MAURER-CARTAN EQUATION IN THE DGLA OF GRADED DERIVATIONS

PAOLO DE BARTOLOMEIS AND ANDREI IORDAN

In the memory of Pierre Dolbeault

Abstract. Let $M$ be a smooth manifold and $\Phi$ a differential 1-form on $M$ with values in the tangent bundle $T_M$. We construct canonical solutions $e^\Phi$ of Maurer-Cartan equation in the DGLA of graded derivations $D^\ast(M)$ of differential forms on $M$ by means of deformations of the $d$ operator depending on $\Phi$. This yields to a classification of the canonical solutions of the Maurer-Cartan equation according to their type: $e^\Phi$ is of finite type $r$ if there exists $r \in \mathbb{N}$ such that $\Phi^r |\Phi|_{FN} = 0$ and $r$ is minimal with this property, where $[\cdot, \cdot]_{FN}$ is the Frölicher-Nijenhuis bracket. A distribution $\xi \subset T_M$ of codimension $k \geq 1$ is integrable if and only if the canonical solution $e^\Phi$ associated to the endomorphism $\Phi$ of $T_M$ which is trivial on $\xi$ and equal to the identity on a complement of $\xi$ in $T_M$ is of finite type $\leq 1$, respectively of finite type $0$ if $k = 1$.

1. Introduction

In [9], one of the last papers of their seminal cycle of works on deformations of differentiable and complex structures, K. Kodaira and D. C. Spencer studied the deformations of multifoliate structures. A $\mathcal{P}$-multifoliate structure on an orientable manifold $X$ of dimension $n$ is an atlas $(U_i, (x_i^\alpha)_{\alpha=1,\ldots,n})$ such that the changes of coordinates verify

$$\frac{\partial x_\alpha^i}{\partial x_\beta^j} = 0 \text{ for } \beta \not\geq \alpha,$$

where $(\mathcal{P}, \succeq)$ is a finite partially ordered set, $\{\alpha\}$ the set of integers $\alpha = 1, 2, \ldots, n$ such there is given a map $\{\alpha\} \mapsto [\alpha]$ of $\alpha$ onto $\mathcal{P}$ and the order relation " $\succeq$ " is defined by $\alpha > \beta$ if and only if $[\alpha] > [\beta]$, $\alpha \sim \beta$ if and only if $[\alpha] = [\beta]$. An usual foliation is the particular case when $\mathcal{P} = \{a, b\}$, $a > b$.

They defined a DGLA structure $(\mathcal{D}^\ast(M), \mu, [\cdot, \cdot])$ on the graded algebra of graded derivations introduced by Frölicher and Nijenhuis in [5] and the deformations of the multifoliate structures are related to the solutions of the Maurer-Cartan equation in this algebra. This was done in the spirit of [11], where A. Nijenhuis and R. W. Richardson adapted a theory initiated by M. Gerstenhaber [6] and proved the connection between the deformations of complex analytic structures and the theory of differential graded Lie algebras (DGLA).
In the paper [1], the authors elaborated a theory of deformations of integrable distributions of codimension 1 in smooth manifolds. Our approach was different of K. Kodaira and D. C. Spencer’s in [9] (see remark 14 of [1] for a discussion). We considered in [1] only deformations of codimension 1 foliations, the DGLA algebra $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$ associated to a codimension 1 foliation on a co-oriented manifold $L$ being a subalgebra of the algebra $(\Lambda^*(L), \delta, \{\cdot, \cdot\})$ of differential forms on $L$. Its definition depends on the choice of a DGLA defining couple $(\gamma, X)$, where $\gamma$ is a 1-differential form on $L$ and $X$ is a vector field on $L$ such that $\gamma(X) = 1$, but the cohomology classes of the underlying differential vector space structure do not depend on its choice. The deformations are given by forms in $\mathcal{Z}^1(L)$ verifying the Maurer-Cartan equation and the moduli space takes in account the diffeomorphic deformations. The infinitesimal deformations along curves are subsets of the first cohomology group of the DGLA $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$.

This theory was adapted to the study of the deformations of Levi-flat hypersurfaces in complex manifolds: we parametrized the Levi-flat hypersurfaces in a complex manifold and we obtained a second order elliptic partial differential equation for an infinitesimal Levi-flat deformation.

In this paper we consider the graded algebra of graded derivations defined by Frölicher and Nijenhuis in [5] with the DGLA structure defined by K. Kodaira and D. C. Spencer in [9]. We construct canonical solutions of the Maurer-Cartan equation in this algebra by means of deformations of the $d$-operator depending on a vector valued differential 1-form $\Phi$ and we give a classification of these solutions depending on their type. A canonical solution of the Maurer-Cartan equation associated to an endomorphism $\Phi$ is of finite type $r$ if there exists $r \in \mathbb{N}$ such that $\Phi^r [\Phi, \Phi]_{FN} = 0$ and $r$ is minimal with this property, where $[\cdot, \cdot]_{FN}$ is the Frölicher-Nijenhuis bracket. We show that a distribution $\xi$ of codimension $k$ on a smooth manifold is integrable if and only if the canonical solution of the Maurer-Cartan equation associated to the endomorphism of the tangent space which is the trivial extension of the $k$-identity on a complement of $\xi$ in $TM$ is of finite type $\leq 1$. If $\xi$ is a distribution of dimension $s$ such that there exists an integrable distribution $\xi^*$ of dimension $d$ generated by $\xi$, we show that there exists locally an endomorphism $\Phi$ associated to $\xi$ such that the canonical solution of the Maurer-Cartan equation associated to $\Phi$ is of finite type less than $r = \min \{ m \in \mathbb{N} : m \geq \frac{d}{s} \}$.

In the case of integrable distributions of codimension 1, we study also the infinitesimal deformations of the canonical solutions of the Maurer-Cartan equation in the algebra of graded derivations by means of the theory of deformations developed in [1].

2. The DGLA of graded derivations

In this paragraph we recall some definitions and properties of the DGLA of graded derivations from [7], [9] (see also [10]).

**Notation 1.** Let $M$ be a smooth manifold. We denote by $\Lambda^* M$ the algebra of differential forms on $M$, by $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$ and by $\Lambda^* M \otimes TM$ the algebra of $TM$-valued differential form on $M$, where $TM$ is the tangent bundle to $M$. In the sequel, we will identify $\Lambda^1 M \otimes TM$ with the algebra $\text{End}(TM)$ of endomorphisms of $TM$ by their canonical isomorphism: for $\sigma \in \Lambda^1 M$, $X, Y \in \mathfrak{X}(M)$, $(\sigma \otimes X)(Y) = \sigma(Y)X$. 
Definition 1. A differential graded Lie algebra (DGLA) is a triple $(V^*, d, [\cdot, \cdot])$ such that:

1) $V^* = \bigoplus_{i \in \mathbb{N}} V^i$, where $(V^i)_{i \in \mathbb{N}}$ is a family of $\mathbb{C}$-vector spaces and $d : V^* \to V^*$ is a graded homomorphism such that $d^2 = 0$. An element $a \in V^k$ is said to be homogeneous of degree $k = \deg a$.

2) $[\cdot, \cdot] : V^* \times V^* \to V^*$ defines a structure of graded Lie algebra i.e. for homogeneous elements we have

$$[a, b] = - (-1)^{\deg a \deg b} [b, a]$$

and

$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg a \deg b} [b, [a, c]]$$

3) $d$ is compatible with the graded Lie algebra structure i.e.

$$d[a, b] = [da, b] + (-1)^{\deg a}[a, db].$$

Definition 2. Let $(V^*, d, [\cdot, \cdot])$ be a DGLA and $a \in V^1$. We say that $a$ verifies the Maurer-Cartan equation in $(V^*, d, [\cdot, \cdot])$ if

$$da + \frac{1}{2} [a, a] = 0.$$  

Definition 3. Let $A = \bigoplus_{k \in \mathbb{Z}} A_k$ be a graded algebra. A linear mapping $D : A \to A$ is called a graded derivation of degree $p = |D|$ if $D : A_k \to A_{k+p}$ and $D(ab) = D(a)b + (-1)^{p \deg a} aD(b)$.

Definition 4. Let $M$ be a smooth manifold. We denote by $\mathcal{D}^*(M)$ the graded algebra of graded derivations of $\Lambda^*M$.

Definition 5. Let $P, Q$ be homogeneous elements of degree $|P|, |Q|$ of $\mathcal{D}^*(M)$. We define

$$[P, Q] = PQ - (-1)^{|P||Q|} QP,$$

$$\nabla P = [d, P].$$

Lemma 1. Let $M$ be a smooth manifold. Then $(\mathcal{D}^*(M), [\cdot, \cdot], \nabla)$ is a DGLA.

Definition 6. Let $\alpha \in \Lambda^* M$ and $X \in \mathfrak{X}(M)$. We define $\mathcal{L}_{\alpha \otimes X}, \mathcal{I}_{\alpha \otimes X}$ by

$$(2.1) \quad \mathcal{L}_{\alpha \otimes X} \sigma = \alpha \wedge \mathcal{L}_X \sigma + (-1)^{|\alpha|} d\alpha \wedge \iota_X \sigma, \, \sigma \in \Lambda^* (M)$$

and

$$(2.2) \quad \mathcal{I}_{\alpha \otimes X} \sigma = \alpha \wedge \iota_X \sigma, \, \sigma \in \Lambda^* (M)$$

where $\mathcal{L}_X$ is the Lie derivative and $\iota_X$ the contraction by $X$.

For $\Phi \in \Lambda^* M \otimes TM$ we define $\mathcal{L}_\Phi, \mathcal{I}_\Phi$ as the extensions by linearity of $(2.1), (2.2)$.

Remark 1. Let $\omega \in \Lambda^2 (M), Z \in \mathfrak{X}(M)$ and $\sigma \in \Lambda^1 (M)$. Then for every $X, Y \in \mathfrak{X}(M)$

$$\mathcal{I}_{\omega \otimes Z} \sigma (X, Y) = (\omega \wedge \iota_Z \sigma) (X, Y) = \sigma (Z) \omega (X, Y) = \sigma ((\omega \otimes Z) (X, Y)).$$

By linearity, for every $\Phi \in \Lambda^2 M \otimes TM, \sigma \in \Lambda^1 (M), X, Y \in \mathfrak{X}(M)$ we have

$$\mathcal{I}_\Phi \sigma (X, Y) = \sigma (\Phi (X, Y)).$$

Lemma 2. For every $\Phi \in \Lambda^k M \otimes TM, \mathcal{L}_\Phi, \mathcal{I}_\Phi \in \mathcal{D}^* (M), |\mathcal{L}_\Phi| = k, |\mathcal{I}_\Phi| = k - 1.$
Notation 2.

$$\mathcal{L}(M) = \{ \mathcal{L}_\Phi : \Phi \in \Lambda^* M \otimes TM \}, \quad \mathcal{I}(M) = \{ \mathcal{I}_\Phi : \Phi \in \Lambda^* M \otimes TM \}.$$ 

In [5] the graded derivations of $\mathcal{L}(M)$ (respectively of $\mathcal{I}(M)$) are called of type $d_*$ (respectively of type $i_*$).

Lemma 3. (1) For every $D \in \mathcal{D}^k(M)$ there exist unique forms $\Phi \in \Lambda^k M \otimes TM, \Psi \in \Lambda^{k+1} M \otimes TM$ such that

$$D = \mathcal{L}_\Phi + \mathcal{I}_\Psi,$$

so

$$\mathcal{D}^* (M) = \mathcal{L}(M) \oplus \mathcal{I}(M).$$

We denote $\mathcal{L}_\Phi = \mathcal{L}(D)$ and $\mathcal{I}_\Psi = \mathcal{I}(D)$

(2) For every $\Phi \in \Lambda^* M \otimes TM$

$$\nabla(-1)^{|\Phi|} \mathcal{I}_\Phi = [\mathcal{I}_\Phi, d] = \mathcal{L}_\Phi.$$ (2.3)

(3) $$\mathcal{L}(M) = \ker \nabla.$$

Notation 3. We denote by $\mathcal{R} : \mathcal{D}^* (M) \to \mathcal{D}^* (M)$ the mapping defined by

$$\mathcal{R}(D) = (-1)^{|D|} \mathcal{I}(D)$$

Remark 2.

$$\text{Id} = \nabla \mathcal{R} + \mathcal{R} \nabla.$$

Indeed for $D = \mathcal{L}_\Phi + \mathcal{I}_\Psi \in \mathcal{D}^* (M)$, by using Lemma[3] we have

$$(\nabla \mathcal{R} + \mathcal{R} \nabla)(D) = (\nabla \mathcal{R} + \mathcal{R} \nabla)(\mathcal{L}_\Phi + \mathcal{I}_\Psi) = \nabla(-1)^{|\Phi|} \mathcal{I}_\Phi + \mathcal{R}(-1)^{|\Phi|} \mathcal{L}_\Psi = \mathcal{L}_\Phi + \mathcal{I}_\Psi = D.$$

Lemma 4. (1) Let $D \in \mathcal{D}^* (M)$. The following are equivalent:

i) $D \in \mathcal{I}(M)$;

ii) $D |\Lambda^0 (M) = 0$;

iii) $D(f \omega) = f D(\omega)$ for every $f \in C^\infty (M)$ and $\omega \in \Lambda^* (M)$.

(2) The mapping $\mathcal{L} : \Lambda^* M \otimes TM \to \mathcal{D}^* (M)$ defined by $\mathcal{L}(\Phi) = \mathcal{L}_\Phi$ is an injective morphism of graded Lie algebras.

Remark 3. $d \in \mathcal{D}^1(M)$ and

$$d = \mathcal{L}_{\text{Id}_\mathcal{T}(M)} = -\nabla_{\text{Id}_\mathcal{T}(M)}.$$ 

By Lemma[3] and the Jacobi identity, for every $\Phi \in \Lambda^k M \otimes TM, \Psi \in \Lambda^l M \otimes TM$ we have

$$\nabla([\mathcal{L}_\Phi, \mathcal{L}_\Psi]) = [d, [\mathcal{L}_\Phi, \mathcal{L}_\Psi]] = [[d, \mathcal{L}_\Phi], \mathcal{L}_\Psi] + (-1)^{|\Phi|} [\mathcal{L}_\Phi, [d, \mathcal{L}_\Psi]]$$

$$= [\nabla \mathcal{L}_\Phi, \mathcal{L}_\Psi] + (-1)^{|\Phi|} [\mathcal{L}_\Phi, \nabla \mathcal{L}_\Psi] = 0,$$

so there exists a unique form $[\Phi, \Psi] \in \Lambda^{k+l} M \otimes TM$ such that

$$[\mathcal{L}_\Phi, \mathcal{L}_\Psi] = \mathcal{L}_{[\Phi, \Psi]}.$$ (2.4)

This gives the following

Definition 7. Let $\Phi, \Psi \in \Lambda^* M \otimes TM$. The Frölicher-Nijenhuis bracket of $\Phi$ and $\Psi$ is the unique form $[\Phi, \Psi]_{FN} \in \Lambda^* M \otimes TM$ verifying (2.4).
Lemma 5. Let $\Phi_1 \in \Lambda^{k_1} M \otimes TM$, $\Phi_2 \in \Lambda^{k_2} M \otimes TM$, $\Psi_1 \in \Lambda^{k_1+1} M \otimes TM$, $\Psi_2 \in \Lambda^{k_2+1} M \otimes TM$. Then
\[ [\mathcal{L}_{\Phi_1} + \mathcal{I}_{\Phi_1}, \mathcal{L}_{\Phi_2} + \mathcal{I}_{\Psi_2}] = \mathcal{L}_{[\Phi_1, \Phi_2]} + \mathcal{I}_{\Phi_1 \Phi_2} - (-1)^{k_1 k_2} \mathcal{I}_{\Phi_2 \Phi_1} \]

In particular
\[ \mathcal{I}_{\Phi_2} \mathcal{I}_{\Phi_1} \mathcal{L}_{\Phi_3} = \mathcal{I}_{\Phi_1 \Phi_2} \mathcal{L}_{\Phi_3} - (-1)^{k_1 k_2} \mathcal{I}_{\Phi_3 \Phi_1 \Phi_2}; \]

\[ \mathcal{L}_{\Phi_3} \mathcal{I}_{\Phi_1} \mathcal{I}_{\Phi_2} = \mathcal{I}_{\Phi_1 \Phi_2} \mathcal{L}_{\Phi_3} - (-1)^{k_1 k_2} \mathcal{I}_{\Phi_3 \Phi_1 \Phi_2}. \]

Definition 8. Let $\Phi \in \Lambda^1 M \otimes TM$. The Nijenhuis tensor of $\Phi$ is $N_\Phi \in \Lambda^2 M \otimes TM$ defined by
\[ N_\Phi (X, Y) = [\Phi X, \Phi Y] + \Phi^2 [X, Y] - \Phi [\Phi X, Y] - \Phi [X, \Phi Y], \quad X, Y \in \mathfrak{X}(M). \]

Proposition 1. Let $\alpha \in \Lambda^k (M)$, $\beta \in \Lambda^l (M)$, $X, Y \in \mathfrak{X}(M)$. Then:

1. \[ [\alpha \otimes X, \beta \otimes Y]_{F^N} = \alpha \wedge \beta \otimes [X, Y] + \alpha \wedge \mathcal{L}_X \beta \otimes Y - \mathcal{L}_Y \alpha \wedge \beta \otimes X + (-1)^k (d\alpha \wedge \iota_X \beta + \iota_Y \alpha \wedge d\beta \otimes X). \]

2. Let $\Phi, \Psi \in \Lambda^1 M \otimes TM$. Then
\[ [\Phi, \Psi]_{F^N} (X, Y) = [\Phi X, \Psi Y] - [\Phi Y, \Psi X] - \Psi [\Phi X, Y] + \Psi [\Phi Y, X] - \Phi [\Psi X, Y] + [\Psi Y, \Phi X] + \frac{1}{2} \Psi (\Phi [X, Y]) - \frac{1}{2} \Phi (\Psi [Y, X]) + \frac{1}{2} \Phi (\Psi [X, Y]) - \frac{1}{2} \Psi (\Phi [Y, X]). \]

In particular
\[ [\Phi, \Phi]_{F^N} (X, Y) = 2 ([\Phi X, \Phi Y] + \Phi^2 [X, Y] - \Phi [\Phi X, Y] - \Phi [X, \Phi Y]) = 2 N_\Phi (X, Y). \]

3. Canonical solutions of Maurer-Cartan equation

Definition 9. Let $\Phi \in \Lambda^1 M \otimes TM$.

a) Let $\sigma \in \Lambda^p M$. We define $\Phi \sigma \in \Lambda^p M$ by $\Phi \sigma = \sigma$ if $p = 0$ and
\[ (\Phi \sigma) (V_1, \ldots, V_p) = \sigma (\Phi V_1, \ldots, \Phi V_p) \text{ if } p \geq 1, \quad V_1, \ldots, V_p \in \mathfrak{X}(M). \]

b) Let $\Psi \in \Lambda^p M \otimes TM$. We define $\Phi \Psi \in \Lambda^p M \otimes TM$ by $\Phi \Psi = \Psi$ if $p = 0$ and
\[ \Phi \Psi (V_1, \ldots, V_p) = \Phi (\Psi (V_1, \ldots, V_p)), \quad V_1, \ldots, V_p \in \mathfrak{X}(M) \text{ if } p \geq 1. \]

Lemma 6. Let $\Phi \in \Lambda^1 M \otimes TM$, $\Psi \in \Lambda^2 M \otimes TM$. Then
\[ \mathcal{I}_{\Phi} \Phi = \Phi \Psi. \]
Proof. It is sufficient to prove the assertion for \( \Psi = \alpha \otimes X, \Phi = \beta \otimes Y, \alpha \in \Lambda^2 (M), \beta \in \Lambda^1 (M), X, Y \in \mathfrak{X} (M) \).

For every \( Z_1, Z_2 \in \mathfrak{X} (M) \) we have

\[
(\beta \otimes Y) (\alpha \otimes X) (Z_1, Z_2) = (\beta \otimes Y) (\alpha (Z_1, Z_2) \otimes X) = \beta (X) \alpha (Z_1, Z_2) \otimes Y.
\]

Since

\[
\mathcal{I}_{\alpha \otimes X} \beta \otimes Y = (\mathcal{I}_{\alpha \otimes X} \beta) \otimes Y = \beta (X) \alpha \otimes Y,
\]

the Lemma is proved. \( \square \)

**Definition 10.** Let \( D \in \mathcal{D}^k (M) \) and \( \Phi \in \Lambda^1 M \otimes TM \) invertible. We define \( \Phi^{-1} D \Phi : \Lambda^* M \rightarrow \Lambda^* M \) by \( \Phi^{-1} D \Phi (\sigma) = \Phi^{-1} D (\Phi \sigma) \).

**Lemma 7.** Let \( D \in \mathcal{D}^k (M) \) and \( \Phi \in \Lambda^1 M \otimes TM \) invertible. Then \( \Phi^{-1} D \Phi \in \mathcal{D}^k (M) \).

**Proof.** Let \( \sigma \in \Lambda^p (M), \eta \in \Lambda^q (M) \). Since \( \Phi (\sigma \wedge \eta) = \Phi \sigma \wedge \Phi \eta \), it follows that

\[
\Phi^{-1} D (\Phi (\sigma \wedge \eta)) = \Phi^{-1} D (\Phi \sigma \wedge \Phi \eta) = \Phi^{-1} \left( D \Phi \sigma \wedge \Phi \eta + (-1)^p k \sigma \wedge D \Phi \eta \right) = \Phi^{-1} D \Phi \sigma \wedge \eta + (-1)^p k \sigma \wedge \Phi^{-1} D \Phi \eta.
\]

\( \square \)

**Notation 4.** Let \( \Phi \in \Lambda^1 M \otimes TM \) such that \( R_\Phi = Id_{TM} + \Phi \) is invertible. Set

\[
d_\Phi = R_\Phi d R_\Phi^{-1},
\]

\[
e_\Phi = d_\Phi - d
\]

and

\[
b (\Phi) = - \frac{1}{2} R_\Phi^{-1} [\Phi, \Phi]_{\mathcal{F}^N}.
\]

The following Theorem is a refinement of results from \( \cite{3} \) and \( \cite{4} \):

**Theorem 1.** Let \( \Phi \in \Lambda^1 M \otimes TM \) such that \( R_\Phi = Id_{TM} + \Phi \) is invertible. Then

\[
e_\Phi = \mathcal{L}_\Phi + \mathcal{I}_{b(\Phi)}.
\]

**Proof.** Since both terms of (3.1) are derivations of degree 1, it is enough to prove (3.1) on \( \Lambda^0 (M) \) and \( \Lambda^1 (M) \).

Let \( f \in \Lambda^0 (M) \) and \( X \in \mathfrak{X} (M) \). Then

\[
d_\Phi f (X) = (R_\Phi d R_\Phi^{-1} f) (X) = (R_\Phi d f) (X) = df (Id_{TM} + \Phi) (X) = df (X) + df (\Phi (X)).
\]

If \( \alpha \in \Lambda^1 (M), X \in \mathfrak{X} (M) \), by (2.2),

\[
\mathcal{I}_{\alpha \otimes Y} (df) (X) = (\alpha \otimes i_Y df) (X) = df (Y) (\alpha (X)) = df (\alpha \otimes Y) X
\]

and by linearity we obtain

\[
\mathcal{I}_{\Phi} (df) (X) = df (\Phi (X)).
\]

So, from (3.2) it follows that

\[
d_\Phi f (X) = df (X) + \mathcal{I}_\Phi (df) (X) = (d + [\mathcal{I}_\Phi, d]) f (X) = (d + \mathcal{L}_\Phi) f (X) = (d + \mathcal{L}_\Phi) f (X).
\]

Since \( \mathcal{I}_\Phi \) is of type \( i_* \), \( \mathcal{I}_\Phi f = 0 \) and therefore (3.1) is verified for every \( f \in \Lambda^0 (M) \).

Let now \( \sigma \in \Lambda^1 (M) \).
We will prove firstly that

\[(3.3) \quad \mathcal{I}_{b(\Phi)}(\sigma)(X, Y) = -\sigma\left(\frac{1}{2}R^{-1}_\Phi N_{R_\Phi}(X, Y)\right).\]

By using Remark 3, we have

\[\mathcal{L}_{Id_{T(M)}, Id_{T(M)}} = [\mathcal{L}_{Id_{T(M)}}, \mathcal{L}_{Id_{T(M)}}] = [d, d] = 0\]

and

\[\mathcal{L}_\Phi, \mathcal{L}_{Id_{T(M)}} = [\mathcal{L}_\Phi, d] = \nabla \mathcal{L}_\Phi = 0.\]

So

\[\mathcal{L}_{\Phi, R_\Phi}_{FN} = [Id_{T(M)} + \Phi, Id_{T(M)} + \Phi]_{FN} = [\Phi, \Phi]_{FN}\]

and by Proposition 1

\[(3.4) \quad 2N_{R_\Phi} = [R_\Phi, R_\Phi]_{FN} = [\Phi, \Phi]_{FN} = 2N_{\Phi}.\]

By Remark 1 it follows that

\[\mathcal{I}_{\Phi, R^{-1}_\Phi}_{FN}(\sigma)(X, Y) = -\sigma\left(\frac{1}{2}R^{-1}_\Phi N_{R_\Phi}(X, Y)\right)\]

and (3.3) is proved.

We will compute now \(\mathcal{L}_\Phi \sigma, d_\Phi \sigma\) and \(\mathcal{I}_{b(\Phi)} \sigma\):

We remark that (3.3) gives \(\mathcal{L}_\Phi \sigma = [\mathcal{I}_\Phi, d](\sigma)\) and thus

\[\mathcal{L}_\Phi \sigma = [\mathcal{I}_\Phi, d](\sigma)(X, Y) = (\mathcal{I}_\Phi d\sigma)(X, Y) - d(\mathcal{I}_\Phi \sigma)(X, Y)\]

\[
\begin{align*}
&= d\sigma(\Phi X, Y) + d\sigma(X, \Phi Y) \\
&\quad - X(\mathcal{I}_\Phi \sigma(Y)) + Y(\mathcal{I}_\Phi \sigma(X)) + (\mathcal{I}_\Phi \sigma)[X, Y] \\
&= (\Phi X)(\sigma(Y)) - Y(\sigma(\Phi X)) - \sigma(\Phi X, Y) \\
&\quad + X(\sigma(\Phi Y)) - (\Phi Y)(\sigma(X)) - \sigma([X, \Phi Y]) \\
&\quad - X(\sigma(\Phi Y)) + Y(\sigma(\Phi X)) + (\Phi ([X, Y])) \\
&= (\Phi X)(\sigma(Y)) - (\Phi Y)(\sigma(X)) + \sigma(\Phi([X, Y]) - [\Phi X, Y] - [X, \Phi Y])).
\end{align*}
\]

\[d_\Phi \sigma(X, Y) = (R_\Phi dR^{-1}_\Phi)(\sigma)(X, Y) = (dR^{-1}_\Phi \sigma)(R_\Phi X, R_\Phi Y)\]

\[= R_\Phi X(\sigma(R_\Phi Y)) - R_\Phi Y(\sigma(R_\Phi X)) - R^{-1}_\Phi \sigma([R_\Phi X, R_\Phi Y])\]

\[= R_\Phi X(\sigma Y) - R_\Phi Y(\sigma X) - R^{-1}_\Phi \sigma([R_\Phi X, R_\Phi Y])\]

\[= X(\Phi X)(\sigma Y) - (\Phi Y)(\sigma X) - \sigma(\Phi([X, Y]) - [\Phi X, Y] - [X, \Phi Y])\]

\[(3.6)\]

By developing (3.3) we have

\[\mathcal{I}_{b(\Phi)} \sigma(X, Y) = -\sigma(R^{-1}_\Phi N_{R_\Phi}(X, Y))\]

\[= \sigma(R^{-1}_\Phi([R_\Phi X, R_\Phi Y]) + R_\Phi[X, Y] - [R_\Phi X, Y] - [X, R_\Phi Y])\]

\[= \sigma(R^{-1}_\Phi([R_\Phi X, R_\Phi Y])) + \sigma([X, Y] + \Phi [X, Y])\]

\[= -\sigma([X, Y] + \Phi [X, Y] + [X, \Phi Y])\]

\[(3.7)\]
Since
\[ d\sigma (X,Y) = X (\sigma Y) - Y (\sigma X) - \sigma [X,Y], \]
by comparing (3.5), (3.6) and (3.7) it follows that (3.1) is verified for each form in \( \Lambda^1 M \) and the Lemma is proved. \( \square \)

**Theorem 2.** Let \( \Phi \in \Lambda^1 M \otimes TM \) such that \( R_\Phi = Id_{T(M)} + \Phi \) is invertible. Then:

i) \( e_\Phi \) is a solution of the Maurer-Cartan equation in \((D^* (M), [\cdot, \cdot], \tau)\).

ii) Let \( \Psi \in \Lambda^2 M \otimes TM \) such that \( D = L_\Phi + I_\Psi \) is a solution of the Maurer-Cartan equation in \((D^* (M), [\cdot, \cdot], \tau)\). Then \( \Psi = b(\Phi) \).

**Proof.**

i) Since \( [d,d] = 0 \), \( [R_\Phi dR_\Phi^{-1}, R_\Phi dR_\Phi^{-1}] = 0 \), and \( [d, R_\Phi dR_\Phi^{-1}] = [R_\Phi dR_\Phi^{-1}, d] \) it follows that
\[ \nabla e_\Phi + \frac{1}{2} [e_\Phi, e_\Phi] = [d, R_\Phi dR_\Phi^{-1} - d] + \frac{1}{2} [R_\Phi dR_\Phi^{-1} - d, R_\Phi dR_\Phi^{-1} - d] \]
\[ = [d, R_\Phi dR_\Phi^{-1} - d] - [d, R_\Phi dR_\Phi^{-1}] = 0. \]

ii) Let \( D = L_\Phi + I_\Psi \), \( \Phi \in \Lambda^1 M \otimes TM \), \( \Psi \in \Lambda^2 M \otimes TM \). By using Lemma 1 and Lemma 5 we have
\[ \nabla D = \nabla I_\Psi = L_\Phi \]
and
\[ [D, D] = [L_\Phi + I_\Psi, L_\Phi + I_\Psi] = L_{[\Phi, \Phi]} + 2 [L_\Phi, I_\Psi] + [I_\Psi, I_\Psi] \]
\[ = L_{[\Phi, \Phi]} + 2 [I_\Phi, I_\Psi] + [I_\Psi, I_\Psi]. \]

so
\[ \nabla \frac{1}{2} [D, D] = L_\Psi + \frac{1}{2} L_{[\Phi, \Phi]} + (2 I_\Phi + 2 I_\Psi) + \frac{1}{2} [I_\Psi, I_\Psi]. \]

It follows that
\[ L \left( \nabla \frac{1}{2} [D, D] \right) = L_\Psi + \frac{1}{2} L_{[\Phi, \Phi]} + L_\Psi \Phi \]
and
\[ J \left( \nabla \frac{1}{2} [D, D] \right) = I_{[\Phi, \Phi]} + \frac{1}{2} [I_\Psi, I_\Psi]. \]

Suppose that \( D = L_\Phi + I_\Psi \) verifies the Maurer-Cartan equation. Then
\[ 0 = L \left( \nabla \frac{1}{2} [D, D] \right) = L \left( \Psi + \frac{1}{2} [\Phi, \Phi] + I_\Psi \Phi \right) \]
Since \( L \) is injective, this implies
\[ \Psi + \frac{1}{2} [\Phi, \Phi] + I_\Psi \Phi = 0. \]

By Lemma 6 we obtain
\[ \Psi + \frac{1}{2} [\Phi, \Phi] + \Phi \Psi = 0, \]
which is equivalent to
\[ \Psi = -\frac{1}{2} (Id_{TM} + \Phi)^{-1} [\Phi, \Phi] = b(\Phi). \]

**Definition 11.** Let \( \Phi \in \Lambda^1 M \otimes TM \) such that \( Id_{T(M)} + \Phi \) is invertible. \( e_\Phi \) is called the canonical solution of Maurer-Cartan equation associated to \( \Phi \).
4. Canonical solutions of finite type of Maurer-Cartan equation

**Theorem 3.** Let $\Phi \in \Lambda^1 M \otimes TM$ small enough such that $\sum_{k=0}^{\infty} (-1)^k \Phi^k = (\text{Id}_{T(M)} + \Phi)^{-1} \in \Lambda^1 M \otimes TM$ and $e_\Phi$ the canonical solution of Maurer-Cartan equation associated to $\Phi$. Then

a) $e_\Phi = \sum_{k=1}^{\infty} \gamma_k$, where $\gamma_k \in \mathcal{D}^1 (M)$ are defined by induction as

$$\gamma_1 = L_\Phi, \quad \gamma_k = (-1)^k \frac{1}{2} \sum_{(p,q) \in \mathbb{N}^2, p+q=k} \mathcal{N} \left( [\gamma_p, \gamma_q] \right), \quad k \geq 2.$$

b) $\gamma_k = (-1)^{k+1} \frac{1}{2} I_{\Phi^{k-2}[\Phi, \Phi]^N}, \quad k \geq 2.$

c) $\mathcal{I}_{b(\Phi)} = \sum_{k=2}^{\infty} \gamma_k$.

**Proof.** We remark that for $r \geq 2$, $\gamma_r \in \mathcal{I} (M)$, so by Lemma 2.6, it follows that $[\gamma_p, \gamma_q] \in \mathcal{I} (M)$ for $p, q \geq 2$. Since $\mathcal{N} \mathcal{I} (M) = 0$ we have $\mathcal{N} [\gamma_p, \gamma_q] = 0$ for $p, q \geq 2$ and so

$$\gamma_r = - (-1)^r \frac{1}{2} \mathcal{N} \left( \left[ \gamma_1, \gamma_{r-1} \right] + \left[ \gamma_1, \gamma_{r-1} \right] \right) = - (-1)^r \mathcal{N} \left( \gamma_1, \gamma_{r-1} \right), \quad r \geq 2.$$

We will show by induction that for every $r \geq 2$

$$\gamma_r = - (-1)^r \frac{1}{2} I_{\Phi^{r-2}[\Phi, \Phi]}.$$

Suppose that for every $r \geq 3$

$$\gamma_{r-1} = - (-1)^k \frac{1}{2} I_{\Phi^{r-3}[\Phi, \Phi]}.$$

Then

$$\mathcal{N} \left[ \gamma_1, \gamma_{r-1} \right] = \mathcal{N} \left[ L_\Phi, - (-1)^{r-1} \frac{1}{2} I_{\Phi^{r-3}[\Phi, \Phi]} \right],$$

and by Lemma 2.6 we have

$$\left[ L_\Phi, I_{\Phi^{r-3}[\Phi, \Phi]} \right] = \left[ I_{\Phi^{r-3}[\Phi, \Phi]^N}, \Phi \right] - (-1)^{r-1} \Phi^r \left( I_{\Phi^{r-3}[\Phi, \Phi]^N} \Phi \right) I_{\Phi^{r-3}[\Phi, \Phi]^N} \Phi.$$

So, from (4.3) and (4.4) we obtain

$$\mathcal{N} \left[ \gamma_1, \gamma_{r-1} \right] = - \frac{1}{2} (-1)^{r-1} \mathcal{N} \left( L_{\Phi^{r-3}[\Phi, \Phi]^N} \Phi \right) = - \frac{1}{2} (-1)^{r-1} \left( -1 \right)^{r-1} \left( -1 \right)^{r-3} \Phi^r \left( I_{\Phi^{r-3}[\Phi, \Phi]^N} \Phi \right) I_{\Phi^{r-3}[\Phi, \Phi]^N} \Phi = - \frac{1}{2} (-1)^r I_{\Phi^{r-3}[\Phi, \Phi]^N} \Phi.$$

But by Lemma 6

$$I_{\Phi^{r-3}[\Phi, \Phi]^N} \Phi = \Phi^{r-2} [\Phi, \Phi]^N,$$

and (4.5) is verified.

It follows that

$$\mathcal{I}_{b(\Phi)} = \sum_{k=0}^{\infty} (-1)^k \Phi^k [\Phi, \Phi]^N = \sum_{k=0}^{\infty} - (-1)^k \frac{1}{2} I_{\Phi^k [\Phi, \Phi]^N} = \sum_{k=2}^{\infty} \gamma_k.$$
By Theorem 3.1
\[ e_\Phi = \mathcal{L}_\Phi + \mathcal{I}_{b(\Phi)} = \sum_{k=1}^{\infty} \gamma_k \]
and the Proposition is proved. \(\square\)

**Definition 12.** Let \( \Phi \in \Lambda^1 M \otimes TM \) small enough such that \( \sum_{k=0}^{\infty} (-1)^k \Phi^k = (Id_{TM} + \Phi)^{-1} \in \Lambda^1 M \otimes TM \) and \( e_\Phi \) the canonical solution of Maurer-Cartan equation associated to \( \Phi \). \( e_\Phi \) is called of finite type if there exists \( r \in \mathbb{N} \) if \( \Phi^r [\Phi, \Phi]_{\mathcal{FN}} = 0 \) and of finite type \( r \) if \( r = \min \{ s \in \mathbb{N} : \Phi^s [\Phi, \Phi]_{\mathcal{FN}} = 0 \} \).

**Remark 4.** Let \( e_\Phi \) the canonical solution of Maurer-Cartan equation corresponding to \( \Phi \in \Lambda^1 M \otimes TM \). Suppose that \( e_\Phi \) is of finite type \( r \). Then
\[ e_\Phi = \sum_{k=1}^{r+1} \gamma_k. \]

**Proposition 2.** Let \( \Phi \in \Lambda^1 M \otimes TM \) such that \( R_\Phi \) is invertible. The following are equivalent:

i) The canonical solution \( e_\Phi \) of Maurer-Cartan equation corresponding to \( \Phi \) is of finite type 0.

ii) \( e_\Phi \) is \( \mathcal{T} \)-closed.

iii) \( d_\Phi \) is \( \mathcal{T} \)-closed.

iv) \( N_\Phi = 0 \).

**Proof:** i) \( \iff \) ii) Suppose that the canonical solution \( e_\Phi \) of Maurer-Cartan equation corresponding to \( \Phi \) is of finite type 0. Then by Remark 4 it follows that
\[ e_\Phi = \gamma_1 = \mathcal{L}_\Phi. \]

and by Lemma 2 it follows that \( e_\Phi \) is \( \mathcal{T} \)-closed.

Conversely, suppose \( \mathcal{T} e_\Phi = 0 \). By using again Lemma 2 it follows that \( e_\Phi \in \mathcal{L}(M) \). In particular \( \mathcal{I}_{b(\Phi)} = 0 \), so \( [\Phi, \Phi]_{\mathcal{FN}} = 0 \).

ii) \( \iff \) iii) We have \( d = \mathcal{I}_{Id_{TM}} \), so \( \mathcal{T} d = 0 \). Since \( e_\Phi = d_\Phi - d \) the assertion follows.

i) \( \iff \) iv) By Proposition 1 \( [\Phi, \Phi]_{\mathcal{FN}} = 2N_\Phi \) so \( N_\Phi = 0 \) if and only if \( [\Phi, \Phi]_{\mathcal{FN}} = 0 \). \(\square\)

By using Proposition 2 we obtain:

**Corollary 1.** Let \( M \) be a smooth manifold and \( J \) an almost complex structure on \( M \). Then the canonical solution associated to \( J \) is of finite type 0 if and only if \( N_J = 0 \), i.e. if and only if \( J \) is integrable.

**Theorem 4.** Let \( M \) be a smooth manifold and \( \xi \subset TM \) a distribution. Let \( \zeta \subset TM \) such that \( TM = \xi \oplus \zeta \) and consider \( \Phi \in \text{End}(TM) \) defined by \( \Phi = 0 \) on \( \xi \) and \( \Phi = \text{Id} \) on \( \zeta \). Then:

1. The canonical solution associated to \( \Phi \) is of finite type 0 if and only if \( \xi \) and \( \zeta \) are integrable.
2. The canonical solution associated to \( \Phi \) is of finite type 1 if and only if \( \xi \) is integrable and \( \zeta \) is not integrable.
3. If \( \xi \) is not integrable then \( \Phi^k [\Phi, \Phi]_{\mathcal{FN}} \neq 0 \) for every \( k \in \mathbb{N} \).
Proof. Let $Y, Z \in \xi$. Since $\Phi^k = \Phi$ for every $k \geq 1$ and $\Phi Y = \Phi Z = 0$, 

$$(\Phi [\Phi, \Phi]_{\mathcal{F}_X})(Y, Z) = \Phi (\Phi [\Phi, \Phi]_{\mathcal{F}_X}(Y, Z)) \quad (4.6)$$

Suppose that the canonical solution associated to $\Phi$ is of finite type $\leq 1$. Then $\Phi [\Phi, \Phi]_{\mathcal{F}_X}(Y, Z) = 0$ for every $Y, Z \in \xi$ and by (4.6) it follows that $[Y, Z] \in \xi$. Therefore $\xi$ is integrable by the theorem of Frobenius.

Suppose now that $\xi$ is not integrable. There exist $Y, Z \in \xi$ such that $[Y, Z] \notin \xi$. By (4.6) we obtain

$$\begin{align*}
(\Phi [\Phi, \Phi]_{\mathcal{F}_X})(Y, Z) &= \Phi^k ([\Phi Y, \Phi Z] + \Phi^{k+2} [Y, Z] - \Phi^{k+1} [\Phi Y, Z] - \Phi^{k+1} [Y, \Phi Z]) \\
&= \Phi ([Y, Z]) \neq 0
\end{align*}$$

for every $k \geq 1$.

Conversely, suppose that $\xi$ is integrable. Then for every $V, W \in \xi$, we have $[V, W] \in \xi$, so $\Phi ([V, W]) = 0$. Since $\Phi (TM) \subset \xi$, for every $V \in TM$, there exist unique $V_\xi \in \xi$ and $V_\zeta \in \zeta$ such that $V = V_\xi + V_\zeta$, and $\Phi V = V_\xi$. Since $\Phi^k = \Phi$ for every $k \geq 1$,

$$\begin{align*}
(\Phi [\Phi, \Phi]_{\mathcal{F}_X})(V, W) &= \Phi ([\Phi, \Phi]_{\mathcal{F}_X}(V, W)) \\
&= \Phi ([\Phi V, \Phi W] + \Phi^2 [V, W] - \Phi [\Phi V, W] - \Phi [V, \Phi W]) \\
&= \Phi ([V_\xi, W_\zeta]) + \Phi ([V_\zeta, W_\xi + W_\zeta]) - \Phi ([V_\zeta, W_\zeta + W_\xi + W_\zeta]) \\
&= \Phi [V_\xi, W_\zeta] + \Phi [V_\zeta, W_\xi] + \Phi [V_\xi, W_\zeta] + \Phi [V_\zeta, W_\xi] \\
&= 0
\end{align*}$$

and it follows that the canonical solution associated to $\Phi$ is of finite type $\leq 1$.

If $\zeta$ is integrable too, $\Phi [V_\zeta, W_\zeta] = [V_\zeta, W_\zeta]$ for every $V, W \in TM$ and

$$\begin{align*}
([\Phi, \Phi]_{\mathcal{F}_X})(V, W) &= [\Phi V, \Phi W] + \Phi^2 [V, W] - \Phi [\Phi V, W] - \Phi [V, \Phi W] \\
&= [V_\xi, W_\zeta] + \Phi [V_\xi + V_\zeta, W_\zeta] \\
&= [V_\zeta, W_\xi + W_\zeta] - \Phi [V_\zeta, W_\zeta] \\
&= 0
\end{align*}$$

so the canonical solution associated to $\Phi$ is of finite type 0.

If $\xi$ is integrable and $\zeta$ is not integrable, there exists $Y, Z \in \zeta$ such that $[Y, Z] \notin \xi$, so

$$\begin{align*}
([\Phi, \Phi]_{\mathcal{F}_X})(Y, Z) &= [\Phi Y, \Phi Z] + \Phi^2 [Y, Z] - \Phi [\Phi Y, Z] - \Phi [Y, \Phi Z] \\
&= [Y, Z] + \Phi [Y, Z] - \Phi [Y, Z] = [Y, Z] - \Phi [Y, Z] \neq 0
\end{align*}$$

and the Theorem follows. \(\square\)

Corollary 2. Let $M$ be a smooth manifold and $\xi \subset TM$ a co-orientable distribution of codimension 1. There exist $X \in \mathfrak{X} (M)$ and $\gamma \in \Lambda^1 (M)$ such $\xi = \ker \gamma$ and $\iota_X \gamma = 1$. We have $T(M) = \xi \oplus \mathbb{R} [X]$ and we consider $\Phi \in \text{End}(TM)$ defined by $\Phi = 0$ on $\xi$ and $\Phi = \text{Id}$ on $\mathbb{R} [X]$. Then the canonical solution associated to $\Phi$ is of finite type 0 if and only if $\xi$ is integrable.

Proof. We apply Proposition 4 for $\eta = \mathbb{R} [X]$ which is obviously integrable. \(\square\)
Theorem 5. Let $M$ be a smooth manifold of dimension $n$ and $\xi, \tau \subset TM$ distributions such that $\xi \subseteq \tau$. We consider $\eta, \zeta \subset TM$ distributions such that $\tau = \xi \oplus \eta$ and $TM = \tau \oplus \zeta$ and let $A : \eta \to \xi$, $B : \eta \to \eta$ such that $\xi = \ker K$, where $K : \tau \to \tau$ is defined by $K = 0$ on $\xi$ and $K = A + B$ on $\eta$. We suppose that there exists a natural number $m \geq 1$ such that $K^m = 0$. Let $\Phi \in \text{End}(TM)$ defined by $\Phi = K$ on $\tau$ and $\Phi = Id$ on $\zeta$. The following are equivalent:

1. $\tau$ is integrable.
2. The canonical solution associated to $\Phi$ is of finite type $\leq m$.

Proof. We have

$$K = \begin{pmatrix}
\dim \xi & \dim \eta \\
\dim \xi & A \\
\dim \eta & B
\end{pmatrix}$$

and

$$\Phi = \begin{pmatrix}
\dim \tau & \dim \zeta \\
\dim \tau & C \\
\dim \zeta & D
\end{pmatrix}.$$

So $K^m = 0$, $\Phi^m = 0$ on $\tau$ and $\Phi^m = Id$ on $\zeta$.

Let $Y, Z \in TM$, $Y = Y_\tau + Y_\zeta$, $Y_\tau = Y_\xi + Y_\eta$, $Z = Z_\tau + Z_\zeta$, $Z_\tau = Z_\xi + Z_\eta$, $Y_\xi, Z_\xi \in \xi$, $Y_\eta, Z_\eta \in \eta$, $Y_\zeta, Z_\zeta \in \zeta$. We have $\Phi^{m+j} = \Phi^m$ for every $j \in \mathbb{N}$, so

$$\Phi^m [Y, Z] = \Phi^m [\Phi Y, \Phi Z] + \Phi^m [Y, \Phi Z] + \Phi^m [\Phi Y, Z] + \Phi^m [Y, Z].$$

(4.7)

Suppose that $\tau$ is integrable. Since $\xi = \ker \Phi$,

$$\Phi Y = \Phi Y_\eta + \Phi Y_\zeta = Ay_\eta + By_\eta + Y_\zeta = C_\tau + Y_\zeta,$$

$$\Phi Z = \Phi Z_\eta + \Phi Z_\zeta = AZ_\eta + BZ_\eta + Z_\zeta = D_\tau + Y_\zeta,$$

where $C_\tau = AY_\eta + BY_\eta \in \tau$ and $D_\tau = AZ_\eta + BZ_\eta \in \tau$. Since $\tau$ is integrable, $[C_\tau, D_\tau] \in \tau = \ker \Phi^m$, so $\Phi^m ([C_\tau, D_\tau]) = 0$ and

$$\Phi^m [Y, Z] = \Phi^m (Y_\tau + Y_\zeta, Z_\tau + Z_\zeta).$$

(4.8)

Similarly, since $[Y_\tau, Z_\tau], [C_\tau, Z_\zeta], [Y_\tau, D_\tau] \in \tau = \ker \Phi^m$

$$\Phi^m [Y, Z] = \Phi^m (Y_\tau + Y_\zeta, Z_\tau + Z_\zeta).$$

(4.9)

Replacing (4.8), (4.9), (4.10), (4.11) in (4.7) we obtain

$$\Phi^m [Y, Z] = \Phi^m ([C_\tau, Z_\zeta]) + \Phi^m ([Y_\zeta, Z_\zeta]) + \Phi^m ([Y_\tau, Z_\zeta]) + \Phi^m ([Y_\tau, Z_\zeta]) = 0.$$
and it follows that the canonical solution associated to $\Phi$ is of finite type $\leq m$.

Conversely, suppose that the canonical solution of the Maurer-Cartan equation associated to $\Phi$ is of finite type $k \leq m$, i.e. $\Phi^m [\Phi, \Phi]_{\mathcal{F}_{\mathcal{A}}} = 0$. We will prove that $[Y, Z] \in \tau = \text{Ker } \Phi^m$ for every $Y, Z \in \tau$ by taking in account several cases.

a) Let $Y, Z \in \xi$. Then $\Phi Y = \Phi Z = 0$ and by using (4.17) we obtain

$$\Phi^m [\Phi, \Phi]_{\mathcal{F}_{\mathcal{A}}} ([Y, Z]) = \Phi^{m+2} [Y, Z] = \Phi^m [Y, Z] = 0.$$ 

So $[Y, Z] \in \tau$.

b) Let $Y \in \xi, Z \in \tau, Z = Z_\xi + Z_\eta$. Then

$$[Y, Z] = [Y, Z_\xi] + [Y, Z_\eta].$$

(4.12)

By a) $[Y, Z_\xi] \in \tau$ and from (4.12) it follows that $[Y, Z] \in \tau$ if and only if $[Y, Z_\eta] \in \tau$.

Since $\Phi Y = 0$, by using (4.7) we have

$$[\Phi, \Phi]_{\mathcal{F}_{\mathcal{A}}} ([Y, Z_\eta]) = [\Phi Y, \Phi Z_\eta] + \Phi^2 [Y, Z_\eta] - \Phi [\Phi Y, Z_\eta] - \Phi [Y, \Phi Z_\eta]$$

and

$$\Phi^m [\Phi, \Phi]_{\mathcal{F}_{\mathcal{A}}} ([Y, Z_\eta]) = \Phi^m [Y, Z_\eta] - \Phi^m [Y, \Phi Z_\eta]$$

$$= \Phi^m ([Y, (\text{Id} - \Phi) Z_\eta]) = 0.$$ 

In particular $[Y, (\text{Id} - \Phi) Z_\eta] \in \tau$ for every $Z_\eta \in \eta$.

But

$$\det (\text{Id} - \Phi|_\eta) = \det (\text{Id} - B) = 1,$$

so for every $X_\eta \in \eta$ there exists $Z_\eta \in \eta$ such that $X_\eta = (\text{Id} - \Phi) Z_\eta$. It follows that $[Y, Z] \in \tau$ for every $Y \in \xi$ and $Z \in \tau$.

c) Let $Y, Z \in \tau, Y = Y_\xi + Y_\eta, Z = Z_\xi + Z_\eta, Y_\xi, Z_\xi \in \xi, Y_\eta, Z_\eta \in \eta$. Since

$$[Y, Z] = [Y_\xi + Y_\eta, Z_\xi + Z_\eta] = [Y_\xi, Z_\xi] + [Y_\eta, Z_\eta] + [Y_\eta, Z_\xi] + [Y_\xi, Z_\eta].$$

and $[Y_\xi, Z_\xi], [Y_\eta, Z_\eta], [Y_\xi, Z_\eta] \in \tau$ it follows that $[Y, Z] \in \tau$ if and only if $[Y_\eta, Z_\eta] \in \tau$.

We have

$$[\Phi, \Phi]_{\mathcal{F}_{\mathcal{A}}} ([Y_\eta, Z_\eta]) = [\Phi Y_\eta, \Phi Z_\eta] + \Phi^2 [Y_\eta, Z_\eta] - \Phi [\Phi Y_\eta, Z_\eta] - \Phi [Y_\eta, \Phi Z_\eta]$$

$$= [AY_\eta + B Y_\eta, AZ_\eta + B Z_\eta] + \Phi^2 [Y_\eta, Z_\eta]$$

$$- \Phi [AY_\eta + B Y_\eta, Z_\eta] - \Phi [Y_\eta, AZ_\eta + B Z_\eta].$$

Since $AY_\eta, AZ_\eta \in \xi$ it follows that $[AY_\eta, AZ_\eta], [BY_\eta, BZ_\eta], [BY_\eta, AZ_\eta], [AY_\eta, AZ_\eta], [Y_\eta, AZ_\eta] \in \tau = \text{ker } \Phi^r$ and

$$\Phi^m [\Phi, \Phi]_{\mathcal{F}_{\mathcal{A}}} ([Y_\eta, Z_\eta]) = \Phi^m [BY_\eta, BZ_\eta] + \Phi^m [Y_\eta, Z_\eta] - \Phi^m [BY_\eta, Z_\eta] - \Phi^m [Y_\eta, BZ_\eta]$$

$$= \Phi^m [Y_\eta, (\text{Id} - B) Z_\eta] - \Phi^m [BY_\eta, (\text{Id} - B) Z_\eta]$$

$$= \Phi^m [(\text{Id} - B) Y_\eta, (\text{Id} - B) Z_\eta] = 0$$

for every $Y_\eta, Z_\eta \in \eta$.

As before, by (4.12) it follows that $\Phi^r [Y_\eta, Z_\eta] = 0$ for every $Y_\eta, Z_\eta \in \eta$ and this implies that $[Y, Z] \in \tau$ for every $Y, Z \in \tau$.

In order to compute the type of the canonical solution of Theorem 5 we need the following elementary lemma:
Lemma 8. Let

\[ K = \begin{pmatrix} s & d-s \\ s \{ & 0 & \phi \\ d-s \{ & 0 & B \end{pmatrix} \]

a \((d,d)\) nilpotent matrix of rank \(d-s > 0, s \geq 1\). Set \( r = \min \{ m \in \mathbb{N} : K^m = 0 \} \).

Then \( r = \min \{ m \in \mathbb{N} : m \geq \frac{d}{s} \} \).

Proof. Since \( K \) is nilpotent of maximal rank we may suppose that

\[ K = \begin{pmatrix} s & d-s \\ s \{ & 0 & \phi \\ d-s \{ & 0 & 0 \end{pmatrix} \]

By induction it follows that if \( d - js > 0 \), we have

\[ K^j = \begin{pmatrix} js & d-js \\ js \{ & 0 & \phi \\ d-js \{ & 0 & 0 \end{pmatrix} \neq 0 \]

and \( K^j = 0 \) for each \( j \in \mathbb{N}^* \) such that \( d - js \leq 0 \).

Notation 5. Let \( \xi \subset TM \) a distribution. We denote by \( \xi^* \) the smallest involutive subset of \( TM \) such that \( \xi \subset \xi^* \). If \( \mathcal{E} = \{X_1, \cdots, X_s\} \) are generators of \( \xi \) on an open subset \( U \) of \( M \), then for every \( x \in U \), \( \xi^*_x \) is the linear subspace of \( T_xM \) generated by \([X_{i_1}, [X_{i_2}, \cdots, X_{i_k}]](x) \), \( k \geq 1 \), \( 1 \leq i_k \leq s \).

Remark 5. If \( \dim \xi^*_x \) is independent of \( x \), \( \xi^* \) is a distribution, but in general \( \dim \xi^*_x \) depends on \( x \). If \( \xi^* \) is a distribution, then \( \xi^* \) is the smallest integrable distribution containing \( \xi \) [22].

Corollary 3. Let \( M \) be a smooth manifold of dimension \( n \), \( \xi \subset TM \) a distribution of dimension \( s \) such that \( \xi^* \) is a distribution of dimension \( d \). Then for every \( x \in M \) there exists a neighborhood \( U \) of \( x \) and \( \Phi \in \Lambda^1(U,TU) \) such that the canonical solution of the Maurer-Cartan equation associated to \( \Phi \) is of finite type \( \leq r \), where \( r = \min \{ m \in \mathbb{N} : m \geq \frac{d}{s} \} \).

Proof. If \( \xi \) is integrable, \( d = s, r = 1 \) and the corollary follows from Theorem 4.

Suppose that \( \xi \) is not integrable, i. e. \( d > s \). For each \( x \in M \) there exists a neighborhood \( U \) of \( x \) and a basis \((X_1, \cdots, X_n)\) of \( TM \) on \( U \) such that \((X_1, \cdots, X_s)\) is a basis of \( \xi \) and \((X_1, \cdots, X_d)\) is a basis of \( \xi^* \) on \( U \).

We define \( \Phi \in \text{End}(TU) \) as \( \Phi X_i = 0, i = 1, \cdots, s \) \( \Phi (X_i) = X_{i-s}, i = s+1, \cdots, d \), \( \Phi (X_i) = X_i, i = d+1, \cdots, n \). Then the matrix of \( \Phi \) in the basis \((X_1, \cdots, X_n)\) is

\[ \Phi = \begin{pmatrix} d & n-d \\ d \{ & K & 0 \\ n-d \{ & 0 & \phi \end{pmatrix} \]

where

\[ K = \begin{pmatrix} s & d-s \\ s \{ & 0 & \phi \\ d-s \{ & 0 & 0 \end{pmatrix} \]

Since \( r \geq 2 \), by Lemma 3 and Theorem 3 the canonical equation solution of the Maurer-Cartan equation associated to \( \Phi \) is of finite type \( \leq r \). \( \square \)
Remark 6. In [2] it is proved that the deformation theory in the DGLA \((\mathcal{D}^* (M), \partial, [\cdot, \cdot])\) is not obstructed but it is level-wise obstructed.

5. Deformations of foliations of codimension 1

Definition 13. By a differentiable family of deformations of an integrable distribution \(\xi\) we mean a differentiable family \(\omega : \mathcal{D} = (\xi_t)_{t \in I} \mapsto t \in I = [-a, a], a > 0\), of integrable distributions such that \(\xi_0 = \omega^{-1}(0) = \xi\).

Remark 7. An integrable distribution \(\xi\) of codimension 1 in a smooth manifold \(L\) is called co-orientable if the normal space to the foliation defined by \(\xi\) is orientable. We recall that \(\xi\) is co-orientable if and only if there exists a 1-form \(\gamma\) on \(L\) such that \(\xi = \ker \gamma\) (see for ex. [7]). A couple \((\gamma, X)\) where \(\gamma \in \Lambda^1 (L)\) and \(X\) is a vector field on \(L\) such that \(\ker \gamma = \xi\) and \(\gamma (X) = 1\) was called a DGLA defining couple in [1].

If \((\xi_t)_{t \in I}\) is a differentiable family of deformations of an integrable co-orientable distribution \(\xi\), then the distribution \(\xi_t\) is co-orientable for \(t\) small enough. So, if \(\xi\) is an integrable co-orientable distribution of codimension 1 in \(L\) and \((\xi_t)_{t \in I}\) is a differentiable family of deformations of \(\xi\) we may consider a DGLA defining couple \((\gamma_t, X_t)\) for every \(t\) small enough such that \(t \mapsto (\gamma_t, X_t)\) is differentiable on a neighborhood of the origin of the origin.

Lemma 9. Let \(L\) be a \(C^\infty\) manifold and \(\xi \subset T \langle L \rangle\) a co-orientable distribution of codimension 1. Let \((\gamma, X)\) be a DGLA defining couple and denote \(\Phi \in \text{End}(TM)\) the endomorphism corresponding to \(\gamma \otimes X \in \Lambda^1 M \otimes TM\). Then \(\Phi\) is defined on \(TM = \xi \oplus \mathbb{R} \langle X \rangle\) as \(\Phi = 0\) on \(\xi\) and \(\Phi = \text{Id} on \mathbb{R} \langle X \rangle\).

Proof. Let \(Y = Y_\xi + \lambda X\) vector fields on \(L\), \(V_\xi \in \xi\), \(\lambda \in \mathbb{R}\). Then

\[
(\gamma \otimes X)(Y) = \gamma(Y)X = \gamma(Y_\xi + \lambda X)X = \lambda X.
\]

\[\square\]

Lemma 10. Let \(L\) be a \(C^\infty\) manifold and \(\xi \subset T \langle L \rangle\) a co-orientable distribution of codimension 1. Let \((\gamma, X)\) be a DGLA defining couple. Then the following are equivalent:

i) \(\xi\) is integrable;

ii) \(d\gamma = -\iota_X d\gamma \wedge \gamma\);

iii) \([\gamma \otimes X, \gamma \otimes X]_{\mathcal{FN}} = 0\).

Proof. i) \(\iff\) ii) is a variant of the theorem of Frobenius and it was proved in [1].

ii) \(\iff\) iii). We have

\[
[\gamma \otimes X, \gamma \otimes X]_{\mathcal{FN}} = \gamma \wedge \mathcal{L}_X \gamma \otimes X - \mathcal{L}_X \gamma \wedge \gamma \otimes X - (d\gamma \wedge \iota_X d\gamma \otimes X + \iota_X \gamma \wedge d\gamma \otimes X)
\]

\[= 2\gamma \wedge \mathcal{L}_X \gamma \otimes X - 2d\gamma \otimes X = 2(\gamma \wedge d\iota_X \gamma + \gamma \wedge \iota_X d\gamma - d\gamma) \otimes X
\]

\[= 2(\gamma \wedge \iota_X d\gamma - d\gamma) \otimes X.
\]

\[\square\]

We recall the following lemma from [1]:

Lemma 11. Let \(L\) be a \(C^\infty\) manifold and \(X\) a vector field on \(L\). For \(\alpha, \beta \in \Lambda^* (L)\), set

\[
(5.1) \quad \{\alpha, \beta\} = \mathcal{L}_X \alpha \wedge \beta - \alpha \wedge \mathcal{L}_X \beta
\]

where \(\mathcal{L}_X\) is the Lie derivative. Then \((\Lambda^* (L), d, \{\cdot, \cdot\})\) is a DGLA.
Proposition 3. Let $L$ be a $C^2$ manifold and $\xi \subset T(L)$ an integrable co-orientable distribution of codimension 1. Let $(\xi_t)_{t \in I}$ be a differentiable family of deformations of $\xi$ such that $\xi_t$ is co-orientable and integrable for every $t \in I$ and let $(\gamma_t, X_t)$ a DGLA defining couple for $\xi_t$ such that $t \mapsto (\gamma_t, X_t)$ is differentiable on $I$. Denote $\gamma = \gamma_0$, $\alpha = \frac{d\gamma}{dt}|_{t=0}$, $X = X_0$, $Y = \frac{dX}{dt}|_{t=0}$. Then
$$\delta \alpha + \mathcal{L}_Y \gamma \wedge \gamma = 0$$
where
$$\delta = d + \{\cdot, \cdot\}$$
and $\{\cdot, \cdot\}$ is defined in (5.7).

In particular $\delta \alpha (V, W) = 0$ for every vector fields $V, W$ tangent to $\xi$.

Proof. Since
$$\gamma_t (X_t) = (\gamma + t\alpha + o(t)) (X + tY + o(t)) = 1 + t (\alpha (X) + \gamma (Y)) + o(t) = 1$$

it follows that

(5.2)
$$\alpha (X) + \gamma (Y) = 0.$$

Denote $\sigma (t) = \gamma_t \otimes X_t \in A^1 M \otimes TM$. By Corollary \[\text{and Lemma } \[\text{the canonical solution of the Maurer-Cartan equation in } (D^* (L), [\cdot, \cdot], \mathcal{I}) \text{ associated to } \sigma (t) \text{ is of finite type 0 for every } t, \text{ so } [\sigma (t), \sigma (t)]_{\mathcal{F}_N} = 0 \text{ for every } t. \text{ We have}
$$\sigma (t) = \gamma_t \otimes X_t = (\gamma + t\alpha + o(t)) \otimes (X + tY + o(t))$$
$$= \gamma \otimes X + t (\alpha \otimes X + \gamma \otimes Y) + o(t)$$

and
$$[\sigma (t), \sigma (t)]_{\mathcal{F}_N} = [\gamma \otimes X, \alpha \otimes X + \gamma \otimes Y]_{\mathcal{F}_N} + 2 (\gamma \otimes X, \alpha \otimes X)_{\mathcal{F}_N} + o(t).$$

By Lemma \[ $[\gamma \otimes X, \alpha \otimes X]_{\mathcal{F}_N} = 0$, so
$$[\sigma (t), \sigma (t)]_{\mathcal{F}_N} = 2 (\gamma \otimes X, \alpha \otimes X + \gamma \otimes Y]_{\mathcal{F}_N} + o(t) = 0$$

and it follows that

(5.3)
$$[\gamma \otimes X, \alpha \otimes X + \gamma \otimes Y]_{\mathcal{F}_N} = 0.$$

But Proposition \[ gives
$$[\gamma \otimes X, \alpha \otimes X]_{\mathcal{F}_N} = \gamma \wedge \mathcal{L}_X \alpha \otimes X - \mathcal{L}_X \gamma \wedge \alpha \otimes X - (d\gamma \wedge \iota_X \alpha \otimes X + \iota_X \gamma \wedge d\alpha \otimes X)$$
$$= -\{\gamma, \alpha\} \otimes X - (\alpha (X) d\gamma \otimes X - d\alpha \otimes X)$$

(5.4)
$$= -\delta \alpha \otimes X - (\alpha (X) d\gamma \otimes X$$

and
$$[\gamma \otimes X, \gamma \otimes Y]_{\mathcal{F}_N} = \gamma \wedge \mathcal{L}_X \gamma \otimes Y - \mathcal{L}_X \gamma \wedge \gamma \otimes X$$
$$-d\gamma \wedge \iota_X Y - \iota_Y \gamma \wedge d\gamma \otimes X$$
$$= \gamma \wedge \iota_X d\gamma \otimes Y - \mathcal{L}_Y \gamma \wedge \gamma \otimes X$$
$$-d\gamma \otimes Y - \gamma (Y) d\gamma \otimes X.$$

By using Lemma \[ it follows that

(5.5)
$$[\gamma \otimes X, \gamma \otimes Y]_{\mathcal{F}_N} = -\mathcal{L}_Y \gamma \wedge \gamma \otimes X - \gamma (Y) d\gamma \otimes X$$

and by (5.3), (5.4) and (5.2) we obtain
$$-\delta \alpha - (\alpha (X) + \gamma (Y)) d\gamma - \mathcal{L}_Y \gamma \wedge \gamma = -\delta \alpha - \mathcal{L}_Y \gamma \wedge \gamma = 0.$$
Remark 8. A smooth hypersurface in a complex manifold is Levi flat if it admits a foliation of codimension 1 by complex manifolds. In [1] the authors studied the deformations of Levi flat hypersurfaces and obtained a second order elliptic differential equation for the infinitesimal deformations, which was used to prove the non existence of of transversally parallelizable Levi flat hypersurfaces in the complex projective plane. In [8] it is proved that the results of this paragraph lead to the same second order elliptic differential equation for the infinitesimal deformations of Levi flat hypersurfaces.

References

1. P. de Bartolomeis and A. Iordan, Deformations of Levi flat hypersurfaces in complex manifolds, Ann. Sci. Ecole Norm. Sup. 48 (2015), no. 2, 281–311.
2. ———, On the obstruction of the deformation theory in the DGLA of graded derivations, Complex and Symplectic Geometry (D. Angella, C. Medori and A. Tomassini, ed.), vol. 21, Springer Indam Series, 2017, pp. 95–105.
3. P. de Bartolomeis and S. V. Matveev, Some remarks on Nijenhuis bracket, formality and Kähler manifolds, Advances in Geometry 13 (2013), 571–581.
4. P. de Bartolomeis and A. Tomassini, Exotic deformations of Calabi-Yau manifolds, Ann. Inst. Fourier 63 (2013), 391–415.
5. A. Frölicher and A. Nijenhuis, Theory of vector valued differential forms. Part I. Derivations of the graded ring of differential forms, Indag. Math. 18 (1956), 338–359.
6. M. Gerstenhaber, On deformation on rings and algebras, Ann. of Math. 79 (1964), 59–103.
7. C. Godbillon, Feuilletages: Etudes géométriques, Birkhäuser Verlag, 1991.
8. A. Iordan, Infinitesimal deformations of Levi flat hypersurfaces, BUMI (2018), In Recent developments in complex and symplectic geometry, Special volume in memory of Paolo de Bartolomeis, to appear.
9. K. Kodaira and D. Spencer, Multifoliate structures, Ann. of Math. 74 (1961), no. 1, 52–100.
10. P. W. Michor, Topics in differential geometry, AMS, Providence, Rhode Island, 2008.
11. A. Nijenhuis and R. W. Richardson, Cohomology and deformations in graded Lie algebra, Bull. A. M. S. 72 (1966), 1–29.
12. H. J. Sussmann, Orbits of families of vector fields and integrability of distributions, Transactions of A. M. S. 180 (1973), 171–188.

Università degli Studi di Firenze, Dipartimento di Matematica e Informatica "U. Dini", Viale Morgagni 67/A I-50134, Firenze, Italia

Sorbonne Université, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586 du CNRS, case 247, 4 Place Jussieu, 75252 Paris Cedex 05, France

E-mail address: andrei.iordan@imj-prg.fr