2-INTERSECTING PERMUTATIONS

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Abstract. In this paper we consider the Erdős-Ko-Rado property for both 2-pointwise and 2-setwise intersecting permutations. Two permutations \( \sigma, \tau \in \text{Sym}(n) \) are t-setwise intersecting if there exists a t-subset S of \( \{1,2,\ldots,n\} \) such that \( S^\sigma = S^\tau \). If for each \( s \in S, s^\sigma = s^\tau \), then we say \( \sigma \) and \( \tau \) are t-pointwise intersecting. We say that \( \text{Sym}(n) \) has the t-setwise (resp. t-pointwise) intersecting property if for any family \( F \) of t-setwise (resp. t-pointwise) intersecting permutations, \( |F| \leq (n-t)! \) (resp. \( |F| \leq (n-t)! \)). Ellis ([“Setwise intersecting families of permutations”. *Journal of Combinatorial Theory, Series A*, 119(4):825849, 2012.]), proved that for \( n \) sufficiently large relative to \( t \), \( \text{Sym}(n) \) has the t-setwise intersecting property. Ellis also conjectured that this result holds for all \( n \geq t \). Ellis, Friedgut and Pilpel [Ellis, David, Elad Friedgut, and Haran Pilpel. “Intersecting families of permutations.” *Journal of the American Mathematical Society* 24(3):649-682, 2011.] also proved that for \( n \) sufficiently large relative to \( t \), \( \text{Sym}(n) \) has the t-pointwise intersecting property. It is also conjectured that \( \text{Sym}(n) \) has the t-pointwise intersecting property for \( n \geq 2t+1 \). In this work, we prove these two conjectures for \( \text{Sym}(n) \) when \( t = 2 \).

1. Introduction

The study of intersecting properties of finite sets is a central theme in extremal combinatorics. A collection or family of subsets \( F \) of \( \{1,2,\ldots,n\} \) is called intersecting if for any \( A, B \in F \), \( A \cap B \neq \emptyset \). In 1961, Erdős, Ko and Rado proved an important result on intersecting families of \( k \)-subsets of \( \{1,2,\ldots,n\} \). The collection of all \( k \)-subsets of \( \{1,2,\ldots,n\} \) is denoted by \( \binom{[n]}{k} \). This result is stated as follows.

**Theorem 1.1** (Erdős-Ko-Rado, [8]). For any positive integer \( n \) and \( k \) such that \( n \geq 2k \), if \( F \) is an intersecting family of \( \binom{[n]}{k} \), then \( |F| \leq \binom{n-1}{k-1} \). Moreover, if \( n > 2k \), then \( |F| = \binom{n-1}{k-1} \) if and only if \( F = F_i = \{ A \in \binom{[n]}{k} \mid i \in A \} \), for some \( i \in \{1,2,\ldots,n\} \).

There are several proofs and extensions of Theorem 1.1 in the literature [3,4,6,8,10,11,15]. In particular, Deza and Frankl [10] extended Theorem 1.1 for permutations. A family of permutations \( F \subseteq \text{Sym}(n) \) is called \( t \)-pointwise intersecting if for any \( \sigma, \tau \in F \), there exists \( i_1, i_2,\ldots,i_t \in \{1,2,\ldots,n\} \) such that \( \sigma(i_\ell) = \tau(i_\ell) \), for all \( \ell \in \{1,2,\ldots,t\} \). It is proven in [10] that if \( F \) is a family of 1-pointwise intersecting permutations of the symmetric group \( \text{Sym}(n) \), then \( |F| \leq (n-1)! \). In 2003, Cameron and Ku [2], independently Larose and Malvenuto [16], proved that the only intersecting families of permutations meeting the bound are cosets of a

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stabilizer of a point. In 2009, Godsil and Meagher [12] gave an algebraic proof of this result; the work in this paper uses a similar algebraic approach.

We can also consider intersecting families of permutations from a specific permutation group, rather than all of Sym(n). For an arbitrary t, we say that a permutation group G has the t-pointwise intersecting property if any family F of t-pointwise intersecting permutations, is no larger than the maximum size of a pointwise stabilizer of a t-set. Deza and Frankl’s result [10] proves that Sym(n) has the t-pointwise intersecting property.

A natural extension of this type of result is to consider the setwise action of the permutations. The family of permutations F is called t-setwise intersecting if for any σ, τ ∈ F, there exists S ∈ \binom{[n]}{t} such that Sσ = Sτ. The stabilizer of a t-set is an example of a t-setwise intersecting family of permutations. We say that a permutation group G has the t-setwise intersecting property if any family F of t-setwise intersecting permutations, is no larger than the maximum size of a setwise stabilizer of a t-set.

Note that 1-pointwise intersecting and 1-setwise intersecting are equivalent, so we simply call this property intersecting.

In particular, Sym(n) has the t-setwise (resp. t-pointwise) intersecting property if for any family F of t-setwise (resp. t-pointwise) intersecting permutations, |F| ≤ (n − t)!t! (resp. |F| ≤ (n − t)!).

It was also conjectured in [10] that for n sufficiently large with respect to t, a t-pointwise intersecting family F of Sym(n) is such that |F| ≤ (n − t)!t!. This conjecture was proved by Ellis et al. [7] using spectral methods and the representation of the symmetric group.

**Theorem 1.2 (Ellis, Friedgut, Pilpel [7]).** For n sufficiently large with respect to t, if a family of permutations F of Sym(n) is t-pointwise intersecting, then |F| ≤ (n − t)!t!. Moreover, |F| = (n − t)! if and only if F is a coset of the stabilizer of t elements from \{1, 2, . . . , n\}.

There is a similar result for t-setwise intersection. It was proved in 2011 by Ellis [6].

**Theorem 1.3 (Ellis, [6]).** For n sufficiently large with respect to t, if a family of permutations F of Sym(n) is t-setwise intersecting, then |F| ≤ t!(n − t)!. Moreover, |F| = t!(n − t)! if and only if F is a coset of a stabilizer of a t-subset of \{1, 2, . . . , n\}.

The proof of Theorem 1.3 uses similar arguments to the proof of the Deza-Frankl conjecture in [7]. In both proofs the result holds for n sufficiently large relative to t and exact bounds for n are not given. It is conjectured that for t-pointwise intersection the correct lower bound on n is 2t + 1, while for t-setwise intersection it is conjectured in [6] that the result holds for all n ≥ t.

In this paper we will prove that the conjectured exact lower bound on n hold in both cases for t = 2. We also give a characterization of the sets that meet the bound—before we can describe this characterization, we need to define some terms.

The regular module of Sym(n) is the complex vector space with basis Sym(n)—the elements of this module can be thought of as vectors of length |Sym(n)|. For example, the characteristic vector for a subset S ⊆ Sym(n) is the length-|Sym(n)| vector with the g-entry equal to 1 if g ∈ S and 0 otherwise is a vector in this module. This module can be identified with the vector space \mathbb{C}[Sym(n)] which has the structure of a left \mathbb{C}[Sym(n)]-module by left multiplication. So \mathbb{C}[Sym(n)] can also be identified with a subalgebra of the \mathbb{C}[Sym(n)] = |Sym(n)| \times |Sym(n)|-matrices.
It is well-known that each irreducible character of \( \text{Sym}(n) \) corresponds to a partition \( \lambda \vdash n \) (denoted by \( \chi^\lambda \)) for each such irreducible character, let \( E_\lambda \) be the \( |\text{Sym}(n)| \times |\text{Sym}(n)| \) matrix with the \((g, h)\)-entry equal to \( \chi^\lambda(hg^{-1}) \). We call the image \( E_\lambda \) (considered as a linear operator on \( \mathbb{C}[\text{Sym}(n)] \)) the \( \lambda \)-module and denote if by \( V_\lambda \).

In [12] it is shown that the characteristic vector of any maximum intersecting set in \( \text{Sym}(n) \) is a vector in \( V_n \oplus V_{n-1,1} \). In general, if \( G \) is any 2-transitive group, then the permutation module is the sum of two irreducible representations; the trivial representation and one denoted by \( \phi \). In [17] it is shown that for any 2-transitive group and the characteristic vector of any maximum intersecting set of permutations \( S \) in \( G \) lies in the sum of the trivial and the \( \phi \)-module. This is called the EKR-module property.

In this paper, we will show that the characteristic vectors of the maximum 2-setwise and 2-pointwise intersecting sets in \( \text{Sym}(n) \) each lie in a specific module. We will use spectral methods and the representation theory of the symmetric group in our proof for Theorem 1.4 and Theorem 1.5.

We state our first theorem on the 2-setwise action as follows.

**Theorem 1.4.** Let \( n \geq 2 \). If \( F \) is a family of 2-setwise intersecting permutations, then \( |F| \leq 2(n - 2)! \). Moreover, for \( n \geq 4 \), if \( S \) is a maximum 2-setwise intersecting family, then its characteristic vector \( \chi_S \) is in \( V_n \oplus V_{n-1,1} \oplus V_{n-2,2} \).

Our second theorem is a similar result for 2-pairwise intersecting permutations. We state it as follows.

**Theorem 1.5.** Let \( n \geq 5 \). If \( F \) is a 2-pointwise intersecting family of permutations, then \( |F| \leq (n - 2)! \). In addition, if \( S \) is a maximum 2-pointwise intersecting family of \( \text{Sym}(n) \), then \( \chi_S \) in \( V_n \oplus V_{n-1,1} \oplus V_{n-2,2} \oplus V_{n-2,1,1} \).

We organize this paper as follows. In Section 2 and Section 3 we recall some basics on the symmetric group \( \text{Sym}(n) \) and combinatorial objects called weighted adjacency matrix. Section 4 is devoted to the proof of Theorem 1.4. In Section 5 we give the proof for Theorem 1.5.

## 2. Background

### 2.1. Derangement graphs.

One of the techniques to prove the EKR property for a group is to use the derangement graph. The set of derangements of a permutation group \( G \leq \text{Sym}(n) \) is the set of all permutations of \( G \) without fixed points. We denote by \( \text{Der}(G) \) the set of all derangements of the permutation group \( G \). The derangement graph \( \Gamma_G \) is the undirected graph with vertex set \( G \), where two permutations \( g, h \in G \) are adjacent if and only if \( hg^{-1} \in \text{Der}(G) \). For the case where \( G \) is symmetric group we denote \( \Gamma_{\text{sym}(n)} \) by \( \Gamma_n \).

If \( F \) is an intersecting family of \( G \), then in \( \Gamma_G \), the elements of \( F \) form an independent set or coclique. Therefore, a transitive group has the EKR property if and only if the size of a maximum coclique is at most \( \frac{|G|}{n} \).

Given a graph \( X \), we denote by \( \alpha(X) \) and \( \omega(X) \) respectively the maximum size of a coclique and maximum size of a clique of \( X \). The following result is well-known and a proof can be found in [14, Section 2.1].

**Lemma 2.1** (Clique-coclique bound). Let \( X \) be a vertex-transitive graph on \( n \) vertices. Then \( \alpha(X)\omega(X) \leq n \).
Corollary 2.2. The symmetric group $\text{Sym}(n)$ has the EKR property.

Proof. First we observe that $\omega(\Gamma_n)$ is at most the degree, which is $n$ in this case. Moreover, the rows of a Latin square of order $n$ correspond to a set of permutations that forms a clique of size $n$ in $\Gamma_n$. Using Lemma 2.1, we have

$$\alpha(\Gamma_n) \leq \frac{n!}{n} = (n-1)!.$$ 

Hence, $\text{Sym}(n)$ has the EKR property. $\square$

This approach does not work in general for 2-setwise or 2-pointwise intersection. To see this, consider $\text{Sym}(n)$ with 2-pointwise intersection; a clique in the corresponding derangement graph would be a sharply 2-transitive subset of $\text{Sym}(n)$. Since such a set does not exist for all $n$, the clique-coclique bound will not always hold with equality. Rather we will use the ratio bound, as was used in [6,7,13]. The ratio bound is also known as the Delsarte bound or Hoffman bound, we refer the reader to [13] for a proof. We use $\mathbf{1}$ to denote the all ones vector and $J$ for the all ones matrix (the sizes will be clear from context).

Lemma 2.3 (Ratio bound). Let $X$ be a $d$-regular graph having $n$ vertices. Let $\lambda_{\text{min}}$ be the minimum eigenvalue of the adjacency matrix of $X$. Then we have

$$\alpha(X) \leq \frac{n}{1 - \frac{d}{\lambda_{\text{min}}}}.$$ 

Moreover, if $S$ is an independent set of size $\frac{n}{1 - \frac{d}{\lambda_{\text{min}}}}$ with characteristic vector $v_S$, then

$$v_S - \frac{|S|}{n} \mathbf{1}$$

is a $\lambda_{\text{min}}$-eigenvector.

We will actually use a generalization of the ratio bound which we state and prove once we give a definition. For a graph $X$ on $n$ vertices, a real symmetric $n \times n$ real matrix $A = (a_{i,j})$ with constant row and column sum is a weighted adjacency matrix (or a pseudo-adjacency matrix) for $X$ if $a_{i,j} = 0$ whenever $i \not\sim_X j$ (note that $a_{i,j}$ could be $0$ for adjacent vertices $i$ and $j$).

Theorem 2.4 (Weighted Ratio Bound). Let $X$ be a connected graph. Let $A$ be a weighted adjacency matrix for $X$ with constant row and column sum $d$.

If the least eigenvalue of $A$ is $\tau$, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$ 

Further, if equality holds for some coclique $S$ with characteristic vector $v_S$, then

$$v_S - \frac{|S|}{|V(X)|} \mathbf{1}$$

is an eigenvector with eigenvalue $\tau$. $\square$
Proof. Set $v = |V(X)|$ and $s = |S|$ and denote $A(X)$ by $A$. Let $\{1, w_2, \ldots, w_n\}$ be an orthonormal basis of real eigenvectors for $A$ with $Aw_i = \lambda_i$.

Define

$$M = A - \tau I - \frac{d - \tau}{v} J.$$ 

Then $M 1 = 0$, and for $w_i$

$$Mw_i = (\lambda_i - \tau)w_i$$

since $w_i$ is orthogonal to the all ones vector. Thus all the eigenvalues of $M$ are non-negative, and $M$ is positive semidefinite. Hence for any vector $x$

$$0 \leq x^T Mx = x^T Ax - \tau x^T x - \frac{d - \tau}{v} x^T Jx.$$ 

Let $x$ be the characteristic vector of a coclique of size $s$, then $x^T Ax = 0$ and Equation (1) simplifies to

$$0 \leq x^T Ax - \tau s - \frac{d - \tau}{v} s^2.$$ 

Hence

$$\tau s \leq \frac{d - \tau}{v} s^2$$

and the inequality follows. If equality holds, then $x^T Mx = 0$; since $M$ is positive semidefinite, this implies that $Mx = 0$. Therefore

$$(A - \tau I)x = \frac{d - \tau}{v} Jx = \frac{s}{v}(d - \tau) 1 = (A - \tau I) \frac{s}{v} 1$$

setting $x = vS - \frac{s}{v} 1$ implies the second claim. \hfill \Box

2.2. Eigenvalues of normal Cayley graphs. Let $G$ be a group and let $C$ be an inverse closed subset of $G \setminus \{1\}$. A Cayley graph Cay($G, C$) is graph whose vertex set is the set $G$, and two group elements $g$ and $h$ are adjacent if $hg^{-1} \in C$. A Cayley graph is called normal if the set $C$ is closed under conjugation. That is, for any $g \in G$, we have $gCg^{-1} \subseteq C$.

The derangement graph for a group is a Cayley graph, in particular, $\Gamma_G = \text{Cay}(G, \text{Der}(G))$. Since Der($G$) is closed under conjugation, the derangement graph is a normal Cayley graph.

In this subsection, we review some properties of Cayley graphs and give a formula for the eigenvalues of normal Cayley graphs.

The eigenvalues of a Cayley graph Cay($G, C$) can be obtained from the irreducible characters of the group $G$. We present this result in the following lemma which is usually attributed to Babai [1], or Diaconis and Shahshahani [5]; a proof may be found in [12, Section 11.12].

**Lemma 2.5 (Babai [1]).** Let $\text{Irr}(G)$ be the set of all irreducible representations on a group $G$. If $X = \text{Cay}(G, C)$ is a normal Cayley graph, then eigenvalues of the adjacency matrix of $X$ are given by

$$\xi_{\chi} = \frac{1}{\chi(\text{id})} \sum_{g \in C} \chi(g),$$

for $\chi \in \text{Irr}(G)$. The multiplicity of $\xi$ is given by $\sum m_{\chi}^2$ where $m_{\chi}$ is the dimension of $\chi$, and the sum is taken over all irreducible representations $\chi$ with $\xi_{\chi} = \xi$. 

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We will use a weighted adjacency matrix that is formed by taking a linear combination of matrices from the conjugacy class association scheme on Sym($n$). Next we show how to calculate the eigenvalues of such a matrix.

For $\rho$ a partition of $n$, let $C_{\rho}$ represent the conjugacy classes of Sym($n$) with shape $\rho$, and define the $n! \times n!$ matrix

$$A_{\rho}[g, h] = \begin{cases} 1 & \text{if } hg^{-1} \in C_{\rho}, \\ 0 & \text{otherwise.} \end{cases}$$

The set of matrices $A = \{A_{\rho}\}_{\rho \vdash n}$ form an association scheme called the conjugacy class scheme on Sym($n$)—in particular $A_{\rho}$ is the matrix in this association scheme corresponding to the conjugacy class of permutations with shape $\rho$ (see [14, Section 3.3] for details about this association scheme). For a conjugacy class of derangements $C_{\rho}$, the matrix $A_{\rho}$ is the adjacency matrix of Cay(Sym($n$), $C_{\rho}$), and by Lemma 2.5 the eigenvalues of $A_{\rho}$ are given by

$$\xi = \sum_{\rho \vdash n} \omega_{\rho} \frac{|C_{\rho}|}{\chi(\text{id})} \chi(g_{\rho})$$

where $g_{\rho} \in C_{\rho}$ and $\chi$ is an irreducible representation of Sym($n$) [13].

We will consider weighted adjacency matrices of the derangement graphs of Sym($n$) with the 2-setwise and the 2-pointwise actions. The adjacency matrices we consider have a constant weight on each conjugacy class; hence we only consider matrices in the conjugacy class association scheme for Sym($n$). The next result gives a formula for the eigenvalues of such a weighted adjacency matrix.

**Lemma 2.6.** Let $\{C_{\rho} \mid \rho \vdash n\}$ be the conjugacy classes of Sym($n$), let $A_{\rho}$ be the matrix of the conjugacy classes scheme of Sym($n$) that corresponds to $C_{\rho}$. Let $A = \sum_{\rho \vdash n} \omega_{\rho} A_{\rho}$, where $(\omega_{\rho})_{\rho \vdash n} \subseteq \mathbb{R}$. Then the eigenvalues of $A$ are

$$\xi = \sum_{\rho \vdash n} \omega_{\rho} \frac{|C_{\rho}|}{\chi(\text{id})} \chi(g_{\rho})$$

where $g_{\rho} \in C_{\rho}$ and $\chi$ is an irreducible representation of Sym($n$).

If the conjugacy class $C_{\rho}$ is a conjugacy class of derangements under the action, then it is called a derangement class. For the setwise action on 2-sets, a derangement class is any class $C_{\rho}$ in which $\rho$ has no parts of size 2. For the pointwise action, a derangement class is any class with a corresponding partition that contains at most one part of size 1.

3. **Representations of Sym($n$)**

In this section we state the results on the representation theory of the symmetric group that we need. We do not prove these results, but we refer the reader to Sagan [18] or another such book.

The degree of a representation is given by the value of the character on the identity. For the symmetric group, the degree of an irreducible representation can be computed via the well-known hook length formula.

If $\lambda \vdash n$, then a pair $(i, j)$ is a node of the Young diagram of $\lambda$ if the path downward $j$ cells from the top leftmost cell and then rightward $i$ cells ends on a cell of $\lambda$. By $(i, j) \in \lambda$, we mean $(i, j)$ is a node of $\lambda$. For a node $(i, j)$ define

$$H_{i,j} = \{(u, j) \in \lambda \mid i \leq u\} \cup \{(i, v) \in \lambda \mid j \leq v\}$$

and $h_{i,j} = |H_{i,j}|$. 

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\(\rho = (n)\) corresponding to the conjugacy classes of \(\text{Sym}(n)\)

\[\chi^\lambda(\text{id}) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}}\]

where the product is taken over all nodes in the Young’s diagram of \(\lambda\).

We will need to consider the low-dimensional representations of \(\text{Sym}(n)\) separately from the ones with higher dimension. We will not give a proof to this since a similar result is proved in [14].

**Lemma 3.2.** Let \(n \geq 13\). If \(\chi^\lambda\) is an irreducible representation of \(\text{Sym}(n)\) of dimension less than \(2^{\left\lfloor \frac{n+1}{2} \right\rfloor}\), then \(\lambda\) is one of the following: \([n], [1^n], [n-1,1], [2,1^{n-2}], [n-2,2], [2^2,1^{n-4}], [n-2,1^2]\) or \([3,1^{n-3}]\).

Next we state the well-known recursive formula for calculating the value of a character of \(\text{Sym}(n)\) on a conjugacy class. Let \(\chi^\lambda_\rho\) be the value of the character \(\chi^\lambda\) on an element of the conjugacy class \(C_\rho\). A composition is an unordered partition. We will use square brackets for the partitions corresponding to irreducible representations, and round bracket for partitions corresponding to the conjugacy classes of \(\text{Sym}(n)\).

**Lemma 3.3** (Murnaghan-Nakayama Rule). If \(\lambda \vdash n\) and \(\rho = (\rho_1, \rho_2, \ldots, \rho_k)\), then

\[\chi^\lambda_\rho = \sum_{\xi \in \text{RH}_{\rho_1}(\lambda)} (-1)^{\ell(\xi)} \chi^{\lambda\xi}_{\rho_1\rho_2}\]

where \(\text{RH}_{\rho_1}(\lambda)\) is the set of all rim hooks with \(\rho_1\) cells of \(\lambda\), and \(\ell(\xi)\) is the number of rows the rim hook spans minus one.

We will build weighted adjacency matrices for the derangement graphs for \(\text{Sym}(n)\) with the setwise and the pointwise action. We will then calculate the eigenvalues of these matrices and prove that the ratio bound holds with equality. To calculate the eigenvalues, we will need to determine the value of the irreducible representations on specific conjugacy classes. In our weighting, we weight many of the conjugacy classes to be 0, so we only need to consider the values of the irreducible representations that have a non-zero weight. The next result gives the values of the irreducible representation on these specific conjugacy classes.

**Lemma 3.4.** The irreducible characters of \(\text{Sym}(n)\) that do not vanish on the conjugacy classes \(C(n), C(n-1,1), C(n-3,3), C(n-4,3,1), C(n-2,2), C(n-3,2,1)\) are given in Table 3.

We end this section with considering two representations of the symmetric group. The Young’s subgroup \(\text{Sym}(n-2) \times \text{Sym}(2) = \text{Sym}([n-2,2])\) is the setwise stabilizer of a 2-set. Thus the setwise action of \(\text{Sym}(n)\) sets of size 2 has representation

\[\text{ind}_{\text{Sym}(n)}(1_{\text{Sym}([n-2,2])}) = \chi^{[n]} + \chi^{[n-1,1]} + \chi^{[n-2,2]}\]

Similarly, the Young’s subgroup \(\text{Sym}(n-2) \times \text{Sym}(1) \times \text{Sym}(1) = \text{Sym}([n-2,1,1])\) is the pointwise stabilizer of a 2-set. Thus the pointwise action of \(\text{Sym}(n)\) on sets of size 2 has representation

\[\text{ind}_{\text{Sym}(n)}(1_{\text{Sym}([n-2,1,1])}) = \chi^{[n]} + 2\chi^{[n-1,1]} + \chi^{[n-2,2]} + \chi^{[n-2,1,1]}\].
Both of these decompositions follow from calculating the Kostka numbers.

We will construct a weighted adjacency matrix for the derangement graph for each action and the ratio bound will hold with equality for this matrix. In the weighted adjacency matrix that we construct, the non-trivial representations in the decomposition of each of the representation of the permutation action will be exactly the representations that achieve the minimal eigenvalue. This fact will show that the characteristic vectors of any maximum coclique will lie in a specific Sym($n$)-module.

4. Proof of Theorem 1.4

In this section we will give the weighted adjacency matrix for the derangement graph of Sym($n$) with the 2-setwise action; this matrix has the form

$$A = \omega_1 A_{(n)} + \omega_2 A_{(n-1,1)} + \omega_3 A_{(n-3,3)} + \omega_4 A_{(n-4,3,1)}$$

for some positive numbers $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$ which are to be determined. We are choosing only four of the derangement classes in the association scheme (namely the conjugacy classes with cycle types ($n$), ($n-1,1$), ($n-3,3$) and ($n-4,3,1$)) to have a non-zero weighting. The sizes of the conjugacy classes with cycle types ($n$), ($n-1,1$), ($n-3,3$) and ($n-4,3,1$) are, respectively,

$$\alpha = (n-1)!,$$
$$\beta = n(n-2)!,$$
$$\gamma = 2\left(\frac{n}{3}\right)(n-4)!,$$
$$\delta = 8\left(\frac{n}{4}\right)(n-5)!$$

We choose our weighting so that the following three conditions hold:

1. the trivial representation gives the eigenvalue $\left(\frac{n}{2}\right) - 1$;
2. the nontrivial irreducible characters that are in the decomposition of the permutation action (namely $\chi^{[n-1,1]}$ and $\chi^{[n-2,2]}$) have eigenvalue $-1$; and
3. all other representations give eigenvalues strictly between $\left(\frac{n}{2}\right) - 1$ and $-1$.

It is straightforward to calculate the eigenvalues of the adjacency matrices for the four conjugacy classes we have chosen corresponding to the irreducible representations in the decomposition of the permutation action. This values are in the following table.

| Representation | $A_{(n)}$ | $A_{(n-1,1)}$ | $A_{(n-3,3)}$ | $A_{(n-4,3,1)}$ |
|---------------|-----------|---------------|---------------|---------------|
| $\chi^{[n]}$  | $\alpha$  | $\beta$       | $\gamma$      | $\delta$      |
| $\chi^{[n-1,1]}$ | $-\frac{\alpha}{n-1}$ | 0             | $-\frac{\gamma}{n-1}$ | 0             |
| $\chi^{[n-2,2]}$ | 0         | $-\frac{2\beta}{n(n-3)}$ | 0             | $-\frac{2\delta}{n(n-3)}$ |

Table 1. Eigenvalues of $A_{(n)}$, $A_{(n-1,1)}$, $A_{(n-3,3)}$ and $A_{(n-4,3,1)}$ afforded by $\chi^{[n]}$, $\chi^{[n-1,1]}$ and $\chi^{[n-2,2]}$.

By Lemma 2.6 to find weightings that satisfy conditions (1) and (2) above we need to solve the following three linear equations.
\[ \omega_1 \alpha + \omega_2 \beta + \omega_3 \gamma + \omega_4 \delta = \binom{n}{2} - 1 \]

(4)

\[ -\omega_1 \alpha - \omega_3 \gamma = -(n - 1) \]

\[ -\omega_2 \beta - \omega_4 \delta = -\frac{n(n - 3)}{2}. \]

This linear system has infinitely many solutions with two free variables. The general solution to (4) has the form

\[
\begin{align*}
\omega_1(s,t) &= \frac{1}{\alpha} (-\gamma s + (n - 1)) \\
\omega_2(s,t) &= \frac{1}{\beta} (-\delta t + \frac{n(n - 3)}{2}) \\
\omega_3(s,t) &= s \\
\omega_4(s,t) &= t.
\end{align*}
\]

Thus, expressed in these two free variables, any eigenvalue of \( A \) is of the form

\[
\xi_{\chi} = \frac{1}{\chi(\text{id})} \left( ((n - 1) - \gamma s) \chi(n) + \left( \frac{n(n - 3)}{2} - \delta t \right) \chi(n-1,1) + \gamma s \chi(n-3,3) + \delta t \chi(n-4,3,1) \right)
\]

for \( \chi \in \text{Irr}(\text{Sym}(n)) \). In other words, eigenvalues of \( A \) are functions of the parameters \( s \) and \( t \). For the remaining part of the proof, we shall write the eigenvalues for an irreducible character \( \chi \) in function of the parameters \( t \) and \( s \) (that is, \( \xi_{\chi}(s,t) \)). We will choose values of \( s \) and \( t \) so that all the eigenvalues of \( A \) satisfy all three conditions listed above.

To do this, we will define a polytope \( \mathcal{P} \) and show for any values of \( (s,t) \) in \( \mathcal{P} \), the weightings \( \omega_i(s,t) \) give a matrix \( A \) that satisfies the three conditions above. We then apply Theorem 2.4 to this \( A \), which shows that Theorem 1.4 holds.

Define \( (\mathcal{P}) \) to be the polytope that is the intersection of the following halfspaces of \( \mathbb{R}^2 \)

\[
\begin{align*}
2\gamma x - 2\delta y + \left( \frac{n - 1}{2} \right) - (n - 1) + 2 &< 0 \\
2\gamma x - 2\delta y + \left( \frac{n - 1}{2} \right) - (n - 1) &> 0 \\
0 &< \gamma x < n - 1, \\
0 &< \delta y < \frac{n(n - 3)}{2}.
\end{align*}
\]

The polytope \( (\mathcal{P}) \) is non-empty since the first two equations are those of parallel lines and they intersect the rectangle formed by the last two equations. Note that the final two equations imply the following result.

**Lemma 4.1.** For any \( (s,t) \in \mathcal{P} \), the weightings \( \omega_1(s,t), \omega_2(s,t), \omega_3(s,t) \) and \( \omega_4(s,t) \) are positive.

Next we will determine the eigenvalues of \( A \) so that Theorem 2.4 can be applied.
Lemma 4.2. Let $n \geq 11$. For any $(s, t) \in \mathcal{P}$, the eigenvalues of the matrix
\[
A = \omega_1(s, t)A(n) + \omega_2(s, t)A(n-1,1) + \omega_3(s, t)A(n-3,3) + \omega_4(s, t)A(n-4,3,1)
\]
are in $[-1, (n/2) - 1]$. Moreover, the eigenvalues for $\chi^{[n-1,1]}$ and $\chi^{[n-2,2]}$ are the only ones equal to $-1$.

Proof. Let $\chi \in \text{Irr}(\text{Sym}(n))$ be such that $\chi(\text{id}) > (n/2)$. Using the triangle inequality and Lemma 4.2 on (5)

\[
|\xi_\chi| \leq \frac{1}{\binom{n}{2}} \left( ((n-1) - \gamma s) |\chi(n)| + \left( \frac{n(n-3)}{2} - \delta t \right) |\chi(n-1,1)| + \gamma s |\chi(n-3,3)| + \delta t |\chi(n-4,3,1)| \right)
\]

\[
\leq \frac{1}{\binom{n}{2}} \left( ((n-1) - \gamma s) + \left( \frac{n(n-3)}{2} - \delta t \right) + \gamma s + \delta t \right)
\]

\[
= \frac{1}{\binom{n}{2}} \left( (n-1) + \frac{n(n-3)}{2} \right)
\]

\[
= \frac{1}{\binom{n}{2}} \left( \left( \frac{n}{2} \right) - 1 \right)
\]

\[
< 1.
\]

Using Lemma 3.2, the eigenvalues for irreducible characters of degree less than $\binom{n}{2}$ are

\[
\xi_{\chi^{[n]}} = \left( \frac{n}{2} \right) - 1,
\]

\[
\xi_{\chi^{[n-1,1]}} = \xi_{\chi^{[n-2,2]}} = -1
\]

\[
\xi_{\chi^{[n-2,1,2]}} = \frac{2}{n-2}
\]

\[
\xi_{\chi^{[3,n-3]}} = \frac{2(-1)^{n-1}}{n-2} + (-1)^n \frac{2\gamma s}{\binom{n-2}{1}}
\]

\[
\xi_{\chi^{[2,n-4]}} = (-1)^{n-1} + (-1)^n \frac{4\delta t}{n(n-3)}
\]

\[
\xi_{\chi^{[2,n-2]}} = (-1)^n + (-1)^{n-1} \frac{2\gamma s}{n-1}
\]

\[
\xi_{\chi^{[1,n]}} = (-1)^n \left( \left( \frac{n}{2} \right) - 2(n-1) - 1 + 2\gamma s - 2\delta t \right)
\]

One can immediately see that, with the exception of $\xi_{\chi^{[n]}}$, these eigenvalues are all strictly less than $\left( \frac{n}{2} \right) - 1$.

So we need to show that the eigenvalues $\xi_{\chi^{[3,n-3]}}, \xi_{\chi^{[2,n-4]}}, \xi_{\chi^{[2,n-2]}}$ and $\xi_{\chi^{[1,n]}}$ are strictly greater than $-1$ whenever $(s, t) \in \mathcal{P}$.

When $n$ is even, the eigenvalues are larger than $-1$ if

(1) $2\gamma s - 2\delta t + \binom{n-1}{2} - (n-1) > 0$,

(2) $\gamma s < n-1$,

(3) $0 < t$
When \( n \) is odd, the eigenvalues are larger than \(-1\) if

1. \( 2\gamma s - 2\delta t + \binom{n-1}{2} - (n-1) + 2 < 0, \)
2. \( 0 < s, \)
3. \( \delta t < \frac{n(n-3)}{2}. \)

Combining the cases when \( n \) is even and odd, we obtain exactly the equations of \((P)\). Since \((P)\) is not empty, all the eigenvalues are greater than \(-1\) for any \((s, t) \in P\). This completes the proof. \( \square \)

**Proof of Theorem 1.4.** For \( 2 \leq n \leq 10 \), we use Sagemath \([19]\) to prove the result. For \( n \geq 11 \), we use Theorem 2.4 we have

\[ \alpha(\Gamma_{\text{Sym}(n)}) \leq \alpha(X) \leq \frac{n!}{1 - \binom{n}{2}^{-1}} = 2(n-2)!. \]

This result shows that equality holds in Theorem 2.4; thus, if \( S \) is a maximum 2-setwise intersecting set, then \( v_S - \left(\binom{n}{2}\right)^{-1} 1 \) is a \(-1\)-eigenvector for \( A \). Since the only irreducible representations that give the least eigenvalue \(-1\) are \( \chi^{[n-1,1]} \) and \( \chi^{[n-2,2]} \), we have the following corollary.

**Corollary 4.3.** For \( n \geq 4 \), any characteristic vector of a maximum 2-setwise intersecting set of permutations in \( \text{Sym}(n) \) is in the module \( V_{[n]} \oplus V_{[n-1,1]} \oplus V_{[n-2,2]} \).

5. **Proof of Theorem 1.5**

In this section we prove Theorem 1.5 by constructing a weighted adjacency matrix for the derangement graph of \( \text{Sym}(n) \) with the 2-pointwise action. This weighted adjacency matrix will be a linear combination of the adjacency matrices in the conjugacy classes scheme for \( \text{Sym}(n) \) corresponding to the conjugacy classes with cycle types \((n), (n-1,1), (n-2,2), (n-3,3), (n-3,2,1)\) and \((n-4,3,1)\). In particular, we set

\[ A = \omega_1 A_{(n)} + \omega_2 A_{(n-1,1)} + \omega_3 A_{(n-2,2)} + \omega_4 A_{(n-3,3)} + \omega_5 A_{(n-3,2,1)} + \omega_6 A_{(n-4,3,1)}. \]

Similar to the proof of Theorem 1.4 we will find \( (\omega_i) \) for \( i = 1, \ldots, 6 \) so that the following three conditions hold:

1. the trivial representation gives the eigenvalue \( 2\binom{n}{2} - 1; \)
2. the nontrivial irreducible characters that are in the decomposition of the 2-pointwise permutation action (namely \( \chi^{[n-1,1]}, \chi^{[n-2,2]} \) and \( \chi^{[n-2,1,1]} \)) have eigenvalue \(-1; \) and
3. all other representations give eigenvalues strictly between \( 2\binom{n}{2} - 1 \) and \(-1.\)

Define

\[ \alpha = (n-1)!, \quad \beta = n(n-2)!, \quad \gamma = \binom{n}{2}(n-3)!, \]
\[ \delta = 2\binom{n}{3}(n-4)!, \quad \mu = 3\binom{n}{3}(n-4)!, \quad \nu = 8\binom{n}{4}(n-5)!. \]

These numbers are respectively the sizes of the conjugacy classes with cycle type \((n), (n-1,1), (n-2,2), (n-3,3), (n-3,2,1)\) and \((n-4,3,1)\). The following table gives the
eigenvalues values of the matrices $A_n$, $A_{n-1,1}$, $A_{n-2,2}$, $A_{n-3,3}$, $A_{n-3,2,1}$ and $A_{n-4,3,1}$ in the conjugacy class association scheme corresponding to these irreducible characters.

| Representation | $A_n$ | $A_{n-1,1}$ | $A_{n-2,2}$ | $A_{n-3,3}$ | $A_{n-3,2,1}$ | $A_{n-4,3,1}$ |
|---------------|-------|-------------|-------------|-------------|-------------|-------------|
| $\chi^{[n]}$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\mu$ | $\nu$ |
| $\chi^{[n-1,1]}$ | $-\frac{\alpha}{n-1}$ | $0$ | $-\frac{\beta}{n-1}$ | $-\frac{\gamma}{n-1}$ | $0$ | $0$ |
| $\chi^{[n-2,2]}$ | $0$ | $-\frac{2\beta}{n(n-3)}$ | $\frac{2\gamma}{n(n-3)}$ | $0$ | $0$ | $-\frac{2\nu}{n(n-3)}$ |
| $\chi^{[n-2,1,1]}$ | $\frac{\alpha}{n(n-1)}$ | $0$ | $0$ | $\frac{\delta}{n(n-1)}$ | $-\frac{\mu}{n(n-1)}$ | $0$ |

**Table 2.** Eigenvalues afforded by $\chi^{[n]}$, $\chi^{[n-1,1]}$, $\chi^{[n-2,2]}$ and $\chi^{[n-2,1,1]}$ on $A_n$, $A_{n-1,1}$, $A_{n-2,2}$, $A_{n-3,3}$, $A_{n-3,2,1}$ and $A_{n-4,3,1}$.

Using Lemma 2.6, it is straight-forward to calculate the eigenvalues of $A$ afforded by $\chi^{[n]}$, $\chi^{[n-1,1]}$, $\chi^{[n-2,2]}$ and $\chi^{[n-2,1,1]}$ as functions of $\omega_i$ where $i = 1, \ldots, 6$. Thus, in order to satisfy conditions (1) and (2) above, the $\omega_i$ must satisfy the following system of linear equations.

\[
\begin{align*}
\alpha \omega_1 + \beta \omega_2 + \gamma \omega_3 + \delta \omega_4 + \mu \omega_5 + \nu \omega_6 &= n(n-1) - 1 \\
-\alpha \omega_1 - \gamma \omega_3 - \delta \omega_4 &= -(n-1) \\
-\beta \omega_2 + \gamma \omega_3 - \nu \omega_6 &= -\frac{n(n-3)}{2} \\
\alpha \omega_1 + \delta \omega_4 - \mu \omega_5 &= -\left(\frac{n-1}{2}\right).
\end{align*}
\]

(7)

The system (7) has infinitely many solutions with three free variables. The following is a general solution to (7).

\[
\begin{align*}
\omega_1(r, s, t) &= \frac{1}{\alpha} \left( \binom{n}{2} - 1 - \beta r - \nu t - \delta s \right), \\
\omega_2(r, s, t) &= r, \\
\omega_3(r, s, t) &= \frac{1}{\gamma} \left( 1 - \binom{n-1}{2} + \beta r + \nu t \right), \\
\omega_4(r, s, t) &= s, \\
\omega_5(r, s, t) &= \frac{1}{\mu} \left( \binom{n}{2} + \binom{n-1}{2} - 1 - \beta r - \nu t \right), \\
\omega_6(r, s, t) &= t
\end{align*}
\]

(8)

for $r, s, t \in \mathbb{R}$. 

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The eigenvalue of $A$ corresponding to $\chi$, denoted by $\xi_{\chi}(r, s, t)$, as a function of $r$, $s$, and $t$ is the following.

\[
\xi_{\chi}(r, s, t) = \frac{1}{\chi(\text{id})} \left( \left( \left( \frac{n}{2} \right) - 1 - \beta r - \delta s - \nu t \right) \chi(n) + \beta r \chi(n-1,1) 
+ \left( 1 - \left( \frac{n-1}{2} \right) + \beta r + \nu t \right) \chi(n-2,2) + \delta s \chi(n-3,3) 
+ \left( \left( \frac{n}{2} \right) + \left( \frac{n-1}{2} \right) - 1 - \beta r - \nu t \right) \chi(n-3,2,1) + \nu t \chi(n-4,3,1) \right).
\]

In particular, we have

\[
\begin{align*}
\xi_{\chi[n]} &= (-1)^n \left( 4\beta r + 2\nu t + 2\delta s + 3 - 2 \left( \frac{n}{2} \right) - 2 \left( \frac{n-1}{2} \right) \right) \\
\xi_{\chi[2,n-2]} &= \frac{(-1)^n}{n-1} \left( \left( \frac{n}{2} \right) + \left( \frac{n-1}{2} \right) - 2 - 2\beta r - 2\delta s - 2\nu t \right) \\
\xi_{\chi[2,2,n-4]} &= (-1)^{n-1} + (-1)^n \frac{4\nu t}{n(n-3)} \\
\xi_{\chi[3,1,n-3]} &= (-1)^n + (-1)^n \frac{2\delta s}{\left( \frac{n-1}{2} \right)}.
\end{align*}
\]

We will distinguish the cases $n$ even and $n$ odd, for each case we will pick values for $r$, $s$ and $t$ so that the matrix $A$ satisfies the conditions in Theorem 2.4.

5.1. **Subcase 1: $n$ even.** For this case we set $s = 0$; this removes the conjugacy class with cycle type $(n - 3, 3)$ from the weighted adjacency matrix. With $s = 0$ and $n$ is even, we can calculate the eigenvalues for the eight irreducible characters of degree less than $2\left( \frac{n+1}{2} \right)$ as follows.

\[
\begin{align*}
\xi_{\chi[n]} &= 2 \left( \frac{n}{2} \right) - 1 \\
\xi_{\chi[n-1,1]} &= \xi_{\chi[n-2,2]} = \xi_{\chi[n-2,1,2]} = -1 \\
\xi_{\chi[3,1,n-3]} &= 1 \\
\xi_{\chi[2,2,n-4]} &= -1 + \frac{4\nu t}{n(n-3)} \\
\xi_{\chi[2,1,n-2]} &= \frac{1}{n-1} \left( \left( \frac{n}{2} \right) + \left( \frac{n-1}{2} \right) - 2 - 2\beta r - 2\nu t \right) \\
\xi_{\chi[n]} &= 4\beta r + 2\nu t + 3 - 2 \left( \frac{n}{2} \right) - 2 \left( \frac{n-1}{2} \right)
\end{align*}
\]
Lemma 5.1. Let $(P')$ be the polytope obtained by the following equations of halfspaces of $\mathbb{R}^2$
\[
(P') \begin{cases} 
2\beta x + \nu y - \left(\frac{n}{2}\right) - \left(\frac{n-1}{2}\right) + 2 > 0 \\
\beta x + \nu y + 1 - \left(\frac{n}{2}\right) < 0 \\
0 < y.
\end{cases}
\]
This polytope is defined so that the eigenvalues $\xi_{\chi^{[2^2,1^{n-4}]}}$, $\xi_{\chi^{[2,1^{n-2}]}}$ and $\xi_{\chi^{[1^n]}}$ are all strictly greater than $-1$. We let the reader verify that $(P')$ is a triangle (without the boundary) with coordinates
\[
\left(\frac{1}{\beta} \left(\frac{n-1}{2} - 1\right), \frac{1}{\nu} (n-2)\right), \quad \left(\frac{1}{\beta} \left(\frac{n}{2} - 1\right), 0\right), \quad \left(\frac{1}{\beta} \left(\frac{n-1}{2} - 1 + \frac{n-1}{2}\right), 0\right).
\]
In particular, $(P')$ is non-empty.

Next, we prove that for any $(r,t) \in P'$, the weightings $w_i$ are all non-negative.

**Lemma 5.1.** The weighting $\omega_i(r,0,t)$ is non-negative for any $i = 1, \ldots, 6$, and $(r,t) \in P'$.

**Proof.** Let $(r,t) \in P'$. From the equations of $P$, we have
\[-\beta x - \nu y > 1 - \left(\frac{n}{2}\right), \quad \beta x + \nu y > 2\left(\frac{n}{2}\right) + 2\left(\frac{n-1}{2}\right) - 2 - \beta x.\]
Using these relations and the fact that $\beta x \in [(n-1)/2, (n)/2 - 1]$, one can derive that the weightings are indeed positive. \hfill \square

Next, we prove that the eigenvalues of the weighted adjacency matrix are also in the correct range whenever $(r,t) \in P'$.

**Lemma 5.2.** Let $n \geq 13$ and even. If $(r,t) \in P'$ the eigenvalues of the weighted adjacency matrix $A$ defined in (6) are in the interval $[-1,2(n/2) - 1]$. In addition, the only irreducible character giving the eigenvalue $2(n/2) - 1$ is $\chi_{[n]}$; and the only irreducible characters giving eigenvalue $-1$ are $\chi_{[n-1,1]}$, $\chi_{[n-2,2]}$ and $\chi_{[n-2,1^2]}$.

**Proof.** First we show that the statement holds for all irreducible characters $\chi$ of $\text{Sym}(n)$ with $\chi(\text{id}) > 2\left(\frac{n+1}{2}\right)$. On each of the conjugacy classes that we consider, the value of any irreducible representation is bounded by 1. For any $(r,t) \in P'$ we have the following bound.
\[
|\xi_{\chi}(r,0,t)| \leq \frac{1}{2\left(\frac{n+1}{2}\right)} \left(\left(\frac{n}{2} - 1 - \beta r - \nu t\right) |\chi(n)| + \beta r |\chi(n-1,1)| + \left(1 - \left(\frac{n-1}{2}\right) + \beta r + \nu t\right) |\chi(n-2,2)| + \left(\frac{n}{2} + \left(\frac{n-1}{2}\right) - 1 - \beta r - \nu t\right) |\chi(n-3,2,1)| + \nu t |\chi(n-4,3,1)|\right)
= \frac{2\left(\frac{n}{2}\right) - 1}{2\left(\frac{n+1}{2}\right)} < 1.
\]
Thus the eigenvalue of $A$ for $\chi$ is bounded in absolute value by 1 and the statement holds for all irreducible characters except $[n], [1^n], [n-1, 1], [2, 1^{n-2}], [n-2, 2], [2^2, 1^{n-4}], [n-2, 1^2]$ or $[3, 1^{n-3}]$.

It is straightforward to see that the non-trivial character all give eigenvalue less than $2\binom{n}{2} - 1$.

Finally, Equations $\text{(10)}$ and the definition of $\mathcal{P}'$ show that the statement holds for all other irreducible characters. □

5.2. **Subcase 2: $n$ odd.** If $n$ is odd, we first note that in the expression of the eigenvalue for $[3, 1^{n-3}]$, the value of $s$ must to be negative for the inequality $\xi_{[3,1^n-3]} > -1$ to hold. We will only use five conjugacy classes, so we drop conjugacy classes with cycle type $(n-4, 3, 1)$ by making $t = 0$. As in the previous case, we will consider the irreducible representations with degree less than $2\binom{n+1}{2}$ and greater than $2\binom{n+1}{2}$ separately. The eigenvalues belonging to representations with degree less than $2\binom{n+1}{2}$ are

$$
\xi_{\chi^{[n]}} = 2\binom{n}{2} - 1.
$$

$$
\xi_{\chi^{[n-1, 1]}} = \xi_{\chi^{[n-2, 2]}} = \xi_{\chi^{[n-2, 1^2]}} = -1
$$

$$
\xi_{\chi^{[3, 1^{n-3}]}} = -1 - \frac{2\delta_s}{n-1}
$$

$$
\xi_{\chi^{[2, 1^{n-4}]}} = 1
$$

$$
\xi_{\chi^{[2, 1^{n-2}]}} = \frac{-1}{n-1}\left(\binom{n}{2} + \binom{n-1}{2} - 2 - 2\beta r - 2\delta s\right)
$$

$$
\xi_{\chi^{[n]}} = -\left(4\beta r + 2\delta s + 3 - 2\binom{n}{2} - 2\binom{n-1}{2}\right)
$$

Let $(\mathcal{P}'')$ be the polytope of $\mathbb{R}^2$ defined as follows:

$$(\mathcal{P}'') = \left\{ \begin{array}{l}
2\beta x + \delta y + 1 - \binom{n}{2} - \binom{n-1}{2} < 0 \\
\beta x + \delta y + 1 - \binom{n-1}{2} > 0 \\
y < 0.
\end{array} \right.$$ 

Just as in the case when $n$ is even, this polytope is defined so that the eigenvalues $\xi_{\chi^{[2, 1^{n-4}]}}$, $\xi_{\chi^{[2, 1^{n-2}]}}$ and $\xi_{\chi^{[1^n]}}$ are all strictly greater than $-1$. The polytope $(\mathcal{P}'')$ is a triangle (without the boundary) with coordinates

$$
\left(\frac{1}{\beta}\binom{n}{2}, -\frac{n}{\delta}\right), \left(\frac{1}{\beta}\left(\binom{n-1}{2} - 1\right), 0\right), \left(\frac{1}{2\beta}\left(\binom{n}{2} + \binom{n-1}{2} - 1\right), 0\right).
$$

In the next lemma, we state that all weightings, except $\omega_4$, are positive for any $(r, s) \in \mathcal{P}''$. We give the statement of this without a proof since it is straightforward.

**Lemma 5.3.** For $i \in \{1, 2, 3, 5, 6\}$ and for $(r, s) \in \mathcal{P}'$, we have $\omega_i(r, s, 0) \geq 0$.

As in the even case, the eigenvalues of the weighted adjacency matrix are in the required range whenever $(r, s) \in \mathcal{P}''$. We omit the proof as it is identical to the case where $n$ is even.
Lemma 5.4. Let \( n \geq 13 \) and odd. For \((r, s) \in \mathcal{P}''\), the eigenvalues of the weighted adjacency matrix \( A \) defined in (6) are in \([-1, 2\binom{n}{2} - 1]\). Moreover, the only irreducible characters giving eigenvalue \(-1\) are \( \chi^{[n-1,1]} \), \( \chi^{[n-2,1]} \) and \( \chi^{[n-2,2]} \).

Proof. Lemma 3.2 gives the eight irreducible representations of \( \text{Sym}(n) \) with degree less than \( 2\binom{n+1}{2} \). The polytope \( \mathcal{P}'' \) is defined so that this result holds for these eight irreducible representations.

Next assume \( \chi \) is an irreducible character of \( \text{Sym}(n) \) with \( \chi(\text{id}) > 2\binom{n+1}{2} \). On each of the conjugacy classes that we consider, the value of any irreducible representation is bounded by 1. Noting that \((r, s) \in \mathcal{P}''\) implies \( \delta s \in (-n, 0) \), we have the following bound.

\[
|\xi(\chi, s, 0)| \leq \frac{1}{\chi(\text{id})} \left( \binom{n}{2} - 1 - \beta r - \delta s \right) |\chi(n)| + \beta r |\chi(n-1,1)| + \left( 1 - \binom{n-1}{2} + \beta r \right) |\chi(n-2,2)| \\
+ |\delta s| \left( \binom{n}{2} - 1 - \beta r - \delta s \right) |\chi(n-3,3)| + \left( \binom{n}{2} + \binom{n-1}{2} - 1 - \beta r \right) |\chi(n-3,2,1)| \\
\leq \frac{1}{2\binom{n+1}{2}} \left( \binom{n}{2} - 1 - \beta r + n \right) + \beta r + \left( 1 - \binom{n-1}{2} + \beta r \right) \\
+ n + \left( \binom{n}{2} + \binom{n-1}{2} - 1 - \beta r \right) \\
\leq \frac{1}{2\binom{n+1}{2}} \left( 2\binom{n}{2} - 1 + 2n \right) \\
< \frac{n^2 + n - 1}{n^2 + n} \\
< 1.
\]

Thus the eigenvalue corresponding to any irreducible representation with degree greater than or equal to \( 2\binom{n+1}{2} \) is strictly between \(-1\) and 1, so the result holds. \( \square \)

The proof of Theorem 1.5 follows from Lemma 5.2 and Lemma 5.4 using the ratio bound on \( \Gamma_n \) with the weighted adjacency matrix \( A \).

Proof of Theorem 1.5. If \( 5 \leq n \leq 12 \), we use Sagemath [19] to confirm that the result holds. For \( n \geq 13 \), we prove the result by using Theorem 2.4. Therefore, if \( n \) is even, then \( \alpha(\Gamma_{\text{Sym}(n)}) \leq \alpha(X_1) = (n-2)! \), and similarly, if \( n \) is odd, then \( \alpha(\Gamma_{\text{Sym}(n)}) \leq \alpha(X_2) = (n-2)! \).

We finish this section by proving a conjecture of Godsil and Meagher ([12], Conjecture 7.3).

Corollary 5.5. Let \( n \geq 5 \). If \( \chi \) is the characteristic vector of a maximum 2-pointwise intersecting family of \( \text{Sym}(n) \), then \( \chi \in V_{[n]} \oplus V_{[n-1,1]} \oplus V_{[n-2,2]} \oplus V_{[n-2,1^2]} \).

6. Further work

In this paper we construct weighted adjacency matrices for the derangement graphs of \( \text{Sym}(n) \) for two different actions. This work proves that the conjectured lower bounds on \( n \) for each action are indeed the correct bound when \( t = 2 \). It is also interesting that this work also shows that there are infinitely many weighted adjacency matrices that would work in the ratio bound. We leave the reader with two open problems.
Problem 6.1. In Theorem 1.4 and Theorem 1.5, we only proved that any characteristic vector of a maximum 2-setwise and 2-pointwise intersecting family are in their respective permutation module. Characterize the maximum intersecting families for both type of intersections.

Problem 6.2. Prove that the $t$-setwise intersecting property of $\text{Sym}(n)$ holds for any $3 \leq t \leq n$ and for $n \geq 2t + 1$, prove that $\text{Sym}(n)$ has the $t$-pointwise intersecting property.

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### Table 3. Values of irreducible characters on conjugacy classes $C_{(n-3,3)}$, $C_{(n-4,3,1)}$, $C_{(n)}$, $C_{(n-1,1)}$, $C_{(n-2,2)}$ and $C_{(n-3,2,1)}$.

| Representation | Range of $k$ | $C_{(n-3,3)}$ | $C_{(n-4,3,1)}$ | $C_{(n)}$ | $C_{(n-1,1)}$ | $C_{(n-2,2)}$ | $C_{(n-3,2,1)}$ |
|---------------|-------------|--------------|----------------|----------|-------------|-------------|----------------|
| $\chi_{[n]}$ | -           | 1            | 1              | 1        | 1           | 1           | 1              |
| $\chi_{[n-1,1]}$ | -           | -1          | 0              | -1       | 0           | -1          | 0              |
| $\chi_{[n-2,2]}$ | -           | 0           | -1             | 0        | -1          | 1           | 0              |
| $\chi_{[n-2,1^*]}$ | -           | 1           | 0              | 1        | 0           | 0           | -1             |
| $\chi_{[n-3,3]}$ | -           | 1           | 1              | 0        | 0           | -1          | 0              |
| $\chi_{[n-3,2,1]}$ | -           | -1          | 0              | 0        | 0           | 0           | 1              |
| $\chi_{[n-4,1^*]}$ | -           | 0           | -1             | -1       | 0           | 0           | 0              |
| $\chi_{[n-4,2^*]}$ | -           | 1           | 0              | 0        | 0           | -1          | -1             |
| $\chi_{[n-5,2^*]}$ | -           | 0           | 1              | 0        | -1          | 0           | 0              |
| $\chi_{[n-6,2^*]}$ | -           | -1          | -1             | 0        | 0           | 0           | 1              |
| $\chi_{[n-k-4,4,1^*]}$ | $0 \leq k < n - 8$ | $(-1)^{k+1}$ | 0              | 0        | 0           | 0           | ($-1)^{k+1}$   |
| $\chi_{[n-k-5,5,1^*]}$ | $0 \leq k < n - 10$ | $(-1)^{k+1}$ | 0              | 0        | 0           | 0           | 0              |
| $\chi_{[n-k-5,3,2,1^*]}$ | $0 \leq k < n - 8$ | $(-1)^{k+3}$ | 0              | 0        | 0           | 0           | 0              |
| $\chi_{[n-k-6,2^*]}$ | $0 \leq k < n - 8$ | $(-1)^{k+3}$ | 0              | 0        | 0           | 0           | ($-1)^{k+4}$   |
| $\chi_{[n-k-6,3,1^*]}$ | $0 \leq k < n - 9$ | $(-1)^{k+1}$ | 0              | 0        | 0           | 0           | 0              |
| $\chi_{[n-k-8,2^*]}$ | $0 \leq k < n - 10$ | $(-1)^{k+4}$ | 0              | 0        | 0           | 0           | 0              |
| $\chi_{[n-k-1^*]}$ | $4 \leq k \leq n - 5$ | 0              | 0              | $(-1)^k$ | 0           | 0           | 0              |
| $\chi_{[n-k-2,2,1^*]}$ | $3 \leq k \leq n - 6$ | 0              | 0              | 0        | $(-1)^{k+1}$ | 0           | 0              |
| $\chi_{[4^2,1^*}]$ | -           | $(-1)^n$ | $(-1)^n-7$ | 0        | 0           | 0           | ($-1)^{n-7}$   |
| $\chi_{[4,3,1^{n-7}]}$ | -           | 0           | $(-1)^n$ | $(-1)^n-5$ | 0           | 0           | ($-1)^{n-6}$   |
| $\chi_{[4,2,1^{n-6}]}$ | -           | 0           | $(-1)^n$ | $(-1)^n$ | 0           | 0           | 0              |
| $\chi_{[2^3,1^{n-6}]}$ | -           | $(-1)^n$ | 0              | 0        | 0           | $(-1)^n$ | ($-1)^{n-6}$   |
| $\chi_{[3,2,1^{n-5}]}$ | -           | $(-1)^n$ | 0              | 0        | 0           | $(-1)^n$ | ($-1)^{n-5}$   |
| $\chi_{[2^2,1^{n-5}]}$ | -           | $(-1)^n$ | 0              | 0        | 0           | $(-1)^n$ | ($-1)^{n-5}$   |
| $\chi_{[1^5]}$ | -           | $(-1)^n$ | 0              | 0        | 0           | $(-1)^n$ | ($-1)^{n-4}$   |

**APPENDIX A.**

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