FORMATION OF SINGULARITIES IN COLD ION DYNAMICS

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Abstract. We propose a criterion for the singularity formation of the pressureless Euler-Poisson system equipped with the Boltzmann relation, which describes the dynamics of cold ions in an electrostatic plasma. Under the proposed criterion, we prove that the smooth solutions develop a $C^1$ blow-up in a finite time and obtain their temporal blow-up rates. In general, it is known that smooth solutions to nonlinear hyperbolic equations fail to exist globally in time when the gradient of initial velocity is negatively large. In contrast, our blow-up condition does not require the largeness of gradient of velocity, and our result particularly implies that the smooth solutions can break down even if the gradient of initial velocity is trivial.

Keywords: Euler-Poisson system; Boltzmann relation; cold ion; singularity

1. Introduction

We consider the pressureless Euler-Poisson system in a non-dimensional form:

\begin{align}
\rho_t + (\rho u)_x &= 0, \\
u_t + uu_x &= -\phi_x, \\
-\phi_{xx} &= \rho - e^\phi,
\end{align}

where $\rho > 0$, $u$ and $\phi$ are the unknown functions of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ representing the ion density, the fluid velocity for ions, and the electric potential, respectively.

The Euler-Poisson system (1.1) is a fundamental fluid model describing the dynamics of cold ions in an electrostatic plasma. We briefly discuss important physical assumptions imposed on (1.1). In the one-fluid model (1.1), the electron density $\rho_e$ is assumed to satisfy the Boltzmann relation

\begin{equation}
\rho_e = e^\phi.
\end{equation}

Based on the physical fact that the electron mass $m_e$ is much smaller than the ion mass $m_i$, i.e., $m_e/m_i \ll 1$, the relation (1.2) can be formally derived from the two-fluid model of ions and electrons by neglecting the momentum of electrons under the massless electron assumption. We refer to [9] for a rigorous justification of the massless electron limit. On the other hand, taking account of the fact that the ion temperature $T_i$ is much smaller than the electron temperature $T_e$, the ion pressure is often neglected to simplify the model. In other words, the pressureless Euler-Poisson system (1.1) is an ideal model for cold ions (a plasma with $T_i/T_e \ll 1$). We refer to [4, 6] for more detailed physicality of the model.

The Euler-Poisson system equipped with the Boltzmann relation (1.2) is often employed to study various phenomena of plasma such as plasma sheaths [8, 19], KdV limit [1, 11], KP-II and Zakharov-Kuznetsov limits [14, 20]. In particular, effort has been made to

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mathematically justify the phenomena of plasma solitons by showing existence \cite{5, 15, 21} and linear stability of traveling solitary waves \cite{2, 12}.

A question of global existence or finite time blow-up of smooth solutions naturally arises in the study of large-time dynamics of the Euler-Poisson system. To the best of our knowledge, no global well-posedness of smooth solutions is known except that of \cite{10} for 3D case with the isothermal pressure law showing that smooth irrotational flow can persist globally. On the other hand, (1.1) is shown to have the smooth solutions leaving the class of $C^1$ in a finite time when the initial velocity has negatively large gradient at some point \cite{16}.

In the present work, we establish a criterion for singularity formation of (1.1), under which we show the smooth solutions develop a $C^1$ blow-up in a finite time along with the temporal blow-up rates. In general, it is known that smooth solutions to nonlinear hyperbolic equations fail to exist globally in time when the gradient of initial velocity is negatively large. Roughly speaking, this means that if the given initial data is near the shock waves, then the corresponding solutions develops into the shock waves. In contrast, our blow-up condition does not require the largeness of gradient of the initial velocity. In particular, our results demonstrate that $C^1$ norm of velocity blows up even if the initial velocity has trivial gradient. From a physical point of view, this phenomenon is caused by the effect of the electrostatic repulsive force. For instance, when the initial density is locally lower than the background density, i.e., ion density is locally rarefied, the electrostatic potential is determined in a way that the fluid momentum with negative gradient is generated at later times, resulting in the finite-time singularity formation. (See the numerical simulations of specific examples in Section 3.)

1.1. Main result. We consider the Euler-Poisson system (1.1) around a constant state, i.e., $(\rho, u, \phi) \to (1, 0, 0)$ as $|x| \to \infty$. It is known that the system (1.1) admits a unique smooth solution locally in time for the smooth initial data, for instance, $(\rho_0, u_0) \in H^2(\mathbb{R}) \times H^3(\mathbb{R})$. Furthermore, as long as the smooth solution exists, the energy

$$H(t) := \int_{\mathbb{R}} \frac{1}{2} \rho u^2 + \frac{1}{2} |\partial_x \phi|^2 + (\phi - 1)e^\phi + 1 \, dx$$

is conserved, that is,

$$H(t) = H(0).$$

We refer to \cite{14} for more details.

To state our main theorem, let us define a function $f_- : (\mathbb{R}) \to [0, \infty)$ by

$$f_- (z) := \int_z^0 \sqrt{2 ((s - 1)e^s + 1)} \, ds \quad \text{for} \quad z \in (-\infty, 0].$$

By inspection, we see that $f_-$ is well-defined since $(s - 1)e^s + 1$ is nonnegative, it is strictly decreasing in $(-\infty, 0]$, and it has the inverse function $f_-^{-1} : [0, +\infty) \to (-\infty, 0]$.

**Theorem 1.1.** For the initial data satisfying

$$\exp \left( f_-^{-1}(H(0)) \right) > 2\rho_0(\alpha) \quad \text{for some} \quad \alpha \in \mathbb{R},$$

the maximal existence time $T_*$ for the smooth solution to the Euler-Poisson system (1.1) is finite. In particular,

$$\lim_{t \to T_*} \sup_{x \in \mathbb{R}} \rho(x, t) = +\infty \quad \text{and} \quad \inf_{x \in \mathbb{R}} u_x(x, t) \approx \frac{1}{t - T_*}.$$
for all \( t < T_* \) sufficiently close to \( T_* \).

Our main theorem demonstrates that singularities in solutions to \((1.1)\) can occur in a finite time if the initial density at some point is small compared to the initial energy. In fact, the negativity of the initial velocity gradient is not required.

We remark that there is a fairly wide class of the initial data satisfying the condition \((1.4)\). From the elliptic estimates for the Poisson equation \((1.1c)\), we have (Appendix 4.1)

\[
0 \leq H(0) \leq \frac{\sup_{\alpha \in \mathbb{R}} \rho_0}{2} \int_{\mathbb{R}} |u_0|^2 \, dx + \frac{1}{\inf_{\alpha \in \mathbb{R}} \rho_0} \int_{\mathbb{R}} (\rho_0 - 1)^2 \, dx =: C(\rho_0, u_0).
\]

On the other hand, since \( \lim_{\zeta \to 0} \frac{f^{-1}(\zeta)}{-1 - (\zeta)} = 0 \), for any given constant \( 0 < c < 1/2 \), there is \( \delta_c > 0 \) such that \( \zeta < \delta_c \) implies \( \exp(f^{-1}(\zeta)) > 2c \). Thus, \((1.4)\) holds for all initial data satisfying \( \inf_{\rho_0} = c \in (0, 1/2) \) and \( C(\rho_0, u_0) < \delta_c \ll 1 \). In particular, one can take \( u_0 \equiv 0 \). We shall give more specific examples and their numerical experiments exhibiting the singularities in Section 3.

We discuss some difficulties of our problem and related results. Along the characteristic curve \( x(\alpha, t) \) associated with the fluid velocity \( u \), issuing from an initial point \( \alpha \in \mathbb{R} \) (see (2.1)), one can easily obtain from \((1.1)\) that

\[
\dot{\rho} = -u_x \rho, \quad \dot{u}_x = -u_x^2 + \rho - e^\phi,
\]

where \( \dot{\cdot} := \partial_t + u \partial_x \). The behavior of \( \rho \) and \( u_x \) depends not only on the initial data, but the potential \( \phi \) along the characteristic curve due to the nonlocal nature of the system \((1.1)\). This makes the problem challenging, and also distinguishes the aforementioned blow-up mechanism for \((1.1)\) from those for different types of the Euler-Poisson systems, for instance, the one with constant background density \([7]\). The most relevant study to our result is that of \([16]\), where smooth solutions to \((1.1)\) are shown to blow up when the initial data satisfies \( \partial_x u_0 \leq -\sqrt{2\rho_0} \) at some point, i.e., the gradient of velocity is large negatively compared to the density. In fact, the result is obtained by discarding \( e^\phi \) in \((1.6)\) and solving the resulting (closed) system of differential inequalities for \( \rho \) and \( u_x \). For this reason, the criterion, which is described only by the local quantities of the initial data, is not sharp, i.e., it only provides a sufficient condition resulting from the local structure of \((1.1)\) alone dropping \( e^\phi \). Our analysis takes account of the non-local structure. As such, the blow up criterion \((1.4)\) involves the non-local quantity.

To overcome the non-locality issue of the problem, by making use of the energy conservation, we first show that the amplitude of \( \phi \) is bounded uniformly in \( x \) and \( t \) as long as the smooth solution exists (Lemma 2.1) and that this uniform bound can be controlled only by the size of initial energy \( H(0) \).

Next, we define

\[
w(\alpha, t) := \frac{\partial_x}{\partial \alpha}(\alpha, t)
\]

and derive a second-order ODE \((2.7)\) for \( w \). Using Lemma 2.1, we find that \( w \) vanishes at a finite time \( T_* \) if and only if the solution blows up in the \( C^1 \) topology, i.e., \( u_x \searrow -\infty \) as \( t \nearrow T_* \) at a non-integrable order in time \( t \) (Lemma 2.2). Our goal is then to find some sufficient conditions guaranteeing \( w \) vanishes in a finite time. By applying Lemma 2.1 we employ a comparison argument for the differential inequality to study the behavior of \( w \).

The derivation of \((2.7)\) is motivated by the well-known fact that the Riccati equation can be reduced to a second-order linear ODE \([13]\, pp.23–25\). This formulation is also employed in \([7]\) for the 1D pressureless Euler-Poisson system with constant doping profile. In this
case, the associated ODE is explicitly solvable, and thus, the so-called critical threshold is observed. We refer to [17, 18] for 2D case. We also refer to [3] for singularity formation of the Euler-Poisson system with nonlocal alignment forces. An interesting open question is whether such critical threshold is available for the Euler-Poisson system with the Boltzmann relation (1.2).

2. Proof of Theorem 1.1

In this section, we present all the necessary lemmas, and prove Theorem 1.1 at the end of the section.

2.1. Uniform bound of \( \phi \). We first show the uniform boundedness of \( \phi(x,t) \) in \( x \) and \( t \).

Let us define the functions

\[
    f(z) := \begin{cases} 
    f_+(z) := \int_0^z \sqrt{2U(s)} \, ds & \text{for } z \geq 0, \\
    f_-(z) := \int_z^0 \sqrt{2U(s)} \, ds & \text{for } z \leq 0, 
    \end{cases}
\]

where \( U(s) := (s-1)e^s + 1 \) is nonnegative for all \( s \in \mathbb{R} \) and satisfies

\[
    U(s) \to +\infty \text{ as } s \to +\infty, \quad U(s) \to 1 \text{ as } s \to -\infty
\]

(see Figure 1). Hence, \( f_+ \) and \( f_- \) have the inverse functions \( f_+^{-1} : [0, +\infty) \to [0, +\infty) \) and \( f_-^{-1} : [0, +\infty) \to (-\infty, 0] \), respectively. Furthermore, \( f \) is of \( C^1(\mathbb{R}) \).

**Lemma 2.1.** As long as the smooth solution to (1.1) exists for \( t \in [0, T] \),

\[
    f_-^{-1}(H(0)) \leq \phi(x,t) \leq f_+^{-1}(H(0)) \quad \text{for all } (x,t) \in \mathbb{R} \times [0, T].
\]

![Figure 1](image-url)
Proof. Since \( f \in C^1(\mathbb{R}) \) and \( f \geq 0 \), we have that for all \( t \geq 0 \) and \( x \in \mathbb{R} \),
\[
0 \leq f(\phi(x,t)) = \int_{-\infty}^{x} f'(\phi(y,t)) \phi_y \, dy \\
\leq \int_{-\infty}^{x} |f'(\phi(y,t))| \phi_y \, dy \\
\leq \int_{-\infty}^{\infty} U(\phi) \, dy + \frac{1}{2} \int_{-\infty}^{\infty} |\phi_y|^2 \, dy \\
\leq H(t) = H(0),
\]
where the last equality holds due to the energy conservation \( (1.3) \). This completes the proof. \( \square \)

2.2. Second order ODE along characteristic curves. For \( u \in C^1 \), the characteristic curves \( x(\alpha, t) \) are defined as the solution to the ODE
\[
(2.1) \quad x' = u(x(\alpha, t), t), \quad x(\alpha, 0) = \alpha \in \mathbb{R}, \quad t \geq 0,
\]
where \( ' := d/dt \) and the initial position \( \alpha \) is considered as a parameter. Since \( x(\alpha, t) \) is differentiable in \( \alpha \), we obtain from (2.1) that
\[
(2.2) \quad w' = u_x(x(\alpha, t), t)w, \quad w(\alpha, 0) = 1, \quad t \geq 0,
\]
where
\[
w = w(\alpha, t) := \frac{\partial x}{\partial \alpha}(\alpha, t).
\]

We show that \( w \) satisfies a certain second-order ordinary differential equation. By integrating (1.1b) along \( x(\alpha, t) \), we obtain that
\[
(2.3) \quad x' = u(x(\alpha, t), t) = u_0(\alpha) - \int_{0}^{t} \phi_x(x(\alpha, s), s) \, ds.
\]
Differentiating (2.3) in \( \alpha \),
\[
(2.4) \quad w' = \partial_\alpha u_0(\alpha) - \int_{0}^{t} \phi_{xx}(x(\alpha, s), s) w(\alpha, s) \, ds.
\]
Since the RHS of (2.4) is differentiable in \( t \), so is the LHS. Hence, we get
\[
(2.5) \quad w'' = -\phi_{xx} w = (\rho - e^\phi) w,
\]
where we have used (1.1c). On the other hand, using (1.1a) and (2.2), we obtain that
\[
(\rho(x(\alpha, t), t) w(\alpha, t))' = -\rho u_x w + \rho u_x w = 0,
\]
which yields
\[
(2.6) \quad \rho(x(\alpha, t), t) w(\alpha, t) = \rho_0(\alpha).
\]

Finally, combining (2.2), (2.5), (2.6), we see that \( w(\alpha, t) \) satisfies the second-order non-homogeneous equation
\[
(2.7) \quad w'' + e^{\phi(x(\alpha, t))} w = \rho_0(\alpha), \quad w(\alpha, 0) = 1, \quad w'(\alpha, 0) = u_{0x}(\alpha).
\]
2.3. Blow-up criterion. From (2.6), it is obvious that for each \( \alpha \in \mathbb{R} \),
\[
0 < w(\alpha, t) < +\infty \quad \iff \quad 0 < \rho(x(\alpha, t), t) < +\infty,
\]
\[
\lim_{t \to T_*} w(\alpha, t) = 0 \quad \iff \quad \lim_{t \to T_*} \rho(x(\alpha, t), t) = +\infty.
\]

Using Lemma 2.1, we show that \( \sup_{x \in \mathbb{R}} |\rho(x, t)| \) and \( \sup_{x \in \mathbb{R}} |u_x(x, t)| \) blow up at the same time, if one of them blows up at a finite time \( T_* \).

**Lemma 2.2.** Suppose that the smooth solution to (1.1) exists for all \( 0 \leq t < T_* < +\infty \). Then the following statements hold.

1. For each \( \alpha \in \mathbb{R} \), the following holds true:
\[
\lim_{t \to T_*} w(\alpha, t) = 0 \quad \text{if and only if} \quad \lim_{t \to T_*} u_x(x(\alpha, t), t) = -\infty.
\]

2. If one of (2.8)–(2.9) holds for some \( \tilde{\alpha} \in \mathbb{R} \), then there are uniform constants \( c_0, c_1 > 0 \) such that
\[
c_0 \frac{t}{T_*} < u_x(x(\tilde{\alpha}, t), t) < c_1 \frac{t}{T_*}
\]
for all \( t < T_* \) sufficiently close to \( T_* \).

**Remark 1.** (1) By integrating (2.2), we obtain
\[
w(\alpha, t) = \exp \left( \int_0^t u_x(x(\alpha, s), s) \, ds \right).
\]
While it is easy by (2.11) to see that (2.8) implies (2.9), the converse is not obvious since one cannot exclude the possibility that \( u_x \) diverge in some other earlier time, say \( T_0 < T_* \) with an integrable order in \( t \), for which we still have \( w(\alpha, T_0) > 0 \). For the proof of the converse and obtaining the blow-up rate (2.10), Lemma 2.1, the uniform boundedness of \( \phi \) will be crucially used.

2. From (2.15) and (2.16), we see that if \( w'(T_0) < 0 \), the vanishing (or blow-up) order of \( w \) (or \( \rho \)) is \((t - T_0)^{-1}\) and if \( w'(T_0) = 0 \), the vanishing (or blow-up) order of \( w \) (or \( \rho \)) is \((t - T_*^2)^{-2}\). \( \frac{d}{dt} \)

**Proof.** We suppress the parameter \( \alpha \) for notational simplicity. We first make a few basic observations. By the assumption, we have that \( w(t) > 0 \) for all \( t \in [0, T_*] \). From (2.7) and the fact that \( e^{\phi} w(t) > 0 \), we obtain that
\[
w''(t) < w''(t) + e^{\phi(x(\alpha, t), t)} w(t) = \rho_0,
\]
for which we integrate (2.12) in \( t \) twice to deduce that \( w(t) \) is bounded above on \( [0, T_*] \). This together with (2.7) and Lemma 2.1 implies that \( |w''(t)| \) is bounded on the interval \([0, T_*]\). Using this for
\[
w'(t) - w'(s) = \int_s^t w''(t) \, d\tau,
\]
we see that \( w'(t) \) is uniformly continuous on \([0, T_*]\). Hence, we see that the following limit
\[
w'(T_0) := \lim_{t \to T_*} w'(t) \in (-\infty, +\infty)
\]
exists. In a similar fashion, one can check that

$$ w(T_*) := \lim_{t \nearrow T_*} w(t) \in [0, +\infty). $$

We prove the first statement. It is obvious from (2.11) that (2.8) implies (2.9). To show that (2.9) implies (2.8), we suppose

$$ \lim_{t \nearrow T_*} w(t) > 0. $$

Then, since $w(0) = 1$, $w(t)$ has a strictly positive lower bound on $[0, T_*]$. From (2.9), we may choose a sequence $t_k$ such that $u_x(t_k) \rightarrow -\infty$ as $t_k \nearrow T_*$. Now using (2.2), we obtain that

$$ u_x(t_k)w(t_k) - u_x(s)w(s) = w'(t_k) - w'(s) = \int_s^{t_k} w''(\tau) d\tau, $$

which leads a contradiction by letting $t_k \nearrow T_*$. Hence, (2.8) holds.

Now we prove the second statement. Due to the first statement, it is enough to assume that (2.8) holds for some $\tilde{\alpha} \in \mathbb{R}$. From (2.7) and Lemma 2.1, we see that (2.8) implies (2.13)

$$ \lim_{t \nearrow T_*} w''(t) = \rho_0 > 0. $$

Since $w(t) > 0$ on $[0, T_*], (2.8)$ also implies that

$$ (2.14) \quad w'(T_*) = \lim_{t \nearrow T_*} w'(t) \leq 0. $$

By the fundamental theorem of calculus, one has $w'(t) = w'(\tau) + \int_\tau^t u''(s) ds$ for all $t, \tau \in [0, T_*]$. Then taking the limit $\tau \nearrow T_*$ and integrating once more, we obtain that for $t < T_*$,

$$ (2.15) \quad w'(t) = w'(T_*) + \int_{T_*}^t u''(s) ds, $$

$$ w(t) = w'(T_*)(t - T_*) + \int_{T_*}^t w''(s)(t - s) ds. $$

Using (2.13), we have that for all $t < T_*$ sufficiently close to $T_*$,

$$ (2.16) \quad 2\rho_0(t - T_*) < \int_{T_*}^t w''(s) ds < \frac{\rho_0}{2}(t - T_*), $$

$$ \frac{\rho_0}{4}(T_* - t)^2 < \int_{T_*}^t w''(s)(t - s) ds < \rho_0(T_* - t)^2. $$

Thanks to (2.14), we note that either $w'(T_*) < 0$ or $w'(T_*) = 0$ holds. Combining (2.15)–(2.16), we conclude that if $w'(T_*) < 0$, then

$$ 1/2 < (t - T_*)u_x = (t - T_*)\frac{w'}{w} < 2, $$

and if $w'(T_*) = 0$, then

$$ 1 < (t - T_*)u_x = (t - T_*)\frac{w'}{w} < 8. $$

This completes the proof of (2.10). \qed
2.4. Proof of main theorem. Now we are ready to prove our main theorem.

Proof of Theorem 1.1. We consider the equation (2.7) with \( \alpha \in \mathbb{R} \), for which (1.4) holds. Suppose that the smooth solution to (1.1) exists for all \( t \in [0, +\infty) \). Then, thanks to Lemma 2.2 we must have

\[
(2.17) \quad w(\alpha, t) > 0 \quad \text{for all} \quad t \in [0, +\infty).
\]

Combining (2.7) and Lemma 2.1, we have that for all \( t \in [0, +\infty) \),

\[
(2.18) \quad w''(t) + aw(t) \leq b, \quad w(0) \geq 1,
\]

where we let

\[ w(t) = w(\alpha, t), \quad a := \exp \left( f^{-1}(H(0)) \right), \quad b := \rho_0(\alpha) \]

for notational simplicity. We notice that the inequality \( w(0) \geq 1 \) is allowed in (2.18). In what follows, we show that there exists a finite time \( T^* > 0 \) such that \( \lim_{t \to T^*} w(\alpha, t) = 0 \).

This contradicts to (2.17), and hence finishes the proof of Theorem 1.1.

We consider two disjoint cases, call them Case A and Case B for \( w'(0) \leq 0 \) and \( w'(0) > 0 \), respectively.

Case A: We first consider the case \( w'(0) \leq 0 \). We claim that \( b - aw(t) = 0 \) for some \( t \).

Suppose to the contrary that \( b - aw(t) \neq 0 \) for all \( t \geq 0 \). Since \( b - aw(0) < 0 \) from (1.4) and \( w(0) \geq 1 \), we have

\[
(2.19) \quad b - aw(t) < 0 \quad \text{for all} \quad t \in [0, +\infty).
\]

Combining (2.18)–(2.19), we see that \( w''(t) < 0 \) for all \( t \). From this and \( w'(0) \leq 0 \), we have that \( w'(t) \to c \in [-\infty, 0) \) as \( t \to +\infty \), which implies that \( w(t) \to -\infty \) as \( t \to +\infty \). This is a contradiction to (2.19). This proves the claim.

Then, by the continuity of \( w \), we can choose the minimal \( T_1 > 0 \) such that

\[
(2.20) \quad b = aw(T_1).
\]

Hence there holds \( w''(t) \leq b - aw(t) < 0 \) for all \( t \in (0, T_1) \), which in turn implies

\[ w'(t) = \int_0^t w''(s) \, ds + w'(0) < 0 \quad \text{for all} \quad t \in (0, T_1]. \]

Now we split the proof further into two cases:

(2.21a) (i) \( w'(t) < 0 \) on \( (0, T_1] \) and \( w'(t) \) has a zero on \( (T_1, +\infty) \),

(2.21b) (ii) \( w'(t) < 0 \) for all \( t > 0 \).

Case (i): We choose the minimal \( T_2 > T_1 \) satisfying

\[
(2.22) \quad w'(T_2) = 0.
\]

Then, \( w'(t) < 0 \) for \( t \in (0, T_2) \). It suffices to show that \( w(T_2) \leq 0 \) since this implies that \( w(t) = 0 \) for some \( t \in (0, T_2] \) as desired.

We shall show that \( w(T_2) \leq 0 \) by contradiction. Suppose not, i.e., \( w(T_2) > 0 \). Then since \( w \) decreases on \( [T_1, T_2] \), we have

\[
(2.23) \quad 0 < w(T_2) < w(T_1) = b/a,
\]
where the equality is from \((2.20)\). Multiplying \((2.18)\) by \(w' \leq 0\), and then integrating over \([0,t]\), we obtain that for \(t \in [0,T_2]\),

\[
\frac{|w'(t)|^2}{2} \geq -a \left( \frac{(w(t))^2 - |w(0)|^2}{2} \right) + b(w(t) - w(0)) + \frac{|w'(0)|^2}{2}.
\]

Here we define a function \(g(w) := -a \left( \frac{w^2 - |w(0)|^2}{2} \right) + b(w - w(0)) + \frac{|w'(0)|^2}{2}\). We see that

\[
g(0) = \frac{a|w(0)|^2}{2} - w(0)b + \frac{|w'(0)|^2}{2} \geq \frac{a}{2} - b + \frac{|w'(0)|^2}{2} > 0,
\]

where we have used the assumption \(w(0) \geq 1\) and \((1.4)\) for the last two inequalities, respectively. By inspection, one can check that the function \(g(w)\) is strictly increasing on \([0, b/a]\).

Using this together with \((2.25)\), we have

\[
g(w) \geq g(0) > 0 \quad \text{for all } w \in [0, b/a].
\]

Combining \((2.22)-(2.26)\), we have

\[
0 = \frac{|w'(T_2)|^2}{2} \geq g(w(T_2)) > 0,
\]

which is a contradiction.

**Case (ii)**: We first claim that \(\limsup_{t \to \infty} w'(t) = 0\). If not, i.e., \(\limsup_{t \to \infty} w'(t) \neq 0\), then thanks to \((2.21b)\), we have \(\limsup_{t \to \infty} w'(t) < 0\). This implies \(w(t) = 0\) for some \(t > 0\), which is a contradiction to \((2.17)\).

On the other hand, since \(w\) is monotonically decreasing on \((0, \infty)\) thanks to \((2.21b)\), we see that \(w_\infty := \lim_{t \to \infty} w(t)\) exists and \(w_\infty \in [0, b/a]\) by \((2.20)\). Similarly as in obtaining \((2.24)\), we multiply \((2.18)\) by \(w'(t) \leq 0, t \in [0, \infty)\), and then integrate the resultant over \([0,t]\) to obtain that \((2.24)\) holds for \(t \in [0, \infty)\). Since \(0 = \limsup_{t \to \infty} w'(t) = \liminf_{t \to \infty} |w'(t)|\), we arrive at

\[
0 = \liminf_{t \to \infty} |w'(t)|^2/2 \geq \liminf_{t \to \infty} g(w(t)) = g(w_\infty) \geq g(0) > 0,
\]

where we have used \((2.26)\) for the last inequality. This is absurd, which completes the proof for **Case A**.

**Case B**: Now we consider the case \(w'(0) > 0\). We claim that \(w'(t) = 0\) for some \(t > 0\). If not, i.e., \(w'(t) > 0\) for all \(t \geq 0\), we have

\[
w''(t) \leq b - aw(t) \leq b - aw(0) < 0.
\]

This implies that \(w'(t) \to -\infty\) as \(t \to +\infty\), which is a contradiction to the assumption that \(w'(t) > 0\) for all \(t \geq 0\).

By the continuity of \(w'(t)\), there is a minimal number \(T_0 > 0\) such that \(w'(T_0) = 0\). Since \(w'(t) > 0\) for \(t \in [0, T_0)\), we see that \(w(T_0) \geq w(0) \geq 1\). Now one can apply the same argument as **Case A** to conclude that \(w(t)\) has a zero on the interval \([T_0, +\infty)\). This completes the proof of Theorem \(1.1\) \(\square\)

We remark that, following the proof of Theorem \(1.1\), one obtains an interesting lemma concerning the existence of zeros of second-order linear differential inequality (see Appendix \(4.2\).
3. Numerical experiments

In this section, we present numerical examples concerning our theoretical result in Theorem 1.1. Referring to [15], the implicit pseudo-spectral scheme is employed to solve (1.1) numerically on periodic domains for numerical convenience.

We demonstrate three cases (see Table 1). In the case (a), the condition (1.4) holds, and \( \rho \) and \( u_x \) are expected to blow up after \( t = 2.3 \) in Figure 2. This supports our Theorem 1.1. In the case (b), although the condition (1.4) is not satisfied, the solutions are bound to break down after \( t = 2.7 \) in Figure 3. This indicates that the blow-up condition (1.4) has a room to be improved. Lastly, in the case (c), the condition (1.4) is not satisfied, and the smooth solutions seem to persist for \( t \in [0, 20] \) in Figure 4.

\[
\begin{array}{|c|c|c|c|}
\hline
\rho_0(x) & \text{(a)} & \text{(b)} & \text{(c)} \\
\hline
H(0) & 1 - 0.7 \text{sech}(3x) & 1 - 0.7 \text{sech}(2x) & 1 - 0.3 \text{sech}(2x) \\
\hline
\exp(-F(H(0))) & 0.6448 & 0.5390 & 0.7585 \\
\hline
\text{Blow-up condition (1.4)} & \text{Hold} & \text{Not hold} & \text{Not hold} \\
\hline
\text{Numerical results} & \text{Figure 2} & \text{Figure 3} & \text{Figure 4} \\
\hline
\end{array}
\]

Table 1. \( \rho_0 \) is the initial density function. The initial velocity \( u_0 \) are given as identically zero function for all cases. \( H(0) \) is the energy defined in (1.3) for the initial data.

4. Appendix

4.1. Proof of inequality (1.5).

Proof. We use the following elliptic estimates (see [14]):

\[
(4.1a) \quad K_- := \inf_{x \in \mathbb{R}} \rho \leq e^\phi \leq \sup_{x \in \mathbb{R}} \rho =: K_+ \quad \text{for all } x \in \mathbb{R},
\]

\[
(4.1b) \quad \int_{\mathbb{R}} |\phi_x|^2 + \frac{K_-}{2} |\phi|^2 \, dx \leq \frac{1}{2K_-} \int_{\mathbb{R}} |\rho - 1|^2 \, dx.
\]

Using the Poisson equation (1.1c), we have that

\[
\int (\rho - 1) \phi \, dx = \int |\phi_x|^2 + (e^\phi - 1) \phi \, dx
\]

\[
= \int |\phi_x|^2 + (\phi - 1)e^\phi + 1 + (e^\phi - 1 - \phi) \, dx
\]

\[
\geq \int \frac{1}{2} |\phi_x|^2 + (\phi - 1)e^\phi + 1 \, dx
\]

since \( e^\phi - 1 - \phi \geq 0 \). Using Young’s inequality and (4.1b), we obtain

\[
\int (\rho - 1) \phi \, dx \leq \frac{K_-}{2} \int |\phi|^2 \, dx + \frac{1}{2K_-} \int |\rho - 1|^2 \, dx
\]

\[
\leq \frac{1}{K_-} \int |\rho - 1|^2 \, dx.
\]

Combining (4.1a), (4.2) and (4.3), we obtain the inequality (1.5). \( \square \)
4.2. **Zeros of second-order differential inequality.** Following the proof of Theorem 1.1, one obtains the following lemma:
Figure 3. Numerical plots of $\sup_x |\rho(x,t) - 1|$ (left) and $\sup_x |u_x(x,t)|$ (right) for the case (b). Although the condition (1.4) does not hold, $\rho$ and $u_x$ are expected to eventually blow up at a finite time.

Figure 4. Numerical plots of $\sup_x |\rho(x,t) - 1|$ (left) and $\sup_x |u_x(x,t)|$ (right) for the case (c). $\sup_x |\rho - 1|$ and $\sup_x |u_x|$ keep oscillating and decreasing as times $t$ goes by.

Lemma 4.1. Let $a$ and $b$ be positive constants. Suppose $w(t)$ satisfies

$$w'' + aw \leq b$$

for all $t \geq T_0$ and $w(T_0) \geq 1$. If $a/2 > b$ and

$$a|w(T_0)|^2 - w(T_0)b + \frac{|w'(T_0)|^2}{2} > 0,$$

then $w(t)$ has a zero on the interval $(T_0, +\infty)$.

The authors are not aware of any literature addressing the existence of zeros of second-order linear differential inequality with the coefficient $a > 0$ and constant nonhomogeneous term $b$. We finish this subsection with some remarks regarding Lemma 2.3.
Remark 2. (1) For the case of the differential equation \( w'' + aw = b \),
\[
\frac{a|w(0)|^2}{2} - w(0)b + \frac{|w'(0)|^2}{2} \geq 0
\]
is a necessary and sufficient condition in order for \( w \) to have a zero on \([0, +\infty)\).

(2) One needs the restriction \( a/2 > b \) (or \( a/2 \geq b \)) in Lemma 4.1. If \( a/2 < b \), then the solution to \( w'' + aw = b \) with \( w(0) = 1 \) and \( w'(0) = 0 \) has no zero since (4.5) is not satisfied. For another example, we consider the equation
\[
w'' + aw = b - e^{-t}, \quad t \in [0, +\infty),
\]
where \( a, b > 0 \) are constants. Since the general solution of (4.6) is
\[
w(t) = \alpha \cos \sqrt{a}t + \beta \sin \sqrt{a}t + \frac{b}{a} - \frac{e^{-t}}{a + 1},
\]
we have
\[
w(0) = \alpha + \frac{b}{a} - \frac{1}{a + 1}, \quad w'(0) = \sqrt{a} \beta + \frac{1}{a + 1}.
\]
Since
\[
\min_{t \geq 0} w(t) \geq \min_{t \geq 0} \left( \alpha \cos \sqrt{a}t + \beta \sin \sqrt{a}t \right) + \min_{t \geq 0} \left( \frac{b}{a} - \frac{e^{-t}}{a + 1} \right)
\]
\[
= -\sqrt{\alpha^2 + \beta^2} + \frac{b}{a} - \frac{1}{a + 1},
\]
\( w(t) \) has no zero on \([0, +\infty)\) provided that
\[
-\sqrt{\alpha^2 + \beta^2} + \frac{b}{a} - \frac{1}{a + 1} > 0.
\]
We choose \( b = 1/3 \) and \( a > 0 \) sufficiently small such that
\[
\frac{1}{2(a + 1)^2} > b - \frac{a}{2} > \frac{a}{a + 1}.
\]
For \( w(0) = 1 \) and \( w'(0) = \frac{1}{a + 1} \), the first inequality of (4.8) is equivalent to (4.4) and the second inequality of (4.8) is equivalent to (4.7). On the other hand, \( b > a/2 \) holds.

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