LP-based Approximation for Personalized Reserve Prices

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Abstract

We study the problem of computing personalized reserve prices in eager second price auctions without having any assumption on valuation distributions. Here, the input is a dataset that contains the submitted bids of \( n \) buyers in a set of auctions and the goal is to return personalized reserve prices \( r \) that maximize the revenue earned on these auctions by running eager second price auctions with reserve \( r \). We present a novel LP formulation to this problem and a rounding procedure which achieves a \((1 + 2(\sqrt{2} - 1)e^{\sqrt{2}-2})^{-1} \approx 0.684\)-approximation. This improves over the \( \frac{1}{2} \)-approximation algorithm due to Roughgarden and Wang. We show that our analysis is tight for this rounding procedure. We also bound the integrality gap of the LP, which bounds the performance of any algorithm based on this LP.

1 Introduction

Second price (Vickrey) auctions with reserves have been prevalent in many marketplaces such as online advertising markets \([GLMN17, PLPV16, CS14]\). A key parameter of this auction format is its reserve price, which is the minimum price at which the seller is willing to sell an item. While we have a full understanding of the optimal reserve prices when the buyers’ valuation distributions are, for example, i.i.d. and regular \([Mye81]\), there are many practical applications including online advertising markets in which these assumptions fail to hold \([GLMN17, CLMN14]\). Furthermore, there are empirical and theoretical evidence that highlight the significance of setting personalized reserve prices for the buyers in order to maximize the revenue \([EOS07, OS11, BGL+18]\).

We study the problem of optimizing personalized reserve prices in second price auctions when the buyer valuations can be correlated. There are two different ways that personalized reserve prices can be applied in the second price auctions: lazy and eager \([DRY15]\). In the lazy version, we first determine the potential winner and then apply the reserve prices. In the eager version, we first apply the reserve prices and then determine the winner. In this work, we focus on optimizing eager reserve prices because (i) while the optimal lazy reserve prices can be computed exactly in polynomial time, they have worse revenue performance both in theory and practice, and (ii) eager reserves perform better in terms of social efficiency for similar revenue levels \([PLPV16]\).

To optimize the eager reserve prices, we take a data-driven approach as suggested in the literature \([PLPV16, RW16]\). The input in this setting is a history of the buyers’ submitted bids/valuations over multiple runs of an auction and the goal, roughly speaking, is to set a personalized reserve price \( r_b \) for each buyer \( b \) such that the total revenue obtained on the same data set according to these reserve prices is maximized (see Section 2 for the formal definition). While the problem is APX-hard \([RW16]\), the state-of-the-art algorithm of Roughgarden and Wang \([RW16]\)

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1The seminal work of \([Mye81]\) shows that when the buyers’ valuation distributions are i.i.d. and regular, the monopoly price defined as \( \arg \max_r \ r \cdot (1 - F(r)) \) for \( F \) being the buyers’ valuation distribution is optimal.
achieves a 1/2-approximation which itself improves over an earlier 1/4-approximation algorithm by Paes Leme, Pál and Vassilvitskii [PLPV16]. Our main result is an algorithm with a significantly improved approximation factor:

**Theorem** (formally as Theorem 1). *There exists a randomized polynomial time algorithm that given a dataset, outputs a vector of reserve prices whose expected revenue is a 0.684-approximation of that of the optimal value.*

The known algorithms of the literature are all greedy and only take into account the two highest bids in each auction. Another limitation of these algorithms is that the reserve price for each buyer is computed in isolation. That is, the reserve price for a buyer only depends on the bids of the auctions in which the buyer submits the highest bid. In fact, [RW16] argue that these limitations are precisely what prevent their algorithm from obtaining any guarantee better than 1/2. We bypass this bound by a careful analysis of a rounding technique for a natural linear programming formulation of the problem proposed in this work.

The optimal data-driven reserve prices solve an *offline optimization problem*, i.e., given a table of bid data, it computes the optimal reserve prices in retrospect. Such an approach, which is inspired by practice, does not need the knowledge of valuations/bids distributions. Suppose that there is a distribution over buyers’ valuations/bids and the goal is compute the optimal prices by having access to samples from that distribution [MM14, HMR18]. This setting is called *batch learning* in [RW16]. Using the machinery developed by Morgenstern and Roughgarden [MR15], by solving the data-driven offline optimization problem on the dataset with $\Omega(|B| \log |B|/\epsilon^2)$ auctions, we can obtain $1 - \epsilon$ fraction of the maximum revenue of any eager second price auction that one could have hoped to obtain by knowing the valuation distribution. This implies that the data-driven approach leads to approximately optimal reserve prices in the batch learning setting.

If the value distributions are independent, an improved approximation to personalized reserves are known via techniques like the correlation gap [Yan11, CHMS10] and prophet inequalities [KS78, HK81, ACK18, EHL17, BGL+18, CSZ19] (to cite a few). The latest result is 0.675-approximation by Correa et al [CSZ19]. Although those results are typically states as an approximation ratio with respect to the (stronger) Myerson revenue benchmark, those are also the best-known approximation ratios with respect to the optimal reserve prices for independent distributions.

**Our result and techniques**  Our main contribution is a polynomial-time 0.684-approximation algorithm for the data-driven reserve prices problem with *correlated* distributions, improving over the 1/2-approximation of [RW16]. This implies a $(0.684 - \epsilon)$-algorithm for the batch learning version of the problem using the reduction in [MR15]. It also implies a polynomial time $(0.684 - \epsilon)$-algorithm for independent distributions, which beats the best approximation known via prophet techniques$^2$.

To overcome the limitation of algorithms by [PLPV16] and [RW16], we present an algorithm called “**Profile-based LP-Rounding**”, Pro-LPR for short, that takes advantage of a concise representation of the solution space. This representation, that we call profile space, is inspired by how revenue is computed in the eager auctions. Working with the profile space enables us to consider all the bids in an auction, not only the highest and second highest bids, to set the reserve prices.

$^2$While we provide a better guarantee against the optimal reserves, our technique does not provide approximation guarantees with respect to the optimal auction as prophet inequalities do.
It further allows us to describe the optimal solution by a polynomial-size integer program. By relaxing the integrality constraints on the variables of the integer program, we construct a linear program (LP). The fractional solution of the LP is then rounded to obtain the reserve prices. The final reserve price of the algorithm is the best of the zero reserves and the reserves obtained from rounding the solution of the LP. The most technically challenging step in the analysis is to bound the approximation ratio. This is done via careful probabilistic analysis of the rounding procedure which leads to a non-linear mathematical program bounding the ratio. Our last step is to use techniques from non-linear optimization to bound the solution of the mathematical program. We would like to emphasize that our analysis of our algorithm is tight in a sense that there is an example for which our algorithm cannot get an approximation factor better than $0.684$.

Finally, we point out that the performance of our algorithm is evaluated against the optimal value of the LP, which is an upper bound on the maximum revenue. By analyzing the integrality gap of the LP, we show that no algorithm can obtain more a $0.828$ fraction of the optimal value of the LP; see Theorem 3. This highlights that our algorithm is evaluated against a powerful benchmark and despite that, it obtains $0.684$ fraction of this powerful benchmark.

**Other related work** Our work relates and contributes to the broad literature on revenue-maximizing mechanisms in a single-item environment. Within that line of work, our paper is in the intersection of two major sub-streams: (i) reserve price optimization and (ii) auction design for correlated valuations. Most of the reserve price optimization literature has been devoted to the case where valuations are independent, see Hartline and Roughgarden [HR09], Yan [Yan11], Dhagwatnotai et al [DRY15] and more recently, a very fruitful line of work on posted-price and reserve-price optimization via prophet inequalities [KS78, HK81, ACK18, EHLM17, BGL+18, CSZ19].

This work on auction design for correlated distributions pioneered by Ronen [Ron01] and Ronen and Saberi [RS02]. The positive and negative results were later improved by Dobzinski et al [DFK11] and Papadimitriou and Pierrakos [PP11]. Our paper departs from this line work in the sense that we do not try to approximate the optimal incentive-compatible auction, but instead, we try to approximate the best auction in the subclass of second price auctions with reserves. Note that this is the auction format adopted by most online marketplaces, including online display advertising markets.

The rest of the paper is organized as follows. In Section 2, we define the model. Section 3 presents a high level view of the results and techniques. In Section 4, we provide our LP, which will be used as our benchmark. In Section 5, we present the LP-base algorithm and show its performance guarantee. Section 7 provides the proof of the integrality gap and Section 6 shows that our analysis is tight. We conclude in Section 8.

## 2 Preliminaries and Problem Statement

There are $n$ buyers participating in a set of single-item eager second price auctions. Let $A$ and $B$ respectively denote the set of auctions and buyers. For any buyer $b \in B$, and for any auction $a \in A$, we are given a non-negative number $\beta_{a,b}$ which indicates the bid of buyer $b$ in auction $a$. Let $r_b$ be the personalized reserve price of buyer $b \in B$. Then, given the bids $\{\beta_{a,b}\}_{b \in B}$ in auction $a \in A$ and reserve prices $r = \{r_b\}_{b \in B}$, the eager second price (ESP) auction works as follows.

- First, any buyer $b$ with $\beta_{a,b} < r_b$ is eliminated. Let $S_a = \{b : \beta_{a,b} \geq r_b\}$ be the set of buyers who clear their reserve prices in auction $a$. 

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- When set $S_a$ is nonempty, the item is allocated to buyer $b^*_a = \arg \max_{b \in S_a} \{ \beta_{a,b} \}$ who has the highest bid among all the buyers in set $S_a$ and is charged

$$\Rev_a(r) := \max \left\{ r_{b^*_a}, \max_{b \in S_a, b \neq b^*_a} \{ \beta_{a,b} \} \right\}.$$ 

Note that $S_a$ and $b^*_a$ are implicitly depend on reserve prices $r$. Any other buyers $b \in B, b \neq b^*_a$ are not charged. Further, when set $S_a$ is empty, the item is not allocated and $\Rev_a(r) = 0$.

Note that the reserve prices are the same across all the auctions $a \in A$. However, each buyer $b$ is assigned a personalized reserve price $r_b$. Given the dataset of bids $\{ \beta_{a,b} \}_{a \in A, b \in B}$, our goal here is to find personalized reserve prices that maximize revenue of the auctioneer. See the introduction section for a discussion on the nice properties of this data-driven optimization. Formally, we would like to solve the following optimization problem:

$$\text{ESP}^* = \max_{r \in \mathbb{R}^n} \Rev(r) := \sum_{a \in A} \Rev_a(r).$$ \quad (\text{ESP-OPT})$$

Note that, without loss of generality, we assume that the optimal reserve price for buyer $b$ is equal to one of his submitted bids $\{ \beta_{a,b} \}_{a \in A}$. Let $R = \{0, \infty\} \cup \{ \beta_{a,b} \}_{a \in A, b \in B}$. Then, Problem ESP-OPT can be rewritten as $\max_{r \in \mathbb{R}^n} \sum_{a \in A} \Rev_a(r)$, which leads to a search space of size $|R|^n$.

### 3 Results and Techniques

The main result of the paper is a randomized algorithm that returns an 0.684-approximation solution for Problem ESP-OPT.

**Theorem 1 (Main Theorem).** There exists a randomized polynomial time algorithm that given a dataset $\{ \beta_{a,b} \}_{a \in A, b \in B}$, outputs a vector of eager reserve prices whose expected revenue is a 0.684-approximation of that of the optimal value of Problem ESP-OPT, denoted by $\text{ESP}^*$.

To find an approximate solution, the overall idea is to construct an LP whose objective function at its optimal solution provides an upper bound for $\text{ESP}^*$. The LP that takes advantage of a concise representation of the solution space, has a polynomial number of variables and constraints. Then, we use a rounding technique to transform the optimal solution of the LP to a vector of reserve prices. We show that if we consider the reserve prices obtained from the rounding technique and the vector of all-zero reserve prices and choose the one with the maximum revenue, we obtain the desired approximation factor. In Theorem 2, we further show that our analysis of our approximation factor is tight. That is, we provide an example for which our algorithm obtains exactly 0.684 fraction of the optimal value of the LP, i.e., the upper bound on for $\text{ESP}^*$. Finally, in Theorem 3, we bound the integrality gap of the LP. This characterization shows that no algorithm can obtain more than 0.828 fraction of the LP.

### 4 Linear Program

The main challenge in designing an LP formulation for this problem is to find a concise representation of the solution space. Instead of considering all possible assignments of reserves to
buyers, we will consider only partial assignments in which we only specify the reserve prices of two buyers. We will call such partial assignment a profile. Formally, a profile is a tuple \( p = (b_1, b_2, r_1, r_2) \in B \times B \times R \times R \), which represents an assignment of reserve \( r_1 \) to buyer \( b_1 \) and reserve \( r_2 \) to buyer \( b_2 \). If it is the case that the reserves are below the corresponding bids in an auction \( a \), i.e. \( r_1 \leq \beta_{a,b_1} \) and \( r_2 \leq \beta_{a,b_2} \), then no matter how the assignment of the remaining reserves, the revenue of this partial assignment is at least \( \max\{r_1, \beta_{a,b_2}\} \) for \( \beta_{a,b_1} \geq \beta_{a,b_2} \). We also note that given any vector of reserve prices \( r \), the revenue that can be obtained from \( r \) only depends on the reserve price of the highest and second highest bidders that clear the reserve prices.

Next, we formally define the notion of valid profile and show that the ESP-OPT problem can be relaxed to finding the best consistent distribution over valid profiles in each auction. To define valid profiles, we assume that the data has two additional buyers \( b_0 \) and \( b_{00} \) who always bid zero which means \( b_{00}, b_0 \in B \). We further elaborate on this.

**Definition 4.1 (Valid Profiles).** We define the set of valid profiles for auction \( a \) as the set \( \mathcal{P}_a \) consisting of all tuples \( (b_1, b_2, r_1, r_2) \in B \times B \times R \times R \) satisfies the following conditions:

1. Bid of buyer \( b_1 \) is greater than or equal to that of buyer \( b_2 \); that is, \( \beta_{a,b_1} \geq \beta_{a,b_2} \).
2. Buyer \( b_1 \) clears his reserve; that is, \( \beta_{a,b_1} \geq r_1 \).
3. Buyer \( b_2 \) clears his reserve; that is, \( \beta_{a,b_2} \geq r_2 \).

For any given \( p \in \mathcal{P}_a \), we define \( \text{Rev}_a(p) := \max(\beta_{a,b_2}, r_1) \).

By adding buyers \( b_0 \) and \( b_{00} \) to \( B \), we can define valid profiles to represent the cases in which less than two buyers cleared their reserve prices. We present the cases with one (respectively zero) cleared buyer with valid profile of \( (b_1, b_0, r_1, 0) \) (respectively \( (b_0, b_{00}, 0, 0) \)).

Note that we abuse notation and use \( \text{Rev}_a(\cdot) \) for both revenue from reserves and revenue from profiles. The following lemma (which follows from the preceding discussion) states that reserve price vectors can always be mapped to a profile with the same revenue.

**Lemma 4.2.** Given a vector of reserve prices \( r \) and an auction \( a \), there is a valid profile \( p = (b_1, b_2, r_1, r_2) \) such that \( \text{Rev}_a(r) = \text{Rev}_a(p) \). Such a profile \( p \) is called the profile corresponding to reserve price vector \( r \).

We are now ready to describe our LP.

**Decision variables of the LP:** The LP will have two sets of variables:

1. For any auction \( a \in A \) and any valid profile \( p \in \mathcal{P}_a \), define a variable \( s_{a,p} \geq 0 \) such that \( \sum_{p \in \mathcal{P}_a} s_{a,p} \leq 1 \). This variable represents a probability distribution over valid profiles in auction \( a \). We refer to \( \{s_{a,p} | a \in A, p \in \mathcal{P}_a\} \) as a profile-solution.

2. For any buyer \( b \in B \) and reserve price \( r \in R \), define a variable \( q_{b,r} \geq 0 \) such that \( \sum_{r \in R} q_{b,r} = 1 \). This variable represents the probability that buyer \( b \) is assigned a reserve price of \( r \).

Finally, we add constraints relating \( s_{a,p} \) and \( q_{b,r} \) which will ensure the consistency of probability distributions across all profiles. To define this set of constraints, for every \( b \in B \), \( a \in A \), and \( r \in R \), we define set

\[
\mathcal{Q}_{b,r,a} := \{ p = (b_1, b, r_1, r) : p \in \mathcal{P}_a \} \cup \{ p = (b, b_2, r, r_2) : p \in \mathcal{P}_a \},
\]  

(1)
which corresponds to all valid profiles of auction \( a \) that assign reserve \( r \) to buyer \( b \). A natural constraint to add is that the total probability assigned to profiles in \( Q_{b,r,a} \) is at most the probability that buyer \( b \) is assigned to reserve price \( r \). That is,
\[
\sum_{p \in Q_{b,r,a}} s_{a,p} \leq q_{b,r}.
\]

Finally, we can put it all together in the following LP:

\[
\begin{align*}
\max_{q,s} & \quad \sum_{a \in A} \sum_{p \in P_a} s_{a,p} \cdot \text{Rev}_a(p) \\
\text{s.t.} & \quad \sum_{p \in P_a} s_{a,p} \leq 1 \quad \forall a : a \in A \\
& \quad \sum_{p \in Q_{b,r,a}} s_{a,p} \leq q_{b,r} \quad \forall a, b, r : b \in B, r \in R, a \in A \\
& \quad \sum_{r \in R} q_{b,r} = 1 \quad \forall b : b \in B \\
& \quad s_{a,p} \geq 0 \quad \forall a, p : a \in A, p \in P_a
\end{align*}
\]

We start by noting that the LP is a relaxation of the ESP-OPT problem:

**Lemma 4.3** (Upper bound on Revenue). The solution of Profile-LP is an upper bound to ESP\(^*\), i.e., the optimal value of Problem ESP-OPT.

**Proof.** Given reserve prices \( r^* \) such that \( \text{ESP}^* = \sum_a \text{Rev}_a(r^*) \), we construct a feasible solution to the LP as follows. For each \( a \in A \), we let \( s_{a,p} = 1 \) for the profile \( p \) corresponding to \( r^* \) (according to lemma 4.2) and \( s_{a,p} = 0 \) for all remaining profiles. Further, we let \( q_{b,r^*} = 1 \) and \( q_{b,r} = 0 \) for all remaining reserves. It is straightforward to verify that it is a feasible solution to the Profile-LP and that \( \sum_{a \in A} \sum_{p \in P_a} s_{a,p} \cdot \text{Rev}_a(p) = \text{ESP}^* \).

\[\square\]

### 5 Profile-based LP-rounding (Pro-LPR) Algorithm

In this section, we present an algorithm, called Profile-based LP-rounding (Pro-LPR), that uses the optimal solution of (Profile-LP), \( s^* \), to devise reserve prices. Our rounding procedure is as follows:

- Construct reserve prices \( r^\mathcal{R} \). To do so, for each buyer \( b \in B \), independently sample reserve price \( r \in R \) with probability proportional to \( q_{b,r} \).
- Let \( z \) be the vector of all zero reserves. Output the best of \( r^\mathcal{R} \) and \( z \), i.e.,
\[
\begin{align*}
\text{r}^\text{out} & \leftarrow \arg \max_{r \in \{z, r^\mathcal{R}\}} \text{Rev}(r).
\end{align*}
\]

Now we analyze the rounding procedure and show that \( \mathbb{E}[\text{Rev}(\text{r}^\text{out})] \) is at least a 0.684 fraction of the solution of the Profile-LP and hence at most 0.684 · ESP\(^*\). One of the biggest strengths of our LP formulation is that it allows the analysis to decouple the effect of rounding for each individual auction.
Lemma 5.1 (Two Conditions). Let \( s^* \) and \( q^* \) be the optimal solution of (Profile-LP) and \( r^R \) be a random reserve price obtained from the rounding procedure. If there exists a constant \( c > 0 \) such that for any \( t \geq 0 \) and any auctions \( a \in a \), we have
\[
\sum_{\{p : p \in P_a, \text{Rev}(p) \geq t\}} s^*_{a,p} - \text{Pr}[\text{Rev}_a(r^R) \geq t] \leq 0 \quad \text{for } t > \beta_a^{(2)} \tag{2}
\]
\[
\sum_{\{p : p \in P_a, \text{Rev}(p) \geq t\}} s^*_{a,p} - \text{Pr}[\text{Rev}_a(r^R) \geq t] \leq c \quad \text{for } t \leq \beta_a^{(2)}, \tag{3}
\]
then Pro-LPR algorithm is a \((1 + c)^{-1}\)-approximation. That is, it obtains at least \((1 + c)^{-1}\) fraction of the optimal value of Problem ESP-OPT. Here, \( \beta_a^{(2)} \) is the second highest bid in auction \( a \) and \( \text{Rev}_a(r^R) \) is the revenue in auction \( a \) under reserve prices \( r^R \).

Proof of Lemma 5.1. By integrating over \( t \) in Equations (2) and (3) and adding them up, we get
\[
\int_{\beta_a^{(2)}}^{\infty} \left( \sum_{\{p : p \in P_a, \text{Rev}(p) \geq t\}} s^*_{a,p} - \text{Pr}[\text{Rev}_a(r^R) \geq t] \right) dt \\
+ \int_0^{\beta_a^{(2)}} \left( \sum_{\{p : p \in P_a, \text{Rev}(p) \geq t\}} s^*_{a,p} - \text{Pr}[\text{Rev}_a(r^R) \geq t] \right) dt \leq c \cdot \beta_a^{(2)}. \tag{4}
\]
This is simplified as follows
\[
\sum_{p \in P_a} s^*_{a,p} \text{Rev}_a(p) - E[\text{Rev}_a(r^R)] \leq c \cdot \beta_a^{(2)}. \tag{4}
\]
Then, if \( \sum_{a \in A} \beta_a^{(2)} = x \cdot ESP^* \), Equation (4) leads to
\[
\sum_{a \in A} \sum_{p \in P_a} s^*_{a,p} \text{Rev}_a(p) - \sum_{a \in A} E[\text{Rev}_a(r^R)] \leq c \cdot \sum_{a \in A} \beta_a^{(2)} = c \cdot x \cdot ESP^* \tag{5}
\]
Here \( x \in (0, 1] \). Note that by Lemma 4.3, the optimal value of Problem ESP-OPT, denoted by \( ESP^* \), is upper bounded by \( \text{Rev}(s^*) \). That is,
\[
ESP^* \leq \text{Rev}(s^*) = \sum_{a \in A} \sum_{p \in P_a} s^*_{a,p} \text{Rev}(p). \tag{6}
\]
Further, the revenue of Pro-LPR algorithm, i.e., \( E[\text{Rev}(r^{out})] \), is lower bounded by
\[
E[\text{Rev}(r^{out})] \geq \max \left( \sum_{a \in A} \beta_a^{(2)}, E[\text{Rev}(r^R)] \right). \tag{7}
\]
To see why this holds note that Pro-LPR algorithm returns the best of reserve price \( r^R \) and all zero prices, where the revenue under all zero prices is the sum of the second highest highest bids \( \sum_{a \in A} \beta_a^{(2)} \). By using Equations (6) and (7) in (5), we have
\[
ESP^* - E[\text{Rev}(r^{out})] \leq c \cdot x \cdot ESP^*. \tag{7}
\]
Invoking Equation (7) again, we have
\[
E[\text{Rev}(r^{out})] \geq \sum_{a \in A} \beta_a^{(2)} = x \cdot ESP^*. \tag{7}
\]
Putting these together, we have

$$\mathbb{E}[\text{Rev}(r^{\text{out}})] \geq \max \{ x, 1 - cx \} \cdot \text{ESP}^* \geq \frac{1}{1+c} \cdot \text{ESP}^*, $$

which is the desired result. \[\square\]

By Lemma 5.1, to complete the proof of the main theorem it suffices to prove that Equation (2) holds for any auction \( a \in A \), and find a constant \( c \) that satisfies Equation (3). The following lemma shows that Equations (2) holds.

**Lemma 5.2 (First Condition Holds).** Let \( s^* \) denote an optimal solution of (Profile-LP) and let \( s^R \) be the profile-solution associated with the vector of reserve prices \( r^R \), defined in Pro-LPR Algorithm. For any auction \( a \in A \), we have

$$\sum_{\{p: p \in P_a, \text{Rev}(p) \geq t\}} s^*_a \cdot p - \Pr[\text{Rev}_a(r^R) \geq t] \leq 0 \quad \text{for } t > \beta_a^{(2)}. $$

(8)

**Proof.** The first term in the l.h.s. of (8) can be written as

$$\sum_{\{p: p \in P_a, \text{Rev}(p) \geq t\}} s^*_a \cdot p = \sum_{\{p: p \in P_a, p = (b_a^{(1)}, b_2, r, r^R), r \geq t\}} s^*_a \cdot p \leq \sum_{r \geq t} q_{b_a^{(1)}, r} = \Pr[\text{Rev}_a(r^R) \geq t],$$

(9)

where the first equation holds because revenue of a profile \( p \in P_a \) is \( t > \beta_a^{(2)} \) if and only if the bidder with the highest bid in auction \( a \), i.e., \( b_a^{(1)} \), is assigned a reserve price \( t > \beta_a^{(2)} \) and the bid of this bidder is greater than \( t \). The second equation holds because of the second set of constraints of (Profile-LP). The last equation follows from the construction of reserve prices \( r^R \). Note that Equation (9) verifies condition (8). \[\square\]

### 5.1 Bounding the Constant in the Second Condition

We start by noting that the second condition in Lemma 5.1 holds trivially for \( c = 1 \), which recovers the same approximation factor of 1/2 of [RW16]. For the rest of the paper, we will improve past 1/2 by constructing a non-linear mathematical program to optimize \( c \) and then applying the first order conditions in non-linear programming to bound the optimal solution. In Lemma 5.3, we show that

$$c = \max_{\theta \in [0,1]} \text{OPT}(\theta),$$

where for any real number \( \theta \in [0,1] \), \( \text{OPT}(\theta) \) is defined as follows

$$\text{OPT}(\theta) = \max_{x \geq 0} e^{\theta - 1} \left[ \prod_{i \in [n]} (1 - x_i) + \sum_{i \in [n]} x_i \prod_{j \in [n], j \neq i} (1 - x_j) \right]$$

s.t.

$$\frac{1}{2} \sum_{i \in [n]} x_i = \theta$$

$$x_i \leq \theta, \quad \forall i \in [n],$$

(10)
where \( n \) is the number of buyers. Characterizing \( \text{OPT}(\theta) \) is technically involved and because of that its details is postponed to Section 5.2. There, we show that for any number of buyers \( n \geq 2 \) and any real number \( \theta \in [0,1] \),

\[
\text{OPT}(\theta) \leq 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2} \approx 0.4612.
\]

Then, invoking Lemmas 5.1 and 5.2, this leads to the approximation factor of \( \frac{1}{1+0.4612} \approx 0.6844 \), which is the desired result.

In the next lemma, we formally state the relationship between \( \text{OPT}(\theta) \) and the approximation factor of our algorithm.

**Lemma 5.3 (Second Condition).** Let \( s^* \) denote an optimal solution of Profile-LP and \( r^R \) be the vector of reserve prices, defined in Pro-LPR Algorithm. Let

\[
c = \max_{\theta \in [0,1]} \text{OPT}(\theta).
\]

Then, for any auction \( a \in A \), the following equation holds.

\[
\sum_{\{p: p \in \mathcal{P}_a, \text{Rev}(p) \geq t\}} s^* - \Pr[\text{Rev}_a(r^R) \geq t] \leq c \quad \text{for} \quad t \leq \beta_a^{(2)}.
\]

To show Lemma 5.3, we make use of the following lemma.

**Lemma 5.4.** Given fixed \( x_{1,b}, x_{2,b} \) with \( b \in B \) and \( x_{1,b} + x_{2,b} \leq 1 \), the following inequality holds:

\[
\prod_{b \in B} (1-x_{1,b} - x_{2,b}) + \sum_{b \in B} x_{2,b} \prod_{b' \neq b} (1-x_{1,b'} - x_{2,b'}) \leq \prod_{b \in B} (1-x_{1,b}) + \sum_{b \in B} x_{2,b} \prod_{b' \neq b} (1-x_{2,b'})
\]

The proof of Lemma 5.4 is provided at the end of this section. Before it, we use Lemma 5.4 to prove Lemma 5.3.

**Proof of Lemma 5.3.** We start with a few definitions. Consider a certain auction \( a \in A \) and all of its valid profiles \( p \in \mathcal{P}_a \). Fix some threshold \( t \leq \beta_a^{(2)} \) and an optimal solution of (Profile-LP), denoted by \( s^* \). Consider a buyer \( b \in B \). Then, define

\[
\mathcal{X}'_{1,b} = \{ p = (b, b_2, r_1, r_2) : p \in \mathcal{P}_a, \ r_1 \geq t \}
\]

\[
\mathcal{X}''_{1,b} = \{ p = (b_1, b, r_1, r_2) : p \in \mathcal{P}_a, \ r_1 < t \text{ and } r_2 \geq t \}
\]

\[
\mathcal{X}_{2,b} = \{ p = (b_1, b_2, r_1, r_2) : p \in \mathcal{P}_a, b \in \{b_1, b_2\}, r_1, r_2 < t \text{ and } \beta_{a,b_1} \geq t \}
\]

and then set:

\[
x_{1,b} = \sum_{p \in \mathcal{X}'_{1,b} \cup \mathcal{X}''_{1,b}} s^*_{a,p} \quad \text{and} \quad x_{2,b} = \sum_{p \in \mathcal{X}_{2,b}} s^*_{a,p}
\]

We note that \( \mathcal{X}'_{1,b} \) is the set of all valid profiles \( p = (b, b_2, r_1, r_2) \) in which reserve of buyer \( b \) is at least \( t \). \( \mathcal{X}''_{1,b} \) is the set of all valid profiles \( p = (b_1, b, r_1, r_2) \) in which reserve of buyer \( b_1 \) is less than \( t \) and reserve of buyer \( b \) is greater than or equal to \( t \). Observe that for all the profiles \( p \) in \( \mathcal{X}'_{1,b} \cup \mathcal{X}''_{1,b} \), reserve of buyer \( b \) is at least \( t \). This implies that for all of these profiles, \( \text{Rev}(p) \geq t \). We also note that \( \mathcal{X}_{2,b} \) is the set of all valid profiles \( p = (b_1, b_2, r_1, r_2) \) such that buyer \( b \in \{b_1, b_2\} \).
and bid of buyer $b_2$ is at least $t$. Again, it is easy to see that for any valid profile $p \in \mathcal{X}_{2,b}$, we have $\text{Rev}(p) \geq t$. Finally, we point that while any profile $p$ in $\mathcal{X}_{2,b}$ and $\mathcal{X}_{1,b}' \cup \mathcal{X}_{1,b}''$ has $\text{Rev}(p) \geq t$, by construction, $\mathcal{X}_{2,b}$ and $\mathcal{X}_{1,b}' \cup \mathcal{X}_{1,b}''$ are disjoint. Therefore, we have

$$
\sum_{\{p: p \in P, \text{Rev}(p) \geq t\}} s_{a,p} = \sum_{b \in B} x_{1,b} + \frac{1}{2} \sum_{b \in B} x_{2,b},
$$

where the coefficient $\frac{1}{2}$ accounts for double-counting. That is, while any profile $p$ in $\mathcal{X}_{1,b}' \cup \mathcal{X}_{1,b}''$ contributes to $s_{a,p}$ once, any profile $p$ in $\mathcal{X}_{2,b}$ contributes to $s_{a,p}$ twice.

Define $y_{1,b}$ as the probability that the sampled reserve of buyer $b$, i.e., $r^R_b$, is in $[t, \beta_{a,b}]$ and $y_{2,b}$ as the probability that the sampled reserve $r^R_b$ is in $[0, t]$. By the sampling procedure we know that:

$$y_{1,b} \geq x_{1,b} \quad \text{and} \quad y_{2,b} \geq x_{2,b}.$$ 

Observe that $\text{Rev}(r^R) \geq t$ iff at least one of the two following events happen.

**Event $\mathcal{E}_1$:** There exists a buyer with a reserve of at least $t$ whose bid is cleared.

**Event $\mathcal{E}_2$:** There are at least two buyers with cleared bids greater than or equal to $t$.

Precisely,

$$\Pr[\text{Rev}(r^R) \geq t] = \Pr[\mathcal{E}_1 \text{ or } \mathcal{E}_2] = \Pr[\mathcal{E}_1] + \Pr[\mathcal{E}_2 \text{ and } \bar{\mathcal{E}}_1] = \Pr[\mathcal{E}_1] + \Pr[\bar{\mathcal{E}}_1] \Pr[\mathcal{E}_2|\bar{\mathcal{E}}_1], \quad (12)$$

where

$$\Pr[\mathcal{E}_1] = 1 - \prod_{b \in B} (1 - y_{1,b})$$

and

$$\Pr[\mathcal{E}_2|\bar{\mathcal{E}}_1] = 1 - \prod_{b \in B} (1 - \tilde{y}_{2,b}) - \sum_{b \in B} \tilde{y}_{2,b} \prod_{b' \neq b} (1 - \tilde{y}_{2,b'}) \quad \text{for } \tilde{y}_{2,b} = \frac{y_{2,b}}{1 - y_{1,b}}$$

This gives us

$$\Pr[\mathcal{E}_2 \text{ and } \bar{\mathcal{E}}_1] = \Pr[\bar{\mathcal{E}}_1] \Pr[\mathcal{E}_2|\bar{\mathcal{E}}_1] = \Pr[\bar{\mathcal{E}}_1] - \prod_{b \in B} (1 - y_{1,b} - y_{2,b}) - \sum_{b \in B} y_{2,b} \prod_{b' \neq b} (1 - y_{1,b'} - y_{2,b'}).$$

Thus, by Equation (12), we get

$$\Pr[\mathcal{E}_2 \text{ or } \mathcal{E}_1] = 1 - \prod_{b \in B} (1 - y_{1,b} - y_{2,b}) - \sum_{b \in B} y_{2,b} \prod_{b' \neq b} (1 - y_{1,b'} - y_{2,b'}).$$

Now observe that the expression above, i.e., $\Pr[\mathcal{E}_2 \text{ or } \mathcal{E}_1]$, is increasing in both $y_{1,b}$ and $y_{2,b}$, $b \in B$. To see why $\Pr[\mathcal{E}_2 \text{ or } \mathcal{E}_1]$ is increasing in $y_{2,b}$, note that

$$\frac{\partial(\Pr[\mathcal{E}_2 \text{ or } \mathcal{E}_1])}{\partial y_{2,b}} = \sum_{b' \in B, b' \neq b} y_{2,b'} \prod_{b'' \neq b, b'} (1 - y_{1,b''} - y_{2,b''}) \geq 0.$$ 

This implies that

$$\Pr[\mathcal{E}_2 \text{ or } \mathcal{E}_1] \geq 1 - \prod_{b \in B} (1 - x_{1,b} - x_{2,b}) - \sum_{b \in B} x_{2,b} \prod_{b' \neq b} (1 - x_{1,b'} - x_{2,b')}.$$
We now invoke Lemma 5.4, stated earlier, to get

\[
\Pr[\mathcal{E}_2 \text{ or } \mathcal{E}_1] \geq 1 - \prod_{b \in B} (1 - x_{1,b}) \left[ \prod_{b \in B} (1 - x_{2,b}) + \sum_{b \in B} x_{2,b} \prod_{b' \neq b} (1 - x_{2,b'}) \right].
\]  

(13)

Using Equations (11), (13), and (12), we get

\[
\sum_{\{p:p \in \mathcal{P}_s, \text{Rev}(p) \geq t\}} s^*_a,p - \Pr[\text{Rev}_a(r^R) \geq t] \leq \sum_{b \in B} x_{1,b} + \frac{1}{2} \sum_{b \in B} x_{2,b} - \left( 1 - \prod_{b \in B} (1 - x_{1,b}) \left[ \prod_{b \in B} (1 - x_{2,b}) + \sum_{b \in B} x_{2,b} \prod_{b' \neq b} (1 - x_{2,b'}) \right] \right)
\]  

(14)

It easy to check that for any \(b \in B\), the above expression is non-decreasing in \(x_{1,b}\). This allows that to assume without loss of generality\(^3\) that \(\sum_{b \in B} x_{1,b} + \frac{1}{2} \sum_{b \in B} x_{2,b} = 1\).

As a result, we have

\[
\sum_{\{p:p \in \mathcal{P}_s, \text{Rev}(p) \geq t\}} s^*_a,p - \Pr[\text{Rev}_a(r^R) \geq t] \leq \prod_{b \in B} (1 - x_{1,b}) \left[ \prod_{b \in B} (1 - x_{2,b}) + \sum_{b \in B} x_{2,b} \prod_{b' \neq b} (1 - x_{2,b'}) \right],
\]

where \(\sum_{b \in B} x_{2,b} = 2\theta\), \(\sum_{b \in B} x_{1,b} = 1 - \theta\). Here, \(\theta \in [0, 1]\). To complete the proof, we simply use that: \(\prod_{b \in B} (1 - x_{1,b}) \leq e^{-\sum_{b \in B} x_{1,b}} = e^{\theta - 1}\). Given how we constructed the variables \(x_{2,b}\), we also need \(x_{2,b} \leq \theta\). Hence,

\[
\sum_{\{p:p \in \mathcal{P}_s, \text{Rev}(p) \geq t\}} s^*_a,p - \Pr[\text{Rev}_a(r^R) \geq t] \leq e^{\theta - 1} \left[ \prod_{b \in B} (1 - x_{2,b}) + \sum_{b \in B} x_{2,b} \prod_{b' \neq b} (1 - x_{2,b'}) \right]
\]

where \(\sum_{b \in B} x_{2,b} = 2\theta\) and \(x_{2,b} \leq \theta\) for any \(b \in B\).

\[\square\]

**Proof of Lemma 5.4.** Given a partition of \(B\) in two sets \(B_1, B_2\), define the following function:

\[
\Phi(B_1, B_2) = \prod_{b \in B_1} (1 - x_{1,b})(1 - x_{2,b}) \prod_{b \in B_2} (1 - x_{1,b} - x_{2,b}) + \sum_{b \in B_1 \cup B_2} x_{2,b} \left[ \prod_{b' \in B_1, b' \neq b} (1 - x_{1,b})(1 - x_{2,b}) \prod_{b' \in B_2, b' \neq b} (1 - x_{1,b} - x_{2,b}) \right].
\]

The main claim in the lemma is that \(\Phi(B, \emptyset) \geq \Phi(\emptyset, B)\). We will show that for any \(B_1, B_2\) and \(b \in B_2\), we have

\[
\Phi(B_1, B_2) \leq \Phi(B_1 \cup \{\hat{b}\}, B_2 \setminus \{\hat{b}\})
\]

and the claim will follow by moving the elements from \(B_2\) to \(B_1\) one by one. To simplify notation, define

\[
w = \prod_{b \in B_1} (1 - x_{1,b})(1 - x_{2,b}) \prod_{b \in B_2 \setminus \{\hat{b}\}} (1 - x_{1,b} - x_{2,b})
\]

\(^3\)To see why, suppose that \(\sum_{b \in B} x_{1,b} + \frac{1}{2} \sum_{b \in B} x_{2,b} = 1 - \delta\), where \(\delta > 0\). Then, by replacing \(x_{1,b}\) with \(x_{1,b}' := x_{1,b} + \delta\), the r.h.s. of Equation (14) can only increase. Therefore, without loss of generality, we assume that \(\sum_{b \in B} x_{1,b} + \frac{1}{2} \sum_{b \in B} x_{2,b} = 1\).
Now we can write:

\[
\Phi(B_1, B_2) = w \cdot (1-x_{1,\hat{b}} - x_{2,\hat{b}}) + w \cdot x_{2,\hat{b}} + \sum_{b \in B_2 : b \neq \hat{b}} \frac{w - 1}{1-x_{1,\hat{b}} - x_{2,\hat{b}}} \cdot x_{2,\hat{b}} + \sum_{b \in B_1} w \cdot \frac{1-x_{1,\hat{b}} - x_{2,\hat{b}}}{(1-x_{1,\hat{b}})(1-x_{2,\hat{b}})} \cdot x_{2,\hat{b}}
\]

and

\[
\Phi(B_1 \cup \{\hat{b}\}, B_2 \setminus \{\hat{b}\}) = w \cdot (1-x_{1,\hat{b}})(1-x_{2,\hat{b}}) + w \cdot x_{2,\hat{b}} + \sum_{b \in B_2 : b \neq \hat{b}} \frac{(1-x_{1,\hat{b}})(1-x_{2,\hat{b}})}{1-x_{1,b} - x_{2,b}} \cdot x_{2,b} + \sum_{b \in B_1} w \cdot \frac{(1-x_{1,\hat{b}})(1-x_{2,\hat{b}})}{(1-x_{1,b})(1-x_{2,b})} \cdot x_{2,b}
\]

Our goal here is to show \( \Phi(B_1, B_2) \leq \Phi(B_1 \cup \{\hat{b}\}, B_2 \setminus \{\hat{b}\}) \). We start with comparing the first two terms of \( \Phi(B_1, B_2) \) and \( \Phi(B_1 \cup \{\hat{b}\}, B_2 \setminus \{\hat{b}\}) \):

\[w \cdot (1-x_{1,\hat{b}} - x_{2,\hat{b}}) + w \cdot x_{2,\hat{b}} = w \cdot (1-x_{1,\hat{b}} + x_{1,\hat{b}} x_{2,\hat{b}}) = w \cdot (1-x_{1,\hat{b}})(1-x_{2,\hat{b}}) + w \cdot x_{2,\hat{b}}.\]

We can compare the remaining terms one by one using the fact that:

\[1-x_{1,\hat{b}} - x_{2,\hat{b}} \leq (1-x_{1,\hat{b}})(1-x_{2,\hat{b}}).\]

This concludes that \( \Phi(B_1, B_2) \leq \Phi(B_1 \cup \{\hat{b}\}, B_2 \setminus \{\hat{b}\}) \) as desired.

\[\square\]

### 5.2 Approximation Factor

In this section, we will show that for any given \( \theta \in [0, 1] \), we have

\[\text{OPT}(\theta) \leq 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2},\]

where \( \text{OPT}(\theta) \) is defined in Equation (10). Since the constraints of Program (10) are linear in \( x_i \)'s, the first order conditions of Karush-Kuhn-Tucker (KKT) are a necessary condition for optimality [Ber99]. Let

\[F(x, \theta) = e^{\theta - 1} \left[ \prod_{i \in [n]} (1 - x_i) + \sum_{i \in [n]} x_i \prod_{j \in [n], j \neq i} (1 - x_j) \right].\]

Observe that \( F(x, \theta) \) is the objective function of \( \text{OPT}(\theta) \). Then, according to the KKT conditions, the optimal solution must satisfy the following constraints for some \( \lambda \in \mathbb{R}, \mu, \eta \in \mathbb{R}_+^n \):

\[\nabla_x F(x, \theta) + \lambda \frac{1}{2} - \mu + \eta = 0\]

\[\sum_{i \in [n]} x_i = \frac{1}{2} \theta\]

\[\mu_i (x_i - \theta) = 0, \quad \forall i \in [n]\]

\[\eta_i x_i = 0, \quad \forall i \in [n]\]

\[0 \leq x_i \leq \theta, \quad \forall i \in [n]\]

where \( 1 \in \mathbb{R}^n \) is the vector of all one.

It is enough to show that \( F(x, \theta) \leq 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2} \) for any tuple \((x, \theta, \lambda, \mu, \eta)\) satisfying the KKT conditions. A simple consequence of the KKT condition is the following:
Lemma 5.5. If \((x, \theta, \lambda, \mu, \eta)\) satisfies the KKT conditions for Problem (10), then if \(x_k\) and \(x_t\) are such that \(0 < x_k < \theta\) and \(0 < x_t < \theta\), then \(x_k = x_t\).

Proof. By conditions (17) and (18), we must have \(\mu_k = \eta_k = 0\). Plugging that into condition (15), we get that \(\partial F/\partial x_k + \lambda/2 = 0\). This implies that

\[
\sum_{i \neq k} x_i \prod_{j \neq i, k} (1 - x_j) + \frac{\lambda}{2} = 0.
\]

Let \(Q = \prod_{i \in [n]} (1 - x_i)\) and \(S = \sum_{i \in [n]} x_i\). Then, the above condition can be written as

\[
\frac{Q}{1 - x_k} \sum_{i \neq k} \frac{x_i}{1 - x_i} + \frac{\lambda}{2} = 0 \Rightarrow \frac{Q}{1 - x_k} (S - \frac{x_k}{1 - x_k}) + \frac{\lambda}{2} = 0.
\]

This is further simplified as follows

\[
(SQ + \frac{\lambda}{2}) - (SQ + Q + \lambda)x_k + \frac{\lambda}{2}x_k^2 = 0.
\]

The polynomial \(p(y) := (SQ + \frac{1}{2}) - (SQ + Q + \lambda)y + \frac{\lambda}{2}y^2\) is quadratic with \(\frac{\partial^2 p}{\partial y^2} \geq 0\) and \(p(1) = -Q < 0\). Thus, \(p(y) = 0\) has an unique solution with \(y < 1\). This implies \(x_k\) is uniquely determined as a function of \(S\), \(Q\), and \(\lambda\). By the same argument, \(x_t\) is also a solution to the same equation and hence \(x_k = x_t\).

Lemma 5.5 leads to the following corollary.

Corollary 5.6. We can bound \(\text{OPT}(\theta) \leq \max_{k \in \mathbb{Z}, k \geq 2} \max[\text{OPT}^1(\theta, k), \text{OPT}^2(\theta, k)]\) where:

\[
\text{OPT}^1(\theta, k) = e^{\theta - 1} \left(1 - \frac{2\theta}{k}\right)^{k-1} \left(1 - \frac{2\theta}{k} + 2\theta\right)
\]

\[
\text{OPT}^2(\theta, k) = e^{\theta - 1} \left[\left(1 - \frac{\theta}{k}\right)^k + \theta(1 - \theta)\left(1 - \frac{\theta}{k}\right)^{k-1}\right]
\]

Proof. As stated earlier, in order to maximize the objective function \(\text{OPT}(\theta)\), it is enough to consider feasible solutions \(x\) satisfying the KKT conditions. To do so, we use Lemma 5.5 to narrow down such solutions.

Since for any \(i \in [n]\), \(x_i \leq \theta\) and \(\sum_{i \in [n]} x_i = 2\theta\), we an only have the following three cases:

- **Case 1**: Two variables in the support have value \(\theta\) and by constraint \(\sum_{i \in [n]} x_i = 2\theta\), the rest of them are zero. In that case, \(\text{OPT}(\theta) = \text{OPT}^1(\theta, 2)\).

- **Case 2**: One variable has value \(\theta\) and by Lemma 5.5, the rest \(n - 1 \geq 2\) variables in the support have value \(\theta/(n - 1)\). In that case, \(\text{OPT}(\theta) = \text{OPT}^2(\theta, n - 1)\).

- **Case 3**: All variables in the support are strictly below \(\theta\). In this case, by Lemma 5.5, \(x_i = \theta/n\) for \(n \geq 3\), and the solution is \(\text{OPT}(\theta) = \text{OPT}^1(\theta, n)\).

\[
\square
\]
**Lemma 5.7.** For any $\theta \in [0,1]$ and $k \geq 2$, we have $\text{OPT}^1(\theta, k) \leq 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2}$.

**Proof.** For each $k \geq 0$, define $\theta^*(k) = \arg \max_{\theta \in [0,1]} \text{OPT}^1(\theta, k)$. By solving $\frac{\partial \text{OPT}^1(\theta, k)}{\partial \theta} = 0$ we obtain the following expression for $\theta^*(k)$:

$$k^2(2\theta^*(k) - 1) + 4(k - 1)(\theta^*(k))^2 = 0.$$  

The aforementioned equation has two solutions, only one of which is in $[0,1]$. Thus,

$$\theta^*(k) = \frac{k - \sqrt{k^2 + 4k - 4}}{4 - 4k}. \quad (20)$$

We need to show that for any $k \geq 2$, we have $\text{OPT}^1(\theta^*(k), k) \leq 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2} \approx 0.461$. For $k = 2$, we have $\text{OPT}^1(\theta^*(k), k) = 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2}$. For $k < 40$ we can verify this inequality numerically. For $k \geq 40$, we define and upper bound:

$$U(\theta, k) = \frac{2\theta + 1}{e^{\theta+1}(1 - \frac{2\theta}{k})},$$

and show that for any $\theta \in [0,1]$ and $k \geq 40$:

$$\text{OPT}^1(\theta, k) \leq U(\theta, k) \leq U(\theta, 40) \leq 0.459 < 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2}.$$  

For the first inequality note that:

$$\text{OPT}^1(\theta, k) = e^{\theta-1} \left( 1 - \frac{2\theta}{k} \right)^{k-1} \left( 1 + (k-1) \frac{2\theta}{k} \right) < e^{\theta-1} \left( 1 - \frac{2\theta}{k} \right)^{k} \left( 1 - \frac{2\theta}{k} \right)^{-1} (1 + 2\theta) \leq U(\theta, k). \quad (21)$$

For the second inequality, we use the fact that for any $\theta$, $U(\theta, k)$ is decreasing in $k$. To find an upper-bound for value of $U(\theta, 40) = \frac{(2\theta + 1)}{e^{\theta+1}(1 - \frac{2\theta}{40})}$, we take derivative of that which is

$$\frac{\partial U(\theta, 40)}{\partial \theta} = \frac{20 \left( 2\theta^2 - 39\theta + 21 \right)}{e^{\theta+1}(\theta - 20)^2}.$$  

By solving $\frac{\partial U(\theta, 40)}{\partial \theta} = 0$, we obtain that maximum of $U(\theta, 40)$ is at $\theta = \frac{1}{4} \left( 39 - \sqrt{1353} \right)$ and

$$U \left( \frac{1}{4} \left( 39 - \sqrt{1353} \right), 40 \right) < 0.459.$$  

This completes the proof. \hfill \qed

**Lemma 5.8.** For any $\theta \in [0,1]$ and $k \geq 2$, we have $\text{OPT}^2(\theta, k) \leq 0.46 < 2 \left( \sqrt{2} - 1 \right) e^{\sqrt{2} - 2}$.

**Proof.** Observe that

$$e^{1-\theta} \text{OPT}^2(\theta, k) = \left( 1 - \frac{\theta}{k} \right)^k + \theta(1-\theta) \left( 1 - \frac{\theta}{k} \right)^{k-1} \leq \left( 1 - \frac{\theta}{k} \right)^k + \frac{1}{4} \left( 1 - \frac{\theta}{k} \right)^k = \frac{5}{4} \left( 1 - \frac{\theta}{k} \right)^k, \quad (22)$$

where the first inequality holds because $\max_{\theta \in [0,1]} \theta(1-\theta) = \frac{1}{4}$ and $1 - \frac{\theta}{k} \leq 1$. Finally, note that $e^{\theta-1} \cdot \frac{5}{4}(1 - \frac{\theta}{k})^k$ is decreasing for $\theta \in [0,1]$, Thus, we can bound $\text{OPT}^2(\theta, k)$ by the value of $e^{\theta-1} \cdot \frac{5}{4}(1 - \frac{\theta}{k})^k$ at $\theta = 0$ which is $5/(4e) < 0.46$. \hfill \qed
6 Tightness of the analysis

In this section, we show that the analysis of our algorithm is tight, i.e., we construct an example for which the performance of the algorithm matches the $0.684$ factor approximation.

To make the construction cleaner, we can define the weighted version of our problem in which each auction $a \in A$ has an associated weight $w_a > 0$, and the objective is to maximize $\sum_{a \in A} w_a \cdot Rev_a(r)$. Note that if the weights are integers, this is exactly the same as the original problem, replacing each weighted auction by $w_a$ (unweighted) copies. Even if $w_a$’s are not integers, it is easy to see that the algorithm and the analysis generalize with essentially no change to the weighted case (the only modification involves adding weights to the objective function in the LP). In other words, if the objective were the weighted revenue, we would still get $0.684$ approximation factor by applying a similar algorithm. Furthermore, any lower bound to the weighted case translates to the unweighted case by replacing a weighted auction $a$ by $\lceil Nw_a \rceil$ unweighted copies for some large $N$.

**Theorem 2.** There is a weighted instance $\{w_a\}_{a \in A}, \{\beta_{a,b}\}_{a \in A, b \in B}$ and an optimal LP solution $s, q$ such that

$$\max \left( \mathbb{E} \left[ \sum_a w_a Rev_a(r^x) \right], \sum_a w_a Rev_a(0) \right) \leq 0.684 \cdot Rev(s)$$

**Proof.** Fix $\theta = \sqrt{2} - 1$ and $c = (1 - \theta^2)e^{\theta - 1}$. Consider an instance with three weighted auctions and $n = k + 3$ buyers described by the following table:

| Weights $w_a$ | $\beta_{a,1}$ | $\ldots$ | $\beta_{a,k}$ | $\beta_{a,k+1}$ | $\beta_{a,k+2}$ | $\beta_{a,k+3}$ |
|---------------|---------------|-----------|----------------|-----------------|----------------|----------------|
| $1/(c+1)$     | 1             | $\ldots$  | 1              | 1               | 1              | 0              |
| $c/(c+1)$     | 0             | $\ldots$  | 0              | 0               | 0              | 1 + $\epsilon$ |
| $\epsilon$    | $1 + \epsilon$| $\ldots$  | $1 + \epsilon$| $1 + \epsilon$  | 1 + $\epsilon$ | 0              |

Now, consider the following solution to the Profile-LP. For the first auction,

- the profile $p = (i, 0, 1, 0)$ has $s_{a,p} = (1 - \theta)/k$ for $i \in [k]$. In this profile, the $i$-th buyer is reserve priced at 1 and the second buyer is the dummy buyer.
- the profile $p = (k + 1, k + 2, 0, 0)$ has weight $s_{a,p} = \theta$. In this profile, both buyers $k + 1$ and $k + 2$ have zero reserve prices. Observe that the revenue under this profile is 1 due to the highest second price.

For the second auction, we consider only one profile:

- the profile $p = (k + 3, 0, 1 + \epsilon, 0)$ has $s_{a,p} = 1$. In this profile, the $(k + 3)$-th buyer is reserve priced at 1 and the second buyer is the dummy buyer.

And for the third auction we have:

- the profile $p = (i, 0, 1 + \epsilon, 0)$ has $s_{a,p} = \theta/k$ for $i \in [k]$. In this profile, the $i$-th buyer is reserve priced at $1 + \epsilon$ and the second buyer is the dummy buyer.
• the profile \( p = (k + 1, k + 2, 1 + \epsilon, 1 + \epsilon) \) has weight \( s_{a,p} = 1 - \theta \). In this profile, both buyers \( k + 1 \) and \( k + 2 \) have reserve price \( 1 + \epsilon \) and thus the revenue is \( 1 + \epsilon \).

For this solution, we define the \( q \) variables as follows.

• For buyers \( i \in [k] \), we set \( q_{i,1} = (1 - \theta) / k \) and \( q_{i,1+\epsilon} = 1 - q_{i,1} \).
• For buyers \( i = k + 1, k + 2 \), we set \( q_{i,0} = \theta \) and \( q_{i,1+\epsilon} = 1 - q_{i,1} \).
• For buyer \( k + 3 \), we set \( q_{k+3,1+\epsilon} = 1 \).

It is easy to see that this solution is feasible and that it is the optimal solution Profile-LP. This is so because for any auction, any profile that has a positive weight yield the maximum revenue for that auction. Note that for simplicity in the formulation of revenue, we can remove the terms that are a factor of \( \epsilon \) since they can be arbitrary small and are negligible. Now we argue that the rounding procedure produces a \( 1/(c + 1) \) approximation. First notice that the vector of zero reserves obtains revenue \( 1/c \).

Now, we compute the expected revenue from rounding. After rounding, the reserve of any buyer \( i \in [k] \) is either 1 or \( 1 + \epsilon \), the reserve of buyers \( k + 1 \) and \( k + 2 \) is either zero or \( 1 + \epsilon \), and reserve of buyer \( k + 3 \) is always \( 1 + \epsilon \). Thus, the expected revenue from rounding is given by

\[
\frac{1}{c + 1} \left[ 1 - \left( 1 - \frac{1 - \theta}{k} \right)^k \cdot (1 - \theta^2) \right] + \frac{c}{c + 1},
\]

where the first term is the revenue of first auction and the second term, i.e., \( \frac{c}{c + 1} \), is the revenue of the second auction.\(^4\) To see why the latter holds note that in the first auction, we always get a revenue of one unless none of the first \( k \) buyers have a reserve of one and neither buyers \( k + 1 \) nor buyer \( k + 2 \) have a reserve of zero. As \( k \to \infty \), the expected revenue after rounding becomes:

\[
\frac{1}{c + 1} \left[ 1 - e^{\theta - 1} \cdot (1 - \theta^2) \right] + \frac{c}{c + 1} = \frac{1 - c}{c + 1} + \frac{c}{c + 1} = \frac{1}{c + 1},
\]

where the first equation holds because \( c = (1 - \theta^2)e^{\theta - 1} \). The above equation is the desired result because the optimal revenue is at most 1 and \( 1/(c + 1) = 0.684 \). The latter follows from \( c = (1 - \theta^2)e^{\theta - 1} \) and \( \theta = \sqrt{2} \).

\[\square\]

7 Integrality Gap

In the previous section, we showed an instance for which our algorithm obtains exactly an 0.684-factor of the optimal solution. This can be conceivably be improved by either a better rounding procedure or a smart way to select an optimal LP solution. In this section, we show a bound of 0.828, which says that any rounding procedure for this LP formulation will obtain at most 0.828 of the optimal value of the LP.

\(^4\)We do not include the revenue of the third auction because we would like to take \( \epsilon \) to zero and in that case, the revenue of the third auction approaches zero.
Theorem 3 (Integrality Gap of Profile-LP). There exists a dataset of bids \( \{\beta_{a,b}\}_{a \in A, b \in B} \) for which the integrality gap of the LP is at least \( 2(\sqrt{2} - 1) \approx 0.828 \). That is,

\[
\text{ESP}^* \leq 2(\sqrt{2} - 1)\text{LP}^*
\]

where \( \text{LP}^* \) is the optimal fraction solution of the Profile-LP and \( \text{ESP}^* \) is the optimal integral solution.

Proof. Given \( n \) buyers, an integer \( k > 0 \), \( \delta = 1/k \) and a constant \( \lambda \in (0,1) \) to be determined later, consider an instance built as follows:

• Type one auctions: For any buyer, \( b \in [n] \), we have an auction in which all the bids are zero except the bid of buyer \( b \). Precisely, buyer \( b \) has a bid of \( \lambda n \).

• Type two auctions: For any pair of buyers \( b_1 \) and \( b_2 \), there are \( k \) copies of an auction in which \( b_1 \) and \( b_2 \) bid \( \delta = 1/k \) and the rest of the buyers bid 0. We assume that \( \lambda n > \delta \).

For this instance, consider the fractional solution that assigns \( s_{a,p} = 1/2 \) for any auction \( a \) of type two and profiles \( (b_1, b_0, \delta, 0) \) and \( (b_2, b_0, \delta, 0) \). For the rest of the valid profiles of auction \( a \), we set \( s_{a,p} \) to zero. Note that \( b_1 \) and \( b_2 \) are the buyers with nonzero bids in auction \( a \) and \( b_0 \) is an auxiliary buyer. Moreover, for any auction \( a \) of type one, in which buyer \( b \) has a nonzero bid, we have \( s_{a,p} = 1/2 \) for profile \( p = (b, b_0, \lambda n, 0) \). For the rest of the valid profiles of this auction, we set \( s_{a,p} \) to zero. In this solution for any buyer \( b \), we have \( q_b, \delta = 1/2 \) and \( q_b, \lambda n = 1/2 \). One can simply verify that this solution satisfies all the constraints of the LP and as a result, it is a valid fractional solution. The optimal value of the LP is therefore bounded by:

\[
\text{LP}^* \geq \sum_a \text{Rev}_a(s) = n \cdot \frac{\lambda}{2} + \left( \frac{n}{2} \right) \cdot k \cdot \delta = \frac{1 + \lambda}{2} \cdot n^2 + o(n^2),
\]

where the first term corresponds to the revenue from auctions of type one and the second term corresponds to the revenue of auctions of type two. To bound \( \text{ESP}^* \), we note that in the optimal solution of Problem (ESP-OPT), the reserve of each buyer is either \( \delta \) or \( \lambda n \). Given that the buyers are symmetric, the value of the optimal solution depends only on the number of buyers with each reserve. Let \( t \) be the number of buyers with reserve \( \lambda n \). Then, we can write:

\[
\text{ESP}^* = \max_{0 \leq t \leq n} \left[ t \cdot \lambda n + (n-t) \cdot \delta + \left( \frac{n}{2} \right) - \left( \frac{t}{2} \right) \right].
\]

By taking \( \delta \to 0 \), we obtain,

\[
\text{ESP}^* = \max_{0 \leq t \leq n} \left[ t \cdot \lambda n + \left( \frac{n}{2} \right) - \left( \frac{t}{2} \right) \right].
\]

Since the term inside the maximum is a quadratic function of \( t \), the optimal integral solution should be \( t = \lambda n + o(n) \), since the optimal integral solution \( t \) deviates from the optimal fractional solution (which is \( \lambda n + 1/2 \)) by at most 1. Substituting that in the expression of \( \text{ESP}^* \), we get:

\[
\text{ESP}^* = \frac{1 + \lambda^2}{2} \cdot n^2 + o(n^2).
\]

Taking \( n \to \infty \), we get:

\[
\frac{\text{ESP}^*}{\text{LP}^*} \leq \frac{(1 + \lambda^2)n^2 + o(n^2)}{(1 + \lambda)n^2 + o(n^2)} \to \frac{1 + \lambda^2}{1 + \lambda}.
\]

We can choose the parameter \( \lambda = \sqrt{2} - 1 \) to minimize the above expression, which leads to a ratio of \( 2(\sqrt{2} - 1) \approx 0.828 \). ∎
8 Conclusion

In this paper, we take a data-driven approach to optimize personalized reserve prices in the eager second price auction. We design a polynomial time LP-based algorithm to optimize reserve prices on a given dataset of submitted bids and show that our algorithm obtains more than 0.684 fraction of the optimal revenue. This improves upon the best known approximation factor due to [RW16]. We believe that our data-driven approach as well as our LP-based algorithm can also be applied to a wider class of problems with revenue objective.

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