ADDITIVE SELF-HELICITY AS A KINK MODE THRESHOLD

A. Malanushenko$^1$, D. W. Longcope$^1$, Y. Fan$^2$, and S. E. Gibson$^2$

$^1$Department of Physics, Montana State University, Bozeman, MT 59717, USA
$^2$High Altitude Observatory, National Center for Atmospheric Research, P.O. Box 3000, Boulder, CO 80307, USA

Received 2009 April 16; accepted 2009 July 2; published 2009 August 13

ABSTRACT

In this paper, we propose that additive self-helicity, introduced by Longcope and Malanushenko, plays a role in the kink instability for complex equilibria, similar to twist helicity for thin flux tubes. We support this hypothesis by a calculation of additive self-helicity of a twisted flux tube from the simulation of Fan and Gibson. As more twist gets introduced, the additive self-helicity increases, and the kink instability of the tube coincides with the drop of additive self-helicity, after the latter reaches the value of $H_A/\Phi^2 \approx 1.5$ (where $\Phi$ is the flux of the tube and $H_A$ is the additive self-helicity). We compare the additive self-helicity to twist for a thin subportion of the tube to illustrate that $H_A/\Phi^2$ is equal to the twist number, studied by Berger and Field, when the thin flux tube approximation is applicable. We suggest that the quantity $H_A/\Phi^2$ could be treated as a generalization of a twist number, when the thin flux tube approximation is not applicable. A threshold on a generalized twist number might prove extremely useful studying complex equilibria, just as the twist number itself has proven useful studying idealized thin flux tubes. We explicitly describe a numerical method for calculating additive self-helicity, which includes an algorithm for identifying a domain occupied by a flux bundle and a method of calculating potential magnetic field confined to this domain. We also describe a numerical method to calculate twist of a thin flux tube, using a frame parallelly transported along the axis of the tube.

Key words: instabilities – magnetic fields – Sun: corona – Sun: coronal mass ejections (CMEs) – Sun: flares – Sun: magnetic fields

Online-only material: color figures

1. INTRODUCTION

According to a prevalent model, coronal mass ejections (CMEs) are triggered by current-driven magnetohydrodynamic (MHD) instability related to the external kink mode (Hood & Priest 1979; Török et al. 2004; Rachmeler et al. 2009). The external kink mode, in its strictest form, is a helical deformation of an initially symmetric, cylindrical equilibrium, consisting of helically twisted field lines. The equilibrium is unstable to this instability if its field lines twist about the axis by more than a critical angle, typically close to $3\pi$ radians (Hood & Priest 1979; Baty 2001). The helical deformation leads to an overall decrease in magnetic energy, since it shortens many field lines even as it lengthens the axis.

Equilibria without symmetry can undergo an analogous form of current-driven instability under which global motion lowers the magnetic energy (Bernstein et al. 1958; Newcomb 1960). Such an instability implies the existence of another equilibrium with lower magnetic energy. The spontaneous motion tends to deform the unstable field into a state resembling the lower energy equilibrium. Indeed, it is generally expected that there is at least one minimum energy state from which deformation cannot lower the magnetic energy without breaking magnetic field lines; its energy is the absolute minimum under ideal motion.

Linear stability and instability are determined by the energy change under infinitesimal motions. An equilibrium will change energy only at the second order since first-order changes vanish as a requirement for force balance. Ideal stability demands that no deformation decreases the energy at second order, while instability will result if even one energy-decreasing motion is possible. The infinite variety of possible motions makes it impractical to establish stability in any but the simplest and most symmetric equilibria.

Based on analogy to axisymmetric systems it is expected that general equilibria, including those relevant to CMEs, are probably unstable when some portion of their field lines are twisted about one another by more than some critical angle. This expectation was mentioned in a study by Fan & Gibson (2003) of the evolution of a toroidal flux rope into a pre-existing coronal arcade. They solved time-dependent equations of MHD in a three-dimensional, rectangular domain. Flux tube emergence was simulated by kinematically introducing an isolated toroidal field through the lower boundary. The toroidal field was introduced beneath a pre-existing arcade slowly enough that the coronal response never approached the local Alfvén speed. Fan and Gibson concluded that the system underwent a current-driven instability after a critical amount of the torus had been introduced. They bolstered this claim by performing an auxiliary run where the kinematic emergence was halted and the system was allowed to evolve freely; it settled into an equilibrium.

While twist angle has proven useful in a few cases, it is difficult to demonstrate its utility as a threshold in general, asymmetric equilibria. Indeed, in any but a few very symmetric cases there is no simple, obvious way to define the angle by which the field lines wrap about one another. The local rate of twist is given by the current density, which is after all the source of free energy powering the instability. On the other hand, excessive local current density is not sufficient to drive instability. This fact is illustrated by numerous examples of discontinuous field which are minimum energy states.

It has been suggested that a threshold exists, in general equilibria, for some global quantity such as free magnetic energy or helicity (Zhang et al. 2006; Low 1994). If this is the case then we expect the instability to lower the value of this global quantity so that it falls below the threshold value in the lower energy, stable equilibrium. Magnetic helicity is a logical candidate to
play this role since it is proportional to the total twist angle in cylindrical fields. Relative helicity in particular is a proxy for currents. Helicity is, however, conserved under ideal motion and therefore will not be reduced to a subthreshold value by an ideal instability.

The total helicity of a thin, isolated flux tube can be written as a sum of two terms called twist and writhe (Berger & Field 1984; Moffatt & Ricca 1992),

\[ H = H_T + H_W. \]

The writhe depends on the configuration of the tube’s axis, while the twist depends on the wrapping of field lines about one another. A cylindrical tube has a perfectly straight axis and therefore zero writhe helicity. Any ideal motion which helically deforms the entire flux tube will increase the magnitude of the writhe helicity. Since the motion preserves total helicity, the change in writhe must be accompanied by an offsetting change in twist helicity. If the writhe has the same sign as the initial twist, then the motion will decrease the twist helicity. In cases where the magnetic energy depends mostly on twist, this motion will decrease the magnetic energy (Linton & Antiochos 2002). The straight equilibrium is therefore unstable to an external kink mode.

Topologically, the foregoing properties of magnetic field lines could be compared to the properties of thin closed ribbons. One may introduce twist number, writhe number, and their combination, called linkage number, which is a preserved quantity in the absence of reconnection (Berger & Field 1984; Moffatt & Ricca 1992),

\[ L = Tw + Wr. \]

By analogy to the case of a thin isolated flux tube we consider the twist helicity, rather than the total helicity, to be the most likely candidate for a stability threshold. Indeed, within a thin flux tube, it is possible to derive a net twist angle among field lines and \( H_T = \Phi^2 Tw = \Phi^2 \Delta \theta / 2\pi \), where \( \Phi \) is the total magnetic flux through a cross section of the tube and \( \Delta \theta \) is the net twist angle.

Twist and writhe are, however, defined only in cases of thin, isolated magnetic flux tubes, and can no more set the threshold we seek that the net twist angle can.

Recently, Longcope & Malanushenko (2008) introduced two generalizations of relative helicity applicable to arbitrary subvolumes of a magnetic field. They termed both generalized self-helicity, and the two differed only by the reference field used in their calculation. The one called additive self-helicity (that we denote \( H_A \)) uses a reference field confined to the same subvolume as the original field, and can be interpreted as a generalization of the twist helicity to arbitrary magnetic fields. The additive self-helicity of a thin, isolated flux tube is exactly the twist helicity.

Since the additive self-helicity can be computed for arbitrary magnetic fields, we propose that it (normalized by the squared flux) is the quantity to which current-driven instability sets an upper limit, which could be considered a generalized twist number:

\[ Tw_{\text{gen}} = H_A / \Phi^2. \]  

(1)

The paper is organized as follows. In Section 2, we describe a method for calculating additive self-helicity and \( Tw_{\text{gen}} \) numerically. There are two large and nontrivial parts of this calculation that we describe in Sections 2.1 and 2.2: locating a domain containing a given flux bundle and constructing a potential field in this domain by Jacobi relaxation. In Section 3, we apply the method to a simulation to support our hypothesis, the emerging twisted flux tube from Fan & Gibson (2003). In Section 3.1 we briefly describe this simulation, and then in Section 3.2 we show different embedded domains defined by different subportions of the footpoints. In Section 3.2, we describe how the twist of Berger & Field (1984) could be calculated for those of the domains for which the thin flux tube approximation is applicable. In Section 4, we present the evolution of additive self-helicity, unconfined self-helicity, twist (for “thin” domains), and the integrated helicity flux in the simulation. We demonstrate that \( Tw_{\text{gen}} \) increases corresponding to helicity flux, that it drops after it reaches a certain value (about 1.5), and that this drop coincides with the rapid expansion of the tube due to the kink instability. We also demonstrate that the unconfined self-helicity grows only when helicity flux is nonzero and that it stays constant when kink instability happens. We also show that \( Tw_{\text{gen}} \) corresponds to \( Tw \) when thin flux tube approximation is applicable.

2. NUMERICAL SOLUTIONS

The object of study is a magnetic field \( \mathbf{B}(\mathbf{r}) \) defined in a domain \( D, \mathbf{r} \in D \), that lies on and above the photosphere, \( z \geq 0 \). By domain we understand a volume that encloses the field: \( \mathbf{B} \cdot \mathbf{n} = 0 \) on all boundaries, \( \partial D \), except at the photosphere, where \( \mathbf{B} \cdot \mathbf{n} = B_r(x, y, z) = 0 \). An example of such a volume is the coronal part of an \( \Omega \)-shaped loop. Another example is a field of a dipole in the presence of another dipole (see Figure 1). The self-helicity is given by

\[ H_A(\mathbf{B}, \mathbf{P}(D)) = \int_D (\mathbf{B} - \mathbf{P}) \cdot (\mathbf{A} + \mathbf{A_P}) \, dV, \]

(2)

as defined by Longcope & Malanushenko (2008). Here, \( \mathbf{P} \) is the potential magnetic field, whose normal component matches the normal component of \( \mathbf{B} \) on the boundary \( \partial D \),

\[ \mathbf{P} \cdot \mathbf{n}|_{\partial D} = \mathbf{B} \cdot \mathbf{n}|_{\partial D}, \]

(3)

where \( \mathbf{A} \) and \( \mathbf{A_P} \) are the vector potentials of \( \mathbf{B} \) and \( \mathbf{P} \), respectively (as discussed in Finn & Antonsen 1985, helicity, defined this way is gauge-independent).

Once the self-helicity is known, the twist is given by Equation (1) with \( \Phi \) being the total signed flux of the footpoints of the configuration:

\[ \Phi = \int_{z=0, B_r \geq 0} B_r \, dx \, dy - \int_{z=0, B_r \leq 0} B_r \, dx \, dy. \]

(4)

In the following two sections, we discuss methods of numerically obtaining \( D \), from given footpoints, and \( \mathbf{P} \).

2.1. Finding the Domain

In order to describe the domain on a grid, we introduce the support function:

\[ \Theta(\mathbf{r}) = \begin{cases} 1, & \text{if } \mathbf{r} \in D \\ 0, & \text{if } \mathbf{r} \notin D. \end{cases} \]

This is a function of the given magnetic field \( \mathbf{B} \) and some photospheric area, called the boundary mask. By definition, every field line, initiated at any point on the boundary mask and having the other footpoint somewhere within the mask, is
completely inside the domain $D$. If the field line traced in both
directions from some coronal point ends within the photospheric
mask, then this point also belongs to the domain. In numerical
computations we replace “point” with a small finite volume,
voxel $v_{ijk}$ (three-dimensional pixel). We define a voxel to be
inside $D$ (equivalent to saying $\Theta(v_{ijk}) = 1$), if there is at least
one point inside it that belongs to $D$.

The simplest method of constructing the support function
would be to trace a field line in both directions from every voxel
of the computational grid, set $\Theta = 1$ in the voxel if the footpoints
both terminate in pixels from the boundary regions, and set
$\Theta = 0$ otherwise. This, however, is a very time-consuming
algorithm, especially for a large array of data. Instead, we use an
algorithm which reduces the computational time by tracing field
lines from a subset of voxels. It works by progressively adding
voxels to $\Theta$ adjacent to those already known to belong to $D$.

We add a voxel centered at $r_{ij,k}$ to the domain under two
different circumstances. (1) A field line initialized somewhere
within the volume of the voxel $v_{ijk}$, centered at $r_{ij,k}$, is found
to have both footpoints within the boundary mask. (2) A field
line initiated in some other voxel, and determined to belong to $D$,
passes through some portion of the volume $v_{ijk}$.

Initially, the domain consists only of footpoint voxels, so
the initial step is to trace field lines *initialized at the footpoints*,
assuming that at least some of these lines will lie in the domain.

We illustrate the method on a simplistic case of a potential
magnetic field, confined to a half-space, with $B_z = 0$ ev-
erywhere at the photosphere, except at four pixels, as shown in
Figure 2. We have computed the magnetic field inside a
small box of $15 \times 15 \times 15$ pixels, centered around the pho-
tospheric sources. The boundary mask consists of these four
voxels at the photosphere with nonzero vertical magnetic field.
In this simplistic example the initial guess would be four field
lines, initiated at four footprint voxels, as shown in Figure 2,
left (note that in this particular example a field line, initiated
at one voxel, ends at another voxel within the mask and
thus is the same as the field line, initiated at that another
voxel, so these four initial guesses are really two, not four
field lines). The voxels of the initial guess are shown with
crosses.

In an algorithm, this would be the first step:

*Step 1*: Make the initial guess: trace field lines from the
footpoints.

As soon as an initial step is made, the next step is to assume
that the *immediate neighborhood* of voxels known to be in $D$
are likely to be also in the domain. Thus, in the next (iterative)
search, the following steps are performed:

*Step 2*: Locate voxels on the boundary of the current domain.

*Step 3*: For every voxel on the boundary: trace a field line and
check whether it is in the domain.

If *yes*: Add the voxel to the domain. Add all voxels along
the line to the domain. Exclude them from the boundary (there is
no need to check them again).

If *no*: Mark the voxel as “questionable.” (If there is a field line,
which passes through the voxel and does not belong to the
domain, then at least part of the voxel is outside of the domain.
Since its immediate neighborhood is in the domain, then it is
possible that part of it is also in the domain.)

*Loop*: Repeat Steps 2 and 3 until *all* the voxels in the boundary
are “questionable” and no new voxels are added.

When the iterative search does not find any new voxels, we
make the final check of the boundary voxels. The idea is to
trace field lines from all corners of such “questionable” voxels
to see, which corners (and thus which part of a voxel) belongs to
the domain. We consider this to be optional check, which may
improve the precision of the definition of the domain by at most
one layer of voxels.

This last search may also give information about the normal
to the domain surface. If it is known that some corners of a voxel
are in the domain and some are not, it is possible to approximate
the boundary as a plane separating those two groups of corners.

*Step 4*: optional: For each voxel, marked previously as “ques-
tionable,” check the corners (by tracing field lines) to see which
of them are in the domain and which are not. Keep this infor-
mation.

2.2. Constructing the Confined Potential Field $\mathbf{P}$

Once the domain has been determined, the next step is to
construct the potential magnetic field confined to it. We use
a common relaxation method on a staggered grid in order to
account for the complex boundaries of $D$.

We introduce a scalar potential $\mathbf{B}_p = \nabla \chi$ and look for the
solution of the Laplace’s equation for $\chi$

$$\nabla \cdot \mathbf{B}_p = \nabla^2 \chi = 0.$$
By the definition of $D$, field lines never cross $\partial D$, except at the lower boundary, $z = 0$. Thus, boundary conditions (BCs) for $B_p$ could be written as $B_p \cdot \hat{n} |_{\partial D, z=0} = 0$ and $B_p \cdot \hat{n} |_{\partial D, z>0} = B_z(x, y)$. This is equivalent to Neumann BCs for $\chi$:

$$
\frac{\partial \chi}{\partial n} |_{\partial D, z=0} = 0,
\frac{\partial \chi}{\partial z} |_{\partial D, z=0} = B_z(x, y).
$$

(5)

The Algorithm for the Relaxation Method. We use the Jacobi iterative method (see, for example, LeVeque 1995) to solve for the potential field. Here we briefly summarize the algorithm and further explain in details. The $(n + 1)$th iteration is

1. $\forall \mathbf{r} \in D$: calculate a new iteration $\chi^{n+1}$ as a solution of the equation $\chi^{n+1} - \chi^n = K h^2 \nabla^2 \chi^n$, where $h$ is the grid spacing. The Laplacian $\nabla^2 \chi^n (\mathbf{r})$, found causing standard finite difference methods, is equivalent to an average over some stencil of neighboring points minus the central value; $K$ is a constant that depends on the exact shape of the stencil.
2. $\forall \mathbf{r}_b \in \partial D$: set $\chi^{n+1} (\mathbf{r}_b)$ so as to satisfy BCs.
3. Repeat Steps 1 and 2, until the difference between $\chi^n (\mathbf{r})$ and $\chi^{n+1} (\mathbf{r})$ is sufficiently small in some sense (namely, until $|\chi^{n+1} - \chi^n| < \epsilon$, where $\epsilon$ is a predefined small number).

Staggered Mesh. The functions $B_z(x, y, z)$, $B_y(x, y, z)$, and $B_z(x, y, z)$ are defined on the same mesh points $(x_i, y_j, z_k)$. If we are interested in finding $\chi(x, y, z)$, so that $B_x = \frac{\partial \chi}{\partial x}$, $B_y = \frac{\partial \chi}{\partial y}$, and $B_z = \frac{\partial \chi}{\partial z}$, it is advantageous to define $\chi$ in between the original mesh points and calculate the derivatives using finite difference as follows:

$$
B_x(x_i, y_j, z_k) = \frac{\chi(x_{i+1/2}, y_j, z_k) - \chi(x_{i-1/2}, y_j, z_k)}{x_{i+1/2} - x_{i-1/2}},
$$

and so on for $B_y$ and $B_z$. $\chi$, then, would only be defined in the middle of the faces of cubic voxels, i.e., at points $(i \pm 1/2, j, k)$, $(i, j \pm 1/2)$, and $(i, j, k \pm 1/2)$.

Such a mesh, called a “Cartesian staggered mesh,” is known to have better numerical properties, such as immunity from decoupling of variables and having a smaller numeric dispersion (see, for example, Perot 2000).

The finite difference approximation of a Laplacian at one point can be interpreted as a weighted average over a stencil of several points minus the value at that point. For example, in the two-dimensional case, the second-order approximation to $\nabla^2 \chi(x, y)$ on a uniform Cartesian grid at the point $(x_i, y_j)$ could be computed over a five-point stencil:

$$
\nabla^2 \chi(x_i, y_j) \approx \frac{1}{h^2} (\chi(x_{i-1}, y_j) + \chi(x_{i+1}, y_j) + \chi(x_i, y_{j+1}) + \chi(x_i, y_{j-1}) - 4\chi(x_i, y_j))
$$

(here $h$ is the spacing of the grid). It could be rewritten as

$$
\nabla^2 \chi(x_i, y_j) \approx \frac{1}{4} (\chi(x_{i-1}, y_j) + \chi(x_{i+1}, y_j) + \chi(x_i, y_{j+1}) + \chi(x_i, y_{j-1}) - 4\chi(x_i, y_j))
$$

and

$$
\nabla^2 \chi(x_i, y_j) \approx \frac{h^2}{4} \nabla^2 \chi(x_i, y_j).
$$

The Jacobi method uses this equation to iteratively update the value at the point, constantly assuming $\nabla^2 \chi = 0$. In the case of the five-points stencil, the updated value would be

$$
\chi^{n+1} (x_i, y_j) = \frac{1}{4} (\chi^n (x_{i-1}, y_j) + \chi^n (x_{i+1}, y_j) + \chi^n (x_i, y_{j+1}) + \chi^n (x_i, y_{j-1})).
$$

In our case of a three-dimensional staggered mesh, choosing a stencil becomes more complicated. We propose a 13-point scheme, shown on the right of Figure 3 (black dots). To motivate this stencil, we derive it from the “ unstaggered” one (Figure 3, left, gray dots). In an “ unstaggered” finite differencing scheme, the $(n + 1)$th iteration in the Jacobi method would be expressed as

$$
6\chi^{n+1} (O) = \chi^n (A_1) + \chi^n (A_2) + \chi^n (B_1) + \chi^n (B_2) + \chi^n (C_1) + \chi^n (C_2).
$$

But for the staggered mesh, $\chi$ is undefined at these nodes. This can be resolved by setting $\chi$ at each “gray” point to be equal to the average of its four closest neighbors,
\[ \chi^{[n]}(A_1) = \frac{1}{4} \left[ \chi^{[n]}(SA_1) + \chi^{[n]}(TA_1) + \chi^{[n]}(SA_1) \right] \\
+ \chi^{[n]}(O), \\
\chi^{[n]}(B_1) = \frac{1}{4} \left[ \chi^{[n]}(SB_1) + \chi^{[n]}(TB_1) + \chi^{[n]}(SB_1) \right] \\
+ \chi^{[n]}(O), \\
\chi^{[n]}(C_1) = \frac{1}{4} \left[ \chi^{[n]}(TA_1) + \chi^{[n]}(TA_2) + \chi^{[n]}(TB_1) \right] \\
+ \chi^{[n]}(TB_2). \]

and so on. Then we may substitute this in the original expression and get

\[ 6\chi^{[n+1]}(O) = 2 \times \frac{1}{4} \left[ \chi^{[n]}(TA_1) + \chi^{[n]}(TA_2) + \chi^{[n]}(BA_1) \right] \\
+ \chi^{[n]}(BA_2) \\
+ 2 \times \frac{1}{4} \left[ \chi^{[n]}(TB_1) + \chi^{[n]}(TB_2) + \chi^{[n]}(BB_1) \right] \\
+ \chi^{[n]}(BB_2) \\
+ \frac{1}{4} \left[ \chi^{[n]}(SA_1) + \chi^{[n]}(SA_2) + \chi^{[n]}(SB_1) \right] \\
+ \chi^{[n]}(SB_2) \\
+ 4 \times \frac{1}{4} \chi^{[n]}(O), \]

which is equivalent to

\[ \chi^{[n+1]}(O) = \frac{1}{12} \left[ \chi^{[n]}(TA_1) + \chi^{[n]}(TA_2) + \chi^{[n]}(BA_1) \right] \\
+ \chi^{[n]}(BA_2) \\
+ \frac{1}{12} \left[ \chi^{[n]}(TB_1) + \chi^{[n]}(TB_2) + \chi^{[n]}(BB_1) \right] \\
+ \chi^{[n]}(BB_2) \\
+ \frac{1}{24} \left[ \chi^{[n]}(SA_1) + \chi^{[n]}(SA_2) + \chi^{[n]}(SB_1) \right] \\
+ \chi^{[n]}(SB_2) \\
+ \frac{1}{6} \chi^{[n]}(O). \]

With these weights, the “farthest” nodes $S[AB]_{12}$ have half the influence on the Laplacian, of the “closer” nodes. Note also that the sum of the weights is 1.

Boundary Conditions. BCs (given by Equation (5)) in the staggered mesh are particularly easy if one assumes that the boundary surface passes inside of boundary voxels, rather than on their sides. Suppose, for example, that the boundary plane normal to $\hat{z}$ passes through the center of the voxel $v_{ijk}$. Then the BC for this voxel would be that $B_i (i, j, k) = 0$, or simply $\chi (i, j, k + \frac{1}{2}) = \chi (i, j, k - \frac{1}{2})$.

To motivate such a choice of the boundary, we note that boundary voxels, by definition, are the voxels part of which is inside of $D$ while part is outside. Such a conclusion is made about voxels, some of whose corners are inside of $D$, and some of the corners are outside of $D$ (this information about the domain is obtained in Step 4 of the algorithm, described in Section 2.1). We approximate the boundary inside of each boundary voxel as a plane that passes through the center of the voxel and that separates its “exterior” part from its “interior” part. Such an approximation will err by no more that $1/2\sqrt{3}$ voxel’s length off the real location of the boundary. We also find it easier to work in terms of faces rather than corners, since this is where $\chi$ is defined. (We say that a face is “exterior” to the domain if more than two of its corners are not in the domain, i.e., for a voxel, we say, that if only one corner or only one edge are “exterior,” we do not consider it a subject to BCs).

There are several ways to orient such a boundary plane inside a voxel, based on the behavior of the boundary in the immediate surrounding of the voxel.

1. The voxel has only one face outside of the domain. Then we consider the boundary parallel to that face of the voxel (see Figure 4, left). If, say, the boundary is parallel to the face between faces $A$ and $A_1$ (see Figure 4, bottom left), then the normal field to the boundary is $B \cdot \hat{A}A_1$ (hereafter $\hat{A}A_1$ denotes a unit vector along the line from $A$ to $A_1$, which might be $\pm \hat{x}$, $\pm \hat{y}$, or $\pm \hat{z}$), and BC would be formulated as

\[ \chi_A = 1 \cdot \chi_A + 0 \cdot \chi_B + 0 \cdot \chi_C. \]

2. The voxel has two adjacent faces outside of the domain. Then we approximate the boundary as a plane, that cuts off these two faces, as shown in Figure 4, middle. If faces $A$ and $B$ are outside and faces $A_1$ and $B_1$ are inside of the domain,
then we consider the normal field to be $\mathbf{B} \cdot \frac{1}{\sqrt{2}}(\mathbf{A}_A + \mathbf{B}_B)$ and set BCs as

$$\chi_A = 0 \cdot \chi_A + 1 \cdot \chi_A + 0 \cdot \chi_C,$$

$$\chi_B = 1 \cdot \chi_A + 0 \cdot \chi_B + 0 \cdot \chi_C.$$

3. Similarly, if three mutually adjacent faces of the voxel are outside of the domain (and three others are inside), as shown in Figure 4, right, then, analogously, we assume that the normal field is $\mathbf{B} \cdot \frac{1}{\sqrt{3}}(\mathbf{A}_A + \mathbf{B}_B + \mathbf{C}_C)$ and BCs could be set in the following way:

$$\chi_A = 0 \cdot \chi_A + \frac{1}{2} \cdot \chi_A + \frac{1}{2} \cdot \chi_C,$$

$$\chi_B = \frac{1}{2} \cdot \chi_A + 0 \cdot \chi_B + \frac{1}{2} \cdot \chi_C,$$

$$\chi_C = \frac{1}{2} \cdot \chi_A + \frac{1}{2} \cdot \chi_B + 0 \cdot \chi_C.$$

(Note that in this case there are really three variables and one equation to satisfy; thus, there are different solutions to $\chi$. But each of those solutions would be valid, as long as it satisfies $\mathbf{B} \cdot \hat{n} = 0$.)

4. “Everything else”: the voxel has three or more nonadjacent faces that are outside of the domain, but still is on the boundary. It is considered an extraneous voxel and is removed from the boundary.

3. THE EXPERIMENT

The method described above was tested on a simple quadrupole example, and the values of self-helicity it gives are in a good agreement with theoretical predictions (Longcope & Malanushenko 2008). That work, however, does not consider any sort of stable equilibrium and does not study any kinking instability thresholds, similar to those developed in Hood & Priest (1981).

The objective of the current work is to test whether the parameter $H_A/\Phi^2$ behaves like a total twist in the sense that it has a critical value above which a system is unstable to a global disruption. To do so, we use the numerical simulation of kink instability in an emerging flux tube from Fan & Gibson (2005).

3.1. Simulation Data

The initial configuration is a linear arcade above the photosphere, into which a thick, non-force-free torus was emerged.
Figure 7. Characteristic times for the simulation of Fan & Gibson (2003), different rows correspond to different time (see detailed explanation in the text). First column—$XZ$ slices, the analytical shape of the rising tube is shown beyond the photosphere, solid-dashed line is $\sigma = 1.0$—the formal “edge” of the torus; dotted line is $\sigma = 3.0$. Second column—magnetograms at $z = 0$. Third column—side view of the field lines, initiated at $\sigma = 1.0$ (their footpoints are shown as diamonds in the second column).

in how different portions of the torus, namely, the “core” and the outer layers, behave during the instability.

Our masks are defined to be within the photospheric intersection of the emerging torus, $\sigma \leq \sigma_{\text{max}}$. By choosing different values of $\sigma_{\text{max}}$, we construct domains, containing different portions of the emerging flux tube. The footpoints of domains with different $\sigma_{\text{max}}$ are shown in Figure 8. The shape is distorted with respect to the original cross section of a torus due to reconnection with the arcade, current sheet formation, and due to near horizontality of some field lines.

We found domains for masks with $\sigma_{\text{max}} \in [0.5, 1.0, 2.0]R$ at different times during the emergence. We computed $\Theta(\sigma_{\text{max}}, t)$ and then constructed a potential field confined to it. The results are shown in Figures 8–10.
For each \( t \) and \( \sigma_{\text{max}} \), we calculated vector potentials of the actual field, \( \Theta(t, \sigma_{\text{max}})B(t, \sigma_{\text{max}}) \), and the reference field \( P(t, \sigma_{\text{max}}) \). To do this, we used a gauge in which one of the components of the vector potential (in our case, \( A_z \)) is identically zero. The other two could be found with a straightforward computation:

\[
A_x(x, y, z) = \int_0^z B_y(x, y, z')dz' \\
A_y(x, y, z) = f(x, y) - \int_0^z B_z(x, y, z')dz' \\
f(x, y) = \int_0^x B_z(x', y, 0)dx'.
\]

(7)

In terms of these elements, the additive self-helicity

\[
H_A(t, \sigma_{\text{max}}) = \int_{\Theta(t, \sigma_{\text{max}})} (\Theta B - P) \cdot (A + A_P) dV
\]

is computed.

### 3.3. Measuring Twist in the Thin Flux Tube Approximation

To make contact with previous work, we compare the additive self-helicity to the twist helicity in our flux bundles. It can be shown analytically that in the limit of a vanishingly thin flux tube these quantities are identical. Here we must compute twist helicity for flux bundles of nonvanishing width. We do this in terms of a geometrical twist related to twist helicity.

One cannot really speak of twist, or of an axis, in the domains defined above. First, the thickness and the curvature radius of the flux bundles are comparable to their lengths. Second, the magnetic field and the twist vary rapidly over the cross section of the bundle.

The domains constructed from the smaller masks, \( \sigma_{\text{max}} = 0.5 \) and \( \sigma_{\text{max}} = 1.0 \), may, however, be suitable for approximation as thin tubes. Even in these cases the approximation may suffer near the top part at later times: at \( t = 50 \) the radius of curvature becomes comparable to the width, and later, during kinking the radius of the tube becomes comparable to the length (see Figures 8 and 7).

We define an axis for the flux bundle by first tracing many field lines within it. Then we divide each field line into \( N \) equal segments (\( N \) is the same for all lines) of length \( L_i/N \), where \( L_i \) is the length of the \( i \)th line. If the bundle were an ideal cylinder, the midpoints of the \( n \)th segment from every line would lie on a single plane; provided the bundle is thin the these midpoints will lie close to a plane. We define the \( n \)th point on an axis by the centroid of these approximately coplanar points. The set of \( N \) centroids forms the axis of our tube.

We then define the tangent vector \( \hat{l}_i \) along this axis, and a plane normal to this vector and thus normal to the flux tube (at least in the thin flux tube approximation). If the tube has some twist in it, then the point where one field line intersects the plane will spin about the axis as the plane moves along the tube. Such spinning must be defined relative to a reference vector on the plane which "does not spin." The net angle by which the intersection point spins, relative to the nonspinning vector, is the total twist angle of the tube. In a thin tube, all field lines will spin by the same angle; in our general case, we compute an average angle.

We produce a nonspinning reference vector using an orthonormal triad, arbitrarily defined at one end of the tube, and carried along the axis by parallel transport. For a curve with tangent unit vector \( \hat{l} \), the parallel transport of a vector \( \hat{u} \) means \( \hat{u} - (\partial u/\partial l) \). To implement this numerically an arbitrary unit vector, \( \hat{u}_0 \), is chosen at one end of the axis perpendicular to the tangent, \( \hat{u}_0 \cdot \hat{l}_0 = 0 \). The third member of the triad is \( \hat{v}_0 = \hat{u}_0 \times \hat{l}_0 \). At the next point, \( \hat{u}_1 \) is chosen by projecting \( \hat{u}_0 \) onto a plane normal to \( \hat{l}_1 \) and normalizing it

\[
\hat{u}_1 = \frac{\hat{u}_0 - (\hat{u}_0 \cdot \hat{l}_1)\hat{l}_1}{|\hat{u}_0 - (\hat{u}_0 \cdot \hat{l}_1)\hat{l}_1|}
\]

(see Figure 11). Then \( \hat{v}_1 = \hat{u}_1 \times \hat{l}_1 \), and the procedure is repeated for every segment along the axis. Figure 12 shows an example
Figure 10. Strip plot of the results of the computation. The original data are shown above and the relaxed potential field $P$ below the photosphere. The dotted line indicates slices of the domain $\Theta (\varpi_{\text{max}} = 2.0)$. The magnetogram in the second column and the field lines in the third column are those of $P$. All notation is similar to Figure 7.

of such vectors $\hat{l}$, $\hat{u}$, and $\hat{v}$ computed along the axis of the tube at $t = 58$.

Figure 13 shows the axis of the torus and the behavior of one field line in such a nonspinning reference frame and a field line, traced from the similar location at a later time, after the tube has kinked. Both the trajectory of the field line and the total spin angle demonstrate how the thin tube twist is decreased by kinking.
4. RESULTS

Based on the analogy between $Tw$ and $H_A/\Phi^2$, it would be natural to introduce quantity analogous to $L$ and $Wr$ in a similar way. We propose that $L$ in the general (non-"thin") case might be analogous to the unconfined self-helicity, introduced in Longcope & Malanushenko (2008), and $Wr$ is similar to the helicity of the confined potential field relative to the unconfined potential field. From Equation (3) of Longcope & Malanushenko (2008),

\[
H(B, P_V, \mathcal{V}) \equiv \int_B d^3x B \cdot A - \int_{P_V} d^3x P_V \cdot A_P \\
+ \int_{z=0} dx dy z B_z(x, y, 0) \int_{x_0}^x d\xi (A(\xi') - A_P(\xi'))
\]

(9)

where $x = r(x, y, 0)$ and $P_V$ is a potential field confined to $\mathcal{V}$ that matches BCs $B \cdot \hat{n} |_{\partial D} = P_V \cdot \hat{n} |_{\partial D}$, by plugging it into $H(\Theta_D B, P_V, \mathcal{V})$ and $H(B, P_V, \Theta_D, \mathcal{V})$ and adding them together it immediately follows that

\[
H(\Theta_D B, P_D, \mathcal{V}) + H(P_D, P_V, \Theta_D, \mathcal{V}) = H(\Theta_D B, P_V, \Theta_D, \mathcal{V}),
\]

(10)

where $D \subset \mathcal{V}$ and $\Theta_D$ is a support function of $D$. By $P_D$ we mean the potential field confined to $D$ (and identically zero outside of $D$) that matches BCs $B \cdot \hat{n} |_{\partial D} = P_D \cdot \hat{n} |_{\partial D}$, and by $P_{V,\Theta}$ we mean the potential field, confined to $\mathcal{V}$, that matches BCs $\Theta_D B \cdot \hat{n} = P_{V,\Theta} \cdot \hat{n} |_{\partial D}$. As long as $D$ is fully contained in $\mathcal{V}$, which is constant in time, the quantity $H_{unc,V}/\Phi^2 \equiv H(\Theta_D B, P_{V,\Theta}, \mathcal{V})/\Phi^2$ will behave like $L$ and $H(P_D, P_{V,\Theta}, \mathcal{V})/\Phi^2$ would then behave like $Wr$.

Figure 14 compares the generalized twist number, $H_A/\Phi^2$, with helicity, unconfined to the flux bundle’s volume, but confined to the computational domain of the simulation: $H_{unc,box}$. In this case $\mathcal{V}$ is the computational domain, a rectangular box. The behavior of all quantities matches expectation: $H_{unc,box}/\Phi^2$ increases as the torus emerges, and stays nearly constant after the emergence is complete (the slight decrease is due to the
reconnection with the arcade field). The generalized twist number, \( H_A/\Phi^2 \), also increases with the emergence, but decreases between \( t = 50 \) and \( t = 58 \)—the time when the torus kinks (see Figure 7). For different \( \sigma_{\text{max}} \), the decrease seems to start at a slightly different time.

Figure 14 demonstrates as well that the general behavior of \( H_{\text{unc}, V}/\Phi^2 \) is qualitatively similar whether the volume \( V \) over which unconfined helicity is computed is the computational domain or the half space. To compute the unconfined helicity in the half space, \( H_{\text{unc}, Z_+} \), we integrate the helicity flux in the way described in DeVore (2000) and used in Fan & Gibson (2004). The helicity flux is computed relative to the potential field in half-space, and thus, the helicity flux, obtained in this way, might be considered “confined to a half-space.”

Longcope & Malanushenko (2008) show that \( H_{\text{unc}, \text{box}} = H_{\text{unc}, Z_+} \) when the volumes, \( V \) and \( Z_+ \), and the vertical field, \( B_z(z = 0) \), all share a reflectional symmetry. This situation occurs in the simulation only for \( t \geq 54 \) when the torus is fully emerged and its major axis is at the photosphere. At these times, the vertical component of the field is the toroidal component of the torus, which is symmetric about \( y = 0 \). Due to reconnection with the arcade, however, the footpoints of \( D \) may not share this symmetry, in which case the photospheric field \( \Theta_{y} B_z \) is not precisely symmetric. If the two helicities were ever to coincide, it would be at \( t = 54 \), so we choose the constant of integration by setting \( H_{\text{unc}, \text{box}} = H_{\text{unc}, Z_+} \) at that time. The time histories of both unconfined helicities are plotted in Figure 14. The discrepancy between the two before \( t = 54 \) arises from the nonvanishing helicity of \( P_{V, \theta} \) relative to \( P_{Z_+, \theta} \) owing to a photospheric field, \( B_z \), lacking reflectional symmetry. In spite of the discrepancy, we draw from each curve the same basic conclusion that the kink deformation of \( D \) does not change \( H_{\text{unc}, V} \).

Figure 15 compares the generalized twist number to the traditional twist number described above. The twist number was computed only for the thinner subvolumes of the torus, \( \sigma_{\text{max}} = 0.5 R \) and \( \sigma_{\text{max}} = R \). Figure 15 shows agreement quite well for \( \sigma_{\text{max}} = R \) and less well for \( \sigma_{\text{max}} = 0.5 R \). The reason might be the following: the smaller the subvolume, the fewer points does it have, so that, first, there are fewer field lines to be traced to measure twist, and second, the potential field obtained by relaxation is numerically less precise. Nevertheless, the magnitudes and the general behaviors do agree.

Figure 15 also shows the twist number measured for the potential field in a subvolume \( P \) is zero to measurement error. Note that a significant portion of the torus is emerged, its length is not large enough (relative to the thickness) for the thin tube approximation to be valid. As the twist of the potential field should theoretically be zero (as well as generalized twist), this plot also gives an idea of the magnitude of the error of twist measurements; at most times, the error is less than 15% of the value.
Figure 14. Comparison between $H_A$ (i.e., confined to the volume of the flux tube), $H_{\text{inc, box}}$ (confined to the box in which the original simulation was performed), and $H_{\text{inc, Z+}}$ (confined to half-space), normalized by $\Phi^2$. Vertical dashed line at $t = 54$ indicates the time when the emergence has stopped and all further changes in $T_w$ would be due to kinking and numerical diffusion, and all earlier changes are altered by the emergence of the tube and thus nonzero helicity flux over the surface. For $\sigma_{\text{max}}$ of 2.0 and 1.0, it is clearly visible that (a) after $t = 54$ the unconfined helicities remain nearly constant, while the confined to flux bundle’s $D$, that is, additive self-helicity, decreases due to kinking; (b) before $t = 54$ the difference between $H_{\text{inc, Z+}}$, that is, the integrated helicity flux, and $H_{\text{inc, box}}$ is nonzero. The threshold for $H_A/\Phi^2$ seems to be $-1.7$ for $\sigma_{\text{max}} = 2.0$ and $-1.4$ for $\sigma_{\text{max}} = 1.0$. $\sigma_{\text{max}} = 0.5$ seems to be too noisy to draw reliable conclusions; possible reasons for that are discussed in the text.

Figure 15. Comparison between the generalized twist number (solid line with diamonds) and the “thin tube” classical twist number (dotted line with asterisks) for two subvolumes of a different size. Also, the “classical” twist number for a potential field (dashed line with squares).
5. DISCUSSION

We have demonstrated that, at least in one MHD simulation, the quantity, $T_{w(\text{gen})}$, defined in terms of the additive self-helicity, shows a threshold beyond which the system became dynamically unstable. The simulation we considered, originally studied by Fan & Gibson (2003), is a three-dimensional, numerical solution of the time-dependent, nonlinear evolution of an emerging flux system. The original study established that the system became unstable to a current-driven (kink) mode at some point during its evolution. In this work, we have shown that the quantity $T_{w(\text{gen})}$ increases until the instability ($T_{w(\text{gen})} \approx 1.5$) at which time it drops. This drop occurs as a natural consequence of the instability itself.

The quantity we propose as having a threshold, $T_{w(\text{gen})}$, is computed using a version of the self-helicity previously defined by Longcope & Malanushenko (2008). The present work has provided a detailed method for computing this quantity for any complex bundle of field lines within a magnetic field known on a computational grid. We also demonstrate that for the very special cases when that bundle can be approximated as a thin flux tube, $T_{w(\text{gen})}$ is approximately equal to the traditional twist number, $T_w$. In the case of thin flux tubes which are also dynamically isolated, free magnetic energy is proportional to $(T_w)^2$. Their free energy may be spontaneously reduced if and when it becomes possible to reduce the magnitude of $T_w$ at the expense of the writhe number, $W_r$, of the tube’s axis.

All this supports the hypothesis that $T_{w(\text{gen})}$ could be treated as a generalization of $T_w$. Such a generalization might be extremely useful in predicting the stability of magnetic equilibria sufficiently complex that they cannot be approximated as thin flux tubes. The case we studied, of a thick, twisted torus of field lines (Fan & Gibson 2003), appears to become unstable when $T_{w(\text{gen})}$ exceeds a threshold value between 1.4 and 1.7. This value happens to be similar to the threshold on $T_w$ for uniformly twisted, force-free flux tubes, $T_w \approx 1.6$, as $\Delta \theta \approx 3.3\pi$ (Hood & Priest 1979).

Previous investigations have shown that the threshold on $T_w$ depends on details of the equilibrium such as internal current distribution (Hood & Priest 1979). It is reasonable to expect the same kind of dependence for any threshold on $T_{w(\text{gen})}$, so we cannot claim that $T_{w(\text{gen})} < 1.7$ for all stable magnetic field configurations. To investigate such a claim is probably intractable, but useful insights may be obtained by applying the above analysis to magnetic equilibria whose stability to the current-driven instability is already known. The paucity of closed-form, three-dimensional equilibria in the literature, and far fewer stability analyses of them suggests this may be a substantial undertaking.

REFERENCES

Baty, H. 2001, A&A, 367, 321
Berger, M. A., & Field, G. B. 1984, J. Fluid Mech., 147, 133
Bernstein, I. B., Frieman, E. A., Kruskal, M. D., & Kulsrud, R. M. 1958, Proc. R. Soc. Lond. A, 244, 17
DeVore, C. R. 2000, ApJ, 539, 944
Fan, Y., & Gibson, S. E. 2003, ApJ, 589, L105
Fan, Y., & Gibson, S. E. 2004, ApJ, 609, 1123
Finn, J., & Antonsen, T. M., Jr. 1985, Comments Plasma Phys. Control. Fusion, 9, 111
Hood, A. W., & Priest, E. R. 1979, Solar Phys., 64, 303
Hood, A. W., & Priest, E. R. 1981, Geophys. Astrophys. Fluid Dyn., 17, 297
Le Veque, R. J. 1955, Finite Difference Methods for Ordinary and Partial Differential Equations, Steady-State and Time-Dependent Problems (Philadelphia: SIAM)
Linton, M. G., & Antiochos, S. K. 2002, ApJ, 581, 703
Longcope, D. W., & Malanushenko, A. 2008, ApJ, 674, 1130
Low, B. C. 1994, Phys. Plasmas, 1, 1684
Moffatt, H. K., & Ricca, R. L. 1992, Proc. R. Soc. Lond. A, 439, 411
Newcomb, W. A. 1960, Ann. Phys., 10, 232
Perot, B. 2000, J. Comput. Phys., 159, 58
Rachmeler, L. A., DeForest, C. E., & Kankelborg, C. C. 2009, ApJ, 693, 1431
Török, T., Kliem, B., & Titov, V. S. 2004, A&A, 413, L27
Zhang, M., Flyer, N., & Low, B. C. 2006, ApJ, 644, 575