PRODUCTS OF TOEPLITZ OPERATORS 
ON A VECTOR VALUED BERGMAN SPACE

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Abstract. We give a necessary and a sufficient condition for the boundedness of the Toeplitz product $T_F T_G^*$ on the vector valued Bergman space $L^2_a(C^n)$, where $F$ and $G$ are matrix symbols with scalar valued Bergman space entries. The results generalize those in the scalar valued Bergman space case [4]. We also characterize boundedness and invertibility of Toeplitz products $T_F T_G^*$ in terms of the Berezin transform, generalizing results found by Zheng and Stroethoff for the scalar valued Bergman space [8].

1. Introduction

1.1. Notation. For a measurable function $f: D \rightarrow C^n$ with $\left( \int_D \|f(z)\|^p_{C^n} dA(z) \right)^{\frac{1}{p}} < \infty$, we say that $f \in L^p(D, C^n)$. The vector-valued Bergman space $L^p_a(D, C^n)$ is the intersection of $L^p(D, C^n)$, with the analytic $C^n$-valued functions on $D$ with the usual identification of functions which only differ on sets of measure 0. The norm is given by $\|f\|_{L^p_a(C^n)} = \left( \int_D \|f(z)\|^p_{C^n} dA(z) \right)^{\frac{1}{p}}$, where $dA$ is normalized Lebesgue measure on the unit disk $D$. In the case $p = 2$ this space becomes a Hilbert space with the inner product given by $\langle f, g \rangle = \int_D \langle f(z), g(z) \rangle_{C^n} dA(z)$. $L^p$ and $L^2$ are Banach spaces for $1 \leq p < \infty$. For details see for example [14].

On the scalar valued Bergman space $L^2_a$, the Toeplitz operator with symbol $f \in L^2$ is the densely defined operator $T_f v = P(fv)$, where $P$ is the orthogonal projection from $L^2$ into $L^2_a$ and $v$ is a polynomial. The Toeplitz operator is a multiplication operator composed with an orthogonal projection. The Bergman projection is explicitly given by the following integral:

$$ Pf(w) = \langle f, K_w \rangle = \int_D \frac{f(z)}{(1 - zw)^2} dA(w), $$

where $K_w(z) = \frac{1}{(1 - zw)^2}$ is the reproducing kernel of the Bergman space $L^2_a(D)$. So using this explicit form we can define a Toeplitz operator on a dense subset of $L^2_a$, the polynomials, with symbol in $L^2$ rather than $L^\infty$. We can also see that with a symbol $f \in L^2$ and $v \in L^2_a$, $T_f v(w)$ is well defined point-wise for each $w \in D$.

In [1] Sarason conjectured that a product of Toeplitz operators (defined densely in an appropriate way for analytic functions $f$ and $g$) $T_f T_g^*$ on the Hardy Space $H^2$ is bounded if and only if $|\hat{f}(w)|^2|g|^2(w)$ was uniformly bounded on the disc, $\hat{f}(w)$ being the Poisson integral of $f$. This turned out to be false [19].

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Another conjecture by Sarason dealt with in [3, 4, 7, 8, 9] and [11] was as to when the densely defined operator $T_f T_g^*$ is bounded on $L^2_a$ for $f, g \in L^2_a$. The question was originally posed by Sarason in [11] and a conjecture in section 8 of [4] more explicitly resembles Sarason’s Hardy space case conjecture. This time it is conjectured that the Toeplitz product $T_f T_g^*$ is bounded for analytic $f$ and $g$ on the Bergman space $L^2_a$ if and only if $|f|^2(w)|g|^2(w)$ is uniformly bounded, where $\tilde{f}$ is the Berezin transform of $f$.

The question is investigated in various different cases, such as the weighted Bergman space with standard weights and the Bergman space on the unit ball and polydisk. These papers prove results that approximate to the Bergman space version of Sarason’s conjecture as stated in section 8 of [4].

The purpose of this paper is to investigate products of Toeplitz operators on a Bergman space of vector-valued functions. In the case of the vector valued Bergman space $L^2_a(\mathbb{C}^n)$, we define the Toeplitz operator to be the densely defined composition of multiplication with a matrix valued function and the orthogonal projection from $L^2(\mathbb{C}^n)$ into $L^2_a(\mathbb{C}^n)$. So in this case the symbol $F$ will be a matrix of $L^2$ functions and $T_F v = P(Fv)$, where $v$ is a bounded analytic $\mathbb{C}^n$ valued function. If

$$F = \begin{pmatrix} f_{11} & f_{12} & \cdots \\ f_{21} & \ddots \\ \vdots \\ \end{pmatrix}$$

and $v = (v_1, v_2, \ldots, v_n)$, where $f_{ij} \in L^2$ and $v_i \in H^\infty$, then

$$T_F v = P(Fv) = P \left( \sum_{i=1}^{n} f_{1i} v_i, \sum_{i=1}^{n} f_{2i} v_i, \ldots, \sum_{i=1}^{n} f_{ni} v_i \right) = \begin{pmatrix} T_{f_{11}} & T_{f_{12}} & \cdots \\ T_{f_{21}} & \ddots \\ \vdots \\ \end{pmatrix} v,$$

where each $T_{f_{ij}}$ is a densely defined Toeplitz operator on the scalar Bergman space $L^2_a$. When looking at products of these Toeplitz operators analogous to the treatment in [4] we have products of the form $T_F T_G^*$, where $F$ and $G$ are square matrices of scalar valued Bergman space $L^2_a$ functions.

1.2. **Main Theorems.** The first two main theorems follow, one giving a sufficient condition for the Toeplitz product $T_F T_G^*$ to be bounded and the other a necessary condition. Both are conditions involving the Berezin transform;

**Definition 1.1.** The Berezin transform of a matrix $A$ with $L^2$ entries is the matrix-valued function $B(A)$, where $B(A)(w) = \int (A \circ \phi_w)(z) dA(z)$, $w \in \mathbb{D}$, composition here being composition with each matrix entry. Here, $\phi_w$ is the M{"o}bius transform $z \mapsto \frac{w-z}{1-w\zbar}$. We should also note here that $B(A)(w) = \int A(z) \frac{1-|w|^2}{1-\zbar w} dA(z)$ by a change of variables. Defining the normalized reproducing kernel $k_w(z)$ to be $K_w(z)/\|K_w\|$, we obtain $|k_w(z)|^2 = \frac{(1-|w|^2)^2}{1-\zbar w}$.

Here is our first main result:

**Theorem 1.2.** If for some $\epsilon > 0$ the trace of the matrix $B((F^* F)^{\frac{2+i\epsilon}{2}})(w)B((G^* G)^{\frac{2+i\epsilon}{2}})(w)$ is uniformly bounded for all $w \in \mathbb{D}$, then the Toeplitz product $T_F T_G^*$ is bounded $L^2_a(\mathbb{C}^n) \to L^2_a(\mathbb{C}^n)$. 
We also have the following condition: If there exists $\epsilon > 0$ such that
\[
\left( \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} (\operatorname{Tr}(G(z)F(x)^* F(x)G(z^*)) \frac{2\pi}{2\pi} |k_w(z)|^2 dA(z) \right\} |k_w(x)|^2 dA(x) \right)^{\frac{3}{2}}
\]
is uniformly bounded, then the Toeplitz product $T_F T_{G^*} : L^2_{\mathbb{D}}(\mathbb{C}^n) \to L^2_{\mathbb{D}}(\mathbb{C}^n)$ is bounded.

Here is the necessary condition:

**Theorem 1.3.** If the product of Toeplitz operators $T_F T_{G^*}$ is bounded, then the trace of the matrix $B(F^* F)(w)B(G^* G)(w)$ is uniformly bounded for $w \in \mathbb{D}$.

The next theorem is the other main result presented here, involving a characterization of bounded and invertible Toeplitz products.

**Theorem 1.4.** The Toeplitz product $T_F T_{G^*}$ is bounded and invertible if and only if the trace of the matrix $B(F^* F)(w)B(G^* G)(w)$ is uniformly bounded and there exists $\eta > 0$ with $(FG^* GF^*)(z) > \eta I$ for all $z \in \mathbb{D}$. This last inequality is a matrix inequality.

2. Bounded Toeplitz Products

2.1. A Sufficient Condition (Proof of Theorem 1.2). The technique in [4] for showing a sufficient condition on the boundedness of a Toeplitz product involves an inner product formula that easily generalizes to the vector valued case. So for $g, f \in L^2_{\mathbb{D}}(\mathbb{C}^n)$

\[
\langle f, g \rangle_{L^2_{\mathbb{D}}(\mathbb{C}^n)} = \int_{\mathbb{D}} \langle f(z), g(z) \rangle_{\mathbb{C}^n} dA(z)
\]

\[
= 3 \int_{\mathbb{D}} (1 - |z|^2)^2 \langle f(z), g(z) \rangle_{\mathbb{C}^n} dA(z) + \frac{1}{2} \int_{\mathbb{D}} (1 - |z|^2)^2 \langle f'(z), g'(z) \rangle_{\mathbb{C}^n} dA(z) + \frac{1}{3} \int_{\mathbb{D}} (1 - |z|^2)^3 \langle f''(z), g''(z) \rangle_{\mathbb{C}^n} dA(z).
\]

So to estimate the norm of $T_G T_{F^*}$, we will look at the inner product $\langle T_G T_{F^*} u, v \rangle_{L^2_{\mathbb{D}}(\mathbb{C}^n)}$ with $u, v \in L^2_{\mathbb{D}}(\mathbb{C}^n)$ in the form just given.

Let us start by estimating the term $\langle T_G (u)(w), T_{G^*} (v)(w) \rangle_{\mathbb{C}^n}$.

**Definition 2.1.** For $f, g \in L^2(\mathbb{D})$, define the rank 1 operator $f \otimes g : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ by
\[
(f \otimes g)h = \langle h, g \rangle f
\]
for $h \in L^2(\mathbb{D})$.

Also for $F, G \in M_{n \times n}(L^2(\mathbb{D}))$, define the operator $F \otimes G : L^2(\mathbb{D}, \mathbb{C}^n) \to L^2(\mathbb{D}, \mathbb{C}^n)$ by
\[
(F \otimes G)h = \begin{pmatrix}
\sum_if_{11} \otimes g_{1i} & \cdots & \sum_if_{1i} \otimes g_{ni} \\
\sum_if_{2i} \otimes g_{1i} & \cdots & \sum_if_{2i} \otimes g_{ni} \\
\vdots & \ddots & \vdots \\
\sum_if_{ni} \otimes g_{1i} & \cdots & \sum_if_{ni} \otimes g_{ni}
\end{pmatrix} h
\]
for $h \in L^2(\mathbb{D}, \mathbb{C}^n)$. 
Theorem 2.2.
\[ \langle T_{F^*}(u)(w), T_{G^*}(v)(w) \rangle_{C_n} = \frac{1}{(1 - |w|^2)^2} \int_{D} \langle (Gk_w \otimes Fk_w)u(z), v(z) \rangle_{C_n} dA(z), \]
where \( k_w \) is the normalized reproducing kernel.

Proof.
\[
\langle T_{F^*}(u)(w), T_{G^*}(v)(w) \rangle_{C_n}
\]
\[= \left\langle \int_{D} F^*(z)u(z)K_w(z)dA(z), \int_{D} G^*(\zeta)v(\zeta)K_w(\zeta)dA(\zeta) \right\rangle_{C_n}
\]
\[= \int_{D} \int_{D} \langle G(\zeta)K_w(\zeta)F(z)k_w(z) \rangle^* u(z), v(\zeta) \rangle_{C_n} dA(z)dA(\zeta)
\]
\[= \frac{1}{(1 - |w|^2)^2} \int_{D} \int_{D} \langle G(\zeta)k_w(\zeta)(F(z)k_w(z))^* u(z), v(\zeta) \rangle_{C_n} dA(z)dA(\zeta)
\]
\[= \frac{1}{(1 - |w|^2)^2} \int_{D} \langle (Gk_w \otimes Fk_w)u(\zeta), v(\zeta) \rangle_{C_n} dA(\zeta). \]

\[\square\]

Lemma 2.3.
\[||/(F \otimes G)(G \otimes F)\|_{op} \sim \text{trace}\{(F \otimes G)(G \otimes F)\} = \sum_{r=1}^{n} \sum_{l=1}^{n} \sum_{r=1}^{n} \sum_{l=1}^{n} \langle f_{qr}, f_{ql} \rangle_{L^2} \langle g_{ml}, g_{mr} \rangle_{L^2}. \]

Proof. As \((F \otimes G)(G \otimes F)\) is of finite rank the trace of \((F \otimes G)(G \otimes F)\) will be an equivalent norm. We can express \(F \otimes G\) as a matrix of operators on the scalar Bergman space with the entries \([\sum_{l=1}^{n} f_{dl} \otimes g_{jl}]_{i,j}\). We can then express \((F \otimes G)(G \otimes F)\) in a similar manner;
\[
\sum_{m=1}^{n} \left( \sum_{l=1}^{n} f_{dl} \otimes g_{ml} \right) \left( \sum_{l=1}^{n} g_{ml} \otimes f_{jl} \right)_{i,j}
\]
\[= \sum_{m=1}^{n} \left[ \sum_{r=1}^{n} \sum_{l=1}^{n} \langle f_{jl} \rangle_{L^2} \langle g_{ml}, g_{mr} \rangle_{L^2} \langle f_{ir} \rangle_{i,j} \right]. \]

Noting that we have as an orthonormal basis \(e_{l,m} = (0, \ldots, 0, z^l \sqrt{1 + p}, 0, \ldots)\) i.e. a vector with each coordinate 0 apart from the \(m\)th entry which is the \(l\)th orthonormal basis element of the scalar valued Bergman space. So the trace of the operator \((F \otimes G)(G \otimes F)\) will be
\[
\sum_{p,q} \langle (F \otimes G)(G \otimes F)e_{p,q}, e_{p,q} \rangle
\]
\[= \sum_{q=1}^{n} \sum_{p=1}^{n} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \left( \sum_{r=1}^{n} \sum_{l=1}^{n} \langle z^p \sqrt{1 + p}, f_{ql} \rangle_{L^2} \langle g_{ml}, g_{mr} \rangle_{L^2} \int_{D} f_{qr}(z)z^s \sqrt{1 + p}dA(z) \right). \]

We can write each \(f_{ij}\) as a power series \(\sum_{s=1}^{\infty} a_{s,ij}z^s \sqrt{1 + s}\) and thus this trace becomes
\[ \sum_{q=1}^{n} \sum_{p=1}^{\infty} \sum_{m=1}^{n} \left( \sum_{r=1}^{n} \sum_{l=1}^{n} a_{p,q,l} \langle g_{mt}, g_{mr} \rangle_{L^2} a_{p,q,r} \right) \]

and thus by Parseval's identity the expression for the trace becomes:

\[ \sum_{q=1}^{n} \sum_{m=1}^{n} \left( \sum_{r=1}^{n} \sum_{l=1}^{n} \langle f_{qr}, f_{ql} \rangle_{L^2} \langle g_{mt}, g_{mr} \rangle_{L^2} \right) \]

\[ \Box \]

**Theorem 2.4.** \(||G_k \otimes F_k||_{op} \approx (\text{tr}(B(G^*G)(w)B(F^*F)(w)))^{\frac{1}{2}}\).**

*Proof.* Using Lemma [23] we can see that as \(||G_k \otimes F_k||_{op}\) is equivalent to the square root of the trace of the operator \((G_k \otimes F_k)(F_k \otimes G_k)\) and this is equal to

\[ \sum_{q=1}^{n} \sum_{m=1}^{n} \left( \sum_{r=1}^{n} \sum_{l=1}^{n} \langle f_{qr}, f_{ql} \rangle_{L^2} \langle g_{mt}, g_{mr} \rangle_{L^2}^2 \right) = \text{tr}(B(G^*G)(w)B(F^*F)(w)) \]

we immediately have our result.

\[ \Box \]

**Definition 2.5.** The operator \(P_0\) defined on \(L_p(\mathbb{D})\) is the operator that sends \(f \in L^2\) to the function given by \((P_0 f)(w) = \int_{\mathbb{D}} \frac{f(z)}{|1 - wz|^2} dA(z)\).

Elements from the following two theorems are borrowed from Theorem 3.2 in [4].

**Lemma 2.6.** If we have a scalar valued integrable function \(h\) and a scalar valued Bergman space function \(v\) then for each \(w \in \mathbb{D}\),

\[ \int_{\mathbb{D}} \frac{|xh(x)| |v(x)|}{|1 - wz|^3} dA(x) \leq 2 \left\{ \int_{\mathbb{D}} |h(x)|^{2+\epsilon} \frac{|k_w(x)|^2}{|1 - wz|^2} dA(x) \right\}^{\frac{1}{2+\epsilon}} \left\{ \int_{\mathbb{D}} |v(x)|^{2+\epsilon} dA(x) \right\}^{\frac{1}{2+\epsilon}}. \]

Here, for \(\epsilon > 0, \delta = \frac{2+\epsilon}{1+\epsilon}\).

*Proof.* By Hölder’s inequality,

\[ \int_{\mathbb{D}} \frac{|xh(x)| |v(x)|}{|1 - wz|^3} dA(x) \leq \int_{\mathbb{D}} \frac{|h(x)||1 - wz||v(x)|}{|1 - wz|^4} dA(x) \]

\[ \leq \left\{ \int_{\mathbb{D}} \frac{|h(x)|^{2+\epsilon}}{|1 - wz|^2} dA(x) \right\}^{\frac{1}{2+\epsilon}} \left\{ \int_{\mathbb{D}} \frac{|v(x)|^{2+\epsilon}}{|1 - wz|^2} dA(x) \right\}^{\frac{1}{2+\epsilon}} \]

\[ = \left\{ \int_{\mathbb{D}} \frac{|h(x)|^{2+\epsilon} |k_w(x)|^2}{1 - wz} dA(x) \right\}^{\frac{1}{2+\epsilon}} \left\{ \int_{\mathbb{D}} \frac{|v(x)|^{2+\epsilon} |k_w(x)|^2}{1 - wz} dA(x) \right\}^{\frac{1}{2+\epsilon}}, \]

and our result follows from the fact that

\[ \frac{1 - |w|^2}{|1 - wz|^2} \leq \frac{1 - |w|(1 + |w|)}{|1 - wz|^{2+\epsilon}} \leq \frac{1 - |w|(1 + |w|)}{|1 - wz|^{2+\epsilon}} \leq \frac{(1 - |w|(1 + |w|)}{|1 - wz|}\]

\[ \leq 1 + |w| < 2. \]

\[ \Box \]

Let us now take a look at \(\langle T_{F^*}(u)'(w), T_{G^*}(v)'(w) \rangle_{\mathbb{C}^n}\).
\textbf{Theorem 2.7.} Let \( w \in \mathbb{D} \). Then
\[
| \langle T_{F^*}(u)'(w), T_{G^*}(v)'(w) \rangle_{\mathbb{C}^n} | \leq C \left( \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} (tr(G(z)F(x)^*F(x)G(z)^*))^{\frac{2+n}{2}} \left| \frac{k_w(x)}{1-|w|^2} \right| dA(z) \right\} \right)^{\frac{1}{2+n}}
\times \{(P_0||u||_{\mathbb{C}^n})^\epsilon \} \frac{1}{1-|w|^2} \{(P_0||v||_{\mathbb{C}^n})^\epsilon \} \frac{1}{1-|w|^2}
\]
where \( C \) is a constant, \( \epsilon > 0 \) and \( \frac{1}{\delta} = 1 - \frac{1}{2+\epsilon} \).

\textbf{Proof.} First note that for a function \( u \in L^2_{\alpha}(\mathbb{D}, \mathbb{C}^n) \)
\[
\langle u, K'_w \rangle = u'(w)
\]
so that
\[
| \langle T_{F^*}(u)'(w), T_{G^*}(v)'(w) \rangle_{\mathbb{C}^n} | = \left| \int \int \left\langle F^*(z)u(z)\overline{K'_w(z)}, G^*(x)v(x)\overline{K'_w(x)} \right\rangle_{\mathbb{C}^n} dA(z)dA(x) \right|
\]
\[
\leq \left| \int \int \|G(z)F^*(x)||_{\mathbb{C}^n}||u(z)||||v(x)|| \left| \frac{K_w(z)}{1-\overline{w}z} \right| \left| \frac{K_w(x)}{1-\overline{w}x} \right| dA(z)dA(x) \right|
\]
Then using Lemma 2.6 with \( h(z) = ||G(z)F^*(x)||_{\mathbb{C}^n}||v(x)|| \left| \frac{K_w(x)}{1-\overline{w}x} \right| \), we arrive at the following inequality,
\[
| \langle T_{F^*}(u)'(w), T_{G^*}(v)'(w) \rangle_{\mathbb{C}^n} | \leq 2 \left| \int \left\{ \int \left( ||G(z)F^*(x)||_{\mathbb{C}^n}||v(x)|| \left| \frac{K_w(x)}{1-\overline{w}x} \right| \right)^{2+\epsilon} \left| \frac{k_w(z)}{1-|w|^2} \right| dA(z) \right\}^{\frac{1}{2+\epsilon}} \{(P_0||u||_{\mathbb{C}^n})^\epsilon \} dA(x) \right|
\]
where \( \epsilon > 0 \) and \( \frac{1}{\delta} = 1 - \frac{1}{2+\epsilon} \).

Again using Lemma 2.6 but this time with \( h(x) = \left\{ \int \left( ||G(z)F^*(x)||_{\mathbb{C}^n}^{2+\epsilon} \left| \frac{k_w(z)}{1-|w|^2} \right| dA(z) \right\}^{\frac{1}{2+\epsilon}} \right\} \),
to see that
\[
| \langle T_{F^*}(u)'(w), T_{G^*}(v)'(w) \rangle_{\mathbb{C}^n} | \leq 4 \left| \int \left\{ \int \left( ||G(z)F^*(x)||_{\mathbb{C}^n}^{2+\epsilon} \left| \frac{k_w(z)}{1-|w|^2} \right| dA(z) \right\} \left| \frac{k_w(x)}{1-|w|^2} \right| dA(x) \right\}^{\frac{1}{2+\epsilon}} \{(P_0||v||_{\mathbb{C}^n})^\epsilon \} \{(P_0||u||_{\mathbb{C}^n})^\epsilon \} dA(x) \right|
\]
(This is what we want but we can go a step further and get something that looks even more similar to the analogous result in the scalar case.) Letting the 4 be absorbed into the constant $C$ and using the inequality on matrix norms from [13], theorem IX.2.10 on page 258, we see that

$$\leq \int \left\{ (C \| (G(z) F(z) G(x)) \|_{C^\alpha})^2 \frac{|k_w(z)|^2}{|u|^2} \right\} \left\{ (P_0 \| u \|_{C^\alpha}(w)) \right\}^{\frac{1}{2}} \left\{ (P_0 \| v \|_{C^\alpha}(w)) \right\}^{\frac{1}{2}}$$

which $B$ is the Berezin transform and $C$ is a constant that is possibly different from line to line.

Now let us use the estimates from theorems 2.2 and 2.7 in the inner product formula. Taking our inner product formula

$$\langle T_{F^*}(u), T_{G^*}(v) \rangle_{L^2(C^n)} = \int \langle T_{F^*}(u), T_{G^*}(v) \rangle_{C^n} \ dA(z)$$

$$= \int (1-|z|^2)^2 \langle T_{F^*}(u), T_{G^*}(v) \rangle_{C^n} \ dA(z) + \frac{1}{2} \int (1-|z|^2)^3 \langle T_{F^*}(u), T_{G^*}(v) \rangle_{C^n} \ dA(z) + \frac{1}{3} \int (1-|z|^2)^4 \langle T_{F^*}(u), T_{G^*}(v) \rangle_{C^n} \ dA(z),$$

let’s take the term $\frac{1}{2} \int (1-|z|^2)^2 \langle T_{F^*}(u), T_{G^*}(v) \rangle_{C^n} \ dA(z)$ and estimate its modulus;

$$|\frac{1}{2} \int (1-|z|^2)^2 \langle T_{F^*}(u), T_{G^*}(v) \rangle_{C^n} \ dA(z)|$$

$$\leq \frac{1}{2} \int |C \left( (F^* F) \frac{d}{dz} (w) \right) B \left( (G^* G) \frac{d}{dz} (w) \right)| \frac{1}{2} \frac{1}{(1-|w|^2)^2} \left\{ (P_0 \| u \|_{C^\alpha}(w)) \right\}^{\frac{1}{2}} \left\{ (P_0 \| v \|_{C^\alpha}(w)) \right\}^{\frac{1}{2}} \ dA(w)$$
2. The previous estimate.

By Cauchy-Schwarz, this expression will be less than or equal to

\[
\frac{1}{2} \sup_{w \in \mathbb{D}} \left( Ctr \left( B \left( (F^* F)^{\frac{2s}{3+s}} \right) (w)B \left( (G^* G)^{\frac{2s}{3+s}} \right) (w) \right) \right)^{\frac{1+s}{s}} \times \left\{ \int_{\mathbb{D}} \left| P_0 \right|^2 \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{D}} \left| v \right|^2 \right\}^{\frac{1}{2}} \text{d}A(w)
\]

now as the operator $P_0$ is $L^p$ bounded for $p > 1$, \([2]\), this expression will be less than or equal to

\[
\frac{1}{2} \sup_{w \in \mathbb{D}} \left( Ctr \left( B \left( (F^* F)^{\frac{2s}{3+s}} \right) (w)B \left( (G^* G)^{\frac{2s}{3+s}} \right) (w) \right) \right)^{\frac{1+s}{s}} \times \left\{ \int_{\mathbb{D}} \left| P_0 \right|^2 \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{D}} \left| v \right|^2 \right\}^{\frac{1}{2}} \text{d}A(w)
\]

Estimating the term

\[
\frac{1}{3} \int_{\mathbb{D}} (1 - |z|^2)^3 |\langle T'_{F^*}(u), T_{G^*}(v) \rangle| \text{d}A(z)
\]

from the inner product formula is similar.

Finally let us estimate $3 \int_{\mathbb{D}} (1 - |z|^2)^2 |\langle T'_{F^*}(u), T_{G^*}(v) \rangle| \text{d}A(z)$, We can see from \([2, 2]\) that

\[
|\langle T'_{F^*}(u), T_{G^*}(v) \rangle| \leq \left( \int_{\mathbb{D}} \langle Gk_u \otimes Fk_v, u \rangle \text{d}A(z) \right) \left( \int_{\mathbb{D}} \langle Gk_v \otimes Fk_u, v \rangle \text{d}A(z) \right)
\]

by Hölder.

This is then less than or equal to

\[
\sup_{w \in \mathbb{D}} \left( \text{tr} (B(G^* G)(w)B(F^* F)(w)) \right)^{\frac{1+s}{s}} \left\{ \int_{\mathbb{D}} \left| P_0 \right|^2 \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{D}} \left| v \right|^2 \right\}^{\frac{1}{2}} \text{d}A(w)
\]
Noting that the operator \( F \) norm take the square root of the trace of the operator \( \phi \). The expression is then equal to

\[
\left\{ \int_D \int_D \left( (G(x)G(x))^\frac{2n}{n+1} (F(z)F(z))^\frac{2n}{n+1} \right) |k_w(z)|^2 |k_w(x)|^2 dA(x)dA(z) \right\} \frac{2n}{n+1}
\]

by Theorem IX.2.10 on page 258 of [13] and similar steps as before. This final expression is then equal to

\[
\left\{ \text{tr} \left( \int_D \int_D ((G(x)G(x))^\frac{2n}{n+1} (F(z)F(z))^\frac{2n}{n+1} |k_w(z)|^2 |k_w(x)|^2 dA(x)dA(z) \right) \right\} \frac{2n}{n+1}
\]

Also by Lemma 2.3 we can see that this will be equal to

\[
\text{tr} (B \left( \left( (G(x)G(x))^\frac{2n}{n+1} (F(z)F(z))^\frac{2n}{n+1} \right) (w) \right) \left( (F(z)F(z))^\frac{2n}{n+1} (w) \right) \frac{2n}{n+1}.
\]

Note that here we can use the same reasoning to see that if

\[
\left( \int_D \int_D \left( \text{tr} (G(z)F(x)G(x)F(x))^\frac{2n}{n+1} |k_w(z)|^2 dA(z) \right) |k_w(x)|^2 dA(x) \frac{2n}{n+1} \right)
\]

is uniformly bounded for some \( \epsilon > 0 \) then our Toeplitz product \( T_F T_G^* \) will be invertible. Note that these last inequalities show that the sufficient condition is stronger than the necessary condition.

2.2. A Necessary Condition.

**Proof of Theorem 1.3.** In [7] (see also [4] for a different approach) Park shows that for functions \( f \) and \( g \) in the scalar Bergman space \( L_2^a \) the operator \( f \otimes g \) defined by \( f \otimes gh = \langle h, g \rangle f \) with \( h \in L_2^a \) is equal to the following,

\[
T_F T_G - 2T_z T_F T_G T_z + T_z^2 T_F T_G T_z^2.
\]

Using this result in the vector valued case, we can see that

\[
F \otimes G = \left( \sum f_{11} \otimes g_{11} \quad \sum f_{11} \otimes g_{21} \quad \ldots \right)
\]

is equal to

\[
T_F T_G^* - 2T_z T_F T_G T_z + T_z^2 T_F T_G T_z^2.
\]

Let us estimate the norm of the operator \((F \circ \phi_w) \otimes (G \circ \phi_w)\), where \(F \circ \phi_w\) is the matrix-valued function

\[
\begin{pmatrix}
    f_{11} \circ \phi_w & f_{12} \circ \phi_w & \ldots \\
    f_{21} \circ \phi_w & \ldots \\
    \vdots & \vdots & \ddots
\end{pmatrix}
\]

Noting that the operator \((F \circ \phi_w) \otimes (G \circ \phi_w)\) is of finite rank we can as an equivalent norm take the square root of the trace of the operator \((F \circ \phi_w \otimes G \circ \phi_w) (G \circ \phi_w \otimes F \circ \phi_w)\) by Lemma 2.3.

Also by Lemma 2.3 we can see that this will be equal to

\[
\sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{l=1}^n \langle f_{qr} \circ \phi_w, f_{ql} \circ \phi_w \rangle_{L^2} \langle g_{ml} \circ \phi_w, g_{mr} \circ \phi_w \rangle_{L^2}
\]

\[
= \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{l=1}^n B(f_{qr} f_{ql})(w) B(g_{ml} g_{mr})(w),
\]

in the scalar Bergman space.
which is equal to the trace of the matrix $B(F^*F)(w)B(G^*G)(w)$.

Let $U_w$ be the unitary operator on our vector valued $L^2$ space given by $U_w = (f \circ \phi_w)k_w$. It is well known that $T_{F \circ \phi_w} U_w = U_w T_F$. So $T_{F \circ \phi_w} = U_w T_F U_w^*$, and thus

\[
\{ tr(B(F^*F)(w)B(G^*G)(w)) \}^{\frac{1}{2}} = C || F \circ \phi_w \otimes G \circ \phi_w ||_{op}
\]

\[
= || T_{F \circ \phi_w} T_{G \circ \phi_w} - 2T_z T_{F \circ \phi_w} T_{G \circ \phi_w} T_z + T_z^2 ||_{op} \]

\[
= || (U_w T_F U_w^* - 2T_z U_w T_F U_w^* T_z + T_z^2 U_w T_F U_w^*) ||_{op} \]

\[
= || (U_w T_F T_G U_w^* - 2T_z U_w T_F T_G U_w^* T_z + T_z^2 U_w T_F T_G U_w^*) ||_{op} \]

\[
= || (U_w T_F T_G - 2T_{\phi_w} T_F T_G T_{\phi_w}^* + T_{\phi_w}^2 T_F T_G T_{\phi_w}^2) ||_{op} \]

\[
= || U_w T_F T_G - 2T_{\phi_w} T_F T_G T_{\phi_w}^* + T_{\phi_w}^2 T_F T_G T_{\phi_w}^2 ||_{op} \]

\[
= || | U_w T_F T_G - 2T_{\phi_w} T_F T_G T_{\phi_w}^* + T_{\phi_w}^2 T_F T_G T_{\phi_w}^2 ||_{op} \]

\[
\leq || U_w T_F T_G - 2T_{\phi_w} T_F T_G T_{\phi_w}^* + T_{\phi_w}^2 T_F T_G T_{\phi_w}^2 ||_{op} \]

We can now use the triangle inequality on the operator $U_w T_F T_G - 2T_{\phi_w} T_F T_G T_{\phi_w}^* + T_{\phi_w}^2 T_F T_G T_{\phi_w}^2$ as in [3] to get our result, using that $|| T_{\phi_w} || \leq 1$. \qed

In the following we will be working with square matrices $F$ and $G$ with entries from the scalar valued Bergman space $L^2_a(D)$. Where it is not explicitly stated otherwise, this will be the case. When we refer to a matrix being less than another matrix, $F < G$, we mean in the sense of Löwner partial ordering of matrices. See [13], [16] and [17] for details on this.

3. BOUNDED AND INVERTIBLE TOEPLITZ PRODUCTS

3.1. A reverse Hölder inequality. We will now develop some of the theory needed to show a reverse Hölder inequality used to characterize the matrices of analytic functions, $F$ and $G$, such that the Toeplitz product $T_F T_G^*$ is bounded and invertible on the vector valued Bergman space. Compare this next lemma with Lemma 4.6 in [8].

Lemma 3.1. If $F(z)$ is invertible for all $z \in D$, then

\[
(F^*(w)F(w)) \leq B(F^*F)(w),
\]

\[
(F^*(z)F(z)) \leq \eta_b B(F^*F)(w)
\]

and

\[
(F^{-1}(w)F^{-1}(w)) \leq B(F^{-1}F^{-1})(w)
\]

\[
(F^{-1}(z)F^{-1}(z)) \leq \eta_b B(F^{-1}F^{-1})(w),
\]

when $z \in D(w, s)$ the pseudohyperbolic disk with radius $0 < s < 1$ and centre $w$ and $\eta_b$ is a constant dependent only on $s$.

Proof. Let $e$ be an arbitrary vector. Then for $F \in L^2_a(C^n)$,

\[
\langle F(u)e, F(u)e \rangle = \left< \int F(z)\overline{K_u(z)}dA(z)e, \int F(z)\overline{K_u(z)}dA(z)e \right>
\]

\[
= || \int F(z)\overline{K_u(z)}dA(z)e ||_{C^n}^2
\]

\[
\leq \int \langle Fe, Fe \rangle dA(z) || K_u ||_{L^2}^2 = \left< \int F^*(z)F(z)dA(z)e, e \right> || K_u ||_{L^2}^2.
\]
So if \( u \in D(0, s) \)
\[
F^*(u)F(u) \leq \int F^*(z)F(z)dA(z)||K_u||_2^2 \leq \int F^*(x)dA(z) \frac{1}{(1-s^2)^2}.
\]
If \( z \in D(w, s) \) then \( z = \phi_w(u) \) for some \( u \in D(0, s) \) thus
\[
F^*(z)F(z) = F^*(\phi_w(u))F(\phi_w(u)) \leq \int F^*(\phi_w(x))F(\phi_w(x))dA(z) \frac{1}{(1-s^2)^2} = B(F^*F)(w) \frac{1}{(1-s^2)^2}.
\]
Now let us show that \( F^{-1}(w)F^{-1}(w) \leq B(F^{-1}F^{-1})(w) \).
\[
\langle F^{-1}(w)e, F^{-1}(w)e \rangle = \langle F^{-1}(\phi_w(0))e, F^{-1}(\phi_w(0))e \rangle
\]
and we arrive at the conclusion that \( F^{-1}(w)F^{-1}(w) \leq B(F^{-1}F^{-1})(w) \) in a similar manner to before.
So for \( z \in D(w, s) \) we know that \( F^{-1}(w)F^{-1}(w) \leq B(F^{-1}F^{-1})(w) \) and \( F^*(z)F(z) \leq B(F^*F)(w) \frac{1}{(1-s^2)^2} \). The other inequalities follow from applying the same procedure to \( F^{-1}F^{-1} \) instead of \( F^*F \).

\[ \Box \]

**Lemma 3.2.** If there exists \( \eta \) such that \( F(z)G(z)^*G(z)F(z)^* > \eta I \) for all \( z \in \mathbb{D} \) and \( \text{tr}(B(G^*G)(w)B(F^*F)(w)) \) is uniformly bounded on \( \mathbb{D} \), then
\[
||B(F^{-1}(F^*)^{-1}(w))^{\frac{1}{2}}B(F^*F)(w)^{\frac{1}{2}}||
\]
is uniformly bounded on \( \mathbb{D} \).

**Proof.** Let us suppose that \( F(w)G(w)^*G(w)F(w)^* > \eta I \) for all \( w \in \mathbb{D} \). Then \( B(G^*G)(w) \geq G(w)^*G(w) \geq \eta(F(w)^*F(w))^{-1} \). The key inequality here is \( G(w)^*G(w) \geq \eta(F(w)^*F(w))^{-1} \), as this implies that \( B(G^*G)(w) \geq \eta B((F^*F)^{-1})(w) \) and so
\[
(B(F^*F)(w))^{\frac{1}{2}}B(G^*G)(w)(B(F^*F)(w))^{\frac{1}{2}} \geq \eta (B(F^*F)(w))^{\frac{1}{2}}B((F^*F)^{-1}(w))(B(F^*F)(w))^{\frac{1}{2}}.
\]
Thus as \( ||(B(F^*F)(w))^{\frac{1}{2}}B(G^*G)(w)(B(F^*F)(w))^{\frac{1}{2}}|| < M \) for all \( w \),
\[
\text{tr}(B(F^{-1}(F^*)^{-1})(w)B(F^*F)(w)) \leq \frac{1}{\eta}C||(B(F^*F)(w))^{\frac{1}{2}}B(G^*G)(w)(B(F^*F)(w))^{\frac{1}{2}}|| < \eta CM
\]

and so

\[ \|B(F^{-1}(F^*)^{-1})(w)B(F^*)^{1/2}\|^2 \]
\[ = \|B(F^{-1}(F^*)^{-1})(w)B(F^*)^{1/2}B(F^{-1}(F^*)^{-1})(w)B(F^*)^{1/2}\| \]
\[ \leq Ctr(B(F^{-1}(F^*)^{-1})(w)B(F^*)^{1/2}) \]
\[ \leq \frac{1}{\eta} C\|B(F^*)^{1/2}B(G^*)B(F^*)^{1/2}\| \]

where \( C \) and \( \eta \) are constants independent of \( w \).

Definition 3.3. A dyadic rectangle \( Q_{j,k,l} \) is a subset of the unit disk of the form

\[ \{ z = re^{i\theta} : (k-1)2^{-j} \leq r \leq k2^{-j}, (l-1)2^{1-j} \leq \theta \leq l2^{1-j}\pi \} , \]

where \( j, k, l \) are non negative integers and \( k, l \leq 2^j \).

Figure 1. Two nested dyadic rectangles in the unit disk.

Lemma 3.4. There exists \( 0 < r < 1 \) such that for all dyadic rectangles \( Q \) with positive distance to the boundary \( Q \subset D(z_Q, r) \). Here, \( D \) is the pseudohyperbolic disk and \( z_Q \) is the centre of the dyadic rectangle \( Q \).

Proof. This is just Proposition 4.7 in [8]. □
Lemma 3.5. If

\[ \sup_{w \in D} ||B(F^{-1}(F^{*})^{-1})(w)^{\frac{1}{2}} B(F^{*}F)(w)^{\frac{1}{2}}|| < \infty, \]

then

\[ \sup_{Q \text{dyadic}} \left\| \left\{ \frac{1}{|Q|} \int_{Q} (F^{*}F)dA(z) \right\}^{\frac{1}{2}} \left\{ \frac{1}{|Q|} \int_{Q} (F^{-1}F^{*-1})dA(z) \right\}^{\frac{1}{2}} \right\| < \infty. \]

Proof. If the dyadic rectangle Q is the whole disk, then as \( \int_{D} F^{*}FdA(z) = B(F^{*}F)(0) \) and \( \int_{D} F^{-1}F^{*-1}dA(z) = B(F^{-1}F^{*-1})(0) \), we see that

\[ \left\| \left( \int_{D} F^{*}FdA(z) \right)^{\frac{1}{2}} \left( \int_{D} F^{-1}F^{*-1}dA(z) \right)^{\frac{1}{2}} \right\| = \left\| B(F^{-1}(F^{*})^{-1})(0)^{\frac{1}{2}} B(F^{*}F)(0)^{\frac{1}{2}} \right\| \]

Now let us suppose that our dyadic rectangle Q has a positive distance from the boundary. By Lemma 3.4 our rectangle Q will be strictly contained in a pseudo-hyperbolic disk \( D(z_{Q}, R) \), \( z_{Q} \) being the centre of our dyadic rectangle and R being the same for each dyadic rectangle. Thus by Lemma 3.1

\[ (F^{-1}(z)F^{-1*}(z)) \leq \eta B(F^{-1}F^{*-1})(z_{Q}) \]

and

\[ (F^{*}(z)F(z)) \leq \eta B(F^{*}F)(z_{Q}) \]

for all \( z \) in our pseudo-hyperbolic disk \( D(z_{Q}, R) \). Here the constant \( \eta \) will only be dependent on \( R \) which is the same for all of these dyadic rectangles.

Thus using the fact that if \( A, B \) and \( C \) are positive matrices such that \( A \leq B \) then \( C^{\frac{1}{2}}AC^{\frac{1}{2}} < C^{\frac{1}{2}}BC^{\frac{1}{2}} \) and \( tr(C^{\frac{1}{2}}AC^{\frac{1}{2}}) < tr(C^{\frac{1}{2}}BC^{\frac{1}{2}}) \), we can deduce the following series of inequalities from our hypothesis:

\[ \left\| \left\{ \frac{1}{|Q|} \int_{Q} (F^{*}F)dA(z) \right\}^{\frac{1}{2}} \left\{ \frac{1}{|Q|} \int_{Q} (F^{-1}F^{*-1})dA(z) \right\}^{\frac{1}{2}} \right\|^2 \]
\[
\frac{1}{|Q|} \int_Q (F^* F) dA(z) \leq \frac{1}{|Q|} \int_Q (F^{-1} F^{-1}) dA(z) \leq \frac{1}{|Q|} \int_Q (F^* F) dA(z)
\]

by Lemma 4.5 in [8].

Note that \(|z_0| = 1\) for each dyadic rectangle not touching the boundary.

We can then combine these and take the trace to see that

\[
\text{tr} \left( \left\{ \frac{1}{|Q|} \int_Q (F^* F) dA(z) \right\}^{\frac{1}{2}} \frac{1}{|Q|} \int_Q (F^{-1} F^{-1}) dA(z) \right) \leq 4 \text{tr} \left( \left\{ B(F^{-1} F^{-1})(z_Q) \right\}^{\frac{1}{2}} \frac{1}{|Q|} \int_Q (F^* F) dA(z) \right)
\]

\[
\leq 16 c^2 \text{tr} \left( \left\{ B(F^{-1} F^{-1})(z_Q) \right\}^{\frac{1}{2}} B(F^* F)(z_Q) \right)
\]

Noting that \(C \) is a constant that possibly changes from line to line and is dependent on the dimension of \(C^n \) only. \(M \) will be dependent only on the uniform bound of \(B(F^{-1} F^{-1})(w)B(F^* F)(w) \), the dimension we are working in and the constant \(R \) which is the same for each dyadic rectangle not touching the boundary.

What happens when we have a dyadic rectangle that touches the boundary but is not the whole disk? We can see that the centre of the rectangle \(z_Q \) is at a distance of at least \(1/2 \) from the centre, i.e. \(|z_Q| \geq 1/2 \). Then

\[
B(F^* F)(z_Q) = \int_{D^*} F^*(z)F(z)|z_{z_0}|^2(z) dA(z)
\]

\[
\geq \int_Q F^*(z)F(z)|z_{z_0}(z)|^2 dA(z) \geq \frac{c}{(1 - |z_Q|)^2} \int_Q F^*(z)F(z) dA(z)
\]

by Lemma 4.5 in [8].

We can also see in this case that \(|Q| = 8|z_Q|(1 - |z_Q|)^2 \) and so

\[
B(F^* F)(z_Q) \geq \frac{4c}{|Q|} \int_Q F^*(z)F(z) dA(z).
\]

We can do the same for \(F^{-1} F^{-1} \) to get that

\[
B(F^{-1} F^{-1})(z_Q) \geq \frac{4c}{|Q|} \int_Q F^{-1}(z)F^{-1}(z) dA(z).
\]

We can then combine these and take the trace to see that

\[
\text{tr} \left( \left\{ \frac{1}{|Q|} \int_Q (F^* F) dA(z) \right\}^{\frac{1}{2}} \frac{1}{|Q|} \int_Q (F^{-1} F^{-1}) dA(z) \right) \leq 4 \text{tr} \left( \left\{ B(F^{-1} F^{-1})(z_Q) \right\}^{\frac{1}{2}} \frac{1}{|Q|} \int_Q (F^* F) dA(z) \right)
\]

\[
\leq 16 c^2 \text{tr} \left( \left\{ B(F^{-1} F^{-1})(z_Q) \right\}^{\frac{1}{2}} B(F^* F)(z_Q) \right)\]
Theorem 3.6. If for \( L_f \) averaging operators, \( Q \) angles of such functions \( F \), see [15] for a similar notion of matrix weights. We will now find a characterization of such functions \( F \) in terms of the boundedness of certain averaging operators on the function space \( L^2(F^*F) \).

Proof. Let \( R \) be the subspace \( \{ \chi_Q \frac{1}{|Q|^\frac{1}{2}} e : e \in \mathbb{C}^n \} \). We can see that the orthogonal projection from \( L^2(D, \mathbb{C}^n) \) onto \( R \) is given by \( P_Q : f \mapsto \chi_Q \frac{1}{|Q|^\frac{1}{2}} \int_Q f(z) dA(z) \). So we want to show that these projections are uniformly bounded with respect to the \( L^2(F^*F) \) norm. Clearly,

\[
\| P_Q \|_{L^2(F^*F)} = \sup \left\{ \frac{\| P_Q f \|_{L^2(F^*F)} }{\| f \|_{L^2(F^*F)} } : \| f \|_{L^2(F^*F)} \neq 0 \right\}
\]

If we let \( S \) denote the orthogonal complement of \( R \) in \( L^2 \), then \( f = f_1 + f_2 \), where \( f_1 \in R \) and \( f_2 \in S' = S \cap L^2(F^*F) \). Thus the expression for the norm of the projection will become

\[
\sup_{\{ f_1 + f_2 \in L^2(F^*F) : \| f \|_{L^2(F^*F)} \neq 0 \}} \left\{ \frac{\| f_1 \|_{L^2(F^*F)} }{\| f_1 + f_2 \|_{L^2(F^*F)} } \right\}
\]

\[
= \sup_{\{ e \in \mathbb{C}^n : e \neq 0 \}} \left\{ \frac{\| \chi_Q \frac{1}{|Q|^\frac{1}{2}} e \|_{L^2(F^*F)} }{\inf_{\{ f_2 \in S \}} \| \chi_Q \frac{1}{|Q|^\frac{1}{2}} e + f_2 \|_{L^2(F^*F)} } \right\}
\]

\[
= \sup_{\{ e \in \mathbb{C}^n : e \neq 0 \}} \left\{ \frac{\| \chi_Q \frac{1}{|Q|^\frac{1}{2}} e \|_{L^2(F^*F)} }{\text{dist}_{L^2(F^*F)} (\chi_Q \frac{1}{|Q|^\frac{1}{2}} e, S') } \right\}.
\]

So let us take a look at \( \text{dist}_{L^2(F^*F)} (\chi_Q \frac{1}{|Q|^\frac{1}{2}} e, S') \).

\[
\text{dist}_{L^2(F^*F)} (\chi_Q \frac{1}{|Q|^\frac{1}{2}} e, S') = \text{dist}_{L^2} \left( (F^*F)^\frac{1}{2} \chi_Q \frac{1}{|Q|^\frac{1}{2}} e, (F^*F)^\frac{1}{2} S' \right)
\]

\[
= \sup_{h \in (F^*F)^\frac{1}{2} S' : \| h \| = 1} \left\langle (F^*F)^\frac{1}{2} \chi_Q \frac{1}{|Q|^\frac{1}{2}} e, h \right\rangle,
\]

where \( M' \) is independent of \( Q \).\]
\( (F^*F)^{-1} \) exists as we have the \( A_2 \) condition. Note that \( (F^*F)^{\frac{1}{2}} S' \perp (F^*F)^{\frac{1}{2}} R \).

Then we can see that:

\[
dist_{L^2(F^*F)} \left( \chi_0 \frac{1}{|Q|^\frac{1}{2}} e, S' \right) = \sup_{h \in ((F^*F)^{-\frac{1}{2}} R) : ||h|| = 1} \left( (F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^\frac{1}{2}} e, h \right)
\]

\[
= \sup_{g \in C^\infty((F^*F)^{-1}) : ||g|| \leq 1} \left( (F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^\frac{1}{2}} e, (F^*F)^{-\frac{1}{2}} \chi_Q \frac{1}{|Q|^\frac{1}{2}} g \right)
\]

\[
= \sup_{g \in C^\infty((F^*F)^{-1}) : ||g|| \leq 1} \left( (F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^\frac{1}{2}} e, (F^*F)^{-\frac{1}{2}} \chi_Q \frac{1}{|Q|^\frac{1}{2}} g \right)
\]

Let us now put this equivalent expression for the distance back into our expression for the norm of the projection in \( L^2(F^*F) \):

\[
||P_Q||_{L^2(F^*F) \to L^2(F^*F)} = \sup_{\{e \in C^\infty : e \neq 0\}} \left\{ \frac{||\chi_Q \frac{1}{|Q|^\frac{1}{2}} e||_{L^2(F^*F)}}{dist_{L^2(F^*F)} \left( \chi_0 \frac{1}{|Q|^\frac{1}{2}} e, S' \right)} \right\}
\]

\[
= \sup_{\{e \in C^\infty : e \neq 0\}} \left\{ \frac{||\chi_Q \frac{1}{|Q|^\frac{1}{2}} e||_{L^2(F^*F)}}{||\left\{ \frac{1}{|Q|} \int_Q F^{-1} F^{s-1} \right\}^{-\frac{1}{2}} e ||} \right\}
\]

\[
= \sup_{\{e \in C^\infty : e \neq 0\}} \left\{ \frac{||\left\{ \frac{1}{|Q|} \int_Q F^{s} \right\}^{\frac{1}{2}} e ||}{||\left\{ \frac{1}{|Q|} \int_Q F^{-1} F^{s-1} \right\}^{-\frac{1}{2}} e ||} \right\}
\]

\[
= || \left\{ \frac{1}{|Q|} \int_Q F^{s} \right\}^{\frac{1}{2}} \left\{ \frac{1}{|Q|} \int_Q F^{-1} F^{s-1} \right\}^{\frac{1}{2}} ||.
\]
Lemma 3.7. If the averaging operators \( g \mapsto \chi_Q \frac{1}{|Q|} \int_Q g(z) dA(z) \) are uniformly bounded on \( L^2(|f|^2) \) over \( Q \) dyadic, then \( |f|^2 \) has the scalar \( A_2 \) condition.

Proof. Again we can see that the averaging operator \( g \mapsto \chi_Q \frac{1}{|Q|} \int_Q g(z) dA(z) \) is the projection \( P : L^2 \to \chi_Q \frac{1}{|Q|} \mathbb{C} \). We are working as before on the dense subset \( L^2(\mathbb{C}) \cap L^2(|f|^2) \). If we assume that \( \frac{1}{|f|^2} \) is bounded then we can as before show that

\[
dist_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^2} z, S') = \left| \left[ \int_D \frac{1}{|f|^2} \chi_Q \frac{1}{|Q|} \right]^{-\frac{1}{2}} \right| = \left| \left[ \int_D \frac{1}{|f|^2} \chi_Q \frac{1}{|Q|} \right]^{-\frac{1}{2}} \right|
\]

where \(|z| = 1\).

So if we drop this assumption on \( \frac{1}{|f|^2} \) but instead use \( \frac{1}{|f|^2 + \epsilon} \) for \( \epsilon > 0 \), then we can see that

\[
dist_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^2} z, S') = \lim_{\epsilon \to 0} \dist_{L^2(|f|^2 + \epsilon)}(\chi_Q \frac{1}{|Q|^2} z, S')
\]

\[
= \lim_{\epsilon \to 0} \left| \left[ \frac{1}{|Q|} \int_Q \frac{1}{|f|^2 + \epsilon} \right]^{-\frac{1}{2}} \right|
\]

where \( S' \) is the intersection of the orthogonal complement of \( \chi_Q \frac{1}{|Q|^2} \mathbb{C} \) with \( L^2(|f|^2) \) and \(|z| = 1\).

As the norm of our bounded projection \( P \) is

\[
\sup_{z \in \mathbb{C} : z \neq 0, |z| = 1} \left\{ \frac{\|\chi_Q \frac{1}{|Q|^2} z\|_{L^2(|f|^2)}}{\dist_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^2} z, S')} \right\} = \sup_{z \in \mathbb{C} : z \neq 0, |z| = 1} \left\{ \frac{\|\chi_Q \frac{1}{|Q|^2} z\|_{L^2(|f|^2)}}{\dist_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^2} z, S')} \right\}^{-1}
\]

we know that \( \dist_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^2} z, S') \) is nonzero for nonzero \( z \) and hence

\[
\lim_{\epsilon \to 0} \left| \left[ \frac{1}{|Q|} \int_Q \frac{1}{|f|^2 + \epsilon} \right]^{-\frac{1}{2}} \right| < \infty
\]

and so by the Monotone Convergence Theorem

\[
\left| \left[ \frac{1}{|Q|} \int_Q \frac{1}{|f|^2} \right]^{-\frac{1}{2}} \right| < \infty
\]

and

\[
\dist_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^2} z, S') = \left| \left[ \frac{1}{|Q|} \int_Q \frac{1}{|f|^2} \right]^{-\frac{1}{2}} \right|
\]

where \(|z| = 1\).

Thus

\[
\|P\|_{L^2(|f|^2)} = \sup_{z \in \mathbb{C} : z \neq 0, |z| = 1} \left\{ \frac{\|\chi_Q \frac{1}{|Q|^2} z\|_{L^2(|f|^2)}}{\dist_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^2} z, S')} \right\}
\]

\[
= \sup_{z \in \mathbb{C} : z \neq 0, |z| = 1} \left\{ \left| \left[ \frac{1}{|Q|} \int_Q |f|^2 \right]^{\frac{1}{2}} \right| \right\} = \left\{ \left| \left[ \frac{1}{|Q|} \int_Q |f|^2 \right]^{\frac{1}{2}} \right| \right\} = \left\{ \left| \left[ \frac{1}{|Q|} \int_Q |f|^2 \right]^{\frac{1}{2}} \right| \right\}
\]
which is uniformly bounded as required.

□

Compare the next lemma with Lemma 3.6 in [15].

Lemma 3.8. If \( F^*F \) has the \( A_2 \) condition, then \( \text{trace}(F^*F) \) has the scalar \( A_2 \) condition.

Proof. We will show that each element on the diagonal of \( F^*F \) has the scalar \( A_2 \) condition. We can then deduce that the sum of these will also have the \( A_2 \) condition. Firstly we know that if \( F^*F \) has the \( A_2 \) condition, then the operators \( f \mapsto \chi_Q \frac{1}{|Q|} \int_Q f(z)dA(z) \) are uniformly bounded on \( L^2(F^*F) \), \( f \in L^2(\mathbb{C}^n) \).

So if we take \( g \in L^2(\mathbb{D}) \cap L^2(\mathbb{D}, \langle F^*F(0, \ldots, 1, \ldots, 0), (0, \ldots, 1, \ldots, 0) \rangle) \) where \( \langle F^*F(0, \ldots, 1, \ldots, 0), (0, \ldots, 1, \ldots, 0) \rangle \) is the scalar valued function \( z \mapsto \langle F^*F(z)(0, \ldots, 1, \ldots, 0), (0, \ldots, 1, \ldots, 0) \rangle \). Then note that \( g(0, \ldots, 1, \ldots, 0) \mapsto \chi_Q \frac{1}{|Q|} \int_Q g(z)(0, \ldots, 1, \ldots, 0)dA(z) \) is uniformly bounded between \( L^2(\mathbb{D}, \mathbb{C}^n) \), this implies that \( g \mapsto \chi_Q \frac{1}{|Q|} \int_Q g(z)dA(z) \) is uniformly bounded with respect to the scalar measure \( \langle F^*F(0, \ldots, 1, \ldots, 0), (0, \ldots, 1, \ldots, 0) \rangle \), which will be whatever diagonal element of \( F^*F \) we want. Thus by the previous lemma the trace of \( F^*F \) will have the scalar \( A_2 \) condition.

□

Compare this next lemma with Lemma 4.9 in [8], Lemma 2.5 in [10] and also 1.7 on page 196 of [18].

Lemma 3.9. If a scalar valued function \(|f|^2\) has the \( A_2 \) condition and for some \( 0 < \delta < 1 \) then for each dyadic rectangle \( Q \) and \( E \subset Q \) such that \(|E| \leq \delta|Q|\) we have that \( \mu(E) \leq \lambda \mu(Q) \) for some \( 0 < \lambda < 1 \) where \( d\mu = |f|^2dA \) and \( \lambda \) only depends on \( \delta \) and the \( A_2 \) constant of \(|f|^2\).

Proof.

\[
|Q/E|^2 = \left\{ \int_{Q/E} |f||f|^{-1}dA \right\}^2 \leq \left\{ \int_{Q/E} |f|^2dA \right\} \left\{ \int_{Q/E} |f|^{-2}dA \right\}
\]

\[
\leq \left\{ \int_{Q/E} |f|^2dA \right\} \left\{ \int_{Q} |f|^{-2}dA \right\} \leq \left\{ \int_{Q/E} |f|^2dA \right\} C|Q|^2 \left\{ \int_{Q} |f|^2dA \right\}^{-1}
\]

by our \( A_2 \) condition on \(|f|^2\) this equals

\[
\left\{ \int_{Q} |f|^2dA \right\} \left\{ \int_{Q/E} |f|^2dA \right\} \leq C \left\{ \int_{Q} |f|^2dA \right\} \left\{ \int_{Q/E} |f|^2dA \right\}^{-1}
\]

\[
= C \left( 1 - \left\{ \int_{Q} |f|^2dA \right\} \left\{ \int_{Q/E} |f|^2dA \right\}^{-1} \right)|Q|^2,
\]

so we know that

\[
|Q/E|^2 \leq C|Q|^2 \left( 1 - \frac{\mu(E)}{\mu(Q)} \right)
\]

and thus

\[
\frac{|Q/E|^2}{|Q|^2} \leq C \frac{\mu(Q/E)}{\mu(Q)}.
\]
Now we know that $\frac{|E|}{|Q|} \leq \delta < 1$ from our hypothesis, this implies that $\frac{|Q/E|}{|Q|} \geq 1 - \delta > 0$. So we can now deduce that

$$0 < \frac{(1 - \delta)^2}{C} \leq \frac{1}{C} \frac{|Q/E|^2}{|Q|^2} \leq \frac{\mu(Q/E)}{\mu(Q)}.$$ 

This lets us now see that

$$1 = \frac{\mu(Q)}{\mu(Q)} = \frac{\mu(Q/E) + \mu(E)}{\mu(Q)} \geq \frac{\mu(E)}{\mu(Q)} + \frac{(1 - \delta)^2}{C}$$

and hence

$$\frac{\mu(E)}{\mu(Q)} \leq 1 - \frac{(1 - \delta)^2}{C}. \text{ \hspace{1cm} \square}$$

The following lemma will be crucial to our application of the $A_2$ condition.

**Lemma 3.10.** If $F^*F$ has the $A_2$ condition and $J$ is a strictly positive matrix then $J F^*F J$ will have the $A_2$ condition. The $A_2$ constant of $J F^*F J$ will depend on the $A_2$ bound of $F^*F$ and the dimension only.

**Proof.**

$$\| \left( \frac{1}{|I|} \int_I J F^*FJ \right)^{\frac{1}{2}} \left( \frac{1}{|I|} \int_I (J F^*FJ)^{-1} \right)^{\frac{1}{2}} \| = \| \left( \frac{1}{|I|} \int_I J F^*FJ \right)^{\frac{1}{2}} \left( \frac{1}{|I|} \int_I (J F^*FJ)^{-1} \right)^{\frac{1}{2}} \| \leq Ctr \left( \left( \frac{1}{|I|} \int_I (F^*F)^{-1} \right)^{\frac{1}{2}} \left( \frac{1}{|I|} \int_I F^*F \right)^{\frac{1}{2}} \right)^2$$

(where the constant $C$ depends only on the dimension.)

$$= Ctr \left( \left( \frac{1}{|I|} \int_I (F^*F)^{-1} \right)^{\frac{1}{2}} \left( \frac{1}{|I|} \int_I F^*F \right)^{\frac{1}{2}} \right) \leq C' \| \left( \frac{1}{|I|} \int_I (F^*F)^{-1} \right)^{\frac{1}{2}} \left( \frac{1}{|I|} \int_I (F^*F)^{-1} \right) \| \| F \|_{L^2}.$$ 

$C'$ again depending only on the dimension, thus giving us our result. \hspace{1cm} \square

**Definition 3.11.** The dyadic maximal operator $M_\Delta$ is defined by

$$(M_\Delta f)(w) = \sup_{w \in Q} \frac{1}{|Q|} \int_Q |f(z)|dA(z),$$

where the $Q$ are dyadic rectangles and $f \in L^2$.

**Theorem 3.12.** (The Calderon-Zygmund Decomposition Theorem.) Let $f \in L^1(\mathbb{D})$, if we have $t > 0$ such that the set $\Lambda = \{ z \in \mathbb{D} : M_\Delta f(z) > t \}$ is not the whole of $\mathbb{D}$, then we can decompose $\Lambda$ into a disjoint union of dyadic intervals $Q_i$ such that $t < \frac{1}{|Q_i|} \int_{Q_i} |f(z)|dA(z) < 8t$. 

Proof. The proof of this is exactly as in [6] and [8]. \qed

Compare this next lemma with Proposition 4.14 in [8].

Lemma 3.13. The trace of $F^*F$ satisfies the following:

1. \[ \text{tr}(F^*F) \leq M_\Delta \text{tr}(F^*F) \] on $\mathbb{D}$ and

2. \[ \int_{\mathbb{D}} \text{tr}(F^*(z)F(z))dA(z) \leq M_\Delta \text{tr}(F^*(0)) \leq (4/3)^2 \int_{\mathbb{D}} \text{tr}(F^*(z)F(z))dA(z). \]

Proof. (1) This follows from Proposition 4.14 in [8]. We just need to note that $\text{tr}(F^*F)$ is continuous and the proof works as it is.

(2) $\mathbb{D}$ is a dyadic rectangle containing 0 so

\[ M_\Delta \text{tr}(F^*(0)) \geq \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \text{tr}(F^*(z)F(z))dA(z) = \int_{\mathbb{D}} \text{tr}(F^*(z)F(z))dA(z). \]

Let us take a dyadic rectangle $Q$ containing 0 which is not the unit disk. We know that $Q$ will be contained in the pseudohyperbolic disk $D(0, \frac{1}{2})$.

Let $e \in \mathbb{C}^n$, then as $F \in L^2_a(\mathbb{C}^n)$,

\[ \langle F(u)e, F(u)e \rangle = \|F(u)e\|_{\mathbb{C}^n}^2. \]

\[ = \| \int F(z)K_u(z)dA(z)e \|_{\mathbb{C}^n}^2 \leq \left\{ \int \|F(z)e\| \|K_u(z)dA(z)\right\}^2 \]

\[ \leq \int \langle Fe, Fe \rangle dA(z) \|K_u(z)\|_{L^2}^2 = \left\langle \int F^*(z)F(z)dA(z)e, e \right\rangle \|K_u(z)\|_{L^2}^2. \]

So

\[ F^*(u)F(u) \leq \int F^*(z)F(z)dA(z)\|K_u\|_{L^2}^2 \leq \int F^*(x)F(x)dA(x) \frac{1}{(1 - \frac{1}{2})^2} \]

\[ = \left( \frac{4}{3} \right)^2 \int F^*(x)F(x)dA(x) \]

on each $Q$ containing 0 which is not $\mathbb{D}$.

So

\[ \text{tr}(F^*(u)F(u)) \leq \text{tr} \left( \left( \frac{4}{3} \right)^2 \int F^*(x)F(x)dA(x) \right) \]

for $u \in Q$. Hence

\[ \frac{1}{|Q|} \int_Q \text{tr}(F^*(z)F(z))dA(z) \]

\[ \leq \left( \frac{4}{3} \right)^2 \int_{\mathbb{D}} \text{tr}(F^*(x)F(x))dA(x) \]
and so
\[
M_\Delta \text{tr}(F^*F)(0) \leq (4/3)^2 \int_B \text{tr}(F^*F(z))dA(z).
\]

The proof of the following theorem follows the lines of Theorem 2.1 in [6] and Theorem 4.1 in [8]. It contains the key to the proof of Theorem 1.4 i.e. the reverse Hölder property.

**Theorem 3.14.** If \(F^*F\) satisfies \(A_2\), then there exists \(\epsilon > 0\) such that \(\int (\text{tr}(F^*F(z)))^{1+\epsilon}dA(z) \leq C \int (\text{tr}(F^*F(z)))dA(z)^{1+\epsilon}\) with \(C\) and \(\epsilon\) dependent only on the \(A_2\) constant.

**Proof.** For each \(k\) define
\[
E_k = \left\{ z \in \mathbb{D} : M_\Delta (\text{tr}(F^*F))(z) > 2^{4k+1} \int_D (\text{tr}(F^*F(z)))dA(z) \right\}.
\]

By Lemma 3.13 we can see that
\[
M_\Delta \text{tr}(F^*F)(0)\)
\[
\leq (4/3)^2 \int_D \text{tr}(F^*F(z))dA(z) < 2^{4k+1} \int_D \text{tr}(F^*F(z))dA(z)
\]
for all \(k\). So we know that each \(E_k\) is not the whole disk (as 0 is not contained in it) and hence we can do a Calderon-Zygmund decomposition. So for each \(E_k\) we have a disjoint union of dyadic rectangles \(Q_i\) whose union is equal to \(E_k\) and
\[
2^{4k+1} \int_D (\text{tr}(F^*F(z)))dA(z) < \frac{1}{|Q_i|} \int_{Q_i} \text{tr}(F^*F(z))dA(z)
\]
\[
< 4^{4k+1} \int_D (\text{tr}(F^*F(z)))dA(z).
\]

Two inequalities we will use from this are:
\[
|Q_i| < 2^{-4k-1} \left\{ \int_D (\text{tr}(F^*F(z)))dA(z) \right\}^{-1} \int_{Q_i} \text{tr}(F^*F(z))dA(z)
\]
and
\[
\int_{Q_i} \text{tr}(F^*F(z))dA(z) < |Q_i|2^{4(k+1)} \int_D (\text{tr}(F^*F(z)))dA(z).
\]

We now take a maximal dyadic rectangle \(Q\) in \(E_{k-1}\) (which is larger than \(E_k\)) and note that
\[
|E_k \cap Q| = \sum_{Q_i \subset Q} |Q_i|
\]
(where the \(Q_i\) denote the maximal dyadic rectangles in \(E_k\))
\[
< \sum_{Q_i \subset Q} 2^{-4k-1} \left\{ \int_D (\text{tr}(F^*F(z)))dA(z) \right\}^{-1} \int_{Q_i} \text{tr}(F^*F(z))dA(z)
\]
\[
\leq 2^{-4k-1} \left\{ \int_D (\text{tr}(F^*F(z)))dA(z) \right\}^{-1} \int_Q \text{tr}(F^*F(z))dA(z)
\]
due to the dyadic decomposition of \(E_k\).
But as $Q$ is also part of a Calderon-Zygmund decomposition (this time for $E_{k-1}$) we can also see that

$$\int_Q tr(F^*(z)F(z))dA(z) < |Q|2^{4k} \int_{D} (tr(F^*(z)F(z)))dA(z).$$

Putting the last two inequalities together we see that

$$|E_k \cap Q| < 2^{-4k-1} \left\{ \int_{D} (tr(F^*(z)F(z)))dA(z) \right\}^{-1} |Q|2^{4k} \int_{D} (tr(F^*(z)F(z)))dA(z) = \frac{1}{2}|Q|.$$

We are now in a position to use Lemma 3.9 as $tr(F^*(z)F(z))$ satisfies the scalar $A_2$ condition and $|E_k \cap Q| \leq \frac{1}{2}|Q|$. So with $\frac{1}{2}$ being our $\delta$ in 3.9 we can deduce that

$$\mu(E_k \cap Q) < \lambda \mu(Q)$$

for some $0 < \lambda < 1$ independent of $k$, with $d\mu(z) = tr(F^*(z)F(z))dA(z)$. We can now sum over all maximal dyadic rectangles in $E_{k-1}$ and see that

$$\mu(E_k) = \sum_{Q} \mu(E_k \cap Q) < \lambda \sum_{Q} \mu(Q) = \lambda \mu(E_{k-1}).$$

Let us take a moment here to note that $\lambda$ depends only on our $A_2$ bound of $tr(F^*(z)F(z))$, (we can see this from Lemma 3.9), and that this $A_2$ bound is controlled by the matrix $A_2$ bound for $F^*F$ and the dimension.

We have established that for each $k \geq 1$, $\mu(E_k) < \lambda \mu(E_{k-1})$ and so

$$\mu(E_k) < \lambda^k \mu(E_0) = \lambda^k \int_{E_0} tr(F^*(z)F(z))dA(z) < \lambda^k \int_{D} tr(F^*(z)F(z))dA(z).$$

Now let us move on and look at $\int_{D} tr(F^*(z)F(z))^{1+\epsilon}dA(z)$ for some $\epsilon > 0$. From Lemma 3.13 we know that $tr(F^*F)(z) \leq M_{\Delta} tr(F^*F)(z)$ on the disk so

$$\int_{D} tr(F^*(z)F(z))^{1+\epsilon}dA(z) \leq \int_{D} tr(F^*(z)F(z)) \{M_{\Delta} tr(F^*F)(z)\}^\epsilon dA(z)$$

$$= \int_{x:M_{\Delta} tr(F^*F)(x) \leq \int_{D} tr(F^*F)(z))dA(z)} tr(F^*(z)F(z)) \{M_{\Delta} tr(F^*F)(z)\}^\epsilon dA(z) + \sum_{k} \int_{E_k - E_{k+1}} tr(F^*(z)F(z)) \{M_{\Delta} tr(F^*F)(z)\}^\epsilon dA(z)$$

$$\leq \left\{ \int_{D} tr(F^*F)(z) \right\}^{1+\epsilon} + \sum_{k} 2^{4(k+1)+1} \left\{ \int_{D} tr(F^*F)(z)dA(z) \right\}^{1+\epsilon} \mu(E_k)$$

$$\leq \left\{ \int_{D} tr(F^*F)(z) \right\}^{1+\epsilon} + \sum_{k} 2^{4(k+1)+1} \left\{ \int_{D} tr(F^*F)(z)dA(z) \right\}^{1+\epsilon}$$

$$\leq \left\{ \int_{D} tr(F^*F)(z) \right\}^{1+\epsilon} + \lambda \int_{D} tr(F^*F)(z)dA(z).$$
Lemma 3.17. If the trace of a positive matrix $F$ is less than some constant $\lambda > 0$ and the dimension, then the Toeplitz product $T_G T_{F^*}$ is invertible.

Proof. $F^* G^* F > \eta I$ implies that $G^{-1} F^{-1} F^* G^{-1}$ is bounded and so the operator $T_{G^{-1} F^{-1}} = T_{F^{-1} G^{-1}}$ is bounded. It remains to note that

$$\langle T_F T_{G^*} \rangle T_{G^{-1} F^{-1}} (k_w, 0, 0, \ldots) = F(k_w, 0, 0, \ldots)$$

and

$$T_{G^{-1} F^{-1}} (T_F T_{G^*})(k_w, 0, 0, \ldots) = (k_w, 0, 0, \ldots),$$

and that these also hold for $(0, \ldots, k_w, \ldots)$. This implication holds because the linear spans of $\{F(0, \ldots, k_w, \ldots)\}$ and $\{(0, \ldots, k_w, \ldots)\}$ form dense subspaces. \hfill \square

Lemma 3.18. If the trace of a positive matrix $A$ is less than some constant $\lambda > 0$ then $A < CI$ for some constant $C > 0$ depending only on $\lambda$ and the dimension, $I$ being the identity matrix.

Proof. Trivial. \hfill \square

Proof of Theorem 1.3. From Lemma 3.9 we know that $F^* F$ satisfies our $A_2$ condition. Then by Corollary 3.15

\begin{equation}
\int_{\mathbb{D}} (tr((G^* G)(x)))^{\frac{1}{2}} (F^* F)(z) ((G^* G)(x))^{\frac{1}{2}})^{1+\epsilon} dA(z) \leq C \int (tr((G^* G)(x)))^{\frac{1}{2}} (F^* F)(z) ((G^* G)(x))^{\frac{1}{2}}) dA(z)^{1+\epsilon}
\end{equation}

holds for all $x \in \mathbb{D}$ with some $\epsilon > 0$ and a constant $C$ independent of $x$. Note here that we need to use the fact that $G^* G$ is strictly positive.
We can also see that $G^*G$ satisfies our $A_2$ condition, so a similar reverse Hölder will hold;
\[
\int \left( \int \left( \int (F^*F)(z) \right) \right)^{\frac{1}{2}} (G^*G)(x) \left( \int (F^*F)(z) \right)^{\frac{1}{2}} dA(x) \right) dA(x)
\]
\[
\leq C \left( \int \left( \int \left( \int (F^*F)(z) \right) \right)^{\frac{1}{2}} (G^*G)(x) \left( \int (F^*F)(z) \right)^{\frac{1}{2}} dA(x) \right)
\]
\[
\leq C \left( \int \left( \int (F^*F)(z) \right) \right)^{\frac{1}{2}} (G^*G)(x) \left( \int (F^*F)(z) \right)^{\frac{1}{2}} dA(x) \right)
\]
\[
= C \left( \int \left( (G^*G)(x) \right) \right)^{\frac{1}{2}} \left( \int (F^*F)(z) dA(z) \right)^{\frac{1}{2}} dA(x)
\]
\[
= C \left( \int \left( (G^*G)(x) \right) \right)^{\frac{1}{2}} \left( \int (F^*F)(z) dA(z) \right)^{\frac{1}{2}} dA(x)
\]

So let us set $\epsilon = \min \{\epsilon, \epsilon'\}$.

Thus integrating both sides of the reverse Hölder inequality \[3.1\] with respect to $x$, we get
\[
\int \int \left( \int (F^*F)(z) \right)^{\frac{1}{2}} (G^*G)(x) \left( \int (F^*F)(z) \right)^{\frac{1}{2}} dA(x) \right) dA(x)
\]
\[
\leq C \int \left( \int \left( \int (F^*F)(z) \right) \right)^{\frac{1}{2}} (G^*G)(x) \left( \int (F^*F)(z) \right)^{\frac{1}{2}} dA(x) \right)
\]
\[
\leq C \int \left( (G^*G)(x) \right) \left( \int (F^*F)(z) dA(z) \right) dA(x)
\]
\[
= C \left( \int \left( (G^*G)(x) \right) \right) \left( \int (F^*F)(z) dA(z) \right) dA(x)
\]

and so as $G^*G$ also has the $A_2$ condition, we can use our reverse Hölder again to see that this last expression is less than or equal to
\[
C \left( \int \left( (G^*G)(x) \right) \right) \left( \int (F^*F)(z) dA(z) \right) dA(x)
\]
\[
\leq C \left( \int (F^*F)(z) dA(z) \right) dA(x)
\]

where as usual $C$ is a constant that possibly changes from line to line.

By the Möbius invariance of the Berezin transform \[20\] page 143 we see that
\[
\int \int \left( \int (F^*F)(z) \right)^{\frac{1}{2}} (G^*G)(x) \left( \int (F^*F)(z) \right)^{\frac{1}{2}} dA(x) \right) dA(x)
\]
\[
\leq C \left( \int (B(G^*G)(w)) \right)^{\frac{1}{2}} B(F^*F)(w) (B(G^*G)(w)) \right)^{\frac{1}{2}} dA(x)
\]
\[
\leq C \left( \int (B(G^*G)(w)) \right)^{\frac{1}{2}} B(F^*F)(w) (B(G^*G)(w)) \right)^{\frac{1}{2}} dA(x)
\]

Hence by Theorem \[12\] we can see that the Toeplitz product $T_F T_{G^*}$ is bounded. The invertibility of this Toeplitz product follows from Lemma \[3.16\]

"⇒" If $T_F T_{G^*}$ is bounded and invertible, we know from Theorem \[1.3\] that
\[
\int (B(F^*F)(w)B(G^*G)(w)) \right)^{\frac{1}{2}} dA(x)
\]
\[
\leq C \left( \int (B(G^*G)(w)) \right)^{\frac{1}{2}} B(F^*F)(w) (B(G^*G)(w)) \right)^{\frac{1}{2}} dA(x)
\]

for all vectors $e \in \mathbb{C}^n$. We know that $T_F T_{G^*} k_w = F(z) G^*(w) k_w(z)$ and so we deduce that $G(w) B(F^*F)(w) G^*(w) > \eta I$. From the fact that $\| (T_F T_{G^*})^* \|$ is also
bounded below we can see that $F(w)B(G^*G)(w)F^*(w) > \eta I$. From these we deduce the following:

$$B(G^*G)(w) > \eta F^{-1}(w)F^{*-1}(w)$$

and

$$B(F^*F)(w) > \eta G^{-1}G^{*-1}(w)$$

which lets us see that

$$\{G^{-1}G^{*-1}(w)\}^\frac{1}{2} B(G^*G)(w) \{G^{-1}G^{*-1}(w)\}^\frac{1}{2} > \eta \{G^{-1}G^{*-1}(w)\}^\frac{1}{2} F^{-1}(w)F^{*-1}(w) \{G^{-1}G^{*-1}(w)\}^\frac{1}{2}$$

and also

$$\{B(G^*G)(w)\}^\frac{1}{2} B(F^*F)(w) \{B(G^*G)(w)\}^\frac{1}{2} > \eta \{B(G^*G)(w)\}^\frac{1}{2} G^{-1}(w)G^{*-1}(w) \{B(G^*G)(w)\}^\frac{1}{2},$$

thus

$$\text{tr}(B(G^*G)(w)B(F^*F)(w)) = \text{tr}(\{B(G^*G)(w)\}^\frac{1}{2} B(F^*F)(w) \{B(G^*G)(w)\}^\frac{1}{2})$$

$$> \eta \text{tr}(\{B(G^*G)(w)\}^\frac{1}{2} G^{-1}G^{*-1} \{B(G^*G)(w)\}^\frac{1}{2}))$$

$$= \eta \text{tr}(\{G^{-1}G^{*-1}(w)\}^\frac{1}{2} B(G^*G)(w) \{G^{-1}G^{*-1}(w)\}^\frac{1}{2}))$$

$$> \eta^2 \text{tr}(\{G^{-1}G^{*-1}(w)\}^\frac{1}{2} F^{-1}(w)F^{*-1}(w) \{G^{-1}G^{*-1}(w)\}^\frac{1}{2})).$$

Thus as $\text{tr}(B(G^*G)(w)B(F^*F)(w))$ is uniformly bounded, $\text{tr}(G^{*-1}(w)F^{-1}(w)F^{*-1}(w)G^{-1}(w))$ is uniformly bounded, by $\lambda$, say, and so

$$G^{*-1}(w)F^{-1}(w)F^{*-1}(w)G^{-1}(w) < \lambda I,$$

which gives us that $F(w)G^*(w)G(w)F^*(w) > \frac{1}{\lambda} I$.

\[\square\]

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