A three-parameter model for the neutrino mass matrix

W. Grimus\textsuperscript{(1)}* and L. Lavoura\textsuperscript{(2)}†

\textsuperscript{(1)} Fakultät für Physik, Universität Wien
Boltzmanngasse 5, A–1090 Wien, Austria

\textsuperscript{(2)} Universidade Técnica de Lisboa and Centro de Física Teórica de Partículas
Instituto Superior Técnico, 1049-001 Lisboa, Portugal

24 May 2007

Abstract

Using the type-II seesaw mechanism with three Higgs doublets $\phi_\alpha$ ($\alpha = e, \mu, \tau$) and four Higgs triplets, we build a model for lepton mixing based on a 384-element horizontal symmetry group, generated by the permutation group $S_3$ and by six $\mathbb{Z}_2$ transformations. The charged-lepton mass matrix is diagonal; the symmetries of the model would require all the three masses $m_\alpha$ to be equal, but different vacuum expectation values of the $\phi_\alpha$ allow the $m_\alpha$ to split. The number of parameters in the Majorana neutrino mass matrix $M_\nu$ depends on two options: full breaking of the permutation group $S_3$, or leaving a $\mu-\tau$ interchange symmetry intact; and hard or spontaneous violation of $CP$. We discuss in detail the case with the minimal number of three parameters, wherein $M_\nu$ is real, symmetric under $\mu-\tau$ interchange, and has equal diagonal elements. In that case, $CP$ is conserved in lepton mixing, atmospheric neutrino mixing is maximal, and $\theta_{13} = 0$; moreover, the type of neutrino mass spectrum and the absolute neutrino mass scale are sensitive functions of the solar mixing angle.

*E-mail: walter.grimus@univie.ac.at
†E-mail: balio@cftp.ist.utl.pt
1 Introduction

There are two puzzles associated with neutrinos: why are their masses so much smaller than those of the charged fermions, and why does the lepton mixing matrix feature large mixing angles—for reviews see [1]—in contrast to the quark mixing matrix. It is possible that both puzzles are solved through the same mechanism. In this paper we envisage the type-II seesaw mechanism as a possible solution[1]. We use horizontal symmetries to enforce certain features of lepton mixing, in particular maximal atmospheric neutrino mixing and $\theta_{13} = 0$. In order to achieve this, we enlarge the scalar sector of the Standard Model by adding to it four Higgs triplets and by using altogether three Higgs doublets. Our model has a permutation group $S_3$ together with six cyclic symmetries $\mathbb{Z}_2$, which commute with each other but not with $S_3$; the result is a large discrete symmetry group with 384 elements. This setting allows to obtain four different neutrino mass matrices, depending on the assumed breaking of the horizontal symmetries and of the symmetry $CP$. Amazingly, by breaking the horizontal symmetries softly by terms of dimension two, while leaving a residual $\mu-\tau$ interchange symmetry to be broken at low energy in the charged-lepton sector, we arrive at a viable neutrino mass matrix with only three real parameters.

In section 2 we make a general discussion of the type-II seesaw mechanism with an arbitrary number of Higgs doublets and triplets. Our model, with its multiplets, symmetries, and Lagrangian is explained in section 3. In section 4 we investigate in detail the most predictive case of a three-parameter neutrino mass matrix. A generalization thereof is considered in section 5. The conclusions are presented in section 6.

2 The type-II seesaw mechanism

We first review the type-II seesaw mechanism [2, 3] for small neutrino masses. We assume the existence—in the electroweak theory—of several Higgs doublets $\phi_\alpha$ with hypercharge 1/2, together with several Higgs triplets $\Delta_i$ with hypercharge 1. Let the neutral components of the $\phi_\alpha$ have vacuum expectation values (VEVs) $v_\alpha$ and the neutral components of the $\Delta_i$ have VEVs $\delta_i$. Just because of the hypercharge symmetry, the vacuum potential $V_0$ must be of the form

$$V_0 = (\mu^2_\phi)_{\alpha\beta} v_\alpha^* v_\beta + (\mu^2_\Delta)_{ij} \delta_i^* \delta_j + (t_{i\alpha\beta})_{\delta_i^*} v_\alpha v_\beta + \text{c.c.} + \lambda_{ij\delta\delta} \delta_i^* \delta_j,$$

(1)

The matrices $\mu^2_\phi$ and $\mu^2_\Delta$ are Hermitian and, likewise, the $\lambda$ coefficients must obey various conditions in order that $V_0$ should be real. The VEVs of the triplets are determined by

$$0 = \frac{\partial V_0}{\partial \delta_i} = (\mu^2_\Delta)_{ij} \delta_j + t_{i\alpha\beta} v_\alpha v_\beta + 2\lambda_{ij\delta\delta} \delta_i^* \delta_j + \lambda_{ij\delta\delta} v_\alpha^* v_\beta \delta_j.$$

(2)

1 In our model we do not allow for a type-I seesaw mechanism; we assume right-handed neutrino singlets not to exist.

2 We use the summation convention.
Contrary to $\mu^2$, we assume the matrix $\mu^2$ to be positive definite so that, in the absence of the $t_{\alpha\beta}$ terms, the only solution to equations (2) would be for all the $\delta_i$ to vanish. The VEVs $v_\alpha$ are of order of the electroweak scale $v \approx 174$ GeV, or smaller. Assuming the $t_{\alpha\beta}$ to be of order $M$ and the eigenvalues of $\mu^2$ to be of order $M^2$, where $M$ is a mass scale much larger than $v$ \cite{3}, the approximate solution to equations (2) is given by \cite{5}

$$\delta_i \approx -\left(\mu^2\right)^{-1}_{ij} t_{j\alpha\beta} v_\alpha v_\beta. \quad (3)$$

From equation (3), the $\delta_i$ are of order $v^2/M \ll v$. If, furthermore, all the $\lambda$ coefficients are of order unity or smaller, then the approximate solution (3) will be corrected on its right-hand side only by terms suppressed by a factor $v^2/M^2 \ll 1$.

Under an $SU(2)$ gauge transformation, the left-handed lepton doublets $D_L\alpha$ transform as $D_L\alpha \rightarrow W D_L\alpha$ while the Higgs triplets transform as $\Delta_i \rightarrow W \Delta_i W^\dagger$, where $W$ is an $SU(2)$ matrix. Therefore, the Higgs triplets have Yukawa couplings of the form $D_L^\dagger \alpha C^{-1} \epsilon \Delta_i D_L\beta$, where $C$ is the charge-conjugation matrix in Dirac space and $\epsilon$ is the $2 \times 2$ antisymmetric matrix in gauge-$SU(2)$ space. The VEVs $\delta_i$ being very small, the above Yukawa couplings generate very small neutrino mass terms $\delta_i v_{L\alpha} C^{-1} \nu_{L\beta}$, of order $v^2/M$ times a typical Yukawa-coupling constant. The neutrino masses being of order 0.1 eV, $M$ could easily be of order $10^{14}$ GeV \cite{3}, thus fully justifying the approximate solution (3).

### 3 The model

Our model follows closely, in the symmetries that it utilizes, a previous model of ours \cite{6}. We have three left-handed lepton doublets $D_L\alpha$, three right-handed charged-lepton singlets $\alpha_R$, and three Higgs doublets $\phi_\alpha$ ($\alpha = e, \mu, \tau$) \cite{3}. There are four Higgs triplets, $\Delta_\alpha$ and $\Delta_4$ \cite{3}

The symmetries of the model consist of a permutation group $S_3$ acting simultaneously on all indices $\alpha$, three $Z_2$ symmetries

$$Z^{(1)}_\alpha: \quad \phi_\alpha \rightarrow -\phi_\alpha, \quad \alpha_R \rightarrow -\alpha_R, \quad (4)$$

and another three $Z_2$ symmetries

$$Z^{(2)}_\alpha: \quad D_{L\alpha} \rightarrow -D_{L\alpha}, \quad \alpha_R \rightarrow -\alpha_R, \quad \text{and } \Delta_\beta \rightarrow -\Delta_\beta \text{ iff } \beta \neq \alpha. \quad (5)$$

Notice that $\Delta_4$ is invariant under all these symmetries. In appendix A we make a study of the full symmetry group of our model. The Yukawa Lagrangian invariant under all these symmetries is

$$\mathcal{L}_{Yukawa} = -y_1 D_{L\alpha}^\dagger \phi_\alpha \alpha_R + \frac{1}{2} y_1 D_{L\alpha}^T C^{-1} \epsilon \Delta_4 D_{L\alpha} + y_2 \left(D_{L\alpha}^T C^{-1} \epsilon \Delta_\mu D_{L\tau} + D_{L\mu}^T C^{-1} \epsilon \Delta_\tau D_{L\alpha} + D_{L\tau}^T C^{-1} \epsilon \Delta_\epsilon D_{L\mu} \right) + \text{H.c.} \quad (6)$$

\cite{3} Constraints on multi-Higgs doublet models from electroweak precision tests are not very stringent: Higgs bosons with large $ZZ$ couplings must have an average mass in the range allowed for the mass of the Standard Model Higgs boson \cite{4}.

\cite{4} The scalar content of our model resembles that of the $A_4$ model of \cite{7}. However, in that model, three gauge triplets are used instead of our $\Delta_4$. 

3 Constrains on multi-Higgs doublet models from electroweak precision tests are not very stringent: Higgs bosons with large $ZZ$ couplings must have an average mass in the range allowed for the mass of the Standard Model Higgs boson \cite{4}. 

4 The scalar content of our model resembles that of the $A_4$ model of \cite{7}. However, in that model, three gauge triplets are used instead of our $\Delta_4$. 

3
Thus, the charged-lepton mass matrix is automatically diagonal, the charged lepton \( \alpha \) having mass \( m_\alpha = |y_0 v_\alpha| \). On the other hand, the neutrino mass matrix is

\[
\mathcal{M}_\nu = \begin{pmatrix}
y_1 \delta_4 & y_2 \delta_\tau & y_2 \delta_\mu \\
y_2 \delta_\tau & y_1 \delta_4 & y_2 \delta_e \\
y_2 \delta_\mu & y_2 \delta_e & y_1 \delta_4
\end{pmatrix},
\]

(7)

all its diagonal matrix elements being equal.

Due to the symmetries of our model, the coupling constants \( t_{i\alpha\beta} \) of the previous section assume the very simple form

\[
t_{i\alpha\beta} = t \delta_{i4} \delta_{\alpha\beta}.
\]

(8)

Hence, from equation (3),

\[
\delta_i = -tv_\alpha v_\alpha (\mu_\Delta^2)^{-1}.
\]

(9)

Ordering the triplet fields as \((\Delta_e, \Delta_\mu, \Delta_\tau, \Delta_4)\), the symmetries of our model would enforce

\[
\mu_\Delta^2 = \text{diag}(\mu_1^2, \mu_2^2, \mu_1^2, \mu_2^2),
\]

(10)

which is not satisfactory since it would lead, through equation (9), to \( \delta_e = \delta_\mu = \delta_\tau = 0 \). We must have \((\mu_\Delta^2)^{-1} \neq 0 \) for \( i = e, \mu, \tau \). In order to solve this problem, we assume the symmetries of the model to be broken softly, only by terms of dimension two. Without any residual symmetry, this means that both matrices \( \mu_\Delta^2 \) and \( \mu_\phi^2 \) become fully general, while all other couplings remain unchanged.

However, in order to simplify our model and render it more predictive, we may assume that the interchange symmetry \( \mu \leftrightarrow \tau \) is kept unbroken in \( \mu_\Delta \) and \( \mu_\phi \). Then,

\[
(\mu_\Delta^2)^{-1} = \begin{pmatrix}
a & b & b & c \\
* & d & e & f \\
* & e & d & f \\
* & f* & f* & g
\end{pmatrix}
\]

(11)

(\( a, d, e, \) and \( g \) are real), so that \( \delta_e = -tcv_\alpha v_\alpha \), \( \delta_\mu = \delta_\tau = -tfv_\alpha v_\alpha \), and the neutrino mass matrix is \( \mu-\tau \) symmetric. This immediately leads to the predictions \( \theta_{23} = \pi/4 \) and \( \theta_{13} = 0 \). The \( \mu-\tau \) interchange symmetry is supposed to be spontaneously broken through the VEVs of the Higgs doublets: \( v_\mu \neq v_\tau \). The Higgs potential is rich enough to allow for this outcome—in appendix B we demonstrate this by working out a simplified case. Of course, the spontaneous breaking at the electroweak scale of the \( \mu-\tau \) interchange symmetry will seep, through radiative corrections, into the rest of the theory, so that at loop level the matrix \((\mu_\Delta^2)^{-1}\) will not any more be of the form in equation (11), and then \( \delta_\mu \neq \delta_\tau \). But, both because this is a loop effect, and because it is a correction of order of the ratio of the electroweak scale to the much larger mass terms in \( \mu_\Delta^2 \), we may expect \( \delta_\mu - \delta_\tau \) to remain negligible.

In a further simplification of our model, we may also assume \( CP \) violation to be spontaneous: the matrix \((\mu_\Delta^2)^{-1}\) is then real, but the VEVs \( v_\alpha \) display non-trivial relative phases. Then, \( \delta_e, \delta_\mu, \delta_\tau, \) and \( \delta_4 \) will all have the same phase—the phase of \( v_\alpha v_\alpha \). That
phase may be rephased away from $\mathcal{M}_\nu$, so that the neutrino mass matrix becomes real. Thus, spontaneous $CP$ breaking in our model yields the remarkable outcome that, even though there is $CP$ violation, it remains absent from the mass matrices and from lepton mixing.

One thus obtains the following four possibilities:

1. The general case, in which $CP$ violation is hard and $\mu-\tau$ symmetry is allowed to be broken in $\mu^2_\Delta$. Then,

\[
\mathcal{M}_\nu = \begin{pmatrix} m & p e^{i\psi} & q e^{i\chi} \\
p e^{i\psi} & m & r e^{i\rho} \\
q e^{i\chi} & r e^{i\rho} & m \end{pmatrix},
\]

with real $m$, $p$, $q$, and $r$. This case should not be very predictive, since it has seven parameters to predict nine observables—three neutrino masses, three lepton mixing angles, one CKM-type phase, and two Majorana phases.

2. The case in which $\mu-\tau$ symmetry is allowed to be broken in $\mu^2_\Delta$ but $CP$ violation is spontaneous. Then, $\psi$, $\chi$, and $\rho$ in (12) vanish. There is no $CP$ violation in lepton mixing. The four parameters $m$, $p$, $q$, and $r$ allow one to predict six observables—three neutrino masses and three lepton mixing angles.

3. The case in which $\mu-\tau$ interchange symmetry is preserved in $\mu^2_\Delta$, while $CP$ violation is allowed to be hard. Then,

\[
\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\
y & x & w \\
y & w & x \end{pmatrix},
\]

with complex parameters $x$, $y$, and $w$. There are in this case five parameters—three moduli and two phases.

4. The most predictive case, in which $CP$ violation is spontaneous and $\mu-\tau$ interchange symmetry is preserved down to the electroweak scale. The neutrino mass matrix is the one in equation (13) but with real $x$, $y$, and $w$. The neutrino mass matrix has only three parameters.

4 The three-parameter neutrino mass matrix

In this section we concentrate on case 4 of the previous section, i.e. on the neutrino mass matrix of equation (13) with real $x$, $y$, and $w$. The algebra of the diagonalization of a general $\mu-\tau$ symmetric neutrino mass matrix has been worked out in [9], and we only need to adapt it to the simpler case 4. In the following, the solar mixing angle—which is defined to be in the first quadrant—is denoted $\theta$, the neutrino masses are $m_{1,2,3}$, the

\footnote{This is not an original situation; in the classical Branco model of $CP$ violation [12], spontaneous $CP$ breaking also does not find a way into the quark mixing matrix.}
solar mass-squared difference is \( \Delta m_{\odot}^2 = m_2^2 - m_1^2 > 0 \), and the atmospheric mass-squared difference is
\[
\Delta m_{\text{atm}}^2 = \left| m_3^2 - \frac{m_1^2 + m_2^2}{2} \right| = \epsilon \left( m_3^2 - \frac{m_1^2 + m_2^2}{2} \right), \tag{14}
\]
where \( \epsilon = +1 \) indicates a normal neutrino mass ordering and \( \epsilon = -1 \) an inverted ordering.

Equations (3.9)–(3.11) and (3.15) of [9] yield, respectively,
\[
\begin{align*}
m_3 &= \left| x - w \right|, \tag{15} \\
m_{1,2}^2 &= \frac{x^2 + 4y^2 + (x + w)^2 \mp \Delta m_{\odot}^2}{2}, \tag{16} \\
\tan 2\theta &= \frac{2\sqrt{2}|y|}{w} \sign (2x + w), \tag{17} \\
\Delta m_{\odot}^2 \cos 2\theta &= w(2x + w). \tag{18}
\end{align*}
\]

Experimentally we know that \( \theta \) is in the first octant. Hence \( \tan 2\theta > 0 \) and equations (17) and (18) both give
\[
\sign (2x + w) = \sign w. \tag{19}
\]

Again from equations (17) and (18),
\[
\begin{align*}
|y| &= \frac{|w| \tan 2\theta}{2\sqrt{2}}, \tag{20} \\
x &= \frac{\Delta m_{\odot}^2 \cos 2\theta - w^2}{2w}. \tag{21}
\end{align*}
\]

From equations (14), (15), and (16), we find
\[
\epsilon \Delta m_{\text{atm}}^2 = \frac{w^2}{2} - 3xw - 2y^2. \tag{22}
\]

Inserting equations (20) and (21) into equation (22), we obtain the value of \( w \):
\[
w^2 = \frac{4\epsilon \Delta m_{\text{atm}}^2 + 6\Delta m_{\odot}^2 \cos 2\theta}{8 - \tan^2 2\theta}. \tag{23}
\]

Since \( \Delta m_{\text{atm}}^2 \gg \Delta m_{\odot}^2 \), the numerator of equation (23) has the sign of \( \epsilon \); hence its denominator must also have the sign of \( \epsilon \). That denominator vanishes when \( \sin^2 \theta = 1/3 \), i.e. when \( \theta \) is just the Harrison–Perkins–Scott (HPS) solar mixing angle [10]. We thus conclude that, in our model,

- if the neutrino mass spectrum is normal, then the solar mixing angle is smaller than its HPS value;
- if the neutrino mass spectrum is inverted, then \( \sin^2 \theta > 1/3 \).

This remarkable result relates the type of neutrino mass spectrum to the value of the solar mixing angle.
From equations (15), (16), (20), and (21),

\[ m_{1,2} = \frac{1}{2} \left( \frac{|w|}{\cos 2\theta} + \frac{\Delta m^2_\odot \cos 2\theta}{|w|} \right) \quad \text{and} \quad m_3 = \frac{3|w|}{2} - \frac{\Delta m^2_\odot \cos 2\theta}{2|w|}. \] (24)

This gives the overall scale of the neutrino masses. Since \(|w|\) diverges when \(\tan^2 2\theta \to 8\), we see that in our model the neutrino mass spectrum becomes quasi-degenerate when the solar mixing angle approaches its Harrison–Perkins–Scott value.

One easily sees the reason why our model displays a singularity when \(\sin^2 \theta = 1/3\). The most general neutrino mass matrix leading to HPS lepton mixing is

\[
\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & x + u & y - u \\ y & y - u & x + u \end{pmatrix}. \] (25)

Our \(\mathcal{M}_\nu\) in equation (13) has equal diagonal matrix elements. Hence, if it were to accept \(\sin^2 \theta = 1/3\), it would have to correspond to \(u = 0\) in equation (25). But the \(\mathcal{M}_\nu\) of equation (25) with \(u = 0\) leads to two equal neutrino masses, hence it is unrealistic.

Experimentally \(\sin^2 \theta\) is close to 1/3, therefore there is the danger that our neutrino masses are too large and saturate the cosmological bound [13]. As a numerical exercise, we take the 1\(\sigma\) bound on solar mixing from [14]:

\[ 0.27 < \sin^2 \theta < 0.32 \quad \Leftrightarrow \quad 3.73 < \tan^2 2\theta < 6.72. \] (26)

The mean value of \(\theta\) is given by \(\sin^2 \theta = 0.30\) and \(\tan^2 2\theta = 5.25\). Note that the upper 2\(\sigma\) limit \(\sin^2 \theta = 0.36\) gives \(\tan^2 2\theta = 11.76\), which is already significantly larger than 8. Thus, there is experimentally ample room for the neutrino masses to be sufficiently small. This happens because \(\tan^2 2\theta\) is a rapidly varying function of \(\theta\).

In figure 1 we have plotted \(m_1, m_3,\) and \(m_1 + m_2 + m_3\) against \(\sin^2 \theta\) in our model. We have used the best-fit values \(\Delta m^2_{\text{atm}} = 2.5 \times 10^{-3}\) eV\(^2\) and \(\Delta m^2_\odot = 7.9 \times 10^{-5}\) eV\(^2\) from [13]; for \(\theta\) we have used the 3\(\sigma\) bounds \(0.24 < \sin^2 \theta < 0.40\), from the same source.

Another important observable is \(m_{\beta\beta}\), the effective mass relevant for neutrinoless 2\(\beta\) decay. This is equal to the modulus of the \((e,e)\) matrix element of \(\mathcal{M}_\nu\), i.e., in our case, to \(|x|\). Thus,

\[ m_{\beta\beta} = \frac{|w|}{2} - \frac{\Delta m^2_\odot \cos 2\theta}{2|w|}. \] (27)

Since

\[ |w| \approx 2 \sqrt{\frac{\Delta m^2_{\text{atm}}}{|8 - \tan^2 2\theta|}} \gg \sqrt{\Delta m^2_\odot \cos 2\theta}, \] (28)

we see that in our model we have the relation

\[ m_{\beta\beta} \approx m_3/3. \] (29)

\(\text{A different model in which the neutrino mass spectrum is normal or inverted depending on whether }\sin^2 \theta\text{ is smaller or larger than }1/3,\text{ and the neutrinos become degenerate in the limit }\sin^2 \theta \to 1/3,\text{ has been suggested in }[15].\)
Figure 1: Plot of $m_1$, $m_3$, and $\sum_i m_i$ as a function of $\sin^2 \theta$ in our three-parameter model. The values of $\Delta m^2_{\text{atm}}$ and $\Delta m^2_{\odot}$ have been fixed at $2.5 \times 10^{-3}$ and $7.9 \times 10^{-5}$, respectively, in eV$^2$. The dashed vertical line indicates the singularity at $\sin^2 \theta = 1/3$.

This same relation has recently been obtained in a different model [16].

One may ask oneself whether the neutrino mass matrix displays any characteristic texture in the limit $\tan^2 2\theta \to 8$. A glance at equations (20), (21), and (23) allows one to conclude that, in that limit, all the matrix elements of $\mathcal{M}_\nu$ diverge. Moreover, from equations (21) and (20), respectively, we obtain

\[
\frac{x}{w} \to -\frac{1}{2}, \quad \frac{|y|}{|w|} \to 1
\]

for $\tan^2 2\theta \to 8$, from where the texture of $\mathcal{M}_\nu$ in that limit can be read off.

5 Extension to the complex case

In this section we investigate what happens when one allows the neutrino mass matrix of equation (13) to have complex matrix elements. Does the intriguing feature of quasi-degenerate neutrinos in the limit of HPS mixing, found in the previous section for the case of real matrix elements, still hold true? We shall see that it does not; indeed, the general neutrino mass matrix (13) does not seem to have much predictive power beyond $U_{e3} = 0$ and maximal atmospheric neutrino mixing.
The symmetric matrix
\[
\mathcal{M}_\nu = \begin{pmatrix}
x & y & y \\
y & z & w \\
y & w & z \\
\end{pmatrix}
\]

is diagonalized in the following way:
\[
U^T \mathcal{M}_\nu U = \text{diag} (m_1, m_2, m_3),
\]

the matrix \( U \) being unitary while the \( m_j \) \((j = 1, 2, 3)\) are real and non-negative. Due to the special form of \( \mathcal{M}_\nu \), wherein \((\mathcal{M}_\nu)_{12} = (\mathcal{M}_\nu)_{13}\) and \((\mathcal{M}_\nu)_{22} = (\mathcal{M}_\nu)_{33}\), \( U \) is of the form
\[
U = \text{diag} \left( e^{i\varphi}, e^{i\vartheta}, e^{i\vartheta} \right) \begin{pmatrix} c & s & 0 \\ -r s & r c & r \\ -r s & r c & -r \end{pmatrix} \text{diag} \left( e^{i\Sigma_1}, e^{i\Sigma_2}, e^{i\Sigma_3} \right),
\]

where \( c = \cos \theta \), \( s = \sin \theta \), and \( r = 2^{-1/2} \). From equations (31)–(33), we find
\[
x = e^{-2i\varphi} \left( c^2 m_1 e^{-2i\Sigma_1} + s^2 m_2 e^{-2i\Sigma_2} \right),
\]
\[
z = \frac{e^{-2i\vartheta}}{2} \left( s^2 m_1 e^{-2i\Sigma_1} + c^2 m_2 e^{-2i\Sigma_2} + m_3 e^{-2i\Sigma_3} \right).
\]

We define \( \bar{m}_j \equiv m_j e^{-2i\Sigma_j} \) for \( j = 1, 2, 3 \). We also define \( \chi \equiv 2(\vartheta - \varphi) \). Then, the condition \( x = z \), which makes the \( \mathcal{M}_\nu \) of equation (31) identical with the one of equation (13), is equivalent to
\[
\bar{m}_1 \left( 2c^2 e^{i\chi} - s^2 \right) + \bar{m}_2 \left( 2s^2 e^{i\chi} - c^2 \right) - \bar{m}_3 = 0.
\]

Thus, the condition \( x = z \) is equivalent to the existence of four phases \( \chi \) and \( \Sigma_{1,2,3} \) such that the condition (36) is satisfied. That condition states that it is possible to draw a triangle in the complex plane, the sides of that triangle having lengths \( \sqrt{A} m_1 \), \( \sqrt{B} m_2 \), and \( m_3 \), where
\[
A = 4c^4 + s^4 - 4c^2 s^2 \cos \chi,
\]
\[
B = 4s^4 + c^4 - 4c^2 s^2 \cos \chi.
\]

Therefore, one may eliminate the phases \( \Sigma_{1,2,3} \) from condition (36) by writing the sole “triangle inequality” [17]
\[
A_1 m_1^4 + B_1 m_2^4 + m_3^4 - 2 \left( AB m_1^2 m_2^2 + A m_1^2 m_3^2 + B m_2^2 m_3^2 \right) \leq 0.
\]

Using
\[
m_1^2 = m_3^2 - \epsilon \Delta m_{\text{atm}}^2 - \frac{1}{2} \Delta m_{\odot}^2,
\]
\[
m_2^2 = m_3^2 - \epsilon \Delta m_{\text{atm}}^2 + \frac{1}{2} \Delta m_{\odot}^2,
\]

the inequality (38) takes the form
\[
k_1 m_1^4 + 2k_2 m_3^2 + k_0 \leq 0,
\]
where

\[
\begin{align*}
    k_4 &= 1 - 2(A + B) + (A - B)^2, \\
    k_2 &= [A + B - (A - B)^2] \epsilon \Delta m_{\text{atm}}^2 + \frac{1}{2} (A - B) (1 - A - B) \Delta m_{\odot}^2, \\
    k_0 &= (A - B) \epsilon \Delta m_{\text{atm}}^2 + \frac{1}{2} (A + B) \Delta m_{\odot}^2.
\end{align*}
\]

Since \( k_0 > 0 \), the inequality \((40)\) does not tolerate \( m_3 = 0 \); hence, there is a non-trivial lower bound on the neutrino masses. We want to find the numerical value of that bound. Using the values of \( A \) and \( B \) in equations \((37)\), one finds that

\[
\begin{align*}
    k_4 &= -16c^2 s^2 (1 - \cos \chi), \\
    k_2 &= 2 (-2 + 13c^2 s^2 - 4c^2 s^2 \cos \chi) \epsilon \Delta m_{\text{atm}}^2 + 3 (c^2 - s^2) (-2 + 5c^2 s^2 + 4c^2 s^2 \cos \chi) \Delta m_{\odot}^2, \\
    k_0 &= \left[ 3 (c^2 - s^2) \epsilon \Delta m_{\text{atm}}^2 + \frac{1}{2} (5 - 10c^2 s^2 - 8c^2 s^2 \cos \chi) \Delta m_{\odot}^2 \right]^2.
\end{align*}
\]

The case of real \( x, y, \) and \( w \) corresponds to \( \cos \chi = +1 \). In (and only in) that case, the left-hand side of the inequality \((40)\) becomes linear in \( m_3^2 \); besides, in that case \( k_2 \) vanishes when \( c^2 s^2 = 2/9 \), thereby generating singularities at the points \( s^2 = 1/3 \) and \( s^2 = 2/3 \), as we saw in the previous section.

For \( \cos \chi \neq +1 \), \( k_4 \) is negative. Since \( k_0 \) is always positive, the inequality \((40)\) then yields

\[
m_3^2 \geq \frac{\sqrt{k_2^2 + |k_4| k_0 + k_2}}{|k_4|} \equiv L.
\]

The task now consists in finding the minimum value of \( L \) as a function of \( \cos \chi \) (and of \( \epsilon = \pm 1 \)); that minimum value provides the lower bound on \( m_3^2 \). It is easy to convince oneself that \( L \) always has its minimum when \( \cos \chi = -1 \), for all experimentally allowed values of \( s^2, \Delta m_{\text{atm}}^2, \) and \( \Delta m_{\odot}^2 \). Computing \( L \) as a function of \( s^2 \) for fixed \( \cos \chi = -1, \Delta m_{\text{atm}}^2 = 2.5 \times 10^{-3} \) eV, and \( \Delta m_{\odot}^2 = 7.9 \times 10^{-5} \) eV, we conclude the following:

- When the neutrino mass spectrum is normal, i.e. when \( \epsilon = +1 \), the minimum value of the lowest neutrino mass, \( m_1 \), hardly varies with \( s^2 \). One has \( m_1 > 1.679 \times 10^{-2} \) eV for \( s^2 = 0.24 \) and \( m_1 > 1.665 \times 10^{-2} \) eV for \( s^2 = 0.40 \).

- When the neutrino mass spectrum is inverted, i.e. when \( \epsilon = -1 \), the minimum value of the lowest neutrino mass, \( m_3 \), varies strongly as a function of \( s^2 \). One has \( m_3 > 2.9 \times 10^{-2} \) eV for \( s^2 = 0.24 \), \( m_3 > 9 \times 10^{-3} \) eV for \( s^2 = 0.40 \).

Thus, the mass matrix of equation \((13)\) with complex \( x, y, \) and \( w \) is not very predictive: it only allows one to derive a rather mild lower bound on the neutrino masses. There is also no prediction for the effective mass \( m_{\beta\beta} \), except for the rather trivial bounds

\[
|m_1^2 c^2 - m_2^2 s^2| \leq m_{\beta\beta} \leq |m_1^2 c^2 + m_2^2 s^2|.
\]
6 Conclusions

In this paper we have constructed an extension of the Standard Model with three Higgs doublets \( \phi_\alpha \) and four scalar gauge triplets \( \Delta_\alpha \) and \( \Delta_4 \). The scalar triplets generate a type-II seesaw mechanism, thus explaining the smallness of the neutrino masses. We have employed a large horizontal symmetry group \( G \), generated by the permutation group \( S_3 \) of the indices \( \alpha \) and by six cyclic groups \( \mathbb{Z}_2 \). After spontaneous symmetry breaking, the charged-lepton mass matrix is diagonal; the different VEVs of the \( \phi_\alpha \) allow for different charged-lepton masses \( m_\alpha (\alpha = e, \mu, \tau) \). In order to obtain a realistic neutrino mass matrix \( M_\nu \), we additionally allow for soft breaking of \( G \), through terms of dimension two in the scalar potential. A crucial feature of our model is the equality among the diagonal entries of \( M_\nu \)—this is one of the reasons for the predictiveness of the model.

There are two relevant options: breaking \( G \) softly in the mass matrix of the scalar triplets either fully or keeping a \( \mu \leftrightarrow \tau \) symmetry intact; and having either hard or spontaneous \( CP \) breaking. Our model has the interesting property that spontaneous \( CP \) violation has no effect on \( M_\nu \), i.e. it does not generate any physical phases in lepton mixing. The most predictive scenario combines the preservation of \( \mu-\tau \) interchange symmetry with spontaneous \( CP \) violation, in which case we arrive at a viable neutrino mass matrix which has only three (real) parameters. This neutrino mass matrix leads to the usual predictions of \( \mu-\tau \) symmetric neutrino mass matrices, namely maximal atmospheric mixing and \( \theta_{13} = 0 \)—hence no \( CP \) violation in neutrino oscillations. Besides, the \( CP \) property mentioned before also prevents Majorana phases in our case.

The solar mixing angle \( \theta \) is undetermined. Our three-parameter neutrino mass matrix predicts the neutrino masses \( m_j \) as functions of the two mass-squared differences and of \( \theta \). For \( \sin^2 \theta < 1/3 \), which seems to be preferred by the data, we have a normal spectrum, while for \( \sin^2 \theta > 1/3 \) the neutrino mass spectrum is inverted. When \( \sin^2 \theta \rightarrow 1/3 \) all the \( m_j \) diverge—see figure [1]. As for the effective mass \( m_{\beta\beta} \) of neutrinoless \( 2\beta \) decay, our three-parameter mass matrix predicts \( m_{\beta\beta} \approx m_3/3 \).

Acknowledgements: We thank Ernest Ma for suggestions and discussions that contributed significantly to this work. W.G. is grateful to H. Urbantke for discussions on finite groups. The work of L.L. was supported by the Portuguese Fundação para a Ciência e a Tecnologia through the projects POCTI/FNU/44409/2002, POCTI/FP/63415/2005, and U777–Plurianual.

A The group structure of our model

In this appendix we attempt a mathematical description of the full symmetry group of our model and of its irreducible representations (irreps). Clearly, \( S_3 \) commutes neither with the \( z^{(1)} \) of equation (4) nor with the \( z^{(2)} \) of equation (5), thus the full symmetry group is rather complicated.

Let us define

\[
\begin{align*}
n_1 &= \text{diag}(-1, 1, 1), & n_2 &= \text{diag}(1, -1, 1), & n_3 &= \text{diag}(1, 1, -1),
\end{align*}
\]

(A1)
\[
c_+ = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad c_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (A2)
\]
\[
t_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (A3)
\]

Then,
\[
N = \{ \mathbb{1}, n_1, n_2, n_3, n_1n_2, n_2n_3, n_3n_1, -\mathbb{1} \} \quad (A4)
\]
forms an Abelian group isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). Also,
\[
\hat{S}_3 = \{ \mathbb{1}, c_+, c_-, t_1, t_2, t_3 \} \quad (A5)
\]
forms a three-dimensional (reducible) representation of \( S_3 \).

Let us call \( G \) the symmetry group utilized in this paper. \( G \) may be defined to be the group of the \( 6 \times 6 \) matrices
\[
\begin{pmatrix} ms & 0 \\ 0 & ns \end{pmatrix}, \quad m, n \in N, \quad s \in \hat{S}_3. \quad (A6)
\]
This defining reducible representation of \( G \) may be called the \( 6 \). Clearly, \( G \) has \( 8 \times 8 \times 6 = 384 \) elements.\(^7\) Calling \((m, n, s)\) the abstract element of \( G \) which is represented in the \( 6 \) by the matrix of equation (A6), the group multiplication law is
\[
(m_1, n_1, s_1) (m_2, n_2, s_2) = (m_1s_1m_2s_1^{-1}, n_1s_1n_2s_1^{-1}, s_3s_2). \quad (A7)
\]
From this group multiplication law it follows that \( G \) has eight one-dimensional irreps:
\[
\mathbf{1}^{(p,q,r)} : (m, n, s) \rightarrow (\det m)^p (\det n)^q (\det s)^r, \quad \text{with} \quad p, q, r \in \{0, 1\}. \quad (A8)
\]
It is obvious from equation (A6) that the matrices \( ms \) give a three-dimensional irrep of \( G \), and similarly with the matrices \( ns \). The matrices \( mns \) give one further three-dimensional irrep of \( G \), since
\[
m_1n_1s_1m_2n_2s_2 = m_3n_3s_3, \quad \text{with} \quad m_3 = m_1s_1m_2s_1^{-1}, \quad n_3 = n_1s_1n_2s_1^{-1}, \quad \text{and} \quad s_3 = s_1s_2 \quad (A9)
\]
complies with the multiplication law (A7). Thus, \( G \) has 24 three-dimensional irreps:
\[
\mathbf{2}_1^{(p,q,r)} : (m, n, s) \rightarrow (\det m)^p (\det n)^q (\det s)^r ms, \quad (A10)
\]
\[
\mathbf{2}_2^{(p,q,r)} : (m, n, s) \rightarrow (\det m)^p (\det n)^q (\det s)^r ns, \quad (A11)
\]
\[
\mathbf{2}_3^{(p,q,r)} : (m, n, s) \rightarrow (\det m)^p (\det n)^q (\det s)^r mns, \quad (A12)
\]
with \( p, q, r \in \{0, 1\} \).

\(^7\)The \( 8 \times 6 = 48 \) matrices \( ms \), where \( m \in N \) and \( s \in \hat{S}_3 \), form the Coxeter group \( B_3 \). (We thank E. Ma for drawing our attention to Coxeter groups.) We may write \( G = N \times B_3 = (N \times N) \rtimes S_3 \), the symbol \( \rtimes \) denoting a semi-direct product.
The three-dimensional representations of $G$ that we employ in our model are

\[
\begin{align*}
ms & \text{ for } (\phi_e, \phi_\mu, \phi_\tau), \\
\text{mns} & \text{ for } (e_R, \mu_R, \tau_R), \\
ns & \text{ for } (D_{Le}, D_{L\mu}, D_{L\tau}), \\
(\det n) \ns & \text{ for } (\Delta_e, \Delta_\mu, \Delta_\tau).
\end{align*}
\] (A13)

Our group $G$ also has two and six-dimensional irreps, which are not used in our model. Next we include, for completeness, their construction.

The group $S_3$ has a two-dimensional irrep $D_2$, generated by

\[
t_1 \rightarrow \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad t_2 \rightarrow \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix},
\] (A14)

where $\omega = (-1 + i\sqrt{3}) / 2$. Note that $(\det s) D_2(s)$ is isomorphic to $D_2(2)$:

\[
(\det s) D_2(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D_2(s) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (A15)

Therefore, $G$ has four two-dimensional irreps:

\[
2^{(p,q)} : \quad (m, n, s) \rightarrow (\det m)^p (\det n)^q D_2(s), \quad \text{with } p, q \in \{0, 1\}.
\] (A16)

The remaining irreps of $G$ are four six-dimensional ones,

\[
6^{(p,q)} : \quad (m, n, s) \rightarrow (\det m)^p (\det n)^q D_6(m, n, s), \quad \text{with } p, q \in \{0, 1\}.
\] (A17)

The irrep $D_6(m, n, s)$ is found in the decomposition of the product of the irreps $ms$ and $ns$. Suppose there is a space $\mathbb{C}^3$ spanned by $e_{1,2,3}$ transforming like $ms$, and another space $\mathbb{C}^3$ spanned by $e'_{1,2,3}$ transforming like $ns$. Then, the space spanned by the $e_k \otimes e'_k$ ($k = 1, 2, 3$) transforms like $mns$, while the $e_j \otimes e'_k$ with $j \neq k$ span a space which transforms like $D_6(m, n, s)$. It can be shown that this representation $D_6(m, n, s)$ of $G$ is irreducible, and also that it is equivalent to $(\det s) D_6(m, n, s)$.

The group $G$ has the interesting property that it has no faithful irreps. It is obvious that the irreps with dimensions three and lower are not faithful. The six-dimensional irreps are not faithful either, as we now explain. Defining the elements $a$ and $b$ of $G$ by $a = (-1, 1, 1)$ and $b = (1, -1, 1)$, then $a$, $b$, and $ab$ generate the subgroups $\mathbb{Z}_2^{(a)}$, $\mathbb{Z}_2^{(b)}$, and $\mathbb{Z}_2^{(ab)}$ of $G$, respectively. The isomorphisms

\[
6^{(0,0)} \cong G / \mathbb{Z}_2^{(ab)}, \quad 6^{(1,0)} \cong G / \mathbb{Z}_2^{(a)}, \quad 6^{(0,1)} \cong G / \mathbb{Z}_2^{(b)}, \quad 6^{(1,1)} \cong G / \left(\mathbb{Z}_2^{(a)} \times \mathbb{Z}_2^{(b)}\right)
\] (A18)

are easy to demonstrate. Thus, none of the six-dimensional irreps represents $G$ faithfully.

B  Spontaneous breaking of the $\mu-\tau$ symmetry

Let us consider a simplified model with only two VEVs, $v_\mu$ and $v_\tau$. We assume the following symmetries:

\[
\begin{align*}
z_1 : \quad & v_\mu \rightarrow -v_\mu, \quad v_\tau \rightarrow v_\tau; \\
z_2 : \quad & v_\mu \rightarrow v_\mu, \quad v_\tau \rightarrow -v_\tau; \\
z_3 : \quad & v_\mu \leftrightarrow v_\tau.
\end{align*}
\] (B1)
The symmetries $z_{1,2}$ are assumed to be softly broken by terms of dimension two, while $z_3$ is assumed to be exactly conserved. For the sake of clarity we also assume all coefficients to be real. Then,

\[ V_0 = a \left( |v_\mu|^2 + |v_\tau|^2 \right) + b \left( v_\mu v_\tau + v_\tau^* v_\mu \right) + \lambda \left( |v_\mu|^2 + |v_\tau|^2 \right)^2 + \lambda' |v_\mu|^2 |v_\tau|^2. \]  

(B2)

Only the $b$ term breaks $z_{1,2}$ softly.

Without loss of generality we take $v_\mu$ to be real and positive, writing

\[ v_\mu = \nu \cos \phi, \quad v_\tau = \nu \sin \phi e^{i\alpha}, \]  

(B3)

with $\nu > 0$ and $\phi$ in the first quadrant. Then we obtain

\[ V_0 = a \nu^2 + b \nu^2 \sin 2\phi \cos \alpha + \lambda \nu^4 + \frac{\lambda' \nu^2}{4} \sin^2 2\phi. \]  

(B4)

We require that

\[ 0 = \frac{\partial V_0}{\partial (2\phi)} = (\nu^2 \cos 2\phi) \left( b \cos \alpha + \frac{\lambda' \nu^2}{2} \sin 2\phi \right). \]  

(B5)

The solution $\cos 2\phi = 0$ corresponds to $|v_\mu| = |v_\tau|$ and is undesirable. But there is another solution,

\[ \sin 2\phi = -\frac{2b \cos \alpha}{\lambda' \nu^2}, \]  

(B6)

which we adopt. Since the minimization of $V_0$ in equation (B4) with respect to $\alpha$ leads to $b \cos \alpha = -|b|$ being negative, we must assume $\lambda'$ to be positive. If

\[ |b| \ll \frac{\lambda' \nu^2}{2}, \]  

(B7)

which corresponds to the soft-breaking term being very small, then $\sin 2\phi \ll 1$ and $|v_\mu| \ll |v_\tau|$ can be realized.

References

[1] M. Maltoni, T. Schwetz, M.A. Tórtola and J.W.F. Valle, Status of global fits to neutrino oscillations, New J. Phys. 6 (2004) 122 [hep-ph/0405172];
G.L. Fogli, E. Lisi, A. Marrone and A. Palazzo, Global analysis of three-flavor neutrino masses and mixings, Prog. Part. Nucl. Phys. 57 (2006) 742 [hep-ph/0506083].

[2] J. Schechter and J.W.F. Valle, Neutrino masses in $SU(2) \times U(1)$ theories, Phys. Rev. D 22 (1980) 2227;
G. Lazarides, Q. Shafi and C. Wetterich, Proton lifetime and fermion masses in an $SO(10)$ model, Nucl. Phys. B 181 (1981) 287;
R.N. Mohapatra and G. Senjanović, Neutrino masses and mixings in gauge models with spontaneous parity violation, Phys. Rev. D 23 (1981) 165;
R.N. Mohapatra and P. Pal, Massive neutrinos in physics and astrophysics (World Scientific, Singapore, 1991), p. 127.
[3] E. Ma and U. Sarkar, *Neutrino masses and leptogenesis with heavy Higgs triplets*, Phys. Rev. Lett. **80** (1998) 5716 [hep-ph/9802445].

[4] J.R. Espinosa and J.F. Gunion, *A no-lose theorem for Higgs searches at a future linear collider*, Phys. Rev. Lett. **82** (1999) 1084 [hep-ph/9807275];
J.F. Gunion, H.E. Haber and R. Van Kooten, *Higgs physics at the linear collider*, in *Linear collider physics in the new millenium*, eds. K. Fujii, D. Miller and A. Soni, World Scientific [hep-ph/0301023].

[5] W. Grimus and L. Lavoura, *On a model with two zeros in the neutrino mass matrix*, J. Phys. G **31** (2005) 693 [hep-ph/0412283].

[6] W. Grimus and L. Lavoura, *A model realizing the Harrison–Perkins–Scott lepton mixing matrix*, JHEP **01** (2006) 018 [hep-ph/0509239].

[7] M. Hirsch, A. Villanova del Moral, J.W.F. Valle and E. Ma, *Predicting neutrinoless double beta decay*, Phys. Rev. D **72** (2005) 091301; Err. ibid. D **72** (2005) 119904 [hep-ph/0507148].

[8] T. Fukuyama and H. Nishiura, *Mass matrix of Majorana neutrinos*, hep-ph/9702253; R.N. Mohapatra and S. Nussinov, *Bimaximal neutrino mixing and neutrino mass matrix*, Phys. Rev. D **60** (1999) 013002 [hep-ph/9809415];
E. Ma and M. Raidal, *Neutrino mass, muon anomalous magnetic moment, and lepton flavor nonconservation*, Phys. Rev. Lett. **87** (2001) 011802; Err. ibid. **87** (2001) 159901 [hep-ph/0102255];
C.S. Lam, *A 2–3 symmetry in neutrino oscillations*, Phys. Lett. B **507** (2001) 214 [hep-ph/0104116];
K.R.S. Balaji, W. Grimus and T. Schwetz, *The solar LMA neutrino oscillation solution in the Zee model*, Phys. Lett. B **508** (2001) 301 [hep-ph/0104035];
E. Ma, *The all-purpose neutrino mass matrix*, Phys. Rev. D **66** (2002) 117301 [hep-ph/0207352].

[9] W. Grimus and L. Lavoura, *Softly broken lepton numbers and maximal neutrino mixing*, JHEP **07** (2001) 045 [hep-ph/0105212];
W. Grimus and L. Lavoura, *Softly broken lepton numbers: an approach to maximal neutrino mixing*, Acta Phys. Pol. B **32** (2001) 3719 [hep-ph/0110041].

[10] P.F. Harrison and W.G. Scott, *μ–τ reflection symmetry in lepton mixing and neutrino oscillations*, Phys. Lett. B **547** (2002) 219 [hep-ph/0210197].

[11] For specific schemes and additional references, see for instance
Y. Koide, *Universal texture of quark and lepton mass matrices with an extended flavor 2 ↔ 3 symmetry*, Phys. Rev. D **69** (2004) 093001 [hep-ph/0312207];
W. Grimus, A.S. Joshipura, S. Kaneko, L. Lavoura, H. Sawanaka and M. Tanimoto, *Non-vanishing U_{e3} and \cos 2\theta_{23} from a broken Z_2 symmetry*, Nucl. Phys. B **713** (2005) 151 [hep-ph/0408123];
W. Grimus and L. Lavoura, *S_3 × Z_2 model for neutrino mass matrices*, JHEP **08** (2005) 013 [hep-ph/0504153];
R.N. Mohapatra and W. Rodejohann, *Broken $\mu$–$\tau$ symmetry and leptonic CP violation*, Phys. Rev. D 72 (2005) 053001 [hep-ph/0507312];

Y.H. Ahn, S.K. Kang, C.S. Kim and J. Lee, *$\mu$–$\tau$ symmetry and radiatively generated leptogenesis*, Phys. Rev. D 75 (2007) 013012 [hep-ph/0610007].

[12] G.C. Branco, *Spontaneous CP nonconservation and natural flavor conservation: a minimal model*, Phys. Rev. D 22 (1980) 2901.

For a pedagogical introduction, see

G.C. Branco, L. Lavoura and J.P. Silva, *CP violation* (Oxford University Press, 1999).

[13] See for instance

S. Hannestad and G. Raffelt, *Cosmological mass limits on neutrinos, axions, and other light particles*, JCAP 04 (2004) 008 [hep-ph/0312154];

Ø. Elgarøy and O. Lahav, *Neutrino masses from cosmological probes*, New J. Phys. 7 (2005) 61 [hep-ph/0412075];

A. Goobar, S. Hannestad, E. Mörtsell and H. Tu, *A new bound on the neutrino mass from the SDSS baryon acoustic peak*, JCAP 06 (2006) 019 [astro-ph/0602155];

M. Fukugita, K. Ichikawa, M. Kawasaki and O. Lahav, *Limit on the neutrino mass from the WMAP three year data*, Phys. Rev. D 74 (2006) 027302 [astro-ph/0605362];

and references therein.

[14] T. Schwetz, *Global fits to neutrino oscillation data*, talk presented at SNOW 2006, Stockholm, 2–6 May 2006, Phys. Scripta T 127 (2006) 1 [hep-ph/0606060].

[15] S.-L. Chen, M. Frigerio and E. Ma, *Hybrid seesaw neutrino masses with $A_4$ family symmetry*, Nucl. Phys. B 724 (2005) 423 [hep-ph/0504181].

[16] L. Lavoura and H. Kühböck, *Predictions of an $A_4$ model with a five-parameter neutrino mass matrix*, Mod. Phys. Lett. A 22 (2007) 181 [hep-ph/0610050].

[17] G.C. Branco, L. Lavoura and J.P. Silva, *CP violation* (Oxford University Press, 1999), p. 167.