G₁ Cosmologies with Gravitational and Scalar Waves

Ruth Lazkoz

Astronomy Unit, School of Mathematical Sciences
Queen Mary & Westfield College, London E1 4NS U.K.

and

Fisika Teorikoaren Saila
Euskal Herriko Unibertsitatea
644 P.K., 48080 Bilbao, Spain

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Abstract

I present here a new algorithm to generate families of inhomogeneous massless scalar field cosmologies. New spacetimes, having a single isometry, are generated by breaking the homogeneity of massless scalar field G₂ models along one direction. As an illustration of the technique I construct cosmological models which in their late time limit represent perturbations in the form of gravitational and scalar waves propagating on a non-static inhomogeneous background. Several features of the obtained metrics are discussed, such as their early and late time limits, structure of singularities and physical interpretation.

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I. INTRODUCTION

The high degree of isotropy observed in the Universe on large scales today is usually combined with the Copernican principle to justify the assumption of homogeneity on the same scales. However, there is no known reason to assume that the Universe was isotropic nor homogeneous at very early epochs. The puzzling question of how the Universe might have evolved from an initially irregular state into the current isotropic and apparently homogeneous state lacks a complete answer at present. To date, several regularization mechanisms have been put forward, such as Misner’s chaotic cosmological program [1,2], the standard inflationary scenario [3–6] and, more recently, the alternative pre-Big Bang inflationary scenario [7]. However, none of these is completely satisfactory, and in general one cannot know for certain which range of initial conditions could have allowed the Universe evolve into its present form. Such scenarios can only provide one with partial indications of what initial conditions would have led a generic universe into the one observed at present. One way to maximize the amount of information obtainable from any cosmological program, such as those mentioned above, relies on studying the evolution of models with as many degrees of freedom as possible. This idea has motivated the special attention paid to inhomogeneous cosmological models in the last decades (see Krasinski [8] for a review).

In general, attempts to obtain new inhomogeneous metrics involve some symmetry assumption, so that the full implications of the non-linearity of the theory are compromised to some extent. In addition, non-vacuum spacetimes are required to have a physically meaningful matter content, so that they portray realistic situations. Motivated by the possibility of the existence of non-trivial massless scalar fields in the early universe, I will concern myself here with cosmological solutions to the Einstein equations induced by such matter sources. In particular, I present a new algorithm to generate families of massless scalar field $G_1$ cosmologies, i.e. time-dependent spacetimes with a single isometry. These new sets of metrics will be generated starting from generalized vacuum Einstein-Rosen spacetimes, which admit an Abelian group of isometries $G_2$ acting transitively on spacelike surfaces. In
the last decades there has been intensive study of $G_2$ vacuum and matter filled cosmological models and several major reviews on the subject have been written [9,10,8].

Given the large number of known $G_2$ cosmologies and the various available techniques to generate further ones, algorithms transforming such spacetimes into $G_1$ metrics represent powerful tools for generating new inhomogeneous solutions. The new algorithm, which I shall present, displays the nice property of reducing the symmetry while keeping the type of matter source unaltered. Nonetheless, the input and output solutions may not admit the same physical interpretation or even share some of the same relevant features. For this reason, if one wishes to grasp the physical meaning of every new solution, an independent analysis of it will have to be carried out.

Here in I construct and analyze $G_1$ models representing the propagation of gravitational and matter waves on a non-spatially flat background. The study of primordial inhomogeneities in the form of waves is an active area of research. This is motivated by the fact that wave-like primordial perturbations originating from vacuum fluctuations during inflation may be responsible for structure formation. Unlike other types of inhomogeneities formed in the early universe, they would have remained nearly unaltered up to the present, and therefore allowing the possibility of their detection.

The exact inhomogeneous $G_1$ spacetimes seeded by gravitational and matter waves studied here represent a generalization of more symmetric configurations considered by Charach and Malin [11]; in those models the background hosting the waves was homogeneous. In the context of colliding plane waves, $G_2$ diagonal spacetimes with waves of scalar and gravitational nature have also been considered. Space-times such as those studied by Wu [12] or any solution generated by the methods of Barrow [13] and Wainwright et al. [14] could be taken as starting points to construct new $G_1$ metrics modeling interactions of waves on a curved background.

Furthermore, interest in this generation procedure is not restricted to the relativistic framework; solutions to Einstein’s equations with a massless scalar field (in what follows m.s.f.) may be used to generate solutions to alternative theories of gravity, such as Brans-
Dicke theory or string theory in its low energy limit. In the latter case, one could even take those spacetimes to generate new solutions with other massless modes in the characteristic spectrum of the theory.

The plot of the paper is as follows: First I introduce the $G_1$ massless scalar field solution generating algorithm itself. Then, I construct new inhomogeneous metrics starting from an infinite dimensional family of solutions which in the WKB limit admit an interpretation in terms of waves. It will be shown that at early times these solutions behave like a Belinskii-Khalatnikov generalized model \cite{15} with homogeneity broken along one spatial direction. The structure of spacelike singularities of the new solutions in the early time limit will be analyzed as well, and the special features due to the presence of the matter source will be indicated. Next I will consider the solutions’ high frequency limit and show that they can be thought of in the same physical terms as their $G_2$ counterparts. In particular the new cosmological models represent a spacetime with a spatially inhomogeneous background filled with a non-static inhomogeneous scalar field and a null fluid of “gravitons” and “scalar particles.” As time grows the null fluid’s contribution to the energy-momentum tensor grows faster than that associated with the homogeneous part of the scalar field generating the background geometry. Thus, the model evolves into an inhomogeneous generalization of the cosmological model of Doroshkevich, Zeldovich and Novikov (DZN) \cite{16}. Finally, the main conclusions are outlined.

II. SOLUTION GENERATION ALGORITHM

Basically, the new generating technique is a prescription to break homogeneity along one direction in $G_2$ m.s.f. cosmologies, ending up with new spacetimes possessing a single isometry but having the same type of matter content. Remarkably, the pioneering investigations on matter filled universes of Einstein-Rosen type, carried out respectively by Tabensky and Taub \cite{17} and Liang \cite{18} considered models with non interacting scalar fields, even though at the time there was no clear physical motivation.
According to a conjecture due to Belinskii, Lifshitz and Khalatnikov (BLK) \[19–24\], \(G_2\) metrics seem to be specially relevant for the description of the early universe, as such solutions give the leading approximation to a general solution near the singularity at \(t = 0\). Their claim has very recently found the support of numerical results \[25\]. In particular BLK considered approximate Einstein-Rosen solutions and performed an analysis of their local behavior in the early and late time regime. An interesting result reached in the course of those investigations, which is specially relevant for the present paper, was the prediction of a high frequency gravitational wave regime in the late epochs of the Universe.

Before going any further it is convenient to explain how \(G_2\) spacetimes induced by a m.s.f. can be generated starting from vacuum solutions to Einstein’s equations with the same symmetry. For the sake of simplicity, the discussion will be restricted here to a particular case of a well-known general procedure \[26,11,27,13,14,28,29\]. The generic diagonal line-element with \(G_2\) symmetry will be taken as a starting point:

\[
\begin{align*}
    ds_v^2 &= e^{f_v(t,z)}(-dt^2 + dz^2) + G_v(t,z) \left( e^{p_v(t,z)} dx^2 + e^{-p_v(t,z)} dy^2 \right),
\end{align*}
\]

where the subscript \(v\) stands for vacuum. A new solution \(g_{\mu\nu}\) of the Einstein equations with a massless scalar field \(\varphi\) as a source and line-element

\[
\begin{align*}
    ds^2 &= e^{f(t,z)}(-dt^2 + dz^2) + G(t,z) \left( e^{p(t,z)} dx^2 + e^{-p(t,z)} dy^2 \right),
\end{align*}
\]

can be obtained by the following transformations:

\[
\begin{align*}
    G &= G_v, \quad (3a) \\
    p &= B p_v + C \log G_v, \quad (3b) \\
    f &= f_v + E p_v + F \log G_v, \quad (3c) \\
    \varphi &= A p_v + D \log G_v; \quad (3d)
\end{align*}
\]

provided that the constants \(A, B, C, D, E\) and \(F\) are subject to the constraints:

\[
BC + 2AD = E, \quad (4a)
\]
\[ C^2 + 2D^2 = 2F, \quad (4b) \]
\[ B^2 + 2A^2 = 1. \quad (4c) \]

The conditions (4) arise by demanding the following are satisfied:

\[ R_{\mu\nu} = \varphi_{,\mu\nu}, \quad (5a) \]
\[ g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi = 0. \quad (5b) \]

In principle a large number of new m.s.f. \( G_2 \) cosmologies can be obtained by simply applying the procedure sketched above to any of the representatives of the populated family of vacuum Einstein-Rosen spacetimes. However, generating m.s.f. solutions with a lower degree of symmetry is a more cumbersome task. At this point I would like to draw attention to a method given by Feinstein et al. \[30] which allows one to generate families of solutions with a two-dimensional degree of inhomogeneity and a self-interaction term for the massless field of the form

\[ V = V_0(\lambda) e^{-\lambda \varphi}. \quad (6) \]

With these procedure new metrics are obtained by introducing an \( x \)-dependent conformal factor on the input \( G_2 \) metric, and where in general the potential term only vanishes for \(|\lambda| = 6\). This difficulty in canceling the self-interaction term for the scalar field can be traced back to the high degree of symmetry of the \( x \)-dependence in the models considered in ref. \[30].

In order to find a prescription for introducing an additional degree of symmetry in m.s.f. \( G_2 \) metrics without switching on a potential in the process, I have considered the possibility of having a more general \( x \)-dependence in the metric. In particular, I have sought metrics of the form:

\[ ds^2 = \Omega(x) e^{f(t, z)} (-dt^2 + dz^2) + G(t, z) \left( e^{p(t, z)} dx^2 + \Xi(x) e^{-p(t, z)} dy^2 \right), \quad (7) \]

and made the following ansatz for the scalar field:
\[ \tilde{\varphi}(t, z, x) = \varphi(t, z) + \Lambda(x) . \] (8)

Note that the massless scalar field case included in the solutions of Feinstein et al. is also a particular case of the models here. Since the requirement is that no potential should arise in the transformations, the equations that must hold are:

\[ \tilde{R}_{\mu \nu} = \tilde{\varphi}_{,\mu} \tilde{\varphi}_{,\nu} , \] (9a)

\[ \tilde{g}^{\mu \nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{\varphi} = 0 . \] (9b)

Explicitly, equation (9a) is equivalent to the set of equations:

\[ \tilde{R}_{00} = R_{00} + e^{f - p} \frac{\Xi_x \Omega_x + 2 \Xi \Omega_{xx}}{4 G \Xi} = \tilde{\varphi}_{,t}^2 , \] (10a)

\[ \tilde{R}_{11} = R_{11} - e^{f - p} \frac{\Xi_x \Omega_x + 2 \Xi \Omega_{xx}}{4 G \Xi} = \tilde{\varphi}_{,z}^2 , \] (10b)

\[ \tilde{R}_{01} = R_{01} = \tilde{\varphi}_{,t} \tilde{\varphi}_{,z} , \] (10c)

\[ \tilde{R}_{22} = R_{22} + \frac{\Xi_x^2 - 2 \Xi \Xi_{xx}}{4 \Xi^2} + \frac{\Omega_x^2 - 2 \Omega \Omega_{xx}}{2 \Omega^2} = \tilde{\varphi}_{,x}^2 , \] (10d)

\[ \tilde{R}_{12} = \frac{G_x \Omega_x}{2 G \Omega} + \frac{\Xi_x p_x}{2 \Xi} = \tilde{\varphi}_{,x} \tilde{\varphi}_{,x} , \] (10e)

\[ \tilde{R}_{02} = \frac{G_t \Omega_x}{2 G \Omega} + \frac{\Xi_x p_t}{2 \Xi} = \tilde{\varphi}_{,t} \tilde{\varphi}_{,x} , \] (10f)

\[ \tilde{R}_{33} = R_{33} + e^{-2 p} \frac{\Xi_x^2 - 2 \Xi \Xi_{xx}}{4 \Xi} - e^{-2 p} \frac{\Xi_x \Omega_x}{2 \Omega} = 0 ; \] (10g)

whereas equation (11) in explicit form reads

\[ \frac{g^{\mu \nu} \nabla_\mu \nabla_\nu \varphi}{\Omega} + e^{-p} \left( \frac{\Omega \Xi^{1/2} \tilde{\varphi}_{,x}}{\Xi^{1/2}} \right)_x = 0 . \] (11)

Inspection of the equations (11) indicates that the three \( x \)-dependent metric functions must have the form:

\[ \Omega(x) = x^k , \] (12a)

\[ \Xi(x) = x^n , \] (12b)

\[ \Lambda(x) = m \log |x| ; \] (12c)

subject to the following constraints on the parameters \( k, n \) and \( m \):
\[ k(2k + n - 2) = 0, \]  
(13a)  
\[ n(2k + n - 2) = 0, \]  
(13b)  
\[ k + nC = 2Dm, \]  
(13c)  
\[ nB = 2Am, \]  
(13d)  
\[ 2m^2 = 4k - 3k^2. \]  
(13e)  

In this case, the Klein-Gordon equation (13b) reduces to

\[ m(2k + n - 2) = 0, \]  
(14)

which is automatically satisfied provided that (13a,13b) and (13e) hold. Note that the parameter \( k \) must be non-negative and not larger than 4/3. The \( x \)-dependent term of the scalar field will be maximum for \( k = 2/3 \). Consistency of the solutions is reflected by the fact that for the following choice:

\[ m = n = k = 0. \]  
(15)

the set of equations (13-14) is satisfied for any value of \( A, B, C \) and \( D \), and the input \( G_2 \) m.s.f is recovered. On the one hand, for \( m \neq 0 \) and \( k \neq 1 \), if \( k \) and \( C \) are taken as free parameters, one can parameterize the solution’s constants in the form:

\[ n = 2 - 2k, \]  
(16a)  
\[ m = \text{sign}(m)\frac{\sqrt{4k - 3k^2}}{\sqrt{2}}, \]  
(16b)  
\[ A = \sqrt{2}\text{sign}(A)\left|\frac{1 - k}{2 - k}\right|, \]  
(16c)  
\[ B = \text{sign}(Am)\left|\frac{1 - k}{2 - k}\right|\frac{\sqrt{4k - 3k^2}}{1 - k}, \]  
(16d)  
\[ D = \text{sign}(m)\frac{k + (2 - 2k)C}{\sqrt{8k - 6k^2}}, \]  
(16e)  
\[ E = \text{sign}(Am)\left|\frac{2 - 2k}{2 - k}\right|\frac{C(2 - k)^2 + (2 - 2k)k}{\sqrt{4k - 3k^2}}, \]  
(16f)  
\[ F = \frac{C^2}{2} + \frac{(k + (2 - 2k)C)^2}{8k - 6k^2}. \]  
(16g)
Four different subcases can be distinguished, depending on the choice of the sign of $A$ and $m$. On the other hand, in the particular case $m \neq 0$, $k = 1$, the constants take the values:

$$\sqrt{2|m|} = \sqrt{2|D|} = |B|$$  \hspace{1cm} (17a)

$$A = 0$$  \hspace{1cm} (17b)

$$C^2 + 1 = 2F$$  \hspace{1cm} (17c)

$$E = \text{sign}(B)$$  \hspace{1cm} (17d)

In general the metric $\tilde{g}_{\mu\nu}$ admits only one Killing and one homothetic vector, namely:

$$\xi_{kv} = \frac{\partial}{\partial y}$$  \hspace{1cm} (18a)

$$\xi_{hv} = y \frac{\partial}{\partial y},$$  \hspace{1cm} (18b)

where by definition:

$$\mathcal{L}_{\xi_{kv}} g_{\mu\nu} = 0,$$  \hspace{1cm} (19a)

$$\mathcal{L}_{\xi_{hv}} g_{\mu\nu} = 2g_{\mu\nu};$$  \hspace{1cm} (19b)

It is known that a matter source and the geometry it induces need not share the same symmetry properties unless it is a perfect fluid [31,32]. This property is usually referred to as the “inheritance problem”. Inspection of the metrics obtained here show that for $k = 4/3$ the $x$-dependent term of the scalar field is absent, even though the metric depends on that coordinate in a non-trivial way. This is equivalent to:

$$\mathcal{L}_{\xi_{mc}} T_{\mu\nu} = 0$$  \hspace{1cm} (20a)

$$\mathcal{L}_{\xi_{mc}} g_{\mu\nu} \neq 0;$$  \hspace{1cm} (20b)

where

$$\xi_{mc} = \frac{\partial}{\partial x}$$

and the vector $\xi_{mc}$ is a so-called matter collineation [10,33,32].

Bearing in mind the similarity between the new algorithm and the one of [30], one might wonder whether geometries like (7) can be also seeded by a scalar field with an exponential
potential. Such a situation may be shown to be only possible if \( n = k \), which is nothing but the case already found by Feinstein et al. In order to prove this, let us consider the case where the generic geometry (7), under the constraint (12), is induced by an exponential potential \( \tilde{V}(\tilde{\varphi}) = V_0(\lambda) e^{-\lambda \tilde{\varphi}} \). In this case the field equations are:

\[
\tilde{R}_{\mu \nu} = \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} \tilde{\varphi} + \tilde{g}_{\mu \nu} V_0 e^{-\lambda \tilde{\varphi}}, \quad (21a)
\]
\[
\tilde{\nabla}^\mu \tilde{\nabla}_\mu \tilde{\varphi} = -\frac{\partial \tilde{V}(\tilde{\varphi})}{\partial \tilde{\varphi}}. \quad (21b)
\]

It is only necessary to look at the equations for \( \tilde{R}_{00} \) and \( \tilde{R}_{33} \) to realize the following constraint must hold:

\[
k (2 - 2 k - n) = n (2 - 2 k - n) = 4 V_0 \neq 0. \quad (22)
\]

Compatibility of the latter set of equations in the case of a non vanishing potential requires \( n = k \), or in other words, that the \( x \)-dependence of the metric is given by a global conformal factor as in the case studied by [30].

It is important to note here that the generation technique does not restrict the character of the gradient of metric function \( G(t, z) \) of the input metric. That function determines the local behavior of the spacetime, and its gradient can be globally timelike, spacelike, null or vary from point to point. Although in what follows I am focusing on a case with a timelike character of the gradient of \( G(t, z) \), a window is left open for the study of other physically appealing cases.

Moreover, even though our generating prescription has been used to break homogeneity of an input m.s.f. solution with a single degree of inhomogeneity, it is also possible to construct an equivalent algorithm that would transform certain static spacetimes into non-static ones. One should start with a m.s.f. model with two commuting Killing vectors, one of them being timelike, and then generalize it by introducing time dependent factors in the metric and scalar field as I have done here.
III. COSMOLOGIES WITH GRAVITATIONAL AND SCALAR WAVES

After having outlined our method to generate uniparametric families of $G_1$ cosmologies, I shall now construct the counterparts of a family of $G_2$ cosmologies found by Charach and Malin \[11\]. That set of metrics, which can be thought of as inhomogeneous sinusoidal perturbations of the well-known Bianchi I spacetimes, represent the propagation of gravitational and scalar waves on an anisotropically expanding flat background. Simpler models of that sort, not including scalar degrees of freedom, were studied earlier on by Berger \[25\].

In this paper I am only considering Charach and Malin’s solutions in the asymptotic limits $t \sim 0$ and $jt \gg 1$ for any value of $j$. It is convenient, however, to give here the full expressions of the metric functions of the vacuum solution from which they were derived, namely:

\[ G_v = t, \quad (23a) \]
\[ p_v = \beta \log t + \sum_{j=1}^{\infty} \cos[j(z - z_n)] \alpha_j J_0(j t), \quad (23b) \]
\[ f_v = \frac{\beta^2 - 1}{2} + \beta \sum_{j=1}^{\infty} \alpha_j \cos[\alpha_j(z - z_j)] J_0(j t) J_1(j t) + \frac{t^2}{4} \sum_{j=1}^{\infty} j^2 \{ [\alpha_j J_0(j t)]^2 + [\alpha_j J_1(j t)]^2 \} \]
\[ - \frac{t}{2} \sum_{j=1}^{\infty} j \alpha_j^2 \cos^2[\alpha_j(z - z_j)] J_0(j t) J_1(j t) + \frac{t}{2} \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \frac{l j}{l^2 - j^2} \]
\[ \times \left[ \sin[l(z - z_l)] \sin[j(z - z_j)] \left[ l \alpha_l \alpha_j J_1(l t) J_0(j t) - j \alpha_l \alpha_j J_0(l t) J_1(j t) \right] \right. \]
\[ + \cos[l(z - z_l)] \cos[j(z - z_j)] \left[ j \alpha_l \alpha_j J_1(l t) J_0(j t) - l \alpha_l \alpha_j J_0(l t) J_1(j t) \right] \} \]. \quad (23c) \]

Charach and Malin’s solutions were obtained by breaking the spatial homogeneity of Belinskii-Khalatnikov homogeneous solution \[13\], which also has a m.s.f. as a seed. The gravitational and scalar degrees of freedom of those spacetimes satisfy linear wave equations, with the form of cylindrically symmetric waves propagating on Minkowski spacetime, for which the spatial and temporal coordinates have been interchanged. Since the solutions are of the standing wave form they are fully compatible with the $S^1 \otimes S^1 \otimes S^1$ topology (threetorus). However, the $G_1$ counterparts of those spacetimes cannot have the same topology, $x$ cannot be a cyclic coordinate in this case due to the presence of the term proportional to
log \(|x|\) in the scalar field \(\tilde{\varphi}\).

### A. Early time behavior and singularities

From the analysis of the new \(G_1\) solution’s early time behavior it can be determined whether spacelike singularities arise at the time origin \(t = 0\). A look at expressions (23) shows that in the case under discussion, the periodic inhomogeneities can be neglected in the very first stages of that spacetime’s evolution. Our metrics will be then \(G_2\) inhomogeneous generalizations of the cosmological models of Doroshkevich-Zeldovich-Novikov (DZN) [16], which have three commuting Killing vectors. In this limit the metric and scalar field read

\[
\tilde{\varphi} = \varphi_0 \log t + m \log |x| \tag{24b}
\]

where

\[
\begin{align*}
\epsilon_1 &= t^{f_0}, \\
\epsilon_2 &= t^{1+p_0}, \\
\epsilon_3 &= t^{1-p_0},
\end{align*} \tag{25a-c}
\]

and

\[
\begin{align*}
p_0 &= B\beta + C, \\
f_0 &= \frac{\beta^2 - 1}{2} + E\beta + F, \\
\varphi_0 &= A\beta + D. 
\end{align*} \tag{26a-c}
\]

In addition, the following relationship holds:

\[
f_0 = \frac{p_0^2 - 1}{2} + \varphi_0^2. \tag{27}
\]

Following Charach and Malin, the metric can be rewritten using a synchronous set of coordinates in which the new time coordinate \(\tau\) is defined
\[ d\tau = \sqrt{\epsilon_1(t)}dt; \quad (28) \]

this way the metric can be recognized as a simple inhomogeneous generalization of a Belinskii-Khalatnikov \[\text{solution.}\]

In broad terms, the presence of a spacelike singularity at early times will be reflected in the behavior of the curvature invariants \( R, R_{\mu\nu}R^{\mu\nu} \) and \( R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta} \). Due to the inhomogeneous character of the metric it is possible in principle to have a conspiracy between the parameters, so that on certain hypersurfaces some those invariants are identically null and therefore do not reveal the presence of a singularity in the spacetime. Since the Kretchmann scalar \( K = R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta} \) is however non-null and positive everywhere I will use this in the search for singularities. In the case here it explicitly reads:

\[
K = \frac{3}{8} f_0^2 \left[ (1 + p_0^2)^2 + 2 (f_0 + 1)^2 \right] + \frac{3}{16} t^2 f_0^4 \left[ (1 - p_0)^2 (1 + f + p)^2 + (1 + p)^2 (1 + f - p)^2 \right].
\]

Since (27) holds it can be seen that \( K \) is singular at \( t = 0 \) for any value of the three free parameters of the solution, indicating thus the generic presence of a spacelike singularity at the time origin.

It is also possible to study the singularity structure of the solutions in a more refined way, in particular by studying the expansion along each spatial axis. In general, the behavior will strongly depend on the values of the parameters \( k, C \) and \( \beta \). It can be shown analytically that for large enough \( \beta \) there will necessarily be contraction along the \( z \) axis, moreover if \( k > 1 \) that will be the case regardless the value of \( \beta \) and \( C \). Another fact that can be easily checked is the impossibility of having simultaneous contraction along axes \( x \) and \( y \). Three main types of singular behavior can be distinguished:

a) **Point-like singularities (Quasi-Friedmann behavior)**

All three spatial directions shrink as the initial time \( t = 0 \) is approached; or explicitly \( \lim_{t \to 0} \epsilon_i = 0 \ \forall i \). Depending on how many directions have the same expansion rate the behavior will be completely anisotropic, axially symmetric or isotropic.
b) Finite lines
This type of singular behavior occurs when in the vicinity of \( t = 0 \) one of the spatial directions neither expands nor contracts with time; in other words, it is said that the direction \( i \) is a finite line if \( \lim_{t \to 0} \epsilon_i = 1 \). The subcases can be classified according to which the direction behaves in that way. In general, there will be a single finite line, though it is possible to have particular cases in which a second finite line exists. However, it is impossible to have such behavior along all three directions.

c) Infinite lines (Quasi Kasner regime)
An infinite line along the \( i \) direction exists when \( \lim_{t \to 0} \epsilon_i = \infty \). Again, three cases can be seen to occur, depending on which is the axis displaying that feature. For some particular values of the parameters the maximum allowed number of two infinite lines can be reached.

Since the singular behavior of the metrics depends on three parameters it becomes rather complicated to represent graphically the structure of singularities in the general case. For that reason I will restrict myself to cases in which the value of one of the parameters is fixed, namely \( C \). In the pictures here two different sets of lines can be distinguished. On the one hand, the black lines in each plot correspond to the curves along which a spatial direction becomes a finite line. In the region delimited by the black continuous line and the axes, a infinite line type of singularity arises along direction \( z \). Here one can see how for \( C = 0.5 \) it is possible to have simultaneously the same behavior along directions \( y \) and \( z \). For \( k > 1 \) in the region above the black dashed line there is contraction along direction \( y \), this behavior gets reversed for \( k < 1 \). Similarly, for \( \beta \) values less than those along the dashed-dotted line contraction takes place, and the contrary happens for \( k > 1 \). The points where two black lines intersect correspond to having two finite lines. On the other hand, the grey lines represent the curves along which two spatial directions display the same expansion rate. The fact that at a given point the three grey lines intersect reflects the possibility of having isotropic expansion, and for the two \( C \) values chosen, that point lies in the \( k > 1 \) region.
FIG. 1. Structure of singularities of the inhomogeneous generalization of Belinskii-Khalatnikov’s model for $C = 0.5$ (l.h.s. figure) and $C = -0.5$ (r.h.s. figure) with $\text{sign}(A m)$. On the one hand, the black lines indicate the value of $\beta$ as a function of $k$ for which a given direction behaves as a finite line. The black continuous, dashed and dashed-dotted lines correspond to a finite line along $z$, $y$ or $x$ directions respectively. On the other hand, the grey lines indicate the value of $\beta$ as a function of $k$ for which two spatial directions have the same expansion rate. The grey continuous, dashed and dashed-dotted lines correspond respectively to $\epsilon_2 = \epsilon_3$, $\epsilon_1 = \epsilon_3$ and $\epsilon_1 = \epsilon_2$.

B. WKB limit

With regard to the physical interpretation of the new solutions constructed here, it is their high-frequency limit which turns out to be most interesting. This limit, also called the WKB regime, corresponds to the regime in which the time elapsed since the beginning of the Universe is much larger than the period of any perturbation mode. By taking $jt \gg 1$, for every value of $j$, in the normal mode expansions of the scalar field and metric functions, Charach and Malin were able to show that the relativistic solutions taken here as input,
represent scalar and gravitational waves propagating on an spatially flat background. In this limit, such universes are causally connected because the particle horizon is larger than the wavelength of any of the modes of the independent degrees of freedom, namely the transverse part of the gravitational field $p$ and the scalar field $\tilde{\phi}$.

In this vein, it will be proved here in two different ways that the $G_1$ counterparts to Charach and Malin’s cosmological models can also be thought of in terms of waves propagating on a non-static background. Thus, the physical interpretation is not spoiled in the process of homogeneity breaking. A direct consequence of the additional degree of inhomogeneity present in the $G_1$ solutions, as compared to their more symmetric counterparts, is that the background on which the waves live is not spatially flat. Cosmological backgrounds perturbed by waves have also been considered in a series of paper by Centrella and Matzner [34–36] who studied collisions of plane gravitational waves in these settings.

Let us consider now the late time expressions for the metric functions and the scalar field of the $G_1$ spacetimes obtained by applying the new technique to the cosmological models given by expressions (23):

\[ p \sim p_0 \log t + B \bar{p}(t, z), \]  
\[ f \sim f_0 \log t + \left( \frac{t}{2\pi} \right) \sum_{j=1}^{\infty} j \alpha_j^2, \]  
\[ \tilde{\phi} \sim \varphi_0 \log t + A \bar{p} + m \log |x|, \]  
\[ \bar{p} = \sum_{j=1}^{\infty} \sqrt{\frac{2}{j\pi t}} \alpha_j \cos(\frac{j}{\pi} t - \frac{\pi}{4}) \cos[j(z - z_j)]. \]

Since in this regime $\bar{p} \ll 1$, the metric $\tilde{g}_{\mu\nu}$ can be split into a background $\tilde{\eta}_{\mu\nu}$ plus a perturbation metric $\tilde{h}_{\mu\nu}$, that is:

\[ \tilde{g}_{\mu\nu} \sim \tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}, \]  
\[ \tilde{\eta}_{\mu\nu} = \text{diag} \left( -x^{k} f_0 e^{f_1 t}, x^{k} f_0 e^{f_1 t}, t^{1+p_0}, x^{n} t^{1-p_0} \right), \]  
\[ \tilde{h}_{\mu\nu} = \text{diag} \left( 0, 0, t^{1+p_0} B \bar{p}, -x^{n} t^{1-p_0} B \bar{p} \right). \]
Here, in addition to (26), I have made the following definition:

\[ f_1 = \sum_{j=1}^{\infty} \frac{j \alpha_j^2}{\pi} (B^2 + 2A^2) . \]  

(32)

A peculiarity regarding the perturbations on the scalar and gravitational degrees of freedom is that for \( 0 < k < 1 \) they will be on phase whereas for \( 1 < k < 4/3 \) they will be phase-shifted by \( \pi \). Nonetheless, it will be seen later that whatever the value of \( k \) the scalar and gravitational perturbations contribute constructively to the energy momentum tensor.

A proof of the assertion made above regarding the non spatial flatness of the spacetimes described by the background metric \( \tilde{\eta}_{\mu\nu} \), is provided by the expression of the spatial curvature of the \( t = constant \) three dimensional hypersurfaces:

\[ (3,\eta) R = \frac{m^2}{2x^2 p_0 + 2} . \]  

(33)

Let us now proceed to analyze the \( \tilde{h}_{\mu\nu} \) tensor and thereby show that it represents the gauge-invariant perturbations of a background spacetime \( \tilde{\eta}_{\mu\nu} \); or in other words, that it satisfies the wave equation \( \tilde{\nabla}^\gamma \tilde{\nabla}_\gamma \tilde{h}_{\mu\nu} = 0 \), or equivalently, following [34], that:

\[ \tilde{h}^\gamma_\gamma = 0 , \]  

(34)

\[ \tilde{\nabla}^\gamma \tilde{h}_{\gamma\lambda} = 0 , \]  

(35)

with \( \gamma, \lambda = 2, 3 \), and where the D’Alambertian and the covariant derivative must be calculated using the corresponding background metric \( \tilde{\eta}_{\mu\nu} \).

Since it is straightforward to see that the trace-free condition (34) is satisfied by construction, the problem reduces to satisfying (35), which is equivalent to requiring:

\[ 1 \]  

\[ ^{1} \text{Though the factor } B^2 + 2A^2 \text{ equates to unity in the simple case I am dealing with, it has been deliberately introduced in the definition of } f_1; \text{ so that the trail of the separate contributions to the energy momentum tensor of the graviton and scalar field pair can be followed. Had I considered the general case of the procedure to generate a } G_2 \text{ massless scalar field solution, then } f_1 \neq \sum_{j=1}^{\infty} j \alpha_j^2/(2 \pi) . \]
\[ \tilde{h}_{22,x} - 2 \tilde{\Gamma}_{x}^{2} \tilde{h}_{22} = 0, \quad (36a) \]
\[ \tilde{h}_{33,x} - 2 \tilde{\Gamma}_{x}^{3} \tilde{h}_{33} = 0. \quad (36b) \]

On the one hand, expression (36a) is identically null because neither \( \tilde{\eta}_{22} \) nor \( \tilde{h}_{22} \) are \( x \)-dependent. On the other hand, it can be seen that (36b) is also satisfied by just having in mind that:

\[ \tilde{h}_{33} = B \tilde{\eta}_{33}, \quad (37a) \]
\[ \tilde{\Gamma}_{x}^{3} = \left( \log \sqrt{\tilde{\eta}_{33}} \right)_{x}. \quad (37b) \]

Once it has been proved that the wave equation \( \tilde{\nabla}^{\gamma} \tilde{\nabla}_{\gamma} \tilde{h}_{\mu\nu} = 0 \) holds one can properly refer to the tensor \( \tilde{h}_{\mu\nu} \) as describing metric perturbations in the form of gravitational waves.

In order to give additional arguments in favor of the interpretation of the solution in terms of waves propagating on a non flat background, I shall follow Charach and Malin [11] and analyze the energy-momentum tensor of the background metric \( \tilde{\eta}_{\mu\nu} \). It will be shown that the stress-energy tensor is naturally manifested in terms of two components. One of these corresponds to a null fluid, supporting thus the interpretation suggested above; while the other term corresponds to an inhomogeneous massless scalar field with no \( z \)-dependence.

In particular,

\[ (n) \tilde{T}_{\mu}^{\nu} = (1) \tilde{T}_{\mu}^{\nu} + (2) \tilde{T}_{\mu}^{\nu}, \quad (38) \]

where \( (1) \tilde{T}_{\mu}^{\nu} \neq 0 \) and \( (2) \tilde{T}_{\mu}^{\nu} = 0 \). Explicitly

\[ (1) \tilde{T}_{0}^{0} = - \tilde{t}^{- (1 + p_{0})} \frac{m^{2}}{2 x^{2}} - \tilde{t}^{- (2 + f_{0})} \frac{1 + 2 f_{0} - p_{0}^{2}}{4 e^{f_{1} t} x^{k}}, \quad (39a) \]
\[ (1) \tilde{T}_{1}^{1} = - \tilde{t}^{- (1 + p_{0})} \frac{m^{2}}{2 x^{2}} + \tilde{t}^{- (2 + f_{0})} \frac{1 + 2 f_{0} - p_{0}^{2}}{4 e^{f_{1} t} x^{k}}, \quad (39b) \]
\[ (1) \tilde{T}_{2}^{0} = \tilde{t}^{- (1 + p_{0})} \frac{2 p_{0} (k - 1) - k}{2 e^{f_{1} t} x^{1+k}}, \quad (39c) \]
\[ (1) \tilde{T}_{2}^{2} = \tilde{t}^{- (1 + p_{0})} \frac{m^{2}}{2 x^{2}} + \tilde{t}^{- (2 + f_{0})} \frac{1 + 2 f_{0} - p_{0}^{2}}{4 e^{f_{1} t} x^{k}}, \quad (39d) \]
\[ (1) \tilde{T}_{3}^{3} = - \tilde{t}^{- (1 + p_{0})} \frac{m^{2}}{2 x^{2}} + \tilde{t}^{- (2 + f_{0})} \frac{1 + 2 f_{0} - p_{0}^{2}}{4 e^{f_{1} t} x^{k}}, \quad (39e) \]
\[ (2) \tilde{T}_{0}^{\mu} = -t^{-(1+f_{0})} \frac{f_{1}}{2 e^{f_{1}t} x^{k}}, \quad (39f) \]
\[ (2) \tilde{T}_{1}^{\mu} = t^{-(1+f_{0})} \frac{f_{1}}{2 e^{f_{1}t} x^{k}}. \quad (39g) \]

I will now proceed to give an interpretation for the \((1)\tilde{T}^{\nu}_{\mu}\) term. The Klein-Gordon equation for a scalar field \(\tilde{\psi}\), calculated using the metric \(\tilde{\eta}_{\mu\nu}\), takes the form

\[
\left( \frac{(x \tilde{\psi}, x)}{x^{1-k}} \right)_{x} - \left( \frac{(t \tilde{\psi}, t)}{e^{f_{1}t} f_{0} - p_{0}} \right) = 0, \quad (40)
\]

a solution of which is

\[
\tilde{\psi} = \varphi_{0} \log t + m \log |x|. \quad (41)
\]

The energy momentum tensor for the field \(\tilde{\psi}\) propagating on the spacetime \(\tilde{\eta}_{\mu\nu}\) yields \((1)\tilde{T}^{\nu}_{\mu}\) exactly. That being so, one can conclude that the exact solution obtained after applying the generating technique to \((23)\) asymptotically evolves into a solution with a single degree of inhomogeneity; that is, the sinusoidal inhomogeneities along the \(z\)-axis vanish with time.

In other words, \((1)\tilde{T}^{\nu}_{\mu}\) corresponds to the energy-momentum tensor of the exact solution one would obtain by switching off the periodic inhomogeneities along \(z\).

On the other hand, the traceless term \((2)\tilde{T}^{\nu}_{\mu}\) can be shown to account for waves. It can also be separated into two parts, namely:

\[
(2)\tilde{T}^{\nu}_{\mu} = (GW)\tilde{T}^{\nu}_{\mu} + (SW)\tilde{T}^{\nu}_{\mu},
\]

\[
(GW)\tilde{T}^{\nu}_{\mu} = \sum_{j=-\infty}^{\infty} \frac{B^{2} \alpha_{j}^{2}}{4 |j|} \kappa_{\mu j} \kappa_{\nu j}, \quad (42a)
\]

\[
(SW)\tilde{T}^{\nu}_{\mu} = \sum_{j=-\infty}^{\infty} \frac{A^{2} \alpha_{j}^{2}}{2 |j|} \kappa_{\mu j} \kappa_{\nu j}, \quad (42c)
\]

The null vector \(\kappa_{\mu j}\) is defined by

\[
\kappa_{\mu j} = \frac{1}{\sqrt{\pi t}} (|j|, j, 0, 0). \quad (43)
\]

It is clear that \((2)\tilde{T}^{\nu}_{\mu}\) corresponds to a null fluid describing a collisionless flow of "gravitons" and "scalar particles". It is easy to see that in the case under discussion gravitational waves will be absent only if the metric has no \(x\)-dependence.
I has been shown that the new solutions represent, in their WKB limit, waves propagating on an inhomogeneous non-static spacetime. In our case the only difference with respect to the case studied by Charach and Malin is that the background is not spatially flat.

As far as the further evolution of the model is concerned, the presence of the additional degree of inhomogeneity in the model plays a crucial role. The null fluid’s contribution to the energy-momentum tensor dominates the one due to the homogeneous part of the scalar field, and in the $t \gg 1$ limit the energy-momentum tensor has a term accounting for the waves, plus another corresponding to the $x$-dependent term in the scalar field. The scalar curvature of the background in this limit will be given by:

$$ (\eta) R \sim \frac{m^2}{x^2 t^{1+p_0}}, \quad (44) $$

so that this universe can either become singular at $t = \infty$ or flat, depending on the sign of $1 + p_0$. The possibility of having a curvature spacelike singularity at late times is entirely due to the inhomogeneous character of the background.

Another proof of the interpretation of the solutions in terms of a null fluid propagating on the background $\tilde{\eta}_{\mu\nu}$ can be given. Let us calculate the energy-momentum tensor that corresponds to the scalar field $\tilde{\varphi}$ in the late time limit and just retain terms up to the order $t^{-1}$. Under this restriction the only non-null terms of the energy momentum tensor are $\tilde{T}_{00}$ and $\tilde{T}_{11}$, which are given by:

$$ \tilde{T}_{00} = -\tilde{T}_{11} = \left(\frac{A^2}{\pi t}\right) \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \alpha_l \alpha_j \sqrt{l j} \left\{ \sin[l(z - z_l)] \sin[j(z - z_j)] \cos[l(t - \frac{\pi}{4})] \cos[j(t - \frac{\pi}{4})] ight. \\
- \left. \cos[l(z - z_l)] \cos[j(z - z_j)] \sin[l(t - \frac{\pi}{4})] \sin[j(t - \frac{\pi}{4})] \right\} + \mathcal{O}(t^{-2}). \quad (45) $$

Averaging $\tilde{T}_{00}$ over a region $0 \leq t \leq 2\pi$, $0 \leq z \leq 2\pi$ on the $(t,z)$-plane one obtains:

$$ \langle \tilde{T}_{00} \rangle = -\langle \tilde{T}_{11} \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \langle \tilde{T}_{00} \rangle \, dz \, dt = \frac{1}{2\pi} \sum_{j=1}^{\infty} j \, A^2. \quad (46) $$

In the same approximation it can be seen that

$$ \langle \tilde{T} \rangle = \langle \tilde{T}_2^2 \rangle = \langle \tilde{T}_3^3 \rangle = \mathcal{O}(t^{-2}). \quad (47) $$
So, essentially it has been found that

$$\langle \tilde{T}_{\mu\nu} \rangle \equiv (SW) \tilde{T}_{\mu\nu}. \quad (48)$$

Summarizing, both methods have yielded the result that for \( t \gg 1 \), and up to the order \( t^{-1} \), the energy-momentum of the m.s.f. can be reduced to the null fluid form.

The WKB regime of the solutions under discussion admits a reformulation in terms of the density of particles contributing to the modes of two fields. I shall strictly follow Charach’s approach here [38], which consists of performing a quasi-classical treatment based on the geometrical optics energy-momentum tensor. A family of Lorentz local frames is introduced so that the density of particles in each normal mode can be defined through:

$$\lambda^{(a)} = \sqrt{|\bar{\eta}_{\nu\nu}|} \delta^{(a)}_{\nu}, \text{ (no summation over } \nu) \quad (49)$$

where \( \bar{\eta}_{\nu\nu} \) represents the components of the inhomogeneous generalization of the DZN metric. A set of observers corresponding to this tetrad are characterized by the 4-velocity:

$$u^\nu \equiv \lambda^{\nu}_{(0)} = \left( \sqrt{\bar{\eta}^{00}}, 0, 0, 0 \right). \quad (50)$$

Let us consider now equations (42b,42c), which give the WKB stress-energy tensors of the cosmological model with gravitational and scalar wave perturbations, namely:

$$(GW)\tilde{T}_{\mu\nu j} = \frac{B^2 \alpha_j^2}{4 |j|} \kappa_{\mu j} \kappa_{\nu j} \quad (51a)$$

$$(SW)\tilde{T}_{\mu\nu j} = \frac{A^2 \alpha_j^2}{2 |j|} \kappa_{\mu j} \kappa_{\nu j}. \quad (51b)$$

The density of scalar and gravitational particles in the \( n \)-th mode is given by

$$\rho_j^S = \frac{\tilde{T}^{(0)}_{\mu\nu j}}{h \kappa_{\mu j}} = \frac{A^2 \alpha_j^2}{2 t \sqrt{x^k e^j}} \quad (52a)$$

$$\rho_j^G = \frac{\tilde{T}^{(0)}_{\mu\nu j}}{h \kappa_{\mu j}} = \frac{B^2 \alpha_j^2}{4 t \sqrt{x^k e^j}}, \quad (52b)$$

where \( \kappa_{\mu\nu} \) is a null vector with dimensions of length. Besides, since the description here is based on units \( c = G = 1 \), the Planck constant has dimensions of (length)$^2$, where
\( h \sim 10^{-66} \text{cm}^2 \). The density does not depend on the direction along which the particles propagate, that is:

\[
\rho^S_j = \rho^S_{-j} \tag{53a}
\]

\[
\rho^G_j = \rho^G_{-j}. \tag{53b}
\]

In the \( G_1 \) models considered here, the volume of the spatial sections \( t = \text{constant} \) is not finite; for that reason the total number of particles in each mode of the two degrees of freedom has no upper bound.

In light of this reformulation it was suggested that one should regard the evolution of \( G_2 \) models filled with waves as describing a process of transforming the initial inhomogeneities along \( z \) into quanta of various fields. Clearly, this interpretation’s validity is extendible to models with just one isometry, such as those constructed in this paper.

**IV. CONCLUSIONS**

Before I finish, I will summarize the main results. I have presented the first method to generate uniparametric families of general relativistic cosmologies having a two-dimensional inhomogeneity and a m.s.f. as a source. In the context of either General Relativity or alternative theories of gravity, one can generate a large number of new inhomogeneous cosmologies using this algorithm, where moreover the only input needed is any of the many known vacuum relativistic cosmologies with two commuting Killing vectors.

It has also been shown that this technique allows one to construct families of cosmologies which represent waves propagating on a spatially curved cosmological background. The spacelike singularity structure of these solutions has been studied, and several peculiarities due to the matter content have been elucidated.
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