Complete graph asymptotics for the Ising and random cluster models on 5D grids with cyclic boundary

P. H. Lundow and K. Markström

Department of mathematics and mathematical statistics,
Umeå University, SE-901 87 Umeå, Sweden
(Dated: August 12, 2014)

Abstract

The finite size scaling behaviour for the Ising model in five dimensions, with either free or cyclic boundary, has been the subject for a long running debate. The older papers have been based on ideas from e.g. field theory or renormalization. In this paper we propose a detailed and exact scaling picture for critical region of the model with cyclic boundary. Unlike the previous papers our approach is based on a comparison with the existing exact and rigorous results for the FK-random-cluster model on a complete graph. Based on those results we identify several distinct scaling regions in an $L$-dependent window around the critical point. We test these predictions by comparing with data from Monte Carlo simulations and find a good agreement. The main feature which differs between the complete graph and the five dimensional model with free boundary is the existence of a bimodal energy distribution near the critical point in the latter. This feature was found by the same authors in an earlier paper in the form of a quasi-first order phase transition for the same Ising model.

*Electronic address: per.hakan.lundow@math.umu.se
†Electronic address: klas.markstrom@math.umu.se
I. INTRODUCTION

The Ising model is one of the most studied models in statistical physics. We now have a well developed mathematical theory for it’s behaviour in 2-dimensions [1-3], extensive numerical work in 3-dimensions, and it is known that the upper critical dimension for the model if $d = 4$. For $d > 4$ it is known that the model takes on it’s mean field critical exponents in the thermodynamic limit. For finite systems far less is known rigorously and there has been a long running debate on differences in scaling for systems with free and cyclic boundaries [4-15]. In a previous paper [16] we found an hitherto overlooked way in which for $d = 5$ the behaviour of the model depends strongly on the boundary condition, in fact so strongly that at the critical point the difference is not believed to vanish as the side $L$ of the system grows. This conclusion has been questioned [17] by a group of authors which has earlier been promoting an alternative scaling picture for the case with free boundary conditions. However, in a recent paper we revisited this question with much larger system sizes and found a good agreement with our earlier results, and the standard scaling picture, where the susceptibility of the model with free boundary grows as $L^2$ and for cyclic boundary grows as $L^{5/2}$.

In view of these results a natural question is why we should see a different behaviour for the case with cyclic boundary, and what the exact form of the scaling behaviour for this case should be for $d$ above the critical dimension. Our approach to the answer of this question is via the Random-cluster model. The Fortuin-Kasteleyn random cluster model, or RC-model for short, is a natural extension of the Ising, Potts and edge percolation models, all captured by varying $q$, one of its two parameters, the other $p$ corresponding to the temperature in the first two models and the edge probability in the last. Many properties of the Ising model have direct interpretations in terms of the numbers and sizes of connected clusters in the RC-model. In particular the susceptibility corresponds to the average cluster size.

For percolation, the case $q = 1$ of the RC-model, Aizenman [18] conjectured that the largest clusters should scale as $L^{2d/3}$ for $d > 6$ with cyclic boundary, instead of $L^4$ for free boundary. One of the reasons for this conjecture was that this gives the same scaling as for the largest connected in the Erdős-Renyi, or ER for short, random graph with $N$ vertices at it’s critical point, where the largest component has size proportional to $N^{2/3}$. Note that the ER-random graph can be seen as the percolation model on the complete graph on $N$ vertices.
This conjecture was proved in [19, 20], and in [21–23] it was proved that asymptotics of the same type as on the complete graph can be expected on a wider class of finite graphs as well.

As we have pointed out percolation is a special case of the RC-model and the latter has also been studied on the complete graph by mathematicians. In [24] the critical probability was identified, the exponential asymptotics of the partition function was studied, and it was proved that the phase transition is of second order for \(0 \leq q \leq 2\) and of first order for \(q > 2\). Later [25] a more detailed study of the cluster structure was carried out, and it was found that there are three ranges of \(q\) with distinct behaviour \(q < 2\), \(q > 2\) and \(q = 2\).

Our aim in this paper is to compare the behaviour of the largest and second largest cluster in the RC-model for \(q = 2\), corresponding to the Ising model, for 5-dimensional lattices with side \(L\) and cyclic boundary. For the case \(q = 2\) [25] identified no less than five distinct scaling regions near the critical probability for the complete graph, each with its own asymptotic behaviour for \(L_1\) and \(L_2\), the sizes of the largest and second largest clusters respectively. Some of the regions are difficult to study, since in order to obtain correct scaling they would require far larger graphs than those used here. Thus we have focused on three cases: i) the high-temperature case (fixed \(K\) for \(K < K_c\)), ii) near \(K_c\) at fixed \(\lambda < 0\), and iii) near \(K_c\) at fixed \(\kappa\), where \(\kappa\) and \(\lambda\) are different \(L\)-dependent couplings. As we shall see, for these regions we have an excellent agreement between the scaling for the complete graph and for the 5-dimensional Ising model with cyclic boundary.

II. TERMINOLOGY, DEFINITIONS AND SAMPLING DETAILS

For a graph \(G = G(V,E)\) on \(n = |V|\) vertices and \(m = |E|\) edges the random cluster model’s partition function is

\[
Z_{RC}(G;p,q) = \sum_{A \subseteq E} p^{|A|} (1 - p)^{|E| - |A|} q^{c(A)}
\]

where \(c(A)\) is the number of (connected) components, or clusters as we will call them, of the graph \(G(V,A)\), i.e. the graph with vertex set \(V\) and edge set \(A\). Note that \(0 < p < 1\) and \(q > 0\) are parameters to the distribution. The Ising model without an external field, on the
other hand, has the partition function

\[ Z_I(G; K) = \sum_{s \in \{\pm 1\}^V} \exp(KU(s)) \]  

where the sum is taken over all functions \( s \) from \( V \) to \( \pm 1 \). Thus vertex \( i \) has spin \( s_i \). Here the energy is \( U(s) = \sum_{ij \in E} s_i s_j \), summed over all edges in \( G \). The parameter \( K = 1/T \) is the dimensionless coupling, or inverse temperature.

The two models are actually equivalent for \( q = 2 \) when setting \( p = 1 - \exp(-2K) \). Assuming we have a state \( A \subseteq E \) from the random cluster distribution in Equation (1), we can obtain a state \( s \) from the Ising distribution (2) in the following way: for each component, pick a spin \( \pm 1 \) uniformly at random and assign this spin to all vertices of the component. Going in the other direction is also easy. Starting with a state \( s \) from the Ising distribution at coupling \( K \), let \( A = \emptyset \). Now add each satisfied edge, i.e. edges \( ij \) with \( s_i s_j = 1 \), to the set \( A \) with probability \( p = 1 - \exp(-2K) \). For more information on this see [3].

This second scheme is a surprisingly efficient way of obtaining random cluster data. Just start up your trusted Ising state generator, whether it be Metropolis, heat-bath or Wolff cluster [26], and convert the Ising states to correctly distributed random cluster states.

All graphs studied here are 5-dimensional grid graphs with periodic boundary conditions, i.e. cartesian products of five cycles on \( L \) vertices, so that \( n = L^5 \) and \( m = 5L^5 \), and we have used \( L = 4, 6, 8, 10, 12, 16, 20 \) and \( 24 \). We collected the data by generating Ising states with the Wolff cluster method [27] and then converting them to random cluster states using the scheme described above.

Throughout we use the critical coupling \( K_c = 0.113915 \) [16], thus corresponding to the random cluster critical probability \( p_c \approx 0.203740 \), and denote one scaled temperature by \( \kappa = n^{1/2}(K - K_c)/K_c \) and another by \( \lambda = n^{1/3}(K - K_c)/K_c \). Recall that the critical temperature for a complete graph on \( n \) vertices approaches zero as \( K_c \sim 1/n \) and hence \( p_c \sim 2/n \). We will denote a scaled probability by \( \varepsilon = (p - p_c)/p_c \).

The \( k \)th central moment of a distribution is denoted by \( \sigma_k \), the mean value by \( \langle \cdots \rangle \) and the variance by \( \text{var}(\cdots) = \sigma_2 \). The standard deviation is then \( \sigma = \sqrt{\sigma_2} \). The median is written \( \tilde{\cdots} \). Given a random cluster state \( A \) the largest and second largest cluster size, i.e. the number of vertices in these clusters, is denoted \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) respectively.
A. Scaling for the complete graph

We will here give a very brief description of some of the relevant scaling results from [25] for the complete graph on \( n \) vertices. Where relevant further details will be given in later sections. As proven in [24] the critical probability is given by \( p_c = \frac{2}{n} \). Now if \( p/p_c \rightarrow a \neq 1 \) then the model is not critical and depending on whether \( a \) is less than 1 or greater than 1 we see behaviour corresponding to the high- and low-temperature regions respectively, with only small clusters in the first case and a large cluster, linear in \( n \) sized, plus some small ones in the latter case.

If \( p/p_c \rightarrow 1 \) we are inside the critical window and need a finer parameterization of \( p \). We thus assume that \( n/2p = 1 + \epsilon \), where \( \epsilon \) can depend on \( n \). We now see five distinct regions inside the critical window:

1. if \( n^{1/3} \epsilon \rightarrow -\infty \) then, asymptotically, all clusters are trees and small.
2. if \( n^{1/3} \epsilon \rightarrow c < 1 \), where \( c \) is a constant, then \( \mathcal{L}_1 \) is roughly of order \( n^{2/3} \).
3. if \( n^{1/3} \epsilon \rightarrow 0 \) but \( n^{1/2} \epsilon \rightarrow -\infty \) there exists a unique largest component and it’s size is \( n^c \), for a value \( 2/3 < c < 3/4 \).
4. if \( n^{1/2} \epsilon \rightarrow c \), where \( c \) is a constant, then \( \mathcal{L}_1 \) is of order \( n^{3/4} \) and \( \mathcal{L}_2 \) is bounded by \( O(\log n \sqrt{n}) \).
5. if \( n^{1/2} \epsilon \rightarrow \infty \) and \( \epsilon = o(1) \) then \( \mathcal{L}_1 \) is of order \( n\sqrt{3} \epsilon \) and \( \mathcal{L}_2 \) of order \( \frac{\log n^{2\epsilon^3}}{\epsilon} \).

III. THE HIGH-TEMPERATURE REGION

Strictly speaking, the complete graph version of the high-temperature case only requires that \( \epsilon n^{1/3} \rightarrow -\infty \). Of course, a fixed \( \epsilon < 0 \) will satisfy this condition. Hence, for the 5D case we will simply test the case of a fixed \( K \) for \( K < K_c \).

In [25] it was shown that \( \mathcal{L}_1 \) is distributed as an extreme-value distribution [28]. These have a density function of the form

\[
f(x) = \exp \left( \frac{a-x}{b} - \exp \left( \frac{a-x}{b} \right) \right)
\]

given some parameters \( a, b \). Extreme value distributions have skewness \( 12\sqrt{6} \zeta(3)/\pi^2 = 1.139547... \) and kurtosis \( 27/5 = 5.4 \). In Figure 1 we show how the skewness and excess
kurtosis scales with $L$ and how they approach these values for two different temperatures. The right plot of Figure 1 shows the distribution of $L_1$ at $K = 0.08$ for $L = 16$. In short, we have good reason to think that the complete graph behaviour also holds for 5d in this case.

![Figure 1](image)

**FIG. 1**: (Colour online) Left: skewness $\sigma_3/\sigma_2^{3/2}$, pointing to 1.1395, and excess kurtosis $\sigma_4/\sigma_2^2 - 3$, pointing to 2.4, of the distribution of $L_1$ at $K = 0.03$ (blue circles) and $K = 0.08$ (red squares) plotted versus $1/L$ for $L = 4, 6, 8, 10, 12, 16, 20, 24$. The fitted curves are second degree polynomials. The black points at $x = 0$ are the extreme value distribution values. Right: Distribution of $L_1$ for $L = 16$ at $K = 0.08$ (points) and a fitted density function of Equation (3) with parameters $a = 80.66$ and $b = 10.80$.

**IV. NEGATIVE SCALED TEMPERATURE $\lambda$**

Here we consider the case of a fixed negative scaled temperature, $n^{1/3}(K - K_c)/K_c = \lambda < 0$. This falls under the complete graph case $\varepsilon n^{1/3} \to a < 0$. In [25] it was shown that for this particular case the two largest clusters behave almost surely as

$$\frac{n^{2/3}}{\omega(n)} \leq L_2 \leq L_1 \leq n^{2/3}\omega(n)$$

for all functions $\omega(n)$ tending to infinity. Since $\omega(n)$ is allowed to grow as slowly as we like, thus boxing in $L_1$ and $L_2$, it is quite possible that in fact $L_1, L_2 \propto n^{2/3}$, though with different prefactors. Let us test this for the 5D case. In Figure 2 we show the normalised mean $L_1$ and $L_2$ for different negative $\lambda$. The right plot shows a zoom-in for $L_2$.

Note here that $\langle L_1 \rangle /n^{2/3} \to \infty$ when $\lambda \to 0^-$ whereas the $L_2$ counterpart actually has a local maximum. The right panel of Figure 2 shows a zoom-in for $L_2$, both the sampled data for $L \leq 24$ and also an estimated limit function based on fitting second degree polynomials.

6
to the values at $\lambda$ versus $1/L$. The $\langle L_2 \rangle/n^{2/3}$ has a distinct local maximum for all $L \geq 6$ and its location may possibly have zero as limit, approaching it very slowly. It would take considerably bigger graphs to settle that question. However, a rough estimate gives that the limit has the local maximum 0.59 at $\lambda = -0.17$. To conclude this section, we find that the 5D behaviour matches that of the complete graph case.

![Graph](image)

FIG. 2: (Colour online) Left: $\langle L_1 \rangle/n^{2/3}$ (above) and $\langle L_2 \rangle/n^{2/3}$ (below) versus $\lambda$ for $L = 6, 8, 10, 12, 16, 20, 24$. Right: $\langle L_2 \rangle/n^{2/3}$ for $L = 6, 8, 10, 12, 16, 20, 24$ and $\infty$ (see text).

V. SCALED TEMPERATURE $\kappa$

The next case is that of $n^{1/2}(K - K_c)/K_c = \kappa$, for constant $\kappa$. This fits under the complete graph case in [25] of $\varepsilon n^{1/2} \rightarrow c$ for real constants $c$. In this region the complete graph has different scalings for $L_1$ and $L_2$ and we treat them separately in the following two subsections.

A. The largest cluster

In the complete graph case it was shown [25] that

$$\lim_{n \to \infty} n^{3/4} \Pr(L_1 = \lfloor an^{3/4} \rfloor) = \frac{\exp(-a^4/12 + a^2 c/2)}{\int_0^\infty \exp(-x^4/12 + x^2 c/2) dx}$$

and in fact $a$ can be replaced by $a(n)$ with some positive limit value $a$. Note that the $L_1$-distribution essentially has the same form as the Ising magnetisation distribution of a complete graph, which is $\exp(-4a^4/3 + 2\kappa a^2)$, see [29], except that $L_1$ has support only for $a \geq 0$. This follows since the spin-state coming from an RC-state is obtained by assigning
a random spin value to each cluster in the RC-state. Above all this means that \( \langle L_1 \rangle \sim n^{3/4} \) for constant \( \kappa \). In Figure 3 we show the normalised mean \( \langle L_1 \rangle / n^{3/4} \) and the normalised variance \( \text{var}(L_1) / n^{3/2} \) over the interval \(-8 \leq \kappa \leq 8\). The plot also shows the complete graph values computed from Equation (5) when setting \( c = \kappa \). Note how close the complete graph values are to their 5D counterparts.

The limit of the mean \( L_1 \) is easily found by plotting them for some \( \kappa \) versus \( 1/L \) and then fitting a polynomial to the points. There is some very mild correction to scaling at work for \( \kappa \geq 0 \) but it is easily captured by a second degree polynomial. A slightly more careful analysis on the case \( \kappa = 0 \), i.e. at \( K_c \), based on fitting second degree polynomials to all 4-subsets of the data points for \( L \geq 6 \), results in a median value \( \langle L_1 \rangle / n^{3/4} \rightarrow 1.1266(6) \), and the error corresponds to the interquartile range. The complete graph value at \( \kappa = 0 \) is 0.9098….

![FIG. 3: (Colour online) Left: normalised largest cluster size \( \langle L_1 \rangle / n^{3/4} \) plotted versus scaled temperature \( \kappa \) for \( L = 6, 8, 10, 12, 16, 20, 24 \) and the complete graph case. Right: normalised variance of the largest cluster size \( \text{var}(L_1) / n^{3/2} \) plotted versus scaled temperature \( \kappa \) for \( L = 6, 8, 10, 12, 16, 20, 24 \) and the complete graph case.](image)

What about the distribution of \( L_1 \)? Figure 4 shows a normalised form of the \( L_1 \)-distribution, \( f_1(x) = n^{3/4} \Pr(L_1 = \ell) \) where \( x = \ell / n^{3/4} \), for a few values of \( \kappa \) with \( L = 8 \). Between roughly \( 0 < \kappa < 1.8 \) the distribution actually goes through a bimodal phase and this property is not found in the complete graph case of Equation (5). However, it agrees with the finding in [16] that for the Ising model with cyclic boundary in 5d the energy distribution at the critical temperature becomes bimodal. The second plot in Figure 4 shows the distribution at \( \kappa = 0.44 \) for \( L = 6, 8, 10, 12, 16 \) and it clearly shows the distribution retaining its bimodal form with increasing \( L \).
The skewness and kurtosis of the $L_1$-distribution are shown in Figure 5 together with the complete graph case. Clearly there are distinct limits for each $\kappa$, occasionally with some mild corrections to scaling, easily captured by a second degree polynomial. Perhaps unexpectedly, the data suggest a limit value of the skewness of about $-2.1$ as $\kappa \to -\infty$. There is no conflict in the existence of such a limit and our earlier claim of the skewness taking the extreme value distribution value of 1.1395... in the high-temperature case. It is simply a matter of taking limits in the right order, i.e. $\lim_{\kappa \to -\infty} \lim_{L \to \infty} \sigma_3 / \sigma_2^{3/2} \approx 2.1$ and the kurtosis limit is approximately 10. This is in fact also the behaviour of the complete graph case though it takes different values. As $\kappa \to -\infty$ its skewness approaches 0.9952... and the kurtosis has the limit 3.869... The two models agree well on the case $\kappa = 0$ though. Here the 5D case has skewness 0.458(6) and kurtosis 2.365(4) while the complete graph values are 0.4427... and 2.4446... respectively.

B. The second largest cluster

In the complete graph case it was shown [25] that

$$\lim_{n \to \infty} \Pr \left( L_2 \leq \frac{\sqrt{n} \log n}{2a^2} \right) = \frac{\int_a^\infty \exp(-x^4/12 + x^2c/2)dx}{\int_0^\infty \exp(-x^4/12 + x^2c/2)dx}$$

(6)

for $\varepsilon n^{1/2} \to c$. Note that $a$ may be replaced by function $a(n)$ having some limit $a$. This gives a density function
Consider the skewness and kurtosis of the $\mathcal{L}_1$-distribution this must eventually show up in some higher moment, and indeed it does. $\kappa$ grows as $2^L$ of the figure shows a normalised form of the graphs to get anything close to the complete graph case. For $L \geq 6$, 8, 10, 12, 16, 20, 24 and complete graph case. Note that for $\kappa < 0$ though the corrections to scaling are quite significant for $L = 6$, 8, 10, 12, 16, 20, 24 and complete graph case.

$$
\lim_{n \to \infty} \sqrt{n} \log n \Pr (\mathcal{L}_2 = [b \sqrt{n} \log n]) = \frac{\exp \left( \frac{-1}{2b^2} + \frac{c}{b} \right)}{2 \sqrt{2} b^{3/2} \int_{0}^{\infty} \exp \left( -x^4 / 12 + x^2 c / 2 \right) dx} \quad (7)
$$

which implies an infinite expectation value. We will thus instead consider the median value. So, when we move close enough to $p_c$, from $n^{1/3} \varepsilon \to c$ to $\varepsilon n^{1/2} \to c$, the second largest cluster drops in size from $n^{2/3}$ to $\sqrt{n} \log n$. This also seems to be the case for the 5D case though the corrections to scaling are quite significant for $\kappa < 0$.

Figure 5 shows the normalised median $\tilde{\mathcal{L}}_2 / (\sqrt{n} \log n)$ versus $\kappa$ for 5D and the complete graph case which demonstrates this effect. Note that for $\kappa < 0$ it would require enormous graphs to get anything close to the complete graph case. For $\kappa > 0$, however, the two curves quickly agree on the complete graph behaviour, if not on the actual value. The right plot of the figure shows a normalised form of the $\mathcal{L}_2$-distribution, $f_2(x) = \sqrt{n} \log n \Pr (\mathcal{L}_2 = \ell)$ with $x = \ell / (\sqrt{n} \log n)$, for a range of $L$ together with the complete graph distribution in Equation (7) at $\kappa = 2.6$ where the two cases largely agree.

If the 5D $\mathcal{L}_2$-distribution has a fat tailed distribution like the complete graph case of Equation (7) this must eventually show up in some higher moment, and indeed it does. Consider the skewness and kurtosis of the $\mathcal{L}_2$-distribution in Figure 7. Both show a divergent behaviour around $\kappa \approx 2.5$. In fact, a very rough estimate suggests that the peak skewness grows as $2L^{2/5}$ and the peak kurtosis as $10L^{4/5}$. 

---

**FIG. 5:** (Colour online) Left: skewness $\sigma_3 / \sigma_2^{3/2}$ of the $\mathcal{L}_1$-distribution plotted versus scaled temperature $\kappa$ for $L = 6$, 8, 10, 12, 16, 20, 24 and complete graph case. Right: kurtosis $\sigma_4 / \sigma_2^2$ of the $\mathcal{L}_1$-distribution plotted versus scaled temperature $\kappa$ for $L = 6$, 8, 10, 12, 16, 20, 24 and complete graph case.
FIG. 6: (Colour online) Left: normalised median second largest cluster size \( \tilde{L}_2/(\sqrt{n}\log n) \) plotted versus scaled temperature \( \kappa \) for \( L = 4, 6, 8, 10, 12, 16, 20, 24 \) and the complete graph case. The inset shows the complete graph median for a wider range of \( \kappa \). Right: normalised distribution \( f_2(x) \) (see text) of \( L_2 \), for \( L = 6, 8, 10, 12, 16 \) and the complete graph case of Equation (7) at \( \kappa = 2.63 \).

FIG. 7: (Colour online) Left: skewness \( \sigma_3/\sigma_2^{3/2} \) of the \( L_2 \)-distribution plotted versus \( \kappa \) for \( L = 6, 8, 10, 12, 16, 20, 24 \). Right: kurtosis \( \sigma_4/\sigma_2^2 \) of the \( L_2 \)-distribution plotted versus \( \kappa \) for \( L = 6, 8, 10, 12, 16, 20, 24 \).

VI. CONCLUSIONS

Our aim has been to compare the scaling of the sizes of the largest and second largest clusters for the 5D random cluster model with that seen for the corresponding value of \( p \) for the complete graph. For the complete graph there are two non-critical regions and inside the critical window five distinct scaling regions. Due to the limitations coming from the range of system sizes for which we can simulate the model we have chosen to work with the high temperature region and two of the regions from the critical window.
In each of the tested regions we have found that the scaling from the complete graph agrees well with the observed values from the 5D model, and in many cases not only the scaling but also the probability distributions agree well. The most notable exception is the bimodal distribution of $L_1$ for $\kappa$ close to 0 for the 5D model. This feature corresponds to the bimodal energy distribution seen for the Ising model on the same graphs in [16], which also clearly separates the 5D model with cyclic boundary from the case with free boundary and the infinite system thermodynamic limit.

We expect this agreement to hold for $d > 5$ as well. If we in the scaling for $L_1$ and $L_2$ on the complete graph case replace $n$ by $V = L^d$ we conjecture that we get the correct scaling for the $d$-dimensional case with cyclic boundary.

VII. ACKNOWLEDGEMENTS

The computations were performed on resources provided by the Swedish National Infrastructure for Computing (SNIC) at High Performance Computing Center North (HPC2N) and at Chalmers Centre for Computational Science and Engineering (C3SE).

[1] S. Smirnov, Ann. of Math. (2) 172, 1435 (2010).
[2] D. Chelkak and S. Smirnov, Invent. Math. 189, 515 (2012).
[3] G. Grimmett, The random-cluster model (Springer, 2004).
[4] K. Binder, Z. Phys. B 61, 13 (1985).
[5] K. Binder, M. Nauenberg, V. Privman, and A. P. Young, Phys. Rev. B 31, 1498 (1985).
[6] K. Binder, in Computational Methods in Field Theory, edited by H. Gausterer and C. Lang (Springer Berlin Heidelberg, 1992), vol. 409 of Lecture Notes in Physics, pp. 59–125.
[7] J. Rudnick, H. Guo, and D. Jasnow, J. Stat. Phys. 41, 353 (1985).
[8] J. Rudnick, G. Gaspari, and V. Privman, Phys. Rev. B 32, 7594 (1985).
[9] C. Rickwardt, P. Nielaba, and K. Binder, Ann. Physik 506, 483 (1994).
[10] K. K. Mon, Europhys. Lett. 34, 399 (1996).
[11] G. Parisi and J. J. Ruiz-Lorenzo, Phys. Rev. B 54, R3698 (1996).
[12] E. Luijten and H. W. J. Blöte, Phys. Rev. Lett. 76, 1557 (1996).
[13] H. W. J. Blöte and E. Luijten, Europhys. Lett. 38, 565 (1997).
[14] E. Luijten, K. Binder, and H. Blöte, Eur. Phys. J. B 9, 289 (1999).
[15] J. L. Jones and A. P. Young, Phys. Rev. B 71, 174438 (2005).
[16] P. H. Lundow and K. Markström, Nucl. Phys. B 845, 120 (2011).
[17] B. Berche, C. Chatelain, C. Dhall, R. Kenna, R. Low, and J.-C. Walter, J. Stat. Mech. 2008, P11010 (2008).
[18] M. Aizenman, Nucl. Phys. B 485, 551 (1997).
[19] M. Heydenreich and R. van der Hofstad, Comm. Math. Phys. 270, 335 (2007).
[20] M. Heydenreich and R. van der Hofstad, Probab. Theory Related Fields 149, 397 (2011).
[21] C. Borgs, J. T. Chayes, R. van der Hofstad, G. Slade, and J. Spencer, Random Struct. Algorithms 27, 137 (2005).
[22] C. Borgs, J. T. Chayes, R. van der Hofstad, G. Slade, and J. Spencer, Ann. Probab. 33, 1886 (2005).
[23] C. Borgs, J. T. Chayes, R. van der Hofstad, G. Slade, and J. Spencer, Combinatorica 26, 395 (2006).
[24] B. Bollobás, G. Grimmett, and S. Janson, Probab. Theory Related Fields 104, 283 (1996).
[25] M. J. Luczak and T. Luczak, Random Struct. Algorithms 28, 215 (2006).
[26] M. E. J. Newman and G. T. Barkema, Monte Carlo methods in statistical physics (The Clarendon Press Oxford University Press, New York, 1999), ISBN 0-19-851796-3; 0-19-851797-1.
[27] U. Wolff, Phys. Rev. Lett 62, 361 (1989).
[28] L. de Haan and A. Ferreira, Extreme Value Theory: An Introduction, Springer Series in Operations Research and Financial Engineering (Springer, 2006).
[29] P. H. Lundow and A. Rosengren, Phil. Mag. 90, 3313 (2010).