A CLASSIFICATION OF COHERENT STATE REPRESENTATIONS OF UNIMODULAR LIE GROUPS

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1. INTRODUCTION

Let $G$ be a connected Lie group and $(\pi, \mathcal{H})$ a unitary representation of $G$ on a complex Hilbert space $\mathcal{H}$. Throughout we shall assume that $(\pi, \mathcal{H})$ is nontrivial in the sense that $\dim \mathcal{H} > 1$. By a coherent state orbit (CS orbit for short) for $(\pi, \mathcal{H})$ we mean a complex orbit of $G$ on the projective space $\mathbb{P}(\mathcal{H})$ (which is equipped with a natural structure of an (infinite-dimensional in general) Kaehler manifold (cf. [L])). We call $(\pi, \mathcal{H})$ a coherent state representation (CS representation for short) if (1) it admits a CS orbit, (2) is irreducible and (3) has (at most) discrete kernel, and we call $G$ a CS group if it possesses CS representations.

The purpose of this note is to announce a complete classification of connected unimodular CS groups and their CS representations (Theorems 1 and 2 below). This generalizes the results of Enright-Howe-Wallach [EHW] and Jakobsen [J] on the classification of unitary highest weight (or holomorphic) representations of reductive groups (which coincide with the CS representations as we have shown in [L]). The proofs are "geometric," the main tool being the recent structure theory of homogeneous Kaehler manifolds due to Dorfmeister and Nakajima [DN].

In physics, any orbit on $\mathbb{P}(\mathcal{H})$ is called a system of coherent states in the sense of Perelomov (see [P] and the references therein).

Of particular importance are symplectic coherent state orbits; in many cases such an orbit may be interpreted as the classical phase space of the system whose quantum phase space is $\mathbb{P}(\mathcal{H})$. Such an embedding of the classical phase space into the quantum one is the starting point of Berezin's quantization (see [B1] and [B2];

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see also [T] for a comparison of Berezin's quantization with the Kostant-Souriau geometric quantization) and the "quantization of states" proposed recently by Odzijewicz (see [O1] and [O2]). In both theories, the case of complex orbits plays a distinguished role. On one hand, "complex" coherent states are in a sense closest to the classical states [P] and on the other, we may apply in this case powerful techniques of complex analysis (with Bergman type reproducing kernels playing an essential role).

Thus there is a strong physical motivation for studying CS representations.

2. Basic properties of CS representations

Here the term CS representation refers to a \((n, \mathcal{H})\) which has property (1) but not necessarily (2) and (3).

**Proposition 1** [L]. *Any CS orbit has a natural structure of a Hamiltonian G-space and the corresponding moment mapping takes it diffeomorphically onto an integral coadjoint orbit with Kaehler (i.e. positive totally complex) polarization.*

There is a natural holomorphic line bundle \(E\) over \(P(\mathcal{H})\) whose fiber at \([v] = Cv\) is the dual \([v]^*\). The linear dual \(\mathcal{H}^*\) of \(\mathcal{H}\) is naturally isomorphic to the space of holomorphic sections of \(E\). Given a CS orbit \(G \cdot [v]\) corresponding to a CS representation \((\pi, \mathcal{H})\), we get a natural map from \(\mathcal{H}^*\) to the space \(\Gamma(G \cdot [v], L)\) of holomorphic sections of \(L\), the restriction of \(E\) to \(G \cdot [v]\).

**Proposition 2.** The following are equivalent.

(i) \(v\) is a cyclic vector for \((\pi, \mathcal{H})\).

(ii) The map \(\mathcal{H}^* \to \Gamma(G \cdot [v], L)\) is injective.

(iii) \((\pi, \mathcal{H})\) is irreducible.

The implications (i) \(\Rightarrow\) (ii) and (iii) \(\Rightarrow\) (i) are clear, and (ii) \(\Rightarrow\) (iii) can be deduced from a well-known theorem of Kobayashi [K].

3. Three special cases

It turns out that the case of a general unimodular group can be reduced to three special cases, which we shall now briefly discuss.

3.1. Heisenberg groups. Let \(H_n\) be a \((2n + 1)\)-dimensional Heisenberg group (not necessarily simply connected). Identify the (multiplicative) group \(X(C)\) of unitary characters of the center \(C\) of \(H_n\) with an (additive) subgroup of the dual \(c^*\) of the Lie
algebra of $C$. The infinite-dimensional irreducible unitary representations of $H_n$ are in 1-1 correspondence with the nonzero elements $\lambda$ of $X(C)$, $(\beta, \mathcal{F}_\lambda)$ being the unique (up to equivalence) representation with $\lambda$ as central character (or, in other terms, the unique representation corresponding, via Kirillov's bijection, to the integral coadjoint orbit $\mathcal{O}_\lambda$ determined by $\lambda$). It is well known that any $(\beta, \mathcal{F}_\lambda)$ is a CS representation. Any of the CS orbits on $\mathbf{P}(\mathcal{F}_\lambda)$ is mapped by its moment onto $\mathcal{O}_{-\lambda}$. This establishes a 1-1 correspondence between these orbits and Kaehler polarizations of $\mathcal{O}_\lambda$ which, in turn, are in 1-1 correspondence with points of the Siegel space $\mathcal{G}_n$ (i.e. the Hermitian symmetric space $\text{Sp}(2n, \mathbf{R})/\text{U}(n)$).

Next we consider reductive groups. We shall say that a reductive group is of \textit{compact} (resp. \textit{noncompact}) type if its Lie algebra is so.

3.2. \textbf{Groups of compact type} [KS]. Any such group is a CS group and any of its nontrivial representations is a CS representation. For any CS representation, there is exactly one CS orbit, namely the orbit through a highest weight line. Geometrically, these orbits are compact simply connected homogeneous Kaehler manifolds (i.e. flag manifolds).

3.3. \textbf{Groups of noncompact type} [L]. Such a group is a CS group if and only if it is of \textit{Hermitian type} (i.e. the symmetric space $\mathcal{D}$ associated with it is of Hermitian type). CS representations are the highest weight representations. Again the orbit through a highest weight line is the unique CS orbit for a given CS representation. Geometrically, it is a holomorphic fiber bundle over $\mathcal{D}$ (equipped with one of its invariant complex structures) with flag manifolds as fibers.

4. \textbf{Homogeneous Kaehler manifolds}

Our approach to the problem of classifying CS groups is based on Dorfmeister-Nakajima theorem [DN] (which gives an affirmative answer to a long standing conjecture of Vinberg and Gindikin). For our purposes, it is convenient to state it as follows. \textit{Every homogeneous Kaehler manifold $X$ has a holomorphic double fibration $X$ \hspace{1cm} \downarrow$ \hspace{1cm} \inferior, $M \rightarrow \mathcal{D}$, where $M$ is a homogeneous Kaehler manifold without flat homogeneous Kaehler submanifolds and the fibers of $X \rightarrow M$ are flat.}
homogeneous Kaehler manifolds (i.e. they are of the form $\mathbb{C}^n/\Gamma$, where $\Gamma$ is a discrete subgroup of $\mathbb{C}^n$ and the Kaehler metric is induced by the standard Kaehler metric on $\mathbb{C}^n$), $\mathcal{D}$ is a homogeneous bounded domain and the fibers of $M \to \mathcal{D}$ are flag manifolds. Such a double fibration is unique and is preserved by all automorphisms of $X$.

5. STRUCTURE OF A CS ORBIT

Now suppose $(\pi, \mathcal{H})$ is a CS representation of $G$ and $X = G \cdot [v] \subset \mathbb{P}(\mathcal{H})$ is a CS orbit such that neither its flat fibers nor $\mathcal{D}$ reduce to points. The fact that $X$ is a Hamiltonian $G$-space implies that these flat fibers are isomorphic to some $\mathbb{C}^n$ and coincide with the orbits of a Heisenberg group $N$ (of dimension $2n + 1$) which is contained in $G$ as a normal subgroup. Let $J_N$ denote the moment mapping of the corresponding Hamiltonian action of $N$ on $X$. Since the orbits of this action are symplectic, the symplectic reduction theorem (see [AM]) implies that $J_N(X)$ is a single coadjoint orbit $\mathcal{O}_\lambda$.

$N$ being a normal subgroup of $G$, there is a homomorphism

$$\bar{\rho}: G \to \text{Aut}(N), \ g \mapsto \text{Int} \ g|_N$$

(where $\text{Int} \ g$ denotes the inner automorphism of $G$ corresponding to $g$), which factors through $N$ to give a homomorphism

$$\rho: S = G/N \to \text{Out} \ N = \text{Aut} \ N/\text{Int} \ N.$$

It is clear that $\bar{\rho}(S) \subset (\text{Aut} \ N)_\lambda$, the stabilizer of $\mathcal{O}_\lambda$ (or $\lambda$) in $\text{Aut} \ N$ (which is the same for all $\lambda \neq 0$), and, consequently, $\rho(S) \subset (\text{Aut} \ N)_\lambda/\text{Int} \ N \cong \text{Sp}(2n, \mathbb{R})$.

Being a complex submanifold of $X$, each $N$-orbit carries a Kaehler polarization which is mapped by $J_N$ into a Kaehler polarization of $\mathcal{O}_\lambda$. We thus get a $\rho$-equivariant map from the orbit space $M = X/N$ to the space of all Kaehler polarizations of $\mathcal{O}_\lambda$, i.e. the Siegel space $\mathcal{G}_n$. It can be shown that this map is holomorphic. Hence it factors through the compact fibers of $M$ to give a $\rho$-equivariant holomorphic map

$$\rho_\mathcal{D}: \mathcal{D} \to \mathcal{G}_n.$$

6. CLASSIFICATION OF UNIMODULAR CS GROUPS

From now on we assume that $G$ is unimodular (and nonreductive). Using the results of the preceding section it is not hard to
show that then $S = G/N$ is also unimodular and so is its quotient $S/N_\mathfrak{g}$ which acts effectively on $\mathcal{D}$. Moreover, $N_\mathfrak{g}$ is of compact type (here the assumption that $\pi$ has discrete kernel is essential). Now a theorem of Hano [Ha] asserts that if a unimodular Lie group acts effectively and transitively on a bounded domain, then it is semisimple and the domain is symmetric. Thus $S/N_\mathfrak{g}$ is semisimple and, consequently, $S$ is reductive and of Hermitian type. It follows that $N$ coincides with the nilradical (the maximal connected nilpotent normal Lie subgroup) of $G$.

We have sketched the proof of the “only if part” of the following.

**Theorem 1.** A connected unimodular (nonreductive) Lie group $G$ is a CS group if and only if it satisfies the following conditions.

(i) The nilradical $N$ of $G$ is isomorphic to a Heisenberg group $H_n$.

(ii) The reductive group $S = G/N$ is either of compact or of Hermitian type and its image under the natural homomorphism $\rho: S \to \text{Out}(N)$ is contained in $\text{Sp}(2n, \mathbb{R})$.

(iii) If $S$ is of Hermitian type, there exists a $\rho$-equivariant holomorphic map from the Hermitian symmetric space $\mathcal{D}$ associated with $S$ to the Siegel space $\mathfrak{S}_n$.

That this theorem really classifies unimodular CS groups follows from the results of Satake (see [S2]) who classified $\rho$-equivariant holomorphic maps $\mathcal{D} \to \mathfrak{S}_n$ (this classification is closely related to the classification of Howe’s reductive dual pairs in $\text{Sp}(2n, \mathbb{R})$ (cf. [Ho]).

7. Classification of CS Representations

Irreducible unitary representations of the groups which occur in Theorem 1 have been classified by Satake [S1]. Using his results and the results of the preceding sections (with Proposition 2 playing an essential role) we can complete the proof of Theorem 1 and also prove the following.

**Theorem 2.** Suppose $G$ has properties (i)–(iii) of Theorem 1. For any nonzero $\lambda \in X(C)$, let $(\sigma_\lambda, \mathcal{F}_\lambda)$ be a projective representation of $G$ obtained by composing the (projective) metaplectic representation of $(\text{Aut } N)_\lambda$ (associated with $(\beta_\lambda, \mathcal{F}_\lambda)$) with $\hat{\rho}$ and let $\alpha$ be its cocycle ($\alpha$ does not depend on $\lambda$ and can be considered as a cocycle on $S = G/N$). Let $(\pi_1, \mathcal{E})$ be an irreducible projective
unitary representation of \( S \) with the following properties:

(i) its cocycle is \( \alpha^{-1} \);

(ii) its kernel \( \ker \pi_1 \) is contained in \( N_H \) (cf. §6);

(iii) the corresponding representation of \( S/\ker \pi_1 \) is a (projective) CS representation.

Then \( (\pi, \mathcal{H}) \), where \( \mathcal{H} = \mathcal{E} \otimes \mathcal{F}_\lambda \) (Hilbert tensor product) and

\[
\pi(g) = \hat{\pi}_1(g) \otimes \sigma_\lambda(g) \quad \text{for } g \in G,
\]

\( \hat{\pi}_1 \) being the composition of \( \pi_1 \) and the projection \( G \to S \), is a (linear) CS representation of \( G \) and any CS representation of \( G \) is of this form.

REFERENCES

[AM] R. Abraham and J. Marsden, *Foundations of mechanics*, 2nd ed., Benjamin/Cummings, Reading, MA, 1978.

[B1] F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1116–1175; English transl. in Math. USSR Izv. 38 (1974).

[B2] —, *Quantization in complex symmetric spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 363–402; English transl. in Math. USSR Izv. 39 (1975).

[DN] J. Dorfmeister and K. Nakajima, *The fundamental conjecture for homogeneous Kaehler manifolds*, Acta Math. 161 (1988), 23–70.

[EHW] T. Enright, R. Howe and N. Wallach, *A classification of unitary highest weight modules*, Representation Theory of Reductive Groups (P. Trombi, ed.), Progress in Math., vol. 40, Birkhäuser, Boston, 1983, pp. 97–143.

[Ha] J. I. Hano, *On Kaehlerian homogeneous spaces of unimodular Lie groups*, Amer. J. Math. 79 (1957), 885–900.

[Ho] R. Howe, *\( \theta \)-series and invariant theory*, Automorphic Forms, Representations and \( L \)-functions, Proc. Sympos. Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 275–285.

[J] H. P. Jakobsen, *Hermitian symmetric spaces and their unitary highest weight modules*, J. Funct. Anal. 52 (1983), 385–412.

[K] S. Kobayashi, *Irreducibility of certain unitary representations*, J. Math. Soc. Japan 20 (1968), 638–642.

[KS] B. Kostant and S. Sternberg, *Symplectic projective orbits*, New Directions in Applied Mathematics, Springer-Verlag, Berlin and New York, 1982, pp. 81–84.

[L] W. Lisiecki, *Kaehler coherent state orbits for representations of semisimple Lie groups*, Ann. Inst. H. Poincaré Phys. Théor. 53 (1990), 245–258.

[O1] A. Odzijewicz, *On reproducing kernels and quantization of states*, Comm. Math. Phys. 114 (1988), 577–597.

[O2] —, *On the notion of mechanical system* (to appear).

[P] A. M. Perelomov, *Generalized coherent states and their applications*, Springer-Verlag, Berlin and New York, 1986.
[S1] I. Satake, Unitary representations of a semi-direct product of Lie groups on \( \mathfrak{g} \)-cohomology spaces, Math. Ann. 190 (1971), 177–202.

[S2] ——. Algebraic structures of symmetric domains, Princeton Univ. Press, Princeton, NJ, 1980.

[T] G. M. Tuynman, Studies in geometric quantization, Ph.D. thesis, Amsterdam, 1987.

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