Deformations of KdV and soliton collisions

H Blas\textsuperscript{1}, R Ochoa\textsuperscript{2} and D Suarez\textsuperscript{2}

\textsuperscript{1}Instituto de Física
Universidade Federal de Mato Grosso
Av. Fernando Correa, N\textsuperscript{0} 2367
Bairro Boa Esperança, Cep 78060-900, Cuiabá - MT - Brazil
\textsuperscript{2}Facultad de Ciencias
Universidad Nacional de Ingeniería
Av. Tupac Amaru, N\textsuperscript{0} 210, Rimac, Lima-Perú

E-mail: blas@ufmt.br

Abstract. Some deformations of the integrable Korteweg-de Vries model (KdV) are associated to several towers of infinite number of asymptotically conserved charges. It has been shown that the standard KdV also exhibits infinite number of anomalous charges. In [9] there have been verified numerically the degrees of modifications of the charges around the soliton interaction regions, by computing numerically some representative anomalies, related to lowest order quasi-conservation laws, depending on the deformation parameters \(\{\epsilon_1, \epsilon_2\}\), such that the standard KdV is recovered for \(\epsilon_1 = \epsilon_2 = 0\). Here we present the numerical simulations for some values of the pair \(\{\epsilon_1, \epsilon_2\}\) around \(\epsilon_1 \approx 0, \epsilon_2 \approx 0\), and show that the collision of two and three solitons are elastic. The KdV-type equations are quite ubiquitous and find many applications in several areas of non-linear science.

1. Introduction

Integrable systems and the existence of infinite number of conserved charges are intimately related. However, certain non-linear field theories with solitary wave solutions are not integrable. Recently, it has been performed certain deformations of integrable models such that they possess solitary waves with similar properties to the true solitons. The quasi-integrability concept has recently been introduced for certain deformations of the sine-Gordon, Toda, Bullough-Dodd, KdV, non-linear Schrödinger (NLS) and supersymmetric sine-Gordon models \cite{1, 2, 4, 3, 5, 6, 7, 10, 11, 12} in the context of their anomalous zero-curvature representations. The main properties are the presence of infinite number of asymptotically conserved charges and the elastic scattering of the solitons for a set of deformation parameters, and for a variety of soliton parameters such as velocity, amplitude, width, etc. The space-time integration of the anomalies are shown to vanish analytically in special cases; i.e. for N-soliton configurations provided that they possess definite parity under a special space-time inversion symmetry. Remarkably, the presence of novel towers of infinite number of anomalous charges was recently uncovered by direct construction in the context of deformed sine-Gordon and KdV models \cite{8, 9}, respectively. Novel infinite towers of anomalous charges emerge even for the standard sine-Gordon model and KdV models, respectively.

Moreover, in \cite{8, 9} it has recently been developed the so-called Riccati-type pseudo-potential approach to quasi-integrability, and shown that the anomalous conservation laws of \cite{1, 2} are,
in fact, exact conservation laws, i.e. they become simply the higher order derivatives of the energy-momentum charges. In addition, there have been uncovered infinite set of exact non-local conservation laws associated to a linear system formulation of the deformed sine-Gordon and KdV models [8, 9].

In this paper we will examine by numerical simulations of 2-soliton and 3-soliton collisions the vanishing of the lowest order anomalies associated to the novel towers of anomalous conservation laws of the deformed KdV model presented in [9], for values of the deformation parameters around \(\{\epsilon_1 \approx 0, \epsilon_2 \approx 0\}\). In order to perform the numerical simulations we will use the methods discussed in [9].

This paper is organized as follows: The next section presents the model and the one-soliton general solution. In sec. 3 we present the quasi-conservation laws and their relevant anomalies. In 4 we present the results of our simulations. In 5 we present some discussions.

2. A particular deformation of the KdV model

Let us consider the model studied in [6, 9] as a deformed KdV equation. It involves the real scalar field \(u\) and the auxiliary fields \(w\) and \(v\) with eq. of motion

\[
 u_t + u_x + \left[ \frac{\alpha}{2} u^2 + \epsilon_2 \frac{\alpha}{4} w_x v_t + u_{xx} - \epsilon_1 (u_{xt} + u_{xx}) \right]_x = 0,
\]

such that

\[
 u = w_t \quad \text{(2)}
\]
\[
 u = v_x. \quad \text{(3)}
\]

The real parameters \(\epsilon_1\) and \(\epsilon_2\) are the deformation parameters and \(\alpha\) is an arbitrary real parameter. For \(\epsilon_1 = \epsilon_2 = 0\) one has the integrable KdV model.

Consider the space-time reflection around a given fixed point \((x_\Delta, t_\Delta)\)

\[
 \mathcal{P} : (\tilde{x}, \tilde{t}) \rightarrow (-\tilde{x}, -\tilde{t}); \quad \tilde{x} = x - x_\Delta, \quad \tilde{t} = t - t_\Delta.
\]

The transformation \(\mathcal{P}\) defines a shifted parity \(\mathcal{P}_s\) for the spatial variable and the delayed time reversal \(\mathcal{T}_d\) for the time variable.

In the quasi-integrability approach it is assumed that the \(u\)-field solution of the deformed KdV model evaluated on the \(N\)-soliton solution, viz. \(u_{N_{-sol}}\), is even under the transformation \(\mathcal{P}\)

\[
 \mathcal{P}(u_{N_{-sol}}) = u_{N_{-sol}}. \quad \text{(5)}
\]

Therefore, according to (2)-(3), one has

\[
 \mathcal{P}(v_{N_{-sol}}) = -v_{N_{-sol}}, \quad \mathcal{P}(w_{N_{-sol}}) = -w_{N_{-sol}}, \quad \mathcal{P}(q_{N_{-sol}}) = q_{N_{-sol}}. \quad \text{(6)}
\]

The \(N\)-soliton solution of the standard KdV satisfying the above parity symmetries have been constructed in [9].

For the deformed KdV (1) one has the general 1-soliton solution [9]

\[
 u_{II} = \frac{6}{\alpha} \frac{k^2}{(2 + \epsilon_2)(a^2 - \epsilon_1 k^2)} \text{sech}^2 \left[ \frac{1}{2a} (kx - w_2 t + ) \right], \quad \text{(7)}
\]

where \(w_2 = \frac{a^2 k + (1-\epsilon_1) k^3}{a^2 - \epsilon_1 k^2}\), and \(a, k\) are arbitrary real parameters. This is a new general form of 1-soliton solution which can not be found by the usual Hirota method.
3. Novel asymptotically conserved charges

The next quasi-conservation laws have been defined in [9].

First, the higher order moments become

\[
\frac{d}{dt} \tilde{q}_a^{(n)} = \tilde{\beta}^{(n)}, \quad n = 3, 4, 5, \ldots \\
\tilde{q}_a^{(n)} = \frac{\alpha^n}{2^{2n+1}3^n} \int_{-\infty}^{+\tilde{x}} dx u^n, \quad \tilde{\beta}^{(n)} = \frac{\alpha^n}{2^{2n+1}3^n} \int_{-\infty}^{+\tilde{x}} dx [-(n-1)u^{n-2}u_xv_t].
\]

(8)

(9)

We assume the parity symmetry (5)-(6) for the corresponding fields; so, the time integrated anomalies \( \tilde{\beta}^{(n)} \) vanish for \( \tilde{t} \to +\infty, \tilde{x} \to +\infty \)

\[
\int_{-\tilde{t}}^{+\tilde{t}} dt \tilde{\beta}^{(n)} = \frac{\alpha^n}{2^{2n+1}3^n} \int_{-\tilde{t}}^{+\tilde{t}} dt \int_{-\tilde{x}}^{+\tilde{x}} dx [-n(n-1)u^{n-2}u_xv_t].
\]

(10)

Integrating in time (8) and setting \( \tilde{x} \to +\infty \) one can write

\[
\tilde{q}_a^{(n)}(+\tilde{t}) = \tilde{q}_a^{(n)}(-\tilde{t}), \quad n = 2, 3, 4, \ldots
\]

(11)

provided that the vanishing of the time-integrated anomaly (10) is assumed. So, the higher order moments of (8) become asymptotically conserved charges. In sec. 4, we will numerically simulate the anomaly \( \tilde{\beta}^{(3)} \) in (8) for 2-soliton and 3-soliton scatterings.

Second, consider the quasi-conservation law

\[
\partial_t [u^2 + \epsilon_1 u_x^2] + \partial_x [u^2 + \frac{2\alpha}{3} u^3 + 2uu_{xx} - u_x^2 - 2\epsilon_1 uu_{xt} + \frac{\alpha\epsilon_2}{2} u w_x v_t] = \frac{\alpha\epsilon_2}{2} w_x v_t u_x.
\]

(12)

Notice that for \( \epsilon_2 = 0 \) the r.h.s. of (12) vanishes; so, this equation turns out to be an exact conservation law. For \( \epsilon_2 \neq 0 \) one can define an asymptotically conserved charge from the quasi-conservation law (12) as

\[
\frac{d}{dt} \tilde{Q}_a^{(2)} = \tilde{\alpha}_2
\]

(13)

\[
\tilde{Q}_a^{(2)} = \frac{\alpha^2}{2^{3}5^2} \int_{-\tilde{x}}^{+\tilde{x}} dx [u^2 + \epsilon_1 u_x^2]; \quad \tilde{\alpha}_2 = \frac{\alpha^3\epsilon_2}{2^{6}3^2} \int_{-\tilde{x}}^{+\tilde{x}} w_x v_t u_x dx,
\]

(14)

where the r.h.s. defines the anomaly \( \tilde{\alpha}_2 \). The time-integrated anomaly vanishes provided that the fields satisfy the parity (5)-(6), i.e.

\[
\int_{-\tilde{t}}^{+\tilde{t}} \tilde{\alpha}_2 dt = \frac{\alpha^3\epsilon_2}{2^{6}3^2} \int_{-\tilde{t}}^{+\tilde{t}} dt \int_{-\tilde{x}}^{+\tilde{x}} dx w_x v_t u_x
\]

(15)

\[
= 0.
\]

(16)

So, integrating in time (13) and setting \( \tilde{x} \to +\infty \) one can write for \( \tilde{t} \to +\infty \)

\[
\tilde{Q}_a^{(2)}(+\tilde{t}) = \tilde{Q}_a^{(2)}(-\tilde{t}),
\]

(17)

provided that the vanishing of the time-integrated anomaly (15) is used. In sec. 4, we will numerically simulate the anomaly \( \tilde{\alpha}_2 \) in (13)-(14) for 2-soliton and 3-soliton scatterings.
Third, one has the quasi-conservation law
\[
\partial_t \left[ \frac{\alpha}{3} u^3 + u^2 + (\epsilon_1 - 1)u_x^2 \right] + \partial_x \left[ \frac{36}{\alpha^2} F^2 - \epsilon_1 u_t^2 - 2(\epsilon_1 - 1)u_t u_x \right] = -\frac{\alpha \epsilon_2}{2} w_x v_t u_t. \tag{18}
\]
\[
F \equiv X + \frac{\alpha}{6} u + \frac{\alpha}{6} \left( \frac{\alpha}{2} u^2 + u_{xx} \right). \tag{19}
\]
For \( \epsilon_2 = 0 \) it provides an exact conservation law. One can define the asymptotically conserved charge
\[
\frac{d}{dt} Q^{(3)}_a = \alpha_3 \tag{20}
\]
\[
Q^{(3)}_a = \frac{\alpha^2}{2^6 3^2} \int_{-\infty}^{+\infty} \left[ \frac{\alpha}{3} u^3 + u^2 + (\epsilon_1 - 1)u_x^2 \right] dx, \quad \alpha_3 = -\frac{\alpha^3 \epsilon_2}{2^6 3^2} \int_{-\infty}^{+\infty} w_x v_t u_t dx. \tag{21}
\]
Since the anomaly \( \alpha_3 \) vanishes when integrated in space-time for fields satisfying the parity symmetry (5)-(6), following similar steps as above, one can define the asymptotically conserved charge
\[
Q^{(3)}_a (\tilde{t} \to \infty) = Q^{(3)}_a (\tilde{t} \to -\infty). \tag{22}
\]

In sec. 4, we will numerically simulate the anomaly \( \alpha_3 \) in (20)-(21) for 2-soliton and 3-soliton scatterings.

Fourth, taking into account the definition \( u = -\frac{q}{\alpha} q_{xt} \) and the eq. of motion (1) one can write
\[
\partial_t \left[ q_{xx}^2 - 2q_t q_{xxx} - (1 - \epsilon_1) q_{xxx}^2 + 2\epsilon_1 q_{xx} q_{xxt} \right] + \partial_x \left[ 2q_{xx} H + 2q_t q_{xx} - q_{xt}^2 - \epsilon_1 q_{xxt}^2 \right] = \mathcal{H}_1 \tag{23}
\]
\[
\mathcal{H}_1 = -4 q_{xx} [2q_{xt}^2 + \epsilon_2 q_{xxt} q_{xt}], \quad H = -q_{tt} + 4q_{xt}^2 + 2\epsilon_2 q_{xx} q_{tt} - q_{xxt}^2 + \epsilon_1 (q_{xx} + q_{xxt}). \tag{24}
\]
So, one can define the next asymptotically conserved charge and anomaly
\[
\frac{d}{dt} \bar{Q}_2 = h_1, \tag{25}
\]
\[
\bar{Q}_2 = \frac{2^4}{\alpha^4} \int dx \left[ q_{xx}^2 - 2q_t q_{xxx} - (1 - \epsilon_1) q_{xxx}^2 + 2\epsilon_1 q_{xx} q_{xxt} \right], \quad h_1 \equiv \int dx \mathcal{H}_1. \tag{26}
\]
Taking into account the symmetry property of \( q \) (6), the anomaly \( \mathcal{H}_1 \) exhibits odd parity under (4). In sec. 4, we will numerically simulate the anomaly \( h_1 \) in (25)-(26) for 2-soliton and 3-soliton scatterings.

4. Numerical treatment of the anomalies

In the Figs. 1 and 2 we present the plots of 2-soliton and 3-soliton collisions for three successive times of the deformed KdV (1). We have used the LU decomposition method and considered the time steps and spatial grid as \( \tau = 0.0025 \) and \( h = 0.14 \), and \( \tau = 0.0045 \) and \( h = 0.184 \), for 2-soliton and 3-soliton, respectively. As an initial configuration, in the case of the 2-soliton we have assumed two solitons (7) located far away from each other; similarly, in the case of the 3-soliton one considers three solitons of the general type (7) located far away from each other. So, in the both cases one considers (7) conveniently located some distance apart from each other. In fact, they turn out to be convenient initial conditions, since there are no visible loss of radiation in the corresponding interaction regions.

Below, we compute the anomalies presented above through numerical simulations for the collisions of two and three solitons. The parameter values are \( \epsilon_1 = 0.05, \epsilon_2 = 0.05, \alpha = 4, k_1 = 0.75, k_2 = 0.71, a_1 = a_2 = 1 \) for the two-soliton, and \( \epsilon_1 = 0.05, \epsilon_2 = 0.05, \alpha = 4, k_1 = 0.75, k_2 = 0.71, k_3 = 0.67, a_1 = a_2 = a_3 = 1 \) for three-solitons.
4.1. 2-soliton charges and anomalies
We consider the anomaly in eq. (9) for \( n = 3 \), with density function in the form \(-6uu_xv_t\). The Fig. 3 shows the anomaly density versus \( x \)-coordinate for three successive times, before collision, during collision and after the collision of the 2-soliton presented in Fig. 1. Notice the vanishing of the anomaly and its \( t \)-integrated anomaly functions of \( t \), within numerical accuracy; in fact, the anomaly \( \tilde{\beta}^{(3)}(t) \approx 0 \) within the order of \( 10^{-6} \), whereas the \( t \)-integrated anomaly vanishes within the order of \( 10^{-7} \). Therefore, according to (10) and (11) one can argue that the charge \( q_a^{(3)} \) in (9) is asymptotically conserved, i.e. it satisfies \( q_a^{(3)}(+\tilde{t}) = q_a^{(3)}(-\tilde{t}) \) for large time \( \tilde{t} \).

We simulate the anomaly \( \tilde{\alpha}_2 \) in (14) whose corresponding density is \( w_xv_tu_x \). The Fig. 4 shows \( \tilde{\alpha}_2(t) \) versus \( x \)-coordinate for three successive times, before collision, during collision and after the collision of the 2-soliton of the Fig. 1. The anomaly and its \( t \)-integrated anomaly functions of \( t \) vanish within numerical accuracy; in fact, one has \( \tilde{\alpha}_2(t) \approx 0 \) within the order of
Figure 3. Top Fig: anomaly density of $\tilde{\beta}^{(3)}$ (8); i.e. the function $(-6uu_xv_t)$ vs $x$ for three successive times, $t_i =$ before collision (green), $t_c =$ collision (blue) and $t_f =$ after collision (red), for the 2-soliton of Fig. 1. Bottom Figs: anomaly $\tilde{\beta}^{(3)}(t)$ vs $t$ and the $t-$integrated anomaly $\int_{t_i}^{t_f} \tilde{\beta}^{(3)}$ vs $t$, respectively.

$10^{-8}$, whereas the $t-$integrated anomaly vanishes within the order of $10^{-9}$. Therefore, according to (15) and (17) it follows that the charge $\tilde{q}_a^{(3)}$ in (14) is asymptotically conserved, i.e. it satisfies $\tilde{Q}_a^{(2)}(\tilde{t}) = \tilde{Q}_a^{(2)}(-\tilde{t})$ for large time $\tilde{t}$.

We simulate the anomaly $\alpha_3$ in (21) whose corresponding density is $w_xv_tv_t$. The Fig. 5 presents $\alpha_3$ versus $x-$coordinate for three successive times, before collision, during collision and after the collision of the 2-soliton presented in the Fig. 1. The anomaly and its $t-$integrated anomaly functions of $t$ vanish within numerical accuracy; in fact, one has $\alpha_3 \approx 0$ within the order of $10^{-8}$, whereas the $t-$integrated anomaly vanishes within the order of $10^{-10}$. Therefore, according to (22) one can say that the charge $Q_a^{(3)}$ in (21) is asymptotically conserved, i.e. one has $Q_a^{(3)}(\tilde{t}) = Q_a^{(3)}(-\tilde{t})$ for large time $\tilde{t}$.

Similarly, we will simulate the anomaly $h_1$ in (26) with density $-4q_{xxx}(2q_{xt}^2 + \epsilon_2 q_{xx}q_t)$. The Fig. 6 shows the behavior of $h_1$ versus $x-$coordinate for three successive times, before collision, during collision and after the collision of the 2-soliton presented in the Fig. 1. Notice the vanishing of the anomaly and its $t-$integrated anomaly functions of $t$, within numerical accuracy; in fact, one has $h_1 \approx 0$ within the order of $10^{-8}$, whereas the $t-$integrated anomaly vanishes within the order of $10^{-10}$. Therefore, as in the previous discussions, it follows that the charge $Q_2$ in (26) is asymptotically conserved, i.e. one has $Q_2(+\tilde{t}) = Q_2(-\tilde{t})$ for large time $\tilde{t}$.

4.2. 3-soliton charges and anomalies

In the Fig. 7 we consider the anomaly $\tilde{\beta}^{(3)}$ in eq. (9), with density function $-6uu_xv_t$. We plot the anomaly density versus $x-$coordinate for three successive times, before collision, during collision and after the collision of the 3-soliton presented in the Fig. 2. The anomaly and its $t-$integrated anomaly functions of $t$ vanish within numerical accuracy; in fact, the anomaly
Figure 4. Top Fig: anomaly density of $\tilde{\alpha}_2$ in (14); i.e. the function $(w_x v_x u_x) \, vs \, x$ for three successive times, $t_i =$ before collision (green), $t_c =$ collision (blue) and $t_f =$ after collision (red), for the 2-soliton of Fig. 1. Bottom Figs: anomaly $\tilde{\alpha}_2$ vs $t$ and the $t-$integrated anomaly $\int_{t_i}^t \tilde{\alpha}_2 \, vs \, t$, respectively.

Figure 5. Top Fig: anomaly density of $\alpha_3$ in (21); i.e. the function $(w_x v_x u_x) \, vs \, x$ for three successive times, $t_i =$ before collision (green), $t_c =$ collision (blue) and $t_f =$ after collision (red), for the 2-soliton of Fig. 1. Bottom Figs: anomaly $\alpha_3$ vs $t$ and the $t-$integrated anomaly $\int_{t_i}^t \alpha_3 \, vs \, t$, respectively.
Figure 6. Top Fig: anomaly density of $h_1$ in (26); i.e. the function $-4q_{xxx}(2q_{xt}^2 + \epsilon_2 q_{xx}q_{tt})$ vs $x$ for three successive times, $t_i =$ before collision (green), $t_c =$ collision (blue) and $t_f =$ after collision (red), for the 2-soliton of Fig. 1. Bottom Figs: anomaly $h_1$ vs $t$ and the $t-$integrated anomaly $\int_{t_i}^{t_f} h_1$ vs $t$, respectively.

$\tilde{\beta}^{(3)}(t) \approx 0$ within the order of $10^{-6}$, whereas the $t-$integrated anomaly vanishes within the order of $10^{-7}$. Therefore, according to (10) and (11) one has that the charge $\tilde{q}_a^{(3)}$ in (9) is asymptotically conserved for the collision of three solitons, i.e. $\tilde{q}_a^{(3)}(+\tilde{t}) = \tilde{q}_a^{(3)}(-\tilde{t})$ for $\tilde{t} \to \infty$.

In Fig. 8 we simulate the anomaly $\tilde{\alpha}_2$ in (14) with density $w_xv_xu_x$. It is plotted $\tilde{\alpha}_2(t)$ versus $x-$coordinate for three successive times, before collision, during collision and after the collision of the 3-soliton of the Fig. 2. The anomaly and its $t-$integrated anomaly functions of $t$ vanish within numerical accuracy; in fact, one has $\tilde{\alpha}_2(t) \approx 0$ within the order of $10^{-8}$, whereas the $t-$integrated anomaly vanishes within the order of $10^{-9}$. Therefore, according to (15) and (17) the charge $\tilde{Q}_a^{(2)}$ in (14) is asymptotically conserved for the collision of three solitons, i.e. $\tilde{Q}_a^{(2)}(+\tilde{t}) = \tilde{Q}_a^{(2)}(-\tilde{t})$ for large $\tilde{t}$.

Fig. 9 presents the anomaly $\alpha_3$ in (21) with density $w_xv_tu_t$. It presents the behavior of $\alpha_3$ versus $x-$coordinate for three successive times, before collision, during collision and after the collision of the 3-soliton presented in the Fig. 2. The anomaly and its $t-$integrated anomaly functions of $t$ vanish within numerical accuracy; in fact, one has $\alpha_3 \approx 0$ within the order of $10^{-8}$, whereas the $t-$integrated anomaly vanishes within the order of $10^{-10}$. Therefore, according to (22), the charge $Q_a^{(3)}$ in (21) is asymptotically conserved for the collision of three solitons, i.e. $Q_a^{(3)}(+\tilde{t}) = Q_a^{(3)}(-\tilde{t})$ for large $\tilde{t}$.

Finally, Fig. 10 shows the anomaly $h_1$ in (26) with density function $-4q_{xxx}(2q_{xt}^2 + \epsilon_2 q_{xx}q_{tt})$. It shows the behavior of $h_1$ versus $x-$coordinate for three successive times, before collision, during collision and after the collision of the 3-soliton presented in the Fig. 2. The anomaly and its $t-$integrated anomaly functions of $t$ vanish within numerical accuracy; in fact, one has $h_1 \approx 0$ within the order of $10^{-8}$, whereas the $t-$integrated anomaly vanishes within the order of $10^{-9}$. Therefore, the charge $\bar{Q}_2$ in (26) is asymptotically conserved for the collision of three solitons,
Figure 7. Top Fig: anomaly density of $\tilde{\beta}^{(3)}(3)$; i.e. the function $(-6uu_x v_t)$ vs $x$ for three successive times, $t_i =$ before collision (green), $t_c =$ collision (blue) and $t_f =$ after collision (red), for the 3-soliton collision of Fig. 2. Bottom Figs: anomaly $\tilde{\beta}^{(3)}(t)$ vs $t$ and the $t-$integrated anomaly $\int_{t_i}^{t_f} \tilde{\beta}^{(3)}$ vs $t$, respectively.

Figure 8. Top Fig: anomaly density of $\tilde{\alpha}_2$ in (14); i.e. the function $(w_x v_t u_x)$ vs $x$ for three successive times, $t_i =$ before collision (green), $t_c =$ collision (blue) and $t_f =$ after collision (red), for the 3-soliton of Fig. 2. Bottom Figs: anomaly $\tilde{\alpha}_2$ vs $t$ and the $t-$integrated anomaly $\int_{t_i}^{t_f} \tilde{\alpha}_2$ vs $t$, respectively.
Figure 9. Top Fig: anomaly density of $\alpha_3$ in (21); i.e. the function $(w_x v_t u_t)$ vs $x$ for three successive times, $t_i =$ before collision (green), $t_c =$ collision (blue) and $t_f =$ after collision (red), for the 3-soliton of Fig. 2. Bottom Figs: anomaly $\alpha_3$ vs $t$ and the $t-$integrated anomaly $\int_{t_i}^{t_f} \alpha_3$ vs $t$, respectively.

i.e. $\bar{Q}_2(\tilde{t}) = \bar{Q}_2(-\tilde{t})$ for large $\tilde{t}$.

The vanishing of the anomalies and time-integrated anomalies are true for the above lowest order quasi-conservation laws but it has also been shown that it is true for the next order anomalies [9].

5. Discussions and some conclusions

We have verified through numerical simulations of soliton collisions, in sec. 4, the conservation properties of the lowest order charges appearing in the towers of quasi-conservation laws defined in [9], for the 2-soliton and 3-soliton collisions. In this contribution we present the simulations for the parameter values $\{\epsilon_1 \approx 0; \epsilon_2 \approx 0\}$. In our numerical simulations presented in the Figs 3-10 we have observed that the non-trivial lowest order anomalies, and their $t-$integrated anomalies, vanish for the 2-soliton and 3-soliton collisions. So, our numerical simulations allow us to argue that for 2-soliton and 3-soliton configurations the relevant charges are exactly conserved, within numerical accuracy.

In view of the recent results, on deformations of sine-Gordon and KdV [8, 9], one can inquire about the various properties of the quasi-integrable systems studied in the literature, such as the deformations of the non-linear Schrödinger, Bullough-Dodd, Toda and SUSY sine-Gordon systems [3, 10, 11, 4, 5, 7]. In particular, the relationships between the anomalous charges, present in the integrable, as well as in the non-integrable models and more specific structures, such as an infinite number of (non-local) exact conservation laws deserve clarifications. So, they deserve careful considerations in the future in the lines discussed above.
Figure 10. Top Fig: anomaly density of \( h_1 \) in (26); i.e. the function \(-4q_{xxx}(2q_{xt}^2+\epsilon q_{xx}q_{tt}) \) vs \( x \) for three successive times, \( t_i = \) before collision (green), \( t_c = \) collision (blue) and \( t_f = \) after collision (red), for the 3-soliton of Fig. 3. Bottom Figs: anomaly \( h_1 \) vs \( t \) and the \( t \)-integrated anomaly \( \int_{t_i}^{t_f} h_1 \) vs \( t \), respectively.

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