An analytical model for describing sequential initiation and simultaneous propagation of multiple fractures in hydraulic fracturing shale oil/gas formations

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Funding information
National Natural Science Foundation of China, Grant/Award Number: 51874252

Abstract
Frac-driven interactions (FDI) cause sharp declines in gas and oil production rates followed by slow recovery or no recovery. How to minimize the FDI is an open problem for researchers and engineers to solve in the energy industry. An analytical model is presented in this paper for simulating simultaneous propagations of multiple fractures to identify factors affecting FDI. Result of sensitivity analyses with the model indicates that increasing fluid injection rate can depress the growth of long fractures and thus reduce FDI intensity. Using dilatant fracturing fluids can slow down the growth of long fractures and thus reduce the chance of FDI. In order to better apply the model to solving the FDI problem, future research should include determination of the critical fracture width in different types of shale gas/oil formations.

KEYWORDS
frac bashing, frac hits, frac-driven interactions, hydraulic fracturing, multi-stage, shale gas

1 | INTRODUCTION

The term Frac-driven interactions (FDI) was defined by the hydraulic fracturing community to unify the previous terms “frac hits” and “frac bashing” in the oil and gas industry. It is a situation where active hydraulic fractures in a new well intersect passive fractures in an existing well, as illustrated in Figure 1. Wells’ response to the FDI is a sharp decline in gas and oil production rates followed by a slow recovery or no recovery, and this effect is usually not predictable.\textsuperscript{1} Daneshy and King\textsuperscript{2} present a thorough review of the FDI issue regarding the causes, consequences, and mitigation techniques. The currently used techniques of reducing FDI intensity are those measures that can maintain or increase the pressure in the passive wells prior to hydraulic fracturing the new well. These measures include shut-ins, long-term shut-in, temporarily abandon, frac and flow, small preloads, high-rate water-defensive frac, and refracturing. However, none of the measures is found consistently effective. How to minimize the FDI is an open problem for researchers and engineers to solve. The purpose of this study is to develop an analytical method for identifying engineering factors affecting FDI. Result of sensitivity analyses with the model indicates that increasing fluid injection rate can depress the growth of long fractures and thus reduce FDI intensity. Using dilatant fracturing fluids can slow down the growth of long fractures and thus reduce the chance of FDI. In order to better apply the model to solving the FDI problem, future research should include determination of the critical fracture width in different types of shale gas/oil formations.
Recent field investigations by Ali et al.\(^9\) include cleat orientation from ground mapping and image log studies for in situ stress analysis for coal bed methane exploration in the South Karanpura Coalfield of India. Field investigations were also carried out by Das and Chatterjee\(^11\) for mapping of pore pressure, in situ stress, and brittleness in unconventional shale reservoir of Krishna–Godavari Basin.

A new mathematical model was developed in this study to describe the simultaneous propagation of multiple fractures initiated from a perforation cluster. Factors affecting the length of fracture are identified via sensitivity analysis. The result confirmed the latest work of Feng et al.\(^9\) in that fracture population increases with fluid injection rate. This study further reveals the effect of fluid properties on the fracture length and thus the tendency of FDI. Dilatant fluids with shear-thickening characteristics should be employed in hydraulic fracturing to mitigate FDI in shale gas/oil formations.

### 2 | MATHEMATICAL MODEL

An analytical model was developed in this study to reveal factors affecting fracture length and thus development of FDI. Derivation of the model is given in Appendices. Resultant equations in the model are presented in this section.

Considering propagation of two fractures initiated from a perforation cluster at different times. If the fracturing fluid injection rate is set constant, the distance of propagation of the first fracture can be expressed as:

\[
x = \left( \frac{2n+2}{2n+1} C_1 \right)^{n+1/2} t^{n+1/2} \tag{1}
\]

where \(x\) is fracture length of the first fracture in ft, \(n\) is fluid flow behavior index, \(t\) is fluid injection time in second, and the constant \(C_1\) is defined by

\[
C_1 = \left[ \frac{372.25 g q_i (P_p - P_t)}{K h_f} \right]^{1/2} \left( \frac{0.5 q_i}{(3 + 1/n) h_f} \right)^{1/2} \tag{2}
\]

where \(g\) is gravitational acceleration factor (32.17 lbm·ft/lbf·s\(^2\)), \(q_i\) is the fracturing fluid injection rate per half cluster in ft\(^3\)/s, \(P_p\) is the pressure in the perforation cluster in lb/ft\(^2\), \(P_t\) is the pressure at fracture tip in lb/ft\(^2\), \(K\) is the consistency coefficient of fracturing fluid in cp, and \(h_f\) is constant fracture height in ft.

The dynamic average fracture width (in ft) is given by

\[
w = \frac{\left( \frac{2n+2}{2n+1} C_1 \right)^{n+1/2}}{h_f} q_i t^{1/2} \tag{3}
\]
When the width of the first fracture reaches a critical value $w_c$, the pressure inside the fracture induces the second fracture. According to Equation (3), the time at which the second fracture is initiated is given by

$$ t_c = \left\{ \frac{w_c h_1}{q_1 \left( \frac{2n+2}{2n+1} \right) + \left( \frac{2n+2}{2n+1} C_1 \right)^{\frac{2n+1}{2n+2}}} \right\}^{\frac{2n+2}{2n+1}} (4) $$

According to Equation (1), the length of the first fracture when the second fracture is initiated is given by

$$ x_c = \left( \frac{2n+2}{2n+1} C_1 \right)^{\frac{2n+1}{2n+2}} (5) $$

Assuming that the first fracture remains open with a constant width $w_c$ after the second fracture is created, the total propagation distance of the first fracture can be expressed as

$$ x_1 = x_c + \left( x_c^\frac{n+1}{n} + \frac{n+1}{n} C_1 t \right)^\frac{1}{n} (6) $$

where $t$ is the time starting from the initiation of the second fracture and

$$ C'_1 = \left[ \frac{372.25gw_1 (P_p - P_t)}{K} \right]^\frac{1}{2} \left( \frac{0.5w_1}{3+1/n} \right) (7) $$

where the average width of the first fracture $w_1$ after initiation of the second fracture is assumed to be the critical fracture length $w_c$.

After the second fracture is created, the fluid flow rate in the first fracture drops and is expressed by

$$ q_1 = h_1 w_1 C'_1 \left( x_c^\frac{n+1}{n} + \frac{n+1}{n} C'_1 t \right)^{-\frac{1}{n}} (8) $$

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**FIGURE 2** Flowchart for calculation of simultaneous propagation of two fractures
The fluid injection rate in the second fracture is thus expressed as

\[ q_2 = q_i - q_1 = q_i - h_1 w_1 C_1 \left( \frac{x_1^{n+1}}{n} + \frac{n+1}{n} C_1 t \right) \]  

(9)

The propagation distance of the second fracture is given by

\[ x_2 = \left( \frac{2n+2}{2n+1} C_2 \right)^{\frac{2n+1}{2n+2}} \frac{2n+1}{2n+2} \]  

(10)

where \( C_2 \) is defined by

\[ C_2 = \left[ \frac{372.25 g q_2 (P_p - P_t)}{K h f} \right]^{\frac{1}{n+1}} \left( \frac{0.5 q_2}{3 + 1/n h f} \right)^{\frac{1}{n+1}} \]  

(11)

Following the same procedure, the initiation and propagation of the third and more fractures can be mathematically described. Figure 2 presents a flowchart to help understand the methodology of analyzing the simultaneous propagation of two fractures.

### 3 | SENSITIVITY ANALYSIS

The controllable parameters during hydraulic fracturing are injection rate and rheological properties of the fracturing fluid. Rheological properties of different types of fluids are generally differentiated by the value of the flow behavior index \( n \). For the conventional slick water made from polyacrylamide, the \( n \)-value is less or equal to 1, while for dilatant fluids, the \( n \)-value is greater than 1, and 1.2 is considered as a typical value. Model-predicted fracture length was sensitized to these two parameters using a basic data set presented in Table 1.

#### 3.1 | Case 1—Conventional slick water at low injection rate

Figure 3 presents model-calculated propagation and width development of the first fracture before the second fracture is created with a half-cluster fluid injection rate of 5 bpm and \( n = 1 \). This figure shows that fracture width widens fast in the beginning and slows down to minimize flow friction naturally and fracture keeps propagating almost linearly as fluid injection time goes on. In 20 minutes, the fracture propagates to 1480 ft while the average fracture width reaches 0.06 inch. Suppose the second fracture is created at this moment due to the rock stress induced by the opening

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**TABLE 1** A basic data set for sensitivity analysis

| Parameter                                | Value          |
|------------------------------------------|----------------|
| Pressure in perforation cluster          | 5500 psi       |
| Pressure at fracture tip                 | 5000 psi       |
| Overall density of fracturing fluid      | 165.36 lbm/ft³ |
| Fracturing fluid injection rate to half cluster | 5 ~ 10 bpm     |
| Fracture height                          | 100 ft         |
| Coefficient of consistency \( (K) \)     | 2 cp           |
| Fluid behavior index \( (n) \)           | 1.0 ~ 1.2      |

**FIGURE 3** Model-calculated propagation and width development of the first fracture before the second fracture is created \((q_i = 5 \text{ bpm}, n = 1)\)
of the first fracture, Figure 4 shows model-calculated flow rate and continued propagation of the first fracture after the second fracture is created. This figure indicates that the first fracture will continue grow by 2000 ft in the next 60 minutes. This means that the first fracture will propagate 3480 ft in 80 minutes.

Figure 5 demonstrates model-calculated propagation and width development of the second fracture. It shows that fracture width widens fast in the beginning and slows down to minimize flow friction naturally and the fracture keeps propagating almost linearly as fluid injection time goes on. In 60 minutes, the fracture propagates to 3400 ft while the average fracture width reaches 0.06 inch.

3.2 | Case 2—Conventional slick water at high injection rate

Figure 6 presents model-calculated propagation and width development of the first fracture before the second fracture is created with a half-cluster fluid injection rate of 10 bpm and n = 1. Similar to the Case 1, fracture width widens fast in the beginning and slows down to minimize flow friction naturally and fracture keeps propagating almost linearly as fluid injection time goes on. In 5 minutes, the fracture propagates to 740 ft while the average fracture width reaches 0.06 inch. Suppose the second fracture is created at this moment due to the rock stress induced by the opening of the first fracture, Figure 7 shows model-calculated flow rate and continued propagation of the first fracture after the second fracture is created. This figure indicates that the first fracture will continue grow by 1000 ft in the next 15 minutes. This means that the first fracture will propagate 1740 ft in 20 minutes.

Figure 8 demonstrates model-calculated propagation and width development of the second fracture. It shows that fracture width widens fast in the beginning and slows down to minimize flow friction naturally and the fracture keeps propagating almost linearly as fluid injection time goes on. In 15 minutes, the fracture propagates to 1820 ft while the average fracture width reaches 0.06 inch.

3.3 | Case 3—Dilatant fluid at low injection rate

Figure 9 presents model-calculated propagation and width development of the first fracture before the second fracture is created with a half-cluster fluid injection rate of 5 bpm and n = 1.2. Similar to Cases 1 and 2, fracture width widens fast in the beginning and slows down to minimize flow friction naturally and fracture keeps propagating almost linearly as fluid injection time goes on. In 5 minutes, the fracture propagates to 360 ft while the average fracture width reaches 0.06 inch. Suppose the second fracture is created at this moment due to the rock stress induced by the opening of the first fracture, Figure 10 shows model-calculated flow rate and continued propagation of the first fracture after the second fracture is created. This figure indicates that the first fracture will continue grow by 600 ft in the next 18 minutes. This means that the first fracture will propagate 960 ft in 23 minutes.

Figure 11 demonstrates model-calculated propagation and width development of the second fracture. It shows that fracture width widens fast in the beginning and slows down to minimize flow friction naturally and the fracture keeps propagating almost linearly as fluid injection time goes on.
In 18 minutes, the fracture propagates to 920 ft while the average fracture width reaches 0.06 inch.

3.4 Case 4—Dilatant fluid at high injection rate

Fracture propagation was carried out with a half-cluster fluid injection rate of 10 bpm and \( n = 1.2 \) in Case 4. The total length of fracture 1 was predicted to be 410 ft in 4.8 minutes. This and the results of Cases 1, 2, and 3 are summarized in Table 2 for comparison. It is observed from Table 2 that short fractures can be created with high fluid injection rates using dilatant fluids.

4 DISCUSSION

The sensitivity analyses indicate that use of high flow rate in hydraulic fracturing can reduce fracture length and thus increase fracture population for a given amount of fluid.
volume. This is explained by the fact that high flow rate creates high flow friction and thus high resistance to flow, reducing the growth rate of long fractures. The observation is consistent with the model prediction given by Feng et al.\textsuperscript{9}

Industry experiences from massive fracturing operation on shale gas/oil wells also confirm this trend.

The sensitivity analyses also show that use of dilatant fracturing fluids (n > 1) can reduce fracture length and thus increase fracture population for a given amount of fluid volume. This is explained by shear-thickening property of the dilatant fluids. The viscosity of fluid is high in long fractures, slowing down the growth of fracture, while the viscosity of fluid is low in short fractures, promoting the growth of fracture.

It is understood that the analytical model can be erroneous if the assumed conditions in model derivation are not met. The major assumption is that the fracture takes a rectangular shape with a constant height. This assumption should be valid when the fracture length is significantly greater than the pay zone thickness.

The width of the first fracture may slightly drop at the moment when the second fracture is suddenly created, but it should open up soon after the pressure in the second fracture increases. The sustainable fracture width is controlled by
the net pressure (stress). Without knowing the net pressure change along the fracture, a pressure-dependent width model is not possible. As an approximation, this study assumes that the first fracture remains open with the critical width after the second fracture is created. The critical fracture width of 0.06 inch assumed in the example calculation is based on the 0.0666 inch from the following equation by Rahman and Rahman\textsuperscript{12} for the maximum fracture width at the wellbore:

\[
w_f = 9.15 \left( \frac{1 + 2.14n}{n} \right)^{\frac{1}{3n+2}} K^{\frac{1}{2n+2}} \left( \frac{q_i}{2} \right)^{\frac{1}{2n+2}} E^{\frac{1}{2n+2}}
\]

where \(w_f\) (m) is the maximum fracture width under a constant flow rate of \(q_i\) (m\(^3\)/s), \(K\) is the consistency index (Pa s\(^n\)), \(n\) is the fluid flow consistency index, \(E\) is the Young’s modulus of formation rock (Pa), \(x_i\) is the half fracture length (m), and \(h_i\) is the fracture height (m). However, the method for predicting the critical fracture width should be further investigated. It is suggested this parameter is determined from the pressure profile recorded during the hydraulic fracturing process for individual shale gas/oil formations.

It is understood that the theoretical model should be fully validated before it is used in quantitative analyses.
Unfortunately with limited size of core samples where the friction pressure along the short fracture is very low, it is very difficult to create multiple fractures in laboratories for model validation. Also fracturing simulators, if they exist, should be employed for model validation. This work is left for other investigators to do in the future. Nevertheless, the model may be used in qualitative analyses for factor identification before it is fully validated for quantitative analysis in fracturing design.

5  |  CONCLUSIONS

An analytical model is presented in this paper for simulating simultaneous propagations of multiple fractures. Sensitivity analyses with the model allow for drawing the following conclusions.

1. Increasing fluid injection rate causes more friction and thus more resistance to flow in long fractures. Short fractures can be created using high injection rate of fracturing fluids. Therefore, Frac-driven interaction (FDI) can be mitigated with high-rate fracturing.

2. Dilatant fracturing fluids (n > 1) have shear-thickening property. Their viscosity is high in long fractures, slowing down the growth of fracture, and low in short fractures, promoting the growth of fracture. Frac-driven interaction can be mitigated using dilatant fracturing fluids.

3. Future research should include determination of the critical fracture width in different types of shale gas/oil formations. This parameter may be determined from the pressure profile recorded during the hydraulic fracturing process for individual shale gas/oil formations.

ACKNOWLEDGMENTS

This research was supported by the NSFC under Award Number 51874252.

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APPENDIX 1

DERIVATION OF MATHEMATICAL MODEL FOR PROPAGATION OF A SINGLE FRACTURE

For the first hydraulic fracture initiated from a cluster of perforations depicted in Figure 12,

Hydraulics gives the following relation:

\[ P_p - P_t = \frac{2f \rho_L v^2 x}{g w} \]  

(A1)

where \( P_p \) is the pressure at the perforation cluster in lbf/ft²; \( P_t \) is the pressure at the fracture tip in lbf/ft²; \( f \) is the Fanning friction factor; \( \rho_L \) is the liquid density in lbm/ft³; \( g \) is the gravitational acceleration factor (32.17 lbm-ft/lbf-s²); \( v \) is the velocity of fluid (or the fracture propagation rate) in ft/s; \( x \) is the fracture propagation length in ft; and \( w \) is the fracture width in ft.13

If the fracturing fluid is injected into only one fracture, the injection rate \( q_i \) is expressed as

\[ q_i = w h_f v \]  

(A2)

where \( h_f \) is the fracture height. This equation gives

\[ w = \frac{q_i}{h_f v} \]  

(A3)

Substituting Equation (A3) into Equation (A1) and rearranging the latter, we have

\[ P_p - P_t = \frac{2f \rho_L h_f}{g q_i} v^3 x \]  

(A4)

The friction factor for laminar flow is expressed as

\[ f = \frac{16}{N_{Re}} \]  

(A5)

where the Reynolds number for Power Law fluids is expressed by Dodge and Metzner14 as

\[ N_{Re} = \frac{1912 \rho_L v^{2-n}}{K} \left( \frac{0.5w}{3+n^{-1}} \right)^n \]  

(A6)

where \( K \) is the consistency index in cp and \( n \) is flow behavior index. Substituting Equations (A3), (A5), and (A6), into Equation (A4) gives

FIGURE A1 Top view of a hydraulic fracture initiated from a cluster of perforations

How to cite this article: Xiao D, Guo B, Zhang X. An analytical model for describing sequential initiation and simultaneous propagation of multiple fractures in hydraulic fracturing shale oil/gas formations. Energy Sci Eng. 2019;7:1514–1526. https://doi.org/10.1002/ese3.421

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APPENDIX 2

Derivation of mathematical model for propagation of two fractures

Two hydraulic fractures are depicted in Figure 13. Once the second fracture is created, we assume that the width of the first fracture will remain its constant value $w_1$.

Propagation of the first fracture

Hydraulics equation for the first fracture is

$$P_p - P_1 = \frac{2f \rho_v v_1^2 (x_0 + x_1)}{g w_1}$$

(A16)

where $x_0$ is the length of fracture 1 when fracture 2 was created and $x_1$ is the length increase of fracture 1 after fracture 2 was created. Assuming laminar flow, the friction factor is

$$f = \frac{16}{N_{Re}}$$

(A17)

where the Reynolds number for Power Law fluids is expressed by Dodge and Metzner \(^{14}\) as

$$N_{Re} = 11912 \frac{\rho_v^{\frac{1}{2-n}} K}{\frac{3}{3+n-1}}$$

(A18)

Substituting Equations (A17) and (A18) into Equation (A16), we have

$$P_p - P_1 = \frac{K v_1''}{372.25 g w_1} \left( \frac{3 + n^{-1}}{0.5 w_1} \right)^n (x_0 + x_1)$$

(A19)
The velocity of fluid $v_1$ is expressed as

$$v_1 = \frac{dx_1}{dr}$$  \hspace{1cm} (A20)

Substituting Equation (A20) into Equation (A19) and rearranging the latter, we have

$$\frac{dx_1}{dr} = \left[ \frac{372.25gw_1 (P_p - P_i)}{K (x_0 + x_1)} \right]^{\frac{1}{n+1}} \left[ \frac{0.5w_1}{3+n^{-1}} \right]$$  \hspace{1cm} (A21)

or

$$(x_0 + x_1)^{\frac{1}{n+1}} \frac{dx_1}{dr} = C_1' t$$  \hspace{1cm} (A22)

where

$$C_1' = \left[ \frac{372.25gw_1 (P_p - P_i)}{K} \right]^{\frac{1}{n+1}} \left[ \frac{0.5w_1}{3+n^{-1}} \right]$$  \hspace{1cm} (A23)

Integration of Equation (A22) yields

$$\int_0^{x_1} (x_0 + x_1)^{\frac{1}{n+1}} dx_1 = \int_0^t C_1' dt$$  \hspace{1cm} (A24)

or

$$\left( \frac{n}{n+1} \right) (x_0 + x_1)^{\frac{n+1}{n+1}} - \left( \frac{n}{n+1} \right) x_0^{\frac{n+1}{n+1}} = C_1' t$$  \hspace{1cm} (A25)

which gives

$$x_1 = x_0 + \left( x_0^{\frac{n+1}{n+1}} + \frac{n+1}{n} C_1' t \right)^{\frac{n}{n+1}}$$  \hspace{1cm} (A26)

and

$$v_1 = \frac{dx_1}{dr} = C_1' \left( x_0^{\frac{n+1}{n+1}} + \frac{n+1}{n} C_1' t \right)^{\frac{n}{n+1}}$$  \hspace{1cm} (A27)

The flow rate in the first fracture is expressed as

$$q_1 = h_i w_1 v_1 = h_i w_1 C_1' \left( x_0^{\frac{n+1}{n+1}} + \frac{n+1}{n} C_1' t \right)^{\frac{n}{n+1}}$$  \hspace{1cm} (A28)

**Propagation of the second fracture**

Hydraulics gives the following relation:

$$P_p - P_i = \frac{2f \rho_i h_i}{g w_2} v_2^2 x_2$$  \hspace{1cm} (A29)

The width of fracture 2 is given by material balance:

$$w_2 = \frac{q_2}{h_j v_2}$$  \hspace{1cm} (A30)

where the fluid injection rate in fracture 2 is expressed as

$$q_2 = q_i - q_i - h_i w_1 C_1' \left( x_0^{\frac{n+1}{n+1}} + \frac{n+1}{n} C_1' t \right)^{\frac{1}{n+1}}$$  \hspace{1cm} (A31)

Substituting Equation (A30) into Equation (A29) and rearranging the latter, we have

$$P_p - P_i = \frac{2f \rho_i h_i}{g q_2} v_2^2 x_2$$  \hspace{1cm} (A32)

For laminar flow, the friction factor is expressed as

$$f = \frac{16}{N_{Re}}$$  \hspace{1cm} (A33)

where the Reynolds number for Power Law fluids is expressed by Dodge and Metzner\(^{14}\) as

$$N_{Re} = 11912 \frac{\rho_L^{1-\frac{n}{2}}}{K} \left( \frac{0.5w_1}{3+n^{-1}} \right)^n$$  \hspace{1cm} (A34)

Substituting Equations (A30), (A33), and (A34) into Equation (A32), we have

$$P_p - P_i = \frac{Kh_i}{372.25g q_2} \left( \frac{3+n^{-1}}{0.5w_1} \right)^n v_2^{n+1} x_2$$  \hspace{1cm} (A35)

or

$$P_p - P_i = \frac{Kh_i}{372.25g q_2^{n+1}} v_2^{2n+1} x_2$$  \hspace{1cm} (A36)

The velocity of fluid $v_2$ is expressed as

$$v_2 = \frac{dx_2}{dr}$$  \hspace{1cm} (A37)

Substituting Equation (A37) into Equation (A36) and rearranging the latter, we have

$$\frac{dx_2}{dr} = \left[ \frac{372.25g q_2^{n+1} (P_p - P_i)}{Kh_i x_2^{n+1}} \right]^{\frac{1}{n+1}} \left[ \frac{0.5}{3+n^{-1}} h_i \right]^{\frac{1}{n+1}}$$  \hspace{1cm} (A38)

Substituting Equation (A31) into Equation (A30) and rearranging the latter, we have

$$\frac{1}{x_2^{2n+1}} \frac{dx_2}{dr} = C_2$$  \hspace{1cm} (A39)
where

\[ C_2 = \left[ \frac{372.25 g q^{n+1} (P_p - P_t)}{K h_I x_2} \right]^{\frac{n}{n+1}} \left[ \frac{0.5}{(3+n^{-1}) h_I} \right]^{\frac{n-1}{n+1}} \] (A40)

Integration of Equation (A39) gives

\[ x_2 = \left( \frac{2n+2}{2n+1} C_2 \right)^{\frac{2n+1}{2n+2}} t^{\frac{2n+1}{2n+2}} \] (A41)

Substituting Equation (A42) into Equation (A30) results in

\[ v_2 = \frac{dx_2}{dt} = \left( \frac{2n+2}{2n+1} \right) \left( \frac{2n+2}{2n+1} C_2 \right)^{\frac{2n+1}{2n+2}} t^{\frac{2n+1}{2n+2}} \] (A42)

\[ w_2 = \frac{1}{h_I} q_2 t^{\frac{1}{2n+2}} \] (A43)