Minimal Network Coding for Multicast

Kapil Bhattad∗, Niranjan Ratnakar†, Ralf Koetter‡, and Krishna R. Narayanan∗
∗Texas A & M University, College Station, TX. Email: kbhattad,krn@ee.tamu.edu
†University of Illinois, Urbana Champaign, IL. Email: ratnakar,koetter@uiuc.edu

Abstract—We give an information flow interpretation for multicasting using network coding. This generalizes the fluid model used to represent flows to a single receiver. Using the generalized model, we present a decentralized algorithm to minimize the number of packets that undergo network coding. We also propose a decentralized algorithm to construct capacity achieving multicast codes when the processing at some nodes is restricted to routing. The proposed algorithms can be coupled with existing decentralized schemes to achieve minimum cost multicast.

I. INTRODUCTION

In their seminal work, Ahlswede et al [1] showed that if the nodes in the network are allowed to perform network coding rather than just routing then the max flow min cut bound on the multicast capacity is achievable. Li et al [2] showed that linear codes are sufficient to achieve the multicast capacity. Since then several techniques have been proposed to design codes that achieve the multicast capacity. Among them, the idea of random network coding seems very promising. Ho et al [3], [4] propose a scheme in which data is collected in the form of packets of, say, length n. These packets are then treated as elements of a finite field of size $q = 2^n$ (assuming that the data is in bits) and they show that if the messages on the outgoing edges of every node are set to be a random linear combination of the messages received along the incoming edges over a finite field of size q then the probability that the resulting code is not a valid multicast code is $O(1/q)$. (We call a multicast valid if the destination nodes can decode the data.) Therefore a valid multicast code can be designed with very high probability by random coding over a large field.

Random network coding by itself could be inefficient in terms of network resources. Since the scheme is completely distributed and there is no communication between the nodes, each node sends messages on all its outgoing edges in the process using up all the available bandwidth. But, this problem can be solved. In [5], [6] Lun et al proposed a distributed algorithm which can be used, for example, to find a sub network that minimizes link usage costs while having the same multicast capacity as the given network. Random network coding can be employed on this sub network to achieve the multicast capacity.

In general, minimal cost network coding solutions are of practical interest. The cost to be minimized may depend on the network and application at hand. For example, if a router that employs network coding is expensive we will want to minimize the number of nodes that perform network coding. In optical networks the operation of computing linear combination of inputs may require conversion from optical signals to electrical signals which is expensive and hence we may want to minimize the number of packets that undergo network coding. Random network coding as such would result in schemes where every node performs network coding. In this paper, we will address the problem of minimal cost network coding where the cost is the number of packets that need to be network coded. We also consider the problem of finding minimum cost solutions when some of the nodes are restricted to perform only routing. The multicast capacity for a special case of this problem when all the nodes only route has been studied in [7]. We will refer to nodes employing network coding by network coding nodes and nodes restricted to routing by routing nodes.

In [5], the authors consider costs such as bandwidth and delay and investigate minimum cost multicast. However, the results in [5] cannot be directly used to solve the problems considered in this paper because the fluid model used to represent flows to individual receivers cannot be used when some of the nodes are restricted to routing. It is also not possible to differentiate between the operations of network coding and routing at a node by only looking at the input and output flows of that node. The main contribution of this paper is to give a new information flow based interpretation for the multicast flow and use this model to set up optimization problems that can be solved in a distributed manner.

The optimization problem formulated in this paper has a complexity that grows exponentially with the number of receivers but in many applications like video conferencing the number of receivers is quite small and hence these algorithms can be of practical use.

In section II we give the notation used in the paper. We present the new information flow model in section III. In section IV we set up the optimization problems and finally conclude in section V.

II. NOTATION

We represent a network by a directed graph $G = (V, E)$, where $V$ is the set of vertices (nodes) and $E$ is the set of edges (links). The capacity of edge $e \in E$ is given by $C(e)$. For each node $v \in V$ we define sets $E_{in}(v)$ and $E_{out}(v)$ as the set of all edges that come into $v$ and that go out of $v$ respectively.

We consider a multicast problem with one source $S \subset V$ and $K$ receivers in the set $D \subset V$. We assume that $D = \{1, 2, \cdots, K\}$. For convenience we define two sets $\mathcal{P}$ and $\mathcal{Q}$ where $\mathcal{P}$ is the power set of $D$ (neglecting the
empty set) and $Q$ is a set containing all collections of two or more disjoint sets in $P$. For example, when $K = 3$, $P = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$ and $Q = \{\{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{2\}, \{3\}\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}\}$. We fix the ordering in $P$ and $Q$ and represent the $i$-th element in $P$ and the $j$-th element in $Q$ by $P_i$ and $Q_j$ respectively.

With each edge $e \in E$ and a coding scheme we associate a $2K - 1$ length information flow vector $X_e$, where the $i$-th element denoted by $x_e(P_i)$ represents the amount of information common to and only common to receivers in the set $P_i$ that flows through the edge $e$. We define $I_k(X_e) = \sum_{i,k \in P_i} x_e(P_i)$ as the amount of flow along edge $e$ in the flow decomposition of receiver $k$. The definitions will be made precise in Section III.

It is sometimes convenient to assume that the edges capacities and the flow vectors are integers. This assumption is justified since we can always consider the network over multiple time instances.

### III. INFORMATION FLOW

In a multicast setup, any multicast solution can be decomposed into flows to individual receivers [1], [2]. The flows to different receivers could overlap. Overlapping flows indicates that the data sent along the overlapping part of the flow has to be eventually conveyed to all the receivers whose flows overlap.

The main idea here is to partition the flows to the individual receivers as components of the form $x_e(P_i)$. We formally do this in the rest of the section. We define $x_e((k_1, k_2, \cdots , k_j))$ for an edge $e$ as the amount of overlap in the flows along $e$ from the source node to the receiver nodes $k_1, k_2, \cdots , k_j$ and that does not overlap with any other flow for any other receiver.

To identify the overlapping flows, consider a network obtained by expanding the original network by replacing each edge $e$ by parallel edges, $e_1', \cdots , e_{C(e)}'$, of unit capacity (assuming edge capacities are integers). The expanded network also supports the same rate (h, also assumed to be an integer) as the original network and hence $h$ edge disjoint paths from source to receiver $k$ for each $k$ can be found [1], [2]. The paths to the different receivers could have overlapping edges. For an edge $e'$ in the expanded network, let $P_i$ be the set of all receivers that have edge $e'$ in one of their paths. The element $x_{e'}(P)$ in the information flow vector for $e'$ is then 1 for $P = P_i$ and 0 otherwise. If no paths pass through $e'$ its information flow vector is zero. The information flow vector for edge $e$ in the original network is the sum of the information flow vectors of the parallel edges $e_1', \cdots , e_{C(e)'}$.

To keep the notation brief we also use $x_e((k_1, k_2, \cdots , k_j))$ with $k_1 < k_2 < \cdots < k_j$ to represent $x_e((k_1, k_2, \cdots , k_j))$. It is easy to see that the flow to receiver $k$ along edge $e$ is given by $\sum_{i,k \in P_i} x_e(P_i)$. Since this is a function of $X_e$, for each $k$, we represent it by $I_k(X_e)$. We show the flow vector and information flow vector for some multicast networks in Example 1.

#### Example 1

Consider the network shown in Fig. 1a. A code that achieves the multicast capacity is shown in Fig. 1b. The flows to the two receivers are shown in Fig. 1b. In Fig. 1c the information flow vector for each edge is shown. The information flow vector is (1,0,0) when the edge carries data at unit rate for receiver 1, is (0,1,0) when the edge carries data at unit rate for receiver 2 and (0,0,1) when the edge carries data at unit rate meant for both the receivers. In Fig. 1d we show a code over two time instances that achieves the routing capacity of the network [7] and in Fig. 1e we show the corresponding flows. We note that the edge between node 4 and node 3 has flow for both the receivers but they are not overlapping flows. It is easy to verify in Fig. 1f, 1g, 1h, and 1i that $I_1(X_e)$ and $I_2(X_e)$ gives the amount of flow along edge $e$ to receiver 1 and 2 respectively.

The amount of data flowing along an edge $e$ is the sum of the elements of $X_e$. Since each edge has a capacity constraint, we have the following constraint on $X_e$.

$$\text{sum}(X_e) = b^T X_e \leq C(e) \quad \forall e \in E$$

where $b$ is the all one vector of length $2K - 1$. We denote the constraint in (1) as the edge constraint.

#### Theorem 1

In any multicast network that supports a rate $h$, we can find $X_e$ for every edge $e \in E$ satisfying the edge constraint such that

$$\sum_{e \in E_{in}(v)} I_k(X_e) = \sum_{e \in E_{out}(v)} I_k(X_e) \quad \forall k, v \in V - \{S, k\}$$

$$\sum_{e \in E_{in}(S)} I_k(X_e) = \sum_{e \in E_{out}(k)} I_k(X_e) = h \quad \forall k$$

**Proof:** It is possible to decompose any multicast code into $h$ flows to individual receivers [1]. Consider the $X_e$’s and $I_k(X_e)$’s corresponding to one such flow decomposition. $I_k(X_e)$ is the amount of flow along edge $e$ in the flow decomposition of receiver $k$. The equations in (2) claim that in the flow decomposition of receiver $k$, the flow coming into any intermediate node is equal to the flow coming out of the
node, the flow coming out of the source node is \( h \), and, the flow into receiver \( k \) is \( h \). These are well known properties of the flow decomposition [1].

It is convenient to define \( X^v \) and \( X^w \) for node \( v \in V \) as

\[
\sum_{e \in E}(v) X_e \ 	ext{ and } \sum_{e \in E}(v) X_e
\]

respectively. To keep the notation brief we will drop the superscript \( v \) in discussions involving just one node. Since \( I \) is a linear function of \( X \), the conditions in (2) reduce to

\[
I_k(X^v) = I_k(X^w) \forall k, \forall v \in V - \{S, k\}
I_k(X^w) = h \forall k
\]

We will call the necessary conditions in (3) as flow constraints.

In Fig. 1 we can easily see that the edge and the flow constraints are satisfied.

### A. Routing, Replicating and Network Coding

Let us take a closer look at the different operations that occur in a node in a multicast network. In Fig. 1 we see that there are three different operations that happen at a node. The first and simplest operation is when a packet is routed to one of the output edges. The second type of operation is replication in which multiple copies of the packet are sent along different edges. The third operation, network coding, refers to the case when two or more packets are combined into one packet. We will see that these three operations are sufficient to represent any necessary processing being done at the node but before that we need to understand what the different operations represent.

We first look at routing and replication. Each packet that comes into a node has an associated set of receivers \( Q \subset D \). The packet has to eventually reach each node in \( Q \). When it gets routed onto one of the output edges, the packet on the output edge still has to reach all nodes in \( Q \). In terms of the information flows to the various receivers, this corresponds to the case when overlapping flows or a simple flow passes through a node and continues unaffected.

When a packet gets replicated, then each copy of the packet on the output edge has to reach nodes in \( P_i \) a subset of \( Q \). \( P_i \) has to be a subset of \( Q \) since the packet has to reach only nodes in \( Q \). Since the packet has to reach all nodes in \( Q \) we have \( \cup P_i = Q \). Moreover, the same packet does not need to reach the same destination along two different paths. Therefore the \( P_i \)'s are disjoint \( P_{i1} \cap P_{i2} = \emptyset \ \forall i_1 \neq i_2 \). In the flow decomposition replication corresponds to the point where two or more overlapping flows diverge.

For example, consider the node 3 in Fig. 1. The incoming packet has to be sent to both node 1 and node 2. \( X^v_{in} = [0, 0, 1] \). The node replicates it and forwards it to two edges. Along one of the edges that packet reaches node 1 \( X^{(3,1)}_{in} = [1, 0, 0] \) and the packet sent on the other edge is meant for node 2 \( X^{(3,2)}_{in} = [0, 1, 0] \). At the output \( X^v_{out} = [1, 1, 0] \).

This concept becomes clearer when we look at the relationship between \( X_{in} \) and \( X_{out} \). We consider the case for two and three destinations and then generalize the results. When there are two destinations replication occurs only when a packet meant for both destinations is replicated and sent on two different paths, one path for each receiver node. \( x_{in}(1, 2) \) represents the average number of packets coming in per unit time that need to go to both 1 and 2. If \( r(r \geq 0) \) of these packets are duplicated and transmitted per unit time we have

\[
\begin{align*}
\text{x}_{out}(1) &= \text{x}_{in}(1) + r \\
\text{x}_{out}(2) &= \text{x}_{in}(2) + r \\
\text{x}_{out}(1, 2) &= \text{x}_{in}(1, 2) - r
\end{align*}
\]

Now consider the case with three destinations. Similar to the two receiver case, a packet meant for two destinations can get replicated to produce two packets for the two destinations. Let \( r_1, r_2 \) and \( r_3 \) represent the amount of replication corresponding to flows to receiver sets \{1, 2\}, \{1, 3\}, and \{2, 3\} respectively. When packets meant for all three receivers replicate, they split the flow in four possible ways \{1,2,3\}, \{1,2,3\}, \{2,1,3\} and \{3,1,2\}. Let \( r_4, r_5 \) and \( r_7 \) represent the number of packets replicated per unit time corresponding to the four cases. The relation between the \( X_{in} \) and \( X_{out} \) is therefore given by

\[
\begin{align*}
\text{x}_{out}(1) &= \text{x}_{in}(1) + r_1 + r_2 + r_4 + r_5 \\
\text{x}_{out}(2) &= \text{x}_{in}(2) + r_1 + r_3 + r_4 + r_6 \\
\text{x}_{out}(3) &= \text{x}_{in}(3) + r_2 + r_3 + r_4 + r_7 \\
\text{x}_{out}(1, 2) &= \text{x}_{in}(1, 2) - r_1 + r_7 \\
\text{x}_{out}(1, 3) &= \text{x}_{in}(1, 3) - r_2 + r_6 \\
\text{x}_{out}(2, 3) &= \text{x}_{in}(2, 3) - r_3 + r_5 \\
\text{x}_{out}(1, 2, 3) &= \text{x}_{in}(1, 2, 3) - r_4 - r_5 - r_6 - r_7
\end{align*}
\]

We note that each of the \( r \)'s are \( \geq 0 \). Moreover, if all the \( r \)'s equal 0 then only routing is performed at a node. In the general case we will have a routing variable \( r_j \) associated with every set \( Q_j \) corresponding to flow for receivers in the set \( \cup Q_j \). We denote the set of routing variables \( r_j \)’s by \( R \). The general equation is

\[
x_{out}(P_i) = x_{in}(P_i) + \sum_{j : P_i \subseteq Q_j} r_j - \sum_{j : \cup Q_j = P_i} r_j
\]

Any node that is restricted to routing/replicating has to satisfy (6). We will call this constraint on \( X_{in} \) and \( X_{out} \) as routing constraint. Note that although we call the variable \( r_j \)'s as routing variables they actually correspond to replication. Also when we say a node is a routing node we allow for replication at that node.

The third type of operation is network coding. This happens at nodes where two or more flows merge. Similar to the routing variables we define a set of network coding variables \( N \) where element \( n_j \) represents the amount of flow meant for each set of receivers \( Q \subseteq Q \) that merges to form one \( n_j \) flow that has to reach all receivers in the set \( \cup Q_j \). \( n_j \) packets are network coded to form \( n_j \) packets. It is easy to see that for a network coding node the relationship between \( X_{in} \) and \( X_{out} \) has to be of the form

\[
x_{out}(P_i) = x_{in}(P_i) + \sum_{j : P_i \subseteq Q_j} r_j - \sum_{j : \cup Q_j = P_i} r_j
\]

- \( \sum_{j : P_i \subseteq Q_j} n_j \) - \( \sum_{j : \cup Q_j = P_i} n_j
\]
which reduces to

\[ x_{\text{out}}(P_i) = x_{\text{in}}(P_i) + \sum_{j: P_i \in Q_j} (r_j - n_j) - \sum_{j: \cup Q_j \in Q_i, Q = P_i} (r_j - n_j) \]  

(8)

We note that it is sufficient to consider variables \( r_j - n_j \) but we retain both for now. We will refer to the conditions in \( B \) as node constraints.

In the following theorem, for any pair of \( X_{\text{in}} \) and \( X_{\text{out}} \) that satisfy the flow constraints, we show that the operations at the node can be decomposed into routing, replicating and network coding operations and hence these operations are sufficient to represent any processing done at the node.

**Theorem 2:** The relationship between \( X_{\text{in}} \) and \( X_{\text{out}} \) for any valid operation at the node can be expressed in terms of routing variables \( R \) and network coding variables \( N \) such that each element of \( R \) and \( N \) is \( \geq 0 \).

**Proof:** We will give a particular solution satisfying all the conditions. The main idea used in constructing the particular solution is that all packets meant for more than one receiver can be replicated to produce packets such that each packet is meant for one receiver. They can then be suitably network coded to get the desired output information flow vector.

Consider a set of receivers \( P_i \) and corresponding set \( Q(P_i) \) the set of all singleton subsets of \( P_i \). For every set \( P_i \in P \) containing two or more elements set \( r_j = x_{\text{in}}(P_i) \) and \( n_j = x_{\text{out}}(P_i) \) where \( Q_j = Q(P_i) \). Set all other routing and network coding variables to 0. We will show that this solution satisfies the constraints in \( B \). On substituting for the routing and network coding variables that have been set to 0, for all non singleton sets \( P_i \) the constraints in \( B \) reduce to

\[ x_{\text{out}}(P_i) = x_{\text{in}}(P_i) - r_j + n_j, \quad Q_j = Q(P_i) \]

which is exactly the flow constraint on information flow to the receiver \( k \) (Eq. \( B \) and hence is satisfied.

**Theorem 3:** Given a network \( G = (V, E) \), flow vectors \( X_e \) for each edge \( e \in E \) and routing and network coding variables \( R^v \) and \( N^v \) for each node \( v \in V \) such that the edge, flow and node constraints are satisfied, we can construct a valid multicast code that performs routing and network coding as specified by \( R^v \) and \( N^v \).

**Proof:** We prove the theorem by replacing each node in the network by a network that has routing and network coding nodes corresponding to the variables \( R^v \) and \( N^v \) such that there is no loss in the multicast rate.

With every node \( v \in V \) associate a set of \( (2^k - 1) \) nodes where each new node, \( v(P_i) \), corresponds to one set of receivers \( P_i \in P \). For every set \( P_i \in P \) connect all the \( x_{\text{in}}(P_i) \) incoming edges and the \( x_{\text{out}}(P_i) \) outgoing edges of node \( v \) carrying data for receivers in and only in set \( P_i \) as input and output edges to the node \( v(P_i) \).

Corresponding to each non zero routing variable \( r^v_j \) construct \( r^v_j \) nodes, each node having exactly one incoming edge coming from node \( v(\cup P_i \in Q_j, P_i) \) and \( |Q_j| \) outgoing edges that are connected as inputs to nodes in \( \{v(P_i) : P_i \in Q_j\} \). Corresponding to each non zero network coding variable \( n^v_j \) construct \( n^v_j \) nodes with each node having one input edge from every node in \( \{v(P_i) : P_i \in Q_j\} \) and one output edge that is connected as input to node \( v(\cup P_i \in Q_j, P_i) \).

Now the number of incoming edges to node \( v(P_i) \) is \( x_{\text{in}}(P_i) + \sum_{j: P_i \in Q_j} r^v_j + \sum_{j: \cup Q_j \in Q_i, Q = P_i} n^v_j \) and the number of outgoing edges is \( x_{\text{out}}(P_i) + \sum_{j: P_i \in Q_j} n^v_j + \sum_{j: \cup Q_j \in Q_i, Q = P_i} r^v_j \). From \( B \) the number of incoming edges is equal to the number of outgoing edges. Randomly connect the set of input edges and the set of output edges of node \( v(P_i) \) in a one to one manner and delete node \( v(P_i) \).

It is easy to see that this construction procedure replaces each node by a network that maintains the same flows and hence there is no loss in rate.

In the construction procedure provided in the proof for Theorem 3 the network that replaces each node could have cycles. These cycles are formed when a packet meant for a set of receivers \( P_i \) goes through a series of network coding and routing operations to get back a packet meant for \( P_i \) itself. Clearly the involvement of this packet in those operations is unnecessary. All cycles correspond to unnecessary operations and hence can be removed. We note that cycles within a node will be absent in solutions that minimizes the number of network coding operations. The construction procedure provided can be used along with ideas of random network coding [3], [4] to construct multicast codes corresponding to the given information flow vectors.

**IV. Optimization**

Since any solution to the set of linear equations specified by \( A, B \) and \( C \) corresponds to a network coding solution, we can use the set of equations to obtain a network coding solution in order to minimize a “cost” associated with the network code. The problem can be stated as follows:

minimize Cost

subject to

\[ x_e(P_i) \geq 0 \quad \forall P_i \in P, \quad \forall e \in E, \]

\[ r^v_j \geq 0, n^v_j \geq 0 \quad \forall j, \quad \forall v \in V \]

**Edge Constraints:**

\[ \sum_{P_i \in P} x_e(P_i) \leq C(e) \quad \forall e \in E \]

**Node Constraints:**

\[ x_{\text{out}}(P_i) = x_{\text{in}}(P_i) + \sum_{j: P_i \in Q_j} (r^v_j - n^v_j) \]

\[ \sum_{j: \cup Q_j \in Q_i, Q = P_i} (r^v_j - n^v_j) \quad \forall P_i \in P \quad \forall v \in V \]

\[ I_k(X_{\text{out}}) = h \quad \forall k \]

\[ I_k(X_{\text{in}}^k) = h \quad \forall k \]

(9)
where
\[ x_{in}^v(P_i) = \sum_{e \in E_{in}(v)} x_e(P_i), \quad x_{out}^v(P_i) = \sum_{e \in E_{out}(v)} x_e(P_i) \]
and \[ I_k(X_e) = \sum_{j: k \in P_j} x_e(P_j) \]
Note that we have dropped some of the flow constraints in (4) as they are satisfied automatically if the node constraints in (3) are satisfied.

In the remainder of the section, we list a few natural cost criteria.

1) **Number of Network Coding nodes.** Since additional coding capabilites are required at a node in order to perform network coding, it is potentially of interest to minimize the number of nodes performing network coding. Using Theorem 4 it follows that network coding needs to be performed at a node \( v \) only if \( n^v_i > 0 \) for some \( i \). Since \( n^v_i \geq 0 \), this condition is equivalent to \( \sum_v n^v_i > 0 \). Thus, the number of nodes in the network performing network coding is \( \sum_{v \in V} I(\sum_v n^v_i > 0) \), which we choose as the cost function.

However, note that for \( n^v_i \geq 0 \), the function \( \sum_{v \in V} I(\sum_v n^v_i > 0) \) is a concave function and the problem becomes one of minimizing a concave function over a convex set. This solution might admit local minima and standard convex minimization techniques cannot be used to solve this problem. We relax this problem and investigate minimizing the number of network coding operations and minimizing the number of packets involved in network coding in the following problems.

2) **Number of network coding operations.** In this problem we investigate minimizing the number of network coding operations at a node \( v \). From Theorem 4 it follows that network coding operations (linear encoding of packets) need to be performed corresponding to each \( n^v_i \). Thus the number of network coding operations at node \( v \) is \( \sum_v n^v_i \). We define this quantity as the **amount** of network coding. Thus, the cost function in this problem is given by \( \sum_{v \in V} \sum_v n^v_i \).

3) **Number of packets involved in network coding.** In this problem we investigate minimizing the number of packets over which network coding is performed at a node \( v \). This is particularly relevant in optical networks when a conversion from optical signals to electrical signals is involved in order to encode the packets. We conjecture that the cost function is given by \( \sum_{v \in V} \sum_v \max(A^v_i, 0) \) where \( A^v_i = \sum_{j: P \in Q_j} v_i - \lambda_{ij}^v \) \( \sum_{j: v \in Q_j, Q \in P} (v_i - \max_i \lambda_{ij}^v) \). \( \lambda_{ij} \) represents the number of packets meant for receivers \( P_j \) that participate in network coding and that are obtained by routing packets meant for \( v \in Q_j \). From the definition it follows that \( 0 \leq \lambda_{ij} \leq r_j \).

4) **Minimum resource cost.** In the setup considered in [5], each edge \( e \) is associated with a cost function \( f_e(z_e) \) when the data rate on \( e \) is \( z_e \). The net cost associated with the network is then given by \( \sum_e f_e(z_e) \). This cost was minimized over the set of equations specified by equations (1) and (2) in [5]. The same approach can be applied in the setting where only certain nodes are allowed to perform network coding. The restriction that a node \( v \) can perform only routing can be imposed by further constraining the equations in (9) by \( n^v_i = 0 \) for all \( i \).

5) **Maximum rate.** The problem considered here is one of maximizing \( h \) constrained to (9) and additionally the set of equations \( n^v_i = 0 \) for all \( i \) and nodes \( v \) which are restricted to routing.

Note that the problems 2, 3, 4 (if the cost function \( f_e() \) is linear) and 5 are linear problems and can be solved by standard linear programming approaches. It remains to be investigated if the decentralized subgradient optimization suggested in [5] can be applied to these problems. To the end of providing decentralized solutions to these linear problem, we consider the approach suggested by [5] in which a linear function \( ax \) is approximated by a strictly convex function \( (ax)^{1+\alpha} \) where \( \alpha > 0 \) is chosen small enough for a valid approximation. This makes the problem a convex optimization problem which can be solved in a decentralized manner by a modified version of the primal-dual algorithm used in [5]. We do not prove this due to lack of space. The main idea in the proof is to show that the edge and node constraints involved are local in the sense of involving variables of the neighbouring edges or nodes and then follow the same steps as used in [5].

Problem 4 is a convex optimization problem if the function \( f_e() \) is convex. If we further assume that the function \( f_e() \) is strictly convex, it follows that problem 4 admits a unique solution. Further, it can be shown that the primal-dual algorithm used in [5] can be modified to solve problem 4 in a decentralized manner. Again we do not prove this due to lack of space.

V. Conclusion

In this paper, we presented a new Information flow model to represent multicast flows. Using this model we set up optimization problems and presented distributed algorithms to minimize costs like number of packets undergoing network coding and amount of network coding. We also showed that this approach can be used to minimize network costs like link usage when some nodes are restricted to routing.

REFERENCES

[1] R. Ahlswede, N. Cai, S.-Y. R. Li and R. W. Yeung, “Network information flow,” IEEE Trans. on Inform. Theory, vol. 46, pp. 1204-1216, 2000.

[2] S.-Y. R. Li, R. W. Yeung, and N. Cai, “Linear network coding,” IEEE Trans. on Inform. Theory, vol. 49, pp. 371-381, 2003.

[3] T. Ho, M. Medard, J. Shi, M. Effros and D. R. Karger, “On Randomized Network Coding”, 41st Annual Allerton Conference on Communication Control and Computing, Oct. 2003.

[4] T. Ho, R. Koetter, M. Medard, M. Effros, J. Shi, and D. Karger, “Toward a Random Operation of Networks”, submitted to IEEE Trans. on Inform. Theory.

[5] D. S. Lun, N. Ratnakar, R. Koetter, M. Medard, E. Ahmed, and H. Lee “Achieving minimum-cost multicast: A decentralized approach based on network coding”, Proc. IEEE INFOCOM 2005, Mar. 2005.

[6] D. S. Lun, M. Medard, T. Ho, R. Koetter, “Network coding with a cost criterion”, Proc. 2004 International Symposium on Information Theory and its applications (ISITA 2004), Oct. 2004.

[7] J. Cannons, R. Dougherty, C. Freiling, and K. Zeger, “Network Routing Capacity”, submitted to IEEE/ACM Trans. on Networking.