The local discontinuous Galerkin method on layer-adapted meshes for time-dependent singularly perturbed convection-diffusion problems

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Abstract

In this paper we analyze the error as well for the semi-discretization as the full discretization of a time-dependent convection-diffusion problem. We use for the discretization in space the local discontinuous Galerkin (LDG) method on a class of layer-adapted meshes including Shishkin-type and Bakhvalov-type meshes and the implicit $\theta$-scheme in time. For piecewise tensor-product polynomials of degree $k$ we obtain uniform or almost uniform error estimates with respect to space of order $k + 1/2$ in some energy norm and optimal error estimates with respect to time. Our analysis is based on careful approximation error estimates for the Ritz projection related to the stationary problem on the anisotropic meshes used. We discuss also improved estimates in the one-dimensional case and the use of a discontinuous Galerkin discretization in time. Numerical experiments are given to support our theoretical results.

Keywords: singular perturbation, layer-adapted mesh, error estimate, local discontinuous Galerkin method

AMS subject classifications: 65M15, 65M12, 65M60

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1 Introduction

We consider the two-dimensional singularly perturbed problem

\[ u_t - \varepsilon \Delta u + a \cdot \nabla u + bu = f, \quad \text{in } \Omega \times (0, T], \quad (1a) \]

\[ u|_{t=0} = u_0, \quad \text{in } \Omega, \quad (1b) \]

\[ u|_{\partial \Omega} = 0, \quad \text{for } t \in (0, T], \quad (1c) \]

where \( \Omega = (0, 1)^2 \), \( T > 0 \), \( a = a(x, y), b = b(x, y), f = f(x, y, t), u_0 = u_0(x, y) \) are sufficiently smooth and

\[ a = (a^1(x, y), a^2(x, y)) \geq (\alpha_1, \alpha_2), \quad b - \frac{1}{2} \nabla \cdot a \geq \beta, \quad (x, y) \in \Omega, \quad (2) \]

for some positive constants \( \alpha_1, \alpha_2 \) and \( \beta \).

Such singularly perturbed problems arise in many applications. They serve as a model problem which helps to understand the behavior of numerical methods in the presence of layers for more complex problems like the linearized Navier-Stokes equations with high Reynolds number, the drift-diffusion equations of semiconductor device modeling, and the Black-Scholes equation from financial modeling [11].

The solution of such problems usually exhibits rapid changes along the boundary, i.e., the solution forms boundary layers. The use of a standard Galerkin finite element method yields unsatisfactory numerical solution in that situation. Even on an adapted mesh small oscillations occur, and it can be difficult to solve the associated discrete algebraic system efficiently [9].

Therefore, many stabilised finite element methods were developed. Several such methods are described in [11]. The methods described in [11], for instance, streamline-diffusion finite element method (SDFEM), local projection stabilization (LPS), continuous interior penalty (CIP) and discontinuous Galerkin (DG) methods like non-symmetric interior penalty DG (NIPG) and symmetric interior penalty DG (SIPG) use the primal weak formulation for the stationary problem. Instead, we use the flux formulation, especially the local discontinuous Galerkin (LDG) method.

The LDG method is a kind of finite element methods which was first proposed as a generalization of the discontinuous Galerkin method for a convection-diffusion problem [13]. This method inherits many advantages of the DG method and can simulate the acute change of singular solution very well [17]. The LDG method has also already been used to solve singularly perturbed problems. In [16], Xie et al. demonstrated that the LDG solution does not produce any oscillations on a uniform mesh. In [2, 3], Cheng et al. realized a local stability analysis and proved double-optimal local error estimates for two explicit fully-discrete LDG methods on quasi-uniform meshes. In [18], Zhu et al. performed a uniform convergence analysis of the LDG method on a standard Shishkin mesh for a stationary problem.

However, as far as the authors’ know, there exist no results concerning the uniform convergence of the fully-discrete LDG method on general layer-adapted meshes. Our paper is characterized by the following features:
1. In the literature, often Shishkin meshes are used for the discretization of singularly perturbed problems due to the simple structure of the mesh. However, this mesh gives non-optimal error estimates up to a logarithmic factor which gets more and more influence for an increasing degree of the polynomials used. We consider meshes defined by an adequately chosen mesh-generating function in the layer region which are uniform outside the layer region. Thus we allow as well Shishkin-type (S-type) meshes as Bakhvalov-type (B-type) meshes, introduced in [10]. Our unified analysis yields optimal error estimates for S-type meshes, the results for B-type meshes are almost optimal.

2. The key ingredient in our analysis is the use of the Ritz projection related to the stationary problem. The approximation error estimates obtained on our class of anisotropic meshes are new. In the analysis, three two-dimensional local Gauss-Radau projections as well as their stability and approximation property play an important role. It is worthy to point out that a modified superconvergence property and a stability result of [18] are particularly used to derive the error estimates on the considered class of meshes.

3. For simplicity, we use an implicit θ-scheme for the discretization in time and establish (almost) optimal error estimates for the fully discrete LDG-scheme. In the literature, also BDF methods (backward differencing) [6] and Runge-Kutta methods [14] are used. Since the spatial mesh size is extremely small in the boundary layer region, explicit time discretizations in [2, 3] are not suitable. Alternatively, one can also use a DG method for the discretization in time, as described in the Appendix (see also [7], where general stabilization methods for the primal weak formulation are combined with DG in time).

The paper is organized as follows. In Section 2 we present the LDG method for the given singularly perturbed problem [1]. Layer-adapted meshes are introduced in Section 3. In Section 4 we define a Ritz projection and estimate its approximation errors. Based on this projection, we realize error estimates as well for the semi-discrete as for the fully-discrete LDG method in Section 5. In Section 6 we present some numerical experiments. In the last section, we sketch improved results for the one-dimensional case and the DG discretization in time.

2 The LDG method

In this section we present the semi-discrete and the fully-discrete LDG method for [1]. Let \( \Omega_N = \{ K_{ij} \}_{i=1,2,...,N_x}^{j=1,2,...,N_y} \) be a rectangular tessellation of \( \Omega \) with elements \( K_{ij} = I_i \times J_j \), where \( I_i = (x_{i-1}, x_i) \) and \( J_j = (y_{j-1}, y_j) \). Denote by \( h_{x,i} = x_i - x_{i-1}, \) \( h_{y,j} = y_j - y_{j-1}, \) \( h_{ij} = \max\{h_{x,i}, h_{y,j}\} \), \( h_{ij} = \min\{h_{x,i}, h_{y,j}\} \), \( h = \max_{K_{ij} \in \Omega_N} h_{ij} \) and
\[ h = \min_{K_{ij} \in \Omega} h_{ij}. \] Define the discontinuous finite element space as

\[ \mathcal{W}_N = \{ v \in L^2(\Omega) : v|_K \in \mathcal{Q}^k(K), K \in \Omega_N \}, \]

where \( \mathcal{Q}^k(K) \) denotes the space of the tensor-product polynomials of degree \( k \) in each variable on \( K \). \( \mathcal{W}_N \) is contained in the broken Sobolev space

\[ \mathcal{H}^1(\Omega_N) = \{ v \in L^2(\Omega) : v|_K \in H^1(K), K \in \Omega_N \}, \]

whose functions allow discontinuity across element interfaces. For \( v \in \mathcal{H}^1(\Omega_N) \) and \( y \in J_j, j = 1, 2, \ldots, N_y \), we use \( v^{\pm} = \lim_{x \to x^\pm} v(x, y) \) to express the traces evaluated from the right element \( K_{i+1,j} \) and the left element \( K_{ij} \). The jump on the vertical edge is thus denoted by \( [v]_{i,y} = v^{+}_{i,y} - v^{-}_{i,y} \) for \( i = 1, 2, \ldots, N_x - 1 \), \( [v]_{0,y} = v^{+}_{0,y} \) and \( [v]_{N_x,y} = -v^{-}_{N_x,y} \). Analogously, for \( x \in I_i \) and \( i = 1, 2, \ldots, N_x \), we can define the traces \( v^{\pm}_x \) and the jumps \( [v]_{x,j}, j = 0, 1, \ldots, N_y \) on the horizontal edges.

Rewrite (1) into an equivalent first order system

\[ u_t - p_x - q_y + a^1 u_x + a^2 u_y + bu = f, \quad \varepsilon^{-1} p = u_x, \quad \varepsilon^{-1} q = u_y, \quad \text{in} \ \Omega. \]  

Let \( \langle \cdot, \cdot \rangle_D \) be the inner product in \( L^2(D) \). Then, the semi-discrete LDG method reads: For any \( t \in (0, T] \), find such that in each element \( K_{ij} \) the following variational equations

\[
\langle u_t, \psi \rangle_{K_{ij}} + \langle (b - a^1_x - a^2_y) u, \psi \rangle_{K_{ij}} \\
- \langle a^1 u - p, \psi_x \rangle_{K_{ij}} + \langle a^1_{i,y} \tilde{u}_{i,y} - \tilde{p}_{i,y}, \psi_{i,y} \rangle_{J_j} = \langle a^1_{i-1,y} \tilde{u}_{i-1,y} - \tilde{p}_{i-1,y}, \psi_{i-1,y} \rangle_{J_j} \\
- \langle a^2 u - q, \psi_y \rangle_{K_{ij}} + \langle a^2_{x,j} \tilde{u}_{x,j} - \tilde{q}_{x,j}, \psi_{x,j} \rangle_{I_i} = \langle a^2_{x-1,j} \tilde{u}_{x-1,j} - \tilde{q}_{x-1,j}, \psi_{x-1,j} \rangle_{I_i} = \langle f, \psi \rangle_{K_{ij}},
\]

\[
\varepsilon^{-1} \langle p, s \rangle_{K_{ij}} + \langle u, s_x \rangle_{K_{ij}} - \langle \tilde{u}_{i,y}, s_{i,y} \rangle_{J_j} + \langle \tilde{u}_{i-1,y}, s_{i-1,y} \rangle_{J_j} = 0,
\]

\[
\varepsilon^{-1} \langle q, r \rangle_{K_{ij}} + \langle u, r_x \rangle_{K_{ij}} - \langle \tilde{u}_{x,j}, r_{x,j} \rangle_{I_i} + \langle \tilde{u}_{x,j-1}, r_{x,j-1} \rangle_{I_i} = 0,
\]

hold for any \( z = (\psi, s, r) \in \mathcal{W}_N^3 \). Here the numerical fluxes are defined by

\[
\hat{p}_{i,y} = \begin{cases} 
  p^{+}_{i,y}, & i = 0, 1, \ldots, N_x - 1, \\
  p_{N_x,y} - \lambda_{N_x,y}^+ n_{N_x,y}, & i = N_x,
\end{cases}
\]

\[
\tilde{u}_{i,y} = \begin{cases} 
  0, & i = 0, N_x, \\
  u^+_{i,y}, & i = 1, 2, \ldots, N_x - 1,
\end{cases}
\]

\[
\tilde{u}_{i,y} = \begin{cases} 
  0, & i = 0, \\
  u^-_{i,y}, & i = 1, 2, \ldots, N_x,
\end{cases}
\]

for \( y \in J_j \) and \( j = 1, 2, \ldots, N_y \). We use \( \lambda_{N_x,y}^+ = \varepsilon/h_{x,N_x} \). Analogously, for \( x \in I_i \) and \( i = 1, 2, \ldots, N_x \), we can define \( \hat{q}_{x,j}, \tilde{u}_{x,j} \) and \( \tilde{u}_{x,j}, j = 0, 1, \ldots, N_y \).
To complete the definition of the LDG method, we take the initial value \( u(0) = \pi u_0 \) as the local \( L^2 \)-projection of \( u_0 \). That means, \( \pi u_0 \) is the unique function in \( W_N \), such that in each element \( K_{ij} \) it holds
\[
\langle \pi u_0, v \rangle_{K_{ij}} = \langle u_0, v \rangle_{K_{ij}}, \quad \forall v \in Q^h(K_{ij}). \tag{6}
\]

For notational convenience, we use \( \langle w, v \rangle = \sum_{K_{ij} \in \Omega_N} \langle w, v \rangle_{K_{ij}} \) and write the above LDG method into a compact form:
For any \( t \in (0, T] \), find \( w = (u, p, q) \in W_N^3 \), such that
\[
\langle u_t, v \rangle + B(w; z) = \langle f, v \rangle \quad \forall z = (v, s, r) \in W_N^3, \tag{7}
\]
where
\[
B(w; z) = T_1(w; z) + T_2(u; z) + T_3(w; v) + T_4(u; v), \tag{8}
\]
with
\[
\begin{align*}
T_1(w; z) &= \varepsilon^{-1} [\langle p, s \rangle + \langle q, v \rangle] + \langle (b - a_1^1 - a_2^y)u, v \rangle, \\
T_2(u; z) &= \langle u, s_x \rangle + \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} \langle u_{i,y}, [s^i]_{i,y} \rangle J_j + \langle u, s_y \rangle + \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \langle u_{x,j}, [v^x]_{x,j} \rangle L_i, \\
T_3(w; v) &= \langle p, v_x \rangle + \sum_{i=1}^{N_x} \left[ \sum_{j=0}^{N_y-1} \langle p^+_{i,y}, [v^i]_{i,y} \rangle J_j - \langle p^+_{i,y}, [v^i]_{i,y} \rangle J_j \right] \\
&\quad + \langle q, v_y \rangle + \sum_{j=0}^{N_y-1} \left[ \langle q^+_{x,j}, [v^x]_{x,j} \rangle L_i - \langle q^+_{x,j}, [v^x]_{x,j} \rangle L_i \right], \\
T_4(u; v) &= -\langle a^1 u, v_x \rangle - \sum_{j=1}^{N_y} \left[ \sum_{i=1}^{N_x} \langle u^{i,y}_{i,y}, [u^i_{i,y}]_{i,y} \rangle J_j - \langle \lambda_{N_x,y} [u^i_{i,y}]_{i,y} \rangle J_j \right] \\
&\quad - \langle a^2 u, v_y \rangle - \sum_{j=1}^{N_y} \left[ \sum_{i=1}^{N_x} \langle a^2 u^{i,y}_{i,y}, [v^x]_{x,j} \rangle L_i - \langle \lambda_{N_x,y} [v^x]_{x,j} \rangle L_i \right].
\end{align*}
\]

Induced by (8), we define an energy norm \( ||w||_E^2 = B(w; w) \), that means,
\[
||w||_E^2 = \varepsilon^{-1} ||p||^2 + \varepsilon^{-1} ||q||^2 + ||(b - a_1^1/2 - a_2^y/2)^{1/2}u||^2 \\
+ \sum_{j=1}^{N_y} \left[ \sum_{i=0}^{N_x} \frac{1}{2} \langle a^1_{i,y}, [u^i_{i,y}]_{i,y} \rangle J_j + \langle \lambda_{N_x,y} [u^i_{i,y}]_{i,y} \rangle J_j \right] \\
+ \sum_{i=1}^{N_x} \left[ \sum_{j=0}^{N_y} \frac{1}{2} \langle a^2_{x,j}, [u^x_{x,j}]_{x,j} \rangle L_i + \langle \lambda_{N_x,y} [u^x_{x,j}]_{x,j} \rangle L_i \right]. \tag{9}
\]
Next we introduce a fully-discrete LDG method, which is the combination of the semi-discrete LDG method (7) and an implicit \(\theta\)-scheme with \(\theta \in [1/2, 1]\).

Let \(M\) be a positive integer and \(0 = t^0 < t^1 < \cdots < t^M = T\) be an equidistant partition of \([0, T]\). Define the time interval \(K^m = (t^{m-1}, t^m], \ m = 1, 2, \ldots, M\), with the mesh width \(\Delta t = \frac{t^m - t^{m-1}}{M}\) satisfying \(M\Delta t = T\). Denote by \(v^m = v(t^m)\) and \(v_{\theta}^m = \theta v^m + (1 - \theta)v^{m-1}\).

The fully-discrete LDG \(\theta\)-scheme for (1) reads:

Take \(U^0 = \pi u_0\) as before. For any \(m = 1, 2, \ldots, M\), find \(W^m = (U^m, P^m, Q^m) \in W^3_N\) such that

\[
\left\langle \frac{U^m - U^{m-1}}{\Delta t}, v \right\rangle + B(W^m, \theta; z) = \left\langle f^m, \theta, v \right\rangle,
\]

holds for any \(z = (v, s, r) \in W^3_N\), where \(B(\cdot; \cdot)\) is defined in (8).

### 3 Layer-adapted meshes

In this section, we introduce a class of layer-adapted meshes based on some precise information on the exact solution of (1) and its derivatives.

**Proposition 3.1.** Assume \(u \in C^{l+\kappa} \times (0, T)\) with some adequate positive integer \(l\) and \(0 < \kappa < 1\). Moreover, assume the existence of a decomposition of the solution into a smooth term \(S\) and layer components

\[
u = S + E_{21} + E_{12} + E_{22},
\]

where \(S\) and the layer components satisfy

\[
|\partial_x^i \partial_y^j \partial_t^m S(x, y, t)| \leq C, \quad (12a)
\]

\[
|\partial_x^i \partial_y^j \partial_t^m E_{21}(x, y, t)| \leq C\varepsilon^{-i}e^{-\alpha_1(1-x)/\varepsilon}, \quad (12b)
\]

\[
|\partial_x^i \partial_y^j \partial_t^m E_{12}(x, y, t)| \leq C\varepsilon^{-j}e^{-\alpha_2(1-y)/\varepsilon}, \quad (12c)
\]

\[
|\partial_x^i \partial_y^j \partial_t^m E_{22}(x, y, t)| \leq C\varepsilon^{-(i+j)}e^{-[\alpha_1(1-x) + \alpha_2(1-y)]/\varepsilon}, \quad (12d)
\]

for positive integers \(i, j\) and \(m\) with \(i + j + 2m \leq l\). Here \(C > 0\) is a constant independent of \(\varepsilon\).

Shishkin proved the existence of such a decomposition under certain conditions (smoothness of the data, strong compatibility), see also [11], Part III, Chapter 4.

Our class of layer-adapted meshes is now constructed as follows. For the notational simplification, we assume that \(\alpha_1 = \alpha_2 = \alpha\), \(N_x = N_y = N\). Let \(N \geq 2\) be an even integer. We introduce the mesh points

\[
0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \quad 0 = y_0 < y_1 < \cdots < y_{N-1} < y_N = 1,
\]

6
and consider a tensor-product mesh with mesh points \((x_i, y_j)\). Since both meshes have the same structure we only describe the mesh in \(x\)-direction.

The mesh is equidistant on \([0, x_{N/2}]\) with \(N/2\) elements, but gradually divided on \([x_{N/2}, 1]\) with \(N/2\) elements, where \(x_{N/2} = 1 - \tau\) with

\[
\tau = \min \left\{ \frac{1}{2}, \frac{\sigma \varepsilon}{\alpha} \varphi \left( \frac{1}{2} \right) \right\}. \tag{13}
\]

Here \(\sigma > 0\) is a user-chosen parameter and the function \(\varphi\) satisfies

\[
\varphi(0) = 0, \quad \varphi' > 0, \quad \varphi'' \geq 0. \tag{14}
\]

Assume that \(\varepsilon \leq N^{-1}\) throughout the paper and it is so small that (13) can be replaced by \(\tau = \alpha^{-1} \sigma \varepsilon \varphi \left( \frac{1}{2} \right)\) as otherwise the problem can be analyzed in a classical manner.

The mesh points are given by \(x_i = \lambda(i/N) \ (i = 0, 1, \ldots, N)\) with the mesh generating function

\[
\lambda(t) = \begin{cases} 
2(1 - \tau)t, & \text{for } t \in [0, \frac{1}{2}], \\
1 - \frac{\sigma \varepsilon}{\alpha} \varphi(1 - t), & \text{for } t \in \left[ \frac{1}{2}, 1 \right].
\end{cases} \tag{15}
\]

Introduced in [10], meshes with \(\varphi(1/2) = \ln N\) are called Shishkin-type meshes (S-type), meshes with \(\varphi(1/2) = \ln(1/\varepsilon)\) are Bakhvalov-type meshes (B-type). Remark that the original Bakhvalov mesh has a continuously differentiable mesh generating function.

In the analysis of numerical methods on our class of meshes the so called mesh characterizing function \(\psi\), defined by \(\psi = e^{-\varphi}\), plays an important role.

In Table 1, we list three often used layer-adapted meshes, Shishkin-meshes (S-mesh), Bakhvalov-Shishkin meshes (BS-mesh) and a Bakhvalov-type mesh (B-type mesh) together with \(\psi\) and the important quantity \(\max |\psi'|\), which arises in error estimates. Figure 1 illustrates the generating functions on these meshes and the generated meshes, when \(\varepsilon = 10^{-2}\) and \(N = 16\) are chosen. For a survey concerning layer-adapted meshes see also [8].

| S-mesh | BS-mesh | B-type mesh |
|--------|---------|-------------|
| \(\varphi(t)\) | \(2t \ln N\) | \(- \ln \left[ 1 - 2(1 - N^{-1})t \right]\) |
| \(\psi(t)\) | \(N^{-2t}\) | \(1 - 2(1 - \varepsilon)t\) |
| \(\max |\psi'|\) | \(C \ln N\) | \(C\) |

As mentioned already, the two-dimensional layer-adapted mesh is obtained by the tensor product principle. We define the following subregions:

\[
\Omega_{11} = (0, 1 - \tau) \times (0, 1 - \tau), \quad \Omega_{21} = (1 - \tau, 1) \times (0, 1 - \tau),
\]

\[
\Omega_{12} = (0, 1 - \tau) \times (1 - \tau, 1), \quad \Omega_{22} = (1 - \tau, 1) \times (1 - \tau, 1).
\]
Obviously, the elements in $\Omega_{11}$ are uniform with a mesh size of order $N^{-1}$, while the elements in $\Omega_{12}$ and $\Omega_{21}$ are highly anisotropic.

In the sequel, we state two preliminary lemmas, which reflect the smallness of a weighted layer-function and the information about the mesh size for the meshes considered. Since $h_{x,i} = h_{y,i}$, $i = 1, 2, \ldots, N$, we simple use $h_i$ to represent one of them.

**Lemma 3.1.** Denote by $\Theta_i = \min\{h_i/\varepsilon, 1\}e^{-\alpha(1-x_i)/\sigma\varepsilon}$ for $i = N/2 + 1, \ldots, N$. There exists a constant $C > 0$ independent of $\varepsilon$ and $N$ such that

\begin{align}
\max_{N/2+1 \leq i \leq N} \Theta_i &\leq CN^{-1} \max |\psi'|, \\
\sum_{i=N/2+1}^{N} \Theta_i &\leq C.
\end{align}

**Proof.** Let $t_i = i/N$. For any $i = N/2 + 1, \ldots, N$, one has

\[
e^{-\frac{\alpha(1-x_i)}{\sigma\varepsilon}} = e^{-\psi(1-t_i)} = e^{\ln(1-t_i)} = \psi(1-t_i),
\]

\[
h_i = \frac{\sigma\varepsilon}{\alpha} [\varphi(1-t_{i-1}) - \varphi(1-t_i)] = \frac{\sigma\varepsilon}{\alpha} \int_{1-t_i}^{1-t_{i-1}} \varphi'(s)ds = \frac{\sigma\varepsilon}{\alpha} \int_{1-t_i}^{1-t_{i-1}} -\frac{\psi'(s)}{\psi(s)}ds
\]

\[
\leq \frac{\sigma\varepsilon}{\alpha} \int_{1-t_i}^{1-t_{i-1}} -\psi'(s)ds = \frac{\sigma\varepsilon}{\alpha} \frac{\psi(1-t_i) - \psi(1-t_{i-1})}{\psi(1-t_{i-1})},
\]

where we used the monotone decrease property of $\psi = e^{-\varphi}$ due to $\varphi' > 0$. Therefore,
one obtains

\[
\Theta_i \leq C \min \left\{ \frac{\psi(1 - t_i) - \psi(1 - t_{i-1})}{\psi(1 - t_{i-1})}, 1 \right\} \psi(1 - t_i) \\
= C \min \left\{ 1 + \frac{\psi(1 - t_i) - \psi(1 - t_{i-1})}{\psi(1 - t_{i-1})}, 1 + \frac{\psi(1 - t_{i-1})}{\psi(1 - t_i) - \psi(1 - t_{i-1})} \right\} \\
\cdot \left[ \psi(1 - t_i) - \psi(1 - t_{i-1}) \right]
\]

\[
\leq C \left[ \psi(1 - t_i) - \psi(1 - t_{i-1}) \right],
\]

due to the trivial inequality \(1 < \min\{1 + 1/x, 1 + x\} \leq 2\) for \(x > 0\). This leads immediately to (16). \(\Box\)

**Lemma 3.2.** For our three types of layer-adapted meshes we have \(h_1 = h_2 = \cdots = h_{N/2}\) and \(h \geq C\varepsilon N^{-1} \max |\psi'|\). Furthermore,

\[
h_{N/2+1} = h_{N/2+2} = \cdots = h_N, \quad \text{for a S-mesh,} \tag{17}
\]

\[
i = N/2 + 1, \ldots, N, \quad 1 \geq \frac{h_{i+1}}{h_i} \geq C, \quad \text{for a BS-mesh,} \tag{18}
\]

\[
i = N/2 + 2, \ldots, N, \quad 1 \geq \frac{h_{i+1}}{h_i} \geq C, \quad \text{for a B-type mesh,} \tag{19}
\]

\[
i = 1, 2, \ldots, N/2, \quad h_{N/2+i} \geq \frac{\sigma \varepsilon}{\alpha i+1}, \quad \text{for a B-type mesh,} \tag{20}
\]

where \(C > 0\) is a constant independent of \(\varepsilon\) and \(N\).

**Proof.** It is obvious that \(h_1 = h_2 = \cdots = h_{N/2}\). Moreover, (17)-(19) can be verified easily, see also [12].

By the definition (15), assumption (14) and \(\psi = e^{-\varphi}\), one has

\[
h \geq C\varepsilon N^{-1} \min |\varphi'| = C\varepsilon N^{-1} |\varphi'(0)| = C\varepsilon N^{-1} |\psi'(0)| = C\varepsilon N^{-1} \max |\psi'|. \tag{21}
\]

Furthermore, on the B-type mesh, it holds

\[
h_{N/2+i} = \frac{\sigma \varepsilon}{\alpha} \ln \left( \frac{1 - 2(1 - \varepsilon)(1 - t_{N/2+i})}{1 - 2(1 - \varepsilon)(1 - t_{N/2+i-1})} \right) = \frac{\sigma \varepsilon}{\alpha} \ln \left( 1 + \frac{2(1 - \varepsilon)}{N\varepsilon + 2(1 - \varepsilon)(i - 1)} \right) \\
\geq \frac{\sigma \varepsilon}{\alpha} \ln \left( 1 + \frac{1}{i} \right) \geq \frac{\sigma \varepsilon}{\alpha i+1}, \quad i = 1, 2, \cdots, N/2,
\]

because \(0 < N\varepsilon \leq 1 \leq 2(1 - \varepsilon)\) and \(\ln(1 + x) \geq x/(1 + x)\) for any \(x > 0\). \(\Box\)
4 A Ritz projection

In this section we introduce a Ritz projection related to the stationary problem and establish its approximation properties.

For this purpose, we define three two-dimensional local Gauss-Radau projections. To be specific, for each element $K_{ij} \in \Omega_N$ and any $z \in C(K_{ij})$, projections $\Pi^{-} z, \Pi_{x}^{+} z, \Pi_{y}^{+} z \in Q^{k}(K_{ij})$ are defined as

$$
\begin{align*}
\Pi_{ij}(\Pi^{-} z) & = \int_{K_{ij}} z v dx dy, \quad \forall v \in Q^{k-1}(K_{ij}), \\
\Pi_{i}(\Pi^{-} z)_{i,y} & = \int_{I_i} z_{i,y} v dy, \quad \forall v \in P^{k-1}(J_i), \\
\Pi_{ij}(\Pi_{x}^{+} z) & = \int_{K_{ij}} z_{i,x} v dx, \quad \forall v \in P^{k-1}(I_i), \\
(\Pi^{-})(z)(x_{i}, y_{j}) & = z(x_{i}, y_{j}).
\end{align*}
$$

Using these definitions and Lemma 2.14 of [1] we fix the $L^{\infty}$-stability property and the approximation error estimates

$$
\begin{align*}
\|\Phi z\|_{L^{\infty}(K_{ij})} & \leq C \|z\|_{L^{\infty}(K_{ij})}, \\
\|z - \Phi z\|_{L^{\infty}(K_{ij})} & \leq C \left[ h_{i}^{k+1} \|\partial x^{k+1} z\|_{L^{\infty}(K_{ij})} + h_{j}^{k+1} \|\partial y^{k+1} z\|_{L^{\infty}(K_{ij})} \right],
\end{align*}
$$

where $\Phi \in \{\Pi^{-}, \Pi_{x}^{+}, \Pi_{y}^{+}\}$.

The next lemma states some approximation error estimates for $\zeta_u = u - \Pi^{-} u$, $\zeta_p = p - \Pi_{x}^{+} p$ and $\zeta_q = q - \Pi_{y}^{+} q$.

**Lemma 4.1.** Let $\sigma \geq k + 1$. Then it holds

$$
\begin{align*}
\|\zeta_u\|_{\Omega_{ii}} & \leq C N^{-(k+1)}, \quad \|\zeta_u\|_{L^{\infty}(\Omega_{\Omega_{ii}})} \leq C (N^{-1} \max |\psi'|)^{k+1}, \\
\sum_{j=1}^{N} \| (\zeta_u)_{N,y} \|_{J_{j}}^{2} + \sum_{i=1}^{N} \| (\zeta_u)_{x,N} \|_{I_{i}}^{2} & \leq C (N^{-1} \max |\psi'|)^{2(k+1)}, \\
\sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \| (\zeta_u)_{i,y} \|_{J_{j}}^{2} + \| (\zeta_u)_{x,j} \|_{I_{i}}^{2} \right] & \leq C (N^{-1} \max |\psi'|)^{2k+1}, \\
\sum_{j=1}^{N} \sum_{i=0}^{N} \langle 1, [\zeta_u]_{i,y}^{2} \rangle_{J_{j}} + \sum_{i=1}^{N} \sum_{j=0}^{N} \langle 1, [\zeta_u]_{x,j}^{2} \rangle_{I_{i}} & \leq C (N^{-1} \max |\psi'|)^{2k+1}, \\
\varepsilon^{-\frac{1}{2}} \|\zeta_q\| + \varepsilon^{-\frac{1}{2}} \|\zeta_p\| & \leq C (N^{-1} \max |\psi'|)^{k+1}, \\
\sum_{i=1}^{N} \| (\zeta_q)_{x,N} \|_{I_{i}}^{2} + \sum_{j=1}^{N} \| (\zeta_p)_{N,y} \|_{J_{j}}^{2} & \leq C (N^{-1} \max |\psi'|)^{2(k+1)},
\end{align*}
$$

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where $C > 0$ is independent of $\varepsilon$ and $N$.

Proof. Let us first prove \textbf{(26a)} based on the decomposition $u = S + E_{21} + E_{12} + E_{22}$. Obviously, it holds for the smooth component $S$ due to the approximation property \textbf{(25b)} and the estimates \textbf{(12)}.

Next we have to study each layer component separately. For $K_{ij} \in \Omega_{11} \cup \Omega_{12}$, one obtains from the $L^\infty$-stability \textbf{(25a)}

$$\|\zeta_{E_{21}}\|_{L^\infty(K_{ij})}^2 \leq C \|E_{21}\|_{L^\infty(K_{ij})}^2 \leq C e^{-2\alpha(1-x_i)/\varepsilon} \leq C e^{-2\alpha\tau/\varepsilon} \leq CN^{-2(k+1)}, \quad (27a)$$
due to $\sigma \geq k + 1$ and $\varphi(1/2) \geq \ln N$. For $K_{ij} \in \Omega_{21} \cup \Omega_{22}$, one obtains from the stability \textbf{(25a)} and the approximation property \textbf{(25b)}

$$\|\zeta_{E_{21}}\|_{L^\infty(K_{ij})}^2 \leq C \min \left\{ \|E_{21}\|_{L^\infty(K_{ij})}^2, \frac{h_i}{\varepsilon} \right\} e^{-2\alpha(1-x_i)/\varepsilon} \leq C \|\zeta_{E_{22}}\|_{L^\infty(K_{ij})}^2 \leq C(\max \{\psi'\})^{2(k+1)}, \quad (27b)$$

where we used \textbf{(12)}, $h_i/\varepsilon \geq CN^{-1} \geq Ch_{ij}$, $\sigma \geq k + 1$ and Lemma 3.1. Similarly, we estimate $\zeta_{E_{12}}$.

For $K_{ij} \in \Omega_{11} \cup \Omega_{12} \cup \Omega_{21}$, one also has $\|\zeta_{E_{22}}\|_{L^\infty(K_{ij})}^2 \leq C N^{-2(k+1)}$. For $K_{ij} \in \Omega_{22}$, one obtains

$$\|\zeta_{E_{22}}\|_{L^\infty(K_{ij})}^2 \leq C \min \left\{ \|E_{22}\|_{L^\infty(K_{ij})}^2, \frac{h_i}{\varepsilon} \right\} e^{-2\alpha(1-x_i)/\varepsilon} \leq C \|\zeta_{E_{12}}\|_{L^\infty(K_{ij})}^2 \leq C(N^{-1} \max \{\psi'\})^{2(k+1)}, \quad (27c)$$

where we used a trivial inequality $\min \{1, a + b\} \leq \min \{1, a\} + \min \{1, b\}$ for any real number $a, b > 0$ and Lemma 3.1. Summarizing, one gets \textbf{(26a)}.

Next we prove \textbf{(26b)} and \textbf{(26c)}. Noticing that $\Pi^- = \pi_x^- \otimes \pi_y^-$, where $\pi_x^-$ is one dimensional Gauss-Radau projection in the $x$-direction (see (A.7)). Thus, for any $i = 1, 2, \ldots, N$, one has $(\zeta_u)_{i,y} = u_{i,y} - \pi_y^-(u_{i,y})$, see [5]. According to the decomposition $u_{i,y} = S(x_i, y, t) + E(x_i, y, t)$ and the estimates $|\partial^l_y \partial^m_t S(x_i, y, t)| \leq C$ and $|\partial^l_y \partial^m_t E(x_i, y, t)| \leq C \varepsilon^{-1} e^{-\alpha(1-y)/\varepsilon}$, one easily derives

$$\sum_{j=1}^{N} \|\zeta_{(\zeta_u)_{i,y}}\|_{L^2}^2 \leq C \left[ N^{-2(k+1)} + N^{-1} \left( \max \{\psi'\} \right)^{2k+1} \right]$$

by using the solution decomposition, the $L^\infty$-stability and the $L^\infty$-approximation property of one-dimensional Gauss-Radau projections. Similarly, one can bound $\sum_{i=1}^{N} \|\zeta_{(\zeta_u)_{i,x}}\|_{L^2}^2$. Then \textbf{(26b)} and \textbf{(26c)} follow immediately.
To prove (26d), one starts from the following inequality:

$$\sum_{j=1}^{N} \sum_{i=0}^{N} \langle 1, [\zeta_u]_{i,y}^2 \rangle_{J_j} \leq 2 \left[ \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{J_j} [(\zeta_u)_{i-1,y}^+]^2 dy + \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{J_j} [(\zeta_u)_{i,y}^-]^2 dy \right].$$

For the first term one notices

$$\sum_{j=1}^{N} \sum_{i=1}^{N} \int_{J_j} [(\zeta_u)_{i-1,y}^+]^2 dy \leq \sum_{j=1}^{N} \sum_{i=1}^{N} h_j \|\zeta_u\|_{L^\infty(K_{ij})}^2$$

and proceeds as in (26a) using (16b). For the second term one uses (26c). Similarly, the remaining inequalities of (26) can be proved analogously, we omit the details.

Theorem 4.1. Let $\sigma \geq k + 2$. There exists a constant $C > 0$ independent of $\varepsilon$ and $N$ such that

$$\|u_t - F_1 u_t\| + \|w - Fw\|_E \leq CQ^* N^{-(k+1/2)},$$

where

$$Q^* = \begin{cases} \ln N \ln(1/\varepsilon) & \text{for a S-mesh}, \\ 1 & \text{for a BS-mesh}, \\ \max \left\{ \sqrt{N^{-1} \ln(1/\varepsilon)}, 1 \right\} & \text{for a B-type mesh}. \end{cases}$$

Proof. We start from $w - Fw = \zeta - \delta$ with

$$\zeta = (\zeta_u, \zeta_p, \zeta_q) = (u - \Pi^- u, p - \Pi^+_x p, q - \Pi^+_y q),$$

$$\delta = (\delta_u, \delta_p, \delta_q) = (F_1 u - \Pi^- u, F_2 p - \Pi^+_x p, F_3 q - \Pi^+_y q) \in W^3_N.$$

By [28], one has

$$B(\delta; z) = B(\zeta; z) \quad \forall z = (v, s, r) \in W^3_N.$$
(1) We shall show that
\[ B(\zeta; z) \leq CQ^* N^{-(k+1/2)} \| z \|_z \]
for any test function \( z \in \mathcal{W}_N^3 \), where
\[ \|z\|_z \equiv C \left[ \| z \|_E^2 + N^{-1} \| \alpha \cdot \nabla v \|_{\Omega_1}^2 + \sum_{K_{ij} \in \Omega \setminus \Omega_1} \tilde{h}_{ij} \| \nabla v \|_{K_{ij}}^2 \right]^{1/2}. \]

Here \( \tilde{h}_{ij} = \min\{h_{ij}, h_{i+1,j}, h_{i,j+1}\} \) and \( \tilde{h}_G = \min_{K_{ij} \in G} \tilde{h}_{ij} \).

To get (33), we bound each term in
\[ B(\zeta; z) = T_1(\zeta; z) + T_2(\zeta_u; z) + T_3(\zeta; v) + T_4(\zeta_u; v). \]

Using the Cauchy-Schwarz inequality, (26a) and (26e) yield
\[ T_1(\zeta; z) \leq C \left( \varepsilon^{-1/2} \| \zeta_p \| + \varepsilon^{-1/2} \| \zeta_t \| + \| \zeta_u \| \right) \| z \|_E \]
\[ \leq C (N^{-1} \max |\psi'|)^{k+1} \| z \|_E. \]

By the definitions (23)-(24), Cauchy-Schwarz inequality and (26f), one obtains easily
\[ T_3(\zeta; v) = - \sum_{j=1}^N \left\langle (\zeta_p)_{N,y}, v_{N,y}^+ \right\rangle_{I_j} - \sum_{i=1}^N \left\langle (\zeta_q)_{x,N}, v_{x,N}^- \right\rangle_{I_i} \]
\[ \leq C (N^{-1} \max |\psi'|)^{k+1} \| z \|_E. \]

We are left to bound \( T_2(\zeta_u; z) \) and \( T_4(\zeta_u; v) \). Define on each element \( K_{ij} \) the bilinear forms as
\[ D^{1}_{ij}(\zeta_u; v) = \left\langle \zeta_u, v_x \right\rangle_{K_{ij}} - \left\langle (\zeta_u)_{i-1,y}, v_{i-1,y} \right\rangle_{J_j} + \left\langle (\zeta_u)_{i-1,y}, v_{i-1,y}^+ \right\rangle_{J_j}, \]
\[ D^{2}_{ij}(\zeta_u; v) = \left\langle \zeta_u, v_y \right\rangle_{K_{ij}} - \left\langle (\zeta_u)_{x,j}, v_{x,j}^- \right\rangle_{I_i} + \left\langle (\zeta_u)_{x,j}, v_{x,j}^+ \right\rangle_{I_i}. \]

Take \( D^{1}_{ij}(\zeta_u; v) \) as an example. On one hand, there holds the superconvergence property
\[ |D^{1}_{ij}(\zeta_u; v)| \leq C \sqrt{\frac{\tilde{h}_{ij}}{h_{ij}}} \left[ h_{ij}^{k+2} \| \partial_x^{k+2} u \|_{L^\infty(K_{ij})} + h_{ij}^{k+2} \| \partial_y^{k+2} u \|_{L^\infty(K_{ij})} \right] \| v \|_{K_{ij}}, \]
for any \( v \in Q^k(K_{ij}) \), which follows directly from Lemma 4.8 of [18]. On the other hand, it holds
\[ |D^{1}_{ij}(\zeta_u; v)| \leq C \sqrt{\frac{\tilde{h}_{ij}}{h_{ij}}} \| u \|_{L^\infty(K_{ij})} \| v \|_{K_{ij}}, \]

(36a)
which is derived from the Cauchy-Schwarz inequality and inverse inequalities.

Since \( u = 0 \) on \( \partial \Omega \), one gets

\[
    T_2(\zeta_u, z) = \sum_{K_{ij} \in \Omega_N} D^1_{ij}(\zeta_u, s) + \sum_{K_{ij} \in \Omega_N} D^2_{ij}(\zeta_u, s).
\]

Now we focus on estimating \( \sum_{K_{ij} \in \Omega_N} D^1_{ij}(\zeta_u, \delta_p) \) based on the decomposition

\[
    u = \sum_{\varphi \in \{S, E_{21}, E_{12}, E_{22}\}} \varphi.
\]

For \( \varphi = S \), using \( h_i \geq CeN^{-1} \geq Ch_j \), one gets from (36a)

\[
    \sum_{K_{ij} \in \Omega_N} D^1_{ij}(\zeta_s, s) \leq C \sum_{K_{ij} \in \Omega_N} \sqrt{\frac{h_j}{h_i}} \left[ h_i^k \| \partial^k \varphi \|_{L^\infty(K_{ij})} + h_j^k \| \partial^k \varphi \|_{L^\infty(K_{ij})} \right] \| s \|_{K_{ij}}
\]

\[
    \leq CN^{-(k+1)} \| s \|_{\Omega_{11}} + C \left[ \sum_{K_{ij} \in \Omega_{11}} h_j/h_i N^{-2(k+2)} \right]^{1/2} \| s \|_{\Omega_1(\Omega_{11})}
\]

\[
    \leq CN^{-(k+1)} \| z \|_E.
\]

For \( \varphi = E_{21} \), using (36b) and \( \sigma \geq k + 2 \) yields

\[
    \sum_{K_{ij} \in \Omega_{11} \cup \Omega_{12}} D^1_{ij}(\zeta_s, s) \leq C \sum_{K_{ij} \in \Omega_{11} \cup \Omega_{12}} \sqrt{\frac{h_j}{h_i}} \| \varphi \|_{L^\infty(K_{ij})} \| s \|_{K_{ij}}
\]

\[
    \leq C \sum_{K_{ij} \in \Omega_{11} \cup \Omega_{12}} \varepsilon^{-1/2} e^{-\alpha(1-x_i)/\varepsilon} \| s \|_{K_{ij}} \leq CN^{-(k+1)} \| z \|_E.
\]

Additionally, we introduce the notation \( \Omega_x = \Omega_{21} \cup \Omega_{22} \).

Using (36), \( \sigma \geq k + 2 \) and Lemma 3.1 gives

\[
    \sum_{K_{ij} \in \Omega_x} D^1_{ij}(\zeta_s, s) \leq C \sum_{K_{ij} \in \Omega_x} \sqrt{\frac{h_j}{h_i}} \min \left\{ h_i^{k+2} \| \partial^k \varphi \|_{L^\infty(K_{ij})} + h_j^{k+2} \| \partial^k \varphi \|_{L^\infty(K_{ij})} \right\} \| s \|_{K_{ij}}
\]

\[
    \leq C \sum_{K_{ij} \in \Omega_x} \left( \varepsilon \max |\psi'| \right)^{-1/2} \min \left\{ 1, \left( \frac{h_i}{\varepsilon} \right)^{k+2} \right\} e^{-\alpha(1-x_i)/\varepsilon} \| s \|_{K_{ij}}
\]

\[
    \leq C \sum_{K_{ij} \in \Omega_x} \left( \varepsilon \max |\psi'| \right)^{-1/2} \Theta_i^{k+2} \| s \|_{K_{ij}}
\]

\[
    \leq C \max |\psi'|^{-1/2} \left( \sum_{j=1}^N \sum_{i=N/2+1}^N \Theta_i \right)^{1/2} \max_{N/2+1 \leq i \leq N} \Theta_i^{k+3/2} \left( \varepsilon^{-1/2} \| s \| \right)
\]

\[
    \leq C \left( N^{-1} \max |\psi'| \right)^{k+1} \| z \|_E.
\]
here we used $h_j/h_i \leq h_j/h \leq C(\varepsilon \max |\psi'|)^{-1}$ by Lemma 3.2. Analogously, one can bound $\sum_{K_{ij} \in \Omega_N} D_{ij}(\zeta, s)$ for $\varphi = E_{12}, E_{22}$.

Another term in $T_2(\zeta_u; z)$ can be bounded similarly. Consequently,

$$T_2(\zeta_u; z) \leq C(N^{-1} \max |\psi'|)^{k+1} || z ||_E.$$  \hspace{1cm} (37)

Now we turn to bound the convection term $T_4(\zeta_u; v)$, which needs a different treatment. Define

$$\Omega_{11}^* = \begin{cases} 
\Omega_{11}, & \text{for S-type meshes,} \\
(0, x_{N/2+1}) \times (0, y_{N/2+1}), & \text{for a B-type mesh.} 
\end{cases} \hspace{1cm} (38)$$

We introduce by $\Omega_f = \Omega_{11}^* \setminus \Omega_{11}$. Let $|\Omega_f|$ be the measure of the domain $\Omega_f$. Then $|\Omega_f| = 0$ for the S-type meshes and $|\Omega_f| \leq C \varepsilon \ln(1/\varepsilon)$ for a B-type mesh. By Lemma 3.2, one has also $\tilde{h}_{\Omega_f} \geq C \varepsilon$ for a B-type mesh.

Using (26a) and (17)-(19), we have

$$\langle a^1 \zeta_u, v_x \rangle + \langle a^2 \zeta_u, v_y \rangle = \langle \zeta_u, a \cdot \nabla v \rangle 
\leq C \left[ N \| \zeta_u \|^2_{\Omega_{11}^*} + \sum_{K_{ij} \in \Omega_f} \tilde{h}_{ij}^{-1} \| \zeta_u \|^2_{K_{ij}} + \sum_{K_{ij} \in \Omega \setminus \Omega_{11}^*} \tilde{h}_{ij}^{-1} \| \zeta_u \|^2_{K_{ij}} \right]^{1/2} || z ||_z 
\leq C \left\{ N^{-2(k+1)} + \left( \tilde{h}_{\Omega_f}^{-1} |\Omega_f| + \sum_{K_{ij} \in \Omega \setminus \Omega_{11}^*} \frac{h_i h_j}{\min\{h_i, h_j\}} \right) (N^{-1} \max |\psi'|)^{2(k+1)} \right\}^{1/2} || z ||_z 
\leq C Q^* N^{-(k+1/2)},$$

where $Q^*$ is given by (30).

As for the other terms in $T_4(\zeta_u; v)$, one can estimate

$$\left( \sum_{i=1}^N \sum_{j=1}^N \| (\zeta_u)_{i,j}^- \|^2_{J_j} \right)^{1/2} || z ||_E \leq C(N^{-1} \max |\psi'|)^{k+1/2} || z ||_E \hspace{1cm} (39)$$

due to (26c). Consequently,

$$T_4(\zeta_u; v) \leq C Q^* N^{-(k+1/2)} || z ||_z.$$  \hspace{1cm} (40)

Summarizing, we obtain (33).

(2) Now we start to prove (29). Taking $z = \delta$ in (32), one gets

$$|| \delta ||_z = B(\delta; \delta) = B(\zeta; \delta).$$  \hspace{1cm} (41)

Slightly modifying the proof of Lemma 4.6 in [18], we can state stability with respect to the norm $|| \cdot ||_z$, i.e.,

$$\sup_{z \in W_N^3} \frac{B(w;z)}{|| z ||_z} \geq C || w ||_z,$$  \hspace{1cm} (42)
here $C > 0$ is independent of $\varepsilon$ and $N$. Thus one derives from (32) and (33)
\[ \|\delta\|_E \leq \|\delta\|_z \leq C \sup_{z \in W_N^3} \frac{B(\delta; z)}{\|z\|_z} = C \sup_{z \in W_N^3} \frac{B(\zeta; z)}{\|z\|_z} \leq CQ^* N^{-(k+1/2)}. \] (43)

Since $\|\zeta\|_E$ can be estimated from (26), we can bound $\|w - f w\|_E$ from (43) and the triangle inequality.

Using (26a), one has
\[ \|u - F_1 u\| \leq \|\zeta_u\| + \|\delta_u\| \leq C[(N^{-1} \max |\psi'|)^{k+1} + \|\delta\|_E] \leq CQ^* N^{-(k+1/2)}. \]

If we differentiate (28) with respect to $t$, and take into account that $u_t, p_t, q_t$ satisfy similar estimates as $u, p$ and $q$, we can get estimate for $\|u_t - F_1 u_t\|$ analogously as before. This completes the proof. \[ \square \]

5 Error estimates

In this section we use Theorem 4.1 to derive error estimates as well for the semi-discrete as the fully-discrete LDG method.

**Theorem 5.1.** Let $w = (u, p, q) = (u, \varepsilon u_x, \varepsilon u_y)$ where $u$ is the exact solution of (1) satisfying Proposition 3.1. Let $w = (u, p, q) \in W_N^3$ be the numerical solution of semi-discrete LDG method (4). Then, it holds
\[ \|(u - u)(T)\| + \int_0^T \|w - w\|_E \, dt \leq C(1 + T)Q^* N^{-(k+1/2)}, \] (44)
where $Q^*$ is given by (30) and $C > 0$ is a constant independent of $\varepsilon$ and $N$.

**Proof.** Denote the numerical error by $e = w - \mathcal{W}$, and divide into the two parts $\eta$ and $\xi$ with $e = \eta - \xi$, where $\eta = w - f w$ and $\xi = w - f w$. Here $f w$ is the Ritz projection of $w$ defining in (28). The consistence of the numerical fluxes in the LDG method leads to the error equation
\[ \langle e_{u,t}, v \rangle + B(e; z) = 0 \quad \forall z = (v, s, r) \in W_N^3. \] (45)

Taking $z = \xi$ yields
\[ \frac{1}{2} \frac{d}{dt} \|\xi_u\|^2 + B(\xi; \xi) = \langle \eta_{u,t}, \xi_u \rangle, \] (46)
since $B(\eta; \xi) = 0$ by the definition (28). Using (2), we conclude
\[ B(\xi; \xi) = \|\xi\|_E^2 \geq \beta \|\xi_u\| \|\xi\|_E. \]

By (46) and the Cauchy-Schwarz inequality one gets
\[ \|\xi_u\| \left[ \frac{d}{dt} \|\xi_u\| + \beta \|\xi\|_E \right] \leq \frac{1}{2} \frac{d}{dt} \|\xi_u\|^2 + B(\xi; \xi) = \langle \eta_{u,t}, \xi_u \rangle \leq \|\eta_{u,t}\| \|\xi_u\|. \]
Canceling $\|\xi_u\|$ on both sides results in
\[
\frac{d}{dt} \|\xi_u\| + \beta \frac{1}{2} \|\xi\|_E \leq \|\eta_{u,t}\| \leq CQ^* N^{-(k+1/2)}
\]
due to Theorem 4.1. Taking into account the choice of the initial value we have
\[
\|\xi_u(T)\| + \beta \int_0^T \|\xi\|_E \, dt \leq \|\xi_u(0)\| + \int_0^T CQ^* N^{-(k+1/2)} \, dt \\
\leq C(1 + T)Q^* N^{-(k+1/2)}.
\]
(47)
The use of Theorem 4.1 and the triangle inequality give finally the estimate of Theorem 4.1.

Next we study the error of the fully discrete scheme.

**Theorem 5.2.** The error of fully-discrete LDG scheme (10) satisfies
\[
\|u(T) - U^M\| + \Delta t \sum_{m=1}^M \| (w - W)^{m,\theta}\|_E \leq C(1 + T)(\Delta t)^\nu + Q^* N^{-(k+1/2)}),
\]
(48)
where $Q^*$ is given by (30), $\nu = 2$ for $\theta = 1/2$ and $\nu = 1$ for $1/2 < \theta \leq 1$. Here $C > 0$ is a constant independent of $\varepsilon$ and $N$.

**Proof.** Define $f^m = f(w(t^m))$ and
\[
\eta^m = w^m - f^m, \quad \xi^m = W^m - f^m \in W^3_N.
\]
For the exact solution it holds
\[
\langle u^{m,\theta}, \nu \rangle + B(w^{m,\theta}; z) = \langle f^{m,\theta}, \nu \rangle \quad \forall z = (\nu, s, \tau) \in W^2_N.
\]
(49)
Subtracting this identity from (10) and noticing the relation
\[
B(\eta^{m,\theta}; z) = B(w^{m,\theta} - f w^{m,\theta}; z) = 0
\]
leads us to the the error equation
\[
\left< \frac{\xi^m - \xi^{m-1}}{\Delta t}, \nu \right> + B(\xi^{m,\theta}; z) = \langle L^m, \nu \rangle, \quad z = (\nu, s, \tau) \in W^2_N,
\]
(50)
where
\[
L^m = \left( u^{m,\theta} - \frac{u^m - u^{m-1}}{\Delta t} \right) + \frac{\eta^m - \eta^{m-1}}{\Delta t}.
\]
(51)
Using a Taylor expansion we get
\[
\begin{align*}
\frac{u^m - u^{m-1}}{\Delta t} &= \left( \theta - \frac{1}{2} \right) \Delta t u^m_t + \left( \frac{1 - \theta}{2} u^m_{ttt}(s_1) - \frac{1}{6} u^m_{ttt}(s_2) \right) (\Delta t)^2, \\
\frac{\eta^m - \eta^{m-1}}{\Delta t} &= u^m_t(s_3) - F_1 u^m_t(s_3),
\end{align*}
\]
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here the \( s_i (i = 1, 2, 3) \) are intermediate points belonging to \((t^{m-1}, t^m)\). By (12) and Theorem 4.1 we obtain

\[
\| L^m \| \leq C \left[ (\theta - \frac{1}{2}) \Delta t + (\Delta t)^2 + Q^* N^{-(k+1/2)} \right] \equiv E_{N, \Delta t}.
\]

We use the stability result

\[
B(\xi^{m, \theta}, \xi^{m, \theta}) = \|\xi^{m, \theta}\|_E^2 \geq \beta^{\frac{1}{2}} \|\xi^{m, \theta}\| \|\xi^{m, \theta}\|_E.
\]

The application of the Cauchy-Schwarz inequality, \( \theta \geq 1/2 \) and the triangle inequality imply

\[
\left\langle \frac{\xi_u^{m} - \xi_u^{m-1}}{\Delta t}, \xi_u^{m, \theta} \right\rangle = \left( \Delta t \right)^{-1} \left\langle \xi_u^{m} - \xi_u^{m-1}, \theta \xi_u^{m} + (1 - \theta) \xi_u^{m-1} \right\rangle
\]

\[
\geq \left( \Delta t \right)^{-1} \left[ \theta \|\xi_u^{m}\|^2 - (2\theta - 1) \left\langle \xi_u^{m}, \xi_u^{m-1} \right\rangle -(1 - \theta) \|\xi_u^{m-1}\|^2 \right]
\]

\[
\geq \left( \Delta t \right)^{-1} \left[ \theta \|\xi_u^{m}\|^2 - (2\theta - 1) \|\xi_u^{m}\| \|\xi_u^{m-1}\| - (1 - \theta) \|\xi_u^{m-1}\| \right]
\]

\[
= \left( \Delta t \right)^{-1} \left( \|\xi_u^{m}\| - \|\xi_u^{m-1}\| \right) \left( \theta \|\xi_u^{m}\| + (1 - \theta) \|\xi_u^{m-1}\| \right)
\]

\[
= \left( \Delta t \right)^{-1} \left( \|\xi_u^{m}\| - \|\xi_u^{m-1}\| \right) \|\xi_u^{m, \theta}\|.
\]

Inserting \( z = \xi \) in (50) yields

\[
\|\xi_u^{m, \theta}\| \left[ \left( \Delta t \right)^{-1} \left( \|\xi_u^{m}\| - \|\xi_u^{m-1}\| \right) + \beta^{\frac{1}{2}} \|\xi_u^{m, \theta}\|_E \right] \leq \| L^m \| \|\xi_u^{m, \theta}\| \leq E_{N, \Delta t} \|\xi_u^{m, \theta}\|.
\]

or

\[
\left( \Delta t \right)^{-1} \left( \|\xi_u^{m}\| - \|\xi_u^{m-1}\| \right) + \beta^{\frac{1}{2}} \|\xi_u^{m, \theta}\|_E \leq E_{N, \Delta t}.
\]

Summing over \( m = 1, 2, \ldots, M \) leads to

\[
\|\xi_u^M\| + \beta^{\frac{1}{2}} \sum_{m=1}^{M} \|\xi_u^{m, \theta}\|_E \Delta t \leq \|\xi_u^0\| + \Delta t \sum_{m=1}^{M} E_{N, \Delta t} \leq C(1 + T)E_{N, \Delta t},
\]

due to the choice of the initial value. By Theorem 4.1 and the triangle inequality we finally get the result stated in Theorem 5.2 \( \square \)

### 6 Numerical experiments

In this section, we present some numerical experiments for the fully-discrete LDG Crank-Nicolson scheme, i.e., for \( \theta = 1/2 \). All the calculations were realized in MATLAB R2015B. The systems of linear equations resulting from the discrete problems were solved by LU-decomposition. All integrals were evaluated using the 5-point Gauss-Legendre quadrature rule.
Let $T = 1$ and $\Omega = (0,1)^2$. We consider the test problem
\[ u_t - \varepsilon \Delta u + u_x + u_y + u = f(x,y,t), \quad (53) \]
where $f$ and the initial-boundary conditions are chosen in such a way that the exact solution is given by
\[ u(x,y,t) = e^t \sin(\pi xy)(1 - e^{-(1-x)/\varepsilon})(1 - e^{-(1-y)/\varepsilon}). \quad (54) \]

The fully-discrete LDG method is implemented on the three layer-adapted meshes listed in Table 1. We take $\sigma = k + 2$ and compute the $L^2$-error $\| u(T) - U(T) \|$ and the energy-norm error $\Delta t \sum_{m=1}^{M} \| (w - \mathbb{W})^m,\|_E$, respectively. The numerical convergence order is obtained from the formula
\[ r_2 = \frac{\log e_N - \log e_{2N}}{\log 2} \quad \text{or} \quad r_S = \frac{\log e_N - \log e_{2N}}{\log p_S}, \]
where $e_N$ is the error using $N$ elements in the $x$- and $y$-direction. For the Shishkin mesh we use a scaling with $p_S = 2 \ln N / \ln(2N)$ to compute the numerical convergence order with respect to the power of $\ln N/N$.

The three numerical solutions and the absolute errors on a S-mesh, BS-mesh and B-type mesh are plotted in Figure 2 when $\varepsilon = 10^{-3}$, $k = 1$ and $N = 64$. From this picture, we can see that the LDG method with these meshes gives well-behaved numerical solutions. The solution on the S-mesh produces the largest errors, while the solutions on the BS-mesh and B-type mesh have a comparable numerical performance.

We test also the numerical convergence rate in space. Set $\varepsilon = 10^{-8}$ and compute the $L^2$-error and energy-error for $k = 1, 2$. To obtain a balanced space-time error, we take the time step as $\Delta t = N^{-1}$ for $k = 1$ and $\Delta t = N^{-1.5}$ for $k = 2$. In Tables 2, 3 one observes convergence rates of order $k+1$ and $k+1/2$ for the $L^2$-norm error and the energy-norm error, which imply that the numerical error behaves as the energy-norm error estimate in Theorem 5.1.

To test the convergence rate in time, we take $k = 3$ and $N = 128$ such that the temporal error is dominant. Decrease the time step from 0.5 to 0.0625 and compute the $L^2$-error and the energy-error, respectively. In the Tables 4, 5 a second order convergence rate is clearly observed which corresponds to the theoretical prediction of Theorem 5.2.

Finally we investigate the robustness of the LDG method with respect to the small parameter $\varepsilon$. Fix $k = 1$, $N = 128$ we vary $\varepsilon$ from $10^{-4}$ to $10^{-11}$. Table 6 shows that both errors are almost constant.

**Appendix**

In the Appendix, we sketch some improved results in the one-dimensional case in space and comment the alternative of a DG discretization in time.
A.1 The one-dimensional case

We examine the LDG method for the one-dimensional singularly perturbed problem

\[ u_t - \varepsilon u_{xx} + a(x)u_x + b(x)u = f(x,t), \quad \text{in } (0,1) \times (0,T), \quad (A.1a) \]

\[ u(x,0) = u_0(x), \quad \text{in } [0,1], \quad (A.1b) \]

\[ u(0,t) = u(1,t) = 0, \quad \text{for } t \in (0,T). \quad (A.1c) \]

Assume that \( u \) can be decomposed as (see [11])

\[ u(x,t) = S(x,t) + E(x,t), \]

with

\[ \left| \frac{\partial_x^j \partial_t^m S(x,t)}{x} \right| \leq C, \quad \left| \frac{\partial_x^j \partial_t^m E(x,t)}{x} \right| \leq C\varepsilon^{-j}e^{-\alpha(1-x)/\varepsilon} \quad \forall j, m. \quad (A.2) \]

Define the discontinuous finite element space

\[ \mathcal{V}_N = \{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}^k(I_j), j = 1, \ldots, N \}, \]

where \( \mathcal{P}^k(I_j) \) denotes the space of polynomials in \( I_j \) of degree \( k \geq 0 \).

The fully-discrete LDG \( \theta \)-scheme reads:

Let \( \mathbb{U}^0 = \pi u_0 \) be the one-dimensional \( L^2 \) projection of \( u_0 \). For any \( m = 1, 2, \ldots, M \), find the numerical solution \( \mathbb{W}^m = (\mathbb{U}^m, \mathbb{Q}^m) \in \mathcal{V}_N^2 \) such that

\[ \left\langle \frac{\mathbb{U}^m - \mathbb{U}^{m-1}}{\Delta t}, v \right\rangle + B(\mathbb{W}^m, \mathbb{V}; x) = \left\langle f^m, v \right\rangle, \quad (A.3) \]
Table 2: $L^2$-error and convergence rate in space.

| $k$ | $N$ | S-mesh $L^2$-error | BS-mesh $L^2$-error | B-type mesh $L^2$-error |
|-----|-----|----------------|-----------------|-----------------|
|     |     | $r_2$ | $r_2$ | $r_2$ | $r_2$ | $r_2$ |
| 1   | 4   | 1.64e-01 | - | 1.62e-01 | - | 1.59e-01 | - |
| 8   | 1.14e-02 | 1.90 | 1.90 | 1.87 |
| 16  | 2.93e-03 | 1.97 | 1.96 | 1.96 |
| 32  | 7.41e-04 | 1.98 | 1.98 | 1.98 |
| 64  | 1.86e-04 | 1.99 | 1.99 | 1.99 |
| 128 | 2.09e-03 | 2.91 | 2.93 | 2.89 |
| 16  | 2.75e-04 | 2.93 | 2.93 | 2.93 |
| 32  | 3.52e-05 | 2.97 | 2.97 | 2.97 |
| 64  | 4.45e-06 | 2.98 | 2.98 | 2.98 |
| 128 | 5.60e-07 | 2.99 | 2.99 | 2.99 |

holds for any $z = (v, r) \in V^2_N$. Here $B(\cdot; \cdot)$ is defined analogously as in the two-dimensional case. The related energy norm is

$$||w||_E^2 = \varepsilon^{-1} ||q||^2 + ||(b - a_x/2)^{1/2} u||^2 + \sum_{j=0}^{N} \frac{1}{2} a_j [u]^2_j + \lambda_N[u]^2_N.$$  \hspace{1cm} (A.4)

A careful analysis shows that in 1D an improved error estimate is possible:

**Theorem 6.1.** The errors of fully-discrete LDG $\theta$-scheme (A.3) satisfy

$$\|u(T) - \mathbb{U}^M\| \leq C(1 + T)(\Delta t)^{\nu} + (N^{-1} \max |\psi'|)^{k+1}),$$  \hspace{1cm} (A.5a)

$$\Delta t \sum_{m=1}^{M} \|\|w - \mathbb{W}\|^m,\theta\|_E \leq C(1 + T)(\Delta t)^{\nu} + (N^{-1} \max |\psi'|)^{k+1/2}),$$  \hspace{1cm} (A.5b)

where $\nu = 2$ for $\theta = 1/2$ and $\nu = 1$ for $1/2 < \theta \leq 1$. Here $C > 0$ is a constant independent of $\varepsilon$ and $N$.

The proof is based on two improved approximation error estimates for the Ritz projection:

$$\|\|w - \mathcal{F}w\|_E \leq C(N^{-1} \max |\psi'|)^{k+1/2},$$  \hspace{1cm} (A.6a)

$$\|u_t - \mathcal{F}_1 u_t\| \leq C(N^{-1} \max |\psi'|)^{k+1}.$$  \hspace{1cm} (A.6b)
Table 3: Energy-error and convergence rate in space.

| k  | N | S-mesh energy-error | r_S | BS-mesh energy-error | r_2 | B-type mesh energy-error | r_2 |
|----|----|---------------------|-----|---------------------|-----|--------------------------|-----|
| 1  | 4  | 4.57e-01            | -   | 3.77e-01            | -   | 4.65e-01                 | -   |
| 8  | 2.65e-01 | 1.89               | 1.52e-01 | 1.32               | 1.68e-01 | 1.47               | 1.60e-02 | 1.47               |
| 16 | 1.46e-01 | 1.48               | 5.76e-02 | 1.39               | 6.07e-02 | 1.48               |
| 32 | 7.35e-02 | 1.46               | 2.12e-02 | 1.44               | 2.17e-02 | 1.48               |
| 64 | 3.46e-02 | 1.48               | 7.64e-03 | 1.47               | 7.74e-03 | 1.49               |
| 128| 1.55e-02 | 1.49               | 2.73e-03 | 1.49               | 2.75e-03 | 1.49               |
| 2  | 4  | 1.29e-01            | -   | 7.32e-02            | -   | 1.50e-01                | -   |
| 8  | 6.99e-02 | 2.13               | 1.79e-02 | 2.03               | 2.44e-02 | 2.62               |
| 16 | 2.86e-02 | 2.21               | 3.70e-03 | 2.27               | 4.28e-03 | 2.51               |
| 32 | 9.46e-03 | 2.35               | 7.06e-04 | 2.39               | 7.59e-04 | 2.50               |
| 64 | 2.73e-03 | 2.44               | 1.30e-04 | 2.44               | 1.35e-04 | 2.50               |
| 128| 7.18e-04 | 2.47               | 2.34e-05 | 2.47               | 2.38e-05 | 2.50               |

Table 4: $L^2$-norm error and convergence rate in time.

| $\Delta t$ | S-mesh $L^2$-error | r_2 | BS-mesh $L^2$-error | r_2 | B-type mesh $L^2$-error | r_2 |
|------------|---------------------|-----|---------------------|-----|--------------------------|-----|
| 0.5        | 7.35e-03            | -   | 7.35e-03            | -   | 7.35e-03                 | -   |
| 0.25       | 1.80e-03            | 2.03| 1.80e-03            | 2.03| 1.80e-03                 | 2.03|
| 0.125      | 4.53e-04            | 1.99| 4.53e-04            | 1.99| 4.53e-04                 | 1.99|
| 0.0625     | 1.13e-04            | 2.00| 1.13e-04            | 2.00| 1.13e-04                 | 2.00|

For that orthogonality and the exact collocations of the two one-dimensional Gauss-Radau projections $\pi^{\pm}$ used are important:

\[
\begin{align*}
\langle \pi^- z, v \rangle_{I_j} &= \langle z, v \rangle_{I_j}, \quad \forall v \in \mathcal{P}^{k-1}(I_j), \\
(\pi^- z)^j &= z_j^-.
\end{align*}
\]

\[
\begin{align*}
\langle \pi^+ z, v \rangle_{I_j} &= \langle z, v \rangle_{I_j}, \quad \forall v \in \mathcal{P}^{k-1}(I_j), \\
(\pi^+ z)^{j-1} &= z_{j-1}^+.
\end{align*}
\]

A.2 DG time-discretization

Instead of the $\theta$-scheme one can also use a DG time-discretization. We present a corresponding result for the one-dimensional problem in space.

Let $r \geq 0$ be the polynomial order of the elements in time, and the discrete
Table 5: Energy-norm error and convergence rate in time.

| $\Delta t$ | S-mesh energy-error | BS-mesh energy-error | B-type mesh energy-error |
|-----------|---------------------|----------------------|-------------------------|
| 0.5       | 7.35e-03            | 7.35e-03             | 7.35e-03                |
| 0.25      | 1.85e-03 1.99       | 1.85e-03 1.99        | 1.85e-03 1.99           |
| 0.125     | 4.63e-04 2.00       | 4.62e-04 2.00        | 4.62e-04 2.00           |
| 0.0625    | 1.21e-04 1.94       | 1.15e-04 2.00        | 1.15e-04 2.00           |

Table 6: $L^2$-norm error and energy-norm error for different $\varepsilon$.

| $\varepsilon$ | S-mesh $L^2$-error | BS-mesh $L^2$-error | B-type mesh $L^2$-error | S-mesh energy-error | BS-mesh energy-error | B-type mesh energy-error |
|---------------|---------------------|----------------------|-------------------------|---------------------|----------------------|-------------------------|
| $10^{-4}$     | 1.91e-04 1.86e-04   | 1.85e-03 1.85e-04    | 1.55e-02 2.73e-03       | 2.73e-03            |
| $10^{-5}$     | 1.87e-04 1.86e-04   | 1.86e-04 1.85e-04    | 1.55e-02 2.73e-03       | 2.75e-03            |
| $10^{-6}$     | 1.86e-04 1.86e-04   | 1.86e-04 1.86e-04    | 1.55e-02 2.73e-03       | 2.75e-03            |
| $10^{-7}$     | 1.86e-04 1.86e-04   | 1.86e-04 1.86e-04    | 1.55e-02 2.73e-03       | 2.75e-03            |
| $10^{-8}$     | 1.86e-04 1.86e-04   | 1.86e-04 1.86e-04    | 1.55e-02 2.73e-03       | 2.75e-03            |
| $10^{-9}$     | 1.86e-04 1.86e-04   | 1.86e-04 1.86e-04    | 1.55e-02 2.73e-03       | 2.75e-03            |
| $10^{-10}$    | 1.86e-04 1.86e-04   | 1.86e-04 1.86e-04    | 1.55e-02 2.73e-03       | 2.75e-03            |
| $10^{-11}$    | 1.86e-04 1.86e-04   | 1.86e-04 1.86e-04    | 1.55e-02 2.73e-03       | 2.74e-03            |

function space $V^{\Delta t}_N$ be given by

$$V^{\Delta t}_N \equiv \{ V \in L^2(0,T;V_N) : V|_{K^m} \in \mathcal{P}^r(K^m,V_N), m = 1,2,\ldots,M \}.$$  \hspace{1cm} (A.8)

Here $\mathcal{P}^r(K^m,V_N)$ denotes the space of $V_N$-valued polynomials of degree $r$ on the time interval $K^m$. For a function $V \in V^{\Delta t}_N$, we define at $t = t^m \pm 0$ the one-sided limits and the jump by $V_{m,\pm} = \lim_{t \to t^m \pm 0} V(t)$, $[V]^m = V^m_+ - V^m_-.$

Then, the space-time DG scheme reads:

Find the discrete solution $W = (U,Q) \in V^{\Delta t}_N \times V^{\Delta t}_N$ such that

$$\int_{K^m} (\langle U_t, V \rangle + B(W,Z))dt + \langle [V]^{m-1,-}_{m+1}, V^{m-1,+} \rangle = \int_{K^m} \langle f, V \rangle dt$$  \hspace{1cm} (A.9)

holds for any test function $Z = (V,R) \in V^{\Delta t}_N \times V^{\Delta t}_N$, $m = 1,2,\ldots,M$. For the initial value we take $U^{0,-} = \pi u_0$ as before.

Following the ideas of [7, 15] and the estimation (A.6) for Ritz projection, we can establish the following error estimates:

**Theorem 6.2.** Let $w = (u,q) = (u,\varepsilon u_\nu)$ where $u$ is the solution of (A.1) satisfying (A.2). Let $W = (U,Q) \in V^{\Delta t}_N \times V^{\Delta t}_N$ be the numerical solution given by (A.9).
Then the errors satisfy
\[
\|u(T) - U(T)\| \leq C((\Delta t)^{r+1} + (N^{-1} \max |\psi'|)^{k+1}), \tag{A.10a}
\]
\[
\left( \sum_{m=1}^M \int_{K^m} \|w - W\|_{L^2}^2 \, dt \right)^{1/2} \leq C((\Delta t)^{r+1} + (N^{-1} \max |\psi'|)^{k+1/2}), \tag{A.10b}
\]
where \( C > 0 \) is a constant independent of \( \varepsilon \) and \( N \).

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