On Viviani’s Theorem and its Extensions

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Abstract

Viviani’s theorem states that the sum of distances from any point inside an equilateral triangle to its sides is constant. We consider extensions of the theorem and show that any convex polygon can be divided into parallel segments such that the sum of the distances of the points to the sides on each segment is constant. A polygon possesses the CVS property if the sum of the distances from any inner point to its sides is constant. An amazing result, concerning the converse of Viviani’s theorem is deduced; Three non-collinear points which have equal sum of distances to the sides inside a convex polygon, is sufficient for possessing the CVS property. For concave polygons the situation is quite different, while for polyhedra analogous results are deduced.

Key words: distance sum function, CVS property, isosum segment, isosum cross section.

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1 Introduction

Let \( \mathcal{P} \) be a polygon or polyhedron, consisting of both boundary and interior points. Define a distance sum function \( V: \mathcal{P} \to \mathbb{R} \), where for each point \( P \in \mathcal{P} \) the value \( V(P) \) is defined as the sum of the distances from the point \( P \) to the sides (faces) of \( \mathcal{P} \).

We say that \( \mathcal{P} \) has the constant Viviani sum property, abbreviated by the ”CVS property”, if and only if the function \( V \) is constant.

Viviani (1622-1703), who was a student and assistant of Galileo, discovered the theorem which states that equilateral triangles have the CVS property. The theorem can be easily proved by an area argument; Joining a point \( P \) inside the triangle to its vertices divides it into three parts, the sum of their areas will be equal to the area of the original one. Therefore, \( V(P) \) will be equal to the height of the triangle and the theorem follows. The importance of Viviani’s theorem
may be derived from the fact that his teacher Torricelli (1608-1647) used it to locate the Fermat point of a triangle [2, pp. 443].

Samelson [6, pp. 225] gave a proof of Viviani’s theorem that uses vectors and Chen & Liang [1, pp. 390-391] used this vector method for proving the converse of the theorem: If inside a triangle there is a circular region in which $V$ is constant then the triangle is equilateral.

Kawasaki [3, pp. 213], by a proof without words, uses only rotations to establish Viviani’s theorem. There is an extension of the theorem to all regular polygons, by the area method: All regular polygons have the CVS property. There is also an extension of the theorem to regular polyhedrons, by a volume argument: All regular polyhedra have the CVS property. Kawasaki, Yagi and Yanagawa [4, pp. 283] gave a different proof for the regular tetrahedron.

What happens for general polygons and polyhedra?

Surprisingly, there is a strict correlation between Viviani’s theorem, its converse and extensions to linear programming.

This correlation is manifested by the following main result:

**Theorem 1.1** (a) Any convex polygon can be divided into parallel segments such that $V$ is constant on each segment.

(b) Any convex polyhedron can be divided into parallel cross sections such that $V$ is constant on each cross section.

These segments or cross sections, on which $V$ is constant, will be called isosum layers (or definitely isosum segments and isosum cross sections). They are formed by the intersection of the feasible region and a suitable family of parallel lines (planes). The value of the function $V$ will increase when passing, in some direction, from one isosum layer into another, unless the feasible region has the CVS property.

The correlation soon will be clear. Each linear programming problem is composed of an objective function and a feasible region (see for example [5] or [7]). Moreover, the objective function divides the feasible region into isoprofit layers, these layers are parallel and consist of points on which the objective function has constant value. Furthermore, moving in some direction will increase the value of the objective function unless it is constant in the feasible region.

Because of this correlation we thus conclude the following amazing result, concerning the converse of Viviani’s theorem.

**Theorem 1.2** (a) If $V$ takes equal values at three non-collinear points, inside a convex polygon, then the polygon has the CVS property.

(b) If $V$ takes equal values at four non-coplanar points, inside a convex polyhedron, then the polyhedron has the CVS property.

The theorem tells us that measuring the distances from the sides of three non-collinear points, inside a convex polygon, is sufficient for determining if the polygon has the CVS property. Likewise, measuring the distances from the faces of four non-coplanar points, inside a convex polyhedron, is sufficient for determining if the polyhedron possesses the CVS property.

We then end with the following beautiful conclusions.
Corollary 1.3  
(a) If there is an isometry of the plane which fixes the polygon but not an isosum segment, then the polygon has the CVS property.

(b) If a convex polygon possesses a rotational symmetry, around a central point, then the polygon has the CVS property.

(c) If a convex polygon possesses a reflection symmetry across an axis \( l \), then the polygon has the CVS property or otherwise the isosum segments are perpendicular to \( l \).

While for polyhedra we have,

Corollary 1.4  
(a) If there is an isometry of the space which fixes the polyhedron but not an isosum cross section, then the polyhedron has the CVS property.

(b) If a convex polyhedron possesses two rotational symmetries, around different axes, then the polyhedron has the CVS property.

From Corollary 1.3, one can deduce that all regular polygons have the CVS property. Besides, any parallelogram has this property, since it possesses a rotational symmetry around its centroid by an angle of 180°.

Obviously, the existence of two reflection symmetries, by different axes, of a polygon will imply a rotational symmetry and hence the polygon must own the CVS property.

Moreover, for triangles and quadrilaterals, the existence of a rotational symmetry characterizes the possessing of CVS property, since in these cases the polygons would be only equilateral triangles and parallelograms.

Consequently, an \( n \)-gon, for \( n \geq 5 \), that does not possess the CVS property must have at most one symmetry which is the reflection symmetry.

Analogously, by Corollary 1.4, all regular polyhedra and regular prisms have the CVS property. Likewise, any parallelepiped has the CVS property. Since it possesses three rotational symmetries by an angle of 180°, around the axes, each passing through the centroids of a pair of parallel faces.

On the other hand, the property of possessing a symmetry does not characterize all polygons (polyhedra) satisfying the CVS property. In section IV. we will validate the existence of a polygon with only one reflection symmetry, an asymmetric polygon and a polyhedron with a reflection symmetry only, which possess the CVS property.

We will proceed as follows. In section II., we first introduce a linear programming problem for general triangles. The main statement will be;

"A triangle has the CVS property if and only if it is equilateral, if and only if there are three non-collinear points inside the triangle that have equal sum of distances from the sides".

Then we deal with general convex polygons and polyhedrons and the proof of theorems (1.1), (1.2). Here we shall rely on methods from analytic geometry because the use of coordinates. This allows us to determine the line (plane) which the isosum layers are parallel to.

In section III. we see what happens for concave polygons and polyhedra, then in section IV. we compute some examples.
It is worthy mentioning that these results can be stated and generalized for $n$-dimensional geometry.

2 Convex Polygons and Polyhedra

2.1 The case of triangle: linear programming approach

Given a triangle $\triangle ABC$ let $a_1, a_2, a_3$ be the lengths of the sides $BC, AC, AB$ respectively. Let $P$ be a point inside the triangle and let $h_1, h_2, h_3$ be the distances (the lengths of the altitudes) of the point $P$ to the three sides respectively, see Figure 1.

For $1 \leq i \leq 3$, let $x_i = \frac{h_i}{\sum_{i=1}^{3} h_i}$, where as previously defined, $\sum_{i=1}^{3} h_i = V(P)$. Clearly, for each $1 \leq i \leq 3$, we have $0 \leq x_i \leq 1$ and $\sum_{i=1}^{3} x_i = 1$. Denote $x = (x_1, x_2, x_3)$ and consider the linear function in three variables $F(x) = \sum_{i=1}^{3} a_i x_i$. Now, this function is closely related to the function $V$. Accurately, $F(x) = \sum_{i=1}^{3} a_i x_i = \frac{\sum_{i=1}^{3} a_i h_i}{\sum_{i=1}^{3} h_i} = \frac{2S}{V(P)}$, where $S$ is the area of the triangle.

Consequently, $F(x) = \sum_{i=1}^{3} a_i x_i$ takes equal values in a subset of points of the feasible region if and only if the function $V$ takes equal values at the corresponding points inside the triangle.

Thus we may define the following linear programming problem:

The objective function is:

$$F(x) = \sum_{i=1}^{3} a_i x_i$$

subject to the following constraints:

$$\left\{ \begin{array}{l}
\sum_{i=1}^{3} x_i \leq 1 \\
x_i \geq 0, \quad 1 \leq i \leq 3
\end{array} \right.$$ 

Now, solving the problem means maximizing or minimizing the objective function in the feasible region, and this optimal value must occur at some corner point. For that, one may use the simplex method, which is simple in this case because the simplex tableau contains only two rows. But rather than using this algebraic method, and since we are not seeking for optimal values, and for better understanding the correlation with Viviani’s theorem, we use a geometric method. The feasible region will be the right pyramid with vertex $(0, 0, 1)$ and basis vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and the isoprofit planes are obtained by taking $F(x) = \sum_{i=1}^{3} a_i x_i = c$, where $c$ is constant, see Figure 2.

These isoprofit planes will meet the face $\sum_{i=1}^{3} x_i = 1$ across a line which in its turn will correspond to an isosum segment inside the triangle $\triangle ABC$. The value of the function $V$ will be constant for all points on the isosum segment. Moreover, moving in a perpendicular direction to the isosum segments will increase the value of $V$. Hence, $V$ does not take the same value at any three non-collinear points inside the triangle.
This is true in general, except for one case where the isopoint planes, \( \sum_{i=1}^{3} a_i x_i = c \), are parallel to the face \( \sum_{i=1}^{3} x_i = 1 \). This exceptional case occurs if and only if the vectors \((a_1, a_2, a_3)\) and \((1, 1, 1)\) are linearly dependent i.e.,

\[
(a_1, a_2, a_3) = \lambda (1, 1, 1),
\]

and this happens if and only if the triangle is equilateral. Thus proving the following theorem.

**Theorem 2.1**

(a) Any triangle can be divided into parallel segments such that \( \mathcal{V} \) is constant on each segment.

(b) The following conditions are equivalent

- The triangle \( \triangle ABC \) has the CVS property.
- There are three non-collinear points, inside the triangle, at which \( \mathcal{V} \) takes the same value.
- \( \triangle ABC \) is equilateral.

### 2.2 Extension for convex polygons: analytic geometry approach

Inspired by the geometric method of the linear programming problem defined in the previous subsection, we attack the general case by using analytic geometry techniques.

Firstly, we aim to prove Theorems 1.1(a) and 1.2(b).

Given a polygon with \( n \) sides, we embed it in a Cartesian plane. Suppose that its sides lie on lines with equations:

\[
\alpha_i x + \beta_i y + \gamma_i = 0. \tag{1}
\]

Since the convex polygon lies on the same side of line (1) then the expression \( \alpha_i x + \beta_i y + \gamma_i \) has unchanging sign, for all \( P = (x, y) \) inside the polygon.

Thus the distance, \( h_i \), of the point \( P \) from each side of the polygon is given by the equation:

\[
h_i = (-1)^{\varepsilon_i} \frac{\alpha_i x + \beta_i y + \gamma_i}{\sqrt{\alpha_i^2 + \beta_i^2}},
\]

where \( \varepsilon_i \in \{0, 1\} \).

Therefore, the function \( \mathcal{V} \) is given by a linear expression

\[
\mathcal{V}(x, y) = \sum_{i=1}^{n} (-1)^{\varepsilon_i} \frac{\alpha_i x + \beta_i y + \gamma_i}{\sqrt{\alpha_i^2 + \beta_i^2}}. \tag{2}
\]
Letting this sum equals some constant $c$, we get the equation of the line which the isosum segments are parallel to:

$$\sum_{i=1}^{n} (-1)^{\varepsilon_i} \frac{\alpha_i x + \beta_i y + \gamma_i}{\sqrt{\alpha_i^2 + \beta_i^2}} = c. \quad (3)$$

If the point $P = (x, y)$ is restricted to be inside the polygon then for different values of $c$, we get parallel segments. On each such segment, the function $\mathcal{V}$ takes the constant value $c$. This proves Theorem 1.1 (a).

The equation given by (2) is independent of $(x, y)$ if and only if the variable part of the function $\mathcal{V}$ vanishes. But then $\mathcal{V}$ is constant and so the polygon has the CVS property.

If there are three non-collinear points inside the polygon at which $\mathcal{V}$ takes the same value then there exist two different isosum segments on which the function $\mathcal{V}$ takes the same value. This happens if and only if $\mathcal{V}$ is constant and so the polygon once again has the CVS property. Hence Theorem 1.2(a) is proved.

We turn now to proving Corollary 1.3.

(a) If there is an isometry of the plane which fixes the polygon but not some isosum layer, then this will assure the existence of three non-collinear points inside the polygon at which $\mathcal{V}$ takes equal values. That’s because an isometry preserves distances and the sets of boundary and inner points. Thus Theorem 1.2(a) yields that the polygon has the CVS property.

(b) Follows from part (a).

(c) If the isosum segments are not fixed by the reflection, then according to part (a) the polygon has the CVS property. Otherwise, if fixed, then the isosum segments must be perpendicular to the reflection axis.

### 2.3 Extension for convex polyhedra

The proof for a Polyhedron is similar up to minor modifications. The faces lie on planes with equations:

$$\alpha_i x + \beta_i y + \gamma_i z + \delta_i = 0,$$

and the linear function $\mathcal{V}$ becomes:

$$\mathcal{V}(x, y, z) = \sum_{i=1}^{n} (-1)^{\varepsilon_i} \frac{\alpha_i x + \beta_i y + \gamma_i z + \delta_i}{\sqrt{\alpha_i^2 + \beta_i^2 + \gamma_i^2}},$$

where $\varepsilon_i \in \{0, 1\}$.

The same argument yields the result for the polyhedron (part (b) of Theorems 1.1 and 1.2).

The proof of Corollary 1.4 is similar to corollary 1.3:

(a) Follows from part (b) of Theorem 1.2.

(b) Two rotational symmetries of a polyhedron around different axes, guarantee the existence of four non-coplanar points inside the polyhedron at which $\mathcal{V}$ takes equal values. By Theorem 1.2 (b), the result follows.
3 Concave Polygons and Polyhedra

The situation for concave polygons and polyhedra is quite different. Theorems 1.1 and 1.2 are no longer valid. Moreover, concave polygons and polyhedra don’t have the CVS property. Surprisingly, with a little more elaborate effort, one might rather find a generality of Theorem 1.1. Concerning polygons, the crucial point which makes the difference is that the points inside a concave polygon are no longer lie on the same side of each boundary line. This was a key for defining the distance sum function $V$.

We turn to giving an example to illustrate the theme. Let $ABCD$ be the concave kite with vertices $(0, 8), (-6, 0), (0, 2.5), (6, 0)$ respectively. Let $l_1, l_2, l_3, l_4$ be the lines containing the sides $AB, BC, CD, DA$ and $E, F$ be the intersection points of $l_1, l_3$ and $l_2, l_4$ respectively, see Figure 3. Then the concave kite $ABCD$ is divided into three distinct convex polygonal regions, namely; $AECF, EBC$ and $FCD$. All points inside any region lie on the same side of each $l_i, 1 \leq i \leq 4$.

To explain that, note that any line $l$ divides the plane into two half-planes “$O_l$” which contains the origin and its complement ”$O_l^c$” . The following table shows the location of the points inside any region relative to the lines $l_i, 1 \leq i \leq 4$.

| region  | $l_1$ | $l_2$ | $l_3$ | $l_4$ |
|---------|-------|-------|-------|-------|
| AECF    | $O_{l_1}$ | $O_{l_2}^c$ | $O_{l_3}$ | $O_{l_4}$ |
| EBC     | $O_{l_1}$ | $O_{l_2}$ | $O_{l_3}$ | $O_{l_4}$ |
| FCD     | $O_{l_1}$ | $O_{l_2}$ | $O_{l_3}^c$ | $O_{l_4}$ |

Note that any two neighboring regions, having a common edge, differ only in one entry in this table. For instance, the points of AECF and EBC lie on the same side of $l_1, l_2, l_4$ and on opposite sides of $l_3$. The implementation is that the distance sum function $V$ will be a split function composed of three components;

$$V_{ABCD}(P) = \begin{cases} 
  V_{AECF}(P), & \text{P inside AECF} \\
  V_{EBC}(P), & \text{P inside EBC} \\
  V_{FCD}(P), & \text{P inside FCD}
\end{cases}$$

where each component is given by a linear expression as in equation (2), definitely;
This says that each region can be divided into parallel isosum segments but in a different direction. In our example, the isosum segments of $AECF$ are parallel to a line of the form $y = c$, the isosum segments of $EBC$ are parallel to the line $100x + 156y = 0$ and the isosum segments of $DCF$ are parallel to the line $100x − 156y = 0$. Moreover, two isosum segments from neighboring regions which meet at a point on the common edge, define three non-collinear points with the same distance sum from the sides of $ABCD$, see Figure 3.

These ideas can be generalized to any concave polygon or polyhedron. Thus we have the following theorem;

**Theorem 3.1**

(a) Any concave polygon can be divided into convex polygonal regions such that each region can be divided into parallel isosum segments. Moreover, isosum segments of neighboring regions have different directions.

(b) There are three non-collinear points inside a concave polygon which have equal distance sum from the sides.

(c) Any concave polyhedron can be divided into convex polyhedral regions such that each region can be divided into parallel isosum cross sections. Moreover, isosum cross sections of neighboring regions have different directions.

(d) There are four non-coplanar points inside a concave polyhedron which have equal distance sum from the faces.

(e) Concave polygons and polyhedra do not possess the CVS property.

**Proof.** We shall outline the proof for polygons.

(a) Let $P$ a concave polygon. Extend the sides and construct all possible intersection points in $P$ of the boundary lines, see Figure 4. In this way one gets a partition of $P$ into $m$ convex polygonal regions $P_1, P_2, ..., P_m$. Note that each side of any $P_j$ lie on a boundary line of $P$. The points inside any two neighboring regions $P_i, P_j$ with a common edge $LM$, lie on the same side of each boundary line of $P$ except one, namely, the boundary line $l_{LM}$ containing $LM$, see Figure 4. We claim that the conjugation of these two neighboring regions $P_i \cup P_j$, will remain a convex polygonal region. This is true since the two sides of $P_i$ and $P_j$ which meet at $L$ (or $M$) lie on the same boundary line of $P$, by the construction (in Figure 4 the points $K, L, D$ are collinear, likewise $I, M, D
are collinear). Thus all points of \( P_i \cup P_j \) lie on the same side of the boundary lines of \( P \) except \( l_{LM} \).

Now, the distance sum function \( V \) of \( P \) is defined as a split function;

\[
V_P(P) = \begin{cases} 
V_{P_1}(P), & P \text{ inside } P_1 \\
V_{P_2}(P), & P \text{ inside } P_2 \\
\vdots & \\
V_{P_m}(P), & P \text{ inside } P_m 
\end{cases}
\]

The points on a common edge of neighboring regions can be attached to any one of both. Each convex polygonal region has parallel isosum segments according to the linear expression which defines \( V_{P_i}, 1 \leq i \leq m \). Isosum segments of neighboring regions \( P_i, P_j \) have different directions since the linear expressions of \( V_{P_i}, V_{P_j} \) differ only in one sign as explained above.

(b) Take any point \( P \) on a common edge of two neighboring regions \( P_i, P_j \) and let \( s_1, s_2 \) be two isosum segments of \( P_i, P_j \) respectively, issuing from \( P \). Let \( Q_1, Q_2 \) two points on \( s_1, s_2 \) respectively. Since \( s_1, s_2 \) have different directions then \( P, Q_1, Q_2 \) are non-collinear and have the same distance sum from the sides.

c) The distance sum functions \( V_{P_i}, V_{P_j} \) of two neighboring regions \( P_i, P_j \) have the same linear expressions with the same signs except one; \( V_{P_i} = \sum_{i=1}^{m} (a_k x + b_k y + c_k) \) and \( V_{P_j} = \sum_{i=1}^{m} (a_k x + b_k y + c_k) - (a_0 x + b_0 y + c_0) \). Thus \( V_P \) cannot be constant on \( P \).

4 Examples

In the following we compute the equations of the isosum layers for particular polygons and polyhedra. It is an easy matter to check the results, by constructing the figures using a computer program such as the geometer sketchpad or wingeom.

Example 4.1 (Kite)

Construct a kite in the Cartesian plane with vertices, \( A = (0, \beta), B = (-\alpha, 0), C = (0, \gamma), D = (\alpha, 0) \), where, \( \alpha, \beta > 0 \) and \( \gamma < 0 \).

The equations of the lines containing the sides \( AB, BC, CD, DA \), respectively are:

\[-\alpha y + \beta x + \alpha \beta = 0 \]
\[-\alpha y + \gamma x + \alpha \gamma = 0 \]
\[\alpha y + \gamma x - \alpha \gamma = 0 \]
\[\alpha y + \beta x - \alpha \beta = 0 \].
Thus the function $V$ is given by:

$$V_{ABCD} = -\frac{\alpha y + \beta x + \alpha \beta}{\sqrt{\alpha^2 + \beta^2}} - \frac{\alpha y + \gamma x + \alpha \gamma}{\sqrt{\alpha^2 + \gamma^2}} + \frac{\alpha y + \gamma x - \alpha \gamma}{\sqrt{\alpha^2 + \gamma^2}} - \frac{\alpha y + \beta x - \alpha \beta}{\sqrt{\alpha^2 + \beta^2}}. $$

Equivalently,

$$V_{ABCD} = -\frac{2\alpha y}{\sqrt{\alpha^2 + \beta^2}} + \frac{2\alpha \beta}{\sqrt{\alpha^2 + \beta^2}} + \frac{2\alpha y}{\sqrt{\alpha^2 + \gamma^2}} - \frac{2\alpha \gamma}{\sqrt{\alpha^2 + \gamma^2}}. \quad (4)$$

As a result we see that the isosum segments have simple equations, namely, $y = c$. Obviously, equation (4), shows that the $V_{ABCD}$ is independent of variables if and only if $\beta = \gamma$. This means that the polygon is in fact a parallelogram and hence, it owns the CVS property. On the other side, if $\beta \neq \gamma$, then the kite does not own the CVS property, but it can be divided into isosum segments which are parallel to the diagonal $BD$, which is consistent with Corollary 1.3 (c).

**Example 4.2 (Isosceles triangle)**

In the previous example if one considers $P$ as a point inside the isosceles triangle $ABD$ then,

$$V_{ABD} = -\frac{\alpha y + \beta x + \alpha \beta}{\sqrt{\alpha^2 + \beta^2}} - \frac{\alpha y + \beta x - \alpha \beta}{\sqrt{\alpha^2 + \beta^2}} + y.$$

Equivalently,

$$V_{ABD} = \frac{(\sqrt{\alpha^2 + \beta^2} - 2\alpha)y}{\sqrt{\alpha^2 + \beta^2}} + \frac{2\alpha \beta}{\sqrt{\alpha^2 + \beta^2}}.$$

The function $V_{ABD}$ is independent of variables if and only if $\beta = \sqrt{3}\alpha$. In which case the triangle is in fact equilateral and hence it has the CVS property.

If $\beta \neq \sqrt{3}\alpha$, then it can be divided into isosum segments parallel to the base, which is consistent with Corollary 1.3 (c)

**Example 4.3 (Quadrilateral)**

Let be given the quadrilateral with vertices $(0, 0), (3, 0), (1, 2), (0, 1)$. Direct computations will show that the isosum segments are parallel to the line $y = (1 + \sqrt{2})x$, see Figure 5. This quadrilateral does not have the CVS property.

**Example 4.4 (Pentagon with a reflection symmetry)**
The pentagon $ABCDE$ with vertices $A = (0, 3 + \sqrt{3}), B = (-1, 3), C = (-1, 0), D = (1, 0), E = (1, 3)$ and sides $2, 3, 2, 3, 2$ respectively, owns the CVS property although it has one reflection symmetry and no rotational symmetries. It is a straightforward matter, to show the possessing of the CVS property, since it is composed of a rectangle and an equilateral triangle.

In general, suppose that $ABCDE$ is a pentagon where $ABE$ is an equilateral triangle with side-length $a$ and height $h$, and $BCDE$ is a rectangle with side-lengths $a, b$. Then the distance sum of an inner point from its sides is $a + b + h$. This is obvious if the point is inside the triangle. If the point is inside the rectangle then observe that it is located on the base of another equilateral triangle obtained by extending the sides $AB, AE$ and the result easily follows.

**Example 4.5 (Asymmetric pentagon)**

Construct any pentagon $ABCDE$ with angles $70^\circ, 110^\circ, 130^\circ, 60^\circ$ and $170^\circ$. Here again it is a straightforward matter, to show that $ABCDE$ possess the CVS property, since it is composed of a parallelogram and an equilateral triangle. This pentagon has no symmetries.

Observe that in this case, the distance sum of an inner point from the sides is $a + b + h$, where $a, b$ are the distances between the opposite sides of the parallelogram and $h$ is the height of the equilateral triangle.

**Example 4.6 (Equianangular polygon)**

Any equiangular polygon $\mathcal{P}$ with $n$ sides has the CVS property. This can be demonstrated by the following argument; Locate inside the equiangular polygon a regular $n$-gon, $\mathcal{P}_r$. Rotate $\mathcal{P}_r$ around its centroid until one of its sides gets parallel to one side of $\mathcal{P}$. But then all corresponding sides of both polygons get parallel. Let $\nu_\mathcal{P}, \nu_\mathcal{P}_r$ be the distance sum functions defined on $\mathcal{P}$ and $\mathcal{P}_r$ respectively. Then for any point $P$ inside $\mathcal{P}_r$ we have,

$$\nu_\mathcal{P}(P) = \nu_\mathcal{P}_r(P) + c,$$

where $c$ represents the sum of distances between the parallel sides of $\mathcal{P}$ and $\mathcal{P}_r$.

Since $\nu_\mathcal{P}_r$ is constant inside $\mathcal{P}_r$ and also $c$ is constant then $\nu_\mathcal{P}$ is constant inside $\mathcal{P}_r$. By Theorem 1.2(a), $\nu_\mathcal{P}$ is constant on $\mathcal{P}$.

**Example 4.7 (Polyhedron with a reflection symmetry)**

Construct the right prism for which the pentagon $ABCDE$ from the previous example congruent to its bases. This prism has only one symmetry, namely, the reflection symmetry by the plane parallel to its two bases and passes in the middle. This polyhedron has the CVS property. Since vertically the sum of the two altitudes to the bases from any inner point is constant and equals to the height of the prism and, horizontally, the sum of the altitudes to the faces is also constant by the previous example.
Example 4.8 (Pyramid with a rotational symmetry around one axis)

Construct in the space, the rectangular pyramid with vertices

\[ A = (0, 0, \alpha), \quad B = (\beta, 0, 0), \quad C = (0, \gamma, 0), \quad D = (-\beta, 0, 0) \text{ and } E = (0, -\gamma, 0), \]

where \( \alpha, \beta, \gamma > 0 \).

The faces \( ABC, ACD, ADE \) and \( AEB \) lie on the following planes respectively:

\[
\begin{align*}
\alpha \gamma x + \alpha \beta y + \beta \gamma z &= \alpha \beta \gamma \\
-\alpha \gamma x + \alpha \beta y + \beta \gamma z &= \alpha \beta \gamma \\
-\alpha \gamma x - \alpha \beta y + \beta \gamma z &= \alpha \beta \gamma \\
\alpha \gamma x - \alpha \beta y + \beta \gamma z &= \alpha \beta \gamma.
\end{align*}
\]

Letting \( \Delta = \sqrt{\alpha^2 \gamma^2 + \alpha^2 \beta^2 + \beta^2 \gamma^2} \) then the altitudes from the point \( P = (x, y, z) \) to the faces are given by:

\[
\begin{align*}
\hspace{1cm} h_{BCDE} &= z \\
\hspace{1cm} h_{ABC} &= -\frac{1}{\Delta}(\alpha \gamma x + \alpha \beta y + \beta \gamma z - \alpha \beta \gamma) \\
\hspace{1cm} h_{ACD} &= -\frac{1}{\Delta}(\alpha \gamma x + \alpha \beta y + \beta \gamma z - \alpha \beta \gamma) \\
\hspace{1cm} h_{ADE} &= -\frac{1}{\Delta}(\alpha \gamma x - \alpha \beta y + \beta \gamma z - \alpha \beta \gamma) \\
\hspace{1cm} h_{AEB} &= -\frac{1}{\Delta}(\alpha \gamma x - \alpha \beta y + \beta \gamma z - \alpha \beta \gamma).
\end{align*}
\]

Thus

\[
\mathcal{V}_{ABCDE} = (1 - \frac{4 \beta \gamma}{\Delta})z + \frac{4 \alpha \beta \gamma}{\Delta}.
\]

Evidently, the isosum layers are parallel to the base of the pyramid and it has the \( CVS \) property if and only if

\[
\alpha = \frac{\sqrt{15} \beta \gamma}{\sqrt{\beta^2 + \gamma^2}}.
\]

Figure 6 shows a special case where, \( \beta = \gamma = 1 \) and \( \alpha = \sqrt{15/2} \).

5 Concluding Remarks

- Corollary 1.3 part (c) can be extended for polyhedra but the property of being perpendicular to the reflection plane would not uniquely determine the isosum cross sections.

- It is possible to state an algebraic necessary and sufficient condition for the \( CVS \) property, using the expression of \( \mathcal{V} \) given in (2). But a geometric one is more favorable.

Thus the substantial question is how can one characterizes, geometrically, all those polygons and polyhedra that satisfy the \( CVS \) property?
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Figure 1: A point $P$ inside the triangle with distances $h_1$, $h_2$, $h_3$. 
Figure 2: The feasible region with an isoprofit plane: $F(x) = 0$.

Figure 3: The isosum segments have different directions and $X, Y, Z$ are non-collinear points with equal distance sum from the sides of $ABCD$. 
Figure 4: Partitioning the concave polygon $ABCDEFGH$ into convex polygonal regions.

Figure 5: The isosum segments are parallel to the line $y = (1 + \sqrt{2})x$. 
Figure 6: A square pyramid which has the CVS property.