Bifurcation and instability problems in vortex wakes

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Abstract. A number of instability and bifurcation problems related to the dynamics of vortex wake flows are addressed using various analytical tools and approaches. We discuss the bifurcations of the streamline pattern behind a bluff body as a vortex wake is produced, a theory of the universal Strouhal-Reynolds number relation for vortex wakes, the bifurcation diagram for “exotic” wake patterns behind an oscillating cylinder first determined experimentally by Williamson & Roshko, and the bifurcations in topology of the streamlines pattern in point vortex streets. The Hamiltonian dynamics of point vortices in a periodic strip is considered. The classical results of von Kármán concerning the structure of the vortex street follow from the two-vortices-in-a-strip problem, while the stability results follow largely from a four-vortices-in-a-strip analysis. The three-vortices-in-a-strip problem is argued to be relevant to the wake behind an oscillating body.

1. Introduction
Vortex street wakes are very common in the Reynolds number range from about 100 to 400. Even at much higher Reynolds numbers vortex street wakes may arise, where the vortices now have considerable turbulent fine structure. Understanding the structure and dynamics of vortex wakes is of interest in its own right and as a way of understanding the forces that act on the objects and structures forming such wakes.

While vortex streets had been observed qualitatively for many years, the seminal work of T. von Kármán in 1911-12 [15, 16, 17] provided the first theory of these structures. We return to the kind of modeling introduced by von Kármán in Section 4. In Section 2 we first consider what one can say about the structure of a vortex street from a study of the topology of streamlines in the wake flow immediately behind the bluff body. In Section 3 we then address two bifurcation problems arising in the dynamics of vortex wakes: (i) the well-known relation between the Strouhal number for vortex shedding into the wake and the Reynolds number of the wake-generating flow (see Fig. 3), and (ii) the intriguing bifurcation diagram found experimentally by Williamson & Roshko [34] for vortex wake structure behind an oscillating cylinder (see Fig. 5). Finally, in Section 5, we consider the topology of streamlines in wakes generated by a periodically repeated vortex pair.

This paper represents a combined report on three individual contributions to the Symposium, viz. “Streamline Topology in Vortex Wakes” by M.A.S. and M.B., “Bifurcation of Streamline
Figure 1. Essentials of the bifurcation diagram for the normal form (1). A slice $c_{1,2} = \text{constant}$ in the three-parameter space $c_{0,2}, c_{1,2}, c_{0,3}$ is shown. The smooth closed curves represent possible sequences of topologies in the periodic domain. Reproduced from [5].

Patterns – Methods and Applications” by M.B., and the keynote address “Vortex Dynamics of Wakes” by H.A. The authors have been collaborating on these problems for some time and the Symposium offered a welcome opportunity to articulate elements of our approach.

2. The streamline topology of vortex wake formation

While the dynamics of a developed vortex wake is our main concern, it is of basic interest to understand the creation of the wake structure as $Re$ is increased from very low values. For the flow around a circular cylinder, the process is well described [33, 10]: For $Re < 5$ the flow is attached to the cylinder, and for $5 < Re < 45$ two steady counter-rotating vortices exist behind the cylinder. At $Re = 45$ the flow becomes unsteady in a Hopf bifurcation and one enters the vortex-shedding regime. Hence, in the early stages of the wake development, there are two basic bifurcation phenomena occurring: A topological bifurcation at $Re = 5$, where the steady flow field changes its topology but not its stability, and a dynamic bifurcation at $Re = 45$ where a new family of periodic flows are created.

Topological bifurcations of flow fields are efficiently described with concepts from the theory of dynamical systems. Considering the differential equations for the streamlines as a dynamical system – here of dimension two – bifurcation theory can be applied to describe changes in the streamline pattern under variation of external parameters. This approach has, for instance, successfully been used to locate different regimes of vortex breakdown [9, 7]. For an overview of the method see [6]. Here we review recent research on a topological approach to the creation of vortex wakes [5, 8].

The topological analysis is local in nature and commences by locating a value of $Re$ where a degenerate critical point exists in the flow. For the flow around a circular cylinder we take this to be $Re = 5$ where the steady vortices are created, and a degenerate critical point exists
on the back of the cylinder. It can be shown that a Taylor expansion of the streamfunction to order 5 in the spatial variables suffices to qualitatively describe the degenerate flow. Many of the low-order terms in this expansion vanish due to the symmetry of the flow. Adding all the small terms which are not present at the exact degeneracy as perturbations, an unfolding is obtained. Using a non-linear normal form transformation a number of terms can be eliminated, and one ends with the simple form for the streamfunction

\[
\psi = y^2 (c_{0.2} + c_{1.2} x + c_{0.3} y + xy + x^3).
\]  

Here \( x \) is a coordinate along the cylinder wall, \( y \) is the downstream coordinate, and \( c_{0.2}, c_{1.2}, c_{0.3} \) are small perturbation parameters. The interesting feature of the normal form of the unfolding is that all flows which are perturbations of the degenerate flow at \( Re = 5 \) locally are equivalent to one of the flows represented by this three-parameter family. Hence, by analyzing the bifurcations of (1), a complete catalog of possible streamline patterns near the vortex creation is obtained.

Such a bifurcation analysis is a relatively straightforward matter. We refer to [8] for the details. The results are summarized in Fig. 1. It turns out that no bifurcation occurs for \( c_{1.2} > 0 \), so only a typical slice for \( c_{1.2} < 0 \) is shown. The diagram shows a partition of the parameter space into regions with different streamline topologies. At the origin we find the symmetric separated flow. Moving away from this point, the topology changes as the bifurcation curves are crossed.

If we assume that the dynamic Hopf bifurcation at \( Re = 45 \) occurs close to the topological bifurcation at \( Re = 5 \), the bifurcation diagram in Fig. 1 also contains all possible instantaneous streamline patterns in the periodic regime close to the bifurcation. The periodic flow at a given \( Re > 45 \) is represented by a closed curve in the diagram, which can be considered a topological phase space for the flow. For \( Re \) just above 45, the amplitude of the periodic flow is very small, and must be represented by a small orbit encircling the symmetric steady state. It appears that there is only one way such a curve can generically be placed in the diagram, as indicated by the small ellipse in Fig. 1. As \( Re \) is increased the amplitude of the periodic solution also increases, the closed orbit grows, and may enter other regions in the bifurcation diagram. One such orbit is shown as the large ellipse.

Numerical computations confirm the existence of these two scenarios [8] as shown in Fig. 2. In the first scenario the streamline topology changes between two symmetric states with an attached recirculation zone and a detached saddle-loop. In the second scenario the detached saddle loop moves downstream and disappears in a topological bifurcation where the saddle and the centre merge. At a later time, the same structure is re-created closer to the cylinder.

While the first scenario exists only in an exceedingly small \( Re \)-range close to the Hopf bifurcation point and is probably not experimentally observable, the second scenario persists, with minor variations, to \( Re \) well over 200 [5].

While the numerical verification of the above scenario is of course valid for the circular cylinder only, the supporting analysis is of a very general nature. Since the normal form (1) encompasses all possible perturbations of the degenerate flow at the creation of the steady vortices, it also includes perturbations of the body shape. For bodies that are close to the circular cylinder in shape, the same sequences of topologies of the instantaneous streamlines are expected, even if the actual shape of the streamlines will change. For bodies deviating further from the circular cylinder, other regions of the bifurcation diagram may be explored. The streamline patterns for the flow around an inclined elliptical cylinder found numerically by Dennis & Young [11] are covered by the present theory, even for quite slender ellipses at high angles of incidence. A substantial deviation of the body shape is probably needed to escape the topological phase space in Fig. 1.
Figure 2. Numerically obtained temporal sequences of streamline patterns for the flow around a circular cylinder. (a) $Re = 45.6$, corresponding to the small ellipse in Fig. 1. (b) $Re = 100$, corresponding to the large ellipse in Fig. 1. Reproduced from [5].

3. Bifurcation problems in the dynamics of vortex wake formation
Empirically one finds the Strouhal number, a non-dimensional shedding frequency, to depend on the Reynolds number of the wake-producing flow according to the formula

$$St = 0.2175 - \frac{5.1064}{Re},$$

(2)
for what we may call the “laminar” regime (regime I in Fig. 3; up to $Re \approx 200$), and according to
\begin{equation}
St = 0.212 - \frac{2.7}{Re},
\end{equation}
for larger values of $Re$, say 400 and higher. The latter segment includes the limiting value (0.212) of the Strouhal number at high Reynolds number. In this second regime (regime II in Fig. 3) the flow does not just respond with one frequency, but the Strouhal number corresponds to the frequency with the most energy.

There is also a transition regime (regime III in Fig. 3) where the curve seems to break. This regime reflects three-dimensional vortex motion in the wake. Since this paper is solely concerned with the two-dimensional dynamics of the vortex wake, we shall not have anything to say about this regime.

We may approach this problem in the following way: First, dimensional analysis of the Navier-Stokes equations gives us that $St$ is a function of $Re$, i.e., $St = f(Re)$ where $f$ is to be determined from the equations of motion and may depend on the shape of the body. Second, based on an analogy to the phenomenology of phase transitions or on general ideas of bifurcation theory, we expect that close to the bifurcation the Strouhal number will depend as a power law on the deviation of the “order parameter” $1/Re$ from its “critical” value at the bifurcation, i.e., we expect for $Re \approx Re_{crit}$ that
\begin{equation}
St = A \left( \frac{1}{Re_{crit}} - \frac{1}{Re} \right)^{\alpha},
\end{equation}
where $Re_{crit}$ is the bifurcation Reynolds number (non-universal), $A$ is a non-universal coefficient, but $\alpha$ is a universal exponent. Experiment suggests that $\alpha = 1$ which in turn suggests that a “mean-field theory” of the phenomenon should be applicable.
The equation to which we [26] have applied such a “mean-field theory” is the two-dimensional vorticity equation,

\[
\frac{\partial \zeta}{\partial t} + \mathbf{V} \cdot \nabla \zeta = \nu \nabla^2 \zeta.
\] (5)

In [26] we have estimated the terms in this equation as follows:

\[
\frac{\partial \zeta}{\partial t} \approx f \Delta \zeta, \quad \mathbf{V} \cdot \nabla \zeta \approx U \frac{\Delta \zeta}{d}, \quad \nu \nabla^2 \zeta \approx \nu \frac{\Delta \zeta}{d^2}.
\] (6)

Here \( f \) is the shedding frequency, which defines a natural time scale for the flow, \( U \) is the free stream velocity and \( d \) the diameter of the cylinder. The quantity \( \Delta \zeta \) gives the scale of vorticity fluctuations in the emerging wake. There are points of large vorticity, primarily in the vortices that are forming to make up the vortex street, and there are points of smaller vorticity in sheets and other “background” flow structures that are ultimately swept up into the vortices.

In [26] it is argued, based on careful examination of the vortex wake formation process in a numerical simulation [28], that the viscous term acts exclusively to spread out and impede vortex formation, i.e., in the vorticity balance in the near wake this term is a sink when writing the vorticity balance. Further, part of the advective term on the right hand side acts to assemble the vortices (while the rest simply advects the vorticity downstream). This is a source term for vortex generation. Based on the order of magnitude estimates just given, and physical reasoning to determine the signs of the various contributions, the vorticity equation is recast in the following form (in terms of orders of magnitude with signs):

\[
f \Delta \zeta = k_a U \frac{\Delta \zeta}{d} - k_d \nu \frac{\Delta \zeta}{d^2},
\] (7)

where \( k_a \) and \( k_d \) are dimensionless parameters that would require a more comprehensive analysis for their determination. After cancellation of \( \Delta \zeta \) from all terms and multiplication by \( d/U \), this relation is precisely of the form of the empirical Strouhal-Reynolds number relation.

We will not enter into a deeper discussion here but simply suggest that the correct and full approach to this problem requires determination of a similarity solution for the vorticity that, somehow, applies to vortex shedding.

A different but very interesting bifurcation problem arises when one considers vortex wake patterns behind an oscillating cylinder. In the experiment two key control parameters are available, viz. the amplitude and frequency of oscillation of the wake-producing cylinder. Williamson & Roshko [34] provided an experimental bifurcation diagram for wakes produced by a cylinder oscillating normally to an oncoming uniform free stream. Different wake formation patterns were identified and delineated in a plot that has as its abscissa the wavelength of the oscillatory motion of the cylinder and as its ordinate the amplitude of that same oscillation (both coordinates non-dimensionalized by the cylinder diameter). Wake formation modes were labeled by “S” for singlet, “P” for pair, and various combinations thereof. For example, the wake in Fig. 4 would be “S+P”. The bifurcation diagram of Williamson & Roshko is shown as the blurred background in Fig. 5.

We have provided some theoretical ideas to “rationalize” the structure of the Williamson-Roshko diagram. A crude approximation suggests that the dividing lines between the various regimes in the diagram are radial and circumferential. There appear to be more vortices shed per cycle as one goes farther out radially from the origin in the diagram. In [27] the undulatory motion of the cylinder is considered to be a sequence of rectilinear “strokes” interrupted by sharp turns. During any “stroke” the cylinder sheds much as it would in an oncoming steady uniform stream. However, there is one complication: Since the experiment is conducted by giving a constant unidirectional velocity to the cylinder, the speed along the actual path varies. Hence
Figure 4. “Exotic” vortex street wake behind an oscillating cylinder (courtesy of C. H. K. Williamson).

the Reynolds number varies and, because of the Strouhal-Reynolds number relation, the shedding frequency varies. Apart from this effect – which is similar to driving an oscillator with a slightly varying forcing frequency – the length of the rectilinear “stroke” determines how many vortices are shed. Thus, we have both an effect of the amplitude and of the wavelength of the oscillation on how many vortices are shed per “stroke”. Since the number of vortices that are recognized in the wake patterns in terms of pairs and singlets is an integer, there is a “quantization” of the resulting wake as a function of the continuously variable control parameters (i.e., wavelength and amplitude of oscillation). This quantization can be expressed by the formula

\[ \frac{\lambda}{D} St \, E \left( -\left(2\pi A / \lambda \right)^2 \right) = n \frac{\pi}{2} \]  

(8)

For a derivation see [27]. Here \( \lambda \) and \( A \) are, respectively, the wavelength and amplitude of oscillation of the cylinder, \( St \) is the Strouhal number corresponding to the Reynolds number for the free stream according to the \( St - Re \) relation, and \( n \) is an integer. The function \( E \) is the complete elliptic integral of the second kind. The curves (8) correspond to the radial delimiting lines in the Williamson-Roshko bifurcation diagram in Fig. 5. When plotted in that diagram, the correspondence is surprisingly good with no adjustable parameters.

A convincing theory for the radial lines that divide shedding regimes in the Williamson-Roshko diagram is harder to come by. These delineations appear to be related to a threshold tolerance of the vortex shedding process to variations in Reynolds number (and, hence, in the corresponding Strouhal number) during the oscillatory motion of the cylinder. The instantaneous Reynolds number for flow about the cylinder varies in the course of its motion because the streamwise velocity is held constant in the experiment. Hence the speed along the undulatory path must vary and it is this speed relative to the fluid that sets the shedding. However, what such a threshold might be is difficult to tell without a more detailed quantitative model of the shedding process itself. The radial lines in Fig. 5 were drawn by choosing values for the threshold to give a best fit to the experimentally observed lines. There is some general level of agreement, but having to arbitrarily choose or fit threshold values is clearly unsatisfactory. Furthermore, some of the “radial” lines in Fig. 5 do not extend across the circumferential regimes. There is currently no explanation for this feature.

4. **Point vortex models of vortex street wakes**

The point vortex model originated with Helmholtz’s seminal 1858 paper on vortex dynamics [20]. The most elegant statement arises if one concatenates the \( x \)- and \( y \)-coordinates of the vortices into complex positions \( z_\alpha = x_\alpha + iy_\alpha, \alpha = 1, 2, \ldots, N \). Then the equations of motion take the
Figure 5. Theoretical contours superimposed on the Williamson-Roshko bifurcation diagram [34] (background, blurred). The radial contours, Eq. (8), have no adjustable parameters. After [27].

The form

$$\dot{z}_\alpha^* = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \quad (9)$$

Here the $\Gamma_\beta$ are the circulations of the vortices, invariant in time by Helmholtz’s theory – even better, maybe, by Kelvin’s circulation theorem – the asterisk on the left hand side denotes complex conjugation, the dot differentiation with respect to time, and the prime on the summation symbol reminds us to skip the singular term $\beta = \alpha$.

Helmholtz gave the solution of the two-vortex problem, where he showed that two vortices would have orbits on concentric circles, which in the special case of a vortex pair degenerate to translation along parallel lines.

A major formal development of the theory was provided by Kirchhoff [21], who in his lectures on theoretical physics, published in several editions starting in 1876, showed that the point vortex equations could be recast in Hamilton’s canonical form. The Hamiltonian nature of the point vortex equations immediately leads to important insights about the availability of integrals of the motion and, in turn, about integrability of the $N$-vortex problem. Thus, the invariance of the Hamiltonian to translation and rotation of coordinates, and its independence of time, leads to the integrals linear and angular impulse. It may then be shown that on the infinite plane the integrals taken together with the Hamiltonian itself assure the integrability of the three-vortex problem for arbitrary vortex strengths. Poincaré realized as much in his lectures of 1891-92 [25] (see also [23]). Apparently this was not of sufficient interest to him and he never returned to the problem. Actually some 15 years before Poincaré’s work the three-vortex problem had been completely solved by the young Swiss mathematician W. Gröbli, whose 1877 thesis [14] was for some reason
overlooked\(^1\) for about a century. Even the “revival” of Gröbli’s work in an important paper [32] by J. L. Synge for the inaugural issue of the *Canadian Journal of Mathematics* in 1949 failed to introduce the solution of this three-body problem into the mainstream of fluid mechanics. For a review of the history of the three-vortex problem and Gröbli’s contribution see [3].

While the special case \(\Gamma_1 + \Gamma_2 + \Gamma_3 = 0\) is formally covered by the above works, it admits a more complete discussion as provided by Rott [29] and Aref [1]. This analysis shows that the relative separation of two of the vortices, say vortices 1 and 2, i.e., \(Z = z_1 - z_2\), evolves as if it were the position of a fictitious passive particle in the field of three *fixed* vortices. The strengths and locations of the three fixed vortices are given by the strengths of the original three vortices and the linear impulse of the original three-vortex system. Thus, if the original three vortices have strengths \(\Gamma_1, \Gamma_2, \Gamma_3\), the three fixed vortices in the advection problem have strengths that are in the proportion \(\Gamma_1^{-1}:\Gamma_2^{-1}:\Gamma_3^{-1}\).

The reduction of the problem from three points corresponding to the three original vortices, to one point corresponding to an advected particle is reminiscent of what happens in the Kepler problem of celestial mechanics, where the motion of two interacting mass points is decomposed into a trivial center-of-mass motion and a relative motion. It leads to the following scenario: There is the *physical plane* where the motion of the three vortices takes place, i.e., the vortex positions \(z_1, z_2, z_3\) “live” in this plane. There is a *phase plane* where the advection of the fictitious particle takes place, i.e., \(Z\) evolves in this plane. For three vortices on the infinite plane the advection problem in the phase plane is relatively simple. There are four distinct regimes of motion. Three of these arise in the obvious way through two of the vortices being closer to one another than to the third vortex, and hence moving as if in a “bound state”. The fourth regime corresponds to truly “collective states” where all three vortices interact continuously.

The solution method for three vortices on the infinite plane can be extended to the problem of three vortices in a domain with periodic boundary conditions as was first shown in [4, 31]. The case of vortices in a periodic strip is immediately applicable to vortex wakes. One has to stipulate that \(\Gamma_1 + \Gamma_2 + \Gamma_3 = 0\) just as on the infinite plane. (In the case of vortices in a periodic parallelogram the periodicity of the flow assures that the sum of the “base” vortices in the basic parallelogram is zero.) The equations of motion for vortices in a periodic strip of width \(L\) are

\[
\dot{z}_\alpha = \frac{1}{2L} \sum_{\beta=1}^{N} \Gamma_\beta \cot \left[ \frac{\pi}{L} (z_\alpha - z_\beta) \right],
\]

\[\text{(10)}\]
equations that appear to have been first published in 1928 by Friedmann & Polubarinova [13].

With the wisdom of hindsight one may say that von Kármán’s theory of the structure of the vortex street follows from (10) with \(N = 2\) and \(\Gamma_1 = -\Gamma_2 = \Gamma\). Thanks to later work by Domm [12], the necessary condition in von Kármán’s theory of the stability of vortex streets follows from (10) with \(N = 4\) and \(\Gamma_1 = \Gamma_2 = -\Gamma_3 = -\Gamma_4 = \Gamma\). (Probably the most accessible account of von Kármán’s theory for the modern reader is the exposition in [22].) In brief, von Kármán’s theory of the vortex street shows, first, that the only two-vortex-per-strip configurations that propagate downstream are the symmetric and the staggered configuration. From the two-vortex version of (10) one easily deduces that a \(\pm \Gamma\) pair in a periodic strip propagates with velocity

\[
U - iV = \frac{-\Gamma}{2L} \cot \left[ \frac{\pi}{L} (z_+ - z_-) \right].
\]

\[\text{(11)}\]

For the velocity to be real, i.e., in order to have \(V = 0\) in (11), the cotangent must be pure imaginary. This implies \(\Re(z_+ - z_-) = 0\) or \(\Re(z_+ - z_-) = L/2\). The first possibility corresponds to symmetric vortex streets, the second to staggered vortex streets.

\(\text{1} \) This happened in spite of references to it in Kirchhoff’s lectures (2nd ed.) [21] and in Lamb’s well known text [22].
Von Kármán next considered the stability of these two types of configurations. He did, in essence, two stability calculations, in both cases working with infinite rows of vortices. In the first he simply perturbed one vortex keeping all the others fixed. This calculation showed that the symmetric configuration was always linearly unstable and the staggered configuration was linearly unstable unless the ratio of \( b = \Re(z_+ - z_-) \) and the inter-vortex distance in each row, \( h \), has a certain value. (To avoid confusion we use a new symbol, \( h \), for the distance between vortices in either row because for, say, four-vortices-in-a-strip the period of the strip, \( L \), is related to the inter-vortex distance by \( L = 2h \), whereas \( L = h \) for the two-vortices-per-strip case.) In fact, in his first attempt von Kármán produced the erroneous result \( \sinh(\pi b/h) = \sqrt{2} \). (The reason for this “error” is that when perturbing just one vortex one is adding linear momentum and kinetic energy to the system being perturbed. The appropriate criterion arises from perturbations that do not add linear momentum or energy.) The correct result, which von Kármán quickly produced as well, and which is today known as his famous stability criterion for vortex streets reads

\[
\sinh \frac{\pi b}{h} = 1. \tag{12}
\]

Domm’s treatment [12] shows that even for this “magic” width-to-spacing ratio the point vortex street is still unstable in second order perturbation theory. This result was obtained working with the infinite two-row system in [19].

The main thrust of our work on more complex or “exotic” vortex wakes is to apply the solution for three-vortices-in-a-strip to model these in the same spirit that von Kármán modeled steady vortex streets by the two-vortices-in-a-strip solutions. An experimental picture of an “exotic” wake with three vortices shed per cycle is shown in Fig. 4. It is a tenet of vortex wake dynamics, apparently true but difficult to prove, that the total circulation of all vortices shed during one cycle is zero. This applies also to such cases as a cylinder oscillating normally to an oncoming uniform flow. A recent paper by Ponta and two of the present authors [2] gives a rather thorough exposition of our ideas so we shall be content with a brief summary.

In the extension of the solution for three vortices with sum of circulations equal to zero to periodic boundary conditions [4, 31] one finds, once again, that the problem can be “reduced” to an advection problem for the relative position of two of the vortices, say again \( Z = z_1 - z_2 \). In the periodic strip case, however, the advecting system of vortices consists of three rows of advecting vortices, not just three vortices. The vortices in each of the three rows are identical, and their circulations are in the proportion \( \Gamma_1^{-1} : \Gamma_2^{-1} : \Gamma_3^{-1} \). The position of the “base vortex” in each row is given exactly as in the unbounded plane case in terms of the linear impulse of the system and the circulations. It turns out that if the ratio of the circulations is rational, the three rows of advecting vortices fit into a periodic strip with a width that is a multiple of the period \( L \) of the strip in the physical plane. If the ratios are irrational, the three rows of advecting vortices have no common period and we are faced with advection by an infinite system of stationary vortices. Once the \( Z \)-motion is known, the determination of \( z_1, z_2 \) and \( z_3 \) from \( Z \) requires an additional quadrature.

An advection problem in the phase plane arises once again but this time with a more complicated structure of the various regimes of motion than in the unbounded plane case. There are, in general, many more regimes for \( Z \) to wander through and thus many more regimes for the vortex motion itself. This provides the first qualitative conclusion: Vortex wakes with three (and, thus, presumably with more than three) vortices shed per cycle provide a considerably richer variety of wake patterns than the classical vortex street wakes (and we include under this rubric both the von Kármán street and its oblique “cousins” found subsequently by Dolaptschiew and Maue, cf. [24]). Furthermore, so far as we can tell, the richness in the dynamical structure of the three-vortices-in-a-strip solution is only partially reflected in the known experimental and numerical results.
The phase plane diagrams reveal a multitude of relative equilibria with three vortices per period, none of which have been observed experimentally or in numerical simulations. Somewhat surprisingly, these can be determined analytically [30]. Since they correspond to saddle points in the phase plane diagram, they are all linearly unstable which may explain why they do not seem to occur even as transients in experimental images of vortex wakes. However, a thorough analysis of such images has yet to be undertaken, since we have only recently understood what to look for.

It should also be possible to generalize the Kármán drag law, that was derived for the ordinary vortex street [17], to a certain class of more complicated vortex wakes. Work is in progress on this topic. Experimental results suggest considerable richness in the structure of the drag force versus the frequency of oscillation of the cylinder. It would be interesting to produce such results using the simple wake models considered here.

5. The streamline topology of vortex street wakes

The periodic point vortex street model provides a mathematically tractable representation of flow structure in the mid-wake region. Vortex configurations in relative equilibrium translate uniformly, thus generating steady streamline patterns in the appropriate co-moving frame of reference. The topology of these streamline patterns, particularly bifurcations in the topology, lead insight into fluid transport through the mid-wake region.

The classical vortex streets, with two oppositely-signed vortices generated each shedding cycle, always translate uniformly with (complex) velocity \( U - iV \) given by (11) for any possible choice of intervortex spacing. This case includes the symmetric and staggered streets considered by von Kármán [15, 16, 17] and the oblique streets considered by Dolaptchiev and Maue, cf. [24]. In a frame of reference moving with the vortices, the velocity generated at \( z = x + iy \) by a \( \pm \Gamma \) pair in a strip of width \( L \) is

\[
\dot{z}^+ = \frac{\Gamma}{2L} \cot \left[ \frac{\pi}{L} (z - z_+) \right] - \frac{\Gamma}{2L} \cot \left[ \frac{\pi}{L} (z - z_-) \right] - (U - iV).
\]

Streamlines in this flow are given by level curves of the streamfunction

\[
\psi = -\frac{\Gamma}{2\pi} \ln \left| \sin \left[ \frac{\pi}{L} (z - z_+) \right] \right| + \frac{\Gamma}{2\pi} \ln \left| \sin \left[ \frac{\pi}{L} (z - z_-) \right] \right| - 3 \Im [(U - iV)z]
\]

\[
= \frac{\Gamma}{4\pi} \ln \left[ \frac{\sin^2 \left[ \pi(x - x_-)/L \right] + \sinh^2 \left[ \pi(y - y_-)/L \right]}{\sin^2 \left[ \pi(x - x_+)/L \right] + \sinh^2 \left[ \pi(y - y_+)/L \right]} \right] + Vx - Uy.
\]

Changes in the intervortex spacing \( z_+ - z_- = a + ib \) can produce changes in the streamline topology. Since \( \psi \) is invariant under the transformations \( z_+ \rightarrow z_+^*, z_- \rightarrow z_-^*, z \rightarrow z^* \) and \( z_+ \rightarrow L - z_+, z_- \rightarrow -z_-, z \rightarrow -z \), we need to consider only those configurations with \( 0 \leq h \leq L/2 \) and \( b \geq 0 \). In this region \( U \geq 0 \), so that the vortex street moves in the positive \( x \)-direction (in a fixed frame of reference).

The staggered von Kármán vortex street, with \( b = \frac{L}{4}\log(1 + \sqrt{2}) \) according to (12), has the streamline pattern shown in Fig. 6(a) (only the dividing streamlines are shown). Although this vortex configuration is linearly stable, the corresponding streamline topology is structurally unstable, as it contains heteroclinic connections between the stagnation points. Perturbations in \( a \) and \( b \) result in changes in the streamline structure. Such perturbations can produce patterns that are structurally stable with homoclinic loops only, such as that shown in Fig. 6(b). There are also a (infinite) number of other structurally unstable patterns in which there are heteroclinic connections between the upper stagnation point in one strip and the lower stagnation point in a different strip. To enumerate these vortex configurations, we assign an integer index \( n \) to a
Figure 6. (a)–(d) Streamline patterns in the co-moving frame for a two-vortex street. Three strip widths are shown for each case. In (c) and (d), representative heteroclinic connections are shown with dashed lines; these configurations correspond to the intersection of the stability curve with the bifurcation curve having index (c) $n = 1$ and (d) $n = 2$. (e) Bifurcation curves for indices $-4 \leq n \leq 4$. The dashed lines are the curves for $n < 0$, the light solid lines are the curves for $n \geq 0$, and the heavy line is the stability curve given by (15). The symbol ($\Box$) marks the parameter values for the streamline pattern in (b).

configuration if the upper stagnation point is connected to the lower stagnation point $n$ strips away. For example, the streamline pattern in Fig. 6(c) has $n = 1$. Each index $n$ gives rise to a bifurcation curve in the $(a, b)$-plane, as illustrated in Fig. 6(e). For $n < 0$ the vortex configurations behave as two interacting shear layers rather than a bluff body wake, and we do not consider these cases here.

The bifurcation curves for $n \geq 0$ correspond to wake-like vortex configurations. For our discussion we will focus on those configurations that satisfy the general stability criterion [24]

\[
\sinh \frac{\pi b}{L} = \sin \frac{\pi a}{L};
\]

we show this stability curve along with the bifurcation curves in Fig. 6(e). At $(a, b) = (0, 0)$ the stability curve is tangent to the bifurcation curve with index 0. The bifurcation curves with $n \geq 1$ intersect the stability curve at points $(a_n, b_n)$ (see table 1). As $n \to \infty$, the bifurcation curves accumulate on $a = L/2$.

The streamline patterns for $0 < a < a_1$ (with $b$ defined by (15)) are structurally stable and
Table 1. Approximate parameter values \((a_n, b_n)\) for the intersections of the stability curve (15) and the bifurcation curves with index \(n\).

| \(n\) | \(a_n/L\)       | \(b_n/L\)       | streamline pattern |
|-------|-----------------|-----------------|--------------------|
| 1     | 0.45481872      | 0.27828066      | Fig. 6(c)          |
| 2     | 0.47347651      | 0.27976831      | Fig. 6(d)          |
| 3     | 0.48116388      | 0.28015579      |                    |
| 4     | 0.48538411      | 0.28031263      |                    |
| \(\infty\)| 0.5             | 0.28054993      | Fig. 6(a)          |

have the topological structure illustrated in Fig. 6(b), for which \(a = 3L/8\). As these vortex streets translate obliquely down and to the right, fluid is advected along three types of paths: (i) those that wrap around a single vortex and are trapped there for all time, (ii) those that pass directly through the street, and (iii) those that wind between a vortex pair while passing through the street. Taking \(a \rightarrow a_1\) causes more of the fluid passing through the street to wind between a vortex pair; when \(a = a_1\), as in Fig. 6(c), all of the ambient fluid that enters the wake moves on an ‘S’-shaped path around a vortex pair as it passes through the street. When \(a = a_2\), as in Fig. 6(d), fluid is wrapped between two vortex pairs. The ambient fluid that is transported through the wake is wrapped between an increasing number of vortex pairs as \(n\) increases. However, as \(n\) increases the volume of ambient fluid that passes through the street decreases, and in the limit \(n \rightarrow \infty\) the symmetric von Kármán vortex street is isolated from the surrounding ambient fluid.

These results appear to have implications for transport and mixing in vortex wakes. While it is true that a real wake consists of only a finite number of vortices, and thus the bifurcations associated with large \(n\) cannot be realized, it seems quite possible that naturally occurring (or perhaps forced) perturbations would generate the bifurcations for moderate \(n\). In these cases the wake is slightly oblique, and ambient fluid is transported into the wake where it is stretched and folded by the vortices. One expects this mixing of the wake with the surrounding fluid to impact the dissipation of quantities such as heat and momentum away from the vortex cores. These results thus motivate further analysis of transport in oblique vortex wakes.

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