Large Deviations for processes on half-line

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Abstract. We consider a sequence of processes $X_n(t)$ defined on half-line $0 \leq t < \infty$. We give sufficient conditions for Large Deviation Principle (LDP) to hold in the space of continuous functions with metric

$$
\rho_\kappa(f, g) = \sup_{t \geq 0} \left\{ \frac{|f(t) - g(t)|}{1 + t^{\frac{1}{1+\kappa}}} \right\}, \quad \kappa \geq 0.
$$

LDP is established for Random Walks, Diffusions, and CEV model of ruin, all defined on the half-line. LDP in this space is "more precise" than that with the usual metric of uniform convergence on compacts.

Key words. Large Deviations, Random Walk, Diffusion processes, CEV model.

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§1. Introduction

In this work we derive sufficient conditions for a sequence $\{X_n\}_{n=1}^\infty$ of stochastic processes $X_n(t); 0 \leq t < \infty$, to satisfy the Large Deviation Principle (LDP) in the space of continuous functions, which we denote by $C$.

In the recent literature \cite{1}, \cite{2}, \cite{3} the space $C$ is considered with the metric

$$
\rho^{(P)}(f, g) := \sum_{k=1}^{\infty} 2^{-k} \min\{ \sup_{0 \leq t \leq k} |f(t) - g(t)|, 1 \}.
$$

\cite{4} (Theorem 2.6) gives sufficient conditions for $X_n$ to satisfy LDP in the space $(C, \rho^{(P)})$.

As noted in \cite{4}, convergence $f_n \to f$ in metric $\rho^{(P)}$ is equivalent to convergence in $C[0, T]$ with uniform metric for any $T \geq 0$. A considerable drawback of metric $\rho^{(P)}$ is that it is "not sensitive" to behaviour of functions as $t \to \infty$.

We consider the space $C$ with metric

$$
\rho(f, g) = \rho_\kappa(f, g) := \sup_{t \geq 0} \frac{|f(t) - g(t)|}{1 + t^{\frac{1}{1+\kappa}}},
$$

for a fixed $\kappa \geq 0$.

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As we shall see in §2, the LDP in the space \((\mathbb{C}, \rho)\) is "more precise" than the LDP in \((\mathbb{C}, \rho^{(P)})\).

In this work we treat continuous processes on infinite interval. As we envisage, a treatment of discontinuous processes on infinite interval will need essentially different to \(\rho\) metric, (see [5], for the LDP for Compound Poisson processes on infinite interval).

The paper is organised as follows:

Sufficient conditions for LDP in the space \((\mathbb{C}, \rho)\) are given in §2, Theorem 2.1. We also compare Theorem 2.1 and Theorem 2.6 of [4], and show that Theorem 2.1 is more precise. Next we demonstrate Theorem 2.1 on different kind of processes. In §3 LDP and Moderate Deviation Principle are obtained for Random Walks on half-line. We also give LDP for Winner process in the space \((\mathbb{C}, \rho)\) and an example showing that the metric \(\rho = \rho_{\kappa}\) is preferable to \(\rho^{(P)}\). §4 is devoted to LDP for diffusion processes on half-line. We obtain LDP in space \((\mathbb{C}, \rho)\) for solutions of stochastic differential equations with various conditions on the coefficients. In §5 we give LDP for a diffusion model of ruin (a CEV model in finance).

§2. Main Result

Denote by \(C_0 \subset \mathbb{C}\) the class of functions \(f \in \mathbb{C}\), such that \(f(0) = 0, \lim_{t \to \infty} \frac{f(t)}{1+t^{1+\kappa}} = 0\). It is obvious that \((\mathbb{C}, \rho)\) is a complete separable metric (Polish) space.

For any \(T \in (0, \infty)\) denote by \(C[0,T]\) the metric space of real continuous functions \(f = f(t); 0 \leq t \leq T\), with metric

\[
\rho_T(f, g) := \sup_{0 \leq t \leq T} \frac{|f(t) - g(t)|}{1 + t^{1+\kappa}},
\]

where \(\kappa \geq 0\) is fixed.

We say that in space \(C[0,T]\) there is a (good) rate function

\[
I_0^T = I_0^T(f) : \mathbb{C}[0,T] \to [0, \infty],
\]

if: (i) it is lower semi-continuous: for any \(f \in \mathbb{C}[0,T]\)

\[
\lim_{f_n \to f} I_0^T(f_n) \geq I_0^T(f); \tag{2.1}
\]

(ii) for any \(r \geq 0\) the set

\[
B_{T,r} := \{f \in \mathbb{C}[0,T] : I_0^T(f) \leq r\}
\]

is a compact in \(\mathbb{C}[0,T]\).

For a non-empty set \(B \subset \mathbb{C}[0,T]\) let

\[
I_0^T(B) := \inf_{f \in B} I_0^T(f), \quad I_0^T(0) := \infty.
\]

\((f)_{T,\varepsilon}\) and \((B)_{T,\varepsilon}\) denote \(\varepsilon\)-neighbourhood in metric \(\rho_T\) in space \(\mathbb{C}[0,T]\) of \(f \in \mathbb{C}[0,T]\) and measurable set \(B \subset \mathbb{C}[0,T]\) respectively. The interior and the closure of a measurable set \(B \subset \mathbb{C}[0,T]\) denote by \((B)_{T}\) and \([B]_{T}\), respectively.

Note that lower semi-continuity (2.1) can be written as: for any \(f \in \mathbb{C}[0,T]\)

\[
\lim_{\varepsilon \to 0} I_0^T((f)_{T,\varepsilon}) = I_0^T(f). \tag{2.2}
\]
It is obvious that (2.1) and (2.2) are equivalent.

We shall keep the same notations as defined above also for an arbitrary space $\mathbb{M}$ with metric $\rho = \rho_{\mathbb{M}}$.

For a function $f \in \mathbb{C}$, $f(T)$ denotes its projection on $\mathbb{C}[0, T]$,

$$f(T) = f(T)(t) := f(t); \quad 0 \leq t \leq T.$$ 

Let now $X_n(t); \quad t \in [0, \infty)$, be a sequence of processes in space $\mathbb{C}_0$. We shall assume in the remainder of §2 that the following conditions hold:

I. For any $T \in (0, \infty)$ processes $X_n(T)$ satisfy LDP in space $\mathbb{C}[0, T]$ with good rate function $I_0^T$, i.e. for any measurable set $B \subset \mathbb{C}[0, T]$

$$\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n(T) \in B) \leq -I_0^T((B)_T),$$

$$\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n(T) \in B) \geq -I_0^T((B)_T).$$

Moreover, for any $f \in \mathbb{C}[0, T]$ there is $g = g_f \in \mathbb{C}_0$, such that $g(T) = f$, and for any $U \geq T$ it holds

$$I_0^U(g(U)) = I_0^{(T)}(f).$$

(2.3)

Remark that condition (2.3) means that one can extend any $f \in \mathbb{C}[0, T]$ for $t > T$ such that the rate function will stay the same. It is natural to call the function $g = g_f$ most likely extension of $f$ beyond $[0, T]$.

II. For any $r \geq 0$

$$\lim_{T \to \infty} \sup_{f \in B_r} \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} = 0,$$

where

$$B_r := \{ f \in \mathbb{C} : \lim_{T \to \infty} I_0^T(f(T)) \leq r \}.$$ 

III. For any $N < \infty$ and $\varepsilon > 0$ there is $T = T_{N, \varepsilon} < \infty$ such that

$$\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(\sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} > \varepsilon) \leq -N.$$ 

It is known (see e.g. [6] or [7]), that LDP implies local LDP: for any $f \in \mathbb{C}[0, T]$

$$-I_0^T(f) \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n(T) \in (f)_{T, \varepsilon}) \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n(T) \in (f)_{T, \varepsilon}) \geq -I_0^T(f).$$

For $U \geq T$, with obvious notations, we have for $f \in \mathbb{C}$

$$\{ X_n^{(U)} \in (f(U))_{U, \varepsilon} \} \subset \{ X_n^{(T)} \in (f(T))_{T, \varepsilon} \},$$

therefore

$$-I_0^U(f(U)) \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n^{(U)} \in (f(U))_{U, \varepsilon}) \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n^{(U)} \in (f(U))_{U, \varepsilon}) \geq -I_0^U(f(U)).$$
Thus we established that for $U \geq T$, $I_0^T(f^{(T)}) \leq I_0^U(f^{(U)})$, i.e. the rate function $I_0^T(f^{(T)})$ of argument $T$ is non-decreasing in $T$. Therefore for any $f \in \mathbb{C}$ there exists limit
\[
\lim_{T \to \infty} I_0^T(f^{(T)}) =: I(f).
\]
(2.4)

In what follows it will be shown (see Lemma 2.1), that $I(f)$ is a good rate function in the space $(\mathbb{C}, \rho)$.

For a non-empty set $B \subset \mathbb{C}$ Let
\[
I(B) := \inf_{f \in B} I(f).
\]
For $B = \emptyset$ let $I(B) = \infty$.

For $B \subset \mathbb{C}[0, T]$ denote by
\[
\overline{B}^{(T)} := \{ g \in \mathbb{C}_0 : g^{(T)} \in B \}.
\]
we show that the following equality holds:
\[
I_0^T(B) = I(\overline{B}^{(T)}).
\]
(2.5)

Indeed, for any $\varepsilon > 0$ let $f \in B$ be such that
\[
I_0^T(f) \leq I_0^T(B) + \varepsilon.
\]
Then due to (2.3) there is $g \in \mathbb{C}_0$ such that $g^{(T)} = f$ (consequently $g \in \overline{B}^{(T)}$) with $I(g) = I_0^T(f)$. Therefore
\[
I_0^T(B) + \varepsilon \geq I_0^T(f) = I(g) \geq I(\overline{B}^{(T)}).
\]
Since $\varepsilon > 0$ is arbitrary,
\[
I_0^T(B) \geq I(\overline{B}^{(T)}).
\]
(2.6)

Let now $g \in \overline{B}^{(T)}$ such that
\[
I(g) \leq I(\overline{B}^{(T)}) + \varepsilon.
\]
Then $g^{(T)} \in B$ with $I_0^T(g^{(T)}) \leq I(g)$. Therefore
\[
I(\overline{B}^{(T)}) + \varepsilon \geq I(g) \geq I_0^T(g^{(T)}) \geq I_0^T(B),
\]
and
\[
I_0^T(B) \leq I(\overline{B}^{(T)}).
\]
(2.7)

Inequalities (2.6), (2.7) now prove equality (2.5).

For a $\varepsilon > 0$ $(f)_\varepsilon$, $(B)_\varepsilon$ denote $\varepsilon$-neighborhood of $f \in \mathbb{C}$, and set $B \subset \mathbb{C}$, respectively.

**Lemma 2.1.** The rate function $I(f)$ (see definition 2.4) is a good rate function, i.e. for any $r \geq 0$ the set
\[
B_r := \{ f \in \mathbb{C} : I(f) \leq r \}
\]
is a compact in $\mathbb{C}$ and
\[
\lim_{\varepsilon \to 0} I((f)_\varepsilon) = I(f).
\]
(2.8)
Proof. We show that if \( f_n \to f \), then
\[
\lim_{n \to \infty} I(f_n) \geq I(f).
\] (2.9)

For any \( N < \infty, \varepsilon > 0 \) there is \( T = T_{N, \varepsilon} < \infty \) such that
\[
I_T(f) \geq \min\{I(f), N\} - \varepsilon.
\]

Since \( f_n \to f, \rho_T(f_n, f) \to 0 \). The rate function \( I_T(f) \) is lower semi-continuous in \((C[0,T], \rho_T)\) (due to condition I), therefore
\[
\lim_{n \to \infty} I(f_n) \geq \lim_{n \to \infty} I_T(f_n) \geq I_T(f) \geq \min\{I(f), N\} - \varepsilon.
\]

Since \( N < \infty \) and \( \varepsilon > 0 \) are arbitrary, the latter implies (2.9). As noted earlier, (2.9) implies lower semi-continuity (2.8).

We show next that the set \( \mathbf{B}_r \) is completely bounded. For any \( \varepsilon > 0 \) due to condition II there is \( T = T_r < \infty \) such that for any \( f \in \mathbf{B}_r \)
\[
\sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon.
\] (2.10)

Denote
\[
\mathbf{B}_r^{(T)} := \{ f(T) : f \in \mathbf{B}_r \},
\]
so that
\[
\mathbf{B}_r^{(T)} \subset \mathbf{B}_{T,r}.
\]

Since by I the set \( \mathbf{B}_{T,r} \) is a compact in \( C[0,T] \), it is possible to find finite \( \varepsilon \)-net:
\[
\mathbf{B}_r^{(T)} \subset \bigcup_{i=1}^M (f_i)_{T,\varepsilon}.
\]

Now for \( f \in C[0,T] \) define \( \overline{f}^{(T)} \in \mathcal{C}_0 \) as
\[
\overline{f}^{(T)}(t) := \begin{cases} f(t), & \text{if } 0 \leq t \leq T, \\ f(T), & \text{if } t \geq T \end{cases}
\]

For any \( f \in \mathbf{B}_r \) there is \( i \in \{1, \cdots, M\} \) such that
\[
\sup_{0 \leq t \leq T} \frac{|f(t) - \overline{f}_i^{(T)}(T)|}{1 + t^{1+\kappa}} < \varepsilon < 3\varepsilon.
\]

We have for this \( i \) due to (2.10)
\[
\sup_{t \geq T} \frac{|f(t) - \overline{f}_i^{(T)}(T)|}{1 + t^{1+\kappa}} \leq \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} + \sup_{t \geq T} \frac{|f(T)|}{1 + t^{1+\kappa}} + \sup_{t \geq T} \frac{|f(T) - \overline{f}_i(T)|}{1 + t^{1+\kappa}} \leq 3\varepsilon,
\]
therefore the collection \( \overline{f}_1, \cdots, \overline{f}_M \) represents a \( 3\varepsilon \)-net in the set \( \mathbf{B}_r \). Thus we have shown that the set \( \mathbf{B}_r \) is completely bounded in \( \mathcal{C}_0 \).

From lower semi-continuity of \( I(f) \), established earlier, it follows that \( \mathbf{B}_r \) is closed in \( \mathcal{C}_0 \). Since a closed completely bounded subset of a Polish space is a compact (see [8], Theorem 3, p. 109), we have shown that \( \mathbf{B}_r \) is a compact in \( \mathcal{C}_0 \), thus completing the proof of Lemma 2.1.
The sequence \( X_n \) satisfies LDP in the space \( \mathbb{C} \) with rate function \( I(f) \), i.e., for any measurable \( B \subset \mathbb{C} \)

\[
\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in B) \leq -I([B]),
\]

(2.11)

\[
\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in B) \geq -I((B)),
\]

(2.12)

where \([B], (B)\) is the closure and the interior of \( B \), respectively.

Note that if for a measurable set \( B \) \( I([B]) = I((B)) \), then, obviously, \( I([B]) = I((B)) = I(B) \) and inequalities (2.11), (2.12) can be replaced by equality

\[
\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in B) = -I(B).
\]

Hence the difference

\[
D(B) := I((B)) - I([B]) \geq 0
\]

describes precision of LDP: the smaller the difference the more precise is the theorem. In [4] (Theorem 2.6) sufficient conditions are given for a sequence \( X_n \) to satisfy LDP in the space \((\mathbb{C}, \rho^{(P)})\).

Assume, that conditions of both theorems 2.1 and 2.6 in [4] are satisfied, and compare their statements. It follows that the rate functions in both theorems are the same. This is because projections \( X_n^{(T)} \) on \([0, T]\) satisfy LDP in the space \( \mathbb{C}[0, T] \) with uniform metric and rate function \( I_0^T(f) \) (common for both theorems), and rate functions on half-line in both theorems are defined by using \( I_0^T(f) \) for all \( T > 0 \). Therefore we can compare these theorems by comparing differences

\[
D(B) := I((B)) - I([B]) \quad \text{and} \quad D^{(P)}(B) := I((B)^{(P)}) - I([B]^{(P)}),
\]

where \([B]^{(P)}, (B)^{(P)}\) are the closure and the interior of \( B \) in metric \( \rho^{(P)} \), respectively. Let’s do the comparison.

As noted earlier, (see also [4]), \( \rho^{(P)}(f_n, f) \to 0 \) as \( n \to \infty \), is equivalent to \( \rho_T(f_n, f) \to 0 \) for any \( T > 0 \). Therefore \( \rho(f_n, f) \to 0 \) implies \( \rho^{(P)}(f_n, f) \to 0 \). The opposite is not true, as the following example demonstrates.

\[
f_n(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq n, \\ -n^2 - 2n + t(n + 2), & \text{if } n \leq t \leq n + 1, \\ n + 2, & \text{if } t > n + 1. \end{cases}
\]

Then, clearly, \( f_0 = f_0(t) \equiv 0 \) and

\[
\rho(f_n, f_0) \to 1, \quad \rho^{(P)}(f_n, f_0) \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus

\[
[B] \subset [B]^{(P)}, \quad (B)^{(P)} \subset (B),
\]

therefore

\[
I([B]^{(P)}) \leq I([B]), \quad I((B)) \leq I((B)^{(P)}),
\]

so that we have always \( D(B) \leq D^{(P)}(B) \). In §3 we give an example of a measurable set \( B \) satisfying simultaneously

\[
I([B]) = I((B)) \in (0, \infty), \quad I([B]^{(P)}) = 0.
\]
Due to condition (2.14) now follows from (2.15) by using (2.16), (2.17).

For the proof of Theorem 2.1 we need two Lemmas (Lemma 2.2 and Lemma 2.3).

**Lemma 2.2.** For any \( f \in C_0 \), \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in (f)_{\varepsilon}) \leq -I((f)_{2\varepsilon}), \tag{2.13}
\]

\[
\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in (f)_{\varepsilon}) \geq -I(f). \tag{2.14}
\]

**Proof** (i). First we prove the bound from below (2.13) (as it is also used in the proof of upper bound (2.13)). If \( I(f) = \infty \), then (2.13) is trivially satisfied. Let now \( I(f) < \infty \). For any \( T \in (0, \infty) \) there holds the inclusion

\[ \{X_n \in (f)_{\varepsilon}\} \supset A_n(T) \cap B_n(T) \cap C_n(T) \cap D(T) = A_n(T) \cap B_n(T) \cap D(T), \]

where

\[ A_n(T) := \{ \sup_{0 \leq t \leq T} \frac{|X_n(t) - f(t)|}{1 + t^{1+\kappa}} < \varepsilon \}, \quad B_n(T) := \{ \sup_{t \geq T} \frac{|X_n(t) - X_n(T)|}{1 + t^{1+\kappa}} < \varepsilon/4 \}, \]

\[ C_n(T) := \{ \frac{|X_n(T)|}{1 + T^{1+\kappa}} < \varepsilon/4 \}, \quad D(T) := \{ \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon/2 \}. \]

For a large \( T \) the event \( D(T) \) is a certainty (due to \( I(f) < \infty \)). Therefore there exists \( T_0 < \infty \), such that for all \( T \geq T_0 \) it holds that

\[ P(X_n \in (f)_{\varepsilon}) \geq P(A_n(T) \cap B_n(T)) \geq P(A_n(T)) - P(\overline{B_n}(T)), \tag{2.15} \]

where \( \overline{B_n}(T) \) is a complement of \( B_n(T) \). Due to condition III there is \( T \geq T_0 \) such that

\[ \lim_{n \to \infty} \frac{1}{n} \ln P(\overline{B_n}(T)) \leq -2I(f), \tag{2.16} \]

and for this \( T \) due to I we have

\[ \lim_{n \to \infty} \frac{1}{n} \ln P(A_n(T)) \geq -I_0^n(f(T)) \geq -I(f). \tag{2.17} \]

(2.14) now follows from (2.15) by using (2.16), (2.17).

(ii). Now we prove the upper bound (2.13). It is obvious that for any \( T \in (0, \infty) \)

\[ P(X_n \in (f)_{\varepsilon}) \leq P(X_n^{(T)} \in (f(T))_{\varepsilon}). \]

Due to condition I for any \( \delta > 0 \)

\[ L(\varepsilon) := \lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in (f)_{\varepsilon}) \leq \lim_{n \to \infty} \frac{1}{n} \ln P(X_n^{(T)} \in (f(T))_{\varepsilon}) \leq \]

\[ \leq -I_0^n((f(T))_{T,\varepsilon}) \leq -I_0^n((f(T))_{T,\varepsilon+\delta}). \]

For any \( T \in (0, \infty) \) and chosen \( \varepsilon \) and \( \delta \), in this way we have the inequality

\[ L(\varepsilon) \leq -I_0^n((f(T))_{T,\varepsilon+\delta}). \tag{2.18} \]
Choose now $T < \infty$ so large, that simultaneously the following holds:

$$\sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \delta; \quad (2.19)$$

$$\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in R(T, \varepsilon)) \leq -N, \quad (2.20)$$

where $N < \infty$ is arbitrary, and

$$R(T, \varepsilon) := \{g \in \mathbb{C}_0 : \sup_{t \geq T} \frac{|g(t)|}{1 + t^{1+\kappa}} > \varepsilon\}.$$

Denote

$$\overline{(f(T))}_{T,\varepsilon+\delta} := \{g \in \mathbb{C}_0 : g^{(T)} \in (f(T))_{T,\varepsilon+\delta}\}.$$

Due to property (2.5) (which was obtained from (2.3) in condition I) we have

$$I_T ((f(T))_{T,\varepsilon+\delta}) = I((f(T))_{T,\varepsilon+\delta}),$$

therefore due to (2.18)

$$L(\varepsilon) \leq -I((f(T))_{T,\varepsilon+\delta}). \quad (2.21)$$

Take an arbitrary $g \in (f(T))_{T,\varepsilon+\delta}$. Then either

$$\sup_{t \geq T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} < \varepsilon + 2\delta,$$

and then

$$g \in (f)_{\varepsilon+2\delta}; \quad (2.22)$$

either

$$\sup_{t \geq T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} \geq \varepsilon + 2\delta, \quad (2.23)$$

and then

$$\sup_{t \geq T} \frac{|g(t)|}{1 + t^{1+\kappa}} \geq \varepsilon + \delta, \quad (2.24)$$

and

$$g \in R(T, \varepsilon). \quad (2.25)$$

To clarify deduction of (2.24) from (2.23), note that if the inequality (2.21) is not true, then the opposite holds

$$\sup_{t \geq T} \frac{|g(t)|}{1 + t^{1+\kappa}} < \varepsilon + \delta,$$

and due to (2.19)

$$\sup_{t \geq T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} \leq \sup_{t \geq T} \frac{|g(t)|}{1 + t^{1+\kappa}} + \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon + \delta + \delta = \varepsilon + 2\delta,$$

which contradicts (2.23). We have proved (see (2.22) and (2.25)), that

$$\overline{(f(T))}_{T,\varepsilon+\delta} \subset (f)_{\varepsilon+2\delta} \cup R(T, \varepsilon).$$
From the latter we obtain

\[ I((f(T))_{T,\varepsilon} + \delta) \geq \min\{I((f)_{\varepsilon + 2\delta}), I(R(T, \varepsilon))\}. \quad (2.26) \]

Further, due to (2.20)

\[ -N \geq \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n \in R(T, \varepsilon)) \geq \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n \in R(T, \varepsilon)) \geq -I(R(T, \varepsilon)), \]

where the last inequality for an open set \( R(T, \varepsilon) \) follows from the established lower bound (2.14). Therefore

\[ I(R(T, \varepsilon)) \geq N, \]

and, in view of (2.26),

\[ I((f(T))_{T,\varepsilon} + \delta) \geq \min\{I((f)_{\varepsilon + 2\delta}), N\}. \]

Going back to (2.21), we obtain the inequality

\[ L(\varepsilon) \leq -\min\{I((f)_{\varepsilon + 2\delta}), N\}, \]

in which \( \delta > 0 \) and \( N < \infty \) are arbitrary. Taking \( 2\delta = \varepsilon \) and sending \( N \) to \( \infty \), we obtain the required upper bound

\[ L(\varepsilon) \leq -I((f)_{2\varepsilon}). \]

Lemma 2.2 is now proved.

Local LDP for \( X_n \) in \( C_0 \) follows from Lemma 2.2:

**Corollary 2.1.** For any \( f \in C_0 \)

\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n \in (f)_\varepsilon) \leq -I(f), \]

\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n \in (f)_\varepsilon) \geq -I(f). \]

**Lemma 2.3.** For any \( N < \infty \) and \( \varepsilon > 0 \) there is a finite collection of \( f_1, \ldots, f_M \in C_0 \) such that

\[ \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n \notin \cup_{i=1}^M (f_i)_\varepsilon) \leq -N. \]

**Proof.** Denote by

\[ R_T(\varepsilon) := \{ f \in C_0 : \sup_{t \geq T} \frac{|f(t)|}{1 + t^{1+\kappa}} \leq \varepsilon \}. \]

Then due to condition III there is \( T < \infty \) such that

\[ \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n \notin R_T(\varepsilon)) \leq -N. \quad (2.27) \]

For this \( T \) due to condition I the process \( X_n^{(T)} \) satisfies LDP in the space \( C[0, T] \). Therefore for a chosen \( N \) by a theorem of Puhalskii (see [1]) there is a compact \( K \subset C[0, T] \) such that

\[ \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n^{(T)} \notin K) \leq -N. \]
For a given $\varepsilon > 0$ take a finite $\varepsilon$-net $f_1, \cdots, f_M \in \mathbb{C}[0, T]$ in $K$:

$$K \subset \bigcup_{i=1}^{M} (f_i)_{T, \varepsilon}.$$ 

Then

$$\lim_{n \to \infty} \frac{1}{n} \ln P(X_n^{(T)} \notin \bigcup_{i=1}^{M} (f_i)_{T, \varepsilon}) \leq -N. \quad (2.28)$$

Denote for all $i = 1, \cdots, M$

$$f_i(t) := \begin{cases} f_i(t), & \text{if } 0 \leq t \leq T, \\ f_i(T), & \text{if } t \geq T. \end{cases}$$

Define the set $M_\varepsilon := \{i \in \{1, \cdots, M\} : |f_i(T)| \leq 2\varepsilon\}$. Then

$$P := P(X_n \notin \bigcup_{i=1}^{M} (f_i)_{3\varepsilon}) \leq P(X_n \notin R_T(\varepsilon)) + P(X_n \in R_T(\varepsilon), X_n^{(T)} \notin \bigcup_{i=1}^{M} (f_i)_{T, \varepsilon}) + P(X_n \in R_T(\varepsilon), X_n^{(T)} \in \bigcup_{i=1}^{M} (f_i)_{T, \varepsilon}, X_n \notin \bigcup_{i=1}^{M} (f_i)_{3\varepsilon}) =: P_1 + P_2 + P_3.$$

We bound $P_3$ as follows:

$$P_3 \leq \sum_{i \in M_\varepsilon} P(\sup_{t \geq T} |X_n(t) - f_i(T)| > 3\varepsilon) \leq M \sup_{t \geq T} |X_n(t)| \leq M P(X_n \notin R_T(\varepsilon)).$$

Since

$$P_1 \leq P(X_n \notin R_T(\varepsilon)),$$

we obtain

$$P \leq P(X_n^{(T)} \notin \bigcup_{i=1}^{M} (f_i)_{T, \varepsilon}) + (M + 1)P(X_n \notin R_T(\varepsilon)). \quad (2.29)$$

Using bounds (2.27) and (2.28) with (2.29), we obtain the required inequality

$$\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \notin \bigcup_{i=1}^{M} (f_i)_{3\varepsilon}) \leq -N.$$ 

Lemma 2.3 is now proved.

We proceed now to prove Theorem 2.1.

Proof of Theorem 2.1. (i). By Lemma 2.2 and 2.3 for any measurable set $B \subset \mathbb{C}_0$ and $\varepsilon > 0$ it holds (see e.g. [6] or [7])

$$\lim_{n \to \infty} \frac{1}{n} \ln P(X_n \in B) \leq -I((B)_\varepsilon).$$

As it is known (see e.g. [6] or [7]), that a good rate function $I(f)$ satisfies

$$\lim_{\varepsilon \to \infty} I((B)_\varepsilon) = I([B]),$$

the upper bound (2.11) is proved.

(ii). Lower bound (2.12) follows from (2.14) of Lemma 2.2. Theorem 2.1 is proved.
§3. Large Deviations for Random Walks

3.1. Large Deviation Principle for Random Walks on half-line.

Let $\xi$ be a non-degenerate random variable satisfying the following condition $[C_\infty]$. For any $\lambda \in \mathbb{R}$

$\mathbb{E}e^{\lambda \xi} < \infty$.

Denote

$\Lambda(\alpha) := \sup_{\lambda} \{\lambda \alpha - A(\lambda)\}, \quad A(\lambda) := \ln \mathbb{E}e^{\lambda \xi}$,

the deviation function of $\xi$. It is a convex non-negative lower-semicontinuous function with a single zero at $\alpha = 0$, (see e.g. [6] or [9]).

Denote

$S_0 := 0, \quad S_k := \xi_1 + \cdots + \xi_k$ for $k \geq 1$,

where $\{\xi_n\}$ is a sequence of i.i.d. copies of $\xi$. Consider a random piece-wise linear function $s_n = s_n(t) \in \mathbb{C}$, going through the nodes

$\left(\frac{k}{n}, \frac{S_k}{x}\right), \quad k = 0, 1, \cdots$,

where $x = x(n)$ is a fixed sequence of positive constants such that $x \sim n$ as $n \to \infty$. Rate function corresponding to the process $s_n$, define as

$I(f) := \begin{cases} \int_0^\infty \Lambda(f'(t))dt, & f(0) = 0, \ f \text{ is absolutely continuous,} \\ +\infty, \text{otherwise.} \end{cases}$

**Theorem 3.1.** Assume $[C_\infty]$. Then $s_n$ satisfies LDP in space $(\mathbb{C}, \rho_\kappa)$ for $\kappa = 0$ with rate function $I$.

In the heart of the proof of the theorem is the following

**Lemma 3.1.** Assume conditions of Theorem 3.1 as well as $a := \mathbb{E}\xi = 0$. Then $s_n$ satisfies LDP in space $(\mathbb{C}, \rho_\kappa)$ with $\kappa = 0$.

We postpone the proof of the Lemma and now show how Theorem 3.1 follows from it.

**Proof** of Theorem 3.1. In what follows superscript $(0)$ denotes quantities for the centered random variable $\xi^{(0)} := \xi - a$. The deviation function for $\xi^{(0)}$ is given by

$\Lambda^{(0)}(\alpha) = \Lambda(\alpha + a)$.

Therefore the rate function for $s_n^{(0)}$, is given by

$I^{(0)}(f) = I(f + e_a)$,

where $e_a = e_a(t) := at; \ t \geq 0$. Clearly, $s_n = s_n^{(0)} + e_a$,

$\mathbb{P}(s_n \in B) = \mathbb{P}(s_n^{(0)} \in B - e_a)$,

where $B - e_a := \{f - e_a : \ f \in B\}$. It is obvious that

$[B - e_a] = [B] - e_a, \quad (B - e_a) = (B) - e_a$,

implying

$I^{(0)}([B - e_a]) = I([B]), \quad I^{(0)}((B - e_a)) = I((B))$. 
Hence the LDP for $s_n^{(0)}$ with rate function $I^{(0)}$ implies LDP for $s_n$ with rate function $I$. Theorem 3.1 is proved.

We proceed to prove Lemma 3.1.

Proof of Lemma 3.1. Theorem 3.1 consists in checking conditions I–III of Theorem 2.1. Condition I follows from the LDP for $s_n$ in $C[0,1]$ (e.g. [6] or [9] or [10]).

Proof of II. By $[C_{\infty}]$, with $E\xi = 0$ it follows that there exists a non-decreasing continuous function $h(t); t \geq 0$, such that for some $\delta > 0$, $h(t) = \delta t$, if $0 \leq t \leq 1$, $\lim_{t \to \infty} h(t) = \infty$, and that for all $\alpha \in \mathbb{R}$ the following inequality holds

$$\Lambda(\alpha) \geq h(|\alpha|)|\alpha|. \quad (3.1)$$

Indeed, for $\alpha \to 0$ (see e.g. [6] or [9])

$$\Lambda(\alpha) \sim \frac{\alpha^2}{2\sigma^2}; \quad (3.2)$$

for any $\lambda > 0, \alpha > 0$

$$\Lambda(\alpha) \geq \lambda \alpha - A(\lambda), \quad \Lambda(-\alpha) \geq \lambda \alpha - A(-\lambda).$$

Therefore

$$\lim_{|\alpha| \to \infty} \frac{\Lambda(\alpha)}{|\alpha|} \geq \lambda,$$

so that

$$\lim_{|\alpha| \to \infty} \frac{\Lambda(\alpha)}{|\alpha|} = \infty. \quad (3.3)$$

(3.1) follows from (3.2) and (3.3). Denote by

$$f_T(t) := tf(T), \quad t \in [0,T].$$

Function $f_T$ "straightens" function $f$ on $[0,T]:$

$$I_T^0(f) \geq I_T^0(f_T) = \int_0^T \Lambda\left(\frac{f(T)}{T}\right)dt = T\Lambda\left(\frac{f(T)}{T}\right).$$

Therefore by (3.1) for $f \in B_r$

$$r \geq T\Lambda\left(\frac{f(T)}{T}\right) \geq T\frac{|f(T)|}{T}h\left(\frac{|f(T)|}{T}\right),$$

so that

$$\frac{|f(T)|}{T} \leq \frac{r}{Th\left(\frac{|f(T)|}{T}\right)}. \quad (3.4)$$

For $c := \sqrt{\frac{r}{T}}, \frac{c}{\sqrt{T}} \leq 1$ assume that

$$\frac{|f(T)|}{T} \geq \frac{c}{\sqrt{T}}.$$

It follows from (3.4) and (3.1) that

$$\frac{|f(T)|}{T} \leq \frac{r}{T h\left(\frac{c}{\sqrt{T}}\right)} = \frac{r}{T \delta \frac{c}{\sqrt{T}}} = \frac{c}{\sqrt{T}}.$$
The latter gives: for $T \geq c^2 = \frac{\varepsilon}{\delta}$

$$|f(T)| \leq \sqrt{\frac{r}{\delta}} \sqrt{T}.$$  

Clearly, $B_r = \overline{B}_r$. Therefore we have established that

$$\sup_{f \in \mathcal{F}_{T_k}} \sup_{t \geq T} \frac{|f(t)|}{1 + t} \leq \sqrt{\frac{r}{\delta}} \sqrt{T} \leq \sqrt{\frac{r}{\delta}} \frac{1}{\sqrt{T}}.$$  

Condition II is now proved.

Check now III. For $T_n := \max\{\frac{k}{n} \leq T : k = 1, 2, \ldots\}$, we have

$$P\left(\sup_{t \geq T_n} \frac{|s_n(t)|}{1 + t} > \varepsilon/2\right) + P\left(\sup_{t \geq T_n} \frac{|s_n(T_n)|}{1 + T_n} > \varepsilon/2\right) =$$

$$P\left(\sup_{t \geq T_n} \frac{|s_n(t) - s_n(T_n)|}{1 + t} > \varepsilon/2\right) + P\left(\sup_{t \geq T_n} \frac{|s_n(T_n)|}{1 + T_n} > \varepsilon/2\right) \leq$$

$$P\left(\sup_{k \geq 1} \frac{|s_n(T_n)|}{T + \frac{k}{n}} > \varepsilon/2\right) + P\left(\frac{|s_n(T_n)|}{T_n} > \varepsilon/2\right) =: P_1(n) + P_2(n).$$

To bound $P_1(n)$ use the exponential Chebyshev’s (Chernoff’s) inequality (see e.g. [6]):

$$P_1(n) \leq \sum_{k \geq 1} P\left(\frac{|S_k|}{x(T + \frac{k}{n})} > \varepsilon/2\right) \leq$$

$$\sum_{k \geq 1} P\left(\frac{S_k}{x(T + \frac{k}{n})} > \varepsilon/2\right) + \sum_{k \geq 1} P\left(\frac{S_k}{x(T + \frac{k}{n})} < -\varepsilon/2\right) \leq$$

$$\sum_{k \geq 1} e^{-k \Lambda(R)} + \sum_{k \geq 1} e^{-k \Lambda(-R)},$$

where $R := \frac{\varepsilon}{4}(T + \frac{k}{n})$. Since for all $n$ and $T$ large enough

$$R \geq \varepsilon/4, \quad kR \geq T\varepsilon/4 + k\varepsilon/4,$$

we have due to (3.1) for $\varepsilon/4 \in (0,1)$

$$k\Lambda(\pm R) \geq kRh(R) \geq (T\varepsilon/4 + k\varepsilon/4)\delta \varepsilon/4 = T\frac{\delta \varepsilon^2}{16} + k\frac{\delta \varepsilon^2}{16}.$$  

Therefore

$$P_1(n) \leq 2e^{-T\delta_1} \sum_{k \geq 1} e^{-k\delta_1} = C_1e^{-T\delta_1},$$

where $\delta_1 := \frac{\delta \varepsilon^2}{16}, \quad C_1 := 2\frac{e^{-\delta_1}}{1 - e^{-\delta_1}}$.

Similarly we obtain the bound

$$P_2(n) \leq C_2e^{-T\delta_2}.$$
for some $\delta_2 > 0$, $C_2 < \infty$. Hence condition III holds and proof of Lemma 3.1 is complete.

3.2. Moderate Deviation Principle for Random Walks on half-line. Let random piece-wise linear function $s_n = s_n(\cdot) \in C$ be defined as before by the sums $S_k$ of independent random variables distributed as $\xi$. Let $\xi$ have zero mean $E\xi = 0$ and assume Cramer’s condition $[C_0]$. For some $\delta > 0$

$$E e^{\delta|\xi|} < \infty.$$  

Let a sequence $x = x(n)$, used in the construction of $s_n$, satisfy

$$\frac{x}{\sqrt{n}} \to \infty, \quad \frac{x}{n} \to 0 \quad \text{if} \quad n \to \infty.$$

The rate function for $s_n$, is defined as

$$I_0(f) := \begin{cases} \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} (f'(t))^2 dt, & \text{if} \quad f(0) = 0, \quad f \text{ is absolutely continuous} \\
\infty & \text{otherwise,} \end{cases}$$

where $\sigma^2 := E\xi^2$.

**Theorem 3.2.** Let $E\xi = 0$ and condition $[C_0]$ holds. Then $s_n$ satisfies LDP with speed $\frac{x^2}{n}$ and rate function $I_0$ in space $(C, \rho_\kappa)$ with $\kappa = 0$, i.e. for any measurable set $B \subset C$

$$\lim_{n \to \infty} \frac{n}{x^2} P(s_n \in B) \leq -I_0([B]),$$

$$\lim_{n \to \infty} \frac{n}{x^2} P(s_n \in B) \geq -I_0((B)).$$

Similarly to the proof of Lemma 3.1, the proof of Theorem 3.2 consists in checking that I – III hold, replacing $n$ by $\frac{x^2}{n}$. In all other details the proof is the same.

Condition I is verified with help of [11] and [12]. Condition II is obvious. Only condition III requires a clarification, done by using the following form of Kolmogorov’s inequality ([13], p. 295, lemma 11.2.1):

**Lemma 3.2.** For any $x \geq 0$, $y \geq 0$, $n \geq 1$

$$P(\max_{1 \leq m \leq n} |S_m| \geq x + y) \leq \frac{P(|S_n| \geq x)}{\min_{1 \leq m \leq n} P(|S_m| \leq y)}.$$

An upper bound for

$$P := P(\sup_{t \geq T} \frac{|s_n(t)|}{1 + t} \geq \varepsilon)$$

is obtained by using:

$$P \leq \sum_{K \geq T} P(\sup_{K \leq t \leq K+1} |s_n(t)| \geq K\varepsilon) \leq$$

$$\sum_{K \geq T} P(\sup_{K \leq t \leq K+1} |s_n(t) - s_n(K)| \geq K\varepsilon/2) + \sum_{K \geq T} P(|s_n(K)| \geq K\varepsilon/2),$$

so that

$$P \leq \sum_{K \geq T} P_1(K, n) + \sum_{K \geq T} P_2(K, n),$$

(3.5)
where

\[ P_1(K, n) := \mathbf{P}(\sup_{0 \leq t \leq 1} |s_n(u)| \geq K\varepsilon/2), \quad P_2(K, n) := \mathbf{P}(|s_n(K)| \geq K\varepsilon/2). \]

We bound \( P_2(K, n) \) using exponential Chebyshev’s inequality:

\[ P_2(K, n) = \mathbf{P}\left( \frac{|S_{nK}|}{nK} \geq \frac{x\varepsilon}{2n} \right) \leq e^{-nK\Lambda(\frac{x\varepsilon}{2n})} + e^{-nK\Lambda(-\frac{x\varepsilon}{2n})}. \]

Since for all \( n \) large enough \( \frac{x\varepsilon}{2n} \leq 1 \), then by (3.1)

\[ nK\Lambda(\pm \frac{x\varepsilon}{2n}) \geq \frac{x^2}{nK}\delta_1, \quad \delta_1 := \frac{\delta\varepsilon^2}{4}. \]

Therefore

\[ \sum_{K \geq T} P_2(K, n) \leq 2 \frac{e^{-\frac{x^2}{nT}\delta_1}}{1 - e^{-\frac{x^2}{n}\delta_1}}. \] (3.6)

Bound now \( P_1(K, n) \) using Kolmogorov’s inequality (Lemma 3.2) and exponential Chebyshev’s inequality (Chernoff)

\[ P_1(K, n) = \mathbf{P}(\max_{1 \leq m \leq n} \frac{|S_m|}{xK} \geq \varepsilon/4 + \varepsilon/4) \leq \frac{1}{c} \mathbf{P}(\frac{|S_n|}{xK} \geq \varepsilon/4), \]

where

\[ c := \min_{1 \leq m \leq n} \mathbf{P}(\frac{|S_m|}{xK} < \varepsilon/4). \]

Since

\[ c = \min_{1 \leq m \leq n} \mathbf{P}(\frac{|S_m|}{xK} < \varepsilon/4) \geq \min_{1 \leq m \leq n} \mathbf{P}(\frac{|S_m|}{\sqrt{m}} < \frac{xT}{\sqrt{n}\varepsilon/4}) \to 1 \]

as \( n \to \infty \), for \( n \) large enough

\[ P_1(K, n) \leq 2\mathbf{P}(\frac{|S_n|}{xK} \geq \varepsilon/4) \leq 2e^{-n\Lambda(\frac{xK\varepsilon}{n}\varepsilon/4)} + 2e^{-n\Lambda(-\frac{xK\varepsilon}{n}\varepsilon/4)}. \]

By (3.1) for large enough \( n \) and some \( \delta_1 > 0 \) we have

\[ n\Lambda(\pm \frac{xK}{n}\varepsilon/4) \geq \frac{x^2}{nK}\delta_1, \]

therefore

\[ P_1(K, n) \leq 4e^{-\frac{x^2}{nT}\delta_1}, \]

\[ \sum_{K \geq T} P_1(K, n) \leq 4 \frac{e^{-\frac{x^2}{nT}\delta_1}}{1 - e^{-\frac{x^2}{n}\delta_1}}. \] (3.7)

Applying (3.6), (3.7) to (3.5), we have for \( T \geq \frac{N}{\delta_1} \) the required inequality in III:

\[ \lim_{n \to \infty} \frac{1}{n} \ln P \leq -N. \]

Thus condition III is proved.
3.3. LDP for Wiener process on half-line. Consider Wiener process $w = w(t)$ on $[0, \infty)$. Denote

$$w_n = w_n(t) := \frac{1}{\sqrt{n}} w(t), \quad t \geq 0.$$ 

Since conditions I–III are easily checked, then LDP follows from Theorem 2.1 with rate function

$$I(f) = \begin{cases} \frac{1}{2} \int_0^\infty (f'(t))^2 dt, & \text{if } f(0) = 0, \text{ } f \text{ is absolutely continuous}, \\ \infty & \text{otherwise}. \end{cases}$$

**Theorem 3.3.** The process $w_n$ satisfies LDP in $(C, \rho)$ with $k = 0$ with good rate function $I$.

Next result is given in [4] (Theorem 2.7)

**Theorem 3.4.** The process $w_n$ satisfies LDP in $(C, \rho^{(P)})$ with good rate function $I$.

To compare Theorems 3.3 and 3.4 take a set $B$ as

$$B = (f_0)_1 := \{ g \in C : \sup_{t \geq 0} \frac{|g(t)|}{1 + t} \geq 1 \}, \quad f_0 = f_0(t) \equiv 0.$$ 

Since it is a compliment to an open set $(f_0)_1$, it is closed in $(C, \rho)$, and therefore

$$I([B]) = I(B) = \inf_{g \in B} I(g).$$

Infimum is taken over absolutely continuous functions in $B$. Therefore by using Cauchy-Bunyakovsky inequality

$$1 \leq \sup_{t \geq 0} \frac{|g(t)|}{1 + t} = \sup_{t \geq 0} \frac{|f_0 t g'(s)ds|}{1 + t} \leq \sup_{t \geq 0} \left( \int_0^t (g'(s))^2ds \right)^{1/2} \sup_{t \geq 0} \frac{t^{1/2}}{1 + t} = \frac{\sqrt{2I(g)}}{2}.$$ 

This gives that $I(g) \geq 2$ for all $g \in B$.

We show next that there is an $f \in B$ such that $I(f) = 2$. This $f$ is given by

$$f(t) := \begin{cases} 2t, & \text{if } 0 \leq t \leq 1, \\ 2, & \text{if } t \geq 1. \end{cases}$$

It is easy to see that $f \in B$ and $I(f) = 2$. Therefore, $I([B]) = I(B) = 2$.

It remains to show that $I((B)) = 2$. To this end consider a sequence of functions

$$f_n(t) := \begin{cases} (2 + 1/n)t, & \text{if } 0 \leq t \leq 1, \\ 2 + 1/n, & \text{if } t \geq 1. \end{cases}$$

Clearly, $f_n \in (B)$ for all integer $n$ and we have, $I([B]) = I(B) = I((B)) = 2$. Hence,

$$\lim_{n \to \infty} \frac{1}{n} \ln P(w_n \in B) = -2.$$ 

Consider now $[B]^{(P)}$, the closure of $B$ in metric $\rho^{(P)}$. By taking $g_n(t) = \frac{t^2}{n}$ it is easy to see that $g_n \in B$ for all $n$ and $\lim_{n \to \infty} \rho^{(P)}(g_n, f_0) = 0$. Therefore, $f_0 \in [B]^{(P)}$. Therefore $I([B]^{(P)}) = 0$, and the upper bound in Theorem 3.4 for the set $B$ is trivial, which does not allow to find logarithmic asymptotic of the required probability.
§4. Large Deviations for Diffusion Processes on half-line

4.1. Zero drift. Consider a stochastic process \(X_n(t), t \geq 0\), defined on the stochastic basis \((\Omega, \mathcal{F}, \mathbb{F}_t, \mathbb{P})\) that is an Itô integral with respect to Wiener process \(w(t)\).

\[
X_n(t) = x_0 + \frac{1}{\sqrt{n}} \int_0^t \sigma_n(\omega, s) dw(s),
\]

where \(\sigma_n(\omega, t)\) is \(\mathbb{F}_t\)-adapted and such that the Itô integral is defined.

**Lemma 4.1.** Let for some \(\lambda > 0\) and all \(t \geq 0, \ n \geq 0\)

\[
\sigma^2_n(\omega, t) \leq \lambda \text{ a.s.} \quad (4.1)
\]

Then for any \(N < \infty\) and \(\varepsilon > 0\) there exists \(T = T_{N,\varepsilon} < \infty\) such that

\[
\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}\left(\sup_{t \geq T} \left| X_n(t) \right| \leq \varepsilon \right) = -N. \quad (4.2)
\]

**Proof.** For \(T > 1 + \sqrt{\frac{2|\omega_0|}{\varepsilon}} - 1\) we have

\[
\mathbb{P}\left(\sup_{t \geq T} \left| X_n(t) \right| > \varepsilon \right) \leq \mathbb{P}\left(\sup_{t \geq T} \frac{1}{1 + t} \left| \frac{1}{\sqrt{n}} \int_0^t \sigma_n(\omega, s) dw(s) \right| > \frac{\sqrt{n\varepsilon}}{2} \right) \leq \mathbb{P}\left(\sup_{t \geq T} \frac{1}{t} \int_0^t \sigma_n(\omega, s) dw(s) \right| > \frac{\sqrt{n\varepsilon}}{2} \right) \leq \sum_{r=1}^{\infty} \mathbb{P}\left(\sup_{t \in [Tr, Tr+1)} \left| \int_0^t \sigma_n(\omega, s) dw(s) \right| > \frac{Tr\sqrt{n\varepsilon}}{2} \right) = \sum_{r=1}^{\infty} P_r. \quad (4.3)
\]

We bound \(P_r\) from above as follows. For any \(c > 0\) we have

\[
P_r = \mathbb{P}\left(\sup_{t \in [0, Tr+1]} \exp\left\{c \int_0^t \sigma_n(\omega, s) dw(s)\right\} > \exp\left\{\frac{cTr\sqrt{n\varepsilon}}{2}\right\}\right) \leq \mathbb{P}\left(\sup_{t \in [0, Tr+1]} \exp\left\{c \int_0^t \sigma_n(\omega, s) dw(s)\right\} > \exp\left\{\frac{cTr\sqrt{n\varepsilon}}{2}\right\}\right) + \mathbb{P}\left(\sup_{t \in [0, Tr+1]} \exp\left\{-c \int_0^t \sigma_n(\omega, s) dw(s)\right\} > \exp\left\{\frac{cTr\sqrt{n\varepsilon}}{2}\right\}\right) =: P_{r,1} + P_{r,2}. \quad (4.4)
\]

We proceed to bound \(P_{r,1}\). For ease of notation we drop arguments in \(\sigma_n(\omega, t)\).

Using (4.1) we have

\[
P_{r,1} = \mathbb{P}\left(\sup_{t \in [0, Tr+1]} \exp\left\{c \int_0^t \sigma_n dw(s) + \frac{c^2}{2} \int_0^t \sigma_n^2 ds\right\} > \exp\left\{\frac{cTr\sqrt{n\varepsilon}}{2}\right\}\right) \leq \mathbb{P}\left(\sup_{t \in [0, Tr+1]} \exp\left\{c \int_0^t \sigma_n dw(s) - \frac{c^2}{2} \int_0^t \sigma_n^2 ds\right\} > \exp\left\{cTr\sqrt{n\varepsilon} - c^2\lambda T(r+1)\right\}\right).
\]
By Doob’s martingale inequality for
\[ M(t) = \exp \left\{ c \int_0^t \sigma_n(\omega, s)dw(s) - \frac{c^2}{2} \int_0^t \sigma_n^2(\omega, s)ds \right\}, \]
we have
\[ P_{r,1} \leq \frac{\mathbb{E}M(t)}{\exp \left\{ cTr\sqrt{n\varepsilon} - c^2\lambda T(r+1) \right\}} = \exp \left\{ -\frac{cTr\sqrt{n\varepsilon} - c^2\lambda T(r+1)}{2} \right\}. \]

Taking \( c = \frac{\sqrt{n\varepsilon T}}{2\lambda T(r+1)} \) we obtain
\[ P_{r,1} \leq \exp \left\{ -\frac{Tr\sqrt{n\varepsilon}}{8\lambda T(r+1)} \right\} \leq \exp \left\{ -\frac{Trn^2\varepsilon^2}{16\lambda} \right\}. \quad (4.5) \]

\( P_{r,2} \) is bounded exactly the same. It follows from (4.3), (4.4), (4.5) that
\[ \mathbb{P} \left( \sup_{t \geq T} \frac{|X_n(t)|}{1 + t} > \varepsilon \right) \leq 2 \sum_{r=1}^{\infty} \exp \left\{ -\frac{Trn^2\varepsilon^2}{16\lambda} \right\} = \frac{2 \exp \left\{ -\frac{Trn^2\varepsilon^2}{16\lambda} \right\}}{1 - \exp \left\{ -\frac{Trn^2\varepsilon^2}{16\lambda} \right\}} \leq 4 \exp \left\{ -\frac{Trn^2\varepsilon^2}{16\lambda} \right\}. \quad (4.6) \]

Using inequality (4.6) we have
\[ \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P} \left( \sup_{t \geq T} \frac{|X_n(t)|}{1 + t} > \varepsilon \right) \leq -\frac{T\varepsilon^2}{16\lambda}. \]

It follows now that for \( T > 1 \lor \left( \frac{2|x_0|}{\varepsilon} - 1 \right) \lor \frac{16\lambda N}{\varepsilon^2} \)
\[ \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P} \left( \sup_{t \geq T} \frac{|X_n(t)|}{1 + t} > \varepsilon \right) \leq -N, \]
which proves (4.2).

Let now \( X_n \) solve a stochastic differential equation (SDE)
\[ X_n(t) = x_0 + \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s))dw(s) \quad (4.7) \]
on half-line \([0, \infty)\).

**Theorem 4.1.** Let \( \sigma(x) \) be a measurable function of real argument \( x \), such that for some \( \lambda \geq 1 \) and all \( x \in \mathbb{R} \)
\[ \frac{1}{\lambda} \leq \sigma^2(x) \leq \lambda. \quad (4.8) \]

Let the Lebesgue measure of discontinuities of \( \sigma \) be zero.

Then \( X_n \) satisfies LDP in space \((\mathbb{C}, \rho_n)\) with \( \kappa = 0 \) and good rate function:
\[ I(f) = \begin{cases} \frac{1}{2} \int_0^\infty \frac{(f(t))^2}{\sigma^2(f(0))}dt, & \text{if } f(0) = x_0, \ f \text{ is absolutely continuous}, \\ \infty, & \text{otherwise}. \end{cases} \]
Proof. Existence of weak solution in (4.7) follows e.g. from Proposition 1 of [14]. Condition I holds by Theorem 1 in [13] and by extending \( f \) for \( t \geq T \) by its value at \( T \). Consider the rate function
\[
I_0^T (f(T)) = \begin{cases} \frac{1}{2} \int_0^T \frac{(f'(t))^2}{\sigma^2(f(t))} dt, & \text{if } f(0) = x_0, \text{ if } f \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}
\]

We verify condition II. Using (4.8) and applying Cauchy-Bunyakovskii inequality, we have
\[
\lim_{T \to \infty} \sup_{f \in B_r} \sup_{t \geq T} \frac{|f(t)|}{1 + t} = \lim_{T \to \infty} \sup_{f \in B_r} \sup_{t \geq T} \frac{1}{1 + t} \left| \int_0^t f'(s) ds \right| \leq \lim_{T \to \infty} \left( \int_0^T (f'(s))^2 ds \right)^{1/2} \leq \lim_{T \to \infty} \sup_{f \in B_r} \frac{1}{T^{1/2}} \left( \lambda \int_0^T \frac{(f'(s))^2}{\sigma^2(f(s))} ds \right)^{1/2} = \lim_{T \to \infty} \sup_{f \in B_r} \frac{\sqrt{2\lambda}}{T^{1/2}} \sqrt{I_0^T (f(T))} \leq \lim_{T \to \infty} \frac{\sqrt{2\lambda}}{T^{1/2}} = 0,
\]
where \( B_r := \{ f \in C : \lim_{T \to \infty} I_0^T (f(T)) \leq r \} \).

Condition III follows from (4.8) and Lemma 4.1. The Theorem now is proved.

4.2. Non-zero drift. Consider solution of SDE on half-line \([0, \infty)\)
\[
X_n(t) = x_0 + \int_0^t a(X_n(s)) ds + \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s)) dw(s). \tag{4.9}
\]

Lemma 4.2. Suppose there exists \( \lambda > 0 \) such that for all \( y \in R \)
\[
|a(y)| + \sigma^2(y) \leq \lambda. \tag{4.10}
\]

Then for any \( \kappa > 0 \), \( N < \infty \) and \( \varepsilon > 0 \) there exists \( T = T_{N, \varepsilon, \kappa} < \infty \) such that
\[
\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P} \left( \sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} > \varepsilon \right) \leq -N. \tag{4.11}
\]

Proof. Using condition (4.10) for \( T > 1 \lor (\frac{4|x_0|}{\varepsilon} - 1)^{\frac{1}{1+\kappa}} \lor (\frac{4\lambda}{\varepsilon})^{\frac{1}{1+\kappa}} \) we obtain
\[
\mathbb{P} \left( \sup_{t \geq T} \frac{|X_n(t)|}{1 + t^{1+\kappa}} > \varepsilon \right) \leq \mathbb{P} \left( \sup_{t \geq T} \frac{1}{1 + t^{1+\kappa}} \left( |x_0| + \int_0^t a(X_n(s)) ds + \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s)) dw(s) \right) > \varepsilon \right) \leq \mathbb{P} \left( \sup_{t \geq T} \frac{1}{1 + t^{1+\kappa}} \left( |x_0| + \lambda t + \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s)) dw(s) \right) > \varepsilon \right) \leq \mathbb{P} \left( \sup_{t \geq T} \frac{1}{t^{1+\kappa}} \int_0^t \sigma(X_n(s)) dw(s) > \frac{\sqrt{n} \varepsilon}{2} \right) \leq \mathbb{P} \left( \sup_{t \geq T} \frac{1}{t} \int_0^t \sigma(X_n(s)) dw(s) > \frac{\sqrt{n} \varepsilon}{2} \right).
\]

Further argument repeats verbatim Lemma 4.1. Thus (4.11) is proved.
Теорема 4.2 Let \( a(y) \) and \( \sigma(y) \) are functions of real argument such that for some \( \lambda \geq 1 \) and all \( y, x \in \mathbb{R} \)

\[
|a(y) - a(x)| + |\sigma(y) - \sigma(x)| \leq \lambda|y - x| \tag{4.12}
\]

\[
\frac{1}{\lambda} \leq \sigma^2(y) \leq \lambda, \quad |a(y)| \leq \lambda. \tag{4.13}
\]

Then for any given \( \kappa > 0 \) the sequence \( X_n \) satisfies LDP in \( (\mathbb{C}, \rho_\kappa) \) with good rate function:

\[
I(f) = \begin{cases} 
\frac{1}{2} \int_0^\infty \frac{(f'(t) - a(f(t)))^2}{\sigma^2(f(t))} dt, & \text{if } f(0) = x_0, \ f \text{ is absolutely continuous}, \\
\infty, & \text{otherwise}.
\end{cases}
\]

Proof. Existence of strong solution in (4.9) is assured by Theorem 1 in [15]. Condition I follows from [16] and that it is possible to extend \( f \) for \( t \geq T \) by solution of differential equation:

\[
g'(t) = a(g(t)), \quad g(T) = f(T), \ t \geq T.
\]

Condition II is verified similarly to that in Theorem 4.1. From (4.12), (4.13) and Lemma 4.2 follows condition III. The Theorem is now proved.

§5. Large Deviations for CEV model on half-line

Consider \( X_n(t), t \geq 0 \), that solves the following SDE (also known as the Constant Elasticity of Variance model, CEV).

\[
X_n(t) = 1 + \int_0^t \mu X_n(s)ds + \frac{1}{n^{1-\gamma}} \int_0^t \sigma(X_n(s))\gamma dw(s),
\]

where \( \mu \) and \( \sigma \) are arbitrary constants, \( \gamma \in [1/2, 1), n > 0 \). Existence and uniqueness of strong solution is given e.g. in [17] and [18].

Lemma 5.1. For any \( N < \infty \) and \( \varepsilon > 0 \) there exists \( T = T_{N, \varepsilon} < \infty \) such that

\[
\lim_{n \to \infty} \frac{1}{n^{2(1-\gamma)}} \ln \mathbb{P} \left( \sup_{t \geq T} \frac{X_n(t)}{e^{\mu t} (1 + t)^{1-\gamma}} > \varepsilon \right) \leq -N. \tag{5.1}
\]

Proof. Denote \( \tau = \inf\{t \geq 0 : X_n(t) = 0\} \wedge \infty \). Using Itô’s formula for \( (X_n(t)e^{-\mu t})^{1-\gamma} \), \( t \in [0, \tau] \), we have

\[
(X_n(t)e^{-\mu t})^{1-\gamma} = 1 + \frac{1}{n^{1-\gamma}} \int_0^t \sigma(1 - \gamma)e^{-\mu(1-\gamma)s}dw(s) -
\]

\[
- \frac{1}{2n^{2(1-\gamma)}} \int_0^t \gamma(1 - \gamma) \frac{\sigma^2 e^{-\mu(1-\gamma)s}}{(X_n(s))^{1-\gamma}}ds, \quad t \in [0, \tau).
\]

Since \( X_n(t) \) is non-negative with probability 1

\[
(X_n(t)e^{-\mu t})^{1-\gamma} \leq 1 + \frac{1}{n^{1-\gamma}} \int_0^t \sigma(1 - \gamma)dw(s) \leq
\]

Then $\Theta$ and $| \sigma | \leq 1$. Since $X_n(t) \equiv 0$ for $t \geq \tau$, the above inequality trivially holds for $t \in [0, \infty)$.

Therefore for $T > 2/e^{1-\gamma}$

$$
P \left( \sup_{t \geq T} \frac{X_n(t)}{e^{\mu (1 + \gamma)^{-1}}} > \varepsilon \right) = P \left( \sup_{t \geq T} \frac{X_n(t)e^{-\mu t}}{1 + t} > e^{1-\gamma} \right) \leq \frac{1}{n^{1-\gamma}} \left\{ \int_0^t \frac{\sigma(1 - \gamma)}{e^{\mu (1 - \gamma) s}} dw(s) \right\} \leq \frac{1}{n^{1-\gamma}} \left\{ \int_0^t \frac{\sigma(1 - \gamma)}{e^{\mu (1 - \gamma) s}} dw(s) \right\} \leq \frac{e^{1-\gamma}}{2}
$$

Since $|\sigma(1 - \gamma)e^{-\mu (1 - \gamma)s}| \leq 1$, using Lemma 4.1 we obtain (5.1).

Denote by $C^+$ the set of functions in $f \in C$, such that $f(0) = 1$, $f(t) \geq 0$ for all $t \geq 0$.

Define a metric in $C^+$ by

$$\rho^{\mu,\gamma}(f, g) := \sup_{t \geq 0} \frac{|f(t) - g(t)|}{e^{\mu (1 + t)^{1-\gamma}}}
$$

It is obvious that the space $(C^+, \rho^{\mu,\gamma})$ is Polish.

**Theorem 5.1.** The process $X_n$ satisfies LDP in space $(C^+, \rho^{\mu,\gamma})$ with rate $\frac{1}{n^{\gamma(1-\gamma)}}$ and good rate function:

$$I(f) = \begin{cases} \frac{\Theta(f)}{2} \int_0^T \frac{(f'(t) - \mu f(t))^2}{(f(t))^{2-\gamma}} dt, & \text{if } f(0) = 1, \text{ } f \text{ is absolutely continuous}, \\ \infty, & \text{otherwise}, \end{cases}
$$

and $\Theta(f) = \inf \{ t : f(t) = 0 \}$

Proof. Condition I follows from [17] and that one can extend $f$ for $t \geq T$ by solution of differential equation:

$$g'(t) = \mu g(t), \quad g(T) = f(T), \quad t \geq T.
$$

We verify condition II. Let

$$\lim_{T \to \infty} I^T(f(T)) = \frac{r^2}{2} < \infty.
$$

Write $f(t)$ as $f(t) = e^{\mu}g(t)$, $g(0) = 1$. Then

$$I^T(f(T)) = \frac{1}{2} \int_0^T e^{2\mu(1-\gamma)t} \left( \frac{g'(t)}{g^\gamma(t)} \right)^2 dt.
$$

Denote

$$u^2(t) = e^{2\mu(1-\gamma)t} \left( \frac{g'(t)}{g^\gamma(t)} \right)^2, \quad \lim_{T \to \infty} \int_0^T u^2(t) dt = r^2. \tag{5.2}
$$

Then

$$\frac{g'(t)}{g^\gamma(t)} = e^{-\mu(1-\gamma)t} u(t), \quad g(0) = 1.
Solving, we have
\[ g^{1-\gamma}(t) = 1 + (1 - \gamma) \int_0^t e^{-\mu(1-\gamma)s} u(s) ds. \]

Using Cauchy-Bunyakovskii inequality and (5.2) we have
\[ |g^{1-\gamma}(t)| \leq 1 + (1 - \gamma) \left( \int_0^t e^{-2\mu(1-\gamma)s} ds \int_0^t u^2(s) ds \right)^{1/2} \leq 1 + r \left( \frac{1 - \gamma}{2\mu} \right)^{1/2}. \]

Hence
\[ |f(t)| \leq e^{\mu t} \left( 1 + r \left( \frac{1 - \gamma}{2\mu} \right)^{1/2} \right)^{1 - \gamma}. \]

It now follows that
\[ \lim_{T \to \infty} \sup_{T \geq t} \frac{|f(t)|}{e^{\mu t} (1 + t)^{1 - \gamma}} = 0. \]

Condition III follows from Lemma 5.1. Theorem 5.1 is now proved.

References

[1] Puhalskii A.A. Large deviations and idempotent probability. Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 119. Chapman and Hall/CRC, Boca Raton, FL, 2001.

[2] Dembo A. Zeitouni O. Large Deviations Techniques and Applications. Springer, 2nd edition, 1998.

[3] Feng J. Kurtz T. Large deviations for stochastic processes. American Mathematical Society, 2006.

[4] Puhalskii A.A. Large deviations for stochastic processes LMS/EPSRC Short Course: Stochastic Stability, Large Deviations and Coupling Methods, Heriot-Watt University, Edinburgh, 4-6 September 2006.

[5] Dobrushin R.L. Pecherskij E.A., Large deviations for random processes with independent increments on infinite intervals // Probl. Inf. Transm. 34, No.4, p. 354-382, 1998.

[6] Borovkov A.A. Asymptotic Analysis of Random Walks. Quikly Decreasing Jumps, Fizmathlit, Moscow, 2013. (in Russian)

[7] Borovkov A.A. Mogulski A.A. On large deviation principles in metric spaces // Sibirsk. Mat. Zh. v. 51, p. 1251-1269, 2010.

[8] Kolmogorov A.N. Fomin S.V. Elements of the theory of functions and functional analysis, Nauka, Moscow, 1976, 543 p. (in Russian)

[9] Borovkov A.A. Mogulski A.A. Large deviations and testing statistical hypothesis, Nauka, Novosibirsk, 1992, 222 p. (in Russian)

[10] Borovkov A.A. Boundary problems for random walks and large deviations in function spaces // Theor. Probability Appl., v. 12, No. 4, p. 575-595, 1967.
[11] Mogulski A.A. Large Deviations for Trajectories of Multi-Dimensional Random Walks // Theory Probab. Appl., v. 21, No. 2, p. 300–315, 1976.

[12] Borovkov A.A. Mogulski A.A. Moderately large deviation principles for trajectories of random walks and processes with independent increments // Theory Probab. Appl., v. 58, No. 4, p. 648–671, 2013. (in Russian)

[13] Borovkov A.A. Probability Theory, Moscow, Editorial URSS, 2009, 470 p. (in Russian)

[14] Kulik A.M., Soboleva D.D. Large deviations for one-dimensional SDE with discontinuous diffusion coefficient, Theory of Stochastic Processes 18(34) (2012), no. 1, 101–110.

[15] Gihman I.I. Skorokhod A.V. The stochastic differential equations, Kiev, Naukova dumka, 1968, 355 p. (in Russian)

[16] Freidlin M., Wentzell A. Random Perturbations of Dynamical Systems, Springer-Verlag, New York, 1998.

[17] Klebaner F. Liptser R. Asymptotic Analysis of Ruin in the Constant Elasticity of Variance Model // Theory Probab. Appl., v. 55, No 2, p. 291–297, 2011.

[18] F. Delbaen and H. Shirakawa. A note on option pricing for the constant elasticity of variance model. Asia-Pacific Financial Markets, 9(2):85–99, 2002.