IWASAWA THEORY AND F-ANALYTIC LUBIN-TATE
(ϕ, Γ)-MODULES

by

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Abstract. — Let K be a finite extension of Qp. We use the theory of (ϕ, Γ)-modules in the Lubin-Tate setting to construct some corestriction-compatible families of classes in the cohomology of V, for certain representations V of Gal(Qp/K). If in addition V is crystalline, we describe these classes explicitly using Bloch-Kato’s exponential maps. This allows us to generalize Perrin-Riou’s period map to the Lubin-Tate setting.

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Introduction

Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $G_K = \text{Gal}(\overline{\mathbb{Q}}_p/K)$. In this article, we use the theory of $(\varphi, \Gamma)$-modules in the Lubin-Tate setting to construct some classes in $H^1(K, V)$, for “$F$-analytic” representations $V$ of $G_K$. If in addition $V$ is crystalline, we describe these classes explicitly using Bloch and Kato’s exponential maps and generalize Perrin-Riou’s period map to the Lubin-Tate setting.

We now describe our constructions in more detail, and introduce some notation which is used throughout this paper. Let $F$ be a finite Galois extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$ and maximal ideal $\mathfrak{m}_F$, let $\pi$ be a uniformizer of $\mathcal{O}_F$ and let $k_F = \mathcal{O}_F/\pi$ and $q = \text{Card}(k_F)$. Let $L_T$ be the Lubin-Tate formal group $[\text{LT65}]$ attached to $\pi$. We fix a coordinate $T$ on $L_T$, so that for each $a \in \mathcal{O}_F$ the multiplication-by-$a$ map is given by a power series $[a](T) = aT + O(T^2) \in \mathcal{O}_F[T]$. Let $\log_{L_T}(T)$ denote the attached logarithm and $\exp_{L_T}(T)$ its inverse for the composition. Let $\chi_\pi : G_F \to \mathcal{O}_F^\times$ be the attached Lubin-Tate character. If $K$ is a finite extension of $F$, let $K_n = K(L_T[\pi^n])$ and $K_\infty = \cup_{n \geq 1} K_n$ and $\Gamma_K = \text{Gal}(K_\infty/K)$.

Let $A_F$ denote the set of power series $\sum_{i \in \mathbb{Z}} a_i T^i$ with $a_i \in \mathcal{O}_F$ such that $a_i \to 0$ as $i \to -\infty$ and let $B_F = A_F[1/\pi]$, which is a field. It is endowed with a Frobenius map $\varphi_\pi : f(T) \mapsto f([\pi](T))$ and an action of $\Gamma_F$ given by $g : f(T) \mapsto f([\chi_\pi(g)](T))$. If $K$ is a finite extension of $F$, the theory of the field of norms (\cite{FW79a, FW79b} and \cite{Win83}) provides us with a finite unramified extension $B_K$ of $B_F$. Recall \cite{Fon90} that a $(\varphi, \Gamma)$-module over $B_K$ is a finite dimensional $B_K$-vector space endowed with a compatible Frobenius map $\varphi_\pi$ and action of $\Gamma_K$. We say that a $(\varphi, \Gamma)$-module over $B_K$ is étale if it has a basis in which $\text{Mat}(\varphi_\pi) \in \text{GL}_d(A_K)$. The relevance of these objects is explained by the result below (see \cite{Fon90, KR09}).

Theorem. — There is an equivalence of categories between the category of $F$-linear representations of $G_K$ and the category of étale $(\varphi, \Gamma)$-modules over $B_K$.

Let $B_K^\dagger$ denote the set of power series $f(T) \in B_F$ that have a non-empty domain of convergence. The theory of the field of norms again provides us \cite{Mat95} with a finite extension $B_K^\dagger$ of $B_K^\dagger$. We say that a $(\varphi, \Gamma)$-module over $B_K$ is overconvergent if it has a basis in which $\text{Mat}(\varphi_\pi) \in \text{GL}_d(B_K^\dagger)$ and $\text{Mat}(g) \in \text{GL}_d(B_K^\dagger)$ for all $g \in \Gamma_K$. If $F = \mathbb{Q}_p$, every étale $(\varphi, \Gamma)$-module over $B_K$ is overconvergent \cite{CC98}. If $F \neq \mathbb{Q}_p$, this is no longer the case \cite{FX13}. Let us say that an $F$-linear representation $V$ of $G_K$ is $F$-analytic if for all embeddings $\tau : F \to \overline{\mathbb{Q}}_p$, with $\tau \neq \text{Id}$, the representation $C_p \otimes_{\overline{\mathbb{Q}}_p} V$ is trivial (as a semilinear $C_p$-representation of $G_K$). The following result is known \cite{Ber16}.
Theorem. — If $V$ is an $F$-analytic representation of $G_K$, it is overconvergent.

Another source of overconvergent representations of $G_K$ is the set of representations that factor through $\Gamma_K$ (see §1.3). Our first result is the following (theorem 1.3.1).

Theorem A. — If $V$ is an overconvergent representation of $G_K$, there exists an $F$-analytic representation $X_{\text{an}}$ of $G_K$, a representation $Y_\Gamma$ of $G_K$ that factors through $\Gamma_K$, and a surjective $G_K$-equivariant map $X_{\text{an}} \otimes_F Y_\Gamma \to V$.

We next focus on $F$-analytic representations. Let $B_{\text{rig},F}^\dagger$ denote the Robba ring, which is the ring of power series $f(T) = \sum_{i \in \mathbb{Z}} a_i T^i$ with $a_i \in F$ such that there exists $\rho < 1$ such that $f(T)$ converges for $\rho < |T| < 1$. We have $B_{F}^\dagger \subset B_{\text{rig},F}^\dagger$. The theory of the field of norms again provides us with a finite extension $B_{\text{rig},K}^\dagger$ of $B_{\text{rig},F}^\dagger$. If $V$ is an $F$-linear representation of $G_K$, let $D(V)$ denote the $(\varphi, \Gamma)$-module over $B_K$ attached to $V$. If $V$ is overconvergent, there is a well defined $(\varphi, \Gamma)$-module $D(V)$ over $B_K^{\dagger}$ attached to $V$, such that $D(V) = B_K \otimes_{B_K^{\dagger}} D(V)$. We call $D_{\text{rig}}(V)$ the $(\varphi, \Gamma)$-module over $B_{\text{rig},K}^\dagger$ attached to $V$, given by $D_{\text{rig}}(V) = B_{\text{rig},K}^{\dagger} \otimes_{B_K^{\dagger}} D(V)$.

The ring $B_{\text{rig},K}^\dagger$ is a free $\varphi_q(B_{\text{rig},K}^\dagger)$-module of degree $q$. This allows us to define a map $\psi_q : B_{\text{rig},K}^\dagger \to B_{\text{rig},K}^{\dagger}$ that is a $\Gamma_K$-equivariant left inverse of $\varphi_q$, and likewise, if $V$ is an overconvergent representation of $G_K$, a map $\psi_q : D_{\text{rig}}(V) \to D_{\text{rig}}(V)$ that is a $\Gamma_K$-equivariant left inverse of $\varphi_q$.

The main result of this article is the construction, for an $F$-analytic representation $V$ of $G_K$, of a collection of maps

$$h_{K_{L},V}^1 : D_{\text{rig}}(V)^{\psi_q = 1} \to H^1(K_{L}, V),$$

having a certain number of properties. For example, these maps are compatible with corestriction: $\text{cor}_{K_{L}/K_{L}} \circ h_{K_{L},V}^1 = h_{K_{L},V}^1$ if $n \geq 1$. Another property is that if $F = \mathbb{Q}_p$ and $\pi = p$ (the cyclotomic case), these maps coincide with those constructed in [CC99] (and generalized in [Ber03]).

If now $K = F$ and $V$ is a crystalline $F$-analytic representation of $G_F$, we give explicit formulas for $h_{\text{an},V}$ using Bloch and Kato’s exponential maps [BK90]. Let $V$ be as above, let $D_{\text{cris}}(V) = (B_{\text{cris},F} \otimes_F V)^{G_F}$ (note that because the $\otimes$ is over $F$, this is the identity component of the usual $D_{\text{cris}}$) and let $t_\pi = \log_{\text{LT}}(T)$. Let $\{u_n\}_{n \geq 0}$ be a compatible sequence of primitive $\pi^n$-torsion points of LT. Let $B_{\text{rig},F}^\dagger$ denote the positive part of the Robba ring, namely the ring of power series $f(T) = \sum_{i \geq 0} a_i T^i$ with $a_i \in F$ such that $f(T)$ converges for $0 \leq |T| < 1$. If $n \geq 0$, we have a map $\varphi_q^{-n} : B_{\text{rig},F}^\dagger \to F_n[t_\pi]$ given by $f(T) \mapsto f(u_n \oplus \exp_{\text{LT}}(t_\pi/\pi^n))$. Using the results of [KR09], we prove that there is a
natural \((\varphi, \Gamma)\)-equivariant inclusion \(D^1_{\rig}(V)^{\psi_q=1} \to B^+_{\rig,F}[1/t\pi] \otimes_F D_{\cris}(V)\). This provides us, by composition, with maps \(\varphi^{-n}_q : D^1_{\rig}(V)^{\psi_q=1} \to F_n(t_n) \otimes_F D_{\cris}(V)\) and \(\partial_V \circ \varphi^{-n}_q : D^1_{\rig}(V)^{\psi_q=1} \to F_n \otimes_F D_{\cris}(V)\) where \(\partial_V\) is the "coefficient of \(t_\varphi^0\)" map. Recall finally that

we prove that if \(x \in "Lubin-Tate" Kummer theory . Recall that if \(L \to \mathbb{Q}_p\) the exponential map \(\log\) constructed in \([\text{Ber03}]\) using the same method as that of \([\text{PR94}]\) gives us a map \(z \mapsto \log_{\psi_q}(z)\). The first result is as follows (theorem 3.3.1).

**Theorem B.** — If \(V\) is as above and \(y \in D^1_{\rig}(V)^{\psi_q=1}\), then

\[
\exp^{\psi_q=1}_{F_n,V^*}(h^1_{F_n,V}(y)) = \begin{cases} 
q^{-n} \partial_y (\varphi^{-n}_q(y)) & \text{if } n \geq 1 \\
(1 - q^{-1} \varphi^{-1}_q) \partial_y (y) & \text{if } n = 0.
\end{cases}
\]

Let \(\nabla = t_x \cdot d/dt\), let \(\nabla_i = \nabla - i\) if \(i \in \mathbb{Z}\) and let \(h \geq 1\) be such that \(\Fil^h D_{\cris}(V) = D_{\cris}(V)\). We prove that if \(y \in (B^+_{\rig,F} \otimes_F D_{\cris}(V))^{\psi_q=1}\), then \(\nabla h^{-1} \circ \nabla_0(y) \in D^1_{\rig}(V)^{\psi_q=1}\), and we have the following result (theorem 3.3.2).

**Theorem C.** — If \(V\) is as above and \(y \in (B^+_{\rig,F} \otimes_F D_{\cris}(V))^{\psi_q=1}\), then

\[
h^1_{F_n,V}(\nabla h^{-1} \circ \nabla_0(y)) = (-1)^{h-1}(h-1)! \begin{cases} 
\exp_{F_n,V}(q^{-n} \partial_y (\varphi^{-n}_q(y))) & \text{if } n \geq 1 \\
\exp_{F,V}((1 - q^{-1} \varphi^{-1}_q) \partial_y (y)) & \text{if } n = 0.
\end{cases}
\]

Using theorems B and C, we give in \([\text{Ber03}]\) a Lubin-Tate analogue of Perrin-Riou’s “big exponential map” \([\text{PR94}]\) using the same method as that of \([\text{Ber03}]\) which treats the cyclotomic case. It will be interesting to compare this big exponential map with the “big logarithms” constructed in \([\text{Fou05}]\) and \([\text{Fou08}]\).

It is also instructive to specialize theorem C to the case \(V = F(\chi_\pi)\), which corresponds to “Lubin-Tate” Kummer theory. Recall that if \(L\) is a finite extension of \(F\), Kummer theory gives us a map \(\delta : \mathrm{LT}(m_L) \to H^1(L, F(\chi_\pi))\). When \(L\) varies among the \(F\), these maps are compatible: the diagram

\[
\begin{array}{ccc}
\mathrm{LT}(m_{F_{n+1}}) & \xrightarrow{\delta} & H^1(F_{n+1}, V) \\
\downarrow \mathrm{Tr}_{F_{n+1}/F_n} & & \downarrow \mathrm{cor}_{F_{n+1}/F_n} \\
\mathrm{LT}(m_{F_n}) & \xrightarrow{\delta} & H^1(F_n, V)
\end{array}
\]

commutes. Let \(S\) denote the set of sequences \(\{x_n\}_{n \geq 1}\) with \(x_n \in m_{F_n}\) and such that \(\mathrm{Tr}_{F_{n+1}/F_n}(x_{n+1}) = [q/\pi](x_n)\) for \(n \geq 1\). We prove that \(S\) is big, in the sense that (if \(F \neq \mathbb{Q}_p\)) the projection on the \(n\)-th coordinate map \(S \otimes_{\mathbb{Q}_p} F \to F_n\) is onto (this would not be the case if we did not have the factor \(q/\pi\) in the definition of \(S\)). Furthermore, we prove that if \(x \in S\), there exists a power series \(f(T) \in (B^+_{\rig,F})^{\psi_q=1/\pi}\) such that

\[
f(T) = \sum_{n \geq 1} a_n x^n,
\]
\[
\frac{f(u_n)}{d/dt}(T) \in \left( B_{rig,F}^+ \right)^{\psi_n = 1} \]
and the following holds (Theorem 3.4.5), where \( u \) is the basis of \( F(\chi) \) corresponding to the choice of \( \{ u_n \}_{n \geq 0} \).

**Theorem D.** — We have \( h_{F_n,F(\chi)}^1 \left( \frac{d}{d \pi}(f(T)) \cdot u \right) = \left( q/\pi \right)^{-n} \cdot \delta(x_n) \) for all \( n \geq 1 \).

In the cyclotomic case, there is \( \text{Col}_{[79]} \) a power series \( \text{Col}_\pi(T) \) such that \( \text{Col}_\pi(u_n) = x_n \) for \( n \geq 1 \). We then have \( f(T) = \log \text{Col}_\pi(T) \), and theorem D is proved in \( \text{CC}_{99} \). In the general Lubin-Tate case, we do not know whether there is a “Coleman power series” of which \( f(T) \) would be the \( \log_{LT} \). This seems like a non-trivial question.

It would be interesting to compare our results with those of \( \text{SV}_{17} \). The authors of \( \text{SV}_{17} \) also construct some classes in \( H^1(K,V) \), but start from the space \( D(V(\chi \cdot \chi_{\text{cyc}})^{\psi_q = \pi/q}) \). In another direction, is it possible to extend our constructions to representations of the form \( V \otimes_{F} Y_T \) with \( V \) \( F \)-analytic and \( Y_T \) factoring through \( \Gamma_K \), and in particular recover the explicit reciprocity law of \( \text{Tsu}_{04} \)?

### 1. Lubin-Tate \((\varphi, \Gamma)\)-modules

In this chapter, we recall the theory of Lubin-Tate \((\varphi, \Gamma)\)-modules and classify overconvergent representations.

**1.1. Notation.** — Let \( F \) be a finite Galois extension of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O}_F \), and residue field \( k_F \). Let \( \pi \) be a uniformizer of \( \mathcal{O}_F \). Let \( d = [F : \mathbb{Q}_p] \) and \( e \) be the ramification index of \( F/\mathbb{Q}_p \). Let \( q = p^f \) be the cardinality of \( k_F \) and let \( F_0 = W(k_F)[1/p] \) be the maximal unramified extension of \( \mathbb{Q}_p \) inside \( F \). Let \( \sigma \) denote the absolute Frobenius map on \( F_0 \).

Let \( \text{LT} \) be the Lubin-Tate formal \( \mathcal{O}_F \)-module attached to \( \pi \) and choose a coordinate \( T \) for the formal group law, such that the action of \( \pi \) on \( \text{LT} \) is given by \( [\pi](T) = T^q + \pi T \). If \( a \in \mathcal{O}_F \), let \( [a](T) \) denote the power series that gives the action of \( a \) on \( \text{LT} \). Let \( \log_{LT}(T) \) denote the attached logarithm and \( \exp_{LT}(T) \) its inverse. If \( K \) is a finite extension of \( F \), let \( K_n = K(\text{LT}[\pi^n]) \) and let \( K_\infty = \cup_{n \geq 1} K_n \). Let \( H_K = \text{Gal}(\overline{\mathbb{Q}}_p/K) \) and \( \Gamma_K = \text{Gal}(K_\infty/K) \). By Lubin-Tate theory (see \( \text{LT}_{65} \)), \( \Gamma_K \) is isomorphic to an open subgroup of \( \mathcal{O}_F^\times \) via the Lubin-Tate character \( \chi_\pi : \Gamma_K \to \mathcal{O}_F^\times \).

Let \( n(K) \geq 1 \) be such that if \( n \geq n(K) \), then \( \chi_\pi : \Gamma_{K_n} \to 1 + \pi^n \mathcal{O}_F \) is an isomorphism, and \( \log_p : 1 + \pi^n \mathcal{O}_F \to \pi^n \mathcal{O}_F \) is also an isomorphism.

Since \( \log_{LT}(T) \) converges on the open unit disk, it can be seen as an element of \( B_{rig,F}^+ \) and denote it by \( t_\pi \). Recall that \( g(t_\pi) = \chi_\pi(g) \cdot t_\pi \) if \( g \in G_K \) and that \( \varphi_q(t_\pi) = t_\pi \). Let
We have \(\partial = d/dt_\pi\) so that \(df(T) = a(T) \cdot df(T)/dT\), where \(a(T) = (d \log_{LT}(T)/dT)^{-1} \in \mathcal{O}_\mathcal{F}[T]^{-1}\).

We have \(\partial \circ g = \chi(g) \cdot g \circ \partial\) if \(g \in \Gamma_K\) and \(\partial \circ \varphi_q = \pi \cdot \varphi_q \circ \partial\).

Recall that \(\mathcal{B}_{\text{rig},F}^1\) denotes the Robba ring, the ring of power series \(f(T) = \sum_{i \in \mathbb{Z}} a_i T^i\) with \(a_i \in F\) such that there exists \(\rho < 1\) such that \(f(T)\) converges for \(\rho < |T| < 1\). We have \(\mathcal{B}_F^1 \subset \mathcal{B}_{\text{rig},F}^1\) and by writing a power series as the sum of its plus part and its minus part, we get \(\mathcal{B}_{\text{rig},F}^1 = \mathcal{B}_F^1 + \mathcal{B}_{\text{rig},F}^1\).

Each ring \(R \in \{\mathcal{B}_{\text{rig},F}^1, \mathcal{B}_{\text{rig},F}^+, \mathcal{B}_F^+, \mathcal{B}_F\}\) is equipped with a Frobenius map \(\varphi_q : f(T) \mapsto f([\pi](T))\) and an action of \(\Gamma_F\) given by \(g : f(T) \mapsto f([\chi(g)](T))\). Moreover, the ring \(R\) is a free \(\varphi_q(R)\)-module of rank \(q\), and we define \(\psi_q : R \rightarrow R\) by the formula \(\varphi_q(\psi_q(f)) = 1/q \cdot \text{Tr}_{\mathcal{B}_F^1/\varphi_q(R)}(f)\). The map \(\psi_q\) has the following properties (see for instance §2A of [FX13] and §1.2.3 of [Col16]): \(\psi_q(x \cdot \varphi_q(y)) = \psi_q(x) \cdot y\), the map \(\psi_q\) commutes with the action of \(\Gamma_F\), \(\partial \circ \psi_q = \pi^{-1} \cdot \psi_q \circ \partial\) and if \(f(T) \in \mathcal{B}_{\text{rig},F}^1\) then \(\varphi_q \circ \psi_q(f) = 1/q \cdot \sum_{z \in \text{LT}([\pi])} f(T \oplus z)\). If \(M\) is a free \(R\)-module with a semilinear Frobenius map \(\varphi_q\) such that \(\text{Mat}(\varphi_q)\) is invertible, then any \(m \in M\) can be written as \(m = \sum_i r_i \cdot \varphi_q(m_i)\) with \(r_i \in R\) and \(m_i \in M\) and the map \(\psi_q : m \mapsto \sum_i \psi_q(r_i) \cdot m_i\) is then well-defined. This applies in particular to the rings \(\mathcal{B}_{\text{rig},K}^1, \mathcal{B}_{\text{rig},K}^+, \mathcal{B}_K^1, \mathcal{B}_K\) and to the \((\varphi, \Gamma)\)-modules over them.

### 1.2. Construction of Lubin-Tate \((\varphi, \Gamma)\)-modules.

A \((\varphi, \Gamma)\)-module over \(\mathcal{B}_K\) (or over \(\mathcal{B}_{\text{rig},K}^1\) or over \(\mathcal{B}_{\text{rig},K}^1\)) is a finite dimensional \(\mathcal{B}_K\)-vector space \(D\) (or a finite dimensional \(\mathcal{B}_K^1\)-vector space or a free \(\mathcal{B}_{\text{rig},K}^1\)-module of finite rank respectively), along with a semilinear Frobenius map \(\varphi_q\) whose matrix (in some basis) is invertible, and a continuous, semilinear action of \(\Gamma_K\) that commutes with \(\varphi_q\).

We say that a \((\varphi, \Gamma)\)-module \(D\) over \(\mathcal{B}_K\) is étale if \(D\) has a basis in which \(\text{Mat}(\varphi_q)\) is invertible, and a continuous, semilinear action of \(\Gamma_K\) that commutes with \(\varphi_q\).

We say that a \((\varphi, \Gamma)\)-module \(D\) is overconvergent if there exists a basis of \(D\) in which the matrices of \(\varphi_q\) and of all \(g \in \Gamma_K\) have entries in \(\mathcal{B}_K^1\). This basis then generates a \(\mathcal{B}_K^1\)-vector space \(D^1\) which is canonically attached to \(D\). If \(V\) is a \(p\)-adic representation, we say that it is overconvergent if \(D(V)\) is overconvergent, and then \(D^1(V)\) denotes the corresponding \((\varphi, \Gamma)\)-module over \(\mathcal{B}_K^1\). The main result of [CC98] states that if \(F = \mathbb{Q}_p\), then every étale \((\varphi, \Gamma)\)-module over \(\mathcal{B}_K\) is overconvergent (the proof is given for \(\pi = p\),

**Theorem 1.2.1.** The functors \(V \mapsto D(V) = (B \otimes_F V)^{H_K}\) and \(D \mapsto (B \otimes_{\mathcal{B}_K} D)^{\varphi_q=1}\) give rise to mutually inverse equivalences of categories between the category of \(F\)-linear representations of \(G_K\) and the category of étale \((\varphi, \Gamma)\)-modules over \(\mathcal{B}_K\).
but it is easy to see that it works for any uniformizer). If $F \neq \mathbb{Q}_p$, some simple examples (see [FX13]) show that this is no longer the case.

Recall that an $F$-linear representation of $G_K$ is $F$-analytic if $C_p \otimes_F V$ is the trivial $C_p$-semilinear representation of $G_K$ for all embeddings $\tau \neq \text{Id} \in \text{Gal}(F/\mathbb{Q}_p)$. This definition is the natural generalization of Kisin and Ren’s notion of $F$-crystalline representation. Kisin and Ren then show that if $K \subset F_\infty$, and if $V$ is a crystalline $F$-analytic representation of $G_K$, the $(\varphi, \Gamma)$-module attached to $V$ is overconvergent (see §3.3 of [KR09]; they actually prove a stronger result, namely that the $(\varphi, \Gamma)$-module attached to such a $V$ is of finite height).

If $D^\dagger_{\text{rig}}$ is a $(\varphi, \Gamma)$-module over $B^\dagger_{\text{rig}, K}$, and if $g \in \Gamma_K$ is close enough to 1, then by standard arguments (see §2.1 of [KR09] or §1C of [FX13]), the series $\log(g) = \log(1 + (g - 1))$ gives rise to a differential operator $\nabla_g : D^\dagger_{\text{rig}} \to D^\dagger_{\text{rig}}$. The map $v \mapsto \exp(v)$ is defined on a neighborhood of 0 in Lie $\Gamma_K$; the map $\text{Lie} \Gamma_K \to \text{End}(D^\dagger_{\text{rig}})$ arising from $v \mapsto \nabla_{\exp(v)}$ is $\mathbb{Q}_p$-linear, and we say that $D^\dagger_{\text{rig}}$ is $F$-analytic if this map is $F$-linear (see §2.1 of [KR09] and §1.3 of [FX13]).

If $V$ is an overconvergent representation of $G_K$, we let $D^\dagger_{\text{rig}}(V) = B^\dagger_{\text{rig}, K} \otimes_{B^\dagger_K} D^\dagger(V)$. The following is theorem D of [Ber16].

**Theorem 1.2.2.** — The functor $V \mapsto D^\dagger_{\text{rig}}(V)$ gives rise to an equivalence of categories between the category of $F$-analytic representations of $G_K$ and the category of étale $F$-analytic Lubin-Tate $(\varphi, \Gamma)$-modules over $B^\dagger_{\text{rig}, K}$.

In general, representations of $G_K$ that are not $F$-analytic are not overconvergent (see §1.3), and the analogue of theorem 1.2.2 without the $F$-analyticity condition on both sides does not hold.

### 1.3. Overconvergent Lubin-Tate $(\varphi, \Gamma)$-modules

By theorem 1.2.2 there is an equivalence of categories between the category of $F$-analytic representations of $G_K$ and the category of étale $F$-analytic Lubin-Tate $(\varphi, \Gamma)$-modules over $B^\dagger_{\text{rig}, K}$. The purpose of this section is to prove a conjecture of Colmez that describes all overconvergent representations of $G_K$.

Any representation $V$ of $G_K$ that factors through $\Gamma_K$ is overconvergent, since $H_K$ acts trivially on $V$ so that $D(V) = B_K \otimes_F V$ and therefore $D(V)$ has a basis in which $\text{Mat}(\varphi_q) = \text{Id}$ and $\text{Mat}(g) \in \text{GL}_d(O_F)$ if $g \in \Gamma_K$. If $X$ is $F$-analytic and $Y$ factors through $\Gamma_K$, $X \otimes_F Y$ is therefore overconvergent. We prove that any overconvergent representation of $G_K$ is a quotient (and therefore also a subobject, by dualizing) of some representation of the form $X \otimes_F Y$ as above.
Theorem 1.3.1. — If $V$ is an overconvergent representation of $G_K$, there exists an $F$-analytic representation $X$ of $G_K$, a representation $Y$ of $G_K$ that factors through $\Gamma_K$, and a surjective $G_K$-equivariant map $X \otimes_F Y \to V$.

Proof. — Recall (see §3 of [Ber16]) that if $r > 0$, then inside $B_{\text{rig}, K}^1$ we have the subring $B_{\text{rig}, K}^{1, r}$ of elements defined on a fixed annulus whose inner radius depends on $r$ and whose outer radius is 1, and that $(\varphi, \Gamma)$-modules over $B_{\text{rig}, K}^1$ can be defined over $B_{\text{rig}, K}^{1, r}$ if $r$ is large enough, giving us a module $D_{\text{rig}}^{1, r}(V)$. We also have rings $B_{K}^{1, r}$ of elements defined on a closed annulus whose radii depend on $r \leq s$. One can think of an element of $B_{\text{rig}, K}^{1, r}$ as a compatible family of elements of $\{B_{K}^{1, r}\}_{I}$ where $I$ runs over a set of closed intervals whose union is $[r; +\infty[$. In the rest of the proof, we use this principle of glueing objects defined on closed annuli to get an object on the annulus corresponding to $B_{\text{rig}, K}^{1, r}$.

Choose $r > 0$ large enough such that $D_{\text{rig}}^{1, r}(V)$ is defined, and $s \geq qr$. Let $D^{[r, s]}(V) = B_{K}^{[r, s]} \otimes_{B_{\text{rig}, K}^{1, r}} D_{\text{rig}}^{1, r}(V)$. If $a \in O_F$, and if $\text{val}_p(a) \geq n$ for $n = n(r, s)$ large enough, the series $\exp(a \cdot \nabla)$ converges in the operator norm to an operator on the Banach space $D^{[r, s]}(V)$. This way, we can define a twisted action of $\Gamma_K$ on $D^{[r, s]}(V)$, by the formula $h \ast x = \exp(\log_p(\chi(h)) \cdot \nabla)(x)$. This action is now $F$-analytic by construction.

Since $s \geq qr$, the modules $D^{[r, r, q^n s]}(V)$ for $m \geq 0$ are glued together (using the idea explained above) by $\varphi_q$ and we get a new action of $\Gamma_K$ on $D_{\text{rig}}^{1, r}(V) = D^{[r, +\infty]}(V)$ and hence on $D_{\text{rig}}^{1}(V)$. Since $\varphi_q$ is unchanged, this new $(\varphi, \Gamma)$-module is étale, and therefore corresponds to a representation $W$ of $G_K$. The representation $W$ is $F$-analytic by theorem 1.2.2 and its restriction to $H_K$ is isomorphic to $V$.

Let $X = \text{ind}_{G_K}^{G_K} W$. By Mackey’s formula, $X|_{H_K}$ contains $W|_{H_K} \simeq V|_{H_K}$ as a direct summand. The space $Y = \text{Hom}(\text{ind}_{G_K}^{G_K} W, V)^{H_K}$ is therefore a nonzero representation of $\Gamma_K$, and there is an element $y \in Y$ whose image is $V$. The natural map $X \otimes_F Y \to V$ is therefore surjective. Finally, $X$ is $F$-analytic since $W$ is $F$-analytic. \hfill \QED

By dualizing, we get the following variant of theorem 1.3.1.

Corollary 1.3.2. — If $V$ is an overconvergent representation of $G_K$, there exists an $F$-analytic representation $X$ of $G_K$, a representation $Y$ of $G_K$ that factors through $\Gamma_K$, and an injective $G_K$-equivariant map $V \to X \otimes_F Y$.

1.4. Extensions of $(\varphi, \Gamma)$-modules. — In this section, we prove that there are no non-trivial extensions between an $F$-analytic $(\varphi, \Gamma)$-module and the twist of an $F$-analytic $(\varphi, \Gamma)$-module by a character that is not $F$-analytic. This is not used in the rest of the paper, but is of independent interest.
If $\delta : \Gamma_K \to \mathcal{O}_F^\times$ is a continuous character, and $g \in \Gamma_K$, let $w_\delta(g) = \log \delta(g)/\log \chi_\pi(g)$. Note that $\delta$ is $F$-analytic if and only if $w_\delta(g)$ is independent of $g \in \Gamma_K$.

We define the first cohomology group $H^1(D)$ of a $(\varphi, \Gamma)$-module $D$ as in §4 of [FX13]. Let $D$ be a $(\varphi, \Gamma)$-module over $B^\dagger_{\text{rig}, K}$. Let $G$ denote the semigroup $\varphi_q^{\text{rig}} \times \Gamma_K$ and let $Z^1(D)$ denote the set of continuous functions $f : G \to D$ such that $(h - 1)f(g) = (g - 1)f(h)$ for all $g, h \in G$. Let $B^1(D)$ be the subset of $Z^1(D)$ consisting of functions of the form $g \mapsto (g - 1)y$, $y \in D$ and let $H^1(D) = Z^1(D)/B^1(D)$. If $g \in G$ and $f \in Z^1$, then $[h \mapsto (g - 1)f(h)] = [h \mapsto (h - 1)f(g)] \in B^1$. The natural actions of $\Gamma_K$ and $\varphi_q$ on $H^1$ are therefore trivial.

If $D_0$ and $D_1$ are two $(\varphi, \Gamma)$-modules, then $\text{Hom}(D_1, D_0) = \text{Hom}_{B^\dagger_{\text{rig}, K}-\text{mod}}(D_1, D_0)$ is a free $B^\dagger_{\text{rig}, K}$-module of rank $\text{rk}(D_0) \text{rk}(D_1)$ which is easily seen to be itself a $(\varphi, \Gamma)$-module. The space $H^1(\text{Hom}(D_1, D_0))$ classifies the extensions of $D_1$ by $D_0$. More precisely, if $D$ is such an extension and if $s : D_1 \to D$ is a $B^\dagger_{\text{rig}, K}$-linear map that is a section of the projection $D \to D_1$, then $g \mapsto s - g(s)$ is a cocycle on $G$ with values in $\text{Hom}(D_1, D_0)$ (the element $g(s) \in \text{Hom}(D_1, D)$ being defined by $g(s)(g(x)) = g(s(x))$ for all $g \in G$ and all $x \in D_1$). The class of this cocycle in the quotient $H^1(\text{Hom}(D_1, D_0))$ does not depend on the choice of the section $s$, and every such class defines a unique extension of $D_1$ by $D_0$ up to isomorphism.

**Theorem 1.4.1.** — If $D$ is an $F$-analytic $(\varphi, \Gamma)$-module, and if $\delta : \Gamma_K \to \mathcal{O}_F^\times$ is not locally $F$-analytic, then $H^1(D(\delta)) = \{0\}$.

**Proof.** — If $g \in \Gamma_K$ and $x(\delta) \in D(\delta)$ with $x \in D$, we have

$$\nabla_g(x(\delta)) = \nabla(x)\delta + w_\delta(g) \cdot x(\delta).$$

If $g, h \in \Gamma_K$, this implies that $\nabla_g(x(\delta)) - \nabla_h(x(\delta)) = (w_\delta(g) - w_\delta(h)) \cdot x(\delta)$. If $\delta \in H^1(D(\delta))$ and $g \in \Gamma_K$, then $g(\delta) = \delta$ and therefore $\nabla_g(\delta) = 0$. The formula above shows that if $k \in \Gamma_K$, then $\nabla_g(f(k)) - \nabla_h(f(k)) = (w_\delta(g) - w_\delta(h)) \cdot f(k)$, so that $0 = (\nabla_g - \nabla_h)(\delta) = (w_\delta(g) - w_\delta(h)) \cdot \delta$, and therefore $\delta = 0$ if $\delta$ is not locally analytic. \hfill $\square$

## 2. Analytic cohomology and Iwasawa theory

In this chapter, we explain how to construct classes in the cohomology groups of $F$-analytic $(\varphi, \Gamma)$-modules. This allows us to define our maps $h^1_{K, \nu}$.

### 2.1. Analytic cohomology. — Let $G$ be an $F$-analytic semigroup and let $M$ be a Fréchet or LF space with a pro-$F$-analytic (§2 of [Ber16]) action of $G$. Recall that this
means that we can write $M = \lim\limits_{\leftarrow} \lim\limits_{i \leftarrow j} M_{ij}$ where $M_{ij}$ is a Banach space with a locally analytic action of $G$. A function $f : G \to M$ is said to be pro-$F$-analytic if its image lies in $\lim\limits_{\leftarrow} M_{ij}$ for some $i$ and if the corresponding function $f : G \to M_{ij}$ is locally $F$-analytic for all $j$.

The analytic cohomology groups $H^1_{\text{an}}(G, M)$ are defined and studied in §4 of [FX13] and §5 of [Col16]. In particular, we have $H^0_{\text{an}}(G, M) = M^G$ and $H^1_{\text{an}}(G, M) = Z^1_{\text{an}}(G, M)/B^1_{\text{an}}(G, M)$ where $Z^1_{\text{an}}(G, M)$ is the set of pro-$F$-analytic functions $f : G \to M$ such that $(g - 1)f(h) = (h - 1)f(g)$ for all $g, h \in G$ and $B^1_{\text{an}}(G, M)$ is the set of functions of the form $g \mapsto (g - 1)m$.

Let $M$ be a Fréchet space, and write $M = \lim\limits_{\leftarrow} M_n$ with $M_n$ a Banach space such that the image of $M_{n+j}$ in $M_n$ is dense for all $j \geq 0$.

**Proposition 2.1.1.** — We have $H^1_{\text{an}}(G, M) = \lim\limits_{\leftarrow} H^1_{\text{an}}(G, M_n)$.

**Proof.** — By definition, we have an exact sequence

$$0 \to B^1_{\text{an}}(G, M_n) \to Z^1_{\text{an}}(G, M_n) \to H^1_{\text{an}}(G, M_n) \to 0.$$ 

It is clear that $B^1_{\text{an}}(G, M) = \lim\limits_{\leftarrow} B^1_{\text{an}}(G, M_n)$ and that $Z^1_{\text{an}}(G, M) = \lim\limits_{\leftarrow} Z^1_{\text{an}}(G, M_n)$, since these spaces are spaces of functions on $G$ satisfying certain compatible conditions. The Banach spaces $B^1_{\text{an}}(G, M_n)$ satisfy the Mittag-Leffler condition: $B^1_{\text{an}}(G, M_n) = M_n/M^G_n$ and the image of $M_{n+j}$ in $M_n$ is dense for all $j \geq 0$. This implies that the sequence

$$0 \to \lim\limits_{\leftarrow} B^1_{\text{an}}(G, M_n) \to \lim\limits_{\leftarrow} Z^1_{\text{an}}(G, M_n) \to \lim\limits_{\leftarrow} H^1_{\text{an}}(G, M_n) \to 0$$

is exact, and the proposition follows. 

In this paper, we mainly use the semigroups $\Gamma_K$, $\Gamma_K \times \Phi$ where $\Phi = \{\phi^n_q, n \in \mathbb{Z}_{\geq 0}\}$ and $\Gamma_K \times \Psi$ where $\Psi = \{\psi^n_q, n \in \mathbb{Z}_{\geq 0}\}$. The semigroups $\Phi$ and $\Psi$ are discrete and the $F$-analytic structure comes from the one on $\Gamma_K$.

**Definition 2.1.2.** — Let $G$ be a compact group and let $H$ be an open subgroup of $G$. We have the corestriction map $\text{cor} : H^1_{\text{an}}(H, M) \to H^1_{\text{an}}(G, M)$, which satisfies $\text{cor} \circ \text{res} = [G : H]$. This map has the following equivalent explicit descriptions (see §2.5 of [Ser94] and §II.2 of [CC99]). Let $X \subset G$ be a set of representatives of $G/H$ and let $f \in Z^1_{\text{an}}(H, M)$ be a cocycle.

1. By Shapiro’s lemma, $H^1_{\text{an}}(H, M) = H^1_{\text{an}}(G, \text{ind}^G_H M)$ and $\text{cor}$ is the map induced by $i \mapsto \sum_{x \in X} x \cdot i(x^{-1})$;
2. if $M \subset N$ where $N$ is a $G$-module and if there exists $n \in N$ such that $f(h) = (h - 1)(n)$, then $\text{cor}(f)(g) = (g - 1)(\sum x \in X x n)$;
3. if $g \in G$, let $\tau_g : X \to X$ be the permutation defined by $\tau_g(x)H = gxH$. We have $\text{cor}(f)(g) = \sum x \in X \tau_g(x) \cdot f(\tau_g(x)^{-1}gx)$.

If $g \in \Gamma_K$, let $\ell(g) = \log_p \chi_x(g)$. If $M$ is a Fréchet space with a pro-$F$-analytic action of $\Gamma_K$ and if $g \in \Gamma_K$ is such that $\chi_x(g) \in 1 + 2p\mathcal{O}_F$, then $\lim_{n \to \infty} (g^p^n - 1)/(p^n \ell(g))$ converges to an operator $\nabla$ on $M$, which is independent of $g$ thanks to the $F$-analyticity assumption. If $c : \Gamma_K \to M$ is an $F$-analytic map, let $c'(1)$ denote its derivative at the identity.

**Proposition 2.1.3.** — If $M$ is a Fréchet space with a pro-$F$-analytic action of $\Gamma_K$, the map $c \mapsto c'(1)$ induces an isomorphism $H^1_{\text{an}}(\Gamma_K, M) = (M/\nabla M)^{\Gamma_K}$, under which $\text{cor}_{L/K}$ corresponds to $\text{Tr}_{L/K}$.

**Proof.** — Assume for the time being that $M$ is a Banach space. We first show that the map induced by $c \mapsto c'(1)$ is well-defined and lands in $(M/\nabla M)^{\Gamma_K}$. The map $c \mapsto c'(1)$ from $Z^1_{\text{an}}(\Gamma_K, M) \to M$ is well-defined, and if $c(g) = (g - 1)m$, then $c'(1) = \nabla m$ so that there is a well-defined map $H^1_{\text{an}}(\Gamma_K, M) \to M/\nabla M$. If $h \in \Gamma_K$ then $(h - 1)c'(1) = \lim_{g \to 1} (h - 1)c(g)/\ell(g) = \lim_{g \to 1} (g - 1)c(h)/\ell(g) = \nabla c(h)$ so that the image of $c \mapsto c'(1)$ lies in $(M/\nabla M)^{\Gamma_K}$.

The formula for the corestriction follows from the explicit descriptions above: if $h \in \Gamma_L$ then $\tau_h(x) = x$ so that $\text{cor}(c)(h) = \sum x \in X x \cdot c(h)$ and

$$\text{cor}(c)'(1) = \lim_{h \to 1} \text{cor}(c)(h)/\ell(h) = \sum x \in X x \cdot c'(1) = \text{Tr}_{L/K}(c'(1)).$$

We now show that the map is injective. If $c'(1) = \nabla m$, then the derivative of $g \mapsto c(g) - (g - 1)m$ at $g = 1$ is zero and hence $c(g) = (g - 1)m$ on some open subgroup $\Gamma_L$ of $\Gamma_K$ and $c = [L : K]^{-1} \text{cor}_{L/K} \circ \text{res}_{K/L}(c) = 0$.

We finally show that the map is surjective. Suppose now that $y \in (M/\nabla M)^{\Gamma_K}$. The formula $g \mapsto (\exp(\ell(g)\nabla) - 1)/\nabla \cdot y$ defines an analytic cocycle $c_L$ on some open subgroup $\Gamma_L$ of $\Gamma_K$. The image of $[L : K]^{-1} c_L$ under $\text{cor}_{L/K}$ gives a cocycle $c \in H^1_{\text{an}}(\Gamma_K, M)$ such that $c'(1) = y$.

We now let $M = \varprojlim_n M_n$ be a Fréchet space. The map $H^1_{\text{an}}(\Gamma_K, M) \to (M/\nabla M)^{\Gamma_K}$ induced by $c \mapsto c'(1)$ is well-defined, and in the other direction we have the map $y \mapsto c_y$:

$$(M/\nabla M)^{\Gamma_K} \to \varprojlim_n (M_n/\nabla M_n)^{\Gamma_K} \to \varprojlim_n H^1_{\text{an}}(\Gamma_K, M_n) \to H^1_{\text{an}}(\Gamma_K, M).$$

These two maps are inverses of each other. \qed
Remark 2.1.4. — Compare with the following theorem (see [Tam15], corollary 21): if $G$ is a compact $p$-adic Lie group and if $M$ is a locally analytic representation of $G$, then $H^i_{\text{an}}(G,M) = H^i(\text{Lie}(G), M)^G$.

2.2. Cohomology of $F$-analytic $(\varphi, \Gamma)$-modules. — If $V$ is an $F$-analytic representation, let $H^1_{\text{an}}(K,V) \subset H^1(K,V)$ classify the $F$-analytic extensions of $F$ by $V$. Let $D$ denote an $F$-analytic $(\varphi, \Gamma)$-module over $B_{\text{rig}, K}^\dagger$, such as $D_{\text{rig}}^\dagger(V)$.

Proposition 2.2.1. — If $V$ is $F$-analytic, then $H^1_{\text{an}}(K,V) = H^1_{\text{an}}(\Gamma_K \times \Phi, D_{\text{rig}}^\dagger(V))$.

Proof. — The group $H^1_{\text{an}}(\Gamma_K \times \Phi, D_{\text{rig}}^\dagger(V))$ classifies the $F$-analytic extensions of $B_{\text{rig}, K}^\dagger$ by $D_{\text{rig}}^\dagger(V)$, which correspond to $F$-analytic extensions of $F$ by $V$ by theorem 1.2.2.

Theorem 2.2.2. — If $D$ is an $F$-analytic $(\varphi, \Gamma)$-module over $B_{\text{rig}, K}^\dagger$ and $i = 0, 1$, then $H^i_{\text{an}}(\Gamma_K, D^{\psi_q=0}) = 0$.

Proof. — Since $B_{\text{rig}, F}^\dagger \subset B_{\text{rig}, K}^\dagger$, the $B_{\text{rig}, K}^\dagger$-module $D$ is a free $B_{\text{rig}, F}^\dagger$-module of finite rank. Let $\mathcal{R}_F$ denote $B_{\text{rig}, F}^\dagger$ and let $\mathcal{R}_{C_\Phi}$ denote $C_p \hat{\otimes}_F B_{\text{rig}, F}^\dagger$ the Robba ring with coefficients in $C_p$. There is an action of $G_F$ on the coefficients of $\mathcal{R}_{C_\Phi}$ and $\mathcal{R}_{C_\Phi}^G = \mathcal{R}_F$.

Theorem 5.5 of [Col16] says that $H^i_{\text{an}}(\Gamma_K, (\mathcal{R}_{C_\Phi} \otimes \mathcal{R}_F D)^{\psi_q=0}) = 0$. For $i = 0$, this implies our claim. For $i = 1$, it says that if $c : \Gamma_K \rightarrow D^{\psi_q=0}$ is an $F$-analytic cocycle, there exists $m \in (\mathcal{R}_{C_\Phi} \otimes \mathcal{R}_F D)^{\psi_q=0}$ such that $c(g) = (g-1)m$ for all $g \in \Gamma_K$. If $\alpha \in G_F$, then $c(g) = (g-1)\alpha(m)$ as well, so that $\alpha(m) - m \in ((\mathcal{R}_{C_\Phi} \otimes \mathcal{R}_F D)^{\psi_q=0})^{G_K} = 0$. This shows that $m \in ((\mathcal{R}_{C_\Phi} \otimes \mathcal{R}_F D)^{\psi_q=0})^{G_F} = D^{\psi_q=0}$.

Corollary 2.2.3. — The groups $H^i_{\text{an}}(\Gamma_K \times \Phi, D)$ and $H^i_{\text{an}}(\Gamma_K \times \Psi, D)$ are isomorphic for $i = 0, 1$.

Proof. — If $i = 0$, then we have an inclusion $D^{\psi_q=1, \Gamma_K} \subset D^{\psi_q=1, \Gamma_K}$. If $x \in D^{\psi_q=1, \Gamma_K}$, then $x - \varphi_q(x) \in D^{\psi_q=0, \Gamma_K} = \{0\}$ by theorem 2.2.2 so that $x = \varphi_q(x)$ and the above inclusion is an equality.

Now let $i = 1$. If $f \in Z^i_{\text{an}}(\Gamma_K \times \Phi, D)$, let $Tf \in Z^i_{\text{an}}(\Gamma_K \times \Psi, D)$ be the function defined by $Tf(g) = f(g)$ if $g \in \Gamma_K$ and $Tf(\varphi_x) = -\varphi_q(f(\varphi_x))$.

If $f \in Z^i_{\text{an}}(\Gamma_K \times \Psi, D)$ and $g \in \Gamma_K$, then $(\varphi_q \psi_q - 1)f(g) \in D^{\psi_q=0}$ and the map $g \mapsto (\varphi_q \psi_q - 1)f(g)$ is an element of $Z^i_{\text{an}}(\Gamma_K, D^{\psi_q=0})$. By theorem 2.2.2 applied once for existence and once for unicity, there is a unique $m_f \in D^{\psi_q=0}$ such that $(\varphi_q \psi_q - 1)f(g) = (g-1)m_f$. Let $Uf \in Z^i_{\text{an}}(\Gamma_K \times \Phi, D)$ be the function defined by $Uf(g) = f(g)$ if $g \in \Gamma_K$ and $Uf(\varphi_q) = -\varphi_q(f(\varphi_q)) + m_f$. 

It is straightforward to check that $U$ and $T$ are inverses of each other (even at the level of the $\mathbb{Z}_p(1)$) and that they descend to the $\mathbb{H}_1^1$.

**Theorem 2.2.4.** — The map $f \mapsto f(\psi_q)$ from $\mathbb{Z}_p^1(\Gamma_K \times \Psi, D)$ to $D$ gives rise to an exact sequence:

$$0 \to \mathbb{H}_1^1(\Gamma_K, D^{\psi_q=1}) \to \mathbb{H}_1^1(\Gamma_K \times \Psi, D) \to \left( \frac{D}{\psi_q - 1} \right)^{\Gamma_K}$$

**Proof.** — If $f \in \mathbb{Z}_p^1(\Gamma_K \times \Psi, D)$ and $g \in \Gamma_K$, then $(g-1)f(\psi_q) = (\psi_q-1)f(g) \in (\psi_q-1)D$ so that the image of $f$ is in $(D/(\psi_q - 1))^{\Gamma_K}$. The other verifications are similar.

2.3. The space $D/(\psi_q - 1)$. — By theorem 2.2.4 in the previous section, the cokernel of the map $\mathbb{H}_1^1(\Gamma_K, D^{\psi_q=1}) \to \mathbb{H}_1^1(\Gamma_K \times \Psi, D)$ injects into $(D/(\psi_q - 1))^{\Gamma_K}$. It can be useful to know that this cokernel is not too large. In this section, we bound $D/(\psi_q - 1)$ when $D = B_{\text{rig},F}$, with the action of $\varphi_q$ twisted by $a^{-1}$, for some $a \in F^\times$.

**Theorem 2.3.1.** — If $a \in F^\times$, then $\psi_q - a : B_{\text{rig},F}^+ \to B_{\text{rig},F}^+$ is onto unless $a = q^{-1}\pi^m$ for some $m \in \mathbb{Z}_{\geq 1}$, in which case $B_{\text{rig},F}^+/(\psi_q - a)$ is of dimension 1.

In order to prove this theorem, we need some results about the action of $\psi_q$ on $B_{\text{rig},F}^+$.

Recall that the map $\vartheta = d/dt_\pi$ was defined in §1.1.

**Lemma 2.3.2.** — If $a \in F^\times$, then $a \varphi_q - 1 : B_{\text{rig},F}^+ \to B_{\text{rig},F}^+$ is an isomorphism, unless $a = \pi^{-m}$ for some $m \in \mathbb{Z}_{\geq 0}$, in which case

$$\ker(a \varphi_q - 1 : B_{\text{rig},F}^+ \to B_{\text{rig},F}^+) = \mathcal{F}t_\pi^m$$

$$\text{im}(a \varphi_q - 1 : B_{\text{rig},F}^+ \to B_{\text{rig},F}^+) = \{ f(T) \in B_{\text{rig},F}^+ \mid \vartheta^m(f)(0) = 0 \}.$$**Proof.** — This is lemma 5.1 of [FX13].

**Lemma 2.3.3.** — If $m \in \mathbb{Z}_{\geq 0}$, there is an $h(T) \in (B_{\text{rig},F}^+)^{\psi_q=0}$ such that $\vartheta^m(h)(0) \neq 0$.

**Proof.** — We have $\psi_q(T) = 0$ by (the proof of) proposition 2.2 of [FX13]. If there was some $m_0$ such that $\vartheta^m(T)(0) = 0$ for all $m \geq m_0$, then $T$ would be a polynomial in $t_\pi$, which it is not. This implies that there is a sequence $\{ m_i \}$ of integers with $m_i \to +\infty$, such that $\vartheta^m(T)(0) \neq 0$, and we can take $h(T) = \vartheta^{m_i-m}(T)$ for any $m_i \geq m$.

**Corollary 2.3.4.** — If $a \in F^\times$, then $\psi_q - a : B_{\text{rig},F}^+ \to B_{\text{rig},F}^+$ is onto.

**Proof.** — If $f(T) \in B_{\text{rig},F}^+$ and if we can write $f = (1 - a \varphi_q)g$, then $f = (\psi_q - a)(\varphi_q(g))$. If this is not possible, then by lemma 2.3.2 there exists $m \geq 0$ such that $a = \pi^{-m}$ and $\vartheta^m(f)(0) \neq 0$. Let $h$ be the function provided by lemma 2.3.3. The function $f -$
Lemma 2.3.6. — If $a^{-1} \in q \cdot \mathcal{O}_F$, then $\psi_q - a : B_{\text{rig},F}^1 \to B_{\text{rig},F}^1$ is onto.

Proof. — We have $B_{\text{rig},F}^1 = B_{\text{rig},F}^1 + B_F^1$ (by writing a power series as the sum of its plus part and of its minus part) and by corollary 2.3.4, $\psi_q - a : B_{\text{rig},F}^1 \to B_{\text{rig},F}^1$ is onto. Take $f(T) \in B_F^1$, choose some $r > 0$ and let $B_{F,(0,r]}^1$ be the set of $f(T) \in B_F^1$ that converge and are bounded on the annulus $0 < \text{val}_p(x) \leq r$. It follows from proposition 1.4 of [Col16] that if $n \gg 0$, then $\psi_q^n(f) \in B_{F,(0,r]}^1$ and by proposition 2.4(d) of [FX13], the sequence $(q/\pi \cdot \psi_q)^n(f)$ is bounded in $B_{F,(0,r]}^1$. The series $\sum_{n \geq 0} a^{-1-n} \psi_q^n(f)$ therefore converges in $B_{F,(0,r]}^1$, and we can write $f = (\psi_q - a)g$ where $g = a^{-1}(1 - a^{-1}\psi_q)^{-1}f = \sum_{n \geq 0} a^{-1-n} \psi_q^n(f)$. 

Let $\text{Res} : B_{\text{rig},F}^1 \to F$ be defined by $\text{Res}(f) = a_{-1}$ where $f(T)dt_\pi = \sum_{n \in \mathbb{Z}} a_n T^n dT$. The following lemma combines propositions 2.12 and 2.13 of [FX13].

Lemma 2.3.5. — The sequence $0 \to F \to B_{\text{rig},F}^1 \xrightarrow{\partial} B_{\text{rig},F}^1 \xrightarrow{\text{Res}} F \to 0$ is exact, and $\text{Res}(\psi_q(f)) = \pi/q \cdot \text{Res}(f)$.

Proof of theorem 2.3.1. — Since $\partial \circ \psi_q = \pi^{-1} \psi_q \circ \partial$, the map $\partial$ induces a map:

$$(\text{Der}) \quad \frac{B_{\text{rig},F}^1}{\psi_q - a} \xrightarrow{\partial} \frac{B_{\text{rig},F}^1}{\psi_q - a \pi}.$$

Take $x \in B_{\text{rig},F}^1$ such that $\text{Res}(x) = 1$. We have $\text{Res}((\psi_q - a\pi)x) = \pi/q - a\pi$. If $a \neq q^{-1}$, this is non-zero and if $f \in B_{\text{rig},F}^1$, proposition 2.3.6 allows us to write $f = \partial g + \text{Res}(f)/(\pi/q - a\pi) \cdot (\psi_q - a\pi)x$. This implies that (Der) is onto if $a \neq q^{-1}$.

Combined with lemma 2.3.5, this implies that $B_{\text{rig},F}^1/(\psi_q - a) = 0$ if $a$ is not of the form $q^{-1} \pi^m$ for some $m \in \mathbb{Z}_{>1}$.

When $a = q^{-1}$, we have an exact sequence

$$\frac{B_{\text{rig},F}^1}{\psi_q - q^{-1}} \xrightarrow{\partial} \frac{B_{\text{rig},F}^1}{\psi_q - q^{-1} \pi} \xrightarrow{\text{Res}} F \to 0,$$

which now implies that $B_{\text{rig},F}^1/(\psi_q - q^{-1} \pi) = F$, generated by the class of $x$.

We now assume again that $a \neq q^{-1}$ and compute the kernel of (Der). If $f \in B_{\text{rig},F}^1$ is such that $\partial f = (\psi_q - a\pi)g$, then $\text{Res}(\partial f) = \text{Res}(\psi_q - a\pi)g = (\pi/q - a\pi)\text{Res}(g)$, so that $\text{Res}(g) = 0$ and we can write $g = \partial h$. We have $\partial(f - (\psi_q - a)h) = 0$, so that $f = (\psi_q - a)h + c$, with $c \in F$. By corollary 2.3.4, there exists $b \in B_{\text{rig},F}^1$ such that $(\psi_q - a)(b) = c$, so that $f = (\psi_q - a)(h + b)$ and (Der) is bijective. We then have, by induction on $m \geq 1$, that $B_{\text{rig},F}^1/(\psi_q - q^{-1} \pi^m) = F$, generated by the class of $\partial^m(x)$. 

$\square$
Remark 2.3.7. — More generally, we expect that the following holds: if $D$ is a $(\varphi, \Gamma)$-module over $B_{rig,K}^1$, the $F$-vector space $D/(\psi_q - 1)$ is finite dimensional.

2.4. The operator $\Theta_b$. — The power series $F(X) = X/((\exp(X) - 1)$ belongs to $Q_p[X]$ and has a nonzero radius of convergence. If $M$ is a Banach space with a locally $F$-analytic action of $\Gamma_K$ and $h \in \Gamma_K$ is close enough to 1, then

$$\nabla \frac{h}{h - 1} = \frac{\nabla}{\exp(\ell(h) \nabla) - 1} = \ell(h)^{-1} F(\ell(h) \nabla)$$

converges to a continuous operator on $M$. If $g \in \Gamma_K$, we then define

$$\nabla \frac{1}{1 - g} = \frac{\nabla}{1 - g^n} \cdot \frac{1 - g^n}{1 - g}.$$ 

This operator is independent of the choice of $n$ such that $g^n$ is close enough to 1, and can be seen as an element of the locally $F$-analytic distribution algebra acting on $M$.

If $M$ is a Fréchet space, write $M = \varprojlim_i M_i$ and define operators $\nabla \frac{i}{1 - g}$ on each $M_i$ as above. These operators commute with the maps $M_j \to M_i$ (because $n$ can be taken large enough for both $M_i$ and $M_j$). This defines an operator $\nabla \frac{i}{1 - g}$ on $M$ itself. The definition of $\nabla \frac{i}{1 - g}$ extends to an LF space with a pro-$F$-analytic action of $\Gamma_K$.

Assume that $K$ contains $F_1$ and let $r(K) = f + \text{val}_p([K : F_1])$. For example, $p^{(F_n)} = q^n$ if $n \geq 1$. Assume further that $K$ contains $F_n(K)$, so that $\chi_K : \Gamma_K \to \mathcal{O}_F^*$ is injective and its image is a free $\mathbb{Z}_p$-module of rank $d$. If $b = (b_1, \ldots, b_d)$ is a basis of $\Gamma_K$ (that is, $\Gamma_K = \mathbb{Z}_p^{b_1} \cdots \mathbb{Z}_p^{b_d}$), then let $\ell^*(b) = \ell(b_1) \cdots \ell(b_d)$ and

$$\Theta_b = \ell^*(b) \cdot \frac{\nabla^d}{(b_1 - 1) \cdots (b_d - 1)}.$$

Lemma 2.4.1. — If $K = F_n$ and $m \geq 0$ and $x \in F_{m+n}$, then

$$\Theta_b(x) = q^{-m-n} \cdot \text{Tr}_{F_{m+n}/F_n}(x).$$

Proof. — Since $\nabla = \lim_{k \to \infty} (b^k - 1)/p^k \ell(b)$, we have

$$\Theta_b = \lim_{k \to \infty} q^n p^{kd} \cdot \frac{(b_1^k - 1) \cdots (b_d^k - 1)}{(b_1 - 1) \cdots (b_d - 1)}.$$ 

The set $\{b_1^{a_1} \cdots b_d^{a_d}\}$ with $0 \leq a_i \leq p^k - 1$ runs through a set of representatives of $\Gamma_n/\Gamma_n^{p^k} = \Gamma_n/\Gamma_n^{ek}$ so that

$$\frac{1}{q^n p^{kd}} \cdot \frac{(b_1^k - 1) \cdots (b_d^k - 1)}{(b_1 - 1) \cdots (b_d - 1)} = \frac{1}{q^n p^{ek}} \text{Tr}_{F_{n+ek}/F_n} = \frac{1}{q^n p^{ek}} \cdot \text{Tr}_{F_{n+ek}/F_n}.$$ 

The lemma follows from taking $k$ large enough so that $ek \geq m$. 

For $i \in \mathbb{Z}$, let $\nabla_i = \nabla - i$. 

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Lemma 2.4.2. — If $b$ is a basis of $\Gamma_{F_n}$ and if $f(T) \in (\mathcal{B}_{rig,F}^+)_{\psi_q=0}$, then $\Theta_b(f(T)) \in (t_{\pi}/\varphi_q^n(T)) \cdot \mathcal{B}_{rig,F}^+$, and if $h \geq 2$ then $\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b(f(T)) \in (t_{\pi}/\varphi_q^n(T))^h \cdot \mathcal{B}_{rig,F}^+$.

Proof. — If $m \geq 1$, then by lemma 2.4.1 and using repeatedly the fact (see §1.1) that $\varphi_q \circ \psi_q(f) = 1/q \cdot \sum_{z \in \mathbb{LT}[\pi]} f(T \oplus z)$,
\[
\Theta_b(f(u_{n+m})) = 1/q^{m+n} \cdot \text{Tr}_{F_{m+n}/F_n} f(u_{m+n}) = \psi_q^m(f)(u_n) = 0.
\]
This proves the first claim, since an element $f(T) \in \mathcal{B}_{rig,F}^+$ is divisible by $t_{\pi}/\varphi_q^n(T)$ if and only if $f(u_{n+m}) = 0$ for all $m \geq 1$. The second claim follows easily. \qed

Let $D$ be a $\varphi_q$-module over $F$. Let $\varphi_q^{-n} : \mathcal{B}_{rig,F}^+[1/t_{\pi}] \otimes_F D \to F_n((t_{\pi})) \otimes_F D$ be the map
\[
\varphi_q^{-n} : t_{\pi}^h f(T) \otimes x \mapsto \pi^{nh} t_{\pi}^h f(u_n \oplus \exp_{LT}(t_{\pi}/\pi^n)) \otimes \varphi_q^{-n}(x).
\]
If $f(t_{\pi}) \in F_n((t_{\pi})) \otimes_F D$, let $\partial_D(f) \in F_n \otimes_F D$ denote the coefficient of $t_{\pi}^n$.

Lemma 2.4.3. — If $y \in (\mathcal{B}_{rig,F}^+[1/t_{\pi}] \otimes_F D)_{\psi_q=1}$ and if $m \geq n$, then
\[
q^{-m} \text{Tr}_{F_m/F_n} \partial_D(\varphi_q^{-m}(y)) = \begin{cases} q^n \partial_D(\varphi_q^{-n}(y)) & \text{if } n \geq 1 \\ (1 - q^{-1} \varphi_q^{-1}) \partial_D(y) & \text{if } n = 0. \end{cases}
\]

Proof. — If $y = t_{\pi}^{-\ell} \sum_{k=0}^{+\infty} a_k T^k \in \mathcal{B}_{rig,F}^+[1/t_{\pi}] \otimes_F D$, then (by definition of $\varphi_q^{-m}$)
\[
\varphi_q^{-m}(y) = \pi^{m \ell} t_{\pi}^{-\ell} \sum_{k=0}^{+\infty} \varphi_q^{-m}(a_k)(u_m \oplus \exp_{LT}(t_{\pi}/\pi^m))^k,
\]
and $\psi_q(y) = y$ means that:
\[
\varphi_q(y)(T) = \frac{1}{q} \sum_{[\pi](\omega) = 0} y(T \oplus \omega).
\]
If $m \geq 2$, the conjugates of $u_m$ under $\text{Gal}(F_m/F_{m-1})$ are the $\{\omega \oplus u_m\}_{[\pi](\omega) = 0}$ so that:
\[
\text{Tr}_{F_m/F_{m-1}} \partial_D(\varphi_q^{-m}(y)) = \partial_D \left( \sum_{[\pi](\omega) = 0} \pi^{m \ell} t_{\pi}^{-\ell} \sum_{k=0}^{+\infty} \varphi_q^{-m}(a_k)(\omega \oplus u_m \oplus \exp_{LT}(t_{\pi}/\pi^m))^k \right)
\]
\[
= \partial_D \left( \varphi_q^{-m} \left( \sum_{[\pi](\omega) = 0} y(T \oplus \omega) \right) \right)
\]
\[
= q \partial_D(\varphi_q^{-(m-1)}(y)).
\]
For \( m = 1 \), the computation is similar, except that the conjugates of \( u_1 \) under \( \text{Gal}(F_1/F) \) are the \( \omega \), where \( |\pi|/\omega = 0 \) but \( \omega \neq 0 \), which results in:

\[
\text{Tr}_{F_1/F} \partial_D(\varphi_{q_1}^{-1}(y)) = \partial_D \left( \varphi_{q_1}^{-1} \left( \sum_{|\pi|/\omega = 0 \atop \omega \neq 0} y(T \oplus \omega) \right) \right) = \partial_D(qy - \varphi_{q_1}^{-1}(y)).
\]

\[\square\]

2.5. Construction of extensions. — Let \( D \) be an \( F \)-analytic \((\varphi, \Gamma)\)-module over \( \mathcal{B}_{\text{rig}, K}^\dagger \). The space \( D^{\psi_0=1} \) is a closed subspace of \( D \) and therefore an LF space. Take \( K \) such that \( K \) contains \( F_n(K) \) and let \( b \) be a basis of \( \Gamma_K \).

**Proposition 2.5.1.** — If \( y \in D^{\psi_0=1} \), there is a unique cocycle \( c_b(y) \in Z^1_{\text{an}}(\Gamma_K, D^{\psi_0=1}) \) such that for all \( 1 \leq j \leq d \) and \( k \geq 0 \), we have

\[
c_b(y)(b^j_k) = \ell^*(b) \cdot \frac{b^k_j - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i - 1)}(y).
\]

We then have \( c_b(y)'(1) = \Theta_b(y) \).

**Proof.** — There is obviously one and only one continuous cocycle satisfying the conditions of the proposition. It is \( \mathbb{Q}_p \)-analytic, and in order to prove that it is \( F \)-analytic, we need to check that the directional derivatives are independent of \( j \). We have

\[
\lim_{k \to 0} \frac{c_b(y)(b^j_k)}{\ell(b^j_k)} = \ell^*(b) \cdot \frac{\nabla^d}{\prod_i(b_i - 1)}(y) = \Theta_b(y),
\]

which is indeed independent of \( j \), and thus \( c_b(y)'(1) = \Theta_b(y) \).

\[\square\]

**Lemma 2.5.2.** — If \( n \geq n(K) \) and \( L = K_n \) and \( M = K_{n+e} \) and \( b \) is a basis of \( \Gamma_L \), then \( b^p \) is a basis of \( \Gamma_M \) and \( \text{cor}_{M/L} c_b(y) = c_b(y) \).

**Proof.** — The Lubin-Tate character maps \( \Gamma_L \) to \( 1 + \pi^n \mathcal{O}_F \), and \( \Gamma_M = \Gamma_L^p \) because \( (1 + \pi^n \mathcal{O}_F)^p = 1 + \pi^{n+e} \mathcal{O}_F \). Since \( \{b^1_{k_1} \cdots b^d_{k_d}\} \) with \( 0 \leq k_i \leq p-1 \) is a set of representatives for \( \Gamma_L/\Gamma_M \), and since \( [M : L] = q^e = p^d \), the explicit formula for the corestriction (definition
Lemma 2.5.3. — If $a$ and $b$ are two bases of $\Gamma_K$, then $c_a(y)$ and $c_b(y)$ are cohomologous.

Proof. — If $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$ are in $F^\times$, the Laurent series
\[
\frac{\alpha_1 \cdots \alpha_d \cdot T^{d-1}}{(\exp(\alpha_1 T) - 1) \cdots (\exp(\alpha_d T) - 1)} - \frac{\beta_1 \cdots \beta_d \cdot T^{d-1}}{(\exp(\beta_1 T) - 1) \cdots (\exp(\beta_d T) - 1)}
\]
is the difference of two Laurent series, each having a simple pole at 0 with equal residues, and therefore belongs to $F[[T]]$. Let $a$ and $b$ be two bases of $\Gamma_K$ and take $y \in D_{\psi_0=1}$.

Let $N$ be a $\Gamma_K$-stable Fréchet subspace of $D$ that contains $y$ and write $N = \varprojlim M_j$. Since $M = M_j$ is $F$-analytic, we have $g = \exp(\ell(y)\nabla)$ on $M$ for $g$ in some open subgroup of $\Gamma_K$. Let $k \gg 0$ be large enough such that $a_i^k$ and $b_i^k$ are in this subgroup, and let $\alpha_i = p^k \ell(a_i)$ and $\beta_i = p^k \ell(b_i)$. Taking $k$ large enough (depending on $M$), we can assume moreover that the power series $T/(\exp(T) - 1)$ applied to the operators $\alpha_i \nabla$ and $\beta_i \nabla$ converges on $M$. The element
\[
w = \left(\frac{\alpha_1 \cdots \alpha_d \cdot \nabla^{d-1}}{(\exp(\alpha_1 \nabla) - 1) \cdots (\exp(\alpha_d \nabla) - 1)} - \frac{\beta_1 \cdots \beta_d \cdot \nabla^{d-1}}{(\exp(\beta_1 \nabla) - 1) \cdots (\exp(\beta_d \nabla) - 1)}\right)(y)
\]
of $M$ is well defined. By proposition 2.5.1, we have
\[
c_{a^k}(y)'(1) - c_{b^k}(y)'(1) = \Theta_{a^k}(y) - \Theta_{b^k}(y) = p^{-r(L)}\nabla(w)
\]
where $L$ is the extension of $K$ such that $\Gamma_L = \Gamma_{K_0}^K$. Thus, for $g$ close enough to 1, we have $c_{a^k}(y)(g) - c_{b^k}(y)(g) = (g - 1)(p^{-r(L)}w)$. Lemma 2.5.2 now implies by corestricting that this holds for all $g$, and, by corestricting again, that $c_a(y)$ and $c_b(y)$ are cohomologous in $M$. By varying $M$, we get the same result in $N$, which implies the proposition. □
Lemma 2.5.4. — If $L/K$ is a finite extension contained in $K_{\infty}$, and if $b$ is a basis of $\Gamma_K$ and $a$ is a basis of $\Gamma_L$, then $\text{cor}_{L/K}c_a(y) = c_b(y)$.

Proof. — The groups $\Gamma_K$ and $\Gamma_L$ are both free $\mathbb{Z}_p$-modules of rank $d$, so that by the elementary divisors theorem, we can change the bases $a$ and $b$ in such a way that there exists $e_1, \ldots, e_d$ with $a_i = b_i^{p^{e_i}}$.

Since $\{b_1^{k_1}, \ldots, b_d^{k_d}\}$ with $0 \leq k_i \leq p^{e_i} - 1$ is a set of representatives for $\Gamma_K/\Gamma_L$, and since $[L : K] = p^{e_1 + \cdots + e_d}$, the explicit formula for the corestriction implies

$$\text{cor}_{L/K}(c_a(y))(b_j^k) = \sum_{0 \leq k_i, 0 \leq k_d < p^{e_d} - 1} b_1^{k_1} \cdots b_d^{k_d} \cdot \ell^*(a) \cdot \frac{a_j^{k_j} - 1}{a_j - 1} \cdot \prod_{i \neq j}^{d-1} (a_i - 1) \cdot \nabla_{d-1}(y)$$

$$= \ell^*(b) \cdot \left( \sum_{k_j=0}^{p^{e_j} - 1} \frac{a_j^{k_j} - 1}{a_j - 1} \right) \cdot \left( \prod_{i \neq j}^{d-1} (a_i - 1) \right) \cdot \nabla_{d-1}(y)$$

$$= c_b(y)(b_j^k).$$

\[ \square \]

Definition 2.5.5. — Let $h_{K,V}^1 : D_{\text{rig}}^\dagger(V)^{\psi=1} \to H^1_{\text{an}}(K,V)$ denote the map obtained by composing $y \mapsto c_0(y)$ with $H^1_{\text{an}}(\Gamma_K, D_{\text{rig}}^\dagger(V)^{\psi=1}) \to H^1_{\text{an}}(\Gamma_K \times \Psi, D_{\text{rig}}^\dagger(V))$ (theorem 2.2.2), and with $H^1_{\text{an}}(\Gamma_K \times \Psi, D_{\text{rig}}(V)) \simeq H^1_{\text{an}}(K,V)$ (proposition 2.2.1 and corollary 2.2.3).

Proposition 2.5.6. — We have $\text{cor}_{M/L} \circ h_{M,V}^1 = h_{L,V}^1$ if $M/L$ is a finite extension contained in $K_{\infty}/K_{n(K)}$. In particular, $\text{cor}_{K_{n+1}/K_n} \circ h_{K_{n+1},V}^1 = h_{K_n,V}^1$ if $n \geq n(K)$.

Proof. — This follows from the definition and from lemma 2.5.4 above.

Remark 2.5.7. — Proposition 2.5.6 allows us to extend the definition of $h_{K,V}^1$ to all $K$, without assuming that $K$ contains $F_{n(K)}$, by corestricting.
Theorem 2.5.8. — If \( y \in D_{\text{rig}}^1(V)^{\psi=1} \) and \( K \) contains \( K_{n(K)} \) and \( b \) is a basis of \( \Gamma_K \), then

1. there is a unique \( c_b(y) \in \mathbb{Z}_{\text{an}}^1(\Gamma_K, D_{\text{rig}}^1(V)^{\psi=1}) \) such that for \( k \in \mathbb{Z}_p \),
   \[
   c_b(y)(b_j^k) = \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i - 1)}(y);
   \]
2. there is a unique \( m_c \in D_{\text{rig}}^1(V)^{\psi=0} \) such that \( (\varphi_q - 1)c_b(y)(g) = (g - 1)m_c \) for all \( g \in \Gamma_K \);
3. the \((\varphi, \Gamma)\)-module corresponding to this extension has a basis in which
   \[
   \text{Mat}(g) = \begin{pmatrix}
   * & c_b(y)(g) \\
   0 & 1
   \end{pmatrix}
   \quad \text{if } g \in \Gamma_K, \quad \text{and} \quad \text{Mat}(\varphi_q) = \begin{pmatrix}
   * & m_c \\
   0 & 1
   \end{pmatrix};
   \]
4. if \( z \in \tilde{B}_{\text{rig}}^1 \otimes_F V \) is such that \( (\varphi_q - 1)z = m_c \), then the cocycle
   \[
   g \mapsto c_b(y)(g) - (g - 1)z
   \]
   defined on \( G_K \) has values in \( V \) and represents \( h_{1,K,V}^1(y) \) in \( H_{\text{an}}^1(K,V) \).

Proof. — Items (1), (2) and (3) are reformulations of the constructions of this chapter. Let us prove (4). Let us write the \((\varphi, \Gamma)\)-module corresponding to the extension in (3) as \( D' = D_{\text{rig}}^1(V) \oplus \tilde{B}_{\text{rig},F}^1 \cdot e \). It is an étale \((\varphi, \Gamma)\)-module that comes from the \( p \)-adic representation \( V' = (\tilde{B}_{\text{rig}}^1 \otimes \tilde{B}_{\text{rig},F}^1) \cdot \psi \). We have \( V' = V \oplus F \cdot (e - z) \) as \( F \)-vector spaces since \( \varphi_q(e - z) = e - z \). If \( g \in G_K \), then
   \[
   g(e - z) = e + c_b(y)(g) - g(z) = e - z + c_b(y)(g) - (g - 1)z.
   \]
This proves (4). \( \square \)

Let \( F = \mathbb{Q}_p \) and \( \pi = p = q \), and let \( V \) be a representation of \( G_K \). In §II.1 of \cite{CC99}, Cherbonnier and Colmez define a map \( \text{Log}^\ast_{V,1} : D^1(V)^{\psi=1} \to H^1_{\text{lw}}(K,V) \), which is an isomorphism (theorem II.1.3 and proposition III.3.2 of \cite{CC99}).

Proposition 2.5.9. — If \( F = \mathbb{Q}_p \) and \( \pi = p \), then the map
   \[
   D^1(V)^{\psi=1} \to D_{\text{rig}}^1(V)^{\psi=1} \xrightarrow{\{b_{K_n,V}\}_{n \geq 1}} \lim_n H_{\text{an}}^1(K_n, V) \to \lim_n H^1(K_n, V)
   \]
coincides with the map \( \text{Log}^\ast_{V,1} : D^1(V)^{\psi=1} \to H^1_{\text{lw}}(K,V) \subset \lim_n H^1(K_n, V) \).

Proof. — The map \( \text{Log}^\ast_{V,1} \) is constructed by mapping \( x \in D^1(V)^{\psi=1} \) to the sequence \( (\ldots, \ell_{\psi,n}(x), \ldots) \in \lim_n H^1(K_n, V) \) (see theorem II.1.3 in \cite{CC99} and the paragraph preceding it), where
   \[
   \ell_{\psi,n}(x) = \left[ \sigma \mapsto \ell_{K_n}(\gamma_n) \left( \frac{\sigma - 1}{\gamma_n - 1} x - (\sigma - 1)b \right) \right]
   \]
on $G_{K_n}$ and where (see proposition I.4.1, lemma I.5.2 and lemma I.5.5 of ibid.)

1. $\gamma_n = \gamma_1^{[K_n:K_1]}$ and $\gamma_1$ is a fixed generator of $\Gamma_{K_1}$;
2. $\ell_{K_n}(\gamma_n) = \frac{\log(\gamma_n)}{r(K_n)}$ where $r(K_n)$ is the integer such that $\log(\Gamma_{K_n}) = p^{r(K_n)}\mathbb{Z}_p$;
3. $b \in \mathcal{B}_r \otimes \mathbb{Q}_p V$ is such that $(\varphi - 1)b = a$ and $a \in D^\dagger(V)_{\psi=1}$ is such that $(\gamma_n - 1)a = (\varphi - 1)x$ (using the fact that $\gamma_n - 1$ is bijective on $D^\dagger(V)_{\psi=0}$).

The theorem follows from comparing this with the explicit formula of theorem 2.5.8 \[\Box\]

### 3. Explicit formulas for crystalline representations

In this chapter, we explain how the constructions of the previous chapter are related to $p$-adic Hodge theory, via Bloch and Kato’s exponential maps. Let $\mathcal{B}_{\text{dr}}$ be Fontaine’s ring of periods \[\text{[Fon94]}\] and let $\mathcal{B}_{\text{max}, F}$ be the subring of $\mathcal{B}_{\text{dr}}^+$ that is constructed in §8.5 of \[\text{[Col02]}\] (recall that $\mathcal{B}_{\text{max}, F} = F \otimes_{F_0} \mathcal{B}_{\text{max}}$ where $F_0 = F \cap \mathbb{Q}_p^{\text{ur}}$ and $\mathcal{B}_{\text{max}}$ is a ring that is similar to Fontaine’s $\mathcal{B}_{\text{cris}}$).

We assume throughout this chapter that $K = F$ and that the representation $V$ is crystalline and $F$-analytic.

#### 3.1. Crystalline $F$-analytic representations

If $V$ is an $F$-analytic crystalline representation of $G_F$, let $D_{\text{cris}}(V) = (\mathcal{B}_{\text{max}, F} \otimes_F V)^{G_F}$ (this is the “component at identity” of the usual $D_{\text{cris}}$). By corollary 3.3.8 of \[\text{[KR09]}\], $F$-analytic crystalline representations of $G_F$ are overconvergent. Moreover, if $\mathcal{M}(D) \subset \mathcal{B}_{\text{rig}, F}[1/t_\pi] \otimes_F D$ is the object constructed in §2.2 of ibid., then by §2.4 of ibid., $\mathcal{M}(D_{\text{cris}}(V))$ contains a basis of $D^\dagger(V)$ and $D_{\text{rig}}^\dagger(V) = \mathcal{B}_{\text{rig}, F} \otimes \mathcal{B}_{\text{rig}, F} \mathcal{M}(D_{\text{cris}}(V))$. This implies that $D_{\text{rig}}^\dagger(V) \subset \mathcal{B}_{\text{rig}, F}[1/t_\pi] \otimes_F D_{\text{cris}}(V)$.

**Theorem 3.1.1.** We have $D_{\text{rig}}^\dagger(V)_{\psi=1} \subset \mathcal{B}_{\text{rig}, F}[1/t_\pi] \otimes_F D_{\text{cris}}(V)$.

**Proof.** Take $h \geq 0$ such that the slopes of $\pi^{-h} \varphi_q$ on $D_{\text{cris}}(V)$ are $\leq -d$. Let $E$ be an extension of $F$ such that $E$ contains the eigenvalues of $\varphi_q$ on $D_{\text{cris}}(V)$. We show that $D_{\text{rig}}^\dagger(V)_{\psi=1} \subset t^{-h}_\pi E \otimes_F \mathcal{B}_{\text{rig}, F} \otimes_F D_{\text{cris}}(V)$. Let $e_1, \ldots, e_n$ be a basis of $t^{-h}_\pi E \otimes_F D_{\text{cris}}(V)$ in which the matrix $(p_{i,j})$ of $\varphi_q$ is upper triangular. If $y = \sum_{i=1}^d y_i \otimes \varphi_q(e_i)$ with $y_i \in E \otimes_F \mathcal{B}_{\text{rig}, F}$, then $\psi_q(y) = y$ if and only if $\psi_q(y_k) = p_{k,k} y_k + \sum_{j=k}^d p_{k,j} y_j$ for all $k$. The theorem follows from applying lemma 3.1.2 below to $k = n, n - 1, \ldots, 1$.

**Lemma 3.1.2.** Take $y \in E \otimes_F \mathcal{B}_{\text{rig}, F}^+$ and $\alpha \in F$ such that $\text{val}_\pi(\alpha) \leq -d$. If $\psi_q(y) - \alpha y \in E \otimes_F \mathcal{B}_{\text{rig}, F}^+$, then $y \in E \otimes_F \mathcal{B}_{\text{rig}, F}^+$.

**Proof.** This is lemma 5.4 of \[\text{[FX13]}\].
3.2. Bloch-Kato’s exponentials for analytic representations. — We now recall
the definition of Bloch-Kato’s exponential map and its dual, and give a similar
definition for $F$-analytic representations.

Lemma 3.2.1. — We have an exact sequence

$$0 \to F \to \left(B_{\text{max},F}[1/t\tau]\right)^{\otimes 1} \to B_{d}\!/B_{d}^{+} \to 0.$$ 

Proof. — This is lemma 9.25 of [Co102].

If $V$ is a de Rham $F$-linear representation of $G_K$, we can $\otimes F$ the above sequence with $V$ and we get a connecting homomorphism $\exp_{K,V} : (B_{d}\otimes_F V)^{G_K} \to H^1(K,V)$. Recall that if $W$ is an $F$-vector space, there is a natural injective map $W \otimes_F V \to W \otimes_{Q^p} V$.

Lemma 3.2.2. — If $V$ is $F$-analytic, the map $\exp_{K,V} : (B_{d}\otimes_F V)^{G_K} \to H^1(K,V)$ defined above coincides with Bloch-Kato’s exponential via the inclusion $(B_{d}\otimes_F V)^{G_K} \subset (B_{d}\otimes_{Q^p} V)^{G_K}$, and its image is in $H^1_{an}(K,V)$.

Proof. — Bloch and Kato’s exponential is defined as follows (definition 3.10 of [BK90]): if $\varphi_p$ denotes the Frobenius map that lifts $x \mapsto x^p$ and if $x \in (B_{d}\otimes_{Q^p} V)^{G_K}$, there exists $\tilde{x} \in B_{\text{max},Q^p}^{+} \otimes_{Q^p} V$ such that $\tilde{x} - x \in B_{d}\otimes_{Q^p} V$, and $\exp(x)$ is represented by the cocycle $g \mapsto (g - 1)\tilde{x}$.

Lemma 3.2.1 says that we can lift $x \in (B_{d}\otimes_F V)^{G_K}$ to some $\tilde{x} \in (B_{\text{max},F}[1/t\tau])^{\otimes 1} \otimes F V$ such that $\tilde{x} - x \in B_{d}\otimes_F V \subset B_{d}\otimes_{Q^p} V$. In addition, $B_{\text{max},Q^p}^{+} = F_{1} \otimes_{Q^p} B_{\text{max},Q^p}^{+}$ (see lemma 1.1.11 of [Ber08]) so that $(B_{\text{max},F}[1/t\tau])^{\otimes 1} \subset F \otimes_{Q^p} B_{\text{max},Q^p}^{+}$. We can therefore view $\tilde{x}$ as an element of $B_{\text{max},Q^p}^{+} \otimes_{Q^p} V$, and $\exp_{K,V}(x) = [g \mapsto (g - 1)\tilde{x}] = \exp(x)$.

The construction of $\exp_{K,V}(x)$ shows that the cocycle $\exp_{K,V}(x)$ is de Rham. At each embedding $\tau \neq \text{Id of } F$, the extension of $F$ by $V$ given by $\exp_{K,V}(x)$ is therefore Hodge-Tate with weights 0. This finishes the proof of the lemma.

Recall the following theorem of Kato (see §II.1 of [Kat93]).

Theorem 3.2.3. — If $V$ is a de Rham representation, the map from $(B_{d}\otimes_{Q^p} V)^{G_K}$ to $H^1(K,B_{d}\otimes_{Q^p} V)$ defined by $x \mapsto [g \mapsto \log(\chi_{\text{cyc}}(g))x]$ is an isomorphism, and the dual exponential map $\exp_{K,V^*}^{(1)} : H^1(K,V) \to (B_{d}\otimes_{Q^p} V)^{G_K}$ is equal to the composition of the map $H^1(K,V) \to H^1(K,B_{d}\otimes_{Q^p} V)$ with the inverse of this isomorphism.

Concretely, if $c \in Z^1(K,B_{d}\otimes_{Q^p} V)$ is some cocycle, there exists $w \in B_{d}\otimes_{Q^p} V$ such that $c(g) = \log(\chi_{\text{cyc}}(g)) \cdot \exp_{K,V^*}^{(1)}(c) + (g - 1)(w).$
Corollary 3.2.4. — If \( c \in \mathbb{Z}^1(K, B_{\text{dr}} \otimes_F V) \), and if there exist \( x \in (B_{\text{dr}} \otimes_F V)^{G_K} \) and \( w \in B_{\text{dr}} \otimes_F V \) such that \( c(g) = t(\overline{g}) \cdot x + (g - 1)(w) \), then \( \exp_{K^*V^*(1)}^*(c) = x \).

Proof. — This follows from theorem 3.2.3 and from the fact that \( g \mapsto \log(\chi(\overline{g})/\chi_{\text{cyc}}(\overline{g})) \) is \( B_{\text{dr}} \)-admissible, since \( t_{\pi}/t \in (B_{\text{dr}}^+) \times \) so that \( \log(t_{\pi}/t) \in B_{\text{dr}}^+ \) is well-defined.

\[ \square \]

3.3. Interpolating exponentials and their duals. — Let \( V \) be an \( F \)-analytic crystalline representation. By theorem 3.1.1, we have \( D_{\text{rig}}^+(V)_{\psi=1} \subset B_{\text{rig},F}[1/t_\pi] \otimes_F D_{\text{cris}}(V) \).

Let \( \partial_V \) denote the map \( \partial_D \) of §2.4 for \( D = D_{\text{cris}}(V) \).

Theorem 3.3.1. — If \( y \in D_{\text{rig}}^+(V)_{\psi=1} \), then

\[
\exp_{F_{n+1,V^*}}^*(h_{F_{n+1,V}}^1(y)) = \begin{cases} 
q^{-n}\partial_V(\varphi_q^{-n}(y)) & \text{if } n \geq 1 \\
(1 - q^{-1}\varphi_q^{-1})\partial_V(y) & \text{if } n = 0.
\end{cases}
\]

Proof. — Since the diagram

\[
\begin{array}{ccc}
H^1(F_{n+1}, V) & \xrightarrow{\exp_{F_{n+1,V^*}}^*} & F_{n+1} \otimes_F D_{\text{cris}}(V) \\
\cor_{F_{n+1}/F_{n}} & & \downarrow \text{Tr}_{F_{n+1}/F_{n}} \\
H^1(F_{n}, V) & \xrightarrow{\exp_{F_{n,V^*}}^*} & F_{n} \otimes_F D_{\text{cris}}(V)
\end{array}
\]

is commutative, we only need to prove the theorem when \( n \geq n(F) \) by lemma 2.4.3 and proposition 2.5.8. By theorem 2.5.8 we have

\[
h_{F_{n+1},V^*}^1(y)(b_j^k) = \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}v}{\prod_i (b_i - 1)}(y) - (b_j^k - 1)z,
\]

with \( z \in \tilde{B}_{\text{rig}} \otimes_F V \) so that if \( m \gg 0 \), then \( \varphi_q^{-m}(z) \in B_{\text{dr}} \otimes_F V \) (see §3 of [Ber16] and §2.2 of [Ber02]). Moreover, \( \varphi_q^{-m}(y) \in F_m((t_\pi)) \otimes_F D_{\text{cris}}(V) \).

Let \( W = \{ w \in F_m((t_\pi)) \otimes_F D_{\text{cris}}(V) \} \) such that \( \partial_V(w) = 0 \). The operator \( \nabla \) is bijective on \( W \), and \( F_m((t_\pi)) \otimes_F D_{\text{cris}}(V) \) injects into \( B_{\text{dr}} \otimes_F V \), hence there exists \( u \in B_{\text{dr}} \otimes_F V \) such that

\[
h_{F_{n+1},V^*}^1(y)(b_j^k) = \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}v}{\prod_i (b_i - 1)}(\partial_V(\varphi_q^{-m}(y))) - (b_j^k - 1)u
\]

by lemmas 2.4.1 and 2.3.3. This proves the theorem by corollary 3.2.4.

\[ \square \]

We now give explicit formulas for \( \exp_{F_{n},V^*} \). Take \( h \geq 0 \) such that \( \text{Fil}^{-h}D_{\text{cris}}(V) = D_{\text{cris}}(V) \), and that \( t_{\pi}^h(B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)) \subset D_{\text{rig}}^+(V) \) (in the notation of §2.2 of [KR09], we have \( t_{\pi}^h(B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)) \subset M(D_{\text{cris}}(V)) \)). In particular, if \( y \in (B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V))_{\psi=1} \), then \( \nabla_{h-1} \circ \cdots \circ \nabla_{0}(y) \in D_{\text{rig}}^+(V)_{\psi=1} \).
Theorem 3.3.2. — If \( y \in (B_{\rig,F}^+ \otimes_F D_{\cris}(V))^{\psi_q=1} \), then

\[
h_{F_{n,V}}^1(\nabla_{h-1} \circ \cdots \circ \nabla_0(y))(b_j^k) = (\ell^*(b_j) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i - 1)}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) - (b_j^k - 1)z)
\]

so that \( h_{F_{n,V}}^1(\nabla_{h-1} \circ \cdots \circ \nabla_0(y))(g) = (g - 1)(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - (g - 1)z \) if \( g \in \Gamma_K \).

By lemma 2.4.2, we have

\[
(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)((\varphi_q - 1)y) \in (t_\pi/\varphi_q^n(T))^k(B_{\rig,F}^+ \otimes_F D_{\cris}(V))^{\psi_q=0} \subset D^{\dagger}_{\rig}(V)^{\psi_q=0},
\]

so that in the notation of theorem 2.5.8 \( m_c = (\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)((\varphi_q - 1)y) \).

Since \( (\varphi_q - 1)z = m_c \), we have \( (\varphi_q - 1)((\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - z) = 0 \), and therefore

\[
(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - z \in (B_{\rig}[1/t_\pi])^{\varphi_q=1} \otimes_F V
\]

The ring \( B_{\rig}^\dagger \) contains \( B_{\max,F}^+ \) and the inclusion \( (B_{\max,F}^+[1/t_\pi])^{\varphi_q=1} \subset (B_{\rig}[1/t_\pi])^{\varphi_q=1} \) is an equality (proposition 3.2 of [Ber02]). This implies that

\[
(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - z \subset (B_{\max,F}^+[1/t_\pi])^{\varphi_q=1} \otimes_F V.
\]

Moreover, we have \( z \in B_{\rig}^\dagger \otimes_F V \) so that if \( m \gg 0 \), then \( \varphi_q^{-m}(z) \in B_{\dR}^\dagger \otimes_F V \). In addition, \( \varphi_q^{-m}(y) \) belongs to \( F_{m}[t_\pi] \otimes_F D_{\cris}(V) \), so that \( \varphi_q^{-m}(y) - \partial_V(\varphi_q^{-m}(y)) \) belongs to \( t_\pi F_{m}[t_\pi] \otimes_F D_{\cris}(V) \) and therefore

\[
(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)\left(\varphi_q^{-m}(y) - \partial_V(\varphi_q^{-m}(y))\right) \in t_\pi F_{m}[t_\pi] \otimes_F D_{\cris}(V)
\]

\[
\subset B_{dR}^\dagger \otimes_F V.
\]

We can hence write

\[
h_{F_{n,V}}^1(\nabla_{h-1} \circ \cdots \circ \nabla_0(y))(g) = (g - 1)(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b \circ \partial_V(\varphi_q^{-m}(y)) - (g - 1)u,
\]
with \( u \in B_{dR}^+ \otimes_F V \). The theorem now follows from the fact that
\[
\Theta_b \circ \partial_V (\varphi_q^{-m}(y)) = q^{-n} \partial_V (\varphi_q^{-n}(y)) \in F_n \otimes_F D_{cris}(V)
\]
by lemmas 2.4.2 and 2.4.3, that \( \nabla_{h-1} \circ \cdots \circ \nabla_1 = (-1)^{h-1}(h-1)! \) on \( F_n \otimes_F D_{cris}(V) \), and from the reminders given in §3.2, in particular the fact that \( \exp_{K,V} \) is the connecting homomorphism when tensoring the exact sequence of lemma 3.2.1 with \( V \) and taking Galois invariants.

\[
\square
\]

3.4. Kummer theory and the representation \( F(\chi_\pi) \). — Throughout this section, \( V = F(\chi_\pi) \). Let \( L \subset \mathbb{Q}_p \) be an extension of \( K \). The Kummer map \( \delta : LT(m_L) \to H^1(L,V) \) is defined as follows. Choose a generator \( u = (u_k)_{k \geq 0} \) of \( T \pi \text{LT} = \varprojlim \text{LT}[\pi^k] \). If \( x \in \text{LT}(m_L) \), let \( x_k \in \text{LT}(m_{\mathbb{Q}_p}) \) be such that \([\pi^k](x_k) = x\). If \( g \in G_L \), then \( g(x_k) - x_k \in \text{LT}[\pi^k] \) so that we can write \( g(x_k) - x_k = [c_k(g)](u_k) \) for some \( c_k(g) \in \mathcal{O}_F/\pi^k \). If \( c(g) = (c_k(g))_{k \geq 0} \in \mathcal{O}_F \) then \( \delta(x) = [g \mapsto c(g)] \in H^1(L,V) \).

If \( x \in \text{LT}(m_L) \), let \( \text{Tr}^{LT}_{L/K} \) be defined by \( \text{Tr}^{LT}_{L/K}(x) = \sum_{g \in \text{Gal}(L/K)} g(x) \) where the superscript \( LT \) means that the summation is carried out using the Lubin-Tate addition. If \( F = \mathbb{Q}_p \) and \( LT = G_m \), we recover the classical Kummer map, and \( \text{Tr}^{LT}_{L/K}(x) = N_{L/K}(1+x) - 1 \).

**Lemma 3.4.1.** — We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{LT}(m_{K_{n+1}}) & \xrightarrow{\delta} & H^1(K_{n+1}, V) \\
\text{Tr}^{LT}_{K_{n+1}/K_n} \downarrow & & \downarrow \text{cor}_{K_{n+1}/K_n} \\
\text{LT}(m_{K_n}) & \xrightarrow{\delta} & H^1(K_n, V).
\end{array}
\]

**Proof.** — This is a straightforward consequence of the explicit description of the corestriction map.

Recall that \( \varphi_q \circ \psi_q(f) = \frac{1}{q} \sum_{\omega \in \text{LT}[\pi]} f(T \oplus \omega) \), so that for \( n \geq 1 \):

\[
\psi_q(f)(u_n) = \frac{1}{q} \sum_{\omega \in \text{LT}[\pi]} f(u_{n+1} \oplus \omega) = \frac{1}{q} \text{Tr}_{F_{n+1}/F_n} f(u_{n+1}).
\]

In particular, if \( f(T) \in B^{+}_{\text{rig},F} \) is such that \( \psi_q(f(T)) = 1/\pi \cdot f(T) \) and \( y_n = f(u_n) \), then \( \text{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \).

**Proposition 3.4.2.** — Assume that \( F \neq \mathbb{Q}_p \). If \( \{y_n\}_{n \geq 1} \) is a sequence with \( y_n \in F_n \) and \( \text{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \), there exists \( f(T) \in B^{+}_{\text{rig},F} \) such that \( \psi_q(f(T)) = 1/\pi \cdot f(T) \) and \( y_n = f(u_n) \) for all \( n \geq 1 \).
Proof. — By [Laz62], there exists a power series \( g(T) \in B^+_{\text{rig}, F} \) such that \( g(u_n) = y_n \) for all \( n \geq 1 \). We also have
\[
\psi_q g(0) = \frac{1}{q} g(0) + \frac{1}{q} \text{Tr}_{F_1/F_0} g(u_1),
\]
and since \( q \neq \pi \) (because \( F \neq Q_p \)), we can choose \( g(0) \) such that
\[
\frac{1}{\pi} g(0) = \frac{1}{q} g(0) + \frac{1}{q} \text{Tr}_{F_1/F_0} y_1.
\]
This implies that \( (\psi_q(g) - 1/\pi \cdot g)(u_n) = 0 \) for all \( n \geq 0 \), so that \( \psi_q(g) - 1/\pi \cdot g \in t_\pi \cdot B^+_{\text{rig}, F} \).

It is therefore enough to prove that \( \psi_q - 1/\pi : t_\pi \cdot B^+_{\text{rig}, F} \to t_\pi \cdot B^+_{\text{rig}, F} \) is onto. Since \( \psi_q(t_\pi f) = 1/\pi \cdot t_\pi \psi_q(f) \), this amounts to proving that \( \psi_q - 1 : B^+_{\text{rig}, F} \to B^+_{\text{rig}, F} \) is onto, which follows from corollary 2.3.3.

\[\square\]

**Definition 3.4.3.** — Let \( S \) denote the set of sequences \( \{x_n\}_{n \geq 1} \) with \( x_n \in m_{F_n} \) and \( \text{Tr}^{LT}_{F_{n+1}/F_n}(x_{n+1}) = [g/\pi](x_n) \) for \( n \geq 1 \).

The following proposition says that if \( F \neq Q_p \), then \( S \) is quite large: for any \( k \geq 1 \), the “\( k \)-th component” map \( F \otimes_{O_F} S \to F_k \) is surjective (if \( F = Q_p \), there are restrictions on “universal norms”).

**Proposition 3.4.4.** — Assume that \( F \neq Q_p \). If \( z \in m_{F_k} \), there exists \( \ell \geq 0 \) and \( x \in S \) such that \( x_k = [\pi^\ell](z) \).

**Proof.** — We claim that \( \text{Tr}_{F_{n+1}/F_n}(O_{F_{n+1}}) = \pi O_{F_n} \). Indeed, let \( D \) denote the different. We have (see for instance proposition 7.11 of [Iwa86])
\[
\text{val}_p(D_{F_{n+1}/F_n}) = \frac{1}{e} \left( n + 1 - \frac{1}{q-1} \right) - \frac{1}{e} \left( n - \frac{1}{q-1} \right) = \text{val}_p(\pi).
\]
This implies that \( \text{Tr}_{F_{n+1}/F_n}(O_{F_{n+1}}) = \pi O_{F_n} \) by proposition 7 of Chapter III of [Ser68].

Since \( \pi \) divides \( q/\pi \), this shows that given \( y \in O_{F_k} \), there exists a sequence \( \{y_n\}_{n \geq 1} \) with \( x_n \in O_{F_n} \) such that \( y_k = y \), and \( \text{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \) for \( n \geq 1 \). Take \( \ell_1, \ell_2 \geq 0 \) such that \( \pi^{\ell_1} O_{Q_p} \) is in the domain of \( \exp^{LT} \) and such that \( \pi^{\ell_2} \log^{LT}(z) \in O_{F_k} \).

Let \( y = \pi^{\ell_2} \log^{LT}(z) \). Let \( \{y_n\}_{n \geq 1} \) be a sequence as above, let \( x_n = \exp^{LT}(\pi^{\ell_1} y_n) \) and \( \ell = \ell_1 + \ell_2 \). The elements \( x_k \otimes [\pi^\ell](z) \), as well as \( \text{Tr}^{LT}_{F_{n+1}/F_n}(x_{n+1}) \otimes [q/\pi](x_n) \) for all \( n \), have their \( \log^{LT} \) equal to zero and are in a domain in which \( \log^{LT} \) is injective. This proves the proposition.

If \( x \in S \) and \( y_n = \log^{LT}(x_n) \), then \( y_n \in F_n \) and \( \text{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \), so that by proposition 3.4.2, there exists \( f(T) \in B^+_{\text{rig}, F} \) such that \( \psi_q(f(T)) = \pi^{-1} \cdot f(T) \) and \( y_n = f(u_n) \) for all \( n \geq 1 \). If \( f(T) \in B^+_{\text{rig}, F} \) is such that \( \psi_q(f(T)) = \pi^{-1} \cdot f(T) \), then \( \partial f \in (B^+_{\text{rig}, F})_{\psi_q=1} \) and \( \partial f \cdot u \) can be seen as an element of \( D^\dagger_{\text{rig}}(V)_{\psi_q=1} \).
Theorem 3.4.5. — If \( x \in S \), and if \( f(T) \in B_{\text{rig},F}^+ \) is such that \( f(u_n) = \log_{\text{LT}}(x_n) \) and \( \psi_q(f(T)) = \pi^{-1} \cdot f(T) \), then \( h^1_{F,u,v}(\partial f(T) \cdot u) = (q/\pi)^n \cdot \delta(x_n) \) for all \( n \geq 1 \).

Proof. — Let \( y = f(T) \otimes t_n^{-1}u \), so that \( y \in (B_{\text{rig},F}^+ \otimes F D_{\text{cris}}(V))^{\psi_q = 1} \). By theorem 3.3.2 applied to \( y \) with \( h = 1 \), we have \( h^1_{F,u,v}(\nabla(y)) = \exp_{F,u,v}(q^{-n} \partial_v(\varphi_q^{-n}(y))) \) if \( n \geq 1 \). Since \( \varphi_q^{-n} \circ \partial = \pi^n \cdot \partial \circ \varphi_q^{-n} \), this implies that

\[
h^1_{F,u,v}(\partial f(T) \cdot u) = \exp_{F,u,v}(q^{-n} \partial_v(\varphi_q^{-n}(y))) = (q/\pi)^n \cdot \exp_{F,u,v}(\log_{\text{LT}}(x_n) \cdot u).
\]

By example 3.10.1 of [BK90] and lemma 3.2.2, we have \( \delta(x_n) = \exp_{F,u,v}(\log_{\text{LT}}(x_n) \cdot u) \). This proves the theorem.

Remark 3.4.6. — If \( F = Q_p \) and \( \pi = q = p \) and \( x = \{x_n\}_{n \geq 1} \), this theorem says that \( \text{Exp}^*_{Q_p}((\delta(x)) = \partial \log \text{Col}_F(T) \), which is (ii) of proposition V.3.2 of [CC99] (see theorem II.1.3 of ibid for the definition of the map \( \text{Exp}^*_{Q_p} : H^1_{\text{Iw}}(F,Q_p(1)) \to D_{\text{rig}}^1(Q_p(1))^{\psi_q = 1} \)).

Remark 3.4.7. — If \( x \in S \), then by proposition 3.3.2 there is a power series \( f(T) \) such that \( f(u_n) = \log_{\text{LT}}(x_n) \) for \( n \geq 1 \). Is there a power series \( g(T) \in \mathcal{O}_F[T] \) such that \( g(u_n) = x_n \), so that \( f(T) = \log g(T) \)?

If \( F = Q_p \), such a power series is the classical Coleman power series [Col79]. If \( F \neq Q_p \) and \( x \in S \) and \( z \) is a \([q/\pi]\)-torsion point, and \( k \geq d - 1 \) so that \( z \in F_k \), then the sequence \( x' = \{x'_n\}_{n \geq 1} \) defined by \( x_n' = x_n \) if \( n \neq k \) and \( x'_n = x_k \oplus z \) also belongs to \( S \). This means that we cannot naïvely interpolate \( x \).

3.5. Perrin-Riou’s big exponential map. — In this last section, we explain how the explicit formulas of the previous sections can be used to give a Lubin-Tate analogue of Perrin-Riou’s “big exponential map” [PR94]. Take \( h \geq 1 \) such that \( \text{Fil}^h \text{D}_{\text{cris}}(V) = \text{D}_{\text{cris}}(V) \). If \( f \in B_{\text{rig},F}^+ \otimes F D_{\text{cris}}(V) \), let \( \Delta(f) \) be the image of \( \oplus_{k=0}^{h} \partial^k(f)(0) \) in \( \oplus_{k=0}^{h} D_{\text{cris}}(V)/(1 - \pi^k \varphi_q) \).

Lemma 3.5.1. — There is an exact sequence:

\[
0 \to \bigoplus_{k=0}^{h} D_{\text{cris}}(V)^{\varphi_q = \pi^{-k}} \to \left( B_{\text{rig},F}^+ \otimes F D_{\text{cris}}(V) \right)^{\psi_q = 1} \to \left( B_{\text{rig},F}^+ \otimes F D_{\text{cris}}(V) \right)^{\psi_q = 0} \to D_{\text{cris}}(V) \oplus \bigoplus_{k=0}^{h} D_{\text{cris}}(V)/(1 - \pi^k \varphi_q) \to 0.
\]

Proof. — Note that the map \( \varphi_q \) acts diagonally on tensor products. It is easy to see that \( \ker(1 - \varphi_q) = \oplus_{k=0}^{h} D_{\text{cris}}(V)^{\varphi_q = \pi^{-k}} \), that \( \Delta \) is surjective, and that \( \text{im}(1 - \varphi_q) \subset \ker \Delta \), so we now prove that \( \text{im}(1 - \varphi_q) = \ker \Delta \).


If \( f, g \in B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V) \) and \( f = (1 - \varphi_q)g \), then \( \psi_q(f) = 0 \) if and only if \( \psi_q(g) = g \). It is therefore enough to show that if \( f \in B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V) \) is such that \( \Delta(f) = 0 \), then \( f = (1 - \varphi_q)g \) for some \( g \in B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V) \).

The map \( 1 - \varphi_q : T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V) \to T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V) \) is bijective because the slopes of \( \varphi_q \) on \( T^{h+1}B_{\text{rig},F}^+ \otimes_F D \) are \( > 0 \). This implies that \( 1 - \varphi_q \) induces a sequence

\[
0 \to \bigoplus_{k=0}^h t^k D_{\text{cris}}(V)^{\varphi_q = \pi^{-k}} \to \frac{B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)}{T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)} \xrightarrow{1 - \varphi_q} \frac{B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)}{T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)} \xrightarrow{\Delta} \bigoplus_{k=0}^h D_{\text{cris}}(V). 
\]

We have \( \ker(1 - \varphi_q) = \bigoplus_{k=0}^h t^k D_{\text{cris}}(V)^{\varphi_q = \pi^{-k}} \) and by comparing dimensions, we see that \( \operatorname{coker}(1 - \varphi_q) = \bigoplus_{k=0}^h D_{\text{cris}}(V)/(1 - \pi^k \varphi_q) \). This and the bijectivity of \( 1 - \varphi_q \) on \( T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V) \) imply the claim. \( \square \)

If \( f \in ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^\Delta = 0 \), then by lemma 3.5.1 there exists \( y \in (B_{\text{rig},F}^+)_{\psi_q = 1} \) such that \( f = (1 - \varphi_q)y \). Since \( \nabla_{h-1} \circ \cdots \circ \nabla_0 \) kills \( \bigoplus_{k=0}^{h-1} \otimes \overline{D_{\text{cris}}(V)}^{\varphi_q = \pi^{-k}} \) we see that \( \nabla_{h-1} \circ \cdots \circ \nabla_0(y) \) does not depend upon the choice of such a \( y \) (unless \( D_{\text{cris}}(V)^{\varphi_q = \pi^{-h}} \neq 0 \)).

**Definition 3.5.2.** — Let \( h \geq 1 \) be such that \( \text{Fil}^{-h}D_{\text{cris}}(V) = D_{\text{cris}}(V) \) and such that \( D_{\text{cris}}(V)^{\varphi_q = \pi^{-h}} = 0 \). We deduce from the above construction a well-defined map:

\[
\Omega_{V,h} : ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^\Delta = 0 \to D_{\text{rig},F}^+(V)^{\psi_q = 1},
\]

given by \( \Omega_{V,h}(f) = \nabla_{h-1} \circ \cdots \circ \nabla_0(y) \) where the element \( y \in (B_{\text{rig},F}^+)_{\psi_q = 1} \) is such that \( f = (1 - \varphi_q)y \) and is provided by lemma 3.5.1.

If \( D_{\text{cris}}(V)^{\varphi_q = \pi^{-h}} \neq 0 \), we get a map

\[
\Omega_{V,h} : ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^\Delta = 0 \to D_{\text{rig},F}^+(V)^{\psi_q = 1}/V^{G_F = \chi_k^2}.
\]

Let \( u \) be a basis of \( F(\chi_n) \) as above, and let \( e_j = u^\otimes j \) if \( j \in \mathbb{Z} \).

**Theorem 3.5.3.** — Take \( y \in (B_{\text{rig},F}^+)_{\psi_q = 1} \) and let \( h \geq 1 \) be such that \( \text{Fil}^{-h}D_{\text{cris}}(V) = D_{\text{cris}}(V) \). Let \( f = (1 - \varphi_q)y \) so that \( f \in ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^\Delta = 0 \).

If \( j \in \mathbb{Z} \) and \( h + j \geq 1 \), then

\[
h_{F_n,V(\chi_n^k)}^1(\Omega_{V,h}(f) \otimes e_j) = (-1)^{h+j-1}(h + j - 1)! x \begin{cases} \exp_{F_n,V(\chi_n^k)}(q^n \partial_{V(\chi_n^k)}(\varphi_q^{-n}(\partial^{-j}y \otimes \tau^{-j}e_j))) & \text{if } n \geq 1 \\
- \exp_{F,\hat{V}(\chi_n^k)}((1 - q^{-1} \varphi_q^{-1})\partial_{V(\chi_n^k)}(\partial^{-j}y \otimes \tau^{-j}e_j)) & \text{if } n = 0. \end{cases}
\]
If \( j \in \mathbb{Z} \) and \( h + j \leq 0 \), then

\[
\exp_{F_n,V^{(1-j)}}^*(h_{F_n,V}(\chi^j_n))(\Omega_{V,h}(f) \otimes e_j) = \\
\frac{1}{(-h-j)!} \begin{cases} 
q^{-n} \partial_{V}(\varphi_q^{-n}(\partial^{-j}y \otimes t_j^{-j}e_j)) & \text{if } n \geq 1 \\
(1 - q^{-1} \varphi_q^{-1}) \partial_{V}(\partial^{-j}y \otimes t_j^{-j}e_j) & \text{if } n = 0.
\end{cases}
\]

Proof. — If \( h + j \geq 1 \), the following diagram is commutative:

\[
\begin{array}{ccc}
D_{\rig}^\dagger(V)^{\psi_q=1} & \xrightarrow{\otimes e_j} & D_{\rig}^\dagger(V(\chi^j_n))^{\psi_q=1} \\
\nabla_{h-1} \circ \ldots \circ \nabla_0 \uparrow & & \nabla_{h+j-1} \circ \ldots \circ \nabla_0 \uparrow \\
\left( B_{\rig,F}^\dagger \otimes F D_{\cris}(V) \right)^{\psi_q=1} & \xrightarrow{\partial^{-j} \otimes t_j^{-j}e_j} & \left( B_{\rig,F}^\dagger \otimes F D_{\cris}(V(\chi^j_n)) \right)^{\psi_q=1},
\end{array}
\]

and the theorem is a straightforward consequence of theorem 3.3.2 applied to \( \partial^{-j}y \otimes t_j^{-j}e_j \), \( h + j \) and \( V(\chi^j_n) \) (which are the \( j \)-th twists of \( y \), \( h \) and \( V \)).

If \( h + j \leq 0 \), and \( \Gamma_{F_n} \) is torsion free, then theorem 3.3.1 shows that

\[
\exp_{F_n,V^{(1-j)}}^*(h_{F_n,V}(\chi^j_n))(\nabla_{h-1} \circ \ldots \circ \nabla_0(y) \otimes e_j) = q^{-n} \partial_{V}(\varphi_q^{-n}(\nabla_{h-1} \circ \ldots \circ \nabla_0(y) \otimes e_j))
\]

in \( D_{\cris}(V(\chi^j_n)) \), and a short computation involving Taylor series shows that

\[
\partial_{V}(\varphi_q^{-n}(\nabla_{h-1} \circ \ldots \circ \nabla_0(y) \otimes e_j)) = (-h-j)!^{-1} \partial_{V}(\varphi_q^{-n}(\partial^{-j}y \otimes t_j^{-j}e_j)).
\]

To get the other \( n \), we corestrict. \( \square \)

Corollary 3.5.4. — We have \( \Omega_{V,h}(x) \otimes e_j = \Omega_{V(\chi^j_n),h+j}(\partial^{-j}x \otimes t_j^{-j}e_j) \) and \( \nabla_h \circ \Omega_{V,h}(x) = \Omega_{V,h+1}(x) \).

Remark 3.5.5. — The notation \( \partial^{-j} \) is somewhat abusive if \( j \geq 1 \) as \( \partial \) is not injective on \( B_{\rig,F}^\dagger \) (it is surjective as can be seen by “integrating” directly a power series) but the reader can check that this leads to no ambiguity in the formulas of theorem 3.5.3 above.

If \( F = \mathbb{Q}_p \) and \( \pi = p \), definition 3.5.2 and theorem 3.5.3 are given in §II.5 of [Ber03]. They imply that \( \Omega_{V,h} \) coincides with Perrin-Riou’s exponential map (see theorem 3.2.3 of [PR94]) after making suitable identifications (theorem II.13 of [Ber03]).

Our definition therefore generalizes Perrin-Riou’s exponential map to the \( F \)-analytic setting. We hope to use the results of [Fou05] and [Fou08] to relate our constructions to suitable Iwasawa algebras as in the cyclotomic case.
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