Analysis stability of predator-prey model with Holling type I predation response function and stage structure for predator

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Abstract. This paper developed mathematical models with structural stages from predators, namely small predators and adult predators. The predation function of adult predators follows the Holling I response function according to the characteristics in the ecosystem. In this model, an analysis of the equilibrium value and stability of the interior equilibrium value is carried out. Analysis of the stability of interior equilibrium values with the system linearization method at the value of equilibrium. Four equilibrium values were realized. Each equilibrium value has its own characteristics and conditions. The stability characteristics of equilibrium values were obtained using the Routh-Hurwitz criteria. Also, consider the eigenvalue of the characteristic equation in the Jacobi matrix. We also identify population changes made with cases that allow the ecosystem to work.

1. Introduction
Research on predator-prey models has been carried out by many previous researchers. The most important reference in the predator-prey model is the interaction in an ecosystem of the life of living things. Research carried out for predators with stage-structure, mostly depicts events in the ecosystem. The stage-structure in the predator-prey model intends to see the ecosystem of living things that have a longer growth process. So that it will be divided into immature and mature both on predators or prey on the model.

Thieme discusses a population that uses stage-structure differential equations in predators [1], conducted a predator-prey model specifically on the prey population as a stage-structure [2], examined the interaction between predator-prey and stage-structured populations in the two-predator population [3], The effects of time-delay, in the stage-structure population [4]. Analyzed the effect of time delay on different populations in a model, where the population remained using the stage-structure [5], discussed the analysis of stability and maximum profit values in the predator-prey and stage-structure population models [6].
2. Literature review

\[
\frac{dx}{dt} = x \rho_1 \left(1 - \frac{x}{k_1}\right) - \frac{\beta xz}{\phi + \tau x}
\]

\[
\frac{dy}{dt} = \frac{c \beta xz}{\phi + \tau x} - (\alpha + \delta_1) y
\]

\[
\frac{dz}{dt} = \alpha y - \delta_2 z
\]

with the initial value
\[x(0) > 0; \ y(0) \geq 0; \ z(0) \geq 0\]

Mathematical modeling by Subhas Khajanchi [7], describes three forms of differential equations with small predators \(y\) that are not directly related to prey. Immature does not have reproductive abilities and its survival is very dependent on mature and prey interactions. While for predatory populations that are mature \(z\) by preying on the population of prey \(x\). Predators, in this case, are assumed to possess the ability to reproduce and have direct interaction with prey in the ecosystem. For predation figures from mature, it is \(\beta\) assumed to be relevant to the value of change from mature \(c\).

The population model experienced a reduction by the function response, while the prey population grew logically with \(\rho_1\) as an intrinsic growth value and \(k_1\) as the amount of capacity.

3. Methods

3.1. Literature

Study This literature study was conducted for the first time, in order to identify problems by choosing references that could be the main source of information on the theories and ideas of research ideas. Analysis of the problem of stability on the equilibrium value is the most important thing in solving this modeling problem.

3.2. Model analysis

The analysis is done by looking for equilibrium values and stability of the value of the equilibrium. The modeling equation is a non-linear differential equation, so the model is linearized first. The Jacobian matrix which is formed from the model in the study is used to see the stability of the eigenvalues or by using the Routh-Hurwitz condition.

3.3. Simulation numeric

Simulation numerics are carried out by taking parameters from several references and several assumptions. From this stage, the behavior of the solution curve from the research model will also be seen.

3.4. Analysis of simulation results

Analysis of the results of the simulation on the results of research and discussion is done by determining the equilibrium value and stability up to harvesting value by calculating selective harvesting and economic value.

3.5. Conclusion

Conclusion stage is obtained by analyzing the stability and the resulting simulation.
4. Results and discussion

4.1. Models
In the study of predator-prey models involving the interaction of adult predator populations and prey populations. Some of the assumptions used in this study are as follows.
1) The growth rate of the prey population is the growth rate of logistics.
2) The predator population has a stage-structure, which is immature and mature.
In predatory interactions, the characteristics assumed to follow the response function Holling Type I.

\[
\begin{align*}
\frac{dx}{dt} &= x \rho_1 \left(1 - \frac{x}{k_1}\right) - \beta xz \\
\frac{dy}{dt} &= c \beta xz - (\alpha + \delta_1) y \\
\frac{dz}{dt} &= \alpha y - \phi z^2 - \delta_2 z
\end{align*}
\]  

(1)

with \( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \) is the population prey, immature population and mature population. \( \rho_1 \) is the logistic growth rate for prey. \( \beta \) is the prediction number of the mature population. \( c \) and \( \alpha \) each is the rate of immature change from the results of prey and mature interactions and the efficiency of immature changes to mature. \( \delta_1 \) and \( \delta_2 \) respectively are immature and mature natural mortality rates. \( \phi \) representing mature intraspecific competition coefficients. For the predation function of prey and mature interaction, it follows the Holling I response function, \((\beta xz)\). In the system model (1) dimensionless variables are used without simplification.

**Tabel 1.** Variables definition of the system (2).

| Variable | Definition | Unit |
|----------|------------|------|
| \( x \)  | Population prey | [N]  |
| \( y \)  | Population predator (immature) | [N]  |
| \( z \)  | Population predator (mature) | [N]  |

**Tabel 2.** Values of some parameters in the system (2) from several sources.

| Par. | Values | Unit | References |
|------|--------|------|------------|
| \( \rho_1 \) | 1.5    | [T]  | [5]        |
| \( k_1 \)  | 100    | [N]  | [5]        |
| \( \phi \) | 0.04   | [N] | [4]        |
| \( \beta \) | 0.031  | [N] | [4]        |
| \( c \)  | 0.41   | -    | [5]        |
| \( \alpha \) | 0.08   | [T]  | [4]        |
| \( \delta_1 \) | 0.1    | [T]  | [4]        |
| \( \delta_2 \) | 0.008  | [T]  | [3]        |

4.2. Equilibrium point
Analysis equilibrium analysis will begin by analyzing Eq. (2) Next, we will investigate the equilibrium value of Eq. (2). Each possible equilibrium in model Eq. (2) is \( T_1(0,0,0), T_2(K,0,0), T_3(0,0,L) \) and \( T_4(x',y',z') \).
with

\[ K = k_1, \]
\[ L = \frac{-\delta_2}{\phi}, \]
\[ x' = \frac{k_1 (a \beta \delta_2 + a \phi \rho_1 + \beta \delta_1 \delta_2 + \phi \delta_1 \rho_1)}{\alpha c \beta^2 k_1 + \alpha \phi \rho_1 + \phi \delta_1 \rho_1}, \]
\[ y' = \frac{\rho_1 c \beta k_1 (a \beta k_1 - a \delta_2 - \delta_1 \delta_2) (\beta \delta_2 + \phi \rho_1)}{(\alpha c \beta^2 k_1 + \alpha \phi \rho_1 + \phi \delta_1 \rho_1)^2}, \]
\[ z' = \frac{\rho_1 (a \beta k_1 - \alpha \delta_2 - \delta_1 \delta_2)}{\alpha c \beta^2 k_1 + \alpha \phi \rho_1 + \phi \delta_1 \rho_1}. \]

4.3. Equilibrium stability analysis the stability

Analysis of the equilibrium value is by using the Routh-Hurwitz criteria. The value of the Jacobian matrix will be sought first from the differential equation in the model Eq. (2). The Jacobi matrix model Eq. (2) with \( q_i E_i = 0 \) is

\[ J = \begin{bmatrix} J_{11} & 0 & J_{13} \\ J_{21} & J_{22} & J_{23} \\ 0 & J_{32} & J_{33} \end{bmatrix} \]

with:

\[ J_{11} = \rho_1 - \frac{2 \rho_1 x}{k_1} - \beta z \]
\[ J_{13} = -\beta x \]
\[ J_{21} = c \beta z \]
\[ J_{22} = -\alpha - \delta_1 \]
\[ J_{23} = c \beta x \]
\[ J_{32} = \alpha \]
\[ J_{33} = -2 \phi z - \delta_2 \]

by subsidizing the equilibrium value \( x', y', z' \) then obtained

\[ J_{11} = \rho_1 - \frac{2 \rho_1 (a \beta \delta_2 + a \phi \rho_1 + \beta \delta_1 \delta_2 + \phi \delta_1 \rho_1)}{(ac \beta^2 k_1 + a \phi \rho_1 + \phi \delta_1 \rho_1)} - \frac{\beta \rho_1 (a \beta c k_1 - a \delta_2 - \delta_1 \delta_2)}{(ac \beta^2 k_1 + a \phi \rho_1 + \phi \delta_1 \rho_1)} \]
\[ J_{13} = -\frac{\beta k_1 (a \beta \delta_2 + a \phi \rho_1 + \beta \delta_1 \delta_2 + \phi \delta_1 \rho_1)}{(ac \beta^2 k_1 + a \phi \rho_1 + \phi \delta_1 \rho_1)} \]
\[ J_{21} = -\frac{c \beta \rho_1 (a \beta c k_1 - a \delta_2 - \delta_1 \delta_2)}{(ac \beta^2 k_1 + a \phi \rho_1 + \phi \delta_1 \rho_1)} \]
\[ J_{22} = -\alpha - \delta_1 \]
\[ J_{23} = c \beta k_1 (a \beta \delta_2 + a \phi \rho_1 + \beta \delta_1 \delta_2 + \phi \delta_1 \rho_1) \]
\[ J_{32} = \alpha \]
\[ J_{33} = -\frac{2 \phi \rho_1 (a \beta c k_1 - a \delta_2 - \delta_1 \delta_2)}{(ac \beta^2 k_1 + a \phi \rho_1 + \phi \delta_1 \rho_1)} \]
The Jacobian characteristic equation of the matrix $J(T_4)$ is

$$
\lambda^3 + N_2\lambda^2 + N_2\lambda + N_3 = 0,
$$

with

$$
N_1 = -(J_{11} + J_{22} + J_{33}),
N_2 = J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{23}J_{32},
N_3 = J_{11}J_{23}J_{32} - J_{11}J_{22}J_{33} - J_{13}J_{32}J_{21}.
$$

To ensure the stability of $T_4$ must meet the Routh-Hurwitz criteria, namely $N_1 > 0, N_2 > 0, N_3 > 0$, and $N_4N_2 > N_3$.

- a. $N_1 > 0$
  $$-(J_{11} + J_{22} + J_{33}) > 0,$$
- b. $N_2 > 0$
  $$J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} > J_{23}J_{32},$$
- c. $N_3 > 0$
  $$J_{11}J_{23}J_{32} > J_{11}J_{22}J_{33} + J_{13}J_{32}J_{21},$$
- d. $N_4N_2 > N_3$
  $$-(J_{11} + J_{22} + J_{33})(J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{23}J_{32}) > J_{11}J_{23}J_{32} + -J_{11}J_{22}J_{33} - J_{13}J_{32}J_{21}.$$

Thus, the equilibrium $T_4(x', y', z')$ is said to be asymptotically stable if it meets $-(J_{11} + J_{22} + J_{33}) > 0$, $J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} > J_{23}J_{32}$, $J_{11}J_{23}J_{32} > J_{11}J_{22}J_{33} + J_{13}J_{32}J_{21}$, $-(J_{11} + J_{22} + J_{33})(J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{23}J_{32}) > J_{11}J_{23}J_{32} + -J_{11}J_{22}J_{33} - J_{13}J_{32}J_{21}$.

### 4.4. Simulation numeric

The parameter values used in this simulation are taken several references and some basic assumptions. The parameter values in numerical simulation are $\rho_1 = 1.5, k_1 = 100, \phi = 0.04, \beta = 0.031, c = 0.41, \alpha = 0.08, \delta_1 = 0.1, \delta_2 = 0.008$. The equilibrium value obtained by the $T_4 = (77.72776532, 59.14834353, 10.77688775)$

The Jacobi matrix of $T_4$ is

$$
J(T_4) = \begin{bmatrix}
-1.165916480 & 0 & -2.409560725 \\
0.1369742433 & -0.18 & 0.9879198972 \\
0 & 0.08 & -0.87015102
\end{bmatrix}
$$

The characteristic equation formed from $J(T_4)$ is

$$
\lambda^3 + 2.2160675 \lambda^2 + 1.30198197 \lambda + 0.116871468 = 0.
$$

The eigenvalue obtained is $\lambda_1 = -1.24747324110835, \lambda_2 = -0.85960654381903, \lambda_3 = -0.10897715072615$.

### 4.5. Population dynamics for the efficiency of immature to mature growth

This population dynamics will be seen from several experimental cases, which are based on numerical simulations. The effects of prey will be investigated (x)population changes, immature population changes (y) and change the population mature (z). What is expected is that these changes do not disturb the balance of the ecosystem.
The prey population dynamics from the efficiency of immature to mature changes ($\alpha$), as follows.

**Figure 1.** Population dynamics at $\alpha = 1$.

**Figure 2.** Population dynamics at $\alpha = 0.08$.

**Figure 3.** Population dynamics at $\alpha = 0.04$. 

\[
\alpha \neq 1
\]
Figure 4. Population dynamics at $\alpha = 0.02$.

Figure 5. Population dynamics at $\alpha = 0$.

In each case of picture 1, picture 2, picture 3, picture 4, and picture 5, describe the characteristics of each. This change in efficiency from immature predators to predatory mature, in an ecosystem, is certainly not always constant. Environmental factors and natural disasters are always a consideration. From each case, it was obtained for changes in immature to mature predator efficiency from high efficiency to low efficiency. The population dynamics of prey ($x$) always experience a significant increase. In this case, it can be said that if the amount of immature to mature efficiency gets smaller, the growth of the population of prey ($x$) will be faster. While for the immature population ($y$) and mature predator ($z$), the growth identified is linear and does not undergo significant changes.

5. Conclusion

The prey-predator model with structural stages with the Holling I response function shows stability in the ecosystem. This is indicated by the consistent eigenvalues given either in algebraic calculations or in numerical simulations. Also in population dynamics identified by changes in the value of immature to mature efficiency parameters on the model.

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