Cotangent bundle and micro-supports in mixed characteristic case

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Abstract

For a regular scheme and a prime number $p$, we define the FW-cotangent bundle as a vector bundle on the closed subscheme defined by $p = 0$, under a certain finiteness condition.

For a constructible complex on the étale site of the scheme, we introduce the condition to be micro-supported on a closed conical subset in the FW-cotangent bundle. At the end of the article, we compute the singular supports in some cases.

Let $k$ be a perfect field of characteristic $p > 0$ and let $X$ be a regular noetherian scheme such that the closed subscheme $X_{\mathbb{F}_p}$ defined by $p = 0$ is a scheme of finite type over $k$. For example, $X$ is of finite type over a discrete valuation ring $\mathcal{O}_K$ with residue field $k$ or over $k$ itself. The main purpose of the article is to prepare a framework to study the micro-support for an étale sheaf on $X$ as in the transcendental setting \cite{14} or in the setting of algebraic geometry \cite{4}, by introducing a variant of the cotangent bundle defined on $X_{\mathbb{F}_p}$ in an arithmetic setting.

The key property used in the definition of singular supports in \cite{4} in the geometric case is the local acyclicity of morphisms to other smooth schemes from the scheme where the sheaf is defined. However, as we see in Remark 3.1.6, a simple imitation of the geometric case does not work in the mixed characteristic situation as we do not have sufficiently many morphisms out of a scheme. Instead, we will use the $\mathcal{F}$-transversality introduced in \cite{17} Definition 8.5] of morphisms from other schemes to the scheme where the sheaf is defined, which is known to give a characterization of singular support in the geometric case.

Let $\Lambda$ be a finite field of characteristic $\neq p$. For a separated morphism $h: W \to X$ of finite type of regular noetherian schemes and a constructible complex $\mathcal{F}$ of $\Lambda$-modules on the étale site of $X$, we define the $\mathcal{F}$-transversality (Definition [11.5] as the property for the canonical morphism $c_{\mathcal{F},h}: h^*\mathcal{F} \otimes R^{h!}\Lambda \to R^{h!}\mathcal{F}$ \cite[1.4]{17} to be an isomorphism, similarly as in \cite[Definition 8.5]{17}. We show that the transversality for the direct image is equivalent to the property that the base change morphism is an isomorphism in Proposition [11.82].

We have shown that the sheaf $F\Omega^1_X$ of FW-differentials is a locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$-modules of rank $\dim X$ in \cite{19}. We call the associated vector bundle $FT^\ast X|_{X_{\mathbb{F}_p}}$ on $X_{\mathbb{F}_p}$ the FW-cotangent bundle of $X$. The fiber $FT^\ast X|_x$ at a closed point $x \in X_{\mathbb{F}_p}$ is canonically
identified with the Frobenius pull-back \( F^* T_x^* X \) of the cotangent space that is the vector space \( m_x / m_x^2 \) regarded as a scheme over the residue field \( k(x) \). For a closed conical subset \( C \) of the vector bundle \( F T_x^* X |_{X_{F_{p}}} \) and for a morphism \( h: W \rightarrow X \) of finite type of regular schemes, we define the \( C \)-transversality in Definition 2.2.1.1 similarly as in \([4, 1.2]\).

Using the \( C \)-transversality and the \( F \)-transversality, we define the condition for \( F \) to be micro-supported on \( C \) in Definition 3.1.1. This is a property along the closed subscheme \( X_{F_{p}} \). For example, if \( X \) is of finite type over \( \mathcal{O}_K \) as above, then \( F \) is locally constant on a neighborhood of the closed fiber \( X_{F_{p}} \) if and only if \( F \) is micro-supported on the 0-section \( F T_x^* X |_{X_{F_{p}}} \).

If the smallest closed conical subset of \( F T_x^* X |_{X_{F_{p}}} \) on which \( F \) is micro-supported exists, we call it the singular support \( SSF \) of \( F \). The author does not know how to show the existence in general. We compute the singular support of some sheaves on regular schemes in Proposition 3.2.6.

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1 \( F \)-transversality

In this section, we study properties of morphisms of schemes with respect to complexes on the étale site of a scheme. The transversality is defined as a condition for a canonical morphism for extraordinary pull-back to be an isomorphism. In Section 1.1 after preparing some sortes on the canonical morphism, we establish basic properties on the transversality. In Section 1.2, after recalling basic properties of local acyclicity, we study the relation between the local acyclicity and the transversality.
In this section and Section 3, \( \Lambda \) denotes a finite field of characteristic \( \ell \) invertible on relevant noetherian schemes. The derived categories \( D^+(\cdot, \Lambda) \) of bounded below complexes and \( D^b_\ell(\cdot, \Lambda) \) of constructible complexes are defined as usual.

### 1.1 \( \mathcal{F} \)-transversality

Let \( h: W \rightarrow X \) be a separated morphism of finite type of noetherian schemes and \( \Lambda \) be a finite field of characteristic \( \ell \) invertible on \( X \). The functor \( Rh^!: D^+(X, \Lambda) \rightarrow D^+(W, \Lambda) \) is defined as the adjoint of \( Rh_!: D(W, \Lambda) \rightarrow D(X, \Lambda) \) in \([6\, Théorème 3.1.4.]\). If \( X \) is quasi-excellent, by the finiteness theorem \([15\, Théorème 1.1.1]\), we have a functor \( Rh^!: D^b_c(X, \Lambda) \rightarrow D^b_c(W, \Lambda) \) see also \([8\, Corollaire 1.5]\). Recall that a scheme of finite type over a Dedekind domain with fraction field of characteristic 0 is quasi-excellent by \([11\, Scholie (7.8.3)]\).

Let \( F \in D^+(X, \Lambda) \) and \( G \in D^+(W, \Lambda) \). Then, the adjoint of the morphism \( h^*F \otimes h^*Rh_*G \rightarrow h^*F \otimes G \) induced by the adjunction \( h^*Rh_*G \rightarrow G \) defines a canonical morphism

\[
F \otimes Rh_*G \rightarrow Rh^*(h^*F \otimes G).
\]

If \( h \) is an open immersion and if \( G = h^*G_X \) for some extension of \( G \) on \( X \), \((1.1)\) is identified with the morphism \( F \otimes \mathcal{R}Hom(h_!\Lambda, G_X) \rightarrow \mathcal{R}Hom(h_!\Lambda, F \otimes G_X) \) defined by the product.

Applying the construction \((1.1)\) to a compactification of \( h \) and the extension by 0, a canonical isomorphism

\[
F \otimes Rh_!G \rightarrow Rh_!(h^*F \otimes G),
\]

the projection formula \([7\, (4.9.1)]\) is defined.

**Definition 1.1.1.** Let \( h: W \rightarrow X \) be a separated morphism of finite type of quasi-excellent noetherian schemes. Let \( F \in D^+(X, \Lambda) \).

1. Let \( G \in D^+(X, \Lambda) \). We define a canonical morphism

\[
(1.3) \quad c_{F,G,h} : h^*F \otimes Rh^!G \rightarrow Rh^!(F \otimes G)
\]

to be the adjoint of the composition

\[
Rh_!(h^*F \otimes Rh^!G) \rightarrow F \otimes Rh_!Rh^!G \rightarrow F \otimes G
\]

of the inverse of the isomorphism \((1.2)\) and the morphism induced by the adjunction \( Rh_!Rh^!G \rightarrow G \). For \( G = \Lambda \), we define a canonical morphism

\[
(1.4) \quad c_{F,h} : h^*F \otimes^L Rh^!\Lambda \rightarrow Rh^!F
\]

to be \( c_{F,A,h} \).

**Lemma 1.1.2.** Let \( h: W \rightarrow X \) be a separated morphism of finite type of noetherian schemes. Let \( F \in D^+(X, \Lambda) \).
1. Let $\mathcal{G}, \mathcal{H} \in D^+(X, \Lambda)$. Then, the diagram

$$
\begin{array}{ccc}
\mathcal{G} \otimes \mathcal{H} & \xrightarrow{c_{\mathcal{G}, \mathcal{H}, h}} & \mathcal{H} \\
\downarrow_{c_{\mathcal{G}, \mathcal{H}, h} \otimes 1} & & \downarrow_{c_{\mathcal{F}, \mathcal{G}, \mathcal{H}, h} \otimes 1}
\end{array}
$$

is commutative.

2. Let $g: V \to W$ be a separated morphism of finite type of schemes and let $\mathcal{G} \in D^+(X, \Lambda)$. Then, the diagram

$$
\begin{array}{ccc}
(hg)^\ast \mathcal{F} \otimes R(hg)^! \mathcal{G} & \xrightarrow{c_{\mathcal{F}, \mathcal{G}, h, g}} & R(hg)^! (\mathcal{F} \otimes \mathcal{G}) \\
\downarrow_{c_{h^\ast \mathcal{F}, \mathcal{G}, h} \otimes 1} & & \downarrow_{c_{h^\ast \mathcal{F}, \mathcal{G}, h} \otimes 1}
\end{array}
$$

where the upper vertical arrows are canonical isomorphisms is commutative.

3. Let

$$
\begin{array}{ccc}
X & \leftarrow_h & W \\
\downarrow_f & & \downarrow_{f'} \\
Y & \leftarrow_g & V
\end{array}
$$

be a cartesian diagram of separated morphisms of finite type. Then, the diagram

$$
\begin{array}{ccc}
g^\ast Rf_\ast \mathcal{F} \otimes Rg^! \Lambda & \xrightarrow{c_{Rf_\ast \mathcal{F}, g}} & Rg^! Rf_\ast \mathcal{F} \\
\downarrow & & \downarrow \\
Rf'_\ast h^\ast \mathcal{F} \otimes Rg^! \Lambda & \xrightarrow{c_{h^\ast \mathcal{F}, g}} & Rf'_\ast h^\ast \mathcal{F} \\
\downarrow_{Rf'_\ast (h^\ast \mathcal{F} \otimes f^\ast Rg^! \Lambda)} & & \downarrow_{Rf'_\ast (h^\ast \mathcal{F} \otimes h^\ast \mathcal{H})}
\end{array}
$$

where the arrows without tags are defined by base change morphisms is commutative.

**Proof.** 1. The diagram

$$
\begin{array}{ccc}
Rh_1 \mathcal{G} \otimes \mathcal{H} & \longrightarrow & \mathcal{G} \otimes \mathcal{H} \\
\uparrow_{Rh_1(c_{\mathcal{G}, \mathcal{H}, h})} & & \uparrow \\
Rh_1(Rh_1 \mathcal{G} \otimes h^\ast \mathcal{H}) & \xrightarrow{1.2} & Rh_1 \mathcal{G} \otimes \mathcal{H}
\end{array}
$$
where the arrows without tags are defined by the adjunction is commutative by the definition of $c_{G,H,h}$. Taking the tensor products with $F$, applying the projection formula (1.2) and taking the adjoint, we see that the upper triangle in

\[
\begin{align*}
    h^* F \otimes R^1(G \otimes H) &\xrightarrow{c_{F,G,H,h}} R^1(F \otimes G \otimes H) \\
    \downarrow & \\
    h^* F \otimes R^1G \otimes h^*H &\xrightarrow{c_{F,G,h} \otimes 1} R^1(F \otimes G) \otimes h^*H
\end{align*}
\]

is commutative. The lower triangle is similarly commutative and the assertion follows.

2. The lower quadrangle is commutative by 1. The composition $g^*h^* F \otimes Rg^1Rh^1G \to Rh^1F$ through $Rg^1(h^* F \otimes Rh^1G)$ is the adjoint of $Rh_1Rg_1(g^*h^* F \otimes Rg^1Rh^1G) \to F \otimes Rh_1Rg_1Rg^1Rh^1G$ induced by the adjunction $Rh_1Rg_1Rg^1Rh^1G \to Rh_1Rh^1G \to G$. Since the last morphism is identified with the adjunction $R(hg)_1R(hg)^1G \to G$, the upper pentagon is also commutative.

3. For $G \in D^+(V, \Lambda)$, we consider the diagram

\[
\begin{align*}
    f^*Rg_1(g^*Rf_*F \otimes G) &\xleftarrow{f^*Rg_1} f^*Rf_*F \otimes f^*Rg_1G \longrightarrow F \otimes f^*Rg_1G \\
    \downarrow & \\
    Rf_1f'^*(Rf'_*h^*F \otimes G) &\longrightarrow Rf_1(h^*F \otimes f^*G) \xrightarrow{1.2} F \otimes Rf_1f'^*G
\end{align*}
\]

defined as follows. The vertical arrows are defined by the base change morphisms and the horizontal arrows without labels are defined by adjunction. We see that the diagram is commutative by reducing to the case where $g$ is proper and going back to the definition of (1.2).

We apply (1.8) to $G = Rg^1\Lambda$. Since the composition $f^*Rg_1Rg^1\Lambda \to Rf_1f'^*Rg^1\Lambda \to Rh_1Rh^1\Lambda \to \Lambda$ of the base change morphisms with the adjunction is induced by the adjunction $Rg_1Rg^1\Lambda \to \Lambda$, we obtain a commutative diagram

\[
\begin{align*}
    f^*Rg_1(g^*Rf_*F \otimes Rg^1\Lambda) &\xleftarrow{f^*Rg_1} f^*Rf_*F \otimes f^*Rg_1Rg^1\Lambda \longrightarrow F \\
    \downarrow & \\
    Rf_1f'^*(Rf'_*h^*F \otimes Rg^1\Lambda) &\longrightarrow Rf_1(h^*F \otimes Rh^1\Lambda) \xrightarrow{1.2} F \otimes Rf_1Rh^1\Lambda
\end{align*}
\]

Since the canonical morphism (1.4) is defined as the adjoint of (1.2), we obtain (1.7) by taking the adjoint of (1.9).

\[\square\]

**Lemma 1.1.3.** Let $i: Z \to X$ be a closed immersion of noetherian schemes and let $F, G \in D^+(X, \Lambda)$.

1. We define the slant arrow and the vertical arrow in the diagram

\[
\begin{align*}
    F \otimes i_*R^iG &\xrightarrow{(1.2)} i_*(i^*F \otimes R^iG) \xrightarrow{i_*(c_{F,G,i})} i_*R^i(F \otimes G) \\
    \downarrow & \\
    F \otimes R\text{Hom}(i_*\Lambda, G) &\longrightarrow R\text{Hom}(i_*\Lambda, F \otimes G)
\end{align*}
\]
by the canonical isomorphism \( i_* Ri^! \rightarrow R\text{Hom}(i_! \Lambda, -) \) and the lower horizontal arrow by the product. Then, the diagram (1.10) is commutative.

2. Let \( j: U = X \rightarrow Z \rightarrow X \) be the open immersion of the complement. Then, the exact sequence \( 0 \rightarrow j_! \Lambda \rightarrow \Lambda \rightarrow i_* \Lambda \rightarrow 0 \) defines a commutative diagram

\[
\begin{array}{cccccc}
F \otimes i_* Ri^! G & \longrightarrow & F \otimes G & \longrightarrow & F \otimes Rj_* j^* G & \longrightarrow \\
\downarrow_{i_*(c_{F,G,i})} & \downarrow & \downarrow & & \downarrow_{Rj_* j^* (F \otimes G)} \\
i_* Ri^!(F \otimes G) & \longrightarrow & F \otimes G & \longrightarrow & Rj_* j^*(F \otimes G) & \longrightarrow \\
\end{array}
\]

(1.11)

of distinguished triangles.

**Proof.** 1. By the definition of \( c_{F,G,i} \), the morphism \( i_*(c_{F,G,i}) : i_*(i^* F \otimes Ri^! G) \rightarrow i_* Ri^!(F \otimes G) \) is the unique morphism such that the diagram

\[
\begin{array}{cccccc}
F \otimes i_* Ri^! G & \longrightarrow & F \otimes G & \longrightarrow & F \otimes R\text{Hom}(i_* \Lambda, G) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
i_* (i^* F \otimes Ri^! G) & \longrightarrow & i_* Ri^!(F \otimes G) & \longrightarrow & i_* Ri^!(F \otimes G) & \longrightarrow \\
\end{array}
\]

(1.2)

is commutative. Here the arrows without tag are defined by the adjunction \( i_* Ri^! \rightarrow 1 \). Similarly, the lower horizontal arrow \( F \otimes R\text{Hom}(i_* \Lambda, G) \rightarrow R\text{Hom}(i_* \Lambda, F \otimes G) \) is the unique morphism such that the diagram

\[
\begin{array}{cccccc}
F \otimes i_* Ri^! G & \longrightarrow & F \otimes G & \longrightarrow & F \otimes R\text{Hom}(i_* \Lambda, G) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
F \otimes R\text{Hom}(i_* \Lambda, F \otimes G) & \longrightarrow & F \otimes G & \longrightarrow & R\text{Hom}(i_* \Lambda, F \otimes G) & \longrightarrow \\
\end{array}
\]

is commutative. Here the left vertical arrow is the slant arrow in (1.10) and the right vertical arrow is induced by \( \Lambda \rightarrow i_* \Lambda \). Hence the assertion follows.

2. The exact sequence \( 0 \rightarrow j_! \Lambda \rightarrow \Lambda \rightarrow i_* \Lambda \rightarrow 0 \) defines a commutative diagram

\[
\begin{array}{cccccc}
F \otimes R\text{Hom}(i_* \Lambda, G) & \longrightarrow & F \otimes G & \longrightarrow & F \otimes R\text{Hom}(j_! \Lambda, G) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
R\text{Hom}(i_* \Lambda, F \otimes G) & \longrightarrow & F \otimes G & \longrightarrow & R\text{Hom}(j_! \Lambda, F \otimes G) & \longrightarrow \\
\end{array}
\]

(1.12)

of distinguished triangles. By 1., the left vertical arrow of (1.11) is identified with that of (1.12) and similarly for the right vertical arrows. \( \blacksquare \)

**Lemma 1.1.4.** Let

\[
\begin{array}{ccc}
X & \overset{h}{\leftarrow} & W \\
Y & \overset{g}{\leftarrow} & V \\
f & \downarrow & f' \\
\end{array}
\]

be a cartesian of morphisms of finite type of regular noetherian schemes. If \( f \) and \( g \) are transversal, then the base change morphism \( f^* Rg^! \Lambda \rightarrow Rh^! \Lambda \) is an isomorphism of locally constant complexes.

6
We say that a complex $\mathcal{F}$ is locally constant, if its cohomology sheaf $\mathcal{H}^q\mathcal{F}$ is locally constant for every $q$ and if $\mathcal{H}^q\mathcal{F} = 0$ except for finitely many $q$.

**Proof.** Since the assertion is local, we may assume that the morphism $g$ is a composition of a smooth morphism and a regular immersion. Hence, it suffices to show each case.

Assume that $g$ is a smooth of relative dimension $d$. Then, the adjoint of the trace morphism $Rg_!\Lambda(d)[2d] \to \Lambda$ \cite[Théorème 2.9]{[G]} defines an isomorphism $\Lambda(d)[2d] \to Rg_!\Lambda$ by Poincaré duality \cite[Théorème 3.2.5]{[G]}. Since the formation of the trace morphism commutes with base change, the assertion follows in this case.

Assume that $g$ is a regular immersion of codimension $c$. Then, by the absolute purity \cite[Théorème 3.1.1]{[G]}, the fundamental class $[V]$ defines an isomorphism $\Lambda \to Rg_!\Lambda(c)[2c]$. Since $f$ and $g$ are transversal, further by the absolute purity, the fundamental class $[W] = f^*[V]$ defines an isomorphism $\Lambda \to Rh^!\Lambda(c)[2c]$. Hence the base change morphism $f^*Rg_!\Lambda \to Rh^!\Lambda$ is an isomorphism.

\[\square\]

**Definition 1.1.5.** Let $h: W \to X$ be a separated morphism of finite type of noetherian schemes and let $\mathcal{F} \in D^+(X, \Lambda)$. We say that $h$ is $\mathcal{F}$-transversal if the canonical morphism \cite[(1.4)]{[G]} is an isomorphism.

For a closed immersion $i: Z \to X$ of regular noetherian schemes and a separated morphism $h: W \to X$ of finite type of regular noetherian schemes, we show that $h$ is $i_*\Lambda$-transversal if $h$ and $i$ are transversal in Corollary \cite[1.1.9]{[G]}. If $h$ is also an immersion, if $Z$ and $W$ meets properly, if the reduced part $V$ of $Z \times_X W$ is regular and if the intersection multiplicity $\mu(Z, W)$ is invertible in $\Lambda$, then $h$ is still $i_*\Lambda$-transversal. Hence the converse does not hold.

**Lemma 1.1.6.** Let $h: W \to X$ be a separated morphism of finite type of noetherian schemes and let $\mathcal{F} \in D^+(X, \Lambda)$.

1. If $h: W \to X$ is smooth, then $h$ is $\mathcal{F}$-transversal.
2. If $\mathcal{F}$ is locally constant, then $h$ is $\mathcal{F}$-transversal.

**Proof.**

1. This is exactly the Poincaré duality \cite[Théorème 3.2.5]{[G]}.

2. Since the assertion is étale local, the assertion is reduced to the case where $\mathcal{F} = \Lambda$ by devissage. \[\square\]

**Lemma 1.1.7.** Let $i: Z \to X$ be a closed immersion of noetherian schemes and let $\mathcal{F} \in D^+(X, \Lambda)$.

1. Assume that $Z$ is the union of closed subsets $Z_1, \ldots, Z_n \subset X$ and that for each subset $I \subset \{1, \ldots, n\}$, the immersion $i_I: Z_I = \bigcap_{j \in I} Z_j \to X$ is $\mathcal{F}$-transversal. Then, $i: Z \to X$ is $\mathcal{F}$-transversal.
2. Let $j: U = X - Z \to X$ be the open immersion of the complement. Then, the following conditions are equivalent:
   1. $i: Z \to X$ is $\mathcal{F}$-transversal.
   2. The canonical morphism $\mathcal{F} \otimes Rj_*\Lambda \to Rj_*i_*\mathcal{F}$ is an isomorphism.

**Proof.**

1. The quasi-isomorphism $i_*\Lambda \to [\bigoplus_{j \in I} i_j_*\Lambda \to \cdots \to \bigoplus_{|I| = p} i^*_I\Lambda \to \cdots]$ defines a spectral sequence $E_1^{p,q} = \bigoplus_{|I| = -p} R^p\mathcal{H}om(i_*\Lambda, -) \Rightarrow R^{p+q}\mathcal{H}om(i_*\Lambda, -)$. By
Lemma 1.1.3.1, the assumption implies that the morphisms $\mathcal{F} \otimes R^q\mathcal{H}om(i_!\Lambda, \Lambda) \to R^q\mathcal{H}om(i_!\Lambda, \mathcal{F})$ on $E_1$-terms are isomorphisms. Hence the assertion follows.

2. The assertion follows from Lemma 1.1.3.2 for $\mathcal{G} = \Lambda$. □

**Proposition 1.1.8.** Let $h: W \to X$ be a separated morphism of finite type of noetherian schemes and let $\mathcal{F} \in D^+(X, \Lambda)$. Assume that $h$ is $\mathcal{F}$-transversal.

1. Assume that $Rh^!\Lambda$ is locally constant on a neighborhood $W_1$ of $\text{supp} h^*\mathcal{F}$ such that $W_1 \subset \text{supp} Rh^!\Lambda$. Then, for a separated morphism $g: V \to W$ of finite type of noetherian schemes, the following conditions are equivalent:
   1. $g$ is $h^*\mathcal{F}$-transversal.
   2. $hg$ is $\mathcal{F}$-transversal.

2. Let

\[
\begin{array}{ccc}
X & \xleftarrow{h} & W \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{g} & V
\end{array}
\]

be a cartesian diagram of morphisms of finite type of noetherian schemes. Assume that $g$ is separated and that $Rg^!\Lambda$ is locally constant of support $V$. Further assume that the base change morphism

\[
f'^* \cdot Rg^!\Lambda \to Rh^!\Lambda
\]

is an isomorphism on a neighborhood of $\text{supp} h^*\mathcal{F}$. Then the following conditions are equivalent:

1. The morphism $g: V \to Y$ is $Rf_*\mathcal{F}$-transversal.
2. The base change morphism

\[
g^* \cdot Rf_*\mathcal{F} \to Rf'_* h^*\mathcal{F}
\]

is an isomorphism.

By Lemma 1.1.6.1, Proposition 1.1.8.2 (1)$\Rightarrow$(2) gives a generalization of the smooth base change theorem [3, Corollaire 1.2].

**Proof.** 1. We consider the commutative diagram (1.6) for $\mathcal{G} = \Lambda$. Since $h$ is assumed to be $\mathcal{F}$-transversal, $c_{\mathcal{F}, h}$ is an isomorphism. Since $Rh^!\Lambda$ is locally constant on a neighborhood of the support of $h^*\mathcal{F}$, the morphisms $1 \otimes c_{Rh^!\Lambda, g}$ and $c_{h^*\mathcal{F}, Rh^!\Lambda, g}$ are isomorphisms. Hence $c_{\mathcal{F}, h}$ is an isomorphism if and only if $c_{h^*\mathcal{F}, g} \otimes 1$ is an isomorphism. Further by the assumption on the support of $Rh^!\Lambda$, the latter condition is equivalent to the condition that $c_{h^*\mathcal{F}, g}$ is an isomorphism.

2. We consider the commutative diagram (1.7). By the proper base change theorem or [3 Corollaire 3.1.12.3], the upper right vertical arrow is an isomorphism. Since $h$ is assumed $\mathcal{F}$-transversal, the lower right vertical arrow $Rf'_*(c_{\mathcal{F}, h})$ is an isomorphism. By the assumption on $Rg^!\Lambda$, the upper left vertical arrow is an isomorphism if and only if (1.14) is an isomorphism. Further the arrow labeled (1.1) is an isomorphism. Since (1.13) is assumed to be an isomorphism, the bottom horizontal arrow is an isomorphism. Hence the assertion follows from the commutative diagram (1.7). □
Corollary 1.1.9. 1. Let the assumption be the same as in Proposition 1.1.8.2. Assume further that \( f \) is proper on the support of \( F \). Then, \( g \) is \( Rf_*F \)-transversal.

2. Let \( p: Z \to X \) be a proper morphisms of regular schemes and let \( h: W \to X \) be a separated morphism of finite type of regular schemes. If \( h \) and \( p \) are transversal, then \( h \) is \( Rp_*\Lambda \)-transversal.

Proof. 1. By the assumption that \( f \) is proper on the support of \( F \), the base change morphism (1.14) is an isomorphism by the proper base change theorem. Hence the assertion follows from Proposition 1.1.8.2 (2) ⇒ (1).

2. Let \( Z \xleftarrow{g} V \) \( p \) \( \downarrow \) \( p' \) \( X \xleftarrow{h} W \) be a cartesian diagram. By Lemma 1.1.4, the base change morphism \( p'^*Rh!\Lambda \to Rg!\Lambda \) is an isomorphism. Since \( X \) and \( W \) are regular, the assumption in Proposition 1.1.8.2 that \( Rh!\Lambda \) is locally constant of support \( W \) is satisfied by the absolute purity \([16, \text{Théorème 3.1.1}]\) as in the proof of Lemma 1.1.4. Since \( g \) is \( \Lambda \)-transversal and \( p \) is proper, \( h \) is \( Rp_*\Lambda \)-transversal by 1.

### 1.2 Local acyclicity and \( F \)-transversality

Let \( f: X \to Y \) be a morphism of schemes and \( x \) and \( y \) be geometric points of \( X \) and \( Y \). Let \( f(x) \) denote the geometric point of \( Y \) defined by the composition \( x \to X \to Y \) and let \( X(x) \to Y_{(f(x))} \) be the induced morphism of strict localizations. We call a morphism \( y \to Y_{(f(x))} \) of schemes over \( Y \) a specialization \( f(x) \leftarrow y \) and call \( X(x),y = X(x) \times_{Y_{(f(x))}} y \) the Milnor fiber. For a complex \( F \) of \( \Lambda \)-modules on \( X \), the pull-back by \( X(x),y \to X(x) \) defines a canonical morphism

\[
F_x = R\Gamma(X(x), F|_{X(x)}) \to R\Gamma(X(x),y, F|_{X(x),y}).
\]

**Definition 1.2.1** (cf. \([8, \text{Définition 2.12}]\)). Let \( f: X \to Y \) be a morphism of schemes and \( Z \subset X \) be a closed subset. Let \( \mathcal{F} \) be a complex of \( \Lambda \)-modules on \( X \). We say that \( f \) is locally acyclic relatively to \( \mathcal{F} \) or \( \mathcal{F} \)-acyclic for short along \( Z \) if for every geometric point \( x \) of \( Z \) and for every specialization \( f(x) \leftarrow y \), the canonical morphism \( \mathcal{F}_x \to R\Gamma(X(x),y, F|_{X(x),y}) \) (1.15) is an isomorphism. If \( X = Z \), we drop along \( Z \) in the terminology.

We say that \( f \) is universally \( \mathcal{F} \)-acyclic along \( Z \), if for every morphism \( Y' \to Y \), the base change \( X' \to Y' \) is locally acyclic relatively to the pull-back of \( \mathcal{F} \) along the inverse image \( Z' \subset X' \) of \( Z \).

**Lemma 1.2.2.** Let \( f: X \to Y \) be a morphism of schemes and \( Z \subset X \) be a closed subset. Let \( \mathcal{F} \in D^+(X, \Lambda) \).

1. The following conditions are equivalent.
   (1) \( f \) is \( \mathcal{F} \)-acyclic along \( Z \).
(2) Let $s \leftarrow t$ be a specialization of geometric points of $Y$ such that $t$ is the spectrum of an algebraic closure of the residue field of the point of $Y$ below $t$. Let $Y(s)$ denote the strict localization and let

$$X_s \xrightarrow{i_s'} X \times_Y Y(s) \xleftarrow{j_t'} X_t$$

(1.16)

$$f_s \downarrow \quad f(s) \downarrow \quad f_t \downarrow$$

$$s \xrightarrow{i_s} Y(s) \xleftarrow{j_t} t$$

be the cartesian diagram. Then, the canonical morphism

$$i_s'' F \to i_s'' Rj_{ts}j_t'' F$$

(1.17)

is an isomorphism on the inverse image of $Z$.

2. For a proper morphism $p: X \to P$ of schemes over $Y$, we consider the following conditions:

1. $f$ is $F$-acyclic along $Z$.

2. The morphism $g: P \to Y$ is $R\pi_* F$-acyclic along $p(Z)$.

We have $(1) \Rightarrow (2)$ if $Z = p^{-1}(p(Z))$. If $p$ is finite, we have $(2) \Rightarrow (1)$.

3. The following conditions are equivalent:

1. $f$ is universally $F$-acyclic along $Z$.

2. For every smooth morphism $Y' \to Y$ and for the pull-back $F'$ of $F$ on $X' = X \times_Y Y'$, the base change $f': X' \to Y'$ is $F'$-acyclic along the inverse image $Z' \subset X'$ of $Z$.

Since the local acyclicity is a local property, by locally taking an immersion $X \to \mathbb{A}_Y^n$, the study of local acyclicity is reduced to the case where $f: X \to Y$ is the projection $\mathbb{A}_Y^n \to Y$ by Lemma 1.2.2.

**Proof.** 1. A morphism $y' \to y$ of geometric points of $Y$ is the composition of a limit of smooth morphisms and a homeomorphism in étale topology. Hence for a geometric point $x$ of $X$ and a specialization $f(x) \leftarrow y$, the pull-back $R\pi'(X(x),y,F|_{X(x),y}) \to R\pi'(X(x),y,F|_{X(x),y})$ is an isomorphism by the smooth base change theorem [3 Corollaire 1.2]. Thus, in the definition of local acyclicity, it suffices to consider specializations $f(x) \leftarrow y$ such that $y$ is the spectrum of an algebraic closure of the residue field of the point of $Y$ below $y$.

In the notation of (2), for a geometric point $x$ of $X_s$, the morphism induced by (1.17) on the stalks at $x$ equals $F_x \to R\pi'(X(x),t,F|_{X(x),t})$ (1.15). Hence, the assertion follows.

2. $(1) \Rightarrow (2)$: Let

$$X_s \xrightarrow{i_s'} X \times_Y Y(s) \xleftarrow{j_t'} X_t$$

(1.18)

$$p_s \downarrow \quad p(s) \downarrow \quad p_t \downarrow$$

$$P_s \xrightarrow{i_s''} P \times_Y Y(s) \xleftarrow{j_t''} P_t$$

be the base change of $X \to P$. Then, the isomorphism (1.17) on the inverse image of $Z = p^{-1}(p(Z))$ implies an isomorphism $i_s''^* R\pi(s)_* F \to i_s''^* Rj_{ts}j_t''^* R\pi(s)_* F$ on the inverse image of $p(Z)$ by proper base change theorem.
(2)⇒(1): Let \( z \) be a geometric point of \( Z \) and let \( w = p(z) \leftarrow y \) be a specialization. Then the cospecialization morphism \( p_*F_w \to R\Gamma(P_{(w),y}, p_*F|_{P_{(w),y}}) \) is the direct sum of (1.15) for \( x \in p^{-1}(w) \) since \( p \) is finite. Hence the assertion follows.

3. Since the local acyclicity is a local property preserved by base change by immersions and commutes with limits, the assertion follows. \( \square \)

**Lemma 1.2.3.** Let \( X \) be a noetherian scheme and \( F \in D^b(X, \Lambda) \).

1. If \( F \) is locally constant and if \( f: X \to Y \) is smooth, then \( f \) is \( F \)-acyclic.
2. If \( 1_X: X \to X \) is \( F \)-acyclic along \( Z \) and if \( F \) is constructible, then \( F \) is locally constant on a neighborhood of \( Z \).

**Proof.** 1. By devissage, the assertion follows from the local acyclicity of smooth morphism \([2, \text{Théorème 2.1}]\).

2. For every geometric point \( s \) of \( Z \) and every specialization \( s \leftarrow t \) of geometric points of \( X \), the cospecialization morphism \( F_s \to F_t \) is an isomorphism. Hence the constructible sheaf \( \mathcal{H}^qF \) is locally constant on a neighborhood of every geometric point \( s \) of \( Z \) for every \( q \in \mathbb{Z} \) by \([1, \text{Proposition 2.11}]\). Hence \( F \) is locally constant on a neighborhood of \( Z \). \( \square \)

**Proposition 1.2.4** (cf. \([17, \text{Proposition 8.11}]\)). Let \( f: X \to Y \) be a smooth morphism of regular schemes of finite type over a discrete valuation ring \( \mathcal{O}_K \) and \( Z \subset X \) be a closed subset. Let \( F \) be a constructible complex of \( \Lambda \)-modules on \( X \). Assume that for every separated morphism \( V \to Y \) of regular schemes of finite type over \( \mathcal{O}_K \), the projection \( h: W = X \times_Y V \to X \) is \( F \)-transversal on a neighborhood of \( h^{-1}(Z) \).

1. Let

\[
\begin{array}{c}
X \leftarrow X' \leftarrow W \\
\downarrow f \downarrow f' \downarrow f'_V \\
Y \leftarrow Y' \leftarrow V
\end{array}
\]

be a cartesian diagram of regular schemes of finite type over \( \mathcal{O}_K \). Assume that \( p \) is proper and that \( j: V = Y' \setminus D \to Y' \) is the open immersion of the complement of a divisor \( D \) with simple normal crossings. Then, the composition

\[
F \otimes f^*R(pj)_*\Lambda \to F \otimes R(p'j'_*)_*\Lambda \xrightarrow{\text{1.1}} R(p'j'_*)((p'j')^*F)
\]

where the first morphism is induced by the base change morphism is an isomorphism on a neighborhood of \( Z \).

2. \( f \) is universally \( F \)-acyclic along \( Z \).

For the sake of completeness, we record the proof in \([17]\) with more detail.

**Proof.** 1. Let \( D_1, \ldots, D_n \) be the irreducible components of \( D \). For a subset \( I \subset \{1, \ldots, n\} \), let \( X'_I = X' \times_Y (\bigcap_{i \in I} D_i) \) and let \( i'_I: X'_I \to X' \) be the closed immersion. By the assumption, \( p': X' \to X \) and \( p'i'_I: X'_I \to X \) are \( F \)-transversal on neighborhoods of the inverse images of \( Z \).
Let $F' = p^*F$. Since the assumption on $Rh^i\Lambda$ in Proposition 1.1.8.1 is satisfied by the absolute purity [16 Théorème 3.1.1], the immersions $i^*_{ij}: X'_j \to X'$ are $F'$-transversal on neighborhoods of the inverse images of $Z$ by Proposition 1.1.8.1. Hence by Lemma 1.1.7 the canonical morphism $F' \otimes Rj^*_s\Lambda \to Rj^*_sRj^*_rF'$ (1.1) is an isomorphism on a neighborhood of $p'^{-1}(Z)$. Since $p'$ is proper, we obtain an isomorphism $Rp'(F' \otimes Rj^*_s\Lambda) \to R(p')_*(p')^*F$ on a neighborhood of $Z$.

By the projection formula (1.2), we have a canonical isomorphism $F \otimes Rp'_sRj^*_s\Lambda \to Rp'_s(F' \otimes Rj^*_s\Lambda)$. The base change morphism $F(pj)_*\Lambda \to Rp'_sRj^*_s\Lambda$ is an isomorphism by the smooth base change theorem [3 Corollaire 1.2]. Hence the morphism (1.20) is an isomorphism on a neighborhood of $Z$.

2. It suffices to show that for a smooth morphism $Y' \to Y$, the base change $X' \to Y'$ of $f$ is locally acyclic with respect to the pull-back of $F$ by Lemma 1.2.2.3. Similarly as in the proof of 1., the assumption is satisfied for the pull-back $Y' \to Y$. Hence, by replacing $Y$ by $Y'$, it suffices to show that $f$ is locally acyclic with respect to $F$.

Let $s \leftarrow t$ be a specialization of geometric points of $Y$ as in Lemma 1.2.2.1 and let the notation be as loc. cit. By [3 Theorem 4.1, Theorem 8.2], we may write $t$ as a limit $\lim_{\lambda} U_\lambda$ of the complements $U_\lambda = Y_\lambda \setminus D_\lambda$, in regular schemes $Y_\lambda$ endowed with a proper, surjective and generically finite morphism $p_\lambda: Y_\lambda \to Y$ of divisors $D_\lambda \subset Y_\lambda$ with simple normal crossings. Then, as the limit of (1.20), the canonical morphism

$$F \otimes f_{(s)\ast}Rj^*_sRj^*_t\Lambda \to Rj^*_sRj^*_tF$$

is an isomorphism on the inverse image of $Z$. Since $Y$ is normal, the canonical morphism $\Lambda \to i^*_sRj^*_sRj^*_t\Lambda$ is an isomorphism. Hence the isomorphism (1.21) induces an isomorphism (1.17) on the inverse image of $Z$.

\begin{proof}
By Proposition 1.2.4 applied to $1_X: X \to X$, the identity $1_X: X \to X$ is $F$-acyclic along $Z$. Hence $F$ is locally constant on a neighborhood of $Z$.

We have a partial converse of Proposition 1.2.4 not used in the article.

\begin{proposition} [17 Corollary 8.10]
Let $f: X \to Y$ be a smooth morphism of noetherian schemes and let $F$ be a constructible complex of $\Lambda$-modules on $X$. Let $i: Z \to Y$ be an immersion and let

$$\begin{array}{ccc}
X & \leftarrow^h & W \\
\downarrow & & \downarrow \\
Y & \leftarrow^i & Z
\end{array}$$

be a cartesian diagram. If $f: X \to Y$ is $F$-acyclic, then $h: W \to X$ is $F$-transversal.

For the sake of convenience, we record the proof in [17].
\end{proof}
Proof. We may assume that \( i: Z \to Y \) is a closed immersion. Let \( V = Y - Z \) and consider the cartesian diagram

\[
\begin{array}{ccc}
W & \xrightarrow{h} & X \\
\downarrow{g} & & \downarrow{f} \\
Z & \xrightarrow{i} & Y \\
\end{array}
\]

(1.22)

By [13, Proposition 2.10] applied to the right square, we obtain an isomorphism \( F \otimes f^* Rj_! \Lambda \to Rj'_! j'^* F \). Since \( f \) is smooth, this induces an isomorphism \( F \otimes Rj'_! \Lambda \to Rj'_! j'^* F \) by smooth base change theorem [3, Corollaire 1.2]. Hence the assertion follows by Lemma 1.1.7.2.

Corollary 1.2.7. Let

\[
\begin{array}{ccc}
V' & \xrightarrow{g'} & X' \\
\downarrow{h_v} & & \downarrow{h'} \\
V & \xrightarrow{g} & X \\
\end{array}
\]

be a cartesian diagram of morphisms of finite type of schemes such that \( f: X \to Y \) is smooth and that the vertical arrows are separated. Assume that \( Rh^1 \Lambda \) is locally constant of support \( X' \) and that the base change morphism \( g^* Rh^1 \Lambda \to Rh^1 \Lambda \) is an isomorphism.

Let \( G \) be a constructible complex of \( \Lambda \)-modules on \( V \) and assume that \( f \) is \( Rg_* G \)-acyclic and that \( fg \) is \( G \)-acyclic. Then, the base change morphism

\[
h^* Rg_* G \to Rg'_* h^*_V G
\]

is an isomorphism.

Proof. Since \( f \) is \( Rg_* G \)-acyclic and \( fg \) is \( G \)-acyclic, by Proposition 1.2.6 \( h \) is \( Rg_* G \)-transversal and \( h_v \) is \( G \)-transversal. Hence the assertion follows from Proposition 1.1.8.2.

2 C-transversality

In this section, first we define the FW-cotangent bundle of a regular scheme, as a vector bundle on the closed subscheme defined by \( p = 0 \). Then, we study properties of morphisms with respect to its closed conical subsets corresponding to the transversality and the local acyclicity studied in Section 1.

First in Section 2.1, we recall basic properties of the sheaf \( F\Omega^1_X \) of Frobenius-Witt differentials from [19]. In particular if \( X \) is regular, under a certain finiteness condition, the sheaf \( F\Omega^1_X \) is a locally free \( \mathcal{O}_{X_{F_p}} \)-module of rank \( \dim X \) on \( X_{F_p} = X \times_{\text{Spec} \mathbb{Z}} \text{Spec} F_p \). Under this condition, we define the FW-cotangent bundle \( FT^* X|_{X_{F_p}} \) on \( X_{F_p} \) as the vector bundle associated to the locally free \( \mathcal{O}_{X_{F_p}} \)-module \( F\Omega^1_X \).

We study properties of morphisms with respect to a given closed conical subset in Sections 2.2 and 2.3. In Section 2.2 we study the transversality for morphisms to \( X \).
Section 2.3, we study the acyclicity, which was also called transversality, for morphisms from $X$.

### 2.1 FW-cotangent bundle

**Definition 2.1.1 (19 Definition 1.1).** Let $p$ be a prime number.

1. Define a polynomial $P \in \mathbb{Z}[X, Y]$ by

   \begin{equation}
   (2.1) \quad P = \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} \cdot X^iY^{p-i}.
   \end{equation}

2. Let $A$ be a ring and $M$ be an $A$-module. We say that a mapping $w: A \to M$ is an Frobenius-Witt derivation or FW-derivation for short if the following condition is satisfied: For any $a, b \in A$, we have

   \begin{align*}
   (2.2) & \quad w(a + b) = w(a) + w(b) - P(a, b) \cdot w(p), \\
   (2.3) & \quad w(ab) = b^p \cdot w(a) + a^p \cdot w(b).
   \end{align*}

Definition 2.1.1 is essentially the same as [9, Definition 2.1.1]. We recall some results from [19].

**Lemma 2.1.2.** Let $p$ be a prime number and $A$ be a ring.

1. ([19, Lemma 2.1.1]) There exists a universal pair of an $A$-module $F\Omega^1_A$ and an FW-derivation $w: A \to F\Omega^1_A$.
2. ([19, Corollary 2.3.1]) If $A$ is a ring over $\mathbb{Z}(p)$, we have $p \cdot F\Omega^1_A = 0$.
3. ([19, Corollary 2.3.2]) If $A$ is a ring over $\mathbb{F}_p$, then there exists a canonical isomorphism $F\Omega^1_A \to F^*\Omega^1_A = \Omega^1_A \otimes_A A$ to the tensor product with respect to the absolute Frobenius morphism $A \to A$.

We call $F\Omega^1_A$ the module of FW-differentials of $A$ and $w(a) \in F\Omega^1_A$ the FW-differential of $a \in A$. For a morphism $A \to B$ of rings, we have a canonical $B$-linear morphism $F\Omega^1_A \otimes_A B \to F\Omega^1_B$.

We may sheafify the construction and define $\mathcal{F}\Omega^1$ as a quasi-coherent $\mathcal{O}_X$-module for a scheme $X$. We call $\mathcal{F}\Omega^1_X$ the sheaf of FW-differentials on $X$. If $X$ is a scheme over $\mathbb{Z}(p)$, the $\mathcal{O}_X$-module $\mathcal{F}\Omega^1_X$ is an $\mathcal{O}_{X/F_p}$-module where $X_{F_p} = X \times_{\text{Spec} \mathbb{Z}(p)} \text{Spec} \mathbb{F}_p$. Further if $X$ is noetherian and if $X_{F_p}$ is of finite type over a field of finite $p$-basis, then $\mathcal{F}\Omega^1_X$ is a coherent $\mathcal{O}_{X_{F_p}}$-module by [19, Lemma 4.1.2]. If $X$ is a scheme over $\mathbb{F}_p$, we have a canonical isomorphism

\begin{equation}
(2.4) \quad \mathcal{F}\Omega^1_X \to F^*\Omega^1_X
\end{equation}

to the pull-back by the absolute Frobenius morphism $F: X \to X$, sending $w(a)$ to $da$.

For a morphism $f: X \to Y$ of schemes, we have a canonical morphism

\begin{equation}
(2.5) \quad f^*F\Omega^1_Y \to F\Omega^1_X
\end{equation}
Definition 2.1.7. Let \( k \) be a perfect field of characteristic \( p > 0 \) and let \( X \) be a regular noetherian scheme satisfying the following condition:

\((F)\) \( X_{\mathbf{F}_p} = X \times_{\text{Spec} \ Z} \text{Spec} \mathbf{F}_p \) is a scheme of finite type over \( k \).

Then, we define the FW-cotangent bundle \( FT^*X|_{X_{\mathbf{F}_p}} \) of \( X \) to be the vector bundle on \( X_{\mathbf{F}_p} \) associated with the locally free \( \mathcal{O}_{X_{\mathbf{F}_p}} \)-module \( F\Omega^1_X \) of rank \( \dim X \).
Let \( x \in X_{\mathbb{F}_p} \) be a closed point and let \( T^*_x X \) denote the cotangent space at \( x \) defined as a scheme \( \text{Spec} \, S_{k(x)}(m_x/m^2_x)^\vee \) associated to the \( k(x) \)-vector space \( m_x/m^2_x \). Since \( k(x) \) is perfect, the exact sequence \([2.6]\) defines a canonical isomorphism

\[
F^*T^*_x X \to FT^* X|_x
\]

(2.9)

to the fiber of the FW-cotangent bundle at \( x \) from the pull-back by Frobenius \( F: x \to x \) of \( T^*_x X \). If \( X = X_{\mathbb{F}_p} \), then the FW-cotangent bundle \( FT^* X|_{X_{\mathbb{F}_p}} \) is the pull-back of the cotangent bundle \( T^* X \) by the Frobenius morphism \( F: X \to X \) by \([2.4]\).

Let \( X \to Y \) be a morphism of finite type of regular noetherian schemes satisfying the condition (F) in Definition \([2.1.7]\). Then, the morphism \([2.5]\) defines morphisms

\[
FT^* X|_{X_{\mathbb{F}_p}} \leftarrow^f FT^* Y|_{Y_{\mathbb{F}_p}} \times_{Y_{\mathbb{F}_p}} X_{\mathbb{F}_p} \longrightarrow FT^* Y|_{Y_{\mathbb{F}_p}}
\]

(2.10)

of schemes.

Assume that \( X \to Y \) is smooth and let \( F^*T^* X/Y|_{X_{\mathbb{F}_p}} \) denote the pull-back by the Frobenius \( F: X_{\mathbb{F}_p} \to X_{\mathbb{F}_p} \) of the restriction to \( X_{\mathbb{F}_p} \) of the vector bundle defined \( T^* X/Y \) by the locally free \( \mathcal{O}_X \)-module \( \Omega^1_{X/Y} \). Then, by Proposition \([2.1.3]\) we have an exact sequence

\[
0 \to FT^* Y|_{Y_{\mathbb{F}_p}} \times_{Y_{\mathbb{F}_p}} X_{\mathbb{F}_p} \to FT^* X|_{X_{\mathbb{F}_p}} \to F^*T^* X/Y|_{X_{\mathbb{F}_p}} \to 0
\]

(2.11)

of vector bundles on \( X_{\mathbb{F}_p} \).

Similarly, let \( Z \to X \) be a closed immersion of regular noetherian schemes satisfying the condition (F). Let \( \mathcal{I}_Z \subset \mathcal{O}_X \) be the ideal sheaf and let \( T^*_Z X \) be the conormal bundle defined by the locally free \( \mathcal{O}_Z \)-module \( \mathcal{I}_Z/\mathcal{I}_Z^2 \). Let \( F^*T^*_Z X|_{Z_{\mathbb{F}_p}} \) denote the pull-back by the Frobenius \( F: Z_{\mathbb{F}_p} \to Z_{\mathbb{F}_p} \) of the restriction to \( Z_{\mathbb{F}_p} \) of the vector bundle defined \( T^*_Z X \). Then, by Corollary \([2.1.6]\) we have an exact sequence

\[
0 \to F^*T^*_Z X|_{Z_{\mathbb{F}_p}} \to FT^* X|_{Z_{\mathbb{F}_p}} \to F^*T^*_Z Z|_{Z_{\mathbb{F}_p}} \to 0
\]

(2.12)

of vector bundles on \( Z_{\mathbb{F}_p} \).

### 2.2 \( C \)-transversality

In the rest of this section, we fix a perfect field \( k \) of characteristic \( p > 0 \).

We fix some terminology on closed conical subsets of a vector bundle of a scheme. Let \( V \) be a vector bundle over a scheme \( Y \). We say that a closed subset of \( V \) is conical if it is stable under the action of \( G_{m,Y} \). For a closed conical subset \( C \subset V \), the intersection \( B = C \cap Y \) with the 0-section \( Y \subset V \) regarded as a closed subset of \( Y \) is called the base of \( C \). The base \( B \) equals the image of \( C \) by the projection \( V \to Y \).

We say that a separated morphism \( f: X \to Y \) of finite type of schemes is proper on a closed subset \( Z \subset X \) if for every base change \( f': X' \to Y' \) of \( f \) its restriction to the inverse image \( Z' \subset X' \) is a closed mapping. For a morphism \( V \to V' \) of vector bundles on a scheme \( Y \) and a closed conical subset \( C \) of \( V \), the morphism \( V \to V' \) is proper on \( C \) if and only if the intersection \( C \cap \text{Ker}(V \to V') \) is a subset of the 0-section of \( V \) by \([4]\) Lemma 1.2(ii)].
Definition 2.2.1. Let $X$ be a regular noetherian scheme satisfying the condition (F) in Definition 2.1.7 and let $C \subset FT^*X|_{X_{F_p}}$ be a closed conical subset of the FW-cotangent bundle. Let $h: W \rightarrow X$ be a morphism of finite type of regular schemes.

1. ([11, 12], [17, Definition 3.3]) We say that $h: W \rightarrow X$ is C-transversal if the intersection of $h^*C = C \times_X W \subset FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p}$ with the kernel $\text{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \rightarrow FT^*W|_{W_{F_p}})$ is a subset of the 0-section.

2. Assume that $h$ is C-transversal. Then we define a closed conical subset $h^*C \subset FT^*W|_{W_{F_p}}$ to be the image of $h^*C$ by $FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \rightarrow FT^*W|_{W_{F_p}}$.

Example. Let $Z \subset X$ be a regular closed subscheme. Then a closed conical subset $C = FT^*Z \times_X X_{F_p} \subset FT^*X|_{X_{F_p}}$ is defined by (2.12). In particular, for $Z = X$, the 0-section $FT^*X|_{X_{F_p}} = X_{F_p}$ is a closed conical subset of $FT^*X|_{X_{F_p}}$.

Lemma 2.2.2. Let $X$ be a regular noetherian scheme satisfying the condition (F) in Definition 2.1.7 and let $C \subset FT^*X|_{X_{F_p}}$ be a closed conical subset. Let $h: W \rightarrow X$ be a morphism of finite type of regular schemes.

1. Let $C = FT^*X|_{Z}$ be the restriction to a closed subset $Z \subset X_{F_p}$ of the closed fiber. If $h$ is C-transversal, then $h$ is smooth on a neighborhood of the inverse image $h^{-1}(Z)$.

2. If $C$ is the 0-section of $FT^*X|_{X_{F_p}}$, then $h$ is C-transversal.

3. If $h$ is smooth, for any closed conical subset $C$ of $FT^*X|_{X_{F_p}}$, the morphism $h$ is C-transversal.

Proof. 1. The condition that the intersection of $h^*C = FT^*X|_{Z} \times_{X_{F_p}} W_{F_p} = FT^*X \times_{X_{F_p}} h^{-1}(Z)$ with the kernel $\text{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \rightarrow FT^*W|_{W_{F_p}})$ is a subset of the 0-section means that $F\Omega^1_X \otimes_{O_X} O_W \rightarrow F\Omega^1_W$ is a locally splitting injection on a neighborhood of $h^{-1}(Z)$. By Proposition 2.1.4, this means that $W \rightarrow X$ is smooth on a neighborhood of the inverse image $h^{-1}(Z)$.

2. If $C$ is the 0-section, its intersection with the kernel $\text{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \rightarrow FT^*W|_{W_{F_p}})$ is also the 0-section.

3. If $h$ is smooth, the morphism $FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \rightarrow FT^*W|_{W_{F_p}}$ is an injection by Proposition 2.1.4. Hence for any subset $C \subset FT^*X|_{X_{F_p}}$, its intersection with the kernel $\text{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \rightarrow FT^*W|_{W_{F_p}})$ is a subset of the 0-section. □

Lemma 2.2.3. Let $h: W \rightarrow X$ be a morphism of finite type of regular noetherian schemes satisfying the condition (F) and let $C$ be a closed conical subset of $FT^*X|_{X_{F_p}}$. Assume that $h$ is C-transversal. Then, for a morphism $g: V \rightarrow W$ of finite type of regular noetherian schemes the following conditions are equivalent:

1. The morphism $g$ is $h^*C$-transversal.

2. The composition $hg$ is C-transversal.

If these equivalent conditions are satisfied, we have $(hg)^*C = g^*h^*C$.

Proof. The condition (1) means that the intersection $h^*C \cap \text{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \rightarrow FT^*W|_{W_{F_p}})$ is a subset of the 0-section and further that for $h^*C \subset FT^*W|_{W_{F_p}}$, the intersection $g^*h^*C \cap \text{Ker}(FT^*W|_{W_{F_p}} \times_{W_{F_p}} V_{F_p} \rightarrow FT^*V|_{V_{F_p}})$ is a subset of the 0-section. This means that $(hg)^*C \cap \text{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} V_{F_p} \rightarrow FT^*V|_{V_{F_p}})$ is a subset of the 0-section, namely the condition (2).
The image of $(hg)^*C$ by $FT^*X|_{X_{k_F}} \times_{X_{k_F}} V_{F_p} \to FT^*V|_{V_{k_F}}$ equals the image of $g^*h^*C$ by $FT^*W|_{W_{k_F}} \times_{W_{k_F}} V_{F_p} \to FT^*V|_{V_{k_F}}$. □

The terminology transversality is related to the transversality of morphisms of regular schemes defined as follows.

**Definition 2.2.4.** Let $f: X \to Y$ and $g: V \to Y$ be morphisms of finite type of regular schemes and set $W = X \times_Y V$.

1. Let $w \in W$ and $x \in X, y \in Y, v \in V$ be the images. We say that $f$ and $g$ are transversal at $w$, if $\mathcal{O}_{W,w}$ is regular and if $Tor^q_{\mathcal{O}_Y,y}(\mathcal{O}_{X,x}, \mathcal{O}_{V,v}) = 0$ for $q > 0$.

2. Let $W_1 \subset W$ be an open subscheme. We say that $f$ and $g$ are transversal on $W_1$ if $f$ and $g$ are transversal at every point of $W_1$.

**Example.** Let $Z \subset X$ be a regular closed subscheme and $C = F^*T^*_Z X|_{X_{k_F}} \subset FT^*X|_{X_{k_F}}$ be the closed conical subset defined by the conormal bundle. Then, as we will see in Corollary 2.2.7, a morphism $h: W \to X$ of finite type of regular quasi-excellent noetherian schemes is $C$-transversal if and only if $h: W \to X$ is transversal to $Z \subset X$ on a neighborhood of the closed fiber $W_{F_p}$.

In particular, if $X$ is smooth over a discrete valuation ring $\mathcal{O}_K$ of mixed characteristic with residue field $k$ and if $C = F^*T^*_X X|_{X_k}$ for the closed fiber $Z = X_k$, then the condition that $h: W \to X$ is $C$-transversal means that $W$ is smooth over $\mathcal{O}_K$ on a neighborhood of the closed fiber $W_k$.

**Lemma 2.2.5.** Let $f: X \to Y$ and $g: V \to Y$ be morphisms of finite type of regular schemes and set $W = X \times_Y V$. Let $w \in W$ and $x \in X, y \in Y, v \in V$ be the images.

1. Suppose that $g: V \to Y$ is an immersion. Then, the following conditions are equivalent:
   
   1. $f$ and $g$ are transversal at $w$.
   2. The morphism $T^*_x Y \times_y x \to T^*_x X$ on the cotangent space induces an injection on the subspace $T^*_x Y \times_Y y \subset T^*_y Y \times_y x$.

   2. Suppose that the subset $\text{Reg}(W) \subset W$ consisting of regular points is an open subset. If $f$ and $g$ are transversal at $w \in W$, then $f$ and $g$ are transversal on a neighborhood of $w$.

The condition that $\text{Reg}(W) \subset W$ is an open subset is satisfied if $W$ is of finite type over a Dedekind domain such that the fraction field is of characteristic 0 or a semi-local ring of dimension at most 1 by [11 Corollaire (6.12.6)].

**Proof.** 1. Let $a_1, \ldots, a_r \in \mathcal{O}_{Y,y}$ be a minimal system of generators of $\text{Ker}(\mathcal{O}_{Y,y} \to \mathcal{O}_{V,y})$. Then, the both conditions are equivalent to the condition that $a_1, \ldots, a_r \in \mathcal{O}_{X,x}$ is a part of a regular system of parameters.

   2. Since the $\mathcal{O}_W$-modules $\text{Tor}^q_{\mathcal{O}_Y,y}(\mathcal{O}_X, \mathcal{O}_V) = 0$ are coherent and $w$ is an element of the open subset $\text{Reg}(W)$, the assertion follows. □

Let $f: X \to Y$ be a morphism of finite type of regular noetherian schemes such that $Y_{F_p}$ is of finite type over $k$ and consider the morphisms (2.10). Let $C$ be a closed conical subset of $FT^*X|_{X_{k_F}}$ such that $f: X \to Y$ is proper on the base $B(C)$. Then we define a closed...
conical subset $f_C$ of $FT^*Y|_{Y_{F_p}}$ to be the image by $FT^*X|_{X_{F_p}} \times_{X_{F_p}} Y_{F_p} \to FT^*Y|_{Y_{F_p}}$ of the inverse image of $C$ by $FT^*X|_{X_{F_p}} \times_{X_{F_p}} Y_{F_p} \to FT^*X|_{X_{F_p}}$.

For a closed immersion $i: Z \to X$ of regular noetherian schemes such that $X_{F_p}$ is of finite type over $k$, the closed conical subset $F^*_Z X|_{X_{F_p}}$ defined by the conormal bundle equals $i_C$ for the 0-section $C = FT^*Z|_{Z_{F_p}}$ of $FT^*Z|_{Z_{F_p}}$.

**Proposition 2.2.6.** Let $X, Y$ and $V$ be regular noetherian schemes satisfying the condition (F) and

$$X \xleftarrow{h} W \xrightarrow{f} Y \xleftarrow{g} V$$

be a cartesian diagram of morphisms of finite type. Assume that the subset $\text{Reg}(W) \subset W$ consisting of regular points is an open subset. Let $C$ be a closed conical subset of $FT^*X|_{X_{F_p}}$ such that $f$ is proper on the base $B(C)$. Then, the following conditions are equivalent:

1. The morphism $g$ is $f_C$-transversal.
2. There exists a regular neighborhood $W_1 \subset W$ of the inverse image of the base $B(C)$ such that $f$ and $g$ are transversal on $W_1$ and that the restriction $h_1: W_1 \to X$ is $C$-transversal.

If these equivalent conditions are satisfied, we have $g \circ f_C = f'_1 \circ h_1^\circ C$ for the restriction $f'_1: W_1 \to V$ of $f'$.

**Proof.** (1)⇒(2): Let $x \in B(C) \subset X_{F_p}$ be a closed point and $y = f(x) \in Y_{F_p}$. Since the assertion is étale local, we may assume that the morphism $k(y) \to k(x)$ of residue fields is an isomorphism. There exist an open neighborhood $U \subset X$ of $x \in X$ and a cartesian diagram

$$W \xleftarrow{\gamma} W \times_X U \xrightarrow{f} Q \xrightarrow{g} V$$

$$X \xleftarrow{\gamma} U \xrightarrow{f} P \xrightarrow{g} Y$$

such that $P \to Y$ is smooth and $U \to P$ is a closed immersion. Let $w \in W$ be a closed point above $x$ and $v = f'(w) \in V$. We may also assume that the morphisms $k(y) \to k(v)$ and hence $k(x) \to k(w)$ are isomorphisms. We consider the cartesian diagram

$$T_w^*Q \xleftarrow{\gamma} T_v^*V$$

$$T_x^*X \xleftarrow{\gamma} T_x^*P \xleftarrow{\gamma} T_y^*Y$$

of cotangent spaces and identify their Frobenius pull-backs with the fibers of FW-cotangent bundles by the isomorphism (2.9).

Let $C_x \subset F^*T_x^*P$ and $A_x \subset F^*T_y^*Y$ be the inverse images of $C_x \subset F^*T_x^*X$. Then, by the condition (1), the intersection $A_x \cap \ker(F^*T_y^*Y \to F^*T_v^*V)$ is a subset of the 0-section.

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Since $T^*_y Y \to T^*_x P$ induces an isomorphism $\text{Ker}(T^*_y Y \to T^*_v V) \to \text{Ker}(T^*_x P \to T^*_w Q)$, the intersection $\widetilde{C}_x \cap \text{Ker}(F^*T^*_y Y \to F^*T^*_w V)$ is a subset of the 0-section.

By the exact sequence $0 \to T^*_x P|_x \to T^*_x P \to T^*_x X \to 0$ and $x \in B(C)$, we have $F^*T^*_x P|_x \subset \widetilde{C}_x$. Hence $T^*_x P \to T^*_w Q$ induces an injection on $T^*_x P|_x$. Namely, the morphism $Q \to P$ and the immersion $U \to P$ are transversal on a neighborhood of $w$ by Lemma 2.2.5.

Hence the horizontal arrows of the commutative diagram

\[
\begin{array}{ccc}
T^*_w W & \leftarrow & T^*_v V \\
\uparrow & & \uparrow \\
T^*_x X & \leftarrow & T^*_y Y
\end{array}
\]

induce isomorphisms on the kernels and cokernels of the vertical arrows. Since the intersection of $C$ with $\text{Ker}(F^*T^*_x X \to F^*T^*_w W)$ is also a subset of the 0-section. Namely, $h$ is $C$-transversal on a neighborhood of $w$. Thus $h$ is $C$-transversal on a neighborhood of the inverse image of $B(C)$.

Further an elementary diagram chasing shows that the inverse image of $h^0C|_w$ by $F^*T^*_w W \leftarrow F^*T^*_v V$ equals the image of $A_x$ by $F^*T^*_y Y \to F^*T^*_w V$. Hence we have $g^0f_0C = f_0^0h^0C$.

(2)$\Rightarrow$(1): Let $w \in B(h^0C)$ be a closed point and let $v \in V, x \in X$ and $y \in Y$ be the image. Then, the commutative diagram (2.14) induces an isomorphism $\text{Ker}(T^*_y Y \to T^*_v V) \to \text{Ker}(T^*_x X \to T^*_w W)$ on the kernels. In the same notation, since the intersection of $C_x$ with $\text{Ker}(F^*T^*_x X \to F^*T^*_w W)$ is a subset of the 0-section, the intersection of $A_x$ with $\text{Ker}(F^*T^*_y Y \to F^*T^*_w V)$ is also a subset of the 0-section.

**Corollary 2.2.7.** Let $X, Y$ and $V$ be regular noetherian schemes satisfying the condition (F) and let $W$ be a cartesian diagram of morphisms of finite type. Assume that the subset $\text{Reg}(W) \subset W$ consisting of regular points is an open subset and that $f : X \to Y$ is an immersion. Then, the following conditions are equivalent:

1. The morphism $g$ is $F^*T^*_X Y|_{X_{F_p}}$-transversal.
2. The morphism $g : V \to Y$ is transversal with the immersion $f : X \to Y$ on a neighborhood of $W_{F_p} = V \times_X X_{F_p}$.

**Proof.** It suffices to apply Proposition 2.2.6 together with Lemma 2.2.6 to the 0-section $C$ of $F^*X|_{X_{F_p}}$.

**Definition 2.2.8.** Let $f : U \to X$ be an étale morphism of regular noetherian schemes satisfying the condition (F) and let $C'$ be a closed conical subset of $F^*U$. We identify $F^*U$ with the pull-back $F^*X \times_{X_{F_p}} U_{F_p}$ by the canonical isomorphism induced by $F\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_U \to F\Omega^1_U$ and let $\text{pr}_1 : F^*X \times_{X_{F_p}} U_{F_p} \to F^*X$ be the projection. Then, we define a closed conical subset $f_*C'$ of $F^*X$ to be the union of the closure $\overline{\text{pr}_1(C')} = f(U_{F_p})$ to the image of the complement.
Lemma 2.2.9. Let
\[
\begin{array}{ccc}
V & \longrightarrow & W \\
g & \downarrow & h \\
U & \underset{f}{\longrightarrow} & X
\end{array}
\]
be a cartesian diagram of regular noetherian schemes satisfying the condition (F) such that \( f \) is an étale morphism of finite type. Let \( C' \) be a closed conical subset of \( FT^*U \) and set \( C = f_*C' \subset FT^*X \) as in Definition 2.2.8.

If \( h \) is \( C \)-transversal, then \( h \) is smooth on a neighborhood of \( h^{-1}(X_{F_p} = f(U_{F_p})) \) and \( g \) is \( C' \)-transversal.

Proof. Assume that \( h \) is \( C \)-transversal. Since \( C \supset F^*T^*U \mid X_{F_p} \), the morphism \( h \) is smooth on a neighborhood of \( h^{-1}(X_{F_p} = f(U_{F_p})) \) by Lemma 2.2.2.1. Since \( f^*C \supset C' \), the morphism \( g \) is \( C' \)-transversal by Lemma 2.2.3. \( \square \)

2.3 \( C \)-acyclicity

We keep fixing a perfect field \( k \) of characteristic \( p > 0 \).

Definition 2.3.1. Let \( f: X \to Y \) be a morphism of finite type of regular noetherian schemes satisfying the condition (F) in Definition 2.1.7 and let \( C \) be a closed conical subset of the FW-cotangent bundle \( FT^*X \mid X_{F_p} \). We say that \( f \) is \( C \)-acyclic if the inverse image of \( C \) by the morphism \( FT^*Y \mid Y_{F_p} \times Y_{F_p} \cdot X_{F_p} \to FT^*X \mid X_{F_p} \) is a subset of the 0-section.

The corresponding notion is called \( C \)-transversality in [4, 1.2] and [17, Definition 3.5]. Here to avoid confusion with the \( C \)-transversality for morphisms to \( X \) in Definition 2.2.1.2, in [4, 1.2] and [17, Definition 3.3], we introduce another terminology. We will show in Lemma 2.3.4.2 that for a morphism \( f: X \to Y \) of regular schemes and a closed immersion \( i: Z \to X \) of regular schemes, the morphism \( f \) is \( F^*T^*_Z X \mid X_{F_p} \)-acyclic if and only if the composition \( fi \) is smooth on a neighborhood of \( Z_{F_p} \).

Lemma 2.3.2. Let \( f: X \to Y \) be a morphism of finite type of regular noetherian schemes satisfying the condition (F) and let \( C \) be a closed conical subset of \( FT^*X \mid X_{F_p} \):

1. The following conditions are equivalent:
   (1) \( f \) is \( C \)-acyclic.
   (2) \( f \) is smooth on a neighborhood of the base \( B(C) \subset X_{F_p} \) and the intersection of \( C \subset FT^*X \mid X_{F_p} \) with the image of the morphism \( FT^*Y \mid Y_{F_p} \times Y_{F_p} \cdot X_{F_p} \to FT^*X \mid X_{F_p} \) is a subset of the 0-section.

2. If \( C \) is the 0-section \( F^*T^*_X X \mid X_{F_p} \), the following conditions are equivalent:
   (1) \( f \) is \( C \)-acyclic.
   (2) \( f \) is smooth on the neighborhood of \( X_{F_p} \).

Proof. 1. The condition (1) is equivalent to the conjunction of the following \( (1') \) and \( (1'') \):
   (1') The inverse image of the 0-section by \( FT^*Y \mid Y_{F_p} \times Y_{F_p} \cdot X_{F_p} \to FT^*X \mid X_{F_p} \) on the base \( B(C) \subset X_{F_p} \) is a subset of the 0-sections.
(1') The intersection of $C \subset FT^*X|_{X_{F_p}}$ with the image of the morphism $FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} X_{F_p} \to FT^*X|_{X_{F_p}}$ is a subset of the 0-sections.

The condition (1') means that the morphism $f^*F\Omega^1_Y \to F\Omega^1_X$ is a locally splitting injection on a neighborhood of the base $B(C) \subset X_{F_p}$. Hence the assertion follows from Proposition 2.1.4.

2. For the 0-section $C = F^*T_X^*X|_{X_{F_p}}$, the base $B(C)$ is $X_{F_p}$ and the condition (1'') in the proof of 1 is satisfied. Hence the assertion follows from 1.

**Proposition 2.3.3.** Let $X, Y, V$ be regular noetherian schemes satisfying the condition (F) and let

$$
\begin{array}{ccc}
X & \xleftarrow{h} & W \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{g} & V
\end{array}
$$

be a cartesian diagram of morphisms of finite type. Let $C$ be a closed conical subset of $FT^*X|_{X_{F_p}}$. Then the following conditions are equivalent:

1. $f$ is $C$-acyclic on a neighborhood of the image $h(W_{F_p})$.
2. There exists a regular neighborhood $W_1 \subset W$ of the inverse image of the base $B(C)$ satisfying the following conditions: $f$ and $g$ are transversal on $W_1$, the restriction $h_1: W_1 \to X$ is $C$-transversal and the restriction $f'_1: W_1 \to V$ is $h_1^!C$-acyclic.

**Proof.** First, we show that the both conditions imply that $f$ is smooth on a neighborhood of the intersection $B(C) \cap h(W_{F_p})$. For (1), this follows from Lemma 2.3.2.1. For (2), similarly, $f'_1$ is smooth on a neighborhood of $B(h_1^!(B(C))) = h_1^{-1}(B(C))$. This implies that $f$ is smooth on a neighborhood of $h(h_1^{-1}(B(C))) = B(C) \cap h(W_{F_p})$ since $f$ and $g$ are transversal on $W_1$.

By replacing $X$ by a neighborhood of $B(C) \cap h(W_{F_p})$ smooth over $Y$, we may assume that $f$ is smooth. Then, $W$ is regular and $f$ and $g$ are transversal. By Proposition 2.1.4, we have a commutative diagram

$$
\begin{array}{ccc}
0 \to FT^*V|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p} & \longrightarrow & FT^*W|_{W_{F_p}} & \longrightarrow & F^*T^*W/V|_{W_{F_p}} & \to 0 \\
0 \to FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} & \longrightarrow & FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} & \longrightarrow & F^*T^*X/Y|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} & \to 0
\end{array}
$$

of exact sequences of vector bundles on $W_{F_p}$. Let $C' \subset FT^*W|_{W_{F_p}}$ be the image of $h^*C = C \times_{X_{F_p}} W_{F_p} \subset FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p}$ and let $A \subset FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}$ and $A' \subset FT^*V|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}$ be their inverse images.

Since the right vertical arrow is an isomorphism, the lower left arrow induces an isomorphism $\operatorname{Ker}(FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p} \to FT^*V|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}) \to \operatorname{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \to FT^*W|_{W_{F_p}})$. Hence $A \subset FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}$ is a subset of the 0-section if and only if $A' \subset FT^*V|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}$ and $h^*C \cap \operatorname{Ker}(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \to FT^*W|_{W_{F_p}})$ are subsets of the 0-sections and the assertion follows.

**Lemma 2.3.4.** Let $f: X \to Y$ be a morphism of finite type of regular noetherian schemes satisfying the condition (F).
1. Let $C$ be a closed conical subset of $FT^*X|_{X_{F_p}}$ and assume that $f$ is proper on the base $B(C)$. Let $g: Y \to Z$ be a morphism of finite type of regular noetherian schemes such that $Z_{F_p}$ is of finite type over $k$. Then the following conditions are equivalent:

   (1) $g$ is $f_0C$-acyclic.
   (2) $gf$ is $C$-acyclic.

2. Let $p: V \to X$ be a proper morphism of regular schemes and let $C = p_0F^*T^*_vV|_{Y_{F_p}} \subset FT^*X|_{X_{F_p}}$. Then, the following conditions are equivalent:

   (1) $f$ is $C$-acyclic.
   (2) The composition $fp$ is smooth on a neighborhood of $V_{F_p}$.

Proof. 1. Let $x \in X_{F_p}$ be a closed point and $y \in Y_{F_p}$ and $z \in Z_{F_p}$ be the images. Since the assertion is étale local, we may also assume that the morphisms $k(z) \to k(y) \to k(x)$ are isomorphisms.

   Let $A_x$ be the inverse image of $C_x$ by $F^*T^*_xX \leftarrow F^*T^*_yY$. Then, the inverse image $A'_x$ of $C_x$ by $F^*T^*_xX \leftarrow F^*T^*_xZ$ equals the inverse image $A''_x$ of $A_x$ by $F^*T^*_xY \leftarrow F^*T^*_xZ$. Since the condition (1) (resp. (2)) is equivalent to that $A'_x$ (resp. $A''_x$) is a subset of the 0-section for any $x$, the assertion follows.

2. By 1. applied to $p_0F^*T^*_vV|_{Y_{F_p}} = F^*T^*_vX|_{X_{F_p}}$, the condition (1) is equivalent to that the composition $fp$ is $F^*T^*_vV|_{Y_{F_p}}$-acyclic. Hence the assertion follows from Lemma 2.3.2.

Definition 2.3.5. Let $X$ be a regular noetherian scheme satisfying the condition (F) and let $C$ be a closed conical subset of $FT^*X|_{X_{F_p}}$. We say that a pair $(h, f)$ of morphisms $h: W \to X$, $f: W \to Y$ of finite type of regular noetherian schemes such that $Y_{F_p}$ is of finite type over $k$ is $C$-acyclic if the intersection of $(C \times_{X_{F_p}} W_{F_p}) \times_{W_{F_p}} (FT^*Y|_{Y_{F_p}})$ with $(FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p}) \times_{W_{F_p}} (FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p})$ with the kernel $\text{Ker}((h^*, f^*): (FT^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p}) \times_{W_{F_p}} (FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}) \to FT^*W|_{W_{F_p}})$ is a subset of the 0-section.

Lemma 2.3.6. Let $X$ be a regular noetherian scheme satisfying the condition (F) and let $C$ be a closed conical subset of $FT^*X|_{X_{F_p}}$.

1. Let $f: X \to Y$ be a morphism of finite type of regular noetherian schemes satisfying the condition (F). Then, the following conditions are equivalent:

   (1) $f$ is $C$-acyclic.
   (2) $(1_X, f)$ is $C$-acyclic.

2. Let $h: W \to X$ and $f: W \to Y$ be morphisms of finite type of regular noetherian schemes satisfying the condition (F). Then the following conditions are equivalent:

   (1) $(h, f)$ is $C$-acyclic.
   (2) $h$ is $C$-transversal and $f$ is $h_0C$-acyclic.

Proof. 1. Identify the kernel of $(1, f^*)$: $FT^*X|_{X_{F_p}} \times_{X_{F_p}} (FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} X_{F_p}) \to FT^*X|_{X_{F_p}}$ with the image of the injection $(f^*, -1): FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} X_{F_p} \to FT^*X|_{X_{F_p}} \times_{X_{F_p}} (FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} X_{F_p})$. Then the inverse image in $FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} X_{F_p}$ of $C \times_{X_{F_p}} (FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} X_{F_p}) \subset FT^*X|_{X_{F_p}} \times_{X_{F_p}} (FT^*Y|_{Y_{F_p}} \times_{Y_{F_p}} X_{F_p})$ is the same as the inverse image of $C \subset T^*X$ and the assertion follows.
2. Since \( \text{Ker}(h^*F^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p} \to F^*W|_{W_{F_p}}) \times 0 \subset \text{Ker}((h^*, f^*): (F^*X|_{X_{F_p}} \times_{X_{F_p}} W_{F_p}) \times_{W_{F_p}} (F^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}) \to F^*W|_{W_{F_p}}) \), the \( C \)-acyclicity of \((h, f)\) implies the \( C \)-transversality of \( h \). By 1., the \( h^*C \)-acyclicity of \( f \) is equivalent to the condition that the intersection of \( h^*C \times_{W_{F_p}} (F^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}) \) with \( \text{Ker}(F^*Y|_{W_{F_p}} \times_{W_{F_p}} (F^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}) \to F^*W|_{W_{F_p}}) \) is a subset of the 0-section. This condition is equivalent to the \( C \)-acyclicity of \((h, f)\) since \((h^*C \times_{W_{F_p}} (F^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p})) \) is the image of \( h^*C \times_{W_{F_p}} (F^*Y|_{Y_{F_p}} \times_{Y_{F_p}} W_{F_p}) \).

\[ \blacksquare \]

3 Micro-support

We fix a perfect field \( k \) of characteristic \( p > 0 \) and a finite field \( \Lambda \) of characteristic \( \ell \neq p \). We will assume that a regular noetherian scheme \( X \) over \( \mathbb{Z}_{(p)} \) satisfies the condition \((F)\) in Definition 2.1.7.

3.1 Micro-support

Definition 3.1.1. Let \( X \) be a regular noetherian scheme over \( \mathbb{Z}_{(p)} \) satisfying the condition \((F)\) in Definition 2.1.7 and let \( C \) be a closed conical subset of the \( FW \)-cotangent bundle \( F^*X|_{X_{F_p}} \). Let \( F \) be a constructible complex of \( \Lambda \)-modules on \( X \). We say that \( F \) is micro-supported on \( C \) if the following conditions (1) and (2) are satisfied:

1. The intersection of the support \( \text{supp} F \) with the closed fiber \( X_{F_p} \) is a subset of the base \( B(C) \).

2. Every \( C \)-transversal separated morphism \( h: W \to X \) of finite type of regular schemes is \( F \)-transversal on a neighborhood of the closed fiber \( W_{F_p} \).

This definition of micro-support is related to [17] Proposition 8.13 but is different from [1] 1.3. We discuss this point in Remark after Proposition 3.1.3. It is a property on a neighborhood of \( X_{F_p} \). If \( X_Q = X \times_{\text{Spec} \mathbb{Z}_{(p)}} \text{Spec} Q \) is smooth over a field \( K \) of characteristic 0, to cover \( X_Q \), one can use the micro-support of the restriction of \( F \) on \( X_Q \) defined as closed conical subset of the cotangent bundle \( T^*X_Q/K \).

Lemma 3.1.2. Let \( X \) be a regular noetherian scheme over \( \mathbb{Z}_{(p)} \) satisfying the condition \((F)\) and \( F \) be a constructible complex of \( \Lambda \)-modules.

1. \( F \) is micro-supported on \( F^*X|_{X_{F_p}} \).

2. If \( F \) is locally constant on a neighborhood of the closed fiber \( X_{F_p} \), then \( F \) is micro-supported on the 0-section \( F^*T^*_X|_{X_{F_p}} \).

3. Assume that \( X \) is a smooth scheme over \( k \). Let \( C \subset T^*X \) be a closed conical subset and \( F^*C \subset F^*T^*X = F^*X \) be the pull-back of \( C \). Then, \( F \) is micro-supported on \( C \) in the sense of ([1] 1.3, [17] Definition 4.1) if and only if \( F \) is micro-supported on \( F^*C \).

We show the converse of 2 in Corollary 3.1.7.

Proof. 1. Let \( h: W \to X \) be a separated morphism of finite type of regular schemes. If \( h \) is \( FT^*X|_{X_{F_p}} \)-transversal, then \( h \) is smooth on a neighborhood of \( W_{F_p} \) by Lemma 2.2.21. Hence \( h \) is \( F \)-transversal on a neighborhood of \( W_{F_p} \) by Lemma 1.1.61.
2. Let \( h: W \to X \) be a separated morphism of finite type of regular schemes. Then, since \( \mathcal{F} \) is locally constant on a neighborhood of the closed fiber \( X_{F_p} \), \( h \) is \( \mathcal{F} \)-transversal on a neighborhood of \( W_{F_p} \) by Lemma 1.1.6.2.

3. Let \( h: W \to X \) be a separated morphism of finite type of regular schemes. Then, \( h: W \to X \) is a separated morphism of smooth schemes of finite type over \( k \). The morphism \( h: W \to X \) is \( F^*C \)-transversal if and only if \( h: W \to X \) is \( C \)-transversal. Hence the equivalence follows from [17, Proposition 8.13].

**Proposition 3.1.3.** Let \( X \) be a regular scheme over \( \mathbb{Z}_{(p)} \) satisfying the condition (F) and \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules. Let \( C \) be a closed conical subset of \( FT^*X|_{X_{F_p}} \) such that \( \mathcal{F} \) is micro-supported on \( C \).

1. Let \( h: W \to X \) be a separated morphism of finite type of regular schemes. If \( h \) is \( C \)-transversal, then \( h \) is \( \mathcal{F} \)-transversal on a neighborhood of \( W_{F_p} \) and \( h^*\mathcal{F} \) is micro-supported on \( h^*C \).

2. Let \( f: X \to Y \) be a separated morphism of finite type proper on the base \( B(C) \) of regular quasi-excellent noetherian schemes satisfying the condition (F). Then \( Rf_*\mathcal{F} \) is micro-supported on \( f_*C \).

**Proof.** 1. Let \( g: V \to W \) be an \( h^*C \)-transversal separated morphism of finite type of regular noetherian schemes. Then, by Lemma 2.2.3 \( hg \) and \( h \) are \( C \)-transversal. Since \( \mathcal{F} \) is micro-supported on \( C \), \( hg \) and \( h \) are \( \mathcal{F} \)-transversal on neighborhoods of \( V_{F_p} \) and of \( W_{F_p} \) respectively. Hence by Proposition 1.1.8.1, \( g \) is \( h^*\mathcal{F} \)-transversal on a neighborhood of \( V_{F_p} \).

2. Let \( g: V \to Y \) be an \( f_*C \)-transversal separated morphism of finite type of regular noetherian schemes and let

\[
\begin{array}{ccc}
X & \xleftarrow{h} & W \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xleftarrow{g} & V
\end{array}
\]

be a cartesian diagram. Then, \( f \) and \( g \) are transversal on a regular neighborhood \( W_1 \subset W \) of the inverse image of \( B(C) \) and \( h_1 = h|_{W_1} \colon W_1 \to X \) is \( C \)-transversal by Proposition 2.2.6. Since \( \mathcal{F} \) is micro-supported on \( C \), the restriction \( h_1 \colon W_1 \to X \) is \( \mathcal{F} \)-transversal.

Since the intersection of \( \text{supp} \mathcal{F} \) with \( X_{F_p} \) is a subset of \( B(C) \), the intersection of \( A = \text{supp} h^*\mathcal{F} \) with \( W_{F_p} \) is a subset of \( W_1 \). Since the closed set \( A = A \cap W_1 \) does not intersect the closed fiber \( W_{F_p} \), the complement \( V_0 = V - f'(A - A \cap W_1) \) is an open neighborhood of \( V_{F_p} \). By replacing \( V \) by \( V_0 \), we may assume \( A = \text{supp} h^*\mathcal{F} \subset W_1 \). Then, \( h: W \to X \) is \( \mathcal{F} \)-transversal. Since \( f \) and \( g \) are transversal on \( W_1 \), the base change morphism \( f'^*Rg^i\Lambda \to Rh^i\Lambda \) is an isomorphism on \( W_1 \) by Lemma 1.1.3. Hence \( g \) is \( Rf_*\mathcal{F} \)-transversal on \( V \) by Corollary 1.1.9.1.

We show that being micro-supported is an étale local property.

**Lemma 3.1.4.** Let \( X \) be a regular noetherian scheme over \( \mathbb{Z}_{(p)} \) satisfying the condition (F) and let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules.

1. Let \( C \) be a closed conical subset of \( FT^*X|_{X_{F_p}} \) and let \( (f_i: U_i \to X)_{i \in I} \) be an étale covering. Then, the following conditions are equivalent:

\begin{enumerate}
\item \( \mathcal{F} \) is micro-supported on \( C \).
\end{enumerate}
(2) For every \( i \in I \), the pull-back \( f_i^* F \) is micro-supported on \( C_i = f_i^* C \).

2. Let \( f: U \to X \) be an étale morphism of finite type and let \( F_U = f^* F \) be the restriction. Let \( C' \subset FT^* U \) be a closed conical subset and let \( C = f_* C' \) be the closed conical subset defined in Definition 2.2.8. If \( F_U \) is micro-supported on \( C' \), then \( F \) is micro-supported on \( C \).

Proof. 1. The equivalence for the condition (1) in Definition 3.1.1 on the support is verified easily. We show the equivalence for the condition (2) in Definition 3.1.1 on transversality.

(1) \( \Rightarrow \) (2): Let \( i \in I \) and let \( h: W \to U_i \) be a separated morphism of finite type of regular schemes. If \( h \) is \( C_i \)-transversal, then since \( f_i \) is étale, the composition \( f_i \circ h \) is \( C \)-transversal by Lemma 2.2.3. Hence \( f_i \circ h \) is \( F \)-transversal on a neighborhood of the closed fiber and consequently \( h \) is \( F_i \)-transversal on a neighborhood of the closed fiber.

(2) \( \Rightarrow \) (1): Let \( h: W \to X \) be a separated morphism of finite type of regular schemes. Assume \( h \) is \( C \)-transversal. Then, for every \( i \in I \), the base change \( h_i: W_i = W \times_X U_i \to U_i \) is \( C_i \)-transversal by Lemma 2.2.3. Hence \( h_i \) is \( F_i \)-transversal on a neighborhood of the closed fiber for every \( i \in I \) and consequently \( h \) is \( F \)-transversal on a neighborhood of the closed fiber.

2. The condition (1) in Definition 3.1.1 on the support is verified easily. We show that the condition (2) in Definition 3.1.1 on transversality is satisfied. Let \( h: W \to X \) be a separated morphism of finite type of regular noetherian schemes. Assume that \( h: W \to X \) is \( C \)-transversal. Then, by Lemma 2.2.9 the morphism \( h: W \to X \) is smooth on a neighborhood \( W_1 \) of \( h^{-1}(X_{F_p} - f(U_{F_p})) \). Hence \( h|_{W_1}: W_1 \to X \) is \( F \)-transversal. Further, by Lemma 2.2.9 the base change \( W \times_X U \to U \) of \( h \) is \( C' \)-transversal. Since \( F_U \) is micro-supported on \( C' \), the morphism \( W \times_X U \to U \) is \( F_U \)-transversal on a neighborhood of the closed fiber. Since \( W_1 \) and \( W \times_X U \) form an étale covering of \( W \), the morphism \( h \) is \( F \)-transversal on a neighborhood of the closed fiber.

In the rest of this subsection, let \( O_K \) be a discrete valuation ring with residue field \( k \) such that the fraction field \( K \) is of characteristic 0. Recall that \( O_K \) is excellent. For a scheme \( X \) over \( O_K \), the closed fiber \( X_k = X \times_{\text{Spec } O_K} \text{Spec } k \) has the same underlying set as \( X_{F_p} = X \times_{\text{Spec } F_p} \text{Spec } F_p \).

Proposition 3.1.5. Let \( h: W \to X \) and \( f: W \to Y \) be morphisms of regular schemes of finite type over \( O_K \). Let \( F \) be a constructible complex of \( \Lambda \)-modules and \( C \) be a closed conical subset of \( FT^* X|_{X_k} \). Suppose that \( F \) is micro-supported on \( C \). If the pair \((h,f)\) is \( C \)-acyclic, then \( f: W \to Y \) is \( h^* F \)-acyclic along \( W_k \).

Proof. By Lemma 2.3.6, \( h: W \to X \) is \( C \)-transversal and \( f: W \to Y \) is \( h^* C \)-acyclic. Since \( F \) is micro-supported on \( C \), the pull-back \( h^* F \) is micro-supported on \( h^* C \) by Proposition 3.1.3. Hence by replacing \( X \) by \( W \), we may assume \( W = X \).

Since \( f: X \to Y \) is \( C \)-acyclic, the morphism \( f \) is smooth on a neighborhood of the intersection \( B(C) \cap X_k \supset \text{supp } F \cap X_k \) by Lemma 2.3.2. Hence, we may assume \( f: X \to Y \) is smooth.

Let \( V \to Y \) be a separated morphism of regular schemes of finite type over \( O_K \). Then the projection \( p: U = V \times_Y X \to X \) is \( C \)-transversal by Proposition 2.3.3. Hence \( p \) is.
$\mathcal{F}$-transversal on a neighborhood of $U_k$. Thus by Proposition 1.2.4, $f$ is $\mathcal{F}$-acyclic along $X_k$.

**Remark 3.1.6.** The conclusion of Proposition 3.1.5 is an analogue of the original condition defining the micro-support in [4, 1.3]. In the geometric case, this is shown to be equivalent in [17, Proposition 8.13] to the condition analogous to that in Definition 3.1.1. However the following example shows that the condition is too weak in the setting of this article.

Let $X$ be a smooth scheme over $\mathcal{O}_K$ and $C = F^*T^*_X|_{X_k}$ be the conormal bundle of the closed fiber. Let $(h, f)$ be a $C$-acyclic pair of morphisms of regular schemes of finite type over $\mathcal{O}_K$. Then, since $h: W \to X$ is transversal to the immersion $X_k \to X$, the closed fiber $W_k$ is regular and $W$ is smooth over $\mathcal{O}_K$ on a neighborhood of $W_k$ by Lemma 2.3.6.2 and Corollary 2.2.7. Since $f: W \to Y$ is $F^*T^*_W|_{W_k}$-acyclic, further by Lemma 2.3.6.2 and Lemma 2.3.4.2, the morphism $W \to Y$ is smooth on a neighborhood of $W_k$ and $W_k \to Y$ is also smooth. This means that $W_k$ is empty. Thus any $F$ satisfies the conclusion of Proposition 3.1.5.

**Corollary 3.1.7.** Let $X$ be a regular scheme of finite type over $\mathcal{O}_K$ and $F$ be a constructible complex of $\Lambda$-modules. Then, the following conditions are equivalent.

1. $F$ is locally constant on a neighborhood of the closed fiber $X_k$.
2. $F$ is micro-supported on the 0-section $F^*T^*_X|_{X_k}$.

**Proof.** (1)$\Rightarrow$(2) is proved in Lemma 3.1.2.2.

(2)$\Rightarrow$(1): By Proposition 3.1.5 applied to $(1_X, 1_X)$, the identity $1_X: X \to X$ is $F$-acyclic along $X_k$. Hence $F$ is locally constant on a neighborhood of $X_k$ by Lemma 1.2.3.2.

### 3.2 Singular support

**Definition 3.2.1.** Let $X$ be a regular noetherian scheme over $\mathbb{Z}_{(p)}$ satisfying the condition (F) in Definition 2.1.7 and $F$ be a constructible complex of $\Lambda$-modules on $X$. We say that a closed conical subset $C \subset FT^*X|_{X_{\mathbb{F}_p}}$ of the FW-cotangent bundle is the singular support $SSF$ of $F$ if the following condition is satisfied: For any closed conical subset $C' \subset FT^*X|_{X_{\mathbb{F}_p}}$, $F$ is micro-supported on $C'$ if and only if $C \subseteq C'$.

If the singular support $SSF$ exists, it is the intersection of closed conical subsets $C \subset FT^*X|_{X_{\mathbb{F}_p}}$ on which $F$ is micro-supported. The author does not know how to prove the existence of the singular support in general. We compute the singular supports in some cases.

**Lemma 3.2.2.** Let $X$ be a regular noetherian scheme over $\mathbb{Z}_{(p)}$ such that $X_{\mathbb{F}_p}$ is of finite type over $k$ and $F$ be a constructible complex of $\Lambda$-modules on $X$. We consider the following condition:

1. $X$ is of finite type over $\mathcal{O}_K$ for a discrete valuation field $K$ of characteristic 0 with residue field $k$.

We consider the following conditions:

1. $SSF$ is the 0-section $FT^*_X|_{X_{\mathbb{F}_p}}$. 

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(2) $\mathcal{F}$ is locally constant on a neighborhood of $X_{F_p}$ and $X_{F_p}$ is a subset of the support of $\mathcal{F}$.
We have $\⇒(1)$. If (DVR) is satisfied, we have $(1)⇒(2)$.
2. We consider the following conditions:
(1) $SS\mathcal{F} = \emptyset$.
(2) $\mathcal{F} = 0$ on a neighborhood of $X_{F_p}$.
We have $(2)⇒(1)$. If (DVR) is satisfied, we have $(1)⇒(2)$.
3. Assume that $X = \text{Spec} \mathcal{O}_K$ for a discrete valuation ring $\mathcal{O}_K$ as in (DVR). If $\mathcal{F}$ is not locally constant, we have $SS\mathcal{F} = FT^*X|_{X_k}$.

Recall that a discrete valuation ring $\mathcal{O}_K$ as in (DVR) is excellent by [Scholie (7.8.3)].

Proof. 1. $(1)⇒(2)$: Since $\mathcal{F}$ is micro-supported on the 0-section $F^*T^*X|_{X_{F_p}}$, by Corollary 3.1.7, $\mathcal{F}$ is locally constant on a neighborhood of $X_{F_p}$. After replacing $X$ by a neighborhood of $X_{F_p}$, we may assume that $\mathcal{F}$ is locally constant. Then, the support $Z = \text{supp}\mathcal{F}$ is an open and closed subset of $X$ and $\mathcal{F}$ is micro-supported on the 0-section $T^*_ZX|_{X_{F_p}}$ on $Z$.
By the minimality of the singular support, we have $T^*_ZX|_{X_{F_p}} = T^*_X|_{X_{F_p}}$ and $X_{F_p} \subset Z$.

$(2)⇒(1)$: Since $\mathcal{F}$ is locally constant on a neighborhood of $X_{F_p}$, by Lemma 3.1.2, $\mathcal{F}$ is micro-supported on the 0-section $F^*T^*X|_{X_{F_p}}$. Suppose $\mathcal{F}$ is micro-supported on a closed conical subset $C \subset F^*X|_{X_{F_p}}$. Since $X_{F_p} \subset \text{supp}\mathcal{F}$, we have $X_{F_p} \subset B(C)$. This is equivalent to $F^*T^*_X|_{X_{F_p}} \subset C$ and we obtain $F^*T^*_X|_{X_{F_p}} = SS\mathcal{F}$.

2. $(1)⇒(2)$: Since the intersection $\text{supp}\mathcal{F} \cap X_{F_p}$ is a subset of $SS\mathcal{F} = \emptyset$, we have $X_{F_p} \subset X = \text{supp}\mathcal{F}$ and the condition $(2)$ holds.

$(2)⇒(1)$: Since every separated morphism $h: W \to X$ is $\mathcal{F}$-transversal on a neighborhood of $W_{F_p}$ and since the intersection $\text{supp}\mathcal{F} \cap X_{F_p}$ is empty, $\mathcal{F}$ is micro-supported on $\emptyset$.

3. By Lemma 3.1.2, $\mathcal{F}$ is micro-supported on $F^*X|_{X_k}$. Suppose that $\mathcal{F}$ is micro-supported on a closed conical subset $C \subset F^*X|_{X_k}$. Since $F^*X|_{X_k}$ is a line bundle over the point $\text{Spec} k$, $C$ is either $\emptyset$, the 0-section $F^*T^*_X|_{X_k}$ or $F^*X|_{X_k}$ itself. Since $\mathcal{F}$ is not locally constant, by the contraposition of Corollary 3.1.7 $(2)⇒(1)$, $\mathcal{F}$ is not micro-supported on the 0-section $F^*T^*_X|_{X_k}$.

We show that being singular support is a local property.

**Lemma 3.2.3.** Let $X$ be a regular noetherian scheme over $\mathbb{Z}_{(p)}$ satisfying the condition (F) and let $\mathcal{F}$ be a constructible complex of $\mathcal{A}$-modules. Let $C$ be a closed conical subset of $F^*X|_{X_{F_p}}$ and let $(f_i: U_i \to X)_{i \in I}$ be an étale covering. We consider the following conditions:

(1) $C = SS\mathcal{F}$.
(2) For every $i \in I$ and the pull-backs $C_i = f_i^*C$ and $\mathcal{F}_i = f_i^*\mathcal{F}$, we have $C_i = SS\mathcal{F}_i$.

We have $(2)⇒(1)$. If $(f_i: U_i \to X)_{i \in I}$ is a Zariski covering, we have $(1)⇒(2)$ conversely.

Proof. Since the equivalence of the condition to be micro-supported is proved in Lemma 3.1.4, we show the implications for the minimality.

$(1)⇒(2)$: Let $i \in I$ and let $C' \subset F^*U_i$ be a closed conical subset on which $\mathcal{F}_i$ is micro-supported. Then, by Lemma 3.1.3, $\mathcal{F}$ is micro-supported on $f_i^*C'$ in the notation of
Definition 2.2.8. Hence, we have \( f_i \circ C' \supset SSF = C \). If \( f_i : U_i \to X \) is an open immersion, we have \( C' \supset f_i^* \circ C = C_i \).

(2) \( \Rightarrow \) (1): Let \( C' \subset FT^*X \) be a closed conical subset on which \( F \) is micro-supported. Then, for every \( i \in I \), the pull-back \( F_i \) is micro-supported on \( f_i^* C' \) by Lemma 3.1.4. Hence, we have \( f_i^* C' \supset f_i^* \circ C \). Since \( (U_i \to X)_{i \in I} \) is an étale covering, we have \( C' \supset C \).

We compute the singular supports of some sheaves on regular schemes of finite type over a discrete valuation ring.

Lemma 3.2.4. Let \( K \) be a discrete valuation field of characteristic 0 such that the residue field \( k \) is a perfect field of characteristic \( p > 0 \). Let \( h : W \to X \) be a finite surjective morphism of regular flat schemes of finite type over \( \mathcal{O}_K \) such that the morphism \( W_K \to X_K \) on the generic fiber is étale. Assume that the reduced parts \( D = X_{k,\text{red}} \) and \( E = W_{k,\text{red}} \) of the closed fibers are irreducible and are smooth of dimension \( \geq 1 \) over the residue field \( k \).

Assume that the following condition is satisfied:

1. The cokernel of the canonical morphism \( F\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_E \to F\Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{O}_E \) of locally free \( \mathcal{O}_E \)-modules is locally free of rank 1.
2. The direct image \( C = \pi_* FT^*_W \mathcal{W} \mid_E \subset FT^*_X \mid_D \) of the 0-section is the image of the sub line bundle \( \text{Ker}(FT^*_X \mid_D \times_D E \to FT^*_W \mathcal{W} \mid_E) \) of \( FT^*_X \mid_D \times_D E \).
3. Further assume that the following condition is satisfied:

   (2) The finite morphism \( E \to D \) is purely inseparable of degree \( \geq 1 \).

Then, for each closed point \( x \in D \) and for the point \( w \in E \) above \( x \), there exists a regular subscheme \( Z \subset W \) of codimension 1 containing \( w \) and flat over \( \mathcal{O}_K \) satisfying the following conditions:

The composition \( Z \to W \to X \) is unramified. The pull-back \( C \times_{X_{\text{Fp}}} w \subset FT^*_X \times_{X_{\text{Fp}}} w \) of the fiber at \( x \) equals the fiber of the kernel of the surjection \( FT^*_X \times_{X_{\text{Fp}}} Z \to FT^*_Z \).

Proof. 1. Since the \( \mathcal{O}_E \)-linear morphism \( F\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_E \to F\Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{O}_E \) of locally free \( \mathcal{O}_E \)-modules of the same rank has the cokernel of rank 1, the kernel is also locally free of rank 1. Hence the assertion follows.

2. Let \( n = \dim \mathcal{O}_{X,x} \). Since \( E \to D \) is assumed to be purely inseparable, the residue field \( k(w) \) is a purely inseparable extension of a perfect field \( k(x) \) and hence the morphism \( k(x) \to k(w) \) is an isomorphism. By the assumption on the rank of the cokernel and by Proposition 21.3, the \( k(x) \)-linear mapping \( m_x/m_x^2 \to m_w/m_w^2 \) induced by \( \mathcal{O}_{X,w} \to \mathcal{O}_{W,w} \) is of rank \( n - 1 \).

Take an element of \( m_w/m_w^2 \) not contained in the image of \( m_x/m_x^2 \) and take its lifting \( f \in m_w \) not divisible by a prime element \( t \) defining the divisor \( E \subset W \). Then, a regular closed subscheme \( Z \) of codimension 1 of a neighborhood of \( w \) is defined by \( f \). Let \( z \) denote \( w \in W \) regarded as a point of \( Z \). Since \( f \) is not divisible by \( t \), we may assume that \( Z \) is flat over \( \mathcal{O}_K \).

Since \( f \in m_w/m_w^2 \) is not contained in the image of \( m_x/m_x^2 \), the induced morphism \( m_x/m_x^2 \to m_w/((f) + m_w^2) = m_z/m_z^2 \) is a surjection. Hence further shrinking \( Z \) if necessary, we may assume that \( Z \to X \) is unramified. Since the kernel of the surjection \( m_x/m_x^2 \to m_z/m_z^2 \) equals the kernel of \( m_x/m_x^2 \to m_w/m_w^2 \), the last condition on the fibers is satisfied. 

\[ \square \]
We show that some concrete examples of Kummer coverings satisfy the assumptions in Lemma 3.2.4. Let $K$ be a discrete valuation field as in Lemma 3.2.4 containing a primitive $p$-th root of 1. Let $X$ be a regular flat scheme of finite type over $\mathcal{O}_K$ and assume that the reduced part $D = X_{k, \text{red}}$ is smooth over the residue field $k$. Let $L$ be the local field at the generic point of $D$ and let $e = \text{ord}_L p \geq p - 1$ be the absolute ramification index.

Lemma 3.2.5. Let $\pi \in \Gamma(X, \mathcal{O}_X)$ be a uniformizer of the divisor $D = X_{k, \text{red}} \subset X$ and let $u \in \Gamma(X, \mathcal{O}_X^\times)$ be a unit. Let $1 \leq n < \frac{pe}{p-1}$ be an integer congruent to 0 or 1 modulo $p$ and set $n = pm$ or $n = pm + 1$ respectively. In the case $n = pm$, assume that $u$ defines locally a part of a basis of $\Omega^1_D$. Define a Kummer covering $V \to U = X_K$ by $v^n = 1 + u\pi^n$.

1. The normalization $\pi: W \to X$ in $V$ is regular. The reduced closed fiber $E = W_{k, \text{red}}$ is smooth over $k$ and the finite morphism $E \to D$ is purely inseparable.

2. The cokernel $\text{Coker}(F \Omega^*_X \otimes_{\mathcal{O}_X} \mathcal{O}_E \to F \Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{O}_E)$ is an invertible $\mathcal{O}_E$-module.

3. Assume $n = pm$. If $e = m+1$, let $\pi'$ denote the uniformizer $p/\pi^m$. Then, the kernel of the canonical morphism $F \pi^* X \otimes_D E \to F \pi^* W|_E$ is a line bundle spanned by

$$\begin{cases}
w(u) - u \cdot w(\pi') & \text{if } p = 2 \text{ and } e = m + 1, \\
w(u) & \text{otherwise.}
\end{cases}$$

Proof. 1. Since the assertion is local, we may assume that $X = \text{Spec} A$ is affine. We show that the normalization $B$ of $A$ is generated by $t = (v - 1)/\pi^m$. By the assumption $n < \frac{pe}{p-1}$, we have $e + m > pm$ and the polynomial $(1 + \pi^m T)^p - 1 \in A[T]$ is divisible by $\pi^m$. Define a monic polynomial $F \in A[T]$ by $1 + \pi^m F = (1 + \pi^m T)^p$. Since $F \equiv T^p \mod \pi A$ and since $u$ is a unit, in the case $n = pm + 1$, the equation $F = \pi u$ is an Eisenstein equation. In the case $n = pm$, the reduction of the equation $F = u \pi A$ gives $T^p = u$. In this case $du$ is a part of a basis of $\Omega^1_D$ by the assumption. Hence by setting $v = 1 + \pi^m t$ where $t \in B$ denotes the class of $T$, we obtain $B = A[T]/(F - u\pi)$ or $B = A[T]/(F - u)$ respectively.

The reduced part $E$ is defined by $t$ or $\pi$ according to $n = pm + 1$ or $n = pm$ respectively. Hence $E$ is smooth over $k$ and the finite morphism $E \to D$ is purely inseparable of degree 1 or $p$ respectively.

2. By Corollary 2.1.6, we have a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & F^* N_{D/X} \otimes_{\mathcal{O}_D} \mathcal{O}_E \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F^* N_{E/W} \\
\end{array}
\begin{array}{ccc}
& \longrightarrow & F \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_E \\
\downarrow & & \downarrow \\
& \longrightarrow & F \Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{O}_E \\
\end{array}
\begin{array}{ccc}
\longrightarrow & & \longrightarrow 0 \\
\longrightarrow & & \longrightarrow 0
\end{array}
$$

of exact sequences of locally free $\mathcal{O}_E$-modules. In the case $n = pm + 1$, the right vertical arrow is an isomorphism since $E \to D$ is an isomorphism. Further the left vertical arrow is 0 since the ramification index is $p$. In the case $n = pm$, the left vertical arrow is an isomorphism since the ramification index is 1. Further the cokernel of the right vertical arrow is locally free of rank 1 since $E \to D$ is a purely inseparable covering defined by $T^p = u$ and $du$ is a part of a basis of $\Omega^1_D$. Hence the assertion follows.
3. We compute the polynomial $F \mod \pi^2$. Recall that we have $e + m > pm$. Since $e$ is divisible by $p - 1$, the equality $e + m = pm + 1$ holds if and only if $p = 2$ and $e = m + 1$. Hence the coefficients of $T^i$ for $i = 1, \ldots, p - 1$ in the polynomial $F$ are divisible by $\pi^2$ except $F = T^2 + 2/\pi^m \cdot T$ in the exceptional case.

Thus, except the exceptional case, we have a congruence $F \equiv T^p \mod \pi^2$ and hence the kernel is spanned by $w(u)$. In the exceptional case, we have $t^2 + \pi' t = u$ for $\pi' = 2/\pi^m$. Hence $w(u)$ is sent to $t^2 \cdot w(\pi') = u \cdot w(\pi')$.

**Proposition 3.2.6.** Let $K$ be a discrete valuation field of characteristic 0 such that the residue field $k$ is a perfect field of characteristic $p > 0$. Let $X$ be a regular flat scheme of finite type over $O_K$ such that the reduced part $D = X_{k,\text{red}}$ is irreducible and is smooth over the residue field $k$.

Let $F_U$ be a locally constant constructible sheaf of $\Lambda$-modules on the generic fiber $U = X_K$ and let $F = j_! F_U$ be the $0$-extension for the open immersion $j : U \to X$. Let $V \to U$ be a finite étale Galois covering of Galois group $G$ such that the pull-back $F_V$ is constant and let $\pi : W \to X$ be the normalization of $X$ in $V$.

Assume that $W$ is regular and that the reduced part $E = W_{k,\text{red}}$ is also irreducible and smooth over the residue field $k$. Assume that the order of $G$ is invertible in $\Lambda$ and that $F_U$ corresponds to a non-trivial irreducible representation $M$ of $G$.

1. The canonical morphism $\mathcal{F} = j_! F_U \to R j_* F_U$ is an isomorphism.
2. Assume that conditions (1) and (2) in Lemma $3.2.4$ are satisfied. Then, the singular support $SS \mathcal{F}$ equals the direct image $C = \pi_* FT^*_W W|_{W_k}$ of the $0$-section.

**Proof.** 1. By the assumption that the order of $G$ is invertible in $\Lambda$ and that $M$ is an irreducible representation, the locally constant sheaf $F_U$ is isomorphic to a direct summand of $\pi_K_* \Lambda$ where $\pi_K : V = W_K \to U = X_K$ is the restriction of $\pi$.

Let $j_W : W_K \to W$ be an open immersion of the generic fiber. Since $W$ is regular and the reduced part of the closed fiber $W_k$ is a regular divisor, we have isomorphisms $\Lambda \to j_{W*} \Lambda$, $\Lambda_E(-1) \to R^1 j_{W*} \Lambda$ and $R^q j_{W*} \Lambda = 0$ for $q \neq 0, 1$ by the absolute purity [16, THÉORÈME 3.1.1]. Similarly, we have isomorphisms $\Lambda \to j_* \Lambda$, $\Lambda_D(-1) \to R^1 j_* \Lambda$ and $R^q j_* \Lambda = 0$ for $q \neq 0, 1$. Since $E \to D$ induces a homeomorphism on the étale site by the assumption, the canonical morphism $\Lambda_E \to \pi_* \Lambda_E$ is an isomorphism. Hence, for the cokernel $G = \text{Coker}(\Lambda_X \to \pi_* \Lambda_W)$, the canonical morphisms $j_! j^* G \to G \to R j_* j^* G$ are isomorphisms.

Since $M$ is a non-trivial irreducible representation of a semi-simple algebra $\Lambda[G]$, the corresponding sheaf $\mathcal{F}$ is a direct summand of $j^* G$. Hence the canonical morphism $\mathcal{F} = j_! F_U \to R j_* F_U$ is an isomorphism.

2. Since $\mathcal{F}$ is a direct summand of $\pi_* \Lambda_W = \Lambda_X \oplus G$, by Proposition $3.1.3$ 2, the constructible sheaf $\mathcal{F}$ is micro-supported on $C = \pi_* FT^*_W W|_{W_k}$.

Suppose $\mathcal{F}$ is micro-supported on a closed conical subset $C'$. It suffices to prove $C \subset C'$. Let $x \in X_{F_p}$ be a closed point, let $h : Z \to X$ be an unramified morphism as in Lemma $3.2.4$ and let $z \in Z$ be the unique point above $x$. Since $Z \to X$ factors through $Z \to W$, the restriction $\mathcal{F}_{Z \cap U}$ is constant. Hence the morphism $h$ is not $\mathcal{F}$-transversal by the contraposition of Proposition $1.1.8$ 2 (1)⇒(2) and 1. Since $\mathcal{F}$ is micro-supported on $C'$, the morphism $h$ is not $C'$-transversal, on any open neighborhood of $z \in Z$. 

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The kernel $L = \text{Ker}(FT^*X \times_{X_{\mathbb F_p}} Z_{\mathbb F_p} \to FT^*Z)$ is a line bundle on $Z_{\mathbb F_p}$. The intersection $C'_1 = h^*C' \cap L \subset FT^*X \times_{X_{\mathbb F_p}} Z_{\mathbb F_p}$ is a closed conical subset of $L$. Let $Z_1 = \{y \in Z_{\mathbb F_p} \mid C'_{1,y} = L_y\}$ be the image by the projection of the complement $C'_1 = (C'_1 \cap Z_{\mathbb F_p})$ of the 0-section. Since $C'_1 \subset L$ is a closed conical subset, the image $Z_1 \subset Z_{\mathbb F_p}$ is a closed subset. Since the restriction $Z \to X$ of $h$ is $C'$-transversal, the complement $Z \setminus Z_1$ is not an open neighborhood of $z$. Namely, we have $z \in Z_1$ and hence $C'_{1,z} = L_z$ is a subset of $C'_{z'}$.

Since $L_z = C_z = C_x \times_x z$ by the condition on $Z$, we get $C_x \subset C'_x$ for each closed point $x \in X_k$. Thus we have $C \subset C'$ as required.

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