Dirac Field on Moyal-Minkowski Spacetime and Non-Commutative Potential Scattering

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Dedicated to Klaus Fredenhagen
on the occasion of his 61st birthday

Abstract. The quantized free Dirac field is considered on Minkowski spacetime (of general dimension). The Dirac field is coupled to an external scalar potential whose support is finite in time and which acts by a Moyal-deformed multiplication with respect to the spatial variables. The Moyal-deformed multiplication corresponds to the product of the algebra of a Moyal plane described in the setting of spectral geometry. It will be explained how this leads to an interpretation of the Dirac field as a quantum field theory on Moyal-deformed Minkowski spacetime (with commutative time) in a setting of Lorentzian spectral geometries of which some basic aspects will be sketched. The scattering transformation will be shown to be unitarily implementable in the canonical vacuum representation of the Dirac field. Furthermore, it will be indicated how the functional derivatives of the ensuing unitary scattering operators with respect to the strength of the non-commutative potential induce, in the spirit of Bogoliubov’s formula, quantum field operators (corresponding to observables) depending on the elements of the non-commutative algebra of Moyal-Minkowski spacetime.

1 Introduction

There are several theoretical arguments leading to the hypothesis that spacetime coordinates may, at extremely short length scales, no longer be described by continuous, mutually commuting numbers, but by non-commutative quantities, so that spacetime coordinates of events become subject to uncertainty relations characteristic of quantum physics. In a somewhat more technical sense, this means that the commutative algebra containing the coordinate functions of a spacetime manifold in the sense of general relativity
is to be replaced by a non-commutative algebra. There is a fair number of publications devoted to giving arguments to this effect and we shall make no attempt at reviewing this issue; for the best motivation, in our opinion, see [18].

For more than two decades by now, various approaches to non-commutative spacetimes have been investigated. In the context of physically motivated investigations, the procedure consists in replacing either the numerical coordinate functions on spacetime by elements of a non-commutative algebra, together with (more or less, physically motivated) commutation relations for a set of generating elements, or in replacing the spacetime symmetry group by a suitable deformation (e.g., a Hopf algebra), or both. Some of these non-commutative spacetime models correspond to Lorentzian spacetime structure, some others to Riemannian. It is then attempted to formulate physical theories (of matter or of gauge fields) over such spacetimes, patterned either after classical field theory, after quantum mechanics, or quantum field theory. (We refrain from listing the quite extended literature that is available on this matter and refer instead to the review [11] for comments and references.) Some of the results obtained along that route offer interesting and promising perspectives. However, it is quite difficult, at present, to compare all these various approaches [10, 4, 24, 25, 36]. Their interpretation is often problematic, in particular when the underlying non-commutative spacetime geometry corresponds to Riemannian metric signature. It would be very much desirable to have a mathematical and conceptual framework which allows to stage a discussion as to which of the various proposals for non-commutative spacetime geometries appear more favourable than others.

While it is very likely that such a discussion won’t lead to definite conclusions in the absence of experimental evidence for non-commutative spacetime structure, it would still be a valuable step to have a mathematical and conceptual framework broad enough so that the various approaches can be systematically compared. Concerning Riemannian non-commutative geometries, it appears that the general approach by spectral geometry, due to Alain Connes, provides such a framework in principle [12, 13, 14, 23]. It incorporates many of the known examples of Riemannian non-commutative manifolds. Moreover, the structural results which have been obtained in this approach — among them, but not only, the result that a spectral geometry with a commutative “function”- algebra actually corresponds to a Riemannian manifold — lend further to its support. Furthermore, one can give in this approach a certain reformulation of the current standard model of elementary particle physics [15]. This reformulation is, however, not a full quantum field theory, and cannot account for all processes of elementary particle physics in the same way as a quantum field theory. The conclusion may be — and we shall in fact take this point of view which we share, amongst others, with [18] — that a combination of non-commutative spacetime geometry and quantum field theory is a most promising candidate for
describing processes at extremely short distances and high energies, up to (and possibly including) Planck scale.

Now, the closest connection to the physical geometry of spacetime in quantum field theory is seen when spacetime has Lorentzian signature. In keeping with what we just mentioned, it would then be of interest to have a framework of spectral geometry corresponding to Lorentzian metric structure which includes, at least in approximation, those of the known non-commutative examples of Lorentzian spacetimes for which a good physical interpretation can be given. This is by no means a humble request as it is not at all straightforward to generalize the setting of spectral geometry from Riemannian to Lorentzian signature. While some proposals have been made [28, 29, 31, 40], they still seem to lack certain important ingredients and results. What appears to be lacking is a concept of covariance (see [34] for discussion), and structural results like in the Riemannian case. Endeavour in this regard shall be brought to the fore elsewhere [33], where a new approach to Lorentzian spectral geometry will be developed.

Accepting for the time being that the framework to appear in [33] yields a viable general setting for non-commutative Lorentzian geometry, the next question is if there is a general method of constructing quantum field theories over such Lorentzian spectral geometries. To wit, the ways of assigning quantum field theories with non-commutative spacetime models proposed up to now are, by and large, ad hoc, and not based on a systematic general method. It is therefore of importance to see if there is a systematic way to assign quantum field theories to abstractly described Lorentzian spectral geometries, and to identify the meaning of their observables.

In this publication, we attempt some first steps along these lines. We shall present a very superficial sketch of some elements of the setting of Lorentzian spectral geometry anticipated to be set out in detail in [33]. We discuss an abstract way of assigning a quantum field theory to any Lorentzian spectral geometry. In the case of a “classical” spacetime, with the usual commutative algebra of coordinate functions, this method amounts to assigning the quantized linear Dirac field to that spacetime. Part of our discussion will address the construction and meaning of observables in this setting. Much of this will actually be carried out at a much more concrete level by means of an example: The Dirac field on the Moyal-deformed non-commutative version of Minkowski spacetime. It will turn out that the spectral geometrical (“spectral triple”) data of Moyal-Minkowski spacetime agree with those of ordinary Minkowski spacetime except that the commutative algebra of functions on Minkowski spacetime — which may be taken to be Schwartz-type functions, $\mathcal{S}(\mathbb{R}^n)$ — is replaced by the algebra of Schwartz-type functions $\mathcal{S}(\mathbb{R}^n)_\star$, where the pointwise multiplication is replaced by the Moyal product (with commutative time). This indeed fits into the setting of spectral geometry as was demonstrated in detail in [20]. One can assign to these spectral data abstractly a quantized Dirac field es-
sentially by second quantization (or, more specifically, CAR-quantization) of the other spectral data, given by a Hilbert space carrying a Dirac operator etc. These quantizations agree for both usual Minkowski spacetime and Moyal-Minkowski spacetime since they depend on the same spectral data. A difference is only noticeable when the algebras of coordinate functions are brought into play. A very natural way to let them enter is via a scattering process.

Let us explain this at the level of the Dirac field on usual Minkowski spacetime. Consider the coupling of the Dirac field to an external scalar potential \( V = V(x^0, \vec{x}) \) where \( x^0 \) is the time-coordinate and \( \vec{x} \) denotes the spatial coordinates with respect to some chosen Lorentz frame. Assume that the potential is of Schwartz type and has finite support in time, i.e. it is different from zero only for \( x^0 \)-coordinates lying in some finite interval. Then the scattering of the Dirac field is described by a unitary S-matrix, \( S_V \), in the vacuum representation of the free Dirac field [1, 42]. Re-writing the scalar potential \( V \) as \( c(x^0, \vec{x}) = V(x^0, \vec{x}) \), and accordingly, \( S_c = S_V \), one can form the functional derivative \( \Phi(c) = -i \frac{d}{d\lambda}|_{\lambda=0} S_{\lambda c} \) of the scattering matrix with respect to the strength of the scalar scattering potential. This is a special case of “Bogoliubov’s formula” [6], whose content is, roughly speaking, the idea that observable quantum fields can be obtained from scattering matrices by functional differentiation with respect to the interaction strength. In the case of the Dirac field on Minkowski spacetime, one finds that \( \Phi(c) =: \psi^+ \psi : (c) \), i.e. the Wick-ordered operator standing for “absolute square of field strength”, which is an observable quantum field, as opposed to the quantized Dirac field operators \( \psi(f) \) labelled by spinor fields \( f \), which aren’t directly observable.

For the Dirac field on Moyal-Minkowski spacetime, one can proceed in a similar fashion, but now replacing the ordinary scalar potentials by “non-commutative” scalar potentials, where \( V = V(x^0, \vec{x}) \) is coupled to spinor fields not by pointwise multiplication at each spacetime point, but by Moyal-multiplication. (By the nature of Moyal-multiplication, this yields a non-local interaction of the Dirac field with the external potential.) We will show that, under certain, general conditions, there will then again be unitary S-matrices \( S^M_c = S^M_V \) (\( V = V_c \)) describing scattering by such non-commutative potentials in the vacuum representation of the free Dirac field. Furthermore, we also show that the corresponding functional derivatives \( \Phi(c) = -i \frac{d}{d\lambda}|_{\lambda=0} S^M_{\lambda c} \) exist as (essentially) selfadjoint operators. These operators are now in a natural way labelled by the elements \( c \) of the non-commutative algebra \( \mathcal{S}(\mathbb{R}^n) \), of Schwartz functions endowed with Moyal-multiplication. In principle, this method of assigning operators (to be interpreted as observables) to elements in the — possibly non-commutative — algebra of spacetime coordinate functions could be carried over to other models of non-commutative spacetimes, whenever it is possible to have a well-posed scattering process in the indicated sense (which seems to imply
restrictions on the degree of non-commutativity of time-like coordinates, at least asymptotically).

Let us now describe the contents and organization of the present article in more detail. In Sec. 2 we present the theory of the Dirac field. First, the classical Dirac field coupled to an external scalar potential, together with the theory of solutions, will be summarized, on \( n = 1 + s \) dimensional Minkowski spacetime, where \( n \) is even or fulfills the relations \( n = 3, 9 \mod 8 \). This then implies the existence of a “self-dual” charge-conjugation. These considerations draw mainly on material in \[17, 16, 23\]. We also present the abstract CAR \((C^*-algebraic)\) quantization of the Dirac field and summarize the connection between the scattering transformation induced by a scalar scattering potential and the \(C^*-algebraic\) Bogoliubov transformations which they induce, following mainly Araki’s works \[1, 2\]. The scattering transformations will be considered both in covariant form (following ideas in \[9\]), and in the Hamiltonian form at the level of Cauchy-data, since the interplay between both formulations will be useful later on.

Sec. 3 is a short section recapitulating the basics on the vacuum-representation of the Dirac field, both in covariant description and in the Hamiltonian, or Cauchy-data, formulation; and citing results on the unitary implementability of the Bogoliubov transform describing scalar potential scattering from \[32\]. The latter is mainly included for comparison with the non-commutative case treated later.

In Section 4 we discuss the non-commutative algebra \(\mathcal{S}(\mathbb{R}^n)\) of Schwartz functions with the Moyal product (with commutative time). Our presentation draws heavily on \[20\] with some small alterations adapted to our setting.

Sec. 5 contains the main conceptual considerations. In this longer section, we give a sketch of some of the ingredients of the approach to Lorentzian spectral geometry expected to appear in \[33\], illustrating the main points by the example of Moyal-deformed Minkowski spacetime with commutative time. We will discuss the general route of associating to a Lorentzian spectral geometry a quantum field theory, where the observables depend on the elements of the (non-commutative) “function-”algebra in the spectral geometric data, via the procedure of abstract CAR quantization and Bogoliubov’s formula, as indicated above, in some detail. (Incidentally, a formally similar set-up appears in \[22\], but in this reference, the quantum field transformations are not linked to any dynamical process like scattering.) Also some speculations about a possible general structure of quantum field theories over Lorentzian spectral geometries will appear in Sec. 5.

In Section 6 we investigate the solution properties of the Dirac equation with a non-commutative scalar potential (we investigate two such potentials obtained by Moyal-multiplying scalar functions with spinor fields). Since the time coordinate, in a chosen Lorentz frame, is still commutative, we can formulate the Cauchy-problem and establish its well-posedness. There are unique advanced and retarded fundamental solutions with respect to
the chosen Lorentzian frame. This is implied by (in fact, equivalent to) a uniquely solvable initial value problem in the Hamiltonian formulation of the Dirac equation, which we solve by constructing the Dyson series for the time-dependent interaction Hamiltonians (at the one-particle level). Correspondingly, we construct the one-particle scattering transformations, which induce Bogoliubov-transformations on the CAR-algebra of the free Dirac field, describing the scattering of the field by the non-commutative potential. This discussion parallels the discussion of the usual scalar potential scattering in many formal respects, but at several points, different arguments are required due to the non-local character of the non-commutative potential with respect to spatial coordinates.

The main results will be presented in Sec. 7. It will be proved that the Bogoliubov-transformations describing non-commutative potential scattering are unitarily implementable in the vacuum presentation of the free quantized Dirac field on Minkowski spacetime. In order to prove this, we make significant use of an earlier result by Langmann and Mickelsson [30] who developed a sufficient criterion for unitary implementability that can be applied in the case considered here. We show that this criterion is fulfilled. Furthermore, one can differentiate the Bogoliubov transformation with respect to the strength of the scattering potential as mentioned above. This leads to a derivation on the CAR-algebra, which we show to be induced by an essentially selfadjoint operator \( \Phi(c) \) in the vacuum-representation of the Dirac field. This is the precise form of the relation \( \Phi(c) = -i d/d\lambda|_{\lambda=0} S^M_\lambda \).

Finally, in Sec. 8, we derive the action of the derivative of the commutative and non-commutative scattering Bogoliubov transformations with respect to the potential strength on the generating elements of the Dirac field algebra (the field operators), drawing on the results of Sec. 6. Together with the relation \( \Phi(c) =: \psi^+ \psi : (c) \), which will be proved in Appendix A, this result finally hints at the operational meaning of \( \Phi(c) \), and illustrates how an interpretation of quantum field observables on a non-commutative spacetime may be reached at in more general situations.

There is a short conclusion and outlook in Sec. 9.

2 The Dirac Field

We start our discussion by summarizing some of the essentials on Dirac spinors and Dirac representations as far as required for describing the quantized Dirac field on \( n = 1 + s \) dimensional Minkowski spacetime. In doing so, we proceed quite leisurely; most of our presentation relies on [17], [2], [1], [3], [22], [23], [16]. We refer to these references for proofs of the statements appearing in this section.

Minkowski spacetime of dimension \( n = 1 + s \) will be described as \( \mathbb{R}^n \).
with the Minkowskian metric

$$\eta = (\eta_{\mu\nu})_{\mu,\nu=0}^{s} = \text{diag}(1, -1, \ldots, -1)$$

where the entry $-1$ appears $s$ times. [The opposite signature convention would in some respects suit the NCG context better, but we find it convenient to stick to the convention which is more common in QFT.] For any given $n = 1 + s \in \mathbb{N}$, $s \geq 1$, we set

$$N = N(n) = \begin{cases} 2^{n/2} & : n \text{ even} \\ 2^{(n-1)/2} & : n \text{ odd} \end{cases} \quad (1)$$

Then we refer to a collection $(\gamma_0, \gamma_1, \ldots, \gamma_s)$ of $N \times N$-matrices as a set of Dirac matrices if the relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \mathbf{1} \quad (\mu, \nu = 0, 1, \ldots, s)$$

$$\gamma_0^* = \gamma_0, \quad \gamma_k^* = -\gamma_k \quad (k = 1, \ldots, s)$$

are fulfilled. A set of Dirac matrices thus corresponds to an irreducible Dirac representation of the complexified Clifford algebra $\mathbb{C}l_{1,s}$; it exists for all $n \geq 2$.

We shall from now on restrict ourselves to dimensions

$$n \text{ even or } n = 3, 9 \text{ mod } 8. \quad (2)$$

For these values of $n$, it is possible to find a charge conjugation operator $C : \mathbb{C}^N \to \mathbb{C}^N$ for the Dirac matrices $(\gamma_0, \gamma_1, \ldots, \gamma_s)$; this means that $C$ is an antilinear involution ($C^2 = \mathbf{1}$) satisfying

$$C \gamma_\mu = -\gamma_\mu C, \quad (3)$$

whence, restriction to dimensions $n$ with (2) has the advantage that one can quantize Dirac fields with quite arbitrary (real) potentials in the “self-dual formalism” at the level of spinor fields only, without need to use a “doubled” system of spinor (and co-spinor) fields. The resulting simplification is convenient later when discussing the quantized Dirac field on Moyal-deformed spacetime.

Let $(\gamma_0, \gamma_1, \ldots, \gamma_s)$ be a set of Dirac matrices with charge conjugation $C$. Then we denote by

$$D_V = (-i\partial + m) + V \quad (4)$$

the Dirac operator (with fixed constant mass $m > 0$) with potential term $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$. $D_V$ acts on $f \in C^\infty(\mathbb{R}^n, \mathbb{C}^N)$ according to

$$(D_V f)(x) = (-i\partial + m)f(x) + V(x)f(x) \quad (x \in \mathbb{R}^n),$$
i.e. $V$ acts as a (scalar) multiplication operator, and writing $f^A(x)$ for the components of $f(x)$ regarded as a column vector, the explicit definition of $(-i\partial + m)$ is given by

$$((-i\partial + m)f)^A(x) = -i\gamma^A_B \frac{\partial}{\partial x^\mu} f^B(x) + mf^A(x) \quad (x \in \mathbb{R}^n),$$

(5)

where $\gamma^\mu = \eta^{\mu\nu} \gamma_\nu$ with $(\eta^{\mu\nu}) = \text{diag}(1, -1, \ldots, -1)$, and with $\gamma^A_B$ denoting the matrix entries of $\gamma^\mu$. We also make use of the summation convention so that doubly appearing indices are understood as being summed over.

On $C^\infty_0(\mathbb{R}^n, \mathbb{C}^N)$ we can introduce the sesquilinear form

$$\langle f, h \rangle = \int_{\mathbb{R}^n} \gamma_{0AB} \bar{f}^B(x) h^A(x) d^n x \quad (f, h \in C^\infty_0(\mathbb{R}^n, \mathbb{C}^N)),$$

(6)

where $\gamma_{0AB}$ are the matrix elements of $\gamma_0$. The charge conjugation $C$ is a skew conjugation for this sesquilinear form, that is,

$$\langle Cf, Ch \rangle = -\langle h, f \rangle \quad (f, h \in C^\infty_0(\mathbb{R}^n, \mathbb{C}^N)).$$

(7)

Since we will also use the description of solutions to the Dirac equation in terms of their Cauchy data, we have cause to introduce also the following objects. Let $t \in \mathbb{R}$ and define the $x^0 = t$ hyperplane

$$\Sigma_t = \{x = (x^0, x^1, \ldots, x^s) \in \mathbb{R}^n : x^0 = t\}$$

in $n = 1 + s$ dimensional Minkowski spacetime. We introduce the Hilbert space $D_t = L^2(\Sigma_t, \mathbb{C}^N)$ with canonical scalar product

$$(v, w)_{D_t} = \int_{\Sigma_t} \bar{v}^A(x) \delta_{AB} w^B(x) d^s x \quad (v, w \in D_t).$$

Each $D_t$ is canonically isomorphic to $L^2(\mathbb{R}^s, \mathbb{C}^N)$. Note that the charge conjugation $C$ induces a conjugation, denoted by the same symbol $C$, on each $D_t$, i.e. it holds that

$$(Cv, Cw)_{D_t} = (w, v)_{D_t} \quad (v, w \in D_t).$$

For a subset $G$ of $n$ dimensional Minkowski spacetime we define, following usual convention, $J^\pm(G)$ as the causal future$(+)/past(−)$ set of $G$, defined as consisting of all points that can be reached from $G$ by smooth future/past directed causal curves. We say that an open subset $G$ of $n$ dimensional Minkowski spacetime is hyperbolic if for each pair of points $x, y \in G$ the set $J^+(x) \cap J^-(y)$ is a subset of $G$. Examples of hyperbolic subsets are neighbourhoods $G$ of $\Sigma_t$ of the form $G = \{(x^0, x^1, \ldots, x^s) : t_+ > x^0 > t_-\}$ where $t_+ > t$ and $t_- < t$, or sets $G$ of the form $G = \text{int}(J^+(x) \cap J^-(y))$ where $y$ lies in the open interior of $J^+(x)$.

Now we collect some well-known results (well-known mainly in the context of the quantized Dirac field on curved spacetimes) on the existence and uniqueness of advanced and retarded fundamental solutions for the Dirac operator $D_V$. 

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Proposition 2.1 ([17], [3])

(a) \( \langle D_V f, h \rangle = \langle f, D_V h \rangle \) \( (f, h \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)) \)

(b) There is a unique pair of linear maps

\[ R^\pm_V : C_0^\infty(\mathbb{R}^n, \mathbb{C}^N) \rightarrow C^\infty(\mathbb{R}^n, \mathbb{C}^N) \]

having the properties

\[ D_V R^\pm_V f = f = R^\pm_V D_V f \quad \text{and} \quad \text{supp} \ R^\pm_V f \subset \mathcal{J}^\pm(\text{supp} \ f) \quad (f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)). \]

\( R^+_V \) is called advanced (+)/retarded (-) fundamental solution of \( D_V \).

(c) \( C R^+_V = R^-_V C \)

(d) Writing \( R_V = R^+_V - R^-_V \), the form

\[ (f, h)_V = \langle f, i R_V h \rangle \quad (8) \]

is a sesquilinear form on \( C_0^\infty(\mathbb{R}^n, \mathbb{C}^N) \), and \( C \) is a conjugation for this form:

\[ (Cf, Ch)_V = (h, f)_V = (\overline{f}, \overline{h})_V, \quad (f, h \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)). \]

(e) For each \( t \in \mathbb{R} \) it holds that

\[ (f, h)_V = (P_t R_V f, P_t R_V h)_D, \quad (f, h \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)), \]

where \( P_t : C^\infty(\mathbb{R}^n, \mathbb{C}^N) \rightarrow C^\infty(\Sigma_t, \mathbb{C}^N) \) is the map given by

\[ P_t : \varphi \mapsto \varphi(t, \cdot) \]

for \( \varphi : (x^0, x) \mapsto \varphi(x^0, x) \) in \( C^\infty(\mathbb{R}^n, \mathbb{C}^N) \), \( x = (x^1, \ldots, x^n) \). Hence, \( (\cdot, \cdot)_V \) is positive-semidefinite on \( C_0^\infty(\mathbb{R}^n, \mathbb{C}^N) \).

(f) The Cauchy-problem for the Dirac-equation \( D_V \varphi = 0 \) is well-posed: Given any Cauchy-hyperplane \( \Sigma_t \) and Cauchy-data \( w \in \mathcal{S}(\Sigma_t, \mathbb{C}^N) \), there is a unique \( \varphi \in C^\infty(\mathbb{R}^n, \mathbb{C}^N) \) such that

\[ D_V \varphi = 0 \quad \text{and} \quad P_t \varphi = \varphi|_{\Sigma_t} = w. \]

Furthermore, the solution \( \varphi \) fulfills the causal propagation property in the sense that

\[ \text{supp} \ \varphi \subset J(\text{supp} \ w). \]
(g) Let $E_V$ be the subspace of all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ so that $(f, f)_V = 0$, and let $K_V$ be the Hilbert space arising as completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)/E_V$ with respect to the scalar product induced by $(\cdot, \cdot)_V$ (which will be denoted by the same symbol). The quotient map $C_0^\infty(\mathbb{R}^n, \mathbb{C}^N) \to C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)/E_V$ will be written

$$f \mapsto [f]_V.$$ 

Then for each $t \in \mathbb{R}$, the map

$$Q_{V,t} : [f]_V \mapsto P_t R_V f$$

extends to a unitary map from $K_V$ onto $\mathcal{D}_t$.

(h) Let $G$ be a hyperbolic subset of $n$ dimensional Minkowski spacetime, and suppose that $V_1$ and $V_2$ are real-valued, $C^\infty$, and that $V_1 = V_2$ on $G$. Then

$$R_{V_1}^\pm f = R_{V_2}^\pm f \text{ on } G \text{ for all } f \in C_0^\infty(G, \mathbb{C}^N).$$

Sketch of proof

(a) This is a straightforward calculation.

(b) This is proved using the same argument as for Theorem 2.1 in [17], which applies also in the presence of a real scalar potential $V$, together with the existence and uniqueness result for fundamental solutions of hyperbolic wave operators, which can be found (in far greater generality than needed here) in [3].

(c) This is a consequence of the uniqueness of the $R_{V}^\pm$ together with $CD_V = D_V C$.

(d) The only non-obvious part $(Cf, Ch)_V = (h, f)_V$ of the claim follows easily from (c), equation (7) and the relation

$$\langle R_V f, h \rangle = - \langle f, R_V h \rangle,$$

which is shown in the proof of Theorem 2.1 in [17].

(e) The argument is the same as for Proposition 2.4 (d) in [17].

(f) This statement is proved analogously to Thm. 2.3 in [17]. It is proved there for the case that the Cauchy-data are $C_0^\infty$. However, existence and uniqueness of a distributional solution is proved in Prop. 2.4 in [17] for distributional Cauchy-data. The smoothness of the solution in case of Cauchy-data that are of Schwartz type can be proved by making use of the causal propagation property of the solutions (i.e. supp $\varphi \subset \mathcal{J}(\text{supp } \varphi)$) in combination with a partition of unity argument.
(g) In view of (i), what remains to be checked is the surjectivity of $Q_{V,t}$. To see this, let $w \in C_0^\infty(\Sigma_t, \mathbb{R}^N)$, and let $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{C}^N)$ be the solution of $D_V \varphi = 0$ having Cauchy-data $w$ on $\Sigma_t$, i.e. $P_t \varphi = w$. We will construct some $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ so that $P_t R_V f = w$. To this end, we take two further Cauchy-hyperplanes, $\Sigma_{\pm}$, with

$$\Sigma_{\pm} = \{ x = (x^0, \ldots, x^s) : x^0 = t \pm 1 \}.$$

Then we can consider the open sets

$$G_{\pm} = \text{int}(J_{\pm}(\Sigma_{\pm})) = \{ x = (x^0, \ldots, x^s) : \pm x^0 > t \mp 1 \}.$$

The sets $G_{\pm}$ form an open covering of $\mathbb{R}^n$. Let $\chi_{\pm}$ be a $C^\infty$ partition of unity of $\mathbb{R}^n$ subordinate to the covering. It is easy to see that the functions $\chi_{\pm}$ can be chosen in such a way that they depend only on $x^0$, and we will assume that this choice has been made (although this is not relevant at this point; see however the proof of Prop. 6.3 (g) later). Then one has

$$D_V(\chi_+ \varphi) = -D_V(\chi_- \varphi),$$

and owing to the support properties of $\chi_{\pm}$, one concludes that both $D_V(\chi_{\pm} \varphi)$ have support contained in $G_{\pm} = \{ (x^0, \ldots, x^s) : t + 1 \geq x^0 \geq t - 1 \}$. One the other hand, since we also have $\text{supp} \ \varphi \subset J(\text{supp} \ w)$ and since $\text{supp} \ w$ was assumed to be compact, this implies that both $D_V(\chi_{\pm} \varphi)$ are $C_0^\infty$. Setting now $f = D_V(\chi_+ \varphi)$, it holds that $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$, and moreover, we see that $D_V(R_V f - \varphi) = 0$. However, we also have that $R_V f = R_+ V f - R_- V f$, and owing the support properties of $R_+ V$, on $\mathbb{R}^n \setminus G_+ = \{ (x^0, \ldots, x^s) : x^0 > t + 1 \}$ it holds that $R_+ V f = \chi_+ V f = \chi_+ \varphi = \varphi$. This means that $P_t (R_V f - \varphi) = 0$ for all real $\tau > t + 1$ and hence, since $(R_V f - \varphi)$ is a $C^\infty$ solution of the Dirac equation with $C_0^\infty$ Cauchy-data, one actually concludes that $(R_V f - \varphi) = 0$ on all of $\mathbb{R}^n$. Hence we have shown that there is some $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ with $R_V f = \varphi$, implying $P_t R_V f = P_t \varphi$. This shows that the range of $Q_{V,t}$ is dense, and by its isometric property, $Q_{V,t}$ is actually surjective.

(h) The spacetime region $G$, endowed with the Minkowski metric (and standard spin structure) is a globally hyperbolic spacetime. Given a smooth real-valued $V : G \to \mathbb{R}$ as potential function, one can define the “intrinsic” Dirac operator of $G$, $D_{V|G} : C^\infty(G, \mathbb{C}^N) \to C^\infty(G, \mathbb{C}^N)$ by $D_{V|G} f = D_V f$, $f \in C^\infty(G, \mathbb{C}^N)$, which is nothing but the canonical restriction of $D_V$ onto $G$. According to [17] (cf. also [3]), there are unique advanced/retarded fundamental solutions $R_{V|G}^\pm : C_0^\infty(G, \mathbb{C}^N) \to C^\infty(G, \mathbb{C}^N)$ for $D_{V|G}$. Now, if $V_1 = V_2 = V$ on
G, then the appropriate restrictions of $R^\pm_{V_1}$ and $R^\pm_{V_2}$ onto $C^\infty_0(G, \mathbb{C}^N)$ (more precisely, the maps $f \mapsto (R^\pm_{V_j})^\circ_G$, $f \in C^\infty_0(G, \mathbb{C}^N)$, $j = 1, 2$) have the same properties as the map $R^\pm_{V_j}|_G$. Hence, by the uniqueness statement, these restrictions must be equal to $R^\pm_{V_j}|_G$. □

Starting from $(K_V, C)$, the Hilbert space $K_V$ with conjugation $C$, one can form, following [2], the corresponding self-dual CAR-algebra $\mathcal{F}(K_V, C)$. It is defined as follows: One introduces a $*$-algebra generated by symbols $B(\xi) = B_V(\xi), \xi \in K_V$, subject to the relations

\[
B(\xi)^* = B(C\xi),
\]

\[
B(\xi_1)^*B(\xi_2) + B(\xi_2)B(\xi_1)^* = 2(\xi_1, \xi_2)_V \mathbf{1},
\]

\[
\xi \mapsto B(\xi) \text{ is complex linear},
\]

where $\mathbf{1}$ is an algebraic unit. One can show that the resulting $*$-algebra admits a unique $C^*$-norm, and $\mathcal{F}(K_V, C)$ is the completion of that $*$-algebra with respect to the $C^*$-norm. Therefore, $\mathcal{F}(K_V, C)$ is a $C^*$-algebra. Writing $\Psi(f) = \Psi_V(f) = B_V([f]_V)$ for $f \in C^\infty_0(\mathbb{R}^n, \mathbb{C}^N)$, $\mathcal{F}(K_V, C)$ is generated by “abstract field operators” $\Psi(f)$, which are $\mathbb{C}$-linear and obey the relations

\[
\Psi(f)^* = \Psi(Cf),
\]

\[
\{\Psi(f)^*, \Psi(h)\} = 2(f, h)_V \mathbf{1},
\]

\[
\Psi(D_V f) = 0.
\]

The construction of $\mathcal{F}(K_V, C)$ can also be carried out, in analogous manner, for “local subspaces” of $K_V$. For this purpose, let $G$ be a hyperbolic subset of $n$ dimensional Minkowski spacetime. For $f \in C^\infty_0(G, \mathbb{C}^N)$, we introduce the equivalence class

\[
[f]^G_V = \{f + h : h \in C^\infty_0(G, \mathbb{C}^N), R_V h = 0\}.
\]

As before, the space of the $[f]^G_V$ carries a scalar product $(\cdot, \cdot)_V$ in the same fashion as $C^\infty_0(\mathbb{R}^n, \mathbb{C}^N)/E_V$ (denoted by the same symbol as there is no danger of confusion). The resulting Hilbert-space completion will be denoted by $K^G_V$. Again, the charge conjugation $C$ induces a conjugation on $K^G_V$ as well. Whence, we can form the self-dual CAR-algebra $\mathcal{F}(K^G_V, C)$, which is the $C^*$-algebra generated by symbols $B^G_V([f]^G_V)$, $[f]^G_V \in K^G_V$, obeying relations akin to those fulfilled by the $B([f]_V)$ above.

**Lemma 2.2** Suppose that $G$ is a hyperbolic neighbourhood of a Cauchy hyperplane in $n$ dimensional Minkowski spacetime. Moreover, suppose that $V_1$ and $V_2$ are two smooth, real-valued potentials which coincide on the region $G$. Then
(a) The map
\[ u^{G}_{V_1, V_2} : [f]_{V_1}^{G} \mapsto [f]_{V_2}, \quad f \in C_0^\infty(G, \mathbb{C}^N) \]
extends to a unitary between \( \mathcal{K}^{G}_{V_1} \) and \( \mathcal{K}_{V_2} \) commuting with the charge conjugation \( C \). 

(b) There is a \( * \)-algebra isomorphism 
\[ \alpha^{G}_{V_1, V_2} : \mathfrak{F}(\mathcal{K}^{G}_{V_1}, C) \rightarrow \mathfrak{F}(\mathcal{K}_{V_2}, C) \]
induced by 
\[ \alpha^{G}_{V_1, V_2}(B^{G}_{V_1}([f]_{V_1}^{G})) = B_{V_2}([f]_{V_2}), \quad f \in C_0^\infty(G, \mathbb{C}^N). \]

Proof

(a) In view of (1) of Proposition 2.1, \( R^{G}_{V_1} f = R_{V_2} f \) on \( G \) for all \( f \in C_0^\infty(G, \mathbb{C}^{N}) \). Using the definition of \( (\cdot, \cdot)_{V} \), this implies that the map \( u^{G}_{V_1, V_2} \) is isometric. To show that the map is surjective, let \( h \in C_0^\infty(\mathbb{R}^{n}, \mathbb{C}^N) \). Since \( \mathcal{G} \) is an open neighbourhood of a Cauchy surface, there is some \( f \in C_0^\infty(G, \mathbb{C}^N) \) such that \( R_{V_2}(f - h) = 0 ([3]) \) and hence \( [f]_{V_2} = [h]_{V_2} \).

(b) This is a straightforward consequence of the fact that \( u^{G}_{V_1, V_2} \) is a unitary intertwining the action of \( C \), see [1] (or [7],[8]). \( \square \)

We will now make use of the Lemma. Suppose that a smooth scalar (real) potential \( V \) is given on \( n \) dimensional Minkowski spacetime, having support contained in the time-slice \( \{ (x^0, x^1, \ldots, x^s) : \lambda_- < x^0 < \lambda_+ \} \) for some real numbers \( \lambda_- < \lambda_+ \). Then one can consider the regions 
\[ G_+ = \{ (x^0, x^1, \ldots, x^s) : x^0 > \lambda_+ + \frac{1}{2} \} \text{ and} \]
\[ G_- = \{ (x^0, x^1, \ldots, x^s) : x^0 < \lambda_- - \frac{1}{2} \}. \]

They form hyperbolic neighbourhoods of the Cauchy hyperplanes 
\[ \Sigma_+ = \{ (x^0, x^1, \ldots, x^s) : x^0 = \lambda_+ + 1 \} \text{ and} \]
\[ \Sigma_- = \{ (x^0, x^1, \ldots, x^s) : x^0 = \lambda_- - 1 \}, \]
respectively. Lemma [2.2] then warrants the following \( C^* \)-algebraic isomorphisms:
\[ \alpha_0 = \alpha^{G_\pm}_{0,0} : \mathfrak{F}(\mathcal{K}^{G_\pm}_{0,0}, C) \rightarrow \mathfrak{F}(\mathcal{K}_0, C) \]
\[ B^{G_\pm}_{0}([f]_{0}^{G_\pm}) \mapsto B_{0}([f]_{0}), \quad f \in C_0^\infty(G_{\pm}, \mathbb{C}^N). \]
\[ \alpha_{V}^{\pm} = \alpha_{0,V}^{G_{\pm}} : \mathfrak{F}(\mathcal{K}_{0}^{G_{\pm}}, \mathbb{C}) \to \mathfrak{F}(\mathcal{K}_{V}, \mathbb{C}) \]
\[ B_{0}^{G_{\pm}}([f]^{G_{\pm}}) \mapsto B_{V}([f]_{V}), \quad f \in C_{0}^{\infty}(G_{\pm}, \mathbb{C}^{N}). \]

Since these maps are isomorphisms, they can be combined into an automorphism

\[ \beta_{V} : \mathfrak{F}(\mathcal{K}_{0}, \mathbb{C}) \to \mathfrak{F}(\mathcal{K}_{0}, \mathbb{C}), \]
\[ \beta_{V} = \alpha_{0,-} \circ \alpha_{V,-}^{-1} \circ \alpha_{V,+} \circ \alpha_{0,=}^{-1}. \quad (11) \]

This isomorphism is reminiscent of a similar object defined in Section 4 of [9], and it has similar properties. Its significance is that it describes the scattering of the quantized Dirac field by the classical potential \( V \) at the level of a \( C^{*} \)-algebraic Bogoliubov transformation. In order to see this more clearly, we will discuss how \( \beta_{V} \) relates to the perhaps more familiar scattering formalism in terms of time-evolution on the Cauchy data.

For this purpose, let us first revisit \( \beta_{V} \). We have

\[ \beta_{V}(B_{0}([f]_{0})) = B_{0}(U_{V}[f]_{0}), \quad (12) \]

where \( U_{V} \) is the unitary given by

\[ U_{V} = u_{0,-} \circ u_{V,-}^{-1} \circ u_{V,+} \circ u_{0,=}^{-1} \]

and where, similarly as for the isomorphisms above, we have used the abbreviations

\[ u_{0,=} = u_{0,0}^{G_{+}}, \quad u_{V,=} = u_{0,V}^{G_{+}}. \]

The action of the succession of unitaries on the right hand side of the defining equation of \( U_{V} \) can be described as follows:

\[ [f]_{0} \mapsto [f]_{0}^{G_{+}} \mapsto [f^{G_{+}}]_{0} \mapsto [f^{G_{+}}]_{V} \mapsto [f^{G_{+}}]_{0} \mapsto [f^{G_{+}}]_{0} \quad (13) \]

In this chain of mappings, \( f^{G_{+}} \) is any element in \( C_{0}^{\infty}(G_{+}, \mathbb{C}^{N}) \) such that \( R_{0}(f - f^{G_{+}}) = 0 \), and \( f^{G_{-}} \) is any element in \( C_{0}^{\infty}(G_{-}, \mathbb{C}^{N}) \) such that \( R_{V}(f^{G_{+}} - f^{G_{-}}) = 0 \).

Turning to the description of the quantized Dirac field in terms of its Cauchy data, we recall that \( \mathcal{D}_{0} = L^{2}(\Sigma_{0}, d^{x}x) \), where \( \Sigma_{0} \) is the \( x^{0} = 0 \) Cauchy hyperplane. We have also seen that the charge conjugation \( C \) acts as a complex conjugation on \( \mathcal{D}_{0} \). Hence one can associate to \( \mathcal{D}_{0} \) and \( C \) the CAR-algebra \( \mathfrak{F}(\mathcal{D}_{0}, \mathbb{C}) \) with generators \( B_{\mathcal{D}_{0}}(v) \), \( v \in \mathcal{D}_{0} \), linear in \( v \), and with the relations

\[ B_{\mathcal{D}_{0}}(v)^{*} = B_{\mathcal{D}_{0}}(Cv), \quad \{B_{\mathcal{D}_{0}}(v)^{*}, B_{\mathcal{D}_{0}}(w)\} = 2(v, w)\mathbb{1}. \quad (14) \]
Writing $Q_0$ for $Q_{V,0}$ in the case of $V = 0$, $Q_0 : K_0 \to D_0$, $[f]_0 \mapsto P_0 R_0 f$ is a unitary intertwining the actions of $C$ on the respective Hilbert spaces. Consequently ([(2)], there is a canonical isomorphism $\varrho : \mathfrak{F}(K_0, C) \to \mathfrak{F}(D_0, C)$ of CAR-algebras induced by

$$\varrho(B_0([f]_0)) = B_{D_0}(Q_0([f]_0)).$$

(15)

On $\mathfrak{F}(D_0, C)$, we can introduce two types of time evolutions, one corresponding to a vanishing potential $V = 0$ in the Dirac equation (the “free” dynamics), and another corresponding to a non-vanishing $C^\infty$ potential term $V$ in the Dirac equation (the “interacting” dynamics). These dynamical evolutions will be defined on the Cauchy-data space $D_0$. To this end, we define on $C^\infty_0(\mathbb{R}^s, \mathbb{C}^N)$ the operators

$$\begin{align*}
(H_0 f)(x^1, \ldots, x^s) &= \left( i\gamma^0 \gamma^k \frac{\partial}{\partial x^k} + \gamma^0 m \right) f(x^1, \ldots, x^s) \\
(H_V(t) f)(x^1, \ldots, x^s) &= \left( i\gamma^0 \gamma^k \frac{\partial}{\partial x^k} + \gamma^0 m + \gamma^0 V(t) \right) f(x^1, \ldots, x^s),
\end{align*}$$

where $f(x^1, \ldots, x^s)$ is regarded as column vector on which the $\gamma$-matrices act by matrix multiplication. These operators are symmetric with respect to the scalar product $(\cdot, \cdot)_{D_0}$, and even essentially selfadjoint under very general conditions on (the real-valued) $V(t)$ (e.g. see Theorem 1.1 of [42], and Theorem X.69 of [35]). Moreover, it is easy to check that the operators anti-commute with the charge conjugation $C$,

$$C H_0 = -H_0 C, \quad C H_V(t) = -H_V(t) C.$$  

(17)

There is hence a continuous unitary group $T_t$, $t \in \mathbb{R}$, on $D_0$ such that

$$\frac{1}{i} \left. \frac{d}{dt} T_t \right|_{t=0} v = H_0 v, \quad v \in C^\infty_0(\mathbb{R}^s, \mathbb{C}^N).$$

There is also a continuous family of unitarities $T_{s,t}^{(V)}$, $s, t \in \mathbb{R}$, so that

$$T_{r,s}^{(V)} \circ T_{s,t}^{(V)} = T_{r,t}^{(V)}, \quad T_{t,t}^{(V)} = 1 \quad \text{and} \quad \frac{1}{i} \left. \frac{d}{ds} T_{s,t}^{(V)} \right|_{s=t} v = H_V(t) v, \quad v \in C^\infty_0(\mathbb{R}^s, \mathbb{C}^N).$$

(18)

Let us indicate that the existence of the family $T_{s,t}^{(V)}$ with the said properties is implied by the well-posedness of the Cauchy problem for the Dirac equation: For each $v \in C^\infty_0(\mathbb{R}^s, \mathbb{C}^N) \subset D_t$ there is a unique solution $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{C}^N)$ to the Dirac equation

$$D_V \varphi = 0.$$
having initial data \( v \) on \( \Sigma_t \), i.e.
\[
P_t \phi = \phi|_{\Sigma_t} = v.
\]

The solution property is equivalent to
\[
\frac{1}{i} \frac{d}{dt} P_t \phi = H_V(t) P_t \phi.
\] (19)

On the other hand, the uniqueness statement implies that there is a map \( T_{t,t'}^{(V)} : P_{t'} \phi \mapsto P_t \phi \) with the properties \( T_{t,t'}^{(V)} \circ T_{t',t''}^{(V)} = T_{t,t''}^{(V)} \) and \( T_{t,t}^{(V)} = 1 \).

And \( \frac{1}{i} \frac{d}{ds} \big|_{s=t} T_{s,t}^{(V)} = H_V(t) \) on \( C_0^\infty(\mathbb{R}^n, \mathbb{C}^N) \) then follows from (19). The unitarity of \( T_{t,t'}^{(V)} \) is implied by Proposition 2.1 (e). We note also that
\[
CT_t = T_t C \quad \text{and} \quad CT_{t,t'}^{(V)} = T_{t,t'}^{(V)} C
\] (20)
on account of (17). Therefore, \( T_t \) and \( T_{t,t'}^{(V)} \) give rise to CAR-algebra automorphisms \( \tau_t \) and \( \tau_{t,t'}^{(V)} \) of \( \mathcal{F}(D_0, C) \) induced by
\[
\tau_t(B_{D_0}(v)) = B_{D_0}(T_t v),
\]
\[
\tau_{t,t'}^{(V)}(B_{D_0}(v)) = B_{D_0}(T_{t,t'}^{(V)} v).
\]

As before, we will now assume that the potential \( V \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \) has support contained in the set \( \{x^0, x^1, \ldots, x^s : \lambda_- < x^0 < \lambda_+ \} \) for some real numbers \( \lambda_- < \lambda_+ \). The scattering operator for the Dirac equation at the level of Cauchy data on \( \Sigma_0 \) is the operator
\[
T_{sc}^{(V)} = \lim_{t' \to -\infty} T_{t,t'}^{-1} \circ T_{t,t'}^{(V)} \circ T_{t'}
\] (21)
on \( D_0 \). The restriction on the time-support of \( V \) implies that the limit (21) is reached as soon as \( t' > \lambda_+ \) and \( t < \lambda_- \), so that
\[
T_{sc}^{(V)} = T_{t,t'}^{-1} \circ T_{t,t'}^{(V)} \circ T_{t'} \quad \text{for} \quad t' > \lambda_+, t < \lambda_-.
\] (22)

We denote by \( \tau_{sc}^{(V)} \) the corresponding scattering morphism on \( \mathcal{F}(D_0, C) \) given by
\[
\tau_{sc}^{(V)}(B_{D_0}(v)) = B_{D_0}(T_{sc}^{(V)} v).
\] (23)

Now we want to demonstrate that
\[
\varrho \circ \beta_V = \tau_{sc}^{(V)} \circ \varrho.
\] (24)

Thus we aim at showing
\[
Q_0 \circ U_V = T_{sc}^{(V)} \circ Q_0.
\] (25)
To prove this, we write the action of $Q_0^{-1} \circ T_{sc}^{(V)} \circ Q_0$ on an element $[f]_0 \in \mathcal{K}_0$ in the following form:

$[f]_0 \xrightarrow{Q_0} P_0 R_0 f \xrightarrow{T_{t'}} P_t R_0 f \xrightarrow{(\ast 1)} P_t R_0 h^{G_+} \xrightarrow{(\ast 2)} P_t R_V h^{G_+} \xrightarrow{T_{t',t' \to t}^{(V)}} P_t R_V h^{G_+} \xrightarrow{(\ast 3)} P_t R_V h^{G_-} \xrightarrow{(\ast 4)} P_t R_0 h^{G_-} \xrightarrow{T_{t \to 0}} P_0 R_0 h^{G_-} \xrightarrow{Q_0^{-1}} [h^{G_-}]_0$

In this succession of maps, at $(\ast 1)$ an element $h^{G_+} \in C_0^\infty(G_+, \mathbb{C}^N)$ is chosen so that $R_0 h^{G_+} = R_0 f$. At $(\ast 2)$, it is used that $R_V h^{G_+} = R_0 h^{G_+}$ on $G_+$ because of the support properties of the functions $V$ and $h^{G_+}$, cf. Proposition 2.1 (h). At $(\ast 3)$, an element $h^{G_-} \in C_0^\infty(G_-, \mathbb{C}^N)$ is chosen so that $R_V h^{G_+} = R_0 h^{G_-}$. Then at $(\ast 4)$, it is again used that $R_0 h^{G_-} = R_V h^{G_-}$ on $G_-$ owing to the support properties of $V$ and $h^{G_-}$. Comparing (13) and (26), one can see that the specifications of $f^{G_\pm}$ and $h^{G_\pm}$ are such that one can may even choose (starting from the same given $f$) $f^{G_\pm} = h^{G_\pm}$, and this then proves the relation (25). Summarizing, we have proved

**Lemma 2.3** The morphism $\beta_V$ of $\mathfrak{F}(\mathcal{K}_0, C)$ defined in (11) and the scattering morphism $\tau_{sc}^{(V)}$ describing the potential scattering of the quantized Dirac field at the level of the Cauchy-data CAR-algebra $\mathfrak{F}(\mathcal{D}_0, C)$ are intertwined by the CAR-algebra isomorphism $\Phi : \mathfrak{F}(\mathcal{K}_0, C) \to \mathfrak{F}(\mathcal{D}_0, C)$ defined in (15), i.e. it holds that

$\Phi \circ \beta_V = \tau_{sc}^{(V)} \circ \Phi$.

One advantage of working with $\beta_V$ is that it can be associated to localization in spacetime: It acts trivially outside of $J(\text{supp } V)$. This is our next assertion.

**Proposition 2.4** Let $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ have support causally disjoint from $\text{supp } V$, i.e. $\text{supp } f \cap J(\text{supp } V) = \emptyset$. Then

$\beta_V(\Psi_0(f)) = \Psi_0(f)$.

**Proof** According to Proposition 2.1 (h), if $\text{supp } f \cap J(\text{supp } V) = \emptyset$, then $R_0 f = R_V f$ on $\mathbb{R}^n \setminus J(\text{supp } V)$, since $\mathbb{R}^n \setminus J(\text{supp } V)$ is a hyperbolic region in $n$ dimensional Minkowski spacetime. On the other hand, $\text{supp } f \cap J(\text{supp } V) = \emptyset$ is equivalent to $J(\text{supp } f) \cap \text{supp } V = \emptyset$. Now, $R_V f$ is a solution to $(D + V) R_V f = 0$, and $\text{supp } R_V f \subset J(\text{supp } f)$, thus $\text{supp } R_V f \cap \text{supp } V = \emptyset$, implying that $D R_V f = 0$. Consequently, $R_0 f$ and $R_V f$ are both solutions to the Dirac equation with vanishing potential $V = 0$,
and coincide in the neighbourhood of a Cauchy surface for $n$ dimensional Minkowski spacetime (which is implied by $R_0 f = R_V f$ on $\mathbb{R}^n \setminus J(\text{supp} V)$ and $\text{supp} R_0 f \cup \text{supp} R_V f \subset J(\text{supp} f)$). This implies that $R_0 f = R_V f$ on all of $n$ dimensional Minkowski spacetime. Now consider the map

$$U_V : [f]_0 \longrightarrow [f^{G+}]_{0}^{G+} \longrightarrow [f^{G+}]_V \longrightarrow [f^{G-}]_{0}^{G-} \longrightarrow [f^{G-}]_0. \quad (27)$$

In this succession of mappings, $f^{G+}$ is any element in $C_0^\infty(G_+, \mathbb{C}^N)$ so that $R_0 (f - f^{G+}) = 0$, and $f^{G-}$ is any element in $C_0^\infty(G_-, \mathbb{C}^N)$ so that $R_V (f^{G+} - f^{G-}) = 0$. However, since $R_0 f = R_V f$, it holds that $R_V (f - f^{G+}) = R_0 f - R_V f^{G+} = R_0 f - R_0 f^{G+} = 0$ on $G_+$, hence $R_V (f - f^{G+}) = 0$ on $G_+$, and hence $R_V (f - f^{G+})$ is equal everywhere on $n$ dimensional Minkowski spacetime. Furthermore, $R_V f^{G-} = R_V f^{G+}$, from which $R_V (f - f^{G-}) = 0$ obtains. On the other hand, we also have $R_V f^{G-} = R_0 f^{G-}$ on $G_-$, and $R_V f = R_0 f$, thus $R_0 f = R_0 f^{G-}$ on $G_-$, and hence everywhere on $n$ dimensional Minkowski spacetime. This shows that $[f]_0 = [f^{G-}]_0$ and therefore, $U_V [f]_0 = [f]_0$. In view of (12), this yields the claimed proposition. □

3 Scattering of the Dirac field in the vacuum representation and implementability of the scattering transformation

The Hamilton operator $H_0$ defined in (16) is essentially selfadjoint on $C_0^\infty(\mathbb{R}^s, \mathbb{C}^N) \subset L^2(\mathbb{R}^s, \mathbb{C}^N)$ (see Theorem 1.1 of [42]). Therefore, its self-adjoint extension, again denoted by $H_0$, possesses a spectral decomposition, and we denote by $p_+$ the spectral projection of $H_0$ corresponding to the spectral interval $(0, \infty)$. Since the mass term $m$ in the Dirac equation has been assumed to be strictly greater than 0, $p_+$ projects in fact on the spectral values in $[m, \infty)$ and the orthogonal projector $p_- = 1 - p_+$ coincides with the spectral projector of the spectral interval $(-\infty, -m]$. Owing to $CT_t = T_t C$ for all $t \in \mathbb{R}$, it holds that

$$C p_+ = p_- C.$$

Thus, $p_+$ is a basis projection in the sense of [2]. To this basis projection one can associate a pure, quasifree state $\omega^{p_+}$ on $\mathfrak{F}(\mathcal{D}_0, C)$ whose two-point function is given by

$$\omega^{p_+}_2 (B_{\mathcal{D}_0} (u)^* B_{\mathcal{D}_0} (w)) = (u, p_+ w)_{\mathcal{D}_0}, \quad u, w \in \mathcal{D}_0.$$

The state can be pulled back by $\varrho$ to a pure, quasifree state

$$\omega^{\text{vac}} = \omega^{p_+} \circ \varrho.$$

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on $\mathfrak{F}(K_0, C)$. This state is actually just the usual ($\mathfrak{P}_+^\dagger(n)$-invariant) vacuum state on $\mathfrak{F}(K_0, C)$. Writing

$$e_+ = Q_0^{-1} p_+ Q_0,$$

its GNS-representation $(\mathcal{H}^{\text{vac}}, \pi^{\text{vac}}, \Omega^{\text{vac}})$ can be realized as follows:

$$\mathcal{H}^{\text{vac}} = \mathcal{F}_+(e_+(K_0)),$$

is the Fermionic Fock space over the one-particle Hilbert space $e_+(K_0)$ ($e_+$ projects on the “positive frequency” solutions of the Dirac equation), $\Omega^{\text{vac}} = (1, 0, 0, \ldots)$ the Fock vacuum vector,

$$\pi^{\text{vac}}(\Psi_0(f)) = A(e_+C[f]_0) + A^+(e_+[f]_0),$$

where $A(\chi)$ and $A^+(\chi)$ denote, respectively, the Fermionic annihilation and creation operators of a $\chi$ in the one-particle Hilbert space. We will sometimes use the notation

$$\psi(f) = \pi^{\text{vac}}(\Psi_0(f)), \quad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N),$$

for the field operators of the quantized Dirac field in the vacuum representation.

For several reasons, it is important to investigate the question of unitary implementability of the scattering transformation in the vacuum representation. In the situation at hand, this is the question if there exists a unitary operator $S_V : \mathcal{H}^{\text{vac}} \to \mathcal{H}^{\text{vac}}$ such that

$$S_V \pi^{\text{vac}}(\Psi_0(f)) S^{-1}_V = \pi^{\text{vac}}(\beta_V(\Psi_0(f))), \quad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N). \quad (28)$$

This issue has been investigated for the Dirac field on Minkowski spacetime by several authors in various publications that have appeared over the last decades. The result is that there is such an operator, or “$S$-matrix”, provided that the potential $V$ is sufficiently regular and sufficiently fast decaying. A sufficient condition to this end, which is convenient for comparison with developments presented later in this article, is the following

**Proposition 3.1** If $V$ is in $\mathcal{S}(\mathbb{R}^n, \mathbb{R})$ (the class of Schwartz functions) and if $V$ has compact support with respect to the time-coordinate $x^0$, then there is a unitary operator $S_V$ on $\mathcal{H}^{\text{vac}}$ implementing the potential scattering morphism $\beta_V$ in the vacuum representation, i.e. relation (28) holds.

This is, however, a very specialized version of results which have been obtained previously. We make no attempt to review these results here, but mention the following. It is quite obvious that one may generalize the result by dropping the compact support of $V$ in time, relaxing the smoothness requirement and replacing the rapid decay conditions by suitable conditions
of integrability. Furthermore, one can generalize \( V \) to a matrix-valued function as long as the resulting Hamilton operator \( H_V(t) \) remains essentially selfadjoint and still fulfills

\[
CH_V(t) = -H_V(t)C.
\]

Generalizations of this type have been considered by Palmer \[32\], and he has found that the S-matrix \( S_V \) implementing the scattering transformation exists, if \( \|\partial_t^\alpha \hat{V}(t, \cdot)\|_{L^q(\mathbb{R}^s)} \) is integrable over \( t \in \mathbb{R} \) for all \( 1 \leq q < 2 + \varepsilon \) and for all \( 0 \leq \alpha < s/2 + \varepsilon \). \( \hat{V} \) denotes the Fourier transform of \( V \) with respect to the spatial variables \( x^1, \ldots, x^s \). We refer to \[32\] for further details, and also for references to related, earlier work.

4 Moyal Minkowski spacetime

As it is usually introduced, \( n = 1 + s \) dimensional Minkowski spacetime gets Moyal-deformed if one postulates the following commutation relations between the coordinates:

\[
[x_\mu, x_\nu] = i\theta_{\mu\nu} \quad (\mu, \nu = 0, \ldots, s)
\]

with \( \theta \) being some antisymmetric, real \((n \times n)\)-matrix. Of course, this stems from the idea of generalizing the behaviour of the quantum mechanical position operators \( x_\mu \) originating in motivations like \[15\] (restricting event localization by incorporating the uncertainty principle in general relativity) and \[41\] (string theory). Alternatively one can implement the relations \[29\] by changing the product structure on the spacetime manifold such that

\[
[x_\mu, x_\nu]_* = x_\mu * x_\nu - x_\nu * x_\mu = i\theta_{\mu\nu} \quad (\mu, \nu = 0, \ldots, s)
\]

is fulfilled between the coordinate chart functions \( x_\mu \) of the manifold. Thereby \( * \) is the non-commutative Moyal product. But let us make this last point more precise now.

Let \( q, p \in \mathbb{N}_0 \), with \( p = 2l \) for \( l \in \mathbb{N}_0 \), and let \( \theta > 0 \). Then we define the \((q + p) \times (q + p)\)-matrix

\[
M = M_\theta = \frac{\theta}{2} \begin{bmatrix}
0_{q \times q} & 0_{q \times p} \\
0_{p \times q} & 0_{p \times l} & 0_{l \times l} \\
0_{l \times p} & 0_{l \times l} & 0_{l \times l} \\
0_{p \times l} & -1_{l \times l} & -1_{l \times l}
\end{bmatrix}
\]

having the \( 2l \times 2l \)-dimensional standard symplectic matrix in the lower right corner, and zeros everywhere else. With this notation, we introduce the
Moyal product

\[ c \star_{(q,p)} g(x) = \frac{1}{(2\pi)^{q+p}} \int \int c(x - Mu)g(x + v)e^{-iu \cdot v}d^q u d^p v, \quad x \in \mathbb{R}^{q+p}, \]

for (complex-valued) Schwartz functions \( c, g \in \mathcal{S}(\mathbb{R}^{q+p}) \). By \( u \cdot v \) we denote the standard Euclidean scalar product of vectors \( u, v \in \mathbb{R}^{q+p} \). One can show, either directly or by adapting the arguments of [20], that \( c \star_{(q,p)} g \) is again in \( \mathcal{S}(\mathbb{R}^{q+p}) \) and that the product \( c \star_{(q,p)} g \) is jointly continuous in \( c \) and \( g \) with respect to the usual test-function topology on \( \mathcal{S}(\mathbb{R}^{q+p}) \).

In the case that \( q = 0 \), \( M = M_\theta \) is invertible, and then one has

\[ c \star_{(0,p)} g(x) = \frac{1}{(2\pi)^p} \int \int c(x - u)g(x + v)e^{-iu \cdot v}d^p u d^p v, \]

which is the usual Moyal product investigated in several references (see [20], [21]). In the other extreme case, \( p = 0 \), one finds

\[ c \star_{(q,0)} g(x) = \frac{1}{(2\pi)^q} \int \int c(x)g(x + v)e^{-iu \cdot v}d^q u d^q v = c(x)g(x), \]

i.e. the product \( c \star_{(q,0)} g \) coincides with the usual pointwise product of functions.

In the general case, it is straightforward to check that

\[ (c \otimes \varphi) \star_{(q,p)} (g \otimes \xi) = (c \star_{(q,0)} g) \otimes (\varphi \star_{(0,p)} \xi) \quad (30) \]

for \( c, g \in \mathcal{S}(\mathbb{R}^q) \) and \( \varphi, \xi \in \mathcal{S}(\mathbb{R}^p) \). Together with the continuity of \( \cdot \star_{(q,p)} \cdot \) in both entries and the fact that \( \mathcal{S}(\mathbb{R}^{q+p}) = \mathcal{S}(\mathbb{R}^q) \otimes \mathcal{S}(\mathbb{R}^p) \) topologically, this shows that the product \( \star_{(q,p)} \) is associative and furnishes an algebra product on \( \mathcal{S}(\mathbb{R}^{q+p}) \), because these properties are known for \( \star_{(q,0)} \) and \( \star_{(0,p)} \). Furthermore, the standard complex conjugation induces a \( \ast \)-involution on \( \mathcal{S}(\mathbb{R}^{q+p}) \) with respect to the product \( \star_{(q,p)} \). We denote this by \( c \mapsto c^* = \overline{c} \).

As a \( \ast \)-involution, it has the property

\[ c^* \star_{(q,p)} g^* = (g \star_{(q,p)} c)^*. \]

With the algebra product \( \star_{(q,p)} \) and the complex conjugation as a \( \ast \)-involution, \( \mathcal{S}(\mathbb{R}^{q+p}) \) is turned into a \( \ast \)-algebra which we denote by \( \mathcal{S}^M_{\star_{(q,p)}} \).

By (30), we have

\[ \mathcal{S}^M_{\star_{(q,p)}} = \mathcal{S}^M_{\star_{(q,0)}} \otimes \mathcal{S}^M_{\star_{(0,p)}}, \quad (31) \]

which holds also in the topological sense.

One can adapt the arguments in [20] to observe that the product \( \star_{(q,p)} \) can be extended to much larger spaces of functions and even distributions. An important case is that one factor in \( c \star_{(q,p)} g \) is in \( \mathcal{S}(\mathbb{R}^{q+p}) \) and the other is in \( L^2(\mathbb{R}^{q+p}) \). Again we consider this situation first for \( q = 0 \). Using Lemma
Carrying out the substitution $(w, v, x)$ becomes $c^\star_{2,12}$ of [20] (resp. reference [43] therein, which is [21] here), it holds that $c^\star_{(0,p)} g$ is in $L^2(\mathbb{R}^p)$ if both $c$ and $g$ are in $L^2(\mathbb{R}^p)$. One can thus also define the operator of left Moyal multiplication on $L^2(\mathbb{R}^p)$,

$$L_c : g \mapsto c^\star_{(0,p)} g, \quad g \in L^2(\mathbb{R}^p),$$

for $c \in L^2(\mathbb{R}^p)$. It is proved in [21] that this operator is bounded, more precisely, that

$$\|L_c g\|_{L^2} \leq \frac{1}{(2\pi\theta)^{p/2}} \|c\|_{L^2} \|g\|_{L^2}. \quad (32)$$

The same estimate holds then also for the operator of right multiplication by $c \in L^2(\mathbb{R}^p)$ given by

$$R_c : g \mapsto g^\star_{(0,p)} c, \quad g \in L^2(\mathbb{R}^p),$$

since $\|R_c g\|_{L^2} = \|L_c \tilde{g}\|_{L^2}$ and $\|	ilde{g}\|_{L^2} = \|g\|_{L^2}$, where the overlining denotes complex conjugation. For $p = 0$, as $c^\star_{(q,0)} g = c \cdot g = g \cdot c = g^\star_{(q,0)} c$ is just the usual pointwise product of functions, one has

$$\|c^\star_{(q,0)} g\|_{L^2} = \|g^\star_{(q,0)} c\|_{L^2} \leq \|c\|_{L^\infty} \|g\|_{L^2},$$

where $\| \cdot \|_{\infty}$ is the supremum norm. This entails that for $c = c_q \otimes c_p$ with $c_q \in \mathcal{S}(\mathbb{R}^q)$ and $c_p \in \mathcal{S}(\mathbb{R}^p)$, the operators

$$L_c : g \mapsto c^\star_{(q,p)} g, \quad \text{and} \quad R_c : g \mapsto g^\star_{(q,p)} c, \quad g \in L^2(\mathbb{R}^q)$$

are bounded operators whose operator norms are not greater than $\frac{1}{(2\pi\theta)^{p/2}} \|c_q\|_{L^\infty} \|c_p\|_{L^2}$. Since each $c \in \mathcal{S}(\mathbb{R}^{q+p})$ can be approximated by a sequence $\sum_{j=1}^N c_{q,j} \otimes c_{p,j}$ as $N \to \infty$, so that for all of the Schwartz norms $\| \cdot \|_s$ there holds $\sum_{j=1}^\infty \|c_{q,j} \otimes c_{p,j}\|_s < \infty$, it follows that $L_c$ and $R_c$ are bounded operators on $L^2(\mathbb{R}^{q+p})$ for all $c \in \mathcal{S}(\mathbb{R}^{q+p})$. Furthermore, we put on record here the following hermiticity property of $L_c$ and $R_c$.

**Lemma 4.1** Let $c \in \mathcal{S}^M_{(q,p)}$ and let $\varphi, \psi \in L^2(\mathbb{R}^{q+p})$. Then

$$
\begin{align*}
(c^\star_{(q,p)} \varphi, \psi)_{L^2} & = (\varphi, c^\star_{(q,p)} \psi)_{L^2}, \quad (33) \\
(\varphi^\star_{(q,p)} c, \psi)_{L^2} & = (\varphi, c^\star_{(q,p)} c^\star)_{L^2}. \quad (34)
\end{align*}
$$

**Proof** Consider first the case $q = 0$. Then

$$
\begin{align*}
(c^\star_{(q,p)} \varphi, \psi)_{L^2} & = \frac{1}{(2\pi\theta)^p} \int c(w) \varphi(v) e^{-i(x-w) \cdot M^{-1}(x-v)} \psi(x) d^p w d^p v d^p x, \quad (35) \\
(\varphi, c^\star_{(q,p)} \psi)_{L^2} & = \frac{1}{(2\pi\theta)^p} \int \varphi(x) c(y) \psi(z) e^{i(x-y) \cdot M^{-1}(x-z)} d^p y d^p z d^p x. \quad (36)
\end{align*}
$$

Carrying out the substitution $(w, v, x) \mapsto (y, x, z)$, the right hand side of (35) becomes

$$
\frac{1}{(2\pi\theta)^p} \int c(y) \varphi(x) e^{-i(z-y) \cdot M^{-1}(z-x)} \psi(z) d^p y d^p x d^p z. \quad (37)
$$
Thus one can see that (37) coincides with (36) upon noticing that, using the anti-symmetry of $M^{-1}$,

$$(z - x) \cdot M^{-1}(z - y) = -x \cdot M^{-1}z + x \cdot M^{-1}y - z \cdot M^{-1}y$$

coincides with

$$(x - y) \cdot M^{-1}(x - z) = -x \cdot M^{-1}z + y \cdot M^{-1}z - y \cdot M^{-1}x.$$ 

This proves (33) in the case $q = 0$, and (34) is proved analogously. Then we notice that (33) and (34) are obviously correct for $p = 0$. Therefore we obtain, using the tensor product decomposition of $\star_{(q,p)}$ as in (30),

$$\left( \varphi_q \otimes \varphi_p, (c_q \otimes c_p)^* \star_{(q,p)} (\psi_q \otimes \psi_p) \right)_{L^2}$$

$$= \left( \varphi_q \otimes \varphi_p, (c^*_q \star_{(q,0)} \psi_q) \otimes (c^*_p \star_{(0,p)} \psi_p) \right)_{L^2}$$

$$= \left( (c_q \star_{(q,0)} \varphi_q) \otimes (c_p \star_{(0,p)} \varphi_p), \psi_q \otimes \psi_p \right)_{L^2}$$

whenever $c_q \in \mathcal{S}^M_{(q,0)}$, $c_p \in \mathcal{S}^M_{(0,p)}$ and $\varphi_q, \psi_q \in L^2(\mathbb{R}^q)$, $\varphi_p, \psi_p \in L^2(\mathbb{R}^p)$. This implies (33). Relation (34) is proved analogously. □

5 The Dirac field on Moyal-deformed Minkowski spacetime as a Lorentzian spectral geometry — general discussion

We will now embark on a — rather informal — discussion on the setting in which we wish to view the quantized Dirac field on Moyal-deformed Minkowski spacetime.

Assume that $n \geq 2$, $n = 1 + s$, and assume the restrictions on $n$ made before in (2). Let $q + p = n$ where $p$ is even. Let $C^\infty(\mathbb{R}^n, \mathbb{C}^N)$, $N = N(n)$ as in (1), denote the space of smooth spinor fields on flat Minkowski spacetime $\mathbb{R}^n = \mathbb{R}^{1+s}$ as introduced in section 2. We can introduce a scalar product on the spinors given by

$$\langle \psi, \eta \rangle = \int_{\mathbb{R}^n} \bar{\psi}^A(x) \delta_{AB} \eta^B(x) d^n x$$

for $\psi = (\psi^A)_A = 1$, $\eta = (\eta^A)_A = 1$ in $L^2(\mathbb{R}^n) \otimes \mathbb{C}^N$. Let $\mathcal{H} = \mathcal{H}_n$ denote the Hilbert space of square-integrable spinors $L^2(\mathbb{R}^n) \otimes \mathbb{C}^N$, carrying the scalar product (38). Then $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^N) \cong \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}^N$ is a dense subspace of $\mathcal{H}$. The algebra $\mathcal{S}^M_{(q,p)}$ can act from the left or the right on $\mathcal{H}$; an explicit representation of the left action is

$$(L_c \psi)^A = c \star_{(q,p)} \psi^A$$

for $c \in \mathcal{S}^M_{(q,p)}$.
for $\psi = (\psi^A)_{A=1}^N$ in $\mathcal{H}$. We denote by $A^M$ the represented algebra $L_{\mathcal{H}_M^{(q,p)}}$. Thus, we have a $\ast$-algebra of bounded linear operators, $A^M$, acting on $\mathcal{H}$, (cf. last section), and if $p \neq 0$, then this algebra is non-commutative. Furthermore, we have the usual Dirac operator $D$ defined in (1), whereas we set $D = D_0$ for potential $V = 0$ here, acting on a dense domain in $\mathcal{H}$; for convenience, we shall take this domain to be $C^\infty_0(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N$.

The said data $A^M, \mathcal{H}, D$ are reminiscent of the data of a spectral triple in the spectral triple approach to non-commutative geometry by Connes [12],[13], and in fact, this is how we would like to think of them. There are, however, a few technical obstructions to doing so, since the original spectral geometry approach generalizes compact Riemannian spin geometries, while in our case $A^M$ is a non-commutative deformation of an algebra of functions over the non-compact $\mathbb{R}^n$, and $D$ is the Dirac operator of a metric of Lorentzian signature. This means that a modified structure needs to be provided in order to attain a spectral geometry generalization of non-compact Lorentzian spin geometries of comparable strength as in the compact, Riemannian case. This endeavour will be carried out elsewhere [33], we report here only about some of the important ingredients in a rather non-technical manner, and largely tailored to our Moyal spacetime case at hand.

We begin by noting that structural elements in addition to $A^M, \mathcal{H}, D$ are needed already in the Riemannian spectral geometry framework. What is required is an anti-unitary involution $C$ on $\mathcal{H}$, playing the role of a charge conjugation, and in our Moyal-setting, $C$ will in fact be defined as in (3). Additionally, one needs an operator $\gamma$ on $\mathcal{H}$ which induces an orientation, and in our Moyal-case at hand, $\gamma = \gamma_0\gamma_1 \cdots \gamma_s$ is the product of Dirac matrices, acting on $L^2$-spinors in $\mathcal{H}$ by matrix multiplication from the left.

Supposing for a moment (for the purpose of comparison) that $A^M, \mathcal{H}, D, C, \gamma$ were describing a compact (non-commutative) Riemannian spin geometry in the framework of spectral geometry — which actually is not the case — then the just listed items would be required to fulfill important structural properties, such as:

(i) $A^M$ is a unital pre-$C^*$-algebra

(ii) $D$ is hermitean and elliptic, and $(D - \lambda I)^{-n}$ is in a suitable Schatten class for $\lambda \notin \text{spec } D$

(iii) a series of (anti-)commutation relations between $A^M, D, C$ and $\gamma$

(iv) certain “regularity” conditions on $A^M$ and $D$ (including domain conditions)

(See [23] for a detailed exposition of the required properties.)
Now in the present case, where $A^M, \mathcal{H}, D, C, \gamma$ actually derive from Moyal-deformed Minkowski spacetime, several of these properties, in particular (i) and (ii), no longer hold, but need to be replaced by suitable generalizations. We won’t discuss here the appropriateness of the generalizations envisaged (see [33]), but only give a few indications of their nature. $A^M$ is not a unital algebra, so one needs, as a further datum, a unitalization $A^M \supset A^M$, where $A^M$ is a unital pre-$C^*$-algebra. The work [20] contains an extended discussion on the best choice of $A^M$ in the Riemannian Moyal-algebra case (actually, for $q = 0$), and since this discussion concerns mainly topological aspects of the non-commutative space as opposed to its metric structure, the results of this apply here as well.

In [20], $A^M$ is constructed as follows. Let $c$ be a $C^\infty$ function on $\mathbb{R}^p$ which is bounded together with all of its derivatives. Then define the operator (cf. (39))

$$L_c : \psi \mapsto L_c \psi$$

for all $\psi \in \mathcal{H} = L^2(\mathbb{R}^p) \otimes \mathbb{C}^N$. This is a bounded operator with respect to the operator norm on $L^2(\mathbb{R}^p) \otimes \mathbb{C}^N$. The $*$-algebra generated by these operators is taken as $A^M$. Note that

$$L_{c_1} L_{c_2} \psi = L_{c_1 \ast (0, p)} c_2 \psi$$

when $c_1 \ast (0, p) c_2$ is defined, and likewise $L_c^* = L_c^*$. One can opt for this choice of $A^M$ also in the case of $\ast_{(q, p)}$.

Another modification is needed for (ii). Already in the non-compact Riemannian case, $(D - \lambda I)^{-n}$ is not compact for resolvent values $\lambda$ of $D$, but this can be remedied by requiring that $a(D - \lambda I)^{-n}$ is in a suitable Schatten class for $a \in A^M$. However, in the Lorentzian case, $D$ is not elliptic, and thus $a(D - \lambda I)^{-n}$ is non-compact. Moreover, $D$ is not hermitean with respect to the $L^2$ scalar product.

A way to get around this difficulty is to introduce another element of structure in the form of a further linear, bounded operator $\beta : \mathcal{H} \rightarrow \mathcal{H}$. This operator carries the information of a “time-like” direction and thereby encodes the Lorentzian metric signature; in our case, $\beta = \gamma_0$, acting as (matrix) multiplication operator on the spinors. The characteristic properties of $\beta$, besides $\beta^2 = 1$ and suitable Clifford relations with $C$ and $\gamma$, are

$$\beta D = D^* \beta$$

on the $C^\infty$-domain of $D$, and that

$$\langle D \rangle = \sqrt{\frac{1}{2}(D^* D + DD^*)}$$

is an elliptic operator so that $a(\langle D \rangle - \lambda I)^{-n}$ is in a suitable Schatten class for resolvent values $\lambda$ of $\langle D \rangle$ and $a \in A^M$. (The adjoint $D^*$ is defined with respect to the scalar product of $\mathcal{H}$.) Whence, the collection of objects

$$A^{M, I} \supset A^M, \mathcal{H}, D, \beta, C, \gamma$$
in combination with a list of relations and conditions that will be discussed in detail in [33], can be viewed as a “Lorentzian spectral triple” (LOST), i.e. the generalization of spectral geometry from Riemannian to Lorentzian signature. As we have outlined, Moyal-deformed Minkowski spacetime can be fit into this setting.

If one now contends that non-commutative Lorentzian spacetimes are described in terms of LOSTs with data $A^{M,I} \supset A^M, \mathcal{H}, D, \beta, \gamma$, one is faced with the question as to what a quantum field theory on a LOST should be, and how such quantum field theories can, on one hand, be constructed, and on the other hand, be interpreted. A fairly immediate idea is this: Since a Hilbert space $\mathcal{H}$ with a Dirac operator $D$ and a charge conjugation $C$ acting in it are part of the data describing a LOST, one may define the Dirac field on a LOST as an abstract CAR algebra corresponding to these data.

One must remember, however, that the Hilbert space $\mathcal{H}$ does not play the role of the Hilbert space $\mathcal{K} (= \mathcal{K}_V, V = 0)$ in Proposition [21] describing the space of equivalence classes of smooth, compactly supported elements in $L^2(\mathbb{R}^n) \otimes \mathbb{C}^N$ modulo the kernel of the operator $R = R^+ − R^−$ (where $R^\pm$ are the advanced/retarded fundamental solutions of $D$). Nevertheless, the Hilbert space structure of $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^N$ is used to obtain a Hilbert space structure on the set of equivalence classes.

In the case of a general LOST, it is at present not clear how to characterize advanced and retarded fundamental solutions of $D$. One of the difficulties is caused by the circumstance that “advanced” and “retarded” refer to localization properties which are notoriously difficult to capture in non-commutative geometry. This notwithstanding let us, for the time being, suppose that we have a LOST where advanced and retarded fundamental solutions of $D$ are given as quadratic forms on a suitable domain $D$ contained in the joint $C^\infty$-domain of $D$ and $D^*$. Abusing notation, we will denote these quadratic forms by

$$f, h \mapsto (f, R^\pm h), \quad f, h \in \mathcal{D}.$$  

The fundamental solution property amounts to the condition

$$(D^*f, R^\pm h) = (f, h) = (f, R^\pm Dh)$$

for all $f, h \in \mathcal{D}$.

Guided by the example of the Dirac field on commutative Minkowski spacetime, one is led to the assumption that

$$(f, h)_{(R)} = e^{i\delta} [(\beta f, R^+ h) − (\beta f, R^- h)]$$

defines, upon choice of a suitable phase $\delta$, a scalar product on $\mathcal{D}/ \ker(\cdot, \cdot)_{(R)}$. [At present it is not clear if such a property can actually be proved under suitable additional “regularity” conditions on LOSTS, or if this is genuinely
an extra assumption; but in our Moyal spacetime example in the next section we will see that this property is fulfilled.] With this assumption, one can define the Hilbert space $K(R)$ as the completion of $\mathcal{D}/\ker(\cdot,\cdot)_{(R)}$ with respect to $(\cdot,\cdot)_{(R)}$. Under these circumstances, the conjugation $C$ on $\mathcal{D}$ induces a conjugation $C$ on $K(R)$ via $C[f]_{(R)} = [Cf]_{(R)}$. Thus, one has a Hilbert space $K(R)$ with a conjugation $C$ on it. One can therefore define the associated CAR-algebra $\mathfrak{F}(K(R),C)$ in a manner completely analogous to the example of the free Dirac field on Minkowski spacetime, cf. Section 2. That is, $\mathfrak{F}(K(R),C)$ is generated by $B([f]_{(R)}), f \in \mathcal{D}$, which are linear in $[f]_{(R)}$, and subject to the relations
\[
B([f]_{(R)})^* = B(C[f]_{(R)}),
\]
\[
\{B([f]_{(R)})^*, B([h]_{(R)})\} = 2([f]_{(R)],[h]_{(R)})_1,
\]
\[
B([Df]_{(R)}) = 0.
\]
At this stage, one has constructed abstractly a quantum field theory on a non-commutative geometry described by a LOST and some additional structure. The quantum field theory was then essentially obtained by second quantization. The question arises how such a quantum field theory should be interpreted.

Regarding this point, let us specialize to the case that $\mathcal{A}^M$ is the Moyal-deformed algebra of functions on Minkowski spacetime, and $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^N$, $\mathcal{D} = C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N$, with the Dirac operator as in (5). This means that $\mathcal{H}$ and $D$ are the same as in the case of commutative, "undeformed" Minkowski spacetime, just the domain $\mathcal{D}$ has changed, but this does not lead to a significant modification. As will be explained in the next section, there will again be uniquely determined advanced and retarded fundamental solutions $R^\pm$ of $D$. The CAR-algebra $\mathfrak{F}(K(R),C)$ one obtains in this case coincides with $\mathfrak{F}(\mathcal{K},C)$ defined in Section 2 except that $K(R)$ is larger than $\mathcal{K}$ owing to the fact that $\mathcal{D}$ is taken larger than it was in the case of commutative spacetime. This difference would, however, disappear in the vacuum representation of the Dirac field (defined with respect to the time-translations) upon passing to von Neumann algebras in that representation. Thus, the von Neumann algebras of the CAR-algebras of the Dirac field, in vacuum representation, constructed either for classical Minkowski spacetime, or for Moyal-Minkowski spacetime, both coincide.

It is therefore worth contemplating if the sketched way of "abstract" quantization of the LOST corresponding to Moyal-deformed Minkowski spacetime leads to anything different from the usual quantized Dirac field on usual Minkowski spacetime. We argue that this is indeed the case. One must remember that, in operational terms, a quantum field theory — on a classical spacetime — is described by an assignment of observables to spacetime regions and that the physical content of the theory lies mainly in the localization properties of the observables (and their algebraic relations) relative
to each other, see [26], [27] and discussion further below. We must, in the case of Moyal-Minkowski spacetime, specify the observables of the quantum field theory we have defined, and study their localization properties in connection with the algebraic structure of the Moyal-Minkowski-algebra $\mathcal{A}^M$.

In the vacuum representation $(\mathcal{H}^{\text{vac}}, \pi^{\text{vac}}, \Omega^{\text{vac}})$ of $\mathcal{F}(\mathcal{K}_0, C)$, we have defined the field operators

$$\psi(f) = \pi^{\text{vac}}(\Psi_0(f)), \quad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N).$$

These operators do not correspond directly to observable quantities since they fulfill anticommutativity upon spacelike separation of the test-spinors $f$. Therefore, one needs to build operators corresponding to observables from the $\psi(f)$. A common choice is to take operators of the form $\psi(f_1)^*\psi(f_2)$ as building blocks for observables. Then $\psi(f_1)^*\psi(f_2)$ commutes with $\psi(h_1)^*\psi(h_2)$ if the supports of $f_1$ and $f_2$ are spacelike separated from the supports of $h_1$ and $h_2$.

Certain operators arising as limits of linear combinations of such operators have interesting properties. Among them is the Wick-product $:\psi^+\psi:(c)$ which is indexed by scalar testing functions $c \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. One may define $:\psi^+\psi:(c)$ as follows. Take two finite families of spinors, $c_\mu$ and $\eta_\mu (\mu = 1, \ldots, L)$ in $\mathbb{C}^N$, with the property that $\sum_{\mu=1}^L c_\mu^A \eta_\mu^B = \frac{1}{2} \gamma_0^{AB}$ (the matrix entries of $\gamma_0$). Then define, for $q_1$ and $q_2$ in $C_0^\infty(\mathbb{R}^n, \mathbb{R})$, the operator

$$:\psi^+\psi(q_1 \otimes q_2) = \sum_{\mu=1}^L \psi(q_1 c_\mu)^*\psi(q_2 \eta_\mu).$$

The map $q_1 \otimes q_2 \rightarrow :\psi^+\psi(q_1 \otimes q_2)$ defines a real-linear operator-valued distribution and thus extends to $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. Let $j_\epsilon$ be a family of real-valued functions in $C_0^\infty(\mathbb{R}^n)$ approaching the $\delta$-measure peaked at 0 for $\epsilon \rightarrow 0$, and set, for $q_1, q_2 \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$,

$$F_\epsilon(x, y) = q_1(x)q_2(y)j_\epsilon(x - y) \quad (x, y \in \mathbb{R}^n).$$

Moreover, denote by $\mathcal{W} \subset \mathcal{H}^{\text{vac}}$ the dense subspace generated by $P\Omega^{\text{vac}}$ where $P$ ranges over all polynomials in the $\psi(f)$ with $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ (including the case that $P$ has degree zero, i.e. $P$ is a multiple of 1). With these conventions, we define

$$:\psi^+\psi:(c)\chi = \lim_{\epsilon \rightarrow 0} :\psi^+\psi(F_\epsilon)\chi - (\Omega^{\text{vac}}, :\psi^+\psi(F_\epsilon)\Omega^{\text{vac}})\chi$$

for all $\chi \in \mathcal{W}$ and $c(x) = q_1(x)q_2(x) \quad (x \in \mathbb{R}^n)$. It turns out (see Sec. 7 and Appendix A) that $:\psi^+\psi:(c)$ is an essentially selfadjoint operator on $\mathcal{W}$ which furthermore turns out to be independent of the choices made for $c^\mu$ and $\eta_\mu (\mu = 1, \ldots, L)$. The $:\psi^+\psi:(c)$ are local operators in the sense
that: $\psi^+\psi : (c_1)$ commutes with $\psi^+\psi : (c_2)$ if the supports of $c_1$ and $c_2$ are spacelike separated. For $c \geq 0$, $\psi^+\psi : (c)$ can be interpreted as the observable of (squared) field strength density weighted with the function $c$.

An interesting property of $\psi^+\psi : (c)$, proven in Appendix A, is

$$[\psi^+\psi : (c), \psi(f)] = -i\psi(cR_0f) \tag{40}$$

for all $c \in C_0^\infty(\mathbb{R}^n)$ and all $f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}^N$. On the other hand, we will also show in Sec. 8 that, identifying $c$ with the scalar potential in the discussion of potential scattering in Sec. 6, there holds

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} \pi_{\text{vac}}(\beta_{\lambda c}\Psi_0(f))) = \frac{d}{d\lambda}\bigg|_{\lambda=0} S_{\lambda c}\psi(f)S_{\lambda c}^{-1} = \psi(cR_0f) \tag{41}$$

for all $c \in C_0^\infty(\mathbb{R}^n)$ and $f \in C_0^\infty(\mathbb{R}^{n}, \mathbb{C}^N)$. In view of (40) and (41), the observables $\psi^*\psi : (c)$ are identified as $-i\frac{d}{d\lambda}\bigg|_{\lambda=0} S_{\lambda c}$ where $S_c$ is the scattering matrix corresponding to the localized scattering potential $c$. This connection between localized observables and the derivative of the scattering matrix of a localized interaction with respect to the interaction strength is, of course, long known, especially in the context of perturbative interacting quantum field theory, and often goes by the name “Bogoliubov’s formula” [6].

We now wish to point out that one can obtain in a similar manner observables for the quantized Dirac field on Moyal-deformed Minkowski spacetime employing Bogoliubov’s formula. The precise mathematical discussion of the considerations we present here will be given in the next section. In the case of the Dirac field on usual Minkowski spacetime, the scattering matrix $S_V \equiv S_c$ was constructed for the Dirac operator $D_V = D + V$ where the potential term was $Vf = cf$, $cf$ meaning the usual pointwise (and component-wise) multiplication of a scalar function $c$ with a spinor-field $f$. We should now recall that classical Minkowski spacetime is also described by the structure of a LOST. The data for the LOST corresponding to classical Minkowski spacetime coincide with the data for the LOST of Moyal-Minkowski spacetime, except that instead of the non-commutative algebra $A^M = \mathcal{S}_*(q, p)$ we have the commutative algebra $A^{\text{Min}} = C_0^\infty(\mathbb{R}^n)$ of scalar functions on spacetime. The map $c \mapsto c\varphi$, $\varphi \in \mathcal{H} = L^2(\mathbb{R}^n, \mathbb{C}^N)$ produces a faithful representation of $A^{\text{Min}}$ on the Hilbert-space of square-integrable spinor fields. For the case of Moyal-Minkowski spacetime, one can regard the potential term $V$ in a similar light, and define, for $\varphi \in \mathcal{H}$, for instance

$$V\varphi = L_c\varphi + R_c\varphi = c\ast\varphi + \varphi\ast c \tag{42}$$

with real-valued $c$ in $A^M = \mathcal{S}_*(q, p)$. In the next section we will show that in the case $q = 1, p = 2l > 0$, i.e. when the Moyal-deformed Minkowski spacetime has no non-trivial commutation relations between time- and space-coordinates, there is a Bogoliubov-transformation $\beta^M_\psi$ on the CAR-algebra.
$\mathcal{F}(K = K_{(R_0)}, C)$ describing scattering by the non-commutative potential $V$ given in (42). (This needs mild further assumptions on $c$, see Sec. 6 for details.) Furthermore, we will show that this scattering transformation is unitarily implementable in the vacuum-representation $(H^\text{vac}, \pi^\text{vac}, \Omega^\text{vac})$, so that there is a unitary operator $S^M_V$ with

$$S^M_V \pi^\text{vac}(\Psi_0(f))(S^M_V)^{-1} = \pi^\text{vac}(\beta_V^M(\Psi_0(f)))$$

for all $f \in C^\infty_0(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{C}^N$. Consequently, one can formally define the derivative

$$\Phi(c) = -i \left. \frac{d}{d\lambda} \right|_{\lambda=0} S^{M}_{\lambda V}$$

which, following the ideas underlying Bogoliubov’s formula alluded to just before, would correspond to an observable quantity. In Sec. 7 we will in fact show that

$$\frac{d}{d\lambda} \left. \right|_{\lambda=0} S^{M}_{\lambda V} \psi(f) S^{M}_{\lambda V}^{-1} = [i\Phi(c), \psi(f)] = \psi(\mathcal{V} R_0 f)$$

for all $f \in C^\infty_0(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{C}^N$ with an essentially selfadjoint operator $\Phi(c)$ on $\mathcal{W}$. One may therefore identify $\Phi(c)$ with the derivative $-i \left. \frac{d}{d\lambda} \right|_{\lambda=0} S^{M}_{\lambda V}$, in the sense that (44) holds.

In the case of usual Minkowski spacetime, the assignment $c \mapsto \psi^+\psi : (c)$, where $c$ is a scalar $C^\infty_0$ test-function on spacetime, has the typical properties of an observable quantum field of Wightman type [39]. The support of the test-function $c$ limits the localization of the observable $\psi^+\psi : (c)$, which is reflected by the relations (40) and (41) and the fact that the changes of states which $\beta^\lambda c$ induces are localized in the support of $c$. In the algebraic approach to quantum field theory [26, 27], one therefore considers the $*$-algebras $\mathfrak{A}(O)$ generated by all observable quantum field operators $\psi^+\psi : (c)$ where the support of $c$ is contained in the spacetime region $O$.

Then one obtains an assignment $O \mapsto \mathfrak{A}(O)$ of spacetime regions to operator algebras with the two characteristic properties of

**Isotony:** $O_1 \subset O_2 \implies \mathfrak{A}(O_1) \subset \mathfrak{A}(O_2)$

**Locality:** $O_1 \perp O_2 \implies [F_1, F_2] = 0$ for $F_j \in \mathfrak{A}(O_j)$ ($j = 1, 2$)

where $O_1 \perp O_2$ means that the spacetime regions are causally separated, i.e. there is no causal curve joining them.

---

1Two things should be noted here. (1) Actually, $\mathfrak{A}(O)$ would have to be defined as algebraically generated by all observable quantum field operators smeared with test-functions supported in $O$; we use $\psi^+\psi$ as a placeholder for any observable quantum field at this point. (2) In the algebraic approach to quantum field theory it is customary to define $\mathfrak{A}(O)$ as algebraically generated by the bounded functions of quantum field operators smeared with test-functions supported in $O$; here, our $\mathfrak{A}(O)$ are algebras of unbounded operators.
According to the algebraic approach to quantum field theory, a quantum field theoretical model is basically characterized by a map \( O \mapsto \mathfrak{R}(O) \) with these properties (see [27, 26, 37]), describing especially the localization of observables of the quantum system under consideration on a “classical” spacetime with commutative coordinate functions. Let us now discuss some, however vague, ideas how this may be generalized to quantum field theories on non-commutative spacetimes, where again we stay at the level of the Dirac field on Moyal-Minkowski spacetime. The scattering by a non-commutative potential furnishes the assignment \( c \mapsto \Phi(c) \) of (43). We interpret \( \Phi(c) \) as an observable, and hence we have an assignment of elements \( c \) in the non-commutative algebra \( \mathcal{A}^M \) to (unbounded) operators in \( \mathcal{H}^{\text{vac}} \). The \( c \) now carries the information about the spacetime localization of \( \Phi(c) \), but due to the non-commutativity of \( \mathcal{A}^M \), this is subject to uncertainties. In particular, in general \( \Phi(c_1) \) and \( \Phi(c_2) \) won’t commute anymore if the supports of \( c_1 \) and \( c_2 \), viewed as test-functions, are causally separated. Therefore, if one defines the algebras \( \mathfrak{R}^M(O) \) as being generated by the \( \Phi(c) \) where \( c \) has support in \( O \), then the assignment \( O \mapsto \mathfrak{R}^M(O) \) is clearly different from \( O \mapsto \mathfrak{R}(O) \) as defined above for usual Minkowski spacetime, and thus we see that we derive indeed a different system of observables from the scattering morphisms via Bogoliubov’s formula in the non-commutative case, without an obvious locality structure.

Nevertheless, one may attempt to mimic the algebraic approach to quantum field theory in a generalized form, upon forming algebras of observables \( \mathfrak{R}^M(\mathcal{P}) \) labelled by subsets \( \mathcal{P} \) of \( \mathcal{A}^M \), understanding that \( \mathfrak{R}^M(\mathcal{P}) \) be generated by the \( \Phi(c) \) with \( c \in \mathcal{P} \). It is not clear at this stage what structure these subsets should have, e.g. if they should be subalgebras of \( \mathcal{A}^M \). In comparison to the classical case, what seems to be required is a partial ordering on the collection of chosen \( \mathcal{P} \), and a concept of causal separation [37]. An idea could be to choose the \( \mathcal{P} \) as sets of (approximate) projections, inspired by the situation on classical spacetime, where a subset \( O \) may be identified with its characteristic function, which is a projection in the commutative algebra of coordinate functions. The ordering relation may then be taken as operator ordering. It is more difficult to capture the concept of causal separation. In the case of classical Minkowski spacetime, the supports of two \( C_0^\infty \) test-functions \( c_1 \) and \( c_2 \) are causally separated if and only if \( i\langle c_1 f, R_m c_2 h \rangle = 0 \) for all spinor fields \( f \) and \( h \), where \( R_m \) is the causal propagator for the Dirac equation for any mass term \( m \) (corresponding to \( R_V \) for \( V = m \), cf. eqn. (8)). By assumption, the causal propagator is available in our non-commutative setup as the quadratic form \( (\cdot,\cdot)_{(R)} \) on the domain \( \mathcal{D} \subset \mathcal{H} \), and thus one may characterize the causal disjointness of two subsets \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) of \( \mathcal{A}^M \) with the help of this quadratic form for any mass term. Ideally, one might want to define \( \mathcal{P}_1 \perp \mathcal{P}_2 \) to be equivalent to \( \langle c_1 f, c_2 h \rangle_{(R)} = 0 \) for all \( c_j \in \mathcal{P}_j \) and all \( f, h \in \mathcal{D} \), provided this is compatible with the ordering relation. If that cannot be had, the second best option
would be to define \( P_1 \perp P_2 \) as meaning that \((c_1, c_2)_R\) is, in a suitable sense, “small” compared to \((\ldots)_R\) — a sort of “infinitesimal” quantity in the sense of spectral geometry.

Supposing that suitable forms of a partial ordering relation \( P_1 \leq P_2 \) and a causal separation relation \( P_1 \perp P_2 \) have been found for suitably chosen subsets \( P \) of \( A^M \), it seems well possible that the generalized version of a quantum field theory on non-commutative spacetime in the operator algebraic setting may take the shape of an assignment \( P \mapsto \mathfrak{R}^M(P) \), where the \( \mathfrak{R}^M(P) \) are operator algebras, subject to the relations of

\[
\begin{align*}
\text{Isotony:} & \quad P_1 \leq P_2 \quad \Rightarrow \quad \mathfrak{R}^M(P_1) \subset \mathfrak{R}^M(P_2) \\
\text{Locality:} & \quad P_1 \perp P_2 \quad \Rightarrow \quad [F_1, F_2] = 0 \quad \text{for} \quad F_j \in \mathfrak{R}^M(P_j) \quad (j = 1, 2).
\end{align*}
\]

Actually, it could happen that the condition of locality ought to be relaxed requiring only that \([F_1, F_2]\) is in a suitable sense “small” compared to \( F_1 \) and \( F_2 \) if \( P_1 \perp P_2 \), similar in spirit to the possibly generalized condition of causal separation. Admittedly, this is at present all speculation, and a careful study of examples is required before a clear picture of the basic structure of quantum field theory on (Lorentzian) non-commutative spacetime will emerge.

### 6 The Dirac field on Moyal-deformed Minkowski spacetime – the model

Now our intention is to follow the lines of Section 2 under the modifications of using \( n = 1 + s = q + p \) dimensional Moyal-deformed Minkowski spacetime and a suitable different potential term for the Dirac operator.

The spacetime of interest (with dimension \( n = 1 + s = q + p \)) will be described as in section 4 with the exception that we restrict ourselves to Moyal matrices \( M \) of the more specialized form

\[
M = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & & \\
0 & M_{(q+p-1)\times(q+p-1)} & \\
\end{bmatrix}_{(q+p)\times(q+p)}
\]

(45)

i.e. the first row and the first column shall vanish.

Nothing is changed (cf. Section 2) in the manner of how we define the algebra of Dirac matrices \((\gamma_0, \gamma_1, \ldots, \gamma_s)\) and the charge conjugation \( C \). Again the Dirac operator \((m > 0 \ \text{constant})\) acting on \( C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \) is denoted by

\[
D_V = (-i\gamma + m) + V.
\]

But now the “potential term” operator \( V \) acting on \( f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \) is not just the multiplication operator multiplying \( f \) with a scalar function,
but one of the following operators:

\[(i) \quad (Vf)^A(x) = (V_i f)^A(x) = (c *_{(q,p)} f^A)(x) + (f^A *_{(q,p)} c)(x) \quad (46)\]
\[(ii) \quad (Vf)^A(x) = (V_{ii} f)^A(x) = (c *_{(q,p)} f^A *_{(q,p)} c)(x), \quad (47)\]

where \(c \in C_0^\infty(\mathbb{R}, \mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s, \mathbb{R})\) is a function of the form

\[c(x) = a(t)b(x), \quad (48)\]

with \(a \in C_0^\infty(\mathbb{R}, \mathbb{R}), \ b \in \mathcal{S}(\mathbb{R}^s, \mathbb{R}), \ t = x^0, \ \overline{x} = (x^1, \ldots, x^s)\). We aim at presenting an analogue of Proposition \[2.1\] for the potential operators \(V = V_i\) or \(V = V_{ii}\) which describe “scattering by a time-dependent, spatially non-commutative potential”.

It is useful, at this point, to consider first the Cauchy-data version of the dynamical problem. As in \[10\], we have the free Hamiltonian

\[(H_0 f)(\overline{x}) = \left( i\gamma^0 \gamma^k \frac{\partial}{\partial x^k} + \gamma^0 m \right) f(\overline{x}) \quad (49)\]

and the Hamiltonian with time-dependent interaction term,

\[(H_V(t) f)(\overline{x}) = \left( i\gamma^0 \gamma^k \frac{\partial}{\partial x^k} + \gamma^0 m + \gamma^0 V(t) \right) f(\overline{x}), \quad (50)\]

acting on \(f \in \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N\). Here \(V(t)\) stands for the operators

\[V_i(t) : \quad f \mapsto V_i(t) f, \quad f \in \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N, \quad (51)\]
\[(V_i(t)f)^A(\overline{x}) = \quad a(t)(b *_{(q-1,p)} f^A(\overline{x}) + f^A *_{(q-1,p)} b(\overline{x})) \quad (52)\]

or

\[V_{ii}(t) : \quad f \mapsto V_{ii}(t) f, \quad f \in \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N, \quad (53)\]
\[(V_{ii}(t)f)^A(\overline{x}) = \quad a(t)^2(b *_{(q-1,p)} f^A *_{(q-1,p)} b(\overline{x})). \quad (52)\]

By the assumptions made on \(a\) and \(b\) above, \(V(t) = V_i(t)\) and \(V(t) = V_{ii}(t)\) are bounded operators on \(L^2(\mathbb{R}^s, \mathbb{C}^N)\). As in the case of a scalar potential, we have again that \(CH_V(t) = -H_V(t)C\), which is a consequence of the easily checked equation

\[C R_c = L_c C, \quad (54)\]

obviously in the case of \(V_i\) and under additional usage of the associativity of the Moyal product in the case of \(V_{ii}\). And \(H_V(t)\) is a symmetric (in fact essentially selfadjoint) operator on \(\mathcal{S}(\mathbb{R}^s, \mathbb{C}^N) \subset L^2(\mathbb{R}^s, \mathbb{C}^N)\). As in the case considered before in Section \[2\] a smooth function \(\varphi \in C^\infty(\mathbb{R}^s, \mathbb{C}^N)\) is a solution of

\[D_V \varphi = 0 \quad (55)\]
if and only if
\[ \frac{1}{i} \frac{d}{dt} P_t \varphi = H_V(t) P_t \varphi. \]
with
\[ P_t \varphi = \varphi|_{\Sigma_t}. \]
Establishing existence and uniqueness of the Cauchy problem for the Dirac equation (55) is therefore equivalent to proving existence and uniqueness of solutions for the initial value problem
\[ \frac{1}{i} \frac{d}{dt} v_t = H_V(t) v_t, \quad v_t|_{t=0} = w. \]
This will be our next auxiliary result.

**Proposition 6.1** (a) There is a unique family of unitaries \( T_{t,t'}^{(V)} \) on \( L^2(\mathbb{R}^s, \mathbb{C}^N) \), strongly continuous in \( t \) and \( t' \), so that
\[ T_{t,t'}^{(V)} \circ T_{t',s}^{(V)} = T_{t,s}^{(V)}, \quad T_{t,t}^{(V)} = 1, \] (56)
and
\[ \frac{1}{i} \frac{d}{dt} T_{t,0}^{(V)} w = H_V(t) w \] (57)
for all \( w \in L^2(\mathbb{R}^s, \mathbb{C}^N) \). Moreover, \( T_{t,t'}^{(V)} \) maps \( \mathcal{S}(\mathbb{R}^s, \mathbb{C}^N) \) into itself.

(b) Given \( w \in \mathcal{S}(\mathbb{R}^s, \mathbb{C}^N) \), the map
\[ (t, t', \underline{x}) \mapsto T_{t,t'}^{(V)} w(\underline{x}) \in \mathbb{C}^N \quad ((t, t', \underline{x}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^s) \]
is jointly \( C^\infty \) in all variables.

(c) The Cauchy-problem for the Dirac-equation \( D_V \varphi = 0 \) with the potential term \( V = V_{(i)} \) or \( V = V_{(ii)} \) is well-posed in the following sense. For any given \( w \in \mathcal{S}(\mathbb{R}^s, \mathbb{C}^N) \) and \( t' \in \mathbb{R} \) there is a unique \( \varphi \in C^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \) such that \( D_V \varphi = 0 \) and
\[ P_{t'} \varphi = w. \]

**Proof** Part (a). We shall work in the interaction picture, i.e. we obtain \( T_{t,t'}^{(V)} \) as
\[ T_{t,t'}^{(V)} = e^{itH_0} \tilde{T}_{t,t'}^{(V)} e^{-it'H_0}, \] (58)
where \( \tilde{T}_{t,t'}^{(V)} \) is the Dyson series for
\[ \tilde{U}(t) = e^{itH_0} \gamma^0 V(t) e^{-itH_0}, \]

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meaning that
\[ \tilde{T}_n^{(V)}(t, t') = \sum_{n=0}^{\infty} \tilde{T}_n^{(V)}(t, t'), \]
where \( \tilde{T}_n^{(V)}(t, t') \) is iteratively defined by
\[ \tilde{T}_0^{(V)}(t, t') = 1, \quad \tilde{T}_n^{(V)}(t, t') = \frac{1}{i} \int_{t'}^{t} \tilde{U}(r) \tilde{T}_{n-1}^{(V)}(r, t') dr. \]

Since the operators \( V(t) \) and \( \gamma^0 V(t) \) are bounded operators (with uniform bound in \( t \)) on \( L^2(\mathbb{R}^s, \mathbb{C}^N) \), and \( H_0 \) is essentially selfadjoint, one can rely on Theorem X.69 in [35] to see that \( T_{t,t'}^{(V)} \) is a family of unitaries with the required properties, provided that \( T_{t,t'}^{(V)} \) maps \( \mathcal{S}(\mathbb{R}^s, \mathbb{C}^N) \) into itself. To show this, we note that (cf. [12], Theorem 1.2 and Appendix 1.D)

\[ (e^{itH_0} f)(\bar{x}) = \int e^{ikx} \left( \cos (|\tilde{H}(k)| t) - (\gamma^0 \gamma^k p_k + i\gamma^0 m) \frac{\sin (|\tilde{H}(k)| t)}{|\tilde{H}(k)|} \right) \hat{f}(p) \frac{dp}{(2\pi)^{s}}, \]

where \( \hat{f} \) is the Fourier transform of \( f \), \( p = (p_1, \ldots, p_s) \in \mathbb{R}^s \), and \( |\tilde{H}(k)| = \sqrt{|k|^2 + m^2} \). This shows that \( e^{itH_0} f \) is in \( \mathcal{S}(\mathbb{R}^s, \mathbb{C}^N) \) for \( t \in \mathbb{R} \) and that, moreover, \( e^{itH_0} f \) is \( C^\infty \) in \( t \) with respect to the \( \mathcal{S} \)-topology. In the next step, we note that

\[ \tilde{T}_n^{(V)}(t'', t') = \left( \frac{1}{i} \right)^n \int_{t'}^{t''} \int_{t'}^{t_0} \cdots \int_{t'}^{t_1} \tilde{U}(t_n) \cdots \tilde{U}(t_1) dt_1 \cdots dt_n. \]

We set
\[ \nu f = \nu(i) f = \gamma^0 (b \ast (q-1,p)) f + f \ast (q-1,p) b, \quad V = V(i) \tag{59} \]
\[ \nu f = \nu(ii) f = \gamma^0 (b \ast (q-1,p)) f \ast (q-1,p) b, \quad V = V(ii). \tag{60} \]

Then \( V(t)f = a(t)\nu f \), and
\[ \tilde{T}_n^{(V)}(t'', t') f = \left( \frac{1}{i} \right)^n \int_{t'}^{t''} \int_{t'}^{t_n} \cdots \int_{t'}^{t_2} a(t_n) \cdots a(t_1) f(n)(t^{(n)}) dt_1 \cdots dt_n, \]
where
\[ f(n)(t^{(n)}) = e^{it_n H_0} \nu e^{i(t_{n-1}-t_n) H_0} \nu \cdots e^{i(t_1-t_2) H_0} \nu e^{-it_1 H_0} f. \]

This implies that, given any pair of multi-indices \( \alpha, \beta \in \mathbb{N}_0^s \), we obtain an estimate of the form

\[ \left\| \chi^\alpha D^\beta \tilde{T}_n^{(V)}(t'', t') f \right\|_{L^2} \leq C_n(t''', t') \frac{2^n}{n!} \max_{t_j \in [t'', t']} \left\| \chi^\alpha D^\beta f(n)(t^{(n)}) \right\|_{L^2} \tag{61} \]

\[ ^2\text{For the remainder of this proof, } D^\beta = (-i\partial/\partial x^1)^{\beta_1} \cdots (-i\partial/\partial x^s)^{\beta_s}; \text{ here, } D \text{ is not to be confused with the Dirac operator.} \]

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with the constant
\[ C_a(t'', t') = \max_{t \in \mathbb{R}} |a(t)||t'' - t'|. \]

Now we will show that there is a constant \( C_{b,\alpha,\beta}(t'', t') \) and a constant \( m(\alpha, \beta) \) such that
\[
\max_{t_j \in [t'', t']} \left\| x^\alpha D^\beta f^{(n)}(t^{(n)}) \right\|_{L^2} \leq C_{b,\alpha,\beta}(t'', t')^n \sup_{|\gamma| \leq 2m(\alpha, \beta)} \| D^\gamma f \|_{L^2}. \tag{62}
\]

The proof will be given by induction on \( n \), and we will only treat the case \( v = v_{(i)} \) (corresponding to \( V = V_{(i)} \)) as the other case \( v = v_{(ii)} \) is completely analogous. The inductive proof will only be needed for the special case \( x^\alpha = 1 \) (i.e. all \( \alpha_j = 0 \)) which makes it more transparent, and the result will be used in the proof of the general case. Let \( \beta \in \mathbb{N}_0 \) be a multi-index and define \( B_{[\beta]} = \sup_{|\gamma| \leq |\beta|} \| D^\gamma b \|_{L^2} \). We want to prove by induction that there is a constant \( F > 0 \) with
\[
\left\| D^\beta f^{(n)}(t^{(n)}) \right\|_{L^2} \leq 2^n \cdot 2^{|\beta|} F^n B_{[\beta]} \sup_{|\gamma| \leq |\beta|} \| D^\gamma f \|_{L^2}. \tag{63}
\]

Let us show that this is correct for \( n = 1 \):
\[
f^{(1)}(t_1) = e^{it_1 H_0} v e^{-it_1 H_0} f = e^{it_1 H_0} \gamma^0 (b \star (q-1,p) (e^{-it_1 H_0} f) + (e^{-it_1 H_0} f) \star (q-1,p) b). \]

The Leibniz rule for coordinate derivatives applies with respect to the Moyal product \( \star = \star_{(q-1,p)} \):
\[
\frac{\partial}{\partial x_j} (h \star g) = \left( \frac{\partial}{\partial x_j} h \right) \star g + h \star \left( \frac{\partial}{\partial x_j} g \right), \quad h, g \in \mathcal{S}(\mathbb{R}^s). \tag{64}
\]

Since the coordinate derivatives \( \frac{\partial}{\partial x_j} \) commute with \( e^{it H_0} \), we hence obtain
\[
D^\beta f^{(1)}(t_1) = \sum_{k=1}^{g_{[\beta]}} e^{it_1 H_0} \gamma^0 \left( (D^\beta(k) b) \star (D^\beta''(k) e^{-it_1 H_0} f) + (D^\beta(k) e^{-it_1 H_0} f) \star (D^\beta''(k) b) \right), \tag{65}
\]

where \( \beta'(k) \) and \( \beta''(k) \) are suitable multi-indices with \( \beta_j'(k) + \beta_j''(k) = \beta_j \). Using the fact that \( \|h \star g\|_{L^2} \leq F \|h\|_{L^2} \|g\|_{L^2} \) for \( h, g \in \mathcal{S}(\mathbb{R}^s) \), with \( F = (2\pi \theta)^{-p/2} \), one deduces
\[
\left\| D^\beta f^{(1)}(t_1) \right\|_{L^2} \leq 2 \cdot 2^{|\beta|} F B_{[\beta]} \sup_{|\gamma| \leq |\beta|} \| D^\gamma f \|_{L^2}, \tag{66}
\]

\[^3\]The case \( \beta'(k_1) = \beta'(k_2) \) and \( \beta''(k_1) = \beta''(k_2) \) for some \( k_1 \neq k_2 \) typically occurs in our sum decomposition of multiple derivatives of a product. Usually, this is written as a sum over fewer terms, occurring with a multiplicity expressed by binomial coefficients.
In order to conclude that the validity of \((63)\) for some \(n \in \mathbb{N}\) implies the relation \((63)\) with \(n + 1\) in place of \(n\), we note that
\[
 f^{(n+1)}(t^{(n+1)}) = e^{it_{n+1}H_0}v e^{-it_{n+1}H_0} f^{(n)}(t^{(n)})
 = e^{it_{n+1}H_0}e^0 \left( b \ast (e^{-it_{n+1}H_0} f^{(n)}(t^{(n)})) \right)
 + (e^{-it_{n+1}H_0} f^{(n)}(t^{(n)})) \ast b.
\]
Hence, relation \((65)\) continues to hold under the simultaneous replacements
\( f^{(1)}(t_1) \mapsto f^{(n+1)}(t^{(n+1)}) \), \( e^{\pm it_{n+1}H_0} \mapsto e^{\pm it_{n+1}H_0} \) and \( f \mapsto f^{(n)}(t^{(n)}) \). Therefore, \((66)\) also holds when making these replacements, leading to the estimate
\[
\left\| D^\beta f^{(n+1)}(t^{(n+1)}) \right\|_{L^2} \leq 2 \cdot 2^{2|\beta|} F B_{|\beta|} \sup_{|\gamma| \leq |\beta|} \left\| D^\gamma f^{(n)}(t^{(n)}) \right\|_{L^2}
 \leq 2^{n+1} \cdot 2^{2|\beta|(n+1)} F^{n+1} B^{n+1}_{|\beta|} \sup_{|\gamma| \leq |\beta|} \left\| D^\gamma f \right\|_{L^2},
\]
where the induction hypothesis \((63)\) was used in the second inequality. This proves by induction that \((63)\) holds for all \(n \in \mathbb{N}\).

Turning to the general case, the first observation is that, given multi-indices \(\alpha, \beta \in \mathbb{N}_0^n\) and a finite real interval \([t'', t']\), there are constants \(\Gamma_{\alpha, \beta}(t'', t') > 0\) and \(m(\alpha, \beta) > 0\) such that
\[
\left\| x^\alpha D^\beta e^{itH_0} \psi \right\|_{L^2} \leq \Gamma_{\alpha, \beta}(t'', t') \sum_{|\rho|, |\delta| \leq m(\alpha, \beta)} \left\| x^\rho D^\delta \psi \right\|_{L^2}
\]
holds for all \(\psi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^N)\) and all \(t \in [t'', t']\), where \(\rho, \delta\) are multi-indices. The second observation is that the action of multiplication by a coordinate function \(x^j\) on a Moyal product can be decomposed as follows (cf. [21]):
There are numbers \(\varepsilon(j)\) which may take the values 0, 1 or \(-1\), and for each coordinate \(x^j\) there is a coordinate \(x^{\varepsilon(j)}\), such that
\[
x^j(h \ast g) = h \ast (x^j g) + \frac{i \varepsilon(j) \theta}{2} \frac{\partial h}{\partial x^{\varepsilon(j)}} \ast g
 = (x^j h) \ast g - \frac{i \varepsilon(j) \theta}{2} h \ast \frac{\partial g}{\partial x^{\varepsilon(j)}}\tag{67}
\]
for all \(h, g \in \mathcal{S}(\mathbb{R}^n)\). Now we define:
\[
M_{\alpha, \beta} = \sup_{|\gamma|, |\sigma| \leq m(\alpha, \beta)} \left\| x^\gamma D^\sigma b \right\|_{L^2}
\]
\[
C_{b, \alpha, \beta}(t'', t') = 2 \cdot 2^{2m(\alpha, \beta)} FT_{\alpha, \beta}(t'', t') m(\alpha, \beta) \frac{1}{2} M_{\alpha, \beta} \left( 1 + \frac{|\theta|}{2} \right)^{m(\alpha, \beta)}
\]
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and we will show that (62) holds with these definitions for all \( n \in \mathbb{N} \). It holds that

\[
\left\| x^{\alpha} D^{\beta} f^{(n+1)} (t^{(n+1)}) \right\|_{L^2} \leq \Gamma_{\alpha \beta} (t^\prime, t^\prime) \sum_{|\rho| \leq m(\alpha, \beta)} \| x^{\rho} D^{\delta} e^{-it_{n+1}H_0} f^{(n)} (t^{(n)}) \|_{L^2} \tag{68}
\]

for all \( t_{n+1} \in [t^\prime, t^\prime] \) and all \( n \in \mathbb{N}_0 \) (with \( f^{(0)}(t^{(0)}) = f \)). Now we use (64) and (67) to conclude that we can write

\[
x^{\rho} D^{\delta} e^{-it_{n+1}H_0} f^{(n)} (t^{(n)})
\]

\[
= \sum_{l=1}^{2|\rho|} \sum_{k=1}^{2|\delta|} (\mu(l, k) \gamma^0 (D^{\rho}(l) D^{\delta}(k) e^{-it_{n+1}H_0} f^{(n)} (t^{(n)})) * (x^{\rho}(l) D^{\delta}(k) b) + \nu(l, k) \gamma^0 (x^{\rho}(l) D^{\delta}(k) b) * (D^{\rho}(l) D^{\delta}(k) e^{-it_{n+1}H_0} f^{(n)} (t^{(n)})),
\]

where \( \mu(l, k), \nu(l, k) \) are complex numbers and \( \rho(l), \rho(l), \delta(k), \delta(k) \) are suitable multi-indices, where \( \delta_j(k) + \delta^0_j(k) = \delta_j; \ |\rho(l)|, |\rho(l)| \leq |\rho| \), and \( |\mu(l, k)|, |\nu(l, k)| \leq \left( 1 + \frac{|\theta|}{2} \right)^{|\alpha|} \). Thus we obtain for the sum on the right hand side of (69) the \( L^2 \)-norm bound

\[
2 \cdot 2|\rho| \cdot 2|\delta| \left( 1 + \frac{|\theta|}{2} \right)^{|\alpha|} \sup_{|\gamma| \leq |\rho|, |\sigma| \leq |\delta|} \| x^{\gamma} D^{\sigma} b \|_{L^2} \cdot \sup_{|\gamma| \leq |\rho| + |\delta|} \| D^{\gamma} f \|_{L^2},
\]

using (63) again; inserting into (68) yields

\[
\left\| x^{\alpha} D^{\beta} f^{(n+1)} (t^{(n+1)}) \right\|_{L^2} \leq \Gamma_{\alpha \beta} (t^\prime, t^\prime) \sum_{|\rho| \leq m(\alpha, \beta)} \left( 1 + \frac{|\theta|}{2} \right)^{m(\alpha, \beta)} \sup_{|\gamma| \leq |\rho| + |\delta|} \| D^{\gamma} f \|_{L^2}
\]

with the above definitions. This proves that (62) holds for all \( n \in \mathbb{N} \).

In combination with (61), we have thus proved that for each given \( f \in \mathcal{S}(\mathbb{R}, \mathbb{C}^N) \), the Dyson series \( \sum_{n=0}^{\infty} T^{(V)}_n (t', t') f \) converges in all Schwartz norms and thus yields again an element in \( \mathcal{S}(\mathbb{R}, \mathbb{C}^N) \). Therefore, \( T^{(V)}_{t, t'} \) also maps \( \mathcal{S}(\mathbb{R}, \mathbb{C}^N) \) into itself and thence has (as mentioned, by Thm. X.69 in [35]) the properties claimed in statement (a) of the Lemma.
Part (b). The arguments showing the claimed property are quite standard in view of the estimates given to establish part (a), so we will mainly sketch them. Let \( \mu \) be any \( C^\infty \)-valued function on \( \mathbb{R} \times \mathbb{R} \) and denote by \( Y \) the function

\[
(t, t', \underline{x}) \mapsto \mu(t, t') T^{(V)}_{t, t'} w_0(x) = Y(t, t', \underline{x}).
\]

Since \( (t, t') \mapsto T^{(V)}_{t, t'} w \in L^2(\mathbb{R}^s, \mathbb{C}^N) \) is continuous, \( Y \) is in \( L^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^s, \mathbb{C}^N) \). Thus we need only show that, if \( \Delta_\pm \) denotes the Laplacian in \( s+2 \) dimensions, \( (1 - \Delta_\pm) Y \) is again in \( L^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^s, \mathbb{C}^N) \) for all \( J \in \mathbb{N} \); the claimed statement on smoothness then follows by Sobolev’s Lemma (cf. Thm. IX.24 in [35]). In turn, the required property follows from the fact that \( T^{(V)}_{t, t'} \) maps \( \mathscr{D}(\mathbb{R}^s, \mathbb{R}^N) \) into itself and that, as established in the proof of (a) or following immediately thereof,

\[
\begin{align*}
(i) \quad & \frac{\partial}{\partial t} T^{(V)}_{t, t'} w = iH_V(t) T^{(V)}_{t, t'} w, \\
(ii) \quad & x^\alpha \partial^\beta H_V(t) x^{\alpha'} \partial^{\beta'} w, \quad \text{is a continuous operator on} \\
& \mathscr{D}(\mathbb{R}^s, \mathbb{C}^N) \quad \text{uniformly in} \quad t \quad \text{ranging over compact intervals.}
\end{align*}
\]

Part (c). It follows form parts (a) and (b) that there exits for any given Cauchy-datum \( w \in \mathscr{D}(\mathbb{R}^s, \mathbb{C}^N) \) a solution \( \varphi \in C^\infty(\mathbb{R}) \otimes \mathscr{D}(\mathbb{R}^s) \otimes \mathbb{C}^N \) of \( D_V \varphi = 0 \) with \( P_t \varphi = w \).

Recalling the definition

\[
(v, w)_D = \int_{\mathbb{R}^s} \delta_{AB}(x) w^h(x) d^n x
\]

for \( v, w \in L^2(\mathbb{R}^s, \mathbb{C}^N) \), one finds

\[
\frac{d}{dt}(P_t \varphi, P_t \psi)_D = (iH_V(t) P_t \varphi, P_t \psi)_D + (P_t \varphi, iH_V(t) P_t \psi)_D = 0
\]

for all \( \varphi, \psi \in C^\infty(\mathbb{R}) \otimes \mathscr{D}(\mathbb{R}^s) \otimes \mathbb{C}^N \) which are solutions of the equations \( D_V \varphi = 0 \) and \( D_V \psi = 0 \). Hence, in particular, if for two solutions \( \varphi \) and \( \psi \) there holds \( (P_t (\varphi - \psi), P_t (\varphi - \psi))_D = 0 \) for some real \( t' \), then it follows that \( (P_t (\varphi - \psi), P_t (\varphi - \psi))_D = 0 \) for all real \( t \). This shows that, if \( \varphi \) and \( \psi \) have the same Cauchy-datum on some Cauchy-hyperplane \( \Sigma_t \), then actually \( \varphi = \psi \). \( \Box \)

On \( C^\infty(\mathbb{R}) \otimes \mathscr{D}(\mathbb{R}^s) \otimes \mathbb{C}^N \) we can introduce the sesquilinear form

\[
\langle f, h \rangle = \int_{\mathbb{R}^n} \gamma_{0AB}(f^B(x, h_A(x)) d^n x
\]

\[
= \int_{\mathbb{R}^n} \gamma_{0AB}(f^B(x) h_A(x)) d^n x,
\]

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where the last equality follows from the tracial property of the Moyal product for \( q = 0 \) (Lemma 2.1 (v) in [20]) and its obvious generalization to arbitrary \((q,p)\) due to the trivial case \( p = 0 \) and the tensor product structure [31]. Therefore we still have

\[
\langle Cf, Ch \rangle = -\langle h, f \rangle \quad (f, h \in \mathcal{C}_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N).
\]

We recall the definitions \( \Sigma_t = \{ x = (x^0, x^1, \ldots, x^s) \in \mathbb{R}^n : x^0 = t \} \), \( D_t = L^2(\Sigma_t, \mathbb{C}^N) \),

\[
(v, w)_D = \int_{\Sigma_t} \delta_{AB}(\bar{v}\,^A\,(q,p)\,w\,^B)(x)\,d^s x
= \int_{\Sigma_t} \bar{v}\,^A(x)\delta_{AB}w\,^B(x)\,d^s x \quad (v, w \in D_t).
\]

and the property \((Cv, Cw)_D = (w, v)_D\) of a conjugation \( C\) induced on each \( D_t \) by the charge conjugation \( C \) (same symbol).

For a subset \( G \) of \( n \) dimensional Moyal-deformed Minkowski spacetime the sets \( J^\pm(G) \) and the notion of hyperbolicity are defined in exactly the same way as in section 2.

Now we transfer the results collected in Proposition 2.1 to our new setting. For this purpose the following definition is needed.

**Definition 6.2** Let \( K \) be a non-empty subset of \( \mathbb{R}^n \). Then let

\[
\kappa_-(K) = \inf\{ x^0 : x = (x^0, x) \in K \}, \quad \kappa_+(K) = \sup\{ x^0 : x = (x^0, x) \in K \},
\]

and define

\[
T^+(K) = \{(x^0, x) \in \mathbb{R}^n : x^0 \geq \kappa_-(K) \},
T^-(K) = \{(x^0, x) \in \mathbb{R}^n : x^0 \leq \kappa_+(K) \}.
\]

![Figure 1: Sketch of the regions \( T^\pm(K) \)](image)

**Proposition 6.3**

\( \langle D_V f, h \rangle = \langle f, D_V h \rangle \quad (f, h \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N) \)

(b) There is a unique pair of continuous linear maps

\[
R^\pm_V : C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \to C^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N
\]
having the properties

\[ D_V R_V^\pm f = f = R_V^\pm D_V f \] and

\[ \text{supp } R_V^\pm f \subset T^\pm(\text{supp } f) \quad (f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N). \]

(c) \( CR_V^\pm = R_V^\pm C \)

(d) Writing \( R_V = R_V^+ - R_V^- \), the form

\[ (f,h)_V = \langle f,iR_V h \rangle \]

is a sesquilinear form on \( C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \), and \( C \) is a conjugation for this form:

\[ (Cf,Ch)_V = (h,f)_V = \overline{(f,h)_V} \quad (f,h \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N). \]

(e) For each \( t \in \mathbb{R} \) it holds that

\[ (f,h)_V = \langle P_t R_V f, P_t R_V h \rangle_D, \quad (f,h \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N), \]

where \( P_t : C^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \rightarrow \mathcal{S}(\Sigma_t, \mathbb{C}^N) \) is the map given by

\[ P_t : \varphi \mapsto \varphi(t,\cdot) \]

for \( \varphi : \mathbb{R}^n \rightarrow \mathbb{C}^N \) in \( C^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \), \( x_0 = (x^1, \ldots, x^s). \)

Hence, \((\cdot,\cdot)_V \) is positive-semidefinite on \( C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \).

(f) Let \( E_V \) be the subspace of all \( f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \) so that \( (f,f)_V = 0 \), and let \( \mathcal{K}_V \) be the Hilbert space arising as completion of \( (C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N) / E_V \) with respect to the scalar product induced by \((\cdot,\cdot)_V \) (which will be denoted by the same symbol). The quotient map \( C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \rightarrow (C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N) / E_V \) will be written

\[ f \mapsto [f]_V. \]

Then for each \( t \in \mathbb{R} \), the map

\[ Q_{V,t} : [f]_V \mapsto P_t R_V f \]

extends to a unitary map from \( \mathcal{K}_V \) onto \( D_t \).

(g) Let \( G \) be an open time-slice of \( n \) dimensional Moyal-deformed Minkowski spacetime, i.e. \( G = \{(x^0, x^1, \ldots, x^s) : \lambda_1 < x^0 < \lambda_2 \} \) for some real numbers (or infinite) \( \lambda_1 < \lambda_2 \), and suppose that \( V_1 \) and \( V_2 \) are potentials of the form described by (46), (48), and that \( V_1 = V_2 \) on \( G \). Then

\[ R_{V_1}^\pm f = R_{V_2}^\pm f \text{ on } G \text{ for all } f \in C_0^\infty((\lambda_1, \lambda_2)) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N. \]
Sketch of proof

(a) Clearly the only difference compared to Proposition 2.1 (a) is the partial claim \((Vf, h) = \langle f, Vh \rangle\). To show this, calculate (only for the case \(V = V(\iota)\); the other one is completely analogous)

\[
(Vf, h) = \int_{\mathbb{R}^n} \gamma_{0AB}(Vf)^B(x)h^A(x)d^n x = \gamma_{0AB}((Vf)^B, h^A)_{L^2}
\]

(b) To prove this, the fundamental solutions will be constructed explicitly. Using the notation \(f_{\nu}(.) = f(t', .)\), define for \(f \in C^\infty_0(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}^N\),

\[
(R^\pm_V f)(t, x) = \pm i\gamma^0 \int \theta(\pm(t - t'))T_{t', t}^{(V)} f_{\nu}(x) dt',
\]

where \(\theta\) is the Heaviside step function. Note that the integral is well-defined since \(f_{\nu}(x)\) has compact support in \(t'\). By Lemma 6.1 it follows that \((t, x) \mapsto (R^\pm_V f)(t, x)\) is \(C^\infty\), and of Schwartz type with respect to \(x\). Using standard arguments, and exploiting the properties of \(T_{t', t}^{(V)}\) given in Lemma 6.1, one proves that \(D_V R^\pm_V f = f = R^\pm_V D_V f\). The next step consists in showing that \(\text{supp}(R^\pm_V f) \subset T^{\pm}(\text{supp} f)\). To this end, suppose that \((x_0, \xi) \notin T^+(\text{supp} f)\). Then \(x_0 < \kappa_-(\text{supp} f)\) and therefore

\[
\theta(x_0 - t')T_{t', t}^{(V)} f_{\nu}(\xi) = 0
\]

for all values of \(t, t'\) and \(\xi\). To see this, note that if \(t' \geq x_0\), then \(\theta(x_0 - t') = 0\), and if \(t' < x_0 < \kappa_-(\text{supp} f)\), then \(f_{\nu}(.) = 0\) by the definition of \(\kappa_-(\text{supp} f)\). Hence, it holds that \((R^+_V f)(x_0, \xi) = 0\) if \((x_0, \xi) \notin T^+(\text{supp} f)\), implying that \(\text{supp}(R^+_V f) \subset T^+(\text{supp} f)\). The inclusion \(\text{supp}(R^+_V f) \subset T^-(\text{supp} f)\) can be shown in an analogous manner. The uniqueness property of the fundamental solutions follows by a standard argument owing to the well-posedness of the Cauchy-problem for the Dirac-equation \(D_V \varphi = 0\).

(c) This is a consequence of the uniqueness of the \(R^\pm_V\) together with \(CD_V = D_V C\).

(d) Analogous to Proposition 2.1 the crucial point is the validity of \(\langle R_V h, f \rangle = -\langle h, R_V f \rangle\), for which the argument is again similar to the proof of Thm. 2.1 in [17].
(e) The argument is the same as in Proposition 2.1.

(f) The proof is identical to the corresponding statement (g) of Proposition 2.1. The modification lies in the generalized class of Cauchy data. The choice of a partition of unity $\chi_\pm$ depending only on the time-coordinate $x^0$ is needed at this point.

(g) Apart from the modified assumption on the subset $G$, providing an adjusted time-direction behaviour for the non-commutative case, this can obviously proved the same way as Proposition 2.1, (h). □

Analogous to Section 2 the self-dual CAR-algebra $\mathfrak{H}(\mathcal{K}_V, C)$ is generated by the $C$-linear “abstract field operators” $\Psi(f) = \Psi_V(f) = B_V([f]_V)$, for $f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N$, obeying the relations

\[
\Psi(f)^* = \Psi(Cf), \\
\{\Psi(f)^*, \Psi(h)\} = 2(f, h)_V 1, \\
\Psi(D_V f) = 0.
\]

Note that because of the trivial action of $V$ with respect to the first coordinate (the time) it holds that $D_V f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N$.

Again this construction can be carried out for “local subspaces” of $\mathcal{K}_V$ as well (cf. Section 2), and we get $\mathfrak{H}(\mathcal{K}_V^G, C)$ generated by $B_{V}^G([f]_V^G)$, but this time only for an open time-slice $G$ of $n$ dimensional Moyal-deformed Minkowski spacetime and no longer for arbitrary hyperbolic subsets. Being mainly a consequence of Proposition 2.1 (h), Lemma 2.2 can be transferred almost unchanged.

\textbf{Lemma 6.4} Suppose that $G$ is a hyperbolic neighbourhood of a Cauchy hyperplane in $n$ dimensional Moyal-deformed Minkowski spacetime, of the form as in Proposition 6.3 (g). Moreover, suppose that $V_1$ and $V_2$ are two potentials of type (46), (48), which coincide on the region $G$. Then

(a) The map $u_{V_1, V_2}^G : [f]_{V_1}^G \mapsto [f]_{V_2}$, $f \in C_0^\infty((\lambda_1, \lambda_2)) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N$

extends to a unitary between $\mathcal{K}_V^G$ and $\mathcal{K}_V$ commuting with the charge conjugation $C$.

(b) There is a $*$-algebra isomorphism

$\alpha_{V_1, V_2}^G : \mathfrak{H}(\mathcal{K}_V^G, C) \to \mathfrak{H}(\mathcal{K}_V, C)$

induced by

$\alpha_{V_1, V_2}^G (B_{V_1}^G([f]_{V_1}^G)) = B_{V_2}([f]_{V_2}), \quad f \in C_0^\infty((\lambda_1, \lambda_2)) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N$. 

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Since our potential operator \( V \) (see (46), (48)) was chosen to be a compactly supported multiplication operator with respect to the time coordinate, we can maintain exactly the same geometrical setting as in Section 2 involving the same time-slice \( \{(x^0, x^1, \ldots, x^s) : \lambda_- < x^0 < \lambda_+\} \) for some real numbers \( \lambda_- < \lambda_+ \) and the same regions \( G_+, G_- \) being hyperbolic neighbourhoods of the Cauchy hyperplanes \( \Sigma_+, \Sigma_- \). As a result we again arrive at an automorphism

\[
\beta_V : \mathfrak{F}(K_0, C) \to \mathfrak{F}(K_0, C),
\beta_V = \alpha_{0,-} \circ \alpha_{V,-}^{-1} \circ \alpha_{V,+} \circ \alpha_{0,+}^{-1}.
\]  

(70)

As in Sec. 2 we can again define \( T_{sc}^{(V)} \) as the scattering transformation on \( \mathcal{D}_0 \simeq L^2(\Sigma_0, \mathbb{C}^N) \) \( (\Sigma_0 \simeq \mathbb{R}^s) \), the space of Cauchy data for the Dirac equation at coordinate-time \( t = 0 \), by setting

\[
T_{sc}^{(V)} = T_{t,t'}^{-1} \circ T_{t',0}^{(V)} \circ T_{t'},
\]

for \( t > \lambda_+, t' < \lambda_- \) (recall that the interval \([\lambda_-, \lambda_+]\) is the time-support of the potential term \( V \)). As before in Sec. 2 \( T_t \) denotes the “free” evolution of the Dirac equation without potential term, coinciding with \( T_{t,0}^{(V)} \big|_{V=0} \). In consequence one obtains, exactly as in eqn. (23), an induced automorphism \( \tau_{sc}^{(V)} \) on the CAR-algebra \( \mathfrak{F}(\mathcal{D}_0, C) \), given by

\[
\tau_{sc}^{(V)}(B_{\mathcal{D}_0}(v)) = B_{\mathcal{D}_0}(T_{sc}^{(V)} v), \quad v \in \mathcal{D}_0,
\]

where the \( B_{\mathcal{D}_0}(v) \) are the generators of \( \mathfrak{F}(\mathcal{D}_0, C) \). Again, there is a canonical identification between the CAR algebras \( \mathfrak{F}(K_0, C) \) and \( \mathfrak{F}(\mathcal{D}_0, C) \),

\[
\varrho(B_0([f]_0)) = B_{\mathcal{D}_0}(Q_0([f]_0)).
\]

In the next section we will study the problem of unitary implementability of \( \beta_V \) in the Fock-vacuum-representation of \( \mathfrak{F}(K_0, C) \), and the following Lemma, which is the counterpart of Lemma 2.3 for the case of potential \( V \) defined as in (46) and (47), guarantees that unitary implementability of \( \tau_{sc}^{(V)} \) in the vacuum representation is just the equivalent problem.

**Lemma 6.5** The morphism \( \beta_V \) of \( \mathfrak{F}(K_0, C) \) defined in (70) and the scattering morphism \( \tau_{sc}^{(V)} \) (defined like (23)) describing the potential scattering of the quantized Dirac field at the level of the Cauchy-data CAR-algebra \( \mathfrak{F}(\mathcal{D}_0, C) \) (with \( \mathcal{D}_0 = L^2(\Sigma_0, \mathbb{C}^N) \)) are intertwined by the CAR-algebra isomorphism \( \varrho : \mathfrak{F}(K_0, C) \to \mathfrak{F}(\mathcal{D}_0, C) \) defined in (15), i.e. it holds that

\[
\varrho \circ \beta_V = \tau_{sc}^{(V)} \circ \varrho. \quad (71)
\]

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7 Moyal-deformed Minkowski spacetime: Scattering of the Dirac field in the vacuum representation and implementability of the scattering transformation

In the present section we will prove unitary implementability of the scattering transformation $\beta_\tau$ on $\mathfrak{F}(K_0, C)$ for the “Moyal-Minkowski-potentials” $V$ defined in (49) and (48), in the Fock-vacuum representation $(\mathcal{H}_{\text{vac}}, \pi_{\text{vac}}, \Omega_{\text{vac}})$ of the vacuum state $\omega_{\text{vac}}$ on $\mathfrak{F}(K_0, C)$. Owing to Lemma 6.5 this is equivalent to the problem of unitary implementability of the scattering transformation $\tau_{\text{sc}}^{(V)}$ on $\mathfrak{F}(D_0, C)$ in the Fock-vacuum representation $(\mathcal{H}^{p_+}, \pi^{p_+}, \Omega^{p_+})$ of $\omega^{p_+}$ (where $\omega^{p_+} = \omega^{p_+} \circ g$), the pure, quasifree ground state on $\mathfrak{F}(D_0, C)$ with respect to the time-evolution induced by the Hamiltonian $H_0$ of (49) on the domain $\mathcal{D}(\mathbb{R}^8, \mathbb{C}^N) \subset L^2(\mathbb{R}^8, \mathbb{C}^N)$. Recall that $p_+$ is the spectral projection of $H_0$ corresponding to the spectral interval $[m, \infty)$, and that the conjugation $C$ intertwines $p_+$ and $1 - p_+$ and hence $p_+$ is a basis projection on $(D_0, C)$ according to [2]. A well-established criterion for unitary implementability of $\tau_{\text{sc}}^{(V)}$ in the Fock-vacuum representation is that $[p_+, T_{\text{sc}}^{(V)}]$ is Hilbert-Schmidt. If and only if this is the case, then there is a unitary operator $S_{\tau_{\text{sc}}^{(V)}}$ on $\mathcal{H}^{p_+}$ such that

$$S_{\tau_{\text{sc}}^{(V)}} \pi^{p_+} (B_{D_0}(v)) S_{\tau_{\text{sc}}^{(V)}}^{-1} = \pi^{p_+} \left( \tau_{\text{sc}}^{(V)} (B_{D_0}(v)) \right) = \pi^{p_+} (B_{D_0} (T_{\text{sc}}^{(V)} v)), \quad v \in D_0,$$

(72)

which is just what it means to say that $\tau_{\text{sc}}^{(V)}$ is unitarily implementable.

The condition that $[p_+, T_{\text{sc}}^{(V)}]$ is Hilbert-Schmidt is equivalent to the condition that $[\varepsilon, T_{\text{sc}}^{(V)}]$ is Hilbert-Schmidt as an operator on $L^2(\mathbb{R}^8, \mathbb{C}^N)$, where $\varepsilon = \text{sign}(H_0) = H_0 / |H_0|$ is the sign function of $H_0$ in the sense of the functional calculus, since $p_+ = (1 + \varepsilon) / 2$. In an interesting work, Langmann and Mickelsson [30] have shown that certain conditions on the potential term $V(t)$ (cf. eqn. (50)) are sufficient to conclude that $[\varepsilon, T_{\text{sc}}^{(V)}]$ is Hilbert-Schmidt. Their argument is interesting as it involves a non-local regularization of the interacting dynamics which nevertheless leads to the same scattering transformation $T_{\text{sc}}^{(V)}$. We refer to [30] for details and present only the relevant conditions, adapted to our notation.

Let the interaction potential $W(t) = \gamma^0 V(t)$ in the Hamiltonian $H_V(t)$ of eqn. (50) have the following properties:

(I) $W(t)$ is a bounded operator on $L^2(\mathbb{R}^8, \mathbb{C}^N)$ for each $t \in \mathbb{R}$, such that $t \mapsto W(t)$ is $C^\infty$.

(II) There is a core for $H_0$, contained in the $C^\infty$-domain of $H_0$, which is left invariant by all $W(t)$ and $(\partial_t)^k W(t)$ ($k \in \mathbb{N}$) and by all $C^\infty$ func-
tions of \( H_0 \) which, together with all their derivatives, are polynomially bounded.

(III) There is some \( \nu \in \mathbb{N}_0 \) so that \( |H_0|^{-\nu}W(t) \) and \( |H_0|^{-\nu}(\partial_t)^kW(t), k = 1, \ldots, \nu, \) are Hilbert-Schmidt operators for all \( t \).

(IV) Defining \( \delta_{|H_0|}(A) = [|H_0|, A] \), it holds that \( \delta^n_{|H_0|}(W(t)) \) and \( \delta^n_{|H_0|}((\partial_t)^kW(t)), k = 1, \ldots, \nu, \) are bounded operators on \( L^2(\mathbb{R}^s, \mathbb{C}^N) \) for all \( n \in \mathbb{N} \) \( (t \in \mathbb{R}) \).

(V) \( |H_0|^{-\nu}\delta^n_{|H_0|}(W(t)) \) and \( |H_0|^{-\nu}\delta^n_{|H_0|}((\partial_t)^kW(t)), k = 1, \ldots, \nu, \) are Hilbert-Schmidt operators for all \( n \in \mathbb{N} \) \( (t \in \mathbb{R}) \).

(VI) \( W(t) = 0 \) if \( t < \lambda_- \) and if \( t > \lambda_+ \) for some real numbers \( \lambda_- < \lambda_+ \).

We cite the result relevant for our purposes.

**Theorem 7.1 (Langmann and Mickelsson [30])** If the interaction term \( W(t) = \gamma^0V(t) \) in \( H_V(t) \) (cf. (50)) satisfies the conditions (I) ... (VI), then \( [\varepsilon, T_{sc}^{(V)}] \) is Hilbert-Schmidt, and hence \( \tau_{sc}^{(V)} \) is implementable in the vacuum-representation (\( \mathcal{H}^{p+}, \pi^{p+}, \Omega^{p+} \)) of the Dirac field.

Consequently, what we will now set out to demonstrate is

**Proposition 7.2** Let \( V(t) \) be any of the \( V_{(i)}(t) \) or \( V_{(ii)}(t) \) defined in (52) and (53), with \( a \in C_0^\infty(\mathbb{R}, \mathbb{R}) \) and \( b \in \mathcal{S}(\mathbb{R}^s, \mathbb{R}) \). Then \( W(t) = \gamma^0V(t) \) fulfills the criteria (I) ... (VI) above. Therefore, \( \tau_{sc}^{(V)} \) is unitarily implementable in the vacuum representation, so that there is a unitary \( S_{\tau_{sc}^{(V)}} \) on \( \mathcal{H}^{p+} \) such that (72) holds.

**Proof** Observing that (cf. (52), (53))

\[
W(t)f = \tilde{a}(t)v f \quad (f \in L^2(\mathbb{R}^s, \mathbb{R}^N))
\]

with \( \tilde{a}(t) = a(t) \) for \( V = V_{(i)} \) and \( \tilde{a}(t) = a(t)^2 \) for \( V = V_{(ii)} \), the time-dependence of \( W(t) \) is trivial in the context of conditions (I) ... (VI), and they need only be checked for the time-independent operator \( v \), which is

\[
v f = v_{(i)} f = \gamma^0(L_b f + R_b f) \quad \text{or} \quad \gamma^0(L_b R_b f).
\]

We note also that the multiplication with \( \gamma^0 \) is irrelevant for checking the conditions (I) ... (VI) since \( \gamma^0 \) commutes with \( |H_0| \). Thus, the conditions need only be checked for \( L_b, R_b \) and \( L_b R_b \). A quick inspection shows that the conditions are algebraic in the sense that, if they hold for \( L_b \) and \( R_b \), then they hold also for the operator product \( L_b R_b \). As will become clear from the
Fourier-representations of $L_b$ and $R_b$ (see below), checking the conditions for $R_b$ is completely analogous to the case of $L_b$, so it is sufficient to show that the conditions are fulfilled for the operator $L_b$.

Let, for $g \in L^2(\mathbb{R}^s, \mathbb{C})$, the Fourier-transform be defined by

$$(Fg)(\mathbf{k}) = \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} g(y) e^{-i\mathbf{k} \cdot \mathbf{y}} \, dy,$$  \hspace{1cm} (73)

this definition is extended componentwise to elements in $L^2(\mathbb{R}^s, \mathbb{C}^N)$. It is easy to see that

$$F_{L_b} F^{-1} \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \hat{b}(\mathbf{k} - \mathbf{u}) e^{i\mathbf{k} \cdot \mathbf{u}} \hat{g}(\mathbf{u}) \, d^n \mathbf{u}, \hspace{1cm} (74)$$

$$F_{R_b} F^{-1} \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \hat{b}(\mathbf{k} - \mathbf{u}) e^{-i\mathbf{k} \cdot \mathbf{u}} \hat{g}(\mathbf{u}) \, d^n \mathbf{u}, \hspace{1cm} (75)$$

where $\hat{b}$ is the block-entry $M_{(q+p-1) \times (q+p-1)}$ in the matrix (45). Moreover, one finds

$$F_{H_0} F^{-1} \hat{g}(\mathbf{k}) = \tilde{H}(\mathbf{k}) \hat{g}(\mathbf{k}), \hspace{1cm} (76)$$

$$F_{|H_0|^k} F^{-1} \hat{g}(\mathbf{k}) = |\tilde{H}(\mathbf{k})|^k \hat{g}(\mathbf{k}) \quad (k \in \mathbb{R}),$$

with

$$\tilde{H}(\mathbf{k}) = -i\gamma^0 \gamma^j \mathbf{k}_j + \gamma^0 m, \quad |\tilde{H}(\mathbf{k})| = (|\mathbf{k}|^2 + m^2)^{1/2} \quad (k \in \mathbb{R}^s). \hspace{1cm} (77)$$

This implies

$$F_{\delta_{|H_0|}^n(L_b)} F^{-1} \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} (|\tilde{H}_0(\mathbf{k})| - |\tilde{H}_0(\mathbf{u})|)^n \hat{b}(\mathbf{k} - \mathbf{u}) e^{i\mathbf{k} \cdot \mathbf{u}} \hat{g}(\mathbf{u}) \, d^n \mathbf{u} \hspace{1cm} (78)$$

for all $n \in \mathbb{N}$ and all $\hat{g} \in \mathcal{S}(\mathbb{R}^s, \mathbb{C}^N)$, and similarly

$$F_{|H_0|^{-n}} \delta_{|H_0|}^n(L_b) F^{-1} \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \frac{|\tilde{H}_0(\mathbf{k})| - |\tilde{H}_0(\mathbf{u})|^n}{|\tilde{H}_0(\mathbf{k})|^n} \hat{b}(\mathbf{k} - \mathbf{u}) e^{i\mathbf{k} \cdot \mathbf{u}} \hat{g}(\mathbf{u}) \, d^n \mathbf{u}. \hspace{1cm} (79)$$

The discussion in Sec. 4 shows that $L_b$ is bounded. Moreover, we see from the Fourier-representation of $H_0$ that $\mathcal{S}(\mathbb{R}^s, \mathbb{C}^N)$ is a core with the properties demanded in (II). In view of what we observed previously, (IV) is proved once we have shown that $\delta_{|H_0|}^n(L_b)$ is a bounded operator for all $n \in \mathbb{N}$. We use that, given $n$, there are constants $\alpha, \beta > 0$ such that

$$|\tilde{H}_0(\mathbf{k})| - |\tilde{H}_0(\mathbf{u})|^n \leq \alpha |\mathbf{k} - \mathbf{u}|^{2n} + \beta \quad (\mathbf{k}, \mathbf{u} \in \mathbb{R}^s). \hspace{1cm} (80)$$

Now the integral kernel in (78) is actually a matrix, and owing to (80), each of its entries has a modulus which can be bounded by

$$\alpha |(-\Delta^n b)(\mathbf{k} - \mathbf{u})| + \beta |b(\mathbf{k} - \mathbf{u})|,$$  \hspace{1cm} (81)
where \( \Delta \) denotes the Laplace operator. This shows that \( F\delta^n_{|H_0|}(L_b)F^{-1} \) has an operator norm which can be dominated by a constant times

\[
\sup_k \left( \int_{\mathbb{R}^s} [\alpha|(-\Delta^n b)(k-w)| + \beta|b(k-w)|]^2 \, d^s k \right)^{1/2} \tag{82}
\]

which is finite since \( b \) is of Schwartz type. It remains to check conditions (III) and (V). To this end, we observe that the integral kernel in (79) is a matrix where each entry has a modulus which, for given \( n \) and \( \nu \), can be estimated by a constant times

\[
\frac{1}{(|k|^2 + m^2)^{\nu/2}} [\alpha|(-\Delta^n b)(k-w)| + \beta|b(k-w)|]. \tag{83}
\]

One can obviously choose \( \nu \) large enough so that this expression is, for each \( n \), square integrable over \((k, u) \in \mathbb{R}^s \times \mathbb{R}^s\), using again that \( b \) is a Schwartz function. This shows that a number \( \nu \) can be chosen large enough so that \( F|H_0|^{-\nu}\delta^n_{|H_0|}(L_b)F^{-1} \) is Hilbert-Schmidt for all \( n \) (including \( n = 0 \), ie. \( F|H_0|^{-\nu}L_bF^{-1} \)).

Finally we remark that condition (VI) is clearly fulfilled since it was assumed that \( a \in C_0^\infty(\mathbb{R}, \mathbb{R}) \).

We will also show in this chapter that the generator of the S-matrix \( S_{\tau V} \) with respect to variations of \( V \) exists as an essentially selfadjoint operator in the sense of derivations. Using the fact that \( S_{\tau V} \) and \( S_{\tau V}^M \) are intertwined by a unitary establishing the equivalence between \( \pi_{\text{vac}} \) and \( \pi_{\text{p+}} \circ \partial \), this will allow the conclusion that also the generator of the S-matrix \( S_{\tau V}^M \) with respect to variations of \( V \) exists as an essentially selfadjoint operator on a suitable domain.

In preparing the proof of the assertion we aim to establish, we need an auxiliary result.

**Proposition 7.3** Let \( V \) be any of the operators \( V_{(0)} \), \( V_{(i)} \) or \( V_{(ii)} \), where

\[
V_{(0)} f(x) = c(x) f(x) \quad (f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^N), \ x \in \mathbb{R}^n),
\]

and \( V_{(i)} \) and \( V_{(ii)} \) are defined as (46) and (47), with \( c(x) = a(x^0) b(x) \) for any \( a \in C_0^\infty(\mathbb{R}, \mathbb{R}) \) and \( b \in \mathcal{S}(\mathbb{R}^s, \mathbb{R}), \ x = (x^0, x) \in \mathbb{R}^{1+s} = \mathbb{R}^n \).

Then

(a) Defining

\[
d T^{(V)}_{\tau V} v = -i \left. \frac{d}{d\lambda} \right|_{\lambda=0} T^{(\lambda V)}_{\tau V} v \quad (v \in \mathcal{S}(\mathbb{R}^s, \mathbb{C}^N)), \tag{84}
\]

it holds that

\[
d T^{(V)}_{\tau V} v(x) = -\int_{-\infty}^{\infty} \tilde{a}(t) e^{iH_0 t} v(x) e^{-iH_0 t} v(x) \, dt \quad (x \in \mathbb{R}^s), \tag{85}
\]

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where \( v \) is either of the operators \( v_{(0)}, v_{(i)} \) or \( v_{(ii)} \), with \( v_{(i)} \) and \( v_{(ii)} \) defined in (59) and (60) and

\[
v_{(0)} v(x) = \gamma_0 b(x) v(x),
\]

and with \( \tilde{a}(t) = a(t) \) in the cases \( V = V_{(0)}, V_{(i)} \), while \( \tilde{a}(t) = a(t)^2 \) in case \( V = V_{(ii)} \). The operator \( dT_{sc}^{(V)} \) is bounded and selfadjoint on \( L^2(\mathbb{R}^s, \mathbb{C}^N) \).

(b) The commutator

\[
[dT_{sc}^{(V)}, p_+]
\]

is a Hilbert-Schmidt operator on \( L^2(\mathbb{R}^s, \mathbb{C}^N) \). Here, \( p_+ \) denotes again the spectral projection of \( H_0 \) corresponding to the spectral interval \([m, \infty)\).

Before we give the proof of that Proposition (see towards the end of this Section), we explain how this result allows it to conclude the statements made in Sec. 5, which re-appear below in the eqns. (88), (89) and (91).

Recall that \( \mathcal{H}_{p_+} = \mathcal{F}_+(p_+ L^2(\mathbb{R}^s, \mathbb{C}^N)) \) is the Fermionic Fock-space with one-particle space \( p_+ L^2(\mathbb{R}^s, \mathbb{C}^N) \). For \( v \in L^2(\mathbb{R}^s, \mathbb{C}^N) \), define the field operators

\[
\psi(v) = A(p_+ C v) + A^+(p_+ v), \quad v \in D_0 \equiv L^2(\mathbb{R}^s, \mathbb{C}^N),
\]

where \( A \) and \( A^+ \) denote the Fermionic annihilation and creation operators.

In other words,

\[
\psi(v) = \pi^{p_+}(B_{D_0}(v)). \quad (86)
\]

By \( \mathcal{F} \) we denote the \( * \)-algebra generated by all \( \psi(v) \) and the unit operator, and we set \( \mathcal{W} = \mathcal{F} \Omega^{p_+} \). Now consider an orthonormal basis \( \{\chi_j^\pm\}_{j \in \mathbb{N}} \) of \( p_+ L^2(\mathbb{R}^s, \mathbb{C}^N) \); then \( \chi_j^- = C \chi_j^+ \) is an orthonormal basis of \( p_- L^2(\mathbb{R}^s, \mathbb{C}^N) \). If \( [dT_{sc}^{(V)}, p_+] \) is Hilbert-Schmidt, we can form the operator

\[
:G(dT_{sc}^{(V)}) := \lim_{k \to \infty} G_k(dT_{sc}^{(V)}) - (\Omega^{p_+}, G_k(dT_{sc}^{(V)}) \Omega^{p_+})
\]

upon defining

\[
G_k(dT_{sc}^{(V)}) = \sum_{j=1}^{k} \left( \psi(dT_{sc}^{(V)} \chi_j^+)^* \psi(\chi_j^+) + \psi(dT_{sc}^{(V)} \chi_j^-)^* \psi(\chi_j^-) \right).
\]

According to Sec. 10 in [32] (cf. also [11]), \( :G(dT_{sc}^{(V)}) : \) defines an essentially selfadjoint operator on \( \mathcal{W} \) (to see the hermiticity, use that \( CdT_{sc}^{(V)} = -dT_{sc}^{(V)} C \)), and it holds that

\[
[:G(dT_{sc}^{(V)}), \psi(v)] = \psi(dT_{sc}^{(V)} v) \quad (87)
\]
for all \( v \in L^2(\mathbb{R}^s, \mathbb{C}^N) \). Moreover, \( G(dT_{sc}^{(V)}) \) is actually independent of the choice of \( \{ \chi_j^\pm \}_{j \in \mathbb{N}} \). (Note that in the notation of \([32, 11]\), \( G(dT_{sc}^{(V)}) \) would be written \( dT_{sc} \psi^* \psi \)). On the other hand we have, by eqn. (72) and owing to (86), the relation

\[
\left. \frac{d}{d\lambda} \right|_{\lambda=0} S_{\tau_{sc}}(\lambda V) \psi(v) S_{\tau_{sc}}^{-1} = \psi \left( \left. \frac{d}{d\lambda} \right|_{\lambda=0} T_{sc}^{(\lambda V)} v \right) = \psi(idT_{sc}^{(V)} v) \quad (88)
\]

for all \( v \in L^2(\mathbb{R}^s, \mathbb{C}^N) \), resulting in

\[
\left. \frac{d}{d\lambda} \right|_{\lambda=0} S_{\tau_{sc}}(\lambda V) \psi(v) S_{\tau_{sc}}^{-1} = [i : G(dT_{sc}^{(V)}) : \psi(v)] \quad (89)
\]

for all \( v \in L^2(\mathbb{R}^s, \mathbb{C}^N) \).

In view of \( \omega^{\text{vac}} = \omega^{p^+} \circ \varrho \) with the morphism \( \varrho \) in \([15, 71]\), there is a canonical unitary operator \( U : \mathcal{H}^{\text{vac}} \rightarrow \mathcal{H}^{p^+} \) so that

\[
U \pi^{\text{vac}}(A) U^{-1} = \pi^{p^+} \circ \varrho(A) \quad (A \in \mathfrak{F}(\mathcal{K}_0, C)) \quad \text{and} \quad U\Omega^{\text{vac}} = \Omega^{p^+}. \quad (90)
\]

It is easy to see that \( U\mathcal{W} = \overline{\mathcal{W}} \) where \( \mathcal{W} \) has been introduced as domain for \( : \psi^+ \psi : (c) \) in Sec. 5. Furthermore, defining

\[
S_V^M = U^{-1} S_{\tau_{sc}}(\lambda V) U,
\]

and using (71), one can see that (72), which was proven in Prop. 7.2, is equivalent to

\[
S_V^M \pi^{\text{vac}}(A)(S_V^M)^{-1} = \pi^{\text{vac}}(\beta_V(A)) \quad (A \in \mathfrak{F}(\mathcal{K}_0, C)).
\]

Setting

\[
\Phi(c) = U^{-1} : G(dT_{sc}^{(V)}) : U,
\]

it holds that \( \Phi(c) \) is essentially selfadjoint on \( \mathcal{W} \), and by eqn. (88), there holds

\[
\left. \frac{d}{d\lambda} \right|_{\lambda=0} S_V^M \psi(f) S_V^M = [i \Phi(c), \psi(f)] = \psi(V R_0 f) \quad (91)
\]

for all \( f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N \). Thus we have established the statements announced in Sec. 5.

A further remark is in order here. We do not directly establish the relation \(-i \left. \frac{d}{d\lambda} \right|_{\lambda=0} S_{M \lambda \tau_{sc}}^M = \Phi(c) \). Notice that the unitary \( S_{M \lambda \tau_{sc}}^M \) implementing the scattering transformation is not uniquely determined, but only determined up to a phase, i.e. if \( S_{M \lambda \tau_{sc}}^M \) is another choice of unitary implementer of the scattering matrix, then \( S_{M \lambda \tau_{sc}}^M(S_{M \lambda \tau_{sc}}^M)^{-1} = e^{ir(\lambda)} \) with some real-valued function \( r(\lambda) \). However, if \( \lambda \mapsto S_{M \lambda \tau_{sc}}^M \) is indeed differentiable at \( \lambda = 0 \) (upon a suitable choice of \( \lambda \mapsto r(\lambda) \)), then it follows that its derivative with respect to \( \lambda \) at \( \lambda = 0 \) in fact equals the above defined \( \Phi(c) \) up to an additive multiple of the
unit operator. This is a consequence of \([91]\) together with the fact that the \(*\)-algebra generated by \(I\) and the \(\psi(f)\) acts irreducibly in the vacuum representation. The additive constant may be compensated for by a re-definition of the phase function \(r(\lambda)\).

**Proof of Proposition 7.3**

In order to show that \( [dT_{sc}^{(V)}, p_+] \) is Hilbert-Schmidt, it is sufficient to prove that \( p_+ dT_{sc}^{(V)} p_- \) is Hilbert-Schmidt. It holds that

\[
p_+ dT_{sc}^{(V)} p_- = \int_{-\infty}^{\infty} \hat{a}(t) e^{iH_0 t} p_+ v p_- e^{-iH_0 t} dt,
\]

implying

\[
Fp_+ dT_{sc}^{(V)} p_- F^{-1} = \int_{-\infty}^{\infty} \hat{a}(t) F e^{-iH_0 t} p_+ F^{-1} F v F^{-1} F p_- e^{iH_0 t} F^{-1} dt,
\]

where \(F\) is the Fourier-transform as in (73). In view of the Fourier-expressions for \(H_0\) (cf. (76), (77)), one has

\[
Fe^{iH_0 t} p_+ F^{-1} = \hat{H}_0 (k) \hat{p}_+ (k) \hat{g}(k)
\]

where \(\hat{p}_+ (k)\) is an \(N \times N\)-matrix valued projection. By the properties of \(H_0\) and the resulting \(\hat{H}_0 (k)\), it follows that there is a smooth family of unitary matrices \(U(k)\) diagonalizing \(\hat{H}_0 (k)\) and with the property that

\[
U(k) \hat{p}_+ (k) U(k)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv P_+,
\]

\[
U(k) \hat{p}_- (k) U(k)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \equiv P_-,
\]

where \(1\) denotes the \(N/2 \times N/2\)-unit matrix (recall that \(N\) is even). Since \(U(k) \hat{H}_0 (k) U(k)^{-1}\) is diagonal and \(\hat{H}_0 (k)^* \hat{H}_0 (k) = |\hat{H}_0 (k)|^2 1_{N \times N}\) is a multiple of the \(N \times N\)-unit matrix, the eigenvalues of \(U(k) \hat{H}_0 (k) U(k)^{-1}\) are \(\pm |\hat{H}_0 (k)|\), and one obtains

\[
U(k) e^{i\hat{H}_0 (k) t} \hat{p}_+ (k) U(k)^{-1} = e^{-i |\hat{H}_0 (k)| t} P_+,
\]

\[
U(k) e^{-i\hat{H}_0 (k) t} \hat{p}_- (k) U(k)^{-1} = e^{-i |\hat{H}_0 (k)| t} P_-.
\]

Using the Fourier-representations of \(R_b\) and \(L_b\) given in (74) and (75), it furthermore follows that

\[
F v F^{-1} \hat{g}(k) = \int \hat{v}(k, \ell) \hat{g}(\ell) d^s \ell \quad (g \in L^2 (\mathbb{R}^s, \mathbb{C}^N))
\]

with a smooth, bounded, \(N \times N\)-matrix valued function \(\hat{v}(k, \ell)\). Taking together all these observations, we find

\[
Fp_+ dT_{sc}^{(V)} p_- F^{-1} \hat{g}(k) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^s} \tilde{a}(t) e^{-i |\hat{H}_0 (k)| + |\hat{H}_0 (\ell)|} \hat{p}_+ (k) \hat{v}(k, \ell) \hat{p}_- (\ell) \hat{g}(\ell) d^s \ell dt
\]

\[
= \sqrt{2\pi} \int_{\mathbb{R}^s} (\tilde{F} \tilde{a})(|\hat{H}_0 (k)| + |\hat{H}_0 (\ell)|) \hat{p}_+ (k) \hat{v}(k, \ell) \hat{p}_- (\ell) \hat{g}(\ell) d^s \ell.
\]

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The Fourier-transform \((F\hat{a})\) of \(\hat{a}\) is in the Schwartz-class while the modulus of \(\hat{p}_+ (k, \ell) \hat{p}_- (k, \ell)\) is continuous and uniformly bounded; therefore the integral kernel of the last integral is clearly \(L^2\) in the \((k, \ell)\) variables, proving that \(Fp_+dT_{sc}^{(V)}p_-F^{-1}\) and hence \(p_+dT_{sc}^{(V)}p_-\) is Hilbert-Schmidt.

8 Bogoliubov’s formula

In this section, we will derive the expressions for \(d/d\lambda|_{\lambda=0} \beta_{\lambda V}\) that we alluded to in Sec. 5 (cf. eqns. \((41)\) and \((44)\)). We proceed using the geometrical setting from the last part of Section 6 (which has also been investigated already in the commutative case in Section 2). Under consideration is, respectively, one of the following “potential” operators:

\[
\begin{align*}
(0) \quad (Vf)^A(x) &= (V_0f)^A(x) = c(x)f^A(x) \\
(i) \quad (Vf)^A(x) &= (V_if)^A(x) = (c*(q,p)f^A)(x) + (f^A*(q,p)c)(x) \\
(ii) \quad (Vf)^A(x) &= (V_{ii}f)^A(x) = (c*(q,p)f^A*(q,p)c)(x),
\end{align*}
\]

where \(c \in C^\infty_0(\mathbb{R}, \mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s, \mathbb{R})\) is a function of the form

\[c(x) = a(t)b(x),\]

with \(a \in C^\infty_0(\mathbb{R}, \mathbb{R}), b \in \mathcal{S}(\mathbb{R}^s, \mathbb{R}), t = x^0, x = (x^1, \ldots, x^s)\). These operators act on \(f \in C^\infty_0(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N\) and \((q+p) \times (q+p)\) is the dimension of the matrix \(M\), which still shall be of the form

\[
M = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & & \\
0 & M_{(q+p-1)\times (q+p-1)} & \\
\end{bmatrix}_{(q+p)\times (q+p)}
\]

These potentials act non-trivially only inside the time-slice \(\{(x^0, x^1, \ldots, x^s) : \lambda_- < x^0 < \lambda_+\}\) for some real numbers \(\lambda_- < \lambda_+\). We recall the definitions of the regions \(G_+, G_-\),

\[
G_+ = \{(x^0, x^1, \ldots, x^s) : x^0 > \lambda_+ + \frac{1}{2}\}
\]

\[
G_- = \{(x^0, x^1, \ldots, x^s) : x^0 < \lambda_- - \frac{1}{2}\}
\]

forming hyperbolic neighbourhoods of the Cauchy hyperplanes

\[
\Sigma_+ = \{(x^0, x^1, \ldots, x^s) : x^0 = \lambda_+ + 1\},
\]

\[
\Sigma_- = \{(x^0, x^1, \ldots, x^s) : x^0 = \lambda_- - 1\}
\]

respectively.

With these assumptions, we obtain the following result, the proof of which makes use of Proposition 6.1 Lemma 6.2 (commutative case) and Proposition 6.3 Lemma 6.4 (non-commutative case).
Theorem 8.1 It holds that

\[
\frac{d}{d\lambda} \bigg|_{\lambda=0} \beta_{\lambda V}(\Psi_0(f)) = \begin{cases} 
\Psi_0(c R_0 f) \\
\Psi_0(c \ast_{(q,p)} R_0 f + (R_0 f) \ast_{(q,p)} c) \\
\Psi_0(c \ast_{(q,p)} (R_0 f) \ast_{(q,p)} c),
\end{cases}
\]

\(\lambda\) being a real parameter, for the respective choices of the operator \(V = V_{(0)}, V_{(i)}, V_{(ii)}\).

Proof The priority of this proof lies on the non-commutative cases. The commutative case can be carried out along the same lines. We recall the automorphism \(\beta_{\lambda V}\) from (70)

\[
\beta_{\lambda V} : \mathfrak{F}(\mathcal{K}_0, C) \rightarrow \mathfrak{F}(\mathcal{K}_0, C),
\]

\(\beta_{\lambda V} = \alpha_{0,-} \circ \alpha_{\lambda V,-}^{-1} \circ \alpha_{\lambda V,+} \circ \alpha_{0,+}^{-1},\)

together with

\[
\beta_{\lambda V}(B_0([f])_0) = B_0(U_{\lambda V} [f]_0),
\]

where \(U_{\lambda V}\) is the unitary given by

\[
U_{\lambda V} = u_{0,-} \circ u_{\lambda V,-}^{-1} \circ u_{\lambda V,+} \circ u_{0,+}^{-1},
\]

and where, similarly as for the isomorphisms above, we have used the abbreviations

\[
u_{0,\pm} = u_{0,\pm}^{G}, \quad u_{\lambda V,\pm} = u_{\lambda V,\pm}^{G}.
\]

These equations arise from Lemma 6.4. The proof relies now on exactly the same strategy as the one for the very similar Theorem 4.3 of [9]. With that in mind we start by finding more explicit expressions for the inverses in the chain of mappings

\[
U_{\lambda V} : [f]_0 \xrightarrow{u_{0,+}^{-1}} [f^G_+]_0 \xrightarrow{u_{\lambda V,+}^{-1}} [f^G]_{\lambda V} \xrightarrow{u_{\lambda V,-}^{-1}} [f^-_{\lambda V}]_0 \xrightarrow{u_{0,-}} [f^-]_0,
\]

where \(f^G_+\) is any element in \(C_0^\infty((\lambda_+, \infty)) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N\) such that \(R_0(f - f^G_+) = 0\), and \(f^-_\lambda\) is any element in \(C_0^\infty((\lambda_-, \infty)) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N\) such that \(R_{\lambda V}(f^G_+ - f^-_\lambda) = 0\). According to [9], we choose

\[
f^G_+ = -D_0 \chi^\text{ret}_+ R_0 f, \quad f^-_\lambda = -D_{\lambda V} \chi^\text{ret}_- R_{\lambda V} f^G_+,
\]

(96)

where \(\chi^\text{ret}_\pm\) are defined as follows: It has been demanded that the open regions \(G_\pm\) contain Cauchy hyperplanes \(\Sigma_\pm\). Then there are two pairs of further Cauchy surfaces in \(G_\pm\), namely \(\Sigma^\text{adv}_\pm\) in the timelike future of \(\Sigma_\pm\) and \(\Sigma^\text{ret}_\pm\) in the timelike past of \(\Sigma_\pm\). Thus

\[
O_\pm = [J^- (\Sigma^\text{adv}_\pm) \cap J^+ (\Sigma^\text{ret}_\pm)]^\circ
\]

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are open neighbourhoods of \( \Sigma_{\pm} \) and \( \overline{\Sigma}_{\pm} \subseteq G_{\pm} \). Now a partition of unity \( \{ \chi_{\pm}^{\text{adv}}, \chi_{\pm}^{\text{ret}} \} \) is introduced on \( \mathbb{R}^n \) with \( \chi_{\pm}^{\text{adv}} = 0 \) on \( J^- (\Sigma_{\pm}) \) and \( \chi_{\pm}^{\text{ret}} = 0 \) on \( J^+ (\Sigma_{\pm}) \).

The crucial point lies in having to ensure that \( f^{G_{\pm}} \) both lie in the the domain of \( R_{\lambda V} \) in view of the weakened support properties of the fundamental solutions \( R_{\lambda V}^{\pm} \) (cf. Proposition 6.3(3)) compared to the situation in [9].

Obviously it holds

\[
D_{\lambda V} \chi_{\pm}^{\text{adv}} R_{\lambda V} f = -D_{\lambda V} \chi_{\pm}^{\text{ret}} R_{\lambda V} f. \tag{97}
\]

since \( D_{\lambda V} R_{\lambda V} f = 0 \) and \( \chi_{\pm}^{\text{adv}} + \chi_{\pm}^{\text{ret}} = 1 \). The left hand side of \( (97) \) vanishes on \( J^- (\Sigma_{\text{adv}}) \) and the right hand side on \( J^+ (\Sigma_{\text{adv}}) \). Thus we can conclude from \( (97) \) that both \( D_{\lambda V} \chi_{\pm}^{\text{adv}} R_{\lambda V} f \) and \( -D_{\lambda V} \chi_{\pm}^{\text{ret}} R_{\lambda V} f \) lie in \( C_0^\infty (\mathbb{R}) \otimes \mathcal{S} (\mathbb{R}^8) \) and, hence, in the domain of \( R_{\lambda V} \) for any \( \lambda \). This shows immediately that \( f^{G_{\pm}} \) lies in the domain of any \( R_{\lambda V} \), and, iterating the argument, the same holds for \( f^{G^{-}} \).

Putting the definitions of \( (96) \) into the chain of mappings composing \( U_{\lambda V} \) results in

\[
U_{\lambda V} [f]_0 = [D_{\lambda V} \chi_{\pm}^{\text{ret}} R_{\lambda V} D_{\lambda V} \chi_{\pm}^{\text{ret}} R_{0} f]_0.
\]

In the following we would like to abbreviate formally \( \delta = \frac{d}{d\lambda} \big|_{\lambda=0} \). We calculate

\[
\delta U_{\lambda V} [f]_0 = \delta [D_{\lambda V} \chi_{\pm}^{\text{ret}} R_{\lambda V} D_{0} \chi_{\pm}^{\text{ret}} R_{0} f]_0 \\
= -[\delta D_{\lambda V} \chi_{\pm}^{\text{ret}} R_{0} f]_0 + [D_{0} \chi_{\pm}^{\text{ret}} (\delta R_{\lambda V}) D_{0} \chi_{\pm}^{\text{ret}} R_{0} f]_0,
\]

since \( R_{0} D_{0} \chi_{\pm}^{\text{ret}} \varphi = -\varphi \). It is easy to see that \( \delta D_{\lambda V} \) and \( \chi_{\pm}^{\text{ret}} \) have disjoint supports, and thus

\[
\delta U_{\lambda V} [f]_0 = [D_{0} \chi_{\pm}^{\text{ret}} \delta R_{\lambda V} D_{0} \chi_{\pm}^{\text{ret}} R_{0} f]_0.
\]

\( R_{\lambda V} = R_{\lambda V}^{+} - R_{\lambda V}^{-} \) implies

\[
\chi_{\pm}^{\text{ret}} R_{\lambda V} D_{0} \chi_{\pm}^{\text{ret}} \varphi = \chi_{\pm}^{\text{ret}} R_{\lambda V} D_{0} \chi_{\pm}^{\text{ret}} \varphi - \chi_{\pm}^{\text{ret}} R_{\lambda V} D_{0} \chi_{\pm}^{\text{ret}} \varphi,
\]

whereof the first term on the right hand side vanishes, since \( \text{supp} \chi_{\pm}^{\text{ret}} \subseteq J^- (G_{-}) \), \( \text{supp} R_{\lambda V}^{+} D_{0} \chi_{\pm}^{\text{ret}} \subseteq \overline{T}^{-} (G_{+}) \) and \( \overline{T}^{-} (G_{+}) \cap J^- (G_{-}) = \emptyset \). Hence

\[
\delta U_{\lambda V} [f]_0 = [-D_{0} \chi_{\pm}^{\text{ret}} \delta R_{\lambda V} D_{0} \chi_{\pm}^{\text{ret}} R_{0} f]_0.
\]

And this equals

\[
[D_{0} \chi_{\pm}^{\text{ret}} R_{0} \delta D_{\lambda V} R_{0}^{+} D_{0} \chi_{\pm}^{\text{ret}} R_{0} f]_0,
\]

because of the following deduction:

\[
R_{\lambda V}^{+} D_{\lambda V} = 1 \implies (\delta R_{\lambda V}^{-}) D_{0} + R_{0}^{+} (\delta D_{\lambda V}) = 0 \\
\implies \delta R_{\lambda V}^{-} D_{0} R_{0}^{-} = -R_{0}^{+} \delta D_{\lambda V} R_{0}^{-} \\
\implies \delta R_{\lambda V}^{-} = -R_{0} \delta D_{\lambda V} R_{0}^{-}.
\]
Support arguments lead to $\chi_{\lambda V}^{\text{ret}} R_0^+ \delta D_{\lambda V} = 0$ and $\delta D_{\lambda V} R_0^+ D_0 \chi_{\lambda V}^{\text{ret}} \varphi = 0$ and therefore

$$
\delta U_{\lambda V}[f]_0 = [D_0 \chi_{\lambda V}^{\text{ret}} R_0 \delta D_{\lambda V} R_0 D_0 \chi_{\lambda V}^{\text{ret}} R_0 f]_0 = [\delta D_{\lambda V} R_0 f]_0,
$$
since $R_0 D_0 \chi_{\lambda V}^{\text{ret}} = -1$.

Obviously $\delta D_{\lambda V} = V$, which is just one of the three choices of a potential operator. Hence

$$
\delta \beta_{\lambda V}(\Psi_0(f)) = \delta \beta_{\lambda V}(B_0([f]_0)) = \delta B_0(U_{\lambda V}[f]_0) = B_0([VR_0 f]_0) = \Psi_0(V R_0 f).
$$

\(\square\)

9 Conclusion and outlook

We have given a brief sketch of how a simple quantum field theory on Moyal-Minkowski spacetime can be constructed in such a way as providing a model for the construction of quantum field theories on more general Lorentzian non-commutative spacetimes in a setting inspired by spectral geometry. For this quantum field theory — which is the quantized Dirac field — we have seen that a construction of observable field operators labelled by elements of the deformed function algebra of Moyal-Minkowski space can be derived, via Bogoliubov’s formula, from the S-matrix describing scattering of the usual Dirac field on Minkowski spacetime by a non-commutative scalar potential. Again, it is feasible that this procedure can be generalized to more general Lorentzian non-commutative spacetimes.

However, it is certainly inappropriate, at this stage, to judge the generality of the method. Moyal-Minkowski spacetime with commutative time is a very simple and very special non-commutative geometry whose physical relevance is not compelling, to say the least. On the other hand, due to the quite unusual and counter-intuitive properties of non-commutative geometries, one surely needs examples as one’s guidance towards developing physical theories in non-commutative geometries, such as quantum field theory. This clearly shows a dilemma: The examples for non-commutative geometries that are manageable are probably un-physical and may therefore do a very poor job as a guidance when attempting to find some central principles, while without such principles, it is hard to judge which non-commutative geometries are related to physics. Nevertheless, one can probably do better, and try and investigate our method of construction of quantum field theories and their observables for non-commutative spacetimes that have a greater physical appeal, such as developed in [15] and [5], for example. Even if this appears to be a considerably more difficult task, we think it is worth an attempt.
A The Action of the Wick square

In this Appendix we will prove that the Wick-square acts as a derivation on the Dirac field operators in the same way as the derivative of the S-matrix with respect to the scalar scattering potential. The assumptions, wherever not spelled out in detail, are those stated in Sec. 5.

Proposition A.1 Let $e_\mu$ and $\eta_\mu$ ($\mu = 1, \ldots, L$) be elements in $\mathbb{C}^N$, chosen such that $\sum_{\mu=1}^L \bar{e}_\mu A \eta_\mu = \frac{i}{4} \gamma_0^{AB}$. Define for $q_1, q_2 \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$,

$$\psi^+ \psi(q_1 \otimes q_2) = \sum_{\mu=1}^L \psi(q_1 e_\mu)^* \psi(q_2 \eta_\mu)$$

and whence,

$$: \psi^+ \psi : (c) = \lim_{\epsilon \to 0} \psi^+ \psi(F_\epsilon) - (\Omega^{\text{vac}}, \psi^+ \psi(F_\epsilon) \Omega^{\text{vac}}) 1$$

on $\mathcal{W}$ with $F_\epsilon(x,y) = q_1(x)q_2(y)j_\epsilon(x-y)$ and $c(x) = q_1(x)q_2(x)$ ($x, y \in \mathbb{R}^N$), where $\lim_{\epsilon \to 0} \int q(x)j_\epsilon(x-y) d^nx = q(y)$ ($q \in C_0^\infty(\mathbb{R}^n)$).

Then

$$[ : \psi^+ \psi : (c), \psi(f) ] = -i \psi(c R_0 f) \quad (f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N).$$

Moreover, $: \psi^+ \psi : (c)$ is independent of the choice of families $e_\mu, \eta_\mu \in \mathbb{C}^N$ fulfilling $\sum_{\mu=1}^L \bar{e}_\mu A \eta_\mu = \frac{i}{4} \gamma_0^{AB}$.

Proof Using the relations of the generators of the CAR algebra, it follows that

$$\sum_{\mu} (\chi_1, [\psi(q_1 e_\mu)^* \psi(q_2 \eta_\mu), \psi(f)] \chi_2) \tag{98}$$

$$= 2 \sum_{\mu} \{ (\chi_1, \psi(C q_1 e_\mu) \chi_2)(C q_2 \eta_\mu, f)_0 - (\chi_1, \psi(q_2 \eta_\mu) \chi_2)(q_1 e_\mu, f)_0 \}$$

holds for all vectors $\chi_1, \chi_2$ in the dense domain $\mathcal{W} \subset \mathcal{H}^{\text{vac}}$ and for all $f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N$. We recall here the definition

$$(f, h)_0 = i \int \gamma_0^{AB} T^B(x)(R_0 h)^A(x) d^nx$$

(for $V = 0$, see Prop. 6.3 or respectively 2.1 and eqn. 6). One can show either directly, or by falling back onto general arguments [19], that for each pair of vectors $\chi_1, \chi_2 \in \mathcal{W}$ there are smooth functions $\xi_A$ on $\mathbb{R}^n$ ($A = 1, \ldots, N$) such that

$$(\chi_1, \psi(f) \chi_2) = \int \xi_A(y) f^A(y) d^ny \quad (f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^s) \otimes \mathbb{C}^N). \tag{99}$$
With this notation, the right hand side of (98) assumes the form
\[ 2i \sum_{\mu} \int \int q_1(y)q_2(x)\xi_A(y)(Ce_\mu)^A(y)\gamma_{0BD}C\eta_\mu^B (R_0 f)^D(x) \ d^n x \ d^n y \]  
\[ - 2i \sum_{\mu} \int \int q_1(x)q_2(y)\xi_A(y)\eta_\mu^A\gamma_{0D'}B'f_{\mu}^{B'} (R_0 f)'D'(x) \ d^n x \ d^n y \]

Using the defining property \( \sum_\mu e_\mu^A \eta_\mu^B = \frac{1}{4} \gamma_{0}^{AB} \) and \( \gamma_0^2 = 1 \), the second integral of (100) simplifies to
\[ -\frac{i}{2} \int \int \bar{q}_1(x)q_2(y)\xi_A(y)(R_0 f)^A(x) \ d^n x \ d^n y . \]  
(101)

In a similar manner, using also the relations \( C^2 = 1 \) and (7), one can check that the first integral in (100) simplifies to
\[ -\frac{i}{2} \int \int q_1(y)q_2(x)\xi_A(y)(R_0 f)^A(x) \ d^n x \ d^n y , \]  
(102)
so that (100) becomes equal to
\[ -\frac{i}{2} \int \int (q_1(y)q_2(x) + q_2(y)q_1(x))\xi_A(y)(R_0 f)^A(x) \ d^n x \ d^n y , \]  
(103)
observing that \( q_1 \) and \( q_2 \) are real-valued. Replacing here \( q_1 \otimes q_2 \) by \( F_\epsilon \) and taking the limit \( \epsilon \to 0 \) turns the last expression into
\[ -i \int \xi_A(x)c(x)(R_0 f)^A(x) \ d^n x . \]  
(104)

On using (99), we have therefore proved the first claim of the Proposition.

To see the independence of the definition of \( \psi^+ \psi : (c) \) of the mentioned choices, we note that the commutator formula for \( \psi^+ \psi : (c) \), together with the fact that the *-algebra generated by \( 1 \) and all the \( \psi(f) \) acts irreducibly, fixes \( \psi^+ \psi : (c) \) up to addition of a multiple of the unit operator \( 1 \). On the other hand, by construction we have \( (\Omega_{\text{vac}}, \psi^+ \psi : (c)\Omega_{\text{vac}}) = 0 \), so that the scalar multiple in question must vanish in the vacuum state, which implies that it is zero. This demonstrates the claimed independence of the definition of \( \psi^+ \psi : (c) \) of the possible choices for \( e_\mu \) and \( \eta_\mu \).

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