Algebro-geometric approach in the theory of integrable hydrodynamic type systems

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Abstract

The algebro-geometric approach for integrability of semi-Hamiltonian hydrodynamic type systems is presented. This method is significantly simplified for so-called symmetric hydrodynamic type systems. Plenty interesting and physically motivated examples are investigated.

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1 Introduction

The theory of the integrable hydrodynamic type systems

\[ u_i^t = v_j^i(u)u_j^x, \quad i, j = 1, 2, ..., N \]  

was established by S.P. Novikov, B.A. Dubrovin (see [6]) and S.P. Tsarev (see [31]). This differential-geometric approach has been developed also by E.V. Ferapontov (see, for instance, [7], [10] and other references therein), I.M. Krichever (see, for instance, [18] and other references therein), O.I. Mokhov (see, for instance, [23] and other references therein) and by the author (see, for instance, [28] and other references therein). Also, B.A. Dubrovin and I.M. Krichever (see, for instance, [5], [19] and other references therein), Yu. Kodama and J. Gibbons (see, for instance, [14], [16] and other references therein) used algebro-geometric approach in the theory of integrable hydrodynamic type systems, which are dispersionless limits of integrable dispersive equations (or in most general case the Whitham equations, i.e. obtained by the Whitham averaging method of multi-phase solutions of dispersive systems). Thus, all corresponding information like the Riemann surfaces, a quasi-momentum and a quasi-energy can be reconstructed. Then (as usual in the algebro-geometric approach) generating functions of conservation laws and commuting flows can be found automatically.

This paper is devoted to the algebro-geometric approach for hydrodynamic type systems, whose origin is unknown. For simplicity in this paper we restrict our consideration on symmetric hydrodynamic type systems, because just in this case a generating function of conservation laws in fact is given in advance. In all other cases derivation of a generating function of conservation laws is separate and complicated computational problem, which will be investigated in details in another publication devoted integrable hydrodynamic chains. The next step is a computation of the equation of the Riemann surface. The corresponding linear ODE system can be solved if it is invariant under a some Lie group symmetry. For instance, if the symmetric hydrodynamic type system is homogeneous (this is a typical formulation of physically motivated examples), then the corresponding generating function of conservation laws and the equation of the Riemann surface are homogeneous too.

Moreover, the generalized hodograph method established by S.P. Tsarev in [31] is based on a concept of the Riemann invariants. In this paper we suggest an alternative approach based on a conservative form of hydrodynamic type systems. Most physically motivated problems are given in such form.

The paper is organized in the following order. In the second section a semi-Hamiltonian (integrability) property for hydrodynamic type systems is reformulated for a conservative form. The method allowing immediately to construct generating function of conservation laws and the corresponding Riemann surface is established. In the third section we
briefly describe Tsarev’s observations improving plenty calculations. In the fourth section the chromatography system as the most interesting and very complicated example of symmetric hydrodynamic type systems is investigated. In the fifth section the generalized hodograph method adopted to a conservative form is considered in details. Several different sub-classes of hydrodynamic type systems are presented. In the sixth section homogeneous hydrodynamic type systems are considered. In such case a computation of the Riemann surface can be found in quadratures. In the seventh section integrable hydrodynamic chains as a natural generalization of the symmetric hydrodynamic type systems are discussed. In the eighth section the Hamiltonian chromatography hydrodynamic type system is considered. Corresponding hydrodynamic chain is found as well its local Hamiltonian structure. In the ninth section the linearly degenerate case is investigated. In the conclusion the main open problem is considered.

2 Symmetric hydrodynamic type systems

The hydrodynamic type system (1) written in the Riemann invariants

$$r^i_t = \mu^i(r) r^i_x, \quad i = 1, 2, ..., N$$

is integrable by the generalized hodograph method (see [31]) iff the semi-Hamiltonian condition

$$\partial_j \frac{\partial_k \mu^i}{\mu^k - \mu^i} = \partial_k \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j \neq k$$

is valid identically (here $\partial_k \equiv \partial / \partial r^k$; the semi-Hamiltonian condition also can be written in arbitrary field variables $u^k$, see [27]). Then the hydrodynamic type system (1) has infinitely many conservation laws and commuting flows

$$r^i_y = w^i(r) r^i_x,$$

parameterized by $N$ arbitrary functions of a single variable.

**Remark:** Characteristic velocities $w^i(r)$ satisfy the linear PDE system (see [31])

$$\partial_k w^i = \frac{\partial_k \mu^i}{\mu^k - \mu^i} (w^k - w^i), \quad i \neq k,$$

which cannot be solved explicitly in general case, because this is a linear system with variable coefficients. However, in this paper we are able to avoid this problem, because the generalized hodograph method can be formulated via arbitrary field variables (see the beginning of the section 6). The algebro-geometric approach is based on the concept of the Riemann surface, where all conservation laws and commuting flows can be found (see below) via the Riemann invariants as well as conservation law densities $u^k$, which appear in this framework in a natural way. The system (5) has the general solution parameterized by $N$ arbitrary functions of a single variable. The generalized hodograph method established by S.P. Tsarev (see [31]) leads to the general solution written in implicit form as an algebraic system

$$x + \mu^i t = w^i, \quad i = 1, 2, ..., N$$
for the semi-Hamiltonian hydrodynamic type system (2). In this paper several tools useful for constructing of commuting flows are suggested in the section 5.

Plenty hydrodynamic type systems are known from physical applications, which have the very special conservative form

\[ u_i^t = \partial_x \psi(u^1, u^2, ..., u^N; u^i), \tag{7} \]

where the sole function \( \psi(u^1, u^2, ..., u^N; p) \) is invariant under permutation of the first \( N \) entries (also we restrict our consideration on the case, when the function \( \psi(u; p) \) is nonlinear with respect to \( p \), see details in the last section 9). We call such hydrodynamic type systems the symmetric hydrodynamic type systems. The theory presented below can be easily extended on more general symmetric classes, for instance on

\[ u_i^t = \partial_x \psi(u^1, u^2, ..., u^N; v^1, v^2, ..., v^M; u^i), \]

\[ v_i^t = \partial_x g(u^1, u^2, ..., u^N; v^1, v^2, ..., v^M; v^i), \]

where \( N \) and \( M \) are arbitrary integers. Moreover, we shall demonstrate below (in a set of examples) that this is not necessary restriction. However, in the symmetric case this theory is very effective.

Let us introduce the matrix \( A_{ik}^j \) given by

\[ A_{ik}^j (u; p) = \left( \frac{\partial \psi}{\partial p} \right)_{p=u^i} - \frac{\partial \psi}{\partial u^i} \right) \delta_{ik}^j + \frac{\partial \psi}{\partial u^i} \right)_{p=u^k} \tag{8} \]

and formulate the phenomenological algebro-geometric approach.

**Statement 1:** If the symmetric hydrodynamic type system (7) is integrable, then this system has the generating function of conservation laws

\[ p_i = \partial_x \psi(u^1, u^2, ..., u^N; p). \tag{9} \]

If the generating function of conservation laws (9) is consistent with the hydrodynamic type system (7), then

\[ \frac{\partial p}{\partial u^i} = B_{ik}^j \frac{\partial \psi}{\partial u^k}, \tag{10} \]

where the matrix \( B_{ik}^j (u; p) \) is an inverse matrix to the matrix \( A_{ik}^j \) (see (8)). This is \( N \) ODE’s of the first order for every fixed index \( i \), where any of them can be written in the form \( dy/dx = f(x, y) \).

Let us introduce the function \( \lambda(u^1, u^2, ..., u^N; p) \) determined by \( N \) linear PDE’s of the first order (see (10))

\[ A_{ik}^j \frac{\partial \lambda}{\partial u^k} + \frac{\partial \psi}{\partial u^i} \frac{\partial \lambda}{\partial p} = 0. \tag{11} \]

**Definition 1:** The function \( \lambda(u; p) \) satisfying (11) is said to be the equation of the Riemann surface.

If the linear PDE system (11) has an integration factor then the equation of the Riemann surface can be found in quadratures

\[ d\lambda = \frac{\partial \lambda}{\partial p} \left[ dp - B_{ik}^j \frac{\partial \psi}{\partial u^k} du^k \right]. \]
If, for instance, the hydrodynamic type system (7) is homogeneous (see examples below), then functions \( \psi(u; p) \) and \( \lambda(u; p) \) are homogeneous too. Then the integration factor can be found by using the Euler theorem

\[
\lambda = p \frac{\partial \lambda}{\partial p} + u^k \frac{\partial \lambda}{\partial u^k}.
\]

**Statement 2:** A deformation of the Riemann surface determined by the equation \( \lambda(u; p) \) satisfies the Gibbons equation

\[
\lambda_t - \frac{\partial \psi}{\partial p} \lambda_x = \frac{\partial \lambda}{\partial p} [p_t - \partial_x \psi(u; p)].
\]

The Gibbons equation (first introduced in [11]) has three distinguish features:

1. if one fixes \( \lambda = \text{const} \) (free parameter), then one obtains (9),
2. if one fixes \( p = \text{const} \) (free parameter), then one obtains the kinetic equation (a collisionless Vlasov equation) written in so-called Lax form

\[
\lambda_t = \{ \lambda, \hat{H} \} = \frac{\partial \lambda}{\partial x} \frac{\partial \hat{H}}{\partial p} - \frac{\partial \lambda}{\partial p} \frac{\partial \hat{H}}{\partial x},
\]

where \( \hat{H} = \psi(u; p) \).

3. if one choose coordinates, which are the Riemann invariants \( r^i \) \((i = 1, 2, ..., N)\) determined by the condition \( \partial \lambda / \partial p = 0 \) (see (11)), then the corresponding hydrodynamic type system (7) can be written in the diagonal form (2)

\[
r^i_t = \frac{\partial \psi}{\partial p} |_{p=p^i} r^i_x, \quad i = 1, 2, ..., N,
\]

where the corresponding values \( p^i \) can be expressed via these Riemann invariants \( r^k \). In this algebro-geometric construction the Riemann invariants are the branch points \( r^i = \lambda |_{\partial \lambda / \partial p = 0} \) of the Riemann surface (exactly as it is in the Whitham theory, see [5] and [19]).

**Remark:** The characteristic velocities \( \mu^k \) of hydrodynamic type system (1) can be found from algebraic system

\[
\det |v^i_k(u) - \mu \delta^i_k| = 0.
\]

All of them must be distinct in agreement with Tsarev’s assumptions (see [31]). Thus, if the hydrodynamic type system (7) is integrable by the generalized hodograph method, then the values \( p^i \) are given by (see (13))

\[
\mu^i(u) \equiv \frac{\partial \psi}{\partial p} |_{p=p^i},
\]

where characteristic velocities are determined from (14)

\[
\det A^i_k(u; p) = 0.
\]
Suppose the hydrodynamic type system (7) is semi-Hamiltonian. Then such system must have \( N \) series of conservation laws, which can be obtained by the substitution of the formal series
\[
p^{(k)} = u^k + \lambda u^k(u) + \lambda^2 w^k(u) + \ldots
\]
in (9). The compatibility conditions of the first \( N \) extra conservation laws
\[
\partial_t v^i(u) = \partial_x \left[ v^i(u) \frac{\partial \psi^i}{\partial p} \bigg|_{p=u^i} \right]
\]
with the hydrodynamic type system (7) are equivalent the semi-Hamiltonian property.

**Main statement:** The symmetric hydrodynamic type system (7) is semi-Hamiltonian iff the compatibility condition
\[
\partial_i (\partial_k p) = \partial_k (\partial_i p)
\]
is fulfilled.

**Comment:** Computation of this compatibility condition can be made in the coordinates \( u^k \) (see (7)) or in the Riemann invariants (see (13) and details in the Conclusion). In both cases the nonlinear PDE system in involution coincides with integrability criterion for hydrodynamic type systems following from existence of \( N \) conservation laws and vanishing Haantjes tensor (see [15], [27]).

## 3 Tsarev’s observations

Let us consider the dispersionless limit of the vector NLS (see [34])
\[
u^i_t = \partial_x \left( \frac{(u^i)^2}{2} + \sum \eta^k \right), \quad \eta^i_t = \partial_x (u^i \eta^i), \quad i = 1, 2, \ldots, N.
\]
The eigenvalue–eigenfunction problem (cf. (16)) is
\[
\begin{vmatrix}
(u^i - \mu) \delta_{ik} & 1 \\
\eta^i \delta_{ik} & (u^i - \mu) \delta_{ik}
\end{vmatrix}
\begin{vmatrix}
q^i \\
s^i
\end{vmatrix} = 0,
\]
Thus,
\[
q^i = \frac{1}{\mu - u^i} \sum s^k, \quad s^i = \frac{\eta^i}{(\mu - u^i)^2} \sum s^k.
\]
The **first Tsarev observation** [32] is that the sum of the last \( N \) equations yields an expression determining characteristic velocities \( \mu^k \) (see (14) and (16)) via very compact formula
\[
1 = \sum \frac{\eta^n}{(\mu - u^n)^2}.
\]
The **second Tsarev observation** [32] is that the above expression can be integrated once with respect to \( \mu \)
\[
\lambda = \mu + \sum \frac{\eta^n}{\mu - u^n}
\]
**Comment:** This equation of the Riemann surface can be obtained directly from the spectral problem for the vector NLS (see [34]). The characteristic velocities \( \mu^k \) are determined by the condition \( \partial \lambda / \partial \mu = 0 \), where the Riemann invariants \( r^i = \lambda \big|_{\partial \lambda / \partial \mu = 0} \equiv \ldots \)
\( \lambda \vert_{\mu=\mu^i} \) are branch points of the Riemann surface (see [11]). A deformation of the Riemann surface is described by the Gibbons equation (see [11] again)

\[
\lambda_t - \mu \lambda_x = \frac{\partial \lambda}{\partial \mu} \left[ \mu_t - \partial_x \left( \frac{\mu^2}{2} + \sum \eta^n \right) \right],
\]

which connects (18) with (20).

The third Tsarev observation [32] is that the flat diagonal metric of the hydrodynamic type system (18) written in the Riemann invariants (2)

\[ r_i^i = \mu^i(r) r_x^i, \quad i = 1, 2, ..., 2N \]

is given by (see also [31])

\[
g_{ii} = \operatorname{res}_{\lambda=r_i} \left( \frac{\partial \lambda}{\partial \mu} \right)^2 d\mu.
\]

Indeed, since the Hamiltonian structure of the dispersionless limit of the vector NLS is

\[ u_i^i = \partial_x \frac{\partial h}{\partial \eta_i}, \quad \eta_i^i = \partial_x \frac{\partial h}{\partial u_i}, \]

the diagonal metrics (in Riemann invariants)

\[
g_{ii} = \sum_{k=1}^{N} \frac{\partial r_i^i}{\partial u_k} \frac{\partial r_i^i}{\partial \eta_k}
\]

is given by

\[
g_{ii} = 2 \sum_{k=1}^{N} \frac{\eta_k}{(\mu^i - u^k)^2},
\]

where (see (20) and cf. (19))

\[
\frac{\partial r_i^i}{\partial u_k} = \frac{\eta_k}{(\mu^i - u^k)^2}; \quad \frac{\partial r_i^i}{\partial \eta_k} = \frac{1}{\mu^i - u^k}.
\]

Thus, \( g_{ii} = \partial^2 \lambda / \partial \mu^2 \vert_{\mu=\mu^i} \) in agreement with (22).

### 4 Generalized chromatography

The chromatography process (see, for instance, [10]) is described by the hydrodynamic type system

\[
u_i^i = \partial_x \frac{(u^i)\alpha}{[1 + \sum \gamma_k (u^k)^\beta]^{\varepsilon}}, \quad i = 1, 2, ..., N
\]

where \( \alpha, \beta, \varepsilon \) and \( \gamma_i \) are constants. All results presented in this section generalize results from [10] (see formulas 1, 3, 4, 5). This hydrodynamic type system has the obvious couple of conservation laws

\[
\partial_t \left[ \sum \gamma_k (u^k)^{\beta - \alpha + 1} \right] = (\beta - \alpha + 1) \partial_x \left[ \frac{\alpha - \beta \varepsilon}{\beta (1 - \varepsilon)} \Delta^{1 - \varepsilon} - \Delta^{-\varepsilon} \right],
\]
\[ \partial_t \Delta^{\varepsilon-\beta \varepsilon+\alpha} = \partial_x \left[ \frac{\beta(\alpha+\varepsilon-\beta \varepsilon)}{\alpha+\beta-1} \Delta^{1-\beta \varepsilon-\alpha} \sum \gamma_n(u^n)^{\alpha+\beta-1} \right], \]

where \( \Delta = 1 + \sum \gamma_k(u^k)^\beta. \)

If \( \alpha = \beta - 1, \) then the hydrodynamic type system (23) is Hamiltonian (see [6], [28])

\[ u^i_t = \frac{1}{\gamma_i} \partial_x \frac{\partial h}{\partial u^i}, \quad (24) \]

where the momentum density is \( \Sigma \gamma_k(u^k)^2 \) and the Hamiltonian density is \( h = \Delta^{1-\varepsilon}. \)

Suppose the above hydrodynamic type system is integrable for some values of constants \( \alpha, \beta, \varepsilon \) and \( \gamma_i. \) Then the generating function of conservation laws given by

\[ p_t = \partial_x \frac{p^\alpha}{\Delta^\varepsilon} \]

should exist. It is easy to check that the compatibility conditions \( \partial_i(\partial_k p) = \partial_k(\partial_i p) \) are valid iff \( \alpha = \beta \varepsilon, \) where (see (10))

\[ \frac{\partial p}{\partial u^i} = \frac{\gamma_i(u^i)^{\beta-1} p^\alpha}{(u^i)^{\alpha-1} - p^\beta \varepsilon - 1} \left[ \sum \frac{\gamma_n(u^n)^{\alpha+\beta-1}}{(u^n)^{\alpha-1} - p^\beta \varepsilon - 1} - \frac{\alpha \Delta - 1}{\beta \varepsilon} \right]. \quad (25) \]

All first derivatives (see (11))

\[ \frac{\partial \lambda}{\partial u^i} = \varphi(u, p) p^{\beta \varepsilon} \frac{\gamma_i(u^i)^{\beta-1}}{(u^i)^{\beta \varepsilon - 1} - p^{\beta \varepsilon - 1}}, \quad \frac{\partial \lambda}{\partial p} = \varphi(u, p) \left[ 1 - p^{\beta \varepsilon - 1} \sum \frac{\gamma_n(u^n)^\beta}{(u^n)^{\beta \varepsilon - 1} - p^{\beta \varepsilon - 1}} \right] \]

are determined up to integration factor \( \varphi(u, p), \) which is not found yet. Then the equation of the Riemann surface \( \lambda(u, p) \) can be found in quadratures

\[ d\lambda = \varphi(u, p) \left( dp + \frac{q^{\beta \varepsilon - 1}}{\beta \varepsilon - 1} \sum \frac{\gamma_n(u^n)^{\beta}}{u^n - 1} dw^n \right), \]

if the integration factor is \( \varphi(p) = p^{1-\beta} \) (here we use the substitutions \( p = q^{1-\beta} \) and \( u^n = (w^n q)^{\frac{1}{\beta \varepsilon - 1}} \)). Then, the integrable hydrodynamic type system (23)

\[ u^i_t = \frac{1}{\gamma_i} \partial_x \frac{(u^i)^{\beta \varepsilon}}{1 + \sum \gamma_k(u^k)^\beta}, \quad i = 1, 2, ..., N \quad (26) \]

is connected with the equation of the Riemann surface

\[ \lambda = \frac{p^{-\beta}}{\beta} + \frac{1}{\beta \varepsilon - 1} \sum \gamma_k(w^k)^\delta F(1, \delta, \delta + 1, w^k), \]

where \( _2F_1(a, b, c, z) \) is a hyper-geometric function and \( \delta = \beta/(\beta \varepsilon - 1). \) If \( \delta = m/n, \) where \( m \) and \( n \) are integers, then the equation of the Riemann surface \( \lambda(u, p) \) can be found in elementary functions.
Remark: If $\beta \to \infty$, then the hydrodynamic type system (26) reduces to

$$u^i_t = \partial_x \left[ e^{u^i} \left( 1 + \sum \gamma_k e^{u^k} \right) \right], \quad i = 1, 2, \ldots, N. \tag{27}$$

The corresponding equation of the Riemann surface is

$$\lambda = e^{-p} + \frac{1}{\varepsilon} \sum \gamma_n (w^n)^{1/\varepsilon} F(1, \frac{1}{\varepsilon}, 1, w^n),$$

where $p = \ln q$ and $u^n = \ln(w^n q)$. Moreover, if $\varepsilon = 1$, the above hydrodynamic system has the local Hamiltonian structure (24) with the Hamiltonian density $h = \ln \Delta$. The above equation of the Riemann surface reduces to

$$\lambda = e^{-p} - \sum \gamma_n \ln(e^{u^n - p} - 1). \tag{28}$$

Remark: If $\varepsilon \to 0$, then the hydrodynamic type system (26) reduces to

$$u^i_t = \partial_x \ln \left( \frac{(u^i)^{\beta}}{1 + \sum (u^k)^{\beta}} \right), \quad i = 1, 2, \ldots, N,$$

where the constants $\gamma_k$ are removed by appropriate scaling of the field variables $u^k$. The corresponding equation of the Riemann surface is

$$\lambda = \frac{p - \beta}{\beta} - \sum (w^n)^{\beta} F(1, -\beta, 1 - \beta, w^n),$$

where $u^k = w^k p$. If $\beta = 1$, the corresponding hydrodynamic type system (26) has the local Hamiltonian structure (24) with the Hamiltonian density

$$h = \sum u^m (\ln u^m - 1) - (1 + \sum u^m) [\ln(1 + \sum u^n) - 1].$$

5 The generalized hodograph method

If the hydrodynamic type system (1) is semi-Hamiltonian, then the general solution parameterized by $N$ arbitrary functions of a single variable is given in an implicit form by the algebraic system (cf. (6); see [31])

$$x \delta^i_k + t u^i_k(u) = w^i_k(u), \tag{29}$$

where $w^i_k(u)$ are characteristic velocities of an arbitrary commuting flow.

However, in this paper above, we present the approach producing the generating function of conservation laws only. Thus, we need to extend this mechanism to produce the generating function of conservation laws and commuting flows simultaneously. In this paper we restrict our consideration on three sub-cases connected with the Egorov conjugate curvilinear coordinate nets (see [28]), with the orthogonal coordinate nets (see [31]) and with so-called “mirrored” conjugate nets (see below). The general case will be considered elsewhere.
Nevertheless, the first step is a description of $N$ series of conservation laws (see (17)). They can be obtained by expansion in the Bürmann–Lagrange series (see, for instance, [22]) at the vicinity of each singular point.

**Theorem 1 [22]:** The analytic function

$$y = y_1(x-x_0) + y_2(x-x_0)^2 + y_3(x-x_0)^3 + ...$$

can be inverted ($y(x) \rightarrow x(y)$) as the Bürmann–Lagrange series

$$x = x_0 + x_1 y + x_2 y^2 + x_3 y^3 + ..., $$

whose coefficients are

$$x_n = \frac{1}{n!} \lim_{x \rightarrow x_0} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{x-x_0}{y}\right)^n, \hspace{1cm} n = 1, 2, ... \tag{30}$$

For example, the so-called “waterbag” hydrodynamic type system (see [14], [16])

$$a_i^t = \partial_x \left(\frac{(a_i^t)^2}{2} + \sum \varepsilon_k a^k\right) \tag{31}$$

is connected with the equation of the Riemann surface

$$\lambda = p - \sum \varepsilon_k \ln(p - a^k). \tag{32}$$

The main (so-called “Kruskal”) series of conservation law densities can be obtain by substitution of the Taylor series

$$p = \lambda - \frac{H_0}{\lambda} - \frac{H_1}{\lambda^2} - \frac{H_2}{\lambda^3} - ... \tag{33}$$

into the above expression if $\Sigma \varepsilon_k = 0$. If $\Sigma \varepsilon_k \neq 0$, then at first, the above equation of the Riemann surface must be replaced on

$$\lambda - \sum \varepsilon_k \ln \lambda = p - \sum \varepsilon_k \ln(p - a^k), \tag{34}$$

because the Gibbons equation is invariant under scaling $\lambda \rightarrow \tilde{\lambda}(\lambda)$. In both cases $H_k$ are polynomials with respect to field variables $a^n$.

Also, the “waterbag” hydrodynamic type system has $N$ infinite series of conservation laws. At first let us rewrite the equation of the Riemann surface in the form

$$\lambda = (p-a^i)e^{-p/\varepsilon_i} \prod_{k \neq i} (p-a^k)^{\varepsilon_k/\varepsilon_i}$$

for any fixed index $i$. Then the infinite series of conservation laws (17)

$$p^{(i)} = a^i + h_1^{(i)}(a)\lambda + h_2^{(i)}(a)\lambda^2 + h_3^{(i)}(a)\lambda^3 + ... \tag{35}$$
can be obtained with the aid of Bürmann–Lagrange series (see [22]), which coefficients are determined by (see (30))

\[ h_n^{(i)} = \frac{1}{n!} \frac{d^{n-1}}{d(a^i)^{n-1}} \left( e^{a^i / \varepsilon_i} \prod_{k \neq i} (a^i - a^k)^{-n \varepsilon_k / \varepsilon_i} \right), \quad n = 1, 2, \ldots \]

Thus, the first conservation laws are

\[ h_1^{(i)} = e^{a^i / \varepsilon_i} \prod_{k \neq i} (a^i - a^k)^{-\varepsilon_k / \varepsilon_i}, \quad h_2^{(i)} = \frac{e^{2a^i / \varepsilon_i}}{\varepsilon_i} \left( 1 - \sum_{n \neq i} \frac{\varepsilon_n}{a^i - a^n} \right) \prod_{k \neq i} (a^i - a^k)^{-2 \varepsilon_k / \varepsilon_i}, \ldots \]

### 5.1 Quasi-symmetric form

The dispersionless limit of the vector NLS (18) is a degeneration of the “waterbag” hydrodynamic type system (31). A substitution of the expansion (see, for instance, [2])

\[ \tilde{u}^{(k)} = u^k + \eta^k / \varepsilon^k + \ldots \text{ in (32)} \]

\[ \lambda = \mu - \sum_{k=1}^{N} \varepsilon_k \ln \frac{\mu - \tilde{u}^k}{\mu - u^k} \]

yields (20) if \( \varepsilon^k \to \infty \).

Let us consider again the generating function of conservation laws (see (21))

\[ \mu_t = \partial_x \left( \frac{\mu^2}{2} + A^0 \right), \quad (36) \]

for the Benney hydrodynamic chain (see [1])

\[ A_x^k = A_x^{k+1} + kA_x^{k-1}A_x^0, \quad k = 0, 1, 2, \ldots \quad (37) \]

and substitute \( N \) Taylor series (35)

\[ \mu^{(i)} = a^i + \lambda b^i + \lambda^2 c^i + \ldots \quad (38) \]

1. Suppose the function \( A^0 \) depends on \( N \) field variables \( a^k \) only. Then corresponding hydrodynamic type system is

\[ a^i_t = \partial_x \left( \frac{(a^i)^2}{2} + A^0(a) \right). \]

However, this hydrodynamic type system is integrable iff the function \( A^0(a) \) satisfies some nonlinear PDE system

\[ (a^i - a^k) \partial_{ik} A^0 = \partial_k A^0 \partial_i \left( \sum \partial_n A^0 \right) - \partial_i A^0 \partial_k \left( \sum \partial_n A^0 \right), \quad i \neq k, \]

\[ (a^i - a^k) \frac{\partial_{ik} A^0}{\partial_i A^0 \partial_k A^0} + (a^k - a^j) \frac{\partial_{jk} A^0}{\partial_j A^0 \partial_k A^0} + (a^j - a^i) \frac{\partial_{ij} A^0}{\partial_i A^0 \partial_j A^0} = 0, \quad i \neq j \neq k, \]
which is a consequence of the compatibility conditions \( \partial_i (\partial_k \mu) = \partial_k (\partial_i \mu) \), where

\[
\partial_i \mu = \frac{\partial_i A^0}{\mu - a^i} \left( \sum \frac{\partial_n A^0}{\mu - a^n} - 1 \right)^{-1}.
\]

Several such particular choices are described above.

2. Suppose the function \( A^0 \) depends on \( N \) field variables \( a^k \) and \( M \) field variables \( b^k \) (where \( M \) must be not exceed \( N \)) only. Then corresponding hydrodynamic type system is

\[
\begin{align*}
a^i_t &= \partial_x \left( \frac{(a^i)^2}{2} + A^0(a, b) \right), \quad i = 1, 2, ..., N, \\
b^j_t &= \partial_x (a^j b^j), \quad j = 1, 2, ..., M,
\end{align*}
\]

Simplest such example is (18). This procedure can be extended on other auxiliary field variables from (38). For instance, the third such sub-case is

\[
\begin{align*}
a^i_t &= \partial_x \left( \frac{(a^i)^2}{2} + A^0(a, b, c) \right), \quad i = 1, 2, ..., N, \\
b^j_t &= \partial_x (a^j b^j), \quad j = 1, 2, ..., M, \\
c^k_t &= \partial_x \left[ a^k c^k + \frac{1}{2} (b^k)^2 \right], \quad k = 1, 2, ..., K,
\end{align*}
\]

where \( K \leq M \leq N \).

3. Suppose the function \( A^1 \) (see (37)) is a function of the field variables \( a^k \) only. Then corresponding hydrodynamic type system is

\[
A^0_t = \partial_x A^1(a), \quad a^i_t = \partial_x \left( \frac{(a^i)^2}{2} + A^0 \right), \quad i = 1, 2, ..., N. \tag{39}
\]

Suppose the function \( A^2 \) is a function of the field variables \( a^k \) only. Then corresponding hydrodynamic type system is

\[
A^0_t = \partial_x A^1, \quad A^1_t = \partial_x \left( A^2(a) + \frac{(A^0)^2}{2} \right), \quad a^i_t = \partial_x \left( \frac{(a^i)^2}{2} + A^0 \right), \quad i = 1, 2, ..., N.
\]

Thus, in general case one can consider \( M \) first moments \( A^k \) and \( K \) first sets \( a^{i_k}, b^{i_k}, c^{i_k}, \) ... (where the index \( i_k \) run all values from 1 up to \( N_k, k = 1, 2, ..., K \)) as field variables of corresponding hydrodynamic type systems. In such case just the latest moment \( A^M \) depends on sets \( a^{i_1}, b^{i_2}, c^{i_3}, \) ...

Nevertheless, the theory established above is still working for these hydrodynamic type systems, because all of them have the same generating function of conservation laws (36).

**Example:** The dispersionless limit of the vector Yajima–Oikawa system (see [28]) is the hydrodynamic type system (cf. (18) and (39))

\[
A^0_t = \partial_x \sum \eta^n, \quad u^k_t = \partial_x \left( \frac{(u^k)^2}{2} + A^0 \right), \quad \eta^k_t = \partial_x (u^k \eta^k), \quad k = 1, 2, ..., N. \tag{40}
\]
The eigenvalue–eigenfunction problem is
\[
\begin{vmatrix}
-\mu & 0 & 1 \\
1 & (u^i - \mu)\delta_{ik} & 0 \\
0 & \eta^i\delta_{ik} & (u^i - \mu)\delta_{ik}
\end{vmatrix}
\begin{vmatrix}
r \\
q^i \\
s^i
\end{vmatrix} = 0.
\]
Thus,
\[
q^i = \frac{r}{\mu - u^i}, \quad s^i = \frac{\eta^i}{(\mu - u^i)^2}r,
\]
where \(2N + 1\) eigenvalues are solutions of the discriminant equation
\[
\mu = \sum \frac{\eta^k}{(\mu - u^k)^2}.
\]
In accordance with the Tsarev observations, let us integrate this expression once. Indeed,
\[
\lambda = \frac{\mu^2}{2} + A^0 + \sum \frac{\eta^k}{\mu - u^k}
\]
is the equation of the Riemann surface connected with the hydrodynamic type system (40). The factorized form of this equation
\[
\lambda = \frac{1}{2} \left( \mu - \sum u^k + \sum a^k \right) \frac{\prod_{n=1}^{N+1} (\mu - a^n)}{\prod_{k=1}^{N} (\mu - u^k)}
\] (41)
yields \(2N + 1\) independent series of conservation laws
\[
\mu^{(k)} = u^k + \lambda \eta^k(u, a) + \lambda w^k(u, a) + ..., \quad k = 1, 2, ..., N,
\]
\[
\mu^{(n)} = a^n + \lambda b^n(u, a) + \lambda c^n(u, a) + ..., \quad n = 1, 2, ..., N + 1,
\]
whose coefficients can be found with the aid of the Bürmann–Lagrange expansion (see (30)).

**Remark:** While the conservation law densities \(\eta^k\) are coefficients of the first \(N\) series, the function \(A^0\) is a coefficient of the Kruskal series for (41) at infinity \(\lambda \to \infty, \mu \to \infty\)
\[
\lambda = \frac{\mu^2}{2} + A^0 + \frac{A^1}{\mu} + ..., \quad \mu \to \infty
\]
where \(A^1 \equiv \Sigma \eta^n\) (see (39) and (40)).

### 5.2 Hamiltonian formalism

The Hamiltonian formalism of hydrodynamic type systems was established in [6] for a local case, and developed later for a nonlocal case in [7], [9]. The generating function of
conservation laws for the symmetric hydrodynamic type systems (7) is given by (9). Suppose for simplicity without lost of generality that the field variables \(u^k\) are flat coordinates. Then (7) can be written in the symmetric Hamiltonian form

\[
\frac{d u_i}{d t} = \partial_x \left( \varepsilon_i \frac{\partial h}{\partial u_i} + \gamma_i \sum_{k \neq i} \gamma_k \frac{\partial h}{\partial u_k} \right),
\]

where the Hamiltonian density \(h\) is a symmetric expression under an index permutation (\(\varepsilon_i\) and \(\gamma_i\) are some constants). Thus, the generating function of commuting flows is given by

\[
\frac{d u_i}{d \tau} = \partial_x \left( \varepsilon_i \frac{\partial \rho}{\partial u_i} + \gamma_i \sum_{k \neq i} \gamma_k \frac{\partial \rho}{\partial u_k} \right).
\]

Then infinitely many particular solutions are given by the generalized hodograph method (29)

\[
x \delta^i_k + \frac{d}{d u^k} \left( \varepsilon_i \frac{\partial h}{\partial u^i} + \gamma_i \sum_{m \neq i} \gamma_m \frac{\partial h}{\partial u^m} \right) = \sum_{n=1}^{\infty} \sum_{s=1}^{N} \sigma_{sn} \frac{\partial}{\partial u^k} \left( \varepsilon_i \frac{\partial h^{(s)}_n}{\partial u^i} + \gamma_i \sum_{m \neq i} \gamma_m \frac{\partial h^{(s)}_n}{\partial u^m} \right),
\]

where \(\sigma_{sn}\) are arbitrary constants.

**Example:** The Hamiltonian exponential chromatography (27)

\[
u_i = \partial_x \frac{e^{u_i}}{1 + \sum \gamma_k e^{u_k}}, \quad i = 1, 2, ..., N
\]

is connected with the equation of the Riemann surface (28) by the Gibbons equation

\[
\lambda_t - \frac{\Delta}{\lambda_x} = \partial \lambda \frac{\partial p}{\partial x} \left[ p_t - \partial_x e^p \right].
\]

The Kruskal series of conservation law densities can be found by the application of the Bürmann–Lagrange series (30) at the vicinity \(\lambda \to 0, q \to 0\), where \(q = \exp(-p)\). In this case the coefficients \(q_n\) of the inverse series

\[
q = q_1 \lambda + q_2 \lambda^2 + q_3 \lambda^3 + q_4 \lambda^4 + ...
\]

are determined by

\[
q_n = \frac{1}{n!} \lim_{q \to 0} \frac{d^{n-1}}{dq^{n-1}} \left( \frac{q}{q - \sum \gamma_k \ln(1 - qe^{u^k})} \right)^n, \quad n = 1, 2, ...
\]

Then the Kruskal conservation law densities can be found from \(p = -\ln q\)

\[
\tilde{p} = p + \ln \lambda = -\ln[1 + q_2 \lambda + q_3 \lambda^2 + q_4 \lambda^3 + ...].
\]

For instance, \(p_1 = \ln \Delta, p_2 = \Delta^{-3} \sum \gamma_n e^{2u_n} \).

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The Kruskal series can be extended in the opposite direction \( \lambda \to \infty, q \to \infty \). At first, the equation of the Riemann surface (28) must be replaced by (cf. (34))

\[
\lambda - \sum \gamma_k \ln \lambda = s - \sum \gamma_k \ln(s + \sum \gamma_m u^m - e^{-u^k}),
\]

where \( s = \exp(-p) - \Sigma \gamma_m u^m \). Then the Kruskal conservation law densities \( p_{-k} \) can be found exactly as for the Benney hydrodynamic chain (37) (see (34)) by substitution (cf. (33))

\[
s = \lambda + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^3} + ... \tag{42}
\]
in the above formula. Then \( p_{-k} \) are coefficients of the series

\[
\tilde{\phi} = p + \ln \lambda = \ln \left( 1 + \frac{1}{\lambda} \sum \gamma_k u^m + \frac{C_1}{\lambda^2} + \frac{C_2}{\lambda^3} + \frac{C_3}{\lambda^4} + ... \right). \tag{42}
\]

For instance, \( p_{-1} = \Sigma \gamma_m u^m, p_{-2} = \Sigma \gamma_m e^{-u^m} + (\Sigma \gamma_m u^m)^2/2 \).

Any semi-Hamiltonian hydrodynamic type system has \( N \) series of conservation laws. The equation of the Riemann surface (28) can be written in the form

\[
\lambda = (e^{-p} - e^{-u^i}) \exp \left( u^i - \frac{e^{-p}}{\gamma_i} + \sum_{k \neq i} \gamma_k \ln(e^{u^k} - p - 1) \right), \quad p \to u^i.
\]

Then the coefficients \( q_k \) of the Bürmann–Lagrange series are given by (30)

\[
q_n = \lim_{n!Q \to \exp(-u^i)} \frac{d^{n-1}}{dq^{n-1}} \left( \frac{q - e^{-u^i}}{\lambda(q)} \right)^n, \quad n = 1, 2, ...
\]

where \( q = \exp(-p) \). Finally, \( N \) series of conservation law densities \( p^{(i)} \) can be found from the series

\[
\tilde{p}^{(i)} = p^{(i)} + \ln \lambda = - \ln[q_1 + q_2 \lambda + q_3 \lambda^2 + q_4 \lambda^3 + ...].
\]

For instance, the first \( N \) nontrivial conservation law densities are

\[
p^{(i)} = e^{-u^i} - \sum_{k \neq i} \gamma_k \ln(e^{u^k} - u^i - 1). \tag{42}
\]

Commuting flows of this Hamiltonian chromatography system are

\[
u^{i_k}_{\cdot} = \frac{1}{\gamma_i} \partial_{x} \frac{\partial h_k}{\partial u^i}, \quad k = 0, \pm 1, \pm 2, ...,
\]

\[
u^{i_{k,n}}_{\cdot} = \frac{1}{\gamma_i} \partial_{x} \frac{\partial h_{k,n}}{\partial u^i}, \quad k = 1, 2, ..., \quad n = 1, 2, ..., N,
\]

where \( h_k \) are Kruskal conservation law densities and \( h_{k,n} \) are \( N \) series of conservation law densities. The first \( N \) commuting flows

\[
u^{i}_{\cdot k} = \frac{1}{\gamma_k} \partial_{x} \left( \sum_{n \neq i} \gamma_n e^{u^n} - e^{u^k} - e^{-u^k} \right), \quad u^{i}_{\cdot k} = \partial_{x} \frac{e^{u^i}}{e^{u^k} - e^{u^i}}, \quad i \neq k
\]

are given by (42).
5.3  *Mirrored* curvilinear conjugate coordinate systems

If the semi-Hamiltonian property (3) is valid, then the diagonal metric \( g_{ii} \) can be introduced by

\[
\frac{\partial_{k} \mu^{i}}{\mu^{k} - \mu^{i}} = \partial_{k} \ln H_{i}, \quad i \neq k.
\]

(43)

Following G. Darboux (see [4]), let us introduce the rotation coefficients of the conjugate curvilinear coordinate nets

\[
\beta_{ik} = \frac{\partial_{i} H_{k}}{H_{i}}, \quad i \neq k.
\]

(44)

Then any solution \( \tilde{H}_{k} \) of the linear problem (see [31])

\[
\partial_{i} \tilde{H}_{k} = \beta_{ik} \tilde{H}_{i}, \quad i \neq k
\]

(45)
determines commuting flow to (2), where characteristic velocities \( w^{i} \) are connected with \( \tilde{H}_{i} \) by the Combescure transformation \( w^{i} = \tilde{H}_{i}/H_{i} \). Any solution of the conjugate linear problem

\[
\partial_{i} \psi_{k} = \beta_{ki} \psi_{i}, \quad i \neq k
\]

(46)
determines conservation law density

\[
\partial_{i} a = \psi_{i} H_{i}.
\]

(47)

**Definition 2 [28]:** The conjugate curvilinear coordinate net determined by symmetric rotation coefficients \( \beta_{ik} = \beta_{ki} \) is called the Egorov conjugate curvilinear coordinate net.

**Theorem 2 [28]:** If the integrable hydrodynamic type system (2) has the couple of conservation laws

\[
a_{t} = b_{x}, \quad b_{t} = c_{x},
\]

(48)
then corresponding conjugate curvilinear coordinate net is Egorov.

**Remark [28]:** This couple is unique for each given Lame coefficients \( H_{k} \).

**Corollary [28]:** Any commuting flow (4) has similar couple of conservation laws (48)

\[
a_{y} = h_{x}, \quad h_{y} = f_{x},
\]

where \( a \) is the unique potential of the Egorov metric for all commuting flows.

**Definition 3:** Two conjugate curvilinear coordinate nets determined by the rotation coefficients \( \beta_{ik} \) and \( \bar{\beta}_{ik} = \beta_{ki} \) is called *mirrored* conjugate curvilinear coordinate nets.

**Theorem 3:** If one integrable hydrodynamic type system (2) has the conservation law

\[
a_{t} = b_{x}
\]

(49)
such that another integrable hydrodynamic type system

\[
r_{y}^{i} = \tilde{\mu}^{i}(r)r_{z}^{i}
\]

(50)
has the couple of conservation laws

\[
a_{y} = c_{z}, \quad b_{y} = B_{z},
\]

(51)
then corresponding conjugate curvilinear coordinate nets are mirrored.

**Proof:** The characteristic velocities $v^i$ of the hydrodynamic type system (2) are (see (47))

$$\mu^i = \frac{H^{(1)}_i}{H_i} = \frac{\psi_i H^{(1)}_i}{\psi_i H_i} = \frac{\partial_i b}{\partial_i a},$$

where $H^{(1)}_i$ is a solution of the linear system (45). Let us consider such hydrodynamic type system (50), whose characteristic velocities $\tilde{\mu}^i$ are (see (47))

$$\tilde{\mu}^i = \frac{\psi^{(1)}_i}{\psi_i} = \frac{H_i \psi^{(1)}_i}{H_i \psi_i} = \frac{\partial_i c}{\partial_i a} = \frac{H^{(1)}_i \psi^{(1)}_i}{H^{(1)}_i \psi_i} = \frac{\partial_i B}{\partial_i b},$$

where $\psi^{(1)}_i$ is a solution of the conjugate linear system (46). The rotation coefficients can be found in two steps (43) and (44). Indeed, $\bar{\beta}_{ik} = \beta_{ki}$.

**Remark:** The construction described above is symmetric. Thus, the second conservation law of the first hydrodynamic type system (2)

$$c_t = C_x$$

is given by quadratures

$$dc = \sum \psi^{(1)}_i H_i dr^i, \quad dC = \sum \psi^{(1)}_i H^{(1)}_i dr^i.$$

**Corollary:** Any conservation law

$$P_y = Q_z$$

of the hydrodynamic type system (50) determines the corresponding commuting flow in the conservative form

$$a_\tau = P_x$$

or in the Riemann invariants (see (4))

$$r_i^\tau = \frac{H^{(2)}_i}{H_i} r^i_x$$

of the hydrodynamic type system (2), where $H^{(2)}_i$ is some solution of the linear system (45), $\partial_i P = H^{(2)}_i \psi_i$ and $\partial_i Q = H^{(2)}_i \psi^{(1)}_i$.

Thus, if two symmetric hydrodynamic type systems are related by the above link, then the generating function of the second hydrodynamic type system (50) determines the generating function of commuting flows for the first hydrodynamic type system (2).

### 5.4 Chromatography system

The integrable chromatography system (26) has the couple of conservation laws

$$\partial_t \left[ \sum \gamma_k (u^k)^{\beta - \beta \epsilon + 1} \right] = (\beta \epsilon - \beta - 1) \partial_x \Delta^{-\epsilon},$$
\[
\partial_t \Delta^{1/\beta} = \partial_x \left[ \frac{\left( \frac{\beta \varepsilon}{\beta \varepsilon + \beta - 1} \right)^{1 - \beta - \beta \varepsilon}}{\Delta^{1/\beta}} \sum \gamma_n (u^n)^{\beta \varepsilon + \beta - 1} \right].
\]

1. The Egorov sub-case. If \( \beta \varepsilon = -1 \), then this is the Egorov hydrodynamic type system (see (48))

\[
u^i_t = \partial_x \left[ 1 + \sum \gamma_k (u^k)^{\beta} \right]^{1/\beta}, \quad i = 1, 2, ..., N
\]

where the potential of the Egorov metric (see [28]) is

\[a = \sum \gamma_k (u^k)^{\beta + 2}.
\]

2. The local Hamiltonian sub-case. If \( \beta (1 - \varepsilon) = 1 \), then chromatography system (26) can be written in the Hamiltonian form (24). If \( \beta \to \infty \), the exponential chromatography (27) \( (\varepsilon = 1) \) also has the Hamiltonian form (24), where the Hamiltonian density is \( h = \ln \Delta \).

3. The nonlocal Hamiltonian sub-case (associated with constant curvature metric, see [9], [24]). If \( \beta = 2 \), then the conservation law density \( q = 1 - \sqrt{1 + \sum \gamma_k (u^k)^2} \) is the momentum density (see [24]) of the nonlocal Hamiltonian structure

\[
u^i_t = \partial_x \left( (\bar{g}^{ik} - u^i u^k) \frac{\partial \tilde{h}}{\partial u^k} + u^i \tilde{h} \right),
\]

where diagonal matrix elements \( \bar{g}_{ik} = -\gamma_i \delta_{ik} \) (where \( \delta_{ik} \) is the Kronecker symbol) and the Hamiltonian density

\[\tilde{h} = -\frac{\Delta^{-\varepsilon}}{2\varepsilon + 1} \sum \gamma_k (u^k)^{2\varepsilon + 1}.
\]

4. The general (mirrored) sub-case. If two hydrodynamic type systems (26) are related via mirrored conjugate curvilinear coordinate nets, then they must have one common conservation law density (see (49) and (51)). Let us prove that such conservation law density is

\[a = \sum \gamma_k (u^k)^{\beta - \beta \varepsilon + 1}.
\]

If the another chromatography system (26)

\[
\tilde{u}^i_y = \partial_z \left[ \frac{(\tilde{u}^i)^{\tilde{\beta}}}{\left( 1 + \sum \gamma_k (\tilde{u}^k)^{\tilde{\beta}} \right)^{\tilde{\beta}} \varepsilon} \right], \quad i = 1, 2, ..., N
\]

has the same conservation law density

\[a = \sum \gamma_k (\tilde{u}^k)^{\tilde{\beta} - \tilde{\beta \varepsilon} + 1},
\]

then transformation \( u \to \tilde{u}, \beta \to \tilde{\beta}, \varepsilon \to \tilde{\varepsilon} \) can be found from two other obvious assumptions (see (49) and (51))

\[u^i = (\tilde{u}^i)^\delta, \quad \Delta^{-\varepsilon} = \tilde{\Delta}^{1/\tilde{\beta}}.
\]
Result:
\[ \delta = -\frac{1}{\beta \varepsilon}, \quad \bar{\varepsilon} = -\frac{1}{\beta}, \quad \bar{\beta} = -\frac{1}{\varepsilon}. \]

Since the second chromatography system (55) has the generating function of conservation laws

\[ P_y = \partial_x \left( \frac{P^\frac{1}{\beta \varepsilon}}{1 + \sum \gamma_k (u^k)^\beta^{-1/\beta}} \right), \]

then the generating function of commuting flows is given by (52)

\[ a_x = \partial_x p^{-\beta \varepsilon}, \quad (56) \]

where the relationship

\[ P = p^{-\beta \varepsilon} \]

is obtained by a comparison the equations of the Riemann surface (which are equivalent) for both chromatography systems.

The integrable chromatography system (26) in the Riemann invariants has the form (13)

\[ r_t^i = \beta \varepsilon \frac{(p^t)^{\beta \varepsilon - 1}}{\Delta^\varepsilon} r_x^i \]

Since this hydrodynamic type system has two conservation laws

\[ p_t = \partial_x \frac{p^{\beta \varepsilon}}{\Delta^\varepsilon}, \quad a_t = (\beta \varepsilon - 1) \partial_x \Delta^{-\varepsilon}, \]

then one can obtain, respectively

\[ \partial_t p = \frac{p^{\beta \varepsilon}}{p^{\beta \varepsilon - 1} - (p^t)^{\beta \varepsilon - 1}} \frac{\partial \ln \Delta}{\beta}, \quad \partial_t a = \frac{1 + \beta - \beta \varepsilon}{\beta} (p^t)^{1-\beta \varepsilon} \partial_t \ln \Delta. \]

Thus, the generating function of commuting flows (56) in the Riemann invariants (53)

\[ r_t^i = \sigma \frac{(p^t)^{\beta \varepsilon - 1}}{p[(p^t)^{\beta \varepsilon - 1} - p^{\beta \varepsilon - 1}]} r_x^i \]

is connected with the Gibbons equation

\[ \lambda_t(\zeta) - \frac{\beta \varepsilon}{1 + \beta - \beta \varepsilon} p(\zeta)[(p^t)^{\beta \varepsilon - 1} - p^{\beta \varepsilon - 1}(\zeta)] \lambda_x \]

\[ = \frac{\partial \lambda}{\partial p(\lambda)} \left[ \partial_t(\zeta) p(\lambda) + \frac{\beta \varepsilon}{(\beta \varepsilon - 1)(1 + \beta - \beta \varepsilon)} \partial_x \left( \frac{p(\lambda)}{p(\zeta)} \right)^{\beta \varepsilon} F \left( 1, \sigma, \sigma + 1, \frac{p^{\beta \varepsilon - 1}(\lambda)}{p^{\beta \varepsilon - 1}(\zeta)} \right) \right], \]

where we use the notation

\[ \sigma = \frac{\beta \varepsilon}{\beta \varepsilon - 1}. \]
Then the generating function of commuting flows written via the physical field variables $u^k$ (cf. (26)) is

$$u^i_{r(\zeta)} = -\frac{\beta \varepsilon}{(\beta \varepsilon - 1)(1 + \beta - \beta \varepsilon)} \partial_x \left[ \left( \frac{u^i}{p(\zeta)} \right)^{\beta \varepsilon} F \left( 1, \sigma, \sigma + 1, \frac{(u^i)^{\beta \varepsilon - 1}}{p^{\beta \varepsilon - 1}(\zeta)} \right) \right].$$

Substituting the formal series $\partial_{r(\zeta)} = \partial_{i_0} + \zeta \partial_{i_1} + \zeta^2 \partial_{i_2} + \ldots$ and respectively the generating function of conservation law densities expanded in the series (35), one can obtain infinitely many generating functions of conservation laws for all commuting flows. For instance, the first $N$ such functions are

$$p_{i_0} = \partial_x \left[ \left( \frac{p}{u^i} \right)^{\beta \varepsilon} F \left( 1, \sigma, \sigma + 1, \frac{p^{\beta \varepsilon - 1}}{(u^i)^{\beta \varepsilon - 1}} \right) \right].$$

**Remark:** Suppose that some $N$ component hydrodynamic type system contains $N - 1$ equations (see the above generation function of conservation laws)

$$u^k_{i_0} = \partial_x \left[ \left( \frac{u^k}{u^i} \right)^{\beta \varepsilon} F \left( 1, \sigma, \sigma + 1, \frac{(u^k)^{\beta \varepsilon - 1}}{(u^i)^{\beta \varepsilon - 1}} \right) \right], \quad k \neq i.$$

Then such hydrodynamic type system is integrable if its $N$th equation satisfies some extra conditions. This hydrodynamic type system is not symmetric like (7). However, the algebro-geometric approach still is valid.

Thus, if any given hydrodynamic type system contains $N - 1$ equations

$$u^k = \partial_x \psi \left( u; \frac{u^k}{u^1} \right), \quad k \neq 1,$$

then one should substitute the Taylor series (35) in the generating function of conservation laws

$$p_t = \partial_x \psi \left( u; \frac{p}{u^1} \right).$$

The $N$th equation will be obtained by the limit (see such example at the end of the sub-section “Hamiltonian formalism”)

$$u^1_t = \partial_x \left[ \lim_{\varepsilon \to 0} \psi \left( u; 1 + \varepsilon \frac{h^1(u)}{u^1} \right) \right].$$

All other computations are exactly as in the symmetric case.

### 5.5 Reciprocal transformations

The concept “reciprocal transformation” was introduced by S.A. Chaplygin (see, for instance, [29] and [30])

$$dz = A(u)dx + B(u)dt, \quad dy = C(u)dx + D(u)dt,$$

where two arbitrary conservation laws $A_t = B_x$ and $C_t = D_x$ preserve the gas dynamic equations changing adiabatic index only. The gas dynamics is the first known example in
the theory of integrable hydrodynamic type systems. In this sub-section we show that the integrable chromatography system (26) is invariant under couple of different reciprocal transformations. Thus, infinite sets of such systems are related by a *chain* of reciprocal transformations, which are described below.

1. The first such reciprocal transformation is very simple

\[ dz = dt, \quad dy = dx. \]

Then the integrable chromatography system (26) reduces to the chromatography system

\[ w^i_y = \partial_z \left[ \frac{(w^i)^{\frac{1}{\beta}}}{1 - \sum \gamma_k(w^k)^{1/\varepsilon}} \right]^{1/\beta}, \quad i = 1, 2, ..., N, \]

where

\[ u^i = (w^i)^{\frac{1}{\beta} \Delta^{-1/\beta}}, \quad \Delta = \Delta^{-1}. \]

Thus, two chromatography systems (26) \((\beta, \varepsilon, \gamma_k)\) and \((1/\varepsilon, 1/\beta, -\gamma_k)\) are related by the transformation \(x \leftrightarrow t\).

Remark: The first conservation law

\[ \partial_t \left[ \sum \gamma_k(u^k)^{\beta - \beta \varepsilon + 1} \right] = (\beta \varepsilon - \beta - 1) \partial_x \Delta^{-\varepsilon} \]
transforms into the second conservation law

\[ \partial_y \Delta^{1/\beta} = \partial_z \left[ \frac{\beta \varepsilon}{\beta \varepsilon + \beta - 1} \Delta^{1-\beta \beta \varepsilon} \sum \gamma_n(u^n)^{\beta \varepsilon + \beta - 1} \right] \]
and vice versa.

2. The reciprocal transformation

\[ dz = \Delta^{1/\beta} dx + \left[ \frac{\beta \varepsilon}{\beta \varepsilon + \beta - 1} \Delta^{1-\beta \beta \varepsilon} \sum \gamma_n(u^n)^{\beta \varepsilon + \beta - 1} \right] dt, \quad dy = \beta \varepsilon dt \]
connects the chromatography system (26) and another symmetric system

\[ v^i_y = \partial_z \left[ \frac{(v^i)^{\beta \varepsilon}}{\beta \varepsilon + \beta - 1} \Delta^{1-\beta \beta \varepsilon} \sum \gamma_n(u^n)^{\beta \varepsilon + \beta - 1} \right], \quad i = 1, 2, ..., N, \]

where \( v^i = u^i \Delta^{-1/\beta}. \)

However, the chromatography system (26) has the commuting flow

\[ u^i_1 = \partial_x \left[ \frac{(u^i)^{2-\beta \varepsilon}}{2 - \beta \varepsilon} + \frac{u^i}{\beta - \beta \varepsilon + 1} \sum \gamma_n(u^n)^{\beta \varepsilon + \beta - 1} \right], \]

(58)

It is easy to verify by the compatibility conditions \( \partial_t(u^i_1) = \partial_x(u^i_1) \). Then the hydrodynamic type system (57) has the commuting flow

\[ v^i_1 = \partial_x \left[ \frac{(v^i)^{2-\beta \varepsilon}}{2 - \beta \varepsilon} \right]^{1/\beta}, \quad i = 1, 2, ..., N \]
which is the chromatography system (26) again. Thus, two chromatography systems (26) \((\beta, \varepsilon, \gamma_k)\) and \((\beta / 2 - \varepsilon, -\gamma_k)\) are related by the above reciprocal transformation, where \( \Delta = \Delta^{-1} = 1 - \sum \gamma_n(u^n)^{\beta} \).

Applying both reciprocal transformations iteratively, one can construct a link between the chromatography systems (26) with the distinct indexes \( \beta \) and \( \varepsilon \).
6 Homogeneous hydrodynamic type systems

Another hydrodynamic type system

\[ u^i_t = \partial_x \left( (u^i)^\beta \prod (u^n)^{\gamma_n} \right) \]

arising in the chromatography (see formula 6 in [10], \( \beta \) and \( \gamma_n \) are arbitrary constants) is invariant under scaling of field variables \( u^k \to cu^k \) (and appropriate scaling of independent variable \( t \) or \( x \)). We call such hydrodynamic type systems as homogeneous. Plenty physically interested hydrodynamic type systems belong to this class.

The existence of the corresponding generating function of conservation laws

\[ p_t = \partial_x \left( p^\beta \prod (u^n)^{\gamma_n} \right) \]

yields (see (10))

\[ \frac{\partial p}{\partial u^i} = \frac{\gamma_i p^\beta}{\beta u^i [p^\beta - (u^i)^\beta - 1]} \left( \frac{1}{\beta} \sum \frac{\gamma_k (u^k)^{\beta-1}}{p^\beta - (u^i)^\beta - 1} - 1 \right)^{-1}. \]

Also, the equation of the Riemann surface \( \lambda(u, p) \) must be invariant under extended scaling \( u^k \to cu^k, \ p \to cp \). Since (see (11))

\[ \frac{\partial \lambda}{\partial u^i} = \frac{\gamma_i u^i^\beta}{\beta u^i [p^\beta - (u^i)^\beta - 1]} \left( 1 - \frac{1}{\beta} \sum \frac{\gamma_k (u^k)^{\beta-1}}{p^\beta - (u^i)^\beta - 1} \right)^{-1} \frac{\partial \lambda}{\partial p}, \]

then the equation of the Riemann surface \( \lambda(u, p) \) can be found in quadratures

\[ \ln \lambda = \ln p + \frac{1}{(\beta - 1)(\beta + \sum \gamma_m)} \sum \gamma_k \ln \frac{(u^k)^{\beta-1}}{p^\beta - (u^i)^\beta - 1}, \]

where we used the Euler theorem

\[ \lambda = p\lambda_p + \sum u^k \lambda_k. \]

Since the Gibbons equation (12) is invariant under point transformation \( \lambda \to \tilde{\lambda}(\lambda) \), then the above equation of the Riemann surface \( \lambda(u, p) \) can be written in the form

\[ \lambda = q^{\beta + \sum \gamma_m} \prod \left( 1 - \frac{q}{w^k} \right)^{-\gamma_k}, \]

where \( q = p^\beta - 1, \ w^k = (u^k)^{\beta-1}. \)

6.1 The Kodama hydrodynamic type system

The Kodama hydrodynamic type system (see [16])

\[ a_t^k = \partial_x \left[ \frac{1}{2} \sum_{m=1}^k a^m a^{k+1-m} + \delta^{k,1} a^N \right], \quad k = 1, 2, \ldots, N, \]
where $\delta_{ik}$ is the Kronecker symbol, is a homogeneous (The Euler operator is $\hat{E} = \Sigma(N + k - 2)a^k\partial_k$) hydrodynamic reduction of the Benney hydrodynamic chain (37). Thus, the corresponding generating function of conservation laws is (36) (the generating function of conservation laws for any integrable hydrodynamic chain is unique for all its hydrodynamic reductions). The Kodama hydrodynamic type system can be obtained from the above generating function by substitution of the Taylor series

$$p = a^1 + \lambda^2 + \lambda^3 + \ldots + \lambda^{N-1}a^N + \lambda^Nh_1(a) + \lambda^{N+1}h_2(a) + \lambda^{N+2}h_3(a) + \ldots$$

Then $A^0 \equiv a^N$ and $h_k(a)$ are some polynomial conservation law densities. For instance, $h_1(a) = \Sigma a^k a^{N+1-k}/2$ is a momentum, $h_2(a)$ is the Hamiltonian, where the Kodama hydrodynamic type system has bi-Hamiltonian structure, and the first of them is

$$a^k_t = \partial_x \frac{\partial h_2}{\partial a^{N+1-k}}, \quad k = 1, 2, \ldots, N.$$ 

Without lost of generality let us restrict our consideration on the three component case

$$u_t = \partial_x \left( \frac{u^2}{2} + w \right), \quad \upsilon_t = \partial_x (uw), \quad w_t = \partial_x \left( uw + \frac{\upsilon^2}{2} \right).$$

Thus, the Gibbons equation (see (12) and (36)) is

$$\lambda_t - p\lambda_x = \frac{\partial \lambda}{\partial p} \left[ p_t - \partial_x \left( \frac{p^2}{2} + w \right) \right].$$

Then one has

$$\lambda_u = \frac{w(p - u) + \upsilon^2}{\Delta} \lambda_p, \quad \lambda_{\upsilon} = \frac{v(p - u)}{\Delta} \lambda_p, \quad \lambda_w = \left[ \frac{(p - u)^3}{\Delta} - 1 \right] \lambda_p,$$

where

$$\Delta = (p - u)^3 - w(p - u) - \upsilon^2.$$

Since, the Kodama hydrodynamic type system is homogeneous, then the function $\lambda(u, \upsilon, w, p)$ must be homogeneous. Then $\lambda = (2p\partial_p + 2u\partial_u + 3v\partial_v + 4w\partial_w)\lambda$ up to some insufficient constant factor (degree of homogeneity). Thus, the equation of the Riemann surface can be found in quadratures. For instance,

$$\partial_p \ln \lambda = \frac{2\Delta}{(p - u)[2p^3 - 4up^2 + 2(u^2 + w)p - 2uw + v^2]}.$$ 

Then (cf. [16]) the equation of the Riemann surface for the Kodama hydrodynamic type system is

$$\lambda = p + \frac{w}{p - u} + \frac{\upsilon^2}{2(p - u)^2}.$$ 

**Remark:** The same procedure can be repeated for any $N$ component Kodama hydrodynamic type system. Moreover, suppose the determinant (14) is computed and written in the factorized form

$$\prod_{k=1}^{N} (p - p^k(a))$$

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for an arbitrary \( N \), then the equation of the Riemann surface is given by rational function (see [16])

\[
\lambda = \int \prod_{k=1}^{N} \frac{p - p^k(a)}{p - a^1} dp \equiv p + \sum_{k=1}^{N-1} \frac{B_k(a)}{(p - a^1)^k},
\]

(59)

where \( B_k \) are some polynomials with respect to flat coordinates \( a^n \). These coefficients \( B_k \) can be found by substitution the above formula in (11). The corresponding linear system is

\[
\partial_n B_{k+1} = \sum_{m=n+1}^{N} a^{m+1-n} \partial_m B^k, \quad k = 1, 2, ..., N - 2, \quad n = 2, 3, ..., N - 1,
\]

where \( B^1 \equiv a^N \) and

\[
\sum_{m=n+1}^{N-1} a^{m+1-n} \partial_m B^{N-1} = 0, \quad \partial_N B^n = 0, \quad n = 2, 3, ..., N - 1.
\]

**Remark:** In the symmetric case (7) all derivatives \( \lambda_k = \ldots \lambda_p \) can be found immediately, but in the above case for each \( N \), one must (step by step) compute all above derivatives consequently. If the given hydrodynamic type system is non-symmetric, if the generating function of conservation laws (as in the above example) is not given a priori, then derivation of the equation of the Riemann surface becomes the very complicated computational problem.

**Remark:** The equation of the Riemann surface (59) can be written in the totally factorized form

\[
\lambda = (p - a^1)^{1-N} \prod_{k=1}^{N} (p - b^k),
\]

where \((N - 1)a^1 = \Sigma b^k(a)\). Substituting the Taylor series (17) \( p^{(k)} = b^k + \lambda c^k(b) + \ldots \) in (36) yields the Kodama hydrodynamic type system written in the symmetric form

\[
b_k^i = \partial_x \left[ \frac{(b_k^i)^2}{2} + \frac{1}{2} \sum (b_k^i)^2 - \frac{1}{2(N - 1)} \left( \sum b_k^i \right)^2 \right].
\]

(60)

### 6.2 Cubic Hamiltonian hydrodynamic type system

The **cubic Hamiltonian** hydrodynamic type system (24) with the Hamiltonian density

\[
h = \frac{1}{6} \sum \gamma_k(u^k)^3 + \beta \sum \gamma_k u^k \sum \gamma_n(u^n)^2 + \varepsilon \left( \sum \gamma_k u^k \right)^3
\]

is equivalent to homogeneous hydrodynamic type system (cf. (60))

\[
a_i^i = \partial_x \left[ \frac{(a_i^i)^2}{2} + \alpha \sum \gamma_k (a^k)^2 + \delta \left( \sum \gamma_k a^k \right)^2 \right]
\]

(61)

under the transformation \( a^i = u^i + 2\beta \Sigma \gamma_k u^k \).
Following the recipe given above, one can verify that the hydrodynamic type system (61) is integrable iff
\[ \delta = -\frac{2\alpha^2}{1+2\alpha \sum \gamma_n} \iff \varepsilon = -\frac{2\beta^2/3}{1+2\beta \sum \gamma_n} \]
and is connected with the equation of the Riemann surface
\[ \lambda = \left( p - \frac{2\alpha}{1+2\alpha \sum \gamma_n} \sum \gamma_m a^m \right)^{1+2\alpha \sum \gamma_n} \prod (p - a^k)^{-2\alpha \gamma_k}. \]

**Remark:** The mechanical interpretation. Let us consider the classical Hamilton’s system
\[ \dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \]
where the Hamiltonian is \( H = p^2/2 + V(x,t) \). We seek the Hamiltonian system \( \ddot{x} = -V_x \) possessing the first integral (“energy constant surface”); see [17]
\[ \lambda = V_0 + V_1 \dot{x} + V_2 \dot{x}^2 + ... + V_{N-2} \dot{x}^{N-2} + V_N \dot{x}^N, \]  
(62)
where \( V_k(x,t) \) are some functions and \( N \) is arbitrary. Differentiating this equation with respect to \( t \) yields the hydrodynamic type system (61) (cf. (60))
\[ u^k_t = -\partial_x \left( \frac{(u^k)^2}{2} - \frac{1}{2(N+1)} \left[ \left( \sum u^m \right)^2 + \sum (u^m)^2 \right] \right), \]
where the field variables \( u^k \) are coefficients of the polynomial (62) written in the factorized form
\[ \lambda = (\dot{x} + \sum u^m) \prod (\dot{x} - u^k) \]
and the “potential energy” is given by the symmetric expression
\[ V = -\frac{1}{2(N+1)} \left[ \left( \sum u^m \right)^2 + \sum (u^m)^2 \right]. \]

Let us replace (62) on an arbitrary dependence
\[ \lambda = \lambda(u; \dot{x}), \]
where \( u^k(x,t) \) are some functions. Differentiating of the above equation with respect to \( t \), one can obtain (see (1), (14); cf. (11))
\[ \lambda_\mu \partial_\mu V(u) = (v^k_i(u) - \mu \delta^k_i) \lambda_k, \]
where we use the equation of the Riemann surface \( \lambda = \lambda(u; \mu) \). In the symmetric case (7) the above system reduces to (see (15), (8); cf. (11))
\[ \lambda_\mu \partial_\mu V(u) = A^k_i(u, \mu) \lambda_k, \]
which should be connected with (11) by the substitution (see (15))
\[ \mu = \frac{\partial \psi}{\partial p}. \]
6.3 The ideal gas dynamics

The ideal gas dynamics

\[ u_t = \partial_x \left( \frac{u^2}{2} + \frac{v^\beta}{\beta} \right), \quad v_t = \partial_x (uv) \]  

is connected with the equation of the Riemann surface (see [3])

\[ \lambda = \frac{1}{4} [p^\beta + \beta u + v^\beta p^{-\beta}] \]  

The corresponding Gibbons equation is

\[ \lambda_t - (p^\beta + u)\lambda_x = \frac{\partial \lambda}{\partial p} \left[ p_t - \partial_x \left( \frac{p^{\beta+1}}{\beta+1} + up \right) \right]. \]  

Remark: The equation of the Riemann surface (64) is unique for all values of index $\beta$. Indeed, introducing the Riemann invariants

\[ r^{1,2} = u \pm \frac{2}{\beta} v^{\beta/2}, \]

the ideal gas dynamics (63) can be written in the diagonal form

\[ r^{1,2}_t = \frac{1}{2} [r^1 + r^2] \pm \frac{\beta}{2} (r^1 - r^2) r^{1,2}_x. \]

Then the equation of the Riemann surface (64) in the Riemann invariants is

\[ \lambda = \frac{\nu}{4} + \frac{r^1 + r^2}{2} + \frac{(r^1 - r^2)^2}{4\nu}, \]  

where

\[ \nu = p^\beta. \]

Remark: Two infinite series of conservation laws can be obtained with the aid of the Bürmann–Lagrange series (see the next section) at the vicinity of two zeros of the equation of the Riemann surface

\[ 4\lambda = \frac{[\nu + (\sqrt{r^1} + \sqrt{r^2})^2] [\nu + (\sqrt{r^1} - \sqrt{r^2})^2]}{\nu}, \]

while the Kruskal series can be derived at the infinity ($\lambda \to \infty, \nu \to \infty$) and at the vicinity of another singular point ($\lambda \to \infty, \nu \to 0$).

6.4 Whitham averaged Sinh-Gordon equation

A one-phase solution of the Sinh–Gordon

\[ u_{xt} = \sinh u \]
averaged by the Whitham approach is the two component hydrodynamic type system (see [33])

\[ r^i_t = \mu^i(r)r^i_x, \tag{67} \]

where the differentials of the quasi-momentum and the quasi-energy, respectively,

\[ dp = \frac{\lambda - <\lambda>}{\sqrt{\lambda(r^1 - \lambda)(r^2 - \lambda)}} d\lambda, \quad dq = \frac{\frac{1}{\lambda} - <\frac{1}{\lambda}>}{\sqrt{\lambda(r^1 - \lambda)(r^2 - \lambda)}} d\lambda, \]

determine characteristic velocities

\[ \mu^{1,2}(r) = \left. \frac{dq(\lambda)}{dp(\lambda)} \right|_{\lambda=r^{1,2}} = \frac{1}{\sqrt{r^1 r^2}} \left[ 1 - (s^2) \frac{K(s)}{E(s)} \right]^{\pm 1}, \]

where \( K(s) \) and \( E(s) \) are complete elliptic integrals of the first and second kind, respectively; \( s^2 = r^2/r^1 \) is elliptic module, and

\[ <a> \equiv \frac{1}{T} \int_0^{r^2} \frac{ad\lambda}{\sqrt{\lambda(r^1 - \lambda)(r^2 - \lambda)}}, \quad T = \int_0^{r^2} \frac{d\lambda}{\sqrt{\lambda(r^1 - \lambda)(r^2 - \lambda)}}. \]

However, this hydrodynamic type system can be enclosed in the framework presented in the previous sub-section. Two next theorems can be proved by straightforward calculation.

**Theorem 4:** The Gibbons equation

\[ \lambda_t - \frac{d\tilde{q}}{d\nu} \frac{d\nu}{d\tilde{p}} \lambda_x = \frac{\partial \lambda}{\partial \tilde{p}} (\tilde{p}_t - \tilde{q}_x) \]

connects the averaged (by the Whitham approach) one-phase solution of the Sinh-Gordon equation (67) with the equation of the Riemann surface (66), where

\[ \tilde{p} = \frac{\nu}{2} + r^1 + r^2 + \left[ \frac{r^1 + r^2}{2} - <\lambda> \right] \ln \nu, \quad \tilde{q} = \ln \frac{\nu + (\sqrt{r^1} - \sqrt{r^2})^2}{\nu + (\sqrt{r^1} + \sqrt{r^2})^2} - <\lambda> \ln \nu, \]

where

\[ <\lambda> = r^1 \left[ 1 - \frac{K(s)}{E(s)} \right]. \]

**Theorem 5:** The generating function of commuting flows in the Riemann invariants

\[ r^i_{\tau(\zeta)} = \left( \ln \nu(\zeta) + g^{ij} \frac{\nu^j + 2P}{\nu^i - \nu(\zeta)} \right) r^i_x \]

is connected with the Gibbons equation

\[ \lambda_{\tau(\zeta)} - \left( \ln \nu(\zeta) + \frac{3\nu^2(\lambda) + 4(r^1 + r^2)\nu(\lambda) + (r^1 - r^2)}{(\nu(\lambda) + 2P)(\nu(\lambda) - \nu(\zeta))} \right) \lambda_x = \frac{\partial \lambda}{\partial \tilde{p}(\lambda)} \left[ \partial_{\tau(\zeta)} \tilde{p}(\lambda) - \partial_x Q(\lambda, \zeta) \right], \]
where the potential $\mathbf{P}$ of the Egorov metric $g_{ij} = \partial_i \mathbf{P}$ is
\[
\mathbf{P} = \frac{r^1 + r^2}{2} - \langle \lambda \rangle,
\]
the Egorov metric is
\[
g_{11} = \frac{E(s)/K^2(s)}{2(1 - s^2)}, \quad g_{22} = -\frac{[1 - s^2 - E(s)/K(s)]^2}{2s^2(1 - s^2)}
\]
and
\[
Q(\lambda, \zeta) = 2(r^1 + r^2) + \frac{1}{2} \nu(\lambda) + \frac{1}{2} \nu(\zeta) + \mathbf{P} \ln \nu(\lambda) \ln \nu(\zeta) + [r^1 + r^2 + \frac{1}{2} \nu(\zeta)] \ln \nu(\lambda)
\]
\[+ [r^1 + r^2 + \frac{1}{2} \nu(\lambda)] \ln \nu(\zeta) + 2(\lambda + \zeta) \ln[\nu(\lambda) - \nu(\zeta)] - 2\lambda \ln \nu(\zeta) - 2\zeta \ln \nu(\lambda).
\]

7 Integrable hydrodynamic chains

The Gibbons equation for the hydrodynamic type system (58)
\[
\lambda_{t^1} - (p^{1-\beta\varepsilon} + \bar{a}) \lambda_x = \frac{\partial \lambda}{\partial p} \left[ p_{t^1} - \partial_x \left( \frac{p^{2-\beta\varepsilon}}{2 - \beta\varepsilon} + \bar{a}p \right) \right],
\]
where $\bar{a} = a/(\beta - \beta\varepsilon + 1)$ (see (54)), is exactly the same as the Gibbons equation for the ideal gas dynamics (65).

Introducing the moments
\[
B^k = \frac{1}{(1 - \beta\varepsilon)(k + 1) + \beta} \sum \gamma_i(u^i)^{(1-\beta\varepsilon)(k+1)}\beta, \quad k = 0, 1, 2, ...
\]
the hydrodynamic type system (58) can be rewritten as the Kupershmidt hydrodynamic chain (see [20])
\[
B^k_{t^1} = B^k_{x} + B^0 B^k_{x} + [(1 - \beta\varepsilon)(k + 1) + \beta]B^k B^0_{x}, \quad k = 0, 1, 2, ...
\]
connected with the Gibbons equation (68), where
\[
\bar{a} = \frac{B^0}{1 - \beta\varepsilon}.
\]

Suppose the moments $B^k$ of the Kupershmidt hydrodynamic chain are some functions of $N$ field variables $u^k$, then $N$ component hydrodynamic reduction
\[
u_{t^1} = \partial_x \left( \frac{(u^i)^{2-\beta\varepsilon}}{2 - \beta\varepsilon} + \frac{B^0(u)}{1 - \beta\varepsilon} u^i \right)
\]
is an integrable hydrodynamic type system (7) iff the function $B^0(u)$ satisfies some nonlinear PDE system, which is consequence of the compatibility condition $\partial_i(\partial_k p) = \partial_k(\partial_i p)$, where (cf. (25))
\[
\partial_i p = \frac{\partial_i B^0(u)}{(u^i)^{1-\beta\varepsilon} - p^{1-\beta\varepsilon}} \left[ 1 - \beta\varepsilon + \sum \frac{\partial_k B^0(u)}{(u^k)^{1-\beta\varepsilon} - p^{1-\beta\varepsilon}} \right]^{-1}.
\]
If \( B^0(u) = (1 - \beta \varepsilon) \sum_k u^k \beta^{\varepsilon+1} / (\beta - \beta \varepsilon + 1) \), then (71) reduces to (25). The compatibility condition \( \partial_t(\partial_x p) = \partial_x(\partial_t p) \) creates a nonlinear PDE system on function \( B^0(u) \) only. Its solution is parameterized by \( N \) arbitrary functions of a single variable. Thus, any symmetric hydrodynamic type system (7) can be used for derivation of corresponding integrable hydrodynamic chain, which have the same Gibbons equation. At the same time, all hydrodynamic reductions can be written in a similar symmetric form, but with another dependence of r.h.s. functions like \( B^0(u) \) with respect to field variables \( u^k \). If somebody will be able to solve corresponding nonlinear PDE system (which is known in the Riemann invariants as the Gibbons–Tsarev system, see [13]), then infinitely many symmetric hydrodynamic type systems (7) will be produced.

**Corollary:** Let us consider the generating function of conservation laws (see (70))

\[
p_{t^i} = \partial_x \left[ p^{\alpha+1} + \frac{B^0(u)}{\alpha} p \right]
\]

and introduce the parameter \( \alpha = 1 - \beta \varepsilon \). Then \( N + 1 \) parametric family \( (N \) parameters \( \gamma_k \) and \( \beta \) for each fixed index \( \alpha \)) of hydrodynamic reductions of the hydrodynamic chain (69)

\[
B_{t^i}^k = B_{x^i}^{k+1} + B^0 B_x^k + [\alpha(k + 1) + \beta] B_x^k B_x^0, \quad k = 0, 1, 2, ...
\]

is a set of the hydrodynamic type systems (70)

\[
u_{t^i} = \partial_x \left( \frac{(u^i)^{\alpha+1}}{\alpha + 1} + \sum_k \frac{\gamma_k (u^k)^{\alpha + \beta}}{\alpha + \beta} u^i \right),
\]

which are distinct for every value of index \( \beta \) (all above hydrodynamic chains are equivalent for each fixed index \( \alpha \) and for any value of the index \( \beta \), see details in [26]).

**Remark:** Introducing the moments

\[
C^k = \frac{1}{(\beta \varepsilon - 1) k + \beta} \sum \gamma_i (u^i)^{(\beta \varepsilon - 1) k + \beta}, \quad k = 0, 1, 2, ...
\]

the hydrodynamic type system (26) can be rewritten as the Kupershmidt hydrodynamic chain (see [21])

\[
C_{t^i}^k = \beta \varepsilon (1 + \beta C^0)^{-\varepsilon} C_{x^i}^{k+1} [((\beta \varepsilon - 1)(k + 1) + \beta] C^{k+1} [(1 + \beta C^0)^{-\varepsilon}]x, \quad k = 0, 1, 2, ...
\]

In this section we proved that hydrodynamic type systems (26) and (58) are commuting flows, then the corresponding \( (B \) and \( C \)) hydrodynamic chains are commute with each other. All other details can be found in [26].

### 8 Hamiltonian chromatography system

Another generalization of the chromatography system (cf. (23) and [10]) is given by the Hamiltonian hydrodynamic type system

\[
a_i^t = \partial_x \frac{\partial h}{\partial a^i}, \quad i = 1, 2, ..., N,
\]
where the Hamiltonian density $h(\Delta)$ and $\Delta = \Sigma z_k(a^k)$. If this system is diagonalizable (see (2))

$$r_i = \mu^i(r_i), \quad i = 1, 2, ..., N,$$

then it is integrable (see [6]). Thus, we are looking for corresponding transformation $r^i(a)$. A direct computation yields (the indexes of the Riemann invariants $r^k$ and characteristic velocities $\mu^i$ are omitted for simplicity below)

$$\frac{\partial r}{\partial z_i} = \frac{\varphi}{\zeta - V_i}, \quad \rho(\Delta) = \sum \frac{V_n}{\zeta - V_n'},$$

where $V_k(z_k) = \frac{z_k'}{2}$, $(\ln h')'\rho(\Delta) = 1/2$, $\varphi = \rho^{-1}(\Delta)\Sigma V_k\partial r/\partial z_k$ and $\mu = \zeta h'$. The Riemann invariants exist iff the compatibility conditions $\partial_i(\partial_k r) = \partial_k(\partial_i r)$ are fulfilled, where $\partial_i \equiv \partial/\partial z_i$. Eliminating $\varphi$ and its first derivatives from the compatibility conditions, one can obtain the integrability condition

$$q_i(\partial q_k - \partial_k q_i) + q_k(\partial_j q_i - \partial_i q_j) + q_i(\partial_k q_j - \partial_j q_k) = 0$$

for every three distinct indexes $i, j, k$, where $q_i = (\zeta - V_i')^{-1}$. Taking into account

$$\partial \zeta = \left( \frac{V_iV_i''}{\zeta - V_i'} + \frac{V'_i}{\zeta - V_i'} - \rho'(\Delta) \right) \left[ \sum \frac{V_m}{(\zeta - V_m')^2} \right]^{-1},$$

the above integrability condition reduces to sole ODE

$$VV'' = (1 + \alpha)V'^2 + \beta V' + \gamma,$$

where $V_i \equiv V(z_i)$, $z_i \equiv z(a^i)$ and $\rho(\Delta) = \alpha \Delta + \delta$ ($\alpha, \beta, \gamma, \delta$ are arbitrary constants). Thus, the Hamiltonian chromatography system

$$a_i = \partial_x [h'(\Delta)z'(a^i)]$$

is integrable if $h = e^\Delta$ ($\alpha = 0$), $h = \ln \Delta$ ($\alpha = -1/2$) and $h = \Delta^\varepsilon$ (for all other values $\alpha$; $\varepsilon$ is an arbitrary constant).

Then the Gibbons equation

$$\lambda_t - z''(p)h'(\Delta)\lambda_x = \frac{\partial \lambda}{\partial p} (p_t - \partial_x [z'(p)h'(\Delta)])$$

is determined by the equation of the Riemann surface $\lambda(a; p)$, which can be found in quadratures

$$d\lambda = z^{1+4\alpha}(p) \exp[\beta \int \frac{dz}{V}] \left[ z''(p) \sum \frac{dz_n}{z''(p) - V_n'} + 2 \left( \alpha \Delta + \delta - \sum \frac{V_n}{z''(p) - V_n'} \right) dp \right],$$

where $\zeta = z''(p)$.

**Remark:** Introducing the moments

$$A^k = \sum \int V_i^n dz_i$$

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the Hamiltonian chromatography system (74) can be rewritten as the Hamiltonian hydrodynamic chain

\[ A_k^t = \sum_{n=0}^{k+1} F_n^k(A_x) A_n^t, \quad k = 0, 1, 2, \ldots \]

determined by the Hamiltonian density \( h(A^0) (h = e^{A^0}, h = \ln A^0, h = (A^0)^\varepsilon) \) and by the Poisson bracket

\[ \{A^k, A^n\} = [B_{k,n} \partial_x + \partial_x B_{n,k}] \delta(x - x'), \]

where (here we use (73))

\[ B_{k,n} = [2(1 + \alpha)k + 1]A^{k+n+1} + 2\beta k A^{k+n} + 2\gamma k A^{k+n-1}. \]

9 Exceptional (linearly degenerate) case

In this paper the algebro-geometric approach for integrability of hydrodynamic type systems is established. However, this approach is most effective just in case of the symmetric hydrodynamic type systems possessing any symmetry operator (see section 6). Then the generating function of conservation laws for the symmetric hydrodynamic type systems can be found immediately; the integration factor in the computation of the Riemann surface can be found just if a corresponding hydrodynamic type system is invariant under some Lie group of symmetries (like homogeneity).

Nevertheless, this algebro-geometric approach can be used in all other more general and complicated cases, except possibly hydrodynamic type systems which are hydrodynamic reductions of linear-degenerate hydrodynamic chains (see [8], [25]). These hydrodynamic type systems usually can be written explicitly in the Riemann invariants (see [25]) and the conservation law fluxes of their generating functions of conservation laws are linear functions with respect to conservation law density \( p \) (see [25])

\[ p_t = \partial_x [p(\nu(r, \lambda))]. \quad (75) \]

For instance, in the limit \( \beta \varepsilon = 1 \), the integrable chromatography system (26)

\[ u^i_i = \varepsilon_i \partial_x \frac{u^i}{[1 + \sum \gamma_k (u^k)^{\varepsilon/\varepsilon}]} \quad i = 1, 2, \ldots, N \]

can be written explicitly in the Riemann invariants (see [10], the formula 5; also [25]; the parameters \( \varepsilon_i \) and \( \gamma_k \) do not affect on explicit expressions of characteristic velocities in the Riemann invariants)

\[ r_i^i = \frac{(\prod r^m)^\varepsilon}{r^i_x}, \quad i = 1, 2, \ldots, N. \]

Its generating function of conservation laws is (see [25])

\[ p_t = \partial_x \left( p \frac{(\prod r^m)^\varepsilon}{\lambda} \right). \]
In the same limit $\beta \varepsilon = 1$, the hydrodynamic type system (58)

$$u^i_t = \delta_i \partial_x \left[ u^i \left( 1 + \varepsilon \sum \gamma_n (u^n)^{1/\varepsilon} \right) \right]$$

in the Riemann invariants is (the parameters $\delta_i$ and $\gamma_k$ do not affect on explicit expressions of characteristic velocities in the Riemann invariants)

$$r^i_t = \left( r^i - \varepsilon \sum r^m \right) r^i_x.$$ 

Its generating function of conservation laws is (see [25])

$$p_t = \partial_x \left[ p \left( \lambda - \varepsilon \sum r^m \right) \right].$$ 

In this section we consider hydrodynamic type system written explicitly in the Riemann invariants

$$r^i_t = v^i(r)r^i_x, \quad i = 1, 2, ..., N,$$

such that the characteristic velocities $v^i(r)$ are determined by the unique function $v(r;\lambda)$

$$v^i(r) = v(r;\lambda)|_{\lambda=r^i}. \quad (76)$$

Then the semi-Hamiltonian criterion (3) reduces to

$$\partial_j \frac{\partial_i v}{v_i - v} = \partial_i \frac{\partial_j v}{v_j - v}, \quad i \neq j. \quad (77)$$

Suppose this hydrodynamic type system has the generating function of conservation laws (75). Then

$$\partial_t \ln p = \frac{\partial_i v}{v_i - v}$$

and the semi-Hamiltonian criterion (77) is satisfied automatically. Moreover, the generating function of conservation law densities $p$ in such case can be found explicitly

$$p = \exp \left[ \int \sum \frac{\partial_k v}{v_k - v} dr^k \right].$$

Thus, any commuting flow

$$r^i_t = w^i(r)r^i_x$$

has similar generating function of conservation laws

$$p_r = \partial_x [pw(r;\lambda)],$$

where $w^i(r) = w(r,\lambda)|_{\lambda=r^i}$.

**Remark:** $N$ phase solutions of the nonlinear equations (like KdV, NLS, Sinh-Gordon) averaged by the Whitham approach are hydrodynamic type systems belong to the class presented in this section (76)

$$r^i_t = \frac{dq}{dp}|_{\lambda=r^i} r^i_x,$$

where the Abelian holomorphic differentials of the quasi-momentum $dp$ and the quasi-energy $dq$ are determined on the Riemann surfaces of genus $N$ (see, for instance, details in [18]). Thus, corresponding hydrodynamic chains should be linear degenerate.
10 Conclusion and outlook

In this paper we suggest universal (except linearly degenerate case) approach for integrable (semi-Hamiltonian) symmetric hydrodynamic type systems. This approach is very effective because the generating function of conservation laws in such case is unique, this is a consequence of the construction ((7) $\rightarrow$ (9)). In all other cases the first step is a computation of the generating function of conservation laws.

Let us illustrate a complexity of this problem on two component hydrodynamic type system (for simplicity and without lost of generality we can restrict our consideration on two component case only). Suppose we have nonlinear elasticity equation (this is nothing but the ideal gas dynamic written in the Lagrangian coordinates, while (63) is written in the Euler coordinates), which is commuting flow to ideal gas dynamics

$$v_y = u_x,$$

$$u_y = \partial_x \frac{v^{\beta-1}}{\beta - 1}.$$  \hfill (78)

This hydrodynamic type system is written in non-symmetric form. Thus, we do not know in advance the generating function of conservation laws in this case. Nevertheless, obviously, we must seek such generating function in the form

$$p_y = \partial_x \psi(u, v, p).$$ \hfill (79)

However, if such generating function will be found, then this generating function is a generating function for whole hydrodynamic chain, is not for nonlinear elasticity only. Thus, one should seek $N$ component hydrodynamic type systems written in the Riemann invariants (13) compatible with the above generating function. Thus, we have

$$\partial_i p = \frac{\partial \psi}{\partial u} \partial_i u + \frac{\partial \psi}{\partial \nu} \partial_i \nu,$$

Here is a crucial point of the general construction. From the first equation of the nonlinear elasticity equation (78) one can obtain (take into account (13))

$$\left. \frac{\partial \psi}{\partial \nu} \right|_{p=p^i} \partial_i \nu = \partial_i u.$$ \hfill (80)

From the second equation one gets

$$\left. \frac{\partial \psi}{\partial \nu} \right|_{p=p^i} \partial_i \nu = v^{\beta-2} \partial_i u.$$ \hfill (81)

Thus, we have two choices

$$\partial_i p = \left. \frac{\partial \psi}{\partial u} \right|_{p=p^i} \partial_i u + \left. \frac{\partial \psi}{\partial \nu} \right|_{p=p^i} \partial_i \nu,$$

$$\partial_i p = \left. \frac{\partial \psi}{\partial u} \right|_{p=p^i} + v^{2-\beta} \left. \frac{\partial \psi}{\partial \nu} \right|_{p=p^i} \partial_i \nu.$$ \hfill (82)

These two options are no longer equivalent (in comparison with the symmetric case), because we ignore the identity (which is consequence of (80) and (81))

$$\left( \left. \frac{\partial \psi}{\partial \nu} \right|_{p=p^i} \right)^2 = v^{\beta-2},$$

The generating function for the whole hydrodynamic chain is unique. Thus, one should seek $N$ component hydrodynamic type systems written in the Riemann invariants (13) compatible with the above generating function.
which is valid for ideal gas dynamics only. Since we consider the generating function of
conservation laws (79) for whole hydrodynamic chain, then depending on this choice, we
shall be able to construct two different hydrodynamic chains. Thus, for instance, the non-
linear elasticity equation (78) can be embedded into different Kupershmidt hydrodynamic
chains (69) and (72).

Example: The dispersionless limit of the Boussinesq system (see (78), β = 3; [12], [25])
\begin{align*}
v_y &= u_x, \\
u_y &= \partial_x (u^2/2)
\end{align*}
(82)
satisfies the Gibbons equation
\[
\lambda_y - \frac{v^2}{\tilde{p}^2} \lambda_x = \frac{\partial \lambda}{\partial \tilde{p}} \left[ p_y + \partial_x \frac{v^2}{2\tilde{p}^2} \right],
\]
where \( \mu = p^3 \) and the equation of the Riemann surface is (cf. (64))
\[
\lambda = \mu + 3u + \frac{\nu^3}{\mu}.
\]
(83)
Simultaneously, the simplest reduction of the Benney moment chain (see [12]) is again
the dispersionless limit of the Boussinesq system determined by the Gibbons equation
\[
\lambda_y - \tilde{\mu} \lambda_x = \frac{\partial \lambda}{\partial \tilde{\mu}} \left[ \tilde{\mu}_y - \partial_x \left( \frac{\tilde{\mu}^2}{2} - \nu \right) \right],
\]
where the equation of the Riemann surface is
\[
\lambda = -\tilde{\mu}^3 + 3\nu \tilde{\mu} + 3u.
\]
(84)
The substitution \( \tilde{\mu} = -(p + \nu/p) \) in this cubic equation yields exactly (83)
\[
\lambda = p^3 + 3u + \frac{\nu^3}{p^3}.
\]
Factorizing the cubic polynomial (84)
\[
\lambda = -\tilde{p}(\tilde{p} - u^1)(\tilde{p} - u^2),
\]
where \( \tilde{p} = \tilde{\mu} + (u^1 + u^2)/3 \), the dispersionless limit of the Boussinesq system (82)
\[
u_y^1 = \frac{1}{6} \partial_x [(u^1)^2 - 2u^1 u^2], \\
u_y^2 = \frac{1}{6} \partial_x [(u^2)^2 - 2u^1 u^2]
\]
satisfies the Gibbons equation
\[
\lambda_y - \left( \tilde{p} - \frac{u^1 + u^2}{3} \right) \lambda_x = \frac{\partial \lambda}{\partial \tilde{p}} \left[ \tilde{p}_y - \partial_x \left( \frac{\tilde{p}^2}{2} - \frac{u^1 + u^2}{3} \tilde{p} \right) \right].
\]

Remark: Generating functions of conservation laws
\[
p_y = -\partial_x \frac{v^2}{2\tilde{p}^2}, \\
\tilde{p}_y = \partial_x \left( \frac{\tilde{p}^2}{2} - \frac{u^1 + u^2}{3} \tilde{p} \right)
\]

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have different sets of conservation law densities $H_k$, which coincide for nonlinear elasticity equation (82) up to insufficient factors.

In another paper we present a classification of integrable hydrodynamic chains based on the concept of generating functions of conservation laws. For instance, all generating functions of conservation laws (79) can be found. Thus, at least two of them are connected with the ideal gas dynamics (63); each two component hydrodynamic type system (1) must be connected with some function $\psi(u, v, p)$.

We cannot suggest the recipe how to construct this function $\psi(u^1, u^2, ..., u^N; p)$ for any a priori given hydrodynamic type system (1) in general case. However, for any given function $\psi(u^1, u^2, ..., u^N; p)$ we are able to reconstruct a corresponding hydrodynamic type system (1) together with its commuting flows.

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References

[1] D.J. Benney, Some properties of long non-linear waves, Stud. Appl. Math., 52 (1973) 45-50.

[2] L.V. Bogdanov, B.G. Konopelchenko, Symmetry constraints for dispersionless integrable equations and systems of hydrodynamic type, Phys. Lett. A, 330 (2004) 448–459.

[3] J.C. Brunelli, A. Das, A Lax description for polytropic gas dynamics, Phys. Lett. A 235 No. 6 (1997) 597–602.

[4] G. Darboux, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes, Paris (1910).

[5] B.A. Dubrovin, Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models, Comm. Math. Phys., 145 (1992) 195-207. B.A. Dubrovin, Geometry of 2D topological field theories, Lecture Notes in Math. 1620, Springer-Verlag (1996) 120-348.

[6] B.A. Dubrovin, S.P. Novikov, Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, Soviet Math. Dokl., 27 (1983) 665–669. B.A. Dubrovin, S.P. Novikov, Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, Russian Math. Surveys, 44 No. 6 (1989) 35–124.
[7] E.V. Ferapontov, Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, Amer. Math. Soc. Transl. (2), 170 (1995) 33-58.

[8] E.V. Ferapontov, K.R. Khusnutdinova, On integrability of (2+1)-dimensional quasi-linear systems, Comm. Math. Phys. 248 (2004) 187-206, E.V. Ferapontov, K.R. Khusnutdinova, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen. 37 No. 8 (2004) 2949 - 2963.

[9] E.V. Ferapontov, O.I. Mokhov, Nonlocal Hamiltonian operators of hydrodynamic type that are connected with metrics of constant curvature, Russian Math. Surveys, 45 No. 3 (1990) 218–219.

[10] E.V. Ferapontov, S.P. Tsarev, Systems of hydrodynamic type that arise in gas chromatography. Riemann invariants and exact solutions, (Russian) Mat. Model. 3 No. 2 (1991) 82-91.

[11] J. Gibbons, Collisionless Boltzmann equations and integrable moment equations, Physica D, 3 (1981) 503-511.

[12] J. Gibbons, Yu. Kodama, Solving dispersionless Lax equations. In N. Ercolani et al., editor, Singular limits of dispersive waves, v. 320 of NATO ASI Series B, page 61. Plenum (1994) New York.

[13] J. Gibbons, S.P. Tsarev, Reductions of the Benney equations, Phys. Lett. A 211 (1996) 19-24. J. Gibbons, S.P. Tsarev, Conformal maps and reductions of the Benney equations, Phys. Lett. A 258 (1999) 263-271.

[14] J. Gibbons, L.A. Yu, The initial value problem for reductions of the Benney equations, Inverse Problems 16 No. 3 (2000) 605-618, L.A. Yu, Waterbag reductions of the dispersionless discrete KP hierarchy, J. Phys. A: Math. Gen., 33 (2000) 8127–8138.

[15] J.K. Haantjes, On $X_{m-1}$-forming sets of eigenvectors, Indagationes Mathematicae 17 (1955) 158-162.

[16] Yu. Kodama, A method for solving the dispersionless KP equation and its exact solutions. Phys. Lett. A, 129 No. 4 (1988) 223-226. Yu. Kodama, A solution method for the dispersionless KP equation, Prog. Theor. Phys. Supplement. 94 (1988) 184.

[17] V.V. Kozlov, Polynomial integrals of dynamical systems with one-and-a-half degrees of freedom. (Russian) Mat. Zametki 45 No. 4 (1989) 46–52; translation in Math. Notes 45 No. 3-4 (1989) 296–300.

[18] I.M. Krichever, The averaging method for two-dimensional "integrable" equations, Funct. Anal. Appl. 22 No. 3 (1988) 200-213, I.M. Krichever, Spectral theory of two-dimensional periodic operators and its applications, Russian Math. Surveys 44 No. 2 (1989) 145-225.
[19] I.M. Krichever, The dispersionless equations and topological minimal models, Comm. Math. Phys., 143 No. 2 (1992) 415-429. I.M. Krichever, The $\tau$-function of the universal Whitham hierarchy, matrix models and topological field theories, Comm. Pure Appl. Math. 47 (1994) 437-475.

[20] B.A. Kupershmidt, Deformations of integrable systems, Proc. Roy. Irish Acad. Sect. A 83 No. 1 (1983) 45-74. B.A. Kupershmidt, Normal and universal forms in integrable hydrodynamical systems, Proceedings of the Berkeley-Ames conference on nonlinear problems in control and fluid dynamics (Berkeley, Calif., 1983), in Lie Groups: Hist., Frontiers and Appl. Ser. B: Systems Inform. Control, II, Math Sci Press, Brookline, MA, (1984) 357-378.

[21] B.A. Kupershmidt, Hydrodynamic chains of Pavlov’s class, accepted in Phys. Lett. A.

[22] M.A. Lavrentiev, B.V. Shabat, Metody teorii funktsii kompleksnogo peremennogo (Russian) [Methods of the theory of functions of a complex variable] Third corrected edition Izdat. “Nauka”, Moscow (1965) 716 pp.

[23] O.I. Mokhov, Compatible nonlocal Poisson brackets of hydrodynamic type and related integrable hierarchies. (Russian) Teoret. Mat. Fiz. 132 No. 1 (2002) 60–73; translation in Theoret. and Math. Phys. 132 No. 1 (2002) 942–954. O.I. Mokhov, The Liouville canonical form of compatible nonlocal Poisson brackets of hydrodynamic type, and integrable hierarchies. (Russian) Funktsional. Anal. i Prilozhen. 37 No. 2 (2003) 28–40; translation in Funct. Anal. Appl. 37 No. 2 (2003) 103–113.

[24] M.V. Pavlov, Integrable systems and metrics of constant curvature, Journal of Nonlinear Mathematical Physics. No. 9 (2002) Supplement 1, 173-191.

[25] M.V. Pavlov, Integrable hydrodynamic chains, J. Math. Phys. 44 No. 9 (2003) 4134-4156.

[26] M.V. Pavlov, The Kupershmidt hydrodynamic chains and lattices.

[27] M.V. Pavlov, S.I. Svinolupov, R.A. Sharipov, An invariant criterion for hydrodynamic integrability, Funktsional. Anal. i Prilozhen. 30 (1996) 18-29; translation in Funct. Anal. Appl. 30 (1996) 15-22.

[28] M. V. Pavlov, S.P. Tsarev, Three-Hamiltonian structures of the Egorov hydrodynamic type systems, Funct. Anal. Appl., 37 No. 1 (2003) 32-45.

[29] C. Rogers, W. F. Shadwick, Bäcklund Transformations and their Applications, Academic Press (1982) NewYork.

[30] B.L. Rozhdestvenski, N.N. Yanenko, Systems of quasilinear equations and their applications to gas dynamics. Translated from the second Russian edition by J. R. Schumengerber. Translations of Mathematical Monographs, 55. American Mathematical Society, Providence, RI, 1983; Russian ed. Nauka, (1968) Moscow.
[31] S.P. Tsarev, On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl., 31 (1985) 488–491. S.P. Tsarev, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya 37 No. 2 (1991) 397–419.

[32] S.P. Tsarev, private communications (1985).

[33] G.B. Whitham, Linear and Nonlinear Waves, Wiley–Interscience, (1974) New York.

[34] V.E. Zakharov, Benney’s equations and quasi-classical approximation in the inverse problem method, Funct. Anal. Appl., 14 No. 2 (1980) 89-98. V.E. Zakharov, On the Benney’s Equations, Physica 3D (1981) 193-200.