Homogenization and multicontinuum models for high contrast composites

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Abstract. We consider the high contrast periodic composite materials having high ratio $\omega >> 1$ of physical constants of the components such as the heat conductivity or Young’s moduli. The ratio of the characteristic microscopic size and the characteristic macroscopic size is a small parameter $\varepsilon$. If the component having high constants has the shape of small isolated particles then the homogenized (macroscopic, effective) model demonstrates a loss of the effective wave velocity. If the topology of the component having high constants corresponds to several (at least two) connected periodic sets disconnected one from other and if $\varepsilon^2 \omega >> 1$ or $\varepsilon^2 \omega = \text{const}$ then the macroscopic model is described by a multicontinuum, i.e. several continua in each point.

1. Introduction

The high contrast composites have components with high ratio of mechanical properties (for example, Young moduli or conductivities). Such materials have atypical properties which cannot be obtained in natural materials. Namely, they may have a considerable loss of the wave velocity compared to the wave velocity in the components, in some cases they are described by multicontinuum models (several continuums coexisting in each point of the space) or by frequency dependent models presenting the micro-resonance effects.

2. Loss of the effective velocity in composites with dispersed isolated heavy stiff inclusions.

Consider first the non-stationary elasticity equations in a composite material having the following periodic structure. Let $\varepsilon$ be the ratio of the microscale size and the macroscale size. A periodic cell having the edge $\varepsilon$ contains one stiff and heavy particle occupying a ball inside the cell, while the filament is soft and light. The ratio of the Young moduli of components as well as the ratio of densities is a large parameter $\omega >> 1$. Namely, let the density of inclusions be $\omega \rho_i$ and the density of the filament be $\rho_m$, let the Young’s modulus of inclusions be $\omega E_i$ and the Young’s modulus of the filament be $E_m$. Introduce the Poisson’s ratios of the inclusions $\nu_i$ and of the filament $\nu_m$. Define the density $\rho(y)$ and the elasticity moduli $a_{ijkl}(y)$ as a function of a fast variable $y=x/\varepsilon$ equal to $\omega \rho_i$ and to $0.5 \omega E_i (1+\nu_i)$ if $x$ belongs to inclusions and respectively $\rho_m$ and $0.5 E_m (1+\nu_m)$ if $x$ belongs to the filament.

The wave propagation in such elastic medium is described by equations

$$\rho(x/\varepsilon) u_{tt}^k - (a_{ijkl}(x/\varepsilon) u_{ij})_{x} = 0$$

(summing up over $i,j,r,l$).

The homogenization of this model by the two stage homogenization technique [1] gives the effective

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density of order of $\omega >> 1$ while the stiffness is of order $O(1)$. It means that the effective wave velocity is of order $(1/\omega)^{1/2} << 1$. It means that if there are some dissipative factors in the soft component then they will act for a longer time while the wave slows down. Such dissipative factors may be, for example, the friction between the parts of the soft material, modeled by the Robin type boundary conditions at the boundaries of periodic holes in the soft phase or a viscoelastic behavior of the soft phase, with short or long memory.

3. When the macroscopic model of a high contrast composite is a multicontinuum?

Another effect of the high contrast properties of the components is the multicontinuum models. Such models at macroscopic level are asymptotically derived from the microscopic description of a highly heterogeneous medium. To our knowledge the first asymptotic derivations of such models appear in [12],[14], [18]-[21] and later such models were obtained by means of two-scale convergence [1]-[3]. In particular, the double porosity model, (see [5] (introduced), [2], [8] (justified)) is one of examples, for wave propagation see [13]. In [23] we derive a multicontinuum wave propagation model in a laminated beam with contrasting stiffness and density of the layers using technique of multicomponent homogenization introduced in [19], the 3D theory of wave propagation in such media was developed in [20]. Consider an $\varepsilon$-periodic structure consisting of three periodic sets $B_1$, $B_2$, $B_3$ of thin cylinders parallel to the Cartesian axes $x_1$, $x_2$, $x_3$ respectively. The cylinders of the sets $B_1$, $B_2$, $B_3$ have no common points. Let the conductivity $K$ be equal to positive constants $\omega K_1$, $\omega K_2$, $\omega K_3$ in $B_1$, $B_2$, $B_3$ respectively and $K_\rho >> 0$ out of the union of $B_1$, $B_2$ and $B_3$. Let the density have a similar structure: $\omega \rho_1$, $\omega \rho_2$, $\omega \rho_3$ in $B_1$, $B_2$, $B_3$ and $\rho_0 >> 0$ out of the union of $B_1$, $B_2$ and $B_3$. Consider the wave equation in such a structure. Then the asymptotic behavior of the solution depends on the parameter $\varepsilon^2 \omega$. If this parameter is small ($<< 1$) then the structure can be approximated by a classical homogenized wave equation [20] with the effective conductivities $\omega K_i$, $\omega K_{\rho i}$, $\omega K_{\rho i}$ in three coordinate directions; here $\rho_i$ are the volume ratio of the set $B_i$; the effective density is $\omega \rho_i + \omega \rho_{\rho i} + \rho_0 \theta_i$, $i=1,2,3$. If $\varepsilon^2 \omega >> 1$ then the effective macroscopic model is a multicontinuum: each set of highly conductive cylinders “lives” independently of other sets, so that in every point $x$ there co-exist three one-dimensional continua described by three independent equations

$$\omega \rho_i \theta_i u_{i\theta} - \omega K_i \theta_{i\theta} u_{i\varepsilon} + \rho_0 \theta_i u_{i\varepsilon} = 0, i=1,2,3.$$ 

Finally, if $\varepsilon^2 \omega = \eta$ is constant of order 1 then these equations become weakly coupled:

$$\omega \rho_i \theta_i u_{i\theta} - \omega K_i \theta_{i\theta} u_{i\varepsilon} + \rho_0 \theta_i u_{i\varepsilon} = 0, i=1,2,3 \text{ (summing up over $k$)},$$

where the coefficients $\rho_i$ are defined by some special cell problems (see [20]). Such multicontinua demonstrate special atypical properties, in particular, they may have many different wave velocities (as many as there are sets of cylinders).

A similar effect takes place for composites having several disconnected three-dimensionally spaced stiff components or for a laminated rod with stiff and heavy layers alternating with the soft light layers [23]. In any case the multicontinuum effects take place for special topologies of the components, and they are absent in the case of isolated particles described in the first part of the paper.

Finally, some special effects are related to the frequency dependent materials. The spectral analysis of the high contrast composites was provided in the classical paper [25]. Later the theory was developed in papers [10], [11]. The sufficient conditions for the spectral gaps were pointed out.

4. Conclusion

The high contrast composites have atypical properties. Choosing an appropriate topology of the components and ratio of physical constants of components we can get special effects in the macroscopic behavior of the material. In particular, sufficient conditions for multicontinua at the macroscopic level is the presence of at least two unconnected (or weakly connected) highly conductive (very stiff) periodic sets crossing the whole material and the relation $\varepsilon^2 \omega > 1$ or $\varepsilon^2 \omega = \text{const.}$
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