The central charge of supersymmetric $AdS_5$ solutions of type IIB supergravity

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We show that generic supersymmetric $AdS_5$ solutions of type IIB supergravity admit a canonical contact structure. This structure determines the central charge of the dual field theory and the conformal dimension of operators dual to supersymmetric wrapped D3-branes. Hence both quantities can be calculated using incomplete information about the solutions, allowing us to prove that they are rational numbers for solutions with a $U(1)$ $R$-symmetry, in agreement with field theory expectations. We also discuss related Duistermaat-Heckman integrals and localization formulae.

INTRODUCTION

In its simplest setting, the AdS/CFT correspondence is an equivalence between $d$-dimensional conformally invariant quantum field theory (CFT) and string theory (or M-theory) in a ten- (or eleven-) dimensional background of the form $AdS_{d+1} \times Y$, where $AdS_{d+1}$ is $(d+1)$-dimensional anti-de Sitter space and $Y$ is a compact, internal manifold. Generically, the weakly coupled limit of the string/M-theory background describes a strongly coupled limit of the CFT. This remarkable duality has provided powerful new insights into strong coupling phenomena, with potential applications to condensed matter and nuclear physics systems.

One focus has been to study superconformal field theories (SCFTs) in four dimensions with $N = 1$ supersymmetry. Such SCFTs are invariant under an abelian $R$-symmetry which encodes considerable information about the theory. For example, the anomalous dimensions of a special class of operators $O$, known as chiral and anti-chiral primaries, are given, exactly, by their $R$-charges via $\Delta(O) = \frac{1}{2d} R(O)$. Furthermore, the central charges, which determine the conformal anomaly, are fixed by the $R$-symmetry via [1]

$$a = \frac{1}{32} \left( 3 \mathrm{Tr} R^3 - \mathrm{Tr} R \right), \quad c = \frac{1}{32} \left( 9 \mathrm{Tr} R^3 - 5 \mathrm{Tr} R \right) \quad (1)$$

where the trace is over the $R$-charges of the fermions in the theory.

An interesting subclass of such SCFTs have a dual description in terms of type IIB string backgrounds of the form $AdS_5 \times Y$. In the supergravity limit, requiring the background to be supersymmetric implies particular restrictions on the geometry of $Y$ [2]. The dimension of certain chiral operators of the SCFT can then be deduced via Kaluza-Klein (KK) reduction of the supergravity modes on $Y$, while others can be computed by analysing supersymmetric sub-manifolds of $Y$ on which branes can wrap. In this limit, the SCFTs necessarily have $a = c$, the value of which may be calculated by determining Newton’s constant, $G_5$, in the effective five-dimensional KK reduced theory [3].

The $R$-symmetry of the SCFT corresponds to a Killing vector field $\xi$ on $Y$, which has a canonical construction given the background is supersymmetric [2]. If the orbits of this Killing vector field all close the $R$-symmetry is $U(1)$, while if they do not it is $\mathbb{R}$. In light of the comments above, one would like to understand in general how the $R$-symmetry Killing vector can be used to calculate quantities of interest for the dual SCFT, in particular, $a$ and $\Delta(O)$. Indeed, the $a$ central charge is an important quantity in a $d = 4$ CFT, being (conjecturally) a count of the massless degrees of freedom in the theory.

To date, the most widely studied class of dual solutions are in a particular class where only the metric and the self-dual five-form flux $F_5$ in the supergravity theory are non-vanishing. In this case $Y$ is necessarily a Sasaki-Einstein manifold $SE_5$. By definition, the metric cone over $SE_5$ is Calabi-Yau and the $AdS_5$ solutions arise from the near-horizon limit of D3-branes sitting at the apex of this cone. The central charge is inversely proportional to the volume of $SE_5$ [4]. In this case, $\xi$ is known as the Reeb vector field, and using the Sasaki-Einstein geometry, several formulae for $a$ and $\Delta(O)$ in terms of $\xi$ can indeed be derived [5, 6]. These formulae are significant because they allow one to calculate important physical quantities without requiring the full explicit solution. Although rich classes of explicit $SE_5$ solutions are known, their construction relies on the solutions having a large amount of symmetry and they comprise a small subset of all solutions.

In this letter we show [12] that such formulae still arise in the case of generic supersymmetric $AdS_5$ type IIB solutions. This is rather remarkable since the derivations in [4, 6] for the $SE_5$ case used the Calabi-Yau property of the cone which no longer holds in general. Our analysis only excludes the special case when $F_5$ vanishes. Physically, this means we require that the D3-brane charge is non-vanishing and this includes all known $AdS_5$ type IIB solutions.

Our key observation is that, remarkably, supersymmetry implies that these solutions all have a canonical contact structure. A contact structure on $Y$ is a one-form $\sigma$ such that the five-form $\sigma \wedge d\sigma \wedge d\sigma$ is nowhere-vanishing.
Equivalently there is a natural symplectic structure on the cone over $Y$. We will find that, up to a universal constant of proportionality, the central charge $a$ is the “contact volume” of $Y$. In addition, the conformal dimension $\Delta(\mathcal{O}_{\Sigma_3})$ of the chiral primary operator $\mathcal{O}_{\Sigma_3}$ dual to a D3-brane wrapped on a supersymmetric submanifold $\Sigma_3 \subset Y$ is given by the contact volume of $\Sigma_3$. More precisely,

$$\frac{a_{N=4}}{a} = \frac{1}{(2\pi)^2} \int_Y \sigma \wedge d\sigma \wedge d\sigma,$$

and

$$\Delta(\mathcal{O}_{\Sigma_3}) = \frac{2\pi N \int_{\Sigma_3} \sigma \wedge d\sigma}{\int_Y \sigma \wedge d\sigma \wedge d\sigma},$$

where $a_{N=4} = N^2/4$ is the (large $N$) central charge for $SU(N) \mathcal{N} = 4$ super Yang-Mills theory, and $N$ is the quantized D3-brane charge. We expect that the existence of this contact structure will be important in extending other results from the $SE_5$ case to generic $AdS_5$ solutions.

**SUPERSYMMETRIC TYPE IIB $AdS_5$ SOLUTIONS**

Our main results follow directly from the analysis of generic supersymmetric $AdS_5$ solutions of type IIB supergravity given in [2]. The ten-dimensional metric, in Einstein frame, is

$$g_E = e^{2\Delta} (g_{AdS} + g_Y),$$

where $g_Y$ is a Riemannian metric on the compact internal five-manifold $Y$, and $\Delta$ is a function on $Y$. The $AdS_5$ metric is normalized to have unit radius, $R_{\mu\nu}^{AdS} = -4g_{\mu\nu}^{AdS}$. The remaining IIB fields are the dilaton $\phi$, the NS three-form field strength $H = dB$, and the RR potentials $C = C_0 + C_2 + C_4$, with field strengths $F = F_1 + F_3 + F_5 = (d-H)\wedge C$. It is useful to introduce the complex combinations

$$P = \frac{1}{2} e^{\phi} F_1 + \frac{1}{2} d\phi,$$

$$G = -e^{-\phi/2} H - i e^{\phi/2} F_3,$$

together with $Q = -\frac{1}{2} e^{\phi} F_1$. All of these fields are taken to be pull-backs of forms on $Y$, so as to preserve the $SO(4,2)$ symmetry, with the exception of the self-dual five-form $F_5$, which necessarily takes the form

$$F_5 = f (\text{vol}_{AdS} + \text{vol}_Y),$$

where $f$ is a constant. The conventions for the volume forms $\text{vol}_{AdS}$ and $\text{vol}_Y$ are given in [2]. To match standard contact structure conventions, it will be useful here to take $\text{vol}_Y = -\text{vol}_{AdS}$ as defining our orientation for integrating five-forms on $Y$.

As explained in detail in [2], a supersymmetric $AdS_5$ solution is specified by two $Spin(5)$ spinors $\xi_1, \xi_2$ on $Y$ satisfying the following system of equations

$$0 = D_m \xi_1 + \frac{1}{2} (e^{-4\Delta} f - 2) \gamma_m \xi_1 + \frac{1}{2} e^{-2\Delta} G_{mnp} \gamma^{mnp} \xi_2$$

$$0 = \bar{D}_m \xi_2 - \frac{1}{2} (e^{-4\Delta} f + 2) \gamma_m \xi_2 + \frac{1}{2} e^{-2\Delta} G^{*}_{mnp} \gamma^{mnp} \xi_1,$$

$$0 = (\gamma^m \partial_m \Delta - \frac{1}{2} f e^{-4\Delta} + i) \xi_1 - \frac{1}{48} e^{-2\Delta} \gamma_{mnp} G_{mnp} \xi_2$$

$$0 = (\gamma^m \partial_m \Delta + \frac{1}{2} f e^{-4\Delta} + i) \xi_2 - \frac{1}{48} e^{-2\Delta} \gamma_{mnp} G_{mnp} \xi_1$$

$$0 = \gamma^m P_m \xi_2 + \frac{1}{24} e^{-2\Delta} \gamma_{mnp} G_{mnp} \xi_1$$

$$0 = \gamma^m P^*_m \xi_1 + \frac{1}{24} e^{-2\Delta} \gamma_{mnp} G_{mnp} \xi_2.$$

Here $\gamma_m$ generate the Clifford algebra for $(Y, g_Y)$, with $\gamma_12345 = +1$, and $D_m = (\nabla_m - \frac{1}{2} Q_m)$. The equations may be rewritten in terms of differential constraints on scalars, one-forms and two-forms that are constructed as bilinears in the spinors [2]. We shall introduce and study two particular such one-forms in the next section.

**CONTACT STRUCTURE**

We begin with the one-form bilinear

$$K_{5,m} = \frac{1}{2} (\xi_1 \gamma_m \xi_1 + \xi_2 \gamma_m \xi_2).$$

Defining $\xi^m \equiv g_Y^{mn} K_{5,n}$, it was shown in [2] that $\xi$ is Killing, preserves all of the bosonic fields, and is therefore the canonical vector field dual to the $R$-symmetry of the $\mathcal{N} = 1$ SCFT. We next consider the spinor bilinears

$$K_{4,m} = \frac{1}{2} (\xi_1 \gamma_m \xi_1 - \xi_2 \gamma_m \xi_2),$$

$$iV_{5,m} = \frac{1}{2} (\xi_1 \gamma_m \xi_1 - \xi_2 \gamma_m \xi_2).$$

Supersymmetry implies [2] that

$$e^{-4\Delta} d(e^{4\Delta} K_4) = -2V.$$

We claim that

$$\sigma \equiv \frac{4}{7} e^{4\Delta} K_4$$

is a contact one-form on $Y$. In particular, one can readily show that

$$\sigma \wedge d\sigma \wedge d\sigma = \frac{128}{7} e^{8\Delta} \text{vol}_Y.$$

Notice this calculation makes sense only if $f \neq 0$, or equivalently $F_5 \neq 0$. (We shall determine the quantization condition on $f$, which is related to the D3-brane charge, in the next section.) With this assumption, $\sigma \wedge d\sigma \wedge d\sigma$ is nowhere-zero, and so, by definition, $\sigma$ is a contact form.

Furthermore, again using the results of [2], we have

$$1 = \xi_1 d\sigma, \quad 0 = \xi_2 d\sigma,$$

which shows that $\xi$ is also the unique “Reeb vector field” associated with the contact structure.
If \((Y, \sigma)\) is a contact manifold, the product \(X = \mathbb{R}_{>0} \times Y\) has a natural symplectic structure with symplectic two-form, by definition closed and non-degenerate,
\[
\omega = \frac{1}{2} \text{d}(r^2 \sigma) ,
\]
where \(r > 0\) is a coordinate on \(\mathbb{R}_{>0}\). Note that writing the unit \(\text{AdS}_5\) metric in Poincaré coordinates, the metric \(\ Greeks \) can also be rewritten in the form of a warped supersymmetric \(\mathbb{R}^{3,1} \times X\) solution
\[
g_E = e^{2\Delta} r^2 g_{\mathbb{R}^{3,1}} + e^{2\Delta} r^{-2} g_X ,
\]
where \(g_{\mathbb{R}^{3,1}}\) is the flat metric on \(\mathbb{R}^{3,1}\) and the six-dimensional metric on \(X\) is given by the cone metric \(g_X = dr^2 + r^2 g_Y\). In the \(SE_5\) case, \(g_X\) is Calabi-Yau and \(\omega\) is the corresponding Kähler form. In contrast, in the generic case, there is no longer such a simple relation between \(g_X\) and \(\omega\): in fact \(X\) is a special kind of generalized complex geometry which will be discussed in [7].

**CENTRAL CHARGE**

The central charge of the dual SCFT is related to the five-dimensional Newton constant \(G_5\) [3]. The latter, in turn, was computed in appendix E of [2], and is given by the supergravity formula
\[
G_5 = \frac{\kappa^3_0}{8\pi V_5} , \text{ where } V_5 \equiv \int_Y e^{8\Delta} \text{vol}_Y .
\]
In string theory the five-form flux \(F_5 + H \wedge C_2 = dC_4\) is quantized. Specifically, we have
\[
N = \frac{1}{(2\pi)^4} \int_Y (F_5 + H \wedge C_2) ,
\]
and after a calculation we find
\[
N = -\frac{f}{(2\pi)^4} \int_Y \frac{1}{\sin^2 \zeta} \text{vol}_Y ,
\]
where \(\sin \zeta \equiv \frac{1}{2}(\xi_1 - \xi_2) = \frac{1}{2} e^{-2\Delta}, \) with the last equality taken from [2]. Here the spinors are normalized so that \(\frac{1}{2}(\xi_1 + \xi_2) = 1\), as in [2]. Combining (16) and \(\xi_2\) and using \(2\kappa^2_0 = (2\pi)^4 g^2_Y\) leads to
\[
G_5 = \frac{8V_5}{\pi^2 f^2 N^2} .
\]
Finally, using equation (12) together with the formula for the central charge \(a = \pi/8G_5\) and the fact that \(a_{N=4} = N^2/4\), we arrive at the remarkably simple formula (13).

**SOME APPLICATIONS**

The formula (13) has some immediate applications.

Suppose that the Reeb vector field \(\xi\) is quasi-regular, meaning that its orbits all close. Its flow then defines a \(U(1)\) action on \(Y\), which is dual to a \(U(1)\) \(R\)-symmetry (more generally the orbits of \(\xi\) need not all close, as happens, for example, for infinitely many of the \(Y^{p,q}\) Sasaki-Einstein five-manifolds [8]). Then, using formula (14), we may prove that the central charge is always a rational number. Notice that this is predicted from the dual \(N = 1\) SCFT, using (15) and the fact that the \(R\)-charges are all rational numbers if the \(R\)-symmetry is \(U(1)\).

To proceed, let \(\chi\) denote the canonically normalized generator of the \(U(1)\) action, so that \(\xi = k\chi\) for some constant \(k \in \mathbb{R}_{>0}\). Since \(\xi\) is nowhere-zero [2], the \(U(1)\) action generated by \(\chi\) is locally free (all isotropy groups are finite), and the orbit space will in general be an orbifold \(M\). Thus \(Y\) is the total space of a \(U(1)\) principal orbibundle \(\mathcal{L}\) over \(M\). Then the relations (13) imply that \(k\sigma\) is a connection one-form on this bundle. We thus compute
\[
\frac{k^3}{(2\pi)^4} \int_Y \sigma \wedge d\sigma \wedge d\sigma = \int_M c_1^3(\mathcal{L}) \in \mathbb{Q} .
\]
Here we have used the fact that the Chern numbers of an orbibundle are rational. Finally, the constant \(k\) is also necessarily rational. This follows, for example, since the scalar \(S \equiv \frac{1}{2} \xi_2 \xi_1\) has charge \(-3\) under \(\xi\) [2], whereas it must have an integer charge \(-n\) under \(\chi\); thus \(k = 3/n \in \mathbb{Q}\), and it follows from (14) that \(a \in \mathbb{Q}\).

In general, we may also rewrite (14) in terms of the symplectic structure on \(X\) as
\[
\frac{a_{N=4}}{a} = \frac{1}{(2\pi)^4} \int_X e^{-r^2/2} \frac{\omega^3}{3!} .
\]
This is a Duistermaat-Heckman integral, with Hamiltonian function \(\mathcal{H} = r^2/2\) for the Reeb vector field \(\xi\), so \(d\mathcal{H} = -\xi_\omega\). The power of the formula (15) is that it may be evaluated using localization [6]. This is easiest to explain when the solution is toric, meaning that there is a \(U(1)^3\) action on \(Y\) under which \(\sigma\) (and hence \(\omega\) under the lift to \(X\)) is invariant. In this case there is a moment map \(\mu : X \to \mathbb{R}^3\), whose image is a strictly convex rational polyhedral cone. As explained in [6], the integral (15) may be evaluated by taking any simplicial resolution \(\mathcal{P}\) of this cone, and evaluating
\[
\frac{1}{(2\pi)^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!} = \sum_{\text{vertices } p \in \mathcal{P}} \frac{1}{\text{vertices } i = 1} \frac{1}{\prod_{\text{edges } u_{i}^p} \langle \xi, u_{i}^p \rangle} ,
\]
where \(u_{i}^p\), \(i = 1, 2, 3\), are the three edge vectors of the moment polytope \(\mathcal{P}\) at the vertex point \(p\). The vertices of \(\mathcal{P}\) correspond to the \(U(1)^3\) fixed points of a symplectic toric resolution \(X_P\) of \(X\). Thus, remarkably, these results of [6] hold in general, even when there are non-trivial fluxes turned on and \(X\) is not Calabi-Yau.
CONFORMAL DIMENSION OF BPS BRANES

Static probe D3-branes that wrap supersymmetric three-submanifolds $\Sigma_3 \subset Y$ are dual to chiral primary operators in the dual SCFT. We claim that the conformal dimension $\Delta(\mathcal{O}_{\Sigma_3})$ of the operator is given by $\frac{2}{3}$ to the results of [3, 6] for the special Sasaki-Einstein case, is sufficient.

The generic type IIB $AdS_5$ solutions considered here, like those in [1, 4], have non-vanishing D3-brane charge. It is natural to wonder what happens for the special case where this is no longer true (assuming that such solutions exist). The corresponding formulae would have to be quite different since there is no longer a contact structure. It is also known that there are rich classes of supersymmetric $AdS_5 \times M_4$ solutions of $D = 11$ supergravity that are dual to $\mathcal{N} = 1$ SCFTs [11]. To obtain the geometric formulae for this class will also require new ideas, since now the cone geometry is seven-dimensional.

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CONCLUDING REMARKS

The fact that several key properties of $\mathcal{N} = 1$ SCFTs are determined by the $R$-symmetry of the theory should be reflected in the geometry of the dual supergravity background. For the case of generic $AdS_5 \times Y$ solutions of type IIB supergravity, we have shown that the relevant object is a canonical contact structure on $Y$, determined by the Reeb vector field. This led to succinct and universal formulae for calculating the central charge and certain conformal dimensions, without knowing the full solution. Potentially, as in the $SE_5$ case, the contact structure is rich enough to encode other properties of the field theory. For instance, the principal of $a$-maximization suggests that the Reeb vector field itself might be determined from general geometric data. For the class of type IIB solutions considered here, it may be that knowing only the symplectic structure of the cone $X$, together with some other (integral) data of the solution, analogous