Precise asymptotics for large deviations of integral forms

Xiangfeng Yang*
Grupo de Física Matemática
Universidade de Lisboa
Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

September 13, 2012

Abstract

For suitable families of locally infinitely divisible Markov processes \( \{ \xi_t \}_{0 \leq t \leq T} \) with frequent small jumps depending on a small parameter \( \epsilon > 0 \), precise asymptotics for large deviations of integral forms \( \mathbb{E}^\epsilon \left[ \exp \{ \epsilon^{-1} F(\xi^\epsilon) \} \right] \) are proved for smooth functionals \( F \). The main ingredient of the proof in this paper is a recent result regarding the asymptotic expansions of the expectations \( \mathbb{E}^\epsilon [G(\xi^\epsilon)] \) for smooth \( G \). Several connections between these large deviation asymptotics and partial integro-differential equations are included as well.

Keywords and phrases: Crâmer’s transformation, large deviations, normal deviations, locally infinitely divisible, compensating operators

AMS 2010 subject classifications: Primary 60F10, 60F17; secondary 60J75, 35C20

1 Introduction

The study of large deviations in limit theorems can be formulated as follows. Let \( X \) be a metric space with metric \( \rho \), and \( \mu^\epsilon \) be a family of probability measures on \( X \) depending on a parameter \( \epsilon > 0 \). Suppose there is a point \( x_* \in X \) such that for any \( \delta > 0 \) and small \( \epsilon \), \( \mu^\epsilon \{ y : \rho(x_*, y) < \delta \} \) have overwhelming probabilities: \( \lim_{\epsilon \to 0} \mu^\epsilon \{ y : \rho(x_*, y) < \delta \} = 1 \). Problems on large deviations are concerned with the limiting behavior as \( \epsilon \to 0 \) of the infinitesimal probabilities \( \mu^\epsilon (A) \) for measurable

*xyang2@tulane.edu
sets $A \subseteq \mathbb{X}$ that are situated at a positive distance from point $x_*$. Problems concerning asymptotics as $\epsilon \to 0$ of integrals in the form $\int_X f_\epsilon(x)\mu^\epsilon(dx)$ also belong to large deviations if the main part of such integrals for small $\epsilon$ is due to the values of $x$ far away from point $x_*$. This paper deals with the later (asymptotics of integrals).

Classical large deviation problems are about empirical means $\bar{S}_n = \sum_{i=1}^n \xi_i/n$ of random variables $\xi_i$. In general, results obtained deal with asymptotics up to logarithmic equivalence $\ln \mathbb{P}\{\bar{S}_n \in A\}$ or $\ln \mathbb{E}\exp\{nf(\bar{S}_n)\}$, and we call these results rough large deviations, see [44], [11], [14] and [46]. If we assume that the random variables $\xi_i$ are real-valued and independent identically distributed (i.i.d.), Cramér in [10] made use of limit theorems on normal deviations (asymptotic expansions in limit theorem for i.i.d. random variables) and proved a precise large deviation result: $\mathbb{P}\{\bar{S}_n > a\} \sim \frac{1}{\sqrt{n}} \exp\{-nI(a)\}$ for $a > 0$, some constant $c$ and a rate function $I(x)$ provided $\xi_i$ are non-lattice having zero mean and finite moment generating function. He used what we will call Cramér’s transformation to define a new distribution $\tilde{\mu}(dx) = e^{z_0x}\mu(dx)/\int e^{z_0x}\mu(dx)$ for some $z_0$ so that new random variables $\tilde{\xi}_i$ corresponding to $\tilde{\mu}$ have mean $a$. If more conditions are assumed on $\xi_i$, then Cramér derived precise asymptotics for large deviation probabilities $\mathbb{P}\{\bar{S}_n > a\} = \exp\{-nI(a)\}(\sum_{1 \leq i \leq N} l_in^{-i/2} + o(n^{-N/2}))$ for an integer $N$ depending on the moments of $\xi_i$ (see also [9], [26] and the references therein for related works). If we use $\mu^n$ to denote the distributions of $\bar{S}_n$, then results concerning integrals $\int_X f_n(x)\mu^n(dx)$ with $f_n(x) = \exp\{nf(x)\}$ can be obtained similarly in the form $\int_X \exp\{nf(x)\}\mu^n(dx) = \exp\{n[f(x_0) - I(x_0)]\}(\sum_{0 \leq i \leq M} k_in^{-i/2} + o(n^{-M}))$ provided $\max[f(x) - I(x)]$ is reached uniquely at $x_0$ for some integer $M$ depending on the smoothness of $f(x)$. If $\xi_i$ are not independent or the moment generating function doesn’t exist, similar precise large deviations can be also obtained (see for instance [31], [32], [28] and the references therein). For related treatments on other types of sequences of random variables (such as randomly indexed sums), we refer to [29] and [34]. Precise large deviations are also called in the literature as sharp (or exact) large deviations.

When we study large deviations for stochastic processes $\xi^\epsilon$ defined on probability spaces $(\Omega, \mathcal{F}, \mathbb{P}^\epsilon)$, one usually investigates the asymptotics up to logarithmic equivalence $\ln \mathbb{P}^\epsilon\{\xi^\epsilon \in A\} \sim g(\epsilon, A)$ or $\ln \mathbb{E}^\epsilon f_\epsilon(\xi^\epsilon) \sim g(\epsilon, f_\epsilon)$ (we use $\mathbb{E}^\epsilon$ to denote the expectation with respect to the probability measure $\mathbb{P}^\epsilon$). More precisely, if $(\mathcal{X}, \mathcal{B})$ denotes a function space with a metric and the Borel $\sigma$-algebra $\mathcal{B}$, then the family $\{\xi^\epsilon\}$ is said to satisfy the large deviation principle with a normalized action functional.
$S(x)$ on $(X, \mathcal{B})$ if for every Borel measurable set $\Gamma \in \mathcal{B}$,

$$
- \inf_{x \in \Gamma} S(x) \leq \liminf_{\epsilon \to 0} \epsilon \ln \mathbb{P}^{\epsilon} \{\xi^\epsilon \in \Gamma\} \leq \limsup_{\epsilon \to 0} \epsilon \ln \mathbb{P}^{\epsilon} \{\xi^\epsilon \in \Gamma\} \leq - \inf_{x \in \Gamma} S(x) \quad (1.1)
$$

where $S(x)$ takes values in $[0, +\infty]$ such that each level set $\Phi(s) := \{x \in X : S(x) \leq s\}$ is compact ($s \geq 0$). The normalized action functional $S(x)$ is also called a rate function in the literature. Here we also consider the large deviation principle as rough large deviations. We refer to [11], [44], [30], [24], [1] and [27] for the large deviation principles for various classes of stochastic processes. Of note, references [1] and [27] study processes with jumps, which will be included in this paper.

The following identity, to be called as Varadhan’s integral lemma according to [11], was derived in [41] from (1.1)

$$
\lim_{\epsilon \to 0} \epsilon \ln \mathbb{E}^\epsilon \exp \left\{ \epsilon^{-1} F(\xi^\epsilon) \right\} = \max_{x \in X} [F(x) - S(x)] \quad (1.2)
$$

for every bounded and continuous functional $F(x)$ on $X$. If the metric space $X$ is regular enough, then (1.2) and (1.1) are equivalent, see Section 3.3 in [25], [7] and [11]. Related works were considered in [12]. Two questions arise here. First, it is natural to expect precise large deviation probabilities from (1.1) for suitable stochastic processes. This direction has been extensively studied, such as for random walks, actual aggregate loss processes, prospective-loss processes, (fractional) Ornstein-Uhlenbeck processes, Gaussian quadratic forms, Markov chains and so on (see [37], [33], [35], [2], [3], [4], [5], [21] and [22]). Second, it is natural to expect precise large deviations of integral forms from (1.2) for more regular $F$ such as what we had for sums of i.i.d. random variables. Namely, we want to specify the conditions on $F$ and $\xi^\epsilon$ under which $\mathbb{E}^\epsilon \exp \left\{ \epsilon^{-1} F(\xi^\epsilon) \right\}$ has precise asymptotics.

Not many references can be found along this direction, and below is a summary.

Indeed, for the family of stochastic processes $\{\sqrt{\epsilon}W_t\}_{t \in [0,T]}$, where $\{W_t\}$ is the standard Wiener process, it was proved by Schilder in [38] that the following precise asymptotics hold

$$
\mathbb{E}^\epsilon \left[ \exp \left\{ \epsilon^{-1} F(\sqrt{\epsilon}W) \right\} \right] = \exp \left\{ \epsilon^{-1}[F(\phi_0) - S(\phi_0)] \right\} \left( \sum_{0 \leq i \leq s/2} K_i \cdot \epsilon^i + o(\epsilon^{s/2}) \right) \quad (1.3)
$$

for a positive integer $s$ depending on the smoothness of $F$, where the normalized action functional $S(\phi) = \frac{1}{2} \int_0^T \phi'(t)^2 dt$ for absolutely continuous $\phi$ and $S(\phi) = \infty$ for other $\phi$. We note that the trajectory metric space $X$ here is the continuous function space $C[0, T]$, and $F(\phi_0) - S(\phi_0)$ indicates the maximum of $F(\phi) - S(\phi)$ is reached uniquely at $\phi_0$. The proof of (1.3) made use of many particular properties of Wiener processes such as $d\mu_{\phi, \sqrt{\epsilon}W}/d\mu_W$ and the distribution of $\max_{t \in [0, T]} |W_t|$.
In 1970s, Dubrovskii and Wentzell showed precise large deviation of the first order, i.e. $s = 0$ in (1.3), for suitable Markov processes by a transformation similar to Cramér’s. The general precise asymptotics can’t be obtained due to the lack of tools which will be explained below, see [13] and [44]. Ellis and Rosen in 1980s derived precise asymptotics in the form (1.3) for Gaussian probability measures by suitable technical arguments based on Hilbert spaces, see [16], [17], [18], [36] and [23].

The purpose of this paper is to study precise asymptotics for large deviations in form (1.3) for a wide class of families of stochastic processes including diffusion processes, pure jump processes, deterministic processes and the mixture processes of them - locally infinitely divisible processes.

Most of the proofs regarding the precise large deviations mentioned above made use of a transformation $\tilde{\mathbb{P}}^\epsilon(A) = \int_A \pi^\epsilon d\tilde{\mathbb{P}}$ ($\pi^\epsilon$ is chosen such that $\tilde{\mathbb{P}}^\epsilon(\Omega) = 1$) in order that the main part of $\mathbb{E}^\epsilon [\exp\{\epsilon^{-1} F(\xi^\epsilon)\}]$ for small $\epsilon$ is due to the set of paths in a neighborhood of $\phi$ which has large $\tilde{\mathbb{P}}^\epsilon$ probability. We call such a transformation the generalized Cramér’s transformation which will be used also in this paper.

Back to the classical precise asymptotics for large deviations $\int_{\mathbb{R}} \exp\{nf(x)\} \mu^n(dx)$ on i.i.d. random variables, Cramér’s main tools are the asymptotic expansions on normal deviations for $\sqrt{n}S_n$ in the form $F_{\sqrt{n}S_n}(x) = F_\infty(x) + \sum_{i=1}^k P_i(x)n^{-i/2} + o(n^{-k/2})$ where $F_\infty(x)$ is the limiting distribution of $F_{\sqrt{n}S_n}(x)$ (the distribution function of the random variable $\sqrt{n}S_n$). Equivalently, for smooth function $g(x)$, the normal deviations take the following form

$$\mathbb{E}g(\sqrt{n}S_n) = \mathbb{E}g(Y) + \sum_{i=1}^k p_in^{-i/2} + o(n^{-k/2})$$

(1.4)

where $Y$ is the random variable corresponding to the distribution function $F_\infty(x)$. It is well-known that a family of stochastic processes $\eta^\epsilon$ converges weakly to a process $\eta$ as probability measures on the trajectory function space $X$ if for any continuous and bounded functional $G(x)$ on $X$

$$\mathbb{E}^\epsilon G(\eta^\epsilon) = \mathbb{E}G(\eta) + o(1).$$

The exact order for $o(1)$ is generally unknown. Thus, if one wants to follow the idea of Cramér on random variables to derive precise asymptotics of large deviations for stochastic processes by using the asymptotic expansions on normal deviations for stochastic processes, then the first step would be to obtain normal deviations for stochastic processes, namely,

$$\mathbb{E}^\epsilon G(\eta^\epsilon) = \mathbb{E}G(\eta) + \sum_{i=1}^k P_\epsilon \epsilon^{i/2} + o(\epsilon^{k/2}).$$

(1.5)
But normal deviations (1.5) are far from clear until a recent result [47], and we refer to [45] and the references therein for closely related works. The method of deriving precise asymptotics of large deviations from precise normal deviations for stochastic processes seems to appear for the first time in this paper.

In Section 1.1 we give the definition of a locally infinitely divisible process and list several related concepts. The main result of this paper is contained in Section 2, where some examples are also included. After appropriate recall from [47] on normal deviations for stochastic processes in Section 3.1, we present the proof of our main theorem in the rest of Section 3.

As related problems, in Section 4 we study the connections between precise asymptotics for large (or normal) deviations and for the solutions to partial integro-differential equations

\[
\begin{align*}
\frac{\partial}{\partial t} u^\epsilon(t, x) &= \frac{\epsilon}{2} a(t, x) \Delta u^\epsilon(t, x) + b(t, x) \nabla u^\epsilon(t, x) + \epsilon^{-1} c(x) u^\epsilon(t, x) \\
&\quad + \epsilon^{-1} \int_{\mathbb{R}} [u^\epsilon(t, x + \epsilon u) - u^\epsilon(t, x) - \epsilon u \nabla u^\epsilon(t, x)] \nu_{t,x}(du) \\
u^\epsilon(0, x) &= g(x)
\end{align*}
\]

under suitable smooth and growth conditions on \(a, b, c\) and \(g\). For instance, if \(c = 1, a(t, x) = a(x), b(t, x) = b(x), 0 < \inf_x a(x) \leq \sup_x a(x) < \infty, \nu_{t,x}(du) = u^2 1_{\{|u| \leq 1\}}(du)\), the smooth functions \(a(x), b(x)\) and \(g(x)\) are bounded together with their derivatives \(d^j a/dx^j, d^j b/dx^j\) and \(d^j g/dx^j\), then the precise asymptotics for the solution \(u^\epsilon(t, x)\) for fixed \((t, x)\) is (with \(n\) being an arbitrary integer)

\[
u^\epsilon(t, x) = e^{t/\epsilon} \left[ \sum_{k=0}^n k_i(x) \epsilon^{k/2} + o(\epsilon^{n/2}) \right], \quad \text{for constants } k_i \text{ depending on } x.
\]

1.1 Locally infinitely divisible processes

If \((\xi_t, \mathbb{P}_{s,x}), t \in [s, T]\), is a real-valued Markov process (the subscript \(s,x\) means the process starts from \(x\) at time \(s\)), we use \(P^{s,t}, 0 \leq s \leq t \leq T\), to denote the corresponding multiplicative family of linear operators acting on functions according to the formula

\[
P^{s,t} f(x) = \mathbb{E}_{s,x} f(\xi_t),
\]

where \(\mathbb{E}_{s,x}\) is the expectation with respect to probability measure \(\mathbb{P}_{s,x}\). The compensating operator \(\mathfrak{A}\) of this Markov process, taking functions \(f(t, x)\) to functions of the same two arguments, is defined by

\[
P^{s,t} f(t, \cdot)(x) = f(s, x) + \int_s^t P^{s,u} \mathfrak{A} f(u, \cdot)(x) du \quad (1.6)
\]
under suitable assumptions on the measurability in \((t, x)\) of \(A f(t, x)\), where \(P^{s,t} f(t, \cdot)(x)\) means that \(P^{s,t}\) is applied to the function \(f(t, x)\) in its second argument \(x\), and \(P^{s,u} A f(u, \cdot)(x)\) means that \(P^{s,u}\) is applied to function \(g(u, x) := A f(u, x)\) in its second argument \(x\). If some measurability conditions are imposed on the process \(\xi_t(\omega)\), then \(1.6\) is equivalent to that

\[
 f(t, \xi_t) - \int_s^t A f(u, \xi_u) du
\]

is a martingale with respect to the natural family of \(\sigma\)-algebras and every probability measure \(P_{s,x}\). Of course, compensating operator \(A\) is not defined uniquely. Different versions are such that \(A f(u, \xi_u)\) coincide almost surely except on a set of time argument \(u\) of zero Lebesgue measure.

We say \(A_t\) is the generating operator of our process \((\xi_t, P_{s,x})\) if for \(s \leq t\),

\[
P^{s,t} f(x) = f(x) + \int_s^t P^{s,u} A f(x) du
\]

for suitable \(f\). Also a generating operator has different versions. For a wide class of Markov processes, a version of the compensating operator \(A\) of process \(\xi_t\) for smooth functions \(f(t, x)\) is given by

\[
 A f(t, x) = \frac{\partial f}{\partial t}(t, x) + A_t f(t, \cdot)(x),
\]

where generating operator \(A_t\) acts on functions of the spatial argument \(x\) only.

For each fixed \(\epsilon > 0\), let \((\xi_t^\epsilon, P_{0,x}^\epsilon), t \in [0, T]\), be a one-dimensional process with jumps whose trajectories are right continuous with left limits. We assume that the generating operator of \(\xi_t^\epsilon\) is

\[
 A_t^\epsilon f(x) = \epsilon^{-1} \int_R \left[ f(x + \epsilon u) - f(x) - \epsilon u f'(x) \right] \nu_{t,x}(du) + \alpha_t(t, x) f'(x) + \frac{\epsilon}{2} a(t, x) f''(x) \quad (1.7)
\]

for functions \(f\) that are bounded and continuous together with their first and second derivatives, and that a version of its compensating operator is given by \(A f(t, x) = \frac{\partial f}{\partial t}(t, x) + A_t f(t, \cdot)(x)\) for bounded functions \(f(t, x)\) that are absolutely continuous in \(t\), twice continuously differentiable in \(x\) for fixed \(t\) with bounded derivatives \(\partial f / \partial t, \partial f / \partial x\) and \(\partial^2 f / \partial x^2\). In order to make sense of the integral in \(A_t^\epsilon f(x)\) and also for the purpose of the proof, throughout this paper we impose two conditions on measures \(\nu_{t,x}: \int u^2 \nu_{t,x}(du) < \infty\) for every \((t, x)\), and there is a bounded support \(K\) for all \(\nu_{t,x}(\cdot)\), i.e., \(\nu_{t,x}(K^c) \equiv 0\). The family \(\{\xi_t^\epsilon\}\) is the underlying family of stochastic processes in this paper, and we call them \(locally\ infinitely\ divisible\ processes\) with bounded support, see also \[45\] and \[44\]. We note that this family contains diffusion processes, pure jump processes, deterministic processes and the processes coming from the mixture of them.
For each $\epsilon$, the process $\xi^\epsilon$ makes jumps of size $\epsilon \cdot u$, according to the rate measure $\epsilon^{-1} \nu_{t,x}(du)$, and moves with velocity $\alpha(t,x) - \int \nu_{t,x}(du)$ between the jumps. We define the cumulant $G^\epsilon(t,x;z)$ of $(\xi^\epsilon, \mathbb{P}_{0,x})$ by, for $t \in [0,T], x,z \in \mathbb{R},$

$$G^\epsilon(t,x;z) = \epsilon \alpha(t,x) + \frac{\epsilon}{2} a(t,x)z^2 + \epsilon^{-1} \int_{\mathbb{R}} (e^{\epsilon u} - 1 - \epsilon u) \nu_{t,x}(du).$$

Here $G^\epsilon(t,x;z)$ is well defined because of two conditions we imposed on $\nu_{t,x}$, and it satisfies $G^\epsilon(t,x;z) = \epsilon^{-1}G_0(t,x;\epsilon z)$, where

$$G_0(t,x;z) = \epsilon \alpha(t,x) + \frac{1}{2} a(t,x)z^2 + \int_{\mathbb{R}} (e^{zu} - 1 - zu) \nu_{t,x}(du).$$

Let $H_0(t,x;u), G_0(t,x;z)$ be coupled by the Legendre transformation in the third argument,

$$H_0(t,x;u) = \sup_{z \in \mathbb{R}} [zu - G_0(t,x;z)].$$

For an absolutely continuous function $\phi_0$ (which will be specified later as a maximizer) we define $z^\epsilon(t) = \epsilon^{-1}z_0(t), z_0(t) = \partial H_0(t, \phi_0(t); \phi_0'(t))$ and generalized Cramér’s transformation:

$$\mathbb{P}_{0,x}^{z^\epsilon}(A) := \int_A \pi^\epsilon(0,T)d\mathbb{P}_{0,x},$$

with $\pi^\epsilon(0,T) = \exp \left\{ \epsilon^{-1} \int_0^T z_0(t)d\xi^\epsilon_t - \epsilon^{-1} \int_0^T G_0(t, \xi^\epsilon_t; z_0(t))dt \right\}$. For each $\epsilon > 0$, this transformation gives us a new probability measure $\mathbb{P}_{0,x}^{z^\epsilon}$ if we assume $\pi^\epsilon(0,T)$ is a martingale as a process in $T$ with respect to $\mathbb{P}_{0,x}$. For each $\epsilon > 0$, under $\mathbb{P}_{0,x}^{z^\epsilon}$ it turns out $\xi^\epsilon$ is again a jump process with compensating operator (see [44], Section 2.2.2),

$$\mathcal{A} z^\epsilon f(t,x) = \frac{\partial f}{\partial t}(t,x) + \frac{\partial G_0}{\partial z}(t,x;z_0(t)) \frac{\partial f}{\partial x}(t,x) + \epsilon \frac{1}{2} a(t,x) \frac{\partial^2 f}{\partial x^2}(t,x)$$

$$+ \epsilon^{-1} \int_{\mathbb{R}} \left[ f(t,x + \epsilon u) - f(t,x) - \epsilon \frac{\partial f}{\partial u}(t,x) \right] e^{z_0(t)u} \nu_{t,x}(du). \quad (1.8)$$

Let us define the normalized action functional as follows

$$S(\phi) = S_{0,T}(\phi) = \int_0^T H_0(t, \phi(t); \phi'(t))dt$$

for absolutely continuous function $\phi$, otherwise $S(\phi) = +\infty$. At the end of this section, we introduce several notations which are needed for our formulation of the main theorem. Let $\phi_0$ be continuously differentiable,

$$G^\epsilon_0(t,x;z) = z \left[ \alpha(t,\phi_0(t) + x) - \int_{\mathbb{R}} u \nu_{t,\phi_0(t)+x}(du) - \phi'_0(t) \right]$$

$$+ \frac{1}{2} a(t,\phi_0(t) + x)z^2 + \int_{\mathbb{R}} (e^{zu} - 1) e^{z_0(t)u} \nu_{t,\phi_0(t)+x}(du).$$

7
and $H_0^*(t, x; u)$ be the Legendre transformation of $G_0^*(t, x; z)$ in the third argument. For simplicity, throughout this paper we will only consider $\mathbb{P}_{0,0}^x, \mathbb{P}_{0,0}^{x'}$, and use symbols $\mathbb{P}^x, \mathbb{P}^{x'}$ for short. The following symbols are also used,

$$
\alpha^1(t, x) = \frac{\partial G_0}{\partial z}(t, x; z_0(t)), \quad \alpha^2(t, x) = a(t, x) + \int u^2 e^{z_0(t)u} v_{t,x}(du);
$$

$$
\alpha^j(t, x) = \int u^j e^{z_0(t)u} v_{t,x}(du), \quad \beta^j(t, x) = \int |u|^j e^{z_0(t)u} v_{t,x}(du), \quad j \geq 3.
$$

\[ \text{(1.9)} \]

1.2 Functional derivatives

Let us include in this section the function spaces related to the trajectory spaces of our stochastic processes and the corresponding functional derivatives. We use $D_0[0, T]$ to denote the space of all functions defined on $[0, T]$ vanishing at 0 which are right continuous with left limits; $C^1_0[0, T]$ the space of all continuously differentiable functions on $[0, T]$ vanishing at 0; and $W^{1,2}_0[0, T]$ the space of absolutely continuous functions vanishing at 0 having square integrable derivatives. In the space $D_0[0, T]$, the uniform norm $||\phi|| = \sup_{0 \leq t \leq T} |\phi(t)|$ will be used. Throughout this paper, we understand the differentiability of a functional $F(\phi)$ on $D_0[0, T]$ as Fréchet differentiability. Moreover, we assume that the derivatives $F^{(j)}(\phi)(\delta_1, \ldots, \delta_j)$ can be represented as integrals of the product $\delta_1(s_1) \cdots \delta_j(s_j)$ with respect to some signed measures, denoted by $F^{(j)}(\phi; \nu)$:

$$
F^{(j)}(\phi)(\delta_1, \ldots, \delta_j) = \int_{[0, T]^j} \delta_1(s_1) \cdots \delta_j(s_j) F^{(j)}(\phi; ds_1 \cdots ds_j).
$$

\[ \text{(1.10)} \]

The norm of the signed measure is defined by

$$
||F^{(j)}|| := \sup_{x[0, T] \in D_0[0, T]} \left| F^{(j)}(x[0, T]; \nu) \right| ([0, T]^j).
$$

The notation $F^{(j)}(\phi)(y[0, T]^{\otimes j})$ stands for the $j$-th derivative $F^{(j)}(\phi)(y[0, T], \ldots, y[0, T])$ of the functional $F$ at point $\phi[0, T]$ in directions $y[0, T]$. 

2 The main theorem and examples

From now on, an integer $s \geq 2$ will be used. To precisely state our main result, we make a list of assumptions on two pairs $G_0, H_0$ and $G_0^*, H_0^*$ introduced in Section 1.1. The first five general assumptions (A)-(E) can be found in [44]. Let $p(t, x; z)$ and $q(t, x; u)$ be coupled by the Legendre transformation in the third arguments.
Furthermore, assume \(\alpha\) and continuous in \(u\) respectively, and their first and second derivatives be continuous with respect to \((t, x, z)\).

**2.1 The main theorem**

For a continuous functional \(F\) on \(D_0[0, T]\) which is bounded above, let the maximum of functional \(F - S\) be attained at a unique function \(\phi_0 \in C^1_0[0, T]\), and the assumption (F) be fulfilled. Furthermore, assume that the assumptions (A)-(E) are satisfied for \(p(t, x; z) = G_0(t, x; z)\) and \(p(t, x; z) = G_0^*(t, x; z)\).

**Theorem 2.1.** Let \((\xi^*_t, \mathbb{P}^*)\), \(t \in [0, T]\), be a family of one-dimensional locally infinitely divisible processes with bounded support introduced in Section 1.1. For a continuous functional \(F\) on \(D_0[0, T]\) which is bounded above, let the maximum of functional \(F - S\) be attained at a unique function \(\phi_0 \in C^1_0[0, T]\), and the assumption (F) be fulfilled. Furthermore, assume that the assumptions (A)-(E) are satisfied for \(p(t, x; z) = G_0(t, x; z)\) and \(p(t, x; z) = G_0^*(t, x; z)\).

Suppose \(F\) is \(s+1\) times differentiable at all points \(\phi\) in a neighborhood of \(\phi_0\) and \(F(2)(\phi_0)(x, x) < S^{(2)}(\phi_0)(x, x)\) for any non-zero function \(x \in W^{1,2}_0[0, T]\). For all \(\phi\) in this neighborhood of \(\phi_0\), any \(x[0, T], x[0, T] \in D_0[0, T]\), we assume

\[
F(2)(\phi)(x[0, T], x[0, T]) + \int_0^T (x(t))^2 \frac{\partial^2 G_0}{\partial x^2}(t, \phi(t), z_0(t)) dt \leq 0,
\]

\[
\left| F^{(i)}(\phi)(x_1[0, T], \cdots, x_i[0, T]) \right| \leq p \left[ x_1(T) \cdots x_i(T) \right]^m + \prod_{j=1}^i \left( 1 + \int_0^T |x_j(t)|^n dt \right)
\]

(2.1)
with $2 \leq i \leq s + 1$ and some constants $m, n, p \geq 1$. Another continuous and bounded functional $H$ is assumed to be $s - 1$ times differentiable at all points $\phi$ in a neighborhood of $\phi_0$, and

$$\left| H^{(i)}(\phi)(x_1[0, T], \cdots, x_i[0, T]) \right| \leq p^i \left[ |x_1(T)| \cdots |x_i(T)|^{m'} + \prod_{j=1}^{i} \left( 1 + \int_0^T |x_j(t)|^{n'} \, dt \right) \right]$$

(2.2)

with $1 \leq i \leq s - 1$ and some positive constants $m', n', p' \geq 1$, for all $x_i[0, T] \in D_0[0, T]$.

Then as $\epsilon \to 0$, the following precise asymptotics hold

$$\mathbb{E}^\epsilon \left[ H(\xi^\epsilon) \exp\{\epsilon^{-1} F(\xi^\epsilon)\} \right] = \exp\left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0)] \right\} \sum_{0 \leq i \leq (s-2)} K_i \cdot \epsilon^{i/2} + o\left( \epsilon^{(s-2)/2} \right)$$

(2.3)

where the coefficients $K_i$ are determined by $F, H$ and their derivatives at $\phi_0$; in particular,

$$K_0 = H(\phi_0) \cdot \mathbb{E} \left( \exp\{Q(2, \eta)\} \right),$$

$$K_1 = C_{01} H(\phi_0) + C_{11} + \mathbb{E} \left[ \exp\{Q(2, \eta)\} \left( Q(3, \eta) H(\phi_0) + H^{(1)}(\phi_0)(\eta) \right) \right]$$

with the constants $C_{ij}$ depending on $F, H$ and $\phi_0$, the process $\eta$ being a Gaussian diffusion having diffusion coefficient $A(t) = \frac{\partial^2 G_0}{\partial x^2}(t, \phi_0(t); z_0(t))$ and drift coefficient $B(t, x) = x \cdot \frac{\partial^1 G_0}{\partial x} (t, \phi_0(t); z_0(t))$, and

$$Q(n, x[0, T]) = \frac{1}{n!} F^{(n)}(\phi_0)(x[0, T], \cdots, x[0, T]) + \int_0^T \frac{1}{n!} (x(t))^n \frac{\partial^n G_0}{\partial x^n} (t, \phi_0(t); z_0(t)) \, dt.$$

Remark: (1). The constants $C_{ij}$ are determined from the asymptotic expansions for normal deviations formulated in (III) of Section 3.3. The finiteness of the coefficients $K_i$ is proved in Section 3.3.

(2). In the spatial case when $\xi^\epsilon = \sqrt{\epsilon} W$, each coefficient $K_i$ in (2.3) with an odd index $i$ is equal to zero because of the symmetry of the distribution of $W$.

(3). If the initial position of $\xi^\epsilon$ is $x$ instead of 0, then we think of $\xi^\epsilon - x$ as a new process with zero initial position and

$$\mathbb{E}^\epsilon_{0,x} \Phi(\xi^\epsilon) = \mathbb{E}^\epsilon_{0,0} \Phi(\xi^\epsilon - x).$$

2.2 Examples

Example 1. Let us consider a family of one-dimensional pure jump processes $\xi^\epsilon_t, t \in [0, 1]$, with generating operator given by

$$A^\epsilon_t f(x) = \epsilon^{-1} \int_{\mathbb{R}} (f(x + \epsilon u) - f(x)) \nu_t(x)(du),$$
where \( \nu_{t,x}(du) = \frac{1}{2} (\delta_1(du) + \delta_{-1}(du)) \) with \( \delta_1(\cdot) \) (resp. \( \delta_{-1}(\cdot) \)) denoting the probability measure concentrating at point 1 (resp. -1). For each \( \epsilon \), the trajectories of \( \xi^\epsilon \) are step functions with finitely many steps on \([0, 1]\). This process makes jumps of size \( \pm \epsilon \) according to the rate \( \frac{1}{2} \epsilon^{-1} \).

The most probable trajectory for \( \xi^\epsilon \) as \( \epsilon \to 0 \) is identically zero. To see this, we first note that

\[
G_0(t, x; z) = \int_{\mathbb{R}} (e^{zu} - 1) \nu_{t,x}(du) = \frac{1}{2} (e^z + e^{-z} - 2);
\]

\[
H_0(t, x; u) = \sup_{z \in \mathbb{R}} [zu - G_0(t, x; z)] = u \ln \left( u + \sqrt{u^2 + 1} \right) + 1 - \sqrt{u^2 + 1};
\]

\[
S(\phi) = \int_0^1 H_0(t, \phi(t); \phi'(t)) dt, \text{ for absolutely continuous } \phi \text{ in } D_0[0, 1].
\]

As a function of \( u \), \( H_0(t, x; u) \) is positive except at \( u = 0 \), strictly increasing on \((0, \infty)\), and strictly decreasing on \((-\infty, 0)\). Thus, in order to make \( S(\phi) = 0 \), it is required \( \phi'(t) = 0 \) almost everywhere with respect to Lebesgue measure. But \( \phi(t) \) is absolutely continuous and \( \phi(0) = 0 \), it follows that \( \phi(t) \equiv 0 \). This proves that the most probable trajectory is zero.

Let the functional \( F \) on \( D_0[0, 1] \) be

\[
F(\phi) = \int_0^1 (\phi(t) - \phi^2(t)) dt,
\]

and \( H(\phi) \equiv 1 \). We need to show that \( \max(F - S) \) is attained at a unique function \( \phi_0 \in C^1_0[0, 1] \), that is, the following variational problem

\[
\max_{\phi \in C^0[0, 1]} \int_0^1 \left[ \phi(t) - \phi(t)^2 - \left( \phi'(t) \ln \left( \frac{\phi'(t) + \sqrt{\phi'(t)^2 + 1}}{\phi'(t) - \sqrt{\phi'(t)^2 + 1}} \right) + 1 - \sqrt{\phi'(t)^2 + 1} \right) \right] dt, \tag{2.4}
\]

has an unique (nonzero) solution. The existence and uniqueness for a nonzero solution of (2.4) are shown in the Appendix. All other conditions of Theorem 2.1 can be easily checked.

We could have considered some wider families of processes and more general functionals (e.g. \( F(\phi[0, T]) = h \left( \int_0^T g(\phi(s)) ds \right) \)), and each time we will have to verify the existence and uniqueness of the solution for the corresponding variational problem.

**Example 2.** Suppose \( \xi^\epsilon_t, t \in [0, 1] \) is a family of one-dimensional pure jump processes with generating operator

\[
A^\epsilon_t f(x) = \epsilon^{-1} \int_{\mathbb{R}} (f(x + \epsilon u) - f(x)) \nu_{t,x}(du),
\]

where \( \nu_{t,x}(du) = r(x)\delta_1(du) + l(x)\delta_{-1}(du) \) and \( r(x) = l(x) = \sin(x) + 2 \). It is easy to get

\[
G_0(t, x; z) = \int_{\mathbb{R}} (e^{zu} - 1) \nu_{t,x}(du) = r(x)(e^z - 1) + l(x)(e^{-z} - 1).
\]

11
Again we consider functional $F(\phi) = \int_0^1 (\phi(t) - \phi^2(t)) \, dt$. The existence and uniqueness of the variational problem $\max_{\phi \in C^1_{0,1}} [F(\phi) - S(\phi)]$ can be similarly obtained as Example 1. Now we check $G_0^*(t, x; z)$ satisfies conditions (A)-(E). Let us recall that in this example

$$G_0^*(t, x; z) = -z[\phi_0(t) + r(\phi_0(t) + x) - l(\phi_0(t) + x)] + r(\phi_0(t) + x)(e^z - 1)e^{\phi_0(t)} + l(\phi_0(t) + x)(e^{-z} - 1)e^{-\phi_0(t)}.$$  

For conditions (A) and (B): we choose $\tilde{G}_0(z) = C(|z| - 1)$ with a positive constant $C$, then $H_0(u) = Ch(|u|/C)$, where function $h(y) = y \ln y - y + 1$ for $y \geq 1$, and $= 0$ for $0 \leq y < 1$. For condition (C): for any fixed $h > 0$, if $(t, x)$ and $(s, y)$ are close enough, then

$$G_0^*(t, x; (1 - h)z) - (1 - h)G_0^*(s, y; z) \leq h. \quad (2.5)$$

To see (2.5), note that for large $z \to \infty$ or $z \to -\infty$, the left hand side of (2.5) goes to $-\infty$ uniformly in $t$ and $x$. This means that we just need to consider bounded $z$, which proves (2.5). Conditions (D) and (E) are easy to be checked.

3 Proof of Theorem 2.1

As mentioned in the introduction, we will use precise normal deviations for stochastic processes in our proof. The normal deviations needed here are not for the processes $\xi^\varepsilon$, but for another family of processes $\eta^\varepsilon$ related to $\xi^\varepsilon$.

Because of assumption (F) in Section 2, the ordinary differential equation $x'(t) = \alpha^1(t, x(t))$ with an initial condition $x(0) = 0$ has a unique solution with $\alpha^1$ defined in (1.9). Furthermore, this unique solution can be proved to be $\phi_0$ from Legendre transformation. Let us set

$$\eta^\varepsilon_t = e^{-1/2}(\xi^\varepsilon_t - \phi_0(t)).$$

Note that here the initial point $\eta^\varepsilon_0 = 0$. More generally, we consider an initial point $\eta^\varepsilon_0 = x$ in this section in order to fully exhibit the normal deviations. It was proved in [45] that as $\varepsilon \to 0$ the family $\eta^\varepsilon$ under $\mathbb{P}_{0,x}^{\varepsilon^\prime}$ converges weakly to a process $\eta$, which is a Gaussian diffusion process on the real line with generating operator

$$A^1_2 f(x) = \alpha^1_2(t, \phi_0(t)) \cdot x \cdot f'(x) + \frac{1}{2} \alpha^2_2(t, \phi_0(t)) \cdot f''(x), \quad (3.1)$$

where the subscript 2 means differentiation in the second spatial argument.
3.1 Results on normal deviations

Before the statement of normal deviations, let us recall a differential operator $A_1$ which was defined in [45] for functionals $G$ on $D[0, T]$ (consisting of right continuous functions with left limits):

$$A_1 G(x[0, T]) = \sum_{k=1}^{3} \int_{[0, T]^k} \Gamma^k_1(x[0, T]; s_1, \ldots, s_k)G^{(k)}(x[0, T]; ds_1 \cdots ds_k)$$

where

$$\Gamma^1_1(x[0, T]; s_1) = \frac{1}{2} \int_{0}^{s_1} \alpha_1^2(t, \phi_0(t))x(t) \exp \left\{ \int_{0}^{s_1} \alpha_2^1(v, \phi_0(v))dv \right\} dt,$$

$$\Gamma^2_1(x[0, T]; s_1, s_2) = \frac{1}{2} \int_{0}^{\min\{s_1, s_2\}} \alpha_2^2(t, \phi_0(t))x(t) \exp \left\{ \sum_{i=1}^{2} \int_{0}^{s_i} \alpha_2^1(v, \phi_0(v))dv \right\} dt,$$

$$\Gamma^3_1(x[0, T]; s_1, s_2, s_3) = \frac{1}{6} \int_{0}^{\min\{s_1, s_2, s_3\}} \alpha_3^3(t, \phi_0(t)) \exp \left\{ \sum_{i=1}^{3} \int_{0}^{s_i} \alpha_2^1(v, \phi_0(v))dv \right\} dt.$$

A crucial functional in [45] and [47] is defined through a conditional expectation on the past path

$$f(t, x[0, t]) = \mathbb{E}_{t, x[0, t]} G(\eta), \quad t \in [0, T].$$

**Theorem 3.1** (Theorem 5.2 in [47]). Let a functional $G(x[0, T])$ on $D[0, T]$ be $3(s - 2)$ times differentiable with the following conditions:

(I). there is a constant $C > 0$ such that for all $x[0, T], y[0, T] \in D[0, T]$

$$|G(x[0, T])| \leq C \left( 1 + |x(T)|^s + \int_{0}^{T} |x(t)|^s dt, \right.$$

$$\left| G^{(i)}(x[0, T]) (y[0, T]^{\otimes i}) \right| \leq (1 + ||y||^i) C \left( 1 + |x(T)|^{s-2} + \int_{0}^{T} |x(t)|^{s-2} dt, \right) 1 \leq i \leq 3(s - 2),$$

(II). $G^{(i)}(x[0, T]) (I_{[t, T]} \delta, \ldots, I_{[t, T]} \delta), 1 \leq i \leq 3(s - 2)$, are continuous with respect to $x[0, T]$ uniformly as $x[0, T]$ changes over an arbitrary compact subset of $D[0, T]$, $t$ over $[0, T]$, and $\delta[0, T]$ over the set of Lipschitz continuous functions with constant 1, $||\delta|| \leq 1$.

Then as $\epsilon \to 0$, under the assumptions of Theorem 3.1 the precise normal deviations hold

$$\mathbb{E}^{s_\epsilon}_{0, x} G(\eta^\epsilon) = \mathbb{E}_{0, x} G(\eta) + \sum_{i=1}^{s-2} \epsilon^{i/2} \mathbb{E}_{0, x} A_i G(\eta) + o(\epsilon^{(s-2)/2}) \quad (3.2)$$

where $A_1$ is a third-order differential operator defined before, $A_2$ is a sixth-order differential operator.
given by

\[ A_2 G(x[0, T]) = \int_0^T A_1 \tilde{G}(x[0, t]) \, dt + \int_0^T \left[ \frac{1}{3!} \alpha_{222}^2(t, \phi_0(t)) x(t)^2 f^{(1)}(t, x[0, t](I_{(t)}) + \frac{1}{4} \alpha_{22}^2(t, \phi_0(t)) x(t)^2 f^{(2)}(t, x[0, t])(I_{(t)}^{\otimes 2}) + \frac{1}{4} \alpha_{22}^2(t, \phi_0(t)) x(t)^2 f^{(3)}(t, x[0, t])(I_{(t)}^{\otimes 3}) + \frac{1}{4!} \alpha^4(t, \phi_0(t)) f^{(4)}(t, x[0, t])(I_{(t)}^{\otimes 4}) \right] \, dt \]

with

\[ \tilde{G}(x[0, t]) = \frac{1}{2} \alpha_{22}^2(t, \phi_0(t)) x^2(t) f^{(1)}(t, x[0, t])(I_{(t)}) + \frac{1}{2} \alpha_{22}^2(t, \phi_0(t)) x(t) f^{(2)}(t, x[0, t])(I_{(t)}^{\otimes 2}) + \frac{1}{6} \alpha^3(t, \phi_0(t)) f^{(3)}(t, x[0, t])(I_{(t)}^{\otimes 3}), \]

and \( A_3, \cdots, A_{n-2} \) are suitable differential operators defined through derivatives of \( f(t, x[0, t]) \).

### 3.2 Large deviations for \( \epsilon^{1/2} \eta^\epsilon \)

It can be easily seen that the process \( \eta^\epsilon = \epsilon^{-1/2}(\xi^\epsilon - \phi_0) \) under the measure \( \mathbb{P}^{z, \epsilon} \), has compensating operator

\[
\mathcal{A}^\epsilon f(t, x) = \frac{\partial f}{\partial t}(t, x) + \epsilon^{-1/2} \left[ \frac{\partial G_0}{\partial z}(t, \phi_0(t) + \epsilon^{1/2} x; z_0(t)) - \phi_0'(t) \right] \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} a(t, \phi_0(t) + \epsilon^{1/2} x) \frac{\partial^2 f}{\partial x^2}(t, x)
\]

\[ + \epsilon^{-1} \int_{\mathbb{R}} \left[ f(t, x + \epsilon^{1/2} u) - f(t, x) - \epsilon^{1/2} \frac{\partial f}{\partial x}(t, x) \cdot u \right] e^{z_0(t)u} \nu_{t, \phi_0(t) + \epsilon^{1/2} x}(du), \]

and cumulant

\[ G^\epsilon(t, x; z) = z \cdot \epsilon^{-1/2} \left[ \frac{\partial G_0}{\partial z}(t, \phi_0(t) + \epsilon^{1/2} x; z_0(t)) - \phi_0'(t) \right] + \frac{1}{2} a(t, \phi_0(t) + \epsilon^{1/2} x) z^2
\]

\[ + \epsilon^{-1} \int_{\mathbb{R}} \left[ e^{z\sqrt{\epsilon} u} - 1 - z \sqrt{\epsilon} u \right] e^{z_0(t)u} \nu_{t, \phi_0(t) + \epsilon^{1/2} x}(du). \]

The limiting process \( \eta \) has compensating operator given by

\[
\mathcal{A}^0 f(t, x) = \frac{\partial f}{\partial t}(t, x) + x \cdot \frac{\partial^2 G_0}{\partial z \partial x}(t, \phi_0(t); z_0(t)) \cdot \frac{\partial f}{\partial x}(t, x) + \frac{1}{2} \frac{\partial^2 G_0}{\partial z^2}(t, \phi_0(t); z_0(t)) \cdot \frac{\partial^2 f}{\partial x^2}(t, x),
\]

and cumulant

\[ G^0(t, x; z) = z x \cdot \frac{\partial^2 G_0}{\partial z \partial x}(t, \phi_0(t); z_0(t)) + \frac{1}{2} z^2 \cdot \frac{\partial^2 G_0}{\partial z^2}(t, \phi_0(t); z_0(t)). \]
In this case, the Legendre transformation of $G^\eta(t, x; z)$ in $z$ becomes
\[ H^\eta(t, x; u) = \frac{1}{2} \frac{\partial^2 H_0}{\partial u^2}(t, \phi_0(t); \phi_0'(t)) \left( u - x \frac{\partial^2 G_0}{\partial z \partial x}(t, \phi_0(t); z_0(t)) \right)^2. \]

It then follows from Section 5.2.6 of [44] that the normalized action functional for the family of processes $\epsilon^{1/2} \eta$ is
\[ I^\eta(f(\cdot)) = \int_0^T H^\eta(t, f(t); f'(t)) dt. \]

An important tool we need for the proof of Theorem 2.1 is the following convergence
\[ \mathbb{P}^{z^*} \left\{ ||\eta^\epsilon|| \geq \epsilon^{-1/2} \right\} \to 0 \text{ exponentially fast as } \epsilon \downarrow 0. \] (3.4)

3.2.1 Proof of (3.4)

Let us consider the family of processes $\zeta^\epsilon = \epsilon^{1/2} \eta^\epsilon = \xi^\epsilon - \phi_0$ with respect to probabilities $\mathbb{P}^{z^*}$.

Since $\eta^\epsilon$ converge weakly to $\eta$, it is expected that the most probable trajectory of $\zeta^\epsilon$ as $\epsilon \downarrow 0$ is path zero. Then the exponential convergence to zero of $\mathbb{P}^{z^*} \left\{ ||\eta^\epsilon|| \geq \epsilon^{-1/2} \right\} = \mathbb{P}^{z^*} \left\{ ||\zeta^\epsilon|| \geq 1 \right\}$ follows provided a large deviation result for $\zeta^\epsilon$ is proved. To be precise, we now prove that a large principle holds for $\zeta^\epsilon$ under the assumptions of Theorem 2.1.

First it is straightforward to compute the cumulant of $\zeta^\epsilon$:
\[
G^\epsilon(t, x; z) = z \left[ \frac{\partial G_0}{\partial z}(t, \phi_0(t) + x; z_0(t)) - \phi_0'(t) \right] + \frac{\epsilon}{2} a(t, \phi_0(t) + x) z^2 
+ \epsilon^{-1} \int_\mathbb{R} [e^{z\epsilon u} - 1 - z\epsilon u] e^{z_0(t)u} \nu_{t, \phi_0(t) + x}(du),
\]

which satisfies $\epsilon G^\epsilon(t, x; \epsilon^{-1} z) = G_0^*(t, x; z)$. Then by taking into account the assumptions (A)-(F), we can deduce a large deviation principle with the following normalized action functional
\[ S^*_{0, T}(\phi) = \int_0^T H_0^*(t, \phi(t); \phi'(t)) dt, \]

for absolutely continuous function $\phi$ (see Theorem 3.2.1 in [44] for details).

Now let us consider a closed set $A$ in $D_0[0, T]$ given by $A = \{ x[0, T] : ||x[0, T]|| \geq 1 \}$. The large deviation principle for $\zeta^\epsilon$ gives that, for any small $\gamma > 0$, there is $\epsilon_0$ such that for all $\epsilon \in (0, \epsilon_0)$,
\[ \mathbb{P}^{z^*} \left\{ ||\eta^\epsilon|| \geq \epsilon^{-1/2} \right\} = \mathbb{P}^{z^*} \left\{ ||\zeta^\epsilon|| \geq 1 \right\} = \mathbb{P}^{z^*} \left\{ \zeta^\epsilon \in A \right\} \leq \exp \left\{ -\epsilon^{-1} \inf_{\phi \in A} S^*_{0, T}(\phi) - \gamma \right\}. \]

So (3.4) is proved if $C_A := \inf_{\phi \in A} S^*_{0, T}(\phi) > 0$. 

15
Note that $C_A$ is reached at some point $\phi_A$. This is because $A$ is closed and any level set $\{ \phi \in D_0[0, T] : S^*_{0,T}(\phi) \leq s \}$ is compact for any $s > 0$. Now we set $C_A = S^*_{0,T}(\phi_A)$. If $S^*_{0,T}(\phi_A) = 0$, then $H_0'(t, \phi_A(t); \phi_A'(t)) = 0$ almost everywhere with respect to Lebesgue measure, i.e., for all $z \in \mathbb{R}$, $z \cdot \phi_A'(t) - G_0'(t, \phi_A(t); z) \leq 0$, almost all $t$.

Thus for almost all $t$,

$$
\phi_A'(t) \leq \lim_{z \to 0} z^{-1} G_0'(t, \phi_A(t); z) = \left[ \frac{\partial G_0}{\partial z}(t, \phi_0(t) + \phi_A(t); z_0(t)) - \phi_0'(t) \right]
+ \lim_{z \to 0} z^{-1} \int_{\mathbb{R}} (e^{zu} - 1 - zu) e^{z_0(t)} u \nu(t, \phi_0(t) + \phi_A(t); z_0(t)) - \phi_0'(t),
$$

where the fact that the second limit is equal to zero is from the assumption that $\nu_t, x$ have a bounded support $K$. Similarly,

$$
\phi_A'(t) \geq \lim_{z \to 0} z^{-1} G_0'(t, \phi_A(t); z) = \frac{\partial G_0}{\partial z}(t, \phi_0(t) + \phi_A(t); z_0(t)) - \phi_0'(t),
\text{thus } \phi_A'(t) = \frac{\partial G_0}{\partial z}(t, \phi_0(t) + \phi_A(t); z_0(t)) - \phi_0'(t). \text{ Taking the initial condition } \phi_A(0) = 0 \text{ into account, we deduce that } \phi_A \equiv 0. \text{ But this is a contradiction with } ||\phi_A|| \geq 1, \text{ so } S^*_{0,T}(\phi_A) \neq 0, \text{ i.e. } C_A > 0.
$$

### 3.3 Taylor’s expansions and estimates

We start this section with a technical lemma which suggests that when we consider the precise asymptotics for large deviations, the part away from the most probable path can be simply dropped.

**Lemma 3.1.** Let the family $\xi^\epsilon$ satisfy a large deviation principle with a normalized action functional $S$, and $F$ be a continuous measurable functional on $D_0[0, T]$. Suppose $F$ is bounded above and the difference $F - S$ attains its maximum at a unique function $\phi_0 \in D_0[0, T]$. Then for any $h > 0$, there is a $\gamma > 0$ such that as $\epsilon \to 0$,

$$
\mathbb{E}^\epsilon \left[ 1_{\{ ||\xi^\epsilon - \phi_0|| \geq h \}} \exp\left\{ \epsilon^{-1} F(\xi^\epsilon) \right\} \right] = o \left( \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0) - \gamma] \right\} \right).
$$

The proof of this lemma is contained in the Appendix. It follows from this lemma that for any $h > 0$, there exists some $\gamma > 0$ such that as $\epsilon \to 0$,

$$
\mathbb{E}^\epsilon \left[ H(\xi^\epsilon) \exp\left\{ \epsilon^{-1} F(\xi^\epsilon) \right\} \right] = \mathbb{E}^\epsilon \left[ 1_{\{ ||\xi^\epsilon - \phi_0|| < h \}} H(\xi^\epsilon) \exp\left\{ \epsilon^{-1} F(\xi^\epsilon) \right\} \right] + o \left( \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0) - \gamma] \right\} \right).
$$
Noticing that \( \exp\{-e^{-1}\gamma\} \) tends to zero exponentially fast, we thus only focus on the first part over the set \( \{||\xi - \phi_0|| < h\} \). Simple calculations yield

\[
E^\varepsilon \left[ 1_{\{||\xi - \phi_0|| < h\}} H(\xi) \exp\{e^{-1}F(\xi)\} \right] = E^\varepsilon \left[ 1_{\{||\eta|| < e^{-1/2}h\}} H(\xi_0 + \varepsilon^{1/2} \eta) \exp \left\{ e^{-1} F(\xi_0 + \varepsilon^{1/2} \eta) - e^{-1/2} \int_0^T z_0(t)d\eta(t) \right\} \right]
\]

(3.5)

Now we apply Taylor’s expansion for \( F \) at \( \phi_0 \) up to order \( s \) with an integral’s remainder (IR) (see [8] for details),

\[
F(\phi_0 + \varepsilon^{1/2} \eta) = F(\phi_0) + \frac{\varepsilon^{1/2}}{s!} F^{(s)}(\phi_0)(\eta, \ldots, \eta) + IR_1,
\]

\[
IR_1 = e^{-\frac{1}{2s}} \int_0^1 \frac{(1 - u)^s}{s!} F^{(s+1)}(\phi_0 + u\varepsilon^{1/2} \eta)(\eta, \ldots, \eta) du.
\]

For \( G_0 \), Taylor’s expansion at \( \phi_0(t) \) in the second argument up to order \( s \) with an integral’s remainders yields

\[
\int_0^T G_0(t, \phi_0(t) + \varepsilon^{1/2} \eta_t; z_0(t)) dt =
\int_0^T \left( G_0(t, \phi_0(t); z_0(t)) + \varepsilon^{1/2} \eta_t \frac{\partial G_0}{\partial x}(t, \phi_0(t); z_0(t)) + \cdots + \frac{(\varepsilon^{1/2} \eta_t)^s}{s!} \frac{\partial^s G_0}{\partial x^s}(t, \phi_0(t); z_0(t)) \right) dt + IR_2
\]

where

\[
IR_2 = \int_0^T \left( \varepsilon^{1/2} \eta_t \right)^{s+1} \int_0^1 \frac{(1 - u)^s}{s!} \frac{\partial^{s+1} G_0}{\partial x^{s+1}}(t, \phi_0(t) + u\varepsilon^{1/2} \eta_t; z_0(t)) du dt.
\]
We use these expansions to replace the exponential in (3.5) to get

\[
\exp \left\{ \epsilon^{-1} F(\phi_0 + \epsilon^{1/2} \eta^f) - \epsilon^{-1/2} \int_0^T z_0(t) d\eta^f_t \right\}
\]

\[-\epsilon^{-1} \int_0^T \left( z_0(t) \phi'_0(t) - G_0(t, \phi_0(t) + \epsilon^{1/2} \eta^f_t; z_0(t)) \right) dt \right\}
\]

\[
= \exp \left\{ \epsilon^{-1} \left[ F(\phi_0) - \int_0^T (z_0(t) \phi'_0(t) - G_0(t, \phi_0(t); z_0(t)) \right) dt \right\}
\]

\[+ \epsilon^{-1/2} \left[ F^{(1)}(\phi_0)(\eta^f) - \int_0^T z_0(t) d\eta^f_t + \int_0^T \eta^f_t \frac{\partial G_0}{\partial x}(t, \phi_0(t); z_0(t)) dt \right]
\]

\[+ \frac{1}{2!} F^{(2)}(\phi_0)(\eta^f, \eta^f) + \int_0^T \frac{1}{2!} (\eta^f_t)^2 \frac{\partial^2 G_0}{\partial x^2}(t, \phi_0(t); z_0(t)) dt \]

\[+ \epsilon^{1/2} \left[ \frac{1}{3!} F^{(3)}(\phi_0)(\eta^f, \eta^f, \eta^f) + \int_0^T \frac{1}{3!} (\eta^f_t)^3 \frac{\partial^3 G_0}{\partial x^3}(t, \phi_0(t); z_0(t)) dt \right] + \cdots
\]

\[+ \epsilon^{\frac{s-2}{2}} \left[ \frac{1}{s!} F^{(s)}(\phi_0)(\eta^f, \cdots, \eta^f) + \int_0^T \frac{1}{s!} (\eta^f_t)^s \frac{\partial^s G_0}{\partial x^s}(t, \phi_0(t); z_0(t)) dt \right]
\]

\[+ \epsilon^{-1}(IR_1 + IR_2) \]

\[
= \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0)] + \frac{1}{2!} F^{(2)}(\phi_0)(\eta^f, \eta^f) + \int_0^T \frac{1}{2!} (\eta^f_t)^2 \frac{\partial^2 G_0}{\partial x^2}(t, \phi_0(t); z_0(t)) dt \right\}
\]

\[+ \epsilon^{1/2} \left[ \frac{1}{3!} F^{(3)}(\phi_0)(\eta^f, \eta^f, \eta^f) + \int_0^T \frac{1}{3!} (\eta^f_t)^3 \frac{\partial^3 G_0}{\partial x^3}(t, \phi_0(t); z_0(t)) dt \right] + \cdots
\]

\[+ \epsilon^{\frac{s-2}{2}} \left[ \frac{1}{s!} F^{(s)}(\phi_0)(\eta^f, \cdots, \eta^f) + \int_0^T \frac{1}{s!} (\eta^f_t)^s \frac{\partial^s G_0}{\partial x^s}(t, \phi_0(t); z_0(t)) dt \right]
\]

\[+ \epsilon^{-1}(IR_1 + IR_2) \}
\]

\[
= \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0)] + Q(2, \eta^f) \right\} \times
\]

\[
\exp \left\{ \epsilon^{1/2} Q(3, \eta^f) + \cdots + \epsilon^{\frac{s-2}{2}} Q(s, \eta^f) + \epsilon^{-1}(IR_1 + IR_2) \right\}.
\]

For the second exponential function in the last equality of (3.6), we apply \( e^a = 1 + a + \cdots + a^{s-2}/(s-2)! + \frac{\theta(a-1)!}{(s-1)!} a^{s-1} \) with \( 0 \leq \theta(a) \leq 1 \), then formula (3.6) is equal to

\[
(3.6) = \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0)] + Q(2, \eta^f) \right\} \times
\]

\[
\left\{ 1 + \epsilon^{1/2} Q(3, \eta^f) + \epsilon \left( Q(4, \eta^f) + \frac{1}{2} Q(3, \eta^f)^2 \right) + \cdots + \epsilon^{\frac{s-2}{2}} \ell(\epsilon, \eta^f) + \Re(\epsilon, \eta^f) \right\}
\]

where the coefficient \( \ell(\epsilon, \eta^f) \) of \( \epsilon^{s-2}/2 \) is a functional of \( \eta^f \) and depends on \( \epsilon \). Special attention needs to be paid to the structure of the remainder term \( \Re(\epsilon, \eta^f) \). There are two different aspects \( \Re(\epsilon, \eta^f) = \Re_1(\epsilon, \eta^f) + \Re_2(\epsilon, \eta^f) \), where the first \( \Re_1(\epsilon, \eta^f) \) can be bounded by powers of \( \eta^f \), while the
second $\Re_2(\epsilon, \eta^f)$ involves a part $\epsilon^{\theta(\alpha)-a}$. By taking conditions (2.1)-(2.2) into account,

$$|\ell(\epsilon, \eta^f)| + |\Re_1(\epsilon, \eta^f)| \leq \epsilon^{\frac{\alpha}{2}} \cdot c \cdot \left( 1 + |\eta_T|^k + \int_0^T |\eta_t|^k \, dt \right),$$

for some nonnegative constants $c$ and $k$. The second one $\Re_2(\epsilon, \eta^f)$ on the set $\{||\eta^f|| < \epsilon^{-1/2}\}$ can be estimated as

$$|\Re_2(\epsilon, \eta^f)| \leq \epsilon^{\frac{\alpha}{2}} \cdot c \cdot \left( 1 + |\eta_T|^k + \int_0^T |\eta_t|^k \, dt \right) e^{h||\eta^f||^2}.$$

Hölder’s inequality and the fact $\mathbb{E}^{\epsilon^*} 1_{\{||\eta^f|| < \epsilon^{-1/2}\}} e^{h||\eta^f||^2} < C < \infty$ uniformly in $\epsilon$ (see Section 5.2.6 in [44]) suggest that we only need to take care of $\Re_1(\epsilon, \eta^f)$.

Taylor’s expansion for $H(\phi_0 + \epsilon^{1/2}\eta^f)$ at $\phi_0$ up to $(s-2)$-derivative gives

$$H(\phi_0 + \epsilon^{1/2}\eta^f) = H(\phi_0) + \epsilon^{1/2} H^{(1)}(\phi_0)(\eta^f) + \epsilon^{\frac{s-2}{2}} \frac{(s-2)!}{(s-2)!} H^{(s-2)}(\phi_0)(\eta^f, \ldots, \eta^f) + IR_3,$$

$$IR_3 = \epsilon^{\frac{s-2}{2}} \int_0^1 \frac{(1-u)^{s-2}}{(s-2)!} H^{(s-1)}(\phi_0 + u\epsilon^{1/2}\eta^f)(\eta^f, \ldots, \eta^f) \, du,$$

Now we combine (3.6) and (3.7) to rewrite (3.5) on the set $\{||\eta^f|| < \epsilon^{-1/2}\}$ as follows,

$$\mathbb{E}^\epsilon \left[ 1_{\{||\eta^f|| < \epsilon^{-1/2}\}} H(\xi^f) \exp\{\epsilon^{-1} F(\xi^f)\} \right] = \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0)] \right\} \times \mathbb{E}^{\epsilon^*} \left\{ \exp\{Q(2, \eta^f)\} \cdot \left( H(\phi_0) + \epsilon^{1/2} \left[ Q(3, \eta^f) H(\phi_0) + H^{(1)}(\phi_0)(\eta^f) \right] 
+ \epsilon \left( Q(4, \eta^f) + \frac{1}{2} [Q(3, \eta^f)]^2 \right) H(\phi_0) + H^{(2)}(\phi_0)(\eta^f, \eta^f) + Q(3, \eta^f) H^{(1)}(\phi_0)(\eta^f) 
+ \cdots + \epsilon^{\frac{s-2}{2}} \tilde{\ell}(\epsilon, \eta^f) \right) \right\}$$

$$- \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0)] \right\} \times \mathbb{E}^{\epsilon^*} \left\{ \exp\{Q(2, \eta^f)\} \cdot \left( H(\phi_0) + \epsilon^{1/2} \left[ Q(3, \eta^f) H(\phi_0) + H^{(1)}(\phi_0)(\eta^f) \right] 
+ \epsilon \left( Q(4, \eta^f) + \frac{1}{2} [Q(3, \eta^f)]^2 \right) H(\phi_0) + H^{(2)}(\phi_0)(\eta^f, \eta^f) + Q(3, \eta^f) H^{(1)}(\phi_0)(\eta^f) 
+ \cdots + \epsilon^{\frac{s-2}{2}} \tilde{\ell}(\epsilon, \eta^f) \right) \right\} + \exp \left\{ \epsilon^{-1} [F(\phi_0) - S(\phi_0)] \right\} \cdot \mathbb{E}^{\epsilon^*} \left\{ 1_{\{||\eta^f|| < \epsilon^{-1/2}\}} \exp\{Q(2, \eta^f)\} \times \Re_1(\epsilon, \eta^f) \right\}$$
for two functionals $\tilde{\ell}(\epsilon, \eta^f)$ and $\tilde{\ell}(\epsilon, \eta^f)$. Now it becomes quite clear that Theorem 2.1 is proved if the following (I)-(III) are proved:

(I).

$$\mathbb{E}^{\epsilon^*} \left\{ 1_{\{||\eta^f|| < \epsilon^{-1/2}\}} \exp\{Q(2, \eta^f)\} \times \Re_1(\epsilon, \eta^f) \right\} = o \left( \epsilon^{\frac{s-2}{2}} \right).$$ (3.9)
(II). Each term over the set \( \{ ||\eta'|| \geq \epsilon^{-1/2} h \} \) in (3.8) tends to zero exponentially fast.

(III). The expectations \( \mathbb{E}^\epsilon \) in (3.8) without \( \{ ||\eta'|| \geq \epsilon^{-1/2} h \} \) have precise asymptotic expansions. Obviously, this is the place where normal deviations for stochastic processes are used. To make (III) more precise, we need to show the following asymptotic expansions for normal deviations.

\[
\mathbb{E}^\epsilon \left\{ \exp\{Q(2, \eta')\} \right\} = \mathbb{E} \left\{ \exp\{Q(2, \eta)\} \right\} + \epsilon^{1/2} C_0 + \epsilon C_0 + \cdots + o(\epsilon^{4/2}),
\]

\[
\mathbb{E}^\epsilon \left\{ \exp\{Q(2, \eta')\} \right\} \left[ Q(3, \eta')H(\phi_0) + H^{(1)}(\phi_0)(\eta') \right] = \mathbb{E} \left\{ \exp\{Q(2, \eta)\} \right\} \left[ Q(3, \eta)H(\phi_0) + H^{(1)}(\phi_0)(\eta) \right] + \epsilon^{1/2} C_{11} + \epsilon C_{12} + \cdots + o(\epsilon^{4/2}),
\]

\[
\mathbb{E}^\epsilon \left\{ \exp\{Q(2, \eta')\} \right\} \left[ \left( Q(4, \eta') + \frac{1}{2}[Q(3, \eta')]^2 \right) H(\phi_0) + H^{(2)}(\phi_0)(\eta', \eta') + Q(3, \eta')H^{(1)}(\phi_0)(\eta') \right]
= \mathbb{E} \left\{ \exp\{Q(2, \eta)\} \right\} \left[ \left( Q(4, \eta) + \frac{1}{2}[Q(3, \eta)]^2 \right) H(\phi_0) + H^{(2)}(\phi_0)(\eta, \eta) + Q(3, \eta)H^{(1)}(\phi_0)(\eta) \right] + \epsilon^{1/2} C_{21} + \epsilon C_{22} + \cdots + o(\epsilon^{4/2})
\]

and so on, where \( C_{ij} \) are constants which can be determined by Theorem 3.1 in Section 3.1. If we replace all terms in (3.8) by these asymptotic expansions for normal deviations, then we get

\[
\mathbb{E}^\epsilon \left[ H(\xi') \exp\{\epsilon^{-1} F(\xi')\} \right] = \mathbb{E}^\epsilon \left[ ||\xi' - \phi_0|| < h; H(\xi') \exp\{\epsilon^{-1} F(\xi')\} \right]
+ o(\exp\{\epsilon^{-1}[F(\phi_0) - S(\phi_0) - \gamma]\})
\]

\[
= \exp \left\{ \epsilon^{-1}[F(\phi_0) - S(\phi_0)] \right\} \cdot \sum_{0 \leq i \leq (s-2)} K_i \epsilon^{i/2} + o \left( \epsilon^{(s-2)/2} \right),
\]

which is exactly (2.3). That is, Theorem 2.1 is proved if (I), (II), (III) and the finiteness of \( K_i \) are proved.

### 3.4 Proofs of (I)-(III)

#### 3.4.1 Proof of (I)

It is clear that (I) will be proved if for any positive integer \( k \),

\[
\mathbb{E}^\epsilon \left\{ \exp\{Q(2, \eta')\} \right\} \left\{ \frac{1}{\epsilon^{-1/2}} \left( 1 + ||\eta'||^k + \int_0^T ||\eta'^t||^k \, dt \right) \right\} = o \left( \epsilon^{4/2} \right).
\]

In order to prove (3.10), we establish a lemma first.
Lemma 3.2. Under the assumption (F), for any positive \( k \), there are constants \( c_1 \) and \( c_2 \) (depending only on \( k \)) such that for all \( t \in [0, T] \), \( 1 > \epsilon > 0 \),

\[
\mathbb{E}^{z^t} \left[ (\eta_n^k)^k \right] \leq t \cdot c_1 + c_1 \cdot c_2 \cdot \int_0^t s \cdot e^{c_2(t-s)} ds.
\]

Proof. We consider a sequence of functions \( f_n(x) = \frac{x^k}{1+(x/n)^k} \) with an even positive integer \( k \). Then, according to (3.3), the generating operator \( A^\nu_t \) applied to \( f_n \) gives

\[
A^\nu_t f_n(x) = \epsilon^{-1/2} \left[ \frac{\partial G_0}{\partial z}(t, \phi_0(t) + \epsilon^{1/2} x; z_0(t)) - \phi_0'(t) \right] f_n'(x) + \frac{1}{2} a(t, \phi_0(t) + \epsilon^{1/2} x) f_n''(x)
\]

\[
+ \epsilon^{-1} \int_{\mathbb{R}} \left[ f_n(x + \epsilon^{1/2} u) - f_n(x) - \epsilon^{1/2} f_n'(x) \cdot u \right] e^{z_0(t)u} \nu_{t, \phi_0(t) + \epsilon^{1/2} x}(du)
\]

\[
= \frac{\partial^2 G_0}{\partial z \partial x}(t, \phi_0(t) + \theta_1 \epsilon^{1/2} x; z_0(t)) \cdot x \cdot f_n'(x)
\]

\[
+ \left( \int_{\mathbb{R}} \frac{u^2 e^{z_0(t)u} \nu_{t, \phi_0(t) + \epsilon^{1/2} x}(du)}{2} + a(t, \phi_0(t) + \epsilon^{1/2} x) \right)
\]

\[
+ \cdots + \frac{1}{k!} \int_{\mathbb{R}} e^{\frac{k-2}{2} u^2} f_n^{(k)}(x + \theta_2 \epsilon^{1/2} u) e^{z_0(t)u} \nu_{t, \phi_0(t) + \epsilon^{1/2} x}(du),
\]

which is less than or equal to \( c_3 + c_4 \cdot f_n(x) \), since \( f_n^{(k)} \) is bounded, and \( x \cdot f_n'(x), f_n'', f_n''' \), \ldots, \( f_n^{(k-1)} \) are bounded by \( c_5 + c_6 \cdot f_n(x) \). So

\[
\mathbb{E}^{z^t} [f_n (\eta_n^k)] \leq f_n(0) + \int_0^t \left( c_1 + c_2 \cdot \mathbb{E}^{z^s} [f_n (\eta_n^k)] \right) ds.
\]

Applying Gronwall’s lemma with such nonnegative \( \mathbb{E}^{z^t} [f_n (\eta_n^k)] \), we obtain

\[
\mathbb{E}^{z^t} [f_n (\eta_n^k)] \leq t \cdot c_1 + c_1 \cdot c_2 \cdot \int_0^t s \cdot e^{c_2(t-s)} ds,
\]

then the proof is done by sending \( n \) to infinity. \( \square \)

From this lemma, we have \( \sup_{\epsilon \in (0,1)} \mathbb{E}^{z^t} \left[ (\eta_T^k)^k \right] < \infty \) and \( \sup_{\epsilon \in (0,1)} \mathbb{E}^{z^t} \left[ \int_0^T (\eta_t^k)^k dt \right] < \infty \) for any integer \( k \). These together with the fact \( \exp \{ Q(2, \eta^t) \} \leq 1 \) yield (3.10).

3.4.2 Proof of (II)

It is immediate that the first term goes to zero exponentially fast,

\[
\mathbb{E}^{z^t} \left\{ 1_{\{||\eta^t|| \geq \epsilon^{-1/2} k\}} \exp \{ Q(2, \eta^t) \} \cdot H(\phi_0) \right\} \to 0 \text{ exponentially fast,}
\]

which is from (3.3). Every other term has an upper bound by using estimates (2.1)-(2.2):

\[
c \cdot \mathbb{E}^{z^t} \left\{ 1_{\{||\eta^t|| \geq \epsilon^{-1/2} k\}} \epsilon^{\frac{k-2}{2}} \left( 1 + |\eta_T^k|^k + \int_0^T |\eta_t^k|^k dt \right) \right\}, \tag{3.11}
\]
for some nonnegative constants $c$ and $k$. By applying Hölder’s inequality to (3.11) and using Lemma 3.2, it follows each term over the set $\{ ||\eta'|| \geq e^{-1/2}h \}$ in (3.8) tends to zero exponentially fast.

3.4.3 Proof of (III)

(a) Proof of the first expansion

The task is to prove

$$E^z \{ \exp \{ Q(2, \eta') \} \} = E \{ \exp \{ Q(2, \eta) \} \} + \epsilon^{1/2} C_{01} + \epsilon C_{02} + \cdots + \epsilon^{(s-2)/2} C_{0(s-s)} + o(\epsilon^{(s-2)/2}).$$

Denoting

$$\bar{F}(x[0, T]) = \exp \{ Q(2, x[0, T]) \} = \exp \left\{ \frac{1}{2} F(2)(\phi_0)(x[0, T], x[0, T]) \right\},$$

we need to check such $\bar{F}(x[0, T])$ satisfies all conditions of Theorem 3.1 in Section 3.1.

Claim 1. $\bar{F}(x[0, T])$ is $3(s-2)$ times differentiable. It is easy to see that $\bar{F}$ is infinitely differentiable. Furthermore, the derivatives can be computed as follows

$$\bar{F}^{(1)}(x[0, T])(\delta) = \lim_{h \to 0} h^{-1} \left[ \bar{F}(x[0, T] + h\delta) - \bar{F}(x[0, T]) \right],$$

$$\bar{F}^{(2)}(x[0, T])(\delta_1, \delta_2) = \lim_{h \to 0} h^{-1} \left[ \bar{F}^{(1)}(x[0, T] + h\delta_2)(\delta_1) - \bar{F}^{(1)}(x[0, T])(\delta_1) \right],$$

and so on.

Claim 2. $\bar{F}$ satisfies condition (I) of Theorem 3.1. First we have $|\bar{F}(x[0, T])| \leq 1$. The derivatives of $\bar{F}$ satisfy (I). For instance, here we check for $\bar{F}^{(2)}$.

$$|\bar{F}^{(2)}(x[0, T])(\delta, \delta)| \leq \left( F(2)(\phi_0)(x[0, T], \delta) + \int_0^T x(t)\delta(t) \frac{\partial^2 G_0}{\partial x^2}(t, \phi_0(t); z_0(t))dt \right)^2 + c(||\delta||),$$
where \( c(||\delta||) \) is a constant depending on the uniform norm of \( \delta \). Taking into account the assumptions on \( F^{(2)} \), the above is less than or equal to

\[
p^2 \cdot \left( |x(T)|^{m} + (1 + T||\delta||^n)(1 + \int_0^T |x(t)|^n dt) + ||\delta \frac{\partial^2 G_0}{\partial x^2}|| \int_0^T x(t)^2 dt \right)^2 + c(||\delta||).
\]

We apply Hölder inequality several times to get an upper bound

\[
c_1(||\delta||) \left( 1 + |x(T)|^{2m} + \int_0^T |x(t)|^{2n} dt \right).
\]

Claim 3. \( \tilde{F}^{(i)}(x[0,T]) (I_{[t,T]}\delta, \cdots, I_{[t,T]}\delta), 3 \leq i \leq 3(s-2), \) are continuous with respect to \( x[0,T] \) uniformly as \( x[0,T] \) changes over an arbitrary compact subset of \( D_0[0,T], t \) over \( \{0,T\} \), and \( \delta[0,T] \) over the set of Lipschitz continuous functions with constant 1, \( ||\delta|| \leq 1 \).

It follows from \( [39] \) that any compact subset of \( D_0[0,T] \) is a bounded set in uniform topology. Taking \( ||I_{[t,T]}\delta|| \leq 1 \) into account, Claim 3 is done easily. For instance, the uniform continuity of \( F^{(2)}(\phi_0)(x[0,T], I_{[t,T]}\delta) \) in \( x[0,T] \) can be achieved as follows:

\[
\left| F^{(2)}(\phi_0)(y[0,T], I_{[t,T]}\delta) - F^{(2)}(\phi_0)(x[0,T], I_{[t,T]}\delta) \right| = \left| F^{(2)}(\phi_0)(y[0,T] - x[0,T], I_{[t,T]}\delta) \right| \leq C||y - x||,
\]

where \( C \) is independent of \( x, y, t, \delta \). Uniform continuity of \( F^{(2)}(\phi_0)(x[0,T], x[0,T]) \) in \( x[0,T] \) can be also obtained by

\[
\left| F^{(2)}(\phi_0)(y[0,T], x[0,T]) - F^{(2)}(\phi_0)(x[0,T], x[0,T]) \right| \\
\leq \left| F^{(2)}(\phi_0)(y[0,T] - x[0,T], y[0,T]) \right| + \left| F^{(2)}(\phi_0)(y[0,T] - x[0,T], x[0,T]) \right| \\
\leq C \cdot (||y|| + ||x||) \cdot ||y - x|| \leq C \cdot C_1 \cdot ||x - y||,
\]

where \( C \) is independent of \( x, y \), and \( C_1 \) can be chosen independently of \( x, y \), because a compact set is a bounded set in uniform topology.

(b) Proofs of the second expansion and the other expansions

The second expansion is

\[
\mathbb{E}^{\epsilon} \left\{ \exp\{Q(2, \eta')\} \left[ Q(3, \eta')H(\phi_0) + H^{(1)}(\phi_0)(\eta') \right] \right\} \\
= \mathbb{E} \left\{ \exp\{Q(2, \eta)\} \left[ Q(3, \eta)H(\phi_0) + H^{(1)}(\phi_0)(\eta) \right] \right\} + \epsilon^{1/2}C_{11} + \epsilon C_{12} + \cdots + o(\epsilon^{(s-3)/2}).
\]
We use $\hat{F}$ to denote

$$\hat{F}(x[0, T]) = \exp \{Q(2, x[0, T])\} [Q(3, x[0, T]) + H^{(1)}(\phi_0)(x[0, T])]$$

$$= \hat{F}(x[0, T]) \cdot [Q(3, x[0, T]) + H^{(1)}(\phi_0)(x[0, T])].$$

**Claim 1.** $\hat{F}(x[0, T])$ is infinitely differentiable. The first two derivatives are

$$\hat{F}^{(1)}(x[0, T])(\delta) = \lim_{h \to 0} \frac{1}{h} \left[ \hat{F}(x[0, T] + h\delta) - \hat{F}(x[0, T]) \right]$$

$$= \hat{F}^{(1)}(x[0, T])(\delta) \cdot [Q(3, x[0, T]) + H^{(1)}(\phi_0)(x[0, T])] + \frac{3}{3!} \hat{F}(x[0, T])$$

$$\times \left( F^{(3)}(\phi_0)(x[0, T], x[0, T], \delta) + \int_0^T x^2(t)\delta(t) \frac{\partial^3 G_0}{\partial x^3}(t, \phi_0(t); z_0(t))dt + H^{(1)}(\phi_0)(\delta) \right),$$

$$\hat{F}^{(2)}(x[0, T])(\delta_1, \delta_2) = \lim_{h \to 0} \frac{1}{h^2} \left[ \hat{F}(x[0, T] + h\delta_2)(\delta_1) - \hat{F}(x[0, T])(\delta_1) \right]$$

$$= \hat{F}^{(2)}(x[0, T])(\delta_1, \delta_2) \cdot [Q(3, x[0, T]) + H^{(1)}(\phi_0)(x[0, T])]$$

$$+ \frac{3}{3!} \sum_{1 \leq i \neq j \leq 2} \hat{F}^{(1)}(x[0, T])(\delta_j) \left( F^{(3)}(\phi_0)(x[0, T], x[0, T], \delta_i) \right. $$

$$\left. + \int_0^T x^2(t)\delta_i(t) \frac{\partial^3 G_0}{\partial x^3}(t, \phi_0(t); z_0(t))dt + H'(\phi_0)(\delta_i) \right)$$

$$+ \frac{6}{3!} \hat{F}(x[0, T]) \left( F^{(3)}(\phi_0)(x[0, T], \delta_1, \delta_2) + \int_0^T x(t)\delta_1(t)\delta_2(t) \frac{\partial^3 G_0}{\partial x^3}(t, \phi_0(t); z_0(t))dt \right).$$

**Claim 2.** $\hat{F}$ satisfies condition (I) of Theorem 3.1. We notice that $Q(3, x[0, T])$ satisfies

$$|Q(3, x[0, T])| \leq \frac{1}{3!} \left( F^{(3)}(\phi_0)(x[0, T], x[0, T], x[0, T]) + \int_0^T \frac{1}{3!} x^3(t) \frac{\partial^3 G_0}{\partial x^3}(t, \phi_0(t); z_0(t))dt \right)$$

$$\leq \frac{p}{3!} \left[ |x(T)|^{3m} + \left( 1 + \int_0^T |x(t)|^n dt \right)^3 \right] + \frac{1}{3!} \left\| \frac{\partial^3 G_0}{\partial x^3} \right\| \cdot \int_0^T |x(t)|^3 dt.$$ 

This together with (2.2) imply that the first part of (I) is fulfilled. For the second part of (I) on derivatives of $\hat{F}$, similar arguments can be applied. We can also prove the following

**Claim 3.** $\hat{F}^{(i)}(x[0, T])(I_{[t, T]}\delta, \cdots, I_{[t, T]}\delta), 3 \leq i \leq 3(s - 3)$, are continuous with respect to $x[0, T]$ uniformly as $x[0, T]$ changes over an arbitrary compact subset of $D_0[0, T], t$ over $[0, T]$, and $\delta[0, T]$ over the set of Lipschitz continuous functions with constant 1 and $||\delta|| \leq 1$.

The other expansions are proved in the same way.
3.5 Finiteness of the coefficients

First it is obvious that $K_0$ is finite since $H$ is bounded and $\exp\{Q(2, \eta)\} \leq 1$. For the rest, we will use Theorem 2.3.1 in [44] to prove each finiteness. The very first requirement of Theorem 2.3.1 is that $G^0(t, x; z)$ satisfies condition (A) in Section 2. Let us recall $G^0(t, x; z)$:

$$G^0(t, x; z) = zx \cdot \frac{\partial^2 G_0}{\partial z \partial x} (t, \phi_0(t); z_0(t)) + \frac{1}{2} z^2 \cdot \frac{\partial^2 G_0}{\partial z^2} (t, \phi_0(t); z_0(t)),$$

which doesn’t satisfies condition (A) obviously because of the linear term in $x$. So we turn to consider the following transformation

$$\tilde{\eta}_t = \exp \left\{ - \int_0^t g(s) ds \right\} \eta_t,$$

where $g(t) = \frac{\partial^2 G_0}{\partial z \partial x} (t, \phi_0(t); z_0(t))$. Straightforward computation will give us

$$\mathfrak{A} \tilde{f}(t, x) = \frac{\partial f}{\partial t} (t, x) + \frac{1}{2} z^2 \cdot \frac{\partial^2 G_0}{\partial z^2} (t, \phi_0(t); z_0(t)) \cdot \frac{\partial^2 f}{\partial x^2} (t, x) \exp \left\{ -2 \int_0^t g(s) ds \right\},$$

$$G^\tilde{\eta}(t, x; z) = \frac{1}{2} z^2 \cdot \frac{\partial^2 G_0}{\partial z^2} (t, \phi_0(t); z_0(t)) \exp \left\{ -2 \int_0^t g(s) ds \right\},$$

$$H^\tilde{\eta}(t, x; u) = \frac{1}{2} \frac{\partial^2 H_0}{\partial u^2} (t, \phi_0(t); \phi'_0(t)) \exp \left\{ 2 \int_0^t g(s) ds \right\} u^2.$$

Now $G^\tilde{\eta}(t, x; z)$ satisfies condition (A). We will apply Theorem 2.3.1 restricting ourself to $G^\tilde{\eta}(t, x; z)$. It is then shown that each finiteness can be deduced from this.

3.5.1 Finiteness of $K_1$

We now show an auxiliary result.

**Lemma 3.3.** For any positive integer $j$,

$$\mathbb{E}(||\tilde{\eta}||^j) < \infty.$$  \hspace{1cm} (3.12)

**Proof.** The normalized action functional for the family of processes $\sqrt{\epsilon} \cdot \tilde{\eta}$ is

$$I^\tilde{\eta}(f(\cdot)) = \int_0^T H^\tilde{\eta}(t, f(t); f'(t)) dt.$$

Let us consider, for some positive $\alpha$, positive integer $m$,

$$\Phi^\tilde{\eta}_0(m) = \left\{ f \in D_0[0, T] : I^\tilde{\eta}(f(\cdot)) \leq m \right\};$$

$$\Phi^\tilde{\eta}_0(m)_{+\alpha \sqrt{m}} : \alpha \sqrt{m} - \text{neighbourhood of } \Phi^\tilde{\eta}_0(m).$$
The space $D_0[0,T]$ decomposes into the union

$$\Phi_0^\gamma(1)_{+a} \cup \bigcup_{m=1}^\infty \Phi_0^\gamma(m+1)_{+a\sqrt{m+1}} \setminus \Phi_0^\gamma(m)_{+a\sqrt{m}},$$

thus we have

$$E(||\eta||^2) \leq \sum_{m=0}^\infty E\left\{ \eta \in \Phi_0^\gamma(m+1)_{+a\sqrt{m+1}} \setminus \Phi_0^\gamma(m)_{+a\sqrt{m}} : ||\eta||^2 \right\} \leq \sum_{m=0}^\infty P\left\{ \eta \notin \Phi_0^\gamma(m)_{+a\sqrt{m}} \right\} \sup \left\{ ||f||^2 : f \in \Phi_0^\gamma(m+1)_{+a\sqrt{m+1}} \right\}. \tag{3.13}$$

We first analyze the supremum term in (3.13). Let us recall a fact that, for any fixed $m$ such that $\sup_{x < d} \eta(x)$ satisfies condition $\eta$.

$$|x|^l \leq Ae^{a2}, \quad \text{for all } x \in \mathbb{R}. \tag{3.14}$$

Thus, for any $a > 0$, any positive integer $j$, there is some $A = A(a,j) > 0$ such that

$$||f||^2 \leq A \exp\left\{ a||f||^2 \right\}.$$

And for any $f \in \Phi_0^\gamma(m+1)_{+a\sqrt{m+1}}$, we can choose a small $a$ such that (such $a$ can be chosen independent of $m$ by using Lemma 5.2.5 on in [44])

$$a||f||^2 \leq \frac{1}{3}(m+1).$$

So the supremum term can be estimated as follows

$$\sup \left\{ ||f||^2 : f \in \Phi_0^\gamma(m+1)_{+a\sqrt{m+1}} \right\} \leq A \exp\left\{ \frac{1}{3} \cdot (m+1) \right\}. \tag{3.15}$$

Now we analyze the probabilities in (3.13) by using Theorem 2.3.1 in [44]. We check all the conditions of Theorem 2.3.1 as follows. We suppose $\sup_{t} |\frac{\partial^2 G^\gamma}{\partial z^2}(t,\phi_0(t);z_0(t)) \exp\left\{ -2 \int_0^t g(s)ds \right\} | \leq c$ for some constant $c$. Let us consider a constant $Z$, an integer $n$, whose values will be determined a little later. We set $\epsilon_2 = \frac{mZ^2\epsilon^2}{6l}, t_1 = \frac{iT}{n}, \Delta t_{\min} = \Delta t_{\max} = \frac{T}{n}, k = 2, z(1) = \sqrt{m}Z, z(2) = -\sqrt{m}Z, d(1) = d(2) = mZ^2c, \delta' = \frac{\alpha\sqrt{m}}{3}, A = m,

$$U_0 = \{ u : z(j) \cdot u < d(j), j = 1,2 \} = (-\sqrt{mZc}, \sqrt{mZc}).$$

Now we define a small $\epsilon_1$ such that $\epsilon_1(2-\epsilon_1) + T\epsilon_1(3-\epsilon_1) + \frac{Z^2\epsilon^2}{6}(1-\epsilon_1) \leq \frac{Z^2\epsilon^2}{3}. \text{ Firstly we know } G^\gamma \text{ satisfies condition } A \text{ with } G(z) = \frac{1}{2}z^2c. \text{ Secondly } G^\gamma \text{ has the property:}

$$G^\gamma(t,y; (1-\epsilon_1)z) \leq (1-\epsilon_1)G^\gamma(s,x;z),$$

26
for $t,s$ which are close enough (this can be guaranteed by choosing a large $n$). Finally we can approximate $H^\vec{y}$ on $U_0$ by tangent lines from below with any accuracy. More precisely, we can obviously find some $z_0\{1\}, \cdots, z_0\{N\}$ such that

$$
\sup_{u \in [-1,1]} \left( H^\vec{y}(t,x;u) - \max_{1 \leq j \leq N} \left[ z_0\{j\} u - G^\vec{y}(t,x;z_0\{j\}) \right] \right) \leq \frac{\kappa}{6T}.
$$

For general $u \in U_0$, we set $z_j = \sqrt{mZc}z_0\{j\}, 1 \leq j \leq N$ (here $N$ can be chosen independent of $m$). For short, we will use $h_1(t) = \frac{\partial^2 H^\vec{y}}{\partial u^2}(t,\phi_0(t);\phi'_0(t)) \exp\{2 \int_0^t g(s) ds\}$ and $h_2(t) = \frac{\partial^2 H_0}{\partial z^2}(t,\phi_0(t);z_0(t)) \exp\{-2 \int_0^t g(s) ds\}$. Then

$$
\sup_{u \in U_0} \left( H^\vec{y}(t,x;u) - \max_{1 \leq j \leq N} \left[ z_j u - G^\vec{y}(t,x;z_j) \right] \right) = \sup_{u \in (-\sqrt{mZc},\sqrt{mZc})} \left( \frac{1}{2} h_1(t) u^2 - \max_{1 \leq j \leq N} \left[ z_j u - z_j^2 h_2(t) \right] \right)
$$

$$
= mZ^2c^2 \cdot \sup_{u \in (-\sqrt{mZc},\sqrt{mZc})} \left( \frac{1}{2} h_1(t) \left( \frac{u}{\sqrt{mZc}} \right)^2 - \max_{1 \leq j \leq N} \left[ z_j \frac{u}{\sqrt{mZc}} - \frac{1}{2} \left( \frac{z_j}{\sqrt{mZc}} \right)^2 h_2(t) \right] \right)
$$

$$
\leq \frac{mZ^2c^2 \cdot \kappa}{6T} = \epsilon_2.
$$

All conditions of Theorem 2.3.1 are thus checked. Applying this theorem with $\delta' \geq \frac{T}{\alpha} \sqrt{mZc}$, i.e. $n \geq \frac{2nTc}{\alpha}$, we get

$$
P \left\{ \vec{y} \notin \Phi_0^\vec{y}(m) + \alpha \sqrt{m} \right\} \leq 4n \exp \left\{ \frac{T}{n} \left[ \frac{Z^2 c m}{2} - Z^2 c m \right] \right\} + N^n \exp \left\{ -m + m\epsilon_1(2 - \epsilon_1) + T \left( m\epsilon_1(3 - \epsilon_1) + \frac{mZ^2c^2 \kappa}{6T}(1 - \epsilon_1) \right) \right\}
$$

$$
= 4n \exp \left\{ -m \left[ \frac{Z^2 c}{2n} \right] \right\} + N^n \exp \left\{ -m \left[ 1 - \epsilon_1(2 - \epsilon_1) - T\epsilon_1(3 - \epsilon_1) - \frac{Z^2 c^2 \kappa}{6}(1 - \epsilon_1) \right] \right\}
$$

$$
\leq 4n \exp \left\{ -m \left[ \frac{Z^2 c}{2n} \right] \right\} + N^n \exp \left\{ -m \left( 1 - \frac{Z^2 c^2 \kappa}{3} \right) \right\}, \text{ definition of } \epsilon_1
$$

$$
= 4n \exp \left\{ -m \right\} + N^n \exp \left\{ -m \left( 1 - \frac{Z^2 c^2 \kappa}{3} \right) \right\}, \text{ choose } Z = \sqrt{\frac{2n}{T c}}, n \geq \frac{18T}{\alpha^2 c}.
$$
Noticing that $\frac{Z_2^2 \kappa^3}{3}$ can be as small as possible by choosing a small $\kappa$, we thus assume $1 - \frac{Z_2^2 \kappa^3}{3} > 1/2$. Then it follows

$$P\left\{ \tilde{\eta} \notin \Phi_0^\ast(m)_{+\alpha \sqrt{m}} \right\} \leq 4n \exp \{-m\} + N^n \exp \left\{ -\frac{m}{2} \right\}.$$ 

Let us now go back to (3.13) combining above estimate and (3.15)

$$E(||\tilde{\eta}||^j) \leq \sum_{m=0}^{\infty} \left( 4n \exp \{-m\} + N^n \exp \left\{ -\frac{m}{2} \right\} \right) A \exp \left\{ \frac{1}{3} \cdot (m + 1) \right\}$$

$$\leq Ae^{1/3} (4n + N^n) \sum_{m=0}^{\infty} \exp \left\{ -\frac{m}{6} \right\} < \infty.$$ 

By observing the transformation $\tilde{\eta}_t = \exp \left\{ -\int_0^t g(s) ds \right\} \eta_t$, we immediately derive

$$E(||\eta||^j) \leq Be^{1/3} (4n + N^n) \sum_{m=0}^{\infty} \exp \left\{ -\frac{m}{6} \right\} < \infty.$$ 

It is clear that $K_1$ can be bounded by expectation of $c_1 + c_2 \cdot ||\eta||^j$ for some $j, c_1$ and $c_2$, from which finiteness of $K_1$ follows according to Lemma 3.3.

### 3.5.2 Finiteness of the rest of the coefficients

Since all derivatives of $F$ and $H$ are bounded symmetric linear functionals, we can use Lemma 3.3 to prove the finiteness of the rest of the coefficients.

### 4 Connections with partial integro-differential equations

The connections are between large (or normal) deviations and solutions to

\[
\begin{aligned}
\frac{\partial}{\partial t} u^\varepsilon (t, x) &= \frac{\varepsilon}{2} a(t, x) \Delta u^\varepsilon (t, x) + b(t, x) \nabla u^\varepsilon (t, x) + \varepsilon^{-1} c(x) u^\varepsilon (t, x) \\
&\quad + \varepsilon^{-1} \int_{\mathbb{R}} \left[ u^\varepsilon (t, x + \varepsilon u) - u^\varepsilon (t, x) - \varepsilon u \nabla u^\varepsilon (t, x) \right] \nu_{t,x}(du) \\
u^\varepsilon (0, x) &= g(x)
\end{aligned}
\] (4.1)

over $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. More precisely, it is expected that

\[ u^\varepsilon (t, x) = E_{0,x}^\varepsilon \left[ g(\xi^\varepsilon_t) \exp \left\{ \varepsilon^{-1} \int_0^t c(\xi^\varepsilon_s) ds \right\} \right], \] (4.2)
then the precise asymptotics for large deviations (or normal deviations) developed in Theorem 2.1 (or Theorem 3.1) can be applied. Of course, formula (4.2) is not always true unless suitable conditions are imposed. In the first part of this section, we prove (4.2) for a special case when \( c(x) \) is a constant. Then from Theorem 2.1 it follows

\[
u(t, x) = e^{t/\varepsilon} \cdot \left[ \sum_{k=0}^{n} k_i(x) (e^{k/2}) + o(e^{n/2}) \right].
\]

The second part of this section is on the study of precise asymptotics of \( u(t, x) \) in more general settings.

### 4.1 The specific case

In (4.1), we set \( c = 1 \) and \( \nu_{t,x}(du) = u^21_{(|u|\leq 1)}(du) \). What is more, we assume \( a(t, x) = a(x), b(t, x) = b(x), \) and \( 0 < \inf_x a(x) \leq \sup_x a(x) < \infty \), the smooth functions \( a(x), b(x) \) and \( g(x) \) are bounded together with their derivatives \( d^i a/dx^i, d^j b/dx^j \) and \( d^j g/dx^j \). In this case, we consider a family of jump processes \( \xi_\varepsilon \) with generating operators

\[
A^\varepsilon f(x) = \frac{\varepsilon}{2} a(x) f''(x) + b(x) f'(x) + \varepsilon^{-1} \int_{-1}^{1} [f(x + \varepsilon u) - f(x)] u^2 du
\]

for continuous bounded \( f \) together with its first and second derivatives. From the theory of semi-groups, the function

\[
v^\varepsilon(t, x) := \mathbb{E}_{0,x}^\varepsilon f(\xi^\varepsilon_t)
\]

is the unique solution to the problem, for \( f \) in the domain of \( A^\varepsilon \),

\[
\begin{cases}
\frac{\partial}{\partial t} v^\varepsilon(t, x) = A^\varepsilon v^\varepsilon(t, x), \\
v^\varepsilon(0, x) = f(x).
\end{cases}
\]

Now it is easy to see that \( u^\varepsilon(t, x) := e^{t/\varepsilon} \cdot \mathbb{E}_{0,x}^\varepsilon f(\xi^\varepsilon_t) \) is the unique solution of

\[
\begin{cases}
\frac{\partial}{\partial t} u^\varepsilon(t, x) = A^\varepsilon u^\varepsilon(t, x) + \varepsilon^{-1} u^\varepsilon(t, x), \\
u^\varepsilon(0, x) = f(x).
\end{cases}
\]

The conditions imposed on \( a, b \) and \( g \) are mainly for the smooth and growth assumptions in Theorem 2.1, such as (F) and (2.2). The condition \( c(x) = 1 \) is crucial in this special case since it forces the \( \max[F - S] \) is reached uniquely at \( \phi_0 \equiv 0 \). This example should be considered as the asymptotics for
normal (not large) deviations since the main part of the integral \(4.2\) is due to the most probable sample path (which is identically zero). Asymptotics for large (not normal) deviations can be seen below.

4.2 In more general settings

Let \(\xi\) now be the locally infinitely divisible family of processes considered in Theorem 2.1 satisfying all the assumptions. Then the corresponding partial integro-differential equation is \((4.1)\) with \(b(t,x)\) replaced by \(\alpha(t,x)\). In order to show \((4.2)\), it is natural to impose suitable conditions on two new functions \(c(x)\) and \(g(x)\). What is more, more conditions on the processes are also expected. This leads to a theorem borrowed from \([42]\).

**Theorem 4.1** (Section 10.3 in \([42]\)). Let \(\xi\) be uniformly stochastically continuous, the function \(g(x)\) be in the domain of the generating operator \(A_t\), and the function \(c(x)\) be bounded uniformly continuous. Then the function given by \((4.2)\) is the unique solution of \((4.1)\).

For \((4.2)\), we need further assumptions in order to apply Theorem 2.1. For instance, it is assumed that \(\max \left[\int_0^t c(\phi(s)) - H_0 ds\right]\) is reached uniquely at non-zero \(\phi_0\).

5 Appendix

5.1 Compensating operators after transformations

In preceding sections, we presented many compensating operators after transformations without any proofs. In order to show the method, we give the details for deriving the compensating operator appeared in Section 3.5. There, Gaussian process \(\eta\) was considered with compensating operator

\[
\mathfrak{A} f(t,x) = \frac{\partial f}{\partial t}(t,x) + x \cdot \frac{\partial^2 G_0}{\partial z \partial x}(t,\phi_0(t);z_0(t)) \cdot \frac{\partial f}{\partial x}(t,x) + \frac{1}{2} \frac{\partial^2 G_0}{\partial z^2}(t,\phi_0(t);z_0(t)) \cdot \frac{\partial^2 f}{\partial x^2}(t,x).
\]

for \(f(t,x)\) which is bounded and continuous together with its first derivatives in \(t\) and \(x\) and its second derivative in \(x\). The following transformation was used

\[
\bar{\eta}_t = \exp \left\{- \int_0^t g(s) ds\right\} \eta_t, \quad t \in [0,T].
\]
After such a transformation, we give the details in this section that \( \tilde{\eta} \) has compensating operator given by

\[
\tilde{\mathfrak{A}}^{\tilde{\eta}} f(t, x) = \frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 G_0}{\partial z^2}(t, \phi_0(t); z_0(t)) \cdot \frac{\partial^2 f}{\partial x^2}(t, x) \exp \left\{ -2 \int_0^t g(s)ds \right\}
\]

for the same class of functions \( f \).

The generating operator \( A^\eta_t \) of process \( \eta \) is \( A^\eta_t f(x) = \mathfrak{A}^\eta f(t, x) \) for \( f(t, x) = f(x) \). Let us assume the starting position of process \( \eta \) is \( \eta_s = x \). From definitions of compensating operator and generating operator, we have the following two equalities:

\[
P^s_t f(x) - f(x) = \int_s^t P^s_v A^\eta_v f(x) dv,
\]

\[
P^s_t f(t, \cdot)(x) - f(s, x) = \int_s^t P^s_v \mathfrak{A}^\eta v f(v, \cdot)(x) dv,
\]

for suitable \( f(x) \) and \( f(t, x) \), where \( P^s_t \) is the multiplicative family of operators of Markov process \( \eta \) given by \( P^s_t f(x) = \mathbb{E}^\eta_{s,x} f(\eta_t) \). The following two connections between multiplicative families of \( \eta \) and \( \tilde{\eta} \) are easily derived:

\[
P^s_t f(x) = \mathbb{E}^\eta_{s,x} f(\eta_t) = \mathbb{E}^{\tilde{\eta}}_{s,x} \exp(-\int_s^t g(u)du) f\left( \tilde{\eta}_t \exp\left\{ \int_0^t g(u)du \right\} \right)
\]

\[
= P^s_t \tilde{\eta} G(t, \cdot) \left( x \exp\left\{ -\int_s^t g(u)du \right\} \right), \quad G(t, x) = f \left( x \exp\left\{ \int_0^t g(u)du \right\} \right),
\]

\[
P^s_t f(x) = \mathbb{E}^{\tilde{\eta}}_{s,x} f(\tilde{\eta}_t) = \mathbb{E}^\eta_{s,x} \exp(-\int_0^t g(u)du) f\left( \eta_t \exp\left\{ -\int_0^t g(u)du \right\} \right)
\]

\[
= P^s_t \tilde{\eta} F(t, \cdot) \left( x \exp\left\{ \int_0^t g(u)du \right\} \right), \quad F(t, x) = f \left( x \exp\left\{ -\int_0^t g(u)du \right\} \right).
\]

Now we look for \( A^\tilde{\eta}_t \) in the following way.

\[
P^s_t f(x) - f(x) = P^s_t \tilde{\eta} F(t, \cdot) \left( x \exp\left\{ \int_0^t g(u)du \right\} \right) - F \left( s, x \exp\left\{ \int_0^t g(u)du \right\} \right)
\]

\[
= \int_s^t P^s_v \tilde{\mathfrak{A}}^\eta v F(v, \cdot) \left( x \exp\left\{ \int_0^t g(u)du \right\} \right) dv
\]

\[
= \int_s^t P^s_v \tilde{\mathfrak{A}}^\eta v g(v, \cdot) \left( x \exp\left\{ \int_0^t g(u)du \right\} \right) dv, \quad \text{set } h(v, x) = \tilde{\mathfrak{A}}^\eta v F(v, \cdot)(x)
\]

\[
= \int_s^t P^s_v \tilde{\mathfrak{A}}^\eta v h(v, \cdot)(x) dv, \quad \text{where } \tilde{h}(v, x) = h \left( v, x \exp\left\{ \int_0^t g(u)du \right\} \right).
\]

It is straightforward to compute

\[
h(v, x) = \frac{1}{2} \frac{\partial^2 G_0}{\partial z^2}(v, \phi_0(v); z_0(v)) \cdot f'' \left( x \exp\left\{ -\int_0^v g(u)du \right\} \right) \cdot \exp \left\{ -2 \int_0^v g(u)du \right\},
\]

31
thus
\[ A_\tilde{g}(x) = \tilde{h}(t, x) = h \left( t, x \exp \{ \int_0^t g(u) du \} \right) \]
\[ = \frac{1}{2} \frac{\partial^2 G_0}{\partial z^2}(t, \phi_0(t); z_0(t)) \cdot f''(x) \cdot \exp \left\{ -2 \int_0^v g(u) du \right\}. \]

5.2 Proof of Lemma 3.1

Proof. The same conclusion with a continuous and bounded functional \( F \) was given in [44] without a proof. For completeness, we first present the proof for continuous and bounded \( F \) and then extend the argument to include the continuous functional \( F \) which is only bounded above.

We take
\[ \gamma = \left[ F(\phi_0) - S(\phi_0) - \max_{\|\phi_0 - \phi\| \geq \delta} [F(\phi) - S(\phi)] \right] / 2. \]
Here the \( \max_{\|\phi_0 - \phi\| \geq \delta} [F(\phi) - S(\phi)] \) is reached at some point \( \phi_1 \in X \). To see this, we note that
\[ \sup_{\|\phi_0 - \phi\| \geq \delta} [F(\phi) - S(\phi)] = \sup_{\phi \in A} [F(\phi) - S(\phi)] \]
where
\[ A = \left\{ \phi \in D_0[0, T] : \|\phi_0 - \phi\| \geq \delta \text{ and } S(\phi) \leq \sup_{x \in X} F(x) - [F(\tilde{\phi}) - S(\tilde{\phi})] \right\} \]
given a fixed \( \tilde{\phi} \) such that \( \|\phi_0 - \tilde{\phi}\| \geq \delta \) and \( |F(\tilde{\phi}) - S(\tilde{\phi})| < \infty \). The compactness of \( A \) implies that \( \sup_{\phi \in A} [F(\phi) - S(\phi)] \) attains its maximum at some \( \phi_1 \).

We first assume that \( F \) is bounded, thus \( F \) can be split into finitely many parts as
\[ F(x) \in \bigcup_{i=-k}^k [i\gamma/4, (i+1)\gamma/4]. \]

It then follows
\[ \int_{\{\|\xi^\epsilon - \phi_0\| \geq \delta\}} \exp \{ F(\xi^\epsilon) / \epsilon \} d\mathbb{P}^\epsilon \]
\[ \leq \sum_{i=-k}^k \int_{\{\|\xi^\epsilon - \phi_0\| \geq \delta, F(\xi^\epsilon) \in [i\gamma/4, (i+1)\gamma/4]\}} \exp \{ F(\xi^\epsilon) / \epsilon \} d\mathbb{P}^\epsilon \]
\[ \leq \sum_{i=-k}^k \exp \{ (i+1)\gamma/4 \} \cdot \mathbb{P}^\epsilon \{\|\xi^\epsilon - \phi_0\| \geq \delta, F(\xi^\epsilon) \in [i\gamma/4, (i+1)\gamma/4] \}. \]

On each set \( \{\|\xi^\epsilon - \phi_0\| \geq \delta, F(\xi^\epsilon) \in [i\gamma/4, (i+1)\gamma/4] \} \),
\[ S(\xi^\epsilon) \geq F(\xi^\epsilon) - [F(\phi_1) - S(\phi_1)] \geq i\gamma/4 - [F(\phi_1) - S(\phi_1)], \]
then according to large deviation principle, for small enough $\epsilon$,

$$
\mathbb{P}^{\epsilon}\{\|\xi^{\epsilon} - \phi_{0}\| \geq \delta, F(\xi^{\epsilon}) \in [i\gamma/4, (i+1)\gamma/4]\}
\leq \exp \left\{-\inf_{\{\phi: \|\xi^{\epsilon} - \phi_{0}\| \geq \delta, F(\phi) \in [i\gamma/4, (i+1)\gamma/4]\}} S(\phi)/\epsilon + \gamma/(4\epsilon) \right\}
\leq \exp \{[F(\phi_{1}) - S(\phi_{1})]/\epsilon - i\gamma/(4\epsilon) + \gamma/(4\epsilon)\}.
$$

Therefore,

$$
\int_{\{\|\xi^{\epsilon} - \phi_{0}\| \geq \delta\}} \exp \{F(\xi^{\epsilon})/\epsilon\} d\mathbb{P}^{\epsilon}
\leq \sum_{i=-k}^{k} \exp \{(i+1)\gamma/(4\epsilon)\} \cdot \exp \{[F(\phi_{1}) - S(\phi_{1})]/\epsilon - i\gamma/(4\epsilon) + \gamma/(4\epsilon)\}
= \sum_{i=-k}^{k} \exp \left\{\frac{1}{\epsilon} \left[F(\phi_{0}) - S(\phi_{0}) - \frac{3\gamma}{2}\right]\right\} = o \left(\exp \{[F(\phi_{0}) - S(\phi_{0}) - \gamma]/\epsilon\}\right).
$$

Now we assume $F$ to be bounded above, i.e., $M := \sup_{x \in D_{0}[0,T]} F(x) < \infty$. Let us define a sequence of truncated functionals $G^{N}$ as follows

$$
G^{N}(x) = \begin{cases} 
F(x) & \text{if } F(x) \geq -N, \\
-N & \text{if } F(x) < -N.
\end{cases}
$$

Taking into account the fact that $F \leq G$, it is clear that the proof is complete if we can prove $G^{N} - S$ attains its maximum uniquely at $\phi_{0}$ for large $N$ given the condition that $F - S$ attains its maximum uniquely at $\phi_{0}$. We will argue this by contradiction. Suppose for every large $N$, there is a point $\phi_{N} \in X$ different from $\phi_{0}$ such that

$$
G^{N}(\phi_{N}) - S(\phi_{N}) \geq G^{N}(\phi_{0}) - S(\phi_{0}) = F(\phi_{0}) - S(\phi_{0}) \text{ for large } N. \quad (5.1)
$$

Noticing that

$$
\sup_{\phi \in X} [G^{N}(\phi) - S(\phi)] = \sup_{\phi \in B} [G^{N}(\phi) - S(\phi)]
$$

for a compact set $B = \{\phi \in X : S(\phi) \leq M - |F(\phi_{0}) - S(\phi_{0})|\}$, we have $\{\phi_{N}\} \subseteq B$. Then there must be a limiting point $\hat{\phi}$ of a subsequence of $\{\phi_{N}\}$ (we still denote the subsequence as $\phi_{N}$),

$$
\lim_{N \to \infty} \phi_{N} = \hat{\phi}.
$$

From (5.1), it follows

$$
F(\phi_{N}) \leq -N
$$

33
because of the uniqueness of the maximizer of $F - S$. Now we take the limit

$$F(\hat{\phi}) = \lim_{N \to \infty} F(\phi_N) \leq \lim_{N \to \infty} -N = -\infty,$$

which is impossible. \hfill \Box

### 5.3 On a variational problem

In this section, we show the existence and uniqueness of the variational problem of Example 1 in Section 2.2 (note that $T = 1$ in Example 1)

$$\max_{\phi \in C^1_0[0,T]} \int_0^T \left[ \phi(t) - \phi'(t)^2 - \left( \phi'(t) \ln \left( \phi'(t) + \sqrt{\phi'(t)^2 + 1} \right) + 1 - \sqrt{\phi'(t)^2 + 1} \right) \right] dt.$$ 

The proof of uniqueness of our problem is standard and is included in Section 5.3.1. For existence, many references deal with problems having two fixed boundaries and satisfying coercivity assumption (see (5.7) in Section 5.3.2), our problem fails to meet these two requirements. A proof for the existence is given in Section 5.3.2 mainly based on nice properties of the functional $F(\phi) - S(\phi)$ and the analysis on absolutely continuous function space.

#### 5.3.1 Uniqueness

For short, let us define,

$$H(u) = u \ln \left( u + \sqrt{u^2 + 1} \right) + 1 - \sqrt{u^2 + 1}, \quad (5.2)$$

$$v(\phi) = \int_0^T \left[ \phi(t) - \phi'(t)^2 - H(\phi'(t)) \right] dt. \quad (5.3)$$

Let $f(x, y) = H(y) + x^2 - x$, then the variational problem becomes

$$\alpha = \max_{\phi \in C^1_0[0,T]} \int_0^T \left[ \phi(t) - \phi'(t)^2 - H(\phi'(t)) \right] dt = -\min_{\phi \in C^1_0[0,T]} \int_0^T f(\phi(t), \phi'(t))dt. \quad (5.4)$$

Now suppose $\phi_1$ and $\phi_2$ are two minimizers of problem (5.4). Let $w(t) = [\phi_1(t) + \phi_2(t)]/2$, then on one hand, $\int_0^T f(w(t), w'(t))dt \geq -\alpha$; on the other hand, convexity of $f$ yields

$$\int_0^T f(w(t), w'(t))dt = \int_0^T f \left( \frac{1}{2}(\phi_1(t), \phi_1'(t)) + \frac{1}{2}(\phi_2(t), \phi_2'(t)) \right) dt \leq \frac{1}{2} \int_0^T f(\phi_1(t), \phi_1'(t))dt + \frac{1}{2} \int_0^T f(\phi_2(t), \phi_2'(t))dt = -\alpha,$$
which indicates that \( w(t) \) is also a minimizer of (5.4). From equality

\[
\int_0^T \left[ \frac{1}{2} f(\phi_1(t), \phi'_1(t)) + \frac{1}{2} f(\phi_2(t), \phi'_2(t)) - f(w(t), w'(t)) \right] \, dt = -\frac{1}{2} \alpha - \frac{1}{2} \alpha + \alpha = 0 \tag{5.5}
\]

where the integrand of (5.5) is always nonpositive (from convexity of \( f \)), we have

\[
\frac{1}{2} f(\phi_1(t), \phi'_1(t)) + \frac{1}{2} f(\phi_2(t), \phi'_2(t)) = f(w(t), w'(t)), \quad \text{for all } t \in [0, T].
\]

Rewrite above identity as follows

\[
\frac{1}{2} \phi_1^2(t) + \frac{1}{2} \phi_2^2(t) - \left( \frac{\phi_1(t) + \phi_2(t)}{2} \right)^2 = H(\frac{\phi'_1(t) + \phi'_2(t)}{2}) - \left( \frac{1}{2} H(\phi'_1(t)) + \frac{1}{2} H(\phi'_2(t)) \right). \tag{5.6}
\]

If there were a point \( t_0 \in [0, T] \) such that \( \phi_1(t_0) \neq \phi_2(t_0) \), then left hand side of (5.6) would be strictly \(< 0 \) (which is from strict convexity of function \( x^2 \)), while the right hand side is always \( \geq 0 \) from convexity of \( H \). This contradiction proves that (5.4) has at most one minimizer.

### 5.3.2 Existence

When one deals with the existence of variational problems, the following coercivity condition is in general assumed: for all \( p, z \in \mathbb{R} \),

\[
H(p) - (z - z^2) \geq \alpha |p|^q - \beta, \quad \exists \alpha > 0, \beta \geq 0, q > 1. \tag{5.7}
\]

(see Section 8.2 in [20], or see [40] for the case \( q = 2 \)). But this condition is not satisfied for our problem since \( \lim_{|p| \to \infty} \frac{H(p) - (z - z^2)}{|p|^q} = 0 \) for any fixed \( z \). What is more, calculus of variations in references were given in general with two fixed boundaries: \( \phi(0) = A \) and \( \phi(T) = B \). But our problem has one movable boundary \( \phi(T) \).

We define a space \( AC_0[0, T] \) consisting of absolutely continuous functions on \( [0, T] \) vanishing at zero:

\[
AC_0[0, T] = \{ f : [0, T] \to \mathbb{R} \text{ being absolutely continuous with } f(0) = 0 \}.
\]

The existence of our variational problem is solved in the following way. We first prove the existence for \( \max_{\phi \in AC_0[0, T]} v(\phi) \), which implies that this variational problem coincides with a two fixed boundary problem

\[
\max_{\phi \in AC_0[0, T]} v(\phi), \text{ for some } c.
\]

Then we show \( C^1 \) regularity of the maximizer by means of two fixed boundary variational results, which immediately implies the existence of \( \max_{\phi \in C_1^1[0, T]} v(\phi) \).
Existence of $\max_{\phi \in AC_0[0,T]} v(\phi)$

Obviously, we can find some $\phi_* \in AC_0[0,T]$ with $|v(\phi_*)| < T/4$ (for instance $\phi_* \equiv 0$). We define a subset $A$ of $AC_0[0,T]$,

$$A = \left\{ \phi \in AC_0[0,T] : \int_0^T H(\phi'(t)) dt \leq \frac{T}{4} - v(\phi_*) \right\}.$$

Then

$$\sup_{\phi \in AC_0[0,T]} v(\phi) = \sup_{\phi \in A} v(\phi). \tag{5.8}$$

To see (5.8), we notice that for any $\phi \notin A$,

$$\int_0^T H(\phi'(t)) dt > \frac{T}{4} - v(\phi_*), \text{ then } v(\phi_*) > \frac{T}{4} - \int_0^T H(\phi'(t)) dt \geq v(\phi).$$

Let us write

$$\alpha = \sup_{\phi \in A} v(\phi),$$

and let $\{\phi_n(t)\}_{n \geq 1} \subseteq A$ be chosen such that

$$\lim_{n \to \infty} v(\phi_n) = \alpha, \quad \text{and} \quad \lim_{n \to \infty} \max_{0 \leq t \leq T} |\phi_n(t) - \phi_0(t)| = 0 \text{ for some } \phi_0 \in AC_0[0,T].$$

The reason why we can choose such a sequence $\phi_n$ is from the fact that $A$ is compact in $AC_0[0,T]$ according to the following Lemma 5.1 (after passing to a subsequence). We now show $v(\phi_0) = \alpha$.

In fact, $v(\phi_0) \leq \alpha$ is trivial. Lower semi-continuity of $-v(\cdot)$ in Lemma 5.1 gives

$$v(\phi_0) \geq \limsup_{n \to \infty} v(\phi_n) = \alpha.$$

So (5.8) can be rewritten as

$$\max_{\phi \in AC_0[0,T]} v(\phi) = \max_{\phi \in A} v(\phi),$$

which proves the existence of $\max_{\phi \in AC_0[0,T]} v(\phi)$.

**Lemma 5.1.** A defined above is compact in $AC_0[0,T]$ and $-v(\phi)$ is lower semi-continuous in $AC_0[0,T]$ in uniform topology.
Proof. We will finish the proof in several steps. First we show $A$ is an absolutely euqicontinuous family of functions: for any $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that whenever finitely many non-overlapping intervals $\sum_i(t_i - s_i) \leq \delta$, then

$$\sum_i |\phi(t_i) - \phi(s_i)| < \epsilon, \quad \forall \phi \in A.$$ \hfill (5.9)

To see (5.9), first we have a nice property for $H$:

$$H(p) \geq |p| \cdot \ln \left( |p| + \sqrt{p^2 + 1} \right) + 1 - \sqrt{2} |p| - \sqrt{2}, \quad \text{for all } p \in \mathbb{R}. \hfill (5.10)$$

Then $\lim_{|p| \to \infty} H(p)/|p| = \infty$, so there is some $P(\epsilon) > 0$, such that when $|p| > P$,

$$H(p)/|p| \geq 2 \left[ \frac{1}{4} - v(\phi^*) \right]/\epsilon.$$ 

Let $\delta(\epsilon) = \epsilon/(2P)$, then for all $\phi \in A$,

$$\frac{1}{4} - v(\phi^*) \geq \int_0^T H(\phi'(t))dt \geq \sum_i \int_{s_i}^{t_i} H(\phi'(t))dt \geq \sum_i \int_{s_i}^{t_i} \frac{H(\phi'(t))}{|\phi'(t)|} |\phi'(t)|1_{\{|\phi'(t)| > P\}}(t)dt$$

$$\geq \sum_i \int_{s_i}^{t_i} |\phi'(t)|1_{\{|\phi'(t)| > P\}}(t)dt \cdot 2 \left[ \frac{1}{4} - v(\phi^*) \right]/\epsilon,$$

so

$$\epsilon/2 \geq \sum_i \int_{s_i}^{t_i} |\phi'(t)|1_{\{|\phi'(t)| > P\}}(t)dt = \sum_i \int_{s_i}^{t_i} |\phi'(t)|dt - \sum_i \int_{s_i}^{t_i} |\phi'(t)|1_{\{|\phi'(t)| \leq P\}}(t)dt,$$

$$\Rightarrow \sum_i |\phi(t_i) - \phi(s_i)| \leq \sum_i \int_{s_i}^{t_i} |\phi'(t)|dt \leq \epsilon/2 + \sum_i \int_{s_i}^{t_i} |\phi'(t)|1_{\{|\phi'(t)| \leq P\}}(t)dt$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon.$$ 

The second step is to prove lower semi-continuity of $-v(\cdot)$. Let $\phi_n \in AC_0[0, T]$ be a family of absolutely continuous functions such that $\max_{0 \leq t \leq T} |\phi_n(t) - \phi_\infty(t)| \to 0$ as $n \to \infty$. It turns out that $\phi_\infty$ is also absolutely continuous. More precisely, according to absolute equicontinuity, for any $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that if $\sum_i(t_i - s_i) < \delta$, then $\sup_n \sum_i |\phi_n(t_i) - \phi_n(s_i)| < \epsilon$. Sending $n \to \infty$ we get $\sum_i |\phi_\infty(t_i) - \phi_\infty(s_i)| < \epsilon$, which proves the absolute continuity of $\phi_\infty$. Now we show the lower semi-continuity of $\int_0^T H(\phi(t))dt$. Let $0 = t_0 < t_1 \cdots < t_k = T$, Jensen’s
inequality implies
\[ \liminf_{n \to \infty} \int_0^T H(\phi_n'(t)) dt = \liminf_{n \to \infty} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} H(\phi_n'(t)) dt \]
\[ \geq \liminf_{n \to \infty} \sum_{i=0}^{k-1} (t_{i+1} - t_i) H \left( \frac{\phi_n(t_{i+1}) - \phi_n(t_i)}{t_{i+1} - t_i} \right) = \sum_{i=0}^{k-1} (t_{i+1} - t_i) H \left( \frac{\phi(0)(t_{i+1}) - \phi(0)(t_i)}{t_{i+1} - t_i} \right) \]
\[ = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} H(\Psi(t)) dt, \quad \text{where } \Psi(t) = \frac{\phi_{\infty}(t_{i+1}) - \phi_{\infty}(t_i)}{t_{i+1} - t_i} \text{ for } t_i \leq t < t_{i+1} \]
\[ = \int_0^T H(\Psi(t)) dt. \]

Now let a sequence \( \Delta_m \) of partitions be infinitely small, then the corresponding functions \( \Psi_m(t) \) converge to \( \phi'_\infty(t) \) almost everywhere (because of absolute continuity of \( \phi_\infty \)). Using continuity of \( H \) and Fatou’s lemma we get
\[ \int_0^T H(\phi'_\infty(t)) dt \leq \liminf_{n \to \infty} \int_0^T H(\phi'_n(t)) dt, \]
which gives us the lower semi-continuity of \( \int_0^T H(\phi(t)) dt \). Then the lower semi-continuity of \( -v(\cdot) \) is from lower semi-continuity of \( \int_0^T H(\phi(t)) dt \).

The last step will present the compactness of \( \mathcal{A} \) in \( AC_0[0, T] \). Lower semi-continuity of \( \int_0^T H(\phi(t)) dt \) shows \( \mathcal{A} \) is closed in \( AC_0[0, T] \). What’s more, the equicontinuity in step one and the fact all functions in \( \mathcal{A} \) have zero initial value imply that \( \mathcal{A} \) is pre-compact in \( C_0[0, T] \), thus \( \mathcal{A} \) is compact in \( AC_0[0, T] \).

\[ \square \]

**\( C^1 \) regularity of a maximizer of** \( \max_{\phi \in AC_0[0, T]} v(\phi) \)

Let us consider two fixed boundaries problem as follows
\[ g(c) := \max_{\substack{\phi \in AC_0[0, T] \\ \phi(T) = c}} v(\phi). \]

First we note that \( g(c) \) is well defined because of the existence of a maximizer of \( v(\phi) \) under restrictions \( \phi \in AC_0[0, T] \) and \( \phi(T) = c \). Clarke and Vinter in their paper [9] showed several powerful regularity theorems under pretty mild hypotheses by using nonsmooth analysis.

More precisely, Clarke and Vinter in [9] considered the basic problem in the calculus of variation, which is to minimize
\[ J(\phi) := \int_0^T -L(\phi(t), \phi'(t)) dt \]
over the class of absolutely continuous functions $\phi$ having two fixed boundaries $\phi(0) = A$ and $\phi(T) = B$. Assuming $L$ satisfies suitable conditions, they proved that if there is a $\phi(t)$ solving the variational problem, then at every point $t \in [0, T]$, the function $\phi$ is $C^\infty$ in a neighborhood of $t$, see Theorem 2.1 and Corollary 3.1 in [9]. Our functional $\phi(t) - \phi(t)^2 - H(\phi'(t))$ has nice properties which make it satisfy all the hypotheses in [9], so we immediately deduce the $C^1$ regularity (actually $C^\infty$ regularity) of the maximizer of $g(c)$.

Now, according to Section 5.3.2

$$\max_{\phi \in AC_0[0,T]} v(\phi) = \max_{\phi \in AC_0[0,T]} v(\phi) \quad \text{for some (possibly not unique) } c \in \mathbb{R}.$$  

because of the existence of $\max_{\phi \in AC_0[0,T]} v(\phi)$.

Acknowledgment

The author wishes to thank Professor Alexander Wentzell for his guidance and suggestions on this work, and Portuguese Science Foundation project (PTDC/MAT/120354/2010) for the support.

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