Laplace hyperfunctions via Čech Dolbeault cohomology

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Abstract

The paper studies several properties of Laplace hyperfunctions introduced by H. Komatsu in the one dimensional case and by the authors in the higher dimensional cases from the viewpoint of the relative Čech Dolbeault cohomology theory, which enables us, for example, to construct the Laplace transformation and its inverse in a simple and systematic way. We also give some applications to a system of PDEs with constant coefficients.

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1 Introduction

A Laplace hyperfunction on the one dimensional space was first introduced by H. Komatsu ([5], [6], [7]) to justify the operational calculus for arbitrary functions without any growth condition at infinity. After his great success of the one dimensional Laplace hyperfunctions, the authors of this paper established an Oka type vanishing theorem (Theorem 3.7 [2]) and an edge of the wedge type theorem (Theorem 3.12 [3]) for the sheaf of holomorphic functions of several variables with exponential growth at infinity. Thanks to these fundamental theorems, we were succeeded in defining the sheaf $\mathcal{B}^{\exp}$ of Laplace hyperfunction of several variables as a local cohomology groups along the radial compactification $D_{\mathbb{R}^n} = \mathbb{R}^n \sqcup S^{n-1}$ of $\mathbb{R}^n$ with coefficients in the sheaf $\mathcal{O}^{\exp}$ of holomorphic functions with exponential growth, and also showing that $\mathcal{B}^{\exp}$ is a soft sheaf (Corollary 5.9 [3]).

Since a Laplace hyperfunction is defined as an element of the local cohomology group, to understand its concrete expression we need some interpretation of the local cohomology group, which is done by usually considering its Čech representation through the relative Čech cohomology group or more generally its “intuitive representation” introduced in [9] Section 4 (see Subsection 4.3 also).

Recently T. Suwa in [11] and [12] proposed another method to compute a local cohomology group by using a soft resolution of a coefficient sheaf, which is called the relative Čech Dolbeault cohomology. This implies, in particular, the sheaf of Sato’s hyperfunction can be computed with the famous Dolbeault resolution of holomorphic functions by using the relative Čech Dolbeault cohomology theory. In fact, N. Honda, T. Izawa and T. Suwa [11] studies Sato’s hyperfunctions from the viewpoint of Čech Dolbeault cohomology theory and finds that several operations to a hyperfunction such as the integration of a hyperfunction along fibers, etc. have very simple and easily understandable descriptions in this framework because a hyperfunction is represented by a pair $(\mu_1, \mu_{01})$ of $C^\infty$-differential forms.

The purpose of this paper is to study Laplace hyperfunctions from the viewpoint of Čech Dolbeault cohomology theory, which gives us several advantages to their treatments like the case of Sato’s hyperfunctions. To make this point more clear, we briefly explain, as such an example, an inverse Laplace transformation $I_L$ in the framework of Čech Dolbeault cohomology: It is given by a quite simple form (see Definition 7.0.3 for details)

$$I_L\omega(f) = \left[ \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\gamma^*} \rho(\omega)(\text{Im} \, \zeta/|\text{Im} \, \zeta|, z) e^{\zeta z} f(\zeta) d\zeta \right],$$

where $\gamma^*$ is an appropriate real $n$-dimensional chain asymptotic to $\sqrt{-1}\mathbb{R}^n$ and a pair $\rho(\omega)(\theta, z)$ of $C^\infty$-differential forms represents, roughly speaking, the constant function 1.
in the relative Čech Dolbeault cohomology group on $S^{n-1} \times D_{\mathbb{C}^n}$ which also satisfies the support condition
\[
\text{supp}(\rho(\omega)) \subset \{ (\theta, z) \in S^{n-1} \times \mathbb{C}^n; \langle \theta, \text{Im} z \rangle > 0 \} \subset S^{n-1} \times D_{\mathbb{C}^n}.
\]

Here $D_{\mathbb{C}^n} = \mathbb{C}^n \sqcup S^{2n-1}$ is the radial compactification of $\mathbb{C}^n$, and see Definition 2.2.2 for the symbol $\wedge(\bullet)$. Note that the above support condition for $\rho(\omega)$ guarantees the convergence of the integral. The existence of such a kernel $\rho(\omega)e^{\zeta z}$ with the desired support condition is crucial in the definition of the inverse Laplace transformation, which comes from the fact that in Čech Dolbeault cohomology group the support of a representative can be cut off in a desired way.

Furthermore, as was seen in the proof of Lemma 7.0.5, we can estimate the support of a Laplace hyperfunction $\mathcal{L}_\omega(f)$ by using the fact that any derivative of $\rho(\omega)$ becomes zero as a cohomology class because it is cohomologically constant. In this way Čech Dolbeault cohomology theory gives us several new ideas and methods in a study of Laplace hyperfunctions.

The paper is organized as follows: In Section 2, after a short review of Čech Dolbeault cohomology theory, we introduce several geometrical notations which are used through the whole paper. Then we establish the fundamental de-Rham and Dolbeault theorems in Section 3 and give the definition of the sheaf of Laplace hyperfunctions in Section 4. We also give several expressions of Laplace hyperfunctions via Čech cohomology and Čech Dolbeault cohomology in the same section. The one of important facts in hyperfunction theory is the notion of boundary values of holomorphic functions. We construct a boundary value morphism for Laplace hyperfunctions in Section 5. The Laplace transformation and its inverse in the framework of Čech Dolbeault cohomology are defined in Sections 6 and 7, and the fact that they are inverse to each other is shown in Section 8. The last section gives some applications to a system of PDEs with constant coefficients.

2 Preparations

Through the paper, we use the language of the derived categories: Notations Mod$(\mathbb{Z})$, Mod$(\mathbb{Z}_X)$, $C^+(\text{Mod}(\mathbb{Z}_X))$, $K^+(\text{Mod}(\mathbb{Z}_X))$, $D^+(\text{Mod}(\mathbb{Z}_X))$, etc. are the same as those in the book [4], for example, Mod$(\mathbb{Z})$ denotes the category of abelian groups, Mod$(\mathbb{Z}_X)$ the category of sheaves on $X$ of abelian groups, $C^+(\text{Mod}(\mathbb{Z}_X))$ the category of complexes bounded below of sheaves on $X$ of abelian groups, and $D^+(\text{Mod}(\mathbb{Z}_X))$ is the subcategory consisting of complexes bounded below of the derived category of Mod$(\mathbb{Z}_X)$. We sometimes write $\mathcal{F} \in \text{Mod}(\mathbb{Z}_X)$ instead of $\mathcal{F} \in \text{Ob}(\text{Mod}(\mathbb{Z}_X))$.

2.1 A relative Čech derived complex

In this subsection, we briefly recall the definition of a relative Čech derived complex. For details, refer the readers to [1]. Let $X$ be a locally compact and $\sigma$-compact Hausdorff space and $K$ its closed subset, and let $\mathcal{S} = \{ U_i \}_{i \in \Lambda}$ be a finite open covering of $X$ and $\Lambda'$ a subset of $\Lambda$ such that $\mathcal{S}' = \{ U_i \}_{i \in \Lambda'}$ ($\Lambda' \subset \Lambda$) becomes an open covering of $X \setminus K$. For $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_k) \in \Lambda^{k+1}$, we set
\[
U_\alpha = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}.
\]
Let $\mathcal{F} \in \text{Mod}(\mathbb{Z}_X)$. We denote by $C(S, S'; \mathcal{F})$ the relative Čech complex of $\mathcal{F}$ with respect to the pair $(S, S')$ of coverings, that is, $C(S, S'; \mathcal{F})$ is the complex

$$\cdots \overset{\delta^{k-1}}{\longrightarrow} C^k(S, S'; \mathcal{F}) \overset{\delta^k}{\longrightarrow} C^{k+1}(S, S'; \mathcal{F}) \overset{\delta^{k+1}}{\longrightarrow} C^{k+2}(S, S'; \mathcal{F}) \overset{\delta^{k+2}}{\longrightarrow} \cdots .$$

Here $C^k(S, S'; \mathcal{F})$ consists of alternating sections $\{s_\alpha\}_{\alpha \in \Lambda^{k+1}}$ with $s_\alpha \in \mathcal{F}(U_\alpha)$ and $s_\alpha = 0$ if $\alpha \in (\Lambda')^{k+1}$, and the differential $\delta^k$ is defined by

$$\delta^k(\{s_\alpha\}_{\alpha \in \Lambda^{k+1}})_\beta = \sum_{i=1}^{k+2} (-1)^{i+1} s_{\beta \setminus \alpha_i}|_{U_\beta} \quad (\beta \in \Lambda^{k+2}),$$

where $\beta \setminus \alpha_i$ denotes the sequence such that the $i$-th element of $\beta$ is removed.

Let $\mathcal{F} \in C^+(\text{Mod}(\mathbb{Z}_X))$ be a complex with bounded below of sheaves of $\mathbb{Z}$-modules

$$\cdots \overset{d^{k-1}}{\longrightarrow} \mathcal{F}^k \overset{d^k}{\longrightarrow} \mathcal{F}^{k+1} \overset{d^{k+1}}{\longrightarrow} \mathcal{F}^{k+2} \overset{d^{k+2}}{\longrightarrow} \cdots .$$

Then we denote by $C(S, S')(\mathcal{F}^\bullet)$ the single complex associated with the double complex

$$\cdots \overset{d^{q-1}}{\longrightarrow} C^p(S, S'; \mathcal{F}^q) \overset{d^q}{\longrightarrow} C^p(S, S'; \mathcal{F}^{q+1}) \overset{d^{q+1}}{\longrightarrow} C^p(S, S'; \mathcal{F}^{q+2}) \overset{d^{q+2}}{\longrightarrow} \cdots ,$$

that is, the complex is given by

$$C^k(S, S')(\mathcal{F}^\bullet) = \bigoplus_{p+q=k} C^p(S, S'; \mathcal{F}^q)$$

and, for $\omega = \bigoplus_{p+q=k} \omega^{p,q} \in C^k(S, S')(\mathcal{F}^\bullet)$,

$$d^k_{C(S, S')(\mathcal{F}^\bullet)}(\omega) = \bigoplus_{p+q=k+1} (\delta^{q-1}(\omega^{p-1,q}) + (-1)^p d^{q-1}(\omega^{p,q-1})).$$

Let $\mathcal{F} \in \text{Mod}(\mathbb{Z}_X)$ and let $i : \mathcal{F} \to \mathcal{F}^\bullet$ be a resolution of $\mathcal{F}$ by soft sheaves, that is, $\mathcal{F}^\bullet \in C^+(\text{Mod}(\mathbb{Z}_X))$ consists of soft sheaves and the morphism $i$ of complexes is quasi-isomorphic. Then we sometimes call the complex $C(S, S')(\mathcal{F}^\bullet)$ the relative Čech derived complex of $\mathcal{F}$ (with respect to the pair $(S, S')$ of coverings). In particular, if $X$ is a complex manifold and $\mathcal{O}_X^{(0,\bullet)}$ is the Dolbeault complex which is a soft resolution of the sheaf $\mathcal{O}_X$ of holomorphic functions on $X$, then we say $C(S, S')(\mathcal{O}_X^{(0,\bullet)})$ to be the relative Čech Dolbeault complex.

**Theorem 2.1.1** ([1]). Under the above situation, there exists the canonical isomorphism in $\text{D}^+(\text{Mod}(\mathbb{Z}))$:

$$R\Gamma_K(X; \mathcal{F}) \simeq C(S, S')(\mathcal{F}^\bullet).$$

**Example 2.1.2.** If we take

$$\mathcal{V} = \{V_0 = X \setminus K, V_1 = X\}, \quad \mathcal{V}' = \{V_0\}.$$
as coverings of $X$ and $X \setminus K$, then the complex $C(V, V')(\mathcal{F}^*)$ becomes quite simple as follows:

$$C^k(V, V')(\mathcal{F}^*) = \mathcal{F}^k(V_1) \oplus \mathcal{F}^{k-1}(V_{01}),$$

where $V_{01} = V_0 \cap V_1$, and $d^k_{C(V, V')(\mathcal{F}^*)}$ is given by

$$\mathcal{F}^k(V_1) \oplus \mathcal{F}^{k-1}(V_{01}) \ni (\omega_1, \omega_{01}) \mapsto (d^k\omega_1, \omega_1|_{V_{01}} - d^{k-1}\omega_{01}) \in \mathcal{F}^{k+1}(V_1) \oplus \mathcal{F}^k(V_{01}).$$

This complex is often denoted by $C(X, X \setminus K)(\mathcal{F}^*)$, and its $k$-th cohomology group is also written by $H^k(X, X \setminus K; \mathcal{F})$ if $\mathcal{F}^*$ is a soft resolution of $\mathcal{F}$, which is isomorphic to $H^k_K(X; \mathcal{F})$ by the above theorem.

### 2.2 Radial compactification

Let $M$ be an $n$-dimensional real vector space with the norm $|\cdot|$ and $E = M \otimes_{\mathbb{R}} \mathbb{C}$. We denote by $\mathbb{D}_E$ (resp. $\mathbb{D}_M$) the radial compactification $E \sqcup S^{2n-1}$ (resp. $M \sqcup S^{n-1}$) of $E$ (resp. $M$) as usual (see Definition 2.1 [3]). Note that $\mathbb{D}_M = \overline{M}$ holds, where $\overline{M}$ is the closure of $M$ in $\mathbb{D}_E$. We also set $M_{\infty} = \mathbb{D}_M \setminus M$ and $E_{\infty} = \mathbb{D}_E \setminus E$. Through the paper, we use the following identification

$$E_{\infty} = S^{2n-1} = (E \setminus \{0\})/\mathbb{R}^+, \quad M_{\infty} = S^{n-1} = (M \setminus \{0\})/\mathbb{R}^+.$$

In particular, $\zeta \in E_{\infty}$ is sometimes identified with a unit vector in $E$.

We define an $\mathbb{R}^+$-action on $\mathbb{D}_E$ by, for $\lambda \in \mathbb{R}^+$ and $x \in \mathbb{D}_E$,

$$\lambda x = \begin{cases} \lambda x & \text{if } x \in E, \\ x & \text{if } x \in E_{\infty}. \end{cases}$$

The $\mathbb{R}^+$-action on $\mathbb{D}_M$ is defined to be the restriction of the one in $\mathbb{D}_E$ to $\mathbb{D}_M$. And we also define an addition for $a \in M$ (resp. $a \in E$) and $x \in \mathbb{D}_M$ (resp. $x \in \mathbb{D}_E$) by

$$a + x = \begin{cases} a + x & \text{if } x \in M \ (\text{resp. } x \in E), \\ x & \text{if } x \in M_{\infty} \ (\text{resp. } x \in E_{\infty}). \end{cases}$$

**Definition 2.2.1.** A subset $K$ in $\mathbb{D}_M$ is said to be a cone with vertex $a \in M$ in $\mathbb{D}_M$ if there exists an $\mathbb{R}^+$-conic set $L \subset \mathbb{D}_M$ such that

$$K = a + L = \{a + x \in \mathbb{D}_M; x \in L\}.$$

Here, if $L$ is an empty set, we set $a + L = \emptyset$ for convenience.

The notion of a cone in $\mathbb{D}_E$ is similarly defined. We often need to extend an open subset in $E$ to the one in $\mathbb{D}_E$.

**Definition 2.2.2.** Let $V$ be an open subset in $E$, we define the open subset $\widehat{V}$ in $\mathbb{D}_E$ by

$$\widehat{V} = \mathbb{D}_E \setminus (E \setminus V).$$

Note that we sometimes write $\sim V$ instead of $\widehat{V}$. For an open subset $U$ in $M$, we can define an open subset $\widehat{U}$ in $\mathbb{D}_M$ in the same way as that in $\mathbb{D}_E$. 

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Lemma 2.2.3. Let $V$ be an open subset in $E$. Then $\hat{V}$ is the largest open subset $W$ in $D_E$ with $V = W \cap E$.

**Proof.** Let $W$ be the largest open subset with $W \cap E = V$. Clearly we have $W \supset \hat{V}$. Let us show the converse inclusion: $W \cap E = V$ implies $E \setminus V \subset D_E \setminus W$. Since $D_E \setminus W$ is closed, we have $E \setminus \hat{V} \subset D_E \setminus W$, which shows $W \subset \hat{V}$. □

In Definition 3.4 of [2], we introduced the notion that an open subset $U$ in $D_E$ is regular at $\infty$. In this paper, we call such an open subset “1-regular at $\infty$” to distinguish it from the similar notion for a closed subset defined below.

**Definition 2.2.4.** A closed subset $F \subset D_E$ is said to be regular (at $\infty$) if $\overline{F \cap E} = F$ holds.

**Lemma 2.2.5.** Let $K \subset D_E$ be a closed cone with vertex $a$. Then $K$ becomes regular if and only if the equivalence

$$\pi_{E_\infty}(x) \in K \cap E_\infty \iff a + x \in K$$

holds for any $x \in E \setminus \{0\}$. Here $\pi_{E_\infty} : E \setminus \{0\} \to E_\infty = (E \setminus \{0\})/\mathbb{R}_+$ is the canonical projection.

**Proof.** Assume $\overline{K \cap E} = K$. Let $x_\infty = \pi_{E_\infty}(x) \in K \cap E_\infty$ with $x \neq 0$. It follows from the assumption that there exists a sequence $\{x_k\} \subset K \cap E$ such that $x_k \to x_\infty (k \to \infty)$, which implies $(x_k - a)/|x_k - a|$ converges to $x/|x|$. Since $K$ is a closed cone with the vertex $a$, we have $(x_k - a)/|x_k - a| + a \in K$, and hence, we get $x/|x| + a \in K$ by taking $k \to \infty$. This implies $x + a \in K$. Conversely, if $x + a \in K (x \neq 0)$. Then we have $\mathbb{R}_0 x + a \subset K$, which implies $\pi_{E_\infty}(x) \in K \cap E$. Hence we get $\pi_{E_\infty}(x) \in K$.

We show $\overline{K \cap E} = K$ under the equivalence condition of the lemma. Since $K$ is closed, we have $\overline{K \cap E} \subset K$. It suffices to show that $x_\infty \in \overline{K \cap E}$ holds for $x_\infty \in K \cap E_\infty$. If we identify $x_\infty$ as a unit vector in $E$, then we have $x_\infty + a \in K$ by the equivalence condition of the lemma. Therefore, we have $\mathbb{R}_0 x_\infty + a \subset K \cap E$, which implies $x_\infty \in \overline{K \cap E}$. □

Note that, for example, the set consisting of the only one point in $E_\infty$ is a closed cone in our definition, however, which is not regular.

**Lemma 2.2.6** (Lemma 3.5 [2]). Let $K \subset D_E$ be a closed cone with vertex $a$. The conditions below are equivalent:

1. $K$ is regular.
2. $\overline{(E \setminus K)} = D_E \setminus K$ holds.
3. $D_E \setminus K$ is a 1-regular at $\infty$ (for the definition of 1-regularity, see Definition 3.4 [2]).

**Proof.** It follows from the definition of $\overline{(*)}$ that we have

$$\overline{(E \setminus K)} = D_E \setminus (K \cap E).$$

Hence the conditions 1. and 2. are equivalent.

Let us show 3. implies 1. By the definition, “$D_E \setminus K$ being 1-regular” is equivalently saying that

$$K \cap E_\infty = \text{cl}_{\infty}^1(K)$$
holds. Since we have $\text{clos}_\infty^1(K) \subset (K \cap E) \cap E_\infty$, we get

$$K \cap E_\infty = \text{clos}_\infty^1(K) \subset (K \cap E) \cap E_\infty \subset K \cap E_\infty,$$

which shows $K \cap E = K$. Since $\mathbb{D}_E \setminus K$ is an open cone with vertex $a$, the implication 2. to 3. immediately follows from the Lemma 3.5 [2].

The following definition are often used through the paper: For open subsets $U \subset \mathbb{D}_M$ and $\Gamma \subset M$, define an open subset $U \times \sqrt{-1} \Gamma$ in $\mathbb{D}_E$ by

$$U \times \sqrt{-1} \Gamma = \cap ((U \cap E) \times \sqrt{-1} \Gamma) \subset \mathbb{D}_E. \quad (2.1)$$

Let $M^*$ and $E^*$ be dual vector spaces of $M$ and $E$, respectively. Then we can define the radial compactification $\mathbb{D}_{M^*}$ and $M^*_\infty$ (resp. $\mathbb{D}_{E^*}$ and $E^*_\infty$) for a vector space $M^*$ (resp. $E^*$) in the same way as those of $\mathbb{D}_M$ and $M_\infty$ (resp. $\mathbb{D}_E$ and $E_\infty$).

We also define the open subset $\hat{V}$ in $\mathbb{D}_{E^*}$ for an open subset $V$ in $E^*$ in the same way as that in $\mathbb{D}_E$, that is,

$$\hat{V} = \mathbb{D}_{E^*} \setminus (E^* \setminus V). \quad (2.2)$$

Now we introduce the subset $N^*_pc(Z)$ in $E^*_\infty$ and the canonical projection $\varpi_{M^*_\infty}$ as follows: The canonical projection $\varpi_{M^*_\infty}: E^*_\infty \setminus \sqrt{-1}M^*_\infty \rightarrow M^*_\infty$ is defined by

$$E^*_\infty \setminus \sqrt{-1}M^*_\infty = ((M^* \setminus \{0\}) \oplus \sqrt{-1}M^*)/\mathbb{R}_+ \xrightarrow{\varpi_{M^*_\infty}} (M^* \setminus \{0\})/\mathbb{R}_+ = M^*_\infty, \quad (2.3)$$

which is induced from the canonical projection $E^* = M^* \oplus \sqrt{-1}M^* \rightarrow M^*$, that is, $\varpi_{M^*_\infty}$ is given by

$$E^*_\infty \setminus \sqrt{-1}M^*_\infty \ni \xi + \sqrt{-1} \eta ((\xi, \eta) \in S^{2n-1}, \xi \neq 0) \mapsto \xi/|\xi| \in M^*_\infty.$$ 

Let $Z$ be a subset in $\mathbb{D}_E$.

**Definition 2.2.7.** The subset $N^*_pc(Z)$ in $E^*_\infty$ is defined by

$$\{ \zeta \in E^*_\infty; \text{Re} \langle z, \zeta \rangle > 0 (\forall z \in Z \cap E_\infty) \}.$$

Note that $N^*_pc(Z)$ is an open subset in $E^*_\infty$. Further, for $Z \subset \mathbb{D}_M \subset \mathbb{D}_E$, we see that $N^*_pc(Z) \cap \sqrt{-1}M^*_\infty \neq \emptyset$ holds if and only if $Z \cap M_\infty = \emptyset$ (i.e., $Z$ is a compact set in $M$).

**Definition 2.2.8.** We say that $Z$ is properly contained in a half space of $\mathbb{D}_E$ with direction $\zeta \in E^*_\infty$ if there exists $r \in \mathbb{R}$ such that

$$Z \subset \{ z \in E; \text{Re} \langle z, \zeta \rangle > r \}, \quad (2.4)$$

where $\zeta$ is regarded as a unit vector in $E^*$. If a subset $Z$ is properly contained in a half space of $\mathbb{D}_E$ with some direction, then $Z$ is often said to be a proper subset in $\mathbb{D}_E$.

Then it is easy to see:

**Lemma 2.2.9.** Let $\zeta \in E^*_\infty$ and $Z \subset \mathbb{D}_E$. The $Z$ is properly contained in a half space of $\mathbb{D}_E$ with direction $\zeta$ if and only if $\zeta \in N^*_pc(Z)$. 

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Proof. The implication $\Rightarrow$ is clear. We will show the converse implication. Take an open subset $\Omega$ in $E_\infty$ such that
\[ \bar{Z} \cap E_\infty \subset \Omega \subset \bar{\Omega} \subset \{ z \in E_\infty; \Re\langle z, \zeta \rangle > 0 \}. \]
Then, since $(\bar{Z} \cap E_\infty) \cap (E_\infty \setminus \Omega) = \emptyset$ holds, there exists $R > 0$ such that
\[ \{ t\omega \in E; t \geq R, \omega \in E_\infty \setminus \Omega \} \cap \bar{Z} = \emptyset, \]
from which we have
\[ \bar{Z} \subset \bar{\{ z \in E; \Re\langle z, \zeta \rangle > -R|\zeta| \}}. \]
\[ \Box \]

Example 2.2.10. Let $G$ be an $\mathbb{R}_+\text{-conic}$ closed subset in $E$ and $a \in E$. Set $K = a + G \subset \mathbb{D}_E$. Then we have
\[ N^*_p(K) = N^*_p(G) = \bar{( \text{int } G^\circ )} \cap E_\infty^*, \]
where $G^\circ$ is the dual cone of $G$ in $E^*$, that is,
\[ G^\circ = \{ \zeta \in E^*; \Re \langle z, \zeta \rangle \geq 0 \ (\forall z \in G) \}. \]

3 Several variants of de-Rham and Dolbeault complexes of exponential type on $\mathbb{D}_E$

Let $V$ be an open subset in $\mathbb{D}_E$ and $f$ a measurable function on $V \cap E$. We fix a coordinate system $z = x + \sqrt{-1}y$ of $E$ in what follows.

We say that $f$ is of exponential type (at $\infty$) on $V$ if, for any compact subset $K$ in $V$, there exists $H_K > 0$ such that $|\exp(-H_K|z|) f(z)|$ is essentially bounded on $K \cap E$, i.e.,
\[ ||\exp(-H_K|z|) f(z)||_{L^\infty(K \cap E)} < +\infty. \] (3.1)

Set
\[ \mathcal{Q}_{\mathbb{D}_E}(V) := \left\{ f \in C^\infty(V \cap E); \text{Any higher derivative of } f \text{ with respect to variables } z \text{ and } \bar{z} \right. \]
\[ \left. \text{is of exponential type on } V \right\}. \]
Then it is easy to see that $\{ \mathcal{Q}_{\mathbb{D}_E}(V) \}_V$ forms the sheaf $\mathcal{Q}_{\mathbb{D}_E}$ on $\mathbb{D}_E$. The following easy lemma is crucial in our theory:

Lemma 3.0.1. The sheaf $\mathcal{Q}_{\mathbb{D}_E}$ is fine, in particular, it is a soft sheaf.

Let $\mathcal{Q}_{\mathbb{D}_E}^{(p,q)}$ denote the sheaf on $\mathbb{D}_E$ of $(p, q)$-forms with coefficients in $\mathcal{Q}_{\mathbb{D}_E}$, that is, $f \in \mathcal{Q}_{\mathbb{D}_E}^{(p,q)}(V)$ is written by
\[ \sum_{|I|=p, |J|=q} f_{I,J}(z) dz_I \wedge d\bar{z}_J \]
with $f_{I,J}(z) \in \mathcal{Q}_{\mathbb{D}_E}(V)$, and set
\[ \mathcal{Q}_{\mathbb{D}_E}^{(k)} = \bigoplus_{p+q=k} \mathcal{Q}_{\mathbb{D}_E}^{(p,q)}. \]
Now we define the de-Rham complex $\mathcal{D}_E$ on $\mathbb{D}_E$ with coefficients in $\mathcal{Q}$ by

$$0 \longrightarrow \mathcal{D}_E^{(0)} \xrightarrow{d} \mathcal{D}_E^{(1)} \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{D}_E^{(2n)} \longrightarrow 0,$$

and the Dolbeault complex $\mathcal{D}_E^{(p,\bullet)}$ on $\mathbb{D}_E$ by

$$0 \longrightarrow \mathcal{D}_E^{(p,0)} \xrightarrow{\bar{\partial}} \mathcal{D}_E^{(p,1)} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \mathcal{D}_E^{(p,n)} \longrightarrow 0.$$

Let $\mathcal{O}_{\mathbb{D}_E}$ (resp. $\mathcal{O}_{\mathbb{D}_E}^{(p)}$) denote the sheaf of holomorphic functions (resp. $p$-forms) of exponential type (at $\infty$) on $\mathbb{D}_E$, that is,

$$\mathcal{O}_{\mathbb{D}_E}^{(p)}(V) = \{ f \in \mathcal{D}_E^{(p,0)}(V) ; \bar{\partial} f = 0 \}.$$

The following proposition can be shown by the similar arguments as those in the proof of the usual de-Rham and Dolbeault theorems with bounds.

**Proposition 3.0.2.** Both the canonical morphisms of complexes below are quasi-isomorphic:

$$\mathbb{C}_{\mathbb{D}_E} \longrightarrow \mathcal{D}_E^{(\bullet)}; \quad \mathcal{O}_{\mathbb{D}_E}^{(p)} \longrightarrow \mathcal{D}_E^{(p,\bullet)}.$$

We show, in [2], the Oka type vanishing theorem of holomorphic functions of exponential type on a Stein domain. Hence the above proposition immediately concludes:

**Corollary 3.0.3 (2, Theorem 3.7).** Assume that $V \cap E$ is Stein and that $V$ is 1-regular at $\infty$. Then we have the quasi-isomorphism

$$\mathcal{O}_{\mathbb{D}_E}^{(p)}(V) \longrightarrow \mathcal{D}_E^{(p,\bullet)}(V).$$

Furthermore, the edge of the wedge type theorem of exponential type has been also established in our previous papers:

**Theorem 3.0.4 (3, Theorem 3.12, Proposition 4.1).** The complexes $\mathcal{R}_E\Gamma_{\mathbb{D}_E}(\mathcal{O}_{\mathbb{D}_E}^{(p)})$ and $\mathcal{R}_E\Gamma_{\mathbb{D}_E}(\mathcal{D}_E^{(p,\bullet)})$ are concentrated in degree $n$. Furthermore, $\mathcal{H}_E^n(\mathcal{D}_E^{(p,\bullet)})$ is isomorphic to $\mathcal{D}_E^{(p,\bullet)}$.

In subsequent sections, we need to extend our de-Rham theorem to the one with a parameter. Let $T$ be a real analytic manifold and set $Y := T \times \mathbb{D}_E$ and $Y_\infty = T \times (\mathbb{D}_E \setminus E)$. We denote by $p_T : Y \rightarrow T$ (resp. $p_{\mathbb{D}_E} : Y \rightarrow \mathbb{D}_E$) the canonical projection to $T$ (resp. $\mathbb{D}_E$).

Let $W$ be an open subset in $Y$ and $f(t, z)$ a measurable function on $W \setminus Y_\infty$. We say that $f(t, z)$ is of exponential type on $W$ if, for any compact subset $K$ in $W$, there exists $H_K > 0$ such that $| \exp(-H_K |z|) f(t, z)|$ is essentially bounded on $K \setminus Y_\infty$.

Now we introduce the set $\mathcal{L}\mathcal{D}_Y(W)$ consisting of a locally integrable function $f(t, z)$ on $W \setminus Y_\infty$ satisfying the condition that any higher derivative (in the sense of distributions, for example) of $f(t, z)$ with respect to the variables $z$ and $\bar{z}$ is a locally integrable function of exponential type on $W$. Then, in the same way as in $\mathcal{D}_E^{(p,\bullet)}$, the family $\{ \mathcal{L}\mathcal{D}_Y(W) \}_W$ forms the sheaf $\mathcal{L}\mathcal{D}_Y$ on $Y$ which is fine. Let $\mathcal{L}\mathcal{D}_Y^{(k)}$ denotes the sheaf on $Y$ of $k$-forms with respect to the variables in $E$, and let us define the de-Rham complex $\mathcal{L}\mathcal{D}_Y^{(\bullet)}$ by

$$0 \longrightarrow \mathcal{L}\mathcal{D}_Y^{(0)} \xrightarrow{d_\mathcal{D}} \mathcal{L}\mathcal{D}_Y^{(1)} \xrightarrow{d_\mathcal{D}} \ldots \xrightarrow{d_\mathcal{D}} \mathcal{L}\mathcal{D}_Y^{(2n)} \longrightarrow 0,$$
where $d_{\mathcal{D}_E}$ is the differential on $\mathcal{D}_E$.

Let $\mathcal{E}\mathcal{D}_Y$ be the subsheaf of $\mathcal{L}\mathcal{D}_Y$ consisting of a $C^\infty$-function (with respect to all the variables $t$, $z$ and $\bar{z}$) whose any higher derivative also belongs to $\mathcal{L}\mathcal{D}_Y$. Then we have also the de-Rham complex $\mathcal{E}\mathcal{D}_Y^\bullet$:

$$0 \to \mathcal{E}\mathcal{D}_Y^{(0)} \xrightarrow{d_{\mathcal{D}_E}} \mathcal{E}\mathcal{D}_Y^{(1)} \xrightarrow{d_{\mathcal{D}_E}} \cdots \xrightarrow{d_{\mathcal{D}_E}} \mathcal{E}\mathcal{D}_Y^{(2n)} \to 0.$$

We denote by $\mathcal{L}_{\text{loc},T}^\infty$ (resp. $\mathcal{E}_T$) the sheaf of $L_{\text{loc}}^\infty$-functions (resp. $C^\infty$-functions) on $T$. Then the following proposition follows from the same arguments as those of a usual de-Rham complex.

**Proposition 3.0.5.** We have the quasi-isomorphisms

$$p_T^{-1} \mathcal{L}_{\text{loc},T}^\infty \to \mathcal{L}\mathcal{D}_Y^\bullet \quad \text{and} \quad p_T^{-1} \mathcal{E}_T \to \mathcal{E}\mathcal{D}_Y^\bullet.$$

We also have

**Proposition 3.0.6.** Let $\mathcal{F}$ be a sheaf of $\mathbb{Z}$-modules on $T$. The complexes $R\Gamma_{p_{\mathcal{D}_E}(\mathbb{D}_M)}(p_T^{-1} \mathcal{F})$ is concentrated in degree $n$, and we have the canonical isomorphism

$$\tilde{p}_T^{-1} \mathcal{F} \cong \mathcal{F} \otimes_{p_{\mathcal{D}_E}(\mathbb{D}_M)} \mathcal{O}_{p_{\mathcal{D}_E}(\mathbb{D}_M)/Y} \to \mathcal{H}_{p_{\mathcal{D}_E}(\mathbb{D}_M)}^n(p_T^{-1} \mathcal{F}),$$

where $\tilde{p}_T : p_{\mathcal{D}_E}(\mathbb{D}_M) = T \times \mathbb{D}_M \to T$ is the canonical projection.

**Proof.** Since $\mathbb{D}_M$ has an open neighborhood $U$ in $\mathbb{D}_E$ which is topologically isomorphic to $\mathbb{D}_M \times \mathbb{R}^n$, we may replace $\mathbb{D}_E$ with $U = \mathbb{D}_M \times \mathbb{R}^n$, and we have the commutative diagram of topological spaces

$$\begin{array}{ccc}
T & \xrightarrow{p_T} & Y = T \times \mathbb{D}_M \times \mathbb{R}^n \\
\downarrow{\tilde{p}_T} & & \downarrow{\pi} \\
p_{\mathcal{D}_E}(\mathbb{D}_M) & = T \times \mathbb{D}_M & p_{\mathcal{D}_E}^{-1}(\mathbb{D}_M) = T \times \mathbb{D}_M \\
\downarrow{id} & & \downarrow{id}, \\
\end{array}$$

where $i(t, x) = (t, x, 0)$ and $\pi(t, x, y) = (t, x)$. Then, for a sheaf $\mathcal{F}$ on $T$, we have a chain of isomorphisms

$$R\Gamma_{p_{\mathcal{D}_E}(\mathbb{D}_M)}(p_T^{-1} \mathcal{F}) \cong i^! p_T^{-1} \mathcal{F} \cong i^! \pi^{-1} \tilde{p}_T^{-1} \mathcal{F}$$

$$\cong i^! \pi^{-1} p_T^{-1} \mathcal{F} \otimes i^{-1} \mathcal{O}_{Y/p_{\mathcal{D}_E}(\mathbb{D}_M)}[-n]$$

$$\cong \tilde{p}_T^{-1} \mathcal{F} \otimes i^{-1} \mathcal{O}_{Y/p_{\mathcal{D}_E}(\mathbb{D}_M)}[-n].$$

The last isomorphism comes from the fact $\pi \circ i = id$, which also implies

$$\mathcal{O}_{Y/p_{\mathcal{D}_E}(\mathbb{D}_M)}[-n] \cong \mathbb{Z}_{p_{\mathcal{D}_E}(\mathbb{D}_M)}.$$ 

This completes the proof. \qed

**Corollary 3.0.7.** Let $W$ be an open subset in $Y$ and $s \in \mathcal{H}^n_{p_{\mathcal{D}_E}(\mathbb{D}_M)}(W; p_T^{-1} \mathcal{L}_{\text{loc},T}^\infty)$, and let $\Delta$ be a subset in $\bar{W} := W \cap p_{\mathcal{D}_E}^{-1}(\mathbb{D}_M)$. Assume the conditions below:
1. \( \tilde{p}_T(\tilde{W}) \setminus \tilde{p}_T(\Delta) \) is a set of measure zero in \( T \).

2. For any \( q \in \Delta \), the stalk \( s_q \in H^{n}_{p^{\tilde{p}_T}(D_M)}(p^{\tilde{p}_T}_{T} \mathcal{L}_{loc,T}^\infty) \) of \( s \) is zero.

3. The set \( \tilde{p}_T^{-1}(\tilde{W}) \cap \tilde{W} \) is connected for any \( q \in \tilde{W} \).

Then \( s \) is zero.

**Proof.** We have the commutative diagram, for any point \( q \in \tilde{W} \),

\[
\begin{array}{ccc}
H^n_{p^{\tilde{p}_T}(D_M)}(W; p^{\tilde{p}_T}_{T} \mathcal{L}_{loc,T}^\infty) & \cong & \Gamma(\tilde{W}; \tilde{p}^{-1}_T \mathcal{L}_{loc,T}^\infty) \\
\downarrow & & \downarrow \\
H^n_{p^{\tilde{p}_T}(D_M)}(p^{\tilde{p}_T}_{T} \mathcal{L}_{loc,T}^\infty) & \cong & \Gamma(\tilde{p}_T(\tilde{W}); \mathcal{L}_{loc,T}^\infty)
\end{array}
\]

Hence \( s \) can be regarded as an \( L_{loc}^\infty \)-function on \( \tilde{p}_T(\tilde{W}) \). Then, by the assumption, \( s \) is zero on \( \tilde{p}_T(\Delta) \). Hence \( s \) is almost everywhere zero, and thus, \( s \) is zero as an \( L_{loc}^\infty \)-function. This completes the proof.

We can also define the Dolbeault complex with a parameter in the same way as \( \mathcal{L}_Y^{(p,\bullet)} \).

Let \( \mathcal{L}_Y^{(p,q)} \) and \( \mathcal{E}_Y^{(p,q)} \) be the sheaves of \( (p,q) \)-forms on \( \mathbb{D} \) with coefficients in \( \mathcal{L}_Y \) and \( \mathcal{E}_Y \), respectively. Then we define the Dolbeault complex \( \mathcal{L}_Y^{(p,\bullet)} \) with a parameter on \( Y \) by

\[
0 \to \mathcal{L}_Y^{(p,0)} \xrightarrow{\partial} \mathcal{L}_Y^{(p,1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{L}_Y^{(p,n)} \to 0,
\]

and \( \mathcal{E}_Y^{(p,\bullet)} \) on \( Y \) by

\[
0 \to \mathcal{E}_Y^{(p,0)} \xrightarrow{\partial} \mathcal{E}_Y^{(p,1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{E}_Y^{(p,n)} \to 0.
\]

Then by standard arguments we have

**Proposition 3.0.8.** Both the canonical morphisms of complexes below are quasi-isomorphic:

\[
\mathcal{L}_Y^{(p,0)} \to \mathcal{L}_Y^{(0,\bullet)}, \quad \mathcal{E}_Y^{(p,0)} \to \mathcal{E}_Y^{(0,\bullet)}.
\]

Here \( \mathcal{L}_Y^{(p,0)} \) and \( \mathcal{E}_Y^{(p,0)} \) are the subsheaves of \( \mathcal{L}_Y \) and \( \mathcal{E}_Y \) consisting of sections which are holomorphic with respect to the variables \( z \), respectively.

## 4 Various expressions of Laplace hyperfunctions

Let \( M \) be an \( n \)-dimensional real vector space with the norm \( | \cdot | \) and \( E = M \otimes_{\mathbb{R}} \mathbb{C} \). Recall that \( \mathbb{D}_E \) (resp. \( \mathbb{D}_M \)) denotes the radial compactification \( E \sqcup S^{2n-1} \) (resp. \( M \sqcup S^{n-1} \)) of \( E \) (resp. \( M \)). Let \( U \) be an open subset in \( \mathbb{D}_M \), and \( V \) an open subset in \( \mathbb{D}_E \) with \( V \cap \mathbb{D}_M = U \).

**Definition 4.0.1.** The sheaf on \( \mathbb{D}_M \) of \( p \)-forms of Laplace hyperfunctions is defined by

\[
\mathcal{B}_{\mathbb{D}_M}^{(p)} := H_{\mathbb{D}_M}^{n}(\mathcal{O}_{\mathbb{D}_E}^{exp,(p)}) \otimes_{\mathbb{D}_M} \mathcal{O}_{\mathbb{D}_M/\mathbb{D}_E},
\]

where \( \mathcal{O}_{\mathbb{D}_M/\mathbb{D}_E} \) is the relative orientation sheaf over \( \mathbb{D}_M \), that is, it is given by \( \mathcal{H}_M^n(\mathbb{Z}_{\mathbb{D}_E}) \).

It follows from Theorem 3.0.4 that we have

\[
\mathcal{B}_{\mathbb{D}_M}^{(p)}(U) = H_U^n(V; \mathcal{O}_{\mathbb{D}_E}^{exp,(p)}) \otimes_{\mathbb{D}_M} \mathcal{O}_{\mathbb{D}_M/\mathbb{D}_E}(U).
\]

In particular, \( \{ H_U^n(V; \mathcal{O}_{\mathbb{D}_E}^{exp,(p)}) \}_U \) forms a sheaf on \( \mathbb{D}_M \).

The above cohomology groups have several equivalent expressions. We briefly recall those definitions which will be used in this paper.
4.1 Representation by relative Čech Dolbeault cohomology groups

We first give a representation of a Laplace hyperfunction by the relative Čech Dolbeault cohomology groups. Set \( V_0 = V \setminus U, V_1 = V \) and \( V_{01} = V_0 \cap V_1 \) as usual. Then define the coverings

\[
\mathcal{V}_U = \{V_0, V_1\}, \quad \mathcal{V}'_U = \{V_0\}.
\]

The complex \( C(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(n, \bullet)}) \) is called the relative Čech Dolbeault complex of exponential type (see Subsection 2.1 for the definition of the functor \( C(\mathcal{V}_U, \mathcal{V}'_U)(\bullet) \)):

\[
0 \to C^0(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(n, \bullet)}) \overset{\partial}{\to} C^1(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(p, \bullet)}) \overset{\partial}{\to} \ldots \overset{\partial}{\to} C^n(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(p, \bullet)}) \to 0,
\]

where \( \partial \) is used to denote the differential of this complex. In the same way, the complex \( C(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}) \)

\[
0 \to C^0(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}) \overset{D}{\to} C^1(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}) \overset{D}{\to} \ldots \overset{D}{\to} C^{2n}(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}) \to 0,
\]

where \( D \) is used to denote the differential of this complex, is called the relative Čech de-Rham complex of exponential type.

As was seen in the previous section, \( \mathcal{L}_{\mathbb{D}_E}^{(\bullet)} \) and \( \mathcal{L}_{\mathbb{D}_E}^{(p, \bullet)} \) are soft resolutions of \( \mathbb{C}_{\mathbb{D}_E} \) and \( \mathbb{C}_{\mathbb{D}_E}^{\exp,(p)} \), respectively. Hence the following theorem immediately follows from Theorem 2.1.1.

**Theorem 4.1.1.** There exist the canonical isomorphisms in \( \mathbf{D}^+(\text{Mod}(\mathbb{Z})) \):

\[
\text{R} \Gamma_U(V; \mathbb{C}_{\mathbb{D}_E}) \simeq C(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}), \quad \text{R} \Gamma_U(V; \mathbb{C}_{\mathbb{D}_E}^{\exp,(p)}) \simeq C(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(p, \bullet)}).
\]

It follows from the theorem that we have

\[
\mathcal{L}_{\mathbb{D}_M}^{\exp,(p)}(U) \simeq H^n(C(\mathcal{V}_U, \mathcal{V}'_U)(\mathcal{L}_{\mathbb{D}_E}^{(p, \bullet)})) \otimes \mathbb{Z}_{\mathbb{D}_M(U)} \text{ or } \mathbb{D}_M/\mathbb{D}_M(U). \tag{4.1}
\]

This implies that any Laplace hyperfunction \( u \in \mathcal{L}_{\mathbb{D}_M}^{\exp,(p)}(U) \) is represented by a pair \((\omega_1, \omega_0)\) of \( C^\infty \)-forms which satisfies the following conditions 1. and 2.

1. \( \omega_1 \in \mathcal{L}_{\mathbb{D}_E}^{(p,n)}(V) \) and \( \omega_0 \in \mathcal{L}_{\mathbb{D}_E}^{(p,n-1)}(V \setminus U) \)

2. \( \bar{\partial} \omega_0 = \omega_1 \) on \( V \setminus U \).

**Remark 4.1.2.** Let \( \mathcal{S} = \{S_i\}_{i \in A} \) be a finite open covering of \( V \), and let \( \Lambda' \subset \Lambda \). Assume \( \mathcal{S}' = \{S_i\}_{i \in A'} \) is an open covering of \( V \setminus \Lambda' \). Then, as did in Subsection 2.1 C(\( \mathcal{S}', \mathcal{S}')(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}) \) (resp. \( C(\mathcal{S}, \mathcal{S}')(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}) \)) denotes the relative Čech Dolbeault complex (resp. the relative Čech de-Rham complex) of exponential type with respect to the pair \((\mathcal{S}, \mathcal{S}')\) of coverings.

For these complexes, we also have the isomorphisms

\[
\text{R} \Gamma_U(V; \mathbb{C}_{\mathbb{D}_E}) \simeq C(\mathcal{S}, \mathcal{S}')(\mathcal{L}_{\mathbb{D}_E}^{(\bullet)}), \quad \text{R} \Gamma_U(V; \mathbb{C}_{\mathbb{D}_E}^{\exp,(p)}) \simeq C(\mathcal{S}, \mathcal{S}')(\mathcal{L}_{\mathbb{D}_E}^{(p, \bullet)}).
\]
4.2 Representation by relative Čech cohomology groups

Next we give a representation of a Laplace hyperfunction by the relative Čech cohomology groups. We first recall the Grauert type theorem for existence of a Stein open neighborhood of an open subset in $\mathbb{D}_M$.

**Theorem 4.2.1** (Theorem 4.10 [9]). Let $\Omega$ be an open cone in $M$, and let $V \subset \mathbb{D}_E$ be an open neighborhood of $\hat{\Omega}$. Then we can find an open set $W \subset \mathbb{D}_E$ such that

1. $\hat{\Omega} \subset W \subset V$.
2. $W \cap E$ is a Stein open subset.
3. $W$ is 1-regular at $\infty$.

By taking the above theorem into account, we assume that, in this subsection, $V$ is 1-regular at $\infty$ and $V \cap E$ is a Stein open subset. We also set $U = V \cap \mathbb{D}_M$. Let $\eta_0, \ldots, \eta_{n-1}$ be linearly independent vectors in $M^*$ so that $\{\eta_0, \ldots, \eta_{n-1}\}$ forms a positive frame of $M^*$. Set $\eta_n := -(\eta_0 + \cdots + \eta_{n-1}) \in M^*$ and

$$S_k := \{z = x + \sqrt{-1}y \in E; \ z \in V, \langle y, \eta_k \rangle > 0\} \quad (k = 0, 1, \ldots, n).$$

For convenience, we set $S_{n+1} = V$. Let $\Lambda = \{0, 1, 2, \ldots, n+1\}$ and set, for any $\alpha = (\alpha_0, \ldots, \alpha_k) \in \Lambda^{k+1}$,

$$S_\alpha := S_{\alpha_0} \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}.$$ 

We define coverings of $V$ and $V \setminus U$ by

$$\mathcal{S} := \{S_0, S_1, \ldots, S_{n+1}\}, \quad \mathcal{S}' := \{S_0, \ldots, S_n\}.$$ 

Since $S_\alpha \cap E$ is a Stein open subset and $S_\alpha$ is 1-regular at $\infty$ for any $\alpha \in \Lambda^{k+1}$, by the theory of the relative Čech cohomology, we have the isomorphism

$$H^n_U(V; \mathcal{O}_{\mathbb{D}_E}^{\exp(p)}) \simeq H^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{O}_{\mathbb{D}_E}^{\exp(p)})).$$

Let $\Lambda^{k+1}_*$ be the subset in $\Lambda^{k+1}$ consisting of $\alpha = (\alpha_0, \ldots, \alpha_k)$ with

$$\alpha_0 < \alpha_1 < \cdots < \alpha_k = n+1.$$ 

Then we obtain

$$H^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{O}_{\mathbb{D}_E}^{\exp(p)})) \simeq \bigoplus_{\alpha \in \Lambda^{k+1}_*} \mathcal{O}_{\mathbb{D}_E}^{\exp(p)}(S_\alpha).$$

Hence, any hyperfunction $u$ has a representative $\bigoplus_{\alpha \in \Lambda^{k+1}_*} f_\alpha$ which is a formal sum of $(n+1)$-holomorphic functions of exponential type defined on each $S_\alpha$ ($\alpha \in \Lambda^{n+1}_*$).

Note that the Čech representation and the Čech Dolbeault representation of Laplace hyperfunctions are linked by the following diagram whose morphisms are all quasi-isomorphisms.

$$C(\mathcal{S}, \mathcal{S}')(\mathcal{O}_{\mathbb{D}_E}^{\exp(p)}) \xrightarrow{\alpha_1} C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)}) \xleftarrow{\alpha_2} C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)}),$$

(4.2)

where the middle complex is the Čech Dolbeault one associated with the covering $(\mathcal{S}, \mathcal{S}')$, $\alpha_1$ is induced from the canonical morphism $\mathcal{O}_{\mathbb{D}_E}^{\exp(p)} \to \mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)}$ of Dolbeault complexes and $\alpha_2$ follows from the fact that $\mathcal{S}$ is a finer covering of $\mathcal{V}_U$. 

Furthermore, let $i : \mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)} \to \mathcal{L}^\bullet$ be a flabby resolution of the complex $\mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)}$, i.e., $i$ is a morphism of complexes which is quasi-isomorphic and $\mathcal{L}^\bullet$ is the complex of flabby sheaves on $\mathbb{D}_E$. Then we have the commutative diagram

$$
\begin{array}{ccc}
\Gamma_U(V; \mathcal{L}^\bullet) & \xrightarrow{\alpha_2} & C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{L}^\bullet) \\
\downarrow & & \downarrow \\
C(\mathcal{S}, \mathcal{S}')(\mathcal{L}^\bullet) & \xrightarrow{i} & C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{L}^\bullet) \\
\downarrow & & \downarrow \\
C(\mathcal{S}, \mathcal{S}')(\mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)}) & \xrightarrow{\alpha_1} & C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)})
\end{array}
$$

where all the morphisms are quasi-isomorphisms. Hence we have obtained the (canonical) isomorphisms between cohomology groups:

$$
\mathcal{B}_{\mathbb{D}_M}^{\exp,(p)}(U) = H^n_U(V; \mathcal{D}_{\mathbb{D}_E}^{\exp,(p)})
$$

$$
H^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{D}_{\mathbb{D}_E}^{\exp,(p)})) \xrightarrow{\alpha_1} H^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)})) \xrightarrow{\alpha_2} H^n(C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{D}_{\mathbb{D}_E}^{(p, \bullet)}))
$$

In what follows, all the cohomology groups are identified through these canonical isomorphisms.

## 4.3 Generalization of Čech representations

Representation by Čech cohomology groups can be generalized to the much more convenient one, that is “intuitive representation” of Laplace hyperfunctions introduced in [9]. Let us briefly recall this representation. Let $U$ be an open subset in $\mathbb{D}_M$, and let $\Gamma$ be an $\mathbb{R}_+$-conic connected open subset in $\mathbb{D}_E$.

### Definition 4.3.1 ([9] Definition 4.8). An open subset $W \subset \mathbb{D}_E$ is said to be an infinitesimal wedge of type $U \hat{\times} \sqrt{-1}\Gamma$ if and only if for any $\mathbb{R}_+$-conic open subset $\Gamma'$ properly contained in $\Gamma$ there exists an open neighborhood $O \subset \mathbb{D}_E$ of $U$ such that

$$(U \hat{\times} \sqrt{-1}\Gamma') \cap O \subset W.$$
where $\Gamma$ runs through all the $\mathbb{R}_+$-conic connected open subsets in $M$ (in particular, $\Gamma$ is non-empty).

Define the quotient vector space

$$\hat{H}^n(\mathcal{O}_{\mathcal{D}_E}^{\exp}(\mathcal{W}(U))) := \left( \bigoplus_{W \in \mathcal{W}(U)} \mathcal{O}_{\mathcal{D}_E}^{\exp}(W) \right) / \mathcal{R},$$

where $\mathcal{R}$ is a $\mathbb{C}$-vector space generated by elements

$$f \oplus (-f|_{W_2}) \in \mathcal{O}_{\mathcal{D}_E}^{\exp}(W_1) \oplus \mathcal{O}_{\mathcal{D}_E}^{\exp}(W_2)$$

for any $W_2 \subset W_1$ in $\mathcal{W}(U)$ and any $f \in \mathcal{O}_{\mathcal{D}_E}^{\exp}(W_1)$.

**Theorem 4.3.3** ([9] Theorem 4.9). Let $U$ be an open cone in $\mathcal{D}_M$ such that $^\wedge(U \cap M) = U$. Then there exists a family $b_W = \{b_W\}_{W \in \mathcal{W}(U)}$ of morphisms $b_W : \mathcal{O}_{\mathcal{D}_E}^{\exp}(W) \to \mathcal{B}_{\mathcal{D}_M}^{\exp}(U)$ ($W \in \mathcal{W}(U)$) which satisfies

$$b_{W_1}(f) = b_{W_2}(f|_{W_2}) \quad \text{in } \mathcal{B}_{\mathcal{D}_M}^{\exp}(U)$$

for any $W_2 \subset W_1$ in $\mathcal{W}(U)$ and any $f \in \mathcal{O}_{\mathcal{D}_E}^{\exp}(W_1)$. Furthermore the induced morphism

$$b_W : \hat{H}^n(\mathcal{O}_{\mathcal{D}_E}^{\exp}(\mathcal{W}(U))) \to \mathcal{B}_{\mathcal{D}_M}^{\exp}(U)$$

becomes an isomorphism.

**Remark 4.3.4.** If $W \in \mathcal{W}(U)$ is cohomologically trivial, that is, it satisfies the condition A2. given in Subsection 5.1, then $b_W$ coincide with the boundary value map functorially constructed in Subsection 5.1 (see also Subsection 3.1 in [9], where the boundary value map was constructed in a functorial way).

Now let us consider the problem under the situation given in the previous subsection, that is, open subsets $U$ and $V$, coverings $\mathcal{S}$ and $\mathcal{S}'$, and vectors $\{\eta_k\}$ are those ones already given in the previous subsection. Additionally we also assume $^\wedge(U \cap M) = U$.

Then there exists the canonical isomorphism

$$\iota_{IC} : \hat{H}^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{O}_{\mathcal{D}_E}^{\exp})) \to \hat{H}^n(\mathcal{O}_{\mathcal{D}_E}^{\exp}(\mathcal{W}(U)))$$

which is defined by

$$\bigoplus_{\alpha \in \Lambda_{n+1}} \mathcal{O}_{\mathcal{D}_E}^{\exp}(S_\alpha) \ni [(f_\alpha)_\alpha] \mapsto (-1)^n \bigoplus_{\alpha} \text{sgn}(\alpha)f_\alpha \in \hat{H}^n(\mathcal{O}_{\mathcal{D}_E}^{\exp}(\mathcal{W}(U))),$$

where, for $\alpha = (\alpha_0, \cdots, \alpha_n)$, the $\text{sgn}(\alpha)$ is 1 if the vectors $\eta_{\alpha_0}, \eta_{\alpha_1}, \cdots, \eta_{\alpha_n-1}$ form a positive frame in $M$ and it is $-1$ otherwise.

The following lemma follows from the construction of $b_W$ (see Definition 3.14 and Theorem 3.15 [9]).

**Lemma 4.3.5.** The morphism $\iota_{IC}$ makes the following diagram commutative:

$$\begin{array}{ccc}
\hat{H}^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{O}_{\mathcal{D}_E}^{\exp})) & \xrightarrow{\iota_{IC}} & \hat{H}^n(\mathcal{O}_{\mathcal{D}_E}^{\exp}(\mathcal{W}(U))) \\
\sim & & b_W \\
& & \mathcal{B}_{\mathcal{D}_M}^{\exp}(U)
\end{array}$$

where all the morphism are isomorphic.
5 Boundary value map in $\mathbb{D}_E$

One of the important features in hyperfunction theory is a boundary value map, by which we can regard a holomorphic function of exponential type on an wedge along $\mathbb{D}_M$ as a Laplace hyperfunction. We construct, in this section, the boundary value map in the framework of the relative Čech Dolbeault cohomology.

Let $U$ be an open subset in $\mathbb{D}_M$ and $V$ an open subset in $\mathbb{D}_E$ such that $V \cap \mathbb{D}_M = U$. Let $\Omega$ be an open subset in $\mathbb{D}_E$.

5.1 Functorial construction

We first construct the boundary value map in a functorial way. For an open subset $W$ in $\mathbb{D}_E$ and a complex $F$ of sheaves on $W$, we define its dual on $W$ by

$$D_W(F) := R\text{Hom}_C(W, C)$$

Note that, for a complex $F$ of sheaves on $\mathbb{D}_E$, we have $D_{\mathbb{D}_E}(F)|_W = D_W(F|_W)$. We assume:

A1. $U \subset \Omega$.

A2. $\Omega$ is cohomologically trivial in $V$, that is,

$$D_{\mathbb{D}_E}(\mathbb{C}|_\Omega)|_V \simeq \mathbb{C}|_V,$$  
$$D_{\mathbb{D}_E}(\mathbb{C}|_\Omega)|_V \simeq \mathbb{C}|_V.$$

Through this subsection, we always assume conditions A1. and A2. Following Schapira’s construction (see Section 11.5 in [4]) of a boundary value morphism, we can construct the corresponding one for a Laplace hyperfunction as follows: Let $j_V : V \to \mathbb{D}_E$ be the canonical inclusion. By the assumption, we have the canonical morphism on $V$

$$j_V^{-1}(\mathbb{C}|_\Omega) \to j_V^{-1}(\mathbb{C}|_{\mathbb{D}_M}).$$

It follows from the assumption that we have

$$D_V(j_V^{-1}(\mathbb{C}|_\Omega)) \simeq j_V^{-1}(\mathbb{C}|_\Omega),$$  
$$D_V(j_V^{-1}(\mathbb{C}|_{\mathbb{D}_M})) \simeq j_V^{-1}(\mathbb{C}|_{\mathbb{D}_M} \otimes or_{\mathbb{D}_M/\mathbb{D}_E}[-n]).$$

Hence, applying the functor $D_V(\bullet)$ to the above morphism, we obtain the canonical morphism

$$j_V^{-1}(\mathbb{C}|_{\mathbb{D}_M} \otimes or_{\mathbb{D}_M/\mathbb{D}_E}[-n]) \to j_V^{-1}(\mathbb{C}|_\Omega).$$

Now applying the functor $R\text{Hom}_{\mathbb{C}^V}(\bullet, j_V^{-1}(\mathbb{E}^\text{exp}_{\mathbb{D}_E}))$ to the above morphism and taking the 0-th cohomology groups, we have obtained the boundary value map

$$b_\Omega : \mathbb{E}^\text{exp}_{\mathbb{D}_E}(\Omega \cap V) \to \mathbb{E}^\text{exp}_{\mathbb{D}_M}(U).$$

5.2 Čech Dolbeault construction of a boundary value map

The construction of a boundary value map for Laplace hyperfunctions in the framework of the relative Čech Dolbeault cohomology is the almost same as that for hyperfunctions done in the paper [1]. First recall the coverings

$$\mathcal{V}_U = \{V_0, V_1\}, \quad \mathcal{V}_U' = \{V_0\}$$
of $V$ and $V \setminus U$, where $V_0 = V \setminus U$, $V_1 = V$ and $V_{01} = V_0 \cap V_1$. We now construct the boundary value morphism

$$b_\Omega : \mathcal{O}^{\exp}_{\mathbb{D}_E} (\Omega) \longrightarrow \mathcal{H}^n (C(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(0, \bullet)}_{\mathbb{D}_E})) \otimes \mathbb{Z}_\mathcal{D}_M (U) \text{ or } \mathcal{D}(\mathbb{D}_E)_M (U)$$

in the framework of the relative Čech Dolbeault cohomology.

Let us first recall the morphism of complexes

$$\rho : C(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(\bullet)}_{\mathbb{D}_E}) \rightarrow C(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(0, \bullet)}_{\mathbb{D}_E}),$$

which is defined by the projection to the space of anti-holomorphic forms, that is,

$$C^k(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(\bullet)}_{\mathbb{D}_E}) \ni \sum_{|I| + |J| = k} f_{I,J} dz^I \wedge d\bar{z}^J \mapsto \sum_{|J| = k} f_{\emptyset,J} d\bar{z}^J \in C^k(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(0, \bullet)}_{\mathbb{D}_E}).$$

Then we have

**Lemma 5.2.1.** The following diagram commutes:

$$\begin{array}{ccc}
\mathbb{R} \Gamma_U (V; \mathcal{C}_{\mathbb{D}_E}) & \longrightarrow & \mathbb{R} \Gamma_U (V; \mathcal{O}^{\exp}_{\mathbb{D}_E}) \\
\downarrow & & \downarrow \\
C(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(\bullet)}_{\mathbb{D}_E}) & \longrightarrow & C(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(0, \bullet)}_{\mathbb{D}_E})
\end{array}$$

where the top horizontal arrow is the morphism associated with the canonical sheaf morphism $\mathcal{C}_{\mathbb{D}_E} \rightarrow \mathcal{O}^{\exp}_{\mathbb{D}_E}$.

Let us take a section $1 \in \mathcal{H}^n_U (V; \mathcal{Z}_{\mathbb{D}_E})$ such that, for each $x \in U$, the stalk $1_x$ of $1$ at $x$ generates $\mathcal{H}^n_{\mathcal{D}_M} (\mathcal{Z}_{\mathbb{D}_E})_x$ as a $\mathbb{Z}$-module. Note that we have, in each connected component of $U$, two choices of such a $1$, i.e., either $1$ or $-1$. Then the canonical sheaf morphism $\mathcal{Z}_{\mathbb{D}_E} \rightarrow \mathcal{C}_{\mathbb{D}_E}$ induces the injective morphism

$$\mathcal{H}^n_U (V; \mathcal{Z}_{\mathbb{D}_E}) \rightarrow \mathcal{H}^n_U (V; \mathcal{C}_{\mathbb{D}_E}).$$

Note that we still denote by $1$ the image in $\mathcal{H}^n_U (V; \mathcal{C}_{\mathbb{D}_E})$ of $1$ by this morphism.

Now we assume the following conditions to $\Omega$.

**B1.** The canonical inclusion $(V \setminus \Omega) \setminus \mathcal{D}_M \hookrightarrow (V \setminus \Omega)$ gives a homotopical equivalence.

The following lemma can be proved in the same way as that in Lemma 7.10 in [1].

**Lemma 5.2.2.** Assume the conditions A1 and B1. Then there exists $\tau = (\tau_1, \tau_{01}) \in C^n(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(\bullet)}_{\mathbb{D}_E})$ which satisfies the following conditions:

1. $D\tau = 0$ and $[\tau] = 1$ in $\mathcal{H}^n (C(\mathcal{V}_U, \mathcal{V}_{U^\prime})(\mathcal{Q}^{(\bullet)}_{\mathbb{D}_E}))$.
2. $\text{supp}_{V_0} (\tau_{01}) \subset \Omega$ and $\text{supp}_{V_1} (\tau_1) \subset \Omega$. 

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Now we assume the conditions A1 and B1, and let $\tau = (\tau_1, \tau_0)$ be the one given in the above Lemma. Then we can define the morphism

$$b_\Omega : D^E_\Omega(\Omega) \rightarrow H^n(C(V_U, V_U')(D^E_\Omega(\Omega))) \otimes_{D^M(U)} or_{D^M/D^E(U)}.$$

(5.1)

by

$$b_\Omega(f) = [f \rho(\tau)] \otimes 1 \quad (f \in D^E_\Omega(\Omega)).$$

(5.2)

**Lemma 5.2.3.** The above $b_\Omega$ is well-defined.

To avoid a higher jet as an $\Omega$, we also introduce the following condition

**B2.** For any point $x \in D_M$, there exist an open neighborhood $W \subset D_E$ of $x$ and a non-empty open cone $\Gamma \subset M$ such that

$$((W \cap M) \times \sqrt{-1}\Gamma) \cap W \subset \Omega.$$

Note that the condition B2 implies A1. We also introduced the localized version of the condition B1.

**B1'.** For any point $x \in D_M$, there exist a family $\{V_\lambda\}_{\lambda \in \Lambda}$ of fundamental open neighborhoods of $x$ in $V$, for which the canonical inclusion $(V_\lambda \setminus \Omega) \setminus D_M \hookrightarrow (V_\lambda \setminus \Omega)$ gives a homotopical equivalence.

The following theorem can be shown in the same way as that in Appendix A. in [1].

**Theorem 5.2.4 (Theorem A.2 [1]).** Assume the conditions A2, B1, B1' and B2. Then the boundary value morphism constructed in the functorial way and the one in this subsection coincide. To be more precise, the following diagram commutes:

$$\begin{array}{ccc}
D^E_\Omega(\Omega) & \xrightarrow{b_\Omega} & D^M(U) \\
\downarrow{b_\Omega} & & \downarrow \quad \\
H^n(C(V_U, V_U')(D^E_\Omega(\Omega))) \otimes_{D^M(U)} or_{D^M/D^E(U)} & & \\
\end{array}$$

Furthermore, for any $W \in W(U)$, we can find $\tilde{W} \in W(U)$ such that $\tilde{W} \subset W$ and it satisfies the conditions given in the above theorem. Hence, by Remark 4.3.4 we have the following corollary.

**Corollary 5.2.5.** For any $\Omega \in W(U)$ satisfying the condition A2, we have the commutative diagram below:

$$\begin{array}{ccc}
D^E_\Omega(\Omega) & \xrightarrow{b_\Omega} & D^M(U) \\
\downarrow{b_\Omega} & & \downarrow \quad \\
\hat{H}^n(D^E_\Omega(W(U))) & & \\
\end{array}$$

where $b_\Omega$ is given by Theorem 4.3.3 and the down right arrow is just the embedding $D^E_\Omega(\Omega) \ni f \mapsto f \in \hat{H}^n(D^E_\Omega(W(U)))$. 

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Now we give a concrete construction of \( \tau \) in a specific case.

**Example 5.2.6.** Let \( U = \mathbb{D}_M \) and \( V \subset \mathbb{D}_E \) be an open neighborhood of \( U \). Let \( \eta_0, \ldots, \eta_\ell \) be a family of unit vectors in \( M^* \) such that for any \( k + 1 \) choices \( \eta_{\alpha_0}, \ldots, \eta_{\alpha_k} \) of vectors which are linearly dependent, the cone

\[
\mathbb{R}_+ \eta_{\alpha_0} + \mathbb{R}_+ \eta_{\alpha_1} + \cdots + \mathbb{R}_+ \eta_{\alpha_k}
\]

should contain a line (i.e., it becomes a non-proper cone). Furthermore, we also assume that

\[
\sum_{k=0}^\ell \mathbb{R}_+ \eta_k = M^*.
\]

Note that a typical family of such vectors is, for example,

- \( n \) linearly independent unit vectors \( \xi_0, \ldots, \xi_{n-1} \) with \( \xi_\alpha = -(\xi_0 + \cdots + \xi_{n-1})/|\xi_0 + \cdots + \xi_{n-1}| \).
- a family of \( 2n \) unit vectors \( (\pm 1_0, 0, \cdots, 0), \ldots, (0, 0, \cdots, 0, \pm 1) \).

Let \( \Gamma_k (k = 0, \cdots, \ell) \) be open subsets in \( M \) defined by

\[
\Gamma_k = \{ y \in M; \langle y, \eta_k \rangle > 0 \}.
\]

Then, by the conditions for \( \{ \eta_k \} \), for any \( \alpha = (\alpha_0, \cdots, \alpha_k) \) with \( \eta_{\alpha_0}, \cdots, \eta_{\alpha_k} \) being linearly dependent (which is always satisfied if \( k \geq n \)), we have

\[
\Gamma_{\alpha_0} \cap \Gamma_{\alpha_2} \cap \cdots \cap \Gamma_{\alpha_k} = \emptyset.
\]

Further we also have

\[
\bigcup_{k=0,\ldots,\ell} \Gamma_k = M \setminus \{0\}.
\]

Let \( H_k (k = 0, \cdots, \ell) \) be an \( \mathbb{R}_+ \)-conic open convex subset in \( M \) satisfying the conditions below:

1. \( H_k \subset \Gamma_k \ (k = 0, 1, \cdots, \ell) \).
2. \( H_0 \cup H_1 \cup \cdots \cup H_\ell = M \setminus \{0\} \).

Then we choose \( \ell + 1 \) sections \( \varphi_0, \ldots, \varphi_\ell \) in \( \mathcal{D}_{\mathbb{D}_E}(V \setminus U) \) which satisfies

1. Set \( S_k := (U \wedge \sqrt{-1}H_k) \cap V \). Then \( \text{supp}(\varphi_k) \subset S_k \) holds for \( k = 0, \ldots, \ell \).
2. \( \varphi_0 + \varphi_1 + \cdots + \varphi_\ell = 1 \) on \( V \setminus U \).

Set

\[
\Lambda_{s+1}^{k+1} = \{ \alpha = (\alpha_0, \cdots, \alpha_k) \subset \{0, 1, \cdots, \ell + 1\}; \alpha_0 < \alpha_1 < \cdots < \alpha_k = \ell + 1 \}
\]

and set \( S_{\ell+1} = V \). For \( \alpha = (\alpha_0, \cdots, \alpha_k) \in \Lambda_{s+1}^{k+1} \), we define

\[
S_\alpha = S_{\alpha_0} \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}
\]
as usual and for \( \alpha = (\alpha_0, \cdots, \alpha_n) \in \Lambda^n_{\ast+1} \), we also define

\[
\text{sgn}(\alpha) = \begin{cases} +1 & (\xi_{\alpha_0}, \cdots, \xi_{\alpha_{n-1}} \text{ is a positive frame in } M^*), \\ -1 & \text{(otherwise)}. \end{cases}
\]

Now for \( \alpha = (\alpha_0, \cdots, \alpha_n) \in \Lambda^n_{\ast+1} \), we define

\[
\tau^\alpha_{01} = (-1)^n(n - 1)! \text{sgn}(\alpha)\chi_{S_\alpha} d\varphi_{\alpha_0} \wedge \cdots \wedge d\varphi_{\alpha_{n-2}},
\]

where \( \chi_Z \) is the characteristic function of the set \( Z \). We can see the following facts by the same reasoning as that of Example 7.14 in [1].

1. \( \tau^\alpha := (0, \tau^\alpha_{01}) \) belongs to \( C^n(\mathcal{V}_U, \mathcal{V}_U')((\mathcal{Q}_{DE}^{(\bullet)}) \).

2. \( D\tau^n = 0 \) and \( [\tau^n] = 1 \) in \( H^n(C(\mathcal{V}_U, \mathcal{V}_U')((\mathcal{Q}_{DE}^{(\bullet)})) \). Here we choose \( 1 \) so that it gives the standard positive orientation of \( M \).

3. \( \text{supp}_{V \setminus U}(\tau^\alpha_{01}) \subset S_\alpha \). This follows from the fact that, on \( S_\alpha \), we have \( \varphi_{\alpha_0} + \cdots + \varphi_{\alpha_{n-1}} = 1 \).

Hence this \( \tau^\alpha \) satisfies all the desired properties described in Lemma 5.2.2 when \( \Omega = S_\alpha \). Note that we have

\[
\rho(\tau^\alpha) = (0, (-1)^n(n - 1)! \text{sgn}(\alpha)\chi_{S_\alpha} \bar{\partial}\varphi_{\alpha_0} \wedge \cdots \wedge \bar{\partial}\varphi_{\alpha_{n-2}}).
\]

In particular, for \( f \in \mathcal{E}_{DE}^{\exp}(\Omega) \), we have

\[
b_\Omega(f) = [f \rho(\tau^\alpha)] = [(0, (-1)^n(n - 1)! \text{sgn}(\alpha)f(z)\chi_{S_\alpha} \bar{\partial}\varphi_{\alpha_0} \wedge \cdots \wedge \bar{\partial}\varphi_{\alpha_{n-2}})].
\]

In what follows, we additionally assume that \( V \) is 1-regular and \( V \cap E \) is Stein. We define the coverings \( \mathcal{S} \) and \( \mathcal{S}' \) of \( (V, V \setminus U) \) by

\[
\mathcal{S} = \{S_0, \cdots, S_{\ell+1}\}, \quad \mathcal{S}' = \{S_0, \cdots, S_{\ell}\},
\]

where

\[
S_k = (U \wedge \sqrt{-1}H_{k+1}) \cap V \quad (k = 0, 1, \cdots, \ell)
\]

and \( S_{\ell+1} = V \). We also define the coverings

\[
\mathcal{V}_U = \{V \setminus U, V\}, \quad \mathcal{V}_U' = \{V \setminus U\}.
\]

Let us consider the diagram whose morphisms are all isomorphic:

\[
H^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{E}_{DE}^{\exp})) \xrightarrow{\alpha_1^n} H^n(C(\mathcal{S}, \mathcal{S}')((\mathcal{Q}_{DE}^{(0, \bullet)}))) \xleftarrow{\alpha_2^n} H^n(C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{Q}_{DE}^{(0, \bullet)})).
\]

Let \( f \in \mathcal{E}_{DE}^{\exp}(S_\alpha) \). Then it follows from Lemma A.5 [1] that, by repeated applications of the well-known Wiel procedure, we can find

\[
((\alpha_2^n)^{-1} \circ \alpha_1^n)([f]) = [(0, (n - 1)! f(z)\chi_{S_\alpha} \bar{\partial}\varphi_{\alpha_0} \wedge \cdots \wedge \bar{\partial}\varphi_{\alpha_{n-2}})] = (-1)^n\text{sgn}(\alpha)b_{S_\alpha}(f).
\]
Hence we have obtained the following commutative diagram:

\[
\begin{array}{ccc}
\hat{H}^{n}(\mathcal{E}_{\mathcal{D}_{E}}^{\text{exp}}(W(U))) & \xrightarrow{b} & \hat{H}^{n}(C(\mathcal{V}_{U}, \mathcal{V}_{U}')(\mathcal{D}_{\mathcal{D}_{E}}^{0}(\bullet))) \\
\iota_{IC} \downarrow & & \downarrow (\alpha_{2} )^{-1}\circ \alpha_{1} \\
H^{n}(\mathcal{E}_{\mathcal{D}_{E}}^{\text{exp}}(\mathcal{W}(U))) & & H^{n}(C(\mathcal{V}_{U}, \mathcal{V}_{U}')(\mathcal{D}_{\mathcal{D}_{E}}^{0}(\bullet)))
\end{array}
\]

(5.6)

where all morphisms are isomorphic and \( b : \hat{H}^{n}(\mathcal{E}_{\mathcal{D}_{E}}^{\text{exp}}(W(U))) \to H^{n}(C(\mathcal{V}_{U}, \mathcal{V}_{U}')(\mathcal{D}_{\mathcal{D}_{E}}^{0}(\bullet))) \) is defined by the boundary value morphisms

\[\mathcal{E}_{\mathcal{D}_{E}}^{\text{exp}}(W) \ni f \mapsto b_{W}(f) \in H^{n}(C(\mathcal{V}_{U}, \mathcal{V}_{U}')(\mathcal{D}_{\mathcal{D}_{E}}^{0}(\bullet))) \quad (W \in \mathcal{W}(U)).\]

Furthermore, \( \iota_{IC} \) is given by

\[\bigoplus_{\alpha \in \mathcal{A}_{\mathcal{E}}^{\mathcal{D}_{E}}(S_{\alpha})} [((f_{\alpha})_{\alpha}) \mapsto (-1)^{n} \bigoplus_{\alpha} \text{sgn}(\alpha) f_{\alpha} \in \hat{H}^{n}(\mathcal{E}_{\mathcal{D}_{E}}^{\text{exp}}(W(U)))\]

as in Lemma 4.3.5.

6 Laplace transformation \( \mathcal{L} \) for hyperfunctions

6.1 Preparation

Let \((z_{1} = x_{1} + \sqrt{-1}y_{1}, \ldots, z_{n} = x_{n} + \sqrt{-1}y_{n})\) be a coordinate system of \( E \). Hereafter, we fix the orientation of \( M \) and \( E \) so that \( \left\{ \frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{2}}, \ldots, \frac{\partial}{\partial x_{n}} \right\} \) gives the positive orientation on \( M \), and \( \left\{ \frac{\partial}{\partial y_{1}}, \ldots, \frac{\partial}{\partial y_{n}}, \frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{n}} \right\} \) give the one on \( E \).

Remark 6.1.1. The above orientation of \( E \) is different from the usual standard orientation of \( \mathbb{C}^{n} \), where \( \left\{ \frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial y_{1}}, \frac{\partial}{\partial x_{2}}, \frac{\partial}{\partial y_{2}}, \ldots, \frac{\partial}{\partial x_{n}}, \frac{\partial}{\partial y_{n}} \right\} \) is taken to be a positive frame.

We say that the boundary \( \partial D \) of a subset \( D \subset \mathbb{D}_{E} \) is (partially) smooth if \( \partial D \cap E \) is (partially) smooth. Note that, when the boundary \( \partial D \) is smooth, the orientation of \( \partial D \) is determined so that the outward-pointing normal vector of \( \partial D \) followed by a positive frame of \( \partial D \) determines the positive orientation of \( E \).

Let \( h : E^{*}_{\infty} \to (-\infty) \cup \mathbb{R} \) be an upper semi-continuous function, and let \( W \) be an open subset in \( \mathbb{D}_{E^{*}} \) and \( f \) a holomorphic function on \( W \cap E^{*} \).

Definition 6.1.2. We say that \( f \) is of infra-\( h \)-exponential type (at \( \infty \)) on \( W \) if, for any compact set \( K \subset W \) and any \( \epsilon > 0 \), there exists \( C > 0 \) such that

\[e^{\epsilon |\zeta|} h(\pi_{E^{*}_{\infty}}(\zeta)) |f(\zeta)| \leq C e^{\epsilon |\zeta|} \quad (\zeta \in K \cap (E^{*} \setminus \{0\})) ,\]

where \( \pi_{E^{*}_{\infty}} : E^{*} \setminus \{0\} \to (E^{*} \setminus \{0\})/\mathbb{R}_{+} = E^{*}_{\infty} \) is the canonical projection, i.e., \( \pi_{E^{*}_{\infty}}(\zeta) = \zeta/|\zeta| \), and we set \( e^{-\infty} = 0 \) for convenience. In particular, we say that \( f \) is simply called of infra-exponential type if \( h \equiv 0 \).
Define a sheaf on \( E_\infty^* \) by, for an open subset \( \Omega \) in \( E_\infty^* \),
\[
\mathcal{O}^{\text{inf}}_{E_\infty^*}(\Omega) := \lim_{\rightarrow W} \{ f \in \mathcal{O}(W \cap E^*); f \text{ is of infra-exponential type on } W \},
\]
where \( W \) runs through open neighborhoods of \( \Omega \) in \( D_{E^*} \). Then the family \( \{ \mathcal{O}^{\text{inf}}_{E_\infty^*}(\Omega) \}_\Omega \) forms the sheaf \( \mathcal{O}^{\text{inf}}_{E_\infty^*} \) on \( E_\infty^* \). Similarly we define the sheaf \( \mathcal{O}^{\text{inf-h}}_{E_\infty^*} \) on \( E_\infty^* \) by, for an open subset \( \Omega \subset E_\infty^* \),
\[
\mathcal{O}^{\text{inf-h}}_{E_\infty^*}(\Omega) := \lim_{\rightarrow W} \{ f \in \mathcal{O}(W \cap E^*); f \text{ is of infra-h-exponential type on } W \},
\]
where \( W \) runs through open neighborhoods of \( \Omega \) in \( D_{E^*} \).

We also introduces the sheaf \( \mathcal{A}^{\text{exp}}_{\mathcal{D}_M} := \mathcal{A}^{\text{exp}}_{\mathcal{D}_E}|_{\mathcal{D}_M} \) of real analytic functions of exponential type and the one \( \mathcal{V}^{\text{exp}}_{\mathcal{D}_M} \) of real analytic volumes of exponential type. The latter sheaf is defined by
\[
\mathcal{V}^{\text{exp}}_{\mathcal{D}_M} = \mathcal{A}^{\text{exp}}_{\mathcal{D}_E}|_{\mathcal{D}_M} \otimes_{\mathcal{O}_{\mathcal{D}_M}} \mathcal{O}_{\mathcal{D}_M},
\]
where \( \mathcal{O}_{\mathcal{D}_M} := (j_M)_* \mathcal{O}_M \) with \( j_M : M \hookrightarrow \mathcal{D}_M \) being the canonical inclusion. Note that we can also define the orientation sheaf \( \mathcal{O}_E \) on \( E \) by \( (j_E)_* \mathcal{O}_E \) with the canonical inclusion \( j_E : E \hookrightarrow \mathcal{D}_E \), for which we have the canonical isomorphism
\[
\mathcal{O}_{\mathcal{D}_M/\mathcal{D}_E} \otimes \mathcal{O}_{\mathcal{D}_M} \simeq \mathcal{O}_{\mathcal{D}_E}|_{\mathcal{D}_M}. \tag{6.1}
\]

Let \( K \) be a subset in \( \mathcal{D}_E \). Then we define the support function \( h_K(\zeta) : E_\infty^* \rightarrow \{ \pm \infty \} \cup \mathbb{R} \) by
\[
h_K(\zeta) = \begin{cases} +\infty & \text{if } K \cap E \text{ is empty,} \\ \inf_{z \in K \cap E} \text{Re} \langle z, \zeta \rangle & \text{otherwise,} \end{cases} \tag{6.2}
\]
where we identify \( \zeta \in E_\infty^* \) with a unit vector in \( E^* \). Note that if \( K \) is properly contained in a half space of \( \mathcal{D}_E \) with direction \( \zeta_0 \in E_\infty^* \) (which is equivalently saying \( \zeta_0 \in N_{pc}^*(K) \)) and if \( K \cap E \) is non-empty, then the subset
\[
\overline{K} \cap \{ z \in E; \text{Re} \langle z, \zeta_0 \rangle = h_K(\zeta_0) \}
\]
is a compact set in \( E \). The following lemma easily follows from the definition.

**Lemma 6.1.3.** Let \( K \subset \mathcal{D}_E \) with \( N_{pc}^*(K) \neq \emptyset \). Then \( N_{pc}^*(K) \) is a connected open subset in \( E_\infty^* \). Furthermore, the function \( h_K(\zeta) \) is upper semi-continuous on \( E_\infty^* \), in particular, it is continuous on \( N_{pc}^*(K) \) and \( h_K(\zeta) > -\infty \) there.

**Remark 6.1.4.** In the above lemma, if \( K \subset \mathcal{D}_M \), then we have
\[
N_{pc}^*(K) = \begin{cases} \varpi_{M_\infty}^{-1}(N_{pc}^*(K) \cap M_\infty) & (\overline{K} \cap M_\infty \neq \emptyset), \\ \varpi_{M_\infty}^{-1}(N_{pc}^*(K) \cap M_\infty) \cup \sqrt{-1}M_\infty = E_\infty^* & (\overline{K} \cap M_\infty = \emptyset) \end{cases}
\]
and \( h_K(\zeta) \) is continuous on \( N_{pc}^*(K) \cup \sqrt{-1}M_\infty^* \) (for the definition of \( \varpi_{M^*} \), see (23)).
6.2 Laplace transformation

Let $K$ be a closed subset in $\mathbb{D}_M$ such that $N^*_p(K)$ is non-empty. Take $\xi_0 \in N^*_p(K) \cap M^*$ and an open neighborhood $V$ of $K$ in $\mathbb{D}_E$. Set $U := \mathbb{D}_M \cap V$ and coverings

$$\mathcal{V}_K := \{V_0 := V \setminus K, V_1 := V\}, \quad \mathcal{V}_K' := \{V_0\}.$$  

(6.3)

In what follow, we assume that $U$ and $V$ are connected for simplicity. Note that we have

$$\Gamma_K(U; \mathcal{B}_\mathbb{D}_M^{exp} \otimes \mathcal{D}_\mathbb{D}_M^{exp} \mathcal{Y}_\mathbb{D}_M^{exp}) \simeq H^n(C(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{Z}_\mathbb{D}_E^{(n,\bullet)})) \otimes \text{or}_{\mathbb{D}_M/\mathbb{D}_E}(U) \otimes \text{or}_{\mathbb{D}_M}(U).$$

Let

$$u = \tilde{u} \otimes \text{or}_{\mathbb{D}_M/\mathbb{D}_E}(U) \otimes \text{or}_{\mathbb{D}_M}(U)$$

where $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M} \in \Gamma_K(U; \mathcal{B}_\mathbb{D}_M^{exp} \otimes \mathcal{D}_\mathbb{D}_M^{exp} \mathcal{Y}_\mathbb{D}_M^{exp})$, and let $\nu = (\nu_1, \nu_{01}) \in C^n(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{Z}_\mathbb{D}_E^{(n,\bullet)})$ be a representative of $\tilde{u}$, i.e., $\tilde{u} = [\nu]$.

Here we may assume that $a_{\mathbb{D}_M/\mathbb{D}_E}$ and $a_{\mathbb{D}_M}$ are generators in each orientation sheaf. Hence, through the canonical isomorphism (6.1), the section $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M}$ determines the orientation of $E$. We perform the subsequent integrations under this orientation.

Remark 6.2.1. If $\text{or}_{\mathbb{D}_M/\mathbb{D}_E}$ gives the orientation so that $\{dy_1, \ldots, dy_n\}$ is a positive frame and if $\text{or}_M$ gives the orientation so that $\{dx_1, \ldots, dx_n\}$ is a positive one. Then $\{dy_1, \ldots, dy_n, dx_1, \ldots, dx_n\}$ becomes a positive frame under the orientation determined by $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M}$.

Definition 6.2.2. The Laplace transform of $u$ with a Čech Dolbeault representative $\nu = (\nu_1, \nu_{01}) \in C^n(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{Z}_\mathbb{D}_E^{(n,\bullet)})$ is defined by

$$L_D(u)(\zeta) := \int_{D \cap \overline{E}} e^{-\zeta} \nu_1 - \int_{\partial D \cap \overline{E}} e^{-\zeta} \nu_{01},$$

(6.4)

where $D$ is a contractible open subset in $\mathbb{D}_E$ with a good boundary (see the remark below) such that $K \subset D \subset \overline{D} \subset V$ and it is properly contained in a half space of $\mathbb{D}_E$ with direction $\xi_0$.

Note that the orientation of $D$ and $\partial D$ is taken in the usual way, that is, the orientation of $D$ is that of $E$, and the one of $\partial D$ is determined so that the outward pointing normal vector of $D$ and a positive frame of $\partial D$ form that of $E$.

Remark 6.2.3. Recall the definition of a piecewise $C^\infty$ submanifold with boundary in $E$ introduced in Definition 5.10 [10]. We use, through the paper, a slightly weaker definition of a piecewise $C^\infty$ submanifold with boundary so that a cone whose boundary is smooth except for the vertex in $E$ also belongs to such a class of submanifolds.

Let $Z' \subset Z$ be subsets in $E$. $Z$ is called a partially $C^\infty$ submanifold with boundary $Z'$ if there exist a triangulation $(K, h)$ of $E$ and subcomplexes $L' \subset L$ of $K$ which satisfy the conditions below:

1. $h(|L|) = Z$ and $h(|L'|) = Z'$ and $|L|$ is a PL submanifold of $|K|$ with boundary $|L'|$.

2. Any simplex $\sigma$ of $L$ satisfies the following conditions, where we set $\ell_\sigma = \dim \sigma$ and $H_\sigma$ denotes the $\ell_\sigma$ dimensional affine space containing $\sigma$:
(a) \( h|_\sigma : \sigma \to E \) is a Lipschitz mapping.

(b) \( h|_{\sigma^*} : \sigma \setminus \partial\sigma \to E \) is a \( C^\infty \) mapping of rank \( \ell \), and it extends to a \( C^\infty \) mapping of rank \( \ell \) on an open neighborhood of \( \sigma^* \) in \( H_{\sigma} \). Here we set
\[
\sigma^* = \sigma \setminus \bigcup_{\tau < \sigma, \dim \tau \leq \ell - 2} \tau.
\]

Note that the Stokes formula holds for a partially \( C^\infty \) submanifolds with boundary in the same way as those in the case of “a piecewise \( C^\infty \) manifold with boundary” ([11]) and “a standard chain” introduced by Whitney (see Chapter III [13]).

Now let us define that a open subset \( D \subset \mathbb{D}_E \) has a good boundary: \( D \) has a good boundary if there exist \( C > 0 \) and \( R > 0 \) such that the following conditions are satisfied.

- For any \( r > R \), the sets \( \overline{D} \cap B_r \) and \( \partial D \cap B_r \) are compact partially \( C^\infty \) submanifolds with boundaries \( \partial(\overline{D} \cap B_r) \) and \( \partial D \cap \partial B_r \) in \( E \), respectively. Furthermore,
\[
|\partial(\overline{D} \cap B_r)|_{2n-1} < C r^{2n-1}, \quad |\partial D \cap \partial B_r|_{2n-2} < C r^{2n-2}
\]
hold. Here \( |\cdot|_d \) denotes the \( d \) dimensional Hausdorff measure of a Borel set in \( E \) and
\[
B_r = \{ x + \sqrt{-1} y \in E; |x| \leq r \}.
\]

In what follows, we always assume a chain \( D \) of Laplace integral has a good boundary.

Before ending this remark, we explain how to make a chain \( D \) with a good boundary of the integration of Laplace transform in a convenient way. Let \( K \subset \mathbb{D}_M \) be a non-empty regular closed cone. Now let us take a 1-regular open cone \( W \subset \mathbb{D}_M \) with the vertex \( a \in M \) such that \( K \subset W \) and \( W \cap M \) has a smooth boundary except for the vertex and choose a smooth function \( \rho = (\rho_1, \rho_2, \ldots, \rho_n) : W' \cap M \to \mathbb{R}^n \) \((W' \subset \mathbb{D}_M \) is an open neighborhood of \( \overline{W} \)) satisfying

1. \( \rho(x) \in \mathbb{R}^n_+ \ (x \in W \cap M) \) and there exists \( C > 0 \) such that \( \sum_{1 \leq i \leq n} |\frac{\partial \rho}{\partial x_i}(x)| \leq C \) on \( W \cap M \).

2. there exist \( \epsilon > 0 \) and an open neighborhood \( \widetilde{W} \) of \( K \cap M_\infty \) in \( W \) such that \( |\rho(x)| \geq \epsilon |x| \ (x \in \widetilde{W} \cap M) \) holds.

Then we define
\[
D(W, \rho) = \{ x + \sqrt{-1} y \in E; x \in W, |y_k| < \rho_k(x) \ (k = 1, \ldots, n) \}.
\]
Clearly \( D(W, \rho) \) is an open neighborhood of \( K \) in \( \mathbb{D}_E \). Conversely, for any open neighborhood \( V \) of \( K \) in \( \mathbb{D}_E \), we have \( K \subset \overline{D(W, \rho)} \subset V \) by suitable choices of \( W \) and \( \rho \). Note that \( D(W, \rho) \) has a good boundary in the above sense. As a special case of \( D(W, \rho) \), if we take
\[
W = \{ (y_1, \ldots, y_n) \in M; y_k > 0 \ (k = 1, 2, \ldots, n) \}
\]
and
\[
\rho(x) = \sigma x
\]
for some \( \sigma > 0 \), then \( D(W, \rho) \cap E \) becomes the product of one dimensional cone in \( \mathbb{C} \).
\[
\bigwedge \{ |y_1| < \sigma x_1 \} \times \cdots \times \{ |y_n| < \sigma x_n \}
\]
Remark 6.2.4. In what follows, we write $\int_D e^{-z\xi}\nu_1$ instead of $\int_{D\cap E} e^{-z\xi}\nu_1$, etc., for simplicity.

Set $z = x + \sqrt{-1}y$ and $\zeta = \xi + \sqrt{-1}\eta$. We may assume $\xi_0 = (1, 0, \ldots, 0)$, and we write $x = (x_1, x')$ and $\xi = (\xi_1, \xi')$. Then there exist $b \in \mathbb{R}$ and $\kappa > 0$ such that

$$D \subset \{z = x + \sqrt{-1}y; |x'| + |y| \leq \kappa(x_1 - b)\}.$$ 

Furthermore, it follows from the definition of $\nu$ that there exist $H > 0$ and $C \geq 0$ such that $|\nu_{01}| \leq Ce^{Hz_1}$ on a neighborhood of $\partial D$ and $|\nu_1| \leq Ce^{Hz_1}$ on a neighborhood of $\overline{D}$. Hence, if $z \in D$, we have

$$|e^{-z\xi}\nu_1| \leq Ce^{-x_1\xi_1+y_1+Hz_1} \leq Ce^{-x_1\xi_1+\kappa(|\xi'|+|\eta|)(x_1-b)+Hz_1},$$

from which the integral $\int_D e^{-z\xi}\nu_1$ converges if $\xi_1$ is sufficiently large. We also have the same conclusion for $\int_{\partial D} e^{-z\xi}\nu_01$.

Lemma 6.2.5. $\mathcal{L}_D(u)$ is holomorphic at points $\zeta = R\xi_0$ if $R > 0$ is sufficiently large. Furthermore, $\mathcal{L}_D(u)$ is independent of the choices of a representative $\nu$ of $u$ and $D$ of the integral. Here we identify $\xi_0$ with the corresponding unit vector in $M^*$.

Proof. The convergence of the integration is already shown above. We first show $\mathcal{L}_D(\vec{\varphi}) = 0$ for $\tau = (\tau_1, \tau_{01}) \in C^{n-1}(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{D}_E^{(\cdot, \ast)})$.

$$\mathcal{L}_D(\vec{\varphi}) = \int_D e^{-z\xi}\overline{\varphi}d\tau_1 - \int_{\partial D} e^{-z\xi}(\tau_1 - \overline{\varphi}\tau_{01}) = \int_D d(\overline{\varphi}d\tau_1) - \int_{\partial D} e^{-z\xi}(\tau_1 - d\tau_{01}) = (\int_D d(\overline{\varphi}d\tau_1) - \int_{\partial D} e^{-z\xi}d\tau_1 + \int_{\partial D} d(\overline{\varphi}d\tau_{01}) = 0,$$

where the last equality comes from the Stokes formula. Hence the Laplace integral does not depend on the choices of representative of $u$.

Next we will show the Laplace integral is independent of the choices of $D$. Let $\varphi \in \mathcal{D}_{\mathcal{D}_{E}}(\mathbb{D}_E)$ which satisfies

1. supp($\varphi$) $\subset D$,

2. $\varphi = 1$ on $W \cap E$ for an open neighborhood $W$ of $K$ in $\mathbb{D}_E$,

and define

$$\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_{01}) = (\varphi\nu_1 + \partial\varphi \wedge \nu_01, \varphi\nu_{01}).$$

Since we have

$$\nu - \tilde{\nu} = \overline{\varphi} ((1 - \varphi)\nu_{01}, 0),$$

representatives $\nu$ and $\tilde{\nu}$ give the same cohomology class. Hence, as the support of $\tilde{\nu}$ is contained in $D$, we have obtained

$$\mathcal{L}_D(\nu) = \mathcal{L}_D(\tilde{\nu}) = \int_D e^{-z\xi}\tilde{\nu}_1 = \int_D e^{-z\xi}\tilde{\nu}_1 = \int_E e^{-z\xi} (\varphi\nu_1 + \partial\varphi \wedge \nu_{01}). \quad (6.5)$$

The last expression does not depend on $D$. This show the claim. □
Due to the above lemma, in what follows, we write \( \mathcal{L}(\bullet) \) instead of \( \mathcal{L}_D(\bullet) \). Then, thanks to the expression (6.5) and the integration by parts, we get:

**Corollary 6.2.6.** For \( u \in \Gamma_K(U; \mathcal{F}_D^{\exp} \otimes \mathcal{V}_D^{\exp}) \) and \( v \in \Gamma_K(U; \mathcal{F}_D^{\exp}) \), we have the formulas

\[
\frac{\partial}{\partial \zeta_k} \mathcal{L}(u) = \mathcal{L}(-x_k u), \quad \zeta_k \mathcal{L}(v \, dx \otimes a_D) = \mathcal{L} \left( \frac{\partial v}{\partial x_k} \, dx \otimes a_D \right) \quad (k = 1, 2, \ldots, n).
\]

Note that, by the definition of \( \varpi_{M^*} \) given in (2.3), we have, for \( \xi_0 \in M^*_\infty \),

\[\varpi_{M^*_\infty}^{-1}(\xi_0) = \{ \xi_0 + \sqrt{-1} \eta \in E^*; \eta \in M^* \}/\mathbb{R}_+ \subset E^*_\infty \setminus \sqrt{-1} M^*_\infty.\]

**Proposition 6.2.7.** Assume \( K \cap M \) is non-empty. For any \( a \in K \cap \{ x \in M; \langle x, \xi_0 \rangle = h_K(\xi_0) \} \), any \( \epsilon > 0 \) and any compact subset \( L \) in \( \varpi_{M^*_\infty}^{-1}(\xi_0) \), there exist \( C > 0 \) and an open neighborhood \( W \subset \mathbb{D}_{E^*} \) of \( L \) such that

\[|e^{i \zeta} \mathcal{L}(u)(\zeta)| \leq Ce^{\eta |\zeta|} \quad (\zeta \in W \cap E^*).\]

**Proof.** Take a point \( \zeta_0 = (\xi_0 + \sqrt{-1} \eta_0)/|\xi_0 + \sqrt{-1} \eta_0| \in E^*_\infty \). In what follows, we sometimes identify a point in \( E^*_\infty \) with a unit vector in \( E^* \). Denote by \( B_\delta(\zeta_0) \) an open ball with radius \( \delta > 0 \) and center at \( \zeta_0 \).

Since \( K \) is properly contained in a half space of \( \mathbb{D}_M \) with direction \( \xi_0 \), there exist \( \delta_1 > 0 \), \( \sigma_1 > 0 \), a relatively compact open neighborhood \( O \subset M \) of \( K \cap \{ x \in M; \langle x - a, \xi_0 \rangle = 0 \} \) and an \( \mathbb{R}_+ \)-conic proper closed set \( G \subset \mathbb{D}_M \) such that

\[K \subset O \cup (a + \text{int}(G)),\]

\[O \subset \{ x \in M; |\langle x - a, \xi \rangle| < \epsilon/2 \} \quad (\xi \in B_{\delta_1}(\zeta_0) \cap M^*_\infty),\]

and

\[\langle x, \xi \rangle \geq \sigma_1 |x| \quad (x \in G \cap M, \xi \in B_{\delta_1}(\zeta_0) \cap M^*_\infty).\]

For \( \delta_2 > 0 \), define open subsets \( D_O \) in \( E \) and \( D_G \) in \( \mathbb{D}_E \) by

\[D_O = \left\{ z = x + \sqrt{-1} y \in E; x \in O, \ |y| < \frac{\epsilon}{2 \max\{1, 2|\eta_0|\}} \right\},\]

\[D_G = \left\{ z = x + \sqrt{-1} y \in E; x \in a + \text{int}(G), \ |y| < \delta_2 \text{dist}(x, M \setminus (a + G)) \right\} \cdot\]

If we take \( \delta_2 > 0 \) sufficiently small, there exists \( \sigma_2 > 0 \) such that

\[\text{Re} \langle z - a, t \zeta \rangle \geq \sigma_2 t |z - a| \quad (t \in \mathbb{R}_+, \ z \in D_G \cap E, \ \zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty)\]

holds. Note that we also have

\[|\text{Re} \langle z - a, t \zeta \rangle| < \epsilon t \quad (t \in \mathbb{R}_+, \ z \in D_O, \ \zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty).\]

As in the proof of Lemma 6.2.5, we take a \( \varphi \in \mathcal{A}_{\mathbb{D}_E}(\mathbb{D}_E) \) and set

\[\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_{01}) = (\varphi \nu_1 + \tilde{\varphi} \varphi \wedge \nu_{01}, \varphi \nu_{01}).\]

Then we have

\[\mathcal{L}(u)(\zeta) = \int_E e^{-z \zeta} \tilde{\nu}_1.\]
Furthermore, by taking $\varphi$ suitably, we may assume
\[ \text{supp} \tilde{\nu}_1 \subset D_O \cup D_G. \]

Therefore, for $t \in \mathbb{R}_+$ and $\zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty$, we get
\[
|e^{ta\zeta}L(u)(t\zeta)| = \left| \int_E e^{-t(z-a)\zeta} \tilde{\nu}_1 \right| \leq \left| \int_{D_O} e^{-t(z-a)\zeta} \tilde{\nu}_1 \right| + \left| \int_{D_G} e^{-t(z-a)\zeta} \tilde{\nu}_1 \right|.
\]

Then, it is easy to see that there exists a positive constant $C_1 > 0$ such that, for any $\zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty$ and $t \in \mathbb{R}_+$, we have
\[
\left| \int_{D_O} e^{-t(z-a)\zeta} \tilde{\nu}_1 \right| \leq C_1 e^{t}. \]

Furthermore, since there exist a constant $C_2, H > 0$ such that
\[
|\tilde{\nu}_1| \leq C_2 e^{H|z-a|} \quad (z \in D_G \cap E),
\]
we get, for $\zeta \in B_{\delta_2}(\zeta_0) \cap E^*_\infty$ and $t \in \mathbb{R}_+$,
\[
\left| \int_{D_G} e^{-t(z-a)\zeta} \tilde{\nu}_1 \right| \leq C_2 \int_{D_G} e^{(-\sigma_2t+H)|z-a|} d\mu,
\]
where $d\mu$ denotes the Lebesgue measure on $E$. Hence the last integral converges if $t$ is sufficiently large, which completes the proof. \qed

we have the following corollary as a consequence of the proposition:

**Corollary 6.2.8.** Assume $K \cap M$ is non-empty. Then we have $L(u) \in \hat{\Theta}^{\inf-h_K}_{E^*_\infty}(N_{pc}^*(K))$.

**Proof.** First assume that $K \cap M_\infty \neq \emptyset$. In this case, we see that $N_{pc}^*(K) = \overline{\omega}(N_{pc}^*(K) \cap M_\infty)$ and that $h_K(\zeta)$ is upper semi-continuous. Hence the result comes from the above proposition.

Next assume that $K \cap M_\infty = \emptyset$, which implies $K$ is a compact set in $M$. Note that $N_{pc}^*(K) = E^*_\infty$ holds. Then we can take a relatively compact open subset in $E$ as $D$ which is sufficiently close to $K$, and the result immediately follows. \qed

Let $G \neq \emptyset$ be an $\mathbb{R}_+$-conic proper closed subset in $M$ and $a \in M$. We denote by $G^o \subset E^*$ the dual cone of $G$ in $E^*$, that is,
\[ G^o := \{ \zeta \in E^*; \text{Re} \langle \zeta, x \rangle \geq 0 \text{ for any } x \in G \}. \]

Assume $K = \{a\} + G \subset \mathbb{D}_M$. Since $N_{pc}^*(K) = \overline{\text{int} G^o} \cap E^*_\infty$ and $h_K(\zeta) = \text{Re} a\zeta$ on $N_{pc}^*(K)$ hold (here we write $a\zeta = \langle a, \zeta \rangle$), the corollary immediately implies the following theorem.

**Theorem 6.2.9.** Under the above situation, $e^{a\zeta}L(u)(\zeta)$ belongs to $\hat{\Theta}^{\inf}_{E^*_\infty}(\overline{\text{int} G^o} \cap E^*_\infty)$.  

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6.3 Several equivalent definitions of Laplace transform

We give, in this subsection, several equivalent definitions of Laplace transform previously defined for various expressions of a Laplace hyperfunction. The following proposition is quite important to obtain a good Čech representation of a Laplace hyperfunction with compact support. Recall the definition of a regular closed subset given in Definition 2.2.4 and the one of an infinitesimal wedge in Definition 4.3.1. Recall also that we use the word “1-regular at $\infty$” to indicate the notion “regular at $\infty$” introduced in Definition 3.4 [2].

**Proposition 6.3.1.** Let $K \neq \emptyset$ be a regular closed cone in $\mathbb{D}_M$ and let $\eta \in M^*_\infty$. Then we can find an open subset $S \subset \mathbb{D}_E \setminus K$ such that

1. $S$ is an infinitesimal wedge of type $M \hat{x} \sqrt{-1} \Gamma$, where $\Gamma = \{ y \in M; \langle y, \eta \rangle > 0 \}$.
2. $S \cap E$ is a Stein open subset and $S$ is 1-regular at $\infty$.
3. $S$ is an open neighborhood of $\mathbb{D}_M \setminus K$ in $\mathbb{D}_E$.

**Proof.** The proof is the almost same as that of Theorem 4.10 [9]. For reader’s convenience, we briefly explain how to construct the desired $S$. We may assume that the vertex of $S$ is the origin and $\eta = (1, 0, \cdots, 0)$. Let $\sigma$ be a sufficiently small positive number and set, for $\xi \in M$,

$$\varphi_\xi(z) = (z_1 - (\xi_1 + \sqrt{-1}\sigma|\xi|))^2 + (z_2 - \xi_2)^2 + \cdots + (z_n - \xi_n)^2 + \sigma^2|\xi|^2.$$ 

Note that

$$\text{Re} \varphi_\xi(z) > 0 \iff (y_1 - \sigma|\xi|)^2 + y_2^2 + \cdots + y_n^2 < \sigma^2|\xi|^2 + |x - \xi|^2.$$ 

Then, by the same reasoning as in the proof of Theorem 4.10 [9], the set

$$O = \text{Int} \left( \bigcap_{\xi \in K} \{ z \in E; \text{Re} \varphi_\xi(z) > 0 \} \right)$$

is an $\mathbb{R}_+$-conic Stein open subset, and hence, $\hat{O}$ is 1-regular at $\infty$. Define $S$ by modifying $O$ near the origin:

$$S = \hat{\text{Int}} \left( \left( \bigcap_{\xi \in K, |\xi| \geq 1} \{ z \in E; \text{Re} \varphi_\xi(z) > 0 \} \right) \bigcap \left( \bigcap_{\xi \in K, |\xi| < 1} \{ z \in E; \text{Re} \psi_\xi(z) > 0 \} \right) \right),$$

where

$$\psi_\xi(z) = (z_1 - (\xi_1 + \sqrt{-1}\sigma))^2 + (z_2 - \xi_2)^2 + \cdots + (z_n - \xi_n)^2 + \sigma^2.$$ 

Since $\hat{O}$ and $S$ coincide in an open neighborhood of $E_\infty$, the $S$ is still 1-regular at $\infty$ and $S \cap E$ is a Stein open subset. We can easily confirm that $S$ satisfies the rest of required properties in the proposition.
6.3.1 Laplace transform for Čech representation

We give here several examples to compute the Laplace transform of a Čech representative of a Laplace hyperfunction. The following lemma is needed for subsequent examples.

**Lemma 6.3.2.** Let $K \subset \mathbb{D}_M$ be a non-empty regular closed cone and $\Omega \subset \mathbb{D}_M$ an open neighborhood of $K$. Then there exist an open subset $U' \subset \mathbb{D}_M$ with smooth boundary satisfying $K \subset U' \subset \overline{U'} \subset \Omega$ and a smooth function $\varrho_{U'} : M \to \mathbb{R}$ such that

1. $\sum_{k=1}^n \left| \frac{\partial \varrho_{U'}}{\partial x_k} \right|$ is bounded on $M$.
2. there exists an open subset $T \subset \mathbb{D}_M$ with $\partial U' \subset T$ satisfying $\varrho_{U'}(x) = \text{dist}(x, M \setminus U') \quad (x \in \overline{U'} \cap T) \cap M).

Furthermore, there exists $C > 1$ such that

$$C^{-1} \text{dist}(x, M \setminus U') \leq \varrho_{U'}(x) \leq C \text{dist}(X, M \setminus U') \quad (x \in \overline{U'} \cap M).$$

**Proof.** By making an open cone with smooth boundary except for the vertex $a \in M$ (see Proposition 2.10 and Corollary 2.11 [8]) and then by modifying the cone near $a$, we can find the open subset $U' \subset \mathbb{D}_M$ with smooth boundary such that $K \subset U' \subset \overline{U'} \subset \Omega$ which also satisfies the additional condition: there exists $R > 0$ such that

for any $c > 0$ and $x \in U'$ with $|x - a| > R$ and $|c(x - a)| > R \Rightarrow c(x - a) + a \in U'$.

Since $\partial U'$ is smooth and $U'$ satisfies the above additional condition, the function

$$\tau(x) = \begin{cases} \text{dist}(x, M \setminus U') & (x \in \overline{U'} \cap M), \\ -\text{dist}(x, U') & (x \in M \setminus U') \end{cases}$$

is smooth on $T \cap M$ for an open neighborhood $T \subset \mathbb{D}_M$ of $\partial U'$. Taking open sets in $\mathbb{D}_M$

$$\overline{U'} \cap T \subset U_1 \subset U_1 \subset U_2 \subset \overline{U_2} \subset U_3 \subset \overline{U_3} \subset U'.
$$

Let $\varphi_1$ is a $C^\infty$ function on $M$ with $0 \leq \varphi_1 \leq 1$ and bounded derivatives on $M$ such that $\varphi_1(x) = 0$ in an open neighborhood of $(\overline{U'} \setminus T) \cap M$ and $\varphi_1(x) = 1$ in an open neighborhood of $(\overline{U'} \setminus U_1) \cap M$. In the same way, let $\varphi_2$ is a $C^\infty$ function on $M$ with $0 \leq \varphi_2 \leq 1$ and bounded derivatives on $M$ such that $\varphi_2(x) = 0$ in an open neighborhood of $(\overline{U'} \setminus U_3) \cap M$ and $\varphi_1(x) = 1$ in an open neighborhood of $\overline{U_2} \cap M$. Then, for $\epsilon > 0$, we define

$$\varrho_{U'}(x) = \varphi_1(x) \tau(x) + \varphi_2(x) (\epsilon - n \psi(x) / \epsilon) \ast \text{dist}(x, X \setminus U'))$$

where $\psi(x)$ is a $C^\infty$ function on $M$ with $0 \leq \psi \leq 1$, $\text{supp}(\psi) \subset \{|x| < 1\}$ and $\int_M \varphi(x) dx = 1$. If we take $\epsilon > 0$ sufficiently small, then $\varrho_{U'}(x)$ satisfies required conditions.\[\square\]

**Example 6.3.3.** Let $K \neq \emptyset$ be a closed cone in $\mathbb{D}_M$ which is regular and proper, and let $\eta_0, \ldots, \eta_{n-1}$ be linearly independent vectors in $M^*$ so that $\{\eta_0, \ldots, \eta_{n-1}\}$ forms a positive frame of $M^*$. Set $\eta_n := -(\eta_0 + \cdots + \eta_{n-1}) \in M^*$.

Then, by applying Proposition 6.3.1 to the vector $\eta_k$, we obtain $S_k$ satisfying the conditions in the proposition with $\eta = \eta_k \ (k = 0, \ldots, n)$. Since $S_0 \cup \cdots \cup S_n \cup \mathbb{D}_M$ is an open neighborhood of $\mathbb{D}_M$, it follows from Theorem 4.10 [9] that we can take an open neighborhood $S \subset \mathbb{D}_E$ of $\mathbb{D}_M$ such that
1. \( S \cap E \) is a Stein open subset and it is 1-regular at \( \infty \).

2. \( \{S_0 \cap S, S_1 \cap S, \ldots, S_n \cap S\} \) is a covering of the set \( S \backslash K \).

For simplicity, we set \( S_{n+1} := S \). Let \( \Lambda = \{0,1,2,\ldots,n+1\} \) and set, for any \( \alpha = (\alpha_0, \ldots, \alpha_k) \in \Lambda^{k+1} \),
\[
S_\alpha := S_{\alpha_0} \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}.
\]
We already defined the covering \((\mathcal{V}_K, \mathcal{V}_K')\) of \((S, S \backslash K)\) in (6.3) with \( V = S \). We also define another covering of \((S, S \backslash K)\) by
\[
S := \{S_0, S_1, \ldots, S_{n+1}\}, \quad S' := \{S_0, \ldots, S_n\}.
\]
Then, by the theories of the relative \( \check{\text{C}} \)ech and the relative \( \check{\text{C}} \)ech Dolbeault cohomologies, we have
\[
H^n_K(S; \mathcal{O}_{\overline{\mathbb{D}}_E}^{\exp(n)}) \simeq H^n(C(S, S') (\mathcal{O}_{\overline{\mathbb{D}}_E}^{\exp(n)})) \simeq H^n(C(S, S') (\mathcal{O}_{\overline{\mathbb{D}}_E}^{(n, *)})) \simeq H^n(C(\mathcal{V}_K, \mathcal{V}_K') (\mathcal{O}_{\overline{\mathbb{D}}_E}^{(n, *)})).
\]
Let \( \Lambda^{k+1}_* \) be the subset in \( \Lambda^{k+1} \) consisting of \( \alpha = (\alpha_0, \ldots, \alpha_k) \) with
\[
\alpha_0 < \alpha_1 < \cdots < \alpha_k = n + 1.
\]
We take proper open subset \( U' \subset \mathbb{D}_M \) with \( K \subset U' \) and \( g_{U'}(x) \) given in Lemma 6.3.2. Assume \( \epsilon > 0 \) is sufficiently small. Then we define closed subsets in \( E \) by
\[
\sigma_{n+1} := \bigcap_{0 \leq k \leq n} \{z = x + \sqrt{-1}y \in E; x \in U' \cap E, \langle y, \eta_k \rangle < \epsilon g_{U'}(x) \} \cap E
\]
and, for \( 0 \leq k \leq n \),
\[
\sigma_k := \{z = x + \sqrt{-1}y \in E; x \in U' \cap E, \langle y, \eta_k \rangle > \epsilon g_{U'}(x) \} \cap E.
\]
We may assume that, by taking \( \epsilon > 0 \) sufficiently small,
\[
\overline{\sigma_{n+1} \cap \sigma_k} \subset S_k \quad (k = 0, 1, \ldots, n+1) \tag{6.6}
\]
holds in \( \mathbb{D}_E \). For any \( \alpha = (\alpha_0, \ldots, \alpha_k) \in \Lambda^{k+1}_* \), we also define
\[
\sigma_{\alpha} := \sigma_{\alpha_0} \cap \sigma_{\alpha_1} \cap \cdots \cap \sigma_{\alpha_k}.
\]
Here we determine the orientation of \( \sigma_{\alpha} \) in the following way:

1. \( \sigma_{n+1} \) has the same orientation as the one of \( E \).

2. For \( k > 0 \) and \( \alpha \in \Lambda^{k+1}_* \), the vectors \((-\eta_{\alpha_0}), (-\eta_{\alpha_1}), \ldots, (-\eta_{\alpha_{k-1}})\) followed by the the positive frame of \( \sigma_\alpha \) form a positive frame of \( E \). Note that \( \alpha_k = n + 1 \) as \( \alpha \in \Lambda^{k+1}_* \).

**Remark 6.3.4.** The above 2. is equivalently saying that, for a point \( x \) in the smooth part of \( \sigma_\alpha \) and taking points \( x_j \in \text{int}(\sigma_\alpha) \) \( (j = 0, 1, \cdots, k) \) sufficiently close to \( x \), the positive frame of \( \sigma_\alpha \) at \( x \) is determined so that the vectors \( \overrightarrow{xx_0}, \overrightarrow{xx_1}, \cdots, \overrightarrow{xx_{k-1}} \) and the positive frame of \( \sigma_\alpha \) at \( x \) form that of \( E \) at \( x \).
Then, for any \( \alpha \in \Lambda^{k+1} \) which contains the index \( n+1 \), we can define \( \sigma_\alpha \) with orientation by extending the above definition in the alternative way, that is, \( \sigma_\alpha = 0 \) if the same index appears twice in \( \alpha \), and otherwise

\[
\sigma_\alpha = \text{sgn}(\alpha, \tilde{\alpha}) \sigma_{\tilde{\alpha}},
\]

where \( \tilde{\alpha} \in \Lambda^{k+1}_s \) is obtained by a permutation of \( \alpha \) and \( \text{sgn}(\alpha, \tilde{\alpha}) \) denotes the signature of this permutation.

Now let us consider the Čech Dolbeault complex \( C^*(\mathcal{S}, \mathcal{S}')(\mathcal{D}(n, \bullet)_E) \) for the covering \((\mathcal{S}, \mathcal{S}')\). Then, for any

\[
\omega = \{ \omega_\alpha \}_{0 \leq k \leq n, \alpha \in \Lambda^{k+1}_s} \in \bigoplus_{0 \leq k \leq n} C^k(\mathcal{S}, \mathcal{S}'; \mathcal{D}(n, n-k)_E) = C^*(\mathcal{S}, \mathcal{S}')(\mathcal{D}(n, \bullet)_E),
\]

we define the Laplace transform of \( \omega \) by

\[
I(\omega) := \sum_{0 \leq k \leq n} \sum_{\alpha \in \Lambda^{k+1}_s} \int e^{-z_\alpha} \omega_\alpha.
\]

By our convention of orientation of \( \sigma_\alpha \) and the fact

\[
\dim \mathbb{R} \sigma_{n+1} \cap \{ z = x + \sqrt{-1}y \in E; \ x \in \partial U' \} < n,
\]

we have, for any \( \alpha \in \Lambda^{k+1}_s \),

\[
\partial \sigma_\alpha = \sum_{0 \leq j \leq n+1} \sigma_{[\alpha j]},
\]

where \([\alpha j]\) denotes a sequence in \( \Lambda^{k+2} \) whose last element is \( j \).

Hence it follows from Stokes’s formula that we obtain

\[
I(\overline{\partial} \omega) = 0 \quad (\omega \in C^{n-1}(\mathcal{S}, \mathcal{S}')(\mathcal{D}(n, \bullet)_E)).
\]

As a matter of fact, for \( \omega_\alpha \in \mathcal{D}(n, n-k-1)_E(S_\alpha) \) with \( \alpha \in \Lambda^{k+1}_s \), we have

\[
e^{-z_\alpha} \overline{\partial} \omega_\alpha = (-1)^k \overline{\partial}(e^{-z_\alpha} \omega_\alpha) + \delta(e^{-z_\alpha} \omega_\alpha) = (-1)^k d(e^{-z_\alpha} \omega_\alpha) + \delta(e^{-z_\alpha} \omega_\alpha),
\]

and thus, by noticing \( \sigma_{[j a]} = (-1)^{k+1} \sigma_{[a j]} \),

\[
I(\overline{\partial} \omega_\alpha) = (-1)^k \int_{\sigma_\alpha} d(e^{-z_\alpha} \omega_\alpha) + \sum_{j=0}^{n+1} \int_{\sigma_{[j a]}} e^{-z_\alpha} \omega_\alpha
\]

\[
= (-1)^k \sum_{j=0}^{n+1} \int_{\sigma_{[j a]}} e^{-z_\alpha} \omega_\alpha + \sum_{j=0}^{n+1} \int_{\sigma_{[j a]}} e^{-z_\alpha} \omega_\alpha = 0.
\]

Summing up, if \( \omega \) and \( \omega' \) in \( C^n(\mathcal{S}, \mathcal{S}')(\mathcal{D}(n, \bullet)_E) \) give the same cohomology class, we have

\[
I(\omega) = I(\omega').
\]

Now let us consider the canonical quasi-isomorphisms of complexes

\[
C(\mathcal{S}, \mathcal{S}')(\mathcal{D}(n, \bullet)_E^{\exp(n)}) \xrightarrow{\beta_1} C(\mathcal{S}, \mathcal{S}')(\mathcal{D}(n, \bullet)_E) \xrightarrow{\beta_2} C(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{D}(n, \bullet)_E),
\]

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where $\beta_1$ is induced from the resolution $\mathcal{O}_{\mathbb{D}_E}^{\exp,(n)} \to C(S, S')(\mathcal{D}_E^{(n, \bullet)})$ and $\beta_2$ is due to the fact that $(S, S')$ is a covering finer than $(V_K, V'_K)$. It is easy to see, for $\nu_2 \in C^n(V_K, V'_K)(\mathcal{D}_E^{(n, \bullet)})$ with $\partial \nu_2 = 0$,

$$\mathcal{L}([\nu_2]) = I(\beta_2(\nu_2)).$$

Let $\nu_1 = \{\nu_{1,\alpha}\}_{\alpha \in \Lambda_{n+1}} \in C^n(S, S')(\mathcal{D}_E^{\exp,(n)})$ with $\delta \nu_1 = 0$. If $\beta_1(\nu_1)$ and $\beta_2(\nu_2)$ give the same cohomology class in $H^n(C(S, S')(\mathcal{D}_E^{(n, \bullet)}))$, by the above reasoning, we get

$$I(\beta_1(\nu_1)) = I(\beta_2(\nu_2)) = \mathcal{L}([\nu_2]).$$

It follows from the definition of $I(\bullet)$ that we have

$$I(\beta_1(\nu_1)) = \sum_{\alpha \in \Lambda_{n+1}} \int_{\sigma_\alpha} e^{-zζ} \nu_{1,\alpha}.$$  

Furthermore, each integration can be rewritten to

$$\int_{\sigma_\alpha} e^{-zζ} \nu_{1,\alpha} = (-1)^n \text{sgn}(\det(\eta_{\alpha_0}, \ldots, \eta_{\alpha_{n-1}})) \int_{L_\alpha} e^{-zζ} \nu_{1,\alpha}, \quad (6.7)$$

where $L_\alpha$ is a real $n$-chain in $E$

$$L_\alpha = \{ z = x + \sqrt{-1}y \in E; x \in \overline{U'} \cap M, y = \rho_\alpha(x) \} \quad (6.8)$$

with a smooth function $\rho_\alpha : \overline{U'} \cap M \to M$ satisfying the conditions

1. $\rho_\alpha(x) = 0$ for $x \in \partial U' \cap M$,
2. $L_\alpha \subset S_\alpha$ in $\mathbb{D}_E$,
3. $\sum_{k=1}^n \left| \frac{\partial \rho_\alpha}{\partial x_k}(x) \right|$ is bounded on $\overline{U'} \cap M$,

and its orientation is the same as the one of $U'$.

Summing up, for a Čech representation $\{\nu_{1,\alpha}\}_{\alpha \in \Lambda_{n+1}}$ of a Laplace hyperfunction $u = [\tau]$, its Laplace transform is given by

$$\mathcal{L}(u) = (-1)^n \sum_{\alpha \in \Lambda_{n+1}} \text{sgn}(\det(\eta_{\alpha_0}, \ldots, \eta_{\alpha_{n-1}})) \int_{L_\alpha} e^{-zζ} \nu_{1,\alpha}. \quad (6.9)$$

**Remark 6.3.5.** In our settings, the last index of a covering is assigned to the one for an open neighborhood $S$ of $\mathbb{D}_M$, i.e., $S_{n+1} = S$. In usual hyperfunction theory, however, the first index 0 is assigned to it, i.e., $S_0 = S$. This is the reason why the factor $(-1)^n$ appeared in the above expression.

**Example 6.3.6.** Now we consider another useful example. Set

$$\Gamma_{+n} = \{ y = (y_1, \cdots, y_n) \in M; y_k > 0 \ (k = 1, 2, \cdots, n) \}$$
and \( K = \overline{\Gamma_{+n}} \) in \( \mathbb{D}_M \). Let \( S \subset \mathbb{D}_E \) be an open neighborhood of \( \mathbb{D}_M \) such that \( S \cap E \) is a Stein open subset and \( S \) is \( 1 \)-regular at \( \infty \). Define, for \( k = 0, 1, \cdots, n-1 \),

\[
S_k := \{ z = (z_1, z_2, \cdots, z_n) \in S; z_{k+1} \in \mathbb{C} \setminus \mathbb{R}_{\geq 0} \} \subset \mathbb{D}_E.
\]

Set \( S_n = S \). Then

\[
S = \{ S_0, S_1, \ldots, S_n \}, \quad S' = \{ S_0, \ldots, S_{n-1} \}
\]

are coverings of \( (S, S \setminus K) \).

Define the \( n \times n \) matrix \( B := (1 + \epsilon)I - \epsilon C \) for sufficiently small \( \epsilon > 0 \), where \( I \) is the identity matrix and \( C \) is the \( n \times n \) matrix with entries being all 1. We define the \( \mathbb{R} \)-linear transformation \( T \) on \( E = M \times \sqrt{-1}M \) by

\[
x + \sqrt{-1}y \in E \rightarrow Bx + \sqrt{-1}y \in E.
\]

Let \( \gamma \subset \mathbb{C} \) be the open subset defined by

\[
\gamma := \{ z = x + \sqrt{-1}y \in \mathbb{C}; |y| < \epsilon(x + \epsilon) \}.
\]

Then we introduce real \( 2n \)-dimensional chains in \( E \) by

\[
\sigma_n := \overline{T(\gamma \times \cdots \times \gamma)} \cap E,
\]

and, for \( k = 0, \ldots, n-1 \),

\[
\sigma_k := \overline{T(\mathbb{C} \times \cdots \times (\mathbb{C} \setminus \gamma) \times \cdots \times \mathbb{C})} \cap E.
\]

Note that \( \overline{\sigma_n} \) is a neighborhood of \( K \) in \( \mathbb{D}_E \). One should aware that, however, \( \gamma \times \gamma \times \cdots \times \gamma \) is not.

Set \( \Lambda = \{ 0, 1, \ldots, n \} \), and \( \Lambda_{k+1}^k \) is the subset of \( \Lambda^{k+1} \) consisting of an element \( (\alpha_0, \alpha_1, \cdots, \alpha_k) \) with

\[
\alpha_0 < \alpha_1 < \cdots < \alpha_k = n.
\]

Then, for any \( \alpha = (\alpha_0, \ldots, \alpha_k) \in \Lambda_{k+1}^k \), the orientation of \( \sigma_\alpha := \sigma_{\alpha_0} \cap \sigma_{\alpha_1} \cap \cdots \cap \sigma_{\alpha_k} \) is determined in the following way:

1. \( \sigma_n \) has the same orientation as the one of \( E \).

2. the outward-pointing normal vector of \( \sigma_{\alpha_0} \), that of \( \sigma_{\alpha_1} \), \cdots, that of \( \sigma_{\alpha_{k-1}} \) followed by the the positive frame of \( \sigma_\alpha \) form a positive frame of \( E \).

Note that, for any \( \alpha \in \Lambda^{k+1} \) which contains the index \( n \), we can define \( \sigma_\alpha \) with orientation by extending the above definition in the alternative way as did in the previous example.

For any \( \alpha \in \Lambda_{k+1}^k \), we have

\[
\partial \sigma_\alpha = \sum_{0 \leq j \leq n} \sigma_{[\alpha, j]},
\]

where \( [\alpha, j] \) is the sequence in \( \Lambda^{k+2} \) whose last element is \( j \). Therefore the rest of argument goes in the same way as in Example 6.3.3 and we finally obtain, for \( u = [\tau] \in \).
forms a positive frame of cone in $D_M$ of linearly independent vectors in $M^*$, for which the sequence $\eta_0, \eta_1, \cdots, \eta_{n-1}$ of vectors forms a positive frame of $M^*$.

Let us consider another kind of Čech covering: Let $\overrightarrow{\gamma}$ be the set consisting of $\alpha<n$ with $\eta = \eta_{k, \pm}$. Set $S_n = S$ and coverings

$$S := \{S_{0, \pm}, \cdots, S_{n-1, \pm}, S_n\}, \quad S' := \{S_{\pm}, \cdots, S_{n-1, \pm}\}.$$  

Then, we take open subsets $S$ and $S_{\pm}$ $(k = 0, 1, \cdots, n-1)$ in the same way as those in Example 6.3.3 by using Proposition 6.3.1 with $\eta = \eta_{k, \pm}$. Set $S_n = S$ and coverings

$$S := \{S_{0, \pm}, \cdots, S_{n-1, \pm}, S_n\}, \quad S' := \{S_{\pm}, \cdots, S_{n-1, \pm}\}.$$  

Let $\Lambda$ be the set consisting of “$n$” and pairs “$(i, e)$” with $i \in \{0, 1, \cdots, n-1\}$ and $e \in \{+, -\}$. We define the linear order $<$ on $\Lambda$ by:

(a) $\alpha < n$ for any $\alpha \in \Lambda \setminus \{n\}$.

(b) $(i, e_i) < (j, e_j)$ if $i < j$ or if $i = j$ and $e_i = +$ and $e_j = -$.

Let $\Lambda_{k+1}$ be the subset in $\Lambda_{k+1}^*$ consisting of $\alpha = (\alpha_0, \cdots, \alpha_k)$ with

$$\alpha_0 < \alpha_1 < \cdots < \alpha_k = n.$$  

Furthermore, let $\Lambda_{k+1}$ be the subset in $\Lambda_{k+1}^*$ consisting of

$$\alpha = ((i_0, e_0), \cdots, (i_{k-1}, e_{k-1}), n) \in \Lambda_{k+1}^*$$  

with $i_0 < i_1 < \cdots < i_{k-1}$. For $\alpha \in \Lambda_{k+1}^*$, the subset $S_\alpha$ is defined as usual, that is,

$$S_\alpha = S_{\alpha_0} \cap \cdots \cap S_{\alpha_k}.$$  

Note that, in this example, the open subset $S_\alpha$ is not necessarily empty for $\alpha \in \Lambda_{k+1}^* \setminus \Lambda_{k+1}^*$ with $k > n$.

We take a proper open subset $U' \subset D_M$ with $K \subset U'$ and $\varrho_U(x)$ given in Lemma 6.3.2. Assume $\epsilon > 0$ is sufficiently small. Then we define closed subsets in $E$ by

$$\sigma_n = \bigcap_{0 \leq k \leq n-1} \{z = x + \sqrt{-1} y \in E; x \in U' \cap E, |\langle y, \eta_k \rangle| < \epsilon \varrho_U(x)\} \cap E$$  

and, for $0 \leq k \leq n-1$,

$$\sigma_{(k, \pm)} = \{z = x + \sqrt{-1} y \in E; x \in U' \cap E, \pm \langle y, \eta_k \rangle > \epsilon \varrho_U(x)\} \cap E.$$
Note that $\sigma^n \cap \sigma_\alpha \subset S_\alpha$ holds for $\alpha \in \Lambda$ if $\epsilon$ is sufficiently small. Then, in the same way as in the previous example, we can define $\sigma_\alpha$ for $\alpha \in \Lambda^{k+1}_{**}$ and determine its orientation. For any
\[
\omega = \{\omega_\alpha\}_{0 \leq k \leq n, \alpha \in \Lambda^{k+1}_{**}} \in \bigoplus_{0 \leq k \leq n} C^k(S, S'; \mathcal{D}^{(a,n-k)}_{D_E}) = C^n(S, S')(\mathcal{D}^{(n,\bullet)}_{D_E}),
\]
we define the Laplace transform of $\omega$ by
\[
I(\omega) := \sum_{0 \leq k \leq n} \sum_{\alpha \in \Lambda^{k+1}_{**}} \int_{\sigma_\alpha} e^{-z\zeta} \omega_\alpha,
\]
for which one should aware that the sum ranges through indices only in $\Lambda^{k+1}_{**} \subset \Lambda^{k+1}_*$. We have, for any $\alpha = ((i_0, e_0), \cdots, (i_{k-1}, e_{k-1}), n) \in \Lambda^{k+1}_{**}$,
\[
\partial\sigma_\alpha = \sum_{j \notin \{i_0, \cdots, i_{k-1}, n\}, \epsilon = \pm} \sigma_{[\alpha(j, \epsilon)]},
\]
where $[\alpha(j, \epsilon)]$ is a sequence in $\Lambda^{k+2}$ whose last element is $(j, \epsilon)$. The important fact here is that $\partial\sigma_\alpha$ ($\alpha \in \Lambda^{k+1}_{**}$) does not contain any cell $\sigma_\beta$ with $\beta \in \Lambda^{k+2}_{**} \setminus \Lambda^{k+2}_{**}$. Hence, by Stokes’s formula, we still obtain
\[
I(\tilde{\partial}\omega) = 0 \quad (\omega \in C^{n-1}(S, S')(\mathcal{D}^{(n,\bullet)}_{D_E})).
\]
As a matter of fact, if $\alpha \in \Lambda^{k+1}_{*} \setminus \Lambda^{k+1}_{**}$, then $I(\tilde{\partial}\omega_\alpha) = 0$ for $\omega_\alpha \in \mathcal{D}^{(n,n-k-1)}_{D_E}(S_\alpha)$ because $\tilde{\partial}\omega_\alpha$ (resp. $\delta\omega_\alpha$) does not contain a non-zero term with an index in $\Lambda^{k+1}_{**}$ (resp. $\Lambda^{k+2}_{**}$). If $\alpha = ((i_0, e_0), \cdots, (i_{k-1}, e_{k-1}), n) \in \Lambda^{k+1}_{**}$, then we have for $\omega_\alpha \in \mathcal{D}^{(n,n-k-1)}_{D_E}(S_\alpha)$
\[
I(\tilde{\partial}\omega_\alpha) = (-1)^k\int_{\sigma_\alpha} d(e^{-z\zeta} \omega_\alpha) + \sum_{j \notin \{i_0, \cdots, i_{k-1}, n\}, \epsilon = \pm} \int_{\sigma_{[\alpha(j, \epsilon)]}} e^{-z\zeta} \omega_\alpha
\]
\[
= (-1)^k \sum_{j \notin \{i_0, \cdots, i_{k-1}, n\}, \epsilon = \pm} \int_{\sigma_{[\alpha(j, \epsilon)]}} e^{-z\zeta} \omega_\alpha + \sum_{j \notin \{i_0, \cdots, i_{k-1}, n\}, \epsilon = \pm} \int_{\sigma_{[\alpha(j, \epsilon)]}} e^{-z\zeta} \omega_\alpha = 0.
\]
The rest of argument is the same as the one in the previous example: For $u = [\tau] \in \Gamma_K(D_M; \mathcal{D}^{(n,\bullet)}_{D_M} \otimes \mathcal{D}^{(n,\bullet)}_{D_M})$ and its relative Čech representative
\[
\nu = \bigoplus_{\alpha \in \Lambda^{n+1}_{**}} \nu_\alpha \in C^n(S, S')(\mathcal{D}^{(n,\bullet)}_{D_E}) \quad \text{with } \delta\nu = 0,
\]
we obtain
\[
\mathcal{L}(u) = (-1)^n \sum_{\alpha \in \Lambda^{n+1}_{**}} \text{sgn}(\alpha) \int_{L_\alpha} e^{-z\zeta} \nu_\alpha. \quad (6.12)
\]
Here, for $\alpha = ((0, e_0), \cdots, (n-1, e_{n-1}), n) \in \Lambda^{n+1}_{**}$, we set $\text{sgn}(\alpha) = e_0^2 e_1 \cdots e_{n-1}$ and $L_\alpha$ is the real $n$-chain in $E$
\[
L_\alpha = \{z = x + \sqrt{-1} y \in E; \ x \in \overline{U'} \cap M, \ y = \rho_\alpha(x)\} \quad (6.13)
\]
with a smooth function $\rho_\alpha : \overline{U'} \cap M \to M$ satisfying the conditions
1. \( \rho_{\alpha}(x) = 0 \) for \( x \in \partial U' \cap M \),
2. \( \overline{L_{\alpha}} \subset S_{\alpha} \) in \( D_{E} \),
3. \( \sum_{k=1}^{n} \left| \frac{\partial \rho_{\alpha}(x)}{\partial x_k} \right| \) is bounded on \( \overline{U'} \cap M \),

and its orientation is the same as the one of \( U' \).

### 6.3.2 Laplace transform whose chain is of product type

Let us consider the Laplace transformation of a Laplace hyperfunction \( u \) whose support is contained in \( \Gamma_{+n} \subset D_{M} \). Here \( \Gamma_{+n} = \{(x_1, \ldots, x_n) \in M; x_k > 0 \ (k = 1, 2, \ldots, n)\} \). In this case, one can expect the path of the integration to be the product \( \gamma_1 \times \cdots \times \gamma_n \) of the one dimensional paths \( \gamma_k \). However, we cannot take such a path unless the support of \( u \) is contained in a cone strictly smaller than \( \Gamma_{+n} \). In this subsection, we show that a chain of product type can be taken as an integral path of the Laplace transformation if the condition \( \text{supp}(u) \setminus \{0\} \subset \Gamma_{+n} \) is satisfied.

Let \( K \neq \emptyset \) be a regular closed cone in \( D_{M} \) satisfying

\[
K \setminus \{0\} \subset \widehat{\Gamma_{+n}}. \tag{6.14}
\]

Let \( \epsilon > 0 \) and Let \( D_{k} \subset \mathbb{C} \) be an open subset satisfying that \( \widehat{D_{k}} \) has a good boundary (see Remark [6.2.3] for the definition of a good boundary), \( \mathbb{R}_{+} \subset D_{k} \) and

\[
D_{k} \subset \{z = x + \sqrt{-1}y \in \mathbb{C}; |y| < \epsilon(x + \epsilon)\}.
\]

Set

\[
D = D_{1} \times D_{2} \times \cdots \times D_{n} \subset E.
\]

One should aware that \( \widehat{D} \) is not an open neighborhood of \( \Gamma_{+n} \) in \( D_{E} \). However, since \( \widehat{D} \) becomes an open neighborhood of \( K \) in \( D_{E} \) because of (6.14), we can compute its Laplace transform by

\[
\mathcal{L}(u)(\zeta) := \int_{D} e^{-z\zeta} \nu_{1} - \int_{\partial D} e^{-z\zeta} \nu_{01}
\]

for a Laplace hyperfunction \( u = [\nu_{1}, \nu_{01}] \) \( ((\nu_{1}, \nu_{01}) \in C^{n}(V_{K}, V'_{K})(\mathcal{D}_{\mathbb{D}_{E}})) \) with support in \( K \).

Let \( \eta_{k, \pm} = (0, \ldots, \pm 1, \ldots, 0) \ (k = 0, \ldots, n - 1) \) be a unit vector whose \( (k + 1) \)-th element is \( \pm 1 \). Recall the definitions of \( \Lambda_{k+1}^{*} \) and \( \Lambda_{k+1}^{*+1} \) given in Example 6.3.7 and let us introduce open subsets \( S, S_{k, \pm} \) and the pair \((S', S')\) of coverings of \((S, S \setminus K)\) in the same way as those in Example 6.3.7. Set \( \sigma_{n} = \overline{D} \cap E \) and, for \( k = 0, \ldots, n - 1 \),

\[
\sigma_{k, \pm} = \{z = (z_1, \ldots, z_n) \in E; z_{k+1} \in \mathbb{C} \setminus D_{k+1}, \pm \text{Im} z_{k+1} \geq 0\} \cap E.
\]

Then, as we did in the example, we define the Laplace transform by

\[
I(\omega) := \sum_{0 \leq k \leq n} \sum_{\alpha \in \Lambda_{k+1}^{*}} \int_{\sigma_{\alpha}} e^{-z\zeta} \omega_{\alpha}
\]

for \( \omega = \{\omega_{\alpha}\}_{0 \leq k \leq n, \alpha \in \Lambda_{k+1}^{*}} \in \sum_{0 \leq k \leq n} C^{k}(S, S'; \mathcal{D}_{\mathbb{D}_{E}}^{(n-k)}) = C^{n}(S, S')(\mathcal{D}_{\mathbb{D}_{E}}^{(n, \bullet)}) \).
Note that we have, for \( \alpha = ((i_0, \epsilon_0), \ldots, (i_{k-1}, \epsilon_{k-1}), n) \in \Lambda_k^{k+1} \),
\[
\partial \sigma_{\alpha} = \sum_{j \notin \{i_0, \ldots, i_{k-1}, n\}, \epsilon = \pm} \sigma_{[\alpha (j, \epsilon)]} + \sum_{j \in \{i_0, \ldots, i_{k-1}, n\}, \epsilon = \pm} \sigma_{[\alpha (j, \epsilon)]}.
\]
Define \( \pi : \mathbb{C}^n \to \mathbb{C} \) to be \( \pi_j(z_1, \ldots, z_n) = z_j+1 \). Since \( \pi_j(\sigma_{j, \pm} \cap \sigma_{j, \mp} \cap \sigma_n) \) \((j = 0, 1, \ldots, n-1)\) consists of the one point, for \( j \in \{i_0, \ldots, i_{k-1}\} \) and \( \epsilon = \pm \), the restriction of the holomorphic \( n \)-form \( dz \) to \( \sigma_{[\alpha (j, \epsilon)]} \) becomes 0 and we get
\[
\int_{\sigma_{[\alpha (j, \epsilon)]}} e^{-z \cdot \tau} = 0
\]
for an \((n, n-k-1)\)-form \( \tau \). Therefore we still have the same Stokes formula as the one in Example 6.3.7
\[
\int_{\sigma_{\alpha}} e^{-z \cdot \overline{\tau}} = \sum_{j \notin \{i_0, \ldots, i_{k-1}, n\}, \epsilon = \pm} \int_{\sigma_{[\alpha (j, \epsilon)]}} e^{-z \cdot \tau},
\]
and hence, we obtain
\[
\mathcal{I}(\overline{\omega}) = 0 \quad (\omega \in C^{n-1}(\mathcal{S}, \mathcal{S}')(\mathscr{D}_{DE}^{(n, \bullet)})).
\]
Summing up, let \( u = [\tau] \in \Gamma_K(\mathbb{D}_M; \mathscr{R}_{DM}^{exp, \nu} \otimes \mathcal{F}^{exp, \nu}_{DM}) \) and let
\[
\nu = \bigoplus_{\alpha \in \Lambda_k^{n+1}} \nu_{\alpha} \in C^{n}(\mathcal{S}, \mathcal{S}')(\mathcal{E}_{DE}^{exp, (n)}) \quad \text{with} \quad \delta \nu = 0
\]
be its \( \check{\text{C}} \)ech representative, that is, through the diagram of isomorphisms
\[
H^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{E}_{DE}^{exp, (n)})) \xrightarrow{\beta_2} H^n(C(\mathcal{S}, \mathcal{S}')(\mathcal{D}_{DE}^{(n, \bullet)})) \xrightarrow{\epsilon_2} H^n(C(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{D}_{DE}^{(n, \bullet)})),
\]
\( \tau \) and \( \nu \) satisfy \( [\tau] = ((\beta_{2}^{-1} \circ \beta_{1}^{n})([\nu])) \). We have obtained
\[
\mathcal{L}(u) = (-1)^n \sum_{\alpha \in \Lambda_k^{n+1}} \text{sgn}(\alpha) \int_{\gamma_{\alpha}} e^{-z \cdot \nu_{\alpha}}, \quad (6.15)
\]
where, for \( \alpha = ((0, \epsilon_0), \ldots, (n-1, \epsilon_{n-1}), n) \in \Lambda_k^{n+1} \), we set \( \text{sgn}(\alpha) = \epsilon_0 \epsilon_1 \cdots \epsilon_{n-1} \),
\[
\gamma_{\alpha} = (\partial D_1 \times \partial D_2 \times \cdots \times \partial D_n) \bigcap \Gamma_{\alpha}, \quad \Gamma_{\alpha} = \{ z = (z_1, \ldots, z_n) \in E ; \epsilon_k \text{Im} z_{k+1} > 0 \quad (k = 0, 1, \ldots, n-1) \}
\]
and the orientation of \( \gamma_{\alpha} \) is chosen to be the same as the one in \( M \).

### 6.4 Reconstruction of a representative

By the same arguments as in the previous examples, we have a formula to reconstruct the corresponding \( \check{\text{C}} \)ech representative from a \( \check{\text{C}} \)ech Dolbeault representative of a Laplace hyperfunction.

Recall the definition of \( \Lambda_{k}^{n+1} \) and \( \Lambda_{**}^{n+1} \) given in Example 6.3.7. Set
\[
\Gamma_{\alpha} := \{ x \in M ; \epsilon_k x_{k+1} > 0 \quad (k = 0, \ldots, n-1) \}
\]
for any \( \alpha = ((0, \epsilon_0), (1, \epsilon_1), \ldots, (n - 1, \epsilon_{n-1}), n) \in \Lambda^{n+1}_{**} \). In particular, we denote by \( +^n \) the sequence \( ((0, +), (1, +), \ldots, (n - 1, +), n) \). Thus \( \Gamma_{+n} \) denotes the first orthant in \( M \).

Let \( K \subset \mathbb{D}_M \) be a regular closed cone such that \( K \cap M \) is a non-empty convex set, and \( V \subset \mathbb{D}_E \) an open cone such that \( V \) is 1-regular at \( \infty \) and \( V \cap E \) is a Stein open subset. Note that, since \( V \) is an open cone, the fact that \( V \) is 1-regular at \( \infty \) is equivalent to saying that \( \sim (V \cap E) = V \). We also assume

\[
K \setminus \{0\} \subset \Gamma_{+n} \subset \Gamma_{+n} \subset V. \tag{6.17}
\]

Set \( U = V \cap \mathbb{D}_M \) and we also assume \( \sim (U \cap M) = U \). Let \( \hat{H}^n (\mathcal{E}^{\exp}_E(W(U))) \) denote the intuitive representation of Laplace hyperfunctions on \( U \subset \mathbb{D}_M \).

**Remark 6.4.1.** In this subsection, we assume that \( \mathcal{W}(U) \) consists of an infinitesimal wedge which satisfies the condition B1. in Section 5. For such a family \( \mathcal{W}(U) \) of restricted open subsets, still Theorem [1.3.3] holds.

Then, we define \( b : \hat{H}^n (\mathcal{E}^{\exp}_E(W(U))) \rightarrow H^n(C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})) \) by

\[
\mathcal{E}^{\exp}_E(W) \ni f \mapsto b_W(f) \in H^n(C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})) \quad (W \in \mathcal{W}(U)),
\]

where \( \mathcal{V}_U = \{V \setminus U, V\} \), \( \mathcal{V}_U' = \{V \setminus U\} \) and \( b_W \) is the boundary value map \( (5.2) \).

Recall that the isomorphism \( \mathcal{V}_W : \hat{H}^n (\mathcal{E}^{\exp}_E(W(U))) \rightarrow \Gamma(U; \mathcal{B}^{\exp}_M) \) was given in Theorem [1.3.3] for which we have the commutative diagram (see Theorem [5.2.4] and Corollary [5.2.5]):

\[
\begin{array}{ccc}
\hat{H}^n (\mathcal{E}^{\exp}_E(W(U))) & \xrightarrow{\sim} & H^n(C(\mathcal{V}_U, \mathcal{V}_U')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})) \\
\downarrow \cong & & \downarrow \cong \\
\Gamma(U; \mathcal{B}^{\exp}_M) & \rightarrow & \Gamma(K(U; \mathcal{B}^{\exp}_M))
\end{array}
\]

where \( \mathcal{V}_K = \{V \setminus K, V\} \) and \( \mathcal{V}_K' = \{V \setminus K\} \), the morphisms \( \iota \) are injective and all the other morphisms are isomorphic. Set

\[
\hat{H}^n_K(\mathcal{E}^{\exp}_E(W(U))) := \{u \in \hat{H}^n (\mathcal{E}^{\exp}_E(W(U))); \text{ Supp}(b_W(u)) \subset K\}.
\]

Then the morphism \( b \) induces the isomorphism

\[
b_K : \hat{H}^n_K(\mathcal{E}^{\exp}_E(W(U))) \xrightarrow{\sim} H^n(C(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})).
\]

Now we give the inverse of \( b_K \) concretely. Let \( u \in \Gamma_K(U; \mathcal{B}^{\exp}_M) \) and \( \tau = (\tau_1, \tau_01) \in C^n(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E}) \) be its representation. Define

\[
h_{\tau}(z) = \frac{1}{(2\pi \sqrt{-1})^n} \left( \int_D \tau_1(w)e^{(z-w)a}dw - \int_{\partial D} \tau_01(w)e^{(z-w)a}dw \right),
\]

where \( \frac{1}{w - z} \) denotes \( \frac{1}{(w_1 - z_1) \cdots (w_n - z_n)} \), the vector \( a \) and the domain \( D \) are as follows:
1. $D$ is a contractible open subset in $\mathbb{D}_E$ with a good boundary $\partial D$ (see Remark 6.2.3 for the good boundary) which satisfies

$$K \subset D \subset \overline{D} \subset V$$

and

$$(D \cap E) \subset \bigcup_{k=1}^{n} \{w \in E; |w_k - z_k| > \delta\}$$

for some $\delta > 0$. Furthermore, $D$ is properly contained in an half space of $\mathbb{D}_E$ with direction $\frac{1}{\sqrt{n}}(1, 1, \ldots, 1)$.

2. $a = R(1, 1, \ldots, 1)$, where $R > 0$ is sufficiently large so that the integrals converge.

Note that the orientation of $D$ is the same as the one of $E$, and that of $\partial D$ is determined so that the outward-pointing normal vector of $\partial D$ followed by a positive frame of $\partial D$ form a positive frame of $E$.

Then it is easy to check that $h_\tau(z)$ remains unchanged when we take another $D$ and representative $\tau$ of $u$ if the integral converges for the same $a$. Hence, by deforming $D$ suitably (here keep $D$ unchanged near $K \cap E$, and hence, we do not need to change $a$ in this deformation), we find that $h_\tau(z)$ belongs to $\mathcal{O}_{\text{exp}}(\mathbb{D}_E(\Omega))$, where

$$\Omega := \bigwedge\{z = x + \sqrt{-1}y \in E; y_1y_2\cdots y_n \neq 0\}.$$

For $\alpha \in \Lambda^{n+1}$, set $\Omega_{\alpha} := M\bigwedge\sqrt{-1}\Gamma_{\alpha} \subset \mathbb{D}_E$. Note that we have

$$\Omega := \bigcup_{\alpha \in \Lambda^{n+1}} \Omega_{\alpha}.$$

Now we define the inverse $b_{K}^\dagger$ of $b_{K}$ by

$$u = [\tau] \mapsto \bigoplus_{\alpha \in \Lambda_{n+1}^+} \text{sgn}(\alpha) h_\tau(z)|_{\Omega_{\alpha}} \in \mathcal{H}^n(\mathcal{O}_{\text{exp}}(\mathbb{D}_E(\mathcal{W}(U))), \quad (6.18)$$

where $\text{sgn}(\alpha) = \epsilon_0\epsilon_1\cdots\epsilon_{n-1}$ for $\alpha = ((0, \epsilon_0), (1, \epsilon_1), \ldots, (n-1, \epsilon_{n-1}), n) \in \Lambda_{n+1}^+$. 

**Lemma 6.4.2.** $b_{K}^\dagger$ is independent of the choices of $a = R(1, \ldots, 1)$ if $R > 0$ is sufficiently large.

**Proof.** Let $a' = (R', R, \ldots, R)$ with $R' > R$. It is enough to show that $b_{K}^\dagger(u)$ gives the same result for both the $a$ and $a'$ because a general case is obtained by the repetition of application of this result. Clearly we have

$$\left(\int_D \frac{\tau_1(w)e^{(z-w)a'}}{w-z} dw - \int_{\partial D} \frac{\tau_01(w)e^{(z-w)a'}}{w-z} dw\right) -$$

$$\left(\int_D \frac{\tau_1(w)e^{(z-w)a}}{w-z} dw - \int_{\partial D} \frac{\tau_01(w)e^{(z-w)a}}{w-z} dw\right)$$

$$= (R' - R) \left(\int_D \int_0^1 \frac{\tau_1(w)e^{(z-w)(\alpha'+(1-t)a)}}{w'-z'} dt dw' - \int_{\partial D} \int_0^1 \frac{\tau_01(w)e^{(z-w)(\alpha'+(1-t)a)}}{w'-z'} dt dw'\right),$$
where \( z' = (z_2, \ldots, z_n) \) and \( w' = (w_2, \ldots, w_n) \). Since the last integral denoted by \( \tilde{h}(z) \) hereafter belongs to \( \mathcal{O}^\exp_{\mathcal{D}_E}(\Omega') \) with

\[
\Omega' := \{ z = x + \sqrt{-1}y \in E; y_2 \cdots y_n \neq 0 \}
\]

we have \( \bigoplus_\alpha \text{sgn}(\alpha)\tilde{h}(z)|_{\Omega_\alpha} = 0 \) in \( \hat{H}^n(\mathcal{O}^\exp_{\mathcal{D}_E} (\mathcal{W}(U))) \). This shows the result.

**Theorem 6.4.3.** \( b_K \) and \( b_K^\dagger \) are inverse to each other.

**Proof.** We use the same notations as those in Subsection 6.3.2, where we take an open subset \( V \) as \( S \). Hence, the pair \((S, S')\) are coverings of \((V, V \setminus K)\). Set

\[
Q_{k,\epsilon} = \{ y = (y_1, \cdots, y_n) \in M; \epsilon y_{k+1} > L^{-1}|y| \} \quad (k = 0, 1, \cdots, n-1, \epsilon = \pm)
\]

for sufficiently large \( L > 0 \) and set

\[
T_{k,\epsilon} = U \tilde{\times} \sqrt{-1}Q_{k,\epsilon}.
\]

Let \( T \subset \mathbb{D}_E \) be an open neighborhood of \( U \) such that \( T \cap \mathbb{D}_M = U \), \( T \) is 1-regular at \( \infty \) and \( T \cap E \) is a Stein open subset. Furthermore, by shrirking \( T \) if necessary, we may assume \( T \subset S \) and

\[
T_{k,\epsilon} \cap T \subset S_{k,\epsilon} \cap S \quad (k = 0, 1, \cdots, n-1, \epsilon = \pm).
\]

Set also \( T_n = T \) and define the pair \((T, T')\) of coverings of \((T, T \setminus \mathbb{D}_M)\) by

\[
T = \{T_{0,+}, T_{0,-}, \cdots, T_{n-1,+}, T_{n-1,-}, T_n\}, \quad T' = \{T_{0,+}, T_{0,-}, \cdots, T_{n-1,+}, T_{n-1,-}\}.
\]

Using these coverings, we have the commutative diagram of complexes, where the horizontal arrows are all quasi-isomorphisms:

\[
\begin{array}{ccc}
C(S, S')(\mathcal{O}^\exp_{\mathcal{D}_E}) & \xrightarrow{\beta_1} & C(S, S')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E}) & \xrightarrow{\beta_2} & C(V_K, V_K')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E}) \\
\quad \downarrow \iota_1 \quad & & \quad \downarrow \iota \quad & & \quad \downarrow \iota_2 \quad \\
C(T, T')(\mathcal{O}^\exp_{\mathcal{D}_E}) & \xrightarrow{\alpha_1} & C(T, T')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E}) & \xrightarrow{\alpha_2} & C(V_U, V_U')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})
\end{array}
\]

Then by taking \( n \)-th cohomology groups we get

\[
\begin{array}{ccc}
H^n(C(S, S')(\mathcal{O}^\exp_{\mathcal{D}_E})) & \xrightarrow{\beta_1^n} & H^n(C(S, S')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})) & \xrightarrow{\beta_2^n} & H^n(C(V_K, V_K')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})) \\
\quad \downarrow \iota_1^n \quad & & \quad \downarrow \iota^n \quad & & \quad \downarrow \iota_2^n \quad \\
H^n(C(T, T')(\mathcal{O}^\exp_{\mathcal{D}_E})) & \xrightarrow{\alpha_1^n} & H^n(C(T, T')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E})) & \xrightarrow{\alpha_2^n} & H^n(C(V_U, V_U')(\mathcal{L}^{(0,\bullet)}_{\mathcal{D}_E}))
\end{array}
\]

where all the horizontal arrows are isomorphic and all the vertical arrows are injective.

Before entering the proof, we first confirm several fundamental facts which are needed in the proof:
Lemma 4.3.3 still holds by the same reasoning, that is, the canonical isomorphism

$$
\iota_{IC} : H^n(C(\mathcal{T}, \mathcal{T}')(\mathcal{E}^{\exp}_{\mathcal{D}_E})) \overset{\sim}{\longrightarrow} \hat{H}^n(\mathcal{E}^{\exp}_{\mathcal{D}_E}(\mathcal{W}(U)))
$$

is given by

$$
\left[ \bigoplus_{\alpha \in \Lambda_n^{+1}} g_\alpha \right] \mapsto (-1)^n \bigoplus_{\alpha \in \Lambda_n^{+1}} \sgn(\alpha) g_\alpha |_{T_\alpha}.
$$

We have the commutative diagram below as in (5.6).

$$
\begin{array}{ccc}
H^n(C(\mathcal{T}, \mathcal{T}')(\mathcal{E}^{\exp}_{\mathcal{D}_E})) & \xrightarrow{\iota_{IC}} & \hat{H}^n(\mathcal{E}^{\exp}_{\mathcal{D}_E}(\mathcal{W}(U))) \\
\end{array}
$$

The morphism $\iota_1$ is induced from the restriction of coverings, that is, for $(f_\alpha)_{\alpha \in \Lambda_n^{+1}} \in C^n(\mathcal{T}, \mathcal{T}')(\mathcal{E}^{\exp}_{\mathcal{D}_E})$, we have

$$
\iota_1^n([f_\alpha]_{\alpha \in \Lambda_n^{+1}}) = ([f_\alpha |_{T_\alpha}]_{\alpha \in \Lambda_n^{+1}}) \quad \text{in } H^n(C(\mathcal{T}, \mathcal{T}')(\mathcal{E}^{\exp}_{\mathcal{D}_E})).
$$

Note also that, for $\alpha \in \Lambda_n^{+1} \setminus \Lambda_n^{*+1}$, we have always $T_\alpha = \emptyset$ but $S_\alpha$ is not necessarily empty.

Since $b_K$ is an isomorphism, it suffices to show $b_K^\dagger$ is the inverse of $b_K$, i.e.,

$$
b_K^\dagger \circ b_K = \text{id} \quad \text{in } \hat{H}^n(\mathcal{E}^{\exp}_{\mathcal{D}_E}(\mathcal{W}(U))).
$$

To see the above formula, for any cocycle $f = (f_\alpha)_{\alpha \in \Lambda_n^{+1}} \in C^n(\mathcal{T}K \mathcal{S}, \mathcal{S}')(\mathcal{E}^{\exp}_{\mathcal{D}_E})$, it suffices to show the equality

$$
(b_K^\dagger \circ b_K \circ \iota_{IC} \circ \iota_1^n)([f]) = (\iota_{IC} \circ \iota_1^n)([f]) \quad \text{in } \hat{H}^n(\mathcal{E}^{\exp}_{\mathcal{D}_E}(\mathcal{W}(U))).
$$

Then it follows from the above fundamental facts that we have

$$
(b_K^\dagger \circ b_K \circ \iota_{IC} \circ \iota_1^n)([f]) = (b_K^\dagger \circ (\iota_1^n)^{-1} \circ b \circ \iota_{IC} \circ \iota_1^n)([f])
$$

$$
= (b_K^\dagger \circ (\iota_1^n)^{-1} \circ (\alpha_1^n)^{-1} \circ \alpha_1^n \circ \iota_1^n)([f]) = (b_K^\dagger \circ (\beta_1^n)^{-1} \circ \beta_1^n)([f]).
$$

Here $(\iota_1^n)^{-1}$ denotes the inverse on $\text{Im}(\iota_1^n)$. Hence it is enough to see the equality

$$
(b_K^\dagger \circ (\beta_1^n)^{-1} \circ \beta_1^n)([f]) = (\iota_{IC} \circ \iota_1^n)([f]) \quad \text{in } \hat{H}^n(\mathcal{E}^{\exp}_{\mathcal{D}_E}(\mathcal{W}(U))),
$$

whose concrete form is as follows:

$$
\bigoplus_{\alpha \in \Lambda_n^{+1}} \sgn(\alpha) h_\tau(z) |_{\Omega_\alpha} = (-1)^n \bigoplus_{\alpha \in \Lambda_n^{+1}} \sgn(\alpha) f_\alpha |_{T_\alpha} \quad \text{in } \hat{H}^n(\mathcal{E}^{\exp}_{\mathcal{D}_E}(\mathcal{W}(U))), \quad (6.19)
$$

where $\tau = (\tau_1, \tau_{01}) \in C^n(\mathcal{V}_K, \mathcal{V}_K')(\mathcal{D}_{\mathcal{D}_E}^{(0,\bullet)})$ is given by $[\tau] = ((\beta_2^n)^{-1} \circ \beta_1^n)([f])$.
Let us show \((6.19)\). By applying the same arguments as in Subsection \(6.3.2\) to the integral \(h_\tau(z)\), we have
\[
h_\tau(z) = \sum_{\alpha \in \Lambda_{++}^{n+1}} \frac{(-1)^n \text{sgn}(\alpha)}{(2\pi\sqrt{-1})^n} \int_{\Phi_\alpha(L(z))} \frac{f_\alpha(z)e^{(z-w)a}}{w-z} \, dw \quad (z \in \Omega \cap E),
\]
where, for any \(\alpha = ((0,\epsilon_0), \cdots, (n-1,\epsilon_{n-1}), n) \in \Lambda_{++}^{n+1}\), the mapping \(\Phi_\alpha : \mathbb{C}^n \to \mathbb{C}^n\) is defined by
\[
\Phi_\alpha(x_1 + \sqrt{-1}y_1, \cdots, x_n + \sqrt{-1}y_n) = (x_1 + \epsilon_0\sqrt{-1}y_1, x_2 + \epsilon_1\sqrt{-1}y_2, \cdots, x_n + \epsilon_{n-1}\sqrt{-1}y_n)
\]
and
\[
L(z) = \ell_+(z_1) \times \ell_+(z_2) \times \cdots \times \ell_+(z_n).
\]
Here, for \(z_0 = x_0 + \sqrt{-1}y_0 \in \mathbb{C}\) with \(y_0 \neq 0\), the path \(\ell_+(z_0) \subset \mathbb{C}\) is defined as follows: Let \(\gamma \subset \mathbb{D}_C\) be a domain with a good boundary such that it contains the real half line \(\{z = x + \sqrt{-1}y \in \mathbb{C}; x \geq \min\{0, 2x_0\}, y = 0\} \subset \mathbb{D}_C\) and two points \(x_0 \pm \sqrt{-1}y_0 \in \mathbb{C}\) are outside \(\gamma\). Then we set
\[
\ell_+(z_0) = \partial \gamma \cap \{z \in \mathbb{C}; \text{Im} z \geq 0\}.
\]
Furthermore, the orientation of \(\ell_+(z)\) is the same as that of the real axis.

In the same way, we define \(\ell_-(z_0) \subset \mathbb{C}\) by taking the domain \(\gamma\) as in the case of \(\ell_+(z_0)\). However, in this case, we take \(\gamma\) so that the two points \(x_0 \pm \sqrt{-1}y_0\) are also contained in \(\gamma\). For any \(\beta = ((0,\epsilon_0), (1,\epsilon_1), \cdots, (n-1,\epsilon_{n-1}), n) \in \Lambda_{++}^{n+1}\), we set
\[
L_\beta(z) := \ell_{\epsilon_0}(z_1) \times \ell_{\epsilon_1}(z_2) \times \cdots \times \ell_{\epsilon_{n-1}}(z_n)
\]
and
\[
g_{\alpha,\beta}(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Phi_\beta(L_\beta(z))} \frac{f_\alpha(z)e^{(z-w)a}}{w-z} \, dw.
\]
It follows from the Cauchy integral formula that
\[
\sum_{\alpha \in \Lambda_{++}^{n+1}} \text{sgn}(\alpha)g_{\beta,\alpha}(z) = \text{sgn}(\beta)f_\beta(z) \quad (z \in T_\beta, \beta \in \Lambda_{++}^{n+1}). \quad (6.20)
\]
For \(\alpha = ((0,\epsilon_0), (1,\epsilon_1), \cdots, (n-1,\epsilon_{n-1}), n)\) and \(\beta = ((0,\eta_0), (1,\eta_1), \cdots, (n-1,\eta_{n-1}), n)\) in \(\Lambda_{++}^{n+1}\), we define
\[
\alpha \cdot \beta = ((0,\epsilon_0\eta_0), (1,\epsilon_1\eta_1), \cdots, (n-1,\epsilon_{n-1}\eta_{n-1}), n) \in \Lambda_{++}^{n+1}.
\]
Remember that \(+^n\) denotes \(((0, +), (1, +), \cdots, (n-1, +), n)\). If \(\beta \in \Lambda_{++}^{n+1}\) is different from \(+^n\), then \(g_{\alpha,\beta}|_{T_\alpha}\) and \(g_{\alpha,+^n}|_{T_{\alpha,\beta}}\) can analytically extend to some common infinitesimal wedge in \(\mathbb{D}_E\) and they coincide there. Hence we have, for any \(\alpha, \beta \in \Lambda_{++}^{n+1}\),
\[
g_{\alpha,\beta}|_{T_\alpha} = g_{\alpha,+^n}|_{T_{\alpha,\beta}} \quad \text{in} \quad \hat{H}^n(\Theta_{\mathbb{D}_E}(\mathcal{W}(U))),
\]
from which we have obtained in \(\hat{H}^n(\Theta_{\mathbb{D}_E}(\mathcal{W}(U)))\)
\[
(-1)^n \bigoplus_{\alpha \in \Lambda_{++}^{n+1}} \text{sgn}(\alpha)h_\tau(z)|_{\Omega_\alpha} = \bigoplus_{\alpha \in \Lambda_{++}^{n+1}} \sum_{\beta \in \Lambda_{++}^{n+1}} \text{sgn}(\alpha)\text{sgn}(\beta)g_{\beta,+^n}|_{T_\alpha} = \bigoplus_{\alpha \in \Lambda_{++}^{n+1}} \sum_{\beta \in \Lambda_{++}^{n+1}} \text{sgn}(\alpha \cdot \beta)g_{\beta,\alpha,\beta}|_{T_\beta} = \bigoplus_{\beta \in \Lambda_{++}^{n+1}} \text{sgn}(\beta)f_\beta|_{T_\beta}.
\]
This completes the proof. \(\square\)
7 Laplace inverse transformation \( \mathcal{L} \)

Let \( S \) be a connected open subset in \( M^*_\infty \) and \( a \in M \). Note that a connected subset is, in particular, non-empty. Recall the definition of the map \( \varpi_{M^*_\infty} \) given in (2.3), for which we have

\[
\varpi_{-1,M^*_\infty}(S) = \{ \xi + \sqrt{-1}\eta \in E^*; \xi \in S, \eta \in M^* \}/\mathbb{R}_+ \subset E^*_\infty \setminus \sqrt{-1}M^*_\infty.
\]

Here we identify a point in \( M^*_\infty \) with a unit vector in \( M^* \).

Let \( h : M^*_\infty \to \{ \pm \infty \} \cup \mathbb{R} \) be an upper semi-continuous function such that \( h(\xi) \) is continuous on \( S \) and \( h(\xi) > -\infty \) there. Now we extend \( h \) to the one on \( E^*_\infty \) in the following canonical way: Define \( \tilde{h}(\zeta) : E^*_\infty \to \{ \pm \infty \} \cup \mathbb{R} \) by, for \( \zeta = \xi + \sqrt{-1}\eta \in E^*_\infty \ ((\xi, \eta) \in S^{2n-1}) \),

\[
\tilde{h}(\zeta) = \begin{cases} 
0 & (\zeta \in \sqrt{-1}M^*_\infty), \\
|\xi|h(\varpi_{M^*_\infty}(\zeta)) & (\zeta \in E^*_\infty \setminus \sqrt{-1}M^*_\infty),
\end{cases}
\]

where we set \( \pm \infty \times c = \pm \infty \) for \( c > 0 \). Note that \( \tilde{h} \) is also upper semi-continuous on \( E^*_\infty \) and continuous on \( \varpi_{-1,M^*_\infty}(S) \setminus \sqrt{-1}M^*_\infty \).

Let \( f \in \mathcal{O}_{E^*_\infty}^{\inf}(-\tilde{h}(\varpi_{M^*_\infty}(S))) \). It follows from the definition of \( \mathcal{O}_{E^*_\infty}^{\inf}(-\tilde{h}) \) that we can find continuous functions \( \psi : S \times [0, \infty) \to \mathbb{R}_{\geq 0} \) and \( \varphi : [0, \infty) \to \mathbb{R}_{\geq 0} \) satisfying the following conditions:

1. For any compact subset \( L \subset S \), the function \( \sup_{\xi \in L} \psi(\xi, \lambda) \) is an infra-linear function of the variable \( \lambda \) and \( f \) is holomorphic on an open subset \( W_\psi \cap E^* \), where

\[
W_\psi := \sim \{ \zeta = \lambda \xi + \sqrt{-1}\eta \in E^*; \eta \in M^*, \xi \in S, \lambda > \psi(\xi, |\eta|) \} .
\]

Note that we identify a point in \( M^*_\infty \) with a unit vector in \( M^* \) here.

2. \( \varphi(t) \) is a continuous infra-linear function on \( [0, \infty) \) such that

\[
|f(\zeta)| \leq e^{-|\xi|\tilde{h}(\pi_{E^*_\infty}(\zeta)) + \psi(|\xi|)} = e^{-|\xi|\tilde{h}(\pi_{E^*_\infty}(\zeta)) + \varphi(|\xi|)} \quad (\zeta = \xi + \sqrt{-1}\eta \in W_\psi \cap E^*) ,
\]

where \( \pi_{E^*_\infty} : E^* \setminus \{ 0 \} \to (E^* \setminus \{ 0 \})/\mathbb{R}_+ = E^*_\infty \) (resp. \( \pi_{M^*_\infty} : M^* \setminus \{ 0 \} \to M^*_\infty \)) is the canonical projection and we also set \( e^{+\infty} = +\infty, e^{-\infty} = 0 \).

We also define an \( n \)-dimensional real chain in \( E^* \) by

\[
\gamma^* := \{ \zeta = \xi + \sqrt{-1}\eta \in E^*; \eta \in M^* \setminus \{ 0 \}, \xi = \psi_{\xi_0}(|\eta|) \xi_0 \} .
\]

Here \( \xi_0 \in S \) and \( \psi_{\xi_0}(\lambda) \) is a smooth function on \( [0, \infty) \) which is monotonically increasing and has bounded derivatives on \( [0, \infty) \). Further it is infra-linear with \( \psi_{\xi_0}(\lambda) > \psi(\xi_0, \lambda) \ (\lambda \in [0, \infty)) \) and \( \psi_{\xi_0}(\lambda)/(|\psi(\xi_0, \lambda) + 1|) \to \infty (\lambda \to \infty) \). Note that the orientation of \( \gamma^* \) is chosen to be the same as that of \( \sqrt{-1}M^*_\infty \).

**Example 7.0.1.** The following situation is the most important one considered in the paper: Let \( K \) be a regular closed subset in \( \mathbb{D}_M \) such that \( N^*_{pc}(K) \cap M^*_\infty \) is connected (in particular, non-empty). Then we set \( S = N^*_{pc}(K) \cap M^*_\infty \) and we also set

\[
h(\xi) = h_K(\xi) = \inf_{x \in K \cap M} \langle x, \xi \rangle \quad (\xi \in M^*_\infty)
\]
if $K \cap M \neq \emptyset$ and set $h(\xi) = +\infty$ if $K \cap M = \emptyset$. In this case, we have

$$N^*_p(K) = \begin{cases} \varpi_{M_\infty}^{-1}(S) & (K \cap M_\infty \neq \emptyset), \\ E^*_\infty = \varpi_{M_\infty}^{-1}(S) \cup \sqrt{-1}M^*_\infty & (K \cap M_\infty = \emptyset), \end{cases}$$

and

$$\hat{h}( \zeta ) = \inf_{x \in K \cap M} \Re \langle x, \zeta \rangle \quad (\zeta \in E^*_\infty)$$

if $K \cap M \neq \emptyset$ and $\hat{h}(\zeta) = +\infty$ otherwise. Furthermore, $\hat{h}(\zeta)$ is upper semi-continuous on $E^*_\infty$ and continuous on $N^*_p(K) \cup \sqrt{-1}M^*_\infty$.

Now we consider the de-Rham theorem with a parameter in Section 3, for which we take $T = S^{n-1} = \{ \eta \in M^*; |\eta| = 1 \}$ and $Y = S^{n-1} \times \mathbb{D}_E$. Define coverings

$$\mathcal{W} = \{ W_0 = Y \setminus p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M), W_1 = Y \}, \quad \mathcal{W}' = \{ W_0 \}$$

with $W_{01} = W_0 \cap W_1$. Recall the isomorphisms given in Proposition 3.0.6

$$\Gamma(T; \mathcal{L}_{loc,T}^\infty) = \Gamma(Y; p_T^{-1}(\mathcal{L}_{loc,T}^\infty)) \sim \overset{\text{H}^n}{\overset{\text{Im}}{\text{H}}} \overset{\text{p}_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}{\text{p}_T^{-1}(\mathcal{L}_{loc,T}^\infty)}(Y) = \overset{\text{H}^n}{\overset{\text{C}(\mathcal{W}, \mathcal{W}')}{\text{C}(\mathcal{W}, \mathcal{W}')}},$$

and set

$$\Omega := \{ (\theta, z) \in S^{n-1} \times E; \langle \theta, \text{Im} z \rangle > 0 \} \subset Y.$$

Let $j : \Omega \hookrightarrow Y$ be the canonical open inclusion. Then we can take a specific $\omega = (\omega_1, \omega_{01}) \in C^m(\mathcal{W}, \mathcal{W}')(\mathcal{L}_{Y}^{(\bullet)})$ satisfying the following conditions:

D1. $D_{\mathbb{D}_E}\omega = 0$ and $[\omega]$ is the image of a constant function $1 \in \Gamma(T; \mathcal{L}_{loc,T}^\infty)$ through the above isomorphisms.

D2. We have supp$_{\mathcal{W}'}(\omega_1) \subset \Omega$ and supp$_{\mathcal{W}_{01}}(\omega_{01}) \subset \Omega$.

The existence of the above $\omega$ comes from the following lemma:

**Lemma 7.0.2.** The canonical morphisms

$$C(\mathcal{W}, \mathcal{W}')(j;i,j^{-1}(\mathcal{L}_{Y}^{(\bullet)})) \longrightarrow C(\mathcal{W}, \mathcal{W}')(\mathcal{L}_{Y}^{(\bullet)}),$$

$$C(\mathcal{W}, \mathcal{W}')(j;i,j^{-1}(\mathcal{E}_{Y}^{(\bullet)})) \longrightarrow C(\mathcal{W}, \mathcal{W}')(\mathcal{E}_{Y}^{(\bullet)})$$

are quasi-isomorphic.

**Proof.** Let $\mathcal{F}$ be a $\mathcal{L}_{loc,T}^\infty$ or $\mathcal{E}_T$, and let $i : Y \setminus \Omega \rightarrow Y$ denote the closed embedding. Then the above isomorphism is equivalent to the following isomorphism:

$$\mathcal{R}\Gamma p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)(Y; j;i,j^{-1}p_T^{-1}\mathcal{F}) \longrightarrow \mathcal{R}\Gamma p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)(Y; p_T^{-1}\mathcal{F}),$$

which comes from the fact

$$\mathcal{R}\Gamma p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)(Y; i_*i^{-1}p_T^{-1}\mathcal{F}) \simeq 0.$$  \hspace{1cm} (7.4)
The fact itself can be shown by the following argument: Let us consider the distinguished triangle

\[ \mathbf{R}\Gamma_{\mathcal{D}_E}(\mathbb{D}_M)(Y; i_*i^{-1}p_T^{-1}\mathcal{F}) \to \mathbf{R}\Gamma(Y \setminus \Omega; p_T^{-1}\mathcal{F}) \stackrel{\beta}{\to} \mathbf{R}\Gamma((Y \setminus \Omega) \setminus \mathcal{D}_E(\mathbb{D}_M); p_T^{-1}\mathcal{F}) \xrightarrow{\pm 1} \]

Under the commutative diagram below,

\[
\begin{array}{ccc}
(Y \setminus \Omega) \setminus \mathcal{D}_E(\mathbb{D}_M) & \xrightarrow{\iota} & Y \setminus \Omega \\
\downarrow p_T & & \downarrow p_T \\
T & & T
\end{array}
\]

the morphism \( \iota \) gives a homotopical equivalence over \( T \), and hence, it follows from Corollary 2.7.7 (i) [KS] that the morphism \( \beta \) is isomorphic. This implies (7.4). The proof has been completed.

Note that we will give a concrete construction of such an \( \omega \) later. Recall the standard coverings

\[ \mathcal{V}_{\mathbb{D}_M} = \{V_0 = \mathbb{D}_E \setminus \mathbb{D}_M, V_1 = \mathbb{D}_E\}, \quad \mathcal{V}_{\mathbb{D}_M}' = \{V_0\}, \]

and the morphism \( \rho = \{\rho_k\} : \mathcal{L}\mathcal{D}_Y^{(\bullet)} \to \mathcal{L}\mathcal{D}_Y^{(0,\bullet)} \) of complexes which is the projection to the space of anti-holomorphic forms, that is, each \( \rho_k : \mathcal{L}\mathcal{D}_Y^{(k)} \to \mathcal{L}\mathcal{D}_Y^{(0,k)} \) is defined by

\[
\sum_{|I|=i, |J|=j, i+j=k} f_{I,J}(\theta, z)dz^J \wedge d\bar{z}^J \mapsto \sum_{|J|=k} f_{0,J}(\theta, z)d\bar{z}^J.
\]

Note that the following diagram commutes

\[
\begin{array}{ccc}
\mathbf{R}\Gamma_{\mathcal{D}_E}(\mathbb{D}_M)(Y; p_T^{-1}\mathcal{L}\mathcal{O}^{\infty}_{\text{loc},T}) & \xrightarrow{\rho} & \mathbf{R}\Gamma_{\mathcal{D}_E}(\mathbb{D}_M)(Y; \mathcal{L}\mathcal{O}^{\exp}_Y) \\
\downarrow & & \downarrow \\
C(W, W')(\mathcal{L}\mathcal{D}_Y^{(\bullet)}) & \xrightarrow{\rho} & C(W, W')(\mathcal{L}\mathcal{D}_Y^{(0,\bullet)})
\end{array}
\]

where vertical arrows are quasi-isomorphic.

Let us take an \( \omega = (\omega_1, \omega_{01}) \in C^n(W, W')(\mathcal{L}\mathcal{D}_Y^{(\bullet)}) \) which satisfies the conditions D1. and D2.

**Definition 7.0.3.** The Laplace inverse transform \( \mathcal{I}\mathcal{C} \) is given by

\[
\mathcal{I}\mathcal{C}(f) = ([\mathcal{I}\mathcal{C}(fd\zeta)] \otimes a_{\mathbb{D}_M/\mathbb{D}_E}) \otimes \nu_{\mathbb{D}_M}
\]

with

\[
\mathcal{I}\mathcal{C}(fd\zeta) := \left(\frac{1}{2\pi \sqrt{-1}}\right)^n \int_\gamma \rho(\omega)(\frac{\eta}{|\eta|}, z) e^{\xi z} f(\zeta)d\zeta
\]

\[
= \left(\frac{1}{2\pi \sqrt{-1}}\right)^n \left(\int_{\gamma^*} \rho_n(\omega_1)(\frac{\eta}{|\eta|}, z) e^{\xi z} f(\zeta)d\zeta, \int_{\gamma^*} \rho_{n-1}(\omega_{01})(\frac{\eta}{|\eta|}, z) e^{\xi z} f(\zeta)d\zeta\right).
\]
Lemma 7.0.4. We have

1. The integration $\mathcal{I}_\omega(f \, d\zeta)$ converges and it belongs to $C^n(V_{\mathcal{D}_M}, V_{\mathcal{D}_M}')(\mathfrak{D}_E^{(0,\bullet)})$. Furthermore, $\overline{\partial}(\mathcal{I}_\omega(f \, d\zeta)) = 0$ holds.

2. $\mathcal{I}_\omega(f \, d\zeta)$ does not depend on the choices of $\omega$.

Proof. Since the support of $\omega_0$ (resp. $\omega_1$) is a closed subset in $W_{01}$ (resp. $W_1$) and $Y_\infty = S^{n-1} \times E_\infty$ is compact, we have the following facts:

1. There exist an open neighborhood $O \subset \mathbb{D}_E$ of $\mathbb{D}_M$ and $\delta > 0$ such that
   
   $\text{supp}_{W_1}(\omega_1) \subset (S^{n-1} \times (\mathbb{D}_E \setminus O)) \cap \{ (\eta, z) \in S^{n-1} \times E; \langle \eta, \text{Im} z \rangle > \delta |\text{Im} z| \}$.

2. For any open neighborhood $O \subset \mathbb{D}_E$ of $\mathbb{D}_M$, there exists $\delta > 0$ such that
   
   $\text{supp}_{W_{01}}(\omega_{01}) \cap (S^{n-1} \times (\mathbb{D}_E \setminus O)) \subset \{ (\eta, z) \in S^{n-1} \times E; \langle \eta, \text{Im} z \rangle > \delta |\text{Im} z| \}$.

The fact $\mathcal{I}_\omega(f \, d\zeta) \in C^n(V_{\mathcal{D}_M}, V_{\mathcal{D}_M}')(\mathfrak{D}_E^{(0,\bullet)})$ and the claim 1. easily follows from these facts. As a matter of fact, for example, the integral

$$\int_{\gamma^*} \rho_{n-1}(\omega_0)(\eta | \eta|, z) e^{\zeta zf(\zeta)} d\zeta$$

is shown to be a form on $V_0$ with the desired growth condition in the following way: Let $O \subset \mathbb{D}_E$ be an open neighborhood of $\mathbb{D}_M$. Then, by the above fact 2. there exists $\delta > 0$ such that we have on $E \setminus O$

$$\int_{\gamma^*} \rho_{n-1}(\omega_0)(\eta | \eta|, z) e^{\zeta zf(\zeta)} d\zeta = \int_{\hat{\gamma}^*(z)} \rho_{n-1}(\omega_0)(\eta | \eta|, z) e^{\zeta zf(\zeta)} d\zeta,$$

where

$$\hat{\gamma}^*(z) = \gamma^* \cap \{ \xi \in \mathbb{C}^n; \langle \xi, \text{Im}(z)/|\text{Im}(z)| \rangle > \delta |\text{Im}(z)| \}.$$ 

For any $z = x + \sqrt{-1}y \in E \setminus O$, where we may assume $|y| > \delta' \max \{|x|, 1\}$ holds for some $\delta' > 0$, and for any $\zeta = \xi + \sqrt{-1}\eta \in \hat{\gamma}^*(z)$, we have

$$\log |e^{\zeta zf(\zeta)}| \leq \psi_{\xi_0}(\eta)(\xi_0, x) - \langle \eta, y \rangle - \psi_{\xi_0}(\eta)h(\xi_0) + \varphi(|\zeta|)$$

$$\leq \psi_{\xi_0}(\eta)(\xi_0, x) - h(\xi_0) - \delta |\eta||y| + \varphi(|\zeta|).$$

(7.5)

Set $\delta'' = \max \{ \delta''', |h(\xi_0)| \}$. Then the last term is estimated by:

$$\log |e^{\zeta zf(\zeta)}| \leq \psi_{\xi_0}(\eta)(\delta''|y| + |h(\xi_0)|) - \delta |\eta||y| + \varphi(|\zeta|)$$

$$\leq \delta'' \psi_{\xi_0}(\eta)(|y| + 1) - \delta |\eta||y| + \varphi(|\zeta|).$$

Since $\psi_{\xi_0}(t)$ and $\varphi(t)$ are of infra-linear, for any $\epsilon > 0$, there exists $M > 0$ such that

$$\psi_{\xi_0}(|\eta|) \leq M + \epsilon |\eta|, \quad \varphi(|\zeta|) \leq M + \epsilon |\eta|.$$
Hence we have
\[ \log |e^{\xi} f(\zeta)| \leq \delta^n M |y| + ((\delta^n \epsilon - \delta)|y| + (\delta^n + 1)\epsilon)|\eta| + (\delta^n + 1)M, \]
which implies that, for a sufficiently small \( \epsilon > 0 \), the integral converges on \( E \setminus O \) with the desired growth condition. Since \( O \) is an arbitrary open neighborhood of \( \mathbb{D}_M \), we can get the conclusion.

Now let us show the claim 2. Let \( \omega' \) be another choice of \( \omega \). Then, by the Lemma 7.0.2 we can find \( \omega^{-1} \in C^{n-1}(\mathcal{W}, \mathcal{W}')(\mathcal{T} \mathcal{D}_Y^b) \) such that
\[ \rho(\omega) - \rho(\omega') = \rho(D_{\mathbb{D}_E} \omega^{-1}) = \mathcal{I} \mathcal{D}_E \rho(\omega^{-1}). \]
Since \( \omega^{-1} \) satisfies the same support conditions as those for \( \omega \), the integration \( \mathcal{I}_E \omega^{-1}(fd\zeta) \) which is defined by replacing \( \omega \) with \( \omega^{-1} \) in the definition of \( \mathcal{I}_E(fd\zeta) \) also converges. Hence we have
\[ \mathcal{I}_E(fd\zeta) - \mathcal{I}_E(\omega'(fd\zeta)) \to \mathcal{I} \mathcal{D}_E \omega^{-1}(fd\zeta). \]
This completes the proof. \( \square \)

**Lemma 7.0.5.** The \( \mathcal{I}_E(f) \) is independent of the choice of \( \xi_0 \) and \( \psi_{\xi_0} \) which appear in the definition of \( \gamma^* \). As a consequence, we have
\[ \text{supp}(\mathcal{I}_E(f)) \subset \bigcap_{\xi_0 \in S} \{ x \in M; \langle x, \xi_0 \rangle \geq h(\xi_0) \}. \tag{7.6} \]

**Proof.** We first assume \( n > 1 \). Let us consider the commutative diagram below:
\[
\begin{align*}
\Gamma(T; \mathcal{L}_T^{\infty}) & = \Gamma(Y; \tilde{p}_T^{-1} \mathcal{L}_T^{\infty}) \twoheadrightarrow H^n_{\tilde{p}_T^{-1}(\mathbb{D}_E)}(Y; \mathcal{D}_Y) = H^n(C(W, W')(\mathcal{L}_Y^b)) \\
\Gamma(T; \mathcal{E}_T) & = \Gamma(Y; \tilde{p}_T^{-1} \mathcal{E}_T) \twoheadrightarrow H^n_{\tilde{p}_T^{-1}(\mathbb{D}_E)}(Y; \mathcal{E}_Y^b) = H^n(C(W, W')(\mathcal{E}_Y^b)),
\end{align*}
\]
where all the horizontal arrows are isomorphisms and every vertical arrow is injective. Furthermore, the bottom horizontal arrows are morphisms of \( \mathcal{D}_T \)-modules. Hence we can take \( \omega = (\omega_1, \omega_0) \in C^n(W, W')(\mathcal{E}_Y^b) \) that is a representative of the image of \( 1 \in \Gamma(T; \mathcal{E}_T) \) by the bottom horizontal arrows. It follows from Lemma 7.0.2 that the \( \omega \) is assumed to satisfy the following conditions:

1. \( \text{supp}_{\mathcal{W}_1}(\omega_1) \subset \Omega \) and \( \text{supp}_{\mathcal{W}_0}(\omega_0) \subset \Omega. \)
2. For any vector fields \( \nu \) on \( T \), we have \( \nu[\omega] = 0 \) since \( [\omega] \) is the image of 1 and the bottom horizontal morphisms in the commutative diagram are \( \mathcal{D}_T \)-linear.

Let \( (\theta_1, \cdots, \theta_n) \) be a homogeneous coordinate system of \( S^{n-1} \), and let \( \pi : M^* \setminus \{0\} \to S^{n-1} \) a smooth map defined by
\[ (\eta_1, \cdots, \eta_n) \mapsto \left( \frac{\eta_1}{|\eta|}, \frac{\eta_2}{|\eta|}, \cdots, \frac{\eta_n}{|\eta|} \right), \]
which induces the morphism of vector bundles
\[ \pi' : T(M^* \setminus \{0\}) \to (M^* \setminus \{0\}) \times_{S^{n-1}} TS^{n-1}. \]
By restricting the base space of the above bundle map to \( S^{n-1} \subset M^* \setminus \{0\} \), we get the morphism of vector bundles
\[
\varphi : TM^*|_{S^{n-1}} \to TS^{n-1},
\]
by which we define the vector fields \( \nu_k \) on \( T = S^{n-1} \) as
\[
\nu_k = \varphi(\frac{\partial}{\partial \eta_k}|_{S^{n-1}}) \quad (k = 1, 2, \ldots, n).
\]
(7.8)

Then, since \( \nu_k[\omega] = 0 \) holds, it follows from Lemma [0.2] that there exists \( \tilde{\omega}_k = (\tilde{\omega}_{k,1}, \tilde{\omega}_{k,01}) \in C^{n-1}(\mathcal{W}, \mathcal{W}')(\mathcal{E}\mathcal{D}^{*(\gamma)}) \) with
\[
\text{supp}_{\mathcal{W}}(\tilde{\omega}_{k,1}) \subset \Omega \quad \text{and} \quad \text{supp}_{\mathcal{W}_{01}}(\tilde{\omega}_{k,01}) \subset \Omega,
\]
such that
\[
\nu_k \omega = D_{\mathcal{E}\mathcal{D}} \tilde{\omega}_k,
\]
from which we have (\( \zeta = \xi + \sqrt{-1} \eta \))
\[
\frac{\partial}{\partial \xi} \left( \rho(\omega)(\eta/|\eta|, z) \right) = \frac{\sqrt{-1}}{|\eta|} \rho(\nu_k \omega)(\eta/|\eta|, z) = \frac{\sqrt{-1}}{|\eta|} \rho(D_{\mathcal{E}\mathcal{D}} \tilde{\omega}_k)(\eta/|\eta|, z) = \sqrt{-1} \gamma(\tilde{\omega}_k)(\eta/|\eta|, z).
\]
(7.9)

Let us consider \((\xi_0, \psi_{00})\) and \((\xi_1, \psi_{01})\), which generate the \( n \)-dimensional chains \( \gamma_0 \) and \( \gamma_1 \), respectively. Then, by taking a continuous path \( s(\lambda) (\lambda \in [0, 1]) \) in \( S \) with \( s(0) = \xi_0 \) and \( s(1) = \xi_1 \), we define an \((n+1)\)-dimensional chain \( \bar{\gamma}^* \) by
\[
\bar{\gamma}^* := \{ \xi + \sqrt{-1} \eta \in E^* ; \xi = ((1 - \lambda) \psi_{00}(|\eta|) + \lambda \psi_{01}(|\eta|)) s(\lambda), \quad 0 \leq \lambda \leq 1, \quad \eta \in M^* \setminus \{0\} \}.
\]
Here we may assume \( \bar{\gamma}^* \subset W_\psi \). In fact, we first consider the pair of chains generated by \((\xi_0, \psi_{00})\) and \((\xi_0, g)\) where \( g \) is taken to be a sufficiently large infra-linear function. Then consider the pair of chains generated by \((\xi_0, g)\) and \((\xi_1, g)\) and finally that by \((\xi_1, \psi_{01})\) and \((\xi_1, g)\).

By noticing that the function \( \frac{1}{|\eta|} \) on \( M^* \setminus \{0\} \) is integrable near the origin if \( n > 1 \) and that each \( \tilde{\omega}_k \) satisfies the same support condition as that for \( \omega \), it follows from the Stokes formula that we obtain
\[
\int_{\bar{\gamma}^*} f(\zeta) \bar{\partial}_\zeta (\rho(\omega)(\eta/|\eta|, z)) e^{\xi z} d\zeta
\]
\[
= \int_{\gamma_1} f(\zeta) \rho(\omega)(\eta/|\eta|, z) e^{\xi z} d\zeta - \int_{\gamma_0} f(\zeta) \rho(\omega)(\eta/|\eta|, z) e^{\xi z} d\zeta.
\]
It follows from (7.3) that we have
\[
\int_{\bar{\gamma}^*} f(\zeta) \bar{\partial}_\zeta (\rho(\omega)(\eta/|\eta|, z)) e^{\xi z} d\zeta = \bar{\partial} \int_{\bar{\gamma}^*} f(\zeta) e^{\xi z} \sum_{k=1}^{n} \frac{\partial}{\partial \xi_k} (\rho(\omega)(\eta/|\eta|, z)) d\zeta_k \wedge d\zeta
\]
\[
= \bar{\partial} \int_{\bar{\gamma}^*} f(\zeta) e^{\xi z} \frac{\sqrt{-1}}{|\eta|} \sum_{k=1}^{n} \rho(\tilde{\omega}_k)(\eta/|\eta|, z) d\zeta_k \wedge d\zeta.
\]
Hence the Laplace transform of \( f \) with the chain \( \gamma^*_0 \) and the one with the chain \( \gamma^*_1 \) give the same cohomology class.

Let us show (7.6) in the lemma. Fix \( \xi_0 \in S \) and take a sufficiently large \( \ell > 0 \) so that \( \psi_{\xi_0}(t) \leq t + \ell \) holds for \( t \in [0, \infty) \). Let us consider the \( n \)-dimensional chain \( \gamma^*_\epsilon \) for \( 1 \geq \epsilon > 0 \)
\[
\gamma^*_\epsilon := \left\{ \xi + \sqrt{-1} \eta \in E^*; \xi = (\epsilon^{-1} |\eta| + \ell) \xi_0, \eta \in M^* \setminus \{0\} \right\}
\]
and the \((n + 1)\)-dimensional chain
\[
\tilde{\gamma}^*_\epsilon = \left\{ \xi + \sqrt{-1} \eta \in E^*; \xi = \left((1 - \lambda)\psi_{\xi_0}(|\eta|) + \lambda(\epsilon^{-1} |\eta| + \ell)\right) \xi_0 \right\}.
\]
Note that \( \tilde{\gamma}^*_\epsilon \subset W_{\psi} \) holds for \( 1 \geq \epsilon > 0 \). Then, on \( \{z \in E; \Re\langle \xi_0, z \rangle < h(\xi_0)\} \), by taking the estimate (7.5) into account, we have
\[
\int_{\tilde{\gamma}^*_\epsilon} f(\zeta) \overline{\varphi}(\rho(\omega)(|\eta|, z)) e^{iz} d\zeta = \int_{\gamma^*_\epsilon} f(\zeta) \rho(\omega)(|\eta|, z) e^{iz} d\zeta - \int_{\gamma^*_\epsilon} f(\zeta) \rho(\omega)(|\eta|, z) e^{iz} d\zeta,
\]
where all the integrals converge. Hence, by letting \( \epsilon \to 0 + 0 \), we get
\[
\int_{\tilde{\gamma}^*_{0+0}} f(\zeta) \overline{\varphi}(\rho(\omega)(|\eta|, z)) e^{iz} d\zeta = \int_{\gamma^*_{0}} f(\zeta) \rho(\omega)(|\eta|, z) e^{iz} d\zeta.
\]
Here the \((n + 1)\)-dimensional chain \( \tilde{\gamma}^*_{0+0} \) is
\[
\tilde{\gamma}^*_{0+0} := \{\xi + \sqrt{-1} \eta \in E^*; \xi = \lambda \xi_0, \lambda \geq \psi_{\xi_0}(|\eta|), \eta \in M^* \setminus \{0\}\}
\]
and all the integrals still converge. This implies that, as the left hand side of the above equation gives the zero cohomology class in \( \tilde{\gamma} \{z \in E; \Re\langle \xi_0, z \rangle < h(\xi_0)\} \), and thus, \( \text{supp}(\mathcal{L}(f)) \) is contained in \( \{x \in M; \Re\langle \xi_0, x \rangle \geq h(\xi_0)\} \). Since we can take any vector in \( S \) as \( \xi_0 \), we have concluded the second claim of this lemma when \( n > 1 \).

Now we consider the case \( n = 1 \). In this case, \( S^{n-1} \) consists of only two points \( \{+1, -1\} \). Hence it follows from the definition of \( \omega \) that \( \tau = \omega(1, z) \) (resp. \( \tau = \omega(-1, z) \)) satisfies the conditions in Lemma 5.2.2 with \( \Omega = \Omega^1_+ \) (resp. \( \Omega = \Omega^1_- \)), where
\[
\Omega^1_\pm = \{z \in \mathbb{C}; \pm \text{Im} z > 0\} \subset \mathbb{D}_\mathbb{C}.
\]
Hence we have obtained
\[
\mathcal{L}_\omega(f \overline{d\zeta}) = b_{\Omega^1_+} \left( \frac{1}{2\pi \sqrt{-1}} \int_{\gamma^* \cap \Omega^1_+} e^{iz} f(\zeta) d\zeta \right) - b_{\Omega^1_-} \left( \frac{1}{2\pi \sqrt{-1}} \int_{\gamma^* \cap \Omega^1_-} e^{iz} f(\zeta) d\zeta \right)
\]
for which we can easily see the claims of the lemma. This completes the proof.

As an immediate application of the above lemma, we have the following corollary. Recall that, for a subset \( G \subset M \), we define
\[
G^o = \{\zeta \in E^*; \Re\langle \zeta, x \rangle \geq 0 \ (\forall x \in G)\}.
\]
Corollary 7.0.6. Let \( a \in M \) and \( G \neq \emptyset \) be an \( \mathbb{R}_+ \)-conic proper closed convex subset in \( M \). Furthermore, we also assume that \( G^0 \cap M^*_{\infty} \) is connected when \( n = 1 \). Set \( K = a + G \subset \mathbb{D}_M \) and let \( e^a \zeta \) \( g(\zeta) \in \mathcal{G}_{E_{\infty}^0}(N_{pc}(K)) = \mathcal{G}_{E_{\infty}^0}(^\circ/int G^0) \cap E_{\infty}^0). \) Then we have \( \text{supp}(\mathcal{L}(g)) \subset K. \) (7.10)

In fact, the corollary follows from the lemma by taking \( S = N_{pc}(K) \cap M^*_{\infty} \) and \( h(\xi) = a \xi \) and by noticing the facts that \( \overline{\mathcal{M}}_{M_{\infty}^*}(S) = N_{pc}(K) \) (resp. \( \overline{\mathcal{M}}_{M_{\infty}^*}(S) \cup \bigwedge M_{\infty}^* = N_{pc}(K) = E_{\infty}^* \)) holds if \( K \cap M_{\infty} \neq \emptyset \) (resp. \( K \cap M_{\infty} = \emptyset \)) and that \( S \) is connected.

7.1 Concrete construction of \( \omega \)

Now we give a method to construct \( \omega \) concretely. Let \( \mathcal{O} \) be a subset in \( S^{n-1} = \{ \xi \in M^*; |\xi| = 1 \} \), and let \( \theta_k : \mathcal{O} \rightarrow S^{n-1} \subset M^* \) \((k = 1, \ldots , n)\) be continuous maps on \( \mathcal{O} \). Set, for \( \xi \in \mathcal{O} \),

\[
\kappa(\xi) := \bigcap_{k=1}^n \{ x \in M; \langle x, \theta_k(\xi) \rangle > 0 \} \subset M.
\]

We assume that there exists \( \delta > 0 \) satisfying

C1. \( S^{n-1} \setminus \mathcal{O} \) is measure zero.

C2. \( \kappa(\xi) \subset \{ x \in M; \langle x, \xi \rangle > \sigma|x|\} \) for any \( \xi \in \mathcal{O} \).

C3. Let \( A(\xi) \) be an \( n \times n \)-matrix \((\theta_1(\xi), \ldots , \theta_n(\xi))\). Then \( \text{det}(A(\xi)) \geq \delta \) for any \( \xi \in \mathcal{O} \).

Note that the condition C2 is equivalent to the following C2’:

C2’. Set \( G(\xi) := \sum_{k=1}^n \mathbb{R}_+ \theta_k(\xi) \). Then we have

\[
\text{dist}(\xi, \mathbb{R}^n \setminus G(\xi)) > \delta \quad (\xi \in \mathcal{O}).
\]

In fact, C2’ implies

\[
\left\{ \tau \in \mathbb{R}^n; \left| \frac{\tau}{|\tau|} - \xi \right| \leq \frac{\delta}{2} \right\} \subset G(\xi).
\]

Then, by taking the dual of the above sets and by noticing \( \kappa(\xi) = \text{int } G(\xi)^{\circ} \), we can obtain C2.

Let \( \varphi_0(z), \ldots , \varphi_n(z) \) be in \( \mathcal{Q}_{\mathcal{D}_E}(\mathbb{D}_E \setminus \mathbb{D}_M) \) which are given in Example 5.2.6 with \( U = \mathbb{D}_M \), \( V = \mathbb{D}_E \);

\[
\eta_k = \begin{cases} 
(k + 1)\text{-th} \, \text{component} & (k = 0, \ldots , n - 1), \\
1 & (k = n) 
\end{cases} \quad \eta_k = -\eta_0 + \cdots + \eta_{n-1}/|\eta_0 + \cdots + \eta_{n-1}| \\
\eta_0 = 0, \ldots , 0, 1, 0, \ldots , 0 \\
\eta_k = 0, \ldots , 0, 1, 0, \ldots , 0 \\
\eta_k = 0, \ldots , 0, 1, 0, \ldots , 0 \\
\eta_k = 0, \ldots , 0, 1, 0, \ldots , 0
\]

and \( H_k = \Gamma_k = \{ y \in M; \langle y, \eta_k \rangle > 0 \} \) \((k = 0, 1, \ldots , n)\). Using these \( \varphi_k \)'s, we define \( \omega_{01} \) by

\[
\omega_{01}(\xi, z) := (-1)^n(n - 1)! \chi_{S_{01} \ldots n-1}(tA(\xi)z) \tilde{\partial}_z(\varphi_0(tA(\xi)z)) \wedge \cdots \wedge \tilde{\partial}_z(\varphi_{n-2}(tA(\xi)z))
\]

where \( S_{01} \ldots n-1 \) is also given in Example 5.2.6. Then, by the same reasoning as that of Example 7.14 in [1] and Corollary 3.0.7, we have
Lemma 7.1.1. Thus constructed $\omega = (0, \omega_{01})$ satisfies the conditions D1. and D2. described before Lemma 7.0.2.

We give some examples of such a family $\theta_k$'s.

Example 7.1.2. Let $\chi$ be a triangulation of $S^{n-1}$, and let $\{\sigma_{\lambda}\}_{\lambda \in \Lambda}$ be the set of $(n - 1)$-cells of $\chi$. For each $\lambda \in \Lambda$, we take linearly independent $n$-vectors $\nu_{\lambda,1}, \cdots, \nu_{\lambda,n} \in M^*$ which satisfies

$$\overline{\sigma_{\lambda}} \subset \sum_{k=1}^{n} \mathbb{R}^+ \nu_{\lambda,k},$$

and $\det A_{\lambda} > 0$ for the constant matrix $A_{\lambda} := (\nu_{\lambda,1}, \nu_{\lambda,2}, \cdots, \nu_{\lambda,n})$. Note that such a family of constant vectors always exists if each $\sigma_{\lambda}$ is sufficiently small. Furthermore, we may assume the frame $\nu_{\lambda,1}, \nu_{\lambda,2}, \cdots, \nu_{\lambda,n}$ determine the positive orientation in $M^*$ for each $\lambda$, Then, we set $O := \cup_{\lambda \in \Lambda} \sigma_{\lambda}$ and, for $k = 1, \cdots, n$, define $\theta_k(\xi)$ on $O$ by

$$\theta_k(\xi) = \nu_{\lambda,k} \quad (\xi \in \sigma_{\lambda}).$$

Clearly these $O$ and $\theta_k$'s satisfy the conditions C1, C2 and C3.

Example 7.1.3. Assume $M^*$ has an inner product. Let $p$ be a point in $S^{n-1}$ and set $O := S^{n-1} \setminus \{p\}$. Then $O$ becomes contractible, and hence, there exists a continuous orthogonal frame $\hat{\theta}_1(\xi), \cdots, \hat{\theta}_n(\xi) \in M^*$ on $O$. Here we may assume $\hat{\theta}_1(\xi) = \xi$. Set, for some $\delta > 0$,

$$\theta_1(\xi) := \hat{\theta}_1(\xi) + \delta \hat{\theta}_1(\xi),$$

$$\theta_2(\xi) := \hat{\theta}_2(\xi) + \delta \hat{\theta}_1(\xi),$$

$$\vdots$$

$$\theta_{n-1}(\xi) := \hat{\theta}_{n-1}(\xi) + \delta \hat{\theta}_1(\xi),$$

$$\theta_n(\xi) := -(\hat{\theta}_2 + \cdots + \hat{\theta}_{n}(\xi)) + \delta \hat{\theta}_1(\xi).$$

Then these $O$ and $\theta_k$'s satisfy the conditions C1, C2 and C3.

Let us compute $\mathcal{I}\mathcal{C}$ when $\omega$ comes from Example 7.1.2. In this case, on each $\sigma_{\lambda}$, $\omega_{01}(\xi, z)$ does not depend on the variables $\xi$. Hence we obtain

$$\mathcal{I}\mathcal{C}(f) := \left[\left(\frac{1}{2\pi \sqrt{-1}}\right)^n \left(0, \sum_{\lambda \in \Lambda} \tau_{01,\lambda} \int_{\gamma_{\lambda}^*} f(\zeta) e^{i\zeta} d\zeta\right)\right] \otimes a_{D/M} \otimes \nu_{D/M}. \quad (7.11)$$

Here

$$\gamma_{\lambda}^* := \{\xi = \xi + \sqrt{-1}\eta \in E^*; \eta \in \mathbb{R}+\sigma_{\lambda}, \xi = \psi_{0}\eta, \xi_0\},$$

and

$$\tau_{01,\lambda}(z) := (-1)^n(n - 1)! \chi_s_{01-\cdots-1}(^tA_{\lambda}z) \tilde{\partial}(\varphi_1(^tA_{\lambda}z)) \wedge \cdots \wedge \tilde{\partial}(\varphi_{n-1}(^tA_{\lambda}z)),$$

where the constant matrix $A_{\lambda}$ is given by $(\nu_{\lambda,1}, \cdots, \nu_{\lambda,n})$ and the orientation of the chain $\gamma_{\lambda}^*$ is induced from the one of $\sqrt{-1}M^*$ through the canonical projection $E^* = M^* \times$
\(\sqrt{-1}M^* \to \sqrt{-1}M^*\). Then, as we see in Example 5.2.6, \(\tau_\lambda := (0, \tau_{01, \lambda})\) satisfies the conditions in Lemma 5.2.2. Hence, by the definition of the boundary value map explained in Subsection 5.2, we have

\[
\mathcal{L}(f) = \sum_{\alpha \in \Lambda} b_{\Omega_\lambda} \left( \left( \frac{1}{2\pi \sqrt{-1}} \right)^{n_k} \int_{\gamma} f(\zeta)e^{\zeta \cdot \zeta}d\zeta \right) \otimes \nu_{\mathbb{D}_M} \in H^n(C(\mathbb{V}_{\mathbb{D}_M}, \mathbb{V}_{\mathbb{D}_M}')(\mathcal{Q}^{(n,*)}_{\mathbb{D}_E})),
\]

where \(\Omega_\lambda := M \tilde{\gamma} \sqrt{-1} \Gamma_\lambda\) with \(\Gamma_\lambda := \bigcap_{k=1}^n \{ y \in M ; \langle y, \nu_{\lambda,k} \rangle > 0 \}\).

Let \(\Lambda = \{+1, -1\}\). For \(\alpha = (\alpha_1, \cdots, \alpha_n) \in \Lambda^n\), we define

\[
\Gamma_\alpha := \{ x = (x_1, \cdots, x_n) \in M ; \alpha_k x_k > 0 \ (k = 1, \cdots, n) \},
\]
\[
\Gamma_\alpha^* := \{ \eta = (\eta_1, \cdots, \eta_n) \in M^* ; \alpha_k \eta_k > 0 \ (k = 1, \cdots, n) \}.
\]

We denote by \(+^n \in \Lambda^n\) (resp. \(-^n \in \Lambda^n\)) the multi-index in \(\Lambda^n\) whose entries are all +1 (resp. -1). Hence, \(\Gamma_{+^n}\) (resp. \(\Gamma_{-^n}\)) designates the first orthant of \(M\) (resp. \(M^*\)).

Let \(G \neq \emptyset\) be an \(\mathbb{R}_+\)-conic proper closed convex subset in \(M\) and \(a \in M\). Furthermore, we also assume that \(G^o \cap M^*_\infty\) is connected if \(n = 1\). Set \(K = a + G \subset \mathbb{D}_M\) and let \(f \in e^{-\alpha G^\text{int} E^\text{inf}_\infty (N^*_\text{pr}(K))} \in \mathbb{D}_M\). Suppose that \(G \setminus \{0\} \subset \Gamma_{+^n}\). Then \(f\) is holomorphic on \(W_\psi \cap E^*\) given in (7.1) with \(S = N^*_\text{pr}(K) \cap M^*_\infty\) and \(h(\xi) = a\xi\), and it satisfies (7.2) there. It follows from the assumption \(G \setminus \{0\} \subset \Gamma_{+^n}\) that we can find \(a^* = (a^*_1, \cdots, a^*_n) \in M^*\) such that the open subset \(W_\psi\) given in (7.1) satisfies

\[
a^* + \Gamma_{+^n} \subset W_\psi.
\]

Because of this fact, we can take a specific real \(n\)-chain \(\tilde{\gamma}^* \subset E^*\) defined below which enjoys some good properties:

\[
\tilde{\gamma}^* := \{ \zeta = \xi + \sqrt{-1} \eta \in E^* ; \eta \in M^* \setminus \{0\}, \xi = a^* + \hat{\psi}(\eta) \left( \frac{\eta_1}{|\eta|}, \frac{\eta_2}{|\eta|}, \cdots, \frac{\eta_n}{|\eta|} \right) \},
\]

where \(\hat{\psi}(\eta)\) is a continuous infra-linear function on \([0, \infty)\) which satisfies \(\hat{\psi}(0) = 0\) and \(\tilde{\gamma}^* \subset W_\psi\). Note that the orientation of \(\tilde{\gamma}^*\) is the same as that of \(\sqrt{-1}M^*\). For \(\alpha \in \Lambda^n\), we also define

\[
\tilde{\gamma}_\alpha^* := \{ \zeta = \xi + \sqrt{-1} \eta \in E^* ; \eta \in \Gamma_{\alpha^*}, \xi = a^* + \hat{\psi}(\eta) \left( \frac{\eta_1}{|\eta|}, \frac{\eta_2}{|\eta|}, \cdots, \frac{\eta_n}{|\eta|} \right) \}.
\]

We can replace the chain \(\gamma^*\) of \(\mathcal{L}_\omega\) in Definition 7.0.3 with the above chain \(\tilde{\gamma}^*\), which is guaranteed by the same proof as that in Lemma 7.0.5. Therefore, we have obtained

**Lemma 7.1.4.** Under the above situation, we can take the chain \(\tilde{\gamma}^*\) as the chain of the Laplace inverse integral of \(f\). In particular, we have

\[
\mathcal{L}(f) = \sum_{\alpha \in \Lambda^n} b_{\Omega_{\lambda, \alpha}} \left( \left( \frac{1}{2\pi \sqrt{-1}} \right)^{n_k} \int_{\tilde{\gamma}_\alpha} f(\zeta)e^{\zeta \cdot \zeta}d\zeta \right) \otimes \nu_{\mathbb{D}_M} \in H^n(C(\mathbb{V}_{\mathbb{D}_M}, \mathbb{V}_{\mathbb{D}_M}')(\mathcal{Q}^{(n,*)}_{\mathbb{D}_E})),
\]

where \(\Omega_{\alpha} := M \tilde{\gamma} \sqrt{-1} \Gamma_{\alpha} \subset \mathbb{D}_E\).
Note that each integral

\[ h_\alpha(z) := \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\tilde{\gamma}_\alpha} f(\zeta)e^{\xi\zeta}d\zeta \]  

(7.16)

belongs to \( \mathcal{O}_{\mathbb{D}_E}^{\text{exp}}(\Omega_\alpha) \). We will now explain an advantage of this expression: Set

\[ \Omega := \hat{\cdot}\{(\mathbb{C} \setminus \mathbb{R}_{\geq 0}) \times (\mathbb{C} \setminus \mathbb{R}_{\geq 0}) \times \cdots \times (\mathbb{C} \setminus \mathbb{R}_{\geq 0})\} \subset \mathbb{D}_E. \]

**Proposition 7.1.5.** For any \( \alpha \in \Lambda^n \), the \( \text{sgn}(\alpha)h_\alpha(z) \in \mathcal{O}_{\mathbb{D}_E}^{\text{exp}}(\Omega_\alpha) \) analytically extends to the same holomorphic function in \( \mathcal{O}_{\mathbb{D}_E}^{\text{exp}}(\Omega) \). Here we set \( \text{sgn}(\alpha) = \alpha_1\alpha_2\cdots\alpha_n \).

**Proof.** Let \( \beta \) be the subset in \( \{1, \ldots, n\} \), and set

\[
\Omega_{\alpha,\beta} := \Omega_\alpha \bigcap \{z \in \mathbb{E}; \Re z_k < 0 (k \in \beta)\}
\]

\[
= \hat{\cdot}\{z = x + \sqrt{-1}y \in \mathbb{E}; x_k < 0 (k \in \beta), \alpha_jy_j > 0 (j = 1, 2, \ldots, n)\}
\]

and

\[
\hat{\Omega}_{\alpha,\beta} := \hat{\cdot}\{z = x + \sqrt{-1}y \in \mathbb{E}; x_k < 0 (k \in \beta), \alpha_jy_j > 0 (j \notin \beta)\}.
\]

Clearly we have

\[ \Omega_{\alpha,\beta} \subset \hat{\Omega}_{\alpha,\beta}, \quad \Omega = \bigcup_{\alpha\in\Lambda^n, \beta\subset\{1,2,\ldots,n\}} \hat{\Omega}_{\alpha,\beta}. \]

Let us define the continuous function \( \tilde{\gamma}_{\alpha,\beta} : [0, 1] \times \Gamma_\alpha^* \to \mathbb{E}^* \) by

\[ \tilde{\gamma}_{\alpha,\beta}(s, \eta) := \xi + \sqrt{-1}\eta \quad (\eta \in \Gamma_\alpha^*, s \in [0, 1]). \]

Here

\[ \xi = \alpha^* + \left((1 - \delta_{\beta,1}(s))\hat{\psi}(\eta) + \delta_{\beta,1}(s)|\eta| \right) |\eta_1|, \ldots, \left((1 - \delta_{\beta,n}(s))\hat{\psi}(\eta) + \delta_{\beta,n}(s)|\eta| \right) |\eta_n| \]

and

\[ \tilde{\eta} = (1 - \delta_{\beta,1}(s))\eta_1, \ldots, (1 - \delta_{\beta,n}(s))\eta_n, \]

where \( \delta_{\beta,k}(s) = s \) if \( k \in \beta \) and \( \delta_{\beta,k}(s) = 0 \) otherwise. Since \( \tilde{\gamma}_{\alpha,\beta}(0, \Gamma_\alpha^*) = \tilde{\gamma}_\alpha^* \) holds, we have

\[ \partial \tilde{\gamma}_{\alpha,\beta}([0, 1], \Gamma_\alpha^*) = -\tilde{\gamma}_\alpha^* + \tilde{\gamma}_{\alpha,\beta}(1, \Gamma_\alpha^*) - \tilde{\gamma}_{\alpha,\beta}([0, 1], \partial \Gamma_\alpha^*). \]

Let \( z \) be a point in \( \Omega_{\alpha,\beta} \). Then, as \( f \) is holomorphic, we have

\[
0 = \int_{\tilde{\gamma}_{\alpha,\beta}([0, 1], \Gamma_\alpha^*)} d(f(\zeta)e^{\xi\zeta}d\zeta) = \int_{\partial \tilde{\gamma}_{\alpha,\beta}([0, 1], \Gamma_\alpha^*)} f(\zeta)e^{\xi\zeta}d\zeta,
\]

which implies

\[
\int_{\tilde{\gamma}_{\alpha,\beta}(1, \Gamma_\alpha^*)} f(\zeta)e^{\xi\zeta}d\zeta - \int_{\tilde{\gamma}_{\alpha,\beta}([0, 1], \partial \Gamma_\alpha^*)} f(\zeta)e^{\xi\zeta}d\zeta = \int_{\tilde{\gamma}_{\alpha,\beta}([0, 1], \partial \Gamma_\alpha^*)} f(\zeta)e^{\xi\zeta}d\zeta.
\]

Note that

\[ \tilde{\gamma}_{\alpha,\beta}([0, 1], \partial \Gamma_\alpha^*) = \bigcup_{k=1}^n \left( \tilde{\gamma}_{\alpha,\beta}([0, 1], \partial \Gamma_\alpha^*) \cap \{\zeta_k = a_k^*\} \right) \]
holds. By noticing $d\zeta_k = 0$ on each real $n$-chain $\tilde{\gamma}_{\alpha,\beta}^{*}([0,1], \partial \Gamma^*) \cap \{\zeta_k = a^*_k\}$, we get
\[
\int_{\tilde{\gamma}_{\alpha,\beta}^{*}([0,1], \partial \Gamma^*)} f(\zeta) e^{\zeta \cdot z} d\zeta = 0,
\]
from which
\[
\int_{\tilde{\gamma}_{\alpha}^{*}([0,1], \partial \Gamma^*)} f(\zeta) e^{\zeta \cdot z} d\zeta = \int_{\tilde{\gamma}_{\alpha}^{**}(1, \Gamma^*)} f(\zeta) e^{\zeta \cdot z} d\zeta
\]
follows. It is easy to see that the last integral belongs to $\mathcal{O}(\exp(\overline{\Omega}_{\alpha,\beta}^*))$. Hence, by taking arbitrary $\beta \subset \{1, \ldots, n\}$, we see that $\text{sgn}(\alpha)h_\alpha(z)$ analytically extends to $\bigcup_{\alpha,\beta} \overline{\Omega}_{\alpha,\beta}$. In particular, on $\overline{\Omega}_{\alpha,\beta}$ with $\beta = \{1, \ldots, n\}$, i.e.,
\[
\overline{\Omega}_{\alpha,\beta} = \{z = x + \sqrt{-1}y \in E; x_k < 0 \ (k = 1, \ldots, n)\},
\]
$\text{sgn}(\alpha)h_\alpha(z)$ coincides with the integration on the real domain
\[
\left(\frac{1}{2\pi \sqrt{-1}}\right)^n \int_{a^*+\Gamma^*_+} f(\xi) e^{\xi \cdot z} d\xi,
\]
which does not depend on the index $\alpha \in \Lambda^n$. Therefore, all the analytic extensions of $\text{sgn}(\alpha)h_\alpha$ coincide on this domain, and thus, they form the holomorphic function of exponential type on the domain
\[
\bigcup_{\alpha,\beta} \overline{\Omega}_{\alpha,\beta} = \Omega.
\]
This completes the proof.

\section{8 Laplace inversion formula}

This section is devoted to proof for the Laplace inversion formula, that is, $\mathcal{L}$ and $\mathcal{I}\mathcal{L}$ are mutually inverse.

\textbf{Theorem 8.0.1.} Let $G \neq \emptyset$ be an $\mathbb{R}_+$-conic proper closed convex subset in $M$ and $a \in M$. Set $K = a + G \subset \mathbb{D}_M$. Furthermore, we also assume that $N_{pc}^*(K) \cap M^*_\infty$ is connected when $n = 1$. Then the Laplace transformation
\[
\mathcal{L} : \Gamma_K(\mathbb{D}_M; \mathcal{B}_{\mathbb{D}_M}^{\exp} \otimes \mathcal{A}_{\mathbb{D}_M}^{\exp} \mathcal{V}_{\mathbb{D}_M}^{\exp}) \to e^{-a\zeta} \mathcal{O}_{E_\infty}^\inf (N_{pc}^*(K))
\]
and the inverse Laplace transformation
\[
\mathcal{I}\mathcal{L} : e^{-a\zeta} \mathcal{O}_{E_\infty}^\inf (N_{pc}^*(K)) \to \Gamma_K(\mathbb{D}_M; \mathcal{B}_{\mathbb{D}_M}^{\exp} \otimes \mathcal{A}_{\mathbb{D}_M}^{\exp} \mathcal{V}_{\mathbb{D}_M}^{\exp})
\]
are inverse to each other.

\textbf{Remark 8.0.2.} For $K$ and $G$ in the above theorem,
\[
N_{pc}^*(K) = N_{pc}^*(G) = \overline{\text{int} \ G^\circ} \cap E^*_\infty
\]
hold, where $G^\circ$ is the dual cone of $G$ in $E^*$, that is,
\[
G^\circ = \{\zeta \in E^*; \text{Re} \langle \zeta, x \rangle \geq 0 \ (\forall x \in G)\}.
\]
Thanks to Corollary 6.2.8 and Lemmas 7.0.5 and 8.0.4, the following corollary immediately follows from Theorem 8.0.1.

**Corollary 8.0.3.** Let $K$ be a regular closed subset in $\mathbb{D}_M$ satisfying that $K \cap M$ is convex and $N^*_p(K) \cap M^*_\infty$ is connected (in particular, non-empty). Then the Laplace transformation

$$\mathcal{L} : \Gamma_K(\mathbb{D}_M; \mathcal{B}^{\text{exp}}_\mathbb{D}M \otimes \mathcal{B}_M^\text{exp} \mathcal{V}_\mathbb{D}^\text{exp}) \to \mathcal{O}_{E^\infty}(N^*_p(K))$$

and the inverse Laplace transformation

$$\mathcal{IL} : \mathcal{O}_{E^\infty}(N^*_p(K)) \to \Gamma_K(\mathbb{D}_M; \mathcal{B}^{\text{exp}}_\mathbb{D}M \otimes \mathcal{B}_M^\text{exp} \mathcal{V}_\mathbb{D}^\text{exp})$$

are inverse to each other.

**Lemma 8.0.4.** Let $K$ be a regular closed subset in $\mathbb{D}_M$. Assume that $K$ is convex and that $N^*_p(K)$ is also non-empty. Then we have

$$K = \bigcap_{\xi \in N^*_p(K) \cap M^*_\infty} \{x \in M; \langle x, \xi \rangle \geq h_K(\xi)\}.$$  

*Proof.* If $K$ is an empty set, both the sides in the above equality become empty sets as $h_K(\xi) = +\infty$, and hence, we may assume $K \neq \emptyset$. It is enough to show that, for any $x_0 \in M$ with $x_0 \notin K$, there exists a hypersurface $L$ in $M$ passing through $x_0$ such that $K$ and $L$ are disjoint in $\mathbb{D}_M$.

Since $N^*_p(K)$ is not empty, we can take $\xi_0 \in N^*_p(K) \cap M^*_\infty$ and $r \in \mathbb{R}$ such that

$$K \subset \{x \in M; \langle x, \xi_0 \rangle > r\}.$$  

Set

$$L_{\xi_0} := \{x \in M; \langle x, \xi_0 \rangle = r\}.$$  

We may assume $x_0 \in \{x \in M; \langle x, \xi_0 \rangle > r\}$ from the beginning.

Since $K \cap M$ is convex, we can find a hypersurface $L$ which separates $x_0$ and $K$ in $M$. The claim follows if $L$ also separates them in $\mathbb{D}_M$. Hence we may assume that $L \cap K \cap M^*_\infty$ is non-empty, from which we conclude that the both normal vectors of $L$ are not in $N^*_p(K)$, and thus, we have $\dim(L \cap L_{\xi_0}) = n - 2$.

We can take the hypersurface $\tilde{L}$ in $M$ which passes $x_0$ and $L \cap L_{\xi_0}$. Then the hypersurface $\tilde{L}$ has the required properties, which completes the proof.  

**8.1 The proof for $\mathcal{L} \circ \mathcal{IL} = \text{id}$.**

Let $f \in e^{-a^\zeta} \mathcal{O}_{E^\infty}(\Lambda \cap E^*_\infty)). By a coordinate transformation, we may assume that $a = 0$ and $G \subset \bar{\Gamma}^n \cup \{0\}$ from the beginning (see (7.13) for the set $\bar{\Gamma}^n$). Let $\Lambda = \{+1, -1\}$, and let $h_\alpha(z)$ ($\alpha \in \Lambda^n$) be a holomorphic function defined in (7.16). Then, by Lemma 7.1.4 we have

$$\mathcal{IL}(f) = \sum_{\alpha \in \Lambda^n} b_{\alpha_\zeta}(h_\alpha(z)) \otimes \nu_{\mathbb{D}_M}.$$
Note that \( \text{Supp}(\mathcal{IL}(f)) \subset \overline{G} \subset \Gamma_{+n} \cup \{0\} \) hold. It follows from Proposition 7.1.5 that we can compute the Laplace transform of \( \mathcal{IL}(f) \) by the formula given in Example 6.3.6. Hence, by noticing (5.6), we have

\[
(\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = \frac{1}{(2\pi \sqrt{-1})^n} \sum_{\alpha \in \Lambda^n} \text{sgn}(\alpha) \int_{\gamma_\alpha} dz \int_{\gamma_\alpha^*} f(\zeta) e^{(\zeta - \tilde{\zeta})z} d\zeta.
\]

Here we take \( \epsilon > 0 \) sufficiently small and \( \gamma_\alpha \subset E \) is given by

\[
\{ z = b + (B_\epsilon + \sqrt{-1}\epsilon A_\alpha)x \mid x \in \Gamma_{+n} \},
\]

where the diagonal matrix

\[
A_\alpha = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
& \vdots & \\
0 & \alpha_{n-1} & 0 \\
0 & 0 & \alpha_n
\end{pmatrix},
\]

\( b = -c(1, 1, \ldots, 1) \in \Gamma_{-n} \) with a sufficiently small \( c > 0 \) and \( B_\epsilon \) is given in Example 6.3.6. The \( \gamma_\alpha^* \subset E^* \) is given by

\[
\{ \zeta = a^* + \xi(\delta I + \sqrt{-1}A_\alpha) ; \xi \in \Gamma_{+n}^* \},
\]

where \( I \) is the identity matrix, \( \epsilon > \delta > 0 \) and \( a^* = a(1, 1, \cdots, 1) \in \Gamma_{+n}^* \) for a sufficiently large \( a > 0 \). Note that the orientation of \( \gamma_\alpha \) and \( \gamma_\alpha^* \) are determined by those of the parameter spaces \( \Gamma_{+n} \) and \( \Gamma_{+n}^* \), respectively.

**Remark 8.1.1.** The above integral does not depend on the choice of \( \epsilon > 0 \) if it is sufficiently small, and we make \( \epsilon \) tend to 0 later.

In what follows, we may assume that \( \tilde{\zeta} \in E^* \) is in a sufficiently small open neighborhood of \( a^* + \Gamma_{+n}^* \) and that \( |\tilde{\zeta}| \) is large enough. As a matter of fact, if we could show \( (\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = f(\tilde{\zeta}) \) for such a \( \tilde{\zeta} \), the claims follows from the unique continuation property of \( f \).

When \( z \in \gamma_\alpha \) and \( \zeta \in \gamma_\alpha^* \), we have

\[
\text{Re}(\zeta - \tilde{\zeta})z = -\text{Re} \tilde{\zeta}z + \text{Re} \zeta z
\]

\[
= \left( \langle a^* - \text{Re} \tilde{\zeta}, b + B_\epsilon x \rangle + \epsilon \sum_{k=1}^n \alpha_k x_k \text{Im} \tilde{\zeta}_k \right) + \delta(\xi, b) + \left( \delta(\xi, B_\epsilon x) - \epsilon \langle \xi, x \rangle \right).
\]

Note that, for \( x \in \Gamma_{+n} \) and \( \xi \in \Gamma_{+n}^* \), we have

\[
\left( \delta(\xi, B_\epsilon x) - \epsilon \langle \xi, x \rangle \right) \leq -\min\{\epsilon - \delta, \epsilon \delta\} |x||\xi| \leq 0.
\]

Hence the above integration absolutely converges and, by the Fubini’s theorem, we obtain

\[
(\mathcal{L} \circ \mathcal{IL})(f)(\tilde{\zeta}) = \frac{1}{(2\pi \sqrt{-1})^n} \sum_{\alpha \in \Lambda^n} \text{sgn}(\alpha) \int_{\gamma_\alpha^*} f(\zeta) d\zeta \int_{\gamma_\alpha} e^{(\zeta - \tilde{\zeta})z} dz.
\]
Then, if $\zeta$ is quite near $a^*$ and $\tilde{\zeta} \in E^*$ belongs to a sufficiently small open neighborhood of $a^* + \Gamma_{+a}^*$ and if $|\tilde{\zeta}|$ is large enough, we get
\[
\int_{\gamma_a^*} e^{(\zeta-\tilde{\zeta})z} dz = \det(Q_{\alpha,\epsilon}) \int_{\Gamma_{+a}} e^{(\zeta-\tilde{\zeta})(b+Q_{\alpha,\epsilon}x)} dx = \frac{\det(Q_{\alpha,\epsilon}) e^{(\zeta-\tilde{\zeta})b}}{(\zeta - \tilde{\zeta})Q_{\alpha,\epsilon}},
\]
where
\[
\frac{1}{(\zeta - \tilde{\zeta})Q_{\alpha,\epsilon}} := \frac{1}{\prod_{k=1}^n e_k(\zeta - \tilde{\zeta})Q_{\alpha,\epsilon}}.
\]
Here $e_k$ is the unit row vector whose $k$-th entry is 1 and
\[
Q_{\alpha,\epsilon} = B_\epsilon + \sqrt{-1} \epsilon A_\alpha.
\]
By uniqueness of the analytic continuation, the above formula holds at any point $\zeta$ in a neighborhood of the chain $\gamma_a^*$, and hence, we have
\[
(L \circ \mathcal{I})(f)(\zeta) = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\alpha \in \Lambda^n} \text{sgn}(\alpha) \det(Q_{\alpha,\epsilon}) \int_{\gamma_a^*} \frac{f(\zeta) e^{(\zeta-\tilde{\zeta})b}}{(\zeta - \tilde{\zeta})Q_{\alpha,\epsilon}} d\zeta.
\]
Now if we could show that there exist $s > \delta$ and a complex open neighborhood $T \subset \mathbb{C}$ of $(0, s)$ such that the denominator of the integrand in the above integral does not vanish when $\zeta \in \gamma_a^*$ and $\epsilon \in T$ ($\delta$ and other constants are fixed, where we do not keep the condition $\epsilon > \delta$ anymore), then the above integral becomes an analytic function of $\epsilon$ ($\epsilon > 0$), and thus, it turns out to be a constant function of $\epsilon$ due to Remark 8.11. Hence, by letting $\epsilon$ to 0, we have obtained
\[
(L \circ \mathcal{I})(f)(\zeta) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \sum_{\alpha \in \Lambda^n} \text{sgn}(\alpha) \int_{\gamma_a^*} \frac{f(\zeta) e^{(\zeta-\tilde{\zeta})b}}{\zeta - \tilde{\zeta}} d\zeta
\]
which is clearly equal to $f(\zeta)$ by the Cauchy integral formula.

Let $g(\zeta, \eta)$ be the first element of the vector $Q_{\alpha,\epsilon} - \eta$, and let us show $g(\zeta, \eta) \neq 0$ for any $\zeta \in \gamma_a^*$ and for any $\eta$ contained in a sufficiently small neighborhood of the point $R(1, 1, \cdots, 1)Q_{\alpha,\epsilon}$ with a sufficiently large $R > 0$. Set $\epsilon = \epsilon' + \sqrt{-1} \epsilon''$ for a sufficiently small $\epsilon' > 0$ and $\epsilon'' \in \mathbb{R}$ with $|\epsilon''| < \delta \epsilon'/2$. The real part of $g(\zeta, \eta)$ is, for $\zeta = a^* + \xi(\delta E + \sqrt{-1} A_\alpha)$ with $\xi \in \Gamma_{+a}$,
\[
(\delta - \epsilon' - \alpha_1 \epsilon'')\xi_1 - ((\delta \epsilon' - \alpha_2 \epsilon'')\xi_2 + \cdots + (\delta \epsilon' - \alpha_n \epsilon'')\xi_n) + (1 - (n - 1) \epsilon') - \alpha_1 \epsilon'')a - \text{Re } \eta_1
\]
and its imaginary part is
\[
\alpha_1((1 + \epsilon' - \alpha_1 \epsilon'')\xi_1 + (\epsilon' - (n - 1) \alpha_1 \epsilon'')a) - ((\alpha_2 \epsilon' + \delta \epsilon'')\xi_2 + \cdots + (\alpha_n \epsilon' + \delta \epsilon'')\xi_n) - \text{Im } \eta_1.
\]
If $\text{Re } g(\zeta, \eta) = 0$, then we have
\[
(\delta \epsilon' - \alpha_2 \epsilon'')\xi_2 + \cdots + (\delta \epsilon' - \alpha_n \epsilon'')\xi_n = (\delta - \epsilon' - \alpha_1 \epsilon'')\xi_1 - (\text{Re } \eta_1 - (1 - (n - 1) \epsilon') - \alpha_1 \epsilon'')a,
\]
which gives the estimate
\[
(\delta \epsilon' - |\epsilon''|)(\xi_2 + \cdots + \xi_n) \leq (\delta - (\epsilon' - |\epsilon''|))\xi_1 - (\text{Re } \eta_1 - a),
\]
that is, we have obtained
\[(\epsilon' - \delta^{-1}|\epsilon''|)(\xi_2 + \cdots + \xi_n) \leq (1 - \delta^{-1}(\epsilon' - |\epsilon''|))\xi_1 - \delta^{-1}(\Re \eta_1 - a).\]

Hence, when \(\Re g(\zeta, \eta) = 0\), we get
\[|\Im g(\zeta, \eta)| \geq (1 + \epsilon' - |\epsilon''|)(\xi_1 - (\epsilon' + \delta|\epsilon''|)(\xi_2 + \cdots + \xi_n)) - |\Im \eta_1| \geq \ell(\epsilon', \epsilon'')(\xi_1 + \delta^{-1}\left(\frac{\epsilon' + \delta|\epsilon''|}{\epsilon' - \delta^{-1}|\epsilon''|}\right)(\Re \eta_1 - a) - |\Im \eta_1|,
\]
where
\[\ell(\epsilon', \epsilon'') = (1 + \epsilon' - |\epsilon''|) - \left(\frac{\epsilon' + \delta|\epsilon''|}{\epsilon' - \delta^{-1}|\epsilon''|}\right)(1 - \delta^{-1}(\epsilon' - |\epsilon''|)).\]

Note that, for each \(\epsilon' > 0\), we have \(\ell(\epsilon', \epsilon'') > 0\) if \(|\epsilon''|\) is sufficiently small. In what follows, we consider the case for such an \(\epsilon = \epsilon' + \sqrt{-1}\epsilon''\). When \(\eta\) is contained in a sufficiently small neighborhood of the point \(R(1, 1, \cdots, 1)Q_{a, \epsilon}\), we have
\[\Re \eta_1 \sim R - ((n - 1)\epsilon' + \alpha_1\epsilon'')R, \quad \Im \eta_1 \sim (\alpha_1\epsilon' - (n - 1)\epsilon'')R.\]

Hence, if \(R\) is sufficiently large, \(\Im g(\zeta, \eta)\) never becomes zero. This completes the proof.

### 8.2 The proof for \(IL \circ L = \text{id}\).

Let \(G\) be an \(\mathbb{R}_+\)-conic proper closed convex subset in \(M\) and \(a \in M\). Set \(K = \overline{a + G} \subset \mathbb{D}_M\). Then we take an open convex cone \(V \subset \mathbb{D}_E\) containing \(K\). Let \(u \in \Gamma_K(\mathbb{D}_M; \mathcal{E}_{\mathbb{D}_E} \otimes \mathcal{F}_{\mathbb{D}_M})\) with a representative \(\nu = (\nu_1, \nu_0) \in C^n(V_K, V_{K'})(\mathcal{L}^{(n, \bullet)}_{\mathbb{D}_E})\). We will show \((IL \circ L)(u) = u\).

By a coordinate transformation, we may assume \(a = 0\) and
\[\overline{G \setminus \{0\}} \subset \overline{\Gamma_+^{\alpha}} \subset \overline{\Gamma_+^\alpha} \subset V\]
from the beginning. Then, it follows from Lemma \([7.1.4]\) that we get
\[(IL \circ L)(u) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \sum_{\alpha \in \Lambda_\alpha} b_{\Omega_\alpha} \left(\int_{\hat{\gamma}_\alpha^*} L(u) e^{\xi\bar{\zeta}} d\zeta\right).
\]

Set
\[g_\alpha(\tilde{z}) := \int_{\hat{\gamma}_\alpha^*} L(u) e^{\xi\bar{\zeta}} d\zeta.
\]

It follows from Proposition \([7.1.5]\) that \(g_\alpha\) extends to a holomorphic function on \(\Omega\) of exponential type. Here
\[\Omega = \overline{((C \setminus \mathbb{R}_\geq) \times (C \setminus \mathbb{R}_\geq) \times \cdots \times (C \setminus \mathbb{R}_\geq)} \subset \mathbb{D}_E\]
We first consider \(g_\alpha(\tilde{z})\) at a point in \(\Gamma_-^\alpha \times \sqrt{-1}\Gamma_\alpha\). Let us take \(\tilde{z}\) in \(\Gamma_-^\alpha \times \sqrt{-1}\Gamma_\alpha\) and fix it. Then, at this \(\tilde{z}\), we can deform the \(n\)-chain \(\tilde{\gamma}_\alpha^*\) to
\[\left\{\begin{array}{l}
\zeta = \xi + \sqrt{-1}\eta' \in E^*; \quad \eta'_k = \alpha_k \eta_k \ (k = 1, \ldots, n), \\
\xi = a^* + \epsilon' \eta; \quad \eta \in \Gamma_+^\alpha
\end{array}\right\}
\]
with \(a^* \in \Gamma_+^\alpha\) and \(\epsilon' > 0\). Here the orientation of the modified chain \(\tilde{\gamma}_\alpha^*\) is the same as the original one and we assume \(|a^*|\) to be sufficiently large.
Now, since $\zeta$ runs in $\tilde{\gamma}^*_\alpha$, the real $2n$-chain $D$ of the integration
\[ \mathcal{L}(u)(\zeta) := \int_D e^{-z\zeta} \nu_1(z) - \int_{\partial D} e^{-z\zeta} \nu_{01}(z), \]
can be taken so that it is an open subset in $\mathbb{D}_E$ with a good boundary such that
\[ \overline{G} \subset D \subset \{z = x + \sqrt{-1}y; x \in b + \Gamma', |y| < \epsilon \text{ dist}(x, M \setminus (b + \Gamma'))\}, \]
where $b = -\epsilon (1, \ldots, 1)$ and $\Gamma' \subset M$ is an $\mathbb{R}_+$-conic open convex cone such that
\[ \overline{G} \setminus \{0\} \subset \Gamma' \subset \Gamma \setminus \{0\} \subset \Gamma_+. \]
Note that, if $\zeta = \xi + \sqrt{-1}\eta \in \tilde{\gamma}^*_\alpha$ and $z = x + \sqrt{-1}y \in D \cap E$, we have
\[ \text{Re}(\tilde{z} - z) = (\text{Re} \tilde{z} - x)a^* - \sum_{k=1}^n \alpha_k (\text{Im} \tilde{z}_k) \eta_k + \left( \epsilon'(\text{Re} \tilde{z} - x) \eta + \sum_{k=1}^n \alpha_k y_k \eta_k \right). \]
If $\epsilon > 0$ is sufficiently small and $\text{Re} \tilde{z} \in b + \Gamma_-$, by noticing
\[ |y| < \epsilon(|b| + |x|) \quad (z = x + \sqrt{-1}y \in D \cap E), \]
we can easily see that, for any $z = x + \sqrt{-1}y \in D \cap E$ and $\zeta = \xi + \sqrt{-1}\eta \in \tilde{\gamma}^*_\alpha$,
\[ \epsilon'(\text{Re} \tilde{z} - x) \eta + \sum_{k=1}^n \alpha_k y_k \eta_k = \left( \epsilon'(\text{Re} \tilde{z} - x) + (\alpha_1 y_1, \ldots, \alpha_n y_n) \right) \eta \leq 0 \]
holds. Hence, the double integral in $g_\alpha$ absolutely converges and we can apply Fubini’s theorem to $g_\alpha$, from which we get
\[ g_\alpha(\tilde{z}) = \int_D \nu_1(z) \int_{\tilde{\gamma}^*_\alpha} e^{(\tilde{z} - z)\zeta} d\zeta - \int_{\partial D} \nu_{01}(z) \int_{\tilde{\gamma}^*_\alpha} e^{(\tilde{z} - z)\zeta} d\zeta. \]
Now let us consider the integral $\int_{\tilde{\gamma}^*_\alpha} e^{(\tilde{z} - z)\zeta} d\zeta$. If $\text{Re}(\tilde{z} - z) \in \Gamma_-$ and $|\text{Im}(\tilde{z} - z)|$ is sufficiently small, then we can deform the $n$-chain to the one in $M^*$ as was done in the proof of Proposition 7.1.3, we have
\[ \int_{\tilde{\gamma}^*_\alpha} e^{(\tilde{z} - z)\zeta} d\zeta = \text{sgn}(\alpha) \int_{a^* + \Gamma^*_n} e^{(\tilde{z} - z)\xi} d\xi = \text{sgn}(\alpha) \frac{e^{(\tilde{z} - z)a^*}}{z - \tilde{z}}, \]
where $\text{sgn}(\alpha) = \alpha_1 \alpha_2 \cdots \alpha_n$. Note that, by the unique continuation property,
\[ \int_{\tilde{\gamma}^*_\alpha} e^{(\tilde{z} - z)\zeta} d\zeta = \text{sgn}(\alpha) \frac{e^{(\tilde{z} - z)a^*}}{z - \tilde{z}} \]
holds at a point where the integral is defined. Summing up, we have obtained
\[ g_\alpha(\tilde{z}) = \text{sgn}(\alpha) \left( \int_D \frac{e^{(\tilde{z} - z)a^*} \nu_1(z)}{z - \tilde{z}} - \int_{\partial D} \frac{e^{(\tilde{z} - z)a^*} \nu_{01}(z)}{z - \tilde{z}} \right) \]
if $\tilde{z} \in \Gamma_- \times \sqrt{-1} \Gamma_. \alpha$. By deforming $D$ appropriately, we see that the integrals in the right-hand side converge on $M \times \sqrt{-1} \Gamma_. \alpha$, and hence, the above equation also holds there. It follows from Theorem 6.4.3 that we have
\[ \left( \frac{1}{2\pi \sqrt{-1}} \right)^n \sum_{\alpha \in \Lambda^n} \text{sgn}(\alpha) b_\alpha \left( \int_D \frac{e^{(\tilde{z} - z)a^*} \nu_1(z)}{z - \tilde{z}} - \int_{\partial D} \frac{e^{(\tilde{z} - z)a^*} \nu_{01}(z)}{z - \tilde{z}} \right) = [\nu] = u. \]
This completes the proof.
9 Application to PDE with constant coefficients

Let $\mathcal{R}$ be the polynomial ring $\mathbb{C}[\zeta_1, \cdots, \zeta_n]$ on $E^*$ and $\mathfrak{D}$ the ring $\mathbb{C}[\partial_{x_1}, \cdots, \partial_{x_n}]$ of linear differential operators on $M$ with constant coefficients. We denote by $\sigma$ the principal symbol map from $\mathfrak{D}$ to $\mathcal{R}$, that is,

$$\mathfrak{D} \ni P(\partial) = \sum c_\alpha \partial^\alpha \mapsto \sigma(P)(\zeta) = \sum_{|\alpha| = \text{ord}(P)} c_\alpha \zeta^\alpha \in \mathcal{R}.$$ 

For an $\mathfrak{D}$-module $M = \mathfrak{D}/\mathfrak{I}$ with the ideal $\mathfrak{I} \subset \mathfrak{D}$, we define the closed subset in $E^*_\infty$

$$\text{Char}_{E^*_\infty}(\mathfrak{M}) = \{\zeta \in E^*_\infty; \sigma(P)(\zeta) = 0 \ (\forall P \in \mathfrak{I})\}.$$ 

Here we identify a point in $E^*_\infty$ with a unit vector in $E^*$.

Recall that $\{f_1, \cdots, f_\ell\}$ $(f_k \in \mathcal{R})$ is said to be a regular sequence over $\mathcal{R}$ if and only if the conditions below are satisfied:

1. $(f_1, \cdots, f_\ell) \neq \mathcal{R}$.
2. For any $k = 1, 2, \cdots, \ell$, the $f_k$ is not a zero divisor on $\mathfrak{R}/(f_1, \cdots, f_{k-1})$.

The following theorem is fundamental in the theory of operational calculus: Let $P_1(\partial), \cdots, P_\ell(\partial)$ be in $\mathfrak{D}$, and define the $\mathfrak{D}$-module

$$\mathfrak{M} = \mathfrak{D}/(P_1(\partial), \cdots, P_\ell(\partial)).$$

**Theorem 9.0.1.** Let $K$ be a regular closed subset in $\mathbb{D}_M$. Assume that $K \cap M$ is convex and $N^*_\text{pc}(K) \cap M^*_\infty$ is connected (in particular, it is non-empty), and that $P_1(\zeta), \cdots, P_\ell(\zeta)$ form a regular sequence over $\mathcal{R}$. Then the condition

$$N^*_\text{pc}(K) \cap \text{Char}_{E^*_\infty}(\mathfrak{M}) = \emptyset$$

implies

$$\text{Ext}^k_\mathfrak{D}(\mathfrak{M}, \Gamma_K(\mathbb{D}_M; \mathscr{B}_\mathbb{D}_M^{\text{exp}})) = 0 \ (k = 0, 1).$$

**Proof.** Let $\mathcal{F}$ be a sheaf of $\mathbb{Z}$-modules or a $\mathbb{Z}$-module itself and $s_i : \mathcal{F} \rightarrow \mathcal{F} (i = 1, \cdots, \ell)$ a morphism such that $s_i \circ s_j = s_j \circ s_i$ holds for $1 \leq i, j \leq \ell$. Then we denote by $K(s_1, \cdots, s_\ell; \mathcal{F})$ the Koszul complex associated to $(s_1, \cdots, s_\ell)$ with coefficients in $\mathcal{F}$. That is,

$$0 \rightarrow \mathcal{F} \otimes (\Lambda^{\ell}) \xrightarrow{d} \mathcal{F} \otimes (\Lambda^{1}) \xrightarrow{1} \mathcal{F} \otimes (\Lambda^{2}) \xrightarrow{2} \cdots \xrightarrow{\ell} \mathcal{F} \otimes (\Lambda^{\ell}) \rightarrow 0,$$

where $\Lambda$ is a free $\mathbb{Z}$-module of rank $\ell$ with basis $e_1, e_2, \cdots, e_\ell$ and

$$d(f \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^\ell s_j(f) \otimes e_j \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

Since $P_1(\zeta), \cdots, P_\ell(\zeta)$ form a regular sequence, the complex $K(P_1(\partial), \cdots, P_\ell(\partial); \mathfrak{D})[\ell]$ is a free resolution of $\mathfrak{M}$ and we get

$$\text{RHom}_{\mathfrak{D}}(\mathfrak{M}, \Gamma_K(\mathbb{D}_M; \mathscr{B}_\mathbb{D}_M^{\text{exp}})) \simeq K(P_1(\partial), \cdots, P_\ell(\partial); \Gamma_K(\mathbb{D}_M; \mathscr{B}_\mathbb{D}_M^{\text{exp}})).$$

Hence it follows from Corollary 8.0.3 that we have

$$\text{RHom}_{\mathfrak{D}}(\mathfrak{M}, \Gamma_K(\mathbb{D}_M; \mathscr{B}_\mathbb{D}_M^{\text{exp}})) \simeq K(P_1(\zeta), \cdots, P_\ell(\zeta); \Gamma(N^*_\text{pc}(K); \mathcal{O}_{E^*_\infty}^{\text{inf-K}})). \quad (9.1)$$

The lemma below is a key for the theorem:
Lemma 9.0.2. Let $\zeta^* \notin \text{Char}_{E_\infty^*}(\mathfrak{M})$. Then the Koszul complex
\[
K(P_1(\zeta), \ldots, P_\ell(\zeta); (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*})
\] (9.2)
is exact.

Proof. By the definition of $\text{Char}_{E_\infty^*}(\mathfrak{M})$, we can find $h(\zeta)$ and $a_j(\zeta)$ ($j = 1, 2, \ldots, \ell$) in $\mathfrak{M}$ such that
\[
\sigma(h)(\zeta^*) \neq 0, \quad h(\zeta) = \sum_{j=1}^\ell a_j(\zeta) P_j(\zeta).
\]
In particular, as $\sigma(h)(\zeta^*) \neq 0$ holds, $h$ is also invertible in the germ $(\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*}$ of the sheaf $\mathcal{O}_{E_\infty}^{\inf-hK}$ at $\zeta^*$. Set $\Lambda = \{1, 2, \ldots, \ell\}$ and let $s = \{s_k\}$ be a homotopy from $K(P_1, \ldots, P_\ell; (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*})$ to itself. Here
\[
s_k : K^{k+1}(P_1, \ldots, P_\ell; (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*}) \to K^k(P_1, \ldots, P_\ell; (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*})
\]
is given by
\[
s_k(\sum_{\alpha \in \Lambda^{k+1}} f_\alpha(\zeta)e_\alpha) = \sum_{\beta \in \Lambda^k} \sum_{j=1}^\ell a_j(\zeta)f_j(\zeta)e_\beta,
\]
where $e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{k+1}}$ and $j\beta$ is a sequence such that $\beta$ follows $j$. Then, by the simple computation, we can easily get the equality
\[
s \circ d - d \circ s = h.
\]
Here $h : K(P_1, \ldots, P_\ell; (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*}) \to K(P_1, \ldots, P_\ell; (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*})$ is the morphism of complexes defined by
\[
K^k(P_1, \ldots, P_\ell; (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*}) \ni u \mapsto hu \in K^k(P_1, \ldots, P_\ell; (\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*}),
\]
which is an isomorphism because $h(\zeta)$ is invertible on $(\mathcal{O}_{E_\infty^*}^{\inf-hK})_{\zeta^*}$. Therefore the isomorphism $h$ is homotopic to zero, from which we conclude that the complex (9.2) is quasi-isomorphic to zero. This completes the proof of the lemma.

It follows from the lemma that the Koszul complex
\[
K(P_1(\zeta), \ldots, P_\ell(\zeta); \mathcal{O}_{E_\infty^*}^{\inf-hK})
\] (9.3)
of sheaves is exact on $N_{pc}^*(K)$ because of the condition $N_{pc}^*(K) \cap \text{Char}_{E_\infty^*}(\mathfrak{M}) = \emptyset$. Applying the left exact functor $\Gamma(N_{pc}^*(K); \bullet)$ to the complex (9.3), we get a short exact sequence
\[
0 \to \Gamma(N_{pc}^*(K); \mathcal{O}_{E_\infty^*}^{\inf-hK}) \to \Gamma(N_{pc}^*(K); \mathcal{O}_{E_\infty^*}^{\inf-hK}) \otimes (\Lambda \Lambda) \to \Gamma(N_{pc}^*(K); \mathcal{O}_{E_\infty^*}^{\inf-hK}) \otimes (\Lambda \Lambda).
\]
Then, by noticing (9.1), the claim follows from the above short exact sequence.

Corollary 9.0.3. Let $P(\partial) \in \mathfrak{D}$, and let $K$ be a regular closed subset in $\mathbb{D}_M$ satisfying that $K \cap M$ is convex and $N_{pc}^*(K) \cap M_{\infty}^*$ is connected (in particular, non-empty). Then the morphism
\[
\Gamma_K(\mathbb{D}_M; \mathfrak{B}^{exp}_{\mathbb{D}_M}) \xrightarrow{P(\partial) \bullet} \Gamma_K(\mathbb{D}_M; \mathfrak{B}^{exp}_{\mathbb{D}_M})
\]
becomes isomorphic if $\sigma(P)(\zeta) \neq 0$ holds for any $\zeta \in N_{pc}^*(K)$.  

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