Analytic solution for bending-compression/tension members with different moduli

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Abstract. In this paper, based on elastic theory of different tension-compression moduli, formulas for calculation of stress and displacement are obtained for bending-compression/tension members under complex stress and subject to combined loadings. An example is given and the obtained analytical solution is compared with numerical results, showing high accuracy of the obtained analytic solution.

1. Introduction

It is assumed in classical mechanics that the elastic modulus of materials is the same when the material is extended or compressed. However, many materials such as concrete, metal, graphite, and plastic etc., have different tension and compression moduli. Especially for composite materials, which had been developed in recent years, the ratio of their tensile modulus to compression modulus is as high as over four times. Thus, a great error may arise if the same modulus theory in classical mechanics is still adopted. Therefore, it has become a new trend for engineers and researchers to use different tension-compression modulus to study composite materials.

Many researches are reported on the theory of elasticity about different moduli[1-7]. Till now, no analytic solution had been developed. In this paper, the author has deduced an analytic solution for bending-compression/tension members with different modulus under complex stress and subjected to the combined loadings by using flowing coordinate system and phased integration method. The new method proposed is more simple and convenient when compared to the earlier numerical solution, which is featured by heavy calculation and slow convergence speed. Therefore, This paper provides not only a new method for the calculation of the structure but also a new perspective for the study of composite materials with great differences between tension modulus and compression modulus.

2. Material model for different moduli

The stress of same magnitude yields different strain in tension or compression for some of materials, implying that the tension modulus $E_p$ is different from the compression $E_n$. In a word, the relationship between stress and strain is nonlinear. However, that is the only different from the theory of classic mechanics including single modulus to that including different modulus.

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3. Structural model

As shown in figure 1, with one end fixed, a beam is subject to axial force $P$, bending moment $m$, lateral force $F$ and distributed load $q(x)$. The neutral axis of the cross-section will change along x axis since $M(x)$ at any cross section is variable, that is, the height of tension area in arbitrary section is $h_p = f(x)$. Then, the author introduces flowing coordinate system, where the coordinate axis flows once whenever $\Delta x$ is added. Therefore, for any cross section in the yoz plane after flowing, the origin of coordinate will uniformly passes through the neutral axis.

![Figure 1. Structural model.](image1)

![Figure 2. Left end.](image2)

![Figure 3. Member deformation state.](image3)

4. Theoretical analysis

Lateral buckling is generated by the joint action of the distributed load $q(x)$, lateral force $F$, bending moment $m$ and internal force $P$. Its effect is reflected in the bending internal force $M(x), Q(x)$ of any arbitrary section, (See figure 2). Separate the left of an arbitrary section for study, since

$$
\Sigma M = 0, \quad M(x) = m + Fx + \int_0^x q(x)(x - x')dx,
$$

$$
\Sigma Y = 0, \quad Q(x) = F + \int_0^x q(x)dx
$$

Finally, the neutral axis, stress and displacement of the member can be derived under the combined action of $M(x), Q(x)$ and $P$. It is noted that the assumption of pure bending instead of the lateral force bending mode is adopted during the derivation of the normal stress and strain.

5. Derivation of neutral axis

As shown in figure 3, the deformation of bending-compression occurs to the member in the combined action of $M$ and $P$. The form of deformation is the same as that of pure bending, and the only difference is reflected in the $\delta$, which results from the neutral axis moving from the $h/2$ in the pure...
bending state to the compression region with the inclusion of \( P \). Meanwhile, the deformation of the member still conforms to plane cross-section assumption.

Take a segment \( dx \) from the member, the radius of curvature of neutral layer is \( s \), the normal strain of random point whose distance to the coordinate is \( y \) can be expressed as

\[
\varepsilon_n = \frac{(s + y)d\theta - sd\theta}{sd\theta} = \frac{y}{s}
\]

As different modulus is bilinear, applying a physical equation to the tension and compression area respectively gives the corresponding normal stress of any point in arbitrary section

\[
\sigma^p_n = E_p \frac{y}{s}, \quad \sigma^n_n = E_n \frac{y}{s}
\]

Where the subscripts \( p \) and \( n \) represent tension and compression area respectively.

The height of tension area in arbitrary section is \( h_p \) when the member is subject to the combined action, According to the equilibrium condition and Saint-Venant Principle, we have

\[
\int_{h_p}^0 E_n \frac{y}{s} bdy + \int_0^{h_p} E_p \frac{y}{s} bdy = M(x)
\]

\[
\int_{h_p}^0 E_n \frac{y}{s} bdy + \int_0^{h_p} E_p \frac{y}{s} bdy = P
\]

Integrating the two formulation above and simplifying it, we have

\[
2P(E_p - E_n)h_p^3 - 3[M(x)(E_p - E_n) - 2PE_nh]h_p^2
- 6E_nh[M(x) + Ph]h_p + E_nh^2[3M(x) + 2Ph] = 0
\]

Solving equation (5), we have

\[
h_p = \frac{B}{3A} + \frac{(1 - i\sqrt{3})J}{3 \times 2^{2/3} \times A \times 3\sqrt{F + \sqrt{4J^3 + F^2}}} + \frac{(1 + i\sqrt{3})J}{6 \times 2^{1/3} \times A}
\]

Where

\[
A = 2P(E_p - E_n), \quad B = -3[M(x)(E_p - E_n) - 2PE_nh], \quad C = -6E_nh[M(x) + Ph],
\]

\[
D = E_nh^2[3M(x) + 2Ph], \quad J = -B^2 + 3AC, \quad F = -2B^2 + 9ABC - 27A^2D.
\]

As for the axial force at both ends of the member, when it is in tension, \( P \) is positive; when it is in compression, \( P \) is negative.

6. Derivation of normal stress and shear stress

Integrating equation (4.1) and equation (4.2), we have

\[
S = \frac{h_p[E_p h_p^2 + E_n(h - h_p)^3]}{3M(x)}
\]
\[ \sigma_x^p = \frac{3E_pM(x)}{b[E_p h_p^3 + E_n(h-h_p)^3]} \]  
\[ \sigma_y^p = \frac{3E_pM(x)}{b[E_p h_p^3 + E_n(h-h_p)^3]} \]

(8)

Take a micro segment \( dx \) from a beam, the left and right section of segment are subject to the moment of \( M(x), M(x)+dM(x), \) and the shear stress of \( Q(x), Q(x)+dQ(x) \) respectively, subsequently, take the upper of section whose distance to the neutral axis is \( y \) from segment as separate body, as shown in figure 4. Assume the area of both left and right section of separate body is \( A \), the normal stress, which is composed of lateral bending force in each section, can be written as

\[
\begin{align*}
N_1 &= \int_{A_y} \sigma_x^p dA = \int_{A_y} \frac{3E_pM(x)}{b[E_p h_p^3 + E_n(h-h_p)^3]} \cdot y^* dA \\
N_2 &= \int_{A_y} (\sigma_x^p + d\sigma_x^p) dA = \int_{A_y} \frac{3E_p[M(x) + dM(x)]}{b[E_p h_p^3 + E_n(h-h_p)^3]} \cdot y^* dA
\end{align*}
\]

(9.1)

As the shear stress should be equal mutually, that is, \( \tau_{xy}^p = -\tau_{xy}^p \), and \( \sum x=0 \), we have

\[ \tau_{xy}^p bdx = N_2 - N_1. \]

(9.2)

Substituting equation (9.1) and \( Q(x) = \frac{dM(x)}{dx} \) into the formula above, the shear stress in the tension area can be expressed as

\[ \tau_{xy}^p = \frac{3E_p[h^2_p - y^2]}{2b[E_p h_p^3 + E_n(h-h_p)^3]}Q(x), \quad \tau_{xy}^n = \frac{3E_p[(h-h_p)^2 - y^2]}{2b[E_p h_p^3 + E_n(h-h_p)^3]}Q(x) \]

(10)

7. Derivation of displacement

Based on the geometry equation and physical equation, we have

\[ \varepsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \mu \sigma_y), \quad \varepsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \mu \sigma_x), \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1+\mu)}{E} \tau_{xy} \]

(11.1)

Substituting equation (8), equation (10) and \( \sigma_y=0 \) into Equation (11.1), we have

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{3M(x)}{b[E_p h_p^3 + E_n(h-h_p)^3]}, \quad \frac{\partial v}{\partial y} = -\frac{3\mu M(x)}{b[E_p h_p^3 + E_n(h-h_p)^3]}y \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= \frac{3(1+\mu)[h^2_p - y^2]}{b[E_p h_p^3 + E_n(h-h_p)^3]}Q(x) \quad (0 \leq y \leq h_p) \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= \frac{3(1+\mu)[(h-h_p)^2 - y^2]}{b[E_p h_p^3 + E_n(h-h_p)^3]}Q(x) \quad [-(h-h_p) \leq y < 0]
\end{align*}
\]

(11.2)

As shown in figure 3, assume \( q(x)=0 \), the internal force in arbitrary section can be written as

\[ M(x) = (Fx + m), \quad Q(x) = F \]

(11.3)

Substituting equation (11.3) into equation (11.2), then integrating the 1st and the 2nd equation of equation (11.2), we have
\[
\begin{align*}
\{ u &= \frac{3}{b[E_p h_p^3 + E_n (h - h_p)^3]} \left( \frac{1}{2} Fx^2 + mx \right) y + f_1(y) \\
\{ v &= -\frac{3\mu}{2b[E_p h_p^3 + E_n (h - h_p)^3]}(Fx + m)y^2 + f_2(x)
\end{align*}
\] (11.4)

Assuming
\[
[E_p h_p^3 + E_n (h - h_p)^3] = K
\] (11.5)

Differentiating the function \( u \) and \( v \), then substituting them into the 3rd equation of formula (11.2), we have
\[
\begin{align*}
\frac{df_2(x)}{dx} + \frac{3}{bK} \left( \frac{1}{2} Fx^2 + mx \right) + \frac{df_1(y)}{dy} + \frac{3F}{2bK} y^2 (\mu + 2) &= \frac{3F(1 + \mu)}{bK} h_p^2 \\
\end{align*}
\] (11.6)

Assuming
\[
\begin{align*}
\frac{df_2(x)}{dx} + \frac{3}{bK} \left( \frac{1}{2} Fx^2 + mx \right) &= T \\
\frac{df_1(y)}{dy} + \frac{3F}{2bK} y^2 (\mu + 2) &= W
\end{align*}
\] (11.7)

the equation (11.6) can be expressed as
\[
T + W = -\frac{3(1 + \mu)}{bK} Fh_p^2
\] (11.8)

Solving equation (11.7), we have
\[
f_1(y) = W y + \frac{F(\mu + 2)}{2bK} y^3 + d \\
f_2(x) = T x + \frac{3}{bK} \left( \frac{1}{6} Fx^3 + \frac{1}{2} mx^2 \right) + c
\] (11.9)

Substituting equation (11.9) into equation (11.4):
\[
\begin{align*}
\{ u &= \frac{3}{bK} \left( \frac{1}{2} Fx^2 + mx \right) y + W y + \frac{F(\mu + 2)}{2bK} y^3 + d \\
\{ v &= -\frac{3\mu}{2bK} (Fx + m)y^2 + T x + \frac{3}{bK} \left( \frac{1}{6} Fx^3 + \frac{1}{2} mx^2 \right) + c
\end{align*}
\] (12)

The coefficients of \( d, c, T, W \) can be deduced based on the boundary condition of
\[
u|_{y=0} = 0, v|_{x=l} = 0, \frac{\partial v}{\partial x}|_{y=0} = 0 \quad \text{and} \quad (11.8) \quad \text{and simplify it, the formula of displacement can be}
\] expressed as
\[
\begin{align*}
\{ u &= \frac{3y[F(x^2 - l^2) + 2m(x - l) - 2F(2 + \mu)y^2 + 2F(1 + \mu)h_p^2]}{2b[E_p h_p^3 + E_n (h - h_p)^3]} \quad (0 \leq y \leq h_p) \\
\{ u &= \frac{3y[F(x^2 - l^2) + 2m(x - l) - 2F(2 + \mu)y^2 + 2F(1 + \mu)(h - h_p)^2]}{2b[E_p h_p^3 + E_n (h - h_p)^3]} \quad [-(h - h_p) \leq y \leq 0] \\
\{ v &= \frac{3x(Fl^2 + 2ml) - 3\mu y^2 (Fx + m) - x^2 (Fx + m) - l^2 (2Fl + 3m)}{2b[E_p h_p^3 + E_n (h - h_p)^3]}
\end{align*}
\] (13)
8. Example

As shown in figure 5, a cantilever beam is subject to the combined action of moment, axial force, and bending force, the elastic modulus of material is $E = 2.8 \times 10^7$ kpa. Assume $E_p/E_n = 1/3 \sim 3$, calculate the stress and displacement using the same modulus theory in classical mechanics, the different modulus theory proposed in this paper, and finite element method (FEM) respectively.

![Diagram of stress analysis](image1)

**Figure 4** Micro segment sketch body schematic

![Diagram of cantilever beam](image2)

**Figure 5** Example

**Figure 6** The normal stress (compression) varies with $E_p/E_n$

**Figure 7** The displacement along the direction of $x$ (top in tension area) varies with $E_p/E_n$
9. Conclusions

The normal stress in tension area diminishes with the decline of Ep/En, while normal stress in compression area increases with the decrease of Ep/En (See figure 6), which is in agreement with the regularity of “rigidity adjusts the internal force”.

The displacement is very sensitive to Ep/En. The horizontal displacement at the top of the tension area utop decreases with the increase of Ep/En (See figure 7), whereas the horizontal displacement at the bottom of compression area ubottom decreases with the decrease of Ep/En (See figure 8). It is noted that the deflection in the neutral axis $\nu_y=0$ will always increase no matter Ep/En increases or decreases (See figure 9). When $Ep/En=1$, an extreme value occurs on $\nu_y=0$ curve. In summary, the increase of tension modulus lead to the reduction of the displacement in the tension area, likewise, the increase of compression modulus result in the reduction of the displacement in the compression area.

In general, there is a relatively great error between the results derived from same modulus of classical mechanics and those from different modulus, for example, in the case of $Ep/En =1/3 \sim 3$, the
errors of both stress and displacement between two methods reach 38%. (See figure10 and 11.) Therefore, it can be concluded that: For the structure with great difference between tension modulus and compression modulus, it should be calculated and analyzed by the different modulus theory instead of the same modulus theory so that the results can not only better reflect the actual condition but also accurately describe the features and potentialities of the material.

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