DESERIVED CATEGORIES OF ONE-SIDED EXACT CATEGORIES AND THEIR
LOCALIZATIONS

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Abstract. The construction of the derived category of an exact category can be extended to one-sided exact categories in a straightforward way. We discuss some properties of the derived category of a (one-sided) exact category and its embedding into its derived category.

Our main application is the localization of a one-sided exact category $\mathcal{E}$ with respect to a percolating subcategory $A \subseteq \mathcal{E}$. We show that the quotient functor $\mathcal{E} \to \mathcal{E}/A$ lifts to a Verdier localization $D^b(\mathcal{E}) \to D^b(\mathcal{E}/A)$. We apply this to the subcategories $\text{LCA}_D$ and $\text{LCA}_C$ of compact and discrete abelian groups, as subcategories of the category $\text{LCA}$ of locally compact abelian groups. We obtain that the localizations $\text{LCA}_D/\text{LCA}_D$ and $\text{LCA}/\text{LCA}_C$ are exact categories.

We further discuss the exact hull $\mathcal{E}$ and weak idempotent completion $\mathcal{E}$ of a one-sided exact category $\mathcal{E}$, and we show that the canonical embeddings lift to triangle equivalences on the bounded derived categories.

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2010 Mathematics Subject Classification. 18E10, 18E30, 18E35; 22B05.
1. Introduction

Abelian categories and their derived categories provide a framework for homological algebra. There are many meaningful, more general settings in which one can still use some homological machinery. For example, the category of vector bundles on a variety and the category of filtered objects of an abelian category are not abelian but they are exact categories in the sense of Quillen (see [30]).

Quillen exact categories are additive categories together with a distinguished class of kernel-cokernel pairs called conflations. The kernel morphism of a conflation is called an inflation and the cokernel morphism is called a deflation. The axioms of exact categories can be partitioned into two dual sets of axioms: one set only referring to the inflation-side and one set only referring to the deflation-side. Requiring only one of these sets leads to the notion of one-sided exact categories (see [4, 32]). These are additive versions of left/right exact categories [31] and homological categories [6], and are closely related to Waldhausen categories and categories with fibrations [41]).

In this paper, we study the derived category of a one-sided exact category. Our main result (see theorem 1.4 below) shows that the localizations we considered in [17] induce Verdier localization functors between the bounded derived categories.

We now briefly discuss the main parts of this paper.

1.1. Quotients of one-sided exact categories. Quotients of abelian categories are well understood (see [13]). Let $A$ be a Serre subcategory of an abelian category $E$. There is an exact quotient functor $Q: E \to E/A$ which is universal among all exact functors $F: E \to F$ with $F(A) = 0$. These quotients occur often as localizations in algebra and geometry. Similar statements have been developed for exact categories; see for example [10, 34].

In [17], as a generalization, we introduce percolating subcategories of one-sided exact categories. Given a right exact category $E$ and a right percolating subcategory $A$, we construct a quotient $Q: E \to E/A$ which is universal among all exact functors $F: E \to D$ between right exact categories such that $F(A) \cong 0$. This quotient can be described via a localization with respect to a right multiplicative system $S_A$. Dual statements hold for the left exact setting.

In §4, we recall the relevant definitions and results. The main result in that section (combining propositions 4.12 and 4.15) is that, given a right percolating subcategory $A$ of a right exact category $E$, the category $C^b(A)$ is right percolating in $C^b(E)$ and the natural functor $C^b(E) \to C^b(E/A)$ factors through an equivalence $C^b(E)/C^b(A) \cong C^b(E/A)$. This will be used in the proof of theorem 1.4.

1.2. Construction of the derived category. Recall that for an abelian category $A$ the derived category can be constructed from the category of complexes $C(A)$ by inverting all quasi-isomorphisms (i.e., all morphisms between complexes which induce isomorphisms on homology), thus $D(A) = C(A)[S^{-1}]$ where $S$ is the set of all quasi-isomorphisms. One can simplify the construction by factoring the localization as $C(A) \to K(A) \to C(A)[S^{-1}]$, where $K(A)$ is the homotopy category, and the functor $K(A) \to C(A)[S^{-1}]$ is a Verdier localization (see [39] or [21, §1.2]). Specifically, $K(A)/Ac(A) \cong C(A)[S^{-1}] = D(A)$ where $Ac(A)$ is the category of all acyclic complexes (i.e., complexes with zero homology).

In the setting of exact categories, even though the notion of homologies is not readily available, the notion of an acyclic complex is easily expressible within the framework: a complex is acyclic if it can be obtained by splicing conflations (or, equivalently, all differentials are admissible maps and the image of one differential is the kernel of the ensuing one). In [27, lemma 1.1], Neeman shows that the subcategory $Ac(E) \subseteq K(E)$ of acyclic complexes is a triangulated subcategory of $K(E)$ (although, in general, it need not be closed under isomorphisms). The derived category $D(E)$ is then defined as the Verdier localization $K(E)/Ac(E)$, or equivalently, $K(E)/\overline{Ac(E)}$ where $\langle \overline{Ac(E)} \rangle_{thick}$ is the thick closure of $Ac(E)$ in $K(E)$ (see [21, 27]). When $E$ is an abelian category, this construction recovers the usual derived category.

As a one-sided exact category comes equipped with a set of conflations (and hence with acyclic complexes), one can construct the derived category in an analogous way. This was done in [4]. Explicitly, Bazzoni and Crivei show that the category of acyclic complexes $Ac(E)$ is a triangulated subcategory of the homotopy category $K(E)$ and define the derived category as the Verdier localization $K(E)/\overline{Ac(E)}$.

In section 3, we delve deeper into the construction of the derived category and establish some basic properties. We first remark that, if $E$ is a right exact category not satisfying axiom RO*, then the derived
category misses some key desirable properties. As an example, even for a conflation \( X \to Y \to Z \) in \( \mathcal{E} \), the corresponding chain map

\[
\cdots \to 0 \to X \to Y \to 0 \to \cdots
\]

need not be a quasi-isomorphism (for this map to be a quasi-isomorphism, we additionally need that \( Z \to 0 \) is a deflation; see remark 3.4 for more information). If \( \mathcal{E} \) is a right exact category with \( \text{R0}^* \), then this morphism is indeed a quasi-isomorphism (see proposition 3.19). Note that an exact category automatically satisfies \( \text{R0}^* \) and hence this situation does not occur there.

We give some basic properties of the derived category of a one-sided exact category. The following theorem summarizes some of the main results of section 3 (see [29] for the case of an exact category).

**Theorem 1.1.** Let \( \mathcal{E} \) be a right exact category.

1. The natural embedding \( i : \mathcal{E} \to D(\mathcal{E}) \) is fully faithful.
2. For all \( X, Y \in \mathcal{E} \) and \( n > 0 \), we have \( \text{Hom}_{D(\mathcal{E})}(\Sigma^n i(X), i(Y)) = 0 \).

If \( \mathcal{E} \) satisfies \( \text{R0}^* \), then

3. a conflation \( X \to Y \to Z \) in \( \mathcal{E} \) lifts to a triangle \( i(X) \to i(Y) \to i(Z) \to \Sigma i(X) \) in \( D(\mathcal{E}) \).
4. if \( \mathcal{E} \) has enough projectives, the natural functor \( K^{-,b}(\text{Proj}(\mathcal{E})) \to D^b(\mathcal{E}) \) is a triangle equivalence.

We remark that, even when the natural embedding \( i : \mathcal{E} \to D(\mathcal{E}) \) maps conflations to triangles, the essential image of \( i \) need not be extension-closed in \( D^b(\mathcal{E}) \). Moreover, when \( \mathcal{E} \) is weakly idempotent complete and satisfies \( \text{R3} \), the essential image is extension-closed if and only if \( \mathcal{E} \) is two-sided exact (see proposition 6.2).

In §7, we use these ideas to construct the exact hull \( \mathcal{E} \) of a right exact category \( \mathcal{E} \) with \( \text{R0}^* \) as the extension closure of \( i(\mathcal{E}) \) in \( D(\mathcal{E}) \). Based on [11], we endow \( \mathcal{E} \) with the following exact structure: a sequence \( X \to Y \to Z \to X \) is a conflation in \( \mathcal{E} \subset D^b(\mathcal{E}) \) if there is a triangle \( X \to Y \to Z \to \Sigma X \) in \( D^b(\mathcal{E}) \). The following theorem combines the results of section 7.

**Theorem 1.2.** Let \( \mathcal{E} \) be a right exact category satisfying axiom \( \text{R0}^* \). The embedding \( \mathcal{E} \to \mathcal{E} \) is an exact embedding which is 2-universal among exact functors to exact categories.

Moreover, the embedding \( \mathcal{E} \to D^b(\mathcal{E}) \) lifts to a triangle equivalence \( D^b(\mathcal{E}) \simeq D^b(\mathcal{E}) \).

Note that the first part of theorem 1.2 recovers a result from [31]. The second part indicates that taking the exact hull is homologically a gentle operation.

As is remarked in [27], the embedding of an exact category \( \mathcal{E} \to \mathcal{E} \) into its weak idempotent completion (there called the semi-saturation) lifts to a derived equivalence \( D^b(\mathcal{E}) \simeq D^b(\mathcal{E}) \). A similar remark holds for one-sided exact categories; this will be examined in appendix B. The situation is slightly more subtle in the one-sided exact setting, as is indicated by the presence of the axiom \( \text{R3} \) in the following statement (combining proposition B.19 and theorem B.20).

**Theorem 1.3.** Let \( \mathcal{E} \) be a right exact category satisfying axiom \( \text{R0}^* \).

1. The weak idempotent completion \( \widehat{\mathcal{E}} \) has a canonical right exact structure satisfying axiom \( \text{R3} \) such that the embedding \( j_{\mathcal{E}} : \mathcal{E} \to \widehat{\mathcal{E}} \) is 2-universal among exact functors to weakly idempotent complete right exact categories satisfying axiom \( \text{R3} \).
2. The natural embedding \( j_{\mathcal{E}} : \mathcal{E} \to \widehat{\mathcal{E}} \) lifts to a triangle equivalence \( D^b(\mathcal{E}) \simeq D^b(\widehat{\mathcal{E}}) \).

We note that, even though the bounded derived categories of \( \mathcal{E} \) and its weak idempotent completion \( \widehat{\mathcal{E}} \) are equivalent, a similar statement does not hold for its idempotent completion (see [2]).

### 1.3. Localizing with respect to percolating subcategories

In [17], we introduce a percolating subcategory \( \mathcal{A} \) of a one-sided exact category \( \mathcal{E} \) and construct the quotient \( \mathcal{E}/\mathcal{A} \). One of the main results of this paper (theorem 5.5 in the text) states that the quotient functor \( Q : \mathcal{E} \to \mathcal{E}/\mathcal{A} \) induces a Verdier localization functor \( D^b(\mathcal{E}) \to D^b(\mathcal{E}/\mathcal{A}) \). This generalizes [26, theorem 3.2] and [34, proposition 2.6] (our proof follows these references closely).

**Theorem 1.4.** Let \( \mathcal{E} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory.

1. The derived quotient functor \( D^b(\mathcal{E}) \to D^b(\mathcal{E}/\mathcal{A}) \) is a Verdier localization.
If $\mathcal{E}$ satisfies axiom $R0^*$ and $S_A$ is right weakly saturated, then the following is a Verdier localization sequence:

$$D^b(\mathcal{E}) \to D^b(\mathcal{E}) \to D^b(\mathcal{E}/A).$$

Here, $D^b_A(\mathcal{E})$ is the replete (meaning that it is closed under isomorphisms) triangulated subcategory of $D^b(\mathcal{E})$ generated by $A$. When $\mathcal{E}$ is an abelian category, $D^b_A(\mathcal{E})$ is usually defined as the full subcategory of $D^b(\mathcal{E})$ consisting of those objects with homologies in $A$. In general, there is a triangle functor $D^b(\mathcal{E}) \to D^b_A(\mathcal{E})$, but this is not an equivalence. Note that the conditions required in [34, proposition 2.6] do imply that this triangle functor is an equivalence (see also proposition 5.9).

Theorem 1.4 implies that $D^b(\mathcal{E}/A) \cong D^b(D^b(\mathcal{E}))/D^b_A(\mathcal{E})$. Moreover, in appendix A, we show, using the language of Frobenius pairs from [35], that if $D^b_A(\mathcal{E}) \cong D^b(\mathcal{A})$, there is a homotopy fibration of $K$-theory spectra

$$K(A) \to K(\mathcal{E}) \to K(\mathcal{E}/A).$$

As illustrated in [17], even if $\mathcal{E}$ is exact, the localization $\mathcal{E}/A$ need only be one-sided exact. If one prefers to work in the setting of exact categories, one can replace $\mathcal{E}/A$ by its exact hull $\mathcal{E}/\mathcal{A}$ without changing the derived category (cf. theorem 1.2), thus $D^b(\mathcal{E}/\mathcal{A}) \cong D^b(\mathcal{E})/D^b_A(\mathcal{E})$.

1.4. Application to locally compact abelian groups. Our main example concerns the exact category LCA of locally compact (Hausdorff) abelian groups. The subcategory $\text{LCA}_D$ of discrete groups is a right percolating subcategory and the quotient $\text{LCA}/\text{LCA}_D$ satisfies the conditions of theorem 1.4. Moreover, we show in corollary 5.11 that the Verdier localization sequence from theorem 1.4 is equivalent to the sequence

$$D^b(\text{LCA}_D) \to D^b(\text{LCA}) \to D^b(\text{LCA}/\text{LCA}_D).$$

There is a similar sequence concerning the category $\text{LCA}_C$ of compact abelian groups.

In [18], Hoffmann and Spitzweck investigated the homological properties of LCA and showed that $\text{Hom}_{D^b(\text{LCA})}(\text{LCA}, \Sigma^2 \text{LCA}) = 0$. Hence, Pontryagin duality and theorem 1.5 allow us to conclude the following statement.

Corollary 1.6. The categories $\text{LCA}/\text{LCA}_D$ and $\text{LCA}/\text{LCA}_C$ (where the quotients are given by localizations, as described in §4) are exact categories.

Acknowledgments We are grateful to Frederik Caenepeel and Freddy Van Oystaeyen for motivating discussions, and to Francesco Genovese and Ivo Dell’Ambrogio for helpful comments on an earlier draft. The second author is currently a postdoctoral researcher at FWO (12.M33.16N).

2. Preliminaries

Throughout the paper all categories are assumed to be small.

2.1. One-sided exact categories. We recall the definition of a left and a right exact category from [4].

Definition 2.1. Let $\mathcal{C}$ be an additive category and let $A \xrightarrow{f} B \xrightarrow{g} C$ be a kernel-cokernel pair, i.e. $f = \ker g$ and $g = \ker f$. A conflation category $\mathcal{C}$ is an additive category with a chosen class of kernel-cokernel pairs, called conflations, closed under isomorphisms. Given a conflation $A \xrightarrow{f} B \xrightarrow{g} C$, we refer to the map $f$ as an inflation and to the map $g$ as a deflation. Inflations will often be denoted by $\rightarrow$ and deflations by $\twoheadrightarrow$. A map $f : X \to Y$ is called admissible if it admits a deflation-inflation factorization, i.e. $f$ factors as $X \twoheadrightarrow Z \rightarrow Y$.

Let $\mathcal{C}$ and $\mathcal{D}$ be conflation categories. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is called exact if it maps conflations in $\mathcal{C}$ to conflations in $\mathcal{D}$.

Remark 2.2. The dual of a conflation category is again a conflation category in a natural way: such a duality exchanges inflations and deflations.

Definition 2.3. A conflation category $\mathcal{C}$ is called right exact or deflation-exact if it satisfies the following axioms:
R0 The identity morphism 1₀: 0 → 0 is a deflation.
R1 The composition of two deflations is again a deflation.
R2 The pullback of a deflation along any morphism exists and is again a deflation, i.e.

\[
\begin{array}{c}
X \\
\downarrow \\
Z \\
\end{array}
\begin{array}{c}
Y \\
\downarrow \\
W \\
\end{array}
\]

Dually, a conflation category $C$ is called left exact or inflation-exact if the opposite category $C^{\text{op}}$ is right exact. For completeness, a left exact category is a conflation category such that the class of conflations satisfies the following axioms:

L0 The identity morphism 1₀: 0 → 0 is an inflation.
L1 The composition of two inflations is again an inflation.
L2 The pushout of an inflation along any morphism exists and is again an inflation, i.e.

\[
\begin{array}{c}
X \\
\downarrow \\
Z \\
\end{array}
\begin{array}{c}
Y \\
\downarrow \\
W \\
\end{array}
\]

**Definition 2.4.** Let $C$ be a conflation category. In addition to the properties listed in definition 2.3, we will also refer to the following properties:

R0* For any $A \in \text{Ob}(C)$, $A \rightarrow 0$ is a deflation.
R3 If $i: A \rightarrow B$ and $p: B \rightarrow C$ are morphisms in $C$ such that $p$ has a kernel and $pi$ is a deflation, then $p$ is a deflation.
L0* For any $A \in \text{Ob}(C)$, $0 \rightarrow A$ is a deflation.
L3 If $i: A \rightarrow B$ and $p: B \rightarrow C$ are morphisms in $C$ such that $i$ has a cokernel and $pi$ is an inflation, then $i$ is an inflation.

A right exact category satisfying R3 is called strongly right exact. Dually, a left exact category satisfying L3 is called strongly left exact.

**Remark 2.5.**

1. Axioms R3 and L3 are sometimes referred to as Quillen’s obscure axioms (see [9, 38]). Note that axiom R3 implies axiom R0* and axiom L3 implies axiom L0*.
2. An exact category in the sense of Quillen (see [30]) is a conflation category $E$ which is both left and right exact. It is shown in [20, appendix A] that an exact category automatically satisfies the obscure axioms R3 and L3.
3. Let $E$ be a right exact category. Axiom R0* has the following characterization: split kernel-cokernel pairs are conflations if and only if $E$ satisfies axiom R0*. Indeed, if for every $C \in E$, the split kernel-cokernel pair $\xymatrix{C \ar[r] & 0}$ is a conflation, then $C$ satisfies R0*. The converse follows from [4, proposition 5.6].
4. If $E$ is a weakly idempotent complete right exact category, one can drop the requirement that $p$ admits a kernel in axiom R3. Thus $E$ satisfies R3 if and only if for all maps $i: A \rightarrow B$ and $p: B \rightarrow C$ in $E$ such that $pi$ is a deflation, $p$ is a deflation. A dual statement holds for L3 (see [9, proposition 7.6]).

2.2. Localizations and right multiplicative systems. The material of this section is based on [12, 19].

**Definition 2.6.** Let $C$ be any category and let $S \subseteq \text{Mor}(C)$ be any subset of morphisms of $C$. The localization of $C$ with respect to $S$ is a universal functor $Q: C \rightarrow S^{-1}C$ such that $Q(s)$ is invertible, for all $s \in S$. Here, universality means that any functor $F: C \rightarrow D$ for which $F(s)$ is invertible for all $s \in S$, factors uniquely through $Q: C \rightarrow S^{-1}C$.

In this paper, we often consider localizations with respect to right multiplicative systems.

**Definition 2.7.** Let $C$ be a category. A set $S \subseteq \text{Mor}(C)$ is called a right multiplicative system if it has the following properties:

RMS1 For every object $A \in \text{Ob}(C)$, the identity $1_A: A \rightarrow A$ is contained in $S$. The set $S$ is closed under composition.
Every solid diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{t} & & \downarrow{s} \\
Z & \xrightarrow{f} & W
\end{array}
\]

with \( s \in S \) can be completed to a commutative square with \( t \in S \).

For every localization, \( s \in S \) with \( s \circ f = s \circ g \) there exists a \( t \in S \) with target \( X \) such that \( f \circ t = g \circ t \).

Often arrows in \( S \) will be endowed with \( \sim \).

A \textit{left multiplicative system} is defined dually. We say that \( S \) is a \textit{multiplicative system} if it is both a left and a right multiplicative system.

For localizations with respect to a right multiplicative system, we have the following description of the localization.

\textbf{Construction 2.8.} Let \( \mathcal{C} \) be a category and \( S \) a right multiplicative system in \( \mathcal{C} \). We define a category \( S^{-1}\mathcal{C} \) as follows:

1. We set \( \text{Ob}(S^{-1}\mathcal{C}) = \text{Ob}(\mathcal{C}) \).
2. Let \( f_1: X_1 \to Y, s_1: X_1 \to X, f_2: X_2 \to Y, s_2: X_2 \to X \) be morphisms in \( \mathcal{C} \) with \( s_1, s_2 \in S \). We call the pairs \( (f_1, s_1), (f_2, s_2) \in (\text{Mor}\mathcal{C}) \times S \) equivalent (denoted by \( (f_1, s_1) \sim (f_2, s_2) \)) if there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{s_1} & X_1 \\
\downarrow{s_2} & & \downarrow{f_2} \\
X_2 & \xleftarrow{v} & Y
\end{array}
\]

3. \( \text{Hom}_{S^{-1}\mathcal{C}}(X, Y) = \{(f, s) \mid f \in \text{Hom}_{\mathcal{C}}(X', Y), s: X' \to X \text{ with } s \in S\} / \sim \)

4. The composition of \( (f: X' \to Y, s: X' \to X) \) and \( (g: Y' \to Z, t: Y' \to Y) \) is given by \( (g \circ h: X'' \to Z, s \circ u: X'' \to X) \) where \( h \) and \( u \) are given by a commutative diagram

\[
\begin{array}{ccc}
X'' & \xrightarrow{h} & Y' \\
\downarrow{u} & & \downarrow{t} \\
X' & \xleftarrow{f} & Y
\end{array}
\]

as in RMS2.

\textbf{Proposition 2.9.} Let \( \mathcal{C} \) be a category and \( S \) a right multiplicative system in \( \mathcal{C} \).

1. The assignment \( X \mapsto X \) and \( (f: X \to Y) \mapsto (f: X \to Y, 1_X: X \to X) \) defines a localization functor \( Q: \mathcal{C} \to S^{-1}\mathcal{C} \). In particular, for any \( s \in S \), the map \( Q(s) \) is an isomorphism.
2. The localization functor commutes with finite limits.
3. If \( \mathcal{C} \) is an additive category, then \( S^{-1}\mathcal{C} \) is an additive category as well and the localization functor \( Q: \mathcal{C} \to S^{-1}\mathcal{C} \) is additive.

\textbf{Definition 2.10.} Let \( \mathcal{C} \) be any category and let \( S \subseteq \text{Mor}\mathcal{C} \) be any subset.

1. We say that \( S \) satisfies the \textit{2-out-of-3 property} if, for any two composable morphisms \( f, g \in \text{Mor}\mathcal{C} \), the following property holds: if two of \( f, g, fg \) are in \( S \), then so is the third.
2. Let \( Q: \mathcal{C} \to S^{-1}\mathcal{C} \) be the localization of \( \mathcal{C} \) with respect to \( S \). We say that \( S \) is \textit{saturated} if \( S = \{f \in \text{Mor}\mathcal{C} \mid Q(f) \text{ is invertible}\} \).
3. We say that \( S \) is \textit{right weakly saturated} if for any morphism \( f \in \text{Mor}\mathcal{C} \) such that \( Q(f) \) descends to an isomorphism, there exists a map \( s \in S \) such that \( f \circ s \in S \). Dually, \( S \) is \textit{left weakly saturated} if for any \( f \in \text{Mor}\mathcal{C} \) such that \( Q(f) \) is an isomorphism, there exists a map \( s \in S \) such that \( s \circ f \in S \).
2.3. Verdier localizations. We recall some notions and properties of Verdier localizations. Our main references are [24, 28, 39]. Let $\mathcal{T}$ be a triangulated category and $S \subseteq \mathcal{T}$ a full triangulated subcategory. We write $\mathcal{N}(S)$ for the set of morphisms $f: X \to Y$ in $\mathcal{T}$ which fit in a triangle $X \to Y \to S \to \Sigma(X)$ with $S \in \mathcal{S}$.

Proposition 2.11. The set $\mathcal{N}(S)$ is a (left and right) multiplicative system satisfying the 2-out-of-3 property. Furthermore, $\mathcal{N}(S)$ is saturated if and only if $S$ is a thick subcategory of $\mathcal{T}$.

We now define the Verdier localization $\mathcal{T}/S$ as the localization $\mathcal{N}(S)\^{-1}\mathcal{T}$ together with the localization functor $\mathcal{T} \to \mathcal{T}/S$.

Proposition 2.12. Let $S$ be a triangulated subcategory of a triangulated category $\mathcal{T}$.

1. The category $\mathcal{T}/S$ has a unique structure of a triangulated category such that the localization functor $Q: \mathcal{T} \to \mathcal{T}/S$ is a triangle functor.
2. The kernel of $Q: \mathcal{T} \to \mathcal{T}/S$ is the thick closure of $S$.
3. A morphism in $\mathcal{T}$ becomes zero in $\mathcal{T}/S$ if and only if it factors through an object in $S$.
4. Any triangle functor $F: \mathcal{T} \to \mathcal{U}$ satisfying $F(S) = 0$ factors uniquely through $Q: \mathcal{T} \to \mathcal{T}/S$.

A sequence $S \hookrightarrow \mathcal{T} \xrightarrow{i} \mathcal{C}$ is called a Verdier localization sequence if $f \circ i = 0$, and the unique functor $\mathcal{T}/S \to \mathcal{C}$ induced by the previous proposition is an equivalence.

3. Derived categories of right exact categories

We now come to the derived category of a right exact category $\mathcal{E}$. We recall the definition of an acyclic complex and construct the derived category $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}$ as the Verdier quotient $\mathcal{K}(\mathcal{E})/\mathcal{A}(\mathcal{E})$ of the homotopy category by the category of acyclic complexes (see [4]). We discuss the embedding $i: \mathcal{E} \to \mathcal{D}(\mathcal{E})$ and show that, if $\mathcal{E}$ has enough projective objects, then the natural embedding $\mathcal{K}^{\text{a}}(\mathcal{P}(\mathcal{E})) \to \mathcal{D}(\mathcal{E})$ is a triangle equivalence. The main results are summarized in theorem 1.1.

3.1. Definitions. Let $\mathcal{A}$ be an additive category. We denote by $\mathcal{C}(\mathcal{A})$ the additive category of cochain complexes in $\mathcal{A}$ and by $\mathcal{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$. It is well known that $\mathcal{K}(\mathcal{A})$ has the structure of a triangulated category induced by the strict triangles in $\mathcal{C}(\mathcal{A})$, see e.g. [5, 15, 19, 39, 40]. We denote by $\mathcal{C}^{\text{n}}(\mathcal{A}), \mathcal{C}^{\text{m}}(\mathcal{A})$ and $\mathcal{C}^{\text{n}}(\mathcal{A})$ the full subcategories of $\mathcal{C}(\mathcal{A})$ generated by the left bounded, right bounded and bounded complexes, respectively. Similarly, we write $\mathcal{K}^{\text{n}}(\mathcal{A}), \mathcal{K}^{\text{m}}(\mathcal{A})$ and $\mathcal{K}^{\text{n}}(\mathcal{A})$ for the full triangulated subcategories of $\mathcal{K}(\mathcal{A})$ which are homotopic to the left bounded, right bounded and bounded complexes, respectively. We write $\mathcal{C}^{[n,m]}(\mathcal{A})$ for the bounded complexes in $\mathcal{A}$ supported only in degrees $n$ to $m$.

Whenever we write $\mathcal{C}^{\text{n}}(\mathcal{A})$ or $\mathcal{K}^{\text{n}}(\mathcal{A})$ without explicitly specifying $\ast$, the statement holds for both the unbounded and bounded setting.

Definition 3.1.

1. Given a chain map $f^{\ast}: X^{\ast} \to Y^{\ast}$ in $\mathcal{C}^{\ast}(\mathcal{A})$ or $\mathcal{K}^{\ast}(\mathcal{A})$, the mapping cone $\text{cone}(f^{\ast})$ is the complex given by $\text{cone}(f^{\ast})^{n} = X^{n+1} \oplus Y^{n}$ and the differentials $d_{f}^{\ast}$ are given by

\[
\begin{pmatrix}
-d_{1}^{n+1} & 0 \\
 0 & d_{n}^{n+1} & d_{n}^{m}
\end{pmatrix}.
\]

2. Let $C^{\bullet, \bullet}$ be a double complex. We denote the vertical differentials by $d_{v}^{n,m}$ and the horizontal differentials by $d_{h}^{n,m}$. By convention, all squares of the bicomplex $C^{\bullet, \bullet}$ commute. The associated total complex $\text{Tot}(C^{\bullet, \bullet})$ is defined by

\[
(\text{Tot}(C^{\bullet, \bullet}))^{n} = \bigoplus_{i+j=n} C^{i,j}
\]

and the differentials $d_{\text{tot}}$ are given by the rule

\[
d_{\text{tot}} = d_{h} + (-1)^{\text{horizontal degree}}d_{v}.
\]

3. A complex $X^{\ast}$ is called a standard contractible complex if it is of the form:

\[
\cdots \to 0 \to X \xrightarrow{d_{X}} X \to 0 \to \cdots
\]

Remark 3.2. There is a strong connection between cones and totalizations. Given a chain map $f^{\ast}: X^{\ast} \to Y^{\ast}$, one finds the mapping cone as the total complex associated to the double complex defined by placing $X^{\ast}$ in the column of degree $-1$ and $Y^{\ast}$ in the column of degree $0$ (see for example [40, exercise 1.2.8]).

From now on, $\mathcal{E}$ denotes a right exact category.
Definition 3.3. Let \( \mathcal{E} \) be a right exact category. A complex \( X^\bullet \in C(\mathcal{E}) \) is called **acyclic in degree** \( n \) if \( d_X^{n-1} \) factors as

\[
\begin{array}{c}
X_{n-1} \\
\downarrow p^{n-1} \\
\downarrow i^{n-1} \\
\uparrow \ker(d_X^n)
\end{array} \xrightarrow{d_X^{n-1}} X^n
\]

where the deflation \( p^{n-1} \) is the cokernel of \( d_X^{n-2} \) and the inflation \( i^{n-1} \) is the kernel of \( d_X^n \).

A complex \( X^\bullet \) is called **acyclic** if it is acyclic in each degree. The full subcategory of \( C(\mathcal{E}) \) of acyclic complexes is denoted by \( Ac_C(\mathcal{E}) \). We write \( Ac_K(\mathcal{E}) \) for the full subcategory of \( K(\mathcal{E}) \) given by acyclic complexes. We simply write \( Ac(\mathcal{E}) \) for either \( Ac_C(\mathcal{E}) \) or \( Ac_K(\mathcal{E}) \) if there is no confusion. The bounded versions are defined by \( Ac^*(\mathcal{E}) = Ac(\mathcal{E}) \cap C^*(\mathcal{E}) \).

Remark 3.4.

1. If \( \mathcal{E} \) is abelian, then \( X^\bullet \) is acyclic in degree \( n \) if and only if the cohomology group \( H^n(X^\bullet) \) is zero.
2. If \( \mathcal{E} \) is right exact, an acyclic complex \( X^\bullet \) with three consecutive non-zero terms and all other terms being zero gives a conflation in \( \mathcal{E} \). Note, however, that a conflation is not necessarily an acyclic complex. Indeed, the complex \( \cdots \to 0 \to X \to Y \to Z \to 0 \to \cdots \) is acyclic if and only if \( X \to Y \to Z \) is a conflation and \( Z \to 0 \) is a deflation. If \( \mathcal{E} \) satisfies axiom \( R0^+ \), any conflation yields a three-term acyclic complex.
3. While standard contractible complexes are null-homotopic, they are acyclic if and only if the category \( \mathcal{E} \) satisfies axiom \( RO^+ \).

The next lemma (see [4, proposition 7.2]) will be used in the definition of the derived category of \( \mathcal{E} \).

Lemma 3.5. Let \( \mathcal{E} \) be a right exact category. The mapping cone of a chain map \( f^\bullet: X^\bullet \to Y^\bullet \) between acyclic complexes is acyclic. In particular, the category \( Ac_K(\mathcal{E}) \) is a triangulated subcategory (not necessarily closed under isomorphisms) of \( K(\mathcal{E}) \). Similar statements hold for the bounded versions.

Definition 3.6. Let \( \mathcal{E} \) be a right exact category.

1. The (unbounded) derived category \( D(\mathcal{E}) \) is defined as the Verdier localization \( K(\mathcal{E})/Ac_K(\mathcal{E}) \).
2. The bounded derived category \( D^b(\mathcal{E}) \) is defined as the full triangulated subcategory of \( D(\mathcal{E}) \) spanned by bounded complexes up to quasi-isomorphism. The left and right bounded derived categories \( D^-(\mathcal{E}) \) and \( D^+(\mathcal{E}) \) are defined similarly.

Definition 3.7. A chain map \( f^\bullet: A^\bullet \to B^\bullet \) is a called a **quasi-isomorphism** if its mapping cone is homotopy equivalent to an acyclic complex.

The chain map \( f^\bullet: A^\bullet \to B^\bullet \) is called a **weak quasi-isomorphism** if its mapping cone is a direct summand of an acyclic complex up to homotopy equivalence.

Remark 3.8.

1. Let \( \mathcal{E} \) be a right exact category. In the notation of section 2.3, the set of quasi-isomorphisms is denoted by \( N(Ac_K(\mathcal{E})) \) and the set of weak quasi-isomorphisms is given by \( N(Ac_K(\mathcal{E}))_{\text{thick}} \) where \( (Ac_K(\mathcal{E}))_{\text{thick}} \) is the thick closure of \( Ac_K(\mathcal{E}) \) in \( K(\mathcal{E}) \). A chain map in \( K^*(\mathcal{E}) \) becomes an isomorphism in \( D^*(\mathcal{E}) \) if and only if it is a weak quasi-isomorphism.
2. Alternatively, one could define the bounded derived category \( D^b(\mathcal{E}) \) as the Verdier localization \( K^b(\mathcal{E})/ (Ac_K(\mathcal{E}) \cap K^b(\mathcal{E})) \). To show that this definition coincides with the previous definition, it suffices to show that if a roof

\[
\begin{array}{c}
X^\bullet \\
\downarrow \phi \\
\downarrow \phi \\
\uparrow \psi
\end{array} \xrightarrow{Z^\bullet} Y^\bullet
\]

represents a map \( X^\bullet \to Y^\bullet \) in \( D(\mathcal{E}) \) between bounded complexes \( X^\bullet \) and \( Y^\bullet \), one can find an equivalent roof consisting of only bounded complexes. We leave it to the reader to verify that this follows from propositions 3.12 and 3.15 below (see also [21, lemma 11.7]).

In proposition 3.10 below, we give conditions under which \( Ac^*(\mathcal{E}) \subseteq K^*(\mathcal{E}) \) is closed under isomorphisms. We start by stating the following well-known lemma.

Lemma 3.9. Let \( \mathcal{E} \) be a right exact category.
Let \( K^\bullet \to L^\bullet \) be a coretraction in \( K^\bullet(\mathcal{E}) \). The map \( K^\bullet \to L^\bullet \oplus IK^\bullet \) where \( IK = \text{cone}(1_{K^\bullet}) \) is a coretraction in \( C^\bullet(\mathcal{E}) \).

(2) If \( \mathcal{E} \) satisfies axiom \( RO^* \), the complex \( IK^\bullet \) is acyclic.

Proof. The proof is a straightforward adaptation of [20, section 2.3.a]. \qed

**Proposition 3.10.** Let \( \mathcal{E} \) be a right exact category and let \( * \in \{ +, -, b \} \).

(1) If \( Ac^b(\mathcal{E}) \) is a thick subcategory of \( K^b(\mathcal{E}) \), then \( \mathcal{E} \) is weakly idempotent complete.

(2) If \( \mathcal{E} \) satisfies axiom \( R3 \) and is weakly idempotent complete, then \( Ac^b(\mathcal{E}) \) is a thick subcategory of \( K^b(\mathcal{E}) \).

Replacing weakly idempotent complete by idempotent complete, the lemma remains valid for unbounded derived categories.

**Proof.** (1) It suffices to show the statement for the bounded complexes. So assume that \( Ac^b(\mathcal{E}) \) is a thick subcategory of \( K^b(\mathcal{E}) \). Let \( r: B \to A \) be a retraction with section \( s: A \to B \). The complex \( X^\bullet \) given by \( 0 \to A \to B \to r \to A \to 0 \) is null-homotopic, indeed; the following diagram shows that \( 1_{X^\bullet} \) is homotopic to zero:

\[
\begin{array}{ccccccccc}
0 & \to & A & \to & B & \xrightarrow{1_B-se} & B & \xrightarrow{r} & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \xrightarrow{s} & B & \xrightarrow{1_B} & A & \to & 0
\end{array}
\]

It follows that the complex \( X^\bullet \) is isomorphic to the zero complex in \( K^b(\mathcal{E}) \). As the zero complex is acyclic and \( Ac^b(\mathcal{E}) \) is a thick subcategory, we conclude that \( X^\bullet \) is acyclic as well. It follows that \( r \) is a deflation and thus \( r \) has a kernel. This shows that any retraction has a kernel and thus \( \mathcal{E} \) is weakly idempotent complete.

(2) Let \( K^\bullet \to L^\bullet \) be a coretraction in \( K^b(\mathcal{E}) \) with \( L^\bullet \in Ac^b(\mathcal{E}) \). By lemma 3.9, the map \( K^\bullet \to L^\bullet \oplus IK^\bullet \) is a coretraction in \( C^b(\mathcal{E}) \) and \( L^\bullet \oplus IK^\bullet \in Ac^b(\mathcal{E}) \). Moreover, the coretraction \( K^\bullet \to L^\bullet \oplus IK^\bullet \) has a cokernel in \( C^b(\mathcal{E}) \). By axiom \( R3 \) and by [4, proposition 5.9], \( K^\bullet \in Ac^b(\mathcal{E}) \).

The unbounded case is similar to [9, corollary 10.11]. \qed

### 3.2. Truncations of complexes

Truncations are an important tool for studying complexes in an abelian (5) or quasi-abelian (36) setting. The next definition defines such truncations in an additive setting provided that some differentials have kernels (or cokernels).

**Definition 3.11.** Let \( C^\bullet \) be a complex in an additive category \( \mathcal{A} \) such that \( d_C^{n-1}: C^{n-1} \to C^n \) factors as

\[
C^{n-1} \xrightarrow{p^{n-1}} \ker(d^n) \xrightarrow{\iota^{n-1}} C^n
\]

where \( \iota^{n-1} \) is the kernel of \( d_C^n \). The canonical truncation \( \tau^{\leq n} C^\bullet \) is a complex together with a morphism \( \tau^{\leq n} C^\bullet \to C^\bullet \) given by:

\[
\begin{array}{cccccccc}
\tau^{\leq n} C^\bullet & \to & C^{n-3} & \to & C^{n-2} & \to & C^{n-1} & \xrightarrow{p^{n-1}} \ker(d_C^n) & \to & 0 & \to & \cdots \\
\end{array}
\]

and the canonical truncation \( C^\bullet \to \tau^{\geq n+1} C^\bullet \) similarly defined by:

\[
\begin{array}{cccccccc}
C^\bullet & \to & C^{n-3} & \to & C^{n-2} & \to & C^{n-1} & \xrightarrow{p^{n-1}} 0 & \to & \ker(d^n_C) & \xrightarrow{\iota^{n-1}} C^n & \to & \cdots \\
\tau^{\geq n+1} C^\bullet & \to & 0 & \to & 0 & \to & \ker(d^n_C) & \xrightarrow{\iota^{n-1}} C^n & \to & C^{n+1} & \to & \cdots \\
\end{array}
\]

**Proposition 3.12.** Let \( \mathcal{A} \) be an additive category and let \( C^\bullet \in C^\bullet(\mathcal{A}) \). If \( d_C^n \) has a kernel, then the following triangle is a distinguished triangle in \( K^\bullet(\mathcal{A}) \)

\[
\tau^{\leq n} C^\bullet \to C^\bullet \to \tau^{\geq n+1} C^\bullet \to \Sigma(\tau^{\leq n} C^\bullet)
\]

In other words, \( C^\bullet \) is an extension of \( \tau^{\leq n+1} C^\bullet \) by \( \tau^{\leq n} C^\bullet \) in \( K^\bullet(\mathcal{A}) \).
**Proposition 3.15.** Let \( E \in K \) be a right exact category and let \( \alpha : \zeta \xrightarrow{\sim} X^\bullet \) be a quasi-isomorphism. There exists a homotopy equivalence \( \zeta \xrightarrow{\sim} Z^\bullet \) such that \( \zeta \xrightarrow{\sim} Z^\bullet \xrightarrow{\alpha} X^\bullet \) is a congenial quasi-isomorphism.

Moreover, if \( X^\bullet \) is supported on degrees at most \( n \), then \( \tau^{<n} \zeta \to \zeta \) is a quasi-isomorphism.

**Proof.** Since \( \alpha \) is a quasi-isomorphism, there exists a homotopy equivalence \( \mathrm{cone}(\alpha) \xrightarrow{\beta} E^\bullet \) where \( E^\bullet \) is an acyclic complex. We get the following commutative diagram in \( K(E) \) whose rows are triangles:

\[
\begin{array}{cccccccc}
Z^\bullet & \xrightarrow{\alpha} & X^\bullet & \xrightarrow{\mathrm{cone}(\alpha)} & \Sigma Z^\bullet \\
\downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\beta} \\
\Sigma^{-1} \mathrm{cone}(\gamma) & \xrightarrow{\gamma} & X^\bullet & \xrightarrow{\gamma} & E^\bullet & \xrightarrow{\gamma} & \mathrm{cone}(\gamma) \\
\end{array}
\]

Since \( \beta \) is a homotopy equivalence, these triangles are isomorphic. Hence, there exists a homotopy equivalence \( \Sigma^{-1} \mathrm{cone}(\gamma) \xrightarrow{\sim} Z^\bullet \). By the construction of the cone of a morphism, the map \( \zeta^\bullet = \Sigma^{-1} \mathrm{cone}(\gamma) \to X \) is congenial.

Assume now that \( X^\bullet \) is supported on degrees at most \( n \). Without loss of generality, assume that \( n = 0 \). Then \( \zeta^0 \cong X^0 \oplus E^{-1}, \zeta^1 = E^0 \) and \( \alpha^1 : X^0 \oplus E^{-1} \to E^0 \) is given by \( -(\gamma^0 d_{E^1}^0) \). We claim that \( d^1_{\zeta} \) has a kernel.

The map \( \gamma^1 : X^\bullet \to E^\bullet \) induces the commutative diagram from figure 1. The kernel of \( -d^0_{\zeta} = (\gamma^0 d_{E^1}^0) : X^0 \oplus E^{-1} \to E^0 \) is given \( P \to E^{-1} \oplus X \). This shows the claim.
a quasi-isomorphism \( \tau \) represented by a roof \( i \).

Composition \( \tau \).

As the map \( \zeta \) congenial; thus, we have that derived category and truncations, we now examine the canonical embedding.

We first show that Proposition 3.16.

Let \( \mathcal{E} \) be a right exact category. The canonical functor \( i : \mathcal{E} \to D^*(\mathcal{E}) \)

mapping objects to stalk complexes in degree zero is fully faithful.

Proof. We first show that \( i : \mathcal{E} \to D^*(\mathcal{E}) \) is faithful. Let \( f \in \text{Hom}_\mathcal{E}(X, Y) \) be a map such that the induced map \( i(f) : i(X) \to i(Y) \) is zero in \( D^*(\mathcal{E}) \). As the quasi-isomorphisms form a right multiplicative system, we have that \( i(f) \) is zero in \( D^*(\mathcal{E}) \) if and only if there exists a quasi-isomorphism \( g^* : Z^* \to i(X) \) such that \( i(f) \circ g^* \) is zero in \( K^*(\mathcal{E}) \). By proposition 3.15, we may assume that the composition \( i(f) \circ g^* : \zeta^* \to X^* \) is congruent; thus, \( \zeta = \Sigma^{-1} \text{cone}(i(X) \to E^*) \) for an acyclic complex \( E^* \). The situation is as given in figure 1. As the map \( \zeta^* \to i(X) \to i(Y) \) is null-homotopic, there is a map \( t \) such that the following diagram commutes:

\[
\begin{array}{ccc}
E^{-1} \oplus X & \xrightarrow{(0,1)} & X \\
\downarrow \phi & & \downarrow f \\
\ker(d^0_E) & \xrightarrow{-\phi} & Y \\
\end{array}
\]

It follows that \( t \circ i^0 \circ p^{-1} = 0 \) and since \( p^{-1} \) is epic, \( t \circ i^0 = 0 \). On the other hand, \( f = (t \circ i^0) \circ \phi = 0 \). This shows faithfulness.

We now show that the functor \( i : \mathcal{E} \to D^*(\mathcal{E}) \) is full. Consider \( f : i(X) \to i(Y) \) a morphism in \( D^*(\mathcal{E}) \), represented by a roof \( i(X) \xrightarrow{\alpha} Z^* \xrightarrow{\beta} i(Y) \) where \( \alpha \) is a quasi-isomorphism. By proposition 3.15, we obtain a quasi-isomorphism \( \tau^{\leq 0} \xrightarrow{\sim} Z^* \xrightarrow{\beta} Y^* \). Using the notation of figure 1, the composition \( \tau^{\leq 0} \xrightarrow{\sim} Z^* \xrightarrow{\beta} Y^* \)

3.3. The canonical embedding \( i : \mathcal{E} \hookrightarrow \text{D}^b(\mathcal{E}) \). Let \( \mathcal{E} \) be a right exact category. Having defined the derived category and truncations, we now examine the canonical embedding \( i : \mathcal{E} \hookrightarrow \text{D}^b(\mathcal{E}) \). The next proposition says that, as in the abelian or exact case, the category \( \mathcal{E} \) is equivalent to the category of stalk complexes in \( \text{D}^b(\mathcal{E}) \) in degree zero.

**Proposition 3.16.** Let \( \mathcal{E} \) be a right exact category. The canonical functor

\[
i : \mathcal{E} \to \text{D}^*(\mathcal{E})
\]
Proposition 3.17. Let $\tau_i$ be an isomorphism. Let $\rho^{-1}$ be the cokernel of $\iota^{-2}$. It follows from the following commutative diagram in $D^*(E)$ that $i(g) = f$.

$$
\begin{array}{ccc}
E^{-2} & \longrightarrow & 0 \\
\downarrow & & \\
\ker(d_E^{-1}) & \longrightarrow & P \\
\downarrow & \mathrlap{\iota^{-2}} & \downarrow \mathrlap{\gamma} \\
\rho^{-1} & \longrightarrow & Y \\
\downarrow & \mathrlap{\iota^{-2}} & \downarrow \\
X & \longrightarrow & \\
\end{array}
$$

Here the dotted arrow is obtained as $\rho^{-1}$ is the cokernel of $\iota^{-2}$. It follows from the following commutative diagram in $D^*(E)$ that $i(g) = f$.

$$
\begin{array}{ccc}
i(X) & \longrightarrow & i(Y) \\
\downarrow & \mathrlap{\iota} & \downarrow \mathrlap{i(g)} \\
\tau_{<0} \cdot & \mathrlap{\sim} & i(Y) \\
\downarrow & \mathrlap{\iota} & \downarrow \\
Z & \longrightarrow & \\
\end{array}
$$

Under the above embedding, the category $E$ has no negative self-extensions in $D^*(E)$.

**Proposition 3.17.** Let $E$ be a right exact category. Let $X, Y \in \text{Ob}(E)$, then

$$\text{Hom}_{D^*(E)}(i(X), \Sigma^{-n}i(Y)) = 0$$

for all $n > 0$.

**Proof.** Let $n > 0$ and let $g \in \text{Hom}_{D^*(E)}(i(X), \Sigma^{-n}i(Y))$. The morphism $g$ can be represented by a roof $i(X) \xrightarrow{\alpha} Z^* \xrightarrow{\iota} \Sigma^{-n}i(Y)$ where $\alpha$ is a quasi-isomorphism. By proposition 3.15, there is a quasi-isomorphism $\tau_{<0} \cdot \sim \rightarrow Z^*$. It follows that $g$ can be represented by the roof $i(X) \sim \tau_{<0} \cdot \rightarrow \Sigma^{-n}i(Y)$. As $\tau_{<0} \cdot \rightarrow \Sigma^{-n}i(Y)$ is zero, so is $g$. $\square$

The next lemma is a useful tool.

**Lemma 3.18.** Let $C^{\cdot,\cdot}$ be a double complex such that for every $n \in \mathbb{Z}$ the set $\{C^{i,j} \mid i + j = n, C^{i,j} \neq 0\}$ is finite.

1. If all rows or all columns of $C^{\cdot,\cdot}$ are acyclic, then $\text{Tot}^\otimes C^{\cdot,\cdot}$ is acyclic.
2. Let

$$
D^{n,m} = \begin{cases} 
C^{n,m} & \text{if } n \leq k, \\
0 & \text{if } n > k 
\end{cases}
$$

and let

$$
E^{n,m} = \begin{cases} 
0 & \text{if } n \leq k, \\
C^{n,m} & \text{if } i > k 
\end{cases}
$$

be naive truncations of $C^{\cdot,\cdot}$. Then

$$
\text{Tot}^\otimes C^{\cdot,\cdot} = \text{cone}(\Sigma^{-1} \text{Tot}^\otimes D^{\cdot,\cdot} \rightarrow \text{Tot}^\otimes E^{\cdot,\cdot})
$$

where $\text{Tot}^\otimes D^{\cdot,\cdot} \rightarrow \Sigma \text{Tot}^\otimes E^{\cdot,\cdot}$ is the natural map induces by the differential in $C^{\cdot,\cdot}$ (note that there appears an alternating sign in this map).

A similar statement holds for vertical cuts.

**Proof.** The second statement is an exercise in sign bookkeeping. For the first statement, assume that the rows are acyclic. To show that the complex $\text{Tot}^\otimes C^{\cdot,\cdot}$ is acyclic, it suffices to show that it is acyclic for each $n \in \mathbb{Z}$. As in definition 3.3, this is only dependent on a part of the complex:

$$(\text{Tot}^\otimes C^{\cdot,\cdot})^{n-2} \rightarrow (\text{Tot}^\otimes C^{\cdot,\cdot})^{n-1} \rightarrow (\text{Tot}^\otimes C^{\cdot,\cdot})^n \rightarrow (\text{Tot}^\otimes C^{\cdot,\cdot})^{n+1}.$$
As we have assumed that \( \{C^{i,j} \mid i+j = n, C^{i,j} \neq 0\} \) is finite, the statement that the complex \( \text{Tot}^E C^{\bullet, \bullet} \) is acyclic in degree \( n \) depends only on finitely many rows of the bicomplex \( C^{\bullet, \bullet} \). So, consider the bicomplex \( F^{\bullet, \bullet} \) obtained from \( C^{\bullet, \bullet} \) by replacing all other rows by zero. By construction, the sequence \( \text{Tot}^E C^{\bullet, \bullet} \) is acyclic in degree \( n \) if and only if \( \text{Tot}^E F^{\bullet, \bullet} \) is acyclic in degree \( n \). As in the second statement, we find that \( \text{Tot}^E E^{\bullet, \bullet} \) is obtained via consecutive cones. As the rows are acyclic, this shows that \( \text{Tot}^E E^{\bullet, \bullet} \) is acyclic. We conclude that \( \text{Tot}^E C^{\bullet, \bullet} \) is acyclic as well. \( \square \)

Given a right exact category \( E \) one can extend the right exact structure to \( C(E) \) degreewise. The canonical embeddings \( i: E \hookrightarrow D^+(E) \) and \( C^*(E) \hookrightarrow D^+(E) \) are compatible with the right exact structure in the following sense.

**Proposition 3.19.** Let \( E \) be a right exact category satisfying \( R0^* \). A conflation \( X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \) in \( C^*(E) \) induces a triangle

\[
X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \longrightarrow \Sigma X^*
\]

in \( D^*(E) \).

**Proof.** Consider the commutative diagram in \( D^*(E) \)

\[
\begin{array}{ccc}
X^* & \xrightarrow{f^*} & Y^* \\
\downarrow & & \downarrow \\
\text{cone}(f^*) & \xrightarrow{\omega^*} & \Sigma X^* \\
\downarrow & & \downarrow \\
X^* & \xrightarrow{f^*} & Y^* \xrightarrow{g^*} Z^* \\
\end{array}
\]

where \( h^* \) is given by \((0, g^*): X^{i+1} \oplus Y^i \to Z^i \).

It suffices to show that \( h \) is a quasi-isomorphism and hence invertible in \( D^*(E) \), so that the lower sequence fits into the triangle:

\[
X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \xrightarrow{w^* \circ (h^*)^{-1}} \Sigma X^*.
\]

By definition, the cone of \( h^* \) is given by

\[
\begin{pmatrix}
\begin{array}{cc}
X^{n+1} & 0 \\
-\tau f^n & -\tau g^n \\
0 & 0
\end{array}
\end{pmatrix}
\]

\[
\cdots \longrightarrow X^{n+2} \oplus Y^{n+1} \oplus Z^n \xrightarrow{\begin{pmatrix}
\begin{array}{ccc}
\tau f^n & 0 & 0 \\
-\tau f^{n+1} & -\tau g^{n+1} & 0 \\
0 & \tau g^{n+1} & \tau d^n
\end{array}
\end{pmatrix}} X^{n+3} \oplus Y^{n+2} \oplus Z^{n+1} \longrightarrow \cdots
\]

We need to show that this cone is acyclic. Now consider the double complex

\[
\begin{array}{cccc}
X^n & Y^n & Z^n & 0 \\
\downarrow f^n & \downarrow f^{n+1} & \downarrow f^{n+2} & \\
X^{n+1} & Y^{n+1} & Z^{n+1} & 0 \\
\downarrow g^n & \downarrow g^{n+1} & \downarrow g^{n+2} & \\
X^{n+2} & Y^{n+2} & Z^{n+2} & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

By definition, the associated total complex is equal to the cone of \( h^* \). As each column is a conflation and \( E \) satisfies axiom \( R0^* \), remark 3.4 yields that the columns are acyclic complexes. By lemma 3.18 the associated total complex is acyclic. This concludes the proof. \( \square \)

**Remark 3.20.** Example 6.3 shows that the converse in general is false. Indeed, \( iS(2) \to i\tau^{-1}P(2) \to iS(3) \to \Sigma iS(2) \) is a triangle in \( D^b(E) \) but \( S(2) \to \tau^{-1}P(2) \to S(3) \) is not a conflation in \( E \).
3.4. The homotopy category of projectives. We discuss projective and injective objects in a one-sided exact category. We show, if \( \mathcal{E} \) has enough projectives, that \( D^b(\mathcal{E}) \) is triangle equivalent to the homotopy category of projective objects (see proposition 3.27). The following definition is standard.

**Definition 3.21.** Let \( \mathcal{E} \) be a right exact category (or conflation category).

1. An object \( P \in \mathcal{E} \) is called **projective** if \( \text{Hom}(P,-): \mathcal{E} \to \text{Ab} \) is an exact functor. We say that \( \mathcal{E} \) has **enough projectives** if for every object \( M \in \mathcal{E} \) there is a deflation \( P \to M \) where \( P \) is projective.
2. Dually, an object \( f \) is called **injective** if \( \text{Hom}(-, f): \mathcal{E}^o \to \text{Ab} \) is an exact functor. We say that \( \mathcal{E} \) has **enough injectives** if for every object \( M \in \mathcal{E} \) there is an inflation \( M \to I \) where \( I \) is projective.

We write \( \text{Proj} \mathcal{E} \) and \( \text{Inj} \mathcal{E} \) for the full subcategories of projectives and injectives, respectively.

**Proposition 3.22.** Let \( \mathcal{E} \) be a right exact category. The following are equivalent:

1. \( P \) is projective.
2. For all deflations \( f: X \to Y \) and any map \( g: P \to Y \) there exists a map \( h: P \to X \) such that \( g = f \circ h \).
3. Any deflation \( f: X \to P \) is a retraction, i.e. there exist a map \( g: P \to X \) such that \( f \circ g = 1_P \).

**Proof.** The only non-standard implication is \( (3) \implies (2) \). Thus, assume that \( (3) \) holds. Let \( f: X \to Y \) be a deflation and \( g: P \to Y \) a map. By axiom \( \text{L2} \) there exists a pullback square

\[
\begin{array}{ccc}
Q & \overset{a}{\longrightarrow} & P \\
\downarrow{b} & & \downarrow{g} \\
X & \overset{f}{\longrightarrow} & Y
\end{array}
\]

Since \( Q \to P \) is a deflation, there is a corresponding section \( h: P \to Q \). It is straightforward to show that \( f \circ (b \circ h) = g \), establishing (2).

**Proposition 3.23.** Let \( \mathcal{E} \) be a right exact category. The following are equivalent:

1. \( P \) is injective.
2. For all inflations \( f: X \to Y \) and for any map \( g: X \to I \) there exists a map \( h: Y \to I \) such that \( g = h \circ f \).

**Remark 3.24.** In a right exact category, it is not necessarily true that an object \( I \) is injective if all inflations \( I \to X \) are sections. Indeed, the proof of proposition 3.22 cannot be dualized as axiom \( \text{L2} \) is not guaranteed.

An explicit example can be constructed from example 6.3. Indeed, consider the full and additive subcategory \( \mathcal{F} \) of \( \mathcal{U} \) generated by \( S(1), P(2), P(3), S(2) \), and \( S(3) \). The category \( \mathcal{F} \) inherits a right exact structure from \( \mathcal{U} \). The object \( S(2) \) is not injective in \( \mathcal{F} \), but all inflations starting at \( S(2) \) are coretractions.

**Proposition 3.25.** Let \( \mathcal{E} \) be a right exact category. Let \( P^* \in C(\mathcal{E}) \) be a complex of which every entry is projective. For all \( X^* \in C(\mathcal{E}) \), we have \( \text{Hom}_{K(\mathcal{E})}(P^*, X^*) = \text{Hom}_{D(\mathcal{E})}(P^*, X^*) \).

**Proof.** Without loss of generality, we may assume that \( P^n = 0 \) for \( i > 0 \). As in [9, theorem 12.4], the lifting property of projective objects yields that \( \text{Hom}_{K(\mathcal{E})}(P^*, X^*) = 0 \) whenever \( X^* \) is acyclic (and hence also when \( X^* \) is a direct summand of an acyclic complex in \( K(\mathcal{E}) \)).

Let \( f: Y^* \to X^* \) be a quasi-isomorphism in \( K(\mathcal{E}) \), so that \( \text{cone}(f) \) is a direct summand of an acyclic complex in \( K(\mathcal{E}) \). As then \( \text{Hom}_{K(\mathcal{E})}(P^*, \Sigma^n \text{cone}(f)) = 0 \) for all \( n \in \mathbb{Z} \), we find that \( f \) induces a bijection \( \text{Hom}_{K(\mathcal{E})}(P^*, Y^*) \cong \text{Hom}_{K(\mathcal{E})}(P^*, X^*) \). From the description of the maps in \( D(\mathcal{E}) \), we find that the natural map \( \text{Hom}_{K(\mathcal{E})}(P^*, X^*) \to \text{Hom}_{D(\mathcal{E})}(P^*, X^*) \) is a bijection as well.

**Definition 3.26.** We denote by \( K^{b}(\mathcal{E}) \) the full subcategory of \( K(\mathcal{E}) \) consisting of those complexes \( C^* \) bounded in the positive direction (i.e. \( C^i = 0 \), for \( i > 0 \)) and which are quasi-isomorphic to a bounded complex.

**Proposition 3.27.** Let \( \mathcal{E} \) be a right exact category satisfying axiom \( \text{R0}^* \) with enough projectives, then the categories \( K^{b}(\text{Proj}) \) and \( D^{b}(\mathcal{E}) \) are triangle equivalent. As similar statement holds if \( \mathcal{E} \) has enough injectives.

**Proof.** Since each object of \( K^{b}(\text{Proj}) \) is quasi-isomorphic to a bounded complex, it follows that there is an obvious triangle functor \( F: K^{b}(\text{Proj}) \to D^{b}(\mathcal{E}) \). It is shown in proposition 3.25 that this functor \( K^{b}(\text{Proj}) \to D^{b}(\mathcal{E}) \) is fully faithful.
We only need to show that $F$ is essentially surjective. To show this, let $C^\bullet \in \text{Ob}(D^b(E))$ be any object. Since $E$ has enough projectives, each $C^i$ has a projective resolution $P^{i,\bullet}$ (see for example [9, theorem 12.7]). Using projectivity one easily obtains the following commutative diagram

$$
\begin{array}{cccccc}
\cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
0 & P_{m-1} & \cdots & P_{0} & \cdots & 0 \\
\downarrow & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
0 & P_{m-1} & \cdots & P_{0} & \cdots & 0 \\
\downarrow & \cdots & \downarrow & \cdots & \cdots & \downarrow \\
0 & C_m & \cdots & C_0 & \cdots & 0 \\
\end{array}
$$

By lemma 3.18 and remark 3.4, the totalization of this double complex is acyclic. Denote by $P^{*,\bullet}$ the projective part of this double complex. It follows that the cone of the natural map $\text{Tot}^b(P^{*,\bullet}) \to C^\bullet$ is acyclic. Hence, $C^\bullet$ is quasi-isomorphic to $\text{Tot}^b(P^{*,\bullet}) \in \text{Ob}(K^{-b}(\text{Proj}))$. Hence, the functor $F: K^{-b}(\text{Proj}) \to D^b(E)$ is essentially surjective. This establishes that $F$ is a triangle equivalence, as required. \qed

4. Percolating subcategories and complexes

In [17], we considered the localization of a right exact category $E$ with respect to a right percolating subcategory $A$. We showed that the quotient $E/A$ can be obtained by localizing $E$ with respect to a right multiplicative system $S_A$. Moreover, the quotient $E/A$ inherits a natural right exact structure.

The category $C^b(E)$ of bounded cochain complexes in $E$ can be endowed with a right exact structure by extending the structure of $E$ degreewise. In this section we show that $C^b(A)$ is a right percolating subcategory of $C^b(E)$ and that the categories $C^b(E/A)$ and $C^b(E)/C^b(A)$ are equivalent.

4.1. Localization at percolating subcategories. We recall the basic notions and results on localizations of right exact categories at right percolating subcategories. We start with the definition of a percolating subcategory.

**Definition 4.1.** Let $E$ be a conflating category. A non-empty full subcategory $A$ of $E$ satisfying the following four axioms is called a right percolating subcategory of $E$.

- **P1** $A$ is a Serre subcategory, that is:
  
  If $A' \hookrightarrow A \twoheadrightarrow A''$ is a conflating in $E$, then $A \in \text{Ob}(A)$ if and only if $A', A'' \in \text{Ob}(A)$.

- **P2** For all morphisms $C \to A$ with $C \in \text{Ob}(E)$ and $A \in \text{Ob}(A)$, there exists a commutative diagram

  $$
  \begin{array}{ccc}
  A' & \to & A \\
  \downarrow & & \downarrow \\
  C & \to & A \\
  \end{array}
  $$

  with $A' \in \text{Ob}(A)$, and where $C \to A'$ is a deflation.

- **P3** If $i: C \to D$ is an inflation and $p: C \to A$ is a deflation with $A \in \text{Ob}(A)$, then the pushout of $i$ along $p$ exists and yields an inflation and a deflation, that is:

  $$
  \begin{array}{ccc}
  C & \to & D \\
  \downarrow & & \downarrow \\
  A & \leftarrow & B \\
  \end{array}
  $$

- **P4** For all inflations $i: A \to X$ and deflations $p: X \to B$ with $A, B \in A$, there exists objects $A', B' \in A$ such that there exists a commutative diagram:

  $$
  \begin{array}{ccc}
  A & \to & X \\
  \downarrow & & \downarrow \\
  A' & \to & B' \\
  \end{array}
  $$
A left percolating subcategory of a conflation category is defined dually. Following the terminology from [34], a non-empty full subcategory \( \mathcal{A} \) of a right exact category \( \mathcal{E} \) satisfying axioms P1 and P2 (respectively axioms P1 and P2\(^{op} \)) is called right filtering (respectively left filtering). If \( \mathcal{A} \) is a right filtering subcategory of \( \mathcal{C} \) such that the map \( A' \to A \) in axiom P2 can be chosen as a monomorphism, we will call \( \mathcal{A} \) a strongly right filtering subcategory. A right percolating subcategory which is also strongly right filtering will be abbreviated to a strongly right percolating subcategory.

**Remark 4.2.**

1. A subcategory \( \mathcal{A} \) of a right exact category \( \mathcal{E} \) satisfying axioms P1, P2 and P3 such that the map \( A' \to A \) in axiom P2 can be chosen as an inflation, is called an abelian right percolating subcategory in [17]. An abelian right percolating subcategory is, as the name suggests, right percolating (thus axiom P4 is automatic) and abelian. Note that an abelian right percolating subcategory is a strongly right percolating subcategory.

2. If \( \mathcal{E} \) is a (left or right) quasi-abelian category, any percolating subcategory is automatically a strong percolating subcategory (see [17, corollary 7.21]).

3. The notion of a right percolating subcategory is weaker than the notion of a right s-filtering subcategory as introduced by Schlichting ([34]). We refer the reader to [17, section 7.1] for a comparison to the localization theories of Cardenas and Schlichting (see [10, 34]).

The next definition constructs a right multiplicative system \( S_{\mathcal{A}} \). The terminology is based on [10, 34].

**Definition 4.3.** Let \( \mathcal{E} \) be a conflation category and let \( \mathcal{A} \) be a non-empty full subcategory of \( \mathcal{E} \).

1. An inflation \( f: X \to Y \) in \( \mathcal{C} \) is called an \( \mathcal{A}^{-1} \)-inflation if its cokernel lies in \( \mathcal{A} \).

2. A deflation \( f: X \to Y \) in \( \mathcal{C} \) is called a \( \mathcal{A}^{-1} \)-deflation if its kernel lies in \( \mathcal{A} \).

3. A morphism \( f: X \to Y \) is called a weak \( \mathcal{A}^{-1} \)-isomorphism, or simply a weak isomorphism whenever \( \mathcal{A} \) is implied, if it is a finite composition of \( \mathcal{A}^{-1} \)-inflations and \( \mathcal{A}^{-1} \)-deflations.

The set of weak isomorphisms is denoted by \( S_{\mathcal{A}} \).

Given a weak isomorphism \( f \), the composition length of \( f \) is smallest number \( n \) such that \( f \) can be written as a composition of \( n \) \( \mathcal{A}^{-1} \)-inflations and \( \mathcal{A}^{-1} \)-deflations.

**Proposition 4.4.** Let \( \mathcal{E} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory. The composition of \( \mathcal{A}^{-1} \)-deflations (resp. \( \mathcal{A}^{-1} \)-inflations) is again an \( \mathcal{A}^{-1} \)-deflation (resp. \( \mathcal{A}^{-1} \)-inflation). In particular, any weak isomorphism is a finite composition of alternating \( \mathcal{A}^{-1} \)-inflations and \( \mathcal{A}^{-1} \)-deflations.

**Proposition 4.5.** Let \( \mathcal{E} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory. Let \( f: X \to Y \) be an \( \mathcal{A}^{-1} \)-inflation and let \( g: Y \to Z \) be an inflation. The composition \( g \circ f: X \to Z \) is an inflation.

**Definition 4.6.** Let \( \mathcal{E} \) be a right exact category and let \( \mathcal{A} \) be a full right exact subcategory. The quotient of \( \mathcal{E} \) by \( \mathcal{A} \) is a right exact category \( \mathcal{E}/\mathcal{A} \) together with an exact functor \( Q: \mathcal{E} \to \mathcal{E}/\mathcal{A} \), called the quotient functor, such that, for any right exact category \( \mathcal{D} \) and any exact functor \( F: \mathcal{E} \to \mathcal{D} \) with \( F(\mathcal{A}) \equiv 0 \), there exists a unique exact functor \( G: \mathcal{E}/\mathcal{A} \to \mathcal{D} \) such that \( F = G \circ Q \).

The next proposition states that \( S_{\mathcal{A}} \) is a right multiplicative system with a convenient strengthening axiom RMS2.

**Proposition 4.7.** Let \( \mathcal{E} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory. The set \( S_{\mathcal{A}} \) of weak isomorphism is a right multiplicative system. Furthermore, if the map \( s \) in axiom RMS2 is an \( \mathcal{A}^{-1} \)-deflation or an \( \mathcal{A}^{-1} \)-inflation, then one can choose \( t \) to be an \( \mathcal{A}^{-1} \)-deflation or an \( \mathcal{A}^{-1} \)-inflation, respectively.

If \( \mathcal{A} \) is a strongly right percolating subcategory, the square in axiom RMS2 can be chosen as a pullback square.

The following theorem is the main theorem of [17].

**Theorem 4.8.** Let \( \mathcal{E} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory. The category \( S_{\mathcal{A}}^{-1} \mathcal{E} \) is a right exact category where the conflation structure is induced by the localization functor \( Q: \mathcal{E} \to S_{\mathcal{A}}^{-1} \mathcal{E} \). The category \( S_{\mathcal{A}}^{-1} \mathcal{E} \) satisfies the universal property of the quotient \( \mathcal{E}/\mathcal{A} \).

The next proposition gives a useful characterization of the kernel of the localization functor \( Q \).

**Proposition 4.9.** Let \( \mathcal{A} \) be a right percolating subcategory of a right exact category \( \mathcal{E} \). For any \( f: X \to Y \) a map in \( \mathcal{E} \) with \( Q(f) = 0 \), there exists an \( \mathcal{A}^{-1} \)-inflation \( s \) such that \( f \circ s = 0 \).

The following proposition is a version of the 3 × 3-lemma and an apparent strengthening of P4.
Proposition 4.10. Let $A$ be a right percolating subcategory of a right exact category $E$. Any diagram

\[
\begin{array}{ccc}
U & \longrightarrow & Z \\
\downarrow \rho \downarrow f & & \downarrow \gamma \\
X \\ \downarrow i \\
B \\
\end{array}
\]

whose row and column are conflations with $B \in A$, can be embedding in a commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & Z' \\
\downarrow \lambda & & \downarrow \lambda \\
X & \longrightarrow & Z \\
\downarrow i \\
A \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \rho \downarrow f & & \downarrow \gamma \\
Y \\
\downarrow \lambda \\
B' \\
\end{array}
\]

where the rows and columns are conflations.

4.2. Lifting percolating subcategories to complexes. In this subsection, we start with a right percolating subcategory $A$ of a right exact category $E$. We will show that $C^b(A)$ is a right percolating subcategory of the right exact category $C^b(E)$.

Lemma 4.11. Let $E$ be a right exact category and let $A$ be a right percolating subcategory. Let $X^* \in C^b(E)$. For each $n \in \mathbb{Z}$, let $K^n \longrightarrow X^n \longrightarrow A^n$ be a conflation with $A^n \in A$. Then there exists a complex $L^* \in C^b(E)$ such that for each $n \in \mathbb{Z}$ there is an $A^{-1}$-inflation $L^n \longrightarrow K^n$ and such that the induced map $L^* \rightarrow X^*$ is a chain map.

Proof. Applying axiom RMS2 to the composition $K^{n+1} \rightarrow X^{n+1} \rightarrow X^{n+2}$ and $K^{n+2} \rightarrow X^{n+2}$, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
L^{n+1} & \longrightarrow & K^{n+2} \\
\downarrow \iota \downarrow & & \downarrow \iota \\
K^{n+1} & \longrightarrow & K^{n+2} \\
\downarrow \iota \downarrow & & \downarrow \iota \\
X^{n+1} & \longrightarrow & X^{n+2} \\
\end{array}
\]

Here we used proposition 4.7 to obtain an $A^{-1}$-inflation $L^{n+1} \longrightarrow K^{n+1}$. By proposition 4.4, the composition $L^{n+1} \longrightarrow K^{n+1} \longrightarrow X^{n+1}$ is an $A^{-1}$-inflation. Hence applying axiom RMS2 to $K^n \rightarrow X^n \rightarrow X^{n+1}$ and $L^{n+1} \rightarrow X^{n+1}$, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
L^n & \longrightarrow & L^{n+1} \\
\downarrow \iota \downarrow & & \downarrow \iota \\
K^n & \longrightarrow & K^{n+1} \\
\downarrow \iota \downarrow & & \downarrow \iota \\
X^n & \longrightarrow & X^{n+1} \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \iota \downarrow & & \downarrow \iota \\
K^{n+1} & \longrightarrow & K^{n+2} \\
\downarrow \iota \downarrow & & \downarrow \iota \\
X^{n+1} & \longrightarrow & X^{n+2} \\
\end{array}
\]

The composition $L^n \rightarrow L^{n+1} \rightarrow L^{n+2}$ is zero as $X^*$ is a complex and $L^{n+2} \rightarrow X^{n+2}$ is monic.
Since $X^\bullet$ is right bounded, there exists an $m \in \mathbb{Z}$ such that $X^{m+i} = 0$ for all $i > 0$. Since $K^{m+i} \to X^{m+i}$ is monic, $K^{m+i} = 0$ for all $i > 0$. Applying the above procedure from right to left, one obtains the desired complex $L^\bullet \in \mathcal{C}^-(\mathcal{E})$ with $L^{m+i} = 0$ for all $i > 0$. 

**Proposition 4.12.** Let $\mathcal{E}$ be a right exact category and let $\mathcal{A} \subseteq \mathcal{E}$ be a right percolating subcategory.

(1) The category $\mathcal{C}^b(\mathcal{E})$ has a natural right exact structure defined degreewise.

(2) The subcategory $\mathcal{C}^b(\mathcal{A})$ is a right percolating subcategory of $\mathcal{C}^b(\mathcal{E})$.

**Proof.** The first statement is straightforward to check. To show the second statement we need to verify axioms P1, P2, P3 and P4. Axiom P1 is automatic. We now show P2. Let $f^\bullet : X^\bullet \to A^\bullet$ be a map in $\mathcal{C}^b(\mathcal{E})$ with $A^\bullet \in \mathcal{C}^b(\mathcal{A})$. Applying axiom P2 degreewise, we obtain a commutative diagram

$$
\begin{array}{ccc}
K^n & \xrightarrow{i} & K^{n+1} \\
\downarrow & & \downarrow \\
X^n & \xrightarrow{i} & X^{n+1} \\
\downarrow & & \downarrow \\
B^n & \xrightarrow{i} & B^{n+1}
\end{array}
$$

where the downwards arrows $X^i \to B^i \to A^i$ compose to $f^i$ and the sequences $K^i \to X^i \to B^i$ are conflations. The result follows by applying lemma 4.11.

Consider an inflation $i^\bullet : X^\bullet \to Y^\bullet$ and a deflation $p^\bullet : X^\bullet \to A^\bullet$ with $A^\bullet \in \mathcal{C}^b(\mathcal{A})$. Applying axiom P3 degreewise, there exists pushout squares

$$
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
$$

Using the universal property of pushouts, we obtain a map $q^n : Q^n \to Q^{n+1}$ such that the diagram

$$
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^n} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n_Y} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n_A} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n} \quad
\begin{array}{ccc}
X^n & \xrightarrow{i_n} & Y^n \\
\downarrow & & \downarrow \\
A^n & \xrightarrow{\epsilon_n} & Q^n
\end{array}
\xrightarrow{\pi^{n+1} \circ d^n}
$$

is commutative. Moreover, as each $\pi^n$ is a deflation (and, in particular, an epimorphism) and $Y^\bullet$ is a cochain complex, $(Q^\bullet, q^\bullet)$ is a cochain complex. This shows axiom P3.

To show axiom P4, consider maps $A^\bullet \xrightarrow{\iota^\bullet} X^\bullet \xrightarrow{p^\bullet} B^\bullet$ in $\mathcal{C}^b(\mathcal{E})$ with $A^\bullet, B^\bullet \in \mathcal{C}^b(\mathcal{A})$. Let $n \in \mathbb{Z}$ be the largest integers such that $A^n, X^n$ and $B^n$ are not all zero. Proposition 4.10 yields the existence
of a commutative diagram

\[
\begin{array}{cccc}
U^n & \downarrow & \ker(p^n) & \\
A^n & \downarrow & X^n & \to \coker(i^n) \\
B^n & \downarrow & & \\
C^n & \downarrow & D^n & \\
\end{array}
\]

where \(C^n\) and \(D^n\) belong to \(\mathcal{A}\).

Using axiom RMS2 one obtains a commutative diagram

\[
\begin{array}{cccc}
V^{n-1} & \to & U^n & \\
\downarrow & \ker(p^{n-1}) & \downarrow & \\
X^{n-1} & \to & X^n & \\
\downarrow & p^{n-1} & \downarrow & \\
B^{n-1} & \to & D^n & \\
\end{array}
\]

Here \(X^n \to D^n\) is the cokernel of the \(A^{-1}\)-inflation \(V^{n-1} \to X^{n-1}\). We now apply proposition 4.10 to \(A^{n-1} \to X^{n-1}\) and the \(A^{-1}\)-inflation \(V^{n-1} \to X^{n-1}\) to obtain a commutative diagram

\[
\begin{array}{cccc}
U^{n-1} & \downarrow & \to & \\
A^{n-1} & \downarrow & X^{n-1} & \to \coker(i^{n-1}) \\
\downarrow & \to & \coker(i^{n-1}) & \\
C^{n-1} & \downarrow & D^{n-1} & \\
\end{array}
\]
Combining the three diagrams, we obtain a commutative diagram

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
U^{n-1}
\end{array}
\begin{array}{c}
V^{n-1} \quad U^n \\
\downarrow \quad \downarrow \\
X^{n-1} \quad X^n \\
\downarrow \quad \downarrow \\
D^{n-1} \quad D^n
\end{array}
\]

By construction the map \(X^{n-1} \to B^{n-1}\) factors through \(X^{n-1} \to D^{n-1}\).

Alternatively applying axiom RMS2 and proposition 4.10 to the newly obtained \(A^{-1}\)-inflation, one constructs a complex \(D^*\) such that there is a deflation \(A^* \to D^*\) which factors \(X^* \to B^*\). Moreover, the diagrams above define a complex \(C^*\) which is the image of the composition \(A^* \to X^* \to D^*\). This shows axiom P4.

Combining this with theorem 4.8 yields the following corollary.

**Corollary 4.13.** Let \(\mathcal{E}\) be a right exact category and let \(\mathcal{A}\) be a right percolating subcategory. The quotient \(C^b(\mathcal{E})/C^b(\mathcal{A})\) is a right exact category which can be realized as the localization of \(C^b(\mathcal{E})\) with respect to the right multiplicative system \(S_{C^b(\mathcal{A})}\) of weak isomorphisms.

**4.3. The equivalence** \(C^b(\mathcal{E}/\mathcal{A}) \cong C^b(\mathcal{E})/C^b(\mathcal{A})\). We show that the natural map \(C^b(\mathcal{E}) \to C^b(\mathcal{E}/\mathcal{A})\) induced by \(Q: \mathcal{E} \to \mathcal{E}/\mathcal{A}\) factors through an equivalence \(C^b(\mathcal{E})/C^b(\mathcal{A}) \to C^b(\mathcal{E}/\mathcal{A})\).

**Lemma 4.14.** Let \(\mathcal{E}\) be a right exact category and let \(\mathcal{A}\) be a right percolating subcategory.

1. Let \(X^* \in C^{[n, m]}(\mathcal{E}/\mathcal{A})\). There exists a \(Y^* \in C^{[n, m]}(\mathcal{E}/\mathcal{A})\) whose \(i\)-th differential is of the form \((d^*_Y, 1_Y)\) such that \(X^* \cong Y^* \in C^{[n, m]}(\mathcal{E}/\mathcal{A})\). Moreover, one can choose \(Y^*\) such that \(Y^{n+m} = X^{n+m}\).

2. Let \(X^* \in C^{[n, m]}(\mathcal{E}/\mathcal{A})\). There exists a \(Z^* \in C^{[n, m]}(\mathcal{E})\) such that \(Z^*\) represents \(X^*\) in \(C^{[n, m]}(\mathcal{E}/\mathcal{A})\).

3. Let \(X^* \in C^{[n, m]}(\mathcal{E})\). Let \(s^i: Y^i \to X^i\) be weak isomorphisms for all \(n \leq i \leq m\). There is a cochain complex \((Z^*, d^*_Z) \in C^{[n, m]}(\mathcal{E})\), together with weak isomorphisms \(t^i: Z^i \to Y^i\), for all \(n \leq i \leq m\), such that the composition \(s^* \circ t^* : Z^* \to X^*\) is a cochain morphism. Moreover, one can choose \(Z^*\) such that \(Z^{n+m} = Y^{n+m}\).

**Proof.**

1. The complex \(X^*\) has the following form:

\[
\begin{array}{c}
0 \\
\downarrow \\
X^n \\
\downarrow \\
X^{n+1} \\
\downarrow \\
X^{n+2} \\
\downarrow \\
X^{n+m} \\
\downarrow \\
0
\end{array}
\]

Repeatedly applying axiom RMS2, we obtain a commutative diagram:

\[
\begin{array}{c}
Y^{n+m-3} \\
\downarrow \\
Y^{n+m-2} \\
\downarrow \\
Y^{n+m-1} \\
\downarrow \\
X^{n+m} \\
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{c}
X^{n+m-3} \\
\downarrow \\
X^{n+m-2} \\
\downarrow \\
X^{n+m-1} \\
\downarrow \\
X^{n+m} \\
\downarrow \\
0
\end{array}
\]

By construction the sequence \(Y^*\) has the desired form. (Note that \(Y^*\) need not be cochain complex in \(C^b(\mathcal{E})\)).
(2) Let $X^* \in \mathbf{C}^b(\mathcal{E}/A)$ and let $Y^*$ be the sequence as above. Clearly, the composition $Y^{n+m-2} \rightarrow X^{n+m}$ is zero in $\mathcal{E}/A$. Proposition 4.9 yields the existence of a weak isomorphism $Z^{n+m-2} \cong Y^{n+m-2}$ such that the composition $Z^{n+m-2} \rightarrow X^{n+m}$ is zero in $\mathcal{E}$.

Applying axiom RMS2 to $Z^{n+m-2} \cong Y^{n+m-2}$ and $Y^{n+m-3} \rightarrow Y^{n+m-2}$, we obtain a commutative diagram:

\[ \begin{array}{ccc}
U^{n+m-3} & \rightarrow & Z^{n+m-3} \\
\downarrow & & \downarrow \\
Y^{n+m-3} & \rightarrow & Y^{n+m-2} \\
& & \downarrow 0 \\
Y^{n+m-2} & \rightarrow & Y^{n+m-1} \\
& \phi & \\
& X^{n+m} & \rightarrow 0
\end{array} \]

As the composition $U^{n+m-3} \rightarrow Y^{n+m-1}$ descends to zero, proposition 4.9 yields a weak isomorphism $Z^{n+m-3} \rightarrow U^{n+m-3}$ such that the composition $Z^{n+m-3} \rightarrow Y^{n+m-1}$ is zero in $\mathcal{E}$. One can now iteratively construct the complex $Z^*$ by repeatedly applying axiom RMS2 to $Z^{n+m-i} \rightarrow Y^{n+m-i}$ and $Y^{n+m-i-1} \rightarrow Y^{n+m-i}$ to obtain a commutative square:

\[ \begin{array}{ccc}
U^{n+m-i-1} & \rightarrow & Z^{n+m-i} \\
\downarrow & & \downarrow \\
Y^{n+m-i-1} & \rightarrow & Y^{n+m-i} \\
& & \phi
\end{array} \]

The composition $U^{n+m-i-1} \rightarrow Z^{n+m-i+2}$ descends to zero and thus one can apply proposition 4.9 to obtain $Z^{n+m-i-1}$. Proceeding in this fashion one constructs the desired complex $Z^* \in \mathbf{C}^b(\mathcal{E})$.

(3) The proof of this statement is similar to the previous argument. We leave the details to the reader. 

**Proposition 4.15.** Let $\mathcal{E}$ be a right exact category and let $A$ be a right percolating subcategory. The natural functor $\mathbf{C}^b(\mathcal{E}) \rightarrow \mathbf{C}^b(\mathcal{E}/A)$ factors through an equivalence $\Psi: \mathbf{C}^b(\mathcal{E})/\mathbf{C}^b(A) \cong \mathbf{C}^b(\mathcal{E}/A)$.

**Proof.** We write $Q: \mathcal{E} \rightarrow \mathcal{E}/A$ for the quotient functor. The functor $Q$ induces a natural exact functor $\hat{Q}: \mathbf{C}^b(\mathcal{E}) \rightarrow \mathbf{C}^b(\mathcal{E}/A)$ such that $\hat{Q}(\mathbf{C}^b(A)) = 0$. By the universal property of the quotient $\mathbf{C}^b(\mathcal{E})/\mathbf{C}^b(A)$, there exists an induced functor $\Psi: \mathbf{C}^b(\mathcal{E})/\mathbf{C}^b(A) \rightarrow \mathbf{C}^b(\mathcal{E}/A)$. We will show that $\Psi$ is an equivalence of right exact categories.

We first show that $\Psi$ is essentially surjective. Let $X^* \in \mathbf{C}^b(\mathcal{E}/A)$. By lemma 4.14.(2) there is a complex $Z^* \in \mathbf{C}^b(\mathcal{E})$ such that $\Psi(Z^*) = X^*$.

To show that $\Psi$ is full, let $\phi^*: X^* \rightarrow Y^*$ be an arbitrary map in $\mathbf{C}^b(\mathcal{E}/A)$. By lemma 4.14.(2), we may assume that $X^*$ and $Y^*$ belong to $\mathbf{C}^b(\mathcal{E})$. Hence we may represent $\phi^* = (\phi^i, s^i)_{i \in \mathbb{Z}}$ in $\mathcal{E}$ by the following diagram:

\[ \begin{array}{ccccccccc}
\ldots & \rightarrow & X^{n-1} & \rightarrow & X^n & \rightarrow & X^{n+1} & \rightarrow & \ldots \\
& & & & & i & & & \\
& & \phi_{n-1} & \phi_n & \phi_{n+1} & \phi_{n+2} & \ldots \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
& \phi_{n-2} & \phi_{n-1} & \phi_n & \phi_{n+1} & \ldots \\
& & \downarrow & \downarrow & \downarrow & \ldots \\
& & Y^{n-1} & \rightarrow & Y^n & \rightarrow & Y^{n+1} & \rightarrow & \ldots
\end{array} \]

By lemma 4.14.(3), we can replace $L^*$ by a weakly isomorphic complex $Z^* \in \mathbf{C}^b(\mathcal{E})$ such that the induced map $Z^* \rightarrow Y^*$ descends to the morphism $\phi$. This shows that $\Psi$ is full.

It remains to show that $\Psi$ is faithful. To that end, let $\phi^*: X^* \rightarrow Y^*$ be a map in $\mathbf{C}^b(\mathcal{E})$ such that $\Psi(\phi^*) = 0$. Without loss of generality, we may assume that $\phi^i = 0$ for all $i$. It follows that each $\phi^i$ factors as $X^i \rightarrow A^i \rightarrow Y^i$ where $A^i \in A$ and the first map is a deflation. Write $K^i$ for the kernel of the deflation $X^i \rightarrow A^i$. By lemma 4.11 there is a complex $L^*$ and a chain map $L^* \rightarrow X^*$ with a cokernel $B^* \in \mathbf{C}^b(A)$ such that $\phi^i$ factors through $B^i$. This concludes the proof.

**Remark 4.16.** Note that the results in this section can be extended to right bounded complexes, i.e. $\mathbf{C}^b(\mathcal{E}/A) \cong \mathbf{C}^b(\mathcal{E})/\mathbf{C}^b(A)$. For left exact categories this dualizes to left bounded complexes.

5. Derived categories of localizations

Given a right exact category $\mathcal{E}$ and a right percolating subcategory $A \subseteq \mathcal{E}$, we study the derived category $\mathbf{D}^b(\mathcal{E}/A)$. We show that, under mild assumptions, the sequence

$$\mathbf{D}^b_A(\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E}/A)$$
is a Verdier localization sequence.

5.1. Some results on localizations. The following proposition is an adaptation of [16, proposition I.3.4] (see also [12, proposition I.1.3.(iv)]).

**Proposition 5.1.** Let \( L: \mathcal{C} \rightarrow S^{-1}\mathcal{C} \) be a localization. For each category \( \mathcal{D} \), there is a fully faithful functor
\[
- \circ L: \text{Fun}(S^{-1}\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})
\]
whose image consists of those functors \( F: \mathcal{C} \rightarrow \mathcal{D} \) for which \( F(s) \) is invertible (for all \( s \in S \)).

**Proof.** It is clear that the functor \(- \circ L: \text{Fun}(S^{-1}\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})\) is faithful. Moreover, it follows from the universal property that the image is as described. We only need to show that it is full. Let \( F,G: \mathcal{C} \rightarrow \mathcal{D} \) be functors in the image of \(- \circ L\) (thus, \( F = F' \circ L \) and \( G = G' \circ L \) for some \( F,G: S^{-1}\mathcal{C} \rightarrow \mathcal{D} \)) and \( \eta: F \Rightarrow G \) a natural transformation. This describes a functor \( N: \mathcal{C} \rightarrow \mathcal{D}^{-1} \) (where \( \mathcal{D}^{-1} \) is the arrow category of \( \mathcal{D} \)) by \( C \mapsto \left[ F(C) \overset{G(C)}{\rightarrow} G(C) \right] \). As \( F \) and \( G \) map every element of \( S \) to an invertible element, so does \( N \). This implies that \( N \) factors as \( \mathcal{C} \xrightarrow{L} S^{-1}\mathcal{C} \xrightarrow{M} \mathcal{D}^{-1} \). The functor \( M \) then gives the required natural transformation \( F' \Rightarrow G' \).

The following lemma is well known. We provide a proof for the benefit of the reader.

**Lemma 5.2.** Let \( \mathcal{C} \) be any category.

1. Let \( S,T \subset \text{Mor}(\mathcal{C}) \). Let \( N: \mathcal{C} \rightarrow S^{-1}\mathcal{C} \) and \( M: \mathcal{C} \rightarrow T^{-1}\mathcal{C} \) be the corresponding localizations. If \( F: S^{-1}\mathcal{C} \rightarrow T^{-1}\mathcal{C} \) is a functor such that
\[
\begin{array}{ccc}
C & \xrightarrow{M} & T^{-1}\mathcal{C} \\
\downarrow N & & \downarrow \Phi \\
S^{-1}\mathcal{C} & \xrightarrow{F} & \mathcal{C}
\end{array}
\]
commutes, then \( F \) is a localization itself.

2. Let \( S \subset \text{Mor}(\mathcal{C}) \) be a right multiplicative system and \( U \subseteq \text{Mor}(S^{-1}\mathcal{C}) \) any set of morphisms. Let \( F: \mathcal{C} \rightarrow S^{-1}\mathcal{C} \) and \( G: S^{-1}\mathcal{C} \rightarrow U^{-1}(S^{-1}\mathcal{C}) \) be the localizations functors, then the composition \( G \circ F \) is a localization as well.

3. Let \( F: \mathcal{C} \rightarrow S^{-1}\mathcal{C} \) be a localization. Let \( \Phi: \mathcal{D} \rightarrow \mathcal{C} \) be any functor. Let \( T = \{ f \in \text{Mor}(\mathcal{D}) \mid \Phi \circ F(f) \text{ is invertible} \} \). If \( \Phi \) is an equivalence, so is the natural functor \( G: T^{-1}\mathcal{D} \rightarrow S^{-1}\mathcal{C} \).

**Proof.**

1. Define \( U = \{ u \in \text{Mor}(S^{-1}\mathcal{C}) \mid F(u) \text{ invertible} \} \). We will verify that \( F: S^{-1}\mathcal{C} \rightarrow T^{-1}\mathcal{C} \) is a localization with respect to \( U \) by showing it satisfies the corresponding universal property. Therefore, let \( G: S^{-1}\mathcal{C} \rightarrow \mathcal{D} \) be a functor such that \( G(u) \) is invertible for all \( u \in U \). As \( M \) is a localization with respect to \( T \), we know that \( M(t) = F(N(t)) \) is invertible, for all \( t \in T \). By definition, we have that \( N(t) \in U \) and hence \( G \circ N(t) \) is invertible as well. By the universal property of \( T^{-1}\mathcal{C} \) there exists a unique functor \( L: T^{-1}\mathcal{C} \rightarrow \mathcal{D} \) such that \( L \circ M = G \circ N \). As then \( L \circ F \circ N = G \circ N \), the universality of \( N \) implies that \( L \circ F = G \). This finishes the proof.

2. Define \( W = \{ w \in \text{Mor}(\mathcal{C}) \mid GF(u) \text{ invertible} \} \). We will show that \( G \circ F \) satisfies the universal property of the localization with respect to \( W \). Let \( H: \mathcal{C} \rightarrow \mathcal{D} \) be a functor such that \( H(w) \) is invertible for any \( w \in W \). Obviously \( S \subseteq W \), so that the universal property of \( S^{-1}\mathcal{C} \) implies that there exists a unique functor \( K: S^{-1}\mathcal{C} \) such that \( H = K \circ F \). We now show that \( K \) maps every map in \( U \) to an invertible map in \( \mathcal{D} \). Let \( u \in U \). Since \( S \) is a right multiplicative system, we can write \( u = fs^{-1} \) with \( s \in S \) and \( f \) a map in \( \mathcal{C} \). Since \( G(u) \) and \( G(s) \) are invertible, \( G(f) = G(u)G(s) \) is invertible as well. Using that \( F(f) = f \), we infer that \( G \circ F(f) \) is invertible and hence \( f \in W \). Thus, \( H(f) \) is invertible, and it follows that \( K(u) \) is invertible. By the universal property of the localization \( G: S^{-1}\mathcal{C} \rightarrow U^{-1}(S^{-1}\mathcal{C}) \), there exists a unique functor \( L: U^{-1}(S^{-1}\mathcal{C}) \rightarrow \mathcal{D} \) such that \( K = L \circ G \). Putting everything together we find that \( H = L \circ (G \circ F) \). This establishes the required properties.

3. This follows easily from proposition 5.1. □

5.2. **Proof of main theorem.** In the statement of the main theorem we use the following definition.

**Definition 5.3.** Let \( \mathcal{A} \) be a right percolating subcategory of a right exact category \( \mathcal{E} \) satisfying axiom \( \text{RO}^* \). The triangulated subcategory of \( \text{D}^b(\mathcal{E}) \) generated by the image of \( \mathcal{A} \) under the canonical embedding is denoted by \( \text{D}^b_{\mathcal{A}}(\mathcal{E}) \).
**Proposition 5.4.** Let \( \mathcal{E} \) be a right exact category satisfying axiom \( R0^* \) and let \( \mathcal{A} \subseteq \mathcal{E} \) be a right percolating subcategory.

1. If \( X^* \xrightarrow{f^*} Y^* \) is a weak isomorphism (i.e. \( f^* \in S_{C^b(A)} \)), then \( \text{cone}(f^*) \in D^b_{\mathcal{A}}(\mathcal{E}) \).
2. If \( X^* \xrightarrow{f^*} Y^* \) is a weak isomorphism, then \( X^* \in D^b_{\mathcal{A}}(\mathcal{E}) \) if and only if \( Y^* \in D^b_{\mathcal{A}}(\mathcal{E}) \).

**Proof.**

(1) In any triangulated category, the cone of a composition is an extension of the cones of the composition factors (this is an immediate corollary of [28, proposition 1.4.6]). Since \( f^* \) factors as a composition of \( C^b(A)^{-1}\)-deflations and \( C^b(A)^{-1}\)-inflations, it suffices to show that the cones of \( C^b(A)^{-1}\)-deflations (resp. \( C^b(A)^{-1}\)-inflations) belong to \( D^b_{\mathcal{A}}(\mathcal{E}) \). The latter follows immediately from proposition 3.19 and the definition of \( C^b(A)^{-1}\)-deflations (resp. \( C^b(A)^{-1}\)-inflations).

(2) This statement follows from the previous statement. \( \square \)

We now come to the the main theorem; our proof closely resembles the proofs given in [26, theorem 3.2] and [34, proposition 2.6]. The second part of the main theorem requires the right exact category \( \mathcal{E} \) to satisfy axiom \( R0^* \) and requires the right multiplicative system \( S_\mathcal{A} \) to be right weakly saturated (see definition 2.10). In proposition 5.6 below, we give a sufficient condition on the percolating subcategory \( \mathcal{A} \) for \( S_\mathcal{A} \) to be right weakly saturated.

**Theorem 5.5.** Let \( \mathcal{E} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory.

1. The derived quotient functor \( D^b(\mathcal{E}) \to D^b(\mathcal{E}/\mathcal{A}) \) is a Verdier localization.
2. If additionally \( \mathcal{E} \) satisfies axiom \( R0^* \) and \( S_\mathcal{A} \) is right weakly saturated, then the sequence

\[ D^b_{\mathcal{A}}(\mathcal{E}) \to D^b(\mathcal{E}) \to D^b(\mathcal{E}/\mathcal{A}) \]

is a Verdier localization sequence.

**Proof.**

(1) Consider the natural diagram

\[
\begin{array}{ccc}
C^b(\mathcal{E}) & \to & C^b(\mathcal{E})/C^b(\mathcal{A}) \cong C^b(\mathcal{E}/\mathcal{A}) \\
\downarrow & & \downarrow \\
K^b(\mathcal{E}) & \to & K^b(\mathcal{E}/\mathcal{A})
\end{array}
\]

where \( G \) is a triangle functor. Recall from corollary 4.13 that the functor \( C^b(\mathcal{E}) \to C^b(\mathcal{E})/C^b(\mathcal{A}) \) is a localization with respect to a right multiplicative set and that, for an additive category \( C \), the quotient \( C^b(\mathcal{E}) \to K^b(\mathcal{E}) \) is a localization functor (with respect to the Hurewicz model structure, see [14, 3]). It follows from lemma 5.2 that the triangle functor \( G \) is a Verdier localization.

Similarly, it follows from the diagram

\[
\begin{array}{ccc}
K^b(\mathcal{E}) & \to & K^b(\mathcal{E}/\mathcal{A}) \\
\downarrow & & \downarrow \\
D^b(\mathcal{E}) & \to & D^b(\mathcal{E}/\mathcal{A})
\end{array}
\]

that \( F \) is a Verdier localization.

(2) We need to show that \( \ker(F) \) equals the thick closure \( \langle D^b_{\mathcal{A}}(\mathcal{E}) \rangle_{\text{thick}} \) of \( D^b_{\mathcal{A}}(\mathcal{E}) \) in \( D^b(\mathcal{E}) \).

It is clear that \( D^b_{\mathcal{A}}(\mathcal{E}) \subseteq \ker(F) \). As \( \ker(F) \) is a thick subcategory, it follows that \( \langle D^b_{\mathcal{A}}(\mathcal{E}) \rangle_{\text{thick}} \subseteq \ker(F) \). It remains to show the other inclusion.

Let \( X^* \in \ker(F) \). By the above diagram we find that \( G(X^*) \in \langle \text{Ac}^b(\mathcal{E}/\mathcal{A}) \rangle_{\text{thick}} \). It follows that \( X^* \) is a direct summand in \( K^b(\mathcal{E}/\mathcal{A}) \) of an acyclic complex in \( C^b(\mathcal{E}/\mathcal{A}) \). We split the proof into two parts. Assume first that \( X^* \in \text{Ac}^b(\mathcal{E}/\mathcal{A}) \). We proceed by induction on the width \( n \) of the support of \( X^* \in C^b(\mathcal{E}) \).

If \( X^* \) is a stalk complex with \( X \) in degree \( k \), we find that \( X \cong 0 \) in \( \mathcal{E}/\mathcal{A} \) as \( X^* \in \text{Ac}^b(\mathcal{E}/\mathcal{A}) \).

It follows from proposition 4.9 that \( X \in \mathcal{A} \). Thus \( X^* \in D^b_{\mathcal{A}}(\mathcal{E}) \).

Assume that \( X^* \) has support of width \( n \geq 2 \). Without loss of generality assume that \( X^* \in C(1,n)^{-1} \). Since \( X^* \in \text{Ac}^b(\mathcal{E}/\mathcal{A}) \), the differential \( d^*_{X^{-1}} \) is a deflation in \( \mathcal{E}/\mathcal{A} \). By construction,
there is a diagram in $\mathcal{E}$:

$$
\begin{array}{c}
X^{n-1} & \longrightarrow & X^n \\
\downarrow & & \downarrow \\
U^{n-1} & \longrightarrow & V^n \\
\downarrow & & \downarrow \\
Y^{n-1} & \longrightarrow & Y^n
\end{array}
$$

Without loss of generality, we may assume that all vertical arrows are weak isomorphisms (this uses the right weak saturation property). Axiom R2 allows us to take the pullback $P$ of $V^n \rightarrow Y^n$ along $Y^{n-1} \rightarrow Y^n$. We obtain the solid diagram:

$$
\begin{array}{c}
X^{n-1} & \longrightarrow & X^n \\
\downarrow & & \downarrow \\
U^{n-1} & \longrightarrow & V^n \\
\downarrow & & \downarrow \\
Y^{n-1} & \longrightarrow & Y^n
\end{array}
$$

Here we used [17, proposition 3.12] to see that $P \sim Y^{n-1}$ is a weak isomorphism. The dotted arrows are obtained by axiom RMS2 and the right weak saturation.

Applying lemma 4.14.(3) (with $s^n; V^n \sim X^n$, $s^{n-1}; V^{n-1} \sim X^{n-1}$ and $s^{n-1} = 1_{X^{n-1}}$ for all $i \geq 2$), we obtain commutative diagram

$$
\begin{array}{c}
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & V^{n-2} & \longrightarrow & V^{n-1} & \longrightarrow & P & \longrightarrow & V^n & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

with $V^* \in \text{C}^b(\mathcal{E})$. Now consider the commutative diagram

$$
\begin{array}{c}
\cdots & \longrightarrow & V^{n-2} & \longrightarrow & V^{n-1} & \longrightarrow & V^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & V^{n-2} & \longrightarrow & P & \longrightarrow & V^n & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

Denote the lower row by $W^*$. Clearly, $W^*$ is a complex and, by proposition 3.12, it is the extension of $\tau^{s^n}W^*$ and $\tau^{s^{n-1}}W^*$ in $\text{K}^b(\mathcal{E})$. As $\tau^{s^n}W^*$ is acyclic in $\text{K}^b(\mathcal{E})$ and the induction hypothesis yields that $\tau^{s^{n-1}}W^* \in D^b_\delta(\mathcal{E})$, we find that $W^* \in D^b_\delta(\mathcal{E})$. Proposition 5.4 implies that $V^*, X^* \in D^b_\delta(\mathcal{E})$. This shows that $X^* \in D^b_\delta(\mathcal{E})$ if $X^* \in \text{Ac}^b(\mathcal{E}/A)$.

We now show that $X^* \in D^b_\delta(\mathcal{E})$ if $X^*$ is a direct summand in $\text{K}^b(\mathcal{E}/A)$ of an acyclic complex $U^* \in \text{C}^b(\mathcal{E}/A)$. By lemma 3.9, $X^*$ is a summand of the acyclic complex $U^* \oplus IX^*$. By proposition 4.15, $\text{C}^b(\mathcal{E})/\text{C}^b(A) = \text{C}^b(\mathcal{E}/A)$ and thus $X^*$ is a direct summand in $\text{C}^b(\mathcal{E})/\text{C}^b(A)$ of an acyclic complex $V^* \in \text{C}^b(\mathcal{E})$. It follows that there is a coretraction $X^* \rightarrow V^*$ in $\text{C}^b(\mathcal{E})/\text{C}^b(A)$. Hence the following roof represents the identity on $X^* \in \text{C}^b(\mathcal{E})/\text{C}^b(A)$ (the map from $V^* \rightarrow X^*$ in $\text{C}^b(\mathcal{E})/\text{C}^b(A)$ is the corresponding retraction):

$$
\begin{array}{c}
X^* \leftarrow \leftarrow V^* \leftarrow \leftarrow X^*
\end{array}
$$

Clearly the map $Z^* \rightarrow X^*$ is an isomorphism in $\text{C}^b(\mathcal{E})/\text{C}^b(A)$. Since $S_\delta$ is right weakly saturated, we may assume that $Z^* \rightarrow X^*$ belongs to $S_{\text{C}^b(A)}$. By proposition 5.4, cone($Z^* \rightarrow X^*$) $\in D^b_\delta(\mathcal{E})$. It follows that $Z^* \rightarrow X^*$ is an isomorphism in $D^b(\mathcal{E})/D^b_\delta(\mathcal{E})$. On the other hand, as $V^*$ is acyclic, $V^* \in D^b_\delta(\mathcal{E})$ and proposition 5.4 yields that $W^* \in D^b_\delta(\mathcal{E})$. It follows that $Z^* \rightarrow X^*$ is the zero map in $D^b(\mathcal{E})/D^b_\delta(\mathcal{E})$ and hence $X^* \in \text{C}^b(\mathcal{E})/\text{C}^b(A)$. This completes the proof. □

**Proposition 5.6.** Let $\mathcal{E}$ be a right exact category and let $\mathcal{A}$ be a strongly right percolating subcategory. The right multiplicative system $S_\mathcal{A}$ is right weakly saturated.
Proof. Since $0 \in \mathcal{A}$ and $\mathcal{A}$ is strongly right percolating (see definition 4.1), any map $X \to 0$ is a deflation. Hence, $\mathcal{E}$ satisfies axiom $R0^*$. Let $f: X \to Y$ be map in $\mathcal{E}$ such that $Q(f)$ descends to an isomorphism. By definition there exists a map $(g: Y' \to X, s: Y' \tilde{\to} Y)$ such that they fit in a commutative diagram:

\[
\begin{array}{c}
Y' \xrightarrow{g} X \\
\downarrow f \\
Y \xrightarrow{\gamma} M \xrightarrow{\beta} Y
\end{array}
\]

Since $\mathcal{A}$ is strongly right percolating, proposition 4.7 allows us to take the pullback of $f$ along $fgh: M \tilde{\to} Y$. We obtain the commutative diagram:

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow gh \downarrow \gamma \\
M \xrightarrow{\beta} M \\
\downarrow 1_{\mathcal{A}} \downarrow \alpha
\end{array}
\]

Note that it is sufficient to show that $\beta$ is a weak isomorphism. Indeed, in this case, there is a weak isomorphism $\alpha: P \to X$ such that the composition $\alpha \circ f = g \circ \beta$ is a weak isomorphism (this uses that weak isomorphisms are closed under composition), establishing the desired weak saturation. To show that $\beta$ is a weak isomorphism, it suffices to show that $\ker(\beta) \in \mathcal{A}$. Indeed, from the above diagram, we infer that $\beta$ is a retraction and hence a deflation, by $R0^*$.

We first show that $\beta$ has a kernel. Since $Q(f)$ is an isomorphism and $Q$ commutes with pullbacks, $Q(\beta)$ is an isomorphism as well. It follows from $Q(\beta)Q(\gamma) = Q(1_{P})$ that $Q(\gamma)$ is an isomorphism as well. Clearly $(1_{P} - \gamma\beta)\gamma = 0$, hence $Q(1_{P} - \gamma\beta) = 0$. To simplify the notation, we denote the idempotent $(1_{P} - \gamma\beta): P \to P$ by $e$.

Since $\mathcal{A}$ is strongly right percolating, it is idempotent complete. It now follows from [17, lemma 4.14] (based on [34, lemma 1.17.6]) that $P \cong \ker(e) \oplus A$ for some $A \in \mathcal{A}$. Using that $\gamma$ is a monomorphism, we infer that $A = \ker(1 - e) = \ker(\gamma\beta) = \ker(\beta)$. $\square$

5.3. Application to locally compact modules. Let $\text{LCA}$ be the category of locally compact (Hausdorff) abelian groups. Let $R$ be a unital ring endowed with the discrete topology. Denote by $R - \text{LC}$ the category of locally compact left $R$-modules. We furthermore write $R - \text{LC}_D$ and $R - \text{LC}_C$ for the full subcategories of discrete $R$-modules and compact $R$-modules, respectively. Note that $\mathbb{Z} - \text{LC}$ is simply the category $\text{LCA}$. We use similar notations for locally compact right $R$-modules. We recall the following proposition.

**Proposition 5.7.**

1. The categories $R - \text{LC}$ and $\text{LC} - R$ are quasi-abelian categories.
2. The standard Pontryagin duality can be extended to a duality

$$D: R - \text{LC} \to \text{LC} - R$$

which interchanges the discrete and compact modules.
3. The category $R - \text{LC}_D$ is a right abelian percolating subcategory of $R - \text{LC}$, moreover, the right multiplicative system of weak isomorphisms is saturated.
4. The category $R - \text{LC}_C$ is a left abelian percolating subcategory of $R - \text{LC}$, moreover, the left multiplicative system of weak isomorphisms is saturated.

**Proof.** The first two statements are well-known (see for example [25]). For the last two statements, we refer to [17, proposition 6.23]. $\square$

**Remark 5.8.**

1. The exact structure on $R - \text{LC}$ is described as follows: closed injections are inflations and open surjections are deflations.
(2) As noted in [7, example 4], the category \( R - \text{LC}_C \) is in general not left s-filtering, nor is \( R - \text{LC}_C \) a right filtering subcategory of \( R - \text{LC} \). It follows that one cannot use the localization theories of [34, 10] to describe the quotient \( R - \text{LC}/R - \text{LC}_C \). The above proposition shows that this quotient satisfies the conditions of theorems 4.8 and 5.5.

The following proposition is [21, theorem 12.1.b].

Proposition 5.9. Let \( \mathcal{E} \) be an exact category and let \( \mathcal{A} \subseteq \mathcal{E} \) be a fully exact subcategory. Suppose that either of the following conditions hold:

1. \( \mathcal{A} \) is a right filtering subcategory of \( \mathcal{E} \).
2. \( \mathcal{A} \) is a left filtering subcategory of \( \mathcal{E} \).

Then the canonical functor \( \text{D}^b(\mathcal{A}) \to \text{D}^b(\mathcal{E}) \) is fully faithful. In particular, \( \text{D}^b_\mathcal{A}(\mathcal{E}) = \text{D}^b(\mathcal{A}) \).

Remark 5.10. Conditions \( \mathcal{C}2/\mathcal{C}2^{op} \) are related to right/left s-filtering subcategories (see [8, proposition A.2]).

Corollary 5.11. The following are Verdier localization sequences:

\[
\text{D}^b(R - \text{LC}_D) \to \text{D}^b(R - \text{LC}) \to \text{D}^b(R - \text{LC}/R - \text{LC}_D)
\]

and

\[
\text{D}^b(R - \text{LC}_C) \to \text{D}^b(R - \text{LC}) \to \text{D}^b(R - \text{LC}/R - \text{LC}_C).
\]

Proof. To ease notation, we write \( \mathcal{E} = R - \text{LC} \) and \( \mathcal{A} = R - \text{LC}_D \). By proposition 5.7, \( \mathcal{A} \subseteq \mathcal{E} \) satisfies the conditions of theorem 5.5. By proposition 5.9, it suffices to check whether condition \( \mathcal{C}2^{op} \) holds. Note that for each \( D \in \mathcal{A} \), there is a deflation \( R^{\text{op}} \to \mathcal{E} \).

By Pontryagin duality, we conclude that \( \text{D}^b(R - \text{LC}_C) \to \text{D}^b(R - \text{LC}) \to \text{D}^b(R - \text{LC}/R - \text{LC}_C) \) is a Verdier localization as well. \( \square \)

6. \( \text{LCA}/\text{LCA}_D \) and \( \text{LCA}/\text{LCA}_C \) are exact

In general, a quotient \( \mathcal{E}/\mathcal{A} \) where \( \mathcal{E} \) is an exact category and \( \mathcal{A} \) is a right percolating subcategory need not be exact. In this section we provide a sufficient condition (theorem 6.5) such that the quotient is exact. As an application, we find that the quotients \( \text{LCA}/\text{LCA}_D \) and \( \text{LCA}/\text{LCA}_C \) are exact.

6.1. Extension-closed subcategories of derived categories.

Proposition 6.1. Let \( \mathcal{E} \) be a weakly idempotent complete right exact category and let \( i: \mathcal{E} \to \text{D}^b(\mathcal{E}) \) the canonical embedding. The following are equivalent:

1. The category \( \mathcal{E} \) satisfies axiom \( \text{R3} \).
2. A sequence \( X \xrightarrow{f} Y \xrightarrow{g} Z \) in \( \mathcal{E} \) is a conflation if and only if there is a triangle \( i(X) \xrightarrow{i(f)} i(Y) \xrightarrow{i(g)} i(Z) \to \Sigma i(X) \) in \( \text{D}^b(\mathcal{E}) \).

Proof. We first show that (1) implies (2). Assume that \( \mathcal{E} \) satisfies axiom \( \text{R3} \). In particular we have that \( \mathcal{E} \) satisfies axiom \( \text{R0}^{op} \) and thus proposition 3.19 shows that conflations in \( \mathcal{E} \) yield triangles in \( \text{D}^b(\mathcal{E}) \).

Conversely, consider a sequence \( X \xrightarrow{f} Y \xrightarrow{g} Z \) in \( \mathcal{E} \) such that \( i(X) \xrightarrow{i(f)} i(Y) \xrightarrow{i(g)} i(Z) \to \Sigma i(X) \) is a triangle in \( \text{D}^b(\mathcal{E}) \). By proposition 3.16, the functor \( i: \mathcal{E} \to \text{D}^b(\mathcal{E}) \) is fully faithful and hence \( i(g \circ f) \) yields...
Proof. (1) Assume that as in the following diagram:

\[
\begin{array}{ccc}
\text{cone}(j(f)) & \xrightarrow{\alpha} & \Sigma\text{j}(X) \\
\downarrow & & \downarrow \\
\text{j}(Z) & \xrightarrow{\text{j}(g)} & \text{j}(Y)
\end{array}
\]

which becomes an isomorphism in \(D^b(\mathcal{E})\). By remark 3.8, \(\text{cone}(h)\) is acyclic. Note that \(\text{cone}(h)\) is given by the complex

\[
\cdots \longrightarrow 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \longrightarrow \cdots
\]

with \(Z\) in degree 0. We conclude that \(X \xrightarrow{f} Y \xrightarrow{g} Z\) is a conflation in \(\mathcal{E}\). This shows (1)\(\Rightarrow\)(2).

Conversely, assume (2). To establish that \(R3\) holds, we start with two morphisms \(X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z\) such that \(\beta\alpha\) is a deflation. We show that if \(\beta\) has a kernel, then \(\beta\) is a deflation.

We consider the following diagram where the right square is a pullback (see [4, proposition 5.4] or [17, proposition 1.9]):

\[
\begin{array}{ccc}
\ker \gamma & \xrightarrow{\delta} & P \\
\downarrow & & \downarrow \\
\ker \beta \alpha & \xrightarrow{\beta} & X
\end{array}
\]

It follows from the pullback property that \(\delta : P \to X\) is a retraction. As \(\mathcal{E}\) is weakly idempotent complete, this means that \(\ker \delta \in \mathcal{E}\). Suppressing the functor \(i\), we may use [28, lemma 1.4.4] to obtain a diagram in \(D^b(\mathcal{E})\)

\[
\begin{array}{ccc}
\ker \delta & \xrightarrow{\delta} & \ker \beta \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0} & \ker \beta
\end{array}
\]

where the rows and columns are triangles. From this, we infer that there is a conflation \(\ker \beta \to Y \to Z\), as required.

The first part of the following proposition was also shown in [29, §A.8].

**Proposition 6.2.** Let \(\mathcal{E}\) be a right exact category satisfying axiom \(R0^e\) and let \(i : \mathcal{E} \to D^b(\mathcal{E})\) be the canonical embedding.

1. If \(\mathcal{E}\) is exact, then \(i(\mathcal{E})\) is extension closed in \(D^b(\mathcal{E})\).
2. If \(i(\mathcal{E})\) is extension closed in \(D^b(\mathcal{E})\), then \(i(\mathcal{E})\) can be endowed with a natural exact structure such that the inclusion \(\mathcal{E} \hookrightarrow i(\mathcal{E})\) is exact.

   If in addition \(\mathcal{E}\) is weakly idempotent complete and satisfies axiom \(R3\), the right exact category \(\mathcal{E}\) is exact.

**Proof.** (1) Assume that \(\mathcal{E}\) is exact, we will show that \(i(\mathcal{E})\) is extension closed in \(D^b(\mathcal{E})\). Consider a triangle

\[
i(A) \to B^\bullet \to i(C) \to \Sigma i(A)
\]

in \(D^b(\mathcal{E})\) with \(A, C \in \mathcal{E}\). Rotating the triangle we obtain a the map \(\Sigma^{-1}i(C) \to i(A)\) in \(D^b(\mathcal{E})\) which can be represented by a roof \(\Sigma^{-1}i(C) \xrightarrow{\beta^*} Z^\bullet \to i(A)\) in \(K^b(\mathcal{E})\). By proposition 3.15, we obtain a roof \(\Sigma^{-1}i(C) \xrightarrow{\beta^*} \tau^{<1}Q^\bullet \to i(A)\) representing the same map and such that the quasi-isomorphism \(\Sigma^{-1}i(C) \xrightarrow{\beta^*} \tau^{<1}Q^\bullet\) is congenial. By figure 1 (the indices are shifted as we truncate at degree one), the degree zero differential of \(\tau^{<1}Q^\bullet\) factors as \(E^{-1}P_3\ker(d_{E_1}^0) \xrightarrow{\alpha_\gamma} \ker(d_{E_2}^1)\) where

\(g \circ f = 0\). Writing \(j : \mathcal{E} \to K^b(\mathcal{E})\) for the canonical embedding, there is a morphism \(h : \text{cone}(j(f)) \to j(Z)\) as in the following diagram:

\[
\begin{array}{ccc}
j(X) & \xrightarrow{j(f)} & \text{cone}(j(f)) \\
\downarrow & & \downarrow \\
j(Z) & \xrightarrow{j(g)} & \text{j}(Y)
\end{array}
\]


Lemma 6.4. Let $\hom{d_E^1}{E^{-1}}{\ker(d_E^1)}$ is the cokernel of $d_E^{-2} : E^{-2} \to E^{-1}$. It follows that the chain map $\tau^{-1} \xi \beta^* \to i(A)$ induces a map $\ker(d_E^1) \to A$.

Using the exact structure on $\mathcal{E}$ and [9, proposition 2.12], we obtain a conflation

$$\ker(d_E^1) \to \ker(d_E^1) \oplus A \to Q$$

in $\mathcal{E}$. Consider the natural map $\cone(\beta) \to D^*$ where $D^*$ is the complex

$$\cdots \to 0 \to \ker(d_E^1) \to \ker(d_E^1) \oplus A \to 0 \to \cdots$$

with $\ker(d_E^1)$ in degree $-1$. One easily verifies that $\cone(\beta^*) \to D^*$ is a quasi-isomorphism and that $D^*$ is quasi-isomorphic to the stalk complex $i(Q)$. It follows that $B^*$ can be represented by $i(Q)$ in $\mathcal{D}^b(\mathcal{E})$. This shows that $i(\mathcal{E})$ is extension closed in $\mathcal{D}^b(\mathcal{E})$.

(2) Assume that $i(\mathcal{E})$ is extension closed in $\mathcal{D}^b(\mathcal{E})$. By proposition 3.17, $i(\mathcal{E})$ has no negative self-extensions in the triangulated category $\mathcal{D}^b(\mathcal{E})$. Following [11], the triangles in $\mathcal{D}^b(\mathcal{E})$ induce an exact structure on $i(\mathcal{E})$. On the other hand, by axiom $R0^*$ and proposition 3.19, each conflation in $\mathcal{E}$ gives rise to a triangle in $i(\mathcal{E})$. It follows that the inclusion $\mathcal{E} \hookrightarrow i(\mathcal{E})$ is exact.

If $\mathcal{E}$ is weakly idempotent complete and satisfies axiom $R3$, proposition 6.1 yields that $\mathcal{E}$ has the same conflation structure as $i(\mathcal{E})$. In particular, $\mathcal{E}$ is an exact category. \hfill \QED

Example 6.3. Let $\mathcal{U}$ be the category of finite-dimensional representations of the quiver $A_3 : 1 \leftarrow 2 \leftarrow 3$. The category $\mathcal{U}$ can be visualized by its Auslander-Reiten quiver

$$\begin{array}{ccc}
P(3) & \xrightarrow{r} & P(2) \\
S(1) & \xrightarrow{\tau^{-1}} & S(2) & \xrightarrow{\tau^{-1}} & S(3)
\end{array}$$

We endow $\mathcal{U}$ with the structure of a right exact category by taking as conflations all short exact sequences in $\mathcal{U}$ that are not isomorphic to $0 \to S(2) \to \tau^{-1} P(2) \to S_3 \to 0$. We write $\mathcal{E}$ for the category $\mathcal{U}$ endowed with this right exact structure. Clearly, $\mathcal{E}$ does not satisfy $R3$ as the composition $P(3) \to \tau^{-1} P(2) \to S_3$ is a deflation but $\tau^{-1} P(2) \to S_3$ is not. One readily verifies that $\Proj(\mathcal{E}) = \Proj(\mathcal{U})$. By proposition 3.27, $\mathcal{D}^b(\mathcal{E}) \cong R\hom{\mathcal{E}}{\mathcal{U}} = \mathcal{K}^{\mathcal{U}}(\Proj(\mathcal{E})) = D^b(\Proj(\mathcal{U})) \cong \mathcal{D}^b(\mathcal{U})$ are triangle-equivalent categories. It follows that $i(\mathcal{E})$ is extension closed in $\mathcal{D}^b(\mathcal{E})$.

6.2. Obtaining exact localizations.

Lemma 6.4. Let $\mathcal{E}$ be a weakly idempotent complete exact category. Consider the morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$. If

$$\hom{\mathcal{D}^b(\mathcal{E})}{\coker g}{\Sigma^2 \ker f} = 0,$$

then $gf$ factors as $X \xrightarrow{g'} W \xrightarrow{f'} Z$. Moreover, $\ker f \cong \ker f'$ and $\coker g \cong \coker g'$.

Proof. We consider the triangles $\ker f \to X \to Y \to \Sigma \ker f$ and $\Sigma^{-1} \ker g \to Y \to Z \to \coker g$. As $\hom{\mathcal{D}^b(\mathcal{E})}{\coker g}{\Sigma^2 \ker f} = 0$, we may use [28, proposition 1.4.6] to obtain the commutative diagram:

$$\begin{array}{cccc}
0 & \xrightarrow{0} & \coker g & \xrightarrow{0} & 0 \\
\ker f & \xrightarrow{0} & W & \xrightarrow{f'} & Z & \xrightarrow{0} & \Sigma \ker f \\
\ker f & \xrightarrow{0} & X & \xrightarrow{Y} & \Sigma \ker f \\
0 & \xrightarrow{0} & \Sigma^{-1} \coker g & \xrightarrow{0} & \Sigma^{-1} \coker g
\end{array}$$

Proposition 6.2 shows that $W \in i(\mathcal{E}) \subset \mathcal{D}^b(\mathcal{E})$ and proposition 6.1 (recall that an exact category satisfies axiom $R3$) yields that $g' : X \to W$ is an inflation and $f' : W \to Z$ is a deflation. It is obvious that $\ker f \cong \ker f'$ and $\coker g \cong \coker g'$. \hfill \QED
Theorem 6.5. Let $\mathcal{E}$ be a weakly idempotent complete exact category and let $\mathcal{A} \subseteq \mathcal{E}$ be a right percolating subcategory. If
\[
\text{Hom}_{\text{D}^b(\mathcal{E})}(E, \Sigma^2 A) = 0,
\]
for all $E \in \mathcal{E}$ and $A \in \mathcal{A}$, then the right exact structure on $\mathcal{E}/\mathcal{A}$ is an exact structure.

Proof. By [20], a conflation category satisfying axioms R0, R1, R2 and L2 is an exact category. As the first three conditions are guaranteed by Theorem 4.8, we only need to show that axiom L2 holds.

Thus, consider a span $Y \xrightarrow{f} X \xrightarrow{g} Z$ in $\mathcal{E}/\mathcal{A}$ where $g$ is an inflation, thus, $g$ is isomorphic to a morphism $Q(i)$ where $i$ is an inflation in $\mathcal{E}$. This gives the following diagram in $\mathcal{E}/\mathcal{A}$
\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X & \xrightarrow{f} & Y \\
\downarrow{Q(i)} & & \downarrow{g} & & \\
Z' & \xrightarrow{\beta} & Z
\end{array}
\]
where $\alpha, \beta$ are isomorphisms. We now only consider the upper and the left part of this diagram. Writing $f\alpha = (s, h)$, we obtain a diagram
\[
\begin{array}{ccc}
X' & \xleftarrow{s} & X & \xrightarrow{h} & Y \\
\downarrow{i} & & \downarrow{} & & \\
Z'
\end{array}
\]
in $\mathcal{E}$. Since $s$ is a weak isomorphism, it is a finite composition of $\mathcal{A}^{-1}$-inflations and $\mathcal{A}^{-1}$-deflations. If $s$ ends on an $\mathcal{A}^{-1}$-inflation, proposition 4.5 allows us to absorb this $\mathcal{A}^{-1}$-inflation into the inflation $i$.

Hence we may assume that $s$ ends on a deflation, i.e. $s$ factors as $X \xrightarrow{\sim} X'' \xrightarrow{\sim} X'$. By lemma 6.4, the composition $i \circ s_1$ factors as $X'' \xrightarrow{\sim} W \xrightarrow{\sim} Z'$. Iterating this argument on the length of $s$, we obtain a span $Z \xrightarrow{k} X \xrightarrow{h} Y$ in $\mathcal{E}$ which is isomorphic to the original span in $\mathcal{E}/\mathcal{A}$.

Since $\mathcal{E}$ is exact, the pushout $P$ of this span exists in $\mathcal{E}$ and yields an inflation $Y \xrightarrow{\sim} P$. Since the localization functor $Q$ is exact, it commutes with cokernels of inflations. Since the pushout $P$ can be obtained as a cokernel of an inflation in $\mathcal{E}$, the pushout square $XY \xrightarrow{P} Z$ descends to a pushout square in $\mathcal{E}/\mathcal{A}$. Clearly the map $Y \xrightarrow{\sim} P$ descends to an inflation as well. This completes the proof. \qed

Remark 6.6. The condition $\text{Hom}_{\text{D}^b(\mathcal{E})}(E, \Sigma^2 A) = 0$, for all $E \in \mathcal{E}$ and $A \in \mathcal{A}$ is equivalent to the condition $\text{Ext}^2(E, A) = 0$ for all $E \in \mathcal{E}$ and $A \in \mathcal{A}$.

We now apply the previous theorem to the category of locally compact abelian groups $\text{LCA}$. Recall from proposition 5.7 that the quotient category $\text{LCA}/\text{LCA}_D$ is a right exact category and that $\text{LCA}/\text{LCA}_C$ is a left exact category. Theorem 6.5 shows that these categories are, in fact, two-sided exact. The main difficulty is showing the vanishing of $\text{Ext}^2(E, A)$ for all $A \in \text{LCA}_D$ which is proved by Hoffmann and Spitzweck in [18].

Corollary 6.7. The quotients $\text{LCA}/\text{LCA}_D$ and $\text{LCA}/\text{LCA}_C$ are exact categories.

Proof. It is shown in [18, proposition 4.13, proposition 4.15] that $\text{LCA}/\text{LCA}_D$ satisfies the conditions of theorem 6.5. It follows that $\text{LCA}/\text{LCA}_D$ is an exact category. By Pontryagin duality, the localization $\text{LCA}/\text{LCA}_C$ is exact as well. \qed

7. The exact hull

In this section we construct the exact hull $\mathcal{E}$ of $\mathcal{E}$. Explicitly, $\mathcal{E}$ is the extension closure of $i(\mathcal{E}) \subseteq \text{D}^b(\mathcal{E})$.

If $\mathcal{E}$ satisfies axiom R0*, the inclusion map $j : \mathcal{E} \hookrightarrow \mathcal{E}$ is exact and 2-universal among exact functors to exact categories (see proposition 7.7). In theorem 7.15 we show that the embedding $\mathcal{E} \subseteq \text{D}^b(\mathcal{E})$ lifts to a triangle equivalence $\text{D}^b(\mathcal{E}) \xrightarrow{\sim} \text{D}^b(\mathcal{E})$.

7.1. Construction of the exact hull.

Definition 7.1. Let $\mathcal{E}$ be a right exact category and let $i : \mathcal{E} \hookrightarrow \text{D}^b(\mathcal{E})$ be the canonical embedding. We denote the extension closure of $i(\mathcal{E})$ in $\text{D}^b(\mathcal{E})$ by $\mathcal{E}$. The category $\mathcal{E}$ is called the exact hull of $\mathcal{E}$. The composition $\mathcal{E} \hookrightarrow i(\mathcal{E}) \hookrightarrow \mathcal{E}$ is denoted by $j$.

We extend proposition 3.17 to the extension closure $\mathcal{E}$. 

Proposition 7.2. For all $X, Y \in \text{Ob}(\mathcal{E})$ we have that $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(X, \Sigma^{-n}Y) = 0$ for all $n > 0$.

Proof. Let $\mathcal{E}_0 = \iota(\mathcal{E})$. We inductively define $\mathcal{E}_i$ (for $i \geq 0$) as the set of all objects $E \in \text{Ob}(\mathcal{D}^b(\mathcal{E}))$ occurring in a triangle

$$X \rightarrow E \rightarrow Y \rightarrow \Sigma X$$

where $X, Y \in \mathcal{E}_{i-1}$. We now claim that $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(\mathcal{E}_i, \Sigma^{-1}\mathcal{E}_i) = 0$. We proceed by induction. The case where $i = 0$ is shown in proposition 3.17.

Consider now $i > 0$. Let $E_{i-1} \in \mathcal{E}_{i-1}$. Applying the functor $\text{Hom}(E_{i-1}, -)$ to the triangle $X \rightarrow E \rightarrow Y \rightarrow \Sigma X$ as above (thus, with $X, Y \in \mathcal{E}_{i-1}$), yields the long exact sequence:

$$\cdots \rightarrow \text{Hom}_{\mathcal{D}^b(\mathcal{E})}(E_{i-1}, \Sigma^{-n}X) \rightarrow \text{Hom}_{\mathcal{D}^b(\mathcal{E})}(E_{i-1}, \Sigma^{-n}E) \rightarrow \text{Hom}_{\mathcal{D}^b(\mathcal{E})}(E_{i-1}, \Sigma^{-n}Y) \rightarrow \cdots$$

By the induction hypothesis, we have that $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(\mathcal{E}_{i-1}, \Sigma^{-n}\mathcal{E}_{i-1}) = 0$ and hence $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(\mathcal{E}_{i-1}, \Sigma^{-n}E) = 0$. This shows that $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(\mathcal{E}_{i-1}, \Sigma^{-n}\mathcal{E}_i) = 0$.

On the other hand, applying the functor $\text{Hom}(-, \Sigma^{-n}\mathcal{E}_i)$ with $E_i \in \mathcal{E}_i$ to the same triangle yields the long exact sequence

$$\cdots \rightarrow \text{Hom}_{\mathcal{D}^b(\mathcal{E})}(Y, \Sigma^{-n}E_i) \rightarrow \text{Hom}_{\mathcal{D}^b(\mathcal{E})}(E, \Sigma^{-n}E_i) \rightarrow \text{Hom}_{\mathcal{D}^b(\mathcal{E})}(X, \Sigma^{-n}E_i) \rightarrow \cdots$$

Using that $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(\mathcal{E}_{i-1}, \Sigma^{-n}\mathcal{E}_i) = 0$ and $X, Y \in \mathcal{E}_{i-1}$, we find that $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(E, \Sigma^{-n}E_i) = 0$. Hence $\text{Hom}_{\mathcal{D}^b(\mathcal{E})}(\mathcal{E}_i, \Sigma^{-n}\mathcal{E}_i) = 0$. This shows the claim.

The required statement follow from $\mathcal{E} = \bigcup_{i \geq 0} \mathcal{E}_i$.

\[ \square \]

Corollary 7.3. Let $\mathcal{E}$ be a right exact category. The category $\overline{\mathcal{E}} \subseteq \mathcal{D}^b(\mathcal{E})$ has the structure of an exact category, the conilations are given by triangles in $\mathcal{D}^b(\mathcal{E})$ with three consecutive terms in $\overline{\mathcal{E}}$.

If $\mathcal{E}$ satisfies axiom $\text{RO0}^\ast$, the embedding $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is exact.

Proof. Proposition 7.2 shows that $\overline{\mathcal{E}}$ is an extension closed subcategory of $\mathcal{D}^b(\mathcal{E})$ having no negative self-extensions. Following [11], $\overline{\mathcal{E}}$ is an exact category.

If $\mathcal{E}$ satisfies axiom $\text{RO0}^\ast$, proposition 3.19 implies that the embedding $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is exact. \[ \square \]

7.2. A 2-universal property. We now show that the embedding of a right exact category into its exact hull $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is 2-universal (see proposition 7.7). We start by showing that every object in $\overline{\mathcal{E}}$ can be obtained as the cokernel of a monomorphism in $\mathcal{E} \subseteq \overline{\mathcal{E}}$ (see corollary 7.5).

Proposition 7.4. Let $\mathcal{E}$ be a right exact category and let $\overline{\mathcal{E}}$ be its exact hull. Any complex $W^\bullet \in \overline{\mathcal{E}}$ can be represented by a two-term complex $Y^{-1} \rightarrow Y^0$ in $\mathcal{D}^b(\mathcal{E})$.

Proof. Since $\overline{\mathcal{E}}$ is the extension closure of $\mathcal{E}$ in $\mathcal{D}^b(\mathcal{E})$, we have $\overline{\mathcal{E}} = \bigcup_{i \geq 0} \mathcal{E}_i$ where $\mathcal{E}_0 = \mathcal{E}$ and where each $\mathcal{E}_i$ ($i \geq 1$) is given by objects $W^i \in \mathcal{D}^b(\mathcal{E})$ occurring in a triangle $\iota(A) \rightarrow W^i \rightarrow X^i \rightarrow \Sigma i(A)$ with $A \in \mathcal{E}$ and $X^i \in \mathcal{E}_{i-1}$. Let $W^i \in \mathcal{E}_i$, then $\Sigma i(A)$ for some $i \geq 0$. We proceed by induction on $i \geq 0$. If $i = 0$, there is nothing to prove, so assume that $i \geq 1$.

The map $X^i \rightarrow \Sigma i(A)$ in $\mathcal{D}^b(\mathcal{E})$ can be represented by a roof $X^i \overset{\tau_{\leq 0}}{\rightarrow} \Sigma i(A)$ in $\mathcal{K}^b(\mathcal{E})$. By proposition 3.15, we may replace the latter roof by a roof $X^i \overset{\tau_{\leq 0}}{\rightarrow} \Sigma i(A)$ where the quasi-isomorphism $\tau_{\leq 0} \rightarrow X^i$ is a homotopy. Using that $X^i$ is supported on degrees at most 0 (induction hypothesis), we may truncate $i(A)$ to obtain a roof $X^i \overset{\tau_{\leq 0}}{\rightarrow} \Sigma i(A)$. We find a commutative diagram

\[
\begin{array}{ccc}
\iota(A) & \rightarrow & W^i \\
\downarrow & & \downarrow \\
\iota(A) & \rightarrow & X^i \\
\end{array}
\]

\[
\begin{array}{ccc}
\tau_{\leq 0} \iota(A) & \rightarrow & \Sigma i(A) \\
\end{array}
\]

in $\mathcal{D}^b(\mathcal{E})$ with $A \in \mathcal{E}, X^i \in \mathcal{E}_{i-1}$ and whose rows are triangles. The vertical maps are quasi-isomorphisms and the complex $C^i$ is given by $\Sigma^{-1} \text{cone}(\tau_{\leq 0} \iota(A) \rightarrow \Sigma i(A))$. Using the notation from figure 1 for $\tau_{\leq 0} \iota(A)$, we can write the complex $C^i$ as

$$\cdots \rightarrow \rightarrow E^{-3} \rightarrow X^{-1} \rightarrow \Sigma E^{-2} \rightarrow A \rightarrow P \rightarrow 0 \rightarrow \cdots$$

where $A \rightarrow P$ lives in degree 0. We describe the map $X^{-1} \rightarrow P \rightarrow A \rightarrow P$ in more detail.

Since $X^i \rightarrow E^i$ is a chain map and $P$ is obtained via a pullback, there is a unique monomorphism $X^{-1} \rightarrow P$ factoring $X^{-1} \rightarrow X^i$. Since $\tau_{\leq 0} \iota(A) \rightarrow \Sigma i(A)$ is a chain map and $h_E$: $E^{-2} \rightarrow k(d^{-1}_E)$ is the cokernel of $d^{-1}_E$, there is an induced map $\psi: k(d^{-1}_E) \rightarrow A$. The map $E^{-2} \rightarrow P$ factors as $E^{-2} \rightarrow \ker(d^{-1}_E)$.
Now consider the map $\alpha: \tau^{\leq 0} E^\bullet \to C^\bullet$ which is the identity on all degrees except $-1$ and $0$. In degrees $-1$ and $0$, the map is given by the following commutative diagram:

$$
\begin{array}{c}
X^{-1} \oplus E^{-2} \\
\downarrow (0) \\
E^{-2} \xrightarrow{\psi} \ker(d_E^{-1})
\end{array}
$$

Since $\tau^{\leq 0} E^\bullet$ is acyclic, the map $C^\bullet \to \text{cone}(\alpha)$ is a quasi-isomorphism. The reader can verify explicitly that the cone of $\alpha$ is given by a two-term complex $\ker(d_E^{-1}) \oplus X^{-1} \to A \oplus P$. It follows that this complex represents $W^\bullet$. This concludes the proof.

**Corollary 7.5.** For every $Z \in \overline{\mathcal{E}}$, there is a conflation $X \to Y \to Z$ in $\overline{\mathcal{E}}$ with $X, Y \in \mathcal{E}$.

**Proof.** By proposition 7.4, $Z \in \overline{\mathcal{E}} \subseteq D^b(\mathcal{E})$ can be represented by a two term complex $X \xrightarrow{\tilde{f}} Y$. Note that $Z \cong \text{cone}(\iota(f))$. Hence, there is a triangle $i(X) \xrightarrow{\tilde{f}} i(Y) \to Z \to \Sigma i(X)$ in $D^b(\mathcal{E})$. This shows that $i(X) \to i(Y) \to Z$ is a conflation in $\overline{\mathcal{E}}$. □

In the following, we write $\text{Adm}(\mathcal{F})$ for the category of admissible arrows in $\mathcal{F}$ and $\text{Adm}_{\overline{\mathcal{E}}}(\mathcal{E})$ for the category of arrows in $\mathcal{E}$ which become admissible under $j: \mathcal{E} \to \overline{\mathcal{E}}$.

**Corollary 7.6.** The functor $\text{coker}: \text{Adm}_{\overline{\mathcal{E}}}(\mathcal{E}) \to \overline{\mathcal{E}}$ is a lax epimorphism (in the sense of [1]) in the 2-category $\text{CAT}$.

**Proof.** Let $f: X \to Y$ be a map in $\overline{\mathcal{E}}$ such that $X, Y \in \text{im}(\text{coker})$. By axiom R2 we obtain the following solid commutative diagram in $\overline{\mathcal{E}}$:

$$
\begin{array}{ccc}
P & \xrightarrow{g} & X \\
\downarrow f & & \downarrow \text{id} \\
A & \xrightarrow{h} & B & \xrightarrow{\text{id}} & Y
\end{array}
$$

Here we used corollary 7.5 to obtain the lower conflation with $A, B \in \mathcal{E}$. Using corollary 7.5 once more, we find an object $P \in \mathcal{E}$ and a deflation $P \to P''$. Applying corollary 7.5 to the kernel of $P \to X$, we obtain a map $Q \xrightarrow{\delta} P$ such that $h \in \text{Adm}_{\overline{\mathcal{E}}}(\mathcal{E})$ and $\text{coker}(h) \cong X$. The induced map $h \to g$ in $\text{Adm}_{\overline{\mathcal{E}}}(\mathcal{E})$ maps to $f$ under the functor $\text{coker}$. This shows the third property equivalent characterization of a lax epimorphism in [1, theorem 1.1]. □

**Proposition 7.7.** Let $\mathcal{E}$ be a right exact category satisfying axiom R0*. The embedding $j: \mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is 2-universal among exact functors to exact categories.

**Proof.** Let $F: \mathcal{E} \to \mathcal{F}$ be an exact functor to an exact category $\mathcal{F}$. By the universal property of Verdier localizations, there exists a triangle functor $\tilde{F}: D^b(\mathcal{E}) \to D^b(\mathcal{F})$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{F} & D^b(\mathcal{E}) \\
\downarrow F & & \downarrow \tilde{F} \\
\mathcal{F} & \xrightarrow{\mathcal{F}} & D^b(\mathcal{F})
\end{array}
$$

By proposition 6.2, $\mathcal{F}$ is an extension-closed subcategory of $D^b(\mathcal{F})$. It follows that the restriction of $\tilde{F}$ to $\overline{\mathcal{E}}$ maps $\mathcal{E}$ into the essential image of $\mathcal{F} \subseteq D^b(\mathcal{F})$. Let $\overline{\mathcal{F}}$ be the restriction of $\tilde{F}$ to $\overline{\mathcal{E}}$. Clearly $\overline{\mathcal{F}}: \overline{\mathcal{E}} \to \mathcal{F} \subseteq D^b(\mathcal{F})$ is exact and $\overline{\mathcal{F}} \circ j = F$.

Now assume that $G, H: \overline{\mathcal{E}} \to \mathcal{F}$ are two exact functors such that $G \circ j = H \circ j = F$. Using the notation of corollary 7.6, one finds that both $G$ and $H$ give an essentially commutative diagram:

$$
\begin{array}{ccc}
\text{Adm}_{\overline{\mathcal{E}}}(\mathcal{E}) & \xrightarrow{\text{coker}} & \overline{\mathcal{E}} \\
\downarrow F & & \downarrow G \\
\text{Adm}(\mathcal{F}) & \xrightarrow{\text{coker}} & \mathcal{F}
\end{array}
$$

Hence, $G \circ \text{coker} \cong H \circ \text{coker}$. As $\text{coker}: \text{Adm}_{\overline{\mathcal{E}}}(\mathcal{E}) \to \overline{\mathcal{E}}$ is a lax epimorphism, we find that $G$ and $H$ are naturally equivalent. □
Remark 7.8. In [31, proposition I.7.5], Rosenberg employs a different constructing for the exact hull of a right exact category. As both hulls satisfy the same 2-universal property, they are equivalent.

7.3. Derived equivalence of hull. We recall the following definition from [22].

Definition 7.9. Let $\mathcal{T}$ be a $k$-linear triangulated category. We say that $\mathcal{T}$ is algebraic if it is triangle-equivalent to $\mathcal{C}$ for some $k$-linear Frobenius category $\mathcal{C}$ (here, $\mathcal{C}$ denotes the stable category of $\mathcal{C}$).

The following proposition is well known (see [22, section 3.6]).

Proposition 7.10. Full triangulated subcategories and Verdier localizations of algebraic categories are algebraic.

Corollary 7.11. Let $\mathcal{E}$ be a right exact category. The homotopy category $K(\mathcal{E})$ and the derived category $D(\mathcal{E})$ are algebraic triangulated categories.

Proof. We endow the category of complexes $C(\mathcal{E})$ with the structure of a Frobenius exact category by taking as set of conflations these kernel-cokernel pairs which are degreewise split. The projective-injective objects are then the null-homotopic complexes and the corresponding stable category is the homotopy category $K(\mathcal{E})$. It follows from proposition 7.10 that the derived category $D(\mathcal{E})$ is algebraic as well. □

Theorem 7.12. Let $A$ be a Frobenius category, $B$ an additive category and $F: B \to A$ an additive functor such that $\text{Hom}_A(\Sigma^n FA, FB) = 0$, for all $A, B \in B$, and for all $n > 0$.

1. The functor $F$ extends to a triangulated functor $\tilde{F}: K^b(B) \to A$. Moreover, $\tilde{F}$ is fully faithful if and only if $\text{Hom}_A(FA, \Sigma^n FB) = 0$ for all $A, B \in B$ and for all $n > 0$.

2. If the category $B$ is exact and the image of every short exact sequence extends to a triangle, then $\tilde{F}$ decomposes as $K^b(B) \xrightarrow{\text{can}} D^b(B) \xrightarrow{\tilde{F}} A$. Moreover, $\tilde{F}$ is fully faithful if and only if $F$ is fully faithful and for each $A, A' \in B$, $n > 0$ and $f \in \text{Hom}_B(FA, \Sigma^n FA)$, every short exact sequence

$$0 \to A \xrightarrow{j} B \to C \to 0$$

in $B$ satisfies $(\Sigma^n f) \circ j = 0$.

The next proposition is similar to [37, proposition 4.5]. The adaptation to this setting is left to the reader.

Lemma 7.13. Let $\mathcal{E}$ be a right exact category and let $\mathcal{E}'$ be the extension closure of $i(\mathcal{E})$ in $D^b(\mathcal{E})$. The following are equivalent:

1. For all $f \in \text{Hom}_{D^b(\mathcal{E})}(A, \Sigma^n B)$ with $A, B \in \mathcal{E}$ and $n \geq 2$, there exists an inflation $g: B \to C$ in $\mathcal{E}'$ such that the composition $A \xrightarrow{f} \Sigma^n B \xrightarrow{\Sigma^n g} \Sigma^n C$ in $D^b(\mathcal{E})$ is zero.

2. For all $f \in \text{Hom}_{D^b(\mathcal{E})}(A, \Sigma^n B)$ with $A, B \in \mathcal{E}$ and $n \geq 2$, there exists a deflation $h: C \to A$ such that $C \xrightarrow{h} A \xrightarrow{f} \Sigma^n B$ in $D^b(\mathcal{E})$ is zero.

3. For all $f \in \text{Hom}_{D^b(\mathcal{E})}(A, \Sigma^n B)$ with $A, B \in \mathcal{E}$ and $n \geq 2$, $f$ factors in $D^b(\mathcal{E})$ as follows

$$A \to \Sigma A_1 \to \Sigma A_2 \to \ldots \to \Sigma A_{n-1} \to \Sigma^n B$$

for some $A_1, \ldots, A_{n-1} \in \mathcal{E}$.

Corollary 7.14. Let $\mathcal{E}$ be a right exact category and $\mathcal{E}'$ its exact closure. The inclusion functor $\mathcal{E}' \to D^b(\mathcal{E})$ factors through a functor $\text{real}: D^b(\mathcal{E}') \to D^b(\mathcal{E})$ which is an equivalence if and only if one of the equivalent properties of lemma 7.13 holds.

Proof. By proposition 7.10 and corollary 7.11, the category $D^b(\mathcal{E})$ is algebraic, i.e. $D^b(\mathcal{E})$ is triangle equivalent to $\mathcal{C}$ for some Frobenius category $\mathcal{C}$. Hence we may apply theorem 7.12 to the canonical embedding $F: \mathcal{E}' \to D^b(\mathcal{E})$. The functor $\text{real} := \tilde{F}$ is fully faithful if and only if the conditions of lemma 7.13 hold. As $\mathcal{E} \subseteq \mathcal{E}'$, it follows that real is essentially surjective. □
Theorem 7.15. Let $\mathcal{E}$ be a right exact category and $\mathcal{E}^\circ$ its exact hull. The functor \( D^b(\mathcal{E}) \to D^b(\mathcal{E}) \) is a triangle equivalence.

Proof. By corollary 7.14 we only need to show that one of the properties of lemma 7.13 holds. We prove the third property by induction on $n \geq 2$. Let $A, B \in \mathcal{E}$ and let $f \in \text{Hom}_{D^b(\mathcal{E})}(A, \Sigma^n B)$. By definition we have that $\mathcal{E} = \bigcup_{i \geq 0} \mathcal{E}_i$ where $\mathcal{E}_i$ is defined inductively as extensions of objects in $\mathcal{E}_{i-1}$ and $\mathcal{E}_{i-1}$ (where $\mathcal{E}_0 = \mathcal{E}$). It follows that there exists integers $i, j \geq 0$ such that $A \in \mathcal{E}_i$ and $B \in \mathcal{E}_j$. We proceed by an additional induction on $(i, j)$.

Assume first that $i = 0 = j$. The map $f$ can be represented by a roof $A \xrightarrow{\alpha} Z^\bullet \to \Sigma^2 B$ in $K^b(\mathcal{E})$ where $A, B$ are stalk complexes and $\alpha$ is a quasi-isomorphism. Following the notation of proposition 3.15 and figure 1 we obtain a quasi-isomorphism $\tau^{<0} \zeta^* \to Z^\bullet$. The chain map $\tau^{<0} \zeta^* \to \Sigma^2 B$ induces a map $\text{ker}(d^2_E) \to B$ such that

\[
\begin{array}{ccc}
E^{-3} & \xrightarrow{d^2_E} & \text{ker}(d^2_E) \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & Q
\end{array}
\]

commutes. Since $\mathcal{E}$ is exact, there is a pushout square

\[
\begin{array}{ccc}
\text{ker}(d^2_E) & \xrightarrow{d^2_E} & E^{-2} \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & Q
\end{array}
\]

in $\mathcal{E}$. One readily verifies that the composition $\tau^{<0} \zeta^* \to \Sigma^2 B \to \Sigma^2 Q$ is null-homotopic (the homotopy is determined by the map $E^{-2} \to Q$) and thus is zero in $D^b(\mathcal{E})$. Write $R$ for the cokernel of $B \to Q$. It follows that $f: A \to \Sigma^2 B$ factors through $\Sigma R$ as we needed to show.

Consider the factorization $A \xrightarrow{\alpha} \Sigma R \to \Sigma^2 B$. From the triangle $R \to C \to A \xrightarrow{\alpha} \Sigma R$, we obtain a conflation $R \to C \to A$ in $\mathcal{E}$. Note that the composition $C \to A \xrightarrow{\alpha} \Sigma^2 B$ is zero. By corollary 7.5, we know that there is a deflation $p': C' \to A$ in $\mathcal{E}$ such that $C' \in \mathcal{E}$ and $f \circ p' = 0$.

Now assume that $i = 0$ and $j > 0$. By definition of $\mathcal{E}_j$, we obtain a diagram

\[
\begin{array}{c}
A \\
\downarrow \\
\Sigma^2 X \xrightarrow{\Sigma^2 d} \Sigma^2 Y \xrightarrow{\Sigma^2 d} \Sigma^3 X
\end{array}
\]

where $\Sigma^2 X, \Sigma^2 Y \in \mathcal{E}_{j-1}$. By the induction hypothesis there exists a deflation $C \to A$ with $C \in \mathcal{E}_0$ such that the composition $C \to A \to \Sigma^2 Y$ is zero. Hence, there is an induced map $C \to \Sigma^2 X$ factoring $C \to \Sigma^2 B$. Again, the induction hypothesis yields the existence of a deflation $D \to C$ with $D \in \mathcal{E}_0$ such that the composition $D \to C \to \Sigma^2 X$ is zero. By axiom R1, the composition of two deflations is again a deflation, hence we obtained a deflation $D \to A$ such that $D \to A \to \Sigma^2 B$ is zero. This concludes the case $i = 0$ and $j \geq 0$.

Assume that $i > 0$. The argument is dual to the previous one. By definition there is a triangle $X \to A \to Y$ where $X, Y \in \mathcal{E}_{i-1}$. By the induction hypothesis there exists an inflation $\Sigma^2 B \to \Sigma^2 I$ in $\mathcal{E}$ such that the composition $X \to \Sigma^2 I$ is zero. It follows that the map $A \to \Sigma^2 I$ factors through $A \to Y$ via a map $Y \to \Sigma^2 I$. By the induction hypothesis there exists an inflation $\Sigma^2 J \to \Sigma^2 I$ such that the composition $Y \to \Sigma^2 J$ is zero. As the two inflations belong to the exact category $\mathcal{E}$, the composition of the inflations is again an inflation. It follows that $f$ factors as desired in all cases. This concludes the base case $n = 2$.

The reader can verify that the case $n \geq 2$ can be proved similarly. \( \square \)
Appendix A. K-theory

In [35], a framework for the definition of the K-theory spectrum of an exact category is given in terms of Frobenius pairs. As the construction of the derived category of a one-sided exact category is similar to that of an exact category, the same construction can be used to define the K-theory spectrum for one-sided exact categories.

Recall the following definitions from [35].

Definition A.1. (1) A map of Frobenius categories \( F : \mathcal{C} \to \mathcal{D} \) is an exact functor preserving projective-injective objects.

(2) A Frobenius pair \((\mathcal{C}, \mathcal{C}_0)\) is a fully faithful embedding \(\mathcal{C}_0 \hookrightarrow \mathcal{C}\) of Frobenius categories (in particular the projective-injective objects of \(\mathcal{C}_0\) are mapped to the projective-injective objects of \(\mathcal{C}\)).

(3) A map \((\mathcal{C}, \mathcal{C}_0) \to (\mathcal{D}, \mathcal{D}_0)\) of Frobenius pairs is a map \(\mathcal{C} \to \mathcal{D}\) of Frobenius categories such that \(\mathcal{C}_0\) is mapped into \(\mathcal{D}_0\).

Definition A.2. Given a Frobenius pair \((\mathcal{C}, \mathcal{C}_0)\), the induced map \(\mathcal{C}_0 \to \mathcal{C}\) of small triangulated categories is fully faithful. The derived category \(\mathcal{D}(\mathcal{C}, \mathcal{C}_0)\) of the Frobenius pair is defined as the Verdier localization \(\mathcal{L}(\mathcal{C}_0/\mathcal{C})\).

Definition A.3. An exact sequence of Frobenius pairs is a composable pair of maps of Frobenius pairs

\[
(\mathcal{C}', \mathcal{C}_0') \xrightarrow{f} (\mathcal{C}, \mathcal{C}_0) \xrightarrow{g} (\mathcal{C}'', \mathcal{C}_0'')
\]

together with a natural transformation \(\eta\) from \(g \circ f\) to the zero functor \(0 : \mathcal{C}' \to \mathcal{C}''\) such that for each \(\mathcal{C}' \in \mathcal{C}'\) we have that \(\eta_{\mathcal{C}''} : g(f(\mathcal{C}'')) \to 0\) is a weak equivalence in the induced Waldhausen structure of \((\mathcal{C}'', \mathcal{C}_0'')\), and the functor \(\mathcal{D}(\mathcal{C}', \mathcal{C}_0') \to \mathcal{D}(\mathcal{C}, \mathcal{C}_0)\) is fully faithful and the induced functor from the Verdier quotient \(\mathcal{D}(\mathcal{C}, \mathcal{C}_0)/\mathcal{D}(\mathcal{C}', \mathcal{C}_0')\) to \(\mathcal{D}(\mathcal{C}'', \mathcal{C}_0'')\) is cofinal, meaning that the functor is fully faithful and any object in the target is a direct summand of an image of an object in the domain.

For the remainder of this section, let \(\mathcal{E}\) be a right exact category satisfying axiom \(R0^*\) and let \(\mathcal{A}\) be a right percolating subcategory such that the right multiplicative system of weak isomorphisms is right weakly saturated.

Consider the Frobenius pair \((\mathcal{C}^b(\mathcal{E}), \mathcal{A}^b(\mathcal{E}))\) where \(\mathcal{C}^b(\mathcal{E})\) is endowed with the structure of an exact category by declaring conflations to be degreewise split kernel-cokernel pairs. One readily verifies that \((\mathcal{C}^b(\mathcal{E}), \mathcal{A}^b(\mathcal{E}))\) is indeed a Frobenius pair and \(\mathcal{D}(\mathcal{C}^b(\mathcal{E}), \mathcal{A}^b(\mathcal{E})) \simeq \mathcal{D}^b(\mathcal{E})\) as triangulated categories. Similarly, the Frobenius pair \((\mathcal{C}^b(\mathcal{E}/\mathcal{A}), \mathcal{A}^b(\mathcal{E}/\mathcal{A}))\) yields an equivalence of triangulated categories \(\mathcal{D}(\mathcal{C}^b(\mathcal{E}/\mathcal{A}), \mathcal{A}^b(\mathcal{E}/\mathcal{A})) \simeq \mathcal{D}^b(\mathcal{E}/\mathcal{A})\).

The quotient functor \(Q : \mathcal{C}^b(\mathcal{E}) \to \mathcal{C}^b(\mathcal{E}/\mathcal{A})\) induces a functor of Frobenius categories

\[
Q : (\mathcal{C}^b(\mathcal{E}), \mathcal{A}^b(\mathcal{E})) \to (\mathcal{C}^b(\mathcal{E}/\mathcal{A}), \mathcal{A}^b(\mathcal{E}/\mathcal{A}))
\]

with respect to the degreewise split exact structures.

Let \(\mathcal{C}^b_\mathcal{A}(\mathcal{E})\) be the full subcategory of \(\mathcal{C}^b(\mathcal{E})\) consisting of all \(X^\bullet \in \mathcal{C}^b(\mathcal{E})\) such that \(Q(X^\bullet)\) is quasi-isomorphic to the zero complex in \(\mathcal{D}(\mathcal{C}^b(\mathcal{E}/\mathcal{A}), \mathcal{A}^b(\mathcal{E}/\mathcal{A}))\). Note that \((\mathcal{C}^b_\mathcal{A}(\mathcal{E}), \mathcal{A}^b(\mathcal{E}))\) is a Frobenius pair and \(\mathcal{D}(\mathcal{C}^b_\mathcal{A}(\mathcal{E}), \mathcal{A}^b(\mathcal{E}))\) is triangle equivalent to the kernel of \(\mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E}/\mathcal{A})\). By theorem 5.5 \(\mathcal{D}(\mathcal{C}^b_\mathcal{A}(\mathcal{E}), \mathcal{A}^b(\mathcal{E})) \simeq \mathcal{D}^b_\mathcal{A}(\mathcal{E})\) are triangle equivalent.

Combining these ingredients with theorem 5.5, we obtain the following proposition:

Proposition A.4. The sequence

\[
(\mathcal{C}^b_\mathcal{A}(\mathcal{E}), \mathcal{A}^b(\mathcal{E})) \to (\mathcal{C}^b(\mathcal{E}), \mathcal{A}^b(\mathcal{E})) \to (\mathcal{C}^b(\mathcal{E}/\mathcal{A}), \mathcal{A}^b(\mathcal{E}/\mathcal{A}))
\]

is an exact sequence of Frobenius pairs.

We refer the reader to [35, section 12.1] for the definition of K-theory spectra of Frobenius pairs. Combining the above with [35, theorem 9] we obtain the following theorem:

Theorem A.5. The square

\[
\begin{array}{ccc}
\mathcal{K}(\mathcal{C}^b_\mathcal{A}(\mathcal{E}), \mathcal{A}^b(\mathcal{E})) & \longrightarrow & \mathcal{K}(\mathcal{C}^b(\mathcal{E}), \mathcal{A}^b(\mathcal{E})) \\
\downarrow & & \downarrow \\
\ast & \simeq & \mathcal{K}(\mathcal{C}^b_\mathcal{A}(\mathcal{E}), \mathcal{C}^b_\mathcal{A}(\mathcal{E})) \longrightarrow \mathcal{K}(\mathcal{C}^b(\mathcal{E}/\mathcal{A}), \mathcal{A}^b(\mathcal{E}/\mathcal{A}))
\end{array}
\]

is a homotopy Cartesian square of K-theory spectra in which the lower left corner is contractible.
following \[35\], we have the following definition:

**Definition A.6.** Let \( \mathcal{F} \) be a (one-sided) exact category. Following \[35\], the \( K \)-theory spectrum of \( \mathcal{E} \) is defined as

\[
\mathbb{K}(\mathcal{E}) = \mathbb{K}(\mathcal{C}^b(\mathcal{E}), \mathcal{A}c^b(\mathcal{E})).
\]

Using this notation, the square of theorem A.5 can be written as

\[
\begin{array}{ccc}
\mathbb{K}(\mathcal{C}^b(\mathcal{E}), \mathcal{A}c^b(\mathcal{E})) & \rightarrow & \mathbb{K}(\mathcal{E}) \\
\downarrow & & \downarrow \\
\ast & \simeq & \mathbb{K}(\mathcal{C}^b(\mathcal{E}), \mathcal{C}^b(\mathcal{E})) \rightarrow \mathbb{K}(\mathcal{E}/\mathcal{A})
\end{array}
\]

The following proposition is a consequence of theorem A.5 (see also \[35\, proposition 3\]).

**Proposition A.7.** A map \( \varphi : (C, C_0) \rightarrow (C', C'_0) \) of Frobenius pairs such that \( D(\varphi) : D(C, C_0) \xrightarrow{\simeq} D(C', C'_0) \) is a triangle equivalence, induces a homotopy equivalence \( \mathbb{K}(C, C_0) \rightarrow \mathbb{K}(C', C'_0) \).

It is shown in \[33\] that the existence of a triangle equivalence \( D(C, C_0) \xrightarrow{\simeq} D(C', C'_0) \) alone is not sufficient.

**Proposition A.8.** Let \( \mathcal{E} \) be a right exact category satisfying axiom \( R0^* \) and let \( \mathcal{A} \) be a right percolating subcategory such that \( S_\mathcal{A} \) is right weakly saturated. Assume that the natural map \( (\mathcal{C}^b(\mathcal{A}), \mathcal{A}c^b(\mathcal{A})) \rightarrow (\mathcal{C}^b(\mathcal{E}), \mathcal{A}c^b(\mathcal{E})) \) of Frobenius pairs induces a triangle equivalence \( \mathcal{D}^b(\mathcal{A}) \xrightarrow{\simeq} \mathcal{D}^b(\mathcal{E}) \). The sequence \( \mathcal{A} \rightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{A} \) of right exact categories induces a homotopy fibration of \( K \)-spectra

\[
\mathbb{K}(\mathcal{A}) \rightarrow \mathbb{K}(\mathcal{E}) \rightarrow \mathbb{K}(\mathcal{E}/\mathcal{A}).
\]

**Proof.** The result follows by combining theorem A.5 and A.7 and the previous definition. \( \square \)

**Appendix B. The weak idempotent completion**

In this appendix we construct the weak idempotent completion \( \tilde{\mathcal{A}} \) of an additive category \( \mathcal{A} \) as a subcategory of the idempotent completion \( \mathcal{A} \). We provide two descriptions of \( \tilde{\mathcal{A}} \). Firstly, \( \tilde{\mathcal{A}} \) can be constructed as the intersection of all (additive) weakly idempotent complete subcategories of \( \tilde{\mathcal{A}} \) containing \( \mathcal{A} \). Secondly, \( \tilde{\mathcal{A}} \) is obtained by iteratively adding kernels of retractions.

If the category \( \mathcal{E} \) is a right exact category, both the idempotent completion and weak idempotent completion can be endowed with a natural right exact structure extending the original structure. Moreover, we show that \( \mathcal{D}^b(\mathcal{E}) \simeq \mathcal{D}^b(\tilde{\mathcal{E}}) \) as triangulated categories.

**B.1. Idempotent completions.** An additive category \( \mathcal{A} \) is called *idempotent complete* if every idempotent in \( \mathcal{A} \) has a kernel. We recall the construction of the idempotent completion \( \tilde{\mathcal{A}} \) of an additive category \( \mathcal{A} \).

**Construction B.1.** Let \( \mathcal{A} \) be an additive category. We denote by \( \tilde{\mathcal{A}} \) the category whose objects are pairs \( (A, p) \) where \( A \in \text{Ob}(\mathcal{A}) \) and \( p : A \rightarrow A \) is an idempotent in \( \mathcal{A} \). The morphisms are given by

\[
\text{Hom}_{\tilde{\mathcal{A}}}((A, p), (B, q)) = q \circ \text{Hom}_{\mathcal{A}}(A, B) \circ p.
\]

The functor \( i_{\mathcal{A}} : \mathcal{A} \rightarrow \tilde{\mathcal{A}} \) given by mapping \( A \in \mathcal{A} \) to \( (A, 1) \) is an embedding of additive categories. The category \( \tilde{\mathcal{A}} \) is called the *idempotent completion* of \( \mathcal{A} \).

**Remark B.2.** For each idempotent \( p : A \rightarrow A \), one has \( (A, 1) \cong (A, p) \oplus (A, 1 - p) \) in \( \tilde{\mathcal{A}} \).

The following proposition characterizes the idempotent completion in terms of a 2-universal property.

**Proposition B.3.** The embedding \( \mathcal{A} \hookrightarrow \tilde{\mathcal{A}} \) is 2-universal among additive functors to idempotent complete additive categories:

1. For any additive functor \( \mathcal{F} : \mathcal{A} \rightarrow \mathcal{I} \) where \( \mathcal{I} \) is an idempotent complete additive category, there exists a functor \( \tilde{\mathcal{F}} : \tilde{\mathcal{A}} \rightarrow \mathcal{I} \) and a natural isomorphism \( F \xrightarrow{\simeq} \tilde{\mathcal{F}}_{\mathcal{A}} \).

2. For any additive functor \( G : \tilde{\mathcal{A}} \rightarrow \mathcal{I} \) and any natural transformation \( \gamma : F \Rightarrow G_{\mathcal{A}} \) there is a unique natural transformation \( \beta : \tilde{\mathcal{F}} \Rightarrow G \) such that \( \gamma = \beta \circ \alpha \).

Equivalently, the functor \( i_{\mathcal{A}} \) induces an equivalence of functor categories

\[
\text{Hom}(\tilde{\mathcal{A}}, \mathcal{I}) \rightarrow \text{Hom}(\mathcal{A}, \mathcal{I})
\]

for each idempotent complete category \( \mathcal{I} \).
Proposition B.4. Let \( \mathcal{E} \) be a right exact category satisfying axiom \( R0^\mathfrak{e} \) and let \( \tilde{\mathcal{E}} \) be its idempotent completion. Declare a sequence in \( \tilde{\mathcal{E}} \) to be a conflation if it is a direct summand in \( (\tilde{\mathcal{E}})^{-} \) of a conflation in \( (\mathcal{E})^{-} \). The conflations in \( \tilde{\mathcal{E}} \) induce a right exact structure on \( \tilde{\mathcal{E}} \) satisfying axiom \( R3 \). Moreover, the embedding \( \mathcal{E} \hookrightarrow \tilde{\mathcal{E}} \) is an exact embedding which is 2-universal among exact functors to idempotent complete right exact categories satisfying axiom \( R3 \).

Proof. The reader may verify that the structure above endows \( \tilde{\mathcal{E}} \) with the structure of a right exact category satisfying axiom \( R0^\mathfrak{e} \).

We now show that \( \tilde{\mathcal{E}} \) satisfies \( R3 \). Let \( i: A \to B \) and \( p: B \to C \) be maps in \( \tilde{\mathcal{E}} \) such that \( pi \) is a deflation in \( \tilde{\mathcal{E}} \). Consider the composition

\[
\begin{array}{ccc}
B \oplus A & \xrightarrow{(1 \mathbf{0}, -1 \mathbf{1})} & B \oplus A & \xrightarrow{(1 \mathbf{0}, 0 \mathbf{p})} & B \oplus C & \xrightarrow{(1 \mathbf{p}, 0 \mathbf{1})} & B \oplus C & \xrightarrow{(0 \mathbf{1}, C)} & C
\end{array}
\]

The first and third morphism are isomorphisms, the second a is a direct sum of deflations in \( \tilde{\mathcal{E}} \) and the last is a deflation since \( \tilde{\mathcal{E}} \) satisfies axiom \( R0^\mathfrak{e} \). By \( R1 \), the composition is a deflation in \( \tilde{\mathcal{E}} \). Note that the composition is simply \((p \circ 0)\). It follows that \( p \) is a direct summand of a deflation in \( \mathcal{E} \) as we needed to show.

The 2-universal property follows from the additive 2-universal property.

Corollary B.5. Let \( \mathcal{E} \) be an idempotent complete right exact category satisfying \( R0^\mathfrak{e} \). Then \( \mathcal{E} \) satisfies \( R3 \) if and only if the class of conflations is closed under direct summands.

B.2. Weak idempotent completion of additive categories. We now show the existence of a weak idempotent completion of a (small) additive category satisfying a 2-universal property. Later we will give an explicit construction in the style of construction B.1.

Definition B.6. Let \( \mathcal{A} \) be an additive category. A retraction is a map \( r: B \to C \) such that there is a section \( s: C \to B \) in the sense that \( rs = 1_C \). A coretraction is defined dually.

Lemma B.7. Let \( \mathcal{A} \) be an additive category. The following are equivalent:

1. Every retraction has a kernel.
2. Every coretraction has a cokernel.

Definition B.8. An additive category \( \mathcal{A} \) is called weakly idempotent complete if the conditions of the previous lemma hold.

Definition B.9. Let \( \mathcal{A} \) be an additive category. Let \( \hat{\mathcal{A}} \) be the intersection of all weakly idempotent complete additive subcategories of \( \mathcal{A} \) containing \( \mathcal{A} \). We write \( j_A: \mathcal{A} \hookrightarrow \hat{\mathcal{A}} \) for the inclusion.

Proposition B.10. The embedding \( j_A: \mathcal{A} \hookrightarrow \hat{\mathcal{A}} \) is 2-universal among additive functors to weakly idempotent complete additive categories:

1. For any additive functor \( F: \mathcal{A} \to \mathcal{W} \) where \( \mathcal{W} \) is a weakly idempotent complete additive category, there exists a functor \( \hat{F}: \hat{\mathcal{A}} \to \mathcal{W} \) and a natural isomorphism \( F \to \hat{F}j_A \).
2. For any additive functor \( G: \hat{\mathcal{A}} \to \mathcal{W} \) and any natural transformation \( \gamma: F \Rightarrow Gj_A \) there is a unique natural transformation \( \beta: \hat{F} \Rightarrow G \) such that \( \gamma = \beta \circ \alpha \).

Equivalently, the functor \( j_A \) induces an equivalence of functor categories

\[
\text{Hom}(\hat{\mathcal{A}}, \mathcal{W}) \to \text{Hom}(\mathcal{A}, \mathcal{W})
\]

for each weakly idempotent complete category \( \mathcal{W} \).

Proof. It is straightforward to see that the intersection of weakly idempotent complete categories is weakly idempotent complete, it follows that \( \hat{\mathcal{A}} \) is weakly idempotent complete.

Let \( F: \mathcal{A} \to \mathcal{W} \) be an additive functor to a weakly idempotent complete category \( \mathcal{W} \). Consider the following diagram:

![Diagram]

\[
\begin{array}{cccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{W} \\
\hat{\mathcal{A}} & \xrightarrow{j_A} & \mathcal{W} \\
\hat{\mathcal{A}} & \xrightarrow{i\mathcal{W}} & \mathcal{W}
\end{array}
\]
Here $\tilde{F}^{-1}(W)$ is the full and replete subcategory of $\tilde{A}$ generated by all objects which are mapped to $i_W(W) \subseteq \tilde{W}$ under $\tilde{F}$. As additive functors preserve rejections, $\tilde{F}^{-1}(W)$ is weakly idempotent complete. By construction, we have that $\tilde{A} \subseteq \tilde{F}^{-1}(W)$. Denote the composition $\tilde{A} \to \tilde{F}^{-1}(W) \to W$ by $\tilde{F}$. The statement now easily follows from the corresponding 2-universal property of the idempotent completion.

In practice it is convenient to have a explicit construction of a weak idempotent completion. It turns out that an explicit description is more subtle than for the idempotent completion.

**Construction B.11.** Let $\mathcal{A}$ be an additive category and set $\mathcal{A}_0 := i_\mathcal{A}(\mathcal{A})$. Recursively define $\mathcal{A}_n$ as the full and replete subcategory of $\tilde{A}$ generated by

$$\{(A, 1 - p) \in \tilde{A} \mid p = sr \text{ where } r \text{ is a retraction in } \mathcal{A}_{n-1} \text{ and } s \text{ the corresponding section}\}.$$  

Clearly $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for each $n \in \mathbb{N}$. Denote the filtered union $\bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ by $\tilde{\mathcal{A}}$.

**Remark B.12.** (1) The category $\mathcal{A}_n$ equals the full and replete subcategory generated by

$$\{(A, 1 - p) \in \tilde{A} \mid (A, p) \in \mathcal{A}_{n-1}\}.$$  

(2) The object $(A, 1 - p)$ is the kernel of $p$ in $\tilde{A}$ (this follows from the proof of lemma B.13) whereas the object $(A, p)$ is simply the image of $p$ in $\tilde{A}$.

**Lemma B.13.** Let $r : (A, 1 - e) \to (B, 1 - f)$ be a retraction in $\mathcal{A}_n$, then $r$ has a kernel in $\mathcal{A}_{n+1}$. Moreover, each object in $\mathcal{A}_n$ appears as the kernel of a retraction in $\mathcal{A}_{n-1}$. In particular, $\tilde{\mathcal{A}}$ is weakly idempotent complete.

**Proof.** Write $s : (B, 1 - f) \to (A, 1 - e)$ for the corresponding section of $r$. Put $p = sr$, then $p$ is an idempotent in $\mathcal{A}_n$. One can verify explicitly that the map $(A, 1 - p) \to (A, 1 - e)$ induced by the identity on $\mathcal{A}$ is the kernel of $r$ in $\mathcal{A}_{n+1}$.

Now let $(A, 1 - p) \in \mathcal{A}_n$, then $p$ factors as $p = sr$ where $r$ is a retraction in $\mathcal{A}_{n-1}$. The previous statement implies that $(A, 1 - p)$ is the kernel of $r$ in $\mathcal{A}_n$.

To show that $\tilde{\mathcal{A}}$ is weakly idempotent complete, let $r$ be retraction in $\tilde{\mathcal{A}}$. By construction there is a natural number $n$ large enough such that $r$ is a retraction in $\mathcal{A}_n$. By the above, $r$ admits a kernel in $\mathcal{A}_{n+1}$ and thus in $\tilde{\mathcal{A}}$. □

**Proposition B.14.** Let $\mathcal{A}$ be an additive category, then $\tilde{\mathcal{A}} = \tilde{\tilde{\mathcal{A}}}$ as subcategories of $\tilde{\mathcal{A}}$.

**Proof.** By lemma B.13, $\tilde{\mathcal{A}}$ is a weakly idempotent complete subcategory of $\tilde{\mathcal{A}}$ containing $i_\mathcal{A}(\mathcal{A})$. It follows that $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{A}}$. On the other hand, lemma B.13 implies that $\mathcal{A}_{n+1}$ is obtained from $\mathcal{A}_n$ by only adjoining kernels of retractions in $\mathcal{A}_n$. It follows that each $\mathcal{A}_n \subseteq \tilde{\mathcal{A}}$ and thus $\tilde{\mathcal{A}} \subseteq \tilde{\tilde{\mathcal{A}}}$. This concludes to proof. □

The next example shows that $\mathcal{A}_n$ need not be weakly idempotent complete for any $n$, hence the filtered union is required to obtain the weak idempotent completion.

**Example B.15.** Let $k$ be a field and consider the $k$-algebra $A = k^N$. Let $\mathcal{C}$ be the category of finite dimensional $A$-modules. Write $S_i$ for the simple $A$-module $Ae_i$, where $e_i$ is the obvious idempotent. Let $\mathcal{A}$ be the full and replete subcategory of $\mathcal{C}$ generated by $\{S_0, S_i \oplus S_{i+1} \mid i \in \mathbb{N}\}$. The map $S_0 \oplus S_1 \xrightarrow{(1_{S_0} \ 0)} S_0$ belongs to $\mathcal{A}_0$ and its kernel is given by $S_1$. Clearly $S_1 \notin \mathcal{A}_0$ but $S_1 \in \mathcal{A}_1$. Similarly, the map $S_1 \oplus S_2 \xrightarrow{(1_{S_1} \ 0)} S_1$ belongs to $\mathcal{A}_1$, its kernel is given by $S_2$ which belongs to $\mathcal{A}_2$ but does not belong to $\mathcal{A}_1$.

In general the map $S_{n-1} \oplus S_n \xrightarrow{(1_{S_{n-1}} \ 0)} S_{n-1}$ is a retraction in $\mathcal{A}_{n-1}$ with kernel $S_n$ and $S_n \notin \mathcal{A}_{n-1}$. It follows that $\mathcal{A}_{n-1}$ is not weakly idempotent complete.

Note that in this example the weak idempotent completion and idempotent completion equal $\mathcal{C}$.

**B.3. Weak idempotent completion of right exact categories.** Given a right exact category $\mathcal{E}$ satisfying axiom R0*, we endow the weak idempotent completion $\tilde{\mathcal{E}}$ with a right exact structure satisfying a 2-universal property. Moreover, we show that $D^b(\mathcal{E}) \simeq D^b(\tilde{\mathcal{E}})$ as triangulated categories.

We start with an obvious characterization of weakly idempotent complete right exact categories satisfying axiom R0*.

**Proposition B.16.** Let $\mathcal{E}$ be a right exact category satisfying axiom R0*. The following are equivalent:
Lemma B.17. Let $E$ be a right exact category and let $F$ be an extension-closed subcategory (i.e. for any conflation $A \to B \to C$ in $E$, we have that $A,C \in F$ implies $B \in F$). The category $F$ inherits a right exact structure from $E$ such that the inclusion $F \hookrightarrow E$ is a fully exact embedding.

Lemma B.18. Let $E$ be a right exact category satisfying axiom $R0^*$ and let $F$ be a weakly idempotent complete fully exact subcategory of $E$ containing $E$. Then $F$ is an extension-closed subcategory of $E$.

Proof. As $F$ is weakly idempotent complete, remark B.2 implies that for each $L \in E$, one has that $L \in F$ if and only if there exists an object $L' \in F$ such that $L \oplus L' \in E$.

Let $A \to B \to C$ be a conflation in $E$ and assume that $A,C \in F$. By definition, there is a commutative diagram

$$
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}
\quad
\begin{array}{ccc}
B & \to & C \\
\downarrow & & \downarrow \\
Y & \to & Z
\end{array}
$$

in $E$ whose rows are conflations, the second row belongs to $E$ and the vertical maps are coretractions.

Since $E$ is a right exact category, [4, proposition 5.2] or [17, proposition 2.10] yields a commutative diagram

$$
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}
\quad
\begin{array}{ccc}
B & \to & C \\
\downarrow & & \downarrow \\
P & \to & C
\end{array}
\quad
\begin{array}{ccc}
P \oplus C' & \to & C \oplus C' \\
\downarrow & & \downarrow \\
Y & \to & Z
\end{array}
$$

in $E$ such that upper left and lower right squares are bicartesian squares. Since $C \in F$, there exists an $C' \in F$ such that $C \oplus C' \in E$, moreover, the square

$$
\begin{array}{ccc}
P \oplus C' & \to & C \oplus C' \\
\downarrow & & \downarrow \\
Y & \to & Z
\end{array}
$$

is a pullback square in $E$ with $Y,Z,(C \oplus C') \in E$. Since $E$ is a right exact category, the pullback is contained in $E$. Hence $P \oplus C' \in E$.

As $A \in F$, there is an $A' \in F$ such that $A \oplus A' \cong X$. Note that there is a map from the conflation $A' = A' \to 0$ to the conflation $X \to P \oplus C' \to C \oplus C'$. Here, we use $R0^*$ to see that the former is indeed a conflation. Applying the short five lemma [4, lemma 5.3] to the commutative diagram (the middle map is determined by the commutativity of the diagram)

$$
\begin{array}{ccc}
A \oplus A' & \to & B \oplus A' \oplus C' \\
\downarrow & & \downarrow \\
X & \to & P \oplus C'
\end{array}
\quad
\begin{array}{ccc}
B \oplus A' \oplus C' & \to & C \oplus C' \\
\downarrow & & \downarrow \\
P \oplus C' & \to & C \oplus C'
\end{array}
$$

yields that $B \oplus A' \oplus C' \cong P \oplus C' \in E$. Since $(A' \oplus C') \in F$, we conclude that $B \in F$. Note that we used that $E$ satisfies axiom $R0^*$ to see that the upper row of the previous diagram is a conflation. □
Proposition B.19. Let \( \mathcal{E} \) be a right exact category satisfying axiom \( R0^p \). The weak idempotent completion \( \hat{\mathcal{E}} \) has a canonical right exact structure satisfying axiom \( R3 \) such that the embedding \( j_{\mathcal{E}} \) is \( 2 \)-universal among exact functors to weakly idempotent complete right exact categories satisfying axiom \( R3 \).

In particular, each conflation in \( \hat{\mathcal{E}} \) can be realized as a direct summand of a conflation in \( \mathcal{E} \).

Proof. Combining the previous two lemmas, the category \( \hat{\mathcal{E}} \) inherits a right exact structure. By proposition B.4, the idempotent completion satisfies axiom \( R3 \). As the weak idempotent completion \( \hat{\mathcal{E}} \) is a fully exact subcategory of \( \hat{\mathcal{E}} \), axiom \( R3 \) is satisfied. The result follows.

We are now in a position to prove the following theorem.

Theorem B.20. Let \( \mathcal{E} \) be a right exact category satisfying axiom \( R0^p \). The embedding \( j_{\mathcal{E}} : \mathcal{E} \hookrightarrow \hat{\mathcal{E}} \) lifts to a triangle equivalence \( D^b(\mathcal{E}) \cong D^b(\hat{\mathcal{E}}) \).

Proof. The embedding \( j_{\mathcal{E}} : \mathcal{E} \hookrightarrow \hat{\mathcal{E}} \) lifts to a fully faithful triangular functor \( F : K^b(\mathcal{E}) \hookrightarrow K^b(\hat{\mathcal{E}}) \). We claim that \( F \) is essentially surjective. It suffices to show that \( F \) is essentially surjective on stalk complexes.

Let \( X \in K^b(\hat{\mathcal{E}}) \) be a stalk complex. If \( X \in \mathcal{E}_0 \), the statement is trivial, so assume that \( X \in \mathcal{E}_n \) for some \( n \geq 1 \). By lemma B.13, there is a retraction \( f : Y \rightarrow Z \) in \( \mathcal{E}_{n-1} \) such that \( X \cong \ker(f) \) in \( \mathcal{E}_n \). Consider the commutative diagram

\[
\begin{array}{ccccccccc}
\ldots & 0 & X & 0 & 0 & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & 0 & Y & Z & 0 & \ldots \\
\end{array}
\]

in \( K^b(\hat{\mathcal{E}}) \) where the second row lies in \( F(K^b(\mathcal{E}_{n-1})) \). The cone of this map is a split conflation and hence the rows are homotopy equivalent in \( K^b(\hat{\mathcal{E}}) \). It follows that \( F \) is essentially surjective and thus \( K^b(\mathcal{E}) \cong K^b(\hat{\mathcal{E}}) \) as triangulated categories.

Consider the following diagram:

\[
\begin{array}{ccc}
\text{Ac}^b(\mathcal{E}) & \longrightarrow & K^b(\mathcal{E}) \\
\downarrow & & \downarrow \quad p \\
\text{Ac}^b(\hat{\mathcal{E}}) & \longrightarrow & K^b(\hat{\mathcal{E}}) \\
\end{array}
\]

We claim that \( \text{Ac}^b(\mathcal{E}) \) and \( \text{Ac}^b(\hat{\mathcal{E}}) \) have the same thick closure when viewed as subcategories of \( K^b(\hat{\mathcal{E}}) \). Obviously \( \text{Ac}^b(\mathcal{E}) \subseteq \text{Ac}^b(\hat{\mathcal{E}}) \). On the other hand, any bounded acyclic complex in \( \text{Ac}^b(\hat{\mathcal{E}}) \) can be written as a finite extension of conflations in \( \hat{\mathcal{E}} \). By proposition B.19 each such conflation arises as a direct summand of a conflation in \( \mathcal{E} \). The claim follows.

Combining both claims we find that \( D^b(\mathcal{E}) \) and \( D^b(\hat{\mathcal{E}}) \) are triangle equivalent categories.

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