The Complete Hierarchical Locality of the Punctured Simplex Code

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Abstract—This paper presents a new alphabet-dependent bound for codes with hierarchical locality. Then, the complete list of possible localities is derived for a class of codes obtained by deleting specific columns from a Simplex code. This list is used to show that these codes are optimal codes with hierarchical locality.

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I. INTRODUCTION

In modern distributed storage systems (DSSs) failures happen frequently, whence decreasing the number of connections required for node repair is crucial. Locally repairable codes (LRCs) are a subclass of erasure-correcting codes, which allow a small number of failed nodes to be repaired by accessing only a few other nodes. LRCs were introduced in [1], [2] where the codes can locally repair one failure. They were later extended in [3] to be able to locally repair up to \( \delta - 1 \) failures.

An \([n, k, d]\) linear code \( C \) of length \( n \), dimension \( k \), and minimum Hamming distance \( d \), has all-symbol locality \((r, \delta)\) if for all code symbols \( i \in [n] = \{1, \ldots, n\} \), there exists a set \( R_i \subseteq [n] \) containing \( i \) such that \(|R_i| \leq r + \delta - 1 \) and the minimum distance of the restriction of \( C \) to \( R_i \) is at least \( \delta \). We refer to \( C \) as an \((n, k, d, r, \delta)\)-LRC and to the sets \( R_i \) as repair sets or local sets. Related Singleton-type bounds have been derived for various cases in [1]–[3] and the first bound with a fixed code alphabet was obtained in [4] for \( \delta = 2 \). Constructions achieving the Singleton-type bounds and the bound in [4] for \( \delta = 2 \) were provided in [2], [3], [5]–[9].

The authors of [10] proposed the first alphabet-dependent bound on LRCs over the alphabet \( Q \) using an upper bound \( B(n, d) \) on the cardinality of a code of length \( n \) and minimum distance \( d \). The global bound is as follows:

\[
k \leq \left( \frac{n - d + 1}{r + \delta - 1} + 1 \right) \log_q B(r + \delta - 1, \delta).
\]

(1)

Recently, [11] provided a different alphabet-dependent bound for LRCs of the same type as the bound in [4] using the Griesmer bound \( G_q(k, d) := \sum_{i=0}^{k-1} \lfloor d/q^i \rfloor \). The bound has the following form: For any linear \((n, k, d, r, \delta)\)-LRC \( C \) with \( \kappa \) the upper bound on the local dimension of the repair sets,

\[
k \leq \min_{\lambda \in \mathbb{Z}_+} \left\{ \lambda + k_{\text{opt}}^q(n - \mu, d) \right\}
\]

(2)

where \( a, b \in \mathbb{Z} \) such that \( \lambda = a \kappa + b, 0 \leq b < \kappa \) and \( \mu = (a + 1)G_q(\kappa, \delta) - G_q(\kappa - b, \delta) \).

In [12], the authors introduced the notion of codes with hierarchical locality (H-LRCs), which optimizes further the number of nodes contacted for repair according to the number of failures. A 2-level H-LRC is a code where the restrictions to the repair sets are themselves LRCs, thus providing an extra layer of locality. If an H-LRC has locality \([r_1, \delta_1), (r_2, \delta_2)]\), then the number of nodes contacted to repair up to \( \delta_2 - 1 \) failures is at most \( r_2 \) and the number of nodes contacted for repair is at most \( r_1 \) if the number of failures is \( \geq \delta_2 \) and \( \leq \delta_1 - 1 \). This concept can be easily generalized to an arbitrary level of hierarchy. A Singleton-type bound was derived in [12] and constructions attaining the bound were given in [12], [13].

In Section III, we focus on 2-level H-LRCs and show how we can adapt the construction algorithm provided in [12] to obtain an alphabet-dependent bound of the same type as in [4]. By construction, this bound is at least as good as the Singleton-type bound derived in [12].

In Section IV, we study the locality of one particular construction of LRCs presented in [9]. The general idea of this construction is to remove a Simplex code from another Simplex code of higher dimension. It was shown in [9] that these codes achieve the Griesmer bound and are LRCs with \( \delta = 2 \). The goal in this section is to prove the locality for every dimension and \( \delta \) and show that this construction leads to optimal H-LRCs. As the first step, we describe the restrictions of dimension \( k - 1 \) and prove the locality for this dimension using combinatorial techniques. As the second step, we use a recursive argument to get all the restrictions of the constructed codes to closed sets. Our main contribution is the complete list of possible localities for these codes. In particular, this shows that the constructed codes are alphabet-optimal H-LRCs for every dimension. Finally, since a special case of this construction leads to the Reed–Muller codes \( \text{RM}(1, m) \), we obtain as a corollary to our result that the Reed–Muller codes
RM(1, m) are optimal H-LRCs and we derive their locality parameters.

II. PRELIMINARIES

We denote the set \{1, 2, ..., n\} by \([n]\) and the set of all subsets of \([n]\) by \(2^n\). The set of all positive integers including 0 is denoted by \(\mathbb{Z}_+\). For a length-\(n\) vector \(v\) and a set \(I \subseteq [n]\), the vector \(v_I\) denotes the restriction of the vector \(v\) to the coordinates in the set \(I\). A generator matrix of a linear code \(C\) is \(G_C = (g_1 \cdots g_n)\) where \(g_i \in \mathbb{F}_q^n\) is a column vector for \(i \in [n]\). The shortening of a code \(C\) to the set of coordinates \(I \subseteq [n]\) is defined by \(C/I = \{c_{[n]}|_I : c \in C\}\) such that \(c_i = 0\) for all \(i \in I\) and the restriction of a code \(C\) to \(I\) is defined by \(C|_I = \{c_I : c \in C\}\). For convenience, we call the codes obtained by a restriction restricted codes. Two linear codes \(C\) and \(C'\) are called isomorphic if \(C'\) can be obtained by a permutation on the coordinates of the codewords of \(C\).

The Simplex code \(S(m)\), or sometimes \(S_2(m)\), is a linear code over \(\mathbb{F}_q\) obtained via the generator matrix \(G_m\) consisting of all pairwise linearly independent vectors in \(\mathbb{F}_q^n\). The parameters of \(S(m)\) are therefore \([q^m - 1)/(q - 1), m, q^{m-1}\]. Finally, we define a closure operation on the subsets of \([n]\) for linear codes.

**Definition 1.** Let \(C\) be a linear code of length \(n\) and \(I \subseteq [n]\).
The closure operation \(c: 2^n \to 2^n\) is \(c(I) = \{e \in [n] : \dim(C|_{[n]}|_e) = \dim(C|_I)\}\). A set \(I \subseteq [n]\) is a closed set if \(c(I) = I\).

One can think of the closure operation via the generator matrix \(G_C\) of \(C\) where \(c(I)\) is the set of all columns in \(G_C\) contained in the linear span of the columns indexed by \(I\).

Since we will work on Simplex codes which have a lot of combinatorial properties, we will use some tools coming from matroid theory. Matroids have many equivalent definitions in the literature. Here, we choose to present matroids via their rank functions. Much of the contents in this part can be found in more detail in [14].

**Definition 2.** A (finite) matroid \(M = (E, \rho)\) is a finite set \(E\) together with a rank function \(\rho: 2^E \to \mathbb{Z}\) such that for all subsets \(X, Y \subseteq E\),

\[
\begin{align*}
(R.1) & \quad 0 \leq \rho(X) \leq |X|, \\
(R.2) & \quad X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y), \\
(R.3) & \quad \rho(X + Y) \geq \rho(X) + \rho(Y) - \rho(X \cap Y).
\end{align*}
\]

There is a straightforward connection between linear codes and matroids. Any two different generator matrices of \(C\) have the same row space by definition. Therefore, without any inconsistenscy, we can associate a matroid to a linear code via the generator matrix \(G_C\) where \(\rho(X)\) is the rank of the submatrix of \(G_C\) formed by the columns indexed by \(X\).

One way of defining a new matroid from an existing one is obtained by restricting the matroid to one of its subsets. For a given set \(Y \subseteq E\), we define the restriction of \(M\) to \(Y\) to be the matroid \(M|_Y = (Y, \rho|_Y)\) by \(\rho(X) = \rho(Y, X)\) for all subsets \(X \subseteq Y\). The restriction of \(M\) to \(E \setminus Y\) is called the deletion of \(Y\) and is denoted by \(M \setminus Y\). The two previous operations correspond to the restriction and puncturing of a linear code \(C\).

Let \(M = (E, \rho)\) be a matroid. The closure operator \(c: 2^E \to 2^E\) is defined by \(c(X) = \{e \in E : \rho(X \cup e) = \rho(X)\}\). A set \(H \subseteq E\) is a hyperplane if \(c(H) = H\) and \(\rho(H) = \rho(E) - 1\). The collection of all hyperplanes is denoted by \(\mathcal{H}(M)\).

III. BOUND FOR H-LRC

Codes with hierarchical locality were introduced in [12] to optimize further the number of nodes contacted for repair. In this section, we present the definition of H-LRCs with 2-level hierarchy and derive an alphabet-dependent bound for these codes based on [4].

**Definition 3.** Let \(r_2 \leq r_1\) and \(\delta_2 \leq \delta_1\). An \([n, k, d]\) linear code \(C\) is a code with hierarchical locality having locality parameters \([(r_1, \delta_1), (r_2, \delta_2)]\) if for all code symbols \(i \in [n]\), there exists a set \(M_i \subseteq [n]\) containing \(i\) such that

1. \(\dim(C|M_i) \leq r_1\),
2. The minimum distance of \(C|M_i\) is at least \(\delta_1\),
3. \(C|M_i\) is an LRC with \((r_2, \delta_2)\)-locality.

The codes \(C|M_i\) are called middle codes and their restrictions of dimension \(\leq r_2\) and minimum distance \(\geq \delta_2\) are called local codes. Similarly, the middle sets and local sets are the sets \(M_i\) and the sets such that the restrictions to them give the local codes. Notice that, contrary to the standard definition of LRCs, the authors of [12] bound the dimension of the restricted codes instead of the size. It was proven in [12, Theorem 2.1] that any \([n, k, d]\) linear code with hierarchical locality \([(r_1, \delta_1), (r_2, \delta_2)]\) satisfies the following Singleton-type bound

\[
d \leq n - k + 1 - \frac{k - 1}{r_2} (\delta_2 - 1) - \frac{r_1 - 1}{\delta_1 - \delta_2}.
\]

To obtain the alphabet-dependent bound in [4], the authors proved two results: the construction of a restricted code with small dimension and large size and a lemma about shortened codes. The lemma is the following.

**Lemma 1** ([4], Lemma 2). Let \(C\) be an \([n, k, d]\) linear code over \(\mathbb{F}_q\) and \(I \subseteq [n]\) such that \(\dim(C|_I) < k\). Then the shortened code \(C/I\) has parameters \([n - |I|, k - \dim(C|_I), d' \geq d]\).

Therefore, to get an alphabet-dependent bound for H-LRCs, we need to construct a set with an upper bound on the dimension and such that its size uses the hierarchical locality property to be as large as possible. To achieve this requirement, we modify the construction algorithm used in the proof of the Singleton-type bound in [12]. For the proofs of the rest of the results in this paper, we refer to the extended version of this paper [15].

**Lemma 2.** Let \(C\) be an \([n, k, d]\) H-LRC with locality \([(r_1, \delta_1), (r_2, \delta_2)]\) and \(\lambda \in \mathbb{Z}_+\) with \(0 \leq \lambda \leq k\). Then, there exists a set \(I_c\) such that
Theorem 1. Let \( C \) be an \([n, k, d]_q\) H-LRC over \( \mathbb{F}_q \) with locality \([(r_1, \delta_1), (r_2, \delta_2)]\). Then we have
\[
k \leq \min_{\lambda \in \mathbb{Z}_+} \{ \lambda + k_{\text{opt}}^{(r)}(n - \nu, d) \}
\]
where \( \nu = \lambda + \frac{1}{r_2} (\delta_2 - 1) + \frac{1}{r_1} (\delta_1 - \delta_2) \) and \( k_{\text{opt}}^{(r)}(n, d) \) is the largest possible dimension of a code of length \( n \), for a given alphabet size \( q \) and minimum distance \( d \).

Lemma 2 can be seen as a proof of concept that we can modify the algorithms in [12] to obtain an alphabet-dependent bound. Indeed, the algorithm used in the proof of Lemma 2 and the one presented in [12] are equivalent in the sense that if \( \lambda = k - 1 \) and \( I_r \) is the set obtained in Lemma 2, we obtain again the Singleton-type bound (3) via the relation \( d \leq n - |I_r| \). This implies that the bound (4) is at least as good as the Singleton-type bound (3). Moreover, we gain from the bound (4) that H-LRCs achieving any bound on the parameters \([n, k, d]\) only are directly alphabet-optimal H-LRCs by setting \( \lambda = 0 \). The extension of the bound (4) to arbitrary levels of hierarchy is left for future work.

IV. CONSTRUCTION \( S(m) - S(s) \)

In [9], the authors presented four different constructions of linear LRCs with small locality and high availability. The constructions are based on a method developed in [16] where the generator matrix of a code is obtained by deleting specific columns from the generator matrix of a Simplex code. In this section, we are interested in the locality of one particular construction where the deleted columns form again a Simplex code. The objective is to describe the locality parameters for \( \delta > 2 \) and all dimensions. We show that these codes are LRCs for every local dimension implying a complete optimization of the number of nodes contacted for repair according to the number of failures. Moreover, using combinatorial techniques, we establish the complete list of possible locality and show how many symbols are contained in each hierarchical locality. Finally, we prove that these codes are optimal LRCs for all localities and alphabet-optimal 2-level H-LRCs by the new bound (4).

We start by formally defining the construction of these linear LRCs.

Construction 1 ([9] Construction IV). Let \( G_m \) be an \( m \times \frac{q^m - 1}{q - 1} \) generator matrix of the Simplex code \( S(m) \) and \( G_s \) an \( s \times \frac{q^s - 1}{q - 1} \) generator matrix of the Simplex code \( S(s) \) with \( s \leq m \). Let \( G'_s \) be the generator matrix obtained by prepending \( m - s \) zeros to every column of \( G_s \). Let \( G_C \) be the \( m \times \frac{q^m - q^s}{q - 1} \) matrix obtained by deleting the \( \frac{q^s - 1}{q - 1} \) columns of \( G'_s \) from \( G_m \). Then \( G_C \) generates a linear code \( C \) over \( \mathbb{F}_q \) denoted by \( S(m) - S(s) \).

It was proven in [9, Theorem 14] that the code \( S(m) - S(s) \) with \( m \geq 3 \) and \( s \in [2, m - 1] \) is an \( \frac{q^m - 1}{q - 1}, m, q^{m-1} - q^{s-1} \) linear LRC code over \( \mathbb{F}_q \) with locality \( (r = 2, \delta = 2) \) if \( q > 2 \) or if \( q = 2 \) and \( s < m - 1 \), and with locality \( (r = 3, \delta = 2) \) when \( q = 2 \) and \( s = m - 1 \). Moreover, \( S(m) - S(s) \) achieves the Griesmer bound by [9, Lemma 16].

Remark 1. The code \( S(m) - S(m - 1) \) is isomorphic to the Reed–Muller code \( RM(1, m - 1) \).

The following example illustrates Construction 1.

Example 1. Let \( G_4 \) and \( G_2 \) be the generator matrices of the binary Simplex codes \( S_4(2) \) and \( S_2(2) \) respectively. Then \( C = S_4(4) - S_2(2) \) is a binary \([12, 4, 6]\) code generated by the matrix
\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
where the shadowed columns 3, 4, and 10 are deleted.

A. Locality with dimension \( m - 1 \) of \( S(m) - S(s) \)

The goal of this subsection is to obtain the locality of \( S(m) - S(s) \) with a local dimension of \( m - 1 \) for this, we make a detour to matroid theory by studying the relation between the hyperplanes of the matroid associated to the Simplex code \( S(m) \) and the hyperplanes of the matroid associated to \( S(m) - S(s) \). Indeed, the Simplex code has intrinsically a lot of useful combinatorial structures and Construction 1 corresponds to a deletion in matroid theory. Therefore, matroid theory is used here as a tool to understand the closed sets of \( S(m) - S(s) \) of dimension \( m - 1 \) and to construct the local set for every code symbol. We start by giving a map from the hyperplanes of \( S(m) \) to the hyperplanes of \( S(m) - S(s) \).

Proposition 1. Let \( M_S = (E_S, \rho_S) \) be the matroid associated to the Simplex code \( S(m) \) and \( M_C = (E_C, \rho_C) \) the matroid associated to the code \( C = S(m) - S(s) \). Let also \( Y \subseteq E_S \) be such that \( M_S \setminus Y = M_C \). Then, the map
\[
\phi : \{ H \in \mathcal{H}(M_S) \text{ with } Y \neq H \} \rightarrow \mathcal{H}(M_C) \\
H \mapsto H \setminus Y \cap Y
\]
is a bijection.

The map of Proposition 1 gives us the relation between the hyperplanes of \( M_C \) and the hyperplanes of the matroid \( M_S \). We can now completely describe the restrictions of \( C \) to hyperplanes and see that these restrictions are in fact isomorphic to certain codes obtained by Construction 1.

Proposition 2. Let \( C \) be the code \( C = S(m) - S(s) \), \( M_C \) the matroid associated to \( C \), and \( H_C \in \mathcal{H}(M_C) \). Then \( C|_{H_C} \) is either isomorphic to the code \( S(m - 1) - S(s - 1) \) or to \( S(m - 1) - S(s) \).

It remains to show the existence of such closed sets for every code symbol in order to prove that the code \( S(m) - S(s) \) is
an LRC with locality obtained by restrictions to closed sets of dimension \( m - 1 \).

**Theorem 2.** Let \( C \) be the linear code \( C = S(m) - S(s) \) of length \( n \) with \( m \geq 2 \) and \( s \geq 0 \). Then, for all \( e \in [n] \) we have the following:
- If \( s = 0 \), then there is a set \( H \subseteq [n] \) containing \( e \) such that \( C|_H \) is isomorphic to \( S(m - 1) \).
- If \( s = m - 1 \), then there is a set \( H \subseteq [n] \) containing \( e \) such that \( C|_H \) is isomorphic to \( S(m - 1) - S(m - 2) \).
- If \( 1 \leq s \leq m - 2 \), then there exist two sets \( H_1, H_2 \) containing \( e \) with \( H_1 \not\subseteq H_2 \) such that \( C|_{H_1} \) is isomorphic to \( S(m - 1) - S(s) \) and \( C|_{H_2} \) is isomorphic to \( S(m - 1) - S(s - 1) \).

Theorem 2 can be seen as showing the existence of certain hyperplanes while Proposition 2 is of the form of a unicity statement on the parameters size, dimension and minimum distance of the hyperplanes. Therefore, the two combined with Proposition 1 yield the complete characterization of all the hyperplanes of the recursive form of Theorem 2, the local sets can be isomorphic to \( S \) for \( 1 \leq s \leq m - 2 \).

Theorem 2 implies that for all code symbols \( e \in [n] \), there are two sets \( H_1, H_2 \subseteq [n] \) such that \( C|_{H_1} \) is isomorphic to \( S_2(3) - S_2(1) \) and \( C|_{H_2} \) is isomorphic to \( S_2(3) - S_2(2) \). In other words, there are two restrictions \( C|_{H_1}, C|_{H_2} \) containing \( e \) with parameters \([6, 3, 3]\) and \([4, 3, 2]\) respectively. Therefore, \( C \) is an LRC with locality \((r = 4, \delta = 3)\) and also an LRC with locality \((r = 3, \delta = 2)\).

If we apply Theorem 2 to \( S_2(3) - S_2(1) \), we obtain by isomorphism that there exist \( H_3, H_4 \subseteq H_1 \) containing \( e \) such that \( C|_{H_3} \) is isomorphic to \( S_2(2) - S_2(1) \) and \( C|_{H_4} \) is isomorphic to \( S_2(2) - S_2(1) \) is a \([2, 2, 1]\) code and thus does not provide an extra locality. On the other hand, \( S_2(2) \) is a \([3, 2, 2]\) code which implies that \( C \) is also an LRC with locality \((r = 2, \delta = 2)\). Furthermore, by construction of the local sets, \( C \) is an H-LRC with locality \([3, 3, 2]\).

Example 2 illustrates how Theorem 2 can be used to obtain the locality for different dimensions. Moreover, because of the recursive form of Theorem 2, the local sets can be arranged in such a way that we obtain a hierarchical locality.

We break down what happens to the restrictions when we iterate Theorem 2 by considering the restriction types, i.e., the different isomorphic restrictions. Suppose for simplicity that \( m \) is sufficiently large and \( s \) is close to half of \( m \). As illustrated in Figure 1, applying Theorem 2 on the two restriction types of dimension \( m - 1 \) gives three new restriction types of dimension \( m - 2 \), as two of them lead to the same isomorphic code.

Suppose now that, after some iterations of Theorem 2, we obtain the restriction types of dimension \( \kappa \). Let \( a, b \in \mathbb{Z}_+ \) such that all restriction types are of the form \( S(\kappa) - S(i) \) with \( a \leq i \leq b \). Now, the restriction types of dimension \( \kappa - 1 \) can be obtained by applying Theorem 2 on the restriction types of dimension \( \kappa \). Two extremal cases need to be taken into account. If \( a = 0 \), the only restriction type that we get from \( S(\kappa) - S(a) \) is \( S(\kappa - 1) \). If \( b = \kappa - 1 \), the only restriction type is \( S(\kappa - 1) - S(\kappa - 2) \). This is illustrated in Figure 2 where the two dashed boxes represent the conditional new restriction types that exist only if \( a > 0 \) or \( b < \kappa - 1 \). This sketches the high level idea of the proof of the next theorem describing precisely the different restriction types for a given dimension.

**Theorem 3.** Let \( C = S(m) - S(s) \) with \( m \geq 3 \) and \( 0 \leq s \leq m - 1 \). Let \( \kappa \in [2, m - 1] \) and \( i \) be an integer such that \( \max\{0, s - m + \kappa\} \leq i \leq \min\{s, \kappa - 1\} \). Then for all code symbols \( e \in [n] \), there is a set \( F_i \) containing \( e \) such that \( C|_{F_i} \) is isomorphic to \( S(\kappa) - S(i) \).

**Remark 2.** Notice that \( \max\{0, s - m + \kappa\} \leq \min\{s, \kappa - 1\} \).

Therefore, the claim of Theorem 3 is that there always exist such restrictions for all dimensions \( \kappa \in [2, m - 1] \).

As a corollary of Theorem 3, we have that \( C \) is an LRC as long as the restriction types have a minimum distance greater than or equal to 2. We choose to give here the length, dimension and minimum distance of the local codes to avoid confusion on the parameter \( r \) and \( \tau \) in the two definitions of LRCs and H-LRCs.

**Corollary 1.** Let \( C = S(m) - S(s) \) with \( m \geq 3 \) and \( 0 \leq s \leq m - 1 \). Let \( \kappa \in [3, m - 1] \) and \( i \) be an integer such that
max\{0, s − m + κ\} ≤ i ≤ min\{s, κ − 1\}. Then \( C \) is an LRC with local code parameters
\[
\begin{align*}
&\left\{ \frac{q^i - q^{i-1}}{q+1}, q^i - q^{i-1} \right\} & \text{if } i > 0, \\
&\left\{ \frac{q^i - q^{i-1}}{q+1}, q^i \right\} & \text{if } i = 0.
\end{align*}
\]

Furthermore, for \( κ = 2 \), we have the following local parameters:

- If \( 0 ≤ s ≤ m − 2 \), then \( C \) is an LRC with local parameters \([q + 1, 2, q]\).
- If \( q > 2 \) and \( 1 ≤ s ≤ m − 1 \) then \( C \) is an LRC with local parameters \([q, 2, q − 1]\).

Notice that when \( κ = 2 \), we obtain again the locality proven in [9]. We explain next why the list of localities of Corollary 1 is the complete list of possible localities with closed local sets. In Corollary 1, the list of different localities is established by removing the restrictions that lead to codes with minimum distance 1 from the list of Theorem 3. But the list of Theorem 3 relies on consecutive iterations of Theorem 2, where Proposition 2 can be applied at each iteration guaranteeing the uniqueness of the restriction types. Therefore, Theorem 3 contains the complete list of restriction types and Corollary 1 contains the complete list of possible localities with closed local sets.

We now look at a special case of Theorem 3 and Corollary 1 when \( s = m − 1 \) and \( C \) is the Reed–Muller code \( RM(1, m − 1) \).

**Example 3.** Let \( C = S(m) − S(m − 1) \) with \( m ≥ 3 \) corresponding to the Reed–Muller code \( RM(1, m − 1) \). Let \( I_κ \), the set \( I_κ = \{ i : max\{0, s − m + κ\} ≤ i ≤ min\{s, κ − 1\} \} \). Since \( s = m − 1 \), we have \( I_κ = \{ κ − 1 \} \). Hence Theorem 3 and Corollary 1 imply that \( RM(1, m − 1) \) is an LRC with local parameters \([q^κ − 1, q^κ − q^{κ−1}, q^κ − q^{κ−2}] \) for all \( κ \in [3, m − 1] \) and the local codes are isomorphic to the Reed–Muller code \( RM(1, κ − 1) \). Furthermore, if \( q > 2 \), then \( RM(1, m − 1) \) is also an LRC with local parameters \([q, 2, q − 1]\). Moreover, since every locality is obtained by restrictions on the previous local sets, we get that the Reed–Muller codes \( RM(1, m − 1) \) are H-LRCs with \((m − 3)\)-level hierarchy over the binary field and with \((m − 2)\)-level hierarchy over \( F_q \) with \( q > 2 \).

We conclude this section by showing the optimality of \( S(m) − S(s) \) and present a table containing binary codes obtained by Construction 1 and their hierarchical localities.

**Lemma 3.** Let \( C = S(m) − S(s) \) with \( m ≥ 3 \) and \( 0 ≤ s ≤ m − 1 \). Then \( C \) is an optimal LRC for every locality described in Corollary 1 and \( C \) is also an alphabet-optimal 2-level H-LRC for every locality where the local codes are obtained by consecutive iterations of Theorem 2 on the middle codes described in Corollary 1.

The table in Figure 3 represents the binary linear codes obtained by the construction \( S_2(m) − S_2(s) \). The columns are sorted by \( s \in [0, 4] \) and the rows are sorted by \( m \in [2, 6] \). Moreover, the lines describe the locality of dimension \( m − 1 \) obtained by Theorem 2. Therefore, if \( C \) and \( C' \) are two codes in the table such that there exists a path from \( C' \) to \( C \), then \( C \) has locality \( C' \), i.e., each symbol of \( C \) is contained in a restriction isomorphic to \( C' \). Figure 3 gives also the hierarchical locality of a binary code via the paths to smaller codes. Finally, the codes in blue are the binary Reed–Muller codes \( RM(1, m − 1) \).

**References**

[1] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, “On the locality of codeword symbols,” IEEE Transactions on Information Theory, vol. 58, no. 1, pp. 6025–6034, 2012.

[2] D. Papailiopoulos and A. Dimakis, “Locally repairable codes," in International Symposium on Information Theory. IEEE, 2012, pp. 2771–2775.

[3] N. Prakash, G. M. Kamath, V. Lalitha, and P. V. Kumar, “Optimal linear codes with a local-error-correction property,” in International Symposium on Information Theory. IEEE, 2012, pp. 2776–2780.

[4] V. Cadambe and A. Mazumdar, “An upper bound on the size of locally recoverable codes,” in International Symposium on Network Coding, 2013, pp. 1–5.

[5] A. S. Rawat, D. S. Papailiopoulos, A. G. Dimakis, and S. Vishwanath, “Locality and availability in distributed storage,” IEEE Transactions on Information Theory, vol. 62, no. 8, p. 4481–4493, Feb 2016.

[6] A. S. Rawat, O. O. Koyluoglu, N. Silberstein, and S. Vishwanath, “Optimal locally repairable and secure codes for distributed storage systems,” IEEE Transactions on Information Theory, vol. 60, pp. 212–236, 2014.

[7] I. Tamo and A. Barg, “A family of optimal locally recoverable codes,” IEEE Transactions on Information Theory, vol. 60, no. 8, pp. 4661–4676, 2014.

[8] T. Westerbäck, R. Freij-Hollanti, T. Enqvist, and C. Hollanti, “On the combinatorics of locally repairable codes via matroid theory,” IEEE Transactions on Information Theory, vol. 62, pp. 5296–5315, 2016.

[9] N. Silberstein and A. Zeh, “Anticode-based locally repairable codes with high availability,” Designs, Codes and Cryptography, vol. 86, no. 2, pp. 419–445, 2018.

[10] A. Agarwal, A. Barg, S. Hu, A. Mazumdar, and I. Tamo, “Combinatorial-alphabet-dependent bounds for locally recoverable codes,” IEEE Transactions on Information Theory, vol. 64, pp. 4381–4392, 2018.

[11] M. Gurel, R. Freij-Hollanti, T. Westerbäck, and C. Hollanti, “Alphabet-dependent bounds for linear locally repairable codes based on residual codes,” arXiv preprint arXiv:1810.08510, 2018.

[12] B. Sasidharan, G. K. Agarwal, and P. V. Kumar, “Codes with hierarchical locality,” 2015 IEEE International Symposium on Information Theory (ISIT), pp. 1257–1261, 2015.

[13] S. Ballentine, A. Barg, and S. Vladut, “Codes with hierarchical locality from covering maps of curves,” arXiv preprint arXiv:1807.05473, 2018.

[14] R. Freij-Hollanti, C. Hollanti, and T. Westerbäck, “Matroid theory and storage codes: bounds and constructions,” in Network Coding and Subspace Designs. Springer, 2018, pp. 385–425.

[15] M. Gurel and C. Hollanti, “The complete hierarchical locality of the punctured simplex code,” arXiv preprint arXiv:1901.05149, 2019.

[16] P. Farrell, “Linear binary anticodes,” Electronics Letters, vol. 6, no. 13, pp. 419–421, 1970.