RANDOM MANIFOLDS HAVE NO TOTALLY GEODESIC SUBMANIFOLDS

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Abstract. For $n \geq 4$ we show that generic closed Riemannian $n$–manifolds have no nontrivial totally geodesic submanifolds, answering a question of Spivak. An immediate consequence is a severe restriction on the isometry group of a generic Riemannian metric. Both results are widely believed to be true, but we are not aware of any proofs in the literature.

Schoen-Simon showed that every Riemannian manifold admits an embedded minimal hypersurface ([8], cf. also [7]). Intuition suggests that the analogous result about totally geodesic submanifolds is false. In fact, Spivak writes that it

“seems rather clear that if one takes a Riemannian manifold $(N, \langle \cdot, \cdot \rangle)$ ‘at random’, then it will not have any totally geodesic submanifolds of dimension $> 1$. But I must admit that I don’t know of any specific example of such a manifold.” ([9], p. 39)

The existence of specific examples was established by Tsukada in [10], who found some left-invariant metrics on 3–dimensional Lie groups without totally geodesic surfaces. In the present paper, we prove that Spivak’s intuition about generic metrics is correct for compact Riemannian $n$–manifolds with $n \geq 4$.

**Theorem A.** Let $M$ be a compact, smooth manifold of dimension $\geq 4$. For any finite $q \geq 2$, the set of Riemannian metrics on $M$ with no nontrivial immersed totally geodesic submanifolds contains a set that is open and dense in the $C^q$–topology.

Put another way: in a generic Riemannian $n$–manifold with $n \neq 3$, any totally geodesic submanifold is either a geodesic or the whole manifold. We emphasize that this statement applies to all immersed submanifolds—there is no requirement that the submanifolds be closed or complete.

It also seems clear that a random Riemannian metric has no isometries other than the identity. Theorem A yields a simple proof of this for most group actions.

**Corollary B.** Let $M$ be a compact, smooth manifold of dimension $\geq 4$, and let $G$ be a Lie group that acts smoothly and effectively on $M$. For any finite $q \geq 2$, the set of Riemannian metrics on $M$ that are not $G$–invariant contains a set that is open and dense in the $C^q$–topology, provided any of the following hold.

1. A subgroup of $G$ has a fixed point set of dimension $\geq 2$.
2. $G$ has a subgroup $H$ whose fixed point set is 0 or 1 dimensional, and $H$ does not act freely on a sphere.

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To see how this follows from Theorem A, suppose that $G$ acts isometrically and effectively on a Riemannian manifold that has no nontrivial immersed totally geodesic submanifolds. Then the fixed point sets of $G$ and all of its subgroups have dimension $\leq 1$. If a subgroup $H$ has a one dimensional fixed point set, then since no subgroup of $H$ can have a larger fixed point set, $H$ acts freely on any unit normal sphere to its fixed point set. If no subgroup of $G$ has a one dimensional fixed point set, but some subgroup $H$ has a zero dimensional fixed point set, then differentiating $H$ produces a free action on the unit tangent sphere at any fixed point of $H$. In particular, if $G$ has a subgroup, $H$, with a one dimensional fixed point set, then since no subgroup of $H$ can have a larger fixed point set, $H$ acts freely on any unit normal sphere to its fixed point set. If no subgroup of $G$ has a one dimensional fixed point set, but some subgroup $H$ has a zero dimensional fixed point set, then differentiating $H$ produces a free action on the unit tangent sphere at any fixed point of $H$. In particular, if $G$ has a subgroup, $H$, with a one dimensional fixed point set, then since no subgroup of $H$ can have a larger fixed point set, $H$ acts freely on any unit normal sphere to its fixed point set. If no subgroup of $G$ has a one dimensional fixed point set, but some subgroup $H$ has a zero dimensional fixed point set, then differentiating $H$ produces a free action on the unit tangent sphere at any fixed point of $H$. In particular, if $G$ has a subgroup, $H$, with a one dimensional fixed point set, then since no subgroup of $H$ can have a larger fixed point set, $H$ acts freely on any unit normal sphere to its fixed point set. If no subgroup of $G$ has a one dimensional fixed point set, but some subgroup $H$ has a zero dimensional fixed point set, then differentiating $H$ produces a free action on the unit tangent sphere at any fixed point of $H$.

It seems rather easy to construct a deformation that kills the totally geodesic property for a fixed submanifold or a fixed compact family of submanifolds (see, e.g., [2]). Although there are compactness theorems for submanifolds with constrained geometry in, e.g., [3], the space of all submanifolds of a compact Riemannian manifold is not compact. For example, via the Nash isometric embedding theorem, all Riemannian manifolds of any fixed dimension $k$ embed isometrically into a fixed flat $n$–torus if $n >> k$.

To circumvent this difficulty we propose a new concept called partially geodesic. It is defined in terms of the following invariant of self adjoint linear maps.

**Definition.** Let $\Phi : V \rightarrow V$ be a self adjoint linear map of an inner product space $V$. For a subspace $W$ of $V$, we set

$$I_\Phi (W) \equiv \max_{\{w \in W : |w|=1\}} \left| \Phi (w)^{W^\perp} \right|,$$

where $\Phi (w)^{W^\perp}$ is the component of $\Phi (w)$ that is perpendicular to $W$.

Let $V = T_p M$ be a tangent space to a Riemannian manifold $(M, g)$. For $v \in W \subset T_p M$, the Jacobi operator $R_v = R(\cdot, v)v : T_p M \rightarrow T_p M$ is self adjoint with respect to $g$, and if $I_{R_v} (W) \neq 0$ for some $v \in W$, then $W$ is not tangent to any totally geodesic submanifold. This motivates the following concept.

**Definition.** An $l$–plane, $P$, tangent to $M$ is called partially geodesic if and only for all $v \in P$,

$$I_{R_v} (P) = 0.$$

Theorem A is a consequence of

**Theorem C.** Let $M$ be a compact, smooth manifold of dimension $\geq 4$. For any finite $q \geq 2$, there is set of Riemannian metrics on $M$ that is open and dense in the $C^q$–topology that has no partially geodesic $l$–planes for every $l \in \{2, 3, \ldots, n-1\}$.

The $C^q$–closure of the space of Riemannian metrics inside of the space of symmetric $(0, 2)$–tensors is a complete metric space (see, e.g., Theorem 4.4 on page 62 of [4]). Combining this with the Baire Category Theorem, we see that Theorem C is a consequence of the following statement, which, a priori, is weaker.
**Theorem D.** Let $M$ be a compact, smooth manifold of dimension $\geq 4$. For any finite $q \geq 2$ and any $l \in \{2, 3, \ldots, n-1\}$, the set of Riemannian metrics on $M$ with no partially geodesic planes is open and dense in the $C^q$-topology.

We will give a direct proof of Theorem C that is only slightly more involved than the corresponding proof of Theorem D. The direct proof has the advantage of yielding a construction that is a little more explicit. We execute this using reverse induction on $l$ via the following induction statement.

$l$th Partially Geodesic Assertion. Given $l \in \{2, 3, \ldots, n-1\}$, a finite $q \geq 2$, and $\xi > 0$, there is a Riemannian metric $\tilde{g}$ on $M$ that has no partially geodesic $k$-planes for all $k \in \{l, l+1, \ldots, n-1\}$ and satisfies $|\tilde{g} - g|_{C^q} < \xi$.

The rest of the paper is devoted to proving this assertion. To do so, we exploit a principle given in the following lemma.

**Lemma E.** Let $\{g_s\}_{s \geq 0}$ be a smooth family of Riemannian metrics on $M$. Let $R^s$ be the curvature tensor of $g_s$. Let $\mathcal{P}_0$ be the set of partially geodesic $l$-planes for $g_0$, and suppose that for all $k \in \{l+1, \ldots, n-1\}$, $g_0$ has no partially geodesic $k$-planes.

Suppose further that there are $c, s_0 > 0$ and a neighborhood $U_0$ of $\mathcal{P}_0$ so that for every $P \in U_0$, every $s \in (0, s_0)$, and some $g_0$-unit $v \in P$,

$$I_{R^s_0} (P) > cs.$$ \hspace{1cm} (0.0.2)

Then for all sufficiently small $s$, and all $k \in \{l, \ldots, n-1\}$, $(M, g_s)$ has no partially geodesic $k$-planes.

**Proof.** We write $\mathcal{G}_k (M)$ for the Grassmannian of $k$-planes tangent to $M$. Since each $\mathcal{G}_k (M)$ is compact, there is a $\delta > 0$ so that for all $k \in \{l+1, \ldots, n-1\}$ and all $P \in \mathcal{G}_k (M)$ there is a unit $v \in P$ so that

$$I_{R^s_0} (P) > \delta.$$ \hspace{1cm} (0.0.2)

Similarly $\mathcal{G}_l (M) \setminus U_0$ is compact. Thus there is a (possibly different) $\delta > 0$ so that for all $P \in \mathcal{G}_l (M) \setminus U_0$ there is a unit $u \in P$ so that

$$I_{R^s_0} (P) > \delta.$$ \hspace{1cm} (0.0.2)

By combining the previous two displays with Inequality (0.0.2) and a continuity argument, it follows that for all sufficiently small $s$, $(M, g_s)$ has no partially geodesic $l$-planes. \hspace{1cm} $\Box$

In Section 1 we establish notations and conventions. In Section 2 we prove Lemma 2.1 which implies that the $l$th Partially Geodesic Assertion holds locally, in a sense that is quantifiable. This allows us, in Section 3, to piece together various local deformations and complete the proof of Theorem C. We do not use Lemma E explicitly, but the reader will notice that a similar principle is used in our global argument in Section 3.

For a quick overview of the proof, imagine that $v$ and $T$ are tangent to a partially geodesic plane $P$ and $n \perp P$. The strategy is to change $\langle T, n \rangle$ by a function $f$ that has a relatively large 2nd derivative in the $v$-direction. This has the effect of giving $R(T, v) v$ a component
in the $n$–direction. In particular, $P$ is no longer partially geodesic. Since this is a local deformation, $f$ has compact support and necessarily has inflection points. To deal with this, we simultaneously change two components of the metric tensor using two functions whose inflection points occur at different places. Since this construction requires the presence of two distinct orthonormal triples, it only works in dimensions $\geq 4$. Our sense is that a modification of our ideas might also yield a proof of Theorems A and C in dimension 3. In fact, Bryant has outlined a local proof in [2].

Remark. As mentioned above, Theorem A is widely believed to be true. In [1], Berger wrote (without proof) “a generic Riemannian manifold does not admit any such submanifold”.

Hermann states in [5] that Theorem A should be true but that there is little research in this direction.

Remark. In [8], Schoen-Simon showed that every Riemannian $n$–manifold admits an embedded minimal hypersurface. If $n \geq 8$, the Schoen-Simon construction can lead to minimal hypersurfaces with singularities. By contrast, Theorem A rules out the possibility of a generic metric having any totally geodesic submanifold, complete or otherwise. In particular, generic Riemannian manifolds, of dimension $\geq 4$, have no totally geodesic submanifolds with singularities.

Remark. Theorem A asserts that the set of metrics with no totally geodesic submanifolds has nonempty interior. On the other hand, Theorem C says that the set of metrics with no partially geodesic submanifolds is an actual open set in the $C^q$–topology. The latter assertion follows from the observations that $I_{R^q}$ is continuous and that Grassmannians of compact manifolds are compact. It is not clear to us whether the set of metrics with no totally geodesic submanifolds is open. As mentioned above, one difficulty is that the space of isometrically embedded $k$–manifolds in an $n$–manifold is not compact.

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1. Notations and Conventions

Throughout, $(M, g)$ will be a smooth, connected compact Riemannian manifold of dimension $n \geq 4$. We will denote the Levi-Civita connection, curvature tensor, and Christoffel symbols by $\nabla$, $R$, and $\Gamma$, respectively. We adopt the sign convention that $R_{xyyx}$ is the sectional curvature of a plane spanned by orthonormal $x, y \in T_pM$. Thus the Jacobi operator is $R_v = R(\cdot, v)v : T_pM \to T_pM$. For a nearby metric $\tilde{g}$, the corresponding objects will be denoted $\tilde{\nabla}$, $\tilde{R}$, and $\tilde{\Gamma}$, respectively.

Given local coordinates $\{x_i\}_{i=1}^n$, define $\partial_i$ to be the partial derivative in the direction $\frac{\partial}{\partial x_i}$. At times the notation $\partial_{x_i}$ will also be used for the same object. We let $G_l(M)$ denote the Grassmannian of $l$–planes in $M$, and $\pi : G_l(M) \to M$ the projection of $G_l(M)$ to $M$. We fix a Riemannian metric on $G_l(M)$ so that $\pi : G_l(M) \to (M, g)$ is a Riemannian submersion with totally geodesic fibers that are isometric to the Grassmannian of $l$–planes in $\mathbb{R}^n$. For a metric space $X$, $A \subset X$, and $r > 0$, we let

$$B(A, r) \equiv \{x \in X \mid \text{dist} (x, A) < r\}.$$  

For some $l \in \{2, \ldots, n-1\}$ we let $P_0$ be the set of partially geodesic $l$–planes for $g$. 
2. The Local construction

In this section, we prove Lemma 2.1, which can be viewed as a local version of the \( l \)-th Partially Geodesic Assertion. In Section 3, we exploit the fact that \( \mathcal{P}_0 \) is compact and apply Lemma 2.1 successively to each element of a finite open cover \( \{O_i\}_i \) of \( \mathcal{P}_0 \). This will produce a finite sequence of metrics \( g_1, g_2, \ldots, g_k \) where, for example, \( g_2 \) is obtained by applying Lemma 2.1 to \( g_1 \). The idea is that Lemma 2.1 kills the partially geodesic property on \( O_k \) while simultaneously preserving it on \( \bigcup_{i=1}^{k-1} O_i \). In particular, the set of partially geodesic \( l \)-planes for \( g_k \) is contained in \( \bigcup_{i=k+1}^{G} O_i \).

Because of the successive nature of our construction, in Lemma 2.1 we construct a deformation, not of \( g \), but rather of an abstract metric, \( \hat{g} \), that is \( C^2 \)-close to \( g \).

**Lemma 2.1.** Given \( K, \eta > 0, P \in \mathcal{P}_0 \), and sufficiently small \( \varepsilon_0, \rho > 0 \), there is a \( \xi > 0 \) so that if

\[
|g - \hat{g}|_{C^2} < \xi,
\]

then there is a \( C^\infty \)-family of metrics \( \{g_s\}_{s \in [0, \varepsilon_0]} \) so that the following hold.

1. For all \( s \), \( g_s = \hat{g} \) on \( M \setminus B(\pi(P), \rho + \eta) \), and \( g_0 = \hat{g} \).
2. Let \( \sigma(P) \) be the section of \( \mathcal{G}_1(\pi(P), \rho) \) determined by \( P \) via normal coordinates at \( \pi(P) \) with respect to \( g \). For all

\[
\tilde{P} \in \pi^{-1} (B(\pi(P), \rho)) \cap B(\sigma(P), \rho),
\]

there is a \( v \in \tilde{P} \) so that

\[
|\mathcal{I}_{R^s_v}(\tilde{P}) - \mathcal{I}_{R^0_v}(\tilde{P})| > Ks.
\]

Here \( R^s \) and \( R^0 \) are the curvature tensors of \( g_s \) and \( \hat{g} \), respectively.

3. For all \( \tilde{P} \in \mathcal{G}_1(M) \) and all \( v \in \tilde{P} \),

\[
|\mathcal{I}_{R^s_v}(\tilde{P}) - \mathcal{I}_{R^0_v}(\tilde{P})| \leq 2Ks.
\]

4. For all \( \tilde{P} \in \mathcal{G}_1(M) \setminus \{\pi^{-1}(B(\pi(P), \rho + \eta)) \cap B(\sigma(P), \rho + \eta)\} \) and \( w \in \tilde{P} \),

\[
|\mathcal{I}_{R^s_w}(\tilde{P}) - \mathcal{I}_{R^0_w}(\tilde{P})| \leq \varepsilon_0 s.
\]

We will not need Parts 3 and 4 to prove Theorem A, but have included them since they are obtained relatively easily and seem to be of independent interest.

The proof of Lemma 2.1 occupies the rest of this section and starts with some preliminary results.

**Lemma 2.2.** Given \( K, \varepsilon, \eta > 0 \), and \( P \in \mathcal{P}_0 \), there are coordinate neighborhoods \( N \) and \( G \) of \( \pi(P) \) and \( C^\infty \) functions \( f_1, f_2 : M \to \mathbb{R} \) with the following properties.

1. \( \text{dist} (N, M \setminus G) < \eta \).

2. On \( N \), the second partial derivatives in the first coordinate direction satisfy

\[
\max \{|\partial_1 \partial_1 f_1|, |\partial_1 \partial_1 f_2|\} > 2K.
\]
In general, 
\[ \max \{ |\partial_1 \partial_1 f_1|, |\partial_1 \partial_1 f_2| \} \leq 4K. \]

4. For \( j \in \{1, 2, \ldots, n\}, k \in \{2, \ldots, n\}, \) and \( i \in \{1, 2\}, \)
\[ |\partial_j \partial_k f_i| < \varepsilon \]
and
\[ |f_i|_{C^1} < \varepsilon. \]

5. On \( M \setminus G, f_1 = f_2 = 0. \)

This follows by composing the coordinate chart of \( G \) with the functions on Euclidean space given by the next lemma.

**Lemma 2.3.** Let \( \pi_1 : \mathbb{R}^n \rightarrow \mathbb{R} \) be orthogonal projection onto the first factor. Let \( C \) be a compact subset of \( \mathbb{R}^n \) with \( \pi_1(C) = [a, b] \), for \( a, b \in \mathbb{R} \). Given \( K, \varepsilon > 0 \) and a compact set \( \tilde{C} \) with \( C \subset \text{int} (\tilde{C}) \), there are \( C^\infty \) functions \( f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \) with the following properties.

1. On \( C, \)
\[ \max \{ |\partial_1 \partial_1 f_1|, |\partial_1 \partial_1 f_2| \} > 2K. \]

2. In general, 
\[ \max \{ |\partial_1 \partial_1 f_1|, |\partial_1 \partial_1 f_2| \} \leq 4K. \]

3. For \( j \in \{1, 2, \ldots, n\}, k \in \{2, \ldots, n\}, \) and \( i \in \{1, 2\}, \)
\[ |\partial_j \partial_k f_i| < \varepsilon \]
and
\[ |f_i|_{C^1} < \varepsilon. \]

4. On \( \mathbb{R}^n \setminus \tilde{C}, f_1 = f_2 = 0. \)

**Proof.** Let \( \chi : \mathbb{R}^n \rightarrow [0, 1] \) be \( C^\infty \) and satisfy
\[ \chi|_C \equiv 1, \]
\[ \chi|_{\mathbb{R}^n \setminus \tilde{C}} \equiv 0. \]

Let \( M > 1 \) satisfy
\[ |\chi|_{C^2} \leq M. \]

For \( 0 < 5\tau < \varepsilon, \) let \( h_1 : \mathbb{R} \rightarrow \mathbb{R} \) be \( C^\infty \) and satisfy
\[ |h_1|_{C^1} < \frac{\varepsilon}{2M}. \]

In addition, we require that
\[ 3K \leq |h_1''(t)| \leq \frac{7}{2}K, \]
except if
\[ t \in \bigcup_{l \in \mathbb{Z}} \left( \frac{\varepsilon}{10 \cdot MK} - \tau, \frac{\varepsilon}{10 \cdot MK} + \tau \right), \]
which is a union of disjoint, small intervals that contain the inflection points of \( h_1. \) Setting
\[ h_2(t) = h_1(t - 5\tau) \]
and

\[ f_i(p) = \chi(p) \cdot (h_i \circ \pi_1)(p) \]

gives us the desired functions. \qed

Let \( P \in \mathcal{P}_0 \) be as in Lemma 2.1 and have foot point \( p \). If \( l = 2 \), let \( \{v, T, n_3, n_4\} \) be an ordered orthonormal quadruplet at \( p \) with

\[ \{v, T\} \in P \text{ and } \{n_3, n_4\} \text{ normal to } P. \tag{2.3.1} \]

If \( l = n - 1 \), let \( \{v, T_3, T_4\} \) be an ordered orthonormal quadruplet at \( p \) with

\[ \{v, T_3, T_4\} \in P \text{ and } n \text{ normal to } P. \tag{2.3.2} \]

If \( l \in \{3, \ldots, n - 2\} \), let \( \{E_i\}_i \) be an ordered orthonormal quadruplet at \( p \) that satisfies either (2.3.1) or (2.3.2). In either case, extend the ordered quadruplet to a coordinate frame \( \{E_i\}_i \).

Choose \( g_s \) so that with respect to the ordered frame \( \{E_i\}_i \), the matrix of \( g_s - \hat{g} \) is 0 except for the upper \((4 \times 4)\)-block which is

\[ \{g_s - \hat{g}\}_{l,m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & s f_1 & s f_2 \\ 0 & s f_1 & 0 & 0 \\ 0 & s f_2 & 0 & 0 \end{pmatrix} \tag{2.3.3} \]

where to construct \( f_1 \) and \( f_2 \) we apply Lemma 2.2 with \( N = B(\pi(P), \rho) \) and \( G = B(\pi(P), \rho + \eta) \).

To simplify notation, we write \( \tilde{g} \) for \( g_s \) and use \( \tilde{\ } \) for objects associated to \( \tilde{g} \). Recall (see, e.g., [6]) that with respect to \( \{E_i\}_i \), the Christoffel symbols are

\[ \tilde{\Gamma}_{ij,k} \equiv \tilde{g} \left( \nabla_{E_i} E_j, E_k \right) \]

and

\[ \tilde{R}_{ijkl} = \partial_l \tilde{\Gamma}_{jk,i} - \partial_l \tilde{\Gamma}_{ik,j} + \tilde{g}^{\sigma\tau} \left( \tilde{\Gamma}_{ik,\sigma} \tilde{\Gamma}_{jl,\tau} - \tilde{\Gamma}_{jk,\sigma} \tilde{\Gamma}_{il,\tau} \right), \tag{2.3.4} \]

where \( \tilde{g}^{\sigma\tau} \) are the coefficients of the inverse \((\{\tilde{g}\}_{\sigma\tau})^{-1} \) of \( \{\tilde{g}\}_{\sigma\tau} \), and the Einstein summation convention is being used. Combining Lemma 2.2 with the definition of \( \tilde{g} \) gives us

**Proposition 2.4.** The coefficients \( \hat{g}^{l,m} \) and \( \tilde{g}^{l,m} \) of the inverses of \( \{\hat{g}\}_{l,m} \) and \( \{\tilde{g}\}_{l,m} \) satisfy

\[ |\hat{g}^{l,m} - \tilde{g}^{l,m}| < O(\varepsilon). \]

Using Equation (2.3.3) and Lemma 2.2, we will show

**Proposition 2.5.** Writing \( \left( \hat{\Gamma} - \tilde{\Gamma} \right)_{jk,l} \) for \( \hat{\Gamma}_{ijk} - \tilde{\Gamma}_{ijk} \) we have

\[ \left| \left( \hat{\Gamma} - \tilde{\Gamma} \right)_{jk,l} \right| < O(\varepsilon). \tag{2.5.1} \]

Let \( i, j, k, l \) be arbitrary elements of \( \{1, 2, \ldots, n\} \). Then all expressions

\[ \partial_i \left( \hat{\Gamma} - \tilde{\Gamma} \right)_{jk,l} \]
are $\leq O(\varepsilon s)$ except for

$$
\begin{align*}
\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{2v,3} &= \partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{v3,2} = -\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{23,v} \\
\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{v2,3} &= \partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{3v,2} = -\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{32,v}
\end{align*}
$$

(2.5.2)

and

$$
\begin{align*}
\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{2v,4} &= \partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{v4,2} = -\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{24,v} \\
\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{v2,4} &= \partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{4v,2} = -\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{42,v},
\end{align*}
$$

(2.5.3)

where we write $v$ for the first element of our frame to emphasize its special role. The expressions in (2.5.2) and (2.5.3) are $\leq 2Ks$ everywhere, and on $N$,

$$\max \{2.5.2, \, 2.5.3\} \geq Ks.$$

Proof. Inequality (2.5.1) follows from the fact that $|s f_i|_{C^1} < \varepsilon s$.

To prove the remainder note

$$
\partial_l \tilde{\Gamma}_{j,k,l} = \partial E_i \tilde{\Gamma} \left( \nabla E_j E_k, E_l \right)
$$

$$
= \frac{1}{2} \partial E_i \left[ \partial E_k \tilde{\Gamma} (E_j, E_l) + \partial E_j \tilde{\Gamma} (E_i, E_k) - \partial E_i \tilde{\Gamma} (E_k, E_j) \right]
$$

$$
= \frac{1}{2} \partial E_i \left[ \partial E_k \tilde{\Gamma} (E_j - \hat{\Gamma}) (E_i, E_l) + \partial E_j \tilde{\Gamma} (E_i - \hat{\Gamma}) (E_i, E_k) - \partial E_i \tilde{\Gamma} (E_k - \hat{\Gamma}) (E_i, E_j) \right]
$$

$$
+ \frac{1}{2} \partial E_i \left[ \partial E_k \tilde{\Gamma} (E_j, E_i) + \partial E_j \tilde{\Gamma} (E_i, E_k) - \partial E_l \tilde{\Gamma} (E_k, E_l) \right].
$$

Combining this with Lemma 2.2 and the definition of $\tilde{\Gamma}$ gives us

$$
\partial_l \tilde{\Gamma}_{j,k,l} = \partial E_i \tilde{\Gamma} \left( \nabla E_j E_k, E_l \right) + O(\varepsilon s)
$$

$$
= \partial_l \tilde{\Gamma}_{j,k,l} + O(\varepsilon s),
$$

unless the indices correspond to the situation in (2.5.2) or (2.5.3). In the former case,

$$
\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{2v,3} = \frac{1}{2} \partial E_v \left[ \partial E_v \tilde{\Gamma} (E_i - \hat{\Gamma}) (E_2, E_3) + \partial E_2 \tilde{\Gamma} (E_i - \hat{\Gamma}) (E_3, E_v) - \partial E_3 \tilde{\Gamma} (E_i - \hat{\Gamma}) (E_v, E_2) \right]
$$

$$
= \frac{s}{2} \partial E_v \partial E_v (f_1).
$$

In the case of (2.5.3) we have

$$
\partial_v \left( \tilde{\Gamma} - \hat{\Gamma} \right)_{2v,4} = \frac{1}{2} \partial E_v \left[ \partial E_v \tilde{\Gamma} (E_i - \hat{\Gamma}) (E_2, E_4) + \partial E_2 \tilde{\Gamma} (E_i - \hat{\Gamma}) (E_4, E_v) - \partial E_4 \tilde{\Gamma} (E_i - \hat{\Gamma}) (E_v, E_2) \right]
$$

$$
= \frac{s}{2} \partial E_v \partial E_v (f_2).
$$

The result follows by combining the previous three displays with Lemma 2.2.
Proof of Lemma 2.1. Combining Propositions 2.4 and 2.5 with Equation (2.3.4) we see that
\[ \left| \left( \tilde{R} - \hat{R} \right)_{ijkl} \right| \leq O(\varepsilon s), \]
except if the quadruple corresponds, up to a symmetry of the curvature tensor, to either \( \left( \tilde{R} - \hat{R} \right)_{2v3} \) or \( \left( \tilde{R} - \hat{R} \right)_{2v4} \), in which case we have
\[ \left( \tilde{R} - \hat{R} \right)_{2v3} = \partial_v \partial_v (f_1) + O(\varepsilon s) \text{ and} \]
\[ \left( \tilde{R} - \hat{R} \right)_{2v4} = \partial_v \partial_v (f_2) + O(\varepsilon s). \]

Lemma 2.1 follows from the previous two equations and our choices of \( f_1 \) and \( f_2 \), provided \( \varepsilon \) is sufficiently small. \( \square \)

3. The Global Construction

In this section, we prove the \( l^{th} \)-Partially Geodesic Assertion by reverse induction, starting with the case when \( l = n - 1 \). The strategy is to apply Lemma 2.1 successively to the elements of an open cover of \( \mathcal{P}_0 \). When \( l = n - 1 \), this is all that is needed. Otherwise, as in the proof of Lemma 2.1, we note that for each \( k \in \{l + 1, \ldots, n - 1\} \), \( \mathcal{G}_k(M) \) is compact. By our induction hypothesis, there is a \( \delta > 0 \) so that for all \( P \in \mathcal{G}_k(M) \) there is a unit \( v \in P \) with
\[ I_{R^0_{\varepsilon}}(P) > \delta. \]
Thus all sufficiently small deformations of \( g \) have no partially geodesic \( k \)-planes for all \( k \in \{l + 1, \ldots, n - 1\} \). In particular, the \( l^{th} \)-Partially Geodesic Assertion follows from the

Modified \( l^{th} \)-Partially Geodesic Assertion. Given \( l \in \{2, 3, \ldots, n - 1\} \), a finite \( q \geq 2 \), and \( \xi > 0 \), there is a Riemannian metric \( \tilde{g} \) on \( M \) with
\[ |\tilde{g} - g|_{C^q} < \xi \]
that has no partially geodesic \( l \)-planes.

Proof. Given \( K > 0 \), we combine Lemma 2.1 with the compactness of \( \mathcal{P}_0 \) to see that there is a finite open cover \( \{\pi^{-1} (B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i)\}_{i=1}^G \} \) of \( \mathcal{P}_0 \) whose elements satisfy the conclusion of Lemma 2.1. In particular, for each \( i \in \{2, 3, \ldots, G\} \), there is a \( \xi_i > 0 \) so that if
\[ |g - \tilde{g}|_{C^2} < \xi_i, \]
then the conclusion of Lemma 2.1 holds on \( \pi^{-1} (B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i) \). Set
\[ \xi = \min \{\xi_i\}. \]
Since \( \mathcal{G}_l(M) \setminus \left\{ \bigcup_{i=1}^G \pi^{-1} (B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i) \right\} \) is compact, there is an \( \delta > 0 \) so that for all
\[ P \in \mathcal{G}_l(M) \setminus \left\{ \bigcup_{i=1}^G \pi^{-1} (B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i) \right\}, \]

\[ I_{R^0_{\varepsilon}}(P) > \delta. \]
there is a \( v \in P \) so that
\[
|I_{RG}(P)| > \delta. \tag{3.0.1}
\]

We will successively apply Lemma 2.1 to the \( \pi^{-1}(B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i) \) and get a sequence of metrics \( g_1, g_2, \ldots, g_G \). To obtain \( g_1 \), we apply Lemma 2.1 with \( g = \hat{g}, P = P_1 \), and \( \rho = \rho_1 \). This yields a deformation \( g_1 \) of \( g \). Let \( P_s \) be the set of partially geodesic \( l \)-planes for \( g_s \). It follows from Part 2 of Lemma 2.1 that for all sufficiently small \( s \),
\[
P_s \cap \pi^{-1}(B(\pi(P_1), \rho_1)) \cap B(\sigma(P_1), \rho_1) = \emptyset. \tag{3.0.2}
\]

By combining (3.0.1) and (3.0.2), we see that for sufficiently small \( s \),
\[
P_s \subset \bigcup_{i=2}^{G} \pi^{-1}(B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i).
\]

Moreover, by further restricting \( s \), we can ensure that \( g_s \) is close enough to \( g \) in the \( C^2 \)-topology so that
\[
|g_s - g|_{C^2} < \xi.
\]

We let \( g_1 = g_s \) for some \( s \) as above. Assume, by induction, that for some \( k \in \{1, \ldots, G - 1\} \), we have constructed a metric \( g_k \) so that the following hold:

\textbf{(Hypothesis 1)} If \( P_{g_k} \) is the set of \( l \)-dimensional partially geodesic subspaces for \( g_k \), then
\[
P_{g_k} \subset \bigcup_{i=k+1}^{G} \pi^{-1}(B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i).
\]

\textbf{(Hypothesis 2)}
\[
|g_k - g|_{C^2} < \xi.
\]

It follows from Hypothesis 1 that there is an \( \delta > 0 \) so that for all
\[
P \in G_l(M) \setminus \left\{ \bigcup_{i=k+1}^{G} \pi^{-1}(B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i) \right\},
\]
there is a \( v \in P \) so that
\[
|I_{RG}(P)| > \delta. \tag{3.0.3}
\]

Since \( |g_k - g|_{C^2} < \xi \), we can apply Lemma 2.1 with \( \hat{g} = g_k \), \( P = P_{k+1} \), and \( \rho = \rho_{k+1} \). This yields a deformation \( g_s \) of \( g_k \) so that for all sufficiently small \( s > 0 \),
\[
|g_s - g|_{C^2} < \xi.
\]

In other words, Hypothesis 2 holds.

To establish Hypothesis 1, combine Part 2 of Lemma 2.1 with (3.0.3) to see that \( g_s \) has no partially geodesic \( l \)-dimensional subspaces in
\[
G_l(M) \setminus \left\{ \bigcup_{i=k+2}^{G} \pi^{-1}(B(\pi(P_i), \rho_i)) \cap B(\sigma(P_i), \rho_i) \right\},
\]
provided \( s \) is positive and sufficiently small. So Hypothesis 1 also holds. \( \square \)
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