Pseudospin-one particles in the time-periodic dice lattice: a new approach to transport control

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Abstract
The controlling of the transmission in the pseudospin-one Dirac–Weyl systems offers a rich tool to study new concepts of massive Dirac electron tunneling by means of a time-dependent potential. The time-periodic potential is one of the experimental techniques to have more control over the tunneling effect. In this paper, we study the transmission coefficient for different sidebands to obtain total transmission. We show how the super Klein tunneling under special conditions is independent of the incidence angle, oscillation amplitude, frequency, and barrier width. We consider a band gap opening with different locations of the flat band and modulate the resonances by tuning free parameters in our system.

Keywords: transport, dice lattice, pseudospin-one particles

1. Introduction
There has been a growing interest in two-dimensional (2D) materials since the first synthesis of graphene in 2004 [1–10]. Graphene is a one-atom-thick crystal of hexagonal arrangement of carbon atoms which has remarkable properties such as high electronic and thermal conductivities, quantum Hall effect, etc [11–19]. Furthermore, due to the relativistic behavior of electrons, graphene provides a platform to study the predicted relativistic quantum mechanics effects, such as Klein tunneling [20–25]. Besides the great number of theoretical studies, the possibility of creating different 2D nanostructures has progressed due to the rapid development in synthesis and artificial heterostructures growth techniques [26–29].

One of the main techniques which have been established is growing a trilayer superlattice in the (111) direction for creating dice lattice ($\tau_3$ lattice) [30–34]. The dice lattice is a honeycomb lattice with the addition of an extra atom is located at the center of each hexagon which is described by the Dirac–Weyl Hamiltonian [35–38]. The charge carriers at low energies in the dice lattice are described by the Weyl equation with pseudospin $S = 1$. The gapless low-energy band structure of this lattice consists of Dirac cones and an additional flat band. This kind of band structure has important consequences, such as magneto-optical conductivity, Hofstadter butterfly effect, and zero-momentum optical conductivity [39–42]. Furthermore, an attractive feature of dice lattice is that it displays perfect transmission independent of the incident angle through a barrier, which is known as super-Klein tunneling (SKT) [43, 44]. Recently many papers have been devoted to controlling transport property in this lattice. Among the possibilities, applying a time-dependent potential can be selected as an alternative approach to having further control on particle transmission [45–48].

The Floquet theory is used to analyze the time-periodic systems by converting the Hamiltonian into the time-independent one with infinite dimensionality. In this method, we expand the wave function of the combined electron–photon system into Floquet sidebands [49–52]. Extrinsic additional sidebands at energies $E \pm mh\omega$ which correspond to Floquet channels $m = 0, \pm 1, \pm 2, \ldots$ induced by a time-periodic potential have an important effect on transport properties [45, 47, 53–57]. These sidebands arise in the context of the exchange of...
energy quanta between electrons and photons due to the oscillating potential. In fact, the pseudospin-one particle can emit (absorb) photons and drop (jump) to incident channels or the other Floquet channels. A simple method of setting the ac-driven tunneling was first presented by Tien and Gordon by applying an electric field to the superconducting films [58]. Furthermore, the time-periodic potential can be created by applying a small ac signal in the potential barrier causes arising many sidebands because of absorbing or emitting photons. We take into account the presence of a small ac signal with potential \( V_0 \) and a static square potential \( V_0 \) [73]. Accordingly, the potential is given by:

\[
V(x, t) = \begin{cases} 
V_0 + V_{ac} \cos(\omega t) & \text{for } 0 < x < D, \\
0 & \text{otherwise}.
\end{cases}
\]  

(1)

Pseudospin-one particles in dice lattice in the presence of energy band gap \( 2\Delta \) are governed by the following Hamiltonian,

\[
H = v_p S \cdot p + \Delta M + V(x, t),
\]

(2)

where \( S = (S_x, S_y) \) is the \( x \) and \( y \) components of spin-one matrices,

\[
S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}
\]

(3)

and \( M \) describes the possible ways of band gap opening which is given by [44]

\[
M = S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M = \pm U = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(4)

2. Theory

Here, we study the effect of time-periodic potential on the transport property of the pseudospin one Dirac–Weyl system as shown in figure 1. A tentative way for controlling the particle transmission is offered by increasing the number of degrees of freedom in our system. Thus, we study the transport of electrons in the presence of a barrier with time-oscillating height. In this case, we can change the amplitude and frequency of the time-dependent potential to have further control of particle transmission. It is important to note that the time-square potential barrier causes arising many sidebands because of absorbing or emitting photons. We take into account the presence of a small ac signal with potential \( V_{ac} \) and a static square potential \( V_0 \) [73]. Accordingly, the potential is given by:

\[
\text{Figure 1.} \text{ (a) The dice lattice with three sites A, B and C (denoted by red, blue, yellow color) in each unit cell in the presence of time-periodic potential. (b) Scheme of the band structure for the case } (l, L) = (1, 1) \text{ across the time-periodic potential. The green lines indicate the quantized energies } E = \pm m\hbar\omega.
\]

\[
\begin{array}{l}
\text{Table 1. The set of used parameters in equation (12).} \\
\hline
\text{Notation} & \text{Formula} \\
\hline
Z_j & b_j \beta Y_j - b_j^* \gamma_j \\
W_j & b_j \beta \lambda_j + b_j^* \gamma_j \\
F_j & b_j \beta Y_j + b_j^* \gamma_j \\
N_j & b_j \beta Y_j + b_j^* \gamma_j \\
\hline
\end{array}
\]

\]
cyan curve and stars symbols are the numerical results considering Floquet sidebands $V = \omega_0$ where the phases of pseudospin for sideband It is noteworthy that The transmission coefficient for first sidebands as a function of $\alpha$ for normal incident wave with $\phi_0 = 0$ where $E = 100$ meV, $V_0 = 200$ meV, $\Delta = \Delta' = 40$ meV and $D = 100$ nm. The open symbols represent the analytical expression (12) for first sidebands and cyan curve and stars symbols are the numerical results considering Floquet sidebands $m = 0, \pm 1, \ldots, \pm 12$ for the cases: (a) $(1, -1)$, (b) $(0, -1)$ (c) $(1, 0)$ and (d) $(0, 0)$.

It is noteworthy that $M = S_c$ corresponds to the flat band at the middle of the bandgap which is denoting by $L = 0, l = 0$ for inside and outside the barrier respectively. Outside the oscillating potential barrier, the eigenenergies are $E_{fb} = 0$ and $E = m\hbar\omega \pm \sqrt{\Delta^2 + (\nu_i \hbar k_m)^2}$ where the symbol $(fb)$ stands for the flat band. Furthermore, the case where the flat band is located at the bottom (top) of the conduction (valence) band is described by $M = +U$ ($M = -U$) with $l = 1(l = -1)$ for the outside and with $L = 1(L = -1)$ for the inside the potential barrier. In this case, the dispersion relation becomes $E_{fb} = \Delta$ ($E_{fb} = -\Delta$) and $E = m\hbar\omega \pm \sqrt{\Delta^2 + (\nu_i \hbar k_m)^2}$. The linear momentum in $y$ direction is conserved as the Hamiltonian commutes with it and consequently leads to the Snell’s law $\sqrt{E^2 - \Delta^2} \sin \phi_0 = \sqrt{(E - m\hbar\omega - V_0)^2 - \Delta^2} \sin \xi_m$ where the phases of pseudospin for sideband $m = 0, \pm 1, \pm 2, \ldots$ is $\xi_m = \arctan(q_{x,m}/q_{y,m})$ and $\phi_0$ is the angle of incidence. Thus, the $y$ component of wave vector $k$ is given by

$$k_{0,y} = q_{m,y} = \frac{\sqrt{E^2 - \Delta^2}}{\hbar v_F} \sin \phi_0. \quad (5)$$

On the other hand, the $x$ components of the wave vectors for the outside and inside the barrier, respectively are given by:

$$k_{x,m} = \sqrt{\frac{(E - m\hbar\omega)^2 - \Delta^2}{\hbar^2 v_F^2} - k_{0,x}^2}, \quad (6a)$$

$$q_{m,x} = \sqrt{\frac{(E - V_0 - m\hbar\omega)^2 - \Delta^2}{\hbar^2 v_F^2} - k_{0,x}^2}, \quad (6b)$$

The solution of the wave equation $(H + V(x,t))\Phi(r,t) = i\hbar\partial_t \Phi(r,t)$, inside the barrier (region II) can be written as (see the appendix A):

$$\Phi^I(r,t) = \frac{e^{iE t}/\hbar}{2} \sum_{n,m = -\infty}^{\infty} J_n(\alpha) \times \left[ \begin{array}{c} a_{l,m} e^{i\xi_m} \\
 \sum_{n,m = -\infty}^{\infty} J_n(\alpha) \\
 c'_{L,m} e^{-i\xi_m} \end{array} \right] \begin{cases} \text{for } m = 0, \pm 1, \pm 2, \ldots \text{ and outside the barrier } \xi_m \text{ is the angle of incidence} \end{cases}.$$
The corresponding wave functions in the incident and transmitted sides, can be expressed as:

\[
\Phi^I(r, t) = \frac{e^{ik_{y}y}}{2} \sum_{m=-\infty}^{\infty} \left[ \begin{array}{c} a_{l,m} e^{-i\phi_m} \\
- s_{m} b_{l,m} \\
c_{l,m} e^{i\phi_m} \end{array} \right] e^{-ik_{x}x} e^{i\omega t}, \]

and

\[
\Phi^T(r, t) = \frac{e^{ik_{y}y}}{2} \sum_{m=-\infty}^{\infty} \left[ \begin{array}{c} a_{l,m} e^{-i\phi_m} \\
- s_{m} b_{l,m} \\
c_{l,m} e^{i\phi_m} \end{array} \right] e^{-ik_{x}x} e^{i\omega t}, \]

where \( \phi_m = \arctan(k_y/k_{x,m}) \). Here, the band indices \( s_m = \text{sgn}(E - m\hbar \omega) \) and \( s'_m = \text{sgn}(E - V_0 - m\hbar \omega) \) with the positive (negative) sign indicate the conduction (valence) band for the inside and outside the barrier, respectively.

In the outside of barrier, the coefficients \( a_{l,m}, b_{l,m}, c_{l,m} \) are given by:

\[
a_{l,m} = \bar{\rho} \sqrt{1 \pm \frac{\Delta}{E - m\hbar \omega}} + (1 - \bar{\rho}) \left( 1 \mp \frac{\Delta}{E - m\hbar \omega} \right). \]

\[
b_{l,m} = \bar{\rho} \sqrt{2 \left( 1 - \frac{\Delta}{E - m\hbar \omega} \right) + (1 - \bar{\rho}) \sqrt{2 \left( 1 - \frac{\Delta^2}{(E - m\hbar \omega)^2} \right)}}. \]

\[
c_{l,m} = \bar{\rho} \delta_{l,1} + (1 - \bar{\rho}) \left( 1 - \frac{\Delta}{E - m\hbar \omega} \right). \]

According to the above expressions, wave vectors in the time-periodic potential can be imaginary and consequently the wave function becomes evanescent \[74, 75\]. For grazing incidence angles and largest sidebands, the evanescent modes appear in the barrier. We take these evanescent modes into account in the numerical result.
The transmission coefficient \( T_m = |t_m|^2 \) of the scattering problem can be obtained analytically for the first sidebands (see the appendix). In the limiting case when the parameter \( \alpha \) is small, we can obtain the transmission coefficient \( t_j \) (with \( j = \pm 1 \)) for the first sidebands, as follows:

\[
 t_j = \frac{e^{-i\phi_0} b_{j} b_{j+1} b_{j+2}}{b_{j} b_{j+1} b_{j+2} (F_{0} + W_{0} j_{0} (\alpha))} \left( F_{j} b_{j} + b_{j} W_{j} \right) \\
 + W_{0} \left( \frac{b_{j+1} b_{j+2} Z_{j}}{F_{j}} \right) \left( e^{i\phi_0} b_{j+1} b_{j+2} (F_{j} + N_{j}) \right) \left( (F_{j})^2 - (N_{j})^2 \right) \\
 \times \left[ F_{j} Z_{j} e^{-i\phi_0} - W_{0} W_{j} e^{i\phi_0} + \frac{N_{j} b_{j+1} b_{j+2} (Z_{j} W_{j} - F_{j} W_{j})}{F_{j}} \right].
\]

(12)

In order to simplify the notations, we use variables which are summarized in table 1 and the following definitions:

\[
 \gamma_j = \hat{F}_{j} \alpha_{j} \cos(\phi_j) + (1 - \hat{F}_{j}) [i \alpha_{j} \sin(\phi_j) + e^{-i\phi_j}],
\]

(13a)

\[
 \Omega_j = L^2 a_{j}^2 \cos(\xi_j) + (1 - L^2) [i \alpha_{j} \sin(\xi_j) + e^{-i\xi_j}],
\]

(13b)

The transmission coefficient for the first sidebands, associated with the above exposition by considering \(-1 < m - n < 1\) is in perfect agreement with the numerical result, by considering Floquet sidebands \( m = 0, \pm 1, \ldots, \pm 12 \), as displayed in figure 2. We must note that \( t_0 \) in the above expressions denotes the transmission coefficient for the central band and the corresponding transmission coefficient \( T_0 = |t_0|^2 \) for the static barrier takes the form:

\[
 T_0 = \frac{b_{0}^2 (|F_{0} + W_{0}|^2) (\gamma_0 + \gamma_0^*)}{(|W_{0}|^2 - |F_{0}|^2)^2 + 4|F_{0} W_{0}|^2 \sin^2(\phi_0 D)}.
\]

(14)

For the spatial cases \((0,0), (1,-1)\) and \((-1,1)\) with energy \( E = 0.5V_0 \) and \( \Delta = \Delta' \), term \( F_{0} W_{0} \) is equal to zero and \( T = 1 \) for all of the incident angle due to the equality \( k_{s} = q_{i} \). Consequently, in the static barrier and for these cases the transmission is perfect as well as independent of the incident angle and barrier width.

3. Numerical results and discussion

As shown in the previous section, the transmission coefficient for the first sidebands can be easily obtained analytically. On the other hand, a sufficiently large number of sidebands is needed to obtain the total transmission which can be calculated numerically. The total transmission as illustrated in figures 3(a)–(c) is independent of \( \alpha \), incidence angle, and width of the barrier for the case \((0,0)\) and \((1,-1)\) with \( E = V_0/2 \). However, in the cases \((1,1)\) and \((-1,-1)\), the change of \( \alpha \) affects significantly the electron tunneling. figures 3(b) and (c) shows how the tunneling is influenced by the incidence angle and width of the barrier. The resonances occur in the same values of incidence angle and width of the barrier in all of the following cases: \((\pm 1, 0), (0, \pm 1), (1, 1)\). However, the resonant peaks are characterized by changing the width of the barrier for cases, \((\pm 1, 0), (0, \pm 1), (1, 1)\). For more details we show the transmission coefficients \( T_m \) for the \((0,0)\) with \( m = -3, \ldots, 3 \) in figure 3(d) which is obtained by the numerical solution of the linear equation system (11). We obtain similar behavior of the total transmission for the cases \((l_1, l_2)\).
and \((l = L_2, L = l_1)\) with energy \(V_0/E = 2\) and \(\Delta = \Delta'\) because of the symmetrical location of the flat bands and equality of \(k_{x,m} = q_{x,-m}\). As shown in figure 3(d) the transmission probability \(T_m\) is equal to \(T_{-m}\) which means that if a particle absorbs or emits \(m\) photons has the same probability amplitudes to cross the barrier. We notice that for small value of \(\alpha\) the transmission is mainly due to the central band. In this figure, we only show several sidebands, but in the numerical results for the total transmission we used \(m = -12, \ldots, 12\).

On the other hand, we have super Klein tunneling in the static barrier and in the cases \((0,0), (1, -1)\) and \((-1, 1)\), since the term \(W_0 F_0\) is equal to zero in equation (14). The independence of the total transmission coefficient in the presence of time-periodic potential in special cases arises from the fact that absorbing or emitting \(m\) photons have the same probability to cross the barrier.

Figure 4 shows the angular dependence of the total transmission \(T = \sum_{m=-\infty}^{\infty} T_m\). Moreover, one can easily see that for the same effective mass inside and outside the barrier and a specific value \(V_0/E = 2\) in cases \((1, -1)\) and \((0, 0)\), applying time-periodic potential has no visible effect on super Klein tunneling for different incident angles. We also notice that increasing energy \((E > 0.5V_0)\) leads to decreasing of the transmission for large value of incident angle due to the increasing evanescent waves in the potential region. We also consider different effective masses for inside and outside barrier which corresponds to a heterogeneous junction in figure 4(d). This case directly shows more non-zero transmission with \(E > 0.5V\) compared to the case with \(\Delta' = 40\) meV as a consequence of the fact that by decreasing gap in the barrier region, the evanescent modes decrease. In figure 4(f), we consider static barrier at energy \(E = V_0/2\) in the case \((0, 0)\) where we see clearly well-defined resonances for condition \(q_0 D = h\pi\), with \(h\) an integer. As compared to the time-periodic potential (see figure 4(c)), we found that the peaks become sharper in the static case. Our results for the case \((0, 0)\) agree with the limited one, for a static barrier in reference [76]. For the case \((0, 0)\) with \(\Delta' = \Delta\) at energy \((E = 0.5V_0)\) we find perfect transmission independent of incidence angle as expected from equation (14).

In figure 5, we obtain the total transmission probability for the normal incidence as functions of barrier width and energy. At first glance, one can recognize that by tuning the width of barrier and energy, resonances can be characterized. The important point to be highlighted is that Klein tunneling for the normal incident electron in case \((1, -1)\), where \(V_0/E = 2\), is independent of barrier width.

The transmission coefficient for the normal incident electron is shown in figure 6 as a function of the width of the barrier and \(\alpha\). This figure highlights that tunneling is almost independent of free parameters \(D, \alpha\) for cases \((1, -1)\) and \((0, 0)\) where \(V_0/E = 2\) and \(\Delta = \Delta'\) at normal incidence. Further, in the case \((0, -1)\) we can see that resonances in the tunneling occur by tuning \(V_\infty\) and width of the barrier. In order to quantify the transmission properties, we explain the behavior of the transmission as a function of effective mass \(m = \Delta/v_2^2\). As we can see, in the case \((0, -1)\) with \(\alpha = 0\), the transmission excites regular oscillation and the resonances occur because wave vector inside the barrier, satisfies
condition \( q_0 D = h \pi \), with \( h \) an integer, in equation (14). Furthermore, in this case for large \( \alpha \) values (\( V_{ac} = h \omega \alpha \) ) the transmission decreases with increasing barrier width which is the consequence of increasing the size of the barrier. As we can see, in figure 7 the transmission coefficient is again almost independent in the cases \((1, -1)\) and \((0, 0)\) at energy \( E = 0.5V_0 \) for normal incidence. It is clearly shown by these figures that the super Klein tunneling in cases \((1, -1), \) \((-1, 1)\) and \((0, 0)\) with \( \Delta = \Delta' \) and at energy \( E = 0.5V_0 \) is independent of the free parameters in our system. On the other hand, in the case \((1, 1)\), the transmission for the small value of \( \alpha \) is approximately the same as static one due to small value of \( V_{ac} \) as compared to the \( V_0 \). In the time-periodic potential, the total transmission is independent of parameter \( \alpha \) for small \( \Delta \)-values which carriers behave like relativistic particles. The difference of results for different cases arises from the location of flat bands inside and outside the barrier. Consequently, location of flat bands plays an important role in the transmission probability. For the case \((1, -1)\) and \((-1, 1)\) and with \( \Delta = \Delta' \) and \( V_0/E = 2 \) due to the equality of \( a_{m} = a'_{-m}, b_{m} = b'_{-m} \) and \( c_{m} = c'_{-m} \), the transmission coefficient for the sideband \( m \) is approximately equal to the \(-m\) one. On the other hand, the transmission coefficient for side band \( m \) is equal to \(-m\) for the case \((0, 0)\) with \( \Delta = \Delta' \) and \( V_0/E = 2 \) due to the equality of \( a_{m} = a'_{-m}, b_{m} = b'_{-m} \) and \( c_{m} = c'_{-m} \). It is apparent from figure 7 that the electron is fully transmitted for the gap-less band structure of the dice lattice at energy \( E = 0.5V_0 \) and as expected has the same value for all the cases \((l, L)\). Consequently, for the cases where absorbing or emitting \( m \) photons have the same probability to cross the barrier, the total transmission is independent of incidence angle, the width of the barrier, and the ratio \( \alpha \).

4. Conclusion

According to Floquet scattering theory, the incident electrons in the time-periodic potential, scatter into Floquet sidebands due to the exchange of energy between the electrons and the harmonically driven potential. Therefore, it is crucial to know how the transmission can be affected owing to the appearance transitions between sideband states. In this paper we use analytical and numerical techniques to study transmission probability for massive pseudo-spin one particles including periodic time-dependent potential. Our analytical solutions are generally in excellent agreement with numerical results. We explained the difference in the band gap opening by considering the cases corresponding to different locations of the flat band. We show that at energy \( E = 0.5V_0 \) when the transmission coefficient for the sideband \( m \) is approximately equal to the \(-m\) one, applying time periodic potential has no visible effect on super Klein tunneling, which is an important consideration for the design of electronic devices. Furthermore, by tuning free parameters, we found some particular values leading to transmission resonances. The next result to be highlighted is that increasing energy \( E > 0.5V_0 \) leads to decreasing of the transmission for large value of incident angle due to the increasing evanescent modes. We find that a time-periodic barrier, besides the standard parameters such as width and height of the barrier, effective mass etc, provides a more flexible way for controlling the transmission probability. Our results in figures 3(a), 6 and 7 show that the transmission probability strongly depends on the applied frequency of time-periodic potential \( (\omega = V_{ac}/\hbar \alpha) \) for special types of band gap opening. Our results can be relevant for periodic arrays of coupled waveguides to characterize propagation within this
setup. Helical wave-guides arranged in the dice lattice geometry can be considered as an emulation of Floquet scattering of pseudospin-one particles. Finally, our findings may have an important impact on the design and analysis of Floquet topological insulators.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Derivation of the wave functions for inside and outside of the potential regions

Here, we are interested in Floquet solution for inside of the potential region, thus by substituting \( \Phi(r, t) = e^{-iE_{f}t/\hbar} \psi(r, t) \) into the wave equation \((H + V(x, t))\Phi(r, t) = i\hbar\partial_{t}\Phi(r, t)\) we find

\[
(H + V_{0})\chi(r) = E'\chi(r)
\]

(A2a)

\[
(E' - E_{f} + V_{ac}\cos(\omega t))f(t) = i\hbar\partial_{t}f(t)
\]

(A2b)

where \( E' \) is a constant and \( f(0) = 1 \). By imposing the periodic condition \( f(t) = f(t + T) \), we find that the \( E' \) and \( E_{f} \) are related by \( E' = E_{f} = m\hbar\omega \). The solution of the above differential equation for \( f(t) \), can be presented as [47]:

\[
f(t) = e^{-iE'\omega T/\hbar} e^{i\frac{\alpha}{\hbar} \int_{0}^{T} \cos(\omega t') dt'}
\]

\[
= e^{-iE'\omega T/\hbar} \sum_{n=-\infty}^{\infty} J_{n}(\alpha) e^{-i\mathbf{m}nT}, \quad \text{(A3)}
\]

where \( \alpha = V_{ac}/\hbar\omega \) and we use the identity \( e^{-ix} \sin(x) = \sum_{n=-\infty}^{\infty} J_{n}(\alpha) e^{-i\mathbf{m}nT} \). Note that the \( \chi(r) \) is the spinor solution for the static potential difference. As a consequence, the wave functions inside the barrier can be written as:

\[
\Phi^{I}(r, t) = e^{i\tilde{b}_{0}x} e^{-iE_{f}t/\hbar} \sum_{n=-\infty}^{\infty} J_{n}(\alpha) e^{-i\mathbf{m}nT}
\]

\[
\times \left[ \begin{array}{c} a_{l,n} e^{-i\mathbf{m}nT} \\ s_{n}\phi'_{Lm} e^{-i\mathbf{m}nT} \\ c_{Lm} e^{-i\mathbf{m}nT} \end{array} \right] f'_{m} e^{i\mathbf{q}r_{m}T}
\]

\[
+ \left[ \begin{array}{c} a_{l,n} e^{-i\mathbf{m}nT} \\ -s_{n}\phi'_{Lm} e^{-i\mathbf{m}nT} \\ c_{Lm} e^{-i\mathbf{m}nT} \end{array} \right] r'_{m} e^{-i\mathbf{q}r_{m}T}. \quad \text{(A4)}
\]

Furthermore, in the oscillating potential barrier the energy is quantized, the wave-functions outside the barrier involve additional sidebands. Finally, for the outside the barrier, the wave function can be written in terms of the band index as:

\[
\Phi^{I}(r, t) = \frac{e^{i\tilde{b}_{0}y}}{2} \sum_{m=-\infty}^{\infty} \left[ \begin{array}{c} a_{l,m} e^{-i\mathbf{m}nT} \\ s_{m}\phi'_{Lm} e^{-i\mathbf{m}nT} \\ c_{Lm} e^{-i\mathbf{m}nT} \end{array} \right] f'_{m} e^{i\mathbf{q}r_{m}T} e^{-i\mathbf{m}nT/\hbar}, \quad \text{(A5)}
\]

and

\[
\Phi^{II}(r, t) = \frac{e^{i\tilde{b}_{0}y}}{2} \sum_{m=-\infty}^{\infty} \left[ \begin{array}{c} a_{l,m} e^{-i\mathbf{m}nT} \\ -s_{m}\phi'_{Lm} e^{-i\mathbf{m}nT} \\ c_{Lm} e^{-i\mathbf{m}nT} \end{array} \right] f_{m} e^{i\mathbf{q}r_{m}T} e^{-i\mathbf{m}nT/\hbar}, \quad \text{(A6)}
\]

where \( E_{m} = E_{0} + m\hbar\omega \). In order to match the solutions at the boundaries, it is convenient to choose \( E := E_{f} = E_{0} \).

Appendix B. Derivation of the transmission coefficient for the first sidebands

By using the boundary conditions \( S_{x}\Phi^{I}(r, t) = S_{x}\Phi^{II}(r, t) \) and \( S_{x}\Phi^{II}(r, t) = S_{x}\Phi^{II}(r, t) \) at the interfaces, we can find the reflected \( r_{m} \) and transmitted \( t_{m} \) amplitudes in the following system of equations

\[
s_{m}b_{l,0} = s_{m}b_{l,0} + r_{m}j_{m-n}(\alpha), \quad \text{(B1a)}
\]

\[
a_{l,0} e^{-i\mathbf{m}nT} + c_{l,0} e^{i\mathbf{m}nT} + [a_{l,m} e^{i\mathbf{m}nT} + c_{l,m} e^{-i\mathbf{m}nT}] r_{m}
\]

\[
= \sum_{n=-\infty}^{\infty} \{ [a_{l,m} e^{-i\mathbf{m}nT} + c_{l,m} e^{i\mathbf{m}nT}] r_{m} + [a_{l,m} e^{i\mathbf{m}nT} + c_{l,m} e^{-i\mathbf{m}nT}] j_{m-n}(\alpha) \}, \quad \text{(B1b)}
\]

\[
s_{m}b_{l,m} t_{m} e^{i\mathbf{m}nD}
\]

\[
= \sum_{n=-\infty}^{\infty} j_{m-n}(\alpha) [r_{m} e^{i\mathbf{m}nD} - r_{m} e^{-i\mathbf{m}nD}] s_{m}b_{Lm}, \quad \text{(B1c)}
\]

\[
(a_{l,m} e^{-i\mathbf{m}nT} + c_{l,m} e^{i\mathbf{m}nT}) t_{m} e^{i\mathbf{m}nD}
\]

\[
= \sum_{n=-\infty}^{\infty} j_{m-n}(\alpha) [a_{l,m} e^{-i\mathbf{m}nT} + c_{l,m} e^{i\mathbf{m}nT}] r_{m} e^{i\mathbf{m}nD}
\]
To solve the above infinite number of coupled equations, one approximation must be used. For the small value of $\alpha$, we can consider limited sidebands since Bessel functions are small for a large value of $m$. By using finite Floquet sidebands with $m = 0, \pm 1, \ldots, \pm n$ and by writing the above system of equations in a matrix form $TX = \mu$, we can obtain the transmission coefficient. Here, $T, X$ have the following forms:

\begin{equation}
T = [T_1\Lambda_1 \ T_1\Lambda_2 \ T_2\Lambda_3 \ T_2\Lambda_2]^T, \quad (B2)
\end{equation}

and

\begin{equation}
X = (r_u, \ldots, r_u, t'_{-u}, \ldots, t'_{-u}, t_u, \ldots, t_u)^T, \quad (B3)
\end{equation}

with $T_1 = [-I \mathcal{J} \mathcal{J} \mathcal{O}]$ and $T_2 = [\mathcal{O} \mathcal{J} \mathcal{J} \mathcal{J} - I]$. We use the short notations:

\begin{equation}
\mathcal{J} = J_{mn} = J_{m-n}(\alpha), \quad \mu = [\mu_1 \mu_2 \mathcal{O} \mathcal{O}]^T,
\end{equation}

\begin{equation}
\Lambda_1 = \text{diag} \left( -s_m b_m^{w} \big|_{m=-u}^{u} - s'_m b'_m^{u} \big|_{m=-u}^{u} \right) \times \left( s'_m b'_m^{u} \big|_{m=-u}^{u} (s_m b_m)^{w} \big|_{m=-u}^{u} \right),
\end{equation}

\begin{equation}
\Lambda_2 = \text{diag} \left( (a_m e^{i(k_n D)} + c_m e^{-i(k_n D)}) \big|_{m=-u}^{u} \right) \times \left( (a_m e^{i(k_n D)} + c_m e^{-i(k_n D)}) \big|_{m=-u}^{u} \right) \times \left( a_m e^{-i(k_m D)} + c_m e^{i(k_m D)} \big|_{m=-u}^{u} \right) \times \left( a_m e^{-i(k_m D)} + c_m e^{i(k_m D)} \big|_{m=-u}^{u} \right),
\end{equation}

\begin{equation}
\Lambda_3 = \text{diag} \left( s_m b_m \big|_{m=-u}^{u} (s'_m b'_m)^{u} \big|_{m=-u}^{u} \right) \times \left( s'_m b'_m^{u} \big|_{m=-u}^{u} (s_m b_m)^{w} \big|_{m=-u}^{u} \right). \quad (B4)
\end{equation}

and define

\begin{equation}
\mathcal{D} = \text{diag} \left( e^{-i(k_n D)} \big|_{m=-u}^{u} (e^{i(k_n D)})^{w} \big|_{m=-u}^{u} \right) \times \left( (e^{i(k_n D)})^{w} \big|_{m=-u}^{u} (e^{i(k_n D)})^{u} \big|_{m=-u}^{u} \right),
\end{equation}

\begin{equation}
\mu_1 = \left( s_m b_m \delta_m \big|_{m=-u}^{u} \right),
\end{equation}

\begin{equation}
\mu_2 = \left( (a_m e^{-i(k_m D)} + c_m e^{i(k_m D)} \delta_m) \big|_{m=-u}^{u} \right). \quad (B5)
\end{equation}

The solutions of equations system $TX = \mu$ could be obtained for the first sidebands by using the above approximation.

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