Nambu-Jona-Lasinio model in a sphere

Zheng Zhang,1,∗ Chao Shi,2,† and Hongshi Zong1,3,4,‡

1Department of Physics, Nanjing University, Nanjing 210093, China
2College of Material Science and Technology, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
3Nanjing Proton Source Research and Design Center, Nanjing 210093, China
4Joint Center for Particle, Nuclear Physics and Cosmology, Nanjing 210093, China

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We study the chiral phase transition of the two flavor Nambu-Jona-Lasinio (NJL) model in a sphere with the MIT boundary condition. We find that the MIT boundary condition results in much stronger finite size effects than antiperiodic boundary condition. Our work may be helpful to study the finite size effects in heavy-ion collision in a more realistic way.

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I. INTRODUCTION

The finite size effects in Quantum Chromodynamics (QCD) have caused much theoretical interest for more than two decades. The study of finite size effects is important for the high-energy heavy ion collisions (HICs) experiments. It is believed that these experiments could produce the quark-gluon plasma (QGP), a phase of matter believed to exist in the early universe. However, the QGP systems produced by HICs are always in a finite volume. For instance, the sizes of QGP are estimated to be between 2 fm to 10 fm in [1], although the volume of Au-Au and Pb-Pb before freeze-out is about 50 fm³ to 250 fm³ [2, 3]. The finite size effects can modify the phase structure of strong interaction, dislocating critical lines and critical points, and also affect the dynamics of phase transition [1, 4–17]. The finite size effects have been investigated by different methods including chiral perturbation theory [18, 19], quark-meson model [11, 13, 20, 21], Dyson-Schwinger approach [22–25], Polyakov loop extended Nambu-Jona-Lasinio model [26–28] and other non-perturbative renormalization group methods [29]. A review of finite size effects can be found in [30]. In most existing studies of finite size effects in QCD, the systems are usually treated as a box. However, we know that the QGP in HICs is more like a sphere than a box, so for a more realistic calculation, the shape effect should be considered. There are several studies [6, 31–33] treating the system as a sphere, and the MRE method [34] is used. But the MRE method is essentially an approximation method and has some defects. For instance, It normally takes an asymptotic expansion which becomes invalid for very small volume. This limits the studying range of chiral phase transition since generally speaking the chiral symmetry should get restored in a small volume [22, 35]. So we want to find a better method to deal with the finite size effects in a sphere, which is one of the major subjects of this work.

Our method generalizes that dealing with finite size effects in a box, and it works as follows. Usually, we put the system into a box and a spatial boundary condition is required. It results in discretized momenta in the spatial direction. To consider finite size effects, we replace the integral over spatial momenta with a sum over discrete momentum modes. This "brute force method" has no difficulty to be applied for the sphere case, though the calculation is more complex. In this work, we will use this method to study the chiral phase transition of NJL model in a sphere.

This paper is organized as follows: In Section 2, The gap equation of NJL model in a sphere with the MIT boundary condition is derived. The chiral phase transition of the model is presented in Section 3, and a summary is given in Section 4.

II. NJL MODEL IN A SPHERE WITH THE MIT BOUNDARY CONDITION

NJL model is a low energy effective theory of QCD. It has the feature of dynamical chiral symmetry breaking. The Lagrangian of the two flavor NJL model is

\[ \mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi + G [ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma^5 \tau \psi)^2], \]

where \( G \) is the effective coupling and \( m \) is the current quark mass. Here we consider up and down quarks with exact isospin symmetry. In the mean field approximation, the gap equation is given as (only consider the Hartree term)

\[ M = m - 2G \langle \bar{\psi} \psi \rangle, \]

At zero temperature, the condensation [36]

\[ \langle \bar{\psi} \psi \rangle = iN_c N_f \int \frac{d^4 p}{(2\pi)^4} \text{tr} S(p), \]

where \( N_c = 3 \) is the number of colors, \( N_f = 2 \) is the number of flavors, and

\[ S(p) = \frac{\hat{p} \cdot M}{p^2 - M^2}. \]
The trace is taken over the Dirac indices. Apply the three momentum cut-off regulation, the gap equation can be finally written as

\[ M = m + 4GN_cN_f \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{M}{E_p}, \]  

where \( \Lambda \) is the cut-off.

Eq. (5) is the usual gap equation of NJL model in infinite volume. For finite size system, it should be modified. As we stated in the Introduction, three momenta will be discretized within a box under antiperiodic boundary condition. For example, the allowed values of momentum modes in a box under antiperiodic boundary condition are

\[ p_{\text{APBC}}^2 = \frac{4\pi^2}{L_j^2} \sum_{i=1}^{3} (n_i + \frac{1}{2})^2, \quad n_i = 0, \pm 1, \pm 2, \ldots \]  

To modify the gap equation, we only need to replace the integral over momentum with a sum over all allowed momentum modes, i.e.

\[ \int \frac{d^3\bar{p}}{(2\pi)^3} \rightarrow \frac{1}{V} \sum_{p_i}. \]  

This replacement has been used in many works, e.g. [1, 25, 29, 37].

Now we want to follow the above procedure to get the gap equation of NJL model in a sphere. First, we should select a proper boundary condition. In the past studies of finite size effects, periodic and antiperiodic boundary conditions are usually selected. However, it’s hard to define periodic and antiperiodic on a sphere. Dirichlet boundary condition can be defined on a sphere, but for fermions, it is too strict and no solution exists [38]. Here we select the MIT boundary condition, first proposed in the MIT bag model [39, 40]. For a sphere, it can be written as

\[-i\hat{r} \cdot \gamma \psi(t, r, \theta, \phi)|_{r=R} = 0, \]  

where \( \hat{r} \) is the unit vector normal to the sphere surface, and \( \gamma = (\gamma^1, \gamma^2, \gamma^3) \). This boundary condition confines the fermions inside the cavity since it forces the normal component of the fermionic current \( \nabla \gamma \psi \) to be zero at the surface of the cavity [39]. We think the MIT boundary condition is more suitable for the finite size effects study in HICs than (anti)periodic boundary conditions for its confinement character. Note the authors of [41] already investigated the NJL model in a cylinder with the MIT boundary condition. Here we investigate the NJL model in a sphere with also the MIT boundary condition.

Once the boundary condition Eq. (8) has been selected, the discrete values of the momentum can be obtained by solving the equation of motion of the NJL model with Eq. (8). The NJL model after mean field approximation can be seen as a model of free particle with mass \( M \), and its equation of motion is the free Dirac equation. Under spherical MIT boundary condition, the allowed momentum values are given by the following eigen-equations [42]

\[ j_l(x) = \frac{-\text{sgn}(\kappa)}{E + M} \hat{J}_x(pR), \]  

where

\[ \kappa = \pm 1, \pm 2, \ldots \]  

and \( j_l(x) \) is the \( l \)-th ordered spherical Bessel function. The \( p \) stands for \( |\bar{p}| \). With the following replacement

\[ \int \frac{d^3\bar{p}}{(2\pi)^3} \rightarrow \frac{1}{2V} \left( \sum_{p_i, \kappa > 0} + \sum_{p_i, \kappa < 0} \right), \]  

we finally get the gap equation of NJL model in a sphere with the MIT boundary condition. The factor 2 in Eq. (10) comes from the nondegeneracy of \( \kappa + -\kappa \) states. The reader can verify this replacement by direct calculation from Eq. (3).

For finite temperature case, the gap equations (consider both Hartree term and Fock term) in infinite space are [36]

\[ M = m + G \left[ N_c N_f + \frac{1}{2} \frac{M}{\pi^2} \right. \times \int_0^\Lambda dE_p p^2 \left( \tanh \frac{1}{2} \beta \omega_p^+ + \tanh \frac{1}{2} \beta \omega_p^- \right) \right. \]  

\[ + G \int_0^\Lambda dE_p p^2 \left( \tanh \frac{1}{2} \beta \omega_p^+ - \tanh \frac{1}{2} \beta \omega_p^- \right) \]  

\[ \mu' = \mu - G \frac{1}{\pi^2} \int_0^\Lambda dE_p p^2 \left( \tanh \frac{1}{2} \beta \omega_p^+ \right) \]  

where \( \omega_p^{\pm} = E_p \pm \mu' \). As before, use the replacement

\[ \int \frac{4\pi p^2 dp}{(2\pi)^3} \rightarrow \frac{1}{2V} \left( \sum_{p_i, \kappa > 0} + \sum_{p_i, \kappa < 0} \right), \]  

we get the gap equation of NJL model at finite temperature in a sphere with the MIT boundary condition.

III. CHIRAL PHASE TRANSITION OF THE NJL MODEL IN A SPHERE

Solving the gap equation Eq. (5) with replacement Eq. (10) at different radius, we can find how the finite size influence the constituent quark mass at zero temperature and chemical potential. It is well known that spontaneous symmetry breaking can only occur in infinitely large systems in principle [35, 43]. So in our cases, we expect the small size will lead to the restoration of chiral symmetry. In fact, the decrease of the volume has a similar effect with the increase of the temperature, so we expect the constituent quark mass will decrease when the volume decreases. Note these statements have been
confirmed in [37, 44]. Our results are qualitatively consistent with theirs, while quantitative different. Fig. 1 presents the constituent quark mass $M$ as a function of the radius $R$ at zero temperature and chemical potential. The parameters are chosen as $m = 5.5$ MeV, $\Lambda = 631$ MeV, $G\Lambda^2 = 2.02$. We find when the radius reaches about 140 fm, the constituent quark mass gets very close to that in an infinite system. That is to say, a system whose size is above 140 fm can be regarded as an infinitely large system. When the radius is smaller than 5 fm, the finite size effect is so strong that the dynamically generated mass $M$ is almost identical to the current quark mass. Compared with the results of the antiperiodic boundary condition in [37], we can see the MIT boundary condition has much stronger finite size effects than antiperiodic boundary condition. In fact, the confinement character of the MIT boundary condition makes it more like the hard wall boundary condition, which restricts the system more compared to the (anti)periodic boundary condition.

To show the influence of the finite size on the chiral phase transition of NJL model, we solve the gap equation Eq. (11) with the replacement Eq. (12). Fig. 2 presents the constituent masses as functions of temperature at zero chemical potential in spheres with different radius. The parameters are chosen as before. We find at the same temperature, the constituent quark mass in small volume is smaller than that in large volume because the finite size partially restores the chiral symmetry. We notice that when the radius gets below around 7 fm, it seems that there is no chiral phase transition, for the finite size effect is too strong. Considering the diameters of the QGP systems to be several fm, the finite size effects would be strong in these systems. This may lead to important consequences, which should be studied carefully in the future.

IV. SUMMARY

In this work, we apply the widely used "brute force method" to study the finite size effects of the NJL model in a sphere with the MIT boundary condition. The chiral phase transition is investigated in this model and our results are qualitatively consistent with [37]. The MIT boundary condition has much stronger finite size effects than (anti)periodic boundary condition. Since we believe the systems we deal with here are more close to the QGP systems in HICs, we think the finite size effects in HICs may be stronger than former estimations, so it deserves careful study in the future.

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