PERTURBATION EFFECTS FOR THE MINIMAL SURFACE EQUATION WITH MULTIPLE VARIABLE EXponents

Ramezi Alsaeedi*
Department of Mathematics, Faculty of Sciences
King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia

This paper is dedicated to Prof. Vicențiu Rădulescu
on the occasion of his 60th anniversary

Abstract. We are concerned with the existence of nontrivial weak solutions for a class of generalized minimal surface equations with subcritical growth and Dirichlet boundary condition. In relationship with the values of several variable exponents, we establish two sufficient conditions for the existence of solutions. In the first part of this paper, we prove the existence of a non-negative solution. Next, we are concerned with the existence of infinitely many solutions in a symmetric abstract setting.

1. Introduction and abstract setting. The Plateau problem is associated with the study of minimal surfaces. This problem consists in finding a surface with least area in $\mathbb{R}^2$ that spans a given closed curve with smooth boundary. Such a surface is described by solutions of the following minimal surface equation

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |
abla u|^2}} \right) = 0.$$  

This problem was formulated by J.L. Lagrange in 1760 and it is named after the Belgian physicist J. Plateau who experimented with soap films. The concept of mean curvature of a surface was used by J.B. Meusnier in 1776 in his works on minimal surfaces related with the Plateau problem. These problems have been intensively studied in the last few decades, see, e.g., E. Giusti [12], B. Kawohl [14], and M. Struwe [23].

The differential operator in Eq. (1) has been extended into several directions. In the present paper we are interested in a nonlinear problem driven by the generalized mean curvature operator

$$\text{div} \left[ (1 + |
abla u|^2)^{(p(x)-2)/2} \nabla u \right].$$

We refer to Example 5 in the monograph of V. Rădulescu and D. Repovš [22, p. 28].

2010 Mathematics Subject Classification. Primary: 35J93; Secondary: 35J60, 49Q05, 58E05, 58E30.

Key words and phrases. Generalized mean curvature equation, minimal surface, variable exponent, mountain pass geometry, Ekeland variational principle.

* Corresponding author.
The main results in this paper are concerned with the existence of nontrivial solutions for a nonhomogeneous perturbation of the differential operator
\[ \text{div} \left[ (1 + |\nabla u|^2)^{(p(x)-2)/2} \nabla u \right] \]
and corresponding to a power-type reaction term with variable exponent. More precisely, we study the following nonlinear problem
\[ \begin{cases} -\text{div} \left[ (1 + |\nabla u|^2)^{(p(x)-2)/2} \nabla u \right] + |u|^{q(x)-2}u = \lambda |u|^{r(x)-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]
where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 2 \)) is a smooth bounded domain and \( \lambda \) is a positive parameter.

In accordance with the values of the variable exponents \( p, q \) and \( r \), we study two different situations in a subcritical abstract setting. In the first case, we establish the existence of solutions for small perturbations of the reaction term, that is, provided that \( \lambda > 0 \) is small enough. In the second case, we prove that nontrivial solutions do exist for all \( \lambda > 0 \). The proofs rely on variational arguments and essential tools are Ekeland’s variational principle \[9\] and the symmetric mountain pass theorem of A. Ambrosetti and P. Rabinowitz \[3\].

Relevant applications of nonlinear problems with variable exponent are developed in the monographs \[7\], \[8\] and \[22\]. We also refer to R. Alsaedi et al. \[1, 2, 16\], M. Cencelj, D. Repovš and Z. Virk \[6\], Y. Fu and Y. Shan \[11\], P. Pucci and Q. Zhang \[19\], V. Rădulescu \[21\] for recent related results devoted to the mathematical analysis of some problems driven by differential operators with variable exponent.

This paper is organized as follows. In the next section, we recall some basic properties of the Lebesgue and Sobolev spaces with variable exponent. The main results are stated in section 3 while the proofs are developed in sections 4 and 5 of this paper.

**Notation.** for given real numbers \( a \) and \( b \), we denote
\[ a \wedge b := \min\{a, b\} \text{ and } a \vee b := \max\{a, b\}. \]

For two functions \( f, g : D \to \mathbb{R} \) we write
\[ f \ll g \iff \inf\{g(x) - f(x); \ x \in D\} > 0. \]

2. **Function spaces with variable exponent.** In this section we recall some basic facts that concern the Lebesgue and Sobolev function spaces with variable exponent. We refer to V. Rădulescu and D. Repovš \[22\] for proofs and additional results.

Let \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) be a bounded domain with smooth boundary. For every continuous function \( p : \overline{\Omega} \to \mathbb{R} \) we define
\[ p^{-} = \min_{x \in \overline{\Omega}} p(x) \text{ and } p^{+} = \max_{x \in \overline{\Omega}} p(x). \]

Consider the classes
\[ C_{+}(\overline{\Omega}) = \{p \in C(\overline{\Omega}); \ 1 < p^{-}\} \]
and
\[ C = \{p \in C(\overline{\Omega}); \ 2 \leq p^{-} < p^{+} < N\}. \]
For any \( p \in C_+(\bar{\Omega}) \), we define the variable exponent Lebesgue space
\[
L^{p(x)}(\Omega) = \{ u : u \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \}.
\]
Then \( L^{p(x)}(\Omega) \) is a Banach space endowed with the Luxemburg norm, namely
\[
||u||_{p(x)} = \inf \left\{ \mu > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\}.
\]
The function space \( L^{p(x)}(\Omega) \) is reflexive for all \( p \in C \) and continuous functions with compact support are dense in \( L^{p(x)}(\Omega) \).

The inclusion between Lebesgue spaces also generalizes the classical framework, namely if \( p_1, p_2 \) are variable exponents so that \( p_1 \leq p_2 \) in \( \Omega \) then there exists the continuous embedding \( L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega) \).

Let \( L^{p'}(\Omega) \) be the conjugate space of \( L^{p(x)}(\Omega) \), where \( 1/p(x) + 1/p'(x) = 1 \). For any \( u \in L^{p(x)}(\Omega) \) and \( v \in L^{p'(x)}(\Omega) \) the following Hölder-type inequality holds:
\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p} + \frac{1}{p'} \right) |u|_{p(x)} |v|_{p'(x)}.
\] (3)

Consider the mapping \( \rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R} \) defined by
\[
\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} \, dx.
\]
If \( (u_n), u \in L^{p(x)}(\Omega) \) then the following relations are true:
\[
|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p(x)} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-(x)}
\] (4)
\[
|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+(x)} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p(x)}
\] (5)
\[
|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0.
\] (6)

Let \( W^{1,p(x)}(\Omega) \) denote the variable exponent Sobolev space defined by
\[
W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \}.
\]
On \( W^{1,p(x)}(\Omega) \) we may consider one of the following equivalent norms
\[
||u||_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}
\]
or
\[
||u|| = \inf \left\{ \mu > 0 : \int_{\Omega} \left( \left| \nabla u(x) \right|^{p(x)} \left| \mu \right| + \left| u(x) \right|^{p(x)} \left| \mu \right| \right) \, dx \leq 1 \right\}.
\]
We define \( W^{1,p(x)}_{0}(\Omega) \) as the closure of the set of compactly supported \( W^{1,p(x)} \) functions with respect to the norm \( ||u||_{p(x)} \). Equivalently, we can also use the closure of \( C_0^\infty(\Omega) \) in \( W^{1,p(x)}_{0}(\Omega) \). Using the Poincaré inequality, the space \( W^{1,p(x)}_{0}(\Omega) \) can be also defined, in an equivalent manner, as the closure of \( C_0^\infty(\Omega) \) with respect to the norm
\[
||u||_{p(x)} = |\nabla u|_{p(x)}.
\]
As pointed out in [22, p. 13], the space \( W^{1,p(x)}_{0}(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) under the norm \( ||u||_{p(x)} \) even if \( p \) is logarithmic Hölder continuous, that is,
\[
|p(x) - p(y)| \leq \frac{C}{\log |x - y|} \quad \text{for all } x, y \in \bar{\Omega}, \ |x - y| \leq \frac{1}{2}.
\]
The space \( W^{1,p(x)}_0(\Omega), \| \cdot \| \) is a separable and reflexive Banach space. Moreover, if \( p_1, p_2 \) are variable exponents so that \( p_1 \leq p_2 \) in \( \Omega \) then there exists the continuous embedding \( W^{1,p(x)}_0(\Omega) \hookrightarrow W^{1,p_1(x)}_0(\Omega) \).

Set

\[ \theta_{p(x)}(u) = \int_{\Omega} |\nabla u(x)|^{p(x)} \, dx. \quad (7) \]

If \( (u_n), u \in W^{1,p(x)}_0(\Omega) \) then the following properties are true:

\[ \|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^+ \leq \theta_{p(x)}(u) \leq \|u\|_{p(x)}^-, \quad (8) \]

\[ \|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^+ \leq \theta_{p(x)}(u) \leq \|u\|_{p(x)}^-, \quad (9) \]

\[ \|u_n - u\|_{p(x)} \to 0 \iff \theta_{p(x)}(u_n - u) \to 0. \quad (10) \]

Let \( p^* : \overline{\Omega} \to \mathbb{R} \) be the critical Sobolev function associated to \( p \in \mathcal{C} \), namely

\[ p^*(x) = \frac{Np(x)}{N - p(x)} \quad \text{for all } x \in \overline{\Omega}. \]

We recall that if \( p, q \in \mathcal{C} \) and \( q(x) < p^*(x) \) for all \( x \in \overline{\Omega} \) then the embedding \( W^{1,p(x)}_0(\Omega) \hookrightarrow L^{q(x)}(\Omega) \) is compact.

3. **Main results.** Throughout this paper, we are interested in combined effects generated by the competition between the growth of the three variable exponents involved in the generalized mean curvature problem (2). We consider problem (2) under different assumptions concerning the values of \( p, q \) and \( r \) and we establish sufficient conditions for the existence of nontrivial weak solutions. In both cases considered in the present paper, we are concerned with a subcritical abstract setting. The first main result is concerned with the existence of a non-negative solution while the second theorem established in this paper deals with the existence of infinitely many solutions.

We say that \( v \in W^{1,p(x)}_0(\Omega) \setminus \{0\} \) is a solution of problem (2) if for all \( v \in W^{1,p(x)}_0(\Omega) \)

\[ \int_{\Omega} (1 + |\nabla u|^2)^{(p(x)-2)/2} \nabla u \nabla v dx + \int_{\Omega} |u|^{q(x)-2} uv dx = \lambda \int_{\Omega} |u|^{r(x)-2} u dx. \]

We are first concerned with the existence of non-negative solutions of problem (2). More precisely, we study the following nonlinear problem

\[
\begin{cases}
-\text{div} \left[ (1 + |\nabla u|^2)^{(p(x)-2)/2} \nabla u \right] + |u|^{q(x)-2} u = \lambda |u|^{r(x)-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega \\
u \geq 0 & \text{in } \Omega.
\end{cases}
\]  

(11)

In order to establish a sufficient condition for the existence of solutions of problem (11), we assume that \( p \in \mathcal{C} \) and \( q, r \in C_{+}(\overline{\Omega}) \) satisfy the hypothesis

\[ r^- < p^- \land q^- \leq 2 \quad \text{and} \quad q \vee r \leq p^*. \quad (12) \]

Under this assumption, we establish the existence of a non-negative solution in the case of small perturbations of the reaction term, namely if the positive parameter \( \lambda \) is small enough. A related result has been established in R. Alsaedi [1, Theorem 3.1].

**Theorem 3.1.** Assume that \( p \in \mathcal{C} \) and \( q, r \in C_{+}(\overline{\Omega}) \) satisfy hypothesis (12). Then there exists a positive number \( \lambda^* \) such that for all \( \lambda \in (0, \lambda^*) \), problem (11) has at least one solution.
In particular, under hypothesis (12), we deduce that
\[ \inf_{u \in W_0^{1,p(x)}(\Omega)} \left\{ \int_\Omega (1 + |\nabla u|^2)^{(p(x)-2)/2}|\nabla u|^2 dx + |u|_{p(x)}; \ |u|_{r(x)} = 1 \right\} = 0. \]

A counterpart of Theorem 3.1 in the semilinear elliptic case includes problems of the type
\[
\begin{cases}
-\Delta u + u = \lambda |u|^{r-2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega \\
u \geq 0 & \text{in } \Omega,
\end{cases}
\] (13)
where \(1 < r < 2\). We refer to H. Brezis and L. Oswald [5] for a thorough analysis of problem (13). In [5], the existence of the (unique) nontrivial solution of problem (13) relies on a minimization technique. In addition, it is observed that the energy functional to be minimized associated to problem (13) is convex with respect to the variable \(\rho = u^2\). We notice that problem (13) has a solution for all \(\lambda > 0\). However, in our anisotropic case described in Theorem 3.1, we are not able to prove that the solution exists for all \(\lambda > 0\). This comes essentially from the combined growths of the variable exponents \(p, q\) and \(r\), as they are described in hypothesis (12). In problem (11) the presence of variable exponents allow us to establish the existence of solutions only for small values of \(\lambda\). At this stage, we are not able to say what happens for values of \(\lambda\) larger than \(\lambda^*\). This is mainly due to the fact that in our case described by hypothesis (12), we cannot apply coercivity arguments for the energy functional associated to problem (11).

Next, we are interested in the existence of infinitely many solutions of problem (2). The main assumption is
\[ p^+ \vee q^+ < r^- \quad \text{and} \quad r \ll p^*. \] (14)

This hypothesis corresponds to the case where the subcritical reaction term dominates the left-hand side of problem (2). In this case, the associated energy is not coercive but has a mountain pass geometry. The symmetry of the problem implies the existence of infinitely many solutions. However, we cannot assert that these solutions have constant sign or if they are nodal.

**Theorem 3.2.** Assume that \(p \in C\) and \(q, r \in C_+(\Omega)\) satisfy hypothesis (14). Then problem (2) has infinitely many solutions for all \(\lambda > 0\).

A counterpart of Theorem 3.2 in the semilinear elliptic case includes the problem
\[
\begin{cases}
-\Delta u + u = \lambda |u|^{r-2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \(2 < r < 2N/(N - 2)\). This problem goes back to A. Ambrosetti and P. Rabinowitz [3] who used the symmetric version of their mountain pass theorem in order to prove the existence of infinitely many solutions for all \(\lambda > 0\).

In both cases described by Theorems 3.1 and 3.2, the associated energy is \(\mathcal{J}(u) : W^{1,p(x)}_0(\Omega) \to \mathbb{R}\) defined by
\[
\mathcal{J}(u) = \int_\Omega \frac{1}{p(x)} \left[ (1 + |\nabla u|^2)^{p(x)/2} - 1 \right] dx + \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx - \lambda \int_\Omega \frac{1}{r(x)} |u|^{r(x)} dx.
\]

Any of the hypotheses (12) or (14) implies that \(W^{1,p(x)}_0(\Omega)\) is compactly embedded both into \(L^{q(x)}(\Omega)\) and \(L^{r(x)}(\Omega)\), hence \(\mathcal{J}\) is well defined. Moreover, \(\mathcal{J}\) is of
class $C^1$ and its directional derivative is

$$
\langle J'(u), v \rangle = \int_{\Omega} (1 + |\nabla u|^2)^{(p(x)-2)/2} \nabla u \nabla v dx + \int_{\Omega} |u|^{q(x)-2}uv dx - \lambda \int_{\Omega} |u|^{r(x)-2}uv dx,
$$

for all $u, v \in W^{1,p(x)}_0(\Omega)$.

4. Proof of Theorem 3.1. Under hypothesis (12), we show that the energy functional $J$ satisfies only one of the geometric assumptions of the mountain pass theorem. In our arguments we apply some ideas found in the proof of Theorem 2.1 in [17].

By hypothesis (12) and compact embeddings for Lebesgue and Sobolev spaces with variable exponent, there exists a positive constant $C_0$ such that for all $u \in W^{1,p(x)}_0(\Omega)

$$
|u|_{q(x)} \leq C_0 \|u\|_{p(x)} \quad \text{and} \quad |u|_{r(x)} \leq C_0 \|u\|_{p(x)}.
$$

We divide the proof into several steps.

Step 1. There exist positive numbers $R_0$, $a_0$ and $\lambda^*$ such that $J(u) \geq a_0$ for all $u \in W^{1,p(x)}_0(\Omega)$ with $\|u\|_{p(x)} = R_0$ and for all $\lambda \in (0, \lambda^*)$.

We fix $R_0 \in (0, 1)$ such that $R_0C_0 < 1$. Fix $u \in W^{1,p(x)}_0(\Omega)$ with $\|u\|_{p(x)} = R_0$. By (15), it follows that $|u|_{q(x)} < 1$ and $|u|_{r(x)} < 1$. Using relations (5) and (9) we have

$$
J(u) \geq \frac{1}{p^+} \rho_{p(x)}(u) + \frac{1}{q^+} \rho_{q(x)}(u) - \frac{\lambda}{r^-} p_{r(x)}(u)
\geq \frac{1}{p^+} \|u\|^{p^+}_{p(x)} - \frac{\lambda}{r^-} C_0^{r^-} \|u\|^{p^+}_{p(x)}
= \frac{R_0^{p^+}}{p^+} - \frac{\lambda}{r^-} C_0^{r^-} R_0^{-r^-}.
$$

Let

$$
\lambda^* := \frac{r^- R_0^{p^+ - r^-}}{p^+ C_0^{r^-}}.
$$

By relation (16), there exists $a_0 = a_0(R_0) > 0$ such that $J(u) \geq a_0$ for all $u \in W^{1,p(x)}_0(\Omega)$ with $\|u\|_{p(x)} = R_0$ and for all $\lambda \in (0, \lambda^*)$. We also point out that the optimal value for $\lambda^*$ defined by (17) corresponds to $R_0 = \min\{1, C_0^{-1}\}$, hence

$$
\lambda^* := \frac{r^- \min\{1, C_0^{-1}\}^{p^+ - r^-}}{p^+ C_0^{r^-}}.
$$

Step 2. There exist $v \in W^{1,p(x)}_0(\Omega)$ and $t_0 > 0$ such that $J(tv) < 0$ for all $t \in (0, t_0)$.

By assumption (12), there exists $\eta > 0$ such that $r^- + \eta < p^- \wedge q^- \leq 2$. On the other hand, since $r$ is continuous, there exists an open set $\omega$ with $\omega \subset \Omega$ such that

$$
|r(x) - r^-| < \eta \quad \text{for all} \quad x \in \omega.
$$

It follows that

$$
r(x) < r^- + \eta < p^- \wedge q^- \leq 2 \quad \text{for all} \quad x \in \omega.
$$

(18)
Proof of Theorem 3.1 completed. We know that there exists a bounded sequence \( u \) such that \( u \equiv 1 \) in \( \omega \). For all \( t \in (0, 1) \) we have

\[
\mathcal{J}(tv) = \int_{\Omega} \frac{t^2}{p(x)} \left[ 1 + t^2 |\nabla v|^2 \right]^{(p(x)-2)/2} |\nabla v|^2 \, dx \\
+ \int_{\Omega} \frac{t^q(x)}{R^q(x)} |v-q(x)| \, dx - \lambda \int_{\Omega} \frac{t^r(x)}{r(x)} |r(x)| \, dx \\
\leq C_1 t^2 + \frac{t^q}{q} \|\omega\| - \frac{\lambda}{r^+} \int_{\omega} t^r(x) |r(x)| \, dx \\
\leq C_1 t^2 + C_2 t^{q^-} - C_3 t^{r^- + \eta}.
\]

Using (18) we deduce that \( \mathcal{J}(tv) < 0 \) for all \( t > 0 \) sufficiently small.

**Step 3.** Existence of almost critical points.

Steps 1 and 2 show that there exist \( \lambda^* > 0 \) and \( R_0 > 0 \) such that for all \( \lambda \in (0, \lambda^*) \)

\[
\inf \{ \mathcal{J}(u); \|u\|_{p(x)} = R_0 \} > 0
\]

and

\[
m := \inf \{ \mathcal{J}(u); \|u\|_{p(x)} \leq R_0 \} < 0.
\]

Fix \( \lambda \in (0, \lambda^*) \) and

\[
0 < \varepsilon < \inf \{ \mathcal{J}(u); \|u\|_{p(x)} = R_0 \} - \inf \{ \mathcal{J}(u); \|u\|_{p(x)} \leq R_0 \}.
\]

We apply the Ekeland variational principle to \( \mathcal{J} \) restricted to the complete metric space \( B(0, R_0) \subset W_0^{1,p(x)}(\Omega) \). For any \( \varepsilon > 0 \) as above we find \( u_\varepsilon \) such that

\[
\|u_\varepsilon\|_{p(x)} \leq R_0 \quad \text{and} \quad m \leq \mathcal{J}(u_\varepsilon) \leq m + \varepsilon
\]

\[*\]

\[
0 < \mathcal{J}(u) - \mathcal{J}(u_\varepsilon) + \varepsilon \|u - u_\varepsilon\| \quad \text{for all} \ u \neq u_\varepsilon.
\]

With a standard argument, relation \( \[*\] \) shows that \( \|\mathcal{J}'(u_\varepsilon)\| \leq \varepsilon \). Thus, using \( \|[\mathcal{J}'(u_\varepsilon)\] \), we conclude that \( u_\varepsilon \) is an “almost critical point” of \( \mathcal{J} \) for all \( \varepsilon > 0 \) sufficiently small.

**Proof of Theorem 3.1 completed.** We know that there exists a bounded sequence \( (u_n) \subset W_0^{1,p(x)}(\Omega) \) (with \( \|u_n\| \leq R_0 \) for all \( n \)) such that

\[
\|\mathcal{J}'(u_n)\| \to 0.
\]

By reflexivity and compact embeddings, we can assume that, up to a subsequence,

\[
u_n \to u \quad \text{in} \ W_0^{1,p(x)}(\Omega)
\]

\[
u_n \to u \quad \text{in} \ L^q(x)(\Omega) \quad \text{and} \quad L^r(x)(\Omega).
\]

Relation \( \[21\] \) implies that

\[
\langle \mathcal{J}'(u_n), u_n - u \rangle \to 0 \quad \text{as} \ n \to \infty.
\]

We first argue that

\[
\int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) \, dx \to 0 \quad \text{as} \ n \to \infty.
\]

Indeed, by H"older’s inequality \( \[3\] \),

\[
\left| \int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) \, dx \right| \leq 2 |u_n - u|_{q(x)} |u_n|^{q(x)-2} |u_n|_{q(x)/(q(x)-1)} \to 0 \quad \text{as} \ n \to \infty.
\]
With a similar argument we deduce that
\[ \int_{\Omega} |u_n|^{r(x)-2}u_n(u_n - u) \, dx \rightarrow 0 \quad \text{as} \ n \rightarrow \infty. \quad (24) \]

Relations (22), (23) and (24) yield
\[ \int_{\Omega} (1 + |\nabla u_n|^2)^{(p(x)-2)/2} \nabla u_n \nabla (u_n - u) \rightarrow 0 \quad \text{as} \ n \rightarrow \infty. \quad (25) \]

Consider the nonlinear operator \( L : W^{1,p(x)}_0(\Omega) \rightarrow W^{-1,p'(x)}(\Omega) \) defined by
\[ (Lu, v) = \int_{\Omega} (1 + |u|^2)^{(p(x)-2)/2} u \nabla v \quad \text{for all} \ u, v \in W^{1,p(x)}_0(\Omega). \]

With the same arguments as in the proof of Theorem 3.1 of X. Fan and Q. Zhang [10] we deduce that \( L \) is continuous, bounded, strictly monotone, and a mapping of type \((S_+)\), that is, if
\[ u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n \to \infty} (Lu_n - Lu, u_n - u) \leq 0, \]
then \( u_n \rightarrow u \) in \( W^{1,p(x)}_0(\Omega) \). In our case, using relation (25), we conclude that \( u_n \) strongly converges to \( u \) in \( W^{1,p(x)}_0(\Omega) \). We deduce that \( J(u) = m < 0 \) and \( J'(u) = 0 \), hence \( u \) is a nontrivial solution of problem (2). \( \square \)

5. **Proof of Theorem 3.2.** We check the hypotheses of the symmetric mountain pass theorem of A. Ambrosetti and P. Rabinowitz [3, Corollary 2.9]. For the convenience of the reader, we recall this fundamental result in what follows.

**Theorem 5.1.** Let \( E \) be an infinite dimensional Banach space over \( \mathbb{R} \). We assume that \( I \) is an even functional of class \( C^1 \) which satisfies the following hypotheses:

\( (I_1) \) there exist positive numbers \( R_0 \) and \( a_0 \) such that \( I(u) \geq a_0 \) for all \( u \in E \) with \( \|u\| = R_0 \);

\( (I_2) \) if \( u_n \subset E \) with \( I(u_n) \) bounded and \( I'(u_n) \rightarrow 0 \), then \( u_n \) possesses a convergent subsequence;

\( (I_3) \) for any finite dimensional subspace \( F \) of \( E \), the set \( P := \{ u \in F ; \ I(u) \geq 0 \} \) is bounded.

Then \( I \) has infinitely many distinct pairs of critical points.

We first prove the existence of a “chain of mountains” near the origin for the associated energy functional \( J \). Next, we prove that if \( F \subset W^{1,p(x)}_0(\Omega) \) is an arbitrary finite dimensional subspace then the set of points \( u \in F \) with \( J(u) \geq 0 \) is bounded. Finally, we prove that the even functional \( J \) satisfies the Palais-Smale compactness condition.

Let \( C_0 \) be the positive constant defined in (15) and that corresponds to the continuous embeddings \( W^{1,p(x)}_0(\Omega) \hookrightarrow L^{q(x)}(\Omega) \) and \( W^{1,p(x)}_0(\Omega) \hookrightarrow L^{r(x)}(\Omega) \).

We divide the proof into several steps.

**Step 1.** For all \( \lambda > 0 \), there are positive numbers \( R_0 \) and \( a_0 \) such that \( J(u) \geq a_0 \) for all \( u \in W^{1,p(x)}_0(\Omega) \) with \( \|u\|_{p(x)} = R_0 \).

The key argument in the proof is hypothesis (14), more precisely \( p^+ < r^- \). Fix \( \lambda > 0 \) and \( R_0 \in (0,1) \) such that \( R_0 C_0 < 1 \). Let \( u \in W^{1,p(x)}_0(\Omega) \) such that
\[ \|u\|_{p(x)} = R_0. \] By relation (15), we deduce that \(|u|_{q(x)} < 1 \) and \(|u|_{r(x)} < 1 \). Using (5) and (9) we deduce that

\[
\mathcal{J}(u) \geq \frac{1}{p^+} q_{p(x)}(u) + \frac{1}{q^+} \rho_{q(x)}(u) - \frac{\lambda}{r^+} \rho_{r(x)}(u) \\
\geq \frac{1}{p^+} \|u\|_{p(x)}^{p^+} - \frac{\lambda}{r^+} C_0^r \|u\|_{p(x)}^{p^+} \\
= \frac{R_0^{p^+}}{p^+} - \frac{\lambda}{r^+} C_0^r R_0^{r^-} \\
= C_2 R_0^{r^+} + \lambda C_2 R_0^{r^-}.
\]

Thus, since \( \lambda > 0 \) is fixed and \( p^+ < r^- \), we can choose eventually a smaller \( R_0 > 0 \) and some \( a_0 > 0 \) such that \( \mathcal{J}(u) \geq a_0 \) for all \( u \in W^{1,p(x)}(\Omega) \) with \( \|u\|_{p(x)} = R_0 \).

**Step 2.** Let \( F \) be an arbitrary finite dimensional subspace of \( W^{1,p(x)}(\Omega) \) and \( P := \{u \in F ; \mathcal{J}(u) \geq 0\} \). Then \( P \) is bounded for all \( \lambda > 0 \).

We have for all \( u \in W^{1,p(x)}(\Omega) \)

\[
\int_\Omega \frac{1}{p(x)} (1 + |\nabla u|^2)^{p(x)/2} - 1 \, dx \leq \frac{1}{p^-} \int_\Omega (1 + |\nabla u|^2)^{p(x)/2}. \tag{26}
\]

For fixed \( a > 0 \), since \( \lim_{x \to +\infty} (1 + x)^a/(1 + x^a) = 1 \), there exists \( C > 0 \) such that

\[ (1 + x)^a \leq C(1 + x^a) \quad \text{for all } x \geq 0. \]

Thus, relation (26) yields

\[
\int_\Omega \frac{1}{p(x)} (1 + |\nabla u|^2)^{p(x)/2} - 1 \, dx \leq \frac{C}{p^-} q_{p(x)}(u) \\
\leq C_3 + C_4 (|\nabla u|_{p(x)}^{p^-} + |\nabla u|_{p(x)}^{p^+}) \\
= C_3 + C_4 (\|u\|_{p(x)}^{q(x)} + \|u\|_{p(x)}^{q(x)}). \tag{27}
\]

Next, for all \( u \in W^{1,p(x)}(\Omega) \),

\[
\int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx \leq \frac{1}{q^+} (|u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+}) \\
\leq \frac{C_5}{q^+} (\|u\|_{p(x)}^{q^-} + \|u\|_{p(x)}^{q^+}) \\
= C_5 (\|u\|_{p(x)}^{q^-} + \|u\|_{p(x)}^{q^+}). \tag{28}
\]

For fixed \( u \in W^{1,p(x)}(\Omega) \) \( \setminus \{0\} \) we denote

\[ \Omega_1 := \{s \in \Omega ; |u(s)| \geq 1\}. \]

We have

\[
\int_{\Omega_1} \frac{1}{r(x)} |u|^{r(x)} \, dx \geq \frac{1}{r^-} \int_{\Omega_1} |u|^{r(x)} \, dx \geq \frac{1}{r^+} \int_{\Omega_1} |u|^{r^-} \, dx. \tag{29}
\]

Let \( F \) be an arbitrary finite dimensional subspace of \( W^{1,p(x)}(\Omega) \). Then

\[
\left( \int_{\Omega} |u|^{r^-} \, dx \right)^{1/r^-} \quad \text{is a norm in } F,
\]

which is equivalent with the norm \( \|u\|_{p(x)} \). Thus, there exists \( C_6 > 0 \) such that

\[ |u|_{r^-} \geq C_6 \|u\|_{p(x)} \quad \text{for all } u \in F. \]
Returning to relation (29) we obtain for all $u \in F$

$$
\int_{\Omega} \frac{1}{r(x)} |u|^r dx \geq C_7 \|u\|_{p(x)}^{-r}.
$$

Combining relations (27), (28) and (30) we obtain for all $u \in P$

$$
0 \leq J(u) \leq C_3 + C_4(\|u\|_{p(x)}^r + \|u\|_{p(x)}^{r^+}) + C_5(\|u\|_{p(x)}^r + \|u\|_{p(x)}^{r^+}) - C_7 \|u\|_{p(x)}^{-r}. 
$$

Using hypothesis (14) we conclude that the set $P$ is bounded.

**Step 3.** Any Palais-Smale sequence of $J$ is relatively compact in $W^{1,p(x)}(\Omega)$.

Let $(u_n) \subset W^{1,p(x)}_0(\Omega)$ be such that

$$
J(u_n) = O(1) \quad \text{and} \quad \|J'(u_n)\|_{W^{-1,p'(x)}} = o(1) \quad \text{as} \quad n \to \infty.
$$

It follows that

$$
\int_{\Omega} \frac{1}{p(x)} \left[ (1 + |\nabla u_n|^2)^{p(x)/2} - 1 \right] dx + \int_{\Omega} \frac{1}{q(x)} |u_n|^q dx - \lambda \int_{\Omega} \frac{1}{r(x)} |u_n|^r dx = O(1)
$$

and

$$
\int_{\Omega} (1 + |\nabla u_n|^2)^{(p(x)-2)/2} |\nabla u_n|^2 dx + \int_{\Omega} |u_n|^q dx - \lambda \int_{\Omega} |u_n|^r dx = o(\|u_n\|).
$$

We first prove that $(u_n)$ is bounded in $W^{1,p(x)}_0(\Omega)$.

Arguing by contradiction, we can assume that $\|u_n\|_{p(x)} \to \infty$ as $n \to \infty$ and $\|u_n\|_{p(x)} > 1$ for all $n \geq 1$.

Using relation (32) we obtain

$$
\|\varphi_{p(x)}(u_n) + \rho_{q(x)}(u_n) - \lambda \rho_{r(x)}(u_n)\| \leq \|u_n\|_{p(x)}.
$$

Therefore

$$
\lambda \rho_{r(x)}(u_n) \leq \varphi_{p(x)}(u_n) + \rho_{q(x)}(u_n) + \|u_n\|_{p(x)}.
$$

Set

$$
c_0 := \frac{1}{p(x)} + \frac{1}{q(x)} - \frac{1}{r(x)}.
$$

By hypothesis (14) we have $c_0 > 0$. Using (33), relation (31) yields

$$
O(1) = J(u_n) \geq c_0 \varphi_{p(x)}(u_n) + c_0 \rho_{q(x)}(u_n) - \lambda \|u_n\|_{p(x)}
$$

$$
\geq c_0 \varphi_{p(x)}(u_n) - \lambda \|u_n\|_{p(x)}
$$

$$
\geq c_0 \|u_n\|_{p(x)} - \lambda \|u_n\|_{p(x)} \to \infty \quad \text{as} \quad n \to \infty,
$$

which is a contradiction. We conclude that the sequence $(u_n)$ is bounded in $W^{1,p(x)}_0(\Omega)$. Thus, up to a subsequence,

$$
u_n \to u \quad \text{in} \quad W^{1,p(x)}_0(\Omega)
$$

and

$$
u_n \to u \quad \text{in} \quad L^q(x)(\Omega) \text{ and } L^r(x)(\Omega).
$$
On the other hand, since \( \langle J'(u_n), u_n - u \rangle = o(1) \) as \( n \to \infty \) we have
\[
\int_{\Omega} (1 + |\nabla u_n|^2)^{(p(x)-2)/2} \nabla u_n \nabla (u_n - u) dx + \int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) dx - \\
\int_{\Omega} |u_n|^{r(x)-2} u_n (u_n - u) = o(1) \quad \text{as} \ n \to \infty.
\]
Using (35) we deduce that
\[
\int_{\Omega} (1 + |\nabla u_n|^2)^{(p(x)-2)/2} \nabla u_n \nabla (u_n - u) dx = o(1) \quad \text{as} \ n \to \infty. \tag{36}
\]
Next, by (34), we have
\[
\int_{\Omega} (1 + |u|^2)^{(p(x)-2)/2} \nabla u \nabla (u - u) dx = o(1) \quad \text{as} \ n \to \infty. \tag{37}
\]
By Proposition 3.3 in [15] we have
\[
\varrho_{p(x)}(u_n - u) := \int_{\Omega} \frac{1}{|\nabla (u_n - u)|^{p(x)}} \nabla (u_n - u) dx \\
\leq \frac{4^{1-p}}{4^{1-p}} \left( \int_{\Omega} (1 + |\nabla u_n|^2)^{(p(x)-2)/2} \nabla u_n \nabla (u_n - u) dx \right) \\
- \frac{1}{4^{1-p}} \left( \int_{\Omega} (1 + |u|^2)^{(p(x)-2)/2} \nabla u \nabla (u - u) dx. \right) \tag{38}
\]
Combining relations (36), (37) and (38) we deduce that
\[
\varrho_{p(x)}(u_n - u) \to 0 \quad \text{as} \ n \to \infty.
\]
We conclude that \( J \) satisfies the Palais-Smale condition for all \( \lambda > 0 \).

**Proof of Theorem 3.2 completed.** Steps 1–3 show that \( J \) satisfies the hypotheses of Theorem 5.1. We conclude that \( J \) has infinitely many distinct pairs of critical points, hence problem (2) has infinitely many solutions for all \( \lambda > 0 \). \( \square \)

An alternative argument to show that the sequence \( (u_n) \subset W_0^{1,p(x)}(\Omega) \) satisfying (34) and (35) is relatively compact is to follow the methods introduced by L. Boccardo and F. Murat [4, Theorem 2.1]. These tools allow the strong convergence of gradients in suitable Lebesgue spaces for equations driven by Leray-Lions differential operators.

**REFERENCES**

[1] R. Alsaedi, Perturbed subcritical Dirichlet problems with variable exponents, *Electron. J. Differential Equations*, **2016** (2016), Paper No. 295, 12 pp.

[2] R. Alsaedi, H. Mâagli and N. Zeddini, Exact behavior of the unique positive solution to some singular elliptic problem in exterior domains, *Nonlinear Anal.*, **119** (2015), 186–198.

[3] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Anal.*, **14** (1973), 349–381.

[4] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, *Nonlinear Anal.*, **19** (1992), 581–597.

[5] H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal.*, **10** (1986), 55–64.

[6] M. Cencelj, D. Repovš and Z. Virk, Multiple perturbations of a singular eigenvalue problem, *Nonlinear Anal.*, **119** (2015), 37–45.

[7] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011.

[8] D. Edmunds, J. Lang and O. Méndez, *Differential Operators on Spaces of Variable Integrability*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

[9] I. Ekeland, *On the variational principle*, *J. Math. Anal. Appl.*, **47** (1974), 324–353.
[10] X. Fan and Q. Zhang, Existence of solutions for \( p(x) \)-Laplacian Dirichlet problem, *Nonlinear Anal.*, **52** (2003), 1843–1852.

[11] Y. Fu and Y. Shan, On the removability of isolated singular points for elliptic equations involving variable exponent, *Adv. Nonlinear Anal.*, **5** (2016), 121–132.

[12] E. Giusti, On the equation of surfaces of prescribed mean curvature: existence and uniqueness without boundary conditions, *Invent. Math.*, **46** (1978), 111–137.

[13] T. C. Halsey, Electrorheological fluids, *Science*, **258** (1992), 761–766.

[14] B. Kawohl, From \( p \)-Laplace to mean curvature operator and related questions, *Progress in Partial Differential Equations: The Metz Surveys*, 40–56, Pitman Res. Notes Math. Ser., **249**, Longman Sci. Tech., Harlow, 1991.

[15] I. Kim and Y. Kim, Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents, *Manuscripta Math.*, **147** (2015), 169–191.

[16] H. Maagli, R. Alsaedi and N. Zeddini, Bifurcation analysis of elliptic equations described by nonhomogeneous differential operators, *Electron. J. Differential Equations*, **2017** (2017), Paper No. 223, 12 pp.

[17] M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.*, **135** (2007), 2929–2937.

[18] P. Pucci and V. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: A mathematical survey, *Boll. Unione Mat. Ital.*, Series IX, **3** (2010), 543–582.

[19] P. Pucci and Q. Zhang, Existence of entire solutions for a class of variable exponent elliptic equations, *J. Differential Equations*, **257** (2014), 1529–1566.

[20] V. Rădulescu, *Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations. Monotonicity, Analytic, and Variational Methods*, Contemporary Mathematics and Its Applications, vol. 6, Hindawi Publishing Corporation, New York, 2008.

[21] V. Rădulescu, Nonlinear elliptic equations with variable exponent: Old and new, *Nonlinear Analysis: Theory, Methods and Applications*, **121** (2015), 336–369.

[22] V. Rădulescu and D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[23] M. Struwe, *Plateau’s Problem and the Calculus of Variations*, Mathematical Notes, vol. 35, Princeton University Press, Princeton, NJ, 1988.

Received May 2017; revised November 2017.

E-mail address: ramziaisaeedi@yahoo.co.uk