On the Geometry of Electrovacuum Spaces in Higher Dimensions

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Abstract. A classical question in general relativity is classifying regular static black hole solutions of the static Einstein–Maxwell equations (or electrovacuum system). We prove some classification results for an electrovacuum system such that the electric potential is a smooth function of the lapse function. We particularly show that an \( n \)-dimensional locally conformally flat electrovacuum space satisfying (1.8) must be in the Majumdar–Papapetrou class. Moreover, we prove that an \( n \)-dimensional electrovacuum space satisfying (1.7) with fourth-order divergence-free Weyl tensor and zero radial Weyl curvature is locally a warped product manifold with \((n - 1)\)-dimensional Einstein fibers. Finally, a three-dimensional electrovacuum space satisfying (1.7) with a third-order divergence-free Cotton tensor is also classified.

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1. Introduction and Main Results

Static electrovacuum spacetimes model exterior regions of static configurations of electrically charged stars or black holes (see [9,10,12] and the references therein). Equations of motion for an \((n + 1)\)-dimensional Einstein–Maxwell spacetime are given by

\[
(\widehat{\text{Ric}})_{ij} = 2 \left( F_{il} F_{lj} - \frac{1}{2(n-1)} |F|^2 \widehat{g}_{ij} \right); \quad 1 \leq i, j \leq n+1,
\]

where \( F \) represents the electromagnetic field (the Faraday tensor) and \( \widehat{\text{Ric}} \) is the Ricci tensor for the metric \( \widehat{g} \).

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Our main ground is the static spacetime \((\hat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}\) such that \(\hat{g}(x, t) = g(x) - f^2(x)dt^2; \ x \in M\), where \((M^n, g)\) is an open, connected, and oriented Riemannian manifold, and \(f\) is a smooth warped function \([9, 11, 16]\). Considering as electromagnetic field \(F = d\psi \wedge dt + B\), for some smooth function \(\psi\) on \(M\). Here, \(\xi = \frac{\partial}{\partial t}\) is the static Killing field and the magnetic field \(B\) is a 2-tensor on \(M\) such that \(i_\xi B = 0\). We will consider \(n\)-dimensional spatial slices, i.e., the Riemannian manifold \(M\) is orthogonal to the static Killing field. For simplicity, we will assume the dimensionally reduced Einstein–Maxwell equations, i.e., \(B = 0\) (see \([15, \text{Definition 6}]\) and the references therein). If \(\xi\) is strictly timelike in the spacetime, then \(M\) is a complete manifold. If \(\xi\) is null anywhere, the above coordinate system breaks down at the level set \(\{f = 0\}\). In this case, we extend \(M\) to a manifold with a smooth boundary \(\partial M\) containing \(\{f = 0\}\), which could correspond to an event horizon or an ergosurface if the hypersurface is null or timeline, respectively. We say that \((M, g)\) is complete away from the horizon \(\partial M\) if for any sequence of points \(\{p_i\}\) such that \(p_i \to p \in \partial M\) on the metric of \(M\) one has \(f(p_i) \to 0\). Conversely, if \(\{p_i\}\) is a bounded sequence of \(M\) such that \(f(p_i) \to 0\), then the definition of \((M, g)\) implies that a subsequence of \(\{p_i\}\) converges to a point \(p \in \partial M\) \([2]\).

**Definition 1.** Let \((M^n, g)\) be an \(n\)-dimensional smooth Riemannian manifold with \(n \geq 3\) and let \(f, \psi : M \to \mathbb{R}\) be smooth functions satisfying

\[
\begin{align*}
&f \text{Ric} = \nabla^2 f - \frac{2}{f} d\psi \otimes d\psi + \frac{2}{(n-1)f}|\nabla \psi|^2 g, \\
&\Delta f = 2 \left( \frac{n-2}{n-1} \right) \frac{|\nabla \psi|^2}{f}, \\
&\text{div} \left( \frac{\nabla \psi}{f} \right) = 0,
\end{align*}
\]

(1.1)

where \(\text{Ric}, \nabla^2, \text{div}\) and \(\Delta\) are the Ricci and Hessian tensors, the divergence and the Laplacian operator on the metric \(g\), respectively. Furthermore, \(f > 0\) on \(M\). When \(M^n\) has boundary \(\partial M\), we assume in addition that \(f^{-1}(0) = \partial M\). We also refer to \((M^n, g, f, \psi)\) as an electrovacuum (or electrostatic) system (or space). The smooth functions \(f, \psi\), and the manifold \(M^n\) are called lapse function, electric potential, and spatial factor for the static Einstein–Maxwell spacetime, respectively.

We observe that taking the contraction of the first equation and combining it with the second equation in (1.1), we obtain that the scalar curvature, denoted by \(R\), is given by

\[
f^2 R = 2|\nabla \psi|^2.
\]

(1.2)
Second, when $\psi$ is a constant function, the electrostatic system reduces to the static vacuum Einstein equations, i.e.,

$$f \nabla^2 f = 0 \quad \text{and} \quad \Delta f = 0. \quad (1.3)$$

These equations characterize the static vacuum Einstein spacetime which was widely explored in the literature. Furthermore, the most important solution for this system is the Schwarzschild solution. This solution represents a static black hole with mass, but without electric charge or magnetic fields. Therefore, we can see that Definition 1 generalizes the system (1.3) and we will consider the case where $\psi$ and/or $f$ are constant functions as trivial.

In 1918, independently, Nordström and Reissner found a class of exact solutions to the Einstein equation for the gravitational field of a spherical charged mass (see [22] for a wide-ranging discussion about these solutions). The Reissner–Nordström (RN) electrostatic spacetime is one of the most important solutions for the electrostatic system and it can be thought of as a model for a static black hole or a star with electric charge $q$ and mass $m$. The RN spacetime is called subextremal, extremal or superextremal depending on if $m^2 > q^2$, $m^2 = q^2$ or $m^2 < q^2$, respectively. For instance, we have the following RN solution given by the Riemannian manifold $M^n = S_n - 1 \times (r^+, \infty)$ with metric tensor

$$g = \frac{dr^2}{1 - 2mr^{2-n} + q^2r^{2(2-n)}} + r^2g_{S^{n-1}},$$

where $r$ represents the radial coordinate of the Reissner–Nordström black hole. Here, $m^2 \geq q^2$ are constants, and $r^+ > (m + \sqrt{m^2 - q^2})^{1/(n-2)}$. Moreover, the outer horizon for the Reissner–Nordström spacetime is located at $(m + \sqrt{m^2 - q^2})^{1/(n-2)}$, which corresponds to the zero set of the lapse function of the RN manifold. The static horizon is defined as the set where the lapse function for a static manifold is identically zero. This set is physically related with the event horizon, the boundary of a black hole. The RN space is locally conformally flat (see [11] for instance).

It is well known that the lapse function $f$ and the electric potential $\psi$ of an electrovacuum system asymptotic to Reissner–Nordström of total mass $m$ and charge $q$, with suitable inner boundary, satisfy the functional relationship (see [9, Equation A.1] and [16, Lemma 3])

$$f^2 = 1 + 2\frac{n-2}{n-1} \psi^2 - 2\frac{m}{q} \sqrt{2\frac{n-2}{n-1}} \psi. \quad (1.4)$$

Another important electrovacuum solution is the Majumdar–Papapetrou (see [11,12,18]), which is related to an extremal RN solution. The Majumdar–Papapetrou (MP) solution to the Einstein–Maxwell theory represents the static equilibrium of an arbitrary number of charged black holes whose mutual electric repulsion exactly balances their gravitational attraction. A spacetime will be called a standard MP spacetime if the metric tensor is given by

$$\hat{g} = -f^2 dt^2 + f^{-2/(n-2)}(dx_1^2 + \cdots + dx_n^2), \quad (1.5)$$
in Cartesian coordinates \( x = (x_1, \ldots, x_n) \) and \( \hat{M}^{n+1} = (\mathbb{R}^n \setminus \{a_i\}_{i=1}^I) \times \mathbb{R} \), for a finite set of points \( a_i \in \mathbb{R}^n \), where

\[
\frac{1}{f(x)} = 1 + \sum_{i=1}^I \frac{m_i}{r_i^{n-2}}, \quad r_i = |x - a_i|,
\]

(1.6)

for some positive constants \( m_i \), and the electric potential

\[
\pm \sqrt{\frac{2(n-2)}{(n-1)} \psi} = 1 - f,
\]

(see [12] and [16, Lemma 1]).

The classification problem of an electrovacuum spacetime can be stated as follows. Suppose that 

\[
\forall i, j \quad q_i q_j \geq 0,
\]

where \( q_i \) is the charge of the \( i \)-th connected degenerate component of the electric-charged black hole. Then the black hole is either an RN or an MP black hole. Some important and recent results in the literature concerning the classification of electrovacuum spaces (see for instance [10, 16, 18] and their references).

The most common assumption in analyzing and classifying an electrovacuum space is to consider that such space is asymptotically flat (see [9, 11, 16, 18]). It is well known that using the positive mass theorem, we can conclude that the space is conformally flat (i.e., either the Cotton tensor \( C \) in the three-dimensional case or the Weyl tensor \( W \) in higher dimensions is identically zero). We can then use classical calculations to prove that the solution for the electrovacuum system is either MP or RN (we refer to the reader see the main steps in the proof of [11, Theorem 3.6]). Those asymptotic conditions guarantee information about the metric, lapse function, and the electric potential at infinity. Even though this condition is restrictive in the topological sense, it is physically reasonable to study isolated gravitational systems.

Usually, in differential geometry, we often have some conditions over the metric or curvature (or both) in the attempt to classify an arbitrary space. Locally geometric conditions over the curvatures (Riemannian, Ricci, or scalar) have been used in the study and classification of static vacuum spaces (cf. [1, 13, 17]). For instance, it is well known that if a Riemannian manifold has constant scalar curvature and harmonic Weyl curvature (see Eq.1.9), then its curvature tensor should be harmonic (but not necessarily flat). The harmonic Weyl curvature condition is weaker than the locally conformally flat condition (we refer to the reader [7, Remark 1.2]). It is interesting to remember that some classical proofs for the uniqueness of the static Schwarzschild black holes used the conformally flat structure of the static metric to obtain the classification result (cf. [20] and the references therein). Naturally, we can assume weaker integrability conditions on a Riemannian manifold to understand its geometry. Some of our main results were inspired by the idea used by [8] to classify Ricci solitons, where the authors considered that the Weyl tensor is free from
divergence as a hypothesis, which is a weaker assumption than harmonic Weyl curvature. These conditions will be discussed ahead. In this case, considering just a condition over the curvature on the electrovacuum space classification seems insufficient since we lose information about the electric potential and the lapse function.

We recall that an asymptotically flat $n$-dimensional static Einstein–Maxwell space is extremal (i.e., $m = |q|$) if, and only if, the magnetic field is zero and $f = 1 \pm \sqrt{2(n-2)/(n-1)} \psi$, admitting $f = 0$ at $\partial M$ (see Lemma 1 in [16]). Also, in [16, Lemma 3], the authors proved how certain electrovacuum solutions combined with an equation relating $\psi$ and $f$ have implications on the non-existence of magnetic fields. It is worth saying that an extremal RN spacetime contains a unique photon sphere on which light can get trapped, and it has the largest possible ratio of charge to mass (see [9]). The theory of extremal black holes is critical in physics and has very interesting properties. For instance, extremal charged black holes might be quantum mechanically stable, which is consistent with the ideas of cosmic censorship (see [14]). There are also applications of electrovacuum solutions in supergravity theory (see [18]). Moreover, there is evidence that this type of black hole is important to understanding the no-hair theorem (see [5]).

The RN and MP solutions for the electrovacuum system suggest a class of solutions where the electric potential is a smooth function of the lapse function, i.e., $\psi = \psi(f)$. Our first result proves that there is a certain rigidity in this class of solutions.

In what follows, we will consider that the critical set of the lapse function $f$, i.e., $\text{crit}(f) = \{x \in M, \nabla f(x) = 0\}$, is not dense on $M$. Moreover, $|\nabla f| \neq 0$ at $\partial M$ is known as the non-degeneracy condition.

**Theorem 1.** Let $(M^n, g, f, \psi)$, $n \geq 3$, be an electrovacuum space such that $\psi = \psi(f)$. Then, the electric potential (locally) is either

$$\frac{2(n-2)}{n-1} \psi(f)^2 - \frac{4(n-2)}{n-1} \beta \psi(f) + \frac{2(n-2)}{n-1} \beta^2 + \frac{n-1}{n-2} \sigma = f^2$$

or

$$\psi(f) = \beta \pm \sqrt{\frac{(n-1)}{2(n-2)} f},$$

where $\sigma, \beta \in \mathbb{R}$. Moreover, $\sigma = 0$ if and only if $\psi(f)$ is an affine function of $f$.

It is interesting to remark how constants $\sigma$ and $\beta$ given by (1.7) are related with the mass $m$ and electric charge $q$ for a RN solution which satisfies (1.4). A straightforward computation shows us that

$$\beta^2 = \frac{(n-1)}{2(n-2)} \frac{m^2}{q^2} \quad \text{and} \quad \sigma = \frac{(n-2)}{(n-1)} \frac{q^2 - m^2}{q^2}.$$

The above theorem shows us that an electrovacuum system such that $\psi = \psi(f)$ has basically two possible solutions, which are closely related to the RN and MP solutions, respectively. It is also important to highlight that
with the conformal metric $\tilde{g} = f^{2/(n-2)}g$, the inverse of the electric potential $\frac{1}{\psi(f)}$ given by (1.8) is harmonic in the metric $\tilde{g}$. Moreover, $(M^n, \tilde{g})$ is Ricci-flat (see Lemma 3). Then, considering asymptotic conditions, by the positive mass theorem, $(M^n, \tilde{g})$ is isometric to the Euclidean space. Of course, in a three-dimensional case, this is a direct consequence of $(M^n, \tilde{g})$ being Ricci-flat. These facts are important for the classification of extremal electrovacuum solutions. As pointed out in [16, Remark 1] and [18], any suitably regular asymptotically flat black hole solution in the Majumdar–Papapetrou class must have a space isometric to Euclidean space (minus a point for each horizon) and a harmonic function of the form (1.6). In this case, the spacetime is a Majumdar–Papapetrou multi-centered black hole solution (see [18]). We need to emphasize that we are not considering asymptotic conditions, so the positive mass theorem is not necessarily valid here.

The following result proves that an electrovacuum space under a certain hypothesis must necessarily be in the Majumdar–Papapetrou class. It is worth emphasizing that in the following theorem, we are assuming the existence of an open set $\Omega \subseteq M \setminus \partial M$ such that $\Omega \cap \text{crit}(f) = \emptyset$.

**Theorem 2.** Let $(M^n, g, f, \psi)$, $n \geq 3$, be an electrovacuum space satisfying (1.8). Then, the Schouten tensor for the metric $g$ is Codazzi. If $(M^n, g)$ has identically zero Weyl tensor (i.e., locally conformally flat if $n > 3$), then the space must be in the Majumdar–Papapetrou class, i.e., the static spacetime $(\hat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}$ must have (locally) metric tensor given by

$$\hat{g}(x, t) = f^{-2/(n-2)}(dx_1^2 + \cdots + dx_n^2) - f^2 dt^2.$$ 

We observe that if the Schouten tensor is Codazzi then in dimension three $(M^3, g)$ is locally conformally flat metric; whereas in dimension $n > 3$, Eq. 2.4 implies in harmonic Weyl curvature. Codazzi tensors in Riemannian manifolds are important by themselves (see [3, Proposition 16.11]). In addition, if an extremal electrovacuum solution is locally conformally flat, then it is possible to use classical calculations to prove that it is an MP solution (see [11, Proposition 3.4]). Moreover, the extremal case was recently considered in [18], where the author proved that the only asymptotically flat spacetimes with a suitably regular event horizon, in a generalized Majumdar–Papapetrou class of solutions to higher-dimensional Einstein–Maxwell theory, are the standard multi-black holes (1.5).

**Remark 1.** Lucietti [18, Theorem 1] proved that an asymptotically flat higher dimensional $(n > 3)$ extremal electrovacuum space is in the MP class, by requiring a mild extension of the positive mass theorem to manifolds with conical singularities. Furthermore, the author was able to prove that $f$ must be given by (1.6). Remembering that we are not assuming any asymptotic condition.

Now, it remains to consider the electrovacuum solutions in the RN class, i.e., such that the electric potential is given by (1.7). Here, we are considering divergence conditions on Weyl ($W$) and Cotton ($C$) tensors for a static
Einstein–Maxwell spacetime instead of the traditional asymptotic conditions. Divergence conditions on $W$ have been recently explored in several works (see [7,8,17,19] and the references therein). When the divergence of the Weyl tensor is identically zero, i.e.,

$$\text{div}W = 0,$$  \hspace{1cm} (1.9)

we say that the manifold has a harmonic Weyl curvature. It is well known that if the scalar curvature is constant, then harmonic Weyl curvature implies harmonic curvature. This condition is equivalent to zero Cotton tensor in dimension more than 3 (see 2.4).

In what follows, we will consider that a Riemannian manifold $(M^n, g)$ has zero radial Weyl curvature if

$$W(\cdot, \cdot, \cdot, \nabla f) = 0,$$  \hspace{1cm} (1.10)

where $\nabla f$ is the gradient for a smooth function $f: M \rightarrow \mathbb{R}$. This condition was used in [7] and [17] in the study of Einstein-type manifolds, see more details in the references therein.

A straightforward computation from (2.4) shows us that the harmonic Weyl tensor condition is equivalent to the Schouten tensor being Codazzi when $n > 3$. For the sake of simplicity of the next results, we will now adopt this new definition of harmonic Weyl tensor as the terminology whenever necessary.

Now, we are ready to announce our next classification result.

**Theorem 3.** Let $(M^n, g, f, \psi)$, $n \geq 3$, be an electrovacuum space with harmonic Weyl curvature and zero radial Weyl curvature such that $\psi$ is in the Reissner–Nordström class, i.e., such that $\psi$ is given by (1.7) and $\sigma < 0$. Then, around any regular point of $f$, the manifold is locally a warped product with $(n-1)$-dimensional Einstein fibers.

**Remark 2.** In the three-dimensional case, it is important to notice that the Weyl tensor $W$ is identically zero. So, the zero radial Weyl curvature condition is trivial. Moreover, the harmonic Weyl curvature condition must be replaced by a locally conformally flat metric, i.e., $C = 0$.

**Remark 3.** The condition required over $\sigma$ in the above theorem is just to avoid any major technical issues and can be relaxed by considering that

$$\left\{ f = \pm \sqrt{\frac{(n-1)}{(n-2)} \sigma} \right\} \subset M$$

is not dense in $M$.

In this paper, we will provide several results about divergence-free conditions in an electrovacuum space such that $\psi = \psi(f)$. Our goal is to provide a classification for an electrovacuum space having a fourth-order divergence-free Weyl tensor, i.e., $\text{div}^4W = 0$. In the three-dimensional case, the discussion reduces to consider the Cotton tensor free from divergence, i.e., $\text{div}^3C = 0$. We will show that these higher-order divergence conditions can be reduced to
harmonic Weyl curvature conditions (or locally conformally flat curvature in the three-dimensional case), under some additional hypothesis.

The idea is to prove that the higher-order divergence-free conditions can be reduced to harmonic Weyl curvature (or zero Cotton tensor for $n = 3$) using an appropriate divergence formula combined with some cut-off function and then, by integration of such formula, concluding that the Cotton tensor is identically zero, which is a similar strategy used by [4,7,17,19].

Next, as a consequence of Theorem 3 (see also Corollary 3), we get the following result.

**Corollary 1.** Let $(M^n, g, f, \psi)$, $n > 3$, be an electrovacuum space with fourth-order divergence free Weyl curvature and zero radial Weyl curvature such that the electric potential $\psi$ is in the Reissner–Nordström class (i.e., satisfying Eq. (1.7) with $\sigma < 0$). Around any regular point of $f$, if $f$ is a proper function, then the manifold is locally a warped product with $(n-1)$-dimensional Einstein fibers.

In the three-dimensional case, the computations follow closely the same strategy as the above result and also we provide some interesting results reducing the order of divergence for the Cotton tensor. In this way, we obtain the following result.

**Corollary 2.** Let $(M^3, g, f, \psi)$ be an electrovacuum space with third-order divergence free Cotton tensor such that $\psi$ satisfies (1.7) with $\sigma < 0$. Around any regular point of $f$, if $f$ is a proper function, then the manifold is locally a warped product with a one-dimensional base and a constant curvature surface fiber.

The paper is organized as follows. Section 2 introduces the terminology used throughout this paper. In Sect. 3, we present some structural lemmas that will be used in the proof of the main results. Finally, in Sect. 4 we prove the main results.

**2. Background**

In this section, we fix our notation and recall some basic facts and useful lemmas. In particular, we need to remember some special tensors in the study of curvature for a Riemannian manifold $(M^n, g)$, $n \geq 3$. The first one is the Weyl tensor $W$ which is defined by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il})$$

$$+ \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}),$$

where $R_{ijkl}$ denotes the Riemann curvature tensor. The second one is the Cotton tensor given by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}).$$
And finally, considering \( n \geq 4 \), the Bach tensor is defined by

\[
B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R^{kl} W_{ikjl}.
\]  

(2.3)

We observe that the Weyl tensor has the same symmetries of the curvature tensor, that is

\[
W_{ikjl} = -W_{kijl}, \quad W_{ikjl} = -W_{iklj} \quad \text{and} \quad W_{ikjl} = W_{jlik}.
\]

Moreover, we note that the Bach, the Cotton, and the Weyl tensors are totally trace-free in any two indices (see [6] for instance), i.e.,

\[
g^{ij} C_{ijk} = g^{ik} C_{ijk} = g^{jk} C_{ijk} = 0.
\]

(2.4)

When the dimension of \( M \) is \( n = 3 \), then the Weyl tensor \( W_{ijkl} \) vanishes identically and the Cotton tensor \( C_{ijk} = 0 \) if and only if \( (M^3, g_{ij}) \) is locally conformally flat; this fact holds if and only if \( W_{ijkl} = 0 \), considering dimension \( n \geq 4 \). Thus, for \( n \geq 4 \) we have some well-known relations with these tensors and their derivatives (see [6,7,17]). Involving the Weyl and Cotton tensors a straightforward computation yields to

\[
C_{ijk} = -\frac{n-2}{n-3} \nabla^l W_{ijkl}.
\]  

(2.5)

So, if the Cotton tensor vanishes, then the Weyl tensor is harmonic.

Now, for \( n \geq 3 \) combining (2.3) and (2.4) we can rewrite the Bach tensor as

\[
B_{ij} = -\frac{1}{n-2} \nabla^k C_{ikj} + \frac{1}{n-2} R^{kl} W_{ikjl}.
\]  

(2.6)

In particular, (see [6]), in dimension \( n = 3 \), since the Weyl tensor is identically zero, we can conclude that

\[
B_{ij} = \nabla^k C_{kij}.
\]  

(2.7)

This equation leads us to the following fact:

\[
\nabla^k C_{kij} = \nabla^k C_{kji}.
\]

Is convenient to express the divergence for the Bach tensor, which is given by

\[
\nabla^j B_{ij} = \frac{n-4}{(n-2)^2} C_{ijk} R^{jk}.
\]  

(2.8)

Moreover, it is easy to see that

\[
C_{ijk} = -C_{jik}
\]

and

\[
C_{ijk} + C_{jki} + C_{kij} = 0.
\]  

(2.9)

From the contracted second Bianchi identity and from commutation formulas for any Riemannian manifold we can infer that

\[
\nabla^i C_{jki} = 0.
\]

Moreover, remember that

\[
(n-2)C_{ijk} = \nabla_i S_{jk} - \nabla_j S_{ik},
\]

where \( S_{ij} \) is the Ricci tensor.
where $S$ stands for the Schouten tensor of $g$, i.e.,

$$S_{ij} = \frac{1}{n-2} \left( R_{ij} - \frac{R}{2(n-1)} g_{ij} \right).$$

(2.10)

3. Structural Lemmas

Next, motivated by ideas in [4, 7, 17, 19] we obtain some structural lemmas, which are fundamental to proof our results. Note that in a local coordinates system, using (1.2) we can rewrite (1.1) as

$$f R_{jk} = \nabla_j \nabla_k f - \frac{2}{f} \nabla_j \psi \nabla_k \psi + \frac{1}{n-1} f R g_{jk};$$

(3.1)

$$\Delta f = \frac{n-2}{n-1} f R = 2 \left( \frac{n-2}{n-1} \right) \frac{|\nabla \psi|^2}{f};$$

(3.2)

$$0 = \Delta \psi - \frac{1}{f} \langle \nabla f, \nabla \psi \rangle.$$  

(3.3)

Lemma 1. Let $(M^n, g, f, \psi)$, $n \geq 3$, be an electrovacuum system. Then,

$$f C_{ijk} = W_{ijkl} \nabla^l f + \frac{1}{n-2} (R_{jl} \nabla^l f g_{ik} - R_{il} \nabla^l f g_{jk})$$

$$+ \frac{R}{(n-1)(n-2)} (\nabla_i f g_{jk} - \nabla_j f g_{ik})$$

$$- \frac{2}{f^2} [f (\nabla_j \psi \nabla_i \nabla_k \psi - \nabla_i \psi \nabla_j \nabla_k \psi) - \nabla_i f \nabla_j \psi \nabla_k \psi + \nabla_j f \nabla_i \psi \nabla_k \psi]$$

$$+ \frac{n-1}{n-2} (R_{ik} \nabla_j f - R_{jk} \nabla_i f) + \frac{1}{(n-1)f} (\nabla_i |\nabla \psi|^2 g_{jk} - \nabla_j |\nabla \psi|^2 g_{ik}).$$

Proof. We take the derivative of (3.1) to obtain

$$R_{jk} \nabla_i f + f \nabla_i R_{jk} = -\frac{2}{f^2} [f (\nabla_i \nabla_j \psi \nabla_k \psi + \nabla_j \psi \nabla_i \nabla_k \psi) - \nabla_i f \nabla_j \psi \nabla_k \psi]$$

$$+ \nabla_i \nabla_j \nabla_k f + \frac{1}{n-1} \left( \frac{f}{2} R_{ij} \nabla_j \psi + \frac{1}{f} \nabla_i |\nabla \psi|^2 \right) g_{jk}$$

(3.4)

and

$$R_{ik} \nabla_j f + f \nabla_j R_{ik} = -\frac{2}{f^2} [f (\nabla_j \nabla_i \psi \nabla_k \psi + \nabla_i \psi \nabla_j \nabla_k \psi) - \nabla_j f \nabla_i \psi \nabla_k \psi]$$

$$+ \nabla_j \nabla_i \nabla_k f + \frac{1}{n-1} \left( \frac{f}{2} R_{jk} \nabla_i \psi + \frac{1}{f} \nabla_j |\nabla \psi|^2 \right) g_{ik}.$$ 

(3.5)
Subtracting (3.4) from (3.5) and using that the Hessian operator is symmetric, we can deduce that
\[
R_{jk} \nabla_i f - R_{ik} \nabla_j f + f(\nabla_i R_{jk} - \nabla_j R_{ik}) = \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f
\]
\[
+ \frac{f}{2(n-1)}(\nabla_i R_{gjk} - \nabla_j R_{gik})
\]
\[
- \frac{2}{f^2}[f(\nabla_j \psi \nabla_i \nabla_k \psi - \nabla_i \psi \nabla_j \nabla_k \psi) - \nabla_i f \nabla_j \psi \nabla_k \psi + \nabla_j f \nabla_i \psi \nabla_k \psi]
\]
\[
+ \frac{1}{(n-1)f}(\nabla_i |\nabla \psi|^2 g_{jk} - \nabla_j |\nabla \psi|^2 g_{ik}).
\]

It is well known that in any Riemannian manifold we can relate the Riemannian curvature tensor with a smooth function by using the Ricci identity
\[
\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijk} \nabla^l f.
\] (3.6)

Then, replacing the Ricci identity (3.6) and the Cotton tensor (2.2), we infer that
\[
fC_{ijk} = R_{ijkl} \nabla^l f + \frac{1}{(n-1)f}(\nabla_i |\nabla \psi|^2 g_{jk} - \nabla_j |\nabla \psi|^2 g_{ik}) - R_{jk} \nabla_i f + R_{ik} \nabla_j f
\]
\[
- \frac{2}{f^2}[f(\nabla_j \psi \nabla_i \nabla_k \psi - \nabla_i \psi \nabla_j \nabla_k \psi) - \nabla_i f \nabla_j \psi \nabla_k \psi + \nabla_j f \nabla_i \psi \nabla_k \psi].
\]

Now, using the Weyl formula (2.1), we have
\[
fC_{ijk} = W_{ijkl} \nabla^l f + \frac{1}{n-2}(R_{jl} \nabla^l f g_{ik} - R_{il} \nabla^l f g_{jk})
\]
\[
- \frac{R}{(n-1)(n-2)}(g_{ik} \nabla_j f - g_{jk} \nabla_i f)
\]
\[
- \frac{2}{f^2}[f(\nabla_j \psi \nabla_i \nabla_k \psi - \nabla_i \psi \nabla_j \nabla_k \psi) - \nabla_i f \nabla_j \psi \nabla_k \psi + \nabla_j f \nabla_i \psi \nabla_k \psi]
\]
\[
+ \frac{n-1}{n-2}(R_{ik} \nabla_j f - R_{jk} \nabla_i f) + \frac{1}{(n-1)f}(\nabla_i |\nabla \psi|^2 g_{jk} - \nabla_j |\nabla \psi|^2 g_{ik}).
\]

So, the proof is finished. \(\square\)

In the sequel, we define the covariant 3-tensor \(V_{ijk}\) by
\[
V_{ijk} = \frac{1}{n-2}(R_{jl} \nabla^l f g_{ik} - R_{il} \nabla^l f g_{jk}) + \frac{R}{(n-1)(n-2)}(\nabla_i f g_{jk} - \nabla_j f g_{ik})
\]
\[
- \frac{2}{f^2}[f(\nabla_j \psi \nabla_i \nabla_k \psi - \nabla_i \psi \nabla_j \nabla_k \psi) - \nabla_i f \nabla_j \psi \nabla_k \psi + \nabla_j f \nabla_i \psi \nabla_k \psi]
\]
\[
+ \frac{n-1}{n-2}(R_{ik} \nabla_j f - R_{jk} \nabla_i f) + \frac{1}{(n-1)f}(\nabla_i |\nabla \psi|^2 g_{jk} - \nabla_j |\nabla \psi|^2 g_{ik}).
\] (3.7)

The tensor \(V_{ijk}\) was defined similarly to \(D_{ijk}\) in [6].

Note that from a straightforward computation, we observe that the tensor \(V\) has the same symmetries as the Cotton tensor \(C\), i.e.,
\[
V_{ijk} = -V_{jik} \quad \text{and} \quad V_{ijk} + V_{jki} + V_{kij} = 0.
\]
This 3-tensor has a fundamental importance in what follows. From Lemma 1, we have

\[ f_{Cijk} = W_{ijkl} \nabla^l f + V_{ijk}. \]  

(3.8)

In particular, we obtain the following result if we suppose that \( \psi = \psi(f) \) in the Lemma 1.

**Lemma 2.** Let \((M^n, g, f, \psi)\), \(n \geq 3\), be an electrovacuum system such that \( \psi = \psi(f) \). Then,

\[ V_{ijk} = P(R_{il} \nabla^l f g_{jk} - R_{jl} \nabla^l f g_{ik}) + Q(R_{ik} \nabla_j f - R_{jk} \nabla_i f) + U(\nabla_i f g_{jk} - \nabla_j f g_{ik}), \]

(3.9)

where

\[ P = \frac{-1}{n-2} + \frac{2 \dot{\psi}(f)^2}{n-1}, \quad Q = \frac{n-1}{n-2} - 2 \dot{\psi}(f)^2 \]

and

\[ U = \frac{R}{n-1} \left[ \frac{1}{(n-2)(n-1)} - \frac{2 \dot{\psi}(f)^2}{(n-1)(n-2)} + \frac{f \ddot{\psi}(f)}{\psi(f)} \right]. \]

**Proof.** In fact, since \( \psi = \psi(f) \), using (3.1), we obtain

\[ \nabla_k \nabla_i \psi = \ddot{\psi}(f) \nabla_k f \nabla_i f + \dot{\psi}(f) \nabla_k \nabla_i f \]

\[ = \ddot{\psi}(f) \nabla_k f \nabla_i f + f \dot{\psi}(f) R_{ki} + \frac{2}{f} \ddot{\psi}(f)^3 \nabla_k f \nabla_i f - \frac{1}{n-1} f \ddot{\psi}(f) R_{g_{ki}}. \]

Replacing the above equation in (3.7) we can rewrite the 3-tensor \( V \) as

\[ V_{ijk} = \frac{1}{n-2} (R_{jl} \nabla^l f g_{jk} - R_{il} \nabla^l f g_{jk}) \]

\[ + \left[ \frac{R}{(n-1)(n-2)} - \frac{2}{n-1} \dot{\psi}(f)^2 R \right] (\nabla_i f g_{jk} - \nabla_j f g_{ik}) \]

\[ + \left[ \frac{n-1}{n-2} - 2 \dot{\psi}(f)^2 \right] (R_{ik} \nabla_j f - R_{jk} \nabla_i f) \]

\[ + \frac{1}{(n-1)f} (\nabla_i |\nabla \psi|^2 g_{jk} - \nabla_j |\nabla \psi|^2 g_{ik}). \]

(3.10)

Now, by taking the derivative of (1.2) and using (4.1) we deduce that

\[ 4 \ddot{\psi}(f) \dot{\psi}(f) \nabla_i f |\nabla f|^2 + 2 \ddot{\psi}(f)^2 \nabla_i |\nabla f|^2 = 2f R \nabla_i f + f^2 \nabla_i R. \]

Combining \( \nabla \psi = \dot{\psi}(f) \nabla f \) and (3.1), we obtain

\[ 4 \ddot{\psi}(f) \dot{\psi}(f) \nabla_i f |\nabla f|^2 + 4 \ddot{\psi}(f)^2 \left( f R_{il} \nabla_i f + \frac{2}{f} \dot{\psi}(f)^2 \nabla_i f |\nabla f|^2 - \frac{1}{n-1} f R \nabla_i f \right) \]

\[ = 2f R \nabla_i f + f^2 \nabla_i R, \]
this implies that
\[
f^2 \nabla_i R = 4 \ddot{\psi}(f) \dot{\psi}(f) \nabla_i f |\nabla f|^2 + 4 \dddot{\psi}(f)^2 \nabla_i f |\nabla f|^2 - \frac{1}{n-1} f R \nabla_i f \frac{\psi(f)}{\dot{\psi}(f)} \nabla_i f - 2 f R \nabla_i f \nabla_i \nabla \psi.
\]

(3.11)

Then using (1.2) and (3.11), we get
\[
\nabla_i |\nabla \psi|^2 = f R \nabla_i f + \frac{f^2}{2} \nabla_i R
\]
\[
= f R \left( \frac{f \dddot{\psi}(f)}{\dot{\psi}(f)} + \frac{2(n-2)}{n-1} \dot{\psi}(f)^2 - 1 \right) \nabla_i f + 2 f \dddot{\psi}(f)^2 \nabla_i f.
\]

Thus, replacing this equation in (3.10) the result follows. □

On the other hand, by the right conformal change of the metric, we get our next lemma.

Lemma 3. Let \((M^n, g, f, \psi)\), \(n \geq 3\), be an electrovacuum system such that \(\psi = \psi(f)\) is given by (1.8). Then, the Cotton tensor satisfies
\[
(n-2)^2 f C_{ijk} = W_{ijkl} \nabla^k f.
\]

(3.12)

In particular, when \(n = 3\), then \((M^3, g)\) is locally conformally flat, i.e., \(C = 0\).

Proof. We consider the conformal change of the metric
\[
\tilde{g} = f^{\frac{2}{n-2}} g.
\]

From [7, Appendix] the Cotton tensor for metric \(\tilde{g}\) is given by
\[
(n-2) \tilde{C}_{ijk} = (n-2) C_{ijk} - \frac{1}{(n-2)f} W_{ijkl} \nabla^l f.
\]

(3.13)

Moreover, for \(\tilde{g}\) (see [3, p. 58]), we obtain
\[
\tilde{\text{Ric}} = \text{Ric} - \frac{1}{f} \nabla^2 f + \frac{(n-1)}{(n-2)f^2} df \otimes df - \frac{\Delta f}{(n-2)f} g
\]
\[
= \text{Ric} - \frac{1}{f} \nabla^2 f + \frac{(n-1)}{(n-2)f^2} df \otimes df - \frac{R}{(n-1)} g.
\]

(3.14)

where in the last equation we have used (3.2).

Considering \(\psi = \psi(f)\), from 1.1, we get
\[
\tilde{\text{Ric}} = \frac{1}{f^2} \frac{(n-1)}{(n-2)} df \otimes df - \frac{2}{f^2} d\psi \otimes d\psi + \frac{1}{(n-2)f} \left[ \frac{2(n-2)}{(n-1)} |\nabla \psi|^2 - \Delta f \right] f^{\frac{2}{n-2}} \tilde{g}
\]
\[
= \frac{1}{f^2} \frac{(n-1)}{(n-2)} df \otimes df - \frac{2}{f^2} d\psi \otimes d\psi = \frac{1}{f^2} \left[ \frac{(n-1)}{(n-2)} - 2 \psi^2 \right] df \otimes df.
\]

(3.15)
Moreover,
\[ \tilde{R} = \frac{1}{f^2} \left[ \frac{(n-1)}{(n-2)} - 2\dot{\psi}^2 \right] |\tilde{\nabla} f|^2. \]

By hypothesis \( \psi = \psi(f) \) satisfies (1.8), then
\[ 2\dot{\psi}^2 = \frac{(n-1)}{(n-2)}. \]  

(3.16)

Consequently, from (3.15) and (3.16), we conclude that \( (M^n, \tilde{g}) \) is Ricci-flat. In this case, the Schouten tensor for \( \tilde{g} \) is given by
\[ \tilde{S} = \frac{1}{n-2} \left( \tilde{\text{Ric}} - \frac{1}{2(n-1)} \tilde{R}\tilde{g} \right) \]
\[ = \left[ \frac{(n-1)}{(n-2)} - 2\dot{\psi}^2 \right] \left( \frac{1}{f^2} df \otimes df - \frac{|\tilde{\nabla} f|^2}{2(n-1)} \frac{\tilde{\nabla} f}{\tilde{\nabla} f} \right) = 0. \]

This shows that \( \tilde{S} \) is Codazzi, because \( \tilde{S} = 0 \), i.e., \( (\tilde{\nabla}_X \tilde{S})(Y) = (\tilde{\nabla}_Y \tilde{S})(X) \) for all \( X, Y \in TM \). Therefore, the Cotton tensor for the metric \( \tilde{g} \) is identically zero. So, from (3.13) we have
\[ (n-2)^2 f C_{ijk} = W_{ijkl} \nabla^l f. \]

Thus, we conclude our proof. \( \square \)

Now our goal is to obtain a useful formula for the norm of the Cotton tensor involving the divergence of the tensor \( V \). To prove this, we need to show several lemmas.

**Lemma 4.** Let \( (M^n, g, f, \psi) \), \( n \geq 4 \), be an electrovacuum system. Then,
\[ (n-2)B_{ij} = -\nabla^k \left( \frac{V_{ikj}}{f} \right) + \frac{n-1}{n-2} C_{jki} \nabla^k f - \frac{1}{f^2} W_{ikjl} \left( \nabla^k f \nabla^l f - 2\nabla^k \psi \nabla^l \psi \right). \]  

(3.17)

**Proof.** In fact, from (2.5) and (3.8), we can deduce that
\[ (n-2)B_{ij} = -\nabla^k C_{ikj} + R^{kl} W_{ikjl} \]
\[ = -\nabla^k \left( \frac{V_{ikj}}{f} + \frac{W_{ikjl} \nabla^l f}{f} \right) + R^{kl} W_{ikjl} \]
\[ = -\nabla^k \left( \frac{V_{ikj}}{f} \right) - \frac{\nabla^k W_{ikjl} \nabla^l f}{f} + \frac{W_{ikjl} \nabla^k f \nabla^l f}{f^2} \]
\[- \frac{W_{ikjl} \nabla^k \nabla^l f}{f} + R^{kl} W_{ikjl}. \]

Now, using (3.1), we obtain
\[ (n-2)B_{ij} = -\nabla^k \left( \frac{V_{ikj}}{f} \right) - \frac{\nabla^k W_{ikjl} \nabla^l f}{f} + \frac{W_{ikjl} \nabla^k f \nabla^l f}{f^2} \]
\[- \frac{W_{ikjl}}{f} \left( f R^{kl} + \frac{2}{f} \nabla^k \psi \nabla^l \psi - \frac{1}{n-1} f Rg^{kl} \right) + R^{kl} W_{ikjl}. \]
Since the Weyl tensor is trace-free, we have

$$(n - 2)B_{ij} = -\nabla^k \left( \frac{V_{ikj}}{f} \right) - \nabla^k W_{ikjl} \nabla^j f + \frac{1}{f^2} W_{ikjl} (\nabla^k f \nabla^l f - 2\nabla^k \psi \nabla^l \psi).$$

From (2.4), we get the result. \qquad \square

Proceeding, we can use the previous lemma to obtain the following result.

**Lemma 5.** Let $(M^n, g, f, \psi)$, $n \geq 4$, be an electrovacuum system. Then,

$$C_{jki}R^{ik} = (n - 2)\nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) - (n - 2) \frac{1}{f^2} W_{ikjl} \nabla^i \nabla^k f \nabla^j f$$

$$+ 2(n - 2) \frac{W_{ikjl}}{f^2} \nabla^i \psi \nabla^i \nabla^j \psi - 2(n - 2) \frac{W_{ikjl}}{f^3} \nabla^i f \nabla^k \psi \nabla^l \psi \nabla^j f.$$

(3.18)

**Proof.** Taking the divergence of (3.17) and using (2.9), we infer that

$$(n - 2)\nabla^i B_{ij} = -\nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) + \frac{n - 3}{n - 2} \frac{C_{jki}}{f^2} (f \nabla^i \nabla^k f - \nabla^k f \nabla^j f)$$

$$+ \frac{1}{f^2} W_{ikjl} (\nabla^i \nabla^k f \nabla^j f + \nabla^k f \nabla^i \nabla^j f$$

$$- 2\nabla^i \nabla^k \psi \nabla^l \psi - 2\nabla^k \psi \nabla^i \nabla^l \psi)$$

$$+ \frac{1}{f^2} \nabla^i W_{ikjl} (\nabla^k f \nabla^j f - 2\nabla^k \psi \nabla^l \psi)$$

$$- \frac{2}{f^3} W_{ikjl} (\nabla^i f \nabla^k f \nabla^l f - 2\nabla^i f \nabla^k \psi \nabla^l \psi).$$

(3.19)

Since the Hessian is symmetric, then renaming indices and using the symmetries of the Weyl tensor, we deduce

$$2\nabla^i \nabla^k \psi W_{ikjl} = \nabla^i \nabla^k \psi W_{ikjl} + \nabla^k \nabla^i \psi W_{kijl} = \nabla^i \nabla^k \psi (W_{ikjl} + W_{kijl}) = 0.$$

(3.20)

Analogously, we have the same expression for the lapse function $f$, i.e.,

$$\nabla^i \nabla^k f W_{ikjl} = 0.$$

Combining (3.19) and (3.20), we obtain

$$(n - 2)\nabla^i B_{ij} = -\nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) + \frac{n - 3}{n - 2} \frac{C_{jki}}{f^2} (f \nabla^i \nabla^k f - \nabla^k f \nabla^j f)$$

$$+ \frac{4}{f^3} W_{ikjl} \nabla^i f \nabla^k \psi \nabla^l \psi - \frac{1}{f^2} \nabla^i W_{jiki} (\nabla^k f \nabla^l f - 2\nabla^k \psi \nabla^l \psi)$$

$$+ \frac{1}{f^2} W_{ikjl} (\nabla^k f \nabla^i \nabla^l f - 2\nabla^k \psi \nabla^i \nabla^l \psi).$$
Since the Cotton and Weyl tensors are trace-free, using the symmetries of the Weyl tensor, (2.4) and (3.1) we get

\[(n - 2)\nabla^i B_{ij} = -\nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) + \frac{n - 3}{n - 2} C_{jki} R^{ik} + \frac{1}{f} W_{ikjl} R^{il} \nabla^k f \]
\[- \frac{2}{f^2} W_{ikjl} \nabla^k \psi \nabla^l \psi + \frac{2}{f^3} W_{ikjl} \nabla^i f \nabla^k \psi \nabla^l \psi. \tag{3.21}\]

Now, we need to remember some facts. Firstly, \(B_{ij} = B_{ji}, R_{ij} = R_{ji}\) and the Cotton tensor is skew-symmetric, then we have an analogous relation to (3.20), i.e,

\[C_{ikj} R^{ik} = 0. \tag{3.22}\]

Secondly, using (2.8), we infer \(C_{jik} = C_{jki} + C_{kij}\), this implies that \(C_{jik} R^{ik} = C_{jki} R^{ik}\). Thus, from (2.7) and using these observations after renaming the indices, we obtain

\[\nabla^i B_{ij} = \frac{n - 4}{(n - 2)^2} C_{jik} R^{ik} = \frac{n - 4}{(n - 2)^2} C_{jki} R^{ik}. \]

Finally, using the above equation in (3.21) the result holds. \(\Box\)

**Lemma 6.** Let \((M^n, g, f, \psi), n \geq 4,\) be an electrovacuum system. Then,

\[
\frac{1}{2} |\mathbf{C}|^2 + R^{ik} \nabla^j C_{jki} = (n - 2) \nabla^j \nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) - (n - 2) \nabla^j \left[ \frac{1}{f} W_{ikjl} R^{il} \nabla^k f \right] \\
- 2(n - 2) \nabla^j \left[ \frac{W_{ikjl}}{f^3} \nabla^i f \nabla^k \psi \nabla^l \psi \right] \\
+ 2(n - 2) \nabla^j \left[ \frac{W_{ikjl}}{f^2} \nabla^k \psi \nabla^i \nabla^l \psi \right]. \tag{3.23}\]

**Proof.** Taking the divergence of (3.18), we have

\[C_{jki} \nabla^j R^{ik} + R^{ik} \nabla^j C_{jki} = (n - 2) \nabla^j \nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) \]
\[- (n - 2) \nabla^j \left[ \frac{1}{f} W_{ikjl} R^{il} \nabla^k f \right] \]
\[- 2(n - 2) \nabla^j \left[ \frac{W_{ikjl}}{f^3} \nabla^i f \nabla^k \psi \nabla^l \psi \right] \]
\[+ 2(n - 2) \nabla^j \left[ \frac{W_{ikjl}}{f^2} \nabla^k \psi \nabla^i \nabla^l \psi \right]. \tag{3.23}\]

Note that from the symmetries of the Cotton tensor and renaming indices, we get

\[2C_{jki} \nabla^j R^{ik} = C_{jki} \nabla^j R^{ik} + C_{kji} \nabla^k R^{ij} = C_{jki}(\nabla^j R^{ik} - \nabla^k R^{ij}). \tag{3.24}\]
Then, combining (3.23) and (3.24), we can infer that
\[
\frac{1}{2} C_{jki}(\nabla^j R^{ik} - \nabla^k R^{ij}) + R^{ik} \nabla^j C_{jki} = (n - 2) \nabla^j \nabla^k \left( \frac{V_{ikj}}{f} \right) \\
- (n - 2) \nabla^j \left[ \frac{1}{f} W_{ikjl} R^{il} \nabla^k f \right] \\
- 2(n - 2) \nabla^j \left[ \frac{W_{ikjl} \nabla^i f \nabla^k \psi \nabla^l \psi}{f^3} \right] \\
+ 2(n - 2) \nabla^j \left[ \frac{W_{ikjl} \nabla^k \psi \nabla^i \nabla^l \psi}{f^2} \right].
\]

From (2.2) and using that the Cotton tensor is trace-free, we obtain the result. □

4. Proof of the Main Results

In this section, we prove our main results.

4.1. Classification Results

Now we are ready to present the proofs of Theorem 1, Theorem 2, and Theorem 3, which are the main classification results in this present work. They will be stated again here for the sake of the reader’s convenience.

We start with Theorem 1 which shows us how related the electric potential and the lapse function can be. This result was inspired by [16] and [21].

**Theorem 4 (Theorem 1).** Let \((M^n, g, f, \psi), n \geq 3,\) be an electrovacuum space such that \(\psi = \psi(f)\). Then, the electric potential (locally) is either
\[
\frac{2(n - 2)}{n - 1} \psi(f)^2 - \frac{4(n - 2)}{n - 1} \beta \psi(f) + \frac{2(n - 2)}{n - 1} \beta^2 + \frac{n - 1}{n - 2} \sigma = f^2
\]
or
\[
\psi(f) = \beta \pm \sqrt{\frac{(n - 1)}{2(n - 2)}} f,
\]
where \(\sigma, \beta \in \mathbb{R}\). Moreover, \(\sigma = 0\) if and only if \(\psi(f)\) is an affine function of \(f\).

**Proof.** Since \(\psi = \psi(f)\), we obtain
\[
\nabla \psi = \dot{\psi}(f) \nabla f.
\]
(4.1)

Then,
\[
\nabla^2 \psi = \ddot{\psi}(f) df \otimes df + \dot{\psi}(f) \nabla^2 f.
\]

Now, contracting the above equation over \(g^{-1}\), we infer that
\[
\Delta \psi = \ddot{\psi}(f) |\nabla f|^2 + \dot{\psi}(f) \Delta f.
\]
From (3.2) and (4.1), we have

$$\Delta f = \frac{2}{f} \left( \frac{n-2}{n-1} \right) \psi(f)^2 |\nabla f|^2.$$ 

Combining the last equations with (3.3) and (4.1), we get

$$\ddot{\psi}(f) |\nabla f|^2 + 2 \left( \frac{n-2}{n-1} \right) \frac{\dot{\psi}(f)^3 |\nabla f|^2}{f} = \dot{\psi}(f) \frac{|\nabla f|^2}{f}.$$ 

Notice that there is no open subset \( \Omega \) of \( M \) where \( \{ \nabla f = 0 \} \) is dense. By a straightforward computation, we arrive at

$$\dot{h} + 2 \left( \frac{n-2}{n-1} \right) f h^3 = 0,$$

where

$$h = \frac{\dot{\psi}}{f}.$$ 

So, by solving this ODE, we get

$$\dot{\psi}(f) = \pm \frac{f}{\sqrt{2(n-2)(n-1)f^2 - 2\sigma}}; \quad \sigma \in \mathbb{R}. \quad (4.2)$$

By integration, we obtain, either

$$\psi(f) = \beta \pm \frac{(n-1)}{2(n-2)} \sqrt{2 \left( \frac{n-2}{n-1} \right) f^2 - 2\sigma}; \quad \sigma \neq 0, \beta \in \mathbb{R},$$

or

$$\psi(f) = \beta \pm \sqrt{\frac{(n-1)}{2(n-2)} f}; \quad \sigma = 0 \beta \in \mathbb{R}.$$ 

Moreover, from (4.2) we have the following useful identity

$$2\dot{\psi}(f)^2 = \frac{(n-1)f^2}{(n-2)f^2 - (n-1)\sigma}. \quad (4.3)$$

To finish it off, we observe that if \( \sigma = 0 \), then from the above equation, \( \dot{\psi}(f) \) is a constant, this implies that \( \psi(f) \) is an affine function. \( \square \)

In the next result, we prove that an electrovacuum space under a certain hypothesis necessarily must be in the Majumdar–Papapetrou class.

**Theorem 5 (Theorem 2).** Let \((M^n, g, f, \psi), n \geq 3\), be an electrovacuum space satisfying (1.8). Then, the Schouten tensor for the metric \(g\) is Codazzi. If \((M^n, g)\) is locally conformally flat, then the space must be in the Majumdar–Papapetrou class, i.e., the static spacetime \((\hat{M}^{n+1}, \hat{g}) = M^n \times f \mathbb{R}\) must have (locally) metric tensor given by

$$\hat{g}(x, t) = f^{-2/(n-2)}(dx_1^2 + \cdots + dx_n^2) - f^2(x)dt^2; \quad x \in M.$$
Proof. The proof follows from the previous section. In fact, remember that when $\psi$ is an affine function of $f$, we have equation (3.16). Then, from (3.9) we conclude that $P = Q = U = 0$, and so the tensor $V_{ijk}$ is identically zero. Thus, from (3.8) we obtain $fC_{ijk} = W_{ijkl}\nabla^l f$. Immediately, for $n = 3$ the Cotton tensor is identically zero which means that $(M^3, g, f, \psi)$ is locally conformally flat.

Now, considering $n > 3$, from the proof of Lemma 3 we obtain that the Ricci tensor, $\tilde{Ric}$, for the conformal change of the metric $\tilde{g} = f^{2/(n-2)}g$ is identically zero, and so the Cotton tensor $\tilde{C}_{ijk}$. At the same time, using (3.12), we can infer that $(n-2)^2 fC_{ijk} = W_{ijkl} \nabla^l f$, which combined with (3.8) gives us

$$[(n-2)^2 - 1] fC_{ijk} = 0.$$ 

Consequently, the Schouten tensor (2.10) is Codazzi, i.e., $(\nabla X S)(Y) = (\nabla Y S)(X)$ for all $X, Y \in TM$. Furthermore, since $\tilde{Ric}$ is identically zero, we conclude $(M^3, \tilde{g})$ is isometric to $\mathbb{R}^3$.

Using again the conformal change of the metric $\tilde{g} = f^{2/(n-2)}g$ (see [3, page 58]), we have

$$\tilde{R}_{ijkl} = f^{2/(n-2)} \left[ R_{ijkl} - (g_{ik}T_{jl} + g_{jl}T_{ik} - g_{il}T_{jk} - g_{jk}T_{il}) \right], \quad (4.4)$$

where

$$T_{ij} = \frac{1}{n-2} \left( \frac{1}{f} \nabla_i \nabla_j f - \frac{n-1}{(n-2)f^2} \nabla_i f \nabla_j f + \frac{1}{2(n-2)f^2} |\nabla f|^2 g_{ij} \right)$$

$$= \frac{1}{(n-2)} \left( \frac{1}{f} \nabla_i \nabla_j f - \frac{(n-1)}{(n-2)f^2} \nabla_i f \nabla_j f + \frac{R}{2(n-1)} g_{ij} \right).$$

In the last equality, we have used (3.2) and (3.16). Then, from (3.14), we get

$$R_{ij} = \frac{1}{f} \nabla_i \nabla_j f - \frac{(n-1)}{(n-2)f^2} \nabla_i f \nabla_j f + \frac{R}{(n-1)} g_{ij}.$$ 

Combining these two last identities we obtain

$$T_{ij} = \frac{1}{n-2} \left( R_{ij} - \frac{R}{2(n-1)} g_{ij} \right).$$

Note that the tensor $T$ coincides with the Schouten tensor $S$ given by (2.10). If the Weyl tensor for $g$ is identically zero, then from (2.1) we have $g_{ik}T_{jl} + g_{jl}T_{ik} - g_{il}T_{jk} - g_{jk}T_{il} = R_{ijkl}$.

Therefore, replacing the above formula in (4.4), we can conclude that

$$\tilde{R}_{ijkl} = 0.$$ 

Thus, we can say that (locally) $\tilde{g} = \delta$, where $\delta$ is standard Euclidean metric. Hence, we can infer that $g = f^{-2/(n-2)} \delta$. □

Next, we prove Theorem 3 about classification’s result which was inspired by [17].
**Theorem 6** (Theorem 3). Let \((M^n, g, f, \psi)\), \(n \geq 3\), be an electrovacuum space with harmonic Weyl curvature and zero radial Weyl curvature such that \(\psi\) is in the Reissner–Nordström class, i.e., such that \(\psi\) is given by (1.7). Then, around any regular point of \(f\), the manifold is locally a warped product with \((n-1)\)-dimensional Einstein fibers.

**Proof.** We consider an orthonormal frame \(\{e_1, e_2, \ldots, e_n\}\) diagonalizing the Ricci tensor \(\text{Ric}\) at a regular point \(p \in \Sigma = f^{-1}(c)\), with associated eigenvalues \(R_{kk}, k = 1, \ldots, n\), respectively. That is, \(\text{Ric}(\nabla f) = R_{ii}\delta_{ij}\). From Lemma 2, we infer

\[
\nabla_j f [PR_{jj} + QR_{ii} - U] = 0, \quad \forall i \neq j,
\]

where \(P, Q\) and \(U\) are given by (3.9). Without lost of generalization, consider \(\nabla_i f \neq 0\) and \(\nabla_j f = 0\) for all \(i \neq j\). Observe that \(\text{Ric}(\nabla f) = R_{ii}\delta_{ij}\), i.e., \(\nabla f\) is an eigenvector for \(\text{Ric}\). From (4.5), we obtain that \(\lambda = R_{ii}\) and \(\mu = R_{jj}, j \neq i\), have multiplicity 1 and \(n-1\), respectively. Moreover, if \(\nabla_i f \neq 0\) for at least two distinct directions, then using (4.5) we have that \(\mu = R_{11} = \ldots = R_{nn}\) and we also obtain that \(\nabla f\) is an eigenvector for \(\text{Ric}\).

Therefore, in any case, we have that \(\nabla f\) is an eigenvector for \(\text{Ric}\). From the above discussion we can take \(\{e_1 = \frac{\nabla f}{|\nabla f|}, e_2, \ldots, e_n\}\) as an orthonormal frame for \(\Sigma\) diagonalizing the Ricci tensor \(\text{Ric}\) for the metric \(g\).

Now, we note from (3.1) that

\[
fR_{a\ell} \nabla^\ell f = \frac{1}{2} \nabla_a |\nabla f|^2 - \frac{2\psi^2}{f} |\nabla f|^2 \nabla_a f + \frac{Rf}{(n-1)} \nabla_a f; \quad a \in \{2, \ldots, n\}.
\]

Hence, equation (4.6) gives us \(|\nabla f|\) is a constant in \(\Sigma\). Thus, we can express the metric \(g\) in the form

\[
g_{ij} = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta_a d\theta_b,
\]

where \(g_{ab}(f, \theta) d\theta_a d\theta_b\) is the induced metric and \((\theta_2, \ldots, \theta_n)\) is any local coordinate system on \(\Sigma\). We can find a good overview of the level set structure in [6,17].

Remember that \(\{\nabla f = 0\}\) is dense not dense in \(M\). Thus, we consider \(\Sigma\) a connected component of the level surface \(f^{-1}(c)\) (possibly disconnected) where \(c\) is any regular value of the function \(f\). Suppose that \(I\) is an open interval containing \(c\) such that \(f\) has no critical points in the open neighborhood \(U_I = f^{-1}(I)\) of \(\Sigma\). For sake of simplicity, let \(U_I \subset M \setminus \{f = 0\}\) be a connected component of \(f^{-1}(I)\). Then, we can make a change of variables

\[
r(x) = \int \frac{df}{|\nabla f|}
\]

such that the metric \(g\) in \(U_I\) can be expressed by

\[
g_{ij} = dr^2 + g_{ab}(r, \theta) d\theta_a d\theta_b.
\]

Let \(\nabla r = \frac{\partial}{\partial r}\), then \(|\nabla r| = 1\) and \(\nabla f = f'(r) \frac{\partial}{\partial r}\) on \(U_I\). Note that \(f'(r)\) does not change sign on \(U_I\). Moreover, we have \(\nabla_{\partial r} \partial r = 0\).
From (3.1) and the fact that $\nabla f$ is an eigenvector of $\text{Ric}$, then the second fundamental formula on $\Sigma$ is given by

$$h_{ab} = -\langle e_1, \nabla_a e_b \rangle = \frac{\nabla_a \nabla_b f}{|\nabla f|} = \frac{1}{|\nabla f|} \left( fR_{ab} - \frac{Rf}{n-1}g_{ab} \right) = \frac{f}{|\nabla f|} \left( \mu - \frac{R}{n-1} \right) g_{ab} = \frac{H}{n-1} g_{ab},$$

(4.7)

where $H = H(r)$, since $H$ is constant in $\Sigma$. In fact, contracting the Codazzi equation

$$R_{1cb} = \nabla_a h_{bc} - \nabla_b h_{ac}$$

over $c$ and $b$, it gives

$$R_{1a} = \nabla_a (H) - \frac{1}{n-1} \nabla_a (H) = \frac{n-2}{n-1} \nabla_a (H).$$

On the other hand, since $R_{1a} = 0$, we conclude that $H$ is constant in $\Sigma$. For what follows, we fix a local coordinates system

$$(x_1, \ldots, x_n) = (r, \ldots, \theta_n)$$

in $U_I$, where $(\theta_2, \ldots, \theta_n)$ is any local coordinates system on the level surface $\Sigma_c$. Considering that $a, b, c, \cdots \in \{2, \ldots, n\}$, we have

$$h_{ab} = -g(\partial_r, \nabla_a \partial_b) = -g(\partial_r, \Gamma^l_{ab} \partial_l) = -\Gamma^1_{ab}.$$ 

Now, by definition

$$\Gamma^1_{ab} = \frac{1}{2} g^{11} \left( -\frac{\partial}{\partial r} g_{ab} \right) = -\frac{1}{2} \frac{\partial}{\partial r} g_{ab}. $$

Then,

$$\frac{2}{n-1} H(r) g_{ab} = \frac{\partial}{\partial r} g_{ab}. $$

Hence, we can infer that

$$g_{ab}(r, \theta) = \varphi(r)^2 g_{ab}(r_0, \theta), $$

where $\varphi(r) = e^{\frac{1}{n-1} \int_{r_0}^{r} H(s) ds}$ and the level set $\{ r = r_0 \}$ corresponds to the connected component $\Sigma$ of $f^{-1}(c)$.

Now, we can apply the warped product structure (see [3]). Hence, considering

$$(M^n, g) = (I, dr^2) \times_{\varphi} (N^{n-1}, \bar{g}); \quad g = dr^2 + \varphi^2 \bar{g}, $$

we deduce that

$$W_{1a1b} = \frac{1}{n-2} \bar{R}_{ab} - \frac{\bar{R}}{(n-2)(n-1)} g_{ab}. $$

Finally, if $W(\cdot, \cdot, \cdot, \nabla f) = 0$ we obtain that $N$ is an Einstein manifold.
Since 
\[(M^n, g) = (I, dr^2) \times \varphi (N^{n-1}, \tilde{g}); \quad g = dr^2 + \varphi^2 \tilde{g},\]
applying the warped product formulas (see [3, Chapter 9]), the Ricci tensor of 
\((M^3, g)\) is
\[R_{11} = -(n - 1) \frac{\varphi''}{\varphi}, \quad R_{1a} = 0 \quad (4.8)\]
and
\[R_{ab} = \overline{R}_{ab} - [(n - 2)(\varphi')^2 + \varphi \varphi''] \bar{g}_{ab} \quad (a, b \in \{2, 3\}).\]
On the other hand, since
\[R = \varphi^{-2} \overline{R} - (n - 1)(n - 2) \left( \frac{\varphi'}{\varphi} \right)^2 - 2(n - 1) \frac{\varphi''}{\varphi},\]
we get
\[\overline{R} = \varphi^2 R + (n - 1)(n - 2)(\varphi')^2 + 2(n - 1)\varphi \varphi''.\]
Since \(R = 2 \frac{\dot{\psi}(f)^2}{f^2} |\nabla f|^2\) and \(|\nabla f|\) is constant at \(\Sigma\) we get
\[\overline{R} = 2 \varphi^2 \frac{\dot{\psi}(f)^2}{f^2} |\nabla f|^2 + (n - 1)(n - 2)(\varphi')^2 + 2(n - 1)\varphi \varphi''.\]
We can conclude that \(\overline{R}\) does not depend on \(\theta\). Therefore, \(\overline{R}\) is a constant. \(\square\)

### 4.2. Fourth-Order Divergence Free Weyl Tensor

In this subsection, our aim is to prove some integral theorems in dimension \(n \geq 4\) with a fourth-order divergence-free Weyl tensor for an electrovacuum space in the RN class. To that end, we use the lemmas in the previous section. In our results, we are considering subextremal Riemannian manifolds satisfying the zero radial Weyl curvature. Indeed, the fact that electrovacuum space can not be extremal appears naturally in the first theorem of this subsection.

**Theorem 7.** Let \((M^n, g, f, \psi)\), \(n \geq 4\), be an electrovacuum space satisfying (1.1), (1.7) and (1.10). For every \(\phi : \mathbb{R} \rightarrow \mathbb{R}\), \(C^2\) function with \(\phi(f)\) having compact support \(K \subseteq M\). Then,
\[
\frac{1}{2(n-1)^2\sigma} \int_M |C|^2 \phi(f) \left[ (n-1)\sigma - (n-2)f^2 \right]
= -\frac{n-2}{n-3} \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j \nabla^l W_{jkl},
\]
where \(\sigma\) is a non-null constant.

**Remark 4.** It is important to point out that the choice of \(\phi\) in the above theorem should be made in such way that terms like \(\frac{\phi(f)}{f^m}\), where \(m = 1, 2\) or \(3\), will be integrable at \(K\).
Proof. From Lemma 6, we have
\[
\frac{1}{2} |C|^2 \phi(f) + \phi(f) R^{ik} \nabla^j C_{jki} = (n - 2) \phi(f) \nabla^j \nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) 
- (n - 2) \phi(f) \nabla^j \left[ \frac{1}{f} W_{ikjl} R^{jl} \nabla^k f \right] 
- 2(n - 2) \phi(f) \nabla^j \left[ \frac{W_{ikjl}}{f^2} \nabla^i \nabla^k \psi \nabla^l \psi \right] 
+ 2(n - 2) \phi(f) \nabla^j \left[ \frac{W_{ikjl}}{f^2} \nabla^i \psi \nabla^l \psi \right].
\]

Now, integrating by parts leads us to
\[
\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} = -(n - 2) \int_M \phi(f) \nabla^j f \nabla^i \nabla^k \left( \frac{V_{ikj}}{f} \right) 
+ (n - 2) \int_M \phi(f) \nabla^j f \left[ \frac{1}{f} W_{ikjl} R^{jl} \nabla^k f \right] 
- 2(n - 2) \int_M \phi(f) \nabla^j f \left[ \frac{W_{ikjl}}{f^2} \nabla^i \nabla^k \psi \nabla^l \psi \right] 
+ 2(n - 2) \int_M \phi(f) \nabla^j f \left[ \frac{W_{ikjl}}{f^2} \nabla^i \psi \nabla^l \psi \right].
\]

From Lemma 5, we obtain
\[
\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} = - \int_M \phi(f) \nabla^j f C_{jki} R^{ik}.
\]

Using (3.1), we deduce that
\[
\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} = \left( \nabla^i \nabla^k f - \frac{2}{f} \nabla^i \psi \nabla^k \psi + \frac{1}{n - 1} f R_{gik} \right) \nabla^j C_{jki}.
\]

Since the Cotton tensor is totally trace-free, we can infer that
\[
\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} - 2 \int_M \phi(f) \nabla^i \psi \nabla^k \psi \nabla^j C_{jki} 
= - \int_M \phi(f) \nabla^j f \nabla^k \psi C_{jki} + 2 \int_M \frac{\phi(f)}{f^2} \nabla^j f \nabla^i \psi \nabla^k \psi C_{jki}.
\]

Analogously to (3.20), we have the following equation
\[
2 \nabla^i \nabla^k \psi C_{jki} = \nabla^k \nabla^j \psi C_{jki} + \nabla^j \nabla^k \psi C_{kji} = \nabla^k \nabla^j \psi (C_{jki} + C_{kji}) = 0.
\]

(4.9)
Then, using this relation, we get
\[-2 \int_M \frac{\phi(f)}{f^2} \nabla^i \nabla^k \nabla^j \psi \nabla^i \nabla^k \nabla^j C_{jki} = 2 \int_M \left( \frac{\dot{\phi}(f)}{f^2} - \frac{2\phi(f)}{f^3} \right) \nabla^j f \nabla^i \nabla^k \nabla^j \psi C_{jki} \]
\[+ 2 \int_M \frac{\phi(f)}{f^2} \nabla^j \nabla^i \nabla^k \nabla^j \psi C_{jki}.\]

Replacing it in (4.9), since the Cotton tensor is skew-symmetric, renaming indices we obtain
\[\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f} \nabla^i \nabla^k f \nabla^j \psi C_{jki} - 4 \int_M \frac{\phi(f)}{f^2} \nabla^j f \nabla^i \nabla^k \psi C_{jki} \]
\[+ 2 \int_M \frac{\phi(f)}{f^2} \nabla^j \nabla^i \nabla^k \psi C_{jki} = - \int_M \frac{\dot{\phi}(f)}{f} \nabla^j f \nabla^k \nabla^i f C_{jki} \]
\[= \int_M \frac{\dot{\phi}(f)}{f} \nabla^j f \nabla^i f \nabla^k \psi C_{jki} + \int_M \frac{\dot{\phi}(f)}{f} \left( \frac{\phi(f)}{f^2} \right) C_{jki} \nabla^i f \nabla^k f \nabla^j f \]
\[+ \int_M \frac{\phi(f)}{f} \nabla^k \nabla^j f \nabla^i f C_{jki} \]
\[= \int_M \frac{\dot{\phi}(f)}{f} \nabla^j f \nabla^i f \nabla^k \psi C_{jki} \]
\[= - \int_M \frac{\dot{\phi}(f)}{f} \nabla^i f \nabla^k f \nabla^j \psi C_{jki} = - \int_M \frac{\nabla^i \phi(f)}{f} \nabla^k f \nabla^j C_{jki} \]
\[= \int_M \frac{\phi(f)}{f} \nabla^i \nabla^k f \nabla^j \psi C_{jki} - \int_M \frac{\phi(f)}{f^2} \nabla^i f \nabla^k f \nabla^j C_{jki} \]
\[+ \int_M \frac{\phi(f)}{f} \nabla^k \nabla^i \nabla^j C_{jki}.\]

Hence, from (2.9) and the symmetries of the Cotton tensor by integration, we have
\[\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f^2} \nabla^i f \nabla^k f \nabla^j \psi C_{jki} \]
\[= \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki} \]
\[+ 4 \int_M \frac{\phi(f)}{f^3} \nabla^j f \nabla^i \nabla^k \psi C_{jki} \]
\[+ 2 \int_M \frac{\phi(f)}{f^2} \nabla^j \nabla^i \nabla^k \psi C_{jki} = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki} \]
\[+ 2 \int_M \frac{\phi(f)}{f^3} \left( f \nabla^i \psi \nabla^k \nabla^j \psi + 2 \nabla^j f \nabla^k \psi \nabla^i \psi \right) C_{jki}. \quad (4.11)\]
Now, considering that $\psi = \psi(f)$, we deduce
\[
\int_M \frac{\phi(f)}{f^3} (f \nabla^j \psi \nabla^k \nabla^i \psi + 2 \nabla^j f \nabla^k \psi \nabla^i \psi) C_{jki}
\]
\[
= \int_M \frac{\phi(f)}{f^3} [f \dot{\psi} \nabla^j f (\dot{\psi} \nabla^k \nabla^i f + \dot{\psi} \nabla^i f \nabla^k f) + 2 \ddot{\psi} \nabla^j f \nabla^k f \nabla^i f] C_{jki}
\]
\[
= \int_M \frac{\phi(f)}{f^2} \dot{\psi}^2 \nabla^j f \nabla^k \nabla^i f C_{jki}.
\]

Again, from the symmetries of the Cotton tensor and renaming indices, we obtain
\[
\int_M \frac{\phi(f)}{f^3} (f \nabla^j \psi \nabla^k \nabla^i \psi + 2 \nabla^j f \nabla^k \psi \nabla^i \psi) C_{jki} = \int_M \frac{\phi(f)}{f^2} \dot{\psi}^2 \nabla^j f \nabla^k \nabla^i f C_{jki}
\]
\[
= \int_M \frac{\phi(f)}{f^2} \dot{\psi} (f) \nabla^k f \nabla^i f \nabla^j C_{jki}.
\]

Thus, replacing the above equation in (4.11), we get
\[
\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f^2} (1 - 2 \dot{\psi}(f)^2) \nabla^k f \nabla^i f \nabla^j C_{jki}
\]
\[
= \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i f \nabla^j C_{jki}.
\]

(4.12)

From now on, we will analyze just one part of the above equation. Since the Cotton tensor is trace-free and skew-symmetric, another integration by parts gives us
\[
\int_M \frac{\phi(f)}{f^2} (1 - 2 \dot{\psi}(f)^2) \nabla^k f \nabla^i f \nabla^j C_{jki}
\]
\[
= - \int_M \left( \frac{\phi(f)}{f^2} - \frac{2 \phi(f)}{f^3} \right) (1 - 2 \dot{\psi}(f)^2) \nabla^k f \nabla^j f \nabla^i f C_{jki}
\]
\[
+ 4 \int_M \frac{\phi(f)}{f^2} \dot{\psi}(f) \ddot{\psi}(f) \nabla^k f \nabla^j f \nabla^i f C_{jki}
\]
\[
- \int_M \frac{\phi(f)}{f^2} (1 - 2 \dot{\psi}(f)^2) \nabla^j \nabla^k f \nabla^i f C_{jki}
\]
\[
- \int_M \frac{\phi(f)}{f^2} (1 - 2 \dot{\psi}(f)^2) \nabla^k f \nabla^j \nabla^i f C_{jki}
\]
\[
= \int_M \frac{\phi(f)}{f} (1 - 2 \dot{\psi}(f)^2) R^{ji} \nabla^k f C_{kji}.
\]

(4.13)

In the last equality, we have used (3.1) and renamed indices. Now, since $M^n$ has zero radial Weyl curvature and the Cotton tensor is totally trace-free, from
(3.8) and (3.9), we infer
\[ R^{ij} \nabla^k f C_{kji} = \frac{1}{2} C_{kji} (R^{ij} \nabla^k f - R^{ki} \nabla^j f) \]
\[ = -\frac{1}{2Q} C_{kji} V^{kji} \]
\[ = -\frac{1}{2Q} f |C|^2, \]
where \( Q \) is the same as given in Lemma 2, i.e., \( Q = \frac{n-1}{n-2} - 2\psi(f)^2 \). Therefore, we have
\[ \int_M \frac{\phi(f)}{f^2} (1 - 2\psi(f)^2) \nabla^k f \nabla^i f \nabla^j C_{kji} = -\frac{1}{2} \int_M \frac{\phi(f)}{Q} (1 - 2\psi(f)^2) |C|^2 \]
\[ = -\frac{1}{2} \int_M |C|^2 \phi(f) \left[ \frac{(n-2)(1 - 2\psi(f)^2)}{n - 1 - 2(n-2)\psi(f)^2} \right]. \]
Now, from (4.3) we can conclude that
\[ \int_M \frac{\phi(f)}{f^2} (1 - 2\psi(f)^2) \nabla^k f \nabla^i f \nabla^j C_{kji} = -\frac{n-2}{2(n-1)^2 \sigma} \int_M \phi(f) [f^2 + (n-1)\sigma] |C|^2. \]
Replacing it in (4.12), we obtain
\[ \frac{1}{2(n-1)^2 \sigma} \int_M |C|^2 \phi(f) [(n-1)\sigma - (n-2)f^2] = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i f \nabla^j C_{kji}. \]
Using (2.4), the result holds. \( \square \)

Next, we will take an appropriate \( \phi(f) \) satisfying the conditions of integrability in Theorem 7 (cf. Remark 4).

**Theorem 8.** Let \((M^n, g, f, \psi)\), \(n \geq 4\), be an electrovacuum space satisfying (1.1), (1.7) and (1.10) with fourth-order divergence-free Weyl tensor, i.e., \( \text{div}^4 W = 0 \). If \( f \) is a proper function, then the Weyl tensor is harmonic, i.e., \( \text{div} W = 0 \).

**Proof.** Let \( s > 0 \) be a real number fixed, so we take \( \chi \in C^3 \) a real non-negative function defined by \( \chi = 1 \) in \([0, s]\), \( \chi' \leq 0 \) in \([s, 2s]\) and \( \chi = 0 \) in \([2s, +\infty] \). Since \( f \) is a proper function, we have that \( \phi(f) = f^4 \chi(f) \) has compact support in \( M \) for \( s > 0 \). From Theorem 7, we get
\[ \frac{1}{2(n-1)^2 \sigma} \int_M |C|^2 f^4 \chi(f) [(n-1)\sigma - (n-2)f^2] \]
\[ = -\frac{n-2}{n-3} \int_M f^3 \chi(f) \nabla^k f \nabla^i f \nabla^j f \nabla^l W_{jkl} \]
\[ = -\frac{n-2}{4(n-3)} \int_M \chi(f) \nabla^k f^4 \nabla^i f \nabla^j f \nabla^l W_{jkl} \]
\[ = \frac{n-2}{4(n-3)} \int_M \chi(f) f^4 \nabla^k \nabla^i f \nabla^j f \nabla^l W_{jkl} \]
\[ + \frac{n-2}{4(n-3)} \int_M \chi'(f) f^4 \nabla^k f \nabla^i f \nabla^j f \nabla^l W_{jkl}. \]
In the last equality, we use integration by parts. Now, we take $\phi(f) = f^5 \dot{\chi}(f)$ in the Theorem 7 and since $\text{div}^4 W = 0$, we obtain
\[
\frac{1}{2(n-1)^2} \int_M |C|^2 f^4 \chi(f) \left[ (n-1)\sigma - (n-2)f^2 \right] = -\frac{1}{8(n-1)^2} \int_M |C|^2 f^5 \dot{\chi}(f) \left[ (n-1)\sigma - (n-2)f^2 \right].
\]
Hence,
\[
\int_M f^4 |C|^2 \left[ \chi(f) + \frac{1}{4} f \dot{\chi}(f) \right] \left[ (n-1)\sigma - (n-2)f^2 \right] = 0.
\]

Define $M_s = \{ x \in M; f(x) \leq s \}$. We have, by definition, $\chi(f) + \frac{1}{4} f \dot{\chi}(f) = 1$ on the compact set $M_s$. Thus, on $M_s$, since $M$ is subextremal,
\[
0 \leq \int_{M_s} f^4 |C|^2 \left[ (n-2)f^2 - (n-1)\sigma \right] = 0,
\]
i.e., $C = 0$ in $M_s$. Taking $s \to +\infty$, we obtain that $C = 0$ on $M$.

Now, we are ready to present the proof of Corollary 1 that will be stated again here for the sake of the reader’s convenience.

**Corollary 3** (Corollary 1). Let $(M^n, g, f, \psi)$, $n > 3$, be an electrovacuum space with fourth-order divergence free Weyl curvature and zero radial Weyl curvature such that the electric potential $\psi$ is in the Reissner–Nordström class (i.e., satisfying Eq. (1.7)). Around any regular point of $f$, if $f$ is a proper function, then the manifold is locally a warped product with $(n-1)$-dimensional Einstein fibers.

**Proof.** This result follows combining the Theorem 3 with the Theorem 8.

4.3. Third-Order Divergence Free Cotton Tensor

In this subsection, we will return to our results and study them in dimension $n = 3$. Firstly, it is important to point out that lemmas 4, 5 and 6 are not valid in dimension $n = 3$ due to equation (2.4), which was used in their demonstrations. However, we can prove another version of these lemmas in a convenient way. Another point is the fact that Theorem 7 is not valid in dimension $n = 3$, but the main issue here is that the Weyl tensor vanishes in dimension three. Nonetheless, the computations are very much similar to the previous results proved in dimensions more than 3. We will prove all those results for $n = 3$ for the sake of the completeness of the text.

After these considerations, we can proceed with our results. To that end, since the Weyl tensor vanishes identically in dimension $n = 3$, we can observe that equation (3.8) becomes
\[
f C_{ijk} = V_{ijk}.
\]
Consequently, we have the following lemma.

**Lemma 7.** Let $(M^3, g, f, \psi)$ be an electrovacuum space. Then,
\[
C_{kji} R^{ik} = \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]
Proof. In fact, from (2.6) and (4.14), we obtain

\[ B_{ij} = \nabla^k C_{kij} = \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Taking the derivative over \( i \), we have

\[ \nabla^i B_{ij} = \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Since \( n = 3 \), from (2.7), using (2.8) and (3.22) after renamed the indices, we infer

\[ \nabla^i B_{ij} = -C_{jik} R^{ik} = -C_{jki} R^{ik} = C_{kji} R^{ik}. \]

Thus, combining these two last relations the result holds. \( \square \)

Lemma 8. Let \((M^3, g, f, \psi)\) be an electrovacuum space. Then,

\[ \frac{1}{2} |C|^2 + R^{ik} \nabla^j C_{jki} = -\nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Proof. Taking the divergence in Lemma 7, we get

\[ C_{kji} \nabla^j R^{ik} + R^{ik} \nabla^j C_{kji} = \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Using (3.24), we have

\[ \frac{1}{2} C_{kji} (\nabla^j R^{ik} - \nabla^k R^{ij}) + R^{ik} \nabla^j C_{kji} = \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Now, since the Cotton tensor is trace-free, from (2.2) and renaming the indices, we obtain

\[ -\frac{1}{2} C_{kji} C^{kji} - R^{ik} \nabla^j C_{jki} = \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Therefore, the result holds. \( \square \)

Theorem 9. Let \((M^3, g, f, \psi)\) be an electrovacuum space satisfying (1.7). For every \( \phi : \mathbb{R} \to \mathbb{R} \), \( C^2 \) function with \( \phi(f) \) having compact support \( K \subseteq M \). Then,

\[ \frac{1}{8\sigma} \int_M |C|^2 \phi(f)[2\sigma - f^2] = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki}, \]

where \( \sigma \) is a non-null constant.

Proof. The idea is to proceed as in Theorem 7. From Lemma 8, we obtain

\[ \frac{1}{2} |C|^2 \phi(f) + \phi(f) R^{ik} \nabla^j C_{jki} = -\phi(f) \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Hence, upon integrating this expression, we get

\[ \frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} = \int_M \frac{\phi(f)}{f} \nabla^j f \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right). \]

Then, from Lemma 7 and the symmetries of \( C_{ijk} \), we have

\[ \frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} = -\int_M \frac{\phi(f)}{f} \nabla^j f C_{jki} R^{ik}. \]
Now, from (3.1) and the fact that $C_{ijk}$ is trace-free and skew-symmetric we obtain the following identity

$$\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f} (\nabla^i \nabla^k f - 2 \frac{\dot{\psi}(f)^2}{f} \nabla^i f \nabla^k f) \nabla^j C_{jki}$$

$$= - \int_M \frac{\dot{\phi}(f)}{f} \nabla^j f (\nabla^k \nabla^i f - 2 \frac{\dot{\psi}(f)^2}{f} \nabla^i f \nabla^k f) C_{jki}$$

$$= - \int_M \frac{\dot{\phi}(f)}{f} \nabla^j f \nabla^k \nabla^i f C_{jki}$$

$$= \int_M \left( \frac{\ddot{\phi}(f)}{f} - \frac{\dot{\phi}(f)}{f^2} \right) \nabla^j f \nabla^k f \nabla^i f C_{jki}$$

$$+ \int_M \frac{\dot{\phi}(f)}{f} \nabla^k \nabla^i f C_{jki} + \int_M \frac{\dot{\phi}(f)}{f} \nabla^j f \nabla^k f C_{jki}$$

$$= \int_M \frac{\ddot{\phi}(f)}{f} \nabla^j f \nabla^i f \nabla^k f C_{jki}.$$ 

Note that in the last equality we have used (4.10). From now, we rename the indices and, integrating by parts again, we infer

$$\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f} \nabla^i \nabla^k f \nabla^j C_{jki} - 2 \int_M \frac{\dot{\phi}(f)}{f^2} \psi(f)^2 \nabla^i f \nabla^k f \nabla^j C_{jki}$$

$$= - \int_M \frac{\ddot{\phi}(f)}{f} \nabla^j f \nabla^i f \nabla^j C_{jki}$$

$$= \int_M \frac{\ddot{\phi}(f)}{f} \nabla^k \nabla^i f \nabla^j C_{jki} - \int_M \frac{\phi(f)}{f^2} \nabla^k f \nabla^i f \nabla^j C_{jki}$$

$$+ \int_M \frac{\phi(f)}{f} \nabla^k \nabla^i \nabla^j C_{jki}.$$ 

Thus,

$$\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f^2} \psi(f)^2 \nabla^i f \nabla^j C_{jki}$$

$$= \int_M \frac{\phi(f)}{f} \nabla^i f \nabla^j C_{jki}.$$ 

(4.15)

Furthermore, from the proof of Theorem 7 (Eq. 4.13), we get

$$\int_M \frac{\phi(f)}{f^2} (1 - 2 \psi(f)^2) \nabla^k f \nabla^i f \nabla^j C_{jki} = \int_M \frac{\phi(f)}{f} (1 - 2 \psi(f)^2) R^{ji} \nabla^k f C_{kji}.$$ 

Again, as we did in Theorem 7, from (3.9) and (4.14), we have

$$R^{ji} \nabla^k f C_{kji} = - \frac{1}{2Q} f |C|^2.$$
Note that in dimension three, from Lemma 2 and (4.3), we obtain, respectively,
\[ Q = 2(1 - \dot{\psi}(f)^2) \]
and
\[ \dot{\psi}(f)^2 = \frac{f^2}{f^2 - 2\sigma}; \text{ where } \sigma \neq 0. \]

Finally,
\[
\int_M \frac{\phi(f)}{f^2}(1 - 2\dot{\psi}(f)^2)\nabla^k f\nabla^i f\nabla^j C_{jki} = -\frac{1}{8\sigma} \int_M |C|^2 \phi(f) \left[ f^2 + 2\sigma \right].
\]
Therefore, replacing the above equation in (4.15) the result holds. \[\square\]

**Theorem 10.** Let \((M^3, g, f, \psi)\) be an electrovacuum space satisfying (1.7) with \(\sigma < 0\) and third-order divergence-free Cotton tensor, i.e., \(\text{div}^3 C = 0\). If \(f\) is a proper function, then the Cotton tensor is identically zero, i.e., \((M^3, g)\) is locally conformally flat.

**Proof.** Let \(s > 0\) be a real number fixed, and so we take \(\chi \in C^3\) a real non-negative function defined by \(\chi = 1\) in \([0, s]\), \(\chi' \leq 0\) in \([s, 2s]\) and \(\chi = 0\) in \([2s, +\infty]\). Since \(f\) is a proper function, we have that \(\phi(f) = f^4\chi(f)\) has compact support in \(M\) for \(s > 0\). From Theorem 7, we get
\[
\frac{1}{8\sigma} \int_M |C|^2 f^4 \chi(f) \left[ 2\sigma - f^2 \right] = \int_M f^3 \chi(f) \nabla^k f\nabla^i f\nabla^j C_{jki}
\]
\[
= \frac{1}{4} \int_M \chi(f) \nabla^i f^4 \nabla^k \nabla^j C_{jki}
\]
\[
= -\frac{1}{4} \int_M \chi(f) f^4 \nabla^i \nabla^k \nabla^j C_{jki}
\]
\[
+ \frac{1}{4} \int_M \dot{\chi}(f) f^4 \nabla^i f\nabla^k \nabla^j C_{jki}.
\]

In the last equality, we have used integration by parts. Now, since \(\text{div}^3 C = 0\) we take \(\phi(f) = f^5\dot{\chi}(f)\) in Theorem 9 one more time to obtain
\[
\frac{1}{8\sigma} \int_M |C|^2 f^4 \chi(f) \left[ 2\sigma - f^2 \right] = -\frac{1}{32\sigma} \int_M |C|^2 f^5 \dot{\chi}(f) \left[ 4\sigma - f^2 \right],
\]
i.e.,
\[
\int_M f^4 |C|^2 \left[ \chi(f) + \frac{1}{4} f \dot{\chi}(f) \right] \left[ 2\sigma - f^2 \right] = 0.
\]

Let \(M_s\) be defined as in Theorem 8, i.e., \(M_s = \{ x \in M; f(x) \leq s \}\). We have, by definition, \(\chi(f) + \frac{1}{4} f \dot{\chi}(f) = 1\) on the compact set \(M_s\). Thus, on \(M_s\), since \(\sigma < 0\),
\[
0 \leq \int_{M_s} f^4 |C|^2 \left[ f^2 - 2\sigma \right] = 0.
\]
Therefore, \(C = 0\) in \(M_s\). Taking \(s \to +\infty\), we obtain that \(C = 0\) on \(M\). \[\square\]

Finally, we prove Corollary 2, whose statement is as follows.
Corollary 4 (Corollary 2). Let \((M^3, g, f, \psi)\) be an electrovacuum space with third-order divergence free Cotton tensor such that \(\psi\) satisfies (1.7) with \(\sigma < 0\). Around any regular point of \(f\), if \(f\) is a proper function, then the manifold is locally a warped product with a one-dimensional base and a constant curvature surface fiber.

Proof. This result is a consequence of Theorem 3 and Theorem 10. □

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