Invariant classification of metrics using invariant formalism

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Abstract. Metrics obtained by integrating within the generalised invariant formalism are structured around their intrinsic coordinates and this considerably simplifies their invariant classification and symmetry analysis. We illustrate this by presenting a simple and transparent complete invariant classification of the conformally flat pure radiation metrics (except plane waves) in such intrinsic coordinates. By performing this classification we have corrected and completed statements and results by Edgar and Vickers, and by Skea, about the orders of Cartan invariants at which particular information becomes available.

1. Introduction
The integration procedure developed in the GHP formalism [1] has recently been generalised to the GIF (generalised invariant formalism [2]) [3], [4], [5], [6]. Compared to the familiar integration procedures in NP formalism [7] this procedure is much more efficient and avoids detailed complicated gauge calculations which have the potential for errors. It also supplies the metric in a natural form, with coordinates chosen as far as possible, in an intrinsic and invariant manner, permitting a comparatively simple invariant classification procedure, equivalence problem and symmetry investigations.

The equivalence problem is the problem of determining whether the metrics of two spacetimes are locally equivalent, and the original contribution of Cartan [8] directed attention to the Riemann tensor and its covariant derivatives up to \((q + 1)\)th order, \(\mathcal{R}^{q+1}\), calculated in a particular frame. In going from \(\mathcal{R}^q\) to \(\mathcal{R}^{q+1}\) for a particular spacetime, if there is no new functionally independent Cartan scalar invariant and \(\mathcal{R}^q\) and \(\mathcal{R}^{q+1}\) have equal isotropy group, then all the local information that can be obtained about the spacetime is contained in the set \(\mathcal{R}^{q+1}\). The set \(\mathcal{R}^{q+1}\) is called the Cartan scalar invariants and provide the information for an invariant classification of the spacetime.

Two metrics are equivalent and represent the same spacetime if all their respective Cartan scalar invariants in \(\mathcal{R}^{q+1}\) can be equated consistently. It is important to note that although there will be no new information about essential coordinates in the step from \(\mathcal{R}^q\) to \(\mathcal{R}^{q+1}\), there may be other new information, in particular about inconsistencies and also the nature of apparently non-redundant functions (including constants).

A practical method for invariant classification was developed by Karlhede [9], using fixed frames. In this algorithm the number of functionally independent quantities is kept as small as possible.
at each step by putting successively the curvature and its covariant derivatives into canonical form, and only permitting those frame changes which preserve the canonical form. Although the Karlhede algorithm is more efficient than the original procedure proposed by Cartan, it may need to go as far as to $R^7$ [10], and as a consequence, for some spacetimes, long complicated calculations are required, which usually need computer support, e.g., using the programme CLASSI [11], or the Maple-based GRTensor programme. The Karlhede algorithm can be exploited to determine the structure of the isometry group of the spacetime, as well as subclasses within the spacetime which have additional isometries [12]; more recently the scheme has been the basis for an algorithm which determines whether a spacetime admits a homothetic Killing vector [13].

2. Invariant classification of Edgar-Ludwig spacetime in intrinsic coordinates from GIF

Edgar and Vickers [3] have rederived all CFPR spacetimes, which are not plane waves, using GIF, obtaining in coordinates $t, n, a, b$ the metric

$$ds^2 = \left( a(2s(t) - 2am(t) - a^2 - b^2) - e(t)^2 - n^2 \right) dt^2 - 2adtdn + 2ndtda + 2e(t)dtdb - da^2 - db^2$$

(1)

where $m(t), e(t), s(t)$ are non-redundant functions of the coordinate $t$; this form includes the possibility of any of $m(t)$ or $e(t)$ or $s(t)$ being constant. This form represents the most general metrics for CFPR spacetimes (with zero cosmological constant).

We begin by repeating the zeroth, first and second order invariants quoted in [3]. At zeroth order, there is only the one Cartan spinor invariant

$$\Phi = \frac{q^2}{a}$$

(2)

At first order, there are four Cartan spinor invariants

$$\mathbf{p}\Phi = 0, \quad \partial \Phi = \frac{pq^2}{a^2}, \quad \mathbf{q} \Phi = \frac{\bar{p}q^2}{a^2}, \quad \mathbf{v} \Phi = \frac{q^2}{a^2} \left( \frac{qn}{a} + 3pm + 3\bar{p}\tilde{m} \right)$$

(3)

$t$ is a second spinor which is generated in the GIF analysis, $p$ and $q$ are weighted scalar invariants which represent the spin and boost freedom; $q$ is real while complex $p$ satisfies $\bar{p}p = 1/2$.

It is easy to see that we may invert (2) and (3) for $a, p$ and $q$ in terms of Cartan spinor invariants, and also for $(p\mathbf{t} + \bar{p}\tilde{m})$; therefore $t$ is not uniquely determined, (and so neither is $n$) and at this level there clearly would remain the gauge freedom of a one parameter subgroup of null rotations. Since new information about the essential coordinates has arisen, we must go to the next order.

At second order, a complete set of independent Cartan spinor invariants is

$$\mathbf{p}\partial \Phi = 0, \quad \partial \partial \Phi = \frac{2p^2q^2}{a^3}, \quad \partial \mathbf{q} \Phi = \frac{q^2}{a^3}, \quad \mathbf{v} \partial \Phi = 0$$

$$\partial \mathbf{v} \Phi = \frac{pq^2}{a^3} \left( \frac{3qn}{a} + 6pm + 8\bar{p}\tilde{m} \right)$$

$$\mathbf{v} \partial \mathbf{v} \Phi = \frac{q^4}{a^3} \left( s(t) - 2am(t) - \frac{5}{2}a^2 + \frac{1}{2}b^2 + \frac{3n^2}{a} \right) + \frac{3q^2}{a^3} \left( 4p^2\mathbf{t}^2 + 5pp\mathbf{m} + \bar{p}\tilde{m} \right) + 12\frac{q^2n}{a^3}(pm + \bar{p}\tilde{m})$$

(4)
together with complex conjugates. The GIF commutator equations enable us to concentrate on this reduced list of independent invariants.

We can now invert these equations and obtain explicit expressions for the spinor $i$, as well as for $n$ and the scalar combination $(s(t) - 2am(t) + \frac{1}{2}b^2)$, in terms of Cartan invariants. Thus, at second order, if we make this choice of $i$ as our second spinor, we will have fixed the frame completely (there is no isotropy freedom remaining), and we also have determined three essential base coordinates. Moreover, making this choice of $i$ as the second dyad spinor enables us to transfer to the simpler GHP formalism (see [3] for a fuller discussion on when this is possible) since we require only the scalar parts of the remaining non-trivial GHP Cartan invariants.

At third order, if we make this choice of $i$, then the only possibly new independent information will come from the operator $\mathcal{V}'$ acting on the scalar $X$ defined by

$$X = \mathcal{V}'X = \mathcal{V}'(s(t) - 2am(t) + \frac{1}{2}b^2, a, b, c).$$

This gives

$$\mathcal{V}'X = q^6 \left( s''(t) - 2am''(t) - 5\frac{ns'(t)}{a} + 9nm'(t) + be'(t) - 5\frac{nb}{a}e(t) + e(t)^2 \right)$$

where $m''(t)$ and $s''(t)$ denote differentiation twice with respect to $t$, and $e'(t)$ denotes differentiation with respect to $t$.

Solving for a fourth coordinate is a little more complicated, since we have to go to third order, and the details will depend on the nature of the functions $s(t), m(t), e(t)$.

In summary we get the two following cases:
• if at least one of the functions \( m(t), s(t), e(t) \), is not constant then all four essential base coordinates are obtained from GHP Cartan invariants at third order, and the procedure will therefore formally terminate at fourth order.

• when all of the functions \( m(t), s(t), e(t) \) are constants, then only three essential base coordinates can be obtained from Cartan GHP invariants; these are obtained at second order, and the procedure will therefore formally terminate at third order.

Because of the very simple structure of this metric and the close relationship of its coordinates with its GHP Cartan invariants it has been easy to draw conclusions on its classification from a direct examination of its GHP Cartan invariants. Continuing in this manner, the three apparently non-redundant functions \( m(t), s(t), e(t) \) can each be directly identified with a linear combination of the three GHP Cartan invariants \( I_0'I_0', \Re(\partial I_0'I_0'), I_0'I_0'I_0'I_0' \Phi \), and hence are genuinely non-redundant, and cannot be transformed away.

3. Summery

The simplicity and transparency of this GIF version (1), combined with the fact that we are able to carry it out by hand, gives us a clear unambiguous overview of the invariant classification of this class of metrics. The results obtained in Section 2 have some minor, but subtle and interesting, disagreements with the conclusions by Edgar and Vickers [3], and by Skea [14]. The traditional CLASSI analysis of the spacetime in the intrinsic coordinates was carried out in [15] confirming the results in Section 2.

In addition, using this version of the spacetime, we were able to obtain trivially the Killing vector properties, as well as the homothetic Killing vector properties by a straightforward application of the Koutras-Skea algorithm [15].

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