Eccentricity of the nodes of OTIS-cube and Enhanced-OTIS-cube

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Abstract
In this paper we have classified the nodes of OTIS-cube based on their eccentricities. OTIS (optical transpose interconnection system) is a large scale optoelectronic computer architecture, proposed in [1], that benefit from both optical and electronic technologies. We show that radius and diameter of OTIS-$Q_n$ is $n + 1$ and $2n + 1$ respectively. We also show that average eccentricity of OTIS-cube is $(3n/2 + 1)$. In [10], a variant of OTIS-cube, called Enhanced OTIS-cube (E-OTIS-$Q_n$) was proposed. E-OTIS-$Q_n$ is regular of degree $n + 1$ and maximally fault-tolerant. In this paper we have given a classification of the nodes of E-OTIS cube and derived expressions for the eccentricities of the nodes in each class. Based on these results we show that radius and diameter of E-OTIS-$Q_n$ is $n + 1$ and $\lfloor 4n + 4/3 \rfloor$ respectively. We have also computed the average eccentricity of E-OTIS-$Q_n$ for values of $n$ up to 20.

1 Introduction
The optical transpose interconnection system (OTIS), first introduced in [1], was proposed for large scale parallel computer architectures in which the processors are divided into groups. Each group of processors is fabricated on one or more high density chips or modules with in-built electronic interprocessor connections, whereas processors in different groups are interconnected via free space optics. When the distance between modules is more than a few millimeters, the free space optical interconnects offer power, speed, I/O bandwidth and crosstalk advantages over electronic counterparts, and hence are suitable for large scale systems [8]. Since it has been shown in [1] that both the bandwidth and power consumption are minimized when the number of processors in a group is equal to the number of groups, most of the topologies proposed so far, based on OTIS, follows this design guideline [4], [5], [9]. In such an OTIS system, an intergroup link connects processor $p$ of group $g$ to processor $g$ of group $p$. The interconnection topology formed by the intra-group links are called here the factor network. Depending on the factor network, different OTIS-network topologies have been proposed so far, such as OTIS-Mesh, OTIS-Cube, OTIS-star and so on.
A general study of OTIS-Networks based on any factor network topology, is presented in [2]. In [8], a number of results regarding the topological properties of OTIS-k-ary n-cube interconnection networks have been derived which shows the suitability of these topologies for multiprocessor interconnection network. In a more recent work [11], the authors contribute further results in this direction by studying a general fault tolerance property of OTIS networks. It is proved that an OTIS network is maximally fault tolerant if its basis network is connected. They have also proposed a corresponding method for constructing parallel paths between its nodes. Extensive research has been done to develop efficient algorithms for various applications on the OTIS architecture such as selection and sorting [4], matrix multiplication [6], image processing [7], BPC permutation generation [5] etc.

OTIS-cube is a particular member of the general class of OTIS networks where the factor network is a hypercube. An OTIS-cube Q_n consists of 2^n groups with each group having 2^n nodes. It is to be noted that in each group of an OTIS-cube there is one node whose processor number is the same as the group number. So, there is no optical link from such a node. Hence the OTIS-cube (OTIS-Q_n) is not a regular topology, since in this network some nodes are of degree n + 1 and some nodes are of degree n. In [11], the authors raised the question whether the fault-tolerance of OTIS structure can be further improved by pairwise connecting the nodes which do not possess inter-cluster links. The topology proposed in [10], called the enhanced OTIS-cube (E-OTIS-Q_n) does just that. There, it is shown that if the Basis network is hypercube, the fault-tolerance is improved by adding optical links to the nodes which are of degree n in OTIS-Q_n. The diameter of E-OTIS-Q_n is also \(\lfloor \frac{4n+4}{3} \rfloor\) [3], less than the diameter of OTIS-Q_n by a constant factor.

The OTIS structure is an attractive option for multiprocessor systems as it offers the benefits from both optical and electrical technologies. E-OTIS-Q_n retains OTIS-Q_n as a subgraph and thus has almost all the desirable properties of OTIS-cube but contains some additional optical links. The advantage gained by adding those extra links, namely, uniform node degree, reduction in diameter by almost one-third, and improved fault-tolerance, far outweigh the cost. These advantages make E-OTIS-Q_n a suitable architecture for multiprocessor interconnection network.

We have quoted the parts of the paper [3] presenting the algorithm for shortest path routing and the proof of its optimality, as they constitute the basis for our main contribution in this paper. i.e., finding the eccentricities of the nodes of OTIS-Q_n and E-OTIS-Q_n. In recent times there are a number of research works published on several graph properties like eccentricity, proximity, remoteness [12], [13].

The rest of the paper is organized as follows. In the next section we present the definitions and basic properties. The next two sections deal with routing algorithm and eccentricities of the nodes in E-OTIS-Q_n respectively. Section 5 concludes the paper.
2 Definitions and Preliminaries

The binary hypercube $Q_n$ has $N = 2^n$ nodes labeled from 0 to $2^n - 1$. Two nodes are connected by an edge if and only if, their labels, in binary, differ in exactly one bit position. $Q_n$ has degree $n$ and diameter $n$. Given two nodes $u$ and $v$ in $Q_n$, we denote by $H(u, v)$ the Hamming distance between $u$ and $v$, i.e., the number of bit positions in which $u$ and $v$ differ. $H(u, v)$ is also the length of the shortest path between $u$ and $v$ in $Q_n$.

The OTIS-$Q_n$ is composed of $N = 2^n$ node-disjoint subgraphs $Q_n^{(0)}$, $Q_n^{(1)}$, . . . , $Q_n^{(N-1)}$, called groups. Each of these groups is isomorphic to a binary hypercube $Q_n$. A node $<g, x>$ in OTIS-$Q_n$ corresponds to a node of address $x$ in group $Q_n^{(g)}$. We refer to $g$ as the group address of node $<g, x>$ and to $x$ as its processor address.

**Definition 1.** The OTIS-$Q_n$ network is an undirected graph $(V, E)$ given by

$V = \{<g, x> | g, x \in Q_n\}$ and

$E = \{(<g, x>, <g, y>) | (x, y) \text{ is an edge in } Q_n\} \cup \{(<g, x>, <x, g>) | g \neq x \text{ in } Q_n\}$

An intra-group edge of the form $(<g, x>, <g, y>)$ corresponds to an electrical link.

**Definition 2.** The electrical link $(<g, x>, <g, y>)$ is referred to as link $i$ if $x$ and $y$ differ in the $i$th bit.

An inter-group link of the form $(<g, x>, <x, g>)$ corresponds to an optical link.

Some of the properties of OTIS-$Q_n$ are given below [2]:

- The size of OTIS-$Q_n$ is $2^{2n}$.
- The degree of a node $<g, x>$ is $n + 1$, if $g \neq x$ and $n$, if $g = x$.
- The distance between two nodes $<g, x>$ and $<h, y>$, denoted as $d(g, x, h, y)$ is given as:

$$d(g, x, h, y) = \begin{cases} H(x, y), & \text{if } g = h \\ \min \left\{ H(g, h) + H(x, y) + 2, H(x, h) + H(g, y) + 1 \right\}, & \text{otherwise} \end{cases}$$

- The diameter of OTIS-$Q_n$ is $(2n + 1)$.

**Definition 3.** The enhanced OTIS-$Q_n$ is obtained from OTIS-$Q_n$ by connecting every node of the form $<g, g>$ to the node $<\bar{g}, g>$ by an optical link, where $\bar{g}$ is obtained by complementing all the bits of $g$. We refer to the links of these form as E-links.

3 Shortest Path Routing

The routing algorithms for OTIS-$Q_n$ and E-OTIS-$Q_n$ described in this section are as given in [3].
3.1 Routing in OTIS-$Q_n$

The routing algorithm for OTIS-$Q_n$ is taken from [2] but a few observations are added. For a source node $< g, x >$ and destination node $< h, y >$, this algorithm always gives a path of length $d(g, x, h, y)$ defined in section 2.

If $(g = h)$, then $d(g, x, h, y) = H(x, y)$. Otherwise, $d(g, x, h, y)$ is minimum \{l_1, l_2\}, where $l_1 = H(x, h) + H(g, y) + 1$ and $l_2 = H(g, h) + H(x, y) + 2$.

The paths corresponding to $l_1$ and $l_2$ are referred to as path-1 and path-2 respectively.

The authors made the following observations.

Obs 1. If $g = x$ or $h = y$, then $l_1 < l_2$
Obs 2. If $h = x$, then $l_1 < l_2$
Obs 3. If $l_2 < l_1$ then we must have $g \neq x$ and $h \neq x$.

Now the algorithm Route $(g, x, h, y)$ to route from $< g, x >$ to $< h, y >$ in an OTIS-$Q_n$ is presented.

| Algorithm 1: Routing in OTIS-cube |
|-----------------------------------|
| **input**: Nodes $< g, x >$ and $< h, y >$ |
| **output**: Path from $< g, x >$ to $< h, y >$ |
| **begin** |
| if $g = h$ then |
| traverse group $g$ to $< g, y >$ |
| else |
| if $H(g, h) + H(x, y) + 2 < H(x, h) + H(g, y) + 1$ then |
| route2 $(g, x, h, y)$ |
| else |
| route1 $(g, x, h, y)$ |
| **route1** $(g, x, h, y)$ : $< g, x > \rightarrow < g, h > \rightarrow < h, g > \rightarrow < h, y >$ |
| **route2** $(g, x, h, y)$ : $< g, x > \rightarrow < x, g > \rightarrow < x, h > \rightarrow < h, x > \rightarrow < h, y >$ |

Here route1 and route2 correspond to path-1 and path-2 respectively. route2 is valid only if $g \neq x$ and $h \neq x$. Note that when path 2 is followed, $g \neq x$ and $h \neq x$ by obs. 3.

3.2 Routing in E-OTIS-$Q_n$

The E-links which are added to an OTIS-$Q_n$ to get E-OTIS-$Q_n$ are of the form $(< p, p >, < \bar{p}, \bar{p} >)$. It is shown that any shortest path in E-OTIS-$Q_n$ can contain a link of this form at most once.

**Lemma 1.** Any shortest path from a node $< g, x >$ to a node $< h, y >$ in E-OTIS-$Q_n$, can contain at most one E-link.

**Proof.** The proof is by contradiction. Let us suppose a shortest path from $< g, x >$ to $< h, y >$ contains more than one E-link. Suppose the first two
Lemma 2. If \( E \)

Proof. 

\[ \text{occurrences of such E-links are of the form } (< p, p >, < \bar{p}, \bar{p} >) \text{ and and } (< q, q >, < \bar{q}, \bar{q} >). \]

Suppose the path is 

\[ A : < g, x > \rightarrow < p, p > \rightarrow < \bar{p}, \bar{p} > \rightarrow < q, q > \rightarrow < \bar{q}, \bar{q} > \rightarrow < h, y > \]

The length of the partial path from \(< g, x >\) to \(< p, p >\) is \(H(g, p) + H(x, p) + 1\), if \( g \neq p \) and \( H(x, p) \), if \( g = p \). Similarly length of the partial path from \(< \bar{p}, \bar{p} >\) to \(< q, q >\) is \(H(\bar{p}, q) + H(\bar{p}, q) + 1\). Hence length of the partial path from \(< g, x >\) to \(< \bar{q}, \bar{q} >\) is greater than or equal to \(H(g, p) + H(p, x) + 2H(\bar{p}, q) + 2\)

The node \(< \bar{q}, \bar{q} >\) can be reached from \(< g, x >\) without using any E-link and length of that path is \(H(g, \bar{q}) + H(x, \bar{q}) + 1\), if \( g \neq \bar{q} \) and \( H(x, \bar{q}) \), if \( g = \bar{q}. \)

\[ \text{Now } H(g, p) + H(p, x) + 2H(\bar{p}, q) + 2 \]

\[ \geq H(g, \bar{q}) + H(x, \bar{q}) + 2 \]

\[ > H(g, \bar{q}) + H(x, \bar{q}) + 1 \]

Hence, path \( A \) cannot be a shortest path, which is a contradiction. \( \square \)

3.2.1 Minimizing the number of electrical links

The shortest path between \(< g, x >\) and \(< h, y >\) either won’t have any E-link or will have only one E-link. In the later case the path will be of the form 

\[ E : < g, x > \rightarrow < b, b > \rightarrow < \bar{b}, \bar{b} > \rightarrow < h, y > \]

In the later part of this section whenever a path with E-link is considered, it is assumed that the E-link is of the form \(< b, b >, < \bar{b}, \bar{b} >\).

Definition 4. Let \( C_i(g, x, h, y) \) be the number of times electrical link \( i \) is used in the path \( E \). It is called the cost associated with bit \( i \). We can write \( C_i(g, x, h, y) \) as \( C_i(g, x, b, b) + C_i(b, \bar{b}, h, y) \). The \( i^{th} \) bit of \( g \) is denoted as \( g[i] \).

The following lemmas are regarding \( C_i(g, x, b, b) \) and \( C_i(b, \bar{b}, h, y) \) in the path \( E \).

Lemma 2. If \( g[i] \neq x[i] \) then \( C_i(g, x, b, b) = 1 \).

Proof. If \( x[i] \neq b[i] \), \( g[i] = b[i] \) and \( x[i] = b[i] \), \( g[i] \neq b[i] \). That is, The \( i^{th} \) bit needs to be changed either for moving from \(< g, x >\) to \(< g, b >\) or for moving from \(< b, g >\) to \(< b, b >\). \( \square \)

Lemma 3. If \( g[i] = x[i] \) then \( C_i(g, x, b, b) = \begin{cases} 0, & \text{if } g[i] = b[i] \\ 2, & \text{otherwise} \end{cases} \)

Proof. If \( g[i] = b[i] \), the \( i^{th} \) bit need not be changed at all. If \( g[i] \neq b[i] \), the \( i^{th} \) bit needs to be changed twice, once for moving from \(< g, x >\) to \(< g, b >\) and once for moving from \(< b, g >\) to \(< b, b >\). \( \square \)

Lemma 4. If \( h[i] \neq y[i] \) then \( C_i(b, \bar{b}, h, y) = 1 \).

Proof. The proof is similar to that of lemma 2. \( \square \)

Lemma 5. If \( h[i] = y[i] \) then \( C_i(b, \bar{b}, h, y) = \begin{cases} 0, & \text{if } h[i] = \bar{b}[i] \\ 2, & \text{otherwise} \end{cases} \)
Proof. The proof is similar to that of lemma 3.

The lemmas 2 to 5 are applied and \( C_i(g, x, h, y) \) is computed for each of the following cases for \( b[i] = g[i] \) and \( b[i] \neq g[i] \). Whenever \( C_i(g, x, h, y) \) is expressed as sum of two terms, the first term corresponds to \( C_i(g, x, b, b) \) and the second term corresponds to \( C_i(b, b, h, y) \).

**Case 1.** \( g[i] \neq x[i] \) and \( h[i] = y[i] = g[i] \).

Here \( C_i(g, x, b, b) = 1 \) by lemma 2.

By lemma 3, \( C_i(b, b, h, y) = \begin{cases} 0, & \text{if } b[i] \neq g[i] \\ 2, & \text{otherwise} \end{cases} \).

Hence \( C_i(g, x, h, y) = \begin{cases} 1, & \text{if } b[i] \neq g[i] \\ 3, & \text{otherwise} \end{cases} \).

**Case 2.** \( g[i] \neq x[i] \) and \( h[i] = y[i] \neq g[i] \).

If \( b[i] = g[i] \), \( C_i(g, x, h, y) = 1 + 2 = 3 \). If \( b[i] = g[i] \), \( C_i(g, x, h, y) = 1 + 0 = 1 \).

**Case 3.** \( g[i] \neq x[i] \) and \( b[i] \neq y[i] \).

Here lemma 2 applies. So, \( C_i(g, x, h, y) = 1 + 1 = 2 \).

**Case 4.** \( g[i] = x[i] \) and \( h[i] = y[i] = g[i] \).

If \( b[i] = g[i] \), then \( C_i(g, x, h, y) = 0 + 2 = 2 \) by lemma 3 and 5. If \( b[i] \neq g[i] \), then \( C_i(g, x, h, y) = 2 + 0 = 2 \).

**Case 5.** \( g[i] = x[i] \) and \( h[i] = y[i] \neq g[i] \).

Here, lemma 2 and 3 apply. If \( b[i] = g[i] \), then \( C_i(g, x, h, y) = 0 + 0 = 0 \). If \( b[i] \neq g[i] \), then \( C_i(g, x, h, y) = 2 + 2 = 4 \).

**Case 6.** \( g[i] = x[i] \) and \( b[i] \neq y[i] \).

If \( b[i] = g[i] \), then \( C_i(g, x, h, y) = 0 + 1 = 1 \).

If \( b[i] \neq g[i] \), then \( C_i(g, x, h, y) = 2 + 1 = 3 \).

**Finding the optimal b:** The length of the path \( (B) \) is equal to the number of optical links in the path + \( \sum C_i(g, x, h, y) \). So, for finding the shortest path, we have to consider how to minimize \( C_i \) as well as the number of optical links.

Now, if \( b = g \) there is no optical link in the path from \( < g, x > \) to \( < b, b > \). But if \( b \neq g \), then there is one optical link in that part of the path. Summarizing case 1 to 6, we find that it is advantageous to have \( b[i] \neq g[i] \) only in case 1. We define the set \( S_g = \{ i | g[i] \neq x[i], h[i] = y[i] = g[i] \} \) corresponding to case 1. If \( |S_g| = 0 \), we can make \( b = g \) and avoid having an optical link in the partial path \( < g, x > \) to \( < b, b > \). If \( |S_g| \neq 0 \) we cannot avoid optical link in this part of the path. In that case we try to avoid optical link in the part of the path from \( < b, b > \) to \( < h, y > \). We define \( S_h = \{ i | h[i] \neq y[i], g[i] = x[i] = h[i] \} \). By similar logic \( S_h \) is the set of bit positions \( i \) where it is advantageous to have \( b[i] \neq h[i] \). If \( |S_h| = 0 \), we can set \( b = h \) and avoid optical link in the partial path from \( < b, b > \).

3.2.2 routing algorithm

Now the algorithm RTE to obtain the length of the shortest path involving an E-link is presented. RTE returns two values: the length of the path and the value \( b \).
Let ⊕ denote the bitwise exclusive-OR (XOR) operation, ⊙ denote the bitwise equivalence operation (XNOR), and & denote the bitwise AND operation. Let $t$ and $u$ be two $n$-bit binary numbers.

Algorithm 2: Algorithm RTE

```
input : Nodes < g, x > and < h, y >
output: Length of the shortest path involving an E-link, and $b$ where
       <$b, b>, <$b, b$> is the E-link

begin
  $t \leftarrow (g \oplus x) \& (g \odot h) \& (h \odot y)$.
  $u \leftarrow (h \oplus y) \& (g \odot h) \& (g \odot x)$.
  if $t = 0$ then
    $b \leftarrow g$
  else
    if $(u \neq 0)$ then
      $b \leftarrow g \oplus t$
    else
      $b = \overline{h}$
  return $(d(g, x, b, b) + 1 + d(\overline{b}, \overline{b}, h, y), b)$
```

The terms $d(g, x, b, b)$ and $d(\overline{b}, \overline{b}, h, y)$ correspond to the length of the paths $< g, x > \rightarrow < b, b >$ and $< b, b > \rightarrow < h, y >$ respectively and 1 is due to the use of an E-link $< b, b >$ to $< \overline{b}, \overline{b} >$.

Lemma 6. The algorithm RTE always gives the shortest path involving an E-link.

Proof. By lemma 1, a shortest path involving an E-link cannot have more than one E-links. The algorithm RTE uses only one E-link $< b, b >$ to $< \overline{b}, \overline{b} >$. Thus, we only need to show that choice of $b$ is an optimal one. Clearly, $S_g = \{i | t[i] = 1\}$ and $S_h = \{i | u[i] = 1\}$. If $t = 0$, then $|S_g| = 0$, and we can make $b = g$, avoiding optical link while going from $< g, x >$ to $< b, b >$. If $t \neq 0$, then $|S_g| > 0$. For every $i \in S_g$, if we make $b[i] \neq g[i]$ the cost associated with this bit is 1, compared to cost 3 when $b[i]$ is equal to $g[i]$. So, it is advantageous to have $b[i] \neq g[i]$ for every $i \in S_g$. That is achieved by making $b = g \oplus t$. Now we cannot avoid optical link in route $(g, x, b, b)$. In this case, we check if we can avoid optical link in route $(\overline{b}, \overline{b}, h, y)$, which will be possible if $b = h$ i.e., $|S_h| = 0$. If $u = 0$, i.e., $|S_h| = 0$, we set $b = \overline{h}$ avoiding optical link in the partial path $< \overline{b}, \overline{b} >$ to $< h, y >$. Hence, the choice of $b$ is optimal considering both the cost $C_i$ for each bit position $i$ and use of optical links. \qed

Now the algorithm for routing in E-OTIS-Qn which is called Eroute is presented.

Theorem 1. The algorithm Eroute always routes by the shortest path.
Algorithm 3: Algorithm Eroute

**input**: Nodes \(< g, x >\) and \(< h, y >\)

**output**: Shortest path Routing in E-OTIS-cube

begin

\[ L_1 \leftarrow d(g, x, h, y). \]

\[ (L_2, b) \leftarrow \text{RTE}(g, x, h, y). \]

if \( L_1 \leq L_2 \) then

route \((g, x, h, y)\).

else

route \((g, x, b, b)\). follow E-link to \(< \bar{b}, \bar{b}>\).

route \((\bar{b}, \bar{b}, h, y)\).

Proof. The shortest path between two nodes \(< g, x >\) and \(< h, y >\) either does not use an E-link or uses a single E-link. In the former case, the length of the shortest path is \( L_1 \) and routing is by algorithm route. In the later case, the length of the shortest path is \( L_2 \) as found by RTE.

The number of optical links in the computed shortest path is at most 3.

4 Eccentricity of nodes of E-OTIS-\(Q_n\) and its Diameter

We consider a source \(< g, x >\), a destination \(< h, y >\) and a set of three paths between them. We omit the case when \( g = h \), since in that case the distance between \(< g, x >\) and \(< h, y >\) is at most \( n \).

path 1 : \(< g, x >\rightarrow< g, h >\rightarrow< h, g >\rightarrow< h, y >\)

path 2 : \(< g, x >\rightarrow< x, g >\rightarrow< x, h >\rightarrow< h, x >\rightarrow< h, y >\)

path 3 : \(< g, x >\rightarrow< b, b >\rightarrow< \bar{b}, \bar{b} >\rightarrow< h, y >\) where choice of \( b \) is optimal (as found by RTE).

Here, path 1 uses one optical link, path 2 uses two optical links, path 3 uses one E-link and at most two other optical links. Let \( l_1, l_2 \) and \( l_3 \) be the lengths of path 1, path 2, and path 3 respectively. The shortest path between \(< g, x >\) and \(< h, y >\) is of length \( l = \min(l_1, l_2, l_3) \). Let \( A_i, B_i \) and \( C_i \) be the number of times link \( i \) is used in path 1, path 2 and path 3 respectively. We find the value of \( A_i, B_i \) and \( C_i \) for each of the following cases.

**Case 1.** \( g[i] \neq x[i], h[i] = y[i] = g[i] \): Consider path 1. Since \( x[i] \neq h[i] \) and \( g[i] = y[i] \), we need to use link \( i \) while going from \(< g, x >\) to \(< g, h >\) but not while going from \(< h, g >\) to \(< h, y >\). Hence, \( A_i = 1 \). For path 2, we do not need to use link \( i \) while going from \(< x, g >\) to \(< x, h >\) but use it while going from \(< h, x >\) to \(< h, y >\). Thus, \( B_i = 1 \). For path 3, as explained in
upto the eccentricity is minimum for \( H \) increase in value of \( l \) involving an E-link is implies that \( \bar{g}, x \) \( Q \) belonging to the same class have the same eccentricity. We show that for OTIS-Case 2.

Corollary 1. If \( g = 1 \), \( l \) number, and \( x \) is the node number within the group. We classify the nodes based on the value of \( H(g, x) \). All the nodes belong to one of the \( n + 1 \) classes corresponding to the value \( H(g, x) = 0, 1, \ldots, n \). We show that nodes belonging to the same class have the same eccentricity. We show that for OTIS-\( Q \), the eccentricity is maximum for node with \( H(g, x) = 0 \), then it decrease with increase in value of \( H(g, x) \) and is minimum for \( H(g, x) = n \). For E-OTIS-\( Q \), the eccentricity is minimum for \( H(g, x) = 0 \), increases with increase in \( H(g, x) \) upto \( |2n/3| \), and then again decreases and is minimum for \( H(g, x) = n \).

4.1 Eccentricity of nodes in OTIS-cube and Enhanced OTIS-cube

A node in OTIS-\( Q \) or E-OTIS-\( Q \) is represented as \( < g, x > \) where \( g \) is the group number, and \( x \) is the node number within the group. We classify the nodes based on the value of \( H(g, x) \). All the nodes belong to one of the \( n + 1 \) classes corresponding to the value \( H(g, x) = 0, 1, \ldots, n \). We show that nodes belonging to the same class have the same eccentricity. We show that for OTIS-\( Q \), the eccentricity is maximum for node with \( H(g, x) = 0 \), then it decrease with increase in value of \( H(g, x) \) and is minimum for \( H(g, x) = n \). For E-OTIS-\( Q \), the eccentricity is minimum for \( H(g, x) = 0 \), increases with increase in \( H(g, x) \) upto \( |2n/3| \), and then again decreases and is minimum for \( H(g, x) = n \).
The following lemmas (lemma 10 and lemma 11) hold for OTIS-$Q_n$ as well as E-OTIS-$Q_n$.

**Lemma 10.** A node $< g, \bar{g} >$ has eccentricity at most $(n + 1)$.

*Proof.* We consider two paths between $< g, \bar{g} >$ and $< h, y >$. Let $l_i$ denote the length of path $i$.

path 1: $< g, \bar{g} > \rightarrow < g, h > \rightarrow < h, g > \rightarrow < h, y >$

$l_1 = H(g, h) + 1 + H(g, y)$

path 2: $< g, \bar{g} > \rightarrow < g, h > \rightarrow < h, \bar{g} > \rightarrow < h, y >$

$l_2 = 1 + H(g, h) + 1 + H(\bar{g}, y)$

Let $l_{\min} = \min(l_1, l_2)$. Then $l_{\min} \leq (l_1 + l_2)/2$. Substituting $H(\bar{g}, h) = n - H(g, h)$, we get $l_{\min} \leq (2n + 3)/2$. Since $l_{\min}$ must be an integer, $l_{\min} \leq \lceil (2n + 3)/2 \rceil = n + 1$. \hfill $\square$

**Lemma 11.** A node $< g, x >$, has eccentricity at most $(2n + 1 - H(g, x))$.

*Proof.* Since, we can reach $< g, \bar{g} >$ from $< g, x >$ in $n - H(g, x)$ steps, applying lemma 10 we have a path from $< g, x >$ to $< h, y >$ of length $(n - H(g, x) + n + 1) = 2n + 1 - H(g, x)$. \hfill $\square$

**Lemma 12.** In an OTIS-$Q_n$, the eccentricity of a node $< g, x >$ is equal to $2n + 1 - H(g, x)$.

*Proof.* Consider a node $p =< g, x >$ with $H(g, x) = A$. From lemma 11, the eccentricity of this node is less than or equal to $(2n + 1 - A)$.

We consider two cases.

**Case 1:** $A$ is even. Let $q =< h, y >$ be another node such that for the pair $(p, q)$, $|S_6| = n - A$, $|S_7| = A/2$ and $|S_8| = A/2$. Then by lemma 8 and 9 we have

$l_1 = 2|S_6| + 2|S_7| + 1 = 2(n - A) + 2(A/2) + 1 = 2n - A + 1$

$l_2 = 2|S_6| + 2|S_8| + 2 = 2(n - A) + 2(A/2) + 2 = 2n - A + 2$

Here, $l_1 < l_2$, and hence, the eccentricity of node $p$ is $(2n + 1 - A)$.

**Case 2:** $A$ is odd. Let $q$ be such that $|S_7| = (A + 1)/2$, $|S_8| = (A - 1)/2$ and $|S_9| = n - A$. Proceeding as before, $l_1 = 2n - A + 2$, and $l_2 = 2n - A + 1$.

Here, $l_2 < l_1$ and hence, eccentricity of node $p$ is $(2n + 1 - A)$. \hfill $\square$

**Lemma 13.** In an E-OTIS-$Q_n$, for $H(g, x) > \lfloor 2n/3 \rfloor$, the eccentricity of a node $< g, x >$ is equal to $(2n + 1 - H(g, x))$.

*Proof.* Consider a node $p =< g, x >$ with $H(g, x) = A$ and $A > \lfloor 2n/3 \rfloor$. From lemma 11, the eccentricity of this node is less than or equal to $(2n + 1 - A)$.

We consider two cases.

**Case 1:** $A$ is even. Let $q =< h, y >$ be another node such that for the pair $(p, q)$, $|S_6| = n - A$, $|S_7| = A/2$ and $|S_8| = A/2$. Then by lemma 8 and 9 and corollary 11 we have

$l_1 = 2|S_6| + 2|S_7| + 1 = 2(n - A) + 2(A/2) + 1 = 2n - A + 1$
\[ l_2 = 2|S_5| + 2|S_8| + 2 = 2(n - A) + 2(A/2) + 2 = 2n - A + 2 \\
l_3 = 2|S_5| + 2|S_7| + 2|S_8| + 2 = 2(A/2) + 2(A/2) + 2 = 2A + 2 \\
\]

Here, \( l_1 < l_2 \) and as \( A > \lfloor 2n/3 \rfloor \), \( l_1 < l_3 \). Hence, eccentricity of node \( p \) is \((2n + 1 - A)\).

**Case 2:** \( A \) is odd. Let \( q \) be such that \( |S_T| = (A + 1)/2 \), \( |S_8| = (A - 1)/2 \) and \( |S_5| = n - A \). Proceeding as before, \( l_1 = 2n - A + 2 \), \( l_2 = 2n - A + 1 \) and \( l_3 = 2A + 2 \).

Here, \( l_2 < l_1 \) and \( l_2 < l_3 \). Hence, eccentricity of node \( p \) is \((2n + 1 - A)\).

**Lemma 14.** In an E-OTIS-\( Q_n \), the eccentricity of a node \(<g, x>\) is at most \( n + \lfloor \frac{H(g, x)+3}{2} \rfloor \).

**Proof.** Consider a node \( p = <g, x> \) with \( H(g, x) = A \) and \( A < 2n/3 \). Consider another node \( q = <h, y> \). Then by lemma [8][9] and [17] we have,

\[
\begin{align*}
&l_1 \leq T + 2|S_5| + 2|S_7| + 2|S_8| + 3 \\
&l_1 = T + 2|S_6| + 2|S_7| + 1 \\
&l_2 = T + 2|S_6| + 2|S_8| + 2 \\
&\text{We consider two cases.} \\
\end{align*}
\]

**Case 1:** \( |S_T| \leq |S_8| \). Here \( l_1 < l_2 \). Hence, path between \( p \) and \( q \) is of length \( l \leq (l_1 + l_2)/2 \).

\[
\begin{align*}
l_1 + l_3 &= 2(T + |S_5| + |S_6| + |S_T| + |S_8|) + 2|S_T| + 4 \\
&\Rightarrow l_1 + l_3 \leq 2n + |S_T| + |S_8| + 4 \\
&\text{If } |S_5| = 0, \text{ then } l_3 \text{ is reduced by 1 (corollary 1)} \text{ and } l_1 + l_3 \leq 2n + |S_T| + |S_8| + 3. \\
&\text{As } |S_T| + |S_8| \leq A, \text{ then } l_1 + l_3 \leq 2n + A + 3. \\
&\text{If } |S_5| > 0, |S_T| + |S_8| \leq A - 1 \text{ and } l_1 + l_3 \leq 2n + A + 3. \\
\end{align*}
\]

**Case 2:** \( |S_T| > |S_8| \). Here, \( l_2 < l_1 \). Path between \( p \) and \( q \) is of length \( l \leq (l_2 + l_3)/2 \).

\[
\begin{align*}
l_2 + l_3 &= 2n + 2|S_8| + 5 \leq 2n + |S_8| + (|S_T| - 1) + 5 = 2n + |S_8| + |S_T| + 4. \\
&\text{Proceeding as in Case 1, we can show that } l_2 + l_3 \leq 2n + A + 3. \\
&\text{Combining cases 1 and 2, we conclude that eccentricity of } p \text{ is at most } n + \lfloor \frac{H(g, x)+3}{2} \rfloor. \\
\end{align*}
\]

**Lemma 15.** In an E-OTIS-\( Q_n \), for \( H(g, x) \leq \lfloor 2n/3 \rfloor \), the eccentricity of a node \(<g, x>\) is equal to \( n + \lfloor \frac{H(g, x)+3}{2} \rfloor \).

**Proof.** Consider a node \( p = <g, x> \) with \( H(g, x) = A \) and \( A \leq 2n/3 \). From lemma [14] the eccentricity of this node is less than or equal to \( n + \lfloor \frac{A+3}{2} \rfloor \).

To show that eccentricity of node \( p \) is equal to \( n + \lfloor \frac{A+3}{2} \rfloor \) we consider the following cases.

**Case 1:** \( A \) is even : Consider two sub cases

**Case 1a:** \( (2n - A) = 0 \mod 4 \). Let us take \( q \) such that \( |S_T| = A/2, |S_8| = A/2, |S_5| = (2n - 3A)/4 \), and \( |S_6| = (2n - A)/4 \).

Then by corollary [1] and lemma [8][9] we have, \( l_3 = 2(2n - 3A)/4 + A + A + 2 = n + A/2 + 2 \\
l_1 = 2(2n - A)/4 + A + 1 = n + A/2 + 1 \) and \( l_2 > l_1 \).
Hence minimum of $l_1, l_2$ and $l_3$ is $n + (A + 2)/2$. For even $A$, $n + (A + 2)/2 = n + \lfloor \frac{A+3}{2} \rfloor$.

**Case 1b:** $(2n - A) = 2 \mod 4$. Let $|S_7| = A/2$, $|S_8| = A/2$, $|S_9| = \lfloor \frac{2n - 3A}{4} \rfloor$, and $|S_6| = (2n - A + 2)/4$.

Then by corollary 1 and lemma 8, we have, $l_3 = 2(2n - 3A - 2)/4 + A + 1 = n + A/2 + 1$

$l_1 = 2(2n - A + 2)/4 + A + 1 = n + A/2 + 2$ and $l_2 > l_1$.

Hence minimum of $l_1, l_2$ and $l_3$ is $n + (A + 2)/2$. For even $A$, $n + (A + 2)/2 = n + \lfloor \frac{A+3}{2} \rfloor$.

**Case 2:** $A$ is odd.

**Case 2a:** $(2n - A) = 1 \mod 4$. Let $S_3 = 1$, $|S_7| = (A+1)/2$, $|S_8| = (A-1)/2$, $|S_5| = \lfloor \frac{2n - 3A}{4} \rfloor = (2n - 3A - 3)/4$, and $|S_6| = (2n - A - 1)/4$.

Then by corollary 1 and lemma 8, we have, $l_3 = 1 + 2(2n - 3A - 3)/4 + (A - 1) + (A + 1)/2 = n + (A + 3)/2$

$l_2 = 2(2n - A - 1)/4 + (A - 1)/2/2 + 2 = n + (A + 3)/2$ and $l_2 < l_1$.

**Case 2b:** $(2n - A) = 3 \mod 4$. Let $|S_7| = (A + 1)/2$, $|S_8| = (A - 1)/2$, $|S_5| = \lfloor \frac{2n - 3A}{4} \rfloor = (2n - 3A - 1)/4$, and $|S_6| = (2n - A + 1)/4$.

Then by corollary 1 and lemma 8, we have, $l_3 = 2(2n - 3A - 1)/4 + (A - 1) + (A + 1)/2 = n + (A + 3)/2$

$l_2 = 2(2n - A + 1)/4 + (A - 1)/2/2 + 2 = n + (A + 3)/2$ and $l_2 < l_1$.

Hence, combining Cases 1 and 2, we prove that eccentricity of node $<g, x>$ is $n + \lfloor \frac{H(g, x) + 3}{2} \rfloor$ for $H(g, x) \leq \lfloor 2n/3 \rfloor$.

Table 1 compares the eccentricities of an OTIS-$Q_8$ and an E-OTIS-$Q_8$ for different values of $H(g, x)$.

**Remark:** The variation in eccentricity of the nodes is less in E-OTIS-$Q_n$ compared to OTIS-$Q_n$. For $H(g, x)$ close to $n$, the eccentricity values are equal, and for $H(g, x)$ close to zero, the eccentricity of the nodes in E-OTIS-$Q_n$ is almost half of that in OTIS-$Q_n$.

| Table 1. Eccentricity in OTIS-$Q_8$ and E-OTIS-$Q_8$ |
|-----------------|-----------------|-----------------|
| $H(g, x)$     | eccentricity in OTIS | eccentricity in E-OTIS |
|----------------|-------------------|-------------------|
| 0             | 17                | 9                 |
| 1             | 16                | 10                |
| 2             | 15                | 10                |
| 3             | 14                | 11                |
| 4             | 13                | 11                |
| 5             | 12                | 12                |
| 6             | 11                | 11                |
| 7             | 10                | 10                |
| 8             | 9                 | 9                 |

4.2 **Average eccentricity of OTIS-$Q_n$ and E-OTIS-$Q_n**

The following lemma establishes average eccentricity of OTIS-$Q_n$.

**Lemma 16.** *Average eccentricity of OTIS-$Q_n$ is $(3n + 2)/2*
Proof. The eccentricity of a node $<g, x>$ depends only on $H(g, x)$ irrespective of the value of $g$. So, to find the average we can consider only the nodes of one particular group. For a given $g$, the number of nodes $<g, x>$ with $H(g, x) = k$ is $\binom{n}{k}$ and they are of eccentricity $2n + 1 - k$.

Hence sum of the eccentricities of all the nodes in a group is given by
\[
\sum_{k=0}^{n} \binom{n}{k}(2n + 1 - k) = (2n + 1)2^n - n\sum_{k=1}^{n} \binom{n-1}{k-1}
\]
\[
= (2n + 1)2^n - n2^{n-1} = (3n + 2)2^{n-1}
\] (1)

Dividing the sum by total number of nodes in a group i.e, $2^n$, we get the average eccentricity as $(3n + 2)/2$.

Similarly to find the average eccentricity of E-OTIS-$Q_n$, we note that for $H(g, x) < \lfloor 2n/3 \rfloor$, eccentricity of $<g, x>$ is $n + \lfloor \frac{H(g, x)+3}{2} \rfloor$ and for $H(g, x) > 2n/3$, the eccentricity is $2n + 1 - H(g, x)$.

Hence average eccentricity is equal to
\[
\frac{1}{2^n}\left( \sum_{k=0}^{\lfloor 2n/3 \rfloor} \binom{n}{k}(n + \lfloor \frac{k+3}{2} \rfloor) + \sum_{k=\lfloor 2n/3 \rfloor+1}^{n} \binom{n}{k}(2n + 1 - k) \right)
\] (2)

It is not possible to give a simplified expression for eccentricity in this case. We evaluated eccentricity of E-OTIS-$Q_n$ for different values of $n$ and put them on a table below. We also put the value $(3n + 2)/2$ obtained as average eccentricity of OTIS-$Q_n$.

4.3 Diameter of E-OTIS-$Q_n$

We can now use the eccentricity values to find the diameter of E-OTIS-$Q_n$.

Theorem 2. Diameter of E-OTIS-$Q_n$ is equal to $\lfloor \frac{4n+4}{3} \rfloor$.

Proof. Given a node $<g, x>$, in E-OTIS-$Q_n$, its eccentricity is $2n + 1 - H(g, x)$ for $H(g, x) > \lfloor \frac{2n}{3} \rfloor$ by lemma 13. So, for $H(g, x)$ ranging from $n$ to $\lfloor 2n/3 \rfloor + 1$ eccentricity ranges from $n + 1$ to $2n - \lfloor 2n/3 \rfloor$. Again, by lemma 13 for $H(g, x) \leq \lfloor \frac{2n}{3} \rfloor$, the eccentricity of a node $<g, x>$ is equal to $n + \lfloor \frac{H(g, x)+3}{2} \rfloor$. For $H(g, x)$ ranging from $0$ to $\lfloor \frac{2n}{3} \rfloor$ the eccentricity of a node $<g, x>$ ranges from $n + 1$ to $\lfloor \frac{4n+4}{3} \rfloor$. Hence, the proof.

The smaller diameter and eccentricities can have important application in implementing parallel algorithm on E-OTIS-$Q_n$. Since E-OTIS-$Q_n$ has OTIS-$Q_n$ as subgraph, all the algorithms for OTIS-$Q_n$ can be mapped to E-OTIS-$Q_n$ without change. In those algorithm whenever there is a need for routing between nodes, the shortest path routing algorithm developed here can be applied and if there is a need for single node broadcast, the node having lower eccentricity can be chosen as the originator for broadcast and thus reducing the time for broadcast.

Table 2. Average Eccentricity
Average Eccentricity of Average Eccentricity of
OTIS-\(Q_n\) E-OTIS-\(Q_n\)
\[\begin{array}{|c|c|c|}
\hline
n & \text{Average Eccentricity} & \text{Average Eccentricity} \\
   & \text{of} & \text{of} \\
   & \text{OTIS-}\(Q_n\) & \text{E-OTIS-}\(Q_n\) \\
\hline
4 & 7.0 & 5.875 \\
5 & 8.5 & 7.250 \\
6 & 10.0 & 8.516 \\
7 & 11.5 & 9.695 \\
8 & 13.0 & 11.031 \\
9 & 14.5 & 12.297 \\
10 & 16.0 & 13.501 \\
\hline
\end{array}\]

5 Conclusion

For graphs which are node symmetric the eccentricity of all the nodes are same and equal to its diameter. But for graphs which are not node-symmetric, the eccentricity of the nodes can vary. We observe that minimum eccentricity of OTIS-\(Q_n\) and E-OTIS-\(Q_n\) is almost half of the diameter of OTIS-\(Q_n\). In this work, we make a classification of the nodes in OTIS-\(Q_n\) and E-OTIS-\(Q_n\) based on their position within a group. It is shown that the nodes belonging to the same class have the same eccentricity. Specifically, we have shown that the eccentricity of a node \(<g, x>\), in an OTIS-\(Q_n\) is \(2n + 1 - H(g, x)\). For E-OTIS-\(Q_n\), the eccentricity of a node \(<g, x>\) is equal to \(n + \left\lfloor\frac{H(g, x) + 3}{2}\right\rfloor\), when \(H(g, x) \leq \left\lfloor\frac{2n}{3}\right\rfloor\), and \(2n + 1 - H(g, x)\), when \(H(g, x) > \left\lfloor\frac{2n}{3}\right\rfloor\). Similar classification can be attempted on other OTIS networks.

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