Susceptibility of the QCD vacuum to CP-odd electromagnetic background fields

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We investigate two flavor QCD in presence of CP-odd electromagnetic background fields and determine, by means of lattice QCD simulations, the induced effective \( \theta \) term to the first order in \( \vec{E} \cdot \vec{B} \). We employ a rooted staggered discretization and study lattice spacings down to 0.1 fm and Goldstone pion masses around 480 MeV. In order to deal with a positive measure, we consider purely imaginary electric fields and real magnetic fields, then exploiting analytic continuation. Our results are relevant to a description of the effective pseudoscalar QED-QCD interactions.

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Quantum ChromoDynamics (QCD) may contain interactions which violate CP, the symmetry under charge conjugation and parity, corresponding to a term \(-i \theta Q\) in the Euclidean action, where

\[
Q = \int d^4x \varepsilon(x) = \int d^4x \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}(x) \tilde{G}_{\rho\sigma}(x) \tag{1}
\]

is the topological charge operator, \( G^{a}_{\mu\nu} \) is the non-Abelian gauge field strength and \( \tilde{G}^{a}_{\rho\sigma} = \epsilon_{\rho\sigma\mu\nu} G^{a}_{\mu\nu} \). Experimental upper bounds on \( \theta \) are quite stringent, \(|\theta| \leq 10^{-10} \) \[1,2\]. Nevertheless, \( \theta \) related effects play an important role in strong interactions and are generally linked to fluctuations of \( Q \), which affect, through the axial anomaly, the balance of chirality.

A significant interest is related to the possibility that local effective variations of \( \theta \), corresponding to topological charge fluctuations, may induce detectable phenomena in presence of magnetic fields as strong as those produced in the early phases of non-central heavy ion collisions, reaching up to \( 10^{15} \) Tesla at LHC. According to the so-called chiral magnetic effect (CME) \[3,4\], the net unbalance of chirality induced by the topological background would lead, in presence of a magnetic field strong enough to align quark spins, to a net separation of electric charge along the field direction.

The physics of strong interactions in presence of electromagnetic (e.m.) backgrounds has attracted much interest also in relation to the possible effects on the QCD vacuum and on the QCD phase diagram, stimulating many model computations and lattice simulations. Such effects may be relevant in various contexts: magnetic fields as large as of the order of \( 10^{16} \) Tesla may have been produced at the cosmological electroweak phase transition \[4\]; large magnetic fields are also expected in compact astrophysical objects such as magnetars \[7\].

One aspect, emerging from lattice simulations with dynamical fermions \[8,12\] and from model studies \[14,16\], is that e.m. background fields, even if directly coupled only to charged particles, may have a significant influence, via quark loop effects, also on gluon fields. In this study, in an attempt to better clarify such issue, we will investigate how the explicit breaking of some symmetry by the e.m. background field propagates to gluon fields, considering the particular case of CP symmetry.

Let us consider QCD in presence of a constant and uniform e.m. field such that \( F_{\mu\nu} F_{\mu\nu} \propto \vec{E} \cdot \vec{B} \neq 0 \); such background is expected to induce an effective CP-violating interaction in the gluon sector, \( \theta_{\text{eff}} \epsilon^{\mu\nu\rho\sigma} G^{a}_{\mu\nu}(x) \tilde{G}^{a}_{\rho\sigma}(x) \). \( \theta_{\text{eff}} \) must be an odd function of \( \vec{E} \cdot \vec{B} \), hence we can write

\[
\theta_{\text{eff}} \simeq \chi_{\text{CP}} e^2 \vec{E} \cdot \vec{B} + O((\vec{E} \cdot \vec{B})^3) \tag{2}
\]

where \( \chi_{\text{CP}} \) is a sort of susceptibility of the QCD vacuum to CP-breaking e.m. fields.

The effect that we want to study is in some sense complementary to the CME, where a CP-violating non-Abelian background leads to charge separation, hence to an electric field, parallel to a background magnetic field: in that case CP violation propagates from the gluonic to the e.m. sector, i.e. opposite to what we investigate here. In fact, \( \chi_{\text{CP}} \) is directly related of the effective pseudoscalar QED-QCD interaction, \( \chi_{\text{CP}} q(x) e^2 \vec{E} \cdot \vec{B} \) \[17,20\]; in particular, to connect with the notation of Ref. \[20\], we have \( \chi_{\text{CP}} = \kappa/2 \), with \( \kappa \) defined as in Eq. (5) of Ref. \[20\].

The purpose of this study is to furnish a first determination of \( \chi_{\text{CP}} \), based on lattice QCD simulations. To that aim we simulate QCD in presence of uniform e.m. background fields such that \( \vec{E} \cdot \vec{B} \neq 0 \), determining the induced \( \theta_{\text{eff}} \) by studying the topological charge distribution.

The method – Electromagnetic fields enter the QCD lagrangian by modifying the covariant derivative of quarks, \( D_{\mu} = \partial_{\mu} + ig A^{\mu}_{a} T^{a} + iq A_{\mu} \), where \( A_{\mu} \) is the e.m. gauge potential and \( q \) is the quark electric charge. That

\[
\chi_{\text{CP}} \approx \frac{\kappa}{2} \tag{5}
\]
can be implemented by adding proper $U(1)$ phases $u_\mu(n)$ to the $SU(3)$ parallel transports, $U_\mu(n) \rightarrow u_\mu(n)U_\mu(n)$, where $n$ is a lattice site.

A constant and uniform e.m. field, with a single non vanishing component $F_{\mu\nu} = F$, can be realized by a potential $A_\nu = Fx_\nu$ and $A_\rho = 0$ for $\rho \neq \nu$. In presence of periodic boundary conditions (b.c.), as usual in lattice simulations, $F$ must be integer multiple of a minimum quantum

$$f = 2\pi/(qa^2L_\mu L_\nu),$$

and proper b.c. must be chosen for fermions, to preserve gauge invariance $|21|$. The corresponding $U(1)$ links are

$$u_\nu^{(q)}(n) = e^{i a^2 q F n_\nu}; \hspace{1cm} u_\mu^{(q)}(n)|_{n_\mu = L_\mu} = e^{-i a^2 q L_\mu F n_\nu}\hspace{1cm} (4)$$

and $u_\mu(n) = 1$ otherwise, $L_\mu$ being the number of lattice sites in the $\mu$ direction. In presence of various non-vanishing components of $F_{\mu\nu}$, the definition above generalizes by summing, for each $U(1)$ phase, the contributions from the various non-vanishing components of $F_{\mu\nu}$.

We have considered two flavor QCD with fermions discretized in the standard rooted staggered formulation. In the corresponding functional integral, each quark is described by the fourth root of the fermion matrix determinant:

$$Z = \int DU e^{-S_G} \det M^{\frac{1}{2}}[U, q_u] \det M^{\frac{1}{2}}[U, q_d]\hspace{1cm} (5)$$

$$M_{i,j} = \rho_m \delta_{i,j} + \frac{1}{2} \sum_{\nu = 1}^{4} \eta_\nu(i) \left( u_\nu^{(q)}(i) U_\nu(i) \delta_{i,j} - \delta_{i,j} \right) - u_\nu^{(q)}(i - \hat{\nu}) U_\nu(i - \hat{\nu}) \delta_{i,j + \hat{\nu}}\hspace{1cm} (6)$$

$DU$ is the functional integration over the non-Abelian gauge link variables, $S_G$ is the plaquette action, $i$ and $j$ refer to lattice sites and $\eta_\nu(i)$ are the staggered phases. The choice for the quark charges is standard, $q_u = 2|e|/3$ and $q_d = -|e|/3$, leading to a quantization in units of $f = 6\pi/(|e|a^2 L_\mu L_\nu)$ for each component of $F_{\mu\nu}$.

The addition of $U(1)$ phases to the standard $SU(3)$ link variables leaves the spectrum of $M - m I_d$ purely imaginary and symmetric under conjugation: that guarantees $\det M > 0$, hence the feasibility of numerical simulations. However, it is easy to realize that $F_{\mu i} \neq 0$, as defined above, corresponds to a purely imaginary electric field in Minkowski space $|22|,|23|$

In order to have a real electric field, one should introduce imaginary components for the gauge potential in Euclidean space, but that would take the $u_\mu$ variables out of the $U(1)$ group and make the quark determinant complex: such sign problem would hinder numerical simulations. This is expected in view of the phenomenon that we want to explore, since also the formulation of QCD at non-vanishing $\theta$ suffers from a sign problem.

| lattice size | $a$ (fm) | $\beta$ | $am$ | $\chi_{CP}$ (GeV$^{-1}$) |
|-------------|----------|--------|------|------------------|
| $L = 16$    | 0.094    | 5.6286 | 0.008318 | 5.62(55) |
| $L = 16$    | 0.113    | 5.5829 | 0.01048 | 5.39(39) |
| $L = 12, 24$| 0.141    | 5.527  | 0.0146  | 5.54(47) |
| $L = 12, 18$| 0.188    | 5.453  | 0.02627 | 2.89(39) |
| $L = 12, 16$| 0.282    | 5.35   | 0.075   | 1.05(12) |

TABLE I: Lattice parameters and results for $\chi_{CP}$.

A possible strategy, adopted in lattice studies of the electric polarizabilities of hadrons, is to keep the electric field purely imaginary, to avoid the sign problem, then exploiting analytic continuation. Following such approach, we adopt the definition in Eq. $|21|$ for all components of $F_{\mu\nu}$, corresponding to real magnetic fields $\vec{B}$ and imaginary electric fields $\vec{E} = i \vec{E}_I$. As a consequence, we expect a purely imaginary effective parameter $\theta_{eff} = i \theta_{eff}$. On the other hand, working with an imaginary $\theta$ is also one of the possible approaches to the study of $\theta$-dependence in QCD $|24|,|28|$

The presence of an imaginary $\theta$ adds a factor $\exp(i \theta Q)$ to the path integral measure, shifting distribution of $Q$ by an amount which, at the linear order in $\theta$, is governed by the topological susceptibility $\chi$ at $\theta = 0$

$$(Q)_{\theta} \simeq V \chi_{\theta} = (Q^2)_{\theta = 0} \theta \simeq 0,\hspace{1cm} (7)$$

$V$ being the spacetime volume. That suggests to determine the effective $\theta$ produced by a given e.m. field as

$$\theta_{eff} \simeq \langle (\vec{E}_I, \vec{B})/Q^2 \rangle_0 + O((\vec{E}_I, \vec{B})^3)$$

where by $\langle \cdot \rangle_0$ we mean the average taken at zero e.m. field. Corrections to Eq. (8) will be negligible at least in the region of small $\theta_{eff}$ which is relevant to Eq. (2).

In the following we will show that the so defined $\theta_{eff}$ is indeed an odd function of $\vec{E}_I \cdot \vec{B}$ alone and, assuming analyticity for small enough background fields, we shall determine the susceptibility $\chi_{CP}$ defined in Eq. (2) from the small field behavior of $\theta_{eff}$.

Results – We have studied $N_f = 2$ QCD at $T = 0$ for a fixed pseudo-Goldstone pion mass $m_\pi \approx 480$ MeV $|29|$. We have explored different lattice spacings, by tuning the inverse gauge coupling $\beta$ and $am$ according to what reported in Ref. $|30|$, and different symmetric lattice sizes (see Table I). The lattice spacing is not modified by the presence of a purely magnetic background $|10|$: we have verified, by measuring the static quark-antiquark potential, that this is the case also when both $\vec{E}_I$ and $\vec{B}$ are non-zero $|31|$. We have adopted a Rational Hybrid Monte-Carlo (RHMC) algorithm implemented on GPU cards $|32|$, with statistics of $O(10 K)$ molecular dynamics time units for each run.

The total number of about 300 K topological charge measurements compelled us to adopt a relatively cheap
FIG. 1: Monte-Carlo history of $Q$, measured after 30 cooling steps, for two different e.m. backgrounds, on a $16^4$ lattice; $m_s \simeq 480$ MeV and $a \simeq 0.113$ fm.

determination of $Q$, based on a gluonic definition

$$q_L(x) = \frac{-1}{2^{5/2}} \sum_{\mu \nu \rho \sigma = \pm 1} \dot{\epsilon}_{\mu \nu \rho \sigma} \text{Tr}(\Pi_{\mu \nu}(x)\Pi_{\rho \sigma}(x)),$$

measured after cooling [33, 34], i.e. minimization of the gauge action to eliminate ultraviolet (UV) artifacts; $\dot{\epsilon}_{\mu \nu \rho \sigma} = \epsilon_{\mu \nu \rho \sigma}$ for positive directions and $\dot{\epsilon}_{\mu \nu \rho \sigma} = -\epsilon_{(-\mu)\nu\rho\sigma}$. After cooling, the charge is divided by a constant factor $\alpha$, typically very close to one, so that its distribution gets peaked around integer values (see e.g. Fig. 2), then $Q$ is fixed to the closest integer.

For each run we have verified the stability of results against the number of cooling sweeps $n_{\text{cool}}$; then fixing $n_{\text{cool}}$ inside a well defined plateau (e.g. $n_{\text{cool}} = 30$ for the finest and 60 for the coarsest spacing). Cooling is known to provide results comparable with improved fermionic definitions as the continuum limit is approached [34, 35], however, due to the unusual conditions adopted in our investigation, we have checked this fact on subsets of our configurations. For $a = 0.094$ fm and $\vec{E}_1 \cdot \vec{B} \simeq 0.22$ GeV$^4$ we obtain, on a set of 30 decorrelated configurations, $\langle Q_{ov} \rangle / \langle Q_{30} \rangle \simeq 0.81(8)$, where $Q_{ov}$ counts the difference between left and right-handed zero modes of the Dirac overlap operator and $Q_{n}$ refers to $n$ cooling sweeps; on the full set of about 8000 configurations we have instead $\langle Q_{60} \rangle / \langle Q_{15} \rangle \simeq 0.987(9)$. The agreement is reasonable since, at this lattice spacing, small topological objects may not correspond to exact zero modes [36]. At the same $a$, autocorrelation times for $Q$, which grow critically when approaching the continuum limit, are around 100 RHMC trajectories.

As a first illustration of our results, in Fig. 1 we show the Monte-Carlo history of $Q$ for two numerical simulations, performed respectively at zero and non-zero e.m. field, for $a \simeq 0.113$ fm. The non-zero e.m. field corresponds to $\vec{E}_I = \vec{B} = 3 f \vec{z}$, with $f$ defined in Eq. [3], i.e. $\epsilon^2 \vec{E}_I \cdot \vec{B} \simeq 0.47$ GeV$^4$. While in absence of the e.m. background $Q$ fluctuates around zero, as expected by CPT-invariance, fluctuations are shifted towards positive values as $\vec{E}_I \cdot \vec{B} \neq 0$. This is clearer from Fig. 2 where we plot the corresponding distributions of $Q$; in this case we obtain $\langle Q \rangle(\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0 = 1.46(14)$.

In order to better investigate the dependence of $\langle Q \rangle(\vec{E}_I, \vec{B})$ on the background field values, in Fig. 3 we show $\langle Q \rangle(\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0$ for $a \simeq 0.282$ fm and for various combinations of $\vec{E}_I$ and $\vec{B}$, mostly taken parallel to the $z$ axis. All data, when plotted versus $\vec{E}_I \cdot \vec{B}$, fall on the same curve, thus demonstrating that $\theta_{\text{eff}}$ is, within errors, a function of $\vec{E}_I \cdot \vec{B}$ alone, as expected. We have taken different combinations of the fields, even with $\vec{E}_I$ and $\vec{B}$ not parallel to each other, having exactly the same or opposite values for $\vec{E}_I \cdot \vec{B}$, thus checking also that $\theta_{\text{eff}}$ is odd in $\vec{E}_I \cdot \vec{B}$. The dependence is linear in $\vec{E}_I \cdot \vec{B}$ for small fields, then saturating for larger fields, as common to many systems with a linear response to external stimulation. We have found that all data can be nicely fitted by a function

$$\langle Q \rangle(\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0 = a_0 \arctan(a_1 \vec{E}_I \cdot \vec{B});$$

the best fit curve, corresponding to $\chi^2 / \text{d.o.f.} = 0.74$, is shown in Fig. 3.

To discuss finite size and UV cutoff effects, in Fig. 4 we show $\langle Q \rangle(\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0$ for different spacings $a$ and lattice volumes $L^4$. $\langle Q \rangle$ and $\langle Q^2 \rangle_0$ are both derivatives of the free energy with respect to $\theta$, hence they are proportional to $V$ and, apart from possible systematic effects, their ratio should be volume independent. From Fig. 4 we infer that finite size effects are not significant, even on the smallest volumes corresponding to $am_T L \sim 4$. 

FIG. 2: Distribution of the topological charge shown in Fig. 1, $\alpha = 0.947$ (see text).
mass spectrum in our case; moreover, it is realistic to indicate
χηκRef.[20] for χ and the chiral limit, preliminary results obtained for "quad 1") if restricted to 0.05
slopes at 0.05, 0.1, 0.2, 0.4, and 0.6. Significant lattice artifacts may
be related to the determination of χ if a is so coarse that part of the
topological background, created by the influence of the e.m. field, lives close to the UV scale, then
cooling may destroy part of such background. However data obtained for a < 0.15 fm are in good agreement
with each other, especially in the small field region, which is
the one relevant for the determination of χCP.

In order to determine χCP, we have performed best fits of the data in Fig. 3 to the function in Eq. (10),
in a range c2Ei · B < 0.8 GeV4, then considering its slope at Ei · B = 0 and exploiting Eqs. (2) and (3). Results
are reported in Table I. We have verified that each slope is consistent with a direct linear fit, restricted to
a narrow enough region of small Ei · B. An extrapolation to the continuum limit, assuming O(a2) corrections,
yields χCP = 6.09(31) GeV−4 (χ2/d.o.f. ≈ 2.5), when including all data, and χCP = 5.47(78) GeV−4
(χ2/d.o.f. ≈ 0.1) if restricted to a < 0.15 fm. Due to the large artifacts affecting the coarsest lattices, we prefer to
quote last value as our estimate.

Our investigation is still far from the physical region and the chiral limit, preliminary results obtained for β =
5.504 and am = 0.00381 (a ≈ 0.15 fm, mπ ≈ 280 MeV) indicate χCP ≈ 10 GeV−4 (see Fig. 1), suggesting that
χCP increases when decreasing quark masses.

Discussion – The phenomenological estimate given in Ref. [20] for κ = 2 2χCP, which is based on the effective
couplings of the η and η′ mesons to two photons and to two gluons, is χCP ≈ 0.73/(π2fη′3mη′2) ≈ 3 GeV−4;
our estimate is slightly larger, but one should consider the different systematics, including the unphysical quark
mass spectrum in our case; moreover, it is realistic to estimate a 5% systematic uncertainty in our knowledge of
a, leading an additional ∼ 20% uncertainty on χCP. To give an idea of the magnitude of the effect, we estimate
θeff ∼ 10−5 for parallel E and B with eB ∼ 1 GeV2 and eE ∼ 1 MeV2.

Regarding the validity of analytic continuation from imaginary to real electric fields, we notice that, while a
smooth behavior is expected as the imaginary electric field approaches zero, at least for non-zero quark masses,
a uniform and constant real electric field, however small, is instead expected to induce vacuum instabilities. On
the other hand, this is not true if we consider electric fields which are limited in space. Therefore our result
should be applicable to determine the local effective θ parameter produced by smooth enough but spatially lim-
ited CP-violating e.m. fields. It would be interesting in the future to consider the case of smoothly varying fields
explicitly.

Finally, apart from repeating our determination with more physical quark masses and closer to the continuum
limit, it would be interesting to extend our investigation also to finite temperature, especially across and right
above the deconfinement transition, where the determination of the effective pseudoscalar QED-QCD interaction
would be relevant also to the phenomenology of the CME in heavy ion collisions.

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