TAU-FUNCTIONS, TWISTOR THEORY, AND QUANTUM FIELD THEORY

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Abstract. This article is concerned with obtaining the standard tau function descriptions of
integrable equations (in particular, here the KdV and Ernst equations are considered) from the
geometry of their twistor correspondences. In particular, we will see that the quantum field
theoretic formulae for tau functions can be understood as arising from geometric quantization
of the twistor data. En route we give a geometric quantization formulation of Chern-Simons
and WZW quantum field theories using the Quillen determinant line bundle construction and
ingredients from Segal’s conformal field theory. The $\tau$-functions are then seen to be amplitudes
associated with gauge group actions on certain coherent states within these theories that can be
obtained from the twistor description.

1. Introduction

One of the most significant overviews of the theory of integrable systems is that provided by
the grassmanian approach of Sato and its development by the Japanese school into a formulation
based on quantum field theory. The geometric and analytic underpinnings of the grassmanian
approach were further developed in Segal and Wilson (1985). This approach was first used to
bring out the (infinite-dimensional) geometry of ‘equations of KdV type’; the KdV equation itself
as well as $n$-KdV and the KP equation. The central construct in this approach is the $\tau$-function
which serves as a ‘potential’ for the dependent variables which appear in the KP equation (and its
specializations $n$-KdV and KdV). In the paper of Segal and Wilson, the $\tau$-function is constructed
in terms of infinite determinants. The Japanese school interpret these as quantum field theoretic
amplitudes of the form

$$\tau(x, t) = \langle 0 | \exp \{ x \phi_1 + t \phi_2 \} | \psi \rangle$$

for some state $| \psi \rangle$ and operators $\phi_1$ and $\phi_2$ in a two-dimensional quantum field theory with vacuum
state $| 0 \rangle$. Since this foundational work, $\tau$-functions have been introduced in the study of many
other integrable systems.

Another significant (but more recent) unifying idea in the theory of integrable systems originates
in Richard Ward’s observation that many one and two-dimensional integrable systems are symmetry
reductions of the self-dual Yang-Mills equations. Such systems can be classified as reductions of
the self-dual Yang-Mills equations and their theory obtained from the complex geometry of twistor
theory which gives, in effect, the general solution of these equations, Mason & Woodhouse (1996).
In particular twistor methods are applicable to the study of the KdV and $n$-KdV equations, Mason
and Singer (1994).

This paper is one in a series which is devoted to the clarification of links between Ward’s twistor
approach and other pre-existing methods. Its main purpose is to give a geometric account of the
quantum field-theoretic approach of the Japanese school and its relation to the geometry of the
twistor construction. It is a sequel to Mason & Singer (1994), which focussed on the twistor theory
of $n$-KdV equations and is a parallel development to that in Mason, Singer & Woodhouse (2000),
which gave a definition of tau-functions as an infinite dimensional determinant (or cross-ratio).

$^1$The KP equations do not appear to be a reduction of the self-dual Yang-Mills equations with finite dimensional
gauge group. However, it is possible even so to find generalized twistor correspondences for these equations, Mason
(1985), and §§12.6 of Mason and Woodhouse (1996).
The purpose of this paper is to make a more direct contact with quantum field theory and Quillen determinants. It has been written so as to be largely self-contained.

We now give a more detailed outline of the work presented here. In §§2–4 we give an account of the quantum field theories that are relevant to KdV and certain other integrable systems. These theories are versions of Chern–Simons theory and the WZW model, and have been much studied [see the cited works by Felder, Gawedzki, Kupiainen, Gepner and Witten]. However we did not find in the literature a source which deals with them as presented here. Our treatment seems very natural; it is an application of the methods of geometric quantization in an infinite-dimensional setting, combined with Quillen’s determinant line-bundle to give an explicit construction of the Fock space as the space of holomorphic sections of the prequantum line-bundle with inner product obtained from Segal’s formulation of conformal field theory using the gluing of determinants.

The ingredients needed for geometric quantization are: first, the classical phase space $\mathcal{P}$ (a symplectic manifold); second, a choice of real or complex polarisation; and finally a choice of prequantum line bundle $\text{Det} \to \mathcal{P}$, which, as the notation is intended to suggest, turns out to be Quillen’s determinant line bundle. In this paper, we shall always use a complex polarisation, so that $\mathcal{P}$ becomes a Kähler manifold. The bundle $\text{Det}$ is required to admit a $U_1$-connection whose curvature is $i$ times the symplectic form of $\mathcal{P}$.

- Chern–Simons, §2–3: $\mathcal{P} = \mathcal{A}_+$, the space of unitary connections in a trivial bundle over a disc $D_+ \subset \mathbb{CP}^1$; the symplectic form is

$$\Omega(a, b) = \frac{1}{2\pi} \int_{D_+} \text{tr}(a \wedge b);$$

$\mathcal{A}_+$ has a complex structure by identification with the space of $\bar{\partial}$-operators; and the prequantum line-bundle is essentially Quillen’s determinant bundle.

- WZW, §4: $\mathcal{P} = \mathcal{A}_{\text{red}}$ = the space of based loops $\Omega SU_n = \text{LSU}_n/\text{SU}_n$, with its standard homogeneous symplectic form equal to

$$\Omega^{\text{red}}(u, v) = \int_{S^1} \text{tr}(udv)$$

at the identity coset.

The polarisation is given by one of the standard factorization theorems, which gives $\Omega SU_n$ a Kähler structure through the identification with $\text{LSL}_n(\mathbb{C})/\text{L}^+\text{SL}_n(\mathbb{C})$. The prequantum line-bundle is also a determinant bundle and has been constructed in Segal and Wilson (1985), Pressley and Segal (1986).

Given these data, geometric quantization yields a quantum state space $\mathcal{F}$ as the space of ‘square-integrable’ holomorphic sections of $\text{Det}$ over $\mathcal{P}$. (In the Chern-Simons case we will be concerned with sections invariant under the action of the group of based gauge transformations.) This construction is natural in that if a group $\mathcal{G}$ acts compatibly with the symplectic form and polarisation, then the action can be quantized (provided a moment map can be found). If a moment map is obstructed by a cocycle, we obtain a representation of the central extension of $\mathcal{G}$ on $\mathcal{F}$ generated by that cocycle. We will be interested in the actions both of gauge transformations and diffeomorphisms of $D_+$. The moment maps for these symmetries of the phase spaces can also be obtained from Noether’s theorem.

The geometric interpretation of the ‘Japanese formula’ (1.1) has to do with the non-invariance of certain coherent states in $\mathcal{F}$ under symmetries that act holomorphically but not symplectically on $\mathcal{P}$ (these will lie in the complexification of a real group of holomorphic symplectomorphisms). This leads to a first general definition of the $\tau$-function in §3.5. (Another reason why we are interested in these symmetries is that, as the notation is intended to suggest, the phase space $\mathcal{A}_{\text{red}}$ is a reduction of the phase space $\mathcal{A}_+$ by the group of gauge transformations that are the identity on $\partial D_+$.)
The connection with the more standard quantum field theory notation is given in §5. In this context, the coherent states $|\Psi(p)\rangle$ in $\mathcal{F}$ correspond to points $p \in \mathcal{P}$ and $\Psi(p) \in \text{Det}_p^*$. They arise from the operation of evaluation of holomorphic sections at $p$. This defines a linear functional $\mathcal{F} \to \text{Det}_p$ and then multiplication by $\Psi(p)$ gives a complex number. Any such linear functional is given by pairing with some state of $\mathcal{F}$—let that state be denoted $|\Psi(p)\rangle$. (The simplest example of this phenomenon is the geometric quantization of the Riemann sphere $\mathbb{C}P^1$. If we take for the symplectic form $n$ times the area form, then $\mathcal{F}$ is the space of polynomials of degree $n$ (the $n+1$-dimensional irreducible representation of $SU_2$). If $p \in \mathbb{C}P^1$, then $|p\rangle = (pz + 1)^n$ up to scale, the polynomial with an $n$-fold zero at the antipode $-1/p$. The inner product on the Fock space then can be seen to arise from the Segal gluing formulae for determinants.

We now explain how these phase spaces arise naturally in the twistor description of integrable systems such as KdV and the Ernst equations. The general formulation and details of these two examples appear in §§6–8.

In the twistor description of integrable systems, the basic geometric object is a holomorphic vector bundle $\mathcal{E}$, over an auxiliary complex manifold $\mathcal{Z}$ called twistor space. Twistor space $\mathcal{Z}$ is related to ‘space-time’ $\mathcal{M}$ (the space of independent variables of the integrable system in question) through a correspondence which has the property that the points of space-time parameterize a family of $\mathbb{C}P^1$'s (so-called twistor lines) in $\mathcal{Z}$. Now a $\tau$-function can only be defined from the twistorial point of view when additional symmetries are imposed upon $\mathcal{E}$. Technically, the main requirement is that a group of symmetries should act on $\mathcal{Z}$ with generic orbit of (complex) codimension 1. (In the examples presented here, $\mathcal{Z}$ has dimension 2 so we require a 1-dimensional symmetry group. Higher-dimensional examples appear in Mason, Singer and Woodhouse (2000).) The presence of such symmetries allows us to pass, in a natural way, from the bundle $\mathcal{E}$ to a family of holomorphic structures on a fixed bundle $\mathcal{E}$ over $\mathbb{C}P^1$, with an explicit formula for the variation of holomorphic structure with the point in space-time. More explicitly, we can regard this as a family of $\partial$-operators (parameterized by space-time) on a fixed trivial bundle over $\mathbb{C}P^1$, or as a similarly parameterized family of patching functions (clutching functions) in a Čech description of $\mathcal{E}$. The variation of this holomorphic structure with space-time is given by a combination of a (complex) gauge transformation and a diffeomorphism. In the KdV case we need only use gauge transformations, and in the Ernst equation case we need only use a diffeomorphism. Thus we have reached a point of contact with the classical phase spaces described before, for we can regard our family of $\partial$-operators as a finite-dimensional submanifold of $\mathcal{A}_+$, and our family of patching functions as a finite-dimensional submanifold of $\mathcal{A}^{\text{red}}$. Furthermore, the variation of the $\partial$-operators or patching functions in the family is given by a holomorphic symmetry of the phase space so that the family is entirely determined by the symmetry once its initial value is known. Postponing the details until §7, the upshot of this is that we can interpret the right hand side of (1.1) within the geometric quantization framework—$|0\rangle$ and $|\psi\rangle$ are (suitably normalized) coherent states associated to the initial value of the trivial solution and of $\psi$ respectively; while the exponential represents the quantization (representation on $\mathcal{F}$) of the translation from 0 to $(x, t)$.

The construction presented here is based on the geometric definition of the $\tau$-function given by Segal and Wilson (1985). As discussed in Mason & Singer (1994) the twistor description of KdV can be regarded as a generalization of the description of Segal & Wilson. It is a strict generalization because there is a twistor description of any local (holomorphic) solution of KdV, whereas Segal & Wilson only obtain solutions in a certain class (i.e. those with a convergent Baker function—these are, in particular, meromorphic for all complex times). The correspondence between the Segal–Wilson description and the twistor description goes roughly as follows:

| Segal–Wilson | Twistor description |
|---------------|---------------------|
| $W \subset \text{Gr}$ | Representation of $\mathcal{E}$ restricted to a twistor line |
| $\Gamma_+$ | Holomorphic symmetry of $\mathcal{Z}$ |
The main ideas of this paper can be obtained by reading up to §3.3 and then skipping ahead to §4 leaving out any subsequent material concerning Čech representations of bundles.

Finally we note that once the formulae of §3 and §4 have been obtained, one could refer to Mason, Singer & Woodhouse (2000) for applications to twistorial definitions of τ-functions. In that paper such formulae were derived from a slightly different point of view. The reader is also referred to that paper for the twistorial definition of τ-functions of several integrable systems not discussed here.

2. The space of connections as a classical phase-space

In this section we describe in detail the geometry of the space of connections over the disc, in particular its symplectic and complex structure. We consider natural groups of symmetries (and algebras of infinitesimal symmetries) of this space and the extent to which they preserve the symplectic and complex structure. We also discuss the reduction of this phase-space to one closely related to loop groups. In the next section we shall turn to the problem of quantizing this phase-space.

Since connections, ∂-operators and related notions will be in constant use throughout this paper, we begin by recalling these notions, from a point of view close to that of Atiyah & Bott (1982) or Donaldson & Kronheimer (1990).

2.1. Connections and ∂-operators. In this section M is a smooth manifold, and E → M is a complex vector bundle of rank n, with structure group K, a (usually compact) Lie group. A K-connection A on E determines and is determined by the covariant derivative operator

\[ \nabla_A : \Omega^p(M, E) \rightarrow \Omega^{p+1}(M, E); \]

this is a linear differential operator preserving the K-structure and satisfying the Leibnitz rule

\[ \nabla_A (f \otimes s) = df \otimes s + f \otimes \nabla_A s \]

for any smooth function f and section s of E. (Here Ω^p(M, E) denotes the space of smooth p-forms with values in E.) The operator in (2.1) extends in a standard way to define a covariant exterior derivative

\[ d_A : \Omega^p(M, E) \rightarrow \Omega^{p+1}(M, E) \]

and this has curvature \( F_A = d_A^2 \), a 2-form with values in the endomorphisms of E that respect the K-structure, a space we shall write as \( \Omega^2(\mathfrak{k}(E)) \).

The space \( \mathcal{A} = \mathcal{A}(M, E) \) of all K-connections on E is an infinite-dimensional affine space relative to the vector space \( \mathcal{B} = \mathcal{B}(M, E) = \Omega^1(M, \mathfrak{t}(E)) \). In other words the difference of any two connections in \( \mathcal{A} \) is a 1-form with values in \( \mathfrak{t}(E) \). Thus the tangent space \( T_A \mathcal{A} \equiv \mathcal{B} \), for any point A of \( \mathcal{A} \). If \( a \in \mathcal{B} \), then the derivative of \( F_A \) in the direction of \( a \) is given by

\[ \delta_a F_A = d_A a. \]

The gauge group \( \mathcal{K} \) of all automorphisms of E respecting the K-structure acts on \( \mathcal{A} \) by conjugation. If \( g \in \mathcal{K} \), then

\[ \nabla_{g(A)} = g \cdot \nabla \cdot g^{-1} = \nabla_A - \nabla_A g\, g^{-1}. \]

Any element \( u \in \Omega^0(M, \mathfrak{t}(E)) \) determines an infinitesimal gauge transformation \( g = 1 + \varepsilon u \). Inserting this in (2.5) and working to first order in \( \varepsilon \) we obtain the formula \( \delta_u \nabla_A = -d_A u \). The correct interpretation of \( \delta_u \) here is as a vector field on \( \mathcal{A} \), whose value at \( A \) is \( -d_A u \in \mathcal{B} \).

Turning now to \( \partial \)-operators, we assume that M is a complex manifold, so that we can introduce local holomorphic coordinates \( z_j \) near any point. Then the space of complex valued 1-forms \( \Omega^1 \) splits as a direct sum \( \Omega^{(1,0)}(M) \oplus \Omega^{(0,1)}(M) \) generated locally by the \( dz_j \) or the \( d\bar{z}_j \) respectively. (Complex-valued k-forms can similarly be decomposed as \( \oplus_{p+q=k} \Omega^{(p,q)}(M) \).) If now \( E \rightarrow M \) is
a complex vector bundle with complex structure group \( K^c \), a \((K^c)\bar{\partial}\)-operator on \( E \) is a linear differential operator

\[
\Omega^{(0,0)}(M, E) \longrightarrow \Omega^{(0,1)}(M, E)
\]

satisfying a Leibnitz rule as in (2.2) and preserving the \( K^c \) structure of \( E \). (Later on, we shall only need \( K^c = SU(n) \) so that all \( \bar{\partial} \)-operators are required to annihilate a holomorphic \( n \)-form on \( E \).) Extending \( \bar{\partial}_a \) to act on \( \Omega^{(0,p)}(M, E) \), we introduce the algebraic operator \( \bar{\partial}_a^2 \), a \((0, 2)\)-form with values in \( \mathfrak{t}^c(E) \). We say that \( \bar{\partial}_a \) is integrable if \( \bar{\partial}_a^2 = 0 \). Any integrable \( \bar{\partial} \)-operator defines a holomorphic structure on \( E \); the local holomorphic sections are those that are annihilated by \( \bar{\partial}_a \) and there exist enough such local sections to form holomorphic frames near any point of \( M \).

In general, the space \( \mathcal{A}^c = \mathcal{A}^c(M, E) \) of \( K^c \)-\( \bar{\partial} \)-operators on \( E \) is an infinite-dimensional complex affine space relative to the vector space \( \mathcal{B}^c = \mathcal{B}^c(M, E) = \Omega^{(0,1)}(M, \mathfrak{t}(E)) \). The group of complex gauge transformations \( K^c \) acts on \( \mathcal{A}^c \), analogously to (2.3):

\[
\bar{\partial}_g(A) = g \cdot \bar{\partial}_a \cdot g^{-1} = \bar{\partial}_a - \bar{\partial}_a g g^{-1};
\]

This preserves integrability, and indeed two integrable \( \bar{\partial} \)-operators define isomorphic holomorphic structures on \( E \) if they are (complex)-gauge equivalent. If \( u \) is an infinitesimal complex gauge transformation, then it defines a holomorphic \( \bar{\partial} \)-operator on \( \mathcal{A}^c \).

If one fixes a choice of a hermitian structure, a \( \bar{\partial} \)-operator gives rise to a unitary connection. This follows from Chern’s theorem: there is a unique unitary connection whose (0, 1)-part defines any given holomorphic structure. When \( M \) has complex dimension 1 the integrability condition is trivially satisfied so that on a bundle with fixed hermitian structure \( \bar{\partial} \)-operators are in 1–1 correspondence with unitary connections. It follows that if \( K = U(n) \) or \( SU(n) \), then \( \mathcal{A}(M, E) = \mathcal{A}^c(M, E) \). This identification simply maps \( d_A \) to its \((0, 1)\)-part \( d_A^{(0,1)} = \bar{\partial}_A \) and Chern’s result asserts that this map is an isomorphism. Relative to a unitary trivialization such that \( d_A = \bar{\partial}_a + A \), we write \( A = \alpha - \alpha^* \), where \( \alpha \in \mathcal{B}^c \), so that \( \bar{\partial}_A = \bar{\partial}_a + \alpha \), \( \bar{\partial}_A = \bar{\partial} - \alpha^* \).

An important corollary of this identification between the space of \( \bar{\partial} \)-operators with the space of connections is that it leads to a natural action of the group of complex gauge transformations on \( \mathcal{A} \): if \( g \) is such a complex gauge transformation, its action is given by

\[
d_{g(A)} = g \cdot \bar{\partial}_A \cdot g^{-1} + (g^*)^{-1} \cdot \bar{\partial}_A \cdot g^*.
\]

We now study the symplectic geometry of this space.

2.2. Connections on domains in \( \mathbb{C} \). We will be interested in the space of connections on certain domains with boundary in \( \mathbb{C} \). Let \( D_- \) be a finite disjoint union of open discs in \( \mathbb{CP}_1 \) and let \( D_+ \) be the complement of \( D_- \). Then \( D_+ \) is a closed subset of \( \mathbb{CP}_1 \) with non-empty interior \( D_+^0 \). We think of \( D_- \) as being a neighbourhood of certain points ‘at \( \infty \)’ in \( \mathbb{CP}_1 \). Let \( E_+ \rightarrow D_+ \) be the trivial complex vector bundle of rank \( r \) and with structure group \( SU(r) \); let \( \mathcal{A}_+ \) be the space of \( SU(r) \)-connections or equivalently \( SL(r, \mathbb{C}) \)-\( \bar{\partial} \)-operators that are \( C^\infty \) up to the boundary of \( D_+ \).

We can think of \( \mathcal{A}_+ \) as an infinite-dimensional classical phase space, for it carries a natural symplectic form \( \Omega \), given by

\[
\Omega(a, b) = \frac{1}{2\pi} \int_{D_+} \text{tr}(a \wedge b) \text{ for } a, b \in \mathcal{B}_+.
\]

(The normalization factor of \( 2\pi \) will be convenient later.) In this language the complex structure on \( \mathcal{A}_+ \) is a (positive) complex polarization; in other words, \( \mathcal{A}_+ \) is an infinite-dimensional Kähler manifold.
2.2.1. The action of gauge transformations on $A_+$. Because $D_+$ is a manifold with boundary, it is natural to distinguish inside the group $G^c$ of complex gauge transformations that are $C^\infty$ up to the boundary, the normal subgroup $G_0^c$ of based gauge transformations: those that are equal to the identity on the boundary. Similarly we denote by $G \subset G^c$ and $G_0 \subset G_0^c$ the subgroups of unitary gauge transformations. We shall denote by $\mathfrak{g}_0, \mathfrak{g}_0^c, \mathfrak{g}, \mathfrak{g}^c$ the Lie algebras of $G_0, G_0^c, G, G^c$.

With regard to the action of these groups on $A_+$, we have the following

Proposition 1.  
- The actions of $G_0$ and $G$ on $A_+$ preserve $\Omega$ and the complex polarization;
- The actions of $G_0^c$ and $G^c$ on $A_+$ preserve the complex polarization but not $\Omega$.
- If $u \in \mathfrak{g}$, then a Hamiltonian for $u$ is given by

\[
H_u(A) = -\frac{1}{2\pi} \int \text{tr}(F_Au) + \frac{1}{2\pi} \oint \text{tr}(Au).
\]

- The map $u \mapsto H_u(A)$ is not a co-momentum map for $\mathfrak{g}$ since

\[
\{H_u, H_v\} - H_{[u,v]} = \frac{1}{2\pi} \oint \text{tr}(u \, dv).
\]

Instead $u \mapsto H_u(A)$ is a co-momentum map from the central extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ into $C^\infty(A_+)$ with cocycle

\[
c(u,v) = \frac{1}{2\pi} \oint \text{tr}(u \, dv).
\]

However, the cocycle vanishes on $\mathfrak{g}_0$ and $u \mapsto H_u(A)$ is a moment map $\mathfrak{g}_0 \rightarrow C^\infty(A_+)$. 

Remark In (2.8), $\oint$ denotes an integral over $D_+$, $\int$ denotes an integral over $\partial D_+$. In the boundary-term, $A$ appears, which is not gauge-invariant. This indicates that the most natural framework is to use bundles over $D_+$ that are framed over the boundary. This fits in naturally with our requirements in subsequent sections, for we shall want to extend our $\partial$-operators in a standard way to operate on bundles over $\mathbb{C}P^1$. The reader may alternatively take the view that we are working on the product bundle over $D_+$ which provides a preferred gauge in which to write $A$.

Proof. It is clear that $G^c$ preserves the complex polarisation of $A_+$. To verify that $G$ preserves the symplectic structure, it is enough to verify that (2.8) is indeed a Hamiltonian. Without the boundary term, this calculation is in Atiyah–Bott (1982).

To verify that $H_u$ is a Hamiltonian for the vector field $d_Au$ on $A_+$. Let $a$ represent a variation in $A$. Then we have $\delta_a A = a, \delta_a F_A = d_Aa$ and so

\[
\delta_a H_u(A) = -\frac{1}{2\pi} \int \text{tr}(d_Aa \, u) + \frac{1}{2\pi} \oint \text{tr}(a \, u).
\]

Integrating by parts in the first term, using $\text{tr}(d_Aa \, u) = d \text{tr}(a \, u) - \text{tr}(a \wedge d_Au)$, we obtain

\[
\delta_a H_u(A) = \Omega(a, d_Au)
\]

as required. Next we compare the Poisson bracket of $H_u$ and $H_v$, for two elements $u, v$ of $\mathfrak{g}$, with $H_{[u,v]}$. The Poisson bracket is equal to the variation of $\delta_u H_v$, where we have written $\delta_u$ for $\delta_dA_u$. From above, this is simply

\[
\delta_u H_v - H_{[u,v]} = \frac{1}{2\pi} \int \text{tr}(d_Au \wedge d_Av) + \frac{1}{2\pi} \int \text{tr}(F_A[u,v]) - \frac{1}{2\pi} \oint \text{tr}(A[u,v]) = \frac{1}{2\pi} \oint \text{tr}(u \, dv)
\]

using the definition of curvature, $d_A^2 u = [F_A, u]$ and writing $d_A u = du + [A, u]$ in the boundary term. \qed
The subgroups of based gauge transformations play a different role from the full groups of gauge transformations. The unbased transformations will generate the dynamics of our system when we consider the quantization, whereas the subgroups of based gauge transformations will correspond to ‘genuine’ gauge degrees of freedom. All of (the central extension of) \( G^c \) acts on the quantum Hilbert space but only \( G_0^c \) preserves the ‘vacuum state’.

2.2.2. The action of diffeomorphisms on \( A_+ \). We turn now to consider the action of diffeomorphisms on \( A_+ \). The most obvious group that acts consists of diffeomorphisms of \( D_+ \) that are smooth up to the boundary and tangent to it. Working at the infinitesimal level, we introduce the Lie algebra of this, \( \text{Vect}_0(D_+) \) and its complexification \( \text{Vect}_0^c(D_+) \). We shall also need to consider the algebra \( \text{HolVect}(D_+) \) of real vector fields whose \((1,0)\) part is holomorphic in the interior of \( D_+ \). If \( \xi \in \text{Vect}_0(D_+) \), then the action on \( A_+ \) is given by Lie-derivative; the vector field at \( A \in A_+ \) is given by

\[
\mathcal{L}_\xi(A) = d(A(\xi)) + \xi \lrcorner \, dA = d_A A(\xi) + \xi \lrcorner \, F_A.
\]

Here we have written \( A(\xi) \) for the interior product \( \xi \lrcorner \, A \) and the second formula follows from the first by adding and subtracting the term \( \xi \lrcorner \, A \wedge A \). The fact that \( A \) appears explicitly here reflects the need to make a choice of ‘invariant’ trivialization of \( E \) when lifting the action of the diffeomorphism group to \( A \).

Now one can check that a Hamiltonian for this action is given by

\[
H_\xi(A) = \frac{1}{2\pi} \int \text{tr}(F_A A(\xi)) - \frac{1}{4\pi} \oint \text{tr}(A A(\xi)).
\]

The verification is easiest using the second formula for \( \mathcal{L}_\xi(A) \), and also requires the identity \( F_A a(\xi) = a \wedge (\xi \lrcorner \, F_A) \) which is valid in 2 dimensions. Using these, one verifies that

\[
\delta_\alpha H_\xi(A) = \Omega(\alpha, \mathcal{L}_\xi A).
\]

Hence \( \text{Vect}_0(D_+) \) acts symplectically on \( A_+ \) and in this case there is no cocycle. Writing \( A = \alpha d\bar{z} - \alpha^* dz \) and \( \xi = \xi^{1,0}\partial_z + \xi^{0,1}\bar{\partial}_z \), we have

\[
\mathcal{L}_\xi(A) = [\partial_z (\xi^{0,1} \alpha) + \xi^{1,0} \partial_z \alpha - \alpha^* \bar{\partial}_z \xi^{1,0}] d\bar{z} - \text{hermitian conjugate}
\]

so that the complex polarisation is not preserved unless \( \partial_z \xi^{1,0} = 0 \). Hence the subalgebra \( \text{HolVect}(D_+) \) acts preserving the polarisation (but not, in general, the symplectic structure as such vector fields are generically not tangent to the boundary). To summarize:

**Proposition 2.** The algebra \( \text{Vect}(D_+) \) of vector fields on \( D_+ \) acts symplectically on \( A_+ \). The map \( \text{Vect}(D_+) \to C^\infty(A_+) \) given by \( \xi \mapsto H_\xi \) with \( H_\xi \) given by (2.10) is an equivariant moment map. \( \text{Vect}(D_+) \) does not preserve the complex polarisation of \( A_+ \). The algebra \( \text{HolVect}(D_+) \) of vector fields that are smooth up to the boundary of \( D_+ \) and whose \((1,0)\) part is holomorphic in the interior, acts on \( A_+ \) preserving its complex polarisation (but not, in general, the symplectic form).

**Remark:** The above phase space is that for Chern-Simons theory, with action

\[
S[A] = \int \text{tr} \left( A \wedge dA + \frac{2}{3} A^3 \right)
\]

and one can verify, by use of Noether’s theorem etc., that the above symplectic form and Hamiltonians arise from this action.

3. Geometric quantization of \( A_+ \)

Given a classical phase-space, the pre-quantum data consist of a complex line bundle equipped with metric and compatible connection \( \nabla \), such that the curvature of \( \nabla \) is equal to a fixed multiple of the symplectic form. Given also a polarisation, the quantum phase space is defined to be the vector space of sections of the pre-quantum line bundle that are flat along the leaves of the polarisation. In the case of a complex polarisation, this forces one to look at the holomorphic
sections. Our task in this section is to quantize $\mathcal{A}_+$ in this sense and to consider the extent to which the ‘classical symmetries’ of $\mathcal{A}_+$ can be implemented as symmetries of the quantum phase space.

In order to determine pre-quantum data, one can appeal to general theorems asserting their existence under appropriate circumstances. We prefer, however, to define these data explicitly using Quillen’s construction of the determinant line-bundle of a family of $\bar{\partial}$-operators. This theory applies in the first instance to the space of $\bar{\partial}$-operators over a compact Riemann surface. We shall reduce to this case by extending $\bar{\partial}_A \in \mathcal{A}_+$ in a standard fashion to give a discontinuous $\bar{\partial}$-operator on a bundle over $\mathbb{CP}^1$. This is essentially equivalent to the imposition of Atiyah-Patodi-Singer type boundary conditions on $\bar{\partial}_A$ as an operator over $D_+$. The small price we pay for this is the discontinuity in the $\bar{\partial}$-operator over $\mathbb{CP}^1$. We explain in §3.3 why this does not cause any major difficulties.

First, however, we shall give a brief review of Quillen’s construction, and describe in particular how it simplifies when the base is $\mathbb{CP}^1$. The last part of this section is concerned with the implementation of classical symmetries on the quantum phase spaces and contains the formulae needed for our subsequent definition of the $\tau$-function.

3.1. Quillen’s determinant construction. In this section we first review Quillen’s construction of the determinant line-bundle over the space of $\bar{\partial}$-operators over a compact complex 1-dimensional manifold (Riemann surface) $M$, explaining in particular how the construction simplifies when $M = \mathbb{CP}^1$.

Let $E \to M$ be a smooth complex vector bundle of rank $n$ over a compact Riemann surface $M$. As before, $\mathcal{A}$ is the space of all $\bar{\partial}$-operators on $E$. We assume that the generic element of $\mathcal{A}$ is invertible (as a map $\Omega^{0,0}(M, E) \to \Omega^{0,1}(M, E)$), equivalently that the index of any element is 0. By Riemann–Roch this condition is just the constraint $\text{deg}(E) = n(\text{genus}(M) - 1)$.

Quillen shows how to define a holomorphic line-bundle $\text{Det} \to \mathcal{A}$ with a canonical holomorphic section $\sigma$, also denoted det. Intuitively, $\sigma(A)$ is the determinant of $\bar{\partial}_A$; in particular, $\sigma(A) \neq 0$ iff $\bar{\partial}_A$ is invertible and so is non-vanishing at generic points of $\mathcal{A}$.

The next step is to define a hermitian metric on $\text{Det}$; then by Chern’s theorem $\text{Det}$ will acquire the curvature of $\nabla$ compatible with the holomorphic structure. For this, additional choices must be made. Quillen picks hermitian metrics on $E$ and $M$ and defines a hermitian metric on $\text{Det}$ by $\zeta$-function regularization. He gives a formula for $\nabla \sigma$ (reproduced below) and proves that the curvature of $\nabla$ is the standard symplectic form $\Omega$ on $\mathcal{A}$:

$$\frac{i}{2\pi} \Omega(\theta^*, \theta) = \frac{i}{2\pi} \int \text{tr}(\theta^* \wedge \theta)$$

where $\theta \in \mathcal{B} = \Omega^{0,1}(M, \mathfrak{t}(E))$ represents a $(1, 0)$-tangent vector to $\mathcal{A}$. In terms of the local formulae

$$\bar{\partial}_A = dz(\partial_z + \alpha) \quad \partial_A = dz(\partial_z - \alpha^*),$$

Quillen gives the following formula for the covariant derivative of $\sigma$ in the direction $\theta$:

$$\nabla_\theta \sigma(A) = \sigma(A) \int \text{tr}(J_A \wedge \theta) \quad \text{where} \quad J_A = \frac{i}{2\pi} dz(\beta - \alpha^* - \frac{1}{2} \partial \log \rho),$$

this formula being valid at all points $A$ at which $\sigma(A) \neq 0$. In (3.3), $\alpha^*$ is as before, $\rho$ is the local conformal factor for the metric $ds^2 = \rho^2 |dz|^2$ on $M$ and $\beta dz$ is a globally defined 1-form which arises from the expansion near the diagonal of the Schwartz kernel $G_A(z, z')$ of $\bar{\partial}_A^{-1}$

$$G_A(z, z') = \frac{i}{2\pi} \frac{dz'}{z - z'}(1 + (z - z')\beta(z') - (z - z')\alpha(z') + \ldots).$$
3.2. Calculation when $M = \mathbb{CP}^1$. We now take $M = \mathbb{CP}^1$ and $E = \mathbb{C}^n \otimes H^{-1}$ where $H^{-1} \to \mathbb{CP}^1$ is the tautological bundle $\mathbb{C}^2 \to \mathbb{CP}^1$ (dual to the hyperplane bundle $H$) and is the unique line bundle of degree $-1$. Let $A$ be the space of $\bar{\partial}$-operators on $E$. Any element of $A$ has index zero and the generic element is invertible. Note that in this case the space $B$ is equal to $\Omega^{0,1}(\mathbb{CP}^1, \text{End}(E))$ as the twist by $H^{-1}$ cancels.

By Grothendieck’s theorem, if $\bar{\partial}_A$ is such an invertible element, there exists a gauge transformation $g$ such that

$$\bar{\partial}_A = g \bar{\partial}_0 g^{-1},$$

where $\bar{\partial}_0$ is the standard $\bar{\partial}$-operator on $\mathbb{C}^n \otimes \mathcal{O}(-1)$. The main purpose of this section is to use $g$ to simplify Quillen’s formulae.

Now $\bar{\partial}_0^{-1}$ has Schwartz kernel given by

$$G_0 = \frac{i}{2\pi} \frac{dz'}{z - z'}.$$

We have written this in terms of local coordinates $(z, z') \in \mathbb{CP}^1 \times \mathbb{CP}^1$. However, $G_0$ extends canonically to define a smooth section of $\text{pr}^*_1 E \otimes \text{pr}^*_2 [A^{1,0} \otimes E^*]$ over $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ and hence defines canonically an operator

$$\Omega^{0,1}(\mathbb{CP}^1, E) \to \Omega^{0,0}(\mathbb{CP}^1, E).$$

Using the gauge transformation in (3.4), we have that the Schwartz kernel of $\bar{\partial}_A^{-1}$ is just $G_A(z, z') = g(z) \circ G_0(z, z') \circ g(z')^{-1}$ and expanding near the diagonal we find

$$\beta dz = \bar{\partial}gg^{-1}.$$

In what follows we shall restrict to the subspace $A_0$ of operators which differ from $\bar{\partial}_0$ by trace-free elements $B_0$ of $B$. In that case we can assume that $g$ in (3.4) has unit determinant so that $\beta$ in (3.5) is also trace-free.

The Quillen connection requires choices of hermitian structures on $E$ and $\mathbb{CP}^1$ (although the final $\tau$-function formulae will be independent of them). With such choices, the unitary connection corresponding to $\bar{\partial}_A$ is given by $d_A = \bar{\partial}_A + \partial_A$ where

$$\partial_A = (g^*)^{-1} \circ \bar{\partial}_0 \circ g^*$$

and we obtain the formula

$$dz(\beta - \alpha^*) = \partial gg^{-1} + g^{-1} dg^* = g(h^{-1} dh)g^{-1}$$

where $h = g^*g$. Recall that $\text{End}(E)$ is canonically isomorphic to the bundle of endomorphisms of the trivial rank-$r$ bundle over $F_1$. Furthermore, the induced action of $\bar{\partial}_0$ on $\text{End}(E)$ coincides with that of the standard $\bar{\partial}$-operator on the trivial bundle. Thus we drop the distinction between $\bar{\partial}_0$ and $\bar{\partial}$ when acting on $g$, $g^*$ or $h$.

Substituting into (3.3) we obtain the basic formula

$$\nabla_\theta \sigma = \left( \frac{i}{2\pi} \int \text{tr}(h^{-1} \partial h \wedge (g^{-1} \partial g)) \right) \cdot \sigma$$

for any $\theta \in B_0$. Notice that the term in $\rho$ disappears because $\theta$ can be assumed to be trace-free. Other useful formulae are

$$d_A = g \circ \bar{\partial}_0 \circ g^{-1} + g^{-1} \circ \partial_0 \circ g^* = g \circ (\bar{\partial}_0 + \partial_0 + h^{-1} \partial h) \circ g^{-1},$$

so that

$$F_A = g \bar{\partial}_0 g^{-1} + g \bar{\partial}(h^{-1} \partial h)g^{-1}$$

where

$$F_A = \bar{\partial}_A \partial_A + \partial_A \bar{\partial}_A.$$
is the curvature 2-form. By our choice of metric on \( E, F_0 \) is multiple of the identity. Hence (3.8) can also be written:

\[
(3.10) \quad g^{-1}(F_A - F_0)g = \bar{\partial}(h^{-1}\partial h).
\]

3.3. \( A_+ \) as a space of connections with jumps on \( \mathbb{CP}^1 \). Consider now \( \mathbb{CP}^1 = D_+ \cup D_- \) and the standard bundle \( E = \mathbb{C}^n \otimes H^{-1} \) over \( \mathbb{CP}^1 \). Denote by \( E_\pm \) the restrictions of \( E \) to \( D_\pm \). We may identify \( E_+ \) with the trivial bundle over \( D_+ \) in such a way that \( d_0 = d \). (We choose the metrics on \( H \) and \( \mathbb{C}^n \) to restrict to constant metrics over \( D_+ \).) Now given any element \( A \in A_+ \), we may regard it as a connection on \( E_+ \) and extend it by \( d_0 \) over \( D_- \) to the whole of \( E \). We shall denote this 'extension by zero' of \( d_A \) by \( d_A \); it is a connection on \( E \) with a simple jump discontinuity across \( \partial D_+ \). It will be seen later that such operators are required for the simplest formulation of various important ingredients such as the Fock space inner product etc.

We shall assume in what follows that Quillen's construction extends to the \( \bar{\partial} \)-operators with jump discontinuities that result by taking the \( (0,1) \)-part of the extension by zero of \( d_A \) and in particular that (3.3) continues to hold for such operators. As a partial justification for this, observe first that if \( \partial_h \) is a \( \bar{\partial} \)-operator on \( E \) with jump discontinuity at \( \partial D_+ \), then there is a continuous complex gauge transformation \( c \), say, of \( E \) such that \( c \cdot \partial_h \cdot c^{-1} \) is smooth. A sketch of the proof of this is as follows. If we can find, near any point \( p \) of \( \partial D_+ \) a continuous matrix-valued function \( u \) which solves the equation

\[
\bar{\partial}_z(1 + u) + \alpha(1 + u) = 0 \quad \text{near } p
\]

(in the sense of distributions) then the required gauge transformation \( c \) can be obtained by patching such solutions together by a partition of unity. However the usual proof (cf. for example Donaldson and Kronheimer (1990), Chapter 2) yields such a \( u \) that is locally in \( L^q \), for any \( q > 2 \). Since then \( \alpha u \) is also in \( L^q \), the ellipticity of \( \bar{\partial} \) ensures that \( u \) is actually in \( L^q \) near \( p \). By the Sobolev embedding theorem in 2 dimensions, such a \( u \) is continuous if \( q > 2 \).

In particular, the discontinuous \( \bar{\partial} \)-operator \( \partial_h \) is invertible, as an operator between appropriate Sobolev spaces, if and only if there exists a continuous (complex) gauge transformation \( g \) of \( E \) satisfying (3.4). It follows that the formula (3.3) makes sense in this case, the integrand being bounded on \( \mathbb{CP}^1 \). (The 1-form \( \beta \) is smooth away from \( \partial D_+ \), where it has at worst a jump discontinuity.) In order that \( \theta \) represent a tangent vector to \( A_+ \) we must take \( \theta \) to be the extension by zero to \( D_- \) of a 1-form that is smooth in \( D_+ \). Then the integral in (3.3) extends only over \( D_+ \).

It is perhaps worth pointing out that even if \( A \) vanishes near \( \partial D_+ \), so that its extension to \( \mathbb{CP}^1 \) is actually smooth, the gauge transformation of (3.4) does not in general vanish on \( D_- \). In particular the boundary integrals that we shall see below will not generally vanish even in this case.

This formulation is equivalent to considering a family of \( \bar{\partial} \)-operators on a Riemann surface with boundary, using Atiyah–Patodi–Singer boundary conditions to make such operators Fredholm. It would be interesting to consider the analytic issues involved in giving a more systematic derivation of the connection on a determinant line-bundle for this case. However, from now on, we take over (3.3) to calculate the covariant derivative of the section \( \sigma \) of \( \text{Det} \) over \( A_+ \) and use this formula without further comment.

3.4. Quantization and the action of gauge transformations and vector fields. Since \( \text{Det} \) is a holomorphic line bundle with connection whose curvature is the symplectic form, it is the prequantum line bundle. To obtain the quantum Hilbert space one must introduce a polarisation on \( A_+ \) and consider 'polarised' sections of \( \text{Det} \). In this case the polarisation is the complex structure, and so the space of holomorphic sections of \( \text{Det} \) will yield the quantum Hilbert space associated to the classical phase space \( A_+ \). In the following we wish to lift the action on \( A_+ \) of gauge transformations and diffeomorphisms of \( D_+ \) to act on holomorphic sections of \( \text{Det} \). In the case of gauge transformations, we will see that the action is immediately holomorphic and so acts
directly on the quantum Hilbert space. However, in the case of diffeomorphisms, only a subalgebra of vector fields acts holomorphically.

3.4.1. Lifting the action of gauge transformations. First we follow the standard recipe from geometric quantization to lift the action of the Lie algebra $g$ of infinitesimal gauge transformations to $\det$. If $u \in G$, the corresponding vector field on $A_+$ is $\delta_u = -d_A u$. The geometric quantization lift is

$$L_u = \nabla_u - iH_u \quad (3.11)$$

where we have written $\nabla_u = \nabla_{\delta_u}$ in order to simplify the notation.

**Proposition 3.** The recipe (3.11) gives the formula

$$\frac{\mathcal{L}_u \sigma}{\sigma} = -\frac{1}{2\pi i} \oint \text{tr}(dg^{-1} u). \quad (3.12)$$

**Proof.** From (3.6) and (3.4),

$$\frac{\nabla_u \sigma}{\sigma} = \frac{1}{2\pi i} \int \text{tr}(h^{-1} \partial h \wedge g^{-1} \tilde{\partial} A u g) = \frac{1}{2\pi i} \int \text{tr}(h^{-1} \partial h \wedge \tilde{\partial} (g^{-1} u g)).$$

In order to integrate by parts, note

$$d \text{tr}(h^{-1} \partial h g^{-1} u g) = \text{tr}(\tilde{\partial}(h^{-1} \partial h) g^{-1} u g) - \text{tr}(h^{-1} \partial h \wedge \tilde{\partial} (g^{-1} u g)).$$

But from (3.10) and the fact that $u$ is trace-free, the first term on the right hand side is equal to $\text{tr}(F_{\alpha} u)$, one of the terms in the Hamiltonian. On the left hand side use (3.7) to write

$$gh^{-1} \partial h g^{-1} = d_A - g \cdot d \cdot g^{-1} = d_A - d_0 - d_0 g g^{-1}.$$ 

Hence we find

$$\frac{\nabla_u \sigma}{\sigma} = iH_u(A) - \frac{1}{2\pi i} \oint \text{tr}(dg^{-1} u).$$

**Remark** (1) Assuming that the connection on $\det \to A_+$ really is globally defined, this formula is also globally defined since both the connection and Hamiltonian are.

(2) It is clear from the formula that this action of $\tilde{g}$ on $\det$ is holomorphic, for its action upon $\sigma$ is multiplication by a function that is holomorphic on the dense open set of $A_+$ where $\sigma \neq 0$. It follows that one obtains a holomorphic action of $g^c$ also, simply by replacing $u$ by a complex element of $g^c$.

(3) In this set-up we have worked with bundles framed over $D_-$. In the parallel development of this work, Mason, Singer & Woodhouse (2000), this framing is viewed as coming from another solution. We shall not pursue that viewpoint in this paper.

3.5. The first definition of the $\tau$-function. The framed subalgebra $g^c_0$ clearly preserves $\sigma$. We will see that the natural interpretation is of $\sigma$ as a ‘vacuum state’. The framed gauge transformations, being the true degrees of gauge freedom, fix the vacuum state, while the unframed gauge transformations shift it. The $\tau$-function in its most general form is a function on an orbit in the phase space under the action of some submanifold of $G^c$. It is the value of $\sigma$ on that orbit expressed in an invariant frame of $\det$. Later we will give a quantum field theoretic formulation in which it measures the amplitude of the two vacuum states related by a complex gauge transformation.

More precisely, given a submanifold (usually subgroup) of $G^c$ parametrized by $t \mapsto G(t)$, and an initial connection $\alpha$, define

$$\tau(t) = \widehat{G(t)^{-1}} \sigma(G(t)\alpha) \quad (3.13)$$

where $\widehat{G(t)}$ is the action of $G(t)$ lifted to $\det$. This only defines $\tau$ as a function up to an overall constant as it is an element of the fibre of $\det$ at $\tilde{\partial}g(0)\alpha$ which is not canonically trivial. Furthermore,
\( \tau \) is not well defined in general since the action of \( \mathcal{G}^c \) is generally projective, and so we must require that the submanifold of \( \mathcal{G}^c \) must be one on which the central extension \( 2.3 \) splits.

This can be seen more clearly in the infinitesimal version of this definition. Differentiation of \( (3.13) \) leads to a 1-form \( 'd \log \tau' \) on \( \tilde{\mathcal{G}}^c \) given by the formula

\[
(3.14) \quad L_u \sigma = (u \, j \, d \log \tau) \sigma
\]

where \( u = g^{-1} \partial_x g \). The 1-form \( d \log \tau \) always exists, and when it is closed on restriction to some submanifold of \( \tilde{\mathcal{G}}^c \) it defines \( \tau \) on that submanifold up to a constant. However, in spite of the notation, \( d \log \tau \) is not generally closed, instead we have

\[
v \, j \, u \, j \, d \log \tau = c(u, v)
\]

where \( c(u, v) \) is the cocycle \( (2.3) \) for the central extension, and so the submanifold of \( \mathcal{G}^c \) must be one on which the cocycle vanishes.

This is sufficient for the definition of the \( \tau \)-function of the KdV equation and the reader may wish to skip ahead to section 6 for this. In the remainder of this section, we treat the quantization of diffeomorphisms and in the next section we give a similar treatment of the quantization of the reduced phase space \( A^\text{red} = A_+|_{F_A=0}/\mathcal{G}_0 \).

3.6. Lifting the action of vector fields. For the action of the algebra of vector fields \( \text{Vect}_0(D_+) \), we substitute \( X = L_\xi A \) in \( (3.13) \),

\[
(3.15) \quad \frac{2\pi i \nabla_X \sigma}{\sigma} = - \left( \text{tr}(h^{-1} \partial h \wedge (g^{-1} A(\xi)) g) - \int \text{tr}(h^{-1} \partial h \wedge \xi \, j \, \bar{\partial}(h^{-1} \partial h)) \right)
\]

where we have rearranged the first term as in the proof of Proposition 3. In order to integrate by parts in the first term, note the identity

\[
d \text{tr}(h^{-1} \partial h g^{-1} A(\xi) g) = \text{tr}(F_A A(\xi)) - \text{tr}(h^{-1} \partial h \wedge \bar{\partial}(g^{-1} A(\xi) g))
\]

To integrate the second term in \( (3.13) \) by parts, note

\[
d \left( \frac{1}{2} \text{tr}(h^{-1} h_z)^2 \xi^{(1,0)} \wedge dz \right) = \text{tr}(h^{-1} h_z \bar{\partial}(h^{-1} h_z) \xi^{(1,0)} \wedge dz + \frac{1}{2} \text{tr}(h^{-1} h_z)^2 \bar{\partial} \xi^{(1,0)} \wedge dz
\]

where the first term on the right hand side is equal to the second term on the right hand side of \( (3.13) \). Combining these with the formula \( (2.17) \), we obtain

\[
(3.16) \quad \frac{\nabla_X \sigma}{\sigma} - i H_\xi (A) = \frac{1}{4\pi i} \int \text{tr}(h^{-1} h_z)^2 \bar{\partial} \xi^{(1,0)} \wedge dz + \frac{1}{4\pi i} \int \text{tr}(2gh^{-1} \partial h g^{-1} A(\xi) - A A(\xi) - (h^{-1} h_z)^2 \xi^{(1,0)}) dz.
\]

This is simplified, using the fact that, with \( \xi \) tangent to the boundary, \( \xi^{(1,0)}/z' \) is real on the boundary, where \( ' \) denotes the derivative with respect to some parameter along the boundary, and by using \( A = g(h^{-1} \partial_x h) g^{-1} z' - g' g^{-1} \), as follows

\[
(3.17) \quad \frac{L_\xi \sigma}{\sigma} = \frac{1}{4\pi i} \int \text{tr}(g^{-1} (\xi g) g^{-1} dg) + \frac{1}{4\pi i} \int \text{tr}[(h^{-1} h_z)^2] \bar{\partial} \xi^{(1,0)} \wedge dz.
\]

where we have put \( L_\xi \sigma = \nabla_X \sigma - i H_\xi (A) \sigma \).

Remark This action is not holomorphic in general since the integral over \( D_+ \) does not depend holomorphically on \( A \) (indeed the action on \( A_+ \) was not holomorphic either). The prequantum operator therefore does not send polarised (holomorphic) sections to polarised sections and more work needs to be done to quantize the action of a general diffeomorphism.
We will not, however, be interested in the action of general vector fields on the disc, but of \( \text{HolVect}(D_+) \) (i.e. real vector fields \( \xi \) for which \( \xi^{(1,0)} \) is holomorphic). We have

**Proposition 4.** There exists a holomorphic action of \( \text{HolVect}(D_+) \) on the determinant line bundle given by

\[
\frac{L_{\xi} \sigma}{\sigma} = -\frac{1}{4\pi i} \int \text{tr}(g^{-1}(\xi g)^{-1}dg)
\]

**Proof.** This is not completely trivial as vector fields in \( \text{HolVect}(D_+) \) are not in general tangent to the boundary. We get around this by representing \( \text{HolVect}(D_+) \) as a quotient of complexified vector fields that are tangent to the boundary whose \((1,0)\)-part is holomorphic, by complexified vector fields, tangent to the boundary, whose \((1,0)\)-part is zero. The formula can be extended to complex vector fields (i.e. \( \xi = \xi^{(1,0)}\partial_z + \xi^{(0,1)}\partial_{\bar{z}} \) with \( \xi^{(1,0)} \) independent of \( \xi^{(0,1)} \), except on the boundary where \( \xi^{(1,0)}\bar{z}' = \xi^{(0,1)}z' \)) by requiring complex linearity. We can now restrict to the Lie algebra of complex vector fields such that \( \xi^{(1,0)} \) is holomorphic on \( D_+ \). This latter subalgebra acts holomorphically as it is defined by the condition \( \bar{\partial}\xi^{(1,0)} = 0 \), and so our formula simplifies to the boundary integral (3.18) analogous to the case for complex gauge transformations. Thus, naive geometric quantization does quantize the action of this algebra of complex vector fields tangent to \( \partial D_+ \) with holomorphic \((1,0)\) part.

To obtain an action of \( \text{HolVect}(D_+) \), observe that the subalgebra of complex vector fields of the form \( \xi^{(0,1)}\partial_z \), with \( \xi^{(0,1)}|_{\partial D_+} = 0 \), is a Lie algebra ideal, acts trivially and the quotient of the algebra of complex vector fields tangent to \( \partial D_+ \) and with \( \xi^{(1,0)} \) holomorphic by this subalgebra is \( \text{HolVect}(D_+) \). We have therefore obtained the desired holomorphic action of \( \text{HolVect}(D_+) \) on the determinant line bundle. \( \square \)

### 4. Reduction of phase space

All the constructions of the last two sections are invariant under the action of \( \mathcal{G}_0 \) and the final formulae are invariant under the action of \( \mathcal{G}_0^\circ \); it is therefore natural to attempt to reformulate the constructions in terms of a reduced phase-space obtained by the symplectic reduction in the case of \( \mathcal{G}_0 \) or the straight quotient in the case of \( \mathcal{G}_0^\circ \). In the following we consider the case when \( D_+ \) is just a disc. Then these reductions both give the same reduced phase-space \( \mathcal{A}_{\text{red}} = \text{LSU}_n/\text{SU}_n \), where \( \text{LSU}_n = \text{map}(S^1, \text{SU}_n) \) is the loop group of \( \text{SU}_n \), as a consequence of one of the factorization theorems for loop groups. When \( D_+ \) is the complement of more than 1 disc, the symplectic and complex quotients are still the same, and the general theory follows in much the same way, but the connection with loop groups is less relevant. The connection with loop groups arises as follows (see Donaldson 1992 for a full discussion).

From the complex point of view, the reduced phase-space \( \mathcal{A}_{\text{red}} = A_+/\mathcal{G}_0^\circ \). This follows by noting that if \( \bar{\partial}_\alpha \in A_+ \), then there exists a complex gauge transformation \( F \), smooth up to the boundary, such that

\[
\bar{\partial}_\alpha = F \cdot \bar{\partial} \cdot F^{-1}.
\]

This \( F \) is not unique: the group \( \text{LSL}_n(\mathbb{C}) \) of holomorphic maps \( D_+ \to \text{SL}_n(\mathbb{C}) \) that are smooth up to the boundary acts by multiplication on the right, \( F \mapsto FG \) preserving \( (4.1) \). Elements of \( \mathcal{G}_0^\circ \) act on the left leaving invariant the boundary value of \( F \). Furthermore, any two \( F \)'s with the same boundary values are related by an element of \( \mathcal{G}_0^\circ \). Hence \( A_+/\mathcal{G}_0^\circ \) is the homogeneous space \( \text{LSL}_n(\mathbb{C})/\text{LSL}_n(\mathbb{C}) \).

The symplectic reduction of \( A_+ \) is obtained by dividing the zero-set of the moment map \( \mu : A_+ \to T^*_C \mathcal{G}_0 \)

\[
\langle u, \mu(A) \rangle = H_u(A)
\]

for the action of \( \mathcal{G}_0 \) on \( A_+ \), by \( \mathcal{G}_0 \). Note that the boundary term in \( H_u(A) \) is absent if \( u \in \mathfrak{g}_0 \), so the zero-set of \( \mu \) consists exactly of the flat unitary connections on \( D_+ \). Any such connection is
gauge equivalent to the trivial connection \( d \),

\[
(4.2) \quad d_A = \gamma \cdot d \cdot \gamma^{-1}.
\]

This time, \( \gamma \in G \) is determined up to the action of the gauge transformations, \( \gamma \mapsto \gamma \eta \), where \( \eta \) is constant. The group \( G_0 \) acts on the left, fixing the boundary value of \( \gamma \). Hence \( \mu^{-1}(0)/G_0 \) is identified with \( LSL_n/SU_n = \Omega SU_n \).

In the theory of loop groups, Pressley & Segal (1986), we have that

\[
LSL_n(C)/L^+SL_n(C) = SU_n/SU_n,
\]

i.e., for any \( F \) in \( LSL_n(C) \) there is a \( G \) in \( L^+SL_n(C) \) unique up to constants such that \( FG \) is unitary. This shows that the real and complex reductions give the same answer when \( D_+ \) is a standard disc.

The equivalence between the symplectic and complex reduction is true also in our more general situation when the boundary of \( D_+ \) is a disjoint union of circles.

**Remark** If we consider the smaller phase space of \( \bar{\partial} \)-operators with compact support in \( D_+ \), then this correspondence fails. Indeed a flat connection with compact support is gauge equivalent to \( d \) by a gauge transformation that is necessarily constant near \( \partial D_+ \); so all such flat connections are in one orbit of \( G_0 \).

We shall now reduce all the objects so far considered by \( G \). For each object (e.g. symplectic form, Hamiltonian, prequantum data) we shall restrict to the zero-set of \( \mu \) and then check that this restriction is \( G_0 \)-invariant. The end result are explicit formulae for these reduced objects in terms of the loop space \( \Omega SU_n \).

We first give more details of the identification between \( \mathcal{A}^{\text{red}} \) and \( \Omega SU_n \). Fix \( A \in \mu^{-1}(0) \), so that \( A \) is a flat connection in \( \mathcal{A}_+ \). Then there exists \( \gamma \in G \), unique up to multiplication by a (constant) element of \( SU_n \), with \( d_A = \gamma \cdot d \gamma^{-1} \) so that

\[
\mu^{-1}(0) = G/SU_n
\]

where \( SU_n \) here denotes the subgroup of constant gauge transformations. Thus any flat infinitesimal deformation of \( A \) is given by a gauge transformation and so

\[
(4.3) \quad T_A \mu^{-1}(0) = \{-d_A u : u \in \mathfrak{g} = T_e G \}.
\]

Note that the tangent space of \( G \) at a point \( \gamma \) is identifiable with the Lie algebra \( \mathfrak{g} \), \( \mathfrak{g} \mapsto T_\gamma G \), by \( u \mapsto u \gamma \) and this is compatible with (4.3) (in physicists’ notation \( \delta_A \gamma = u \)). One can check that the left-action of \( G \) on \( G/SU_n \) coincides with the gauge action of \( G \) on \( \mu^{-1}(0) \).

Inside \( T_A \mu^{-1}(0) \), there is the vertical tangent space \( T_A^V \) which is generated by the \( G_0 \)-gauge orbits. By definition

\[
T_A^V = \{-d_A u : u \in \mathfrak{g}_0 \}.
\]

The quotient \( T_A \mu^{-1}(0)/T_A^V = T_{[A]} \mathcal{A}^{\text{red}} \), is thus identified with the space of boundary values \(-d_A u |_{\partial D_+} \). The exact sequence of tangent spaces

\[
0 \to T_A^V \to T_A \mu^{-1}(0) \to T_{[A]} \mathcal{A}^{\text{red}} \to 0
\]

gets identified with the exact sequence

\[
0 \to \mathfrak{g}_0 \to \mathfrak{g}/\mathfrak{su}_n \to \mathfrak{gsu}_n/\mathfrak{su}_n \to 0.
\]

We now compute the reduced symplectic form \( \Omega^{\text{red}} \) on \( \mathcal{A}^{\text{red}} \). With the above identifications, we take tangent vectors \(-d_A u \) and \(-d_A v \) at \( A \) corresponding to elements \( u, v \in \mathfrak{g} \). Then

\[
(4.4) \quad \Omega(d, u, d, v) = \frac{1}{2\pi} \int tr(d, u \wedge d, v) = \frac{1}{2\pi} \int tr(u, d, v) = -\frac{1}{2\pi} \int tr(v, d, u)
\]

where here and below we have used \( u \) and \( v \) to denote \( u |_{\partial D_+} \) and \( v |_{\partial D_+} \) respectively. This is equivalent to the standard symplectic form on \( \mathcal{A}^{\text{red}} \) from Pressley–Segal (1986) (p. 147). There this form is written down at the identity coset in \( \mathcal{A}^{\text{red}} \) and propagated as a left-invariant form.
over the whole of $\mathcal{A}^{\text{red}}$. This agrees with the above formula since, starting with the Pressley–Segal definition,
\[
\Omega^\text{red}_\gamma(u_\gamma, v_\gamma) = \Omega^\text{red}_\gamma(\gamma^{-1}u_\gamma, \gamma^{-1}v_\gamma) = \int \text{tr}(\gamma^{-1}u_\gamma d(\gamma^{-1}v_\gamma)) = \int \text{tr}(udv)
\]
and the last expression is exactly as in (4.4).

Similarly for the Hamiltonians we find, for the action of gauge transformations,
\[
H^\text{red}_u(\gamma) = -\frac{1}{2\pi} \oint \text{tr}(d\gamma \gamma^{-1}u)
\]
from (2.8) and for the action of diffeomorphisms from (2.10)
\[
H^\text{red}_\xi(\gamma) = -\frac{1}{4\pi} \oint \text{tr}(d\gamma \gamma^{-1} \cdot \xi(\gamma)\gamma^{-1}).
\]

To summarize,

**Proposition 5.** The reduced symplectic form $\Omega^\text{red}$ on $\mathcal{A}^{\text{red}}$ is given by
\[
\Omega^\text{red}_\gamma(u, v) = \frac{1}{2\pi} \oint \text{tr}(udv).
\]
The action of $\text{LSU}_n$ upon $\mathcal{A}^{\text{red}}$ is generated by the Hamiltonian
\[
H_u(\gamma) = -\frac{1}{2\pi} \oint \text{tr}(d\gamma \gamma^{-1}u);
\]
the action of $\text{Vect}(S^1)$ is generated by the Hamiltonian
\[
H_\xi(\gamma) = -\frac{1}{4\pi} \oint \text{tr}(d\gamma \gamma^{-1} \cdot \xi(\gamma)\gamma^{-1}).
\]

**Remark** The phase-space $\mathcal{A}^{\text{red}}$ is that for Wess-Zumino-Witten theory in two dimensions. The Lagrangian does not have a local, invariantly defined formula, but its first variation does:
\[
\delta L = \int_D \text{tr}(g^{-1}\delta g(d(g^{-1}\delta g))).
\]
The symplectic form and Hamiltonians can be derived from this Lagrangian by standard applications of Noether’s theorem.

### 4.1. Quantization of the reduced space.

We now proceed to analyze the reduction of the prequantum data, i.e., the Quillen connection on $\text{Det}$ and the lifts of actions of gauge transformations and diffeomorphisms for the reduced phase space.

We first simplify Quillen’s formula for $J_A$ in (3.3). Writing $dA = \gamma \cdot \partial \gamma^{-1}$ for the flat connection $A$ as before, we have $\alpha^* = -\partial \gamma \gamma^{-1}$ but to determine $\beta$ we need the gauge transformation $g$ in (3.4). This amounts to solving
\[
\gamma \cdot \bar{\partial} \cdot \gamma^{-1} = g \cdot \bar{\partial} \cdot g^{-1} \text{ in } D_+, \bar{\partial} = g \cdot \bar{\partial} \cdot g^{-1} \text{ in } D_-.
\]
(Strictly, we should have $\bar{\partial}_0$ here.) From these equations, $g$ is holomorphic in $D_-$ and $g^{-1}\gamma$ is holomorphic in $D_+$. Thus $g$ is determined by the Birkhoff factorization
\[
\gamma = gg^+_+ \text{ on } \partial D_+,
\]
where $g_+ \in L^+ \text{SL}_n(\mathbb{C}), g \in L^- \text{SL}_n(\mathbb{C}); g_+$ and $g$ are the positive and negative-frequency parts of $\gamma$ and $g$ is then continued over $D_+$ by requiring (3.3) to hold on $D_+$.

Using this we obtain
\[
\beta - \alpha^* = \partial g g^{-1} + \partial \gamma \gamma^{-1}
\]
or in terms of $g_+$ and $\gamma$,
\[
\beta - \alpha^* = \gamma(g_+ g_+^{-1})\gamma^{-1}.
\]
Now if $u \in \mathfrak{g}$ is used as before to define a tangent vector $-d_A u$ to $\mu^{-1}(0)$, we get the formula
\[
\nabla_u \sigma \sigma = -\frac{1}{2\pi} \int \text{tr}[\gamma(\partial g_+ g_+)^{-1} \gamma^{-1} \wedge (-d_A u)].
\]
To simplify this, note that we can replace $d_A$ by $\bar{\partial}_A$; writing this derivative out in terms of $\gamma$, we get
\[
\nabla_u \sigma \sigma = \frac{1}{2\pi} \int \text{tr}(\partial g_+ g_+^{-1}) \wedge (\bar{\partial}(\gamma^{-1} u\gamma)).
\]
Since $g_+$ is holomorphic in $D_+$ the obvious integration by parts reduces this to a boundary integral,
\[
(4.6) \quad \nabla_u \sigma \sigma = -\frac{1}{2\pi} \oint (dg_+ g_+^{-1})^{\gamma^{-1} u\gamma} = -\frac{1}{2\pi} \oint (dg g^{-1} - d\gamma g^{-1})u\gamma.
\]
This vanishes if $u \in \mathfrak{g}_0$; it follows that the prequantum data descends to $A^{\text{red}}$. Moreover, the vacuum-state $\sigma$ also descends to define a vacuum state in the reduced theory. We shall denote this by $\sigma$, rather than by $\sigma^{\text{red}}$. Notice that (4.6) depends only on the boundary value of $u$ and $\gamma$, so it could be used as a new definition of pre-quantum data over $A^{\text{red}} = \Omega SU_n$. We have proved

**Proposition 6.** The symplectic reduction of Quillen’s connection yields a connection on the determinant line-bundle over $\Omega SU_n$ whose curvature is $\Omega^{\text{red}}$.

Although this is probably well known ($A^{\text{red}}$ and its determinant bundle have been well studied) we are not aware of a previous occurrence of such a result in the literature. It gives a direct link between Quillen’s construction and the determinant line-bundle over $\Omega SU_n$. The latter is usually defined by expressing $\Omega SU_n$ as a grassmanian and restricting the determinant line-bundle of the grassmanian of Hilbert space.

Using the same recipe as before for the lift of an action, we find that the variation of $\sigma(-\sigma^{\text{red}})$ under the action of $LSU_n$ on $\Omega SU_n$ is given by:
\[
(4.7) \quad \mathcal{L}_u \sigma \sigma = -\frac{1}{2\pi} \oint (dgg^{-1}u)
\]
where $g$ is the ‘negative-frequency part’ of $\gamma$. Similarly for diffeomorphisms, we get
\[
\mathcal{L}_\xi \sigma \sigma = -\frac{1}{2\pi} \oint \text{tr}[\xi d\gamma g^{-1} \xi^{-1} \gamma^{-1} - \frac{i}{4} d\gamma g^{-1} \xi \gamma^{-1}].
\]
This can be simplified almost exactly as for the previous calculation for the action of the diffeomorphism group: cf. (4.14) and the ensuing calculations. One obtains, finally,
\[
(4.8) \quad \mathcal{L}_\xi \sigma \sigma = -\frac{1}{4\pi} \oint (g' g^{-1})^2 -(g' + g_+^{-1})^2.
\]
As before, this does not preserve the complex structure in general (for $g_+$ does not depend holomorphically on $\gamma$) but when $\xi$ is the boundary value of a holomorphic vector field in $D_+$, the latter term drops out, leaving a holomorphic lift.

5. **Construction of the Fock space**

The space $\mathcal{F}_1$ of all holomorphic sections of Det over $A_+$ will be too large for the Fock space as we only wish to consider sections that are invariant under $\mathfrak{g}_0$. The space $\mathcal{F}_0$ of holomorphic sections of Det over $A^{\text{red}}$ is the space of such invariant sections, but will now be too large because we need distinguish the subspace $\mathcal{F}$ of ‘square-integrable’ holomorphic sections. We shall obtain $\mathcal{F} \subset \mathcal{F}_0$ as the completion of a dense subset $C \subset \mathcal{F}_0$. The set $C$ will be a set of coherent states (which, in particular, is not a linear subspace of $\mathcal{F}_0$). The inner product is constructed using Segal’s gluing formula for determinants. The following discussion of these matters runs roughly parallel to that of §10 of Segal and Wilson 1985.

Let $\mathbb{C}^p = D_+ \cup D_-$ as before, and assume that $D_+$ are standard discs with common boundary $S^1 = \{z \in \mathbb{C}, |z| = 1\}$. Let $\rho$ be the anti-holomorphic involution of $\mathbb{C}^p$ given by reflection in $S^1$. To simplify this, note that we can replace $d_A$ by $\bar{\partial}_A$; writing this derivative out in terms of $\gamma$, we get
\[
(4.9) \quad \nabla_u \sigma \sigma = \frac{1}{2\pi} \int \text{tr}(\partial g_+ g_+^{-1} \gamma^{-1} \wedge (\bar{\partial}(\gamma^{-1} u\gamma))).
\]
This vanishes if $u \in \mathfrak{g}_0$; it follows that the prequantum data descends to $A^{\text{red}}$. Moreover, the vacuum-state $\sigma$ also descends to define a vacuum state in the reduced theory. We shall denote this by $\sigma$, rather than by $\sigma^{\text{red}}$. Notice that (4.6) depends only on the boundary value of $u$ and $\gamma$, so it could be used as a new definition of pre-quantum data over $A^{\text{red}} = \Omega SU_n$. We have proved

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integrable systems can be obtained by considering reductions of the self-dual Yang-Mills equations, and we have arrived at the QFT formula (3.1). It can be shown (Segal 1989 & 1991) that the determinant lines of the $\tilde{\mathbf{u}}$ are related by
\[ \text{Det}(\tilde{\mathbf{u}}) = \text{Det}(\tilde{\mathbf{u}}_{\alpha}) = \text{Det}(\tilde{\mathbf{u}}_{\alpha'}) \text{ Det}(\tilde{\mathbf{u}}_{\alpha}). \]
In particular
\[ \text{Det}(\tilde{\mathbf{u}}_{\alpha}) = \text{Det}(\tilde{\mathbf{u}}_{\alpha'}) \text{ Det}(\tilde{\mathbf{u}}_{\alpha'}) \]
is canonically the complexification of an oriented real line; the real positive elements are those of the form $\mathbf{u} \otimes u$, for $0 < u \in \text{Det}(\tilde{\mathbf{u}}_{\alpha})$. Furthermore, we have
\[ \text{det}(\tilde{\mathbf{u}}_{\alpha}) > 0 \]
for every $\alpha$.

For fixed $\alpha$,
\[ \text{det}(\tilde{\mathbf{u}}_{\alpha}) \in \text{Det}(\tilde{\mathbf{u}}_{\alpha'}) \text{ Det}(\tilde{\mathbf{u}}_{\alpha'}) \]
depends holomorphically on $\alpha'$ and so defines a ray in $\mathcal{F}_0$. A genuine state arises by fixing a non-zero element $\Psi(\alpha) \in \text{Det}^* (\tilde{\mathbf{u}}_{\alpha})$; since it depends anti-holomorphically on $\alpha$, it is natural to write it as a 'bra' $\langle \Psi(\alpha) |$. The operation of evaluation at $\alpha'$ gives an element of $\text{Det}(\alpha')$ that given $\Psi(\alpha') \in \text{Det}^* (\alpha')$ we can obtain a complex number. This operation can be thought of as evaluation of the 'bra' against the 'ket' $| \Psi(\alpha') \rangle$, and we have the suggestive formula
\[ \langle \Psi(\alpha)|\Psi(\alpha')\rangle = \text{det}(\tilde{\mathbf{u}}_{\alpha}) \Psi(\alpha) \Psi(\alpha') \in \mathbb{C}, \]
and $\langle \Psi(\alpha)|\Psi(\alpha)\rangle$ is positive definite. We see that $\Psi(\alpha)$ plays the role of a type of coherent state corresponding to $\tilde{\mathbf{u}}_{\alpha}$.

We can now define $\mathcal{F}$ to be the completion inside $\mathcal{F}_0$ of the linear span of all the $\Psi(\alpha)$'s with respect to the inner product $\langle \Psi(\alpha)|\Psi(\alpha') \rangle$. This is the Fock space for our theory. Note that $\mathcal{F}$ will not be dense in $\mathcal{F}_0$ as all elements of $\mathcal{F}$ are invariant under the group of based gauge transformations, and this will certainly not be the case for $\mathcal{F}_0$.

Finally we can give the quantum-field-theoretic interpretation of formula (3.14) for the $\tau$-function. The holomorphic section $\sigma$ corresponds to the vacuum 'bra' $\langle \Psi(0) \rangle$, the coherent state based on $\tilde{\mathbf{u}}_{\alpha}$. A family of elements $G(t)$ of either the gauge or diffeomorphism group depending on a parameter $t$ acts on a coherent state vector by sending $| \Psi(\alpha) \rangle$ to $| G(t) \Psi(\alpha) \rangle$ which is the coherent state based on $G(t)\alpha$ with $G(t)\Psi(\alpha) \in \text{Det}^* (\tilde{\mathbf{u}}_{G(t)\alpha})$ given by the lifted action of $G(t)$ on $\text{Det}$. In this context, then,
\[ \tau = \langle \Psi(0)|G(t)\Psi(\alpha)\rangle = (\sigma(G(t)\alpha), G(t)\Psi(\alpha)) = (G(t)^{-1}\sigma(G(t)\alpha), \Psi(\alpha)) \]
where the last two pairings are between elements of $\text{Det}$ and $\text{Det}^*$. Modulo the irrelevant extra constant factor of $\Psi(\alpha)$, this is (3.13) and differentiation with respect to the parameter $t$ leads to the definition (3.14). If we now consider the case where the submanifold of $\mathcal{G}$ is an abelian subgroup with Lie algebra generators $\phi_1$ and $\phi_2$, then with parameters $(x, t)$ we have $g(x, t) = \exp(x\phi_1 + t\phi_2)$ and we have arrived at the QFT formula (3.1).

6. Integrable equations and their twistor description

In this section we review the twistor correspondence for the Bogomolny equations and its reductions appropriate to the KdV equation and the Ernst equations. Although a wider variety of integrable systems can be obtained by considering reductions of the self-dual Yang-Mills equations, reduction of the Bogomolny equations yields many of the most famous examples, the KdV equations, the Ernst equations, the Sine Gordon equation, the nonlinear Schrödinger equation, sigma
The Bogomolny equations and the Ward correspondence. For our purposes, the Bogomolny equations are best defined to be the integrability condition for the Lax pair
\[ L_0 = (\partial_x + A) - \lambda(\partial_v + B), \quad L_1 = (\partial_t + C) - \lambda(\partial_z + D) \]
where the independent variables \((v, x, t)\) are coordinates on \(\mathbb{C}^3\), and the dependent variables \(A, B, C\) and \(D\) are functions on \(\mathbb{C}^3\) with values in the Lie algebra of some gauge group, which will be \(SL(2, \mathbb{C})\) in the examples we will consider. In the context of Lax pairs, the affine Riemann sphere coordinate \(\lambda\) is more commonly known as the ‘spectral parameter’. More invariantly, one should think of \(L_0\) and \(L_1\) as differential operators on a trivial bundle over the correspondence space \(\mathcal{F}\). We will be interested in gauge-equivalence classes of such operators.

The natural symmetry group of the equations is the complex Euclidean group together with dilations associated to the metric \(ds^2 = dx^2 - 2dv \circ dt\). Later we will see that symmetries can be imposed so that the equations reduce to the KdV equation or the Ernst equation.

The Ward correspondence provides a 1:1 correspondence between solutions to the \(SL(2, \mathbb{C})\) Bogomolny equations on \(\mathbb{C}^3\), and rank-2 holomorphic vector bundles \(E \to \mathcal{Z}\) such that \(E\) is trivial over each twistor line\(^2\). This result is standard and will not be proved here (see for example Ward & Wells 1990, Mason & Woodhouse 1996). We shall, however, need some details of the correspondence and so recall briefly how it works.

**Obtaining a bundle on twistor space from a solution to the Bogomolny equations:**

Given a solution to the Bogomolny equations on \(\mathbb{C}^3\), we introduce the associated Lax pair \(L_0\) and \(L_1\) as in equation (6.2). We define a fibre \(EZ\) for \(Z \in \mathcal{Z}\) of the holomorphic vector bundle \(E \to \mathcal{Z}\) to be the space of solutions to the Lax pair over the null-plane in \(\mathbb{C}^3\) corresponding to \(Z\). (The integrability conditions ensure that \(EZ\) is a complex 2-dimensional vector space.) To be consistent with our subsequent conventions, the matrices in the Lax pair will be assumed to be acting on the right.

**Obtaining a solution to the Bogomolny equations from a bundle on twistor space:**

Suppose we are given a holomorphic bundle \(E \to \mathcal{Z}\), trivial on each twistor line. Pull \(E\) back to \(\mathcal{F}\), to obtain a bundle \(\hat{E} = q^*E\) that is canonically trivial over the fibres of \(\mathcal{F} \to \mathcal{Z}\). The pair \(V_0\) and \(V_1\) of vector fields are tangent to these fibres and the canonical triviality means that these have canonical global holomorphic lifts to the bundle, \(L_0\) and \(L_1\). We have assumed that \(E\) is trivial over each \(\mathbb{CP}^1\) in \(\mathcal{Z}\) so that \(\hat{E}\) is trivial over each \(\mathbb{CP}^1\) fibre of \(\mathcal{F} = \mathbb{C}^3 \times \mathbb{CP}^1\) over \(\mathbb{C}^3\) and so we can trivialize \(\hat{E}\) over \(\mathcal{F}\). In such a trivialization \(L_0\) and \(L_1\) will be holomorphic in \(\lambda\) with a simple pole at \(\lambda = \infty\) and so must take the form as given in (6.2). Since \((L_0, L_1)\) are gauge equivalent to \((V_0, V_1)\) in a frame pulled back from \(\mathcal{Z}\), they must commute.

To make this transform more explicit, we must first choose one of the following explicit presentations of \(E \to \mathcal{Z}\).

**Čech presentation:** Cover \(\mathcal{Z}\) with two open sets, \(U_\pm = \{ (\mu, \lambda) | |\lambda|^{\pm1} < 1 + \varepsilon \}\) so that \(E\) is trivial over \(U_+\) and \(U_-\). The bundle is then completely described by the transition function (patching function) defined on \(U_+ \cap U_-\). On \(\mathcal{F}\), the pull-back \(\hat{P}\) of the patching matrix \(P\) defining \(E\) is annihilated by \(V_0\) and \(V_1\). On the other hand, the assumed holomorphic triviality over each twistor line means that \(\hat{E}\) is trivial on \(\mathcal{F}\) so there exist maps \(g_\pm(\lambda; v, x, t)\) holomorphic on \(\pi_1 U_\pm\) respectively with \(\hat{P} = g_- g_+^{-1}\) defining a global frame of \(\hat{E}\) over \(\mathcal{F}\). Operating with \(V_0\) and \(V_1\) gives, using \(V_1 P = 0\),

\[
(6.3) \quad g_-^{-1} V_i g_- = g_+^{-1} V_i g_+ = L_i
\]

so that the \(L_i\) are global over each Riemann sphere. From the form of \(V_0\) and \(V_1\), the \(L_i\) are holomorphic in \(\lambda\) with a simple pole at \(\lambda = \infty\); hence they are linear in \(\lambda\) and have the form of a Bogomolny Lax pair (6.2) thus defining \(A, B, C, D\) as functions only of \((v, x, t)\) as in (6.2). Moreover, (6.3) implies that \(L_i g_\pm = 0\) which in turn implies that the \(L_i\) commute so that \(A, B, C, D\) satisfy the Bogomolny equations.

---

\(^2\)More generally we can restrict the domain to some Stein open set \(U \subset \mathbb{C}^3\) and have such a correspondence with bundles over \(q(p^{-1}(U)) \subset \mathcal{Z}\) trivial over the \(\mathbb{CP}^1\)'s corresponding to points of \(U\).
Dolbeault presentation: Topologically $E$ is a trivial bundle, so we can regard it as the product bundle $\mathbb{C}^2 \times Z$, equipped with a non-trivial $\bar{\partial}$-operator $\bar{\partial}_a = \bar{\partial} + \alpha$, where $\alpha$ is a $(0,1)$-form with values in $\mathfrak{sl}(2, \mathbb{C})$, as in §2.1, satisfying the integrability condition $\bar{\partial}^2 = 0$; in full,

$$\bar{\partial}\alpha + \alpha \wedge \alpha = 0.$$ 

In this description of holomorphic bundles, there is much freedom in the choice of smooth identification of $E$ with $\mathbb{C}^2$. This translates into a gauge freedom in $\alpha$, so that $\bar{\partial}_a$ and $\bar{\partial}_{g(\alpha)} := g\bar{\partial}_a g^{-1}$ define equivalent holomorphic bundles. Explicitly the complex gauge transformations $g : Z \to SL(2, \mathbb{C})$ act on the space of $\bar{\partial}$-operators by the formula

$$g(\alpha) = g\alpha g^{-1} - \bar{\partial}g^{-1}.$$

In the case of holomorphic bundles over $Z$, each fibre of $Z \to \mathbb{CP}^1$ is Stein so that one can choose a gauge which is holomorphic in the fibre direction and so $\alpha$ can be reduced to $a d\bar{X}$. The integrability condition immediately shows that $a$ is holomorphic in $\mu$. This will be a convenient ‘partial gauge fixing’ in what follows; $g$ must now be holomorphic in $\mu$.

We now repeat the previous operations using $\bar{\partial}_a$ instead of the patching description. We pull back the $\bar{\partial}$-operator to $\mathcal{F}$, and denote the pullback by the same symbol $\bar{\partial}_a$. This operates on the product bundle $\mathbb{C}^2 \times \mathcal{F}$ pulled back from $Z$. This operator commutes with the action of $V_0$ and $V_1$ on $\mathbb{C}^2 \times \mathcal{F}$; and the assumption of holomorphic triviality on each line implies that there also exists a gauge transformation $g : \mathcal{F} \to SL(2, \mathbb{C})$ such that $g\bar{\partial}_a g^{-1} = \bar{\partial}$, the trivial $\bar{\partial}$-operator on $\mathbb{C}^2 \times \mathcal{F}$. Now define

$$L_0 = gV_0 g^{-1}, \quad L_1 = gV_1 g^{-1}.$$ 

These commute with $g\bar{\partial}_a g^{-1} = \bar{\partial}$, so are holomorphic on $\mathcal{F}$. On the other hand $L_0$ and $L_1$ have simple poles at $\infty$ in $\lambda$ and so are linear in $\lambda$. Hence they are in the form of the Lax pair (6.2) and clearly commute with each other as $g^{-1}$ is a solution (on the right).

6.3. Imposition of symmetries. The naturalness of the Ward transform ensures that if a solution to the Bogomolny equations admits a symmetry, then the corresponding holomorphic vector bundle on twistor space is invariant under a corresponding motion of $Z$. Such symmetries must be taken from the group $G$ consisting of complex translations and the conformal orthogonal group $CO(3, \mathbb{C})$. Any element of $G$ permutes the null 2-planes so that $G$ acts naturally on $Z$ and $\mathcal{F}$ by biholomorphic transformations. Thus any generator $X$ of $G$ corresponds to holomorphic vector fields $\tilde{X}$ on $Z$ and $\tilde{X}$ on $\mathcal{F}$.

The most general holomorphic vector field defined globally on $Z$ is given by

$$\tilde{X} = (\alpha \lambda^2 + \beta \lambda + \gamma)\partial/\partial\lambda + \left(\frac{1}{2} \alpha \lambda + 3 \beta + \delta\right)\partial/\partial\mu + \hat{\alpha} \lambda^2 + \hat{b} \lambda + \hat{c} \partial/\partial\mu.$$

and the reader may verify that this corresponds to

$$X = a\partial_t + b\partial_x + c\partial_v + \alpha(2v\partial_x + x\partial_t) + \beta(v\partial_v - t\partial_t) + \gamma(x\partial_v + 2t\partial_x) + \delta(v\partial_v + x\partial_x + t\partial_t)$$

on $\mathbb{C}^3$ and to

$$\tilde{X} = X + V$$

on $\mathcal{F}$, where

$$V = (\alpha \lambda^2 + \beta \lambda + \gamma)\partial/\partial\lambda$$

on $\mathcal{F}$. We see from this that the parameters $a, b, c$ yield translations of $\mathbb{C}^3$, $\alpha, \beta, \gamma$ yield (complex) rotations while $\delta$ yields the dilation.

Let $X$ be the generator of an infinitesimal symmetry of $\mathbb{C}^3$ as above. A Lax pair (6.3) will be $X$-invariant if there exists Lie derivative operator $\mathcal{L}_X$, which commutes with the Lax pair as follows:

$$[\mathcal{L}_X, L_0] = \alpha \lambda L_0 - \alpha L_1, \quad [\mathcal{L}_X, L_1] = \gamma L_0 + (\alpha \lambda + \beta)L_1.$$
Here $\mathcal{L}_{\tilde{X}}$ is by definition a linear differential operator on the sections of the bundle over $\mathcal{F}$ with the property

$$\mathcal{L}_{\tilde{X}}(f \otimes s) = (\tilde{X}f) \otimes s + f \otimes \mathcal{L}_{\tilde{X}}s$$

whenever $f$ is a function and $s$ is a section. It follows that in any local gauge, $\mathcal{L}_{\tilde{X}}$ takes the form

$$\mathcal{L}_{\tilde{X}} = \tilde{X} + \Phi$$

for some matrix function $\Phi$. We also assume in this discussion that $\Phi$ is holomorphic in any holomorphic gauge.

In terms of $\mathcal{L}_{\tilde{X}}$, the local invariant gauge is one in which $\mathcal{L}_{\tilde{X}} = \tilde{X}$, and then the commutation conditions show that $X\mathcal{L} = 0$ etc. It is not difficult to show that the Ward transform restricts to give a 1:1 correspondence between $\tilde{X}$-invariant bundles over $Z$ and $X$-invariant solutions of the Bogomolny equations, whenever $X$ and $\tilde{X}$ are corresponding vector fields on $\mathbb{C}^3$ and $Z$.

6.4. **Formula for the variation of the $\bar{\partial}$-operator.** In order to apply the framework of $\mathcal{L}_{\tilde{X}}$, we wish to produce a family of $\bar{\partial}$-operators on a fixed trivialised bundle over $\mathbb{CP}^1$, the family being parametrized by space-time. We also want the variation with respect to the space-time coordinates to be given by meromorphic gauge transformations or diffeomorphisms of $\mathbb{CP}^1$. This can be done naturally in terms of the twistor data if a symmetry has been imposed.

Fix a holomorphic symmetry generated by $X$ on $\mathbb{C}^3$, an $X$-invariant solution to the Bogomolny equations, and corresponding $\tilde{X}$-invariant bundle $E$ on $\mathcal{Z}$. Let $\{\mathbb{C}^2 \times \mathcal{Z}, \bar{\partial}_a\}$ be a Dolbeault representation of $E$ such that $\alpha$ has the canonical form $\alpha = a d\tilde{X}$ discussed in §3.2. When pulled back to $\mathcal{F}$ we obtain a $\bar{\partial}$-operators $\bar{\partial}_a$ on the bundle $\mathbb{C}^2 \times \mathbb{CP}^1$ parametrized by $(v, x, t) \in \mathbb{C}^3$.

In order to see that the variation of $\partial_a$ with respect to the coordinates on $\mathbb{C}^3$ can be represented in terms of gauge transformations and diffeomorphisms, note first that $V_0$ and $V_1$ and the Lie derivative operator $\mathcal{L}_{\tilde{X}}$ commute with $\bar{\partial}_a$. Identifying $\mathcal{F}$ with $\mathbb{C}^3 \times \mathbb{CP}^1$ we have $X = X + V$. For generic $\lambda$, $(X, V_0, V_1)$ span $\mathbb{C}^3$, so a given holomorphic vector field $Y$ on $\mathbb{C}^3$ can be expressed as

$$Y = f_0 V_0 + f_1 V_1 + h X,$$

where $f_0$, $f_1$ and $h$ are meromorphic in $\lambda$. Thus

$$\mathcal{L}_Y \bar{\partial}_a = \mathcal{L}_{hX} \bar{\partial}_a = (\mathcal{L}_{hX} - \mathcal{L}_{hV}) \bar{\partial}_a - [h\Phi, \bar{\partial}_a] = -\mathcal{L}_{hV} \bar{\partial}_a + \bar{\partial}_a (h\Phi),$$

so that

(6.4) 

$$\mathcal{L}_Y \alpha = -\mathcal{L}_{hV} \alpha + \bar{\partial}_a (h\Phi).$$

Thus the derivative of $\bar{\partial}_a$ along $Y$ is given by a combination of Lie dragging along a meromorphic vector field on $\mathbb{CP}^1$ and by the action of an infinitesimal meromorphic gauge transformation. This gives a natural lift of the action of the Lie algebra of holomorphic vector fields on $\mathbb{C}^3$ to act on the space of $\bar{\partial}$-operators on the product bundle $\mathbb{C}^2 \times \mathbb{CP}^1$ by a combination of meromorphic gauge transformations and diffeomorphisms of $\mathbb{CP}^1$.

6.5. **Formula for the variation of the patching function.** One has a similar story in terms of the pullback of the Čech description. We start with $\mathcal{P}$, a patching function with respect to a covering by sets $\{U_+, U_-\}$ for a holomorphic vector bundle with symmetry on $\mathcal{Z}$. The covering can be chosen so that the symmetry has no fixed points on $U_+$ and so the frames for $E$ on $U_+$ can be chosen so that on $U_+$ is invariant. Thus on $U_+ \cap U_-$ we will have $\tilde{X}P = \phi_- P$ with $\phi_-$ holomorphic on $U_-$. The pullback, $\hat{P}$ of $P$ to $\mathcal{F}$ therefore satisfies $V_0 \hat{P} = V_1 \hat{P} = 0$ and $\tilde{X} \hat{P} = (X + V) \hat{P} = \phi_- \hat{P}$. So, as before, we can express the derivative of $\hat{P}$ along a vector field $Y$ on $\mathbb{C}^3$, using $Y = f_0 V_0 + f_1 V_1 + h X$ and $\tilde{X} = X + V$, as follows

$$\hat{Y} \hat{P} = (f_0 V_0 + f_1 V_1 + h X) \hat{P} = h(\tilde{X} - V) \hat{P} = h(-V \hat{P} + \phi_- \hat{P}).$$

Thus space-time translation corresponds to an action on $\hat{P}$ by diffeomorphisms and left multiplication.
7. Definition and formula for the \( \tau \)-function

The \( \tau \)-function was defined in §3.5 on an orbit of a submanifold of the group of gauge transformations to be, in effect, the determinant of the \( \bar{\partial} \) operator on each of the \( \mathbb{CP}^1 \)'s of the orbit of a given \( \bar{\partial} \)-operator under that submanifold. In §6.4 the twistor theory was reformulated so that space-time \( \mathcal{M} \) emerged as the parameter space of a family of holomorphic structures on a given bundle over \( \mathbb{CP}^1 \). Thus \( \mathcal{M} \) can be naturally embedded into the phase spaces of \( \mathcal{N} \) or \( \mathcal{U} \) and the prequantum bundle \( \text{Det} \) can be restricted to it. For \( \sigma \) to yield a function on space-time, we must trivialize \( \text{Det} \) over \( \mathcal{M} \), as in §3.5, by using an invariant trivialization according to the geometric quantization actions of gauge transformations and diffeomorphisms given in the previous sections.

7.1. Dolbeault presentation. In order to apply the framework of §3 we work locally in \( \mathbb{C}^3 \) and decompose \( \mathbb{CP}^1 = D_+ \cup D_- \) so that the poles of \( h \) lie in \( D_- \times U \) for the region in \( U \subset \mathbb{C}^3 \) under consideration. We also restrict our choice of \( \bar{\partial} \)-operator so that the support of \( q^* \alpha \) lies in \( D_+ \times U \).

The tau function \( \tau(v,x,t) \) can now be defined as in §3.5 to be the Quillen determinant of \( \bar{\partial} \alpha \). Using the infinitesimal version (3.14) we have, for \( Y \) a vector field on space-time,

\[
Y \mathcal{L} \text{d log } \tau = \frac{L_Y \sigma}{\sigma}
\]

where \( L_Y \) is as defined in §3. It is straightforward to see that, since the complex gauge transformations and diffeomorphisms in question extend holomorphically over \( D_+ \), the central extension vanishes on the submanifold of \( \mathcal{G}^r \) so that this formula does indeed define the \( \tau \)-function up to constants as desired. We can make this into an explicit formula for the variation in the \( \tau \)-function by applying Proposition 3 for the part associated to the meromorphic gauge transformation \( h\Phi \) and formula (3.18) for the part arising from the meromorphic diffeomorphism (our situation satisfies the conditions required for the simplification leading to that formula) with \( \xi(1,0) = hV \). We obtain

\[
\text{d log } \tau = -\frac{i}{2\pi} \oint h \text{tr} \left( \frac{1}{2}(Vg)^{-1}(dg)g^{-1} + \Phi(dg)g^{-1} \right)
\]

This is equivalent to Proposition 6 (or more directly equation (14)) of Mason, Singer & Woodhouse (2000).

7.2. Cech presentation. Using equations (4.8) and (4.7) these actions can be lifted to the determinant line bundle and, as in the previous subsections, we define the \( \tau \)-function to be the determinant expressed in an invariant frame. The formula for its variation along \( Y \) then follows from equations (4.3) and (4.7). Using the fact that \( hV \) is holomorphic on \( D_+ \) so that the term in \( g^+ \) integrates to zero by Cauchy’s theorem, This gives

\[
Y \mathcal{L} \text{d log } \tau = -\frac{i}{2\pi} \oint h \text{tr} \left( \frac{1}{2}(Vg)^{-1}(dg)g^{-1} + \Phi(dg)g^{-1} \right)
\]

in which, although formally identical to (7.1) above, the terms have the different interpretations as in (4.8) and (4.7).

In both cases then, we obtain the same formula for the \( \tau \)-function in terms of the pullback to \( \mathcal{F} \) of a holomorphic trivialization of \( E \) on \( U_- \) from twistor space.

8. Examples

In this section we apply the above theory to show that the tau-function for the Ernst equation and the KdV equations according to the above definitions are given by the standard formulae (see eg. Segal & Wilson (1985) for KdV and Breitenlohner & Maison (1986) for Ernst). In each example we first describe enough of the special features that arise in the Ward construction to proceed to the calculation of \( \tau \) in terms of the space-time fields using the above theory. The fact we need to do the calculation is that the \( P_\pm \) and \( g_\pm \) are solutions to the Lax pair operators. We give two examples, one in which \( hV = 0 \) and one for which \( g\Phi = 0 \).
8.1. The KdV equations. For the KdV reduction we have the symmetry \( X = \partial/\partial v \) on space-time which lifts horizontally to the spin bundle \( F \) and descends to \( \tilde{X} = \partial/\partial \mu \) on twistor space. The Bogomolny equations with Lax Pair (6.3) reduce to the KdV equation under the further assumption that \( B \) is nilpotent and \( \text{tr}(AB) = 1 \) in which case there exists an invariant gauge in which \( D = 0 \) and

\[
L_0 = \partial_x + \left( q \begin{pmatrix} -1 & -1 \\ p & -q \end{pmatrix} - \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \quad L_1 = \partial_t + C - \lambda \partial_x
\]

The consistency conditions for the Lax pair determine \( p \) and \( C \) in terms of \( q \) and the condition that \( u = \partial_x q \) satisfies the KdV equation:

\[
4\partial_t u - \partial_x^3 u - 6u\partial_x u = 0.
\]

We can make a vector field, \( Y = y_0 \partial_t + y_1 \partial_x + y_2 \partial_v \), say, act on quantities pulled back from twistor space, as in §6.3 as follows. Since the lift \( \tilde{X} \) of \( X = \partial_\mu \) to \( F \) is horizontal (i.e. the \( V \) of §6.3 is zero) the distribution \( \{ \tilde{X}, V_0, V_1 \} \) determined by the symmetry and the twistor distribution is equivalent to the horizontal (over space-time) distribution \( \{ \partial_v, \partial_x, \partial_t \} \) on \( F \). The leaves are \( \lambda = \text{constant} \). The action of \( L_0, L_1 \) and \( L_{\partial_\mu} \) determine a flat connection on these leaves which is singular at \( \lambda = \infty \). In the notation of §6.4 a \( Y \) can be chosen to be a symmetry vector field \( Y = a \partial_t + b \partial_x + c \partial_v \).

On twistor space we can choose a Cech presentation based on a framing on the complement of \( \lambda = \infty \) such that \( L_{\partial_\mu} = \partial/\partial \mu \) and a framing near infinity so that the lift of the symmetry \( \partial/\partial \mu \) to the bundle has the form

\[
(8.1) \quad L_{\partial/\partial \mu} = \frac{\partial}{\partial \mu} + \Phi, \quad \text{where} \quad \Phi = \Phi(\mu, \lambda) = \begin{pmatrix} 0 & 1/\lambda \\ 1 & 0 \end{pmatrix} + O\left(\frac{1}{\lambda^2}\right)
\]

If the patching function relating the two framings is \( P(\mu, \lambda) \) then the symmetry condition is

\[
\frac{\partial P}{\partial \mu} + \Phi P = 0,
\]

and if \( P \) is pulled back to \( F \), this equation implies

\[
\partial_x P + \lambda \Phi P = 0, \quad \text{and} \quad \partial_t P + \lambda^2 \Phi P = 0.
\]

The simplest special case is where

\[
P(\mu, \lambda) = \exp \left( -\mu \begin{pmatrix} 0 & 1/\lambda \\ 1 & 0 \end{pmatrix} \right) P(0, \lambda), \quad \text{so that} \quad \Phi = \Phi(\mu, \lambda) = \begin{pmatrix} 0 & 1/\lambda \\ 1 & 0 \end{pmatrix}
\]

exactly. These bundles correspond to the solutions that Segal and Wilson work with. It is not always possible to obtain this normal form except to finite order in \( 1/\lambda \). We will only assume below that it has this form up to \( O(1/\lambda^2) \).

Alternatively one can choose a Dolbeault representation based on a smooth frame for \( E \) which is holomorphic up the fibres of minitwistor space over \( \mathbb{C}P^1 \) which agrees with the above framings for \( |\lambda| \geq 1 \) but which is not holomorphic for \( |\lambda| \leq 1 \). In this frame we will also have \( L_{\partial_\alpha} = \partial/\partial \mu - \Phi \) where \( \Phi \) has the above form for \( |\lambda| \geq 1 \) but is no longer explicitly holomorphic for \( |\lambda| \leq 1 \). The condition that the \( \partial_\alpha \)-operator be invariant implies, with \( \partial_\alpha = \partial_0 + \alpha \) that

\[
\frac{\partial \alpha}{\partial \mu} = -\partial_\alpha \Phi, \quad \text{so that on} \ F, \ \partial_x \alpha = -\lambda \partial_\alpha \Phi, \quad \text{and} \ \partial_t \alpha = -\lambda^2 \partial_\alpha \Phi.
\]

Therefore \( \partial_x \log \tau \) and \( \partial_t \log \tau \) are obtained by putting \( u \) equal to \( \lambda \Phi \) and \( \lambda^2 \Phi \) (respectively) in (8.12) or (8.6). In particular,

\[
\partial_x \log \tau = \frac{i}{2\pi} \int \text{tr} \left( (dP_-)P_-^{-1} \lambda \Phi \right)
\]




All the terms in this contour integral are holomorphic in $D_-$, except the simple pole associated to $\lambda$, so that it reduces to a residue at $\lambda = \infty$. Expanding $P_- = \sum_{i=0}^{\infty} P_i^l / \lambda^i$ and $\Phi = \sum_{i=0}^{\infty} \Phi_i / \lambda^i$ we obtain

$$
\partial_x \log \tau = \text{tr} \left( P_+^l (P_0^l)^{-1} \Phi_0 \right)
$$

By the definition of the action of the symmetry, we have $\partial_x P_- = - \Phi P_-$. We shall assume that our frame for $E$ has been chosen so that, to $O(1/\lambda^2)$, $\Phi$ has the normal form given in equation (8.1). Furthermore $P_-$ satisfies the Lax system (on the right according to our conventions), so we find

$$
\partial_x P_- = - P_- \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - 1/\lambda \begin{pmatrix} q & -1 \\ p & -q \end{matrix} \right) + O(1/\lambda^2)
$$

so expanding these two equations, we find

$$
\partial_x P_0^l = - \Phi_0 P_0^l = - P_0^l \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{matrix} \right), \quad \partial_x P_1^l = - \Phi_1 P_1^l - \Phi_0 P_0^l = - P_1^l \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) + P_0^l \left( \begin{pmatrix} q & -1 \\ p & -q \end{matrix} \right).
$$

Sorting through the equations, we find

$$
\partial_x \log \tau = q
$$

which yields the standard relation between the $\tau$-function and the solution $u = \partial_x q$ to the KdV equation.

8.2. The Ernst equations. The Ernst equations describe vacuum space-times in general relativity with two (generic) commuting symmetries that are orthogonal to a foliation by 2-surfaces. We consider a Lorentzian metric on $\mathbb{R}^4$ with coordinates $(x^i, r, z)$

$$
\mathrm{d}s^2 = J_{ij} \mathrm{d}x^i \mathrm{d}x^j - \Omega^2 (\mathrm{d}r^2 + \mathrm{d}z^2)
$$

where $J_{ij} = J_{ij}(r, z)$ is a symmetric $2 \times 2$ matrix with $\det(J) = -r^2$. The Einstein vacuum equations on this metric reduce to

$$
\frac{1}{r} \partial_r (r J^{-1} \partial_r J) + \partial_z (J^{-1} \partial_z J) = 0, \quad \text{and} \quad \frac{\partial \log(r\Omega^2)}{\partial w} = \frac{ir}{2} \text{tr} \left( J^{-1} \frac{\partial J}{\partial w} \right)^2 \quad \text{where} \quad w = z + ir.
$$

The first of these is the reduction of the Bogomolny equations with a rotational and $\mathbb{Z}_2$ symmetry in a potential form (Ward 1983). The second determines the conformal factor in terms of $J$ and will be seen to express $\Omega$ as the tau function associated to this system according to the above system (Breitenlohner & Maison 1986).

We now wish to introduce enough of the twistor correspondence in order to calculate the expression for $\tau$ above. We use coordinates $(r, \theta, z)$ on $\mathbb{R}^3$ such that the minitwistor correspondence becomes

$$
\mu = r e^{i\theta} + \lambda z + \lambda^2 r e^{-i\theta}.
$$

The rotational symmetry $\partial / \partial \theta$ lifts to give $\partial_\theta + i \lambda \partial_\lambda$ on the spin bundle $\mathcal{F}$ and descends to give $V = i (\mu \partial_\mu + \lambda \partial_\lambda)$ on twistor space.

In order to eliminate $\theta$ from the formulae, introduce the invariant coordinate on $\mathcal{F}$, $\zeta = e^{-i\theta} \lambda$, and project to the quotient $\tilde{\mathcal{F}}$ of $\mathcal{F}$ by $\partial_\theta + i \lambda \partial_\lambda$ with coordinates $(r, z, \zeta)$ (this is equivalent to expressing the Lax pair in an invariant spin frame). The Lax pair becomes

$$
L_0 = \partial_z - \zeta (\partial_\zeta - \frac{\zeta}{r} \partial_\xi) + J^{-1} \partial_z J, \quad L_1 = \partial_\zeta + \frac{\zeta}{r} \partial_\xi + \zeta \partial_z + J^{-1} \partial_z J
$$

on $\tilde{\mathcal{F}}$ and their consistency is equivalent to the first of the Einstein vacuum equations given above. These can be reformulated to give the operators

$$
\partial_\zeta + \frac{1}{2ir(\zeta - i)} ((\zeta + i) \zeta \partial_\zeta + 2r J^{-1} \partial_\zeta J), \quad \partial_\zeta + \frac{i}{2r(\zeta + i)} ((\zeta - i) \zeta \partial_\zeta + 2i J^{-1} \partial_\zeta J).
$$
The vector field part of these operators give the form for the map from vector fields on $\mathbb{R}^3$ to meromorphic vector fields on $\mathbb{C}P^1$ described in §6.3. Note that the first operator leads to a vector field on $\mathbb{C}P^1$ with a pole at $\zeta = i$ and the second at $\zeta = -i$. We use a covering of the Riemann sphere so that $\zeta = \pm i$ are contained in $D_-$.

The formulae for $d \log \tau$ that are relevant in this case are those for the action of holomorphic vector fields. There is no contribution from $P_+$ (or $g_+$ in the second formulation) as $v$ and $P_+$ are holomorphic over $D_+$. However there is from $P_-$, but since the only singularity in $D_-$ is that associated to the simple pole in $v$, the integral reduces to a residue calculation.

We therefore find from (3.18) or (4.8),

$$\partial_\bar{w} \log \tau = \frac{i}{4\pi} 2\pi i \text{Res} \left( \frac{(\zeta + i)\zeta}{2\pi(\zeta - i)} \text{tr}(P_-^{-1}\partial P_-)^2, \zeta = i \right) = \frac{1}{2\pi i} \text{tr}(P_-^{-1}\partial \zeta P_-)^2|_{\zeta=i}.$$

This last term can be evaluated using the fact that $P_-$ satisfies the Lax pair so that, at $\zeta = i$

$$\partial \zeta P_- = r J^{-1} \partial \bar{w} J P_-$$

thus giving

$$\partial_\bar{w} \log \tau = -\frac{ir}{2} \text{tr}(J^{-1}\partial \bar{w} J)^2.$$

We similarly obtain

$$\partial_w \log \tau = \frac{ir}{2} \text{tr}(J^{-1}\partial w J)^2.$$

We see from this that $\tau$ is equal to $1/r^2 \Omega$ up to a multiplicative constant giving an interpretation to this second part of the vacuum Einstein equations.

9. CONCLUSION AND FURTHER QUESTIONS

In this paper we have described the interrelations that exist between the twistor description of integrable systems, determinants of $\partial$-operators, and 2-dimensional QFT.

There remain a number of interesting questions. Can a similar formulation be found for reductions of the hyper-Kähler equations and their integrable generalizations? Is it possible to connect the field theories described above to the topological field theories involved in the definition of quantum cohomologies whose partition functions are also described by the above $\tau$-functions? Is there a meaningful way to drop the symmetry assumption on the Bogomolny equations (or indeed the self-dual Yang-Mills equations) and still obtain a $\tau$-function description?

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