Asymptotic behavior of second-order multiagent systems with fixed configuration

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Abstract. Second-order protocols for consensus and convergence of agent’s trajectories to a fixed configuration are investigated. It is proved that the asymptotic behavior of the second-order system is uniquely determined by the eigenprojection of the Laplacian matrix of the communication digraph. The presented protocol and the corresponding expression for the asymptotic behavior are generalizations of the protocols for simpler cases.

Introduction
In recent years a significant part of articles on multiagent systems have been devoted to second-order multiagent systems [1-4]. The second order consensus protocol considered in [1] can be called a gathering protocol (rendezvous). More complex protocols of convergence of agent’s trajectories to a fixed configuration have a complex expression, and their asymptotic behavior is of great interest. The present paper is devoted to the study of the asymptotic behavior of a group of moving objects (flock) in second-order multiagent systems. By ordering the dynamical coordinates (positions and velocities), the second-order protocol is presented in such a way that its expression for asymptotic behavior has a more universal form, and a number of other protocols are derived from it as special cases.

The eigenprojection of the Laplacian matrix of the communication digraph is a key tool for studying the asymptotic behavior of a group of agents and has previously been applied to analyze the first order protocols and to regularize consensus protocols with multiple eigenvalues 0 of the Laplacian matrix.

The article is devoted to the multi-agent system of moving agents in a group. However, second-order consensus protocols are also used to improve the stability of the rotor angle of the power system [5, 6]. In [6] the synchronization of generators in power system is performed by a second order protocol with one leader.

1. Notation and auxiliary results
A digraph is called strongly connected if it contains directed paths from every vertex into every other vertex. Every maximal (by inclusion) strong subgraph of a given digraph is called its strong component or bicomponent. A basis bicomponent of a digraph is a bicomponent, such that the digraph does not have any arcs flowing into this bicomponent from outside.

A communication graph $G$ of multiagent system is a loopless directed graph whose $i$-th vertex represents an agent $i$. It has an arc $(j, i)$ with positive weight $w_{ij}$ if and only if agent $j$ affects agent $i$ with weight $w_{ij}$. 
In this article we assume that the first numbers of the vertices are assigned to the vertices of the first basis bicomponent, the next numbers are assigned to the vertices of the next basis bicomponent, etc., and then the remaining numbers are assigned to the rest of the vertices.

1.1. Asymptotic behavior of first-order multiagent systems.
Consider a continuous-time first-order multiagent system [7, 8] described by the system of equations
\[
\dot{x}_i(t) = -\sum_{j=1}^{n} w_{ij}(t) (x_i(t) - x_j(t)), \quad i = 1, \ldots, n
\]
where \(x_i(t)\) is a characteristic of agent \(i\); \(W = (w_{ij})\) is a matrix of influence between agents.

The system (1) can be written as
\[
\dot{x}(t) = -Lx(t),
\]
where \(x(t) = (x_1(t), \ldots, x_n(t))^T\) and \(L\) is the Laplacian matrix of communication digraph \(G\) with
\[
l_{ij} = -w_{ij}, \quad \text{if } j \neq i, \quad \text{and} \quad l_{ii} = \sum_{k \neq i} w_{ik}.
\]

Let \(A\) be an arbitrary square matrix, and denote the image and the kernel of \(A\) by \(\mathcal{R}(A)\) and \(\mathcal{N}(A)\), respectively. Let \(v = \text{ind } A\) be the index of \(A\), i.e., the smallest number \(k \in \{0, \ldots, n\}\) for which
\[
\text{rank}(A^{k+1}) = \text{rank}(A^k) \quad (A_0 = I), \quad \text{where } I \text{ denotes the identity matrix of order } n.
\]
The eigenprojection \(A^v\) of a matrix \(A\) that corresponds to the eigenvalue 0 (or simply the eigenprojection of \(A\)) is a projector (idempotent matrix), such that \(\mathcal{R}(A^v) = \mathcal{N}(A^0)\) and \(\mathcal{N}(A^v) = \mathcal{R}(A^0)\).

The asymptotic behavior of the system (2) is described by the following proposition.

**Proposition 1.** If \(x(t)\) is the solution of (2), then
\[
\lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{-Lt} x(0) = L^v x(0).
\]

Note that the first equality follows from the representation of the solution to a system of differential equations. The second equality is a direct consequence of the forest consensus theorem; see Theorem 1 in [9].

1.2. Asymptotic behavior of second-order consensus protocol.
Consider the second-order differential model [1]
\[
\dot{\xi}_i = \zeta_i, \quad \dot{\zeta}_i = u_i,
\]
\[
u_i = -\sum_{j=1}^{n} a_{ij} \left[ (\dot{\xi}_i - \dot{\xi}_j) + \gamma (\zeta_i - \zeta_j) \right],
\]
where \(\xi_i\) and \(\zeta_i\) denote the position and the velocity of the agent \(i\) in \(\mathbb{R}^d\), respectively, while \(u_i\) is the control, with a scaling factor \(\gamma\).

In the matrix form, this protocol can be written as
\[
\begin{bmatrix}
\ddot{\xi}(t) \\
\dot{\zeta}(t)
\end{bmatrix} = \begin{bmatrix}
0_{n \times n} & I_n \\
-L & -\gamma L
\end{bmatrix}
\begin{bmatrix}
\dot{\xi}(t) \\
\dot{\zeta}(t)
\end{bmatrix}.
\]

The goal of the protocol (4) is to guarantee the convergence \(|\dot{\xi}_i(t) - \dot{\xi}_j(t)| \to 0\) and \(|\zeta_i(t) - \zeta_j(t)| \to 0\) as \(t \to \infty\).

**Theorem 1** [10]. Let \(\gamma\) satisfy the condition\(^1\)

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\(^1\) This condition was obtained in [11].
\[ y^2 > \max_{\mu_i \neq 0} \frac{1}{\text{Re}(\mu_i)} \cdot \frac{\text{Im}^2(\mu_i)}{\text{Im}^2(\mu_i) + \text{Re}^2(\mu_i)} \]

and let the eigenvalue 0 of the Laplacian matrix \( L \) have multiplicity \( m \). If \((\xi(t), \zeta(t))^T\) is a solution of (4), then for a sufficiently large \( t \)
\[ \begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \otimes L^+ \begin{bmatrix} \xi(0) \\ \zeta(0) \end{bmatrix}, \] (5)

where \( L^+ \) is the eigenprojection of \( L \) and \( \otimes \) denotes the Kronecker product.

2. Main results
2.1. Second-order multiagent systems with fixed configuration.
Suppose that \( z \) is represented as
\[ z = \sum_{i=1}^{n} e_i \otimes \left( \xi_i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \zeta_i \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \]
where \( \xi_i \) and \( \zeta_i \) denote the position and the velocity of the agent \( i \) in \( \mathbb{R}^d \), respectively. In physical applications, \( d \) is equal to 2 or 3.

Let \( h \in \mathbb{R}^{2dn} \) be a formation configuration vector and
\[ h = \sum_{i=1}^{n} e_i \otimes h_i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]
where \( h_i \in \mathbb{R}^d \) denotes a position of the \( i \)-th agent.

Consider the second-order differential model [3, 4],
\[ \dot{y} = (I_n \otimes A)y + (L \otimes K)y, \] (6)
where \( y = z - h - 1_n \otimes a \), (for more details, see 4.2 in [3]), \( I_n \) is the \( n \)-by-\( n \) identity matrix, \( L \) is the \( n \)-by-\( n \) Laplacian matrix, and \( K \) is the feedback 2\( d \)-by-2\( d \) matrix, \( K = I_d \otimes \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \). The matrix \( A \) is the constant 2\( d \)-by-2\( d \) matrix
\[ A = I_d \otimes \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}. \]

2.2. Alternative coordinate ordering and asymptotic behavior of second-order systems with fixed configuration.
We reorder the dynamical coordinates (position and velocity) for the linear model (6) in a \( d \)-dimensional space with \( n \) agents as follows: let the first \( d \) coordinates of the vector represent the positions of the first agent, the second group of \( d \) coordinates represent the positions of the second agent, etc. The second half of the coordinates are the velocities of the agents in the same order as for the first half of the coordinates. For example, for the 3-dimensional case, the multiagent system with three agents has the following vector of coordinates (positions and velocities):
\[ (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3)^T. \]

For this reordering of dynamical coordinates, the configuration vector \( h \in \mathbb{R}^{2dn} \) is defined as
\[ h = \sum_{i=1}^{n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes e_i \otimes h_i, \]
where \( h_i \in \mathbb{R}^d \) is a position of agent \( i \) in the flock.

With this numbering of coordinates, the equations of motion of the flock are:

\[
\dot{z} = \left( \begin{array}{c} O_n \\ fL \\ gL + aI_n \end{array} \right) \otimes I_d (z - h) = (M \otimes I_d) (z - h),
\]

where

\[
M = \left( \begin{array}{c} O_n \\ fL \\ gL + aI_n \end{array} \right).
\]

After performing the coordinate change \( y = z - h \), we can write:

\[
\dot{y} = (M \otimes I_d) y.
\]  \( (7) \)

Obviously, the spectrum of system (8) consists of the spectrum of the matrix

\[
\left( \begin{array}{c} O_n \\ fL \\ gL + aI_n \end{array} \right)
\]

with multiplicity \( d \), and each eigenvalue of the linear system (8) can be calculated in terms of \( a, f, g \) and the eigenvalue of \( L \) (see (5.2) in [3] for example).

Note that protocol (4) is a special case of (8) if we put \( \sigma = -1, \alpha = 0 \) and \( d = 1 \).

**Theorem 2.** Let the parameters \( f, g \) and \( \alpha \) be selected such that the system (8) is stabilized, i.e. all eigenvalues of

\[
\left( \begin{array}{c} O_n \\ fL \\ gL + aI_n \end{array} \right)
\]

corresponding to the nonzero eigenvalues of \( L \) have real parts strictly less than zero. Then for large enough \( t \) we have

\[
y(t) \to (e^{At} \otimes L^r \otimes I_d) y(0),
\]  \( (9) \)

where \( A = \left( \begin{array}{cc} 0 & 1 \\ 1 & \alpha \end{array} \right) \).

**Proof.** According to the assumption, each nonzero eigenvalue of \( L \) corresponds to eigenvalues with negative real parts of the matrix \( M \), and each eigenvalue 0 of \( L \) corresponds to two eigenvalues: 0 and \( \alpha \) of \( M \). Let the multiplicity of the eigenvalue 0 for \( L \) be equal to \( m \) and \( L^r = (l^r_{ij}) \) be the eigenprojection of \( L \). For simplicity, we assume that \( l^r_{11}, ..., l^r_{mn} \) are linearly independent columns of the matrix \( L^r \), where the numbering corresponds to the numbers of the basis bicomponents. We normalize these vectors so that their components corresponding to the basis bicomponents are equal to one. The resulting vectors will be denoted by \( q_1, ..., q_m \). Then for each \( i = 1, ..., m \)

\[
\left( \begin{array}{c} O_n \\ fL \\ gL + aI_n \end{array} \right) \left( \begin{array}{c} q_i \\ 0 \\ 0 \end{array} \right) = 0_{2n},
\]

and

\[
\left( \begin{array}{c} O_n \\ fL \\ gL + aI_n \end{array} \right) \left( \begin{array}{c} q_i \\ aq_i \\ 0 \end{array} \right) = a \left( \begin{array}{c} q_i \\ aq_i \end{array} \right).
\]

Note that the index of the matrix \( M \) is equal to one, i.e., \( M \) has \( m \) linearly independent eigenvectors for a zero.

Suppose that \( J_M = S^{-1}MS \) is the Jordan form of \( M \) and

\[
J_M = \left( \begin{array}{ccc} 0 & \cdots & 0 \\ & a & \cdots \\ & & \ddots & a \\ & & & \ddots & a \end{array} \right),
\]

\( (10) \)
where $J_{n-2m}$ is the submatrix consisting of the Jordan blocks of $J_M$ corresponding to the eigenvalues of $M$ with negative real parts. Suppose that

$$S = \begin{pmatrix} q_1 & \cdots & q_m & q_1 & \cdots & q_m & \cdots \\ 0_n & \cdots & 0_n & aq_1 & \cdots & aq_m & \cdots \end{pmatrix},$$

where the first $m$ columns are the eigenvectors for the eigenvalue 0, and the next $m$ columns are the eigenvectors for the eigenvalue $a$.

Let $p_1, \ldots, p_m$ be the linearly independent rows of $L^+$, where the numbering corresponds to the basis bicomponents of $G$. Then

$$(p_i - a^{-1}p_i)(O_n f_L gL + aI_n) = 0_{2m}^T,$$

$$(0_n^T a^{-1}p_i)(O_n f_L gL + aI_n) = a(0_n^T a^{-1}p_i).$$

Let us prove that the left eigenvectors of $M$ form the first $2m$ rows of $S^{-1}$. Suppose that the matrix $S^{-1}$ has the structure

$$S^{-1} = \begin{pmatrix} p_1 & -a^{-1}p_1 \\ \vdots & \vdots \\ p_m & -a^{-1}p_m \\ 0_n^T & a^{-1}p_1 \\ \vdots & \vdots \\ 0_n^T & a^{-1}p_m \\ * & a^{-1}p_m \end{pmatrix},$$

and

$$M = SJ_M S^{-1},$$

$$e^{Mt} = e^{SJ_MS^{-1}t} = S e^{Jt} S^{-1}. \quad (10)$$

Then, for large enough $t$ providing $e^{-\mu t} \to 0$, where $\mu$ is an eigenvalue of $M$ corresponding to a nonzero eigenvalue of $L$, we have

$$e^{Jt} \to \begin{pmatrix} 1 \\ 1 \ e^{at} \\ \vdots \\ \vdots \ e^{at} \\ O_{n-2m} \end{pmatrix}.$$ 

It follows from (10) that

$$e^{Mt} = Se^{Jt} S^{-1} \to S \begin{pmatrix} 1 \\ 1 \ e^{at} \\ \vdots \\ \vdots \ e^{at} \\ O_{n-2m} \end{pmatrix} S^{-1}.$$
Now, consider the asymptotic behavior of the system. It follows from (11) that
\[
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
\otimes (q_1 \cdots q_m) \begin{bmatrix}
1 \\
0
\end{bmatrix}
\otimes \begin{bmatrix}
l_m \\
0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a^{-1}
\end{bmatrix}
\otimes \begin{bmatrix}
p_1 \\
p_m
\end{bmatrix}.
\]
(12)

For $a \neq 0$,
\[
(1 & 0 \\
0 & a)
\begin{bmatrix}
1 & 0 \\
0 & a^{-1}
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
t + \frac{1}{2} \left(\begin{array}{c}
a^2 \\
a^3
\end{array}\right) t^2 + \frac{1}{3!} \left(\begin{array}{c}
a^2 \\
a^3
\end{array}\right) t^3 + \ldots
\end{bmatrix}
\]
(13)

We use the following expression for the eigenprojection of $L$ without proof:
\[
(q_1 \cdots q_m) \begin{bmatrix}
p_1 \\
p_m
\end{bmatrix} = L^+.
\]

It follows from (12) and (13) that for large $t$
\[
e^{Mt} \rightarrow \begin{bmatrix}
0 & 1 \\
0 & a
\end{bmatrix} t \otimes L^+.
\]

Now, consider the asymptotic behavior of the system (8).
According to $M = Sf_{\hat{M}}S^{-1}$, we have
\[
M \otimes I_d = Sf_{\hat{M}}S^{-1} \otimes I_d = (S \otimes I_d)(I_m \otimes I_d)(S^{-1} \otimes I_d).
\]
By (11), it holds that
\[
e^{M \otimes I_d t} = e^{(S \otimes I_d)(I_m \otimes I_d)(S^{-1} \otimes I_d)t} = (S \otimes I_d)e^{(I_m \otimes I_d)t}(S^{-1} \otimes I_d),
\]
and for large $t$
\[
e^{M \otimes I_d t} \rightarrow \begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a^{-1}
\end{bmatrix}
\begin{bmatrix}
I_m \\
0
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_m
\end{bmatrix} \otimes I_d = \begin{bmatrix}
0 & 1 \\
0 & a
\end{bmatrix} t \otimes L^+ \otimes I_d.
\]
Theorem 2 is proved.

**Remark.** After performing the inverse coordinate change, we obtain the desired result for (7).

**Corollary from Theorem 2.** The expression (5) is a particular case of (9) if we put $a = 0$ and $d = 1$.

The proof of the corollary follows from the identity

$$e^{(0 \ 1) t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

**Conclusion**
In this paper we studied the asymptotic behavior of the second-order multi-agent system with fixed configuration. For simpler protocols, it was known that the consensus was determined by the left eigenvector of the eigenvalue 0 of the Laplacian matrix of the communication digraph. In [10], this result was extended to more general protocols, and it was proved that in second-order multiagent systems, the asymptotic behavior was also determined by the eigenprojection of the Laplacian matrix. Second-order multiagent systems corresponding to a group of moving agents have a more complex mathematical model. In this paper, for such systems an explicit expression for the asymptotic behavior was obtained. These results generalize the previously obtained expression for the second-order MAS with rendezvous protocol.

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