Non-positive fermion determinants in lattice supersymmetry

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Abstract

We find that fermion determinants are not generally positive in a recent class of constructions with explicit lattice supersymmetry. These involve an orbifold of supersymmetric matrix models, and have as their target (continuum) theory (2,2) 2-dimensional super-Yang-Mills. The fermion determinant is shown to be identically zero for all boson configurations due to the existence of a zeromode fermion inherited from the “mother theory.” Once this eigenvalue is factored out, the fermion determinant generically has arbitrary complex phase. We discuss the implications of this result for simulation of the models.

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Introductory remarks. Models with exact lattice supersymmetry have been discussed in the literature by a few groups. For example, latticizations of super-Yang-Mills \[1, 2\], supersymmetric quantum mechanics \[3, 4\], the 2d Wess-Zumino model \[5, 4\], and direct constructions in the spirit of the Ginsparg-Wilson relation—as suggested by Lüscher \[6\] and worked out in noninteracting examples \[7, 8\]—have all been considered. In this letter we will be interested in the super-Yang-Mills constructions that lead to a Euclidean lattice theory \[1\]. The method of building such models is based on deconstruction of extra dimensions \[10, 11\]. The corresponding interpretation in terms of the world-volume theory of D-branes has led to the latticizations of 2d, 3d and 4d supersymmetric gauge theories. These lattice constructions are all arrived at by orbifold projections of supersymmetric matrix models; i.e., in each case we quotient a matrix model by some discrete symmetry group of the theory. Degrees of freedom that are not invariant with respect to the combined action of the orbifold generators are projected out. Thus, throughout this letter we will have occasion to speak of “orbifolded” matrix models and “nonorbifolded” matrix models.

A major motivation for efforts to latticize supersymmetric models is that some nonperturbative aspects of supersymmetric field theories are not accessible by the usual techniques, such as holomorphy. One hope of a lattice supersymmetry program of research is that it would lead to, e.g., simulations that would provide further data on supersymmetric field theories, especially those that include super-Yang-Mills. With exact lattice supersymmetry the target (continuum) theory may be obtained in a more controlled fashion. Indeed, in some cases it may be obtained without the need for fine-tuning \[1, 2\].

In this enterprise, it is of great practical importance that the fermion determinant, obtained integrating over the fermion degrees of freedom in the partition function, be positive. For let $\phi$ be the lattice bosons in the theory, and $\psi, \bar{\psi}$ the lattice fermions.

\[1\] An approach very similar to \[8\] has been applied in \[9\], yielding slightly different expressions. 
\[2\] For a detailed discussion, we refer the reader to \[11\]. 
\[3\] For a recent review of existing work on this broad topic, and a complete list of relevant references, see \[12\].
We obtain

\[ Z = \int [d\phi d\bar{\psi} d\psi] \exp \left[ -S_B(\phi) - \bar{\psi} M(\phi) \psi \right] \]

\[ = \int [d\phi] \det M(\phi) \exp \left[ -S_B(\phi) \right] \]

\[ = \int [d\phi] \exp \left[ -S_{\text{eff}}(\phi) \right], \quad (1) \]

\[ S_{\text{eff}}(\phi) = S_B(\phi) - \ln \det M(\phi). \quad (2) \]

A positive \(^5\) \( \det M(\phi) \) for all \( \phi \) allows us to unambiguously calculate in an equivalent bosonic theory with action \( S_{\text{eff}}(\phi) \). Expectation values of operators \( \mathcal{O}(\phi) \) may be obtained using this action:

\[ \langle \mathcal{O}(\phi) \rangle = \frac{\int [d\phi] \mathcal{O}(\phi) \exp \left[ -S_{\text{eff}}(\phi) \right]}{\int [d\phi] \exp \left[ -S_{\text{eff}}(\phi) \right]} \quad (3) \]

Fermionic correlators will simply involve \( M^{-1}(\phi) \) in the operator of interest. In the case of positive \( \det M(\phi) \), the techniques of estimation by Monte Carlo simulation are robust; otherwise the situation is murkier.

We now summarize the content of our work:

- In this letter we show that the lattice theory with (2,2) 2d super-Yang-Mills as its target \(^1\), obtained from orbifolded supersymmetric matrix models, possesses a problematic fermion determinant. Due to a zero-mode fermion, \( \det M(\phi) \equiv 0 \), i.e., for all boson configurations.

- We then suggest how the zero eigenvalue can be factored out in a controlled way in order to exhibit the determinant for the other fermions. For the orbifolded matrix models, we carry out this factorization (numerically) in the special case of a \( U(2) \) gauge theory, and show that it is robust. As an example, we present our results for a \( 2 \times 2 \) lattice.

\(^4\)Here we assume ordinary finite dimensional Berezin integration for the fermion coordinates; thus we avoid subtleties of the Weyl determinant present in continuum theories \(^3\) \(^4\). The class of theories studied here result from a \( 4d \rightarrow 0d \) dimensional reduction, where the Pfaffian one obtains has an equivalent determinant form \(^5\).

\(^5\)Strictly speaking, we require only positive semi-definite \( \det M(\phi) \), provided \( \det M(\phi) = 0 \) occurs only for some subset of the boson configurations; this subset has measure zero. However, if \( \det M(\phi) \equiv 0 \), i.e., for all boson configurations, then \( Z \) is not well-defined. This latter situation is what occurs in the theory that we study here.
We further validate our method by applying it to nonorbifolded $U(k)$ supersymmetric matrix models, which also contain ever-present zeromode fermions, corresponding to the $U(1)$ “gaugino” in the decomposition $U(k) \supset U(1) \times SU(k)$. We correctly reproduce the fermion determinant for the nonorbifolded $SU(k)$ matrix models by our factorization method. As an example, we present our results for $U(5) \supset U(1) \times SU(5)$.

Once the zeromode fermion has been factored out of the orbifolded matrix models studied here, we find that the remaining product of eigenvalues is generically nonzero with arbitrary complex phase.

We explain how this is not in conflict with results in nonorbifolded $SU(k)$ supersymmetric matrix models. (For example, in an appendix of [16] it was shown that the product of eigenvalues for the fermion matrix is positive semi-definite.) It is found that the orbifold procedure lies at the heart of this matter.

We conclude with a discussion of the implications of our results for lattice simulations of the latticized (2,2) 2d super-Yang-Mills theories.

**Zeromode fermion.** Here we will focus on the orbifolded supersymmetric matrix models that have as their target theory (2,2) 2d super-Yang-Mills with $U(k)$ gauge group. The “mother theory” is a nonorbifolded $U(kN^2)$ supersymmetric matrix model. The “daughter theory” is obtained by orbifolding the “mother theory” by a $Z_N \times Z_N$ symmetry group, leaving intact (among other things) a $U(k)^{N^2}$ symmetry group that will become the gauge symmetry of the $N \times N$ lattice theory. The lattice theory is obtained by studying the “daughter theory” about a particular boson configuration that is a stationary point of the action; i.e., it is a point in the moduli space of the “daughter theory.”

The orbifolded matrix model discussed here has been described in detail in [1]; we refer the reader there for further details. For our purpose it suffices to note that in the $U(k)$ case the theory contains bosons $x_\mathbf{m}, y_\mathbf{m}$ that are $k \times k$ complex matrices. The bosons $x_\mathbf{m}, y_\mathbf{m}$ may written in terms of a Hermitian basis

$$T^\mu \in \left\{ \sqrt{\frac{2}{k}} 1_k, T^a \right\}, \quad (T^a)'^{\dagger} = T^a, \quad x_\mathbf{m} = x^\mu \mathbf{T}_\mu, \quad y_\mathbf{m} = y^\mu \mathbf{T}_\mu. \quad (4)$$

\(^6\)Our notation is standard: $\mathbf{m}$ labels sites on a 2d square lattice, so that $\mathbf{m} \in \mathbb{Z} \times \mathbb{Z}$, with $\mathbb{Z}$ the set of integers.
It is always possible to choose the $T^a$ such that
\[ \text{Tr}(T^\mu T^\nu) = 2\delta^{\mu\nu}. \] (5)
Furthermore we define
\[ \text{Tr}(T^\mu T^\nu T^\rho) = 2\sqrt{\frac{2}{k}} t^{\mu\nu\rho}, \] (6)
and note that (underlining implies all permutations are to be taken):
\[ t^{\mu\rho\delta} = \delta^{\mu\rho}. \] (7)
One finds that the fermionic part of action is given by
\[ S_F = \frac{2}{g^2\sqrt{k}}(\alpha^\mu_m, \beta^\mu_m) \cdot M^{\mu\rho}_{m,n} \cdot \left(\lambda^\rho_n, \xi^\rho_n\right). \] (8)
Here $\alpha^\mu_m, \beta^\mu_m, \lambda^\rho, \xi^\rho$ are the lattice fermions, with upper index corresponding to the basis $T^\mu$ introduced above. The fermion matrix $M^{\mu\rho}_{m,n}$ is given by (sum over $\nu$ implied in the entries, $\hat{i}, \hat{j}$ unit vectors):
\[ M^{\mu\rho}_{m,n} = \begin{pmatrix}
    t^{\mu\rho\nu}_{m,n} & -t^{\mu\rho\nu}_{m,n-i} & t^{\mu\rho\nu}_{m,n+j} \\
    t^{\mu\rho\nu}_{m,n-j} & t^{\mu\rho\nu}_{m,n-j} & t^{\mu\rho\nu}_{m,n+i} \\
    t^{\mu\rho\nu}_{m,n+i} & -t^{\mu\rho\nu}_{m,n-i} & t^{\mu\rho\nu}_{m,n-j}
\end{pmatrix}. \] (9)
Here we have introduced the compact notation
\[ t^{\mu\rho}_{m,n} = \delta_{m,n} t^{\mu\rho}. \] (10)
The fermion zero mode is easily established. We consider fermions of the form
\[ \begin{pmatrix}
    \lambda^\rho_n \\
    \xi^\rho_n
\end{pmatrix} = \begin{pmatrix}
    (\delta_{n,0})^\rho \lambda \\
    0
\end{pmatrix}, \quad \forall n. \] (11)
Then in this case
\[ \sum_n \overline{\lambda}_{m,n} (t^{\mu\rho}_{m,n} - t^{\mu\rho}_{m,n-i}) \lambda^\rho_n = \sum_n \overline{\lambda}_{m,n} (t^{\mu\rho}_{m,n} - t^{\mu\rho}_{m,n-i}) \lambda \\
= \overline{\lambda}_{m,n} \sum_n (\delta_{m,n} - \delta_{m,n-i}) = 0. \] (12)
By similar arguments
\[ \sum_n \overline{\xi}_{m,n} (t^{\mu\rho}_{m,n} - t^{\mu\rho}_{m,n-j}) \lambda^\rho_n = 0. \] (13)
Thus (11) is an eigenvector of $M$ with eigenvalue zero. It follows that $\det M(x, y) \equiv 0$, for all configurations of bosons. This clearly poses a difficulty for defining $S_{\text{eff}}(x, y)$. We will shortly address a method to factor out this zero eigenvalue, and study it in some detail for the $U(2)$ case. However, we first make a few remarks on the existence of the identically zero eigenvalue.

The zeromode fermion is nothing but the “zero-momentum” mode of the Fourier transform of a given $\lambda^0_m$:

$$\lambda \equiv \tilde{\lambda}^0_0 = \frac{1}{N} \sum_m \lambda^0_m.$$  \hspace{1cm} (14)

In the formalism introduced in [1], $\lambda^0_m$ appears in the superfield

$$\Lambda^0_m = \lambda^0_m - [\pi^\mu_{m-t} - \pi^\mu_m + \bar{\gamma}^\mu_{m-j} - \bar{\gamma}^\mu_m + i d^0_m] \theta.$$  \hspace{1cm} (15)

Here $d^0_m$ is an auxiliary boson and $\theta$ is an odd (Grassman) superspace coordinate. The zero-momentum part of this supermultiplet is just

$$\Lambda \equiv \sum_m \Lambda^0_m = \lambda + i d \theta, \quad d \equiv \frac{1}{N} \sum_m d^0_m.$$  \hspace{1cm} (16)

That is, the zeromode fermion is in a multiplet that contains just itself and an auxiliary boson. It is a lattice version of a Fermi multiplet [17]. Thus there is no physical zeromode boson that corresponds to the zeromode fermion.

The existence of the zeromode fermion can be understood in terms of the “mother theory,” i.e., the nonorbifolded matrix model. There, the bosons $x, y$ and their conjugates are understood in terms of a “vector boson” $v$:

$$v = v_m \sigma_m = v_0 + i \mathbf{v} \cdot \mathbf{\sigma} = \sqrt{2} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$  \hspace{1cm} (17)

where $v_m$ are Hermitian matrices that are Lie algebra valued in $U(kN^2)$:

$$(v_m)^{i\mu, j\nu'}_{i'\mu', j'\nu'} = v^\alpha_m (T^\alpha)^{i\mu, j\nu'}_{i'\mu', j'\nu'} \quad i, j = 1, \ldots, k, \quad \mu, \nu, \mu', \nu' = 1, \ldots, N.$$  \hspace{1cm} (18)

Note that the indices of generators have been written in such a way that the $U(k)N^2$ subgroup has been manifestly factored out. One should think of one $U(k)$ factor “living” at each site. Similarly, the fermions of the mother theory are given by

$$\psi^{i\mu, j\nu'}_{i'\mu', j'\nu'} = \psi^\alpha (T^\alpha)^{i\mu, j\nu'}_{i'\mu', j'\nu'}, \quad \psi^\alpha = \begin{pmatrix} \lambda^\alpha \\ \xi^\alpha \end{pmatrix}$$  \hspace{1cm} (19)
with a corresponding expression for $\bar{\psi}$. The fermion action in the mother theory takes the form

$$S_F = \frac{1}{g^2} \mathrm{Tr} \left( \bar{\psi} \sigma_m [v_m, \psi] \right). \quad (20)$$

Now note that the diagonal $U(1)_{\text{diag}} \subset U(k N^2)$ fermions do not appear $S_F$. That is,

$$\psi^\alpha (T^\alpha)_{i\mu\nu,j\mu'\nu'} \ni \psi^0 \delta_{i\mu\nu,j\mu'\nu'} \quad (21)$$

and $\psi^0$ disappears because of the commutator. Since $\psi^0$ is a two component fermion, it gives two zeromode fermions of the mother theory, independent of the boson configuration $v_m$. The boson action takes the form

$$S_B = -\frac{1}{4g^2} \mathrm{Tr} \left( [v_m, v_n] [v_m, v_n] \right). \quad (22)$$

Here again, the $U(1)_{\text{diag}}$ boson disappears from the action because of the commutators. It is a zeromode for all boson configurations.

Most of the fields of the mother theory are projected out in the orbifold construction. The projections depend on charges with respect to a $U(1)_{r_1} \times U(1)_{r_2}$ global symmetry group. As it turns out, the $\lambda^\alpha$ fermions are $U(1)_{r_1} \times U(1)_{r_2}$ neutral. It follows that only the diagonal parts with respect to the "site" indices $(\mu\nu, \mu'\nu')$ survive in the orbifolded matrix model ("daughter theory"):\n
$$\lambda^\alpha (T^\alpha)_{i\mu\nu,j\mu'\nu'} \rightarrow \lambda^\alpha (T^\alpha)_{i\mu\nu,j\mu\nu}. \quad (23)$$

But this includes the zeromode fermion of the mother theory. On the other hand, none of the components of $v$ are $U(1)_{r_1} \times U(1)_{r_2}$ neutral. It follows that, in the orbifolded matrix model, they are all off-diagonal\footnote{In the lattice theory, these are interpreted as link fields beginning at the site labeled by $(\mu, \nu)$ and ending at the site labeled by $(\mu', \nu')$.} with respect to the indices $(\mu\nu, \mu'\nu')$. Hence the zeromode bosons of the mother theory are projected out in the orbifolded theory. Similarly, none of the $\xi^\alpha$ fermions of the mother theory are $U(1)_{r_1} \times U(1)_{r_2}$ neutral, so that the second zeromode fermion $\xi^0$ of the mother theory disappears in the projection.

**Deformed $U(k)$**. The zeromode eigenvalue of the orbifolded matrix model can be factored out as follows. We deform the fermion matrix (9) according to

$$M \rightarrow M + \epsilon 1_{N_f} \quad (24)$$
where \( N_f \) is the dimensionality of the fermion matrix and \( \epsilon \ll 1 \) is a deformation parameter that we will eventually take to zero. We factor out the zero mode through the definition

\[
\hat{M}(0) = \lim_{\epsilon \to 0} \hat{M}(\epsilon), \quad \hat{M}(\epsilon) = \epsilon^{-1/N_f}(M + \epsilon 1_{N_f}),
\]

\[
\Rightarrow \quad \det \hat{M}(0) = \lim_{\epsilon \to 0} \epsilon^{-1} \det(M + \epsilon 1_{N_f}). \quad (25)
\]

If this deformation is added to the action, it explicitly breaks the exact lattice supersymmetry and gauge invariance. This infrared regulator could be removed in the continuum limit, say, by taking \( \epsilon a \ll N^{-1} \). Noting that \( L = Na \) is the physical size of the lattice, the equivalent requirement is that \( \epsilon \ll L^{-1} \) be maintained as \( a \to 0 \), for fixed \( L \). Thus in the thermodynamic limit \( (L \to \infty) \), the deformation is removed. The parameter \( \epsilon \) is a soft infrared regulating mass, and is quite analogous to the soft mass \( \mu \) introduced by Cohen et al. in their Eq. (1.2) to control the bosonic zero mode of the theory. In the same way that a finite \( \mu \) does not modify the final results of the renormalization analysis of Section 5 of \( \mu \), our \( \epsilon \) does not modify the result of the quantum continuum limit. The essence of the argument is that we have introduced a vertex that will be proportional to the dimensionless quantity \( g_2^2 \epsilon a^3 \ll g_2^2 a^3/L \), where \( g_2 \) is the 2d coupling constant. Such contributions to the operator coefficients \( C_{O} \) in Eq. (5.2) of \( \mu \) vanish in the thermodynamic limit and because the target theory is super-renormalizable, we are assured that the perturbative power counting arguments are reliable and the correct continuum limit is obtained.

Of course it remains to study the convergence of \( \det \hat{M}(\epsilon) \to \det \hat{M}(0) \). Indeed, for \( U(2) \) lattice theory (i.e., the orbifolded matrix model with \( U(2)^{N^2} \) symmetry) we have performed this analysis numerically for a large number of boson configurations drawn randomly from a Gaussian distribution, as will be detailed below. We find that the convergence is rapid and that a reliable estimate for \( \det \hat{M}(0) \) can be obtained in this way. Furthermore, we find that for \( \epsilon \ll 1 \), the phase of \( \det \hat{M}(\epsilon) \) quickly converges to a constant value, and that it is uniformly distributed throughout the interval \((-\pi, \pi)\) for the random Gaussian boson variables.

As a check of our method, we have studied also the analogous deformation of the nonorbifolded \( U(k) \) supersymmetric matrix models, where the two zeromode fermions in \( \psi^0 \) are present. Indeed, as will be discussed below, we find that

\[
\lim_{\epsilon \to 0} \epsilon^{-2} \det(M_{U(k)} + \epsilon 1_{N_f}) = \det M_{SU(k)}. \quad (26)
\]
That is, we obtain the determinant of the nonorbifolded $SU(k)$ supersymmetric matrix model, which has the zeromode fermions factored out. With appropriate conventions, $\det M_{SU(k)}$ is positive semi-definite.\(^8\)

**Deformed $U(2)$.** Here we specialize to the orbifolded matrix model with $U(2)^{N^2}$ symmetry, which becomes the $U(2)$ gauge invariance of an $N \times N$ lattice theory. In this case we take

\[ T^\mu \in \{1_2, \sigma^a\}, \quad x_m = x_0^m + x_m^a \sigma^a, \quad y_m = y_0^m + y_m^a \sigma^a. \]  

(27)

Then the fermion matrix is given in apply, where in the present case

\[ t^{000} = 1, \quad t^{a00} = 0, \quad t^{ab0} = \delta^{ab}, \quad t^{abc} = i\epsilon^{abc}. \]  

(28)

The lattice theory is obtained by expansion about a point in moduli space:

\[ x_0^m = \frac{1}{a\sqrt{2}} + \cdots, \quad y_0^m = \frac{1}{a\sqrt{2}} + \cdots, \]  

(29)

where $\cdots$ represent the quantum fluctuations. For this reason, in our study of $\det \hat{M}(\epsilon)$ we scan over a Gaussian distribution where $x_0^m, y_0^m$ have a a nonzero mean $1/a\sqrt{2} \equiv 1$. The remainder of the bosons are drawn with mean zero. All bosons are taken from distributions with unit variance.

In Figs. 1 and 2 we display $\ln |\det \hat{M}(\epsilon)|$ and $\arg \det \hat{M}(\epsilon)$ versus $\epsilon$ for the case of $N = 2$, the smallest lattice possible, of size $2 \times 2$. Each line corresponds to a different random draw. It can be seen that the convergence is rapid and reliable. As a check, we have computed the eigenvalues of the undeformed matrix $M$, using the math package Maple, for the same set of random boson configurations. We find that in each case the product of nonzero eigenvalues agrees with $\det \hat{M}(0)$ in magnitude and phase to within at least 5 significant digits. This conclusively demonstrates that the complex determinant is not an artifact of the deformation, but is a property of the nonzero eigenvalues of the undeformed matrix $M$.

For a set of $10^5$ draws on the bosons of this $2 \times 2$ lattice, we have extrapolated to $\epsilon \to 0$ and binned $\arg \det \hat{M}(\epsilon)$ over its range, with bins of size $\pi/100$. In Fig. 3 we show the frequency for each bin, as a fraction of the total number of draws. In the

\[^8\]For some conventions on the $SU(k)$ generators and the overall coefficient in $S_\phi$ of Eq. (20) it is possible that a constant phase (i.e., independent of boson configurations) may be present. However, this factors out of the and is of no concern.
Figure 1: $\ln |\det \hat{M}(\epsilon)|$ versus $\epsilon$ for a sequence of random draws. These results are for the $U(2)$ lattice theory, with $2 \times 2$ lattice.

extrapolation, we decreased $\epsilon$ by powers of 10 until the change in both $\ln |\det \hat{M}(\epsilon)|$ and $\arg \det \hat{M}(\epsilon)$ were both less than $10^{-4}$ per decade. This was done for each draw to get a reliable estimate for $\det \hat{M}(0)$. In each of the $10^5$ draws the $10^{-4}$ per decade criterion was reached well before $\epsilon = 10^{-10}$.

To summarize, once the zeromode eigenvalue is factored out, the product of the nonzero eigenvalues has arbitrary phase. Consequently an ambiguity exists in defining $S_{\text{eff}}(x, y)$ for the orbifolded matrix model theory. We are presently exploring whether or not this can be overcome for the purposes of simulation. At present, however, all we can say is that this difficulty is rather troubling.

Comparison to nonorbifolded $U(k)$ and $SU(k)$ matrix models. As mentioned above, the nonorbifolded supersymmetric $U(k)$ matrix models contain two ever-present zeromode fermions in the $\psi^0$ that appears in (21). Then $\det M_{U(k)} \equiv 0$. However, it is easy enough to just work with the nonorbifolded $SU(k)$ matrix model, so that $\psi^0, \bar{\psi}^0$ are never in the theory to begin with. Then with appropriate conventions $\det M_{SU(k)} \geq 0$. A proof of this result has been given in an appendix of [16].

As a test of our method, we have verified (26) numerically, for a sequence of
Figure 2: $\phi = \arg \det \hat{M}(\epsilon)$ versus $\epsilon$ for a sequence of random draws. Note that crossing over the boundary of the domain $(-\pi, \pi]$ to an equivalent point within that domain is indicated by the nearly vertical lines. These results are for the $U(2)$ lattice theory, with $2 \times 2$ lattice.
Figure 3: Frequency distribution $F(\phi)$ for $\phi = \text{arg det } \hat{M}(0)$, for $10^5$ random (Gaussian) draws, binned into intervals of $\pi/100$. The distribution of $\phi$ is seen to be nearly uniform. These results are for the $U(2)$ lattice theory, with $2 \times 2$ lattice.

random Gaussian draws on the bosons $v_{\alpha}^\alpha$ that appear in (20). In Fig. 4 we show the quantity

$$\Delta(\epsilon) = \ln \left[ \epsilon^{-2} \det(\mathcal{M}(k) + \epsilon 1_{N_f}) \right] - \ln \left| \det \mathcal{M}_{SU(k)} \right|$$

(30)
as a function of $\epsilon$ for the case of $k = 5$. Indeed it can be seen that the convergence as $\epsilon \to 0$ is quite rapid. We find that $|\Delta(10^{-3})| \lesssim 10^{-4}$ and $|\Delta(10^{-4})| \lesssim 10^{-9}$ as a rule.

In Fig. 5 we show the quantity

$$\phi(\epsilon) = \text{arg } \left[ \epsilon^{-2} \det(\mathcal{M}(k) + \epsilon 1_{N_f}) \right]$$

(31)
as a function of $\epsilon$, again for $k = 5$. The value for $SU(5)$ in our conventions is $\phi = 0$. It can be seen that a rapid convergence to this value is obtained.

One might wonder how the generic phases of Fig. 3 are obtained in the orbifolded matrix models, given the positivity of the fermion determinant in the nonorbifolded $SU(k)$ matrix models. Firstly, one should note that the proof given in [10] shows only that for each eigenvalue $\epsilon$ of $M_{SU(k)}$ there exists also an eigenvalue $\epsilon^*$, and that these always come in pairs. The proof relies essentially on the relation

$$\sigma^2 v^\alpha \sigma^2 = \sigma^2 (v_0^\alpha + i v^\alpha \cdot \sigma) \sigma^2 = (v^\alpha)^*.$$
Figure 4: $\Delta(\epsilon)$ versus $\epsilon$ for a sequence of random draws in the nonorbifolded $U(5)$ matrix model.

Figure 5: $\phi$ versus $\epsilon$ for sequence of random draws in the nonorbifolded $U(5)$ matrix model.
On the other hand, the $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold action on the mother theory bosons is generated by

$$v^\alpha(T^\alpha)^{i\rho\nu,j\rho\nu'} \rightarrow e^{-i\pi \sigma_3/N} v^\alpha e^{-i\pi \sigma_3/N} \Omega_{\mu
u}^{\pm1}(T^\alpha)^{i\mu\nu,j\mu\nu'} \Omega^{-1}_{\mu\nu'}$$

where $\Omega = \text{diag}(\omega, \omega^2, \ldots, \omega^N)$, $\omega = \exp(2\pi i/N)$. Thus because of the $\exp(\pm i\pi \sigma_3/N)$ factors, the orbifold projection does not commute with the operations of the proof given in [16]; i.e., Eq. (32). The fermion matrix of the projected theory—i.e., the orbifolded matrix model—lacks many of the eigenvalues of the mother theory; it should come as no surprise that not all eigenvalues are removed in pairs $(e, e^*)$. After all, we already know that only one of the zero eigenvalues is removed from the $U(kN^2)$ mother theory. For this reason it is not at all contradictory that the projected theory has a product of nonzero eigenvalues that is not positive, nor real.

**Discussion.** The existence of a zeromode fermion in the orbifolded matrix models has been reliably handled by our deformation method, and the approach is easily implemented numerically. Analytic methods in nonorbifolded supersymmetric matrix models have also involved deformations of the theory to render quantities of interest well-defined [18]; however, these involve the introduction of auxiliary fields, which are expensive to implement in a simulation. For this reason, we prefer the method described here. In spite of the complex fermion determinant in the orbifolded matrix models, our approach allows for a detailed study of the partition function by Monte Carlo methods, analogous to what has been performed in [15] for nonorbifolded supersymmetric matrix models. It would be interesting to see how expectations based on continuum results might be realized in the present context. For example, in [14] it was shown that 4d $\mathcal{N} = 1$ pure super-Yang-Mills has a positive fermion determinant. Since the (2,2) 2d target theory studied here is a dimensional reduction of this model, it is rather surprising that we find arbitrary complex phase. However, it remains to be seen whether or not the lattice action described here possesses reflection positivity, or whether this is only obtained in the continuum limit. Study of this issue is in progress.

We suspect that the difficulties faced here may be, broadly speaking, related to those faced in defining the phase of the fermion measure in the attempts to realize chiral gauge theories on the lattice with an exact chiral gauge symmetry [19, 20, 21]. In the present context, we have an exact chiral fermionic symmetry: lattice
supersymmetry acts on the $\lambda_m, \xi_m$ but not on the $\alpha_m, \beta_m$ that appear in (3). A resolution of one problem may lead to answers for the other.

In our opinion, the deconstruction approach to lattice supersymmetry remains an exciting topic, whatever difficulties may face attempts to simulate the theory. For example, the relatively simple systems described here provide another context in which the difficulties\footnote{See for example [22, 23] and references therein.} that plague the complex Langevin approach [24, 25] and other complex action techniques might be studied.

We are presently exploring whether or not the complex phase can be overcome for the purposes of simulation. A typical approach would be to compute averages of an operator $O$ from the \textit{re-weighting} identity:

$$\langle O \rangle = \frac{\langle O e^{i\phi} \rangle_{p.q.}}{\langle e^{i\phi} \rangle_{p.q.}}$$  \hspace{1cm} (34)

Here, $\phi = \text{arg det } \hat{M}(0)$, as above, and “p.q.” indicates phase-quenching: expectation values are computed with the replacement $\text{det } \hat{M}(0) \rightarrow |\text{det } \hat{M}(0)|$. Thus the effective bosonic action

$$S'_{\text{eff}} = S_B - \ln |\text{det } \hat{M}(0)|$$  \hspace{1cm} (35)

is used to generate the phase-quenched ensemble by standard Monte Carlo techniques. However, it is well-known that this suffers efficiency problems: the number of configurations required to get an accurate estimate for, say, $\langle \exp(i\phi) \rangle_{p.q.}$, grows like $\exp(\Delta F \cdot N_f^2)$. Here, $\Delta F$ is the difference in free energy densities between the full ensemble and the phase-quenched ensemble. Recall that $N_f$ is the dimensionality of the fermion matrix. On the other hand, if the phase-quenched distribution is highly concentrated near one value of $\phi$, the phase essentially factors out of the partition function and efficient simulations can be done using the phase-quenched ensemble. Thus it is of interest to study the distribution of $\phi$ in the phase-quenched ensemble rather than the Gaussian distribution used here. Research in this direction is currently in progress and we hope to report on it soon.

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References

[1] A. G. Cohen, D. B. Kaplan, E. Katz and M. Unsal, “Supersymmetry on a Euclidean spacetime lattice. I: A target theory with four supercharges,” arXiv:hep-lat/0302017.

[2] D. B. Kaplan, E. Katz and M. Unsal, “Supersymmetry on a spatial lattice,” arXiv:hep-lat/0206019.

[3] S. Catterall and E. Gregory, “A lattice path integral for supersymmetric quantum mechanics,” Phys. Lett. B 487 (2000) 349 [arXiv:hep-lat/0006013].

[4] S. Catterall and S. Karamov, “A two-dimensional lattice model with exact supersymmetry,” Nucl. Phys. Proc. Suppl. 106 (2002) 935 [arXiv:hep-lat/0110071].

[5] S. Catterall and S. Karamov, “Exact lattice supersymmetry: the two-dimensional N = 2 Wess-Zumino model,” Phys. Rev. D 65 (2002) 094501 [arXiv:hep-lat/0108024].

[6] M. Lüscher, “Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation,” Phys. Lett. B 428 (1998) 342 [arXiv:hep-lat/9802011].

[7] T. Aoyama and Y. Kikukawa, “Overlap formula for the chiral multiplet,” Phys. Rev. D 59 (1999) 054507 [arXiv:hep-lat/9803016].

[8] W. Bietenholz, “Exact supersymmetry on the lattice,” Mod. Phys. Lett. A 14 (1999) 51 [arXiv:hep-lat/9807010].

[9] H. So and N. Ukita, “Ginsparg-Wilson relation and lattice supersymmetry,” Phys. Lett. B 457 (1999) 314 [arXiv:hep-lat/9812002].

[10] N. Arkani-Hamed, A. G. Cohen and H. Georgi, “(De)constructing dimensions,” Phys. Rev. Lett. 86, 4757 (2001) arXiv:hep-th/0104005;
[11] C. T. Hill, S. Pokorski and J. Wang, “Gauge invariant effective Lagrangian for Kaluza-Klein modes,” Phys. Rev. D 64, 105005 (2001) [arXiv:hep-th/0104035].

[12] A. Feo, “Supersymmetry on the lattice,” arXiv:hep-lat/0210015.

[13] L. Alvarez-Gaume and E. Witten, “Gravitational Anomalies,” Nucl. Phys. B 234 (1984) 269.

[14] S. D. Hsu, “Gaugino determinant in supersymmetric Yang-Mills theory,” Mod. Phys. Lett. A 13 (1998) 673 arXiv:hep-th/9704149.

[15] W. Krauth, H. Nicolai and M. Staudacher, “Monte Carlo approach to M-theory,” Phys. Lett. B 431 (1998) 31 arXiv:hep-th/9803117.

[16] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, “Large N dynamics of dimensionally reduced 4D SU(N) super Yang-Mills theory,” JHEP 0007 (2000) 013 arXiv:hep-th/0003208.

[17] E. Witten, “Phases of N = 2 theories in two dimensions,” Nucl. Phys. B 403 (1993) 159 arXiv:hep-th/9301042.

[18] G. W. Moore, N. Nekrasov and S. Shatashvili, “D-particle bound states and generalized instantons,” Commun. Math. Phys. 209 (2000) 77 arXiv:hep-th/9803265.

[19] M. Lüscher, “Abelian chiral gauge theories on the lattice with exact gauge invariance,” Nucl. Phys. B 549 (1999) 295 arXiv:hep-lat/9811032.

[20] M. Lüscher, “Weyl fermions on the lattice and the non-abelian gauge anomaly,” Nucl. Phys. B 568 (2000) 162 arXiv:hep-lat/9904009.

[21] M. Lüscher, “Chiral gauge theories revisited,” arXiv:hep-th/0102028.

[22] K. Fujimura, K. Okano, L. Schulke, K. Yamagishi and B. Zheng, “On The Segregation Phenomenon In Complex Langevin Simulation,” Nucl. Phys. B 424 (1994) 675 arXiv:hep-th/9311174.

[23] H. Gausterer and S. Lee, “The Mechanism of complex Langevin simulations,” arXiv:hep-lat/9211050.
[24] G. Parisi, “On Complex Probabilities,” Phys. Lett. B 131 (1983) 393.

[25] F. Karsch and H. W. Wyld, “Complex Langevin Simulation Of The SU(3) Spin Model With Nonzero Chemical Potential,” Phys. Rev. Lett. 55 (1985) 2242.