Variational principle for optimal quantum controls in quantum metrology

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We develop a variational principle to determine the quantum controls and initial state which optimizes the quantum Fisher information, the quantity characterizing the precision in quantum metrology. When the set of available controls is limited, the exact optimal initial state and the optimal controls are in general dependent on the probe time, a feature missing in the unrestricted case. Yet, for time-independent Hamiltonians with restricted controls, the problem can be approximately reduced to the unconstrained case via the Floquet engineering. In particular, we find for magnetometry with a time-independent spin chain containing three-body interactions, even when the controls are restricted to one and two-body interaction, that the Heisenberg scaling can still be approximately achieved. Our results open the door to investigate quantum metrology under a limited set of available controls, of relevance to many-body quantum metrology in realistic scenarios.

Introduction. At the heart of quantum technology, quantum metrology aims at improving the precision of parameter estimation [1–5]. Recent theoretical advances in quantum metrology [6–11] have led to the introduction of optimal control protocols that maximizes the quantum Fisher information (QFI), the key quantity in characterizing the precision in quantum metrology. The use of control protocols for enhanced parameter estimation has been extensively explored in many practical physical setups [8, 9, 11–13].

The identification of the optimal control protocol for parameter estimation [8, 9] can be done exploiting the notion of adiabatic continuation of quantum states. It resembles the engineering of shortcuts to adiabaticity (STA) by counterdiabatic driving [14–18], whereby the system Hamiltonian is supplemented by some controls that enforce adiabatic continuation along a prescribed trajectory. In the context of STA, the auxiliary controls enforce parallel transport along the eigenstates of the uncontrollable system Hamiltonian. By contrast, optimal controls for quantum metrology enforce adiabatic continuation of the eigenstates of an operator defined by the parametric derivative of the estimation Hamiltonian [19]. This is intuitive as such operator guarantees maximal distinguishability of states with a slight change of the estimated parameter. The connection of optimal control for quantum metrology and STA is not only fruitful in providing a geometric justification of the required controls. As we show in this work, it suggests solutions to common challenges.

In STA, the exact identification of the counterdiabatic control protocol is not feasible in many-body systems in which spectral properties of the system Hamiltonian cannot be found, e.g., in problems such as quantum optimization. In addition, the set of available controls in a given experimental setup is often restricted. This generally precludes the implementation of the exact STA protocol. This is particularly important in the many-particle systems when exact counterdiabatic controls involve non-local multiple-body interactions beyond one-body and pair-wise potentials which are hard to realize in practice [20]. In addition, the use of variational methods provides then an alternative, by designing optimal protocols that are realizable with a restricted set of controls [21–29]. Likewise, the exact construction of optimal protocols in many-body quantum metrology [11] (i) requires access to the spectrum of the parametric derivative of the many-body Hamiltonian which is hardly accessible in general, (ii) may require optimal controls that are nonlocal, and hard to implement in the laboratory. Even in single-qubit metrology using a NV center, superconducting qubit, or quantum dot, certain control operations may be hard to implement. It is thus required to develop a new formalism of optimal control for quantum metrology beyond the state of the art [8], by taking into account the ubiquitous limitations on the controls and circumventing the requirement to access the spectral properties of the Hamiltonian derivative. This is the problem solved in this Letter by means of a novel variational approach that relies on a metrological action. We note that our work is unrelated to other variational approaches introduced in quantum metrology that do not involve quantum control [30, 31].

Specifically, we assume that the space of control Hamiltonians is spanned by a set of local basis operators and introduce a metrological action, which includes contributions of the quantum Fisher information and the Schrödinger equation, which is considered as a constraint to the optimization problem. We derive the optimal control conditions and show that it reduces to the unrestricted protocol in Ref. [8] if the basis operators generate the full space of the Hermitian operators. Furthermore, we show that when the control Hamiltonian is restricted, the exact optimal solution for the initial state and the control Hamiltonian depends on the probe time. However, we find that for multiplicative time-independent Hamiltonians, even with limited controls, one may identify approximate optimal controls, which reduce to the unrestricted protocol and therefore become independent of the probe time, using the high frequency expansion in Floquet engineering [32–38]. In particular, we show in an example in magnetometry that...
one can avoid local three-body interaction terms making use only of local two-body control Hamiltonians.

Optimal initial states and controls through variational principle. Consider the quantum estimation of a parameter $\lambda$ in a general Hamiltonian $H_I(t)$. In this setting, one may find that the QFI may decrease as the probe time increases [39]. This motivates the introduction of quantum controls $H_c(t)$ to enhance the QFI [6, 8]. When the quantum controls are introduced, the unitary evolution $U(t)$ is generated by $H_{\text{tot}}(t) = H_I(t) + H_c(t)$. Let the initial time and fixed final probe time be 0 and $t_f$, respectively. The generator for parameter estimation is given by $G_{t_f}[U] = \int_0^{t_f} U^\dagger(\tau) \partial_\tau H_c(\tau) U(\tau) d\tau$ [8] and the quantum Fisher information $I$ is given by the variance of the generator [4], i.e., $I[U] = \text{Var}(G_{t_f}[U])$.

The optimization of the QFI $I[U]$ over the initial state yields the optimal initial state $\langle \varphi_a(t_f) \rangle$ and $\varphi_c(t_f)$, as they are associated with the maximum and minimum eigenvalues $\mu_{\text{act}}(t_f)$ of $G_{t_f}[U]$ [40]. The maximum value of the QFI over the initial states $I_{\text{opt}} = \max_{\{\psi\}} I = \|G_{t_f}[U]\|^2/4$, where the norm of an operator is defined by the difference between its maximum and minimum eigenvalues [5, 41, 42]. Our next goal is to maximize the quantum Fisher information $I[U]$ over all possible unitary dynamics under the condition that $U$ and $H_c$ satisfy the Schrödinger equation $i\hbar U(t) = H_{\text{tot}}(t)U(t)$ as a constraint. We further require the control Hamiltonian $H_c(t)$ to be spanned by a limited set of available linearly independent  

terms is $\{X_{t_f}\}_{i=1}^d$. Denote $\mathcal{V}_c = \text{span}\{X_{t_f}\}_{i=1}^d$ and expand $H_c(t) = \sum_{i=1}^d c_i(t) X_i$. This procedure amounts to removing the constraint on $H_c$, reducing the optimization over $H_c$ to the optimization over $c_i(t)$ [43].

We shall denote $I_0 = \max_{\{\psi\}, H_{\text{tot}}, U} I[U]$. In principle, $I_0$ can be computed through the variational principle by constructing an appropriate action. We note that $I[U]$ is quartic in $U$ since it is quadratic in $G_{t_f}[U]$, which is itself quadratic in $U$. This makes the variational calculus of $I[U]$ with respect to $U$ tedious. To facilitate the calculation, we observe that

$$I_0 = \max_{\{\psi\}, H_{\text{tot}}, U} \left( \langle \varphi_a | G_{t_f}[U] | \varphi_a \rangle - \langle \varphi_b | G_{t_f}[U] | \varphi_b \rangle \right)^2$$  

(1)

under the constraint of Schrödinger equation and the condition that $|\varphi_a, b\rangle$ are normalized. The introduction of two more optimization variables $|\varphi_a, b\rangle$ allows us to remove the square in Eq. (1) and transform the original optimization problem to the following equivalent one, $\max_{\{\psi\}, \{\varphi\}, U, H_c} S_1[\Delta \rho, U]$, under aforementioned constraints, with the “information action” being defined as

$$S_1[\Delta \rho, U] \equiv \int_0^{t_f} \text{Tr} \left\{ \Delta \rho U^\dagger(\tau) \partial_\tau H_c(\tau) U(\tau) \right\} d\tau$$

(2)

and $\Delta \rho \equiv |\varphi_a\rangle \langle \varphi_a | - |\varphi_b\rangle \langle \varphi_b |$. The introduction of two additional auxiliary variables $|\varphi_a, b\rangle$ effectively renders $S_1[U]$ quadratic in $U$, unlike $I[U]$, facilitating the variational calculus with respect to $U$. Upon introducing the Lagrangian multipliers $\mu_{a, b}$ and $\Lambda(\tau)$, we obtain the following "metrological action"

$$S_M(|\varphi_a\rangle, |\varphi_b\rangle, U, H_c) \equiv S_1[\Delta \rho, U] + S_2[U, H_c]$$

$$- \mu_{a} \{ \langle \varphi_a | \varphi_a \rangle - 1 \} - \mu_{b} \{ \langle \varphi_b | \varphi_b \rangle - 1 \}$$

(3)

where the "Schrödinger action" is defined as

$$S_2[U, H_c] \equiv \int_0^{t_f} \text{Tr} \left\{ \Lambda(\tau) [i \hbar U^\dagger(\tau) - H_c(\tau) - H_0(\tau)] \right\} d\tau.$$  

(4)

We emphasize that $|\varphi_a, b\rangle$, $U$ and $H_c$ are independent variables. The optimization over $|\varphi_a, b\rangle$ can be easily implemented by differentiation with respect to them, which yields

$$G_{t_f}[U] |\varphi_a, b\rangle = \mu_a |\varphi_a, a\rangle, \quad a = a, b.$$  

(5)

As shown in Sec. I in [40], in order for $I_0$ to take the global maximum values over $|\varphi_a, b\rangle$, $\mu_{a, b}$ and $|\varphi_a, b\rangle$ must be the maximum and minimum eigenvalues and eigenvectors of $G_{t_f}[U]$, respectively.

Variation with respect to $H_c$ and $U$ gives the trace condition

$$\text{Tr} \{ \Delta \rho(t) X_i \} = 0 \quad \text{for} \quad i \in [1, d_c]$$

and the differential equation $\Delta \rho(t) - i [\Delta \rho(t), H_{\text{tot}}(t)] + i [\Delta \rho(t), \partial_\tau H_c(\tau)] = 0$, with the final condition $\Lambda(t_f) = 0$, where $\Delta \rho(t) = \text{U}(t) \Delta \rho \text{U}^\dagger(t)$. One can solve for $\Lambda(\tau)$ the differential equation and substitute the result into the trace condition to find

$$\text{Tr} \{ X_i \partial_\tau [\Delta \rho(\tau)] \} = 0, \quad i \in [1, d_c].$$  

(6)

Eq. (5) and Eq. (6) are our central results. The form of these equations in the parameter-independent rotating frame can be found in [40]. They give the optimal initial state and optimal dynamics that maximize the QFI when the quantum controls are restricted to the subspace spanned by $\{X_i\}$. In what follows, we discuss their implications and applications.

The unrestricted control and the general feature of exact restricted controls. As a first application of our results, let us assume there is no restriction on the control Hamiltonians, that is, $\{X_i\}$ spans the whole space of traceless Hermitian operators. We shall see how the Pang-Jordan protocol in Ref. [8] is reproduced. In this case, Eq. (6) is equivalent to $\partial_\tau [\Delta \rho(\tau)] = 0$. Taking the time derivative on both sides yields

$$\partial_\tau \partial_\tau [\Delta \rho(\tau)] = -i [\partial_\tau H_c(\tau), \Delta \rho(\tau)] = 0, \quad \forall \tau \in [0, t_f],$$  

(7)

where we use the Liouville equation for $\Delta \rho(\tau)$. Since $|\varphi_a, b\rangle \equiv \text{U}(\tau) |\varphi_a, b\rangle$ are associated with the non-degenerate eigenvalues $\pm 1$ of $\Delta \rho(\tau)$, we conclude that $|\varphi_a, b\rangle(\tau)$ must also be eigenvectors of $\partial_\tau H_c(\tau)$ at all times. That is, $\partial_\tau H_c(\tau) |\varphi_a, b\rangle(\tau) = \nu_{a, b}(\tau) |\varphi_a, b\rangle(\tau), \forall \tau \in [0, t_f]$, where $\nu_{a, b}(\tau)$ is the eigenvalue of $\partial_\tau H_c(\tau)$. It is then straightforward to check that $|\varphi_a, b\rangle$ are eigenvectors of $G_{t_f}[U]$ with eigenvalue $\int_0^{t_f} \nu_{a, b}(\tau) d\tau$. Thus any unitary dynamics that preserves the adiabatic evolution of any pair of eigenstates of $\partial_\tau H_c(\tau)$ satisfies Eq. (7) and is an extremal solution satisfying
\[ \delta S_M = 0. \]

To further maximize the QFI among the manifold of extremal solutions, one needs to further optimize the difference between \( \int_0^t v_x(\tau) \) and \( \int_0^t v_y(\tau) \). This requires \( v_x(\tau) \) and \( v_y(\tau) \) to be the maximal and minimum eigenvalues of \( \partial_t H_\nu(\tau) \) at all times. When the unitary dynamics preserves the adiabatic evolution of all eigenstates of \( \partial_t H_\nu(\tau) \), i.e., \( U(\tau) = \sum_\lambda |\varphi_\lambda(\tau)\rangle \langle \varphi_\lambda(0)| \), where \( |\varphi_\lambda\rangle \) denotes the eigenvectors of \( \partial_t H_\nu(\tau) \), one recovers the Pang-Jordan control Hamiltonian [8] \( H_c(\tau) = i \sum_\lambda |\varphi_\lambda(\tau)\rangle \langle \varphi_\lambda(\tau)| - H_\nu(\tau) \). Note that if there is any level crossing in \( v_x(\tau) \) at some instant time in \([0, t_f]\), the way of labeling the eigenstates and eigenvalues is not unique. Choosing \( v_x(\tau) \) and \( v_y(\tau) \) always as the maximum and minimum eigenvalues of \( \partial_t H_\nu(\tau) \) at all times, the resulting QFI is the greatest among all the different ways of labeling the eigenstates. With this labeling, the first-order time derivative of \( |\varphi_\lambda(\tau)\rangle \) is discontinuous, which results in a \( \delta \)-pulse in the control Hamiltonian \( H_c \). This provides an alternative understanding of the \( \sigma_\tau \)-like pulses in Ref. [8, 9, 13].

In the general case, \( [X_\sigma, Y_\sigma] \) do not span the whole space of Hermitian operators, and the generic optimal solutions \( U(\tau) \) and \( |\varphi_{\sigma, b}\rangle \) implicitly depend on the probe time \( t_f \). This is due to the dependence of the generator \( G_{\sigma, b}[U] \) on \( t_f \), that may make the eigenvectors \( |\varphi_{\sigma, b}\rangle \) determining the optimal initial state also dependent on \( t_f \). The dependence on \( t_f \) for the optimal unitary \( U(\tau) \) can then be seen from Eq. (6). With these observations, we take the derivative with respect to \( t_f \) to obtain the following differential-integral equation

\[
\partial_t G_{\sigma, b} |\varphi_{\sigma, b}\rangle + G_{\sigma, b} \partial_t |\varphi_{\sigma, b}\rangle = \partial_t H_{\sigma, b} |\varphi_{\sigma, b}\rangle + H_{\sigma, b} \partial_t |\varphi_{\sigma, b}\rangle,
\]

where we have suppressed the dependence on \( U \) in the generator \( G_{\sigma, b} \) for simplicity, and \( \sigma = a, b \), \( \partial_t G_{\sigma, b} = i \frac{\partial}{\partial t} U_{\sigma, b}^{\dagger}(t_f) \partial_t U_{\sigma, b}(t_f) + U_{\sigma, b}^{\dagger}(t_f) \partial_t U_{\sigma, b}(t_f) + U_{\sigma, b}(t_f) \partial_t H_{\sigma, b}(t_f) U_{\sigma, b}^{\dagger}(t_f) \), where \( \partial_t U_{\sigma} \) denotes the derivative with respect to the subscript \( t_f \) instead of the one in the parenthesis [40]. Generally, Eq. (8) is difficult to solve analytically, while numerical calculation is tractable. However, if \( U_{\sigma, b}(\tau) \) and \( |\varphi_{\sigma, b}\rangle \) are independent of the subscript variable \( t_f \), Eq. (8) reduces to \( U^{\dagger}(t_f) \partial_t H_{\sigma, b}(t_f) U(t_f) |\varphi_{\sigma, b}\rangle = \partial_t H_{\sigma, b} |\varphi_{\sigma, b}\rangle \). It then follows that \( |\varphi_{\sigma}(t_f)\rangle = U(t_f) |\varphi_{\sigma}\rangle \) is an eigenstate of \( \partial_t H_{\sigma}(t_f) \) for all \( t_f \), with eigenvalue \( \partial_t H_{\sigma} \). The solution reduces again to the Pang-Jordan protocol [8].

This suggests that when the control Hamiltonian is restricted to some non-trivial subspace of the Hermitian operators, such that \( \partial_t \Delta \sigma(\tau) \) does not always vanish on \([0, t_f]\), both the exact optimal controls and the exact optimal initial states depend on the probing time \( t_f \), making it challenging to find them analytically. In particular, one can show that when only \( U(\tau) \) depends on \( t_f \) but \( |\varphi_{\sigma}\rangle \) is independent of \( t_f \), this is the case as there exists a time such that \( \partial_t \Delta \sigma(\tau) \neq 0 [40] \).

**Approximate solution via Floquet engineering.** We next show that for a time-independent Hamiltonian \( H_\nu \), the restricted optimal controls can be approximately engineered by the high-frequency control, known as the Floquet engineering [32–38]. For time-independent Hamiltonians with unrestricted controls, the minimum optimal control Hamiltonian is the one that cancels the parts in \( H_\nu \) that do not commute with \( \partial_t H_{\nu} [8] \). However, if any of the required control Hamiltonians is not available, it is not clear how to reach the Heisenberg scaling, \( t_{\text{opt}}^{\text{HF}} = 4 n^2 \omega^2 \), where \( n \) is the number of probes. Here, we search for a static control Hamiltonian \( H_{\nu, 0} \) and high-frequency driving controls \( H_{\nu}^{(d)}(t) \), where \( H_{\nu}^{(d)}(t) \equiv \sum_0^\infty H_{\nu} e^{i \omega t} \), so that \( H_{\nu}(t) = H_{\nu, 0} + H_{\nu}^{(d)}(t) \), and show that the unavailable controls in the minimum optimal control Hamiltonian can be actually constructed approximately through a high-frequency expansion. Let us first move to the rotating frame associated with \( U(t) = e^{i K t} \), where \( K(t) \) is the so-called kick operator [33]. In it, the total Hamiltonian becomes is given by the Floquet effective Hamiltonian \( H_F \). When the driving frequency is high enough, an expansion in orders of \( 1/\omega \) can be performed [33, 36]. To the first-order of \( 1/\omega \), one finds that \( K(t) = [1/(i\omega)] \sum_{l=1}^\infty \frac{1}{l} \left( H_{c, l} e^{i \omega t} - 1 \right) + O(1/\omega^2) \) [44] and

\[
H_F = H_{\nu, 0} + H_{\nu}^{(l)} \sum_{l=1}^\infty \frac{1}{l} \left| H_{c, l} e^{i \omega t} - 1 \right| + O(1/\omega^2).
\]

One can explicitly show that the kick operator \( K(t) \) consists of the \( H_{c, l} \) with \( l \neq 0 \) and it is independent of the estimation parameter. Thus, the QFI remains unchanged in the Floquet rotating frame. The generator becomes \( G_{\nu, k} = \int_0^\infty e^{-i \omega t} \partial_t H_{\nu} e^{i \omega t} dt [40] \). The key idea is that the unavailable controls in the original static frame can be constructed through the commutator in Eq. (9). This can be best illustrated using a simple qubit example with the Hamiltonian \( H_{\nu, 1} = \lambda \sigma_x / 2 + \Delta \sigma_z / 2 \) and \( \mathcal{V}_c = \{ \sigma_y, \sigma_z \} \). The term that does not commute with \( \partial_t H_{\nu} \) is \( \Delta \sigma_z / 2 \), which is not available in \( \mathcal{V}_c \). Therefore, we consider \( H_{\nu, 0} = \sigma_y^0 \sigma_y + \sigma_z^0 \sigma_z \) and

\[
\text{Figure 1. QFI for various scenarios. (a) a single qubit with the sensing Hamiltonian } H_\nu = \lambda \sigma_x / 2 + \Delta \sigma_z / 2 \text{ (b-d) the spin chain with the sensing Hamiltonian (13). For all figures (a-d) } \lambda = 1 \text{, the frequency of the drive } \omega = 182.67 \text{. The red lines satisfy the AFM condition (11) or (16) with } c_1^2 = c_1^2 = c_2^2 = 10 \text{ when } 1 \leq l \leq 5 \text{ and vanish for } l > 5 \text{. The total simulation time is 5000 times the fundamental periodic } 2\pi/\omega. \text{ For the case of lowest two harmonics in green lines, } c_1^2, c_2^2, c_3^2, c_4^2 \text{ are non-vanishing only when } l = 1, 2 \text{ and takes value 10.}
\]
where we call the amplitude-frequency matching (AFM) condition. The validity of the high-frequency expansion requires that \( \omega \) should be the largest frequency in the original Hamiltonian, i.e., that \( \omega \gg \lambda, \Delta, c_i^x, c_i^y \). Experimentally, for a laser frequency that satisfies this condition, one can always tune the amplitude so that Eq. (11) is satisfied. Conversely, one can also choose a proper laser frequency for given amplitudes so that Eq. (11) is fulfilled. We emphasize that when Eq. (11) is satisfied and the initial state in the Floquet rotating frame is \( |0\rangle + |1\rangle \)/\( \sqrt{2} \), the optimal control conditions Eqs. (5, 6) are approximately satisfied up to the order of \( 1/\omega \) in the Floquet rotating frame. The initial state in the lab and Floquet rotating frames is the same as \( K(0) = 0 \). Going back to the lab frame, we generate a solution to Eqs. (5, 6) when the controls are restricted. This illustrates the power of our approach beyond the Pang-Jordan protocol [8].

A simple choice for the amplitudes involves taking for \( l \geq 1 \) both \( c_i^x \) and \( c_i^y = ic_i^x \) to be real. This yields

\[
\begin{align*}
H_c^{(d)}(t) &= 2 \sum_{l=1}^{\infty} \left[ c_l^x \cos(\omega t)\sigma_x + c_l^y \sin(\omega t)\sigma_y \right].
\end{align*}
\]

(12)

As one can see from Fig. 1(a), for parameters satisfying the AFM condition Eq. (11), the Heisenberg scaling is achieved. As a result, the first-order term in the high-frequency expansion makes it possible to construct the \( \sigma_x \) term by commuting the operators in \( \mathcal{V}_c \). This idea can be generalized to many-body systems, as shown in the following.

**Restricted control in a quantum spin chain.** Consider the sensing of magnetic field using a spin chain

\[
H_1 = \frac{J}{2} \sum_{i=1}^{n} \sigma_i^x \sigma_{i+1}^x + \frac{\Delta}{2} \sum_{i=1}^{n} \sigma_i^+ \sigma_{i+1}^+ \sigma_i^- \sigma_{i+2}^- + \frac{\Delta}{2} \sum_{i=1}^{n} \sigma_i^z,
\]

(13)

which contains both two-body and three-body interactions.

The approximate optimal control fulfills \( c_0^x = 0 \) and

\[
\omega = \frac{8}{\Delta} \sum_{l=1}^{\infty} \text{Im}(c_l^x c_l^x) / l,
\]

(11)

which we call the amplitude-frequency matching (AFM) condition. The validity of the high-frequency expansion requires that \( \omega \) should be the largest frequency in the original Hamiltonian, i.e., that \( \omega \gg \lambda, \Delta, c_i^x, c_i^y \). Experimentally, for a laser frequency that satisfies this condition, one can always tune the amplitude so that Eq. (11) is satisfied. Conversely, one can also choose a proper laser frequency for given amplitudes so that Eq. (11) is fulfilled. We emphasize that when Eq. (11) is satisfied and the initial state in the Floquet rotating frame is \( |0\rangle + |1\rangle \)/\( \sqrt{2} \), the optimal control conditions Eqs. (5, 6) are approximately satisfied up to the order of \( 1/\omega \) in the Floquet rotating frame. The initial state in the lab and Floquet rotating frames is the same as \( K(0) = 0 \). Going back to the lab frame, we generate a solution to Eqs. (5, 6) when the controls are restricted. This illustrates the power of our approach beyond the Pang-Jordan protocol [8].

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\]

(13)

which contains both two-body and three-body interactions.

We assume periodic boundary conditions for simplicity and consider the set of allowed controls \( \mathcal{V}_c = \{ \sigma_i^x, \sigma_i^y \sigma_{i+1}^x \} \), \( (a, b = x, y, z) \), involving only one-body and nearest neighbor two-body operators. When the controls are unrestricted, the minimum optimal control Hamiltonian contains local two-body and local three-body terms. The first-part in the control Hamiltonian would cancel the nearest neighbor two-body terms, i.e., \( H_{c,0} = -J/2 \sum_{i=1}^{n} \sigma_i^+ \sigma_{i+1}^+ \). However, the term consisting of three-body operators cannot be canceled directly through the allowed control set \( \mathcal{V}_c \).

Our goal is to construct the three-body operators using the first-order correction in Eq. (9), that is, we expect to induce \( [H_c, H_{c,0}] \propto \sum_{i=1}^{n} \sigma_i^x \sigma_i^y \sigma_{i+1}^z \). The commutator between one-body and two-body operators cannot produce a three-body operator. To generate a three-body operator, we must commute the two-body operators in \( \mathcal{V}_c \) [40]. Therefore, one can construct \( H_c(t) = \sum_{i=1}^{n} H_{c,i} e^{i H_{c,0} t} \), where \( H_{c,i} = \sum_{i=1}^{n} (c_i^x \sigma_i^x \sigma_{i+1}^x + c_{i+2} \sigma_i^x \sigma_{i+1}^z) \), \( c_{i+2} = c_i^y \) and \( c_i^y = c_i^y \) to ensure the Hermiticity of \( H_c(t) \) [40]. When \( c_i^x \) and \( c_i^y \) are respectively real and purely imaginary, the commutator \( [H_{c,i}, H_{c,-i}] = -4 \sum_{i=1}^{n} \text{Im}(c_i^x c_i^y) \sigma_i^x \sigma_i^y \sigma_{i+1}^x \sigma_{i+2}^z \). The most general choice of the coefficients \( c_i^x \) and \( c_i^y \) which generates the \( \sigma^z \)-type three-body interaction is provided in Sec. V in [40]. For simplicity, we further assume these coefficients are homogeneous across the chain and take \( c_i^y = c_i^y \) and \( c_i^y = ic_i^x \) to be real for \( l \geq 1 \). This yields the high frequency control Hamiltonian

\[
H_c^{(d)}(t) = 2 \sum_{l=1}^{\infty} \sum_{i=1}^{n} \left[ c_l^x \cos(\omega t)\sigma_i^x \sigma_{i+1}^y + c_l^y \sin(\omega t)\sigma_i^y \sigma_{i+1}^x \right],
\]

(14)

while the effective Hamiltonian obtained as the leading term in the high-frequency expansion becomes

\[
H_F = \frac{\lambda}{2} \sum_{i=1}^{n} \sigma_i^x + \frac{\Delta}{2} - \frac{4}{\omega} \sum_{i=1}^{n} \frac{1}{\text{Im}(c_i^x c_i^y)} \sigma_i^x \sigma_i^y \sigma_{i+1}^z.
\]

(15)

The three-body term is then canceled by tuning \( c_i^x \) and \( c_i^y \) such that

\[
\omega = \frac{8}{\Delta} \sum_{l=1}^{\infty} \text{Im}(c_l^x c_l^y) / l.
\]

(16)

As with the qubit case, the optimal initial state in both the lab and Floquet rotating is the GHZ state. Equations (5, 6) are thus approximately satisfied up to the order of \( 1/\omega \). As shown in Fig. 1(b-d), for parameters satisfying the AFM condition (16), the Heisenberg scaling \( c_0^x \) is achieved. Furthermore, even when one just takes the lowest two harmonics in
the driving, QFI is not very far below $I_{HS}$. In Fig. 2, $I_{HS}$ is achieved with Eq. (14) when the parameters satisfy Eq. (16). Again, the precision using only the lowest two harmonics can approach $I_{HS}$ very closely.

Finally, we emphasize that a similar technique can be applied to design more general driving protocols to cancel the effect of other types of three-body interactions [40]. Experimental platforms where three-body interactions either appear naturally or can be potentially engineered - such as NMR system [45, 46], Kitaev spin liquid [47], superconducting circuit [48] and quantum gas systems [49–52] - can be potentially used to test the metrological protocol discussed here.

In summary, we have introduced a variational approach to quantum parameter estimation and derived the optimal control equations under which the precision is optimal when the available control Hamiltonians are limited. This approach readily yields the optimal initial state and Hamiltonian controls, that are generally dependent on the probe time, in contrast with the unconstrained case. The implementation of the constrained optimal protocol in many-body systems can be eased by Floquet engineering, as we have demonstrated in applications to magnetometry. We hope that our results inspire new theoretical and technological advances in quantum metrology with quantum many-body systems. Many questions are open for further investigation, such as determining the ultimate scaling bounds of the QFI under restricted local controls and the application of our method to critically-enhanced quantum metrology [53].

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[43] This is the approach pursued in e.g., the variational approach in shortcut-to-adiabaticity [26, 27]. Another way of handling the constraints on the control Hamiltonian is by introducing the following constraints $f_j(H_c(\tau)) = \text{Tr}\left\{H_c(\tau)X_j\right\} = 0$, $j = d_c+1, \cdots, N$ to disallow the terms $X_j$, where $j = 1, 2 \cdots d_c$. This way of introducing the constraint is the one used in e.g. quantum brachistochrone equation [21, 22]. However, in many-body quantum metrology, the number of disallowed nonlocal operators is much more than the allowed local operators. Therefore the second approach may introduce an intractable number of constraints and we shall pursue the first approach of expanding $H_c$ in terms of basis operators in the main text.

[44] We choose the normalization $K(0) = 0$ to all orders of $1/\omega$, which is different from the normalization $1/T \int_0^T K(t)dt = 0$ used in Ref. [33, 36]. Therefore, the resulting expression of $K(t)$ is different from the one in Ref. [33, 36], up to some irrelevant constant, which does not affect the form of the Floquet effective Hamiltonian $H_F$.

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I. Optimization over the initial state

Our goal is to maximize \( \text{Var}[O] \) over all the quantum states \( |\psi_0\rangle \), where \( O \) is some Hermitian operator. We note the following gauge invariance

\[
O \to \tilde{O} \equiv O - c, \tag{S1}
\]

\[
\text{Var}[O] |\psi_0\rangle \to \text{Var}[\tilde{O}] |\psi_0\rangle = \text{Var}[O] |\psi_0\rangle, \tag{S2}
\]

where \( c \) is a constant. Introducing a Lagrange multiplier to account for the normalization of \( |\psi_0\rangle \), we aim at optimizing the following function

\[
f \equiv \text{Var}[\tilde{O}] |\psi_0\rangle - \mu \langle \psi_0 | \psi_0 \rangle
= \langle \psi_0 | \tilde{O}^2 |\psi_0\rangle - \langle \psi_0 | \tilde{O} |\psi_0\rangle^2 - \mu \langle \psi_0 | \psi_0 \rangle. \tag{S3}
\]

One can choose \( c = \langle \psi_0 | O |\psi_0\rangle \), so that \( f \) becomes

\[
f = \langle \psi_0 | \tilde{O}^2 |\psi_0\rangle - \mu \langle \psi_0 | \psi_0 \rangle. \tag{S4}
\]

Taking derivatives with respect to \( |\psi_0\rangle \), one can easily obtain

\[
\tilde{O}^2 |\psi_0\rangle = \mu |\psi_0\rangle. \tag{S5}
\]

All eigenstates of \( \tilde{O} \) must be an eigenstate of \( \tilde{O}^2 \), but not vice versa. This is because if there is a pair of opposite eigenvalues of \( \tilde{O} \), the eigenstate of \( \tilde{O}^2 \) could be the superposition of the eigenstates of \( \tilde{O} \) corresponding to these two opposite eigenvalues. Therefore, either \( |\psi_0\rangle \) is an eigenstate of \( \tilde{O} \) or \( |\psi_0\rangle \) is the superposition of two eigenstates of \( \tilde{O} \) with opposite eigenvalues. In the former case \( f = 0 \) corresponds to the extremal minimum value. In the latter case, i.e.,

\[
|\psi_0\rangle = \alpha |\psi_a\rangle + \beta |\psi_b\rangle, \tag{S6}
\]

where

\[
O |\psi_a\rangle = \mu_a |\psi_a\rangle, \tag{S7}
\]

\[
O |\psi_b\rangle = \mu_b |\psi_b\rangle, \tag{S8}
\]

\( \mu_a, \mu_b \) are not equal to each other and satisfy

\[
\mu_a + \mu_b - 2 \langle \psi_0 | O |\psi_0\rangle = 0. \tag{S9}
\]
Then, the function \( f \) becomes

\[
f(a, b) = a\mu_a^2 + b\mu_b^2 - (a\mu_a + b\mu_b)^2,
\]

where \( a \equiv |a|^2 \) and \( b \equiv |b|^2 \). We proceed to introduce a Lagrangian multiplier to perform further maximization over \( a \) and \( b \)

\[
g(a, b) = a\mu_a^2 + b\mu_b^2 - (a\mu_a + b\mu_b)^2 + \xi(a + b - 1).
\]

Setting \( \partial_a g(a, b) = \partial_b g(a, b) = 0 \), one finds \( a = b = 1/2 \). Therefore, we find \( f(a, b) = (\mu_+-\mu_-)^2/4 \). Finally, optimizing over all the possible distinct eigenvalues of \( \mathcal{O} \), we find the global maximum values of \( f \). That is, \( f_{\text{max}} = (\mu_+ - \mu_-)^2/4 \), with the corresponding optimal initial state \( |\varphi_0\rangle = (|\varphi_+\rangle + e^{\theta} |\varphi_-\rangle)/\sqrt{2} \), where \( |\varphi_{\pm}\rangle \) correspond to the maximum and minimum eigenvalues \( \mu_{\pm} \), respectively, and \( \theta \) is an arbitrary phase.

**II. VARIATIONAL CALCULUS**

For the variation with respect to \( U \) we have

\[
\delta S = \delta S_M + \delta S_S = 0,
\]

where

\[
\delta S_M = -\int_0^{t_f} \text{Tr} \{\Delta \rho (\tau) \delta U(\tau) U(\tau) \partial_\lambda H_\lambda(\tau) \delta U(\tau)\} d\tau + \int_0^{t_f} \text{Tr} \{\Delta \rho U(\tau) \delta \partial_\lambda H_\lambda(\tau) \delta U(\tau)\} d\tau,
\]

\[
\delta S_S = i\text{Tr} \{U(\tau) \Lambda(\tau) \delta U(\tau)\} \bigg|_{\tau=0}^{\tau=t_f} - i \int_0^{t_f} d\tau \text{Tr} \left\{U(\tau) \Lambda(\tau) \partial_\lambda H_\lambda(\tau) U(\tau) + \frac{d}{d\tau} \left(U(\tau) \Lambda(\tau)\right)\right\} \delta U(\tau)
\]

\[
= i\text{Tr} \{U(\tau) \Lambda(\tau) \delta U(\tau)\} \bigg|_{\tau=0}^{\tau=t_f} - i \int d\tau \text{Tr} \left\{U(\tau) \left[\Lambda U U^\dagger + \Lambda + U U^\dagger \Lambda\right] \delta U(\tau)\right\},
\]

where we have used the fact that \( \delta U^\dagger = -U^\dagger \delta U U^\dagger \) in Eq. (S13). Eq. (S12) yields

\[
\Lambda(\tau) - i[\Lambda(\tau), H_{\text{tot}}(\tau)] + i[\delta \rho(\tau), \partial_\lambda H_\lambda(\tau)] = 0,
\]

where

\[
\Delta \rho(\tau) \equiv U(\tau) \Delta \rho U^\dagger(\tau),
\]

\[
H_{\text{tot}}(\tau) \equiv i\dot{U}(\tau) U^\dagger(\tau) = H_\lambda(\tau) + H_\epsilon(\tau),
\]

with the essential boundary condition

\[
\Lambda(t_f) = 0.
\]

We observe that Eq. (S15) may be solved in terms of the evolution operator because Eq. (S15) is a driven Liouville equation and the Green’s function is the product of unitary evolution operators for the forward and backward directions multiplied by the Heaviside function. To see this in a clear manner, let us note that

\[
\Lambda_{\text{norm}}(\tau) = U(\tau, t_f) \Lambda(t_f) U^\dagger(\tau, t_f) = 0,
\]

where \( U(\tau, s) \) is the evolution operator from time \( s \) to time \( \tau \), which satisfies

\[
i\dot{U}(\tau, s) = H_{\text{tot}}(\tau) U(\tau, s),
\]

where the time derivative is applied to the variable \( \tau \). One can explicitly check that for the impulse applied at time \( s \), the Green’s function is

\[
G(\tau, s) = -U(\tau, s) \Theta(s - \tau) U^\dagger(\tau, s), \quad s \in [0, t_f],
\]
which satisfies
\[
\partial_t \mathcal{G}(\tau, s) = i[\mathcal{G}(\tau, s), H_{\text{tot}}(\tau)] + \delta(\tau - s),
\] (S22)
and the homogeneous initial condition \( \mathcal{G}(t_f, s) = 0 \). Note that we do not contract \( U(\tau, s) \) and \( U^\dagger(\tau, s) \) in Eq. (S21). Therefore, we find
\[
\Lambda(\tau) = \Lambda_{\text{inhom}}(\tau) = i\int_0^\tau U(\tau, s)[\Delta \rho(s), \partial_\lambda H_s(s)] \Theta(s - \tau) U^\dagger(\tau, s) ds
= iU(\tau)[\Delta \rho, \int_\tau^\tau U(s) \partial_\lambda H_s(s) U(s) ds] U^\dagger(\tau)
= iU(\tau)[\Delta \rho, G_{t_f}[U] - G_t[U]] U^\dagger(\tau)
= -iU(\tau)[\Delta \rho, G_t[U]] U^\dagger(\tau),
\] (S23)
where we have used the fact that \( [\Delta \rho, G_{t_f}[U]] = 0 \). We note the alternative concise expression for the generator \[8\]
\[
G_{t_f}[U] = iU^\dagger(\tau) \partial_\lambda U(\tau).
\] (S24)
Substituting Eq. (S24) into Eq. (S23) yields,
\[
\Lambda(\tau) = U(\tau) \Delta \rho U^\dagger(\tau) \partial_\lambda U(\tau) U^\dagger(\tau) - \partial_\lambda U(\tau) \Delta \rho U^\dagger(\tau)
= -[U(\tau) \Delta \rho \partial_\lambda U^\dagger(\tau) + \partial_\lambda U(\tau) \Delta \rho U^\dagger(\tau)] = -\partial_\lambda [\Delta \rho(\tau)],
\] (S25)
where we have used \( \partial_\lambda U(\tau) U^\dagger(\tau) = -U(\tau) \partial_\lambda U^\dagger(\tau) \). Therefore, the trace condition \( \text{Tr} \{\Lambda(\tau) \mathcal{X}_f\} = 0 \) becomes Eq. (6) in the main text.

Mathematically, we note that \( \Delta \rho \) is independent of the estimation of \( \lambda \) from the very beginning when we construct the metrological action Eq. (3) in the main text. Practically, after solving the optimization problem, one may find that the optimal value of \( \Delta \rho \) depends on \( \lambda \), since \( \Delta \rho \) is related the maximum and minimum eigenvectors of \( G_{t_f}[U] \), which depends on \( \lambda \) in general. If this is the case, the optimal initial state should be prepared with the prior knowledge of the estimation parameter, which is very close to the true value of \( \lambda \), as we focus on the ultra-sensitive estimation regime.

III. DEPENDENCE OF \( U_{t_f}(\tau) \) ON THE PROBE TIME \( t_f \)

Let us first calculate explicitly \( \partial_\lambda G_{t_f} \). We note the generator \( G_{t_f} \) has the following two alternative forms
\[
G_{t_f} = iU^\dagger_{t_f}(t_f) \partial_\lambda U_{t_f}(t_f) = \int_0^{t_f} U^\dagger_{t_f}(\tau) \partial_\lambda H_s(\tau) U_{t_f}(\tau) d\tau.
\] (S26)
Then it is straightforward to calculate
\[
\partial_\lambda G_{t_f} = \int_0^{t_f} \partial_\lambda \left[ U^\dagger_{t_f}(\tau) \partial_\lambda H_s(\tau) U_{t_f}(\tau) \right] d\tau + U^\dagger_{t_f}(t_f) \partial_\lambda H_s(t_f) U_{t_f}(t_f)
= \partial_\lambda \left[ iU^\dagger_{t_f}(t_f) \partial_\lambda U_{t_f}(t_f) \right] + U^\dagger_{t_f}(t_f) \partial_\lambda H_s(t_f) U_{t_f}(t_f)
= i\partial_\lambda U^\dagger_{t_f}(t_f) \partial_\lambda U_{t_f}(t_f) + iU^\dagger_{t_f}(t_f) \partial_\lambda U_{t_f}(t_f) + U^\dagger_{t_f}(t_f) \partial_\lambda H_s(t_f) U_{t_f}(t_f)
\] (S27)
where \( \partial_\lambda \) denotes the derivative with respect the subscript \( t_f \) instead of the one in the parenthesis.

When \( |\varphi_{a,t_f}\rangle \) is independent of \( t_f \), after multiplying both sides by \( U_{t_f}(t_f) \) from the left, Eq. (8) in the main text becomes
\[
\partial_\lambda \tilde{G}_{t_f} \langle \varphi_{a,t_f}(t_f) | \rangle = \partial_\lambda \mu_{a,t_f} |\varphi_{a,t_f}(t_f)\rangle, \quad \forall t_f, \ a = \pm,
\] (S28)
where \( |\varphi_{a,t_f}(t_f)\rangle = U_{t_f}(t_f) |\varphi_a\rangle \), and
\[
\partial_\lambda \tilde{G}_{t_f} \equiv U_{t_f}(t_f) \partial_\lambda G_{t_f} U^\dagger_{t_f}(t_f)
= i\partial_\lambda \left[ \partial_\lambda U_{t_f}(t_f) U^\dagger_{t_f}(t_f) \right] + \partial_\lambda H_s(t_f)
= i\partial_\lambda \left[ \partial_\lambda U_{t_f}(t_f) U^\dagger_{t_f}(t_f) \right] + \partial_\lambda H_s(t_f)
= \partial_\lambda \left[ H_s(t_f) + A_{t_f}(t_f) \right].
\] (S29)
Here, \( A_{i,j}(t_j) \) is defined as
\[
A_{i,j}(\tau) \equiv -iU_{i,j}(\tau)\partial_\tau U^\dagger_{i,j}(\tau),
\]  
(S30)
which bears the same form as \( G_{i,j} \) and therefore has the integral representation
\[
A_{i,j}(\tau) = -\int_0^\tau U^\dagger_{i,j}(s)\partial_\tau H_{i,j}(s)U_{i,j}(s)ds.
\]  
(S31)

The physical meaning of \( A_{i,j}(\tau) \) is the following: When \( A_{i,j}(\tau) \) vanishes as in the Pang-Jordan protocol [8], Eq. (S28) implies that \( |\varphi_{i,j}(\tau)\rangle \) is always an eigenstates of \( \partial_\tau H_j(\tau) \), as can be seen from Eq. (7) in the main text, given that \( \Delta \rho_{i,j}(\tau) \) always commutes with \( \partial_\tau H_j(\tau) \) for any \( \tau \in [0, t_f] \). Eq. (S28) indicates that, as long as long as \( [\partial_\tau A_{i,j}(t_j), \partial_\tau H(t_j)] \neq 0, \Delta \rho_{i,j}(t_f) \) does not commute with \( \partial_\tau H_j(t_f) \) in general, which in turn implies that \( \partial_\tau [\Delta \rho_{i,j}(\tau)] \) does not vanishes at all times. Therefore Eq. (6) in the main text becomes non-trivial in the sense that \( X_j \) must be orthogonal to the non-vanishing values of \( \partial_\tau \Delta \rho_{i,j}(\tau) \) at least for some time \( \tau \).

**IV. GENERATOR AND OPTIMAL CONTROL IN PARAMETER-INDEPENDENT ROTATING FRAME**

For a parameter-independent unitary operator \( \mathcal{U}(t) \), consider the interaction frame associated with \( \mathcal{U}(t) \) and make the transformation
\[
\tilde{\mathcal{U}}(t) = \mathcal{U}(t)\mathcal{U}(t).
\]  
(S32)
Then, the Schrödinger equation becomes
\[
i\partial_\tau \tilde{\mathcal{U}}(t) = \tilde{H}_{\text{tot}}(t)\tilde{\mathcal{U}}(t),
\]  
(S33)
where
\[
\tilde{H}_{\text{tot}}(t) = \mathcal{U}(t)H_{\text{tot}}(t)\mathcal{U}^\dagger(t) - i\mathcal{U}(t)\partial_\tau \mathcal{U}^\dagger(t).
\]  
(S34)
Using Eqs. (S32, S34), the generator can be rewritten as
\[
G_{i,j} = \int_0^{t_f} \tilde{\mathcal{U}}^\dagger(\tau)\partial_\tau H_j(\tau)\tilde{\mathcal{U}}(\tau)d\tau
\]
\[
= \int_0^{t_f} \tilde{\mathcal{U}}^\dagger(\tau)\partial_\tau \mathcal{U}(\tau)H_{\text{tot}}(t)\mathcal{U}^\dagger(\tau) - i\mathcal{U}(\tau)\partial_\tau \mathcal{U}^\dagger(\tau)\tilde{\mathcal{U}}(\tau)d\tau
\]
\[
= \int_0^{t_f} \tilde{\mathcal{U}}^\dagger(\tau)\partial_\tau H_{\text{tot}}(t)\tilde{\mathcal{U}}(\tau)d\tau,
\]  
(S35)
where we have used the important fact that \( \mathcal{U}(t) \) is independent of the parameter \( \lambda \). Eq. (6) in the interaction frame can be rewritten as
\[
\text{Tr} [X_j\partial_\lambda [\Delta \rho(\tau)]] = \text{Tr} \{\tilde{X}_j(\tau)\partial_\lambda [\Delta \tilde{\rho}(\tau)]\},
\]  
(S36)
where
\[
\tilde{X}_j(\tau) \equiv \mathcal{U}(\tau)X_j\mathcal{U}^\dagger(\tau),
\]  
(S37)
\[
\Delta \tilde{\rho}(\tau) \equiv \tilde{\mathcal{U}}(\tau)\Delta \rho\tilde{\mathcal{U}}^\dagger(\tau).
\]  
(S38)

**V. COMMUTATORS IN MANY-SPIN SYSTEMS**

In this section, we discuss several properties for many-spin systems. We define the \( k \)-body Hermitian basis operator
\[
X_k^{\alpha_k} = \sigma_{i_1}^{\alpha_{i_1}}\sigma_{i_2}^{\alpha_{i_2}}\cdots\sigma_{i_k}^{\alpha_{i_k}},
\]  
(S39)
where the distinct indices \( i_1 < i_2, \cdots < i_k \) take values in \([1, n]\), \( \alpha_i \in \{x, y, z\} \) and \( \alpha_k \equiv (\alpha_{i_1}, \cdots, \alpha_{i_k}) \). The 0–body operator \( X_0 \) is defined to be a constant. We first discuss a lemma concerning the general property of the commutator between a \( k \)–body operator and a \( l \)–body operator. We denote by \( p \) the number of identical subscripts for two many-body operators \( X_k^{\alpha_k} \) and \( X_l^{\alpha_l} \). For \( p = 0 \), we know \([X_k^{\alpha_k}, X_l^{\alpha_l}] = 0 \). For positive \( p \), we have the following lemma:
Lemma 1. Consider two many-body spin operators, where $X_k = \sigma_i^{\alpha_k} \sigma_j^{\alpha_k} \cdots \sigma_l^{\alpha_k}$ and $X_l = \sigma_i^{\alpha_l} \sigma_j^{\alpha_l} \cdots \sigma_l^{\alpha_l}$. Without loss of generality, assume $i_1 = j_1, \ldots, i_p = j_p$ with the remaining subscripts being distinct and $\alpha_{i_1} = \alpha_{j_1}, \ldots, \alpha_{i_q} = \alpha_{j_q}$ with the remaining $p-q$ pairs of superscripts being distinct, i.e., $\alpha_{i_{q+1}}, \ldots, \alpha_{i_p} \neq \alpha_{j_q}, \ldots, \alpha_{j_p}$. Then,

$$\{X_k, X_l\} = \begin{cases} 0 & p-q = \text{even} \\ X_{k+i-(p-q)} & p-q = \text{odd} \end{cases}, \quad (S40)$$

where $p \in [0, \min(k, l)]$ and $q \in [0, p]$.

Proof. It is straightforward calculate

$$\{X_k, X_l\} = \left[ \prod_{r=1}^{p} (\sigma_i^{\alpha_r} \sigma_j^{\alpha_r}) - \prod_{r=1}^{q} (\sigma_i^{\alpha_r} \sigma_j^{\alpha_r}) \right] \sigma_{i_1}^{\alpha_{i_1}} \cdots \sigma_{i_q}^{\alpha_{i_q}} \sigma_{i_{q+1}}^{\alpha_{i_{q+1}}} \cdots \sigma_{i_p}^{\alpha_{i_p}} \sigma_{i_{q+1}}^{\alpha_{i_{q+1}}} \cdots \sigma_{i_p}^{\alpha_{i_p}}.$$

With the identity $\sigma_i^{\alpha_r} \sigma_j^{\alpha_r} = \delta^{\alpha_r \alpha_r} + i\epsilon_{\alpha_r \alpha_r \beta} \sigma_i^{\beta}$, where $\epsilon_{\alpha \beta \gamma}$ is the Levi-Civita symbol, we find

$$\sigma_i^{\alpha_r} \sigma_j^{\alpha_r} = \begin{cases} 1 & \alpha_i = \alpha_j \\ -\sigma_i^{\alpha_r} \sigma_j^{\alpha_r} & \alpha_i \neq \alpha_j \end{cases}. \quad (S42)$$

Therefore,

$$\prod_{r=1}^{p} (\sigma_i^{\alpha_r} \sigma_j^{\alpha_r}) = \prod_{r=1}^{q} (\sigma_i^{\alpha_r} \sigma_j^{\alpha_r}) = \prod_{r=1}^{q} (\sigma_i^{\alpha_r} \sigma_j^{\alpha_r}) (-1)^{p-q}, \quad (S43)$$

and we conclude that

$$\prod_{r=1}^{p} (\sigma_i^{\alpha_r} \sigma_j^{\alpha_r}) - \prod_{r=1}^{q} (\sigma_i^{\alpha_r} \sigma_j^{\alpha_r}) = \begin{cases} 0 & p-q = \text{even} \\ \prod_{r=1}^{q} \sigma_i^{\alpha_r} & p-q = \text{odd} \end{cases}, \quad (S45)$$

which completes the proof.

We next consider the following commutator

$$\left[ \sum_{i=1}^{n} z_i^{\alpha \beta} \sigma_i^{\alpha \beta}, \sum_{j=1}^{n} w_j^{\gamma \delta} \sigma_j^{\gamma \delta} \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} z_i^{\alpha \beta} w_j^{\gamma \delta} \left[ \sigma_i^{\alpha \beta}, \sigma_j^{\gamma \delta} \right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i^{\alpha \beta} w_j^{\gamma \delta} \left[ \sigma_i^{\alpha \beta}, \sigma_j^{\gamma \delta} \right] + \sum_{i=1}^{n} \sum_{j=1}^{n} z_i^{\alpha \beta} w_j^{\gamma \delta} \sigma_i^{\gamma \delta} \left[ \sigma_j^{\alpha \beta}, \sigma_j^{\alpha \beta} \right]$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} z_i^{\alpha \beta} w_j^{\gamma \delta} \sigma_i^{\gamma \delta} \left[ \sigma_i^{\alpha \beta}, \sigma_j^{\alpha \beta} \right] \sigma_j^{\alpha \beta} \sigma_j^{\alpha \beta} \sigma_j^{\alpha \beta}, \quad (S46)$$

where $z_i^{\alpha \beta}$ and $w_j^{\gamma \delta}$ are complex numbers and we have used

$$\sum_{i=1}^{n} z_i^{\alpha \beta} w_j^{\gamma \delta} \left[ \sigma_i^{\alpha \beta}, \sigma_j^{\gamma \delta} \right] = \sum_{i=1}^{n} z_i^{\alpha \beta} w_j^{\gamma \delta} \left[ \sigma_i^{\alpha \beta}, \sigma_j^{\gamma \delta} \right] \sigma_i^{\alpha \beta} \quad (S47)$$
With Eqs. (S46, S48), one finds thanks to the periodic boundary condition. Using Eq. (S49), it is straightforward to obtain

\[ \left[ \sum_{i=1}^{n} z_{i}^\alpha \sigma_{i}^\alpha \sigma_{i+1}^\alpha + \sum_{j=1}^{n} z_{j}^\beta \sigma_{j}^\beta \sigma_{j+1}^\beta \right] \]

\[ = \sum_{i=1}^{n} z_{i}^\alpha \sigma_{i}^\alpha \sigma_{i+1}^\alpha + \sum_{j=1}^{n} z_{j}^\beta \sigma_{j}^\beta \sigma_{j+1}^\beta \]

\[ = - 2i \epsilon_{\alpha \beta \gamma} \sum_{i=1}^{n} (z_{i}^\alpha \sigma_{i}^\alpha - z_{i}^\beta \sigma_{i}^\beta) \sigma_{i+1}^\gamma \sigma_{i+2}^\beta \]

\[ = 4 \epsilon_{\alpha \beta \gamma} \sum_{i=1}^{n} \text{Im}(z_{i}^\alpha \sigma_{i}^\alpha) \sigma_{i+1}^\gamma \sigma_{i+2}^\beta. \]  

(S48)

With Eqs. (S46, S48), one finds

\[ \left[ \sum_{i=1}^{n} z_{i}^\alpha \sigma_{i}^\alpha \sigma_{i+1}^\alpha + \sum_{j=1}^{n} w_{j}^\gamma \sigma_{j}^\gamma \sigma_{j+1}^\gamma \sum_{j=1}^{n} z_{j}^\beta \sigma_{j}^\beta \sigma_{j+1}^\beta + \sum_{j=1}^{n} w_{j}^\delta \sigma_{j}^\delta \sigma_{j+1}^\delta \right] \]

\[ = 4 \epsilon_{\alpha \beta \gamma} \sum_{i=1}^{n} \text{Im}(z_{i}^\alpha \sigma_{i}^\alpha) \sigma_{i+1}^\gamma \sigma_{i+2}^\beta + 4 \epsilon_{\alpha \beta \gamma} \sum_{i=1}^{n} \text{Im}(w_{i}^\alpha \sigma_{i+1}^\alpha) \sigma_{i+1}^\gamma \sigma_{i+2}^\beta \]

\[ - 4 \epsilon_{\alpha \beta \gamma} \sum_{i=1}^{n} \text{Im}(z_{i}^\alpha \sigma_{i+1}^\alpha) \sigma_{i+1}^\gamma \sigma_{i+2}^\beta - 4 \epsilon_{\alpha \beta \gamma} \sum_{i=1}^{n} \text{Im}(w_{i}^\alpha \sigma_{i+1}^\alpha) \sigma_{i+1}^\gamma \sigma_{i+2}^\beta + X_{\alpha \beta \gamma}^{\epsilon}, \]  

(S49)

where we have used the fact that \([A + B, A^\dagger + B^\dagger] = [A, A^\dagger] + [B, B^\dagger] + ([A, B^\dagger] + \text{h.c.})\) and that

\[ X_{\alpha \beta \gamma}^{\epsilon} \equiv 2i \sum_{i=1}^{n} \text{Im}(z_{i}^\alpha \sigma_{i}^\alpha) \sigma_{i+1}^\gamma \sigma_{i+2}^\beta. \]

(S50)

is at most two-body.

To cancel the three-body interaction \(\sigma_i^\alpha \sigma_{i+1}^\alpha \sigma_{i+2}^\alpha\) in Eq. (13) in the main text, we would like the second term of Eq. (S46) to be the three body operator. One may take \(\alpha = x, \beta = y, \gamma = z\) and \(\delta = x\) in Eq. (S46). This results in

\[ \left[ \sum_{i=1}^{n} z_{i}^x \sigma_{i}^x \sigma_{i+1}^x + \sum_{j=1}^{n} w_{j}^z \sigma_{j}^z \sigma_{j+1}^z \right] = 2i \sum_{i=1}^{n} \text{Im}(z_{i}^x \sigma_{i}^x) \sigma_{i+1}^z \sigma_{i+2}^x. \]  

(S51)

where have used the fact that

\[ [\sigma_i^\alpha \sigma_{i+1}^\alpha, \sigma_j^\gamma \sigma_{j+1}^\gamma] = 0, \]

(S52)

\[ \sigma_i^\alpha [\sigma_{i+1}^\alpha, \sigma_{i+2}^\alpha] = 0, \]

(S53)

according to Eq. (S45). Thus, if we take

\[ H_{c, \text{I}} = \sum_{i=1}^{n} [c_{i}^{x} \sigma_{i}^{x} \sigma_{i+1}^{x} + c_{i}^{x} \sigma_{i}^{x} \sigma_{i+1}^{x}] \]

(S54)

discussed in the main text, according to Eq. (S49), one can easily find

\[ [H_{c, \text{I}}, H_{c, \text{II}}] = 4 \sum_{i=1}^{n} \text{Im}(c_{i}^{x} \sigma_{i}^{x} \sigma_{i+1}^{x} \sigma_{i+2}^{y}) \]

\[ + 4 \sum_{i=1}^{n} \text{Im}(c_{i}^{x} \sigma_{i}^{x} \sigma_{i+1}^{y} \sigma_{i+2}^{x}) \]

\[ - 4 \sum_{i=1}^{n} \text{Im}(c_{i}^{x} \sigma_{i}^{x} \sigma_{i+1}^{x} \sigma_{i+2}^{x}). \]  

(S55)

To cancel the three-body interactions in Eq. (13), we would like the coefficients in the first and second terms on the r.h.s. of Eq. (S55) to vanish, while the coefficient in last term on the r.h.s. does not vanish. To determine the conditions for this to be the case, we first introduce the following lemma
Lemma 2. Given a set of complex number \( \{z_i\}_{i=1}^n \) and periodic boundary condition \( z_{n+1} = z_1 \), the condition

\[
\text{Im}(z_i z_{i+1}^*) = 0, \quad \forall i
\]  

(S56)

is satisfied if and only if

\[
\frac{\text{Re}(z_i)}{\text{Im}(z_i)} = \alpha,
\]

(S57)

with \( \alpha \) independent of the index \( i \).

Proof. Assuming Eq. (S57), Eq. (S56) is obvious. The other direction can be proved by separating \( z_i \) into real and imaginary part, i.e., \( z_i = u_i + iv_i \). Eq. (S56) leads to \( u_i/v_i = u_{i+1}/v_{i+1}, \forall i \). We may thus set \( u_i/v_i = \alpha \), where \( \alpha \) is real constant and independent of the index \( i \). This concludes the proof.

Therefore, according Lemma 2, in order to obtain vanishing coefficients in the first and second terms on the r.h.s. of Eq. (S55), the coefficients should take the following form

\[
e_{li}^{xy} = (\alpha_l^{xy} + i)\nu_{li}^{xy},
\]

\[
e_{li}^{zx} = (\alpha_l^{zx} + i)\nu_{li}^{zx}.
\]

(S58)

(S59)

As a result, we obtain

\[
\text{Im}(e_{li}^{xy} c_{li+1}^{zx}*) = (\alpha_l^{xy} - \alpha_l^{zx})\nu_{li}^{xy}\nu_{li+1}^{zx}.
\]

(S60)

Thus, it is necessary that \( \alpha_l^{zx} \neq \alpha_l^{xy} \) to have non-vanishing \( [H_{c,l}, H_{c,-l}] \). The choice in the main text corresponds to the case where \( \alpha_l^{xy} = \infty \) and \( \alpha_l^{zx} = 0 \), so that \( e_{li}^{xy} \) is real and \( e_{li}^{zx} \) is purely imaginary.

VI. MORE GENERAL DRIVING PROTOCOLS

In this section, we generalize the mechanism of the cancellation of the the \( \sigma_y \)-type interactions discussed in the main text to arbitrary type of three-body interactions, \( J_{\text{ext}}/2 \sum_{i=1}^n \sigma_i^{\lambda} \sigma_i^{\mu} \sigma_i^{\nu} \). We employ Eqs. (S46, S48, S49) and Lemma 2 to construct the high frequency drive \( H^{(d)}(i) \) in the main text. Note that when assuming two-body controls are accessible, any two-body interaction out of the commuting \( H_{c,l} \) and \( H_{c,-l} \) \((l \geq 1)\) can be cancel by adding control to the static control \( H_{c,0} \). Therefore, one should not concern about the last term \( \Delta(x,y) \) on the r.h.s. of Eq. (S49), which is at most two-body interaction according to Lemma 1. Now let us discuss case by case according to the index degeneracy.

1. For three-body interaction \( J_{\text{ext}}/2 \sum_{i=1}^n \sigma_i^{\lambda} \sigma_i^{\mu} \sigma_i^{\nu} \), we take \( \omega = \delta = \kappa, \beta = \lambda \) and \( \gamma = \mu \) in Eq. (S46), where \( \kappa, \mu, \lambda \) are distinct and \( [\sigma_i^{\lambda}, \sigma_i^{\mu}] = 2i\sigma_i^{\delta} \), so that the second term on the r.h.s. generates \( \sigma_i^{\mu} \sigma_i^{\nu} \). Then Eq. (S46) becomes

\[
\left[ \sum_{i=1}^n e_i^{kl} \sigma_i^{\phi} \sigma_i^{\lambda}, \sum_{j=1}^n w_{ij}^{\mu
u} \sigma_j^{\mu} \sigma_j^{\nu}\right] = 2i \sum_{i=1}^n e_i^{kl} w_{ij}^{\mu
u} \sigma_i^{\mu} \sigma_i^{\nu} \sigma_i^{\lambda} + \sum_{i=1}^n w_{ij}^{\mu
u} \sigma_i^{\mu} \sigma_i^{\nu} \sigma_i^{\lambda}.
\]

(S62)

We construct

\[
H_{c,l} = \sum_{i=1}^n \left( e_i^{kl} \sigma_i^{\lambda} \sigma_i^{\lambda} + w_{ij}^{\mu
u} \sigma_i^{\mu} \sigma_i^{\nu} \sigma_i^{\lambda} \right)
\]

(S63)

and assume the coefficients \( c_i^{kl} \) and \( c_i^{\mu
u} \) satisfy Eq. (S57). Then according to Eq. (S49) and Lemma 2, one can easily find

\[
[H_{c,l}, H_{c,-l}] = -4 \sum_{i=1}^n \text{Im}(c_i^{kl} c_{li+1}^{\mu
u})\sigma_i^{\lambda} \sigma_i^{\lambda} \sigma_i^{\nu} \sigma_i^{\nu}.
\]

(S64)

Assuming the coefficients are independent of \( i \), the three-body interaction \( J_{\text{ext}} \sum_{i=1}^n \sigma_i^{\lambda} \sigma_i^{\mu} \sigma_i^{\nu} \) can be cancelled if

\[
\frac{J_{\text{ext}}}{2} = \sum_{i=1}^n i \text{Im}(c_i^{kl} c_{li+1}^{\mu
u}).
\]

(S65)
2. For three-body interaction \(J_{\text{three-body}}/2 \sum_{i=1}^{n} \sigma_i^\alpha \sigma_i^\beta \sigma_i^\gamma\), where \(\alpha \neq \beta\), we take \(\alpha = \delta = \kappa, \beta = \lambda, \gamma = \kappa\) in Eq. (S46), where \(\kappa, \mu, \lambda\) and \(\lambda\) are distinct and \([\sigma_i^\kappa, \sigma_i^\lambda] = 2i\epsilon_{\kappa\mu\lambda} \sigma_i^\mu\). Eq. (S46) becomes

\[
\left[ \sum_{i=1}^{n} \epsilon_{i}^{\kappa} \sigma_i^\alpha \sigma_{i+1}^\alpha, \sum_{j=1}^{n} w_{j}^{\kappa\kappa} \sigma_j^\alpha \sigma_j^\alpha \right] = \lambda^{\kappa\kappa}_{2} + 2i\epsilon_{\kappa\mu\lambda} \sum_{i=1}^{n} \epsilon_{i}^{\kappa} w_{i}^{\kappa\kappa} \sigma_i^\kappa \sigma_{i+1}^\kappa \sigma_{i+2}^\kappa, \tag{S66}
\]

where the two-body interaction \(\lambda^{\kappa\kappa}_{2}\) can be always cancelled through \(H_{c,0}\). One can then construct

\[
H_{c,l} = \sum_{i=1}^{n} \left( c_{i}^{\kappa} \sigma_i^\alpha \sigma_{i+1}^\alpha + c_{i}^{\kappa} \sigma_i^\alpha \sigma_{i+1}^\alpha \right). \tag{S67}
\]

According to Eq. (S49), one can readily find

\[
[H_{c,l}, H_{c,-l}] = 4\epsilon_{\kappa\mu\lambda} \sum_{i=1}^{n} \text{Im}(c_{i}^{\kappa} w_{i}^{\kappa\kappa}) \sigma_i^\alpha \sigma_{i+1}^\alpha \sigma_{i+2} + \lambda^{\kappa\kappa}_{2} \tag{S68}
\]

with a similar AFM condition (S65).

3. For three-body interaction \(J_{\text{three-body}}/2 \sum_{i=1}^{n} \sigma_i^\alpha \sigma_i^\beta \sigma_i^\lambda\), where \(\alpha, \beta, \lambda\), and \(\alpha\) are distinct, we take \(\alpha = \gamma = \kappa, \beta = \delta = \lambda\) in Eq. (S46) and obtain

\[
\left[ \sum_{i=1}^{n} \epsilon_{i}^{\kappa} \sigma_i^\alpha \sigma_{i+1}^\beta, \sum_{j=1}^{n} w_{j}^{\kappa\kappa} \sigma_j^\alpha \sigma_j^\alpha \right] = 2i\epsilon_{\kappa\mu\lambda} \sum_{i=1}^{n} \epsilon_{i}^{\kappa} w_{i}^{\kappa\kappa} \sigma_i^\kappa \sigma_{i+1}^\kappa \sigma_{i+2}^\kappa + 2i\epsilon_{\kappa\mu\lambda} \sum_{i=1}^{n} \epsilon_{i}^{\kappa} w_{i}^{\kappa\kappa} \sigma_i^\kappa \sigma_{i+1}^\kappa \sigma_{i+2}^\kappa = -2i\epsilon_{\kappa\mu\lambda} \sum_{i=1}^{n} \left[ \epsilon_{i}^{\kappa} w_{i}^{\kappa\kappa} \right] \sigma_i^\beta \sigma_{i+1}^\beta \sigma_{i+2}^\beta, \tag{S69}
\]

which suggests us to use the two-body interaction \(\sigma_i^\beta \sigma_{i+1}^\beta\) as the only drive. Indeed, if we take

\[
H_{c,l} = \sum_{i=1}^{n} c_{i}^{\kappa} \sigma_i^\alpha \sigma_{i+1}^\beta, \tag{S70}
\]

for \(l \geq 1\) then according to Eq. (S48), we find

\[
[H_{c,l}, H_{c,-l}] = 4\epsilon_{\kappa\mu\lambda} \sum_{i=1}^{n} \text{Im}(c_{i}^{\kappa} v_{i}^{\kappa\kappa}) \sigma_i^\kappa \sigma_{i+1}^\kappa \sigma_{i+2}^\kappa. \tag{S71}
\]

According to Lemma 2, as long as the ratio \(\text{Re}(c_{i}^{\kappa\kappa})/\text{Im}(c_{i}^{\kappa\kappa}) \equiv \alpha_{i}^{\kappa\kappa}\) varies across the chain, the three-body interaction is non-zero. Denoting

\[
c_{i}^{\kappa\kappa} = (\alpha_{i}^{\kappa\kappa} + i)v_{i}^{\kappa\kappa}, \tag{S72}
\]

we find

\[
\text{Im}(c_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}) = (\alpha_{i}^{\kappa\kappa} - \alpha_{i}^{\kappa\kappa})v_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}. \tag{S73}
\]

A simple example could be \(\alpha_{i}^{\kappa\kappa} = \alpha_{i}^{\kappa\kappa} + \Delta \alpha_{i}^{\kappa\kappa} + v_{i}^{\kappa\kappa} = v_{i}^{\kappa\kappa}\) for \(i = 1, \cdots n - 1\). Therefore \(\alpha_{i}^{\kappa\kappa} = \alpha_{i}^{\kappa\kappa} + i\Delta \alpha_{i}^{\kappa\kappa}\) and

\[
\text{Im}(c_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}) = \Delta \alpha_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}, \tag{S74}
\]

for \(i = 1, \cdots n - 1\). Now we shall choose \(\alpha_{i}^{\kappa\kappa}\) and \(v_{i}^{\kappa\kappa}\) such that

\[
\text{Im}(c_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}) = \left[ \alpha_{i}^{\kappa\kappa} - \alpha_{i}^{\kappa\kappa} - (n - 2)\Delta \alpha_{i}^{\kappa\kappa} \right] v_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa} = \Delta \alpha_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}, \tag{S75}
\]

\[
\text{Im}(c_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}) = \left[ \alpha_{i}^{\kappa\kappa} - \alpha_{i}^{\kappa\kappa} \right] v_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa} = \Delta \alpha_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa} v_{i}^{\kappa\kappa}, \tag{S76}
\]

from which one finds

\[
\alpha_{i}^{\kappa\kappa} = \alpha_{1} - \left( 1 - \frac{n}{2} \right) v_{i}^{\kappa\kappa}, \tag{S77}
\]

\[
v_{i}^{\kappa\kappa} = \frac{2\Delta \alpha_{i}^{\kappa\kappa}}{2 - n}. \tag{S78}
\]
4. Finally, it turns out, up to the first-order of the high frequency expansion, there is no simple scheme to cancel the interaction of the type $\sigma^\epsilon_i \sigma^\epsilon_{i+1} \sigma^\epsilon_{i+2}$. However the combination $1/2 \sum_{i=1}^n (J_{\kappa \lambda \mu} \sigma^\epsilon_i \sigma^\epsilon_{i+1} + J_{\kappa \mu \lambda} \sigma^\mu_i \sigma^\mu_{i+1}) \sigma^\lambda_{i+2}$, where $\kappa$, $\mu$, and $\lambda$ are distinct and $J_{\kappa \kappa \lambda} = J_{\kappa \mu \mu}$, can be cancel in the following simple way. Taking $\alpha = \kappa, \beta = \lambda, \gamma = \mu$, and $\delta = \lambda$, one finds

$$[\sum_{i=1}^n \sigma^\epsilon_i \sigma^\epsilon_{i+1} \sigma^\epsilon_{i+2} + \sum_{j=1}^n \sigma^\mu_j \sigma^\mu_{j+1} \sigma^\mu_{j+2}] = 2i \epsilon_{ijk} \sum_{i=1}^n (\sigma^\lambda_i \sigma^\gamma_i \sigma^\mu_i + \sigma^\mu_i \sigma^\lambda_i \sigma^\gamma_i + \sigma^\gamma_i \sigma^\mu_i \sigma^\lambda_i + \sigma^\lambda_i \sigma^\gamma_i \sigma^\mu_i) + \chi^{\mu \lambda \nu}.$$  

(S80)

So if we take

$$H_{c,i} = \sum_{i=1}^n (\epsilon^{\mu \lambda}_i \sigma^\mu_i \sigma^\lambda_{i+1} + \epsilon^{\mu \lambda}_i \sigma^\mu_i \sigma^\lambda_{i+1}),$$

(S81)

and assume the coefficients $\epsilon^{\mu \lambda}_i$ and $\epsilon^{\mu \lambda}_i$ satisfy Eq. (S57), the according to Eq. (S49), one find

$$[H_{c,i}, H_{c,-i}] = -4 \epsilon_{ijk} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n [\text{Im}(\epsilon^{\mu \lambda}_i \epsilon^{\mu \lambda}_{i+1} \sigma^\mu_i \sigma^\lambda_{i+1} \sigma^\mu_{i+2}) + \text{Im}(\epsilon^{\mu \lambda}_i \epsilon^{\mu \lambda}_{i+1} \sigma^\mu_i \sigma^\lambda_{i+1} \sigma^\mu_{i+2})] + \chi^{\mu \lambda \nu}$$  

(S82)

Assuming $\epsilon^{\mu \lambda}_i$ and $\epsilon^{\mu \lambda}_i$ are homogeneous, since satisfy Eq. (S57), they can take the form $\epsilon^{\mu \lambda}_i = (\alpha_i^{\mu \lambda} + i) u_i^{\mu \lambda}$ and $\epsilon^{\mu \lambda}_i = (\alpha_i^{\mu \lambda} + i)v_i^{\mu \lambda}$ respectively. Therefore

$$\text{Im}(\epsilon^{\mu \lambda}_i \epsilon^{\mu \lambda}_{i+1}) = (\alpha_i^{\mu \lambda} - \alpha_{i+1}^{\mu \lambda}) u_i^{\mu \lambda} v_{i+1}^{\mu \lambda},$$

(S83)

$$\text{Im}(\epsilon^{\mu \lambda}_{i+1} \epsilon^{\mu \lambda}_i) = (\alpha_i^{\mu \lambda} - \alpha_{i+1}^{\mu \lambda}) u_{i+1}^{\mu \lambda} v_i^{\mu \lambda}.$$  

(S84)

Then one can take $u_i = \delta_{i,1}$ and $v_i = 0$ for $i = 1$ and $v_{i+1} = 0$. Then one can easily show

$$\text{Im}(\epsilon^{\mu \lambda}_i \epsilon^{\mu \lambda}_{i+1}) = (\alpha_i^{\mu \lambda} - \alpha_{i+1}^{\mu \lambda}) u_i^{\mu \lambda} v_{i+1}^{\mu \lambda},$$

(S85)

$$4 \epsilon_{ijk} \sum_{i=1}^n (\alpha_i^{\mu \lambda} - \alpha_{i+1}^{\mu \lambda}) u_i^{\mu \lambda} v_{i+1}^{\mu \lambda} = J_{\kappa \kappa \lambda}/2.$$  

(S86)

Similar construction also holds for the three-body interaction $1/2 \sum_{i=1}^n \sigma^\epsilon_i \sigma^\epsilon_{i+1} \sigma^\epsilon_{i+2} + J_{\kappa \mu \lambda} \sigma^\mu_i \sigma^\mu_{i+1} \sigma^\lambda_{i+2})$ with $J_{\kappa \kappa \lambda} = J_{\kappa \mu \mu}$, which will not be discussed here.