Optimization methods for MR image reconstruction

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Abstract—The development of compressed sensing methods [1] for MR image reconstruction [2] led to an explosion of research on models and optimization algorithms for MRI. Roughly 10 years after such methods first appeared in the MRI literature [3], the US FDA approved the commercial use of certain compressed sensing methods [4], [5], making compressed sensing a clinical success story for MRI. This review paper summarizes several key models and optimization algorithms for MR image reconstruction, including both the type of methods that have FDA approval for clinical use, as well as more recent methods being considered in the research community that use data-adaptive regularizers. One impetus for this paper is that “off the shelf” optimization methods have rarely been the best choice for solving optimization problems in MR image reconstruction, due to the large volume of MRI data collected by clinical systems and practical time constraints on processing time. Instead, special purpose algorithms have been devised that exploit the structure of the system model and regularizers used in MRI; this paper strives to collect such algorithms in a single survey. Many of the ideas used in optimization methods for MRI are also useful for solving other inverse problems.

I. INTRODUCTION

A. Scope

Although the paper title begins with “optimization methods,” in practice one first defines a model and cost function, and then applies an optimization algorithm. There are several ways to partition the space of models, cost functions and optimization methods for MRI reconstruction, such as: smooth vs non-smooth cost functions, static vs dynamic problems, single-coil vs multiple-coil data. This paper focuses on the static reconstruction problem because the dynamic case is rich enough to merit its own survey paper [6]. This paper emphasizes algorithms for multiple-coil data (parallel MRI [7], [8]) because modern systems all have multiple channels and advanced reconstruction methods with under-sampling are most likely to be used for parallel MRI scans. Main families of parallel MRI methods include “SENSE” methods that model the coil sensitivities in the image domain [9], [10], “GRAPPA” methods that model the effect of coil sensitivity in k-space [11], and “calibration-less” methods that use low-rank properties [12], [13]. This paper considers all three approaches, emphasizing SENSE methods for simplicity.[1]

B. Measurement model

The signals recorded by the sensors (receive coils) in MR scanners are linear functions of the object’s transverse magnetization. That magnetization is a complicated and highly nonlinear function of the RF pulses, gradient waveforms, and tissue properties, governed by the physics of the Bloch equation [15]–[17]. Quantifying tissue properties using nonlinear models is a rich topic of its own, e.g., [18]–[21], but we focus here on the problem of reconstructing images of the transverse magnetization from MR measurements.

Ignoring noise, a vector \( s \in \mathbb{C}^M \) of signal samples recorded by a MR receive coil is related (typically) to a discretized version \( x \in \mathbb{C}^N \) of the transverse magnetization via a linear Fourier relationship:

\[
s = Fx, \quad F_{ij} = \exp(-i2\pi \vec{\nu}_i \cdot \vec{x}_j), \quad i = 1, \ldots, M, \quad j = 1, \ldots, N,
\]

where \( \vec{\nu}_i \) denotes the k-space sample location of the \( i \)th sample (units cycles/cm) and \( \vec{x}_j \) denotes the spatial coordinates of the center of the \( j \)th pixel (units cm). In the usual case where the pixel coordinates \( \{ \vec{x}_j \} \) and k-space sample locations \( \{ \vec{\nu}_i \} \) are both on appropriate Cartesian grids, matrix \( F \) is square corresponds to the (2D or 3D) discrete Fourier transform (DFT). In this case \( F^{-1} = \frac{1}{N} F^* \) so reconstructing \( x \) from \( s \) is simply an inverse FFT, and that approach is used in many clinical MR scans.

The reconstruction problem becomes more interesting when the k-space sample locations are on a non-Cartesian grid [22], when the scan is “accelerated” by recording \( M < N \) samples, when non-Fourier effects like magnetic field inhomogeneity are considered [23].

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and/or when there are multiple receive coils. In parallel MRI, let $s_l$ denote the samples recorded by the $l$th of of $L$ receive coils. Then one replaces the model (1) with

$$s_l = F C_l x,$$  

where $C_l$ is a $N \times N$ diagonal matrix containing the coil sensitivity pattern of the $l$ coil on its diagonal. Note that $F$ does not depend on $l$; all coils see the same k-space sampling pattern. Stacking up the measurements from all coils and accounting for noise yields the following basic forward model in MRI:

$$[y_1; \ldots; y_L] = y = (I_L \otimes F)C x + \epsilon, \quad C = \begin{bmatrix} C_1 & \ldots & C_L \end{bmatrix},$$  

where $A \in \mathbb{C}^{ML \times N}$ denotes the system matrix, $y \in \mathbb{C}^{ML}$ denotes the measured k-space data, and $x \in \mathbb{C}^N$ denotes the latent image. The noise in k-space is well modeled as complex white Gaussian noise $\mathcal{C}$.

For extensions that consider other physics effects like relaxation and field inhomogeneity, see [16].

The goal in MR image reconstruction is to recover $x$ from $y$ using the model (3). All MR image reconstruction problems are under-determined because the magnetization of the underlying object being scanned is a space-limited continuous-space function on $\mathbb{R}^3$, yet only a finite number of samples are recorded. Nevertheless, the convention in MRI is to treat the object as a finite-dimensional vector $x \in \mathbb{C}^N$ for which $M \geq N$ appropriate Cartesian k-space samples is considered “fully sampled” and any $M < N$ is considered “under sampled.” The term “compressed sensing” in this setting might simply mean that the k-space sampling is $A$ is a wide matrix, i.e., $M < N$, or might imply that the sampling pattern satisfies some sufficient condition for ensuring good recovery of $x$ from $y$. Sampling pattern design is a topic of ongoing interest [25]–[27], with renewed interest in data-driven methods [28]–[30].

The matrix $F$ in (3) is known prior to the scan, because the k-space sample locations $\{p_l\}$ are controlled by the pulse sequence designer. (Calibration methods are sometimes needed for complicated k-space sampling patterns [31].) In contrast, the coil sensitivity maps $\{C_l\}$ depend on the exact configuration of the receive coils for each patient. To use the model (3), one must determine the sensitivity maps from some patient-specific calibration data, e.g., by joint estimation [32]–[36], regularization [37], or subspace methods [38].

II. COST FUNCTIONS AND ALGORITHMS

A. Quadratic problems

When $ML \geq N$, i.e., when the total number of k-space samples acquired across all coils exceeds the number of unknown image pixel values, the linear model (3) is over-determined and it is reasonable to consider an ordinary least-squares estimator

$$\hat{x} = \arg \min_{x \in \mathbb{C}^N} \frac{1}{2} \|Ax - y\|_2^2 = (AA)^{-1} A'y$$

and$$= \left( \sum_{l=1}^L C_l' F' C_l \right)^{-1} \left( \sum_{l=1}^L C_l' F' y \right).$$

In particular, for fully sampled Cartesian k-space data where $F^{-1} = \frac{1}{N} F'$, this least-squares solution simplifies to

$$\hat{x} = \left( \sum_{l=1}^L C_l' C_l \right)^{-1} \left( \sum_{l=1}^L C_l' F' y \right),$$

which is trivial to implement because each $C_l$ is diagonal. This is known as the optimal coil combination approach [7]. For regularly under-sampled Cartesian data, where only every $n$th row of k-space is collected, the matrix $F'F$ has a simple block structure with $n \times n$ blocks that facilitates non-iterative block-wise computation known as SENSE reconstruction [9]. This form of least-squares estimation is used widely in clinical MR systems.

B. Regularized least-squares

For under-sampled problems ($ML < N$) the LS solution (4) is not unique and for non-Cartesian sampling $A$ is often poorly conditioned. Some form of regularization is needed in both cases. Some early MRI reconstruction work used quadratically regularized cost functions leading to optimization problems of the form:

$$\hat{x} = \arg \min_{x \in \mathbb{C}^N} \frac{1}{2} \|Ax - y\|_2^2 + \beta \|Tx\|_2^2,$$  

where $\beta > 0$ denotes a regularization parameter and $T$ denotes a $N \times N$ matrix transform such as finite differences. The conjugate gradient (CG) algorithm is well-suited to such quadratic cost functions [10], [23]. The Hessian matrix $A'TA + \beta T'T$ often is approximately Toeplitz [41], so CG with circulant preconditioning is particularly effective [42]. Although the quadratically regularized least-squares cost function (5) is passé in the compressed sensing era, CG is often used as an inner step when optimizing more complicated cost functions [39].

Coil coupling induces noise correlation between coils that one should first whiten [39]. Often the data from multiple coils is condensed to a smaller number of virtual coils to save computation and memory [40].
C. Edge-preserving regularization

The drawback of the quadratically regularized cost function \( T \) with \( T \) as finite differences is that it blurs image edges. To avoid this blur, one can replace the quadratic regularizer \( \|Tx\|^2_2 \) with a non-quadratic function \( \psi(Tx) \) where typically \( \psi \) is convex and smooth, such as the Huber function \[33\], a hyperbola \[44\], \[45\], or the Fair potential function \( \psi(z) = \delta^2 (|z/\delta| - \log(1 + |z/\delta|)) \), among others \[46\]. Ch. 2] as follows:

\[
\hat{x} = \arg \min_{x \in \mathbb{C}^N} \Psi(x) \triangleq \frac{1}{2} \|Ax - y\|^2_2 + \beta \psi(Tx).
\] (6)

Such methods have their roots in Bayesian methods based on Markov random fields \[47\] \[48\]. The nonlinear CG algorithm is an effective optimization method for cost functions with such smooth edge-preserving regularizers. An interesting alternative is the complex-valued 3MG (majorize-minimize memory gradient) algorithm \[45\]. Another appropriate optimization algorithm is the optimized gradient method (OGM) \[49\]. A first-order method having optimal worst-case performance among all first-order algorithms for convex cost functions with Lipschitz continuous gradients \[50\]. OGM has a convergence rate bound that is twice better than that of Nesterov’s fast gradient method \[51\]. A recent line-search OGM variant is even more attractive \[52\].

Fig. 1 compares two of these methods for the case where \( T \) is finite differences and \( \psi \) is the Fair potential with \( \delta = 0.1 \), which approximates TV fairly closely while being smooth.

D. Sparsity models: synthesis form

Scan time in MRI is proportional to the number of k-space samples recorded. Reducing scan time in MRI can reduce cost, improve patient comfort, and reduce motion artifacts. Reducing the number of k-space samples \( ML \) to well below \( N \), necessitates stronger modeling assumptions about \( x \), and sparsity models are prevalent. Two main categories of sparsity models are the synthesis approach and the analysis approach. In a synthesis model, one assumes \( x = Bz \) for some \( N \times K \) matrix \( B \) where coefficient vector \( z \in \mathbb{C}^K \) should be sparse. In an analysis model, one assumes \( T \hat{x} \) is sparse, for some \( K \times N \) transformation matrix \( T \).

A typical cost function for a synthesis model is

\[
\hat{x} = B\hat{z}, \quad \hat{z} = \arg \min_{z \in \mathbb{C}^K} \frac{1}{2} \|ABz - y\|^2_2 + \beta \|z\|_1, \quad \text{(7)}
\]

where the \( 1 \)-norm is a convex relaxation of the \( \ell_0 \) counting measure that encourages \( z \) to be sparse. The optimization formulation \( \text{(7)} \) is also known as the LASSO problem \[53\] \[54\] and there are numerous algorithms for solving it. The classical approach is the iterative soft thresholding algorithm (ISTA) \[55\], also known as the proximal gradient method (PGM) and proximal forward-backward splitting \[56\], having the simple form

\[
z_{k+1} = \text{soft}(x - D^{-1}B'A'(ABz_k - y, \beta/d)), \quad \text{(8)}
\]

where the soft thresholding function is defined by \( \text{soft}(z, c) = \text{sign}(z) \max(|z| - c, 0) \) and \( D = \text{diag}[d] \) is any positive definite diagonal matrix such that \( D - B'A'AB \) is positive semidefinite \[57\].

Traditionally \( D = \|B'A'AB\|_2 I \), but computing that spectral norm (via the power iteration) requires considerable computation for parallel MRI problems in general. However, for Cartesian sampling, \( F'F \propto NI \) so it suffices to have \( NB'C'CB \preceq D \). Often the sensitivity maps are normalized such that \( C'C = I \) in which case \( NB'B \preceq D \) suffices. If in addition \( B' \) is a Parseval tight frame, then \( B'B \preceq I \) so using \( D = NI \) is appropriate. For non-Cartesian sampling, or non-normalized sensitivity maps, or general choices of \( B \), finding \( D \) is more complicated \[57\].

Although ISTA is simple, it has an undesirably slow \( O(1/k^2) \) convergence bound. This limitation was first overcome by the fast iterative soft thresholding algorithm (FISTA) \[58\] \[59\], also known as the fast proximal gradient method (FPGM) that has an \( O(1/k^2) \) convergence bound. A recent extension of this line of proximal methods is the proximal optimized gradient method (POGM) that has worst-case convergence bound about twice better than that of FISTA/FPGM. \[60\] \[61\]. Both FISTA and POGM are essentially as simple to implement as \( \text{(8)} \). Recent MRI studies have shown POGM converging faster than FISTA, as one would expect based on the convergence bounds \[62\] \[63\] \[64\], particularly when combined with adaptive restart \[61\]. So POGM (with restart) is a recommended method for optimization problems having the form \( \text{(7)} \). This topic remains an active research area with new variants of FISTA appearing recently \[65\]. Table 3 provides POGM pseudo-code for solving composite optimization problems like the MRI synthesis reconstruction model \( \text{(7)} \).

Fig. 2 shows that POGM converges faster than FISTA and ISTA for minimizing \( \text{(7)} \).

E. Sparsity models: analysis form

A potential drawback of the synthesis formulation \( \text{(7)} \) is that \( x \approx Bz \) may be a more realistic assumption than the strict equality \( x = Bz \) when \( z \) is sparse. The analysis approach avoids constraining \( \hat{x} \) to lie in any
such subspace. For an analysis form sparsity model, a typical optimization problem involves a composite cost function consisting of the sum of a smooth term and a non-smooth term:
\[
\hat{x} = \arg\min_x \frac{1}{2} \|Ax - y\|_2^2 + \beta \|Tx\|_1 ,
\]
where \(T\) is a sparsifying operator such as a wavelet transform. When \(T\) is finite differences, the regularizer is called total variation (TV) \([3]\), and combinations of TV and wavelet transforms are useful \([2]\). Although the details are proprietary, presumably the FDA-approved methods for compressed sensing MRI are related to \([9]\).

When \(T\) is invertible, such as an orthogonal wavelet transform, one rewrites the optimization problem \([9]\) as
\[
\hat{x} = T^{-1} \hat{z}, \quad \hat{z} = \arg\min_z \frac{1}{2} \|AT^{-1}z - y\|_2^2 + \beta \|z\|_1 ,
\]
which is simply a special case of \([7]\) with \(B = T^{-1}\).

In the general case \([9]\) where \(T\) is not invertible, the optimization problem is much harder than \([7]\) due to the non-differentiability of the 1-norm with the matrix \(T\). The PGM for \([9]\) is
\[
\hat{x}_{k+1} = \arg\min_x \frac{L}{2} \|x - \hat{x}_k\|_2^2 + \beta \|Tx\|_1 ,
\]
where \(\hat{x}_k \triangleq x_k - \frac{1}{T} A'(Ax_k - y)\) denotes the usual gradient update and the Lipschitz constant is \(L = \|A\|_2^2\).

Unfortunately there is no simple solution for computing the proximity operator in \([9]\) in general, so inner iterative methods are required, typically involving dual formulations \([59], [66]\). This challenge makes PGM and FPGM and POGM less attractive for \([9]\) and has led to a vast literature on algorithms for problems like \([9]\), with no consensus on what is best. The difficulty of \([10]\) is the main drawback of analysis regularization, whereas a possible drawback of the synthesis regularization in \([7]\) is that often \(K > N\) for overcomplete \(B\).

1) Approximate methods:

One popular “work around” option is to “round the corner” of the 1-norm, making smooth approximations like \(|z| \approx \sqrt{|z|^2 + \epsilon}\). This approximation is simply the hyperbola function that has a long history in the edge-preserving regularization literature. All of the gradient-based algorithms mentioned for edge-preserving regularization above are suitable candidates when a smooth function replaces the 1-norm. Smooth functions can shrink values towards zero, but their proximal operators never have a thresholding effect that induces sparsity.
Fig. 2. Comparison of ISTA/FGM, FISTA/FPGM and POGM for single-coil MRI reconstruction with orthogonal discrete wavelet transform sparsity regularizer using the 1-norm. Minimizer \( \hat{x} \) of (7); cost function for (7); NRMSE versus iteration \( k \). FISTA requires about 40% more iterations to converge than POGM, consistent with the 2\times better worst-case bound of POGM.

Fig. 3. POGM method [60] for minimizing \( f(x) + g(x) \) where \( f \) is convex with \( L \)-Lipschitz smooth gradient and \( g \) is convex. See [61] for adaptive restart version.

by setting many values exactly to zero. Whether a thresholding effect is truly essential is an open question.

One way to overcome the challenge of the matrix \( T \) in the 1-norm in (9) is to replace (9) with the following alternative [67]:

\[
\hat{x} = \arg \min_x \frac{1}{2} \|Ax - y\|^2_2 + \beta R_\alpha(x)
\]

where \( \alpha > 0 \). At first glance this formulation appears to enforce sparsity due to the presence of the 1-norm. However, one can solve for \( z \) and substitute back in to show that \( R_\alpha(x) = \psi(Tx, \alpha) \) where \( \psi \) is the Huber function with parameter \( \alpha \), so (11) is simply another example of corner rounding with an approximate 1-norm. One can show \( \frac{1}{\alpha} R_\alpha(x) \to \|Tx\|_1 \) as \( \alpha \to 0 \). A drawback of (11) is that one must choose the additional regularization parameter \( \alpha \) that can affect both the image quality of \( \hat{x} \) and the convergence rate of iterative algorithms for (11).

Another option is to use an iterative reweighted least-squares approach like FOCUSS [68] that approaches the
1-norm in the limit as the number of iterations grows, but is effectively equivalent to a corner-rounded 1-norm for any finite number of iterations. Hereafter we focus on methods that tackle the 1-norm directly without any such approximations.

2) Variable splitting methods:

Variable splitting methods replace (9) with an exactly equivalent constrained minimization problem involving an auxiliary variable such as \( z = T x, \) e.g.,

\[
\hat{x} = \arg \min_x \min_{z: z = T x} \frac{1}{2} \| A x - y \|_2^2 + \beta \| z \|_1.
\]

This approach underlies the split Bregman algorithm [69], various augmented Lagrangian methods [39,70], and the alternating direction multiplier method (ADMM). The augmented Lagrangian for (12) is

\[
L(x, z; \gamma, \mu) = \frac{1}{2} \| A x - y \|_2^2 + \beta \| z \|_1 \\
+ \text{real} \{ \langle \gamma, T x - z \rangle \} + \frac{\mu}{2} \| T x - z \|_2^2,
\]

where \( \gamma \in \mathbb{C}^K \) denotes the vector of Lagrange multipliers, \( \mu > 0 \) is an AL penalty parameter that affects the convergence rate but not the final image \( \hat{x} \). Defining the scaled dual variable \( \eta \triangleq 1/\mu \gamma \) and completing the square leads to the following scaled augmented Lagrangian:

\[
L(x, z; \eta, \mu) = \frac{1}{2} \| A x - y \|_2^2 + \beta \| z \|_1 \\
+ \frac{\mu}{2} \left( \| T x - z + \eta \|_2^2 - \| \eta \|_2^2 \right).
\]

An augmented Lagrangian approach alternates between descent updates of the primal variables \( x, z \) and an ascent update of the scaled dual variable \( \eta \). The \( z \) update is simply soft thresholding:

\[
z_{k+1} = \text{soft}(T x_k + \eta_k, \beta/\mu).
\]

The \( x \) update minimizes a quadratic function:

\[
x_{k+1} = (A' A + \mu T'T)^{-1} (A' y + \mu T' (z_{k+1} + \eta_k)).
\]

A few CG iterations (with an appropriate preconditioner) is a natural choice for approximating the \( x \) update. Finally the \( \eta \) update is

\[
\eta_{k+1} = \eta_k + (T x_{k+1} - z_{k+1}).
\]

The unit step size here ensures dual feasibility [71]. A drawback of variable splitting methods is the need to select the parameter \( \mu \). Adaptive methods have been

3One can think of \( \gamma_k = \text{real} \{ \gamma \} \) and \( \gamma_i = \text{imag} \{ \gamma \} \) as the Lagrange multipliers for the two constraints \( \text{real} \{ T x - z \} = 0 \) and \( \text{imag} \{ T x - z \} = 0 \), and then note that \( \text{real} \{ \langle \gamma, T x - z \rangle \} = \langle \gamma_R, \text{real} \{ T x - z \} \rangle + \langle \gamma_I, \text{imag} \{ T x - z \} \rangle \).

proposed to help with this tuning [71]–[73]. The above updates of \( x \) and \( z \) are sequential; parallel ADMM updates are also possible [74,75].

The conventional variable split in (12) ignores the specific structure of the MRI system matrix \( A \) in (3). Important properties of \( A \) include the fact that \( A' A \) is circulant (for Cartesian sampling) or Toeplitz (for non-Cartesian sampling) and that each coil sensitivity matrix \( C_i \) is diagonal. In contrast, the Gram matrix \( A' A \) for parallel MRI is harder to precondition, though possible [76] [77]. An alternative splitting that simplifies the updates is [39]:

\[
\begin{align*}
\arg \min_x \min_{z \in \mathbb{C}^N} \min_{u \in \mathbb{C}^{K}} &\frac{1}{2} \| F_L u - y \|_2^2 + \beta \| z \|_1 \\
\text{sub. to} &\quad u = C x, \quad z = T v, \quad v = x,
\end{align*}
\]

where \( F_L \triangleq I_L \otimes F \). With this splitting, the \( z \) update again is simply soft thresholding, and the \( u \) update involves the diagonal matrix \( C'C \) which is trivial. The \( v \) update involves the matrix \( T'T \) that is circulant for periodic boundary conditions or is very well suited to a circulant preconditioner otherwise, using simple FFT operations. The \( u \) update involves the matrix \( F'_L F_L \) that is circulant or Toeplitz. This approach exploits the structure of \( A \) to simplify the updates; the primary drawback is that it requires selecting even more AL penalty parameters; condition number criteria can be helpful [39]. Many variations are possible, such as exploiting the fact that \( T'T \) has block tridiagonal structure when \( T \) involves finite differences [75]. Another splitting with fewer auxiliary variables leads to an inner update step that requires solving denoising problems similar to (10) [78].

3) Primal-dual methods:

A key idea behind duality-based methods is the fact:

\[
\| T x \|_1 = \max_{z \in \mathbb{C}^K : \| z \|_\infty \leq 1} \text{real} \{ \langle z, T x \rangle \}.
\]

Thus the (nonsmooth) analysis regularized problem (9) is equivalent to this constrained problem:

\[
\begin{align*}
\arg \min_x &\min_{z \in \mathbb{Z}} \frac{1}{2} \| A x - y \|_2^2 + \beta \text{real} \{ \langle z, T x \rangle \},
\end{align*}
\]

where \( \mathbb{Z} \triangleq \{ z \in \mathbb{C}^K : \| z \|_\infty \leq 1 \} \). The primal-dual methods typically alternate between updating the primal variable \( x \) and the dual variable \( z \), using more convenient alternatives to (14) that involve separate multiplication by \( A \) and by \( A' \) without requiring inner CG iterations. These methods provide convergence guarantees and acceleration techniques that lead to \( O(1/k^2) \) rates [77]–[80] [81]–[85]. A drawback of such methods is they typically require power iterations to find a Lipschitz
F. Patch-based sparsity models

Using (9) with a finite-difference regularizer $R(x) = \|Tx\|_1$ is essentially equivalent to using patches of size $2 \times 1$. It is plausible that one can regularize better by considering larger patches that provide more context for distinguishing signal from noise. There are two primary modes of patch-based regularization: synthesis models and analysis methods.

A typical synthesis approach attempts to represent each patch using a sparse linear combination of atoms from some signal patch dictionary. Let $P_p$ denote the $d \times N$ matrix that extracts the $p$th of $P$ patches (having $d$ pixels) when multiplied by an image vector $x$. Then the synthesis model is that $P_p x \approx D z_p$ where $D$ is a $d \times J$ dictionary and $z_p \in \mathbb{C}^J$ is a sparse coefficient vector for the $p$th patch. Under this model, a natural regularizer is

$$R(x) = \min_{\{z_p\}} \sum_{p=1}^P \frac{1}{2} \|P_m x - D z_p\|_2^2 + \alpha \|z_p\|_1.$$  \hspace{1cm} (15)

The regularizer has an inner minimization over the sparse coefficients $\{z_p\}$, so the overall problem involves both optimizing the image $x$ and those coefficients. This structure lends itself to alternating minimization algorithms.

A typical analysis approach for patches assumes there is a sparsifying transform $\Omega$ such that $\Omega P_p x$ tend to be sparse. Under this model, a natural regularizer is

$$R(x) = \min_{\{z_p\}} \sum_{p=1}^P \frac{1}{2} \|\Omega P_m x - z_p\|_2^2 + \alpha \|z_p\|_1.$$  \hspace{1cm} (16)

Again a double minimization over the image $x$ and the transform coefficients $\{z_p\}$ is needed, so alternating minimization algorithms are natural. The analysis regularizer (16) is jointly convex in $x$ and $\{z_p\}$, whereas the synthesis regularizer (15) is not due to the product $D z_p$. For alternating minimization (block coordinate descent), the update of each $z_p$ is simply soft thresholding, and the update of $x$ is a quadratic problem involving $A' A + \beta \sum_m P_m' \Omega' \Omega P_m$. When the transform $\Omega$ is unitary and the patches are selected with periodic boundary conditions and a stride of one pixel, then this simplifies to $A' A + \beta I$. A few inner iterations of the (preconditioned) CG algorithm is useful for the $x$ update. Under these assumptions, and using just a single gradient descent update for $x$, an alternating minimization algorithm for least-squares with regularizer (16) simply alternates between a denoising step and a proximal gradient step:

$$\tilde{x}_k = \sum_{p=1}^P P_m' \Omega' \text{soft}(\Omega P_m x_k, \alpha)$$

$$x_{k+1} = x_k - (D + \beta I)^{-1} (A' (Ax - y) + \beta \tilde{x}_k).$$

For this algorithm the cost function is monotonically nonincreasing.

G. Adaptive regularization

The patch dictionary $D$ in (15) or the sparsifying transform $\Omega$ in (16) can be chosen based on mathematical models like the discrete cosine transform (DCT), or they can be learned from a population of preexisting training data and then used in (15) or (16) for subsequent patients. A third possibility is to adapt $D$ or $\Omega$ to each specific patient [86], [87]. The “dictionary learning MRI” (DLMRI) approach [86] uses a regularizer of the following form:

$$R(x) = \min_{D \in D} \min_{\{z_p\}} \sum_{p=1}^P \|P_m x - D z_p\|_2^2 + \alpha \|z_p\|_1,$$  \hspace{1cm} (18)

where $D$ is the feasible set of dictionaries (typically constrained so that each atom has unit norm). Now there are three sets of variables to optimize: $x$, $\{z_p\}$, $D$, so alternating minimization methods are well suited. The update of the image $x$ is a quadratic optimization subproblem, the $z_p$ update is soft thresholding, and the $D$ update is simple when considering one atom at a time [88]. This problem is nonconvex because of the $D z_p$ product, but there is some convergence theory for it [88].

The “transform learning MRI” (TLMRI) approach [87] uses a regularizer of this form:

$$R(x) = \min_{\Omega} \min_{\{z_p\}} \sum_{p=1}^P \|\Omega P_m x - z_p\|_2^2 + \alpha \|z_p\|_1 + \gamma r(\Omega),$$

where $r(\Omega)$ enforces or encourages properties of the sparsifying transform such as orthogonality. Again, alternating minimization methods are well suited; the $\Omega$ update involves (small) SVD operations. See [89] for convergence theory and an extension to learning a union of sparsifying transforms.

H. Convolutional regularizers

An alternative to patch-based regularization is to use convolutional sparsity models [90], [91], [92]. A convolutional synthesis regularizer replaces (15) with

$$R(x) = \min_{\{z_k\}} \frac{1}{2} \left\| x - \sum_{k=1}^K h_k \ast z_k \right\|_2^2 + \alpha \|z_k\|_1,$$
where \( \{h_k\} \) is a set of filters learned from training data \([93]\) and \( * \) denotes convolution. Again, alternating minimization algorithms are a natural choice because the \( x \) update is quadratic and the \( z_k \) update is a sparse coding problem for which proximal methods like POGM are well-suited \([24]\).

A convolution analysis regularizer replaces \((16)\) with
\[
R(x) = \min_{\{z_k\}} \sum_{k=1}^{K} \frac{1}{2} \|h_k * x - z_k\|_2^2 + \alpha \|z_k\|_1.
\]

Again, alternating minimization algorithms are effective, where the \( z_k \) update is soft thresholding. One can either learn the filters \( \{h_k\} \) from good quality (e.g., fully sampled) training data, or adapt the filters for each patient by jointly optimizing \( x \), \( \{h_k\} \) and \( \{z_k\} \) using alternating minimization. For such adaptive regularizers, constraints on the filters are essential \([90], [91]\).

### I. Other methods

The summation in \((17)\) is a particular type of patch-based denoising of the current image estimate \( x_k \). There are many other denoising methods, some of which have variational formulations well-suited to inverse problems, but many of which do not, such as nonlocal means (NLM) \([95]\) and block-matching 3D (BM3D) \([96]\). One way to adapt most such denoising methods for image reconstruction is to use a plug-and-play ADMM approach \([97], [98]\) that replaces a denoising step like \((17)\) that originated from an optimization formulation with a general denoising procedure. See also \([99]\).

### J. Non-SENSE methods

The measurement model \((2)\) and \((3)\) has a single latent image \( x \), viewed by each receive coil. An alternate formulation is to define a latent image for each coil \( x_l \triangleq C_l x \) and write the measurement model as \( y_l = F x_l + \varepsilon_l \). For such formulations, the problem becomes to reconstruct the \( L \) images \( X = [x_1 \ldots x_L] \) from the measurements, while considering relationships between those images. Because multiplication by the smooth sensitivity map \( C_l \) in the image domain corresponds to convolution with a small kernel in the frequency domain, any point in k-space can be approximated by a linear combination of its neighbors in all coil data \([11]\). This “GRAPPA modeling” leads to an approximate consistency condition \( \text{vec}(X) \approx G \text{vec}(X) \) where \( G \) is a matrix involving small k-space kernels that are learned from calibration data \([11]\). This relationship leads to “SPIRiT” \([100]\) optimization problems like:
\[
\hat{X} = \arg\min_{X \in \mathbb{C}^{N \times L}} \frac{1}{2} \|FX - Y\|_{\text{Frob}}^2 + \beta_1 \frac{1}{2} \|(G - I) \text{vec}(X)\|_2^2 + \beta_2 R(X),
\]
where \( Y = [y_1 \ldots y_L] \in \mathbb{C}^{M \times L} \) and \( R(X) \) is a regularizer that encourages joint sparsity because all of the images \( \{x_l\} \) have edges in the same locations \([101]\). No sensitivity maps \( C \) are needed for this approach. When \( \beta_2 = 0 \) the problem is quadratic and CG is well suited \([100]\). Otherwise, ADMM is convenient for splitting this optimization problem into parts with easier updates \([102], [103]\). See \([12], [13], [38], [104]\) for subspace and joint sparsity approaches that go further by circumventing finding the calibration matrix \( G \).

### III. Summary

Although the title of this paper is “optimization methods for...” before selecting an optimization algorithm it is far more important (for under-sampled problems) to first select an appropriate cost function that captures useful prior information about the latent object \( x \). The literature is replete with numerous candidate models, each of which often lead to different optimization methods. Nevertheless, common ingredients arise in most formulations, such as alternating minimization (block coordinate descent) at the outer level, preconditioned CG for inner iterations related to quadratic terms, and soft thresholding or other proximal operators for nonsmooth terms that promote sparsity.

This survey has focused on 1-norm regularizers for simplicity, but (nonconvex) \( p \) “norms” with \( 0 < p < 1 \) have also been investigated and appear to be beneficial particularly for very undersampled measurements \([105]\). This survey considers a single image \( x \) but many MRI scan protocols involve several images with different contrast and it may be useful to reconstruct them jointly, e.g., by considering common sparsity or subspace models \([106], [107]–[113]\).

There are many open problems in optimization that are relevant to MRI. The analysis form regularized problem \([2]\) remains challenging, and further investigation of analysis vs synthesis approaches is needed \([114]\). There has been considerable recent progress on finding optimal worst-case methods \([49], [50], [52]\), but these optimality results are for very broad classes of cost functions, whereas the cost functions in MRI reconstruction have particular structure. Finding algorithms with optimal complexity (fastest possible convergence) for MRI-type
cost functions would be valuable both for clinical practice and for facilitating research.

Finally, the current trend is to use convolutional neural network (CNN) methods as methods for post processing under-sampled images, or for direct reconstruction, or as denoising operators. (Finding stable approaches is crucial.) The stochastic gradient descent method (or a variant) currently is the universal optimization tool for training CNN models. Many “deep learning” methods for MRI are based on network architectures that are “unrolled” versions of iterative optimization methods like PGM like PGM [117]–[119] [120], [121]. Thus, familiarity with classical optimization methods for MR image reconstruction is important even in the machine learning era.

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