Complete integrability of the coupled KdV–mKdV system

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Abstract. The coupled KdV–mKdV system arises as the classical part of one of superextensions of the KdV equation. For this system, we prove its complete integrability, i.e., existence of a recursion operator and of infinite series of symmetries.

Introduction

There are several supersymmetric extensions of the classical Korteweg–de Vries equation (KdV) [5, 8, 9]. One of them is of the form (the so-called $N = 2, A = 1$ extension [3])

$$
\begin{align*}
    u_t &= -u_3 + 6uu_1 - 3\phi\psi_2 - 3\psi_2 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uw_1 \\
    & \quad + 6\psi_1w - 6\phi\psi w - 6\phi\psi w_1, \\
    \phi_t &= -\phi_3 + 3\phi u_1 + 3\phi_1 u - 3\psi_2 w - 3\psi_1 w + 3\phi_1 w^2 + 6\psi w w_1, \\
    \psi_t &= -\psi_3 + 3\psi u_1 + 3\psi_1 u + 3\phi_2 w + 3\phi_1 w_1 + 3\psi_1 w^2 + 6\phi w w_1, \\
    w_t &= -w_3 + 3w^2 w_1 + 3uw_1 + 3u_1 w,
\end{align*}
$$

where $u$ and $w$ are classical (even) independent variables while $\phi$ and $\psi$ are odd ones (here and below the numerical subscript at an unknown variable denotes its derivative over $x$ of the corresponding order). Being completely integrable itself, this system gives rise to an interesting system of even equations

$$
\begin{align*}
    u_t &= -u_3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uw_1, \\
    w_t &= -w_3 + 3w^2 w_1 + 3uw_1 + 3u_1 w,
\end{align*}
$$

(1)

which can be considered as a sort coupling between the KdV (with respect to $u$) and the modified KdV (with respect to $w$) equations. In fact, setting $w = 0$, we obtain

$$
    u_t = -u_3 + 6uu_1,
$$

while for $u = 0$ we have

$$
    w_t = -w_3 + 3w^2 w_1.
$$

1991 Mathematics Subject Classification. 58F07, 58G37, 58H15, 58F37.

Key words and phrases. Coupled KdV–mKdV system, complete integrability, recursion operators, symmetries, conservation laws, coverings, deformations, superdifferential equations.

Partially supported by the INTAS grant 96-0793.
In what follows, we prove complete integrability, cf. [2], of system [1] by establishing existence of infinite series of symmetries and/or conservation laws. Toward this end we construct a recursion operator using the techniques of deformation theory introduced in [3] and extensively described and exemplified in [4].

In the first section of the paper the theoretical background is introduced. The second section deals with particular computations and description of basic results.

1. Geometrical and algebraic background

Here we briefly describe the geometrical theory of partial differential equations and algebraic foundations of computational approach to recursion operators [1, 6].

Let \( \pi: E \rightarrow M \) be a locally trivial vector bundle and \( \pi_k: J^k(\pi) \rightarrow M, k = 0, 1, \ldots, \infty \), be the bundles of its \( k \)-jets. A (nonlinear) partial differential equation (PDE) of order \( k \) is a submanifold \( E \subset J^k(\pi), k < \infty \). Its \( l \)th prolongation is a subset \( E^l \subset J^{k+l}(\pi) \). There exist natural mappings \( \pi_{k+l+1,k+l+1}: E^{k+1} \rightarrow E^l \), and \( E \) is said to be formally integrable, if all \( E^l \) are smooth manifolds while \( \pi_{k+l+1,k+l} \) are smooth fiber bundles. Below, only formally integrable equations are considered.

The inverse limit \( E^\infty \subset J^\infty(\pi) \) of the system \( \{ E^l, \pi_{k+l,k+l-1} \} \) is called the infinite prolongation of \( E \) and we consider the bundle \( \pi_\infty: E^\infty \rightarrow M \). This bundle enjoys the following characteristic property:

**Proposition 1** (see [3]). Let \( \pi_i: E_i \rightarrow M, i = 1, 2, \) be two locally trivial vector bundles and \( \Delta: \Gamma(\pi_1) \rightarrow \Gamma(\pi_2) \) be a linear differential operator acting from sections of \( \pi_1 \) to sections of \( \pi_2 \). Then there exists a unique differential operator \( \mathcal{C} \Delta: \Gamma(\pi_\infty^*(\pi_1)) \rightarrow \Gamma(\pi_\infty^*(\pi_2)) \) such that

\[
\mathcal{J}_\infty(s)^* \circ \mathcal{C} \Delta = \Delta \circ \mathcal{J}_\infty(s)^*
\]

for any formal solution \( s \) of the equation \( E \). The correspondence \( \Delta \mapsto \mathcal{C} \Delta \) is \( C^\infty(E^\infty) \)-linear and comiles with the composition of differential operators:

\[
\mathcal{C}(\Delta_2 \circ \Delta_1) = \mathcal{C} \Delta_2 \circ \mathcal{C} \Delta_1,
\]

where \( \Delta_1: \Gamma(\pi_1) \rightarrow \Gamma(\pi_2), \Delta_2: \Gamma(\pi_2) \rightarrow \Gamma(\pi_3) \) are linear differential operators, \( \pi_3: E_3 \rightarrow M \) being a third vector bundle.

As a corollary of Proposition [1], we get

**Proposition 2.** The bundle \( \pi_\infty \) possesses a natural flat connection.

**Proof.** It suffices to take the trivial bundle \( 1_M: M \times \mathbb{R} \rightarrow M \) for the bundles \( \pi_1 \) and \( \pi_2 \) and an arbitrary vector field \( X \) for the operator \( \Delta \). Flatness is a consequence of (3). \( \square \)

Let us denote by \( D(N) \) the \( C^\infty(N) \)-module of vector fields on a manifold \( N \).

**Definition 1.** The connection \( \mathcal{C}: D(M) \rightarrow D(E^\infty) \) is called the Cartan connection on \( E^\infty \).

Denote by \( CD(E^\infty) \subset D(E^\infty) \) the horizontal distribution on \( E^\infty \) with respect to the Cartan connection (the Cartan distribution) and by \( D_\mathcal{C}(E^\infty) \) the normalizer of \( CD(E^\infty) \) in \( D(E^\infty) \). Then, since \( \mathcal{C} \) is flat, \( CD(E^\infty) \) is integrable in a formal Frobenius sense and thus \( CD(E^\infty) \) is an ideal in \( D_\mathcal{C}(E^\infty) \).

**Definition 2.** The quotient Lie algebra \( \text{sym} E = D_\mathcal{C}(E^\infty)/CD(E^\infty) \) is called the algebra of higher symmetries of the equation \( E \).
The Cartan connection in \( \pi_\mathcal{E} \) determines the splitting
\[
\mathbf{D}(\mathcal{E}^\infty) = \mathbf{D}^v(\mathcal{E}^\infty) \oplus \mathbf{C}\mathbf{D}(\mathcal{E}^\infty),
\]
where \( \mathbf{D}^v(\mathcal{E}^\infty) \) denotes the module of \( \pi_\mathcal{E} \)-vertical vector fields, and for any coset \( S \in \mathbf{sym} \mathcal{E} \) there exists a unique vertical representative. In its turn, such a representative is uniquely determined by a section \( \varphi \in \Gamma(\pi_\mathcal{E}^*(\pi)) \) (generating section) satisfying the defining equation
\[
\ell_\mathcal{E} \varphi = 0
\]
and vice versa. Here \( \ell_\mathcal{E} \) is the universal linearization operator for \( \mathcal{E} \) restricted to \( \mathcal{E}^\infty \). Due to this fact, we shall identify the solutions of (5) with higher symmetries of \( \mathcal{E} \).

The connection form \( U_\mathcal{E} \) (the structural element of \( \mathcal{E} \)) of the Cartan connection \( \mathcal{C} \) is an element of the module \( \mathbf{D}^v(\Lambda^1(\mathcal{E}^\infty)) \) of \( \Lambda^1(\mathcal{E}^\infty) \)-valued vertical derivations. Thus, we can introduce an operator \( \partial_\mathcal{E} : \mathbf{D}^v(\Lambda^1(\mathcal{E}^\infty)) \to \mathbf{D}^v(\Lambda^{i+1}(\mathcal{E}^\infty)) \) defined by
\[
\partial_\mathcal{E} \Omega = [U_\mathcal{E}, \Omega], \quad \Omega \in \mathbf{D}^v(\Lambda^1(\mathcal{E}^\infty)),
\]
where \([\cdot, \cdot]\) is the Frölicher–Nijenhuis bracket. Since the Cartan connection is flat, one has \([U_\mathcal{E}, U_\mathcal{E}] = 0\), from where it follows that \( \partial_\mathcal{E} \circ \partial_\mathcal{E} = 0 \). Thus we obtain a complex \( (\mathbf{D}^v(\Lambda^1(\mathcal{E}^\infty)), \partial_\mathcal{E}) \), whose cohomology is called the \( \mathbf{C} \)-cohomology of \( \mathcal{E} \) and is denoted by \( H^*_\mathcal{E}(\mathcal{E}) \). It is easy to see that \( H^*_\mathcal{E}(\mathcal{E}) = \text{sym} \mathcal{E} \), while \( H^1\mathcal{E}(\mathcal{E}) \), by standard reasons, is identified with classes of nontrivial infinitesimal deformations of \( U_\mathcal{E} \) (or, which is the same, of the equation structure).

If \( \Omega \) and \( \Theta \) are elements of \( \mathbf{D}^v(\Lambda^1(\mathcal{E}^\infty)) \) and \( \mathbf{D}^v(\Lambda^j(\mathcal{E}^\infty)) \) respectively, their contraction \( \Omega \cdot \Theta \) is defined as an element of \( (\mathbf{D}^v(\Lambda^{i+1}(\mathcal{E}^\infty))) \) This operation is inherited by the \( \mathbf{C} \)-cohomology groups. In particular, if \( \varphi \in \text{sym} \mathcal{E} \) and \( \mathcal{R} \in H^1\mathcal{E}(\mathcal{E}) \), then \( \varphi \cdot \mathcal{R} = \mathcal{R}\varphi \) is a symmetry again. In other words, the module \( H^1\mathcal{E}(\mathcal{E}) \) acts on the Lie algebra of higher symmetries.

Let \( \mathcal{C}^1(\mathcal{E}^\infty) \subset \Lambda^1(\mathcal{E}^\infty) \) be the submodule of one-forms on \( \mathcal{E}^\infty \) vanishing on the Cartan distribution. Then one has the direct sum decomposition
\[
\Lambda^1(\mathcal{E}^\infty) = \mathcal{C}^1(\mathcal{E}^\infty) \oplus \Lambda^1_\mathbf{h}(\mathcal{E}^\infty)
\]
dual to (\ref{eq:D}), where \( \Lambda^1_\mathbf{h}(\mathcal{E}^\infty) \) is the submodule of horizontal forms. This splitting is also inherited by \( H^1\mathcal{E}(\mathcal{E}^\infty) \) and an element \( \mathcal{R} \in H^1\mathcal{E}(\mathcal{E}^\infty) \) acts nontrivially on \( \text{sym} \mathcal{E} \) if only it corresponds to a derivation from \( \mathbf{D}^v(\mathcal{C}^1(\mathcal{E}^\infty)) \). Moreover, it can be shown that \( \text{im} \ \partial_\mathcal{E} \cap \mathbf{D}^v(\mathcal{C}^1(\mathcal{E}^\infty)) = 0 \) and consequently nontrivial actions can be found by solving the equation
\[
\partial_\mathcal{E}(\mathcal{R}) = 0.
\]
Solutions of (\ref{eq:D}) are called recursion operators for symmetries. Any recursion operator \( \mathcal{R} \) is uniquely determined by an element \( \omega_\mathcal{R} \in \mathcal{C}^1(\mathcal{E}^\infty) \otimes \Gamma(\pi_\mathcal{E}(\pi)) \) satisfying the defining equation
\[
\ell_\mathcal{E}^{[1]}(\omega_\mathcal{R}) = 0,
\]
where \( \ell_\mathcal{E}^{[1]} \) is the extension of the operator \( \ell_\mathcal{E} \) to the module \( \mathcal{C}^1(\mathcal{E}^\infty) \otimes \Gamma(\pi_\mathcal{E}(\pi)) \). If \( \varphi \in \Gamma(\pi_\mathcal{E}(\pi)) \) is a symmetry and \( S_\varphi \in \mathbf{D}^v(\mathcal{E}^\infty) \) is the corresponding vertical vector field, then the action of \( \mathcal{R} \) on \( \varphi \) is given by
\[
\mathcal{R}\varphi = S_\varphi \cdot \omega_\mathcal{R}.
\]
Local coordinates. Let $\mathcal{U} \subset M$ be a coordinate neighborhood in $M$ such that the bundle $\pi$ trivializes over $\mathcal{U}$, $x_1, \ldots, x_n$ be local coordinates in $\mathcal{U}$ and $u^1, \ldots, u^m$ be coordinates along the fiber in a given trivialization. Then the adapted coordinates in $\pi^{-1}(U) \subset J^\infty(\pi)$ are given by the functions $u^j_\sigma$, $j = 1, \ldots, m$, $\sigma = i_1 \ldots i_k$, $1 \leq i_\alpha \leq n$, uniquely defined by
\[
j_\infty(f)^*(u^j_\sigma) = \frac{\partial|\sigma|f^j}{\partial x_{i_1} \ldots \partial x_{i_k}}
\]
for any local section $f = (f^1, \ldots, f^m) \in \Gamma(\pi|U)$. Then the Cartan connection in $J^\infty(\pi)$ is given by
\[
\mathcal{C} \left( \frac{\partial}{\partial x_j} \right) = D_i = \frac{\partial}{\partial x_i} + \sum_{j, \sigma} u^j_\sigma \frac{\partial}{\partial u^j_\sigma},
\]
where $D_i$ are the so-called total derivatives. The structural element in this case is given by the formula
\[
U_{\pi} = \sum_{j, \sigma} \omega^j_\sigma \otimes \frac{\partial}{\partial u^j_\sigma},
\]
where $\omega^j_\sigma = du^j_\sigma - \sum_i u^j_\sigma dx_i$ are the Cartan forms constituting a basis in the module $\Lambda^1(J^\infty(\pi))$. The forms $\omega^j_\sigma$ may be rewritten as $\omega^j_\sigma = d_C u^j_\sigma$, where $d_C$ is the Cartan differential acting on $f \in C^\infty(J^\infty(\pi))$ by
\[
d_C f = \sum \frac{\partial f}{\partial u^j_\sigma} \omega^j_\sigma.
\]
This differential restricts to any submanifold $\mathcal{E} \subset J^\infty(\pi)$ and for infinite prolongations (10) transforms to
\[
U_{\mathcal{E}} = \sum_I d_C u_I \otimes \frac{\partial}{\partial u_I},
\]
where $\{u_I\}$ spans the set of internal coordinates in $\mathcal{E}^\infty$.

If an equation $\mathcal{E} \subset J^k(\pi)$ is given by the system of equalities
\[
\begin{align*}
F^1(x_1, \ldots, x_n, u^j_\sigma, \ldots) &= 0, \\
& \quad \ldots \\
F^r(x_1, \ldots, x_n, u^j_\sigma, \ldots) &= 0,
\end{align*}
\]
then its infinite prolongation is defined by
\[
D_\sigma F^\alpha = 0, \quad \alpha = 1, \ldots, r, \quad 0 \leq |\sigma| < \infty,
\]
where $D_\sigma = D_{i_1} \circ \cdots \circ D_{i_k}$ for $\sigma = i_1 \ldots i_k$. Then the universal linearization operator corresponding to this system is of the form
\[
\ell_F = \left\| \sum_{\sigma} \frac{\partial F^\alpha}{\partial u^j_\sigma} D_\sigma \right\|.
\]
The operator $\ell_{\mathcal{E}}$ is obtained from (12) by rewriting it in internal coordinates.
Example (Evolutionary $1 + 1$ equations). Consider a system $\mathcal{E}$ of evolutionary equations

$$
\begin{align*}
\{ & u_t^1 = F^1(x, t, u^1, \ldots, u^m), \\
& \ldots \ldots \ldots \ldots \ldots \\
& u_t^m = F^m(x, t, u^1, \ldots, u^m),
\end{align*}
$$

(13)

where $x = x_1, t = x_2$ and $u_t^j = \partial^j u^j / \partial x^j$. Then the functions $x, t, u^1, \cdots$ can be taken for internal coordinates on $\mathcal{E}^\infty$. The total derivatives are written down as

$$
D_x = \frac{\partial}{\partial x} + \sum_{i,j} u_{i+1}^j \frac{\partial}{\partial u_i^j}, \quad D_t = \frac{\partial}{\partial t} + \sum_{i,j} D_x^j(F^j) \frac{\partial}{\partial u_i^j}
$$

in these coordinates. The Cartan forms on $\mathcal{E}^\infty$ are

$$
\omega_t^j = du_t^j - u_{i+1}^j dx - D_x^j(F^j) dt,
$$

while the structural element for $\mathcal{E}$ is given by

$$
U_{\mathcal{E}} = \sum_{i,j} \omega_t^j \otimes \frac{\partial}{\partial u_i^j}.
$$

The restriction $\ell_{\mathcal{E}}$ of the universal linearization operator to $\mathcal{E}^\infty$ is of the form

$$
\ell_{\mathcal{E}} = \sum_{i,j} \frac{\partial F^\alpha}{\partial u_i^j} \frac{\partial}{\partial u_i^j} - E D_t,
$$

where $E$ is the identity matrix. So, a vector function $\varphi = (\varphi^1, \ldots, \varphi^m)$, $\varphi^j = \varphi^j(x, t, \ldots, u_t^1, \ldots)$, is a symmetry of $\mathcal{E}$ if and only if

$$
\sum_{i,j} \frac{\partial F^\alpha}{\partial u_i^j} D_x^j(\varphi^j) = D_t \varphi^\alpha
$$

(15)

for all $\alpha = 1, \ldots, m$. The corresponding vector field is given by the formula

$$
\Theta_{\varphi} = \sum_{i,j} D_x^j(\varphi^j) \frac{\partial}{\partial u_i^j}.
$$

In a similar way, recursion operators are determined by vector-valued forms $\omega_R = (\omega_t^1, \ldots, \omega_t^m)$, where $\omega_t^j = \sum_\alpha \psi^j_\alpha \omega_t^\alpha$, $\psi^j_\alpha \in C^\infty(\mathcal{E}^\infty)$, satisfying the equations

$$
\sum_{i,j} \frac{\partial F^\alpha}{\partial u_i^j} D_x^j(\omega_t^\alpha) = D_t \omega_t^\alpha.
$$

(16)

To compute the left and right sides of (16), it suffices to note that

$$
D_x^j \omega_t^j = \omega_t^{j+1}, \quad D_t \omega_t^j = D_x^j d_x F^j = D_x^j \sum_{\alpha, \beta} \frac{\partial F^j}{\partial u_\beta^\alpha} \omega_t^\alpha.
$$

Remark 1. Let $\omega = (\omega^1, \ldots, \omega^m)$, $\omega^j = \sum_\alpha \psi^j_\alpha \omega_t^\alpha$, be a vector-valued Cartan form on $\mathcal{E}^\infty$. Then for any vector-valued function $\varphi = (\varphi^1, \ldots, \varphi^m)$ on $\mathcal{E}^\infty$ the action $R_\omega: \varphi \mapsto R_\omega \varphi$ is defined by $R_\omega \varphi = \Theta_{\varphi} \omega$. In local coordinates, this action is expressed by the formula

$$
(R_\omega \varphi)^j = \sum_{i, \alpha} \psi_i^j \varphi^\alpha.
$$

(17)
Operators of this type (i.e., expressed in terms of total derivatives) are called \textit{C-differential} (or \textit{total differential}) operators.

\textbf{Remark 2.} As it was mentioned above, operators of the form \(\ell \omega\), provided \(\omega\) satisfies (16), take symmetries of the equation at hand to symmetries of the same equation. In other words, one has \(\mathcal{R}_\omega : \ker \ell \mathcal{E} \rightarrow \ker \ell \mathcal{E}\). Under not very restrictive conditions on the equation \(\mathcal{E}\), this is equivalent to the operator equality
\[
\ell \mathcal{E} \circ \mathcal{R}_\omega = \mathcal{A} \circ \ell \mathcal{E},
\]
where \(\mathcal{A}\) is a \(C\)-differential operator. Taking formally adjoint, one obtains
\[
\mathcal{R}_\omega^* \circ \ell \mathcal{E}^* = \ell \mathcal{E}^* \circ \mathcal{A}^*,
\]
which means, that \(\mathcal{A}^*\) is a recursion operator for generating functions of conservation laws \(\mathcal{L}^\dag\).

\textbf{Nonlocal setting.} In practice, when solving (7), one usually finds no nontrivial solutions, though the equation \(\mathcal{E}\) may possess recursion operators. 

\textbf{Example (The Burgers equation).} Consider the equation
\[
u_t = \nu_{xx} + \nu \nu_x.
\]
It is known to possess a recursion operator of the form
\[
\mathcal{R} = D_x + \frac{1}{2}u_0 + \frac{1}{2}u_1D_x^{-1}.
\]
Nevertheless, the only solution of the equation
\[
\ell (1) = (D^2_x + u_0D_x + u_1 - D_t)\omega = 0
\]
for \(\omega = \psi_0\omega_0 + \ldots \psi_k\omega_k\) is \(\alpha\omega_0, \alpha \in \mathbb{R}\), which provides the trivial action \(\mathcal{R}_\omega : \varphi \mapsto \alpha \varphi\).

To resolve this apparent contradiction, let us extend the algebra \(C^\infty(\mathcal{E}^\infty)\) with an additional element \(u_{-1}\) and set
\[
D_xu_{-1} = u_0,
\]
\[
D_tu_{-1} = u_1 + \frac{1}{2}u_0^2,
\]
\[
dc u_{-1} = \omega_{-1} = du_{-1} - u_0 dx - \left( u_1 + \frac{1}{2}u_0^2 \right) dt.
\]
Then, solving (19) for \(\omega = \psi_{-1}\omega_{-1} + \psi_0\omega_0 + \ldots \psi_k\omega_k\), we obtain a two-parametric solution
\[
\omega = \alpha\omega_0 + \beta\Omega, \quad \Omega = \omega_1 + \frac{1}{2}u_0\omega_0 + \frac{1}{2}u_1\omega_{-1}.
\]
Then the action \(\varphi \mapsto \mathcal{R}_\Omega\varphi = \mathcal{E}_\varphi \omega = \mathcal{E}_\varphi \Omega\) coincides exactly with (18).

This example reflects a general scheme of computations which arises in a lot of applications \cite{5} and is used in Section 2. Namely, in search of recursion operators for symmetries we extend the algebra \(C^\infty(\mathcal{E}^\infty)\) with a new set of variables \(u^1, \ldots, u^r, \ldots\) (so-called \textit{nonlocal variables}) and respectively extend the total derivatives to vector fields
\[
\bar{D}_i = D_i + \sum_{\alpha} X_{\alpha}^i \frac{\partial}{\partial w^\alpha}, \quad i = 1, \ldots, n,
\]
where \(\alpha, \beta = 0, 1, \ldots, \).
in such a way that
\[ [\bar{D}_i, \bar{D}_j] \equiv [D_i, X_j] + [X_i, D_j] + [X_i, X_j] = 0, \quad (22) \]
where \( X_l = \sum \alpha X_{\alpha} \partial/\partial w_{\alpha} \).

Having a solution \( X_1, \ldots, X_n \) of (21), we obtain an integrable distribution on the space \( \mathcal{E}^\infty \times \mathbb{R}^N \), where \( N \) is the number of nonlocal variables (the case \( N = \infty \) is included). The projection \( \tau: \mathcal{E}^\infty = \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty \) is called a covering over \( \mathcal{E} \) and \( N \) is called its dimension (for an invariant geometrical definition see \([7]\)). Similar to the local case, we define the Lie algebra \( \text{sym}_{\tau} \mathcal{E} = D_{\mathcal{E}}(\mathcal{E}^\infty)/C_{\mathcal{E}}(\mathcal{E}^\infty) \) of nonlocal \( \tau \)-symmetries.

We introduce Cartan forms
\[ \theta^j = dw^j - \sum_{i=1}^n X^j_i dx_i, \quad j = 1, \ldots, N, \]
corresponding to nonlocal variables on \( \mathcal{E}^\infty \). The module of all Cartan forms on \( \mathcal{E}^\infty \) is denoted by \( C\Lambda^1(\mathcal{E}^\infty) \). We also extend the universal linearization operator \( \ell_\mathcal{E} \) to \( \mathcal{E}^\infty \) just by changing the total derivatives \( D_i \) to \( \bar{D}_i \). Let us now consider two equations, associated to this extension:
\[ \bar{\ell}_\mathcal{E} \phi = 0 \quad \text{and} \quad \bar{\ell}_\mathcal{E}^{[1]} \Omega = 0, \]
where \( \phi \in \Gamma((\pi_\mathcal{E} \circ \tau)^* \pi) \) and \( \Omega \in \Gamma((\pi_\mathcal{E} \circ \tau)^* \pi) \otimes C\Lambda^1(\mathcal{E}^\infty) \). Solutions of the first equation are called \( \tau \)-shadows of nonlocal symmetries, while solutions of the second one are said to be \( \tau \)-shadows of recursion operators in the covering \( \tau \). The following result establishes relations of shadows to symmetries and recursion operators:

**Theorem 1** (see \([3, 7]\)). Let \( \tau: \mathcal{E}^\infty \to \mathcal{E}^\infty \) be a covering. Then:

1. If \( \phi \) is a \( \tau \)-shadow, then there exists a covering
   \[ \bar{\tau}: \mathcal{E}^\infty \to \mathcal{E}^\infty \]
   such that \( \phi \) reconstructs up to a nonlocal \( \bar{\tau} \)-symmetry.
2. If \( \phi \) is a nonlocal \( \tau \)-symmetry and \( \mathcal{R} \) is a recursion operator shadow, then \( \mathcal{R} \phi \) is a symmetry shadow.

**Remark 3.** Among all one-dimensional coverings over a given equation there exists a special class consisting of those ones, for which the fields \( X_1, \ldots, X_n \) in (22) are independent of nonlocal variables (so-called abelian coverings). To any such a covering, one can put into correspondence a differential form on \( \mathcal{E}^\infty \):
\[ \omega_\tau = \sum_{i=1}^n X_i dx_i. \]
This form is closed with respect to the so-called horizontal differential \( d_h = C d \) (cf. Proposition \([1]\)). Vice versa, to any such a form there corresponds a covering of the above mentioned type. Moreover, two closed forms determine the same class in the cohomology group \( H^1(d_h) \) if and only if the corresponding coverings are equivalent. In particular, if \( n = 2 \), the group \( H^1(d_h) \) coincides with the group of conservation laws of the equation \( \mathcal{E} \) \([10]\). Thus, to construct a covering under consideration is the same as to find a conservation law. This fact is used in the computations below.
2. Basic computations and results

In this section we shall discuss the complete integrability of the KdV–mKdV system given in (1), i.e.,
\begin{align*}
u_t &= -u^3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uvw, \\
w_t &= -w_3 + 3w_2w_1 + 3uw_1 + 3w_1w.
\end{align*}
(23)
In order to demonstrate the complete integrability of this system, we shall construct the recursion operator for symmetries of this coupled system, leading to infinite hierarchies of symmetries and, most probably, of conservation laws. Due to the very special form of the final results, it seems that integrability of this system, which looks quite ordinary, has not been discussed before or elsewhere. In order to do this, we shall discuss conservation laws in Subsection 2.1 leading to the necessary nonlocal variables.

In Subsection 2.2 we shall discuss local and nonlocal symmetries of the system, while in Subsection 2.3 we construct the recursion operator or deformation of the equation structure (14).

2.1. Conservation laws and nonlocal variables. Here we shall construct conservation laws for (23) in order to arrive at an abelian covering of the coupled KdV–mKdV system as was shown for Burgers equation (20). So we construct $X = X(x,t,u,...,w...), T = T(x,t,u,...,w...)$ such that
\[D_x(T) = D_t(X),
\]
(24)
and in a similar way we construct nonlocal conservation laws by the requirement
\[\tilde{D}_x(\tilde{T}) = \tilde{D}_t(\tilde{X}),
\]
where $\tilde{D}_*$ is defined by (1); moreover $\tilde{X}, \tilde{T}$ are dependent on local variables $x, t, u,..., w,...$ as well as the already determined nonlocal variables, denoted here by $p_* or p_{*,*}$, which are associated to the conservation laws $(X, T)$ by the formal definition
\[D_x(p_*) = (p_*)_x = X,
\]
\[D_t(p_*) = (p_*)_t = T.
\]
Proceeding in this way, we obtained the following set of nonlocal variables
\[p_{0.1}, p_{0.2}, p_1, p_{1.1}, p_{1.2}, p_{2.1}, p_3, p_{3.1}, p_{3.2}, p_{4.1}, p_5,
\]
(26)
where their defining equations are given by
\begin{align*}
(p_1)_x &= u, \\
(p_1)_t &= 3u^2 + 3uw^2 - u_2 - 3ww_2, \\
(p_{0.1})_x &= w, \\
(p_{0.1})_t &= 3uw + w^3 - w_2, \\
(p_{0.2})_x &= p_1, \\
(p_{0.2})_t &= -6p_3 - u_1, \\
(p_{1.1})_x &= \cos(2p_{0.1})p_1w + \sin(2p_{0.1})w^2, \\
(p_{1.1})_t &= \cos(2p_{0.1})(3p_1uw + p_1w^3 - p_1w_2 + uw_1 - u_1w - w^2w_1) \\
&+ \sin(2p_{0.1})(4uw^2 + w^4 - 2ww_2 + w_1^2).
\end{align*}
\( (p_{1.2})_x = \cos(2p_{0.1}) w^2 - \sin(2p_{0.1}) p_1 \),
\( (p_{1.2})_t = \cos(2p_{0.1})(4u w^2 + w^4 - 2uw_2 + w_2^2) + \sin(2p_{0.1})(-3p_1 w - p_1 w^3 + p_1 w_2 - uw_1 + u w + w^2 w_1), \)
\( (p_{2.1})_x = (4 \cos(2p_{0.1}) p_{1.1} w^2 - 4 \sin(2p_{0.1}) p_1 p_{1.1} w + w(p_1^2 - 2u + w^2))/2, \)
\( (p_{2.1})_t = (4 \cos(2p_{0.1}) p_{1.1} (4u w^2 + w^4 - 2uw_2 + w_2^2) + 4 \sin(2p_{0.1}) p_{1.1}(-3p_1 w - p_1 w^3 + p_1 w_2 - uw_1 + u w + w^2 w_1)
+ 3p_1^2 w + p_1^2 w^3 - p_1^2 w_2 + 2p_1 uw_1 - 2p_1 u w - 2p_1 w^2 w_1 - 8u^2 w
- uw^3 + 2uw_2 - 2u_1 w + 2u_2 w + w^5 + 3w^2 w_2)/2, \)
\( (p_3)_x = (-u^2 - uw^2 + w_2)/2, \)
\( (p_3)_t = (-4u^3 - 9u^2 w^2 + 2uw_2 - 3uw^4 + 11uww_2 - uw_1^2 - u_1 w_1
+ 4u^2 w^2 + 6w^3 w_2 + 3w^2 w_2^2 - uw_1 + w_1 w_3 - w_2^2)/2, \)
\( (p_{3.1})_x = (\cos(2p_{0.1}) w(p_1^2 - 6p_1 u + 39p_1 w^2 - 24p_{1.1} p_1 w + 12p_3 + 6u_1)
+ 2 \sin(2p_{0.1}) w(12p_{1.1} p_{1.1} + 18p_1 w + 2w^3 + 3w_2)
+ 6p_1 u w(-p_1^2 + 2u - w^2))/12, \)
\( (p_{3.2})_x = (2 \cos(2p_{0.1}) w(12p_{1.1} p_{1.1} - 18p_1 w - 2w^3 - 3w_2)
+ \sin(2p_{0.1}) w(p_1^2 - 6p_1 u + 39p_1 w^2 + 24p_{1.1} p_1 w + 12p_3 + 6u_1)
+ 6p_1 u w(-p_1^2 + 2u - w^2))/12, \)
\( (p_{4.1})_x = (8 \cos(2p_{0.1}) w(p_1^2 p_1 + 12p_1 p_{1.1}^2 + 6p_{1.1} p_1 u + 3p_1 p_1 w^2
- 12p_{1.1} p_1 w + 18p_{1.1} w - 4p_{1.1} w^3 - 6p_{1.1} w_2 + 12p_{1.2} p_3 + 6p_{1.2} u)
+ 8 \sin(2p_{0.1}) w(p_1^2 p_{1.1} + 12p_{1.1} p_{1.1}^2 + 6p_{1.1} p_1 u + 3p_1 p_1 w^2
+ 12p_{1.1} p_1 u + 6p_{1.1} u_1 - 18p_{1.2} w + 4p_{1.2} w^3 + 6p_{1.2} w_2)
+ w(-p_1^2 - 24p_1^2 p_{1.1}^2 - 24p_1^2 p_{1.1}^2 + 12p_1^2 u - 6p_1^2 w^2 - 24p_1^2 w
- 24p_1 u + 18p_1 w + 24p_{1.1}^2 w^2 + 18p_{1.1}^2 w - 24p_{1.1}^2 w^2
- 60u^2 + 44uw^2 + 24uw - 13w^4 + 6uw_2))/48, \)
\( (p_5)_x = (12u^3 + 24u^2 w^2 - 6uw_2 + 6uw^4 - 30uw_2 - 3uw^2 - 8w^3 w_2 + 6uw_2)/6. \)

In the previous equations, we skipped explicit formulas for \( (p_{3.1})_t, (p_{3.2})_t, (p_{4.1})_t, \) and \( (p_5)_t \), because they are too massive, though quite important for the setting to be well defined and in order to avoid ambiguities. The reader is referred to the Appendix for them.

It is quite a striking result that functions \( \cos(2p_{0.1}), \sin(2p_{0.1}) \) appear in the presentation of the conservation laws and their associated nonlocal variables.

We should note that \( p_1, p_{0.1}, p_3, p_5 \) arise from local conservation laws and we shall call \( p_1, p_{0.1}, p_3, p_5 \) nonlocalities of first order.

In a similar way we see that \( p_{0.2}, p_{1.1}, p_{1.2} \) arise from nonlocal conservation laws, where their \( x- \) and \( t- \) derivatives are dependent on the first order nonlocalities.

For this reason \( p_{0.2}, p_{1.1}, p_{1.2} \) are called nonlocalities of second order.

Proceeding in this way \( p_{2.1}, p_{3.1}, p_{3.2}, p_{4.1} \) constitute nonlocalities of third order.
2.2. Local and nonlocal symmetries. In this section we shall present results for the construction of local and nonlocal symmetries of system \((23)\). In order to construct these symmetries, we consider the system of partial differential equations obtained by the infinite prolongation of \((23)\) together with the covering by the nonlocal variables

\[
p_{0,1}, p_{0,2}, p_1, p_{1,1}, p_{1,2}, p_{2,1}, p_3, p_{3,1}, p_{3,2}, p_{4,1}, p_5.
\]

So, in the augmented setting governed by \((23)\), their total derivatives and the equations given in Subsection 2.1 we construct symmetries \(Y = (Y^u, Y^w)\) which have to satisfy the symmetry condition

\[
\ell_\xi Y = 0.
\]

From this condition we obtained the following symmetries

\[
Y_{0,1}, Y_{1,1}, Y_{1,2}, Y_{1,3}, Y_{2,1}, Y_{3,1}, Y_{3,2}, Y_{3,3},
\]

where generating functions \(Y_{*,*}^u, Y_{*,*}^w\) are given as

\[
Y_{0,1}^u = 3t(6uu_1 + 6wuw_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2) + xu_1 + 2u,
Y_{0,1}^w = 3t(3uw_1 + 3u_1w + 3w^2w_1 - w_3) + xw_1 + w,
Y_{1,1}^u = u_1,
Y_{1,1}^w = w_1,
Y_{1,2}^u = \cos(2p_{0,1})(2uw - w_2) + \sin(2p_{0,1})(u_1 + 2ww_1),
Y_{1,2}^w = -\cos(2p_{0,1})u - \sin(2p_{0,1})w_1,
Y_{1,3}^u = \cos(2p_{0,1})(u_1 + 2uw_1) + \sin(2p_{0,1})(-2uw + w_2),
Y_{1,3}^w = -\cos(2p_{0,1})w_1 + \sin(2p_{0,1})u,
Y_{2,1}^u = (2\cos(2p_{0,1})(p_{1,1}u_1 + 2p_{1,1}uw_1 - 2p_{1,2}uw + p_{1,2}w_2)
+ 2\sin(2p_{0,1})(-2p_{1,1}uw + p_{1,1}w_2 - p_{1,2}u_1 - 2p_{1,2}w_1)
+ 2p_{1}uw - p_{1}w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)/2,
Y_{2,1}^w = (2\cos(2p_{0,1})(-p_{1,1}w_1 + p_{1,2}u) + 2\sin(2p_{0,1})(p_{1,1}u + p_{1,2}w_1)
- p_{1}u + u_1 + ww_1)/2,
Y_{3,1}^u = (6 uu_1 + 6uw_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2)/3,
Y_{3,1}^w = (3uu_1 + 3u_1w + 3w^2w_1 - w_3)/3,
Y_{3,2}^u = (\cos(2p_{0,1})(-2p_{1}^2uw + p_{1}^2w_2 - 4p_{1}uw_1 - 6p_{1}u_1w - 4p_{1}w^2w_1 + 2p_{1}w_3
+ 8p_{1,1}p_{1,2}u_1 + 16p_{1,1}p_{1,2}uw_1 - 8p_{1,2}^2uw + 4p_{1,2}^2w_2 - 2p_{1,2}u_1 - 8p_{2,1}uw_1
+ 10u_1w + 6uw^3 - 8uw_2 - 14u_1w_1 - 8u_2w - 11w^2w_2 - 14uw^2 + 2w_4)
+ 2\sin(2p_{0,1})(-8p_{1,1}p_{1,2}uw + 4p_{1,1}p_{1,2}w_2 - 2p_{1,2}u_1 - 4p_{1,2}uw_1 + 4p_{1,2}uw
- 2p_{1,2}w_2 + 6uw_1 + 10uw_1w_3 + 3u_1w^2 - u_3 + 2w^3w_1 - 3ww_3 - 5w_1w_2)
+ 4p_{1,2}(2p_{1}uw - p_{1}w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3))/8,
Y_{3,2}^w = (\cos(2p_{0,1})(p_{1}^2u - 2p_{1}u_1 - 2p_{1}uw + 8p_{1,1}p_{1,2}uw + 4p_{1,2}^2u + 4p_{2,1}w_1
- 4u^2 - 3uw^2 + 2u_2 + 4uw_2 + 2w^2)
+ 2\sin(2p_{0,1})(4p_{1,1}p_{1,2}u + 2p_{1,2}^2w_1 - 2p_{2,1}u - 3uw_1 - 3u_1w - 3w^2w_1 + w_3)
+ 4p_{1,2}(-p_{1}u + u_1 + wu_1))/8,
\]
where the components \( R \) is given by

\[
Y_{3,3}^u = (2 \cos(2p_{0,1})(2p_{1,1}^2 u + 4p_{1,1}^2 uw_1 - 4p_{2,1}^2 uw + 2p_{2,1} w - 6u u_1
- 10u_1 w_1 - 3u_1 w^2 + u_3 - 2u^3 w + 3u w_3 + 5u_1 w_2)
+ \sin(2p_{0,1})(-2p_{1,1}^2 uw + p_{1,1}^2 w_2 - 4p_{1,1} w_1 - 6p_{1,1} u w - 4p_{1,1} u^2 w_1 + 2p_{1,1} w_3
- 8p_{1,1}^2 uw + 4p_{1,1}^2 w_2 - 4u_{2,1} u_1 - 8p_{2,1} u w_1 + 10u_2 w + 6u w_3
- 8u w_2 - 14u_1 w - 8u w_2 - 11u^2 w - 14u w_2^2 + 2w_4
+ 4p_{1,1}(2p_{1,1} w - 2p_{1,1} u + 2u_1 w + 2u w_1 - w_3))/8,
\]

\[
Y_{3,3}^w = (2 \cos(2p_{0,1})(-2p_{1,1}^2 w_1 + 2p_{2,1} u + 3u w_1 + 3u u w + 3w^2 w_1 - w_3)
+ \sin(2p_{0,1})(p_{1,1}^2 u - 2p_{1,1} u - 2p_{1,1} u w_1 + 4p_{1,1} u + 4p_{1,1} w_1 - 4u u
- 3u w^2 + 2u_2 + 4u w_2 + 2w_4^2)
+ 4p_{1,1}(-p_{1,1} u + u_1 + w_1))/8.
\]

### 2.3. Recursion operator

Here we present the recursion operator \( \mathcal{R} \) for symmetries for this case obtained as a higher symmetry in the Cartan covering of system of equations \([1]\) augmented by equations governing the nonlocal variables \([26]\). As explained in the previous section, the recursion operator is in effect the deformation of the equation structure \([11]\).

As demonstrated there, this deformation is a form-valued vector field (or a vectorfield-valued one-form) and has to satisfy

\[
\mathcal{F}^{[1]} \mathcal{R} = 0.
\]

In order to arrive at a nontrivial result as was explained for Burgers’ equation too (c.f. Example \([3]\), we have to introduce nonlocal variables

\[
p_{0,1}, \ p_{0,2}, \ p_1, \ p_{1,1}, \ p_{1,2}, \ p_{2,1}, \ p_3, \ p_{3,1}, \ p_{3,2}, \ p_{4,1}, \ p_5
\]

and their associated Cartan contact forms

\[
\omega_{p_{0,1}}, \ \omega_{p_{0,2}}, \ \omega_p, \ \omega_{p_{1,1}}, \ \omega_{p_{1,2}}, \ \omega_{p_{2,1}}, \ \omega_{p_3}, \ \omega_{p_{3,1}}, \ \omega_{p_{3,2}}, \ \omega_{p_{4,1}}, \ \omega_{p_5}.
\]

The final result, which is dependent on the nonlocal Cartan forms

\[
\omega_{p_{0,1}}, \ \omega_p, \ \omega_{p_{1,1}}, \ \omega_{p_{1,2}},
\]

is given by

\[
\mathcal{R} = R_u \frac{\partial}{\partial u} + R_w \frac{\partial}{\partial w} + \ldots
\]

where the components \( R_u, \ R_w \) are given by

\[
R_u = \omega_{u_2}(-1) + \omega_u(4u + u^2) + \omega_{uw}(-2w) + \omega_{uw_1}(-w_1) + \omega_u(3u w - 2w_2)
+ \omega_{p_{1,2}}(-\cos(2p_{0,1})(u_1 + 2uw_1) + \sin(4p_{0,1})(2uw - w_2))
+ \omega_{p_{1,1}}(\cos(2p_{0,1})(-2uw + w_2) - \sin(2p_{0,1})(u_1 + 2uw_1))
+ \omega_{p_{1,2}}(2u_1 + w_2) + \omega_{p_{1,1}}(2p_{1,1} u w - p_{1,1} w_2 + 2u w_2 + 2u w_3 - w_3)),
\]

\[
R_w = \omega_{w_2}(-1) + \omega_u(2u + u^2) + \omega_u(2w)
+ \omega_{p_{1,2}}(\cos(2p_{0,1})u_1 - \sin(2p_{0,1})u)
+ \omega_{p_{1,1}}(\cos(2p_{0,1})u + \sin(2p_{0,1})u_1)
+ \omega_{p_{1,2}}(u_1 + \omega_{p_{0,1}}(-p_{1,1} u + u_1 + w_1)).
\]
We shall now present this result in a more conventional form which appeals to expressions using operators of the form $D_x$ and $D_x^{-1}$. In order to do this, we first split (29) into the so-called local part and nonlocal parts, consisting of terms associated to $\omega_{0,2}$, $\omega_1$, $\omega_{0,1}$, $\omega_0$, and those associated to $\omega_{p,2}$, $\omega_{p,1,2}$, $\omega_{p,1,1}$, respectively. The first part will account for $D_x$ presentation, while the second one accounts for the $D_x^{-1}$ part.

Due to the action of contraction $\mathcal{R}$, the local part is given by the following matrix operator:

$$
\begin{bmatrix}
-D_x^2 + 4u + w^2 & -2wD_x^2 - w_1D_x + 3uw - 2w_2 \\
2w & -D_x^2 + 2u + w^2
\end{bmatrix}.
$$

The nonlocal part will be split into parts associated to $\omega_{p,1}$, $\omega_{p,0,1}$, and $\omega_{p,2}$, $\omega_{p,1,1}$, respectively. The first one is given as

$$
\begin{bmatrix}
(2u_1 + uw_1)D^{-1}_x & (2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)D^{-1}_x \\
w_1D^{-1}_x & (-p_1u + u_1 + uw_1)D^{-1}_x
\end{bmatrix}.
$$

To deal with the last part, let us introduce the notation:

- $A_1 = \cos(2p_0,1)(-2uw + w_2) - \sin(2p_0,1)(u_1 + 2uw_1)$,
- $A_2 = \cos(2p_0,1)u + \sin(2p_0,1)w_1$,
- $B_1 = -\cos(2p_0,1)(u_1 + 2uw_1) + \sin(2p_0,1)(2uw - w_2)$,
- $B_2 = \cos(2p_0,1)w_1 - \sin(2p_0,1)u$,

being the coefficients at $\omega_{p,1,1}$ and $\omega_{p,1,2}$ in (29).

According to the presentations of $(p_{1,1})_x$ and $(p_{1,2})_x$, i.e.,

- $(p_{1,1})_x = \cos(2p_0,1)p_1w + \sin(2p_0,1)w^2$,
- $(p_{1,2})_x = \cos(2p_0,1)w^2 - \sin(2p_0,1)p_1w$,

we introduce their partial derivatives with respect to $p_0,1, p_1$, and $w$ as

- $\alpha_1 = -2p_1w\sin(2p_0,1) + 2w^2\cos(2p_0,1)$,
- $\alpha_2 = w\cos(2p_0,1)$,
- $\alpha_3 = p_1\cos(2p_0,1) + 2w\sin(2p_0,1)$,
- $\beta_1 = -2w^2\sin(2p_0,1) - 2p_1w\cos(2p_0,1)$,
- $\beta_2 = -w\sin(2p_0,1)$,
- $\beta_3 = 2w\cos(2p_0,1) - p_1\sin(2p_0,1)$.

From this we arrive in a straightforward way at the last nonlocal part of the recursion operator, i.e.,

$$
\begin{pmatrix}
A_1D_x^{-1} \alpha_2 D_x^{-1} & A_1D_x^{-1} \alpha_1 D_x^{-1} + \alpha_3 \\
A_2D_x^{-1} \alpha_2 D_x^{-1} & A_2D_x^{-1} \alpha_1 D_x^{-1} + \alpha_3
\end{pmatrix}
+ \begin{pmatrix}
B_1D_x^{-1} \beta_2 D_x^{-1} & B_1D_x^{-1} \beta_1 D_x^{-1} + \beta_3 \\
B_2D_x^{-1} \beta_2 D_x^{-1} & B_2D_x^{-1} \beta_1 D_x^{-1} + \beta_3
\end{pmatrix}
$$

So, in the final form we obtain the recursion operator as

$$
\mathcal{R} = \begin{bmatrix}
-D_x^2 + 4u + w^2 & -2wD_x^2 - w_1D_x + 3uw - 2w_2 \\
2w & -D_x^2 + 2u + w^2
\end{bmatrix}
+ \begin{bmatrix}
(2u_1 + uw_1)D^{-1}_x & (2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)D^{-1}_x \\
w_1D^{-1}_x & (-p_1u + u_1 + uw_1)D^{-1}_x
\end{bmatrix}
+ \begin{bmatrix}
A_1D_x^{-1} \alpha_2 D_x^{-1} & A_1D_x^{-1} \alpha_1 D_x^{-1} + \alpha_3 \\
A_2D_x^{-1} \alpha_2 D_x^{-1} & A_2D_x^{-1} \alpha_1 D_x^{-1} + \alpha_3
\end{bmatrix}
$$
We gave an outline of the theory of deformations of the equation structure of differential equations, leading to the construction of recursion operators for symmetries of such equations. The extension of this theory to the nonlocal setting of differential equations is essential for getting nontrivial results. The theory has been applied to the construction of the recursion operator for symmetries for a coupled KdV–mKdV system, leading to a highly nonlocal result for this system. Moreover the appearance of nonpolynomial nonlocal terms in all results, e.g., conservation laws, symmetries and recursion operator is striking and reveals some unknown and intriguing underlying structure of the equations.

Appendix

Here we present explicit formulas for \((p_{3,1}), (p_{3,2}), (p_{4,1}),\) and \((p_5)):

\[
\begin{align*}
(p_{3,1}) & = \cos(2p_{0,1}) (3p_1^2 uw + p_1^4 w^3 - p_1^4 w_2 + 3p_1^2 uw_1 - 3p_1^2 w_1 - 3p_1^2 w_2 w_1 \\
& - 24p_1^2 w_2 + 105p_1 uw_3 + 6p_1 w_2 - 6p_1 w_1 + 6p_1 w_2 w + 39p_1 w^5 \\
& - 27p_1^2 w_2 - 96p_1 p_1^2 uw - 24p_1 p_1^2 w^4 + 48p_1 p_1^2 uw_2 \\
& - 24p_1 p_1^2 w_2^3 + 36p_3 uw + 12p_3^3 w - 12p_3 w_2 - 12u^2 w_1 \\
& + 48uw w_1 + 39uw w_2 + 3u_1 w^3 - 6u_1 w_2 + 6u_2 w_1 \\
& - 6u_3 w - 9uw^4 w_1 - 12uw^2 w_3 - 18uw_1 w_2 + 6w_1^3)
+ 2\sin(2p_{0,1}) (36p_1 p_1^2 uw + 12p_1 p_1^2 w^3 - 12p_1 p_1^2 w_2 \\
+ 54p_1 uw w_1 + 54p_1 u_1 w^2 + 54p_1^3 w^4 w_1 - 18p_1 uw_3 + 12p_1 p_1^2 w_1 \\
- 12p_1 p_1^2 w_1 w - 12p_1 p_1^2 w_2 w_1 - 9u_2 w^2 + 18uw^4 + 27uw w_2 - 9uw^2 \\
+ 9u_1 w_1 + 3u_2 w_2 + 2u_6 - 11u_3 w_2 - 12uw^2 - 3uw_4 + 3w_1 w_3 - 3w_2^2) \\
+ 6p_2 (-3p_1^2 uw - p_1^2 w^3 + p_1^2 w_2 - 2p_1 w_1 + 2p_1 w_2 + 2p_1 w^2 w_1 \\
+ 8u^2 w + uw w^3 - 2u_2 w_1 - 2u_2 w - w_3 - 3w_2 w_2))/12,
\end{align*}
\]

\[
\begin{align*}
(p_{3,2}) & = \cos(2p_{0,1}) (36p_1 p_1^2 uw + 12p_1 p_1^2 w^3 - 12p_1 p_1^2 w_2 \\
- 54p_1 uw w_1 + 54p_1 u_1 w^2 - 54p_1^3 w^4 w_1 + 18p_1 uw_3 + 12p_1 p_1^2 w_1 \\
- 12p_1 p_1^2 w_1 w - 12p_1 p_1^2 w_2 w_1 + 9u_2 w^2 - 18uw^4 + 27uw w_2 + 9uw^2 \\
- 9u_1 w_1 - 3u_2 w_2 - 2u_6 + 11u_3 w_2 - 12uw^2 w_1 + 3uw_4 - 3w_1 w_3 - 3w_2^2) \\
+ \sin(2p_{0,1}) (3p_1^2 uw + p_1^2 w^3 - p_1^2 w_2 + 3p_1^2 uw_1 - 3p_1^2 w_1 - 3p_1^2 w^2 w_1 \\
- 24p_1^2 w_2 + 105p_1 uw_3 + 6p_1 w_2 - 6p_1 w_1 + 6p_1 w_2 w + 39p_1 w^5 \\
- 27p_1^2 w_2 - 96p_1 p_1^2 uw - 24p_1 p_1^2 w^4 - 48p_1 p_1^2 uw_2 \\
+ 24p_1 p_1^2 w_2^3 + 36p_3 uw + 12p_3^3 w - 12p_3 w_2 - 12u^2 w_1 \\
+ 48uw w_1 + 39uw w_2 + 3u_1 w^3 - 6u_1 w_2 + 6u_2 w_1 - 6uw_3 - 9uw^4 w_1 - 12uw^2 w_3 \\
- 18uw_1 w_2 + 6w_1^3 + 6p_1, l(-3p_1^2 uw - p_1^2 w^3 + p_1^2 w_2 - 2p_1 uw_1 + 2p_1 w_1 w \\
+ 2p_1 w^2 w_1 + 8u^2 w + uw^3 - 2uw_2 + 2u_1 w_1 - 2uw - w_3 - 3w^2 w_2))/12,
\end{align*}
\]
\[(p_{4,1}) = (8 \cos(2p_{0,1})(3p_{1}p_{1,2}u w + p_{1}^3p_{1,2}w^3 - p_{1}^3p_{1,2}w_1 + 3p_{1}^2p_{1,2}uw_1
- 3p_{1}^2p_{1,2}uw - 3p_{1}^2p_{1,2}w^2 - 12p_{1}p_{1,1}p_{1,2}w + 12p_{1}p_{1,1}p_{1,2}w^3
- 12p_{1}p_{1,1}p_{1,2}w_2 - 24p_{1}p_{1,2}w^2 + 3p_{1}p_{1,2}w_3 - 6p_{1}p_{1,2}w_4 - 6p_{1}p_{1,2}w_5 + 3p_{1}p_{1,2}w_6 + 12p_{1}p_{1,2}w_7 + 12p_{1}p_{1,2}w_8 + 12p_{1}p_{1,2}w_9)\]
\[-120w_w^2 + 60w_1^2w_2)/48,\]

\[(p_3)_t = (54u^4 + 180u^3w - 72u^2w_2 + 126u^2w^4 - 282uw_w^2 - 12u^2w_1^2 - 84uw_1w_1 - 174uw_2w^2 + 6uw_4 + 18uw^6 - 300uw_3w_2 - 90uw_w^2w_1^2 + 66uww_4 + 6uw_1w_3 + 48uw_5^2 + 42u_1^2w_2^2 - 6u_1w_3 + 12uw_3^2w_1 + 42uw_1w_1w_2 + 6uw_2^2 - 39uw_3w_4 + 162uw_2w_2 - 48uw_2^2w_1^2 + 48uw_3w_1 + 21uw_1w_2 - 12uw_4^2w_1^2 + 35uw_3^3w_4 + 120uw_2w_1w_3 + 195uw_2^2w_2^2 + 120uw_1w_2w - 6uw_5 - 30uw_4^2 + 6uw_1w_5 - 6uw_2w_4)/6.\]

References

[1] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov, Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, American Mathematical Society, Providence, RI, 1999. Edited and with a preface by Krasil'shchik and Vinogradov.

[2] R. K. Dodd, J. C. Eliebeek, J. D. Gibbons, and H. C. Morris. Solitons and Nonlinear Wave Equations. Academic Press, 1982.

[3] P. H. M. Kersten. Supersymmetries and recursion operators for $N = 2$ supersymmetric KdV-equation, RIMS Kokyuroku, 1150, Kyoto Uniniv., Kyoto, Japan, 2000, pp. 153–161.

[4] I. S. Krasil'shchik. Some new cohomological invariants for nonlinear differential equations. Differential Geom. Appl., 2 (1992) no. 4, 307–350.

[5] I. S. Krasil'shchik and P. H. M. Kersten, Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations, Kluwer Acad. Publ., Dordrecht/Boston/London, 2000.

[6] I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov, Geometry of Jet Spaces and Nonlinear Partial Differential Equations, Gordon and Breach, New York, 1986.

[7] I. S. Krasil'shchik and A. M. Vinogradov. Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Acta Appl. Math., 15 (1989) no. 1-2, 161–209.

[8] S. Krivonos, A. Sorin. Extended $N = 2$ supersymmetric matrix (1, s)-KdV hierarchies. Phys. Lett. A251 (1999) 109.

[9] P. Mathieu. Open problems for the super KdV equation. AARMS-CRM Workshop on Baecklund and Darboux transformations. The Geometry of Soliton Theory. June 4–9, 1999. Halifax, Nova Scotia.

[10] A. M. Vinogradov. Local symmetries and conservation laws, Acta Appl. Math. 3 (1984) 21–78.