A LINEAR SYSTEM ON NARUKI'S MODULI SPACE OF MARKED CUBIC SURFACES

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Abstract. Allcock and Freitag recently showed that the moduli space of marked cubic surfaces is a subvariety of a nine dimensional projective space which is defined by cubic equations. They used the theory of automorphic forms on ball quotients to obtain these results. Here we describe the same embedding using Naruki’s toric model of the moduli space. We also give an explicit parametrization of the tritangent divisors, we discuss another way to find equations for the image and we show that the moduli space maps, with degree at least ten, onto the unique quintic hypersurface in a five dimensional projective space which is invariant under the action of the Weyl group of the root system $E_6$.

Introduction

Recently Allcock, Carlson and Toledo [ACT] studied the moduli space of smooth cubic surfaces using the intermediate jacobian of the cubic threefold which is the triple cover of projective three space branched along a cubic surface. They show that this moduli space, as well as the moduli space of marked cubic surfaces $M^0$ (that is, cubic surfaces with an ordered set of six skew lines) are open subsets of certain 4-ball quotients. The Weyl group $W(E_6)$ of the root system $E_6$ acts on $M^0$ by permuting the markings on any given cubic surface, the quotient variety is the moduli space of cubic surfaces. The quasi projective variety $M^0$ has a natural compactification $\overline{M}$ given by geometrical invariant theory. The projective variety $M$ coincides with the Baily-Borel compactification of the ball quotient. The action of $W(E_6)$ extends to $\overline{M}$.

Using Borcherds’ work on automorphic forms on ball quotients, Allcock and Freitag [AF] found a $W(E_6)$-equivariant embedding of $\overline{M}$ in a nine dimensional projective space. The action of $W(E_6)$ on the projective space is obtained from the unique ten dimensional irreducible linear representation of $W(E_6)$. This map actually already appears in a paper by A. B. Coble published in 1917 [C] (and see also [Y]) where $\overline{M}$ is identified with the moduli space of six points in the projective plane. The same embedding of $\overline{M}$ was also found by Matsumoto and Terasoma [MT] who used the theta constants associated to the intermediate jacobians.

An explicit smooth projective compactification $\mathcal{C}$ (‘the cross ratio variety’) of the moduli space $\mathcal{M}$ with a biregular action of the Weyl group was constructed by Naruki [N]. It is a modification of a toric variety associated to the root system $D_4$. Naruki constructs and studies his model as a subvariety of the product of 270 projective lines, each component of this map is given by a cross ratio (of certain tritangent planes containing a given line on the cubic surface). The Weyl group acts via permutations of these 270 projective lines.

In this paper we explicitly identify the nine dimensional linear system on Naruki’s model $\mathcal{C}$ which defines the map $F$ to $\mathbb{P}^9$ discovered by Coble, Allcock and Freitag (see Theorem 5.7)

$$F: \mathcal{C} \longrightarrow \mathcal{M} \quad (\subset \mathbb{P}^9).$$

We also give explicit formulas for the $W(E_6)$-action on this linear system in section 5.
A tritangent plane of a cubic surface is a plane which cuts out three lines on the surface. If these three lines meet in a point, that point is called an Eckart point. We obtain a nice parametrization, equivariant for the Weyl group of the root system $F_4$, of the 45 divisors in $\mathcal{M}$ which parametrize marked cubic surfaces with an Eckart point, see Theorem 6.3. A study of the linear relations between tritangent planes leads to the discovery that $\mathcal{M}$ is the singular locus of a variety $X$ defined by six quintic polynomials, see 7.10. The group $W(E_6)$ acts on $X$ and it would be very interesting to have a moduli interpretation for $X$.

The Weyl group of $E_6$ is defined as a reflection group on a real six dimensional vector space. Complexifying and projectivizing this vector space one obtains a biregular action of $W(E_6)$ on $\mathbb{P}^5$. In his book [H], Bruce Hunt suggested an identification of the moduli space with the unique $W(E_6)$-invariant quintic hypersurface $I_5$ in $\mathbb{P}^5$. In section 8 we construct a dominant rational map $\Sigma : \mathcal{M} \to \mathbb{P}^5$ which is equivariant for the action of $W(E_6)$ and we show that its image is $I_5$ (Thm. 8.6), but, unfortunately, this map has degree at least 10 (Thm 8.8).

The results of this paper are obtained from computations with rational functions on the toric variety, many of them computer assisted. It does lead to very explicit formulas and parametrizations, somewhat in contrast to the ball quotient approach where the modular forms in question are hard to describe explicitly.

I'm indebted to E. Freitag for suggesting to undertake this study and for many discussions. I would also like to thank him and E. Carlini for assistance with the computations.

1. Cubic surfaces their moduli space

1.1. We briefly recall the basics on cubic surfaces and $E_6$, see [H] and references given there for proofs. We relate this to the modular orthogonal geometry used by Allcock and Freitag.

1.2. The 27 lines. Any smooth cubic surface $S$ has 27 lines and there are sets of six disjoint lines $\{a_1, \ldots, a_6\}$. Blowing down the lines $a_i$ to points $p_i$ defines a birational isomorphism $S \to \mathbb{P}^2$. The images of the other 21 lines on $S$ are the 15 lines $<p_i, p_j>$ and the 6 conics which pass through all six points except one of the $p_i$. The corresponding lines are denoted by $c_{ij}$ and $b_j$. The birational inverse $\mathbb{P}^2 \to S$ is given by the linear system of all cubics passing through the points $p_1, \ldots, p_6$.

1.3. The root system $E_6$. The Picard group of $S$ is isomorphic to $\mathbb{Z}^7$ and a $\mathbb{Z}$-basis is given by the pull-back $l$ of (the divisor class of) a line in $\mathbb{P}^2$ and the classes of the lines $a_i$. The intersection form is determined by

$$l^2 = 1, \quad a_i^2 = -1, \quad l \cdot a_i = 0, \quad a_i \cdot a_j = 0$$

for $i \neq j$. The classes of the lines are

$$c_{ij} = l - (a_i + a_j), \quad b_i = 2l - (a_1 + \ldots + a_i + \ldots + a_6).$$

The canonical class of $S$ is $K_S := -3l + a_1 + \ldots + a_6$ and $K_S^2 = 3$. The class of a hyperplane section of $S$ is $-K_S$. The primitive cohomology of $S$ is thus the orthogonal complement of $K_S$. This $\mathbb{Z}$-module, with the bilinear form $(x, y) := -x \cdot y$, is isomorphic to the root lattice $Q(E_6)$ of the root system $E_6$:

$$Q(E_6) \cong K_S^\perp := \{x \in \text{Pic}(S) : x \cdot K_S = 0 \}.$$
A basis of simple roots of $E_6$ is given by:

$$
\alpha_1 = a_2 - a_1, \quad \alpha_3 = a_3 - a_2, \quad \alpha_4 = a_4 - a_3, \quad \alpha_5 = a_5 - a_4, \quad \alpha_6 = a_6 - a_5,
$$

$$\alpha_2 = l - a_4 - a_5 - a_6.$$

This is a basis of simple roots of $E_6$:

$$
\alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6
$$

\[ \alpha_2 \]

The set $E_6^+$ of positive roots of $E_6$ consists of the 36 elements in $Q(E_6)$ given by

$$
h_{ij} := -a_i + a_j, \quad (i < j) \quad h_{ijk} := l - a_p - a_q - a_r, \quad h := 2l - a_1 - \ldots - a_6,
$$

where $\{i, j, k, l, p, q, r\} = \{1, 2, \ldots, 6\}$. In particular, $\alpha_2 = h_{123}$ and with this convention our notation is compatible with that of [H]. The root system $E_6 := E_6^+ \cup (-E_6^-) \subset Pic(S)$ contains 72 vectors, called roots.

1.4. **The Weyl group** $W(E_6)$. The Weyl group $W(E_6)$ is the subgroup of $GL(Q(E_6))$ generated by reflections in the roots. We denote by $s_i$ the reflection in the hyperplane perpendicular to the root $\alpha_i$. More generally we write $s_\alpha$, with $\alpha \in E_6$, for the reflection in the hyperplane perpendicular to $\alpha$.

1.5. **The orthogonal geometry**. Allcock and Freitag use a non-degenerate quadratic form $Q$ on the vector space $F^5_3$ and its orthogonal group $O(5, 3)$ to describe the combinatorics of the lines on a cubic surfaces and of divisors on the moduli space $M$. The basic facts are ([AF], section 2):

$$
O(5, 3) \cong W(E_6) \times \{\pm 1\},
$$

there are 72 vectors with $Q(x) = -1$, these are called the short roots (note $Q(x) = Q(-x)$). There are 90 vectors with $Q(x) = -2$, the long roots, and there are 80 nonzero vectors with $Q(x) = 0$, called isotropic vectors. (See also [MT], §3.)

1.6. **Boundary divisors of $M$**. If a root is the class of an effective divisor on the blow up of $P^2$, then this effective divisor is a $P^1$ which is contracted to a node on the cubic surface. This set up a correspondence between the set of irreducible divisors in $M$ parametrizing nodal cubic surfaces and $E_6^+$. These divisors are labelled by pairs $\pm x$ of ‘short roots’ in [AF].

The divisor in $M$ corresponding to $\alpha \in E_6^+$ is denoted by $D_\alpha$ (or by $D_{ij}$ if $\alpha = h_{ij}$ etc.). These divisors are the fixed point sets of the corresponding reflections $s_\alpha \in W(E_6)$ in $M$. The reflection $s_\alpha \in Aut(Pic(S))$ may be identified with the Picard-Lefschetz transformation associated to the general nodal cubic surface $S_0$ in $D_\alpha$.

1.7. **Lines and weights**. Let $P(E_6) \subset Q(E_6) \otimes Q$ be the weight lattice of $E_6$:

$$
P(E_6) := \{x \in Q(E_6) \otimes Z \mid (x, y) \in Z, \forall y \in Q(E_6)\}.
$$

The intersection number of the class $c$ of a line on $S$ with a root is an integer, hence $c$ defines an element $x_c \in P(E_6)$. In this way one obtains a $W(E_6)$-orbit of 27 weights (which are also denoted by $a_i$, $b_i$, $c_{ij}$ with $1 \leq i \leq 6$, $1 \leq i < j \leq 6$, cf. [H], § 6.1.3). Note that
1.8. **The tritangent planes and tritangent divisors.** Since hyperplane sections of $S$ correspond to cubics on the $p_i$, it is easy to see that there are 45 planes, the tritangent planes, which intersect $S$ in three lines, in Schl"afli’s notation these are denoted by:

$$ (ij) = \{a_i, b_j, c_{ij}\}, \quad (ij.kl.mn) = \{c_{ij}, c_{kl}, c_{mn}\}, $$

where $i, \ldots, n = \{1, \ldots, 6\}$. Another labelling for the tritangents was given by Cayley and is used by Naruki. The dictionary between the labels is given in [Sp], p.371. The 45 tritangent divisors in $\mathcal{M}$ are written as $D_t$ where $t$ is one of Schl"afli’s labels. The tritangent divisors correspond to pairs $\pm x$ of long roots of $\mathfrak{E}$. Three lines lie in a tritangent plane iff the sum of their classes in $\mathbb{Z}_S$ is minus a fundamental root of $E_6$. Thus $\mathfrak{E}$ is linearly dependent. The orthogonal complement in $\mathfrak{D}$ are written as $D_t$ where $t$ is one of Schl"afli’s labels. The tritangent divisors correspond to pairs $\pm x$ of long roots of $\mathfrak{E}$.

1.9. **The subsystem $D_4$.** An important example is the case that $t = (16) = w$. In that case $t^\perp$ is the $D_4 \subset E_6$ spanned by the simple roots $\alpha_2$, $\alpha_3$, $\alpha_4$ and $\alpha_5$. This root system is discussed in section 4.

1.10. **The $W(F_4)$ and tritangents.** To a tritangent $t$ one associates an element $\gamma(t) \in W(E_6)$ which is the product of the relections in 4 orthogonal roots in $t^\perp \cong D_4$. Thus $\gamma(t)$ is $-I$ on the span of $t^\perp$ and is $+I$ on the orthogonal complement which is the span on the subspace spanned by the weights corresponding to the lines in $t$. For $t = (16) = w$ one may take $\gamma(w) = s_2s_5s_3(s_4s_5s_3s_4)s_2(s_4s_5s_3s_4)$.

The tritangent divisors $D_t$ which correspond (via their $+1$-eigenspace) with the tritangents. The centralizer of a $\gamma(t)$ in $W(E_6)$ is isomorphic to the Weyl group $W(F_4)$. The fixed point set of a $\gamma(t)$ on $\mathcal{C}$ is the tritangent divisor $D_t$ which parametrises cubic surfaces for which the three lines in $t$ meet in one point, called an Eckart point ([N], §8).

### 2. The toric variety

2.1. For general facts on toroidal compactifications we refer to [Fl], for root systems see [Hu].

2.2. **The torus.** The $D_4$-adjoint torus

$$ T \xrightarrow{\cong} (\mathbb{C}^*)^4, \quad t \mapsto (\lambda(t), \mu(t), \nu(t), \rho(t)) $$

comes with a natural identification of its character group $\text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^4$ with the sublattice

$$ M := \langle e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4 \rangle \subset \bigoplus_{i=1}^4 \mathbb{Z}e_i. $$

The lattice $M$, with the scalar product induced by the standard inner product on $\bigoplus \mathbb{Z}e_i$, is the root lattice $Q(D_4)$ of $D_4$. We often use:

$$ \text{Hom}(T, \mathbb{C}^*) \xrightarrow{\cong} M, \quad \lambda \mapsto e_1 - e_2, \quad \mu \mapsto e_3 + e_4, \quad \nu \mapsto e_3 - e_4, \quad \rho \mapsto e_2 - e_3. $$

For $\alpha \in M$ we define a regular function on $T$ by:

$$ f_\alpha := \lambda^a \mu^b \nu^c \rho^d \quad \text{with} \quad \alpha = a(e_1 - e_2) + b(e_2 - e_3) + c(e_3 - e_4) + d(e_3 + e_4) \in M. $$

...
2.3. **The root system.** The root system $D_4$ consists of the following 24 vectors in $M$:

$$D_4 = \{ \pm e_i \pm e_j \in M : \ 1 \leq i < j \leq 4 \}.$$  

The set

$$\Delta_0 := \{ e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4 \} \quad (\subset D_4)$$

is a fundamental system (or base of the root system), that is any root is a linear combination of these 4 vectors with all coefficients either positive (such a root is called positive) or negative. Let $N = M^*$ be the dual lattice of $M$,

$$N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) = \{ x \in (\oplus \mathbb{Z}e_i)^* \otimes \mathbb{Z} R : \langle x, \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in M \},$$

here $\langle ., . \rangle$ is the pairing between $(\oplus \mathbb{Z}e_i)^* \otimes \mathbb{Z} R$ and its dual. Let $\{ \epsilon_1, \ldots, \epsilon_4 \} \subset (\oplus \mathbb{Z}e_i)^* \otimes \mathbb{Z} R$ be the dual basis of $\{ e_1, \ldots, e_4 \}$. Then the basis of $N$ which is dual to $\Delta_0$ is

$$\epsilon_1, \quad (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2, \quad (\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)/2, \quad \epsilon_1 + \epsilon_2 \quad (\in N).$$

2.4. **The Weyl group.** The Weyl group $W(D_4)$ of the root system is the subgroup of $GL(M \otimes \mathbb{R})$ generated by the reflections in the roots (so $s_\alpha(\beta) = \beta - (\beta, \alpha)\alpha$) and $(\ldots)$ is the standard inner product on $\oplus \mathbb{Z}e_i$. This group has 192 elements and is a semidirect product of $S_4$ (permuting the $e_i$) and $(\mathbb{Z}/2\mathbb{Z})^3$ (changes the sign of an even number of the $e_i$). The Weyl group acts simply transitively on the fundamental systems.

The Weyl group acts on $N$ and the 4 elements of the dual basis above are in distinct orbits of lengths 8, 8, 8 and 24 respectively. We define

$$S := \{ \pm \epsilon_1 \} \cup \{ (\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2 \}, \quad R := \{ \pm \epsilon_i \pm \epsilon_j \},$$

$S$ and $R$ each have 24 elements.

2.5. **The Weyl chambers.** The (closed) Weyl chamber $C(\Delta)$ of a fundamental system $\Delta \subset D_4$ is the (maximal) cone in $N \otimes \mathbb{Z} R = (\oplus \mathbb{Z}e_i)^* \otimes \mathbb{Z} R$ defined by:

$$C(\Delta) := \{ x \in N \otimes \mathbb{Z} R : \langle x, \alpha \rangle \geq 0 \ \forall \alpha \in \Delta \}.$$  

If $\Delta = \{ \alpha_1, \ldots, \alpha_4 \}$ then the edges (i.e. the one dimensional faces) of $C(\Delta)$ are the 4 half-lines $R_{\geq 0}\tau_i$ with $\{ \tau_1, \ldots, \tau_4 \}$ the dual basis of $\Delta$. The decomposition

$$N \otimes \mathbb{Z} R = \cup_{\Delta} C(\Delta)$$

is a regular cone decomposition of the vector space $N \otimes \mathbb{Z} R$, it defines in a fan in $N$ whose faces are the faces of the 192 Weyl chambers. This fan has 48 edges which correspond to the elements of $S \cup R$.

2.6. **The toroidal compactification.** Associated to this fan is a toric variety $\tilde{T}$,

$$\tilde{T} = \cup_{\Delta} A(\Delta), \quad A(\Delta) \cong \mathbb{C}^4$$

and the inclusion $T \subset A(\Delta)$ is defined by the inclusion of the rings of regular functions

$$C[A(\Delta)] := \{ f_\alpha : \alpha \in M, \ \langle x, \alpha \rangle \geq 0 \ \forall x \in C(\Delta) \} \hookrightarrow C[T] := C[\lambda^{\pm 1}, \mu^{\pm 1}, \nu^{\pm 1}, \rho^{\pm 1}].$$

For example $C[A(\Delta_0)] = C[\lambda, \mu, \nu, \rho]$. Each edge $R_{\geq 0}\tau$, with $\tau \in S \cup R$, defines a divisor $V(\tau)$ in $\tilde{T}$ ([14], §3.3) and these 48 divisors are the complement of $T$ in $\tilde{T}$:

$$\tilde{T} - T = \cup_{\tau \in S \cup R} V(\tau).$$
The regular functions \( f_\alpha, \alpha \in M \), on \( T \) extend to rational functions on \( \tilde{T} \). The divisor of \( f_\alpha \) is given by:
\[
(f_\alpha) = \sum_\tau n_\tau V(\tau) \quad \text{with} \quad n_\tau := \langle \tau, \alpha \rangle.
\]

### 2.7. Example.

The divisor of \( \lambda = f_{e_1-e_2} \) is given by:
\[
(\lambda) = D^+ - D^- \quad \text{with} \quad \begin{cases} 
D^+_\lambda = V(\epsilon_1) + V(-\epsilon_2) + \sum_{\pm, \pm} V((\epsilon_1 - \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) / 2) + D' \\
D^-_\lambda = V(-\epsilon_1) + V(\epsilon_2) + \sum_{\pm, \pm} V((-\epsilon_1 + \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) / 2) + D''
\end{cases}
\]
where \( D' \) and \( D'' \) are combinations of the divisors \( V(\tau) \) with \( \tau \in R \) with coefficients in \( \{-2, -1, 0, 1, 2\} \).

### 2.8. The cross ratio variety.

Naruki’s (smooth, projective) cross ratio variety \( C \) is obtained from the toric variety \( \tilde{T} \) as follows ([N], §10-12):
\[
\mathcal{M} \leftarrow C \leftarrow \tilde{T} \xrightarrow{\pi''} \tilde{T}'' \xrightarrow{\pi'} \tilde{T}' \xrightarrow{\pi} \tilde{T}.
\]

The map \( \pi_e \) is the blow up of \( \tilde{T} \) in the identity element \( e \in T \). The exceptional divisor \( \pi_e^{-1}(e) \cong \mathbb{P}^3 \) is denoted by \( \mathbb{P}^3_{\pi_e} \). The image in \( \mathcal{M} \) of its strict transform in \( \tilde{T}'' \) is the tritangent divisor \( D_w = D_{(16)} \).

The map \( \pi' \) is the blow up of \( \tilde{T}' \) in the strict transforms in \( \tilde{T}'' \) of the 12 curves in the \( W(D_4) \)-orbit of the curve in \( \tilde{T} \) defined by \( \lambda = \nu = \rho = 1 \). The morphism \( r \) contracts the strict transforms in \( \tilde{T} \) of the 12 exceptional divisors in \( \tilde{T}'' \) to surfaces in \( C \) and is an isomorphism on the complement ([N], Prop. 11.3).

The map \( \pi'' \) is the blow up in the strict transform in \( \tilde{T}'' \) of the 16 surfaces in the \( W(D_4) \)-orbit of \( \mu = \rho = 1 \). The 16 exceptional divisors in \( \tilde{T} \) map under \( r \) to divisors in \( C \), their \( W(E_6) \)-orbit consists of 40 divisors, the other 24 are the images under \( r \) of the strict transforms of the \( V(\tau) \)'s with \( \tau \in R \) ([N], Prop. 11.2). We call these 40 divisors the cusp divisors of \( \tilde{T} \).

There is a morphism \( C \to \mathcal{M} \), where \( \mathcal{M} \) is the moduli space of semistable marked cubic surfaces, which contracts the 40 cusp divisors to points (cf. [N], Introduction and §12), the cusps of \( \mathcal{M} \). The Weyl group \( W(E_6) \) acts biregularly on \( C \) and \( \mathcal{M} \) and the morphism \( C \to \mathcal{M} \) is \( W(E_6) \)-equivariant.

### 3. The \( W(E_6) \)-action on boundary divisors.

#### 3.1. According to Naruki [N], Prop. 11.3’, the boundary \( C - \mathcal{M}^0 \) consists of two \( W(E_6) \)-orbits of divisors, one orbit is formed by the 36 boundary divisors \( D_\alpha \) with \( \alpha \in E_6^- \). The other orbit consists of the 40 cusp divisors and will not be of interest for us. In Naruki’s toroidal construction, the 36 \( D_\alpha \)'s are parametrized by the 12 positive roots \( D^+_4 \) of \( D_4 \) and by the 24 elements of a set of \( S \) (see 2.4) of weights of \( D_4 \). In this section we determine the corresponding \( W(D_4) \)-equivariant bijection between \( E_6/\{ \pm 1 \} \) and \( (D_4/\{ \pm 1 \}) \cup S \), see table 9.2 for the final result.
3.2. To do the required computations, it is sufficient to work on the blow up of $\tilde{T}$ in the origin, rather than on $C$ or $M$, cf. [2.8]. For each positive root $\alpha \in D_1$ the closure in $\tilde{T}$ of the subtorus defined by $f_\alpha = 1$ in $T$ is an irreducible divisor. Since it contains $e$, its pull-back to $\tilde{T}'$ has two irreducible components, one is $P^3_w$ and the other is its strict transform which we will denote by $D^1_\alpha$. The image in $M$ of the strict transform of $D^1_\alpha$ in $\tilde{T}$ is $D_\alpha$, so these twelve divisors are labelled via $D_1 = w^+ \subset E_0$.

The other 24 boundary divisors in $M$ are the images in $M$ of the strict transforms of the $V(\beta)$ with $\beta \in S$, Prop. 11.1). The Weyl group $W(D_4)$ has three orbits on $S$ and it suffices to identify one divisor from each orbit. That is done in the following lemma. The resulting labelling of all 36 divisors is given in table 9.2.

3.3. Lemma. Let $s_1, s_6$ be the reflections in $W(E_6)$ defined by the roots $\alpha_1 = h_{12}, \alpha_6 = h_{56}$ respectively. Then we have:

$$s_1^* D^1_\lambda = V(-\epsilon_2)$$

hence $V(-\epsilon_2) = D_{13}$. Similarly we have:

$$s_6^* D^1_{\lambda \nu \rho} = V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2),$$

$$s_1^* V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2) = V((\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2)$$

and thus $V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2) = D_{26}, V((\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2) = D_{16}$.

Proof. The divisor of the rational function $\lambda - 1$ on $\tilde{T}'$ is

$$(\lambda - 1) = D^1_\lambda + P^3_w - D^1_\lambda,$$

where $D^1_\lambda$ is as in [2.4]. Therefore $s_1^*(\lambda - 1)$ will have exactly two effective components, one being $s_1^* D^1_\lambda$ which must be in the orbit of length 36 and the other will be a tritangent divisor. From [N], p. 13 we have:

$$s_1 : \lambda \mapsto \frac{\lambda \mu \nu \rho (1 - \lambda)}{\lambda \mu \nu \rho^2 - 1}$$

and hence that

$$s_1 : \lambda - 1 \mapsto f_1 := \frac{1 - \lambda^2 \mu \nu \rho^2}{\lambda \mu \nu \rho^2 - 1}.$$ 

Since $\lambda^2 \mu \nu \rho^2 = f_{2e_1}$ (note that $\lambda^2 \mu \nu \rho^2$ is not a root) and $\lambda \mu \nu \rho^2 = f_{e_1 + e_2}$ we see that the denominator has a pole of order one on $V(-\epsilon_2)$ but the numerator has vanishing order zero on that divisor, hence $V(-\epsilon_2)$ must be one of the two effective components of $(f_1)$. The other effective component is defined by $1 - \lambda^2 \mu \nu \rho^2 = 0$, which is the local equation of the tritangent divisor $D_8$ ($\tilde{x} = (26)$, cf. Table 3 of [N]). Note that $\lambda = f_{e_1 - e_2}$ and $e_1 - e_2 = h_{23}$, so $D^1_\lambda = D_{23}$ and that $s_1$ permutes the indices 1 and 2 of an $h_{ij}$, hence $s_1^* D_{23} = D_{13}$ and $s_1^* D_{(16)} = D_{(26)}$.

Using the formulas from [N], p. 13 again we get:

$$s_6 : 1 - \lambda \nu \rho \mapsto f_2 := \frac{1 - \lambda \mu \nu \rho^2}{1 - \mu \nu \rho}.$$ 

Since $\lambda \mu \nu \rho^2 = f_{e_1 + e_2 + e_3 - e_4}$ and $\mu \nu \rho = f_{e_2 + e_3}$, we see that $V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2)$ is one of the two effective components of $(f_2)$. The other component corresponds to the tritangent divisor $D_9 = D_{(15)}$ defined by $1 - \lambda \mu \nu \rho^2 = 0$. Note that $\lambda \nu \rho = f_{e_1 - e_4}$ and $e_1 - e_4 = h_{25}$, so $D^1_{\lambda \nu \rho} = D_{25}$ and that $s_6$ permutes the indices 5 and 6 of an $h_{ij}$, hence $s_6^* D_{25} = D_{26}$ and $s_6^* D_{(16)} = D_{(15)}$. 

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Next we apply $s_1$ to $f_2$ and obtain:

$$s_6 : f_2 \mapsto f_3 := \frac{-\mu\rho(\lambda + \nu - \lambda\nu - \lambda\mu\nu\rho - \lambda^2\mu\nu^2\rho^2)}{(\lambda\mu\rho - 1)(\lambda\nu\rho - 1)}$$

In the open subset $U = A(\Delta_0) = Spec(\mathbb{C}[\lambda, \mu, \nu, \rho])$, this function is zero on $\mu = 0$, which is $V((\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2) \cap U$ (the zero locus in $U$ of the $i$-th element in $\{\lambda, \mu, \nu, \rho\}$ is the divisor corresponding to the $i$-th element in the dual basis). Thus we found one of the two effective components of the divisor of $f_3$. Note that $(\rho = 0) \cap U$ lies in $V(\epsilon_1 + \epsilon_2)$, which is not in the orbit of the 36 divisors and that the third factor of the numerator of $f_3$ defines the tritangent divisor labelled by $\bar{q}_1 = (25)$. Since $s_1$ permutes the indices 1 and 2 of an $h_{ij}$, we get $s_1^*D_{26} = D_{16}$ and $s_1^*D_{(15)} = D_{(25)}$.

3.4. The labelling of these 36 divisors on $\tilde{T}'$ allows us to express various divisors in a convenient manner. For example (cf. 2.7):

$$(\lambda) = D_{13} + D_{26} + D_{346} + D_{246} + D_{256} + D_{345} - D_{12} - D_{36} - D_{126} - D_{346} - D_{356} - D_{245}$$

and similarly:

$$(\lambda - 1) = D_w + D_{23} - D_{12} - D_{36} - D_{126} - D_{346} - D_{356} - D_{245}.$$

4. The CAF-linear system.

4.1. To identify the linear system on the moduli space $\mathcal{M}$ introduced by Coble, Allcock and Freitag and to describe the $W(E_6)$-action on it, we consider two divisors with support in the boundary of the toric variety $\tilde{T}$.

4.2. Definition. Let $R, S \subset N$ be as in section 2.4. We define divisors in $\tilde{T}$ (cf. 2.4) by:

$$D_S := \sum_{\tau \in S} V(\tau), \quad D_R := \sum_{\tau \in R} V(\tau).$$

4.3. Lemma. We have

$$H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R)) = \langle f_0 = 1 \rangle \oplus \langle f_\alpha : \alpha \in D_4 \rangle,$$

in particular, dim $H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R)) = 25$. The divisor $D_S + 2D_R$ is very ample on $\tilde{T}$.

Proof. The space of global sections of the line bundle associated to a divisor $\sum n_\tau V(\tau)$ is spanned by certain $f_\alpha$’s:

$$H^0(\tilde{T}, \mathcal{O}(\sum n_\tau V(\tau))) = \langle f_\alpha : \alpha \in M \text{ and } \langle \tau, \alpha \rangle \geq -n_\tau \rangle.$$

Thus we must find the $\alpha \in M$ with $\langle \tau, \alpha \rangle \geq -1$ for $\tau \in S$ and $\langle \tau, \alpha \rangle \geq -2$ for $\tau \in R$. Let $\alpha = \sum m_i e_i$ with $m_i \in \mathbb{Z}$. Taking $\tau = \pm e_i \in S$ we get $-1 \leq m_i \leq 1$, taking $\tau = (\pm e_1 \pm \ldots \pm e_4)/2 \in S$ we get $-2 \leq \pm m_1 \pm m_2 \pm m_3 \pm m_4 \leq 2$, hence at most two of the $m_i$ are non zero and thus $\alpha = 0, \pm e_i, \pm e_i \pm e_j$ with $i \neq j$. However $\pm e_i \notin M$ and therefore $\alpha$ is either zero or a root. All these $\alpha$ also satisfy $\langle \tau, \alpha \rangle \geq -2$ for $\tau \in R$.

The proof of the very ampleness is standard, cf. [Fu], and since we do not really need it, we omit the proof. \qed
4.4. Divisors near the identity. The functions $x_r := r - 1$ with $r \in \{\lambda, \mu, \nu, \rho\}$ are local coordinates near the identity element $e = (1, 1, 1, 1) \in T$. Any rational function $f$ on $T$ which is regular in $e$ can be developed in a Taylor series:

$$f = f_d + f_{d+1} + \ldots,$$

with $f_k$ homogeneous of degree $k$ and $d \geq 0$. If the polynomial $f_d$ is not identically zero we say that $f$ vanishes to order $d$ in $e \in T$ and we write $m_e(f) = d$, $f_d$ is called the leading term of $f$.

For $\alpha = a(e_1 - e_2) + \ldots + d(e_2 - e_3) \in M - \{0\}$ we have:

$$f_\alpha - 1 = (x_\lambda + 1)^a(x_\mu + 1)^b(x_\nu + 1)^c(x_\rho + 1)^d - 1 = ax_\lambda + bx_\mu + cx_\nu + dx_\rho + H.O.T.$$

hence $f_\alpha - 1$ vanishes to order 1 at $e$ and a product $\prod_{i=1}^m (f_\alpha_i - 1)$ of such functions vanishes to order $m$ at $e$.

4.5. Definition. We define the vector space $V$ of rational functions on $\tilde{T}$ to be the subspace of those global sections of $\mathcal{O}(D_S + 2D_R)$ which vanish to order at least 3 at $e \in \tilde{T}$:

$$V := \{ f \in H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R)) : m_e(f) \geq 3 \}.$$

4.6. Lemma. The dimension of $V$ is 10. A basis for $V$, multiplied by $\lambda\mu\nu\rho^2$, is given in table \[\text{[table]}.\]

**Proof.** Note that 10 is the expected dimension of $V$ since the spaces of constant, linear and quadratic polynomials in 4 variables have dimension 1, 4, 10 respectively. Thus we only have to show that each monomial $x_\lambda^ax_\mu^bx_\nu^cx_\rho^d$ with $a + b + c + d \leq 2$ is the leading term of a function in $H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R))$. Obviously we can use $f_0 = 1$ to get leading term 1 and the $r - 1$ to get leading term $x_r$. For the roots $\alpha = \lambda\rho, \mu\rho, \nu\rho$ the leading term is a linear combination of the leading terms of the $r - 1$'s which we already have. Subtracting these linear terms we get functions with the leading terms $x_\lambda x_\rho, x_\mu x_\rho, x_\nu x_\rho$. The Taylor series of $t - 1$ with $t = \lambda\nu\rho, \mu\nu\rho, \lambda\mu\rho$, give us, modulo the leading terms we already found, the leading terms $x_\lambda x_\nu, x_\mu x_\nu, x_\lambda x_\nu$. To get the $x_\rho^2$ use that

$$r^{-1} = 1 - x_r + x_r^2 - \ldots.$$

Thus we found all the 15 desired leading terms and we conclude that $V$ has codimension 15 in $H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R))$. \[\square\]

4.7. Example. The following function lies in $V$:

$$\nu^{-1}(r - 1)(\lambda\nu\rho - 1)(\mu\nu\rho - 1) = \lambda\mu\nu\rho^2 - \lambda\mu\nu\rho = \lambda\rho + \mu + \nu^{-1} - (\nu\rho)^{-1}.$$

The first expression shows it vanishes to order three in $e$, the second that it is a linear combination of roots, hence it lies in $H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R))$.

5. The action of $W(E_6)$ on the vector space $V$.

5.1. Naruki $\text{[N]}$ defined a biregular action of $W(E_6)$ on $\mathcal{C}$ ($\text{[N]}, \S 5, \text{p. 13}$). We show that this induces an action of $W(E_6)$ on the vector space $V$ defined in \[\text{4.4}.\] The vector space $V$ may be identified, via pull-back $V \cong H^0(\tilde{T}', \mathcal{O}(2D_R + D_S - 3P^3_w))$ where $\tilde{T}'$ is the blow up of $\tilde{T}$ in the identity element $e$ and $P^3_w$ is the exceptional fiber.
The main problem is to find the images of the divisor $D_S - 3P^3_w$ under $s_1, s_6 \in W(E_6)$ and to show that the images are linearly equivalent to this divisor. For this we use the following rational function:

$$C_1 := \frac{(\lambda^2 \mu \nu \rho^2 - 1)^3}{(\lambda - 1)(\lambda \rho - 1)(\lambda \nu \rho - 1)(\lambda \mu \nu \rho^2 - 1)(\lambda \mu \nu \rho - 1)(\lambda \mu \rho - 1)}.$$ 

5.2. Lemma. The rational function $C_1$ on $\tilde{T}$ has divisor

$$(C_1) = 3D_8 - 3P^3_w + \sum_{\pm, i=2}^4 V(\pm \epsilon_i) - D^1_\lambda - D^1_\lambda \rho - D^1_\lambda \nu \rho - D^1_\lambda \mu \nu \rho^2 - D^1_\lambda \mu \nu \rho - D^1_\lambda \mu \rho + D$$

for some divisor $D$ which is a combination of the divisors $V(\tau)$ with $\tau \in R$. Here $D_8$ is the tritangent divisor defined by the strict transform of the zero locus of $\lambda^2 \mu \nu \rho^2 - 1$ in $\tilde{T}$.

Proof. The proof is straightforward using the formula from [3,6] and the examples in the proof of Lemma [5,3], for example

$$(\lambda^2 \mu \nu \rho^2 - 1) = D_8 + P^3_w - 2V(-\epsilon_1) - \sum_{\pm, \pm} V((-\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2),$$

and the divisor of $\lambda - 1$, in $\tilde{T}$, was determined in [2,7].

5.3. For $f \in V$, the composition $f \circ s_1$ does not lie in $V$. However, we will show that the quotient $(f \circ s_1)/C_1$ does lie in $V$. To get an action of all of $W(E_6)$ however, the correct definition for the action of $s_1$ on $V$ is $s_1(f) = -(f \circ s_1)/C_1$.

5.4. Theorem. The action of $W(E_6)$ on $\mathcal{C}$ defines an action of $W(E_6)$ on $V$ by the following formulas:

$$s_i(f) := \begin{cases} 
-(f \circ s_1)/C_1 & \text{if } i = 1, \\
 f \circ s_i & \text{if } 2 \leq i \leq 5, \\
-(f \circ s_6)/C_6 & \text{if } i = 6, 
\end{cases}$$

here the rational maps $s_i : T \to T$ are as defined by Naruki in [N], p. 13 and $C_6 = C_1 \circ \tau$ where $\tau(\lambda, \mu, \nu, \rho) = (\nu, \mu, \lambda, \rho)$.

The representation of $W(E_6)$ on $V$ is its unique 10 dimensional irreducible representation and is denoted by $10_s$ in [F1].

Proof. Recall that $D_S = \sum D_\alpha$ with $\alpha \in E_6 - D_4$ a positive root. Write:

$$D_S = D_S^{(0)} + D_S^{(1)}, \quad D_S^{(0)} = \sum_\alpha D_\alpha$$

where we sum over the positive roots $\alpha \in E_6, \alpha \not\in D_4$ which are fixed under $s_1$. Then $s_1^* D_S = D_S^{(0)} + s_1^* D_S^{(1)}$. Since $s_1^* D_w = D_8$ (cf. the proof of [5,3]), we get:

$$s_1^*(D_S - 3D_w) = D_S^{(0)} + s_1^* D_S^{(1)} - 3D_8.$$

One verifies, using the tables [2,1] and [2,2] and the lemma above, that

$$(C_1) = 3D_8 - 3D_w + D_S^{(1)} - s_1^* D_S^{(1)}$$

hence $s_1^*(D_S - 3D_w) + (C_1) = D_S - 3D_w$. This suggests that $f \mapsto \pm(f \circ s_1)/C_1$ defines an endomorphism of $V$. To check this and to get a $W(E_6)$ representation on $V$, one computes
matrices and checks the defining relations for $W(E_6)$ (we used a computer, note this direct method avoids a detailed discussion of the divisor $D_R$ and verifies that one has to put a ‘$\scriptstyle -$’ sign in the definition of $s_1$ and $s_6$). Since the only representations of $W(E_6)$ of dimension at most 10 are the trivial one, denoted by $1 = 1_p$, the 6 dimensional reflection representation $6_p$, and their tensor products with the determinant representation $1_n$ and $6_n$, it suffices to compute the traces of a reflection $s_i$ (which is 0) and of a product of two commuting reflections (which has trace 2) to prove that $V \cong 10_s$.

5.5. **Table of a basis of $V$.** To obtain functions in $V$, all entries have to be divided by $\lambda \mu \nu \rho^2$. All ten functions are in one $W(E_6)$-orbit.

\[
\begin{align*}
f_1 &= (\lambda \rho - 1)(\mu \rho - 1)(\nu \rho - 1)(\lambda \mu \nu \rho - 1), \\
g_1 &= (\rho - 1)(\lambda \mu \rho - 1)(\lambda \nu \rho - 1)(\mu \nu \rho - 1), \\
f_2 &= (\mu \rho - 1)(\nu \rho - 1)(1 - \lambda^2 \nu \mu \rho^2), \\
g_2 &= (\rho - 1)(\mu \nu \rho - 1)(1 - \lambda^2 \mu \nu \rho^2), \\
f_3 &= (\lambda \rho - 1)(\mu \rho - 1)(1 - \lambda \mu \nu \rho^2), \\
g_3 &= (\rho - 1)(\lambda \mu \rho - 1)(1 - \lambda \mu \nu \rho^2), \\
f_4 &= \rho(\mu \rho - 1)(\lambda + \nu - \lambda \nu - \lambda \nu \rho - \lambda \mu \nu \rho + \lambda^2 \mu \nu \rho^2), \\
g_4 &= \rho(\mu - 1)(\lambda + \nu - \lambda \nu - \lambda \nu \rho - \lambda \mu \nu \rho + \lambda^2 \mu \nu \rho^2), \\
f_5 &= (\lambda \mu \rho - 1)(\mu \nu \rho - 1)(1 - \lambda \nu \rho^2), \\
g_5 &= (\mu \rho - 1)(\lambda \mu \nu \rho - 1)(1 - \lambda \nu \rho^2).
\end{align*}
\]

5.6. **Crosses.** Allcock and Freitag construct a 10 dimensional space $W$ of automorphic forms on the 4-ball ([AF], between 4.3 and 4.4) which defines the map $\mathcal{M} \hookrightarrow \mathbb{P}^9$. The vector space $W$ is spanned by certain automorphic forms which, up to a scalar multiple, can be characterized by the fact that their divisors in the ball-quotient $\mathcal{M}$ are crosses ([AF], Theorem 4.6). A cross is defined to be a divisor

$$D_\alpha + D_\beta + D_\gamma + D_\delta + D_t$$

where $t$ is a tritangent, defining a subroot system $t^+$ of type $D_4$ in $E_6$ (as in [L8]) and $\alpha, \ldots, \delta \in t^+ \cap E_6^+$ are mutually perpendicular (cf. [AF], Definition 3.2). For each tritangent $t$, there are 3 crosses containing $D_t$, thus there are $45 \cdot 3 = 135$ crosses. For example, the crosses associated to $t = (16)$ have $\{\alpha, \ldots, \delta\}$ equal to one of the three sets:

$$\{h_{23}, h_{45}, h_{123}, h_{145}\}, \quad \{h_{24}, h_{35}, h_{124}, h_{135}\}, \quad \{h_{25}, h_{34}, h_{125}, h_{134}\}.$$ 

The following theorem identifies $W$ with $V$ (as spaces of global sections of a line bundle on $\mathcal{M}$).
5.7. Theorem. The rational map \( \tilde{F} : \tilde{T} \to \mathbf{P}^9 \) defined by a basis of the vector space \( V \) defines a \( W(E_6) \)-equivariant morphism
\[
F : \mathcal{C} \to \mathcal{M} \subset \mathbf{P}^9
\]
which blows down the 40 cusp divisors to the 40 cusps. The image of \( F \) is the moduli space \( \mathcal{M} \) which is embedded into \( \mathbf{P}^9 \) via the map defined by Allcock and Freitag.

Proof. Using the results of [AF] and the \( W(E_6) \)-action on \( W \) and \( V \), it suffices to show that there is a function \( f \in V \cong H^0(T', \mathcal{O}(2D_R + D_S - 3D_w)) \) such that the corresponding section has, modulo cusp divisors, a cross as zero divisor in \( \tilde{T}' \). In fact, the exceptional divisors in the blow-ups \( \pi' \) and \( \pi'' \) get blown down in the composition \( \tilde{T} \to \mathcal{C} \to \mathcal{M} \) and under push-pull via \( \mathcal{M} \leftarrow \tilde{T} \to \tilde{T}' \), crosses in \( \mathcal{M} \) correspond to crosses in \( \tilde{T}' \) and cusp divisors in \( \tilde{T}' \) get contracted to points in \( \mathcal{M} \).

Let \( f = f_1 \) in table 5.3, then the divisor in \( \tilde{T}' \) of the corresponding function in \( V \) is:
\[
\left( \frac{f_1}{\lambda \mu \nu \rho^2} \right) = 4D_w + D_{24} + D_{124} + D_{35} + D_{135} - D_S + D'
\]
where \( D' \) is a divisor with support in \( D_R \).

Thus the zero divisor on \( \tilde{T}' \) of the section corresponding to \( f_1 \) is \( D_w + D_{24} + D_{124} + D_{35} + D_{135} + 2D_R - D' \). Note that \( w = (16) \) and that the four roots \( h_{24}, h_{124}, h_{35}, h_{125} \) are in \( D_4 = (16)^- \) and are perpendicular. The remaining part, \( 2D_R - D' \), has support on cusp divisors.

We observe that using the explicit bases of \( V \) and the method of [AF] Corollary 7.3, one can also prove directly that \( F \) factors over \( \mathcal{M} \) and embeds \( \mathcal{M} \) into \( \mathbf{P}^9 \). \( \square \)

5.8. Cross ratios. The basis of \( V \) given in table 5.3 has the property that the quotients \( f_i/g_i \) are double ratios associated to tritangents (see the table 2 of [N]), and we have in fact one double ratio from each \( D_4 \)-orbit:
\[
r(w) = \frac{f_1}{g_1}, \quad r(\bar{x}) = \frac{g_2}{f_2}, \quad r(\bar{z}) = \frac{g_3}{f_3}, \quad r(q_1) = \frac{g_4}{f_4}, \quad r(y) = \frac{g_5}{f_5}.
\]
(For completeness sake: \( w = (16), \bar{x} = (26), \bar{z} = (15), q_1 = (25), y = (16.23.45). \) Note that the last factor in each function in 5.3 is the local equation of the associated tritangent.

The fact that we find one cross ratio from each \( D_4 \) orbit already implies that \( \mathcal{C} \) is birationally isomorphic with \( F(\mathcal{C}) \) (use the argument of [N], § 5.5).

The involution \( \gamma(t) \in W(E_6) \) associated to a tritangent \( t \), see 1.10, has trace -6 on \( V \) (cf. [AF], Table II), hence it has a 2 dimensional space of invariants \( V_t \) in \( V \). There are, up to scalar multiple, 3 functions in \( V_t \) whose divisors are crosses (cf. [AF], Lemma 4.5). The pairs of functions \( f_i, g_i \) span such \( V_t \)'s. The third function in \( V_{(16)} \) is:
\[
h_1 := f_1 - g_1 = \rho(\lambda - 1)(\mu - 1)(\nu - 1)(\lambda \mu \nu \rho^2 - 1).
\]
The stabilizer \( W(F_4) \) of \( t \) acts on \( V_t \) through the action of a dihedral group with 12 elements; the subgroup \( W(D_4) \) (generated by reflections in the long roots) acts a \( S_3 \) and the reflections in the short roots act as \(-1\) on \( V_t \). In fact, the elements \( \sigma_1, \sigma_2 \in W(F_4) \) given by Naruki in [N], §8, p. 16 act as \(-1\) on \( V_{(16)} \).
5.9. **Complex invariants.** In example 4.7 we considered the following function from $V$:

$$f = \nu^{-1} \rho^{-1} (\rho - 1)(\lambda \nu \rho - 1)(\mu \nu \rho - 1).$$

Its divisor satisfies, modulo components with support in $D_R$:

$$(f) + D_S - 3D_w = D_{16} + D_{34} + D_{25} + D_{125} + D_{256} + D_{136} + D_{146} + D_{234} + D_{345}.$$  

The effective divisor on the right is the sum of the $D_\alpha$ where $\alpha$ runs over the positive roots of three mutually perpendicular $A_2$’s:

$$\{h_{16}, h_{125}, h_{256}\}, \quad \{h_{25}, h_{234}, h_{345}\}, \quad \{h_{34}, h_{136}, h_{146}\}.$$  

There are 40 such triples of orthogonal $A_2$’s in $E_6$ which are permuted transitively by $W(E_6)$ ([4], 6.1.5.3; this particular triple is denoted by $[16, 25, 34]$). The corresponding 40 functions in $V$ were considered by Coble who called them complex invariants (cf. [4], p. 340–341), see also [7]. There are $80 = 2 \cdot 40$ functions in the $W(E_6)$-orbit of a complex invariant, the sign of a complex invariant is not well defined.

6. **Images of divisors in $C$**

6.1. We can use Naruki’s model $C$ and the explicit basis of $V$ to study the moduli space $M \subset P^9$. Here we consider various divisors in $M$ as subvarieties of $P^9$, in particular we find a nice parametrization of a tritangent divisor.

6.2. **The boundary divisors.** We consider the image in $P^9$ of one of the 36 boundary divisors $D_\alpha \subset M$ ([4] and section 3). These parametrize cubic surfaces with at least one node. The divisor $D_\alpha$ is the fixed point set of the involution $s_\alpha$. The trace of $s_\alpha$ on $W$ is zero, hence $W$ is the direct sum of two 5-dimensional eigenspaces of $s_\alpha$. Since $F$ is equivariant for $W(E_6)$, $D_\alpha$ will lie in a $P^4$. The centralizer in $W(E_6)$ of the reflection $s_\alpha$ acts on the divisor $D_\alpha$ and on the eigenspaces of $s_\alpha$. This subgroup is isomorphic to $S_6$. For example if $\alpha = h$, one obtains the ‘standard’ $S_6$ generated by all the $s_i$ except $s_2$.

In particular we consider the image of $D_{345} = V(\epsilon_1)$ under $F$. This divisor is defined by $\lambda = 0$ on the open subset $A(\Delta_0) = Spec(C[\lambda, \mu, \nu, \rho])$ of $\tilde{T}$. Since the 10 functions listed in table 5.3 are regular on $A(\Delta_0)$ and do not vanish simultaneously, we can simply take $\lambda = 0$ and determine (the closure of) the image. The image spans only a $P^4$ since the following linear functions vanish on this divisor (in the notation of table 5.3):

$$f_1 - f_2, \quad g_1 - g_2, \quad f_3 - g_5, \quad f_2 - f_4 - g_5, \quad g_2 - g_3 - f_4 + g_4.$$  

The image of $M$ in $P^9$ is defined by cubics (see [AF]), and one can show that the image of a boundary divisor is the Segre cubic hypersurface in this $P^4$ (cf. [4], 3.2).

6.3. **The cusp divisors.** We consider one of the 40 cusp divisors in $C$ (cf. 2.8), for example $V(\epsilon_1 + \epsilon_2)$, note that $\epsilon_1 + \epsilon_2 \in R$. This divisor is defined by $\rho = 0$ in $Spec(C[\lambda, \mu, \nu, \rho])$. Putting $\rho = 0$ in the 10 functions in table 5.3 one finds that the image of this divisor is the point

$$\langle 1 : 1 : 1 : 1 : 1 : 0 : 0 : 1 : 1 \rangle.$$
6.4. Tritangent divisors. The tritangent divisor $D_t$ is the fixed point set of the involution $\gamma(t) \in W(E_6)$. Each $\gamma(t)$ has trace $-6$ on $V$ [Fr], hence it has two eigenspaces, of dimension 2 and 8, in $V$. Since the dimension of the divisor $D_t$ is three we get $D_t \subset \mathbb{P}^7$.

The centralizer of $\gamma(t)$ is isomorphic to $W(F_4)$ and this group acts on both $D_t$ and $\mathbb{P}^7$. We consider the case $t = (16) = w$, hence $D_t$ is birationally isomorphic to the exceptional fiber $\mathbb{P}_w^3$ of the blow up of the torus $T$ in the identity element $e$. Since $e$ is fixed by $W(D_4)$, we get an induced action of $W(D_4)$ on $\mathbb{P}_w^3$, and we will see that this action extends to a linear action of $W(F_4)$.

The root lattice $Q(F_4)$ of $F_4$ is the lattice in $\mathbb{R}^4$ generated by the 48 roots of $F_4$ which are (cf. [Hu], III 12.1) the 24 roots $\pm 3$ and 8, in $t$ consider the case $\sigma$ roots of $F_6.5. Theorem. Any tritangent divisor $D_t$ is $W(F_4)$-equivariantly birationally isomorphic to $\mathbb{P}^3$ via the map

$$\mathbb{P}^3 = \mathbb{P}(Q(F_4) \otimes \mathbb{Z} C) \hookrightarrow D_t \hookrightarrow \mathbb{P}^7$$

given by the linear system of cubics which are zero in the short roots of $F_4$.

Proof. Since $W(E_6)$ acts transitively on the tritangent divisors, it is sufficient to consider the case $t = (16)$. We show that the functions from $V$ give the desired map $\mathbb{P}_w^3 \rightarrow D_{(16)}$.

The local coordinate functions $\lambda - 1, \ldots, \rho - 1$ near $e$ induce projective coordinates $x_\lambda, \ldots, x_\rho$ on $\mathbb{P}_w^3$. Since $\lambda = e_1 - e_2, \ldots, \rho = e_2 - e_3$ it is more convenient to use coordinates $y_i$ with

$$(x_\lambda : x_\mu : x_\nu : x_\rho) = (y_1 - y_2 : y_3 + y_4 : y_3 - y_4 : y_2 - y_3).$$

The group $W(F_4)$ is generated by the subgroup $W(D_4)$ and $\sigma_1, \sigma_2$ given in [N], §8. Using the explicit formulas for the $\sigma_i$ one finds that these act on $\mathbb{P}_w^3$ as reflection in the planes $y_4 = 0$ and $y_1 - y_2 - y_3 - y_4$ respectively. (For example $\sigma_1$ interchanges $\mu$ and $\nu$ and fixes the other roots, thus on $\mathbb{P}_w^3$ it is the linear map which permutes $y_3 + y_4$ and $y_3 - y_4$ and fixes $y_1 - y_2$ and $y_2 - y_3$.) Thus these $\sigma_i$ are reflections in the short roots. This implies that we may identify $\mathbb{P}_w^3$ with $\mathbb{P}(Q(F_4) \otimes \mathbb{Z} C)$.

All functions in $V$ vanish to third order in $e$, but not all vanish to fourth order, hence restricted to $\mathbb{P}_w^3$ the map $F$ is given by the leading terms of third order. Note that $f_1$ and $g_1$ from table 5.3 vanish to order four at $e$, hence the image of $\mathbb{P}_w^3$ spans at most a $\mathbb{P}^7$, as we observed earlier. The leading terms of the other 8 basis functions are cubics which all contain the 12 points:

$$(1 : 0 : 0 : 0)_y, \ldots, (0 : 0 : 0 : 1)_y \quad \text{and} \quad (1 : \pm 1 : \pm 1 : \pm 1)_y.$$ For example, $f_2$ from table 5.3 has leading term (up to sign):

$$(x_\mu + x_\rho)(x_\nu + x_\rho)(2x_\lambda + x_\mu + x_\nu + 2x_\rho) = (y_2 + y_4)(y_2 - y_4)(2y_1).$$

One can verify that the 8 leading cubics are independent and that these 12 points impose independent conditions on the (20 dimensional) space of cubics. Thus the map $\mathbb{P}_w^3 \rightarrow D_{(16)}$ is given by the subspace of cubics vanishing in these points and the image of $\mathbb{P}_w^3$ spans a $\mathbb{P}^7$.

We observe that the twelve basepoints are in three $D_4$-orbits of length four. Two points in two distinct orbits determine a line on which there is a unique line from the third orbit. For
example the points \((0 : 0 : 1)\), \((1 : 1 : 1)\), \((1 : 1 : -1)\) are on a line. In this way we get 16 lines, on each of these there are 3 base points. Actually, the two fourth-order leading terms are zero exactly on these 16 lines.

6.6. **Incidence of tritangent divisors.** The tritangent divisor \(D_{(16)}\) is one of 45 such divisors, recall that \((16) = w = \{a_1, b_6, c_{16}\}\). The remaining 44 tritangent divisors now divide into 4 groups, \(44 = 4 + 4 + 4 + 2 \cdot 16\) as follows. For each \(l \in (16)\), there are 4 other tritangents containing \(l\) (for example, the \(\{a_1, b_i, c_{11}\}\) for \(2 \leq i \leq 5\) are the other tritangents which also contain \(a_1\)).

The remaining 32 tritangents do not have a line in common with \((16)\). These come in pairs as follows. Given one of these 32, say \(\{l_1, l_2, l_3\}\), after a permutation of the indices one has that \(a_1\) and \(l_1\) meet (and \(a_1\) does not meet \(l_2\) and \(l_3\)) and thus there is a line \(m_1\) such that \(\{a_1, l_1, m_1\}\) is a tritangent. Similarly \(b_6\) and \(l_2\) determine a line \(m_2\) and \(c_{16}\) and \(l_3\) determine a line \(m_3\). Now \(\{m_1, m_2, m_3\}\) is another tritangent which has no line in common with \((16)\). For example, \(\{b_6, c_{26}, c_{34}\}\) determines \(c_{15}, c_{26}, c_{34}\). (To see all this, consider a general cubic surface and the planes \(V(16)\) and \(V'\) spanned by the lines in \((16)\) and the \(l_i\) respectively. These planes meet in a line which by assumption does not lie in the cubic surface. Thus this line meets the surface in 3 points and through each of these points there passes exactly one line from \((16)\) and one line from the \(l_i\).)

If the lines in \((16)\) all pass through an Eckart point \(P_w\) and similarly the lines \(l_i\) all pass through an Eckart point \(P'_w\) the line \(L\) spanned by \(P_w\) and \(P'_w\) meets the cubic surface in a third point \(P''\) which is an Eckart point, being the intersection of the \(m_i\). (To see that each \(m_i\) passes through \(P''\), consider the plane spanned by, say, \(a_1\) and \(l_1\); it cuts out \(m_1\) and contains the line \(L\), hence \(m_1\) meets \(L\) in \(P''\), similarly for the other pairs of lines.)

As a consequence, a point in the intersection of two tritangent divisors without a common line will lie in a third tritangent divisor.

6.7. **Tritangent divisors and \(P^3_w\).** The intersections of \(P^3_w\) with the other 44 tritangent divisors are given by the leading terms of their equations. Those tritangent divisors which have a line in common with \(w = \{a_1, b_0, c_{16}\}\) have a linear leading term, in fact one finds the following 12 linear terms:

\[
y_i \quad (1 \leq i \leq 4), \quad \text{and} \quad y_1 \pm y_2 \pm y_3 \pm y_3
\]

where the last 8 come in two \(W(D_4)\)-orbits distinguished by the parity of the number of minus signs. For example, the tritangent \((15) = \bar{z}\) defined by \(\lambda \mu \nu^2 \rho^2 - 1\) has leading term \(y_1 + y_2 + y_3 - y_4\).

The tritangents which do not have a line in common with \(w\) have a leading term of degree two, in fact the two tritangents in a pair have the same leading term (as they should, see the last part of 6.6). These quadrics correspond to the 16 lines in \(P^3_w\) containing 3 of the 12 base points of \(F\) (see \(6.4\)); each line determines a unique quadric by the condition that it contains the other 9 base points (and this quadric will not contain any of the 3 points on the line). For example, the tritangent divisor \(q_1 = \{a_2, b_5, c_{25}\}\) is defined by \(\lambda + \nu - \lambda \nu - \lambda \nu \rho - \lambda \mu \nu \rho + \lambda^2 \mu \nu^2 \rho^2\) has leading term

\[
Q_{25} := y_1^2 + y_2 y_3 - y_2 y_4 - y_3 y_4
\]
which does not contain the 3 colinear points \((1 : 0 : 0 : 0)_y\), \((1 : -1 : -1 : 1)_y\) and \((1 : 1 : 1 : -1)_y\). The other tritangent divisor having the same leading term is \(\bar{q} = \{c_{15}, c_{26}, c_{34}\}\) which is defined by \(1 - \lambda\nu\rho - \lambda\mu\nu\rho^2 + \lambda^2\mu\nu\rho^2 + \lambda\mu\nu^2\rho^2\).

The other 15 quadrics can be obtained from \(Q_{26}\) by the action of \(W(D_4)\), that is, by permuting the coordinates and changing the signs of an even number of the \(y_i\). These quadrics are smooth and hence are isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\).

7. Equations for the moduli space.

7.1. The universal marked cubic surface is embedded in a \(\mathbb{P}^3\)-bundle over \(\mathcal{C}\). Over the moduli space \(\mathcal{M}^0\) of smooth marked cubic surfaces, this bundle is the projectivization of the tangent bundle (\([\text{ACT}]\) § 10). In Naruki’s paper \([\mathcal{N}]\) one finds an explicit cubic polynomial in \(R[X, Y, Z, T]\), with \(R := \mathbb{C}[\lambda, \mu, \nu, \rho]\), which defines the universal family over an open part of \(\mathcal{M}\). He also gives 45 linear forms in \(R[X, Y, Z, T]\) which define the tritangent planes.

We will verify that there are linear relations between these, suitably normalized, linear forms with coefficients which are elements from \(V\) (note that elements from \(V\) are rational functions on \(T\) and thus are in the field of fractions of \(R\)). This allows us to recover the cubic equations found by Allcock and Freitag which define \(\mathcal{M}\). We also find a six dimensional vector space of quintic polynomials, on which \(W(E_6)\) acts via its standard representation, which define a variety \(X \subset \mathbb{P}^9\) whose singular locus contains \(\mathcal{M}\).

7.2. Consider two tritangents which contain a common line. For any point in the interior of \(\mathcal{C}\), the corresponding planes are distinct. However over one of the 36 boundary divisors the planes may coincide. Over a boundary divisor \(D_\alpha\) the 6 pairs of lines in the double six corresponding to \(\alpha\) (cf. \([\mathcal{H}]\)) on the universal marked surface specialize to the six lines through the node of the universal surface over \(D_\alpha\). The reflection \(s_\alpha\) in \(W(E_6)\) interchanges the lines in each of the six pairs and fixes the other 15 lines. Thus if \(s_\alpha\) maps one tritangent set to another, then the lines in the planes and thus the planes themselves will coincide over \(D_\alpha\).

7.3. Lemma. Let \(t_1, t_2\) be two distinct tritangent sets which have a line in common. Then there are exactly two reflections in \(W(E_6)\) which map \(t_1\) to \(t_2\). The corresponding roots in \(E_6\) are perpendicular.

**Proof.** Since \(W(E_6)\) acts transitively on the set of lines, we may assume that the common line is \(b_6\). Then the \(t_i\) are of type \(\{a_i, b_6, c_{16}\}\) with \(1 \leq i \leq 5\) and applying a suitable element of \(W(E_6)\) we may assume that \(t_1 = \{a_1, b_6, c_{16}\}, t_2 = \{a_2, b_6, c_{26}\}\). By inspection of the lists of double sixes in \([\mathcal{H}]\) one finds exactly one double six which contains the pairs \(\{a_1, a_2\}\) and \(\{c_{16}, c_{26}\}\) (it is \(N_{12}\)) and one which contains the pairs \(\{a_1, c_{26}\}\) and \(\{c_{16}, a_2\}\) (it is \(N_{345}\)). Thus only reflections in \(h_{12}\) (which permutes the indices 1 and 2) and in \(h_{345}\) (which interchanges \(a_1 \leftrightarrow c_{26}\) and \(a_2 \leftrightarrow c_{16}\)) permute these two tritangent sets. It is easy to verify that \(h_{12}\) and \(h_{345}\) are perpendicular. \(\square\)

7.4. Given three linear forms \(K, L, M \in R[X, Y, Z, T]\) which define tritangent planes to the universal cubic surface having a line in common, there is a linear relation, with coefficients in \(R\),

\[AK + BL + CM = 0.\]
The next proposition shows that three tritangent planes with a line in common define three crosses. Recall that a cross is a divisor in \( \mathcal{M} \) determined by the choice of a tritangent set \( t \) and on 4 perpendicular roots in \( t^\perp \cong D_4 \). In the example below we then verify that these crosses are the divisors of the coefficients in the linear relation.

### 7.5. Proposition

Let \( t_1, t_2 \) and \( t_3 \) be tritangent sets with a line in common. Then there are crosses \( X_i \) determined by the tritangent sets \( t_i \), the pair of roots whose reflections interchange \( t_j \) and \( t_k \) (with \( \{i, j, k\} = \{1, 2, 3\} \)) and the pair of roots which is perpendicular to all the weights in the union of these three tritangent sets.

**Proof.** Again we use the \( W(E_6) \) action, and so we may assume that \( t_i = \{a_i, b_i, c_i\} \). These span the subspace \( \langle x_1, x_2, x_5, x_6 \rangle \) ([H], 6.1.3) hence only the roots \( h_{45} = x_3 + x_4 \) and \( h_{145} = x_3 + x_4 \) are perpendicular to this subspace. The two roots whose reflections interchange \( t_1 \) and \( t_2 \) are \( h_{12}, h_{345} \). The roots \( h_{12}, h_{345}, h_{45} \) and \( h_{145} \) are orthogonal and lie in the \( D_4 \) perpendicular to the weights in \( t_3 \). Therefore there is a cross \( X_3 \) which is the sum of the tritangent divisor corresponding to \( t_3 \) and the four boundary divisors corresponding to these four roots. Similarly one finds crosses \( X_1 \) and \( X_2 \).

### 7.6. Example

We consider the tritangents which contain the line \( b_6 \). They are:

| set                  | label | local equation | linear form                  |
|----------------------|-------|----------------|-----------------------------|
| \( \{a_1, b_6, c_{16}\} \) | (16) = w | 1              | \( W \)                    |
| \( \{a_2, b_6, c_{26}\} \) | (26) = \bar{x} | \( \lambda^2\mu\nu\rho^2 - 1 \) | \( \lambda X + (\lambda \rho - 1)(\lambda \mu \nu \rho - 1)W \) |
| \( \{a_3, b_6, c_{36}\} \) | (36) = x | \( \mu \nu \rho^2 - 1 \) | \( -X + (\rho - 1)(\mu \nu \rho - 1)W \) |
| \( \{a_4, b_6, c_{46}\} \) | (46) = x | \( -\rho(\mu \nu - 1) \) | \( X \) |
| \( \{a_5, b_6, c_{56}\} \) | (56) = \xi | \( \mu - \nu \) | \( X + \rho(\mu - 1)(\nu - 1)W \) |

The conversion of the labels is given in [S], the equation of the planes is given in [N], Table 1, but we changed the sign of (36) and we multiplied the local equation of (46) by a unit.

We write \( t_i := \{a_i, b_i, c_{16}\} \). Then \( t_2 = s_1(t_2), t_3 = s_3(t_2), t_4 = s_4(t_3) \) and \( t_5 = s_5(t_4) \) where \( s_i \) is the reflection in \( a_i \). The two roots perpendicular to the span of the sets \( t_1, t_2 \) and \( t_3 \) are \( h_{45} \) and \( h_{123} \). The cross \( X_1 \) is then:

\[
X_1 = D_{23} + D_{145} + D_{45} + D_{123} + D_{(16)}
\]

and \( X_2 = s_1(X_1), X_3 = s_3(X_2) \).

Note that \( X_1 \) is the divisor of the section corresponding to

\[
A_1 = \rho(-1 + \lambda)(-1 + \mu)(-1 + \nu)(-1 + \lambda \mu \nu \rho^2)(\lambda \mu \nu \rho^2)^{-1} \quad (\in V),
\]

and that \( A_1 = h_1 \) in [S]. Similarly we define \( A_2 = s_1(A_1), A_3 = s_3(A_2) \in V \).

We define \( L_{16} \in \mathbb{C}(\lambda, \ldots, \rho)[X, W] \) to be the quotient of the linear form defining the tritangent plane \( (16) \) by the local equation of the tritangent divisor \( D_{(16)} \) as listed in the table. One can then verify the following linear relation:

\[
A_1 L_{16} + A_2 L_{26} + A_3 L_{36} = 0.
\]
7.7. Proposition. Let $A_i$ and $L_{ij}$ be as in Example 7.6. Define functions $B_i, \ldots, F_i \in V$ by:

$$B_i = s_4(A_i), \quad C_i = s_3(B_i), \quad D_i = s_1(C_i), \quad E_i = s_5(D_i), \quad F_i = s_5(B_i).$$

Then we have $Mv = 0$ where

$$M = \begin{pmatrix} A_1 & A_2 & A_3 & 0 & 0 \\ B_1 & B_2 & 0 & B_3 & 0 \\ C_1 & 0 & -C_2 & C_3 & 0 \\ 0 & D_1 & D_2 & -D_3 & 0 \\ 0 & E_1 & E_2 & 0 & E_3 \\ F_1 & F_2 & 0 & 0 & -F_3 \end{pmatrix}, \quad v = \begin{pmatrix} L_{16} \\ L_{26} \\ L_{36} \\ L_{46} \\ L_{56} \end{pmatrix}.$$

In particular, $M$ has rank at most three.

Proof. Applying the reflection $s_4$ in $\alpha_4 = h_34$ (which permutes the indices 3 and 4) to the linear relation from Example 7.6, we obtain a relation between the linear forms defining the tritangents corresponding to $t_1 = s_4(t_1)$, $t_2 = s_4(t_2)$ and $t_4 = s_4(t_3)$. One verifies that this is $B_1L_{16} + B_2L_{26} + B_3L_{46} = 0$ with coefficients $B_i = s_i(A_i)$. Similarly, one verifies the other relations. Since each entry of $v$ is of the form $a_iX + b_iW$ we see that $\ker(M)$ contains the two vectors $a = (a_1, \ldots, a_5)$ and $b = (b_1, \ldots, b_5)$. Thus the rank of $M$ is at most $5 - 2 = 3$. \hfill $\Box$

7.8. Equations. To obtain equations for $\mathcal{M} \subset \mathbb{P}^9$ from this proposition, one chooses a basis $X_0, \ldots, X_9$ of $V$. Then each function in $V$ is a linear form in the $X_i$ with coefficients in $\mathbb{C}$. Thus each entry of the matrix $M$ is a linear form in the $X_i$. Since the rank of $M$ is at most 3, the determinant of each $4 \times 4$ submatrix of $M$, which is a degree 4 polynomial in the $X_i$, is identically zero as function on $\mathcal{M}$. Therefore each such determinant gives a, possibly trivial, quartic polynomial in the ideal of $\mathcal{M}$.

7.9. Cubics. To get cubic equations we consider the following submatrix of $M$:

$$N = \begin{pmatrix} A_1 & A_2 & A_3 & 0 \\ B_1 & B_2 & 0 & B_4 \\ C_1 & 0 & -C_3 & C_4 \end{pmatrix}.$$  

The matrix $N$ has rank at most two since $Nw = 0$, where $w = (L_{16}, \ldots, L_{46})$, gives two vectors in $\ker N$ (put $X = 1$, $W = 0$ and $X = 0$, $W = 1$ in $w$). In particular,

$$\det \begin{pmatrix} A_2 & A_3 & 0 \\ B_2 & 0 & B_4 \\ 0 & C_3 & -C_4 \end{pmatrix} = -A_2B_4C_3 + A_3B_2C_4 = 0.$$

The corresponding cubic polynomial in the $X_i$ is not identically zero in $\mathbb{C}[\ldots, X_i, \ldots]$ and is one of those found in [AF] Lemma 6.3. Theorem 6.4 of that paper implies that $\mathcal{M}$ is defined by the $W(E_6)$-orbit of this cubic equation.
7.10. **Quintics.** One verifies that the determinant of the following submatrix of $M$ is a degree 5 polynomial in the $X_i$ which is not identically zero:

$$M_2 = \begin{pmatrix} A_1 & A_2 & A_3 & 0 & 0 \\ C_1 & 0 & -C_3 & C_4 & 0 \\ 0 & D_2 & D_3 & -D_4 & 0 \\ 0 & E_2 & E_3 & 0 & E_5 \\ F_1 & F_2 & 0 & 0 & -F_5 \end{pmatrix}.$$  

By Proposition 7.7 the rank of $M_2$ is at most 3. Therefore the determinant of any 4×4 submatrix of $M_2$ is zero on $\mathcal{M}$. Since the partial derivatives of $\det(M)$ with respect to the $X_i$ are linear combinations of determinants of such submatrices, we conclude that the quintic hypersurface $X$ in $\mathbf{P}V$ defined by $\det(M)$ is singular along moduli space of marked cubic surfaces $\mathcal{M} \subset \mathbf{P}^9$.

Using the $10 \times 5$ matrix obtained from all $\binom{5}{3} = 10$ linear relations between 3 of the 5 tritangent planes containing the line $b_6$, we get $\binom{10}{5}$ quintics, but they are either 0 or the same as $\det(M)$ up to sign. It can be checked that the $W(E_6)$-orbit of such a quintic has 27 elements and that these quintics span a copy of the standard 6-dimensional representation $6_p$ of $W(E_6)$.

8. **Hunt’s Quintic.**

8.1. **Supercrosses.** We show how to construct 27 quintic polynomials, which we call supercrosses, on $V$ which are permuted, up to sign, as the 27 lines on the cubic surface under the action of $W(E_6)$. We show that the supercrosses span a 6-dimensional vector space on which $W(E_6)$ acts as $6_n$ and that they define a rational map

$$\Sigma : \mathcal{M} \rightarrow \mathbf{P}^5$$

which maps the moduli space onto the the unique $W(E_6)$-invariant hypersurface of degree 5 in $\mathbf{P}^5$. This hypersurface was investigated by Hunt in [H].

8.2. The line $a_1$ on a marked cubic surface defines a weight of $E_6$. The roots $\alpha_2, \ldots, \alpha_6$ are perpendicular to this weight and span a root system, of type $D_5$, consisting of $2 \cdot 20 = 40$ roots. In the notation of [H], this system is ‘in standard form’

$$a_1^+ = \{ \pm x_j \pm x_k : 1 \leq j < k \leq 5 \} \cong D_5.$$  

Any line on a cubic surface lies in 5 tritangent planes. The tritangent planes containing $a_1$ are the $(1i) = \{a_1, b_j, c_{1j} \}, 2 \leq j \leq 6$. The three weights corresponding to the three lines in a tritangent are linearly dependent, hence span a line, and the orthogonal complement of the line is a root system of type $D_4$, in fact $a_1 = -(2/3)x_6, b_j = x_{j-1} + (1/3)x_6$, thus

$$\{a_1, b_j, c_{1j}\}^\perp = \{ x_{j-1}, x_6 \}^\perp = \{ \pm x_i \pm x_k : i < k, i, k \in \{1, \ldots, j-1, \ldots, 5\} \} \cong D_4.$$  

Now the main point is that the 20 positive roots which are perpendicular to $a_1$ split in 5 sets of 4 perpendicular roots such that each of the 5 sets is also perpendicular to the weights corresponding to the lines in a tritangent plane containing $a_1$. Thus each line $l$ determines 5
crosses. In the notation of \([\mathbb{P}]\):

\[
\begin{align*}
(12) &= \langle a_1, b_2, c_{12} \rangle = \langle x_1, x_6 \rangle \\
&= \{ \pm x_2 + x_3, \pm x_4 + x_5 \} = \{ h_{34}, h_{56}, h_{134}, h_{156} \} \\
(13) &= \langle a_1, b_3, c_{13} \rangle = \langle x_2, x_6 \rangle \\
&= \{ \pm x_1 + x_4, \pm x_3 + x_5 \} = \{ h_{25}, h_{46}, h_{125}, h_{146} \} \\
(14) &= \langle a_1, b_4, c_{14} \rangle = \langle x_3, x_6 \rangle \\
&= \{ \pm x_1 + x_5, \pm x_2 + x_4 \} = \{ h_{26}, h_{35}, h_{126}, h_{135} \} \\
(15) &= \langle a_1, b_5, c_{15} \rangle = \langle x_4, x_6 \rangle \\
&= \{ \pm x_1 + x_3, \pm x_2 + x_5 \} = \{ h_{24}, h_{36}, h_{124}, h_{136} \} \\
(16) &= \langle a_1, b_6, c_{16} \rangle = \langle x_5, x_6 \rangle \\
&= \{ \pm x_1 + x_2, \pm x_3 + x_4 \} = \{ h_{23}, h_{45}, h_{123}, h_{145} \}
\end{align*}
\]

8.3. The functions \(F_i\). To each cross corresponds a function, up to scalar multiple, in \(V\). Fixing one such function and applying \(W(E_6)\) we find other functions, unique up to sign, whose divisors are crosses. Fix a line \(l\), then we can associate to it the function \(F_l\), unique up to sign, which is the product of the 5 functions in \(V\) corresponding to the 5 crosses associated to \(l\). The divisor of \(F_l\) is then essentially the sum of the 5 tritangent divisors \(D_l\) with \(l \in t\) and the 20 boundary divisors \(D_\alpha\) with \(\alpha \in l^\perp \cap E_6^\circ\). If \(m\) is a line and \(m = \sigma(l)\) for some \(\sigma \in W(E_6)\), we define \(F_m := \det(\sigma)\sigma(F_l)\) where \(\det(\sigma)\) is the determinant of \(\sigma\) in the 6-dimensional reflection representation. The \(F_m\)’s will be called a supercrosses, they are uniquely determined by \(F_l\).

8.4. Proposition. The 27 functions \(F_l\) on Naruki’s cross ratio variety span a 6 dimensional vector space. The Weyl group \(W(E_6)\) acts on this vector space as \(6_n\), the tensor product of the standard 6 dimensional representation with its determinant.

**Proof.** The functions \(F_l\), with scalar factors suitably normalized, satisfy the linear relations \(F_l \pm F_m \pm F_n = 0\) whenever the lines \(l, m, n\) are in a tritangent plane. From this one concludes that they span a space of dimension 6 on which \(W(E_6)\) acts (the relations \(F_{a_i} \pm F_{b_j} \pm F_{c_{ij}} = 0\) imply one can express the \(F_{c_{ij}}\) in terms of the \(F_{a_i}\) and \(F_{b_j}\), now use the relations \(F_{c_{ij}} \pm F_{c_{kl}} \pm F_{c_{mn}}\) to eliminate the \(F_{b_j}\)).

Since reflections in the stabilizer of an \(F_l\) act by as multiplication by \(-1\) on \(F_l\), the representation is the twist of the standard representation.

8.5. The theorem provides us with a \(W(E_6)\)-equivariant rational map

\[
\Sigma : \mathcal{M} \longrightarrow \mathbb{P}^5.
\]

By computing the differential of \(\Sigma\) in some point of \(\mathcal{M}\) we found that it has maximal rank. Hence the (closure of the) image of \(\Sigma\) is a \(W(E_6)\)-invariant hypersurface in \(\mathbb{P}^5\).

8.6. Theorem. The hypersurface \(\Sigma(\mathcal{M}) \subset \mathbb{P}^5\) is Hunt’s quintic, the unique quintic hypersurface which is \(W(E_6)\)-invariant. It is defined by:

\[
I_5 := \sum l \lambda_l^5 = 0
\]

where \(\lambda_l\) is the linear form on \(\mathbb{P}^5\) defined by the \(E_6\)-weight which corresponds to the line \(l\).

**Proof.** We will show that the following sextic relation holds:

\[
\prod_{l \in A} F_l = \prod_{l \in B} F_l
\]

where \(A = \{a_1, \ldots, a_6\}\) and \(B = \{b_1, \ldots, b_6\}\) form a double six of lines. As observed by Naruki (see [\(\mathbb{P}\), p.235]), this equation is reducible, being the product of \(I_5\) and a linear factor which
is the linear form defined by the root corresponding to the double six given by $A$ and $B$. The $W(E_6)$-invariance of the image implies that the image is defined by $I_5$.

The divisors of both sides of the equation are the sum of the $6 \cdot 5 = 30$ tritangent divisors $D_{ij}$ as well as the sum of $6 \cdot 20 = 120$ boundary divisors. We already determined the positive roots in $a_1^+$ above, those in $b_1^+$ are:

$$b_1^+ = \{h_{jk} = -x_{j-1} + x_{k-1} : 2 \leq j < k \leq 6\} \cup \{h_{pqr} : 2 \leq p < q < r \leq 6\}.$$ 

Thus each $h_{ij}$ occurs 4 times whereas each $h_{pqr}$ occurs 3 times in both the left and the right hand side, note that $4 \cdot 15 + 3 \cdot 20 = 120$. Thus, up to scalar multiple, the left and right hand side coincide. Using the reflection $s$ in the root $h$ (note $s(a_i) = b_i$) one finds the equality. 

8.7. Direct computations show that the images of the 36 divisors are 36 points in $\mathbb{P}^5$, these are the roots of $E_6$. The images of the 45 tritangent divisors are the 45 $\mathbb{P}^3$’s in Hunt’s quintic (see the proof of the theorem below).

8.8. **Theorem.** The rational map 

$$\Sigma : \mathcal{M} \longrightarrow I_5$$

has generic degree at least 10.

**Proof.** We verified by machine computation that $\Sigma$ has maximal rank at the point $(\lambda, \mu, \nu, \rho) = (-1, -1, 2, 3) \in T$. This point lies in the intersection of the two tritangent divisors $(12) = \zeta$ defined by $\lambda = \mu$ and $(13) = z$ defined by $\lambda \mu = 1$ (cf. [N] Table 3). These tritangents have the line $a_1$ in common. Since $\Sigma$ is $W(E_6)$-equivariant we conclude that $\Sigma$ has maximal rank at the general point in the intersection of any two tritangent divisors with a line in common.

We consider the restriction of $\Sigma$ to the intersection of the tritangent divisors $D_w = D_{(16)}$ and $D_{(26)}$ which have the line $b_6$ in common. The divisor $D_w$ is birationally isomorphic to $\mathbb{P}^3_w$, the exceptional fiber of the blow up of $T$ in $e$, and we consider the map induced by $\Sigma$ on this $\mathbb{P}^3$. The local equation of $(26) = \bar{x}$ is $\lambda^2 \mu \nu \rho^2 = 1$ and its intersection with $\mathbb{P}^3_w$ is given by $y_1 = 0$. (cf. [6, 7].) Note that $\Sigma$ has maximal rank in a general point of $\mathbb{P}^3_w \cap (y_1 = 0)$.

On $\mathbb{P}^3_w$ the leading terms of the $F_i$ are of degree 15 or 16 (only for $F_{a_1}, F_{b_6}$ and $F_{e_{16}}$), hence the restriction of $\Sigma$ is given by homogeneous polynomials of degree 15 and the image of $\mathbb{P}^3_w$ under $\Sigma$ lies in the intersection of the hyperplanes defined by $a_1$, $b_6$ and $c_{16}$ which is a $\mathbb{P}^3$. After omitting leading terms which are multiples of $y_1$ and dividing the remaining ones by their common factor $y_2y_3y_4$, we found that $\Sigma$ restricts to $\mathbb{P}^3_w \cap (y_1 = 0)$ to give a map

$$\Sigma_r : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

defined by homogeneous polynomials of degree 12. One coordinate function is

$$F_2 := y_3y_4(y_3 - y_4)((y_3 + y_4)(y_2 - y_3y_4)^2(y_2^2 + y_3y_4)^2),$$

the other two, $F_3$ and $F_4$, are obtained by permuting the coordinates cyclically. All these functions satisfy

$$F(y_2, y_3, y_4) = -(y_2y_3y_4)^8 F(y_2^{-1}, y_3^{-1}, y_4^{-1})$$

due to $\Sigma_r$ having degree at least 2.

The inverse image of a general point $(x_2 : x_3 : x_4) \in \mathbb{P}^2$ under $\Sigma_r$ is defined by the two equations, each homogeneous of degree 12:

$$G_1 := x_3F_2 - x_2F_3 = 0, \quad G_2 := x_4F_2 - x_2F_4 = 0.$$
The 0-cycle defined by these equations has degree $12^2 = 144$, but the linear system defined by the $F_i$ has base points. Below we list the base points and their contribution to the intersection multiplicities (determined with computer). Here $\omega$ is a primitive cube root of unity.

\[
\begin{align*}
(0 : 0 : 1), & \quad (0 : 1 : 0), & \quad (1 : 0 : 0), & \quad m_P = 20, \\
(0 : 1 : \pm 1), & \quad (1 : 0 : \pm 1), & \quad (1 : \pm 1 : 0), & \quad m_P = 1, \\
(1 : 1 : -1), & \quad (1 : -1 : 1), & \quad (1 : -1 : 1), & \quad m_P = 9, \\
(1 : 1 : 1), & \quad m_P = 9, \\
(1 : \omega : \pm \omega^2), & \quad (1 : \omega^2 : \pm \omega), & \quad (1 : -\omega : \pm \omega^2), & \quad (1 : -\omega^2 : \pm \omega) & \quad m_P = 4.
\end{align*}
\]

Thus we find that the base points contribute

\[
3 \cdot 20 + 6 \cdot 1 + 3 \cdot 9 + 1 \cdot 9 + 8 \cdot 4 = 134
\]

to the intersection, so there remain 10 points unaccounted for. Since $\Sigma$ has maximal rank in a general point of this $\mathbb{P}^2$, we conclude that the degree of $\Sigma$ is at least 10.

\[\square\]

9. Tables.

The following tables identify the 36 positive roots of $E_6$, in the notation of Hunt [H], with the 12 positive $D_4$ roots, in the notation of Naruki [N], and 24 $D_4$-weights. We also list the functions $f_\alpha$ on $T$ corresponding to the positive roots $\alpha \in D_4$.

9.1.

| roots of $D_4$ | $f_\alpha$ | roots of $E_6$ | roots of $D_4$ | $f_\alpha$ | roots of $E_6$ |
|----------------|-------------|----------------|----------------|-------------|----------------|
| $e_1 - e_2$    | $\lambda$   | $h_{23} = -x_1 + x_2$ | $e_1 + e_2$    | $\lambda \mu \rho$ | $h_{145} = x_3 + x_4$ |
| $e_1 - e_3$    | $\lambda \rho$ | $h_{24} = -x_1 + x_3$ | $e_1 + e_3$    | $\lambda \mu \rho$ | $h_{135} = x_2 + x_4$ |
| $e_1 - e_4$    | $\lambda \mu \rho$ | $h_{25} = -x_1 + x_4$ | $e_1 + e_4$    | $\lambda \mu \rho$ | $h_{134} = x_2 + x_3$ |
| $e_2 - e_3$    | $\rho$      | $h_{34} = -x_2 + x_3$ | $e_2 + e_3$    | $\mu \nu \rho$   | $h_{125} = x_1 + x_4$ |
| $e_2 - e_4$    | $\nu \rho$  | $h_{35} = -x_2 + x_4$ | $e_2 + e_4$    | $\mu \nu \rho$   | $h_{124} = x_1 + x_3$ |
| $e_3 - e_4$    | $\nu$       | $h_{45} = -x_3 + x_4$ | $e_3 + e_4$    | $\mu$           | $h_{123} = x_1 + x_2$ |

9.2.

| $D_4$-weight | $E_6$-root | $D_4$-weight | $E_6$-root | $D_4$-weight | $E_6$-root |
|---------------|------------|---------------|------------|---------------|------------|
| $e_1$         | $h_{345}$  | $(e_1 + e_2 + e_3 + e_4)/2$ | $h_{16}$   | $-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)/2$ | $h_{56}$ |
| $e_2$         | $h_{245}$  | $(e_1 + e_2 - \epsilon_3 - \epsilon_4)/2$ | $h_{236}$ | $-\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)/2$ | $h_{46}$ |
| $e_3$         | $h_{235}$  | $(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)/2$ | $h_{246}$ | $-\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4)/2$ | $h_{36}$ |
| $e_4$         | $h_{234}$  | $(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)/2$ | $h_{256}$ | $(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2$ | $h_{26}$ |
| $-\epsilon_1$| $h_{12}$   | $(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)/2$ | $h_{346}$ | $(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2$ | $h_{126}$ |
| $-\epsilon_2$| $h_{13}$   | $(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4)/2$ | $h_{356}$ | $(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)/2$ | $h_{136}$ |
| $-\epsilon_3$| $h_{14}$   | $(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)/2$ | $h_{456}$ | $(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4)/2$ | $h_{146}$ |
| $-\epsilon_4$| $h_{15}$   | $(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)/2$ | $h$        | $(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)/2$ | $h_{156}$ |
9.3. $W(E_6)$-representations. In the notation of Frame [Fr], the (unique) 10 dimensional representation $V$ of $W(E_6)$ is denoted by $10_s$. One has:

$$\text{Sym}^2(10_s) = 1 + 15_m + 15_q + 24_p,$$
$$\text{Sym}^3(10_s) = 20_s + 2 \cdot 30_m + 2 \cdot 30_p + 80_s,$$
$$\text{Sym}^4(10_s) = 2 \cdot 1 + 3 \cdot 1_n + 3 \cdot 15_m + 4 \cdot 15_q + 20_p + 20_s + \ldots ,$$
$$\text{Sym}^5(10_s) = 2 \cdot 6_p + 2 \cdot 6_n + 15_p + 15_q + 7 \cdot 30_m + 7 \cdot 30_p + \ldots ,$$
$$\text{Sym}^6(10_s) = 5 \cdot 1 + 3 \cdot 1_n + 11 \cdot 15_m + 14 \cdot 15_q + \ldots ,$$

here $6_p$ is the standard 6-dimensional representation and $6_n$ is the tensor product of $6_p$ with its determinant. On $\mathbb{P}^5$ the representations $6_p$ and $6_n$ are the same. In particular, there are two 1-dimensional families of 6-dimensional representations in $S^5V$.

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