Filtration of a highly concentrated suspension in a porous medium

Liudmila Kuzmina¹ and Yuri Osipov²
¹National Research University Higher School of Economics, Moscow, Russia
²Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, Russia
E-mail: yuri-osipov@mail.ru

Abstract. The problems of filtration in porous media are in demand when strengthening foundations and building waterproof walls in rocks. Deep bed filtration of a highly concentrated monodisperse suspension in a homogeneous porous medium with size-exclusion particle retention mechanism is considered. When filtering a suspension in a porous medium, some solid particles get stuck on the porous frame and form a deposit. The concentration of suspended particles injected at the porous medium inlet decreases when moving from inlet to outlet. The mathematical model for a highly concentrated suspension in a porous medium assumes a nonlinear dependence of the deposit growth rate on the concentration of suspended particles. The exact solution to the filtration problem in implicit integral form and the Riemann invariant relating the concentrations of suspended and retained particles are obtained. The problem is solved for a linear filtration function and a general nonlinear concentration function. An asymptotic solution is constructed near the concentrations front of suspended and retained particles. It is shown that the asymptotics is close to the exact solution, the error decreases with increasing order of asymptotic expansions. The asymptotic solution explicitly defines the dependence of the solution on model parameters and can be used to solve the inverse filtration problem.

1. Introduction
Transport and retention of tiny solid particles in porous media are present in many technological processes. During the construction of the foundation to strengthen the fragile soil, liquid concrete is pumped into the rock. The solution is filtered in the pores of the soil and after solidification increases the strength of the foundation [1, 2].

When filtering a suspension in a porous medium, part of the suspended particles is retained in the pores and forms a deposit. Particle blocking is determined by many factors: gravitational and electromagnetic forces, diffusion into dead-end pores, stuck on non-smooth pore walls and others [3, 4]. If the distributions of particle and pore sizes overlap, then the predominant is the mechanical-geometric mechanism of particle retention called size-exclusion: suspended particles freely pass through large pores and get stuck in the throats of small pores with a diameter smaller than the particle size [5, 6].

The mathematical model of filtration includes the equation of mass balance of suspended and retained particles and the kinetic equation of deposit growth [7]. The deposit growth rate is proportional to the filtration function $\Lambda(s)$, which depends on the concentration of the retained particles, and the concentration function $f(c)$ depending on the concentration of suspended particles. With an increase in the retained particles concentration, the number of free vacant pores of small size decreases; therefore, the filtration function $\Lambda(s)$ decreases. Particle retention stops when all small
pores are clogged and the deposit concentration reaches its maximum value \( s = s_m \). Most often, the models use a linear filtration function called the Langmuir coefficient [8].

The form of the concentration function \( f(c) \) varies depending on the concentration of suspended particles in the suspension. At a low concentration, the particles do not interact with each other. In this case, the growth of deposit is proportional to the first degree of concentration and \( f(c) = f_c \). Models with a linear concentration function have been studied in detail. Analytical solutions were obtained for some models; for others, numerical methods are used [9, 10].

If the suspended particles concentration is high, particles interact with each other, and the concentration function is not linear. This article is devoted to the study of a model with a nonlinear concentration function. An exact and asymptotic solution to the filtration problem are constructed. A numerical calculation confirms the proximity of the asymptotics to the exact solution.

2. Mathematical model

In the domain
\[
\Omega = \{ (x,t) : 0 \leq x \leq 1, t \geq 0 \}
\]
consider the system
\[
\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} + \frac{\partial s}{\partial t} = 0, \quad \frac{\partial s}{\partial t} = \left(1 - \frac{s}{s_m}\right) f(c) \tag{1}
\]
with boundary and initial conditions
\[
x = 0: c = 1, \quad t = 0: c = 0, \quad s = 0. \tag{2}
\]

Here \( f(c) \) is a smooth increasing function, \( f(0) = 0 \).

Conditions (2) mean that a suspension of constant concentration is injected at the porous medium inlet \( x = 0 \); at the initial moment \( t = 0 \), the porous medium does not contain suspended and retained particles. The boundary between the suspension and the empty porous medium — the concentrations front of suspended and retained particles — moves from inlet to outlet \( x = 1 \) at a speed \( v = 1 \) and is given by the equation \( t = x \). In the domain \( \Omega_0 = \{ x,t : 0 \leq x \leq 1, \ 0 \leq t \leq x \} \), the solution is zero: \( c = 0, \ s = 0 \); in the domain \( \Omega = \{ x,t : 0 \leq x \leq 1, \ t \geq x \} \), the solution is positive. Since conditions (2) do not coincide at the origin, the solution \( c \) is discontinuous at the concentrations front. The solution \( s \) is continuous in the entire domain \( \Omega \) and is equal to zero at the concentrations front [11].

In the domain \( \Omega_1 \), the solutions of system (1) with conditions (2) and conditions
\[
x = 0: c = 1, \quad t = x: s = 0. \tag{3}
\]
coincide.

3. Exact solution

For a linear filtration function \( f(c) = c \), the exact solution to problem (1), (3) in the domain \( \Omega_1 \) is given by explicit formulas [12]
\[
c = \frac{1}{1 + (e^v - 1)e^{(v-1)s/s_m}}, \quad s = s_m \frac{1 - e^{(v-1)s/s_m}}{1 + (e^v - 1)e^{(v-1)s/s_m}}. \tag{4}
\]

In the general case, the solution is obtained by the method of [13]. According to condition (3), at the concentrations front \( t = x \), equation (1) takes the form
\[
\frac{\partial c}{\partial x} = -f(c).
\] (5)

Solution \( c_0(x) \) of equation (5) with condition (3)

\[
\int_{c_0(x)}^{1} \frac{dc}{f(c)} = x.
\] (6)

To obtain a solution in the domain \( \Omega_1 \), pass to the characteristic variables

\[
\tau = t - x, \quad x = x.
\]

In the domain \( \Omega_1 = \{ x, \tau : 0 \leq x \leq 1, \tau \geq 0 \} \) the system (1), (3) takes the form

\[
\frac{\partial c}{\partial x} = -\left(1 - \frac{s}{s_m}\right)f(c),
\] (7)

\[
\frac{\partial s}{\partial \tau} = \left(1 - \frac{s}{s_m}\right)f(c),
\] (8)

with conditions

\[
x = 0: \quad c = 1,
\] (9)

\[
\tau = 0: \quad s = 0.
\] (10)

Solution of the system (7) - (10) is given by formula

\[
\int_{c_0(x)}^{1} \frac{dc}{(1-c)f(c)} = \frac{\tau}{s_m},
\] (11)

and the Riemann invariant connecting solutions \( c(x, \tau), \ s(x, \tau) \) on the characteristics

\[
s = s_m \frac{c - c_0}{1 - c_0}.
\] (12)

4. Asymptotic solution
To reduce the calculations, we present the derivation of the second-order asymptotics. Assume that the function \( f(c) \) can be expanded in a series at any point \( c_0 : \ 0 < c_0 \leq 1 \)

\[
f(c) = f(c_0) + f'(c_0)(c - c_0) + f''(c_0)(c - c_0)^2/2 + ... .
\] (13)

In the vicinity of the concentrations front \( \tau = 0 \) asymptotic solution of problem (7) - (10) is constructed in a form [14, 15]

\[
c(x, \tau) = c_0(x) + c_1(x)\tau + c_2(x)\tau^2 / 2 + ... ,
\] (14)

\[
s(x, \tau) = s_0(x)\tau + s_1(x)\tau^2 / 2 + s_2(x)\tau^3 / 6 + ... .
\] (15)

Substitute the expansion (14) in the series (13)

\[
f(c) = f(c_0) + f'(c_0)c_1\tau + \left( f''(c_0)c_1^2 + f''(c_0)c_2\right)\tau^2 / 2 + ... ,
\] (16)
and the expansions (14)-(16) – in equations (7), (8)

\[ c'_1 + c'_2 \tau + c_1 \frac{\tau^2}{2} = \left( 1 - \frac{s_1 + s_2 + s_3}{s_m} \right) \left( f(c_0) + f'(c_0)c_1 + f''(c_0)c_2 \right) \frac{\tau^2}{2}, \]

\[ s_1 + s_2 \tau + s_3 \frac{\tau^2}{2} = \left( 1 - \frac{s_1 + s_2 + s_3}{s_m} \right) \left( f(c_0) + f'(c_0)c_1 + f''(c_0)c_2 \right) \frac{\tau^2}{2}. \]

A recurrent system of equations is obtained by grouping terms with equal powers of \( \tau \) and equating them to zero

\[ c'_0 = -f(c_0), \quad s_0 = f(c_0), \]

\[ c'_1 = -f'(c_0)c_1 + s_0 f(c_0), \quad s_1 = f''(c_0)c_1 - \frac{s_1}{s_m} f(c_0), \]

\[ c'_2 = -\left( f''(c_0)c_1^2 + f'(c_0)c_2 \right) + 2 \frac{s_1}{s_m} f'(c_0)c_1 + \frac{s_2}{s_m} f(c_0), \]

\[ s_2 = \left( f''(c_0)c_1^2 + f'(c_0)c_2 \right) - 2 \frac{s_1}{s_m} f'(c_0)c_1 - \frac{s_2}{s_m} f(c_0). \]

The initial conditions for ordinary differential equations follow from condition (9)

\[ x = 0: \quad c_0 = 1, \quad c_1 = 0, \quad c_2 = 0. \]

The first equation of the recurrence system coincides with equation (5); its solution is given by formula (6). Successive solving of recurrence equations yields

\[ c_1 = \frac{1}{s_m} f(c_0)(1 - c_0), \quad s_2 = \frac{f(c_0)}{s_m} \left( f'(c_0)(1 - c_0) - f(c_0) \right), \tag{17} \]

\[ c_2 = \frac{f(c_0)}{s_m^2} \left( (1 - c_0)f'(c_0) - f(c_0) \right), \tag{18} \]

\[ s_3 = \frac{1}{s_m^2} \left( f''(c_0)f(c_0) + f''(c_0)f(c_0)(1 - c_0)^2 - 4f'(c_0)f''(c_0)(1 - c_0) + f^3(c_0) \right). \tag{19} \]

Substitution of the terms (17) - (19) into expansions (14), (15) leads to the second-order asymptotic solution of the problem (7) - (10)

\[ c(x, \tau) = c_0 + \frac{1}{s_m} f(c_0)(1 - c_0) \tau + \frac{f(c_0)}{s_m^2} \left( (1 - c_0)f'(c_0) - f(c_0) \right) \frac{\tau^2}{2}, \tag{20} \]

\[ s(x, \tau) = s_0 + \frac{f(c_0)\tau}{s_m} + \frac{f(c_0)}{s_m^2} \left( f''(c_0)(1 - c_0)^2 + f''(c_0)(1 - c_0)^2 - 4f'(c_0)f''(c_0)(1 - c_0) + f^3(c_0) \right) \frac{\tau^2}{6}. \tag{21} \]

Here the function \( c_0 = c_0(x) \) is given by formula (6).
5. Numerical calculation

Let

\[ f(c) = c + c^2, \quad s_m = 1. \]  \quad (22)

Calculate the integral on the left side of equation (6) with concentration function (22)

\[ c_0(x) = \frac{1}{2e^x - 1}. \]  \quad (23)

A solution to problem (7) - (10) with the concentration function (22) is obtained by calculating the integral on the left side of equation (11)

\[ c(x, \tau) = \frac{1}{\sqrt{1 + 4e^{-\tau} (e^x - 1)}}. \]  \quad (24)

The graphs of exact and asymptotic solutions in the Cartesian variables \( x, t \) are presented in figures 1-3. Since the 4-order asymptotics is close to the exact solution and visually their graphs coincide in figures (a), enlarged fragments are shown in figures (b).

Figure 1 shows the concentration of suspended particles at the porous medium outlet \( x = 1 \): exact solutions for the linear (red) and quadratic (blue) filtration functions and the asymptotics of the second (green) and fourth (yellow) order.

![Figure 1](image1.png)

**Figure 1.** Exact and asymptotic solutions of suspended particles concentration at \( x = 1 \).

In Figure 1 b) the relative error of the fourth-order asymptotics is less than 2%.

Figure 2 shows the concentration of suspended particles at the moment \( t = 1 \): exact solutions for the linear (red) and quadratic (blue) filtration functions and the asymptotics of the second (green) and fourth (yellow) order.

![Figure 2](image2.png)

**Figure 2.** Exact and asymptotic solutions of suspended particles concentration at \( t = 1 \).

In Figure 2 b) the relative error of the fourth-order asymptotics is less than 1%.
Figure 3 shows the concentration of retained particles at the moment $t = 1$: exact solutions for the linear (red) and quadratic (blue) filtration functions and the asymptotics of the second (green) and fourth (yellow) order.

![Figure 3](image)

**Figure 3.** Exact and asymptotic solutions of retained particles concentration at $t = 1$.

In Figure 3 b) the relative error of the fourth-order asymptotics is no more than 1%.

6. Conclusions
A filtration model of a highly concentrated suspension in a porous medium includes the kinetic equation for the deposit growth with a nonlinear concentration function. Exact and asymptotic solutions are constructed. The solution is zero before the concentrations front and is positive behind the front. The asymptotics is close to the exact solution.

Analytical solutions serve to fine-tune experiments and reduce the amount of laboratory research [17, 18].

Asymptotics can be used to solve the inverse filtration problem — finding the unknown concentration function from the suspended particles concentration measured in the laboratory at the outlet of the porous sample [19-21].

For complicated filtration models of a highly concentrated suspension in a porous medium, taking into account change in porosity by the deposit growth, exact solutions are unknown. The construction of asymptotics is an effective way to obtain approximate analytical solutions that specify the dependence of the solution on the model parameters in explicit form. These problems will be considered separately.

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