SUPERCONVERGENCE AND REGULARITY OF DENSITIES IN FREE PROBABILITY

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Abstract. The phenomenon of superconvergence, first observed in the central limit theorem of free probability, was subsequently extended to arbitrary limit laws for free additive convolution. We show that the same phenomenon occurs for the multiplicative versions of free convolution on the positive line and on the unit circle. We also show that a certain Hölder regularity, first demonstrated by Biane for the density of a free additive convolution with a semicircular law, extends to free (additive and multiplicative) convolutions with arbitrary freely infinitely divisible distributions.

1. Introduction

Suppose that \( \{X_n\}_{n \in \mathbb{N}} \) is a free, identically distributed sequence of bounded random variables with zero mean and unit variance. It is known from \cite{22} that the distributions \( \mu_n \) of the central limit averages
\[
\frac{X_1 + \cdots + X_n}{\sqrt{n}}
\]
converge weakly to a standard semicircular distribution. Unlike the classical central limit theorem, it was shown in \cite{7} that the distribution \( \mu_n \) is absolutely continuous relative to Lebesgue measure on \( \mathbb{R} \) for sufficiently large \( n \), and that the densities \( d\mu_n/\,dt \) converge uniformly to \( \sqrt{4 - t^2} \chi_{[-2,2]} / 2\pi \). This unexpected convergence of densities (along with the fact that the support \([a_n, b_n]\) of \( \mu_n \) converges to \([-2, 2]\) and the density is analytic on \((a_n, b_n)\)) was called superconvergence. The uniform convergence of densities was later proved to hold even when the variables \( X_n \) are not bounded \cite{25}. The phenomenon of superconvergence was extended to other limit laws and applied to limit theorems for eigenvalue densities of random matrices (see, for instance, \cite{20, 2}). Eventually, the present authors proved in \cite{13} that uniform convergence of densities holds in the general context of limit laws for triangular arrays with free, identically distributed rows. That is, suppose that \( k_1 < k_2 < \cdots \) is a sequence of positive integers, and for each \( n \) the variables \( \{X_{n,j} : j = 1, \ldots, k_n\} \) are free and identically distributed. Suppose also that the distribution \( \mu_n \) of
\[
X_{n,1} + \cdots + X_{n,k_n}
\]
converges weakly to some nondegenerate distribution \( \mu \). The measure \( \mu \) is infinitely divisible \cite{31} and it is absolutely continuous except for a set \( D_\mu \) that is either empty or a singleton \cite{13} Proposition 5.1. Let \( V \supset D_\mu \) be an arbitrary open

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set in $\mathbb{R}$; $V$ can be taken to be empty if $D_\mu = \emptyset$. Then the result of [13] states that $\mu_n$ is absolutely continuous on $\mathbb{R}\setminus V$ and the density of $\mu_n$ converges uniformly to the density of $\mu$ as $n \to \infty$.

Of course, the results mentioned above can be formulated just as easily in terms of free additive convolution of measures. One purpose of the present note is to prove completely analogous results for free multiplicative convolution of probability measures on $\mathbb{R}_+ = [0, +\infty)$ and on the unit circle $\mathbb{T} = \{e^{it} : t \in \mathbb{R}\}$. The multiplicative results are not simply consequences of the additive ones. In fact, each of the three convolutions has its own analytic apparatus, and in each case an important fact is that the respective Voiculescu transform of an infinitely divisible measure has an analytic extension to a certain domain $D$ (that depends on the type of convolution). In each case, the proof is done first for convolutions of infinitely divisible measures. The general case is then obtained via an approximation of infinitesimal measures by infinitely divisible ones, somewhat analogous to the associated laws used in the classical treatment of limit laws for sums of independent random variables [18]. These infinitely divisible laws are obtained from the subordination properties that hold for free convolutions.

The methods we develop for superconvergence are useful in other contexts as well. We illustrate this by extending results of Biane [14] concerning the density of a free convolution of the form $\mu \boxplus \gamma$, where $\gamma$ is a semicircular distribution. Such a convolution is always absolutely continuous, its density $h$ is continuous and, in fact, locally analytic wherever it is positive. If $h(t) = 0$ for some $t$ and $h(x) \neq 0$ in some interval with an endpoint at $t$, it is shown in [14] that $h(x) = O(|x - t|^{1/3})$ for $x$ close to $t$ in that interval. We show that this result holds if $\gamma$ is replaced by an arbitrary nondegenerate $\boxplus$-infinitely divisible distribution. Of course, in this general context, it may happen that $\mu \boxplus \gamma$ has a finite number of atoms and points at which the density is unbounded. The result holds for all other points where the density vanishes. Analogous results are also proved for the two multiplicative free convolutions.

The remainder of this paper is organized as follows. Sections 2–4 deal with multiplicative convolution on $\mathbb{R}_+$. A section presents preliminaries about this operation, including a new observation analogous to the Schwarz lemma, the next section demonstrates superconvergence, and the last section deals with the possible cusps of the free convolution with an infinitely divisible law. Sections 5–7 follow the same program for multiplicative free convolution on the unit circle $\mathbb{T}$. Finally, Sections 8 and 9 deal with additive free convolution; there is no additive analog of Sections 3 and 6 in the additive case because the corresponding result was already proved in [13]. The reader may however note that the arguments of [13] can be simplified using the present methods. Applications to free convolution semigroups are discussed in the appendix.

2. FREE MULTIPlicative CONVOLUTION ON $\mathbb{R}_+$

We denote by $\mathcal{P}_{\mathbb{R}_+}$ the collection of all probability measures on $\mathbb{R}_+$. The free multiplicative convolution $\boxtimes$ is a binary operation on $\mathcal{P}_{\mathbb{R}_+}$. The mechanics of its calculation involves analytic functions defined on the domains $\mathbb{C}\setminus \mathbb{R}_+$,

\[ \mathbb{H} = \{ x + iy : x, y \in \mathbb{R}, y > 0 \}, \]
and $-\mathbb{H}$. The first of these is the **moment generating function** $\psi_\mu$ of a measure $\mu \in \mathcal{P}_{\mathbb{R}_+}$ defined by
\[
\psi_\mu(z) = \int_{\mathbb{R}_+} \frac{zt}{1-zt} \, d\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
\]
This function satisfies $\psi_\mu(H) \subset H$ and $\psi_\mu((-\infty, 0)) \subset (-1, 0)$ unless $\mu$ is the unit point mass at 0, denoted $\delta_0$, for which $\psi_{\delta_0} = 0$. A closely related function is the $\eta$-**transform** of $\mu$ given by
\[
\eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
\]
We have $\eta_\mu(H) \subset H$ and $\eta_\mu((-\infty, 0)) \subset (-1, 0)$ when $\mu \neq \delta_0$. These transforms are related to the **Cauchy transform** defined by
\[
G_\mu(z) = \int_{\mathbb{R}_+} \frac{d\mu(t)}{z-t}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+,
\]
by the identity
\[
(2.1) \quad \frac{1}{z} G_\mu \left( \frac{1}{z} \right) = \frac{1}{1 - \eta_\mu(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
\]
The Stieltjes inversion formula shows that any of these functions can be used to recover the measure $\mu$. More precisely, the measures
\[
\frac{-1}{\pi} \mathbb{R} G_\mu(x+iy) dx, \quad y > 0,
\]
converge weakly to $\mu$ as $y \downarrow 0$. The boundary values
\[
G_\mu(x) = \lim_{y \downarrow 0} G_\mu(x+iy), \quad x \in \mathbb{R}_+,
\]
exists almost everywhere (with respect to Lebesgue measure) on $\mathbb{R}_+$, and the density $d\mu/dt$ of $\mu$ is equal almost everywhere to $(-1/\pi) G_\mu$ (cf. [21]).

In terms of the $\eta$-transform, the relation (2.1) shows that
\[
(2.2) \quad \frac{1}{x} \frac{d\mu}{dt} \left( \frac{1}{x} \right) = \frac{1}{\pi} - \frac{1}{1 - \eta_\mu(x)},
\]
almost everywhere on $\mathbb{R}_+$, where $\eta_\mu(x)$ is defined almost everywhere as
\[
\eta_\mu(x) = \lim_{y \downarrow 0} \eta_\mu(x+iy).
\]

The collection of functions $\{\eta_\mu : \mu \in \mathcal{P}_{\mathbb{R}_+} \setminus \{\delta_0\}\}$ is described as follows.

**Lemma 2.1.** Let $f : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C}$ be an analytic function. Then there exists $\mu \in \mathcal{P}_{\mathbb{R}_+}$ such that $f = \eta_\mu$ if and only if the following conditions are satisfied:

1. $f(z) = \overline{f(z)}$ for every $z \in \mathbb{C} \setminus \mathbb{R}_+$,
2. $\lim_{z \to 0} f(x) = 0$, and
3. $\arg f(z) \geq \arg z$, $z \in \mathbb{H}$, where the arguments are in $(0, \pi)$.

Equality occurs in (3) for some $z$ precisely when $\mu = \delta_a$ for some $a > 0$, in which case $f(z) = \eta_\mu(z) = az$.

In fact, condition (3) above is superfluous, as can be seen from Lemma [23] which we view as an analog of the Schwarz lemma for analytic functions in the unit disk. (This result plays no role in the remainder of the paper but it will certainly prove to be useful.)
Lemma 2.2. Suppose that $F : \mathbb{C} \setminus \mathbb{R}^+ \to \mathbb{C}$ is analytic, $F(\mathbb{H}) \subset \mathbb{H}$, $F((-\infty, 0)) \subset (-\infty, 0)$, and

$$F(\tau) = \overline{F(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}^+.$$ 

Then there exist constants $\alpha, \beta \in [0, +\infty)$ and a finite Borel measure $\rho$ on $(0, +\infty)$ such that $\int_{(0, +\infty)} dp(t)/t < +\infty$ and

$$F(z) = -\alpha + \beta z + \int_{(0, +\infty)} \frac{z(1 + t^2)}{t(t - z)} d\rho(t), \quad z \in \mathbb{C} \setminus \mathbb{R}^+.$$ 

**Proof.** Since $F(\mathbb{H}) \subset \mathbb{H}$, $F$ has a Nevanlinna representation of the form

$$F(z) = \alpha_0 + \beta z + \int_{\mathbb{R}} \frac{1 + zt}{t - z} d\rho(t), \quad z \in \mathbb{H},$$

with $\alpha_0 \in \mathbb{R}$, $\beta \in [0, +\infty)$, and a finite Borel measure $\rho$ on $\mathbb{R}$ (cf. [1]). Because $F$ is analytic and real-valued on $(-\infty, 0)$, the measure $\rho$ is supported on $[0, +\infty)$. The formula

$$F'(z) = \beta + \int_{(0, +\infty)} \frac{1 + t^2}{(t - z)^2} d\rho(t)$$

shows that $F$ is increasing on $(-\infty, 0)$. Now, $F((-\infty, 0)) \subset (-\infty, 0)$, so $\lim_{z \uparrow 0} F(z) \leq 0$. The monotone convergence theorem yields now

$$\alpha_0 + \int_{(0, +\infty)} \frac{1}{t} d\rho(t) = \lim_{z \uparrow 0} F(z) \leq 0.$$ 

In particular, $\rho(\{0\}) = 0$ and $\rho$ satisfies the condition in the statement. We set

$$\alpha = -\alpha_0 - \int_{(0, +\infty)} \frac{1}{t} d\rho(t),$$

and obtain the formula

$$F(z) = -\alpha + \beta z + \int_{(0, +\infty)} \left[\frac{1 + zt}{t - z} - \frac{1}{t}\right] d\rho(t),$$

valid in the entire region $\mathbb{C} \setminus \mathbb{R}^+$ by reflection. This is easily seen to be precisely the formula in the statement. \qed

Notation: $\Omega_\alpha = \{z \in \mathbb{C} \setminus \mathbb{R}^+ : |\arg z| > \alpha\}$. Here $\alpha \in (0, \pi)$ and the argument takes values in $(-\pi, \pi)$.

The following result is a version of the Schwarz lemma.

Lemma 2.3. Under the conditions of Lemma 2.2, we have $F(\Omega_\alpha) \subset \Omega_\alpha$ for every $\alpha \in (0, \pi)$.

**Proof.** It suffices to prove that $F(\Omega_\alpha \cap \mathbb{H}) \subset \Omega_\alpha \cap \mathbb{H}$. Since $\Omega_\alpha \cap \mathbb{H}$ is a convex cone, the representation formula in Lemma 2.2 reduces the proof to the following three cases:

1. $F(z) = -1$,
2. $F(z) = z$,
3. $F(z) = z/(t - z)$ for some $t > 0$.

The result is trivial in the first two cases. In the third case one observes that $F$ maps $\Omega_\alpha \cap \mathbb{H}$ conformally onto a region $D_\alpha$ bounded by the negative real line and a circle $C$ passing through the origin. Moreover, since $F'(0) > 0$, the tangent to $C$ at 0 is the line $\{\arg z = \alpha\}$. It follows immediately that $D_\alpha \cap \mathbb{H} \subset \Omega_\alpha \cap \mathbb{H}$. \qed
Mapping \( \mathbb{C} \setminus \mathbb{R}_+ \) conformally to a strip by the logarithm, we obtain another version of Schwarz lemma as follows. We set \( S_t = \{ z \in \mathbb{C} : |\Im z| < t \}, t > 0 \).

**Lemma 2.4.** Let \( F : S_1 \to S_1 \) be an analytic function such \( F(S_1 \cap \mathbb{H}) \subset S_1 \cap \mathbb{H} \) and
\[
F(\overline{z}) = \overline{F(z)}, \quad z \in S_1.
\]
Then \( F(S_t) \subset S_t \) for every \( t \in (0, 1) \).

Given a measure \( \mu \neq \delta_0 \) in \( \mathcal{P}_{\mathbb{R}_+} \), the function \( \eta_\mu \) is conformal in an open set \( U \) containing some interval \((\beta, 0)\) with \( \beta < 0 \), and the restriction \( \eta_\mu|U \) has an inverse \( \eta_\mu^{(-1)} \) defined in an open set containing an interval of the form \((\alpha, 0)\) with \( \alpha < 0 \). The free multiplicative convolution \( \mu_1 \boxtimes \mu_2 \) of two measures \( \mu_1, \mu_2 \in \mathcal{P}_{\mathbb{R}_+} \setminus \{\delta_0\} \) is the unique measure \( \mu \in \mathcal{P}_{\mathbb{R}_+} \setminus \{\delta_0\} \) that satisfies the identity
\[
zn^{(-1)}(z) = \eta_{\mu_1}^{(-1)}(z)\eta_{\mu_2}^{(-1)}(z)
\]
for \( z \) in some open set containing an interval \((\alpha, 0)\) with \( \alpha < 0 \) (see \[9\]). (We also have \( \delta_0 \boxtimes \mu = \delta_0 \) for every \( \mu \in \mathcal{P}_{\mathbb{R}_+} \).) Based on the characterization of \( \eta \)-transform, another approach to free convolution is given by the following reformulation of the subordination results in \[15\].

**Theorem 2.5.** For every \( \mu_1, \mu_2 \in \mathcal{P}_{\mathbb{R}_+} \setminus \{\delta_0\} \), there exist unique \( \rho_1, \rho_2 \in \mathcal{P}_{\mathbb{R}_+} \setminus \{\delta_0\} \) such that
\[
\eta_{\mu_1}(\eta_{\rho_1}(z)) = \eta_{\mu_2}(\eta_{\rho_2}(z)) = \frac{\eta_{\mu_1}(z)\eta_{\rho_2}(z)}{z}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
\]
Moreover, we have \( \eta_{\mu_1} \boxtimes \mu_2 = \eta_{\mu_1} \circ \eta_{\rho_1} \). If \( \mu_1 \) and \( \mu_2 \) are nondegenerate, then so are \( \rho_1 \) and \( \rho_2 \).

We recall that a measure \( \mu \in \mathcal{P}_{\mathbb{R}_+} \) is said to be \( \boxtimes \)-infinitely divisible if there exist measures \( \{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_{\mathbb{R}_+} \) satisfying the identities
\[
\underbrace{\mu_n \boxtimes \cdots \boxtimes \mu_n}_{n \text{ times}} = \mu, \quad n \in \mathbb{N}.
\]
Obviously, \( \delta_0 \) is \( \boxtimes \)-infinitely divisible; one can take \( \mu_n = \delta_0 \). It was shown in \[24], \[9\] that a measure \( \mu \in \mathcal{P}_{\mathbb{R}_+} \setminus \{\delta_0\} \) is \( \boxtimes \)-infinitely divisible precisely when the inverse \( \eta_\mu^{(-1)} \) continues analytically to \( \mathbb{C} \setminus \mathbb{R}_+ \) and this analytic continuation has the special form
\[
\Phi(z) = \gamma z \exp \left [ \int_{[0, +\infty]} \frac{1 + tz}{z - t} \, d\sigma(t) \right ], \quad z \in \mathbb{C} \setminus \mathbb{R}_+,
\]
for some \( \gamma > 0 \) and some finite Borel measure \( \sigma \) on the one point compactification of \( \mathbb{R}_+ \). The fraction in the above formula must be interpreted as \(-z\) when \( t = +\infty \). This is, of course, an analog of the classical Lévy-Hinčin formula. The pair \((\gamma, \sigma)\) is uniquely determined by \( \mu \), and every such pair corresponds with a unique \( \boxtimes \)-infinitely divisible measure, sometimes denoted \( \gamma^\sigma \). Another description of the class of functions defined by \[2.4\] is as follows:
\[
\Phi(z) = z \exp(u(z)),
\]
where \( u : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C} \) is an analytic function such that \( u(\mathbb{H}) \subset -\mathbb{H} \) and \( u(\overline{z}) = u(z) \) for all \( z \in \mathbb{C} \setminus \mathbb{R}_+ \). This equivalent description is used in Lemma \[2.1\].

Suppose now that \( \mu \in \mathcal{P}_{\mathbb{R}_+} \) is a nondegenerate \( \boxtimes \)-infinitely divisible measure and that \( \eta_\mu^{(-1)} \) has the analytic continuation given in \[2.4\]. The equation \( \Phi(\eta_\mu(z)) = z \)
holds in some open set and therefore it holds on the entire \(\mathbb{C}\setminus\mathbb{R}_+\) by analytic continuation. In particular, \(\eta_\mu\) maps \(\mathbb{C}\setminus\mathbb{R}_+\) conformally onto a domain \(\Omega_\mu \subset \mathbb{C}\setminus\mathbb{R}_+\) that is symmetric relative to the real line. The domain \(\Omega_\mu\) is easily identified as the connected component of the set \(\{z \in \mathbb{C}\setminus\mathbb{R}_+ : \Phi(z) \in \mathbb{C}\setminus\mathbb{R}_+\}\) containing \((\infty, 0)\). This set and its boundary were thoroughly investigated in [19], and the results are important in the sequel. Because of the symmetry of \(\Omega_\mu\), we consider only the upper half of \(\Omega_\mu\), namely, \(\Omega_\mu \cap \mathbb{H}\). A simple calculation shows that

\[
\Phi(re^{i\theta}) = \gamma \exp[u(re^{i\theta}) + iv(re^{i\theta})],
\]

where the real and imaginary parts \(u\) and \(v\) are given by

\[
u(re^{i\theta}) = \theta \left[1 - \frac{r \sin \theta}{\theta} \int_{0, +\infty} \frac{1 + t^2}{|re^{i\theta} - t|^2} \, d\sigma(t)\right],
\]

for \(r > 0\) and \(\theta \in (0, \pi)\). As noted in [19], a remarkable situation occurs: for fixed \(r > 0\), the function

\[
I_r(\theta) = \frac{r \sin \theta}{\theta} \int_{0, +\infty} \frac{1 + t^2}{|re^{i\theta} - t|^2} \, d\sigma(t), \quad \theta \in (0, \pi],
\]

is continuous, strictly decreasing, and \(I_r(\pi) = 0\). Thus, the set \(\{\theta \in (0, \pi) : I_r(\theta) < 1\}\) is an interval, say

\[
\{\theta \in (0, \pi) : I_r(\theta) < 1\} = (f(r), \pi).
\]

The value \(f(r)\) is 0 precisely when the limit

\[
I_r(0) = \lim_{\theta \downarrow 0} I_r(\theta) = r \int_{0, +\infty} \frac{1 + t^2}{(r - t)^2} \, d\sigma(t)
\]

is at most 1. Otherwise, we have \(I_r(f(r)) = 1\). The following statement summarizes results from [19] Theorem 4.16.

**Theorem 2.6.** Let \(\mu \in \mathcal{P}_{\mathbb{R}_+}\) be a nondegenerate \(\mathbb{E}\)-infinitely divisible measure, let \(\Phi\) defined by (2.4) be the analytic continuation of \(\eta_\mu^{-1}\), let \(I_r : [0, \pi] \to (0, +\infty)\) be defined by (2.7) and (2.8), and let \(f : (0, +\infty) \to [0, \pi)\) be defined by (2.9). Then:

1. \(\eta_\mu\) maps \(\mathbb{H}\) conformally onto \(\Omega_\mu \cap \mathbb{H} = \{re^{i\theta} : r > 0, \theta \in (f(r), \pi)\}\).
2. The function \(f\) is continuous on \((0, +\infty)\) and real analytic on the open set \(\{r : f(r) > 0\}\).
3. The topological boundary of the set \(\Omega_\mu \cap \mathbb{H}\) is \((\infty, 0] \cup \{re^{i\theta} : r > 0\}\).
4. \(\eta_\mu\) extends continuously to the closure \(\overline{\Omega_\mu \cap \mathbb{H}}\), \(\Phi\) extends continuously to the closure \(\overline{\Omega_\mu \cap \mathbb{H}}\), and these extensions are homeomorphisms, inverse to each other. In particular, the function \(h : (0, +\infty) \to (0, +\infty)\) defined by

\[
h(r) = \Phi(re^{i\theta(r)}), \quad r > 0,
\]

is an increasing homeomorphism from \((0, +\infty)\) onto \((0, +\infty)\) and the image \(\eta_\mu((0, +\infty))\) is parametrized implicitly as

\[
\eta_\mu(h(r)) = re^{i\theta(r)}, \quad r > 0.
\]
It is known that \( \mu(\{0\}) = 0 \) for every \( \mathbb{E} \)-infinitely divisible measure \( \mu \in \mathcal{P}_{\mathbb{R}+}\{\delta_0\} \). For such a measure \( \mu \), we can define a measure \( \mu_* \in \mathcal{P}_{\mathbb{R}+}\{\delta_0\} \) such that \( d\mu_*(t) = d\mu(1/t) \). An easy calculation yields the identities

\[
\psi_{\mu_*}(z) = -1 - \psi_{\mu}(1/z), \quad \eta_{\mu_*}(z) = \frac{1}{\eta_{\mu}(1/z)} = \frac{1}{\eta_{\mu}(1/z)} \quad z \in \mathbb{C} \setminus \mathbb{R}_+,
\]

and therefore

\[
\eta_{\mu_*}^{-1}(z) = \frac{1}{\eta_{\mu}^{-1}(1/z)}
\]

for \( z \) in some open set containing \((-\infty, 0)\). It follows that \( \eta_{\mu_*}^{-1} \) has an analytic continuation to \( \mathbb{C} \setminus \mathbb{R}_+ \). In fact, if \( \Phi \) is the continuation of \( \eta_{\mu_*}^{-1} \) given by (2.4), then the function

\[
\Phi_*(z) = \frac{1}{\Phi(1/z)} = \frac{1}{\Phi(1/z)} = \frac{1}{\gamma} z \exp \left[ \int_{[0, +\infty)} \frac{1 + tz}{z - t} d\sigma_*(t) \right], \quad z \in \mathbb{C} \setminus \mathbb{R}_+,
\]

extends \( \eta_{\mu_*}^{-1} \), where \( d\sigma_*(t) = d\sigma(1/t) \) with the convention that \( 1/0 = +\infty \) and \( 1/+\infty = 0 \). Thus, \( \mu_* \) is also infinitely divisible, and the boundary of \( \Omega_{\mu_*} \cap \mathbb{H} \) is described as above using a continuous function \( f_* : (0, +\infty) \to [0, \pi) \). This function and the associated homeomorphism \( h_* : (r, +\infty) \to (0, +\infty) \) are easily seen to satisfy the identities

\[
f_*(r) = f(1/r), \quad h_*(r) = \frac{1}{h(1/r)}, \quad r \in (0, +\infty).
\]

The following result gives estimates for the growth of \( h \) at 0 and \( +\infty \).

**Proposition 2.7.** Let \( \mu, \Phi, \) and \( h \) be as in Theorem 2.6. Then

\[
h(r) \leq \gamma r \exp(\sigma([0, +\infty]) + 2), \quad r \in (0, 1/4),
\]

and

\[
h(r) \geq \gamma r \exp(-\sigma([0, +\infty]) - 2), \quad r \in (4, +\infty).
\]

In particular, \( \lim_{r \to 0} h(r) = 0 \).

**Proof.** Suppose for the moment that the first inequality was proved. Applying the result to the measure \( \mu_* \), we see that

\[
\frac{1}{h(1/r)} = h_*(r) \leq \frac{1}{\gamma} r \exp(\sigma_*([0, +\infty]) + 2)
\]

for \( r < 1/4 \). The second inequality follows after replacing \( r \) by \( 1/r \).

Fix now \( r \in (0, 1/4) \), and use relations (2.5), (2.6), and the fact that \( r^2 - 1 \leq 0 \) to deduce the inequality

\[
|\Phi(re^{i\theta})| \leq \gamma r \exp \left[ r \cos \theta \int_{[0, +\infty]} \frac{1 - t^2}{|re^{i\theta} - t|^2} d\sigma(t) \right], \quad \theta \in (0, \pi).
\]
We distinguish two cases, according to whether $f(r) < \pi/2$ or $f(r) \geq \pi/2$. In the first case, we have $I_r(f(r)) \leq 1$, and thus for $\theta \in (f(r), \pi/2)$ we have
\[
\left| r \cos \theta \int_{[0, \infty]} \frac{1 - t^2}{|re^{i\theta} - t|^2} \, d\sigma(t) \right| \leq r \int_{[0, \infty]} \frac{1 + t^2}{|re^{i\theta} - t|^2} \, d\sigma(t) = \frac{\theta}{\sin \theta} I_r(\theta) \leq \frac{\pi}{2} I_r(f(r)) < 2.
\]
Thus $|h(r)| = \lim_{\theta \downarrow f(r)} |\Phi(re^{i\theta})| \leq \gamma r e^2$, thus verifying the first inequality in this case. In the second case, we observe that for $\theta = f(r)$ we have
\[
\left| r \cos \theta \int_{[0, 1]} \frac{1 - t^2}{|re^{i\theta} - t|^2} \, d\sigma(t) \right| \leq 0,
\]
and thus
\[
\left| r \cos \theta \int_{[0, \infty]} \frac{1 - t^2}{|re^{i\theta} - t|^2} \, d\sigma(t) \right| \leq r \int_{[1, \infty]} \frac{t^2 - 1}{|re^{i\theta} - t|^2} \, d\sigma(t)
\leq r \int_{[1, \infty]} \frac{t^2 - 1}{t - \frac{1}{2}} \, d\sigma(t)
\leq r \int_{[1, \infty]} 2 \, d\sigma(t) \leq \sigma([0, \infty]).
\]
This verifies the inequality in the second case and concludes the proof. \qed

The continuity of the function $\Phi$ on some parts of $\Omega_\mu$ can be established as follows.

**Lemma 2.8.** Let $\mu \in \mathcal{P}_{\mathbb{R}_+}$ be a nondegenerate $\mathcal{E}$-infinitely divisible measure, and let $\tilde{\Phi}$ defined by \ref{eq:19} be the analytic continuation of $\eta_\mu^{-1}$. Set
\[
u(z) = \int_{[0, \infty]} \frac{1 + tz}{z - t} \, d\sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
\]
Suppose that $z_j = r_j e^{i\theta_j} \in \Omega_\mu \cap \mathbb{H}$ and that $\theta_j \leq \pi/2$ for $j = 1, 2$. Then
\[
|u(z_1) - u(z_2)| \leq \frac{\pi}{2} \frac{|z_1 - z_2|}{\sqrt{|z_1 z_2|}}.
\]

**Proof.** We have $\theta_j \in (f(r_j), \pi/2)$, $j = 1, 2$. In particular,
\[
I_{r_j}(\theta_j) \leq I_{r_j}(f(r_j)) \leq 1, \quad j = 1, 2.
\]
Then
\[
|u(z_1) - u(z_2)| = \left| (z_1 - z_2) \int_{[0, \infty]} \frac{(1 + t^2)^{1/2}(1 + t^2)^{1/2} z_1 - t}{z_2 - t} \, d\sigma(t) \right|
\leq |z_1 - z_2| \left[ \int_{[0, \infty]} \frac{1 + t^2}{|z_1 - t|^2} \, d\sigma(t) \right]^{1/2} \left[ \int_{[0, \infty]} \frac{1 + t^2}{|z_2 - t|^2} \, d\sigma(t) \right]^{1/2}
\leq \frac{\pi}{2} \frac{|z_1 - z_2|}{\sqrt{r_1 r_2}},
\]
where we used the Schwarz inequality. \qed
We conclude this section with a few known facts about convolution powers. Given a measure $\nu \in \mathcal{P}_{\mathbb{R}^+} \setminus \{\delta_0\}$ and $k \in \mathbb{N}$, we use the notation
\[
\nu \otimes^k = \nu \otimes \cdots \otimes \nu
\]
for the free multiplicative convolution of $k$ copies of $\nu$. By Theorem 2.3 there exists a measure $\mu \in \mathcal{P}_{\mathbb{R}^+}$ such that $\eta_{\nu \otimes^k} = \eta_{\nu} \circ \eta_{\mu}$. It is shown in [2] that
\[
\Phi(\eta_{\mu}(z)) = z, \quad z \in \mathbb{C} \setminus \mathbb{R}^+,
\]
where
\[
\Phi(z) = \frac{z^k}{\eta_{\nu}(z)^{k-1}}, \quad z \in \mathbb{C} \setminus \mathbb{R}^+,
\]
is easily seen to have the form (2.4). As seen earlier, this means that $\mu$ is in fact $\otimes$-infinitely divisible, and therefore $\eta_{\mu}$ has a continuous (and injective) extension to $(0, +\infty)$. The relation between $\eta_{\mu}$ and $\nu \otimes^k$ can also be written as
\[
(2.11)
\]
As observed in [3] this equality implies that $\eta_{\nu \otimes^k}$ also has a continuous extension to $(0, +\infty)$ and (2.11) remains true for real values of $z$. We use below this identity under the equivalent form
\[
x \eta_{\mu}(1/x)^k = \eta_{\nu \otimes^k}(1/x)^{k-1}, \quad x \in (0, +\infty).
\]
This is of interest because it allows us to calculate the density $d\nu \otimes^k / dt$ in terms of the density of $\mu$. Indeed, rewriting the above identity as
\[
\eta_{\mu}(1/x) \left[x \eta_{\nu}(1/x)^{1/(k-1)} = \eta_{\nu \otimes^k}(1/x), \quad x \in (0, +\infty),
\]
one may be able to argue (as we do in Section 3) that $\eta_{\nu \otimes^k}(1/x)$ is very close to $\eta_{\mu}(1/x)$ if $k$ is large, and then (2.2) allows us to conclude that these two measures have close densities.

3. SUPERCONVERGENCE IN $\mathcal{P}_{\mathbb{R}^+}$

We begin by studying the weak convergence of a sequence of nondegenerate $\otimes$-infinitely divisible measures in $\mathcal{P}_{\mathbb{R}^+}$. Thus, suppose that $\gamma$ and $\{\gamma_n\}_{n \in \mathbb{N}}$ are positive numbers, $\sigma$ and $\{\sigma_n\}_{n \in \mathbb{N}}$ are finite, nonzero Borel measures on $[0, +\infty]$, $\mu$ and $\{\mu_n\}_{n \in \mathbb{N}}$ are nondegenerate $\otimes$-infinitely divisible measures in $\mathcal{P}_{\mathbb{R}^+}$, and the inverses $\eta_{\mu}^{-1}$, $\{\eta_{\mu_n}^{-1}\}_{n \in \mathbb{N}}$ have analytic continuations $\Phi$, $\{\Phi_{\mu_n}\}_{n \in \mathbb{N}}$ given by (2.4) for $\mu$ and by analogous formulas for $\mu_n$ (with $\gamma_n$ and $\sigma_n$ in place of $\gamma$ and $\sigma$). The sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converges weakly to $\mu$ if and only if $\{\sigma_n\}_{n \in \mathbb{N}}$ converges weakly to $\sigma$ and $\lim_{n \to \infty} \gamma_n = \gamma$. (This fact is implicit in the proof of Theorem 4.3 in [11].) When these conditions are satisfied, it is also true that the sequences $\{\eta_{\mu_n}\}_{n \in \mathbb{N}}$ and $\{\Phi_{\mu_n}\}_{n \in \mathbb{N}}$ converge to $\eta_{\mu}$ and $\Phi_{\mu}$, respectively, and the convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{R}^+$.

In order to show that superconvergence occurs, we need to understand the behavior of the functions $f$ and $h$ defined in Section 2 in relation to $\mu$ and that of the functions $f_n$ and $h_n$ associated to $\mu_n$. It is understood that $h_n$ and $h$ are extended to $\mathbb{R}$ so that $h(0) = h_n(0) = 0$.

**Lemma 3.1.** With the above notation, suppose that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converges weakly to $\mu$. Then:
(1) The sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) uniformly on compact subsets of \((0, +\infty)\).

(2) The sequence \( \{h_n\}_{n \in \mathbb{N}} \) converges to \( h \) uniformly on compact subsets of \(\mathbb{R}_+\).

(3) The sequence of inverses \( \{h_n^{-1}\}_{n \in \mathbb{N}} \) converges to \( h^{-1} \) uniformly on compact subsets of \(\mathbb{R}_+\).

(4) The sequence \( \{f_n \circ h_n^{-1}\}_{n \in \mathbb{N}} \) converges to \( f \circ h^{-1} \) uniformly on compact subsets of \((0, +\infty)\).

(5) The sequence \( \{\eta_n\}_{n \in \mathbb{N}} \) converges to \( \eta \) uniformly on compact subsets of \((0, +\infty)\).

Proof. Fix \( r > 0 \), let \( \varepsilon \in (0, \pi - f(r)) \), and let \( J = [r - \delta, r + \delta] \) be such that \( |f(s) - f(r)| < \varepsilon \) for every \( s \in J \). Observe that the compact set

\[
C = \{ se^{i\theta} : \theta \in [f(r) + \varepsilon, \pi], s \in J \}
\]

has the property that \( \Phi(C) \subset \mathbb{H} \). Since \( \Phi_n \) converges to \( \Phi \) uniformly on \( C \), it follows that \( \Phi_n(C) \subset \mathbb{H} \) for sufficiently large \( n \), and thus \( f_n(s) < f(r) + \varepsilon < f(s) + 2\varepsilon \), \( s \in J \), for such \( n \). This proves (1) in case \( f(r) = 0 \). If \( f(r) > 0 \), there exists a positive angle \( \theta_0 \in (f(r) - \varepsilon, f(r)) \) such that \( \Phi(re^{i\theta_0}) \notin \mathbb{H} \). Shrink the number \( \delta \) such that \( \Phi(se^{i\theta_0}) \in \mathbb{H} \) for every \( s \in J \). It follows from uniform convergence that \( \Phi_n(se^{i\theta_0}) \in \mathbb{H} \), \( s \in J \), for sufficiently large \( n \), and thus \( f_n(s) > \theta_0 > f(r) - \varepsilon > f(s) - 2\varepsilon \), \( s \in J \), thus completing the proof of (1).

For (2) and (3), it suffices to prove pointwise convergence because pointwise convergence of continuous increasing functions is automatically locally uniform. Since convergence obviously holds at 0, fix \( r > 0 \). Suppose first that \( f(r) > 0 \). In this case, \( se^{i\theta} \in \mathbb{H} \) for \( s \) in some compact neighborhood of \( r \), and hence \( \Phi_n \) converges uniformly to \( \Phi \) in a neighborhood of \( re^{i\theta} \). By (1), \( \lim_{n \to \infty} f_n(r) = f(r) \), and the local uniform convergence of \( \Phi_n \) yields

\[ h(r) = \Phi(re^{i\theta}) = \lim_{n \to \infty} \Phi_n(re^{i\theta}f_n(r)) = \lim_{n \to \infty} h_n(r), \]

thus proving (1) in this case. Suppose now that \( f(r) = 0 \), and thus \( \lim_{n \to \infty} f_n(r) = 0 \). Assume, for simplicity, that \( f_n(r) < 1 \) for every \( n \in \mathbb{N} \), and define functions \( \Psi_n : (0, \pi/2 - 1] \to \mathbb{C} \) by setting

\[ \Psi_n(\theta) = \Phi_n(re^{i\theta}f_n(r)), \quad 0 < \theta \leq \frac{\pi}{2} - 1, n \in \mathbb{N}. \]

It follows from Lemma [28] that the functions \( \Psi_n \) are uniformly equicontinuous. The local uniform convergence of \( \Phi_n \) to \( \Phi \) shows that \( \Psi_n \) converges pointwise to \( \Phi(re^{i\theta}) \). Now, both \( \Psi_n \) and \( \Phi(re^{i\theta}) \) extend continuously to \( \theta = 0 \) with

\[ \Psi_n(0) = h_n(r), \quad \Phi(r) = h(r). \]

The uniform equicontinuity of \( \Psi_n \) implies that the convergence also holds (even uniformly) for these continuous extensions, and at \( \theta = 0 \) this yields the desired equality \( \lim_{n \to \infty} h_n(r) = h(r) \).

The pointwise convergence of \( h_n^{-1} \) to \( h^{-1} \) follows directly from (2). Indeed, suppose that \( t_0 > 0 \), \( s_0 = h(t_0) \), and \( 0 < \varepsilon < t_0 \). We have \( \lim_{n \to \infty} h_n(t_0 - \varepsilon) = h(t_0 - \varepsilon) \), \( \lim_{n \to \infty} h_n(t_0 + \varepsilon) = h(t_0 + \varepsilon) \), and the open interval \( (h(t_0 - \varepsilon), h(t_0 + \varepsilon)) \) contains \( s_0 \). It follows that the interval \( (h_n(t_0 - \varepsilon), h_n(t_0 + \varepsilon)) \) also contains \( s_0 \) for sufficiently large \( n \), and thus \( h_n^{-1}(s_0) \in (t_0 - \varepsilon, t_0 + \varepsilon) \) for such \( n \). Since \( \varepsilon \) is arbitrarily small, we have \( \lim_{n \to \infty} h_n^{-1}(s_0) = t_0 = h^{-1}(s_0) \).

Finally, (4) and (5) follow from (1) and (3) (see [16, Theorem XII.2.2]). \( \square \)
We are now ready to show that the weak convergence of infinitely divisible measures implies the convergence of the densities of these measures, locally uniformly outside a singleton. We first identify the density of an infinitely divisible $\mu$, for which $\eta_{\mu}^{-1}$ has the continuation $\Phi$ in (2.3), in terms of the functions $f$ and $h$. The fact that the extension of $\eta_{\mu}$ to $(0, +\infty)$ is continuous and injective shows that $A_{\mu} = \{ t \in (0, +\infty) : \eta_{\mu}(t) = 1 \}$ is either empty or a singleton. It is clear from the definition of $f$ that the set $A_{\mu}$ is nonempty precisely when $I_1(0) \leq 1$. If this condition is satisfied, the set $A_{\mu}$ consists of $h(1)$ and $\mu(\{1/h(1)\}) = 1 - I_1(0)$ (cf. [19]). Accordingly, we denote $D_{\mu} = \{1/h(1)\}$ if $I_1(0) \leq 1$, and $D_{\mu} = \emptyset$ otherwise. It follows that $\mu$ is absolutely continuous with a continuous density $p_{\mu} = d\mu/dt$ on $(0, +\infty) \setminus D_{\mu}$. Equations (2.11) and (2.2) give the implicit formula

$$
\frac{1}{h(r)}p_{\mu}\left(\frac{1}{h(r)}\right) = \frac{1}{\pi|1-re^{i\theta}|^2}, \quad r > 0, \ h(r) \notin A_{\mu}.
$$

We record for further use a simple consequence of (3.1). For fixed $r$, the function $|1-re^{i\theta}|$, $\theta \in \mathbb{R}$, achieves its minimum at $\theta = 0$, and thus

$$
\frac{1}{h(r)}p_{\mu}\left(\frac{1}{h(r)}\right) \leq \frac{r}{\pi(1-r)^2}, \quad r \in (0, +\infty) \setminus \{1\},
$$
or, equivalently,

$$
\langle -\rangle_{1/r}p_{\mu}(t) \leq \frac{h^{-1}(1/t)}{\pi(1-h^{-1}(1/t))^2}, \quad t \in (0, +\infty) \setminus D_{\mu}.
$$

**Proposition 3.2.** Let $\mu$ and $\{\mu_n\}_{n \in \mathbb{N}}$ be nondegenerate $\Xi$-infinitely divisible measures in $\mathcal{P}_{2+}$ and let $U$ be an arbitrary open neighborhood of the set $D_{\mu}$; if $D_{\mu} = \emptyset$, take $U = \emptyset$. Then $D_{\mu_n} \subset U$ for sufficiently large $n$, and the functions $t\langle -\rangle_{1/r}p_{\mu_n}(t)$ converge to $t\langle -\rangle_{1/r}p_{\mu}(t)$ uniformly for $t \in (0, +\infty) \setminus U$.

**Proof.** We use the notation established above: $\eta_{\mu_n}^{-1}$ has the analytic continuation $\Phi_n$ determined by the parameters $\gamma_n$ and $\sigma_n$, and $f_n, h_n$ play the roles of $f, h$ for the measure $\mu_n$. The relation $D_{\mu_n} = \{1/h_n(1)\} \subset U$ for large $n$ follows directly from Lemma 3.1(2). We focus on the proof of uniform convergence. We show first that it suffices to prove that $t\langle -\rangle_{1/r}p_{\mu_n}(t)$ converges to $t\langle -\rangle_{1/r}p_{\mu}(t)$ locally uniformly on $(0, +\infty) \setminus U$.

For this purpose, fix $\varepsilon > 0$ and choose $\alpha, \beta \in (0, +\infty)$ such that

$$
\frac{x}{\pi(1-x)^2} < \varepsilon, \quad x \in (0, +\infty) \setminus [\alpha, \beta].
$$

Since $h^{-1}$ is an increasing homeomorphism of $(0, +\infty)$, there exist $a, b \in (0, +\infty)$ such that $h^{-1}(1/b) < \alpha$ and $h^{-1}(1/a) > \beta$. Lemma 3.1 shows that there exists $N \in \mathbb{N}$ such that $h_n^{-1}(1/b) < \alpha$ and $h_n^{-1}(1/a) > \beta$ for $n \geq N$, and hence

$$
t\langle -\rangle_{1/r}p_{\mu_n}(t), t\langle -\rangle_{1/r}p_{\mu}(t) < \varepsilon, \quad t \in (0, +\infty) \setminus [a, b],
$$

by (3.2). It suffices therefore to prove uniform convergence on $[a, b] \setminus U$, and this would follow from local uniform convergence on $(0, +\infty) \setminus D_{\mu}$. For this purpose, it is convenient to write (3.1) in the explicit form

$$
t\langle -\rangle_{1/r}p_{\mu}(t) = \frac{1}{\pi|1-h^{-1}(1/t)e^{i\theta(h^{-1}(1/t))}|^2}, \quad t \notin D_{\mu}.
$$

Suppose that $t_0 \notin D_{\mu}$, and choose a compact neighborhood $W$ of $t_0$ such that

$$
1 - h^{-1}(1/t)e^{i\theta(h^{-1}(1/t))} \neq 0, \quad t \in W.
$$
Lemma 3.1 shows that there exists an integer $N$ such that

$$1 - h_n^{-1}(1/t)e^{i\alpha(h_n^{-1}(1/t))} \neq 0, \quad t \in W, \; n \geq N,$$

and then we conclude from (3.3) (applied to $\mu_n$), and from Lemma 3.1 that $tp_n(t)$ converges to $tp_\mu(t)$ uniformly on $W$. □

An immediate consequence is as follows.

**Corollary 3.3.** Under the conditions of Proposition 3.2, the sequence $\{p_{\mu_n}\}_{n \in \mathbb{N}}$ converges to $p$ locally uniformly on $(0, +\infty) \setminus D_\mu$.

We can now prove a general version of superconvergence.

**Theorem 3.4.** Let $k_1 < k_2 < \cdots$ be positive integers, and let $\mu$ and $\{\nu_n\}_{n \in \mathbb{N}}$ be nondegenerate measures in $P_{\mathbb{R}^+}$ such that $\mu$ is $\mathbb{S}$-infinitely divisible. Suppose that the sequence $\{\nu^{\star k_n}\}_{n \in \mathbb{N}}$ converges weakly to $\mu$. Let $K \subset (0, +\infty) \setminus D_\mu$ be an arbitrary compact set. Then $\nu^{\star k_n}$ is absolutely continuous on $K$ for sufficiently large $n$, and the sequence $\{d\nu^{\star k_n}/dt\}_{n \in \mathbb{N}}$ converges to $d\mu/dt$ uniformly on $K$.

**Proof.** As noted at the end of Section 2, there exist nondegenerate $\mathbb{S}$-infinitely divisible measures $\mu_n \in P_{\mathbb{R}^+}$ such that $\eta_{\mu_n} = \eta_{\mu} \circ \eta_{\mu_n}$ and

$$(3.4) \quad \eta_{\mu_n}(1/x) [\eta_{\mu_n}(1/x)]^{1/(k_n-1)} = \eta_{\mu^{\star k_n}}(1/x), \quad x \in (0, +\infty), \; n \in \mathbb{N}.$$ 

It is known from [11] that the sequence $\{\nu_n\}_{n \in \mathbb{N}}$ converges weakly to $\delta_1$, and therefore the functions $\eta_{\nu_n}(z)$ converge to $z$ uniformly on compact subsets of $\mathbb{C}\setminus \mathbb{R}^+$. Similarly, the functions $\eta_{\mu_n}^{-1}(z)$ converge uniformly to $z$ on compact subsets of $(-\infty, 0)$. Since $\eta_{\nu_n}^{-1}$ converges to $\eta_\mu$ uniformly on compact subsets of $\mathbb{C}\setminus \mathbb{R}^+$, we deduce that $\eta_{\mu_n}(z) = \eta_{\nu_n}^{-1}(\eta_{\mu_n}(z))$ converges to $\eta_\mu$ uniformly on compact subsets of $(-\infty, 0)$. It follows that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converges weakly to $\mu$. By Lemma 3.1(5), $\eta_{\mu_n}(x)$ tends to $\eta_\mu(x)$ uniformly on compact subsets of $(0, +\infty)$, and therefore

$$[x\eta_{\mu_n}(1/x)]^{1/(k_n-1)}$$

converges to 1 uniformly on compact subsets of $(0, +\infty)$ since $k_n \to \infty$. Then (3.4) shows that $\eta_{\mu^{\star k_n}}(1/x)$ converges to $\eta_\mu(1/x)$ uniformly on compact subsets of $(0, +\infty)$. The conclusion of the theorem follows now from (2.2) applied to these measures, as in the proof of Proposition 3.2. □

### 4. Cusp behavior in $P_{\mathbb{R}^+}$

In this section, we describe the qualitative behavior of a convolution $\mu_1 \boxplus \mu_2$, where $\mu_1, \mu_2 \in P_{\mathbb{R}^+}$ are nondegenerate measures and $\mu_2$ is $\mathbb{S}$-infinitely divisible, subject to a mild additional condition. It was shown in [14] how an analytic function argument provides examples in which the density of $\mu \boxplus \nu$, with $\nu$ a semicircle law, can have a cusp behavior at some points. More precisely, if $h$ is the density, then, at some of its zeros $t_0 \in \mathbb{R}$, the ratio $h(t)/|t - t_0|^{1/3}$ is bounded. Then it is shown in [13] that this is the worst possible cusp behavior that such a density can have. Arguments, similar to those in [14], show that the density of $\mu_1 \boxplus \mu_2$ can also be bounded by a cubic root near a zero if $\mu_2$ is the multiplicative analog of the semicircular law. Our purpose in this section is to show that this is the worst possible behavior for such densities if $\mu_2$ is an almost arbitrary $\mathbb{S}$-infinitely divisible measure. The argument proceeds in two steps. First, we work with the case in
which $\mu_2$ is the multiplicative analog of a semicircular measure, thus producing a multiplicative analog of Lemma 5 and Proposition 4 in [14]. For general $\mu_2$, we show that the density of $\mu_1 \boxtimes \mu_2$ can be estimated using a different convolution $\nu_1 \boxtimes \nu_2$, where $\nu_2$ is one of these multiplicative analogs of the semicircular measure, chosen with appropriate parameters.

We recall an observation first made in [14] in the free additive case. (The simple proof is provided for convenience as well as for establishing notation.)

**Lemma 4.1.** Let $\mu_1, \mu_2 \in \mathcal{P}_{\mathbb{R}^+}$ be such that $\mu_2$ is $\mathcal{E}$-infinitely divisible, and let $\rho_1, \rho_2 \in \mathcal{P}_{\mathbb{R}^+}$ be given by Theorem [2.3]. Then $\rho_1$ is $\mathcal{E}$-infinitely divisible.

**Proof.** Let $\Phi$ given by (2.3) be the analytic continuation of $\eta_{\mu_2}^{(-1)}$. Then (2.3) can be written as

$$\Phi(z) = \frac{\eta_{\mu_1}(z)}{\eta_{\mu_2}(z)} \eta_{\mu_1}^{(-1)}(z),$$

and applying this equality with $\eta_{\mu_1}(z)$ in place of $z$, we obtain

$$\Phi(\eta_{\mu_1}(z)) = \frac{\eta_{\mu_1}(z)}{\eta_{\mu_2}(z)} \eta_{\mu_1}^{(-1)}(\eta_{\mu_1}(z)) = \eta_{\rho_1}^{(-1)}(z)$$

for $z \in (\beta, 0)$ and $\beta < 0$. The lemma follows because the function

$$(4.1) \quad \Psi(z) = \frac{\Phi(\eta_{\mu_1}(z))}{\eta_{\rho_1}(z)} z, \quad z \in \mathbb{C} \setminus \mathbb{R}_+,$$

is of the form $z \exp(v(z))$, where

$$v(z) = \log \gamma + \int_{[0, +\infty]} \frac{1 + \eta_{\mu_1}(z)}{\eta_{\mu_1}(z)} - t d\sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}_+,$$

is an analytic function satisfying $v(\mathbb{H}) \subset -\mathbb{H}$ (since $\eta_{\mu_1}(\mathbb{H}) \subset \mathbb{H}$) and $v(\mathbb{H}) = v(z)$ for $z \in \mathbb{C} \setminus \mathbb{R}_+$. \hfill $\square$

With the notation of the preceding lemma, we recall that the domain

$$\Omega_{\rho_1} = \eta_{\rho_1}(\mathbb{H})$$

(for which we denote by $\Omega_{\rho_1} \cap \mathbb{H}$ earlier) can be described as

$$\Omega_{\rho_1} = \{ r e^{i\theta} : r > 0, f(r) < \theta < \pi \}$$

for some continuous function $f : (0, +\infty) \to [0, \pi)$, and that $\eta_{\rho_1}$ extends to a homeomorphism of $\mathbb{H}$ onto $\Omega_{\rho_1}$. It was shown in [19] that $\eta_{\rho_1}$ extends continuously to $\overline{\Omega_{\rho_1}}$ provided that we allow $\infty$ as a possible value. Using, as before, the increasing homeomorphism

$$h(r) = \Psi(re^{i(f(r))}), \quad r \in (0, +\infty),$$

the density $q_{\mu_1 \boxtimes \mu_2}$ of $\mu_1 \boxtimes \mu_2$, relative to the Haar measure $dx/x$ on $(0, +\infty)$, is calculated using the formula

$$(4.2) \quad q_{\mu_1 \boxtimes \mu_2}(1/x) = \frac{1}{\cosh(h(r))}, \quad x = h(r) \text{ and } f(r) > 0,$$

$$0, \quad x = h(r) \text{ and } f(r) = 0.$$

The following proposition examines the density of $\mu_1 \boxtimes \mu_2$ when $\nu_2$ is analogous to the semicircular measure, that is, when $\sigma$ is a point mass at $t = 1$. (The equation [4.3] regarding this density also appeared in [26].)
Proposition 4.2. Suppose that $\beta, \gamma \in (0, +\infty)$, and that $\mu_2 \in \mathcal{P}_{\mathbb{R}_+}$ is such that
\[
\gamma z \exp \left[ \frac{\beta z + 1}{z - 1} \right], \quad z \in \mathbb{C}\setminus\mathbb{R}_+
\]
is an analytic continuation of $\eta_{\mu_2}^{(-1)}$. Let $q_{\mu_2} \in \mathcal{P}$ be the density of $\mu_1 \boxtimes \mu_2$ relative to the Haar measure $dx/x$ and define $k(x) = q_{\mu_2}(1/x)$. Then:

1. $|k'(x)| k(x)^2 \leq 1/(4\pi^2 \beta^2 x)$ for every $x \in (0, +\infty)$ such that $k(x) \neq 0$.
2. If $I \subseteq \mathbb{R}_+$ is an interval with one endpoint $x_0 > 0$, $k(x) > 0$ for $x \in I$, and $k(x_0) = 0$, then
\[
k(x)^3 \leq \frac{3}{4\pi^4 \beta^2} |\log x - \log x_0|, \quad x \in I.
\]

In particular, $k(x)/|x - x_0|^{1/3}$ and $k(x)/|x^{-1} - x_0^{-1}|^{1/3}$ remain bounded for $x \in I$ close to $x_0$.

Proof. Part (2) follows from (1) because
\[
k(x)^3 = \left| \int_{x_0}^{x} 3k(s)^2 k'(s) \, ds \right|.
\]
By (4.1), we have
\[
\Psi(z) = \gamma z \exp \left[ \frac{\beta \eta_{\mu_1}(z) + 1}{\eta_{\mu_1}(z) - 1} \right]
= \gamma z \exp \beta \left[ 1 - \frac{2}{1 - \eta_{\mu_1}(z)} \right]
= e^{-\beta} \gamma z \exp \left[ -2\beta \psi_{\mu_1}(z) \right].
\]
The fact that $\arg \Psi(re^{if(r)}) = 0$ leads to the identity
\[
f(r) = 2\beta \Im \frac{1}{1 - \eta_{\mu_1}(re^{if(r)})} = 2\pi \beta k(re^{if(r)}).
\]
We note for further use that
\[
\Im \frac{1}{1 - \eta_{\mu_1}(re^{if(r)})} = \Im (1 + \psi_{\mu_1}(re^{if(r)})) = \int_{\mathbb{R}_+} \frac{tr \sin(f(r))}{|1 - tr e^{if(r)}|^2} \, d\mu_1(t).
\]
Of course, our estimate applies to points $x = x(r) = \Psi(re^{if(r)})$ such that $f(r) > 0$, and at such points $f$ is real analytic. By the chain rule,
\[
k'(x(r)) = \frac{(d/dr)k(x(r))}{(d/dr)x(r)} = \frac{(1/2\pi \beta) f'(r)}{(x'(r)/x(r))x(r)}
\]
and thus we must find lower estimates for the logarithmic derivative $x'(r)/x(r)$. We have
\[
\left| \frac{(d/dr)\Psi(re^{if(r)})}{\Psi(re^{if(r)})} \right| = \left| \frac{\Psi'(re^{if(r)})}{\Psi(re^{if(r)})} \right| \left| \frac{d(re^{if(r)})}{dr} \right|
= \left| \frac{1}{re^{if(r)}} - 2\beta \psi_{\mu_1}'(re^{if(r)}) \right| \left| e^{if(r)}(1 + irf'(r)) \right|
= \frac{1}{r} \left| 1 - 2\beta re^{if(r)} \psi_{\mu_1}'(re^{if(r)}) \sqrt{1 + r^2 f'(r)^2} \right|.
\]
We now calculate
\[ \psi_{\mu_1}(re^{if(r)}) = \int_{\mathbb{R}^+} \frac{t}{(1 - tre^{if(r)})^2} d\mu_1(t), \]
and use relations (4.3) and (4.4) to see that
\[ 1 - 2\beta re^{if(r)} \psi_{\mu_1}(re^{if(r)}) = 2\beta \frac{1}{f(r)} - 2\beta re^{if(r)} \psi_{\mu_1}(re^{if(r)}) \]
\[ = 2\beta \int_{\mathbb{R}^+} \left[ \frac{1}{f(r)} \frac{tr \sin(f(r))}{1 - tre^{if(r)} - (1 - tre^{if(r)})^2} \right] d\mu_1(t). \]
We now calculate
\[ \mathbb{R} \left[ \frac{1}{f(r)} \frac{tr \sin(f(r))}{1 - tre^{if(r)} - (1 - tre^{if(r)})^2} \right] \]
\[ = tr \left( f(r)[1 - tre^{if(r)}] - f(r)[1 - tre^{if(r)}]^2 \right) \]
\[ = tr \left( f(r)[1 + t^2r^2] \sin(f(r)) - f(r) \cos(f(r)) + f(r) \sin(f(r)) \right) \]
\[ = \frac{t^2r^2 [f(r) - \sin(2f(r))]}{f(r)[1 - tre^{if(r)}]^2}, \]
where we used the fact that \( \sin f - f \cos f \geq 0 \) for \( f \in (0, \pi) \). Thus,
\[ |1 - 2\beta re^{if(r)} \psi_{\mu_1}(re^{if(r)})| \geq 2\beta \int_{\mathbb{R}^+} \mathbb{R} \left[ \frac{1}{f(r)} \frac{tr \sin(f(r))}{1 - tre^{if(r)} - (1 - tre^{if(r)})^2} \right] d\mu_1(t). \]
\[ \geq 2\beta \frac{2f(r) - \sin(2f(r))}{f(r)} \int_{\mathbb{R}^+} \frac{t^2r^2}{[1 - tre^{if(r)}]^2} d\mu_1(t) \]
(Schwarz inequality)
\[ \geq 2\beta \frac{2f(r) - \sin(2f(r))}{f(r)} \left[ \int_{\mathbb{R}^+} \frac{tr}{[1 - tre^{if(r)}]^2} d\mu_1(t) \right]^2 \]
(by (4.3) and (4.4))
\[ = 2\beta \frac{2f(r) - \sin(2f(r))}{f(r)} \left[ \frac{f(r)}{2\beta \sin(f(r))} \right]^2 \]
\[ = 2\beta \frac{2f(r) - \sin(2f(r))}{f(r) \sin^2(f(r))} \left[ \frac{f(r)^2}{2\beta} \right]. \]
A further lower bound is obtained using the inequality \( 2f - \sin(2f) \geq f \sin^2 f \), valid for \( f \in (0, \pi) \). We obtain
\[ \left| \frac{x'(r)}{x(r)} \right| = \left| \frac{(d/dr)\Psi(re^{if(r)})}{\Psi(re^{if(r)})} \right| \]
\[ \geq \frac{f(r)^2}{2\beta} \frac{\sqrt{1 + r^2f(r)^2}}{r}, \]
and finally from (4.3),
\[
|k'(x(r))| = \left| \frac{(1/2\pi \beta)f'(r)}{(x'(r)/x(r))x(r)} \right| \leq \frac{1}{\pi f(r)^2x(r)} \frac{r|f'(r)|}{\sqrt{1 + r^2f''(r)^2}} \leq \frac{1}{\pi f(r)^2x(r)}.
\]

By (4.3), this is precisely the inequality in (1). \(\square\)

**Remark 4.3.** With the notation of the preceding lemma, it is easy to verify that the inequality in (1) is equivalent to
\[
|q'_{\mu_1 \boxplus \mu_2}(x)|\, q_{\mu_1 \boxplus \mu_2}(x)^2 \leq \frac{1}{4\pi^3 \beta^2x}, \quad \text{where} \quad x \in (0, +\infty), \quad q_{\mu_1 \boxplus \mu_2}(x) \neq 0.
\]

One essential observation that allows us to extend the preceding result to more general \(\boxplus\)-infinitely divisible measures \(\mu_2\) is as follows. The density of \(\mu_1 \boxplus \mu_2\) depends largely, via (4.2) on the function \(f\), and thus on the \(\boxplus\)-infinitely divisible measure \(\rho_1\). In many cases, it is possible to find another convolution \(\nu_1 \boxplus \nu_2\), such that \(\eta_{\nu_1 \boxplus \nu_2} = \eta_{\nu_1} \circ \eta_{\nu_2}\) (with the same measure \(\rho_1\)), and such that \(\nu_2\) is a multiplicative analog of the semicircular measure. The verification of the following result is a simple calculation. The details are left to the reader. Note that the existence of the measure \(\nu_1\) below follows from Lemma 2.1.

**Lemma 4.4.** Let \(\mu_1, \mu_2 \in \mathcal{P}_{\mathbb{R}^+}\) be such that \(\mu_2\) is \(\boxplus\)-infinitely divisible, and let
\[
\Phi(z) = \gamma z \exp \left[ \int_{[0, +\infty]} \frac{1 + tz}{z - t} \, d\sigma(t) \right], \quad z \in \mathbb{C} \setminus \mathbb{R}^+,
\]
be an analytic continuation of \(\eta_{\mu_2}^{(-1)}\). Denote by \(\rho_1 \in \mathcal{P}_{\mathbb{R}^+}\) the \(\boxplus\)-infinitely divisible measure such that \(\eta_{\rho_1}^{(-1)}\) has the analytic continuation
\[
\Psi(z) = \gamma z \exp \left[ \int_{[0, +\infty]} \frac{1 + t\eta_{\rho_1}(z)}{\eta_{\rho_1}(z) - t} \, d\sigma(t) \right], \quad z \in \mathbb{C} \setminus \mathbb{R}^+.
\]

Suppose that
\[
\beta = \frac{1}{2} \int_{[0, +\infty]} \frac{1}{1 + t} \, d\sigma(1/t)
\]
is finite and nonzero. Denote by \(\nu_1 \in \mathcal{P}_{\mathbb{R}^+}\) the measure satisfying
\[
\psi_{\nu_1}(z) = \frac{1}{2\beta} \int_{[0, +\infty]} \frac{t\eta_{\mu_1}(z)}{1 - t\eta_{\mu_1}(z)} \left( \frac{1}{1 + t} \right) \, d\sigma(1/t), \quad z \in \mathbb{C} \setminus \mathbb{R}^+,
\]
and denote by \(\nu_2 \in \mathcal{P}_{\mathbb{R}^+}\) the \(\boxplus\)-infinitely divisible measure such that \(\eta_{\nu_2}^{(-1)}\) has the analytic continuation
\[
\gamma' z \exp \left[ \beta \frac{z + 1}{z - 1} \right], \quad z \in \mathbb{C} \setminus \mathbb{R}^+,
\]
where the constant
\[
\gamma' = \gamma \exp \left[ \frac{1}{2} \int_{[0, +\infty]} \frac{1}{1 - t} \, d\sigma(1/t) \right],
\]
Then $\eta_{\mu_1 \boxdot \mu_2} = \eta_{\mu_1} \circ \eta_{\mu_2}$ and $\eta_{\nu_1 \boxdot \nu_2} = \eta_{\nu_1} \circ \eta_{\nu_2}$.

For the final proof in this section, we need some results from [19], which we formulate using the notation established in Lemma 4.1. According to [19, Theorem 4.16], the zero set $\{ \alpha \in (0, +\infty) : f(\alpha) = 0 \}$ can be partitioned into three sets $A, B, C$ defined as follows.

1. The set $A$ consists of those $\alpha \in (0, +\infty)$ such that $\mu_1(\{1/\alpha\}) > 0$ and
   \[
   \int_{[0, +\infty]} \frac{1 + t^2}{(1 - t)^2} d\sigma(t) \leq \mu_1(\{1/\alpha\}).
   \]

2. The set $B$ consists of those $\alpha \in (0, +\infty)$ for which $\eta_{\mu_1}(\alpha) \in \mathbb{R}\setminus\{1\}$ and
   \[
   \left[ \int_{\mathbb{R}_+} \frac{\alpha t}{(1 - \alpha t)^2} d\mu_1(t) \right] \left[ \int_{[0, +\infty]} \frac{1 + t^2}{(\eta_{\mu_1}(\alpha) - t)^2} d\sigma(t) \right] \leq \frac{1}{(1 - \eta_{\mu_1}(\alpha))^2}.
   \]

3. Finally, $\alpha \in C$ provided that $\eta_{\mu_1}(\alpha) = \infty$ and
   \[
   \left[ \int_{\mathbb{R}_+} \frac{d\mu_1(t)}{(1 - \alpha t)^2} \right] \left[ \int_{[0, +\infty]} (1 + t^2) d\sigma(t) \right] \leq 1.
   \]

The density of $\mu_1 \boxdot \mu_2$ is continuous everywhere, except on the finite set
\[
\{1/\Psi(\alpha) : \alpha \in A\} \cup \{0\}.
\]
If $x \in (0, +\infty)$ is an atom of $\mu_1 \boxdot \mu_2$ then $\eta_{\mu_1}(1/x) \in A$.

**Theorem 4.5.** Let $\mu_1, \mu_2 \in \mathcal{P}_{\mathbb{R}_+}$ be two nondegenerate measures such that $\mu_2$ is $\boxdot$-infinitely divisible, and let
\[
\Phi(z) = \gamma z \exp \left[ \int_{[0, +\infty]} \frac{1 + tz}{z - t} d\sigma(t) \right], \quad z \in \mathbb{C}\setminus\mathbb{R}_+,
\]
be an analytic continuation of $\eta_{\mu_2}^{-1}$. Suppose that $\sigma(0, +\infty) > 0$. If $I \subset (0, +\infty)$ is an open interval with an endpoint $x_0 > 0$ such that $1/\eta_{\mu_1}(1/x_0)$ is not an atom of $\mu_1$, and $q_{\mu_1 \boxdot \mu_2}(x_0) = 0 < q_{\mu_1 \boxdot \mu_2}(x)$ for every $x \in I$, then $q_{\mu_1 \boxdot \mu_2}(x)/|x - x_0|^{1/3}$ is bounded for $x \in I$ close to $x_0$.

**Proof.** We can always find finite measures $\sigma'$ and $\sigma''$ on $[0, +\infty]$ such that $\sigma = \sigma' + \sigma''$, $\sigma'' \neq 0$ and $\sigma''$ has compact support contained in $(0, +\infty)$. The $\boxdot$-infinitely divisible measures $\mu_2', \mu_2'' \in \mathcal{P}_{\mathbb{R}_+}$, defined by the fact that $\eta_{\mu_2'}^{-1}$ and $\eta_{\mu_2''}$ have analytic continuations
\[
\gamma z \exp \left[ \int_{[0, +\infty]} \frac{1 + tz}{z - t} d\sigma'(t) \right] \quad \text{and} \quad \gamma z \exp \left[ \int_{[0, +\infty]} \frac{1 + tz}{z - t} d\sigma''(t) \right], \quad z \in \mathbb{C}\setminus\mathbb{R}_+,
\]
respectively, satisfy the relation $\mu_2' \boxdot \mu_2'' = \mu_2$, and thus $\mu_1 \boxdot \mu_2 = \mu_1' \boxdot \mu_2''$, where $\mu_1'' = \mu_1 \boxdot \mu_2'$. There exist additional $\boxdot$-infinitely divisible measures $\rho_1', \rho_2' \in \mathcal{P}_{\mathbb{R}_+}$ such that $\eta_{\mu_1'} = \eta_{\mu_1} \circ \eta_{\rho_1'}$ and $\eta_{\mu_2' \boxdot \mu_2''} = \eta_{\mu_2'} \circ \eta_{\rho_2'}$. Clearly, $\eta_{\rho_1'} = \eta_{\rho_1'} \circ \eta_{\rho_2'}$, and we argue that $1/\eta_{\rho_1'}(1/x_0)$ is a real number but not an atom of $\mu_1'$. Indeed, letting $z \to 1/x_0$ in the inequality
\[
\arg \eta_{\rho_1'}(z) = \arg(\eta_{\rho_1'}(\eta_{\rho_1'}(z))) \geq \arg \eta_{\rho_1'}(z), \quad z \in \mathbb{H},
\]
the hypothesis $\eta_{\rho_1}(1/x_0) \in (0, +\infty)$ implies that $\eta_{\rho_1}(1/x_0) \in (0, +\infty)$. Suppose, to get a contradiction, that $1/\eta_{\rho_1}(1/x_0)$ is an atom of $\mu_1''$. Then, as seen in [5],
\[
1/\eta_{\rho_1}(1/x_0) = 1/\eta_{\rho_1}(\eta_{\rho_1}(1/x_0))
\]

is necessarily an atom of $\mu_1$, contrary to the hypothesis.

The above construction shows that the hypothesis of the theorem also holds with $\mu_1''$, $\mu_2''$, and $\rho_1''$ in place of $\mu_1, \mu_2$, and $\rho_1$, respectively. Moreover, it is obvious that $\int_{[0, +\infty]} ((t^2 + 1)/t) \, d\sigma(t) < +\infty$. Therefore we may, and do, assume that the additional hypothesis $\int_{[0, +\infty]} ((t^2 + 1)/t) \, d\sigma(t) < +\infty$ is satisfied. In particular, the hypothesis of Lemma 4.4 is satisfied. With the notation of that lemma, Proposition 4.2 shows that it suffices to prove that $q_{\nu_1 \boxplus \nu_2}(x)/q_{\mu_1 \boxplus \mu_2}(x)$ is bounded away from zero for $x \in I$ close to $x_0$. For this purpose, we write points $x \in (0, +\infty)$ as $x = 1/\Psi(re^{i\beta})$. In particular, $x_0 = 1/\Psi(r_0e^{i\beta})$ and $f(r_0) = 0$. The fact that $1/\eta_{\rho_1}(1/x_0)$ is not an atom for $\mu_1$ implies that $r_0 \in B \cup C$. The formula (4.2) and the definition of $\nu_1$ yield
\[
q_{\nu_1 \boxplus \nu_2}(x) = \frac{1}{\pi} \Im \frac{1}{1 - \eta_{\nu_1}(re^{i\beta})} = \frac{1}{\pi} \Im \eta_{\nu_1}(re^{i\beta})
\]
\[
= \frac{1}{2\pi \beta} \Im \left[ \int_{[0, +\infty]} \frac{t \eta_{\nu_1}(re^{i\beta})}{1 - t \eta_{\nu_1}(re^{i\beta})} \left( t + \frac{1}{t} \right) \, d\sigma(1/t) \right]
\]
\[
= \frac{\Im \eta_{\nu_1}(re^{i\beta})}{2\pi \beta} \int_{[0, +\infty]} \frac{1}{|l - \eta_{\nu_1}(re^{i\beta})|^2} \, d\sigma(t).
\]

Since we also have
\[
q_{\mu_1 \boxplus \mu_2}(x) = \frac{1}{\pi} \Im \frac{1}{1 - \eta_{\mu_1}(re^{i\beta})} = \frac{1}{\pi} \Im \eta_{\mu_1}(re^{i\beta})
\]
we deduce that
\[
\frac{q_{\nu_1 \boxplus \nu_2}(x)}{q_{\mu_1 \boxplus \mu_2}(x)} = \frac{|1 - \eta_{\mu_1}(re^{i\beta})|^2}{2\beta} \int_{[0, +\infty]} \frac{1}{|t - \eta_{\mu_1}(re^{i\beta})|^2} \, d\sigma(t).
\]

Letting $x \to x_0$, so $r \to r_0$, we see that
\[
\liminf_{x \to x_0, x \in I} \frac{q_{\nu_1 \boxplus \nu_2}(x)}{q_{\mu_1 \boxplus \mu_2}(x)} \geq \frac{|1 - \eta_{\mu_1}(r_0)|^2}{2\beta} \int_{[0, +\infty]} \frac{1}{|t - \eta_{\mu_1}(r_0)|^2} \, d\sigma(t)
\]
if $r_0 \in B$, and
\[
\liminf_{x \to x_0, x \in I} \frac{q_{\nu_1 \boxplus \nu_2}(x)}{q_{\mu_1 \boxplus \mu_2}(x)} \geq \frac{1}{2\beta} \int_{[0, +\infty]} (1 + t^2) \, d\sigma(t)
\]
if $r_0 \in C$. In either case, the lower estimate is strictly positive.

\[\square\]

**Remark 4.6.** In the above proof, we show that $q_{\mu_1 \boxplus \mu_2}(x) = O(q_{\nu_1 \boxplus \nu_2}(x))$ as $x \to x_0$, $x \in I$. It is also true that $q_{\nu_1 \boxplus \nu_2}(x) = O(q_{\mu_1 \boxplus \mu_2}(x))$ as $x \to x_0, x \in I$. To see this, we observe that the definition of $f$ implies the equality
\[
f(r) = \Im \eta_{\mu_1}(re^{i\beta}) \int_{[0, +\infty]} \frac{1}{|t - \eta_{\mu_1}(re^{i\beta})|^2} \, d\sigma(t).
\]
Thus, the reciprocal of the fraction in (4.6) can be rewritten as
\[
\frac{q_{\mu_1 \boxtimes_{\mathbb{C}} \mu_2}(x)}{q_{\nu_1 \boxtimes_{\mathbb{C}} \nu_2}(x)} = \frac{2\beta \Re \eta_{\mu_1}(re^{if(r)})}{1 - \eta_{\mu_1}(re^{if(r)})^2} \frac{f(r)}{\Re(\eta_{\mu_1}(re^{if(r)})[1 - \eta_{\mu_1}(re^{if(r)})]^2)}
\]
\[
= \frac{2\beta r \sin(f(r)) \Re \eta_{\mu_1}(re^{if(r)})}{f(r)} \Re(\eta_{\mu_1}(re^{if(r)})[1 - \eta_{\mu_1}(re^{if(r)})]^2)
\]
\[
= \frac{2\beta r \sin(f(r)) \Re \psi_{\mu_1}(re^{if(r)})}{f(r)} \Re(\eta_{\mu_1}(re^{if(r)})[1 - \eta_{\mu_1}(re^{if(r)})]^2)
\]
\[
= \frac{2\beta r \sin(f(r))}{f(r)} \int_{\mathbb{R}^+} \frac{t}{1 - tre^{if(r)}} d\mu_1(t).
\]

Letting \( x \to x_0 \) yields
\[
\liminf_{x \to x_0, x \in \mathbb{D}} \frac{q_{\mu_1 \boxtimes_{\mathbb{C}} \mu_2}(x)}{q_{\nu_1 \boxtimes_{\mathbb{C}} \nu_2}(x)} \geq 2\beta \int_{\mathbb{R}^+} \frac{tr_0}{(1 - tr_0)^2} d\mu_1(t) > 0.
\]

Note that the quantity on the right hand side is in fact finite. This is immediate if \( r_0 \in B \), and it follows from the identity
\[
\int_{\mathbb{R}^+} \frac{tr_0}{(1 - tr_0)^2} d\mu_1(t) = \int_{\mathbb{R}^+} \frac{1}{(1 - tr_0)^2} d\mu_1(t) - \int_{\mathbb{R}^+} \frac{1}{1 - tr_0} d\mu_1(t)
\]
if \( r_0 \in C \).

**Remark 4.7.** There are cases, other than those of Proposition 4.2 in which the set \( \{x_0 > 0 : 1/\eta_{\mu_1}(1/x_0) \in A\} \) is empty, and thus the conclusion of Theorem 4.5 holds on every connected component of the set \( \{x : p_{\mu_1 \boxtimes_{\mathbb{C}} \mu_2}(x) > 0\} \). See Remark 9.7 for a brief discussion in the context of additive free convolution.

5. **Free multiplicative convolution on \( \mathbb{T} \)**

We denote by \( \mathcal{P}_{\mathbb{T}} \) the collection of nondegenerate probability measures on \( \mathbb{T} \). The definition of the moment generating function for a measure \( \mu \in \mathcal{P}_{\mathbb{T}} \) is analogous to the one used for \( \mathcal{P}_{\mathbb{R}^+} \), but the domain is now the unit disk \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \):
\[
\psi_{\mu}(z) = \int_{\mathbb{D}} \frac{tz}{1 - tz} d\mu(t), \quad z \in \mathbb{D}.
\]
The \( \eta \)-transform of \( \mu \) is the function
\[
\eta_{\mu}(z) = \frac{\psi_{\mu}(z)}{1 + \psi_{\mu}(z)}, \quad z \in \mathbb{D}.
\]
The collection \( \{\eta_{\mu} : \mu \in \mathcal{P}_{\mathbb{T}}\} \) is simply the set of all analytic functions \( f : \mathbb{D} \to \mathbb{D} \) that satisfy \( f(0) = 0 \). If we denote by
\[
H_{\mu}(z) = \int_{\mathbb{T}} \frac{t + z}{t - z} d\mu(t), \quad z \in \mathbb{D},
\]
the Herglotz integral of \( \mu \), and we define \( \mu_\ast \in \mathcal{P}_{\mathbb{T}} \) by \( d\mu_\ast(t) = d\mu(t)/t \), we have
\[
H_{\mu_\ast}(z) = 1 + 2\psi_{\mu}(z) = \frac{1 + \eta_{\mu}(z)}{1 - \eta_{\mu}(z)}, \quad z \in \mathbb{D}.
\]
Since \( \Re H_{\mu_\ast} \) is the Poisson integral of \( \mu_\ast \), we deduce that the measures
\[
\frac{1}{2\pi} \Re \int_{0}^{2\pi} \frac{1 + \eta_{\mu}(re^{it})}{1 - \eta_{\mu}(re^{it})} dt, \quad t \in [0, 2\pi), r \in (0, 1),
\]
converge weakly to \(d\mu(e^{it})\) as \(r \uparrow 1\). In particular, the density of \(\mu\) relative to arclength measure on \(T\) is given almost everywhere by

\[
p_{\mu}(z) = \frac{1}{2\pi} \Re \frac{1 + \eta_{\mu}(\overline{z})}{1 - \eta_{\mu}(\overline{z})}, \quad z \in T,
\]

where

\[
\eta_{\mu}(z) = \lim_{r \uparrow 1} \eta_{\mu}(rz), \quad z \in T,
\]

exists almost everywhere as shown by Fatou [17]. In many cases of interest, the function \(\eta_{\mu}\) extends continuously to \(T\), and thus \(\mu\) is absolutely continuous on the set \(\{ z \in T : \eta_{\mu}(\overline{z}) \neq 1 \}\).

The \(\eta\)-transform is used in the description of free multiplicative convolution on the subset \(P_{\mu}^{*}\) of \(P_{\mu}\) consisting of those nondegenerate measures \(\mu\) with the property that \(\int_{T} t \, d\mu(t) \neq 0\). If \(\mu \in P_{\mu}^{*}\), we have \(\eta_{\mu}(0) \neq 0\), and thus \(\eta_{\mu}\) has an inverse \(\eta_{\mu}^{-1}\) that is a convergent power series in a neighborhood of zero. The free multiplicative convolution of two measures \(\mu_1, \mu_2 \in P_{\mu}^{*}\) is characterized by the identity \(2.3\) that is now true in some neighborhood of zero. The following theorem is a reformulation of results of [13].

**Theorem 5.1.** For every \(\mu_1, \mu_2 \in P_{\mu}^{*}\), there exist unique \(\rho_1, \rho_2 \in P_{\mu}^{*}\) such that

\[
\eta_{\mu_1}(\eta_{\rho_1}(z)) = \eta_{\mu_2}(\eta_{\rho_2}(z)) = \frac{\eta_{\mu_1}(z)\eta_{\rho_2}(z)}{z}, \quad z \in D.
\]

Moreover, we have \(\eta_{\mu_1} \circ \eta_{\mu_2} = \eta_{\mu_1} \circ \eta_{\rho_1}\).

The concept of \(\varnothing\)-infinite divisibility for measures in \(P_{\mu}\) is defined as for \(P_{\mu}^{*}\). The normalized arclength measure is the only \(\varnothing\)-ininitely divisible measure in \(P_{\mu}\). The other \(\varnothing\)-ininitely divisible measures are described by results of [21]. Suppose that \(\mu \in P_{\mu}^{*}\) is \(\varnothing\)-ininitely divisible. Then the function \(\eta_{\mu}^{-1}\) has an analytic continuation \(\Phi\) to \(D\) satisfying

\[
\Phi(0) = 0, \quad |\Phi(z)| \geq |z|, \quad z \in D.
\]

Conversely, every analytic function \(\Phi : D \to \mathbb{C}\) that satisfies (5.2) is the analytic continuation of \(\eta_{\mu}^{-1}\) for some \(\varnothing\)-ininitely divisible measure \(\mu \in P_{\mu}^{*}\). Of course, the identity

\[
\Phi(\eta_{\mu}(z)) = z
\]

extends by analytic continuation to arbitrary \(z \in D\), and thus \(\eta_{\mu}\) is a conformal map if \(\mu\) is \(\varnothing\)-ininitely divisible. Some further information about this case is summarized below (see [3]).

**Proposition 5.2.** Let \(\mu \in P_{\mu}^{*}\) be \(\varnothing\)-ininitely divisible, and let \(\Phi : D \to \mathbb{C}\) be the analytic continuation of \(\eta_{\mu}^{-1}\). Then:

1. The domain \(\Omega_{\mu} = \eta_{\mu}(D)\) is starlike relative to the origin.
2. The function \(\eta_{\mu}\) extends to a homeomorphism of \(\overline{D}\) onto \(\overline{\Omega_{\mu}}\).
3. We have \(\Omega_{\mu} = \{ z \in D : |\Phi(z)| < 1 \}\).
4. If \(|\eta_{\mu}(t)| < 1\) for some \(t \in T\) then \(\eta_{\mu}\) continues analytically to a neighborhood of \(t\).

The functions \(\Phi\) that satisfy (5.2) can be written as

\[
\Phi(z) = \gamma z \exp H_{\sigma}(z), \quad z \in D,
\]
where $\gamma \in \mathbb{T}$ and $\sigma$ is a finite, positive Borel measure on $\mathbb{T}$. The parameters $(\gamma, \sigma)$ are uniquely determined by $\Phi$ (or by $\mu$) and (5.3) is an analog of the Lévy-Hinčin formula in classical probability. (Recall that $H_\sigma$ denotes the Herglotz integral of $\sigma$.) This representation of $\Phi$, along with part (3) of the above lemma, allow us to give an alternative description of $\eta_{\mu}|\mathbb{T}$. We have

$$|\Phi(r\zeta)| = r \exp \Re H_\sigma(r\zeta) = r \exp \left[ \int_{\mathbb{T}} \frac{1 - r^2}{|t - r\zeta|^2} \, d\sigma(t) \right], \quad r \in (0, 1), \zeta \in \mathbb{T},$$

and thus

$$\log |\Phi(r\zeta)| = [1 - T(r\zeta)] \log r,$$

where

$$T(r\zeta) = \frac{r^2 - 1}{\log r} \int_{\mathbb{T}} \frac{d\sigma(t)}{|t - r\zeta|^2}, \quad r \in (0, 1), \zeta \in \mathbb{T}.$$

We also set

$$T(\zeta) = \lim_{r \to 1} T(r\zeta) = 2 \int_{\mathbb{T}} \frac{d\sigma(t)}{|t - \zeta|^2}.$$

We conclude that $r\zeta \in \Omega_\mu$ precisely when $T(r\zeta) < 1$. Since $\Omega_\mu$ is starlike relative to 0 (a fact that also follows because $T(r\zeta)$ is an increasing function of $r$ for fixed $\zeta$; see [26, Lemma 3.1]) we conclude that, for each fixed $\zeta \in \mathbb{T}$, the set

$$\{ r \in (0, 1) : T(r\zeta) < 1 \}$$

is an interval $(0, R(\zeta))$. We summarize some of the properties of the function $R$ below.

**Lemma 5.3.** [19, Proposition 4.1] Suppose that $\mu \in \mathcal{P}_\mathbb{T}$ is $\mathbb{Z}$-infinitely divisible. With the notation introduced above, we have:

1. The function $R$ is continuous.
2. $\Omega_\mu = \{ r\zeta : \zeta \in \mathbb{T}, 0 \leq r < R(\zeta) \}$ and $\partial \Omega_\mu = \{ R(\zeta) \zeta : \zeta \in \mathbb{T} \}$.
3. $R(\zeta) < 1$ if and only if $T(\zeta) > 1$, in which case $T(R(\zeta) \zeta) = 1$. The inequality $T(R(\zeta) \zeta) \leq 1$ holds for every $\zeta \in \mathbb{T}$.

The following result is analogous to Lemma 2.8. A similar estimate could be derived from [13, (4.20)].

**Lemma 5.4.** Suppose that $\mu \in \mathcal{P}_\mathbb{T}$ is $\mathbb{Z}$-infinitely divisible. With the notation introduced above, we have

$$|dH_\sigma/dz| \leq 8\sigma(\mathbb{T}) + 2, \quad z \in \Omega_\mu.$$

**Proof.** Direct calculation yields

$$|dH_\sigma/dz| = \left| \int_{\mathbb{T}} \frac{2t}{(t - z)^2} \, d\sigma(t) \right| \leq 2 \int_{\mathbb{T}} \frac{d\sigma(t)}{|t - z|^2}.$$

Since $T(z) \leq 1$ for $z \in \Omega_\mu$, we have

$$\int_{\mathbb{T}} \frac{d\sigma(t)}{|t - z|^2} \leq \frac{\log |z|}{|z|^2 - 1} < \frac{1}{2|z|} \leq 1,$$

if $|z| \geq 1/2$. If $|z| < 1/2$, we have $|t - z| \geq 1/2$ for $t \in \mathbb{T}$, and the estimate

$$\int_{\mathbb{T}} \frac{d\sigma(t)}{|t - z|^2} \leq 4\sigma(\mathbb{T})$$

yields the desired result. $\square$
The discussion of convolution powers in $\mathcal{P}_T^*$ is best carried out for real exponents rather than just integer ones. Suppose that $\nu \in \mathcal{P}_T^*$ satisfies $\int_{T} t \, d\nu(t) > 0$ and $\eta_\nu$ has no zeros in $\mathbb{D} \setminus \{0\}$. Fix $k \in (1, +\infty)$ and set

$$\Phi(z) = z \left( \frac{z}{\eta_\nu(z)} \right)^{k-1}, \quad z \in \mathbb{D}.$$ 

We have

$$\eta'_\nu(0) = \int_{T} t \, d\nu(t) > 0,$$

and the power above is chosen such that $\Phi'(0) > 0$. The Schwarz lemma shows that $|\Phi(z)| \geq |z|$ for $z \in \mathbb{D}$, and therefore there exists a $\boxtimes$-infinitely divisible measure $\mu \in \mathcal{P}_T^*$ such that $\Phi$ is an analytic continuation of $\eta^{(-1)}_\mu$. We can then define the convolution power $\nu^{\boxtimes k}$ by setting

$$\eta_\nu^{\boxtimes k} = \eta_\nu \circ \eta_\mu.$$

If $k$ is an integer, the measure $\nu^{\boxtimes k}$ is in fact equal to the free multiplicative convolution of $k$ copies of $\nu$. The analog of (2.11) also holds in this context, but it must be written so the powers make sense:

$$\left( \frac{\eta_\nu(z)}{z} \right)^k = \left( \frac{\eta_\nu(z)}{z} \right)^{(k-1)}^{1/(k-1)}, \quad z \in \mathbb{D};$$

equivalently,

$$\eta_\nu^{\boxtimes k}(z) = \eta_\mu(z) \left( \frac{\eta_\mu(z)}{z} \right)^{1/(k-1)}, \quad z \in \mathbb{D}.$$ 

As in the real case, the function $\eta_\nu^{\boxtimes k}$ extends continuously to the closure $\overline{\mathbb{D}}$ [3]. This construction of real powers fails if $\eta_\nu(z) = 0$ for some $z \in \mathbb{D} \setminus \{0\}$. Suppose however that $\int_{T} t \, d\nu(t) > 0$. The $\eta$-transform of the measure $\nu^{\boxtimes 2} = \nu \boxtimes \nu$ has no zeros other than 0, and therefore one can define

$$\nu^{\boxtimes k} = (\nu \boxtimes \nu)^{k/2}$$

provided that $k > 2$. These considerations can be carried out for arbitrary measures in $\mathcal{P}_T^*$ by choosing an arbitrary determination of the power $(z/\eta_\nu(z))^{k-1}$. If $k$ is not an integer, there may be infinitely many versions of $\nu^{\boxtimes k}$, but each of them can be obtained from the others by appropriate rotations.

6. Superconvergence in $\mathcal{P}_T$

The weak convergence of $\boxtimes$-infinitely divisible measures is equivalent to certain convergence properties of the $\eta$-transforms and of their inverses. We record the result from [8, Proposition 2.9]. The equivalence between (1) and (5) below is implicit in the proof of Theorem 4.3 from [12].

**Proposition 6.1.** [8, 12] Suppose that $\mu$ and $\{\mu_n\}_{n \in \mathbb{N}}$ are $\boxtimes$-infinitely divisible measures in $\mathcal{P}_T^*$. Denote by $\Phi$ and $\{\Phi_n\}_{n \in \mathbb{N}}$ the analytic continuations to $\mathbb{D}$ of the functions $\eta^{(-1)}_\mu$ and $\{\eta^{(-1)}_{\mu_n}\}_{n \in \mathbb{N}}$, and represent these functions as in (5.3), using $(\gamma_n, \sigma_n)$ for the parameters corresponding to $\mu_n$. The following conditions are equivalent:

1. The sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converges weakly to $\mu$.
2. The sequence $\{\eta_{\mu_n}\}_{n \in \mathbb{N}}$ converges pointwise to $\eta_\mu$ on $\mathbb{D}$.
The sequence \( \{\eta_{\mu_n}\} \) converges to \( \eta_{\mu} \) uniformly on the compact subsets of \( \Omega \).

The sequence \( \{\Phi_n\} \) converges to \( \Phi \) uniformly on the compact subsets of \( \Omega \).

The sequence \( \{\gamma_n\} \) converges to \( \gamma \) and the sequence \( \{\sigma_n\} \) converges weakly to \( \sigma \).

In preparation for the analog of Lemma 3.1, we suppose that \( \mu, \mu_n, \Phi, \Phi_n \) are as in the preceding result, and we consider the continuous functions \( R, R_n : T \rightarrow (0,1] \) such that

\[
\Omega_n = \{rt : t \in T, r \in [0,R(t))\}, \quad \Omega_{\mu_n} = \{rt : t \in T, r \in [0,R_n(t))\}.
\]

We also consider the homeomorphisms \( h, h_n : T \rightarrow T \) defined by

\[
h(t) = \frac{\eta_{\mu}(t)}{|\eta_{\mu}(t)|}, \quad h_n(t) = \frac{\eta_{\mu_n}(t)}{|\eta_{\mu_n}(t)|}, \quad t \in T, n \in \mathbb{N}.
\]

The existence of these (orientation preserving) homeomorphisms is a consequence of the fact that \( \Omega_{\mu_n} \) is starlike with respect to 0, and of the fact that \( \eta_{\mu_n} \) extends to a homeomorphism of \( \overline{\Omega_{\mu_n}} \) onto \( \overline{\Omega_{\mu_n}} \). Observe that we have

\[
\eta_{\mu}(t) = R(h(t))h(t), \quad \eta_{\mu_n}(t) = R_n(h_n(t))h_n(t) \quad t \in T, n \in \mathbb{N}.
\]

**Lemma 6.2.** With the above notation, suppose that the sequence \( \{\mu_n\} \) converges weakly to \( \mu \). Then:

1. The sequence \( \{\mu_n\} \) converges to \( \mu \) uniformly on \( T \).
2. The sequence \( \{h_n\} \) converges to \( h \) uniformly on \( T \).
3. The sequence of inverses \( \{h_n^{-1}\} \) converges to \( h^{-1} \) uniformly on \( T \).
4. The sequence \( \{R_n \circ h_n\} \) converges to \( R \circ h \) uniformly on \( T \).
5. The sequence \( \{\eta_{\mu_n}(t)\} \) converges to \( \eta_{\mu}(t) \) uniformly on \( T \).

**Proof.**

1. Since \( T \) is compact, it suffices to show that, for every \( t_0 \in T \) and for every \( \varepsilon > 0 \) there exist \( N \in \mathbb{N} \) and an arc \( V \subset T \) containing \( t_0 \) in its interior such that

\[
R(t) - \varepsilon < R_n(t) < R(t) + \varepsilon, \quad t \in V, n \geq N.
\]

Fix \( t_0 \) and \( \varepsilon \) and chose a compact neighborhood \( V \) of \( t_0 \) such that \( |R(t) - R(t_0)| < \varepsilon/2 \) for \( t \in V \). Thus,

\[
|\Phi((R(t_0) - \varepsilon/2)t)| < 1, \quad t \in V.
\]

The uniform convergence of \( \Phi_n \) to \( \Phi \) on the set \( \{(R(t_0) - \varepsilon/2)t : t \in V\} \) shows that there exists \( N_1 \) such that

\[
|\Phi_n((R(t_0) - \varepsilon/2)t)| < 1, \quad t \in V, n \geq N_1,
\]

and thus

\[
R_n(t) > R(t_0) - \frac{\varepsilon}{2} > R(t) - \varepsilon, \quad t \in V, n \geq N_1.
\]

If \( R(t_0) + \varepsilon/2 \geq 1 \), the inequality \( R_n(t) < R(t) + \varepsilon \) is automatically satisfied for \( t \in V \). If \( R(t_0) + \varepsilon/2 < 1 \), we observe that

\[
|\Phi((R(t_0) + \varepsilon/2)t)| > 1, \quad t \in V,
\]

and we choose \( N_2 \) such that

\[
|\Phi_n((R(t_0) + \varepsilon/2)t)| > 1, \quad t \in V, n \geq N_2.
\]
Thus,

$$R_n(t) < R(t_0) + \frac{\varepsilon}{2} < R(t) + \varepsilon, \quad t \in V, n \geq N_2,$$

so it suffices to choose $N = \max\{N_1, N_2\}$.

(3) It suffices to prove pointwise convergence. We observe that $h_n^{(-1)}(t) = \Phi_n(R_n(t)t)$. Since the measures $\sigma_n$ converge weakly, the sequence $\{\sigma_n(T)\}_{n \in \mathbb{N}}$ is bounded. Lemma 5.4 shows that the restrictions $\Phi_n|_{\Omega_\mu_n}$ are equicontinuous. These facts, along with (1), imply the desired pointwise convergence.

(2) This follows directly from (3). Then (4) and (5) follow as in the proof of Lemma 3.1.

As in the case of $\mathbb{R}_+$, the $\eta$-transform of an $\mathbb{F}$-infinitely divisible measure $\mu \in \mathcal{P}_\mathbb{F}^+$ may take the value 1 at most once on $T$. If $\eta_\mu(t) = 1$, we write $D_\mu = \{t\}$, otherwise $D_\mu = \emptyset$. The measure $\mu$ is absolutely continuous relative to arclength measure on $T \setminus D_\mu$.

We can now use the preceding result and (6.1) to prove the analog of Proposition 3.2 for the circle. The details are left to the interested reader.

**Proposition 6.3.** Let $\mu$ and $\{\mu_n\}_{n \in \mathbb{N}}$ be $\mathbb{F}$-infinitely divisible measures in $\mathcal{P}_\mathbb{F}^+$ such that $\mu_n$ converges weakly to $\mu$. Let $K \subset T \setminus D_\mu$ be an arbitrary compact set. Then $D_\mu_n \subset T \setminus D_\mu$ for sufficiently large $n$, and the densities $p_\mu_n$ of $\mu_n$ relative to arclength measure converge to $p_\mu$ uniformly on $K$. If $D_\mu = \emptyset$, we can take $K = T$.

Finally, we derive a superconvergence result.

**Theorem 6.4.** Let $\{k_n\}_{n \in \mathbb{N}} \subset [2, +\infty)$ be a sequence with limit $+\infty$, and let $\mu$ and $\{\nu_n\}_{n \in \mathbb{N}}$ be measures in $\mathcal{P}_\mathbb{F}^+$ such that $\mu$ is $\mathbb{F}$-infinitely divisible and $\int_T t \, d\nu_n(t) > 0$ for every $n \in \mathbb{N}$. Suppose that the sequence $\{\nu_n^{\otimes k_n}\}_{n \in \mathbb{N}}$ converges weakly to $\mu$. Let $K \subset T \setminus D_\mu$ be an arbitrary compact set. Then $\nu_n^{\otimes k_n}$ is absolutely continuous on $K$ for sufficiently large $n$, and the densities $p_n$ of $\nu_n^{\otimes k_n}$ relative to arclength measure converge to $p_\mu$ uniformly on $K$. If $D_\mu = \emptyset$, we can take $K = T$.

**Proof.** We first replace $\nu_n$ by $\nu_n \otimes \nu_n$ and $k_n$ by $k_n/2$. After this substitution we may assume that $\eta_\nu$ does not vanish on $T$ and the convolution powers can be calculated as in Section 5 using analytic subordination. Thus, there exist $\mathbb{F}$-infinitely divisible measures $\mu_n \in \mathcal{P}_\mathbb{F}$ satisfying the equations

$$\eta_{\mu_n^{\otimes k_n}}(z) = \eta_{\mu_n}(z) \left( \frac{\eta_{\mu_n}(z)}{z} \right)^{1/(k_n - 1)}, \quad z \in \mathbb{T}, \ n \in \mathbb{N},$$

and

$$\eta_{\nu_n^{\otimes k_n}} = \eta_{\nu_n} \circ \eta_{\mu_n}, \quad n \in \mathbb{N}.$$

As in the case of $\mathbb{R}_+$, the measures $\nu_n$ necessarily converge to $\delta_1$ as $n \to \infty$, and thus $\eta_{\nu_n}(z)$ converges to $z$ uniformly for $z$ in a compact subset of $T$. The inverses $\eta_{\nu_n}^{(-1)}$ converge uniformly to the identity function for $z$ in a neighborhood of 0, and therefore

$$\eta_{\mu_n} = \eta_{\nu_n}^{(-1)} \circ \eta_{\nu_n}^{\otimes k_n}$$

converge uniformly on a neighborhood of 0 to $\eta_\mu$. We conclude that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converges weakly to $\mu$. Lemma 6.2 implies now that the functions $\eta_{\mu_n}$
converge to \( \eta_{\mu} \) uniformly on \( \overline{D} \), and therefore the functions

\[
\left( \frac{\eta_{\mu_n}(z)}{z} \right)^{1/(k_n-1)}, \quad z \in \overline{D},
\]

converge uniformly to 1. Formula (6.1) implies now that the sequence \( \{\eta_{n,\pi_k}\}_{n \in \mathbb{N}} \) converges to \( \eta_\mu \) uniformly on \( \overline{D} \). The desired conclusion is now obtained easily by applying (5.1) to these measures.

\[ \square \]

7. Cusp behavior in \( \mathcal{P}_T \)

This section is the counterpart of Section 4 for \( \mathbb{T} \). Thus, we consider the qualitative behavior of a convolution \( \mu_1 \boxtimes \mu_2 \), where \( \mu_1, \mu_2 \in \mathcal{P}_T^\circ \) are nondegenerate measures and \( \mu_2 \) is \( \mathbb{R} \)-infinitely divisible. Of course, all \( \mathbb{R} \)-infinitely divisible measures in \( \mathcal{P}_T \) belong to \( \mathcal{P}_T^\circ \), with the exception of the normalized arclength measure \( m \). For this measure, we have \( \mu \boxtimes m = m, \mu \in \mathcal{P}_T \), so \( m \) is the analog of the measure \( \delta_0 \in \mathcal{P}_{\mathbb{R}^+} \), and indeed it has the same moment sequence.

We start with the analog of Lemma 4.1

**Lemma 7.1.** Let \( \mu_1, \mu_2 \in \mathcal{P}_T^\circ \) be such that \( \mu_2 \) is \( \mathbb{R} \)-infinitely divisible, and let \( \rho_1, \rho_2 \in \mathcal{P}_{\mathbb{R}^+} \) be given by Theorem 5.1. Then \( \rho_1 \) is \( \mathbb{R} \)-infinitely divisible.

**Proof.** Let \( \Phi \) given by (6.3) be the analytic continuation of \( \eta_{\mu_2}^{-1} \) to \( \mathbb{D} \). Thus,

\[
\Phi(z) = zF(z), \quad z \in \mathbb{D},
\]

where \( F \) satisfies \( |F(z)| \geq 1 \) for \( z \in \mathbb{D} \). Then the analog of (2.3) for \( \mathcal{P}_T^\circ \) can be written as

\[
F(z)\eta_{\mu_1}^{-1}(z) = \eta_{\mu_1 \boxtimes \mu_2}^{(-1)}(z),
\]

and applying this equality with \( \eta_{\mu_1}(z) \) in place of \( z \), we obtain

\[
F(\eta_{\mu_1}(z))z = \eta_{\mu_1 \boxtimes \mu_2}(\eta_{\mu_1}(z)) = \eta_{\rho_1}^{-1}(z)
\]

for \( z \) in some neighborhood of zero. The lemma follows because the function

\[
G(z) = F(\eta_{\mu_1}(z)), \quad z \in \mathbb{D},
\]

also satisfies the inequality \( |G(z)| \geq 1 \) for \( z \in \mathbb{D} \). \[ \square \]

With the notation of the preceding lemma, we recall that the domain

\[
\Omega_{\rho_1} = \eta_{\rho_1}(\mathbb{D})
\]

can be described as

\[
\Omega_{\rho_1} = \{ rt : t \in \mathbb{T}, 0 \leq r < R(t) \}
\]

for some continuous function \( R : \mathbb{T} \to (0,1) \), and that \( \eta_{\rho_1} \) extends to a homeomorphism of \( \overline{\mathbb{D}} \) onto \( \overline{\Omega_{\rho_1}} \). Using, the analytic continuation

\[
\Psi(z) = zG(z), \quad z \in \mathbb{D},
\]

of \( \eta_{\rho_1}^{-1} \), the map

\[
\Psi(R(t)t), \quad t \in \mathbb{T},
\]

is a homeomorphism between \( \mathbb{T} \) and \( \partial \Omega_{\rho_1} \). The density \( p_{\mu_1 \boxtimes \mu_2} \) of \( \mu_1 \boxtimes \mu_2 \), relative to arclength measure \( 2\pi dm \) on \( \mathbb{T} \), is calculated using the formula

\[
p_{\mu_1 \boxtimes \mu_2}(\xi) = \begin{cases} \frac{1}{\Psi(R(t)t)^{1-\eta_{\mu_1 \boxtimes \mu_2}(R(t)t)}}, & \text{if } \xi = \frac{1}{\Psi(R(t)t)} \text{ and } R(t) < 1, \\ 0, & \text{if } \xi = \frac{1}{\Psi(R(t)t)} \text{ and } R(t) = 1. \end{cases}
\]
As noted earlier, this density is real analytic at all points where it is nonzero. Using the Herglotz formula for analytic functions with a positive real part, we write the function $F$ above as

$$F(z) = \gamma \exp(H_\sigma(z)), \quad z \in \mathbb{D},$$

where $|\gamma| = 1$ and $\sigma$ is a finite, positive Borel measure on $\mathbb{T}$. The appropriate analog of the semicircular measure is obtained when $\sigma$ is a point mass at $1 \in \mathbb{T}$, that is,

$$F(z) = \gamma \exp \left[ \beta \frac{1 + z}{1 - z} \right], \quad z \in \mathbb{D},$$

for some $\gamma \in \mathbb{T}$ and $\beta > 0$. The following proposition examines the density of $\mu_1 \boxtimes \mu_2$ when $\mu_2$ is one of these measures. (The formula (7.3) below also appeared in [26].) We use the notation $p'$ for the derivative $dp(e^{i\theta})/d\theta$ if $p$ is a differentiable function defined on some open subset of $\mathbb{T}$.

**Proposition 7.2.** Suppose that $\mu_1, \mu_2 \in \mathcal{P}_0^\mathbb{T}$ are nondegenerate measures, and that $\mu_2$ is such that

$$\gamma z \exp \left[ \beta \frac{1 + z}{1 - z} \right], \quad z \in \mathbb{D},$$

is an analytic continuation of $\eta_\mu^{(-1)}$ for some $\gamma \in \mathbb{T}$ and $\beta \in (0, +\infty)$. Let $p_{\mu_1, \mu_2}$ denote the density of $\mu_1 \boxtimes \mu_2$ relative to the arclength measure $2\pi dm$. Then:

1. \( p'_{\mu_1, \mu_2}(\xi) \left| p_{\mu_1, \mu_2}(\xi) \right|^2 \leq \frac{7}{(8\pi^3\beta^3)} \) for every $\xi \in \mathbb{T}$ such that $0 < p_{\mu_1, \mu_2}(\xi) \leq \log 2/(2\pi\beta)$.
2. \( p'_{\mu_1, \mu_2}(\xi) \leq \frac{7}{\pi\beta} \) for every $\xi \in \mathbb{T}$ such that $p_{\mu_1, \mu_2}(\xi) \geq \log 2/(2\pi\beta)$.
3. If $I \subset \mathbb{T}$ is an arc with one endpoint $\xi_0$, $p_{\mu_1, \mu_2}(\xi) > 0$ for $\xi \in I$, and $p_{\mu_1, \mu_2}(\xi_0) = 0$, then

$$p_{\mu_1, \mu_2}(\xi) \leq \frac{2}{\pi\beta} |\xi - \xi_0|^{1/3}$$

for $\xi \in I$ close to $\xi_0$.

**Proof.** Part (3) follows from (1) by integration since $\sqrt{21/4} < 2$ and $\ell(\xi, \xi_0) < 2|\xi - \xi_0|$ if $\xi$ is close to $\xi_0$; here $\ell(\xi, \xi_0)$ denotes the length of the (short) arc joining $\xi$ and $\xi_0$.

As seen in the preceding lemma, $\eta_\mu^{(-1)}$ has the analytic continuation

$$\Psi(z) = \gamma z \exp \left[ \beta u(z) \right], \quad z \in \mathbb{D},$$

where

$$u(z) = \frac{1 + \eta_{\mu_1}(z)}{1 - \eta_{\mu_1}(z)} = 1 + 2\psi_{\mu_1}(z)$$

$$= \int_{\mathbb{T}} \left[ 1 + \frac{2\xi z}{1 - \xi z} \right] d\mu_1(\xi)$$

$$= \int_{\mathbb{T}} \left[ 1 + \xi z \right] d\mu_1(\xi)$$

$$= \int_{\mathbb{T}} \left[ \xi + z \right] d\mu_1(1/\xi)$$
is precisely the Herglotz integral of the measure $d\mu_1(1/\xi) = d\mu_1(\bar{\xi})$. Thus, when the boundary of the domain $\Omega_{\mu}$ is parametrized as $z(t) = R(t)t$, $t \in \mathbb{T}$, we have
\begin{equation}
\beta \int_{\mathbb{T}} \frac{d\mu_1(\bar{\xi})}{|\xi - z(t)|^2} = \frac{\log R(t)}{R(t)^2 - 1}
\end{equation}
whenever $R(t) < 1$. (We will use implicitly the easily established inequality
\[
\log r \over r^2 - 1 < \frac{1}{2r},
\]
valid for $r \in (0,1)$. In fact the function
\[
\frac{2r\log r}{r^2 - 1}
\]
is increasing for $r \in (0,1)$ and it tends to 1 at $r = 1$.) Setting
\[
f(t) = \frac{1}{\Psi(z(t))},
\]
for $R(t) < 1$, we see from (7.1) and (7.2) that
\begin{equation}
p_{\mu_1 \mu_2}(f(t)) = \frac{1}{2\pi} \Re \frac{1 + \eta \mu_1(z(t))}{1 - \eta \mu_1(z(t))}
\end{equation}
\[
= \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z(t)|^2}{|\xi - z(t)|^2} d\mu_1(\bar{\xi})
\]
\[
= \frac{1}{2\pi} \beta \int_{\mathbb{T}} \frac{1 - |z(t)|^2}{|\xi - z(t)|^2} d\mu_1(\bar{\xi}) = -\frac{\log R(t)}{2\pi \beta}.
\]
As in the case of $\mathbb{R}_+$, this allows us to use the chain rule for our estimates. We begin with the derivative of $f$ that can be estimated as
\[
|f'(t)| = \left| \frac{f'(t)}{f(t)} \right| = \left| \frac{\Psi'(z(t))}{\Psi(z(t))} \right| |z'(t)|.
\]
Here, $\Psi'$ is the usual complex derivative of $\Psi$,
\[
\left| \frac{\Psi'(z)}{\Psi(z)} \right| = \left| \frac{1}{z} + \beta u'(z) \right| = \left| \frac{1}{z} + \beta \int_{\mathbb{T}} \frac{2\xi}{(\xi - z)^2} d\mu_1(\bar{\xi}) \right|,
\]
so using (7.2) we obtain
\[
\left| \frac{\Psi'(z(t))}{\Psi(z(t))} \right| = \left| \frac{1}{R(t)} \right| + \beta \int_{\mathbb{T}} \frac{2\xi R(t)t}{(\xi - R(t)t)^2} d\mu_1(\bar{\xi})
\]
\[
\geq \frac{1}{R(t)} \left[ 1 - \frac{2R(t) \log R(t)}{R(t)^2 - 1} \right].
\]
For the second factor, we have
\[
z'(e^{i\theta}) = \frac{d}{d\theta} R(e^{i\theta}) e^{i\theta} = [R'(e^{i\theta}) + iR(e^{i\theta})] e^{i\theta},
\]
and thus
\[
|z'(t)| = \sqrt{R(t)^2 + R'(t)^2}.
\]
Putting these together, we see that
\[
|f'(t)| \geq \sqrt{1 + \left( \frac{R'(t)}{R(t)} \right)^2 \left[ 1 - \frac{2R(t) \log R(t)}{R(t)^2 - 1} \right]}
\]
\[
\geq \left| \frac{R'(t)}{R(t)} \right| \left[ 1 - \frac{2R(t) \log R(t)}{R(t)^2 - 1} \right].
\]
Since
\[
\left| \frac{R'(t)}{R(t)} \right| = |(\log R)'(t)|,
\]
formula (2.3) yields the estimate
\[
|p'_{\mu_1 \otimes \mu_2}(f(t))| = \left| \frac{|(\log R)'(t)|}{2\pi \beta} f'(t) \right|
\]
\[
\leq \frac{1}{2\pi \beta} \frac{1}{1 - \frac{2R(t) \log R(t)}{R(t)^2 - 1}}.
\]
The inequality \(p_{\mu_1 \otimes \mu_2}(f(t)) > (\log 2)/2\pi \beta\) amounts to \(R(t) < 1/2\), and the preceding estimate yields
\[
|p'_{\mu_1 \otimes \mu_2}(f(t))| \leq \frac{1}{2\pi \beta} \frac{1}{1 - \frac{2R(t) \log R(t)}{R(t)^2 - 1}} < \frac{7}{\pi \beta},
\]
thus verifying (2). Finally, we have
\[
|p'_{\mu_1 \otimes \mu_2}(f(t))| p_{\mu_1 \otimes \mu_2}(f(t))^2 \leq \frac{1}{(2\pi \beta)^3} \frac{\log^2 R(t)}{1 - \frac{2R(t) \log R(t)}{R(t)^2 - 1}}
\]
and the fact that
\[
\frac{\log^2 r}{1 - \frac{2r \log r}{r^2 - 1}}
\]
is less than 7 for \(r \in (1/2, 1)\) yields (1). \(\Box\)

Next, we state an analog of Lemma 4.3. The verification is a simple calculation.

**Lemma 7.3.** Let \(\mu_1, \mu_2 \in \mathcal{P}_+^\alpha\) be two nondegenerate measures such that \(\mu_2\) is \(\mathcal{D}\)-infinitely divisible, and let
\[
\Phi(z) = \gamma z \exp H_\sigma(z), \quad z \in \mathbb{D},
\]
be an analytic continuation of \(\eta_{\mu_2}^{(-1)}\). Assume that \(\int_{\mathbb{T}} t \, d\sigma(t) \neq 0\). Denote by \(\rho_1 \in \mathcal{P}_+^\alpha\) the \(\mathcal{D}\)-infinitely divisible measure such that \(\eta_{\rho_1}^{(-1)}\) has the analytic continuation
\[
\Psi(z) = \gamma z \exp \left[ \int_{\mathbb{T}} \frac{t + \mu_1(z)}{t - \mu_1(z)} \, d\sigma(t) \right], \quad z \in \mathbb{D}.
\]
Set \(\beta = \sigma(\mathbb{T})\), denote by \(\nu_1 \in \mathcal{P}_+^\alpha\) the measure satisfying
\[
\psi_{\nu_1}(z) = \frac{1}{\beta} \int_{\mathbb{T}} \frac{t \mu_1(z)}{1 - t \mu_1(z)} \, d\sigma(t), \quad z \in \mathbb{D},
\]
and denote by \(\nu_2 \in \mathcal{P}_+^\alpha\) the \(\mathcal{D}\)-infinitely divisible measure such that \(\eta_{\nu_2}^{(-1)}\) has the analytic continuation
\[
\gamma z \exp \left[ \frac{1 + z}{1 - z} \right], \quad z \in \mathbb{D}.
\]
Then \(\eta_{\mu_1 \otimes \mu_2} = \eta_{\mu_1} \circ \eta_{\rho_1}\) and \(\eta_{\nu_1 \otimes \nu_2} = \eta_{\nu_1} \circ \eta_{\rho_1}\).
In preparation for the final proof in this section, we recall some facts demonstrated in [19, Theorem 4.5]. Suppose that \( \mu_1, \mu_2, \) and \( \rho_1 \) are as in the preceding lemma, and that the domain \( \Omega_{\rho_1} \) is described as
\[
\Omega_{\rho_1} = \{ rt : t \in \mathbb{T}, 0 \leq r < R(t) \}
\]
for some continuous function \( R : \mathbb{T} \to (0, 1] \). The map \( \eta_{\mu_1} \) extends continuously to the closure \( \overline{\Omega_{\rho_1}} \), and the set
\[
\partial \Omega_{\rho_1} \cap \mathbb{T} = \{ t \in \mathbb{T} : R(t) = 1 \}
\]
can be partitioned into two subsets \( A \) and \( B \) described as follows.

1. \( A \) consists of those points \( t \in \mathbb{T} \) for which \( \mu_1((t)) > 0 \) and
\[
\frac{\mu_1((t))}{2} \geq \int_{\mathbb{T}} \frac{d\sigma(\xi)}{|1 - \xi|^2}.
\]

2. \( B \) consists of those \( t \in \mathbb{T} \) for which \( \eta_{\mu_1}(t) \in \mathbb{T}\setminus\{1\} \),
\[
c = \liminf_{z \to t} \frac{1 - |\eta_{\mu_1}(z)|}{1 - |z|} < +\infty,
\]
and
\[
c \int_{\mathbb{T}} \frac{d\sigma(\xi)}{|\eta_{\mu_1}(t) - \xi|^2} \leq 1.
\]

**Theorem 7.4.** Let \( \mu_1, \mu_2 \in \mathcal{P}_c^\infty \) be two nondegenerate measures such that \( \mu_2 \) is \( \boxtimes \)-infinitely divisible and satisfies the hypothesis of Lemma 7.3. Suppose that \( \Gamma \subset \mathbb{T} \) is an open arc with an endpoint \( \xi_0 \), \( \sigma_{\mu_1}(\xi_0) = 0 < \sigma_{\mu_1}(\xi) \) for every \( \xi \in \Gamma \), and—using the notation of Lemma 7.3—\( 1/\eta_{\mu_1}(1/\xi_0) \) is not an atom of \( \mu_1 \). Then \( \sigma_{\mu_1, \mu_2}(\xi)/|\xi - \xi_0|^{1/3} \) is bounded for \( \xi \in \Gamma \) close to \( \xi_0 \).

**Proof.** Using the notation of Lemma 7.3 we observe that
\[
\{ \xi \in \mathbb{T} : \sigma_{\mu_1, \mu_2}(\xi) > 0 \} = \{ 1/\Psi(R(t)t) : R(t) < 1 \}
\]
\[
= \{ \xi \in \mathbb{T} : \sigma_{\nu_1, \nu_2}(\xi) > 0 \}.
\]
By Proposition 7.2, the conclusion of the Theorem is true if \( \mu_1 \) and \( \mu_2 \) are replaced by \( \nu_1 \) and \( \nu_2 \), respectively. It will therefore suffice to prove that the ratio \( \sigma_{\nu_1, \nu_2}(\xi)/\sigma_{\nu_1, \nu_2}(\xi) \) is bounded away from zero for \( \xi \) close to \( \xi_0 \). The hypothesis implies that the number \( \alpha = 1/\eta_{\nu_1}(1/\xi_0) \) belongs to the set \( B \) described before the statement of the theorem. Using the usual parametrization \( z = \eta_{\nu_1}(1/\xi) \), the relations \( \eta_{\nu_1, \nu_2} = \eta_{\nu_1} \circ \eta_{\nu_2} \) and \( \eta_{\nu_1, \nu_2} = \eta_{\nu_1} \circ \eta_{\nu_2} \) yield
\[
\gamma z \exp \left[ \int_{\mathbb{T}} \frac{t + \eta_{\nu_1}(z)}{t - \eta_{\nu_1}(z)} d\sigma(t) \right] = \Psi(z) = \gamma z \exp \left[ \frac{1 + \eta_{\nu_1}(z)}{1 - \eta_{\nu_1}(z)} \right].
\]
Equating the absolute values of these quantities yields
\[
\int_{\mathbb{T}} \frac{1 - |\eta_{\nu_1}(z)|^2}{|t - \eta_{\nu_1}(z)|^2} d\sigma(t) = \beta \int_{\mathbb{T}} \frac{1 - |\eta_{\nu_1}(z)|^2}{|1 - \eta_{\nu_1}(z)|^2},
\]
or, equivalently
\[
\left[ \Re \frac{1 + \eta_{\nu_1}(z)}{1 - \eta_{\nu_1}(z)} \right] \int_{\mathbb{T}} \frac{1 - |\eta_{\nu_1}(z)|^2}{|t - \eta_{\nu_1}(z)|^2} d\sigma(t) = \beta \Re \frac{1 + \eta_{\nu_1}(z)}{1 - \eta_{\nu_1}(z)}.
\]
Applying (7.1) we rewrite this as

\[ \frac{p_{\nu_1 \boxplus \nu_2} (\xi)}{p_{\mu_1 \boxplus \mu_2} (\xi)} = \frac{1}{\beta} \int_T \frac{|1 - \eta_{\mu_1} (z)|^2}{|t - \eta_{\mu_1} (z)|^2} \, d\sigma(t), \quad \xi \in \Gamma. \]

The desired result follows now from the definition of the set \( B \) and an application of Fatou’s lemma.

**Remark 7.5.** With the notation of the preceding proof, we have \(|\Psi(z)| = 1\) for the relevant points \( z \), implying further that

\[ \int_T \frac{|1 - \eta_{\mu_1} (z)|^2}{|t - \eta_{\mu_1} (z)|^2} \, d\sigma(t) = \frac{|1 - \eta_{\mu_1} (z)|^2 \log |z|}{|\eta_{\mu_1} (z)|^2 - 1}. \]

It follows that

\[ \frac{p_{\mu_1 \boxplus \mu_2} (\xi)}{p_{\nu_1 \boxplus \nu_2} (\xi)} = \beta \frac{|\eta_{\mu_1} (z)|^2 - 1}{|1 - \eta_{\mu_1} (z)|^2 \log |z|}, \]

and it is easily seen that this ratio is also bounded away from zero near \( \xi_0 \).

8. Free additive convolution on \( \mathcal{P}_R \)

The free additive convolution \( \boxplus \) is a binary operation defined on \( \mathcal{P}_R \), the family of all probability measures on \( \mathbb{R} \). The Cauchy transform of a measure \( \mu \in \mathcal{P}_R \), already seen in Section 2, is defined by

\[ G_{\mu} (z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}, \quad z \in \mathbb{H}, \]

and the density \( d\mu/dt \) of \( \mu \) is equal almost everywhere to \((-1/\pi) \Im G_{\mu} (x)\), where the boundary limit

\[ G_{\mu} (x) = \lim_{y \downarrow 0} G_{\mu} (x + iy), \quad x \in \mathbb{R}, \]

exists almost everywhere on \( \mathbb{R} \). The reciprocal Cauchy transform

\[ F_{\mu} (z) = \frac{1}{G_{\mu} (z)}, \quad z \in \mathbb{H}, \]

maps \( \mathbb{H} \) to itself, and the collection \( \{ F_{\mu} : \mu \in \mathcal{P}_R \} \) consists precisely of those analytic functions \( F : \mathbb{H} \rightarrow \mathbb{H} \) with the property that

\[ \lim_{y \uparrow \infty} \frac{F(iy)}{iy} = 1. \]

As seen, for instance, in [2], these functions have a Nevanlinna representation of the form

\[ F(z) = \gamma + z - N_{\sigma} (z), \quad z \in \mathbb{H}, \]

where \( \gamma \in \mathbb{R} \) and

\[ N_{\sigma} (z) = \int_{\mathbb{R}} \frac{1 + tz}{z - t} \, d\sigma(t) \]

for some finite positive Borel measure \( \sigma \) on \( \mathbb{R} \). This integral representation implies that

\[ \Im F(z) \geq \Im z, \quad z \in \mathbb{H}. \]

Given a measure \( \mu \in \mathcal{P}_R \), the function \( F_{\mu} \) is conformal in an open set \( U \) containing \( \{ iy : y \in (\alpha, +\infty) \} \) for some \( \alpha > 0 \), and the restriction \( F_{\mu} | U \) has an inverse \( F_{\mu}^{(-1)} \) defined in an open set containing another set of the form \( \{ iy : y \in (\beta, +\infty) \} \) with
\( \beta > 0 \). The free additive convolution \( \mu_1 \boxplus \mu_2 \) of two measures \( \mu_1, \mu_2 \in \mathcal{P}_\mathbb{R} \) is the unique measure \( \mu \in \mathcal{P}_\mathbb{R} \) that satisfies the identity

\[
(8.1) \quad z + F_\mu^{(-1)}(z) = F_{\mu_1}^{(-1)}(z) + F_{\mu_2}^{(-1)}(z)
\]

for \( z \) in some open set containing \( iy \) for \( y \) large enough (see [3]). The analog of Theorems 2.5 and 5.1 is as follows.

**Theorem 8.1.** For every \( \mu_1, \mu_2 \in \mathcal{P}_\mathbb{R} \), there exist unique \( \rho_1, \rho_2 \in \mathcal{P}_\mathbb{R} \) such that

\[
F_{\mu_1}(F_{\rho_1}(z)) = F_{\mu_2}(F_{\rho_2}(z)) = F_{\rho_1}(z) + F_{\rho_2}(z) - z, \quad z \in \mathbb{H}.
\]

Moreover, we have \( F_{\mu_1 \boxplus \mu_2} = F_{\mu_1} \circ F_{\rho_1} \). If \( \mu_1 \) and \( \mu_2 \) are nondegenerate, then so are \( \rho_1 \) and \( \rho_2 \).

It was shown in [23, 9] that a measure \( \mu \in \mathcal{P}_\mathbb{R} \) is \( \boxplus \)-infinitely divisible precisely when the inverse \( F_\mu^{(-1)} \) continues analytically to \( \mathbb{H} \) and this analytic continuation has the Nevanlinna form

\[
(8.2) \quad \Phi(z) = \gamma + z + N_\sigma, \quad z \in \mathbb{H},
\]

for some \( \gamma \) and \( \sigma \). The functions described by (8.2) can also be characterized by

\[
\lim_{y \uparrow +\infty} \frac{\Phi(iy)}{iy} = 1 \quad \text{and} \quad \Im \Phi(z) \leq \Im z, \quad z \in \mathbb{H}.
\]

Suppose now that \( \mu \in \mathcal{P}_\mathbb{R} \) is \( \boxplus \)-infinitely divisible and that \( F_\mu^{(-1)} \) has the analytic continuation given in (8.2). The equation \( \Phi(F_\mu(z)) = z \) holds in some open set and therefore it holds on the entire \( \mathbb{H} \) by analytic continuation. In particular, \( F_\mu \) maps \( \mathbb{H} \) conformally onto a domain \( \Omega_\mu \subset \mathbb{H} \) that can be described as

\[
\Omega_\mu = \{ z \in \mathbb{H} : \Phi(z) \in \mathbb{H} \}.
\]

As in the multiplicative cases, this domain can also be identified with \( \{ x + iy : y > f(x) \} \) for some continuous function \( f : \mathbb{R} \rightarrow [0, +\infty) \). The map \( F_\mu \) extends continuously to the closure \( \overline{\Omega_\mu} \), \( \Phi \) extends continuously to \( \overline{\Omega_\mu} \), and these two extensions are homeomorphisms, inverse to each other. We refer to [19] for the details.

9. Cusp behavior in \( \mathcal{P}_\mathbb{R} \)

We are now ready for the counterpart of Sections 4 and 7 in the context of the free additive convolution. Thus, we study the density of a measure of the form \( \mu_1 \boxplus \mu_2 \), where \( \mu_1, \mu_2 \in \mathcal{P}_\mathbb{R} \) and \( \mu_2 \) is \( \boxplus \)-infinitely divisible. The following result is essentially contained in [14] and the brief argument is included here to establish notation.

**Lemma 9.1.** Let \( \mu_1, \mu_2 \in \mathcal{P}_\mathbb{R} \) be such that \( \mu_2 \) is \( \boxplus \)-infinitely divisible, and let \( \rho_1, \rho_2 \in \mathcal{P}_\mathbb{R} \) be given by Theorem 8.1. Then \( \rho_1 \) is \( \boxplus \)-infinitely divisible.

**Proof.** Let \( \Phi(z) = \gamma + z + N_\sigma(z) \) given by (8.2) be the analytic continuation of \( F_{\mu_2}^{(-1)} \) to \( \mathbb{H} \). Then (8.1) can be rewritten as

\[
F_{\mu_1}^{(-1)}(z) + \gamma + N_\sigma(z) = F_{\mu_1 \boxplus \mu_2}^{(-1)}(z)
\]

in a neighborhood of infinity. Replacing \( z \) by \( F_{\mu_1}(z) \) yields

\[
\gamma + z + N_\sigma(F_{\mu_1}(z)) = F_{\mu_1 \boxplus \mu_2}^{(-1)}(F_{\mu_1}(z)),
\]
and therefore the function
\[ \Psi(z) = \gamma + z + N_\sigma(F_{\mu_1}(z)), \quad z \in \mathbb{H}, \]
is an analytic continuation of \( F_{\mu_1}^{(-1)} \), thus establishing the conclusion of the lemma.

The density \( p_{\mu_1 \boxplus \mu_2} \) of \( \mu_1 \boxplus \mu_2 \) relative to Lebesgue measure has already been studied in [14] for the special case in which \( \mu_2 \) is a semicircular law, that is, the measure \( \sigma \) is a point mass at 0. The following result is [14, Corollary 5].

**Proposition 9.2.** With the notation above, suppose that \( \sigma = \beta \delta_0 \) for some \( \beta > 0 \).

If \( I \subset \mathbb{R} \) is an open interval with an endpoint \( x_0 \) such that \( p_{\mu_1 \boxplus \mu_2}(x_0) = 0 < p_{\mu_1 \boxplus \mu_2}(x) \) for every \( x \in I \), then
\[ p_{\mu_1 \boxplus \mu_2}(x) \leq \left[ \frac{3}{4\pi^2 \beta^2} |x - x_0| \right]^{1/3}, \quad x \in I. \]

In order to extend this result to general \( \boxplus \)-infinitely divisible measures \( \mu_2 \), we proceed as in the multiplicative cases. Thus, we construct another convolution, this time with a semicircular measure, with the property that the two convolutions share the same subordination function.

**Lemma 9.3.** Let \( \mu_1, \mu_2 \in \mathcal{P}_\mathbb{R} \) be such that \( \mu_2 \) is \( \boxplus \)-infinitely divisible, and let
\[ \Phi(z) = \gamma + z + N_\sigma(z), \quad z \in \mathbb{H}, \]
be the analytic continuation of \( F_{\mu_2}^{(-1)} \). Denote by \( \rho_1 \in \mathcal{P}_\mathbb{R} \) the \( \boxplus \)-infinitely divisible measure such that \( F_{\mu_2}^{(-1)} \) has the analytic continuation
\[ \Psi(z) = \gamma + z + N_\sigma(F_{\mu_1}(z)), \quad z \in \mathbb{H}. \]

Suppose that \( \beta = \int_{\mathbb{R}} (1 + t^2) d\sigma(t) \) is finite, and set \( \gamma' = \gamma + \int_{\mathbb{R}} t d\sigma(t) \). Denote by \( \nu_1 \in \mathcal{P}_\mathbb{R} \) the probability measure satisfying
\[ G_{\nu_1}(z) = \frac{1}{\beta} \int_{\mathbb{R}} \frac{1 + t^2}{F_{\rho_1}(z) - t} d\sigma(t), \quad z \in \mathbb{H}, \]
and let \( \nu_2 \in \mathcal{P}_\mathbb{R} \) be the semicircular measure such that
\[ \gamma' + z + \frac{\beta}{z}, \quad z \in \mathbb{H}, \]
is an analytic continuation of \( F_{\nu_2}^{(-1)} \). Then \( F_{\mu_1 \boxplus \mu_2} = F_{\nu_1 \circ \rho_1} \) and \( F_{\nu_1 \boxplus \nu_2} = F_{\nu_1} \circ F_{\rho_1} \).

**Proof.** The final assertion of the lemma is an easy verification that \( F_{\mu_1 \boxplus \mu_2} \circ F_{\mu_1} = F_{\nu_1 \boxplus \nu_2} \circ F_{\nu_1} \). One has to verify however that the measure \( \nu_1 \) actually exists, and that amounts to showing that the reciprocal
\[ F(z) = \frac{\beta}{\int_{\mathbb{R}} \frac{1 + t^2}{F_{\nu_1}(z) - t} d\sigma(t)} \]
maps \( \mathbb{H} \) to itself and that
\[ \lim_{y \uparrow +\infty} \frac{F(\text{i}y)}{\text{i}y} = 1. \]
These facts follow from the corresponding properties of the function \( F_{\mu_1} \). \( \square \)
We will use a decomposition, analogous to those for multiplicative convolutions on \( \mathbb{R}^+ \) and \( \mathbb{T} \). With the notation of the preceding lemma, represent the domain \( \Omega_{\rho_1} = F_{\rho_1}(\mathbb{H}) \) as
\[
\Omega_{\rho_1} = \{x + iy : x \in \mathbb{R}, y > f(x)\},
\]
where \( f : \mathbb{R} \to [0, +\infty) \) is a continuous function. We recall from [19] that \( G_{\mu_1} \) and \( G_{\nu_1} \) extend continuously to the closure \( \overline{\Omega_{\rho_1}} \) provided that \( \infty \) is allowed as a possible value. It was shown in [19, Theorem 3.6] that the set \( \partial \Omega_{\rho_1} \cap \mathbb{R} = \{\alpha \in \mathbb{R} : f(\alpha) = 0\} \) can be partitioned into three sets \( A, B, \) and \( C \) described as follows.

1. \( A \) consists of those points satisfying \( \mu_1(\{\alpha\}) > 0 \) and
\[
\int_{\mathbb{R}} \left\{ 1 + \frac{1}{t^2} \right\} d\sigma(t) \leq \mu_1(\{\alpha\}).
\]
2. \( B \) is characterized by the conditions \( G_{\mu_1}(\alpha) \in \mathbb{R} \setminus \{0\} \) and
\[
\left[ \int_{\mathbb{R}} \frac{1 + t^2}{(1 - tG_{\mu_1}(\alpha))^2} d\sigma(t) \right] \left[ \int_{\mathbb{R}} \frac{d\mu_1(t)}{(\alpha - t)^2} \right] \leq 1.
\]
3. \( C \) consists of those \( \alpha \) satisfying \( G_{\mu_1}(\alpha) = 0 \) and
\[
\text{var}(\mu_2) \int_{\mathbb{R}} \frac{d\mu_1(t)}{(\alpha - t)^2} \leq 1,
\]
where
\[
\text{var}(\mu_2) = \int_{\mathbb{R}} t^2 d\mu_2(t) - \left[ \int_{\mathbb{R}} t d\mu_2(t) \right]^2
\]
denotes the variance of \( \mu_2 \).

In each of the preceding inequalities, the improper integrals converge. Equality in each case is achieved precisely when \( F_{\rho_1} \) has an infinite Julia-Carathéodory derivative at the point \( \Psi(\alpha) \).

The set \( A \) is always finite unless \( \mu_2 \) is a degenerate measure. Moreover, if \( t \) is an atom of \( \mu_1 \boxtimes \mu_2 \), then \( F_{\rho_1}(t) \in A \).

We note for further use an alternative way to write the inequalities defining the sets \( B \) and \( C \) [19, Remark 3.7]. For this purpose, we use the Nevanlinna representation
\[
F_{\mu_1}(z) = c + z - N_{\lambda}(z), \quad z \in \mathbb{H},
\]
where \( c \in \mathbb{R} \) and \( \lambda \) is a finite Borel measure on \( \mathbb{R} \). The inequality in the definition of \( B \) can be replaced by
\[
\left[ \int_{\mathbb{R}} \frac{1 + t^2}{(F_{\mu_1}(\alpha) - t)^2} d\sigma(t) \right] \left[ 1 + \int_{\mathbb{R}} \frac{1 + t^2}{(\alpha - t)^2} d\lambda(t) \right] \leq 1,
\]
and the inequality in the definition of \( C \) can be replaced by
\[
(1 + \alpha^2)\lambda(\{\alpha\}) \geq \text{var}(\mu_2).
\]
In particular, every point \( \alpha \in B \) must satisfy
\[
(9.1) \int_{\mathbb{R}} \frac{1 + t^2}{(F_{\mu_1}(\alpha) - t)^2} d\sigma(t) < 1.
\]
It is also the case that \( C \) is a discrete subset of \( \mathbb{C} \).
Theorem 9.4. Suppose that $\mu_1, \mu_2 \in \mathcal{P}_R$ are nondegenerate measures such that $\mu_2$ is $\boxplus$-infinitely divisible, and let $\rho_1 \in \mathcal{P}_R$ satisfy $F_{\mu_1 \boxplus \mu_2} = F_{\mu_1} \circ F_{\mu_1}$. Let $I \subseteq R$ be an open interval with an endpoint $x_0$ such that $F_{\rho_1}(x) \in H$ for every $x \in I$ and $F_{\rho_1}(x_0)$ is real but not an atom of $\mu_1$. Denote by $p_{\mu_1 \boxplus \mu_2}$ the density of $\mu_1 \boxplus \mu_2$ relative to Lebesgue measure. Then $p_{\mu_1 \boxplus \mu_2}(x)/|x - x_0|^{1/2}$ is bounded for $x \in I$ close to $x_0$.

Proof. Suppose that
\[ c + z + N_\sigma(z), \quad z \in H, \]
is the analytic continuation of $F_{\mu_2}^{(-1)}$ to $H$, where $c \in R$ and $\sigma$ is a nonzero (because $\mu_2$ is nondegenerate) finite measure on $R$. As in the case of $R^+$, we can always find finite measures $\sigma', \sigma''$ on $R$ such that $\sigma'' \neq 0$ has compact support and $\sigma = \sigma' + \sigma''$. Define two $\boxplus$-infinitely divisible measures $\mu_2', \mu_2'' \in \mathcal{P}_R$ by specifying that $F_{\mu_2}^{(-1)}$ and $F_{\mu_2'}^{(-1)}$ have analytic continuations
\[ c + z + N_{\sigma'}(z) \quad \text{and} \quad z + N_{\sigma''}(z), \quad z \in H, \]
respectively. Since $\mu_2' \boxplus \mu_2'' = \mu_2$, we get $\mu_1 \boxplus \mu_2 = \mu_1' \boxplus \mu_2'$ and $\mu_1'' = \mu_1 \boxplus \mu_2''$. There exist additional $\boxplus$-infinitely divisible measures $\rho_1', \rho_1'' \in \mathcal{P}_R$ such that $F_{\rho_1'} = F_{\mu_1} \circ F_{\rho_1}, F_{\mu_1 \boxplus \mu_2} = F_{\rho_1'} \circ F_{\rho_1''}$. Clearly, $F_{\rho_1} = F_{\rho_1'} \circ F_{\rho_1''}$, and we argue that $F_{\rho_1''}(x_0)$ is a real number but not an atom of $\mu_1''$. Indeed, the inequality
\[ \Im F_{\rho_1}(z) = \Im(F_{\rho_1'}(F_{\rho_1''}(z))) \geq \Im(F_{\rho_1''}(z)), \quad z \in H, \]
and the hypothesis that $F_{\rho_1}(x_0) \in R$, shows as $z \to x_0$ that $F_{\rho_1''}(x_0) \in R$. Suppose, to get a contradiction, that $F_{\rho_1''}(x_0)$ is an atom of $\mu_1''$. Then, as seen in [10], $F_{\rho_1'}(F_{\rho_1''}(x_0))$ is necessarily an atom of $\mu_1$, contrary to the hypothesis.

The above construction shows that the hypothesis of the theorem also holds with $\mu_1', \mu_2'$, and $\rho_1'$ in place of $\mu_1, \mu_2$, and $\rho_1$, respectively. Moreover the measure $\sigma''$ has a finite second moment. Therefore it suffices to prove the theorem under the additional hypothesis that $\sigma$ has a finite second moment. Under this hypothesis, Lemma 9.3 applies and provides measures $\nu_1$ and $\nu_2$. Since the set \{ $x \in R : p_{\mu_1 \boxplus \mu_2}(x) > 0$ \} is described in terms of the measure $\rho_1$, namely,
\[ \{ x \in R : p_{\mu_1 \boxplus \mu_2}(x) > 0 \} = \{ x : F_{\rho_1}(x) \in H \}, \]
we have
\[ \{ x \in R : p_{\mu_1 \boxplus \mu_2}(x) > 0 \} = \{ x \in R : p_{\rho_1 \boxplus \rho_2}(x) > 0 \}. \]
By Proposition 8.2, it suffices to show that the ratio $p_{\mu_1 \boxplus \mu_2}(x)/p_{\mu_1 \boxplus \mu_2}(x)$ is bounded away from zero for $x \in I$ close to $x_0$. The two densities are evaluated in terms of the values of $G_{\rho_1}$ and $G_{\mu_1}$ on $\partial \Omega_{\mu_1}$:
\[ p_{\mu_1 \boxplus \mu_2}(x) = -\frac{1}{\pi} \Im G_{\rho_1}(F_{\rho_1}(x)) \]
\[ = -\frac{1}{\pi \beta} \int_R \Im \left[ \frac{G_{\mu_1}(F_{\rho_1}(x))}{1 - tG_{\mu_1}(F_{\rho_1}(x))} \right] (1 + t^2) d\sigma(t) \]
\[ = -\frac{\Im G_{\mu_1}(F_{\rho_1}(x))}{\pi \beta} \int_R \frac{1 + t^2}{|1 - tG_{\mu_1}(F_{\rho_1}(x))|^2} d\sigma(t) \]
\[ = \frac{p_{\mu_1 \boxplus \mu_2}(x)}{\beta} \int_R \frac{1 + t^2}{|1 - tG_{\mu_1}(F_{\rho_1}(x))|^2} d\sigma(t). \]
The hypotheses that $p_{\mu_1\oplus\mu_2}(x_0) = 0$ and $F_{\mu_1}(x_0)$ is not an atom of $\mu_1$ imply $F_{\mu_1}(x_0) \in B \cup C$. Using the Fatou’s lemma, we conclude that

$$
\liminf_{x \to x_0, x \in I} \frac{p_{\mu_1\oplus\mu_2}(x)}{p_{\mu_1\oplus\mu_2}(x)} \geq \frac{1}{\beta} \int_{\mathbb{R}} \frac{1 + t^2}{1 - tG_{\mu_1}(F_{\mu_1}(x))^2} \, d\sigma(t)
$$

thus concluding the proof.

Remark 9.5. With the notation of the preceding proof, it is also true that

$$
\liminf_{x \to x_0, x \in I} \frac{p_{\mu_1\oplus\mu_2}(x)}{p_{\mu_1\oplus\mu_2}(x)} \geq \beta \int_{\mathbb{R}} \frac{d\mu_1(t)}{(F_{\mu_1}(x_0) - t)^2},
$$

in which the improper integral converges because $F_{\mu_1}(x_0) \in B \cup C$. To verify this, we use the parametrization $\partial_{\mu_1} = \{s + i f(s) : s \in \mathbb{R}\}$ to write

$$\{F_{\mu_1}(x) : x \in I\} = \{s + i f(s) : s \in J\},$$

where $J$ is an interval where $f$ is positive and it has one endpoint $\alpha = F_{\mu_1}(x_0) \in \mathbb{R}$ such that $f(\alpha) = 0$. The fact that $F_{\mu_1}^{(-1)}$ has zero imaginary part on $J$ yields the equation

$$f(s) + \int_{\mathbb{R}} \frac{\Im G_{\mu_1}(s + i f(s))}{|1 - tG_{\mu_1}(s + i f(s))|^2} (1 + t^2) \, d\sigma(t) = 0, \quad s \in J.$$

Using this in the above formula for densities, we obtain

$$\frac{p_{\mu_1\oplus\mu_2}(x)}{p_{\mu_1\oplus\mu_2}(x)} = \beta \frac{\Im G_{\mu_1}(s + i f(s))}{f(s)} = \beta \int_{\mathbb{R}} \frac{d\mu_1(t)}{(t-s)^2 + f(s)^2}.$$

We can now apply the Fatou’s lemma as $s \to \alpha$.

Remark 9.6. The two limits inferior above are actual limits precisely when $F'_{\mu_1}(x_0) = +\infty$. Indeed, as seen above, this condition is equivalent to

$$\left[ \frac{1 + t^2}{1 - tG_{\mu_1}(F_{\mu_1}(x_0))^2} \, d\sigma(t) \right] \left[ \int_{\mathbb{R}} \frac{d\mu_1(t)}{(F_{\mu_1}(x_0) - t)^2} \right] = 1.$$ 

Remark 9.7. When $x_0$ is assumed to be a zero of the density $p_{\mu_1\oplus\mu_2}$, it is easy to see that $F_{\mu_1}(x_0)$ is an atom of $\mu_1$ if and only if $F_{\mu_1}(x_0) \in A$. In many cases, the collection $\{x_0 : F_{\mu_1}(x_0) \in A\}$ is not empty. This happens, of course, when $\mu_1$ has no atoms. This also occurs when

$$\int_{\mathbb{R}} \left\{ 1 + \frac{1}{t^2} \right\} \, d\sigma(t) \in [1, +\infty].$$

Indeed, in this case the set $A$ is empty (provided, of course, that $\mu_1$ is not degenerate and so its atoms cannot have measure 1).

Example 9.8. Let $\mu_1$ be an arbitrary nondegenerate measure in $\mathcal{P}_\mathbb{R}$, and let $\mu_2$ be the standard $(0,1)$ normal distribution. It was shown in [4] that $\mu_2$ is $\square$-infinitely divisible. We denote by $\sigma$ the associated measure that provides the analytic continuation of $F_{\mu_2}^{(-1)}$. Since

$$-\Im G_{\mu_2}(x) = \pi p_{\mu_2}(x) = \sqrt{\frac{\pi}{2}} e^{-x^2/2}, \quad x \in \mathbb{R},$$

the continuous extension of $F_{\mu}$ to $\mathbb{R}$ has no zeros and (see \cite{13} Proposition 5.1))
\[
\int_{\mathbb{R}} \frac{1 + t^2}{(x-t)^2} \, d\sigma(t) > 1, \quad x \in \mathbb{R}.
\]
This inequality, along with \cite{11}, implies that $A = B = \emptyset$ for the convolution $\mu_1 \boxplus \mu_2$. Moreover, since $C$ is a discrete set, the measure $\mu_1 \boxplus \mu_2$ is absolutely continuous with support equal to $\mathbb{R}$. If $C$ is not empty and $\alpha \in C$, there is an open interval $I$ centered at $x_0 = F_{\mu_2}^{-1}(\alpha)$ such that $p_{\mu_1 \boxplus \mu_2}(x)/|x-x_0|^{1/3}$ is bounded for $x \in I \setminus \{x_0\}$. Suppose, for instance, that $\mu_1 = \frac{1}{2}(\delta_1 + \delta_{-1})$ or the absolutely continuous measure with density
\[
\frac{3}{8} \left[ t^2 \chi_{[-1,1]}(t) + \frac{1}{t^2} \chi_{\mathbb{R} \setminus [-1,1]}(t) \right].
\]
In both cases, $\alpha = 0$ is the unique solution of the equation $G_{\mu_1}(\alpha) = 0$ under the constraint
\[
\int_{\mathbb{R}} \frac{d\mu_1(t)}{(\alpha - t)^2} \leq 1.
\]
Moreover, we have $F_{\mu_1}(F_{\mu_1}^{-1}(0)) = +\infty$ because the equality in the above constraint is achieved and thus, by Remark 9.6, $p_{\mu_1 \boxplus \mu_2}$ is comparable to $p_{\mu_1 \boxplus \mu_2}$ in $I$.
To obtain an example in which $F_{\mu_1}(F_{\mu_1}^{-1}(0))$ is finite, one can take $\mu_1$ to be the absolutely continuous measure with density
\[
\frac{3}{14} \left[ t^2 \chi_{[-1,1]}(t) + |t|^{-3/2} \chi_{\mathbb{R} \setminus [-1,1]}(t) \right].
\]

**Appendix A. Free convolution semigroups**

We write $\mu = \nu_{\boxplus}^{\gamma,\sigma}$ to indicate that $\mu$ is the $\boxplus$-infinitely divisible measure determined by $\gamma$ and $\sigma$ through the free Lévy-Hinčin formula. The notation $\nu_{\boxplus}^{\gamma,\sigma}$ is used in the multiplicative situation for the same purpose. Measures in this section are assumed to be nondegenerate.

Given a measure $\mu = \nu_{\boxplus}^{\gamma,\sigma}$, recall that the map $F_{\mu}$ is injective on $\mathbb{R}$ and hence it has at most one zero $t_{\mu}$. The point $t_{\mu}$ exists if and only if
\[
I = \int_{\mathbb{R}} \frac{1 + t^2}{t^2} \, d\sigma(t) \leq 1,
\]
and if this condition is satisfied, we have $\mu \{t_{\mu}\} = 1 - I$ \cite{13} Proposition 5.1]. Note that
\[
\mu = \mu_1 \boxplus \mu_2, \quad \text{where} \quad \mu_1 = \mu_2 = \nu_{\boxplus}^{\gamma,\sigma}.
\]
If the point $t_{\mu}$ does not exist (or, if the point $t_{\mu_1}$ relative to $\mu_1$ does not exist), then the set $A$ associated with this decomposition, as described in Section 9, would be empty and therefore Theorem 9.4 applies to the density $p_{\mu}$ of $\mu$. Thus, we have proved the following result.

**Proposition A.1.** Let $\mu = \nu_{\boxplus}^{\gamma,\sigma}$ be a freely infinitely divisible law such that
\[
\int_{\mathbb{R}} \frac{1 + t^2}{t^2} \, d\sigma(t) > 1.
\]
Suppose that $p_{\mu} > 0$ on an open interval $I$, and that $p_{\mu}(t_0) = 0$ at an endpoint $t_0$ of $I$. Then we have $p_{\mu}(t) = O\left(|t - t_0|^{1/3}\right)$ for $t$ sufficiently close to $t_0$.\vspace{1cm}
Given a measure \( \nu \in \mathcal{P}_\mathcal{R} \), we write the map \( F_\nu \) in its Nevanlinna integral form

\[
F_\nu(z) = a + z - N_\nu(z), \quad z \in \mathbb{H},
\]

and for each \( \beta > 1 \), we define \( \gamma = (1 - \beta)a \), \( \sigma = (\beta - 1)r \), and

\[
\Phi(z) = \gamma + z + N_\sigma(z), \quad z \in \mathbb{H},
\]

so that \( \Phi \) is the analytic continuation of \( F_\mu^{(-1)} \) where \( \mu = \nu^{\gamma,\sigma} \). There exists a unique measure \( \nu_\beta \in \mathcal{P}_\mathcal{R} \) such that

\[
F_{\nu_\beta} = F_\nu \circ F_\mu \quad \text{in} \quad \mathbb{H}.
\]

The family \( \{ \nu_\beta : \beta > 1 \} \) forms a free additive convolution semigroup in the sense that \( \nu_{\beta_1 + \beta_2} = \nu_{\beta_1} \boxplus \nu_{\beta_2} \) for all \( \beta_1, \beta_2 > 1 \). Since

\[
\Phi(z) = \beta z + (1 - \beta)F_\nu(z),
\]

the function \( F_\nu \) has a continuous extension (still denoted by \( F_\nu \)) to the image set \( F_\mu (\mathbb{R}) \). Thus, the composition \( F_\nu \circ F_\mu : \mathbb{H} \cup \mathbb{R} \to \mathbb{H} \cup \mathbb{R} \) is a continuous extension of \( F_{\nu_\beta} \) that satisfies the identity

\[
t = \beta F_\mu(t) + (1 - \beta)F_\nu(F_\mu(t)), \quad t \in \mathbb{R}.
\]

Then the parameterization

\[
F_\mu(t) = x + if(x)
\]

allows us to represent the density \( p_{\nu_\beta} = -\pi^{-1} \Im G_{\nu_\beta} \) of \( \nu_\beta \) as follows:

\[
p_{\nu_\beta}(t) = \frac{\beta f(x)}{\pi(\beta - 1)|F_\nu(F_\mu(t))|^2}, \quad \text{where } t \in \mathbb{R}, \ F_\nu(F_\mu(t)) \neq 0.
\]

The next result characterizes the zeros of \( p_{\nu_\beta} \) in terms of \( \nu \).

**Proposition A.2.** The set \( \{ \alpha \in \mathbb{R} : f(\alpha) = 0 \} \) is partitioned into two sets \( A \) and \( B \), defined as follows.

1. The set \( A \) consists of points \( \alpha \in \mathbb{R} \) such that \( \nu(\{ \alpha \}) > 0 \) and \( \nu(\{ \alpha \}) \geq 1 - \beta^{-1} \).

2. The set \( B \) consists of those \( \alpha \in \mathbb{R} \) such that \( F_\nu(\alpha) \in \mathbb{R} \setminus \{ 0 \} \) and

\[
F_\nu(\alpha)^2 \int_{\mathbb{R}} \frac{d\nu(t)}{(\alpha - t)^2} \leq \frac{\beta}{\beta - 1}.
\]

**Proof.** It was proved in [\( \square \) Proposition 5.1] that \( f(\alpha) = 0 \) if and only if the Julia-Carathéodory derivative \( \Phi'(\alpha) \) exists in \( (-\infty, 0] \); in which case we have

\[
\Phi'(\alpha) = \beta + (1 - \beta)F_\nu'(\alpha).
\]

It follows that

\[
\{ \alpha \in \mathbb{R} : f(\alpha) = 0 \} = \{ \alpha : F_\nu'(\alpha) \text{ exists in the interval } (0, \beta/(\beta - 1)) \}
\]

and

\[
= \left\{ \alpha : F_\nu(\alpha) = 0, \ 0 < F_\nu'(\alpha) \leq \frac{\beta}{\beta - 1} \right\} \cup \left\{ \alpha : F_\nu(\alpha) \in \mathbb{R} \setminus \{ 0 \}, \ 0 < F_\nu'(\alpha) \leq \frac{\beta}{\beta - 1} \right\}.
\]

The two disjoint sets in the last union are precisely \( A \) and \( B \), for \( F_\nu'(\alpha) = 1/\nu(\{ \alpha \}) \) if \( F_\nu(\alpha) = 0 \), and

\[
F_\nu'(\alpha) = \lim_{\varepsilon \downarrow 0} \frac{F_\nu(\alpha + i\varepsilon) - F_\nu(\alpha)}{i\varepsilon} = F_\nu(\alpha)^2 \int_{\mathbb{R}} \frac{d\nu(t)}{(\alpha - t)^2}
\]

when \( F_\nu(\alpha) \in \mathbb{R} \setminus \{ 0 \} \). \( \square \)
If $F_{\nu}(F_{\mu}(t)) = 0$, then (A.1) and (A.2) show that $F_{\mu}(t) = t/\beta$ and $f(t/\beta) = 0$, implying further that $F_{\nu}(t) \in A$. It follows that the density $p_{\nu_{\beta}}$ is continuous everywhere on $\mathbb{R}$, expect possibly on the finite set $\{\Phi(\alpha) : \alpha \in A\}$.

**Theorem A.3.** Let $\{\nu_{\beta} : \beta > 1\}$ be the free additive convolution semigroup generated by $\nu \in \mathcal{P}_{\mathbb{R}}$ and $\mu = \nu_{\beta}^{|\sigma|$ as above, and we assume that

$$
\int_{\mathbb{R}} \frac{1 + t^2}{t^2} \, d\sigma(t) > 1.
$$

Suppose that $p_{\nu_{\beta}} > 0$ on an open interval $I$, and that $p_{\nu_{\beta}}(t_0) = 0$ at an endpoint $t_0$ of $I$. Then we have $p_{\nu_{\beta}}(t) = O\left(|t - t_0|^{1/3}\right)$ for $t$ sufficiently close to $t_0$.

**Proof.** The hypothesis on $\sigma$ implies that $0 \notin F_{\mu}(\mathbb{R})$ and the set $A$ in Proposition A2 is empty. So, both $F_{\mu}(t_0)$ and $F_{\nu}(F_{\mu}(t_0))$ are nonzero. Since the density $p_{\mu} = -\pi^{-1}3G_{\mu}$ of $\mu$ is given by

$$
p_{\mu}(t) = \frac{f(x)}{\pi |F_{\mu}(t)|^2}, \quad t \in \mathbb{R},
$$

we can rewrite the formula (A.3) as

$$
p_{\nu_{\beta}}(t) = \frac{\beta}{\beta - 1} \frac{|F_{\mu}(t)|^2}{|F_{\nu}(F_{\mu}(t))|^2} p_{\mu}(t), \quad t \in I \cup \{t_0\}.
$$

The result follows from Proposition A1. \hfill \square

Following the same arguments, one can easily prove the analogs of Propositions A1 and A2 and Theorem A3 for free multiplicative convolution. We shall present these results below and leave the details to the reader. The first result is a direct consequence of Theorems 4.5 and 7.4.

**Proposition A.4.** We have the following cusp regularity for infinitely divisible laws.

1. Let $\mu = \nu_{\beta}^{\gamma,\sigma}$ be an $\mathfrak{B}$-infinitely divisible measure on $\mathbb{R}_+$ such that

$$
\sigma((0, +\infty)) > 0 \quad \text{and} \quad \int_{[0, +\infty)} \frac{1 + t^2}{(1 - t)^2} \, d\sigma(t) > 1.
$$

If the density $p_{\mu}$ of $\mu$ vanishes at an endpoint $t_0$ of an open interval $I \subset (0, +\infty)$ and $p_{\mu} > 0$ on $I$, then $p_{\mu}(t) = O\left(|t - t_0|^{1/3}\right)$ for $t$ close to $t_0$.

2. If $\mu = \nu_{\beta}^{\gamma,\sigma}$ is an $\mathfrak{B}$-infinitely divisible measure on $\mathbb{T}$ such that

$$
\int_{\mathbb{T}} t \, d\sigma(t) \neq 0, \quad 2 \int_{\mathbb{T}} \frac{d\sigma(t)}{|1 - t|^2} > 1
$$

$p_{\mu}(t_0) = 0$ at an endpoint $t_0$ of an open arc $I \subset \mathbb{T}$, and $p_{\mu} > 0$ on $I$, then $p_{\mu}(t) = O\left(|t - t_0|^{1/3}\right)$ for $t$ close to $t_0$.

As shown in [3], the construction of free multiplicative powers in $\mathcal{P}_{\mathbb{R}_+}$ actually goes beyond the integer case. Thus, given a measure $\nu \in \mathcal{P}_{\mathbb{R}_+}$ and a real number $\beta > 1$, there exist unique measures $\nu_{\beta}, \mu \in \mathcal{P}_{\mathbb{R}_+}$ such that $\mu = \nu_{\beta}^{\gamma,\sigma}$ and

$$
\eta_{\nu_{\beta}}(z) = \eta_{\nu}(\eta_{\mu}(z)), \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
$$
The map \( \eta_\nu \) extends continuously to \( \eta_\mu ((0, +\infty)) \), and so does the map \( \eta_\nu \) to \((0, +\infty)\). More importantly, these two extensions never take \( \infty \) or \( 0 \) as a value. The relation (2.11) persists when \( k \) is replaced by \( \beta \) and \( z = 1/t, \ t \in (0, +\infty) \), that is,

\[ t\eta_\mu(1/t)^\beta = \eta_{\nu_\beta}(1/t)^{\beta-1}, \quad t \in (0, +\infty), \]

where the principal branch of the power function is used. Following (2.10), we parameterize each \( \eta_\mu(1/t) \) by

\[ \eta_\mu(1/t) = re^{if(r)}, \]

then the relation between \( \eta_\mu \) and \( \eta_{\nu_\beta} \) shows that the density \( p_{\nu_\beta} \) of \( \nu_\beta \) is given by

\[ tp_{\nu_\beta}(t) = \frac{1}{\pi} (tr)^{\beta-1} \frac{r \sin \left( \frac{\beta}{\pi} f(r) \right)}{|1 - \eta_\nu \left( re^{if(r)} \right)|^2}, \quad r > 0, \ \eta_\nu \left( re^{if(r)} \right) \neq 1. \]

The next result is an analog of Proposition A2. The essence of its proof is that \( f(\alpha) = 0 \) if and only if \( \eta_\nu(\alpha) \in (0, +\infty) \) and the Julia-Carathéodory derivative \( \eta_\nu'(\alpha) \) exists and satisfies \((\beta - 1)\alpha \eta_\nu'(\alpha) \leq \beta \eta_\nu(\alpha)\). (This equivalence was already shown implicitly in the proof of Proposition 5.2 of [3].)

**Proposition A.5.** The set \( \{ \alpha > 0 : f(\alpha) = 0 \} \) is partitioned into two sets \( A \) and \( B \), defined as follows.

1. The set \( A \) consists of points \( \alpha > 0 \) such that \( \nu(\{1/\alpha\}) > 0 \) and \( \nu(\{1/\alpha\}) \geq 1 - \beta^{-1} \).
2. The set \( B \) consists of those \( \alpha > 0 \) such that \( \eta_\nu(\alpha) \in (0, +\infty) \setminus \{1\} \) and

\[ |1 - \eta_\nu(\alpha)|^2 \int_{\mathbb{R}_+} \frac{\alpha t}{(1 - \alpha t)^2} d\nu(t) \leq \frac{\beta \eta_\nu(\alpha)}{\beta - 1}. \]

The cusp behavior of \( p_{\nu_\beta} \) is our next result. The key ingredient of its proof is the identity

\[ p_{\nu_\beta}(t) = (tr)^{\beta-1} \frac{\sin \left( \frac{\beta}{\pi} f(r) \right)}{\sin(f(r))} \frac{|1 - re^{if(r)}|^2}{|1 - \eta_\nu \left( re^{if(r)} \right)|^2} p_\mu(t). \]

**Theorem A.6.** Let \( \{\nu_\beta : \beta > 1\} \) be the free multiplicative convolution semigroups generated by \( \nu \in \mathcal{P}_{\mathbb{R}_+} \) and \( \mu = \nu_{1/\sigma}^{\nu/\sigma} \) as above, and we assume that

\[ \sigma ((0, +\infty)) > 0 \quad \text{and} \quad \int_{[0, +\infty)} \frac{1 + t^2}{(1 - t)^2} d\sigma(t) > 1. \]

Suppose that \( p_{\nu_\beta} > 0 \) on an open interval \( I \subset (0, +\infty) \), and that \( p_{\nu_\beta}(t_0) = 0 \) at an endpoint \( t_0 \) of \( I \). Then we have \( p_{\nu_\beta}(t) = O \left( |t - t_0|^{1/3} \right) \) for \( t \) sufficiently close to \( t_0 \).

Finally, we turn to the free multiplicative convolution semigroups in \( \mathcal{P}_{\mathbb{T}} \). Let \( \nu_k = \nu_{\mathbb{Z}^k}, k \in (1, +\infty), \) be the free convolution powers discussed at the end of Section 5. Let \( \mu = \frac{\nu_0^{\gamma/\sigma}}{\mathbb{Z}} \) be the \( \mathbb{Z} \)-infinitely divisible law such that \( \eta_{\nu_k} = \eta_\nu \circ \eta_\mu \) and

\[ \eta_{\nu_k}(z) = \eta_\mu(z) \left( \frac{\eta_\mu(z)}{z} \right)^{1/\gamma}, \quad z \in \mathbb{T}. \]
At a point $\xi \in \mathbb{T}$, we parameterize the value $\eta_\mu (\xi)$ by

$$\eta_\mu (\xi) = R(t) t$$

to get the density $p_{\nu_k}$ of $\nu_k$ as follows:

$$p_{\nu_k} (\xi) = \frac{1 - R(t) \frac{2k}{2\pi [1 - \eta_\nu (R(t))]} t}{2\pi [1 - \eta_\nu (R(t))]} \quad \text{where } t \in \mathbb{T}, \eta_\nu (R(t)) \neq 1.$$ 

We have a characterization of zeros of $p_{\nu_k}$ as follows (cf. [3 Proposition 5.3]).

**Proposition A.7.** The set \{ $t \in \mathbb{T}$ : $R(t) = 1$ \} is partitioned into two sets $A$ and $B$, defined as follows.

1. The set $A$ consists of points $t \in \mathbb{T}$ such that $\nu (\{ t \}) > 0$ and $\nu (\{ t \}) \geq 1 - k^{-1}$.
2. The set $B$ consists of those $t \in \mathbb{T}$ such that $\eta_\nu (t) \in \mathbb{T} \setminus \{ 1 \}$, the Julia-Carathéodory derivative $\eta'_\nu (t)$ exists, and

$$|\eta'_\nu (t)| \leq \frac{k}{k - 1}.$$ 

Finally, we use the relation

$$p_{\nu_k} (\xi) = p_\mu (\xi) \frac{1 - R(t) t^2}{|1 - \eta_\nu (R(t))|^2} \quad \frac{1 - R(t) \frac{2k}{2\pi [1 - \eta_\nu (R(t))]} t}{1 - \eta_\nu (R(t))}$$

to conclude the following result.

**Theorem A.8.** Let \{ $\nu_k : k > 1$ \} be the free multiplicative convolution semigroup generated by $\nu \in \mathcal{P}_\mathbb{T}^*$ and $\mu = \nu_0^{\otimes \sigma}$ as above, and we assume that

$$\int_{\mathbb{T}} t \, d\sigma(t) \neq 0 \quad \text{and} \quad 2 \int_{\mathbb{T}} \frac{d\sigma(t)}{|1 - t|^2} > 1.$$ 

Suppose that $p_{\nu_k} > 0$ on an open arc $\Gamma$, and that $p_{\nu_k} (t_0) = 0$ at an endpoint $t_0$ of $\Gamma$. Then we have $p_{\nu_k} (t) = O (|t - t_0|^{1/3})$ for $t$ sufficiently close to $t_0$.

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