An Energy-based Damage Model for Concrete Structures under Cyclic Loading

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Abstract

An anisotropic damage model is developed by introducing nonlinear unloading/linear reloading branches on the principal axis of damage to simulate the hysteretic behaviors of concrete structures subjected to cyclic loading. The nonlinear unloading branch is defined as a power function and an energy-based evolution rule of damage is implemented into the definition of linear reloading one. Two independent damage variables, one for tension and the other for compression, are defined as a function of the ratio of accumulated dissipating energy to fracture energy to reflect the stiffness degradation caused by tensile cracking and compressive crushing. The calibrating procedures for key parameters are presented based on the stress-strain response obtained from the uniaxial cyclic tension or compression. A cyclic compressive test is analyzed with this model. The calculated responses are consistent with the experimental ones and reflect the stiffness degradation, the accumulation of irreversible deformation, and hysteretic behaviors. The results show the anisotropic damage model is applicable to the nonlinear analysis of concrete structure under cyclic loading.

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Keywords: Concrete; Anisotropic damage model; Stiffness degradation; Hysteretic behavior.

1. INTRODUCTION

It is an essential yet challenging task to analyze the mechanical response of concrete structure under general loading conditions because developing a constitutive model which reasonably reflects nonlinear...
behaviors of concrete is still difficult. At the macroscopic scale, the phenomenological behavior of concrete has been modeled by classical plasticity, e.g. (Feenstra and de Borst 1996; Papanikolaou and Kappos 2007), and damage mechanics, such as (Badel et al. 2007; Jirasek and Grassl 2008). The former is difficult to represent the degradation of material stiffness observed in concrete tests while the latter, without inelastic (or plastic) strain, is insufficient to simulate the irreversible deformation. Combining the advantages of classical plasticity and damage mechanics, (Lee and Fenves 1998; Voyiadjis et al. 2008) developed plastic-damage models which use the flow theory of plasticity and damage mechanics to represent irreversible deformation and stiffness degradation respectively. Studies have demonstrated that these models can effectively simulate the nonlinear behavior of concrete structure under monotonic loading. However, plastic-damage models are so complicated that only a few simple ones have sufficient efficiency to conduct dynamic or cyclic analysis of complex concrete structure. Moreover, these models utilize linear stress branches to represent unloading/reloading responses, which results in the difficulty of modeling the hysteretic behavior of concrete and corresponding energy dissipation.

An anisotropic damage model for cyclic loading is developed based on conventional rotating crack approach. Similar to the concept proposed by (Palermo and Vecchio 2003), nonlinear unloading/linear reloading branches are defined along the rotating damage axis to model the hysteretic behavior in tension and compression.

2. The constitutive relation of anisotropic damage model

Following presuppositions are used to develop the constitutive model of concrete:

a) The strain is decomposed into elastic part and inelastic part for each material point, and the stress and elastic strain satisfies linear elasticity;

b) There are three mutually orthogonal axes of material which always keep aligned with the directions of principal strain (and stress) during loading histories;

c) Damage constitutive relations are formulated as functions of principal stress and inelastic strain on the rotating damage axis, which are mutually decoupled for all material axes.

2.1. Mathematical formulation of the anisotropic damage model

As shown in Figure 1, axis \( y_1 \) is the principal direction on which damage occurs. According to the approach proposed by (Bazant 1983) and (Willam et al. 1987), constitutive relations is derived as:

\[
\Delta \sigma = D_{\text{tan}} \Delta \varepsilon = T^T D' T \Delta \varepsilon
\]
\[ \Delta \sigma = \begin{bmatrix} \Delta \sigma_{11} & \Delta \sigma_{22} & \Delta \sigma_{33} & \Delta \sigma_{12} & \Delta \sigma_{13} & \Delta \sigma_{23} \end{bmatrix}^T \]

\[ \Delta \epsilon = \begin{bmatrix} \Delta \epsilon_{11} & \Delta \epsilon_{22} & \Delta \epsilon_{33} & \Delta \gamma_{12} & \Delta \gamma_{13} & \Delta \gamma_{23} \end{bmatrix}^T \]

where \( D_{an} \) and \( D' \) are tangent material stiffness matrices in global coordinates \( \alpha x_1x_2x_3 \) and local ones \( \gamma y_1y_2y_3 \) respectively; \( T \) is the transformation matrix of engineering strain from \( \alpha x_1x_2x_3 \) to \( \gamma y_1y_2y_3 \).

The formulation of matrix \( D' \) is:

\[
D' = \begin{bmatrix} \bar{D}_n & 0 \\ 0 & \bar{D}_s \end{bmatrix}, \quad \bar{D}_n = \left( \bar{C}_e + \bar{C}_c \right)^{-1}, \quad \bar{C}_c = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}
\]

\[
\bar{C}_c = \begin{bmatrix} \left( \frac{ds_1}{de_1} \right)^{-1} & 0 & 0 \\ 0 & \left( \frac{ds_2}{de_2} \right)^{-1} & 0 \\ 0 & 0 & \left( \frac{ds_3}{de_3} \right)^{-1} \end{bmatrix}, \quad \bar{D}_s = \begin{bmatrix} \frac{s_1-s_2}{2(e_1-e_2)} & 0 & 0 \\ 0 & \frac{s_1-s_3}{2(e_1-e_3)} & 0 \\ 0 & 0 & \frac{s_2-s_3}{2(e_2-e_3)} \end{bmatrix}
\]

where \( E \) is the elastic modulus of concrete and \( \nu \) denotes the Poisson ratio; \( s_i, e_i \) and \( e_i^{in} (i=1,3) \) are the principal stress, principal strain and its inelastic part on axis \( y_i \) in coordinates \( \gamma y_1y_2y_3 \). According to assumption c), the relations between principal stress and inelastic strain are decoupled for all material axes and then \( D' \) can be easily determined by \( s_i \) and the derivative \( ds_i/de_i^{in} \). For simplicity, it is assumed that damage occurs on one material axis during loading histories.

### 2.2. Formula of stress branches on the damage axis

Accounting for different damage states caused by tensile cracking and compressive crushing, two damage evolution laws are introduced to model distinct nonlinear behaviors of concrete in tension and compression. However, the formulations of these two damage laws are similar. Because there is no energy dissipation caused by the stress-elastic strain relation, it is suitable to define stress branches in the form of principal stress and inelastic strain.

![Figure 2 Stress branches: a) loading branch; b) unloading branch; c) reloading branch.](image-url)
2.2.1. The stress branch of loading

As shown in Figure 2a, a piecewise linear function which connects characteristic data points of experimental response is used to define the loading stress branch.

\[
\begin{align*}
\{ \sigma_i^n \} & = \begin{cases} 
\sigma_i^n + k_i^n \left( e_i^n - a_i^n \right) & a_i^n < e_i^n < a_{i+1}^n \quad (i < n) \\
\sigma_i^n & \quad a_i^n \leq e_i^n 
\end{cases} \\
k_i^n &= \frac{\sigma_{i+1}^n - \sigma_i^n}{a_{i+1}^n - a_i^n}, \\
\kappa &= \begin{cases} 
1 & \text{if } e_i^n > 0, \ s_i > 0 \\
0 & \text{if } e_i^n < 0, \ s_i < 0 
\end{cases}
\end{align*}
\]

where \( \kappa \in \{ t, c \} \) is the state variable with \( \kappa = t \) for tension and \( \kappa = c \) for compression, \( (\alpha_i^n, \sigma_i^n) \) are the absolute values of inelastic strain and corresponding stress of the \( i \)th point.

2.2.2. The stress branch of unloading

Concrete tests by (Bahn and Hsu 1998; Yankelevsky and Reinhardt 1989) show that irrecoverable deformation occurs even after external loads are removed. The residual strain, defined as plastic offset strain by (Palermo and Vecchio 2003), is the amount of irrecoverable damage caused by crushing, compressing of internal voids and micro-crack opening. Herein, the unloading branch is defined as follows which accounting for the influence of the plastic offset strain \( e_p^r \):

\[
\begin{align*}
\{ \sigma_i^m \} & = \sigma_i^m + B_i^x \left| e_m^i - e_i^m \right| + C_i^x \left| e_m^i - e_i^m \right|^\alpha \\
B_i^x & = \beta_i^x \left[ \alpha_i^x \left| \sigma_m^i \right| e_m^i - e_p^i \right]^{-1} - H_i^x \left| e_m^i - e_p^i \right|^{-\alpha_i^x} \\
C_i^x & = \beta_i^x \left[ H_i^x \left| e_m^i - e_p^i \right| - \left| \sigma_m^i \right| \right] \\
\beta_i^x & = \frac{1}{\left( 1 - \alpha_i^x \right) \left| e_m^i - e_p^i \right|^\alpha_i^x}
\end{align*}
\]

in which \( e_i^m, e_i^m \) and \( s_i \) are the present inelastic strain and principal stress on the damage axis \( y_1 \), \( e_m^i \) and \( \sigma_m^i \) denote the previously maximum inelastic strain and corresponding stress at the onset of unloading, \( H_i^x \) represents the tangent modulus at a zero stress and exponent \( \alpha_i^x \) (\( 0 < \alpha_i^x < 1 \)) is material constant determined by tests.

2.2.3. The stress branch of reloading

The reloading branch shown in Figure 2c is defined as equation (5)

\[
\begin{align*}
\{ \sigma_i^a \} & = \sigma_i^a + H \left| e_i^a - e_i^m \right| \\
H & = \frac{1 - d_k^x}{d_k^x}, \quad 0 < d_k^x < 1
\end{align*}
\]

where \( e_i^m \) and \( \sigma_i^a \) are the inelastic strain and corresponding stress at the onset of reloading; \( H \) is the slope of reloading branch; \( d_k \) denotes the tensile or compressive damage variables.
2.2.4. The stress branches of partial unloading and reloading

The definition of partial unloading is identical to that of unloading except variables $e_m^0$ and $\sigma_m$ are substituted with $e_m^e$ and $\sigma_e$ which denote the inelastic strain and corresponding stress at the onset of partial unloading. Similarly, the formula of partial reloading branch is obtained from equation (5) by replacing $e_a^m$ and $\sigma_a$ with $e_a^e$ and $\sigma_e$ that are the inelastic strain and corresponding stress at the onset of partial reloading. For simplicity, there is no need to repeat the formula.

2.3. Definitions of tensile and compressive damage variables

The definition of damage variable $d_{\kappa}$, $\kappa\in\{t,c\}$ depends on the density of accumulated dissipating energy $\Psi^\kappa$ and fracture energy density $g^\kappa_f$ in pure tension or compression. As shown in Figure 3a, the density of fracture energy is defined as the area of the shadow region:

$$g^\kappa_f = \int_{0}^{e^\kappa_f} s_t d\epsilon^m$$

where $e^\kappa_f$ is the inelastic strain corresponding to complete damage in tension or compression, $G^\kappa_f$ is a material property representing the tensile fracture energy or its counterpart in the compressive state, and $l_\kappa$ denotes the size of localization zone determined by the characteristic length of element to maintain objective results at the structural level. However, it is still an open issue whether the counterpart of the fracture energy in compressive failure is a material property.

![Figure 3](image)

The accumulated dissipating energy per unit volume $\Psi^\kappa$ is defined as the summation of energy dissipation during previously cyclic loading. As illustrated in Figure 3b, $\Psi^\kappa$ is:

$$\Psi^\kappa = \Psi_1^\kappa + \Psi_2^\kappa + \Psi_3^\kappa + \cdots$$

where $\Psi_1^\kappa$, $\Psi_2^\kappa$ and $\Psi_3^\kappa$ denote the energy dissipated in loading, unloading/reloading and partial unloading/reloading branches respectively. Thus, the damage variable $d_{\kappa}$ is defined as:

$$d_{\kappa} = (\kappa^\kappa)^{\theta} ; \quad \kappa_{\kappa} = \frac{\Psi^\kappa}{g^\kappa_f}$$
in which $\kappa_\ell$ is a dimensionless variable defined as the ratio of accumulated dissipating energy to fracture energy, and $\theta_\ell$ denotes damage exponent which is material property.

3. The calibration of key parameters

In this model, $e^\kappa_p$, $H^\kappa_i$ and $\theta^\kappa$ are key parameters. The stress-strain responses under uni-axial tension or compression, shown in Figure 4, are used to calibrate these parameters.

3.1. Damage exponent $\theta^\kappa$

The accumulated dissipating energy per unit volume $\Psi^{\kappa(i)}$ and the reloading stiffness $E^{\kappa(i)}$ for the $i$th cycle can be obtained from the stress-strain relation shown in Figure 4a. The ratio of dissipating energy to fracture energy, $\kappa^{\kappa(i)}$, and damage factor, $d^{\kappa(i)}$, are defined as:

$$\kappa^{\kappa(i)} = \frac{\Psi^{\kappa(i)}}{g^\kappa_i}, \quad d^{\kappa(i)} = 1 - \frac{E^{\kappa(i)}}{E}, \quad i = 1, n, \quad \varepsilon(1,c)$$

where $g^\kappa_i$ is the fracture energy density in tension or its counterpart in compression, $E$ denotes the initial elastic modulus of concrete. Based on equation (8) and the data $(\kappa^{\kappa(i)}, d^{\kappa(i)})$, $i = 1, \cdots n$ of the $n$ loading cycles, the damage exponent $\theta^\kappa$ can be obtained from a regression analysis.

3.2. The plastic offset strain $e^c_p$

Experimental results show that the plastic offset strain increases with strain. Herein, the plastic offset strain is defined by equation (10):

$$\frac{e^c_p}{\varepsilon^0} = c + k \left( \frac{e^{in}_c}{\varepsilon^0_c} \right) \quad \text{if} \quad e^{in}_c > 0$$

$$\frac{e^c_p}{\varepsilon^0} = c + m \left( \frac{e^{in}_c}{\varepsilon^0_c} \right)^2 \quad \text{if} \quad e^{in}_c < 0$$

Figure 4 The experimental responses for calibrating key parameters: a) the relation of stress and total strain; b) the relation of stress and inelastic strain.
where $\varepsilon_t^0$ and $\varepsilon_c^0$ are strains corresponding to tensile and compressive strengths under uniaxial loading; $c_t$ and $k_t$ are tensile fitting constants, and $c_c$, $k_c$ and $m_c$ are compressive ones. Obviously, these fitting constants can be calibrated from regression analyses based on the experimental data $(\varepsilon_t^{in(i)}, \varepsilon_p^{xt(i)})$, $i = 1, \cdots, n$ obtained from the stress-inelastic strain relation shown in Figure 4b.

### 3.3. The tangent modulus at the terminal of unloading $H_t^c$

The tangent modulus at a zero stress of unloading $H_t^k$ is defined as:

$$
\frac{H_t^k}{E} = \begin{cases} 
A_c \cdot \exp \left[ B_c \left( \frac{\varepsilon_t^{in}}{\varepsilon_c^0} \right) \right] & \text{if } \frac{H_t^k}{E} < 0.9 \\
0.9 & \text{if } \frac{H_t^k}{E} \geq 0.9
\end{cases}
$$

where $A_c$ and $B_c$ with $\kappa \in (t, c)$ are fitting constants, and $\varepsilon_c^0$ is the strain at the peak stress under uniaxial monotonic tension or compression. Similarly, constants $A_c$ and $B_c$ are also calculated from the regression analysis using the data $(\varepsilon_t^{in(i)}, H_t^{xi(i)})$, $i = 1, \cdots, n$ obtained from the stress-inelastic strain relation shown in Figure 4b.

### 4. Numerical simulations of concrete tests

(Karsan and Jirsa 1969) tested concrete specimens, $82.6 \times 82.6 \times 82.6$ mm$^3$ in dimension, in monotonic and cyclic compression. An 8-node hexahedral element shown in Figure 5 is utilized to model the tested specimen. In addition, special constraints on node 1-4 and displacement loadings on node 5-8 are prescribed for the element to simulate the uniaxial state of stress.

![Figure 5 Finite element mesh, constraints and displacement loading of Karsan’s test.](image)

Material properties are: Young’s modulus $E=31.7$ GPa, Poisson’s ratio $\nu=0.2$, compressive strength $f_c=-27.6$ MPa, the strain corresponding to compressive strength $\varepsilon_c^0 = 2.5 \times 10^{-3}$, compressive fracture energy density $\gamma_f^c = 0.20$ N/mm$^2$ and the exponent of unloading branch $\beta=0.5$. Figure 6 shows the stress branch of loading and calibrated key parameters $\varepsilon_c^0$, $H_t^c$ and $\theta_c$.
Figure 6 Material parameters used to model the Karsan’s test: a) the stress branch of loading; b) the fitting of $\varepsilon_p^c$; c) the fitting of $H_1^c$; d) the fitting of $\theta_c$.

Figure 7 Stress-strain responses and $d_c-\kappa_c$ relation: a) comparison of numerical and experimental results; b) the relation of $d_c$ and $\kappa_c$. 
Figure 7a compares the calculated stress-strain relation with that obtained from the experiment. In addition to $T_c=0.2678$, two analyses with $T_c=0.20, 0.35$ are conducted to investigate the influence of $T_c$ on the accuracy of the model. The numerical responses obtained with the calibrated parameters are in the closest agreement with those observed in the experiment. The results show that the damage model can reflect the residual deformation, nonlinear unloading branches, degraded stiffness and hysteretic behavior of concrete under cyclic loading. Linear reloading branch, however, is approximate because the experimental one appears remarkable nonlinearity in the vicinity of the loading branch.

The results also show that the damage exponent $T_c$ has substantial influence on the computational accuracy of the present model, especially on the reloading behavior. As shown in Figure 7a, the calculated reloading stiffness increases with $T_c$ during the first four load cycles while it decreases with $T_c$ during the last three ones. This phenomenon is illustrated by the $\delta_x-\kappa_x$ relation in Figure 7b. At the first load cycle, all three cases obtain identical $\kappa_x$. The higher $T_c$, therefore, is prescribed, the less damage factor $d_c$ becomes and the stiffer the reloading branch is obtained. Nevertheless, the stiffer reloading branch results in more energy dissipation and the damage variable $\kappa_x$ increases with $T_c$ from the second load cycle to the eighth one. As a consequence, the combined effects of $T_c$ and $N_c$ gradually reduce the difference in $d_c$ from the first load cycle to the fourth one. This difference nearly vanishes at the fifth load cycle and converse correlation between $d_c$ and $T_c$ occurs during the last three ones.

5. CONCLUSIONS

An anisotropic damage model is developed for analyzing the response of concrete structure under cyclic loading. A cyclic compressive test is analyzed by the model and the numerical results draw following conclusions:

a) The stress-strain response calculated by the numerical analysis is similar to that obtained by the experiment. Numerical response reflects the damage-induced degradation of stiffness, the accumulation of residual deformation, energy dissipation due to hysteretic behavior.

b) The calculated responses of unloading and reloading are consistent with the experimental ones, which indicates that the hysteretic behavior can be approximately modeled using the nonlinear unloading/linear reloading branches and it is feasible to define the tensile or compressive damage variable $d$ as a function of the ratio of the accumulated dissipating energy to the fracture energy.

c) The result calculated with calibrated parameters is in the closest agreement with that obtained in the experiment, which means the proposed calibrating procedure is feasible.

From an engineering point of view, the proposed damage model is applicable to simulating the nonlinear behavior of concrete under cyclic loading.

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