(1) \).

Let \( S = \{1, 2, \ldots, d\} \) be a set of \( d \) distinct colors. We randomly color the edges of \( G \) with colors from \( S \). Fix two distinct vertices \( u \) and \( v \). Let \( P \) be a path of length \( d \) connecting \( u \) and \( v \). The probability that \( P \) becomes a rainbow path under the random coloring is

\[
q := d!/d^d \geq (d/e)^d/d^d \geq 4^{-d},
\]

by Stirling’s formula. Since there are at least \( c_1 \log n \) such paths and they are all edge-disjoint, we can upper-bound the probability that at most \( k - 1 \) of them become rainbow paths by

\[
\binom{c_1 \log n}{k-1} (1 - q)^{c_1 \log n - (k-1)} \leq \binom{c_1 \log n}{c_0 \log n} (1 - 4^{-d})^{(c_1 - c_0) \log n} \leq 2^{c_1 \log n \cdot H(c_0/c_1)} \cdot 2^{-(c_1 - c_0) \log n} = n^{-(4^{-d}(c_1 - c_0) - c_1 \cdot H(c_0/c_1))},
\]

where the second inequality follows from the fact that

\[
\left( \frac{m}{\alpha m} \right) \leq 2^{m \cdot H(\alpha)}
\]

for all constants \( \alpha \in (0, 1) \) and sufficiently large \( m \), \( H \) being the binary entropy function defined as

\[ H(\epsilon) = \epsilon \log(1/\epsilon) + (1 - \epsilon) \log(1/(1 - \epsilon)), \]

and that

\[ 1 - x \leq e^{-x} \leq 2^{-x}, \text{ for all } x \geq 0. \]

It is easy to verify that \( \log x \leq \sqrt{x} \) whenever \( x \geq 100 \). Also, since \( 1 + x \leq e^x \leq 2^{2x} \), we have \( \log(1 + x) \leq 2x \) for all \( x > -1 \). Recalling that \( c_1 = 2^{10d}c_0 > 200c_0 \), we get

\[
H(c_0/c_1) = (c_0/c_1) \log(c_1/c_0) + (1 - c_0/c_1) \log(1 + c_0/(c_1 - c_0)) \leq (c_0/c_1) \sqrt{c_1/c_0} + (1 - c_0/c_1) \cdot 2c_0/(c_1 - c_0) = \sqrt{c_0/c_1} + 2c_0/c_1 \leq 3 \sqrt{c_0/c_1}.
\]

We thus have

\[
4^{-d}(c_1 - c_0) - c_1 \cdot H(c_0/c_1) \geq 4^{-d}(c_1 - c_0) - 3 \sqrt{c_1c_0} = 4^{-d}c_1(1 - 2^{-10d}) - 3 \sqrt{2^{-10d} \cdot c_1^2} \geq 2^{-2d-1}c_1 - 2^{-5d+2}c_1 \geq c_1 \cdot 2^{-2d-2} = c_0 \cdot 2^{10d} \cdot 2^{-2d-2} > 100,
\]

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since $c_0 \geq 1$ and $d \geq 2$. Therefore, the probability that there exist at most $k - 1$ rainbow paths between $u$ and $v$ is at most
\[
n^{-d(c_1-c_0) - c_1 H(c_0/c_1))} < n^{-100}.
\]

By the Union Bound, with probability at least
\[
1 - \binom{n}{2} n^{-100} \geq 1 - n^{-90},
\]
every two distinct vertices of $G$ have at least $k$ internally vertex-disjoint rainbow paths connecting them. In particular, there exists a $d$-coloring of the edges of $G$ under which $G$ becomes $k$-rainbow-connected, implying that $rc_k(G) \leq d$. This concludes the proof of Theorem \[4\].

We now prove Theorem \[5\].

of Theorem \[2\]. Let $p = C(\log n)^{1/d}/n^{(d-1)/d}$ where $C = 2^{20}c_0$. Let $V$ be the set of all vertices in $G(n,p)$. For every $S \subseteq V$ and $u \in S$, let $A(S,u)$ be the event that $u$ is adjacent to at least $pn/10(= Cn^{1/d}(\log n)^{1/d}/10)$ distinct vertices in $V \setminus S$. The following lemma is needed for our proof.

Lemma 2. For every $S, u$ such that $u \in S$ and $|S| \leq d \cdot (pn/10)^{d-1}$,
\[
\Pr[A(S,u)] \geq 1 - 2^{-\Omega(n^{1/d})}.
\]

Proof. Fix $S \subseteq V$ with $|S| \leq d \cdot (pn/10)^{d-1}$, and $u \in S$. We have
\[
|V \setminus S| \geq n - d \cdot (pn/10)^{d-1} = n - d \cdot (C/10)^{d-1} n^{(d-1)/d} (\log n)^{(d-1)/d} \geq n/2,
\]
for all sufficiently large $n$. Take $T$ to be any subset of $V \setminus S$ of cardinality $n/2$. Let $X$ be the random variable counting the number of neighbors of $u$ inside $T$. It is obvious that $X$ can be expressed as the sum of $n/2$ independent random variables, each of which taking 1 with probability $p$ and 0 with probability $1 - p$. Thus $E[X] = pn/2$. By the Chernoff-Hoeffding Bound (see e.g. Theorem 4.2 of \[14\]), we have
\[
\Pr[X < (1 - 4/5)pn/2] \leq \exp(-(1/2)(4/5)^2 (pn/2)) = 2^{-\Omega(n^{1/d})},
\]
which gives precisely what we want. 

We now continue the proof of Theorem \[5\]. Fix $u, v \in V, u \neq v$. Let $S_0 = \{u\}$. A $t$-ary tree with a designated root is a tree whose non-leaf vertices all have exactly $t$ children. Consider the following process of “choosing” a $(pn/10)$-ary tree of depth $d - 1$ rooted at $u$:

1. Let $i \leftarrow 1$ and $S_i \leftarrow \emptyset$.

2. For every vertex $w \in S_{i-1}$ (in an arbitrary order), choose $pn/10$ distinct neighbors of $w$ from the set $V \setminus (\{v\} \cup \bigcup_{j=0}^{i-1} S_j)$, and add them to $S_i$. (Note that $S_i$ is updated every time after the processing of a vertex $w$, so that one vertex cannot be chosen and added to $S_i$ more than once. This ensures that at the end of this step, $|S_i| = (pn/10)^i$.)

3. Let $i \leftarrow i + 1$. If $i \leq d - 1$ then go to Step 2, otherwise stop.
Of course the process may fail during Step 2, since with nonzero probability \( w \) will have no neighbor in \( V \setminus (\{v\} \cup \bigcup_{j=0}^{i} S_j) \). (In fact, with nonzero probability the graph becomes empty.) However, noting that at any time during the process,

\[
|\{v\} \cup \bigcup_{j=0}^{i} S_j| \leq 1 + \sum_{j=0}^{d-1} (pn/10)^j \leq d \cdot (pn/10)^{d-1}, \text{ for all sufficiently large } n,
\]

we can deduce from Lemma 2 that every execution of Step 2 fails with probability at most \( 2^{-\Omega(n^{1/d})} \).

Since Step 2 can be conducted for at most \( d \cdot (pn/10)^{d-1} \) times, we obtain that, with probability at least

\[
1 - d \cdot (pn/10)^{d-1} \cdot 2^{-\Omega(n^{1/d})} = 1 - 2^{-\Omega(n^{1/d})},
\]

the process will successfully terminate. At the end of the process, the sets \( S_0, S_1, \ldots, S_{d-1} \) naturally induces a \((pn/10)\)-ary tree \( T \) of depth \( d - 1 \) rooted at \( u \), with \( S_i \) being the collection of vertices in the \( i \)-th level. The number of leaves in \( T \) is exactly \( |S_{d-1}| = (pn/10)^{d-1} \).

Now we assume that \( T \) has been successfully constructed. Let \( Y \) be a random variable denoting the number of neighbors of \( v \) inside \( S_{d-1} \). (Recall that \( v \notin S_{d-1} \).) It is clear that

\[
E[Y] = p \cdot |S_{d-1}| = p^d n^{d-1}/10^{d-1} = 10 \cdot (C/10)^d \log n.
\]

As before, using the Chernoff-Hoeffding Bound, we have

\[
\Pr[Y < (C/10)^d \log n] \leq \exp(-1/2)(9/10)^2(C/10)^d \cdot 10 \log n \leq n^{-10},
\]

for our choice of \( C \).

It is clear that each neighbor \( v' \) of \( v \) inside \( S_{d-1} \) induces a length-\( d \) path between \( u \) and \( v \) (by simply combining the path from \( u \) to \( v' \) in tree \( T \) and the edge \((v', v)\)). The problem is that these paths may not be internally vertex-disjoint. We next address this issue.

For every \( w \in S_1 \), denote by \( T_w \) the subtree of \( T \) of depth \( d - 2 \) rooted at \( w \). Call these \( T_w \) vice-trees. Clearly every vice-tree contains \((pn/10)^{d-2}\) leaves.

The reason for defining such vice-trees is that, by simple observations, any two leaves of \( T \) that belong to different vice-trees must correspond to edge-disjoint root-to-leaf paths in \( T \). Thus, to establish a large number of internally vertex-disjoint paths between \( u \) and \( v \), it suffices to show that we can find many neighbors of \( v \) inside \( S_{d-1} \) that belong to distinct vice-trees.

For each vice-tree \( T_w \), let \( B_w \) be the event that \( v \) has at least \( 10d \) neighbors inside the set of leaves of \( T_w \). Noting that \( T_w \) has exactly \((pn/10)^{d-2}\) leaves, we have

\[
\Pr[B_w] \leq \left( \frac{(pn/10)^{d-2}}{10d} \right)^{10d} = \left( \frac{(Cn^{1/d}(\log n)^{1/d}/10)^{d-2}}{10d} \right)^{10d} \cdot \left( \frac{C(\log n)^{1/d}}{n^{(d-1)/d}} \right)^{10d} \leq \left( \frac{(Cn^{1/d}(\log n)^{1/d}/10)^{d-2}}{n^{(d-1)/d}} \right)^{10d} \leq C^{10d} (C/10)^{10d(d-2)}(\log n)^{10(d-1)} n^{-10} \leq O(n^{-9}).
\]
By applying the Union Bound, we obtain
\[
\Pr[ \bigvee_{w \in S_1} B_w ] \leq \frac{pn}{10} \cdot O(n^{-9}) \leq O(n^{-7}).
\]

Combined with previous results, we deduce that with probability at least
\[
1 - 2^{-\Omega(n^{1/d})} - n^{-10} - O(n^{-7}) \geq 1 - O(n^{-6}),
\]
the following three events simultaneously happen:

1. The tree $T$ is successfully constructed.
2. $v$ has at least $(C/10)^d \log n$ neighbors that are leaves of $T$.
3. Every vice-tree $T_w$ contains at most $10d$ leaves that are neighbors of $v$.

When all these three events happen, we can choose $(\frac{C}{10}d/10d) \log n$ neighbors of $v$, every two of which come from different vice-trees. This immediately leads to $(\frac{C}{10}d/(10d)) \log n \geq 2^{10d} c_0 \log n$.

Using the Union Bound again gives us that, with probability at least
\[
1 - \frac{n}{2} \cdot O(n^{-6}) = 1 - n^{-\Omega(1)},
\]
every two distinct vertices have at least $2^{10d} c_0 \log n$ internally vertex-disjoint paths of length $d$ connecting them. The proof of Theorem 5 is thus completed.

3 Rainbow-coloring Random Graphs

In this section we prove Theorem 2.

As explained in Theorem 2, first note that for every $G$ with at least 2 vertices, $rc_k(G) = 1$ if and only if $k = 1$ and $G$ is a clique, which can be easily checked. Thus, in the following we assume w.l.o.g. that $rc_k(G(n,p)) \geq 2$.

It is easy to see that there exists a (unique) integer $d \geq 2$ such that $(d - 2)/(d - 1) \leq \epsilon < (d - 1)/d$. We have $p = \omega ((\log n)^{1/d}/n^{(d-1)/d})$, which, by Theorem 4 implies that $rc_k(G(n,p)) \leq d$ almost surely holds. Moreover, a scrutiny into the proof of Theorem 5 tells us that for such $p$, a random coloring of $G(n,p)$ using $d$ colors will make it rainbow-$k$-connected with probability $1 - n^{-\Omega(1)}$. Thus, it suffices for us to show that with probability $1 - o(1)$, $rc_k(G(n,p)) \geq d - 1$.

This is trivial for $d \leq 3$, since we have assumed that $rc_k(G(n,p)) \geq 2$. When $d \geq 4$, we have $p = o ((\log n)^{1/(d-2)}/n^{(d-3)/(d-2)})$. Due to Theorem 3, $G(n,p)$ with such $p$ almost surely satisfies $rc_k(G(n,p)) \geq d - 1$.

Hence, we have shown that with probability $1 - o(1)$, a random coloring with $d$ colors will make $G(n,p)$ rainbow-$k$-connected and the number of colors used is at most one more than the optimum, where the probability is taken over both the randomness of $G(n,p)$ and that of the algorithm. This completes the whole proof.
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