1. Introduction

Quotients of complex projective or affine varieties by linear actions of complex reductive groups can be constructed and studied using Mumford’s geometric invariant theory (GIT) [14, 33, 36]. Given a linear action on a complex projective variety $X$ of a linear algebraic group $G$ which is not reductive, the graded algebra of invariants is not necessarily finitely generated, and even if it is finitely generated, so that there is a GIT quotient $X//G$ given by the associated projective variety, the geometry of this GIT quotient is hard to describe. When $G$ is reductive then $X//G$ is the image of a surjective morphism $\phi : X^{ss} \to X//G$ from an open subset $X^{ss}$ of $X$ (consisting of the semistable points for the linear action), and $\phi(x) = \phi(y)$ if and only if the closures of the $G$-orbits of $x$ and $y$ meet in $X^{ss}$. When $G$ is not reductive $\phi : X^{ss} \to X//G$ can still be defined in a natural way but it is not in general surjective, and indeed its image is not in general an algebraic variety, even when the algebra of invariants is finitely generated [16].

In this paper we prove finite generation of the algebras of invariants for a class of linear actions of suitable non-reductive groups on projective and affine varieties, and give a geometric construction for their GIT quotients.

Our leading example and main motivation is the Demailly-Semple algebra of invariants, which we studied in [5], where the reparametrisation group $\mathbb{G}_n$ consisting of $n$-jets of germs of biholomorphisms of $(\mathbb{C},0)$ acts on the jet bundle $J_n(X)$ over a complex manifold $X$ for some positive integer $n$. The fibre of $J_n(X)$ over $x \in X$ is the space of $n$-jets of germs at the origin of holomorphic curves $f : (\mathbb{C},0) \to (X,x)$, and polynomial functions on $J_n(X)$ are algebraic differential operators $Q(f',\ldots,f^{(n)})$, called jet differentials. The reparametrisation group $\mathbb{G}_n$ acts fibrewise on the bundle $E_n(X)$ of jet differentials. Here $\mathbb{G}_n$ is the semi-direct product $\mathbb{U}_n \rtimes \mathbb{C}^*$ of its unipotent radical $\mathbb{U}_n$ with $\mathbb{C}^*$, and is a

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subgroup of $\text{GL}(n)$ which has the upper triangular form

$$\mathbb{G}_n \cong \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & \alpha_1^2 & \cdots \\ 0 & 0 & \alpha_1^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1^n \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \right\}$$

where the entries above the leading diagonal are polynomials in $\alpha_1, \ldots, \alpha_n$, and $\mathbb{U}_n$ is the subgroup consisting of matrices of this form with $\alpha_1 = 1$.

Jet bundles and jet differentials have played a central role in the history of hyperbolic varieties and the Kobayashi conjecture on the non-existence of holomorphic curves in compact complex manifolds of generic type. The use of jet differentials can be traced back to the work of Bloch [8], Cartan [9], Ahlfors [1], Green and Griffiths [13], Siu [46], and their ideas were extended in the seminal paper of Demailly [10], and recently used by Diverio, Merker and Rousseau [11] and the first author [4]. The basic observation is that any entire curve $f : \mathbb{C} \rightarrow X$ must automatically satisfy all algebraic differential equations $Q(f', \ldots, f^{(n)}) = 0$ arising from global jet differential operators $Q \in H^0(X, E_n(X) \otimes O(-A))$ which vanish on some ample divisor $A$. In [10] Demailly suggested using jet differentials invariant under reparametrisation and formulated

**Conjecture 1.1** (Demailly, 1990). The algebra $E_n(X)^{U_n}$ of $U_n$-invariant jet differentials is finitely generated.

As a special case of our main theorem (Theorem 1.3 below) we prove

**Theorem 1.2.** $\mathbb{G}_n$ is a Grosshans subgroup of the complex general linear group $\text{GL}(n)$, so that every linear action of $\mathbb{G}_n$ which extends to a linear action of $\text{GL}(n)$ has a finitely generated algebra of invariants.

In particular this gives an affirmative answer to the Demailly conjecture.

The algebra of invariant jet differentials has been widely studied in hyperbolic algebraic geometry, but with only sporadic results about finite generation. Rousseau ([42]) and Merker ([31] [32]) showed that when both $n$ and $\dim X$ are small then this algebra is finitely generated, and in [32] Merker provided an algorithm which produces finite sets of generators when they exist for any $\dim X$ and $n$. In [5] the authors put forward a proof of the Demailly conjecture, but we have discovered some gaps in that proof, which is more complicated than the proof in this paper of the more general result given by Theorem 1.3 below. Here we use a different affine embedding to make the Grosshans Criterion (see §3) work in a much more general context.

In this paper we study a more general class of groups of a similar form to that of $\mathbb{G}_n$. Let $U$ be a unipotent subgroup of $\text{SL}(n)$ with a semi-direct product

$$\hat{U} = U \rtimes \mathbb{C}^*$$
where \( \mathbb{C}^* \) acts on the Lie algebra of \( U \) with all its weights strictly positive. We assume that \( U \) and \( \hat{U} \) are upper triangular subgroups of \( \text{GL}(n) \) which are ‘generated along the first row’ in the sense that there are integers \( 1 = \omega_1 < \omega_2 \leq \omega_3 \leq \cdots \leq \omega_n \) and polynomials \( v_{i,j}(\alpha_1, \ldots, \alpha_n) \) in \( \alpha_1, \ldots, \alpha_n \) with complex coefficients for \( 1 < i < j \leq n \) such that

\[
\hat{U} = \left\{ \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\
0 & \alpha_1^{\omega_2} & p_{2,3}(\alpha) & \cdots & p_{2,n}(\alpha) \\
0 & 0 & \alpha_1^{\omega_3} & \cdots & p_{3,n}(\alpha) \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \alpha_1^{\omega_n}
\end{pmatrix} : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^* \times \mathbb{C}^{n-1} \right\}
\]

and \( U \) is the subgroup of \( \hat{U} \) where \( \alpha_1 = 1 \). Note that if \( U \) is any unipotent complex linear algebraic group which has an action of \( \mathbb{C}^* \) with all weights strictly positive, then \( U \) can be embedded in \( \text{GL}(\text{Lie}(U \times \mathbb{C}^*)) \) via its adjoint action on the Lie algebra \( \text{Lie}(U \times \mathbb{C}^*) \) as the unipotent radical of a subgroup \( \hat{U} \) of this form which is generated along the first row.

We will call this the adjoint form of \( U \). However there exists a standard criterion, known as the Grosshans principle \([24, 22]\) (see §3), for proving the finite generation of an algebra of invariants \( O(G)^H \), where \( G \) is a complex reductive group and \( H \subset G \) is observable in the sense that

\[
H = \{ g \in G : f(gx) = f(x) \text{ for all } x \in X \text{ and } f \in O(G)^H \}.
\]

In this case the finite generation of \( O(G)^H \) is equivalent to the existence of a finite-dimensional \( G \)-module \( V \) and some \( v \in V \) such that \( H = G_v \) is the stabiliser of \( v \) and \( \dim(G \cdot v) \leq \dim(G) - 2 \). Such a subgroup \( H \) is called a Grosshans subgroup of \( G \), and then any linear action of \( H \) which extends to a linear action of \( G \) has a finitely generated algebra of invariants.

In order to use this criterion when \( \hat{U} \) has the explicit form above, in §4 we obtain an explicit embedding of the quasi-affine variety \( \text{GL}(n)/\hat{U} \) (which can also be identified with the quotient of \( \text{SL}(n)/U \) by a finite central subgroup of \( \hat{U} \)) in the Grassmannian \( \text{Grass}_n(\text{Sym}^{\omega}(\mathbb{C}^n)) \) of \( n \)-dimensional linear subspaces of \( \text{Sym}^{\omega}(\mathbb{C}^n) \)

\[
\text{Sym}^{\omega}(\mathbb{C}^n) = \mathbb{C}^n \oplus \text{Sym}^{\omega_2}(\mathbb{C}^n) \oplus \cdots \oplus \text{Sym}^{\omega_n}(\mathbb{C}^n)
\]

where \( \text{Sym}^{k}(\mathbb{C}^n) \) is the \( k \)th symmetric product of \( \mathbb{C}^n \). Using Plücker coordinates we thus obtain an explicit projective embedding \( \text{GL}(n)/\hat{U} \hookrightarrow \mathbb{P}(\wedge^n\text{Sym}^{\omega}(\mathbb{C}^n)) \). In fact \( \text{GL}(n)/\hat{U} \) embeds into the open affine subset of \( \mathbb{P}(\wedge^n\text{Sym}^{\omega}(\mathbb{C}^n)) \) where the coordinate corresponding to the one-dimensional summand \( \wedge^n\mathbb{C}^n \) of \( \wedge^n\text{Sym}^{\omega}(\mathbb{C}^n) \) does not vanish.

In §5 we study the important special cases of our construction which were the motivation for this work. We observe that the curvilinear component of the punctual Hilbert scheme of \( n \) points on \( \mathbb{C}^d \) can be identified with the closure of the image \( J_n(1,d)/\mathbb{G}_n \).
in Grass$_n$(Sym$^ω\mathbb{C}^d$) via an embedding similar to that in §4, where $ω = (1, 2, \ldots, n)$ and $J_n(1, d)$ is the set of $n$-jets of germs $\mathbb{C} \to \mathbb{C}^d$ with the reparametrisation action of $\mathbb{G}_n$. Thus the curvilinear Hilbert schemes can be thought of as compactifications of non-reductive GIT quotients.

We will see in §5 that it is possible for a boundary orbit of $GL(n)/\mathbb{G}_n$ in this Grassmannian (and indeed in the affine open subset of the Grassmannian where the coordinate corresponding to the one-dimensional summand $\wedge^n \mathbb{C}^n$ of $\wedge^n \text{Sym}^{\omega} \mathbb{C}^n$ does not vanish) to have codimension 1, so the Grosshans principle is not applicable here. We therefore modify this construction in §6 to get an $SL(n)$-equivariant affine embedding of $SL(n)/U$ in an affine space

\[ W_U = W_{1,s_1} \oplus W_{2,s_2} \oplus \ldots \oplus W_{n,s_n} \]

where $s_1, \ldots, s_n$ are chosen such that $s_i > i(s_{i-1} + 1 + \omega_2 + \ldots + \omega_i - 1)$ and $s_i$ is congruent to one modulo $1 + \omega_2 + \ldots + \omega_i + 1$ for $i = 1, \ldots, n$. The main technical theorem of this paper, Theorem 6.5, tells us that the complement of $SL(n)/U$ in its closure $\overline{SL(n)/U}$ in $W_U$ has codimension at least two. From this we obtain the main result of this paper as follows.

**Theorem 1.3.** Assume $\hat{U} = U \times \mathbb{C}^*$ be a subgroup of $GL(n)$ which is generated along its first row with positive weights $1 = \omega_1 < \omega_2 \leq \ldots \leq \omega_n$ as at (1) above. Then $U$ is a Grosshans subgroup of $SL(n)$ and $\hat{U}$ is a Grosshans subgroup of $GL(n)$.

As a corollary we get the following general structure theorem for complex linear algebraic groups:

**Corollary 1.4.** Let $U$ be any unipotent complex linear algebraic group with an action of $\mathbb{C}^*$ with strictly positive weights. Then the adjoint form of $U$ is a Grosshans subgroup of $GL(\text{Lie}(U \times \mathbb{C}^*))$.

Our construction moreover gives an explicit description of a finite set of generators for the algebra of invariants $O(SL(n))^U$ (and similarly for $O(GL(n))^U$).

**Theorem 1.5.** The closure $\overline{SL(n)/U} \subseteq W_U$ of $SL(n)/U$ in the affine space $W_U$ is isomorphic to the canonical affine completion

\[ \text{SL}(n)/\hat{U} = \text{Spec} O(SL(n))^U \]

of $SL(n)/U$, so that the algebra of invariants $O(SL(n))^U$ is generated by the linear coordinates on $W_U$.

More precisely we have a surjective homomorphism of algebras from the polynomial algebra $O(W_U)$ generated by the linear coordinates on $W_U$ to $O(SL(n))^U$, and this surjection is $SL(n)$-equivariant. This implies in turn that if $X$ is any complex projective variety with an action of $U$ which is linear with respect to an ample line bundle $L$ on $X$ and extends to a linear action of $SL(n)$, then the algebra of invariants

\[ \bigoplus_{m \geq 0} H^0(X, L^\otimes m)^U \cong \bigoplus_{m \geq 0} \left( H^0(X, L^\otimes m) \otimes O(SL(n))^U \right)^{SL(n)} \]
is finitely generated. Indeed it is a quotient of the algebra of invariants

\[ \bigoplus_{m \geq 0} \left( H^0(X, L^m) \otimes O(W_0) \right)^{SL(n)} \]

and so is generated by the coordinates on the reductive GIT quotient \((X \times W_0)/\!// SL(n)\), which can be determined using the representation theory of \(SL(n)\) from the decompositions of \( \bigoplus_{m \geq 0} H^0(X, L^m) \) and \( W_0 \) as sums of irreducible representations of \(SL(n)\). Similarly if \(X\) is a \(ffi\)ne with a linear action of \(U\) which extends to a linear action of \(SL(n)\), then \(O(X)^U\) is finitely generated. In each case the associated GIT quotient can be identified with the GIT quotient of the product of \(X\) and \(SL(n)/\!// U\) by the reductive group \(SL(n)\):

\[ X/\!// U \cong (X \times SL(n)/\!// U)/\!// SL(n) \subseteq (X \times W_0)/\!// SL(n). \]

The geometry of the canonical affine completion \(SL(n)/\!// U\) of \(SL(n)/U\), and the projective completion \(GL(n)/\!// U\) of \(GL(n)/U\) in the Grassmannian \(Gr_m(\text{Sym}^n \mathbb{C}^n)\) can be explored using the method of variation of GIT (VGIT) \cite{15, 47}. This leads to a (partial) resolution of singularities of \(GL(n)/\!// U\) of the form \(SL(n) \times_{B_n} W_{\mathbb{P}}(B_{[n-1]})\) where \(B_n\) is a Borel subgroup of \(SL(n)\) and \(W_{\mathbb{P}}(B_{[n-1]})\) is an iterated blow-up of a weighted projective space \(W_{\mathbb{P}}(B_{[n-1]})\).

The layout of this paper is as follows. \(\S 2\) reviews classical geometric invariant theory and some non-reductive GIT very briefly, and \(\S 3\) contains a short summary of what is needed from the theory of Grosshans subgroups. In \(\S 4\) we describe an embedding of \(GL(n)/\!// U\) into \(Gr_m(\text{Sym}^n \mathbb{C}^n)\). In \(\S 5\) we recall the original motivation for this work from global singularity theory and jet differentials and discuss the strong link between jet differentials and the curvilinear component of Hilbert schemes of points. \(\S 6\) describes the affine embedding \(SL(n)/U \hookrightarrow W_0\), and we study the boundary orbits in \(\S 7\), giving a description of the boundary components in Theorem \(7.4\). The proof of the main technical result, Theorem \(6.5\), follows in \(\S 8\), and in \(\S 9\) we describe its implications for geometric descriptions of algebras of invariants and of non-reductive GIT quotients. Finally in \(\S 10\) we study the geometry of the canonical affine completion \(SL(n)/\!// U\) of \(SL(n)/U\) and the closure \(GL(n)/\!// U\) of \(GL(n)/\!// U\) in the Grassmannian \(Gr_m(\text{Sym}^n \mathbb{C}^n)\).

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2. Classical and non-reductive geometric invariant theory

Let \(X\) be a complex quasi-projective variety on which a complex reductive group \(G\) acts linearly; that is, there is a line bundle \(L\) on \(X\) (which we will assume to be ample) and a lift \(\hat{L}\) of the action of \(G\) to \(L\). Then \(y \in X\) is said to be semistable for this linear action if there exists some \(m > 0\) and \(f \in H^0(X, L^m)^G\) not vanishing at \(y\) such that the
open subset

\[ X_f = \{ x \in X \mid f(x) \neq 0 \} \]

is affine \((X_f)\) is automatically affine if \(X\) is projective or affine), and \(y\) is *stable* if also \(f\) can be chosen so that the action of \(G\) on \(X_f\) is closed with all stabilisers finite. The open subset \(X^{ss}\) of \(X\) consisting of semistable points has a quasi-projective categorical quotient \(X^{ss} \to X//G\), which restricts to a geometric quotient \(X^s \to X^s//G\) of the open subset \(X^s\) of stable points (see [33] Theorem 1.10). When \(X\) is projective then \(X_f\) is affine for any nonzero \(f \in H^0(X,L^\otimes m)^G\) (since we are assuming \(L\) to be ample), and there is an induced action of \(G\) on the homogeneous coordinate ring

\[ \hat{O}_L(X) = \bigoplus_{m \geq 0} H^0(X,L^\otimes m) \]

of \(X\). The subring \(\hat{O}_L(X)^G\) consisting of the elements of \(\hat{O}_L(X)\) left invariant by \(G\) is a finitely generated graded complex algebra because \(G\) is reductive, and the GIT quotient \(X//G\) is the associated projective variety \(\text{Proj}(\hat{O}_L(X)^G)\) [14,33,36]. When \(X\) is affine and the linearisation of the action of \(G\) is trivial then the algebra \(O(X)^G\) of \(G\)-invariant regular functions on \(X\) is finitely generated and \(X^{ss} = X\) and \(X//G = \text{Spec}(O(X)^G)\) is the affine variety associated to \(O(X)^G\).

Suppose now that \(H\) is any linear algebraic group, with unipotent radical \(U \unlhd H\) (so that \(H/U\) is reductive), acting linearly on a complex projective variety \(X\) with respect to an ample line bundle \(L\). Then the scheme \(\text{Proj}(\hat{O}_L(X)^H)\) is not in general a projective variety, since the graded complex algebra of invariants

\[ \hat{O}_L(X)^H = \bigoplus_{m \geq 0} H^0(X,L^\otimes m)^H \]

is not necessarily finitely generated, and geometric invariant theory (GIT) cannot be extended immediately to this situation (cf. [16,17,20,21,27,49]). However in some cases it is known that \(\hat{O}_L(X)^U\) is finitely generated, which implies that

\[ \hat{O}_L(X)^H = (\hat{O}_L(X)^U)^{H/U} \]

is finitely generated, and then the *enveloping quotient* in the sense of [16] is given by

\[ X//H = \text{Proj}(\hat{O}_L(X)^H). \]

Moreover, there is a morphism

\[ q : X^{ss} \to X//H, \]

where \(X^{ss}\) is defined as in the reductive case above, which restricts to a geometric quotient

\[ q : X^s \to X^s//H \]

for an open subset \(X^s \subset X^{ss}\). In such cases we have a GIT-like quotient \(X//H\) and we would like to understand it geometrically. However there is a crucial difference here from the case of reductive group actions, even though we are assuming that the
invariants are finitely generated: the morphism \(X^{ss} \to X/\!/H\) is not in general surjective, so we cannot describe \(X/\!/H\) geometrically as \(X^{ss}\) modulo some equivalence relation.

In this paper we will study the situation when \(U\) is a unipotent group with a one-parameter group of automorphisms \(\lambda : \mathbb{C}^* \to \text{Aut}(U)\) such that the weights of the induced \(\mathbb{C}^*\) action on the Lie algebra \(\mathfrak{u}\) of \(U\) are all strictly positive. Then we can form the semidirect product
\[
\hat{U} = \mathbb{C}^* \rtimes U
\]
given by \(\mathbb{C}^* \times U\) with group multiplication
\[
(z_1, u_1)(z_2, u_2) = (z_1 z_2, (\lambda(z_2^{-1})(u_1))u_2).
\]

Note that the centre of \(\hat{U}\) is finite and meets \(U\) in the trivial subgroup, so we have an inclusion given by the composition
\[
U \hookrightarrow \hat{U} \to \text{Aut}(\hat{U}) \hookrightarrow \text{GL}(\text{Lie}(\hat{U})) = \text{GL}(\mathbb{C} \oplus \mathfrak{u})
\]
where \(\hat{U}\) maps to its group of inner automorphisms and \(\mathfrak{u} = \text{Lie}(U)\). Thus we find that \(U\) is isomorphic to a closed subgroup of the reductive group \(G = \text{SL}(\mathbb{C} \oplus \mathfrak{u})\) of the form
\[
U = \left\{ \begin{pmatrix}
1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\
0 & 1 & p_{2,3}(\alpha_2, \ldots, \alpha_n) & \cdots & p_{2,n}(\alpha_2, \ldots, \alpha_n) \\
0 & 0 & 1 & \cdots & p_{3,n}(\alpha_2, \ldots, \beta_n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} : \alpha_2, \ldots, \alpha_n \in \mathbb{C} \right\}
\]
where \(n = 1 + \dim U\) and \(p_{i,j}(\alpha_2, \ldots, \alpha_n)\) is a polynomial in \(\alpha_2, \ldots, \alpha_n\) with complex coefficients for \(1 \leq i < j \leq n\). Our aim is to study linear actions of subgroups \(U\) and \(\hat{U}\) of \(\text{GL}(n)\) of this form (but with the embedding in \(\text{GL}(n)\) not necessarily induced by the adjoint action on the Lie algebra of \(\hat{U}\)) which extend to linear actions of \(\text{GL}(n)\) itself, by finding explicit affine embeddings of the quasi-affine varieties \(\text{GL}(n)/\hat{U}\).

**Remark 2.1.** Since \(\text{GL}(n)\) is the product of \(\text{SL}(n)\) and the subgroup \(\mathbb{C}^*\) in \(\hat{U}\) we have a natural identification of \(\text{GL}(n)/\hat{U}\) with \(\text{SL}(n)/(\hat{U} \cap \text{SL}(n))\). Also \(\hat{U} \cap \text{SL}(n)\) is the semidirect product of \(U\) with the finite group \(F = \mathbb{C}^* \cap \text{SL}(n)\). By [22] Corollary 2.8 \(\text{SL}(n)/U\) is quasi-affine variety (so \(U\) is observable subgroup of \(\text{SL}(n)\)); hence \(\text{GL}(n)/\hat{U} \simeq (\text{SL}(n)/U)/F\) is also quasi-affine.

3. **Grosshans subgroups**

Recall that a subgroup \(H\) of a linear algebraic group \(G\) over an algebraically closed field \(k\) is called a Grosshans subgroup of \(G\) if the algebra of invariants \(\mathcal{O}(X)^H\) is finitely generated for every affine \(G\)-variety \(X\). Recall also that \(H\) is an observable subgroup of \(G\) if
\[
H = \{ g \in G : f(gx) = f(x) \text{ for all } x \in G \text{ and } f \in \mathcal{O}(X)^H \}.
\]
Proposition 3.1 (Grosshans Criterion [22, 24]). Let \( G \) be a reductive group over an algebraically closed field \( k \), and \( H \) an observable subgroup of \( G \). Then the following conditions are equivalent:

1. \( H \) is a Grosshans subgroup of \( G \).
2. The algebra \( O(G)^H \) is finitely generated, where \( H \) acts on \( G \) via right translation.
3. There is a finite-dimensional \( G \)-module \( V \) and some \( v \in V \) such that \( H = G_v \) is the stabiliser of \( v \) (and therefore \( G/H \) is isomorphic to the orbit \( G \cdot v \) of \( v \) in \( V \)) and \( \dim(G \cdot v \setminus G \cdot v) \leq \dim(G \cdot v) - 2 \).

4. Embeddings in Grassmannians

Let \( U \) be a unipotent subgroup of the complex special linear group \( SL(n) \) and let \( \hat{U} = U \times \mathbb{C}^n \) be a subgroup of the complex general linear group \( GL(n) \) which is a \( \mathbb{C} \)-extension of \( U \) such that the weights of the \( \mathbb{C} \) action on \( \text{Lie}(U) \) are all strictly positive. Let us suppose also that \( U \) and \( \hat{U} \) are upper triangular subgroups of \( GL(n) \) which are generated along the first row; that is, there are integers \( 1 = \omega_1 < \omega_2 \leq \omega_3 \leq \cdots \leq \omega_n \) and polynomials \( p_{i,j}(\alpha_1, \ldots, \alpha_n) \) in \( \alpha_1, \ldots, \alpha_n \) with complex coefficients for \( 1 < i < j \leq n \) such that

\[
\hat{U} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & \alpha_1^{\omega_2} & p_{2,3}(\alpha) & \cdots & p_{2,n}(\alpha) \\ 0 & 0 & \alpha_1^{\omega_3} & \cdots & p_{3,n}(\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1^{\omega_n} \end{pmatrix} : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \times \mathbb{C}^{n-1} \right\}
\]

and

\[
U = \left\{ \begin{pmatrix} 1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & 1 & p_{2,3}(\alpha) & \cdots & p_{2,n}(\alpha) \\ 0 & 0 & 1 & \cdots & p_{3,n}(\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \alpha = (1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^{n-1} \right\}.
\]

This implies that the Lie algebra \( u = \text{Lie}(U) \) has a similar form:

\[
u = \left\{ \begin{pmatrix} 0 & a_2 & a_3 & \cdots & a_n \\ 0 & 0 & q_{2,3}(a) & \cdots & q_{2,n}(a) \\ 0 & 0 & 0 & \cdots & q_{3,n}(a) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a = (a_2, \ldots, a_n) \in \mathbb{C}^{n-1} \right\}
\]

where the \( q_{i,j} \) are linear forms in the parameters \( a = (a_2, \ldots, a_n) \in \mathbb{C}^{n-1} \) satisfying the following properties:

1. \( q_{i,j} = 0 \) for \( i \leq j \).
(ii) Let $\hat{\omega}_i = \omega_i - 1$ for $i = 1, \ldots, n$ be the weights of the adjoint action of the subgroup $C^*$ of $\hat{U}$ on $\hat{u} = \text{Lie} \hat{U}$, so that $\hat{\omega}_1 = 0$ and $\hat{\omega}_i > 0$ if $i = 2, \ldots, n$. Then the $C^*$-action makes $u = \text{Lie} U$ into a graded Lie algebra, and therefore

\[
q_{i,j}(a_2, \ldots, a_n) = \sum_{\ell, \omega_\ell + \omega_j = s_j} c_{ij}^\ell a_\ell
\]

for some structure coefficients $c_{ij}^\ell \in \mathbb{C}$.

**Proposition 4.1.** Let the weighted degree of $\alpha_s$ be $\deg(\alpha_s) = \omega_s$ for $1 \leq s \leq n$. Then

(i) the polynomial $p_{i,j}(\alpha)$ which is the $(i, j)$th entry of the element of $\hat{U}$ parametrised by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \times \mathbb{C}^{n-1}$ is homogeneous of degree $\omega_i$ in $\alpha_1, \ldots, \alpha_n$;

(ii) $p_{i,j}(\alpha)$ is weighted homogeneous of degree $\omega_j$ in $\alpha_1, \ldots, \alpha_n$.

**Proof.** The first (respectively second) part of the statement follows from the fact that $\hat{U}$ is closed under multiplication by its subgroup

\[
C^* = \left\{ \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_1^{\omega_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \alpha_1^{\omega_n} \end{pmatrix} : \alpha_1 \in \mathbb{C}^* \right\}
\]

on the left (respectively right). $\square$

**Remark 4.2.** It follows immediately from this proposition that if $j \geq i \geq 2$ then

\[p_{i,j}(\alpha) = p_{i,j}(\alpha_1, \ldots, \alpha_{j-1})\]

depends only on $\alpha_1, \ldots, \alpha_{j-1}$.

### 4.1. The construction.

For a vector of positive integers $\omega = (\omega_1, \ldots, \omega_n)$ we introduce the notation

\[\text{Sym}^\omega \mathbb{C}^n = \mathbb{C}^n \oplus \text{Sym}^{\omega_2}(\mathbb{C}^n) \oplus \cdots \oplus \text{Sym}^{\omega_n}(\mathbb{C}^n),\]

where $\text{Sym}^s(\mathbb{C}^n)$ is the $s$th symmetric power of $\mathbb{C}^n$. Any linear group action on $\mathbb{C}^n$ induces an action on $\text{Sym}^\omega \mathbb{C}^n$.

The most straightforward way to find an algebraic description of the quotient $\text{GL}(n)/\hat{U}$ is to find a $\text{GL}(n)$-module $W$ with a point $w \in W$ whose stabiliser is $\hat{U}$. Then the orbit $\text{GL}(n) \cdot w$ is isomorphic to $\text{GL}(n)/\hat{U}$ as a quasi-affine variety, and its closure $\text{GL}(n) \cdot w$ in $W$ is an affine completion of $\text{GL}(n)/\hat{U}$, while its closure in a projective completion of $W$ is a compactification of $\text{GL}(n)/\hat{U}$.

**Theorem 4.3.** Let $\hat{U} = U \ltimes \mathbb{C}^*$ be a $\mathbb{C}^*$ extension of a unipotent subgroup $U$ of $\text{SL}(n)$ with positive weights $1 = \omega_1 < \omega_2 \leq \cdots \leq \omega_n$ and a polynomial presentation. Fix a basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ and define

\[v_n = [e_1 \wedge (e_2 + e_1^{\omega_2}) \wedge \cdots \wedge (e_j + \sum_{i=2}^{j} p_{i,j}(e_1, \ldots, e_{j-1})) \wedge \cdots \wedge (e_n + \sum_{i=2}^{n} p_{i,n}(e_1, \ldots, e_n))]
\]
Then the stabiliser \( \text{GL}(n)_{v_n} \) of \( v_n \) in \( \text{GL}(n) \) is \( \hat{U} \).

**Corollary 4.4.** The map \( \phi_n : \text{GL}(n) \to \mathbb{P}[\wedge^n \text{Sym}^\omega \mathbb{C}^n] \) which sends a matrix with column vectors \( v_1, \ldots, v_n \) to the point

\[
(v_1, \ldots, v_n) \mapsto [v_1 \wedge (v_2 + v_1^{\omega_2}) \wedge \ldots \wedge (v_n + \sum_{i=2}^n p_i(v_1, \ldots, v_n))]
\]

is invariant under right multiplication of \( \hat{U} \) on \( \text{GL}(n) \) and \( \text{GL}(n) \)-equivariant with respect to left multiplication on \( \text{GL}(n) \) and the induced action on \( \mathbb{P}[\wedge^n \text{Sym}^\omega \mathbb{C}^n] \). It therefore defines a \( \text{GL}(n) \)-equivariant embedding

\[
\phi_n : \text{GL}(n)/\hat{U} \hookrightarrow \text{Grass}_n(\text{Sym}^\omega \mathbb{C}^n).
\]

**Remark 4.5.** Note that the image of the embedding \( \phi_n : \text{GL}(n) \to \mathbb{P}[\wedge^n \text{Sym}^\omega \mathbb{C}^n] \) lies in the open affine subset defined by the non-vanishing of the coordinate in \( \wedge^n \text{Sym}^\omega \mathbb{C}^n \) corresponding to the one-dimensional summand \( \wedge^n \mathbb{C}^n \) of \( \wedge^n \text{Sym}^\omega \mathbb{C}^n \) spanned by \( e_1 \wedge \ldots \wedge e_n \).

**Proof of Theorem 4.3** First we prove that \( \hat{U} \) is contained in the stabiliser \( \text{GL}(n)_{v_n} \). For \( (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \times \mathbb{C}^{n-1} \) let

\[
u(\alpha_1, \ldots, \alpha_n) = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \\
0 & \alpha_1^{\omega_2} & p_{2,3}(\alpha) & \ldots & p_{2,n}(\alpha) \\
0 & 0 & \alpha_1^{\omega_3} & \ldots & p_{3,n}(\alpha) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \alpha_1^{\omega_n}
\end{pmatrix} \in \hat{U}
\]

denote the element of \( \hat{U} \) determined by the parameters \( (\alpha_1, \ldots, \alpha_n) \) and for an \( n \)-tuple of vectors \( v = (v_1, \ldots, v_n) \in (\mathbb{C}^n)^{\omega_n} \) forming the columns of the \( n \times n \)-matrix \( A \in \text{GL}(n) \) we similarly define the matrix

\[
u(A) = \nu(v_1, \ldots, v_n) = \begin{pmatrix}
v_1 & v_2 & v_3 & \ldots & v_n \\
0 & v_1^{\omega_2} & p_{2,3}(v) & \ldots & p_{2,n}(v) \\
0 & 0 & v_1^{\omega_3} & \ldots & p_{3,n}(v) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & v_1^{\omega_n}
\end{pmatrix} \in M_{n \times n}(\text{Sym}^\omega \mathbb{C}^n)
\]

with entries in \( \text{Sym}^\omega \mathbb{C}^n \). Then the map \( \phi \) in (5) is the composition

\[
\phi(v_1, \ldots, v_n) = (u \circ \pi)(v_1, \ldots, v_n)
\]

where the rational map \( \pi : M_{n \times n}(\text{Sym}^\omega \mathbb{C}^n) \to \text{Grass}_n(\text{Sym}^\omega \mathbb{C}^n) \) restricts to a morphism on an open subset of \( M_{n \times n}(\text{Sym}^\omega \mathbb{C}^n) \) containing the image of \( u : \text{GL}(n) \to M_{n \times n}(\text{Sym}^\omega \mathbb{C}^n) \).
Now, since $\hat{U}$ is a group, the $(i, j)$ entry of the product of two elements is the polynomial $p_{i,j}$ in the entries of the first row of the product; that is,

$$u(\beta_1, \ldots, \beta_n)u(\alpha_1, \ldots, \alpha_n) = u(\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \sum_{m=1}^n p_{m,n}(\alpha_1, \ldots, \alpha_n)\beta_m)$$

for any $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$. This implies that

$$u(e_1, \ldots, e_n)u(\alpha_1, \ldots, \alpha_n) = u(\alpha_1e_1, \alpha_2e_2, \ldots, \sum_{m=1}^n p_{m,n}(\alpha_1, \ldots, \alpha_n)e_m)$$

where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{C}^n$. However, the $n$-tuple $(\alpha_1e_1, \alpha_2e_2 + \alpha_2e_1, \ldots, \sum_{m=1}^n p_{m,n}(\alpha_1, \ldots, \alpha_n)e_m) \in (\mathbb{C}^n)^\otimes n$ on the right hand side forms the columns of the matrix $u(\alpha_1, \ldots, \alpha_n)$, so we arrive at

$$(7) \quad u(e_1, \ldots, e_n)u(\alpha_1, \ldots, \alpha_n) = u(u(\alpha_1, \ldots, \alpha_n)\cdot e_1, \ldots, u(\alpha_1, \ldots, \alpha_n)\cdot e_n).$$

Since $u(\alpha_1, \ldots, \alpha_n)$ lies in the standard Borel subgroup $B_n$ of $GL(n)$, the matrices $u(e_1, \ldots, e_n)$ and $u(e_1, \ldots, e_n)u(\alpha_1, \ldots, \alpha_n)$ represent the same element in $Grass_n(Sym^n \mathbb{C}^n)$; that is, in $Grass_n(Sym^n \mathbb{C}^n)$ we have

$$p_n = \pi(u(e_1, \ldots, e_n)) = \pi(u(\alpha_1, \ldots, \alpha_n)) = \pi(u(\alpha_1, \ldots, \alpha_n)\cdot e_1, \ldots, u(\alpha_1, \ldots, \alpha_n)\cdot e_n)$$

which completes the proof that $\hat{U} \subseteq GL(n)_{p_n}$.

It remains to prove that $GL(n)_{p_n} \subseteq \hat{U}$. Suppose that $g = (g_{ij})_{i,j=1}^n \in GL(n)_{p_n}$; we want to show that $g \in \hat{U}$. For $1 \leq m \leq n$ let

$$g^{\otimes m} = (g_{ij})_{i,j=1}^m \in GL(m)$$

be the upper left $m \times m$ block of $g$. Recall that by Remark 4.2 if $j \geq i \geq 2$ then $p_{i,j}(\alpha_1, \ldots, \alpha_n) = p_{i,j}(\alpha_1, \ldots, \alpha_{j-1})$ depends only on $\alpha_1, \ldots, \alpha_{j-1}$. We will prove by induction on $m$ that

$$g^{\otimes m} = u(g_{11}, g_{12}, \ldots, g_{1m})$$

This is clear for $m = 1$ since $g^{\otimes 1} = (g_{11}) = u(g_{11})$. Suppose that it is true for some $m < n$. Since $g \in GL(n)_{p_n}$ the Plücker coordinates

$$e_1 \wedge (e_2 + e_1^{\otimes 2}) \wedge \ldots \wedge \sum_{i=1}^n p_{i,n}(e_1, \ldots, e_n)$$

of $p_n$ agree up to multiplication by a nonzero scalar with the Plücker coordinates

$$ge_1 \wedge (ge_2 + ge_1^{\otimes 2}) \wedge \ldots \wedge \sum_{i=1}^n p_{i,n}(ge_1, \ldots, ge_n)$$
of \( g \nu_n \), where \( g e_j = \sum_{i=1}^n g_{ij} e_i \) and \( p_i, j(g e_1, \ldots, g e_n) \in \text{Sym}^{(\mathbb{C}^n) \subseteq \text{Sym}^{(\mathbb{C}^n)} \subseteq \text{Sym}^{(\mathbb{C}^n)}} \). By the inductive hypothesis we have

\[
 g_{ij} = p_{i, j}(g_{11}, \ldots, g_{1j})
\]

for \( 1 \leq i \leq m \) and \( 1 \leq j \leq m \), so with our previous notation

\[
 g^{m} = u(g_{11}, \ldots, g_{1m}) \in \hat{U}
\]

holds, and therefore \( g^{m} \) fixes \( \nu_m \); thus

\[
 \nu_m = \pi(\mu(e_1, \ldots, e_m)) = \pi(u(g^{m} e_1, \ldots, g^{m} e_m)).
\]

In coordinates this means that

\[
 \cdot e_1 \wedge (e_2 + e_1^{(\omega)}) \wedge \ldots \wedge \sum_{i=1}^n p_i, m(e_1, \ldots, e_m)
\]

agrees up to multiplication by a nonzero scalar with

\[
 g e_1 \wedge (g e_2 + g e_1^{(\omega)}) \wedge \ldots \wedge \sum_{i=1}^n p_i, m(g e_1, \ldots, g e_m).
\]

Therefore

\[
 e_1 \wedge (e_2 + e_1^{(\omega)}) \wedge \ldots \wedge \sum_{i=1}^n p_i, m(e_1, \ldots, e_m) \wedge \sum_{i=1}^{m+1} p_{i, m+1}(e_1, \ldots, e_{m+1}) \wedge \ldots \wedge \sum_{i=1}^n p_{i, n}(e_1, \ldots, e_n)
\]

and

\[
 e_1 \wedge (e_2 + e_1^{(\omega)}) \wedge \ldots \wedge \sum_{i=1}^n p_i, m(e_1, \ldots, e_m) \wedge \sum_{i=1}^{m+1} p_{i, m+1}(g e_1, \ldots, g e_{m+1}) \wedge \ldots \wedge \sum_{i=1}^n p_{i, n}(g e_1, \ldots, g e_n)
\]

agree up to multiplication by a nonzero scalar. Applying the identification

\[
 (\bigwedge_{i=1}^n V_i) = \bigoplus_{p_1 + \ldots + p_n = n} (\wedge^{p_1} V_1 \otimes \ldots \otimes \wedge^{p_n} V_n),
\]

with \( V_1 = \bigwedge_{i=1}^{m+1} (\mathbb{C}^n \oplus \text{Sym}^{(\mathbb{C}^n)} \oplus \cdots \oplus \text{Sym}^{(m+1)} \mathbb{C}^n) \) and

\[
 V_{n-m} = \text{Sym}^{(n-m)} \mathbb{C}^n
\]

we get a natural GL(n)-equivariant projection to the direct summand corresponding to \( p_1 = m + 1, p_2 = \ldots = p_{n-m} = 1 \) given by

\[
 \pi : \bigwedge_{i=1}^n \text{Sym}^{(\mathbb{C}^n)} \to \bigwedge_{i=1}^{m+1} (\mathbb{C}^n \oplus \text{Sym}^{(\mathbb{C}^n)} \oplus \cdots \oplus \text{Sym}^{(m+1)} \mathbb{C}^n) \otimes \text{Sym}^{(m+2)} \otimes \cdots \otimes \text{Sym}^{(n)} \mathbb{C}^n
\]

which takes \( e_1 \wedge (e_2 + e_1^{(\omega)}) \wedge \ldots \wedge \sum_{i=1}^n p_{i, n}(e_1, \ldots, e_n) \) to

\[
 e_1 \wedge (e_2 + e_1^{(\omega)}) \wedge \ldots \wedge \sum_{i=1}^n p_i, m(e_1, \ldots, e_m) \wedge \sum_{i=1}^{m+1} p_{i, m+1}(e_1, \ldots, e_{m+1}) \otimes e_1^{(m+2)} \otimes \cdots \otimes e_1^{(n)}.
\]
This must agree up to multiplication by a nonzero scalar with the projection
\[
\pi \left( g e_1 \wedge (g e_2 + g e_1^{\omega_2}) \wedge \ldots \wedge \sum_{i=1}^{m} p_{i,m}(e_1, \ldots, e_m) \wedge \sum_{i=1}^{n} p_{i,n}(g e_1, \ldots, g e_n) \right) =
\]
\[
e_1 \wedge (e_2 + e_1^{\omega_2}) \wedge \ldots \wedge \sum_{i=1}^{m+1} p_{i,m+1}(g e_1, \ldots, g e_{m+1}) \otimes q_{m+2} \otimes \ldots \otimes q_n
\]
for some \( q_j \in \text{Sym}^{\omega_j} \mathbb{C}^n \) for \( m + 2 \leq j \leq n \). It follows from this that
\[
\lambda e_1 \wedge (e_2 + e_1^{\omega_2}) \wedge \ldots \wedge \sum_{i=1}^{m+1} p_{i,m+1}(e_1, \ldots, e_{m+1}) =
\]
\[
e_1 \wedge (e_2 + e_1^{\omega_2}) \wedge \ldots \wedge \sum_{i=1}^{m+1} p_{i,m+1}(g e_1, \ldots, g e_{m+1}),
\]
for some nonzero scalar \( \lambda \).

Now, \( g^{<m} = u(g_{11}, \ldots, g_{1m}) \) and therefore by (7)
\[
u(g^{<m} e_1, \ldots, g^{<m} e_m) = \nu(e_1, \ldots, e_n) \cdot u(g_{11}, \ldots, g_{1m}).
\]
But if \( m + 1 \geq i \geq 2 \) then \( p_{i,m+1}(\alpha_1, \ldots, \alpha_m) \) is a polynomial in \( \alpha_1, \ldots, \alpha_m \), and does not depend on \( \alpha_{m+1}, \ldots, \alpha_n \). Therefore
\[
p_{i,m+1}(g e_1, \ldots, g e_{m+1}) = g e_{m+1} = \sum_{i=1}^{n} g_{i,m+1} e_i.
\]
Substituting this into (9) we arrive at the equation
\[
\lambda \cdot \left( e_1 \wedge (e_2 + e_1^{\omega_2}) \wedge \ldots \wedge \sum_{i=1}^{m+1} p_{i,m+1}(e_1, \ldots, e_{m+1}) \right) =
\]
\[
e_1 \wedge (e_2 + e_1^{\omega_2}) \wedge \ldots \wedge \left( \sum_{i=2}^{m+1} \sum_{j=1}^{n} p_{i,m+1}(g_{11}, \ldots, g_{1m+1}) p_{i,j}(e_1, \ldots, e_{m+1}) + \sum_{j=2}^{n} g_{s,m+1} e_i \right).
\]

There is another \( \text{GL}(n) \)-equivariant projection to the direct summand corresponding to \( V_1 = \text{Sym}^{\omega_1} \mathbb{C}^n \) and \( p_1 = 2, p_2 = \ldots = p_m = 1 \) in (8), given by
\[
\rho : \bigwedge^{m+1} (\mathbb{C}^n \oplus \text{Sym}^{\omega_1} \mathbb{C}^n \oplus \ldots \oplus \text{Sym}^{\omega_m} \mathbb{C}^n) \to \bigwedge^2 \mathbb{C}^n \otimes \text{Sym}^{\omega_2} \mathbb{C}^n \otimes \ldots \otimes \text{Sym}^{\omega_m} \mathbb{C}^n
\]
which takes the left hand side of (11) to
\[
\lambda (e_1 \wedge e_{m+1}) \otimes e_1^{\omega_2} \otimes \ldots \otimes e_1^{\omega_m}.
\]
and the right hand side to
\[(e_1 \wedge (\Sigma_{i=2}^{m}(p_{s,m+1}(g_{11}, \ldots, g_{1,m+1}) - g_{s,m+1})e_1 + g_{m+1,m+1}e_{m+1})) \otimes e_1^{\omega_1} \otimes \ldots \otimes e_1^{\omega_m}.\]
These two are equal, so we obtain
\[g_{s,m+1} = p_{s,m+1}(g_{11}, \ldots, g_{1,m+1}) \text{ for } s \neq 1, m + 1 \text{ and } \lambda = g_{m+1,m+1}.\]

Note that the right hand side of (11) is independent of \(b_{1,m+1}\), which can be chosen arbitrarily, as we expect. Finally, for \(s = m + 1\), we take the third \(GL(n)\)-equivariant projection corresponding to \(V_i = \text{Sym}^\omega \mathbb{C}^n\) and \(p_1 = \ldots = p_n = 1\) in (8), given by
\[g_{m+1,m+1} = p_{m+1,m+1}(b_{11}, \ldots, b_{1,m+1}).\]
and Theorem 4.3 is proved.

4.2. Changing the basis of \(\mathfrak{u}\). We observed in Proposition 4.1 that the left-right multiplication action of the subgroup \(C^*\) of \(\hat{U}\) implies that the polynomial entry \(p_{i,j}(\alpha)\) of an element of \(\hat{U}\) with parameters \(\alpha\) in the first row has degree \(i\) and weighted degree \(\omega_j\) in \(\alpha\). Similarly we have a bigrading on \(\text{Sym}^\omega \mathbb{C}^n\) as follows: the Lie algebra \(\mathfrak{u} = \text{Lie}(U)\) decomposes into eigenspaces for the adjoint action of \(\text{Lie}(\mathbb{C}^*) = \mathbb{C}z = \mathfrak{u}_i\) as
\[\mathfrak{u} = \bigoplus_{i \neq 1} \mathfrak{u}_i,\]
where \(z \in \mathfrak{u}_1 \setminus \{0\}\) and
\[\mathfrak{u}_i = \{x \in \mathfrak{u} : [x, z] = (\tilde{\omega}_i - 1)x\}\]
if \(\tilde{\omega}_1, \ldots, \tilde{\omega}_r\) are the different weights among \(\omega_1, \ldots, \omega_n\). This induces a decomposition
\[\text{Sym}^\omega \mathbb{C}^n = \mathbb{C}^{n} \oplus \mathfrak{u}_2 \otimes \text{Sym}^{\omega_2} \mathbb{C}^n \oplus \ldots \oplus (\mathfrak{u}_r \otimes \text{Sym}^{\omega_r} \mathbb{C}^n)\]
of \(\text{Sym}^\omega \mathbb{C}^n\). Let \(\text{Sym}_A^\omega \mathbb{C}^n = \bigoplus_{\omega_1 + \ldots + \omega_n = b} (\mathfrak{u}_i \ldots \mathfrak{u}_k) \subseteq \text{Sym}^\omega \mathbb{C}^n\) and define
\[\text{Sym}_A^\omega \mathbb{C}^n = \bigoplus_{i,j=1}^r (\mathfrak{u}_i \otimes \mathfrak{u}_j \otimes \text{Sym}^{\tilde{\omega}_j} \mathbb{C}^n).\]
The image of the embedding \(\phi_n\) of \(GL(n)/\hat{U}\) sits in the subset \(\text{Grass}_n(\text{Sym}_A^\omega \mathbb{C}^n)\) of \(\text{Grass}_n(\text{Sym}^\omega \mathbb{C}^n)\), and the group
\[\widetilde{GL}(\mathfrak{u}) = C^* \times GL(\mathfrak{u}_2) \times \ldots \times GL(\mathfrak{u}_r) \subset GL(\mathfrak{u})\]
acts on \(\text{Sym}_A^\omega \mathbb{C}^n\) via conjugation and thus on \(\text{Grass}(n, \text{Sym}_A^\omega \mathbb{C}^n)\). If \(g \in \widetilde{GL}(\mathfrak{u})\) then the subgroup
\[g^{-1} \hat{U} g\]
of $\text{GL}(n)$ with Lie subalgebra $g^{-1}u g$ has the same form as $\hat{U}$ and so we can compare the corresponding embeddings $\phi_n$ of $\text{GL}(n)/\hat{U}$ and $\text{GL}(n)/g^{-1}\hat{U} g$ in Grass$(n, \text{Sym}^\omega C^n)$; let us denote these by $\phi_{\hat{U}}$ and $\phi_{g^{-1}\hat{U} g}$. The linear forms in the first row of $g^{-1}u g$ (and the same linear forms in the first row of $g^{-1}\hat{U} g$) are linearly independent, and give parameters $b_1, \ldots, b_n$ for the group and its Lie algebra. The corresponding embedding is then $\phi_{g^{-1}\hat{U} g}$, and we have

**Proposition 4.6.** A linear change of basis of $\hat{u}$ by any element of $\tilde{\text{GL}}(u)$ does not change the closure of the image of the embedding $\phi_{\hat{U}}$ of $\text{GL}(n)/\hat{U}$ into the Grassmannian Grass$(n, \text{Sym}^\omega C^n)$ up to isomorphism.

**Proof.** This follows from the commutativity of the diagram

$$
\begin{array}{ccc}
\text{GL}(n)/\hat{U} & \xrightarrow{\phi_{\hat{U}}} & \text{Grass}(n, \text{Sym}^\omega C^n) \\
\downarrow{\text{conj}(g)} & & \downarrow{\text{conj}(g) \circ (g_{11} \cdot g^{-1})} \\
\text{GL}(n)/g^{-1}\hat{U} g & \xrightarrow{\phi_{g^{-1}\hat{U} g}} & \text{Grass}(n, \text{Sym}^\omega C^n),
\end{array}
$$

where

1. the left vertical $\text{conj}(g)$ is the conjugation action sending the coset $\hat{U} h \in \text{GL}(n)/\hat{U}$ to $(g^{-1}\hat{U} g)(g^{-1} h g) = g^{-1}\hat{U} h g \in \text{GL}(n)/g^{-1}\hat{U} g$;
2. the right vertical map is the composition of the multiplication by the scalar $g_{11}$ and the matrix $g^{-1}$ on $C^n$, and conjugation with $g \in \text{GL}(u)$ on $\text{Sym}^\omega C^n$.

\[\square\]

5. **Singularities, jet differentials and curvilinear Hilbert schemes**

Following the Grosshans principle (see §3), we are looking for an embedding of $\text{GL}(n)/\hat{U}$ in an affine space $W$, where the boundary components have codimension at least 2. In this section we will study an important example of a group of the form $\hat{U}$ and its projective embedding $\phi_{\hat{U}} : \text{GL}(n)/\hat{U} \hookrightarrow \text{Grass}_n(\text{Sym}^\omega C^n)$ given by Theorem 4.3 whose image is contained in the affine open subset of the Grassmannian Grass$_n(\text{Sym}^\omega C^n)$ where the coordinate corresponding to $\wedge^n C^n$ is nonzero. Here the codimension-2 property does not hold, which means that our affine embedding will need to be modified; this will be done in §6.

The example we will study in this section is given by $\hat{U} = \mathcal{G}_n \leq \text{GL}(n)$, where as in the introduction $\mathcal{G}_n$ is the group of polynomial reparametrisations of $n$-jets of holomorphic germs $(C, 0) \rightarrow (C, 0)$. This group plays a central role in global singularity theory [2] and in the recent history of hyperbolic varieties [10, 11, 26, 46]. We will see that the compactification $\text{GL}(n)/\mathcal{G}_n$ constructed in §4 as the closure of an orbit of $\text{GL}(n)$
with stabiliser $\mathbb{G}_n$ in a Grassmannian Grass$_s(Sym^n\mathbb{C}^n)$ is isomorphic to the so-called curvilinear component of the punctual Hilbert scheme on $\mathbb{C}^n$ \cite{3, 7}.

5.1. **Singularity theory in a nutshell** \cite{2, 3, 6, 19, 29, 37, 41}. Let $J_n(m, l)$ denote the space of $n$-jets of holomorphic map germs from $\mathbb{C}^m$ to $\mathbb{C}$ mapping the origin to the origin. This is a finite dimensional complex vector space, and there is a complex linear composition of jets

$$J_n(m, l) \otimes J_n(l, p) \to J_n(m, p).$$

Let $J^{reg}_n(m, l)$ denote the open dense subset of $J_n(m, l)$ consisting of jets whose linear part is regular (that is, of maximal rank). Note that

$$\mathbb{G}_n = J^{reg}_n(1, 1)$$

becomes a group under composition of jets, and it acts via reparametrisation on $J_n(1, n)$.

If $z$ denotes the standard complex coordinate on $\mathbb{C}$, then elements of the vector space $J_n(1, 1)$ can be identified with polynomials of the form $p(z) = \alpha_1 z + \ldots + \alpha_n z^n$ with coefficients in $\mathbb{C}$, so $\{z, z^2, \ldots, z^n\}$ is a natural basis for $J_n(1, 1)$ over $\mathbb{C}$. The composition of $p(z)$ with $q(z) = \beta_1 z + \ldots + \beta_n z^n$ is

$$(p \circ q)(z) = (\alpha_1 \beta_1)z + (\alpha_2 \beta_1 + \alpha_1^2 \beta_2)z^2 + \ldots$$

which corresponds (with respect to the basis $\{z, z^2, \ldots, z^n\}$) to multiplication on the right by the matrix

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \\
0 & \alpha_1^2 & 2\alpha_1 \alpha_2 & \ldots & 2\alpha_1 \alpha_{n-1} + \ldots \\
0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2 \alpha_{n-2} + \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \alpha_1^n \\
\end{pmatrix}
$$

(14)

where the polynomial in the $(i, j)$ entry is

$$p_{i,j}(\alpha_1, \ldots, \alpha_n) = \sum_{\ell_1 + \ell_2 + \ldots + \ell_i = j} \alpha_{\ell_1} \alpha_{\ell_2} \ldots \alpha_{\ell_i}.$$

Thus the subgroup $\mathbb{G}_n$ of $\text{GL}(n)$ is an extension by $\mathbb{C}^*$ of its unipotent radical $\mathbb{U}_n$, and both $\mathbb{G}_n$ and $\mathbb{U}_n$ are generated along the first row and have the form (2) with weights $1, 2, \ldots, n$. We can think of the quotient $J^p(1, n)/\mathbb{G}_n$ as the moduli space of $n$-jets of entire holomorphic curves in $\mathbb{C}^n$.

Global singularity theory studies global and local behavior of singularities of holomorphic maps between complex manifolds; \cite{2} is a standard reference. For a holomorphic map $f : M \to N$ with $f(p) = q \in N$ the local algebra is $A(f) = m_p/f^* m_q$; if $m_p$ is a finite $m_q$-module, then $p$ is an isolated singularity. For a complex nilpotent algebra $A$ with $\dim_{\mathbb{C}} A = n$ we define

$$\Sigma_A(m, l) = \{ f \in J_n(m, l) : A(f) \cong A \}$$
to be the subset of $J_n(m,l)$ consisting of germs with local algebra at the origin isomorphic to $A$; these are known as the $A$-singularity germs. There is a natural hierarchy of singularities where for two algebras $A$ and $A'$ of the same dimension $n$ we have

$$A > A' \text{ if } \Sigma_A(m,l) \subset \Sigma_{A'}(m,l) \text{ for } l \gg m.$$ 

When $A_n = \mathbb{Z}[z]/z^{n+1}$ is the nilpotent algebra generated by one element, the corresponding singularities are the so-called $A_n$-singularities (also known as Morin singularities or curvilinear singularities). These vanish to order $n$ in some direction, giving us the geometric description

$$\Sigma_{A_n}(m,l) = \{ \psi \in J_n(m,l) : \exists \gamma \in J_n(1,m) \text{ such that } \gamma \circ \psi = 0 \}.$$ 

If $\psi \in J_n(m,l)$ and a test curve $\gamma_0 \in J_n(1,m)$ exists with $\gamma_0 \circ \psi = 0$, then there is a whole family of such test curves. Indeed, for any $\beta \in J_n^{\text{reg}}(1,1)$, the curve $\beta \circ \gamma_0$ is also a test curve, and in fact if $\psi \in J_n^{\text{reg}}(m,l)$ then we get all test curves $\gamma \in J_n(1,m)$ with $\gamma \circ \psi = 0$ in this way. This description of the curvilinear jets using the so-called ‘test-curve model’ goes back to Porteous, Ronga and Gaffney \cite{19,37,41}.

This means that the regular part of $\Sigma_{A_n}(m,l)$ fibres over the quotient $J_n^{\text{reg}}(1,m)/G_n$, which can be thought of as representing moduli of $n$-jets of holomorphic germs in $\mathbb{C}^m$. We can identify $J_n(1,m)$ with the set $M_{mxn}(\mathbb{C})$ of $m \times n$ complex matrices by putting the $i$th derivative of $\gamma \in J_n(1,m)$ into the $i$th column of the corresponding matrix, and then $J_n^{\text{reg}}(1,m)$ consists of the matrices in $M_{mxn}(\mathbb{C})$ with nonzero first column. Therefore when $m = n$ the quotient $J_n^{\text{reg}}(1,n)/G_n$ contains $GL(n)/G_n$ as a dense open subset.

In \cite{6} the first author and Szenes use this model of the Morin singularities and the machinery of equivariant localization to compute some useful invariants of $A_n$ singularities: their Thom polynomials. These ideas were later generalised in \cite{25,40}.

The hierarchy of singularities is only partially understood, but there are well-known singularity classes in the closure of the $A_n$-singularities (for details see \cite{2,39}). In particular, for $n = 4$, the so called $I_{a,b}$ singularities with $a + b = 4$ are defined by the algebra

$$A_{L,b} = (x,y)/(xy, x^a + y^b)$$

and it is well known (see \cite{39,40}) that

$$\Sigma_{I_{a,b}}(m,l) \subset \Sigma_{A_{L,b}}(m,l)$$

has codimension 1 in $\Sigma_{A}(m,l)$. But as we have just seen, a dense open subset of $\Sigma_{A_{L,b}}(4,l)$ fibres over $GL(4)/G_4$, and the latter is embedded via $\phi_4$ (see Corollary 4.4) into Grass$_4$(Sym$^n \mathbb{C}^m$) where $\omega = (1,2,3,4)$ as at \cite{14}. When $l = 1$, then in fact

$$\Sigma_{A_{L,b}}(4,1) = \phi_4(GL(4) \subset \text{Grass}_4(\text{Sym}^n \mathbb{C}^m),$$
because the fibres are trivial. So it follows that $\Sigma_{1,2}(4, 1)$ lies in the boundary of $\overline{\phi_4(\text{GL}(4))}$ and has codimension one. In fact

$$p_{2,2} = \lim_{t \to 0} \begin{pmatrix} t & t^{-2} & -t^{-5} & 0 \\ 0 & 1 & -2t^{-3} & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot p_n = e_1 \wedge e_2 \wedge (e_3 + e_1^2) \wedge (e_4 + e_1 e_3 + e_2^2 + e_1^3)$$

sits in $\Sigma_{1,2}(4, 1)$ and its orbit has codimension 1 in $\overline{\phi_4(\text{GL}(4))}$. Indeed it can be checked by direct computation that the stabiliser of $p_{2,2}$ is

$$\left\{ \begin{pmatrix} t & a & b & c \\ 0 & t^{3/2} & -2t^{3/2}a & d \\ 0 & 0 & t^2 & tb + a^2 \\ 0 & 0 & 0 & t^3 \end{pmatrix} : t \in \mathbb{C}^*, a, b, c, d \in \mathbb{C} \right\}$$

which has dimension 5, whereas the stabiliser $\mathcal{O}_{26}$ of $p_4$ in $\text{GL}(4)$ has dimension 4.

5.2. Invariant jet differentials and the Demailly bundle. Jet differentials have played a central role in the study of hyperbolic varieties. Their contribution can be traced back to the work of Bloch [8], Cartan [9], Ahlfors [11], Green and Griffiths [13], Siu [46], whose ideas were extended in the seminal paper of Demailly [10], and recently used by Diverio, Merker and Rousseau [11] and the first author in [4] to prove the Green Griffiths conjecture for generic projective hypersurfaces of high order; see also the survey papers [26, 10, 12] for more details.

Let

$$f : \mathbb{C} \to X, \ t \to f(t) = (f_1(t), f_2(t), \ldots, f_d(t))$$

be a curve written in local holomorphic coordinates $(z_1, \ldots, z_d)$ on a complex manifold $X$, where $d = \dim(X)$. Let $J_n(X)$ be the $n$-jet bundle over $X$ of holomorphic curves, whose fibre $(J_n(X))_x$ at $x \in X$ is the space of $n$-jets of germs at $x$ of holomorphic curves in $X$. This fibre can be identified with $J_n(1, d)$. The group of reparametrisations $\mathcal{O}_n = J_n^{\text{reg}}(1, 1)$ acts fibrewise on $J_n(X)$, and the action is linearised as at (14). For $\lambda \in \mathbb{C}^*$ we have

$$(\lambda \cdot f)(t) = f(\lambda \cdot t), \ \text{so} \ \lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Polynomial functions on $J_n(X)$ correspond to algebraic differential operators called jet differentials; these have the form

$$Q(f', f'', \ldots, f^{(k)}) = \sum_{a_1, a_2, \ldots, a_n, \in \mathbb{N}^n} a_{a_1, a_2, \ldots, a_n} \left( f(t) \right)^{a_1} \left( f'(t) \right)^{a_2} \cdots \left( f^{(n)}(t) \right)^{a_n},$$

where $a_{a_1, a_2, \ldots, a_n}(z)$ are holomorphic coefficients on $X$ and $t \mapsto f(t)$ is the germ of a holomorphic curve in $X$. Here $Q$ is homogeneous of weighted degree $m$ under the $\mathbb{C}^*$ action if and only if

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)})$$

for every $\lambda \in \mathbb{C}$.
Definition 5.1. (i) (Green-Griffiths [13]) Let \( E_{n,m}^{GG} \) denote the sheaf on \( X \) of jet differentials of order \( n \) and weighted degree \( m \).

(ii) (Demailly, [10]) The bundle of invariant jet differentials of order \( n \) and weighted degree \( m \) is the subbundle \( E_{n,m}^{GG} \) of \( E_{n,m}^{GG} \) whose elements are invariant under the action of the unipotent radical \( U_n \) of the reparametrisation group \( \mathbb{G}_n \) and transform under the action of \( \mathbb{G}_n \) as

\[
Q((f \circ \phi'), (f \circ \phi''), \ldots, (f \circ \phi)^{(n)}) = \phi'(0)^m Q(f', f'', \ldots, f^{(n)})
\]

for \( \phi \in \mathbb{G}_n \).

Thus the fibres of the Demailly bundle \( \bigoplus_{m \geq 0} E_{n,m} \) are isomorphic to \( \mathbb{C}[J_n(1, d)]^{\mathbb{G}_n} \), where \( U_n \) is the unipotent radical of \( \mathbb{G}_n \). Demailly in [10] conjectured that this algebra of invariant jet differentials is finitely generated (Conjecture [11] above). This will follow from Theorem [12].

5.3. Curvilinear Hilbert schemes. In this subsection we identify the closure \( J_n(1, d)/\mathbb{G}_n \) of \( J_n(1, d)/\mathbb{G}_n \) embedded in \( \text{Grass}_n(\oplus_{i=1}^n \text{Sym}^i \mathbb{C}^d) \) with the curvilinear component of the \( n \)-point punctual Hilbert scheme on \( \mathbb{C}^d \); this geometric component of the punctual Hilbert scheme on \( \mathbb{C}^d \) is thus the compactification of a non-reductive quotient.

Hilbert schemes of points on surfaces form a central object of geometry and representation theory and have a huge literature (see for example [34, 7]). Recently many interesting connections between Hilbert schemes of points on planar curve singularities and the topology of their links have been discovered [43, 44, 45, 30]. However, much less is known about Hilbert schemes or punctual Hilbert schemes on higher dimensional manifolds.

As above let \( \mathcal{G}_n = J_n^{\text{reg}}(1, 1) \) denote the group of \( n \)-jets of reparametrisation germs of \( \mathbb{C} \), which acts on the space \( J_n^{\text{reg}}(1, d) \) of \( n \)-jets of germs of curves \( f : (\mathbb{C}, 0) \to (\mathbb{C}^d, 0) \) with nonzero linear part. As in §4 we have a map

\[
\phi : J_n^{\text{reg}}(1, d) \to \text{Grass}_n(\oplus_{i=1}^n \text{Sym}^i \mathbb{C}^d),
\]

\[
(v_1, \ldots, v_n) \mapsto [v_1 \wedge (v_2 + v_2^2) \wedge \ldots \wedge (\sum_{a_1 + a_2 + \ldots + a_i = n} v_{a_1} v_{a_2} \ldots v_{a_i})]
\]

where \( v_i \in \mathbb{C}^d \) is the degree \( i \) part of the germ in \( J_n^{\text{reg}}(1, d) \), so that \( v_1 \neq 0 \). This map is invariant under the action of \( \mathcal{G}_n = J_n^{\text{reg}}(1, 1) \) on the left, and gives us an embedding

\[
J_n^{\text{reg}}(1, d)/\mathcal{G}_n \hookrightarrow \text{Grass}_n(\oplus_{i=1}^n \text{Sym}^i \mathbb{C}^d).
\]

Let us denote the closure of the image of this embedding by \( X_{n,d} \), so that

\[
X_{n,d} = J_n^{\text{reg}}(1, d)/\mathcal{G}_n.
\]

In fact \( X_{n,d} \) is the curvilinear component of the punctual Hilbert scheme of \( n \) points on \( \mathbb{C}^d \). Let \( (\mathbb{C}^d)^{[n]} \) denote the Hilbert scheme of \( n \) points on \( \mathbb{C}^d \); that is, the set of zero-dimensional subschemes of \( \mathbb{C}^d \) of length \( n \). The punctual Hilbert scheme \( (\mathbb{C}^d)^{[n]}_0 \) consists of those zero-dimensional subschemes of \( \mathbb{C}^d \) of length \( n \) which are supported at the
When \( d \geq 3 \) the punctual Hilbert scheme is highly non-reduced and its irreducible components are not known in general. The geometric (also called curvilinear) component of \( (\mathbb{C}^d)_0^{[n]} \) is its irreducible component which contains the curvilinear subschemes, defined as follows.

**Definition 5.2.** Let \( m = (x_1, \ldots, x_d) \subset O_{\mathbb{C}^d,0} \) denote the maximal ideal of the local ring at the origin. The punctual curvilinear Hilbert scheme is defined as the closure of the set of curvilinear subschemes; that is, those which vanish on a curve up to order \( n \):

\[
C_d^{[n]} = \overline{\{ I \subset m : m/I \cong t\mathbb{C}[t]/t^{n+1} \}}.
\]

Since \( \text{Sym}^{\leq n}\mathbb{C}^d = m/m^{n+1} = \bigoplus_{i=1}^n \text{Sym}^i\mathbb{C}^d \) consists of function-germs of degree \( \leq n \), the punctual Hilbert scheme sits naturally in its Grassmannian \( \rho : (\mathbb{C}^d)_0^{[n]} \hookrightarrow \text{Grass}(n, \text{Sym}^{\leq n}\mathbb{C}^d) \)

\[ I \mapsto m/I. \]

Curvilinear subschemes have test curves; that is, map germs \( \gamma \in J^{\leq n}(1,d) \) on which they vanish up to order \( n \), so that \( \gamma(\mathbb{C}) \subseteq \text{Spec}(m/I) \). Such a test curve is unique up to polynomial reparametrisation of \( (\mathbb{C},0) \). Therefore the image of \( \phi \) is the same as the image of \( \rho \), so they have the same closure \( X_{n,d} \). Thus we have

**Proposition 5.3.** For \( d, n \in \mathbb{Z}^>0 \)

\[
C_d^{[n]} = X_{n,d} = \phi(J^{\text{reg}}_n(1,d)/\mathbb{C}_n).
\]

When \( d = 2 \) the curvilinear component \( C_2^{[n]} \) is dense in \( (\mathbb{C}^2)_0^{[n]} \), and therefore the full punctual Hilbert scheme is equal to the closure of the image of \( \phi \).

**Corollary 5.4.** \( (\mathbb{C}^2)_0^{[n]} = X_{n,2} \) for any \( n \in \mathbb{Z}^>0 \).

This description of the curvilinear component becomes particularly useful when \( n \leq d \) so that the number of points is not more than the dimension \( d \). In this case, the curvilinear component \( C_d^{[n]} \) is the closure of a \( \text{GL}(n) \)-orbit in the Grassmannian \( \text{Grass}_n(\text{Sym}^{\leq n}\mathbb{C}^d) \).

In fact, for any fixed basis \( \{e_1, \ldots, e_d\} \) of \( \mathbb{C}^d \), we have

\[
X_{n,d} = \text{GL}(n) \cdot e_{n,d}
\]

where

\[
e_{n,d} = e_1 \wedge (e_2 \oplus e_1^2) \wedge \ldots \wedge (\sum_{a_1+\ldots+a_l=n, l \leq d} e_{a_1} \ldots e_{a_l}).
\]

This follows when \( n \leq d \) from the fact that \( \phi \) is \( \text{GL}(n) \)-equivariant, but for \( n > d \) it cannot be true as the dimension of the quotient is larger than the dimension of \( \text{GL}(n) \).

In particular, when \( d = n \) we have \( \text{GL}(n) \subseteq J^{\text{reg}}_n(1,n) \), and an embedding

\[
\text{GL}(n)/\mathbb{C}_n \subseteq \text{Grass}_n(\text{Sym}^{\leq n}\mathbb{C}^d)
\]
and the closure of the image $X_{n,d} = C^{[n]}_d$ is the curvilinear component of the punctual Hilbert scheme of $n$ points on $\mathbb{C}^d$. Moreover the case when $d < n$ can be reduced to the case when $n = d$ using the following identifications:

$$J^ {\text{reg}}_n(1,n)/\mathbb{G}_n = (\text{GL}(n) \times \mathbb{G}_nJ^ {\text{reg}}_n(1,d))/\text{GL}(n) = ((\text{GL}(n)/\mathbb{G}_n) \times J^ {\text{reg}}_n(1,d))/\text{GL}(n).$$

### 6. Affine Embeddings of $\text{SL}(n)/U$

Let us now return to the situation in §4 where $\hat{U}$ and $U$ are subgroups of $\text{GL}(n)$ of the form described at (2). In §4 we embedded $\text{GL}(n)/\hat{U}$ in the Grassmannian

$$\text{Grass}_n(\text{Sym}^\omega \mathbb{C}^n) \subseteq \mathbb{P}(\Lambda^n(\text{Sym}^\omega \mathbb{C}^n))$$

as the $\text{GL}(n)$ orbit of

$$p_n = \phi_n(e_1, \ldots, e_n) = [e_1 \wedge (e_2 + e_1^\omega) \wedge \ldots \wedge (\sum_{i=1}^n p_i(e_1, \ldots, e_n))] \in \mathbb{P}[\Lambda^n(\text{Sym}^\omega \mathbb{C}^n)],$$

and observed at Remark 4.5 that the image of this embedding lies in the open affine subset defined by the non-vanishing of the coordinate in $\mathbb{P}(\Lambda^n(\text{Sym}^\omega \mathbb{C}^n))$ corresponding to the one-dimensional summand $\Lambda^n \mathbb{C}^n$ of $\Lambda^n(\text{Sym}^\omega \mathbb{C}^n)$ spanned by $e_1 \wedge \ldots \wedge e_n$. In §5 we saw that there exist examples where the image has codimension-one boundary components which meet this affine open subset, and therefore the Grosshans principle (see §3) is not applicable in this situation.

Combining the isomorphism

$$\text{GL}(n)/\hat{U} \cong \text{SL}(n)/\text{SL}(n) \cap \hat{U} = \text{SL}(n)/U \rtimes F_\omega$$

where $F_\omega = \mathbb{C}^* \cap \text{SL}(n)$ is a finite subgroup of $\hat{U}$ of order $1 + \omega_2 + \cdots + \omega_n$ with the embedding $\phi_n$ gives us an embedding of $(\text{SL}(n)/U)/F_\omega$ in $\Lambda^n(\text{Sym}^\omega \mathbb{C}^n)$ as the $\text{SL}(n)$ orbit of

$$p_n = e_1 \wedge (e_2 + e_1^\omega) \wedge \ldots \wedge (\sum_{i=1}^n p_i(e_1, \ldots, e_n)).$$

Here $\text{SL}(n) \cap \hat{U}$ is the semi-direct product $U_\omega \rtimes F_\omega$ of $U$ by the finite group $F_\omega$ of $\ell_\omega$th roots of unity in $\mathbb{C}$ for $\ell_\omega = 1 + \omega_2 + \cdots + \omega_n$, embedded in $\text{SL}(n)$ as

$$z \mapsto \begin{pmatrix} z & 0 & \ldots & 0 \\ 0 & z^{\omega_2} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & z^{\omega_n} \end{pmatrix} \in \text{SL}(n).$$

(15)

In this section we will look for affine embeddings of $\text{SL}(n)/U$ in affine spaces of the form

$$W_{s,K} = \Lambda^s(\text{Sym}^\omega \mathbb{C}^n) \otimes (\mathbb{C}^n)^{\otimes K}$$

for suitable $s, K$, and direct sums of such spaces, and study their closures.
Remark 6.1. Note that from the isomorphism $\text{GL}(n)/\hat{U} \cong (\text{SL}(n)/U)/F_\omega$ where $F_\omega$ is finite we will be able to deduce immediately that $\hat{U}$ is a Grosshans subgroup of $\text{GL}(n)$ with the canonical affine completion $\text{GL}(n)/\hat{U}$ of $\text{GL}(n)/\hat{U}$ given by

$$\text{GL}(n)/\hat{U} \cong (\text{SL}(n)/U)/F_\omega,$$

once we have proved that $U$ is a Grosshans subgroup of $\text{SL}(n)$.

Lemma 6.2. Let $K = M(1 + \omega_2 + \ldots + \omega_n) + 1$ for some natural number $M$. Then the point

$$\rho_n \otimes e_1^\otimes K \in \wedge^n(\text{Sym}^\omega \mathbb{C}^n) \otimes (\mathbb{C}^n)^\otimes K$$

where

$$\rho_n = e_1 \wedge (e_2 + e_1^\omega) \wedge \ldots \wedge (\sum_{i=1}^n \rho_{in}(e_1, \ldots, e_n)) \in \wedge^n(\text{Sym}^\omega \mathbb{C}^n)$$

has stabiliser $U$ in $\text{SL}(n)$.

Proof. By Theorem 4.3 the stabiliser of

$$[\rho_n] \in \mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n)) \cong \mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n) \otimes (\mathbb{C}e_1)^\otimes K) \subseteq \mathbb{P}(W_{n,K})$$

in $\text{GL}(n)$ is $\text{GL}(n)[\rho_n] = U \times \mathbb{C}^* = \hat{U}$, so the stabiliser of

$$\rho_n \otimes e_1^\otimes K \in \wedge^n(\text{Sym}^\omega \mathbb{C}^n) \otimes (\mathbb{C}^n)^\otimes K$$

is contained in $\hat{U}$. Moreover by the proof of Theorem 4.3 the stabiliser of $\rho_n \otimes e_1^\otimes K$ contains $U$. Finally the element of $\mathbb{C}^* \subseteq \text{GL}(n)[\rho_n]$, given at (15) above acts on $\rho_n \otimes e_1^\otimes K$ as multiplication by

$$z^{1+\omega_2+\ldots+\omega_n+K} = z^{(M+1)(1+\omega_2+\ldots+\omega_n)+1}$$

and has determinant 1 if and only if $z^{1+\omega_2+\ldots+\omega_n} = 1$, so it lies in $\text{SL}(n)$ and fixes $\rho_n \otimes e_1^\otimes K$ if and only if $z = 1$. □

The same argument gives us

Corollary 6.3. Let

$$\rho_s = e_1 \wedge (e_2 + e_1^\omega) \wedge \ldots \wedge (\sum_{i=1}^s \rho_{is}(e_1, \ldots, e_n)) \in \wedge^n(\text{Sym}^\omega \mathbb{C}^n),$$

and choose integers $s_1, \ldots, s_n$ such that $s_i$ and $1 + \omega_2 + \ldots + \omega_i$ are coprime. Then the point

$$\rho_{s_1, \ldots, s_n} = (\rho_1 \otimes e_1^{\otimes s_1}) \oplus \ldots \oplus (\rho_n \otimes e_1^{\otimes s_n}) \in W_{s_1, \ldots, s_n} = W_{1,s_1} \oplus W_{2,s_2} \oplus \ldots \oplus W_{n,s_n}$$

has stabiliser $U$ in $\text{SL}(n)$.

We can rephrase this as in Corollary 4.4.
Corollary 6.4. The map \( \phi : \text{SL}(n) \to W_{s_1,\ldots,s_n} \) defined by

\[
\phi(v) = \left( \phi_1(v) \otimes v_1^\otimes s_1 \right) \oplus \cdots \oplus \left( \phi_n(v) \otimes v_n^\otimes s_n \right)
\]

for \( v = (v_1,\ldots,v_n) \in \text{SL}(n) \) defines an \( \text{SL}(n) \)-equivariant embedding \( \text{SL}(n)/U \hookrightarrow W_{s_1,\ldots,s_n} \).

The main technical result of this paper is the following theorem.

Theorem 6.5. Let \( n \geq 3 \) and let \( s_1,\ldots,s_n \) be positive integers which satisfy the conditions:

- (i) \( s_i > i(s_{i-1} + 1 + \omega_2 + \cdots + \omega_{i-1}) \) for \( i = 1,\ldots,n \).
- (ii) \( s_i = m_i(1 + \omega_2 + \cdots + \omega_i) + 1 \) with some \( m_i \in \mathbb{Z}^+ \) for \( i = 1,\ldots,n \).

Then the orbit of \( v_{s_1,\ldots,s_n} \) under the natural action of \( \text{SL}(n) \) on \( W_{s_1,\ldots,s_n} \) is isomorphic to \( \text{SL}(n)/U \), and the complement of \( \text{SL}(n) v_{s_1,\ldots,s_n} \) in its closure \( \overline{\text{SL}(n) v_{s_1,\ldots,s_n}} \) in \( W_{s_1,\ldots,s_n} \) has codimension at least two.

By the Grosshans Criterion (see §3) Theorem 6.5 is an immediate corollary of Theorem 6.3 when \( n \geq 3 \) (and is well known when \( n = 2 \) and trivial when \( n = 1 \)). The next two sections will be devoted to proving Theorem 6.5.

7. Boundary components of \( \text{SL}(n)/U \) in \( W_{s_1,\ldots,s_n} \)

In this section we will study the boundary of the orbit \( \text{SL}(n) v_{s_1,\ldots,s_n} \) in the affine space \( W_{s_1,\ldots,s_n} = W_{s_1} \oplus W_{s_2} \oplus \cdots \oplus W_{s_n} \), as defined in Corollary 6.3. By this corollary the \( \text{SL}(n) \)-orbit of \( v_{s_1,\ldots,s_n} \) in \( W_{s_1,\ldots,s_n} \) is isomorphic to \( \text{SL}(n)/U \).

Recall that

\[
U = \begin{pmatrix}
1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\
0 & 1 & p_{23}(1, \alpha_2) & \cdots & p_{2n}(1, \alpha_2, \ldots, \alpha_n) \\
0 & 0 & 1 & \cdots & p_{3n}(1, \alpha_2, \ldots, \alpha_n) \\
& & & \ddots & \\
0 & 0 & \cdots & 1 & p_{n-1n}(1, \alpha_2, \ldots, \alpha_n) \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} : \alpha_2,\ldots,\alpha_n \in \mathbb{C}
\]

is generated along its first row.

Let \( B_n \subset \text{SL}(n) \) denote the standard upper triangular Borel subgroup of \( \text{SL}(n) \) which stabilises the filtration \( \mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \cdots \subset \mathbb{C}^n \). Since \( \text{SL}(n)/B_n \) is projective we have

\[
\text{SL}(n) \cdot v_{s_1,\ldots,s_n} = \text{SL}(n)B_n \cdot v_{s_1,\ldots,s_n}.
\]

Remark 7.1. Note that \( B_n = B_{n-1}' \cdot U_n \) where the Borel subgroup \( B_{n-1}' \) of \( \text{GL}(n-1) = \text{GL}(\mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \cdots \oplus \mathbb{C}e_n) \) is embedded diagonally in \( \text{SL}(n) \) via

\[
A \mapsto \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}.
\]
Since $U_n$ stabilises $p_{s_1, \ldots, s_n}$,
\[ B_n \cdot p_{s_1, \ldots, s_n} = B'_{n-1} \cdot p_{s_1, \ldots, s_n}. \]
Recall (see (4)) that the stabiliser in $\text{SL}(n)$ of the distinguished point
\[ p_n = e_1 \wedge (e_2 + e_1^{(2)}) \wedge \ldots \wedge (\sum_{i=1}^n p_i(e_1, \ldots, e_n)) \in \wedge^n \text{Sym}^\omega \mathbb{C}^n \]
is $\text{SL}(n) \cap \hat{U} = U \times F_\omega$, where $F_\omega$ is a finite subgroup of the subgroup $\mathbb{C}^*$ of $\hat{U} = U \rtimes \mathbb{C}^*$.

We can modify the induced action of the Borel subgroup $B_n \subset \text{SL}(n)$ on $\wedge^n \text{Sym}^\omega \mathbb{C}^n$ by multiplying by any character of $\text{SL}(n)$. In particular for any $m > 0$ and $b \in B_n$ we can define a modified action given for $p \in \wedge^n \text{Sym}^\omega \mathbb{C}^n$ by
\[ b \cdot m p = b'^{(m)} \cdot (b \cdot p). \]
This can be identified with the action of $B_n$ on $(\mathbb{C}^e_1)^\otimes \wedge^n \text{Sym}^\omega \mathbb{C}^n \subseteq (\mathbb{C}^n)^\otimes \wedge^n \text{Sym}^\omega \mathbb{C}^n$. The stabiliser of $p_n$ under this $m$-shifted action is the semidirect product $U \times F_{\omega, m}$ of $U$ with a finite subgroup $F_{\omega, m}$ of $\mathbb{C}^*$, and the orbits $B_n \cdot^m p_n$ are in general different for different choices of $m$. However, all these shifted actions induce the same action on the projective space $\mathbb{P}(\wedge^n \text{Sym}^\omega \mathbb{C}^n)$.

**Definition 7.2.** A point in $B_n \cdot^m p_n \setminus B_n \cdot^m p_n$ will be called an $m$-boundary point. We will call a point $[q] \in \mathbb{P}(\wedge^n \text{Sym}^\omega \mathbb{C}^n)$ $m$-small if $[q] = \lim_{m \to \infty} [b^{(m)} \cdot p_n]$ and $\lim_{m \to \infty} b^{(m)} \cdot p_n = 0$ for some sequence $b^{(m)} \in B_n$. If $[q] = \lim_{m \to \infty} [b^{(m)} \cdot p_n]$ is a boundary line, then we take the top left $(i \times i)$ subgroup $b_i^{(m)}$ of $b^{(m)}$ and define the limit line $[q_i] = \lim_{m \to \infty} [b_i^{(m)} \cdot p_i]$. We say that $q_i \in \otimes_{j=1}^i \text{Sym}^\omega \mathbb{C}^n$ represents the line $[q_i] \in \mathbb{P}(\wedge^n \text{Sym}^\omega \mathbb{C}^n)$.

Our next aim is to prove the following characterisation of the boundary of the $B_n$-orbit $B_n p_{s_1, \ldots, s_n} \equiv B_n / U$ in the affine space $W_{s_1, \ldots, s_n} = W_{1,s_1} \oplus W_{2,s_2} \oplus \ldots \oplus W_{n,s_n}$ as defined in Corollary 6.3.

**Remark 7.3.** Since $\text{SL}(n)$ is the product of its maximal compact subgroup $\text{SU}(n)$ and the Borel subgroup $B_n$, this theorem will give us a characterisation of the boundary of the $\text{SL}(n)$-orbit $\text{SL}(n) p_{s_1, \ldots, s_n} \equiv \text{SL}(n) / U$ in the affine space $W_{s_1, \ldots, s_n} = W_{1,s_1} \oplus W_{2,s_2} \oplus \ldots \oplus W_{n,s_n}$ as defined in Corollary 6.3. Since $\text{SU}(n)$ is compact the boundary of $\text{SL}(n) p_{s_1, \ldots, s_n}$ will just be the $\text{SL}(n)$-sweep of the boundary of $B_n p_{s_1, \ldots, s_n}$.

**Theorem 7.4.** Assume that $n \geq 2$ and $s_1, \ldots, s_n$ satisfy the conditions of Theorem 6.5. Let $x \in \overline{B_n p_{s_1, \ldots, s_n}} / B_n p_{s_1, \ldots, s_n}$ be a nonzero boundary point. Then there exist
\[ 1 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_l < \beta_l \leq n \text{ and } b^{(m)} \in \text{SL}(n) \]
such that the $i$th coordinate of $x$ has the following form:
\[ x_i = \begin{cases} (q_1^{(m)} \wedge e_{\alpha_1+1} \wedge e_{\alpha_2+2} \wedge \ldots \wedge e_{\beta_1}) \otimes e_1^{(m)} & \text{for } \alpha_r < i \leq \beta_r \\ 0 & \text{otherwise} \end{cases} \]
where \( a_{i,n} \) is some representative of \([a_{i,n}] = \lim_{m \to \infty} [b_{i,n} \cdot v_{i,n}]\).

Moreover if \( \beta_i = n \) then \( a_i \) is an \( s_n \)-boundary point and if \( \beta_i < n \) then \( [a_i] \) is \( s_n \)-small, in the sense of Definition 7.2.

**Proof.** We have

\[
x = x_1 \oplus \ldots \oplus x_n \in B_n v_{i_1, \ldots, i_n} \subseteq W_{i_1} \oplus \ldots \oplus W_{i_n}.
\]

as above, so there is a sequence of matrices

\[
(17) \quad b^{(m)} = \begin{pmatrix}
    b^{(m)}_{11} & b^{(m)}_{12} & \cdots & b^{(m)}_{1n} \\
    0 & b^{(m)}_{22} & \cdots & b^{(m)}_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & b^{(m)}_{nn}
\end{pmatrix} \in B_n \subseteq \text{SL}(n)
\]

such that \( b^{(m)}_n \rightarrow x \), and therefore

\[
b^{(m)}(v_n \otimes e_1^{\otimes s_n}) \rightarrow x_n \quad \text{as} \quad m \rightarrow \infty.
\]

Now expanding the wedge product in the definition of \( v_n \) we get

\[
b^{(m)}(v_n) = e_1 \wedge \ldots \wedge e_n + \ldots + (b^{(m)}_{11})^{1+\omega_2+\ldots+\omega_n} e_1 \wedge e_1^{\omega_2} \wedge \ldots \wedge e_1^{\omega_n}
\]

while

\[
b^{(m)}(e_1^{\otimes s_n}) = (b^{(m)}_{11})^{s} e_1^{\otimes s_n},
\]

so by considering the coefficient of \((e_1 \wedge \ldots \wedge e_n) \otimes e_1^{\otimes s_n}\) we see that \((b^{(m)}_{11})^{s}\) tends to a limit in \( \mathbb{C} \) as \( m \rightarrow \infty \). Thus, by replacing the sequence \((b^{(m)})^{s}\) with a subsequence if necessary, we can assume that

\[
b^{(m)}_{11} \rightarrow b^{(\infty)}_{11} \in \mathbb{C}
\]

as \( m \rightarrow \infty \).

Assume first that \( b^{(\infty)}_{11} \neq 0 \). We will show that in this case \( x \) lies in the orbit \( B_n v_{i_1, \ldots, i_n} \).

We have that for \( 1 \leq t \leq n \)

\[
b^{(m)}(v_t \otimes e_1^{\otimes s_t}) = (b^{(m)}_{11})^{s_t} (b^{(m)}_{11})^{s} \otimes e_1^{\otimes s_t} \rightarrow x_t
\]

and \( b^{(m)}_{11} \rightarrow b^{(\infty)}_{11} \in \mathbb{C} \setminus \{0\} \) as \( m \rightarrow \infty \), so

\[
(18) \quad (b^{(m)}_{11})^{s_t} b^{(m)}(v_t) \rightarrow v^{\infty}_t \in \land'(\text{Sym}^{\omega_i} \mathbb{C}^n)
\]

as \( m \rightarrow \infty \), and then \( x_t = v^{\infty}_t \otimes e_1^{\otimes s_t} \). Here

\[
(19) \quad b^{(m)} v_t = b^{(m)}_{11} e_1 \wedge (b^{(m)}_{22} e_2 + (b^{(m)}_{11})^{\omega_2} e_1^{\omega_2}) \wedge \ldots \wedge (b^{(m)}_{ni} e_i + b^{(m)}_{n+1-i} e_{n+1-i}) + \ldots
\]

\[
\ldots + b^{(m)}_{11} e_1 + \sum_{s=2}^{n-1} p_s (b^{(m)}_{s+1,1} e_s + \ldots + b^{(m)}_{s+1,1} e_s) + \ldots + b^{(m)}_{11} e_1) + (b^{(m)}_{11})^{\omega_1} e_1^{\omega_1}.
\]

Now take \( t = n \), and look at the coefficient of

\[
e_1 \wedge e_1^{\omega_2} \wedge \ldots \wedge e_1^{\omega_n-1} \wedge e_j \wedge e_1^{\omega_{n+1}} \wedge \ldots \wedge e_1^{\omega_n}
\]
in $b^{(m)}(p_n)$ when $1 \leq j \leq i \leq n$; we see that
\[(b^{(m)}_{11})^{1+\omega_2+\ldots+\omega_{n-1}+\omega_{n+1}+\ldots+\omega_b} b^{(m)}_{ji}\]
tends to a limit in $\mathbb{C}$ as $m \to \infty$, and so since $b^{(\infty)}_{11} \neq 0$
\[b^{(m)}_{ji} \to b^{(\infty)}_{ji} \in \mathbb{C}.\]
Also $b^{(m)}_{11} b^{(m)}_{22} \ldots b^{(m)}_{kk} = 1$ for all $m$, so $b^{(\infty)}_{11} b^{(\infty)}_{22} \ldots b^{(\infty)}_{kk} = 1$, so $b^{(m)} \to b^{(\infty)} \in \text{SL}(n)$. Therefore $x = b^{(\infty)}(p_{i_1,\ldots,i_n})$ lies in the orbit of $p_{i_1,\ldots,i_n}$ as required.

So it remains to consider the case when $b^{(\infty)}_{11} = 0$. Notice that this automatically implies that $x_1 = \lim_{m \to \infty} (b^{(m)}_{11} x_1 + 1 e^{S_1}) = 0$, so the first coordinate of $x$ is zero. If $x = 0$ then its stabiliser is $\text{SL}(n)$ which has dimension $n^2 - 1 \geq n + 1$, so we can assume that $x \neq 0$.

For a partition $\mathbf{i} = [i_1, \ldots, i_l]$ let
\[\Sigma^\omega \mathbf{i} = \omega_{i_1} + \ldots + \omega_{i_l}\]
denote the $\omega$-weighted sum of its elements and define $e_1 = e_{i_1} \ldots e_{i_l} \in \text{Sym}^\omega \mathbb{C}^n$.

**Lemma 7.5.** In the notation of Theorem 7.4 suppose that $1 \leq a \leq n$ and that the coefficient $A_{i_1,\ldots,i_a}$ of
\[e_1 \wedge e_{i_1} \wedge \ldots \wedge e_{i_a}\]
with $1 \leq \Sigma^\omega \mathbf{i}_2 \leq \Sigma^\omega \mathbf{i}_3 \leq \ldots \leq \Sigma^\omega \mathbf{i}_a \leq \omega_a$
in $p^{\infty}_a \in \wedge^a(\text{Sym}^\omega \mathbb{C}^n)$ is nonzero, where $x_a = p^{\infty}_a \otimes e_{i_1}^a$. Then $\mathbf{i}_a = [a]$ (that is, $e_{i_a} = e_a$).

**Proof.** To prove Lemma 7.5 recall from (18) that
\[(b^{(m)}_{11})^{x_a} b^{(m)}_{a} \to p^{\infty}_a \in \wedge^a(\text{Sym}^\omega \mathbb{C}^n)\]
as $m \to \infty$. Therefore by the hypotheses of the lemma
\[0 \neq A_{i_1,\ldots,i_a} = \lim_{m \to \infty} (b^{(m)}_{11})^{x_a} \sum_{\sigma \in S_a} b^{(m)}_{\kappa(1),1} \ldots b^{(m)}_{\kappa(a),a},\]
where $b^{(m)}_{\kappa(j),j}$ is a homogeneous polynomial in the entries of $b^{(m)}$, which is the coefficient of $e_{i_1}$ in the $j$th wedge-factor in $b^{(m)}_{i}$. (see the equation (19)). Now
\[\sum_{\sigma \in S_a} b^{(m)}_{\kappa(1),1} \ldots b^{(m)}_{\kappa(a),a} = \sum_{b=1}^{a} b^{(m)}_{\kappa(b),a} \sum_{\sigma \in S_a, \sigma(a) = b} b^{(m)}_{\kappa(1),1} \ldots b^{(m)}_{\kappa(a-1),a-1},\]
and without loss of generality, replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that if $1 \leq b \leq a$ then
\[b^{(m)}_{\kappa(b),a} \sum_{\sigma \in S_a, \sigma(a) = b} b^{(m)}_{\kappa(1),1} \ldots b^{(m)}_{\kappa(a-1),a-1}\]
Suppose in the notation of Theorem 7.4 that $x\in\mathbb{C}\cup\{\infty\}$ as $m\to\infty$. So there is some $1 \leq b \leq a$ such that

$$0 \neq \lim_{m\to\infty} (b^{(m)}_{11})^{s_{b-1}} \sum_{\sigma\in\mathcal{S}_a,\sigma(\alpha)=b} b^{(m)}_{\sigma(1),1} \cdots b^{(m)}_{\sigma(a-1),a-1} = \lim_{m\to\infty} (b^{(m)}_{11})^{s_{b-1}} \sum_{\sigma\in\mathcal{S}_a,\sigma(\alpha)=b} b^{(m)}_{\sigma(1),1} \cdots b^{(m)}_{\sigma(a-1),a-1},$$

where

$$b^{(m)}_{i_{1},1} \cdots b^{(m)}_{i_{a-1},a-1} \rightarrow \infty$$

as $m \to \infty$.

Since $A_{i_{1}, \ldots, i_{b-1}, b, 1, \ldots, a} \in \mathbb{C}$, we have

$$\lim_{m\to\infty} (b^{(m)}_{11})^{s_{b-1}} b^{(m)}_{i_{b},a} \neq 0. \tag{20}$$

We will prove that $i_b = a$. Since $\sum_{\alpha} i_{\alpha} \leq \omega_n$, this implies $b = a$ and Lemma 7.5.

Assume that $i_b = [j_1, \ldots, j_l]$ with $j_1 \leq j_2 \leq \ldots \leq j_l < a$. By definition

$$b^{(m)}_{i_{b},a} = \sum_{\omega_{j_1} + \ldots + \omega_{j_l} = \omega_a} b^{(m)}_{j_{i_1}i_1} \cdots b^{(m)}_{j_{i_l}i_l}$$

and by (20) there are indices $j_1, \ldots, j_l$ such that

$$\lim_{m\to\infty} (b^{(m)}_{11})^{s_{b-1}} b^{(m)}_{j_{i_1}i_1} \ldots b^{(m)}_{j_{i_l}i_l} \neq 0. \tag{21}$$

Now, since $U$ is generated along its first row, we can assume that $b^{(m)}_{i_1} = 0$ for $2 \leq i \leq n$ (see Remark 7.1, and then (using the notation $U_i = [1, \ldots, 1]$ for a vector with $i$ 1’s)

$$A_{i_1, l_{i_2}, \ldots, l_{i_{b-1}}, j_1, \ldots, A_{i_1, l_{i_2}, \ldots, l_{i_{b-1}}, j_l} = \lim_{m\to\infty} (b^{(m)}_{11})^{s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}} (b^{(m)}_{11})^{(1+s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}) + \ldots + (1+s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}) b^{(m)}_{j_{i_1}i_1} \cdots b^{(m)}_{j_{i_l}i_l} = \lim_{m\to\infty} (b^{(m)}_{11})^{s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}+s_{b-1}+(1+s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}) + \ldots + (1+s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}) + s_{b-1} - s_a \in \mathbb{C}.$$  

By (21) $(b^{(m)}_{11})^{s_{b-1}} b^{(m)}_{j_{i_1}i_1} \ldots b^{(m)}_{j_{i_l}i_l}$ tends to a nonzero limit, so

$$\lim_{m\to\infty} (b^{(m)}_{11})^{s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}+s_{b-1}+(1+s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}) + \ldots + (1+s_{l_{i_1}+s_{i_2}+\ldots+s_{i_l}}) + s_{b-1} - s_a} \in \mathbb{C}.$$

But this is a contradiction, since $\lim_{m\to\infty} b^{(m)}_{11} = 0$ and the exponent is negative:

$$s_{l_1} + s_{l_2} + \ldots + s_{l_l} + (1 + \omega_2 + \ldots + \omega_{l_{i-1}}) + \ldots + (1 + \omega_2 + \ldots + \omega_{l_{i-1}}) + s_{b-1} < 0$$

by the conditions on $s_1, \ldots, s_n$ in Theorem 6.5. This proves Lemma 7.5.

**Lemma 7.6.** Suppose in the notation of Theorem 7.4 that $x = x_1 \otimes \cdots \otimes x_n \in B_n \otimes_{L_{i_1}} \otimes_{L_{i_2}} \otimes_{L_{i_{b-1}}} \otimes_{L_{i_{b}}} \otimes L_{i_{b+1}} \otimes \cdots \otimes L_{i_{a-1}} \otimes L_{i_a} \otimes L_{i_{a+1}} \otimes \cdots \otimes L_{i_k}$ has $x_i = v_i^a \otimes e_i^a$ for each $i$ and $v_i^a \neq 0$, $v_i^a \otimes e_i^b \neq 0$, for some $a \leq b \leq n$. Let $c$ be an integer with $a \leq c \leq b$. If the coefficient $A_{i_1, \ldots, i_{k-1}} \neq 0$ then $i_a = [a], i_{a+1} = [a+1], \ldots, i_c = [c]$. 
Proof. We will prove this lemma by backwards induction on $a$. For $a = b \leq n$ the statement follows from Lemma 7.5. Given the result for all $b$ satisfying $a \leq b \leq n$, we will prove it for all $b$ satisfying $a - 1 \leq b \leq n$ by contradiction. Assume that for some $b$ with $a - 1 \leq b \leq n$

\[ p_{a-1}^\infty \neq 0, \ldots, p_b^\infty \neq 0, \]

and that

\[ A_{i_1, \ldots, i_{a-1}; [a], \ldots, [b]} \neq 0 \text{ with } i_{a-1} \neq [a - 1] \]

holds. By definition

\[ A_{i_1, \ldots, i_{a-1}; [a], \ldots, [b]} = \lim_{m \to \infty} \left( b_{11}^{(m)} \right)^{a_0-a_{a-1}} b_{bb}^{(m)} b_{b-b-1}^{(m)} \cdots b_{aa}^{(m)} (b_{11}^{(m)})^{a_0-a_{a-1}} \sum_{\sigma \in S_{a-1}} b_{\sigma(1), 1}^{(m)} \cdots b_{\sigma(a-1), a-1}^{(m)} \]

tends to $A_{i_1, \ldots, i_{a-1}}$ is nonzero. On the other hand, since $p_{a-1}^\infty \neq 0$, Lemma 7.5 tells us that $A_{j_1, \ldots, j_{a-2}, [a-1]} \neq 0$ for some partitions $j_1, \ldots, j_{a-2}$. Then

\[ A_{j_1, \ldots, j_{a-2}, [a-1], \ldots, [b]} = \lim_{m \to \infty} \left( b_{11}^{(m)} \right)^{a_0-a_{a-1}} b_{bb}^{(m)} b_{b-b-1}^{(m)} \cdots b_{aa}^{(m)} (b_{11}^{(m)})^{a_0-a_{a-1}} \sum_{\sigma \in S_{a-1}} b_{\sigma(1), 1}^{(m)} \cdots b_{\sigma(a-1), a-1}^{(m)} \]

tends to $A_{j_1, \ldots, j_{a-2}, [a-1]} \neq 0$

Therefore

\[ \lim_{m \to \infty} \left( b_{11}^{(m)} \right)^{a_0-a_{a-1}} b_{bb}^{(m)} b_{b-b-1}^{(m)} \cdots b_{aa}^{(m)} \in \mathbb{C}, \]

and (22) and (23) together tell us that

\[ A_{i_1, \ldots, i_{a-1}} \neq 0 \]

which contradicts Lemma 7.5 since $i_{a-1} \neq [a - 1]$. Thus Lemma 7.6 is proved. □

Lemma 7.6 gives us the first part of Theorem 7.4. To prove the second part assume that $x = \lim_{m \to \infty} b^{(m)} \cdot p_{s_1, \ldots, s_n}$ where

\[ x = x_1 \oplus \ldots \oplus x_n \in \mathcal{B}_{n+1} p_{s_1, \ldots, s_n} = \mathcal{B}_{n+1} p_{s_1, \ldots, s_n} \subset W_{1, s_1} \oplus \ldots \oplus W_{n, s_n}. \]

Therefore there exist finite limits

\[ q_{\alpha r}^i = \lim_{m \to \infty} (b_{11}^{(m)})^{r_{\alpha+1, \alpha+1}} \cdots b_{ii}^{(m)} (b_{\alpha r}^{(m)} \cdot p_{\alpha r}) \text{ for } \alpha < i \leq \beta r. \]

Since $b^{(m)} \in \text{SL}(n)$ we have

\[ \det b_{\alpha r}^{(m)} b_{\alpha+1, \alpha+1}^{(m)} \cdots b_{nn}^{(m)} = 1 \text{ for all } m. \]

For $r = l$ and $i = n$ this gives us

\[ q_{\alpha n}^n = \lim_{m \to \infty} (b_{11}^{(m)})^{r_{\alpha+1, \alpha+1}} \cdots b_{nn}^{(m)} (b_{\alpha n}^{(m)} \cdot p_{\alpha n}) = \lim_{m \to \infty} (b_{11}^{(m)})^{s_0} \left( \frac{b_{\alpha n}^{(m)} \cdot p_{\alpha n}}{\det b_{\alpha n}^{(m)}} \right), \]
where division by \( \det b_{i_t}^{(m)} \) on the right hand side is defined as multiplication by an element in the stabiliser of \([a_{i_t}]\), that is,

\[
\frac{p_{i_t}^{(m)} \cdot p_{i_t}}{\det b_{i_t}^{(m)}} = b_{i_t}^{(m)} \left( \begin{array}{c}
\det(b_{i_t}^{(m)})^{\omega_1} & \cdots \\
\vdots & \ddots \\
\det(b_{i_t}^{(m)})^{\omega_n} & \cdots & \det(b_{i_t}^{(m)})^{\omega_n} \\
\end{array} \right)^{-1}.
\]

However the matrix on the right hand side lies in \( SL(n) \), so if \( \beta_i = n \) then \( q_{i_t}^{n} \) is indeed an \( s_{r} \)-boundary point. If \( \bar{\xi}_n = 0 \) then by definition \([a_{i_t}]\) is \( s_{r} \)-small. So the conditions are necessary, and Theorem 7.4 is proved. \( \square \)

8. Proof of Theorem 6.5

In this section we will complete the proof of Theorem 6.5 from which Theorem 1.3 will follow immediately. To finish the proof of Theorem 6.5 assume that \( x = x_1 \oplus \ldots \oplus x_n \in \tilde{B}_n \cdot v_{s_1,\ldots,s_n} \) is a nonzero boundary point of the orbit \( B_n \cdot v_{s_1,\ldots,s_n} \) in the affine space \( W_{s_1,\ldots,s_n} \) where \( n \geq 3 \). Then by Theorem 7.4 there exist

\[
1 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_i < \beta_i \leq n \text{ and } b_{i_t}^{(m)} \in SL(n)
\]
such that each \( x_i \) has the form

\[
x_i = \begin{cases} (q_{i_t}^{r} \wedge e_{i_t+1} \wedge e_{i_t+2} \wedge \ldots \wedge e_i) \otimes e_1^{\otimes n_i} & \text{for } \alpha_r < i \leq \beta_r \\ 0 & \text{otherwise} \end{cases}
\]

where \( q_{i_t}^{r} \) is a representative of \([a_{i_t}] = \lim_{m \to \infty} [b_{i_t}^{(m)} \cdot p_{i_t}]\), and if \( \beta_i = n \) then \( q_{i_t}^{n} \) is an \( s_{n} \)-boundary point, while if \( \beta_i < n \) then \([a_{i_t}]\) is \( s_{r} \)-small, in the sense of Definition 7.2.

Consider the subset \( \mathcal{F}(\alpha_1,\ldots,\alpha_i,\beta_1,\ldots,\beta_i) \) of the boundary of \( \tilde{B}_n \cdot v_{s_1,\ldots,s_n} \) consisting of \( x = x_1 \oplus \ldots \oplus x_n \in \tilde{B}_n \cdot v_{s_1,\ldots,s_n} \). Since \( SL(n) \cdot v_{s_1,\ldots,s_n} = SL(n)B_n \cdot v_{s_1,\ldots,s_n} \), to prove Theorem 6.5 it suffices to show that, for each choice of \((\alpha_1,\ldots,\alpha_i,\beta_1,\ldots,\beta_i)\) satisfying

\[
1 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_i < \beta_i \leq n,
\]

the \( SL(n) \)-sweep \( SL(n) \cdot \mathcal{F}(\alpha_1,\ldots,\alpha_i,\beta_1,\ldots,\beta_i) \) of \( \mathcal{F}(\alpha_1,\ldots,\alpha_i,\beta_1,\ldots,\beta_i) \) has codimension at least two in \( SL(n) \cdot v_{s_1,\ldots,s_n} \).

Suppose that \( x = x_1 \oplus \ldots \oplus x_n \in \mathcal{F}(\alpha_1,\ldots,\alpha_i,\beta_1,\ldots,\beta_i) \) has the form (25). For each \( i \) such that \( \alpha_i + 1 \leq i \leq \beta_i \) we have

\[
q_{i_t}^{i} = \sum_{1 \leq i_1, \ldots, i_{k_i} \leq \alpha_i, 0 \leq \omega_{i_t}} A_{i_1,\ldots,i_{k_i}}^{i} (e_1 \wedge e_{i_1} \wedge \ldots \wedge e_{i_{k_i}})
\]

for some coefficients \( A_{i_1,\ldots,i_{k_i}}^{i} \in \mathbb{C} \). The inequality \( \sum i_{\alpha_i} \leq \omega_{i_t} \) ensures that if \( \alpha_i \in i_{\alpha_i} \), then \( i_{\alpha_i} = [\alpha_i] \). This means that the terms in \( x \) involving \( e_{\alpha_i} \) have the following special form

\[
(e_1 \wedge e_{i_1} \wedge \ldots \wedge e_{k_{i_{\alpha_i}}-1} \wedge e_{\alpha_i} \wedge e_{\alpha_i+1} \wedge \ldots \wedge e_{\alpha_{i_{\alpha_i}}+j}) \otimes e_1^{\otimes \omega_{i_t}}
\]
where $1 \leq j \leq \beta_1 - \alpha_1$. Therefore for $\alpha_l > 1$ the transformation $T_{\alpha_l} : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$T_{\alpha_l}(e_i) = e_i \text{ for } i \neq \alpha_l, \quad T_{\alpha_l}(e_{\alpha_l}) = e_{\alpha_l} + e_{\alpha_l+1}$$

fixes $x$.

When $\alpha_l = 1$ then $a_1 = e_1$ and therefore

$$x = 0 \oplus \xi_2(e_1 \wedge e_2) \otimes e_1^{\otimes 2} \oplus \cdots \oplus \xi_n(e_1 \wedge \cdots \wedge e_n) \otimes e_1^{\otimes n}$$

with some constants $\xi_2, \ldots, \xi_n$. Note that $\xi_n = 0$ necessarily holds as $b^{(m)} \in \text{SL}(n)$ and therefore the coefficient of $e_1 \wedge \cdots \wedge e_n$ is $1$ in $b^{(m)} \cdot p_n$, but $\lim_{m \to \infty} b^{(m)}_{11} = 0$ and therefore $\xi_n = \lim_{m \to \infty} (b^{(m)}_{11})^n = 0$. The dimension of $\mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)$ is therefore at most $n - 2$ (the number of nonzero coefficients $\xi_2, \ldots, \xi_{n-2}$), so this has codimension at least two in the closure of the orbit $B_n \cdot \mathcal{p}_{s_1, \ldots, s_n}$ whose dimension is $(\frac{n+1}{2}) - n = n(n-1)/2 \geq n$ when $n \geq 3$. Hence $\text{SL}(n) \cdot \mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)$ has codimension at least two in $\text{SL}(n) \cdot \mathcal{p}_{s_1, \ldots, s_n}$ when $\alpha_l = 1$.

Now suppose that $\alpha_l > 1$, so that the transformation $T_{\alpha_l} : \mathbb{C}^n \to \mathbb{C}^n$ maps

$$\mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)$$

to itself, as does $B_n$. If $S$ is any subgroup of $\text{SL}(n)$ which maps $\mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)$ to itself then

$$\dim(\text{SL}(n) \cdot \mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)) \leq \dim(\mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)) + \dim(\text{SL}(n)) - \dim(S).$$

But $\mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)$ lies in the boundary of $B_n \cdot \mathcal{p}_{s_1, \ldots, s_n}$ and therefore

$$\dim(\mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)) \leq \dim(B_n \cdot \mathcal{p}_{s_1, \ldots, s_n}) - 1$$

and it is stabilised by both $T_n$ and $B_n$, so by a subgroup $S$ of $\text{SL}(n)$ of dimension at least $\dim(B_n) + 1$. It follows that

$$\dim(\text{SL}(n) \cdot \mathcal{F}(\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)) \leq \dim(B_n \cdot \mathcal{p}_{s_1, \ldots, s_n}) + \dim(\text{SL}(n)) - \dim(B_n) - 2$$

as required.

This completes the proof of Theorem 6.5 and thus of Theorem 1.3.

9. A geometric description of the algebra of invariants $O(\text{SL}(n))^U$

Theorem 6.5 allows us to give a geometric description of the invariant algebra $O(\text{SL}(n))^U$. Recall that in Section 8 we constructed an embedding

$$\phi : \text{SL}(n)/U \hookrightarrow W_{s_1, \ldots, s_n}$$

of $\text{SL}(n)/U$ in an affine space

$$W_{s_1, \ldots, s_n} = \bigoplus_{i=1}^n \wedge^i (\text{Sym}^\omega \mathbb{C}^n) \otimes (\mathbb{C}^n)^{\otimes s_i}$$
where \( s_1, \ldots, s_n \) satisfy the conditions of Theorem 6.5. This map sends a point of \( \text{SL}(n)/U \) represented by a matrix in \( \text{SL}(n) \) with column vectors \( v = (v_1, \ldots, v_n) \) to
\[
\phi(v) = (\phi_1(v) \otimes v_1^{\otimes s_1}) \oplus \ldots \oplus (\phi_n(v) \otimes v_1^{\otimes s_n}),
\]
where (see (5))
\[
\phi_j(v) = v_1 \wedge (v_2 + v_1^{\otimes s_2}) \wedge \ldots \wedge (v_j + \sum_{i=2}^j p_{i,j}(v_1, \ldots, v_j)) \in \wedge^j(\text{Sym}^{s_j} C^n).
\]

It follows from Theorem 6.5 that the boundary components of the image of \( \phi \) in the affine space \( W_{s_1, \ldots, s_n} \) have codimension at least two. Therefore regular functions on \( \text{SL}(n)/U \cong \phi(\text{SL}(n)/U) \) extend to the closure of \( \phi(\text{SL}(n)/U) \). This means that the restriction map to \( O(\text{SL}(n))^U \) from the polynomial algebra \( O(W_{s_1, \ldots, s_n}) \) is surjective, so the algebra of invariants \( O(\text{SL}(n))^U \) is generated by the linear coordinate functions on \( W_{s_1, \ldots, s_n} \). These are the Plücker coordinates on \( \text{Grass}_s(\text{Sym}^{s_j} C^n) \); that is, the initial minors of the matrix
\[
\Phi(v) = \begin{pmatrix}
  v_1 & v_2 & \cdots & v_n \\
  0 & v_1^{\otimes s_2} & \cdots & p_{2n}(v) \\
  \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & \cdots & v_1^{\otimes s_n}
\end{pmatrix}
\]

whose \((i, j)\) entry represents a column vector giving the coordinates of \( p_{i,j}(v) \in \text{Sym}^{s_j} C^n \) with respect to the standard basis. Here a minor \( \Delta_{i_1, \ldots, i_k} \) is called initial if it is spanned by the first \( s \) columns and the rows indexed by the basis elements of \( \text{Sym}^{s_j} C^n \) corresponding to the partitions \( i_1, \ldots, i_s \).

Thus as a consequence of Theorem 6.5 we obtain the following result, and in particular Theorem 1.5.

**Theorem 9.1.** The canonical affine completion
\[
\text{SL}(n)/U = \text{Spec}(O(\text{SL}(n))^U)
\]
of \( \text{SL}(n)/U \) is isomorphic to the closure \( \text{SL}(n) \cdot p_{s_1, \ldots, s_n} \) of the orbit
\[
\text{SL}(n) \cdot p_{s_1, \ldots, s_n} \cong \text{SL}(n)/U
\]
in \( W_{s_1, \ldots, s_n} \). The algebra \( O(\text{SL}(n))^U \) is generated by the pull-backs under \( \phi_i \) of the coordinates on \( \wedge^i(\text{Sym}^{s_j} C^n) \), for \( i = 1, \ldots, n \). These are the initial minors of the matrix \( \Phi(v) \).

Assume now that \( X \) is a complex affine variety with a linear action of \( U \) and that this action extends to a linear \( \text{SL}(n) \)-action. Recall that we have a surjective homomorphism of algebras from the polynomial algebra \( O(W_U) \) generated by the linear coordinates on \( W_U \) to \( O(\text{SL}(n))^U \), and this surjection is \( \text{SL}(n) \)-equivariant. It follows immediately from Theorem 9.1 that the algebra of invariants
\[
O(X)^U \cong (O(X) \otimes O(\text{SL}(n))^U)_{\text{SL}(n)}
\]
is finitely generated, and that the non-reductive GIT quotient
\[ X//U = \text{Spec}(O(X)^U) \]
is isomorphic to the reductive GIT quotient
\[ (X \times \text{SL}(n)//U)//\text{SL}(n) \]
where \( \text{SL}(n)//U \equiv \overline{\text{SL}(n) \cdot p_{s_1, \ldots, s_n}} \) is the canonical affine completion of \( \text{SL}(n)/U \).

Similarly if \( X \) is any complex projective variety with an action of \( U \) which is linear with respect to an ample line bundle \( L \) on \( X \) and extends to a linear action of \( \text{SL}(n) \), then the algebra of invariants
\[ \bigoplus_{m \geq 0} H^0(X, L^\otimes m)^U \cong \bigoplus_{m \geq 0} \left( H^0(X, L^\otimes m) \otimes O(\text{SL}(n))^U \right)^{\text{SL}(n)} \]
is finitely generated, and just as in the affine case the associated non-reductive GIT quotient \( X//U \) is isomorphic to the reductive GIT quotient
\[ (X \times \text{SL}(n)//U)//\text{SL}(n) \]
where \( \text{SL}(n)//U \equiv \overline{\text{SL}(n) \cdot p_{s_1, \ldots, s_n}} \) is the canonical affine completion of \( \text{SL}(n)/U \).

The algebra of invariants
\[ \bigoplus_{m \geq 0} H^0(X, L^\otimes m)^U \cong \bigoplus_{m \geq 0} \left( H^0(X, L^\otimes m) \otimes O(\text{SL}(n))^U \right)^{\text{SL}(n)} \]
is a quotient of the algebra of invariants
\[ \bigoplus_{m \geq 0} \left( H^0(X, L^\otimes m) \otimes O(W^U) \right)^{\text{SL}(n)} \]
and so is generated by the coordinates on the reductive GIT quotient \((X \times W^U)//\text{SL}(n)\), which can be determined using the representation theory of \( \text{SL}(n) \) from the decompositions of \( \bigoplus_{m \geq 0} H^0(X, L^\otimes m) \) and \( W^U \) as sums of irreducible representations of \( \text{SL}(n) \).

Using classical geometric invariant theory [33] \( X//U \) can be described geometrically as the quotient of the open subset \((X \times \text{SL}(n) \cdot p_{s_1, \ldots, s_n})^{ss}\) of \( \text{SL}(n) \)-semistable points of \( X \times \text{SL}(n) \cdot p_{s_1, \ldots, s_n} \) by the equivalence relation \( \sim \) such that \( y \sim z \) if and only if the closures of the \( \text{SL}(n) \)-orbits of \( y \) and \( z \) intersect in \((X \times \text{SL}(n) \cdot p_{s_1, \ldots, s_n})^{ss}\). Alternatively its points can be identified with the closed \( \text{SL}(n) \)-orbits in \((X \times \text{SL}(n) \cdot p_{s_1, \ldots, s_n})^{ss}\).

We will finish this section by describing two examples of the algebra of invariants in the case of jet differentials, and one example in the case of adjoint forms.

Example 9.2. Invariant jet differentials of order 2 in dimension 2. As usual let \( \{e_1, e_2\} \) be the standard basis for \( \mathbb{C}^2 \), and consider the group
\[ G_2 = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_1 \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_2 \in \mathbb{C} \right\} = \mathbb{C}^* \ltimes \mathbb{C}^+ \]
with maximal unipotent $\mathbb{C}^+$ acting on $\mathbb{C}^2$ by translation. Then $\text{Sym}^o \mathbb{C}^n = \mathbb{C}^n \oplus \text{Sym}^2 \mathbb{C}^2$ has an induced basis $\{e_1, e_2, e_1^2, e_1 e_2, e_2^2\}$. Let $x_{ij}$ denote the standard coordinate functions on $\text{SL}(2) \subset (\mathbb{C}^2)^* \otimes \mathbb{C}^2$. Then

$$\phi_1(x_{11}, x_{12}, x_{21}, x_{22}) = (x_{11}, x_{21}),$$

and

$$\phi_2(x_{11}, x_{12}, x_{21}, x_{22}) = (x_{11}, x_{21}) \wedge ((x_{12}, x_{22}) + (x_{11}^2, 2x_{11}x_{21}, x_{21}^2))$$

so $O(\text{SL}(n))^U$ is generated by $x_{11}, x_{21}$ and the $2 \times 2$ minors of

$$\begin{pmatrix}
 x_{11} & x_{21} & 0 & 0 & 0 \\
 x_{12} & x_{22} & x_{11}^2 & 2x_{11}x_{21} & x_{21}^2
\end{pmatrix}.$$ 

Since the determinant is 1 this set of generators reduces to two generators $x_{11}, x_{21}$, as expected since $\text{SL}(2)/\mathbb{C}^+ \cong \mathbb{C}^2 \setminus \{0\}$ and its canonical affine completion $\text{SL}(2)/\mathbb{C}^+$ is $\mathbb{C}^2$.

**Example 9.3. Invariant jet differentials of order 3 in dimension 3.** When $n = 3$ the finite generation of the Demailly-Semple algebra $O((J_3)_3)$ was proved by Rousseau in [42]. Here

$$\mathcal{G}_3 = \left\{ \begin{pmatrix}
 \alpha_1 & \alpha_2 & \alpha_3 \\
 0 & \alpha_1^2 & 2\alpha_1\alpha_2 \\
 0 & 0 & \alpha_3^2
\end{pmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \right\} = \mathbb{C}^* \times U$$

while $\text{Sym}^o \mathbb{C}^n = \mathbb{C}^n \oplus \text{Sym}^2 \mathbb{C}^3 \oplus \text{Sym}^3 \mathbb{C}^3$ has basis $\{e_1, e_2, e_3, e_1^2, e_1 e_2, e_2^2, e_3^2\}$. Let $x_{ij}$ denote the standard coordinate functions on $\text{SL}(3)$. Then

$$\phi_3(x_{11}, \ldots, x_{33}) = (x_{11}, x_{21}, x_{31}) \wedge ((x_{12}, x_{22}, x_{32}) + (x_{11}^2, 2x_{11}x_{21}, x_{21}^2, 2x_{21}x_{31}, 2x_{11}x_{31}, x_{31}^2))$$

$$\wedge ((x_{12}, x_{22}, x_{32}) + (2x_{11}x_{12}, \ldots, 2x_{13}x_{23}) + (x_{31}^3, \ldots, x_{31}^3))$$

So $O(\text{SL}(3))^U$ is generated by those minors of

$$\begin{pmatrix}
 x_{11} & x_{21} & x_{31} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 x_{12} & x_{22} & x_{32} & x_{11}^2 & 2x_{11}x_{21} & \cdots & x_{33}^2 & 0 & 0 & \cdots & 0 \\
 x_{13} & x_{23} & x_{33} & x_{11}x_{12} & x_{11}x_{22} + x_{12}x_{21} & \cdots & x_{31}x_{32} & x_{11}^2 & x_{11}^2 & \cdots & x_{31}^2
\end{pmatrix}$$

whose rows form an initial segment of $\{1, 2, 3\}$, that is the minors $\Delta_{i_1, \ldots, i_s}$ with rows 1, \ldots, $s$ and columns indexed by $i_1, \ldots, i_s$, where $s = 1, 2$ or 3 and $|i_j| \leq 3$.

10. **The geometry of the canonical affine completion $\text{SL}(n)//U$ and the closure $\text{GL}(n)/\ddot{U}$ embedded in $\text{Grass}_n(\text{Sym}^o \mathbb{C}^n)$**.

In the last section (see Theorem 9.1) we saw that the canonical affine completion $\text{SL}(n)//U$ of $\text{SL}(n)/U$ is isomorphic to the closure in

$$W_{s_1, \ldots, s_n} = W_{1, s_1} \oplus W_{2, s_2} \oplus \cdots \oplus W_{n, s_n} = \bigoplus_{i=1}^n (\text{Sym}^o \mathbb{C}^n) \otimes (\mathbb{C}^n)^{\otimes i}$$
of the $\text{SL}(n)$-orbit of the point
\[ p_{s_1, \ldots, s_n} = (p_1 \otimes e_1^{s_1}) \oplus \ldots \oplus (p_n \otimes e_1^{s_n}), \]
where
\[ p_x = e_1 \wedge (e_2 + e_1^{s_2}) \wedge \ldots \wedge (\sum_{i=1}^n p_i(e_1, \ldots, e_n)) \in \wedge^s(\text{Sym}^\omega \mathbb{C}^n). \]
Moreover, since $\text{SL}(n)/B_n$ is projective, we have
\[ \text{SL}(n)/U \cong \overline{\text{SL}(n)p_{s_1, \ldots, s_n}} = \text{SL}(n)(\overline{B_n p_{s_1, \ldots, s_n}}) \]
where $B_n p_{s_1, \ldots, s_n}$ is the closure in $W_{s_1, \ldots, s_n}$ of the orbit of $p_{s_1, \ldots, s_n}$ under the standard Borel subgroup $B_n$ of $\text{SL}(n)$. Since $U$ is a subgroup of $B_n$ this gives us a birational morphism
\[ \text{SL}(n) \times_{B_n} \overline{B_n p_{s_1, \ldots, s_n}} \to \text{SL}(n)/U \]
and we can understand the geometry of $\text{SL}(n)/U$ by studying the geometry of $\overline{B_n p_{s_1, \ldots, s_n}}$ and the fibres of this birational morphism over $\text{SL}(n)/U$. Theorem 7.4 tells us that the geometry of $\overline{B_n p_{s_1, \ldots, s_n}}$ is close-ly related to the geometry of the closure in the Grassman- nian $\text{Grass}_n(\text{Sym}^\omega \mathbb{C}^n)$ of the $B_n$-orbit of $[p_n]$, or equivalently the closure of the $\text{SL}(n)$-orbit
\[ \text{SL}(n)[p_n] = \text{SL}(n)\overline{B_n[p_n]} \subseteq \text{Grass}_n(\text{Sym}^\omega \mathbb{C}^n) \subset \mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n)). \]
Recall that since the subgroup $\mathbb{C}^*$ of $\hat{U}$ fixes $[p_n]$, we have
\[ \text{SL}(n)[p_n] = \text{GL}(n)[p_n] \]
and since the stabiliser of $[p_n]$ in $\text{GL}(n)$ is $\hat{U}$ we have
\[ \text{GL}(n)[p_n] \cong \text{GL}(n)/\hat{U} \cong \text{SL}(n)/\hat{U} \cap \text{SL}(n) \]
where $\hat{U} \cap \text{SL}(n) = U \rtimes F$ is the semidirect product of $U$ and a finite subgroup $F$ of $\text{SL}(n)$. Let us introduce the notation
\[ \Sigma \omega = 1 + \omega_1 + \ldots + \omega_n \]
and consider the subgroup
\[ \hat{U} = \left\{ \begin{pmatrix} \frac{1}{2} \Sigma \omega & \alpha_2 & \ldots & \alpha_n \\ 0 & \frac{1}{2} \Sigma \omega & \ldots & p_2, n(\alpha_2, \ldots, \alpha_n) \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \frac{1}{2} \Sigma \omega \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \right\} 
\]
of $\text{SL}(n)$ which is the semidirect product of $U$ and the diagonal $\mathbb{C}^*$ in $\text{SL}(n)$ normalising $U$ and $\hat{U}$. The action of $\text{SL}(n) \times \hat{U}$ on $\text{SL}(n)$ and $\text{GL}(n)$ by left and right multiplication induces actions of $\text{SL}(n) \times (\hat{U} / U) \cong \text{SL}(n) \times \mathbb{C}^*$ on $\text{SL}(n) / U$ and on $\text{GL}(n) / \hat{U} \cong (\text{SL}(n) / U) / F$. These actions extend to linear actions on $W_{s_1, \ldots, s_n}$ and on $\mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n))$ restricting to $\text{Grass}_n(\text{Sym}^\omega \mathbb{C}^n)$. 
We can use the induced $\mathbb{C}^*$ action on the projective variety $\text{SL}(n)[v_n]$ to study its geometry using Thaddeus’ technique of variation of GIT (abbreviated as VGIT) \cite{15, 47}. Consider the diagonal action of $\mathbb{C}^*$ on
\[ \mathbb{P}^1 \times \text{SL}(n)[v_n] \subseteq \mathbb{P}^1 \times \mathbb{P}^n(\wedge^n(\text{Sym}^\omega \mathbb{C}^*)) \]
where $\mathbb{C}^*$ acts on $\mathbb{P}^1$ with weights 0 and 1. We can linearise this action with respect to the line bundle
\[ L_{p,q} = O_{\mathbb{P}^1}(p) \otimes O_{\mathbb{P}^n(\wedge^n(\text{Sym}^\omega \mathbb{C}^*))}(q) \]
where $p$ and $q$ are positive integers; in fact, for the purposes of GIT replacing a linearisation with a positive tensor power of itself has no effect, so we can take $p$ and $q$ to be positive rational numbers. Finally we can twist this linearisation by multiplying by a character $t \mapsto t^r$ of $\mathbb{C}^*$ for any $r \in \mathbb{Z}$, and indeed as above we can take $r \in \mathbb{Q}$.

Now let us fix $p \gg q$ and consider the GIT quotient $(\mathbb{P}^1 \times \text{SL}(n)[v_n])//\mathbb{C}^*$ with respect to this linearisation for varying $r \in \mathbb{Q}$. The theory of variation of GIT \cite{15, 47} tells us that $\mathbb{Q}$ can be subdivided into finitely many chambers separated by walls (in this case just rational numbers determined by the weights of the action of $\mathbb{C}^*$ on the fibres of the line bundle $L_{p,q}$ at the fixed points in $\mathbb{P}^1 \times \text{SL}(n)[v_n]$ of the $\mathbb{C}^*$ action) such that the GIT quotients $(\mathbb{P}^1 \times \text{SL}(n)[v_n])//\mathbb{C}^*$ are independent of $r$ up to isomorphism for $r$ in the interior of any chamber, and change in a prescribed way as $r$ moves across a wall.

For fixed $p \gg q > 0$ the walls are of the form
\[ \{qr_j : j \in J\} \cup \{p + qr_j : j \in J\} \]
where $\{r_j : j \in J\}$ are the weights of the action of the subgroup $\mathbb{C}^*$ of $\hat{U}$ on $\wedge^n(\text{Sym}^\omega \mathbb{C}^*)$. For $t \in \mathbb{C}^*$ the matrix
\[
\begin{pmatrix}
    t^{1-\frac{1}{\omega_n}} & & \\
    & t^{\omega_2-\frac{1}{\omega_n}} & \\
    & & \ddots \\
    & & & t^{\omega_n-\frac{1}{\omega_n}}
\end{pmatrix} \in \hat{U}
\]
acts on $e_1 \wedge \ldots \wedge e_n \in \wedge^n(\text{Sym}^\omega \mathbb{C}^*)$ as multiplication by
\[ \omega_1 + \ldots + \omega_n = \sum_{k=1}^n \sum_{j \in J_k} (\omega_j - \frac{1}{n}\omega) \]
So we can identify the set $\{r_j : j \in J\}$ of weights with the set of all rationals of this form.

We will assume that $\frac{p}{q}$ is sufficiently large that $p + qr_{j_1} > qr_{j_2}$ holds for any $j_1, j_2 \in J$. Then if $r \in \mathbb{Q}$ is chosen so that
\[ p + qr_{j_1} > r > qr_{j_2} \]
for all $j_1, j_2 \in J$, it will follow from the Hilbert-Mumford criterion (see \cite{33, 36, 48}) that
\begin{equation}
\label{27}
(\mathbb{P}^1 \times \text{SL}(n)[v_n])//\mathbb{C}^* = (\mathbb{P}^1 \times \text{SL}(n)[v_n])^{\text{Isr}}//\mathbb{C}^*
\end{equation}
where \((\mathbb{P}^1 \times \text{SL}(n)[P_n])^{x,r} = \mathbb{C}^* \times \text{SL}(n)[P_n]\). Thus
\[
(\mathbb{P}^1 \times \text{SL}(n)[P_n])/\mathbb{C}^* \cong \text{SL}(n)[P_n]
\]
for such a choice of \(r\). Since \(1 < \omega_2 \leq \ldots \leq \omega_n\), the weight vector in \(\wedge^n(\text{Sym}^\omega(\mathbb{C}^n))\) with smallest possible weight is \(e_1 \wedge e_{\omega_2} \wedge \ldots \wedge e_{\omega_n}\) with weight
\[
r_{\min} = \Sigma \omega(1 - \frac{1}{n}\Sigma \omega).
\]
(28)

Now choose \(r \in \mathbb{Q}\) such that
\[
qr_{\min} < r < qr_j
\]
holds for all \(r_j \neq r_{\min}\). Again we have (27), but now the Hilbert-Mumford criterion for (semi) stability gives us the following characterisation of the semistable set.

**Lemma 10.1.** If \(x \in \mathbb{P}^1, y \in \text{SL}(n)[P_n]\) then \((x, y) \in (\mathbb{P}^1 \times \text{SL}(n)[P_n])^{x,r}\) if and only if
1. \(x \neq \infty\),
2. if \(x = 0\) then \(y\) does not lie in the linear subspace of \(\mathbb{P}(\wedge^n(\text{Sym}^\omega(\mathbb{C}^n)))\) spanned by the vectors \(e_{i_1} \wedge e_{i_2}^{\omega_2} \wedge \ldots \wedge e_{i_n}^{\omega_n}\) where \(i_1, \ldots, i_n \in \{1, \ldots, n\}\), and
3. the coefficient of \(y\) corresponding to some basis vector of the form \(e_{i_1} \wedge e_{i_2}^{\omega_2} \wedge \ldots \wedge e_{i_n}^{\omega_n}\) is nonzero.

Let \(U_{i_1,\ldots,i_n}\) denote the open subset of \(\mathbb{P}(\wedge^n(\text{Sym}^\omega(\mathbb{C}^n)))\) described in (3) above; that is, \(y \in U_{i_1,\ldots,i_n}\) if and only if its coefficient corresponding to \(e_{i_1} \wedge e_{i_2}^{\omega_2} \wedge \ldots \wedge e_{i_n}^{\omega_n}\) is nonzero.

Then to understand the GIT quotient \((\mathbb{P}^1 \times \text{SL}(n)[P_n])/\mathbb{C}^*\) in this case, we need to understand the intersection
\[
\text{SL}(n)[P_n]^{(0)} = \text{SL}(n)[P_n] \cap \bigcup_{i_1,\ldots,i_n \in \{1,\ldots,n\}} U_{i_1,\ldots,i_n}.
\]
Since \(\text{SL}(n)/B_n\) is projective, we have
\[
\text{SL}(n)[P_n]^{(0)} = \text{SL}(n)B_n[P_n]^{(0)}
\]
where
\[
B_n[P_n]^{(0)} = B_n[P_n] \cap \bigcup_{i_1,\ldots,i_n \in \{1,\ldots,n\}} U_{i_1,\ldots,i_n} = B_n[P_n] \cap U_{1,1,\ldots,1}
\]
since the coefficient of any \(y \in B_n[P_n]\) corresponding a basis vector \(e_{i_1} \wedge e_{i_2}^{\omega_2} \wedge \ldots \wedge e_{i_n}^{\omega_n}\) is \(e_1 \wedge e_1^{\omega_2} \wedge \ldots \wedge e_1^{\omega_n}\) is zero.

**Definition 10.2.** Let
\[
B_{(n-1)} = \begin{pmatrix}
  b_{11} & 0 & \cdots & 0 \\
  0 & b_{22} & \cdots & b_{2n} \\
  \vdots & 0 & \ddots & \vdots \\
  0 & 0 & \cdots & b_{nn}
\end{pmatrix} : b_{ij} \in \mathbb{C}, b_{11} \ldots b_{nn} = 1 \leq \text{SL}(n)
\]
and let $B_{[n-1]} \subseteq M_{n \times n}(\mathbb{C})$ be the set of matrices of the same form but without the condition $b_{11} \ldots b_{nn} = 1$.

Note that

$$B_n[v_n] = B_{(n-1)}[v_n]$$

since $U$ stabilises $[v_n]$ and $B_n = B_{(n-1)} U$.

**Lemma 10.3.** The morphism $B_{[n-1]} \rightarrow B_n[v_n]$ given by $b \mapsto b[v_n]$ extends to a surjective morphism $\beta : B^{(0)}_{[n-1]} \rightarrow B_n[v_n]$, where

$$B^{(0)}_{[n-1]} = \{ b \in B_{[n-1]} : b_{11} \neq 0 \}.$$

**Proof.** The image of the restriction to $B^{(0)}_{[n-1]}$ of the morphism $B_{[n-1]} \rightarrow \wedge^n(\text{Sym}^\omega \mathbb{C}^n)$ given by $b \mapsto b[v_n]$ does not contain any points where the coefficient corresponding to the basis vector $e_1 \wedge e_1^{\omega_2} \wedge \ldots \wedge e_1^{\omega_n}$ is zero since $b_{11} \neq 0$ for all $b \in B^{(0)}_{[n-1]}$. Thus this induces a morphism

$$\beta : B^{(0)}_{[n-1]} \rightarrow B_n[v_n] \subseteq \mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n))_{e_1 \wedge \ldots \wedge e_1^{\omega_n}}.$$

$B_n[v_n]$ is invariant under the action of $\mathbb{C}^\ast$ which fixes $e_1 \wedge \ldots \wedge e_1^{\omega_n}$, and is the affine cone over the image of $B_n[v_n] \setminus \{ e_1 \wedge \ldots \wedge e_1^{\omega_n} \}$ in the weighted projective space $W\mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n))/\mathbb{C}(e_1 \wedge \ldots \wedge e_1^{\omega_n}) = (\mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n))_{e_1 \wedge \ldots \wedge e_1^{\omega_n}} \setminus \{ e_1 \wedge \ldots \wedge e_1^{\omega_n} \})/\mathbb{C}^\ast$.

We have

$$\beta^{-1}(\{ e_1 \wedge \ldots \wedge e_1^{\omega_n} \}) = \left\{ \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} : b_{11} \in \mathbb{C}^\ast \right\},$$

and the restriction of $\beta$ to $B_n[v_n] \setminus \beta^{-1}(\{ e_1 \wedge \ldots \wedge e_1^{\omega_n} \})$ is invariant under the $\mathbb{C}^\ast$ action above and under scalar multiplication, so induces a morphism $\hat{\beta}$ from the weighted projective space

$$(B_n[v_n] \setminus \beta^{-1}(\{ e_1 \wedge \ldots \wedge e_1^{\omega_n} \}))/\mathbb{C}^\ast = B_{[n-1]}/(\mathbb{C}^\ast)^2 = \mathbb{P}(B_{[n-1]})/\mathbb{C}^\ast$$

to $W\mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n))/\mathbb{C}(e_1 \wedge \ldots \wedge e_1^{\omega_n})$. The image of $\hat{\beta}$ is projective and hence the image of $\beta$ is a closed subvariety of the affine cone

$$\mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n))_{e_1 \wedge \ldots \wedge e_1^{\omega_n}}$$

over $W\mathbb{P}(\wedge^n(\text{Sym}^\omega \mathbb{C}^n))/\mathbb{C}(e_1 \wedge \ldots \wedge e_1^{\omega_n})$ as required. \hfill \square

**Remark 10.4.** Let $M_{[1,n-1]} \subseteq M_{n \times n}(\mathbb{C})$ be the set of $n \times n$ matrices over $\mathbb{C}$ with nonzero first column. Then the surjective morphism $\beta : B^{(0)}_{[n-1]} \rightarrow B_n[v_n]$ of Lemma 10.3 extends to an $\text{SL}(n)$-equivariant surjective morphism

$$M_{[1,n-1]} \rightarrow \text{SL}(n)[v_n]^{(0)} = \text{SL}(n)B_n[v_n]^{(0)}$$
Corollary 10.5. If $r \in \mathbb{Q}$ satisfies $qr_{\min} < r < qr_j$ for all $r_j \neq r_{\min}$ then the morphism $\beta$ of Lemma 10.3 induces an $(\text{SL}(n) \times \mathbb{C}^*)$-equivariant surjective birational morphism

$$\tilde{\beta} : \text{SL}(n) \times B_n \quad \text{W}^\mathbb{P}(B_{[n-1]}) \rightarrow (\mathbb{P}^1 \times \text{SL}(n)[\bar{p}_n])//\mathbb{C}^*$$

where $W^\mathbb{P}(B_{[n-1]} = (B_{[n-1]} \setminus \{0\})//\mathbb{C}^*$ is a weighted projective space.

Proof. By Lemma 10.1 we have

$$(\mathbb{P}^1 \times \text{SL}(n)[\bar{p}_n])//\mathbb{C}^* = (\mathbb{C} \times \text{SL}(n)B_n[\bar{p}_n]) \setminus \{(0, \text{SL}(n)(e_1 \wedge \cdots \wedge e_1^{\omega_n}))\} //\mathbb{C}^*$$

and $\beta$ induces a birational morphism to this from

$$\text{SL}(n) \times B_n (\mathbb{C} \times (B_{[n-1]}//\mathbb{C}^* \setminus (0, \mathbb{C}^* \left\{ 1 \ 0 \ \cdots \ 0 \right\}) /\mathbb{C}^* \cong \text{SL}(n) \times B_n (B_{[n-1]} \setminus \{0\})//\mathbb{C}^*$$

Remark 10.6. Here $W^\mathbb{P}(B_{[n-1]} = (B_{[n-1]} \setminus \{0\})//\mathbb{C}^*$ is the weighted projective space $(B_{[n-1]} \setminus \{0\})//\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $B_{[n-1]}$ as

$$t \cdot \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & b_{23} & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & b_{23} & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & t & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{\omega_n} \end{pmatrix}.$$ 

There is a birational map from $W^\mathbb{P}(B_{[n-1]}$ to $\text{SL}(n)[\bar{p}_n])$ which is the composition of $\tilde{\beta}$ and the birational map between the GIT quotients $(\mathbb{P}^1 \times \text{SL}(n)[\bar{p}_n])//\mathbb{C}^*$ for one choice of $r$ as in this corollary and another such that $(\mathbb{P}^1 \times \text{SL}(n)[\bar{p}_n])//\mathbb{C}^* \cong \text{SL}(n)[\bar{p}_n]$. Using Proposition 4.1 we find that this birational map takes the orbit of

$$\begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & b_{22} & b_{23} & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \in B_{[n-1]}$$

when $t \neq 0$ to

$$[e_1 \wedge (b_{22}e_2 + t^{\omega_n}e_1^{\omega_n}) \wedge \cdots \wedge \left( \sum_{i=1}^{\omega_n} t^{\omega_n-1} p_{in}(e_1, b_{22}e_2, \cdots, b_{nn}e_n + \cdots + b_{2n}e_2) \right)];$$

it is not in general well defined when $t = 0$.

This corollary leads us towards an explicit geometric description of the GIT quotient $(\mathbb{P}^1 \times \text{SL}(n)[\bar{p}_n])//\mathbb{C}^*$ for some choice of $r \in \mathbb{Q}$. However we have already noted that for another choice of $r$ we have $(\mathbb{P}^1 \times \text{SL}(n)[\bar{p}_n])//\mathbb{C}^* \cong \text{SL}(n)[\bar{p}_n]$. Using the theory of variation of GIT [47] [15] we can relate the geometry of the GIT quotient $(\mathbb{P}^1 \times \text{SL}(n)[\bar{p}_n])//\mathbb{C}^*$
as $r$ moves across the walls between the chambers where the geometry of the quotient is constant.

Recall that for GIT purposes we can use a tensor power of an ample line bundle $L_{p,q}$ to embed $\mathbb{P}^1 \times \text{SL}(n)(v_p)$ in a projective space, and the GIT quotient $(\mathbb{P}^1 \times \text{SL}(n)(v_p))/\mathbb{C}^*$ is then embedded in the corresponding GIT quotient of the ambient projective space. So first let us consider the variation of GIT picture for the action of $\mathbb{C}^*$ on a projective space $\mathbb{P}(V)$ where

$$V = \bigoplus_{\rho \in R} V_{\rho}$$

with weight $\rho$ on $V_{\rho}$ and $\text{dim } V_{\rho_0} = 1$ for $\rho_0 \in R$ satisfying $\rho_0 < \rho$ for all $\rho \in R \setminus \{\rho_0\}$. For any $r \in \mathbb{Q}$ let $\mathbb{P}(V)/\mathbb{C}^r$ denote the GIT quotient of $\mathbb{P}(V)$ by the linear action of $\mathbb{C}^r$ with respect to the ample line bundle $O_{\mathbb{P}(V)}(d)$ twisted by the character $t \mapsto t^d$ for any positive integer $d$ such that $dr \in \mathbb{Z}$. If $\rho \in R$ let $\mathbb{P}(V)/\rho \mathbb{C}^*$ be the GIT quotient $\mathbb{P}(V)/\rho \mathbb{C}^*$ where $r \in \mathbb{Q}$ lies in the chamber immediately to the right or left of $\rho$. Then $\mathbb{P}(V)/\rho_0 + \mathbb{C}^*$ is the weighted projective space

$$\mathbb{P}(V) = \left( \bigoplus_{\rho > \rho_0} V_{\rho} \setminus \{0\} \right) / \mathbb{C}^*$$

where $\mathbb{C}^*$ acts with weight $\rho - \rho_0$. Moreover if $\sigma \in R$ is not minimal (that is, $\sigma \neq \rho_0$) and not maximal, then the semistable subsets of $\mathbb{P}(V)$ with respect to the linearisations $\sigma^-$, $\sigma$ and $\sigma^+$ are given by

1. $\sum_{\rho \in R} V_{\rho} \in \mathbb{P}(V)^{s,s^-}$ if and only if there exist $\rho_1, \rho_2 \in R$ such that $\rho_1 < \sigma \leq \rho_2$ and $v_{\rho_1} \neq 0 \neq v_{\rho_2}$;

2. $\sum_{\rho \in R} V_{\rho} \in \mathbb{P}(V)^{s,s}$ if and only if there exist $\rho_1, \rho_2 \in R$ such that $\rho_1 \leq \sigma \leq \rho_2$ and $v_{\rho_1} \neq 0 \neq v_{\rho_2}$;

3. $\sum_{\rho \in R} V_{\rho} \in \mathbb{P}(V)^{s,s^+}$ if and only if there exist $\rho_1, \rho_2 \in R$ such that $\rho_1 \leq \sigma < \rho_2$ and $v_{\rho_1} \neq 0 \neq v_{\rho_2}$.

Furthermore $\mathbb{P}(V)/\sigma \mathbb{C}^* = \mathbb{P}(V)^{s,s^\pm} / \mathbb{C}^*$ while $\mathbb{P}(V)/\sigma \mathbb{C}^* = \mathbb{P}(V)^{s,s} / \sigma$ where $x \sim \sigma y$ for $x, y \in \mathbb{P}(V)^{s,s}$ if and only if $\mathbb{C}^* x \cap \mathbb{C}^* y \cap \mathbb{P}(V)^{s,s} \neq 0$.

We have birational surjective morphisms

$$h^{(\sigma)} : \mathbb{P}(V)/\sigma \mathbb{C}^* \to \mathbb{P}(V)/\sigma \mathbb{C}^*$$

which are isomorphisms except over $\mathbb{P}(V)$, on which $\mathbb{C}^*$ acts trivially and which is semistable (but not stable) for the linearisation corresponding to $\sigma$ but is not semistable for the linearisations $\sigma^\pm$. The inverse image $(h^{(\sigma)})^{-1}(\mathbb{P}(V))$ can be identified with the quotient by $\mathbb{C}^*$ of $\mathbb{P}(V)^{s,s} \setminus \mathbb{P}(V)^{s,s^\pm}$ and the fibres of $h^{(\sigma)}$ over points in $\mathbb{P}(V)$ are weighted projective spaces.

Let us use the notation $[\sum_{\rho \in R} V_{\rho}]_{\sigma\pm}$ for the image of $\sum_{\rho \in R} V_{\rho} \in \mathbb{P}(V)^{s,s\pm}$ in $\mathbb{P}(V)/\sigma\pm \mathbb{C}^*$. The fibre product

$$h^{(\sigma)} : \mathbb{P}(V)/\sigma \mathbb{C}^* \to \mathbb{P}(V)/\sigma \mathbb{C}^*$$
of $h_2^{(r)}$ and $h_2^{(r_0)}$ given by
\[ P(V)\big//_{\sigma}C^* = \left\{ (y, z) \in P(V)\big//_{\sigma_-}C^* \times P(V)\big//_{\sigma_+}C^* : h_2^{(r)}(y) = h_2^{(r_0)}(z) \right\} \]
factors as
\[ \xymatrix{ P(V) \ar[r]^g \ar[dr]_{\Gamma} \ar[dd] & P(V)\big//_{\sigma}C^* \ar[dl]_{\Lambda} \ar[dd] \ar[dr]_{\Theta} \ar[dl]_{\Psi} \ar[dd] \ar[dr]_{\Xi} \ar[dl]_{\Omega} \ar[dd] \ar[dr]_{\Phi} \ar[dl]_{\Psi} \ar[dd] \ar[dr]_{\Xi} \ar[dl]_{\Theta} \ar[dd] \ar[dr]_{\Gamma} \ar[dl]_{\Lambda} \ar[dd] & P(V)\big//_{\sigma}C^* \ar[r] \ar[dl]_{\Omega} \ar[dd] & P(V) \big//_{\sigma_-}C^* \ar[dr]_{\Xi} \ar[dd] \ar[dr]_{\Psi} \ar[dl]_{\Theta} \ar[dd] \ar[dr]_{\Gamma} \ar[dl]_{\Lambda} \ar[dd] \ar[dr]_{\Phi} \ar[dl]_{\Psi} \ar[dd] \ar[dr]_{\Xi} \ar[dl]_{\Theta} \ar[dd] \ar[dr]_{\Gamma} \ar[dl]_{\Lambda} \ar[dd] \ar[dr]_{\Phi} \ar[dl]_{\Psi} \ar[dd] \ar[dr]_{\Xi} \ar[dl]_{\Theta} \ar[dd] \ar[dr]_{\Gamma} \ar[dl]_{\Lambda} \ar[dd] & P(V)\big//_{\sigma_+}C^* \ar[r] & P(V) \big//_{\sigma}C^* } \]

If $\sum_{p \in R} v_p \in P(V)_{\mu,\sigma_-}$ and $\sum_{p \in R} \bar{v}_p \in P(V)_{\mu,\sigma_+}$ we have
\[ \left( \sum_{p \in R} v_p \right)_{\sigma_-}, \left( \sum_{p \in R} \bar{v}_p \right)_{\sigma_+} \in P(V)\big//_{\sigma}C^* \]
if and only if there exist $t_-, t_+ \in \mathbb{C}$ such that $\sum_{p \in R} (t_-)^{\mu - \rho} v_p$ and $\sum_{p \in R} (t_+)^{\mu - \sigma} \bar{v}_p$ exist and are equal and nonzero, where for $\sum_{p \in R} t^{\mu - \rho} v_p$ to exist when $t \in \mathbb{C}$ means that if $t = 0$ then $v_p = 0$ whenever $\rho < \sigma$ and $\sum_{p \in R} t^{\mu - \sigma} \bar{v}_p$ is $v_\sigma$.

The birational surjective morphism $g_0^{(r)} : P(V)\big//_{\sigma}C^* \to P(V)\big//_{\sigma}C^*$ is an isomorphism over the dense open subset $P(V)_{\mu,\sigma_-} \cap P(V)_{\mu,\sigma_+}/C^*$ of $P(V)\big//_{\sigma}C^*$, which is the complement of the subset $(h_2^{(r)})^{-1}(P(V)_{\sigma})$ represented by those $\sum_{p \in R} v_p \in \mathbb{C}^N$ with $v_{\rho_1} \neq 0 \neq v_\sigma$ for some $\rho_1 < \sigma$ but $v_{\rho_2} = 0$ for all $\rho_2 > \sigma$. The fibre of $g_0^{(r)}$ over such a $\sum_{p \in R} v_p$ is the weighted projective space $W\mathbb{P}(\bigoplus_{p \in \mathbb{C}} V_p)$ represented by all $\sum_{p \in R} w_p$ with $w_\sigma = v_\sigma$ and $w_{\rho_2} \neq 0$ for some $\rho_2 > \sigma$ and $w_\rho = 0$ if $\rho < \sigma$.

Repeating this construction gives us for any $\sigma_0 \in R \setminus \{ \rho_0 \}$
\[ P(V)\big//_{[\rho_0 +, \sigma_0-]}C^* = \left\{ \left( \sum_{p \in R} v_p^{(\rho_0-)} \right)_{\rho_0 < \sigma} \in \bigcap_{\rho_0 < \sigma < 0} P(V)\big//_{\sigma}C^* : \sigma_0 < \rho < \sigma_0 \Rightarrow \exists t_0^{(\rho_0-)}, t_0^{(\rho_0-)} \in \mathbb{C} \text{ such that} \right. \]
\[ \sum_{p \in R} (t_-)^{\rho - \sigma} v_p^{(\rho_0-)} \text{ and } \sum_{p \in R} (t_+)^{\rho - \sigma} v_p^{(\rho_0+)} \text{ exist, are equal and nonzero} \} \]

The projections $g_{[\rho_0 +, \sigma_0-]} : P(V)\big//_{[\rho_0 +, \sigma_0]}C^* \to P(V)\big//_{\rho_0 +}C^*$ and $g_{[\rho_0 +, \sigma_0-]} : P(V)\big//_{[\rho_0 +, \sigma_0]}C^* \to P(V)\big//_{\sigma_0 -}C^*$ are birational surjective morphisms which
are both isomorphisms over the dense open subset

\[ \bigcap_{\rho_0 < \rho \leq \sigma_0} \mathbb{P}(V)^{\rho, \rho} / C^* \]

of \( \mathbb{P}(V)/_{\rho_0 + \sigma_0} C^* \) and \( \mathbb{P}(V)/_{\sigma_0} C^* \). By [47] Theorem 3.5 \( g^{(r)} : \mathbb{P}(V)/_{\sigma} C^* \to \mathbb{P}(V)/_{\sigma_0} C^* \) is the blow-up of \( \mathbb{P}(V)/_{\sigma_0} C^* \) along an ideal sheaf \( I_{\sigma}/_{\sigma_0} C^* \) on \( \mathbb{P}(V)/_{\sigma_0} C^* \) supported on \((h^{(r)})^{-1}(\mathbb{P}(V))\), and it follows that \( g^{[\rho + \sigma_0, \rho_0]} : \mathbb{P}(V)/_{\rho_0 + \sigma_0} C^* \to \mathbb{P}(V)/_{\rho_0 + \sigma_0} C^* \) is an iterated blow-up along a sequence of ideal sheaves.

A similar picture is given by the theory of variation of GIT relating the geometry of the GIT quotients \((\mathbb{P}^1 \times \text{SL}(n)[\mathbb{P}_n])//_{qr} C^* \) for \( r \in \mathbb{Q} \), although more care needs to be taken here since there is in general singular. Now the walls we need to consider are given by \( qr_j \) where \( r_j \) is a weight of the action of the subgroup \( C^* \) of \( \bar{U} \) on \( \text{Sym}^n C^* \) which is strictly larger than the smallest weight \( r_{\min} \) defined in (28). For each such weight \( r_j \), in order to compare the geometry of the GIT quotients \((\mathbb{P}^1 \times \text{SL}(n)[\mathbb{P}_n])//_{qr_j} C^* \) for \( r \) very slightly greater than and less than \( qr_j \), we need to understand the connected components of the fixed point set \( \text{SL}(n)[\mathbb{P}_n]^{C^*} \) of the \( C^* \) action on \( \text{SL}(n)[\mathbb{P}_n] \) which lie in the linear subspace of \( \mathbb{P}(\mathbb{A}^n(\text{Sym}^n C^*)) \) on which \( C^* \) acts with weight \( r_j \). Let

\[ \text{SL}(n)[\mathbb{P}_n]^{C^*} \]

denote the union of these connected components. Then we have birational morphisms

\[
\begin{align*}
&\mathbb{P}^1 \times \text{SL}(n)[\mathbb{P}_n]//_{qr_j} C^* \\
\xrightarrow{(h^{(r)})^{-1}} &\mathbb{P}^1 \times \text{SL}(n)[\mathbb{P}_n]//_{qr_j} C^* \\
\xleftarrow{(h^{(r)}_n)^{-1}} &\mathbb{P}^1 \times \text{SL}(n)[\mathbb{P}_n]//_{qr_j} C^*
\end{align*}
\]

which are isomorphisms except over the fixed point set \( \{0\} \times \text{SL}(n)[\mathbb{P}_n]^{C^*} \) in \( \mathbb{P}^1 \times \text{SL}(n)[\mathbb{P}_n] \). This fixed point set is semistable (but not stable) for the linearisation corresponding to \( qr_j \), but is unstable for the linearisations \( qr_j \pm \). Here \((h^{(r)}_n)^{-1}(\{0\} \times \text{SL}(n)[\mathbb{P}_n]^{C^*}) \) is the quotient by \( C^* \) of the points \((x, y) \in \mathbb{C} \times \text{SL}(n)[\mathbb{P}_n] \) such that \((x, y) \notin \{0\} \times \text{SL}(n)[\mathbb{P}_n]^{C^*} \) but \( t^{\pm 1}(x, y) \) converges to a point in \( \{0\} \times \text{SL}(n)[\mathbb{P}_n]^{C^*} \) as \( t \in \mathbb{C} \) converges to 0. Thus

\[
(h^{(r)}_n)^{-1}(\{0\} \times \text{SL}(n)[\mathbb{P}_n]^{C^*}) = \\
\left\{ (0, y) \in \{0\} \times \text{SL}(n)[\mathbb{P}_n] \setminus \text{SL}(n)[\mathbb{P}_n]^{C^*} \mid \lim_{t \to 0^+} ty \in \text{SL}(n)[\mathbb{P}_n]^{C^*} \right\} / C^*
\]
and

\[(h^i)^{-1}(0) \times \overline{\text{SL}(n)[p_n]} = \left\{ (x, y) \in (C \times \text{SL}(n)[p_n]) \setminus (0) \times \overline{\text{SL}(n)[p_n]} \mid \lim_{t \to 0} y \in \overline{\text{SL}(n)[p_n]} \right\} / C^*.

Thus the geometry of the fixed point sets \(\overline{\text{SL}(n)[p_n]}\) and the behaviour of the \(C^*\) action in a neighbourhood of these fixed point sets determines the geometry of the GIT quotients \((\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})/q \times C^*\) and thus of the closure \(\overline{\text{SL}(n)[p_n]}\) in \(\mathbb{P}(\wedge^n(\text{Sym}^n C^n))\). As we observed at the start of this section, using this and Theorem 7.4 we can hope to describe the geometry of the canonical affine completion \(\overline{\text{SL}(n)}// U\) of \(\text{SL}(n)// U\).

We can also consider the fibre product

\[h^j : (\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})/q_j \times C^* \to (\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^*

of \(h^k_j\) and \(h^j_k\) given by

\[\{(y, z) \in (\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^* \times (\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^* : h^j(y) = h^j_k(z)\}.

In fact (cf. [47] Theorem 3.5) such fibre products may be reducible, and it is more useful to consider the irreducible component of this fibre product which dominates \((\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^*\); let us denote this by

\[\overline{\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]}}// q_j C^*

The restriction of \(h^j\) to \((\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^*\) factors as

\[(\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^*\]

\[\overline{\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]}}// q_j C^*\]

\[\overline{\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]}}// q_j C^*\]

\[\overline{\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]}}// q_j C^*\]

\[(32)\]

where \(g^{(j)}_{\pm}\) and \(h^{(j)}_{\pm}\) are surjective birational morphisms. By [47] Theorem 3.5 the morphisms \(g^{(j)}_{\pm} \to (\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^*\) are the blow-ups of \((\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^*\) along ideal sheaves \(I^{(j)}_{\pm}// q_j C^*\) supported on \((h^{(j)}_{\pm})^{-1}(0) \times \overline{\text{SL}(n)[p_n]}\). These ideal sheaves \(I^{(j)}_{\pm}// q_j C^*\) are the induced sheaves of invariants on \((\mathbb{P}^1 \times \overline{\text{SL}(n)[p_n]})// q_j C^*\) generated by
$H^0(\mathbb{P}^1 \times SL(n)[p_n], O_{\mathbb{P}^1} \otimes O_{\mathbb{P}^1(Sym^nC^*)}(dq))^\mathbb{C}$ for a suitable large positive integer $d$, where the action of $\mathbb{C}^*$ on $O_{\mathbb{P}^1} \otimes O_{\mathbb{P}^1(Sym^nC^*)}(dq))$ is twisted by $qr_j \pm$.

Repeating this construction for every wall between $qr_{\min}^+$ and $p + qr_{\min}^-$ and using [10.5] will give us birational morphisms

\[ (\mathbb{P}^1 \times SL(n)[p_n]) // \sim_{qr_{\min}^+ + p + qr_{\min}^-} \mathbb{C}^* \]

where $g_+$ and $g_-$ are surjective birational morphisms which are iterated blow-ups along the pullbacks of the ideal sheaves $I_j \otimes \mathbb{C}^*$ which are supported on the proper transforms of $(h^{(j)})^{-1}(0) \times SL(n)[p_n]$, for $j \in J$ with $r_j \neq r_{\min}$. From this we obtain an $SL(n) \times \mathbb{C}^*$-equivariant birational morphism

\[ g : SL(n) \times B_n \to SL(n)[p_n] \subset \mathbb{P}(\Lambda^n(Sym^\alpha C^*)) \]

where $\mathbb{P}(B_{[n-1]})$ is an iterated blow-up of the weighted projective space $\mathbb{P}(B_{[n-1]})$.

**Remark 10.7.** We can make this iterated blow-up $\mathbb{P}(B_{[n-1]})$ of $\mathbb{P}(B_{[n-1]})$ and surjective birational morphism

\[ \mathbb{P}(B_{[n-1]}) \to B_n[p_n] \subset \mathbb{P}(\Lambda^n(Sym^\alpha C^*)) \]

more explicit using Remark [10.6]. We can write

\[ e_1 \wedge (b_{22}e_2 + e_1^{a_2}) \wedge \cdots \wedge \left( \sum_{i=1}^n p_{in}(e_1, b_{22}e_2, \ldots, b_{mn}e_n + \cdots + b_{2n}e_2) \right) \]

as

\[ \sum_{i_1 + \cdots + i_n = a} \sum_{j_1 \in I(a_{i_1}, 1)} \cdots \sum_{j_n \in I(a_{i_n}, n)} \psi^{i_1 \ldots i_n}_{j_1 \ldots j_n}(b_{22}, b_{23}, \ldots, b_{mn}) e_{j_1}^{a_{i_1}} \otimes \cdots \otimes e_{j_n}^{a_{i_n}} \]

\[ \in \Lambda^n(Sym^\alpha C^*) = \bigoplus_{i_1 + \cdots + i_n = a} \Lambda^{i_1}(C^*) \otimes \cdots \otimes \Lambda^{i_n}(Sym^\alpha C^*) \]

where $[e_{j}^{[a,i]} : j \in I(a,i)]$ is the standard basis for $\Lambda^i(Sym^\alpha C^*)$ for any $a, i \in \mathbb{N}$ and $\psi^{i_1 \ldots i_n}_{j_1 \ldots j_n}(b_{22}, b_{23}, \ldots, b_{mn})$ is a polynomial in $b_{22}, b_{23}, \ldots, b_{mn}$. Then the birational map from
$W\mathcal{P}(B_{[n-1]})$ to $\text{SL}(n)[v_n]$ described in Remark [10.6] takes

$$
\begin{bmatrix}
t & 0 & \cdots & 0 \\
0 & b_{22} & b_{23} & \cdots & b_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & b_{nn}
\end{bmatrix} \in W\mathcal{P}(B_{[n-1]})
$$

with $t \neq 0$ to

$$[e_1 \wedge (b_{22}e_2 + t^{\omega_2-1}e_1^{\omega_2}) \wedge \cdots \wedge \left(\sum_{i=1}^n t^{\omega_i-1} p_{in}(e_1, b_{22}e_2, \ldots, b_{mn}e_n + \cdots + b_{2n}e_2)\right)]$$

$$= \left[ \sum_{i_1 + \cdots + i_n = n} t^{i_1 \omega_1 + \cdots + i_n \omega_n} \sum_{j_1 \in I(\omega_1, 1)} \cdots \sum_{j_n \in I(\omega_n, n)} \psi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(b_{22}, b_{23}, \ldots, b_{mn}) e_{j_1}^{[\omega_1, i_1]} \otimes \cdots \otimes e_{j_n}^{[\omega_n, i_n]} \right]$$

for any constant $c \in \mathbb{Q}$. Note that the set $\{r_j : j \in J\}$ of weights for the action of the subgroup $\mathbb{C}^*$ on $\wedge^n (\text{Sym}^{\omega_n} \mathbb{C}^n)$ is a translate in $\mathbb{Q}$ of $\{i_1 \omega_1 + \cdots + i_n \omega_n : i_1 + \cdots + i_n = n\}$.

Let $R$ be the set of $\rho = \{i_1 \omega_1 + \cdots + i_n \omega_n : i_1 + \cdots + i_n = n\}$ such that there exists $\psi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(b_{22}, b_{23}, \ldots, b_{mn})$ not identically zero with $j_\ell \in I(\omega_\ell, \ell)$ for $1 \leq \ell \leq n$ and $\rho = i_1 \omega_1 + \cdots + i_n \omega_n$. If $\rho \in R$ then

$$V_\rho = \bigoplus_{i_1 + \cdots + i_n = n, i_1 \omega_1 + \cdots + i_n \omega_n = \rho} \wedge^{i_1}(\mathbb{C}^n) \otimes \cdots \otimes \wedge^{i_n}(\text{Sym}^{\omega_n} \mathbb{C}^n) \subseteq \wedge^n(\text{Sym}^{\omega_n} \mathbb{C}^n)$$

is the corresponding weight space for the action of $\mathbb{C}^*$ on $\wedge^n(\text{Sym}^{\omega_n} \mathbb{C}^n)$.

Recall that the weighted projective space $\overline{W\mathcal{P}}(B_{[n-1]})$ is the quotient of $B_{[n-1]} \setminus \{0\}$ by a $\mathbb{C}^*$-action with weights $1, \omega_2 - 1, \ldots, \omega_n - 1$. The iterated blow-up $\text{WPR}(B_{[n-1]})$ can be obtained by blowing up $\overline{W\mathcal{P}}(B_{[n-1]})$ successively along the pullbacks of the ideal sheaves $\mathcal{I}_j^+//_{q_j, j} \mathbb{C}^*$ for $j \in J$ such that $r_j \neq r_{\min}$, or equivalently as the quotient by the induced $\mathbb{C}^*$-action on the result of blowing up $B_{[n-1]} \setminus \{0\}$ successively along the associated pullbacks of the ideal sheaves $\mathcal{I}_j^+//_{q_j, j} \mathbb{C}^*$, in decreasing order of $r_j$. If $\rho_j$ is the corresponding element of $R$ then the ideal sheaf $\mathcal{I}_j^+//_{q_j, j} \mathbb{C}^*$ is given by the $\mathbb{C}^*$-invariants in the ideal sheaf generated by the polynomial $r$ together with the polynomials

$$\psi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(b_{22}, b_{23}, \ldots, b_{mn})$$

satisfying $i_1 + \cdots + i_n = n$ and $i_1 \omega_1 + \cdots + i_n \omega_n < \rho_j$ as well as $j_\ell \in I(\omega_\ell, \ell)$ for $\ell = 1, \ldots, n$.

10.1. **Partial resolutions of singularities.** The iterated blow-up $\overline{W\mathcal{P}}(B_{[n-1]})$ of the weighted projective space $\overline{W\mathcal{P}}(B_{[n-1]})$ is in general singular (see §10.2 below). However we can include additional blow-ups in the construction of $\overline{W\mathcal{P}}(B_{[n-1]})$ from $\overline{W\mathcal{P}}(B_{[n-1]})$ to obtain an iterated blow-up $\overline{W\mathcal{P}}(B_{[n-1]})$ of $\overline{W\mathcal{P}}(B_{[n-1]})$ with a surjective birational morphism

$$\overline{W\mathcal{P}}(B_{[n-1]}) \to B_n[v_n] \subseteq \mathbb{P}(\wedge^n(\text{Sym}^{\omega_n} \mathbb{C}^n)))$$
such that the centre of each blow-up in the construction of \( W \mathcal{P}(B_{[n-1]}) \) from \( W \mathcal{P}(B_{[n-1]}) \) has only finite quotient singularities. Then the induced morphism

\[
\text{SL}(n) \times_{B_n} W \mathcal{P}(B_{[n-1]}) \to \text{SL}(n)[p_\kappa] = \overline{\text{GL}(n)[p_\kappa]}
\]

is a partial resolution of singularities of the compactification \( \text{SL}(n)[p_\kappa] \) of \( \text{SL}(n)/(\hat{U} \cap \text{SL}(n)) = \text{GL}(n)/\hat{U} \) (in the sense of, for example, [28]): it is a surjective birational morphism and \( \text{SL}(n) \times_{B_n} W \mathcal{P}(B_{[n-1]}) \) has only finite quotient singularities.

10.2. **Example: jet differentials and curvilinear Hilbert schemes on \( \mathbb{C}^3 \).** In this final subsection we demonstrate the computational power of our construction for the example studied in [§5.3] when \( n = d = 3 \) and also Example 9.3; that is, when

\[
\hat{U} = \mathbb{G}_3 = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \alpha_2^2 & 2\alpha_1\alpha_2 \\ 0 & 0 & \alpha_1^3 \end{pmatrix} : \alpha_1 \in \mathbb{C}^\times, \alpha_2, \alpha_3 \in \mathbb{C} \right\}
\]

is the group of 3-jets of reparametrisations of \( \mathbb{C}^3 \) at the origin. Recall from [§5.3] that \( \text{SL}(3) \cdot p_\kappa \subset \mathbb{P}(\wedge^3 \text{Sym}^3 \mathbb{C}^3) \cong \mathbb{C}^3_{\text{sl}} \) can be identified with the curvilinear punctual Hilbert scheme of 3 points on \( \mathbb{C}^3 \). Following Remark [10.7] we can describe the iterated blow-up \( W \mathcal{P}(B_{[2]}) \) of \( W \mathcal{P}(B_{[2]}) \) and the surjective birational morphism

\[
\beta : \overline{W \mathcal{P}(B_{[2]})} \to \overline{B_2[p_2]} \subset \mathbb{P}(\wedge^3 \text{Sym}^3 \mathbb{C}^3)
\]

explicitly. Here \( \omega_1 = 1, \omega_2 = 2, \omega_3 = 3 \) and therefore \( W \mathcal{P}(B_{[2]}) \) is the weighted projective space \( \mathbb{P}[1, 1, 2, 2] \) with coordinates \( t, b_{22}, b_{23}, b_{33} \) respectively. The birational map defined in Remark [10.6] takes the point \( [t : b_{22} : b_{23} : b_{33}] \in \mathbb{P}[1, 1, 2, 2] \) to

\[
e_1 \land (b_{22}e_2 + te_1^2) \land (b_{33}e_3 + b_{23}e_2 + 2tb_{22}e_1e_2 + t^2e_1^3) =
\]

\[
t^0(b_{22}b_{33}e_1 \land e_2 \land e_3) + t^1(b_{33}e_1 \land e_1^2 \land e_3 + b_{23}e_2 \land e_1^2 \land e_2) +
\]

\[
+ t^2(b_{22}e_1 \land e_2 \land e_1^3 + b_{22}e_2 \land e_1^2 \land e_1 e_2) + t^3(e_1 \land e_1^2 \land e_1^3),
\]

and therefore the associated ideals are

\[
I_3 = (t, b_{22}, b_{33}, b_{23}); I_2 = (t, b_{33}, b_{23}); I_1 = (t, b_{22}b_{33}).
\]

Since \( \text{Spec}I_3 = \emptyset \) in \( \mathbb{P}[1, 1, 2, 2] \), we blow up along \( I_3, I_2, I_1 \). In fact the second blow-up along \( (t, b_{22}b_{33}) \) can be replaced by a blow-up along \( (t, b_{22}, b_{33}) \) followed by a blow-up along \( (t, b_{22}) \) and \( (t, b_{33}) \) to get an iterated blow-up

\[
\overline{W \mathcal{P}(B_{[2]})} \subset \mathbb{P}[1, 1, 2, 2] \times \mathbb{P}[1, 2, 2] \times \mathbb{P}[1, 1, 2] \times \mathbb{P}[1, 1] \times \mathbb{P}[1, 2]
\]

of \( W \mathcal{P}(B_{[2]}) \) which has only finite quotient singularities, and an induced surjective birational morphism

\[
\tilde{\beta} : \text{SL}(3) \times_{B_3} \overline{W \mathcal{P}(B_{[2]})} \to \overline{\text{SL}(3)[p_3]} \subset \mathbb{P}(\wedge^3 \text{Sym}^3 \mathbb{C}^3)
\]

which is a partial resolution of singularities of the curvilinear punctual Hilbert scheme of 3 points on \( \mathbb{C}^3 \).
To describe $\beta$ explicitly let us use weighted homogeneous coordinates $[t^{(1)} : b_{23}^{(1)} : b_{33}^{(1)}]$ on $\mathbb{P}[1, 2, 2]$ and similarly on $\mathbb{P}[1, 1, 2], \mathbb{P}[1, 1]$ and $\mathbb{P}[1, 2]$. We first blow up along $\text{Spec} \mathcal{O}$.

(1) On the affine chart where $t^{(1)} \neq 0$ we have $b_{33} = tb_{33}^{(1)}, b_{23} = tb_{23}^{(1)}$ and substituting this into (31) we get

$$e_1 \land (b_{22}e_2 + te_1^2) \land (b_{33}^{(1)}e_3 + b_{22}^{(1)}e_2 + b_{22}e_1e_2 + e_1^2)$$

After the second blow-up along the ideal $(b_{22}, t)$ we get a well-defined morphism into the projective space $\mathbb{P}(\Lambda^3 \text{Sym}^3 \mathbb{C}^3)$ with a nonzero term involving $e_1 \land (b_{22}e_2 + te_1^2) \land e_1^2$.

(2) Similarly, on the affine chart where $b_{33}^{(1)} \neq 0$ we have $t = b_{33}t^{(1)}, b_{23} = b_{33}b_{23}^{(1)}$, and substituting into (31) we get

$$e_1 \land (b_{22}e_2 + te_1^2) \land (e_3 + b_{22}^{(1)}e_2 + 2t^{(1)}b_{22}e_1e_2 + (t^{(1)})te_1^3)$$

which extends to a well-defined morphism into the projective space $\mathbb{P}(\Lambda^3 \text{Sym}^3 \mathbb{C}^3)$ after blowing up along the ideal $(b_{22}, t)$ with a nonzero term involving $e_1 \land (b_{22}e_2 + te_1^2) \land e_3$.

(3) Finally, on the affine chart where $b_{23}^{(1)} \neq 0$ we have $t = b_{23}t^{(1)}, b_{33} = b_{23}b_{33}^{(1)}$, and substituting into (31) we get

$$e_1 \land (b_{22}e_2 + te_1^2) \land (b_{33}^{(1)}e_3 + e_2 + 2t^{(1)}b_{22}e_1e_2 + t^{(1)}te_1^3) =$$

$$= (e_1 \land te_1^2 \land e_2) + (e_1 \land (b_{22}e_2 + te_1^2) \land (b_{33}^{(1)}e_3 + 2t^{(1)}b_{22}e_1e_2 + (t^{(1)})te_1^3))$$

Next, blowing up along $(t, b_{22}, b_{33}^{(1)})$ we get a morphism as follows:

(a) When $t^{(1)} \neq 0$ we have $b_{22} = tb_{22}^{(1)}, b_{33} = tb_{33}^{(2)}$ and

$$(e_1 \land e_1^2 \land e_2) + (e_1 \land (b_{22}e_2 + te_1^2) \land (b_{33}^{(2)}e_3 + 2t^{(1)}b_{22}e_1e_2 + (t^{(1)})e_1^3))$$

which is already well-defined.

(b) When $b_{22} \neq 0$ we have $t = t^{(2)}b_{22}, b_{33} = b_{23}b_{33}^{(2)}$ and

$$(e_1 \land t^{(2)}e_1^2 \land e_2) + (e_1 \land (b_{22}e_2 + te_1^2) \land (b_{33}^{(2)}e_3 + 2t^{(1)}b_{22}e_1e_2 + (t^{(1)})e_1^3))$$

This becomes well-defined when we blow up along $(t^{(1)}, b_{33}^{(2)})$.

(c) When $b_{33}^{(1)} \neq 0$ we have $t = t^{(2)}b_{33}^{(1)}, b_{22} = b_{23}b_{33}^{(1)}$ and

$$(e_1 \land t^{(2)}e_1^2 \land e_2) + (e_1 \land (b_{22}e_2 + te_1^2) \land (e_3 + 2t^{(1)}b_{22}e_1e_2 + (t^{(1)})e_1^3))$$

and now after blowing up along $(t, b_{22})$ we get a well-defined morphism, as the coefficient of $e_1 \land (b_{22}e_2 + te_1^2) \land e_3$ is nonzero.

This gives us an explicit surjective birational morphism

$$\tilde{\beta} : \text{SL}(3) \times_{B} \text{W}_{\mathbb{P}}(B_{[2]}) \rightarrow \text{SL}(3)[p_3]$$
which is a partial resolution of singularities of the curvilinear punctual Hilbert scheme of 3 points on $\mathbb{C}^3$.

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