On the Banach space isomorphism type of AF C*-algebras and their triangular subalgebras

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ABSTRACT

It is shown that all the approximately finite dimensional C*-algebras which are not of Type I are isomorphic as Banach spaces. This generalises the matroid case given previously by Arazy. Analogous results are obtained for various families of triangular subalgebras of AF C*-algebras. In addition the classification of various continua of Type I AF C*-algebras is discussed.
A C*-algebra is approximately finite (or AF) if it is a closed union of finite-dimensional C*-subalgebras. Those AF C*-algebras for which the finite-dimensional subalgebras can be taken to be full matrix algebras are known as matroid C*-algebras. J. Arazy [1] has shown that with the exception of the algebra \( K \) of compact operators, which is the unique matroid algebra with separable dual, all (infinite-dimensional) matroid C*-algebras are isomorphic as Banach spaces. We generalise this by showing that all the AF C*-algebras which are not of Type I are isomorphic. In particular we see that a simple AF C*-algebra is either isomorphic to \( K \) or to the Fermion algebra \( F \), the unital matroid C*-algebra \( \lim_{\rightarrow} M_{2^k} \). We also comment on the classification of AF C*-algebras of Type I.

A similar analysis is given for various distinguished triangular subalgebras of AF C*-algebras and it is this non-self-adjoint context that motivated the present study. In particular the refinement limit algebras \( \rho - \lim_{\rightarrow} T_{n_k} \) are all isomorphic to the model algebra \( T_{2^\infty} = \rho - \lim_{\rightarrow} T_{2^k} \), and the standard limit algebras are all isomorphic to the model algebra \( S_{2^\infty} = \sigma - \lim_{\rightarrow} T_{2^k} \). From this we are able to deduce that all the (proper) alternation algebras are isomorphic. It seems probable that \( T_{2^\infty} \) and \( S_{2^\infty} \) are not isomorphic. Some evidence in support of this is that there are no “natural” complemented contractive injections \( F \to S_{2^\infty} \) and for this reason the methods of this paper cannot be applied. In fact it seems likely that non-self-adjoint subalgebras of AF C*-algebras provide an interesting diversity of Banach spaces. This is to be expected in view of the analogies that exist between triangular algebras and function spaces. Also, in an associated nonselfadjoint context, Arias [2] has recently shown that there are nonisomorphic nest subalgebras (perhaps uncountably many) in the trace class.

The proofs below are straightforward and self-contained. In particular we do not require the C*-algebraic classification of the matroid algebras or the AF C*-algebras (given in Dixmier [3] and Elliott [7] respectively). The key
step in the self-adjoint case is to show that if $B$ is an AF C*-algebra whose Bratteli diagram has a particular property, which we call the Fermion property, then for any other AF C*-algebra $A$ there exists a complemented linear contractive injection $\gamma : A \to B$. This injection is obtained by constructing an infinite commuting diagram whilst at the same time constructing a commuting system of left inverses for the partial injections. This ensures that in the limit the closed space $\gamma(A)$ is complemented in terms of a completely contractive linear map. This form of explicit construction lends itself to natural modifications to deal with triangular subalgebras of AF C*-algebras.

As this paper was being prepared Simon Wasserman pointed out to the author that the isomorphism of non Type I AF algebras has also been obtained in a different way by Kirchberg as a consequence of his work on nuclearly embeddable C*-algebras and exact C*-algebras. In fact, Kirchberg deduces that all separable nuclear non Type I C*-algebras are isomorphic. The rather deep result that leads to this isomorphism is that separable unital nuclear C*-algebras are unitally completely isometrically embeddable in the Fermion algebra. The proof of this, as well as the simplified proof given by Wasserman [17], is quite involved with C*-algebra theory and contrasts markedly with our approach for the simpler case of AF C*-algebras.

I would like to thank Jonathan Arazy for telling me about his classification paper during the Durham Symposium on The Geometry of Banach Spaces and Operator Algebras in 1992. I would also like to thank Gordon Blower, Graham Jameson and Przemek Wojtaszczyk for some helpful conversations.
1 AF algebras with The Fermion property

The Fermion algebra (or CAR algebra) is the unital matroid C*-algebra $F = \lim \rightarrow M_{2k}$ with Bratteli diagram

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A general AF C*-algebra, with presentation $A = \lim \rightarrow A_k$, has an associated Bratteli diagram in which, similarly, multiple edges between two vertices indicate, through their multiplicity, the multiplicity of the partial embedding between the summands associated with the vertices. We shall say that the Bratteli diagram has the Fermion property if there is a sequence of vertices $v_1, v_2, \ldots$ associated with summands of $A_{n_1}, A_{n_2}, \ldots$, respectively, with $n_k$ an increasing sequence, such that there is more than one vertical path between each consecutive pair $v_i, v_{i+1}$. That is, there is a sequence of partial embeddings $A_{n_k} \rightarrow A_{n_{k+1}}$, with nonzero compositions, each of multiplicity at least two. Thus, the Pascal triangle Bratteli diagram has the Fermion property, whereas the following diagram does not:
We now show that two AF C*-algebras whose Bratteli diagrams have the Fermion property are linearly homeomorphic. The next lemma, which is analogous to Lemma 2.11 of [1], is the key result required in the proof.

Let $A$ and $B$ be finite-dimensional C*-algebras with chosen matrix unit systems $\{e_{ij} : (i, j) \in I\}$ and $\{f_{ij} : (i, j) \in J\}$ respectively. Assume that $I, J$ are block diagonal subsets of $\{1, \ldots, m\}^2$ for some $m$. Let $a = (a_{ij})$ belong to $A$. A linear map $\gamma : A \to B$ is said to be of compression type with respect to these systems if $\gamma$ is a (block diagonal) direct sum of maps of the form $\alpha \circ \beta$ where $\beta : A \to M_n$ is given by

$$\beta((a_{ij})) = (a_{k_s,k_t})_{s,t=1}^n,$$

where $\{k_1, \ldots, k_n\}^2 \subseteq I$, and where $\alpha : M_n \to B$ is a multiplicity one algebra injection of the form

$$\alpha((b_{s,t})) = (b_{l_s,l_t})_{s,t=1}^n.$$
where \( \{l_1, \ldots, l_n\}^2 \subseteq J \). If, additionally, \( \{l_1, \ldots, l_n\} \) and \( \{t_1, \ldots, t_n\} \) are ordered subsets \( (l_1 < l_2 \text{ etc}) \), then we refer to \( \gamma \) as an ordered compression type map.

Let \( A = A_1 \oplus \ldots \oplus A_r \) where \( A_1, \ldots, A_r \) are the matrix algebra summands of \( A \) and let \( \gamma : A \to B \) be a map of compression type, as above, with \( \gamma = \gamma_1 \oplus \ldots \oplus \gamma_p \), where \( \gamma_1, \ldots, \gamma_p \) are the elementary summands of \( \gamma \). The map \( \gamma \) has isometric restriction to \( A_1 \) if (and only if) there is a summand \( \gamma_i \) which is isometric on \( A_1 \). It follows that \( \gamma \) is isometric if (and only if) the summands can be reordered and relabelled as \( (\gamma_1 \oplus \ldots \oplus \gamma_r) \oplus (\gamma_{r+1} \oplus \ldots \oplus \gamma_p) \) so that \( \gamma' = \gamma_1 \oplus \ldots \oplus \gamma_r \) is a (multiplicity one) algebra injection of \( A \) into \( B \). With such a relabelling there is an associated contractive left inverse map \( \delta : B \to A \), of compression type, satisfying \( \delta \circ \gamma = id_A \). The map \( \delta \) is the composition of compression onto the range of \( \gamma' \), followed by the inverse map of \( \gamma' \) restricted to its range.

Changing notation, let \( A = \lim \to (A_k, \phi_k) \), and \( B = \lim \to (B_k, \psi_k) \) be presentations of the AF C*-algebras \( A \) and \( B \) where the maps \( \phi_k \) and \( \psi_k \) are isometric C*-algebra injections. Assume that the matrix unit systems \( \{e_{ij}^k : (i, j) \in I_k\} \) and \( \{f_{ij}^k : (i, j) \in J_k\} \) have been chosen for \( A_k \) and \( B_k \) respectively, so that each map \( \phi_k \) and \( \psi_k \) maps matrix units to sums of matrix units, and, for the moment, assume that \( A \) is unital and that the embeddings \( \phi_k \) are unital. Assume furthermore that the Bratteli diagram for the system \( \{B_k, \psi_k\} \) has the Fermion property. By composing maps, forming a subsystem from such compositions, and relabelling, we may assume that \( B_k = M_{r_k} \oplus B'_k \) and that the partial embedding of \( \psi_k \) from \( M_{r_k} \) into \( M_{r_{k+1}} \) has multiplicity at least two.
Lemma 1.1. With the assumptions above there is a commuting diagram

\[
\begin{array}{ccc}
A_1 & \phi_1 & A_2 \\
\gamma_1 & & \gamma_2 \\
B_{n_1} & \theta_1 & B_{n_2}
\end{array}
\]

where each map \( \gamma_k \) is an isometric linear map of compression type relative to the given matrix unit system, and where \( \theta_1, \theta_2, \ldots \) are compositions of the given embeddings \( \psi_1, \psi_2, \ldots \). Furthermore there are linear contractions \( \delta_k : B_{n_k} \rightarrow A_k \), of compression type, satisfying \( \delta_k \circ \gamma_k = \text{id} \), such that the diagram

\[
\begin{array}{ccc}
A_1 & \phi_1 & A_2 \\
\delta_1 & & \delta_2 \\
B_{n_1} & \theta_1 & B_{n_2}
\end{array}
\]

commutes. In particular there exists an isometric injection \( \gamma : A \rightarrow B \) and a contractive map \( \delta : B \rightarrow A \) such that \( \gamma \circ \delta \) is a contractive projection onto the range of \( \gamma \).

Proof: Using the Fermion property for \( B \) choose \( n_1 \) large enough so that there exists a multiplicity one linear isometry \( \gamma_1 : A_1 \rightarrow B_{n_1} \), of compression type, with range in the summand \( M_{r_1} \) of the decomposition \( B_{n_1} = M_{r_1} \oplus B'_{n_1} \). We may assume that \( \gamma_1 \) has the form \( \gamma_1(a) = [a \oplus 0_s] \oplus \{0\} \), that is that the partial embedding of \( \gamma_1 \) from \( A_1 \) into \( M_{r_1} \) has a proper zero summand \( 0_s \). (We indicate the distinguished first summand of \( B_{n_2} \) with square brackets.
and the remaining summands are grouped in braces.) For $n_2 > n_1$, to be chosen, the composed map $\theta_1 : B_{n_1} \to B_{n_2}$ has partial embeddings $\sigma_1 : M_{r_1} \to M_{r_2}$, $\tau_1 : B'_{n_1} \to M_{r_2}$, $\tau_2 : B_{n_1} \to B'_{n_2}$, and by relabelling the matrix units of $M_{r_2}$ we may assume that $\sigma_1$ is of standard type, that is,

$$\sigma_1(b) = b \oplus \ldots \oplus b \oplus 0 \quad \text{(b appearing t times)}.$$  

(The zero summand may be absent.) Using the Fermion property hypothesis we may choose $n_2$ so that $t$ is arbitrarily large. Note that the composition $\theta_1 \circ \gamma_1$ has the form

$$a \to [\sigma_1(a \oplus 0_s) \oplus \tau_1(0)] \oplus \{\tau_2(\gamma_1(a))\}.$$  

Consider now the given map $\phi_1 : A_1 \to A_2$. Let

$$A_1 = A_{1,1} \oplus \ldots \oplus A_{1,p}, \quad A_2 = A_{2,1} \oplus \ldots \oplus A_{2,q}$$

be the matrix algebra decompositions. Relabelling matrix units of $A_2$ we may assume that $\phi_1$ is given in a standard form with respect to the matrix unit systems, that is,

$$\phi_1 : A_{1,1} \oplus \ldots \oplus A_{1,p} \to A_{2,1} \oplus \ldots \oplus A_{2,q}$$

where the summand of $\phi_1(a_1 \oplus \ldots \oplus a_p)$ in the matrix summand $A_{2,t}$ is

$$k_{1,t} \sum_{1}^{k_{1,t}} \oplus (\sum_{1}^{k_{1,t}} a_1) \oplus \ldots \oplus (\sum_{1}^{k_{1,t}} a_p),$$

with the understanding that some of these summands may be absent. The integer $k_{s,t}$, for $1 \leq s \leq p, 1 \leq t \leq q$, is the multiplicity of the partial embedding for $\phi_1$ from $A_{1,s}$ to $A_{2,t}$.
We now construct \( \gamma_2 : A_2 \to B_{n_2} \) as an isometric linear multiplicity one injection of compression type, as suggested by the following diagram:

\[
\begin{array}{ccc}
  a & \rightarrow & [\Sigma_{1,1} \oplus \ldots \oplus \Sigma_{p,1}] \oplus \ldots \oplus [\Sigma_{1,q} \oplus \ldots \oplus \Sigma_{p,q}] \\
  \gamma_1 & \downarrow \phi_1 & \gamma_2 \\
  [a \oplus 0_s] \oplus 0 & \rightarrow & [\sigma_1(a \oplus 0_s) \oplus \tau_1(0)] \oplus \tau_2(\gamma_1(a))
\end{array}
\]

Using the Fermion property choose \( n_2 \) large enough so that

\[ t \geq k_{i,1} + \ldots + k_{i,q} \text{, } 1 \leq i \leq p. \]

These inequalities guarantee that there exists a one to one correspondence of the summands of \( \phi_1(a) \) with some of the appropriate nonzero summands of \( \sigma_1(a \oplus 0_s) \). The point of this observation is that there is a natural multiplicity one algebra injection \( \gamma_2' \) from \( A_2 \) to \( M_{r_2} \) (of compression type) which respects this correspondence. Construct \( \gamma_2 \) by adding extra summands to \( \gamma_2' \) to obtain a linear isometry, of compression type, satisfying \( \gamma_2(\phi_1(a)) = \theta_1(\gamma_1(a)) \) for all \( a \) in \( A_1 \). Since \( \phi_1 \) is isometric this is possible.

Define \( \delta_1 = \gamma_1^{-1} \circ \eta_1 \) where \( \eta_1 \) is the compression map onto the range of \( \gamma_1 \), noting that \( \gamma_1^{-1} \) is well-defined on this range. Similarly define \( \delta_2 \) in terms of \( \gamma_2 \). Thus, \( \delta_2 = (\gamma_2)^{-1} \circ \eta_2 \) where \( \eta_2 \) is the compression onto the range of \( \gamma_2 \). To see the important equality \( \phi_1 \circ \delta_1 = \delta_2 \circ \theta_1 \), let \( b \in B_{n_1} \), and let \( \eta_1(b) = [a \oplus 0_s] \oplus 0 \). By the construction of \( \gamma_2 \) and \( \delta_2 \) we have

\[ \delta_2(\theta_1(b)) = \delta_2(\theta_1(\eta_1(b))). \]
This is because the domain of \( \delta_2 \) is \textit{subordinate} to the nonzero summands of \( \sigma_1(a \oplus 0_s) \), and \( \theta_1(b - \eta_1(b)) \) vanishes on these summands. Thus

\[
\delta_2(\theta_1(b)) = \delta_2(\theta_1(\eta_1(b))) = \delta_2(\theta_1([a \oplus 0_s] \oplus 0)) = \phi_1(a) = \phi_1(\delta_1(b)).
\]

Repeating the arguments above obtain inductively isometric maps \( \gamma_3, \gamma_4, \ldots \), with distinguished contractive left inverses \( \delta_3, \delta_4, \ldots \). Indeed, note that after relabelling the matrix units of \( B_{n_2} \) the map \( \gamma_2 : A_2 \to M_{r_2} \oplus B'_{n_2} \) can be written in the form

\[
a \to [a \oplus 0 \oplus \gamma_{2,1}(a)] \oplus (\gamma_{2,2}(a)),
\]

and so the construction of \( \gamma'_3, \delta_3, \gamma_3 \) is obtained in exactly the same way as \( \gamma'_2, \delta_2, \gamma_2 \). \( \square \)

The lemma shows that there is a complemented isometric linear injection \( A \to B \). If \( A \) is not unital then \( A \) has a complemented isometric linear injection into its unitisation \( A' \), and so the nonunital case follows on consideration of the composition \( A \to A' \to B \).

The next lemma also appears in Arazy’s paper. For completeness we give a proof. Write \( X \approx Y \) if \( X \) and \( Y \) are linearly homeomorphic Banach spaces.

\textbf{Lemma 1.2.} \( F \approx c_0(F) \).

\textit{Proof:} Realise \( F \) as the direct limit \( \varinjlim (M_{2^k}, \rho_k) \) where \( \rho_k : a \to a \otimes I_2 \). Define a natural injection \( \beta : c_0(F) \to \tilde{F} \) which is suggested by the following inclusion diagram.
More precisely let $F_0$ be the subspace $\lim \to (M^0_{2k}\rho_k)$, of codimension one given by the subsystem determined by the subspaces

$$M^0_{2k} = \{(a_{ij}) \in M_{2k} : a_{2k,2k} = 0\}.$$

Define $c_0(F) \to F_0$ as follows. Identify the first copy of $F$ in $c_0(F) = F \oplus F \oplus \ldots$ with $p_1F_0p_1$ where $p_1 = e_{1,1}$ in $M_2$. (Identify $e_{1,1}$ with its image in the limit.) Identify the second copy with $p_2F_0p_2$, where $p_2 = e_{33}$ in $M_{2^2}$, and so on. The resulting inclusion $c_0(F) \to F$ has range which is the range of the projection $E : F_0 \to F_0$ given by $E(a) = \lim_k(\sum_{j=1}^k p_ja p_j)$. Thus $c_0(F)$ is complemented in $F_0$, and hence in $F$. Thus $F \approx c_0(F) \oplus X \approx c_0(F) \oplus c_0(F) \oplus X \approx c_0(F) \oplus F \approx c_0(F)$. \hfill \Box

The proof of the next theorem now reduces to a routine application of the Pelczynski decomposition method.

**Theorem 1.3.** Let $A$ and $B$ be AF $C^*$-algebras given by direct systems whose Bratteli diagrams have the Fermion property. Then $A$ and $B$ are isomorphic as topological vector spaces.

**Proof:** We may assume that $B = F$. By Lemma 1.1, and the remarks concerning the unital case, there exist contractive injective complemented maps $A \to F$ and $F \to A$. Thus, by Lemma 1.2,
\[ c_0(A) \to c_0(F) \approx F \to A. \]

Hence, just as with \( F \), we have \( A \approx c_0(A) \oplus Y \approx c_0(A) \oplus c_0(A) \oplus Y \approx c_0(A) \oplus A \approx c_0(A). \)

Consider now the fact that \( A \approx F \oplus Z \) for some closed subspace \( Z \) of \( A \), and obtain \( A \approx F \oplus Z \approx c_0(F) \oplus Z \approx c_0(F) \oplus F \oplus Z \approx F \oplus A \). Similarly, \( F \approx F \oplus A \), and so \( F \approx A \). \(\square\)

Let \( A = \lim_\to A_k \) be an (infinite-dimensional) AF C*-algebra, with a corresponding Bratteli diagram, which is simple. This means that for each vertex \( v \) of the diagram, at level \( k \), there is a lower level \( m \) such that there exist downward paths from \( v \) to all the vertices at level \( m \). (See Bratteli [1].) In particular any two vertices at a given level have downward paths that meet in a common vertex. This weaker property is precisely the Bratteli diagram criterion for the triviality of the centre of \( A \). Suppose additionally, that the Bratteli diagram fails to have the Fermion property. Then there must exist a vertex with a unique downward path. For otherwise there is repeated branching and convergence characteristic of the Fermion property.

The unique downward path determines a subsystem of \( A \) which defines a subalgebra \( J \) which is isomorphic to \( \mathcal{K} \) or \( M_n \) for some \( n \). Since \( J \) is in fact an ideal, and \( A \) is simple, it follows that \( A = \mathcal{K} \). Thus we have obtained

**Corollary 1.4.** Let \( A \) be a simple (infinite-dimensional) approximately finite C*-algebra. Then \( A \approx \mathcal{K} \) or \( A \approx F \).

A C*-algebra is said to be of Type I if its star representations generate Type I von Neumann algebras. Also it is known that this is equivalent to
the apparently weaker assertion that factorial star representations are Type I. (See, for example, [3].) Using this we can strengthen the last corollary.

**Corollary 1.5.** Let \( A \) be an approximately finite \( C^* \)-algebra which is not Type I. Then \( A \) is isomorphic to \( F \) as a linear topological vector space.

**Proof:** Let \( A = \lim_{\rightarrow} A_k \), with Bratteli diagram without the Fermion property. We show that the factorial representations of \( A \) are Type I. Note first that if \( \pi : A \to L(H) \) is a factorial representation, then \( \ker \pi \) is an ideal, and \( A/\ker \pi \) is an AF \( C^* \)-algebra with Bratteli diagram obtained as a subdiagram of the diagram for \( A \). (See Bratteli [4].) Since this subdiagram also fails to have the Fermion property we may as well assume that \( \ker \pi = \{0\} \). With this assumption it follows that the centre of \( A \) must be trivial. By the argument preceding Corollary 4 it follows that \( A \) possesses an ideal which is isomorphic to \( K \) and which is associated with a vertex of the diagram that has a unique descending path. Let \( p \) be a minimal projection in the matrix summand corresponding to this vertex. Then \( p \) is minimal in \( A \), and so \( \pi(p) \) is minimal in \( \pi(A)'' \). Thus the factor \( \pi(A)'' \) is Type I. \( \square \)
Type I AF C*-algebras

We comment on the isomorphism types of the (infinite dimensional) Type I AF C*-algebras.

In the separable dual case the following three algebras present themselves:

(i) $c_0$, the space of diagonal compact operators,

(ii) $\mathcal{R} = (\sum_{k=1}^{\infty} \oplus M_k)_{c_0}$,

(iii) $\mathcal{K}$, the compact operators.

To see that these are not isomorphic we can distinguish (i) from (ii) and (iii) by noting that all bounded maps from $c_0$ to $\ell^2$ are 2-summing. A simple proof can be found in Pisier’s notes [12]. On the other hand matrix realisations provide ”top row” maps $\mathcal{K} \to \ell^2$, $\mathcal{R} \to \ell^2$ which are not 2-summing. Alternatively, it can be seen in Hamana [10] and Chu and Iochum [5] that (i) has the Dunford Pettis property whereas (ii) and (iii) do not. To distinguish $\mathcal{R}$ and $\mathcal{K}$ one can note that the first space has a dual space with the Schur property, that weakly convergent sequences are norm convergent whereas the trace class operators do not. (See [10] and [5].)

If an AF C*-algebra has a separable dual space then it is easy to see that it does not have the Fermion property and this limits the possibilities for the type of Bratteli diagram. The Bratteli diagrams in this case fall naturally into three types, namely, Type (i), in which there is a uniform bound on the sizes of the matrix summands, Type (ii), in which for every path in the diagram there is a uniform bound on the sizes of the associated summands, and Type (iii), being the rest. In the particular context of a diagram with finite width the associated algebra has a finite composition series and from this it can be shown to be isomorphic to $\mathcal{K}$ as a Banach space.

Bessaga and Pelczynski [3] have classified the spaces $C(S)$ with $S$ countable. For each isomorphism type there is a countable ordinal $\alpha$ with $C(S) \cong$
\( C(\alpha), \) and \( C(\beta) \approx C(\alpha) \) if and only if \( \alpha \leq \beta < \alpha^\omega \) or \( \beta \leq \alpha < \beta^\omega \). In particular there is a continuum of isomorphism types of abelian AF C*-algebras with diagrams of type (i). One would expect there to be similar continua for the algebras whose diagrams are of type (ii) and type (iii).

Turning to the algebras whose dual space is not separable there are, in the first instance, six natural C*-algebras to consider.

(iv) \( C(K) \), with \( K \) a Cantor space,
(v) \( C(K) \oplus \mathcal{R} \),
(vi) \( C(K) \otimes \mathcal{R} \),
(vii) \( C(K) \oplus \mathcal{K} \),
(viii) \( (C(K) \otimes \mathcal{R}) \oplus \mathcal{K} \),
(ix) \( C(K) \otimes \mathcal{K} \).

These algebras are associated with six different types of Bratteli diagram. Thus, a Type (iv) diagram has uncountably many paths and a uniform bound on the sizes of matrix summands. A Type (v) diagram has uncountably many paths, is not of Type (iv), has bounded matrix sizes on each path, and for each integer \( n \) has only countably many paths on which the matrix sizes exceed any \( n \). Similar descriptions hold for the remaining types, culminating in Type (ix) for which there are uncountably many paths each with unbounded matrix sizes. Moreover each Bratteli diagram without the Fermion property, which is not of Types (i), (ii) or (iii) is one of these six types. It seems plausible that isomorphic AF C*-algebras have Bratteli diagrams of the same diagram type, and within some of these diagram types one would expect there to be added ordinal type complexities as in the separable dual case. (For example one might expect there to be a continua of Banach space types of the form \( C(K) \oplus (C(\alpha) \otimes \mathcal{K}) \).
With regard to the duals of AF C*-algebras, Wojtaszczyk has shown that there are just three separable duals, namely the duals of the algebras (i), (ii), and (ii). It seems reasonably to conjecture that there are precisely nine dual spaces of Type I AF C*-algebras.

2 Triangular subalgebras

There are three well-known families of triangular subalgebras of UHF C*-algebras, namely the refinement algebras \( \lim \rightarrow (T_{n_k}, \rho_k) \), the standard algebras \( \lim \rightarrow (T_{n_k}, \sigma_k) \) and the alternation algebras, \( \lim \rightarrow (T_{n_k}, \alpha_k) \). The (unital) embeddings determining these limits have the form

\[
\rho_k((a_{ij})) = (a_{ij}I_{t_k}), \quad \sigma_k(a) = I_{t_k} \otimes a,
\]

where \( t_k \) is the multiplicity of the embedding, and in the alternation case \( \alpha_k \) alternates between these two types. We shall prove the following theorem.

Theorem 2.1. (i) The standard limit algebras are isomorphic as Banach spaces. (ii) The refinement limit algebras are isomorphic as Banach spaces. (iii) The alternation limit algebras are isomorphic as Banach spaces.

Another well-known class consists of the various “refinement with twist” limits \( \lim \rightarrow (T_{n_k}, \tau_k) \), where \( \tau_k \) agrees with \( \rho_k \) on all the standard matrix units \( e_{ij} \) of \( T_{n_k} \), with the exception of those superdiagonal matrix units in the last column. For these

\[
\tau_k(e_{i,n_k}) = e_{i,n_k} \otimes u_k
\]

where \( u_k \) is a permutation unitary in \( M_{t_k} \). It was shown in Hopenwasser
and Power [11] that these algebras provide uncountably many algebra isomorphism classes, distinct from the refinement limits. On the other hand we have

**Theorem 2.2.** If $A$ is a refinement with twist algebra, as above, then as a Banach space, $A$ is isomorphic to the model refinement algebra $T_{2\infty}$.

The algebras above are examples of (canonical regular) triangular subalgebras of UHF C*-algebras. In a different direction one can generalize the standard embedding limit algebras by considering ordered Bratteli diagrams ([14], [13].) A typical such embedding has the form

$$\beta : T_{q_1} \oplus \ldots \oplus T_{q_s} \to T_{p_1} \oplus \ldots \oplus T_{p_r}$$

where

$$\beta : a_1 \oplus \ldots \oplus a_s \to (\sum_{j=1}^{t_{1}} \oplus b_{1,j}) \oplus \ldots \oplus (\sum_{j=1}^{t_{r}} \oplus b_{r,j})$$

and where each $b_{k,l}$ is one of the summands $a_1, \ldots, a_s$. The summations here mean block diagonal direct sums. In the nonunital case one also allows the $b_{k,l}$ to be zero summands. For a simple example, consider the embedding $\beta_1$ from $T_2 \oplus T_3 \oplus T_4$ to $T_7 \oplus T_6 \oplus T_5$ given by

$$\beta_1 : a \oplus b \oplus c \to (b \oplus c) \oplus (a \oplus c) \oplus (a \oplus b).$$

This embedding is not inner conjugate to

$$\beta_2 : a \oplus b \oplus c \to (b \oplus c) \oplus (c \oplus a) \oplus (a \oplus b)$$

and the difference can be indicated by ordered Bratteli diagrams. A consequence of this diversity is the existence of uncountably many nonisomorphic
limit algebras with the same generated C*-algebra. Once again, however, they correspond to a unique Banach space type.

**Theorem 2.3.** Let $A$ be a triangular limit algebra determined by an ordered Bratteli diagram which has the Fermion property. Then, as a Banach space, $A$ is isomorphic to the model algebra $S_{2\infty}$.

Recall the definition of a linear map $\gamma : M_n \to M_m$ which is of ordered compression type and note that such a map has a restriction $\gamma : T_n \to T_m$. Using direct sums of such maps define ordered compression type maps $T_n \to T_{p_1} \oplus \ldots \oplus T_{p_r}$ and use these to define general ordered compression type maps $\gamma = T_{q_1} \oplus \ldots \oplus T_{q_s} \to T_{p_1} \oplus \ldots \oplus T_{p_r}$. If $\gamma(a_1 \oplus \ldots \oplus a_s)$ has at least one complete summand $a_i$, for each $i$, then $\gamma$ is isometric. In this case it follows, as in section 2, that $\gamma$ has an associated left inverse.

*The Proof of Theorem 2.3.* Note first that the conclusions of Lemma 1.1 hold in the triangular context wherein we make the following new assumptions: $A = \lim \to (A_k, \varphi_k)$ and $B = \lim \to (B_k, \psi_k)$ are limit algebras determined by ordered Bratteli diagrams and the diagram for $B$ has the Fermion property. To adapt the proof we use ordered compression maps in place of compression maps. The argument is virtually the same but notationally awkward since we cannot make simplifying reorderings of summands by relabelling matrix units in the codomain. The first occasion for this is the expression for $\theta_1 \circ \gamma_1(a)$. (The map $\gamma_1$ can be chosen to be of ordered compression type.) However $\theta_1 \circ \gamma_1(a)$ is an ordered direct sum such that, in the first summand $T_{r_2}$ of $B_{n_2}$, for suitably large $n_2$, there appear many summands which are copies of the summands of $a$. Thus, as before, there is an association of the summands of $\phi_1(a)$ with some of the summands of $\theta_1 \circ \gamma_1(a)$, if $n_1$ is large enough.
However we assume additionally that this association respects the order in which summands of $\phi_1(a)$ appear in each summand $A_{2,i}$. As before define a multiplicity one injection $\gamma'_2$ of ordered compression type, from $A_2$ to $B_{n_2}$, which respects this correspondence. The extension $\gamma_2$ of $\gamma'_2$ and the left inverse $\delta_2$ of $\gamma_2$ (and $\gamma'_2$) are defined as before, and once again the desired commuting diagrams follow upon iterating this procedure. In the case that $A$ is nonunital obtain a complemented unital isometric injection $A \to B$ by considering the unitisation of $A$.

The argument of Lemma 1.2 also serves to give a natural complemented injection $c_0(S_{2\infty}) \to S_{2\infty}$. (The diagram of Lemma 1.2, however, is not appropriate for standard embeddings.) The remainder of the proof follows as before.

\begin{flushright}
\Box
\end{flushright}

The proof of Theorem 2.1 (ii). This follows a similar scheme. Let $A = \lim(A_k, \phi_k)$ be a refinement limit algebra and let $T_{2\infty} = \lim(B_k, \psi_k)$ be the $2^\infty$ refinement limit algebra. To obtain the appropriate version of Lemma 1.1 first choose $n_1$ large enough so that there is a multiplicity one ordered compression type map $\gamma_1 : A_1 \to B_{n_1}$ given by $\gamma_1(a) = a \oplus 0$ (block diagonal direct sum). The matrix $\phi_1(a)$ has the form $(a_{ij}I_t)$ where $t$ is the multiplicity of $\phi_1$. Choose $\theta_1 : B_{n_1} \to B_{n_2}$, a composition of consecutive maps $\psi_1, \psi_2, \ldots$ so that the multiplicity of $\theta_1$ exceeds that of $\phi_1$. Thus

$$\phi_1((a_{ij})) = (a_{ij}I_t)$$

and

$$\theta_1(\gamma_1((a_{ij}))) = (a_{ij}I_s) \oplus 0$$

where $s > t$. There is now natural multiplicity one isometric (algebra) injection $\gamma'_2 : A_2 \to B_{n_2}$ with the property that
\[ \gamma'_2(a_{ij} I_t) = (a_{ij} (I_t \oplus 0_{s-t})) \oplus 0. \]

This (as before) does not yet give a commuting square. Nevertheless we can add multiplicity one summands of ordered compression type to create \( \gamma_2 \), an extension of \( \gamma'_2 \), so that \( \gamma(a_{ij} I_t) = (a_{ij} I_s) \oplus 0 \), and thus we obtain the first commuting square of Lemma 1.1 in this context. Furthermore, \( \gamma_1 \) has a natural left inverse \( \delta_1 \), and, as before, \( \gamma'_2 \) can be used in the definition of a left inverse \( \delta_2 \) for \( \gamma_2 \), which extends \( \delta_1 \) in the obvious way. The construction of the desired maps \( \gamma_3, \delta_3, \gamma_4, \delta_4, \ldots \) is obtained similarly.

We have \( T_{2^\infty} \approx c_0(T_{2^\infty}) \), by the argument of Lemma 1.2 (the diagram is appropriate this time), and the proof is completed as before. \( \square \)

The proof of Theorem 2.1 (iii) Let \( S \) and \( r \) be the generalised integers associated with the triangular limit algebras \( S_s \) and \( T_r \). Consider the subalgebra \( S_s \ast T_r \) of \( B_s \otimes B_r \) given by

\[ S_s \ast T_r = (S_s \cap S_s^*) \otimes T_r \oplus (S_s^0 \otimes B_r) \]

where \( B_s = C^*(S_s) \) and \( B_r = C^*(T_r) \) are the generated C*-algebras, and where \( S_s^0 \) is the strictly upper triangular subalgebra of \( S_s \). In the terminology of [15] and [16] this algebra is the lexicographic product of the ordered pair \( S_s, T_r \). If \( s \) and \( r \) are not finite then this product coincides with the proper alternation algebra for the pair \( r, s \). This formula forms the basis of the proof.

Note first that \( S_s \ast T_r \) is bicontinuously isomorphic to the \( \ell^\infty \) direct sum of the component spaces above. Also \( S_s \cap S_s^* \) is C*-algebraically isomorphic to \( C(K) \), where \( K \) is a Cantor space. From the linear isomorphism \( T_r \approx T_{2^\infty} \) we obtain
\((S_s \cap S_s^*) \otimes T_r \approx C(K) \otimes T_r \approx C(K) \otimes T_{2\infty} \approx (S_s \cap S_s^*) \otimes T_{2\infty}\)

It remains then to show that \(S_r^0 \otimes B_r\) and \(S_{2\infty} \otimes B_{2\infty}\) are bicontinuously isomorphic. However the maps of Lemma 1.1 and its non-self-adjoint variants respect tensor products. For example, the isometric map \(\gamma : A \to B\) of Lemma 1.1 also provide complemented isometric injections \(\gamma \otimes id : A \otimes D \to B \otimes D\), where \(D\) is a closed subspace of an AF C*-algebra (for example) and the tensor product is the injective, or spatial, tensor product. And so we obtain the needed equivalences

\[S_r^0 \otimes B_r \approx S_{2\infty}^0 \otimes B_r \approx S_{2\infty}^0 \otimes B_{2\infty}\]

and the proof follows.

\(\square\).

\(\textbf{The proof of Theorem 2.2.}\) Let \(A = \lim(T_{n_k}, \tau_k)\) be a refinement with twist limit algebra. Let \(A_0 = \lim(T_{n_k}^-, \tau_k) = \lim(T_{n_k}^-, \rho_k)\) where \(T_{n_k}^- = \text{span}\{e_{i,j} : j < N_k\}\). Also, let \(\mathcal{T} = \lim(T_{n_k}, \rho_k)\) be the associated refinement algebra. Then \(T_{n_k} \approx T_{n_k}^- \oplus \ell^2_{n_k}\). That is, the map

\[a \to a(1 - e_{n_k,n_k}) \oplus ae_{n_k,n_k}\]

is a bicontinuous linear isomorphism to the \(\ell^\infty\) direct sum. From the commuting diagram

\[
\begin{array}{ccc}
T_{n_1} & \xrightarrow{\tau_1} & T_{n_2} & \xrightarrow{\tau_2} & \ldots \\
| & | & | & | & \\
T_{n_1}^- \oplus \ell^2_{n_1} & \xrightarrow{\tau_1} & T_{n_2}^- \oplus \ell^2_{n_2} & \xrightarrow{\tau_2} & \ldots \\
\end{array}
\]
we obtain a bicontinuous isomorphism $A \approx A_0 \oplus \ell^2$. Similarly $\mathcal{T} \approx A_0 \oplus \ell^2$ and so the theorem now follows from Theorem 2.1. 

**Remark 2.4.** With respect to injections which map matrix units to sums of matrix units the Banach space $S_{2^{\infty}}$ is not as injective as $\mathcal{T}_{2^{\infty}}$. More specifically if $\phi : M_2 \to T_{m_1}$ and $\phi' : M_n \to T_{m_2}$ are isometric linear maps which map matrix units to sums of matrix units, and if $i : M_2 \to M_n$ is a unital C*-algebra injection, and if $\phi' \circ i = \sigma \circ \phi$, then it can be shown that $2n \leq m_1$. This suggests that there are no complemented injections of the Fermion algebra in $S_{2^{\infty}}$, whereas it is clear that there are such injections for $\mathcal{T}_{2^{\infty}}$.

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