OPTIMIZATION PROBLEM FOR EXTREMALS OF THE TRACE INEQUALITY IN DOMAINS WITH HOLES

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Abstract. We study the Sobolev trace constant for functions defined in a bounded domain $\Omega$ that vanish in the subset $A$. We find a formula for the first variation of the Sobolev trace with respect to hole. As a consequence of this formula, we prove that when $\Omega$ is a centered ball, the symmetric hole is critical when we consider deformation that preserve volume but is not optimal for some case.

1. Introduction and Main Results.

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$ and $1 < p < \infty$. We denote by $p^*$ the critical exponent for the Sobolev trace immersion given by $p^* = p(N - 1)/(N - p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$.

For any $A \subset \Omega$, which is a smooth open subset, we define the space $W_{1,p}^1(\Omega) = C_\infty^0(\Omega \setminus A)$, where the closure is taken in $W_{1,p}^{-}$-norm. By the Sobolev Trace Theorem, there is a compact embedding

$$W_{1,p}^1(\Omega) \hookrightarrow L^q(\partial \Omega),$$

for all $1 < q < p^*$. Thus, given $1 < q < p^*$, there exist a constant $C = C(q,p)$ such that

$$C \left\{ \int_{\partial \Omega} |u|^q \, dS \right\}^{\frac{p}{q}} \leq \int_{\Omega} |\nabla u|^p + |u|^p \, dx.$$ 

The best (largest) constant in the above inequality is given by

$$S_q(A) := \inf_{u \in W_{1,p}^1(\Omega) \setminus W_{0,1}^1(\Omega)} \left\{ \int_{\Omega} |\nabla u|^p + |u|^p \, dx \right\} \left\{ \int_{\partial \Omega} |u|^q \, dS \right\}^{\frac{p}{q}}.$$ 

By (1.1), there exist an extremal for $S_q(A)$. Moreover, an extremal for $S_q(A)$ is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \setminus A, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial \Omega, \\ u = 0 & \text{on } \partial A, \end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the usual $p$-laplacian, $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative and $\lambda$ depends on the normalization of $u$. When $\|u\|_{L^q(\partial \Omega)} = 1$ we have that $\lambda = S_q(A)$. Moreover, when $p = q$ problem (1.3) becomes homogeneous and therefore is a nonlinear eigenvalue problem. In this case, the first eigenvalue of (1.3) coincides with the best Sobolev trace constant $S_q(A) = \lambda_1(A)$ and it is shown...
in [9] that it is simple (see also [3]). Therefore, if \( p = q \), the extremal for \( S_p(A) \) is unique up to constant factor. In the linear setting, i.e. when \( p = q = 2 \), this eigenvalue problem is known as the Steklov eigenvalue problem, see [10].

It is the purpose of this article to analyze the dependance of the Sobolev trace constant \( S_q(A) \) with respect to variations on the set \( A \). To this end, we compute the so-called shape derivative of \( S_q(A) \) with respect to regular perturbations of the hole \( A \).

Let \( V : \mathbb{R}^N \to \mathbb{R}^N \) be a regular (smooth) vector field, globally Lipschitz, with support in \( \Omega \) and let \( \psi_t : \mathbb{R}^N \to \mathbb{R}^N \) be defined as the unique solution to

\[
\left\{ \begin{array}{ll}
\frac{d}{dt} \psi_t(x) = V(\psi_t(x)) & t > 0 \\
\psi_0(x) = x & x \in \mathbb{R}^N.
\end{array} \right.
\]

We have

\[
\psi_t(x) = x + tV(x) + o(t) \quad \forall x \in \mathbb{R}^N.
\]

Now, we define \( A_t := \psi_t(A) \subset \Omega \) for all \( t > 0 \) and

\[
S_q(t) = \inf_{u \in W^{1,p}_0(\Omega) \setminus W^{1,p}_0(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left( \int_{\partial \Omega} |u|^q \, dS \right)^{\frac{p}{q}}}.
\]

Observe that \( A_0 = A \) and therefore \( S_q(0) = S_q(A) \).

In [2] Fernández Bonder, Groisman and Rossi analyze this problem in the linear case \( p = q = 2 \) and prove that \( S_2(t) \) is differentiable with respect to \( t \) at \( t = 0 \) and it holds

\[
\left. \frac{d}{dt} S_2(t) \right|_{t=0} = -\int_{\partial A} \left( \frac{\partial u}{\partial \nu} \right)^2 \langle V, \nu \rangle \, dS,
\]

where \( u \) is a normalized eigenfunction for \( S_2(A) \) and \( \nu \) is the exterior normal vector to \( \Omega \setminus \overline{A} \).

Furthermore, in the case that \( \Omega \) is the ball \( B_R \) with center 0 and radius \( R > 0 \) the authors show that a centered ball \( A = B_r, r < R \), is critical in the sense that \( S_2(A) = 0 \) when considering deformations that preserves volume and that this configuration is not optimal.

We say that hole \( A^* \) is optimal for the parameter \( \alpha, 0 < \alpha < |\Omega| \), if \( |A^*| = \alpha \) and

\[
S_q(A^*) = \inf_{A \subset \Omega, |A| = \alpha} S_q(A).
\]

Therefore there is a lack of symmetry in the optimal configuration.

Here we extend these results to the more general case \( 1 < p < \infty \) and \( 1 < q < p^* \).

Our method differs from the one in [2] in order to deal with the nonlinear character of the problem.

Our first result states

**Theorem 1.1.** Suppose \( A \subset \overline{\Omega} \) is a smooth open subset and let \( 1 < q < p^* \). Then, with the previous notation, we have that \( S_q(t) \) is differentiable at \( t = 0 \) and there exists \( u \) a normalized extremal for \( S_q(A) \) such that

\[
S_q'(0) = -\int_{\partial A} \left( \frac{\partial u}{\partial \nu} \right)^p \langle V, \nu \rangle \, dS,
\]

where \( S_q'(0) = \left. \frac{d}{dt} S_q(t) \right|_{t=0} \) and \( \nu \) is the exterior normal vector to \( \Omega \setminus \overline{A} \).
Remark 1.2. If $u$ is an extremal for $S_q(A)$ we have that $|v|$ is also an extremal associated to $S_q(A)$. Then in the previous theorem we can suppose that $u \geq 0$ in $\Omega$. Moreover, by [8], we have that $u \in C^{1,\alpha}(\overline{\Omega})$ and if $\Omega$ satisfies the interior ball condition for all $x \in \partial \Omega$ then $u > 0$ on $\partial \Omega$, see [11].

In the case that $\Omega = B_R$, we have the next result

Theorem 1.3. Let $\Omega = B_R$ and let the hole be a centered ball $A = B_r$. Then, if $1 < q \leq p$, this configuration is critical in the sense that $S_q'(B_r) = 0$ for all deformations $V$ that preserve the volume of $B_r$.

But, if $q$ is sufficiently large, the symmetric hole with a radial extremal is not an optimal configuration. In fact, we prove

Theorem 1.4. Let $r > 0$ and $1 < p < \infty$ be fixed. Let $R > r$ and

$$Q(R) = \frac{1}{S_p(B_r)^{\frac{r}{R}}} \left( 1 - \frac{N - 1}{R} S_p(B_r) \right) + 1.$$

If $q > Q(R)$ then the centered hole $B_r$ is not optimal.

Finally, to study the asymptotic behavior of $Q(R)$

Proposition 1.5. The function $Q(R)$ has the following asymptotic behavior

$$\lim_{R \to r} Q(R) = 1^- \quad \text{and} \quad \lim_{R \to +\infty} Q(R) = p$$

Observe that $Q(R) < 1$ for $R$ close to $r$ and therefore the symmetric hole with a radial extremal is not an optimal configuration for $R$ close to $r$.

2. Proof of Theorem 1.1

2.1. Preliminary Results. The proof of Theorem 1.1 require some technical results. In this subsection we use some ideas from [4].

Given $u \in W^{1,p}_A(\Omega) \setminus W^{1,p}_0(\Omega)$ we consider $v = u \circ \psi_t$, so $v \in W^{1,p}_A(\Omega) \setminus W^{1,p}_0(\Omega)$ and $\nabla v^T = T \psi_t' \nabla (u \circ \psi_t)^T$, where $\psi_t'$ denotes the differential matrix of $\psi_t$ and $T A$ is the transpose of matrix $A$. Thus, by the change of variables formula, we have that

$$\int_{\Omega} |\nabla u|^p + |u|^p \, dx = \int_{\Omega} \{ |T \psi_t'|^{-1} \nabla v^T|^p + |v|^p \} J(\psi_t) \, dx,$$

here $J(\psi_t)$ is the usual Jacobian of $\psi_t$. Moreover, since $\text{supp}(V) \subset \Omega$, we have that

$$\int_{\partial \Omega} |u|^q \, dS = \int_{\partial \Omega} |v|^q \, dS.$$

In [5] are proved the following asymptotic formulas

$$[\psi_t']^{-1} = I d - t V'(x) + o(t),$$

$$J(\psi_t)(x) = 1 + t \text{div} V(x) + o(t).$$

Then, by (2.7) and (2.8), we have

$$\int_{\Omega} |v|^p J(\psi_t) \, dx = \int_{\Omega} |v|^p \{ 1 + t \text{div} V + o(t) \} \, dx$$

$$= \int_{\Omega} |v|^p \, dx + t \int_{\Omega} |v|^p \text{div} V \, dx + o(t)$$
Therefore, we can rewrite (1.5) as

\[ \int_{\Omega} |T[\psi']^{-1}\nabla v^p J(\psi_t) \, dx = \int_{\Omega} |[I - t T V' + o(t)]\nabla v^p \{1 + t \text{div} V + o(t)\} \, dx \]

\[ = \int_{\Omega} |\nabla v - t T V' \nabla v^p + o(t)|^p \{1 + t \text{div} V + o(t)\} \, dx, \]

since

\[ |\nabla v - t T V' \nabla v^p + o(t)|^p = |\nabla v|^p - pt|\nabla v|^{p-2} \langle \nabla v, T V' \nabla v^p \rangle + o(t) \]

we obtain that

\[ \int_{\Omega} |T[\psi']^{-1}\nabla v^p J(\psi_t) \, dx = \int_{\Omega} |\nabla v|^p \, dx + t \int_{\Omega} |\nabla v|^p \text{div} V \, dx \]

\[ - pt \int_{\Omega} |\nabla v|^{p-2} \langle \nabla v, T V' \nabla v^p \rangle \, dx + o(t). \]

Thus, we conclude

\[ \int_{\Omega} |\nabla v|^p + |u|^p \, dx = \int_{\Omega} \{ |T[\psi']^{-1}\nabla v^p |^p + |v|^p \} J(\psi_t) \, dx \]

\[ = \int_{\Omega} |v|^p \, dx + t \int_{\Omega} |\nabla v|^p \, dx + t \int_{\Omega} \{ |\nabla v|^p + |v|^p \} \text{div} V \, dx \]

\[ - pt \int_{\Omega} |\nabla v|^{p-2} \langle \nabla v, T V' \nabla v^p \rangle \, dx + o(t). \]

Therefore, we can rewrite (1.5) as

\[ (2.9) \quad S_q(t) = \inf_{v \in W_A^1, \Omega \cap W_0^{1,p} (\Omega)} \{ \rho(v) + t \gamma(v) \} \]

where

\[ \rho(v) = \frac{\int_{\Omega} |\nabla v|^p + |v|^p \, dx}{\left( \int_{\partial \Omega} |v|^q \, dS \right)^{p/q}}, \]

and

\[ \gamma(v) = \frac{\int_{\Omega} |\nabla v|^p + |v|^p \} \text{div} V \, dx - p \int_{\partial \Omega} |\nabla v|^{p-2} \langle \nabla v, T V' \nabla v^p \rangle \, dx}{\left( \int_{\partial \Omega} |v|^q \, dS \right)^{p/q}} + O(t). \]

Given \( t \geq 0 \), let \( v_t \in W_A^{1,p} (\Omega) \setminus W_0^{1,p} (\Omega) \) such that \( \|v_t\|_{L^q(\partial \Omega)} = 1 \) and

\[ S_q(t) = \varphi(t) + t \phi(t), \]

where

\[ \varphi(t) = \rho(v_t) \quad \text{and} \quad \phi(t) = \gamma(v_t) \quad \forall t \geq 0. \]

We observe that \( \varphi, \phi : \mathbb{R}_{\geq 0} \to \mathbb{R} \)

**Lemma 2.1.** The function \( \phi \) is nonincreasing.

**Proof.** Let \( 0 \leq t_1 \leq t_2 \). By (2.9), we have that

\[ (2.10) \quad \varphi(t_2) + t_1 \phi(t_2) \geq S_q(t_1) = \varphi(t_1) + t_1 \phi(t_1) \]

\[ (2.11) \quad \varphi(t_1) + t_2 \phi(t_1) \geq S_q(t_2) = \varphi(t_2) + t_2 \phi(t_2). \]

Subtracting (2.10) from (2.11), we get

\[ (t_2 - t_1) \phi(t_1) \geq (t_2 - t_1) \phi(t_2). \]
Since $t_2 - t_1 \geq 0$, we obtain
\[
\phi(t_1) \leq \phi(t_2).
\]
This ends the proof. \hfill \Box

**Remark 2.2.** Since $\phi$ is nonincreasing, we have
\[
\phi(t) \leq \phi(0) \quad \forall t \geq 0,
\]
and there exists
\[
\phi(0^+) = \lim_{t \to 0^+} \phi(t).
\]

**Corollary 2.3.** The function $\varphi$ is nondecreasing.

**Proof.** Let $0 \leq t_1 \leq t_2$. Again, by (2.9), we have that
\[
\varphi(t_2) + t_1 \phi(t_2) \geq S_q(t_1) = \varphi(t_1) + t_1 \phi(t_1)
\]
so
\[
\varphi(t_2) - \varphi(t_1) \geq t_1 (\phi(t_1) - \phi(t_2)).
\]
Since $0 \leq t_1 \leq t_2$, by Lemma 2.1, we have that $\phi(t_1) - \phi(t_2) \geq 0$. Then
\[
\varphi(t_2) - \varphi(t_1) \geq 0
\]
that is what we wished to prove. \hfill \Box

Now we can prove that $S_q(t)$ is continuous at $t = 0$.

**Theorem 2.4.** The function $S_q(t)$ is continuous at $t = 0$, i.e.,
\[
\lim_{t \to 0^+} S_q(t) = S_q(0).
\]

**Proof.** Given $t \geq 0$ so, by Corollary 2.3,
\[
S_q(t) - S_q(0) = \varphi(t) + t\phi(t) - \varphi(0) \geq t\phi(t).
\]
On the other hand, by (2.9), we have that
\[
S_q(t) \leq \varphi(0) + t\phi(0) = S_q(0) + t\phi(0).
\]
Then
\[
t\phi(t) \leq S_q(t) - S_q(0) \leq t\phi(0).
\]
Thus, by Remark 2.2,
\[
\lim_{t \to 0^+} S_q(t) - S_q(0) = 0.
\]
This finishes the proof. \hfill \Box

Thus, from Remark 2.2 and Theorem 2.4, we obtain the following corollary:

**Corollary 2.5.** The function $\varphi$ is continuous at $t = 0$, i.e.,
\[
\lim_{t \to 0^+} \varphi(t) = \varphi(0).
\]

**Proof.** We observe that
\[
\varphi(t) - \varphi(0) = S_q(t) - S_q(0) - t\phi(t)
\]
then, by Remark 2.2 and Theorem 2.4,
\[
\lim_{t \to 0^+} \varphi(t) - \varphi(0) = 0.
\]
That proves the result. \hfill \Box
Finally, we prove the following:

**Theorem 2.6.** The function \( \varphi \) is differentiable at \( t = 0 \) and

\[
\frac{d \varphi}{dt}(0) = 0.
\]

**Proof.** Let \( 0 < r < t \). By (2.9), we get

\[
S_q(r) = \varphi(r) + r\phi(r) \leq \varphi(t) + r\phi(t),
\]

and

\[
S_q(t) = \varphi(t) + t\phi(t) \leq \varphi(r) + t\phi(r).
\]

So

\[
\frac{r}{t}(\phi(r) - \phi(t)) \leq \frac{\varphi(t) - \varphi(r)}{t} \leq \phi(r) - \phi(t)
\]

hence, taking limits when \( r \to 0^+ \), by Remark 2.2 and Corollary 2.1, we have that

\[
0 \leq \frac{\varphi(t) - \varphi(0)}{t} \leq \phi(0^+) - \phi(t).
\]

Now, taking limits when \( t \to 0^+ \), and again by Remark 2.2, we get

\[
\lim_{t \to 0^+} \frac{\varphi(t) - \varphi(0)}{t} = 0
\]

as we wanted to show. \( \square \)

2.2. **Proof of Theorem 1.1.** We proceed in three steps.

**Step 1.** We show that \( S_q(t) \) is differentiable at \( t = 0 \) and

\[
S_q'(0) = \phi(0^+).
\]

We have that

\[
\frac{S_q(t) - S_q(0)}{t} = \frac{\varphi(t) - \varphi(0)}{t} - \phi(t).
\]

Then, by Remark 2.2 and Theorem 2.6,

\[
S_q'(0) = \lim_{t \to 0^+} \frac{S_q(t) - S_q(0)}{t} = \phi(0^+).
\]

**Step 2.** We show that there exists \( u \) extremal for \( S_q(A) \) such that \( \|u\|_{L^q(\partial \Omega)} = 1 \) and

\[
\phi(0^+) = \int_\Omega (|\nabla u|^p + |u|^p) \text{div} V \, dx - p \int_\Omega |\nabla u|^{p-2} \langle \nabla u, TV' \nabla u \rangle \, dx.
\]

By Theorem 2.1

(2.13) \( \|v_t\|_{W^{1,p}(\Omega)}^p = \varphi(t) - \varphi(0) = S_q(0) \) when \( t \to 0^+ \).

Then there exists \( u \in W^{1,p}(\Omega) \) and \( t_n \to 0^+ \) when \( n \to \infty \) such that

(2.14) \( v_{t_n} \to u \) weakly in \( W^{1,p}(\Omega) \),

(2.15) \( v_{t_n} \to u \) strongly in \( L^p(\partial \Omega) \),

(2.16) \( v_{t_n} \to u \) a.e. in \( \Omega \).

By (2.15) and (2.16), \( u \in W^{1,p}_A(\Omega) \) and \( \|u\|_{L^q(\partial \Omega)} = 1 \) and by (2.14)

\[
S_q(0) = \lim_{n \to \infty} \|v_{t_n}\|_{W^{1,p}(\Omega)}^p \geq \|u\|_{W^{1,p}(\Omega)}^p \geq S_q(0),
\]
then
\begin{equation}
S_q(0) = \| u \|^p_{W^{1,p}(\Omega)}.
\end{equation}
Moreover, by (2.13), (2.14) and (2.17), we have that
\[ v_{t_n} \to u \text{ strongly in } W^{1,p}(\Omega). \]

Therefore
\[
\phi(0^+) = \lim_{n \to \infty} \phi(v_{t_n}) = \int_\Omega (|\nabla u|^p + |u|^p) \text{div} V \, dx - p \int_\Omega |\nabla u|^{p-2} \langle \nabla u, T V' \nabla u^T \rangle \, dx.
\]

**Step 3.** Finally, we show that
\[
S_q'(0) = \int_\Omega (|\nabla u|^p + |u|^p) \text{div} V \, dx - p \int_\Omega |\nabla u|^{p-2} \langle \nabla u, T V' \nabla u^T \rangle \, dx
= - \int_{\partial A} \frac{\partial u}{\partial \nu}^p \langle V, \nu \rangle \, dS.
\]

To show this we require that \( u \in C^2 \). However, this is not true. Since \( u \) is an extremal for \( S_q(A) \) and \( \| u \|_{L^q(\Omega)} = 1 \), we known that \( u \) is weak solution to
\[
\begin{cases}
-\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega \setminus \overline{A}, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = S_q(A) |u|^{q-2} u & \text{on } \partial \Omega, \\
u = 0 & \text{on } \partial A,
\end{cases}
\]
and by [8] we get that \( u \) belongs to the class \( C^{1,\delta} \) for some \( 0 < \delta < 1 \).

In order to overcome this difficulty, we proceed as follows. We consider the regularized problem
\begin{equation}
\begin{cases}
div((|\nabla u|^2 + \varepsilon^2)^{(p-2)/2}) + |u|^p - 2 u \varepsilon = 0 & \text{in } \Omega \setminus \overline{A}, \\
|\nabla u|^2 + \varepsilon^2 (p-2) \frac{\partial u}{\partial \nu} = S_q(A) |u|^{q-2} u & \text{on } \partial (\Omega \setminus \overline{A}),
\end{cases}
\end{equation}
It is well known that the solution \( u^\varepsilon \) to (2.18) is of class \( C^2,\rho \) for some \( 0 < \rho < 1 \) (see [6]).

Then, we can perform all of our computations with the functions \( u^\varepsilon \) and pass to the limit as \( \varepsilon \to 0 \) at the end.

We have chosen to work formally with the function \( u \) in order to make our arguments more transparent and leave the details to the reader. For a similar approach, see [4].

Since
\[
\begin{align*}
div(|u|^p V) &= |u|^p \text{div} V + p|u|^{p-2} u \langle \nabla u, V \rangle, \\
div(|\nabla u|^p V) &= |\nabla u|^p \text{div} V + p|u|^{p-2} \langle \nabla u \nabla^2 u, V \rangle,
\end{align*}
\]
we have that
\[
\int_\Omega (|\nabla u|^p + |u|^p) \text{div} V \, dx = \int_\Omega \text{div}(|u|^p V + |\nabla u|^p V) \, dx
- p \int_\Omega \{ |u|^{p-2} u_0 \langle \nabla u, V \rangle + |\nabla u|^{p-2} \langle \nabla u \nabla^2 u, u V \rangle \} \, dx.
\]
Integrating by parts, we obtain
\[
\int_{\Omega} \text{div}( |u|^p V + |\nabla u|^p V) \, dx = \int_{\partial \Omega} (|u|^p + |\nabla u|^p) \langle V, \nu \rangle \, dS - \int_{\partial A} (|u|^p + |\nabla u|^p) \langle V, \nu \rangle \, dS
\]
\[
= - \int_{\partial A} |\nabla u|^p \langle V, \nu \rangle \, dS.
\]
where the last equality follows from the fact that supp(V) \subset \Omega and u = 0 on \partial A.

Thus
\[
S_q'(0) = - \int_{\partial A} |\nabla u|^p \langle V, \nu \rangle \, dS - p \int_{\Omega} |u|^{p-2} u \langle \nabla u_0, V \rangle \, dx
\]
\[
- p \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, T V' \nabla u + T D^2 u V^T \rangle \, dx
\]
\[
= - \int_{\partial A} |\nabla u|^p \langle V, \nu \rangle \, dS - p \int_{\Omega} |u_0|^{p-2} u \langle \nabla u, V \rangle \, dx
\]
\[
- p \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla (\langle \nabla u, V \rangle) \rangle \, dx.
\]
Since u is a weak solution of (1.3) as \( \lambda = S_q(0) \) and supp(V) \subset \Omega we have
\[
S_q'(0) = - \int_{\partial A} |\nabla u|^p \langle V, \nu \rangle \, dS.
\]
Then, noticing that \( \nabla u = \frac{\partial u}{\partial \nu} \nu \), the proof is complete. \( \Box \)

3. LACK OF SYMMETRY IN THE BALL

In this section we consider the case where \( \Omega = B_R \) and \( A = B_r \) with \( r < R \) and show Theorem 1.3, Theorem 1.4 and Proposition 1.5. The proofs are based on the argument of [2] and [7] adapted to our problem. In order to simplify notations, we write \( S_q(r) \) instead of \( S_q(B_r) \).

First we prove Theorem 1.3, for this we need the following proposition

**Proposition 3.1.** Let \( 1 < q < p \). The nonnegative solution of (1.3) is unique.

**Proof.** Suppose that there exist two nonnegative solutions \( u \) and \( v \) of (1.3). By Remark 1.2 it follows that \( u, v > 0 \) on \( \partial \Omega \). Let \( v_n = v + \frac{1}{n} \) with \( n \in \mathbb{N} \), using first Picone’s identity (see [1]) and the weak formulation of (1.3) we have
\[
0 \leq \int_{B_R} |\nabla u|^p \, dx - \int_{B_R} |\nabla v_n|^{p-2} \nabla v_n \nabla \left( \frac{u^p}{v_n^{p-1}} \right) \, dx
\]
\[
= \int_{B_R} |\nabla u|^p \, dx - \int_{B_R} |\nabla u|^{p-2} \nabla u \nabla \left( \frac{u^p}{v_n^{p-1}} \right) \, dx
\]
\[
= - \int_{B_R} u^p \, dx + \lambda \int_{\partial B_R} u^q \, dS + \int_{B_R} v^{p-1} \frac{u^p}{v_n^{p-1}} \, dx - \lambda \int_{\partial B_R} v^q \frac{u^p}{v_n^{p-1}} \, dS
\]
\[
\leq \lambda \int_{\partial B_R} u^q \, dS - \lambda \int_{\partial B_R} v^{q-1} \frac{u^p}{v_n^{p-1}} \, dS.
\]
Thus, by the Monotone Convergence Theorem,

\[
0 \leq \int_{\partial B_R} u^q \, dS - \int_{\partial B_R} v^q \frac{u^{p-1}}{v^{p-1}} \, dS = \int_{\partial B_R} u^q (u^{q-1} - v^{q-1}) \, dS.
\]

Note that the role of \(u\) and \(v\) in the above equation are exchangeable. Therefore, subtracting we get

\[
0 \leq \int_{\partial B_R} (u^q - v^q) (u^{q-1} - v^{q-1}) \, dS.
\]

Since \(q < p\) we have that \(u \equiv v\) on \(\partial B_R\). Then, by uniqueness of solution to the Dirichlet problem, we get \(u \equiv v\) in \(B_R\).

\[\Box\]

**Remark 3.2.** As the problem (1.3) is rotationally invariant, by uniqueness we obtain that the nonnegative solution of (1.3) must be radial. Therefore, if \(\Omega = B_R\), \(A = B_r\) and \(1 < q \leq p\) we can suppose that the extremal for \(S_q(r)\) found in the Theorem 1.1 is nonnegative and radial.

Now we can prove the Theorem 1.3,

**Proof of Theorem 1.3.** We consider \(\Omega = B_R\), \(A = B_r\) and \(1 < q \leq p\). By Theorem 1.3 and Remark 3.2 there exist a nonnegative and radial normalized extremal for \(S_q(r)\) such that

\[
S_q'(0) = - \int_{\partial B_r} \frac{\partial u}{\partial \nu} \langle V, \nu \rangle \, dS.
\]

Since \(u\) is radial

\[
\frac{\partial u}{\partial \nu} \equiv c \text{ on } \partial B_r,
\]

where \(c\) is a constant.

Thus, using that we are dealing with deformations \(V\) that preserves the volume of the \(B_r\), we have that

\[
S_q'(0) = -c^p \int_{\partial B_r} \langle V, \nu \rangle \, dS = c^p \int_{B_r} \text{div}(V) \, dx = 0.
\]

\[\Box\]

To prove Theorem 1.4, we need two previous results.

**Proposition 3.3.** Let \(r > 0\) fixed. Then, there exists a positive radial function \(u_0\) such that

\[
\begin{aligned}
-\Delta_p u + |u|^{p-2} u &= 0 \quad \text{in } \mathbb{R}^N \setminus B_r, \\
u &= 0 \quad \text{on } \partial B_r.
\end{aligned}
\]

This \(u_0\) is unique up to a constant factor and for any \(R > r\) the restriction of \(u_0\) to \(B_R\) is the first eigenfunction of (1.3) with \(q = p\).

**Proof.** For \(R > r\), let \(u_R\) be the unique solution of the Dirichlet problem

\[
\begin{aligned}
\Delta_p u_R = |u_R|^{p-2} u_R \quad \text{in } B_R \setminus \overline{B_r}, \\
u(R) &= 1, \\
u(r) &= 0.
\end{aligned}
\]

...
Then, by uniqueness, \( u_R \) is a nonnegative and radial function. Moreover, by the regularity theory and maximum principle we have \( \frac{\partial u_R}{\partial \nu}(r) \neq 0 \) (see [8, 11]). Thus, for any \( R > r \), we define the restriction of \( u_0 \) by

\[
u_0 = \frac{u_R}{\nu R} \frac{\partial u_R}{\partial \nu}(r).
\]

By uniqueness of the Dirichlet problem, it is easy to check that \( u_0 \) is well defined and is a nonnegative radial solution of (3.19). Furthermore, by the simplicity of \( S_p(r) \), \( u_0 \) is the eigenfunction associated to \( S_p(r) \) for every \( R > r \).

**Proposition 3.4.** Let \( v \) be a radial solution of (1.3). Then \( v \) is a multiple of \( u_0 \). In particular any radial minimizer of (1.2) is a multiple of \( u_0 \).

**Proof.** Let \( a > 0 \) be such that \( v = au_0 \) on \( \partial B(0,R) \). Then \( v \) and \( au_0 \) are two solutions to the Dirichlet problem \( \Delta_p w = w^{p-1} \) and \( w = v \) on \( \partial (B_R \setminus B_r) \). Hence, by uniqueness, we have that \( v = au_0 \) in \( B_R \).

**Remark 3.5.** If \( 1 < q < p \) then the solution of (1.3), by Remark 3.2 and Proposition 3.4, is a multiple of \( u_0 \).

Now we can deal with the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Let \( R > r \) be fixed and consider \( u_0 \) to be the nonnegative radial function given by Proposition 3.3 such that \( u_0 = 1 \) on \( \partial B_R \). Then, by Proposition 3.4, it is enough to prove that \( u_0 \) is not a minimizer for \( S_p(r) \) when \( q > Q(R) \).

First let us move this symmetric configuration in the \( x_1 \) direction. For any \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^N \) we denote \( x_t = (x_1 - t, x_2, \ldots, x_N) \) and define

\[
U(t)(x) = u_0(x_t)
\]

Observe that \( U \) vanishes in \( A_t := B_r(te_1) \) (the ball with center \( te_1 \) and radius \( r \)) a subset of \( B_R \) of the same measure of \( B_r \) for all \( t \) small.

Consider the function

\[
h(t) = \frac{f(t)}{g(t)}
\]

where

\[
f(t) = \int_{B_R} |\nabla U|^p + Up^p \, dx \quad \text{and} \quad g(t) = \left( \int_{\partial B_R} U^q \, dS \right)^{\frac{p}{q}}.
\]

We observe that \( h(0) = 0 \) and since \( h \) is an even function, we have \( h'(0) = 0 \). Now,

\[
h''(0) = \left. \frac{f''g^2 - f'g'g'' - 2f'gg' - 2fgg''}{g^3} \right|_{t=0}.
\]

Next we compute these terms. First, since \( u_0 \) is the first eigenfunction of (1.3) with \( q = p \) and \( u_0 = 1 \) on \( \partial B_R \) we get

\[
f(0) = S_p(r)|\partial B_R| \quad \text{and} \quad g(0) = |\partial B_R|^{\frac{p}{q}}.
\]

Thus, by Gauss–Green’s Theorem and using the fact that \( u_0 \) is radial, we get

\[
f'(0) = -\int_{B_R} \frac{\partial}{\partial x_1} (|\nabla u_0|^p + u_0^p) \, dx = \int_{\partial B_R} (|\nabla u_0|^p + u_0^p) \nu_1 \, dS = 0.
\]
Again, since $u_0$ is radial,

$$g'(0) = \frac{p}{q} \left( \int_{\partial B_R} u^q \, dS \right)^{\frac{q}{q-1}} \left( \int_{\partial B_R} \frac{\partial u^q}{\partial x_1} \, dS \right) = 0.$$  

Finally, using that $u_0 = 1$ on $\partial B_R$, we obtain

$$g''(0) = p |\partial B_R|^{\frac{q}{q-1}} \int_{\partial B_R} (q - 1) \left( \frac{\partial u_0}{\partial x_1} \right)^2 \, dS$$

and, by the Gauss–Green’s Theorem

$$f''(0) = p \int_{\partial B_R} \left( \frac{1}{2} |\nabla u_0|^{p-2} \frac{\partial |\nabla u_0|^2}{\partial x_1} + \frac{1}{p} \frac{\partial u_0^p}{\partial x_1} \right) \, dS$$

$$= p \int_{\partial B_R} \left( \frac{1}{2} |\nabla u_0|^{p-2} \frac{\partial |\nabla u_0|^2}{\partial x_1} + \frac{1}{p} \frac{\partial u_0^p}{\partial x_1} \right) \nu_1 \, dS.$$  

Then

$$h''(0) = \frac{p}{N |\partial B_R(0)|^{p/q}} \left[ \int_{\partial B_R} \left( \frac{1}{2} |\nabla u_0|^{p-2} \frac{\partial |\nabla u_0|^2}{\partial \nu} + \frac{1}{p} \frac{\partial u_0^p}{\partial \nu} \right) \, dS \right]$$

$$- S_p(r) \int_{\partial B_R} (q - 1) \left| \nabla u_0 \right|^2 + \Delta u_0 \, dS.$$  

Thus, since $u_0$ is radial, we get

$$h''(0) = \frac{p}{N |\partial B_R(0)|^{p/q}} \left[ \int_{\partial B_R} \left( \frac{1}{2} |\nabla u_0|^{p-2} \frac{\partial |\nabla u_0|^2}{\partial \nu} + \frac{1}{p} \frac{\partial u_0^p}{\partial \nu} \right) \, dS \right]$$

$$- S_p(r) \int_{\partial B_R} (q - 1) |\nabla u_0|^2 + \Delta u_0 \, dS.$$  

Now, by definition, $u_0(x) = u_0(|x|)$ and $\alpha$ satisfies

$$\left(s^{N-1} |u'_0|^{p-1} u_0'\right)' = s^{N-1} u_0'^{p-1} \quad \forall s > r$$

with $u_0(R) = 0$ and $u_0(r) = 0$, moreover, by Proposition 3.3, we have

$$u_0'(s)^{p-1} = S_p(r) u_0(s)^{p-1} \quad \forall s > r.$$  

Then

$$\frac{1}{2} |\nabla u_0|^{p-2} \frac{\partial |\nabla u_0|^2}{\partial \nu} + \frac{1}{p} \frac{\partial u_0^p}{\partial \nu} = S_p(r) \frac{\frac{p^+}{p-1}}{p-1} \left( 1 - \frac{N - 1}{R} S_p(r) \right) + S_p(r) \frac{p^-}{p-1}$$

and

$$S_p(r) \left[ (q - 1) |\nabla u_0|^2 + \Delta u_0 \right] = (q - 1) S_p(r) \frac{p^+}{p-1} + S_p(r) \frac{\frac{p^+}{p-1}}{p-1} \left( 1 - \frac{N - 1}{R} S_p(r) \right)$$

$$+ \frac{N - 1}{R} S_p(r) \frac{p^+}{p-1}.$$  

Therefore

$$h''(0) = \frac{p S_p(r)^{\frac{1}{q-1}}}{N |\partial B_R|^{\frac{q}{q-1}}} \left[ 1 - (q - 1) S_p(r) \frac{p^+}{p-1} - \frac{N - 1}{R} S_p(r) \right].$$
Thus, if \( q > Q(R) \) we get that \( h''(0) < 0 \) and so 0 is a strict local maxima of \( \psi \). So we have proved that
\[
S_q(r) = h(0) > h(t) \geq S_q(B_r(\epsilon_1))
\]
for all \( t \) small. Therefore a symmetric configuration is not optimal.

To finish the paper we prove Proposition 1.5.

**Proof of Proposition 1.5.** We proceed in two step.

**Step 1.** First we show that, for \( R > r \), \( S_p(R,r) = S_p(r) \) verifies the differential equation
\[
\frac{\partial S_p}{\partial R} = -\frac{N - 1}{R} S_p + 1 - (p - 1) S_p^\frac{p-1}{p}
\]
with the condition
\[
S_p|_{R=r} = +\infty.
\]

Again we consider \( u_0(x) = u_0(|x|) \) the nonnegative radial function given by Proposition 3.3. Thus, for all \( R > r \), we get
\[
\begin{align*}
(p - 1) (u_0')^{p-2} u_0'' + \frac{N - 1}{R} (u_0')^{p-1} &= u_0^{-1}, \\
u_0'(R)^{p-1} &= S_p u_0(R)^{p-1}, \\
u_0(r) &= 0.
\end{align*}
\]
Then
\[
S_p = \left( \frac{u_0'(R)}{u_0(R)} \right)^{p-1}.
\]

Thus
\[
\begin{align*}
\frac{\partial S_p}{\partial R} &= (p - 1) \left( \frac{u_0'(R)}{u_0(R)} \right)^{p-2} \frac{u_0''(R) u_0(R) - u_0'(R)^2}{u_0(R)^2} \\
&= (p - 1) \left( \frac{u_0'(R)}{u_0(R)} \right)^{p-2} \frac{u_0''(R)}{u_0(R)} - (p - 1) S_p^\frac{p-1}{p} \\
&= (p - 1) \left( \frac{u_0'(R)}{u_0(R)} \right)^{p-2} \frac{u_0''(R)}{u_0(R)^{p-1}} - (p - 1) S_p^\frac{p-1}{p} \\
&= 1 - \frac{N - 1}{R} S_p - (p - 1) S_p^\frac{p-1}{p}.
\end{align*}
\]

On the other hand, since (by definition) \( \frac{\partial u_0}{\partial \nu} \equiv 1 \) on \( \partial B_r \), we get that \( u'(r) = 1 \). Then
\[
\lim_{R \to r} S_p = \lim_{r \to r} \left( \frac{u_0'(R)}{u_0(R)} \right)^{p-1} = +\infty.
\]

Now, it is easy to check that \( \lim_{R \to r} Q(R) = 1^- \).

**Step 2.** Finally, we prove that
\[
\lim_{R \to +\infty} Q(R) = p.
\]

We begin differentiating (3.20) to obtain
\[
\frac{\partial^2 S_p}{\partial R^2} = \frac{N - 1}{R^2} S_p - \frac{N - 1}{R} \frac{\partial S_p}{\partial R} - p S_p^\frac{p-1}{p} \frac{\partial S_p}{\partial R}.
\]
Then, since $S_p > 0$, at any critical point ($S_p' = 0$) we have that $S_p'' > 0$. Thus, $S_p$ has at most one critical point, which is a minimum. If $S_p$ has a minimum, then there exist $R_0 > r$ such that $S_p'(R_0) = 0$. Moreover, since $S_p''(R) \neq 0$ for any $R \neq R_0$ and $S_p \rightarrow +\infty$ as $R \rightarrow r$ and by (3.20), we get that $S_p' < 0$ for all $r < R < R_0$ and $S_p' > 0$ for all $R > R_0$. Thus, using again (3.20) we have that $S_p^{p-1} < \frac{1}{p-1}$ for all $R > R_0$. Then $S_p$ is strictly increasing as a function of $R$ and bonded for all $R > R_0$. Consequently $S_p' \rightarrow 0$ as $R \rightarrow +\infty$. It follows, by (3.20), that $S_p^{\frac{p}{p-1}} \rightarrow \frac{1}{p-1}$ as $R \rightarrow +\infty$. On the other hand using (1.6) and (3.20) we see that

$$S_p = (Q(R) - p)S_p^{\frac{p}{p-1}}.$$  

So, if $S_p$ has a minimum, we get that $Q(R) > p$ for all $R > R_0$ and $Q(R) \rightarrow p^+$ as $R \rightarrow +\infty$. Now, if $S_p$ has not critical points so $S_p' \neq 0$ for all $R > r$ and using that $S_p \rightarrow +\infty$ as $R \rightarrow r$ and (3.20) we get that $S_p' < 0$ for all $R > r$. Consequently, in this case, $S_p$ is strictly decreasing and therefore $S_p' \rightarrow 0$ as $R \rightarrow +\infty$ and by (3.20) we have that $S_p \rightarrow \frac{1}{p-1}$ as $R \rightarrow +\infty$. Then, if $S_p$ has not critical points, we get $Q(R) < p$ and $Q(R) \rightarrow p^-$ as $R \rightarrow +\infty$. \hfill $\square$

Acknowledgements. I want to thank J. Fernández Bender for his throughout reading of the manuscript that help us to improve the presentation of paper.

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