THE C*-ALGEBRA OF A TWISTED GROUPOID EXTENSION

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ABSTRACT. This written version of a talk given in July 2020 at the Western Sydney Abend seminar and based on the joint work [6] gives a decomposition of the C*-algebra of a locally compact groupoid with Haar system, possibly endowed with a twist, in presence of a normal subgroupoid. The natural expression of this result uses Fell bundles over groupoids. When the normal subgroupoid and the twist over it are abelian, one obtains another twisted groupoid C*-algebra.

1. INTRODUCTION.

The Mackey normal subgroup analysis (also called the Mackey machine) describes the representations of a group $G$ in terms of a normal subgroup $S$ and the quotient $H = G/S$. A semidirect product of groups $G = S \rtimes H$ such as the group of rigid motions or the Poincaré group is the simplest example. From the C*-algebraic perspective, it gives a description of $C^*(G)$ as a crossed product. In the simple case of a semidirect product, we easily have

$$C^*(S \rtimes H) = C^*(S) \rtimes H = C^*(H, C^*(S))$$

A semidirect product is a trivial extension. When the extension is not trivial, a twist appears:

**Theorem 1.1 (Green [5]).**

$$C^*(G) = C^*(G, C^*(S), \tau_S)$$

where the right handside is a twisted crossed product.

When $S$ is abelian, one can go one step further, namely use the Gelfand transform

$$C^*(S \rtimes H) = C^*(H, C^*(S)) = C^*(H, C_0(\hat{S})) = C^*(\hat{S} \rtimes H)$$

The last term is a groupoid C*-algebra, where the groupoid $\hat{S} \rtimes H$ has less isotropy than the initial group $S \rtimes H$. One may want to iterate the process. It is then necessary to extend the Mackey machine to a groupoid $G$ rather than a group. The original motivation was the analysis of nilpotent group C*-algebras. It is limited here to the example of the Heisenberg group, which is presented in the last section.

This article is a written version of a talk I gave at the Western Sydney Abend seminar in July 2020. It is based on a joint work [6] with M. Ionescu, A. Kumjian, A. Sims and D. Williams, whom I thank for a stimulating and enjoyable collaboration. The situation considered in the present version is more general than that of [6], since the initial groupoid may be twisted. While most proofs are the same as in [6], it seems preferable to give a separate presentation of the general result because it requires some changes all along.

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*Key words and phrases.* groupoid extension, twist, Mackey machine, Fell bundle.
Note however the case of a twisted groupoid extension considered here can be deduced directly from the untwisted case considered in [6]. This is done in [7, Proposition 3.5]. There is an attempt to make this article self-contained but the reader is directed to [6] (and also [7]) for some proofs. We also refer to [6] for unexplained notation. Here is a notation which is frequently used: given two maps \( p : X \to T \) and \( q : Y \to T \) with the same range, their fibre product over \( T \) is denoted by \( X \ast Y \) when there is no ambiguity about the maps.

2. Fell bundles and groupoid C*-dynamical systems

**Definition 2.1.** A groupoid extension is a short exact sequence of groupoids

\[ S \to G \to H \]

with common unit space \( G(0) \). Equivalently, an extension of the groupoid \( H \) is a surjective homomorphism \( \pi : G \to H \) such that \( \pi(0) : G(0) \to H(0) \) is a bijection (we shall assume that \( G(0) = H(0) \) and that \( \pi(0) \) is the identity map). We shall write \( \dot{\gamma} = \pi(\gamma) \) when there is no ambiguity about the projection map.

Then \( S = \ker(\pi) \) is a subgroup bundle of the isotropy group bundle \( G' \). Moreover, it is normal in the sense that for all compatible pair \((\gamma, s) \in G \ast S\), \( \gamma s \gamma^{-1} \) belongs to \( S \). Note that subgroupoids which are normal in this sense are necessarily subgroup bundles of the isotropy group bundle. Note also that \( H \) is naturally isomorphic to the quotient groupoid \( G/S \). Since the normal subgroupoid \( S \) of \( G \) determines the extension, we shall often use the terminology of normal subgroupoid rather than extension. Then \( H \) is the quotient groupoid. We assume that \( H \) and \( G \) are locally compact Hausdorff groupoids and that \( \pi \) is continuous and open. In particular \( S \) is a closed normal subgroupoid of \( G \). We also assume that \( H \) has a Haar system \( \alpha \) and that \( S \) has a Haar system \( \beta \). There is a homomorphism \( \delta : G \to \mathbb{R}_+^\ast \) such that for all \( \gamma \in G \), \( \gamma \beta(\gamma) \gamma^{-1} = \delta(\gamma) \beta^r(\gamma) \). This homomorphism is called the modular cocycle of the extension. Its cohomology class does not depend on the choice of \( \beta \). For all \( x \in G(0) \), its restriction to the group \( S_x \) is the modular function of \( S_x \). It is continuous (see [6, Lemma 2.4]). Given \( \alpha \) and \( \beta \), we define the Haar system \( \lambda \) for \( G \) by the formula:

\[
\int f(\gamma) d\lambda^x(\gamma) = \int_H \int_S f(\gamma t) d\beta^{s(\gamma)}(t) d\alpha^x(\dot{\gamma})
\]

**Definition 2.2.** Under these assumptions, we say that \( S \to G \to H \) is a locally compact groupoid extension with Haar systems.

It will be convenient in the sequel to define an extension with Haar system as a pair \((G, S)\) where \( S \) is a closed normal subgroupoid of the locally compact groupoid \( G \) which admits a Haar system and such that the quotient groupoid \( H = G/S \) admits a Haar system.

**Definition 2.3.** \([10]\) A groupoid C*-dynamical system (or dynamical system for short) is a triple \((G, S, \mathcal{A})\) where \( G \) is a locally compact groupoid, \( S \) is a closed normal subgroupoid, and \( \mathcal{A} \) is an upper semi-continuous bundle of C*-algebras over \( G(0) \) endowed with a continuous action \( G \ast \mathcal{A} \to \mathcal{A} \) such that \( S \) is unitarily implemented in the multiplier.
algebra bundle $M(A)$, meaning the existence of a bundle homomorphism $\chi$ from $S$ to the unitary bundle of $M(A)$, such that

(i) the map $S \ast A \to A$ sending $(s, a)$ to $\chi(s) a$ is continuous;
(ii) $s \ast a = \chi(s) a \chi(s)^{-1}$ for all $(s, a) \in S \ast A$;
(iii) $\chi(\gamma s \gamma^{-1}) = \gamma \cdot \chi(s)$ for all $(\gamma, s) \in G \ast S$.

In [16] Section 3, it was assumed that the kernel $S$ was abelian, in the sense that it was a bundle of abelian groups. However, as shown in [9], this assumption is not necessary and most results of [16] remain valid.

To distinguish bundles of algebras (or of linear spaces) from algebras, algebra bundles will usually be denoted by calligraphic letters such as $A$ while algebras will be denoted by Roman letters such as $A$. For example, if $p : G \to X$ is a bundle of groupoids with Haar systems, $C_c(G)$ and $C^*(G)$ denote the bundles with respective fibers $C_c(G(x))$ and $C^*(G(x))$. On the other hand, $C_\ast(G)$ and $C^\ast(G)$ denote the usual $\ast$-algebras of the groupoid $G$, which are the sectional algebras of the above bundle. Groupoid dynamical systems fit into the more general framework of Fell bundles over groupoids.

**Definition 2.4.** [22] [10] A Fell bundle over a groupoid is a bundle $B \to H$ where $H$ is a locally compact groupoid and $B$ is an upper semi-continuous bundle of Banach spaces over $H$ endowed with a continuous multiplication $B \ast B \to B$ and a continuous involution $B \to B$ satisfying the $C^\ast$-algebra axioms whenever they make sense.

This definition implies that the fibers $B_x$ over $x \in H(0)$ become $C^\ast$-algebras, the fibres $B_h$ over $h \in H$ become $(B_{r(h)}, B_{s(h)})$-$C^\ast$-bimodules and $h \mapsto B_h$ is functorial. One says that the Fell bundle is saturated if the $B_h$’s are equivalence $C^\ast$-bimodules.

A groupoid $C^\ast$-dynamical system $(G, S, A)$ as above gives rise to a Fell bundle $B$ over $H = G/S$. This is [12] Example 7.3 which we recall now. We form $A \ast G = \{(a, \gamma) \in A \times G : a \in A_{r(\gamma)}\}$.

We let $S$ act on it by $s(a, \gamma) = (a \chi(s^{-1}), s \gamma)$ and consider the quotient $B = (A \ast G)/S$. The image of $(a, \gamma)$ in $B$ is denoted by $[a, \gamma]$. The bundle map $p : B \to H$ sends $[a, \gamma]$ to $\pi(\gamma)$ where $\pi : G \to G/S$ is the quotient map. A choice of $\gamma$ in $\pi^{-1}(h)$ gives a Banach space isomorphism $[a, \gamma] \mapsto a$ from $B_h$ to $A_{r(h)}$. The multiplication in $B$ is given by $[a, \gamma][b, \gamma'] = [a(\gamma, b), \gamma \gamma']$ and the involution by $[a, \gamma]^* = [\gamma^{-1}, a^*, \gamma^{-1}]$.

**Lemma 2.1.** [6] Lemma 1.5 The above bundle $B \to H$ is a saturated Fell bundle.

**Definition 2.5.** This bundle $B \to G/S$ is called the Fell bundle of the groupoid $C^\ast$-dynamical system $(G, S, A)$.

Recall the construction of the crossed products (see [16] Section 3 and [22]). Let $(G, S, A)$ be a groupoid $C^\ast$-dynamical system. We assume that $H = G/S$ has a Haar system $(\alpha^x)_{x \in G(0)}$ and that $S$ has a Haar system $(\beta^x)_{x \in G(0)}$. One first form the $\ast$-algebra $C_\ast(G, S, A)$. Its elements are continuous functions $f : G \to A$ such that...
(i) \( f(\gamma) \) belongs to \( A(\tau(\gamma)) \) for all \( \gamma \in G \);
(ii) \( f(s\gamma) = f(\gamma)\chi(s^{-1}) \) for all \((s, \gamma) \in S \ast G \);
(iii) \( f \) has compact support modulo \( S \).

The product and the involution are respectively given by

\[
f * g(\gamma) = \int f(\tau) [\tau.g(\tau^{-1}\gamma)] d\alpha^{r(\tau)}(\hat{\tau})
\]

and

\[
f^*(\gamma) = \gamma.(f(\gamma^{-1}))^*
\]

The crossed product \( C^*\)-algebra \( C^*(G, S, A) \) is the completion of \( C_c(G, S, A) \) for the full \( C^*\)-norm.

On the other hand, the sectional \( C^*\)-algebra of a Fell bundle \( B \) over a locally groupoid \( H \) endowed with a Haar system \( \alpha \) is constructed from the *-algebra \( C_c(H, B) \) whose elements are continuous compactly supported sections \( F : H \to B \). The product and the involution are respectively given by

\[
F * G(h) = \int F(\eta)G(\eta^{-1}h)d\alpha^{r(h)}(\eta)
\]

and

\[
F^*(h) = F(h^{-1})^*
\]

Again, the \( C^*\)-algebra \( C^*(H, B) \) is obtained as the \( C^*\)-completion for the full \( C^*\)-norm.

**Proposition 2.2.** [6, Section 1.4] Let \( (G, S, A) \) be a groupoid \( C^*\)-dynamical system where \( S \) and \( H = G/S \) have a Haar system and let \( B \twoheadrightarrow H \) be its Fell bundle. Then the \( C^*\)-algebras \( C^*(G, S, A) \) and \( C^*(H, B) \) are canonically isomorphic.

**Proof.** This is just a sketch of the proof. We refer the reader to [6] and the references given there. Let \( f \in C_c(G, S, A) \). Note that \([f(\gamma), \gamma] \in B_{\pi(\gamma)} \) depends on \( h = \pi(\gamma) \) only. Call this element \( F(h) \). Then check that

- \( F \) belongs to \( C_c(H, B) \);
- \( f \mapsto F \) is a *-homomorphism;
- this *-homomorphism extends to an isomorphism \( C^*(G, S, A) \to C^*(H, B) \).

\[ \square \]

**Remark 2.1.** The notion of groupoid Fell bundle \( (H, B) \) generalizes that of groupoid \( C^*\)-dynamical system \( (G, S, A) \), where \( H = G/S \) as above. A Fell bundle over a groupoid \( H \) is sometimes called an action by \( C^*\)-correspondences. The sectional \( C^*\)-algebra \( C^*(H, B) \) is then called its crossed-product \( C^*\)-algebra (see for example [1]).

3. Mackey analysis of a twisted groupoid \( C^*\)-algebra.

3.1. **Twists.** We have given earlier the general notion of an extension. The following special case has been introduced by Kumjian in [9] in the framework of groupoids.
Definition 3.1. A central groupoid extension

\[ G^{(0)} \times T \twoheadrightarrow \Sigma \rightarrow G \]

where \( T \) is the group of complex numbers of module 1, is called a twist. Then, we say that \((G, \Sigma)\) is a twisted groupoid.

We need to distinguish arbitrary extensions as above and twists, because they do not play the same role in this study. While a twisted groupoid is denoted by \((G, \Sigma)\) where \( \Sigma \) is the middle term, an arbitrary extension will be determined by a closed normal subgroupoid and denoted for example by \((G, S)\) where \( S \) is the kernel of the extension.

3.2. Twisted extensions. Let \((G, \Sigma)\) be a twisted groupoid. Then \((\Sigma, G^{(0)} \times T, G^{(0)} \times C)\) is a groupoid dynamical system with the action \( \sigma(s(\sigma), a) = (r(\sigma), a) \) for \((\sigma, a) \in \Sigma \times C\) and \( \chi(x, \theta) = \theta \) for \((x, \theta) \in G^{(0)} \times T\). If \( G \) is a locally compact groupoid with Haar system, we can construct the crossed product \( C^*\)-algebra, which we denote by \( C^*G, \Sigma \) rather than \( C^*(\Sigma, G^{(0)} \times T, G^{(0)} \times C)\) and which we call the twisted groupoid \( C^*\)-algebra. The principle of our version of Mackey analysis is to decompose this \( C^*\)-algebra when \( G \) possesses a closed normal subgroupoid \( S \) endowed with a Haar system. This is a strong condition, which is often not satisfied by the isotropy bundle itself.

Definition 3.2. We call \((G, \Sigma, S)\) a twisted extension with Haar systems when \((G, \Sigma)\) is a twisted groupoid, \( S \) is a closed normal subgroupoid of \( G \) with Haar system and \( G/S \) has a Haar system.

In the sequel, we shall denote \( H = G/S \) the quotient groupoid and \( \pi : G \rightarrow H \) the quotient map. We shall denote by \((\alpha^x)_{x \in G^{(0)}}\) [resp. \((\beta^x)_{x \in G^{(0)}}\)] the Haar system of \( H \) [resp. \( S \)]. We denote by \((\lambda^x)_{x \in G^{(0)}}\) the Haar system of \( G \) described earlier. The following diagram summarizes the situation:

\[
\begin{array}{ccc}
G^{(0)} \times T & \xrightarrow{\pi} & G^{(0)} \times T \\
\Sigma \downarrow & & \downarrow \Sigma' \\
G & \xrightarrow{\pi} & H \\
S \downarrow & & \downarrow \pi \\
G & \rightarrow & H
\end{array}
\]

3.3. The tautological Fell bundle. There is a Fell bundle \( L \rightarrow H \) naturally associated to a twisted extension \((G, \Sigma, S)\). The construction below is a particular case of the construction of Section 6 of [1], where the authors consider the more general framework of fibrations. The idea is very clear: the twisted groupoid \((G, \Sigma)\) defines a \( C^*\)-category over the groupoid \( H \). We first note that \( \Sigma \) defines a Fell line bundle \( L = \Sigma \otimes_T C \) over \( G \), with multiplication \( (\sigma_1 \otimes \lambda_1)(\sigma_2 \otimes \lambda_2) = \sigma_1 \sigma_2 \otimes \lambda_1 \lambda_2 \) and involution \((\sigma \otimes \lambda)^* = \sigma^{-1} \otimes \bar{\lambda}\).

In the next section, we shall view a section of \( L \rightarrow G \) as a function \( f : \Sigma \rightarrow C \) such that \( f(\theta \tau) = f(\tau)\overline{f(\theta)} \) for \((\theta, \tau) \in T \times \Sigma\), but here we use the line bundle framework. We associate to the unit \( x \in H^{(0)} \) the \( C^*\)-algebra \( C_x = C^*G/S_{x (\Sigma_S)} \). We want to associate to the arrow \( h \in H \) a suitable completion \( C_h \) of \( C_x(G(h), L(h)) \) where \( G(h) = \pi^{-1}(h) \) and \( L(h) \) is the restriction to \( G(h) \) of the line bundle \( L \). Let us write \( \Sigma(h) = \pi^*_{\Sigma}(h) \). Since \( \Sigma \) is a right
principal $\Sigma|_{S}$-space, a choice of $\tau \in \Sigma(h)$ defines a homeomorphism $L(\tau) : \Sigma(s(h)) \to \Sigma(h)$ sending $\sigma$ to $\tau \sigma$. In fact, this defines a line bundle isomorphism, still denoted by $L(\tau)$, from the line bundle $\mathcal{L}(s(h)) \to S_{s(h)}$ to the line bundle $\mathcal{L}(h) \to G(h)$, hence an isomorphism $L(\tau)^* : C_c(G(h), \mathcal{L}(h)) \to C_c(S_{s(h)}, \mathcal{L}(s(h))) \subseteq C^*(S_{s(h)}, \Sigma|_{S_{s(h)}})$. The norm $\|L(\tau)^*(f)\|$ of $f \in C_c(G(h), \mathcal{L}(h))$ depends on $h$ only and we write it $\|f\|_h$. We let $C_h$ be the completion of $C_c(G(h), \mathcal{L}(h))$ with respect to this norm. We define $\mathcal{C}$ as the disjoint union $\mathcal{C} = \bigsqcup_{h \in H} C_h$. We take $\Gamma = C_c(G, \mathcal{L})$ as a fundamental family of continuous sections to define its topology. One checks that the following conditions are satisfied.

(i) $\Gamma$ is a linear subspace of the complex linear space $\Pi_{h \in H} C_h$;
(ii) for all $h \in H$, the evaluation map $\text{ev}_h : \Gamma \to C_h$ has a dense image;
(iii) for all $\xi \in \Gamma$, the norm map $h \mapsto \|\xi(h)\|_h$ is upper semicontinuous;
(iv) if $\eta \in \Pi_{h \in H} C_h$ satisfies: for all $h \in H$ and all $\epsilon > 0$, there exists $\xi \in \Gamma$ and a neighborhood $U$ of $h$ in $H$ such that $\|\eta(h') - \xi(h')\|_h' \leq \epsilon$ for all $h' \in U$, then $\eta$ belongs to $\Gamma$.

Before describing the product and the involution of $\mathcal{C}$, it is useful to define a system of measures for the map $\pi : G \to H$. Such a system of measures already appears in [21] in the transitive case. Recall that, by assumption, the group bundle $S$ is equipped with a Haar system $\beta = (\beta_x)_{x \in G(0)}$. We extend it to a $\pi$-system by left invariance: for $h \in H$, we set $\beta^h = \gamma \beta^{\pi(h)}$, where $\gamma$ is an arbitrary element of $G(h)$; it depends on $h$ only. It is a continuous $\pi$-system of measures. It is left-invariant: for $(h, h') \in H(2)\pi$ and $\gamma \in G(h)$, $\gamma \beta^{h'} = \beta^{hh'}$. Note also the relation $(\delta(\gamma) \beta^h)^{-1} = \beta^{h^{-1}}$. Given $(h_1, h_2) \in H(2)$, $f_1 \in C_c(G(h_1), \mathcal{L}(h_1))$, $f_2 \in C_c(G(h_2), \mathcal{L}(h_2))$ and $\gamma \in G(h_1 h_2)$, we define

$$f_1 \ast_\beta f_2(\gamma) = \int f_1(\gamma') f_2(\gamma^{-1} \gamma) d\beta^{h_1}(\gamma')$$

and for $\gamma \in G(h^{-1})$,

$$f^*(\gamma) = f(\gamma^{-1})^*$$

One checks that these operations extend to $\mathcal{C}$ and turn it into a Fell bundle over $H$. We can also use the fact proved below that $\mathcal{C}$ is isomorphic to the Fell bundle of a groupoid dynamical system to complete the proof.

**Definition 3.3.** The Fell bundle $\mathcal{C}$ constructed above is called the tautological Fell bundle of the twisted extension $(G, \Sigma, S)$.

The following result is a particular case of [1, Theorem 6.2].

**Theorem 3.1 (Buss-Meyer [1]).** Let $(G, \Sigma, S)$ be a locally compact twisted groupoid extension with Haar systems. Let $H = G/S$ be the quotient groupoid. Then the twisted groupoid $C^*$-algebra $C^*(G, \Sigma)$ is canonically isomorphic to the sectional $C^*$-algebra $C^*(H, \mathcal{C})$ of the tautological Fell bundle.

**Proof.** Recall that the topology of $\mathcal{C}$ has been defined by the fundamental family of continuous sections $\Gamma$. The bijective map which associates to $f \in C_c(G, \mathcal{L})$ the corresponding
section \( j(f) \in \Gamma \) is an isomorphism of \(*\)-algebras: let \( f, g \in C_c(G, \mathcal{L}) \). On one hand,

\[
f *_\lambda g(\gamma) = \int_{G(\gamma')} f(\gamma')g(\gamma'^{-1}\gamma)d\lambda^{(\gamma)}(\gamma') = \int_{H(\gamma')} \int_{G(\gamma')} f(\gamma')g(\gamma'^{-1}\gamma)d\beta^{\gamma'}(\gamma')d\alpha^{(\gamma)}(\gamma')
\]

On the other hand, for \( h \in H \),

\[
(j(f) *_{\alpha} j(g))_h = \int_{H^{(h)}} j(f)(h') *_{\beta} j(g)(h'^{-1}h)d\alpha^{(h)}(h')
\]

Hence, for \( \gamma \in G(h) \)

\[
(j(f) *_{\alpha} j(g))_h(\gamma) = \int_{H^{(\gamma)}} \int_{G(\gamma')} f(\gamma')g(\gamma'^{-1}\gamma)d\beta^{\gamma'}(\gamma')d\alpha^{(\gamma)}(\gamma')
\]

The involution in \( C_c(G, \mathcal{L}) \) and in \( C_c(H, \mathcal{C}) \) are given by the same formula. The \(*\)-homomorphism \( j : C_c(G, \Sigma) \to C^*(H, \mathcal{C}) \) extends to \( C^*(G, \Sigma) \) by continuity. It is surjective since its range contains the dense subalgebra \( \Gamma \). To show its injectivity, one shows by using the disintegration theorem \cite[Théorème 4.1]{10} that for every representation \( L \) of \( C^*(G, \Sigma) \), there exists a representation \( L' \) of \( C^*(H, \mathcal{C}) \) such that \( L = L' \circ j \). \( \square \)

3.4. **The tautological groupoid dynamical system.** The tautological Fell bundle of the twisted extension \((G, \Sigma, S)\) is in fact the Fell bundle of the following groupoid dynamical system \((\Sigma, \Sigma|_S, C^*(S, \Sigma|_S))\). It can be shown just as in the untwisted case (see for example \cite[Theorem 5.5]{11}) that the twisted group bundle \((S, \Sigma|_S)\) with Haar system \( \beta \) defines an upper semi-continuous bundle of \( C^*\)-algebras over \( G^{(0)} \), with fibres \( C^*(S_x, \Sigma|_S) \) which we denote by \( C^*(S, \Sigma|_S) \). Moreover, when the groups are amenable, it is a continuous bundle. It is endowed with an action \( \Sigma \ast C^*(S, \Sigma|_S) \to C^*(S, \Sigma|_S) \), where

\[
(\tau.f)(\sigma) = \delta(p(\tau))f(\tau^{-1}\sigma\tau), \quad \text{for} \quad \tau \in \Sigma, \ f \in C_c(S_{h(\tau)}, \Sigma|_{S_{h(\tau)}}), \ \sigma \in p^{-1}(S_{h(\tau)}).
\]

The introduction of \( \delta \) is necessary in order to preserve the convolution product of the twisted group \( C^*\)-algebras \( C^*(S_x, \Sigma|_S) \).

**Proposition 3.2.** The triple \((\Sigma, \Sigma|_S, C^*(S, \Sigma|_S))\) is a groupoid \( C^*\)-dynamical system, which we call the tautological groupoid dynamical system of the twisted extension.

**Proof.** The continuity of the action map \( \Sigma \ast C^*(S, \Sigma|_S) \to C^*(S, \Sigma|_S) \) is proved just as in \cite[Proposition 2.7]{6}. The action of \( \Sigma|_S \) on \( C^*(S, \Sigma|_S) \) is implemented by the bundle homomorphism \( \chi : \Sigma|_S \to UMC^*(S, \Sigma|_S) \) which associates to \( \sigma \in p^{-1}(S_x) \) the canonical unitary \( \chi(\sigma) \) in the multiplier algebra of \( C^*(S_x, p^{-1}(S_x)) \). Explicitly, for \( f \in C_c(S_x, \Sigma|_S) \) and \( \sigma, \tau \in p^{-1}(S_x) \),

\[
(\chi(\sigma)f)(\tau) = \delta^{1/2}(p(\sigma))f(\sigma^{-1}\tau)
\]

One checks just as in \cite[Section 2]{6} that the conditions of Definition \cite[2.3]{2} are satisfied. \( \square \)

**Theorem 3.3.** Let \((G, \Sigma, S)\) be a locally compact twisted groupoid extension with Haar systems. Its tautological Fell bundle \( \mathcal{C} \) is isomorphic to the Fell bundle \( \mathcal{B} \) of its tautological groupoid dynamical system.
Proof. As before, we denote by \( \mathcal{L} = \Sigma \otimes_T \mathcal{C} \) the Fell line bundle over \( G \) associated with \( \Sigma \). Here, the sections of this bundle are viewed as functions \( f : \Sigma \to \mathcal{C} \) such that \( f(\theta \tau) = f(\tau) \overline{\theta} \) for all \( (\theta, \tau) \in T \times \Sigma \). Recall that \( \mathcal{B} = (\mathcal{C}^*(\Sigma, \Sigma|_S) \ast \Sigma) / \Sigma|_S \). Let \( \tau \in \Sigma \) and \( h = \pi_\Sigma(\tau) \). Given \( f \in C_c(G(h), \mathcal{L}(h)) \), we define \( \rho_\tau(f) \in C_c(G(r(h)), \mathcal{L}(r(h))) \) by

\[
\rho_\tau(f)(\sigma) = \delta^{1/2}(p(\tau))f(\sigma \tau), \quad \forall \sigma \in p^{-1}(G(r(h))
\]

Then \((\rho_\tau(f), \tau) \in \mathcal{C}^*(\Sigma, \Sigma|_S) \ast \Sigma \) and for all \( \sigma \in \Sigma|_S \), we have:

\[
(\rho_{\sigma \tau}(f), \sigma \tau) = (\rho_\tau(f)\chi(\sigma^{-1}), \sigma \tau) = \sigma(\rho_\tau(f), \tau)
\]

Therefore, for all \( h \in H \), there is a map \( j_h : C_c(G(h), \mathcal{L}(h)) \to B_h \) such that

\[
j_h(f) = [\rho_\tau(f), \tau]
\]

where \( \pi_\Sigma(\tau) = h \). By definition of the norm on \( C_c(G(h), \mathcal{L}(h)) \), \( j_h \) extends to a Banach space isomorphism from \( C_h \) onto \( B_h \) and that this defines a bundle isomorphism \( j \) from \( \mathcal{C} \) to \( \mathcal{B} \). Then one deduces from [4] Propositions 13.6 and 13.7 that \( j \) is a Banach bundle isomorphism. Let us check that \( j \) preserves the product and the involution. Let \( (h_1, h_2) \in H^{(2)} \). Choose \( \tau_i \in \Sigma \) such that \( \pi_\Sigma(\tau_i) = h_i \) for \( i = 1, 2 \). Let \( f_i \in C_c(G(h_i), \mathcal{L}(h_i)) \) for \( i = 1, 2 \). The equality

\[
j_{h_1 h_2}(f_1 \ast_\beta f_2) = j_{h_1}(f_1)j_{h_2}(f_2)
\]

amounts to the equality

\[
\rho_{\tau_1 \tau_2}(f_1 \ast_\beta f_2) = \rho_{\tau_1}(f_1)[\tau_1 \ast \rho_{\tau_2}(f_2)]
\]

We have for \( \sigma \in p^{-1}(G(r(h_1))) \):

\[
\rho_{\tau_1 \tau_2}(f_1 \ast_\beta f_2)(\sigma) = \delta^{1/2}(p(\tau_1 \tau_2))(f_1 \ast_\beta f_2)(\sigma_1 \tau_2) = \delta^{1/2}(p(\tau_1 \tau_2)) \int f_1(\tau) f_2(\tau^{-1} \sigma_1 \tau_2) d_\beta^{h_1}(\hat{\tau})
\]

where \( \hat{\tau} = p(\tau) \). On the other hand

\[
\rho_{\tau_1}(f_1)[\tau_1 \ast \rho_{\tau_2}(f_2)](\sigma) = \int \rho_{\tau_1}(f_1)(\sigma')[\tau_1 \ast \rho_{\tau_2}(f_2)](\sigma'^{-1} \sigma) d_\beta^{r(h_1)}(\hat{\sigma}')
\]

\[
= \int \delta^{1/2}(\hat{\tau}_1) f_1(\sigma_1 \tau_1) d(\hat{\tau}_1) \delta^{1/2}(\hat{\tau}_2) f_2(\tau_1^{-1} \sigma_1 \tau_2) d_\beta^{r(h_1)}(\hat{\sigma}')
\]

\[
= \delta^{1/2}(\hat{\tau}_1) \delta^{1/2}(\hat{\tau}_2) \int f_1(\sigma_1 \tau_1) f_2(\tau_1^{-1} \sigma_1 \tau_2) d(\hat{\tau}_1) \delta(\hat{\tau}_2) d_\beta^{r(h_1)}(\hat{\sigma}')
\]
This last integral is of the form \( \int g(s\dot{\tau})\delta(\dot{\tau})d\beta^{r(h_1)}(s) \) where \( g(\dot{\tau}) = f_1(\tau)f_2(\tau^{-1}\sigma_{\tau_1}\tau_2) \).

The change of variable \( s = \dot{\tau}_1t\dot{\tau}_1^{-1} \) gives

\[
\int g(s\dot{\tau})\delta(\dot{\tau})d\beta^{r(h_1)}(u) = \int g(\dot{\tau}_1t)d\beta^{h_1}(t)
\]

which is the desired equality.

Let \( f \in C_c(G(h), \mathcal{L}(h)) \). The equality \( (j_h(f))^* = j_{h^{-1}}(f^*) \)

amounts to the equality

\[
\tau^{-1}(.\rho_\tau(f))^* = \rho_{\tau^{-1}}(f^*)
\]

where \( \pi_{\Sigma}(\tau) = h \). For \( \sigma \in p^{-1}(G(s(h))) \), we have

\[
\tau^{-1}(\rho_\tau(f))^*(\sigma) = \delta(\dot{\tau}^{-1})(\rho_\tau(f))^*(\tau\sigma\tau^{-1}) = \delta(\dot{\tau}^{-1})(\rho_\tau(f)(\tau\sigma^{-1}\tau^{-1}))^* = \delta(\dot{\tau}^{-1})\delta^{1/2}(\dot{\tau})f(\tau\sigma^{-1})^* = \delta^{1/2}(\dot{\tau}^{-1})f(\tau\sigma^{-1})^*
\]

On the other hand,

\[
\rho_{\tau^{-1}}(f^*)(\sigma) = \delta^{1/2}(\dot{\tau}^{-1})f^*(\sigma\tau^{-1}) = \delta^{1/2}(\dot{\tau}^{-1})f(\tau\sigma^{-1})^*
\]

\[\square\]

**Corollary 3.4.** Let \((G, \Sigma, S)\) be a locally compact twisted groupoid extension with Haar systems. Then the twisted groupoid C*-algebra \( C^*(G, \Sigma) \) is isomorphic to the crossed product C*-algebra \( C^*(\Sigma, \Sigma|_S, C^*(S, \Sigma|_S)) \) of the dynamical system \((\Sigma, \Sigma|_S, C^*(S, \Sigma|_S))\).

**Remark 3.1.** This result can be obtained directly (without introducing the tautological Fell bundle) by using the disintegration theorem of [16] (or rather its generalization to a non abelian extension). This is the road followed in [6], where this result (in the untwisted case) appears as Theorem 2.11. The authors also consider the reduced C*-algebras. The main advantage of the Fell bundle is that it gives simple and natural formulas.

### 4. Abelian Fell Bundles

When the normal subgroupoid \( S \) in the locally compact twisted groupoid extension with Haar systems \((G, \Sigma, S)\) is abelian and the restriction \( \Sigma|_S \) of the twist is also abelian, one can go one step further by using the Gelfand transform for the bundle of abelian C*-algebras \( C^*(S, \Sigma|_S) \). Again, instead of doing this step directly, we make a detour via abelian Fell bundles.
4.1. Abelian Fell bundles. The structure of saturated abelian Fell bundles over a groupoid has been established by V. Deaconu, A. Kumjian and B. Ramazan [2 Theorem 5.6]. It is a direct consequence of the well-known following facts about Morita equivalence of commutative C*-algebras.

Lemma 4.1. [14 Appendix A]

(i) Let $X,Y$ be locally compact spaces, $\varphi : Y \to X$ a homeomorphism and $\mathcal{L}$ a hermitian line bundle over $X$ (with scalar product linear in the first variable). Then $(A = C_0(X), E = C_0(X, \mathcal{L}), B = C_0(Y))$ is an imprimitivity bimodule where

(a) for $(a, \xi, b) \in A \times E \times B$,

$$(b\xi)(x) = b(x)\xi(x) \quad (\xi b)(x) = b(\varphi^{-1}(x))\xi(x)$$

(b) for $(\xi, \eta) \in E \times E$,

$$A(\xi|\eta)(x) = (\xi(x)|\eta(x))_x \quad (\xi, \eta)_E(y) = (\eta(\varphi(y))|\xi(\varphi(y)))_{\varphi(y)}$$

(ii) Conversely, every imprimitivity bimodule $(A,E,B)$, where the C*-algebras $A$ and $B$ are abelian, is isomorphic to an imprimitivity bimodule

$$(C_0(X), C_0(X, \mathcal{L}), C_0(Y))$$

with $(X,Y,\mathcal{L})$ as in (i). The homeomorphism $\varphi : Y \to X$ is uniquely determined by the relation $\xi b = \alpha(b)\xi$ for all $\xi \in E$ and $b \in B$ and $\alpha(b) = b \circ \varphi^{-1}$. The quadruple $(X,Y,\varphi,\mathcal{L})$, which is unique up to isomorphism, is called the spatial realization of the Morita equivalence.

(iii) The spatial realization of the composition $(A, E \otimes_B F, C)$ of two Morita equivalences $(A, E, B)$ and $(B, F, C)$ having $(X,Y,\varphi,\mathcal{L})$ and $(Y,Z,\psi,\mathcal{M})$ as spatial realizations is the quadruple $(X, Z, \varphi \circ \psi, \mathcal{L} \otimes_X \varphi^*\mathcal{M})$.

(iv) If the imprimitivity bimodule $(A,E,B)$ admits the spatial realization $(X,Y,\varphi,\mathcal{L})$, then its inverse $(B,E^*,A)$ admits the spatial realization $(Y,X,\varphi^{-1},\varphi^*\mathcal{L})$, where $\mathcal{L}$ is the conjugate line bundle.

Proof. The assertions (i), (iii) and (iv) are straightforward. A direct proof of the assertion (ii) is given in [2]. It can also be obtained by introducing the imprimitivity algebra and by using [17], to which we refer the reader for unexplained notations. There is a C*-algebra $C$ and complementary full projections $p,q \in M(C)$ such that $(A, E, B)$ is isomorphic to $(pCp, pCq, qCq)$. It is easily checked that $D = pCp + qCq$ is a Cartan subalgebra of $C$ having the map $c \mapsto pc + qc$ as an expectation onto it. The spectrum $Z$ of $D$ is the disjoint union of the open subsets $X$ and $Y$, which are respectively the spectra of $A$ and $B$. Thus, according to [17 Theorem 5.9], there is an isomorphism $\Phi$ from $C$ onto $C^*_r(G(D), \Sigma(D))$, the reduced groupoid C*-algebra of the Weyl twist $(G(D), \Sigma(D))$, which carries $D$ onto $C_0(Z)$. The subsets $pCp, pCq, qCp$ and $qCq$ are contained in the normalizer $N(D)$ of $D$. Since $X$ and $Y$ are open, every germ of a partial homeomorphism induced by an element of the normalizer can be obtained by an element of one of these subsets. The partial homeomorphisms obtained from elements of $pCp$ and $qCq$ are partial identity maps. Let $n \in pCq$ and $y \in Y$ such that $n^*n(y) > 0$. Then the germ of $\alpha_n$ at $y$ does not depend on $n$. Indeed, let $m \in pCq$ such that $m^*m(y) > 0$. Without changing
the germ of $\alpha_m$ at $y$, we may assume that the closed support of $m^*m$ is contained in an open neighborhood of $y$ on which $n^*n \geq \epsilon > 0$. Then, there exists $b \in B$ with $b(y) \neq 0$ and $n^*m = (n^*n)b$. Then, the equality $(nn^*)m = (nn^*)nb$ implies that $\alpha_m$ and $\alpha_n$ have the same germ at $y$. Moreover, since the projection $q$ is full, every $y \in Y$ is the domain of some normalizer $n \in pCq$. Therefore there exists a unique homeomorphism $\varphi : Y \to X$ such that $\alpha_n(y) = \varphi(y)$ for all $n \in pCq$ and $y \in Y$ such that $n^*n(y) > 0$. This shows that $G(D)$ is the graph of the equivalence relation on $Z = X \uplus Y$ whose classes are $\{y, \varphi(y)\}$ for $y \in Y$ and $\{x, \varphi^{-1}(x)\}$ for $x \in X$. It is a closed subset of $Z \times Z$. Let $F$ be the Fell line bundle associated to the $\Sigma(D)$. The restriction $\mathcal{L}$ of $F$ to the subset $\{(x, \varphi^{-1}(x)), x \in X\}$, which we identify to $X$, is a hermitian line bundle with scalar product $(\xi|\eta) = \xi\eta^*$. Then $\Phi$ maps isomorphically $(pCp, pCq, qCq)$ onto $(C_0(X), C_0(X, \mathcal{L}), C_0(Y))$.

We give now a construction of a saturated abelian Fell bundle over a groupoid. It will turn out that every saturated abelian Fell bundle over a groupoid can be constructed in this fashion. Let $(H, \alpha)$ be a locally compact groupoid with Haar system and let $Z$ be a right locally compact $H$-space. As usual, we write $s : Z \to H^{(0)}$ the projection and $Z_x = s^{-1}(x)$ for $x \in H^{(0)}$. Let $\Sigma$ be a twist over the semi-direct product $Z \rtimes H$. We denote by $\mathcal{L}$ the associated Fell line bundle. Then, for each $h \in H$, we obtain by restriction to $Z_{r(h)} \times \{h\}$ a line bundle $\mathcal{L}_h$ over $Z_{r(h)}$ and consider the space of continuous sections vanishing at infinity $B_h := C_0(Z_{r(h)}, \mathcal{L}_h)$ endowed with the sup-norm. We turn $\mathcal{B} := \bigcup_{h \in H} B_h$ into a Banach bundle with $C_c(Z \rtimes H, \mathcal{L})$ as fundamental space of continuous sections, where we identify $f \in C_c(Z \rtimes H, \mathcal{L})$ with the section $h \mapsto f_h$, where $f_h \in C_c(Z_{r(h)}, \mathcal{L}_h)$ is defined by $f_h(z) = f(z, h)$. Given $(h, h') \in H^{(2)}$, we define the product

$$B_h \otimes B_{h'} \to B_{hh'} : b \otimes b' \mapsto b(hb')$$

where, for $z \in Z_{r(h)}$, $(b(hb'))(z) = b(z)b'(zh)$. For $h \in H$, we define the involution

$$B_h \to B_{h^{-1}} : b \mapsto b^*$$

where, for $z \in Z_{s(h)}$, $b^*(z) = (b(zh^{-1}))^*$. 

**Proposition 4.2.** Let $\Sigma$ be a twist over the semi-direct product $Z \rtimes H$, where $(H, \alpha)$ is a locally compact groupoid with Haar system and $Z$ a right locally compact $H$-space $Z$. Construct $\mathcal{B}$ as above. Then

(i) $\mathcal{B}$ is a saturated abelian Fell bundle over $H$;
(ii) $C^*(H, \mathcal{B})$ is isomorphic to $C^*(Z \rtimes H, \Sigma)$.

**Proof.** The first assertion is a straightforward verification. For the second assertion, one checks that the map which sends $f \in C_c(Z \rtimes H, \mathcal{L})$ into the section $h \mapsto f_h$ is a *-homomorphism from $C_c(Z \rtimes H, \Sigma)$ to $C_c(H, \mathcal{B})$. It is continuous in the inductive limit topology and has a dense image. Hence it extends to a *-isomorphism from $C^*(Z \rtimes H, \Sigma)$ to $C^*(H, \mathcal{B})$. □

**Definition 4.1.** We say that $\mathcal{B}$ is the Fell bundle of the twisted semi-direct product $(Z \rtimes H, \Sigma)$.

Note that this terminology does not agree with Definition [25] because here $\mathcal{B}$ is a Fell bundle over $H$ and not over $Z \rtimes H$. A coherent terminology would require the notion of
fibration as in \cite{1}. Let us show now that every saturated abelian Fell bundle is the Fell bundle of a twisted semi-direct product.

**Theorem 4.3.** [2, Theorem 5.6] Let $\mathcal{B}$ be a saturated abelian Fell bundle over a locally compact groupoid $H$. Then there exists a right locally compact $H$-space $Z$ and a twist $\Sigma$ over the semi-direct product $Z \rtimes H$ such that $\mathcal{B}$ is isomorphic to the Fell bundle of $(Z \rtimes H, \Sigma)$. The pair $(Z, \Sigma)$ is unique up to isomorphism. We call it the spatial realization of $\mathcal{B}$.

**Proof.** Let $\mathcal{B}$ be a saturated abelian Fell bundle over a locally compact groupoid $H$. Its restriction to $H(0)$ is an abelian $C^*$-bundle $\mathcal{B}(0)$ over $H(0)$. The sectional $C^*$-algebra $C_0(H(0), \mathcal{B}(0))$ is abelian, hence isomorphic to $C_0(Z)$, where $Z$ is its spectrum. The space $Z$ is fibred above $H(0)$. The bundle map, which is continuous, open and onto, is denoted by $s : Z \to H(0)$ and the fibre above $x \in H(0)$ is written $Z_x$. We write $B_x = C_0(Z_x)$. For each $h \in H$, $(B_{r(h)}, B_h, B_{s(h)})$ is a Morita equivalence. We let $(Z_{r(h)}, Z_{s(h)}, \varphi_h, \mathcal{L}_h)$ be its spatial realization. For $z \in Z_{s(h)}$, we define $zh = \varphi_h(z)$. This defines a map $Z \rtimes H \to Z$. Given $(z, h) \in Z \rtimes H$, we write $\mathcal{L}_{(z, h)} := (\mathcal{L}_h)_z$ and define the algebraic line bundle $\mathcal{L} = \bigcup_{(z, h) \in Z \rtimes H} \mathcal{L}_{(z, h)}$ over $Z \rtimes H$. A section $\xi \in C_c(H, \mathcal{B})$ defines a section $\xi$ of $\mathcal{L}$ according to $\xi(z, h) = \xi(h)(z)$. This defines a Banach bundle structure on $\mathcal{L}$. Let us show that $(z, h) \mapsto zh$ is an action map. The relation $(zh)h' = z(hh')$ results from the lemma. The continuity of the action can be obtained by applying the relation

$$b(zh)\tilde{\xi}(z, h) = (\tilde{\xi}b)(z, h)$$

where $b \in C_c(Z) \subset C_c(H(0), \mathcal{B}(0))$ and $\xi \in C_c(H, \mathcal{B})$. The isomorphism $B_h \otimes B_{s(h)} \to B_{hh'}$ defined by the product in the Fell bundle $\mathcal{B}$ gives an isomorphism $\mathcal{L}_h \otimes Z_{r(h)} \to \mathcal{L}_{hh'}$ which defines a product on $\mathcal{L}$. Similarly, the involution $B_h \to B_{h^{-1}}$ gives an involution $\mathcal{L}_h \to \mathcal{L}_{h^{-1}}$ on $\mathcal{L}$. This turns $\mathcal{L}$ into a Fell line bundle over $Z \rtimes H$. We let $\Sigma$ be the unitary bundle of $\mathcal{L}$. Then, by construction, $\mathcal{B}$ is isomorphic to the Fell bundle of $(Z \rtimes H, \Sigma)$. □

**Corollary 4.4.** Let $\mathcal{B}$ be a saturated abelian Fell bundle over a locally compact groupoid $H$. Then the spectrum $Z$ of the sectional algebra $C_0(H(0), \mathcal{B}(0))$ is a right locally compact $H$-space and there exists a twist $\Sigma$ over the semi-direct product $Z \rtimes H$ such that the sectional algebra $C^*(H, \mathcal{B})$ is isomorphic to the twisted groupoid $C^*$-algebra $C^*(Z \rtimes H, \Sigma)$.

### 4.2. Abelian groupoid dynamical system.

We say that a groupoid dynamical system $(G, S, A)$ is abelian when $A$ is a bundle of commutative $C^*$-algebras over $G(0)$. Note that we do not assume here that the groups $S_x$ are abelian. Then the associated Fell bundle $\mathcal{B}$ over $H = G/S$ is abelian and admits the above spatial realization. The twisted semi-direct product $(Z \rtimes H, \Sigma)$ of the previous section admits a convenient description. The sectional $C^*$-algebra $A = C_0(G(0), A)$ is abelian, hence isomorphic to $C_0(Z)$, where $Z$ is the spectrum of $A$. As said earlier, the space $Z$ is fibred above $G(0)$. The bundle map is denoted by $s : Z \to G(0)$ and the fibre above $x \in G(0)$ is written $Z_x$. We have $A = C_0(Z)$ and $A_x = C_0(Z_x)$. The action of $G$ on $A$ induces an action on $Z$ which we write as a right action so that $((\sigma, f))(z) = f(z\sigma)$ where $f \in C_0(Z_{\sigma}) = A_{\sigma}$ and $z \in Z_{\sigma}$. Because the action of $S$ is unitarily implemented and $A$ is abelian, $S$ acts trivially on $A$. Therefore, the action of $G$ is in fact an action of $H = G/S$. This gives the semi-direct product $Z \rtimes H$. Let us describe now the twist $\Sigma$ over this semi-direct product. It is given by a
pushout construction. We first observe that the homomorphism \( \chi : S \to UM(A) \) which implements the restriction of the action to \( S \) gives a map
\[
\chi : Z*S \to T \quad (z,t) \mapsto (\chi(t))(z)
\]
It is a continuous groupoid homomorphism which satisfies \( \chi(z,\gamma t\gamma^{-1}) = \chi(z\gamma,t) \) for all \((\gamma, t) \in G*S\). Here is a general definition.

**Definition 4.2.** Given a groupoid extension \( S \hookrightarrow G \twoheadrightarrow H \) and an \( H \)-bundle of abelian groups \( T \), we say that a group bundle morphism \( \varphi : S \to T \) is equivariant if \( \varphi(\gamma s\gamma^{-1}) = \dot{\gamma}\varphi(s) \) for all \((\gamma, s) \in G*S\).

We give now the general pushout construction (the reader is directed to [7] for a full exposition). It is summarized by the following diagram.

\[
\begin{array}{cccc}
S & \longrightarrow & G & \longrightarrow & H \\
\varphi \downarrow & & \varphi_* \downarrow & & \downarrow \\
T & \longrightarrow & \underline{G} & \longrightarrow & H
\end{array}
\]

Here are the details.

**Proposition 4.5.** Let \( S \hookrightarrow G \twoheadrightarrow H \) be a groupoid extension, let \( T \) be locally compact abelian group bundle endowed with an \( H \)-action and let \( \varphi : S \to T \) be an equivariant group bundle morphism. Then there is an extension \( \underline{T} \hookrightarrow \underline{G} \twoheadrightarrow H \) and a morphism \( \varphi_* : G \to \underline{G} \) that is compatible with \( \varphi \). They are unique up to isomorphism.

**Proof.** We define
\[
T*G = \{ (t, \gamma) \in T \times G, | p_T(t) = r(\gamma) \}
\]
It is a groupoid over \( G^{(0)} \) with multiplication
\[
(t, \gamma)(t', \gamma') = (t(\dot{\gamma}t'), \gamma\gamma')
\]
and inverse
\[
(t, \gamma)^{-1} = (\dot{\gamma}^{-1}(t^{-1}), \gamma^{-1})
\]
Endowed with the relative topology, it is a locally compact topological groupoid. Then \( S \) embeds into it as a closed normal subgroupoid via \( i : S \to T*G \) given by \( i(s) = (f(s^{-1}), s) \).

We define \( \underline{G} := T*G/i(S) \). Equivalently, \( \underline{G} \) is the quotient of \( T*G \) for the left action of \( S \) given by \( s(t, \gamma) = (t\varphi(s^{-1}), \gamma) \). Its elements are of the form \([t, \gamma]\) where \((t, \gamma) \in T*G \) and satisfy \([t, \gamma] = [t\varphi(s^{-1}), s\gamma]\) for \( s \in S_{r(\gamma)} \). Let us spell out its groupoid structure. Its unit space is \( G^{(0)} \) with obvious range and source maps. The multiplication is given by
\[
[t, \gamma][t', \gamma'] = [t(\dot{\gamma}t'), \gamma\gamma']
\]
and its inverse map is given
\[
[t, \gamma]^{-1} = [\dot{\gamma}^{-1}(t^{-1}), \gamma^{-1}]
\]
The map \( \underline{\pi} : \underline{G} \to H \) given by \( \underline{\pi}[t, \gamma] = \pi(\gamma) \) is a surjective homomorphism and \( \underline{\pi}^{(0)} \) is the identity map. Its kernel is identified to \( T \) via the map \( j : T \to \underline{G} \) defined by \( j(t) = [t, p_T(t)] \). The map \( \varphi_* : G \to \underline{G} \) is given by \( \varphi_*(\gamma) = [r(\gamma), \gamma] \) for \( \gamma \in G \).

\( \square \)
**Definition 4.3.** The above extension \( T \mapsto G \to H \) is called the pushout of the extension \( S \mapsto G \to H \) by the morphism \( \varphi : S \to T \).

To apply this construction to our abelian groupoid \( C^* \)-dynamical system \( (G, S, A) \), we first consider the extension

\[
S \mapsto G \to H
\]

Taking the semi-direct product, we obtain a new extension

\[
Z \ast S \mapsto Z \times G \to Z \times H
\]

We view \( Z \times T \) as a group bundle over \( Z \) with the trivial action of \( Z \times H \). The map

\[
\varphi : Z \ast S \to Z \times T \quad \text{given by} \quad \varphi(z, t) = (z, \chi(z, t))
\]

is a group bundle morphism which is equivariant in the above sense. We define the extension

\[
Z \times T \mapsto \Sigma \to Z \times H
\]

as the pushout by this morphism. Explicitly,

\[
\Sigma = \{ [\theta, z, \gamma] : \theta \in T, (z, \gamma) \in Z \times G \}
\]

where

\[
[\theta, z, t\gamma] = [\theta \chi(z, t), z, \gamma], \quad \forall (t, \gamma) \in S \ast G
\]

Note that \( \Sigma \) is a twist over the semi-direct product \( Z \times H \).

**Theorem 4.6.** Let \( (G, S, A) \) a groupoid \( C^* \)-dynamical system where \( A \) is abelian. Let \( Z \) be the spectrum of the abelian \( C^* \)-algebra \( C_0(G^{(0)}, A) \). Then the twisted crossed product \( C^*(G, S, A) \) is isomorphic to \( C^*(Z \times H, \Sigma) \), where \( \Sigma \) is the above twist.

**Proof.** This is a particular case of Corollary 4.4 but we give here an independent proof. We shall identify both \( C_c(G, S, A) \) and \( C_c(Z \times H, \Sigma) \) as \( * \)-algebras of complex-valued functions on \( Z \times G \) and observe that these \( * \)-algebras essentially coincide.

By definition, an element \( f \in C_c(G, S, A) \) is a map \( f : G \to A \), continuous with compact support modulo \( S \) and satisfying \( f(\gamma) \in A_{r(\gamma)} \) and \( f(s\gamma) = f(\gamma)\chi(s^{-1}) \). Writing \( A_{r(\gamma)} \) as \( C_0(Z_{r(\gamma)}) \), we define

\[
f(z, \gamma) = f(\gamma)(z), \quad (z, \gamma) \in Z \times G
\]

One can check that this complex-valued function defined on \( Z \times G \) is continuous, satisfies

\[
f(z, s\gamma) = f(s, \gamma)\chi(z, s^{-1}), \quad (s, \gamma) \in S \ast G
\]

there is a compact subset \( K \) of \( H \) such that \( f(z, \gamma) = 0 \) for all \((z, \gamma) \in Z \times G \) such that \( \gamma \notin K \) and for all \( \epsilon > 0 \), there exists a compact subset \( L \) of \( Z \) such that \( |f(z, \gamma)| \leq \epsilon \) for all \((z, \gamma) \in Z \times G \) such that \( z \notin L \). Conversely, every complex-valued function defined on \( Z \times G \) satisfying these conditions defines an element of \( C_c(G, S, A) \). With this identification of \( C_c(G, S, A) \), the \( * \)-algebra structure is given by

\[
f \ast g(z, \gamma) = \int f(z, \tau)g(z\tau, \tau^{-1}\gamma)d\alpha^*(\gamma)(\dot{\tau}), \quad f^*(z, \gamma) = \overline{f(z, \gamma)}
\]

On the other hand, an element of \( C_c(Z \times H, \Sigma) \) is a continuous function \( f : \Sigma \to C \) which is compactly supported modulo \( T \) (since \( T \) is compact, this equivalent to be compactly supported) and which satisfies \( f[\theta'\theta, z, \gamma] = f[\theta, z, \gamma]\theta'^{-1} \) for \( \theta, \theta' \in T \) and \((z, \gamma) \in Z \times G \).
It is completely determined by its restriction to $\theta = 1$. Thus, with a slight abuse of notation, we write $f(z, \gamma) = f[1, z, \gamma]$. We obtain a complex-valued function $f$ defined on $Z \times G$ which is continuous with compact support and satisfies

$$f(z, s\gamma) = f(z, \gamma)\chi(z, s)^{-1} \quad \forall (s, z, \gamma) \in S \times Z \times G$$

Conversely, given such a function $f$, we retrieve the original element of $C_c(Z \rtimes H, \Sigma)$ by defining $f(\theta, s, z) = f(s, z)\theta^{-1}$. When we express the product and the involution of $C_c(Z \rtimes H, \Sigma)$ in terms of these functions, we obtain the same expressions as above. Thus the elements of $C_c(G, S, A)$ and $C_c(Z \rtimes H, \Sigma)$ are both continuous functions on $Z \times G$ satisfying

$$f(z, s\gamma) = f(z, \gamma)\gamma(z, s)^{-1} \quad \forall (s, z, \gamma) \in S \times Z \times G$$

The only difference between the elements of $C_c(G, S, A)$ and those of $C_c(Z \rtimes H, \Sigma)$ is their supports. Note that, for these functions $f$, the absolute value $|f|$ is defined on $Z \rtimes H$. For $f \in C_c(Z \rtimes H, \Sigma)$, $|f|$ has compact support. Therefore, $f$ belongs to $C_c(G, S, A)$. Thus, we have realized $C_c(Z \rtimes H, \Sigma)$ as a $*$-subalgebra of $C_c(H, \Sigma, A)$. Let us return to the original description of $C_c(G, S, A)$ as the space of compactly supported continuous sections of a Banach bundle $B$ over $G$. $C_c(Z \rtimes G, \Sigma)$ is a linear subspace of $C_c(G, S, A)$. It satisfies conditions (I) and (II) of Proposition 14.6 of [Fell-Doran, vol I, page 139]. Indeed, for $h$ continuous function on $G$ and $f \in C_c(Z \rtimes G, \Sigma)$, the function $hf$ defined by

$$(hf)(z, \gamma) = h(\gamma)f(z, \gamma) \quad \forall (z, \gamma) \in Z \times G$$

belongs to $C_c(Z \rtimes G, \Sigma)$. The fibre $B_h$ of the bundle $B$ can be identified to the Banach space $C_0(Z_r(h))$. In this identification, the evaluation at $h$ of the elements of $C_c(Z \rtimes G, \Sigma)$ gives the whole subspace $C_c(Z_r(h))$, which is dense in $C_0(Z_r(h))$. Therefore, $C_c(Z \rtimes G, \Sigma)$ is dense in $C_c(G, S, A)$ in the inductive limit topology. Since $C_c(G, S, A)$ is complete in the inductive limit topology, representations of $C_c(Z \rtimes G, \Sigma)$ which are continuous in the inductive limit topology extend by continuity. Therefore, the inclusion of $C_c(Z \rtimes G, \Sigma)$ into $C_c(G, S, A)$ gives an isomorphism $C^*(Z \rtimes G, \Sigma) \simeq C^*(G, S, A)$. \hfill $\Box$

### 4.3. Abelian twisted extensions.

After this digression about abelian Fell bundles and abelian groupoid dynamical systems, we return to our initial problem, which is the analysis of a twisted groupoid $C^*$-algebra $C^*(G, \Sigma)$ in presence of a closed normal subgroupoid $S$ having a Haar system. As said earlier, we make a further assumption, whose present form I owe to Alex Kumjian.

**Definition 4.4.** We say that a twisted extension $(G, \Sigma, S)$ is abelian if

(i) the group bundle $S$ is abelian and

(ii) the group bundle $\Sigma\mid_S$ is abelian.

We stated condition (i) for convenience only since it is implied by condition (ii). When the twisted extension is abelian, the $C^*$-algebra $C^*(S, \Sigma\mid_S)$ is abelian and Corollary 3.4 can be completed. This gives our main result.

**Theorem 4.7.** Let $(G, \Sigma, S)$ be a locally compact abelian twisted groupoid extension. Then the twisted groupoid $C^*$-algebra $C^*(G, \Sigma)$ is isomorphic to the twisted groupoid $C^*$-algebra $C^*(Z \rtimes H, \Sigma)$ where $Z$ is the spectrum of $C^*(S, \Sigma\mid_S)$, $H = G/S$ and the twist $\Sigma$ is obtained by a pushout construction.
Proof. We have seen that $C^*(G, \Sigma)$ is isomorphic to the crossed product $C^*$-algebra $C^*(\Sigma, \Sigma|S, C^*(S, \Sigma|S))$ of the tautological dynamical system $(\Sigma, \Sigma|S, C^*(S, \Sigma|S))$. Since the bundle of $C^*$-algebras $C^*(S, \Sigma|S)$ is abelian, we can apply Theorem 4.6. □

For applications, it is necessary to be more explicit about the space $Z$, the action of $H$ on it and the twist $\Sigma$. Recall that we assume that the abelian group bundle $S$ has a Haar system. Therefore, we endow its dual group bundle $\hat{S}$ with the topology of the spectrum of $C^*(S)$, as in [13, Section 3]. Since the abelian group bundle $\Sigma|S$, as an extension of $S$ by $G(0) \times T$ has also a Haar system, its dual group bundle $\hat{\Sigma}|S$ has also a natural locally compact topology. An element of $\hat{S}$ [resp. $\hat{\Sigma}|S$] will be denoted by $(x, \chi)$, where $x$ is a base point and $\chi \in \hat{S}_x$ [resp. $(\hat{\Sigma}|S)_x$]. We first consider the general case of an abelian twist $(S, \Sigma)$.

Definition 4.5. Let $X \times T \rightarrow \Sigma \rightarrow S$ be an abelian twist over an abelian group bundle $S$. Its twisted spectrum is defined as

$$\hat{S}^\Sigma = \{(x, \chi) \in \hat{S} \text{ such that } \chi(\theta) = \theta \forall \theta \in T\}$$

The twisted spectrum $\hat{S}^\Sigma$ is an affine space over the dual group bundle $\hat{S}$: the action of $\hat{S}$ on $\hat{S}^\Sigma$ is the usual multiplication: given $\chi \in \hat{S}^\Sigma_x$ and $\rho \in \hat{S}_x$, $\chi \rho \in \hat{S}^\Sigma_x$ is defined by $(\chi\rho)(\sigma) = \chi(\sigma)\rho(\hat{\sigma})$ for $\sigma \in \Sigma_x$ and where $\hat{\sigma}$ is the image of $\sigma$ in $S_x$.

Remark 4.1. In [3], the authors give a similar description of the twisted spectrum when the twist $\Sigma$ is given by a symmetric 2-cocycle. Then, the twist is obviously abelian. It may be useful to recall here that, according to [8, Lemma 7.2], a Borel 2-cocycle on a locally compact abelian group is trivial if and only if it is symmetric. Moreover, if the topology of the group is second countable, every twist is given by a Borel 2-cocycle. Thus, a twist over a bundle of second countable locally compact abelian groups is abelian if and only if it is pointwise trivial.

Lemma 4.8. Let $(S, \Sigma)$ be an abelian twist over an abelian group bundle $S$. Assume that $S$ has a Haar system $\beta$. Then the twisted spectrum $\hat{S}^\Sigma$ is the spectrum of the abelian $C^*$-algebra $C^*(S, \Sigma)$.

Proof. When $S$ is an abelian group and the twist $\Sigma$ is trivial, this is, by the choice of a trivialization, the well-known result that $\hat{S}$ is the spectrum of $C^*(S)$. The explicit correspondence between $\hat{S}^\Sigma$ and the spectrum of $C^*(S, \Sigma)$ is given by

$$\chi(f) = \int f(\sigma)\chi(\sigma)d\beta(\hat{\sigma})$$

for $\chi \in S^\Sigma$ and $f \in C_c(S, \Sigma)$.

When $S$ is an abelian group bundle, the $C^*$-algebra $C^*(S, \Sigma)$ is the $C^*$-algebra defined by the continuous field of $C^*$-algebras $x \mapsto C^*(S_x, \Sigma|S_x)$. As a set, its spectrum is the disjoint union over $X$ of the above spectra, which is $\hat{S}^\Sigma$. □

This turns $\hat{S}^\Sigma$ into a locally compact space (in fact, a locally compact affine bundle). We now return to our situation. We denote by $\hat{S}^\Sigma$ rather than $\hat{S}^\Sigma|S$ the twisted spectrum.
of \((S, \Sigma|S)\). Let us describe the action of \(H = G/S\) on \(\hat{S}^\Sigma\). The groupoid \(H\) acts on the group bundle \(\Sigma|S\) by conjugation: \(h.\sigma = \tau \sigma \tau^{-1}\), where \(\pi_\Sigma(\tau) = h\). The transposed action on the dual group bundle \(\hat{\Sigma}|S\), defined by \((\chi h)(\sigma) = \chi(\sigma \sigma^{-1})\), preserves the twisted spectrum \(\hat{S}^\Sigma\). It is easily checked that this is the action arising from the action of \(H\) on the bundle of C*-algebras \(C^*(S, \Sigma|S)\).

The above pushout diagram defining the twist \(\Sigma\) becomes:

\[
\begin{array}{ccc}
\hat{S}^\Sigma \times \Sigma |S & \longrightarrow & \hat{S}^\Sigma \times H \\
\varphi & \downarrow & \downarrow \\
\hat{S}^\Sigma \times T & \longrightarrow & \Sigma |S \times T \longrightarrow \hat{S}^\Sigma \times H
\end{array}
\]

where \(\varphi(\chi, \sigma) = (\chi, \chi(\sigma))\) for \((\chi, \sigma) \in \hat{S}^\Sigma \times \Sigma|S\). Explicitly, \(\Sigma\) is the quotient of the groupoid \((\hat{S}^\Sigma \times \Sigma) \times T\) by the equivalence relation

\[(\chi, \sigma \tau, \theta) \sim (\chi, \tau, \chi(\sigma) \theta), \quad \forall \sigma \in \Sigma|S.\]

**Remark 4.2.** In [7, Proposition 3.5], the above Theorem 4.7 (the twisted case) is deduced from the similar result for the untwisted case, established in the previous work [6, Theorem 3.3].

5. An application: deformation quantization

Rieffel has introduced a notion of C*-algebraic deformation quantization and illustrated it by a number of examples in [18]. On the other hand, Ramazan, generalizing Connes’ tangent groupoid, has produced deformation quantization of Lie-Poisson manifolds by using groupoid techniques (see [15, 11]). Our Theorem 4.7 shows that the two approaches agree on some important examples. We consider here the basic example of a symplectic finite-dimensional real vector space \((V, \omega)\). Then, for every \(h \in \mathbb{R}\), \(\sigma_h = e^{ih\omega/2}\) is a \(T\)-valued 2-cocycle on \((V, +)\). It is shown in [19] that \(h \mapsto C^*(V, \sigma_h)\) can be made into a continuous field of C*-algebras and in [20] that it gives a C*-algebraic deformation quantization of the Lie-Poisson manifold \((V, \omega)\). Note that its sectional algebra can be viewed as a twisted groupoid C*-algebra \(C^*(G, \Sigma)\), where \(G\) is the trivial group bundle \(R \times V\) over \(R\) and \(\Sigma\) is the twist defined by the 2-cocycle \(\sigma(h, .) = \sigma_h\). Let \(V = L \oplus L'\) be a direct sum decomposition, where \(L\) and \(L'\) are complementary Lagrangian subspaces. This gives the extension

\[R \times L \rightarrow G \rightarrow R \times L'\]

The abelian group bundle \(S = R \times L\) satisfies the conditions of Definition 3.2 with respect to the twisted groupoid \((G, \Sigma)\). Therefore, by Theorem 4.7, \(C^*(G, \Sigma)\) is isomorphic to \(C^*(Z \rtimes (R \times L'), \Sigma)\), where \(Z\) is the twisted spectrum and the twist \(\Sigma\) is obtained by the pushout construction. Let us determine them explicitly. The action of \(H = R \times L'\) on \(\Sigma|S = R \times L \times T\) is given by

\[(h, y, (h, x, \theta)) = (h, x, e^{-ih\omega(x,y)} \theta), \quad \text{where} \quad h \in \mathbb{R}, \quad y \in L', \quad x \in L, \quad \theta \in T\]
Since $\Sigma_{LS} = R \times L \times T$, the twisted spectrum $Z$ is $R \times \hat{L}$, where $\hat{L}$ denotes the dual group of the abelian locally compact group $L$. The action of $H$ on $Z$ is given by

$$(h, \chi)(h, y) = (h, \chi\varphi_h(y)), \quad \text{where} \quad h \in R, \chi \in \hat{L}, y \in L'$$

and for $h \in R$, $\varphi_h$ is the group homomorphism from $L'$ to $\hat{L}$ such that

$$< \varphi_h(y), x > = e^{-i\theta(x,y)} \quad \text{where} \quad x \in L, y \in L'$$

The semi-direct product $Z \rtimes H$ is a bundle of semi-direct products $\hat{L} \rtimes_h L'$. For $h \neq 0$, $\varphi_h$ is an isomorphism and $\hat{L} \rtimes_h L'$ is isomorphic to the trivial groupoid $\hat{L} \times \hat{L}$. For $h = 0$, we get $\hat{L} \times L'$, where the first term is a space and the second is a group. We use again $\omega$ to identify $L'$ and the dual $L^*$, which is the tangent space of $\hat{L}$. Thus $\hat{L} \times L'$ is isomorphic to the tangent bundle $T\hat{L}$ and the groupoid $Z \rtimes H$ is isomorphic to the tangent groupoid of the manifold $\hat{L}$. One can check that we have an isomorphism of topological groupoids.

It remains to determine the twist $\Sigma$.

**Proposition 5.1.** The above twist is trivial.

**Proof.** By construction, $\Sigma$ is the quotient of $(Z \rtimes \Sigma) \times T$ by the equivalence relation

$$(h, \chi, x + v, \sigma_h(x,v)\varphi_\psi, \chi(x)\varphi_\theta) \sim (h, \chi, v, \psi, \theta)$$

where $h \in R$, $\chi \in \hat{L}$, $x \in L$, $v \in V$ and $\varphi_\psi, \psi, \theta \in T$. The map

$$(Z \rtimes \Sigma) \times T \to (Z \rtimes H) \times T$$

sending $(h, \chi, x + y, \psi, \theta)$ to $(h, \chi, y, \overline{\psi\chi(x)\sigma_h(x,y)}\theta)$ where $h \in R$, $\chi \in \hat{L}$, $x \in L$, $y \in L'$, and $\varphi_\psi, \psi, \theta \in T$ identifies topologically this quotient. This is also a groupoid homomorphism. Therefore, $\Sigma$ is isomorphic to $(Z \rtimes H) \times T$.

The above example can also be presented via the Heisenberg group $\mathcal{H} = V \times R$ with multiplication $(v, s)(w, t) = (v + w, \omega(v, w) + s + t)$. Mackey’s normal subgroup analysis (i.e. Theorem 4.7) applied to the center $\{0\} \times R$ gives the first deformation. The second deformation can be obtained by applying this analysis to the subgroup $L \times R$. In conclusion, we have three isomorphic $C^*$-algebras: $C^*(\mathcal{H})$, $C^*(G, \Sigma)$ and $C^*(Z \rtimes H)$.

**References**

[1] A. Buss and R. Meyer: *Iterated crossed products for groupoid fibrations*, arXiv:1604.02015
[2] V. Deaconu, A. Kumjian and B. Ramazan: *Fell bundles associated to groupoid morphisms*, Math. Scand. 102 (2008) no 2. 305-319.
[3] A. Duwenig, E. Gillaspie and R. Norton: *Analyzing the Weyl construction for dynamical Cartan subalgebras*, arXiv 2010:04137.
[4] J. Fell and R. Doran: *Representations of $*$-Algebras*, vol 1, Academic Press.
[5] P. Green: *The local structure of twisted covariance algebras*, Acta Mat. 140 (1978).
[6] M. Ionescu, A. Kumjian, J. Renault, A. Sims and D. Williams: *$C^*$-algebras of extensions of groupoids by group bundles*, arXiv 2001:01312.
[7] M. Ionescu, A. Kumjian, J. Renault, A. Sims and D. Williams: *Pushouts of group bundles*, in preparation.
[8] A. Kleppner: *Multipliers on abelian groups*, Math. Ann. 158 (1965), 11–34.
[9] A. Kumjian: *On $C^*$-diagonals*, Can. J. Math. 38 (1986).
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[10] A. Kumjian: *Fell bundles over groupoids*, Proc. Amer. Math. Soc. **126**, 4 (1998).

[11] N. P. Landsman and B. Ramazan: *Quantization of Poisson algebras associated to Lie algebroids*, Contemporary Mathematics **282** (2001), 159–192.

[12] P. Muhly: * Bundles over groupoids*, Contemporary Mathematics **282** (2001), 67–82.

[13] P. Muhly, J. Renault and D. Williams: *Continuous trace groupoid C*-algebras III*, Trans. Amer. Math. Soc. **348** (1996), 3621–3641.

[14] I. Raeburn: *On the Picard group of a continuous trace C*-algebra*, Trans. Amer. Math. Soc. **263**, 1 (1981), 183–205.

[15] B. Ramazan: *Quantification par déformation des variétés de Lie-Poisson*, Ph.D thesis, University of Orléans, 1998.

[16] J. Renault: *Représentations des produits croisés d’algèbres de groupoïdes*, J. Operator Theory, **25** (1987), 3–36.

[17] J. Renault: *Cartan subalgebras in C*-algebras*, Irish Math. Soc. Bulletin **61** (2008), 29–63.

[18] M. Rieffel: *Deformation quantization of Heisenberg manifolds*, Comm. Math. Phys. **122** (1989), 531–562.

[19] M. Rieffel: *Continuous fields of C*-algebras coming from group cocycles and actions*, Math. Ann. **283** (1989), 631–643.

[20] M. Rieffel: *Deformation quantization for actions of $\mathbb{R}^d$*, Memoirs of the Amer. Math. Soc. **106**, Number 506, (1993).

[21] J. Westman: *Harmonic analysis on groupoids*, Pacific J. Math., **87** (1980), 389–454.

[22] S. Yamagami: *On primitive ideal spaces of C*-algebras over certain locally compact groupoids*, Mappings of operator algebras (H. Araki and R. Kadison, eds), Progress in Math, Vol. **84**, Birkhäuser, Boston (1991), 199–204.

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