Mathematical modelling of level ice with continental shelf structures interaction

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Abstract. Development of hydrocarbons deposits on the continental shelf is connected to the construction of floating and fixed objects functioning in high seas conditions continuously. For the safety analysis of such structures not only the knowledge of ice load values is needed, but also such of their frequency and time-dependent characteristics. The purpose of the present work is further development of methodology for interaction dynamics analysis between ice bodies and Arctic shelf structures presented previously in [1-4]. The paper illustrates the realization of dynamic method of calculation of ice floe deformation and movement upon contact with inclined planes of floating or fixed structures. The presented methodology can be used in the research or project work connected to finding time-dependent characteristics of ice loads, their statistical consistencies and dependency on structures’ reaction. The ice field behaviour description algorithm is implemented in the Anchored Structures program. The paper presents some results of the comparison study which prove working efficiency of the presented methodologies.

1. Problem statement and restrictions
When designing methodological procedures, we allow the following assumptions:

1. The ice field is presented as an even flat plate of the length $L_0$, width $B_0$ and thickness $h$. Next, the field in the plan is presented as a discreet spatial model where the elementary rectangular fragments interact with each other along the lines connecting the characteristic points (nodes) placed in the intersecting of the rectangular grid (figure 1).

2. While analyzing the strain-stress state the ice field is regarded as a flexible plate of medium thickness upon the elastic foundation loaded along the contact area. During the interaction ice can be deformed longitudinally and cross-sectionally, as well as bend.

3. Ice is also considered an elastic/fragile feature allowing plastic hinges formation at nodal points when being bent. At the same time, ice crack formation can be caused by its compression, bending, torsion or by a combination of deformations. After cracking the elementary ice floe possesses some residual strength causing the load capacity of the crack to reduce gradually.

2. The mathematical ice field deformation model
To analyze ice field deflection and deformation under vertical and horizontal loads of the inclined surface planes of the structure we use the von Karman equation for deflections with higher values than the plate thickness with the addition of membrane forces [5].
Figure 1. The example of ice field lay-out.

\[
\begin{align*}
q(x, y) &= q_m + D \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial^2 w}{\partial x^2 \partial y^2} \right) \\
q_m &= h \left( T_x \frac{\partial^2 w}{\partial x^2} + T_y \frac{\partial^2 w}{\partial y^2} + 2S \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \\
T_x &= K(e_x + \mu e_y) \\
T_y &= K(e_y + \mu e_x) \\
S &= K_1 \omega
\end{align*}
\] (1)

where \( q(x, y) \) is an external distributed vertical load per area unit on the plate (from the ice weight and the buoyancy force); \( D \) – bending rigidity of a plate which equals to \( D = \frac{E_i h^3}{12(1-\mu^2)} \); \( q_m \) – addition to the vertical load connected to the vertical projection of the membrane forces at the final deflection; \( T_x, T_y, S \) – longitudinal, transversal and shear membrane stresses; \( E_i \) – Young’s modulus of ice; \( \mu \) – Poisson’s ratio; \( h \) – ice thickness; \( K \) and \( K_1 \) – plate rigidity to spraining/compression and shearing;

\[
K = \frac{E_i}{1-\mu^2}, \quad K_1 = \frac{K(1-\mu)}{2}.
\]

For discretization purposes the ice filed is divided into rectangular elements (ice floes) of the length \( L \), width \( B \) and thickness \( h \) (the element’s height equals to the ice field thickness \( h \)). The number of elements to be divided in is chosen so that the \( L/h \) proportion is within the range of 3-6 and so that the element is the plate of average thickness.

Inside each element we place a point node where the inertia characteristics of this element and its external loads are concentrated. Each node has three degrees of freedom. Node numbering about X-axis is opposite the ice movement direction, about Y-axis – about the axis (the corresponding indexes are \( i \) and \( j \)). The adjacent nodes about X-axis and Y-axis are connected by links (line segments) which pass the internal forces in the ice field, described in the equation (1).

Let us assume that at each moment in time we know the deflection form and, correspondingly, the coordinates of each node \( X_{ij} = [x_i, y_i, z_i] \). Then the distributed external vertical load from the buoyancy load and ice weight for the node \( ij \) is:

\[
F_{ij}^e = \int_{x_i+L/2}^{x_i+B/2} \int_{y_i-L/2}^{y_i+L/2} q(x, y) dx dy,
\] (2)

here the index \( e \) means external. At the same time, \( q(x, y) \) equals to
\( q(x, y) = \begin{cases} 
 g \left( \frac{h}{2} - z - \rho_x h \right), & -\frac{h}{2} < z < \frac{h}{2}, \\
 gh(\rho_y - \rho_x), & z < -\frac{h}{2}, \\
 -gh\rho_x, & z > \frac{h}{2}, \\
 z = f(x, y), 
\end{cases} \) \hspace{1cm} (3)

\( \rho_x, \rho_y \) - densities of ice and water correspondingly. Three occurrences are 1) ice floe intersecting the water surface, 2) full submersion, 3) full emersion.

The integral equation (2) is found numerically, for instance, using Gauss integration method (4x4 points). The \( z \) coordinates of the required points inside a specific ice floe are calculated as linear combinations of the coordinates of the considered node and four adjacent ones. This level of detail is required as some part of an ice floe can emerge from water and some other part can be fully submerged.

Let us examine the internal forces in the nodes. From equation (1) we get:

\[ F_{ij} = D \int_{y_i-B/2}^{y_i+B/2} \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial^2 w}{\partial x^2 \partial y^2} \right) dx dy. \] \hspace{1cm} (4)

The force \( \hat{F}_{ij} \) is presented as a sum of the three components: \( \hat{F}_{ij} = \hat{F}_{xy} + \hat{F}_{yz} + \hat{F}_{xz} \), according with the number of the additive component. The first two additive components are written as follows:

\[ \begin{align*}
\hat{F}_{ij} &= \hat{F}_{ij} + \hat{F}_{ij} - \hat{F}_{ij}, \\
\hat{F}_{ij} &= \hat{F}_{ij} + \hat{F}_{ij} - \hat{F}_{ij},
\end{align*} \] \hspace{1cm} (5)

where \( \hat{F}_{ij}, \hat{F}_{ij} \) - normal forces, influencing the back and the right faces of the ice floe \( ij \):

\[ \hat{F}_{ij} = \int_{y_i-B/2}^{y_i+B/2} p_y(y) dy, \quad \hat{F}_{ij} = \int_{x_i-B/2}^{x_i+B/2} p_x(x) dy. \] \hspace{1cm} (6)

Here \( p_y(y) \) and \( p_x(x) \) are the loads per unit of length on the back and the right face of element, at the same time:

\[ p_{ij} = \frac{M_{y_{ij}} - M_{y_{i+1j}}}{L}, \quad p_{ij} = \frac{M_{x_{ij}} - M_{x_{ij-1}}}{B}. \] \hspace{1cm} (7)

Here \( M_{y_{ij}} = D_0 N_y \frac{\partial^2 \dot{w}}{\partial x^2} X_{ij}, M_{x_{ij}} = D_0 N_x \frac{\partial^2 \dot{w}}{\partial y^2} X_{ij} \) are bending moments per unit of length in a node about \( Y \) and \( X \) axes; \( N_y \) is a normal vector to the mean plane of the section. Using the normal vector allows calculating moments for any final deflections (inclinations) of the section. The second derivative is calculated numerically using three adjacent nodes \( X_{ij} \) in the corresponding direction. The same numerical method (using three adjacent nodes) is used to calculate integral equations (6).

Then, the third component equals to:

\[ \hat{F}_{ij} = 2D \int_{y_i-B/2}^{y_i+B/2} \int_{x_i-B/2}^{x_i+B/2} \frac{\partial^2 w}{\partial x^2 \partial y^2} dx dy = \hat{F}_{ij} - \hat{F}_{ij} - \hat{F}_{ij} + \hat{F}_{ij}, \] \hspace{1cm} (8)

where \( \hat{F}_{ij} \approx \frac{4D_0}{LB} N_y \left( X_{ij} - X_{i+1j} + X_{ij+1} \right). \) \hspace{1cm} (9)
Let us examine the membrane forces and longitudinal, transversal and shear deformations $\varepsilon_{xij}$, $\varepsilon_{yij}$ and $\omega_{ij}$ connected to them. Assuming the first two deformations to be constant between the adjacent nodes, for the element $ij$ we get:

$$
\begin{align*}
\varepsilon_{xij} &= \tau_{xij} - 1, \\
\varepsilon_{yij} &= \tau_{yij} - 1, \\
\omega_{ij} &= \tau_{xij} \cdot \tau_{yij},
\end{align*}
$$

(10)

where $\tau_{xij} = \frac{1}{L}(X_{ij} - X_{i-1j})$ and $\tau_{yij} = \frac{1}{B}(X_{ij} - X_{ij+1})$ – direction vectors between the adjacent nodes.

Membrane forces per unit of length at the back (index B) and right (index R) node face:

$$
\begin{align*}
P_{Bij} &= -hK(\varepsilon_{xij} + \mu \varepsilon_{yij}), \\
P_{Rij} &= -hK(\varepsilon_{yij} + \mu \varepsilon_{xij}), \\
P_{Bij} &= \frac{1}{2}hK(\omega_{ij} + \omega_{i+1j}), \\
P_{Rij} &= \frac{1}{2}hK(\omega_{ij} + \omega_{ij+1}).
\end{align*}
$$

(11)

Then we use the interpolation of the integral expression from the membrane force per unit of length using the three points, similarly to the bending forces, to calculate the total membrane forces $F_{Bij}, F_{Rij}, S_{Bij}, S_{Rij}$.

The additional normal force from $q_m$ in equation (1) equals to:

$$
F_{xij} = \int_{y_{ij}+\Delta y/2}^{y_{ij}+\Delta y/2} \int_{y_{ij}+\Delta y/2}^{y_{ij}+\Delta y/2} q_m(x,y)dxdy = h \int_{y_{ij}+\Delta y/2}^{y_{ij}+\Delta y/2} \int_{y_{ij}+\Delta y/2}^{y_{ij}+\Delta y/2} (T_x \frac{\partial w^2}{\partial x^2} + T_y \frac{\partial w^2}{\partial y^2} + 2S \frac{\partial^2 w}{\partial x \partial y})dxdy = F_x + F'_x + F''_x,
$$

(12)

$$
F_x = h \int_{y_{ij}+\Delta y/2}^{y_{ij}+\Delta y/2} \int_{y_{ij}+\Delta y/2}^{y_{ij}+\Delta y/2} T_x \frac{\partial w^2}{\partial x} dxdy = F_{Bij} \tau_{xij} - F_{Rij} \tau_{xij-1},
$$

(13)

Evidently, the additional $q_m$ only requires using membrane forces as vectors, considering having real tangent lines in the membrane force calculation point. Then the sum vectors of the membrane and the intersecting forces influencing the sector in fixed coordinate system equal to:

$$
\begin{align*}
F_{0j} &= F_{Bij} - F_{Rij} + F_{Rij+1} - F_{Rij} + S_{Bij} - S_{Rij} + S_{Rij} - S_{Rij+1}, \\
F_{1j} &= F_{Bij} - F_{Bij+1} + F_{Bij+1} - F_{Bij} + F_{Bij} - F_{Bij},
\end{align*}
$$

(14)

Each vector component in equation (14) is based on scalar values calculated above and on the direction vectors:

$$
\begin{align*}
F_{Bij} &= F_{Bij} \tau_{xij}, \\
F_{Rij} &= F_{Rij} \tau_{xij}, \\
S_{Bij} &= \frac{1}{4}S_{Bij}(\tau_{xij} + \tau_{yij+1} + \tau_{yij+1} + \tau_{yij+1}), \\
S_{Rij} &= \frac{1}{4}S_{Rij}(\tau_{xij} + \tau_{xij+1} + \tau_{xij+1} + \tau_{xij+1}),
\end{align*}
$$

(15)

for the membrane forces and
\[
\begin{align*}
\hat{\mathbf{F}}_{Bij} &= \frac{1}{2} \hat{\mathbf{F}}_{Rij} (\mathbf{N}_{ij} + \mathbf{N}_{ij+1}), \\
\hat{\mathbf{F}}_{Rij} &= \frac{1}{2} \hat{\mathbf{F}}_{Rij} (\mathbf{N}_{ij} + \mathbf{N}_{ij+1}), \\
\hat{\mathbf{F}}_{Nij} &= \hat{\mathbf{F}}_{Nij} \mathbf{N}_{ij},
\end{align*}
\]

(16)

for the intersecting forces.

In case of a very tight bending, a closer approximation of the shear forces direction vector is needed than averaging along the four direction vectors in equation (15). And vice versa for the ice floes which are more remote from the ice/structure contact area the bending stresses and vertical movements are insignificant so all the normal forces can be dismissed, leaving only the membrane ones. At the same time, the equations for deformations (15) also significantly simplified (considering the small level of the deformation):

\[
\begin{align*}
\varepsilon_{xij} &= \frac{x_{ij} - x_{ij+1}}{L}, \\
\varepsilon_{yij} &= \frac{y_{ij} - y_{ij+1}}{B}, \\
\alpha_x &= \frac{x_{ij+1} - x_{ij+1}}{2B} + \frac{y_{ij} - y_{ij+1}}{L}.
\end{align*}
\]

This allows to substantially simplify the calculations for the majority of elements and significantly increase the modelling efficiency.

Dissipative forces required to exclude unrestrained resonance oscillation of a specific element equal to:

\[
\mathbf{F}^d_{ij} = \frac{K_d}{4} (\mathbf{V}_{i+1} + \mathbf{V}_{i-1} + \mathbf{V}_{ij+1} + \mathbf{V}_{ij-1} - 4\mathbf{V}_i).
\]

(17)

That suggests that the element decelerates if it moves differently from the adjacent ice floes.

Here \( \mathbf{V}_{ij} \) – velocity vector of the element in the coordinate system fixed with it, \( \mathbf{V}_i = [v_{i1} v_{i2} v_{i3}]^T \).

The dissipation factor is introduced, for instance, from the 1% damping condition of the element at the resonance frequency:

\[
K_d = 0.01 \sqrt{cm} = 0.01Bh_i \rho_c E. 
\]

(18)

As the deformations and the elastic forces of the plate’s element act in directions of natural axis of this element, the set of simultaneous equations of its movement is written in the finite displacement terms:

\[
\begin{align*}
\mathbf{X}_{ij} &= \mathbf{P}_i \mathbf{V}_{ij},  \\
\mathbf{X}_{ij} &= \mathbf{P}_i \mathbf{V}_{ij}, \\
\end{align*}
\]

where \( \mathbf{E} \) – unit diagonal matrix; \( \lambda \) – added masses matrix; \( \mathbf{C}_v \) – diagonal matrix of viscous coefficients of ice resistance in liquid; \( \mathbf{F}^c_{ij} \) – structure contact force vector (if this element is in contact), is considered in detail in [1]; \( \mathbf{F}^e \) – external distributed forces vector (buoyancy force and weight force), \( \mathbf{F}^e_{ij} = \mathbf{k} \mathbf{F}^e_{ij} \); \( \mathbf{k} \) – vertical ort; \( \mathbf{P}_i \) – element’s rotation tensor relative to the fixed coordinate system, \( \mathbf{P}_i = [\mathbf{r}_{i1} \mathbf{r}_{i2} \mathbf{N}_i] \).
The additive components with the product of coefficients of speeds intersection in the equation (19) are disposed of, as the ice movement is slow. The equation (19) will also describe viscous resistance and the added mass more accurately, which are tied to the ice floe dimension.

3. Ice failure
Let us write down the assumptions caused by this model, the ones which we will hereafter consider in the ice failure model construction.

1. Within any specific ice floe the bending rigidity $D_{ij}$ is constant and initially equals $D$.
2. The crack formation in each ice floe resulting from bending begins after bending stress exceeds the stress limit $R$ in the node.
3. The decrease in ice floe load capacity after cracking and further bending results from decrease in bending rigidity $D_{ij}$ irrespective of the direction of bending or its prominence. Otherwise the using of equation system (1) is impossible.
4. For the system cohesiveness let us assume that ice floes, remaining interconnected by hinges, cannot separate even as a result from the total loss of the load capacity due to bending. This is applicable as the ice floe with no strength does not have a significant impact on the general strain-stress state of the field.
5. The separation of the ice floe from the ones in adjacent rows is possible after exceeding the separation stress limit $R$.
6. For the sake of the same cohesiveness the separation is only possible if the separation of the preceding ice floes in the row took place.
7. The separation and the failure of the first floe in a row take place in the following cases:
   a. The back of the ice floe touched the surface of the structure, which it glides along upwards the structure (or downwards in case of the inverse cone).
   b. The ice floe inclination about the X and Y axes exceeded the accepted value.
   c. The ice floe failed completely due to shattering (local failure).

4. Example tasks
To check ice floe deformation modelling, an example of loading of an ice field row with vertical load along the front plane of this field is provided below.

Ice parameters: thickness – 1 m, Young’s modulus – 1 GPa, bending stress limit - $R_f$=0.1 MPa.

Two loading cases are considered: load per unit of length 10 kPa /m (the ice floats on the water surface, there is an analytical solution) and 30 kPa /m (the field is partly submerged, failure is prohibited), figure 2 (a) and (b) correspondingly.

The ice field bending moment (figure 3a) and deflection (figure 3b) are analyzed. The provided graphs show the ice longitudinal length and deflection are normalized by the characteristic length of the fragment (corresponds at the peak point of the bending moment), bending moment per unit of length – by the set load per unit of length and the fragment length, which is [6]:

$$L_i = \frac{\pi}{4} \left( \frac{4D}{\rho g} \right)^{0.25}$$

(20)
Figure 2. Test calculation – (a) ice floats on the water surface, (b) ice submerges into the water.

For the first loading case the received values of deflection and bending moment are 0.5 - 1% accurate comparing to the theoretical solution. If ice is submerged into the water the situation changes radically: peak point of the moment moves twice further away from the ice floe edge, and the increase in deflection is much more substantial than the increase in the load. Figure 4 shows further ice failure.

Figure 3. Dependencies: (a) of bending moment and (b) of ice field deflection from distance to the edge of the field (load application point).

Figure 4. Further ice failure.
5. Conclusions
The main advantages of this mathematical model of an ice field are the following:

1) High computational efficiency – dynamic analysis of ice loading of a structure in the time domain with statistically-valid results can be performed on a personal computer, taking no more than about ten minutes.

2) The method works even with significant bending and inclinations of an ice field, including the cases of full submersion and full emersion of ice.

3) High cohesiveness of the model – even the fields that are considerably damaged by the contact with a structure behave quite stably.

4) Versatility – there are possibilities for floating as well as fixed structures, having generatrixes in forms of right cone as well as an inverse cone and wall-sided ones.

5) Unlike the FEM methods, having a specific set of elements’ characteristics, this method allows flexible adjustment of ice failure models to refine the design model.

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