Chern-Simons approach to three-manifold invariants*

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A new, formal, non-combinatorial approach to invariants of three-dimensional manifolds of Reshetikhin, Turaev and Witten in the framework of non-perturbative topological quantum Chern-Simons theory, corresponding to an arbitrary compact simple Lie group, is presented. A direct implementation of surgery instructions in the context of quantum field theory is proposed. An explicit form of the specialization of the invariant to the group SU(2) is derived, and shown to agree with the result of Lickorish.

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1. Introduction

Four years ago in his famous paper on quantum field theory and the Jones polynomial [1], Edward Witten proposed a new interesting topological invariant of three-dimensional manifolds. An explicit construction of the invariant, using quantum groups, appeared for the first time in a paper of Reshetikhin and Turaev [2]. Other papers presenting re-derivations of this result are more geometrical by nature [3], and use the Temperley-Lieb algebra, as suggested by Lickorish [4–6]. All the approaches are combinatorial. Non-combinatorial possibilities, very straightforward though mathematically less rigorous, are offered by topological quantum field theory [7].

Inspired by the papers [6,8], we aim to propose a new, formal, non-combinatorial derivation of the three-manifold invariants of the Reshetikhin-Turaev-Witten (RTW) type in the framework of non-perturbative (topological) quantum Chern-Simons (CS) gauge theory. The idea is extremely simple, and in principle applies to an arbitrary compact (semi-)simple group G (not only to the SU(2) one). Our invariant is essentially the partition function of CS theory on the manifold $\mathcal{M}_L$, defined via surgery on the framed link L in the three-dimensional sphere $S^3$. Actually, surgery instructions are implemented in the most direct and literal way. The method of cutting and pasting back, which has been successfully applied to two-dimensional Yang-Mills theory [8], is explicitly used in the standard field-theoretical fashion. Roughly speaking, cutting corresponds to fixing, whereas pasting back to identification and summing up the boundary conditions.

As a by-product of our analysis we consider the satellite formula, and derive the Kauffman bracket polynomial invariant of a trivial (with zero framing) unknot for an arbitrary representation of SU(2).
2. General Formalism

Our principal goal is to compute the partition function $Z(\mathcal{M}_L)$ of CS theory on the manifold $\mathcal{M}_L$, defined via (honest/integer) surgery on the framed link $L = \bigcup_{i=1}^{N} K_i$ in $\mathcal{S}^3$, for an arbitrary compact simple (gauge) Lie group $G$. Obviously, the starting point is the partition function of CS theory $Z(\mathcal{S}^3)$ on the sphere $\mathcal{S}^3$ [1]

$$Z(\mathcal{S}^3) = \int e^{ikcs(A)} \mathcal{D}A,$$  \hfill (2.1)

where the functional integration is performed with respect to the connections $A$ modulo gauge transformations, defined on a trivial $G$ bundle on $\mathcal{S}^3$, and $k$ is the level ($k \in \mathbb{Z}^+$). The classical action is the CS secondary characteristic class

$$cs(A) = \frac{1}{4\pi} \int_{\mathcal{S}^3} \text{Tr}(AdA + \frac{2}{3} A^3),$$  \hfill (2.2)

and the expectation value of an observable $\mathcal{O}$ is defined as

$$\langle \mathcal{O} \rangle = \int \mathcal{O} e^{ikcs(A)} \mathcal{D}A.$$

\hfill (2.3)

According to the surgery prescription we should cut out a closed tubular neighbourhood $N_i$ of $K_i$ (a solid torus), and paste back a copy of a solid torus $T$, matching the meridian of $T$ to the (twisted by framing number) longitude on the boundary torus $\partial N_i$ in $\mathcal{S}^3$ [5, 9]. To this end, in the first step, we should fix boundary conditions for the field $A$ on the twisted longitude represented by $K_i$. Since the only gauge-invariant (modulo conjugation) quantity defined on a closed curve is holonomy [8], we associate the holonomy operator $\text{Hol}_{K_i}(A)$ to each knot $K_i$. Thus the symbol

$$Z(\mathcal{S}^3, L; g_1, g_2, \ldots, g_N)$$  \hfill (2.4)

should be understood as the constrained partition function of CS theory, i.e. the values of holonomies along $K_i$ are fixed

$$\text{Hol}_{K_i}(A) = g_i, \quad i = 1, 2, \ldots, N.$$  \hfill (2.5)
Now, we can put
\[ Z(S^3, L; g_1, g_2, \ldots, g_N) = \left\langle \prod_{i=1}^{N} \delta(g_i, \text{Hol}_{K_i}(A)) \right\rangle, \quad (2.6) \]
where \( \delta \) is a (group-theoretic) Dirac delta-function [8]. Its explicit form following from the (group-theoretic) Fourier expansion [10] is
\[ \delta(g, h) = \sum_{\mu \in \hat{G}} \overline{\chi_{\mu}(g)} \chi_{\mu}(h). \quad (2.7) \]

Physical observables being used in CS theory are typically Wilson loops, defined as
\[ W^K_{\mu}(A) = \text{Tr}_{\mu}(\text{Hol}_K(A)) \equiv \chi_{\mu}(\text{Hol}_K(A)), \quad (2.8) \]
where \( \mu \in \hat{G} \) numbers inequivalent irreducible representations (irrep’s) of \( G \), and \( \chi_{\mu} \) is a character. By virtue of (2.7–8)
\[ \delta(g_i, \text{Hol}_{K_i}(A)) = \sum_{\mu \in \hat{G}} \overline{\chi_{\mu}(g_i)} W^K_{\mu_i}(A). \quad (2.9) \]

Inserting (2.9) into (2.6) yields, as a basic building block, the following representation of the constrained partition function
\[ Z(S^3, L; g_1, g_2, \ldots, g_N) = \left\langle \prod_{i=1}^{N} \sum_{\mu_i \in \hat{G}} \overline{\chi_{\mu_i}(g_i)} W^K_{\mu_i}(A) \right\rangle. \quad (2.10) \]

In the second step of our construction, we should paste back the tori matching the pairs of “longitudes” (the twisted longitudes and the meridians), i.e. we should identify and sum up the boundary conditions. Since the interior of a solid torus is homeomorphic to \( S^3 \) with a removed solid torus, actually the meridians play the role of longitudes in analogous cutting procedures for an unknot \( \{ \bigcirc \} \) (with reversed orientation). Thus the partition function of CS theory on \( \mathcal{M}_L \) is
\[ Z(\mathcal{M}_L) = \frac{1}{N_L} \int \prod_{i=1}^{N} dg_i Z(S^3, \bigcirc; g_i^{-1}) Z(S^3, L; g_1, g_2, \ldots, g_N) \]

\[ = \frac{1}{N_L} \int \prod_{i=1}^{N} dg_i \sum_{\mu_i \in \hat{G}} \sum_{\nu_i \in \hat{G}} \frac{\chi_{\mu_i}(g_i^{-1}) \chi_{\nu_i}(g_i)}{\chi_{\nu_i}(g_i)} \langle W_{\mu_i}^{\bigcirc}(A) \rangle \left\langle \prod_{j=1}^{N} W_{\nu_j}^{K_j}(A) \right\rangle, \quad (2.11) \]

where \( N_L \) is a link-dependent normalization constant, and the reversed orientation of the unknots \( \{\bigcirc\} \) (corresponding to the meridians of the pasted back tori) accounts for the power \(-1\) of the group elements \( g_i \). From the orthogonality relations for characters and unitarity of irrep’s, it follows that the three-manifold invariant is of the form

\[ Z(\mathcal{M}_L) = \frac{1}{N_L} \left\langle \prod_{i=1}^{N} \omega_{K_i}(A) \right\rangle, \quad (2.12a) \]

where

\[ \omega_{K_i}(A) \equiv \sum_{\mu_i \in \hat{G}} \langle W_{\mu_i}^{\bigcirc}(A) \rangle W_{\mu_i}^{K_i}(A) \quad (2.13) \]

is an element of the linear skein of an annulus, immersed in the plane as a regular neighbourhood of \( K_i \) [6]. \( \langle W_{\mu}^{\bigcirc}(A) \rangle \) are some computable coefficients depending on \( \mu, k \) and \( G \). Eq. (2.12a) can be easily generalized to accommodate an ordinary link \( \mathcal{L} = \bigcup_{i=1}^{M} K_i \) embedded in \( \mathcal{M}_L \)

\[ \left\langle \prod_{i=1}^{M} W_{\mu_i}^{K_i}(A) \right\rangle_{\mathcal{M}_L} = \frac{1}{N_L} \left\langle \prod_{i=1}^{M} W_{\mu_i}^{K_i}(A) \prod_{j=1}^{N} \omega_{K_j}(A) \right\rangle. \quad (2.12b) \]

We defer the solution of the issue of the determination of the normalization constant \( N_L \) to the end of Sect. 4.
3. The satellite formula

The easiest way to calculate \( \langle W^\odot_{\mu}(A) \rangle \) follows from the satellite formula [11, 12]. In turn, the simplest derivation of the satellite formula on the level of skein relations, in the context of topological field theory, could look as follows. Let us consider the topological-field-theory approach to skein relations, which yields the (quasi-)braiding matrix \( B \) in the form [12, 13]

\[
B = q \sum_{a=1}^{d} t_{\mu}^a \otimes t_{\nu}^a,
\]

where

\[
q = e^{-\frac{2\pi i}{k}},
\]

and \( \mu, \nu \) are two irreps of the \( d \)-dimensional group \( G \). The square of \( B \), the monodromy matrix \( M (M = B^2) \) can be derived, for example, in the framework of the path-integral approach to link invariants (advocated in [13]) as the contribution coming from the intersection of the surface \( S \) corresponding to the representation \( \mu \) and the line \( \ell \) corresponding to \( \nu \). If we double the line \( \ell \), possibly assigning different representations to each of the components, say \( \nu \) and \( \lambda \), there will appear two intersection points and consequently two contributions giving rise to

\[
B = q \sum_{a=1}^{d} t_{\mu}^a \otimes t_{\nu}^a q \sum_{a=1}^{d} t_{\mu}^a \otimes t_{\lambda}^a
\]

\[
= q \sum_{a=1}^{d} (t_{\mu}^a \otimes t_{\nu}^a \otimes 1_{\lambda} + t_{\nu}^a \otimes t_{\mu}^a \otimes 1_{\lambda}) = q \sum_{a=1}^{d} t_{\mu}^a \otimes t_{\nu}^a \otimes 1_{\lambda},
\]

where

\[
t_{\nu}^a \otimes 1_{\lambda} \equiv t_{\nu}^a \otimes 1_{\lambda} + 1_{\nu} \otimes t_{\lambda}^a
\]

is a generator of \( G \) in the product representation \( \nu \otimes \lambda \). Hence we have the satellite formula (at least on the level of skein relations)

\[
W^K_{\mu}(A) W^K_{\nu}(A) \approx W^K_{\mu \otimes \nu}(A),
\]

where “\( \approx \)” means the “weak equality”,

\[
X \approx Y \iff \langle X \rangle = \langle Y \rangle.
\]
The product on LHS of (3.5) should be understood in a “regularized” form, i.e. the both $K$'s should be split up. Obviously, Eq. (3.5) can be readily generalized by induction to any number of factors, whereas RHS of (3.5) can be expanded into irreducible components of the product $\mu \otimes \nu$.

4. SU(2)-invariant

In this section, we derive an explicit form of the specialization of our invariant (2.12) to the group SU(2), and show that it agrees with the result of Lickorish [6].

It appears that a very convenient way of organization of irrep’s of SU(2) group is provided by the polynomials $S_n(x)$, closely related to the Chebyshev polynomials. $S_n(x)$ are defined recursively by the formula

$$S_{n+2}(x) = xS_{n+1} - S_n(x), \quad n = 0, 1, \ldots,$$

(4.1a)

together with the initial conditions

$$S_0(x) = 1, \quad S_1(x) = x.$$  

(4.1b)

Usefulness of $S_n(x)$, in the context of the SU(2) RTW invariant, has been observed in [4, 6]. By virtue of the definition (4.1), $S_n(x)$ expresses $n$-th irrep of SU(2) in terms of powers of the fundamental representation $x$, denoted as 1 henceforth. The explicit solution of (4.1) is

$$S_n(2\cos\alpha) = \frac{\sin((n + 1)\alpha)}{\sin\alpha}.$$  

(4.2)

For the group SU(2) the satellite formula (3.5) now assumes the following elegant form

$$W^K_n(A) = W^K_{S_n(1)}(A) \approx S_n \left(W^K_1(A)\right),$$

(4.3)

whereas the skein relations for the fundamental representation ($n = 1$) of SU(2)

$$q^{\frac{1}{2}}\left\langle \left\{ \right\} \right\rangle - q^{-\frac{1}{2}}\left\langle \left\{ \right\} \right\rangle \approx (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\left\langle \left\{ \right\} \right\rangle,$$

(4.4a)
\[
\left\{ \left\{ \pm 1 \right\} \right\} = -q^{\pm \frac{3}{2}} \left\{ \left\{ 0 \right\} \right\},
\]

(4.4b)

where the integers in (4.4b) mean a framing. Closing the left legs of all the (three) diagrams in (4.4a) with arcs, as well as the right ones, next applying (4.4b), and using the property of locality, we obtain

\[-(q - q^{-1}) \langle W_1^\circ (A) \rangle = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \langle W_1^\circ \circ (A) \rangle = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \langle W_1^\circ (A) \rangle^2. \quad (4.5)\]

Hence

\[
\langle W_1^\circ (A) \rangle = -\left( q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) = -2 \cos \frac{\pi}{k},
\]

(4.6)

and by virtue of the satellite formula (4.3)

\[
\langle W_n^\circ (A) \rangle = S_n \left( -2 \cos \frac{\pi}{k} \right) = (-)^n \frac{\sin \frac{(n+1)\pi}{k}}{\sin \frac{\pi}{k}} = (-)^n \frac{q^{\frac{n+1}{2}} - q^{-\frac{n+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \quad (4.7)
\]

We can observe a remarkable property of (4.7) for \( n = k - 1 \), namely

\[
\langle W_{k-1}^\circ (A) \rangle = 0. \quad (4.8)
\]

It appears that for any \( K \)

\[
\langle \cdots W_{k-1}^K (A) \cdots \rangle = 0. \quad (4.9)
\]

Actually, we are dealing with a tensor algebra of finite order, which can be interpreted as a fusion algebra [14]. In particular, Eq. (4.9) immediately follows from (4.8) for any \( K \) that can be unknotted with corresponding skein relations. Thus we can truncate representations of SU(2) above the value \( k - 2 \), and assume

\[
0 \leq n \leq k - 2, \quad k = 2, 3, \ldots. \quad (4.10)
\]

The final explicit form of \( \omega_K \) for the group SU(2) is then

\[
\omega_K (A) = \sum_{n=0}^{k-2} (-)^n \frac{q^{\frac{n+1}{2}} - q^{-\frac{n+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} S_n \left( W_1^K (A) \right), \quad (4.11)
\]

and agrees with a corresponding expression derived by Lickorish with a combinatorial method [6]. Strictly speaking, \( Z(M_L) \) is invariant with respect to the second Kirby
move $K_2$. It means that it is insensitive to the operation of sliding one of its handles over another one. But up to now we have not considered the issue of the determination of the normalization constant $N_L$. It appears that proper normalization of the partition function $Z(\mathcal{M}_L)$ universally follows from the requirement of its invariance with respect to the first Kirby move $K_1$. Hence the normalization constant $N_L$ can be chosen in the form [6]

$$N_L = \langle \omega_{\odot_{+1}}(A) \rangle^{b_+(L)} \langle \omega_{\odot_{-1}}(A) \rangle^{b_-(L)},$$

(4.12)

where $b_+(L)$ ($b_-(L)$) is the number of positive (negative) eigenvalues of the linking matrix of $L$.

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