A note on the dynamic dominant resource fairness mechanism

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Abstract. Multi-resource fair allocation has been a hot topic of resource allocation. Most recently, a dynamic dominant resource fairness (DRF) mechanism is proposed for dynamic multi-resource fair allocation. In this paper, we prove that the competitive ratio of the dynamic DRF mechanism is the reciprocal of the number of resource types, for two different objectives. Moreover, we develop a linear-time algorithm to find a dynamic DRF solution at each step.

Keywords: Multi-resource fair allocation; Dominant resource fairness; Dynamic dominant resource fairness; Competitive ratio.

1 Introduction

With the ever-growing demand for cloud resources, multi-resource (such as CPUs, memory, and bandwidth) fair allocation became a fundamental problem in cloud computing systems. The traditional slot-based scheduler for state-of-the-art cloud computing frameworks (for example, Hadoop) can lead to poor performance, unfairly punishing certain workloads. Ghodsi et al. [2] proposed a compelling alternative known as the dominant
resource fairness (DRF) mechanism, which is designed for Leontief preferences. DRF is to maximize the minimum dominant share of users, where the dominant share is the maximum share of any resource allocated to that user. DRF is generally applicable to multi-resource environments where users have heterogeneous demands, and is now implemented in the Hadoop Next Generation Fair Scheduler.

In recent years, DRF has attracted much attention and been generalized to many dimensions. Joe-Wong et al. [7] designed a unifying multi-resource allocation framework that captures the trade-offs between fairness and efficiency, which generalizes the DRF measure. Gutman and Nisan [3] situated DRF in a common economics framework, obtaining a general economic perspective. Parkes et al. [5] extended DRF in several ways, including the presence of zero demands and the case of indivisible tasks. Wang et al. [6] generalized the DRF measure into the cloud computing systems with heterogeneous servers. Most recently, Zarchy, Hay and Schapira [8] developed a framework for fair resource allocation that captures such implementation tradeoffs by allowing users to submit multiple resource demands.

DRF uses complete information about the requirements of all agents in order to find the fair solution. However, in reality, agents arrive over time, and we do not know the requirements of forthcoming agents before allocating the resources to the arrived agents. Recently, Kash, Procaccia and Shah [4] introduced a dynamic model of fair allocation and proposed a dynamic DRF mechanism. They mentioned that a dynamic DRF solution can be found by using water-filling algorithm or solving the corresponding linear program. However, the running time of the water-filling algorithm is pseudo-polynomial in worst-case scenario. Although solving a linear program can be done within polynomial time, the running time is high. It is desired to design an efficient algorithm to find a dynamic DRF solution.

In this paper, we further study the dynamic DRF mechanism. The rest of the paper is organized as follows. Section 2 describes the dynamic DRF mechanism. Section 3 gives the competitive ratios analysis of the dynamic DRF mechanism. Section 4 presents a polynomial-time algorithm, which can find a dynamic DRF solution in O(k) time at
every step $k$. Finally, Section 5 concludes the paper and gives the future work.

2 Dynamic Dominant Resource Fairness

Throughout this paper, assume that resources are divisible. In a multi-resource environment, there are $n$ agents and $m$ resources. Each agent $i$ requires $D_{ir}$-fraction of resource $r$ for each task, assuming that $D_{ir} > 0$ for each resource $r$. As defined in [2], the dominant resource of agent $i$ is the resource $r^*_i$ such that $D_{ir^*_i} = \max_r D_{ir}$, and $D_{ir^*_i}$ is called its dominant share. Following [4, 5], the normalized demand vector of agent $i$ is given by $d_i = (d_{i1}, \ldots, d_{im})$, where $d_{ir} = D_{ir}/D_{ir^*_i}$ for each resource $r = 1, \ldots, m$. Clearly, $d_{ir} \leq 1$ and $d_{ir^*_i} = 1$ for each agent $i$.

In the dynamic resource allocation model considered in [4], agents arrive at different times and do not depart. Assume that agent 1 arrives first, and in general agent $k$ arrives after agents 1, $\ldots$, $k-1$, for $k \geq 2$. For convenience, we say that agent $k$ arrives in step $k$. An agent reports its demand which does not change over time when it arrives. Thus, at step $k$, demand vectors $d_1, \ldots, d_k$ are known, and demand vectors $d_{k+1}, \ldots, d_n$ are unknown. At each step $k$, a dynamic DRF mechanism produces an allocation $A^k$ over the agents present in the system, where $A^k$ allocates $A^k_{ir}$-fraction of resource $r$ to agent $i$, subject to the feasibility condition

$$\sum_{i=1}^n A^k_{ir} \leq k/n, \forall r.$$  

(1)

Under the dynamic DRF mechanism, assume that allocations are irrevocable, i.e., $A^k_{ir} \geq A^{k-1}_{ir}$, for every step $k \geq 2$, every agent $i \leq k-1$, and every resource $r$. At every step $k$, assume $A^k$ is non-wasteful, which means that for every agent $i$ there exists $y \in \mathbb{R}^+$ such that for every resource $r$, $A^k_{ir} = y \cdot d_{ir}$. Let $x^k_i$ be dominant share of user $i$ at step $k$, which implies

$$A^k_{ir} = x^k_i \cdot d_{ir}, \text{ for } i = 1, 2, \ldots, k, \text{ and } r = 1, 2, \ldots, m.$$  

(2)

At every step $k$, the dynamic DRF mechanism [4] starts from the current allocation among the present agents 1, $\ldots$, $k$ and keeps allocating resources to agents that have the minimum
dominant share synchronously, until a $k/n$ fraction of at least one resource is allocated. Formally, at every step $k$, the dominant share vector $(x_1^k, \ldots, x_k^k)$ of the dynamic DRF allocation $A^k$ can be obtained by solving the following linear program:

$$\begin{align*}
\text{Maximize} & \quad M^k \\
x_i^k & \geq M^k, \forall i \leq k; \\
x_i^k & \geq x_i^{k-1}, \forall i \leq k - 1; \quad \text{(irrevocable)} \\
\sum_{i=1}^{k} d_{ir}x_i^k & \leq k/n, \forall r. \quad \text{(capacity constraints)}
\end{align*}$$

As shown in [4], the dynamic DRF mechanism satisfies many desired properties. Especially, it satisfies sharing incentives (SI) and dynamic Pareto optimality (DPO). SI means that, for all steps $k$ and all agents $i \leq k$, $x_i^k \geq 1/n$, i.e., when an agent arrives it receives an allocation that it likes at least as much as an equal split of the resources. DPO means that, for all steps $k$, there is a resource $r$ such that $\sum_{i=1}^{k} d_{ir}x_i^k = k/n$, i.e., it should not be possible to increase the allocation of an agent without decreasing the allocation of at least another user, subject to not allocating more that $k/n$ fraction of any resource.

3 Competitive ratios analysis

In [4], the authors analyzed the performance of the dynamic DRF mechanism on real data, for two objectives: the sum of dominant shares (the maxsum objective) and the minimum dominant share (the maxmin objective) of the agent present in the system. In this section, we analyze the performance of the dynamic DRF mechanism in the worst-case scenario. For a maximization problem, the competitive ratio $\rho$ of an online algorithm is the worst-case ratio between the cost of the solution found by the online algorithm and the cost of an optimal solution in an offline setting where all the demands of agents are known [1]. Clearly, $\rho \in [0, 1]$. Similarly, we define the competitive ratio of the dynamic DRF mechanism as the worst-case ratio between the objective value of the dynamic DRF solution $(x_1^k, \ldots, x_k^k)$ and the optimal solution $(\bar{x}_1^k, \ldots, \bar{x}_k^k)$ of instance $I$ under certain objective function. Accordingly, the competitive ratio $CR$ of the dynamic DRF mechanism.
is defined as

\[ CR = \min \min \frac{\text{The objective value of } (x^k_1, \ldots, x^k_k)}{\text{The objective value of } (\hat{x}^k_1, \ldots, \hat{x}^k_k)}. \quad (4) \]

### 3.1 The maxsum objective

When the objective is the sum of dominant shares maximization (maxsum, for short), for a given instance \( I \), the optimal solution \((\hat{x}^k_1, \ldots, \hat{x}^k_k)\) at step \( k \) (≥ 2) in the offline setting can be obtained by solving the following program

\[
\begin{align*}
& \text{Maximize} \sum_{i=1}^{k} x^k_i \\
& \quad \text{subject to} \sum_{i=1}^{k} d_{ir}^k x^k_i \leq k/n, \forall r. \\
& \end{align*}
\]

Accordingly, the competitive ratio \( CR_1 \) of the dynamic DRF mechanism for the maxsum objective can be defined as

\[ CR_1 = \min \min \frac{\sum_{i=1}^{k} x^k_i}{\sum_{i=1}^{k} \hat{x}^k_i}. \quad (6) \]

**Theorem 1.** When the objective is the sum of dominant shares maximization, the competitive ratio of the dynamic DRF mechanism is \( 1/m \), and the ratio is tight.

**Proof.** Since dynamic DRF mechanism satisfies SI, we have \( x^k_i \geq 1/n \) for every agent \( i \leq k \) at step \( k \), which implies that

\[ \sum_{i=1}^{k} x^k_i \geq \frac{k}{n}. \quad (7) \]

Consider the optimal solution \((\hat{x}^k_1, \ldots, \hat{x}^k_k)\) obtained from (5). Clearly, at step \( k \),

\[ \sum_{i \in r^*_i = r} \hat{x}^k_i = \sum_{i \in r^*_i = r} d_{ir^*_i} \hat{x}^k_i \leq \sum_{i=1}^{k} d_{ir} \hat{x}^k_i \leq \frac{k}{n}, \quad (8) \]

for every resource \( r \), following from the fact \( d_{ir^*_i} = 1 \) and the capacity constraint of (5). It implies that

\[ \sum_{i=1}^{n} \hat{x}^k_i \leq \sum_{r=1}^{m} \sum_{i \in r^*_i = r} \hat{x}^k_i \leq \frac{mk}{n}, \quad (9) \]
where the first inequality follows from the fact that each agent has at least one dominant resource. Thus, following (7) and (9), we have

\[
\sum_{i=1}^{k} x^k_i \geq \frac{1}{m},
\]

(10)
i.e., the competitive ratio of the dynamic DRF mechanism is at least \(1/m\).

Next, we will prove that the competitive ratio is tight. Consider a setting with \(m \geq 2\) resources and \(n \gg m\) agents. For \(i = 1, 2, \ldots, n-m\), the demand vector of agent \(i\) is \(d_i = (1, 1, \ldots, 1)\). For agents \(i = n-m+1, n-m+2, \ldots, n\), the demand vectors are \((1, \epsilon, \ldots, \epsilon), (\epsilon, 1, \ldots, \epsilon), \cdots, (\epsilon, \epsilon, \ldots, 1)\), respectively, where \(\epsilon \to 0\) is a small enough number. It is easy to verify that the dynamic DRF mechanism produces a solution with

\[
x^n_i = \frac{1}{n-m+1+\epsilon(m-1)} \to \frac{1}{n-m+1}, \text{ for each agent } i
\]

(11)
at step \(n\). The optimal solution will allocate all resources to last the \(m\) agents, obtaining a solution with

\[
\hat{x}^n_i = \frac{1}{1+\epsilon(m-1)} \to 1, \text{ for } i = n-m+1, n-m+2, \ldots, n,
\]

(12)
and \(\hat{x}^n_i = 0\) for other agents. Thus, the competitive ratio is

\[
\frac{\sum_{i=1}^{k} x^n_i}{\sum_{i=1}^{n} \hat{x}^n_i} \to \frac{n}{m(n-m+1)} = \frac{1}{m \frac{1-m/n+1}{n}}.
\]

(13)
When \(n\) is large enough, the ratio approaches \(1/m\). Thus, the theorem holds.

3.2 The maxmin objective

When the objective is minimum dominant share maximization (maxmin, for short), the optimal solution \((\hat{x}^k_1, \ldots, \hat{x}^k_k)\) at step \(k \geq 2\) in the offline setting can be obtained by solving the following program

\[
\begin{cases}
\text{Maximize} & \min_i x^k_i \\
\sum_{i=1}^{k} d_{ir} x^k_i & \leq k/n, \forall r. \\
\end{cases}
\]

(14)
Actually, $(\ddot{x}_1^k, \ldots, \ddot{x}_k^k)$ is a DRF solution [2, 5], where the dominant shares of all agents are equal. Formally, for a given instance $I$, at every step $k$, $(\ddot{x}_1^k, \ldots, \ddot{x}_k^k)$ is obtained by

$$\ddot{x}_1^k = \cdots = \ddot{x}_k^k = \min_r \frac{k/n}{\sum_{i=1}^k d_{ir}},$$

(15)

following from [3, 5].

Therefore, the competitive ratio $CR_2$ of the dynamic DRF mechanism for the maxmin objective can be defined as

$$CR_2 = \min_{I} \min_k \frac{\min_i x_i^k}{\min_i \ddot{x}_i^k} = \min_{I} \min_k \frac{\min_i x_i^k}{\ddot{x}_k^k} = \min_{I} \min_k \frac{x_k^k}{\ddot{x}_k^k},$$

(16)

where the last equality follows from the fact $\min_i x_i^k = x_k^k$, which can be obtained by Lemma 2 in [4].

**Theorem 2.** When the objective is minimum dominant share maximization, the competitive ratio of the dynamic DRF mechanism is $1/m$. Moreover, no mechanism satisfying DPO can do better than $1/(m-1)$.

**Proof.** At every step $k \in \{2, \ldots, n\}$, since the dynamic DRF mechanism satisfies the SI property, we have

$$x_k^k \geq \frac{1}{n}.\quad (17)$$

By the pigeonhole principle, there exists a source which is the dominant resource for at least $\lceil k/m \rceil$ agents. It implies that the DRF solution $(\ddot{x}_1^k, \ldots, \ddot{x}_k^k)$ satisfies

$$\ddot{x}_k^k \leq \frac{k/n}{\lceil k/m \rceil}.\quad (18)$$

Thus, the competitive ratio of the dynamic DRF mechanism satisfies

$$CR_2 = \frac{x_k^k}{\ddot{x}_k^k} \geq \frac{\lceil k/m \rceil}{k} \geq \frac{1}{m}.\quad (19)$$

Consider a setting with $m$ ($> 2$) resources and $n = m^2 + 1$ agents. For $i = 1, 2, \ldots, m^2$, the demand vector of agent $i$ is defined as

$$d_i = \begin{cases} (1, \epsilon, \ldots, \epsilon), & \text{if } i \equiv 1 \pmod{m} \\ (\epsilon, 1, \ldots, \epsilon), & \text{if } i \equiv 2 \pmod{m} \\ \vdots \\ (\epsilon, \epsilon, \ldots, 1), & \text{if } i \equiv 0 \pmod{m} \end{cases}$$

(20)
where \( \epsilon \to 0 \) is a small enough number. The demand vector of agent \( n = m^2 + 1 \) is \( d_n = (1, 1, \ldots, 1) \). At step \( k = m^2 \), the dynamic DRF solution is

\[
x^k_i = \frac{m^2}{[m + \epsilon(m^2 - m)](m^2 + 1)} \to \frac{m}{m^2 + 1}, \quad \text{for } i = 1, 2, \ldots, k,
\]

following from the assumption of \( \epsilon \). Actually, after the first \( m^2 \) steps, at least \( m^2/(m^2 + 1) \) share of at least one resource \( r^* \) must be exhausted for any dynamic mechanism satisfying the DPO property. It implies that at most \( 1/(m^2 + 1) \) share of resource \( r^* \) is left for the last agent \( n \). Hence,

\[
x^n_n \leq \frac{1}{m^2 + 1},
\]

for any dynamic mechanism satisfying DPO, while the DRF solution \((\ddot{x}_1^n, \ldots, \ddot{x}_n^n)\) satisfies

\[
\ddot{x}_i^n = \frac{1}{m + 1 + \epsilon(m^2 - m)} \to \frac{1}{m + 1}, \quad \text{for } i = 1, 2, \ldots, n.
\]

It implies that, at step \( k = n \), the competitive ratio of any dynamic mechanism satisfying DPO including the dynamic DRF mechanism is at most

\[
\frac{x^n_n}{\ddot{x}_n^n} \to \frac{m + 1}{m^2 + 1} \leq \frac{1}{m - 1}.
\]

Thus, the theorem holds. \( \square \)

4 A linear-time optimal algorithm

Since dynamic DRF is almost optimal as proved in the last section, it is desired to design an efficient algorithm to find an optimal solution for the dynamic DRF mechanism. Although the water-filling algorithm can produce a dynamic DRF solution \([4]\), the running time is pseudo-polynomial \([3]\). Also, we can compute a dynamic DRF solution by solving the linear program (1). However, it is not a strongly polynomial-time algorithm. In this section, we will design a linear-time algorithm to find a dynamic DRF solution. In the proof below, \( M^k \) and \( x^k_i \) refer to the optimal solution of (1) in step \( k \). The following two lemmas in \([4]\) are very useful for designing the faster algorithm.
Lemma 1. At any step $k \in \{1, \ldots, n\}$, it holds that $x_i^k = \max\{M^k, x_i^{k-1}\}$ for all agents $i \leq k$.

Lemma 2. At any step $k \in \{1, \ldots, n\}$, for all agents $i, j$ such that $i < j$, it holds that $x_i^k \geq x_j^k$.

Theorem 3. At any step $k \in \{2, \ldots, n\}$, a dynamic DRF solution can be found within $O(k)$ time.

Proof. Consider an agent $j$. By Lemma 1, we have $x_j^k = \max\{M^k, x_j^{k-1}\}$. If $x_j^k = x_j^{k-1} > M^k$, by Lemma 2, for all agents $i \leq j$, we have $x_i^k = \max\{M^k, x_i^{k-1}\} = x_i^{k-1}$. If $x_j^k = M^k > x_j^{k-1}$, by Lemma 2, for all agents $i \geq j$, we have $x_i^{k-1} \leq x_j^{k-1} < M^k$, which implies that $x_i^k = \max\{M^k, x_i^{k-1}\} = M^k$. Therefore, at any step $k \geq 2$, there is an agent $\tau \leq k$ such that

$$
\begin{cases}
  x_i^k = x_i^{k-1} > M^k, & \text{for } i < \tau; \\
  x_i^k = M^k \geq x_i^{k-1}, & \text{for } \tau \leq i \leq k.
\end{cases} 
$$

Thus, if we know $\tau$, $M^k$ can be obtained by solving the following linear program

$$
\begin{aligned}
\text{Maximize} & \quad M^k \\
\text{subject to} & \quad \sum_{i: \tau \leq i \leq k} d_{ir} M^k + \sum_{i: i < \tau} d_{ir} x_i^{k-1} \leq \frac{kn}{k}, & \text{for } r = 1, 2, \ldots, m.
\end{aligned}
$$

As pointed in [5], this linear program can be rewritten as

$$
M^k = \min_r \frac{k/n - \sum_{i: i < \tau} d_{ir} x_i^{k-1}}{\sum_{i: \tau \leq i \leq k} d_{ir}}.
$$

We are now ready to describe our linear-time algorithm. Our main idea is to find $\tau$ by using a bisection method. At any step $k \geq 2$, consider the agent $l = \lceil (1 + k)/2 \rceil$. Let

$$
\begin{cases}
  \tilde{x}_i^k = x_i^{k-1}, & \text{for } i < l; \\
  \tilde{x}_i^k = x_i^{k-1}, & \text{for } l \leq i \leq k,
\end{cases}
$$

For convenience, let

$$
\begin{aligned}
\alpha_r & = \sum_{i: i < l} d_{ir} x_i^{k-1}, & \forall r; \\
\beta_r & = \sum_{i: l \leq i \leq k} d_{ir}, & \forall r.
\end{aligned}
$$

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Clearly, if $\alpha_r + x_i^{k-1}\beta_r \leq k/n$ for every resource $r$, i.e., $(\bar{x}_1^k, \ldots, \bar{x}_k^k)$ satisfies the capacity constraints in (1), we have $M^k \geq x_i^{k-1}$ and $l \geq \tau$. Otherwise, we have $M^k < x_i^{k-1}$ and $l < \tau$. We distinguish the following two cases:

**Case 1.** $l \geq \tau$. For every agent $i$ satisfying $l \leq i \leq k$, we have $x_i^k = M^k$, as $i \geq l \geq \tau$. Let $\mathcal{AI} = \{i : l \leq i \leq k\}$ be set of known agents with identical dominant share in the optimal solution $(x_1^k, \ldots, x_k^k)$. Next, consider the agent $\lceil (1 + l)/2 \rceil$ as before.

**Case 2.** $l < \tau$. For every agent $i$ satisfying $i < l$, we have $x_i^k = x_i^{k-1}$, as $i < l < \tau$. Let $\mathcal{AS} = \{i : i < l\}$ be set of known agents with same dominant share as in step $k - 1$ in the optimal solution $(x_1^k, \ldots, x_k^k)$. Next, consider the agent $\lceil (l + k)/2 \rceil$ as before.

At every step $k$, the number of unclassified agents in $\{i : i \notin \mathcal{AI}, i \notin \mathcal{AS}\}$ is reduced to half. Finally, all the agents are divided into two subsets $\mathcal{AI}$ and $\mathcal{AS}$, and we will find the $\tau$ and the optimal solution $(x_1^k, \ldots, x_k^k)$. Clearly, the running time of deciding whether $l \geq \tau$ at each iteration is linear in the number of unclassified agents. Thus, the total running time is $O(k + k/2 + k/2^2 + \cdots + 1) = O(k)$, where $m$ is seen as a constant.

The complete algorithm is given as **Linear-time dynamic DRF algorithm** in Appendix.

### 5 Conclusion and Future Work

We have analyzed the competitive ratio of the dynamic DRF mechanism, which shows that the dynamic DRF mechanism is a nearly optimal mechanism satisfying DPO for the maxmin objective. We have described a non-trivial polynomial-time algorithm to find a dynamic DRF allocation, whose running time is linear in the number of present agents at every step, improving the result in [4].

Note that another fair allocation mechanism, called **cautious LP**, is proposed in [4]. Cautious LP achieves near optimal maxmin value at the last step. However, since cautious LP violates the DPO property and allocates too many resources at the last several steps, it is unfair to compare cautious LP with dynamic DRF for the maxmin objective. It is interesting to analyze the competitive ratio of the cautious LP mechanism under different
objectives. Since solving the linear program takes too much time, it is challenging to develop a combinatorial algorithm to find a cautious LP solution as in Section 4.

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Appendix

**Linear-time dynamic DRF algorithm**

1: **Data:** Demand $d_i$, $1 \leq i \leq k$
2: **Result:** Allocation $A^k$ at each step $k$
3: $x_1^1 \leftarrow 1/n$, $A^1_{ir} \leftarrow x_1^r \cdot d_{ir}$, $\forall r$;
4: $k \leftarrow 2$;
5: **while** $k \leq n$ **do**
6: \hspace{1em} **if** $\sum_{i=1}^k d_{ir} x_{ir}^{k-1} \leq k/n$, $\forall r$, **do**
7: \hspace{2em} $\alpha_r \leftarrow 0$, $\beta_r \leftarrow \sum_{i=1}^k d_{ir}$, $\forall r$;
8: \hspace{1em} **else,** **do**
9: \hspace{2em} $LB \leftarrow 1$, $UB \leftarrow k$, $\tau \leftarrow \lceil (LB + UB)/2 \rceil$;
10: \hspace{2em} $\alpha_r \leftarrow \sum_{i=1}^{\tau-1} d_{ir} x_{ir}^{k-1}$, $\beta_r \leftarrow \sum_{i=\tau}^k d_{ir}$, $\forall r$;
11: \hspace{1em} **while** $UB - LB > 1$, **do**
12: \hspace{2em} \hspace{1em} **if** $\alpha_r + \beta_r x_{ir}^{k-1} \leq k/n$, $\forall r$, **do**
13: \hspace{2em} \hspace{2em} $LB \leftarrow LB$, $UB \leftarrow \tau$, $\tau \leftarrow \lceil (LB + UB)/2 \rceil$;
14: \hspace{2em} \hspace{2em} $\alpha_r \leftarrow \alpha_r - \sum_{i=\tau}^{UB-1} d_{ir} x_{ir}^{k-1}$, $\beta_r \leftarrow \beta_r + \sum_{i=\tau}^{UB-1} d_{ir}$;
15: \hspace{2em} \hspace{1em} **else,** **do**
16: \hspace{2em} \hspace{2em} $LB \leftarrow \tau$, $UB \leftarrow UB$, $\tau \leftarrow \lceil (LB + UB)/2 \rceil$;
17: \hspace{2em} \hspace{2em} $\alpha_r \leftarrow \alpha_r + \sum_{i=LB}^{\tau-1} d_{ir} x_{ir}^{k-1}$, $\beta_r \leftarrow \beta_r - \sum_{i=LB}^{\tau-1} d_{ir}$;
18: \hspace{2em} \hspace{1em} **end if**;
19: \hspace{2em} **end while**;
20: **end if**;
21: $M^k \leftarrow \min_r (k/n - \alpha_r)/\beta_r$;
22: $x_{ir}^k \leftarrow \max(x_{ir}^{k-1}, M^k)$, $\forall i \leq k$;
23: $A^k_{ir} \leftarrow x_{ir}^k \cdot d_{ir}$, $\forall i \leq k$;
24: $k \leftarrow k + 1$;
25: **end while**