A VERAGE STRUCTURES ASSOCIATED TO A FINSLER SPACE

RICARDO GALLEGOS TORROMÉ

Abstract. Given a Finsler space \((M,F)\) on a manifold \(M\), the averaging method associates to Finslerian geometric objects affine geometric objects living on \(M\). In particular, a Riemannian metric is associated to the fundamental tensor \(g\) and an affine, torsion free connection is associated to the Chern-Rund connection. As an illustration of the technique, a generalization of the Gauss-Bonnet theorem to Berwald surfaces using the average metric is presented. The parallel transport and curvature endomorphisms of the average connection are obtained. The holonomy group for a Berwald space is discussed. New affine, local isometric invariants of the original Finsler metric. The heredity of the property of symmetric space from the Finsler space to the average Riemannian metric is proved.

1. Introduction

It is notable that many fundamental results can be generalized from the Riemannian to the Finslerian category by properly adapting the proofs, in some cases in a rather straightforward way [4]. The present paper is motivated by the idea of interpreting this general phenomenon from the perspective of equivalence classes in the Finsler category such that each class contains an affine or Riemannian representative. The reason for such expectation is that, if a given proposition can be casted in terms of notions defined in the coset space of equivalent classes, then it can be investigated using affine or Riemannian methods.

The method that we use to define the equivalence relations is by averaging Finslerian objects. These operations transform geometric objects living in the tangent space to geometric objects living in the base manifold \(M\). The average of the fundamental tensor appeared first in [19], in connection with emergent quantum mechanics [8] and in a series of pre-prints of the author (RGT), of which the present work is the latest version. Preliminary versions of the general theory of averaging were also discussed by the author, for instance in [9]. A revision of the theory is presented in this paper. We will show that we can consider the average of the Finslerian metric (in the form of the fundamental tensor), connection, curvatures or the average differential operators. Two geometric operators/structures will be equivalent if they have the same average. It turns out that results stated in terms of the defining properties of the coset space and that can be proved using Riemannian or affine methods automatically upgrade to the Finsler category. In particular, the method is very well suited for two specific types of problems. The first is in applications to Berwald spaces. The second in the application to isometry properties of the Finsler metric and related notions.

In this paper we discuss the general method of average and its application to several geometric operators and structures. These applications are illustrated by several examples. This paper is organized as follows. In section 2, several standard notions of Finsler geometry are introduced. In section 3 we explain the measure used for the averaging operation. The averaging can be casted in terms of integration along vertical fibers of pull-back bundles \(\hat{\pi}^* T^{(p,q)} M\) over the manifold \(TM \setminus \{0\}\).
The averaging associates a Riemannian structure \((M,h)\) to the initial Finsler space \((M,F)\) by performing an average of the components of the fundamental tensor \(g\) on the indicatrix \(I_u\) at each point \(x \in M\). In section 4 we discuss the average connection \((\nabla)\) associated to a linear connection \(\nabla\) on \(\pi^*TM\). We show that for Berwald spaces, the Riemann curvature tensor of the average metric \(h\) can be obtained by averaging the corresponding tensor of the Chern-Rund connection of \(g\). The average metric is used to prove several generalizations of Riemannian results to Berwald spaces. In particular, we have considered a generalization of the Gauss-Bonnet theorem for Berwald surfaces as an example of application of the method. In section 5 the parallel transport operators and curvatures of the average connection \((\nabla)\) are discussed in terms of the corresponding parallel transport and curvatures of the linear connection \(\nabla\) on \(\pi^*TM\). The metrizability of the holonomy of Berwald spaces is discussed. In section 6, isometries are discussed and the proof of the Mayer-Steenrod theorem sketched from the point of view of the averaging method. Symmetric spaces are discussed too from this new framework. The average of the curvature tensor of the connection \(\nabla\) provides new affine isometric invariants of the Finsler metric.

2. Basic notions of Finsler geometry

**Notation.** \((U,x)\) will denote a local coordinate chart of the \(n\)-dimensional manifold \(M\), where a point \(x \in U\) has local coordinates \((x^1,...,x^n)\) and \(U \subset M\) is an open set. \(TM\) is the tangent bundle of \(M\). The slit tangent bundle \(\tilde{\pi} : N \to M\) is the bundle over \(M\) with \(N = TM \setminus \{0\}\). Fixed a local chart \((U,x)\) on the manifold \(M\), a point \(x \in U\) will have coordinates \((x^1,...,x^n)\) and a tangent vector \(y = y^i \frac{\partial}{\partial x^i} \in T_xM\) at \(x \in M\) is determined by its components \(y = (y^1,...,y^n)\) respect to the basis \(\{\frac{\partial}{\partial x^i}\}_{i=1}^n\) of \(T_xM\). Note that we will use Einstein’s convention for up and down equal indices, if anything else is not directly stated. The set of sections of a bundle \(\mathcal{E}\) is denoted by \(\Gamma \mathcal{E}\), except for differential forms that we follow the usual notation. Each local chart \((U,x)\) on \(M\) induces a local chart on \(TM\) denoted by \((TU,x,y)\) such that a point \(u \in TU\) with \(\tilde{\pi}(u) = x\) and corresponding to the tangent vector \(y = y^i \frac{\partial}{\partial x^i} \in T_xM\) has local natural coordinates \((x^1,...,x^n,y^1,...,y^n)\).

**Definition 2.1.** A Finsler space on the manifold \(M\) is a pair \((M,F)\) where \(F\) is a non-negative, real function \(F : M \to [0,\infty[\) such that

- It is smooth in the slit tangent bundle \(N\),
- Positive homogeneity holds: \(F(x,\lambda y) = \lambda F(x,y)\) for every \(\lambda \in [0,\infty[\),
- Strong convexity holds: the Hessian matrix

\[
g_{ij}(x,y) := \frac{1}{2} \frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \quad i,j = 1,...,n
\]

is positive definite on \(N\).

The minimal regularity requirement for the Finsler function \(F\) is to be a \(C^1\)-smooth function on \(N\). However, if the second Bianchi identities are required, then it is necessary for \(F\) to be at least \(C^3\)-smooth on \(N\). The matrix \((g)_{ij} := g_{ij}(x,y)\) is the matrix components of the fundamental tensor \(g\) at the point \(u = (x,y) \in N\).

**Definition 2.2.** Let \((M,F)\) be a Finsler space and \((TU,x,y)\) a local chart induced on \(N\) from the coordinate system \((U,x)\) of \(M\). The components of the Cartan tensor are defined by the collection of functions

\[
A_{ijk} = \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad i,j,k = 1,...,n.
\]
The components \( \{ A_{ijk}, i,j,k = 1, ..., n \} \) are homogeneous functions of degree zero in the coordinates \((y^1, ..., y^n)\) and totally symmetric under permutation of the indices \(i,j,k\). The condition

\[ A_{ijk}(x,y) = 0, \quad \forall y \in T_x M, x \in M \]

characterizes Riemannian geometry among the general class of Finsler geometries.

**Definition 2.3.** Let \((M,F)\) be a Finsler space. The indicatrix over the point \(x \in M\) is the submanifold \(I_x \hookrightarrow T_x M\)

\[ I_x := \{ y \in T_x M \mid F(x,y) = 1 \} \]

The indicatrix \(I_x\) is a compact, strictly convex submanifold of \(T_x M\) \([4, 17]\). We denote by \(I\) the fibered manifold \(\pi : I \rightarrow M\) with \(\pi^{-1}(x) = I_x\) and the base manifold is \(M\).

**2.1. Definition of a non-linear connection of \(N\).** Let us consider the slit bundle \(\tilde{\pi} : N \rightarrow M\).

**Definition 2.4.** A non-linear connection of \(N\) is a distribution \(H \subset TN\) supplementary to the canonical vertical distribution \(V = \ker d\tilde{\pi}\).

Given a Finsler space \((M,F)\), there is defined a non-linear connection in the manifold \(N\). In a local natural coordinate chart \((TU, x, y)\) of \(N\) the collection of local sections

\[ \{ \partial/\partial y^1|_u, ..., \partial/\partial y^n|_u, u \in \tilde{\pi}^{-1}(x), x \in U \} \]

determines a local frame for the vertical distribution \(V\). To obtain a supplementary distribution \(H\) we use a standard local construction \([4]\). First, let us introduce the non-linear connection coefficients \(N^i_j\) by the expression

\[ N^i_j F = \gamma^i_{jk} F - A^i_{jk} \gamma^{k}_{rs} F, \quad i,j,k,r,s = 1, ..., n, \]

where the formal second kind Christoffel’s symbols \(\gamma^i_{jk}(x,y)\) are defined by

\[ \gamma^i_{jk} = \frac{1}{2} g^{is} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^j} \right), \quad i,j,k = 1, ..., n \]

and also \(A^i_{jk} := g^{il} A_{lkj}\) and \(g^{ii} g_{ij} = \delta^i_j\). A tangent basis for \(T_u N\) is determined by the vectors

\[ \left\{ \frac{\delta}{\delta x^1}|_u, ..., \frac{\delta}{\delta x^n}|_u, F \frac{\partial}{\partial y^1}|_u, ..., F \frac{\partial}{\partial y^n}|_u \right\}, \quad \frac{\delta}{\delta x^i}|_u = \frac{\partial}{\partial x^i}|_u - N^i_j \frac{\partial}{\partial y^j}|_u, \quad i,j = 1, ..., n. \]

The collection of local sections

\[ \left\{ \frac{\delta}{\delta x^1}|_u, ..., \frac{\delta}{\delta x^n}|_u, u \in \tilde{\pi}^{-1}(x), x \in U \right\} \]

determines a local frame for the horizontal distribution \(H\) \([2, 4]\). Given \(\tilde{X} \in \Gamma TN\), the horizontal component is denoted by \(H(\tilde{X})\) and the vertical component by \(V(\tilde{X})\). The horizontal lift of tangent vectors is defined by the homomorphism

\[ \iota_u : T_x M \rightarrow T_u N, \quad X = X^i \frac{\partial}{\partial x^i}|_x \mapsto \iota_u(X) = X^i \frac{\delta}{\delta x^i}|_u, \]
for fixed $u \in \pi^{-1}(x)$. For these local horizontal sections (and therefore for any local horizontal section) the relation

$$\frac{\delta}{\delta x^i}|_u \cdot F = 0$$

holds good.

The dual basis associated to the local tangent basis of the dual vector space $T_u^* N$, (Definition 2.5)

$$\begin{cases} dx^1|_u, \ldots, dx^n|_u, \frac{\delta y^1}{F}|_u, \ldots, \frac{\delta y^n}{F}|_u \end{cases}, \quad \frac{\delta y^i}{F}|_u = \frac{1}{F}(dy^i + N^j dx^j)|_u, \quad i, j = 1, \ldots, n.$$

**Definition 2.5.** Let $(M, F)$ be a Finsler space. The fundamental and the Cartan tensors are defined in the natural local coordinate system $(TU, x, y)$ by the equations

1. **The fundamental tensor** is

$$g(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \; dx^i \otimes dx^j,$$

2. **The Cartan tensor** is

$$A(x, y) := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} \frac{\delta y^i}{F} \otimes dx^j \otimes dx^k = A_{ijk} \frac{\delta y^i}{F} \otimes dx^j \otimes dx^k.$$ 

Note that our definition of the Cartan’s tensor (2.8) differs from the usual definition, which is a symmetric tensor $A = A_{ijk} dx^i \otimes dx^j \otimes dx^k$. Such difference is only formal and does not spoil the properties of the Cartan’s tensor in any practical situation.

2.2. **The Chern-Rund connection.** Let us consider the product $N \times TM$ and the canonical projections

$$\hat{\pi}_1 : N \times TM \to N, \quad (u, \xi) \mapsto u, \quad \hat{\pi}_2 : N \times TM \to TM, \quad (u, \xi) \mapsto \xi.$$

The pull-back $\hat{\pi}_1 : \hat{\pi}^* TM \to N$ of the tangent bundle $\pi : TM \to M$ by the projection $\hat{\pi} : N \to M$ is the maximal sub-manifold of the product $N \times TM$ such that $\pi \circ \hat{\pi}_2 (u, \xi) = \hat{\pi} \circ \hat{\pi}_1 (u, \xi)$ holds. It follows that the diagram

$$
\begin{array}{cccc}
\hat{\pi}_1 & \hat{\pi}^* TM & \xrightarrow{\pi} & TM \\
N & \xrightarrow{\hat{\pi}} & M & \\
\end{array}
$$

commutes. Here $\pi_1 : \hat{\pi}^* TM \to N$ and $\pi_2 : \hat{\pi}^* TM \to TM$ are the restrictions of the natural projections $\hat{\pi}_1 : N \times TM \to N$ and $\hat{\pi}_2 : N \times TM \to TM$ to $\hat{\pi}^* TM$. $\pi_1 : \hat{\pi}^* TM \to N$ is a real vector bundle, with fiber over $u = (x, Z) \in N$ isomorphic to $T_x M$. The fiber dimension of $\hat{\pi}^* TM$ is equal to $n = \dim(M)$, while the dimension of the base manifold $N$ is $2n$.

Each tangent field $Z \in T_2 M$ can be pulled-back $\hat{\pi}^* Z$ uniquely by the conditions

- $\pi_1(\hat{\pi}^* Z) = (x, Z) \in N_2$,
- $\pi_2(\hat{\pi}^* Z) = Z$.

These conditions extend pointwise to vector fields. The pull-back of a smooth function $f \in F(M)$ is the smooth function $\hat{\pi}^* f \in F(N)$ such that $\hat{\pi}^* f(u) = f(\hat{\pi}(u))$ for every $u \in N$. Analogously, a pull-back tensor bundle $\hat{\pi}^* T^{(p,q)} M$ can be obtained from each tensor bundle $T^{(p,q)} M$ over $M$.

Let us consider a linear connection on $\hat{\pi}^* TM$. The associated covariant derivative is the operator

$$\nabla : \Gamma \hat{\pi}^* TM \times \Gamma TN \to \Gamma \hat{\pi}^* TM$$
such that

- For every $\tilde{X} \in \Gamma TN$, $S_1, S_2 \in \Gamma \hat{\pi}^* TM$ and $\phi \in \mathcal{F}(M)$ it holds that
\begin{equation}
\nabla_{\tilde{X}}(\hat{\pi}^* \phi S_1 + S_2) = (\tilde{X} \cdot \hat{\pi}^* \phi) S_1 + \phi \nabla_{\tilde{X}} S_1 + \nabla_{\tilde{X}} S_2.
\end{equation}

- For every $\tilde{X}_1, \tilde{X}_2 \in \Gamma TN$, $S \in \Gamma \hat{\pi}^* TM$ and $\hat{\phi} \in \mathcal{F}(\mathcal{N})$ it holds that
\begin{equation}
\nabla_{\tilde{X}_1 + \tilde{X}_2} S = \hat{\phi} \nabla_{\tilde{X}_1} S + \nabla_{\tilde{X}_2} S.
\end{equation}

The Chern-Rund connection $\hat{\nabla}$ is a linear connection on $\hat{\pi}^* TM$ characterized by the following

**Theorem 2.6.** Let $(M, F)$ be a Finsler space. The vector bundle $\hat{\pi}^* TM$ admits a unique linear connection characterized by the collection of connection 1-forms $\{\hat{\omega}^j_i, \ i, j = 1, ..., n\}$ such that the following structure equations hold:

- “Torsion free” condition,
\begin{equation}
d(dx^i) - dx^j \wedge \hat{\omega}^j_i = 0, \quad i, j = 1, ..., n.
\end{equation}

- Almost g-compatibility condition,
\begin{equation}
dg_{ij} - g_{kj} \hat{\omega}^j_i - g_{ik} \hat{\omega}^k_j = 2 A_{ijk} \frac{\delta y^k}{F}, \quad i, j, k = 1, ..., n.
\end{equation}

The torsion free condition is equivalent to the absence of terms containing $\delta y^i$ in the connection 1-forms $\hat{\omega}^j_i$ and also implies the symmetry of the connection coefficients $\Gamma^j_{ik}(x, y)$,
\begin{equation}
\hat{\omega}^j_i(x, y) = \Gamma^j_{ik}(x, y) dx^k, \quad \Gamma^i_{jk}(x, y) = \Gamma^i_{kj}(x, y), \quad i, j, k = 1, ..., n.
\end{equation}

We can characterize the Chern-Rund connection by means of the associated covariant derivative operator $\hat{\nabla}$, with $\tilde{X} \in T_u \mathcal{N}$. The generalized torsion tensor is given by the expression
\begin{equation}
T_{\hat{\nabla}} : \Gamma TM \times \Gamma TM \to \Gamma TM
\end{equation}
\begin{equation}
(X, Y) \mapsto \tau_2(\hat{\nabla} \hat{\pi}^* X) - \tau_2(\hat{\nabla} \hat{\pi}^* Y) - [X, Y],
\end{equation}
where $\tilde{X}$ and $\tilde{Y}$ are the horizontal lifts of the restrictions $X(x), Y(x)$ at a given point $u \in \hat{\pi}^{-1}(x)$. Then the following corollaries are direct consequences of Theorem 2.6,

**Corollary 2.7.** Let $(M, F)$ be a Finsler space. The torsion free condition is equivalent to the following conditions:

1. For any $\tilde{X} \in TN$ and $Y \in M$, the following relation holds,
\begin{equation}
\hat{\nabla}_{V(\tilde{X})} \hat{\pi}^* Y = 0.
\end{equation}

2. Let us consider $X, Y \in \Gamma TM$. Then the following relation holds,
\begin{equation}
T_{\hat{\nabla}}(X, Y) = 0.
\end{equation}

**Proof.** Let $\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^n$ be a local frame on $U$ and $X = e_i = \frac{\partial}{\partial x^i}$, $Y = e_j = \frac{\partial}{\partial y^j}$. Then we have
\begin{equation}
\hat{\nabla}_{V(\tilde{X})} \hat{\pi}^* Y := \hat{\pi}^* e_k w^k_j (V(e_i)) = \hat{\pi}^* e_k \Gamma^k_{aj} dx^a (V(e_i)) = \hat{\pi}^* e_k \Gamma^k_{aj} dx^a \left( \frac{\partial}{\partial y^j} \right) = 0.
\end{equation}
This condition is extended by linearity and by the Leibnitz rule to arbitrary sections $X, Y \in \Gamma TM$. 

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To prove the relation (2.16), let us consider the torsion condition in local coordinates where the local frame \( \{ e_j \} \) on \( \Gamma TM \) commutes, \([e_i, e_j] = 0\). Then from the symmetry in the connection coefficients (2.13) one obtains
\[
\pi_2(\hat{\nabla} e_i \pi^* e_j) - \pi_2(\hat{\nabla} e_j \pi^* e_i) - [e_i, e_j] = \pi_2(\hat{\nabla} e_i \pi^* e_j) - \pi_2(\hat{\nabla} e_j \pi^* e_i) = (\Gamma^k_{ij} - \Gamma^k_{ji}) \pi^* e_k = 0.
\]
This relation is extended by linearity to arbitrary vectors \( X, Y \in \Gamma TM \).

**Corollary 2.8.** Let \((M, F)\) be a Finsler space and \( \tilde{X} \in \Gamma TN \). The almost g-compatibility condition (2.12) is equivalent to the conditions

1. \( \hat{\nabla} \) is metric compatible in the horizontal directions,
\[
\hat{\nabla} H(\tilde{X}) g = 0.
\]
2. \( \hat{\nabla} \) is almost-metric compatible in the vertical directions in the sense that
\[
\hat{\nabla} V(\tilde{X}) g = 2 A(\tilde{X}, \cdot, \cdot)
\]
holds good.

**Proof.** Using local natural coordinates and reading from the expression (2.12), it follows that
\[
\hat{\nabla} g = (dg_{ij} - g_{k}^{ij} \hat{\nabla} w_i - g_{ik}^{\hat{\nabla} w_j}) \pi^* e^i \otimes \pi^* e^j = 2 A_{ijkl} \frac{\delta y^k}{F} \otimes \pi^* e^i \otimes \pi^* e^j,
\]
By the definition of covariant derivative along an horizontal direction and since \( 2 A_{ijkl} \frac{\delta y^k}{F} \) is vertical, the relation (2.17) follows. For the covariant derivative along the vertical component \( V(\tilde{X}) \),
\[
\hat{\nabla} V(\tilde{X}) g := 2 A_{ijkl} \frac{\delta y^k}{F} \cdot V(\tilde{X}) (\pi^* e^i \otimes \pi^* e^j), \forall \tilde{X} \in \Gamma TN,
\]
from where follows the condition (2.18). \( \square \)

The curvature 2-forms associated with a linear connection \( \nabla \) on \( \pi^* TM \) are
\[
\Omega^i_j := dw^i_j - w^k_{ij} \wedge w^i_k, \quad i, j, k = 1, \ldots, n.
\]
In local coordinates, the curvature endomorphisms are decomposed as
\[
\Omega^i_j = \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l + P^i_{jkl} dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q^i_{jkl} \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}
\]
The quantities \( R^i_{jkl}, P^i_{jkl} \) and \( Q^i_{jkl} \) are the hh, hv, and vv-curvature tensor components. For any Finsler space, the hv-curvature endomorphisms of the Chern-Rund connection are identically zero [4]. However, for other linear connections in \( \pi^* TM \) the three curvature types could be different from zero.

3. **RIEMANNIAN AVERAGE METRICS FROM A FINSLER SPACE**

3.1. **Definition of the averaging procedure for Finsler metrics.** The \( n \)-form \( d^n y \) is defined in local coordinates by the expression
\[
d^n y = \sqrt{\det g(x, y)} \frac{\delta y^1}{F} \wedge \cdots \wedge \frac{\delta y^n}{F},
\]
where \( \det g(x, y) \) is the determinant of the fundamental tensor \( (g)_{ij} = g_{ij}(x, y) \). Since \( d^n y \) is invariant by local diffeomorphisms on \( TM \), it defines a section of \( \Lambda^n N \). For each embedding \( i_x : I_x \hookrightarrow N_x \) one consider the volume form on \( I_x \) given by
\[
dvol_x := i_x^* (d^n y \cdot t),
\]
where \( l = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} \) and \( d^a y \cdot l \) is the corresponding contraction. Then the volume function \( \text{vol}(I_x) \) is defined by the expression

\[
\text{vol} : M \to \mathbb{R}^+, \quad x \mapsto \text{vol}(I_x) = \int_{I_x} |\psi|^2(x, y) \, dvol_x,
\]

where the weight factor \( |\psi|^2 : TM \setminus \{0\} \to \mathbb{R}^+ \) is a homogenous of degree zero in \( y \), positive, smooth function on the tangent bundle. The average of a function \( f \in \mathcal{F}(\mathcal{I}) \) is defined by the expression

\[
\langle f \rangle_{\psi}(x) := \frac{1}{\text{vol}(I_x)} \int_{I_x} |\psi|^2(x, y) f(x, y) \, dvol_x.
\]

Given the Finsler space \((M, F)\) let us consider the matrix with components

\[
h_{ij}(\psi, x) := \langle g_{ij}(x, y) \rangle_{\psi}, \quad i, j = 1, \ldots, n,
\]

for each point \( x \in M \). Then we have,

**Proposition 3.1.** Let \((M, F)\) be a Finsler space. Then \( \{h_{ij}(\psi, x)\}_{i,j=1}^n \) are the components of a Riemannian metric

\[
h_{\psi}(x) = h_{ij}(\psi, x) \, dx^i \otimes dx^j, \quad i, j = 1, \ldots, n.
\]

**Proof.** The average \( [3.3] \) of a positive defined, real and symmetric \( n \times n \) matrix is also a positive, real, symmetric matrix. To show this, first we note that

\[
g_{ij}(x)(\hat{y}, \hat{y}) = g_{ij}(x, y)\hat{y}^i\hat{y}^j \geq 0, \quad y, \hat{y} \in T_x M.
\]

This is because \( g_{ij}(x, y) \) for each fixed \( y \in T_x M \) is a positive defined scalar product in \( T_x M \). The symmetric property follows in analogous way. The tensorial character of the expression \( [3.3] \) is also direct. \( \square \)

**Definition 3.2.** Two Finsler spaces \((M, F_1)\), \((M, F_2)\) are said to be \( h\)-equivalent if the corresponding average metrics \( h_1 \) and \( h_2 \) defined by \( [3.3] \) are the same.

\( h\)-equivalence is an equivalence relation in the set of Finsler metrics \( M_F \) defined over \( M \). The equivalence class of \( g \) is denoted by \([g]\). The coset space is defined by \( M_F/\sim \). We call this equivalence relation \( F\)-equivalence.

From now we choose the function \( |\psi|^2 = 1 \), if anything else is not stated. In this case the averaging is called isotropic.

### 3.2. A coordinate-free formula for \( h \)

Let us consider the fiber metric

\[
\hat{g} = g_{ij}(x, y)\hat{\pi}^* dx^i \otimes dx^j, \quad \xi \in \pi_2^{-1}(u), \ u \in \hat{\pi}^{-1}(x).
\]

**Proposition 3.3.** Let \((M, F)\) be a Finsler space. Then the relation

\[
h(x)(X, X) = \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} dvol_x \hat{g} (\hat{\pi}^*|_u(X), \hat{\pi}^*|_u(X)) \right)
\]

holds good for each vector field \( X \in \Gamma TM \).

**Proof.** For each point \( x \in M \), \( h(x) = \langle g_{ij}(x) \rangle_{dx^i|_x} \otimes dx^j|_x \), \( u \in I_x \) and vector field \( X \in \Gamma TM \) one has

\[
h(x)(X, X) = \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} dvol_x g_{ij}(u) \right) X^i X^j
\]

\[
= \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} dvol_x \hat{g}_{ij}(u) \right) X^i X^j
\]

\[
= \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} dvol_x \hat{g} (\hat{\pi}^*|_u(X), \hat{\pi}^*|_u(X)) \right).
\]
4. General form of the average operation and applications

4.1. Average of a family of tensor automorphisms. For each tensor $S_x \in T^{(p,q)}_x M$ with $v \in \pi^{-1}(z)$ and $z \in U \subset M$, the following isomorphism is defined:

$$\pi^* v : \pi^{-1}(v) \to \pi_1^{-1}(v), \quad S_z \mapsto \pi^*_v S_z.$$ 

Consider a family of fiber preserving automorphisms $A := \{A_w : \pi^*_w T^{(p,q)}_x M \to \tilde{\pi}^*_w T^{(p,q)}_x M, \ w \in I_x, x \in M\}$. Then one can define the vector valued integral operation,

$$\langle A \rangle_x : T^{(p,q)}_x M \to T^{(p,q)}_x M$$

where $u = (x,y)$ is a point on the fiber $I_x \hookrightarrow N_x$. This operation is a fiber integration on a sub-manifold $I_x$ of $T_x M$ and with values of $T^{(p,q)}_x M$, for each $x \in M$. The tensor $S_x$ is pulled-back $\{\pi^*_u S_x, u \in N_x \subset N\}$ such that the following diagram commutes,

$$\begin{array}{ccc}
\pi^*_u S_x & \xrightarrow{\pi_1} & S_x \\
\downarrow & & \downarrow \\
I_x & \xrightarrow{\pi} & x
\end{array}$$

for each $x \in M, u \in \pi^{-1}(x)$. The chain of compositions defining the integral operation $\langle A \rangle_x$ is the following. Given $x \in M, u \in \pi^{-1}(x)$ and $S_x \in T^{(p,q)}_x M$,

$$x \mapsto S_x \mapsto \{\pi^*_u S_x\} \mapsto \{A_u(\pi^*_u S_x)\} \mapsto \{\pi_2(A_u(\pi^*_u S_x))\} \mapsto \int_{I_x} \pi_2(A_u(\pi^*_u S_x)) dvol_x.$$

**Definition 4.1.** The average operator of the family of automorphism $A$ is the automorphism

$$\langle A \rangle_x : T^{(p,q)}_x M \to T^{(p,q)}_x M$$

$$S_x \mapsto \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \pi_2|_u A_u \pi^*_u \right) \cdot S_x, \quad u \in \pi^{-1}(x), \forall S_x \in T^{(p,q)}_x M.$$ 

**Proposition 4.2.** The average operator associated with a family of a linear of operators $A$ is a geometric operator on $M$.

**Proof.** Let us consider first sections of the bundle $\pi^* TM$ and two local basis $\{e_i, i = 1, ..., n\}$ and $\{\tilde{e}_i, i = 1, ..., n\}$ of $T_x M$. Then we have

$$\langle A \rangle(S(x)) = \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \pi_2|_u A_u \pi^*_u \right) \cdot S(x)$$

$$= \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \pi_2|_u A_u \pi^*_u \right) \cdot S^i(x) e_i(x)$$

$$= \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \pi_2|_u A_u \pi^*_u \tilde{S}^k(x) \tilde{e}_k(x) dvol_x \right)$$

$$= \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \pi_2|_u A_u \pi^*_u \tilde{S}^k(x) \tilde{e}_k(x) \right).$$

One can extend this calculation to families of homomorphisms of $\pi^* T^{(p,q)}_x M$. \[\square\]
The averaging operation can be extended to families of local operators acting on sections $\pi^*T^{(p,q)}M$ by applying the construction \[123\] in the definition \[111\] to the evaluation of sections $\pi^*_v(S) := (\pi^*)(v)$, where $S \in \Gamma T^{(p,q)}M$ and $v \in U_u$ is a point in the open set $U_u$ containing $u$.

4.2. The average of linear connections on $\pi^*TM$. We can average linear connections on $\pi^*TM$ following the above general method.

**Theorem 4.3.** Let $\nabla$ be a linear connection of the vector bundle $\pi^*TM \to N$. Then there is defined an affine connection of $M$ determined by the covariant derivative of each section $Y \in \Gamma TM$ along each direction $X \in T_xM$,

$$
(\langle \nabla \rangle)_X Y := (\pi_2|_u \nabla_{\pi u(X)} \pi^*_u Y), \quad v \in TU_x \setminus 0,
$$

for each $X \in T_xM$ and $Y \in \Gamma TM$, where $U_x$ is an open neighborhood of $x \in M$.

**Proof.** We check that the properties for a linear covariant derivative hold for $\langle \nabla \rangle$.

1. Using the linearity of the original covariant derivative and the linearity of the averaging operation,

$$
\langle \nabla \rangle_X (Y_1 + Y_2) = (\pi_2|_u \nabla_{\pi u(X)} \pi^*_u (Y_1 + Y_2)) = (\pi_2|_u \nabla_{\pi u(X)} \pi^*_u Y_1) + (\pi_2|_u \nabla_{\pi u(X)} \pi^*_u Y_2),
$$

\[4.3\]

For the second condition we perform the following simplification,

$$
\forall Y_1, Y_2.
$$

for each $X \in T_xM, \lambda \in \mathbb{R}, X \in T_xM$.

2. $\langle \nabla \rangle_X Y$ is a $\mathcal{F}$-linear respect $X$, that is,

$$
\langle \nabla \rangle_{X_1+X_2} Y = (\langle \nabla \rangle_{X_1} Y + \langle \nabla \rangle_{X_2} Y) \quad \text{and} \quad \langle \nabla \rangle_{fX} Y = f(\langle \nabla \rangle_X Y),
$$

holds good for each $Y \in \Gamma TM, v \in \pi^{-1}(z), X, X_1, X_2 \in T_xM$ and $f \in \mathcal{F}$. The first condition is proved by the following short calculation, calculation,

$$
\langle \nabla \rangle_{X_1+X_2} Y = (\pi_2|_u (\nabla_{\pi u(X_1+X_2)} \pi^*_u Y)) = (\pi_2|_u \nabla_{\pi u(X_1)} \pi^*_u Y) + (\pi_2|_u \nabla_{\pi u(X_2)} \pi^*_u Y),
$$

\[4.4\]

For the second condition the proof is similar.

3. The Leibnitz rule holds:

$$
\langle \nabla \rangle_X (\varphi Y) = (d\varphi(X))Y + \varphi(\langle \nabla \rangle_X Y), \quad \forall Y \in \Gamma TM, \quad \varphi \in \mathcal{F}(M), \quad X \in T_xM,
$$

where $d\varphi(X)$ is the action of the 1-form $d\varphi \in \Lambda^1M$ evaluated at $x \in M$ on the tangent vector $X \in T_xM$. In order to prove \[4.5\] we use the following property,

$$
\pi^*_v(\varphi Y) = (\pi^*_v \varphi)(\pi^*_v Y), \quad \forall Y \in TM, \varphi \in \mathcal{F}(M).
$$

Then one obtains the following expressions,

$$
\langle \nabla \rangle_X (\varphi Y) = (\pi_2|_u \nabla_{\pi u(X)} \pi^*_u (\varphi Y)) = (\pi_2|_u \nabla_{\pi u(X)} (\pi^*_u \varphi))\pi^*_u Y
$$

$$
= (\pi_2|_u (\nabla_{\pi u(X)} (\pi^*_u \varphi))\pi^*_u Y) + (\pi_2|_u (\pi^*_u f)\nabla_{\pi u(X)} \pi^*_u (Y))
$$

$$
= (\pi_2|_u (\nabla_{\pi u(X)} (\pi^*_u \varphi))\pi^*_u Y) + (\varphi (\pi_2|_u \nabla_{\pi u(X)} \pi^*_u (Y)) + (X \cdot \varphi)(\pi_2|_u \pi^*_u Y)
$$

\[4.5\]

For the first term we perform the following simplification,

$$
((X \cdot \varphi)(\pi_2|_u \pi^*_u Y)) = (X \cdot \varphi)((\pi_2|_u \pi^*_u Y)) = (X \cdot \varphi)((\pi_2|_u \pi^*_u Y)) Y
$$

$$
= (X \cdot \varphi) Y = (d\varphi(X))Y.
$$
Finally we obtain that
\[
\langle \nabla \rangle_X (\phi Y) = ((\langle \nabla \rangle_X \phi) Y + \phi \langle \nabla \rangle_X Y) = (d\phi(X)) Y + \phi \langle \nabla \rangle_X Y.
\]

We denote the affine connection associated with the above covariant derivative by \( \langle \nabla \rangle \). Then for each section \( Y \in \Gamma TM \), \( \langle \nabla \rangle Y \in \Gamma T^{(1,1)} M \) is given by
\[
\langle \nabla \rangle(X, Y) := \langle \nabla \rangle_X Y, \quad X \in T_x M.
\]
The average covariant derivative is extended to 1-forms by the requirement that it commutes with contractions. Thus for each \( \alpha \in \Lambda^1 M \) and \( X \in \Gamma TM \)
\[
\langle \nabla \rangle_X \alpha(Z) = (\langle \nabla \rangle_X \alpha) \cdot Z + \alpha(\langle \nabla \rangle_X Z)
\]
holds by assumption. Then the extension of the covariant derivative \( \langle \nabla \rangle_X \) to sections of the tensor bundle \( T^{(p,q)} M \to M \) is defined by the rule
\[
\langle \nabla \rangle_X K(X_1, ..., X_s, \alpha_1, ..., \alpha_r) = \langle \nabla \rangle_X K(X_1, ..., X_s, \alpha_1, ..., \alpha_r) - \sum_{i=1}^s K(X_i, ..., \langle \nabla \rangle_X X_i, ..., X_s, \alpha_1, ..., \alpha_r) + \sum_{j=1}^r K(X_1, ..., X_s, \alpha_1, ..., \langle \nabla \rangle_X \alpha_j, ..., \alpha_r).
\]

4.3. General properties of the averaged connection.

**Proposition 4.4.** Let \((M, F)\) be a Finsler space and \( \nabla \) a linear connection on \( \hat{\pi}^* TM \). Then it holds
\[
T_{\langle \nabla \rangle} = \langle T_{\nabla} \rangle.
\]

**Proof.** We can calculate the torsion of the connection \( \langle \nabla \rangle \),
\[
T_{\langle \nabla \rangle}(X, Y) = \langle \pi_2|_u \nabla_{\pi_2(Y)} (\hat{\pi}^* Y) - \langle \pi_2|_u \nabla_{\pi_2(Y)} \hat{\pi}^* X - [X, Y] = \langle \pi_2|_u \nabla_{\pi_2(Y)} (\hat{\pi}^* Y) - \langle \pi_2|_u \nabla_{\pi_2(Y)} \hat{\pi}^* X - \langle \pi_2|_u \hat{\pi}^* Y - \hat{\pi}^* [X, Y])
\]
\[
= \langle \pi_2|_u (\nabla_{\pi_2(Y)} \hat{\pi}^* Y - \nabla_{\pi_2(Y)} \hat{\pi}^* X - \hat{\pi}^* [X, Y])
\]
\[
= \langle T_{\nabla}(X, Y) \rangle.
\]

**Corollary 4.5.** Let \((M, F)\) be a Finsler space with average Chern-Rund connection \( \langle \hat{\pi}^* \nabla \rangle \). Then the torsion \( T_{\langle \hat{\pi}^* \nabla \rangle} \) is zero.

**Definition 4.6.** Two Finsler spaces \((M, F_1)\) and \((M, F_2)\) are \( \Gamma \)-related iff the corresponding average connections are the same.

This is an equivalence relation (\( \Gamma \)-equivalence). \( \Gamma \)-equivalence classes are denoted as \([g]_\Gamma\). Let us remark that the averaging procedure depends on the particular Finsler space \((M, F)\) that one starts and that this equivalence relation is between different types of averaging procedures. Also note that in general the \( \Gamma \)-equivalence relation is different than the \( F \)-equivalence relation.
4.4. Average connection of a Berwald space. In order to avoid cluttering in the notation, the Chern-Rund connection will be denoted simply by $\nabla$ instead of the most specific notation $\text{ch}\nabla$ used until now.

**Definition 4.7.** A Berwald space is a Finsler space such that its Chern-Rund connection also defines an affine connection on $M$.

For a Berwald space the connection coefficients $\text{ch}\Gamma_{jk}^i(x,y)$ depend on $x \in M$ only. Thus, we have the following,

**Theorem 4.8.** For a Berwald space $(M,F)$

- The average of the Chern-Rund connection coincides with the Chern-Rund connection in the sense that
  \[
  (\hat{\pi}^* \langle \nabla \rangle_X S) = \nabla_{\bar{\pi}(X)} \hat{\pi}^* S. \tag{4.8}
  \]

- The average of the Chern-Rund connection coincides with the Levi-Civita connection of the average metric,
  \[
  \langle \nabla \rangle = h \nabla. \tag{4.9}
  \]

**Proof.** The relation (4.8) is direct from the definition of average connection. A detailed proof can be found in [10]. For the proof of relation (4.9), let us first evaluate the covariant derivative of the metric $h$ for $\langle \nabla \rangle$,

\[
\langle \nabla \rangle_X = \frac{\partial}{\partial x^i} h = \left( \frac{\partial h_{jk}}{\partial x^i} - \langle \nabla \rangle^l_{ik} h_{jl} - \langle \nabla \rangle^l_{ij} h_{lk} \right) dx^l \otimes dx^k
\]

\[
= \left( \frac{\partial h_{jk}}{\partial x^i} - ch\Gamma^l_{ik} h_{jl} - ch\Gamma^l_{ij} h_{lk} \right) dx^l \otimes dx^k
\]

\[
= \frac{1}{\text{vol}(I_x)} \left( \int_{I_x} \left( \frac{\partial g_{jk}}{\partial x^i} - ch\Gamma^l_{ik} g_{jl} - ch\Gamma^l_{ij} g_{lk} \right) d\text{vol}_x \right) dx^l \otimes dx^k
\]

Since the horizontal metric compatibility of the Chern-Rund connection (equation (2.18)), the integrand is zero. Therefore,

\[
\langle \nabla \rangle_X = \frac{\partial}{\partial x^i} h = 0, \quad i = 1, \ldots, n.
\]

Moreover, by Corollary 4.5 $\langle \nabla \rangle$ is torsion free. Therefore, $\langle \nabla \rangle$ must be the Levi-Civita connection of $h$. \hfill \square

The geodesic deviation equation for a Finsler space is formally the same than for a Riemannian space. In particular, the Jacobi equation for the Chern-Rund connection [4] is the linear differential equation along the geodesic $X : I \to M$

\[
\nabla_X \nabla_X J + R(X,J)X = 0. \tag{4.10}
\]

It is interesting that if the space $(M,F)$ is of Berwald type, then the equation (4.10) is formally the same than the equation for the geodesics of the average connection,

\[
\langle \nabla \rangle_{\bar{X}} \langle \nabla \rangle_{\bar{X}} \bar{J} + R(\langle \nabla \rangle, \bar{X}, \bar{X}) \bar{X} = 0,
\]

or

\[
\nabla_{\bar{X}} \nabla_{\bar{X}} \bar{J} + R(\bar{X}, \bar{X}) \bar{J} \bar{X} = 0,
\]

where $\bar{X} : \bar{I} \to M$ is a geodesic of the averaged connection $\langle \nabla \rangle$ and $R(\langle \nabla \rangle)$ is the curvature endomorphism. The Jacobi fields $J : I \to X(I)$ and $\bar{J} : \bar{I} \to \bar{X}(\bar{I})$ are the same (at least for small values of the time values of the time parameter, the geodesics
X and $\tilde{X}$ as un-parameterized geodesics, except that they are parameterized by different parameters. Then the condition

$$R^{(\nabla)}(\tilde{X}, J) = R(X, J)$$

must hold for Berwald spaces. Since for a Berwald space $\langle \nabla \rangle = h^{\nabla}$, the following result holds:

**Theorem 4.9.** If $(M, F)$ is a Berwald space, then

$$hR(X, Y) = R(X, Y), \quad X, Y \in \Gamma TM.$$  

(4.11)

**Corollary 4.10.** If $(M, F)$ is a Berwald space, then $hR^i_{\ jkl} = (gR^i_{\ jkl})$.

**Proof.** This is a direct consequence of the above calculation and Theorem 4.9.

A simple calculation shows that in general the average of the curvature endomorphism does not coincide with the curvature endomorphism of the average metric,

$$hR^i_{\ jkl} = h^{im} hR_{mjkl} = h^{im} (gR_{mjkl}) = (h^{im} gR_{mjkl})$$

$$= (h^{im} g_{ms} g^{sa} gR_{sjkl}) = (\partial_s g^{sa} R^s_{\ jkl}),$$

where the tensor $\partial^a_s := h^{im} g_{ms} \neq \delta^a_s$ measures the departure of the fundamental tensor $g$ of being Riemannian. Similarly, in the general case there is no direct relation between the average of the flag curvature of $F$ and the sectional curvatures of $h$, even in the case of constant sign flag curvature spaces.

Let $(M, F)$ be a Berwald space. Then the Riemann tensor of $g$ is a $(0, 4)$-tensor along the map $\tilde{x} : N \to M$ whose components are given in normal coordinates of $g$ by the expression

$$gR_{ijkl} := g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik},$$

where $g_{ij,kl}$ stands for $\partial^2 g_{ij}/\partial x^k \partial x^l$, etc...

**Proposition 4.11.** Let $(M, F)$ be a Berwald space and consider the isometric average metric $h_{ij}$. Then the following relation holds,

$$hR_{ijkl}(x) = (gR_{ijkl}(x, y)).$$

(4.13)

**Proof.** In normal coordinates for $h$, the Riemann tensor $hR_{ijkl}$ is linear on the second derivatives of the components of the curvature tensor of $h$. Then the tensor can be expressed as

$$hR_{ijkl} = h_{il,jk} - h_{ik,jl} + h_{jk,il} - h_{jl,ik},$$

where for instance $h_{il,jk} = \partial^2 h_{ij}(x)/\partial x^k \partial x^l$, etc... From the definition of $h$ it follows that

$$hR_{ijkl} = \left\langle g_{il}, jk \right\rangle - \left\langle g_{ik}, jl \right\rangle + \left\langle g_{jk}, il \right\rangle - \left\langle g_{jl}, ik \right\rangle$$

holds in the normal coordinate chart of $h$. Since the weight factor is $|\psi|^2 = 1$ and the volume function $\text{vol}(I_2)$ is constant for a Berwald space, the partial derivatives can be introduced in the integrals,

$$\langle g \rangle R_{ijkl}(x) = hR_{ijkl} = \left\langle g_{il}, jk \right\rangle - \left\langle g_{ik}, jl \right\rangle + \left\langle g_{jk}, il \right\rangle - \left\langle g_{jl}, ik \right\rangle$$

$$= \left\langle g_{il}, jk \right\rangle - \left\langle g_{ik}, jl \right\rangle + \left\langle g_{jk}, il \right\rangle - \left\langle g_{jl}, ik \right\rangle = \langle gR_{ijkl}(x, y) \rangle,$$

where from the first to the second line in the above expression we have use the equality $\partial^2 \Gamma^i_{\ jk}(x) = h\Gamma^i_{\ jk}(x) = 0$ in normal coordinates of the average metric $h$.\footnote{For a Berwald space, normal coordinate system exists and are $C^2$, see for instance [4].}
In this step is essential the Berwald condition. This formulae has been proved in normal coordinates of \( h \), but since it is an identity between tensor components, it holds in any coordinate system.

As an example of direct application of proposition 4.11 is a generalization of Riemann’s characterization of Euclidean space in terms of curvature. Let \( h_0 \) be the Euclidean metric in \( \mathbb{R}^n \). Then we have

**Corollary 4.12.** Let \((M,F)\) be a Berwald space such that \( g_{ijkl} = 0 \). Then \( M \cong \mathbb{R}^n \) and the fundamental tensor \( g = h_0 + \delta g \), with \( \langle \delta g \rangle = 0 \).

**Proof.** If \( g_{ijkl} = 0 \), then the relation (4.13) implies \( h_{ijkl}(x) = 0 \) and the result follows from the classical result of Riemann [16].

**Gauss-Bonnet theorem for Berwald surfaces.** The construction of the metric \( h \) as an average of the fundamental tensor over the indicatrix opens the possibility to generalize results from Riemannian to Berwald geometry, using directly the Riemannian results. We consider here another example of this technique: a weak version of the Gauss-Bonnet theorem for arbitrary Berwald surfaces. Let \((I,\pi_I,M)\) be the fibered manifold whose fibers are indicatrix over \( M \) and note that for a Berwald space the function \( x \mapsto vol(I_x) \) is constant. Then we have

**Theorem 4.13.** Let \((M,F)\) be a compact Berwald surface with average metric \( h \) and Gaussian curvature \( hK = -hR_{1212} \) in some orthonormal basis of \( h \). Then the following formula holds

\[
\frac{1}{\text{vol}(I_x)} \int_I g_{1212}(x,y) \text{dvol}_x \wedge d\mu(x) = -2\pi \chi(M),
\]

where \( \chi(M) \) is the Euler’s characteristic of \( M \) and \( d\mu \) is the Riemannian volume form of \( h \) on \( M \).

**Proof.** For the Riemannian metric \( h \) one can apply the classical Gauss-Bonnet theorem for compact surfaces \( M \). Thus the relation

\[
\int_M h_{1212}(x) d\mu = -2\pi \chi(M)
\]

holds. Fixed the integration measure as in proposition 4.11 an using an orthonormal frame associated to \( h \), one obtains the relations

\[
hK(x) = -hR_{1212}(x) = -(g_{1212}),
\]

from which the relation (4.14) follows.

5. **The parallel transport of the average connection**

The parallel transport of a linear connection \( \nabla \) along a path \( \gamma : [a,b] \to M \) with \( \gamma(a) = x \) and \( \gamma(b) = z \) is defined as the linear homomorphism

\[
p_{xz}(\gamma) : T_x^{(p,q)}M \to T_z^{(p,q)}M, \quad S_x \mapsto S_z
\]

such that the section \( S(t) \) along \( \gamma \) is a solution of the linear differential equation

\[
\nabla_{\dot{\gamma}(t)} p(\gamma)(S)(t) = 0, \quad p(\gamma)(S)(0) = S(0).
\]

A polygonal approximation \( \gamma \) of \( \gamma \) is determined by a set of points

\[
\{\gamma(0) = x, ..., \gamma(t_i), ..., \gamma(t_{A-1}), \gamma(t_A) = z, \gamma(t_i) \in \gamma([a,b])\}
\]
joined by geodesic segments $\bar{\gamma}_{k,k+1}$ of $F$, with initial and ending points $\gamma(t_k)$ and $\gamma(t_{k+1})$, respectively. One can also consider the case when $t_k - t_{k-1} = \epsilon$. Then the parallel transport operation is

$$p(\hat{\gamma}) := \prod_{k=1}^{A} \circ p_{t_k,t_{k-1}},$$

where the composition of elementary parallel transport $p_{t_k,t_{k-1}}$ is taken along the geodesic segment $\bar{\gamma}_{k,k+1}$ and is given by the endomorphism

$$p_{t_k,t_{k-1}} : T_{\gamma(t_{k-1})}M \to T_{\gamma(t_k)}M,$$

$$X^i e_i \mapsto \left( \delta^i_{j,k-1} X^{i,k-1} - \epsilon \Gamma^i_{j,k-1} X^{i,k-1} \hat{\gamma}^{j,k-1} \right) e_j.$$

$\hat{\gamma}^{j,k-1}$ is the tangent vector at the point $\gamma(k-1)$. Then the double limit $A \to +\infty$ and $\epsilon = t_k - t_{k-1} \to 0$ is taken in this parallel transport operation, under the constraint

$$\lim_{A \to +\infty} \lim_{\epsilon \to 0} \ A \epsilon = b - a.$$

Let us define $\epsilon = \frac{k-1}{A}$. Then the parallel transport of $X \in T_{\gamma(0)}M$ along $\gamma$ between the point $x = \gamma(a)$ and $z = \gamma(b)$ is given by

$$p_{xx}X^j = \lim_{A \to +\infty, \epsilon \to 0} \left( \delta^j_{i,k-1} - \epsilon \Gamma^j_{i,k-1} \hat{\gamma}^{j,k-1} \right) (p_{x\gamma(t_{A-1})}X)^{i,A-1},$$

$$j = 1, \ldots, n,$$

with $(p_{xx}X^j)_{t_0} = X^j_0$, $\lim_{A \to +\infty} \gamma(t_{A-1}) = \gamma(b)$ and $j_A = j$.

Infinitesimally one has the finite difference expression

$$p_{x\gamma(t+\epsilon)}X^j - p_{x\gamma(t)}X^j = \left( \delta^j_{i,k-1} - \epsilon \Gamma^j_{i,k-1} \hat{\gamma}^{j,k-1} \right) (p_{x\gamma(t)}X)^{i,A-1} - (p_{x\gamma(t)}X)^j$$

$$= \epsilon \left( p_{x\gamma(t)} \Gamma^j_{ik} (\gamma(t)) \right) \hat{\gamma}^k(t) (p_{x\gamma(t)}X)^k.$$

For smooth vector fields and connections, one can take the limit

$$\lim_{\epsilon \to 0} \frac{(p_{x\gamma(t+\epsilon)}X^j) - (p_{x\gamma(t)}X)^j}{\epsilon} = -(p_{x\gamma(t)} \Gamma^j_{ik} (\gamma(t)) \hat{\gamma}^k(t) (p_{x\gamma(t)}X)^k),$$

showing that the expression (5.2) is the solution of the parallel transport equation (5.1).

The expression (5.2) applies to the parallel transport of any linear connection. In particular, it can be applied to the average connection,

**Proposition 5.1.** Let $(\Gamma^i_{jk})$ be the connection coefficients of the average connection $(\nabla)$. Then the parallel transport operation is

$$p_{xx}X^j = \lim_{A \to +\infty, \epsilon \to 0} \left( \prod_{k=1}^{A} \right) \left( \delta^j_{i,k-1} - \epsilon \Gamma^j_{i,k-1} (\gamma(t_k)) \hat{\gamma}^{j,k-1} \right) X^i_0, \quad j = 1, \ldots, n.$$

If we explicitly insert the weight factor $|\psi|^2 : N \to \mathbb{R}^+$ in each integration, one obtains the expression for the parallel transport

$$p_{xx}X^j = \lim_{A \to +\infty, \epsilon \to 0} \left( \prod_{k=1}^{A} \frac{1}{\text{vol}(I_{\gamma(t_k-1)})} \left( \int_{I_{\gamma(t_k-1)}} d\text{vol}_2(\gamma(t_k-1)) \right) f(\gamma(t_k),y_{\gamma(t_k)}) \left( \delta^j_{i,k-1} - \epsilon \Gamma^j_{i,k-1} (\gamma(t_k)) \hat{\gamma}^{j,k-1} (\gamma(t_k-1)) \right) X^i_0 \right).$$
In the general case the average of the parallel transport operation does not coincide with the parallel transport of the average connection: the average of the parallel transport of \( \nabla \) involves only one fiber integration, while the parallel transport of the average connection involves a formal infinite number of integral operations along each fiber \( \tilde{\pi}^{-1}(\gamma(t)), \, t \in [a, b] \).

### 5.1. Curvature of the average connection.

Let us consider the curvature endomorphisms for the average connection \( \langle \nabla \rangle \),

\[
R_{\langle \nabla \rangle}(X_1, X_2)Z = (\langle \nabla \rangle X_1 \langle \nabla \rangle X_2 - \langle \nabla \rangle X_2 \langle \nabla \rangle X_1 - \langle \nabla \rangle [X_1, X_2])Z.
\]

Developing this expression in terms of the original connection \( \nabla \) one obtains

\[
R_{\langle \nabla \rangle}(X_1, X_2)(Z) = \frac{1}{\text{vol}(\Sigma_x)} \int_{I_x} \int_{I_x} \text{dvol}_x(v) \text{dvol}_x(u) \pi_2(v) \pi_2(u) \left( \nabla_{\dot{\iota}_u(X_1)} \tilde{\pi}_u^* \pi_2(u) \nabla_{\dot{\iota}_u(X_2)} \tilde{\pi}_u^* Z - \nabla_{\dot{\iota}_u(X_1)} \tilde{\pi}_u^* \pi_2(u) \nabla_{\dot{\iota}_u(X_1)} \tilde{\pi}_u^* Z \\
- \nabla_{\dot{\iota}_u([X_1, X_2])} \tilde{\pi}_u^* Z \right).
\]

It is interesting that the curvature \( R_{\langle \nabla \rangle} \) is not equal to the average curvature of the linear connection \( \nabla \). For instance, the averaged \( hh \)-curvature is

\[
\langle R^\nabla (\iota_u(X_1), \iota_u(X_2)) \rangle Z := \frac{1}{\text{vol}(I_x)} \int_{I_x} \text{dvol}_x \pi_2(u) \nabla_{\dot{\iota}_u(X_1)} \nabla_{\dot{\iota}_u(X_2)} \tilde{\pi}_u^* Z \\
- \frac{1}{\text{vol}(I_x)} \int_{I_x} \text{dvol}_x \pi_2(u) \nabla_{\dot{\iota}_u(X_1)} \tilde{\pi}_u^* Z \\
- \frac{1}{\text{vol}(I_x)} \int_{I_x} \text{dvol}_x \text{dvol}_x \pi_2(u) \nabla_{\dot{\iota}_u([X_1, X_2])} \tilde{\pi}_u^* Z \\
= \frac{1}{\text{vol}(I_x)} \int_{I_x} \text{dvol}_x \pi_2(u) \left( \nabla_{\dot{\iota}_u(X_1)} \nabla_{\dot{\iota}_u(X_2)} \\
- \nabla_{\dot{\iota}_u([X_1, X_2])} \right) \tilde{\pi}_u^* Z.
\]

Therefore, given a linear connection on \( \pi^*TM \), \( \nabla \), there are two notions of average curvature endomorphisms, \( R_{\langle \nabla \rangle}(X_1, X_2) \) and \( \langle R^\nabla (\iota_u(X_1), \iota_u(X_2)) \rangle \). In the general case the tensors \( R_{\langle \nabla \rangle}(X_1, X_2) \) and \( \langle R^\nabla (\iota_u(X_1), \iota_u(X_2)) \rangle \) do not coincide because the covariant derivative \( \nabla_{\dot{\iota}_u(Y)} \) depends on \( u \in N \) for a general Finsler space.

### 5.2. Holonomy of a Berwald space.

For a Berwald space, the Chern-Rund connection lives on the manifold \( M \) and the curvatures of the average connection coincide with the average of the curvature of the original linear connection on \( \pi^*TM \). Therefore, as an application of the Ambrose-Singer theorem on holonomy \[ \text{[1]} \text{[12]} \] and theorem \[ \text{[4.9]} \] it holds the following result. Let us consider the holonomy group \( \text{Hol}(\nabla) \) of the Chern-Rund connection \( \nabla \). Then

**Theorem 5.2.** Let \((M, F)\) be a Berwald space. Then the holonomy group \( \text{Hol}(\nabla) \) is Riemannian.

**Proof.** If the space \((M, F)\) is a Berwald space, then the \( hh \)-curvature endomorphisms are identically zero, \( P = 0 \). Then the result follows from a direct application of the Ambrose-Singer theorem on holonomy and the relation \[ \text{[4.11]} \].

An interesting consequence is that, if the space \((M, F)\) is Berwald, then the average holonomy group is not only an affine holonomy group, but it is indeed metrizable. This is an extension of a theorem from Z. Szabó on metrizability of compact holonomies of Berwald spaces \[ \text{[13]} \].
Theorem 5.3. Let \((M, F)\) be a Berwald space. Then the holonomy of the Chern-Rund connection is metrizable.

6. ISOMETRIES OF THE AVERAGE METRIC

Definition 6.1. Given two Finsler spaces \((M_1, F_1)\) and \((M_2, F_2)\), a base manifold Finsler isometry (or simply a Finsler isometry) is a diffeomorphism \(\Phi : M_1 \to M_2\) such that preserves the Finsler function,

\[
F_2(\Phi(x), d\Phi(y)) = F_1(x, y).
\]

As a direct consequence of this definition and in the case when \(M_1 = M_2\), the components of the fundamental tensor transform locally under an isometry as

\[
(g_1)_{ij}(\tilde{x}(x), \tilde{y}(x, y)) = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} (g_2)_{lk}(x, y).
\]

It turns out that \(^2\) Proposition 6.2. The group of isometries of \((M, F)\) is contained as a closed subgroup of the isometries of \((M, h)\) (in the compact-open topology).

Then it follows that the group of isometries\(^2\) of \((M, F)\) is a Lie group \([3]\) and it is a subgroup of the group of isometries of the average metric \((M, h)\).

Definition 6.3. A Finsler space \((M, F)\) is symmetric if for each point \(x \in M\) there is a Finsler isometry \(\varphi_x : M \to M\) such that

\begin{itemize}
  \item \(\varphi_x(x) = x\),
  \item \((d\varphi_x)_x = -Id_{|T_x M}\).
\end{itemize}

The two conditions of the isometry \(\varphi\) are the same for \(h\) than for \(F\). Then by direct application of proposition \((6.2)\) we have

Theorem 6.4. If \((M, F)\) is a Finsler symmetric space, then \((M, h)\) is a Riemannian symmetric space.

This result holds for locally symmetric spaces and globally symmetric spaces. In particular, we have

Corollary 6.5. If \((M, F)\) is a global symmetric space, then it is an homogeneous space.

Proof. Let us consider the average metric \(h\), which is necessarily globally symmetric. Then by application of the analogous Riemannian result, \(M \cong G/H\), where \(G\) is the isometry group and \(H\) the isotropy group of the average metric \(h\).

It is remarkable that our result applies also to non-reversible Finsler metrics, as our next example shows.

Example 6.6. Let us consider a Randers space \([15]\) of the form \(F_R = \alpha + \beta\), with Finsler function in local coordinates given by the expression \([4]\)

\[
F_R(x, y) = \sqrt{a_{ij} y^i y^j} + b_i y^i,
\]

where \(\alpha = \sqrt{a_{ij} y^i y^j}\) and \(b_i y^i = \beta\) is the action of the 1-form \(b\) on the tangent vector \(y \in T_x M\). The associated fundamental tensor is

\[
g_{ij}^R(x, y) = \frac{F_R}{\alpha} \left( a_{ij} - \frac{y_i y_j}{\alpha} \right) + \frac{y_i}{F_R} \frac{y_j}{F_R}.
\]

\(^2\)Our result is a particular application of the averaged method. Other example is found in \([14]\), where a different average method was applied \([13]\).
On the indicatrix $I_x$ the fundamental tensor is

$$g^r_{ij}(x,y)|_{\alpha+\beta=1} = \frac{1}{\alpha} \left( a_{ij} - \frac{y_i y_j}{\alpha} \right) + y_i y_j.$$  

(6.3)

Formally, the right hand side of the expression does not depend upon the particular details of the 1-form $b$. Thus we can evaluate the average and it will be equal to the Euclidean case: $b = 0$, $\alpha |_{\alpha} = 1$. In particular, if we choose $a_{ij} = \delta_{ij}$, to simplify the argument, then we have

$$\langle g^r_{ij}(x,y) \rangle = \frac{1}{\text{vol}(I_x)} \delta_{ij}. $$  

(6.4)

If the space $(M,F_R)$ is Berwald, then vol$(I_x)$ is constant and the space $(M,F_R)$ is symmetric. Indeed we have that the same conclusion holds for a general space $F_R = \alpha + \beta$ with the function $x \mapsto \text{vol}(I_x)$ constant on $M$ and $(M,\alpha)$ symmetric.

Remark 6.7. This example contrasts with the main result in Remark 6.8. Remarkably, the average metric in example 6.6 is independent on the reversibility condition on the metric $F$ shows that one can have global symmetric spaces which are not Berwald spaces, if the space $M,F$ is not assumed.

Remark 6.8. Remarkably, the average metric in example 6.6 is independent on the 1-form $b$, except for topology of the manifold $M$ that determines the cohomology class of the 1-forms defined on $M$.

6.1. Curvature average isometric invariants. We have briefly considered before the average of a generic curvature endomorphism of the linear connection on $\pi^*TM \nabla$. The first of these average operators is the average hh-curvature endomorphisms, defined before as the endomorphism

$$\langle R \rangle_x(x_1,x_2) : T_xM \to T_xM, \quad Y \mapsto \langle R \rangle_x(x_1,x_2) Y := (R^\pi(e_u(X_1),e_u(X_2)))Y. $$

(6.5)

Let us denote the vertical lift of $X = X^i \frac{\partial}{\partial x^i} \in T_xM$ by $\kappa(X) = X^i \frac{\partial}{\partial y^i} \in V_u$ with $u \in \hat{\pi}^{-1}(x)$. The average hh-curvature in the directions $X_1$ and $X_2$ is the endomorphism

$$\langle P \rangle_x(x_1,x_2) : T_xM \to T_xM, \quad Y \mapsto \langle P \rangle_x(x_1,x_2) Y := (\pi_2 P_u(e_u(X_1),\kappa_u(X_2))\hat{\pi}^*_u Y), $$

(6.6)

with $u \in I_x \subset \hat{\pi}^{-1}(x) \subset N$. Similarly, for the vv-curvature in the case of an arbitrary linear connection on $\hat{\pi}^*TM$, we define the average homomorphisms,

$$\langle Q \rangle_x(x_1,x_2) : T_xM \to T_xM, \quad Y \mapsto \langle Q \rangle_x(x_1,x_2) := (\pi_2 Q_u(\kappa_u(X_1),\kappa_u(X_2))\hat{\pi}^*_u Y), $$

(6.7)

with $u \in I_x \subset \hat{\pi}^{-1}(x) \subset N$. The average endomorphisms (6.6) and (6.7) are defined on the manifold $M$. In the case of a Riemannian metric, the average curvatures $\langle P \rangle_x(x_1,x_2)$ and $\langle Q \rangle_x(x_1,x_2)$ are both zero, for any $X_1, X_2 \in \Gamma TM$.

The Cartan and Chern-Rund connections are invariant under isometries of $F$, since the connections are defined in terms of the Finsler function $F$ and the fundamental tensor $g$ (that determines the isometries). Therefore, if the measure used in the definition of the averaging operation is invariant under isometries of $F$, the endomorphisms $\langle P \rangle(x_1,x_2)$ and $\langle Q \rangle(x_1,x_2)$ are also invariant under the fiber isometries. Then for the Chern-Rund connection $Q = 0$, there are defined global affine isometric invariants of the form

$$\text{Inv}(M) = \int_M d\mu \mathcal{F}_R(\langle R \rangle, h), $$

(6.8)
and also of the form
\begin{equation}
\text{Inv}(M) = \int_M \mu \mathcal{F}_R(\langle R \rangle, h) \, dp(\langle P \rangle, h),
\end{equation}
where \( \mathcal{F}_R(\langle R \rangle, h) \) and \( \mathcal{F}_P(\langle P \rangle, h) \) are scalar functions and the volume form \( \mu \) is the volume form associated to the average Riemannian metric \( \langle g \rangle \). Thus the integrals (6.8) and (6.9) are invariant under isometries.

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E-mail address: rigato39@gmail.com

Departamento de Matemática
Universidade Federal de São Carlos
Brazil.