Constraints on Chaotic Inflation from the *COBE DMR* Results

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**Abstract**

We explore constraints on various forms for the effective potential during inflation based upon a statistical comparison between inflation-generated fluctuations in the cosmic microwave background temperature and the *COBE DMR* results. Fits to the *COBE* $53A + B \times 90A + B$ angular correlation function are obtained using simple analytic forms for the effective potential in chaotic inflation models. Not surprisingly, these fits are optimized for a nearly scale-free fluctuation power spectrum. However, from the $\chi^2$ distribution for the fits we can set upper limits of $n \leq 1.2$ and $n \leq 7$ at the $1\sigma$ and $2\sigma$ confidence levels, respectively, for a $V(\phi) = \lambda\phi^n$ effective potential. Similarly, new limits on parameters for polynomial effective potentials can be determined at the $1\sigma$ and $2\sigma$ confidence levels. The most stringent constraint, however, is on the overall magnitude of the effective potential.

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I. INTRODUCTION

Measurements of the large-scale anisotropy in the cosmic microwave background (CMB) by the Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) experiment have provided important support for the hot big bang model with inflation (e.g. [1]-[2]). A favored explanation for the generation of the observed fluctuations in the CMB temperature is by the expansion of quantum fluctuations of a scalar field during the inflationary epoch [3]. The fact that the observed angular correlation function is more or less consistent with a scale-free Harrison-Zel’dovich spectrum of power on various angular scales is consistent with the predictions of inflationary models. This is particularly true since the fluctuations resolved by COBE are larger than the horizon at recombination and not yet distorted from the inflation-generated spectrum by gravitational clustering on subhorizon scales.

However, the COBE observations are not exactly scale free (i.e. preferring a power law index which deviates slightly from unity), and neither are the predictions of inflation. This is because inflation occurs as the universe is rolling down the effective potential. Thus, the amount of inflation is slightly different for different angular scales. The change in angular anisotropy of the cosmic microwave background from scales of a few degrees to the full sky is, therefore, a measurement of the rate of change in the effective potential during that short interval of inflation for which those angular scales were stretched beyond the apparent horizon. That the amplitude of the fluctuations depends upon the overall amplitude of the effective potential is well known [4]. In this paper we discuss how the functional form of the potential is further constrained from the observed angular correlation function.

Indeed, it is in principle possible to even reconstruct the inflation effective potential from a knowledge of the fluctuation spectrum [5]. However, such a reconstruction is subject to large uncertainties and is probably not possible in the foreseeable future [5]. In this paper, therefore we take a different approach. That is, we consider specific analytic forms for the potential and quantify the range of parameters for these potentials which are statistically compatible with the observations. In this way, we can use the COBE angular correlation function to make a quantitative assessment of possible inflation-generating potentials. Our work also differs from previous studies (e.g. [5], [6]) in that we are using the COBE data only. That is, we do not consider the large scale clustering of galaxies which also contains information on the angular correlation on yet smaller scales. However, since fluctuations on smaller scales have experienced gravitational clustering, this generalization requires a knowledge of the influence of dark-matter components. By restricting ourselves to the COBE data alone, we avoid the necessity to make any assumptions about the nature of dark matter. We also avoid any possible confusion from the mixing of dissimilar data sets.

Moreover, the shape of the inflation effective potential is not known. A large number of different models, with different potentials, some with many free parameters, have been proposed, e.g. [4]. While most models lead to a spectrum which is nearly scale invariant, this scale-invariance is not exact, and some models lead to significant deviations from scale invariance. Of course, the constraint which can be placed upon effective potentials is limited by the uncertainties in the observed correlation function, and effects of cosmic variance [7]. Nevertheless, these data represent the only direct observation of the shape of the inflation effective potential, and the statistical analysis of these data as described here
provides at least some information as to which inflation models can be excluded.

For this study we consider chaotic inflation [8] in particular, with two simple forms for the effective potential,

\[ V(\phi) = \frac{1}{n!} \lambda \phi^n \]  
(1)

and

\[ V(\phi) = \lambda \left( \frac{1}{8} \beta \phi^2 + \frac{1}{3} \alpha \phi^3 + \frac{1}{4} \phi^4 \right), \]  
(2)

where we use Planck units, \( c = m_{Pl} = 1 \). As a function of \( n \) (or \( \alpha, \beta \)) and \( \lambda \), we do a search to find the minimum \( \chi^2 \) and projected confidence limits for the parameters, i.e. we use the goodness of fit to the \textit{COBE} correlation function as a criterion to constrain the degree to which the parameters of these models can be excluded.

**II. METHOD**

The generation of the CMB anisotropy in inflationary models has been discussed extensively in the literature (e.g. [3], [4]). A brief outline of how the results in this paper were calculated is as follows.

The equations governing the evolution of the scalar field and the universal expansion are,

\[ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0 \]  
(3)

and

\[ H^2 \equiv \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi}{3} \left[ V(\phi) + \frac{1}{2} \dot{\phi}^2 \right]. \]  
(4)

For simplicity, we assume instantaneous reheating as the scalar field approaches the minimum of the potential.

The perturbations responsible for the CMB anisotropy originate from quantum fluctuations which are stretched beyond the apparent horizon during inflation. Two kinds of fluctuation can contribute to the CMB. Scalar fluctuations (i.e. fluctuations of the inflaton \( \phi \)) become density perturbations. Tensor fluctuations (i.e. fluctuations of spacetime curvature) become gravitational waves. The waves of interest here have wavelengths of order the horizon scale. The amplitudes for a multipole expansion of the CMB anisotropy can be written,

\[ \langle a_l^2 \rangle = \langle a_l^2 \rangle_S + \langle a_l^2 \rangle_T, \]  
(5)

where the scalar contribution is

\[ \langle a_l^2 \rangle_S = \frac{2l+1}{25\pi} \int_0^{\omega_{\text{max}}} \frac{d\omega}{\omega} j_l(\omega)^2 \frac{H^4}{\dot{\phi}^2}, \]  
(6)

and the tensor contribution is

\[ \langle a_l^2 \rangle_T = 36l(l-1)(l+1)(l+2) \left( \frac{2l+1}{l(l+1)} \right) \int_0^{\omega_{\text{max}}} d\omega \omega F_l(\omega)^2 H^2, \]  
(7)
\[ F_l(\omega) = \int_0^{s_{\text{dec}}} ds \left\{ -j_2[\omega(1-s)] \right\} \left[ \frac{2}{(2l-1)(2l+3)} j_l(\omega s) \right. \\
+ \left. \frac{1}{(2l-1)(2l+1)} j_{l-2}(\omega s) + \frac{1}{(2l+1)(2l+3)} j_{l+2}(\omega s) \right] \]  

where \( \omega_{\text{max}} = 2(1 + Z_{\text{eq}})^{1/2} \) and \( s_{\text{dec}} = 1 - (1 + Z_{\text{dec}})^{-1/2} \). The integrals in eqs. (5-7) are over scales \( \omega \equiv kr_0 \), where \( k \) is the comoving wavelength of the perturbation, and \( r_0 = 2/H_0 \) is the radius of the presently observable universe. We used \( H_0 = 50 \text{ km/s/Mpc} \) and the values \( Z_{\text{eq}} = 10500 \) and \( Z_{\text{dec}} = 1100 \) for the redshifts of matter-radiation equality and of hydrogen recombination, respectively. The quantities \( H^4/\dot{\phi}^2 \) and \( H^2 \) are evaluated at the epoch during inflation, when the scale in question exits the horizon (\( k = H \)).

It should be noted, that the use of \( H^2/\dot{\phi}^2 \) and \( H^2 \) in the integrands for the fluctuation amplitudes is based upon perturbation theory results obtained for exponentially expanding spacetimes, and is thus an approximation. However, the solution of Eqs. (3) and (4), as well as the integrals (6), (7), and (8), were done numerically. Thus, we do not make use of the ‘slow roll over’ approximation.

To compare to the COBE DMR results, we convert from multipole expansion to the angular correlation function with the COBE resolution [1],

\[ C(\theta) = \frac{1}{4\pi} \sum_{l>2} \langle a_l^2 \rangle W_l^2 P_l(\cos \theta), \]  

where

\[ W_l^2 = \exp \left[ -\frac{l(l+1)}{17.8^2} \right]. \]  

The cosmic variance of the correlation function is

\[ \delta C(\theta)^2 = \frac{1}{(4\pi)^2} \sum_{l>2} \frac{2}{2l+1} \langle a_l^2 \rangle^2 \left[ W_l^2 P_l(\cos \theta) \right]^2. \]

We have found that it is sufficient to calculate multipoles up to \( l = 40 \).

### III. RESULTS

Fig. 1 shows the \( 53A + B \times 90A + B \) COBE angular correlation function of Smoot et al. [1] compared with the scale free spectrum, i.e., \( |\delta|^2 \propto k \). We get a fit for a scale free spectrum (including both the cosmic variance and the measurement uncertainties in the weighting) with a minimum \( \chi^2 = 48.0 \). For potentials of the type of eq. (1) we also get a minimum \( \chi^2 = 48.0 \). This occurs for \( n \lesssim 0.1 \). These give spectra almost identical to the scale-free one. For potentials of the type of eq. (2), the extra parameter allows us to find a slightly lower \( \chi^2 = 47.8 \). Since there are 70 measured points, a two-parameter fit of the correct model should have \( \chi^2 \lesssim 68 \). Thus the \( \chi^2 \) per degree of freedom for the fit to the COBE correlation function is exceedingly good and may indicate that there is
less scatter than one would expect from a purely statistical distribution of cosmic variance and experimental error. We note that our minimum $\chi^2$ is slightly less than that quoted in [1] for a pure power-law spectrum with a Gaussian distribution of primordial fluctuations. This difference arises from restricting the current analysis to only the $53A + B \times 90A + B$ cross correlation function which we had available.

For each $n$ or $(\alpha, \beta)$, we have done inflation calculations of the fluctuation spectrum as a function of the potential amplitude, $\lambda$, to find the best fit (minimum $\chi^2$) to the COBE correlation function. Fig. 2 shows $\chi^2$ as a function of $n$ and contours of optimum values for the amplitude $\lambda$ for chaotic inflation models with effective potentials of the form

$$V(\phi) = \frac{1}{n!} \lambda \phi^n.$$  

(12)

Note that larger values for $n$ give a worse fit to the COBE results. A smaller $n$ means a smaller slope in the potential when the relevant scales are generated, and thus a flatter (closer to scale free) spectrum. Indeed, the correlation function for $n = 0.1$ (lowest value on Fig. 2) cannot be distinguished by eye from the completely scale-free case (Fig. 1). Since the scale-free spectrum fits the COBE results so well, a larger deviation from the scale-free spectrum obtained for the larger $n$ leads to a worse fit to the COBE results.

In Fig. 3 we show the correlation function for $n = 4$, which is the usually assumed self coupling [8] in chaotic inflation models and still gives a fairly good fit to the COBE results. Fig. 4 shows an example of a poor fit for which $n = 1000$. Here the inflation-generated correlation function deviates significantly from the COBE correlation function, particularly for angular scales less than 50°.

The confidence limits for the parameters of these fits, $n$ and $\lambda$, are shown as two-parameter contours of $\chi^2$ in Fig. 2. For a two-parameter search, this just corresponds to the locus of points for which $\chi^2$ increases by 2.3 (1$\sigma$ or 68% C.L.) or 6.2 (2$\sigma$ or 95% C.L.) as a function of $n$ and $\lambda$. These regions include values of $n \leq 3.0$ (1$\sigma$) and $n \leq 22$ (2$\sigma$). The confidence limits on $n$ alone, allowing for any $\lambda$, are however tighter [10], corresponding to $\Delta \chi^2 = 1.0$ and 4.0. Thus we obtain the upper limits $n \leq 1.2$ (1$\sigma$ or 68% C.L.) and $n \leq 7$ (2$\sigma$ or 95% C.L.). On the other hand, assuming a fixed $n$, $\lambda$ is determined to better than 25% (1$\sigma$), or 50% (2$\sigma$) accuracy.

It is gratifying that a $\chi^2$ analysis does imply optimum fits for relatively small coupling orders of the scalar field, and that the lower the order the better the fit. This is consistent with what one expects physically since a self coupled scalar field in 4 dimensions is not renormalizable unless $n \leq 4$ [9].

The most general renormalizable potential with just one scalar field can be written,

$$V(\phi) = \lambda \left( \frac{1}{5} \beta \phi^2 + \frac{1}{3} \alpha \phi^3 + \frac{1}{4} \phi^4 \right).$$  

(13)

Although more complicated forms including, for example, one-loop radiative corrections are possible [9,3,4], this form of the potential is sufficiently general for our purpose as it can represent the leading terms in an expansion of a more complicated potential. This term has been previously discussed by Hodges et al. [12] in the context of generating non-Zel’dovich spectra over scales of galactic clustering. Here we apply it to the scales sampled by the COBE correlation function.
As in [12], we can assume with no loss of generality that the initial value for the scalar field is greater than the global minimum. This is equivalent to a negative initial value and an $\alpha \rightarrow -\alpha$ coordinate transformation.

Following Hodges et al., we can exclude immediately values for $\alpha$ and $\beta$ such that,

$$\alpha > 0, \beta < \frac{8}{9} \alpha^2,$$  \hspace{1cm} (14)

and

$$\alpha < 0, \frac{8}{9} \alpha^2 < \beta < \alpha^2,$$ \hspace{1cm} (15)

for which a false-vacuum secondary minimum of $V(\phi)$ occurs for positive nonzero values of the scalar field. Such cases are excluded as, either the universe becomes trapped in the false vacuum and cannot exit inflation, or (for a small false vacuum) they produce unacceptable large-scale structure due to the wall energy associated with nucleated bubbles.

We present our results as a contour plot on the $(\alpha, \beta)$-plane in Fig. 5. As noted by Hodges et al. [12], a significant deviation from a scale-free spectrum occurs only as one approaches the line $\beta = \alpha^2$, $\alpha < 0$ in Fig. 5. This is the upper thick line on the left of figure 5. The thin lines on Fig. 5 identify contours of optimum values of $\lambda$ in the $\beta$-$\alpha$ plane as labeled. Also shown on Figure 5 is the shaded region of excluded parameter space at the 1$\sigma$ level based upon the COBE data. The 2$\sigma$ 95.4% C. L. excluded region is too small to distinguish from the lines on Fig. 5. We, therefore, show this region greatly expanded in Fig. 6 reparameterized [12] as $\beta' = (\beta - \alpha^2)/\alpha^6$ vs. $\alpha$.

In the excluded region, there is a flattening of $V(\phi)$ to the right of the global minimum. This extends the period of inflation near small values of $\phi$ where normally little inflation would occur. This causes a deviation of the correlation function on the largest angular scales relative to the smallest scales. This is evidenced in Fig. 7 which shows the predicted correlation function for the extreme case of $\alpha = -0.9, \beta' = 0.00063$.

IV. CONCLUSIONS

Although there remain large uncertainties from the cosmic variance and from measurement errors of the angular correlation function for fluctuations in the microwave background, we have shown that it is possible to place significant constraints on the effective potential for chaotic inflation based upon the COBE DMR data alone. Based upon a $\chi^2$ analysis of the goodness of fit, and assuming a normal distribution of measurement errors and cosmic variance we conclude that only a chaotic inflation effective potential with relatively low orders of self coupling is consistent with the observed correlation function. This in a small way adds credence to the chaotic inflation scenario with a single scalar field by the simple fact that, if the data had required a large power of $\phi^n$, then the implied potential would not have been renormalizable. We have also shown that the physics of the underlying potential is constrained in that terms in the effective potential which produce too much flattening near the global minimum are excluded by the data at the 1 and 2 $\sigma$ level.

Although most of this discussion has centered on the power-law for the effective potential, it is also worth noting that the strongest constraints are actually upon the overall
amplitude of the effective potential. This is consistently constrained to be $\lesssim 10^{-12}$, although larger values can occur for some polynomial models (cf. Fig. 6).

Ultimately there is a limit as to how much the effective potentials can be constrained due to the unavoidable effects of cosmic variance. Indeed, it should eventually be possible to identify the dispersion in the correlation function due to the cosmic variance alone. This would be a definitive confirmation of the inflationary scenario. Nevertheless, it is also clear that even stronger constraints on the inflation-generating effective potential than those remarked here will be possible as the observed correlation function is better determined.

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Figure Captions

FIG. 1. The *COBE DMR 53A + B × 90A + B* angular correlation function (points) compared with an inflation-generated scale-free spectrum (central solid line). The upper and lower solid lines illustrate the effects of cosmic variance.

FIG. 2. Computed $\chi^2$ (thick line) and optimum $\lambda$ (dashed line) as a function of $n$ for a chaotic inflation potential with $V(\phi) = \lambda \phi^n$. The thin lines enclose the 68% and 95% confidence regions in the $(n, \lambda)$ parameter plane. The scales for $\chi^2$ and $\lambda$ are on the left and right of this figure respectively.

FIG. 3. The *COBE DMR 53A + B × 90A + B* angular correlation function (points) compared with a chaotic inflation $V(\phi) = \lambda \phi^4$ spectrum (central solid line). The upper and lower solid lines illustrate the effects of cosmic variance.

FIG. 4. The *COBE DMR 53A + B × 90A + B* angular correlation function (points) compared with a chaotic inflation $V(\phi) = \lambda \phi^n$ spectrum (central solid line), where $n = 1000$. The upper and lower solid lines illustrate the effects of cosmic variance.

FIG. 5. Excluded regions of the $\alpha$ vs. $\beta$ plane for a polynomial chaotic inflation potential. The lightly shaded region shows the excluded parameter space at the 1$\sigma$ level based upon fits to the *COBE DMR* results. As in [12] we also exclude the darker shaded regions with $\beta < 8\alpha^2/9$, $\alpha > 0$ (lower right), and $8\alpha^2/9 < \beta < \alpha^2$ (upper left), for which there is a secondary minimum in $V(\phi)$ for positive $\phi$. The thin lines show contours of optimum $\lambda$ for each $(\alpha, \beta)$ as labeled.

FIG. 6. Region of excluded parameter space (shaded region) at the 2$\sigma$ level based upon fits to the *COBE DMR* results using a polynomial chaotic inflation potential. Parameters are plotted in the expanded $\alpha$ vs. $\beta' = (\beta - \alpha^2)/\alpha^6$ plane. The thin lines show contours of optimum $\lambda$ for each $(\alpha, \beta)$ as labeled.

FIG. 7. The *COBE DMR 53A + B × 90A + B* angular correlation function (points) compared with a chaotic inflation spectrum (central solid line) generated with a polynomial effective potential with parameters $\alpha$ and $\beta'$ as indicated.