Visualizing quantum phase transitions in the XXZ model via the quantum steering ellipsoid

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The past two decades have witnessed a surge of interest in borrowing tools from quantum information theory to investigate quantum phase transitions (QPTs). The best known examples are entanglement measures whose nonanalyticities at critical points were tied to QPTs in a plethora of physical models. Here, focusing on the XXZ model, we show how QPTs can be revealed through the quantum steering ellipsoid (QSE), which is a geometric tool capable of characterizing both the strength and type of quantum correlations between two subsystems of a compound system. We find that the QSE associated with the XXZ model changes in shape with the QPTs, that is, it is a needle in the ferromagnetic phase, an oblate spheroid in the gapless phase, and a prolate spheroid in the antiferromagnetic phase. This finding offers an example demonstrating the intriguing possibility of unveiling QPTs in a geometrically visible fashion. Some connections between our results and previous ones are discussed.

I. INTRODUCTION

Understanding quantum phase transitions (QPTs) lies at the heart of quantum many-body physics [1]. Traditionally, QPTs are dealt with the Landau-Ginzburg theory, which is built upon the notions of symmetry breaking, order parameter, and correlation length [1]. Despite enormous success in relatively simple systems, the theory fails to capture QPTs exhibited in some complex systems. This can be either due to the difficulty of identifying proper order parameters for systems whose symmetry breaking patterns are unknown or because of the very absence of local order parameters, e.g., for QPTs involving topological order [2–4]. Currently, alternative approaches to characterizations of QPTs are under continuous development.

A line of such developments is to explore tools borrowed from quantum information theory to characterize QPTs, which has received a surge of interest over the past two decades [5]. The starting point of this line is the finding that critical phenomena occurred in complex many-body systems have profound relations with bipartite entanglement of two subsystems [6]. This has motivated a series of studies that make use of entanglement measures like the entanglement of formation [7] and the entropy of entanglement [8] to quantitatively study QPTs for a plethora of physical models [9–31]. Meanwhile, further developments [32–42] have been carried out that establish the links between QPTs and various measures of other types of quantum correlations, such as quantum discord [43–50], Bell nonlocality [51, 52], and steered quantum coherence [53, 54].

These measure-based proposals have proven to be useful in detecting QPTs for a large number of systems. However, it has been found that each measure has its limitations for different types of QPTs. For example, the concurrence was found to be unable to faithfully identify the second-order QPTs in some spin-1/2 models [55, 56], the Kosterlitz-Thouless QPT (KT-QPT) was shown to be undetectable by a number of measures [57–59], and the nonanalytic behavior of quantum discord was argued to be not completely reliable for detecting the critical points in some many-body systems [60]. Therefore, it deserves further effort to find more effective tools for characterizing QPTs.

In this work, we advocate the use of quantum steering ellipsoid (QSE) [61] for characterizing QPTs. Akin to the Bloch sphere which is a geometrical representation of the state space of a qubit, QSE is a faithful representation of an arbitrarily given state of two qubits in the three-dimensional (3D) Euclidean space, up to some local unitary transformations. The key advantage offered by this representation is that both the strength and type of quantum correlations are made manifest in terms of simple geometric entities like the shape and volume of QSE. Note that a measure of quantum correlations is only able to capture the strength of quantum correlations of a specific type, and different types of quantum correlations are described by different kinds of measures. We therefore anticipate that QSE is more effective in characterizing QPTs compared with a measure of quantum correlations adopted previously.

We examine the above idea by inspecting the QSE associated with the XXZ model and find that it changes in shape as well as in volume with the QPTs. As detailed below, in the course of varying the anisotropy parameter of the XXZ model, the shape of the QSE changes successively from a needle to an oblate spheroid and to a prolate spheroid. We show that these three types of shapes correspond to the ferromagnetic phase, the gapless phase, and the antiferromagnetic phase, respectively. This demonstrates that the QSE is capable of revealing QPTs in a geometrically visible fashion, which is, of course, not achievable in the previous measure-based proposals. Moreover, we find that the QSE happens to be

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a sphere at the KT phase transition point [62], indicating that the QSE is able to signal the KT-QPT, which cannot otherwise be detected by a number of measures as shown in Refs. [57–59].

It should be mentioned that there are other proposals of making use of geometric entities to study QPTs, such as the ones based on the fidelity [63–68], the ones based on the Berry phase [69, 70], the ones based on the metric tensor [71, 72], and the one based on the quantum geometric tensor [36] (see the review paper [73] for more information). Generally speaking, all of these proposals are unable to provide a geometrically visible characterization of QPTs as did in our proposal, for the obvious reason that the geometric entities adopted there are defined in a mathematically abstract way. Moreover, although having succeeded in signaling QPTs for various systems, these proposals do not provide much insight into the nature of quantum correlations involved in QPTs. This is contrary to our proposal as well as the aforementioned measure-based proposals.

To be self-contained, we organize the rest of this paper as follows. In Sec. II, we give a brief introduction to QSE. In Sec. III, we describe the XXZ model. In Sec. IV, we present our main results. In Sec. V, we discuss the relations. In Sec. VI, we conclude this work with some necessary remarks on the Berry phase [69, 70], the ones based on the fidelity [63–68], the ones based as the ones based on the geometric entities to study QPTs, such as the ones based on the fidelity [63–68], the ones based on the Berry phase [69, 70], the ones based on the metric tensor [71, 72], and the one based on the quantum geometric tensor [36] (see the review paper [73] for more information). Generally speaking, all of these proposals are unable to provide a geometrically visible characterization of QPTs as did in our proposal, for the obvious reason that the geometric entities adopted there are defined in a mathematically abstract way. Moreover, although having succeeded in signaling QPTs for various systems, these proposals do not provide much insight into the nature of quantum correlations involved in QPTs. This is contrary to our proposal as well as the aforementioned measure-based proposals.

Consider the bipartite situation that Alice and Bob share a two-qubit state $\rho_{AB}$. In general, $\rho_{AB}$ can be expressed as

$$\rho_{AB} = \frac{1}{4} \sum_{\mu,\nu=0}^{3} \Theta_{\mu\nu} \sigma^\mu_A \otimes \sigma^\nu_B,$$

(1)

where $\sigma^\mu_A$ ($\sigma^\nu_B$), $\mu = 0, 1, 2, 3$, stands for the Pauli matrix associated with the Alice’s (Bob’s) qubit, and $\Theta$ denotes a $4 \times 4$ real matrix with elements $\Theta_{\mu\nu} = \text{tr}(\rho_{AB} \sigma^\mu_A \otimes \sigma^\nu_B)$. Note that $\Theta$ can be written in the form of a block matrix as follows:

$$\Theta = \begin{pmatrix} 1 & b^T \\ a & T \end{pmatrix}.$$

(2)

Here, $a$ and $b$ are the Bloch vectors of the reduced density matrices $\rho_A$ and $\rho_B$, respectively. That is, $a = (a_1, a_2, a_3)^T$ with $a_\mu = \text{tr}(\rho_A \sigma^\mu_A)$, and similar expressions hold for $b$. $T$ is a $3 \times 3$ real matrix describing the correlations.

Apparently, performing local measurements on Bob’s side can change or steer the state of the qubit on Alice’s side. The state that Alice’s qubit can be steered to, referred to as the steered state hereafter, is generally different if the measurement performed is different. Upon taking into account all possible local measurements on Bob’s side, Alice can get all the possible steered states on her side. It has been shown that the set of all the possible steered states can be geometrically represented as an ellipsoid in the 3D Euclidean space, i.e., the so-called QSE for Alice [61],

$$\mathcal{E}_A = \left\{ \frac{a + T x}{1 + b \cdot x} | x \leq 1 \right\},$$

(3)

where $x$ is a 3D vector with $x$ denoting its length. Elements of $\mathcal{E}_A$ are simply the Bloch vectors of all the possible steered states. So, the QSE is an ellipsoid inside the Bloch sphere.

The QSE is completely determined by the so-called ellipsoid center and ellipsoid matrix. The ellipsoid center is described by

$$c_A = \frac{a - T b}{1 - b^2},$$

(4)

i.e., the vector pointing from the origin to the ellipsoid center, with $b$ denoting the length of $b$. The ellipsoid matrix reads

$$Q_A = \frac{1}{1 - b^2} \left( T - ab^T \right) \left( I + \frac{bb^T}{1 - b^2} \right) \left( T^T - ba^T \right),$$

(5)

whose eigenvalues $q_i$ determine the lengths $s_i$ of the three semiaxes $s_i$ of the ellipsoid as $s_i = \sqrt{q_i}$. All of the mentioned geometric entities like $c_A$ and $s_i$ are schematically shown in Fig. 1. Note that Eqs. (3), (4), and (5) are only well-defined if $b \neq 1$. If $b = 1$, $\rho_{AB}$ is a product state and $\mathcal{E}_A$ is comprised of a single point determined by the Bloch vector of $\rho_A$.

The strength of quantum correlations may be described by the ellipsoid volume [74]. Roughly speaking, the larger the ellipsoid volume is, the more quantum correlations $\rho_{AB}$ exhibits. In particular, if $\rho_{AB}$ is a product...
state, the QSE is a point and its volume is zero, and if \( \rho_{AB} \) is one of the Bell states, the QSE is exactly the Bloch sphere and achieves its maximal volume over all possible two-qubit states. It has been shown that the volume can be expressed as \([61]\)

\[
V_A = \frac{64\pi}{3} \left| \frac{\det \rho_{AB} - \det \rho_{T_B}}{(1 - b^2)^2} \right|,
\]

where \( T_B \) stands for the partial transpose with respect to Bob’s qubit. This expression indicates that the ellipsoid volume \( V_A \) is closely related to the strength of quantum correlations in view of the well-known Peres-Horodecki criterion \([75, 76]\).

The relations between the QSE and different types of quantum correlations are rich and are still under continuous development \([77–86]\). Here, we shall only mention a few fairly interesting results \([61]\), which link different types of quantum correlations to different shapes as well as volumes of the QSE. It has been shown \([61]\) that \( \rho_{AB} \) has zero discard for \( A \) if and only if \( \mathcal{E}_A \) degenerates to a radial line segment, and \( \rho_{AB} \) has zero discard for \( B \) if and only if \( \mathcal{E}_A \) is 1D and \( b = 2|c_A - a|/l_A \), where \( l_A \) is the length of \( \mathcal{E}_A \). Moreover, \( \rho_{AB} \) is separable if and only if \( \mathcal{E}_A \) fits inside a tetrahedron that fits inside the Bloch sphere. On the other hand, any two-qubit state \( \rho_{AB} \) with \( V_A > 4\pi/81 \) must be entangled, but the converse may not be true, that is, an entangled state may have \( V_A \leq 4\pi/81 \).

Let us end the introduction to QSE by remarking that the QSE \( \mathcal{E}_A \) for Alice is not identical to the QSE \( \mathcal{E}_B \) for Bob in general. To obtain \( \mathcal{E}_B \), we need only to make the substitutions \( a \to b, b \to a \), and \( T \to T^T \) in Eqs. (4) and (5) \([61]\).

### III. SPIN-1/2 XXZ MODEL

The XXZ model is an one-dimensional spin-1/2 chain in which the spins interact with each other through anisotropic Heisenberg interactions. Its Hamiltonian reads

\[
H_{\text{XXZ}} = \sum_{j=1}^{N} S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta S^z_j S^z_{j+1},
\]

where \( S^{\kappa}_j = \sigma^{\kappa}_j / 2 \) (\( \kappa = x, y, z \)), \( \sigma^{\kappa}_j \) are the Pauli spin-1/2 operators on site \( j \), and \( \Delta \) is the anisotropy parameter. Here, we have assumed the periodic boundary condition that \( \sigma^{\kappa}_{j+N} = \sigma^{\kappa}_j \). It is well-known that the model has three phases:

- **\( \Delta > 1 \)**: The system is in the antiferromagnetic phase.

QPTs occur at the boundaries between different phases. Specifically, a first-order QPT occurs at the critical point \( \Delta = -1 \), and the KT-QPT occurs at the critical point \( \Delta = 1 \).

\( H_{\text{XXZ}} \) cannot be diagonalized in general, but its energy spectrum can be obtained by employing Bethe ansatz. The Bethe ansatz solution gives the ground state energy \( e_g \) as \([87, 88]\)

\[
e_g(\Delta) = \begin{cases} \frac{1}{4} & \Delta \leq -1, \\ \frac{1}{4} + \sin \frac{\pi \xi}{2} \int_{-\infty}^{+\infty} dx \frac{1}{\sinh x \sinh \xi x}, & -1 < \Delta < 1, \\ \frac{1}{4} - \ln 2, & \Delta = 1, \end{cases}
\]

where \( \Delta = \cos \pi \xi \). For \( \Delta > 1 \), \( e_g(\Delta) \) is obtained by changing \( \xi \) in Eq. (8). On the other hand, it is worth noting that \( H_{\text{XXZ}} \) respects some symmetries. Two symmetries of particular interest are the discrete parity \( \mathbb{Z}_2 \) symmetry over the plane \( xy \), which transforms \( \sigma^z_i \) into \( -\sigma^z_i \) while keeps \( \sigma^x_i \) and \( \sigma^y_i \) unchanged, i.e.,

\[
\sigma^x_i \to \sigma^x_i, \quad \sigma^y_i \to \sigma^y_i, \quad \sigma^z_i \to -\sigma^z_i,
\]

and the continuous \( U(1) \) symmetry that rotates the spins in the \( xy \) plane by any angle \( \theta \), which transforms \( \sigma^\kappa_i \) as

\[
\sigma^\kappa_i \to \cos \theta \sigma^\kappa_i + \sin \theta \sigma^\kappa_i, \quad \sigma^\kappa_i \to -\sin \theta \sigma^\kappa_i + \cos \theta \sigma^\kappa_i\]

Besides, \( H_{\text{XXZ}} \) respects some other symmetries like the translational symmetry.

Analysis of the correlation functions \( \langle \sigma^{\kappa}_i \sigma^{\kappa}_j \rangle \) for sites \( i \) and \( j \) can be simplified by taking the symmetries into account. It is easy to see that the \( \mathbb{Z}_2 \) symmetry implies that

\[
\langle \sigma^z_i \rangle = 0, \quad \langle \sigma^z_i \sigma^z_j \rangle = 0, \quad \langle \sigma^y_i \sigma^y_j \rangle = 0, \quad \langle \sigma^x_i \sigma^x_j \rangle = 0,
\]

where the state over which the expectation values are taken is assumed to be the ground state of the XXZ model. Here, we have assumed that there is no spontaneous symmetry breaking for the ground state under consideration. This assumption can be made justified for very small systems where the spontaneous symmetry breaking does not happen or by assuming that the ground state considered here is a superposition of the two degenerated ground states with opposite magnetization for the ferro and antiferromagnetic phases. The \( U(1) \) symmetry implies that

\[
\langle \sigma^x_i \rangle = 0, \quad \langle \sigma^y_i \sigma^z_j \rangle = \langle \sigma^z_i \sigma^y_j \rangle, \quad \langle \sigma^x_i \sigma^y_j \rangle = 0.
\]

Taking into account all the symmetry constraints described by Eqs. (13) and (14), we obtain that the reduced
density matrix $\rho_{i,j}$ associated with sites $i$ and $j$ can be written as

$$\rho_{i,j} = \begin{pmatrix} u & 0 & 0 \\ 0 & w & y \\ 0 & y & w \end{pmatrix},$$

(15)

where $u = (1 + \langle \sigma^z_i \sigma^z_j \rangle)/4$, $w = (1 - \langle \sigma^z_i \sigma^z_j \rangle)/4$, and $y = (\langle \sigma^x_i \sigma^x_j \rangle + \langle \sigma^y_i \sigma^y_j \rangle)/4$.

### IV. RESULTS

We show how the QPTs in the XXZ model can be revealed via QSE. Our basic idea is to signal the QPTs through the changes of the shape and volume of the QSE associated with two nearest-neighbor spins of the XXZ model. The reason why our idea shall work is as follows. It is well-known that QPTs are usually associated with some critical changes in the ground state energy of a many-body system due to quantum fluctuations. Moreover, it has been shown [13] that for a Hamiltonian involving only nearest-neighbor interactions, the ground state energy can be obtained from the reduced density matrix of two nearest-neighbor constituents, say, $i$ and $i+1$; that is, the ground state energy is a function of the reduced density matrix $\rho_{i,i+1}$. So, the critical changes in the ground state energy generally lead to some critical changes in $\rho_{i,i+1}$, which shall lead to some critical changes in the shape and volume of the QSE associated with $\rho_{i,i+1}$ in general, since the QSE is a faithful representation of $\rho_{i,i+1}$ [61].

Now, following the above idea, we need to figure out the QSE associated with $\rho_{i,i+1}$. From Eq. (15), it follows that

$$\mathbf{a} = (0, 0, 0)^T, \quad \mathbf{b} = (0, 0, 0)^T,$$

(16)

and

$$T = \begin{pmatrix} \langle \sigma^x_i \sigma^x_{i+1} \rangle & 0 & 0 \\ 0 & \langle \sigma^y_i \sigma^y_{i+1} \rangle & 0 \\ 0 & 0 & \langle \sigma^z_i \sigma^z_{i+1} \rangle \end{pmatrix}.$$  

(17)

Apparently, $\mathbf{a} = \mathbf{b}$ and $T = T^T$, implying that the QSE for the $i$-th spin happens to be identical to that for the $(i+1)$-th spin. This simplifies our following analysis. Substituting Eqs. (16) and (17) into Eqs. (4) and (5), we have

$$e_A = (0, 0, 0)^T,$$

(18)

and

$$Q_A = \begin{pmatrix} \langle \sigma^x_i \sigma^x_{i+1} \rangle^2 & 0 & 0 \\ 0 & \langle \sigma^y_i \sigma^y_{i+1} \rangle^2 & 0 \\ 0 & 0 & \langle \sigma^z_i \sigma^z_{i+1} \rangle^2 \end{pmatrix}.$$  

(19)

Here and henceforth, for the sake of notational consistency, we still adopt the notations like $e_A$, $Q_A$, and $V_A$ for the QSE for the $i$-th spin [or, equivalently, for the $(i+1)$-th spin]. We see from Eqs. (18) and (19) that the QSE is obtained from a unit sphere centered at the origin $(0, 0, 0)^T$ with its three semiaxes shrunk by the amounts of $\langle \sigma^x_i \sigma^x_{i+1} \rangle$, $\langle \sigma^y_i \sigma^y_{i+1} \rangle$, and $\langle \sigma^z_i \sigma^z_{i+1} \rangle$, respectively. Note that it has been found that the correlation functions can be expressed as [89]

$$\langle \sigma^x_i \sigma^x_{i+1} \rangle = \langle \sigma^y_i \sigma^y_{i+1} \rangle = \frac{1}{2} \left[ 4e_g(\Delta) - \Delta \langle \sigma^z_i \sigma^z_{i+1} \rangle \right],$$

(20)

$$\langle \sigma^z_i \sigma^z_{i+1} \rangle = 4 \frac{\partial e_g(\Delta)}{\partial \Delta},$$

(21)

with $e_g(\Delta)$ given by Eq. (8). With the aid of Eqs. (8), (20), and (21), we are able to work out any geometric property of the QSE, which is completely determined by Eqs. (18) and (19).

Figure 2 shows the lengths of the three semiaxes $s_i$ of the QSE with $\rho_{i,i+1}$ as functions of $\Delta$. As can be seen from this figure, $s_1$ is equal to $s_2$ for all the values of $\Delta$. This follows immediately from the equalities connecting $s_i$’s to the correlation functions

$$s_1 = |\langle \sigma^x_i \sigma^x_{i+1} \rangle|, \quad s_2 = |\langle \sigma^y_i \sigma^y_{i+1} \rangle|, \quad s_3 = |\langle \sigma^z_i \sigma^z_{i+1} \rangle|,$$

(22)

and Eq. (20). We observe from Fig. 2 that $s_i$’s abruptly change their values at $\Delta = -1$, which is the first-order QPT point. Another interesting observation is that the lengths of all the three semiaxes are equal at $\Delta = 1$, which is the KT-QPT point. These two points $\Delta = \pm 1$ divide the whole range of the values of $\Delta$ into three regions: (i) $\Delta < -1$, for which $s_1 = s_2 = 0$ and $s_3 = 1$; (ii) $-1 < \Delta < 1$, for which $s_1 = s_2 > s_3$; and (iii) $\Delta > 1$, for which $s_1 = s_2 < s_3$. It is worth noting that these three regions correspond to the three phases listed in the previous section.

Let us inspect the shape of the QSE in these three regions. In the first region which corresponds to the ferromagnetic phase, the QSE is a needle since $s_1 = s_2 = 0$.
Ferromagnetic phase  Antiferromagnetic phase  Gapless phase
First-order QPT  KT-QPT

FIG. 3. Illustrations of different shapes of the quantum steering ellipsoid associated with $\rho_{i,i+1}$ for different phases of the XXZ model. The whole range of the values of $\Delta$ is divided into three regions which correspond to the ferromagnetic phase, the gapless phase, and the antiferromagnetic phase, respectively. The shape of the quantum steering ellipsoid changes with the quantum phase transitions. As indicated in the figure, it is a needle in the ferromagnetic phase, an oblate spheroid in the gapless phase, and a prolate spheroid in the antiferromagnetic phase. Moreover, the quantum steering ellipsoid changes abruptly in shape at the first-order quantum phase transition point $\Delta = -1$ and becomes a sphere at the Kosterlitz-Thouless quantum phase transition point $\Delta = 1$. This establishes a connection between shapes of the quantum steering ellipsoid and different phases of the model, thereby demonstrating the intriguing possibility that quantum phase transitions can be revealed in a geometrically visible fashion.

but $s_3 > 0$. This is pictorially shown in Fig. 3 (see the first Bloch sphere there which contains a green vertical line). The QSE changes in shape abruptly at the first-order QPT point $\Delta = -1$. When entering into the second region which corresponds to the gapless phase, the QSE becomes an oblate spheroid (see the second Bloch sphere in Fig. 3 containing an oblate spheroid colored in green). As the value of $\Delta$ increases from $-1$ to $1$, we observe from Fig. 2 that the difference between the length of $s_1$ ($s_2$) and that of $s_3$ is increasingly small and eventually becomes zero at $\Delta = 1$. This indicates that the QSE is increasingly closer in shape to a sphere and eventually becomes one at the KT-QPT point (see the third Bloch sphere in Fig. 3 containing a sphere colored in green). Keeping on increasing $\Delta$, we finally reach the third region which corresponds to the antiferromagnetic phase. Here, the QSE is a prolate spheroid as $s_1 = s_2 < s_3$ (see the fourth Bloch sphere in Fig. 3 containing a spheroid colored in green). Now, it is clear that the QSE changes in shape with the QPTs. This demonstrates the intriguing possibility that QPTs can be revealed in a geometrically visible fashion.

To gain further insight into the connection between the QSE and the QPTs, we numerically figure out the volume of the QSE in Fig. 4. We observe that the volume $V_A$ of the quantum steering ellipsoid associated with $\rho_{i,i+1}$ as a function of the anisotropy parameter $\Delta$. $V_A$ is zero in the first region $\Delta < -1$. It changes from zero to be larger than zero at the first-order quantum phase transition point $\Delta = -1$, indicating the appearance of quantum correlations. Keeping on increasing $\Delta$, we see that $V_A$ first increases rapidly in the second region $-1 < \Delta < 1$ and then decreases slowly in the third region $\Delta > 1$. Its maximum is attained at the Kosterlitz-Thouless quantum phase transition point $\Delta = 1$. The changes in the value of $V_A$ reflects the changes in the strength of quantum correlations as detailed in the text.

To gain further insight into the connection...
as $\Delta$ increases. So, the strength of quantum correlations becomes increasingly small in the course of increasing $\Delta$ in this region.

V. DISCUSSIONS

So far, we have presented our main finding, which can be thought of as a geometrically visible picture that depicts the connection between the QPTs in the XXZ model and the shapes as well as volumes of the QSE. Note that the QPTs in the XXZ model have been alternatively studied with the aid of a number of measures of quantum correlations in the previous works. In the following, we show that the connection presented here sheds some new lights on these measure-based works. We would like to take the concurrence [56], quantum discord [43, 45], Bell inequality [52] and difference of bond strength [90] as four examples to illustrate the above point.

**Example 1: the concurrence [56]**. The concurrence is an entanglement measure defined as [7]

$$C(\rho_{i,i+1}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (23)$$

where $\lambda_i$’s are the eigenvalues of $\rho_{i,i+1}\rho_{i,i+1}^\dagger$ arranged in descending order, with $\rho_{i,i+1} = \sigma^y \otimes \sigma^y \rho_{i,i+1}^\dagger \sigma^y \otimes \sigma^y$ being the time-reversed density matrix. It has been found that the concurrence $C(\rho_{i,i+1})$ displays some nonanalytical behaviors at $\Delta = -1$ and therefore can be used to signal the first-order QPT of the XXZ model [56]. Using Eq. (22) and the results in Refs. [10, 16, 56], we express $C(\rho_{i,i+1})$ in terms of the correlation functions, we have that

$$C(\rho_{i,i+1}) = \begin{cases} 0, & \Delta \leq -1, \\ \frac{s_1 + s_2 + s_3 - 1}{2}, & \Delta > -1. \end{cases} \quad (24)$$

Equation (24) establishes a connection between the concurrence and the lengths $s_i$ of the three semiaxes of the QSE. Note that $s_i$ abruptly changes its values at $\Delta = -1$, which can be seen from Fig. 2. This means that the derivative of $s_i$ with respect to $\Delta$ is divergent at $\Delta = -1$, which immediately leads to the finding that $C(\rho_{i,i+1})$ is nonanalytical at $\Delta = -1$.

**Example 2: quantum discord [43, 45]**. Quantum discord $Q$ is a measure of nonclassical correlations between two subsystems of a quantum system [91]. Explicitly, it can be expressed as

$$Q(\rho_{i,i+1}) = \min_M \sum_k p_k S(\rho_k^i) + S(\rho_{i,i+1}) - S(\rho_{i,i+1}), \quad (25)$$

Here, $S(\cdot)$ denotes the von Neumann entropy. $M = \{M_k\}$ is a POVM measurement performed on the $(i+1)$-th qubit, with $M_k \geq 0$ and $\sum_k M_k = I$, $p_k$ denotes the probability of getting outcome $k$, and $\rho_k^i$ represents the post-measurement state of the $i$-th qubit. It has been found [43, 45] that $Q(\rho_{i,i+1})$ is nonanalytical at $\Delta = 1$ and therefore can be used to signal the KT-QPT. Note that both $S(\rho_{i+1})$ and $S(\rho_{i+1})$ are smooth functions at $\Delta = 1$. So, nonanalyticities of $Q(\rho_{i,i+1})$ stem from the term $\min_M \sum_k p_k S(\rho_k^i)$. Using the explicit expression of $\min_M \sum_k p_k S(\rho_k^i)$ for a Bell diagonal state [92], we obtain that

$$\min_M \sum_k p_k S(\rho_k^i) = \min \{h(s_1), h(s_2), h(s_3)\}, \quad (26)$$

with

$$h(g) = -\frac{1 + g}{2} \log_2 \frac{1 + g}{2} - \frac{1 - g}{2} \log_2 \frac{1 - g}{2}, \quad (27)$$

where $g \in [0, 1]$. Noting that $h(g)$ is a monotonically decreasing function of $g$, we have that the term $\min_M \sum_k p_k S(\rho_k^i)$ is determined by the longest axis of the QSE. As can be seen from Fig. 2, the longest axis is $s_1$ (or $s_2$) on the left side of $\Delta = 1$ but is $s_3$ on the right side of $\Delta = 1$. It follows immediately that the term $\min_M \sum_k p_k S(\rho_k^i)$ is nonanalytical at $\Delta = 1$, which further leads to the finding that $Q(\rho_{i,i+1})$ is also nonanalytical at $\Delta = 1$.

**Example 3: Bell inequality [52]**. A measure of the Bell nonlocality can be defined as the maximal violation of the CHSH inequality,

$$B(\rho_{i,i+1}) = \max_{m_i, m_j} |\langle B_{CHSH}\rangle|, \quad (28)$$

where $\langle B_{CHSH}\rangle$ is defined as $\langle A_1 \otimes B_1 \rangle + \langle A_2 \otimes B_2 \rangle - \langle A_1 \otimes B_2 \rangle - \langle A_2 \otimes B_1 \rangle$, with $A_i = m_i \cdot \sigma$ and $B_j = n_j \cdot \sigma$. It has been found [52] that the measure $B(\rho_{i,i+1})$ displays some nonanalytical behaviors at $\Delta = \pm 1$ and therefore can be used to signal both the first-order QPT and the KT-QPT. Using Eq. (22) and the results in Ref. [52], which express $B(\rho_{i,i+1})$ in terms of the correlation functions, we have that

$$B(\rho_{i,i+1}) = 2 \max \left\{ \sqrt{s_1^2 + s_2^2}, \sqrt{s_1^2 + s_3^2}, \sqrt{s_2^2 + s_3^2} \right\}. \quad (29)$$

Apparently, $B(\rho_{i,i+1})$ is determined by the longest and second longest axes of the QSE. As can be seen from Fig. 2, these two axes change from $\{s_1, s_3\}$ (or equivalently $\{s_2, s_3\}$) when $\Delta < -1$ to $\{s_1, s_2\}$ when $-1 < \Delta < 1$ and back to $\{s_1, s_3\}$ (or equivalently $\{s_2, s_3\}$) when $\Delta > 1$. That is, one of these two axes changes at $\Delta = \pm 1$. This leads to the finding that $B(\rho_{i,i+1})$ is nonanalytical at $\Delta = \pm 1$.

**Example 4: the difference of bond strength [90]**. The difference of bond strength was introduced to signal first-order QPTs for many-body systems with Hamiltonians of the form $H(\lambda) = H_0 + \lambda H_1$, where $\lambda$ denotes a driving parameter. Apparently, due to the linear dependence of $H(\lambda)$ on $\lambda$, the ground state energy varies linearly with $\lambda$ and can be expressed as $e_\lambda = e_0 + \lambda e_1$. Associated with a first-order QPT, a jump of $e_1$ will appear at the phase transition point $\lambda_t$. This reflects the structural
change of the ground state. It is also expected that a similar jump of $e_0$ appears at $\lambda_t$ and may eliminate the singularity of $e_0(\lambda)$ caused by the jump of $e_1$. To avoid the elimination and therefore detect $\lambda_t$, the difference of bond strength is defined as

$$D = e_0 - \text{sgn}(\lambda_t)e_1,$$

(30)

where the minus sign reflects the spirit of the bond reversal method. Using Eq. (22) and the results in Ref. [90], which express $D$ in terms of the correlation functions, we have that around the first-order phase transition point $\Delta = -1$, $D$ can be expressed as

$$D = \begin{cases} -\frac{1}{4}(s_1 + s_2 - s_3), & \text{if } \Delta < -1; \\ -\frac{1}{4}(s_1 + s_2 + s_3), & \text{if } -1 \leq \Delta < 0. \end{cases}$$

(31)

Equation (31) establishes a connection between the difference of bond strength and the lengths $s_i$ of the QSE. Note that $s_i$ abruptly changes its values at $\Delta = -1$, which can be seen from Fig. 2. This immediately leads to the finding that the difference of bond strength $D$ exhibits a jump at $\Delta = -1$.

VI. CONCLUDING REMARKS

We have shown how the QPTs in the XXZ model can be revealed through the associated QSE. The basic idea is that, since QPTs are tied to critical changes of quantum correlations in general, they shall manifest themselves in terms of geometric entities of QSE. Assisted by the relations between quantum correlations and QSE found earlier [61], we have developed this idea into an explicit connection between different phases in the XXZ model and different types of shapes of the associated QSE as shown in Fig. 3.

According to the connection established here, a QPT in the model occurs whenever the associated QSE changes its shape. Specifically, the QSE changes from a needle to an oblate and to a prolate spheroid when $\Delta$ passes through the first-order QPT point and the KT-QPT point successively. This demonstrates that QPTs can be revealed in a geometrically visible fashion, which, to our knowledge, has never been achieved before. Aided by the established connection, we have shed some lights on a number of previous works, showing that the nonanalyticities of the measures considered there are rooted in the semiaxes of the QSE.

While the present work focuses on the specific XXZ model, we anticipate that the idea of exploring QSE to signal QPTs can find its applications in a more general physical context [93]. This is because QSE is a faithful representation of quantum correlations and therefore any critical change in quantum correlations shall lead to some critical change in QSE. A general framework for connecting QPTs to QSE is beyond the scope of this work but is highly desirable for the following reason. Apart from the advantage of demanding no a priori knowledge of order parameters, the framework is expected to be universally effective in detecting QPTs, since QSE is capable of characterizing both the strength and type of quantum correlations.

We finally remark on the experimental detection of QPTs with QSE. It is well-known that QPTs are assumed to occur at the absolute zero temperature, which is unattainable experimentally because of the third law of thermodynamics. Hence, in practice, one must work at a small temperature as close as possible to the absolute zero temperature, in order to detect QPTs [41, 46, 47, 57]. However, at a small but nonzero temperature, there may exist a gap between theory and experiment so that a well-established theory for detecting QPTs may not work in experiments. Fortunately, the validation of QSE has been verified experimentally in Ref. [86]. This indicates that the connection between QPTs and QSE established here for the XXZ model as well as the general connection to be explored in a future study is likely to hold in experiments.

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[1] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge New York, 1999).
[2] X-G. Wen and Y-S. Wu, “Transitions between the quantum hall states and insulators induced by periodic potentials,” Phys. Rev. Lett. 70, 1501 (1993).
[3] N. Read and D. Green, “Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect,” Phys. Rev. B 61, 10267 (2000).
[4] A. Kitaev, “Anyons in an exactly solved model and beyond,” Ann. Phys. 321, 2 (2006).
[5] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, “Entanglement in many-body systems,” Rev. Mod. Phys. 80, 517 (2008).
[6] A. Osterloh, L. Amico, G. Falci, and R. Fazio, “Scaling of entanglement close to a quantum phase transition,” Nature (London) 416, 608 (2002).
[7] W. K. Wootters, “Entanglement of formation of an arbitrary state of two qubits,” Phys. Rev. Lett. 80, 2245 (1998).
[8] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, “Concentrating partial entanglement by local operations,” Phys. Rev. A 53, 2046 (1996).
[9] T. J. Osborne and M. A. Nielsen, “Entanglement in a sim-
ple quantum phase transition,” Phys. Rev. A 66, 032110 (2002).

[10] S.-J. Gu, H.-Q. Lin, and Y.-Q. Li, “Entanglement, quantum phase transition, and scaling in the XXZ chain,” Phys. Rev. A 68, 042330 (2003).

[11] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, “Entanglement in quantum critical phenomena,” Phys. Rev. Lett. 90, 227902 (2003).

[12] G. Refael and J. E. Moore, “Entanglement entropy of random quantum critical points in one dimension,” Phys. Rev. Lett. 93, 260602 (2004).

[13] L.-A. Wu, M. S. Sarandy, and D. A. Lidar, “Quantum phase transitions and bipartite entanglement,” Phys. Rev. Lett. 93, 250404 (2004).

[14] N. Lambert, C. Emary, and T. Brandes, “Entanglement and the phase transition in single-mode superradiance,” Phys. Rev. Lett. 92, 073602 (2004).

[15] S.-J. Gu, S.-S. Deng, Y.-Q. Li, and H.-Q. Lin, “Entanglement and quantum phase transition in the extended Hubbard model,” Phys. Rev. Lett. 93, 086402 (2004).

[16] S.-J. Gu, G.-S. Tian, and H.-Q. Lin, “Ground-state entanglement in the XXZ model,” Phys. Rev. A 71, 052322 (2005).

[17] T.-C. Wei, D. Das, S. Mukhopadhyay, S. Vishveshwara, and P. M. Goldbart, “Global entanglement and quantum criticality in spin chains,” Phys. Rev. A 71, 060305(R) (2005).

[18] A. Afkhami, P. Giorda, A. Montorsi, and F. Traversa, “Two-point versus multipartite entanglement in quantum phase transitions,” Phys. Rev. Lett. 95, 056402 (2005).

[19] L.-A. Wu, M. S. Sarandy, D. A. Lidar, and L. J. Sham, “Linking entanglement and quantum phase transitions via density-functional theory,” Phys. Rev. A 74, 052335 (2006).

[20] L. C. Venuti, C. D. E. Boschi, and M. Roncaglia, “Long-distance entanglement in spin systems,” Phys. Rev. Lett. 96, 247206 (2006).

[21] T. R. de Oliveira, G. Rigolin, M. C. de Oliveira, and E. Miranda, “Multipartite entanglement signature of quantum phase transitions,” Phys. Rev. Lett. 97, 170401 (2006).

[22] P. Buonsante and A. Vezzani, “Ground-state fidelity and bipartite entanglement in the Bose-Hubbard model,” Phys. Rev. Lett. 98, 110601 (2007).

[23] R. Orús, “Universal geometric entanglement close to quantum phase transitions,” Phys. Rev. Lett. 100, 130502 (2008).

[24] P. Facchi, U. Marzolino, G. Parisi, S. Pascazio, and A. Scardicchio, “Phase transitions of bipartite entanglement,” Phys. Rev. Lett. 101, 050502 (2008).

[25] W. Son, L. Amico, F. Plastina, and V. Vedral, “Quantum instability and edge entanglement in the quasi-long-range order,” Phys. Rev. A 79, 022302 (2009).

[26] C. C. Rulli and M. S. Sarandy, “Entanglement and local extremes at an infinite-order quantum phase transition,” Phys. Rev. A 81, 032334 (2010).

[27] G. De Chiara, L. Lepori, M. Lewenstein, and A. Sanpera, “Entanglement spectrum, critical exponents, and order parameters in quantum spin chains,” Phys. Rev. Lett. 109, 237208 (2012).

[28] M. Hofmann, A. Osterloh, and O. Gühne, “Scaling of genuine multiparticle entanglement close to a quantum phase transition,” Phys. Rev. B 89, 134101 (2014).

[29] S. Sähling, G. Remenyi, C. Paulsen, P. Monceau, V. Saligrama, C. Marin, A. Revcolevschi, L. P. Regnault, S. Raymond, and J. E. Lorenzo, “Experimental realization of long-distance entanglement between spins in antiferromagnetic quantum spin chains,” Nat. Phys. 11, 255 (2015).

[30] A. Bayat, “Scaling of tripartite entanglement at impurity quantum phase transitions,” Phys. Rev. Lett. 118, 036102 (2017).

[31] L. Pezzè, M. Gabbrielli, L. Lepori, and A. Smerzi, “Multipartite entanglement in topological quantum phases,” Phys. Rev. Lett. 119, 250401 (2017).

[32] M. M. Wolf, G. Ortiz, F. Verstraete, and J. I. Cirac, “Quantum phase transitions in matrix product systems,” Phys. Rev. Lett. 97, 110403 (2006).

[33] O. Legeza and J. Sólyom, “Two-site entropy and quantum phase transitions in low-dimensional models,” Phys. Rev. Lett. 96, 116401 (2006).

[34] W.-L. You, Y.-W. Li, and S.-J. Gu, “Fidelity, dynamic structure factor, and susceptibility in critical phenomena,” Phys. Rev. E 76, 022101 (2007).

[35] O. Legeza, J. Sólyom, L. Tincani, and R. M. Noack, “Entropic analysis of quantum phase transitions from uniform to spatially inhomogeneous phases,” Phys. Rev. Lett. 99, 087203 (2007).

[36] L. C. Venuti and P. Zanardi, “Quantum critical scaling of the geometric tensors,” Phys. Rev. Lett. 99, 095701 (2007).

[37] H.-Q. Zhou, R. Orús, and G. Vidal, “Ground state fidelity from tensor network representations,” Phys. Rev. Lett. 100, 080601 (2008).

[38] S. Rachel, N. Laflencie, H. F. Song, and K. L. Hur, “Detecting quantum critical points using bipartite fluctuations,” Phys. Rev. Lett. 108, 116401 (2012).

[39] J. Cui, M. Gu, L. C. Kwek, M. F. Santos, H. Fan, and V. Vedral, “Quantum phases with differing computational power,” Nat. Commun. 3, 812 (2012).

[40] Z.-Y. Sun, Y.-Y. Wu, J. Xu, H.-L. Huang, B.-F. Zhan, B. Wang, and C.-B. Duan, “Characterization of quantum phase transition in the XY model with multipartite correlations and Bell-type inequalities,” Phys. Rev. A 89, 022101 (2014).

[41] G. Karpat, B. Çakmak, and F. F. Fanchini, “Quantum coherence and uncertainty in the anisotropic XY chain,” Phys. Rev. B 90, 104431 (2014).

[42] C. Radhakrishnan, M. Parthasarathy, S. Jambulingam, and T. Byrnes, “Distribution of quantum coherence in multipartite systems,” Phys. Rev. Lett. 116, 150504 (2016).

[43] R. Dillenschneider, “Quantum discord and quantum phase transition in spin chains,” Phys. Rev. B 78, 224413 (2008).

[44] J. Maziero, H. C. Guzman, L. C. Celeri, M. S. Sarandy, and R. M. Serra, “Quantum and classical thermal correlations in the XY spin-1/2 chain,” Phys. Rev. A 82, 012106 (2010).

[45] M. S. Sarandy, “Classical correlation and quantum discord in critical systems,” Phys. Rev. A 80, 022108 (2009).

[46] T. Werlang, C. Trippe, G. A. P. Ribeiro, and G. Rigolin, “Quantum correlations in spin chains at finite temperatures and quantum phase transitions,” Phys. Rev. Lett. 105, 095702 (2010).

[47] T. Werlang, G. A. P. Ribeiro, and Gustavo Rigolin, “Spotlighting quantum critical points via quantum correlations at finite temperatures,” Phys. Rev. A 83, 062334
P. Zanardi, M. Cozzini, and P. Giorda, “Ground state overlap and criticality in the XXZ model,” Phys. Rev. A 92, 022120 (2015).

L. Pirollo, E. Vernier, P. Calabrese, and M. Rigol, “Correlations and diagonal entropy after quantum quenches in XXZ chains,” Phys. Rev. B 95, 054308 (2017).

D.-L. Deng, C. Wu, J.-L. Chen, S.-J. Gu, S. Yu, and C. H. Oh, “Bell nonlocality in conventional and topological quantum phase transitions,” Phys. Rev. A 86, 032305 (2012).

L. Justino and T. R. de Oliveira, “Bell inequalities and entanglement at quantum phase transitions in the XXZ model,” Phys. Rev. A 85, 052128 (2012).

M.-L. Hu, Y.-Y. Gao, and H. Fan, “Steered quantum coherence as a signature of quantum phase transitions in spin chains,” Phys. Rev. A 101, 032305 (2020).

Y.-X. Xie, “Steered quantum coherence and quantum criticality in the XY model with Dzyaloshinsky–Moriya interaction,” Phys. Status Solidi B 258, 2000322 (2020).

J. Vidal, G. Palacios, and R. Mosseri, “Entanglement in a second-order quantum phase transition,” Phys. Rev. A 69, 022107 (2004).

M.-F. Yang, “Reexamination of entanglement and the quantum phase transition,” Phys. Rev. A 71, 030302(R) (2005).

J.-J. Chen, J. Cui, Y.-R. Zhang, and H. Fan, “Coherence susceptibility as a probe of quantum phase transitions,” Phys. Rev. A 94, 022112 (2016).

S. Chen, L. Wang, Y. Hao, and Y. Wang, “Intrinsic relation between ground-state fidelity and the characterization of a quantum phase transition,” Phys. Rev. A 77, 032111 (2008).

B.-L. Ye, L.-Y. Xue, Y.-L. Fang, S. Liu, Q.-C. Wu, Y.-H. Zhou, and C.-P. Yang, “Quantum coherence and quantum Fisher information in the XXZ system,” Phys. E 115, 113690 (2020).

Z.-Y. Sun, L. Li, K.-L. Yao, G.-H. Du, J.-W. Liu, B. Luo, N. Li, and H.-N. Li, “Quantum discord in matrix product systems,” Phys. Rev. A 82, 032310 (2010).

S. Jevtic, M. Pusey, D. Jennings, and T. Rudolph, “Quantum steering ellipsoids,” Phys. Rev. Lett. 113, 020402 (2014).

The finding that the QSE is a sphere at the KT phase transition point is a consequence of the accidental SU(2) symmetry when $\Delta = 1$. It is therefore an interesting result specific to the XXZ model and not a general signature of KT-QPTs.

P. Zanardi and N. Paunkovic, “Ground state overlap and quantum phase transitions,” Phys. Rev. E 74, 031123 (2006).

P. Zanardi, M. Cozzini, and P. Giorda, “Ground state fidelity and quantum phase transitions in free fermi systems,” J. Stat. Mech. 2007, L02002 (2007).

M. Cozzini, P. Giorda, and P. Zanardi, “Quantum phase transitions and quantum fidelity in free fermion graphs,” Phys. Rev. B 75, 014439 (2007).

H.-Q. Zhou and J. P. Barjahtarevic, “Fidelity and quantum phase transitions,” J. Phys. A: Math. Theor. 41, 412001 (2008).

H.-Q. Zhou, J.-H. Zhao, and B. Li, “Fidelity approach to quantum phase transitions: finite-size scaling for the quantum Ising model in a transverse field,” J. Phys. A: Math. Theor. 41, 492002 (2008).

B. Wang, M. Feng, and Z.-Q. Chen, “Berezinskii–kosterlitz-thouless transition uncovered by the fidelity susceptibility in the XXZ model,” Phys. Rev. A 81, 064301 (2010).

A. C. M. Carollo and J. K. Pachos, “Geometric phases and criticality in spin-chain systems,” Phys. Rev. Lett. 95, 157203 (2005).

S.-L. Zhu, “Scaling of geometric phases close to the quantum phase transition in the XY spin chain,” Phys. Rev. Lett. 96, 077206 (2006).

P. Zanardi, P. Giorda, and M. Cozzini, “Information-theoretic differential geometry of quantum phase transitions,” Phys. Rev. Lett. 99, 100603 (2007).

D.-J. Zhang, Q.-h. Wang, and J. Gong, “Quantum geometric tensor in $P^7$-symmetric quantum mechanics,” Phys. Rev. A 99, 042104 (2019).

A. Carollo, D. Valenti, and B. Spagnolo, “Geometry of quantum phase transitions,” Phys. Rep. 838, 1 (2020).

R. McCloskey, A. Ferraro, and M. Paternostro, “Einstein-podolsky-rosen steering and quantum steering ellipsoids: Optimal two-qubit states and projective measurements,” Phys. Rev. A 95, 012320 (2017).

A. Peres, “Separability criterion for density matrices,” Phys. Rev. Lett. 77, 1413 (1996).

M. Horodecki, P. Horodecki, and R. Horodecki, “Separability of mixed states: necessary and sufficient conditions,” Phys. Lett. A 223, 1 (1996).

A. Milne, S. Jevtic, D. Jennings, H. Wiseman, and T. Rudolph, “Quantum steering ellipsoids, extremal physical states and monogamy,” New J. Phys. 16, 083017 (2014).

A. Milne, D. Jennings, and T. Rudolph, “Geometric representation of two-qubit entanglement witnesses,” Phys. Rev. A 92, 012311 (2015).

X. Hu and H. Fan, “Effect of local channels on quantum steering ellipsoids,” Phys. Rev. A 91, 022301 (2015).

S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, “Einstein–Podolsky–Rosen steering and the steering ellipsoid,” J. Opt. Soc. Am. B 32, A40 (2015).

X. Hu, A. Milne, B. Zhang, and H. Fan, “Quantum coherence of steered states,” Sci. Rep. 6, 19365 (2016).

H. C. Nguyen and T. Vu, “Necessary and sufficient condition for steerability of two-qubit states by the geometry of steering outcomes,” Eu 115, 10003 (2016).

Q. Quan, H. Zhu, S.-Y. Liu, S.-M. Fei, H. Fan, and W.-L. Yang, “Steering Bell-diagonal states,” Sci. Rep. 6, 22025 (2016).

S. Cheng, A. Milne, M. J. W. Hall, and H. M. Wiseman, “Volume monogamy of quantum steering ellipsoids for multiqubit systems,” Phys. Rev. A 94, 042105 (2016).

H. C. Nguyen, H.-V. Nguyen, and O. Gühne, “Geometry of Einstein-Podolsky-Rosen correlations,” Phys. Rev. Lett. 122, 240401 (2019).

C. Zhang, S. Cheng, L. Li, Q.-Y. Liang, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, M. J. W. Hall, H. M. Wiseman, and G. J. Pryde, “Experimental validation of quantum steering ellipsoids and tests of volume monogamy relations,” Phys. Rev. Lett. 122, 070402 (2019).

C. N. Yang and C. P. Yang, “One-dimensional chain of anisotropic spin-spin interactions. I. proof of Bethe's hy-
hypothesis for ground state in a finite system,” Phys. Rev. 150, 321 (1966).

[88] C. N. Yang and C. P. Yang, “One-dimensional chain of anisotropic spin-spin interactions. II. properties of the ground-state energy per lattice site for an infinite system,” Phys. Rev. 150, 327 (1966).

[89] M. Shiroishi, “Exact calculation of correlation functions for spin-1/2 Heisenberg chain,” J. Phys. Soc. Jpn. 74, 47 (2005).

[90] Q. Luo, J. Zhao, and X. Wang, “Intrinsic jump character of first-order quantum phase transitions,” Phys. Rev. B 100, 121111(R) (2019).

[91] H. Ollivier and W. H. Zurek, “Quantum discord: A measure of the quantumness of correlations,” Phys. Rev. Lett. 88, 017901 (2001).

[92] M. Shi, F. Jiang, C. Sun, and J. Du, “Geometric picture of quantum discord for two-qubit quantum states,” New J. Phys. 13, 073016 (2011).

[93] Based on our experience, we would like to point out that while QSE is generally capable of unveiling QPTs, different types of QPTs are tied to different characters of QSE.