A CATEGORICAL REDUCTION SYSTEM FOR LINEAR LOGIC

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Abstract. We build calculus on the categorical model of linear logic. It enables us to perform diagram chasing as essentially one-way computations led by rewriting rules. We verify the termination property of the calculus.

1. Introduction

This work starts with a naive idea that diagram chase in category theory may be mechanized, at least in specific cases. Construction of commutative diagrams to show equality of two morphisms is by no means straightforward. The automation of the task will thus be taken favorably. Monoidal categories and symmetric monoidal categories admit coherence theorem ascertaining that any parallel syntactic morphisms are automatically equal [22, 19]. Autonomous categories and ⋆-autonomous categories do not have strict coherence but, through graphical presentation, checking equality of two morphisms can be automated, though intractable [20, 7, 16, 15].

Categorical semantics of type theories provide equivalence between type systems and certain categories [17]. A decisively classic work is the semantics of the simply typed lambda calculus using the cartesian closed category [21]. It shows exact correspondence between $\beta\eta$-equal lambda terms and commutative diagrams coming from an adjunction. Unfortunately, the equivalence is valid only after the process of calculation is ignored. The lambda calculus is a computational system as the name suggests. Equality between two lambda terms can be automatically checked through mechanical computation by $\beta\eta$-reduction. To ensure the equivalence between the calculus and the category, however, we have to identify all terms occurring during the calculation. So dynamic content of the calculus is lost in the categorical semantics.

Existence of computation in the side of the lambda calculus suggests that the corresponding cartesian closed category may well be given a dynamic computational mechanism. The $\beta\eta$-equality in the lambda calculus corresponds to adjunction $(-) \times B \dashv B \to (-)$. For example, the $\beta$-equivalence corresponds to a triangle diagram of the adjunction:

$$
\begin{array}{ccc}
A \times B \xrightarrow{\text{abs}} (B \to A \times B) \times B \\
\downarrow \circ \downarrow \text{ev} \\
A \times B
\end{array}
$$

where $\circ$ denotes that two legs are equal. When we regard this diagram as reduction, we...
modify it as
\[
\begin{array}{c}
A \times B \xrightarrow{\text{abs}} (B \to A \times B) \times B \\
\downarrow \text{ev} \quad \downarrow 1 \\
A \times B
\end{array}
\]

where the 2-cell double arrow means rewriting. The morphism \((\text{abs} \times 1_B); \text{ev}\) contracts to \(1_{A\times B}\). This idea, essentially due to Seely [26], looks sheer natural, but does not seem to be pursued further. Previous works by Seely [26] and Jay [18] construct 2-categories employing ordinary lambda terms, not directly on categories.

We install calculus on the categorical semantics of the linear logic. In the lambda calculus, when an argument of a function is accessed \(n\) times, its \(n\) copies are created to be substituted simultaneously. The duplication process is, however, encapsulated in the \(\beta\)-reduction rule, inseparable from other operations. The linear logic isolates duplication so that the timing and the amount of it can be controlled. The categorical semantics of the linear logic has symmetric monoidal adjunction \((-) \otimes B \dashv B \to (-)\) equipped with comonad \(!A\) and commutative coalgebra structure on it. The \(\beta\)-reduction of the linear logic becomes
\[
\begin{array}{c}
A \otimes B \xrightarrow{\text{abs}} (B \to A \times B) \otimes B \\
\downarrow \text{ev} \quad \downarrow 1 \\
A \otimes B
\end{array}
\]

Alternatively, if we take a \(*\)-autonomous category as its base,
\[
\begin{array}{c}
1 \otimes A \xrightarrow{\gamma} (A \triangleright A^*) \otimes A \\
\downarrow \text{ev} \quad \downarrow \gamma \\
A \triangleright A \triangleright (A^* \otimes A)
\end{array}
\]

Among the defining diagrams of the categorical semantics, twenty-one diagrams are regarded as reduction rules. For example,
\[
\begin{array}{c}
!A \xrightarrow{\delta} !!A \\
\downarrow \gamma \quad \downarrow \delta \\
!!A \xrightarrow{\text{obj}} !!!A
\end{array}
\]

replaces one of the defining commutative diagrams of comonad. The naturality of certain morphisms is also replaced with rewriting rules. For example, naturality of the diagonal of the comonoid gives rise to
\[
\begin{array}{c}
!A \xrightarrow{!f} !B \\
\downarrow d \quad \downarrow d \\
!A \otimes !A \xrightarrow{!f \otimes !f} !B \otimes !B
\end{array}
\]
that realizes duplication. These rules in addition to $\beta\eta$-rules provide twenty-three reduction rules in total. It is our contention that the categorical model of the lambda calculus is too coarse to incorporate a computational system in it. If we clearly separate copying from others using the linear logic, we can directly implement rewriting on a category so that the obtained calculus has desirable properties.

Our purpose is, however, not to transcribe a carbon copy of the type system in a category. We build calculus worth existing in its own right. The linear logic allows finer control on duplication than the lambda calculus, yet the unit of substitutions is coarse. A term is duplicated in one stroke no matter how large it is. To ameliorate the situation graphical reduction systems have been considered [14]. Each link in a graph can be individually duplicated so that optimal efficiency is attained by ultimate usage of sharing. However, graphical systems have a drawback that arbitrary connections of links do not form a syntactically lawful graph in general. Moreover, it is not obvious how to ensure that the graphs occurring in the process of rewriting remains meaningful. A system by Ghani [12] and one by Asperti [1] are examples of calculus inspired by the category theory. The former is term rewriting and the latter is graph rewriting.

Our categorical rewriting system lies between term rewriting and graph rewriting. It enables fine-grained control on resources. We can dissect terms in order to duplicate them piece by piece. A morphism of the form $!f$ corresponds to a box in the linear logic. The linear logic has no function to split boxes, thus a box $!(g; f)$ must be copied as an assembled unit. In contrast our system permits to decompose it into $!g; !f$ to activate partial duplication $!f; d \Rightarrow d; (!f \otimes !f)$ by naturality of the diagonal. A similar property is presented in a system based on lambda terms by Jay [18]. For graph rewriting, in contract, extremely fine control is enabled since copying per link is allowed. However, there is a risk that the intermediate graphs appearing in computation may lose semantical justification. As our system deals with only those which are meaningful as morphisms, the duplicated unit $f$ always keeps semantical meaning. The feature of our calculus is that it is a rewriting system on the entities that have mathematical "meaning" whilst finer control on duplication is enabled than ordinary term rewriting.

Early works that view reductions as 2-cells are [26, 25]. Seely and Jay constructed 2-categories from lambda terms as mentioned above [26, 18]. A graphical system based on the categorical semantics is [1, 2]. The categorical abstract machine [9] is a virtual machine based on categorical combinators [10]. To the author's best knowledge, there are no previous works building computational systems directly on categories.

Our system satisfies the properties that computational systems require. We verify normalizability in this paper. Confluence is discussed in a forthcoming paper.

2. A categorical reduction system

Let us start with the definition of categorical models of the linear logic. Among several equivalent definitions [5, 6, 24], we take the following [23, Prop.25]. A reason for this choice is that we can write down all defining conditions as commutative diagrams.
2.1. Definition. A linear category is a pair of a category \( C \) and a functor \( ! : C \to C \) with the following additional structures:

(i) \( C \) is equipped with either the structure of symmetric monoidal closed category \((C, \otimes, 1, -, \circ)\) or the structure of a \( \star \)-autonomous category \((C, \otimes, \&\!, 1, \bot, (-)^\ast)\).

(ii) \( ! \) is equipped with the structure of a symmetric monoidal functor \((!, \tilde{\varphi}, \varphi_0)\).

(iii) \( ! \) is equipped with the structure of a comonad \((!, \delta, \varepsilon)\) where \( \delta : ! \to !! \) and \( \varepsilon : ! \to Id \) are monoidal natural transformations.

(iv) Objects \( !A \) are equipped with the structure of a commutative comonoid \((!A, d_A, e_A)\) where collectively \( d_A : !A \to !A \otimes !A \) and \( e_A : !A \to 1 \) are monoidal natural transformations in \( A \).

Moreover, these structures are related in the following way:

(v) Each \( d_A \) and each \( e_A \) give rise to coalgebra morphisms when \( !A, 1 \), and \( !A \otimes !A \) are naturally regarded as coalgebras.

(vi) Each \( \delta_A \) is a comonoid morphism.

We consider the model based on the symmetric monoidal closed category if we are interested in the intuitionistic linear logic; the \( \star \)-autonomous category if we focus on the classical linear logic. In this paper, we consider only the linear categories freely generated from a set of atomic objects.

We regard the free linear category as a syntactic structure. Our goal is to develop a dynamic calculus installed directly on the category. In analogy to type systems, objects of the category correspond to types and morphisms to terms. As usual type theories are designed as rewriting systems of terms, our calculus is realized as rewriting systems of morphisms.

If we consider the free linear category based on the symmetric monoidal closed category, the set of objects are generated by

\[
A ::= X \mid 1 \mid A \otimes A \mid A \circ A \mid !A
\]

where \( X \) ranges over a given set of atomic objects. Atomic morphisms are identities and structural isomorphisms:

\[
A \xrightarrow{1_A} A
\]

\[
(A \otimes B) \otimes C \xrightarrow{\alpha_{ABC}} A \otimes (B \otimes C) \quad A \otimes (B \otimes C) \xrightarrow{\alpha_{ABC}^{-1}} (A \otimes B) \otimes C
\]

\[
1 \otimes A \xrightarrow{\delta_A} A \quad A \xrightarrow{\lambda_A^A} 1 \otimes A
\]

\[
A \otimes 1 \xrightarrow{\delta_A} A \quad A \xrightarrow{\mu_A} A \otimes 1
\]

\[
A \otimes B \xrightarrow{\sigma_{AB}} B \otimes A \quad B \otimes A \xrightarrow{\sigma_{AB}^{-1}} A \otimes B
\]

the units and the counits of adjunction:

\[
A \xrightarrow{\text{abs}_{AB}} B \to \circ A \otimes B \quad (B \to \circ A) \otimes B \xrightarrow{\text{ev}_{AB}} A
\]

\(^1\)We use symbols \( \tilde{\varphi}_{AB} : !A \otimes !B \to !(A \otimes B) \) and \( \varphi_0 : 1 \to !1 \). The tilde is added so that these are distinguished if the subscripts are omitted. The naught signifies it to be nullary.
together with the morphisms given in Def. 2.1:

\[
egin{align*}
&!A \otimes !B \xrightarrow{\gamma} !(A \otimes B) \\
&1 \xrightarrow{\sigma_0} !1 \\
&!A \xrightarrow{\delta} !!A \\
&!A \xrightarrow{\varepsilon} A \\
&!A \xrightarrow{d} !A \otimes !A \\
&!A \xrightarrow{e} 1
\end{align*}
\]

The last six is called *algebraic morphisms* for future reference. The set of morphisms is generated from the atomic morphisms by composition \(f; g\) and the functorial operations \(f \otimes g, !f,\) and \(1_B \rightarrow f\). Subscripts are often omitted. Our system will be designed so that subscripts have no significance. It is analogous to ordinary type systems where rules depend only on the shape of terms, not on types.

We introduce a congruence relation over morphisms. Two morphisms equivalent with respect to the congruence are understood to be able to be rewritten to each other. First, we have the axioms of categories and the elementary property of functors:

\[
\begin{align*}
&(f; 1) = f = 1; f \\
&(f; g); h = f; (g; h) \\
&F1 = 1 \\
&F(f; g) = Ff; Fg
\end{align*}
\]

where \(F\) is one of \((-) \otimes (-), B \rightarrow (-), !(-)\). For the tensor product, we appropriately reform the equality as it is a 2-place functor. Second, each structural isomorphism and its inverse are actually inverse:

\[
\begin{align*}
&\alpha; \alpha^{-1} = 1 \\
&\alpha^{-1}; \alpha = 1 \\
&\cdots
\end{align*}
\]

Some of the coherence conditions are regarded as congruence. They are listed in appendix A.1, as they are abundant and well-known. An example is

```
\[
\begin{array}{cc}
& (A \otimes B) \otimes C \\
\sigma \otimes 1 & \sigma \\
B \otimes (A \otimes C) & (B \otimes C) \otimes A
\end{array}
\]
```

in which two legs are regarded to be equivalent.

Next, we consider the case where the base category is \(*\)-autonomous. Among several equivalent definitions known for \(*\)-autonomous categories [4, 16], we adopt the one using the weakly distributive category [8]. The set of objects is generated by

\[
A ::= X \mid 1 \mid \bot \mid A \otimes A \mid A \boxtimes A \mid A^* \mid !A
\]

In addition to the structural isomorphisms for the symmetric monoidal category we include the isomorphisms giving the symmetric monoidal structure on \(\boxtimes\):
Moreover the weak distribution morphisms
\[
A \otimes (B \otimes C) \xrightarrow{\partial_{ABC}} (A \otimes B) \otimes C
\]
are added. The morphism \(\text{abs, ev}\) are removed as \(\to\) exists no more. In place, we add duality morphisms:
\[
1 \xrightarrow{T_A} A \otimes A^* \quad A^* \otimes A \xrightarrow{\tau_A} 1
\]
Six algebraic morphisms are the same. The set of morphisms is generated from the atomic morphisms above by composition \(f \circ g\) and the functorial operations \(f \otimes g, f \otimes A, 1\).

We note that \((-)^*\) is not regarded as a contravariant functor [8]. We distinguish \(A^{**}\) from \(A\).

The congruence relation is similar. The coherence diagrams regarded as equivalence are listed in appendix A.2.

The core of our calculus lies in the following orientation of twenty-three diagrams. These are parts of defining diagrams of the linear category. The orientation of rewriting is denoted by a double arrow. We can rewrite only in the designated direction, whilst between equivalent morphisms we allow rewriting in either direction. The first twenty-one diagrams are common for both the symmetric monoidal closed base and the \(\ast\)-autonomous base. The last two depend on the selected base. In the diagrams, the label \(\sim\) denotes appropriate structural isomorphisms and \(f\) is an arbitrary morphism.
The remaining two rules are interchanged depending on which base category is adopted. If we choose the symmetric monoidal closed category \([20]\),

\[
\begin{align*}
A \otimes B & \xrightarrow{\text{abs}} (B \rightarrow A \otimes B) \otimes B \\
1 & \xrightarrow{i} A \otimes B
\end{align*}
\]

\[
\begin{align*}
B \rightarrow A & \xrightarrow{\text{abs}} (B \rightarrow (B \rightarrow A) \otimes B) \\
1 & \xrightarrow{i} B \rightarrow A
\end{align*}
\]

If we select the \(\ast\)-autonomous category we replace the above by the following two. Therein \(\partial'_{ABC} : (A \nabla B) \otimes C \rightarrow A \nabla (B \otimes C)\) is induced from \(\partial_{ABC}\) by symmetry of tensor and cotensor.

\[
\begin{align*}
1 \otimes A & \xrightarrow{\tau} (A \nabla A^\ast) \otimes A \\
A & \xrightarrow{\nabla} A \nabla (A^\ast \otimes A)
\end{align*}
\]

\[
\begin{align*}
A^\ast \otimes 1 & \xrightarrow{\tau} A^\ast \otimes (A \nabla A^\ast) \\
A^\ast & \xrightarrow{\nabla} (A^\ast \otimes A) \nabla A^\ast
\end{align*}
\]

If we define \(X \rightarrow Y\) with \(Y \nabla X^\ast\), \(\text{abs}_A^B\) with \(A \xrightarrow{\sim} A \otimes 1 \xrightarrow{\tau_{B^\ast}} A \otimes (B^\ast \nabla B^\ast) \xrightarrow{\partial} (A \otimes B) \nabla B^\ast\) and \(\text{ev}_A^B\) with \((A \nabla B^\ast) \otimes B \xrightarrow{\partial} A \nabla (B^\ast \otimes B) \xrightarrow{\tau_{B^\ast}} A \nabla \perp \xrightarrow{\sim} A\), then (22) and (23) for the symmetric monoidal closed base are a consequence of the corresponding rules for the \(\ast\)-autonomous base.\(^2\)

In place of referring rules by numbers, we call them by the shape of redexes. For example, rule (1) is called \(\delta; \delta\)-type, rule (5) \(\delta; \text{id}\)-type, and rule (17) \(\varphi_0; \tilde{\varphi}\)-type. We call (1) through (7) collectively \(\delta\)-type as the redexes start with \(\delta\). Likewise we call (9) through (12) \(\tilde{\varphi}\)-type, and (13) through (17) \(\varphi_0\)-type. If we collectively deal with (1) through (17) starting with one of \(\delta, d, \tilde{\varphi}, \varphi_0\), we call a reduction in the group an algebraic reduction.

We call (18) through (21) naturality reductions. Rule (22) and (23) are called \(\beta\) and \(\eta\) reductions respectively.

Let us check that if we ignore the orientation of rules we obtain the linear category in Def. 2.1. Diagram (17) together with coherence in appendix A.1 asserts that \((!, \tilde{\varphi}, \varphi_0)\) is a symmetric monoidal functor. Diagrams (1), (2), (3) with naturality (18), (19) shows \((!, \delta, \varepsilon)\) is a comonad. We add (9), (13) and (10), (14) by the requirement that natural transformations \(\delta, \varepsilon\) are monoidal. Diagram (8) with coherence in the appendix asserts that \((!A, d_A, e_A)\) is a commutative comonoid. Moreover, (11), (15), (20) says that \(d\) is a monoidal natural transformation, and (12), (16), (21) that \(e\) is a monoidal natural transformation. Diagrams (5) and (7) says that \(d_A, e_A\) are coalgebra maps.

\(^2\)To save space we use a dot to signify the position where a suitable identity is inserted, and omit tensor and cotensor on morphisms.
(4) and (6) says that $\delta_A$ is a comonoid map. If the base is symmetric monoidal closed, (22) and (23) are the adjoint triangle diagrams of $(-) \otimes B \vdash B \multimap (-)$. If the base is $*$-autonomous, the conditions that should be fulfilled by duality morphisms are (22), (23) together with coherence in appendix A.2.

3. Example: local confluence

Let us give several examples of computation in our calculus. We give a few cases of local confluence. Global confluence will be discussed in a forthcoming paper. The selected examples need relatively long chains to resolve diversion by contracting overlapping redexes, i.e., critical pairs.

Let us consider $\delta; !d; \delta$. If we contract $!d; \delta$ first by naturality reduction,

If we contract $\delta; !d$ first,
The next example of critical pairs starts with $\tilde{\varphi}; \delta; !d$. If we contract $\delta; !d$ first,

$$
!A \otimes !A \otimes !B \otimes !B \rightarrow !(A \otimes B)
$$

$$
\tilde{\varphi} \otimes \delta \rightarrow !!(A \otimes B) \otimes !(A \otimes B)
$$

$$
\delta \otimes \delta \otimes \delta \otimes \delta \rightarrow !!(A \otimes B) \otimes !(A \otimes B)
$$

$$
!!A \otimes !!A \otimes !!B \otimes !!B \rightarrow !(A \otimes B) \otimes !(A \otimes B)
$$

$$
\varphi \otimes \varphi \rightarrow !(A \otimes B) \otimes !(A \otimes B)
$$

$$
!(A \otimes B) \otimes !(A \otimes B) \rightarrow !(A \otimes B ! A \otimes !B)
$$

If we contract $\tilde{\varphi}; \delta$ first

$$
!A \otimes !A \otimes !B \otimes !B \rightarrow !(A \otimes B)
$$

$$
\delta \otimes \delta \rightarrow !(A \otimes B)
$$

$$
!!A \otimes !!A \otimes !!B \otimes !!B \rightarrow !(A \otimes B)
$$

$$
\varphi \otimes \varphi \rightarrow !(A \otimes B)
$$

$$
!(A \otimes B) \otimes !(A \otimes B)
$$

$$
!(A \otimes A ! B \otimes B) \rightarrow !(A \otimes !B)
$$

$$
!(A \otimes !B) \otimes !(A \otimes B)
$$

The obtained sequences of morphisms are not exactly equal. By the coherence of symmetric monoidal functors, however, they are equivalent.

4. Comparison to a type theory

We briefly discuss the relation of our calculus to a type system of intuitionistic linear logic. It is intended to justify the design of the calculus. The following comparison shows that our categorical calculus is a refinement of a term calculus. Furthermore, it explains why twenty-three diagrams are oriented in that way.
We use the dual intuitionistic linear logic due to Barber [3], modified slightly. Modality $!A$ is used to decorate types while $\sharp M$ is used instead to decorate terms\(^3\). In the type environment, we use $\sharp x : A$ and $x : A$ to distinguish the intuitionistic part and the linear part instead of partitioning by a semicolon as in the original. The modality $\sharp$ means that $x$ is in the intuitionistic part. So an environment $\Gamma$ is a finite sequence of $\sharp x_i : A_i$ or $x_i : A_i$ where the order has no significance. In the original system the intuitionistic part is strictly separated from the linear part. Instead we use lifting to change the modality of a variable:

$$
\begin{align*}
\Gamma, x : A &\vdash M : B \\
\Gamma, \sharp x : A &\vdash M : B
\end{align*}
$$

Accordingly, we limit the axiom to the shape of $x : A \vdash x : A$. Weakening and contraction are given explicitly:

$$
\begin{align*}
\Gamma &\vdash M : B \\
\Gamma, \sharp x : A &\vdash M : B \\
\Gamma, \sharp x : A, \sharp x'' : A &\vdash M : B \\
\Gamma, \sharp x : A &\vdash M[x/x'][x/x''] : B
\end{align*}
$$

In the contraction rule $M[x/x'x'']$ denotes the operation to substitute $x$ simultaneously for $x'$ and $x''$.

For terms we use postfix notation $M\{\sharp x \mapsto N\}$ in place of the prefix let-operator. It replaces $\text{let } !x \text{ be } N$ in $M$ in Barber’s system. The $\beta$-rule for $\sharp$ is given as $M\{\sharp x \mapsto \sharp N\} \Rightarrow M[N/x]$, and the $\eta$-rule as $\sharp x\{\sharp x \mapsto M\} \Rightarrow M$.

The type system is interpreted in the linear category in a standard way. A type judgement $\Gamma \vdash M : B$ corresponds to a morphism $f : \Gamma \to B$ where $\Gamma$ denotes the sequence of $!A_i$ or $A_i$ connected by $\otimes$. If the type environment contains $\sharp x_i : A_i$ we use $!A_i$, and if it contains $x_i : A_i$ we use $A_i$. Here we associate a morphism to a derivation tree, in order to argue finer rewriting than on terms.

First let us justify the $\beta$-reduction for $\sharp$:

$$
\begin{align*}
\pi_1 : &\Gamma, \sharp x : A \vdash M : B \\
\pi_2 : &\sharp \Delta \vdash K : A \\
\Rightarrow &\Gamma, \sharp \Delta \vdash M\{\sharp x \mapsto \sharp K\} : B
\end{align*}
$$

where $\sharp \Delta$ denotes that all type assignments in the environment has the shape of $\sharp x_i : A_i$. The derivation $\pi_1\sharp^x\pi_2$ is obtained by connecting $\pi_2$ at the place of the axiom involving $x$ in $\pi_1$.

We split cases according to the last rule involving the variable $x$ in $\pi_1$. If the last is lifting:

\(^3\)Simply for immediate viewability. Barber uses $!$ for both.
let $\rho_1$ be interpreted by $\Gamma \otimes A \xrightarrow{\delta} B$ and $\pi_2$ by $!\Delta \xrightarrow{h} A$. For example, if $\Delta$ consists of two types $C_1$ and $C_2$, the derivation before rewriting is interpted by $\Gamma \otimes !C_1 \otimes !C_2 \xrightarrow{\delta \delta} \Gamma \otimes !C_1 \otimes !C_2 \xrightarrow{\varphi} \Gamma \otimes !A \xrightarrow{\varepsilon} \Gamma \otimes A \xrightarrow{f} B$ and the one after rewriting by $\Gamma \otimes !C_1 \otimes !C_2 \xrightarrow{h} \Gamma \otimes A \xrightarrow{f} B$. These are realized by computation in the linear category as

The rules used here are (2)(10)(19). If $\Delta$ is empty (14) is used since $\varphi_0$ is employed in place of $\tilde{\varphi}$. If the last of $\pi_1$ is contraction:

a similar analysis shows that rule (4)(11)(15)(20) are utilized. If the last of $\pi_1$ is weakening

rule (6)(12)(16)(21) are used. If the last of $\pi_1$ is $!$-introduction,

rule (1)(9)(13)(18) are used. When $\Delta$ is empty also (17) is used.

Second we consider the $\eta$-reduction for $\sharp$ modality:
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\[
\begin{align*}
x : A &\vdash x : A \\
\sharp x : A &\vdash \sharp x : A \\
\sharp x : A &\vdash \sharp x : A \\
\Delta &\vdash \Delta \\
\Rightarrow &\vdash 
\end{align*}
\]

The left hand side is interpreted by \(\Delta \vdash K : !A\). Rule (3) is used.

If the type assignment introduced by weakening is deleted by contraction, both are superfluous. Hence we can have the following simplifying rule:

\[
\begin{align*}
\Gamma, \sharp x' : A &\vdash M : B \\
\Gamma, \sharp x : A &\vdash M[x/x'] : B \\
\Rightarrow &\vdash 
\end{align*}
\]

Rule (8) is used.

The rest are interchanging rule of \(\sharp\) modality with weakning and contraction:

\[
\begin{align*}
\sharp \Gamma, \sharp x' : A, \sharp x'' : A &\vdash M : B \\
\sharp \Gamma, \sharp x : A &\vdash M[x/x'] : B \\
\Rightarrow &\vdash 
\end{align*}
\]

The former uses rule (5) and the latter (7)(17).

Finally the \(\beta\)-reduction for abstraction \((\lambda x. M)K \Rightarrow M[K/x]\) corresponds to rule (22) and the \(\eta\)-reduction \(\lambda x. Mx \Rightarrow M\) to rule (23).

Every rule save (17) is used exactly once, as observed from the analysis above. Each rule has an intrinsic role. The orientation of the diagrams is accordingly determined. In addition, a single rewriting step of terms is realized by several steps of categorical rewriting. Therefore we conclude that the categorical calculus is a refinement of the term calculus. Furthermore, as mentioned in Introduction, we permit rewriting \(!f\;g \leadsto !f\;!g\).

So our system incarnates a mechanism to decompose a term and substitute only a subterm obtained by decomposition. Our calculus is refinement also in this sense.

5. Normalizability

We show weak normalization of the categorical reduction system. Hereafter we consider the system based on the \(*\)-autonomous category. A reason for the choice is that naturality of \(\text{abs}_A\) and \(\text{ev}_A\) in the symmetric monoidal closed category is awkward and cumbersome.
to handle. Moreover, the latter is simulated by the former.

In this paper, strong normalization scarcely appears. So if we say simply that something terminates, weak termination is intended, that is, the existence of a finite sequence of contractions leading to a normal form.

A normal form is a morphism that is equivalent to the shape that has no redexes. However, a redex may be produced from none as a consequence of congruence. For example, an obvious normal form $A \xrightarrow{\varepsilon_A} A$ is, by application of $1_{1A} = !1_A$, equivalent to $!A \xrightarrow{1_{1A}} !A \xrightarrow{\varepsilon_A} A$ that has a naturality redex.

A reversible reduction is one of the naturality rules (18) through (21) where $f$ is an identity, a structural isomorphism or its inverse, or their compositions. We can cancel reversible reductions. For example, suppose that $!A \xrightarrow{!f} !B \xrightarrow{!g} !B \otimes !B$ is contracted to $!A \xrightarrow{d_A} !A \otimes !A \xrightarrow{!f \otimes !f} !B \otimes !B$. Then we attach $1_{1A} = !f; !f^{-1}$ in front and transfer $!f^{-1}$ by naturality, restoring a morphism that is equivalent to the original. Reversible reductions are regarded to be inessential.

5.1. Lemma. Only reversible reductions occur in a reduction sequence from a normal form.

Proof. The morphisms equivalent to identities are composed of structural isomorphisms only.

Once a morphism reaches a normal form, we can have only inessential reductions afterward. In the following, we ignore reversible redexes. We assume they are removed by contraction implicitly.

We are not motivated by constructing a graph reduction system, yet it is helpful to use graphs to avoid a nuisance incurred by structural isomorphisms and their coherence. In this paper, we only modestly use graphs, that are introduced informally to enhance intuitive understanding. To discuss confluence in a forthcoming paper, we will rely on full graphical visualization. We will not intend to construct a graph reduction system, though.

As in [7], we represent a morphism $f : A_1 \otimes A_2 \otimes \cdots \otimes A_m \rightarrow B_1 \nabla B_2 \nabla \cdots \nabla B_n$ in the $\ast$-autonomous category as a figure

Each of tensor and cotensor has two gates
We use a double circle in place of $\otimes$ as the latter symbol is not symmetric under a vertical flip. The weak distribution morphism $\partial : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, for example, corresponds to

\[
\begin{array}{c}
A \\
\otimes \\
B \\
\otimes \\
C
\end{array}
\]

Duality morphisms $\tau_A, \gamma_A$ are represented by bends:

\[
\begin{array}{c}
A \\
\uparrow \\
A^* \\
\downarrow \\
A
\end{array}
\]

We add a diode-like symbol to signify which side has the duality star, so that it is restored if the labels attached to wires are omitted. The duality $\beta\eta$-reduction corresponds to the operation straightening double bends:

\[
\begin{array}{c}
A \\
\uparrow \\
A^* \\
\downarrow \\
A
\end{array} \Rightarrow 
\begin{array}{c}
A \\
A^*
\end{array} \Gamma 
\begin{array}{c}
A \\
\uparrow \\
A^* \\
\downarrow \\
A
\end{array} \Rightarrow 
\begin{array}{c}
A^*
\end{array}
\]

For $f : A \to B$, its dual $f^* : B^* \to A^*$ is depicted as

\[
\begin{array}{c}
B^* \\
\uparrow \\
A^*
\end{array}
\]

The verification of weak normalization is based on the standard reducibility method. The following proof strategy is inspired by [13].

5.2. **Definition.** A positive funnel on object $A$ is a set $\mathcal{S}$ of morphisms $X \xrightarrow{f} A$ for varied $X$ satisfying the following two conditions:

(i) Identity $A \xrightarrow{1} A$ is a member of $\mathcal{S}$.

(ii) Each $f \in \mathcal{S}$ terminates.

A negative funnel on $A$ is a set $\mathcal{S}$ of morphisms $A \xrightarrow{f} X$ for varied $X$ satisfying the same conditions (i) and (ii).

We say simply a funnel for either of a positive and a negative funnel. When we mention two funnels in a single statement, they are either both positive or both negative.

5.3. **Definition.** Given an object $A$, let $\mathcal{S}$ be a set of morphisms of the form $X \xrightarrow{f} A$ (resp. $A \xrightarrow{f} X$). The complement $\mathcal{S}^\perp$ is the set of all morphisms $A \xrightarrow{g} Y$ (resp. $Y \xrightarrow{g} A$) subject to the condition that $f; g$ (resp. $g; f$) terminates for all $f \in \mathcal{S}$. 
5.4. Lemma. If $S$ is a positive (negative) funnel on $A$, the complement $S^\perp$ is a negative (resp. positive) funnel.

Proof. Condition (i) of $S^\perp$ follows from (ii) of $S$. Condition (ii) of $S^\perp$ follows from (i) of $S$.

5.5. Lemma. Let $R$ and $S$ be a set of morphisms of the form $X \xrightarrow{f} A$ or of the form $A \xrightarrow{f} X$.

(i) If $R \subseteq S$, then $S^\perp \subseteq R^\perp$.
(ii) $S \subseteq S^{\perp\perp}$.
(iii) $S^\perp = S^{\perp\perp\perp}$.

Proof. Easy.

5.6. Definition. We define sets $R \otimes S$, $R \uplus S$, $!R$, and $R^*$ for funnels $R$ and $S$.

\[
R \otimes S = \{ f \otimes g \mid f \in R, \ g \in S \}
\]
\[
R \uplus S = \{ f \uplus g \mid f \in R, \ g \in S \}
\]
\[
!R = \{!f \mid f \in R\}
\]
\[
R^* = \{ f^* \mid f \in R \}.
\]

The first three sets $R \otimes S$, $R \uplus S$, and $!R$ are funnels, as easily seen. The last $R^*$ is not a funnel, however. In fact, identity $1_A^*$ is not a member of $R^*$, since it does not equal $(1_A)^*$.

5.7. Lemma. Let $A_1 \otimes A_2 \otimes \cdots \otimes A_m \xrightarrow{f} B_1 \uplus B_2 \uplus \cdots \uplus B_n$ be a morphism. Then

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A_1 \ldots A_m
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Proof. As they are symmetric, we verify the first equivalence. Suppose that \( f \) with a bend terminates. If it leads to a normal form where the bend is intact, there is an obvious terminating reduction sequence from \( f \). Otherwise, the bend vanishes by \( \eta \)-reduction as in

\[
\begin{array}{c}
\xrightarrow{\cdot} \\
\end{array}
\Rightarrow
\begin{array}{c}
\xrightarrow{\cdot} \\
\end{array}
\Rightarrow
\begin{array}{c}
\xrightarrow{\cdot} \\
\end{array}
\]

where \( f_0 \) is a normal form. Then we have

\[
\begin{array}{c}
f \\
\xrightarrow{\cdot} \\
\xrightarrow{\cdot} \\
\end{array}
\Rightarrow
\begin{array}{c}
f_0 \\
\end{array}
\]

The last is a normal form unless the bend is connected to another bend, forming a \( \beta \)-redex. If so the contraction of the redex leads to a normal form. The converse is easy. ■

5.8. Lemma. If \( S \) is a negative funnel \((S^*)^\perp\) is a negative funnel. (A similar result holds for positive funnels, though it is not needed.)

Proof. Since every morphism in \( S^* \) terminates by Lem. 5.7, \((S^*)^\perp\) contains an identity. We verify that each morphism \( f \in (S^*)^\perp \) terminates. As \( S \) contains an identity, is a member of \( S^* \). Hence

\[
\begin{array}{c}
\xrightarrow{\cdot} \\
\end{array}
\]

terminates. By Lem. 5.7 we can eliminate the leftmost bend. With a similar argument, we can eliminate the other bend as well. ■

5.9. Definition. A negative funnel \( R^-(A) \) and a positive funnel \( R^+(A) \) are associated to each object \( A \). Definition is by induction on the construction of \( A \).

\[
\begin{array}{ll}
R^-(A) & = \{1_A\}^\perp \\
R^-(1) & = \{1_1\}^\perp \\
R^-(\bot) & = \{1_\bot\}^\perp \\
R^-(A \otimes B) & = (R^+(A) \otimes R^+(B))^\perp \\
R^+(A \nabla B) & = (R^-(A) \nabla R^-(B))^\perp \\
R^-(A^*) & = (((R^-(A))^*)^\perp \\
R^-(!A) & = (!R^+(A))^\perp \\
\end{array}
\]

Only one of \( R^-(A) \) and \( R^+(A) \) is listed above. The other is defined as its complement \((-)^\perp\). For example, \( R^+(!A) \) is \((R^-(!A))^\perp\). Evidently \( R^\pm(A) \) are the complements of each other.
5.10. Lemma. Let \( A \xrightarrow{f} B \) and \( C \xrightarrow{g} D \) be morphisms. The following implications hold:

\[
\begin{align*}
& f \in R^{-}(A) \quad \Rightarrow \quad !f \in R^{-}(!A) \\
& f \in R^{+}(B) \quad \Rightarrow \quad !f \in R^{+}(!B) \\
& f \in R^{-}(A), \ g \in R^{-}(C) \quad \Rightarrow \quad f \otimes g \in R^{-}(A \otimes C) \\
& f \in R^{+}(B), \ g \in R^{+}(D) \quad \Rightarrow \quad f \otimes g \in R^{+}(B \otimes D) \\
& f \in R^{-}(A), \ g \in R^{-}(C) \quad \Rightarrow \quad f \star g \in R^{-}(A \star C) \\
& f \in R^{+}(B), \ g \in R^{+}(D) \quad \Rightarrow \quad f \star g \in R^{+}(B \star D) \\
& f \in R^{-}(A) \quad \Rightarrow \quad f^{*} \in R^{+}(A^{*}) \\
& f \in R^{+}(B) \quad \Rightarrow \quad f^{*} \in R^{-}(B^{*})
\end{align*}
\]

Proof. We verify the case of \(!\). We take arbitrary \(!g\) from \(!R^{+}(A)\). Composition \(g;f\) terminates by hypothesis. So \(!g;f = !(g;f)\) terminates as well. Thus the first assertion follows. The second assertion is a consequence of the inflation property of \((-)^{\perp\perp}\). The rest are similar. For \((-)^{*}\), we note that \(g^{*};f^{*}\) contracts to \((f;g)^{*}\) by \(\beta\)-reduction. \(\blacksquare\)

5.11. Definition. A reducible morphism is a morphism \( A \xrightarrow{f} B \) satisfying that \(g;f;h\) terminates for every pair of \(g \in R^{+}(A)\) and \(h \in R^{-}(B)\).

5.12. Lemma. For a morphism \( A \xrightarrow{f} B \), the following are equivalent:

(i) \( f \) is reducible.

(ii) If \( g \in R^{+}(A) \) then \( g;f \in R^{+}(B) \).

(iii) If \( h \in R^{-}(B) \) then \( f;h \in R^{-}(A) \).

Proof. By definition. \(\blacksquare\)

5.13. Proposition. Each reducible morphism terminates.

Proof. All funnels contain identity. \(\blacksquare\)

Therefore it suffices to show that all morphisms in the free linear category are reducible. We start with easy cases.

5.14. Lemma. The following hold:

(i) A morphism \( !A \xrightarrow{f} B \) is reducible iff \( !g;f \) lies in \( R^{+}(B) \) for every \( g \) in \( R^{+}(A) \).

(ii) A morphism \( A \otimes B \xrightarrow{f} C \) is reducible iff \( (g \otimes h);f \) lies in \( R^{+}(C) \) for every pair of \( g \) in \( R^{+}(A) \) and \( h \) in \( R^{+}(B) \).

(iii) A morphism \( A \xrightarrow{f} B \star C \) is reducible iff \( f; (g \star h) \) lies in \( R^{-}(A) \) for every pair of \( g \) in \( R^{-}(B) \) and \( h \) in \( R^{-}(C) \).

(iv) A morphism \( 1 \xrightarrow{f} A \) is reducible iff \( f \) lies in \( R^{+}(A) \).

(v) A morphism \( B^{*} \xrightarrow{f} C \) is reducible iff \( f^{*} \) lies in \( R^{+}(C) \) for every \( g \) in \( R^{-}(B) \).

Proof. An easy consequence of Lem. 5.12 and definition. \(\blacksquare\)
5.15. **Lemma.** A morphism $!A \otimes !B \xrightarrow{f} C$ is reducible iff $(!g \otimes !h), f$ lies in $R^+(C)$ for every pair of $g \in R^+(A)$ and $h \in R^+(B)$.

**Proof.** By 5.14 $f$ is reducible iff, for every $d$ in $R^+(!A)$, every $e$ in $R^+(!B)$, and every $k$ in $R^-(C)$, morphism $(d \otimes e); f; k$ terminates. By the bending technique 5.7, it means termination of

Since $d$ is arbitrary, it amounts to that

lies in $R^-(!A)$. By Lem. 5.14, it saids that

terminates for every $g \in R^+(A)$. Straightening the bend over $e$, we have succeeded in replacing $d$ with $!g$. Applying the same process to the right wire as well, we obtain the lemma.

5.16. **Proposition.** The following hold:

(i) Identities and structural isomorphisms are reducible.

(ii) The composition of reducible morphisms is reducible.

**Proof.** (i) is obvious. (ii) is an immediate consequence of Lem. 5.12.
5.17. Proposition. The following hold:

(i) If \( f \) is reducible, \( !f \) is reducible.
(ii) If \( f \) and \( g \) are reducible, \( f \otimes g \) is reducible.
(iii) If \( f \) and \( g \) are reducible, \( f \mathbin{\otimes \Box} g \) is reducible.
(iv) If \( f \) is reducible, \( f^* \) is reducible.

Proof. (i) through (iii) is a consequence of Lem. 5.14. We prove (iv). Suppose \( f : A \to B \). By the same lemma, it suffices to show that \( g^*; f^*; h \) terminates for every pair of \( g \in R^- (B) \) and \( h \in R^- (A^*) \). We note that \( g^*; f^* \) contracts to \( (f; g)^* \) by duality \( \beta \)-reduction. Here \( f; g \) lies in \( R^- (A) \), thus \( (f; g)^* \) lies in \( R^+ (A^*) \). So \( (f; g)^*; h \) terminates.

5.18. Proposition. \( \partial \) is reducible.

Proof. Suppose \( \partial : A \otimes (B \mathbin{\otimes \Box} C) \to (A \otimes B) \mathbin{\otimes \Box} C \). We must show that \((e \otimes f); \partial; (g \mathbin{\otimes \Box} h)\) terminates for every \( e \in R^+ (A) \), every \( f \in R^+ (B \mathbin{\otimes \Box} C) \), every \( g \in R^- (A \otimes B) \), and every \( h \in R^- (C) \). For every \( k \in R^+ (B) \), \((e \otimes k); g\) terminates. Thus, by the bedding technique 5.7,

lies in \( R^- (B) \). Hence

\[ e \]
\[ A \]
\[ B \]
\[ g \]

terminates. Straightening the bend over \( e \), we obtain the proposition.

5.19. Proposition. \( \tau_A \) and \( \gamma_A \) are reducible.

Proof. Verification is the same for both cases. We give proof for \( \gamma_A : A^* \otimes A \to \bot \). We must show that \((f \otimes g); \gamma_A\) terminates for every pair of \( f \in R^+ (A^*) \) and \( g \in R^- (A) \). Composition \( f; g^* \) terminates since \( g^* \in R^- (A^*) \) by Lem 5.10. Graphically this composition means

\[ f \]
\[ g \]
Straighten the bend over \( g \).

It remains to verify that algebraic morphisms are reducible. There are technical difficulties that are unable to be covered by the reducibility method. To that end, we introduce several notions.

Let \( X \) range over a chosen family of objects. Later we use the case where the family is a singleton or consists of two elements. We regard \( X \) as if they are atomic objects. We consider two classes of objects generated by the following generative grammar.

\[
A ::= X | A \otimes A | !A \\
B ::= 1 | X | B \otimes B | !B
\]

5.20. Definition.

(i) A composite algebraic morphism \( t \) is a member of the class generated from \( 1_B \) and \( \varphi_0, \varphi_{B,B'}, \delta_B, \varepsilon_B, d_B, e_B \) as well as \( \alpha_{B,B',B''}, \sigma_{B,B'}, \lambda_B, \rho_B \) and their inverses, closed under operations \( t \otimes t', !t \) and composition \( t; t' \).

(ii) A strict composite algebraic morphism \( s \) is a member of the class generated from \( 1_A \) and \( \tilde{\varphi}_{A,A'}, \delta_A, \varepsilon_A, d_A, e_A \) as well as \( \alpha_{A,A',A''}, \sigma_{A,A'} \) and their inverses, closed under operations \( s \otimes s', !s \) and composition \( s; s' \).

Namely composite algebraic morphisms can use everything unrelated to \( \mathcal{N} \) or \((-)^*\) as long as \( X \) is regarded as an atomic object. Strict composite algebraic morphisms preclude \( \varphi_0 \) and the isomorphisms involving \( 1 \). We comment that the target of \( e_A : !A \to 1 \) is not a member of class \( A \). Save this exception, composite algebraic morphisms are between members of class \( B \) and strict composite algebraic morphisms are between members of \( A \).

5.21. Example. \( !X \xrightarrow{\delta_X} !!X \xrightarrow{d_X} !!X \otimes !!X \xrightarrow{\delta_X} !!X \otimes 1 \) is a strict composite algebraic morphism. It contracts to \( !X \xrightarrow{d_X} !!X \otimes !!X \xrightarrow{\delta_X} !!X \otimes 1 \) that is strict composite algebraic. It further contracts to \( !X \xrightarrow{d_X} !!X \otimes 1 \xrightarrow{\varphi_0} !!X \otimes 1 \), that is composite algebraic though it is not strictly composite algebraic for it contains \( \varphi_0 \). Finally it contracts to \( !X \xrightarrow{\delta_X} !!X \xrightarrow{\varphi_0} !!X \otimes 1 \xrightarrow{\varphi_0} !!X \otimes 1 \).

A strict composite algebraic morphism \( s \) that has no naturality redexes at the beginning may create naturality through reduction. For example, \( !X \xrightarrow{\delta} !!X \xrightarrow{d} !!!X \xrightarrow{\delta} !!X \xrightarrow{\varphi_0} !X \xrightarrow{\varphi_0} X \xrightarrow{\varphi_0} 1 \) contracts to \( !X \xrightarrow{\delta} !!X \xrightarrow{\delta} !X \xrightarrow{\delta} 1 \), the latter containing naturality redex \( !\delta \) while the former has none. This happens because rule (1) produces \( !\delta \) wrapped by \((-)\). Likewise rule (9) produces \( !\tilde{\varphi} \). We do not have to consider rule (13) as \( \varphi_0; \delta \) is not allowed in strict composite algebraic morphisms.

5.22. Lemma. Let \( u \) denote one of \( \delta, \varepsilon, d \) and \( e \). Suppose that a strict composite algebraic morphism \( s \) has no naturality redexes other than those of the form \( !f; uA \) where \( f \) consists of \( \delta \) and \( \varphi \) only. Any morphism obtained by contraction of \( s \) satisfies the same property for naturality redexes.
A restricted naturality redex is \( !f; u_A \) where \( f \) consists solely of \( \delta \) and \( \varphi \). The lemma asserts that if the naturality redexes of a strict composite algebraic morphism are restricted then the property is preserved under contraction.

We verify that strict composite algebraic morphisms (strongly) terminate if their naturality redexes are restricted. Although \( X \) may run over a family of two or more objects, the following argument is irrelevant to the number of distinct \( X \). So we describe the case that \( X \) is unique. If there are two or more, each \( X \) should read one of them appropriately. We write \( A = A[X, X, \ldots, X] \) displaying each occurrence of \( X \). We further write \( A = A[X^{x_1}, X^{x_2}, \ldots, X^{x_n}] \). At this stage, \( x_i \) are the labels to distinguish occurrences, but natural numbers are assigned (by technical reason \( x_i \geq 2 \)) as explained below.

Suppose

\[
s : A[X^{y_1}, X^{y_2}, \ldots, X^{y_n}] \to B[X^{x_1}, X^{x_2}, \ldots, X^{x_m}].
\]

Each \( y_i \) is computed by applying a function \(|s|\) determined by the shape of \( s \) to some of \( x_1, x_2, \ldots, x_m \). We show that if \( s \) is contracted by application of a certain type of reduction rules then \( y_1 + y_2 + \cdots + y_n \) strictly decreases. To define \(|s|\) we need auxiliary stuff given below.

5.23. Definition. Let \( x \) denote an occurrence of \( X \) in \( A \). Recursively \( \theta_A(x) \) is defined. If \( A = X \) then we set \( \theta_X(x) = x \). For exponential we set \( \theta_A(x) = 2\theta_A(x) \). Finally we set \( \theta_A(x) = d + \theta_A(x) \) and symmetrically \( \theta_A(x) = d + \theta_A(x) \) where \( d \) denotes the number of occurrences of \( X \) in \( A' \).

This recursive definition is applied to each occurrence of \( X \). For example if \( A = !(X^{x_1} \otimes \! X^{x_2}) \) then \( \theta_A(x_1) = 2(1 + x_1) \) and \( \theta_A(x_2) = 2(1 + 4x_2) \). Observe that \( \theta_A \) is not a single function, the shape of which changes per occurrence. We remark that a structural isomorphism does not affect \( \theta_A \) since it does not change the number of occurrences of \( X \). For example \( \theta_{(A \otimes B) \otimes C}(x) = \theta_{A \otimes (B \otimes C)}(x) \).

5.24. Definition. We define function \(|s|\) for strict composite algebraic morphism \( s \). Its arity depends on the shape of \( s \). Composition of morphisms correspond to composition of functions in the reverse order: \(|s; t| = |s| \circ |t|\). As they are not unary functions in general, the composition should be understood in an appropriate way. For identity morphism, \(|1_A|\) is an identity function. In what follows we write \( A[X^x] \) to display a specific occurrence of \( X \) in \( A \).

\( i \) For \( \delta_A : !A[X^y] \to !A[X^x] \) we associate \( y = |\delta_A|(x) \) where \(|\delta_A|(x) = 2^{\theta_A(x)} \).

\( ii \) For \( d_A : !A[X^y] \to !A[X^x] \otimes !A[X^x] \) we set \( y = |d_A|(x, x') \) where \(|d_A|(x, x') = \theta_A(x) + \theta_A(x') \).

\( iii \) For \( \varepsilon_A : !A[X^y] \to A[X^x] \) we set \( y = |\varepsilon_A|(x) \) where \(|\varepsilon_A|(x) = \theta_A(x) \).
(iv) For \( e_A : !A[X^y] \to 1 \) we set \( y = |e_A|() \) where \(|e_A|() = \theta_A(2) \).

Other morphisms do not modify the labels. For example, if \( \bar{\varphi}_{A,B} : !A[X^y] \otimes !B[X^{y'}] \to !(A[X^z] \otimes B[X^{x'}]) \) then \( y = x \) and \( y' = x' \). Moreover we define as \(|f| = |f| \). For \(|f \otimes g|(x)\), we apply one of \(|f|\) and \(|g|\) according to which side of tensor \( X^x \) lies in.

These functions are applied to each occurrence separately. For instance, when \( A = !(X \otimes!!X) \) and \( d_A : !(X^{y_1} \otimes!!X^{y_2}) \to !(X^{x_1} \otimes!!X^{x_2}) \otimes!(X^{x'_1} \otimes!!X^{x'_2}) \) then \( y_1 = |d_A|(x_1, x'_1) = 2(1 + x_1) + 2(1 + x'_1) \) and \( y_2 = |d_A|(x_2, x'_2) = 2(1 + 4x_2) + 2(1 + 4x'_2) \).

5.25. EXAMPLE. We consider morphisms in Example 5.21. If we label as in \( !X^y \xrightarrow{\delta X} \!X \xrightarrow{d_X} \!X \otimes!!X \xrightarrow{\delta e_X} \!X^z \otimes 1 \), we have \( y = 22^{x+4} \). It contracts to \( \!X^y' \xrightarrow{\delta X'} \!X \xrightarrow{d_X'} \!X \xrightarrow{\delta e_X} \!X^z \otimes 1 \), for which we have \( y' = 2^x + 2^2 \). It further contracts to \( \!X^y'' \xrightarrow{d_X''} \!X \xrightarrow{\delta e_X''} \!X^z \otimes 1 \), for which \( y'' = 2^x + 2 \). Finally it contracts to \( \!X^y''' \xrightarrow{d_X'''} \!X \xrightarrow{\delta e_X'''} \!X^z \otimes 1 \) for which \( y''' = 2^x \). Let us observe \( 2^{2x+4} > 2^x + 2^2 > 2^x + 2 > 2^x \). We will verify this is universally true.

5.26. REMARK. Tranquilli assigns natural numbers to show termination of net rewriting [27]. Although correspondence to the definition is not clear, at least, and naturally, the assignment to diagonal \( d \) is similar.

5.27. LEMMA. Algebraic reduction in a strict composite algebraic morphism decreases the natural numbers involved in the redex.

**Proof.** As \( \varphi \)-type contraction never occurs, it suffices to consider rule (1) through (12). Since \( 1 \) is not involved, in definition \( \theta_{A \otimes B}(x) = d + \theta_A(x) \), the number \( d \) is greater than or equal to 1. We verify several subtle cases, leaving the others to the reader.

Rule (1). Suppose \(!!(A[X^z])\) where \( X^z \) denotes an arbitrary occurrence in \( A \). Then \( |\delta_A !\delta_A|(x) = 2^\theta_A(\theta^2_A(x)) \) while \( |\delta_A \delta_A|(x) = 2^\theta_A(2\theta^2_A(x)) \) as \( \theta_A(x) = 2\theta_A(x) \). Since \( \theta_A(x) < 2\theta_A(x) \) we have \(|\delta_A !\delta_A|(x) < |\delta_A \delta_A|(x) |\).

Rule(3). Suppose that \(!!(A[X^z] \otimes A[X^{z'}])\) displays two occurrences at the corresponding same positions in \( A \). Then \( |d_A (\delta_A \otimes \delta_A) !\delta_A \delta_A|(x, x') = \theta_A(2\theta_A(x') + \theta_A(2\theta_A(x')) \) while \( |\delta_A !\delta_A (x, x') = 2^\theta_A(\theta_A(x) + \theta_A(x')) \). So, putting \( u = \theta_A(x) \) and \( v = \theta_A(x') \), we must show \( \theta_A(2^u) + \theta_A(2^v) < 2\theta_A(u + v) \). This inequality is verified by induction. If \( \theta_A \) is an identity function, the inequality amounts to \( 2^u + 2^v < 2^{u+v} \), which is correct as we have \( u, v \geq 2 \) for which \( \theta_A = 2 \theta_B \) the inequality amounts to \( 2\theta_B(2^u) + 2\theta_B(2^v) < 2^{2\theta_B(2^u+v)} \). By induction hypothesis \( \text{LHS} < 2 \cdot 2^{\theta_B(u+v)} \). By \( 1 < \theta_B(u+v) \) this is smaller than \( \text{RHS} \). If \( \theta_A = d + \theta_B \) the inequality amounts to \( 2d + \theta_B(2^u) + \theta_B(2^v) < 2^d + \theta_B(u+v) \). By induction hypothesis \( \text{LHS} < 2d + 2^\theta_B(2^u+v) \). This is less than or equal to \( 2d + 2^\theta_B(u+v) \leq \text{RHS} \). For the last inequality we use \( 1 \leq d \).

\( \bar{\psi} \)-type rules. These are manipulated uniformly. For example, let us consider rule (9). Suppose \(!!(A[X^z] \otimes B)\). The case where specified \( X \) occurs in \( B \) is similar. Then \(|\delta_A \otimes \delta_B \bar{\psi}_{A,B} !\bar{\psi}_{A,B}|(x) = |\delta_A|(x) = 2^\theta_A(x) \) while \(|\bar{\psi}_{A,B} \delta_A \otimes \delta_B|(x) = |\delta_A \otimes B|(x) = 2^d + \theta_A(x) \) where \( d \) is the number of occurrences of \( X \) in \( B \). Since \( 1 \leq d \) the former is smaller than the latter.
Next, we verify that restricted naturality reductions decrease the assigned natural numbers. We give several lemmata toward it.

5.28. Lemma. Inequality $1 + \theta_A(x) \leq \theta_A(1 + x)$ holds.

Proof. Easy.

5.29. Lemma. Inequality $2\theta_A(x) \leq \theta_A(2d + 2x)$ holds where $d$ is one less than the number of occurrences of $X$ in $A$.

Proof. If we integrate serial applications of tensor, we have

$$\theta_A(x) = 2^{k_0}d_0 + 2^{k_0+k_1}d_1 + \cdots + 2^{k_0+k_1+\cdots+k_{q-1}}d_{q-1} + 2^{k_0+k_1+\cdots+k_q}x.$$ 

It is so, for example, if $A = t^{k_0}(A_0 \otimes t^{k_1}(A_1 \otimes \cdots t^{k_{q-1}}(A_{q-1} \otimes t^{k_q}X)\cdots))$ and each $A_i$ contains $d_i$ occurrences of $X$. We have $d_i \geq 1$. Two numbers $k_0, k_q$ in both ends are non-negative and other $k_i$ is greater than or equal to 1. If $q = 0$, i.e., when $A$ is $!A X$, we have $\theta_A(x) = 2^{k_0}x$. Equality holds in this case as $d = 0$. If $q > 0$ we note $\max\{d_0, d_1, \ldots, d_{q-1}\} \leq d$. Let us observe that $2^{k_0} + 2^{k_0+k_1} + \cdots + 2^{k_0+k_1+\cdots+k_{q-1}} < 2^{k_0+k_1+\cdots+k_{q-1}+1}$ holds as is clear if regarded as numbers in base 2. Therefore we have $2^{k_0}d_0 + 2^{k_0+k_1}d_1 + \cdots + 2^{k_0+k_1+\cdots+k_{q-1}}d_{q-1} < 2^{k_0+k_1+\cdots+k_{q-1}+1}d \leq 2^{k_0+k_1+\cdots+k_{q-1}+k_q}(2d)$. Thus $2\theta_A(x) < 2^{k_0+k_1+\cdots+k_q}(2d) + \theta_A(2x) = \theta_A(2d + 2x)$.

5.30. Lemma. Let $d$ be one less than the number of occurrences of $X$ in $A$. Then $d < \theta_A(x)$ holds.

Proof. The definition of $\theta_A(x)$ sums up all occurrences of $X$ through recursive calls.

5.31. Lemma. Restricted naturality reduction in a strict composite algebraic morphism decreases the natural numbers involved in the redex.

Proof. Consider redex $!f; u_A$. It suffices to prove the case where $f$ consists of single $\delta$ or of single $\tilde{\phi}$. Since the latter is simpler, we prove it first. Namely $f = F(\tilde{\varphi}_{A \otimes A'}) : F(!!A \otimes A') \to F(!!(A \otimes A'))$ for a functor $F$. For example, let us suppose $u = \delta$. We have $|\delta_{F(!A \otimes A')} : F(!!(A \otimes A'))| = 2^{\theta_{F(!!(A \otimes A'))}(x)}$ while $|F(\tilde{\varphi}_{A \otimes A'})| = 2^{\theta_{F((A \otimes A'))}(x)}$. If $X^x$ occurs in $A$ and if $d$ is the number of occurrences of $X$ in $A'$, we have $\theta_{A \otimes u}(x) = d + 2\theta_A(x) < 2(d + \theta_A(x)) = \theta_{(A \otimes A')}(x)$ as $1 \leq d$. Therefore the former is smaller than the latter. We note that this relies only on comparison between $\theta_{A \otimes u}$ and $\theta_{(A \otimes A')}$.

Next we deal with the case $f = F(\delta_A) : F(!A) \to F(!!A)$.

Rule (18). $|\delta_{F(!A)} : !!F(\delta_A)| = 2^{\theta_{F(!!A)}(x)}$ while $|F(\tilde{\varphi}_{A \otimes A'})| = 2^{\theta_{F(!!(A \otimes A'))}(x)}$. So, putting $u = \theta_A(x)$, we must show $\theta_F(2\theta_A(2^u)) < \theta_A(2^{\theta_{F(!!(A' u))}})$. This is verified by induction on $F$. We start with case $\theta_F = 2\theta_A$. By induction hypothesis and Lem. 5.29, (LHS) = $2\theta_A(2^{\theta_A(2^u)}) < \theta_A(2d + 2 \cdot 2^{\theta_A(2^u)})$ where $d$ is one less than the number of occurrences of $X$ in $A$. On the other hand (RHS) = $\theta_A(2^{2\theta_A(2^u)})$. So, if we put $t =$
\( \theta_G(4u) \), it suffices to show \( 2d + 2 \cdot 2^t \leq 2^{2t} \). As \( u \leq t \) we can assume \( 0 \leq d < t \) by Lem. 5.30. If \( d = 0 \) and \( t = 1 \) the inequality is directly checked. Assume \( t \geq 2 \). Then \( 2d + 2 \cdot 2^t < 2^t + 2 \cdot 2^t < 2 \cdot 2^t + 2 \cdot 2^t = 2^{2+t} \leq 2^{2t} \) holds. The next case is \( \theta_F = c + \theta_G \). By induction hypothesis and Lem. 5.28, (LHS) = \( c + \theta_G(2\theta_A(2^u)) \) \( < \theta_A(c + 2\theta_G(4u)) \). Applying 1 + 2 \( t \) \( < 2^{1+t} \) repeatedly, we conclude that it is smaller than \( \theta_A(2^{c + \theta_G(4u)}) \) (RHS). The base case is that \( \theta_F \) is an identity function. By Lem. 5.29, (LHS) = \( 2\theta_A(2^u) \) \( \leq \theta_A(2d+2\cdot2^u) \) where \( d \) is one less than the number of \( X \) in \( A \). On the other hand, (RHS) = \( \theta_A(2^u) \). So it suffices to show \( 2d + 2 \cdot 2^u < 2^{4u} \), for which a sharper result has been verified in the first case.

Rule (19). \( |e_F(A); F(\delta_A)|(x) = \theta_F(2\theta_A(2^{\theta_A(x)})) \) while \( |!F(\delta_A); e_F(A)| = 2^{\theta_A(\theta_F(4\theta_A(x)))} \). So we must prove \( \theta_F(2\theta_A(2^u)) < 2^{\theta_A(\theta_F(4^u))} \) with \( u = \theta_A(x) \). It is verified by induction on \( F \). If \( \theta_F \) is an identity, we show \( 2\theta_A(2^u) < 2^{\theta_A(4^u)} \) by induction on \( A \). If \( \theta_A \) is an identity, \( 2 \cdot 2^u < 2^{4^u} \), that is obvious. If \( \theta_A = d + \theta_B \), inner induction hypothesis implies (LHS) \( < 2d + 2^{\theta_B(4^u)} \). So, for \( t = \theta_B(4^u) \), we show \( 2d + 2^t \leq 2^{d+t} \). We can assume \( 1 \leq d < t \) by Lem. 5.30. Then the inequality is justified as \( 2d + 2^t < 2^d + 2^t < 2^{d+t} \). If \( \theta_A = 2\theta_B \) then by inner induction hypothesis (LHS) \( < 2 \cdot 2^t < 2^{2\theta_B(4^u)} \) (RHS). This finishes the base case. If \( \theta_F = c + \theta_G \) then induction hypothesis (LHS) \( < c + 2^{\theta_A(\theta_F(4^u))} \), which is, by \( 1 + 2 \cdot 2^t \) \( < 2^{1+t} \) and Lem. 5.28, smaller than \( 2^{\theta_A(c + \theta_G(4u))} \) (RHS). If \( \theta = 2\theta_G \) then by induction hypothesis (LHS) \( < 2 \cdot 2^t \) \( < 2^{\theta_A(\theta_G(4^u))} \), which is, by Lem. 5.28, smaller than \( 2^{\theta_A(1+\theta_G(4^u))} \) \( < 2^{\theta_A(2\theta_G(4^u))} \) (RHS).

Rule (20). \( |d_F(A); !F(\delta_A) \otimes !F(\delta_A)|(x, x') = \theta_F(2\theta_A(2^{\theta_A(x)})) + \theta_F(2\theta_A(2^{\theta_A(x)})) \) while \( |!F(\delta_A); d_F(A)| = 2^{\theta_A(\theta_F(4\theta_A(x))) + \theta_F(4\theta_A(x))} \). Putting \( u = \theta_A(x) \) and \( v = \theta_A(x') \), we must show \( \theta_F(2\theta_A(2^u)) + \theta_F(2\theta_A(2^u)) < 2^{\theta_A(\theta_F(4^u) + \theta_F(4^u))} \). It is verified as in the previous case. It turns out to show \( 4d + 2^t \leq 2^{d+t} \) meanwhile. This is true for \( 1 \leq d < t \).

Rule (21). \( |e_F(A)| = \theta_F(2\theta_A(2)) \) while \( |!F(\delta_A); e_F(A)| = 2^{\theta_A(\theta_F(4\theta_A(2))} \). Trivially the former is smaller than the latter.

5.32. Lemma. Let us consider a strict composite algebraic morphism \( s : A[X^y_1, X^y_2, \ldots, X^{y_n}] \to B[X^{x_1}, X^{x_2}, \ldots, X^{x_m}] \). Suppose that \( s \) contracts to \( t : A[X^{y'_1}, X^{y'_2}, \ldots, X^{y'_n}] \to B[X^{x_1}, X^{x_2}, \ldots, X^{x_m}] \) by a reduction sequence where naturality reductions are restricted. Then \( y_i \leq y'_i \) for all \( i \) and strictly \( y_i < y'_i \) for one or more \( i \), provided \( x_j \geq 2 \) for all \( j \).

Proof. Since composition of morphisms are realized by composition of functions and all involving functions are strictly increasing, the local arguments proved in Lem. 5.27 and 5.31 imply the lemma.

5.33. Lemma. Let us consider a strict composite algebraic morphism \( s : A[X^y_1, X^y_2, \ldots, X^{y_n}] \to B[X^{x_1}, X^{x_2}, \ldots, X^{x_m}] \) and a composite algebraic morphism, \( t \) and \( h \) is normal. Two punctuations are called frontiers. For reference, \( s \) is called desme and \( t \) fief. The frontiers are not absolute since definitions (i) and (ii) in Def. 5.20 are not exclusive.

We consider the following condition: \( f = s; t; h \) contains no naturality redaxes except restricted ones and \( t; h \) contains no redaxes other than \( \varphi_0 \)-type. We implicitly assume
that $s$ is a strict composite algebraic morphism, $t$ is a composite algebraic morphism, and $h$ is normal.

5.33. **Lemma.** Suppose that $f = s; t; h$ fulfills the condition above and it contracts to $f'$. Then there is a decomposition $f' = s'; t'; h'$ satisfying the condition. Moreover, for arbitrary $f' = s'; t'; h'$ satisfying the condition, there is a decomposition $f = \tilde{s}; t; h$ satisfying the condition such that $s'$ is a contractum of $\tilde{s}$ or a part of it.

**Proof.** The first assertion means that contraction creates no redexes in the fief except $\varphi_0$-type. The second means that contraction creates no fresh part that can be added to the demesne beyond the area attached thereto at the outset. The following are crucial cases. Suppose $1 \xrightarrow{\varphi_0} !1 \xrightarrow{\delta} !!!1$ contacts to $1 \xrightarrow{\varphi_0} !1 \xrightarrow{h} !!!1$ in the fief. When they are followed by $!!!1 \xrightarrow{\delta} !!!1$ for instance, contraction creates a naturality redex $!\varphi_0; \delta$ in the fief that violates hypothesis. This situation is precluded, however, since, if so, the fief contained a prohibited $\delta$-type redex $\delta; \delta$ beforehand. Next suppose $1 \otimes !A \xrightarrow{\varphi_0} !1 \otimes !A \xrightarrow{\varphi} !(1 \otimes A)$ contracts to $1 \otimes !A \xrightarrow{h} !A \xrightarrow{e} !(1 \otimes A)$ in the fief. Two sides encircling the part may form a new redex, or the right side of the part may be newly attached to the demesne $s$, as the intervening $\varphi_0$ and $\varphi$ vanish. For example, if the right side is $!(1 \otimes A) \xrightarrow{\delta} !!!(1 \otimes A)$ such a trouble may happen. However, if so, the fief has a redex $\varphi; \delta$ that is prohibited by hypothesis. Next consider naturality of $\varphi$. As typical in equivalence of $!!C \otimes !A \xrightarrow{\varphi} !(C \otimes A)$ $!!e$ disappears. Hence all redex crossing the frontier can be engulfed in the demesne by extending $s$. For example, if $\delta; \varepsilon$ crosses the frontier, $\delta$ in the demesne while $\varepsilon$ in the fief, we may enlarge the demesne so that $\varepsilon$ is a part of it. So if we take sufficiently large $\tilde{s}$ then $s'$ is its contractum or a part of it. 

5.34. **Lemma.** If $f = s; t; h$ fulfills the condition introduced before Lem. 5.33, $f$ satisfies the (strong) termination property.

**Proof.** By Lem. 5.33, we can enlarge demesne $s$ at the outset so that subsequent reductions in the demesne are all done in the descendants of $s$. We assign natural numbers to each $X$ in the demesne as in Def. 5.24. By Lem. 5.32 the sum of associated natural numbers $y_i$ strictly decreases by contractions in the demesne. Occasionally $\varphi_0$-type reductions occur in the fief, but they do not alter the associated natural numbers. So the sum reaches constant eventually. Thereafter only $\varphi_0$-type reductions can occur. The sequence of $\varphi_0$-type reductions must be finite since they reduce the number of algebraic morphisms other than $\varphi_0$. For example $\varphi_0; \delta$ contracts to $\varphi_0; !\varphi_0$ where $\delta$ disappears.

5.35. **Proposition.** Algebraic morphisms $\delta, \varepsilon, d, e, \varphi$, $\varphi_0$ are reducible.
Proof. We describe the case of $\delta_A : !A \to !!A$. By Lem. 5.14 it suffices to prove that, for any $f : X \to A$ in $R^+(A)$ and any $g : !!A \to Y$ in $R^-(!!A)$, $!f; \delta_A; g$ terminates. By naturality reduction, it contracts to $\delta_X; !!f; g$. By Lem. 5.10, $!!f$ lies in $R^+(!!A)$. Hence $!!f; g$ contracts to a normal form $h$. Now $\delta_X; h$ satisfies hypothesis of Prop. 5.34 if we set the demesne to be $\delta_X$ and the fief to be empty (an identity). So it terminates. The same argument applies to $\tilde{\varphi}_{X,Y}$ that is a strict composite algebraic morphism in $X, Y$. We use Lem. 5.15. For $\varphi_0$, we take the demesne to be empty and the fief to be $\varphi_0$.

All ingredients are reducible by Prop.5.16, 5.18, 5.19, and 5.35. Moreover, Prop. 5.17 shows that all constructions preserve the property of being reducible. With Prop. 5.13, we conclude weak termination property:

5.36. Theorem. Every morphism terminates.

5.37. Remark. Each morphism terminates under the following specific strategy. First, any $\beta$-redexes are contracted. If no $\beta$ remains, naturality redexes are contracted. Finally, if there are neither $\beta$ nor naturality, the rightmost redexes are contracted. The verification of the theorem remains applicable if we interpret “terminate” as termination under this strategy. The case of dual in Prop. 5.17 as well as Prop. 5.10, and Lem. 5.35 influence the strategy. The rightmost redexes are not unique in general, as both $f$ and $g$ may have redexes in $f \otimes g$ for example.

We conjecture that strong normalizability is fulfilled. We define strong termination to hold if all infinite reduction sequence repeats only reversible reductions after some finite number of reduction steps.

5.38. Remark. We obtain a cartesian closed category if we enforce $!$ to be an identity functor. Tensor turns out to be cartesian product $\times$ and the unit object terminal. Accordingly, we obtain a reduction system for a free cartesian closed category. It contains reduction $e_{AXB} \Rightarrow e_A \times e_B$ and $e_1 \Rightarrow 1$ among others. Note that reduction depends on subscripts, i.e., the shape of objects. This system has a looping reduction sequence $e_{AX1} \Rightarrow e_A \times e_1 \Rightarrow e_A \times 1 \cong e_{AX1}$. In our system, rule (17) blocks this to happen. Moreover, all reduction rules make sense if we omit subscripts.

A. Coherence as congruence

A.1. Symmetric monoidal closed base. We regard two legs of the following diagrams to be equivalent. We denote this by $\circlearrowleft$. First we have coherence of symmetric monoidal category [19]:

\[
\begin{array}{c}
(A \otimes B) \otimes C \otimes D \\
(A \otimes B) \otimes (C \otimes D) \\
A \otimes ((B \otimes C) \otimes D) \\
A \otimes (B \otimes (C \otimes D))
\end{array}
\]

\[
\begin{array}{c}
\alpha \otimes 1 \\
\alpha \\
\alpha \\
1 \otimes \alpha
\end{array}
\]

\[
\begin{array}{c}
1 \otimes 1 \otimes B \\
1 \otimes 1 \\
A \otimes 1 \\
A \otimes B
\end{array}
\]

\[
\begin{array}{c}
\alpha \\
\rho \otimes 1 \\
\circlearrowleft
\end{array}
\]
The following are naturality of structural isomorphisms and $\tilde{\varphi}$:

\[
\begin{align*}
(A \otimes B) \otimes C & \xrightarrow{\sigma} A \otimes (B \otimes C) \\
(B \otimes A) \otimes C & \xrightarrow{\alpha} A \otimes (B \otimes C) \\
(B \otimes (C \otimes A)) & \xrightarrow{\varphi} A \otimes (B \otimes C)
\end{align*}
\]

The following are part of coherence of symmetric monoidal functors [11]. The interaction law of $\varphi_0$ and $\tilde{\varphi}$ becomes one of the rewriting rules. So it is not contained here.

\[
\begin{align*}
(A \otimes B) \otimes C & \xrightarrow{\sigma} A \otimes (B \otimes C) \\
(A' \otimes B') \otimes C' & \xrightarrow{\alpha} A' \otimes (B' \otimes C') \\
(A \otimes B) & \xrightarrow{\varphi} B \otimes A \\
(A' \otimes B') & \xrightarrow{\varphi} B' \otimes A'
\end{align*}
\]

The following are part of the coherence of commutative comonoids. The interaction law of $d$ and $e$ turns to be a rewriting rule.

\[
\begin{align*}
((A \otimes B) \otimes C) & \xrightarrow{\alpha} !A \otimes (B \otimes C) \\
((A \otimes B) \otimes C) & \xrightarrow{\varphi} !(A \otimes B) \\
(A \otimes (B \otimes C)) & \xrightarrow{\sigma} !(A \otimes (B \otimes C))
\end{align*}
\]
Naturality of abs and ev:

\[
\begin{array}{ccc}
A & \xrightarrow{\text{abs}_A^B} & B \rightarrow (A \otimes B) \\
f & \circ & 1 \rightarrow (f \otimes 1) \\
A' & \xrightarrow{\text{abs}_{A'}^B} & B' \rightarrow (A' \otimes B')
\end{array}
\]

\[
\begin{array}{ccc}
B \rightarrow A & \xrightarrow{\text{ev}_A^B} & B \\
(1 \otimes f) \otimes 1 & \circ & f \\
(B \rightarrow A') & \xrightarrow{\text{ev}_{A'}^B} & B' \rightarrow A'
\end{array}
\]

A.2. \(\star\)-AUTONOMOUS BASE. Next, we consider the coherence giving equivalence in case that the \(\star\)-autonomous category is the base category. Since cotensor \(\#\) has a symmetric monoidal structure, we add the following:

\[
\begin{array}{ccc}
((A \# B) \# C) \# D & \xrightarrow{\sim} & A \# (B \# (C \# D)) \\
(A \# (B \# C)) \# D & \xrightarrow{\sim} & (A \# B) \# (C \# D) \\
A \# ((B \# C) \# D) & \xrightarrow{\sim} & A \# (B \# (C \# D))
\end{array}
\]

We add naturality of structural isomorphisms of cotensor and weak distribution:

\[
\begin{array}{ccc}
(A \# B) \# C & \xrightarrow{\sim} & A \# (B \# C) \\
(A \# B) \# C & \xrightarrow{\sim} & (A \# B) \# (C \# A) \\
B \# (A \# C) & \xrightarrow{\sim} & (B \# C) \# A \\
B \# (A \# C) & \xrightarrow{\sim} & (B \# C) \# A
\end{array}
\]

The following are the coherence of weak distribution morphisms [8]. Here \(\partial_{ABC} : (A \# B) \otimes C \rightarrow A \# (B \otimes C)\) is induced from \(\partial_{ABC}\) by symmetry of tensor and cotensor. The
label $\sim$ represents appropriate structural isomorphisms.

These three diagrams are obtained if we upend the sheet on which the former three are written so that the back surface comes to the front, reverse the direction of the arrow, and interchange tensor and cotensor.

The coherence of symmetric monoidal functors and of commutative comonoids is the same as in the case of the symmetric monoidal closed base. We dispense with naturality of abs and ev. We comment that it is derived from naturality of structural isomorphisms and weak distribution if we set $B \rightarrow A = A \S B^*$. 

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