Remark on Wolf’s condition for boundary regularity of Navier-Stokes equations

G Seregin

OxPDE, Mathematical Institute, University of Oxford, Oxford, UK

November 11, 2015

Abstract

We prove Wolf’s regularity condition up to the boundary for solutions to the Navier-Stokes equations satisfying non-slip boundary condition.

1 Introduction

The note is inspired by an interesting result by J. Wolf, see [9]. It reads the following. Let a pair $u$ and $p$ be a suitable weak solution to the Navier-Stokes system in the parabolic cylinder $Q(z_0, R) = B(x_0, R) \times |t_0 - R^2, t_0|$, where $B(x_0, R)$ is a ball of radius $R$ centred at point $x_0 \in \mathbb{R}^3$. There exists a positive constant $\varepsilon$ such that if

$$\frac{1}{R^2} \int_{Q(z_0, R)} |u(z)|^3 dz < \varepsilon$$

then $u \in L_\infty(Q(z_0, R/2))$.

At the first glance, the condition and the result are independent of pressure $p$. But it is wrong impression as one can see from an elementary example in which $R = 1, z_0 = (0, 0)$,

$$u(x, t) = c(t) \nabla h(x), \quad p(x, t) = -c'(t)h(x) + \frac{1}{2} c^2(t) |\nabla h(x)|^2,$$
and $h$ is a harmonic function. If there is no restriction on pressure, then the above assumption does not provide regularity. But of course there is an assumption on the pressure that is hidden in the definition of suitable weak solution. Let me recall it.

**Definition 1.1.** A suitable weak solution $u$ and $p$ to the classical Navier-Stokes system in $Q(z_0, R)$ possess the following properties:

$$u \in L^2_{2,\infty}(Q(z_0, R)) \cap W^{1,0}_2(Q(z_0, R)), \quad p \in L^{3/2}(Q(z_0, R));$$

$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, \quad \text{div} u = 0$$

in $Q(z_0, R)$ in the sense of distributions;

for a.a. $t \in ]t_0 - R^2, t_0[$,

$$\int_{B(x_0, R)} \varphi^2(x, t)|u(x, t)|^2dx + 2 \int_{t_0 - R^2}^t \int_{B(x_0, R)} \varphi^2 |\nabla u|^2 dxds \leq$$

$$\leq \int_{t_0 - R^2}^t \int_{B(x_0, R)} |u|^2(\partial_t \varphi^2 + \Delta \varphi^2) + u \cdot \nabla \varphi^2(|u|^2 + 2p)dxds$$

for all $\varphi \in C_0^\infty(B(x_0, R) \times ]t_0 - R^2, t_0 + R^2[)$.

As we can see, $p$ must have finite $L^{3/2}$-norm. The exponent $3/2$ is convenient but not a unique choice of a function class for the pressure. It would interesting to know how constants in the Wolf’s condition depends on the pressure. For example, the classical Caffarelli-Kohn-Nirenberg condition, see [1], tells that there are two universal positive constants $\varepsilon_1$ and $c_1$ such that if

$$\frac{1}{R^2} \int_{Q(z_0, R)} (|u(z)|^3 + |p - [p]_{B(x_0, R)}|^3\frac{3}{2})dz < \varepsilon_1$$

then

$$|u(z)| \leq \frac{c_1}{R}$$

for all $z = (x, t) \in Q(z_0, R/2)$. The latter condition is also invariant with respect the natural Navier-Stokes scaling $u(x, t) \to \lambda u(\lambda x, \lambda^2 t)$ and $p(x, t) \to \lambda^2 p(\lambda x, \lambda^2 t)$.

Now, we would like to reformulate Wolf’s condition in the above scale invariant style.
Theorem 1.2. Let $u$ and $p$ be a suitable weak solution to the Navier-Stokes equations in $Q(z_0, R)$. Given $M > 0$, there exist positive numbers $\varepsilon_* = \varepsilon_*(M)$ and $c_* = c_*(M)$ such that if two conditions
\[
\frac{1}{R^2} \int_{Q(z_0, R)} |u|^3 \, dx \, dt < \varepsilon_*(M)
\]
and
\[
\frac{1}{R^2} \int_{Q(z_0, R)} |p - [p]_{B(x_0, R)}|^\frac{2}{3} \, dx \, dt < M
\]
hold, then $u$ is H"older continuous in the closure of $Q(z_0, R/2)$. Moreover,
\[
\sup_{z \in Q(z_0, R/2)} |u(z)| \leq \frac{c_*(M)}{R}.
\]

Certainly, Theorem 1.2 implies Wolf’s condition if we let
\[
\varepsilon = \varepsilon_* \left(1 + \frac{1}{R^2} \int_{Q(z_0, R)} |p - [p]_{B(x_0, R)}|^\frac{2}{3} \, dx \, dt\right).
\]

This type of theorems in the case of interior regularity appeared in [6], for further developments, see, for example, [4], [8], and [2].

In the note, we shall study boundary regularity that perhaps cannot be treated by Wolf’s method.

2 Boundary Regularity

We shall study regularity up to a flat part of the boundary only. The following notation will be used in what follows: $B^+(x_0, R) := \{x = (x', x_3) \in \mathbb{R}^3 : x \in B(x_0, R), x_3 < x_3\}$, $B^+(r) := B^+(0, r)$, $B^+ := B^+(1)$. 

Definition 2.1. A suitable weak solution $u$ and $p$ to the classical Navier-Stokes system in $Q^+(z_0, R) = B^+(x_0, R) \times [t_0 - R^2, t_0]$ possess the following properties:

\[
\begin{align*}
&u \in L_{2,\infty}(Q^+(z_0, R)) \cap W^{1,0}_2(Q^+(z_0, R)), \\
&p \in L^2_2(Q^+(z_0, R)); \\
&\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, \\
&\text{div } u = 0
\end{align*}
\]
in \( Q^+(z_0, R) \) in the sense of distributions;

\[ u(x', 0, t) = 0 \]

for all \(|x' - x'_0| < R\) and \( t \in [t_0 - R^2, t_0[\), where \( x' = (x_1, x_2)\);

for a.a. \( t \in [t_0 - R^2, t_0[\),

\[
\int_{B^+(x_0, R)} \varphi^2(x, t)|u(x, t)|^2 dx + 2 \int_{t_0 - R^2}^{t} \int_{B^+(x_0, R)} \varphi^2|\nabla u|^2 dx ds \leq \\
\leq \int_{t_0 - R^2}^{t} \int_{B^+(x_0, R)} |u|^2(\partial_t \varphi^2 + \Delta \varphi^2) + u \cdot \nabla \varphi^2(|u|^2 + 2p) dx ds
\]

for all \( \varphi \in C_0^\infty(B(x_0, R) \times [t_0 - R^2, t_0 + R^2])\).

Our aim is to show the following.

**Theorem 2.2.** Let \( u \) and \( p \) be a suitable weak solution to the Navier-Stokes equations in \( Q^+(z_0, R) \). Given \( M > 0 \), there exist positive numbers \( \varepsilon_\ast = \varepsilon_\ast(M) \) and \( c_\ast = c_\ast(M) \) such that if two conditions

\[
\frac{1}{R^2} \int_{Q^+(z_0, R)} |u|^3 dx dt < \varepsilon_\ast(M)
\]

and

\[
\frac{1}{R^2} \int_{Q^+(z_0, R)} |p - [p]|_{B^+(x_0, R)}|^3 dx dt < M
\]

hold, then \( u \) is Hölder continuous in the closure of \( Q^+(z_0, R/2) \). Moreover,

\[
\sup_{z \in Q^+(z_0, R/2)} |u(z)| \leq \frac{c_\ast(M)}{R}.
\]

We start with the proof of the following auxiliary statement.

**Proposition 2.3.** Let \( u \) and \( p \) be a suitable weak solution to the Navier-Stokes equations in \( Q^+ := Q^+(0, 1) \). Given \( M > 0 \), there exist positive numbers \( \varepsilon = \varepsilon(M) \) and \( c = c(M) \) such that if two conditions

\[
\int_{Q^+} |u|^3 dx dt < \varepsilon(M)
\]
and
\[ \int_{Q^+} |p - [p]_{B^+}|^{\frac{3}{2}} dx dt < M \]

hold, then \( z = 0 \) is a regular point of \( u \) and therefore \( u \) is Hölder continuous in the closure of a parabolic vicinity of \( z = 0 \). Moreover,
\[ |u(0)| \leq c(M). \]

**Proof.** From the local energy inequality with a suitable choice of the cut-off function \( \varphi \), it follows that
\[
|u|_{2, Q^+(r)}^2 := \sup_{-r^2 < t < 0} \int_{B^+(r)} |u(x, t)|^2 dx + \int_{Q^+(r)} |\nabla u|^2 dx dt \leq \\
\leq c(r) \left[ \left( \int_{Q^+} |u|^3 dz \right)^\frac{2}{3} + \int_{Q^+} |u|^3 dz \right] \\
+ \left( \int_{Q^+} |u|^3 dz \right)^\frac{1}{3} \left( \int_{Q^+} |p - [p]_{B^+}|^{\frac{3}{2}} dz \right)^\frac{2}{3} \leq \\
\leq c(r)d(\varepsilon, M)
\]
for any \( r \in ]0, 1[ \), where \( Q^+(r) := Q^+(0, r) \) and
\[
d(\varepsilon, M) := \varepsilon^\frac{2}{3} + \varepsilon + \varepsilon^\frac{1}{3} M.
\]
Using standard multiplicative inequalities, one can show
\[
\|u \cdot \nabla u\|_{\frac{3}{2}, Q^+(\frac{3}{2})} \leq c|u|_{2, Q^+(\frac{3}{2})}^2 \leq cd(\varepsilon, M). \quad (2.1)
\]

Next, as in paper [5], let us pick up a domain \( \Omega \) with smooth boundary such that \( B^+(4/5) \subset \Omega \subset B^+(5/6) \) and consider the following initial boundary value problem
\[
\partial_t v^1 - \Delta v^1 + \nabla q^1 = -u \cdot \nabla u, \quad \text{div } v^1 = 0 \quad (2.2)
\]
in \( Q_0 = \Omega \times ]-(5/6)^2, 0[ \) and
\[
v^1 = 0 \quad (2.3)
\]
on \( \partial^\prime Q_0 = (\Omega \times \{t = -(5/6)^2\}) \cup (\partial \Omega \times [-(5/6)^2, 0]) \). For solutions to (2.2), (2.3), the following estimate is valid:
\[
\|\partial_t v^1\|_{\frac{3}{5}, Q_0} + \|\nabla^2 v^1\|_{\frac{3}{5}, Q_0} + \|\nabla q^1\|_{\frac{3}{5}, Q_0} \leq c|u \cdot \nabla u|_{\frac{3}{5}, Q_0}. \quad (2.4)
\]
Letting $v^2 = u - v^1$ and $q^2 = p - q^1$, we observe that the above introduced functions satisfies the following relations
\[ \partial_t v^2 - \Delta v^2 + \nabla q^2 = 0, \quad \text{div} v^2 = 0 \]
in $B^+ (4/5) \times ] - (4/5)^2, 0[$, and
\[ v^2(x',0,t) = 0 \]
for $|x'| < 4/5$ and $t \in ] - (4/5)^2, 0[$. According to [4] and [7], $q^2$ obeys the estimate
\[ \| \nabla q^2 \|_{9, \frac{8}{3} Q^+ (3/4)} \leq c (\| \nabla v^2 \|_{\frac{8}{3}, Q^+ (4/5)} + \| v^2 \|_{\frac{8}{3}, Q^+ (4/5)}) \]
+ \| q^2 - [q^2]_{B^+ (4/5)} \|_{\frac{8}{3}, Q^+ (4/5)} \leq c (\| \nabla u \|_{\frac{8}{3}, Q^+ (4/5)} + \| u \|_{\frac{8}{3}, Q^+ (4/5)}) +
+ \| p - [p]_{B^+ (4/5)} \|_{\frac{8}{3}, Q^+ (4/5)} + \| \nabla v^1 \|_{\frac{8}{3}, Q^+ (4/5)} + \| v^1 \|_{\frac{8}{3}, Q^+ (4/5)} +
+ \| q^1 - [q^1]_{B^+ (4/5)} \|_{\frac{8}{3}, Q^+ (4/5)} \).

Assuming $0 < \varepsilon < 1$, we find elementary bounds:
\[ \| \nabla u \|_{\frac{8}{3}, Q^+ (4/5)} + \| u \|_{\frac{8}{3}, Q^+ (4/5)} \leq c |u|_{2, Q^+} \leq c(M), \]
\[ \| p - [p]_{B^+ (4/5)} \|_{\frac{8}{3}, Q^+ (4/5)} \leq c \| p - [p]_{B^+} \|_{\frac{8}{3}, Q^+} \leq c M. \]

Next, from (2.1), (2.4), and the elliptic embedding, it follows that:
\[ \| \nabla v^1 \|_{\frac{8}{3}, Q^+ (4/5)} + \| v^1 \|_{\frac{8}{3}, Q^+ (4/5)} \leq c \| \nabla^2 v^1 \|_{\frac{8}{3}, \frac{8}{3}, Q^0} \leq c(M) \]
and
\[ \| q^1 - [q^1]_{B^+ (4/5)} \|_{\frac{8}{3}, Q^+ (4/5)} \leq \| \nabla q^1 \|_{\frac{8}{3}, \frac{8}{3}, Q^+ (4/5)} \leq c(M). \]

So, finally, we find
\[ \| \nabla q^2 \|_{9, \frac{8}{3} Q^+ (3/4)} \leq c(M). \] (2.5)

It is worthy to notice that the right hand side is independent of $\varepsilon$.

We then have for $0 < r < 2/3$
\[ I_\varepsilon (r) := \frac{1}{r^2} \int_{Q^+ (r)} (|u|^3 + |p - [p]_{B^+ (r)}|^3) dz \leq \]

6
\[
\leq c \frac{1}{r^2} \left( \varepsilon + \int_{Q^+(r)} (|q^1 - [q^1]_{B^+(r)}|^2 + |q^2 - [q^2]_{B^+(r)}|^2)dz \right).
\]

By Poincare inequality,
\[
\frac{1}{r^2} \int_{Q^+(r)} |q^1 - [q^1]_{B^+(r)}|^2 dz \leq c \frac{1}{r^2} \int_{-r^2}^{0} \left( \int_{B^+(r)} |\nabla q^1|^2 dx \right)^{\frac{3}{4}} dt \leq c \frac{1}{r^2} \|\nabla q^1\|_{\frac{3}{2}, \frac{3}{2}}^{\frac{3}{4}} \Omega_0 \leq \frac{1}{r^2} d(\varepsilon, M).
\]

For the second part of the pressure, we have the same arguments plus estimate (2.5)
\[
\frac{1}{r^2} \int_{Q^+(r)} |q^2 - [q^2]_{B^+(r)}|^2 dz \leq cr^2 \int_{-r^2}^{0} \left( \int_{B^+(r)} |\nabla q^2|^2 dx \right)^{\frac{3}{4}} dt \leq cr^2 \|\nabla q^2\|_{0, \frac{3}{4}}^{\frac{3}{4}} \Omega^+(\frac{3}{4}) \leq c(M)r^2.
\]

So, we find
\[
I_\varepsilon(r) \leq c \left( \frac{1}{r^2} \varepsilon + \frac{1}{r^2} d(\varepsilon, M) \right) + c(M)r^2 \tag{2.6}
\]

for any \(0 < \varepsilon < 1\) and for any \(0 < r < 2/3\).

Let us state the following condition of local boundary regularity proved in [5].

**Proposition 2.4.** Let \(w\) and \(\pi\) be a suitable weak solution to the Navier-Stokes system in \(Q^+(R)\). There exist two universal positive constants \(\varepsilon_0\) and \(c_0\) such that if
\[
\frac{1}{R^2} \int_{Q^+(R)} (|w|^3 + |\pi - [\pi]_{B^+(R)}|^2)dz < \varepsilon_0,
\]
then the function \(z \mapsto w(z)\) is Hölder continuous in the closure of \(Q^+(R/2)\) and
\[
\sup_{z \in Q^+(R/2)} |w(z)| \leq \frac{c_0}{R}.
\]
Let us select a positive number \( r = r(M) < 1/2 \) such that

\[
 c(M)r^2 < \frac{\varepsilon_0}{2}.
\]

Then we can pick up \( \varepsilon = \varepsilon(M) \) such that

\[
 c\left(\frac{1}{r(M)^2} \varepsilon + \frac{1}{r(M)^2} d(\varepsilon, M)\right) < \frac{\varepsilon_0}{2}.
\]

From Proposition 2.4 and from (2.6), it follows that the function \( z \mapsto u(z) \) is Hölder continuous in the closure of \( Q^+(r(M)/2) \) and

\[
 |u(0)| \leq c(M) = 2c_0/r(M).
\]

\[ \square \]

The scaled version of Proposition 2.3 is as follows.

**Proposition 2.5.** Let \( u \) and \( p \) be a suitable weak solution to the Navier-Stokes equations in \( Q^+(R) \). Given \( M > 0 \), there exist positive numbers \( \varepsilon = \varepsilon(M) \) and \( c = c(M) \) such that if two conditions

\[
 \frac{1}{R^2} \int_{Q^+(R)} |u|^3 dxdt < \varepsilon(M)
\]

and

\[
 \frac{1}{R^2} \int_{Q^+(R)} |p - [p]_{B^+(R)}|^\frac{2}{3} dxdt < M
\]

hold, then \( z = 0 \) is a regular point of \( u \) and therefore \( u \) is Hölder continuous in the closure of a parabolic vicinity of \( z = 0 \). Moreover,

\[
 |u(0)| \leq \frac{c(M)}{R}.
\]

Now, we wish to show the following.

**Proposition 2.6.** Let \( u \) and \( p \) be a suitable weak solution to the Navier-Stokes equations in \( Q^+ \). Given \( M > 0 \), there exist positive numbers \( \varepsilon_1 = \varepsilon_1(M) \) and \( c_1 = c_1(M) \) such that if two conditions

\[
 \int_{Q^+} |u|^3 dxdt < \varepsilon_1(M)
\]

and

\[
 \int_{Q^+} |p - [p]_{B^+(R)}|^\frac{2}{3} dxdt < M
\]

hold, then \( z = 0 \) is a regular point of \( u \) and therefore \( u \) is Hölder continuous in the closure of a parabolic vicinity of \( z = 0 \). Moreover,

\[
 |u(0)| \leq \frac{c_1(M)}{R}.
\]

8
and
\[ \int_{Q^+} |p - [p]_{B^+}|^{\frac{3}{2}} dx dt < M \]
hold, then \( u \) is Hölder continuous in the closure of \( Q^+(1/2) \). Moreover,
\[ \sup_{Q^+(1/2)} |u(z)| \leq c_1(M). \]

Proof. For \( z_0 = (x_0, t_0) \in \overline{Q}^+(1/2) \), we have
\[ \frac{1}{(1/2)^2} \int_{Q^+(z_0, 1/2)} |u|^3 dz \leq 4 \int_{Q^+} |u|^3 dz < 4 \varepsilon_1(M) \]
and
\[ \frac{1}{(1/2)^2} \int_{Q^+(z_0, 1/2)} |p - [p]_{B^+(x_0)}|^{\frac{3}{2}} dz \leq 4c \int_{Q^+} |p - [p]_{B^+}|^{\frac{3}{2}} dz \leq 4cM. \]
We complete the proof by letting
\[ \varepsilon_1(M) = \frac{1}{r} \varepsilon(4cM), \quad c_1(M) = 2c(4cM). \]

\[ \square \]

Now, Theorem 2.2 follows from obvious scaling and shift and from Proposition 2.6.

References

[1] Caffarelli, L., Kohn, R.-V., Nirenberg, L., Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math., Vol. XXXV (1982), pp. 771–831.
[2] Kukavica, I., Rusin, W., Ziane, M., An anisotropic partial regularity criterion for the Navier-Stokes equations, arXiv:1511.02807.
[3] Mikhaylov, A. S. On local regularity for suitable weak solutions of the Navier-Stokes equations near the boundary. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 385 (2010), Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 41, 83–97, 235; translation in J. Math. Sci. (N. Y.) 178 (2011), no. 3, 282291
[4] Seregin, G. A. Some estimates near the boundary for solutions to the non-stationary linearized Navier-Stokes equations. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 271 (2000), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 31, 204–223, 317; translation in J. Math. Sci. (N. Y.) 115 (2003), no. 6, 2820–2831.

[5] Seregin, G.A., Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary, J. math. fluid mech., 4(2002), no.1,1–29.

[6] Seregin, G. A. On the local regularity of suitable weak solutions of the Navier-Stokes equations. (Russian) Uspekhi Mat. Nauk 62 (2007), no. 3(375), 149–168; translation in Russian Math. Surveys 62 (2007), no. 3, 595–614.

[7] Seregin, G. A. A note on local boundary regularity for the Stokes system. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 370 (2009), Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 40, 151–159, 221–222; translation in J. Math. Sci. (N. Y.) 166 (2010), no. 1, 8690

[8] Wang, W., Zhang, Z., On the interior regularity criteria and the number of singular points to the Navier-Stokes equations, J. DAnalyse Math. 123 (2014) no. 1, 139170.

[9] Wolf, J., A new criterion for partial regularity of suitable weak solutions to the Navier-Stokes equations. Advances in mathematical fluid mechanics, 613–630, Springer, Berlin, 2010.