Nonlocal complement value problem for a global in time parabolic equation

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Abstract
The overreaching goal of this paper is to investigate the existence and uniqueness of weak solution of a semilinear parabolic equation with double nonlocality in space and in time variables that naturally arises while modeling a biological nano-sensor in the chaotic dynamics of a polymer chain. In fact, the problem under consideration involves a symmetric integrodifferential operator of Lévy type and a term called the interaction potential, that depends on the time-integral of the solution over the entire interval of solving the problem. Owing to the Galerkin approximation, the existence and uniqueness of a weak solution of the nonlocal complement value problem is proven for small time under fair conditions on the interaction potential.

Keywords Nonlocal operators · Lévy operators · Parabolic equations: IVP, Weak solutions

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1 Introduction

Let $\Omega$ be an open bounded set of $\mathbb{R}^N (N \geq 1)$. For $T > 0$, we are interested in studying the following nonlocal complement value problem

$$
\begin{aligned}
&\partial_t u + \mathcal{L} u + \varphi\left( \int_0^T u(\cdot, \tau) \, d\tau \right) u = 0 \quad \text{in} \quad \Omega_T := \Omega \times (0, T), \\
u = 0 &\quad \text{in} \quad \Sigma := (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
u(\cdot, 0) &= u_0 \quad \text{in} \quad \Omega,
\end{aligned}
$$

where $u = u(x, t)$ is an unknown scalar function, and $\varphi$ a scalar function that will be specified in the sequel. The initial state $u_0 : \Omega \to \mathbb{R}$ is prescribed. Here, we restrict ourselves to a purely integrodifferential operator of Lévy type $\mathcal{L}$, which is a particular type of nonlocal operators acting on a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ as follows

$$
\mathcal{L} u(x) := \text{p.v.} \int_{\mathbb{R}^N} (u(x) - u(y)) \nu(x - y) \, dy, \quad (x \in \mathbb{R}^N),
$$

whenever the right hand side exists and makes sense. Here and henceforward, the function $\nu : \mathbb{R}^N \setminus \{0\} \to [0, \infty)$ is the density of a symmetric Lévy measure. In other words, $\nu \geq 0$ and measurable such that

$$
\nu(-h) = \nu(h) \quad \text{for all} \quad h \in \mathbb{R}^N \quad \text{and} \quad \int_{\mathbb{R}^N} (1 \wedge |h|^2) \nu(h) \, dh < \infty.
$$

Notationally, we write $a \wedge b$ to denote $\min(a, b)$ for $a, b \in \mathbb{R}$. For the sake of simplicity, we also assume that $\nu$ does not vanish on sets of positive measure. To wit, $\nu$ is fully supported on $\mathbb{R}^N$. A prototypical example of an operator $\mathcal{L}$ is the fractional Laplacian $(-\Delta)^s$, which is obtained by taking $\nu(h) = C_{N,s} |h|^{-N-2s}$ for $h \neq 0$ where $s \in (0, 1)$ is fixed and the constant $C_{N,s}$ is given by

$$
C_{N,s} = \frac{2^{2s} s \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{N/2} \Gamma(1-s)}.
$$

The constant $C_{N,s}$ is chosen so that the Fourier relation $\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi)$, $\xi \in \mathbb{R}^N$, holds for all $u \in C_\infty^\infty(\mathbb{R}^N)$. The fractional Laplacian is one of the most heavily studied integrodifferential operators; see for instance [6, 8, 11, 16, 27, 37, 38] for some basics. Additional results related to the fractional Laplacian can be found, in the references [2, 10, 15, 53]. The operator $\mathcal{L}$ in (2) arises naturally in probability theory as the generator of pure Lévy stochastic processes with jump interaction measure $\nu(h) \, dh$. We refer interested readers to [1, 4, 46] for more details on Lévy processes. Recent studies of Integro-Differential Equations(IDEs) involving nonlocal operators of the form $\mathcal{L}$ in (2) can be found in [21, 23]. There exists a substantial
amount of literature on nonlocal problems involving Lévy type operators. For example, see [19, 21, 22, 45] for the study of elliptic problems, see [9, 13, 43, 44] for the regularity of elliptic problems, [20, 26, 36] for the regularity of parabolic.

Our main result (see Theorem 4.2) consists in proving the existence and uniqueness of a weak solution to the problem (1) for $T$ sufficiently small by imposing some conditions on the initial value $u_0$. It should be noted that we do not require any comparability between the operators $\mathcal{L}$ and $(-\Delta)^s$; in the sense that the kernels $\nu$ and $|\cdot|^{-N-2s}$ need not be comparable. Following [50], we do this provided that the potential $\varphi$ satisfies the following assumption which admits functions $\varphi$ that are not convex and not increasing at $\infty$.

Assumption 1.1 The potential $\varphi : \mathbb{R} \to [0, \infty)$ is a continuous non-negative function such that $\varphi(0) = 0$ and $\tau \mapsto \varphi(\tau)\tau$ is a non-decreasing differentiable function whose derivative is bounded on every compact subset of $\mathbb{R}$.

An interesting feature of the problem under consideration is that main equation in (1) contains a nonlocal operator of Lévy type in space and a nonlocal in time term that depends on the integral over the whole interval $(0, T)$ on which the problem is being solved, viz., problem (1) has a global memory, i.e., its depends upon the memory and the future. Note in passing that if $t \in (0, T)$ is the current time then the memory is recorded in $(0, t)$, while the future is recorded in $(t, T)$. The mixture of nonlocal terms (spatial and time variables) appearing in (1) renders the problem somehow fully nonlocal with global memory. For this reason, the problem (1) is termed nonlocal and global in time. It is noteworthy emphasizing that, similar analysis has been carried out in [50] where the Laplace operator $-\Delta$ is used in place of the nonlocal operator $\mathcal{L}$. There are several works in the literature that study parabolic problems with memory which include the integral of the solution from the initial to the current time, e.g., see [12, 32, 52, 54]. To the best of our knowledge, the existing problems with memory in the literature differ from ours. Indeed, on the one side, in our problem, we have to deal with the nonlocality in time variable occurred by the semilinear factor depending on the integral over the whole interval $(0, T)$ appearing in equation (1) whose knowledge demands to know the so called “future” which is $T$. Moreover, the nonlocality in time in the system (1) is governed by a semilinearity due to the potential $\varphi$. On the other side, we have to deal with the nonlocality in spatial variables, due to the nonlocal Lévy operator $\mathcal{L}$. The main novelty of the problem (1) is marked by this double nonlocality in spatial and in time variables rendering the latter the problem somewhat challenging and of particular interest in its own. It is worth emphasizing that the nonlocal problem in (1) as well as its local analogue in space cannot be reduced to known ones by any transformation. We refer the interested reader to [41, 47]) for the studies of some local problems with initial boundary values with memory where the “future” appears in the data. On the other hand, we point out that our problem arises in the local setting while modeling a biological nanosensor in the chaotic dynamics of a polymer chain $s$ also called, a polymer chain in an aqueous solution. In this
model, the density of the probability that a chain occupies certain region of the underlying space allows to describe the position of the chain segment. According to [48], the probability density satisfies with a high accuracy, a certain parabolic equation of the form (1) in which there is a term responsible for the interaction of the chain of polymer segments. For more on the application of the problem (1), we refer interested reader for instance to [48–50] and references therein.

Theoretical motivations of studying the nonlocal model (1) are at least twofold. Firstly, in contrast to the local model, the advantage of the nonlocal model (1) is that, it allows both smooth and non-smooth solutions \( u \) in space variables. Secondly, the nonlocal model can be viewed as an approximation of local model. For example, given \( u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) one can show that \( (-\Delta)^{\epsilon} u(x) \to -\Delta u(x) \) as \( \epsilon \to 1^- \). More generally, we have \( L^{\epsilon} u(x) \to -\Delta u(x) \) as \( \epsilon \to 0 \) (see [21, 25]), where \( L^\epsilon u \) is defined as in (2) with \( \nu \) replaced by \( \nu_\epsilon \) satisfying

\[
\nu_\epsilon \geq 0 \text{ is radial, } \int_{\mathbb{R}^N} (1 \wedge |h|^2) \nu_\epsilon(h)dh = \frac{1}{N}, \quad \text{and } \forall \, \delta > 0 \lim_{\epsilon \to 0} \int_{|h|>\delta} \nu_\epsilon(h)dh = 0. \tag{3}
\]

If we assume that \( \nu \geq 0 \) is radial and satisfies \( \int_{\mathbb{R}^N} (1 \wedge |h|^2) \nu(h)dh = \frac{1}{N} \) then a remarkable example of family \( (\nu_\epsilon)_\epsilon \) satisfying (3), is obtained from the rescaled version \( \nu \) as follows

\[
\nu_\epsilon(h) = \begin{cases} 
\epsilon^{-N-2} \nu(h/\epsilon) & \text{if } |h| \leq \epsilon \\
\epsilon^{-N} |h|^{-2} \nu(h/\epsilon) & \text{if } \epsilon < |h| \leq 1 \\
\epsilon^{-N} \nu(h/\epsilon) & \text{if } |h| > 1.
\end{cases}
\]

In [49], the weak solvability of the problem is proven for the case where \( u \) is a positive bounded function and \( \varphi \) is the so called Flory-Huggins potential, i.e., is a convex increasing function that tends to infinity as its argument approaches a certain positive value. The positiveness is a natural requirement since \( u \) is a density probability. The landmark works in [49, 50] demonstrate that, in the case where only the Laplace operator is involved, the problem (1) is well-posed for sufficiently small time \( T \). As a matter of interest the present work takes the result in [50] to the next stage by using the generator of a pure jump stochastic process of Lévy type, which is a symmetric nonlocal operator of the form \( L^\epsilon \), to prove further results on weak solvability for this type of problem. The rest of the paper is structured as follows. In Section 2, we provide some well-known results and functions spaces which are useful in this paper. In Section 3, we prove auxiliary results which are the milestones to prove our core result. Finally, Section 4 is devoted to the proof of the existence and uniqueness of a weak solution to the problem (1) thereby constituting the main goal of this article. We prove the existence with the aid of the Tychonoff fixed-point theorem and prove the uniqueness for sufficiently small \( T \).
2 Notations and Preliminaries

The purpose of this section is to introduce notations and some preliminary results. Let us collect some basics on nonlocal Sobolev-like spaces in the $L^2$ setting that are generalizations of Sobolev–Slobodeckij spaces and which will be very helpful in the sequel. Let us emphasize that, these function spaces are tailor made for the study of complement value problems involving symmetric Lévy operators of type $\mathcal{L}$. We refer the reader to [23] more extensive discussions on this topic.

From now on, unless otherwise stated, $\Omega \subset \mathbb{R}^N$ is an open bounded set. We also assume that $\nu : \mathbb{R}^N \setminus \{0\} \to [0, \infty)$ has full support, satisfies the Lévy integrability condition, i.e., $\nu \in L^1(\mathbb{R}^N, (1 \wedge |h|^2)dh)$ and is symmetric, i.e., $\nu(h) = \nu(-h)$ for all $h \in \mathbb{R}^N$. We define the space

$$V_\nu(\Omega|\mathbb{R}^N) := \{ u : \mathbb{R}^N \to \mathbb{R} \text{ meas} : \mathcal{E}(u, u) < \infty \},$$

where $\mathcal{E}(\cdot, \cdot)$ is the bilinear form defined by

$$\mathcal{E}(u, v) := \frac{1}{2} \iint_{\Omega} (u(x) - u(y))(v(x) - v(y))v(x - y) \, dy \, dx,$$

where $\Omega(\Omega)$ is the cross-shaped set on $\Omega$ given by

$$\Omega(\Omega) := (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega).$$

We endow the space $V_\nu(\Omega|\mathbb{R}^N)$ with the norm

$$\|u\|_{V_\nu(\Omega|\mathbb{R}^N)} := \left( \int_{\Omega} |u(x)|^2 \, dx + \mathcal{E}(u, u) \right)^{\frac{1}{2}}.$$

In order to study the Dirichlet problem (1) we also need to define the subspace of functions in $V_\nu(\Omega|\mathbb{R}^N)$ that vanishes on the complement of $\Omega$, i.e.,

$$\mathcal{X}_\nu(\Omega|\mathbb{R}^N) := \{ u \in V_\nu(\Omega|\mathbb{R}^N) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega \},$$

where $V_\nu(\Omega|\mathbb{R}^N)$ is defined as in (4). The space $\mathcal{X}_\nu(\Omega|\mathbb{R}^N)$ is clearly a closed subspace of $V_\nu(\Omega|\mathbb{R}^N)$. Furthermore, if $\partial \Omega$ is continuous then [23] smooth functions of compact support $C^\infty_c(\Omega)$ are dense in $\mathcal{X}_\nu(\Omega|\mathbb{R}^N)$. In addition, we have that

$$\|u\|_{\mathcal{X}_\nu(\Omega|\mathbb{R}^N)} = \left( \iint_{\mathbb{R}^N \mathbb{R}^N} (u(x) - u(y))^2 v(x - y) \, dy \, dx \right)^{\frac{1}{2}}$$

defines an equivalent norm on $\mathcal{X}_\nu(\Omega|\mathbb{R}^N)$. Indeed, in virtue of the Poincaré-Friedrichs inequality on $\mathcal{X}_\nu(\Omega|\mathbb{R}^N)$, there exists a constant $C = C(N, \Omega, \nu) > 0$ depending only on $N, \Omega$ and $\nu$ such that

$$\|u\|_{L^2(\Omega)}^2 \leq C\|u\|_{\mathcal{X}_\nu(\Omega|\mathbb{R}^N)}^2$$

for every $u \in \mathcal{X}_\nu(\Omega|\mathbb{R}^N)$. (5)
Remark 2.1 One can observe that the notation $V_{\nu}(\Omega|\mathbb{R}^N)$ is to emphasize that the integral of the measurable map $(x,y) \mapsto (u(x) - u(y))^2v(x - y)$ performed over $\Omega \times \mathbb{R}^N$ is finite. Moreover, the latter is equivalent to that performed over $Q(\Omega)$, that is, $\mathcal{E}(u,u) \propto \iint_{Q(\Omega)} u(x) - u(y))^2v(x - y)dydx$. From the local scenario point of view, it is fair to see the space $V_{\nu}(\Omega|\mathbb{R}^N)$ as the nonlocal replacement of the classical Sobolev space $H^1(\Omega)$, whereas $X_{\nu}(\Omega|\mathbb{R}^N)$ can be viewed as the nonlocal replacement of the classical Sobolev space $H^1_0(\Omega)$.

The aforementioned spaces are Hilbert spaces. Additional, recent finds about these function spaces and their relations with classical Sobolev spaces can be found in [23–25]. Let $(V_{\nu}(\Omega|\mathbb{R}^N)^* \text{ and } (X_{\nu}(\Omega|\mathbb{R}^N))^*$ be the dual spaces of $V_{\nu}(\Omega|\mathbb{R}^N)$ and $X_{\nu}(\Omega|\mathbb{R}^N)$ respectively. We have the following continuous Gelfand triple embeddings

$$X_{\nu}(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega) \hookrightarrow (X_{\nu}(\Omega|\mathbb{R}^N))^* \quad \text{and} \quad V_{\nu}(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega) \hookrightarrow (V_{\nu}(\Omega|\mathbb{R}^N))^*.$$

The next result borrowed from [14, 23, 33], provides sufficient conditions under which the spaces $X_{\nu}(\Omega|\mathbb{R}^N)$ and $V_{\nu}(\Omega|\mathbb{R}^N)$ are compactly embedded in $L^2(\Omega)$.

Theorem 2.2 Assume that $\nu \in L^1(\mathbb{R}^N, 1 \wedge |h|^2)$ and $\nu \notin L^1(\mathbb{R}^N)$. If $\Omega \subset \mathbb{R}^N$ is open and bounded then the embedding $X_{\nu}(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega)$ is compact. Furthermore, the embedding $V_{\nu}(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega)$ is also compact provided that in addition, $\Omega$ has a Lipschitz boundary and $\nu$ satisfies

$$\lim_{\delta \to 0} \frac{1}{\delta^2} \int_{B_\delta(0)} |h|^2 \nu(h) dh = \infty.$$  

(6)

It is worthwhile noticing that we have the natural continuous and dense embeddings

$$L^2(0,T;X_{\nu}(\Omega|\mathbb{R}^N)) \hookrightarrow L^2(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;((X_{\nu}(\Omega|\mathbb{R}^N))^*)^*).$$

Next we introduce the Sobolev type space $H_{\nu}(0,T) := H^1(0,T;X_{\nu}(\Omega|\mathbb{R}^N))$ by

$$H_{\nu}(0,T) = \left\{ \psi \in L^2(0,T;X_{\nu}(\Omega|\mathbb{R}^N)) : \partial_t \psi \in L^2(0,T;((X_{\nu}(\Omega|\mathbb{R}^N))^*)^*) \right\}.$$
The space $H_\nu(0, T)$ (see [39]) is a Hilbert space endowed with the norm given by
\[
\|\psi\|^2_{H_\nu(0, T)} = \|\psi\|^2_{L^2(0, T; X_\nu(\Omega|\mathbb{R}^N))} + \|\partial_t \psi\|_{L^2(0, T; X'_\nu(\Omega|\mathbb{R}^N))}.
\] (7)

As a consequence of Theorem 2.2, we get the compact embedding (9) below.

**Proposition 2.3** Assume that $\nu \notin L^1(\mathbb{R}^N)$ and $\Omega \subset \mathbb{R}^N$ is open bounded. The following are true.

(i) Lions-Magenes Lemma [5, Theorem II.5.12]: the following embedding is continuous
\[
H_\nu(0, T) \hookrightarrow C([0, T]; L^2(\Omega)).
\] (8)

(ii) Lions-Aubin Lemma [5, Theorem II.5.16]: the following embedding is compact
\[
H_\nu(0, T) \hookrightarrow L^2(0, T; L^2(\Omega)).
\] (9)

Now we state the integration by parts formula contained in [17, 23] for smooth functions. Precisely for every $\phi, \psi \in C^\infty_c(\mathbb{R}^N)$ following nonlocal Gauss-Green formula holds true
\[
\mathcal{E}(\phi, \psi) = \int_\Omega \psi \mathcal{L}\phi(x) \, dx + \int_{\mathbb{R}^N \setminus \Omega} \psi(y) \mathcal{N}\phi(y) \, dy,
\] (10)

where, $\mathcal{N}\phi$ denotes the *nonlocal normal derivative* $\mathcal{N}$ of $\phi$ across the boundary of $\Omega$ with respect to $\nu$ and is defined by
\[
\mathcal{N}\phi(x) := \int_\Omega (\phi(x) - \phi(y)) \nu(x - y) \, dy, \quad x \in \mathbb{R}^N \setminus \Omega.
\] (11)

With the aforementioned function spaces at hand, we are now in position to define the notion of weak solutions to the problem (1).

**Definition 2.4** Let $\varphi$ satisfies Assumption 1.1 and $u_0 \in L^2(\Omega)$. A function $u : \Omega_T \rightarrow \mathbb{R}$ is said to be a weak solution of problem (1), if

(i) $u \in L^2(0, T; X_\nu(\Omega|\mathbb{R}^N))$ and $\partial_t u \in L^2(0, T; (X_\nu(\Omega|\mathbb{R}^N))^*)$;

(ii) for every $\psi \in C^1_c([0, T]; X_\nu(\Omega|\mathbb{R}^N))$ (i.e., $\psi(\cdot, T) = 0$), $u$ satisfies $u(\cdot, 0) = u_0$ and
\[
\int_\Omega \partial_t u \psi \, dx + \mathcal{E}(u, \psi) + \int_\Omega \varphi(v) u \psi \, dx = 0 \quad \text{for all } 0 \leq t \leq T.
\] (12)

In particular, we have
\[
\int_0^T \int_\Omega u \partial_t \psi \, dx \, dt + \int_0^T \int_\Omega E(u, \psi) \, dt + \int_0^T \int_\Omega \varphi(v) u \psi \, dx \, dt = \int_\Omega u_0 \psi_0 \, dx.
\]

By the density of \( C^\infty_c(\Omega) \) in \( \mathbb{X}_1(\Omega|\mathbb{R}^N) \), it is sufficient to take \( \psi \in C^1([0, T); C^\infty_c(\Omega)) \) as the test functions in (12). Our proof of the existence of a weak solution to the problem (1) relies upon the following Tychonoff fixed-point Theorem 2.5 which is a generalization of the Brouwer and Schauder fixed-point theorems.

**Theorem 2.5** (Tychonoff [51]) *Let \( X \) be a reflexive separable Banach space. Let \( G \subset X \) be closed convex and bounded set. Any weakly sequentially continuous map \( \pi : G \to G \) has a fixed point.*

We emphasize that when \( G \) is compact and convex, Theorem 2.5 is known as the Schauder fixed-point theorem, while, when \( X \) is of finite dimension it is known as the Brouwer fixed-point theorem.

3 Nonlocal elliptic and parabolic problem

The overreaching goal of this section is to investigate weak solutions to two specific nonlocal problems which is of interest in the proof of our main result. The first problem is an elliptic nonlocal problem and the second one is a parabolic nonlocal problem.

3.1 Nonlocal elliptic problem

Given a measurable function \( f : \Omega \to \mathbb{R} \), we consider the elliptic problem consisting into finding a function \( v : \mathbb{R}^N \to \mathbb{R} \) satisfying of the following problem:

\[
\begin{align*}
\mathcal{L}v + \varphi(v) v &= f \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.
\end{align*}
\]

(13)

Heuristically, the problem (13) results from the evolution problem (1) by integrating with respect to \( t \) from 0 to \( T \). In a sense, the functions \( v \) and \( f \) correspond to \( \int_0^T u(\cdot, t) \, dt \) and \( u_0 - u(\cdot, T) \), respectively. Semilinear problems of type (13) are considered in the classical scenario in [3, 29, 30, 50] with the operator \( \mathcal{L} \) replaced with \( -\Delta \). There, the difficulties with the integrability of the term \( \varphi(v)v \) were handled. In our case, we consider the function \( f \in L^2(\Omega) \), so that we expect more from the solution of the problem such as \( \varphi(v) \in L^2(\Omega) \). We need to introduce the following notation

\[ \chi(\tau) = \varphi(\tau) \tau. \]

A function \( v \in \mathbb{X}_1(\Omega|\mathbb{R}^N) \) is said to be a weak solution of problem (13) if \( \chi(v) \in L^2(\Omega) \) and
Next, we want to show that the above variational problem (14) is well-posed in the sense of Hadamard. In other words, it possesses a unique solution which continuously depends upon the data. Let us start with the following stability lemma.

**Lemma 3.1** Let \( f_i \in L^2(\Omega) \), \( i = 1, 2 \). Assume that \( v_i \in X_v(\Omega; \mathbb{R}^N) \) satisfies

\[
\mathcal{E}(v_i, \psi) + (\chi(v_i), \psi) = (f_i, \psi) \quad \text{for all} \quad \psi \in X_v(\Omega; \mathbb{R}^N).
\]

Then for some constant \( C = C(N, \Omega, \nu) > 0 \) only depending only \( N, \Omega \) and \( \nu \) such that

\[
\|v_1 - v_2\|_{X_v(\Omega; \mathbb{R}^N)} \leq C\|f_1 - f_2\|_{(X_v(\Omega; \mathbb{R}^N))'}.
\]

**Proof** Combining both equation and testing with \( \psi = v_1 - v_2 \) yields

\[
\mathcal{E}(v_1 - v_2, v_1 - v_2) + (\chi(v_1) - \chi(v_2), v_1 - v_2)_{L^2(\Omega)} = (f_1 - f_2, v_1 - v_2)_{L^2(\Omega)}.
\]

Observing that \( \tau \mapsto \chi(\tau) = \varphi(\tau)\tau \) is non-decreasing, is equivalent to saying that

\[
(\chi(\tau_1) - \chi(\tau_2))(\tau_1 - \tau_2) \geq 0 \quad \text{for all} \quad \tau_1, \tau_2 \in \mathbb{R},
\]

the above relation implies

\[
\|v_1 - v_2\|^2_{X_v(\Omega; \mathbb{R}^N)} \leq \|f_1 - f_2\|_{(X_v(\Omega; \mathbb{R}^N))'} \|v_1 - v_2\|_{X_v(\Omega; \mathbb{R}^N)}.
\]

The desired estimate follows from the Poincaré-Friedrichs inequality (5).

The next result reminisces [50, Lemma 1, Section 3.1] in the nonlocal setting.

**Theorem 3.2** Let Assumption 1.1 be in force and let \( f \in L^2(\Omega) \). Then the problem (13) has a unique weak solution \( v \in X_v(\Omega; \mathbb{R}^N) \). Moreover, the following estimates hold true:

(i) \( \mathcal{E}(v, v) \leq C\|f\|^2_{L^2(\Omega)} \) where \( C > 0 \) only depends on \( N, \Omega, \) and \( v \);

(ii) \( \|\varphi(v)\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \);

(iii) \( \|\varphi(v)\|^2_{L^2(\Omega)} \leq \frac{1}{\delta^2}\|f\|^2_{L^2(\Omega)} + |\Omega|, \) with \( \delta > 0 \) only depending on \( \varphi \).

**Proof** Note that the uniqueness immediately follows from Lemma 3.1. We prove the remaining results of Theorem 3.2 in several steps. Our proof follows that of [50, Lemma 1, Section 3.1].

**Step 1:** We are interested in establishing the well-posedness of problem (13) using the Galerkin method which consists into projecting the latter on suitable finite dimensional space. First of all, we mention that bounded functions are dense in \( V_v(\Omega; \mathbb{R}^N) \) and hence in \( X_v(\Omega; \mathbb{R}^N) \). Thus, there is an orthonormal basis \( \{\phi_k\} \) of \( X_v(\Omega; \mathbb{R}^N) \) whose elements are bounded, i.e., \( \phi_k \in L^\infty(\Omega). \)
We emphasize that the inner product in $\mathbb{X}_v(\Omega|\mathbb{R}^N)$ is defined as 
$(\psi_1, \psi_2)_{\mathbb{X}_v(\Omega|\mathbb{R}^N)} = E(\psi_1, \psi_2)$ for $\psi_1, \psi_2 \in \mathbb{X}_v(\Omega|\mathbb{R}^N)$. Let $\mathcal{V}_k$ be the subspace of $\mathbb{X}_v(\Omega|\mathbb{R}^N)$ spanned by the basis functions $\{\phi_1, \ldots, \phi_k\}$. For each $k \in \mathbb{N}$, we claim the existence of a function $v_k \in \mathcal{V}_k$ such that
\[
E(v_k, \psi) + (\chi(v_k), \psi) = (f, \psi) \quad \text{for all} \quad \psi \in \mathcal{V}_k.
\] (16)
We prove this in two different ways. First, note that (16) is equivalent to the minimization problem
\[
\mathcal{J}(v_k) = \min_{w \in \mathcal{V}_k} \mathcal{J}(w) \quad \text{with} \quad \mathcal{J}(w) := \frac{1}{2} E(w, w) + \int_\Omega G(w)dx + \int_\Omega fwdx
\]
where we define the function $G(w) = \int_0^w \chi(\tau)d\tau = \int_0^w \varphi(\tau)\tau d\tau$. Note that $G$ is non-negative since $\varphi(\tau) \geq 0$ and that the mapping $w \mapsto \mathcal{J}(w)$ is continuous on $\mathcal{V}_k$. Furthermore, with the aid of the Poincaré-Friedrichs inequality (5) we find that $\mathcal{J}(w) \to \infty$, as $\|w\|_{\mathbb{X}_v(\Omega|\mathbb{R}^N)} \to \infty$ and $w \in \mathcal{V}_k$. Since $\dim \mathcal{V}_k < \infty$, the existence of a minimizer $v_k \in \mathcal{V}_k$ of $\mathcal{J}$ springs from folklore arguments.

Alternatively, as highlighted in [50], we obtain the existence of $v_k$ using the Brouwer fixed-point theorem as follows. Let $w \in \mathcal{V}_k$, necessarily $\varphi(w)$ is a bounded function since $\phi_k$’s are also bounded. The Lax-Milgram lemma implies there is a unique function $\hat{w} \in \mathcal{V}_k$ such that
\[
E(\hat{w}, \psi) + (\varphi(w)\hat{w}, \psi) = (f, \psi) \quad \text{for all} \quad \psi \in \mathcal{V}_k.
\]
In particular, the Poincaré–Friedrichs inequality (5) yields
\[
E(\hat{w}, \hat{w}) + \int_\Omega \varphi(w)\hat{w}^2 dx \leq \|f\|_{L^2(\Omega)} \|\hat{w}\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \|\hat{w}\|_{\mathbb{X}_v(\Omega|\mathbb{R}^N)}
\]
Thus, letting $R = C \|f\|_{L^2(\Omega)}$, since $\varphi \geq 0$ we obtain the following estimates
\[
\|\hat{w}\|_{\mathbb{X}_v(\Omega|\mathbb{R}^N)} \leq R \quad \text{and} \quad \int_\Omega \varphi(w)\hat{w}^2 dx \leq R^2.
\] (17)
We let $\mathcal{B}_R = \{w \in \mathcal{V}_k : \|w\|_{\mathbb{X}_v(\Omega|\mathbb{R}^N)} \leq R\}$, be the closed ball in $\mathcal{V}_k$ of radius $R$ centered at the origin. Clearly, (17) implies that the mapping $T : \mathcal{V}_k \to \mathcal{B}_R$ with $Tw = \hat{w}$ is well defined. It remains to prove that $T$ is a continuous mapping. Indeed, let $\{w_n\}$ be a sequence in $\mathcal{V}_k$ with $w_n = \lambda_1 \phi_1 + \cdots + \lambda_k \phi_k$ converging in $\mathcal{V}_k$ to a function $w = \lambda_1 \phi_1 + \cdots + \lambda_k \phi_k$, i.e., $\lambda_{\ell,n} \xrightarrow{n \to \infty} \lambda_{\ell}$, $\ell = 1, 2, \ldots, k$. By continuity we have $\varphi(w_n) \xrightarrow{n \to \infty} \varphi(w)$ almost everywhere. In addition, the convergence in $L^2(\Omega)$ also holds, i.e., $\|\varphi(w_n) - \varphi(w)\|_{L^2(\Omega)} \xrightarrow{n \to \infty} 0$ since the continuity gives
\[
sup_{n \geq 0} \|\varphi(w_n)\|_{L^2(\Omega)} < \infty \quad \text{because} \quad sup_{n \geq 0} \|w_n\|_{L^2(\Omega)} < \infty. \]
On the other side, in virtue of the first estimate in (17), the sequence $\{Tw_n\}$ is bounded in finite dimensional space $\mathcal{V}_k$ and thus converges in $\mathcal{V}_k$ up to a subsequence to some $w^* \in \mathcal{V}_k$. Altogether, it follows that, for all $\psi \in \mathcal{V}_k \subset L^\infty(\Omega)$
\[(f, \psi) = \lim_{n \to \infty} \mathcal{E}(\hat{w}_n, \psi) + (q(w_n)\hat{w}_n, \psi) = \mathcal{E}(w_*, \psi) + (q(w)w_*, \psi).\]

The uniqueness of \(\hat{w}\) entails that \(w_* = \hat{w} = Tw\) and hence the whole sequence \(\{Tw_n\}\) converges in \(Tw\) in \(V_k\), which gives the continuity of \(T\). Therefore, by the Brouwer fixed-point theorem, \(T\) has a fixed point \(v_k \in V_k\), i.e., \(v_k = Tv_k\) which clearly satisfies (16) as announced.

To continue, we must show that a subsequence of \(\{v_k\}\) converges in \(L^2(\Omega)\). To do this, we recall that \(R = C \|f\|_{L^2(\Omega)}\) so that from (17) we get the following estimates

\[\|v_k\|_{X_k(\Omega|\mathbb{R}^N)} \leq R \quad \text{and} \quad \int_{\Omega} \chi(v_k) v_k \, dx \leq R^2 \quad \text{for all } k \in \mathbb{N}. \tag{18}\]

Therefore, the sequence \(\{v_k\}\) is clearly bounded in \(X_k(\Omega|\mathbb{R}^N)\). The compactness Theorem 2.2 yields the existence of a subsequence, still denoted by \(\{v_k\}\), converging weakly in \(X_k(\Omega|\mathbb{R}^N)\), strongly in \(L^2(\Omega)\) and almost everywhere in \(\Omega\) to a function \(v\). Wherefore, due to the continuity of \(\chi\), we get

\[\chi(v_k) \to \chi(v) \quad \text{almost everywhere in } \Omega. \tag{19}\]

**Step 2:** Next, we prove that the functions \(\{\chi(v_k)\}\) are uniformly integrable. In view of the estimate (18), for each measurable set \(\Gamma \subset \Omega\) and each \(\Lambda > 0\), we let

\[\Gamma_k^\Lambda = \{x \in \Gamma : |v_k(x)| \geq \Lambda\}\]

so that

\[\int_{\Gamma_k^\Lambda} |\chi(v_k)| \, dx \leq \frac{1}{\Lambda} \int_{\Omega} \chi(v_k) v_k \, dx \leq \frac{R^2}{\Lambda}.\]

Since \(\chi\) is non-decreasing, putting \(\gamma(\Lambda) = \Lambda \max\{\varphi(-\Lambda), \varphi(\Lambda)\}\), we get

\[|\chi(\tau)| \leq \gamma(\Lambda) \quad \text{for all } \tau \in [-\Lambda, \Lambda].\]

Therefore, the following relation holds

\[\int_{\Gamma \setminus \Gamma_k^\Lambda} |\chi(v_k)| \, dx \leq \gamma(\Lambda) |\Gamma|,\]

where \(|\Gamma|\) is the Lebesgue measure of the set \(\Gamma\). These inequalities imply that

\[\int_{\Gamma} |\chi(v_k)| \, dx \leq \frac{R^2}{\Lambda} + \gamma(\Lambda) |\Gamma|.\]

Thus, for an arbitrary \(\varepsilon > 0\), we take \(\Lambda = 2R^2/\varepsilon\) and \(\delta = \varepsilon/(2\gamma(\Lambda))\). Therefore, we find that

\[\sup_{k \geq 1} \int_{\Gamma} |\chi(v_k)| \, dx < \varepsilon\]

for an arbitrary measurable set \(\Gamma \subset \Omega\) such that \(|\Gamma| < \delta\). This, is precisely the uniform integrability of \(\chi(v_k)\). This fact together with (19) and the Vitali convergence theorem (see, e.g., [23, Theorem A.19]) enable us to conclude that \(\chi(v) \in L^1(\Omega)\) and
\( \chi(v_k) \to \chi(v) \) in \( L^1(\Omega) \) as \( k \to \infty \). Now passing to the limit in (16) as \( k \to \infty \) we find that \( v \) satisfies (14), which along with Lemma 3.1, means that \( v \) is a unique weak solution of problem (13).

**Step 3:** We prove the estimates in (i), (ii) and (iii).

The estimate (i) follows from the first inequality in (18) since the weak convergence of \( (v_k)_k \) implies

\[
\mathcal{E}(v, v) \leq \liminf_{k \to \infty} \mathcal{E}(v_k, v_k) \leq C\|f\|^2_{L^2(\Omega)}.
\]

Next, let us consider the truncation \( \tilde{v}_\ell = \max(-\ell, \min(\ell, v)) \), \( \ell \geq 1 \). Then \( \tilde{v}_\ell \in \mathcal{X}_v(\Omega|\mathbb{R}^N) \) since \( (v(x) - v(y)) (\tilde{v}_\ell(x) - \tilde{v}_\ell(y)) \geq |\tilde{v}_\ell(x) - \tilde{v}_\ell(y)|^2 \). The latter inequality and (15) imply \( \mathcal{E}(v, \chi(\tilde{v}_\ell)) \geq \mathcal{E}(\tilde{v}_\ell, \chi(\tilde{v}_\ell)) \geq 0 \). Moreover, \( |\chi(\tilde{v}_\ell)(x) - \chi(\tilde{v}_\ell)(y)| \leq c|\tilde{v}_\ell(x) - \tilde{v}_\ell(y)| \) since \( \chi \) is Lipschitz on \([-\ell, \ell] \) and \( |\tilde{v}_\ell| \leq \ell \). Thus, \( \chi(\tilde{v}_\ell) \in \mathcal{X}_v(\Omega|\mathbb{R}^N) \). Noticing that, \( \chi(v) \chi(\tilde{v}_\ell) \geq \chi^2(\tilde{v}_\ell) \), taking \( \psi = \chi(\tilde{v}_\ell) \) in (14) yields

\[
\|\chi(\tilde{v}_\ell)\|^2_{L^2(\Omega)} \leq \mathcal{E}(v, \chi(\tilde{v}_\ell)) + (\chi(v), \chi(\tilde{v}_\ell)) \leq \|f\|^2_{L^2(\Omega)} \|\chi(\tilde{v}_\ell)\|_{L^2(\Omega)}.
\]

Thus, since \( \{\chi^2(\tilde{v}_\ell)\} \to \chi^2(v) \) a.e. in \( \Omega \), Fatou’s lemma implies the second estimate as follows

\[
\|\chi(v)\|_{L^2(\Omega)} \leq \liminf_{\ell \to \infty} \|\chi(\tilde{v}_\ell)\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.
\]

Finally, by continuity of \( \varphi \), there exists \( \delta > 0 \) such that \( \varphi^2(\tau) \leq 1 \) for all \( \tau \in [-\delta, \delta] \). Hence, letting \( \Gamma_\delta = \{x \in \Omega : |v(x)| \geq \delta\} \), the second estimate implies the third one as follows

\[
\int_{\Omega} \varphi^2(v) \, dx = \int_{\Gamma_\delta} \varphi^2(v) \, dx + \int_{\Omega \setminus \Gamma_\delta} \varphi^2(v) \, dx \leq \frac{1}{\delta^2} \int_{\Omega} \varphi^2(v) \, dx \\
+ \int_{\Omega \setminus \Gamma_\delta} \varphi^2(v) \, dx \leq \frac{1}{\delta^2} \|f\|^2_{L^2(\Omega)} + |\Omega|.
\]

\( \square \)

Next, we define the mapping \( V : L^2(\Omega) \to \mathcal{X}_v(\Omega|\mathbb{R}^N) \) such that, for \( f \in L^2(\Omega) \),

\[
v = V(f) \text{ is the unique weak solution of problem (3.1).} \tag{20}
\]

We derive in the lemma below, some convergence results for the sequence \( \{V(f_k)\} \) which are decisive for the application the Tychonoff fixed-point Theorem 2.5.

**Lemma 3.3** Assume that \( f_k \rightharpoonup f \) weakly \( L^2(\Omega) \) and let \( v_k = V(f_k) \) and \( v = V(f) \). Then it holds that \( v_k \to v \) strongly in \( \mathcal{X}_v(\Omega|\mathbb{R}^N) \) and \( \varphi(v_k) \rightharpoonup \varphi(v) \) weakly in \( L^2(\Omega) \).

**Proof** Let us identify \( f_k - f \) in \( (\mathcal{X}_v(\Omega|\mathbb{R}^N))^* \) with the linear form
By the reflexivity of $\mathcal{X}_t(\Omega|\mathbb{R}^N)$, there exists $w_k \in X_t(\Omega|\mathbb{R}^N)$ (cf. [31, Theorem 2], [35, Chapter 6] or [7, page 60]) such that $\|w_k\|_{X_t(\Omega|\mathbb{R}^N)} \leq 1$ and
$$
\|f_k - f\|_{X_t(\Omega|\mathbb{R}^N)^*} = \int_{\Omega} (f_k(x) - f(x))w_k(x)dx.
$$

According to the compactness Theorem 2.2 we may assume that $\{w_k\}$ strongly converges to some $w$ in $L^2(\Omega)$. Therefore, the weakly convergence of $\{f_k\}$ implies that
$$
\|f_k - f\|_{X_t(\Omega|\mathbb{R}^N)^*} = \int_{\Omega} (f_k(x) - f(x))w_k(x)dx \xrightarrow[k \to \infty]{} 0.
$$

The convergence in $\mathcal{X}_t(\Omega|\mathbb{R}^N)$ follows immediately from Lemma 3.1 since
$$
\|v_k - v\|_{\mathcal{X}_t(\Omega|\mathbb{R}^N)} \leq C\|f_k - f\|_{(X_t(\Omega|\mathbb{R}^N)^*)} \xrightarrow[k \to \infty]{} 0.
$$

On the other hand, we also have the strong convergence of $\{v_k\}$ in $L^2(\Omega)$ and the continuity of $\varphi$ imply that $\{\varphi(v_k)\}$ converges almost everywhere to $\varphi(v)$ up a subsequence. Furthermore, since $\{f_k\}$ is bounded, as in the proof of Lemma 13, one easily gets that
$$
\|\varphi(v_k)\|_{L^2(\Omega)} \leq C \quad \text{for all } k \geq 1,
$$
for a constant $C > 0$ independent on $k$. Thus, $\{\varphi(v_k)\}$ has a further subsequence weakly converging in $L^2(\Omega)$. The Banach-Saks Theorem, see [40, Appendix A] or [42, Proposition 10.8], infers the existence of a further subsequence whose Césaro mean converges strongly in $L^2(\Omega)$ and almost everywhere in $\Omega$ to the same limit. Necessarily, since $\{\varphi(v_k)\}$ converges almost everywhere to $\varphi(v)$, the entire sequence $\{\varphi(v_k)\}$ weakly converges in $L^2(\Omega)$ to $\varphi(v)$.

\[\square\]

3.2 Nonlocal parabolic problem

We consider the following parabolic problem:

$$
\begin{cases}
\partial_t u + \mathcal{L} u + \zeta u = 0 & \text{in } \Omega_T, \\
u = 0 & \text{in } \Sigma, \\
u(\cdot, 0) = u_0, & \text{in } \Omega,
\end{cases}
\tag{21}
$$

where $u_0, \zeta \in L^2(\Omega)$ with $\zeta \geq 0$. We also assume $\nu \not\in L^1(\mathbb{R}^N)$ so that by Theorem 2.2, the embedding $\mathcal{X}_t(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega)$ is compact. Therefore, by the standard Galerkin superposition method (see for instance [23, Section 4.6]), a weak solution $u$ of the problem (21) can be easily obtained in $L^2(0, T; \mathcal{X}_t(\Omega|\mathbb{R}^N) \cap L^2(\Omega, \zeta))$. Here $L^2(\Omega, \zeta)$ is the Hilbert space with the norm

$w \mapsto \int_{\Omega} (f_k(x) - f(x))w(x)dx$. 

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\[ \|u\|^2_{L^2(\Omega, \xi)} = \int_{\Omega} |u(x)|^2 \xi(x) \, dx. \]

We omit the proof as well as various justifications (see also [18, 28]). Another possibility, is to observe that (see [34]) there exists a unique semigroup with generator \( A \) on \( L^2(\Omega) \) associated to the closed bilinear form \( a(u, v) = (u, v)_{\mathcal{X}_v(\Omega; \mathbb{R}^N)} + (u, v)_{L^2(\Omega, \xi)} \), with \( u, v \in \mathcal{X}_v(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \xi) \), such that \( a(u, v) = \langle Au, v \rangle \). Thus \( u(x, t) = e^{-tA}u_0(x) \), \( 0 \leq t \leq T \), is the unique weak solution to (21). The weak solution of problem (21) satisfies the energy estimate:

\[ \frac{1}{2} \|u(\cdot, t)\|^2_{L^2(\Omega)} + \int_0^T \mathcal{E}(u, u) \, dt + \int_0^T \int_{\Omega} \xi u^2 \, dx \, dt \leq \frac{1}{2} \|u_0\|^2_{L^2(\Omega)} \quad (22) \]

for all \( t \in [0, T) \). Besides that, \( \partial_t u \) belongs to the space \( L^2(0, T; (\mathcal{X}_v(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \xi))^*) \), where \((\mathcal{X}_v(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \xi))^* \) is the dual space of \( \mathcal{X}_v(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \xi) \). According to Proposition 2.3 (see (8)) we find that \( u \in C(0, T; L^2(\Omega)) \). Thus, the function \( u_T = u(\cdot, T) \) is well defined and belongs to \( L^2(\Omega) \). Moreover, the estimate (22) holds in particular for \( t = T \).

For each \( \zeta \in L^2(\Omega) \), \( \zeta \geq 0 \), define the mapping \( \mathcal{U} : \zeta \mapsto \mathcal{U}(\zeta) \) where \( \mathcal{U}(\zeta) \in L^2(0, T; (\mathcal{X}_v(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \xi))^*) \) is the unique weak solution of problem (21).

We want to study the continuity of the operators \( \mathcal{U} \) and \( \mathcal{U}_T \) on \( L^2(\Omega) \), with \( \mathcal{U}_T(\zeta)(\cdot) = \mathcal{U}(\zeta)(\cdot, T) \).

**Lemma 3.4** Let \( u_0 \in L^2(\Omega) \) and \( \{\zeta_k\} \) be a sequence of non-negative functions converging weakly in \( L^2(\Omega) \) to a function \( \zeta \). Then \( \mathcal{U}_T(\zeta_k) \rightharpoonup \mathcal{U}_T(\zeta) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \).

**Proof** For brevity, we denote \( u_k = \mathcal{U}(\zeta_k) \) and \( u = \mathcal{U}(\zeta) \). It follows from (22), for all \( k \in \mathbb{N} \),

\[ \frac{1}{2} \|u_k(\cdot, T)\|^2_{L^2(\Omega)} + \int_0^T \mathcal{E}(u_k, u_k) \, dt + \int_0^T \int_{\Omega} \zeta_k u_k^2 \, dx \, dt \leq \frac{1}{2} \|u_0\|^2_{L^2(\Omega)}. \quad (23) \]

Next, consider \( \psi : \mathbb{R}^N \times (0, T) \to \mathbb{R} \) be an arbitrary smooth function such that \( \psi = 0 \) in \( \mathbb{R}^N \setminus \Omega \times (0, T) \). Define \( w_k(x) = \int_0^T u_k(x, t)\psi(x, t) \, dt \) assuming \( \psi(\cdot, t) \in C^\infty_c(\Omega) \) there holds

\[ |\psi(x, t) - \psi(y, t)| \leq \|\psi(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^N)}(1 \wedge |x - y|). \]

Using this inequality yields
|w_k(x) - w_k(y)|^2 = \left| \int_0^T (u_k(x, t) - u_k(y, t))\psi(x, t) \, dt + \int_0^T (\psi(x, t) - \psi(y, t))u_k(y, t) \, dt \right|^2 \\
leq 2T \int_0^T |u_k(x, t) - u_k(y, t)|^2 |\psi(x, t)|^2 \, dt \\
+ 2T \int_0^T |\psi(x, t) - \psi(y, t)|^2 |u_k(y, t)|^2 \, dt \\
leq 2T \sup_{r \in [0, T]} \|\psi(\cdot, t)\|_{W^{1, \infty}(\mathbb{R}^N)}^2 \int_0^T |u_k(x, t) - u_k(y, t)|^2 \\
+ (1 \wedge |x - y|^2)|u_k(y, t)|^2 \, dt.

Integrating both sides over $\Omega \times \mathbb{R}^N$ (see Remark 2.1) with respect to the measure $\nu(x - y) \, dx \, dy$ gives

$$
\mathcal{E}(w_k, w_k) \leq 2T \sup_{r \in [0, T]} \|\psi(\cdot, t)\|_{W^{1, \infty}(\mathbb{R}^N)}^2 \int_0^T \mathcal{E}(u_k, u_k) \, dt \\
+ \int_{\mathbb{R}^N} (1 \wedge |h|^2)\nu(h) \, dh \int_0^T \int_{\Omega} |u_k(\cdot, t)|^2 \, dx \, dt.
$$

Altogether, with the Poincaré inequality (5) and the inequality (23) yield

$$
\frac{1}{2} \left\| u_k(\cdot, T) \right\|_{L^2(\Omega)}^2 + \int_0^T \mathcal{E}(u_k, u_k) \, dt \leq \frac{1}{2} \left\| u_0 \right\|_{L^2(\Omega)}^2 \quad \text{and} \quad \mathcal{E}(w_k, w_k) \leq C_\psi \left\| u_0 \right\|_{L^2(\Omega)}^2
$$

Therefore, $\{w_k\}$ is bounded in $\mathcal{X}_\nu(\Omega|\mathbb{R}^N)$. By the compactness Theorem 2.2 we can assume that $\{w_k\}$ strongly converge to some $w \in L^2(\Omega)$. On the other hand, $\{u_k\}$ and $\{u_k(\cdot, T)\}$ are bounded in $L^2(0, T; \mathcal{X}_\nu(\Omega|\mathbb{R}^N))$ and $L^2(\Omega)$ respectively and hence can be assumed to weakly converge to some $u \in L^2(0, T; \mathcal{X}_\nu(\Omega|\mathbb{R}^N))$ and $h \in L^2(\Omega)$ respectively. In particular, the strong convergence implies that, for every $\phi \in C^\infty_c(\Omega)$,

$$
\int_{\Omega} w(x)\phi(x) \, dx = \lim_{k \to \infty} \int_{\Omega} w_k(x)\phi(x) \, dx = \lim_{k \to \infty} \int_{\Omega} \phi(x) \int_0^T u_k(x, t) \, dt \, dx \\
= \int_{\Omega} \phi(x) \int_0^T u(x, t) \, dt \, dx.
$$

It turns out that $w = \int_0^T u(\cdot, t) \, dt$ a.e. on $\Omega$. By the same token, one gets $h = u(\cdot, T)$ a.e. on $\Omega$.

Once again, the strong convergence of $\{w_k\}$ and the weak of convergence of $\{z_k\}$ in $L^2(\Omega)$ yield

$$
\int_0^T \int_{\Omega} z_k u_k \psi \, dx \, dt \xrightarrow{k \to \infty} \int_0^T \int_{\Omega} z u \psi \, dx \, dt.
$$

(24)

For each $k \geq 1$, by definition of $u_k = \mathcal{N}(z_k)$, we get
By choosing in particular the test function \( \psi(\cdot, T) = 0 \), letting \( k \to \infty \) yields 
\[ w(\cdot, 0) = u_0 \] and 
This means that \( u \) is a weak solution to (21) and by uniqueness, \( u = \mathcal{U}(\zeta) \). The uniqueness of \( u \) implies that the whole sequence \( \{u_k\} \) weakly converges to \( u \) in 
\[ L^2(0, T; X_c(\Omega|\mathbb{R}^N)). \]
Therefore, since \( u_k = \mathcal{U}(\zeta_k) \) and \( u = \mathcal{U}(\zeta) \) are weak solutions, using (24) and the weak convergence, for \( \psi \in L^2(\Omega) \) gives 
\[ \int_{\Omega} (\mathcal{U}_T(\zeta_k) - \mathcal{U}_T(\zeta))(x)\psi(x)dx = \int_0^T \mathcal{E}(u_k - u, \psi) dt 
+ \int_0^T \int_{\Omega} (u_k \zeta_k - u \zeta)(x, t)\psi(x)dxdt \xrightarrow{k \to \infty} 0. \]
That is, the whole sequence \( \{\mathcal{U}_T(\zeta_k)\} \) weakly converges to \( U_T \) in \( L^2(\Omega) \) which is the sought result.

4 Weak solvability and uniqueness of the solution

Armed with the above auxiliaries results, let us turn our attention to the proof of the weak solvability of problem (1). In order to apply the Tychonoff fixed-point Theorem 2.5, we take 
\[ X = L^2(\Omega), \ G = \{w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}\} \] which is clearly closed, convex and bounded. The next result provides the existence of a weak solution to the problem (1).

**Theorem 4.1** Let \( u_0 \in L^2(\Omega) \) and \( T > 0 \). Let the mapping \( \pi : G \to G \) with 
\[ \pi(w) = \mathcal{U}_T(\varphi(v)), \] where \( v = V(u_0 - w) \) (defined as in (20)) is the unique weak solution to (13) with \( f = u_0 - w \). Then \( \pi \) has a fixed point \( u_T \) that is 
\[ u_T = \pi(u_T) = \mathcal{U}_T(\varphi(v)) \quad \text{with} \ v = V(u_0 - u_T). \]
Moreover, \( v = \int_0^T u dt \) and 
\[ u = \mathcal{U}(\varphi(v)) \quad \text{is a weak solution of the problem (1.1).} \]
Proof For \( v = V(u_0 - w) \) with \( w \in G \), we know from Theorem 3.2 that \( \varphi(v) \in L^2(\Omega) \). Thus for \( \zeta = \varphi(v) \geq 0 \), the function \( \mathcal{M}(\zeta) \) satisfies (22) which implies that \( \| \mathcal{M}(\zeta) \|_{L^2(\Omega)} \leq \| u_0 \|_{L^2(\Omega)} \). In particular, \( \| \pi(w) \|_{L^2(\Omega)} \leq \| u_0 \|_{L^2(\Omega)} \) for all \( w \in G \) and thus, \( \pi(G) \subseteq G \). It remains to prove the weak sequential continuity of \( \pi \). Let \( \{ w_k \} \) be an arbitrary sequence in \( G \) that converges to \( w \in G \) weakly in \( L^2(\Omega) \). We need to prove that \( \pi(w_k) \to \pi(w) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \). In virtue of Lemma 3.3, \( v_k = V(u_0 - w_k) \to v = V(u_0 - w) \) strongly in \( X_v(\Omega; \mathbb{R}^N) \) and \( \varphi(v_k) \to \varphi(v) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \), where \( v_k = V(u_0 - w_k) \) and \( v = V(u_0 - w) \). In turn, Lemma 3.4 implies that \( \pi(w_k) = \mathcal{M}_T(\varphi(v_k)) \to \mathcal{M}_T(\varphi(v)) = \pi(w) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \). Thus, according to Theorem 2.5, \( \pi \) has a fixed point \( u_T = \pi(u_T) \). Next, knowing that \( u_T \) is a fixed point of the mapping \( \pi \), we show that \( u = \mathcal{M}(\varphi(v)) \), with \( v = V(u_0 - u_T) \), is a weak solution to the problem (1). Indeed, recall that \( u = \mathcal{M}(\varphi(v)) \) is the unique weak solution to the problem (21) with \( \zeta = \varphi(v) \), i.e., \( u(\cdot, 0) = u_0 \) and for all \( \psi \in C^1_c(0, T; X_v(\Omega; \mathbb{R}^N)) \),

\[
\int_0^T \int_\Omega \partial_t u \psi \, dx \, dt + \int_0^T \int_\Omega E(u, \psi) \, dx \, dt + \int_0^T \int_\Omega \varphi(v) u \psi \, dx \, dt = 0. \tag{25}
\]

Integrating by parts, for \( \psi \in X_v(\Omega; \mathbb{R}^N) \) (time independent) we get

\[
\mathcal{E}\left( \int_0^T u \, dt, \psi \right) + \int_\Omega \varphi(v) \psi \int_0^T u \, dx \, dt = \int_\Omega [u_0 - u_T] \psi \, dx.
\]

Thus, according to Theorem 3.2, \( v = \int_0^T u(x, t) \, dt \) is the unique weak solution to the elliptic problem

\[
\mathcal{L} v + \varphi(v) v = u_0 - u_T \text{ in } \Omega \quad \text{and} \quad v = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \tag{26}
\]

We have shown that \( v = V(u_0 - u_T) = \int_0^T u \, dt \). Therefore, we obtain

\[
u = \mathcal{M}(\varphi\left( \int_0^T u \, dt \right)) \tag{27}
\]

which, according to the relation (25), implies that \( u \) is a weak solution to the problem (1). \( \square \)

The main result of the paper is the following theorem.

**Theorem 4.2** Let \( u_0 \in L^2(\Omega) \), \( T > 0 \) and \( \varphi \) satisfies Assumption 1.1. The problem (1) has a weak solution \( u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; X_v(\Omega; \mathbb{R}^N)) \) such that

\[
\varphi(v) \in L^2(\Omega), \quad \varphi(v) v \in L^2(\Omega), \quad \varphi(v) u^2 \in L^1(\Omega_T), \quad \text{and} \quad u \in C(0, T; L^2(\Omega)),
\]

where \( v = \int_0^T u \, dt \). Moreover, the following estimates hold true.
\[\frac{1}{2} \left\| u \right\|_{L^2(0,T;L^2(\Omega))}^2 + \left\| u \right\|_{L^2(0,T;\mathbb{X}_0(\Omega;\mathbb{R}^N))}^2 + \int_0^T \int_{\Omega} \varphi(v) u^2 \, dx \, dt \leq \frac{1}{2} \left\| u_0 \right\|_{L^2(\Omega)}^2,\]

\[\left\| \partial_t u \right\|_{L^2(0,T;\mathbb{X}_0(\Omega;\mathbb{R}^N) \cap L^2(\Omega;\varphi(v))^*)}^2 \leq \frac{1}{2} \left\| u_0 \right\|_{L^2(\Omega)}^2.\]

**Proof** The existence of a weak solution to the problem (1) follows immediately from Theorem 4.1. Furthermore, mimicking the estimate (22) yields

\[\frac{1}{2} \left\| u \right\|_{L^2(0,T;L^2(\Omega))}^2 + \left\| u \right\|_{L^2(0,T;\mathbb{X}_0(\Omega;\mathbb{R}^N))}^2 + \int_0^T \int_{\Omega} \varphi(v) u^2 \, dx \, dt \leq \frac{1}{2} \left\| u_0 \right\|_{L^2(\Omega)}^2.\]

Now, for each \( \psi \in L^2(0, T; \mathbb{X}_0(\Omega;\mathbb{R}^N) \cap L^2(\Omega, \varphi(v))) \), by definition of \( u \), we have

\[\left| \int_0^T \langle \partial_t u, \psi \rangle dt \right|^2 = -\int_0^T \mathcal{E}(u, \psi) \, dt - \int_0^T \int_{\Omega} \varphi(v) u \psi \, dx \, dt \leq \left( \int_0^T \mathcal{E}(u, u) \, dt \right) + \left( \int_0^T \int_{\Omega} \varphi(v) u^2 \, dx \, dt \right) \left( \int_0^T \int_{\Omega} \varphi(v) \psi^2 \, dx \, dt \right) \leq \frac{1}{2} \left\| u_0 \right\|_{L^2(\Omega)}^2 \left( \int_0^T \mathcal{E}(\psi, \psi) \, dt + \int_0^T \int_{\Omega} \varphi(v) \psi^2 \, dx \, dt \right).\]

This implies that

\[\left\| \partial_t u \right\|_{L^2(0,T;\mathbb{X}_0(\Omega;\mathbb{R}^N) \cap L^2(\Omega;\varphi(v))^*)}^2 \leq \frac{1}{2} \left\| u_0 \right\|_{L^2(\Omega)}^2.\]

Therefore, we have \( u \in L^2(0, T; \mathbb{X}_0(\Omega;\mathbb{R}^N) \cap L^2(\Omega, \zeta)) \) and \( \partial_t u \in L^2(0, T; \mathbb{X}_0(\Omega;\mathbb{R}^N) \cap L^2(\Omega, \zeta))^* \) with \( \zeta = \varphi \left( \int_0^T u \, dt \right) \) which implies that \( \varphi(v) u^2 \in L^1(\Omega_T) \). On the one hand, by definition of \( u = \mathcal{B}(\varphi(v)) \) it follows that \( u \in C(0, T; L^2(\Omega)) \). On the other hand, we know that \( \nu = \int_0^T u \, dt \) is the unique weak solution to the problem (26) and hence from Theorem 3.2 we have \( \varphi(v), \varphi(v) \nu \in L^2(\Omega) \). This ends the proof. \( \square \)

Next, we prove that problem (1) has a unique solution, provided that the initial condition \( u_0 \) is bounded. Before, we need to establish the following maximum principle result.

**Lemma 4.3** Let \( u = u(x,t) \) be a weak solution of the problem (1), i.e., satisfies (12) with \( u_0 \in L^2(\Omega) \cap L^\infty(\Omega) \) then \( \left\| u \right\|_{L^\infty(0,T;L^\infty(\Omega))} \leq \left\| u_0 \right\|_{L^\infty(\Omega)} \) on \( \Omega_T \), i.e., \( \left\| u \right\| \leq \left\| u_0 \right\|_{L^\infty(\Omega)} \) a.e. on \( \Omega_T \).

**Proof** Set \( \Lambda = \left\| u_0 \right\|_{L^\infty(\Omega)} \) and consider the convex function \( F : \mathbb{R} \to [0, \infty) \) defined by

\[F(\tau) = \begin{cases} (\tau + \Lambda)^2 & \text{if } \tau < -\Lambda \\ 0 & \text{if } |\tau| \leq \Lambda \\ (\tau - \Lambda)^2 & \text{if } \tau > \Lambda. \end{cases}\]
So that, \( F(\tau) = 0 \) if and only if \(|\tau| \leq \Lambda\), in particular \( F(u_0) = 0 \) a.e. on \( \Omega \). By convexity, \( F' \) is non-decreasing, i.e., \((F'(\tau_1) - F'(\tau_2))(\tau_1 - \tau_2) \geq 0 \) for all \( \tau_1, \tau_2 \in \mathbb{R} \) in particular, since \( F'(0) = 0 \), we have \( F'(\tau_1) \tau_1 \geq 0 \) for all \( \tau_1 \in \mathbb{R} \). Furthermore, \( F'(u(\cdot, t)) \in \mathcal{X}_T(\Omega) \) because \( u(\cdot, t) \in \mathcal{X}_T(\Omega) \) and one can check that \( F' \) is Lipschitz since \( F'' \) is bounded. Therefore, testing the equation (12) against \( \zeta = F'(u) \) gives
\[
\frac{d}{dt} \int_{\Omega} F(u(x, t)) dx = -\mathcal{E}(u, F'(u)) - \int_{\Omega} \varphi(v)(x) F'(u(x, t)) u(x, t) dx \leq 0.
\]
Since \( F(u_0) = 0 \) almost everywhere on \( \Omega \), integrating the inequality gives
\[
\int_{\Omega} F(u(x, t)) dx \leq 0 \quad \text{for all} \quad 0 \leq t \leq T.
\]
Thus, \( F(u(x, t)) = 0 \) a.e. on \( \Omega_T \), and hence \( |u(x, t)| \leq \Lambda \) a.e. on \( \Omega_T \).

**Proof** Suppose that problem (1) has two weak solutions \( u_1 \) and \( u_2 \), and put
\[
v_i(x) = \int_0^T u_i(x, t) dt, \quad i = 1, 2.
\]
Then \( u = u_1 - u_2 \) is a weak solution to
\[
\begin{aligned}
\partial_t u + \mathcal{L} u + \varphi(v_1) u_1 - \varphi(v_2) u_2 &= 0 \quad \text{in} \quad \Omega_T, \\
u &= 0 \quad \text{in} \quad \Sigma, \\
u(\cdot, 0) &= 0 \quad \text{in} \quad \Omega.
\end{aligned}
\]

The maximum principle in Lemma 4.3, implies that \( |u_i| \leq \Lambda \) a.e. in \( \Omega_T \) and hence \( |v_i| \leq \Lambda T, \quad i = 1, 2 \), a.e. in \( \Omega \). Testing the above equation with \( u \) leads to the following equality:
\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + \mathcal{E}(u, u) + \int_{\Omega} \varphi(v_1) u^2 \, dx + \int_{\Omega} (\varphi(v_1) - \varphi(v_2)) u_2 \, u \, dx = 0
\]
which implies that
\[
\frac{1}{2} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \mathcal{E}(u, u) \, dt \leq \kappa \Lambda \int_0^t \int_{\Omega} |v(x)| |u(x, \tau)| \, dx \, d\tau
\]
for all \( t \in [0, T] \), where \( v = v_1 - v_2 = \int_0^T u(\cdot, \tau) \, d\tau \). Noticing that,
\[
\|v\|_{L^2(\Omega)}^2 = \int_{\Omega} \left( \int_0^T u(x, \tau) \, d\tau \right)^2 \, dx \leq T \int_0^T \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau,
\]
we get
\[
\int_0^T \int_\Omega |v(x)| |u(x, \tau)| \, dx \, d\tau \leq T \left( \int_0^T \|u(\cdot, \tau)\|^2_{L^2(\Omega)} \, d\tau \right)^{1/2} \left( \int_0^T \|u(\cdot, \tau)\|^2_{L^2(\Omega)} \, d\tau \right)^{1/2},
\]

Therefore, we obtain the following inequality for all \( t \in [0, T] \)
\[
\|u(\cdot, t)\|^2_{L^2(\Omega)} \leq 2\kappa \Lambda T \left( \int_0^T \|u(\cdot, \tau)\|^2_{L^2(\Omega)} \, d\tau \right)^{1/2} \left( \int_0^T \|u(\cdot, \tau)\|^2_{L^2(\Omega)} \, d\tau \right)^{1/2}.
\]

(28)

In short we rewrite the above inequality as follows
\[
\varphi'(t) \leq 2\kappa \Lambda T \varphi^{1/2}(T) \varphi^{1/2}(t) \quad \text{with} \quad \varphi(t) = \int_0^t \|u(\cdot, \tau)\|^2_{L^2(\Omega)} \, d\tau.
\]

A routine integration yields that \( \varphi^{1/2}(t) \leq \kappa \Lambda T^2 \varphi^{1/2}(T) \) and, in particular, \( \varphi^{1/2}(T) \leq \kappa \Lambda T^2 \varphi^{1/2}(T) \). The latter inequality holds true only if \( \varphi(T) = 0 \) since \( \kappa \Lambda T^2 < 1 \), which implies that \( u = 0 \).

We now point out the following the closing remark which shows how the function spaces considered in this note extends our studies to a slightly different type of problems.

**Remark 4.5** Analogous results to those obtained in this notes can be established replacing the Dirichlet complement condition \( u = 0 \) in \((\mathbb{R}^N \setminus \Omega) \times (0, T)\), the problem 1 with the Neumann complement condition \( \mathcal{N}u = 0 \) in \((\mathbb{R}^N \setminus \Omega) \times (0, T)\), where \( \mathcal{N}u \) represents the nonlocal normal derivative of \( u \) across as defined in (11). To this end, it is decisive to taking into account the setting of Theorem 2.2, namely that \( \Omega \) is bounded and Lipschitz and that \( v \) satisfies the asymptotic condition (6), in such a way that the compactness of the embedding \( V_v(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega) \) holds true. Wherefrom, one readily obtains (see [21, 23]) the Poincaré type inequality
\[
\|u\|^2_{L^2(\Omega)} \leq C\mathcal{E}(u, u) \quad \text{for all} \ u \in V_v(\Omega|\mathbb{R}^N)^\perp,
\]
for some constant \( C > 0 \) and where \( V_v(\Omega|\mathbb{R}^N)^\perp = \{ V_v(\Omega|\mathbb{R}^N) : \int_\Omega u \, dx = 0 \} \). These observations, alongside of our procedure, allow to replace the space \( X_v(\Omega|\mathbb{R}^N) \) with the space \( V_v(\Omega|\mathbb{R}^N)^\perp \).

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