ON LAGRANGIAN TORI IN K3 SURFACES

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ABSTRACT. Every Maslov-zero Lagrangian torus in a K3 surface has non-trivial homology class. This note aims to extend this result to Lagrangian tori with Maslov indices congruent to zero modulo 4. Conversely, we show that every homologically non-trivial Lagrangian torus is necessarily Maslov-zero.

1. Main result. Let \((X,\omega)\) be a smooth symplectic 4-manifold that is homotopy-equivalent to a complex K3 surface, with the condition that \(c_1(X) = 0\). Let \(L \subset X\) be an embedded Lagrangian torus. Choose an \(\omega\)-compatible almost-complex structure \(J\) on \((X,\omega)\). Since \(L\) is totally real, we have the following natural isomorphisms:

\[
T_X|_L \cong T_L \otimes \mathbb{C}, \quad \Lambda^2_{\mathbb{C}}(T_X)|_L \cong \Lambda^2_{\mathbb{R}}(T_L) \otimes \mathbb{C},
\]

and the latter bundle carries a canonical section since \(L\) is orientable. On the other hand, \(c_1(X) = 0\) implies that \(\Lambda^2_{\mathbb{C}}(T_X)\) has a global section, which is unique up to homotopy since \(H^1(X;\mathbb{Z}) = 0\). We thus obtain two nowhere vanishing sections of \(\Lambda^2_{\mathbb{C}}(T_X)\) over \(L\). They differ by a gauge map \(L \to \mathbb{C}^*\), whose class in \(H^1(L;\mathbb{Z})\) we denote by \(\alpha\). The class \(\mu = 2\alpha\) is referred to as the Maslov class of \(L\).

For the sake of our discussion, it is more convenient to work with the class \(\alpha\). If \(\alpha\) vanishes, then \(L\) is called a Maslov-zero torus. If \((X,\omega)\) is a Kähler K3 surface, then every Maslov-zero Lagrangian torus \(L \subset (X,\omega)\) has non-trivial homology class. Sheridan and Smith (see Theorem 1.4 in \([20]\)) proved this result for specific Kähler symplectic forms on \(X\), using advanced methods of homological mirror symmetry. The theorem’s generalization to arbitrary Kähler forms is due to Entov and Verbitsky, as found in Theorem 1.1 in \([6]\). Their proof reduces the general case to one where the Sheridan-Smith theorem is applicable. This note introduces further generalizations of these results. Specifically, we prove the following:

**Theorem 1.** Let \((X,\omega)\) be a symplectic 4-manifold homotopy-equivalent to a complex K3 surface, and suppose \(c_1(X) = 0\). Let \(L \subset (X,\omega)\) be an embedded Lagrangian torus. Then:

1. If \(L\) is homologically non-trivial, then \(\alpha\) is zero.
2. If \(L\) is nullhomologous modulo 2, then \(\alpha\) is non-zero modulo 2.

Consequently, if the homology class of \(L\) is non-trivial, its modulo 2 reduction is also non-trivial.

One can replace the assumptions that \((X,\omega)\) is homotopy-equivalent to a K3 surface and \(c_1(X) = 0\) with the single assumption that \(X\) is diffeomorphic to a K3 surface. Indeed, if \((X,\omega)\) is a symplectic manifold diffeomorphic to a K3 surface, then \(c_1(X) = 0\), regardless of the choice of \(\omega\). This is explained at the end of \(\S 2\).

Our proof builds on Nemirovski’s work \([13]\), and it is independent of the work \([20]\). In \(\S 6\) we extend this to Lagrangian Klein bottles under the same assumptions on \((X,\omega)\).

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2. Insights from Seiberg-Witten Theory. We briefly introduce Seiberg-Witten invariants, referring readers to [16, 19, 8] for details on four-dimensional gauge theory.

Let \((X, g)\) be a smooth, closed 4-manifold with a Riemannian metric \(g\), \(b_1(X) = 0\), and \(b_2^+(X) > 1\). We choose a spin\(^C\) structure \(s\) on \(X\), which gives rise to spinor bundles \(W^\pm\) and determinant line bundle \(L\). Let \(\mathcal{A}\) denote the space of \(U(1)\)-connections on \(L\), and let \(\mathcal{U}\) be the gauge group associated with \(L\).

\(\mathcal{U}\) consists of smooth maps from \(X\) to \(U(1)\), forming a group since the target space is a group. When \(H^1(X; \mathbb{R}) = 0\), as is the case in this paper, each map \(u \in \mathcal{U}\) can be written as \(u = e^{-if}\) for some smooth function \(f: X \to \mathbb{R}\). The group \(\mathcal{U}\) acts on \(\Gamma(W^+) \times \mathcal{A}\) as follows: for \(u = e^{-if} \in \mathcal{U}\) and \((\varphi, A) \in \Gamma(W^+) \times \mathcal{A}\),

\[
    u \cdot (\varphi, A) = (e^{-if} \varphi, A + 2i df).
\]

If two connections \(A_1, A_2 \in \mathcal{A}\) are gauge equivalent, they differ by a closed 1-form, and so share the same curvature, i.e., \(F_{A_1} = F_{A_2}\). Since the curvature \(F_A\) is gauge-invariant (i.e., \(F_A = F_{u^{-1}Au}\) for all \(u \in \mathcal{U}\)), it follows that the self-dual part of the curvature, \(F^+_A\), is also gauge-invariant. This property is essential for the gauge-invariance of the Seiberg-Witten equations.

On the other hand, if \(H^1(X; \mathbb{R}) = 0\) and \(F_{A_1} = F_{A_2}\), then the difference \(A_1 - A_2\) is not only closed but also exact, meaning it can be expressed as \(2i df\). In this case, the gauge transformation \(e^{-if}\) will map \(A_1\) to \(A_2\). Thus, two connections are gauge-equivalent if and only if they have the same curvature.

Let \(\eta\) be a \(g\)-self-dual form on \(X\). The Seiberg-Witten equations with perturbing term \(\eta\) seek solutions \((\varphi, A)\) and are given by:

\[
    \begin{cases}
        \mathcal{D}_A \varphi = 0, \\
        F^+_A = \sigma(\varphi) + i \eta,
    \end{cases}
\]

where \(\mathcal{D}_A: \Gamma(W^+) \to \Gamma(W^-)\) is the Dirac operator, \(\sigma(\varphi)\) is the squaring map, and \(F^+_A\) is the self-dual part of the curvature of \(A\). The moduli space of solutions, \(M_s\), is defined as:

\[
    M_s = \{(\varphi, A) \in \Gamma(W^+) \times \mathcal{A} \mid (\varphi, A) \text{ is a solution to (2.1)}\} / \sim,
\]

\((\varphi, A) \sim (\varphi', A')\) if \(u \cdot (\varphi', A') = (\varphi, A)\) for some \(u \in \mathcal{U}\).

The moduli space \(M_s\) depends on both the metric \(g\) and the form \(\eta\). As shown in Corollary 3 in [8], \(M_s\) is always compact. For sufficiently generic choices of \(g\) and \(\eta\) (shown in Lemma 5 in [8]), \(M_s\) becomes a smooth manifold of dimension:

\[
    d(s) = \frac{1}{4}(c_1^2(s) - 3 \text{sign}(X) - 2\chi(X)),
\]

where \(c_1(s) = c_1(L)\) is the Chern class of the determinant line bundle, \(\chi(X)\) is the Euler characteristic of \(X\), and \(\text{sign}(X) = b_2^+(X) - b_2^-(X)\) is the signature of \(X\).

We now define a mod 2 version of the Seiberg-Witten invariant for spin\(^C\) structures \(s\) with \(d(s) = 0\). When \(g\) and \(\eta\) are generic, \(M_s\) is zero-dimensional, consisting of a finite number of points. We define the Seiberg-Witten invariant, \(\text{SW}(s)\), as the number of points in \(M_s\), counted modulo 2:

\[
    \text{SW}(s) = \# M_s \mod 2.
\]

For \(b_2^+(X) > 1\), [8] proves that \(\text{SW}(s)\) is independent of the generic choices of \(g\) and \(\eta\).

Assume \(X\) is also a symplectic manifold with a symplectic form \(\omega\) and an \(\omega\)-compatible almost-complex structure \(J\) such that \(g(\cdot, \cdot) = \omega(\cdot, J\cdot)\). As shown in Ch. 7 of [19], every spin\(^C\) structure on
Theorem 2

Theorem by Taubes:

\[
\begin{align*}
W^+ &= L_\varepsilon \oplus (\Lambda^{0.2} \otimes L_\varepsilon), \\
W^- &= \Lambda^{0.1} \otimes L_\varepsilon,
\end{align*}
\]
(2.4)

where \( L_\varepsilon \) is the line bundle on \( X \) with \( c_1(L_\varepsilon) = \varepsilon \in H^2(X; \mathbb{Z}) \). We denote the spin\(^c \) structure defined by (2.4) as \( s_\varepsilon \). Letting \( K_X^* \) denote the anticanonical bundle of \( X \), the determinant line bundle of \( s_\varepsilon \) is given by \( L = K_X^* \otimes L_\varepsilon^2 \). Consequently, \( c_1(s_\varepsilon) = c_1(s_0) + 2\varepsilon \). In terms of \( \varepsilon \), the dimension \( d(s_\varepsilon) \) becomes:

\[
d(s_\varepsilon) = c_1(X) \cdot \varepsilon + \varepsilon \cdot \varepsilon.
\]

This formula for \( d(s_\varepsilon) \) is derived from (2.3) using the Hirzebruch signature formula (also proved by Rokhlin, [7]),

\[
c_1(X)^2 = 3 \text{sign}(X) + 2\chi(X),
\]

and noting that \( c_1(s_0) = c_1(X) \).

Setting \( \varepsilon = 0 \) gives \( d(s_0) = 0 \), and similarly for \( \varepsilon = -c_1(X) \). A key result for our work is the following theorem by Taubes:

**Theorem 2** (Taubes, [21]). Let \((X, \omega)\) be a closed symplectic manifold with \( b_2^+ (X) > 1 \). Then:

1. \( \text{SW}(s_0) = \text{SW}(s_{-c_1(X)}) = 1 \).
2. If \( \text{SW}(s_\varepsilon) \neq 0 \), then \( \varepsilon \) must satisfy the inequality:

\[
\varepsilon \cdot [\omega] \geq 0,
\]

with equality allowed only for \( \varepsilon = 0 \).

Consequently, if \( \varepsilon \) is a non-trivial 2-torsion class, then \( \text{SW}(s_\varepsilon) = 0 \).

**Proof.** For detailed proofs, consult the end-notes of Ch. 10 in [19] or Theorem 3.3.29 in [16]. □

Let us additionally assume that \( X \) satisfies the following topological conditions:

\[
b_1(X) = 0, \quad b_2^+ (X) = 3, \quad b_2^- (X) = 19.
\]

(2.5)

We refer to such a manifold \( X \) as a rational homology K3. This terminology arises because these Betti numbers match those of a K3 surface when considered with rational coefficients. The second key result for our analysis, due to Morgan and Szabó, can be found in [12], specifically in Theorem 1.1 and Remark 2.2.

**Theorem 3** (Morgan-Szabó, [12]). Let \( X \) be a spin 4-manifold and a rational homology K3. Then any spin\(^c \) structure with trivial determinant line bundle has non-trivial Seiberg-Witten invariant.

**Proof.** The proof is provided in [12]. While the original theorem assumes \( X \) to be a homotopy K3 (i.e., simply-connected), the proof holds for the general case as well. Besides (2.5), the only required assumption is that the determinant line bundle of the spin\(^c \) structure is trivial. □

Finally, we show how Taubes’ theorem implies that a symplectic manifold \((X, \omega)\) diffeomorphic to a K3 surface has \( c_1(X) = 0 \). Suppose, for contradiction, that \( c_1(X) \neq 0 \). Then \( c_1(X) \) cannot be a 2-torsion class, as no such classes exist in a K3 surface. By Taubes’ theorem, we have:

\[
\text{SW}(s_0) = 1, \quad \text{SW}(s_{-c_1(X)}) = 1.
\]

Here, the Chern class of \( s_0 \) is \( c_1(X) \), and that of \( s_{-c_1(X)} \) is \(-c_1(X) \). Given the absence of 2-torsion classes, \( X \) must then admit at least two distinct spin\(^c \) structures with non-zero Seiberg-Witten invariants. However, if \( X \) is diffeomorphic to a K3 surface, it can have only one such structure. For a proof, see the end notes of Ch. 10 in [19].
3. Rokhlin and Viro numbers. The following material on the Viro index is well-known (see, e.g., [22, 15, 13]). Let \((X, J)\) be a simply-connected almost-complex 4-manifold, and let \(L \subset X\) be an embedded totally real torus. We assume that the homology class of \(L\) is zero modulo 2 in \(X\), i.e., \([L] = 0 \in H_2(X; \mathbb{Z}_2)\).

Consider a simple closed loop \(\gamma\) on \(L\), a non-vanishing vector field \(\dot{\gamma}\) tangent to \(\gamma\), and a non-vanishing normal vector field \(\nu\) to \(\gamma\) on \(L\). Here, “normal” means \(\nu\) is everywhere transverse to \(\dot{\gamma}\). By pushing off \(\gamma\) in the direction of the field \(J\nu\), we obtain a loop \(\gamma^\# \subset X - L\). The isotopy class of \(\gamma^\#\) in \(X - L\) is well-defined because any two non-vanishing normal vector fields on \(\gamma\) are homotopic through non-vanishing vector fields on the tangent bundle of \(L\) restricted to \(\gamma\).

The Viro index of \(\gamma\) is defined as the linking number modulo 2:

\[
V(\gamma) = \text{lk}(\gamma^\#, L) \in \mathbb{Z}_2.
\]

The linking number is defined as follows. Since \(X\) is simply-connected, there exists an immersed disk \(D\) that bounds \(\gamma\) in \(X\). We choose \(D\) to be transverse to \(L\). Then, the linking number is the number of intersection points between \(D\) and \(L\), modulo 2:

\[
\text{lk}(\gamma^\#, L) = \#(L \cap D) \mod 2.
\]

If \(D'\) is another disk bounding \(\gamma\), then the union of \(D\) and \(D'\) forms an immersed sphere in \(X\) denoted as \(S\). The difference between the number of intersection points of \(L\) with \(D\) and \(D'\) (modulo 2) is equal to the number of intersection points between \(S\) and \(L\) (also modulo 2). Since \(L\) is nullhomologous modulo 2, the latter number is always even. Therefore, the Viro index is well-defined for vanishing homology classes \([L] \in H_2(X; \mathbb{Z}_2)\).

We let \(\alpha \in H^1(L; \mathbb{Z})\) to be half of the Maslov class of \(L\).

**Lemma 1.** Let \((X, J)\) be a simply-connected almost-complex 4-manifold homotopy-equivalent to a complex K3 surface, and suppose \(c_1(X) = 0\). If \(L\) is nullhomologous modulo 2 and the modulo 2 reduction of \(\alpha\) vanishes, then there exists a simple loop \(\gamma\) on \(L\) such that \(V(\gamma) = 1\).

The proof occupies the present section.

\(\Lambda^2_\xi(T_X)|_L\) has a preferred non-vanishing section, which comes from a non-vanishing section of \(\Lambda^2_\xi(T_L)\) through the isomorphism \([11]\). Let \(\sigma\) be a generic extension of this section over the entirety of \(\Lambda^2_\xi(T_X)\). Define:

\[
\Sigma = \{x \in X \mid \sigma(x) = 0\}.
\]

If not empty, \(\Sigma\) is a smooth, oriented, embedded surface in \(X\) disjoint from \(L\). Suppose that \(\gamma\) bounds an embedded (not necessarily orientable) surface \(M \subset X\) such that \(M\) is nowhere-tangent to \(L\) along \(\gamma\) and such that the interior of \(M\) is transverse to both \(L\) and \(\Sigma\). We call \(M\) a membrane for the loop \(\gamma\). Such a surface always exists, since \(X\) is simply-connected.

Let \(N_M = T_X/T_M|_M\) be the normal bundle to \(M\) in \(X\). We define its modulo 2 Euler class, \(d(M) \in \mathbb{Z}_2\), relative to the chosen normal vector field \(\nu\) along \(\gamma = \partial M\). We extend \(\nu\) from a non-vanishing section over \(\gamma\) to a generic section of \(N_M\) over \(M\). \(d(M)\) is the number of zeros of this extension modulo 2, independent of the extension choice.

Denote by \(#(M \cap \Sigma)\) and \(#(M \cap L)\) the intersection numbers (interior) of \(M\) with \(\Sigma\) and \(L\), respectively. Define \(R(M, \sigma) \in \mathbb{Z}_2\) (Rokhlin index of \(M\)) and \(q(\gamma) \in \mathbb{Z}_2\) as:

\[
R(M, \sigma) = d(M) + #(M \cap \Sigma) + #(M \cap L) \mod 2, \quad q(\gamma) = d(M) + #(M \cap L) \mod 2.
\]

Nemirovski’s theorem \([13]\) relates the Rokhlin index, the Viro index, and a membrane.
Theorem 4 (Nemirovski, [13]). Let $M$ be a membrane for $\gamma$ on $L$, chosen so the tangent half-space to $M$ along $\gamma$ is spanned by $\gamma$ and $J\nu$. Then, with this choice of $M$:

$$R(M,\sigma) = 1 + V(\gamma) \mod 2.$$  

(3.1)

Proof. See Lemma 1.13 in [13].

Observe that $\#(M \cap \Sigma)$ coincides with $\alpha(\gamma)$, calculated modulo 2. Therefore, if the modulo 2 reduction of $\alpha$ vanishes, we have the following relationship:

$$R(M,\sigma) = q(\gamma).$$  

(3.2)

As pointed out in [17], if $L$ is a characteristic surface (i.e., nullhomologous modulo 2), then $q(\gamma)$ only depends on the homology class of the loop $\gamma$ in $H_1(L;\mathbb{Z}_2)$ and is independent of the specific choice of membrane $M$. Furthermore, $q$ is a quadratic function over $H_1(L;\mathbb{Z}_2)$ with an associated Arf invariant. This Arf invariant can be calculated as follows: If $e_1$ and $e_2$ form any basis for $H_1(L;\mathbb{Z}_2)$, then the Arf invariant of $q$, denoted by $\text{Arf}(q)$, is defined as:

$$\text{Arf}(q) = q(e_1)q(e_2).$$  

(3.3)

A detailed discussion of the function $q$ and the associated Arf invariant can be found in [9, 17], as well as in the end-notes of Ch. 11 in [19].

Rokhlin and Freedman-Kirby established the following congruence:

$$\text{sign}(X) - [L]^2 = 8 \text{Arf}(q) \mod 16.$$  

(3.4)

A detailed proof of this congruence, along with its background and applications, can be found in [9, 17]. If $X$ is homotopy-equivalent to a K3 surface, then $\text{sign}(X) = -16$. Further, if $L$ is totally real, then $[L]^2 = 0$. Plugging these into (3.4) yields $\text{Arf}(q) = 0$. It follows from (3.3) that there must be a simple loop $\gamma$ on $L$ with $q(\gamma) = 0$. Lemma 1 now follows from (3.1) and (3.2).

Remark 1. Congruence (3.4) admits a generalization to the non-simply-connected case, as discussed in § 2.6 in [4]. Combining with Theorem A in [18], it leads to a generalization of Theorem 1 applicable to symplectic four-tori (Corollary 9.2 in [11]). However, we will not study this generalization further.

4. Luttinger surgery. We assume familiarity with Luttinger surgery (see, e.g., [2, 5, 11, 13, 14]). Let $D$ be a small disk in $\mathbb{R}^2$ with standard Lagrangian torus bundle $\pi: D \times T^2 \to D$ and symplectic form induced by $(T_{\pi^*}^\omega, \omega_0)$. Suitable local coordinates $(x,y)$ on $D$ and $(\theta_x, \theta_y)$ on $T^2$ express $\omega_0$ as $d\theta_x \wedge dx + d\theta_y \wedge dy$. Using polar coordinates $(r,\varphi)$ on $D$ with $(x,y) = (r \cos \varphi, r \sin \varphi)$, define a multi-valued function $f_{m,n}(r,\varphi) = mx\varphi + ny\varphi$, $(m,n) \in \mathbb{Z}^2$, on $N = D - 0$. Though $f_{m,n}$ is not well-defined on $N$, the symplectomorphism $\psi: \pi^{-1}(N) \to \pi^{-1}(N)$ generated by the time-1 Hamiltonian flow of $f_{m,n} \circ \pi$ is well-defined. Denote the $k$th power of $\psi$ as $\psi^k$.

Let $L$ be a Lagrangian torus in $(X,\omega)$. A neighborhood $U$ of $L$ is symplectically identified with $\pi^{-1}(D)$ such that $L$ corresponds to $\pi^{-1}(0)$. Choose a basis for $H_1(T^2;\mathbb{Z})$ and a simple loop $\gamma$ on $L$, represented by $(m,n) \in \mathbb{Z}^2$. (Note that if $\gamma$ is a simple loop, then either $(m,n) = (0,0)$ or $m,n$ are coprime.) Luttinger surgery on $X$ along $L$ with respect to $\gamma$ and $k$ (denoted $X(L,\gamma,k)$) is obtained by removing $U$ from $X$ and gluing it back along $U - L$ via $\psi^k$:

$$X(L,\gamma,k) = ((X - L) \cup U) / \sim,$$

$$x_1 \in X - L \sim x_2 \in U \iff x_1, x_2 \in U - L \text{ and } \psi^k(x_1) = x_2.$$

Since $\psi^k$ is a symplectomorphism, $X(L,\gamma,k)$ carries a natural symplectic form, whose deformation class is independent of the construction choices [2]. We refer to [11], Example 2, for a description of Luttinger surgery using $f_{m,n}$. 

Let us analyze the effect of Luttinger surgery on $H^1$ and $H^2$, specializing to (homotopy) K3 surfaces. For abbreviation, we denote $X(L,\gamma,k)$ as $\tilde{X}$.

**Lemma 2.** If $\pi_1(X)$ is trivial, $L$ is nullhomologous modulo 2 in $X$, and $\gamma$ is such that $V(\gamma) = 1$, then there exists $k \in \mathbb{Z}$ such that $b_1(\tilde{X}) = 0$ and such that $H^2(\tilde{X};\mathbb{Z})$ has non-trivial 2-torsion.

**Proof.** Since $\gamma$ is a simple loop representing a primitive class in $H_1(L;\mathbb{Z})$, there exists a simple loop $\beta$ such that their classes form a basis for $H_1(L;\mathbb{Z})$. In $U-L$, pick a loop $\mu$ such that there exists a nullhomotopy of $\mu$ in $U$ intersecting $L$ at a single point. We call such a loop a meridian for future reference. $H_1(U-L;\mathbb{Z}) = \mathbb{Z}^3$ is generated by $\mu,\gamma\# , \beta\#$.

We consider two cases: (1) nullhomologous $L$ and (2) homologically non-trivial but nullhomologous modulo 2 $L$. We only prove (1), as the argument for (2) is similar.

We have the following piece of the Mayer–Vietoris sequence:

$$\ldots \to H_1(U-L) \to H_1(X-L) \oplus H_1(U) \to H_1(X) \to 0,$$

(4.1)

and a similar sequence for $\tilde{X}-L$ and $\tilde{X}$:

$$\ldots \to H_1(U-L) \to H_1(\tilde{X}-L) \oplus H_1(U) \to H_1(\tilde{X}) \to 0.$$

(4.2)

Since $U-L \subset X-L$, $\mu$ can be considered a loop in $X-L$. Let us show that if $L$ is nullhomologous then $H_1(X-L;\mathbb{Z}) = \mathbb{Z}$, generated by $\mu$. Any loop $\delta$ in $X-L$ bounds an immersed disk $C$ in $X$. By arranging the intersections of $C$ with $L$ to be transverse and removing small neighborhoods of the intersection points, we obtain a sphere with holes $C' \subset C$ that can be isotoped into $X-L$. The boundary of $C'$ consists of $\delta$ and several meridians homologous to $\mu$, implying $\delta$ is homologous to a multiple of $\mu$. To show $\mu$ is non-trivial in $X-L$, assume it is nullhomologous. Then, a surface $C$ in $X-L$ bounding $\mu$, when joined with a nullhomotopy of $\mu$ in $U$, would intersect $L$ once. This is impossible since $L$ is nullhomologous.

If we choose $\mu,\gamma\#, \beta\#$ as a basis for $H_1(U-L)$, $\mu$ as a generator for $H_1(X-L)$, and $\gamma, \beta$ as a basis for $H_1(U)$, then the homomorphism $H_1(U-L) \to H_1(X-L) \oplus H_1(U)$ in (4.1) is the identity matrix. The homomorphism $H_1(U-L) \to H_1(\tilde{X}-L) \oplus H_1(U)$ in (4.2) is given by the matrix:

$$\begin{pmatrix} 1 & p & q \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $p = V(\gamma) \mod 2$ and $q = V(\beta) \mod 2$.

(4.3)

The rank of $H_1(\tilde{X};\mathbb{Z})$ is the corank of this matrix. Choose $k$ such that the above matrix has non-vanishing determinant. On the other hand, (the absolute value of) that determinant is exactly the order of $H_1(\tilde{X};\mathbb{Z})$. Choosing $k$ odd, we arrange so that the order of $H_1(\tilde{X};\mathbb{Z})$ is even. This can be done because $p = 1 \mod 2$. But then there must be elements of order 2 in $H_1(\tilde{X};\mathbb{Z})$, and hence in $H^2(\tilde{X};\mathbb{Z})$. \hfill \Box

**Lemma 3.** If $\pi_1(X)$ is trivial and $L$ has non-trivial homology class, then $b_1(\tilde{X}) = 0$.

**Proof.** We analyze the Mayer–Vietoris sequence (4.2) with rational coefficients. To show $b_1(\tilde{X}) = 0$, it suffices to prove the surjectivity of $H_1(U-L) \to H_1(\tilde{X}-L) \oplus H_1(U)$. From the proof of Lemma 2 we establish that $H_1(\tilde{X}-L)$ is generated by $\mu$. Since $L$ has non-trivial homology, Poincaré duality guarantees an immersed surface $C$ with non-zero intersection number with $L$. By arranging $C$ to be transverse to $L$ and removing small neighborhoods of intersections, we obtain a sphere with holes $C' \subset C$ whose boundary consists of meridians homologous to $\mu$. This implies that a multiple of $\mu$ is
nullhomologous in $\tilde{X} - L$. Since the Mayer-Vietoris sequence \((1.2)\) maps $\gamma^\#,\beta^\#$ to $\gamma,\beta$ in $H_1(U)$, it follows that $H_1(U - L)$ surjects onto $H_1(U)$.  

Let us examine the effect of Luttinger surgery on the Chern class and Euler characteristic.

**Lemma 4.** $\chi(X) = \chi(\tilde{X})$.

**Proof.** $\chi(X)$ depends only on the ranks of $H_k(U), H_k(X - L), H_k(U - L)$, not on the homomorphisms encountered in the Mayer–Vietoris sequence. Since $\tilde{X} - L$ coincides with $X - L$, it follows that $H_k(\tilde{X} - L)$ coincide with $H_k(X - L)$. The groups $H_k(U)$ are the same for both sequences. Thus, $\chi(\tilde{X}) = \chi(X)$.  

**Lemma 5.** If $c_1(X) = 0$, then $c_1(\tilde{X}) = -k\alpha(\gamma)[L] \in H^2(\tilde{X}; \mathbb{Z})$, where $\alpha$ is the Maslov class of $L$ in $X$ and $[L]$ is the dual to $L$ in $\tilde{X}$ (i.e., $[L]$ is the Chern class of the complex line bundle with divisor $L$). Note that the orientation of $L$ does not appear in this formula because it implicitly appears in our definition of Luttinger surgery.

**Proof.** This proof is derived from [2, § 2.2]. Let $\ker d\pi$ denote the vertical tangent bundle for $\pi: U \to D^2$. Since $\pi \circ \psi = \pi$, it follows that $\psi$ induces a bundle morphism $\psi_*: \Lambda^2_{\mathbb{R}}(\ker d\pi) \to \Lambda^2_{\mathbb{R}}(\ker d\pi)$. Using the description of $\psi$ in terms of $f_{m,n}$, one finds a non-vanishing section $\sigma$ of $\Lambda^2_{\mathbb{R}}(\ker d\pi)$ such that $\psi_*\sigma = \sigma$. Explicitly, if we let $v_x$ and $v_y$ be the Hamiltonian vector fields for the functions $x$ and $y$, respectively, then we can set $\sigma = v_x \wedge v_y$. Both $v_x$ and $v_y$ belong to $\ker d\pi$. Moreover, since $\psi$ preserves $x$ and $y$ and is a symplectomorphism, it also preserves $v_x$ and $v_y$.

Choose an almost-complex structure $J$ on $X$. Let $s$ be a section of $\Lambda^2_{\mathbb{C}}(T_X)$ over the whole of $X$. Since $\ker d\pi$ is an orientable Lagrangian plane field over $U$, we have a canonical isomorphism

$$\Lambda^2_{\mathbb{R}}(\ker d\pi) \otimes \mathbb{C} \cong \Lambda^2_{\mathbb{C}}(T_X)|_U. \quad (4.4)$$

By \((4.4)\), $\sigma$ induces a section of $\Lambda^2_{\mathbb{C}}(T_X)|_U$, which we shall also denote by $\sigma$. Then, over $U - L \subset X$, we have $s = g \cdot \sigma$, where $g: U - L \to \mathbb{C}^*$. Letting $\delta \in H^1(U - L; \mathbb{Z})$ be the cohomology class of the mapping $g$, we have $\delta(\gamma^\#) = \alpha(\gamma), \delta(\beta^\#) = \alpha(\beta)$, and $\delta(\mu) = 0$. Here $\mu, \gamma^\#, \beta^\#$ are defined as in the proof of Lemma 2.

Using the identification $\tilde{X} - L \cong X - L$, we endow $\tilde{X} - L$ with the almost-complex structure $J$. Now, consider the subset $U - L \subset \tilde{X}$. Let $\psi^k s$ represent a section of $\Lambda^2_{\mathbb{C}}(T_X)|_{U - L}$, understood with respect to the structure $\psi^k J$. This section is obtained through the pushforward of the section $s$ under $\psi^k$, and it satisfies the relation:

$$\psi^k s = (g \circ \psi^{-k}) \cdot \sigma.$$  

In this formula, $\sigma$ is the section of $\Lambda^2_{\mathbb{C}}(T_X)|_{U - L}$, obtained using the almost-complex structure $\psi^k J$ and the isomorphism \((4.4)\). Additionally, $(g \circ \psi^{-k})$ denotes the pushforward of the function $g$ under $\psi^k$.

Observe that $\psi^k J$ does not extend to cover the entirety of $U$. To fix this, let us consider a tubular neighborhood $V \subset U$ of $L$, which is strictly smaller than $U$. Choose an almost-complex structure $J'$ over the entire region $U$, such that $J' = \psi^k J$ holds on $U - V$. Let $s'$ be a section of $\Lambda^2_{\mathbb{C}}(T_X)|_{U - L}$, understood with respect to the structure $J'$, that is chosen in such a way that $s \equiv s'$ on $U - V$. (Since $U$ and $V$ are homotopy-equivalent, it follows that such an extension of $s|_{U - V}$ exists and is unique up to homotopy.) Now, let $\sigma'$ be the section of $\Lambda^2_{\mathbb{C}}(T_X)|_U$ obtained using the almost-complex structure $J'$ and the isomorphism \((4.4)\). Then, over $U - L \subset \tilde{X}$, we have $s' = g' \cdot \sigma'$. The homotopy class of $g'$ is equal to that of $g \circ \psi^{-k}$ since $g \circ \psi^{-k} = g'|_{U - V}$. Letting $\delta' \in H^1(U - L; \mathbb{Z})$ be the cohomology class of the mapping $g'$, we have $\delta'(\gamma^\#) = \alpha(\gamma), \delta'(\beta^\#) = \alpha(\beta)$, and $\delta'(\mu) = -k\alpha(\gamma)$. 

Therefore, over $U-L \subset \tilde{X}$, we have $s' = (g'/g) \cdot (g \cdot \sigma')$, where $\sigma'$ is a section of $\Lambda^2_c(T_X)$ over $U$, and $s'$ is a section of $\Lambda^2_c(T_X)$ over $\tilde{X} - L$. The class of the mapping $g'/g$ becomes equal to $\delta' - \delta \in H^1(U - L; \mathbb{Z})$, and it satisfies $[\delta' - \delta](\mu) = -k\alpha(\gamma)$ and $[\delta' - \delta](\gamma^\#) = [\delta' - \delta](\beta^\#) = 0$. This completes the proof. □

**Lemma 6.** \(\text{sign}(X) = \text{sign}(\tilde{X})\).

**Proof.** Novikov’s additivity theorem implies that if we glue together two compact oriented 4-manifolds along a connected component of their boundaries, the signature of the resulting manifold does not depend on the choice of the gluing map. For the proof, see the end-notes of Ch. 4 in [19]. □

Luttinger surgery can be extended to Lagrangian Klein bottles. Define \(h: U \to U\) as:

\[
h(\theta_x, \theta_y, x, y) = (-\theta_x + \pi, \theta_y, -x, y).
\]

The map \(h\) has the following properties: (1) \(h\) is a fixed-point free involution; (2) \(h^*\omega_0 = \omega_0\); (3) \(h^*x = -x, h^*y = y\), and \(h\) maps the torus \(L\) to itself. Define \(K\) and \(U_K\) as:

\[
K = L/\sim, \quad x_1 \in L \sim x_2 \in L \iff h(x_1) = x_2;
\]

\[
U_K = U/\sim, \quad x_1 \in U \sim x_2 \in U \iff h(x_1) = x_2.
\]

\(K\) is the resulting Lagrangian Klein bottle, and \(U_K\) is symplectomorphic to a tubular neighbourhood of an embedded Lagrangian Klein bottle in a symplectic 4-manifold.

Setting \(m = 0\) and \(n = 1\) in \(f_{0,n}\), we let \(\psi: U \to U\) be the associated symplectomorphism. Since \(h\) commutes with \(f_{0,1}\), there exists a symplectomorphism \(\psi_K\) such that the following diagram commutes:

\[
\begin{array}{ccc}
U & \xrightarrow{\psi} & U \\
\downarrow h & & \downarrow h \\
U_K & \xrightarrow{\psi_K} & U_K.
\end{array}
\]

Here, the vertical arrows identify \(x\) and \(h(x)\). If \(L\) is an embedded Lagrangian Klein bottle in a symplectic manifold \((X, \omega)\), then a tubular neighborhood of \(K\) is symplectomorphic to \(U_K\). We define \(X(K)\) as the symplectic manifold obtained by performing Luttinger surgery on \(X\) with respect to \(K\). This involves removing \(U_K\) from \(X\) and gluing it back along \(U_K - K\) via \(\psi_K\):

\[
X(K) = ((X - K) \cup U_K) / \sim, \quad x_1 \in X - K \sim x_2 \in U_K \iff x_1, x_2 \in U_K - K \text{ and } \psi_K(x_1) = x_2.
\]

Since \(\psi_K\) is a symplectomorphism, \(X(K)\) carries a natural symplectic form.

Both Lemma 4 and Lemma 6 hold for Luttinger surgery on a Klein bottle, and will be used in the proof of Theorem 5 below. Lemma 4 only relies on the algebraic definition of the Euler characteristic, and Lemma 6 on Novikov’s theorem.

**5. Proof of Theorem 5.** Let us first prove (1). We proceed by contradiction. Assume \(L\) is not a Maslov-zero torus but has a non-trivial homology class. Since \(\tilde{X}\) is not Maslov-zero, there exists a simple loop \(\gamma\) on \(L\) with \(\alpha(\gamma) \neq 0\). We perform Luttinger surgery on \(X\) with respect to \(L, \gamma\), and \(k = 1\) (denoted by \(\tilde{X} = X(L, \gamma, 1)\)). It follows from Lemma 5 that

\[
c_1(\tilde{X}) = -\alpha(\gamma)[L].
\]

Let us show that \(\tilde{X}\) is a rational homology K3. Lemma 3 implies \(b_1(\tilde{X}) = 0\) since \(L\) has a non-trivial homology class. Since \(\tilde{X}\) is closed, \(b_1(\tilde{X}) = 0\) implies \(b_3(\tilde{X}) = 0\). Using Lemma 4, we get \(\chi(\tilde{X}) = \chi(\tilde{X})\), leading to \(2 + b_2(\tilde{X}) = 2 + b_2(\tilde{X})\). This implies the rational Betti numbers of \(\tilde{X}\) match those of \(X\). Lemma 6 then gives \(b_2^+(\tilde{X}) = b_2^+(X)\) and \(b_2^-(\tilde{X}) = b_2^-(X)\).
Let us show that $c_1(\tilde{X}) = 0$. Taubes’ theorem states that

$$\text{SW}(s_{-c_1(X)}) = 1.$$ 

Since $L$ is Lagrangian, $c_1(\tilde{X}) \cdot [\tilde{c}] = 0$. Therefore, by Taubes’ theorem, $c_1(\tilde{X}) = 0$.

Since $c_1(\tilde{X}) = 0$, $\tilde{X}$ is spin. Hence, the intersection form of $\tilde{X}$ must be even and isomorphic to $3H \oplus (-2E_8)$ (see Theorem 5.3 in [10]). Since $c_1(\tilde{X}) = 0$, it follows that $[L] \in H^2(\tilde{X}; \mathbb{Z})$ is a torsion class. Using the three copies of $H$, we can find a triplet of surfaces $C_1, C_2, C_3$ within $\tilde{X}$, each with a positive self-intersection number, and such that they are pairwise disjoint and disjoint from $L$. Given the natural identification between $\tilde{X} - L$ and $X - L$, and the fact that $C_i \subset \tilde{X} - L$, we can also view the surfaces $C_i$ as surfaces in $X$. The following properties of $[C_i]$ and $[L]$, as elements in $H_2(X; \mathbb{Z})$, are immediate:

$$[L]^2 = 0, \quad [C_i]^2 > 0, \quad [C_i] \cdot [L] = 0 \text{ for } i = 1, 2, 3, \quad \text{and } [C_i] \cdot [C_j] = 0 \text{ whenever } i \neq j.$$ 

However, such a configuration of non-zero cycles cannot be realized in a space with signature $(3, 19)$.

Let us now prove (2). We proceed by contradiction again. Assume $L$ is nullhomologous modulo 2, but the modulo 2 reduction of $\alpha$ vanishes. From Lemma 2 there exists a Luttinger surgery result $\tilde{X} = X(L, \gamma, k)$ such that $b_1(\tilde{X}) = 0$ and $H^2(\tilde{X}; \mathbb{Z})$ has 2-torsion. Similar to part (1), Lemma 5 gives $c_1(\tilde{X}) = -k\alpha(\gamma)[L]$, and arguments like before establish that $\tilde{X}$ is a rational homology K3 with $c_1(\tilde{X}) = 0$.

Let $\varepsilon \in H^2(\tilde{X}; \mathbb{Z})$ be a non-trivial 2-torsion class. Since $c_1(\tilde{X}) = 0$ and $\varepsilon$ has order 2, the spin$^c$ structure $s_\varepsilon$ has a trivial determinant line bundle. Using the Morgan-Szabó theorem, we conclude that

$$\text{SW}(s_\varepsilon) \neq 0.$$ 

However, this contradicts Taubes’ theorem. Since $\varepsilon$ has order 2, it follows that $\varepsilon \cdot [\tilde{c}] = 0$. Taubes’ theorem states that such $\varepsilon$ must be zero.

6. On Lagrangian Klein bottles. Congruence (3.14) admits a generalization to unoriented characteristic surfaces. See the papers [7] and [9], both in the same volume, edited by Guillou and Marin. Combining with the result of Morgan-Szabó, it leads to the following Klein bottle version of Theorem 1:

**Theorem 5.** Let $(X, \omega)$ be a symplectic 4-manifold homotopy-equivalent to a complex K3 surface, and suppose $c_1(X) = 0$. An embedded Lagrangian Klein bottle $K \subset (X, \omega)$, if exists, must have non-trivial homology class.

**Proof.** Let us show that if $\tilde{X}$ is obtained from $X$ via Luttinger surgery along a Lagrangian Klein bottle $K$, regardless of whether $K$ is nullhomologous or not, then $\tilde{X}$ is a rational homology K3.

To this end, let us first show that $b_1(\tilde{X}) = 0$. Consider a loop $\delta$ in $\tilde{X}$. Perturb it if necessary to ensure it is disjoint from $K$. Denote $2\delta$ as a loop homotopic to a double of $\delta$. We will show that $2\delta$ is homologically trivial in $\tilde{X} - K$ (equivalent to $X - K$). Since $X$ is simply-connected, there exists an orientable immersed surface $C \subset X$ such that $\partial C = 2\delta$. Arrange $C$ to be transverse to $K$ in $X$. This ensures an even number of intersection points between $C$ and $K$. Using Gompf’s argument (Lemma 4.10 in [3]), we can replace $C$ with another orientable surface $C'$ such that $\partial C' = 2\delta$ but $C'$ is disjoint from $K$. This modification shows that $2\delta$ is nullhomologous in $X - K$, hence in $\tilde{X}$. We have established that $b_1(\tilde{X}) = b_1(X)$. 

Lemma 4 applies to Luttinger surgery with Klein bottles, giving \( b_2(X;Z) = b_2(X;Z) \). Lemma 6 also applies to Luttinger surgery with Klein bottles, resulting in \( b_3(X;Z) = b_3(X;Z) \) and \( b_7(X;Z) = b_7(X;Z) \). We have established that \( X \) is a rational homology K3.

Let us show that \( c_1(X;Z) = 0 \). Since \( c_1(X;Z) = 0 \), then over \( X - K, \Lambda^2_b(T_X) \) has a nowhere-vanishing section \( s \). \( K \) is a homotopy retract of \( U_K \), and a complex line bundle over a non-orientable surface like \( K \) must be trivial. Therefore, \( \Lambda^2_b(T_X) \) is trivial over \( U_K \) and is trivialised over \( \partial U_K \) by \( s \). The obstruction to extending \( s \) from \( \partial U_K \) to the entire \( U_K \) lies in \( H^1(U_K, \partial U_K) \). Let us show that this group is trivial. To begin with, consider the long exact sequence for \( (U_K, \partial U_K) \):

\[
\cdots \to H^0(U_K) \to H^0(\partial U_K) \to H^1(U_K, \partial U_K) \to H^1(U_K) \to H^1(\partial U_K) \to \cdots
\]

Here, the coefficients are to be taken in \( \mathbb{Z} \). It suffices to show that \( H^0(U_K) \to H^0(\partial U_K) \) is surjective and \( H^1(U_K) \to H^1(\partial U_K) \) is injective. The first claim follows since \( \partial U_K \) is connected. To prove the second, consider \( U_K \) as a normal bundle to \( K \) in \( X \), with \( K \) as the zero section, and introduce another section \( K_1 \) as follows. Recall that \( K \) being totally real implies \( T_K \cong U_K \). Since \( T_K \) admits a non-vanishing section, so does \( U_K \). Denote this section as \( K_1 \), and arrange that it lies entirely in \( \partial U_K \). Consider the sequence of maps:

\[
K_1 \to \partial U_K \to U_K \to K,
\]

where \( U_K \to K \) is a deformation retraction, and the rest are inclusions. Suppose \( \varphi \in H^1(U_K; \mathbb{Z}) \) is non-trivial but becomes trivial in \( H^1(\partial U_K; \mathbb{Z}) \). Then \( \varphi \) also vanishes when restricted to \( K_1 \). Since \( K_1 \) is isotopic to \( K \) in \( U_K \), it follows that \( \varphi \) must also vanish on \( K \). However, this leads to a contradiction because \( U_K \) retracts onto \( K \), inducing an isomorphism in cohomology. Hence, the map \( H^1(U_K; \mathbb{Z}) \to H^1(\partial U_K; \mathbb{Z}) \) is indeed injective.

Corollary 2.3 in [13] states that Luttinger surgery along a nullhomologous Lagrangian Klein bottle would result in a \( X \) with \( H_1(\tilde{X};\mathbb{Z}) = \mathbb{Z}_2 \). However, we have already seen that the free part of \( H_1(\tilde{X};\mathbb{Z}) \) vanishes. Combining these results implies that \( H_1(\tilde{X};\mathbb{Z}) \) must have 2-torsion elements. It follows from the universal coefficient theorem that \( H^2(\tilde{X};\mathbb{Z}) \) also has 2-torsion elements. We have also established that \( c_1(\tilde{X}) = 0 \). This leads to a contradiction with the results of Taubes and Morgan-Szabó in the same way as in the proof of Theorem [1]. \( \square \)

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