Off - critical $W_\infty$ and Virasoro Algebras As Dynamical Symmetries Of The Integrable Models

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ABSTRACT

We find an infinite set of new noncommuting conserved charges in a specific class of perturbed CFT’s and present a criterion for their existence. They appear to be higher momenta of the already known commuting conserved currents. The algebra they close consists of two noncommuting $W_\infty$ algebras. We find various Virasoro subalgebras of the full symmetry algebra. It is shown on the examples of the perturbed Ising and Potts models that one of them plays an essential role in the computation of the correlation functions of the fields of the theory.

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1. Noncommuting conserved charges of IM’s

The explicit construction of all the integrals of motion (conserved charges) for a given dynamical problem allows to solve it exactly. An important tool in the realization of such a program is the algebra of the conserved charges and its representations. The power of this symmetries strategy to the problem of the exact solution of $2-D$ integrable models (IM’s) was demonstrated by the recent development of the $2-D$ conformally invariant theories [1]. The key point in the construction of the correlation functions of the fields of this class of IM’s is the appearance of the Virasoro algebra:

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m}$$

and its generalizations - $SU(N)$ - Kac-Moody, super - Virasoro, $W_N, W_\infty, \ldots$, as symmetries of the model. In all these cases the generators of the corresponding algebras are realized as higher momenta of the conserved currents. For example $\mathcal{L}_n$’s are all the momenta of the stress tensor $T_{\mu\nu} = (T, \bar{T})$:

$$\mathcal{L}_n = \oint dz z^{n+1} T(z), \quad \bar{\mathcal{L}}_n = \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}).$$

Since the conformal models belong to the big family of the relativistic integrable models one could wonder whether analogous infinite symmetries algebra approach works in the case of the nonconformal IM’s, say - sin-Gordon, massive fermions, affine Toda models etc. As it is known, the integrability of all these models is based on the existence of an infinite set of conserved charges (CC):

$$P_s = \oint T_{2s} dz - \oint \Theta_{2s-2} d\bar{z}, \quad \bar{P}_s = \oint \bar{T}_{2s} d\bar{z} - \oint \Theta_{2s-2} dz$$

they have. However, the algebra of the $P_s$ ($\bar{P}_s$) is abelian.
\[ [P_s, P_s'] = 0 = [P_s, \bar{P}_s] \] and this is considered to be an obstacle in using these symmetries for the calculation of the exact Green functions of the model. Therefore, the question one has to answer first is whether \( P_s \) exhaust all the conserved charges of these models, \textit{i.e.}

1) are there more (nontrivial) conservation laws?

2) if so, is it the algebra of the new conserved charges \textit{nonabelian}?

Addressing such a question we already have a hint that the answer should be positive (at least for a certain class of IM’s). It is the \textit{important observation} of Thacker and Itoyama \cite{2} that the XY - model in high temperature phase (\textit{i.e.} massive Dirac fermion) possesses on top of the abelian charges (1.2) a large set of noncommuting CC’s. They close two different off-critical Virasoro algebras. The best way of generalizing this result is to find the geometrical origin of these off-critical symmetries and to describe the corresponding class of models having this symmetry. Being far from the full understanding of these infinite symmetries of the IM’s we choose to follow the more practical way of explicit construction of the noncommuting CC’s. Our starting point is the fact that almost all relativistic (nonconformal) IM’s can be represented as an appropriate perturbation of certain conformal models \cite{3}, \textit{i.e.}:

\[
S_{IM} = S_{conf} + g \int \Phi_{\Delta}(z) \bar{\Phi}_{\Delta}(\bar{z}) d^2z. \tag{1.3}
\]

This suggests that the desired new charges (if they exist) should be realized as specific combinations of the higher momenta of the conserved tensors (\( T_{2s}, \bar{T}_{2s}, \Theta_{2s-2} \)):

\[
\mathcal{F}_{2s}^{(n)} = \sum_{k=1}^{s} \left\{ \alpha_k(g) z^{2k-1+n} \bar{z}^{\gamma(k,n)} T_{2k}(z, \bar{z}) + \bar{\alpha}_k(g) \bar{z}^{2k-1+n} z^{\bar{\gamma}(k,n)} \bar{T}_{2k}(z, \bar{z}) \right. \\
+ \beta_k(g) z^{d(k,n)} \bar{z}^{\bar{d}(k,n)} \Theta_{2k-2}(z, \bar{z}) \right\} \tag{1.4}
\]
such that
\[ \bar{\partial}f_{2s}^{(n)} = \partial g_{2s-2}^{(n)}, \]
where \( \alpha_k(g) = \alpha_k g^{\frac{s}{s-k}}, \quad \beta_k(g) = \beta_k g^{\frac{s}{s-k}}. \)

The crucial observation that simplifies the construction of the new conservation laws \( \mathcal{F}_{2s}^{(n)} \) is the following criterion of existing of such quantities: If the conservation laws of the spin - 2s (\( s > 1 \)) tensors \( T_{2s} \) are in the form:

\[
\begin{align*}
\bar{\partial}T_{2s} &= \partial^{2s-1} \Theta + g^p \sum_{l=1}^{s-1} A_l \partial^{2(s-l)-1} T_{2l} \\
\bar{\partial}T_{2s} &= \partial^{2s-1} \Theta + g^p \sum_{l=1}^{s-1} A_l \bar{\partial}^{2(s-l)-1} \bar{T}_{2l} \\
\bar{\partial}T &= \partial \Theta, \\
\bar{\partial}T &= \bar{\partial} \Theta,
\end{align*}
\tag{1.5}
\]

then there exist \( 4s - 3 \) new conserved currents \( \mathcal{F}_{2s}^{(n)} \) (\( n = 1, 2, \ldots, 4s - 3 \)) for each fixed \( s = 1, 2, \ldots \). In words the existence of new conserved charges:

\[
L^{(2s)}_{-n} = \int \mathcal{F}_{2s}^{(n)} dz - \int \mathcal{G}_{2s-2}^{(n)} d\bar{z} \\
\bar{L}^{(2s)}_{-n} = \int \bar{\mathcal{F}}_{2s}^{(n)} - \int \bar{\mathcal{G}}_{2s-2}^{(n)} d\bar{z} \tag{1.6}
\]

is hidden in the specific form of the traces \( \Theta_{2s-2} \) of the traditional conserved currents \( T_{2s} \):

\[
\Theta_{2s-2} = \partial^{2s-2} \Theta + g^p \sum_{l=1}^{s-1} A_l \partial^{2(s-l)-2} T_{2l}.
\]

Instead of general proof we shall show here how our criterion is working on the simplest example. Suppose we have a model such that:

\[
\bar{\partial}T = \partial \Theta, \\
\bar{\partial}T = \bar{\partial} \Theta
\]

and

\[
\bar{\partial}T_4 = \partial^3 \Theta + \beta g^2 \partial T, \\
\bar{\partial}T = \bar{\partial}^3 \Theta + \beta g^2 \bar{\partial} \Theta. \tag{1.7}
\]
We are going to demonstrate that on top of the usual commuting charges:

\[ L_{-1} = \int T dz - \int \Theta d\bar{z}, \quad \bar{L}_{-1} = \int \bar{T} d\bar{z} - \int \Theta dz \]

\[ L_{-3}^{(4)} = \int T_4 dz - \int \Theta_2 d\bar{z}, \quad \bar{L}_{-3}^{(4)} = \int \bar{T}_4 d\bar{z} - \int \Theta_2 dz \]

and the Lorentz rotation generator

\[ L_0 = \int (zT + \bar{z}\Theta)dz - \int (\bar{z}\bar{T} + z\Theta)d\bar{z} \]

five new conserved charges appear. Let us consider the “nonconservation laws” of the following three quantities: \( \bar{z}^2\bar{T}, z\bar{z}T \) and \( z^2T_4 \):

\[
\partial(\bar{z}^2\bar{T}) = \bar{\partial}(\bar{z}^2\Theta) - 2\bar{z}\Theta \\
\bar{\partial}(z\bar{z}T) = zT + \partial(z\bar{z}\Theta) - \bar{z}\Theta \\
\bar{\partial}(z^2T_4) = \partial^3(z^2\Theta) - 6\partial(z\partial\Theta) + \beta g^2\partial(z^2T) - 2\beta g^2zT
\]

A simple algebra leads to the following new conservation law:

\[ \bar{\partial}F^{(1)}_4 = \partial\tilde{\Theta}^{(1)}_2, \quad (1.8) \]

where

\[ F_4^{(1)} = z^2T_4 + 2\beta g^2z\bar{z}T + \beta g^2z^2\Theta \]

\[ \tilde{\Theta}^{(1)}_2 = \beta g^2(z^2T + \bar{z}^2\bar{T}) + 2\beta g^2z\bar{z}\Theta - 6z\partial\Theta + \partial^2(z^2\Theta) \quad (1.9) \]

In the same way one can get the “first momenta” of \( T_4 \) conservation law:

\[ L_{-2}^{(4)} = \int (zT_4 + \beta g^2zT)dz - \int (\beta g^2zT + g^2z\Theta)d\bar{z}. \quad (1.10) \]

The remaining three conserved charges are the “complex conjugated” of \( L_{-1}^{(4)} \) and \( L_{-2}^{(4)} \) and the “third momenta” of \( T_4 \) conservation law \( L_0^{(4)} \) including \( z^3T_4, \bar{z}^3T_4, z^2\bar{z}T \) etc.

In principle one can repeat this procedure for all the \( T_{2s} \)'s and to construct an infinite set of new conserved charges: \( L_{-n}^{(2s)}, \bar{L}_{-n}^{(2s)}, n = 0, 1, \ldots, 2s - 1 \).
2. Symmetries of the off-critical Ising model

Turning back to our problem of constructing noncommuting conserved charges for the IM’s given by (1.3) we have to check whether exist models which satisfy our criterion, i.e. their standard $T_{2s}$ - conservation laws to be in the form (1.5).

The simplest case is the set of models obtained by $\Phi_{\Delta_{1,3}} \bar{\Phi}_{\Delta_{1,3}}$ perturbations of the conformal minimal models $c_p = 1 - \frac{6}{(p+1)(p+2)}$, $\Delta_{1,3}(p) = \frac{p}{p+2}$ (see [3]). They have all the $T_{2s}$, $\bar{T}_{2s}$, $s = 1, 2, \ldots$ conserved. The first model ($p = 2$) of this set is the thermal perturbation of the Ising model which in the continuum limit coincides with the theory of free massive Majorana fermion ($\psi$, $\bar{\psi}$):

$$\bar{\partial}\psi = m\bar{\psi}, \quad \partial\bar{\psi} = -m\psi \quad (2.1)$$

$$T = \frac{1}{2}\psi\partial\psi, \quad \bar{T} = \frac{1}{2}\bar{\psi}\bar{\partial}\bar{\psi}, \quad \Theta = m\bar{\psi}\psi. \quad (2.2)$$

To find the explicit form of $\Theta_{2s-2}$ in this case is better to use the equation of motion (2.1) instead of the conformal perturbative technics. The corresponding conservation tensors of spin $2s$ can be taken in the form $T_{2s} = \psi\partial^{2s-1}\psi$, $s = 2, 3, \ldots$. Simple computations based on the eq. (2.1) leads to the following desired form of the $T_{2s}$ - conservation laws:

$$\bar{\partial}T_4 = \partial^3\Theta + 2m^2\partial T$$
$$\bar{\partial}T_6 = \partial^5\Theta + m^2(\partial T_4 + 4\partial^3 T)$$
$$\bar{\partial}T_8 = 2\partial^7\Theta + m^2(\partial T_6 + 3\partial^3 T_4 + 2\partial^5 T) \quad (2.3)$$

eq$$ etc. The conclusion is that this model satisfies our criterion and therefore it has $4s - 3$ new conservation laws for each $s = 1, 2, \ldots$. The corresponding conserved charges $L_{-n}^{(2s)}$, $\bar{L}_{-n}^{(2s)}$, $0 \leq n \leq 2s - 1$, can be derived by the method we have
demonstrated above on the example of $L_{-1}^{(4)}$ (see eqs. (1.8), (1.9)). For $s = 2$ we have together with the “commuting charges”:

$$L_{-3}^{(4)} = \int T_4 dz - \int (\partial^2 \Theta + 2m^2 T) d\bar{z}, \quad \bar{L}_{-3}^{(4)} = \int \bar{T}_4 d\bar{z} - \int (\bar{\partial}^2 \bar{\Theta} + 2m^2 \bar{T}) dz$$

the new charges $L_{-2}^{(4)}, \bar{L}_{-1}^{(4)}$ given by eqs. (1.9) and (1.10) with $\beta = 2$ and $g = m$, their “conjugated” $\bar{L}_{-2}^{(4)}, \bar{L}_{-1}^{(4)}$ (obtained from (1.9) and (1.10) by the interchange $z \leftrightarrow \bar{z}, T \leftrightarrow \bar{T}, T_4 \leftrightarrow \bar{T}_4$) and the unique $L_{0}^{(4)}$:

$$L_{0}^{(4)} = \int \mathcal{F}_4^{(0)} dz - \int \Theta_4^{(0)} d\bar{z}$$

$$\mathcal{F}_4^{(0)} = z^3 T_4 + 6m^2 z^2 \bar{z} T + 2m^2 z^3 \bar{T} + 3m^2 \bar{z} z \Theta + \bar{\partial}^2 \bar{\Theta} - 9z^2 \partial \Theta$$

$$\Theta_4^{(0)} = z^3 \bar{T} + \ldots \equiv \bar{\mathcal{F}}_4^{(0)}.$$

The CC’s arising from the $T_6, \bar{T}_6$ - conservation laws (2.3) are given by:

$$L_{-5}^{(6)} = \int T_6 dz - \int [2\partial^6 \Theta + m^2 (T_6 + 3\partial^2 T_4 + 2\partial^4 T)] d\bar{z}$$

$$L_{-4}^{(6)} = \int (z T_6 + m^2 \bar{z} T_4) dz - \int (m^2 z T_4 + 2m T^4 - 4m^2 \partial T + m^2 \bar{z} \Theta - \partial^3 \Theta - z \partial^4 \Theta) d\bar{z}$$

etc. Following this method one could keep constructing $L_{-n}^{(2s)}$ for higher spins. However, such a form of the conserved charges is inconvenient both for deriving (or guessing) the general form of $L_{-n}^{(2s)}$ and for computing their algebra as well. For these purposes it is better to have $L_{-n}^{(2s)}$’s as differential operators acting on $\psi$ and $\bar{\psi}$. One can do this in few steps. Starting from eqs. (1.9), (1.10), (2.4) etc., we first exclude the time derivatives $\partial_t \psi$ and then take $t = 0$ ($z = t + x, \bar{z} = t - x, \partial = \partial_x$):

$$L_0 = \frac{1}{2} \int x [\bar{\psi} \partial \psi - \bar{\psi} \partial \bar{\psi} + 2m \bar{\psi} \psi] dx,$$

$$L_{-1} = -\frac{1}{2} \int (\bar{\psi} \partial \psi + m \bar{\psi} \psi) dx$$

$$L_{-1}^{(4)} = \int \left\{ x^2 [\psi \partial^3 \psi - m \bar{\psi} \partial \psi + m^2 \bar{\psi} \psi - m^3 \bar{\psi} \psi] + mx \bar{\psi} \partial \psi + 2m \bar{\psi} \psi \right\} dx$$

(2.5)

etc. The next step is to derive the momentum space form of $L_{-n}^{(2s)}$ by substituting...
the standard creation and annihilation operators $a^\pm(p)$ decomposition of $\psi$ and $\bar{\psi}$:

\[
L_0 = \frac{1}{2} \int \frac{dp}{2\pi} p_0 \left[ (\partial a^+)a^- + (\partial a^-)a^+ \right], \quad L^{(4)}_{-3} = \frac{1}{4} \int \frac{dp}{2\pi} (p_0 - p)^3 a^+a^- \\
L^{(4)}_{-2} = -\frac{1}{4} \int \frac{dp}{2\pi} p_0 (p_0 - p)^2 (a^+\partial a^- + a^-\partial a^+) \\
L^{(4)}_{-1} = \frac{1}{2} \int \frac{dp}{2\pi} \left[ p_0^2 (p_0 - p) (a^+\partial^2 a^- - a^-\partial^2 a^+) + m^2 \frac{5p_0 - p}{4p_0^2} a^+a^- \right]
\] (2.6)

etc. We are prepared now to compute the desired differential form of $L^{(2s)}_{-n}$. Using the canonical anticommutation relations $\{a^+(p),a^-(q)\} = 2\pi\delta(p - q)$ and eqs. (2.6) , we get:

\[
[L_{-1},\psi] = -i\partial\psi, \quad [L_0,\psi] = -i(\bar{z}\bar{\partial} - z\partial - \frac{1}{2})\psi, \quad [L^{(4)}_{-3},\psi] = -i\partial^3\psi \\
[L^{(4)}_{-2},\psi] = -\frac{i}{2} \left[ (\bar{z}\bar{\partial} - z\partial + 3) \right] \partial^2\psi \\
[L^{(4)}_{-1},\psi] = -\frac{i}{2} \left[ (\bar{z}\bar{\partial} - z\partial)(\bar{z}\bar{\partial} - z\partial + 1) + (\bar{z}\bar{\partial} - z\partial - 3)(\bar{z}\bar{\partial} - z\partial - 2) \right] \partial\psi \\
[L^{(6)}_{-2},\psi] = -\frac{i}{2} \left[ (\bar{z}\bar{\partial} - z\partial) + (\bar{z}\bar{\partial} - z\partial - 5) \right] \partial^2\psi.
\] (2.7)

Similar calculation for $\bar{L}^{(4)}_{-k}$ leads to:

\[
[\bar{L}^{(4)}_{-1},\psi] = -\frac{i}{2} \left[ (\bar{z}\bar{\partial} - z\partial + 1)(\bar{z}\bar{\partial} - z\partial + 2) + (\bar{z}\bar{\partial} - z\partial - 2)(\bar{z}\bar{\partial} - z\partial - 1) \right] \bar{\partial}\psi
\]

This form of the $L^{(2s)}_{-k}$’s allows us to make a conjecture about the general form of all the $L^{(2s)}_{-k}$ ( $0 \leq k \leq 2s - 1$ ):

\[
[L^{(2s)}_{-k},\psi] = -\frac{i}{2} \left[ (\bar{z}\bar{\partial} - z\partial)_{2s-1-k} + (\bar{z}\bar{\partial} - z\partial - 2s + 1)_{2s-1-k} \right] \partial^k\psi,
\] (2.8)

where $(A)_p = A(A + 1)\ldots(A + p - 1)$. In order to prove our conjecture we have to be able to derive from (2.8) the integral form of $L^{(2s)}_{-k}$ similar to that of the eqs. (1.9),(1.10),(2.4) and to show that the integrands are conserved quantities.
Fortunately it exists an indirect way to prove that (2.8) are conserved charges. It is related to the answer of the following *important question* we left unanswered up to now: *are the conserved charges* (1.9), (1.10), (2.4) etc. we have constructed, *generators of symmetries of the action* (1.3)? Let us first check whether the simplest nontrivial charge $L_{-2}^{(4)}$ leaves invariant the action:

$$S = \int \left( -\frac{1}{2} \bar{\psi} \partial \psi + \frac{1}{2} \bar{\psi} \partial \bar{\psi} + m \bar{\psi} \psi \right) d^2z \equiv \int \mathcal{L} d^2z. \quad (2.9)$$

By using (2.7) and

$$[L_{-2}^{(4)}, \bar{\psi}] = (\bar{z} \partial - z \partial - \frac{1}{2}) \partial^2 \bar{\psi}$$

one can verify easily that

$$[L_{-2}^{(4)}, \mathcal{L}] = \partial A + \bar{\partial} B.$$

Therefore $L_{-2}^{(4)}$ is a generator of a specific new symmetry of (2.9). The same is true for $\bar{L}_{-2}^{(4)}$. Together with the Lorentz rotation $L_0$ they close an $SL(2, R)$ - algebra. One can repeat this calculation with $L_{-1}^{(4)}$, $\bar{L}_{-1}^{(4)}$, $L_0^{(4)}$, $L_{-2}^{(4)}$ etc. and the result is always that these charges commute with the action (2.9). As it becomes clear from this discussion the proof that $L_{-k}^{(2s)}$ given by eq. (2.8) are conserved charges is equivalent to the following statement: $[L_{-k}^{(2s)}, S] = 0$. To prove it we have to make one more conjecture, namely:

$$[L_{-k}^{(2s)}, \bar{\psi}] = \frac{-i}{2} \left[ (\bar{z} \partial - z \partial + 1)_{2s-1-k} + (\bar{z} \partial - z \partial - 2s + 2)_{2s-1-k} \right] \partial^k \bar{\psi} \quad (2.10)$$

The remaining part of the proof is a straightforward but tedious higher derivative calculus.

To make complete our study of the conserved charges of the off-critical Ising model we have to find the general form of the “conjugated charges” $\bar{L}_{-k}^{(2s)}$. By
arguments similar to the ones presented above we arrive to the following result:

\[ \tilde{L}_{-k}^{(2s)} = \frac{1}{2} \left[ (\bar{z}\bar{\partial} - z\partial + \bar{\alpha} - 2s + k + 2)_{2s-1-k} + (\bar{z}\bar{\partial} - z\partial + \bar{\alpha} + k + 1)_{2s-1-k} \right] \tilde{\partial}^k, \]  

(2.11)

where \( \bar{\alpha} = -1 \) for \( \psi \) and \( \bar{\alpha} = 0 \) for \( \bar{\psi} \).

Our claim is that (2.10) and (2.11) do exhaust all the local symmetries (i.e. local conserved charges) of the action (2.9). An important observation concerning the origin of these symmetries is in order: denoting the Poincare algebra generators by

\[ L_{-1} = \partial, \quad \bar{L}_{-1} = \bar{\partial}, \quad L_0 = \bar{z}\bar{\partial} - z\partial - \frac{1}{2}, \]

it is obvious that the conserved charges (2.10) and (2.11)

\[ L_{-k}^{(2s)} = \frac{1}{2} \left[ (L_0 + \frac{1}{2})_{2s-1-k} + (L_0 - 2s + \frac{3}{2})_{2s-1-k} \right] L_{-1}^k \]
\[ \bar{L}_{-k}^{(2s)} = \frac{1}{2} \left[ (L_0 - 2s + k + \frac{3}{2})_{2s-1-k} + (L_0 + \frac{1}{2} + k)_{2s-1-k} \right] \bar{L}_{-1}^k \]

(0 \leq k \leq 2s - 1), span a specific subalgebra in the enveloping of the Poincare algebra:

\[ \mathcal{EP} = \left\{ L_0^l L_{-1}^m, L_0^l \bar{L}_{-1}^m | L_{-1} L_{-1} \sim m^2 I \right\}. \]

The condition that single out this subalgebra is the invariance of the action (2.9). The open question, however, is about the algebraic meaning of such a condition. The appearance of the \( \mathcal{EP} \)-algebra is not a specific property of the massive Majorana fermion. Studying the conformal limit of \( L_{-k}^{(2s)} \) we have realized that the “conformal” \( W_\infty \) algebra has a specific subalgebra \( \mathcal{PW}_\infty(V) \) spanned by:

\[ L_{-k}^{(2s)} = \frac{1}{2} \left[ (\bar{L}_0 - \frac{1}{2})_{2s-1-k} + (\bar{L}_0 + 2s - \frac{3}{2})_{2s-1-k} \right] L_{-1}^k \]

\[ 0 \leq k \leq 2s - 1, \quad \bar{L}_0 = z\partial + \frac{1}{2} \]

which is a subalgebra of \( \mathcal{EP} \). All the other generators \( L_{-k}^{(2s)} \) (\( k > 2s - 1 \) and \( k < 0 \)) of \( W_\infty \), however, do not belong to \( \mathcal{EP} \). Considering \( W_\infty \) as the largest symmetry
of the corresponding conformal model one can speculate that the only symmetries belonging to \( \mathcal{EP} \) survive after the perturbation.

3. \( W_\infty(V) \) - algebra

Remember that our original motivation was to construct noncommuting conserved charges for certain IM’s. Having at hand the explicit form of (2.10), (2.11) of \( L^{(2s)}_{-k} \) and \( \bar{L}^{(2s)}_{-k} \) we are prepared to compute their algebra. As we have already mentioned \( L^{(4)}_{-2}, L^{(4)}_{-1} \) and \( L_0 \) close an \( SL(2, R) \) algebra. Two more \( SL(2, R) \) algebras are spanned by \( \bar{L}^{(4)}_{-1}, L^{(4)}_{-1}, L_0 \) and \( L^{(4)}_{-1}, L_0 \). Passing to the general case let us first try to find the structure of the “left” algebra, i.e.:

\[
\left[ L^{(2s_1)}_{-k_1}, L^{(2s_2)}_{-k_2} \right] = \sum_{r=1}^{s_1+s_2-1} g_{2r}^{s_1 s_2} (k_1, k_2) L^{2(s_1+s_2-r)}_{-k_1-k_2}.
\]

(3.1)

The simplest way to prove (3.1) and to compute the structure constants \( g_{2r}^{s_1 s_2} (k_1, k_2) \) is based on the following “conformal” decomposition of the generators \( L^{(2s)}_{-k} \) in terms of the conformal generators \( L^{(2s)}_{-k} \):

\[
L^{(2s)}_{-k} = \sum_{l=0}^{2s-1-k} \binom{2s-1-k}{l} (\bar{z} \bar{\partial}^2 + \alpha \bar{\partial})^l L^{(2s)}_{-k-l} (-m^2)^{-l}
\]

(3.2)

\[\alpha = 0 \text{ for } \psi \text{ and } \alpha = 1 \text{ for } \bar{\psi}.\]

The fact that the operators \( S_l = (\bar{z} \bar{\partial}^2 + \alpha \bar{\partial})^l = (\bar{z} \bar{\partial} + \alpha) \bar{\partial}^l \) are commuting, i.e. \([S_l, S_l] = 0\) reduces the computation of the structure constants \( g_{2r}^{s_1 s_2} (k_1, k_2) \) to the conformal ones \( C_{2r}^{s_1 s_2} (k_1, k_2) (k_i \leq 2s_i - 1) \):

\[
\left[ L^{(2s_1)}_{-k_1}, L^{(2s_2)}_{-k_2} \right] = \sum_{r=1}^{s_1+s_2-1} C_{2r}^{s_1 s_2} (-k_1, -k_2) L^{2(s_1+s_2-r)}_{-k_1-k_2}.
\]

(3.3)

Note that (3.3) is a \( \mathcal{PW}_\infty \) subalgebra of the \( W_\infty \) - algebra [4] written in a specific
basis of nonquasiprimary $T_{2s}$ we are using. The remaining part of the proof that

\[ g_{2r}^{s_1s_2}(k_1, k_2) = C_{2r}^{s_1s_2}(-k_1, -k_2) \]

\[ 0 \leq k_i \leq 2s_i - 1 \]

is based on the following property of the conformal structure constants [5]:

\[
C_{2r}^{s_1s_2}(-k_1, -k_2) \left( \begin{array}{c} 2(s_1 + s_2 - r) - k_1 - k_2 - n - 1 \\ n \end{array} \right) = \\
\sum_l \left( \begin{array}{c} 2s_1 - k_1 - 1 \\ l \end{array} \right) \left( \begin{array}{c} 2s_2 - k_2 - 1 \\ n - l \end{array} \right) C_{2r}^{s_1s_2}(-k_1 - l, -k_2 - n + l).
\]

The identical statement holds for the algebra of $\bar{L}_{-k}$’s as well. The conclusion is that the algebra we are looking for has as subalgebras two incomplete 

$(0 \leq k \leq 2s - 1)$ $W_{\infty}$ algebras which do not commute between themselves.

The general structure of the remaining “left - right” commutators:

\[
\left[ \bar{L}_{-k}^{(2s)}, L_{-l}^{(2p)} \right] = \sum_{r=0}^{s+p-k-2} \bar{g}_{2r}^{sp}(k, l)(m^2)^k L_{k-l}^{2(s+p-k-1-r)}
\]

(if $k < l$) is a consequence of the explicit form (2.8) and (2.11) of the generators. In order to calculate $\bar{g}_{2r}^{sp}(k, l)$ we first commute $L_k^l$ and $\bar{L}_l^k$ to the right and then expand the both sides of (3.4) in powers of $X = L_0 + \frac{1}{2}$. In doing this we have to know the coefficients $B_{m,a}^{N}$ in the power expansion of $(X + a)^{N+1}$:

\[
(X + a)^{N+1} = \sum_{m=0}^{N+1} B_{m,a}^{N} X^m.
\]

A simple combinatorial analysis [5] leads to the following form of $B_{m,a}^{N}$:

\[
B_{m,a}^{N} = \frac{1}{3.2^{m+1}} (N + 2 - m) A_{m,a}^{N} (N + 2a).
\]

The $A_{m,a}^{N}$ are certain polynomials of $w = N + 2a$ of degree $m$ satisfying the following
recursion relations [5]:

\[(N+2-m)_m A^{N,a}_m (N+2a) = 2 \sum_{p=0}^{N-m+1} (a+p)(N-p+2-m)_{m-1} A^{N-p-1,a+p+1}_{m-1} (N+2a+p+1)\]

and the differential equation:

\[\frac{d}{dw} A^{N,a}_m (w) = A^{N,a}_{m-1} (w).\]

The general solution is a specific linear combination of the Bernoulli polynomials. The first few are of the form:

\[A^N_a (w) = 6w, \quad A^N_2 (w) = 3w^2 - (N + 2)\]
\[A^N_3 (w) = w^3 - (N + 2)w, \quad A^N_4 (w) = \frac{w^4}{4} - \frac{N + 2}{2}w^2 + \frac{N^2}{N + 1}.\]

The l.h.s. of (3.4) contains 8 terms of the form:

\[(X + a)_{N+1} (X + a)_{M+1} = \sum_{k=0}^{N+M+2} Y^{(N+M+2-k)}_{R} (aN|bM) X^k,\]

where

\[Y^{(m)}_L (aN|bM) = \sum B^{a,N}_k B^{b,M}_{m-k}.\]

Denote by \(Y^{(m)}_L\) the sum of the contributions of all the 8 terms in the l.h.s. The same expansion in the r.h.s. gives:

\[(X + c)_{M+N-2r+1} + (X + d)_{M+N-2r+1} = \sum_m Y^{(m),r}_{R} X^m\]

\[Y^{(m),r}_{R} = B^{c,N-2r+1}_m + B^{d,N-2r+1}_m.\]

Then one can easily derive the following recursive relations for the structure con-
\[ Y_L^{(2n-1)} = 2 \sum_{r=0}^{n-1} \bar{g}_{2r}^{sp}(k|l) Y_R^{(2n-2-2r)} \]  

(3.6)

The first coefficient is quite simple:

\[ \bar{g}_0^{sp}(k|l) = -k(2p - l - 1) - l(2s - k - 1), \]

but the next two \( \bar{g}_2 \) and \( \bar{g}_4 \) are complicated enough and their explicit form does not help us in guessing the form of \( \bar{g}_{2r} \). We shall give here one example of the mixed left-right commutators:

\[
\left[ L^{(2p)}_{-2p+2}, \bar{L}^{(2s)}_{-2s+3} \right] = (m^2)^2(p-1) \left\{ -2(2p + s - \frac{7}{2})L^{2(s-p+1)}_{-2(s-p+1)+3} + 4(p-1)\left[ (s - \frac{1}{2})(-s + \frac{3}{2}) + (s - p + 1)(s - p) \right] \right\}
\]

for \( s > p \).

The remarkable observation by Thacker and Itoyama [2] that the off-critical XY-model has as dynamical symmetries two different Virasoro algebras is a hint to look for Virasoro algebras generated by specific combinations of \( L^{(2s)}_{-k} \) and \( \bar{L}^{(2s)}_{-k} \). We have even an indication where to look for. It is the fact that \( \{ L^{(4)}_{-1}, L_0, \frac{1}{m^2} L^{(4)}_{-2} \} \), \( \{ L^{(4)}_{-1}, L_0, L^{(4)}_{-1} \} \) and \( \{ \bar{L}^{(4)}_{-1}, L_0, \frac{1}{m^2} L_{-1} \} \) are generators of three different \( SL(2, R) \) algebras. To begin with the second one. We have to find the analog of the \( L_2 \)-Virasoro generator (see eq. (1.1) ). An appropriate candidate for this role is \( L_2 = L^{(6)}_{-2} + \alpha L^{(4)}_{-2} \). Simple computations lead us to the conclusion that \( L_{-1} = 1/m^2 L_{-1} \), \( \bar{L}_0 = L_0 \equiv z \bar{\partial} - z \partial - 1/2 \) and

\[
\begin{align*}
L_1 \psi &= L^{(4)}_{-1} - 9 \frac{1}{4} L_{-1} = (z \bar{\partial} - z \partial - \frac{1}{2})(z \bar{\partial} - z \partial - \frac{3}{2}) \partial \psi \\
L_2 \psi &= L^{(6)}_{-2} - 3(\frac{5}{2})^2 L^{(4)}_{-2} = (z \bar{\partial} - z \partial - \frac{1}{2})(z \bar{\partial} - z \partial - \frac{3}{2})(z \bar{\partial} - z \partial - \frac{5}{2}) \partial^2 \psi.
\end{align*}
\]

(3.7)

generate the incomplete Virasoro algebra \( V = \{ L_n, n \geq -1 \} \). The form (3.7) of
\( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) is very suggestive. One can easily verify that \( \mathcal{L}_n \) given by:

\[
\mathcal{L}_n = [\bar{z} \partial - z \partial - \frac{1}{2} j_{n+1} \partial^n, \quad n \geq -1
\]

\[\quad [A]_k = A(A - 1) \ldots (A - k + 1), \quad [A]_0 = 1\]

(note that \( \partial^{-1} \psi = -1/m^2 \bar{\partial} \psi \)) satisfy (1.1). One more incomplete Virasoro algebra \( \bar{\mathcal{V}} \) is generated by:

\[
\bar{\mathcal{L}}_n = [\bar{z} \partial - z \partial + \frac{1}{2} j_{n+1} \partial^n, \quad n \geq -1.
\]

The third Virasoro algebra \( V_c \) spanned by:

\[
\frac{(-m^2)^{1-s}}{2} \mathcal{L}_{2s+2}^{(2s)} = \frac{(-m^2)^{1-s}}{2} (\bar{z} \partial - z \partial - \frac{2s - 1}{2}) \partial^{2s-2} \equiv \frac{(-m^2)^{1-s}}{2} (L_0 - s + 1) \mathcal{L}_{-1}^{2s-2}
\]

\[
\frac{(-m^2)^{1-s}}{2} \bar{\mathcal{L}}_{2s+2}^{(2s)} = \frac{(-m^2)^{1-s}}{2} (\bar{z} \partial - z \partial + \frac{2s - 3}{2}) \partial^{2s-2} \equiv \frac{(-m^2)^{1-s}}{2} (L_0 + s - 1) \bar{\mathcal{L}}_{-1}^{2s-2}
\]

\( s = 1, 2, \ldots \), plays in our opinion the major role for the exact integrability of the \( S_{13}(p) \)- class of IM's. Using once more the formal identity \( \bar{L}_{-1} = -m^2(L_{-1})^{-1} \) we can rewrite the \( V_c \) - generators (3.10) in an unique formula:

\[
V_n = \frac{1}{2} (-m^2)^n (L_0 - n) L_{-1}^{2n}, \quad -\infty \leq n \leq \infty.
\]

As in the case of the massive Dirac fermion [2] one is expecting that \( V_c \) has nonzero central charge. One could see it calculating the commutator

\[
[\mathcal{L}_{-4}^{(6)}, \bar{\mathcal{L}}_{-4}^{(6)}]
\]

and taking care about the right normal ordering of the \( a^\pm \) operators:

\[
\mathcal{L}_{-4}^{(6)} = \int \frac{dq}{2\pi} q_0(q_0 - q)^4 (a^+ \partial a^- + a^- \partial a^+)
\]

\[
\bar{\mathcal{L}}_{-4}^{(6)} = \int \frac{dq}{2\pi} q_0(q_0 + q)^4 (a^+ \partial a^- + a^- \partial a^+)
\]

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The result is:

\[
\left[ L^{(6)}_{-4}, \bar{L}^{(6)}_{-4} \right] = -8m^8 L_0 + m^8. \tag{3.11}
\]

Comparing (3.10) and (3.11) with (1.1) one concludes that:

\[ c = \frac{1}{2}, \]

i.e. the massive Majorana fermion has the same central charge as the massless one. This is in agreement with the Thacker and Itoyama’s result \( c = 1 \) for the Dirac fermions.

This fact allows us to apply the all well-established technology of the highest weight representations, null-vectors etc. to the case of the IM’s.

4. Off-critical Ward identities

An important consequence of the fact that \( L^{(2s)}_{-k} \) and \( \bar{L}^{(2s)}_{-k} \) are generators of the symmetries of the action (2.9) is the following infinite set of Ward identities for the \( n \)-point Green functions of \( \psi(z, \bar{z}) \) and \( \bar{\psi}(z, \bar{z}) \):

\[
\begin{align*}
\left\langle 0 \left| L^{(2s)}_{-k} \prod_{i=1}^{M} \psi(z_i, \bar{z}_i) \prod_{j=1}^{N} \bar{\psi}(z_j, \bar{z}_j) \right| 0 \right\rangle &= 0 \\
\left\langle 0 \left| \prod_{i=1}^{M} \psi(z_i, \bar{z}_i) \prod_{j=1}^{N} \bar{\psi}(z_j, \bar{z}_j) \bar{L}^{(2s)}_{-k} \right| 0 \right\rangle &= 0. \tag{4.1}
\end{align*}
\]

The condition for the invariance of the vacuum: \( L^{(2s)}_{-k} |0\rangle = 0 = (0| \bar{L}^{(2s)}_{-k} \) together with eqs. (2.8), (2.10) and (2.11) lead to the following system of differential equations for \( G_{MN}(z_l, \bar{z}_l) \):

\[
\begin{align*}
\left\{ \sum_{i=1}^{M} \left[ (\bar{z}_i \partial_i - z_i \partial_i)_{2s-1-k} + (\bar{z}_i \partial_i - z_i \partial_i - 2s + 1)_{2s-1-k} \right] \partial_i^k + \\
+ \sum_{j=1}^{N} \left[ (\bar{z}_j \partial_j - z_j \partial_j + 1)_{2s-1-k} + (\bar{z}_j \partial_j - z_j \partial_j - 2s + 2)_{2s-1-k} \right] \partial_j^k \right\} G_{MN}(z_l, \bar{z}_l) &= 0 \tag{4.2}
\end{align*}
\]

A similar set of equations can be obtained from the condition of \( \bar{L}^{(2s)}_{-k} \) - symmetry of \( G_{MN} \). Restricting ourselves to the case of 2-point functions \( (M + N = 2) \) we
are going to demonstrate that the Poincare invariance \((L_{-1}, L_0, \bar{L}_{-1})\) and the new \(SL(2, R)\) symmetries \((L_{-2}^{(4)}, \bar{L}_{-2}^{(4)}, L_0)\) are sufficient to fix uniquely \(G_{20}, G_{02}\) and \(G_{11}\) - functions. The relativistic invariance requires:

\[
G_{20} = m\sqrt{\frac{z}{z}}g_{20}(y), \quad G_{02} = m\sqrt{\frac{z}{z}}g_{02}(y), \quad G_{11} = img_{11}(y), \quad y = m\sqrt{-4z}. 
\]

The condition of \(L_{-2}^{(4)}\) - invariance of \(G_{20}\) leads to the following third order differential equation:

\[
y^3g''_{20} + 2y^2g''_{20} - y(y^2 + 1)g'_{20} - (y^2 + 1)g_{20} = 0.
\]

It happens that one can solve (4.3) in terms of \(K_1(y)\) - Bessel function. This reflects the fact that (4.3) can be obtained as a consequence of the \(K_1\) - Bessel equation:

\[
y^2g''_{20} + yg'_{20} - (y^2 + 1)g_{20} = 0
\]

and a specific third order equation:

\[
y^3g'''_{20} - y(y^2 + 3)g'_{20} + (y^2 + 3)g_{20} = 0.
\]

The eq. (4.4) follows from the standard recursive relations for \(K_{\pm 1}, K_0\) and \(K_2\) - Bessel functions. The \(L_{-2}^{(4)}\) - Ward identity imposes the eq. (4.4) only. Repeating the same analysis for \(G_{02}\) and \(G_{11}\) we find that \(g_{02}(y) = K_1(y)\) and that \(g_{11}\) satisfy the \(K_0\) - Bessel equation:

\[
yg''_{11} + g'_{11} - yg_{11} = 0,
\]

i.e. \(g_{11} = K_0(y)\).

To make complete our discussion of the off-critical Ising model we have to mention that as in the conformal case the WI's (4.1) , (4.2) are fixing uniquely the 2- and 3-point functions only. The calculation of, say, the 4- point function
(using only the symmetries of the model) requires more information about the representations of the algebra (3.1),(3.4) we are using. One could expect that the null-vector conditions for the off-critical Virasoro algebra spanned by $L_{-2s+2}$, $L_0$ and $\bar{L}_{-2s+2}$ will be sufficient to fix uniquely the corresponding 4-point functions ($M + N = 4$).

Continuing this line of arguments it is interesting to derive the $\tau$-function equation for the 2-point function $\langle \sigma \sigma \rangle$ of the “order parameter” field $\sigma(z, \bar{z})$ [6,7] from the corresponding WI’s for $\sigma$ and the null-vector conditions. One could find the infinitesimal $L^{(4)}_{-2}$ transformation of $\sigma$ using the explicit realization of $\sigma$ in terms of $\psi$ and $\bar{\psi}$ [7] or the conformal perturbation technics. The result is quite surprising. Following the analogy with the conformal WI’s one is expecting the commutator $[L^{(4)}_{-2}, \sigma(z, \bar{z})]$ to have a form similar to (2.7) for $\psi$ with some differences in the coefficients of the differential operators. However, the difference with the $[L^{(4)}_{-2}, \psi]$ is more drastic: the new field $\sigma_3(z, \bar{z})$ which is a specific descendant of $\sigma$ coming from the off-critical OPE $T(z, \bar{z})\sigma(w, \bar{w})$ contributes to the commutator [5]. This fact makes the corresponding WI’s much more complicated due to the presence of the new 2-point function $\langle \sigma_3 \sigma \rangle$ together with $\langle \sigma \sigma \rangle$.

It is important to note that the simple differential form (2.8), (2.10), (2.11) of the $L^{(2s)}_{-k}$ is a specific property of the free massive fields only (fermions or bosons). As we shall demonstrate in the case of the off-critical Potts model the action of the $L^{(2s)}_{-k}$ - symmetry (even in its simple ($SL(2, R)$ part) on the interacting fields always involve specific descendents of these fields.
5. Potts model

Our discussion up to now was concentrated on the off-critical symmetries of the Ising model. The goal was to demonstrate on the simplest example how one can construct new conserved charges, how to compute their algebra and how to use this algebra in the calculations of the correlation functions. It is almost evident that one can generalize all these constructions for arbitrary number of free massive fermions and bosons belonging to different representations of $O(n)$ (or $SU(n)$) [8].

The true question however is how one can realize all this program of describing the IM’s in terms of the representations of their off-critical symmetry algebra in the case of interacting fields. Turning back to the set of the IM’s known as $\Phi_{13}$-perturbations of the minimal models, we shall consider the models with even $p$ ($c_p = 1 - \frac{6}{(p+1)(p+2)}$) only. The reason is that in this case the $T_{2s}$ conservation laws has a specific term [3] we need in our constructions. The first such model ($p = 2$) is the thermal perturbation of the Ising model. The next is the so called “kinks perturbation” of the 3-state Potts model [9]:

$$S_{13}(p = 4) = S_{\text{Potts}}(c = 4/5) + g \int \left( \psi_{2/3} \bar{\psi}_{2/3}^+ + \psi_{2/3}^+ \bar{\psi}_{2/3} \right) d^2z.$$  

In order to derive the specific form of the conservation laws for, say, $T_4$ and $T_6$, we cannot use anymore the free massive fermions technics described above. What we can do here (and for each of the models of this class) is: 1) to perform the perturbation expansion around the conformal point ($g = 0$); 2) to use at each order of the perturbation series the conformal WI’s and at the end 3) to resum the perturbation series. To begin with the $T_4$ conservation law. We have to calculate the mean value of $\bar{\partial}T_4$ in the presence of an arbitrary set of other fields:

$$\bar{\partial} \langle T_4(z, \bar{z}) \ldots \rangle = \bar{\partial} \langle T_4(z) \ldots \rangle_{\text{conf}} + g \bar{\partial} \int d^2w \langle T_4(z) \psi_{2/3}(w) \bar{\psi}_{2/3}^+ (\bar{w}) \ldots \rangle +$$
$$+ g^2 \bar{\partial} \int d^2w_1 \int d^2w_2 \langle T_4(z) \psi_{2/3}(w_1) \bar{\psi}_{2/3}^+ (\bar{w}_1) \psi_{2/3}(w_2) \bar{\psi}_{2/3}^+ (\bar{w}_2) \ldots \rangle + \ldots$$
Using the conformal OPE's:
\[ T_4(\psi_{2/3}(0) = \left\{ \frac{\alpha_0}{z^4} + \frac{\alpha_1}{z^3}\partial + \frac{1}{z^2}(\alpha_2L_{-2} + \alpha_3\partial^2) + \frac{1}{z}(L_{-3} + \alpha_4\partial L_{-2} + \alpha_5\partial^3) \right\} \psi_{2/3}(0) + \ldots \]

the level-3 null vector
\[ (L_{-3} + \gamma_1L_{-1}L_{-2} + \gamma_2L_{-1}^3) \psi_{2/3} = 0 \]

one can get read of \( T_4 \) at each order in \( g \). The next step is to take some of the integrals and then to resum the perturbation series. The exact result is:
\[ \langle \bar{\partial}T_4 \ldots \rangle = \langle (a\partial^3\Theta + b\partial L_{-2}\Theta + cg^3\partial T) \ldots \rangle \]

(5.1)

where \( \Theta \) is the trace of the stress-tensor:
\[ \bar{\partial}T = \partial\Theta, \quad \Theta = \frac{g}{3}(\bar{\psi}_{2/3}^+\psi_{2/3} + \bar{\psi}_{2/3}\psi_{2/3}^+) \]

and \( a, b, c \) are fixed numerical constants. Since \( \psi_{2/3} \) as a conformal field has no null vector at the second level:
\[ L_{-2}\psi_{2/3} \neq \frac{9}{14}\partial^2\psi_{2/3} \]

the form (5.1) of the \( T_4 \) conservation law is not the desired one (1.7). According to our criterion (1.5) one can naively conclude that this model has no any nontrivial noncommuting conserved charges. However such a conclusion is wrong and the reason is in the broken \( W_3 \) - symmetry of the model. Using the conformal OPE of the \( W_3 \) current [10]:
\[ W_3(z)\psi_{2/3}(0) = \left\{ \frac{w_0}{z^3} + \frac{1}{z^2}W_{-1} + \frac{1}{z}W_{-2} + \ldots \right\} \psi_{2/3} \]
\[ W_n = \int z^{n+2}W_3(z)dz, \quad w_0 = \frac{2}{9}\sqrt{\frac{26}{15}} \]

one can apply the method of the perturbative expansion described above and to
prove that $W_3$ is not conserved:

$$\bar{\partial}W_3 = w_0 \partial^2 \Theta + \alpha \partial W_{-1} \Theta + \beta W_{-2} \Theta$$

(5.2)

due to the last term in the r.h.s. The crucial point is that the field $\psi_{2/3}$ satisfies the following specific null vector conditions of the bigger $W_3$ algebra:

$$\left(W_{-1} - \frac{1}{2} \sqrt{\frac{26}{15}} L_{-1}\right) \psi_{2/3} = 0, \quad \left(W_{-2} - \frac{6}{13} \sqrt{\frac{26}{15}} (2L_{-1}^2 - \frac{5}{3}L_{-2})\right) \psi_{2/3} = 0.$$

Therefore we can rewrite (5.2) in the following simple form:

$$\bar{\partial}W_3 = A \partial^2 \Theta + B L_{-2} \Theta.$$  

(5.3)

Combining eqs. (5.3) and (5.1) we arrive at the desired spin-4 conservation law:

$$\bar{\partial} \left( T_4 - \frac{b}{B} \partial W_3 \right) = a \partial^3 \Theta + bg^3 \partial T.$$  

(5.4)

Now we are able to construct five new conservation laws. The simplest one is $\bar{\partial} F_4^{(1)} = \bar{\partial} \Theta_4^{(1)}$

$$F_4^{(1)} = zT_4 - \frac{b}{B} \partial W_3 + bg^3 zT, \quad \Theta_4^{(1)} = a \partial^2 (z \Theta) - 3a \partial \Theta + g^3 bz T + abg^4 z \Theta$$

and the corresponding conserved charge has the form:

$$L_{-2}^{(4)} = \int F_4^{(1)} dz - \int \Theta_4^{(1)} d\bar{z}.$$

Comparing eqs. (5.4) and (1.7) we can say that all the charges $L_{-k}^{(4)}$, $\bar{L}_{-k}^{(4)}$ ($k = 0, 1, 2$) can be obtained from the ones of the Ising model by substituting $T_4 \rightarrow T_4 - \frac{b}{B} \partial W_3$, $m^2 \rightarrow g^3$, $\beta \rightarrow c$ etc.
The proof that the $T_6$ conservation law has the form (1.8) is more involved [5] and it requires together with the $W_3$ - nonconservation law (5.3) an analogous nonconservation law for the spin-5 current $W_5 \sim -\alpha \partial^2 W_3$:

$$\bar{\partial}W_5 = \beta_1 \partial^4 \Theta + \beta_2 \partial^2 L_{-2} \Theta + \beta_3 L_{-4} \Theta + \beta_4 L_{-2}^2 \Theta.$$  

Crucial for these constructions is the fact that in this case we have three independent spin-6 tensors, i.e. $T_6^{(1)} \sim T^3$, $T_6^{(2)} \sim (\partial T)^2$ and $T_6^{(3)} \sim W_3^2$.

Having constructed the conserved charges $L_{-k}^{(2s)}$ ($0 \leq k \leq 2s - 1$) we can compute their algebra using the conformal perturbation technics. The simplest but highly nontrivial calculation is the one of the commutators of $L_{-2}^{(4)}$, $\bar{L}_{-2}^{(4)}$ and $L_0$. The surprising fact is that they close again a $SL(2, R)$ algebra as in the case of the Ising model. This is an indication that the conserved charges of the off-critical Potts model could have the same algebra (3.1),(3.4) as the charges of the Ising model. Accepting this as an working hypothesis we have to mention the crucial difference between the off-critical fermion and parafermion. Although they have the common algebra of symmetries the generators of these symmetries $L_{-k}^{(2s)}$ are acting on $\psi$ and $\psi_{2/3}$ in a completely different way:

$$[L_{-3}^{(4)}, \psi_{2/3}(z, \bar{z})] = \left(\frac{3}{2} \partial^3 + \frac{3}{20} \partial T(z, \bar{z})\right) \psi(z, \bar{z}) :$$

$$[L_{-2}^{(4)}, \psi_{2/3}(z, \bar{z})] = \left(g^3 \bar{z} \partial + \left(\frac{4}{5} \partial^2 + \frac{3}{2} T\right) + z \left(\frac{3}{2} \partial^3 + \frac{3}{20} \partial T\right)\right) \psi(z, \bar{z}) :$$

$$[L_{-1}^{(4)}, \psi_{2/3}(z, \bar{z})] = \left(g^3 \left(\frac{4}{3} \bar{z} \partial + 2z \bar{z} \partial - z^2 \bar{\partial}\right) + \frac{13}{10} \partial + 2z \left(\frac{4}{5} \partial^2 + \frac{3}{2} T\right) + z^2 \left(\frac{3}{2} \partial^3 + \frac{3}{20} \partial T\right)\right) \psi(z, \bar{z}) :$$

(5.5)

etc. The corresponding commutators of $L_{-k}^{(6)}$ with $\psi_{2/3}$ contain terms with $T_4$ and $T$ and their derivatives and so on. Therefore the corresponding WI’s equations relate in fact the functions of the parafermions with the functions of their descendents. As one can see from the conformal limit $g \to 0$ this complicated form of the WI’s
is not a specific feature of the off-critical model only. The WI’s for the conformal $CW_\infty$ also include the descendents of the parafermion.

6. Few conjectures

We are prepared now for our main conjecture concerning the symmetries of the IM’s given by the action:

$$S_{13}(p=\text{even}) = S_{conf}(c_p) + g \int \Phi_{13}(z, \bar{z})\bar{\Phi}_{13}(z, \bar{z})d^2z.$$  

All these models have $W_\infty(V)$ (3.1),(3.4) as a symmetry algebra and they belong to different representations of this algebra.

One could try to prove it styding model by model the $S_{13}$ (p=even) set of IM’s in order to understand the general properties of the representations of the $W_\infty$ - algebra. There are many signs that for the practical purposes such as the computation of the correlation functions it is sufficient to classify and explicitely construct the representations of the one of the Virasoro subalgebras of $W_\infty(V)$, namely the one spanned by $L_{-2s+2}$, $L_0$ and $\bar{L}_{-2s+2}$.

The set of models given by $S_{13}$ (p=even) represents only a first member of the infinite family of relativistic IM’s. What one could expect for the other members of this family? Could we realize them as specific representations of certain infinite algebras and how these algebras look like? Our hipothesis for the symmetries of the IM’s is based on their classification according to the admissible spins of the commuting conserved charges $P_s$, $\bar{P}_s$. In the case of $S_{13}(p)$ series we have:

$$s = 1 \pmod{2},$$

i.e. the Coxeter exponents modulo the Coxeter number of $SU(2)$. Their symmetry algebra $W_\infty(V)$ could be considered as a specific deformation of the $PW_\infty$ subalgebra of $W_\infty$ generated by the currents $T_{2s}$. Consider a set of models with
\( s = 1(\text{mod}3) \) as allowed set of spins for \( P_s, \tilde{P}_s \). They are specific perturbations of the minimal models of the \( W_3 \) - algebra [9]. Therefore the adequate conformal \( W_\infty \) algebra is generated by the descendents \( T_{2p}, W_{2p+1} \) of the stress-tensor \( T_2 \) and the spin-3 current \( W_3 \). Then the off-critical infinite algebra \( W_\infty (W_3) \) describing this series of IM’s can be obtained as a deformation of the \( W_3 \) - analog of \( \mathcal{P}W_\infty \), i.e. a specific subalgebra of the conformal \( CW_\infty (W_3) \). A straightforward conjecture is that the IM’s with the spins of \( P_s : s = 1(\text{mod}N) \) can be realized as specific representations of the \( W_\infty (W_N) \) - algebras.

We want to mention in conclusion that one of the important lessons we have learned studing the off-critical properties of the Ising and Potts models is the crucial role of the conformal \( \mathcal{P}W_\infty \) - algebras (and their Virasoro subalgebras) in the description of these models as representations of certain infinite algebra \( W_\infty (V) \). The construction of the relevant conformal \( W_\infty (G) \) algebras is in our opinion the most important part of the ambicious program of classification of the IM’s according to their infinite algebra of symmetries.

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