ALGEBRAIC VECTOR BUNDLES
OVER REAL ALGEBRAIC VARIETIES

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By an affine algebraic variety, we mean in this note a locally ringed space
\((X, R_X)\) which is isomorphic to a ringed space of the form \((V, R_V)\), where \(V\) is
a Zariski closed subset in \(R^n\) and \(R_V\) is the sheaf of rings of regular functions
on \(V\). Recall that \(R_V(V)\) is the localization of the ring of polynomial functions
on \(V\) with respect to the multiplicatively closed subset consisting of functions
vanishing nowhere on \(V\) \([2, 15]\).

Let \(F\) be one of the fields \(R, C\) or \(H\) (quaternions). A continuous \(F\)-vector
bundle \(E\) over \(X\) is said to admit an algebraic structure if there exists a finitely
generated projective module \(P\) over the ring \(R_X(X) \otimes_R F\) such that the \(F\)-
vector bundle over \(X\), associated with \(P\) in the standard way, is \(C^0\) isomorphic
to \(E\).

Our purpose is to study the following

PROBLEM. Characterize continuous \(F\)-vector bundles over \(X\) which admit
an algebraic structure.

This is an old problem, but despite considerable effort, the situation is well
understood only in a few special cases: when \(X\) is the unit sphere \(S^n\) \([4, 16]\),
when \(X\) is the real projective space \(RP^n\) \([5, 7]\) and when \(\dim X \leq 3\) and
\(F = R\) \([8, 9]\) (cf. also \([13]\) for a short survey).

Clearly, \(R. P^n\) with its natural structure of an abstract real algebraic variety
is actually an affine variety and every affine real algebraic variety admits a
locally closed embedding in some \(R. P^n\).

Let us first consider \(C\)-vector bundles.

Let \(X\) be an affine nonsingular real algebraic variety and assume for a mo­
moment that \(X\) is embedded in \(R. P^n\) as a locally closed subvariety. Consider
\(R. P^n\) as a subset of the complex projective space \(CP^n\). Let \(U\) be a Zariski
neighborhood of \(X\) in the set of nonsingular points of the Zariski (complex)
closure of \(X\) in \(CP^n\). Denote by \(H^{even}_{algl}(U, Z)\) the subgroup of the coho­
mology group \(H^{even}(U, Z)\) generated by the cohomology classes which are
Poincaré dual to the homology classes in the Borel-Moore homology group
\(H^{even}_{algl}(U, Z)\) represented by the closed irreducible complex algebraic subvari­
eties of \(U\) (cf. \([3]\)). Let \(H^{even}_{algl}(X, Z)\) be the image of \(H^{even}_{algl}(U, Z)\) via the
restriction homomorphism \(H^{even}_{algl}(U, Z) \rightarrow H^{even}(X, Z)\). One easily checks
that \(H^{even}_{algl}(X, Z)\) does not depend on the choice of \(U\) or the choice of the
embedding of \(X\) in \(R. P^n\).

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THEOREM 1. Let $X$ be an affine nonsingular real algebraic variety and let $\xi$ be a continuous $\mathbb{C}$-vector bundle over $X$. If $\xi$ admits an algebraic structure, then the total Chern class $c(\xi)$ of $\xi$ belongs to $H^\text{even}_{\text{alg}}(X, \mathbb{Z})$. Conversely, $\xi$ admits an algebraic structure, provided that $c(\xi)$ belongs to $H^\text{even}_{\text{alg}}(X, \mathbb{Z})$, $X$ is compact, $\dim X \leq 5$ and $\xi$ is of constant rank.

SKETCH OF PROOF. We can assume that $X$ is a locally closed subvariety in $\mathbb{R}P^n$. Suppose that $\xi$ admits an algebraic structure. Then one can find a Zariski neighborhood $U$ of $X$ in the Zariski closure of $X$ in $\mathbb{C}P^n$ and an algebraic vector bundle $\tilde{\xi}$ over $U$ such that the restriction $\tilde{\xi}|X$ of $\tilde{\xi}$ to $X$ is $C^0$ isomorphic to $\xi$. It easily follows from [3] that $c(\xi)$ belongs to $H^\text{even}_{\text{alg}}(X, \mathbb{Z})$.

If all assumptions of the second part of Theorem 1 are satisfied, then with the help of the Grothendieck formula (cf. [6, p. 151]), one constructs a continuous $\mathbb{C}$-vector bundle $\eta$ over $X$ such that rank $\eta = 2$, $\eta$ admits an algebraic structure and $c(\eta) = c(\xi)$ (here both assumptions, $c(\xi) \in H^\text{even}_{\text{alg}}(X, \mathbb{Z})$ and $\dim X \leq 5$ are essential). Since $\xi$ is of constant rank, $\xi$ and $\eta$ are stably equivalent [12]. The conclusion follows now from [16, Theorem 2.2].

Our next step is the calculation of the groups $H^k_{\text{alg}}(X, \mathbb{Z})$ for a large class of varieties.

THEOREM 2. Let $X$ be a locally closed nonsingular algebraic subvariety of $\mathbb{R}P^n$ and let $X_C$ be the Zariski closure of $X$ in $\mathbb{C}P^n$. Assume that $X_C$ is nonsingular. Then $H^2_{\text{alg}}(X, \mathbb{Z})$ is equal to the image of the restriction homomorphism

$$H^2(\mathbb{R}P^n, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$$

in each of the following two cases:
(a) $2i \leq 2\dim X - n$.
(b) $X_C$ is an ideal theoretic complete intersection in $\mathbb{C}P^n$ and $2i < \dim X$.

SKETCH OF PROOF. Consider the commutative diagram

$$
\begin{array}{ccc}
H^2i(\mathbb{C}P^n, \mathbb{Z}) & \longrightarrow & H^2i(X_C, \mathbb{Z}) \\
\delta \downarrow & & \beta \downarrow \\
H^2i(\mathbb{R}P^n, \mathbb{Z}) & \longrightarrow & H^2i(X, \mathbb{Z})
\end{array}
$$

where all homomorphisms are the restriction homomorphisms. If $\gamma$ is an isomorphism, then $H^2i(X_C, \mathbb{Z}) = H^2i_{\text{alg}}(X_C, \mathbb{Z})$ and $\beta$ maps $H^2i(X_C, \mathbb{Z})$ onto $H^2i_{\text{alg}}(X, \mathbb{Z})$. Moreover, since $\delta$ is an epimorphism, $H^2i_{\text{alg}}(X, \mathbb{Z})$ is equal to the image of $\alpha$.

If (a) is satisfied, then $\gamma$ is an isomorphism by the Lefschetz theorem [1].

If (b) is satisfied, then $\gamma$ is an isomorphism by the Larsen theorem [10].

Notice that if (b) is satisfied and $\dim X$ is odd, then $H^\text{even}_{\text{alg}}(X, \mathbb{Z})$ is completely determined. For even $\dim X$, the situation is more complicated. Indeed, let

$$V^n = \{[x_0, \ldots, x_n, x_{n+1}] \in \mathbb{R}P^{n+1} | x_0^2 + \cdots + x_n^2 = x_{n+1}^2\}.$$ 

Then the Zariski closure of $V^n$ in $\mathbb{C}P^{n+1}$ is nonsingular and the restriction homomorphism $H^\text{even}(\mathbb{R}P^{n+1}, \mathbb{Z}) \rightarrow H^\text{even}(V^n, \mathbb{Z})$ is the zero homomorphism.
On the other hand, $V^n$ is algebraically isomorphic to $S^n$ and hence every continuous $\mathbb{C}$-vector bundle over $V^n$ admits an algebraic structure \cite{4,16}. It follows from Theorem 1 that $H^n_{C-\text{alg}}(V^n, \mathbb{Z})$ is nontrivial, provided that $n$ is even.

The example above indicates that the case in which $\dim X$ is even can only be handled under some additional assumptions.

Denote by $P(n;k)$ the projective space associated with the vector space of all homogeneous polynomials in $\mathbb{R}[x_0,\ldots,x_n]$ of degree $k$. If an element $H$ in $P(n;k)$ is represented by a polynomial $G$, then $V(H)$ will denote the subvariety of $\mathbb{R}P^n$ defined by $G$.

**Theorem 3.** Let $Y$ be a locally closed algebraic subvariety of $\mathbb{R}P^n$, $\dim Y \geq 2$. Assume that the Zariski closure of $Y$ in $\mathbb{C}P^n$ is a nonsingular ideal theoretic complete intersection. Then there exists a nonnegative integer $k_0$ such that, for every integer $k$ greater than $k_0$, one can find a subset $\Sigma_k$ of $P(n;k)$ which is a countable union of proper subvarieties of $P(n;k)$ and has the property that for every $H$ in $P(n;k)\setminus \Sigma_k$, $V(H)$ is either empty or nonsingular and transverse to $Y$ and the group $H^n_{C-\text{alg}}(Y \cap V(H), \mathbb{Z})$ is equal to the image of the restriction homomorphism

$$H^n(\mathbb{R}P^n, \mathbb{Z}) \to H^n(Y \cap V(H), \mathbb{Z}).$$

In particular, if $Y = \mathbb{R}P^n$, then Theorem 3 determines $H^n_{C-\text{alg}}$ for generic algebraic hypersurfaces in $\mathbb{R}P^n$, $n \geq 2$, of sufficiently high degree.

The proof of Theorem 3 is technically more complicated. Besides the Lefschetz theorem Moishezon’s result \cite[Theorem 5.4]{11} also plays an essential role.

Theorems 2 and 3 show that, in many cases, Theorem 1 imposes severe restrictions on continuous $\mathbb{C}$-vector bundles admitting an algebraic structure.

Among several applications of Theorem 3, we want to select only the simplest one.

**Theorem 4.** Let $n$ be a positive integer. Then there exists a $C^\infty$ embedding $h: S^n \to \mathbb{R}^{n+1}$, arbitrarily close in the $C^\infty$ topology to the inclusion map, and a closed nonsingular algebraic subvariety $X$ in $\mathbb{R}^{n+1}$ such that $h(S^n) = X$ and every continuous $\mathbb{C}$-vector bundle over $X$ admitting an algebraic structure is stably trivial. If $n = 4 \pmod 8$, then also every continuous $\mathbb{R}$- or $\mathbb{H}$-vector bundle over $X$ admitting an algebraic structure is stably trivial.

Theorem 4 is interesting in view of the fact that every continuous $\mathbb{F}$-vector bundle over $S^n$ admits an algebraic structure \cite{4,16}. Let us also mention that every continuous stably trivial $\mathbb{F}$-vector bundle admits an algebraic structure \cite[Theorem 2.2]{16}.

The second part of Theorem 4 immediately implies that Shiota’s conjecture \cite[p. 1007]{14} is false over $X$. Shiota has conjectured that a continuous $\mathbb{R}$-vector bundle $\xi$ of constant rank over an affine nonsingular compact real algebraic variety $Y$ admits an algebraic structure if and only if all Stiefel-Whitney classes of $\xi$ are Poincaré dual to the $\mathbb{Z}/2\mathbb{Z}$-homology classes of $Y$ represented by closed algebraic subvarieties of $Y$. He proved the “only if” part of the
conjecture and the "if" part is established in [8, 9] for vector bundles over surfaces and threefolds.

SKETCH OF THE PROOF OF THEOREM 4. Let $G$ be an element in $P(n+1; 2k+2)$ represented by the homogeneous polynomial
\[(x_0^2 + \cdots + x_n^2 - x_{n+1}^2)(x_0^2 + \cdots + x_n^2 + x_{n+1}^2)^k.\]
If we identify $\mathbb{R}^{n+1}$ with a subset of $\mathbb{R}P^{n+1}$ via the map which sends $(x_0, \ldots, x_n)$ to $[x_0, \ldots, x_n; 1]$, then $S^n = V(G)$. By Theorem 3 (applied to $Y = \mathbb{R}P^{n+1}$ and $k$ sufficiently large) together with Theorem 1, there exists an element $H$ in $P(n+1; 2k+2)$ such that $H$ is arbitrarily close to $G$ and for every continuous $\mathbb{C}$-vector bundle $\xi$ over $X = V(H)$, the total Chern class of $\xi$ is equal to 0. Clearly, there exists a $C^\infty$ embedding $h: S^n \to \mathbb{R}^{n+1}$ which is close to the inclusion map and satisfies $h(S^n) = X$. Since $X$ is diffeomorphic to $S^n$, the vector bundle $\xi$ is stably trivial.

The second part of Theorem 4 follows by considering the complexification and the realification of vector bundles and by using the fact that the reduced Grothendieck group of continuous $\mathbb{F}$-vector bundles over $X$ is isomorphic to $Z$.

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