MORE ABOUT CONTINUOUS GABOR FRAMES ON LOCALLY COMPACT ABELIAN GROUPS

Z. HAMIDI, F. ARABYANI-NEYSHABURI, R. A. KAMYABI-GOL, AND M. H. SATTARI

Abstract. For a second countable locally compact abelian (LCA) group $G$, we study some necessary and sufficient conditions to generate continuous Gabor frames for $L^2(G)$. To this end, we reformulate the generalized Zak transform proposed by Grochenig in the case of integer-oversampled lattices, however our formulation rely on the assumption that both translation and modulation groups are only closed subgroups. Moreover, we discuss the possibility of such generalization and apply several examples to demonstrate the necessity of standing conditions in the results. Finally, by using the generalized Zak transform and fiberization technique, we obtain some characterization of continuous Gabor frames for $L^2(G)$ in term of a family of frames in $l^2(\hat{H})$ for a closed co-compact subgroup $H$ of $G$.

1. Introduction

The Zak transform is one of the fundamental tools in both pure and applied mathematics, that was originally introduced by Gelfand [8] due to some problems in differential equations. This transform was studied by weil on locally compact abelian(LCA) groups [22] and by Zak in solid state physics [23]. Later it has been developed by many authors for identifying and characterizing of Gabor frames on $L^2(G)$ [1, 3, 5, 9]. The most research works in this regard are associated with a discrete, co-compact (uniform lattice) subgroup. In particular, Grochenig [9] presented some aspects of Zak transform to analyze uniform lattice Gabor frames. However, in recent years this aspect has been extended to closed subgroups for the characterization of continuous Gabor frames. Indeed, by considering a closed subgroup $H$ of an LCA group $G$ and applying the Zak transform associated to

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H some equivalent conditions for the existence of continuous Gabor frames as the
form \( \{ E_{\ell} T_{\lambda} g \}_{\lambda \in H, \gamma \in H^\perp} \) have been obtained [1, 15].

The main purpose of this paper is to obtain some characterization results of
continuous Gabor frames as \( \{ E_{\ell} T_{\lambda} g \}_{\lambda \in \Lambda, \gamma \in \Gamma} \) on \( L^2(G) \), for closed subgroups \( \Lambda \subseteq G \)
and \( \Gamma \subseteq \widehat{G} \). To this end, we extend and reformulate the idea of integer oversampling
for uniform lattices in [9]. However, our formulation rely on the assumption that
both translation and modulation groups are only closed subgroups and remove some
other limited conditions. Moreover, we discuss the possibility of such generalization.
Finally, by using the generalized Zak transform and fiberization method, we give
some characterizations of continuous Gabor frames for \( L^2(G) \) in term of a family
of frames in \( l^2(\widehat{H}^\perp) \) for a closed co-compact subgroup \( H \) of \( G \).

This paper is organized as follows. In Section 2 we present some basic facts of
locally compact abelian groups and the required definitions of continuous frame
theory. Then we get a sufficient condition for the existence of continuous Gabor frames in \( L^2(G) \). Section 3 is devoted to investigate some equivalent conditions for
continuous Gabor frames in \( L^2(G) \) via generalized Zak transform. Finally, in section
4, we use the fiberization method to survey a relationship between continuous Gabor
frames in \( L^2(G) \) and a family of frames in \( l^2(\widehat{H}^\perp) \) for a closed co-compact subgroup
\( H \) of \( G \).

2. Notations and preliminaries

Throughout this paper, let \( G \) be a second countable locally compact abelian
(LCA) group. It is known that such a group always carries a translation invariant
regular Borel measure called a Haar measure, denoted by \( \mu_G \), and is unique up to
a positive constant. We will use the addition as the group operation and equip
discrete groups with the counting measure. Let \( \widehat{G} \) denote the dual group of \( G \),
then the famous duality theorem of Pontrjagin says that the character group of
\( \widehat{G} \) is topologically isomorphic with \( G \), i.e., \( \widehat{\widehat{G}} \cong G \). The Fourier transform \( \mathcal{F} : L^1(G) \rightarrow C_0(\widehat{G}) \), is defined by

\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_G f(x) \overline{\xi}(x) d\mu_G(x) \quad (\xi \in \widehat{G}).
\]

By the Fourier inversion, we can recover a function from its Fourier transform.
Several different Fourier inversion theorem there exist. One of the most important
states that if \( f \in L^1(G) \) and \( \hat{f} \in L^1(\hat{G}) \), then
\[
\mathcal{F}^{-1}\hat{f}(x) = f(x) = \int_{\hat{G}} \hat{f}(\xi)\xi(x)d\mu_{\hat{G}}(\xi) \quad (a.e \ x \in G).
\]
The Fourier transform can be extended from \( L^1(G) \cap L^2(G) \) to an isometric isomorphism between \( L^2(G) \) and \( L^2(\hat{G}) \), known as the Plancherel transform. See \([7, 13, 14]\).

The mathematical theory for Gabor analysis in \( L^2(G) \) is based on two classes of operators on \( L^2(G) \). The translation by \( \lambda \in G \), denoted by \( T_\lambda \) and is defined as
\[
T_\lambda f(x) = f(x - \lambda) \quad \text{for all } x \in G.
\]
Also, the modulation by \( \gamma \in \hat{G} \), \( E_\gamma \), defined by
\[
E_\gamma f(x) = \gamma(x)f(x) \quad \text{for } x \in G.
\]
These classes of operators are unitary on \( L^2(G) \) and satisfy the following relations;
\[
T_\lambda E_\gamma = \gamma(\lambda)E_\gamma T_\lambda, \quad \mathcal{F}T_\lambda = E_{-\lambda}\mathcal{F} \text{ and } \mathcal{F}E_\gamma = T_\gamma\mathcal{F}.
\]
For a subset \( \Lambda \) of an LCA group \( G \), it’s annihilator defined by
\[
\Lambda^\perp : = \{ \gamma \in \hat{G}; \gamma(\lambda) = 1, \text{ for all } \lambda \in \Lambda \}.
\]
The annihilator is a closed subgroup of \( \hat{G} \) and if \( \Lambda \) is a closed subgroup, then it is proved that \( \hat{\Lambda} \cong \frac{\hat{G}}{\hat{\Lambda}} \) and \( (\frac{\hat{G}}{\Lambda}) \cong \Lambda^\perp \ [7] \). These relations show that for a closed subgroup \( \Lambda \) the quotient \( \frac{G}{\Lambda} \) is compact if and only if \( \Lambda^\perp \) is discrete. See \([7, 13, 19]\) for more details.

We also remind the reader of Weil’s formula; it relates integrable functions over \( G \) with integrable functions on the quotient space \( \frac{G}{\Lambda} \) when \( \Lambda \) is a closed subgroup of \( G \). For a closed subgroup \( \Lambda \) of \( G \) we let \( \pi_\Lambda : G \rightarrow \frac{G}{\Lambda}, \pi_\Lambda(x) = x + \Lambda \) to be the canonical map from \( G \) onto \( \frac{G}{\Lambda} \). If \( f \in L^1(G) \), then \( \dot{x} := \pi_\Lambda(x) \), defined almost everywhere on \( \frac{G}{\Lambda} \), is integrable. Furthermore, when two of the Haar measures on \( G, \Lambda \) and \( \frac{G}{\Lambda} \) are given, then the third can be normalized so that
\[
\int_G f(x)d\mu_G(x) = \int_{\frac{G}{\Lambda}} \int_\Lambda f(x + \lambda)d\mu_\Lambda(\lambda)d\mu_{\frac{G}{\Lambda}}(\dot{x}). \tag{2.1}
\]
If (2.1) holds, then the respective dual measures on \( \hat{G} \), \( \Lambda^\perp \cong \frac{\hat{G}}{\Lambda} \) and \( \frac{\hat{G}}{\Lambda^\perp} \cong \hat{\Lambda} \) satisfy
\[
\int_{\hat{G}} \hat{f}(\xi)d\mu_{\hat{G}}(\xi) = \int_\frac{\hat{G}}{\Lambda^\perp} \int_\Lambda \hat{f}(\xi + \gamma)d\mu_{\Lambda^\perp}(\gamma)d\mu_{\hat{\Lambda}}(\dot{\xi}). \tag{2.2}
\]
Hence, if two of the measures on \( G, \Lambda, \frac{G}{\Lambda}, \hat{G}, \Lambda^\perp \) and \( \frac{\hat{G}}{\Lambda^\perp} \) are given, and these two are not dual measure, then by requiring Weil’s formulas (2.1) and (2.2), all other
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measures are uniquely determined. For a closed subgroup \( \Lambda \) of \( G \), a Borel section or a fundamental domain is a Borel measurable subset \( X \) of \( G \) such that every \( y \in G \) can be uniquely written as \( y = \lambda + x \), where \( \lambda \in \Lambda \) and \( x \in X \). We always equip the Borel section \( X \) of \( G \) and the Haar measure \( \mu_G \mid X \). In \([5]\) it is shown that the mapping \( x \mapsto \rightarrow x + H \) from \((X, \mu_G)\) to \((G/\Lambda, \mu_{G/\Lambda})\) is measure-preserving, and the mapping \( Q(f) = f' \) defined by

\[
Q(f) = f'(x + H) = f(x), \quad x + H \in \frac{G}{\Lambda}, \quad x \in X
\]  

(2.3)
is an isometry from \( L^2(X, \mu_G) \) onto \( L^2(G/\Lambda, \mu_{G/\Lambda}) \) \([5]\).

Assume that \( \Lambda \) is a discrete subgroup. It follows that \( \mu_G(X) \) is finite if and only if, \( \Lambda \) is co-compact, i.e., \( \Lambda \) is a uniform lattice \([4]\). For more information of harmonic analysis on locally compact abelian groups, we refer the reader to the classical books \([7, 13, 14, 19]\).

2.1. Frame theory. The central aspect of this paper is related to continuous frames that was introduced by Ali, Antoine and Cazeau \([2]\). In the following we give the basic definitions and notations of continuous frames.

**Definition 2.1.** Let \( H \) be a complex Hilbert space, and let \((M, \Sigma_M, \mu_M)\) be a measure space, where \( \Sigma_M \) denotes the \( \sigma \)-algebra and \( \mu_M \) the non-negative measure. A family of vectors \( \{f_k\}_{k \in M} \) is called a frame for \( H \) with respect to \((M, \Sigma_M, \mu_M)\) if

(a) the mapping \( M \longrightarrow \mathbb{C}, \ k \longmapsto \langle f, f_k \rangle \) is measurable for all \( f \in H \), and

(b) there exist constants \( A, B > 0 \) such that

\[
A \| f \|^2 \leq \int_M |\langle f, f_k \rangle|^2 d\mu_M(k) \leq B \| f \|^2, \quad (f \in H). \tag{2.4}
\]

The constants \( A \) and \( B \) in (2.4) are called frame bounds. If \( \{f_k\}_{k \in M} \) is weakly measurable and the upper bound in inequality (2.4) holds, then \( \{f_k\}_{k \in M} \) is said to be a Bessel family with bound \( B \). A frame \( \{f_k\}_{k \in M} \) is said to be tight if we can choose \( A = B \); if furthermore \( A = B = 1 \), then \( \{f_k\}_{k \in M} \) is called a Parseval frame.

Also, a family \( \{f_k\}_{k \in M} \) is called minimal if \( f_j \notin \operatorname{span}\{f_k\}_{k \neq j} \) for all \( j \in M \).

To a Bessel family \( \{f_k\}_{k \in M} \) for \( H \), we associate the synthesis operator \( T : L^2(M, \mu_M) \longrightarrow H \) defined by \( T\{c_k\}_{k \in M} = \int_M c_k f_k d\mu_M(k) \) that integral is defined in weak sense and is a bounded linear operator.
Its adjoint operator \( T^* : H \rightarrow L^2(M, \mu_M) \) is called the **analysis operator**, and is given by \( T^*f = \{ \langle f, f_k \rangle \}_{k \in M} \). The frame operator \( S : H \rightarrow H \) is defined as \( S = TT^* \). We remark that the frame operator is the unique operator satisfying

\[
\langle Sf, g \rangle = \int_M \langle f, f_k \rangle \langle f_k, g \rangle \, d\mu_M(k), \quad (f, g \in H)
\]

and is a well-defined, bounded and self-adjoint for any Bessel system \( \{f_k\}_{k \in M} \). Also, it is invertible if and only if \( \{f_k\}_{k \in M} \) is a frame \([2, 18]\).

Let \( P \) be an index set, \( g_p \in L^2(G) \) for all \( p \in P \) and \( H \) be a closed co-compact subgroup of \( G \). The translation invariant system generated by \( \{g_p\}_{p \in P} \) with translation along the closed co-compact subgroup \( H \) is as \( \{T_\eta g_p\}_{h \in H p \in P} \). Also, let for a topological space \( T \), the Borel algebra of \( T \) is denoted by \( B_T \). Then, consider the following standing assumptions of \([15, 16]\):

(I) \((P, \sum_P, \mu_P)\) is a \( \sigma \)-finite measure space,

(II) the mapping \( p \mapsto g_p, (P, \sum_P) \rightarrow (L^2(G), B_{L^2(G)}) \) is measurable,

(III) the mapping \((p, x) \mapsto g_p(x), (P \times G, \sum_P \otimes B_G) \rightarrow (C, B_C) \) is measurable.

The family \( \{g_p\}_{p \in P} \) is called admissible or, when \( g_p \) is clear from the context, simply it is said that the measure space \( P \) is admissible. The nature of these assumptions are discussed in \([16]\). Observe that any closed subgroup \( P_j \) of \( G \) ( or \( \hat{G} \)) with the Haar measure is admissible if \( p \mapsto g_p \) is continuous, e.g., if \( g_p = T_p g \) for some function \( g \in L^2(G) \).

A Gabor system in \( L^2(G) \) with generator \( g \in L^2(G) \) is a family of functions of the form \( \{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma} \), where \( \Gamma \leq \hat{G} \) and \( \Lambda \leq G \).

If both \( \Lambda \) and \( \Gamma \) are closed and co-compact subgroups, the family \( \{E_\gamma T_\lambda g\}_{\gamma \in \Gamma, \lambda \in \Lambda} \) is called a **co-compact Gabor system**; if only one of the sets \( \Lambda \) or \( \Gamma \) are closed and co-compact subgroup, the Gabor system is called **semi co-compact** and it is said to be a uniform lattice Gabor system whenever both \( \Lambda \) and \( \Gamma \) are uniform lattices.

In the following, we are going to derive some sufficient condition for \( \{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma} \) being a frame for \( L^2(G) \). First we need the following result.
Lemma 2.2. [15] Let \( g \in L^2(G) \) and \( \Gamma \subseteq \hat{G} \) be a closed subgroup. Then
\[
\int_{\Gamma} |\langle f, E_\gamma T_\lambda g \rangle|^2 d\mu_\Gamma(\gamma)
= \int_G \int_{\Gamma} f(x) \overline{f(x-\alpha)} T_\lambda g(x) T_\lambda g(x-\alpha) d\mu_\Gamma(\alpha) d\mu_G(x)
\]
for any \( f \in C_c(G) \).

Proposition 2.3. Let \((\Lambda, \mu_\Lambda) \subseteq G\) be an admissible measure space and \( \Gamma \subseteq \hat{G} \) be a closed and co-compact subgroup. Also, let for all \( \alpha \in \Gamma^\perp \) we have \( \text{supp} g \cap \text{supp} T_\alpha g = \emptyset \), up to a set of measure zero in \( G \). If there exist constants \( A, B > 0 \) such that
\[
A \leq \int_{\Lambda} |T_\lambda g(x)|^2 d\mu_\Lambda(\lambda) \leq B, \ a.e. \ (x \in G).
\tag{2.5}
\]
Then \( \{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma} \) is a frame for \( L^2(G) \).

**Proof.** Since \( \Gamma \) is closed and co-compact so \( \Gamma^\perp \) is a discrete subgroup of \( G \). Thus by the assumption and Lemma 2.2 we have
\[
\int_{\Gamma} |\langle f, E_\gamma T_\lambda g \rangle|^2 d\mu_\Gamma(\gamma)
= \int_G \sum_{\alpha \in \Gamma^\perp} f(x) \overline{f(x-\alpha)} T_\lambda g(x) T_\lambda g(x-\alpha) d\mu_G(x)
= \int_G |f(x)|^2 |T_\lambda g(x)|^2 d\mu_G(x)
+ \int_G \sum_{\alpha \neq 0, \alpha \in \Gamma^\perp} f(x) \overline{f(x-\alpha)} T_\lambda g(x) T_\lambda g(x-\alpha) d\mu_G(x)
= \int_G |f(x)|^2 |T_\lambda g(x)|^2 d\mu_G(x),
\]
for every \( f \in C_c(G) \). Therefore,
\[
\int_{\Lambda} \int_{\Gamma} |\langle f, E_\gamma T_\lambda g \rangle|^2 d\mu_\Gamma(\gamma) d\mu_\Lambda(\lambda) = \int_G |f(x)|^2 \int_{\Lambda} |T_\lambda g(x)|^2 d\mu_\Lambda(\lambda) d\mu_G(x)
\]
and consequently by using (2.5)
\[
A \parallel f \parallel^2 \leq \int_{\Lambda} \int_{\Gamma} |\langle f, E_\gamma T_\lambda g \rangle|^2 d\mu_\Lambda(\lambda) d\mu_\Gamma(\gamma) \leq B \parallel f \parallel^2 \tag{2.6}
\]
for all \( f \in C_c(G) \). In addition, \( \Lambda \) and \( \Gamma \) are \( \sigma \)-finite measure spaces and \( C_c(G) = L^2(G) \). Thus, Proposition 2.5 of [18] implies that the inequality (2.6) holds for all \( f \in L^2(G) \), i.e., \( \{E_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma} \) is a frame for \( L^2(G) \). \( \square \)
3. Continuous Zak transform and construction of Gabor frames

In this section, we deal with the generalized Zak transform related to continuous Gabor systems on LCA groups. We extend the idea of integer oversampling to closed subgroups and remove some limited assumptions. Moreover, we discuss the existence conditions and provide equivalent conditions for a continuous Gabor system to be a frame family, orthonormal basis, complete or a minimal family.

Definition 3.1. Let $G$ be an LCA group and $\Lambda$ be a closed subgroup of $G$. For each $f \in C_c(G)$, the mapping $Z_\Lambda f$, defined on $G \times \hat{G}$ as

$$Z_\Lambda f(x, \xi) = \int_\Lambda f(x + \lambda)\xi(\lambda)dm_\Lambda(\lambda)$$

is called the continuous Zak transform of $f$.

It is known that, the continuous Zak transform can be extended to a unitary operator from $L^2(G)$ onto $L^2(M_\Lambda)$, where $M_\Lambda := \frac{G}{\Lambda} \times \frac{\hat{G}}{\Lambda^\perp}$. The next lemma states the basic properties of continuous Zak transform.

Lemma 3.2. [1] Let $\Lambda$ be a closed subgroup of $G$ and $f \in L^2(G)$. Then

(i) Quasi-periodicity: for all $a \in G$ and $\gamma \in \Lambda^\perp$

$$Z_\Lambda f(x + a, \xi) = \xi(a)Z_\Lambda f(x, \xi), \quad Z_\Lambda f(x, \xi + \gamma) = Z_\Lambda f(x, \xi).$$

(ii) Diagonalization: If $(\gamma, \lambda) \in (\Lambda^\perp \times \Lambda)$, then $E_{\gamma, \lambda} f \in L^2(G)$ and

$$Z_\Lambda E_{\gamma, \lambda} f = E_{\lambda, \gamma} Z_\Lambda f,$$

where $E_{\lambda, \gamma}(x, \omega) = \gamma(x)\omega(\lambda)$ for all $(x, \omega) \in G \times \hat{G}$.

Now, let $G$ be an LCA group, $\Lambda \subseteq G$ and $\Gamma \subseteq \hat{G}$ be closed subgroups. Also, let there exists a closed subgroup $H \leq \Lambda$ so that $H^\perp \leq \Gamma$ and $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are countable (or finite). In this case we can choose $\lambda_i \in \Lambda$ so that $\Lambda = \bigcup_{i=1}^\infty (\lambda_i + H)$ and each coset of $\frac{\Lambda}{H}$ contains only one $\lambda_i$. Moreover, there exist $\gamma_j \in \Gamma$, so that $\Gamma = \bigcup_{j=1}^\infty (\gamma_j + H^\perp)$ and each coset of $\frac{\Gamma}{H^\perp}$ contains only one $\gamma_j$. Then the frame operator of the Gabor system $\{E_{\gamma, \lambda} g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$, for a well-fitted window function $g$
on $G$, can be written as follows

$$Sf = \int_{\Lambda} \int_{\Gamma} \langle f, E_{\gamma} T_{\lambda} g \rangle E_{\gamma} T_{\lambda} g d\mu_\Lambda(\lambda) d\mu_\Gamma(\gamma)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{H} \int_{H^\perp} \langle f, E_{\omega, T_{h} T_{\lambda}} E_{\gamma, T_{\lambda}} g \rangle E_{\omega, T_{h} T_{\lambda}} E_{\gamma, T_{\lambda}} g d\mu_H(h) d\mu_{H^\perp}(\omega)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{H} \int_{H^\perp} \langle f, E_{\omega, T_{h}} g_{ij} \rangle E_{\omega, T_{h}} g_{ij} d\mu_H(h) d\mu_{H^\perp}(\omega)$$

for all $f \in L^2(G)$ where

$$g_{ij} = T_{\lambda} E_{\gamma, T_{\lambda}} g.$$  

(3.1)

Thus

$$Z_H Sf = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{H} \int_{H^\perp} \langle Z_H f, Z_H (E_{\omega, T_{h}} g_{ij}) \rangle Z_H (E_{\omega, T_{h}} g_{ij}) d\mu_H(h) d\mu_{H^\perp}(\omega)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{H} \int_{H^\perp} \langle Z_H f, E_{\omega, T_{h} T_{\lambda}} E_{\gamma, T_{\lambda}} g_{ij} \rangle E_{\omega, T_{h} T_{\lambda}} E_{\gamma, T_{\lambda}} g_{ij} d\mu_H(h) d\mu_{H^\perp}(\omega)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{H} \int_{H^\perp} \langle Z_H f, Z_H g_{ij} \rangle (h, \omega) E_{\omega, T_{h} T_{\lambda}} E_{\gamma, T_{\lambda}} g_{ij} d\mu_H(h) d\mu_{H^\perp}(\omega)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Z_H f, Z_H g_{ij}, Z_H g_{ij}$$

$$= \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |Z_H g_{ij}|^2 \right) Z_H f.$$  

The forthcoming theorem, which collects the above computations, shows that the Zak transform on $H$ diagonalize the Gabor frame operator of $\{E_{\gamma} T_{\lambda} g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$. Moreover, the spectrum of the frame operator equals the range of $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |Z_H g_{ij}|^2$.

**Theorem 3.3.** Let $g$, $\Lambda$, $\Gamma$ and $S$ be as above and there exists a closed subgroup $H$ of $G$ so that

$$H \leq \Lambda, \text{ and } H^\perp \leq \Gamma.$$  

(3.2)

Moreover, assume that $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are countable. Then, we obtain $Z_H S Z_H^{-1} F = \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |Z_H g_{ij}|^2 \right) F$, for all $F \in L^2(M_H)$.

As a special case of Theorem 3.3 we record the following corollaries.
Corollary 3.4. Let $G$ be an LCA group, $g \in L^2(G)$, $H, \Lambda \leq G$ and $\Gamma \leq \widehat{G}$ be closed subgroups. Then,

(i) $Z_H(E_{\gamma}T_{\lambda}g)(x,\omega) = \gamma(\lambda)E_{\lambda,\gamma}(x,\omega)Z_Hg(x,\gamma + \omega)$, for all $\lambda \in \Lambda$, $\gamma \in \Gamma$ and a.e. $(x,\omega) \in \Gamma \times \widehat{G}$.

(ii) If the closed subgroup $H$ of $G$ satisfies (3.2), then $Z_H(E_{\gamma}T_{\lambda}g) = E_{\lambda,\gamma}Z_Hg$, for all $\lambda \in \Gamma^+$ and $\gamma \in \Lambda^+$. 

Proof. To show (i), suppose that $g \in L^2(G)$ then

$$Z_H(E_{\gamma}T_{\lambda}g)(x,\omega) = \int_H E_{\gamma}T_{\lambda}g(x+h)\omega(h)d\mu_H(h)$$

$$= \int_H g(x+h-\lambda)\gamma(x)\gamma(h)\omega(h)d\mu_H(h)$$

$$= \int_H g(x+h)(\gamma+\omega)(h)(\gamma+\omega)(\lambda)\gamma(x)d\mu_H(h)$$

$$= (\gamma+\omega)(\lambda)\gamma(x)Z_Hg(x,\gamma+\omega)$$

$$= \gamma(\lambda)E_{\lambda,\gamma}(x,\omega)Z_Hg(x,\gamma+\omega),$$

for all $\lambda \in \Lambda$, $\gamma \in \Gamma$, $x \in G$ and $\omega \in \widehat{G}$.

The proof of (ii) is similar, only it is sufficient to note that in the above computations we have $\gamma(\lambda) = \gamma(h) = 1$, for all $h \in H$, $\lambda \in \Gamma^+$, $\gamma \in \Lambda^+$, by the assumption. □

Corollary 3.5. Let $G$, $g$, $\Lambda$, $\Gamma$, and $H$ satisfy in conditions of Theorem 3.3, and take the sequence $\{g_{ij}\}_{i,j=1}^{\infty}$ as in (3.1). Then, the following statements hold:

(i) $\{E_{\gamma}T_{\lambda}g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is a frame for $L^2(G)$ if and only if there exist constants $A$, $B$ so that $A \leq \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} | Z_Hg_{ij}(x,\omega) |^2 \right) \leq B$, for a.e. $(x,\omega) \in G \times \widehat{G}$.

(ii) $\{E_{\gamma}T_{\lambda}g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is a Parseval frame for $L^2(G)$ if and only if $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} | Z_Hg_{ij}(x,\omega) |^2 = 1$, for a.e. $(x,\omega) \in G \times \widehat{G}$.

(iii) $\{E_{\gamma}T_{\lambda}g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is an orthonormal basis for $L^2(G)$ if and only if $\|g\|_{L^2(G)} = 1$ and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} | Z_Hg_{ij}(x,\omega) |^2 = 1$, for a.e. $(x,\omega) \in G \times \widehat{G}$.

Proposition 3.6. Let $G$ be an LCA group, $g \in L^2(G)$, and $H$ be a closed subgroup of $G$. Then, the following statements hold:

(i) $\{E_{\gamma}T_{\lambda}g\}_{\lambda \in H, \gamma \in H^+}$ is a complete system in $L^2(G)$ if and only if $Z_Hg \neq 0$, a.e.
(ii) If $H$ is a uniform lattice, then \{${E_\gamma T_\lambda g}$\}_{$\lambda \in H, \gamma \in H^\perp$} is a minimal system in $L^2(G)$ if and only if $\frac{1}{Z_H g} \in L^2(M_H)$.

Proof. (i): Let $Z_H g \neq 0$ a.e., to show the Gabor system \{${E_\gamma T_\lambda g}$\}_{$\lambda \in H, \gamma \in H^\perp$} is complete in $L^2(G)$, it is sufficient to prove \{${E_{\lambda, \gamma} Z_H g}$\}_{$\lambda \in H, \gamma \in H^\perp$} is complete in $L^2(M_H)$, by the unitarity of $Z_H$ and Lemma 3.2 (ii). To this end, let $\Phi \in L^2(M_H)$ such that $\langle \Phi, E_{\lambda, \gamma} Z_H g \rangle_{L^2(M_H)} = 0$, for all $\lambda \in H, \gamma \in H^\perp$. So we can write

$$\int \int \frac{\Phi(\alpha, \beta)Z_H g(\alpha, \beta)E_{\lambda, \gamma}(\alpha, \beta)}{d\mu(\alpha) d\hat{\gamma}(\beta)} = \langle \Phi, E_{\lambda, \gamma} Z_H g \rangle_{L^2(M_H)} = 0,$$

for all $\lambda \in H, \gamma \in H^\perp$. Since $\overline{\Phi} Z_H g \in L^1(M_H)$ and the functions in $L^1(M_H)$ are uniquely determined by their Fourier coefficients. Thus, we have $\Phi Z_H g = 0$, a.e.

On the other hand, by the assumption $Z_H g \neq 0$ a.e. that means $\Phi = 0$ a.e., proving the claim.

For the converse, suppose the Gabor system \{${E_\gamma T_\lambda g}$\}_{$\lambda \in H, \gamma \in H^\perp$} is a complete family in $L^2(G)$. Then \{${E_{\lambda, \gamma} Z_H g}$\}_{$\lambda \in H, \gamma \in H^\perp$} is also complete in $L^2(M_H)$. To the contrary, let $\Delta_g = \{(\alpha, \beta) \in M_H : Z_H g(\alpha, \beta) = 0\}$ has positive measure. Put $\Phi = \chi_{\Delta_g}$, that $\chi_{\Delta_g}$ is the characteristic function of $\Delta_g$. Then we obtain $\langle \Phi, E_{\lambda, \gamma} Z_H g \rangle_{L^2(M_H)} = 0$, for all $\lambda \in H, \gamma \in H^\perp$, a contradiction. So, $Z_H g \neq 0$ a.e.

To show (ii), first suppose that $\frac{1}{Z_H g} \in L^2(M_H)$. Since $Z_H$ is surjective, there exists a function $h \in L^2(G)$ such that $\frac{1}{Z_H g} = Z_H h$. Hence, we have

$$\langle E_\gamma T_\lambda g, E_{\gamma'} T_{\lambda'} h \rangle_{L^2(G)} = \langle E_{\lambda, \gamma} Z_H g, E_{\lambda', \gamma'} Z_H h \rangle_{L^2(M_H)} = \langle E_{\lambda, \gamma} Z_H g, E_{\lambda', \gamma'} \frac{1}{Z_H g} \rangle_{L^2(M_H)} = \langle E_{\lambda, \gamma}, E_{\lambda', \gamma'} \rangle_{L^2(M_H)},$$

for all $\lambda, \lambda' \in H$ and $\gamma, \gamma' \in H^\perp$. The assumption that $H$ is a uniform lattice implies that $M_H$ is a compact group and hence \{${E_{\lambda, \gamma}}$\}_{$\lambda \in H, \gamma \in H^\perp$} is an orthonormal basis for $L^2(M_H)$. Therefore, it follows from the above computations that \{${E_\gamma T_\lambda g}$\}_{$\lambda \in H, \gamma \in H^\perp$} has a biorthogonal system and consequently it is minimal. Conversely, let \{${E_\gamma T_\lambda g}$\}_{$\lambda \in H, \gamma \in H^\perp$} is minimal so \{${E_{\lambda, \gamma} Z_H g}$\}_{$\lambda \in H, \gamma \in H^\perp$} is minimal and has a biorthogonal system as \{${\psi_{\lambda, \gamma}}$\}_{$\lambda \in H, \gamma \in H^\perp$} \subseteq $L^2(M_H)$. Fixed $\lambda_0 \in H, \gamma_0 \in H^\perp$,
then \( \psi_{\lambda_0, \gamma_0} \overline{Z_{HG}} \in L^1(M_H) \) and we have
\[
\int \frac{\Phi(\alpha, \beta)}{H} \frac{d\mu_{\overline{HG}}(\alpha, \beta)}{H} d\mu_{\overline{HG}}(\alpha) d\mu_{\overline{HG}}(\beta) = \langle \psi_{\lambda_0, \gamma_0}, \overline{Z_{HG}} E_{\lambda, \gamma} \rangle_{L^2(M_H)} = \delta_{\lambda, \lambda_0} \delta_{\gamma, \gamma_0}
\]
on the other hand \( \langle E_{\lambda, \gamma}, E_{\lambda_0, \gamma_0} \rangle = \delta_{\lambda, \lambda_0} \delta_{\gamma, \gamma_0} \) for all \( \lambda \in H, \gamma \in H^\perp \). so \( \psi_{\lambda_0, \gamma_0} \overline{Z_{HG}} = E_{\lambda_0, \gamma_0} \neq 0 \) thus \( \psi_{\lambda_0, \gamma_0} = \frac{E_{\lambda_0, \gamma_0}}{Z_{HG}} \). In particular, we obtain \( \psi_{c, 1} = \frac{1}{Z_{HG}} \in L^2(M_H) \).

It is worth noticing that for closed subgroups \( \Lambda, \Gamma \) and \( H \) which satisfy (3.2), we have \( \Gamma^\perp \times \Lambda^\perp \subseteq H \times H^\perp \subseteq \Lambda \times \Gamma \). So, if \( Z_H \neq 0 \) a.e. then the Gabor system \( \{E_{\gamma}T_{\lambda}g\}_{\lambda \in \Lambda, \gamma \in \Gamma} \) is complete in \( L^2(G) \). However, as the following examples show, the Gabor system \( \{E_{\gamma}T_{\lambda}g\}_{\lambda \in \Gamma^\perp, \gamma \in \Lambda^\perp} \) is not necessarily complete and in case \( \frac{1}{Z_{HG}} \in L^2(M_H) \), the Gabor system \( \{E_{\gamma}T_{\lambda}\phi\}_{\lambda \in \Lambda, \gamma \in \Gamma} \) is not minimal, in general.

**Example 3.7.** Fix \( 0 < \alpha < 1 \), set \( g(x) = |x|^\alpha \) for \( x \in [-\frac{1}{2}, \frac{1}{2}] \). It is known that the system \( \{T_n E_m g\}_{n, m \in \mathbb{Z}} \) is a Schauder basis for \( L^2(G) \) (but not Riesz basis for \( L^2(G) \), [6]. So this system is minimal and complete. Take \( \Lambda = \frac{1}{2} \mathbb{Z} \), \( \Gamma = \frac{1}{4} \mathbb{Z} \) and \( H = \mathbb{Z} \), then the closed subgroups \( H, \Lambda \) and \( \Gamma \) satisfy (3.2). In addition, the Gabor system \( \{E_{\gamma}T_{\lambda}\phi\}_{\lambda \in H, \gamma \in H^\perp} \) is complete and minimal. However, the Gabor system \( \{E_{\gamma}T_{\lambda}\phi\}_{\lambda \in \Lambda, \gamma \in \Gamma} \) is not minimal and the Gabor system \( \{E_{\gamma}T_{\lambda}\phi\}_{\gamma \in \Gamma^\perp, \lambda \in \Gamma^\perp} \) is not complete.

As another example, suppose that \( G = \mathbb{R} \) and consider the *Gaussian* function \( \phi(x) = e^{-\pi x^2} \). It was already proved in [10] that the Gabor system \( \{E_{m, n} T_{\alpha, \beta} \phi\}_{m, n \in \mathbb{Z}} \) is complete for \( \alpha \beta \leq 1 \) and incomplete for \( \alpha \beta > 1 \). Moreover, \( Z_{2\alpha, \beta} \phi \neq 0 \) a.e. on \( [0, \alpha) \times [0, \frac{1}{\alpha}] \), for all non-zero \( \alpha \in \mathbb{R} \) [12]. Let \( H = 4\mathbb{Z}, \Lambda = 2\mathbb{Z} \) and \( \Gamma = \frac{1}{4} \mathbb{Z} \), then the closed subgroups \( H, \Lambda \) and \( \Gamma \) satisfy (3.2). So, the Gabor system \( \{E_{\gamma}T_{\lambda}\phi\}_{\lambda \in H, \gamma \in H^\perp} \) is complete, and so \( Z_{H} \phi \neq 0 \), a.e., by Proposition 3.6. Although, the Gabor system \( \{E_{\gamma}T_{\lambda}\phi\}_{\gamma \in \Gamma^\perp, \lambda \in \Gamma^\perp} \) is incomplete.

### 3.1. The existence conditions

In what follows, for two given closed subgroups \( \Lambda \) and \( \Gamma \), we discuss the existence of a closed subgroup \( H \) which satisfies (3.2) so that the quotient groups \( \frac{\Lambda}{H} \) and \( \frac{\Gamma}{H^\perp} \) are countable. We first note that, for every LCA group \( G \), the condition \( \Gamma^\perp \subseteq \Lambda \) is necessary for the existence of \( H \). Moreover obviously, for countable groups it can be considered as a necessary and sufficient
condition. Specially, for finite group $G = \mathbb{Z}_L = \{0, \ldots, L - 1\}$, $L \in \mathbb{N}$, it is known that a closed subgroup in $G$ is as $\Lambda = N\mathbb{Z}_L$ where $N \in \mathbb{N}$ so that $N$ is a divisor of $L$. Also, consider closed subgroups $\Gamma = M\mathbb{Z}_L$ and $H = R\mathbb{Z}_L$ so that $M, R \in \mathbb{N}$, are some divisors of $L$. Then the condition (3.2) is equivalent to

$$N \mid R \mid L \text{ and } M \mid L/R.$$  

(3.3)

More precisely, (3.3) is the necessary and sufficient condition for the subgroup $H = R\mathbb{Z}_L$ of $G$ to satisfy (3.2).

**Lemma 3.8.** Assume that $H \leq \Lambda$ are closed subgroups of $G$ so that $\frac{\Lambda}{H}$ is finite. Then $\frac{\Lambda}{H} \cong \frac{H_2}{\Lambda_2}$.

**Proof.** Applying Proposition 4.2.24 of [19] and the fact that any finite group is self-dual we obtain $\frac{\Lambda}{H} \cong \left(\frac{\Lambda}{H}\right) \cong \frac{H_2}{\Lambda_2}$. $\square$

**Example 3.9.** Suppose that $G = \mathbb{Q}_p$, the $p$-adic numbers group, it is known that every non-trivial closed subgroup $H$ of $\mathbb{Q}_p$, is open and compact. Hence, $\mathbb{Q}_p$ is infinite and discrete, so for every non-trivial closed subgroups $H \leq \Lambda$ of $\mathbb{Q}_p$ we imply that $\frac{\Lambda}{H}$ is both discrete and compact and consequently is finite, similarly $\frac{\Gamma}{\Gamma^\perp}$ is finite as well. That means all closed subgroups $H$ with the property $\Gamma^\perp \leq H \leq \Lambda$, fulfill the condition (3.2) a long with finite quotient groups.

**Example 3.10.** Consider $G = \mathbb{R} \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the group of $p$-adic integers, and let $\Lambda, \Gamma$ be two non-trivial closed subgroups of $G$ and $\hat{G}$, respectively. So $\Lambda = \alpha \mathbb{Z} \times \Lambda_2$, for some $\alpha \in \mathbb{R}$ and $\Lambda_2$ is a closed subgroup of $\mathbb{Z}_p$. We show that for every closed subgroup $H$ which satisfies $\Gamma^\perp \leq H \leq \Lambda$, the quotients $\frac{\Lambda}{H}$ and $\frac{\Gamma}{\Gamma^\perp}$ are finite. Indeed $H \leq \Lambda \leq \mathbb{R} \times \mathbb{Z}_p$ implies that $H = m\alpha \mathbb{Z} \times \Lambda_2$, for some $m \in \mathbb{Z}$ and $\Lambda_2 \leq \Lambda_2 \leq \mathbb{Z}_p$. Hence a similar discussion as in Example 3.9 assures that $\frac{\Lambda_2}{\Lambda_2}$ is finite and consequently

$$\frac{\Lambda}{H} = \frac{\alpha \mathbb{Z}}{\alpha \mathbb{Z}} \times \frac{\Lambda_2}{\Lambda_2}$$

is a finite group. Moreover, $\Gamma^\perp \leq H \leq \mathbb{Z}_p$ also follows that $\frac{H}{\Gamma^\perp}$ is finite and so by Lemma 3.8 we obtain $\frac{\Gamma}{\Gamma^\perp} \cong \frac{H}{\Gamma^\perp}$, i.e., $\frac{\Gamma}{\Gamma^\perp}$ is finite, as well.

In the sequel, we show that, $\Gamma^\perp \leq \Lambda$ is not a sufficient condition for the existence of desired $H$ with countable quotient groups, in general.
Example 3.11. Let \( G = \mathbb{R}^n \) and \( \Lambda = \Gamma = \mathbb{R} \times \mathbb{Z}^{n-1} \). Then \( \Gamma \perp \leq \Lambda \) and for every closed subgroup \( H \) so that \( \Gamma \perp \leq H \leq \Lambda \) we can write \( H = H_1 \times H_2 \) where \( H_1 \leq \mathbb{R} \) and \( H_2 = \mathbb{Z}^{n-1} \). If \( H_1 \neq \mathbb{R} \), then \( H_1 = \alpha \mathbb{Z} \) for some \( \alpha \in \mathbb{R} \). Thus \( \Lambda / H_1 \) and \( \Gamma / H_2 \) are uncountable. Moreover, if \( H_1 = \mathbb{R} \) i.e., \( H = \mathbb{R} \times \mathbb{Z}^{n-1} \), then \( \Gamma / H \) is uncountable.

Suppose \( G \) is a compactly generated group of Lie type, that is isomorphic to one of the form \( G = \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^r \times F \) for \( n, m, r \in \mathbb{N} \) and a finite abelian group \( F \). Consider closed subgroups \( \Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3 \times \Lambda_4 \leq G \) and \( \Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3 \times \Gamma_4 \leq \hat{G} \) so that \( \Lambda_1 = \Gamma_1 = \mathbb{R} \times \mathbb{Z}^{n-1} \), and \( \Gamma_i = \Lambda_i \perp \), for \( 2 \leq i \leq 4 \). Then for any closed subgroup \( H \) so that \( \Gamma \perp \leq H \leq \Lambda \) we have that \( H = H_1 \times H_2 \times H_3 \times H_4 \) and so \( H_1 = H_1 \perp \times H_2 \perp \times H_3 \perp \times H_4 \perp \) by lemma 4.2.8 of [19]. Consider \( H_1 \perp < \Lambda_1 = \mathbb{R} \times \mathbb{Z}^{n-1} \), thus by example 3.11, \( \Lambda_1 / H_1 \) is uncountable and consequently \( \Gamma / H_2 \) is uncountable. A similar discussion shows that \( \Gamma / H_3 \) is uncountable, as well. Therefore, for every closed subgroup \( H \) which satisfies \( \Gamma \perp \leq H \leq \Lambda \), we obtain atleast one of \( \Lambda / H \) or \( \Gamma / H \) are uncountable.

Now, we investigate some sufficient condition for the existence of subgroup \( H \) satisfies (3.2) so that \( \Lambda / H \) and \( \Gamma / H \) are finite or countable.

Theorem 3.12. Let \( G \) be an LCA group. Also, let \( \Lambda \) and \( \Gamma \) be subgroups of \( G \) and \( \hat{G} \), respectively so that \( \Gamma \perp \leq \Lambda \). Then the following assertions hold;

(i) If \( \Lambda \) is discrete and \( \Gamma \) is a closed co-compact subgroup, then for every subgroup \( H \) where \( \Gamma \perp \leq H \leq \Lambda \), the quotient groups \( \Lambda / H \) and \( \Gamma / H \) are finite.

(ii) If \( \Lambda \) and \( \Gamma \) are open subgroups, then there exists a closed subgroup \( H \) satisfies (3.2) so that either \( \Lambda / H \) or \( \Gamma / H \) is countable.

(iii) If \( G \) is totally-disconnected and \( \Lambda \), \( \Gamma \) are open subgroups, then there exists a compact subgroup \( H \) satisfies (3.2) so that both \( \Lambda / H \) and \( \Gamma / H \) are countable.

Proof. (i). Consider a subgroup \( H \) of \( G \) such that \( \Gamma \perp \leq H \leq \Lambda \). Then, \( H \) is a discrete subgroup, moreover the assumption assure that \( \Lambda \), \( \Gamma \) and \( H \) are uniform lattices. On the other hand by Proposition 4.2.24 of [19] we have \( \hat{(\Gamma / H)} \cong \hat{\Lambda} / \hat{H} \) and so the duality relationships \( \hat{(\Lambda / H)} \cong \hat{\Lambda} \perp \) and \( \hat{\Gamma} \cong \hat{\Lambda} \perp \) imply that \( \hat{\Lambda} / \hat{H} \) is both compact and discrete. Hence \( \hat{\Lambda} / \hat{H} \) is finite, similarly the quotient group \( \hat{\Gamma} / \hat{H} \) is finite.

(ii). Since \( \Gamma \) is an open subgroup of \( \hat{G} \), the duality relation \( \Gamma \perp \cong \hat{\Gamma} \) implies that \( \Gamma \perp \) is compact. So, by proposition 3.1.5 of [19], there exists a unit neighborhood
V of e (the identity of G) such that \( \Gamma^\perp + V \subseteq \Lambda \). If we take \( H := \Gamma^\perp + \langle V \rangle \), then \( H \) satisfies (3.2) and is open in \( \Lambda \). Thus, \( \frac{\Lambda}{H} \) is discrete and countable but \( \frac{\Gamma}{H} \) is not necessarily countable. On the other hand, the structure of \( V \) assures that there exists a compact subgroup \( N \) of \( G \) such that \( N \subseteq V \), \cite{13}. Set \( H := \Gamma^\perp + N \). Then \( H \) is a compact subgroup of \( G \) and \( \Gamma^\perp \leq H \leq \Lambda \), i.e., \( H \) satisfies (3.2). Moreover,

\[
\frac{\hat{G}}{H} = \frac{\hat{G}}{\Gamma \cap N} \cong (\Gamma^\perp + N)
\]

is discrete and so countable. Therefore, \( \frac{\Lambda}{H} \) is also countable, but not necessarily \( \frac{\Gamma}{H} \).

(iii). Using the proof of (ii) there exists a unit neighborhood \( V \) of e so that \( \Gamma^\perp + V \subseteq \Lambda \). The assumption that G is totally-disconnected implies that there exists an open compact subgroup \( K \) so that \( K \subseteq V \), by Theorem 7.7 of \cite{13}. Put \( H := \Gamma^\perp + K \). Then \( H \) is a compact open subgroup of \( G \) and \( \Gamma^\perp \leq H \leq \Lambda \). Moreover, \( H \) is open in \( \Lambda \). Thus, \( \frac{\Lambda}{H} \) is countable. Also, an analogous discussion shows that \( \frac{\Gamma}{H} \) is countable as well. This completes the proof. \( \square \)

4. Fibrization method

The fiberization technique is closely related to Zak transform methods in Gabor analysis. Let \( H \) be a closed and co-compact subgroup of \( G \) and \( \Omega \subset \hat{G} \) be a Borel section of \( H^\perp \) in \( \hat{G} \), we consider the fiberization mapping \( \mathcal{T} : L^2(G) \longrightarrow L^2(\Omega, l^2(H^\perp)) \), introduced in \cite{5} by

\[
\mathcal{T} f(\omega) = \{ \hat{f}(\omega + \alpha) \}_{\alpha \in H^\perp}, \quad (\omega \in \Omega).
\]

The fiberization is an isometric isomorphic operation as shown in \cite{5}. Furthermore, the frame property of translation-invariant and Gabor system can be characteized in terms of fibers \cite{15}.

\textbf{Theorem 4.1.} Let \( A \) and \( B \) be two positive constants and let \( H \subseteq G \) be a closed, co-compact subgroup, and let \( \{g_p\}_{p \in P} \subseteq L^2(G) \), where \((P, \mu_P)\) is an admissible measure space. Then the following assertions are equivalent.

(i) The family \( \{T_h g_p\}_{h \in H, p \in P} \) is a frame for \( L^2(G) \) with bounds \( A \) and \( B \).

(ii) For almost every \( \omega \in \Omega \), the family \( \{T g_p(\omega)\}_{p \in P} \) is a frame for \( l^2(H^\perp) \) with bounds \( A \) and \( B \), where \( \Omega \) is a Borel section of \( H^\perp \) in \( \hat{G} \).
The next result shows that the frame property of a Gabor system in $L^2(G)$ under certain assumptions is equivalent with the frame property of a family of associated Zak transforms in $l^2(\hat{H}^\perp)$.

**Theorem 4.2.** Let $g \in L^2(G)$, $\Lambda$ and $\Gamma$ be closed subgroups of $G$ and $\hat{G}$ respectively and let $H$ be a closed, co-compact subgroup of $G$ satisfies (3.2). Then there exists a sequence $\{g_{k,u}\}_{k\in\Lambda^+, u\in\frac{1}{\Lambda}}$ in $L^2(G)$ such that following assertions are equivalent.

(i) $\{E_{\gamma}T_\lambda g\}_{\lambda\in\Lambda, \gamma\in\Gamma}$ is a frame for $L^2(G)$ with bounds $A$ and $B$.

(ii) $\{Z_{H^\perp}\hat{g}_{k,u}(\omega, \cdot)\}_{k\in\Lambda^+, u\in\frac{1}{\Lambda}}$ is a frame for $l^2(\hat{H}^\perp)$ with bounds $A$ and $B$, for a.e. $\omega \in \Omega$, where $\Omega$ is a Borel section of $H^\perp$ in $\hat{G}$.

**Proof.** Since $H \leq \Lambda$, so every $\lambda \in \Lambda$ can be written uniquely as $\lambda = t + k$ where $t \in H$ and $k \in \frac{T}{H}$. Also, $\Lambda^\perp \leq \Gamma$ implies that every $\gamma \in \Gamma$ has a unique form such as $\gamma = \mu + u$ where $\mu \in \Lambda^\perp$ and $u \in \frac{1}{\Lambda}$. Thus,

$$\{T_\lambda E_{\gamma}g\}_{\lambda\in\Lambda, \gamma\in\Gamma} = \{T_\lambda E_{\mu}g_{k,u}\}_{t\in H, k\in\frac{T}{H}, \mu\in\Lambda^\perp, u\in\frac{1}{\Lambda}}$$

where $\mathcal{G} := \{g_{k,u}\} = \{T_k E_{\mu}g\}_{t\in H, k\in\frac{T}{H}, \mu\in\Lambda^\perp, u\in\frac{1}{\Lambda}}$. Therefore, applying the fiberization method along with Theorem 4.1 for co-compact subgroup $H$ of $G$, the system $\{T_\lambda E_{\mu}g\}_{t\in H, \mu\in\Lambda^\perp}$ (or equivalently $\{E_{\gamma}T_\lambda g\}_{\lambda\in\Lambda, \gamma\in\Gamma}$) is a frame for $L^2(G)$ if and only if $\{TE_{\mu}g(\omega)\}_{\mu\in\Lambda^\perp}$ is a frame in $l^2(H^\perp)$, for a.e $\omega \in \Omega$ where $\Omega$ is a Borel section of $H^\perp$ in $\hat{G}$. On the other hand, we obtain

$$\{TE_{\mu}g(\omega)\}_{\mu\in\Lambda^\perp} = \{TE_{\mu}g_{k,u}(\omega)\}_{\mu\in\Lambda^\perp, k\in\frac{T}{H}, u\in\frac{1}{\Lambda}}$$

$$= \{(E_{\mu}\hat{g}_{k,u}(\omega + \alpha))_{\alpha\in H^\perp}\}_{\mu\in\Lambda^\perp, k\in\frac{T}{H}, u\in\frac{1}{\Lambda}}$$

$$= \{(T_{\mu}\hat{g}_{k,u}(\omega + \alpha))_{\alpha\in H^\perp}\}_{\mu\in\Lambda^\perp, k\in\frac{T}{H}, u\in\frac{1}{\Lambda}}$$

$$= \{\hat{g}_{k,u}(\omega + \alpha)_{\alpha\in H^\perp}\}_{k\in\frac{T}{H}, u\in\frac{1}{\Lambda}},$$

where the last equality is because of the assumption (3.2). Consider $\psi_{k,u}(\omega) := \hat{g}_{k,u}(\omega + \alpha)$ for all $k \in \frac{T}{H}, u \in \frac{1}{\Lambda}$ and a.e $\omega \in \Omega$. Then the Fourier inversion transform of $\psi_{k,u}(\omega) \in l^2(H^\perp)$ is as follows

$$F^{-1}(\psi_{k,u}(\omega))(\xi) = \sum_{\alpha \in H^\perp} \hat{g}_{k,u}(\omega + \alpha)\alpha(\xi)$$

$$= Z_{H^\perp}\hat{g}_{k,u}(\omega, \xi)$$
for all \( k \in \mathbb{N} \), \( u \in \mathbb{T} \) and a.e., \( \xi \in \mathbb{H}^\perp \) and \( \omega \in \Omega \). Hence, the assertion (i) is equivalent to the system \( \{ Z_{H^\perp} \hat{g}_{ku}(\omega, \cdot) \} \) being a frame for \( l^2(\mathbb{H}^\perp) \), a.e \( \omega \in \Omega \), as required. \( \square \)

In the next corollary, we exploit some connections to the results obtained in [15].

**Corollary 4.3.** Let \( g \in L^2(G) \), \( \Lambda \) be a closed co-compact subgroup of \( G \) and \( \Gamma \) be a closed subgroup of \( \mathbb{G} \) so that \( \Gamma^\perp \leq \Lambda \). Then following assertions are equivalent.

(i) \( \{ E_{\gamma} T_{\lambda} g \}_{\lambda \in \Lambda, \gamma \in \Gamma} \) is a frame for \( L^2(G) \) with bounds \( A \) and \( B \).

(ii) \( \{ \hat{g}(\alpha + \gamma) \}_{\gamma \in \Gamma} \) is a frame for \( l^2(\Lambda^\perp) \) with bounds \( A \) and \( B \), for a.e. \( \alpha \in A \), where \( A \) is a Borel section of \( \Lambda^\perp \) in \( \mathbb{G} \).

(iii) \( A \leq \int_{\mathbb{K}} |Z_{\Lambda^\perp} \hat{g}(\alpha + k, x)|^2 d\mu_{\mathbb{K}}(k) \leq B \), for a.e. \( \alpha \in A \) and \( x \in \mathbb{K} \), where \( \mathbb{A} \) is a Borel section of \( \Lambda^\perp \) in \( \mathbb{G} \).

**Proof.** (i) \( \Leftrightarrow \) (ii): We note that \( (\Gamma, \Sigma_{\Gamma}, \mu_{\Gamma}) \) is an admissible measure space, since \( \Gamma \) is a closed subgroup of \( \mathbb{G} \). Hence, by Proposition 4.5 in [15], (i) is equivalent to \( \{ \hat{g}(\alpha + \gamma + y) \}_{\gamma \in \Gamma} \) being a frame for \( l^2(\Lambda^\perp) \) with bounds \( A \) and \( B \), for a.e. \( \alpha \in A \), where \( A \) is a Borel section of \( \Lambda^\perp \) in \( \mathbb{G} \).

(i) \( \Leftrightarrow \) (iii) Applying the fact that \( \Lambda \) is co-compact and \( \Lambda^\perp \cap \Gamma = \Lambda^\perp \), it is sufficient to take appropriate Haar measures on \( \Gamma, \mathbb{T} \), and put a measure on \( \mathbb{K} \) isometric to \( \mu_{\Gamma/A^\perp} \) in the sense of (2.3). Then the desired result is obtained by Theorem 4.6 in [15]. \( \square \)

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Department of Pure Mathematics, Azarbaijan Shahid Madani University, P.O.Box 53714-161, Tabriz, Iran.
Email address: zohre.hamidi@azaruniv.ac.ir

Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran.
Email address: fahimeh.arabyani@gmail.com

Department of Mathematics, Faculty of Math, Ferdowsi University of Mashhad and Center of Excellence in Analysis on Algebraic Structures (CEAAS), P.O.Box 1159-91775, Mashhad, Iran.
Email address: kamyabi@um.ac.ir

Department of Pure Mathematics, Azarbaijan Shahid Madani University, P.O.Box 53714-161, Tabriz, Iran.
Email address: sattari@azaruniv.edu