Research article

Weilin Zou* and Xinxin Li

Existence results for nonlinear degenerate elliptic equations with lower order terms

https://doi.org/10.1515/anona-2020-0142
Received July 17, 2019; accepted June 17, 2020.

Abstract: In this paper, we prove the existence and regularity of solutions of the homogeneous Dirichlet initial-boundary value problem for a class of degenerate elliptic equations with lower order terms. The results we obtained here, extend some existing ones of [2, 9, 11] in some sense.

Keywords: Degenerate elliptic equations; Lower order terms; Irregular data; $L^1$ data

MSC: 35D30, 35J60, 35J70

1 Introduction

Recently, much attention has been paid to partial differential equations with lower order terms, not only for their physical relevance but also for their mathematical interest. From the mathematical point of view, it is well known that the lower order terms may affect the existence, uniqueness, regularity and asymptotic behavior of solutions to partial differential equations (see e.g. [1, 3–12, 19, 23]). In this paper, we are interested in the existence and regularity of solutions for a class of degenerate elliptic equations with two lower order terms, whose prototype is

\[
\begin{cases}
-\text{div} \left( \frac{a(x, u)\nabla u|^{p-2}\nabla u}{(1 + |u|)^{d(p-1)}} \right) + \nu|u|^{1-1}u = y|\nabla u|^{p-1} + f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega$ is a bounded open subset of $\mathbb{R}^N (1 < p < N)$, $a(x, s)$ is a Carathéodory matrix and $f$ is a measurable function.

As $\theta = 0$, such kind of problems with different lower order terms were studied well in the literature. Without the aim to be complete, let us mention the following relevant works. The quasi-linear case $p = 2$ was treated in [9], where it was shown that the term $\nu|u|^{1-1}u$ was in some sense to guarantee the existence of a solution when the growth of the gradient is superlinear. If there was an $L^1$ interplay between the coefficient of the zero order term and the right-hand side $f$, the existence of bounded solutions was established in [4]. This result was improved in [5] and extended to the parabolic case in [16]. If $p = 2$, $a(x, s)$ was an identity matrix, $f = 0$ and $y = 0$, the existence and non-existence results to problem $(\mathcal{P})$ with two zero order terms were proved in [12]. The case $\nu = 0$ was considered in [1], where it was proved that smallness condition was asked on $f$ to guarantee the existence of a solution. For other related results, see [3], where the existence, multiplicity and non-existence of solutions to a semilinear degenerate elliptic system of Hamiltonian type were proved; see also [19], where a Neumann problem driven by the $p$-Laplacian with singular and convection terms was investigated.

*Corresponding Author: Weilin Zou, College of Mathematics and Information Science, Nanchang Hangkong University, Nanchang, 330063, China. E-mail: wlzou@nchu.edu.cn, Tel.: +8679183863755
Xinxin Li, College of Mathematics and Information Science, Nanchang Hangkong University, Nanchang, 330063, China

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As \( \theta \neq 0 \), it is easy to see that the principal part of problem (\( \mathcal{P} \)) degenerates, hence, a slow diffusion effect may appear as soon as the solution \( u \) becomes large. Such kind of equations could be seen as a reaction model which produces a saturation effect. In case of \( y = 0 \), existence and regularity results of problem (\( \mathcal{P} \)) were established in [2, 13], while the parabolic case of such problem was just treated in [15] (see also [21]). We also mention that the related obstacle problems with \( L^1 \) data were investigated in [24, 25].

When \( p = 2 \) and \( y = 0 \), the author in [11] proved that the zero order term \( v |u|^{p-1} u \) may affect the regularity of solutions of (\( \mathcal{P} \)) under the assumption that \( f \in L^m(\Omega) \) with \( m \geq 1 \). This result was extended to the general case \( p > 1 \) in [10]. For this general case, the stability results were obtained in [14]. Also, it was shown in [7] that the existence of \( W^{1,1}_0(\Omega) \) solutions could be obtained by adding a zero order term. For other relevant papers, see [18, 22] and the references therein.

Motivated by [2, 9, 11], this work studies the regularizing effect of lower order terms on the solutions to problem (\( \mathcal{P} \)) (that is \( v \neq 0 \) and \( y \neq 0 \)). The main results obtained here generalize the previous result of [2, 9, 11] in some sense. The main difficulties are the facts that the differential operator is not coercive on \( W^{1,1}_0(\Omega) \) and the lower order terms have regularizing effects on solutions. To overcome these difficulties, we shall first introduce a class of approximated problems and then establish some estimates for solutions by taking suitable test functions, and finally prove some convergence results to get the existence results.

This paper is organized as follows. In Section 2, we give the assumptions and the main results. In Section 3, we shall prove the main results.

### 2 Assumptions and the main results

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with \( N \geq 2 \), \( 1 < p < N \). Throughout this paper, \( c_i \) or \( \tilde{c}_i \) \((i = 0, 1, 2 \ldots n)\) will denote a positive constant which only depends on the parameters of our problem. For any real number \( \eta > 1 \), we set \( \eta' = \frac{n}{\eta - 1} \) and \( \eta^* = \frac{N \eta}{N - \eta} \). For \( E \subseteq \Omega \), we also denote

\[
\int_E g := \int_E g \, dx, \quad |E| = \text{meas } E.
\]

We shall use the truncation functions \( T_k(s) \) and \( G_k(s) \) defined by

\[
T_k(s) = \max\{-k, \min\{k, s\}\}, \quad G_k(s) = s - T_k(s).
\]

Let us consider the following problem:

\[
\mathcal{P} \begin{cases}
-\text{div}(a(x, u, \nabla u)) = g(x, u) & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \), \( b : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) and \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) are Carathéodory functions satisfying:

There exist constants \( a, \beta, \nu, \gamma \in \mathbb{R}^+ \), and a nonnegative function \( j \in L^p(\Omega) \) such that

\[
a(x, s, \xi) \cdot \xi \geq \frac{\alpha}{(1 + |s|^{p-1})^\theta}|\xi|^p, \quad \text{with } 0 \leq \theta < 1,
\]

\[
|a(x, s, \xi)| \leq \beta |j(x)| + |s|^{p-1} + |\xi|^{p-1}, \quad (2.2)
\]

\[
[a(x, s, \xi) - a(x, s, \zeta)] \cdot [\xi - \zeta] > 0, \quad (2.3)
\]

\[
|b(x, s, \xi)| \leq \gamma |\xi|^p, \quad \text{with } \max\{\frac{p-1}{p}, \frac{1}{p}\} < r < 1,
\]

\[
g(x, s)s \geq \nu |s|^t, \quad \text{with } t > \frac{r + \theta r(p-1)}{1 - r}, \quad (2.5)
\]
for almost every \( x \in \Omega \), for every \( s \in \mathbb{R} \) and for every \( \xi, \zeta \in \mathbb{R}^N \) with \( \xi \neq \zeta \).

Now we give the definition of weak solutions to problem (\( \mathcal{P} \)).

**Definition 2.1.** A measurable function \( u \in W^{1,1}_0(\Omega) \) is called a weak solution of problem (\( \mathcal{P} \)), if \( a(x, u, \nabla u), g(x, u) \) and \( b(x, u, \nabla u) \) are summable functions and

\[
\int_{\Omega} a(x, u, \nabla u) \nabla \varphi + \int_{\Omega} g(x, u) \varphi = \int_{\Omega} b(x, u, \nabla u) \varphi + \int_{\Omega} f \varphi + \int_{\Omega} F \nabla \varphi \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.6}
\]

The main results of this paper are stated as follows.

**Theorem 2.1.** Suppose that assumptions (2.1)-(2.5) hold with \( f \in L^1(\Omega) \) and \( F \in (L^{\frac{p}{p-1}}(\Omega))^N \), then there exists at least a weak solution \( u \in W^{1,q}_0(\Omega) \cap L^t(\Omega) \) of the problem (\( \mathcal{P} \)) for every \( 1 \leq q < q_1 \), where

\[
q_1 = \max\left\{ \frac{pt}{t+(p-1)}, \frac{N(p-1)+(p-1)}{N-1-\theta(p-1)} \right\}.
\]

**Remark 2.1.** The assumption \( F \in (L^{\frac{p}{p-1}}(\Omega))^N \) imposed here is to get the result (3.10) and then to obtain the existence result, see step 2 of the proof of Theorem 2.1. However to get the regularity result \( u \in W^{1,q}_0(\Omega) \cap L^t(\Omega) \), it suffices to assume that \( F \in (L^{\frac{p}{p-1}}(\Omega))^N \), see step 1 of the proof of Theorem 2.1. If \( \theta = 0 \) and \( p = 2 \), the assumption \( F \in (L^{\frac{p}{p-1}}(\Omega))^N \) reduces to \( F \in L^{\frac{p}{p-1}}(\Omega) \), which coincides the result of [9].

**Remark 2.2.** By (2.4) and (2.5) it easy to check that \( \frac{pt}{t+(p-1)} > 1 \) which implies that \( q_1 > 1 \). We remark that the restriction of (2.4) listed here is to simplify the proof and can be replaced by \( \frac{1}{p} < r < 1 \). Indeed, using Young’s inequality, it is easy to see that the above assertion remains true and our results in Theorem 2.1 and Theorem 2.2(see below) are still true in the case that \( \frac{1}{p} < r < 1 \). Thus our results Theorem 2.1 and Theorem 2.2 also cover the results of [9] where \( \theta = 0 \) and \( p = 2 \).

**Theorem 2.2.** Suppose that (2.1)-(2.5) hold, and \( f \in L^m(\Omega) \), where \( 1 < m < \frac{N}{p} \).

1) If \( F = 0 \), then there exists at least a weak solution \( u \in W^{1,q}_0(\Omega) \cap L^m(\Omega) \cap L^{N\frac{p}{p-1}(p-1)\theta(p-1)}(\Omega) \) of the problem (\( \mathcal{P} \)) where \( q = \min\left\{ p, \max\left\{ \frac{ptm}{t+(p-1)}, \frac{pmn^\theta}{(p-1)\theta(p-1)} \right\} \right\} \).

2) If \( F \neq 0 \), then let us assume that \( F \in (L^{\frac{p}{p-1}}(\Omega))^N \). Then there exists a weak solution \( u \in W^{1,q}_0(\Omega) \cap L^m(\Omega) \), where \( b \geq b_0 = \max\left\{ \frac{ptm}{(p-1)(1-\theta)}, \frac{pmn^\theta}{(p-1)\theta(p-1)} \right\} \) and \( q_2 = \min\left\{ p, \frac{ptm}{(p-1)(1-\theta)} \right\} \).

In the case that \( b \geq b_1 = \max\left\{ \frac{ptm}{(p-1)(1-\theta)}, \frac{pmn^\theta}{(p-1)\theta(p-1)} \right\} \), there exists at least a weak solution \( u \in W^{1,q_3}_0(\Omega) \cap L^{\frac{p}{p-1}(p-1)\theta(p-1)}(\Omega) \), where \( q_3 = \min\left\{ p, \frac{pmn^\theta}{(p-1)\theta(p-1)} \right\} \).

Finally, denoting \( q = \max\{q_2, q_3\} \), if \( b \geq \max\{b_0, b_1\} \), then there exists at least a solution \( u \in W^{1,q}_0(\Omega) \cap L^m(\Omega) \cap L^{N\frac{p}{p-1}(p-1)\theta(p-1)}(\Omega) \).

**3 Proof of the main results**

In order to prove the existence results, let us define

\[
b_n(x, s, \xi) = \frac{b(x, s, \xi)}{1 + \frac{1}{q}b(x, s, \xi)}, \quad f_n(x) = \frac{f(x)}{1 + \frac{1}{q}f(x)}, \quad F_n(x) = \frac{F(x)}{1 + \frac{1}{q}F(x)}. \tag{3.1}
\]

We consider the following approximated problems

\[
\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \nabla \varphi + \int_{\Omega} g(x, u_n) \varphi = \int_{\Omega} b_n(x, u_n, \nabla u_n) \varphi + \int_{\Omega} f_n \varphi + \int_{\Omega} F_n \nabla \varphi \quad \forall \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega). \tag{3.2}
\]

The existence of weak solution \( u_n \in W^{1,p}_0(\Omega) \) of (3.2) follows by the classical result of [17].
3.1 Proof of Theorem 2.1.

The proof relies on an approximation procedure which is divided into several steps.

**Step 1:** We prove the following a priori estimates:

\[
\int_{\Omega} |u_n|^{q} \leq c_0 \quad \text{and} \quad \int_{\Omega} |\nabla u_n|^q \leq c_0,
\]

\[
\int_{\Omega} |\nabla G_{\lambda}(u_n)|^q \leq c(k),
\]

for \(1 \leq q < q_1\), where \(c(k)\) does not depend on \(n\) and tends to zero when \(k\) tends to \(+\infty\).

To prove the above estimates, we take \(\varphi = [1 - (1 + |u_n|)^{1-\lambda}]\text{sgn}(u_n)\) as a test function in (3.2), where \(\lambda > 1\) will be chosen later. Using Young’s inequality, it leads to

\[
a(\lambda - 1) \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+\lambda}} + \frac{\nu}{2} \int_{\Omega} |u_n|^t[1 - (1 + |u_n|)^{1-\lambda}]
\]

\[
\leq \frac{a(\lambda - 1)}{2} \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+\lambda}} + \frac{\nu}{2} \int_{\Omega} |u_n|^t[1 - (1 + |u_n|)^{1-\lambda}]
\]

\[
+ \gamma \int_{\Omega} |f|^q + \frac{a(\lambda - 1)}{4} \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+\lambda}} + (\lambda - 1) \left( \frac{4}{\alpha} \right) \frac{1}{\nu} \int_{\Omega} |f|^q.
\]

Choosing \(S > 0\) such that \(1 - (1 + S)^{1-\lambda} = 1/2\), we easily obtain

\[
\frac{1}{2} \int_{\Omega} |u_n|^t \leq \frac{1}{2} \int_{\{x \in \Omega : |u_n| < S\}} |u_n|^t + \frac{1}{2} \int_{\{x \in \Omega : |u_n| > S\}} |u_n|^t \leq \frac{1}{2} \int_{\Omega} |u_n|^t[1 - (1 + |u_n|)^{1-\lambda}].
\]

The above inequality together with (3.5) imply that

\[
\frac{a(\lambda - 1)}{4} \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+\lambda}} + \frac{\nu}{2} \int_{\Omega} |u_n|^t
\]

\[
\leq \left( \frac{\alpha}{2(\lambda - 1)} \right) \frac{1}{\nu} \int_{\Omega} |f|^q + \frac{a(\lambda - 1)}{4} \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+\lambda}} + (\lambda - 1) \left( \frac{4}{\alpha} \right) \frac{1}{\nu} \int_{\Omega} |f|^q.
\]

Observing that \(t > \frac{r + \theta(r-1)}{1-r}\) (see (2.5)), one may choose \(\lambda\) in (3.6) such that \((\theta(r-1)+\lambda)/1-r < t\), i.e. \(1 < \lambda < \max(1/(1-r), \theta(r-1)+1/1-r)\). Hence, we get

\[
\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+\lambda}} \leq c_1 \quad \text{and} \quad \int_{\Omega} |u_n|^t \leq c_1.
\]

Therefore by the two estimates of (3.7), we infer that for \(q \leq \frac{pt}{\theta(p-1)+\lambda} \leq \frac{pt}{\theta(p-1)+1/1-r}\), it results \((\theta(p-1)+\lambda)q/(p-q) \leq t\) and so

\[
\int_{\Omega} |\nabla u_n|^q \leq \int_{\Omega} \frac{|\nabla u_n|^q}{(1 + |u_n|)^{\theta(p-1)+\lambda}q/(p-q)} (1 + |u_n|)^{\theta(p-1)+\lambda}q/p
\]

\[
\leq \frac{c_1^q}{p} \left( \int_{\Omega} (1 + |u_n|)^{\theta(p-1)+\lambda}q/(p-q) \right)^{1-\frac{q}{p}} \leq c_2.
\]

Setting \(\tilde{q} = \frac{N(p-\theta(p-1)-1)}{N-\theta(p-1)-1}\) and \(q_0 = \frac{N(p-\theta(p-1)-1)}{N-\theta(p-1)-1}\), obviously we have \(\tilde{q} < q_0\).
If $q_0 > 1$, then by (2.1), (3.7) and taking $1 < \lambda < p - \theta(p - 1)$, we find that for $1 \leq q < q_0$,

$$C^q \left( \int_{\Omega} |u_n|^q \right)^{\frac{q}{q^*}} \leq \int_{\Omega} |\nabla u_n|^q = \int_{\Omega} \frac{|\nabla u_n|^q}{(1 + |u_n|)^{\theta(p-1) + \lambda q/p}} (1 + |u_n|)^{\frac{(p-1)q}{p}} \leq c_1 \left( \int_{\Omega} (1 + |u_n|)^q \right)^{1 - \frac{q}{q^*}},$$

where $C$ is the Sobolev constant and $q^* = (\theta(p - 1) + \lambda)q/(p - q)$. By the choice of $1 < \lambda < p - \theta(p - 1)$, we observe that $\bar{q}/q^* > 1 - q/p$. Thus by Young’s inequality we easily get

$$\int_{\Omega} |\nabla u_n|^q \leq c_2 für 1 \leq q < q_0. \quad (3.9)$$

Taking $1 < \lambda < \min\{p - \theta(p - 1), \frac{\lambda}{p-1}(p-1)\}$, by (3.7), (3.8) and (3.9), we conclude that (3.3) holds for $1 \leq q < q_0 = \max\{\frac{p}{p-1}, q_0\}$.

If $q_0 \leq 1$, then (3.8) leads to (3.3) for $1 \leq q < \frac{p}{p-1}$. Therefore, for both case $q_0 \leq 1$ and $q_0 > 1$, we conclude that (3.3) holds true.

Finally, the proof of (3.4) is similar at all to that of (3.3), except that the test function is changed to $\varphi = [1 - (1 + |G_k(u_n)|^{1-\lambda})] \text{sgn}(G_k(u_n))$, so we omit the detail here.

**Step 2:** we prove the following convergence result:

$$\|g(x, u_n) - g(x, u)|_{L^1(\Omega)} \to 0. \quad (3.10)$$

To do this, firstly it is easy to derive from the estimates (3.3) that for a subsequence of $\{u_n\}$ (still denote by $\{u_n\}$), and a function $u \in W^{1,q}_0(\Omega) \cap L^1(\Omega)$, such that as $n \to +\infty$ it results:

$$u_n \to u \text{ a.e. in } \Omega \text{ and weakly in } W^{1,q}_0(\Omega) \cap L^1(\Omega). \quad (3.11)$$

Then for fixed $l, d > 0$, taking $T_1 \left[\frac{\lambda}{q}(|u_n| - l)\right] \text{sgn}(u_n)$ as a test function in (3.2), we get by applying Young’s inequality and using (3.3),

$$\frac{a}{d} \int_{\{x \in \Omega: |x| = \bar{u}_n \}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)}} + \int_{\{x \in \Omega: |x| = \bar{u}_n \}} |g(x, u_n)|$$

$$\leq \frac{a}{2d} \int_{\{x \in \Omega: |x| = \bar{u}_n \}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)}} + c_3 \int_{\{x \in \Omega: |x| = \bar{u}_n \}} (1 + |u_n|)^{\frac{(p-1)}{p}} + y \int_{\{x \in \Omega: |x| = \bar{u}_n \}} |\nabla u_n|^p$$

$$+ \int_{\{x \in \Omega: |x| = \bar{u}_n \}} |f| + \frac{a}{2d} \int_{\{x \in \Omega: |x| = \bar{u}_n \}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)}} + c_4 \left( \int_{\{x \in \Omega: |x| = \bar{u}_n \}} |f|^\frac{p}{p-1} \right)^{\frac{p-1}{p}},$$

where $c_3 = (1 - r)(\frac{2d}{p})^r(1-r)$ and $c_4 = \frac{1}{d} \left( \frac{2d}{p} \right)^r(1-r) c_0^\frac{p}{p-1}$.

By (2.5) and (3.3), we have

$$c_3 \int_{A(l)} (1 + |u_n|)^{\frac{p(p-1)}{p}} \leq c_3 \left( \int_{A(l)} (1 + |u_n|)^r \right)^{\frac{p(p-1)}{p}} \leq c_3 |A(l)|^{\frac{p(p-1)}{p}} \leq c_5 \frac{A(l)}{l^{r(p-1)-1}},$$

where $c_5 = 2^\frac{p(p-1)}{p} c_0^\frac{p}{p-1} c_3$ and $A(l) = \{ x \in \Omega : l \leq |u_n| \leq l + d \}$.

The above two inequalities and (2.5) yield

$$\int_{\Omega} |g(x, u_n)| = \int_{\Omega} |g(x, u_n)|_{\{x \in \Omega: |x| \leq \bar{u}_n \}} \leq \int_{\{x \in \Omega: |x| \leq \bar{u}_n \}} |g(x, u_n)|$$

$$\leq \int_{\Omega} |g(x, u_n)|_{\{x \in \Omega: |x| \leq \bar{u}_n \}} + e(l), \quad (3.12)$$
where the term \( \varepsilon(l) \) satisfying \( \lim_{l \to +\infty} \varepsilon(l) = 0 \) is denoted by

\[
\varepsilon(l) = y\|u_n\|_{L^p([\Omega])}^{p^r} \quad \text{for} \quad \{x \in \Omega : |u_n| \geq l + d\} \frac{1}{\varepsilon^p} + \int_{\{x \in \Omega : |u_n| < d\}} |f| + c_5 |A(I)| \frac{\varepsilon(1, d) |u_n|}{\varepsilon^p} + c_4 \left( \int_{\Omega} |F|_{\eta, \eta - \eta} \right)^{\frac{1}{\eta}}.
\]

By applying (3.12), Lebesgue’s dominated convergence theorem and Fatou lemma, it can be concluded that

\[
\int_{\Omega} |g(x, u_n)| \to \int_{\Omega} |g(x, u)|, \quad \text{as} \quad n \to +\infty.
\]

Combining (3.13) with (3.11), we obtain the desired result (3.10).

**Step 3:** End the proof.

To do this, for any \( h > 0 \) let us take \( T_h(u_n) \) as a test function in (3.2), we easily obtain that

\[
\int_{\Omega} |\nabla T_h(u_n)|^p \leq c_6 h^{(p-1)+} + c_7.
\]

Then there exists a subsequence (still denoted \( \{u_n\} \)) such that

\[
T_h(u_n) \rightharpoonup T_h(u) \quad \text{weakly in} \quad W^{1,p}_0(\Omega).
\]

Then use the argument of [20] (see also [6, 8, 9]) one may get

\[
\nabla u_n \rightharpoonup \nabla u \quad \text{a.e. in} \Omega.
\]

By (3.14)-(3.16) and Fatou lemma, we conclude

\[
\int_{\Omega} |\nabla T_h(u)|^p \leq c_8 \quad \text{and} \quad \int_{\Omega} |\nabla T_h(u_n) - \nabla T_h(u)|^p \leq c_9,
\]

which gives

\[
k^p|\Omega \setminus E_{k,n}| \leq \int_{\Omega \setminus E_{k,n}} |\nabla T_h(u_n) - \nabla T_h(u)|^p \leq \int_{\Omega \setminus E_{k,n}} |\nabla T_h(u_n) - \nabla T_h(u)|^q \leq c_9,
\]

where \( E_{k,n} := \{x \in \Omega : |\nabla T_h(u_n)(x) - \nabla T_h(u)(x)| \leq k\} \). Then we obtain

\[
\int_{\Omega} |\nabla T_h(u_n) - \nabla T_h(u)|^q \leq \int_{E_{k,n}} |\nabla T_h(u_n) - \nabla T_h(u)|^q + \int_{\Omega \setminus E_{k,n}} |\nabla T_h(u_n) - \nabla T_h(u)|^q \leq c_{10} k^{q-p},
\]

where we have used the results (3.16)-(3.18) and Hölder inequality. Thus, we get

\[
\lim_{n \to +\infty} \sup_{\Omega} \int_{\Omega} |\nabla T_h(u_n) - \nabla T_h(u)|^q \leq c_{10} k^{q-p} \quad \forall k > 0,
\]

which implies that

\[
\int_{\Omega} |\nabla T_h(u_n) - \nabla T_h(u)|^q \to 0 \quad n \to +\infty.
\]

Combining this result with (3.4), we deduce that

\[
u_n \to u \quad \text{strongly in} \quad W^{1,q}_0(\Omega).
\]

Let \( n \to \infty \) in (3.2), by (2.2), (2.4), (2.5), (3.11) and (3.19), it follows that \( u \) is a weak solution of problem (\( \mathcal{P} \)) in the sense of Definition 2.1. Thus, the proof of Theorem 2.1 is finished.
3.2 Proof of Theorem 2.2.

**Step 1:** We first prove the result when $F = 0$.

Let us take $\varphi = [(1 + |u_n|)^{\delta} - 1]\text{sgn}(u_n)$ in (3.2), where $\delta > 0$ will be chosen later. We get after using Hölder’s inequality and Young’s inequality, 

$$
\frac{a\delta}{2} \int_{\Omega} |\nabla u_n|^p (1 + |u_n|)^{\delta - 1 - \theta(p - 1)} + \int_{\Omega} g(x, u_n)(1 + |u_n|)^{\delta - 1}\text{sgn}(u_n)
$$

$$
\leq \tilde{c}_0 \left( \int_{\{x \in \Omega : |u_n| \leq S_0\}} (1 + |u_n|)^{\frac{\delta - 1 - \theta(p - 1)}{1 - r}} + \int_{\{x \in \Omega : |u_n| > S_0\}} (1 + |u_n|)^{\frac{\delta - 1 - \theta(p - 1)}{1 - r}} \right) + ||f||_{L^m(\Omega)} \left( \int_{\Omega} (1 + |u_n|)^{\delta m} \right)^{\frac{1}{m}}.
$$

(3.20)

Observing that as $\lim_{s \to +\infty} \frac{s^\prime[(1 + s)^{\delta - 1} - 1]}{(1 + s)^\delta} = 1$, there exists $S_0 = S_0(\delta, t) > 0$ such that 

$$
\tilde{c}_1 \frac{(1 + s)^{\delta + t}}{2}, \quad \forall s \geq S_0.
$$

Thus, by (2.5) and (3.20), we obtain

$$
\int_{\Omega} g(x, u_n)(1 + |u_n|)^{\delta - 1}\text{sgn}(u_n) \geq \frac{\nu}{2} \int_{\{x \in \Omega : |u_n| > S_0\}} (1 + |u_n|)^{\delta + t}. \quad (3.21)
$$

By (3.20), (3.21) and notice that $t > [r + \theta r(p - 1)]/(1 - r)$, we infer that

$$
\frac{a\delta}{2} \int_{\Omega} |\nabla u_n|^p (1 + |u_n|)^{\delta - 1 - \theta(p - 1)} + \frac{\nu}{2} \int_{\{x \in \Omega : |u_n| > S_0\}} (1 + |u_n|)^{\delta + t}
$$

$$
\leq \tilde{c}_0 \left( \int_{\{x \in \Omega : |u_n| \leq S_0\}} (1 + |u_n|)^{\frac{\delta - 1 - \theta(p - 1)}{1 - r}} + \int_{\{x \in \Omega : |u_n| > S_0\}} (1 + |u_n|)^{\frac{\delta - 1 - \theta(p - 1)}{1 - r}} \right) + ||f||_{L^m(\Omega)} \left( \int_{\Omega} (1 + |u_n|)^{\delta m} \right)^{\frac{1}{m}}.
$$

(3.22)

By choosing $\delta m' = \delta + t$ (i.e. $\delta = t(m - 1)$ in (3.22), we have

$$
\frac{a\delta}{2} \int_{\Omega} |\nabla u_n|^p (1 + |u_n|)^{\delta - 1 - \theta(p - 1)} + \frac{\nu}{4} \int_{\{x \in \Omega : |u_n| > S_0\}} (1 + |u_n|)^t \leq \frac{\nu}{8} \int_{\{x \in \Omega : |u_n| > S_0\}} (1 + |u_n|)^m + \tilde{c}_2, \quad (3.23)
$$

which implies that

$$
\int_{\Omega} |\nabla u_n|^p (1 + |u_n|)^{(m - 1) - 1 - \theta(p - 1)} \leq \tilde{c}_3 \quad \text{and} \quad \int_{\Omega} (1 + |u_n|)^m \leq \tilde{c}_3.
$$

Obversely, we have

$$
\int_{\Omega} |\nabla u_n|^p \leq \tilde{c}_3, \quad \text{if } t(m - 1) - \theta(p - 1) - 1 \geq 0. \quad (3.24)
$$

Similarly to (3.8), in the case $t(m - 1) - \theta(p - 1) - 1 < 0$ we get

$$
\int_{\Omega} |\nabla u_n|^{q_4} \leq \tilde{c}_4, \quad \text{with } q_4 = \min\{p, \frac{ptm}{t + 1 + \theta(p - 1)}\}. \quad (3.25)
$$

To end this proof in case $F = 0$, we shall choose a different $\delta$ in (3.22). Indeed, similar to (3.21) we infer that for $S_1(\text{independent of } n)$ large enough,

$$
\frac{a\delta}{2} \int_{\Omega} |\nabla u_n|^p (1 + |u_n|)^{\delta - 1 - \theta(p - 1)} = \frac{p^p a\delta}{2[\delta - 1 - \theta(p - 1) + p]^p} \int_{\Omega} \left| \frac{(1 + |u_n|)^{\delta - 1 - \theta(p - 1) - 1}}{p} \right|^p.
$$
Using Young’s inequality, we easily obtain that

\[
\int_{\Omega} \left( 1 + |u_n| \right)^{\frac{p-1}{p}} \geq \tilde{c}_5 \left( \int_{|u_n| > \delta_1} \left( 1 + |u_n| \right)^{\frac{\delta(p-1)}{p}} \right)^{\frac{1}{p'}}. \tag{3.26}
\]

By (3.22) and (3.26), it follows that

\[
\left( \int_{\Omega} \left( 1 + |u_n| \right)^{\frac{\delta(p-1)}{p}} \right)^{\frac{1}{p'}} \leq \tilde{c}_6 + \tilde{c}_6 \left( \int_{\Omega} \left( 1 + |u_n| \right)^{\delta m'} \right)^{\frac{1}{p'}}. \tag{3.27}
\]

Let us choose \( \delta m' = \frac{\delta(p-1)}{p} \) (i.e. \( \delta = \frac{p'(p-1)(1-\theta)}{m' p - p'} \)), then we have \( \delta m' = \frac{m' p'(p-1)(1-\theta)}{m' p - p'} \). Hence, applying Young’s inequality in (3.27), we get

\[
\int_{\Omega} |u_n|^{\delta m'} \leq \tilde{c}_7. \tag{3.28}
\]

Substituting (3.28) into (3.22), we find

\[
\int_{\Omega} |\nabla u_n|^p \left( 1 + |u_n| \right)^{\delta-\delta(p-1)} \leq \tilde{c}_8, \quad \delta = \frac{p'(p-1)(1-\theta)}{m' p - p'}. \tag{3.29}
\]

In the case that \( \delta \geq 1 + \theta(p-1) \) (i.e. \( m \geq \frac{Np}{(1-\theta)(N(p-1)+p'p)}} \)), we have

\[
\int_{\Omega} |\nabla u_n|^p \leq \tilde{c}_8, \tag{3.30}
\]

while in the case that \( \delta < 1 + \theta(p-1) \)(that is \( m < \frac{Np}{(1-\theta)(N(p-1)+p'p)}} \), combining (3.28) with (3.29) and arguing as in (3.8) we obtain

\[
\int_{\Omega} |\nabla u_n|^q \leq \tilde{c}_9 \quad \text{with} \quad q_5 = \frac{p m' p'(p-1)(1-\theta)}{(p-1)[p'(m' - 1) + \theta m'(p - p') + m' - 1]} \tag{3.31}
\]

Finally, by (3.24), (3.25), (3.28), (3.30) and (3.31), arguing as in the step 2 and step 3 we conclude that the result of this theorem holds true when \( F = 0 \).

If \( F \neq 0 \), then in the right-hand side of (3.20) it contains also the following term

\[
\delta \int_{\Omega} F(1 + |u_n|)^{\delta-1} \nabla u_n.
\]

Using Young’s inequality, we easily obtain that

\[
\delta \int_{\Omega} F(1 + |u_n|)^{\delta-1} \nabla u_n \leq \frac{\alpha \delta}{4} \int_{\Omega} |\nabla u_n|^p \left( 1 + |u_n| \right)^{\delta-\delta(p-1)} + \delta \left( \frac{4}{\alpha} \right)^{\frac{1}{p'}} \int_{\Omega} |F|^{\frac{p'}{p'}} \left( 1 + |u_n| \right)^{\delta+\theta-1}.
\]

Thus if \( \delta \leq 1 - \theta \) we obtain

\[
\delta \left( \frac{4}{\alpha} \right)^{\frac{1}{p'}} \int_{\Omega} |F|^{\frac{p'}{p'}} \left( 1 + |u_n| \right)^{\delta+\theta-1} \leq \delta \left( \frac{4}{\alpha} \right)^{\frac{1}{p'}} \int_{\Omega} |F|^{\frac{p'}{p'}},
\]

while if \( \delta > 1 - \theta \) we obtain

\[
\delta \left( \frac{4}{\alpha} \right)^{\frac{1}{p'}} \int_{\Omega} |F|^{\frac{p'}{p'}} \left( 1 + |u_n| \right)^{\delta+\theta-1} \leq \frac{\nu}{16} \int_{\Omega} \left( 1 + |u_n| \right)^{\delta+\theta+1} + \tilde{c}_{10} \int_{\Omega} |F|^{\frac{p'}{p'}} \left( \frac{\nu}{\alpha} \right)^{\frac{1}{p'}}.
\]

As we have said in Remark 2.1, to guarantee the convergence result (3.10), we should assume \( F \in L^{\frac{p}{p'-1}}(\Omega) \). Observe that the assumption \( b \geq b_0 \) is equivalent to require that if \( \delta \leq 1 - \theta \) with \( \delta = \tau(m - 1), \)
then $F \in (L^{\frac{p}{\theta}}(\Omega))^N$; while if $\delta > 1 - \theta$ then $F \in (L^{\frac{p r m}{p-1(1-\theta)}}(\Omega))^N$, where $\frac{p r m}{p-1(1-\theta)} = \frac{p}{p-1} \left( \frac{\delta+1}{\delta+\theta-1} \right)^r$. Similarly, the assumption $b \geq b_1$ is equivalent to require that if $\delta \leq 1 - \theta$ with $\delta = \frac{p^r(p-1)(1-\theta)}{m^r p^r}$, then $F \in (L^{\frac{p}{\theta}}(\Omega))^N$; while if $\delta > 1 - \theta$ then $F \in (L^{\frac{p}{\theta}}(\frac{1}{n}\theta))^{N_{\theta}}$. Thus if $b \geq b_0$, using the previous estimates and proceeding as in the case $F = 0$, we conclude that $u_n$ is equibounded in $W^{1,q}_{0}(\Omega) \cap L^{m}(\Omega)$; while if $b > b_1$ we can conclude that $u_n$ is equibounded in $W^{1,q}_{0}(\Omega) \cap L^{\frac{p^r(p-1-\theta)}{m^r p^r}}(\Omega)$. Hence the assert follows as before. This completes the proof. \qed

Acknowledgment: This work was supported by the National Natural Science Foundation of China(no.11801259, no.11461048), the foundation of Education Department of Jiangxi Province(no.GJJ170604).

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