On the number of vertices in integer linear programming problems

Overview

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Abstract

We give a survey of work on the number of vertices of the convex hull of integer points defined by the system of linear inequalities. Also, we present our improvement of some of these.

Introduction

In the integer linear programming problem, it is required to maximize the linear function \( \sum_{j=1}^{n} c_j x_j \), subject to \( \sum_{j=1}^{n} a_{ij} x_j \leq b_i \) \((i = 1, \ldots, n)\), \( x_j \in \mathbb{Z} \) \((j = 1, \ldots, n)\) where \(a_{ij}, c_j, b_i\) are integers. We can formulate this in the following matrix form:

\[
\begin{align*}
\text{max } & \quad cx \\
\text{subject to } & \quad Ax \leq b, \\
& \quad x \in \mathbb{Z}^n,
\end{align*}
\]

where \(A = (a_{ij}) \in \mathbb{Z}^{m \times n}, b = (b_i) \in \mathbb{Z}^m, c = (c_j) \in \mathbb{Z}^n\). The set of all solutions to (2) is a polyhedron \(P \subseteq \mathbb{R}^n\). Consider the set \(M(A,b) = P \cap \mathbb{Z}^n\) and its convex hull \(P_I\). It is known that \(P_I\) is a polyhedron. If the maximum (1) over the set \(M(A,b)\) associated with (2)3 is finite then it is attained at some vertex of \(P_I\); on the other hand, the facets of \(P_I\) are the strongest regular cuts. Therefore examining a characterization of \(P_I\) can indicate ways of generating effective algorithms or understand why constructing such algorithms are impossible \[EKKS81, Sch86, She97\].

Let \(N(A,b)\) be the set of vertices (extreme points) of the polyhedron \(P_I\). Here we give a survey of work on the cardinality \(|N(A,b)|\). Also, we give one improvement (see Theorem 3).

\[http://www.uic.nnov.ru/~zny\]
It was apparently Rubin [Rub70] who first showed that there is no function \( g(n, m) \) such that for any \( A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m \) it holds \( |N(A, b)| \leq g(n, m) \). In the papers considered below, upper bounds on \( |N(A, b)| \) are (as a rule) functions of three arguments \( n, m, \alpha \) where \( \alpha = \max \{|a_{ij}| (i = 1, \ldots, m; j = 1, \ldots, n)\} \). Another way of upper bounds determination is to use the length of the expression for the coefficients in \( Ax \leq b \). Both ways are equivalent in the following sense. Let

\[
\phi = \max \left\{ 1 + \sum_{j=1}^{n} \left( 1 + \log(1 + |a_{ij}|) \right) + \log(1 + |b_i|) \right\},
\]

then \( \phi \leq (1 + n)(1 + \log(1 + \alpha)) \) and, on the other hand, \( \log(1 + \alpha) \leq \phi \). Hence, if for some function \( g(\cdot, \cdot, \cdot) \) it holds that \( |N(A, b)| \leq g(n, m, \phi) \), then \( |N(A, b)| \leq g\left(n, m, (1 + n)(1 + \log(1 + \alpha))\right) \); on the other hand, if for some function \( f(\cdot, \cdot, \cdot) \) it holds that \( |N(A, b)| \leq f(n, m, \log(1 + \alpha)) \), then \( |N(A, b)| \leq f(n, m, \phi) \).

Together with observation of the number of vertices in \( P_I \) associated with an arbitrary system \( Ax \leq b \), we consider the following important case of the problem. Let \( a_0, a_1, \ldots, a_n \) be natural numbers, \( a = (a_1, \ldots, a_n) \), \( \alpha = \max \{a_0, \ldots, a_n\} \). Denote by \( M(a, a_0) \) the set of integer non-negative solutions to the inequality

\[
\sum_{j=1}^{n} a_j x_j \leq a_0,
\]

let \( N(a, a_0) \) be the set of vertices of the convex hull of \( M(a, a_0) \). The convex hull of \( M(a, a_0) \) is called the knapsack polytope.

We note that Chapter 3 in [She97] is specially devoted to topics considered here. We consider also other results.

Below Conv \{\( v_1, \ldots, v_l \)\} and Cone \{\( v_1, \ldots, v_l \)\} are respectively the convex hull of vectors \( v_1, \ldots, v_l \) in \( \mathbb{R}^n \) and the cone generated by these; \( \lceil \alpha \rceil \) is the maximal integer not exceeding the number \( \alpha \) in \( \mathbb{R} \), \( \log \alpha \) denotes the logarithm of \( \alpha \) to the base 2.

## 1 Upper bounds on the number of vertices

An approach of Shevchenko [She81] has been widely used for determination of upper bounds on the number of vertices in integer linear programming problems. It was developed in [She84], [She92], [CSb], [She97]. The following property lies at the basis of it. We say that a set \( X \subset \mathbb{Z}^n_+ \) has the separation property if for any distinct points \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in \( X \) there is an index \( j \) such that \( 2y_j < x_j \). It has been proved in [She81] that if max \( \{x_j, x \in X\} \leq k_j - 1 \) (\( j = 1, \ldots, n - 1 \)) then

\[
|X| \leq \prod_{j=1}^{n-1} \left( 1 + \log(k_j - 1) \right).
\]  

(4)

Shevchenko’s approach consists of the following steps:

1. a one-to-one mapping of the set \( N(A, b) \) into the set \( X \) with separation property is constructed;
2. quantities $k_j$ are estimated and a bound (4) is written out.

As shown in [She81], the set $N'(A, b)$ of all vertices of the convex hull of $\{x \in \mathbb{Z}^n, Ax = b, x \geq 0\}$ has the separation property. This and (4) lead us to the bound

$$|N'(A, b)| \leq \left(1 + \log n + r \log(\alpha \sqrt{r})\right)^{r-1},$$

(5)

where $r$ is the rank of $A$ and $\alpha$ is the maximum of the moduli of coefficients in the system $Ax = b$.

This result can be applied to an arbitrary system $Ax \leq b$. Using the standard substitution $x'_j = \max\{x_j, 0\}$, $x''_j = -\min\{x_j, 0\}$, $y = A_0 - Ax$ we lead the system $Ax \leq b$ to the form

$$Ax' - Ax'' + y = a_0,$$

$$x' \in \mathbb{Z}^n, x'' \in \mathbb{Z}^n, y \in \mathbb{Z}^m,$$

$$x' \geq 0, x'' \geq 0, y \geq 0.$$  

(6)

It is easy to verify that different points in $N(A, b)$ pass into different vertices of the convex hull of the set of solutions to the system (6). From (5) we get the bound [She81]:

$$|N(A, b)| \leq \left(1 + \log(n + 1) + n \log(\alpha \sqrt{n})\right)^{2n+m-1}.$$

For fixed $n$ and $m$ it has the form of polynomial in $\log \alpha$. This bound was subsequently improved in [CHKM92, CSb, Chi97], and took the form of some polynomial in $m$ and $\log \alpha$ for any fixed $n$ (see below).

For the knapsack problem we introduce the variable $x_{n+1} = a_0 - \sum_{j=1}^{n} a_j x_j$ and use the bound (4). Then we get [She81]:

$$|N(a, a_0)| \leq \prod_{j=1}^{n} \left(1 + \log \left(1 + \frac{a_0}{a_j}\right)\right).$$

(7)

The use of this approach for a square system of linear inequalities see in [She84].

One more approach to estimate the number of vertices in integer linear programming problems was proposed by Hayes and Larman [HL83] and developed by Morgan [Mor91] and Cook et al. [CHKM92]. It can be considered as a variant of Shevchenko’s method. Let $P$ be a polyhedron, defined by the system $Ax \leq b$. We divide $P$ into reflecting sets $P_1, \ldots, P_l$ such that for any two points $x, y$ in $\mathbb{Z}^n \cap P_i$ either $2x - y$ or $2y - x$ belongs to $P_i$. Then it is not hard to verify that no reflecting set in the division of $P$ contains more than one point in $N(A, b)$. Thus, $|N(A, b)| \leq l$. For obtaining a bound it is enough to divide $P$ into the “small” number of reflecting sets.

Let us consider the use of this approach [HL83] for estimating the number of vertices in the knapsack polytope. The polytope

$$P = \left\{ x \geq 0, \sum_{j=1}^{n} a_j x_j \leq a_0 \right\}$$

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is divided into boxes
\[ x = (x_1, \ldots, x_n) \in \mathbb{R}^n, 2^{i_j - 1} \leq x_j < 2^{i_j}, j = 1, \ldots, n, \] where \( i_j \in \mathbb{Z}, \) \( 1 \leq i_j \leq 1 + \log \left( \frac{a_0}{a_j} + 1 \right) \) \( (j = 1, \ldots, n). \) It is obvious that no box contains more than one point in \( \mathcal{N}(a, a_0). \) This again leads us to the bound (7).

As Morgan [Mor91] note, some boxes (8) can fail to contain points in \( \mathcal{N}(a, a_0). \) The reason is that they are “too far” from the facets of \( P. \) For the estimate of \( |\mathcal{N}(a, a_0)| \) it is enough to bound the number of boxes (8) which intersect with the hyperplane \( \sum_{j=1}^{n} x_j c_j = 1, \) where \( c_j = \left\lfloor \frac{a_0}{a_j} \right\rfloor \) \( (j = 1, \ldots, n). \) It is not hard to prove [Mor91] that for \( n \geq 2 \) the number of such boxes is not exceed the quantity
\[ n \log(2n) \left( 1 + \log \left( 1 + \frac{a_0}{\gamma} \right) \right)^{n-1}, \]
where \( \gamma = \min \{|a_j|, j = 1, \ldots, n\}. \) Thus, we have

**Theorem 1** [Mor91] For \( n \geq 2 \) it holds that
\[ |\mathcal{N}(a, a_0)| \leq n \log(2n) \left( 1 + \log \left( 1 + \frac{a_0}{\gamma} \right) \right)^{n-1}. \] (9)

Cook et al. [CHKM92] generalized this approach for the estimate of the number of vertices \( \mathcal{N}(A, b) \) if \( A \in \mathbb{Q}^{m \times n} \) and \( b \in \mathbb{Q}^m \) are arbitrary. If the length of the expression for the each inequality of \( Ax \leq b \) is not more than \( \varphi, \) then \( |\mathcal{N}(A, b)| \leq 2m^n(6n^2\varphi)^{n-1}. \) For systems with integer coefficients we have \( \varphi \leq (n + 1)(1 + \log(1 + \alpha)) \) and consequently
\[ |\mathcal{N}(A, b)| \leq m^n(6n^4 \log(1 + \alpha))^{n-1}. \] (10)
Thus, \( |\mathcal{N}(A, b)| \leq c_n m^n \log^{n-1}(1 + \alpha), \) where \( c_n \) is some quantity depending only on \( n. \)

Chirkov [Chi97] lowers the exponent of \( m \) to \( \lfloor n/2 \rfloor \) in the last estimate. Let \( v_0, \ldots, v_k, v_{k+1}, \ldots, v_r \) be vectors in \( \mathbb{R}^n \) such that the system
\[ (v_0, 1), \ldots, (v_k, 1), (v_{k+1}, 0), \ldots, (v_r, 0) \]
is linear independent. The set
\[ \text{Conv} \{v_0, \ldots, v_k\} + \text{Cone} \{v_{k+1}, \ldots, v_r\} \]
is called the *generalized simplex* with dimension \( r. \) Denote by \( \text{CharCone}(P) \) the *cone of directions* of the polyhedron \( P \) (\( \text{CharCone}(P) \) consists of vectors \( y \) such
that for any \( x \in P \) and \( \gamma > 0 \) the point \( x + \gamma y \) belongs to \( P \). The set of 
generalized simlexes \( S_1, \ldots, S_l \) is called the covering of \( P \) if \( P = \bigcup_{i=1}^{l} S_i \) and for 
any \( i = 1, \ldots, l \) every vertex of \( S_i \) and every edge of \( \text{CharCone}(S_i) \) are a vertex of \( P \) and an edge of \( \text{CharCone}(P) \) respectively. Let 

\[
\xi_n(m) = \left( m - \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \right) + \left( m - \left\lfloor \frac{m}{2} \right\rfloor - 1 \right).
\]

As shown in [Chi], for a polyhedron \( P \) there is a covering containing at most 
\( n! \xi_n(M) \) generalized simplexes. Denote by \( N_i \) the set of vertices of the convex 
hull of integer points in \( S_i \). For the polytope \( P \) we obviously have 
\( N(A, b) \subseteq \bigcup_{i=1}^{l} N_i \). Chirkov [Chi97] bounded the quantity of coefficients in the linear sys-
tem associated with \( S_i \) in terms of \( n \) and \( \alpha \) and estimated 
\( |N_i| \) using (10). These 
led to the following bound.

**Theorem 2** [Chi97]

\[
|N(A, b)| \leq n^7 \xi_n(m) \left( 6 \log(1 + \alpha) + 3 \log n \right)^{n-1}.
\]

From Theorem 2 it follows that \( |N(A, b)| \leq c_n m^{\lfloor n/2 \rfloor} \log^{n-1}(1 + \alpha) \), where \( c_n \) is some quantity depending only on \( n \). In Sect. 3 we show that for any fixed \( n \) this estimate can not be improved with respect to order.

**Note 1** It is known [McM70] that the number of facets of a polyhedron \( P \) in \( \mathbb{R}^n \) with \( v \) vertices is at most \( \xi_n(v) \). From this and estimates above it follows that for any fixed \( n \) the number of facets in \( P \) is bounded above by some poly-
nomial in \( m \) and \( \log(1 + \alpha) \). This result and Lenstra’s algorithm [Len83] allowed 
Shevchenko [She84, She97] to construct a quasipolynomial (i.e. polynomial for 
fixed \( n \)) algorithm for finding all facets and vertices in \( P \).

**Note 2** Chirkov [Chi97] has obtained the upper bounds on the number of ver-
tices in the polytope \( P \), as function of its metric characteristics, i.e. the diame-
ter \( d = \max_{x,y \in P} \max_{j=1,\ldots,n} |x_j - y_j| \) and the volume \( V \). It has be proved that for any fixed \( n \) the quantity \( |N(A, b)| \) is bounded above by some polynomial in \( m \) and \( \log(1 + d) \), with its highest term being \( c_n m^{\lfloor n/2 \rfloor} \log^{n-1}(1 + d) \). If \( P \) has full affine dimension, then an analogous result holds also for the volume of \( P \).

## 2 Upper bounds independent of the right-hand sides

In this section we consider methods for determination of upper bounds on 
\( |N(A, b)| \) independent of \( b \). These bounds are useful if entries of a vector \( b \) are more greater than

\[
\alpha_1 = \max \{|a_{ij}| : (i = 1, \ldots, m; j = 1, \ldots, n)\}.
\]
Let the rank of $A$ be equal to $n$ and the solution set $P$ associated with $Ax \leq b$ be not empty. Then denote by $V(A, b)$ the set of vertices of $P$, and by $\Delta(A)$ the maximum of the moduli of $n$th-order minors of $A$, and by $a_i$ the $i$th row of $A$.

Following [She97], Sect. 3.6.2, let

$$J(v) = \{ i, b_i - a_i v < n \Delta(A) \}, \quad M(v) = \{ x \in \mathbb{Z}^n, a_i x \leq b_i \ \text{for} \ i \in J(v) \}$$

for every vertex $v$ of $P$. Denote by $N(v)$ the set of vertices of the convex hull of $M(v)$. In [She97] it has proved that

$$N(A, b) \subseteq \bigcup_{v \in V(A, b)} N(v).$$

Let $[v]$ denote a vector obtained from $v$ by rounding off the components of $v$. As in [She97], we make the change of variables $x = x' + [v]$ and let $b_i' = b_i - a_i v$. The system of inequalities describing $M(v)$ passes into the system $a_i x' \leq b_i'$, $i \in J(v)$, and, since for $i \in J(v)$ it holds that

$$|b_i'| = |(b_i - a_i v) + (a_i v - a_i [v])| < n \Delta(A) + \frac{na_i}{2},$$

then Theorem 2 leads us to

**Theorem 3**

$$|N(A, b)| \leq c_n m \frac{\lfloor n/2 \rfloor}{n - 1} \log^{n-1}(1 + \alpha_1),$$

where $c_n$ is some quantity depending only on $n$.

Let us now consider the knapsack polytope. For $i = 1, \ldots, n$ we denote by $M_i$ the solution of solutions to the system

$$\sum_{j=1}^{n} a_{ij} x_j \leq a_i, \quad x_j \in \mathbb{Z}, \quad x_j \geq 0 \quad (j = 1, \ldots, i - 1, i + 1, \ldots, n),$$

and by $N_i$ the set of vertices of Conv $M_i$, and set $N_0 = \{0\}$.

**Theorem 4** [She81]

1. $|N_i| \leq \left( \frac{\lfloor \log a_i \rfloor + n - 1}{n - 1} \right)$;

2. if $a_0 \geq \alpha_1 (\alpha_1 - 1)$, then $N(a, a_0) = \bigcup_{i=0}^{n} N_i$ and

$$|N(a, a_0)| \leq 1 + \sum_{j=1}^{n} \left( \frac{\lfloor \log a_i \rfloor + n - 1}{n - 1} \right) \leq 1 + n(\lfloor \log \alpha_1 \rfloor + 1)^{n-1}.$$
3 Lower bounds on the number of vertices

The natural approach for determination of lower bounds on the number of vertices in integer linear programming problems is constructing special examples. Rubin’s work [Rub70] was the first in this area. Rubin found the class of polyhedra with arbitrarily many (but finite) vertices and facets. The kth polytope in the class has k+3 vertices; it is given by the inequality \( F_{2k}x + F_{2k-1}y \leq F_{2k+1}^2 - 1 \), where \( F_s \) is the sth Fibonacci number, i.e. \( F_1 = F_2 = 1, F_s = F_{s-1} + F_{s-2} \) (s = 3, 4, ...).

Veselov and Shevchenko [VS] also investigated the two-dimensional knapsack problem. For naturals \( a, b, c \), we denote by \( \mathcal{N}(a,b,c) \) the set of vertices of the knapsack polytope defined by the inequality \( ax + by \leq c \). Let \( \beta_1 = 1, \beta_2 = 2, \gamma_2 = \gamma'_2 = 1 \), and, for every natural \( s \geq 2 \), let \( \beta_{s+1} = 2\beta_s + \beta_{s-1}, \gamma_{s+1} = \beta_{s+1} - \beta_s + \gamma_s \). We note that

\[
\beta_s = \frac{(1 + \sqrt{2})^s - (1 - \sqrt{2})^s}{2\sqrt{2}}.
\]

**Theorem 5** [VS] If \( a < \beta_{s-1} \), then \( |\mathcal{N}(a,b,c)| < 2s \). If \( a = \beta_{s-1} \), then there are \( b, c \) such that for \( b = \beta_s \) and

\[
c = \begin{cases} 
\gamma_s(\beta_{s-1} + \beta_s) & \text{for even } s, \\
\gamma_{s+1}\beta_{s-1} + \gamma_{s-1}\beta_s & \text{for odd } s
\end{cases}
\]

it holds that \( |\mathcal{N}(a,b,c)| = 2s \).

Thus, \( \beta_{s-1} \) is the minimal value of \( a \) such that there exist \( b, c \) for which \( \beta_{s-1} = a \leq b \leq c \) and \( |\mathcal{N}(a,b,c)| = 2s \). We remark that, by Theorem 5, \( |\mathcal{N}(a,b,c)| \leq 2\log_{1 + \sqrt{2}}(1 + 2\sqrt{2}a) \). Let \( b_s = \min \{ b, b \geq a, |\mathcal{N}(a,b,c)| \geq s \} \). Veselov and Shevchenko proved [VS] that \( b_{2s} = \beta_{2s} \), \( b_{2s+1} = \beta_{s+1} + \beta_{s-1} \). Thus, the upper bounds for the two-dimensional knapsack problem can not be improved. Analogous results has been obtained [VS] for square systems of linear inequalities and for systems of congruences.

For \( n = 3 \), Morgan [Mor91] constructed the class of polytopes with \( m = 5 \) such that \( |\mathcal{N}(A,b)| \) grows proportionally to \( \log^2 \alpha \). Let \( P \) be a polytope, given by the system

\[
\begin{align*}
 x & \geq 0, y \geq 0, z \geq 0, \\
x + \psi y + \theta z & \leq \nu, \\
x + \theta y + \varphi z & \leq \nu,
\end{align*}
\]

where \( \theta = 2\cos \left( \frac{\pi}{7} \right), \varphi = 2\cos \left( \frac{4\pi}{7} \right), \psi = 2\cos \left( \frac{3\pi}{7} \right) \). It is shown [Mor91] that \( |\mathcal{N}(A,b)| \geq \frac{1}{32} \log^2 \nu \).

Another construction is offered in [BHL92]. For any \( n \geq 1 \), Bárány et al. constructed a polyhedron \( P \) with \( m = 2n^2 \) such that \( |\mathcal{N}(A,b)| \geq c_n \varphi^{n-1} \), where \( c_n \) is some positive quantity depending only on \( n \), and \( \varphi \) is the length of the expression for the rational coefficients in a system describing \( P \). (From this it follows that for any fixed \( m \) and \( n \) the bound (2) cannot be improved.)
with respect to order. We note that a stronger result follows from presented below earlier Veselov’s work [Ves]. Chirkov established [Chi95] that the upper bound (2) cannot be improved with respect to order even if only $n$ (but not $m$) is fixed.

We present another example. As shown in [She81], the vertex set $\mathcal{N}(a, a_0)$ of the knapsack polytope with inequality

$$2^{n-1}x_1 + \ldots + 2x_{n-1} + x_n \leq 2^n - 1$$

consists of $2^n$ points. We note that formula (4) yields $\mathcal{N}(a, a_0) \leq n!$. This example shows that upper bounds on $|\mathcal{N}(a, a_0)|$ and $|\mathcal{N}(A, b)|$ must bear (have?) exponential character in $n$.

Another approach, due to Veselov [Ves], consists in constructing a lower bound for the mean number of vertices in an integer linear programming problem. For each vector $a(a_0, a_1, \ldots, a_n)$ with integer non-negative components such that $a_n = 1$ and $0 \leq a_i \leq \Delta - 1$ ($i = 0, \ldots, n - 1$), let us consider the set $\mathcal{M}'(a, \Delta)$ of solutions to the system

$$\sum_{j=1}^{n} a_j x_j \equiv a_0 \pmod{\Delta}, \quad x_j \in \mathbb{Z}, \quad x_j \geq 0 \quad (j = 1, \ldots, n)$$

and the set $\mathcal{N}'(a, \Delta)$ of vertices of the convex hull of $\mathcal{M}'(a, \Delta)$. Then $\varphi(\Delta) = \Delta^{-n} \sum \mathcal{N}'(a, \Delta)$ is the mean number of vertices, where the summation is over all $a$ in the range under consideration.

Veselov [Ves] estimates the number $\sigma(p)$ of $p \in \mathcal{N}'(a, \Delta)$ and finds a bound on $\Delta^{-n} \sum \sigma(p)$, where the summation is over all $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ such that $p_j \geq 0$ ($j = 1, \ldots, n$) and $(p_1 + 1) \cdots (p_n + 1) \leq \Delta$. It is obvious, that $\varphi(\Delta) = \Delta^{-n} \sum \sigma(p)$. The resulting bound has the following form:

$$\varphi(\Delta) \geq \frac{c_n}{(n - 1)^{n-1}} \left(\log \Delta - n - 2 - n \log(n - 1)\right)^{n-1},$$

where $c_n = \left(4n3^n((n - 1)!)^2\right)^{-1}$.

Now let us consider the set $\varphi(\gamma)$ of vectors $(a_0, \ldots, a_{n-1}, \gamma)$ with integer non-negative components such that $a_i \leq \gamma$ ($i = 1, \ldots, n - 1$), $\gamma(\gamma - 1) \leq a_0 < \gamma^2$. The knapsack polytope given by the inequality

$$a_1 x_1 + \ldots + a_{n-1} x_{n-1} + \gamma x_n \leq a_0$$

is considered for any $(a_0, a) \in \varphi(\gamma)$, where $a = (a_1, \ldots, a_{n-1}, \gamma)$. The quantity $\psi(\gamma) = \gamma^{-n} \sum |\mathcal{N}(a, a_0)|$, where the summation is over all $(a_0, a)$ in $\varphi(\gamma)$, is the mean number of vertices in such a polytope. Since $a_0 \geq \gamma(\gamma - 1)$ and $a_i \leq \gamma$ ($i = 1, \ldots, n - 1$), then, by Theorem [3] $|\mathcal{N}(a, a_0)| \geq |\mathcal{N}'|$, where $\mathcal{N}'$ is the set of vertices of the convex hull of integer solutions to the system

$$a_1 x_1 + \ldots + a_{n-1} x_{n-1} + \gamma x_n \leq a_0,$$

$$x_j \geq 0 \quad (j = 1, \ldots, n - 1).$$
It is obvious that $N'$ is mapped one-to-one into the set of vertices of the convex hull of non-negative solutions to the congruence

$$x_0 + a_1x_1 + \ldots + a_{n-1}x_{n-1} \equiv a_0 \pmod{\gamma},$$

or, equivalently,

$$x_0 + a_1x_1 + \ldots + a_{n-1}x_{n-1} \equiv a_0 - \gamma(\gamma - 1) \pmod{\gamma}.$$ 

Thus, $\psi(\gamma) = \varphi(\gamma)$. In particular, this gives us

**Theorem 6** [Ve3] There exist non-negative $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $a_0 \in \mathbb{Z}$ such that $a_j \leq \alpha$ ($j = 0, \ldots, n$) and

$$|N(a_0, a)| \geq c_n \log^{n-1} \alpha,$$

where $c_n$ is some positive quantity depending only on $n$.

This result shows that for any fixed $n$ the order in (9) can not be decrease.

The treated approach was extended by Chirkov and Shevchenko [CSa, Chi95, Chi96] to the case of an arbitrary system $Ax \leq b$. The following theorem holds.

**Theorem 7** [Chi95, Chi96] For any $n$, $m$ and $\alpha$ there exist a polyhedron such that

$$|N(A, b)| \geq \xi_n(m) \frac{\log^{n-1} \alpha}{4^n + 2^m(n-1)!}.$$

Thus, the lower bound on the number of vertices has the following form:

$$|N(A, b)| \geq c_n m_1^{\lfloor n/2 \rfloor} \log^{n-1} \alpha,$$

where $c_n$ is some positive quantity depending only on $n$. This result shows that for any fixed $n$ the bound (2) cannot be improved.

### 4 Final notes and conclusions

Let us sum up. For the number of vertices in the convex hull of integer points in a polyhedron the following bound has been obtained [Chi97]:

$$|N(A, b)| \leq c_n m_1^{\lfloor n/2 \rfloor} \log^{n-1} (1 + \alpha), \quad (12)$$

where $c_n$ is some quantity depending only on $n$. We have shown that this result is valid even if

$$\alpha = \max \{|a_{ij}| (i = 1, \ldots, m; j = 1, \ldots, n)\}.$$

It is established [Chi95] that this bound can not be improved for any fixed $n$ with respect to order, namely, for any $n$, $m$ and $\alpha$ there exist a polyhedron such that

$$|N(A, b)| \geq c'_n m_1^{\lfloor n/2 \rfloor} \log^{n-1} \alpha,$$
where \( c'_n \) is some positive quantity depending only on \( n \). Analogous result is also valid for the knapsack problem [Ves].

The proofs of these lower bounds are not constructive. Examples of polyhedra with the “large” number of vertices in \( P_I \) are known only for few values of \( m \). In particular, for \( n \to \infty \) and for any positive \( c_n \) no sequence of knapsack polyhedrons with at least \( c'_n \log^{n-1} \alpha \) vertices is known.

From the upper bound (12) on the number of vertices in \( P_I \) and from the inequality [McM70], estimating the number of facets in a polyhedron with \( v \) vertices, follows that the number of facets in \( P_I \) is at most

\[
c_n m^{(n/2)^2} \log^{(n-1)/2}(1 + \alpha).
\]

The problem of an attainability of this bound is open.

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