GENERALIZED PARKING FUNCTIONS, DESCENT NUMBERS, AND
CHAIN POLYTOPES OF RIBBON POSETS

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Abstract. We consider the inversion enumerator \( I_n(q) \), which counts labeled trees or, equivalently, parking functions. This polynomial has a natural extension to generalized parking functions. Substituting \( q = -1 \) into this generalized polynomial produces the number of permutations with a certain descent set. In the classical case, this result implies the formula \( I_n(-1) = E_n \), the number of alternating permutations. We give a combinatorial proof of these formulas based on the involution principle. We also give a geometric interpretation of these identities in terms of volumes of generalized chain polytopes of ribbon posets. The volume of such a polytope is given by a sum over generalized parking functions, which is similar to an expression for the volume of the parking function polytope of Pitman and Stanley.

1. Introduction

Let \( T_n \) be the set of all trees on vertices labeled 0, 1, 2, \ldots, \( n \) rooted at 0. For \( T \in T_n \), let \( \text{inv}(T) \) be the number of pairs \( i > j \) such that \( j \) is a descendant of \( i \) in \( T \). Define the \( n \)-th inversion enumerator to be the polynomial

\[
I_n(q) := \sum_{T \in T_n} q^{\text{inv}(T)}.
\]

Another way to define this polynomial is via parking functions. A sequence \((b_1, b_2, \ldots, b_n)\) of positive integers is a parking function of length \( n \) if for all \( 1 \leq j \leq n \), at least \( j \) of the \( b_i \)'s do not exceed \( j \). A classical bijection of Kreweras \[3\] establishes a correspondence between trees in \( T_n \) with \( k \) inversions and parking functions of length \( n \) whose components add up to \( \binom{n+1}{2} - k \). Hence we can write

\[
I_n(q) = \sum_{(b_1, \ldots, b_n) \in \mathcal{P}_n} q^{\binom{n+1}{2} - b_1 - b_2 - \cdots - b_n},
\]

or

\[
\sum_{(b_1, \ldots, b_n) \in \mathcal{P}_n} q^{b_1+b_2+\cdots+b_n-n} = q^{\binom{n}{2}} \cdot I_n(q^{-1}),
\]

where \( \mathcal{P}_n \) is the set of all parking functions of length \( n \). Cayley’s formula states that \(|T_n| = |\mathcal{P}_n| = (n+1)^{n-1}\), hence \( I_n(1) = (n+1)^{n-1} \).

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Here we focus on the formula

\[ I_n(-1) = E_n, \]

where \( E_n \) is the \( n \)-th \textit{Euler number}, most commonly defined as the number of permutations \( \sigma_1 \sigma_2 \ldots \sigma_n \) of \( [n] = \{1, 2, \ldots, n\} \) such that \( \sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \ldots \), called \textit{alternating} permutations. This formula can be obtained by deriving a closed form expression for the generating function \( \sum_{n \geq 0} I_n(q)x^n/n! \) and showing that setting \( q = -1 \) yields \( \tan x + \sec x = \sum_{n \geq 0} E_nx^n/n! \) (see the paper [1] by Gessel or Exercises 3.3.48–49 in [2]).

A direct combinatorial proof was given by Pansiot [3]. In this paper we give two other ways to prove this fact, one of which, presented in Section 2, is an involution argument on the set of all but \( E_n \) members of \( \mathcal{P}_n \). This involution is a special case of a more general argument valid for a broader version of parking functions, which we now describe.

Let \( \vec{a} = (a_1, a_2, \ldots, a_n) \) be a \textit{non-decreasing} sequence of positive integers. Let us call a sequence \( (b_1, b_2, \ldots, b_n) \) of positive integers an \( \vec{a} \)-\textit{parking function} if the increasing rearrangement \( b'_1 \leq b'_2 \leq \cdots \leq b'_n \) of this sequence satisfies \( b'_i \leq a_i \) for all \( i \). Note that \( (1, 2, \ldots, n) \)-parking functions are the regular parking functions of length \( n \). These \( \vec{a} \)-parking functions are \((a_1, a_2 - a_1, a_3 - a_2, \ldots)\)-parking functions in the original notation of Yan [10], but the present definition is consistent with later literature, such as the paper [4] of Kung and Yan.

Let \( \mathcal{P}_{\vec{a}} \) be the set of all \( \vec{a} \)-parking functions, and define

\[ I_{\vec{a}}(q) := \sum_{(b_1, \ldots, b_n) \in \mathcal{P}_{\vec{a}}} q^{b_1 + b_2 + \cdots + b_n - n} \]

(this is the \textit{sum enumerator} studied in [4]). For a subset \( S \subseteq [n - 1] \), let \( \beta_n(S) \) be the number of permutations of size \( n \) with descent set \( S \). In Section 2 (Theorem 2.4) we prove the following generalization of (1):

\[ |I_{\vec{a}}(-1)| = \begin{cases} 0, & \text{if } a_1 \text{ is even;} \\ \beta_n(S), & \text{if } a_1 \text{ is odd,} \end{cases} \]

where

\[ S = \left\{ i \in [n - 1] \mid a_{i+1} \text{ is odd} \right\}. \]

Indeed, for \( \vec{a} = (1, 2, \ldots, n) \) we have \( S = \{2, 4, 6, \ldots\} \cap [n - 1] \), so that \( \beta_n(S) \) counts alternating permutations of size \( n \). The formula (3) arises in a more sophisticated algebraic context in the paper [5] of Pak and Postnikov.

In Section 3 we obtain a geometric interpretation of these results by considering generalized chain polytopes of ribbon posets. Given a subset \( S \subseteq \{2, 3, \ldots, n - 1\} \), define \( u_S = u_1u_2 \ldots u_{n-1} \) to be the monomial in non-commuting formal variables \( a \) and \( b \) with \( u_i = a \) if \( i \notin S \) and \( u_i = b \) if \( i \in S \). Let \( c(S) \) be the composition \((1, \delta_1, \delta_2, \ldots, \delta_{k-1})\) of \( n \), where the \( \delta_i \)'s are defined by \( u_S = a^{\delta_1} b^{\delta_2} a^{\delta_3} b^{\delta_4} \ldots \). For example, for \( n = 7 \) and \( S = \{2, 3, 4\} \) we have \( u_S = abbaa = ab^3a^2 \), so \( c(S) = (1, 1, 3, 2) \). Now define the polytope \( Z_S(d_1, d_2, \ldots, d_k) \), where \( 0 < d_1 \leq d_2 \leq \cdots \leq d_k \) are real numbers, to be the set of all points \((x_1, x_2, \ldots, x_n)\) satisfying the inequalities \( x_j \geq 0 \) for \( j \in [n] \), \( x_1 \leq d_1 \), and

\[ x_{\delta_1+\delta_2+\cdots+\delta_{i-1}+1} + x_{\delta_1+\delta_2+\cdots+\delta_{i-1}+2} + \cdots + x_{\delta_1+\delta_2+\cdots+\delta_i+1} \leq d_i+1 \]
for $1 \leq i \leq k - 1$. Thus to the above example corresponds the polytope $Z_S(d_1, d_2, d_3, d_4)$ in $\mathbb{R}_{\geq 0}^7$ defined by
\[
\begin{align*}
x_1 &\leq d_1; \\
x_1 + x_2 &\leq d_2; \\
x_2 + x_3 + x_4 + x_5 &\leq d_3; \\
x_5 + x_6 + x_7 &\leq d_4.
\end{align*}
\]
We require $1 \notin S$ here to ensure that $\delta_1 \neq 0$, but there is no essential loss of generality because the chain polytope of the poset $Z_S$ is defined by the same relations as $Z_{[n-1]-S}$.

For a poset $P$ on $n$ elements, the chain polytope $C(P)$ is the set of points $(x_1, x_2, \ldots, x_n)$ of the unit hypercube $[0,1]^n$ satisfying the inequalities $x_{p_1} + x_{p_2} + \cdots + x_{p_t} \leq 1$ for every chain $p_1 < p_2 < \cdots < p_t$ in $P$; see [9]. Hence $Z_S(1, 1, \ldots)$ is the chain polytope of the ribbon poset $Z_S$, which is the poset on $\{z_1, z_2, \ldots, z_n\}$ generated by the cover relations $z_i > z_{i+1}$ if $i \in S$ and $z_i < z_{i+1}$ if $i \notin S$. The volume of $C(P)$ equals $1/n!$ times the number of linear extensions of $P$, which in the case $P = Z_S$ naturally correspond to permutations of size $n$ with descent set $S$. Our main result concerning the polytope $Z_S$ is a formula for its volume. For a composition $\gamma$ of $n$, let $K_\gamma$ denote the set of weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of $n$, meaning that $\alpha$ can have parts equal to 0, such that $\alpha_1 + \alpha_2 + \cdots + \alpha_i \geq \gamma_1 + \gamma_2 + \cdots + \gamma_i$ for all $i$. Define $\bar{a}(\gamma)$ to be the sequence consisting of $\gamma_1$ 1’s, followed by $\gamma_2$ 2’s, followed by $\gamma_3$ 3’s, and so on. Then $\alpha$ is in $K_\gamma$ if and only if $\alpha$ is the content of an $\bar{a}(\gamma)$-parking function. (The content of a parking function is the composition whose $i$-th part is the number of components of the parking function equal to $i$.) In Section 3 (Theorem 3.1) we show that
\[
\text{(5)} \quad n! \cdot \text{Vol}(Z_S(d_1, d_2, \ldots, d_k)) = \left| \sum_{\alpha \in K_{\bar{a}(\gamma)}} \left( \begin{array}{c} n \\ \alpha \end{array} \right) \cdot (-1)^{\alpha_1 + \alpha_3 + \cdots} \cdot d_1^{\alpha_1} d_2^{\alpha_2} \cdots d_k^{\alpha_k} \right|,
\]
where $\left( \begin{array}{c} n \\ \alpha \end{array} \right) = \frac{n!}{\alpha_1! \alpha_2! \cdots \alpha_k!}$ and $k = \ell(c(S))$ is the number of parts of $c(S)$. For example, for $n = 5$ and $S = \{1\}$, we have
\[
K_{c(S)} = K_{(1,3,1)} = \{(1,3,1), (1,4,0), (2,2,1), (2,3,0), (3,1,1), (3,2,0), (4,0,1), (4,1,0), (5,0,0)\},
\]
so we get from (5) that
\[
5! \cdot \text{Vol}(Z_{(1)}(d_1, d_2, d_3)) = 20d_1^3 d_2^3 d_3 - 5d_1^4 d_2^2 d_3 - 30d_1^2 d_2^2 d_3 + 10d_1^2 d_2^3.
\]
Setting $d_i = q^{i-1}$ in (5), where we take $q \geq 1$ so that the sequence $d_1, d_2, \ldots$ is non-decreasing, and recalling (2) gives
\[
n! \cdot \text{Vol}(1, q, q^2, \ldots) = \left| \sum_{(b_1, \ldots, b_n) \in P_{\bar{a}(\gamma)}} (-q)^{b_1 + b_2 + \cdots + b_n - n} \right| = |I_{\bar{a}(\gamma)}|(-q)|.
\]
Specializing further by setting \( q = 1 \) yields the identity
\[
|I_{\vec{a}(c(S))}(-1)| = \beta_n(S).
\]
Observe that this identity is consistent with \([4]\). Indeed, the first part of \( c(S) \) is positive, and thus the first element of \( \vec{a}(c(S)) = (a_1, a_2, \ldots, a_n) \) is 1, i.e. an odd number. Comparing the sequence \((a_1, a_2, \ldots, a_n)\) with the letters of the word \( b u_S \) we see that \( a_{i+1} = a_i + 1 \) if the corresponding letters of \( u_S \) are different, and \( a_{i+1} = a_i \) otherwise; in other words, changes of parity between consecutive elements of \((a_1, a_2, \ldots, a_n)\) correspond to letter changes in the word \( b u_S \). (The extra \( b \) in front corresponds to the first part 1 of \( c(S) \).) For example, for \( n = 7 \) and \( S = \{2, 3, 4\} \), we have \( c(S) = (1, 1, 3, 2) \), \( b u_S = b a b^3 a^2 \), and \( \vec{a}(c(S)) = (1, 2, 3, 3, 3, 4, 4) \). It follows that the subset constructed from \( \vec{a}(c(S)) \) according to the rule \([4]\) of an earlier result is \( S \), so the results agree.

Considering once more the case \( S = \{2, 4, 6, \ldots\} \cap [n-1] \), let us point out the similarity between the formula \([5]\) and the expression that Pitman and Stanley \([7]\) derive for the volume of their parking function polytope. This polytope, which we denote by \( \Pi_n(c_1, c_2, \ldots, c_n) \), is defined by the inequalities \( x_1 \geq c_1, x_2 \geq c_2, \ldots, x_i \geq c_i \) for all \( i \in [n] \). The volume-preserving change of coordinates \( y_i = c_n + c_{n-1} + \cdots + c_{n+1-i} - (x_1 + x_2 + \cdots + x_i) \) transforms the defining relations above into \( y_i \geq 0 \) for \( i \in [n] \), \( y_1 \leq c_1 \), and \( y_i - y_{i+1} \geq c_i \) for \( i \in [n-1] \), and these new relations look much like the ones defining \( Z_{\{2, 4, 6, \ldots\}}(c_1, c_2, \ldots, c_n) \): in essence we have here a difference instead of a sum. This similarity somewhat explains the close resemblance of the volume formulas for the two polytopes, as for \( \Pi_n(c_1, c_2, \ldots, c_n) \) we have
\[
n! \cdot \text{Vol}(\Pi_n(c_1, c_2, \ldots, c_n)) = \sum_{(b_1, \ldots, b_n) \in P_n} \prod_{i=1}^{n} c_{b_i} = \sum_{\alpha \in K_{1^n}} \prod_{i=1}^{n} \left( \begin{array}{c} n \\ \alpha \\ i \end{array} \right) c_{\alpha_i}.
\]

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## 2. An involution on \( \vec{a} \)-parking functions

The idea of the combinatorial argument presented in this section was first discovered by the second author and Igor Pak during their work on \([5]\).

Let \( \vec{a} = (a_1, a_2, \ldots, a_n) \) be a non-decreasing sequence of positive integers. As a first step in the construction of our involution on \( \vec{a} \)-parking functions, let \( Y_{\vec{a}} \) be the Young diagram whose column lengths from left to right are \( a_n, a_{n-1}, \ldots, a_1 \). Define a horizontal strip \( H \) inside \( Y_{\vec{a}} \) to be a set of cells of \( Y_{\vec{a}} \) satisfying the following conditions:

(i) for every \( i \in [n] \), the set \( H \) contains exactly one cell \( \sigma_i \) from column \( i \) (we number the columns 1, 2, \ldots, \( n \) from left to right);
(ii) for \( i < j \), the cell \( \sigma_i \) is in the same or in a lower row than the cell \( \sigma_j \).

For a horizontal strip \( H \), let us call a filling of the cells of \( H \) with numbers 1, 2, \ldots, \( n \) proper if the numbers in row \( i \) are in increasing order if \( i \) is odd, or in decreasing order if \( i \) is even (we number the rows 1, 2, \ldots, from top to bottom). Let \( \mathcal{H}_\bar{a} \) denote the set of all properly filled horizontal strips inside \( Y_{\bar{a}} \).

For an \( \bar{a} \)-parking function \( \bar{b} = (b_1, b_2, \ldots, b_n) \), define \( H(\bar{b}) \in \mathcal{H}_\bar{a} \) in the following way. Let \( I_j \subseteq [n] \) be the set of indices \( i \) such that \( b_i = j \). Construct the filled horizontal strip \( H(\bar{b}) \) by first writing the elements of \( I_1 \) in increasing order in the \( |I_1| \) rightmost columns in row 1 of \( Y_{\bar{a}} \), then writing the elements of \( I_2 \) in decreasing order in the next \( |I_2| \) columns from the right in row 2, then writing the elements of \( I_3 \) in increasing order in the next \( |I_3| \) columns from the right in row 3, and so on, alternating between increasing and decreasing order.

**Lemma 2.1.** A sequence \( \bar{b} \) is an \( \bar{a} \)-parking function if and only if the horizontal strip \( H(\bar{b}) \) produced in the above construction fits into \( Y_{\bar{a}} \).

**Proof.** Let \( (b'_1, b'_2, \ldots, b'_n) \) be the increasing rearrangement of \( \bar{b} \). Then the cell of \( H(\bar{b}) \) in the \( i \)-th column from the right belongs to row \( b'_i \). Thus the condition of the lemma is equivalent to \( b'_i \leq a_i \) for all \( i \).

Clearly, the filling of \( H(\bar{b}) \) from the above procedure is proper, and hence \( \bar{b} \leftrightarrow H(\bar{b}) \) is a bijection between \( \mathcal{P}_\bar{a} \) and \( \mathcal{H}_\bar{a} \). We will describe our involution on \( \bar{a} \)-parking functions in terms of the corresponding filled horizontal strips.

Let \( H \) be a properly filled horizontal strip in \( \mathcal{H}_\bar{a} \). In what follows we use \( \sigma_i \) to refer to both the cell of \( H \) in column \( i \) and to the number written in it. Let \( r(\sigma_i) \) be the number of the row containing \( \sigma_i \). We begin by defining the assigned direction for each of the \( \sigma_i \)'s. For the purpose of this definition it is convenient to imagine that \( H \) contains a cell labeled \( \sigma_{n+1} = n + 1 \) in row 1 and column \( n + 1 \), that is, just outside the first row \( Y_{\bar{a}} \) on the right. Let

\[
\epsilon_i = \text{sgn}(\sigma_i - \sigma_{i+1})(-1)^{r(\sigma_i)},
\]

where \( \text{sgn}(x) \) equals 1 if \( x > 0 \), or \(-1 \) if \( x < 0 \). Define the assigned direction for \( \sigma_i \) to be up if \( \epsilon_i = -1 \) and down if \( \epsilon_i = 1 \).

Let us say that \( \sigma_i \) is moveable down if the assigned direction for \( \sigma_i \) is down, \( \sigma_i \) is not the bottom cell of column \( i \), and moving \( \sigma_i \) to the cell immediately below it would not violate the rules of a properly filled horizontal strip. The latter condition prohibits moving \( \sigma_i \) down if there is another cell of \( H \) immediately to the left of it, or if moving \( \sigma_i \) down by one row would violate the rule for the relative order of the numbers in row \( r(\sigma_i) + 1 \). Let us say that \( \sigma_i \) is moveable up if the assigned direction for \( \sigma_i \) is up. Note that we do not need any complicated conditions in this case: if \( \sigma_i \) has another cell of \( H \) immediately to its right, or if \( \sigma_i \) is in the top row, then the assigned direction for \( \sigma_i \) is down.

It is a good time to consider an example. Figure 4 shows the diagram \( Y_{\bar{a}} \) and a properly filled horizontal strip \( H \in \mathcal{H}_\bar{a} \) for \( \bar{a} = (3, 3, 6, 7, 7, 7, 8) \). The horizontal strip shown is \( H(\bar{b}) \),
where \( \vec{b} = (5, 7, 2, 5, 1, 5, 2) \in \mathcal{P}_{\vec{a}}^\ast \). Moveable cells are equipped with arrows pointing in their assigned directions. Note that the assigned direction for \( \sigma_7 = 5 \) is down because of the “imaginary” \( \sigma_8 = 8 \) next to it, but it is not moveable down because the numbers 7, 3, 5 in row 2 would not be ordered properly. The assigned direction for \( \sigma_3 = 4 \) and \( \sigma_4 = 6 \) is also down, but these cells are not moveable down because moving them down would not produce a horizontal strip.

![Figure 1](image)

**Figure 1.** A properly filled horizontal strip with assigned directions for its cells

The validity of the involution we are about to define depends on the following simple but crucial fact.

**Lemma 2.2.** Let \( \sigma_i \) be a moveable cell of \( H \in \mathcal{H}_{\vec{a}} \), and let \( H' \in \mathcal{H}_{\vec{a}} \) be the horizontal strip obtained from \( H \) by moving \( \sigma_i \) by one row in its assigned direction. Then

(a) \( \sigma_i \) is moveable in the opposite direction in \( H' \);

(b) if \( j \neq i \) and \( \sigma_j \) is moveable in \( H \), then \( \sigma_j \) is moveable in the same direction in \( H' \);

(c) if \( \sigma_j \) is not moveable in \( H \), then it is not moveable in \( H' \).

**Proof.** Follows by inspection of the moving rules. \( \square \)

Let \( \mathcal{H}_{\vec{a}} \) be the set of all \( H \in \mathcal{H}_{\vec{a}} \) such that \( H \) has at least one moveable cell (up or down). Define the map \( \psi : \mathcal{H}_{\vec{a}} \rightarrow \mathcal{H}_{\vec{a}} \) as follows: given \( H \in \mathcal{H}_{\vec{a}} \), let \( \psi(H) \) be the horizontal strip obtained from \( H \) by choosing the rightmost moveable cell of \( H \) and moving it by one row in its assigned direction. In view of Lemma 2.2, \( \psi \) is an involution.

For \( \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathcal{P}_{\vec{a}} \) and \( H = H(\bar{b}) \), define \( s(\vec{b}) = s(H) := b_1 + b_2 + \cdots + b_n - n \). Observe that \( s(\vec{b}) \) is the number of cells of \( Y_{\vec{a}} \) that lie above one of the cells of \( H(\bar{b}) \). In the example in Figure 1, we have \( s(\vec{b}) = s(H) = 6 + 4 + 4 + 4 + 1 + 1 + 0 = 20 \). Clearly,
s(H) = s(ψ(H)) ± 1 for \( H \in \tilde{\mathcal{H}}_\vec{a} \). Since \( \psi \) is fixed-point free, it follows that

\[
\sum_{H \in \mathcal{H}_{\vec{a}}} (-1)^{s(H)} = 0,
\]

and that

\[
I_{\vec{a}}(-1) = \sum_{H \in \mathcal{H}_{\vec{a}} - \mathcal{H}_{\vec{a}}} (-1)^{s(H)}
\]

(cf. (2)). It remains to examine the members of \( \mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}} \) in order to evaluate the right hand side of (6).

**Lemma 2.3.** If \( a_1 \) is even, then \( \mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}} = \emptyset \). If \( a_1 \) is odd, then \( \mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}} \) consists of all filled horizontal strips \( H \) in \( Y_{\vec{a}} \) such that the cell \( \sigma_i \) of \( H \) is at the bottom of column \( i \) for all \( i \in [n] \), and the permutation \( \sigma_n \ldots \sigma_2 \sigma_1 \) has descent set

\[
S = \left\{ i \in [n-1] \mid a_{i+1} \text{ is odd} \right\}
\]

(cf. (4)).

**Proof.** Let \( H \in \mathcal{H}_{\vec{a}} \), and consider the cell \( \sigma_n \) in \( H \). Since \( \sigma_n < \sigma_{n+1} = n + 1 \), it follows that \( \sigma_n \) is moveable up if \( r(\sigma_n) \) is even, or moveable down if \( r(\sigma_n) \) is odd, unless in the latter case \( \sigma_n \) is at the bottom of column \( n \). Thus if \( a_1 \) is even, that is, if the rightmost column of \( Y_{\vec{a}} \) has even height, \( \sigma_n \) is always moveable and \( \mathcal{H}_{\vec{a}} = \tilde{\mathcal{H}}_{\vec{a}} \).

Suppose that \( a_1 \) is odd, and let \( H \in \mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}} \). Then no cells of \( H \) are moveable, and hence the assigned direction for every cell is down.

First, let us show that all cells of \( H \) are at the bottom of their respective columns. Suppose it is not so, and choose the leftmost cell \( \sigma_i \) of \( H \) such that the cell immediately below it is in \( Y_{\vec{a}} \). Our choice guarantees that \( \sigma_i \) does not have another cell of \( H \) immediately to its left, so the only way \( \sigma_i \) can be not moveable down is if \( \sigma_{i-1} \) is in column \( i - 1 \) one row below \( \sigma_i \) and \( \text{sgn}(\sigma_{i-1} - \sigma_i) = (-1)^{r(\sigma_{i-1})} \). But in this case the assigned direction for \( \sigma_{i-1} \) is up — a contradiction.

Now let us compute the descent set of \( \sigma_n \sigma_{n-1} \ldots \sigma_1 \). We just proved that \( r(\sigma_{n+1-i}) = a_i \) for all \( i \in [n] \). For \( i \in [n-1] \), we have

\[
1 = \epsilon_{n-i} = \text{sgn}(\sigma_{n-i} - \sigma_{n+1-i})(-1)^{r(\sigma_{n-i})},
\]

and hence \( \sigma_{n+1-i} > \sigma_{n-i} \), that is, \( \sigma_n \sigma_{n-1} \ldots \sigma_1 \) has a descent in position \( i \), if and only if \( r(\sigma_{n-i}) = a_{i+1} \) is odd. \( \square \)

Note that from Lemma 2.3 it follows that for all \( \mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}} \), the value of \( s(H) \) is the same, namely, \( a_1 + a_2 + \cdots + a_n - n \). Combining with (6), we obtain the following theorem (cf. (3)).
Theorem 2.4 (cf. [5]). For a non-decreasing sequence $\bar{a} = (a_1, a_2, \ldots, a_n)$ of positive integers, we have

$$ I_{\bar{a}}(-1) = \begin{cases} 0, & \text{if } a_1 \text{ is even;} \\ (-1)^{a_1+\cdots+a_n-n} \cdot \beta_n(S), & \text{if } a_1 \text{ is odd,} \end{cases} $$

where $S = \{i \in [n-1] \mid a_{i+1} \text{ is odd}\}$, and $\beta_n(S)$ is the number of permutations of $[n]$ with descent set $S$.

3. Generalized chain polytopes of ribbon posets

In this section we prove the formula (5) of Section 1.

Theorem 3.1. For a positive integer $n$, a subset $S \subseteq [n-1]$ such that

$$ c(S) = (1, \delta_1, \delta_2, \ldots, \delta_{k-1}), $$

and a sequence $0 < d_1 \leq d_2 \leq \cdots \leq d_k$ of real numbers, we have

$$ n! \cdot \text{Vol}(\mathcal{Z}_S(d_1, d_2, \ldots, d_k)) = (-1)^{1+\delta_2+\delta_4+\cdots} \sum_{(b_1, \ldots, b_n) \in \mathcal{P}_{\bar{a}(c(S))}} (-1)^{b_1} d_{b_1} \cdot \sum_{\alpha \in K_\alpha(c(S))} \frac{n!}{\alpha} \cdot (-1)^{\alpha_1+\alpha_3+\cdots} d_1^{\alpha_1} \cdots d_k^{\alpha_k}. $$

(7)

Proof. First, note that the expression in the right hand side of (7) is obtained from the middle one by grouping together the terms corresponding to all $\binom{n}{\alpha}$ $\bar{a}$-parking functions of content $\alpha$; each of these terms equals

$$ (-1)^{\alpha_1+2\alpha_2+3\alpha_3+\cdots} d_1^{\alpha_1} \cdots d_k^{\alpha_k} = (-1)^{\alpha_1+\alpha_3+\cdots} d_1^{\alpha_1} \cdots d_k^{\alpha_k}. $$

In what follows we prove the equality between the left and the right hand sides of (7). For $i \in [k]$, let $\rho_i = 1 + \delta_1 + \delta_2 + \cdots + \delta_{i-1}$. The volume of $\mathcal{Z}_S(d_1, d_2, \ldots, d_k)$ can be expressed as the following iterated integral:

(8)

$$ \int_0^{d_1} \int_0^{d_2-x_1} \int_0^{d_2-x_1-x_2} \cdots \int_0^{d_2-x_1-x_2-\cdots-x_{\rho_2-1}} \int_0^{d_3-x_{\rho_2}} \int_0^{d_3-x_{\rho_2}-x_{\rho_2+1}} \cdots \int_0^{d_3-x_{\rho_2}-x_{\rho_2+1}-\cdots-x_{\rho_3-1}} \cdots \int_0^{d_k-x_{\rho_{k-1}}-x_{\rho_{k-1}+1}} \cdots \int_0^{d_k-x_{\rho_{k-1}}-x_{\rho_{k-1}+1}-\cdots-x_{\rho_k-1}} dx_{n} \ dx_{n-1} \cdots dx_1 $$

(Similar integral formulas appear in [4] and in [3] Sec. 18.) Note that the assumption $d_1 \leq d_2 \leq \cdots \leq d_k$ validates the upper limits of those integrals taken with respect to variables $x_2, x_{\rho_2+1}, x_{\rho_3+1}, \ldots, x_{\rho_{k-1}+1}$; for $2 \leq i \leq k - 1$, the condition

$$ x_{\rho_{i-1}+1} \leq d_1 - x_{\rho_{i-1}+1} - x_{\rho_{i-1}+2} - \cdots - x_{\rho_i} $$
Lemma 3.2. For \( \sum \) implies that

\[
d_{i+1} - x_{\rho_{i+1}} \geq 0,
\]

and \( x_1 \leq d_1 \) implies \( d_2 - x_1 \geq 0 \).

For \( \ell \in [n] \), let \( J_\ell \) denote the evaluation of the \( n + 1 - \ell \) inside integrals of (8), that is, the integrals with respect to the variables \( x_n, x_{n-1}, \ldots, x_\ell \).

**Lemma 3.2.** For \( i \in [k] \), we have

\[
J_{\rho_{i+1}} = (-1)^{d_{i+1} + d_{i+2} + \cdots} \sum_{\alpha \in K(0, \delta_{i+1}, \ldots, \delta_{k-1})} (-1)^{\alpha_1 + \alpha_3 + \alpha_5 + \cdots} \frac{1}{\alpha_1! \alpha_2! \cdots} x_{\rho_i}^{\alpha_1} d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots.
\]

**Proof.** We prove the lemma by induction on \( i \), starting with the trivial base case of \( i = k \), in which we have \( J_{\rho_{k+1}} = J_{n+1} = 1 \). Now suppose the claim is true for some \( i \). By straightforward iterated integration one can show that for non-negative integers \( r \) and \( s \),

\[
\int_0^a \int_0^{a-y} \int_0^{a-y_1-\cdots-y_2} y_1^s \, dy_1 \cdots dy_r^y \, dx_r = \frac{s! \, a^{r+s}}{(r+s)!}.
\]

Using (10) to integrate the term of (9) corresponding to a particular \( \alpha \in K(0, \delta_{i+1}, \ldots, \delta_{k-1}) \), we get

\[
\int_0^{d_i - x_{\rho_{i+1}}} \int_0^{d_i - x_{\rho_{i+1}} - x_{\rho_{i+1}} + 1} \cdots \int_0^{d_i - x_{\rho_{i+1}} - x_{\rho_{i+1}} + 1 - \cdots - x_{\rho_{i-1}}} (-1)^{\alpha_1 + \alpha_3 + \cdots} \frac{1}{\alpha_1! \alpha_2! \cdots} \cdot d_{i+1}^{\alpha_1} d_{i+2}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots \cdot dx_{\rho_{i-1}} \cdots \cdot dx_{\rho_{i-1}} + 1
\]

\[
= (-1)^{\alpha_1 + \alpha_3 + \cdots} \frac{1}{\alpha_1! \alpha_2! \cdots} \cdot \frac{\alpha_1! (d_i - x_{\rho_{i-1}})^{\delta_{i-1} + \alpha_1}}{(\delta_{i-1} + \alpha_1)!} \cdot d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots
\]

\[
= (-1)^{\alpha_1 + \alpha_3 + \cdots} \sum_{j, m \geq 0 : j + m = \delta_{i-1} + \alpha_1} \frac{1}{\alpha_1! \alpha_2! \cdots} \cdot (-1)^{j} \cdot \frac{x_{j}^{j} d_{i}^{m}}{j! \, m!} \cdot d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots
\]

(11)

Observe that \( (j, m, \alpha_2, \alpha_3, \ldots) \in K(0, \delta_{i-1}, \delta_{i+1}, \ldots) \) if and only if \( (\alpha_1, \alpha_2, \ldots) \in K(0, \delta_{i}, \delta_{i+1}, \ldots) \), where \( \alpha_1 = j + m - \delta_{i-1} \). Hence summing the above equation over all \( \alpha \in K(0, \delta_{i}, \delta_{i+1}, \ldots) \), we
get
\[
J_{\rho_i-1+1}
= \int_0^{d_1-x_{\rho_i-1}} \int_0^{d_2-x_{\rho_i-1}+1-x_{\rho_i-1}} \cdots \int_0^{d_{\rho_i-1}-x_{\rho_i-1}+1-\cdots-x_{\rho_i-1}} J_{\rho_i+1} \ dx_{\rho_i} \cdots dx_{\rho_i-1+1}
= (-1)^{\delta_i+\delta_{i+2}+\delta_{i+4}+\cdots} \cdot \sum_{(j,m,\alpha_2,\alpha_3,\ldots) \in K(0,\delta_{i-1},\delta_i,\ldots)} (-1)^{j+\alpha_2+\alpha_4+\cdots} \cdot \frac{1}{j! \ \alpha_2! \ \alpha_3! \ \cdots} \cdot x_j^{\alpha_1} d_1^{\alpha_1+1} d_2^{\alpha_2} d_3^{\alpha_3} \cdots
\]

Note that the signs are consistent: taking into account the factor \((-1)^{\delta_i+\delta_{i+2}+\delta_{i+4}+\cdots}\) omitted from (11), the total sign of a term of (11) is
\[
(-1)^{\delta_i+\delta_{i+3}+\delta_{i+5}+\cdots} \cdot (-1)^{j+\alpha_1+\alpha_3+\cdots} = (-1)^{\delta_i+\delta_{i+2}+\delta_{i+4}+\cdots} \cdot (-1)^{j+\alpha_2+\alpha_4+\cdots},
\]
which is true because
\[
\alpha_1 + \alpha_2 + \cdots = \delta_i + \delta_{i+1} + \cdots = n - \rho_i,
\]
and hence all the exponents on both sides add up to \(2(n-\rho_i) + 2j\), i.e. an even number. □

To finish the proof of Theorem 3.1, set \(i = 1\) in Lemma 3.2 and integrate with respect to \(x_n\):

\[
n! \cdot J_n
= n! \int_0^{d_1} J_1 \ dx_1
= (-1)^{\delta_1+\delta_3+\delta_5+\cdots} \cdot \sum_{\alpha \in K(0,\delta_1,\delta_2,\ldots)} (-1)^{\alpha_2+\alpha_4+\alpha_6+\cdots} \cdot \frac{1}{(\alpha_1+1)! \ \alpha_2! \ \alpha_3! \ \cdots} \cdot d_1^{\alpha_1+1} d_2^{\alpha_2} d_3^{\alpha_3} \cdots
\]

\[
= (-1)^{\delta_1+\delta_3+\cdots} \sum_{(\alpha_1+1,\alpha_2,\alpha_3,\ldots) \in K(1,\delta_1,\delta_2,\ldots)} (-1)^{\alpha_2+\alpha_4+\cdots} \left( \frac{n}{\alpha_1+1, \alpha_2, \alpha_3, \ldots} \right) \cdot d_1^{\alpha_1+1} d_2^{\alpha_2} d_3^{\alpha_3} \cdots,
\]
and it is clear that \((\alpha_1+1, \alpha_2, \alpha_3, \ldots) \in K(1,\delta_1,\delta_2,\ldots)\) if and only if \(\alpha \in K(0,\delta_1,\delta_2,\ldots)\). □

References

[1] Gessel, I.: A noncommutative generalization and \(q\)-analogue of the Lagrange inversion formula, *Trans. Amer. Math. Soc.* 257 (1980), no. 2, pp. 455-482
[2] Goulden, I.P. and Jackson, D.M.: *Combinatorial Enumeration*, John Wiley & Sons (1983)
[3] Kreweras, G.: Une famille de polynômes ayant plusieurs propriétés énumératives, *Period. Math. Hung.* 11 (1980), no. 4, pp. 309-320
[4] Kung, J. and Yan, C.H.: Gončarov polynomials and parking functions, *J. Combin. Theory Ser. A* 102 (2003), no. 1, pp. 16-37
[5] Pak, I. and Postnikov, A.: Resolutions for $S_n$-modules associated with rim hooks, and combinatorial applications, *Functional Analysis and its Applications* 28 (1994), no. 2, pp. 132–134
[6] Pansiot, J.-J.: Nombres d’Euler et inversions dans les arbres, *European J. Combin.* 3 (1982), no. 3, pp. 259–262
[7] Pitman, J. and Stanley, R.P.: A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, *Discrete Comput. Geom.* 27 (2002), pp. 603–634
[8] Postnikov, A. and Stanley, R.P.: Chains in the Bruhat order, *preprint*, math.CO/0502363
[9] Stanley, R.P.: Two poset polytopes, *Discrete Comput. Geom.* 1 (1986), pp. 9–23
[10] Yan, C.H.: Generalized parking functions, tree inversions, and multicolored graphs, *Adv. in Appl. Math.* 27 (2001), no. 2–3, pp. 641–670

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