ATOM SPECTRA OF GRADED RINGS AND SHEAFIFICATION IN TORIC GEOMETRY

SEBASTIAN POSUR

ABSTRACT. We prove that the atom spectrum, which is a topological space associated to an arbitrary abelian category introduced by Kanda, of the category of finitely presented graded modules over a graded ring $R$ is given as a union of the homogeneous spectrum of $R$ with some additional points, which we call non-standard points. This description of the atom spectrum helps in understanding the sheafification process in toric geometry: if $S$ is the Cox ring of a normal toric variety $X$ without torus factors, then a finitely presented graded $S$-module sheafifies to zero if and only if its atom support consists only of points in the atom spectrum of $S$ which either lie in the vanishing locus of the irrelevant ideal of $X$ or are non-standard.


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1. Introduction

The sheafification of a finitely presented graded module $M$ over the Cox ring $S$ of a normal toric variety $X$ over $\mathbb{C}$ without torus factors yields a coherent sheaf $\tilde{M}$ on $X$. In the case of the $n$-dimensional projective space $\mathbb{P}^n = \text{Proj}(\mathbb{C}[x_0, \ldots, x_n])$, a finitely presented $\mathbb{Z}$-graded $S$-module $M$ sheafifies to zero if and only if the support of $M$ lies in the vanishing locus of the irrelevant ideal $\langle x_0, \ldots, x_n \rangle$. However, this geometric criterion to check $\tilde{M} \simeq 0$ does not naively generalize from $\mathbb{P}^n$ to $X$: for example, it is impossible to read off from the support of a graded module $M$ over the Cox ring of the weighted
projective space $\mathbb{P}(1, 1, 2)$ whether $M$ sheafifies to zero or not [CLS11, Example 5.3.11].

Cox states in [Cox95] that “there are many nonzero modules which give the zero sheaf” and analyzes this “phenomenon” by giving an algebraic criterion for the special case where $X$ is a simplicial toric variety in [Cox95, Proposition 3.5].

In this paper, we remedy the failure of the geometric criterion for an arbitrary $X$ by applying the very general theory of atom spectra. To an arbitrary abelian category $A$, Kanda [Kan12] defines a topological space $\text{ASpec}(A)$ called the atom spectrum of $A$ whose points consist of atoms, i.e., equivalence classes of so-called monoform objects. For an object $A \in A$, its atom support $\text{ASupp}(A) \subseteq \text{ASpec}(A)$ consists of all the atoms defined by monoform subquotients of $A$. A main theorem on atom spectra states that if all objects in $A$ are noetherian, then an open subclass $U \subseteq \text{ASpec}(A)$ defines a Serre subcategory $\mathcal{A} \subseteq A$ if and only if $\text{ASupp}(A) \subseteq U$, and all Serre subcategories arise in this way from a uniquely determined corresponding $U$.

Specializing to the case $A = S$-mod, the category of finitely presented $S$-modules, it is natural to ask how the open subclass corresponding to the Serre subcategory $\mathcal{A} \subseteq S$-mod looks like.

To answer this question, we first investigate the atom spectrum of the category $R$-mod of finitely presented $G$-graded $R$-modules for a $G$-graded ring $R$ in Section 3, where $G$ denotes an abelian group (see the beginning of that section for our definition of a $G$-graded ring). In Theorem 3.8, we describe the structure of $\text{ASpec}(R) \subseteq \text{ASpec}(R)$ relative to the classical homogeneous spectrum of $R$: any homogeneous prime ideal $p \subseteq R$ and any element $g \in G$ gives rise to a point $\tilde{p}(g)$ in $\text{ASpec}(R)$, and two such points $\tilde{p}(g)$ and $\tilde{q}(h)$ are equal if and only if $p = q$ and $h + G_p = g + G_p$, where $G_p$ is a subgroup of $G$ determined by $p$ (see Definition 3.5). We conclude in Remark 3.10 that we have a disjoint union on a set-theoretic level:

$$\text{ASpec}(R) \simeq \{\tilde{p}(0) \mid p \subseteq R \text{ hom. prime}\} \cup \{\tilde{p}(g) \mid g \notin G_p, \ p \subseteq R \text{ hom. prime}\},$$

where we call all points in that second subset non-standard points.

In Section 4, we apply our findings to the graded Cox ring $S$ of $X$ and remedy our criterion for checking whether $\tilde{M} \simeq 0$ in Theorem 4.5:

**Theorem.** Let $X$ be a normal toric variety over $\mathbb{C}$ without torus factors and with $G$-graded Cox ring $S$ and irrelevant ideal $B(\Sigma)$. A finitely presented $G$-graded $S$-module $M$ sheafifies to zero if and only if

$$\text{ASupp}(M) \subseteq \{\tilde{p}(0) \mid p \in \text{Supp}(S/B(\Sigma))\} \cup \{\text{non-standard points in } \text{ASpec}(S)\}.$$

In conclusion, we can think of the non-standard points as the missing piece of the homogeneous spectrum for understanding when a graded module sheafifies to zero by looking at its support.

**Convention.** Whenever we have an object equipped with a grading by an abelian group $G$, e.g., a graded ring $R$ or a graded $R$-module $M$, we will simplify terminology as follows:
(1) An element \( r \in R \) will always mean a \textit{homogeneous} element. We write \( \deg(r) \in G \) for its degree provided \( r \neq 0 \). We will do the same for elements \( m \in M \).

(2) A prime ideal \( p \subseteq R \) will always mean a \textit{homogeneous} prime ideal. We denote by \( \Spec(R) \) the set of \textit{homogeneous} prime ideals of \( R \).

(3) An \( R \)-module will always mean a \textit{graded} \( R \)-module, and likewise for submodules. By \( R\)-mod we denote the category of finitely presented \( G \)-graded \( R \)-modules. We write \( \Supp(M) \subseteq \Spec(R) \) for the \textit{homogeneous} primes in the support of \( M \).

Whenever we want to address the underlying non-graded ring of \( R \) or non-graded module of \( M \), we write \( |R| \) and \( |M| \), and the adapted notions of element, prime ideal, spectrum, module, and support retain their classical meaning, e.g., by \( \Spec(|R|) \), we address the set of not-necessarily homogeneous prime ideals of \( R \).

In this paper, we will make use of subsets of \( G \) that constitute the degrees of the homogeneous non-zero elements of a \( G \)-graded module \( M \). We set

\[
\deg(M) := \{ \deg(m) \mid m \in M \setminus \{0\} \text{ homogeneous} \} \subseteq G.
\]

If \( M \) is a \( G \)-graded module, its \textbf{shift} by \( g \in G \) is the \( G \)-graded module \( M(g) \) with graded parts

\[
M(g)_h := M_{g+h}
\]

for \( h \in G \).

2. Preliminaries: atom spectra and the classification of Serre subcategories

We give a short introduction to Kanda’s classification of Serre subcategories of an abelian category consisting of noetherian objects [Kan12].

\textbf{Definition 2.1.} Let \( A \) be an abelian category. A \textit{non-zero} object \( A \in A \) is called \textbf{monoform} if the following holds: whenever we are given

1. a chain of subobjects \( 0 \leq B' \leq B \leq A \),
2. a subobject \( C \leq A \),
3. and an isomorphism \( C \cong B/B' \),

then we already have \( C \cong 0 \).

Simple objects in \( A \) are examples for monoform objects. The free abelian group in one generator is an example for a non-simple monoform object in the category of abelian groups.

\textbf{Remark 2.2.} Non-zero subobjects of monoform objects are themselves monoform.

We call two monoform objects \( A, B \in A \) \textbf{atom-equivalent} if they admit a common non-zero subobject. Atom-equivalence defines an equivalence relation on the class of monoform objects in \( A \) [Kan12, Proposition 2.8]. We denote the equivalence class of a monoform object \( A \in A \) by \( \overline{A} \) and call such a class an \textbf{atom} of \( A \).

\textbf{Definition 2.3.} Let \( A \) be an abelian category.
(1) The **atom support** of an object $A \in \mathcal{A}$ is defined as
$$\text{ASupp}(A) := \{ B \mid B \text{ is a monoform subquotient of } A \}$$

(2) The **atom spectrum** of $\mathcal{A}$ is a topological space\(^1\) whose points are given by
$$\text{ASpec}(\mathcal{A}) := \{ B \mid B \text{ is a monoform object in } \mathcal{A} \}$$
and with the classes $\text{ASupp}(A), A \in \mathcal{A}$ as a basis of the topology.\(^2\)

Recall that an object $A \in \mathcal{A}$ is called **noetherian** if every ascending chain of its subobjects eventually stabilizes. Now, we can turn to the geometric picture for the classification of Serre subcategories.

**Theorem 2.4.** Let $\mathcal{A}$ be an abelian category in which every object is noetherian. Then we get a bijection

$$\{ \text{Serre subcategories } C \subseteq \mathcal{A} \}$$

$$\xymatrix{ \text{ASupp} \ar@{<->}[r]^\cong & \text{ASupp}^{-1} \ar@{<->}[l]^\cong \\
\{ \text{open subclasses } U \subseteq \text{ASpec}(\mathcal{A}) \} \ar@{<->}[u]_\subseteq & 
}$$

where we set

$$\text{ASupp}(C) := \bigcup_{C \subseteq C} \text{ASupp}(C)$$

for a Serre subcategory $C \subseteq \mathcal{A}$, and

$$\text{ASupp}^{-1}(U) := \{ A \in \mathcal{A} \mid \text{ASupp}(A) \subseteq U \}$$

for an open subclass $U \subseteq \text{ASpec}(\mathcal{A})$. Under this bijection, the intersection of finitely many Serre subcategories corresponds to the intersection of open subclasses.

**Proof.** The statement about the bijection is [Kan12, Theorem 4.3] for classes instead of sets. Moreover, it is easy to see that $\text{ASupp}^{-1}$ is compatible with intersections, and thus, the same is true for $\text{ASupp}$. \(\Box\)

**Remark 2.5.** We get a geometric picture for the question whether a given object $A \in \mathcal{A}$ lies in a Serre subcategory $C \subseteq \mathcal{A}$:

$$A \in C \iff A \in \text{ASupp}^{-1}(\text{ASupp}(C))$$

$$\iff \text{ASupp}(A) \subseteq \text{ASupp}(C).$$

**Example 2.6.** [Kan12, Proposition 7.2, Remark 7.4] If $R$ is a commutative noetherian ring, and $R$-mod the category of finitely presented $R$-modules, then

$$\text{Spec}(R) \overset{\sim}{\to} \text{ASpec}(R\text{-mod}) : p \mapsto \frac{R}{p}$$

defines a bijection that actually is a homeomorphism of topological spaces if we equip $\text{Spec}(R)$ with the Hochster dual of the Zariski topology, i.e., the open sets in $\text{Spec}(R)$ are

\(^1\)Beware that its underlying class of points might not be a set.

\(^2\) This characterization of the opens in $\text{ASpec}(\mathcal{A})$ is given in [Kan12, Proposition 3.9].
exactly the specialization closed subsets. Furthermore, via this bijection, the atom support of a module \( M \in R\text{-mod} \) corresponds to its support. Thus, if \( C \subseteq R\text{-mod} \) denotes a Serre subcategory, then

\[
M \in C \iff \text{Supp}(M) \subseteq \bigcup_{C \in C} \text{Supp}(C),
\]

a criterion which goes back to Gabriel [Gab62].

3. THE ATOM SPECTRUM OF A GRADED RING

Let \( G \) be an (additively written) abelian group. By a \( G \)-graded ring \( R \), we mean that

1. \( R \) is a commutative ring,
2. \( R \) is equipped with a \( G \)-grading, i.e., a decomposition into abelian groups

\[
R = \bigoplus_{g \in G} R_g
\]

such that \( R_g \cdot R_h \subseteq R_{g+h} \) for all \( g, h \in G \),
3. the subring \( R_0 \) is noetherian,
4. there exist finitely many homogeneous elements \( x_0, \ldots, x_n \in R \) for an \( n \in \mathbb{N}_0 \) such that \( R = R_0[x_0, \ldots, x_n] \).

Note that the underlying ring \( |R| \) is noetherian since it is an algebra of finite type over a noetherian ring.

We are going to describe

\[ \text{ASpec}(R) := \text{ASpec}(R\text{-mod}). \]

We start by exhibiting a special class of monoform objects in \( R\text{-mod} \).

**Lemma 3.1.** Let \( p \in \text{Spec}(R) \) and \( g \in G \). Then \( R/p(g) \) is a monoform object in \( R\text{-mod} \).

**Proof.** The functor that forgets the \( G \)-grading

\[
R\text{-mod} \to |R|\text{-mod}
\]

is faithful and exact, thus, it reflects monoform objects, i.e., \( M \in R\text{-mod} \) is monoform if \( |M| \) is. It follows that \( R/p(g) \) is a monoform object since its underlying non-graded module (over the noetherian ring \( |R| \)) is monoform [Kan12, Proposition 7.1].

Following the notation introduced in [Kan12, Section 6] and generalizing it to the graded case, we write

\[ \widehat{p}(g) := \overline{R/p(g)} \in \text{ASpec}(R) \]

in order to refer to a point in the atom spectrum associated to a prime \( p \subseteq R \) and an element \( g \in G \). If \( g \) is the neutral element, we simply write \( \widehat{p} \).

**Remark 3.2.** Opposed to the non-graded commutative case described in [Kan12, Proposition 7.1], we can have ideals \( \mathfrak{a} \subseteq R \) which are not prime, but such that \( R/\mathfrak{a} \) is monoform. For example, consider the \( \mathbb{Z} \)-graded algebra \( R = \mathbb{Q}[t] \) with \( \deg(t) = 1 \) and \( \mathfrak{a} = \langle t^2 \rangle \). For \( R/\mathfrak{a} \), the only non-trivial subquotient which is not also a subobject is given by \( R/\langle t \rangle \). But there does not exist a submodule of \( R/\mathfrak{a} \) which is isomorphic to \( R/\langle t \rangle \). Nevertheless, since
the submodule of $R/\mathfrak{a}$ which is generated by the class of $t$ is isomorphic to $R/\langle t \rangle(-1)$, we have
\[ R/\mathfrak{a} = \langle t \rangle(-1) \]
in $\text{ASpec}(R)$.

Next, we want to generalize the last observation of Remark 3.2 and see that the class of monoform objects described in Lemma 3.1 suffices to describe all atoms. For this, we need the following well-known structure theorem of finitely presented graded modules.

**Theorem 3.3.** Let $M \in R\text{-mod}$. Then there exist an $r \in \mathbb{N}_0$ and a filtration of $M$ by submodules
\[ 0 = M^0 \leq M^1 \leq \cdots \leq M^r = M \]
such that $M^i/M^{i-1} \cong R/p_i(g_i)$ for $p_i \in \text{Spec}(R)$ and $g_i \in G$, $i = 1, \ldots, r$.

**Proof.** The (non-constructive) proof in the $\mathbb{Z}$-graded case given in [Har77, Proposition I.74] can be used by simply replacing $\mathbb{Z}$ with $G$. \qed

**Corollary 3.4.** Let $M \in R\text{-mod}$ be monoform. Then there exist $p \in \text{Spec}(R)$ and $g \in G$ such that
\[ M = \tilde{p}(g). \]

**Proof.** Theorem 3.3 tells us that $M$ has a submodule of the form $R/p(g)$, which completes the proof. \qed

Thus, every atom of $R\text{-mod}$ is represented by a module of the form $R/p(g)$. Next, we analyze when two such modules give rise to the same atom.

**Definition 3.5.** Let $p \subseteq R$ be a prime. We set
\[ G_p := \langle \text{deg}(R/p) \rangle_{\mathbb{Z}} \leq G. \]

**Lemma 3.6.** Given primes $p, q \subseteq R$ and elements $g, h \in G$. Then
\[ \tilde{p}(g) = \tilde{q}(h) \]
if and only if
\[ p = q \quad \text{and} \quad g + G_p = h + G_p. \]

**Proof.** Since twisting with an element $g \in G$ defines an auto-equivalence of $R\text{-mod}$, it suffices to prove
\[ \tilde{p}(g) = \tilde{q} \quad \iff \quad p = q \quad \text{and} \quad g \in G_p. \]

Assume $\tilde{p}(g) = \tilde{q}$. Then $R/p(g)$ and $R/q$ have a common non-zero cyclic submodule whose annihilator is both $p$ and $q$. Thus, $p = q$. So, we are left to show
\[ \tilde{p}(g) = \tilde{p} \quad \iff \quad g \in G_p. \]

Let $x \in R \setminus p$. Then
\[ R/p(-\text{deg}(x)) \cong \langle x \rangle_R \leq R/p, \]
which implies
\[ \tilde{p}(-\text{deg}(x)) = \tilde{p}. \]
By applying twists and using the transitivity of the atom-equivalence, we obtain
\[ \tilde{p}(g) = \tilde{p} \]
for any \( \mathbb{Z} \)-linear combination \( g \) of elements of the form \( \text{deg}(x) \) for \( x \in R \setminus \mathfrak{p} \), in other words, for all \( g \in G_p \).

Conversely, if \( \tilde{p}(g) = \tilde{p} \) for some \( g \in G \), then \( R/\mathfrak{p}(g) \) and \( R/\mathfrak{p} \) have a common non-zero cyclic subobject which is necessarily of the form \( R/\mathfrak{p}(h) \) for some \( h \in G \). Since \( R/\mathfrak{p}(h) \hookrightarrow R/\mathfrak{p} \), we have \( h \in G_p \). Since \( R/\mathfrak{p}(h) \hookrightarrow R/\mathfrak{p}(g) \), we have \( h = g \in G_p \) and thus \( g \in G_p \).

**Lemma 3.7.** Let \( p \subseteq R \) be a prime, and \( I := \{ i \mid x_i \in \mathfrak{p} \} \). Then
\[ G_p = \langle \text{deg}(x_i) \mid i \notin I \rangle_{\mathbb{Z}}. \]
In particular, \( G_p \) only depends on the set \( I \).

**Proof.** An element \( r \in R/\mathfrak{p} \) can be represented by an \( R_0 \)-linear combination of monomials that only involve elements in \( \{x_0, \ldots, x_n\} \setminus \mathfrak{p} \). In particular, \( \text{deg}(r) \in \langle \text{deg}(x) \mid x \in \{x_0, \ldots, x_n\} \setminus \mathfrak{p} \rangle_{\mathbb{Z}}. \)

Now, we can fully understand the points in \( \text{ASpec}(R) \) and how they are related to the points in \( \text{Spec}(R) \).

**Theorem and Definition 3.8.** Let \( R \) be a \( G \)-graded ring. Then
\[ \text{ASpec}(R) = \{ \tilde{p}(g) \mid p \subseteq R \text{ prime, } g \in G \}. \]

Moreover, the map
\[ \pi_{\text{atom}} : \text{ASpec}(R) \to \text{Spec}(R) : \tilde{p}(g) \mapsto p, \]
which we call the **atom projection**, is well-defined and has the following properties:

1. For every \( p \in \text{Spec}(R) \), we have a bijection
\[ \pi_{\text{atom}}^{-1}(\{p\}) \cong G/G_p : \tilde{p}(g) \mapsto g + G_p. \]
Moreover, the set of factor groups that occur as such fibers, i.e.,
\[ \{ G/G_p \mid p \subseteq R \text{ a prime} \}, \]

is finite.
2. Given \( M \in R\text{-mod} \), then
\[ \pi_{\text{atom}}(\text{ASupp}(M)) = \text{Supp}(M). \]
3. Continuity in the following sense: given \( M \in R\text{-mod} \), then
\[ \pi_{\text{atom}}^{-1}(\text{Supp}(M)) = \{ \tilde{p}(g) \mid p \in \text{Supp}(M), g \in G \} \]
is an open subset of \( \text{ASpec}(R) \).
Proof. This theorem summarizes Lemma 3.1, Corollary 3.4, and Lemma 3.6. Furthermore, from Lemma 3.7, it follows that the number of different $G/G_p$ is bounded by the cardinality of the powerset of $\{x_0, \ldots, x_n\}$. Thus, we only need to prove part 2 and 3 of the assertion.

Part 2 is easy for modules of the form $R/p(g)$ for primes $p \subseteq R$ and elements $g \in G$. The general case follows from the structure Theorem 3.3, and the compatibility of union of atom supports with extensions of objects [Kan12, Proposition 3.3].

To prove continuity, let $M \in R$-mod and $\tilde{p}(g) \in \pi^{-1}_\text{atom}(\text{Supp}(M))$. Then

$$\text{Supp}(R/p(g)) \subseteq \text{Supp}(M)$$

and thus

$$\text{ASupp}(R/p(g)) \subseteq \pi^{-1}_\text{atom}(\text{Supp}(R/p(g))) \subseteq \pi^{-1}_\text{atom}(\text{Supp}(M)).$$

The claim follows since $\text{ASupp}(R/p(g))$ is an open neighborhood of $\tilde{p}(g)$.

The atom projection allows us to think about the atom spectrum of $R$ as an enhancement of the homogeneous spectrum. To foster this intuition, we introduce the following notions.

**Definition 3.9.** Let $R$ be a $G$-graded ring. We call an atom which is of the form $p$ for a prime $p \subseteq R$ a **standard point** in $\text{ASpec}(R)$. All the other atoms, i.e., points of the form $p(g)$ for $g \notin G_p$, are called **non-standard points**. Furthermore, we set

$$U_R := \{\text{non-standard points in ASpec}(R)\}.$$

**Remark 3.10.** Theorem 3.8 allows us to think of $\text{ASpec}(R)$ as a disjoint union:

$$\text{ASpec}(R) \simeq \text{Spec}(R) \cup U_R,$$

where we identify $\text{Spec}(R)$ with the set of all standard points.

**Lemma 3.11.** Let $R$ be a $G$-graded ring. Then $U_R \subseteq \text{ASpec}(R)$ is open.

Proof. Let $\tilde{p}(g)$ be a non-standard point. It suffices to show that $\text{ASupp}(R/p(g)) \subseteq U_R$, so, let $M$ be a monoform subquotient of $R/p(g)$. By Theorem 3.3, $M$ contains a monoform submodule of the form $R/q(h)$ for $q \subseteq R$ a prime and $h \in G$, and $\overline{M} = \tilde{q}(h)$. From

$$\text{deg}(R/q(h)) \subseteq \text{deg}(R/p(g)),$$

we conclude

$$h \in g + G_p \neq G_p$$

and thus

$$h \notin G_p \supseteq G_q.$$

It follows that $\tilde{q}(h)$ is also a non-standard point.

**Example 3.12.** Let $k$ be a field, and let $R = k[x_0, \ldots, x_n]$ be the $\mathbb{Z}$-graded polynomial ring with all $x_i$ of degree 1. Set $m := \langle x_0, \ldots, x_n \rangle \in \text{Spec}(R)$. Then

$$\pi^{-1}_\text{atom}(\{m\}) \simeq \mathbb{Z}$$

and

$$U_R = \{\tilde{m}(i) \mid i \in \mathbb{Z} \setminus \{0\}\}.$$
Example 3.13. Let $k$ be a field, and let $R = k[x_0, x_1, x_2]$ be the $\mathbb{Z}$-graded polynomial ring with $\deg(x_0) = \deg(x_1) = 1$ and $\deg(x_2) = 2$. Set $\mathfrak{m} := \langle x_0, x_1, x_2 \rangle$ and $\mathfrak{p} := \langle x_0, x_1 \rangle$. Then

$$\tau_{\text{atom}}^{-1}(\{\mathfrak{m}\}) \simeq \mathbb{Z} \quad \text{and} \quad \tau_{\text{atom}}^{-1}(\{\mathfrak{p}\}) \simeq \mathbb{Z}/2\mathbb{Z}$$

and

$$U_R = \{ \widehat{\mathfrak{m}}(i) \mid i \in \mathbb{Z}\backslash\{0\} \} \cup \{ \widehat{\mathfrak{p}}(1) \}.$$ 

4. Applications to sheafification in toric geometry

We are interested in the kernel of the sheafification process studied in toric geometry. We give a short introduction to this subject based on [CLS11, Chapter 5], and then show how the atom spectrum of a graded ring sheds new light on it.

For a normal toric variety $X$ over $\mathbb{C}$ without torus factors there exist the following data:

1. A natural number $n \in \mathbb{N}_0$ and an $n+1$-dimensional $\mathbb{C}$-vector space $W = \mathbb{C}^{n+1}$, whose standard basis elements are denoted by $x_0, \ldots, x_n$.
2. An algebraic subgroup $H \subseteq (\mathbb{C}^*)^{n+1}$ acting on $W$ by componentwise multiplication.
3. A non-empty subset $\Sigma$ of the powerset of $\{0, \ldots, n\}$ that is closed under taking subsets.

An element $\sigma \in \Sigma$ gives rise to a monomial

$$x^\sigma := \prod_{i \notin \sigma} x_i$$

in the so-called **Cox ring** of $X$, which is the symmetric algebra

$$S := \text{Sym}(W) = \mathbb{C}[x_0, \ldots, x_n].$$

The action of $H$ on $W$ induces an action on $S$, and since $H$ is reductive, we get a decomposition

$$S = \bigoplus_{g \in G} S_g,$$

where $G := \text{Hom}_{\text{alg.Grp}}(H, \mathbb{C}^*)$ is the group of algebraic characters of $H$. This turns $S$ into a $G$-graded ring with homogeneous generators $x_0, \ldots, x_n$.

Remark 4.1. The action of $H$ on $S$ gives the following geometric interpretation of the group $G/G_p$ (see Definition 3.5) for a prime $p \in \text{Spec}(S)$. Set

$$I := \{ i \mid x_i \in p \} \quad \text{and} \quad \widehat{I} := \{ i \mid x_i \notin p \}.$$ 

We identify $(\mathbb{C}^*)^I$ with the subgroup

$$\{(x_0, \ldots, x_n) \mid \forall i \in \widehat{I} : x_i = 1\} \subseteq (\mathbb{C}^*)^{n+1}.$$ 

Then, for any point

$$p \in \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \mid x_i = 0 \iff i \in I\},$$

we can compute the stabilizer subgroup of $H$ with respect to $p$ as

$$\text{Stab}_H(p) = \text{Stab}_{(\mathbb{C}^*)^{n+1}}(p) \cap H = \mathbb{C}^I \cap H.$$
From the duality of the following pullback and pushout diagrams

\[
\begin{align*}
(C^*)^I & \xleftarrow{(C^*)^{n+1}} Z^{n+1} \\
\text{Stab}_H(p) & \xleftarrow{H} \text{Hom}_Z(-, C^*) \\
\text{G/G}_{p} & \xleftarrow{G/G_{p}} G
\end{align*}
\]

we conclude that \( G/G_{p} \) is the group of algebraic characters of \( \text{Stab}_H(p) \):

\[ G/G_{p} \simeq \text{Hom}_{\text{alg.Grps}} \left( \text{Stab}_H(p), C^* \right). \]

As we have seen in Lemma 3.7, \( G/G_{p} \) only depends on \( I \), not on \( p \) itself.

Next, we define the so-called **irrelevant ideal**

\[ B(\Sigma) := \langle x^{\sigma} \mid \sigma \in \Sigma \rangle_S \]

of the Cox ring. Now, \( X \) can be reconstructed as a variety as a quotient\(^{3}\)

\[ X \simeq (\mathbb{C}^{n+1} \setminus \mathbb{V}(\langle B(\Sigma) \rangle)) / H, \]

where \( \mathbb{V}(\langle B(\Sigma) \rangle) \) denotes the vanishing set of the irrelevant ideal (regarded without its grading), which in particular implies that we have a bijection

\[ \{ \mathbb{C}\text{-points of } X \} \simeq \{ \text{closed orbits of the } H\text{-action on } \mathbb{C}^{n+1} \setminus \mathbb{V}(\langle B(\Sigma) \rangle) \}. \]

**Example 4.2.** For the projective space \( \mathbb{P}^n \), we have \( W = \mathbb{C}^{n+1}, H = C^* \) acting diagonally on \( W \), and \( \Sigma = \mathcal{P}(\{0, \ldots, n\}) \setminus \{0, \ldots, n\} \}, \) where \( \mathcal{P} \) denotes the powerset. Then

\[ S = \mathbb{C}[x_0, \ldots, x_n] \]

is \( G = \text{Hom}(C^*, C^*) \simeq \mathbb{Z} \) graded with \( \text{deg}(x_i) = 1 \) for all \( i = 0, \ldots, n \), and

\[ B(\Sigma) = \langle x_0, \ldots, x_n \rangle. \]

We recover the usual description of \( \mathbb{P}^n \) as a quotient:

\[ \mathbb{P}^n \simeq (\mathbb{C}^{n+1} - \{0\})/C^*. \]

Elements \( \sigma \in \Sigma \) give rise to a cover of \( X \) by affine open subschemes

\[ \text{Spec}(S_{x^{\sigma}})_0 \subseteq X, \]

where \( (S_{x^{\sigma}})_0 \) denotes the homogeneous localization\(^{4}\) of \( S \) at \( x^{\sigma} \). Now, a finitely presented \( G \)-graded module \( M \) over \( S \) induces a coherent sheaf \( \widetilde{M} \) over \( X \) which is uniquely determined by the property

\[ \widetilde{M} |_{\text{Spec}(S_{x^{\sigma}})_0} \simeq (M_{x^{\sigma}})_0, \]

\(^{3}\)In [CLS11, Chapter 5], it is called an almost geometric quotient.

\(^{4}\)The homogeneous localization of a graded ring \( R \) at an element \( x \in R \) consists of all degree 0 elements in the localization \( R_x \).
where \((M_x^\sigma)_0\) denotes the homogeneous localization\(^5\) of \(M\) at \(x^\sigma\). It is called the sheafification of \(M\).

**Example 4.3.** We continue Example 4.2. Given the \(\mathbb{Z}\)-graded \(S\)-module
\[
k \simeq S/\langle x_0, \ldots, x_n \rangle_S,
\]
we clearly have \(\tilde{k} \simeq 0\) since \(k_{x_i} \simeq 0\) for all \(i = 0, \ldots, n\). More generally, we can say that a finitely presented \(\mathbb{Z}\)-graded \(S\)-module \(M\) sheafifies to zero if and only if
\[
\text{Supp}(M) \subseteq \{ \langle x_0, \ldots, x_n \rangle \}.
\]
Note that it does not matter if we consider the support in \(\text{Spec}(S)\) or in \(\text{Spec}(|S|)\). Geometrically, this criterion seems very plausible since it is precisely the origin that is removed from the affine plane \(\mathbb{C}^{n+1}\) in the construction of \(\mathbb{P}^n\). However, we will see in the next example that this criterion does not naively generalize to all toric varieties.

**Example 4.4.** We analyze the weighted projective space \(\mathbb{P}(1, 1, 2)\) [CLS11, Example 5.3.11]. In this case, \(W = \mathbb{C}^3\), \(H = \mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^3 : h \mapsto (h, h, h^2)\), and \(\Sigma = \mathcal{P}((0, 1, 2))\{|\{0, 1, 2\}\}. \)

Then
\[
S = \mathbb{C}[x_0, x_1, x_2]
\]
is \(G = \text{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}\) graded with \(\text{deg}(x_0) = \text{deg}(x_1) = 1\), \(\text{deg}(x_2) = 2\) and
\[
B(\Sigma) = \mathfrak{m} := \langle x_0, x_1, x_2 \rangle.
\]
We recover the usual description of \(\mathbb{P}(1, 1, 2)\) as a quotient:
\[
\mathbb{P}(1, 1, 2) \cong (\mathbb{C}^3 - \{0\})/\{(h, h, h^2) \mid h \in \mathbb{C}^*\}.
\]
Next, we set \(\mathfrak{p} := \langle x_0, x_1 \rangle\) and take a look at the \(S\)-graded module
\[
M := S/\mathfrak{p}.
\]
We have \(\tilde{M} \neq 0\) since \((M_{x_2})_0 \simeq (k[x_2])_0 \simeq k\). However, if we shift \(M\) by 1, then we do have \(\tilde{M}(1) \simeq 0\): localizing \(M\) at \(x_0\) or \(x_1\) yields the zero module, and homogeneous localization at \(x_2\) yields:
\[
(M_{x_2}(1))_0 \simeq (k[x_2])_1 \simeq 0
\]
since \(\text{deg}(k[x_2]) \subseteq 2\mathbb{Z}\).

Since
\[
\text{Supp}(M) = \text{Supp}(M(1)) = \{ \mathfrak{p}, \mathfrak{m} \}
\]
we cannot expect the support of a graded module to give a criterion for the sheafification being zero, as it was the case in Example 4.3.

If we take the atom supports instead, then we can understand the full picture. The preimages of \(\mathfrak{m}\) and \(\mathfrak{p}\) under the atom projection are given by
\[
\pi_{\text{atom}}^{-1}(\langle \mathfrak{p} \rangle) = \{ \tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}(1) \}
\]
\(^5\)The homogeneous localization of a graded module \(M\) over a graded ring \(R\) at an element \(x \in R\) consists of all degree 0 elements in the localization \(M_x\).
and
\[ \pi_{\text{atom}}^{-1}(\{m\}) = \{\tilde{m}(i) \mid i \in \mathbb{Z}\} \]
(see Example 3.13). Now, the atom supports of \( M \) and \( M(1) \) differ quite heavily:
\[
\text{ASupp}(M) = \{\tilde{p}\} \cup \{\tilde{m}(i) \mid i \in 2\mathbb{N}_0\} \neq \{\tilde{p}(1)\} \cup \{\tilde{m}(i) \mid i \in 2\mathbb{N}_0 - 1\} = \text{ASupp}(M(1))
\]
If \( K \) denotes the Serre subcategory of all those modules in \( R\)-mod that sheafify to zero, then the general criterion stated in Remark 2.5 tells us that an \( S \)-module \( N \) sheafifies to zero if and only if
\[
\text{ASupp}(N) \subseteq \text{ASupp}(K).
\]
Since we clearly have \( \pi_{\text{atom}}^{-1}(\{m\}) \subseteq \text{ASupp}(K) \), the only reason for \( \widehat{M} \neq 0 \), but \( \widehat{M}(1) \simeq 0 \) has to be
\[
\widehat{p}(1) \in \text{ASupp}(K) \quad \text{but} \quad \widehat{p} \notin \text{ASupp}(K).
\]
The most apparent difference between these two points is that \( \widehat{p} \) is a standard point while \( \widehat{p}(1) \) is non-standard.

The observations of Example 4.4 motivate our main theorem of this section. Recall that \( U_S \) denotes the set of non-standard points in \( \text{ASpec}(S) \).

**Theorem 4.5.** Let \( X \) be a normal toric variety over \( \mathbb{C} \) without torus factors and with \( G \)-graded Cox ring \( S \) and irrelevant ideal \( B(\Sigma) \). A finitely presented \( G \)-graded \( S \)-module \( M \) sheafifies to zero if and only if
\[
\text{ASupp}(M) \subseteq \{\tilde{p} \mid p \in \text{Supp}(S/B(\Sigma))\} \cup U_S.
\]

We dedicate the subsections of this section to the proof of Theorem 4.5.

**Notation 4.6.** Since the proof of Theorem 4.5 can be given in more general terms (see Theorem 4.16), we let \( G \) be an arbitrary abelian group, and \( R \) be a \( G \)-graded ring in the sense of Section 3.

For the proof of Theorem 4.5, we proceed as follows. By definition, the kernel of the sheafification functor is given as an intersection of kernels of various homogeneous localization functors. In Subsection 4.1, we give a general description of the atom support of the kernel of an exact functor \( F : R\text{-mod} \rightarrow A \) for an arbitrary abelian category \( A \). This description is made more concrete in the case where \( F \) is a homogeneous localization functor in Subsection 4.2. Finally, in Subsection 4.3, we investigate intersections of such atom supports and prove an even more general version of Theorem 4.5.

### 4.1. A general description of the atom support of the kernel of an exact functor.

Given an exact functor
\[ F : R\text{-mod} \rightarrow A \]
into some abelian category \( A \), its kernel is the Serre subcategory
\[ \ker(F) := \{M \in R\text{-mod} \mid F(M) \simeq 0\} \]
of \( R\text{-mod} \).

The goal of this subsection is to prove the following general description of the atom support of \( \ker(F) \).
Theorem 4.7. Given an exact functor $F : R\text{-mod} \to \mathbf{A}$, then
$$\text{ASupp}(\ker F) = \{ \tilde{p}(g) \in \text{ASpec}(R) \mid F(R/p(g)) \simeq 0 \}.$$  

We will need the following lemma for its proof.

Lemma 4.8. Let $U \subseteq \text{Spec}(R)$ be an open set. Then
$$U = \{ \tilde{p}(g) \in \text{Spec}(R) \mid \text{ASupp}(R/p(g)) \subseteq U \}$$

Proof. We denote the set on the right hand side by $U'$.

$U \supseteq U'$: Given $\tilde{p}(g) \in \text{Spec}(R)$ such that $\text{ASupp}(R/p(g)) \subseteq U$, it follows that $\tilde{p}(g) \in U$ since $\tilde{p}(g) \in \text{ASupp}(R/p(g))$.

$U \subseteq U'$: Since $U$ is open, it suffices to prove that given $M \in R\text{-mod}$ such that $\text{ASupp}(M) \subseteq U$, we already have $\text{ASupp}(M) \subseteq U'$. Since a module $M$ admits a finite filtration by modules of the form $R/p(g)$ for primes $p \subseteq R$ and elements $g \in G$ (see Theorem 3.3), and since the union of atom supports is compatible with extensions of objects [Kan12, Proposition 3.3], we can assume $M = R/p(g)$. For every subquotient $N$ of $M$, we have
$$\text{ASupp}(N) \subseteq \text{ASupp}(M) \subseteq U.$$  

But this entails that every point in $\text{ASupp}(M)$ already lies in $U'$.

Caveat 4.9. The previous Lemma 4.8 does not imply that for every point $\tilde{p}(g) \in U$, we have $\text{ASupp}(R/p(g)) \subseteq U$, but only that there exists $h \in g + G_p$ such that $\text{ASupp}(R/p(h)) \subseteq U$ (see Lemma 3.6).

Proof of Theorem 4.7. Using Lemma 4.8, we get
$$\text{ASupp}(\ker F) = \{ \tilde{p}(g) \in \text{ASpec}(S) \mid \text{ASupp}(R/p(g)) \subseteq \text{ASupp}(\ker F) \}.$$  

But we know by Remark 2.5 that
$$\text{ASupp}(R/p(g)) \subseteq \text{ASupp}(\ker F) \iff R/p(g) \in \ker F,$$  

which completes the proof.

4.2. The atom support of the kernel of the homogeneous localization functor. Let $f \in R\setminus\{0\}$. We study the atom support of the kernel $K_f$ of the exact homogeneous localization functor:
$$(-)_0 : R\text{-mod} \to (R_f)_0\text{-mod} : M \mapsto (M_f)_0.$$  

Motivated by Theorem 4.7, which tells us that
$$\text{ASupp}(K_f) = \{ \tilde{p}(g) \mid (R/p(g)_f)_0 \simeq 0 \}$$
we draw our attention to primes $p \subseteq R$ and elements $g \in G$ for which we have $R/p(g) \in K_f$.

Lemma 4.10. Given a prime $p \subseteq R$ and an element $g \in G$, then
$$R/p(g) \in K_f \iff (f \in p) \lor (g \notin \text{deg}(R/p) + \langle \text{deg}(f) \rangle_\mathbb{Z})$$
Proof. If \( f \in \mathfrak{p} \), then \( (R/\mathfrak{p})_f \simeq 0 \), so in particular, its \( g \)-th degree part is zero. If \( f \notin \mathfrak{p} \), then for all \( i \in \mathbb{Z} \), the element \( f^i \in (R/\mathfrak{p})_f \) is not zero, since \( \mathfrak{p} \) is a prime. In particular,
\[
\deg (R/\mathfrak{p}(g)_f) = \deg(R/\mathfrak{p}(g)) + \langle \deg(f) \rangle_Z.
\]
In this case, \( (R/\mathfrak{p}(g)_f)_0 \simeq 0 \) if and only if \( 0 \notin \deg (R/\mathfrak{p}(g)_f) \) if and only if \( 0 \notin \deg(R/\mathfrak{p}(g)) + \langle \deg(f) \rangle_Z \), which is equivalent to the condition in the claim. \( \square \)

Due to Lemma 4.10, we can write \( \text{ASupp}(K_f) \) as a union of two parts:
\[
\text{ASupp}(K_f) = \{ \tilde{p}(g) \ | \ (R/\mathfrak{p}(g)_f)_0 \simeq 0 \} \\
= \{ \tilde{p}(g) \ | \ f \in \mathfrak{p} \} \cup \{ \tilde{p}(g) \ | \ g \notin \deg(R/\mathfrak{p}) + \langle \deg(f) \rangle_Z \}
= \pi^{-1}_{\text{atom}}(\text{Supp}(R/\langle f \rangle)) \cup \{ \tilde{p}(g) \ | \ g \notin \deg(R/\mathfrak{p}) + \langle \deg(f) \rangle_Z \}
\]
The first subset is open by the continuity of the atom projection (Theorem 3.8). The second subset in this union depends only on the degree of \( f \), but not on \( f \) itself.

**Lemma and Definition 4.11.** For every \( d \in G \), the subset
\[
U_d := \{ \tilde{p}(g) \ | \ g \notin \deg(R/\mathfrak{p}) + \langle \deg(f) \rangle_Z \} \subseteq \text{ASpec}(R)
\]
is open.

Proof. Let \( p \subseteq R \) be a prime and \( g \in G \) such that \( 0 \notin \deg(R/\mathfrak{p}(g)) + \langle \deg(f) \rangle_Z \). For any subquotient \( M \) of \( R/\mathfrak{p}(g) \), we have \( \deg(M) \subseteq \deg(R/\mathfrak{p}(g)) \) and thus
\[
0 \notin \deg(M) + \langle d \rangle_Z \subseteq \deg(R/\mathfrak{p}(g)) + \langle d \rangle_Z.
\]
It follows that the open neighborhood \( \text{ASupp}(R/\mathfrak{p}(g)) \) of \( \tilde{p}(g) \) lies in \( U_d \). \( \square \)

In summary, we get the following description of \( \text{ASupp}(K_f) \):

**Theorem 4.12.** The atom support of the kernel \( K_f \) of the homogeneous localization functor \( (-)_0 \) can be written as the following union of open subsets:
\[
\text{ASupp}(K_f) = \pi^{-1}_{\text{atom}}(\text{Supp}(R/\langle f \rangle)) \cup U_{\deg(f)}.
\]

**Remark 4.13.** The open set
\[
\pi^{-1}_{\text{atom}}(\text{Supp}(R/\langle f \rangle))
\]
in Theorem 4.12 corresponds to the kernel of \( (-)_0 \), i.e., localization without taking the 0-th degree part. So, the occurrence of \( U_{\deg(f)} \) is a peculiarity of homogeneous localization.

4.3. **Proof of the main theorem.** In this subsection, we prove Theorem 4.16, which is a variant of the main Theorem 4.5, but with the advantage that it also works outside the context of toric geometry, and that it easily implies the main Theorem 4.5.

Remember that we have \( R = R_0[x_0, \ldots, x_n] \) a \( G \)-graded ring in the sense of Section 3. For a subset \( \sigma \subseteq \{0, \ldots, n\} \), we define \( \bar{\sigma} := \{0, \ldots, n\} \setminus \sigma \) and
\[
x^{\bar{\sigma}} := \prod_{i \notin \sigma} x_i \in R.
\]
Lemma 4.14. Let $\sigma \subseteq \{0, \ldots, n\}$ and assume that $x^\delta \neq 0$ for all $\tau \subseteq \sigma$. Then
\[
\bigcap_{\tau \subseteq \sigma} \text{ASupp}(K_{x^\delta}) = \pi_{\text{atom}}^{-1}(\text{Supp}(R/\langle x^\delta \rangle)) \cup U_R,
\]
where we set $K_{x^\delta} := \ker((-x^\delta)_0)$.

Remark 4.15. Note that the statement of Lemma 4.14 is a peculiarity of homogeneous localization. The intersection of the kernels of the functors $(-x^\delta)$ for all $\tau \subseteq \sigma$, i.e., without taking the 0-th degree part, would simply be equal to $\ker((-x^\delta))$.

Proof of Lemma 4.14. Throughout this proof, we will think of $\text{ASupp}(K_{x^\delta})$ as the union
\[
\text{ASupp}(K_{x^\delta}) = \pi_{\text{atom}}^{-1}(\text{Supp}(R/\langle x^\delta \rangle)) \cup \text{deg}(x^\delta)
\]
that was given by Theorem 4.12. Now, call $I_l$ the left hand side of the equation in the statement of the lemma and $I_r$ the right hand side.

$I_l \supseteq I_r$: Since
\[
\text{Supp}(R/\langle x^\delta \rangle) \subseteq \text{Supp}(R/\langle x^\delta \rangle)
\]
for all $\tau \subseteq \sigma$, it follows that
\[
\pi_{\text{atom}}^{-1}(\text{Supp}(R/\langle x^\delta \rangle)) \subseteq I_l.
\]

Next, let $\tilde{p}(g)$ be a non-standard point. Given $\tau \subseteq \sigma$, we distinguish two cases. First, assume that there exists $j \in \hat{\tau}$ such that $x_j \in p$. In this case,
\[
\tilde{p}(g) \in \pi_{\text{atom}}^{-1}(\text{Supp}(R/\langle x^\delta \rangle)) \subseteq \text{ASupp}(K_{x^\delta}).
\]

Second, assume that $x_j \notin p$ for all $j \in \hat{\tau}$. In that case,
\[
g \notin G_p = \langle \text{deg}(R/p) \rangle_Z \supseteq \text{deg}(R/p) + \langle \text{deg}(x^\delta) \rangle_Z
\]
since $\text{deg}(x^\delta) = \sum_{j \in \hat{\tau}} \text{deg}(x_j) \in \langle \text{deg}(R/p) \rangle_Z$ (note that $x^\delta \neq 0$). Thus,
\[
\tilde{p}(g) \in U_{\text{deg}(x^\delta)} \subseteq \text{ASupp}(K_{x^\delta}).
\]

$I_l \subseteq I_r$: The idea is to prove that given an element of the form
\[
\tilde{p}(g) \in \text{ASupp}(K_{x^\delta}) \setminus I_r,
\]
we can find a $\tau \subseteq \sigma$ such that $\tilde{p}(g) \notin \text{ASupp}(K_{x^\delta})$.

Since $\tilde{p}(g) \in U_{\text{deg}(x^\delta)}$, we may assume that $g$ is chosen such that
\[
g \notin \text{deg}(R/p) + \langle \text{deg}(x^\delta) \rangle_Z.
\]

Since $\tilde{p}(g)$ is a standard point, $g \in G_p$, and thus
\[
g \in G_p \setminus (\text{deg}(R/p) + \langle \text{deg}(x^\delta) \rangle_Z).
\]

Next, we set $\tau := \{i \in \{0, \ldots, n\} \mid x_i \in p\}$. We have $\tau \subseteq \sigma$ since $p \notin \text{Supp}(R/\langle x^\delta \rangle)$ and thus $x_i \notin p$ for all $i \in \hat{\sigma}$. Now, it suffices to prove that
\[
\tilde{p}(g) \notin \text{ASupp}(K_{x^\delta}).
\]
Clearly, $\tilde{p}(g) \notin \pi_{\text{atom}}^{-1}\left(\text{Supp}(R/\langle x^{\hat{\sigma}} \rangle)\right)$ since $x_j \notin p$ for all $j \in \hat{\tau}$. So, let us assume that we have $\tilde{p}(g) \in \mathcal{U}_{\text{deg}(x^{\hat{\sigma}})}$. Then

$$\exists h \in \mathcal{K} + G_p = G_p$$

such that

$$h \notin \text{deg}(R/p) + \langle\text{deg}(x^{\hat{\sigma}})\rangle_{\mathbb{Z}}.$$ 

But we can compute

$$\text{deg}(R/p) + \langle\text{deg}(x^{\hat{\sigma}})\rangle_{\mathbb{Z}} = \langle\text{deg}(x_j) \mid j \in \hat{\tau}\rangle_{\mathbb{N}_0} + \sum_{\tau \in \hat{\tau}} \text{deg}(x^{\tau})_{\mathbb{Z}}$$

$$= \langle\text{deg}(x_j) \mid j \in \hat{\tau}\rangle_{\mathbb{Z}}$$

$$= G_p$$

and thus get $h \notin G_p$, a contradiction. \hfill \square

We are ready to prove our generalized version of Theorem 4.5.

**Theorem 4.16.** Let $R$ be $G$-graded ring (in the sense of Section 3). Let $\Sigma$ be a subset of the powerset of $\{0, \ldots, n\}$ that is closed under taking subsets. Assume that

$$x^{\hat{\sigma}} := \prod_{i \notin \sigma} x_i \neq 0$$

for all $\sigma \in \Sigma$. Then

$$\bigcap_{\sigma \in \Sigma} \text{ASupp} (K_{x^{\hat{\sigma}}}) = \pi_{\text{atom}}^{-1}\left(\text{Supp}(R/B(\Sigma))\right) \cup \mathcal{U}_R,$$

where we set $K_{x^{\hat{\sigma}}} := \ker((-x^{\hat{\sigma}})_0)$ and $B(\Sigma) := \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma\rangle$.

**Proof.**

$$\bigcap_{\sigma \in \Sigma} \text{ASupp} (K_{x^{\hat{\sigma}}}) = \bigcap_{\sigma \in \Sigma} \bigcap_{\tau \subseteq \sigma} \text{ASupp} (K_{x^{\hat{\sigma}}})$$

$$= \bigcap_{\sigma \in \Sigma} \left(\pi_{\text{atom}}^{-1}\left(\text{Supp}(R/\langle x^{\hat{\sigma}} \rangle)\right) \cup \mathcal{U}_R\right)$$

Lemma 4.14

$$= \left(\bigcap_{\sigma \in \Sigma} \pi_{\text{atom}}^{-1}\left(\text{Supp}(R/\langle x^{\hat{\sigma}} \rangle)\right)\right) \cup \mathcal{U}_R$$

$$= \left(\pi_{\text{atom}}^{-1}\left(\bigcap_{\sigma \in \Sigma} \text{Supp}(R/\langle x^{\hat{\sigma}} \rangle)\right)\right) \cup \mathcal{U}_R$$

$$= \pi_{\text{atom}}^{-1}\left(\text{Supp}(R/B(\Sigma))\right) \cup \mathcal{U}_R \hfill \square$$

**Proof of Theorem 4.5.** The kernel $K$ of the sheafification functor is given by

$$K = \bigcap_{\sigma \in \Sigma} \ker((-x^{\hat{\sigma}})_0).$$

Thus,

$$\text{ASupp}(K) = \bigcap_{\sigma \in \Sigma} \text{ASupp} (\ker((-x^{\hat{\sigma}})_0)),$$
and from Theorem 4.16, we conclude
\[ \text{ASupp}(K) = \pi_{\text{atom}}^{-1}(\text{Supp}(S/B(\Sigma))) \cup U_S. \]

Since
\[ \pi_{\text{atom}}^{-1}(\text{Supp}(S/B(\Sigma))) \cup U_S = \{ \hat{p} \mid p \in \text{Supp}(S/B(\Sigma)) \} \cup U_S, \]
we can finish the proof with the geometric criterion stated in Remark 2.5. \qed

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Department of mathematics, University of Siegen, 57068 Siegen, Germany
E-mail address: sebastian.posur@uni-siegen.de