How do you revise your belief set with %£;@*?

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Abstract

In the classic AGM belief revision theory, beliefs are static and do not change their own shape. For instance, if $p$ is accepted by a rational agent, it will remain $p$ to the agent. But such rarely happens to us. Often, when we accept some information $p$, what is actually accepted is not the whole $p$, but only a portion of it; not necessarily because we select the portion but because $p$ must be perceived. Only the perceived $p$ is accepted; and the perception is subject to what we already believe (know). What may, however, happen to the rest of $p$ that initially escaped our attention? In this work we argue that the invisible part is also accepted to the agent, if only unconsciously. Hence some parts of $p$ are accepted as visible, while some other parts as latent, beliefs. The division is not static. As the set of beliefs changes, what were hidden may become visible. We present a perception-based belief theory that incorporates latent beliefs.

1 Introduction

In the classic AGM belief revision theory, what a rational agent is committed to believe [Alchourrón et al., 1985] forms his/her belief set $X$ of formal sentences, which is usually assumed consistent and closed under logical consequences. To $X$ an input $P$, again some (hopefully consistent) sentence, is passed. If $P$ is not in conflict with beliefs in $X$, it is simply incorporated into $X$ without any prior operations. If not, however, the simplistic augmentation leads to inconsistency. In such situations, minimal changes are made to $X$ beforehand. Then the result is a consistent set. Either way, the new belief set is postulated to include $P$ (success postulate), to be consistent unless $P$ itself is inconsistent, and to be closed under logical consequences. This is roughly what it takes in the AGM belief revision.

Some of the subsequent works have felt that there are parts in the AGM theory that may be over-simplified. The success postulate, for instance, has been questioned on the grounds that when one receives a new piece of information, it is hardly the case that he/she accepts it unconditionally [Makinson, 1997].

1. A belief base consists of two types of beliefs. One comprises all that are reachable by the rational agent through logical consequences. This type forms a belief base in the traditional AGM sense. The other comprises all the other beliefs associated to the visible beliefs, which are there, but which are not presently visible to the rational agent. These may be called latent beliefs.  

The interpretation of a latent belief varies. In [Altman, 2006], for instance, it is explained as a belief that can be introspectively

[Alchourrón et al., 1985] [Fermé and Hansson, 1999] [Booth et al., 2012] [Ma and Liu, 2011]. Others have pointed out that the assumption of closure of $X$ by logical consequences should be relaxed, for even though two rational agents both commit to believe the same belief set, they may primarily believe some sub-set of it. They argue that such variance should have an effect on the outcome of a belief revision. Some others point out that the activity of belief revision should be understood more in sequence than as a one-off snapshot [Nayak, 1994].

Rott and Pagnucco, 2000] [Darwiche and Pearl, 1997] [Ma et al., 2010], countenancing that an epistemic state a rational agent is in should not be the same before and after a revision, and that the epistemic state should determine how the next revision is carried out. Still some others reflect that a belief can be not only revised but also updated [Katsuno and Mendelzon, 1992]. In case of belief revision, agents are discovering about the world they are put in. But the world itself may change, in correspondence to which beliefs of a rational agent’s also alter.

To sum up, after so much was achieved in [Katsuno and Mendelzon, 1992], still so much is being investigated around the logical foundation of belief revision. In this work of ours, we consider relation between beliefs and agents’ perception about them. As [Ma et al., 2010] showed in their version, in a perception-based model we can assume that agents accept not $P$ but what $P$ appears to him/her. We retain the promising feature that an agent accepts $P$ only to the extent that his/her belief set allows him/her to see. However, we also argue and consequently model that, in reality, rational agents are still accepting what may potentially come to be $P$, even if only his/her perception of it was originally accepted. Roughly, our belief acceptance model is as follows.

1. A belief base consists of two types of beliefs. One comprises all that are reachable by the rational agent through logical consequences. This type forms a belief base in the traditional AGM sense. The other comprises all the other beliefs associated to the visible beliefs, which are there, but which are not presently visible to the rational agent. These may be called latent beliefs.  

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2. There comes an external information \( \{P\}^\circ \). It is a collection of propositions, and corresponds to an input \( P \) in the traditional AGM sense. Whatever the content of the belief set is, \( P \) out of \( \{P\}^\circ \) is always accepted. \( P \) is therefore the essence of the external information \( \{P\}^\circ \). All the other propositions are attributable to \( P \). How many of them the rational agent sees as that which \( \{P\}^\circ \) is depends on what beliefs are visible to him/her in his/her belief set.

3. Upon acceptance of \( \{P\}^\circ \), therefore, the visible part of \( \{P\}^\circ \) is accepted into the agent’s belief set. If any of them contradicts his/her existing visible beliefs, then a necessary belief contraction first takes place. \( \text{However, the invisible part of } \{P\}^\circ \text{ is also accepted as latent beliefs.} \)

4. The visible part of \( \{P\}^\circ \), meanwhile, may stimulate existing latent beliefs in the agent’s belief set, making them visible. By 1. a belief set in this model is larger than one in the traditional model, since it also contains latent beliefs. By 3. even though only the part of an external information visible to a rational agent is consciously accepted into his/her belief set, the remaining are also unconsciously accepted.

We now look into some details of our model.

**Justification of the essence of external information**

In our perception model, the visible part of the belief set of a rational agent in a way represents his/her mind. The mind determines how an external information appears to him/her. Suppose now, however, that every visible belief in his/her belief set had come originally from the external world. Then there must have been a moment when he/she had no beliefs in his/her belief set. Thence arises a question. Given \( \{P\}^\circ \), what of \( \{P\}^\circ \), provided that it can be believed, does the empty mind accept, and how does he/she do it? Our supposition is that it must be accepted almost axiomatically. Perhaps an example helps here. Suppose a situation where we are instructed to have \( \{\exists + \exists \exists \exists \}^\circ \), or, in the present context, \( \{\exists + \exists \exists \exists \}^\circ \), as our belief. It does not appear at all like a belief. But anyway we are accepting it as such, for we are instructed to do so. Now, acceptance of \( \{P\}^\circ \) in this manner should not reveal so much about that which \( \{P\}^\circ \) is. In case the mind is empty, the revelation should be absolutely minimal. The minimal revelation is the essence of external information. \( P \) for \( \{P\}^\circ \) is chosen to be it.

**Justification of latent beliefs**

If it must be the case that \( \{\exists + \exists \exists \exists \}^\circ \) be categorically accepted as a belief, we, ourselves as a rational agent, go ahead and do so. Suppose that \( \{\exists + \exists \exists \exists \} \) was consciously accepted into our belief set \( X \). Now, suppose that we subsequently search on web (Further Acquisition of Beliefs) to identify that it means ‘The bird eats.’ in English translation. By this time our belief set is \( X' \), and we have a revealed attributive belief to \( \{\exists + \exists \exists \exists \} \), namely, ‘The bird eats.’ and also ‘\( \{\exists + \exists \exists \exists \} \) means The bird eats.’. Evidently, they are a part of that which \( \{\exists + \exists \exists \exists \}^\circ \) is, stimulated to come to the conscious mind by some external information. It would be strange to think that the possibility that they would become visible emerged all of a sudden at \( X' \). The possibility was already there at \( X \) as a potential. This indicates that by accepting an external information, which can be but a view of it in fact, a rational agent unconsciously also accepts the external belief as a whole as potential future consequences, even though much may remain hidden to him/her for a prolonged period. This is the justification of the use of latent beliefs.

**Associations**

Now, because latent beliefs cannot be consciously found, they also cannot be found through logical consequences of visible beliefs. We therefore need to have another kind of links that connect beliefs. For our purposes, they are called associations. Specifically, given three propositions (beliefs) \( P_1, P_2 \) and \( P_3 \), \( P_1 \) is said to be associated to \( P_2 \) through \( P_3 \), provided that neither \( P_2 \) nor \( P_3 \) is a logical consequence of \( P_1 \); that \( P_1 \) is a logical consequence of neither \( P_2 \) nor \( P_3 \), and that \( P_3 \) becomes visible in case \( P_2 \) is held visible in the agent’s belief set. In case \( P_2 \) is not visible, \( P_3 \) is a latent belief. We shall denote the connection by the notation \( P_1(P_2, P_3) \). \( \{P_1\}^\circ \) then comprises \( P_1 \) and certain number of \( \{P_2( P_2, P_3) \) attributive to \( P_1 \). The \( P_1(P_2, P_3) \) is basically \( P_1 \) if an agent can see \( P_3 \). Therefore it might be properer to call \( P_3 \) an attributive belief instead of \( P_1(P_2, P_3) \). But for our purposes, we will simply call the latter an attributive belief (to \( P_1 \)). An attributive belief \( P(P_1, P_2) \), unlike ‘\( P_1 \) implies \( P_2 \), or ‘\( P_2 \) holds if a belief set is revised with \( P_1 \)’ [Darwiche and Pearl, 1997], is inevitably dependent on \( P \), and is a part of \( \{P\}^\circ \).

**Outlines, contributions, related works and other remarks**

To speak of our approach, we represent belief sets and external information both as a set of propositional formulas and of triples of them, and present corresponding postulates to the AGM postulates. There are of course more to be desired. Nonetheless, it is a reasonable point to begin our modelling with. Adaptation to other representations can be sought after later. We formalise our ideas in Section 2. In Section 3, we present postulates for belief expansion, contraction, and revision, and observe an interesting phenomenon that even if visible beliefs and an external information are both consistent, the result of belief revision can lead to an inconsistent belief set. Informally, suppose \( \{\exists + \exists \exists \exists \}^\circ \) again. It is originally accepted into \( X \) as \( \{\exists + \exists \exists \exists \} \). Suppose that \( X \) has (rather irrationally) ‘it is not the case that the bird eats.’. Then of course acceptance of \( \{\exists + \exists \exists \exists \}^\circ \) would be contradictory to one of the beliefs in \( X \). However, it is not contradictory at \( X \), only a non-contradictory view of it is available. However, at \( X' \), some external information reveals a hitherto hidden part of \( \{\exists + \exists \exists \exists \}^\circ \), namely, ‘The bird eats.’. Hence at \( X' \), it turns out that \( X' \) is inconsistent; furthermore it turns out that it (should) have been all along inconsistent from the moment it was accepted. We may say that a potential inconsistency was materialised at \( X' \) through revelation of a latent belief attributable to an existing belief. Compared to the inconsistency in a typical belief revision theory that is caused by a new information contradicting old beliefs, the inconsistency that we have just mentioned may be already in
the old beliefs.

As far as can be gathered from existing works in this research field, the idea that we have gone through, i.e. the incorporation of the kind of latent beliefs and of their revelation through acquisition of beliefs, is new, and in a way models our inspiration which is often triggered by seemingly unrelated information. Typically, agents in belief revision theory accept an external information as that which it really is. But that cannot be generally achievable in our model. [Ma et al., 2010], on the other hand, explicitly look at a perception model, albeit belief revision not on propositions but on possible-world-based epistemic states. However, their model focuses on agents’ interpretation of uncertain inputs, not on the latent beliefs: in their work, belief revision is subject to the perceived inputs, but the parts that are imperceptible at the time of belief revision have no effect on the result of a belief revision. Also, as they note, external information being represented as epistemic states, conjunction of two epistemic states are undefinable in their method. It is also the case, at least for now, that they have dealt only with belief revision, while expansion and contraction are left for a future work. We present all the three, and show the relations between them just as in the AGM theory. It could be that our results may turn out to be also useful in addressing some of the challenges in [Ma et al., 2010]. Several works [Katsuno and Mendelzon, 1991; Benferhat et al., 2005; Ma et al., 2012] consider cases where an agent may have several views about his/her beliefs. But these are concerned with agents’ certainty about their beliefs, not about latent beliefs.

2 Formalisation

Readers are referred to Section 1 for intuition of the key notions. We assume a set of possibly uncountably many atomic propositions. We denote the set by \( \mathcal{P} \), and refer to each element by the letter \( p \) with or without a subscript. More general propositions are constructed from \( \mathcal{P} \) and the logical connectives of propositional classical logic: \( \{ \top, \bot, \land, \lor, \rightarrow, \land, \lor, \land \} \). The subscripts denote the arity. We do not specifically use classical implication \( \supset \), but it is derivable in the usual manner from \( \lor \) and \( \land \), as \( p_0 \supset p_1 \equiv \neg p_0 \lor p_1 \). We denote the set of literals by \( \text{Lit} \). We denote the set of all the propositions by \( \text{Props} \), and refer to each element by \( P \) with or without a subscript. These model beliefs. In the rest, we use the two terms: propositions and beliefs, almost interchangeably. We only prefer to use ‘proposition’ when external information is involved, and ‘belief’ when the proposition is in a belief set. To model the concept of associations between propositions/beliefs, we also define what we call attributional propositions/beliefs. Let us assume that, given any \( O \subseteq 2^{\text{Props}} \), \( L(O) \) is the set of all the propositions that are the logical consequences of any (pairs of) elements in \( O \). We also assume that, given any pair of some sets \( (U_1, U_2) \), \( \pi_0((U_1, U_2)) = U_1 \) and \( \pi_1((U_1, U_2)) = U_2 \). We further assume that the union of two pairs of sets: \( (U_1, U_2) \cup (U_1', U_2') = (U_1 \cup U_1', U_2 \cup U_2') \).

**Definition 1** (Associations and attributional beliefs). We define an association tuple to be a tuple \((\mathcal{I}, X, \text{Assoc})\) where \( \mathcal{I} \) is a mapping from \( \text{Lit} \) to \( 2^{\text{Props}} \times 2^{\text{Props}} \); \( X \) is an element of \( 2^{\text{Props}} \); and \( \text{Assoc} \) is a mapping from \( \text{Props} \) to \( 2^{\text{Props} \times 2^{\text{Props}}} \). Let \( \text{Exc} \) be a mapping from \( \text{Props} \) to \( 2^{\text{Props}} \) such that \( \text{Exc}(P) = \{ P \} \cup \{ P \in \text{Props} \mid P \in L(\{ P \}) \} \). Then \( \mathcal{I} \) is defined to satisfy that, for any \( P \in \text{Lit} \), if either \( P \in \text{Exc}(P) \) or \( P \notin \text{Exc}(P) \), then \( (P, P) \notin \mathcal{I}(P) \). \( \text{Assoc} \) is defined to satisfy (1) that if \( P \) is a tautology, then \( \text{Assoc}(P) = (\emptyset, \emptyset) \); (2) that if \( P \) is inconsistent, i.e. \( \neg P \) is a tautology, then \( \text{Assoc}(P) = (\text{Props}, \text{Props}) \); and (3) that, if neither:

1. \( \text{Assoc}(P) = (\emptyset, \emptyset) \).
2. \( \text{Assoc}(P_1 \land P_2) = (\text{Assoc}(P_1) \cup \text{Assoc}(P_2)) \setminus \mathcal{I}(\text{Exc}(P_1 \land P_2)) \), where \( (U_1, U_2) \setminus U_3 = (U_1', U_2') \) such that \( U_1' \setminus U_3 = U_1 \setminus U_3 \).
3. \( \text{Assoc}(P_1 \lor P_2) = \text{Assoc}(P_1 \land P_2) \) if \( P_1, P_2 \in X \).
4. \( \text{Assoc}(P_1 \lor P_2) = \text{Assoc}(P_1) \) if \( \neg P_1 \in X \) for \( i, j \in \{ 1, 2 \}, i \neq j \).
5. \( \text{Assoc}(P_1 \lor P_2) = \text{Assoc}(P_1), \text{Assoc}(P_2), \text{Assoc}(P_1 \land P_2) \), otherwise.
6. \( \text{Assoc}(\neg (P_1 \land P_2)) = \text{Assoc}(\neg P_1 \lor \neg P_2) \).
7. \( \text{Assoc}(\neg (P_1 \lor P_2)) = \text{Assoc}(\neg P_1 \land \neg P_2) \).

Now, let \( P \) be a proposition which is neither tautological nor inconsistent. We define the set \( \{ (P_1, P_2) \mid (P_1, P_2) \in \text{Assoc}(P) \} \) to be the set of beliefs/propositions attributable to \( P \). We denote the set by \( \text{Cond}(P) \). We denote \( \bigcup_{P \in \text{Props}} \text{Cond}(P) \) simply by \( \text{Cond} \).

Observe that there is no belief attributable to a tautological or an inconsistent belief.

\( \text{Assoc} \) relates pieces of information which are not connected by logical consequences. The definition in (the 3rd, the 4th and the 5th item says that \( \text{Assoc} \) may be nondeterministic for disjunction, depending on the content of \( X \). \( \text{Assoc} \) is defined recursively instead of independently (i.e. \( \text{Assoc}(P) = \mathcal{I}(P) \) for all \( P \in \text{Props} \); and \( \mathcal{I}(P) \) is defined for all \( P \in \text{Props} \)). The reason is that if, say, \( p \) and \( q \) each has certain set of attributive beliefs/propositions, \( U_p \) and respectively \( U_q \), then it is more natural to think that the elements of \( U_p \cup U_q \) are recognised attributive to \( p \land q \). The only exceptions are those beliefs/propositions which are connected to \( p \lor q \) by logical consequences. These are excluded.

We define the set of belief bases to be a pair: \( 2^{\text{Props} \times \text{Cond}} \). Instead of working directly on belief bases, however, just as in the AGM theory we represent a rational agent as a certain belief set. Let us denote the set of all the belief bases by \( \text{BBase} \), and refer to each element in the set by \( B \) with or without a subscript.

**Definition 2** (Belief sets). Let \( Cn \) be a closure operator on \( \text{BBase} \) such that \( Cn(B) \) is the least fixpoint of \( Cn^k(B) \) (\( k \geq 0 \)) defined by:

- \( \pi_0(Cn^0(B)) = L(\pi_0(B)) \).
- \( \pi_0(Cn^{k+1}(B)) = L(\pi_0(Cn^k(B)) \cup A') \).
- \( \pi_1(Cn^k(B)) = \bigcup_{P \in \pi_0(Cn^k(B))} \text{Cond}(P) \), where

\[
A' = \{ P \in \text{Props} \mid (P_1, P_2) \in \pi_1(Cn^k(B)) \} \setminus \{ P \in L(\pi_0(Cn^k(B))) \}
\]
Belief expansion

1. $C_n(B) \vdash \hat{P} = C_n(C_n(B) \cup \hat{P})$.

Belief contraction

1. $C_n(B) \vdash \hat{P} = C_n(C_n(B) \vdash \hat{P})$ (Closure).
2. $\hat{P} \notin C_n(B) \rightarrow \vdash \hat{P} \notin C_n(C_n(B) \vdash \hat{P})$ (Success).
3. $C_n(B) \vdash \hat{P} \subseteq C_n(B)$ (Inclusion).
4. $\hat{P} \notin C_n(B) \rightarrow \vdash C_n(B) \vdash \hat{P} = C_n(B) \vdash \hat{P}$ (Vacuity).
5. $[P_1 \leftrightarrow P_2 \in C_n(\emptyset)] \rightarrow \vdash [C_n(B) \vdash P_1 = C_n(B) \vdash P_2]$ (Extensionality).
6. $C_n(B) \subseteq C_n(B) \vdash \hat{P}$ (Recovery).
7. $[P_1 \notin C_n(B) \vdash (P_1 \land P_2)] \rightarrow \vdash [C_n(B) \vdash (P_1 \land P_2) \subseteq C_n(B) \vdash P_1]$ (Conjunctive inclusion).
8. $\forall \hat{P} \subset (C_n(B) \vdash P_1) \land (C_n(B) \vdash P_2) \subseteq C_n(B) \vdash (P_1 \land P_2)$ (Conjunctive overlap).

Belief revision

1. $C_n(B) \vdash \hat{P} = C_n(C_n(B) \vdash \hat{P})$ (Closure).
2. $\hat{P} \in C_n(B) \vdash \hat{P}$ (Success).
3. $C_n(B) \vdash \hat{P} \subseteq C_n(B) \vdash \hat{P}$ (Inclusion).
4. $\neg \hat{P} \notin C_n(B) \rightarrow \vdash [C_n(B) \vdash \hat{P} = C_n(B) \vdash \hat{P}]$ (Vacuity).
5. $[P_1 \leftrightarrow P_2 \in C_n(\emptyset)] \rightarrow \vdash [C_n(B) \vdash P_1 = C_n(B) \vdash P_2]$ (Extensionality).
6. $C_n(B) \vdash \hat{P}$ is consistent if $\hat{P}$ is consistent (Consistency).
7. $C_n(B) \vdash (P_1 \land P_2) \subseteq (C_n(B) \vdash P_1 + P_2)$ (Super-expansion).
8. $[P_1 \notin C_n(B) \vdash (P_1 \land P_2)] \rightarrow \vdash [(C_n(B) \vdash P_1) \vdash P_1 \subseteq (C_n(B) \vdash P_1 \land P_2)]$ (Sub-expansion).

Figure 1: The AGM postulates for belief expansion, contraction and revision. The last two postulates for belief contraction and for belief revision are supplementary postulates that regulate belief retention. $P_1 \leftrightarrow P_2$ is an abbreviation of $(\neg P_1 \lor P_2) \land (\neg P_2 \lor P_1)$.

$C_n(B)$ is assumed compact. We say that a belief base $B$ is closed iff $B = C_n(B)$. We call any closed belief a belief set. We say that a belief set $C_n(B)$ is consistent iff for any $P \in \text{Props}$, if $P \in \pi_0(C_n(B))$, then $\neg P \notin \pi_0(C_n(B))$, and if $\neg P \in \pi_0(C_n(B))$, then $P \notin \pi_0(C_n(B))$.

No attributive beliefs exist without the proposition which they are allegedly attributing.

Observation 1 (Adequacy). For any belief set $C_n(B)$ and for any $P \in \text{Props}$, if $P \notin C_n(B)$, then $C_n(B)$ does not contain any attributive belief to $P$.

A belief set $C_n(B)$ is the representation of what a rational agent holding it is committed to believe. Meanwhile, just for reference, what a putative irrational agent who sees everything, even beyond what he/she can perceive, would be committed to believe, which we denote by $C_n^*(B)$, is defined in a similar way to $C_n(B)$, but instead of $A^k$, $A^k = \{P_k \mid P_k \in \pi_1(C_n^*(B))\}$ is used. The two do not generally coincide; and we have $C_n(B) \subseteq C_n^*(B)$.

For incoming information from the external world into a belief set, we assume that every piece of such information is a set of propositions with one primary proposition $P$ for some $P$ and other (zero or more) propositions attributive to $P$, i.e. $\text{Cond}(P)$.

Let us denote each external information by $\{P\}^\circ \in (\{P\}, \text{Cond}(P)) \in 2^{\text{Props} \times \text{Cond}}$. Whether the receiving agent notices of any of the attributive beliefs in $\{P\}^\circ$ at the time he/she accepts $\{P\}^\circ$ depends on $C_n(B)$. Let us denote by $\text{Visible}\{\{P\}\}^\circ$ the part of $\{P\}^\circ$ visible to $C_n(B)$. Specifically, $\text{Visible}\{\{P\}\}^\circ$ is defined to be $P \cup \{P_k \mid P_k \in \pi_1(\{P\}^\circ)\}$ and $[P_1 \in \pi_0(C_n(B))]$.

3 Postulates and Representations

We first of all state one postulate about the relation between an association tuple and a belief set.

Adequacy may better be a postulate on its own than be integrated into $C_n$. We choose this way here only because it keeps the number of postulates on par with that in the AGM theory.

1. For any belief set $C_n(B)$, $(I, \pi_0(C_n(B)), \text{Assoc})$ for some $I$ is the association tuple for it.

We have two principles for belief expansion. The AGM postulates for expansion, contraction and revision are listed in

Figure 1 for easy comparisons. $C_n$ is a closure operator by logical consequences; $\hat{B}$ is a set of propositions; $\hat{P}$ is a proposition as an external information, i.e. all in the AGM sense [Alchourrón et al., 1985].

1. $C_n(B) + \{P\}^\circ = C_n(C_n(B) \cup \{P\}^\circ)$ (Augmentation).

Explanation: As in the AGM belief revision theory, belief expansion simply incorporates a provided information, which is then closed by $C_n$.

2. If the association tuple for $C_n(B)$ is $(I, \pi_0(C_n(B)), \text{Assoc})$, then that to $C_n(B) + \{P\}^\circ$ is $(I, \pi_0(C_n(B) + \{P\}^\circ), \text{Assoc})$ (Association update).

The postulates for contraction are as follows.

1. $C_n(B) \vdash \{P\}^\circ = C_n(C_n(B) \vdash \{P\}^\circ)$ (Closure).

Explanation: belief contraction is an operation that guarantees closure by $C_n$.

2. $\forall P_1 \in \text{Visible}_B(\{P\}^\circ), P_1 \notin \pi_0(C_n(\emptyset)) \rightarrow [P_1 \notin \pi_0(C_n(B) \vdash \{P\}^\circ)]$ (Success).

Explanation: If all the propositions that are visible to $C_n(B)$ are not a tautology, then the resultant belief set by belief contraction does not contain any of them.

3. $C_n(B) \vdash \{P\}^\circ \subseteq C_n(B)$ (Inclusion).

Explanation: The resultant belief set by belief contraction is a sub-set of the initial belief set.

4. $\forall P_\alpha \in \text{Visible}_B(\{P\}^\circ), P_\alpha \notin \pi_0(C_n(B)) \rightarrow \vdash C_n(B) \vdash \{P\}^\circ = C_n(B)$ (Vacuity).

Explanation: If all the visible propositions of an external information are either a tautology or a proposition that is not in $C_n(B)$, then contraction of $\{P\}^\circ$ from $C_n(B)$ is still $C_n(B)$.

5. $\lfloor \text{Visible}_B(\{P_1\}^\circ) \rfloor = \lfloor \text{Visible}_B(\{P_2\}^\circ) \rfloor \rightarrow [C_n(B) \vdash \{P_1\}^\circ = C_n(B) \vdash \{P_2\}^\circ]$ (Extensionality).
**Explanation:** If $\text{Visible}_B(\{P_1\}^\circ)$ and $\text{Visible}_B(\{P_2\}^\circ)$ are indistinguishable in content, then contracting $Cn(B)$ by $\{P_1\}^\circ$ or by $\{P_2\}^\circ$ will produce the same belief set.

6. $Cn(B) \subseteq (Cn(B) \div \{P\}^\circ) + (\text{Visible}_B(\{P\}^\circ), \emptyset)$ (Recovery).

**Explanation:** If the contracted belief set is expanded with the beliefs that have been removed, then the belief set contains the original belief set.

7. If the association tuple for $Cn(B)$ is $(I, \pi_0(Cn(B)), \text{Assoc})$, then for $Cn(B) \div \{P\}^\circ$ is $(I, \pi_0(Cn(B) \div \{P\}^\circ), \text{Assoc})$ (Association update).

There is no gratuitous recovery by removing an external information and then adding the same external information. Compare it to Recovery in Figure 4.

**Observation 2.** There is no guarantee that we have $Cn(B) \subseteq (Cn(B) \div \{P\}^\circ) + \{P\}^\circ$.

**Proof.** Suppose $B := \{p_0, p_1, p_2\}, \{p_0(p_1, p_2), p_0(p_2, p_1)\}$ so that $\text{Cond}(p_0) = \{p_0(p_1, p_2), p_0(p_2, p_1)\}$, that $\text{Cond}(p_1) = \text{Cond}(p_2) = \emptyset$ and that none of $\{p_0, p_1, p_2\}$ are a logical consequence of others, or a tautology.

The reason is that a belief set and a contracted belief set of its are not generally identical, which influences how an agent perceives $\{p\}^\circ$; see (Association update).

In line with the AGM representation of the postulates, we define a mapping from belief sets and propositions into belief sets. For any belief set $Cn(B)$ and any external information $\{P\}^\circ$, we say that $Cn(B_1) \subseteq Cn(B)$ is a maximal subset of $Cn(B)$ for $\{P\}^\circ$ iff

1. For any $B \in \text{Visible}_B(\{P\}^\circ), P_1 \notin \pi_0(Cn(B_1))$ if $P_1$ is not a tautology.

2. For any $Cn(B_2), if (Cn(B_2), Cn(B_2) \subseteq Cn(B)$, then there exists some $P_2 \in \text{Visible}_B(\{P\}^\circ)$ such that $P_2 \in \pi_0(Cn(B_2))$.

3. $Cn(B_1) \cup (\text{Visible}_B(\{P\}^\circ), \emptyset) = Cn(B)$.

We define $\triangle(Cn(B), \{P\}^\circ)$ to be the set of all the subsets of $Cn(B)$ maximal for $\{P\}^\circ$. We further define a function $\gamma$, so that, if $\triangle(Cn(B), \{P\}^\circ)$ is not empty, then $\gamma(\triangle(Cn(B), \{P\}^\circ))$ is a sub-set of $\triangle(Cn(B), \{P\}^\circ)$; or if it is empty, it is simply $Cn(B)$.

**Theorem 1** (Representation theorem of belief contraction). Let $Cn(B)$ be a consistent belief set, and let $\{P\}^\circ$ be an external information. Then, it follows that $Cn(B) \div \{P\}^\circ = \bigcap(\gamma(\triangle(Cn(B), \{P\}^\circ)))$.

**Proof.** Note that the postulates modulo association tuple can be reduced down to package contraction [Fuhrmann and Hansson, 1997] of $\text{Visible}_B(\{P\}^\circ)$. However, we show cases of one direction of the proof with details for not very straightforward ones. We show that for any particular belief set as results from $Cn(B) \div \{P\}^\circ$, there exists some particular $\gamma$ such that $Cn(B) \div \{P\}^\circ = \bigcap(\gamma(\triangle(Cn(B), \{P\}^\circ)))$. Suppose, by way of showing contradiction, that there exists some $\alpha$ which is either some belief $P_a$ or some attribute belief $P_a(p, p)$ such that $\alpha \in Cn(B) \div \{P\}^\circ$ and that $\alpha \notin \pi_0(\gamma(\triangle(Cn(B), \{P\}^\circ)))$, where $i = 0$ or 1, depending on which $\alpha$ is. Suppose $\alpha = P_a$. By Closure and Inclusion, we have that $\alpha \in \pi_0(Cn(B))$. Now, we consider two cases: $\alpha$ is a tautology, or otherwise. In the latter case, there are two possibilities: (1) For all $P_a \in \text{Visible}_B(\{P\}^\circ)$, we have either that $P_a$ is a tautology or that $P_a \notin \pi_0(Cn(B))$; (2) There is $P_a \notin \text{Visible}_B(\{P\}^\circ)$ such that $P_a \in Cn(B)$ and that it is not a tautology. For the second case, define Conflict to be \(\bigcup_i \text{for all such } P_i((P_a \div Cn(B)) \div P_i \in \text{Conflict}(B)) = \bigcup_i (\{P_a \div Cn(B) \div P_i \in \text{Conflict}(B)\})\). Then by Success, $\alpha \notin \text{Conflict}$. There are two sub-cases here. If $\alpha \in \text{Conflict}$, then $\alpha \notin \text{Conflict}$. Otherwise, we have that $\alpha \notin \text{Conflict}$, $P_a = \text{Visible}_B(\{P\}^\circ)$, $Cn(B) \div \{P\}^\circ$ on $(\text{Visible}_B(\{P\}^\circ)$, where Visible_B(\{P\}^\circ) is defined to be $(\text{Visible}_B(\{P\}^\circ) \div \{P\}^\circ$ but $\text{Visible}_B(\{P\}^\circ) \div \text{Visible}_B(\{P\}^\circ, \text{Visible}_B(\{P\}^\circ, \text{Visible}_B(\{P\}^\circ)))$ expands the contracted belief set.

1. $Cn(B) \div \{P\}^\circ = Cn(Cn(B) \div \{P\}^\circ)$ (Closure).

**Explanation:** * is an operation that guarantees closure by $Cn$.

2. $\text{Visible}_B(\{P\}^\circ) \subseteq \pi_0(Cn(B) \div \{P\}^\circ)$ (Success 1).

**Explanation:** Any possible proposition in external information is accepted.

3. $\forall P_a \in \text{Visible}_B(\{P\}^\circ).[P_a \notin \text{Conflict}(B)] \rightarrow \gamma(\text{Visible}_B(\{P\}^\circ))$ (Success 2).

**Explanation:** Any contradicting belief to $\text{Visible}_B(\{P\}^\circ)$ is not in the revised belief set.

4. $Cn(B) \div \{P\}^\circ \subseteq Cn(B) + \{P\}^\circ$ (Inclusion).

**Explanation:** The belief set that results from revising a belief set with some external information is a sub-set of the belief set that results from expanding it with the same information.

5. $[\forall P_a \in \text{Visible}_B(\{P\}^\circ).P_a \notin \pi_0(Cn(B))] \rightarrow \gamma(Cn(B) \div \{P\}^\circ) = Cn(B) + \{P\}^\circ$ (Vacuity).

**Explanation:** If no visible propositions of an external information are in conflict with any beliefs in $Cn(B)$, then the result of revising $Cn(B)$ with it is the same as simply expanding $Cn(B)$ with it.

6. $[\gamma(Cn(B)) \div \gamma(Cn(B)) \div \gamma(Cn(B) + \{P\}^\circ) = Cn(B) + \{P\}^\circ$ (Extensionality).

\(^*\)This third condition is derivable from the other two in [Alchourrón et al., 1985]. Here it is not, and is explicitly stated.
Explanation: If one external information is identical with another in content under $C_n$, then revising $C_n(B)$ with either of them leads to the same belief set. Note the difference from Extensionality for $\cdot$. Here latent beliefs also matter.

7. If the association tuple for $C_n(B)$ is $(I, \pi_0(C_n(B)), \mathcal{Assoc})$, then that to $C_n(B) \ast \{P\}^{o}$ is $(I, \pi_0(C_n(B) \ast \{P\}^{o}), \mathcal{Assoc})$ (Association update).

An analogue of Levi identity [Levi, 1977] does not hold here.

**Observation 3.** It is not the case that $C_n(B) \ast \{P\}^{o} = (C_n(B) \div \neg\{P\}^{o}) + \{P\}^{o}$.

Also note the following observation.

**Observation 4 (No AGM consistency upon revision)**. Even if a belief set $C_n(B)$ and all the elements of $\text{Visible}_B(\{P\}^{o})$ are consistent, $C_n(B) \ast \{P\}^{o}$ may be inconsistent.

Proof. Suppose that we have the following belief set: $C_n(\{p_0, \neg p_2\}, \{p_0(p_1, p_2)\})$ and that $\{P\}^{o} = \{(p_1)\} \cup \emptyset$. Suppose that $\text{Cond}(p_0) = \{p_0(p_1, p_2)\}$, that $\text{Cond}(p_1) = \emptyset$, and that none of $p_0, \ldots, p_2$ are a logical consequence of any others.

Hence, we explicitly have the postulate Success 2 for removal of beliefs in $C_n(B)$, instead of the implicit removal by AGM Consistency postulate. Although out of scope of this work, resolution of this kind of inconsistencies that are triggered by revelation of latent parts of existing beliefs is an interesting problem. Despite Observation 3 we have the following identity.

**Theorem 2 (Identity)**. It holds that $C_n(B) \ast \{P\}^{o} = (C_n(B) \div \{\text{Visible}_B(\{P\}^{o}), \emptyset\}) + (\text{Visible}_B(\{P\}^{o}) \cup \{P\}^{o} \cup \text{Cond}(P))$.

Proof. We show cases of one direction. Details are left to readers. Let $Y$ denote $(C_n(B) \div \text{Visible}_B(\{P\}^{o}), \emptyset)) + (\text{Visible}_B(\{P\}^{o}) \cup \text{Cond}(P))$ for space. Show that for any outcome of $Y$, there is an outcome of $C_n(B) \ast \{P\}^{o}$ such that $Y = C_n(B) \ast \{P\}^{o}$. To show contradiction, suppose some $\alpha$ which is either a belief $P_{\alpha}$ or an attributional belief $P_{\alpha}(P_{\alpha}, P_{\alpha})$. Suppose that $\alpha \in \pi_0(Y)$ and that $\alpha \notin C_n(B) \ast \{P\}^{o}$. Three cases here: $\alpha$ is a tautology, $\alpha \in \pi_0(\text{Visible}_B(\{P\}^{o}) \cup \text{Cond}(\{P\}^{o}))$, or otherwise. In the last case, there are sub-cases: either $\forall P_y \in \text{Visible}_B(\{P\}^{o}), P_y \notin \pi_0(C_n(B))$; or otherwise. For the second case, define Conflict to be $L_\pi = \{P_\alpha : P_\alpha \in C_n(B) \equiv \pi_0(L(P_{\alpha})) \equiv \{P_\alpha \in C_n(B) \mid \{L(P_{\alpha}) = L(P_\alpha \cup P_\alpha) \} \wedge \{P_\alpha \in C_n(B) \wedge L(P_{\alpha}) \} \wedge \{P_\alpha \in C_n(B) \wedge L(P_{\alpha}) \} \}$.

By Theorem 1 and Theorem 2 we also gain the representation theorem for belief revision. We may add the following supplementary postulates. These are all adaptations of the AGM supplementary postulates (Figure 1). For any $P_1, P_2, C_n(B)$ and $\text{Visible}_B(\{P\}^{o})$, let us assume for space that $C_n(B) \ast \{P\}^{o} \subseteq C_n(B) \ast \{\emptyset\} \ast \{P\}$ except that all $P_1 \wedge P_2 \in \text{Visible}_B(\{P\}^{o})$ are replaced with $P_3$.

(For belief contraction)

**A** $\forall P_1 \wedge P_2 \in \text{Visible}_B(\{P\}^{o}), (C_n(B) \ast \{\emptyset\} \ast \{P\}^{o}) \to (C_n(B) \ast \{P\}^{o} \subseteq C_n(B) \ast \{\emptyset\} \ast \{P\}^{o})$. (Conjunctive inclusion).

**B** $\forall P_1 \wedge P_2 \in \text{Visible}_B(\{P\}^{o}), (C_n(B) \ast \{\emptyset\} \ast \{P\}^{o}) \to (C_n(B) \ast \{\emptyset\} \ast \{P\}^{o} \subseteq C_n(B) \ast \{P\}^{o})$. (Conjunctive overlap).

(For belief revision)

**A** $\forall P_1 \wedge P_2 \in \text{Visible}_B(\{P\}^{o}), (C_n(B) \ast \{P\}^{o} \subseteq C_n(B) \ast \{P\}^{o} \ast \{P\}^{o}) \to (C_n(B) \ast \{P\}^{o} \ast \{P\}^{o} \ast \{P\}^{o})$. (Super-expansion).

**B** $\forall P_1 \wedge P_2 \in \text{Visible}_B(\{P\}^{o}), (C_n(B) \ast \{P\}^{o} \ast \{P\}^{o} \ast \{P\}^{o}) \to (C_n(B) \ast \{P\}^{o} \ast \{P\}^{o} \ast \{P\}^{o})$. (Sub-expansion).

As in [Alchourrón et al., 1985], addition of these postulates ensure that $\gamma$ selects the best elements under some criteria. Specifically $\gamma(Delta(C_n(B), \{P\}^{o}))$.

4 Conclusion

We have presented a new perception model that incorporates latent beliefs. From the perspective of our postulates, we observe that the AGM postulates (Figure 1) appear a little inconvenient. It is reasonable that $C_n(B) \equiv \pi_0(C_n(B))$. But the point is what the AGM input could be.

1. AGM inputs correspond to visible propositions: In this case, $P \equiv \bigwedge_{P \in \text{Visible}_B(\{P\}^{o})} P$. This then means, from the perspective of our postulates, that AGM theory is not concerned with external information $\{P\}^{o}$, for Visible$_B(\{P\}^{o})$ is not $\{P\}^{o}$. In this case, it is not possible within AGM theory to tell what an external information is from a given input.

2. AGM inputs correspond to the whole external information: In this case, $P \equiv \bigwedge_{P \in \text{Visible}_B(\{P\}^{o})} P$. This means, from the same perspective, that the agents in the AGM theory can completely know each external information as that which it is.

Our work, as well as [Ma et al., 2010], avoids these inconvenience. Unlike in the perception model by [Ma et al., 2010], however, our perception model recognises latent beliefs, and introduced them into the study of belief revision. Expansion of our work with suitable postulates on iterated belief revision [Darwiche and Pearl, 1997] is one obvious direction. Studies into reasonable resolution of the new kind of inconsistencies should be also interesting.
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