Precis of General Relativity∗†

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Abstract

Omitting the motivations and historical connections, and also the
detailed calculations, I state succinctly the principles that determine
the relativistic idealization of a GPS system. These determine the
results that Ashby presents in his tutorial.

A method for making sure that the relativity effects are specified correctly
(according to Einstein’s General Relativity) can be described rather briefly.
It agrees with Ashby’s approach but omits all discussion of how, historically
or logically, this viewpoint was developed. It also omits all the detailed
calculations. It is merely a statement of principles.

One first banishes the idea of an “observer”. This idea aided Einstein
in building special relativity but it is confusing and ambiguous in general
relativity. Instead one divides the theoretical landscape into two categories.
One category is the mathematical/conceptual model of whatever is happen-
ing that merits our attention. The other category is measuring instruments
and the data tables they provide.

For GPS the measuring instruments can be taken to be either ideal SI
atomic clocks in trajectories determined by known forces, or else electromag-
netic signals describing the state of the clock that radiates the signal. Each

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clock maintains its own proper time (but may convert this via software into other information when it transmits). We simplify to assume it transmits its own proper time without random or systematic errors, so that its increments \( d\tau_T \) are simple physical data. Any other clock receiving these signals can record data tables showing the increments \( d\tau_R \) in the SI proper time of the clock at the receiver corresponding to differences \( d\tau_T \) in the proper times encoded in the signal it receives from some other identified transmitting clock. Once conventional zeros of time are identified for each clock, each transmitting and receiving pair produces a data table \( \tau_R(\tau_T) \). These segments of data are to be reproduced by computations from the conceptual model, with any residuals understood on the basis of expected sources of noise and unmodelled phenomena.

A user “fix” or relativistic “event” is the simultaneous reception of signals from four GPS satellites, or its equivalent from short extrapolations from nearly simultaneous signals. This user may not have a reliable clock but should be able to determine the time and position of the event from knowledge of the proper times encoded in the received signals, the identities of the transmitters, and the mathematical/conceptual model that defines the meaning of time and position for this purpose. System software aims to make the user calculations standard and practical, with many of the computational results encoded in the transmitted signals.

What is the conceptual model? It is built from Einstein’s General Relativity which asserts that spacetime is curved. This means that there is no precise intuitive significance for time and position. [Think of a Caesarian general hoping to locate an outpost. Would he understand that 600 miles North of Rome and 600 miles West could be a different spot depending on whether one measured North before West or visa versa?] But one can draw a spacetime map and give unambiguous interpretations. [On a Mercator projection of the Earth, one minute of latitude is one nautical mile everywhere, but the distance between minute tics varies over the map and must be taken into account when reading off both NS and EW distances.] There is no single best way to draw the spacetime map, but unambiguous choices can be made and communicated, as with the Mercator choice for describing the Earth.

The conceptual model for a relativistic system is a spacetime map or diagram plus some rules for its interpretation. For GPS the attached Figure is a simplified version of the map. The real spacetime map is a computer program that assigns map locations \( xyzt \) to a variety of events. In the Figure the \( t \) time axis is vertical, and two of the three space \( xyz \) axes are suggested
horizontally. The wide center swath is the Earth which occupies the same location, centered on the central axis of the map, at all times. Marked on the surface of the Earth is a long spiral representing, e.g., a clock at USNO. The position of this clock as the Earth rotates is described by the coordinates of this curve on the (corresponding conceptual four dimensional) map, \( x(t), y(t), z(t) \) where \( xytz \) are distances measured by a Euclidean ruler on the (conceptual four dimensional) graph paper parallel to its axes. The scale factors needed for interpreting this spacetime map are provided by the metric. In the map projection (coordinate system) from which the GPS model starts (an Earth Centered Inertial coordinate system, ECI) the metric is

\[
d\tau^2 = \left[1 + 2(V - \Phi_0)/c^2\right]dt^2 - \left[1 - 2V/c^2\right](dx^2 + dy^2 + dz^2)/c^2. \tag{1}
\]

Here \( V \) is the Newtonian gravitational potential of the Earth, approximately

\[
V = -(GM/r)[1 - \frac{1}{2}J_2(R/r)^2(3\cos^2\theta - 1)]. \tag{2}
\]

The constant \( \Phi_0 \) is chosen so that a standard SI clock “on the geoid” (e.g., USNO were it at sea level) would give, inserting its world line \( x(t), y(t), z(t) \) into equation (1), just \( d\tau = dt \) where \( d\tau \) is the physical proper time reading of the clock. It is a theorem that if this choice is made for one clock on the geoid it applies to all.

Equation (1) defines not only the gravitational field that is assumed, but also the coordinate system in which it is presented. There is no other source of information about the coordinates apart from the expression for the metric. It is also not possible to define the coordinate system unambiguously in any way that does not require a unique expression for the metric. In most cases where the coordinates are chosen for computational convenience, the expression for the metric is the most efficient way to communicate clearly the choice of coordinates that is being made. Mere words such as “Earth Centered Inertial coordinates” are ambiguous unless by convention they are understood to designate a particular expression for the metric, such as equation (1).

Using equation (1) one can place tic marks along the world line of any clock to show changes in its proper time (which are to be physical changes directly displayed and transmitted by the clock). The computation is just to insert the clock trajectory \( x(t), y(t), z(t) \) to find \( d\tau \) from equation (1) as a thus specified multiple of \( dt \). This applies both to Earth fixed clocks, to satellite clocks, and to clocks with any other motion \( x(t), y(t), z(t) \) that has
been incorporated in the map. The “map” here means a computer program that is designed to produce the trajectories $x(t), y(t), z(t)$ of each modelled object.

The rules for drawing clock world lines or trajectories on the spacetime map (in the computer program) are simplest for dragfree satellites and for electromagnetic signals in vacuum. In these cases the world line must be a (timelike, resp. lightlike) geodesic of the metric (1), i.e., a solution of an ordinary differential equation constructed using the coefficients (scale factors) in equation (1). The electromagnetic signals have the special property that their trajectories also satisfy $d\tau^2 = 0$ in equation (1). By finding a lightlike geodesic that connects one tic mark $\tau_T$ on one clock world line to another mark $\tau_R$ on another clock, the map shows how one entry in the physical data table $\tau_R(\tau_T)$ is computed in the mathematical model. Once the observed data tables are being reproduced adequately in the mathematical model, its assignments of $xyzt$ coordinates to events identify the time and position of those events.

In sum, the $txyz$ time and position values provided by GPS are not simple physical times and positions. Physical times and positions exist but, due to spacetime curvature, cannot be naturally associated with quadruples of numbers. Physical times and positions are identified on a spacetime map by their $xyzt$ map coordinates which depend on the “projection” (coordinate system) chosen in designing that particular map. The ECI map defined by equation (1) is the simplest to describe. More practical maps have been defined in which the space coordinates of geodetic benchmarks on Earth are nearly constant and change only due to tectonic and volcanic activity. To identify such an Earth fixed coordinate system one gives these coordinates as specified functions of those used in the ECI metric. This results in a metric expression different from equation (1) and allows results computed in the ECI coordinate system to be reported in the second coordinate system.

**Figure**

This spacetime diagram shows the Earth, a fixed location (USNO) on the rotating Earth, a satellite orbiting the Earth, and an electromagnetic (EM) signal propagating from an event T on the satellite’s world line to an event R on the USNO world line. Two of the three $xyz$ space axes are indicated. The $t$ time axis is at the center of the Earth. Any point on this diagram or map can be located by its $xyzt$ coordinates which are measured along the
coordinate axes as conventional Cartesian coordinates for points (events) on this map. To deduce physical separations between (nearby) points on the map one must use equation (1) to convert the separations $dx\,dy\,dz\,dt$ read from the map into a physically measurable proper time interval $d\tau$.

References

Neil Ashby, “A tutorial on Relativistic Effects in the Global Positioning System”, NIST Contract No.40RANB9B8112, February 1990.

Neil Ashby, “Relativistic Effects in the Global Positioning System”, NIST Contract No.40RANB9B8112, August 1995.
