BIHARMONIC SEMI-RIEMANNIAN SUBMERSIONS FROM 3-MANIFOLDS

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ABSTRACT. The main interest of the present paper is to prove the dual results for semi-Riemannian submersions, i.e., a semi-Riemannian submersion from a 3-dimensional space form into a surface is biharmonic if and only if it is harmonic. We prove that there is no biharmonic semi-Riemannian submersion from anti-de Sitter space onto a Riemannian manifold. We also give following corollaries.

Corollary 1: Let $M$ be an 3-dimensional semi-Riemannian space form $M^3_1(c)$ of index 1 and $B$ be an 2-dimensional Riemannian manifold. If $c < 0$, $\pi : H^3_1(c) \to B^2$ be a biharmonic semi-Riemannian submersion from the anti-de Sitter space onto a Riemannian manifold. Then, $B^2$ is Kähler manifold holomorphically isometric to $CH^1(4c)$.

Corollary 2: If $\pi : E^3_1 \to B^2$ be a biharmonic semi-Riemannian submersion, then $B = E^2$ and the total space $E^3_1$ is locally decomposed into the product manifold $E^3_1 = R^1_1 \times B$.

1. INTRODUCTION

The theory of Riemannian submersions were introduced by O’Neill [18] and Gray [13]. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. A systematic exposition could be found in the A.Besse’s book [3]. Semi-Riemannian submersions were introduced by O’Neill in his book [19]. Magid [17] classified the semi-Riemannian submersions with totally geodesic fibres from an anti-de Sitter space onto a Riemannian manifold.

A map between Riemannian manifolds is harmonic if the divergence of its differential vanishes. The first major study of harmonic maps has been begun by J. Eells and J. H. Sampson in [11]. Harmonic maps between Riemannian manifolds were applied to board areas in science and engineering including the robotic mechanics (cf. [5], [10]). In [11], Eells and Sampson defined biharmonic maps between Riemannian manifolds as an extension of harmonic maps, and Jiang [14] obtained their first and second variational formulas.

During the last decade important progress has been made in the study of both the geometry and the analytic properties of biharmonic maps. A fundamental problem in the study of biharmonic maps is to classify all proper biharmonic maps between certain model spaces. An example of this is proved indepenently by Chen-Ishikawa [9] and Jiang [14] that every biharmonic surface in a Euclidean 3-space $E^3$ is a minimal surface. In a later paper, Cadeo et al. [6] showed that the theorem remains true if the target Euclidean space is replaced by 3-dimensional hyperbolic space form. Chen and Ishikawa [7] also proved that biharmonic Riemannian surface in $E^3_1$ is a harmonic surface. For

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Riemannian submersions, Wang and Ou stated in [21] that Riemannian submersion from a 3-dimensional space form into a surface is biharmonic if and only if it is harmonic.

The above results give us the motivation for this studying. In this paper we study the biharmonic semi-Riemannian submersions from 3-manifolds. We get especially the following Theorem and Corollaries.

Theorem: Let $M^3$ be a 3-dimensional semi-Riemannian space form $M^3$ of index 1 and $B^2$ be an 2-dimensional Riemannian manifold. If $c > 0$, then there are no biharmonic semi-Riemannian submersions $\pi: M \rightarrow B$.

Corollary 1: Let $M^3$ be a 3-dimensional semi-Riemannian space form $M^3$ of index 1 and $B^2$ be an 2-dimensional Riemannian manifold. If $c < 0$, $\pi: H^3 \rightarrow B^2$ be a biharmonic semi-Riemannian submersion from the anti-de Sitter space onto a Riemannian manifold. Then, $B^2$ is a Kähler manifold holomorphically isometric to $CH^1(4c)$.

Corollary 2: If $\pi: E^3 \rightarrow B^2$ be a biharmonic semi-Riemannian submersion, then $B = E^2$ and the total space $E^3$ is locally decomposed into the product manifold $E^3 = R^1 \times B$.

The main purpose of section §2 to give brief information about semi-Riemannian submersions, biharmonic maps and space forms. In this section we also give some properties of fundamental tensors and fundamental equations then we use these properties to obtain our results. In section §3 we investigate the biharmonicity of a semi-Riemannian submersion from a 3-manifold by using the integrability data of a special orthonormal frame adapted to a semi-Riemannian submersion. Finally we give a complete classification of biharmonic semi-Riemannian submersions from a 3-dimensional semi-Riemannian space form.

2. PRELIMINARIES

2.1. Semi-Riemannian submersions. In this section we recall several notions and results which will be needed throughout the paper.

Let $(M, g)$ be an $m$-dimensional connected semi-Riemannian manifold of index $s$ $(0 \leq s \leq m)$, let $(B, g')$ be an $n$-dimensional connected semi-Riemannian manifold of index $r \leq s$, $(0 \leq r \leq n)$. A semi-Riemannian submersion (see [19]) is a smooth map $\pi: M \rightarrow B$ which is onto and satisfies the following three axioms:

$S1$. $\pi_\ast |_p$ is onto for all $p \in M$;
$S2$. The fibres $\pi^{-1}(b), b \in B$ are semi-Riemannian submanifolds of $M$;
$S3$. $\pi_\ast$ preserves scalar products of vectors normal to fibres.

We shall always assume that the dimension of the fibres $\dim M - \dim B$ is positive and the fibres are connected.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $V$ the vertical distribution and by $H$ the horizontal distribution. The fundamental tensors of a submersion were defined by O’Neill [18], [19]. They are $(1, 2)$-tensors on $M$, given by the formula:

\[
T(E, F) = T_E F = h \nabla_{\nu E} \nu F + \nu \nabla_{\nu E} h F,
\]
\[
A(E, F) = A_E F = \nu \nabla_{h E} h F + h \nabla_{h E} \nu F,
\]
for any $E, F \in X(M)$. Here $\nabla$ denotes the Levi-Civita connection of $(M, g)$. This tensors are called integrability tensors for the semi-Riemannian submersions. We use the $h$ and $\nu$ letters to denote the orthogonal projections on the vertical and horizontal distributions respectively. A vector field $X$ on $M$ is said to be basic if it is the unique horizontal lift.
of a vector field $X_*$ on $B$, so that $\pi_*(X) = X_*$ is horizontal and $\pi$-related to a vector field $X_*$ on $B$. It is easy to see that every vector field $X_*$ on $B$ has a unique horizontal lift $X$ to $M$ and $X$ is basic. The following Lemmas are well known (see [18], [19]).

**Lemma 1.** Let $\pi : (M,g) \rightarrow (B,g')$ be a semi-Riemannian submersion. If $X, Y$ are basic vector fields on $M$, then:

1. $g(X,Y) = g'(X_*,Y_*) \circ \pi$,
2. $h[X,Y]$ is basic, $\pi$-related to $[X_*,Y_*]$.
3. $h(\nabla_X Y)$ is basic vector field corresponding to $\nabla^\pi_{X_*}Y_*$ where $\nabla^\pi$ is the connection on $B$.
4. for any vertical vector field $V$, $[X,V]$ is vertical.

**Lemma 2.** For any $U,W$ vertical and $X,Y$ horizontal vector fields, the tensor fields $T,A$ satisfy:

1. $T_U W = T_W U$,
2. $A_X Y = -A_Y X = \frac{1}{2}g(U,[X,Y])$.

Moreover, if $X$ is basic and $U$ vertical then $h(\nabla_U X) = h(\nabla_U Y) = A_U X$. Notice that $T$ acts on the fibres as the second fundamental form of the submersion and restricted to vertical vector fields and it can be easily seen that $T = 0$ is equivalent to the condition that the fibres are totally geodesic. We call the Riemannian submersion with totally geodesic fiber if $T$ vanishes identically. If $A = 0$ then the horizontal distribution is integrable. Thus we can recall following theorem (see [15]).

**Theorem 1.** Let $\pi : M_{m+n} \rightarrow B^n$ be a semi-Riemannian submersion. If it is totally geodesic and if the horizontal distribution is integrable, then the total space $M$ is locally decomposed into the product manifold $F \times B$.

We define the curvature tensor $R$ of $M$ by $R(E,F) = \nabla_E \nabla_F - \nabla_F \nabla_E - \nabla_{[E,F]}$ for any vector fields $E, F$ on $M$. The semi-Riemannian curvature $(0,4)$ tensor is defined by

$$ R(E,F,G,H) = g(R(E,F)G,H). $$

Let us recall the sectional curvature of semi-Riemannian manifolds for nondegenerate planes. Let $M$ be a semi-Riemannian manifold and $P$ a non-degenerate tangent plane to $M$ at $p$. The number

$$ K_{XY} = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} $$

is independent on the choice of basis $X, Y$ for $P$ and is called the sectional curvature. We use notation $R_{ijkl} = g(R(e_i,e_j)e_k,e_l)$. Next, we can give following Lemma:

**Lemma 3 ([19]).** Let $\pi : (M,g) \rightarrow (B,g')$ be a semi-Riemannian submersion. The $K$ and $K^B$ denote sectional curvatures in $M$ and $B$, respectively. If $X, Y$ are basic vector fields on $M$, then

$$ K^B_{X_* Y_*} = K_{XY} + \frac{3g(A_X Y, A_Y X)}{g(X,X)g(Y,Y) - g(X,Y)^2}. $$

### 2.2. Biharmonic maps and space forms

Let $M^m$ and $B^n$ be semi-Riemannian manifolds of dimensions $m$ and $n$, respectively, and $\varphi : M^m \rightarrow B^n$ a smooth map. We denote
C-Sitter space and anti-de Sitter space, respectively. These spaces with index 1 are called Lorentz space forms. These spaces are complete semi-Riemannian manifolds with index curvature tensor \( \nabla^M \) and \( \nabla^B \) the Levi-Civita connections on \( M^m \) and \( B^n \), respectively. Then the tension field \( \tau(\varphi) \) is a section of the vector bundle \( \varphi^*TB^n \) defined by

\[
\tau(\varphi) = \text{trace}(\nabla^B d\varphi) = \sum_{i=1}^m g(e_i, e_i)(\nabla^B_{e_i} d\varphi(e_i) - d\varphi(\nabla^B_{e_i} e_i)).
\]

Here \( \nabla^B \) and \( \{e_i\} \) denote the induced connection by \( \varphi \) on the bundle \( \varphi^*TB^n \), which is the pull-back of \( \nabla^B \), and a local orthonormal frame field of \( M^m \), respectively. A smooth map \( \varphi \) is called a harmonic map if its tension field vanishes. A map \( \varphi \) is harmonic if and only if it is a critical point of the energy

\[
E(\varphi) = \int \sum_{i=1}^m g(d\varphi(e_i), d\varphi(e_i)) dv
\]

under compactly supported infinitesimal variations, where \( dv \) is the volume form of \( M^m \).

We define the bitension field as follows:

\[
\tau^2(\varphi) = \sum_{i=1}^m g(e_i, e_i)((\nabla^B_{e_i} \nabla^B_{e_i} \varphi - \nabla^B_{\nabla^B_{e_i} e_i}) \tau(\varphi) - R^B(d\varphi(e_i, \tau(\varphi))d\varphi(e_i)),
\]

where \( R^B \) is the curvature tensor of \( B^n \). We say that a smooth map \( \varphi \) is a biharmonic map (or 2-harmonic map) if its bitension field vanishes (see [14, 20]). Harmonic maps are clearly biharmonic. Non harmonic maps are called proper biharmonic maps.

Let \( E_s^n \) be semi-Euclidean n-space with metric given by

\[
g = - \sum_{i=1}^s dx_i^2 + \sum_{j=s+1}^n dx_j^2,
\]

where \( \{x_1, ..., x_n\} \) is the natural coordinate system of \( E_s^n \). Then \( E_s^n \) is a flat semi-Riemannian manifold with index \( s \). We put

\[
S^n_s(c) = \left\{ (x_1, ..., x_{n+1}) \in E_s^{n+1} \mid - \sum_{i=1}^s x_i^2 + \sum_{j=s+1}^{n+1} x_j^2 = \frac{1}{c} \right\},
\]

\[
H^n_s(c) = \left\{ (x_1, ..., x_{n+1}) \in E_s^{n+1} \mid - \sum_{i=1}^{s+1} x_i^2 + \sum_{j=s+2}^{n+1} x_j^2 = \frac{1}{c} < 0 \right\}.
\]

These spaces are complete semi-Riemannian manifolds with index \( s \) of constant curvature \( c \). The semi-Riemannian manifolds \( E_1^n \), \( S^n_1(c) \) and \( H^n_1(c) \) are called Minkowski space, de Sitter space and anti-de Sitter space, respectively. These spaces with index 1 are called Lorentz space forms.

Denote \( n \)-dimensional Lorentz space forms of constant curvature \( c \) by \( M^n_1(c) \). The curvature tensor \( R \) of \( M^n_1(c) \) is given by

\[
R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y),
\]

where \( g \) is the metric tensor of \( M^n_1(c) \).

Now we shall introduce pseudo-complex space form. Let \( g^c \) be the Hermitian form on \( C^{m+1} \) given by

\[
g^c(z, w) = - \sum_{i=0}^s z_i \overline{w_i} + \sum_{i=s+1}^m z_i \overline{w_i}
\]
for \( z = (z_0, \ldots, z_m) \), \( w = (w_0, \ldots, w_n) \in C^{m+1} \). For \( c < 0 \), let \( M(c) \) be the real hypersurface of \( C^{m+1} \) given by \( N(c) = \{ z \in C^{m+1} \mid g'(z, z) = \frac{c}{s} \} \), which is endowed with the induced metric of \( C^{m+1} \), \( ds^2 = -dz_0 \otimes dz_0 - \cdots - dz_s \otimes d\bar{z}^s + d\bar{z}^s + d\bar{z}^{s+1} + \cdots + dz^m \otimes d\bar{z}^m \).

The natural action of \( S^1 = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \) on \( C^{m+1} \) induces an action on \( N(c) \). Let \( CH^m_s(c) = N(c)/S^1 \) endowed with the unique indefinite Kaehler metric of index \( 2s \) such that the projection \( N(c) \to N(c)/S^1 \) becomes a semi-Riemannian submersion (see [2]). \( CH^m_s(c) \) is called the complex pseudo-hyperbolic space.

Let \( N^m_s(4c) \) be a complex space form of complex dimension \( n \), complex index \( s \geq 0 \) and constant holomorphic sectional curvature \( 4c \). The complex index is defined as the complex dimension of the largest complex negative definite vector subspace of the tangent space. If \( s = 1 \), it is called Lorentzian. The curvature tensor \( R \) of \( N^m_s(4c) \) is given by

\[
R(X, Y)Z = c \left\{ \begin{array}{l}
g(Y, Z)X - g(X, Z)Y + g(Z, JY)JX \\
g(Y, JX)Y + 2g(X, JY)JZ
\end{array} \right\},
\]

where \( g \) and \( J \) are the metric tensor and the almost complex structure of \( N^m_s(4c) \), respectively.

Let \( C^n_s \) be the \( n \)-dimensional complex space with complex coordinates \( z_1, \ldots, z_n \), endowed with the metric \( g_{n,s}(z, w) = Re(-\sum_{j=1}^{n} z_j \bar{w}_j + \sum_{i=s+1}^{n} z_i \bar{w}_i) \).

Put

\[
S^{2n+1}_{2s} = \left\{ z \in C^n_s : g_{n+1,s}(z, z) = \frac{1}{c} \right\} \quad \text{for } c > 0,
\]

\[
H^{2n+1}_{2s+1} = \left\{ z \in C^n_s : g_{n+1,s+1}(z, z) = \frac{1}{c} \right\} \quad \text{for } c < 0.
\]

The Hopf fibrations

\[
\pi : S^{2n+1}_{2s} \to CP^n_s(4c) : z \to z \cdot C,
\]

\[
\pi : H^{2n+1}_{2s+1} \to CH^m_s(4c) : z \to z \cdot C,
\]

give \( CP^n_s(4c) \) and \( CH^m_s(4c) \), a unique semi-Riemannian metric of complex index \( s \) and curvature tensor \((2.4)\), such that \( \pi \) is a semi-Riemannian submersion, respectively.

Barros and Romero [2] showed that locally any complex space form \( N^m_s(4c) \) is isometric holomorphically to \( C^n_s, CP^n_s(4c), CH^m_s(4c) \) according to \( c = 0, c > 0 \) or \( c < 0 \).

3. THEOREMS AND PROOFS

In this section, we will prove our Main Theorem and corollaries. Firstly we will recall well known proposition and theorems:

**Proposition 1 ([4])**. \( \varphi \) is a submersion of \( M \) onto \( M' \). If \( \psi : M' \to M'' \) is such that \( \psi \circ \varphi \) is differentiable then \( \psi \) also is differentiable.

**Theorem 2 ([12])**. A semi-Riemannian submersion \( \pi : (M, g) \to (B, g') \) is a harmonic map if and only if each fibre is a minimal submanifold.

**Theorem 3 ([17])**. Let \( \pi : H^{2n+1}_{2s+1} \to B^{2n} \) be a semi-Riemannian submersion with totally geodesic fibres, from the anti-de Sitter space onto Riemannian manifold. Then, \( B^{2n} \) is a Kaehler manifold holomorphically isometric to \( CH^m_s(4c) \).

We will report following theorems which give us the motivation to study on this paper.
Theorem 4 ([7]). Let $x : M \to E^3_s (s = 0, 1)$ be a biharmonic isometric immersion of a Riemannian surface $M$ into $E^3_s$. Then $x$ is harmonic.

Theorem 5 ([22]). If $M$ is a complete biharmonic space-like surface in $S^3_1$ or $R^3_1$, then it must be totally geodesic, i.e., $S^2$ or $R^2$.

Theorem 6 ([21]). Let $(M^3(c), g) \to (B^2, g')$ be a Riemannian submersion from a space form of constant sectional curvature $c$. Then, $\pi$ is biharmonic if and only if it is harmonic, and this holds if and only if it is a harmonic morphism.

We give biharmonicity of a semi-Riemannian submersion from a 3-manifold by using the integrability data of a special orthonormal frame adapted to a semi-Riemannian submersion. This is the main tool we use to prove our Main Theorem.

Let $\pi : (M^3_1, g) \to (B^2, g')$ be a semi-Riemannian submersion. A local orthonormal is said to be adapted to the semi-Riemannian submersion $\pi$ if the vector fields in the frame that are tangent to the horizontal distribution are basic (i.e., they are $\pi$-related to a local orthonormal frame in the base space). Such a frame always exists (cf. e.g., [1]). Let $\{e_1, e_2, e_3\}$ be an orthonormal frame adapted to with $e_3$ being vertical where $g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1$. Then, it is well known (see [18]) that $[e_1, e_3]$ and $[e_2, e_3]$ are vertical and $[e_1, e_2]$ is $\pi$-related to $[e_1, e_2]$, where $\{e_1, e_2\}$ is an orthonormal frame in the base manifold. If we assume that

$$[e_1, e_2] = F_1 e_1 + F_2 e_2,$$

for $F_1, F_2 \in C^\infty(B)$ and use the notations $f_i = F_i \circ \pi$, $i = 1, 2$. Then, we have

$$\begin{align*}
[e_1, e_3] &= k_1 e_3, \\
[e_2, e_3] &= k_2 e_3, \\
[e_1, e_2] &= f_1 e_1 + f_2 e_2 - 2\sigma e_3.
\end{align*}$$

where $k_1, k_2$ and $\sigma \in C^\infty(M)$. Here $f_1, f_2, k_1, k_2$ and $\sigma$ the integrability data of the adapted frame of the semi-Riemannian submersion $\pi$.

Proposition 2. Let $\pi : (M^3_1, g) \to (B^2, g')$ be a semi-Riemannian submersion with the adapted frame $\{e_1, e_2, e_3\}$ and the integrability data $f_1, f_2, k_1, k_2$ and $\sigma$. Then, the semi-Riemannian submersion $\pi$ is biharmonic if and only if

$$\Delta^M_k 1 = -f_1 e_1(k_2) - e_1(k_2 f_1) - f_2 e_2(k_2) - e_2(k_2 f_2) + k_1 k_2 f_1 + k_2^2 f_2 + k_1 \{f_1^2 + f_2^2 - K^B\},$$

$$\Delta^M_k 2 = f_1 e_1(k_1) + e_1(k_1 f_1) + f_2 e_2(k_1) + e_2(k_1 f_2) - k_1 k_2 f_2 - k_1^2 f_1 + k_2 \{f_1^2 + f_2^2 - K^B\},$$

where $K^B = R^B_{1221} \circ \pi = -(-e_1(f_2) + e_2(f_1) + f_1^2 + f_2^2)$ is the Gauss curvature of Riemannian manifold $(B^2, g')$.

Proof. Let $\nabla$ denote the Levi-Civita connection of the semi-Riemannian manifold $(M^3_1, g)$. After a straightforward computation using (3.2) and Koszul formula gives

$$\begin{align*}
\nabla_{e_1} e_1 &= -f_1 e_2, \\
\nabla_{e_1} e_2 &= f_1 e_1 - \sigma e_3, \\
\nabla_{e_1} e_3 &= -\sigma e_2, \\
\nabla_{e_2} e_1 &= -f_2 e_2 + \sigma e_3, \\
\nabla_{e_2} e_2 &= f_2 e_1, \\
\nabla_{e_2} e_3 &= \sigma e_1, \\
\nabla_{e_3} e_1 &= -\sigma e_2 - k_1 e_3, \\
\nabla_{e_3} e_2 &= \sigma e_1 - k_2 e_3, \\
\nabla_{e_3} e_3 &= -k_1 e_1 - k_2 e_2.
\end{align*}$$
The tension of the semi-Riemannian submersion \( \tau \) is given by

\[
\tau(\pi) = \sum_{i=1}^{3} g(e_i, e_i) \left[ \nabla_{e_i}^\pi d\pi(e_i) - d\pi(\nabla_{e_i}^M e_i) \right] = d\pi(\nabla_{e_3}^M e_3) = -k_1 \varepsilon_1 - k_2 \varepsilon_2
\]

After some calculation by using (3.4) we get

\[
\tau^2(\pi) = \sum_{i=1}^{3} g(e_i, e_i) \left\{ \nabla_{e_i}^\pi \nabla_{e_i}^\pi \tau(\pi) - \nabla_{\nabla_{e_i}^\pi e_i}^\pi \tau(\pi) - R^B(d\pi(e_i), \tau(\pi))d\pi(e_i) \right\}
\]

Now we calculate Laplace of \( k_1 \) and \( k_2 \). Since \( \text{grad} k_1 = e_1(k_1)e_1 + e_2(k_1)e_2 - e_3(k_1)e_3 \), we obtain

\[
\Delta^m k_1 = \sum_{i=1}^{3} g(e_i, e_i)g(\nabla_{e_i} \text{grad} k_1, e_i)
\]

Using same calculations for \( k_2 \) we get

\[
\Delta^m k_2 = e_1(e_2(k_2)) + e_2(e_2(k_2)) - e_3(e_3(k_2)) + e_2(k_2)f_1 - e_1(k_2)f_2
\]

from which the proposition follows.

When the integrability data \( k_2 = 0 \) we have the following corollary which will be used later in the paper.

**Corollary 1.** Let \( \pi : (M^3, g) \to (B^2, g') \) be a semi-Riemannian submersion with an adapted frame \( \{e_1, e_2, e_3\} \) and the integrability data \( \{f_1, f_2, k_1, k_2, \sigma\} \) with \( k_2 = 0 \). Then, the semi-Riemannian submersion \( \pi \) is biharmonic if and only if

\[
- \Delta^M k_1 + k_1 \left\{ f_1^2 + f_2^2 - K^B \right\} = 0,
\]

\[
f_1 e_1(k_1) + e_1(k_1 f_1) + f_2 e_2(k_1) + e_2(k_1 f_2) - k_1^2 f_1 = 0.
\]

**Example 1.** We adapted the example which is given in [10] to our example. For \( \varphi(x) = \frac{e^{1+e^x}}{1-e^x} \) and \( \ln \beta(x) = \int \varphi(x)dx \) the semi-Riemannian submersion

\[
\pi : (E^2 \times R^3_1, dx^2 + dy^2 - \beta^{-2}(x)dz^2) \to (E^2, dx^2 + dy^2)
\]

\[
\pi(x, y, z) = (x, y)
\]
is a proper biharmonic map. In fact, the orthonormal frame \( \{ e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}, e_3 = \beta(x) \frac{\partial}{\partial z} \} \) on \( (E^2 \times R_1, dx^2 + dy^2 - \beta^{-2}(x)dz^2) \) is adapted to the semi-Riemannian submersion \( \pi \) with \( d\pi(e_i) = \varepsilon_i, i = 1, 2 \) and \( e_3 \) is being vertical, where \( \varepsilon_1 = \frac{\partial}{\partial x}, \varepsilon_2 = \frac{\partial}{\partial y}, e_3 \) form an orthonormal frame on the base space \( (R^2, dx^2 + dy^2) \). A straightforward computation gives the Lie brackets

\[
[e_1, e_3] = \varphi(x)e_3, \quad [e_1, e_2] = 0, \quad [e_2, e_3] = 0
\]

where \( f = (\ln \beta)_x \). It follows that the integrability data of the semi-Riemannian submersion \( \pi \) are given by

\[
f_1 = f_2 = \sigma = 0, \quad k_1 = f
\]

Substituting these and the curvature \( KE^2 = 0 \) into \( (3.11) \) we conclude that

\[
\Delta^M f = 0
\]

From Corollary 1, we obtain \( \pi \) is biharmonic. Since \( f \) is not vanish, \( \tau(\pi) \) is different from zero. So, \( M \) is proper biharmonic.

The following lemmas will be used to prove the Theorem.

**Lemma 4.** Let \( \pi : M_3^3(c) \to (B^2, g') \) be a semi-Riemannian submersion from a space form of constant sectional curvature \( c \). Then, there exists an orthonormal frame \( \{ e_1, e_2, e_3 \} \) on \( M_3^3(c) \) adapted to the semi-Riemannian submersion such that all the integrability data \( f_1, f_2, k_1, k_2 \) and \( \sigma \) are constant along fibers of \( \pi \), i.e.,

\[
e_3(f_1) = e_3(f_2) = e_3(k_2) = e_3(k_1) = e_3(\sigma) = 0
\]

**Proof.** By definition, \( f_i = F_i \circ \pi \) for \( i = 1, 2 \), so they are constant along the fibers. It remains to show that

\[
e_3(k_2) = e_3(k_1) = e_3(\sigma) = 0
\]

One can easily check that the Jacobi identity applied to the frame \( \{ e_1, e_2, e_3 \} \) yields

\[
2e_3(\sigma) + k_1f_1 + k_2f_2 + e_2(k_1) - e_1(k_2) = 0
\]

By using \( (3.10) \) and the fact that \( M_3^3(c) \) has constant sectional curvature \( c \) gives

\[
\begin{align*}
i) \ R_{1312}^M &= [-e_1(\sigma) + 2k_3\sigma] = 0 \\
ii) \ R_{1313}^M &= [e_1(k_1) - \sigma^2 - k_1^2 + k_2f_1] = c \\
iii) \ R_{1323}^M &= [e_1(k_2) - e_3(\sigma) - k_1f_1 - k_3k_2] = 0 \\
iv) \ R_{1212}^M &= [e_2(f_1) - e_3(f_2) + f_1^2 + f_2^2 - 3\sigma^2] = -c \\
v) \ R_{1223}^M &= [-e_2(\sigma) + 2k_3\sigma] = 0 \\
vi) \ R_{2313}^M &= [e_2(k_1) + e_3(\sigma) + k_2f_2 - k_1k_2] = 0 \\
vii) \ R_{2323}^M &= [-\sigma^2 + e_2(k_2) - k_1f_2 - k_2^2] = c
\end{align*}
\]

Applying \( e_3 \) to both sides of the equation \( iv) \) of \( (3.11) \) and using \( (3.11) \) together with \( e_3e_1 = [e_3, e_1] + e_1e_3 \) and \( e_3e_2 = [e_3, e_2] + e_2e_3 \), we obtain

\[
\sigma e_3(\sigma) = 0,
\]

which implies

\[
e_3(\sigma) = 0.
\]

Using the last equation and applying \( e_3 \) to both sides of the equations \( i) \) and \( v) \) of \( (3.11) \) separately, we get

\[
e_3(k_1) = 0, \quad e_3(k_2) = 0,
\]
Lemma 5. Let \( \pi : (M_1^3(c), g) \to (B^2, \bar{g}') \) be a semi-Riemannian submersion with an adapted frame \( \{e_1, e_2, e_3\} \) and the integrability data \( f_1, f_2, k_1, k_2 \) and \( \sigma \). Then, there exists another adapted orthonormal frame \( \{e'_1, e'_2, e'_3 = e_3\} \) on \( M_1^3(c) \) with integrability data \( f'_1, f'_2, k'_1 = \sqrt{k_1^2 + k_2^2}, k'_2 = 0 \), and \( \sigma' = \sigma \).

Proof. First of all one can choose an orthonormal frame \( \{e_1, e_2, e_3\} \) on \( M_1^3(c) \) adapted to the Riemannian submersion \( \pi \). From Lemma 4 we know that integrability data \( k_1 \) and \( k_2 \) are constant along the fibers of \( \pi \). We know from topology (see Proposition 1) that there exist functions \( \tilde{k}_1 \) and \( \tilde{k}_2 \in C^\infty(B) \) such that \( k_1 = \tilde{k}_1 \circ \pi \) and \( k_2 = \tilde{k}_2 \circ \pi \). Suppose \( e_1, e_2 \) are \( \pi \)-related to \( \varepsilon_1, \varepsilon_2 \) respectively. One can easily see that \( e_1 = \frac{\tilde{k}_1}{\sqrt{\tilde{k}_1^2 + \tilde{k}_2^2}} \varepsilon_1 + \frac{\tilde{k}_2}{\sqrt{\tilde{k}_1^2 + \tilde{k}_2^2}} \varepsilon_2 \) and \( e_2 = \frac{\tilde{k}_1}{\sqrt{\tilde{k}_1^2 + \tilde{k}_2^2}} \varepsilon_1 + \frac{\tilde{k}_2}{\sqrt{\tilde{k}_1^2 + \tilde{k}_2^2}} \varepsilon_2 \) is an orthonormal frame on the base space. Let \( e'_1, e'_2 \) be the horizontal lift of \( \varepsilon_1, \varepsilon_2 \) respectively. Then, one can easily check that the adapted orthonormal frame \( \{e'_1, e'_2, e'_3 = e_3\} \) satisfies the required conditions stated in the lemma.

Now we will give a classification of biharmonic semi-Riemannian submersions.

Theorem 7. Let \( \pi : M_1^3(c) \to (B^2, \bar{g}') \) be a semi-Riemannian submersion from a space form of constant sectional curvature \( c \). Then, \( \pi \) is biharmonic if and only if it is harmonic.

Proof. By Lemma 5, we can choose an orthonormal frame \( \{e_1, e_2, e_3\} \) adapted to the semi-Riemannian submersion with integrability data \( \{f_1, f_2, k_1, k_2, \sigma\} \) with \( k_2 = 0 \). For this frame the curvature (3.11) reduces to

\[
\begin{align*}
    a_1 - e_1(\sigma) + 2k_1\sigma &= 0 \\
    a_2 \left[ e_1(k_1) - \sigma^2 - k_1^2 \right] &= c \\
    a_3 \quad k_1 f_1 &= 0 \\
    a_4 \quad [e_2(f_1) - e_1(f_2) + f_1^2 + f_2^2 - 3\sigma^2] &= -c \\
    a_5 \quad e_2(\sigma) &= 0 \\
    a_6 \quad e_2(k_1) &= 0 \\
    a_7 \quad [-\sigma^2 - k_1 f_2] &= c
\end{align*}
\]

From \( a_3 \) of (3.12), we have either \( k_1 = 0 \) or \( f_1 = 0 \). If \( k_1 = 0 \), then, by (3.5) the tension fields of \( \pi \) vanishes this means the semi-Riemannian submersion is harmonic. If \( f_1 = 0 \) and \( k_1 \neq 0 \) this case can not happen. We will prove this theorem by contradiction.

Case I: \( k_1 \neq 0 \), \( f_1 = 0 \) and \( f_2 = 0 \). In this case, \( a_4 \), \( a_7 \) in (3.12) implies that \( \sigma = c = 0 \). Now substituting \( f_1 = f_2 = \sigma = 0 \) and \( k_2 = 0 \) into biharmonic (3.6) we obtain \( \Delta^M k_1 = 0 \), which, one can easily get by using \( a_2 \), \( a_6 \) of (3.12),

\[
k_1^3 = 0.
\]

It follows that \( k_1 = 0 \) which is a contradiction.

Case II: \( k_1 \neq 0 \), \( f_1 = 0 \) and \( f_2 \neq 0 \). In this case, by using \( f_1 = 0 \) and \( a_5 \), \( a_6 \) and \( a_7 \) of (3.12) to reduce the biharmonic (3.6) into

\[
- \Delta^M k_1 + [k_1 [-c + 3\sigma^2 + f_2^2] = 0,
\]

which completes the proof of the lemma.
where $K^N = c - 3\sigma^2$ obtained from curvature formula for a semi-Riemannian submersion. Using $a_1, a_2$ of (3.12) and after a straightforward calculation yields

$$-\Delta M k_1 = -e_1(e_1(k_1)) + e_1(k_1)f_2 + e_1(k_1)k_1$$

$$-\Delta M k_1 = -5k_1\sigma^2 - k_1^3 - k_1c + f_2(c + \sigma^2 + k_1^2).$$

Substituting this into (3.13) and using $a_7$ we obtain

(3.14) $$k_1[-3\sigma^2 - k_1^2 - 3c] = 0.$$ We accept $k_1 \neq 0$, so (3.14) is equivalent to

(3.15) $$k_1^2 = -3\sigma^2 - 3c.$$ Applying $e_1$ to both sides of (3.15) yields

$$k_1e_1(k_1) = -3\sigma e_1(\sigma).$$

Combining this and $a_1, a_2$ in (3.12) we get

(3.16) $$k_1(k_1^2 + \sigma^2 + c) = -6k_1\sigma^2,$$

By assumption $k_1 \neq 0$, this turned into

$$(k_1^2 + \sigma^2 + c) = -6\sigma^2,$$

or

(3.17) $$k_1^2 = -7\sigma^2 - c.$$ Applying $e_1$ to both sides of (3.16) and again using $a_1, a_2$ in (3.12) we get

(3.18) $$k_1^2 = -15\sigma^2 - c.$$ Combining (3.15), (3.16) with (3.17) we have $k_1 = \sigma = c = 0$. This implies there is a contradiction. Because our assumption is $k_1 \neq 0$. So we complete the proof of the theorem.

By Theorem 7 we get $k_1 = k_2 = 0$. Thus (3.12) is reduced to

(3.19) \begin{align*}
b_1) \quad e_1(\sigma) &= 0 \\
b_2) \quad \sigma^2 &= -c \\
b_3) \quad [e_2(f_1) - e_1(f_2) + f_1^2 + f_2^2 - 3\sigma^2] &= -c \\
b_4) \quad e_2(\sigma) &= 0 \end{align*}

From $b_2$ of (3.19) we get $c \leq 0$. We have following theorem.

**Theorem 8.** Let $M$ be an 3-dimensional semi-Riemannian space form $M^3_1(c)$ of index 1 and $B$ be an 2-dimensional Riemannian manifold. If $c > 0$, then there are no biharmonic semi-Riemannian submersions $\pi : M \to B$.

**Corollary 2.** Let $M$ be an 3-dimensional semi-Riemannian space form $M^3_1(c)$ of index 1 and $B$ be an 2-dimensional Riemannian manifold. If $c < 0$, $\pi : H^3_1(c) \to B^2$ be a biharmonic semi-Riemannian submersion from the anti-de Sitter space onto a Riemannian manifold. Then, $B^2$ is Kaehler manifold holomorphically isometric to $CH^1(4c)$.

**Proof.** From Theorem 7 we know that $k_1 = k_2 = 0$. If we use (3.4) in the first equation of (3.14) we get $T(e_i, e_j) = 0, i \leq i, j \leq 3$. It means that fiber is totally geodesic. By Theorem 3, we get the requested corollary.

For $c = 0$, we obtain following
**Corollary 3.** If $\pi : (E^1_3, -dx^2 + dy^2 + dz^2) \to B^2$ be a biharmonic semi-Riemannian submersion, then $B = (E^2, dy^2 + dz^2)$ and the total space $E^3_1$ is locally decomposed into the product manifold $E^3_1 = R^1_1 \times B$.

**Proof.** Since $k_1 = k_2 = 0$ we know that fiber is totally geodesic. Using (3.4) in the second equation of (2.1), we obtain $A(e_i, e_j) = 0$, $1 \leq i, j \leq 3$. This states that horizontal distribution is integrable. By the help of Theorem 1, we complete the proof of the corollary. □

By the help of Theorem 7 and Corollaries 2, 3 we can easily get the following corollary:

**Corollary 4.** Let $M$ be an 3-dimensional semi-Riemannian space form $M^3_1(c)$ of index 1 and $B$ be an 2-dimensional Riemannian manifold. If $c \leq 0$, there exist no proper biharmonic semi-Riemannian submersion $\pi : M \to B$.

Here we propose the following

**Conjecture 1.** Let $M$ be an $(2n+1)$-dimensional semi-Riemannian space form $M^{2n+1}_1(c)$ of index 1 and $B$ be an 2n-dimensional Riemannian manifold. If $c \leq 0$, there exist no proper biharmonic semi-Riemannian submersion $\pi : M \to B$.

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