Dynamic Critical Behavior of the Chayes–Machta Algorithm for the Random-Cluster Model

I. Two Dimensions

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Abstract

We study, via Monte Carlo simulation, the dynamic critical behavior of the Chayes–Machta dynamics for the Fortuin–Kasteleyn random-cluster model, which generalizes the Swendsen–Wang dynamics for the $q$-state Potts ferromagnet to non-integer $q \geq 1$. We consider spatial dimension $d = 2$ and $1.25 \leq q \leq 4$ in steps of 0.25, on lattices up to $1024^2$, and obtain estimates for the dynamic critical exponent $z_{CM}$. We present evidence that when $1 \leq q \lesssim 1.95$ the Ossola–Sokal conjecture $z_{CM} \geq \beta/\nu$ is violated, though we also present plausible fits compatible with this conjecture. We show that the Li–Sokal bound $z_{CM} \geq \alpha/\nu$ is close to being sharp over the entire range $1 \leq q \leq 4$, but is probably non-sharp by a power. As a byproduct of our work, we also obtain evidence concerning the corrections to scaling in static observables.

Key Words: Random-cluster model, Potts model, Chayes–Machta algorithm, Swendsen–Wang algorithm, cluster algorithm, dynamic critical behavior.

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1 Introduction

Since nontrivial models in statistical mechanics are rarely exactly solvable, Monte Carlo (MC) simulations have become a standard tool for obtaining information on phase diagrams and critical exponents [1–3]. Unfortunately, MC simulations typically suffer from severe critical slowing-down [4,5], so that the computational efficiency tends rapidly to zero as the critical point is approached. More precisely, the autocorrelation (relaxation) time $\tau$ diverges in the critical limit, most often like $\tau \sim \xi^z$, where $\xi$ is the spatial correlation length. The dynamic critical exponent $z$ depends on both the model being investigated and the MC algorithm being used. For local algorithms one typically has $z \approx 2$.

An important advance was made in 1987 with the invention of the Swendsen–Wang (SW) cluster algorithm [6] for simulating the ferromagnetic $q$-state Potts model [7–9] at integer $q \geq 2$. The SW algorithm is based on passing back and forth between the Potts spin representation and the Fortuin–Kasteleyn (FK) bond representation [10–15]. More precisely, one introduces a joint probability distribution [14] of spin and bond variables, whose marginal on the spins (integrating out the bonds) is the Potts spin model and whose marginal on the bonds (integrating out the spins) is the FK random-cluster model [15]; one then updates this joint distribution by alternately applying the two conditional distributions (see Section 2.1 below for details).

Since a local move in one set of variables can have highly nonlocal effects in the other, it is not surprising that the SW algorithm might have less critical slowing-down than the conventional local algorithms. And this is in fact the case: although the SW algorithm does not eliminate critical slowing-down, it does radically reduce it compared to local algorithms. Much effort has therefore been devoted, for both theoretical and practical reasons, to understanding the dynamic critical behavior of the SW algorithm as a function of the spatial dimension $d$ and the number $q$ of Potts spin states. The best information on the dynamic critical exponent $z_{SW}$ prior to the present work is summarized in Table 1. Unfortunately, it is very difficult to develop a physical understanding from the small number of non-trivial “data points” at our disposal: second-order non-mean-field transitions occur only for $(d, q) = (2, 2), (2, 3), (2, 4), (3, 2)$ and $(4, 2)$.

A further advance was made in 1998 by Chayes and Machta (CM) [25], who devised a cluster algorithm for simulating the FK random-cluster model at any real value $q \geq 1$. The idea behind the CM algorithm is very similar to that of SW, but now one starts with the random-cluster (bond) measure and introduces auxiliary color (spin) variables on the sites, so that we again have a joint model of spins and bonds, analogous to the one employed in the SW algorithm. Although the marginal measure on the spins no longer has any obvious physical interpretation when $q$ is noninteger, the joint measure can still be used to construct an efficient cluster algorithm, by alternately applying the conditional distributions exactly as in standard SW. The CM algorithm thus generalizes the SW algorithm and in fact reduces to (a slight variant of) it when $q$ is an integer: see Section 2.2 for details. Indeed, the CM algorithm can be thought of as a “natural”

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1 A continuous (second-order) transition occurs also in the Ising ($q = 2$) model in dimensions $d > 4$, but here the static behavior is mean-field. One expects the dynamic critical exponents likewise to be dimension-independent for $d \geq 4$ (with possible multiplicative logarithmic corrections at $d = 4$).
Table 1: Best estimates of the dynamic critical exponent $z$ for the Swendsen–Wang algorithm prior to the present work. Estimates are taken from [16] for $d = 2$, $q = 2$; [17] for $d = 2$, $q = 3$; [18, 19] for $d = 2$, $q = 4$; [20] for $d = 3$, $q = 2$; and [21–24] for $d = 4$, $q = 2$. Error bars are one standard deviation, and include statistical errors only.

interpolation of the SW algorithm to noninteger $q$ (though unfortunately only for $q \geq 1$). Thus, by using the CM algorithm we can study the dynamic critical behavior of the SW–CM dynamic universality class as a function of the continuous variable $q$ throughout the range $1 \leq q \leq q_c(\mathcal{L})$, where $q_c(\mathcal{L})$ is the maximum $q$ for which the transition is second-order on the given lattice $\mathcal{L}$. This vastly enhances our ability to make theoretical sense of the numerical results.

In the present paper we perform high-precision MC simulations of the random-cluster model on the square lattice ($d = 2$), for $1.25 \leq q \leq 4$ in steps of 0.25, using the CM algorithm. Our main goal is to gain better insight into the SW–CM dynamic universality class in two dimensions by studying the behavior as a function of the continuous variable $q$. We estimate numerically several dynamic critical exponents for each value of $q$, and we attempt to understand their behavior as a function of $q$. As a byproduct we also obtain new information on corrections to scaling in the static quantities. In the companion paper [27] we carry out an analogous study for the simple-cubic lattice ($d = 3$) at $q = 1.5, 1.8, 2.2$; and in a forthcoming paper [28] we will analyze the case of the complete graph (Curie–Weiss model).

One advantage of considering $d = 2$ is of course is that we have available a remarkable amount of information concerning the static behavior of the random-cluster model as a function of $q$. This information includes the exact location [30,31] of the transition point on the square lattice, namely $p_c = \sqrt{q}/(1 + \sqrt{q})$, and the knowledge that the transition is second-order for $0 \leq q \leq 4$ and first-order for $q > 4$. Furthermore, the leading and next-to-leading thermal and magnetic critical exponents are known exactly (though non-

$^2$ We stress that $q_c(\mathcal{L})$ is not necessarily the same for all lattices of a given dimension $d$; the first-order or second-order nature of the transition is a non-universal question. See [26] for further discussion. For the standard two-dimensional lattices (square, triangular and hexagonal) we have $q_c(\mathcal{L}) = 4$.

$^3$ A brief summary of the preliminary results in this trio of papers has appeared in letter form [29].
rigorously) for the entire range $0 \leq q \leq 4$ [32,34]:

\[
y_{T1} = \frac{3g - 6}{g} \\
y_{T2} = \frac{4g - 16}{g} \\
y_{H1} = \frac{(g + 2)(g + 6)}{8g} \\
y_{H2} = \frac{(g - 2)(g + 10)}{8g}
\]

(1.1) \hspace{1cm} (1.2) \hspace{1cm} (1.3) \hspace{1cm} (1.4)

where $g$ is the Coulomb-gas coupling defined by $q = 4 \cos^2(\pi g/4)$ and $2 \leq g \leq 4$. For the standard critical exponents this implies

\[
1/\nu = y_{T1} = \frac{3g - 6}{g} \\
\Delta_1 = -y_{T2} = \frac{16 - 4g}{g} \\
d_F = y_{H1} = \frac{(g + 2)(g + 6)}{8g} \\
\alpha/\nu = 2y_{T1} - d = \frac{4g - 12}{g} \\
\beta/\nu = d - y_{H1} = \frac{(g - 2)(6 - g)}{8g} \\
\gamma/\nu = 2y_{H1} - d = \frac{12 + g^2}{4g}
\]

(1.5) \hspace{1cm} (1.6) \hspace{1cm} (1.7) \hspace{1cm} (1.8) \hspace{1cm} (1.9) \hspace{1cm} (1.10)

(Here $\Delta_1$ is the leading correction-to-scaling exponent, defined e.g. by corrections $\sim L^{-\Delta_1}$ in finite volume at criticality; and $d_F$ is the cluster fractal dimension.) The numerical values of these critical exponents for the values of $q$ employed in our simulations are collected for reference in Table 2.

A key theoretical result concerning the SW dynamics is the Li–Sokal bound [35], which states that

\[
\tau_{\text{int},N}, \tau_{\text{exp},N} \geq \text{const} \times C_H
\]

(1.11)

where $\tau_{\text{int},N}$ and $\tau_{\text{exp},N}$ are, respectively, the integrated and exponential autocorrelation times for the observable $N$ = number of occupied bonds, and $C_H$ is the specific heat. (For the precise definitions of $\tau_{\text{int}}$ and $\tau_{\text{exp}}$, see Section 3.1.) It follows from this lower bound

4 See [19, Appendix A.1] for further discussion and references. Note that there is a typographical error in equation (A.10) of [19], which should read $\Delta_{r,s} = (2(s-r)+sx)^2 - x^2)/(8([2+x])$. We remark that the same formulæ for $4 \leq g \leq 6$ give the exponents $y_{T1}, y_{T2}, y_{H1}, y_{H2}$ of the tricritical Potts model [32,33].
that the SW algorithm cannot completely eliminate critical slowing-down if the specific heat is divergent at criticality. In particular, (1.11) implies the lower bound

$$z_{int,N}, z_{exp} \geq \frac{\alpha}{\nu}$$

(1.12)

for the dynamic critical exponents of the SW algorithm, where $\alpha/\nu$ is the static exponent for the specific heat. Now, simple “Fortuin–Kasteleyn identities” show that $\text{var}(N)/V$ is a specific-heat-like quantity and provides a natural continuation of the notion of specific heat to random-cluster models. It turns out that the Li–Sokal bound (1.11) with this definition of specific heat can be easily extended from the SW to the CM algorithm at arbitrary real $q \geq 1$, as we show in [27, Appendix A].

The physical mechanism underlying the Li–Sokal proof is the slow evolution of $N$, which is an “energy-like” observable. Previous studies in two dimensions for $q = 2, 3, 4$ have shown empirically that the Li–Sokal bound is very close to being sharp for all three values of $q$ [16, Section 6] [17,19]; and we will show here that this is the case also for noninteger $q$ throughout the range $1 \leq q \leq 4$ (see Sections 7.3 and 7.4). For the three-dimensional Ising model, by contrast, the Li–Sokal bound is very far from sharp [20]. Therefore there must be another mechanism, beyond the one captured in the Li–Sokal proof, that is principally responsible for the critical slowing-down in three dimensions. A few years ago, Ossola and Sokal [20] suggested that this as-yet-not-understood mechanism causing slowness might perhaps be somehow related to the typical size of the largest cluster. More specifically, they conjectured that perhaps the SW dynamics for any Potts ferromagnet in any dimension satisfies

$$\tau_{SW} \geq \text{const} \times \frac{L^d}{C_1}$$

(1.13)

where $C_1$ is the expected number of sites in the largest cluster. If true, this would imply the critical-exponent inequality

$$z_{SW} \geq \frac{\beta}{\nu},$$

(1.14)

where $\beta/\nu$ is the static exponent for the magnetization. Indeed, Coddington and Baillie [23] had earlier suggested that the inequality (1.14) holds as an equality for the Ising models in dimensions $d = 2, 3, 4$. The numerical results reported by Ossola and Sokal [20] for the three-dimensional Ising model are consistent with this conjectured equality, though they also present plausible fits consistent with $z_{SW} < \beta/\nu$ (i.e. violation of their conjectured inequality). Finally, an analytical treatment [24] of the SW dynamics for the Ising model on the complete graph suggests that $z_{SW} = \beta/\nu$ in this case also (namely, $z_{SW} = \beta/\nu = 1$).

Alas, by considering the Ossola–Sokal conjecture within the more general framework of the Chayes–Machta dynamics, one can see at a glance that it is probably false! To start with, for $q = 1$ on any lattice, the Swendsen–Wang–Chayes–Machta algorithm reduces to independent sampling for independent bond percolation, so that $z_{CM} = 0$; but in general $\beta/\nu > 0$ for percolation, which shows that the Ossola–Sokal conjecture (1.14) is false for $q = 1$. Moreover, it is reasonable to believe that $z_{CM}$ is a continuous function of $q$: if so, then for $q$ slightly greater than 1 one has either $z_{CM}$ identically zero or else $z_{CM}$ very close
to zero; and in either case one would have $z_{\text{CM}} < \beta/\nu$, so that the Ossola–Sokal conjecture would be violated also for $q$ in some interval above 1. Our numerical results in this paper (see Section 7.3) suggest that in $d = 2$ the Ossola–Sokal conjecture fails for $1 \leq q \lesssim 1.95$; indeed, for $1 \leq q \lesssim 1.6$ we apparently have $z_{\text{CM}} = 0$ exactly, while of course $\beta/\nu > 0$. (But see Section 7.5 for an alternative fit that is compatible with the Ossola–Sokal conjecture.) Likewise, the numerical results for $d = 3$ to be presented in [27] for $q = 1.5$ and $q = 1.8$ strongly suggest that the Ossola–Sokal conjecture is violated for these values of $q$ and hence presumably for all $q \lesssim 2$, but that it holds as a strict inequality $z_{\text{CM}} > \beta/\nu$ when $q = 2.2$ and hence presumably for all $q$ in the range $2 \lesssim q \leq q_c(Z^3)$. Indeed, in $d = 3$ it is found that $z_{\text{CM}}$ is a strongly increasing function of $q$, while $\beta/\nu$ varies very slowly with $q$; the two curves presumably cross somewhere near $q = 2$. Finally, in our forthcoming paper [28] we will extend the method of [24] to cover the CM dynamics on the complete graph for all $q$ in the range $1 \leq q \leq 2$. We find that there are two markedly different behaviors depending on $q$: for $1 \leq q < 2$ we have $z_{\text{CM}} = 0 < \beta/\nu = 1$, while for $q = 2$ we have $z_{\text{CM}} = \beta/\nu = 1$. In particular, the Ossola–Sokal conjecture (1.14) is violated when $1 \leq q < 2$ for the CM dynamics on the complete graph.

It therefore seems that the mechanism causing slowness of SW–CM in three dimensions is not related in any simple way to the size of the largest cluster, contrary to the intuition of Ossola and Sokal. And so we are back to square one: no one seems to have the slightest idea what is the physical mechanism dominating the critical slowing-down of the SW–CM dynamics in dimensions $d \geq 3$.

The present paper is organized as follows: In Section 2 we briefly review the Swendsen–Wang algorithm and then present a simple explanation of the Chayes–Machta algorithm. In Section 3 we define the observables and autocorrelation times that we measured in our simulations, while in Section 4 we discuss our methods of statistical data analysis. In Section 5 we summarize the characteristics of our MC simulations. In Section 6 we analyze our numerical data for the static observables, with an emphasis on detecting corrections to scaling and estimating their exponents and amplitudes. In Section 7 we analyze our numerical data for the dynamic quantities (i.e. autocorrelation times), with an emphasis on estimating the dynamic critical exponent $z_{\text{CM}}$ as a function of $q$; in particular we test the Ossola–Sokal conjecture and the sharpness of the Li–Sokal bound. Finally, in Section 8 we briefly discuss our results and the prospects for future work.

### 2 The Swendsen–Wang and Chayes–Machta algorithms

In this section we describe a family of algorithms for simulating the FK random-cluster model at any real $q \geq 1$, which generalize slightly the original algorithms introduced by Chayes and Machta [25]. We begin (Section [2.1]) by reviewing the Swendsen–Wang

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5 It is of course conceivable that $z_{\text{CM}}$ is discontinuous at $q = 1$: for instance, one might have $z_{\text{CM}} = \beta/\nu$ exactly for all $q$ near 1, but with an amplitude that vanishes as $q \downarrow 1$. In Section 7.5 we will test this scenario against our data.

6 Or at least, no such relation seems to hold for all $q$. It is still conceivable that such a relation might hold for $q = 2$ only, or for all $q \geq 2$. 

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algorithm for simulating the ferromagnetic $q$-state Potts model at integer $q \geq 2$, with emphasis on the perspective afforded by the joint probability distribution of spins and bonds (sometimes known as the Edwards–Sokal coupling [14]). We then present (Section 2.2) our version of the Chayes–Machta algorithm and a proof of its validity. For an alternative (and rather more general) presentation of the Chayes–Machta algorithm, see [27].

2.1 Swendsen–Wang algorithm

Let $G = (V, E)$ be a finite graph with vertex set $V$ and edge set $E$, and let $q$ be an integer $\geq 2$. The $q$-state Potts model on $G$ with nearest-neighbor couplings $\beta = \{\beta_e\}_{e \in E}$ is defined by the Gibbs measure

$$
\pi_{\beta,q}(\sigma) \propto \exp \left( \sum_{e \in E} \beta_e \delta_e(\sigma) \right) \propto \prod_{e \in E} \left[ (1 - p_e) + p_e \delta_e(\sigma) \right],
$$

for $\sigma \in \{1, 2, \ldots, q\}^V$, where $p_e = 1 - e^{-\beta_e}$ for each $e \in E$ and

$$
\delta_e(\sigma) = \delta_{\sigma_e, \sigma_y} \text{ for } e = \langle xy \rangle. \tag{2.2}
$$

One rather natural way to elucidate the Swendsen–Wang (SW) algorithm is via the Edwards–Sokal coupling [14] of the Potts and random-cluster models. Recall first that the random-cluster model [15], introduced by Fortuin and Kasteleyn [10–13], is a correlated bond-percolation model defined on a finite graph $G = (V, E)$ for parameter $q > 0$ (not necessarily an integer) and edge probabilities $p = \{p_e\}_{e \in E}$, by the probability measure

$$
\phi_{p,q}(n) \propto q^{k(n)} B_p(n), \tag{2.3}
$$

for $n \in \{0, 1\}^E$, where

$$
B_p(n) = \prod_{e \in E: n_e=1} p_e \prod_{e \in E: n_e=0} (1 - p_e), \tag{2.4}
$$

and $k(n)$ is the number of connected components (“clusters”) in the spanning subgraph whose edges are those with $n_e = 1$ (we call these “components of $n$” for short). The Potts and random-cluster models are intimately related, in the sense that correlation functions of the Potts spins can be expressed in terms of connectivity functions of the random-cluster model. The Edwards–Sokal coupling is a joint measure on the space of spin and bond configurations, defined by

$$
\mu_{p,q}(n, \sigma) \propto \prod_{e \in E} \left[ (1 - p_e)\delta_{n_e, 0} + p_e \delta_{n_e, 1}\delta_e(\sigma) \right]. \tag{2.5}
$$
It is easy to show \cite{5} that the marginal measure $\sum_n \mu_{p,q}(n, \cdot)$ on the spins is the Potts measure (2.1), and the marginal measure $\sum_\sigma \mu_{p,q}(\cdot, \sigma)$ on the bonds is the random-cluster measure (2.3). Explicit forms for the conditional measures are also easily computed:

\[
\mu_{p,q}(\sigma | n) = \Delta(n, \sigma) \frac{q^{-k(n)}}{B_p(n)} \quad (2.6)
\]

\[
\mu_{p,q}(n | \sigma) = \Delta(n, \sigma) B_p(n) \quad (2.7)
\]

where

\[
\Delta(n, \sigma) = \begin{cases} 
1 & \text{if } \delta_e(\sigma) = 1 \text{ for all } e \text{ with } n_e = 1 \\
0 & \text{otherwise} \end{cases} \quad (2.8)
\]

In words, $\Delta(n, \sigma)$ is the indicator for the event that “we draw occupied bonds only between vertices with the same spin value”.

The SW algorithm simulates the Edwards–Sokal measure (2.5), and hence both the Potts and random-cluster models, by alternately updating the spins conditioned on the bonds using (2.6) and the bonds conditioned on the spins using (2.7). In words, the spin-updating rule (2.6) states that, independently for each cluster, we choose a new spin value uniformly at random from the $q$ possible choices and assign this spin value to all vertices in the cluster; while the bond-updating rule (2.7) states that on each subgraph induced by the vertices of a given spin value we update the bonds via independent bond percolation.

### 2.2 Chayes–Machta algorithm

Consider the random-cluster model defined on a finite graph $G = (V,E)$ by (2.3). Recall that to arrive at the SW algorithm for the Potts model, one introduces auxiliary bond variables and then considers the joint model of Potts spins on the vertices and auxiliary bond variables on the edges. The SW algorithm updates this joint model by alternately applying its two conditional measures (bond variables given spins and spin variables given bonds). Here we do the reverse: starting from the random-cluster model (2.3), we introduce auxiliary color variables on the vertices of $G$ and then consider the joint model of our original bond variables and the new auxiliary color variables; we will then update this joint model by alternately applying its conditional measures. To this end, let us introduce color variables on the sites taking values in some finite set $S$, i.e. $\sigma \in S^V$. For each $\alpha \in S$, we choose a number $q_\alpha \geq 0$, and write $q = \{q_\alpha\}_{\alpha \in S}$. We can now introduce the coupled probability measure

\[
\mu_{p,q}(n, \sigma) \propto \Delta(n, \sigma) B_p(n) \prod_{\alpha \in S} q_\alpha^{k_\alpha(\sigma,n)} \cdot \quad (2.9)
\]

Here $\Delta(n, \sigma)$ is again given by (2.8) but now with $\sigma \in S^V$, i.e. it is the indicator of the event “we draw occupied bonds only between vertices of the same color”; and we denote by $k_\alpha(\sigma,n)$ the number of components of $n$ that are colored $\alpha$ by $\sigma$. Note that $k_\alpha(\sigma,n)$ is well-defined whenever $\Delta(n, \sigma) \neq 0$. From the definition (2.9) we can observe the following:
1) The marginal measure \( \sum_{\mu, \nu} \mu_{\nu}(\cdot, \sigma) \) on the bond variables is the random-cluster measure with parameter \( q = \sum_{\alpha \in S} q_{\alpha} \).

2) The conditional measure of \( \sigma \) given \( n \) is as follows: Independently for each component of \( n \), randomly choose a value \( \alpha \in S \) with probability \( q_{\alpha}/q \) and impose it on all vertices in that component.

3) Every coloring \( \sigma \) defines a partition of the vertex set into \( V = \bigcup_{\alpha \in S} V_{\alpha} \), where \( V_{\alpha} \) is the set of all sites \( v \in V \) colored \( \alpha \) by \( \sigma \). The conditional measure of \( n \) given \( \sigma \) is then simply the product over \( \alpha \in S \) of random-cluster measures with parameter \( q_{\alpha} \) on the induced subgraphs \( G[V_{\alpha}] \). All bonds not lying within a single \( G[V_{\alpha}] \) are forced to be unoccupied.

Now suppose that for each \( \alpha \in S \) and each induced subgraph \( H \subseteq G \), we have a valid Monte Carlo algorithm for updating the random-cluster model with parameter \( q_{\alpha} \) on \( H \). We can then apply the following algorithm to simulate the joint model (2.9), and therefore, via Observation 1, the random-cluster model with parameter \( q = \sum_{\alpha \in S} q_{\alpha} \):

**Algorithm 1 (Random-cluster algorithm).**

1. Given a bond configuration, choose a new color configuration using the conditional measure described in Observation 2.

2. Given the new color configuration, for each \( \alpha \in S \) update the random-cluster model on \( G[V_{\alpha}] \) with parameter \( q_{\alpha} \), using the given algorithm.

Note that for any \( \alpha \in S \), one valid Monte Carlo update is to do nothing (i.e. perform the identity operation), since detailed balance is satisfied trivially. Furthermore, if \( q_{\alpha} \) happens to equal 1, one can use a Bernoulli update, i.e. erase the bonds on \( G[V_{\alpha}] \) and choose new ones via independent bond percolation. But if one has some other way of updating a random-cluster measure with parameter \( q_{\alpha} \), then that is fine too. A number of special cases of Algorithm 1 are now clear:

1. If \( q \) is an integer, \( S = \{1, \ldots, q\} \), \( q_{\alpha} = 1 \) for all \( \alpha \), and one uses Bernoulli updates for every \( \alpha \), then Algorithm 1 is simply the standard SW algorithm.

2. If \( q \) is an integer, \( S = \{1, \ldots, q\} \), \( q_{\alpha} = 1 \) for all \( \alpha \), and one uses Bernoulli updates for \( 1 \leq \alpha \leq k \) and do-nothing updates for \( k + 1 \leq \alpha \leq q \) (where \( k \) is any integer satisfying \( 1 \leq k \leq q \)), then Algorithm 1 is a variant of the SW algorithm in which only the first \( k \) colors are “active”.

3. If \( q \geq 1 \), \( S = \{active, inactive\} \), \( q_{active} = 1 \), \( q_{inactive} = q - 1 \), and one uses a Bernoulli update for \( \alpha = active \) and a do-nothing update for \( \alpha = inactive \), then Algorithm 1 is the original Chayes–Machta algorithm presented in [25].

4. If \( q \geq k \geq 1 \) with \( k \) an integer, \( S = \{1, \ldots, k, inactive\} \), \( q_{\alpha} = 1 \) for \( 1 \leq \alpha \leq k \), \( q_{inactive} = q - k \), and one uses a Bernoulli update for \( 1 \leq \alpha \leq k \) and a do-nothing update for \( \alpha = inactive \), this is the generalized algorithm presented in [25, pp. 482–483].
Note that since each of these cases involves performing Bernoulli updates, they all require \( q \geq 1 \). For a given value of \( q \geq 1 \), Case 4 provides a family of algorithms, indexed by the integer \( k \), for simulating the random-cluster model. We call this “the Chayes–Machta algorithm with \( k \) active colors”. Any choice of \( 1 \leq k \leq \lfloor q \rfloor \) is legitimate, though it seems reasonable to expect that \( k = \lfloor q \rfloor \) should be the most efficient. Indeed, our numerical simulations suggest that in practice the autocorrelation time is approximately proportional to \( 1/k \) (see Section 7.1 below).

3 Observables and autocorrelation times

In this section we recall the definitions of the various autocorrelation times (Section 3.1) and then list the observables that we measured in our simulations (Section 3.2).

3.1 Autocorrelation functions and autocorrelation times

Consider an observable \( \mathcal{O} \) in the random-cluster model, i.e. a real-valued function of \( n \in \{0,1\}^E \). Then a realization of the Chayes–Machta (CM) Markov chain gives rise to a time series \( \mathcal{O}(t) \), where each unit of time corresponds to one step of the CM algorithm. The autocovariance function of \( \mathcal{O} \) is defined to be

\[
C_{\mathcal{O}\mathcal{O}}(t) = \langle \mathcal{O}(0)\mathcal{O}(t) \rangle - \langle \mathcal{O} \rangle^2,
\]

where the expectation is taken in equilibrium. The normalized autocorrelation function of \( \mathcal{O} \) is then

\[
\rho_{\mathcal{O}\mathcal{O}}(t) = \frac{C_{\mathcal{O}\mathcal{O}}(t)}{C_{\mathcal{O}\mathcal{O}}(0)}.
\]

From \( \rho_{\mathcal{O}\mathcal{O}}(t) \) we define the integrated autocorrelation time as

\[
\tau_{\text{int},\mathcal{O}} = \frac{1}{2} \sum_{t=-\infty}^{\infty} \rho_{\mathcal{O}\mathcal{O}}(t)
\]

and the exponential autocorrelation time as

\[
\tau_{\text{exp},\mathcal{O}} = \limsup_{|t| \to \infty} \frac{-|t|}{\log \rho_{\mathcal{O}\mathcal{O}}(t)}.
\]

Finally, the exponential autocorrelation time of the system is defined as

\[
\tau_{\text{exp}} = \sup_{\mathcal{O}} \tau_{\text{exp},\mathcal{O}},
\]

where the supremum is taken over all observables \( \mathcal{O} \). This autocorrelation time thus measures the decay rate of the slowest mode of the system. All observables that are not orthogonal to this slowest mode satisfy \( \tau_{\text{exp},\mathcal{O}} = \tau_{\text{exp}} \).

It is important to remember that there is not just one autocorrelation time, but many: namely \( \tau_{\text{exp}} \) as well as \( \tau_{\text{int},\mathcal{O}} \) for each \( \mathcal{O} \). In all but the most trivial Markov chains these
autocorrelation times are not equal. Correspondingly, there are many dynamic critical exponents: namely \( z_{\text{exp}} \) as well as \( z_{\text{int}, \mathcal{O}} \) for each \( \mathcal{O} \). These exponents may in some cases be equal, but they need not be; this is a detailed dynamical question, and the answer will vary from algorithm to algorithm and model to model.

More information on the principles of Markov-chain Monte Carlo and the relations between these autocorrelation times can be found in [5].

3.2 Observables to be measured

We now take the graph \( G \) to be the \( d \)-dimensional simple-hypercubic lattice of linear size \( L \) with periodic boundary conditions; we denote this graph by \( \mathbb{Z}_L^d \) in order to emphasize that addition of coordinates is always taken mod \( L \). When the precise form of the graph \( G \) is unimportant to our definitions, we denote the vertex set and edge set as simply \( V(G) \) and \( E(G) \), respectively.

For \( x_1, \ldots, x_m \in V(G) \), not necessarily all distinct, we denote by \( \gamma_{x_1 \cdots x_m} \) the indicator for the event that the vertices \( x_1, \ldots, x_m \) are all in the same cluster, i.e.

\[
\gamma_{x_1 \cdots x_m}(n) = \begin{cases} 
1 & \text{if } x_1 \leftrightarrow x_2 \leftrightarrow x_3 \leftrightarrow \cdots \leftrightarrow x_m \text{ in configuration } n \\
0 & \text{otherwise}
\end{cases}
\]  

(3.6)

and \( x \leftrightarrow y \) denotes that \( x \) is connected to \( y \) by at least one path of occupied bonds. In particular, \( \tau(x, y) = \langle \gamma_{xy} \rangle \) is the “two-point connectivity function”, i.e. the probability that \( x \) is connected to \( y \). On \( \mathbb{Z}_L^d \) we write \( \tau(x) = \tau(x, 0) \), and translational invariance gives us \( \tau(x, y) = \tau(x - y) \) for all \( x, y \in \mathbb{Z}_L^d \).

We measured the following observables in our MC simulations:

- The number of occupied bonds

\[
\mathcal{N} = \sum_{e \in E(G)} n_e 
\]  

(3.7)

- The nearest-neighbor connectivity (which is an energy-like observable [17])

\[
\mathcal{E}' = \sum_{(xy) \in E(G)} \gamma_{xy} 
\]  

(3.8)

- The cluster-size moments

\[
S_m = \sum_{C \in K(n)} |C|^m 
= \sum_{x_1, \ldots, x_m \in V(G)} \gamma_{x_1 \cdots x_m}(n),
\]  

(3.9a)

(3.9b)

where the size \( |C| \) of a cluster means the number of vertices. In this work we measured \( S_m \) for \( m = 2, 4, 6, 8; \)
• The size $C_i$ of the $i$th largest cluster. In this work we measured $C_i$ for $i = 1, 2, 3$;

• An observable used to compute the Fourier transform of $\tau(x, y)$ evaluated at the smallest non-zero momentum $(2\pi/L, 0, \ldots, 0)$:

$$
\mathcal{F}' = \frac{1}{d} \sum_{j=1}^{d} \sum_{x, y \in \mathbb{Z}_L^d} \gamma_{xy} e^{2\pi i (x_j - y_j)/L}
$$

(3.10a)

$$
= \frac{1}{d} \sum_{j=1}^{d} \sum_{C \in \mathcal{K}(n)} \left| \sum_{x \in C} e^{2\pi i x_j/L} \right|^2
$$

(3.10b)

From these observables we computed the following quantities:

• The bond density

$$
N = \frac{\langle N \rangle}{B},
$$

(3.11)

where $B = |E|$ is the total number of edges in the graph $G$ (i.e. $B = dL^d$ for $G = \mathbb{Z}_L^d$);

• The connectivity density

$$
E' = \frac{\langle \mathcal{E}' \rangle}{B}
$$

(3.12)

• The specific heat. There are a number of sensible definitions for specific heat, and we considered the following (see Remark 3.1 below):

$$
C_H^{(1)} = \frac{1}{B} \text{var}(N)
$$

(3.13)

$$
C_H^{(2)} = \frac{d}{p^2} [C_H^{(1)} - (1 - p) N]
$$

(3.14)

$$
C_H^{(3)} = \frac{q^2}{(q - 1)^2} C_H^{(2)}
$$

(3.15)

In this paper we used $C_H = C_H^{(2)}$ in agreement with [16,17].

• The expected size of the $i$th largest cluster,

$$
C_i = \langle C_i \rangle
$$

(3.16)

We expect that in the critical region $C_i \sim L^{d-\beta/\nu}$ as $L \to \infty$, for any fixed $i$;

• The susceptibility (see Remark 3.2 below)

$$
\chi = \frac{\langle S_2 \rangle}{L^d}
$$

(3.17)

In the random-cluster model, $\chi$ is the mean size of the cluster containing any specified point.
• The Fourier transform of the two-point connectivity function \( \tau(x) \), evaluated at the smallest nonzero momentum, \((2\pi/L, 0, \ldots, 0)\):

\[
F = \frac{\langle \mathcal{F}' \rangle}{L^d} = \tilde{\tau}(p) \bigg|_{p=(2\pi/L, 0, \ldots, 0)}
\]

(3.18)

• The finite-size second-moment correlation length (see Remark 3.3 below):

\[
\xi = \frac{1}{2 \sin(\pi/L)} \left( \frac{\chi}{F} - 1 \right)^{1/2}
\]

(3.19)

Remark 3.1. The definition \( C_H^{(1)} \) of the specific heat seems quite natural from the perspective of the random-cluster model, in which we view the bond variables as fundamental; in particular, it arises in a very direct way in the proof of the Li–Sokal bound for the CM algorithm [27, Appendix]. The definitions \( C_H^{(2)} \) and \( C_H^{(3)} \), by contrast, are designed to reduce to more familiar expressions in terms of the energy of the Potts spin model when \( q \) is an integer. Specifically, if \( q \) is an integer we have

\[
C_H^{(2)} = \frac{1}{V} \operatorname{var} \left( \sum_{\langle xy \rangle \in E} \delta_{\sigma_x, \sigma_y} \right)
\]

(3.20)

\[
C_H^{(3)} = \frac{1}{V} \operatorname{var} \left( \sum_{\langle xy \rangle \in E} \sigma_x \cdot \sigma_y \right)
\]

(3.21)

where \( \sigma_x \in \{1, 2, \ldots, q\} \) and \( \sigma_x \in \mathbb{R}^{q-1} \) is a Potts spin in the hypertetrahedral representation, so that

\[
\sigma_x \cdot \sigma_y = \frac{q \delta_{\sigma_x, \sigma_y} - 1}{q - 1}.
\]

(3.22)

Remark 3.2. We note that for \( q \in \{2, 3, \ldots\} \), the Fortuin–Kasteleyn identity \( \langle \sigma_x \cdot \sigma_y \rangle = \langle \gamma_{xy} \rangle \) shows that \( \langle \mathcal{M}^2 \rangle = \langle S_2 \rangle \), where \( \mathcal{M}^2 \) is the squared magnetization, and hence \( \chi \) is simply equal to the magnetic susceptibility.

Remark 3.3. In terms of \( \tilde{\tau}(p) \) the definition (3.19) has the general form

\[
\xi = \frac{1}{2 \sin(\pi/L)} \left( \frac{\tilde{\tau}(0, \ldots, 0)}{\tilde{\tau}(2\pi/L, 0, \ldots, 0)} - 1 \right)^{1/2}
\]

(3.23)

and applies equally well to the integer \( q \) case where we interpret \( \tau(x) \) as the spin-spin correlation function. A very nice account of the origin of the definition (3.23) of the finite-size correlation length, relating it to the thermodynamic second-moment correlation length is given in [36, Part III]; see also [37, 38].

The prime on \( \mathcal{F}' \) is employed to distinguish it from another estimator for \( F \), often denoted \( \mathcal{F} \), which can be defined in terms of Potts spins when \( q \in \{2, 3, \ldots\} \): see e.g. [17]. We note that \( \mathcal{F}' \) could profitably be used even in Swendsen–Wang simulations of the Potts model, as an alternative to \( \mathcal{F} \).
Remark 3.4. For any \( q > 0 \) and any \( \langle xy \rangle \in E(G) \) we have the identity
\[
q \langle n_{xy} \rangle = p [(q - 1)\langle \gamma_{xy} \rangle + 1].
\]
(3.24)
This is most easily seen by first considering the Edwards–Sokal joint measure when \( q \) is an integer \( \geq 2 \), and then combining the identities \( \langle n_{xy} \rangle = p(\delta_{\sigma_x, \sigma_y}) \) and \( \langle \gamma_{xy} \rangle = \langle \sigma_x \cdot \sigma_y \rangle \) together with (3.22). See e.g. [5]. This clearly proves (3.24) for all \( q \in \{2, 3, \ldots\} \); and since both sides are rational functions of \( q \), it follows that the equality must hold for all \( q \in \mathbb{C} \). Summing over all edges we then obtain
\[
\langle \mathcal{N} \rangle = p \frac{q - 1}{q} \langle \mathcal{E}' \rangle + \frac{p}{q} B.
\]
(3.25)

As a consistency check on the correctness of our simulations we numerically tested the identity (3.25) to high precision by measuring the observable
\[
\mathcal{N} - p \frac{q - 1}{q} \mathcal{E}' - \frac{p}{q} B.
\]
(3.26)
Clearly this observable should have mean zero, and in all our simulations this was observed to be the case within statistical errors.

4 Statistical analysis of the Monte Carlo data

Consider a generic observable \( \mathcal{O} \) with expectation \( \langle \mathcal{O} \rangle = \mu_\mathcal{O} \). Suppose that, after equilibration, we have a sequence of \( T \) Monte Carlo measurements of \( \mathcal{O} \), denoted \( \{\mathcal{O}_t\}_{t=1}^T \).

Since the Monte Carlo process is assumed equilibrated, the \( \{\mathcal{O}_t\}_{t=1}^T \) are a sample from a stationary stochastic process (also called a stationary time series). In this section we recall the basic principles of statistical time-series analysis, and describe the standard estimators for various quantities associated with \( \mathcal{O} \). For more details, see e.g. [5, 39].

The natural estimator for the expectation \( \mu_\mathcal{O} \) is the sample mean
\[
\overline{\mathcal{O}} = \frac{1}{T} \sum_{t=1}^T \mathcal{O}_t.
\]
(4.1)
This is an unbiased estimator and has a variance
\[
\text{var}(\overline{\mathcal{O}}) = \frac{1}{T^2} \sum_{s,t=1}^T C_{\mathcal{O}\mathcal{O}}(t - s) \quad (4.2a)
\]
\[
= \frac{1}{T} \sum_{t=-(T-1)}^{T-1} \left(1 - \frac{|t|}{T}\right) C_{\mathcal{O}\mathcal{O}}(t) \quad (4.2b)
\]
\[
\approx \frac{1}{T} 2 \tau_{\text{int}, \mathcal{O}} C_{\mathcal{O}\mathcal{O}}(0) \quad \text{for} \quad T \gg \tau_{\text{int}, \mathcal{O}}. \quad (4.2c)
\]
This implies that the variance of \( \overline{\mathcal{O}} \) is a factor \( 2 \tau_{\text{int}, \mathcal{O}} \) larger than it would be if the measurements were uncorrelated. Therefore, in order to obtain accurate error bars for
the estimator of the static quantity $\mu_O$, we must obtain an accurate estimate of the
dynamic quantity $\tau_{\text{int},O}$.

The natural estimator for the autocovariance function is
\[
\hat{C}_{OO}(t) = \frac{1}{T-|t|} \sum_{s=1}^{T-|t|} (O_s - \mu_O)(O_{s+t} - \mu_O)
\]
if the expectation $\mu_O$ is known, and
\[
\hat{C}_{OO}(t) = \frac{1}{T-|t|} \sum_{s=1}^{T-|t|} (O_s - \overline{O})(O_{s+t} - \overline{O})
\]
if $\mu_O$ is unknown. The estimator $\hat{C}_{OO}(t)$ is unbiased, and the bias of $\hat{C}_{OO}(t)$ is of order $1/T$. To leading order for $T \gg \tau_{\text{int},O}$ the covariance matrices of $\hat{C}_{OO}$ and $\hat{C}_{OO}^{\prime}$ are equal and it can be shown [39, 40] that

\[
\text{cov}(\hat{C}_{OO}(t), \hat{C}_{OO}(u)) = \frac{1}{T} \sum_{s=-\infty}^{\infty} [C_{OO}(s)C_{OO}(s+u-t) + C_{OO}(s+u)C_{OO}(s-t) + \kappa(t,s,s+u)] + o(1/T),
\]
where $t, u \geq 0$ and $\kappa$ is the connected 4-point autocorrelation function

\[
\kappa(r,s,t) = \langle (O_i - \mu_O)(O_{i+r} - \mu_O)(O_{i+s} - \mu_O)(O_{i+t} - \mu_O) \rangle
- C_{OO}(r)C_{OO}(t-s) - C_{OO}(s)C_{OO}(t-r) - C_{OO}(t)C_{OO}(s-r).
\]

Similarly, the natural estimator for the autocorrelation function is
\[
\hat{\rho}_{OO}(t) = \frac{\hat{C}_{OO}(t)}{\hat{C}_{OO}(0)}
\]
if $\mu_O$ is known, and
\[
\hat{\rho}_{OO}(t) = \frac{\hat{C}_{OO}(t)}{\hat{C}_{OO}(0)}
\]
if $\mu_O$ is unknown. Both the estimators $\hat{\rho}_{OO}(t)$ and $\hat{\rho}_{OO}(t)$ have bias of order $1/T$. The covariance matrices of $\hat{\rho}_{OO}$ and $\hat{\rho}_{OO}$ are the same to leading order for large $T$. If the process is Gaussian, this covariance matrix is given in the large $T$ limit by [39]

\[
\text{cov}(\hat{\rho}_{OO}(t), \hat{\rho}_{OO}(u)) = \frac{1}{T} \sum_{s=-\infty}^{\infty} [\rho_{OO}(s)\rho_{OO}(s+t-u) + \rho_{OO}(s+u)\rho_{OO}(s-t) + 2\rho_{OO}(t)\rho_{OO}(u)\rho_{OO}(s)] + o(1/T).
\]

If the process is not Gaussian, then there are additional terms proportional to the fourth cumulant $\kappa(s,t,t-u)$. The simplest assumption is to assume the stochastic process to
be “not too far from Gaussian” and simply drop all the terms involving $\kappa$. This is what we will do. If this assumption is not justified, then we are introducing a bias into the estimate of $\text{cov}(\hat{\rho}_{OO}(t), \hat{\rho}_{OO}(u))$.

Finally, we take the estimator for the integrated autocorrelation time to be

$$\hat{\tau}_{\text{int},O} = \frac{1}{2} \sum_{t=-M}^{M} \hat{\rho}_{OO}(t)$$

(4.10)

if $\mu_O$ is known, or the analogous object defined in terms of $\hat{\rho}_{OO}$ is $\mu_O$ is unknown; here $M (\leq T - 1)$ is a suitably chosen number, which we call the window width. One’s first thought might be to use all the data, i.e. take $M = T - 1$; but this turns out to be a very bad estimator, which has a variance of order 1 even as $T \rightarrow \infty$. Loosely speaking (see e.g. [39, 41]), this is because the terms in $\hat{\rho}_{OO}(t)$ with large $t$ (namely, $t \gg \tau_{\exp,O}$) have variance of order $1/T$ that does not vanish as $t$ grows, cf. (4.9), and there are of order $T$ of them. These terms thus contribute much “noise” but very little “signal”, since $\rho_{OO}(t)$ is tiny when $t \gg \tau_{\exp,O}$. To obtain a good estimator, we should instead choose the window width $M$ to be large enough so that we do not lose too much “signal”, i.e. $\rho_{OO}(t)$ is tiny for all $t > M$, but not too much larger than this. In general, a window of width $M$ creates a bias given by

$$\text{bias}(\hat{\tau}_{\text{int},O}) = -\frac{1}{2} \sum_{|t|>M} \rho_{OO}(t) + o(1/T) .$$

(4.11)

The variance of the estimator $\hat{\tau}_{\text{int},O}$ can be computed from (4.9); assuming that $\tau_{\text{int},O} \ll M \ll T$, the final result is

$$\text{var}(\hat{\tau}_{\text{int},O}) \approx \frac{2(2M + 1)}{T} \tau_{\text{int},O}^2 .$$

(4.12)

To choose the window width $M$ we used the automatic windowing algorithm introduced in [41], in which one sets

$$M = \min\{m \in \mathbb{Z} : m \geq c \hat{\tau}_{\text{int},O}(m)\}$$

(4.13)

where $c$ is a suitably chosen constant. If the normalized autocorrelation function is approximately a pure exponential, then a choice in the range $c \approx 6 - 8$ is reasonable. Indeed [17], if we take $\rho_{OO}(t) = e^{-|t|/\tau}$ and minimize the mean-square error

$$\text{MSE}(\hat{\tau}_{\text{int},O}) \equiv \text{bias}(\hat{\tau}_{\text{int},O})^2 + \text{var}(\hat{\tau}_{\text{int},O})$$

(4.14)

using (4.11)/(4.12), we find that the optimal window width is

$$M_{\text{opt}} = \frac{\tau}{2} \log \left( \frac{n}{2\tau} \right) - 1 .$$

(4.15)

For $n/\tau \approx 10^4, 10^5, 10^6, 10^7, 10^8$ with $\tau \gg 1$, we have $M_{\text{opt}}/\tau \approx 4.26, 5.41, 6.56, 7.71, 8.86$, respectively.
In this paper we chose $c = 6$ for the observable $E'$, which has the slowest-decaying autocorrelation function of all the observables we measured and whose autocorrelation function is in fact very close to a pure exponential. We then used the same window width $M$, computed by applying the automatic windowing algorithm to $E'$, for all the other observables. We think that this procedure is preferable to applying the automatic windowing algorithm directly to the other observables, since some of the latter have an autocorrelation decay that is far from a pure exponential (this is especially so for $F'$, $C_2$ and $C_3$).

**Remark 4.1.** If $\{O^{(1)}, \ldots, O^{(m)}\}$ is a family of observables, we estimate composite observables of the form $f(\langle O^{(1)} \rangle, \ldots, \langle O^{(m)} \rangle)$ by $f(\overline{O^{(1)}}^{}, \ldots, \overline{O^{(m)}}^{})$ and approximate the corresponding variance by assuming the fluctuations from the mean are small, so that

$$\text{var } f(\overline{O^{(1)}}^{}, \ldots, \overline{O^{(m)}}^{}) \approx g^2 \text{var } Z,$$

where

$$Z_t = \sum_{i=1}^{m} a_i O^{(i)}_t \quad (4.17)$$

$$a_i = \frac{1}{g(\langle O_1 \rangle, \ldots, \langle O_m \rangle)} \frac{\partial f}{\partial x_i}(\langle O_1 \rangle, \ldots, \langle O_m \rangle) \quad (4.18)$$

and $g = g(\langle O_1 \rangle, \ldots, \langle O_m \rangle)$ is just a convenient common factor of the derivatives of $f$, which may be simply 1 in practice. Obviously in practice we compute $a_i$ using $\overline{O}_i$ in place of $\langle O_i \rangle$. We used this method to compute the estimate and standard error of $\xi$.

In principle we can also estimate the specific heat using this procedure, but in practice it turns out that a better estimator for the specific heat is provided by

$$\hat{C}_H = \frac{1}{B} (\mathcal{N} - \langle \mathcal{N} \rangle)^2.$$  

(4.19)

The mean of (4.19) is clearly $C_H^{(1)}$ [cf. (3.13)], and its variance determines the error bar on the resulting estimate of $C_H^{(1)}$. We prefer this estimator since it avoids computing the quantity $\overline{N^2} - \overline{N^2}$ which can be a numerically perilous object when $\overline{N^2}$ and $\overline{N^2}$ are close in value.

**Remark 4.2.** For the larger values of $L$ we actually performed a number of independent runs rather than one long run. The best estimate of each autocorrelation function $\rho_{OO}(t)$ was then computed, for each fixed $t$, by averaging the $\hat{\rho}_{OO}(t)$ from the individual runs, with weights proportional to the run lengths. The windowing procedure described above was then applied to this best estimate of $\rho_{OO}(t)$ in order to obtain the final value for the integrated autocorrelation time.
5 Description of the simulations

We implemented the Chayes–Machta (CM) algorithm for the random-cluster model on an \( L \times L \) square lattice with periodic boundary conditions. We performed all our runs at the exact critical temperature \( p_c = \sqrt{\gamma}/(1 + \sqrt{\gamma}) \). We studied lattice sizes \( 16 \leq L \leq 1024 \) in powers of 2, and parameters \( 1.25 \leq q \leq 4 \) in steps of 0.25. For each \((q, L)\) pair we studied all values of \( k \) (the number of active colors) in the range \( 1 \leq k \leq \lfloor q \rfloor \). We thus performed 189 runs in total.

For each triplet \((q, k, L)\) we performed between \( 2 \times 10^7 \) and \( 10^8 \) CM iterations: more precisely, we performed \( 10^8 \) iterations for \( 16 \leq L \leq 256 \), \( 5 \times 10^7 \) iterations for \( L = 512 \), and \( 2 \times 10^7 \) iterations for \( L = 1024 \). In terms of the autocorrelation time \( \tau_{\text{exp}} \approx \tau_{\text{int,E'}} \), our total data set at each triplet \((q, k, L)\) ranges in length from \( \approx 7 \times 10^7 \tau \) on the smallest lattices at small \( q \) (i.e. \( L = 16, q = 1.25 \)) to \( \approx 7 \times 10^5 \tau \) on the largest lattices at large \( q \) (i.e. \( L = 1024, q = 4, k = 1 \)); in 90% of cases (171 out of 189 runs) our run length is at least \( 10^5 \tau \). These statistics are high enough to permit a high accuracy in our estimates of the static (error \( \lesssim 0.1\% \)) and dynamic (error \( \lesssim 1\% \)) quantities, except for the largest lattices at large \( q \).

Our results for the principal static observables, obtained by combining the data for all available \( k \) values for each pair \((q, L)\), are reported in Tables 3–7. Our results for the dynamic quantities \( \tau_{\text{int,O}} \) for the case \( k = 1 \) and the most important observables \( O \) are reported in Tables 8–14. The complete set of static and dynamic data, for all observables and all values of \( k \), is contained as a file \texttt{cm2data.tar.gz} in the preprint version of this paper at \texttt{arXiv.org}.

The initial configuration of each run was all-bonds-occupied, except for the run at \((q, k, L) = (3.75, 3, 1024)\), for which it was all-bonds-vacant. The number of iterations discarded at the beginning of each run in order to allow the system to reach equilibrium was \( j \times 10^5 \) for \( j = 1, 2, 3 \) or 4, except for eight runs \((2 \leq q \leq 2.75, L = 256 \text{ and } k = 1, 2)\) for which it was \( 10^7 \). Thus, the discard interval was in all cases at least \( 100 \tau_{\text{int,E'}} \) and in most cases at least \( 1000 \tau_{\text{int,E'}} \). Since our data suggest that \( \tau_{\text{exp}} \) is very close to \( \tau_{\text{int,E'}} \), it follows that our discard interval was in all cases at least \( 100 \tau_{\text{exp}} \), which is more than sufficient for the systematic error from thermalization to be negligible.

Our program was written in Fortran 77 and run on a 1266 MHz Pentium III Tualatin processor using the g77 Fortran compiler. Our program requires approximately \((12 + 26d)L^d\) bytes memory for a lattice of linear size \( L \) in \( d \) dimensions. The CPU time required by our program was approximately \( 0.35 \mu s/\text{iteration/lattice site} \); this value is weakly dependent on the lattice size and on \( k \) and \( q \). In particular, the CPU time per site rises sharply on very small lattices due to the “fixed costs” of the algorithm (i.e. those that

\[\tau_{\text{exp}} \gg \tau_{\text{int,E'}}.\]

---

7 The \texttt{tar} file unpacks to make a directory \texttt{cm2data} with subdirectories \texttt{static} and \texttt{dynamic}. Each of these subdirectories is in turn subdivided according to the value of \( q \) and, in the dynamic case, the value of \( k \). Individual files have names like \texttt{C1Q1.25.txt} or \texttt{tau_C1Q4_K1.txt} and have three fields on each line: \( L \), value and error bar.

8 Please don’t ask why the discard interval was chosen in this erratic way — we don’t remember anymore!

9 Unless there exists a vastly slower mode of which we are unaware, so that in fact \( \tau_{\text{exp}} \gg \tau_{\text{int,E'}}. \)
do not scale with the volume); it rises gradually on lattices \( L \gtrsim 128 \) due to “cache misses” coming from the nonlocal nature of the Chayes–Machta algorithm, when the lattice no longer fits in the computer’s 512 KB cache. The complete set of runs reported in this paper used approximately 14 yr CPU time.

One delicate issue concerns the choice of the pseudo-random-number generator. We learned by bitter experience \cite{42} that linear congruential pseudo-random-number generators can cause systematic errors in Monte Carlo simulations using the Swendsen–Wang (or Chayes–Machta) algorithm, if the lattice size is a multiple of a very large power of 2 and one random number is used per bond in a periodic manner. These systematic errors arise from correlations within a single bond-update half-sweep: see \cite{42} for details. These errors can be eliminated (or at least radically reduced) by updating the bonds in a random order or in an aperiodic manner, and by using a generator of large modulus (e.g. 60 or more bits). In the present project, therefore, we used a linear congruential generator

\[
x_{n+1} = ax_n + c \pmod{m}
\]

with modulus \( m = 2^{64} \), increment \( c = 1 \), and multiplier \( a = 3202034522624059733 \). This multiplier gives good results on the spectral test in low dimensions \cite{43}. Moreover — and perhaps even more importantly — we used an “aperiodic” updating, in which the random-number subroutine is called only if the two spins are equal and of an “active color”, i.e. for an edge \( e = \langle xy \rangle \)

\[
\begin{align*}
  \text{if } \sigma_x = \sigma_y \leq k \text{ then} \\
    \text{if } \text{ran}() \leq p \text{ then} \\
    n_e \leftarrow 1 \\
  \text{else} \\
    n_e \leftarrow 0 \\
  \text{endif} \\
  \text{endif}
\end{align*}
\]

To the best of our knowledge \cite{42} these choices suffice to make the systematic errors negligible.

6 Data analysis: Static quantities

Our general methodology for analyzing the Monte Carlo data (both static and dynamic) is as follows: For each quantity \( Q \) of interest we impose an ansatz of the form

\[
Q(L) = A_1 L^{p_1} + A_2 L^{p_2} + \ldots + A_k L^{p_k}
\]

(or a logarithmic variant thereof), where some of the parameters may be fixed and others free. The precise ansatz will be motivated in each case by finite-size-scaling theory, possibly together with some exactly known exponents. Typically our ansätze will have 2–4 free parameters. We then perform the nonlinear fits corresponding to the chosen ansatz by using MATHEMATICa’s function \texttt{NonlinearModelFit}. As a precaution against
correction-to-scaling terms that we have failed to include in our chosen ansatz, we impose a lower cutoff $L \geq L_{\text{min}}$ on the data points admitted in the fit, and we systematically study the effect on the $\chi^2$ value of increasing $L_{\text{min}}$. In general, our preferred fit for any given ansatz corresponds to the smallest $L_{\text{min}}$ for which the goodness of fit is reasonable and for which subsequent increases in $L_{\text{min}}$ do not cause the $\chi^2$ value to drop by vastly more than one unit per degree of freedom. In practice, by “reasonable” we mean that the confidence level is $\gtrsim 10$–20%\protect\footnote{“Confidence level” is the probability that $\chi^2$ would exceed the observed value, assuming that the underlying statistical model is correct. An unusually low confidence level (e.g. less than 5%) thus suggests that the underlying statistical model is incorrect; in our context it suggests that the terms we have chosen to retain in the ansatz (6.1) are insufficient to explain our data, i.e. there are unincuded corrections to scaling whose contribution is comparable to or larger than the statistical errors in our data.}. We do not allow fits with zero degrees of freedom, since there would then be no way of testing the goodness of fit. As a last step, we consider the effect of including different terms in the ansatz, by comparing the fits obtained with different ansätze.

The exact values of the static critical exponents are of course already known [cf. (1.5)–(1.10)]. Our analysis of the static Monte Carlo data therefore has three major goals:

1. To test how accurately we would be able to estimate the static critical exponents if we did not know them exactly. Here our knowledge of the exact values allows us to evaluate systematic errors (including those due to unknown causes) as well as statistical ones. This study of the static critical exponents serves as a comparison case for our subsequent study of the dynamic critical exponents (Section 7 below). In particular, the behavior of the autocorrelation times is closely related to that of the specific heat, as is evident from the Li–Sokal bound (1.11) and the empirical fact of its near-sharpness.

2. To estimate universal amplitude ratios such as $x^* = \lim_{L \to \infty} \xi(L)/L$.

3. To study the structure of the corrections to scaling. In particular, are we able to see the leading non-analytic correction-to-scaling term $L^{-\Delta_1}$ with exponent given by (1.6)? And what additional corrections are present? Depending on the observable, one might expect to see analytic corrections such as $L^{-1}$ or regular background contributions.

We begin by summarizing the finite-size-scaling (FSS) ansätze that motivate our fits (Section 6.1). We then study successively the correlation length $\xi$ (Section 6.2), the susceptibility $\chi$ (Section 6.3), the largest cluster $C_1$ (Section 6.4), and the specific heat $C_H$ (Section 6.5).

### 6.1 Corrections to finite-size-scaling

Let us work exactly at the critical point, and consider a quantity $Q(L)$ whose asymptotic behavior as $L \to \infty$ is

$$Q(L) = AL^{\psi}(1 + \text{corrections}),$$

(6.2)
where $\psi > 0$ is the leading critical exponent and $A$ is the leading critical amplitude. Then finite-size-scaling (FSS) theory predicts the following types of contributions to the “corrections” in (6.2):

1. “Non-analytic” corrections to scaling $L^{\psi}$ coming from irrelevant scaling fields, i.e. those with $y_i < 0$. The leading such contribution is given by $y_i = y_{T2} = -\Delta_1$ [cf. (1.6)]. There are also higher-order “non-analytic” terms of the general form $L^{y_i+y_j+\ldots}$, but these are so far down as to be undetectable in our data.

2. “Analytic” corrections to scaling coming from nonlinear mixing between the thermal and magnetic scaling fields. The leading such contribution behaves as $L^{y_{T1}-2y_{H1}} = L^{(\alpha-1-\gamma)/\nu}$.

3. A regular background term $L^{-\psi}$ [i.e. a contribution of order $L^{0}$ to $Q(L)$].

4. Corrections with negative integer exponents, i.e. $L^{-1}$, $L^{-2}$, etc. Some theoretical frameworks [44] suggest that these contributions are absent.

Looking at the known exact values (1.5)–(1.10) of these exponents (see Table 2), we can predict the pattern of corrections to scaling for different observables:

- For strongly divergent quantities such as the susceptibility ($\psi = \gamma/\nu \approx 1.75$) and the mean cluster size ($\psi = d - \beta/\nu \approx 1.88$), the strongest correction to scaling at large $L$ is expected to come from the leading irrelevant term $L^{-\Delta_1}$ whenever $q > (3 + \sqrt{5})/2 \approx 2.618$; indeed, this behavior will hold for all $q \gtrsim 1.294$ (resp. $q \gtrsim 1.136$) if the $L^{-1}$ correction is absent. However, for $q \lesssim 1.5$ the irrelevant, regular-background and $L^{-2}$ contributions are all of roughly the same order and will be extremely difficult to disentangle.

- The same pattern is expected to hold for the correlation length, since it is built out of the more fundamental quantities $\chi$ and $F$ [cf. (3.19)] and therefore inherits their corrections to scaling (we expect $F$ to behave like $\chi$ in this context).

- For weakly divergent quantities such as the specific heat for $q \geq 2$, the strongest correction to scaling at large $L$ is expected to come from the regular background term when $2 \leq q < 2 + \sqrt{2} \approx 3.414$ and from the leading irrelevant term when $q > 2 + \sqrt{2}$.

---

11 For the specific heat we have $\psi = \alpha/\nu < 0$ when $q < 2$. The same formulae will apply in this case, but the meanings of “dominant” and “subdominant” will be interchanged.

12 See e.g. [16, Section 3] for a review and citations to the original literature. We are also indebted to Youjin Deng for extremely helpful discussions of these points.

13 See [16, Section 3.2] for further discussion. Interestingly, a recent study [45] of percolation on two-dimensional lattices, using both transfer-matrix and Monte Carlo methods, found no evidence of any $L^{-1}$ correction but found indirect evidence of an $L^{-2}$ correction (namely, an apparent term $L^{-2} \log L$ that could result from mixing between the $L^{-2}$ and $L^{y_{T2}} = L^{-2}$ terms when $q = 1$).
For non-divergent quantities such as the specific heat for \( q < 2 \), the dominant contribution is the regular background, and the leading singular term \( L^{\alpha/\nu} \) becomes the principal subdominant contribution.

One very interesting question concerns the presence or absence of the correction-to-scaling term \( L^{-\Delta_1} = L^{-4/3} \) in the case of the Ising model \( (q = 2) \). Plausible physical arguments \([46]\) suggest that such a correction should be present in at least some models in the Ising universality class, even though special symmetries might cause it to be absent in simple exactly-soluble models such as the nearest-neighbor square-lattice model. A transfer-matrix study of the square-lattice random-cluster model for \( q \approx 2 \) suggested \([46]\) that such a correction is indeed present for all \( q \neq 2 \) but has an amplitude that vanishes linearly with \( q - 2 \) when \( q \to 2 \). If this is the case, then this correction will be observable in derivatives \( d/dq \) of standard Potts-model observables evaluated at \( q = 2 \). Note also that such derivatives can be expressed as bond observables in the random-cluster model; therefore, in this scenario, at least some bond observables in the \( q = 2 \) random-cluster model will exhibit the correction-to-scaling term \( L^{-4/3} \).

6.2 The correlation length \( \xi \)

Finite-size-scaling (FSS) theory predicts that \( \xi(L)/L \) for the random-cluster model on a torus tends to a universal value \( x^*(q) \) as \( L \to \infty \)\(^{14}\) More precisely, we expect the behavior

\[
\frac{\xi}{L} = x^* + BL^{-p} + \ldots \tag{6.3}
\]

where \( p > 0 \) is one of the correction exponents discussed in the preceding subsection. For \( q = 2 \), the exact value of \( x^* \) is known from conformal-invariance theory \([47],[48]\) together with numerical integration \([49]\) to be

\[
x^*(q = 2) = 0.9050488292 \pm 0.0000000004 . \tag{6.4}
\]

For \( q \neq 2 \) the exact value of \( x^* \) is unknown. In this section we will use our Monte Carlo data to estimate the universal values \( x^*(q) \) and to test the agreement of the corrections to scaling with the FSS ansatz \((6.3)\).

In Table \([15]\) we report fits of \( \xi/L \) to the ansätze \( x^* \) and \( x^* + BL^{-p} \). For \( q = 1.25 \) and \( 1.50 \), the fits to both ansätze are excellent, and the estimated values of \( p \) agree closely with the known exact values of \( \Delta_1 \); the correction-to-scaling amplitude \( B \) is positive and more than three standard deviations away from zero. In particular, there is no evidence of any \( L^{-1} \) correction. For \( q = 1.75 \), the behavior is similar but the quality of the fit is poorer (possibly due to a statistical fluctuation that made the \( L = 1024 \) point \( 2\sigma \) too high). For \( 2 \leq q \leq 3.25 \), the fits to a constant are excellent provided that \( L_{\text{min}} \) is chosen large enough, but the fits to the ansatz \( x^* + BL^{-p} \) are mostly badly behaved: either they fail completely to converge (as for \( q = 3.25 \)), or they exhibit estimates for \( p \) that seem much too low (\( q = 2 \)) or much too high (\( q = 2.5, 3 \)) or somewhat too high (\( q = 2.25 \));

\(^{14}\) In general this value depends on the aspect ratio of the torus. Here we are considering only the case of aspect ratio 1.
only \( q = 2.75 \) behaves well in these respects. Also, the estimated correction amplitudes \( B \) are mostly within two standard deviations of zero. Examination of plots of these fits (see Figure 1) suggests that the problem arises because \( \xi/L \) is a very flat and possibly nonmonotonic function of \( L \) at large \( L \) in this range of \( q \).\(^\text{15}\) This suggests that the ansatz (6.3) might be too simple, and that we are seeing the combined effect of two (or more) correction-to-scaling terms of opposite sign. However, our statistical errors are too large for us to resolve this question clearly. For \( 3.5 \leq q \leq 4 \), by contrast, the fits to both ansätze are again excellent, and the estimated values of \( p \) agree well with the known exact values of \( \Delta_1 \) (except at \( q = 4 \), where the corrections to scaling \( \log \log L / \log L \) and \( 1 / \log L \) \(^\text{19}\) are mimicked by a small inverse power of \( L \)). Moreover, the correction-to-scaling amplitude \( B \) is now clearly negative. This suggests that \( B \) passed through zero somewhere in the range \( 2 \lesssim q \lesssim 3.4 \) and was small throughout this range, thereby explaining why our fits to \( x^* + BL^{-p} \) were so unstable.

It is curious that our estimate for \( x^*(q = 2) \) agrees slightly less well in absolute terms with the exact value \(^\text{6.4}\) than the estimate a decade ago by Salas and Sokal \(^\text{49}\), despite the fact that our raw-data error bars are a factor of 3–7 smaller than theirs. Most likely we were the unlucky victim of a statistical fluctuation on our \( L = 512 \) and \( L = 1024 \) data points that placed them roughly \( 2–3\sigma \) and \( 1\sigma \) too low, respectively, thereby causing us to choose \( L_{\text{min}} = 512 \) rather than 256 in the fit to a constant and hence to privilege these discrepant values [see Figure 1(b)].

### 6.3 The susceptibility \( \chi \)

In Table 16 we report fits of the susceptibility \( \chi \) to the pure-power-law ansatz \( AL^{\gamma/\nu} \), just to see how accurately we would be able to estimate the exponent \( \gamma/\nu \) if we did not know its exact value. The last two columns show the deviation of the estimated \( \gamma/\nu \) from the known exact value \(^\text{1.10}\), in absolute terms and in units of its standard deviation.

For small \( q \) we are able to estimate \( \gamma/\nu \) with extraordinary accuracy: both statistical and systematic errors are of order 0.0001 or less. However, as \( q \) grows, both the systematic and the statistical errors grow: the statistical errors grow because the critical slowing-down is becoming gradually worse (see Section 7 below); and the systematic errors grow (even faster than the statistical errors) because the corrections to scaling are becoming much stronger as the correction-to-scaling exponent \( \Delta_1 \) decreases (see Table 2). As a consequence, at \( q \approx 3 \) the statistical and systematic errors are of order 0.0003 and 0.001, respectively; at \( q = 3.75 \) the errors are roughly twice this; and at \( q = 4 \) the statistical and systematic errors are of order 0.0015 and 0.01, respectively. (The large systematic errors at \( q = 4 \) are of course not surprising in view of the multiplicative logarithmic correction \( (\log L)^{-1/8} \) and the additive logarithmic corrections \( \log \log L / \log L, 1 / \log L \), etc. \(^\text{19}\). Indeed, what is somewhat surprising is that the systematic errors are not larger! See Section 6.5 below for the case of the specific heat.)

We next imposed the known exact value \(^\text{1.10}\) of \( \gamma/\nu \) and attempted to extract the corrections to scaling. Table 17 shows fits of \( \chi/L^{\gamma/\nu} \) to the ansätze \( A \) and \( A + BL^{-p} \).

\(^{15}\) Or, perhaps, the statistical errors are just too large, compared to the small “signal”, for us to estimate the corrections to scaling accurately.
Plots for some selected values of $q$ are shown in Figure 2. For $q = 1.25$ and 1.5, the correction to scaling at small $L$ is clearly positive in sign, though very small in absolute magnitude (of order 0.001 at $L = 16$): see Figure 2(a). This suggests that the correction-to-scaling exponent is fairly large, in agreement with the theoretical prediction. However, the fits produce unusually small estimates for the correction exponent $p$; this is a possible behavior of the “effective exponent” when there are two correction-to-scaling terms of opposite sign (e.g. $L^{-1}$ and $L^{-\Delta_1}$). For $q = 1.75$ and 2, the corrections to scaling are small and erratic and it was not possible to fit them to a single inverse power; indeed, for $q = 2$ the behavior may be nonmonotonic in $L$ [see Figure 2(b)], suggesting again the presence of two correction-to-scaling terms of opposite sign. For $q = 3.75$ and 4, the corrections to scaling are small and erratic and it was not possible to fit them to a single inverse power. In summary, for $q \geq 2.25$ we obtain modest evidence that a correction $L^{-\Delta_1}$ is present, with negative amplitude. For $q \leq 2$ we are unable to say much about the corrections to scaling except that the amplitude is probably positive for $q \leq 1.5$ (we say “probably” because it is not clear that the positive contribution at small $L$ comes from the same correction-to-scaling term that is dominant at large $L$). But our data are at least compatible with the scenario that the correction-to-scaling amplitude passes through zero at $q = 2$.

### 6.4 The largest cluster $C_1$

In Table 18 we report fits of the mean size of the largest cluster, $C_1 = \langle C_1 \rangle$, to the pure-power-law ansatz $AL^{d-\beta/\nu}$, just to see how accurately we would be able to estimate the exponent $\beta/\nu$ if we did not know its exact value. The last two columns show the deviation of the estimated $\beta/\nu$ from the known exact value (1.9), in absolute terms and in units of its standard deviation.

The results are extremely similar to those obtained for the susceptibility. For small $q$, both the statistical and the systematic errors are of order 0.0001 or less. As $q$ grows, the statistical errors grow and the systematic errors grow even more. But the results are remarkably good except at $q = 4$.

### 6.5 The specific heat $C_H$

In Table 19 we report fits of the specific heat $C_H$ to the ansätze $AL^{\alpha/\nu}$ and $AL^{\alpha/\nu} + B$. The exact value (1.8) of $\alpha/\nu$ is shown for comparison in the next-to-last column. The last two columns show the deviation of the estimated $\alpha/\nu$ from the known exact value (1.8), in absolute terms and in units of its standard deviation.

For $q \lesssim 2.75$ the fits to a pure power law are horrible. This is not surprising: after all, for $q < 2$ the specific heat increases to a finite value as $L \to \infty$, so the fit will indicate a positive value of $\alpha/\nu$ (but with a ridiculously poor goodness of fit) while the true value of
\( \alpha/\nu \) is actually negative. And for \( 2 < q \lesssim 2.75 \) the additive constant is very important, so a pure power law has poor goodness of fit.

Good fits to \( AL^{\alpha/\nu} + B \) can be obtained for all values of \( q \) other than \( q = 2 \) by appropriate choice of \( L_{\text{min}} \). For \( q = 2 \) the fit fails to converge: initial guesses with \( A > 0, B < 0 \) and \( \alpha/\nu \) slightly positive always get driven to \( \alpha/\nu \downarrow 0 \) with \( A \approx -B \uparrow +\infty \). The fit is therefore suggesting the correct behavior \( C_H \approx A \log L + B \). Indeed, a fit to this ansatz is good already for \( L_{\text{min}} = 64 \) and yields

\[
A = 0.6371(9), \quad B = 0.1732(42) \quad (\chi^2 = 1.49, 3 \text{ DF, CL} = 68%), \quad (6.5)
\]
in good agreement with the known exact values \( A = 2/\pi \approx 0.6366 \) and \( B \approx 0.1778 \) [50, 51].

If we fit instead to \( A \log^2 L + B \log L + C \), we get a good fit for \( L_{\text{min}} = 32 \),

\[
A = -0.0012(6), \quad B = 0.6493(55), \quad C = 0.1415(125) \quad (\chi^2 = 2.36, 3 \text{ DF, CL} = 50%), \quad (6.6)
\]
in which the extremely small value of \( A \) correctly suggests that the \( \log^2 L \) term is absent.

Although the fits to \( AL^{\alpha/\nu} + B \) are good for all \( q \neq 2 \), some of the estimated values of \( \alpha/\nu \) deviate significantly from the known exact value (see the last two columns of Table 19). For \( q < 2 \) the estimates of \( \alpha/\nu \) are surprisingly good, given that we are estimating a subleading singular contribution underneath a nonsingular background. For \( 2.25 \leq q \leq 2.75 \) the estimates of \( \alpha/\nu \) are also excellent. However, for \( q > 3 \) the estimates of \( \alpha/\nu \) deviate by more than three standard deviations from the exact value. This may be due to the effect of corrections to scaling: in particular, those governed by the exponent \( \Delta_1 \), whose known exact value is given by (1.6). Indeed, for \( q > 2 + \sqrt{2} \approx 3.414 \) \((g > 7/2)\) we have \( \Delta_1 < \alpha/\nu \), so that the correction-to-scaling contribution \( L^{\alpha/\nu-\Delta_1} \) is larger than the nonsingular background. For \( 3 \leq q \leq 3.75 \) we therefore tried fits to the ansätze \( AL^{\alpha/\nu} + BL^{\alpha/\nu-\Delta_1} \) and \( AL^{\alpha/\nu} + BL^{\alpha/\nu-\Delta_1} + C \), in which \( \Delta_1 \) is fixed at its known exact value and \( \alpha/\nu \) is free:

(i) For \( q = 3 \) it is silly not to include the constant background, since \( \alpha/\nu - \Delta_1 = -0.4 \ll 0 \); nevertheless, we obtain a good fit to \( AL^{\alpha/\nu} + BL^{\alpha/\nu-\Delta_1} \) when \( L_{\text{min}} = 64 \):

\[
\alpha/\nu = 0.4152(15), \quad A = 1.397(14), \quad B = -2.346(153) \quad (\chi^2 = 0.75, 2 \text{ DF, CL} = 69%). \quad (6.7)
\]

However, the estimate for \( \alpha/\nu \) is 10 standard deviations away from the correct value! If we include the constant background we get a good fit already when \( L_{\text{min}} = 16 \),

\[
\alpha/\nu = 0.4036(34) \quad A = 1.555(44), \quad B = -0.787(188), \quad C = -0.795(152) \quad (\chi^2 = 1.07, 3 \text{ DF, CL} = 78%) \quad (6.8)
\]

and the estimate for \( \alpha/\nu \) is in excellent agreement with the correct answer.

---

\[16\] These values of \( A \) and \( B \) can be extracted from [50] or [51] after translating their conventions to ours. We use the definition \( C_H^{(2)} \) of specific heat [cf. (3.14)]. Salas [51] uses the definition \( C_H^{(3)} \) [cf. (3.15)], which is 4 times ours when \( q = 2 \). Ferdinand and Fisher use \( C_H^{(3)} \) multiplied by \( K^2 \), where \( K \) is the nearest-neighbor coupling in the Ising normalization; the critical point is given by \( K_c = \frac{1}{2} \log(1 + \sqrt{2}) \).
(ii) For $q = 3.25$ the correction-to-scaling contribution is only slightly smaller than the nonsingular background, and it will probably be difficult for the fit to separate the two. The fit to $A L^{\alpha/\nu} + B L^{\alpha/\nu-\Delta_1}$ is good already when $L_{\text{min}} = 16$,

$$\alpha/\nu = 0.4972(7), A = 1.201(5), B = -0.945(19) \quad (\chi^2 = 2.58, 4 \text{ DF, CL = 63\%})$$

but the estimate for $\alpha/\nu$ is almost six standard deviations away from the correct value $\approx 0.5013$. If we include the constant background we also get a good fit when $L_{\text{min}} = 16$,

$$\alpha/\nu = 0.4988(44), A = 1.186(42), B = -1.089(396), C = 0.133(365)$$

$$\quad (\chi^2 = 2.45, 3 \text{ DF, CL = 49\%})$$

but the estimate for the constant $C$ is consistent with zero. The estimate for $\alpha/\nu$ is now less than one standard deviation away from the correct value, but this is principally because the standard deviation has become much larger, not because the estimated value has actually moved much closer to the true value!

(iii) For $q = 3.5$ the correction-to-scaling contribution is now slightly larger than the nonsingular background, and it will again likely be difficult for the fit to separate the two. The fit to $A L^{\alpha/\nu} + B L^{\alpha/\nu-\Delta_1}$ is good when $L_{\text{min}} = 64$,

$$\alpha/\nu = 0.5945(39), A = 0.926(25), B = -0.017(91) \quad (\chi^2 = 0.91, 2 \text{ DF, CL = 64\%})$$

but the estimate for the correction-to-scaling amplitude $B$ is consistent with zero. Indeed, the result from this fit is virtually identical to what was obtained from the ansatz $A L^{\alpha/\nu} + B$, and the estimate for $\alpha/\nu$ is again about four standard deviations away from the correct value $\approx 0.6101$. By contrast, the fit to $A L^{\alpha/\nu} + B L^{\alpha/\nu-\Delta_1} + C$ is good already when $L_{\text{min}} = 16$,

$$\alpha/\nu = 0.6095(67), A = 0.805(43), B = 2.961(368), C = -3.509(390)$$

$$\quad (\chi^2 = 2.32, 3 \text{ DF, CL = 51\%})$$

and the estimate for $\alpha/\nu$ is now in excellent agreement with the correct answer. Interestingly, the estimates for the amplitudes $B$ and $C$ are not consistent with zero; rather, they are strongly nonzero but of opposite signs. Clearly, what happened is that when we performed a fit with a single correction term (whether $B$ or $B L^{\alpha/\nu-\Delta_1}$, which are anyway nearly the same) this pair of correction terms combined to make an “effective” correction term (in the given range of $L$) with a nearly zero amplitude; but this gave a biased estimate of the leading exponent $\alpha/\nu$. What is slightly surprising is that our fit was able to separate the correction-to-scaling contribution $B L^{\alpha/\nu-0.090}$ from the nonsingular background $C$. Perhaps the stunning agreement of the estimated value of $\alpha/\nu$ with the exact answer is a fluke and ought not be taken too seriously.

(iv) For $q = 3.75$ the correction-to-scaling contribution is significantly larger than the nonsingular background. The fit to $A L^{\alpha/\nu} + B L^{\alpha/\nu-\Delta_1}$ is good when $L_{\text{min}} = 64$,

$$\alpha/\nu = 0.7181(90), A = 0.568(42), B = 0.413(76) \quad (\chi^2 = 1.26, 2 \text{ DF, CL = 53\%})$$
and the estimate for $\alpha/\nu$ is only about two standard deviations away from the true value $\approx 0.7376$. By contrast, the fit to $AL^{\alpha/\nu} + BL^{\alpha/\nu - \Delta_1} + C$ is good already when $L_{\text{min}} = 16$:

$$\alpha/\nu = 0.7856(237), \quad A = 0.288(68), \quad B = 1.085(64), \quad C = -1.046(37)$$

$$(\chi^2 = 0.30, 3 \text{ DF}, \text{CL} = 96\%).$$

Here the estimated amplitudes $B$ and $C$ have opposite signs and are apparently nonzero; but the estimate for $\alpha/\nu$ has now far overshot the correct value (it is again two standard deviations away, but with a much larger standard deviation). The poor performance of this two-correction-term fit — in a case where the two terms $BL^{0.388}$ and $C$ should have been much easier to separate than they were for $q = 3.5$ — suggests that the good result obtained for $q = 3.5$ was indeed a fluke and that the mediocre result obtained for $q = 3.75$ is what should ordinarily be expected.

Finally, for $q = 4$ the true leading behavior is known [19] to be $L(\log L)^{-3/2}$, but with corrections to scaling down by $\log \log L/\log L$, $1/\log L$, etc. It is clearly hopeless to try to fit to such an ansatz unless one has data for colossally large values of $L$. It is of course not surprising that fits to $AL^{\alpha/\nu}$ or $AL^{\alpha/\nu} + B$ gave estimates of $\alpha/\nu$ near 0.8, far off from the correct value 1; the factor $(\log L)^{3/2}$ is imitating a power $L^{0.2}$ in our range of $L$.

7 Data analysis: Dynamic quantities

In this section we analyze the dynamic data by the same general methods as were used in the preceding section to analyze the static data. Our main goal is to estimate the dynamic critical exponents $z_{\text{int,O}}$ associated to the integrated autocorrelation times $\tau_{\text{int,O}}$ for various observables $O$.

We proceed as follows: First we discuss the dependence of the autocorrelation times on the number $k$ of active colors (Section 7.1), and we give an overview of the qualitative behavior of the autocorrelation times for different observables (Section 7.2). Then we present a detailed analysis of the dynamic critical exponent $z_{\text{int,E'}}$ (Section 7.3); in particular we discuss the sharpness of the Li–Sokal bound (Section 7.4) and the correctness of the Ossola–Sokal conjecture (Section 7.5). Finally, we analyze briefly the dynamic critical exponent $z_{\text{int,O}}$ for other observables $O$ (Section 7.6).

7.1 Dependence on $k$

We began by analyzing the dependence of $\tau_{\text{int,O}}$ on the number $k$ of active colors. Of course we expect that all values of $k$ lie in the same dynamic universality class: that is, we expect that the ratios of $\tau_{\text{int,O}}$ for different $k$ tend to nonzero finite constants as $L \to \infty$. Moreover, it is intuitively reasonable to think that an update with $k$ active colors does roughly “$k$ times as much work” as an update with one active color; therefore, we expect that $\tau_{\text{int,O}}$ should be roughly proportional to $1/k$.

We tested these expectations by analyzing the ratios $\tau_{\text{int,O}}(1)/\tau_{\text{int,O}}(k)$ as a function of $L$ for each $(q, O)$ and each allowable $k$. In all cases the ratios are fairly close to $k$, as expected; but in general they are not exactly equal to from $k$. The ratios also show some dependence on $L$, but tend as $L \to \infty$ to a limiting value, again as expected. Roughly
speaking, for the smaller values of \( q \) the \( L \)-dependence is fairly strong, and the ratios are comparatively far from \( k \) (which is perhaps not surprising because the values of \( \tau_{\text{int, } O} \) are themselves quite small); the limiting values also appear to be different from \( k \), though this conclusion is only tentative because of the strong corrections to scaling. For the larger values of \( q \) the \( L \)-dependence is weaker, and the ratios are closer to \( k \); in particular, the limiting values are compatible with \( k \) within our statistical errors. In Table 20 we show typical examples of these two behaviors, namely \( O = E' \) with \( q = 2 \) and \( q = 3.25 \). In Table 21 we show our best estimates for the limiting ratios \( \tau_{\text{int, } E'}(1)/\tau_{\text{int, } E'}(k) \), obtained by fitting the ratio to a constant and increasing \( L_{\text{min}} \) until a decent fit is obtained. The behavior for the other observables is qualitatively similar.

Having confirmed that all values of \( k \) lie in the same dynamic universality class, we henceforth analyze the data for each value of \( k \) separately and then compute a weighted average of the resulting exponent estimates.

7.2 Summary of qualitative behavior

Let us begin by summarizing the qualitative behavior of \( \tau_{\text{int, } O} \) for different observables \( O \):

1) For nearly every triplet \((q, k, L)\), we find that \( E' \) is the observable (of those we have measured) that has the largest \( \tau_{\text{int}} \). The only exceptions are \( q = 4, \ L = 16, 1 \leq k \leq 4 \) (for which \( \tau_{\text{int, } S_2} \) is slightly larger than \( \tau_{\text{int, } E'} \)) and \( q = 4, \ L = 1024, \ k = 1 \) (for which \( \tau_{\text{int, } C_1} \) is slightly larger than \( \tau_{\text{int, } E'} \)). But these differences are extremely small and may well represent statistical fluctuations.

2) For every triplet \((q, k, L)\), we find that \( C_2 \) is the observable (of those we have measured) that has the smallest \( \tau_{\text{int}} \).

3) For every triplet \((q, k, L)\) we find that

\[
\tau_{\text{int, } E'} > \tau_{\text{int, } N}.
\] (7.1)

When \( q \) is an integer and \( k = q \), this inequality is easily proved rigorously for the SW algorithm [17], so it is not surprising that it holds here for CM. However, we do not yet have a rigorous proof (not even in the case when \( q \) is an integer and \( k < q \)).

4) In general, the observables \( O \) that we have measured fall into four groups according to their integrated autocorrelation times \( \tau_{\text{int, } O} \):

1. \( E', \ N, \ S_2 \) and \( C_1 \) have the largest values of \( \tau_{\text{int}} \), and they are all fairly close to each other (all are at least \( \approx 0.75 \) times that of \( E' \), and usually much closer);

2. \( F' \) has an intermediate value of \( \tau_{\text{int}} \), of order 0.5–0.9 times that of \( E' \);

3. \( C_3 \) has a slightly lower value of \( \tau_{\text{int}} \), of order 0.4–0.8 times that of \( E' \);

4. \( C_2 \) has the smallest \( \tau_{\text{int}} \), of order 0.35–0.8 times that of \( E' \).

Indeed, for all triplets \((q, k, L)\) we have \( \tau_{\text{int, } O_1} > \tau_{\text{int, } O_2} > \tau_{\text{int, } O_3} > \tau_{\text{int, } O_4} \) whenever \( O_1, O_2, O_3, O_4 \) belong to groups 1,2,3,4, with the exception that for a few triplets at \( q = 4, \ L \geq 256 \) we have \( \tau_{\text{int, } F'} < \tau_{\text{int, } C_3} \).
These behaviors can be better understood by looking at the normalized autocorrelation functions $\rho_{\mathcal{O}}(t)$. A typical example is shown in Figure 3. We see that $\rho_{\mathcal{E}'}(t)$ is nearly a pure exponential, so that $\tau_{\text{int},\mathcal{E}'} \approx \tau_{\text{exp}}$. By contrast, the autocorrelation functions for the observables $\mathcal{O}$ in groups 2, 3 and 4 exhibit an initial fast decay, followed by a decay at the same exponential rate $\tau_{\text{exp}}$ but with an amplitude $A_{\mathcal{O}}$ that is significantly less than 1. What we do not know is whether $A_{\mathcal{O}}$ tends to a nonzero value as $L \to \infty$ (in which case we will have $z_{\text{int},\mathcal{O}} = z_{\text{exp}}$) or tends to zero as an inverse power of $L$ (in which case we will have $z_{\text{int},\mathcal{O}} < z_{\text{exp}}$). See Section 7.6 for further analysis of this question; and see [20, Section 5.2] for a more detailed analysis in the case of the Swendsen–Wang dynamics for the three-dimensional Ising model.

In the following subsections we shall fit $\tau_{\text{int},\mathcal{O}}$, for each observable $\mathcal{O}$, to a variety of ansätze, notably:

- **Fits for $z = 0$:** $\tau_{\text{int},\mathcal{O}} = A$ or $\tau_{\text{int},\mathcal{O}} = A + BL^{-p}$.
- **Fits for $z = 0$ with a multiplicative logarithm:** $\tau_{\text{int},\mathcal{O}} = A\log L + B$ or $\tau_{\text{int},\mathcal{O}} = A\log^2 L + B\log L + C$.
- **Fits for $z > 0$:** $\tau_{\text{int},\mathcal{O}} = AL^z$ or $\tau_{\text{int},\mathcal{O}} = AL^z + B$.

Note that the fits to $A + BL^{-p}$ and $AL^z + B$ are in fact the same fit in different notation!

We shall begin (Sections 7.3–7.5) by focusing on the observable $\mathcal{E}'$, which has the largest autocorrelation time of all the observables we measured. Then (Section 7.6) we shall discuss, more briefly, the other observables, with emphasis on $C_2$.

### 7.3 Dynamic critical exponent $z_{\text{int},\mathcal{E}'}$

In this subsection we fit the integrated autocorrelation time $\tau_{\text{int},\mathcal{E}'}$ to a variety of ansätze in an effort to estimate the dynamic critical exponent $z_{\text{int},\mathcal{E}'}$. We begin by presenting our fits, in order of increasing $q$, without much comment. Then we go back and try to interpret what these fits might be telling us about the dynamic critical behavior of the Chayes–Machta algorithm as a function of $q$.

For $q = 1.25$ the behavior is fairly clear: $\tau_{\text{int},\mathcal{E}'}$ tends to a finite constant as $L \to \infty$. The fits to the ansatz $\tau_{\text{int},\mathcal{E}'} = A$ are horrible (except of course the fit $L_{\text{min}} = 1024$ that has zero degrees of freedom); but if we fit to $\tau_{\text{int},\mathcal{E}'} = A + BL^{-p}$ we get a decent fit for $L_{\text{min}} = 128$:

$$A = 2.19(8), \quad B = -1.61(5), \quad p = 0.213(35) \quad (\chi^2 = 0.49, 1 \text{ DF}, \text{ CL } = 48\%). \quad (7.2)$$

Finally, the fit to $AL^z$ yields an estimated exponent $z \approx 0.05$ but has poor goodness of fit even when $L_{\text{min}} = 256$: $z = 0.053(1), A = 1.27(1)$ with $\chi^2 = 3.76, 1 \text{ DF}, \text{ CL } = 5\%$.

For $q = 1.5$ the fits to a constant $A$ are again horrible; but if we fit to $A + BL^{-p}$ we get a good fit already for $L_{\text{min}} = 32$:

$$A = 17.09(2.96), \quad B = -15.86(2.92), \quad p = 0.034(7) \quad (\chi^2 = 0.82, 3 \text{ DF}, \text{ CL } = 85\%). \quad (7.3)$$
We also tried a fit to $A \log L + B$: a decent fit is obtained for $L_{\text{min}} = 128$, namely

$$A = 0.440(4), \quad B = 1.486(21) \quad (\chi^2 = 0.83, 2 \text{ DF}, \text{ CL} = 66\%) \, .$$

(7.4)

A good fit to $A \log^2 L + B \log L + C$ is obtained already for $L_{\text{min}} = 32$, but with a slightly negative value of $A$ (which is of course impossible for the actual asymptotics): $A = -0.008(2), \quad B = 0.53(2), \quad C = 1.24(4)$ with $\chi^2 = 0.80, 3 \text{ DF}, \text{ CL} = 85\%$. Finally, the fit to $AL^2$ yields an estimated exponent $z \approx 0.10$ but again has poor goodness of fit even when $L_{\text{min}} = 256$: $z = 0.104(2), \quad A = 2.20(2)$ with $\chi^2 = 3.84, 1 \text{ DF}, \text{ CL} = 5\%$.

For $q = 1.75$ (and all larger $q$) the fit to a constant $A$ is again horrible. The fit to $A \log L + B$ is mediocre even when $L_{\text{min}} = 256$: $A = 1.36(2), \quad B = 0.32(12)$ with $\chi^2 = 2.73, 1 \text{ DF}, \text{ CL} = 10\%$. A decent fit to $A \log^2 L + B \log L + C$ is obtained already for $L_{\text{min}} = 16$: $A = 0.037(3), \quad B = 0.915(23), \quad C = 1.689(45)$ with $\chi^2 = 4.14, 4 \text{ DF}, \text{ CL} = 39\%$. The fit to $AL^z$ is good when $L_{\text{min}} = 256$: $z = 0.158(2), \quad A = 3.285(5)$ with $\chi^2 = 0.21, 1 \text{ DF}, \text{ CL} = 65\%$. The fit to $AL^z + B$ is decent already for $L_{\text{min}} = 16$ ($\chi^2 = 4.63, 4 \text{ DF}, \text{ CL} = 33\%$), but the $\chi^2$ drops notably from $L_{\text{min}} = 64$ ($\chi^2 = 2.86$) to $L_{\text{min}} = 128$ ($\chi^2 = 0.33$), so that our preferred fit is $L_{\text{min}} = 128$:

$$z = 0.085(26), \quad A = 9.66(4.42), \quad B = -7.56(4.82) \quad (\chi^2 = 0.33, 1 \text{ DF}, \text{ CL} = 57\%) \, .$$

(7.5)

For $q \geq 2$ we have data from more than one value of $k$. For simplicity we discuss here in words only the case $k = 1$; the other cases are qualitatively similar and are reported in Tables 22 and 23.

For $q = 2$ the fit to $A \log L + B$ is poor (CL < 2%) even when $L_{\text{min}} = 256$. The fit to $A \log^2 L + B \log L + C$ is good already for $L_{\text{min}} = 16$: $A = 0.206(6), \quad B = 1.046(53), \quad C = 2.572(103)$ with $\chi^2 = 1.40, 4 \text{ DF}, \text{ CL} = 84\%$. The fit to $AL^z$ is good for $L_{\text{min}} = 256$: $z = 0.215(3), \quad A = 4.47(9)$ with $\chi^2 = 0.06, 1 \text{ DF}, \text{ CL} = 81\%$. Finally, the fit to $AL^z + B$ is good already for $L_{\text{min}} = 16$:

$$z = 0.145(4), \quad A = 10.4(5), \quad B = -8.4(6) \quad (\chi^2 = 2.32, 4 \text{ DF}, \text{ CL} = 68\%) \, .$$

(7.6)

For $q = 2.25$ the fit to $A \log L + B$ is again poor (CL < 5%) even when $L_{\text{min}} = 256$. The fit to $A \log^2 L + B \log L + C$ is decent already for $L_{\text{min}} = 32$: $A = 0.77(3), \quad B = -0.84(24), \quad C = 7.07(53)$ with $\chi^2 = 3.15, 3 \text{ DF}, \text{ CL} = 37\%$. The fit to $AL^z$ yields an estimated exponent $z \approx 0.29$ but has mediocre goodness of fit even when $L_{\text{min}} = 256$: $z = 0.286(4), \quad A = 5.35(14)$ with $\chi^2 = 2.16, 1 \text{ DF}, \text{ CL} = 14\%$. Finally, the fit to $AL^z + B$ is good already for $L_{\text{min}} = 32$:

$$z = 0.235(8), \quad A = 8.91(61), \quad B = -6.68(86) \quad (\chi^2 = 1.62, 3 \text{ DF}, \text{ CL} = 66\%) \, .$$

(7.7)

For $q = 2.5$ (and all larger $q$) the fit to $A \log L + B$ is poor (CL < 3%) even when $L_{\text{min}} = 256$. The fit to $A \log^2 L + B \log L + C$ is decent for $L_{\text{min}} = 128$: $A = 2.7(3), \quad B = -13.7(31), \quad C = 37.4(8.6)$ with $\chi^2 = 0.66, 1 \text{ DF}, \text{ CL} = 42\%$. However, the error bars on $B$ and $C$ are very large in absolute magnitude (and in particular large compared to the value of $A$), which renders the fit somewhat dubious. The fit to $AL^z$ is good for
\(L_{\text{min}} = 256\): \(z = 0.353(6), A = 6.35(23)\) with \(\chi^2 = 0.02, 1\) DF, CL = 90%. Finally, the fit to \(AL^2 + B\) is excellent already for \(L_{\text{min}} = 32:\)

\[
z = 0.315(8), \ A = 9.01(59), \ B = -6.75(1.00) \quad (\chi^2 = 0.21, 3\) DF, CL = 98%) . 
\] (7.8)

For \(q = 2.75\) the fit to \(A\log^2 L + B\log L + C\) is decent for \(L_{\text{min}} = 64\): \(A = 5.9(3), B = -31.9(2.7), C = 71.0(6.5)\) with \(\chi^2 = 2.56, 2\) DF, CL = 28%. But the error bars on \(B\) and \(C\) are again quite large. The fit to \(AL^2\) is good for \(L_{\text{min}} = 256\): \(z = 0.424(8), A = 7.19(34)\) with \(\chi^2 = 0.07, 1\) DF, CL = 80%. Finally, the fit to \(AL^2 + B\) is excellent already for \(L_{\text{min}} = 16:\)

\[
z = 0.411(5), \ A = 8.19(30), \ B = -4.76(57) \quad (\chi^2 = 1.06, 4\) DF, CL = 90%) . 
\] (7.9)

For \(q = 3\) the fit to \(A\log^2 L + B\log L + C\) is decent for \(L_{\text{min}} = 128\), but with huge error bars: \(A = 16.0(1.5), B = -112.6(16.5), C = 257.5(44.5)\) with \(\chi^2 = 0.77, 1\) DF, CL = 38%. (The same behavior persists for all larger values of \(q\), with even larger values of the coefficients \(A, B, C\) and their error bars; we refrain from reporting the gory results.) The fit to \(AL^2\) is good for \(L_{\text{min}} = 128\): \(z = 0.505(6), A = 7.52(23)\) with \(\chi^2 = 1.18, 2\) DF, CL = 56%. Finally, the fit to \(AL^2 + B\) is excellent already for \(L_{\text{min}} = 32:\)

\[
z = 0.481(10), \ A = 9.08(63), \ B = -6.79(1.68) \quad (\chi^2 = 0.84, 3\) DF, CL = 84%) . 
\] (7.10)

For \(q = 3.25\) the fit to \(AL^2\) is decent for \(L_{\text{min}} = 128\): \(z = 0.590(7), A = 7.66(30)\) with \(\chi^2 = 3.14, 2\) DF, CL = 21%. Finally, the fit to \(AL^2 + B\) is decent for \(L_{\text{min}} = 64:\)

\[
z = 0.558(21), \ A = 9.8(1.4), \ B = -11.9(6.0) \quad (\chi^2 = 2.93, 2\) DF, CL = 23%) . 
\] (7.11)

For \(q = 3.5\) the fit to \(AL^2\) is decent for \(L_{\text{min}} = 128\): \(z = 0.676(10), A = 7.77(39)\) with \(\chi^2 = 1.07, 2\) DF, CL = 58%. Finally, the fit to \(AL^2 + B\) is decent for \(L_{\text{min}} = 64:\)

\[
z = 0.648(25), \ A = 9.5(1.6), \ B = -14.4(8.9) \quad (\chi^2 = 1.15, 2\) DF, CL = 56%) . 
\] (7.12)

For \(q = 3.75\) the fit to \(AL^2\) is good already for \(L_{\text{min}} = 32\): \(z = 0.779(4), A = 7.14(13)\) with \(\chi^2 = 0.94, 4\) DF, CL = 92%. Finally, the fit to \(AL^2 + B\) is excellent already for \(L_{\text{min}} = 16:\)

\[
z = 0.790(9), \ A = 6.7(3), \ B = 3.0(1.5) \quad (\chi^2 = 0.33, 4\) DF, CL = 99%) . 
\] (7.13)

For \(q = 4\) the fit to \(AL^2\) is good for \(L_{\text{min}} = 128\): \(z = 0.916(16), A = 5.44(46)\) with \(\chi^2 = 0.15, 2\) DF, CL = 93%. Finally, the fit to \(AL^2 + B\) is excellent already for \(L_{\text{min}} = 32:\)

\[
z = 0.935(18), \ A = 4.8(5), \ B = 20.5(5.1) \quad (\chi^2 = 0.66, 3\) DF, CL = 88%) . 
\] (7.14)

Let us now comment on what we think these fits show.

For \(q = 1.25\) it seems fairly clear that \(\tau_{\text{int},E'}\) converges as \(L \rightarrow \infty\) to a finite value, i.e. \(\tau_{\text{int},E'} = 0\). For \(q = 1.5\) the behavior is unclear: perhaps \(\tau_{\text{int},E'}\) converges to a finite value, but with extremely strong corrections to scaling (e.g. \(A + BL^{-p}\) with \(p > 0\) extremely
The evidence points weakly towards the first scenario, but a divergence like \( \log L \) or an extremely small positive power of \( L \) is also a possibility. For \( 1.75 \leq q \leq 2.25 \) a reasonable fit is obtained with the ansatz \( A \log^2 L + B \log L + C \); but it seems to us implausible on theoretical grounds that we would have such a logarithmic growth for an entire interval of \( q \). Much more likely is that there exists one value \( q_* \) such that \( z = 0 \) for \( q < q_* \) and \( z > 0 \) for \( q > q_* \), in which case \( \tau_{\text{int},\mathcal{E}'} \) might grow logarithmically at \( q = q_* \) (but only there). Our data suggest that \( q_* \) lies between 1.25 and 1.75; our best guess would be \( \approx 1.6 \), based on linearly interpolating the exponent estimates produced by our fits for \( q = 1.5 \) and \( q = 1.75 \).

For all \( q \geq 1.75 \) we are able to obtain decent fits to the ansätze \( AL^z \) and \( AL^z + B \) with an exponent \( z > 0 \); these fits are reported in Tables 22 and 23, respectively, and the results obtained by averaging over \( k \) are reported in Table 24. For the smaller values of \( q \) in this table, the discrepancy between the fits with and without a constant term \( B \) is fairly large: this is not surprising because \( z \) is fairly small and hence the effect of the constant term is very strong. As \( q \) grows, the discrepancy between the two fits decreases: from \( \approx 0.1 \) at \( q = 1.75 \) and \( \approx 0.07 \) at \( q = 2 \) to approximately zero at \( q = 3.5 \); for \( q > 3.5 \) the discrepancy has the opposite sign but remains small. Correspondingly, the estimated value of \( B \) appears to go through zero (and change sign) at \( q \approx 3.5 \). We have a slight preference for the fits to \( AL^z + B \), for the simple reason that such a constant term must surely be present, if only because the definition of \( \tau_{\text{int},\mathcal{O}} \) is somewhat arbitrary (should one include the contribution \( \frac{1}{2} \) from \( t = 0 \) or not?). But our data are insufficient to resolve clearly the discrepancy between the two fits. We therefore choose to report our results for \( q \geq 1.75 \) in the form

\[
\text{best estimate} \pm \text{statistical error} \pm \text{systematic error}, \tag{7.15}
\]

where “statistical error” denotes the one-standard-deviation error bar from the fit to \( A L^z + B \) (after averaging over \( k \) the fits from the chosen values of \( L_{\text{min}} \)); and “systematic error” is a 68% subjective confidence interval defined as the absolute value of the discrepancy between the fits \( A L^z \) and \( A L^z + B \) (after averaging over \( k \) the fits from the chosen values of \( L_{\text{min}} \)) plus 0.02. The final results are therefore:

\[
\begin{align*}
q = 1.75: & \quad z_{\text{int},\mathcal{E}'} = 0.085 \pm 0.026 \pm 0.094 \tag{7.16a} \\
q = 2.00: & \quad z_{\text{int},\mathcal{E}'} = 0.143 \pm 0.003 \pm 0.092 \tag{7.16b} \\
q = 2.25: & \quad z_{\text{int},\mathcal{E}'} = 0.231 \pm 0.008 \pm 0.071 \tag{7.16c} \\
q = 2.50: & \quad z_{\text{int},\mathcal{E}'} = 0.307 \pm 0.007 \pm 0.066 \tag{7.16d} \\
q = 2.75: & \quad z_{\text{int},\mathcal{E}'} = 0.408 \pm 0.005 \pm 0.036 \tag{7.16e} \\
q = 3.00: & \quad z_{\text{int},\mathcal{E}'} = 0.497 \pm 0.003 \pm 0.033 \tag{7.16f} \\
q = 3.25: & \quad z_{\text{int},\mathcal{E}'} = 0.572 \pm 0.007 \pm 0.035 \tag{7.16g} \\
q = 3.50: & \quad z_{\text{int},\mathcal{E}'} = 0.689 \pm 0.004 \pm 0.024 \tag{7.16h} \\
q = 3.75: & \quad z_{\text{int},\mathcal{E}'} = 0.796 \pm 0.004 \pm 0.032 \tag{7.16i} \\
q = 4.00: & \quad z_{\text{int},\mathcal{E}'} = 0.910 \pm 0.005 \pm 0.032 \tag{7.16j}
\end{align*}
\]
In Figure 4 we plot these estimates versus \( q \), and compare them with the static exponents \( \alpha/\nu \) and \( \beta/\nu \). The Ossola–Sokal conjecture \( z \geq \beta/\nu \) appears to be violated for \( 1 \leq q < 1.95 \) (but see Section 7.5 for an alternative fit that is compatible with the conjecture). The Li–Sokal bound \( z \geq \alpha/\nu \) is obeyed for \( q \neq 4 \) and appears to be non-sharp (see Section 7.4 for a more detailed analysis). The apparent violation of the Li–Sokal bound at \( q = 4 \) is manifestly due to the multiplicative logarithmic corrections: it is known [19] that \( C_H \sim L(\log L)^{-3/2} \) but fits to a power law (with or without a constant background) yield an effective exponent \( \approx 0.8 \) (see the last line of Table 19); it is therefore not surprising that \( \tau_{\text{int},E'} \) shows a similar behavior. If one looks directly at the ratio \( \tau_{\text{int},E'}/C_H \) one finds that the effective exponent \( \approx 0.8 \) (see the last line of Table 19); it is therefore not surprising that \( \tau_{\text{int},E'} \) shows a similar behavior. If one looks directly at the ratio \( \tau_{\text{int},E'}/C_H \) one finds that the Li–Sokal bound (which is after all a rigorous theorem!) is obeyed (see Section 7.4).

For \( q = 1.25 \) and 1.50 our best estimates suggest that \( \tau_{\text{int},E'} \) is bounded as \( L \to \infty \), i.e. that \( z_{\text{int},E'} = 0 \). Our fits to \( A + BL^{-p} \) suggest values for the correction exponent \( p \) [cf. (7.2)/(7.3)], but we do not know how reliable these estimates are.

Since the autocorrelation function \( \rho_{E'E}(t) \) is nearly a pure exponential, we expect that the dynamic critical exponent \( z_{\text{exp}} \) is either exactly equal or almost exactly equal to \( z_{\text{int},E'} \).

### 7.4 Sharpness of Li–Sokal bound

The estimates of \( z_{\text{int},E'} \) summarized in Figure 4 suggest that the Li–Sokal bound \( z_{\text{int},E'} \geq \alpha/\nu \) holds as a strict inequality over the entire range \( 1 \leq q < 4 \), i.e. that it is non-sharp by a power. But this conclusion is weakened by the fact that our fits of the specific heat \( C_H \) give estimates of \( \alpha/\nu \) that deviate significantly from the known exact values (Table 19). It is therefore of interest to study directly the ratio \( \tau_{\text{int},E'}/C_H \), in an effort to fit its behavior as \( L \to \infty \) to one of the following ansätze:

1. Asymptotically constant with additive corrections to scaling \( A + BL^{-p} \).
2. A logarithmic growth, either as \( A \log L + B \) or as \( A \log^p L \).
3. A power-law growth \( AL^p \) or \( AL^p + B \).

Unfortunately our time-series analysis does not produce statistically valid error bars for composite static-dynamic quantities such as \( \tau_{\text{int},E'}/C_H \). We therefore conducted the analysis in this subsection under the crude assumption that the statistical fluctuations on our estimators of \( \tau_{\text{int},E'} \) and \( C_H \) are uncorrelated. In fact it is likely that these fluctuations are positively correlated, so that the true error bars on the ratio \( \tau_{\text{int},E'}/C_H \) are smaller than we have supposed. If so, this means that the true confidence level of our fits is smaller than what we report.

The fits to \( A + BL^{-p} \) with \( p > 0 \) were always bad: either they had a horrible confidence level, or they converged to a value \( p < 0 \) (indicating that the true leading behavior is a power-law growth). We therefore focussed on comparing the logarithmic-growth and power-law-growth scenarios. We chose \( A \log L + B \) and \( AL^p \) as the ansätze in order to compare fits with an equal number of free parameters. The results are shown in

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17 This failure to provide statistically valid error bars is embarrassing. We should have used the batch-means method [17, Section 4.2] to obtain such error bars. Unfortunately, the raw data from our simulations are no longer accessible, so we are unable to conduct such a reanalysis.
We see that both fits are in general good (though the confidence levels may be overestimated as noted above). However, the power-law fits are in general better. In particular, there are no cases in which the power-law fit has a confidence level less than 25%; but there are quite a few cases in which the logarithmic fit exhibits such a low confidence level (sometimes much lower). We therefore conclude that the fits to $\tau_{\text{int},E'/C_H}$ also provide weak evidence that the Li–Sokal bound is non-sharp by a power.

### 7.5 Test of Ossola–Sokal conjecture

In the Introduction we argued that the Ossola–Sokal conjecture $z_{CM} \geq \beta/\nu$ is probably false, on the grounds that it fails for $q = 1$ (where $z_{CM} = 0$ and $\beta/\nu > 0$) and that $z_{CM}$ is presumably a continuous function of $q$. But this latter assumption is far from certain: it is possible, for instance, that $z_{CM} = \beta/\nu$ exactly for all $q$ near 1, but with an amplitude that vanishes as $q \downarrow 1$. In this subsection we would like to test this scenario against our data for $1.25 \leq q \leq 2$.

We first tried fits to $\tau_{\text{int},E} = AL^{\beta/\nu} + B$. We then tried fits to $\tau_{\text{int},E} = AL^{\beta/\nu} + B + CL^{-p}$ where $p > 0$ is fixed and $A, B, C$ are free. Let us report our results from these fits in decreasing order of $q$:

1) For $q = 2$ ($k = 1$) the fit to $AL^{\beta/\nu} + B$ is decent already for $L_{\text{min}} = 32$ ($\chi^2 = 3.90$, 4 DF, CL = 42%) and better for $L_{\text{min}} = 64$:

   $$A = 13.27(7), B = -11.83(13) \quad (\chi^2 = 1.51, 3 \text{ DF}, \text{CL} = 68\%) \ . \tag{7.17}$$

This behavior is not surprising, because our fit to $AL^z + B$ with $z$ free yielded $z = 0.145(4)$ with $L_{\text{min}} = 16$, which is not very far from $\beta/\nu = 0.125$.

In Figure 5 we plot the results of our fits to $AL^{\beta/\nu} + B + CL^{-p}$ as a function of $p$, for $L_{\text{min}} = 16$ and 32:

- For $L_{\text{min}} = 16$ an excellent confidence level is obtained over the whole range $0 < p < 2$ (and indeed beyond), with an optimum at $p \approx 0.692$ (CL = 92%). The estimated amplitudes $A$ are positive and very far from zero; at the optimum we have $A = 13.44(9)$. As $p \to \infty$ the amplitude $A$ tends to a value $\approx 13$, which is close to that obtained from the fit to $AL^{\beta/\nu} + B$.

- For $L_{\text{min}} = 32$ an excellent confidence level is again obtained over the whole range $0 < p < 2$ (and indeed beyond), with an optimum at $p \approx 0.228$ (CL = 84%) but with a very broad peak. The estimated amplitudes $A$ are again positive and very far from zero; at the optimum we have $A = 13.71(30)$. As $p \to \infty$ the amplitude $A$ again tends to $\approx 13$.

2) For $q = 1.75$ the fit to $AL^{\beta/\nu} + B$ is decent already for $L_{\text{min}} = 128$ ($\chi^2 = 2.31$, 2 DF, CL = 32%) and excellent for $L_{\text{min}} = 256$:

   $$A = 5.36(8), B = -2.62(17) \quad (\chi^2 = 0.002, 1 \text{ DF}, \text{CL} = 97\%) \ . \tag{7.18}$$

Once again this behavior is not surprising, because our fit to $AL^z + B$ with $z$ free yielded $z = 0.085(26)$ with $L_{\text{min}} = 128$, which is not very far from $\beta/\nu \approx 0.121$.  

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In Figure 6 we plot the results of our fits to $AL^{\beta/\nu} + B + CL^{-p}$ as a function of $p$, for $L_{\text{min}} = 32$ and 64:

- For $L_{\text{min}} = 32$ a decent confidence level is obtained for $p \lesssim 0.6$, with an optimum at $p \approx 0.056$ (CL = 41%). The estimated amplitudes $A$ are positive and far from zero; at the optimum we have $A = 3.51(24)$. As $p \to \infty$ the amplitude $A$ tends to a value $\approx 5.8$, which is close to that obtained from the fit to $AL^{\beta/\nu} + B$.

- For $L_{\text{min}} = 64$ a decent confidence level is obtained over the entire range $0 < p \leq 2$ (or even beyond), with an optimum at $p \approx 1.321$ (CL = 96%). The estimated amplitudes $A$ are again positive and far from zero, and in fact quite close to those obtained from $L_{\text{min}} = 32$. At the optimum, we have $A = 5.28(7)$.

3) For $q = 1.5$ the fit to $AL^{\beta/\nu} + B$ is poor even for $L_{\text{min}} = 256$ (CL = 3%). In Figure 7 we plot the results of our fits to $AL^{\beta/\nu} + B + CL^{-p}$ as a function of $p$, for $L_{\text{min}} = 32$ and 64:

- For $L_{\text{min}} = 32$ a decent confidence level is obtained for $p \lesssim 0.3$, but with the optimum attained at the ridiculously small value $p \approx 0.004$ (CL = 86%). Also, the estimated amplitude $A$ is negative for $p \lesssim 0.033$, which is obviously impossible. However, for $0.08 \lesssim p \lesssim 0.3$ a decent fit is obtained with an amplitude $A$ that is positive and far from zero.

- For $L_{\text{min}} = 64$ a decent confidence level is obtained for $0 < p \lesssim 0.9$, with an optimum at $p \approx 0.228$ (CL = 86%). The estimated amplitudes $A$ are again negative for $p \lesssim 0.038$ but positive and far from zero for $p \gtrsim 0.1$. At the optimum, we have $A = 1.11(8)$.

4) For $q = 1.25$ the fit to $AL^{\beta/\nu} + B$ is poor even for $L_{\text{min}} = 256$ (CL = 1%). The fits to $AL^{\beta/\nu} + B + CL^{-p}$ are poor (CL < 6%) for $L_{\text{min}} = 32$ and 64 for all $p > 0$. For $L_{\text{min}} = 128$, however, we are able to obtain decent fits over the whole range $0 < p \lesssim 2$, as shown in Figure 8. The optimum lies at $p \approx 0.695$ (CL = 99.9997%). The estimated amplitudes $A$ are negative for $p \lesssim 0.211$ but positive and far from zero for $p \gtrsim 0.4$. At the optimum, we have $A = 0.28(2)$.

The foregoing fits show that the Ossola–Sokal conjecture $z_{\text{CM}} \geq \beta/\nu$ is not ruled out by our data at $1.25 \leq q \leq 2$. Indeed, our data are consistent with the possibility that $z_{\text{CM}} = \beta/\nu$ exactly for $1 < q \lesssim 2$, but with an amplitude that vanishes as $q \downarrow 1$, perhaps proportional to $q - 1$ (the fits for $q = 1.25$ and 1.50 are consistent with this latter behavior).

7.6 Dynamic critical exponents $z_{\text{int, } \mathcal{O}}$ for other $\mathcal{O}$

Let us now look briefly at the dynamic critical exponents $z_{\text{int, } \mathcal{O}}$ for observables $\mathcal{O}$ other than $\mathcal{E}'$. For $\mathcal{O} = \mathcal{N}, \mathcal{S}_2, \mathcal{C}_1$, the values of $\tau_{\text{int, } \mathcal{O}}$ are very close to those of $\tau_{\text{int, } \mathcal{E}'}$, so the estimates of $z_{\text{int, } \mathcal{O}}$ will be nearly the same; little would be gained by going through these fits in detail. Instead, it seems sensible to look at the observable that has the smallest
autocorrelation time, namely $C_2$ — the idea being that if any differences in $z_{\text{int},C}$ between different observables are to be detected, they will be detected here.

From Figure 3 we see that the autocorrelation function of $E'$ is nearly a pure exponential (and this is so for all $q,k,L$), so that $\tau_{\text{int},E'} \approx \tau_{\exp}$ and hence $z_{\text{int},E'} = z_{\exp}$ or nearly so. The autocorrelation function of $C_2$, by contrast, exhibits an initial fast decay, followed by a decay at the same exponential rate $\tau_{\exp}$ as for the other observables but with an amplitude $A_{C_2}$ that is significantly less than 1 (e.g. around 0.4 in the plot shown). The key question is whether this amplitude tends to a nonzero value when $L \to \infty$ (in which case we will have $z_{\text{int},C_2} = z_{\exp}$) or tends to zero as an inverse power of $L$ (in which case we will have $z_{\text{int},C_2} < z_{\exp}$).

In Tables 26 and 27 we show the fits for $\tau_{\text{int},C_2}$ to the ansätze $AL^z$ and $AL^z + B$. In Table 28 we show the results for $z_{\text{int},C_2}$ obtained by averaging over $k$.

The estimates for $z_{\text{int},C_2}$ are indeed less than those for $z_{\text{int},E'}$, by an amount that is $\approx 0.1$ for $q \approx 2$ and tends to zero as $q \uparrow 4$. The question is: Are these differences real, or are they artifacts of corrections to scaling at small $L$? The fact that the differences are smaller for $q \gg 3$, where the autocorrelation times are larger, suggests that perhaps the differences will disappear as $L \to \infty$ but that at small $q$ we have to go to larger $L$ to see this. But this is far from clear; we will only know the truth by doing simulations at significantly larger values of $L$.

8 Discussion

In this paper we have studied the dynamic critical behavior of the Chayes–Machta algorithm as a function of $q$ over the whole range $1.25 \leq q \leq 4$. We have obtained estimates of the dynamic critical exponent $z_{\text{int},E'}$ as a function of $q$: see (7.16), Table 24 and Figure 4. Since the autocorrelation function $\rho_{E'E'}(t)$ is nearly a pure exponential, we also expect that the dynamic critical exponent $z_{\exp}$ is either exactly equal or almost exactly equal to $z_{\text{int},E'}$.

By simultaneously studying the whole range of values of $q$, we were able to gain some insights that would not have been available had we studied only a single value of $q$ (such as the Ising value $q = 2$) or even all integer values of $q$ as in past studies of the Swendsen–Wang algorithm. For instance:

1) The autocorrelation time $\tau_{\text{int},E'}$ at $q = 2$ can be plausibly fit with the ansatz $A \log^2 L + B \log L + C$, suggesting that the Li–Sokal bound might be sharp modulo a logarithm; this agrees with the conclusions of the paper [16], where the data were in fact found to slightly favor the non-sharp-by-a-logarithm ansatz over the non-sharp-by-a-power ansatz. But we now find that the good fit to $A \log^2 L + B \log L + C$ persists over the whole range $1.75 \leq q \lesssim 2.25$. And it seems to us implausible on theoretical grounds that we would have such a logarithmic growth for an entire interval of $q$; rather, we expect that there exists one value $q_*$ such that $z = 0$ for $q < q_*$ and $z > 0$ for $q > q_*$, with a possible (poly)logarithmic growth at $q = q_*$. Our data suggest that $q_*$ lies between 1.25

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18 See [20, Section 5.2] for a more detailed analysis of this kind, for the Swendsen–Wang dynamics for the three-dimensional Ising model.
and 1.75, with our best guess being around 1.6. If this scenario is correct, it follows that the Li–Sokal bound is non-sharp by a power for $q_* < q \leq 2$. This then suggests (but of course does not prove) that it might be non-sharp by a power over the whole interval $q_* < q < 4$ (possibly reverting to non-sharp by a logarithm at $q = 4$).

2) By considering the Ossola–Sokal conjecture $z_{CM} \geq \beta/\nu$ simultaneously for all $q$, we can see immediately that it fails at $q = 1$ (where $z_{CM} = 0$ but $\beta/\nu > 0$) and hence fails also for $q$ near 1 if the dynamic critical exponent $z_{CM}$ is a continuous function of $q$. Indeed, our pure power-law fits suggest that the conjecture fails for $1 \leq q \lesssim 1.95$. (In particular, if $q_* > 1$ as just suggested, then the conjecture fails spectacularly in the interval $1 \leq q < q_*$, where $z_{CM} = 0$ but $\beta/\nu > 0$.) However — and perhaps surprisingly — our data are also compatible with an alternative scenario in which $z_{CM} = \beta/\nu$ exactly for all $q$ near 1, but with an amplitude that vanishes as $q \downarrow 1$ (Section 7.5).

However, the behavior of the Chayes–Machta autocorrelation time for $1 < q < 2$ is still unclear: though our data strongly suggest that $\tau_{int,E'}$ diverges as $L \to \infty$ for $q$ slightly below 2, and is nondivergent as $L \to \infty$ for $q$ slightly above 1, we cannot rule out the possibility that the data at $L \leq 1024$ are misleading and that the true asymptotic behavior is different from what we conjecture. Future work at larger values of $L$ would of course be desirable.

Likewise, though our data suggest that the Li–Sokal bound is non-sharp by a power for $1.6 \lesssim q < 4$, it is also true that the exponent estimates have been dropping over time as data becomes available at larger values of $L$ and as we try ansätze other than a pure power law (see Table 24 for the effect of the ansatz). Our data suggest most clearly the non-sharpness of the Li–Sokal bound (when the non-sharpness is measured in units of the standard deviation of our estimate) at $q = 2.75$ and $q = 3$. At these values of $q$, the critical slowing-down is strong enough that interference from the regular background term is less important than it is at smaller $q$ (i.e. the $AL^z$ and $AL^z + B$ fits show less discrepancy) but modest enough that we can have reasonably good data on large lattices (contrary to the situation at larger $q$); furthermore, the correction-to-scaling exponent $\Delta_1$ is still fairly large. It would therefore be of great interest to perform high-precision simulations at these values of $q$, going to very high values of $L$.

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| $q$ | $\alpha/\nu$ | $\beta/\nu$ | $\gamma/\nu$ | $\Delta_1$ |
|-----|--------------|--------------|--------------|-------------|
| 1.25 | -0.35527 | 0.11118 | 1.77764 | 1.80702 |
| 1.50 | -0.22663 | 0.11678 | 1.76644 | 1.63551 |
| 1.75 | -0.10929 | 0.12131 | 1.75738 | 1.47905 |
| 2.00 | 0.00000 | 0.12500 | 1.75000 | 1.33333 |
| 2.25 | 0.10363 | 0.12798 | 1.74404 | 1.19517 |
| 2.50 | 0.20357 | 0.13034 | 1.73932 | 1.06191 |
| 2.75 | 0.30168 | 0.13212 | 1.73576 | 0.93110 |
| 3.00 | 0.40000 | 0.13333 | 1.73333 | 0.80000 |
| 3.25 | 0.50126 | 0.13393 | 1.73214 | 0.66499 |
| 3.50 | 0.61007 | 0.13377 | 1.73246 | 0.51991 |
| 3.75 | 0.73760 | 0.13242 | 1.73517 | 0.34986 |
| 4.00 | 1.00000 | 0.12500 | 1.75000 | 0.00000 |

Table 2: Exact values of the static critical exponents, rounded to five decimal places.
Table 3: Static data for $\langle N \rangle$, from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, obtained by combining the data for all available $k$ values for each $q$. The quoted error bar corresponds to one standard deviation.
Table 4: Static data for $C_H$, from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, obtained by combining the data for all available $k$ values for each pair $(q, L)$. The quoted error bar corresponds to one standard deviation.
The quoted error bar corresponds to one standard deviation.

Table 5: Static data for $\chi$, from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, obtained by combining the data for all available $k$ values for each pair $(q, L)$. The quoted error bar corresponds to one standard deviation.
Table 6: Static data for $\xi/L$, from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, obtained by combining the data for all available $k$ values for each $q$. The quoted error bar corresponds to one standard deviation.

| $L$  | $q = 1.25$ | $q = 1.50$ | $q = 1.75$ | $q = 2.00$ |
|------|------------|------------|------------|------------|
| 16   | 0.92329(10)| 0.91489(13)| 0.91142(17)| 0.91137(13)|
| 32   | 0.91903(10)| 0.91064(13)| 0.90705(17)| 0.90702(13)|
| 64   | 0.91784(10)| 0.90927(14)| 0.90559(18)| 0.90596(14)|
| 128  | 0.91755(10)| 0.90869(14)| 0.90492(19)| 0.90553(15)|
| 256  | 0.91727(10)| 0.90868(14)| 0.90482(20)| 0.90511(16)|
| 512  | 0.91748(14)| 0.90867(20)| 0.90500(27)| 0.90433(24)|
| 1024 | 0.91729(23)| 0.90855(33)| 0.90595(46)| 0.90467(40)|

| $L$  | $q = 2.25$ | $q = 2.50$ | $q = 2.75$ | $q = 3.00$ |
|------|------------|------------|------------|------------|
| 16   | 0.91427(16)| 0.91952(19)| 0.92604(22)| 0.93484(20)|
| 32   | 0.91015(17)| 0.91564(21)| 0.92363(26)| 0.93262(23)|
| 64   | 0.90856(18)| 0.91384(23)| 0.92198(29)| 0.93200(27)|
| 128  | 0.90866(20)| 0.91375(26)| 0.92226(33)| 0.93190(32)|
| 256  | 0.90774(21)| 0.91372(29)| 0.92091(38)| 0.93187(37)|
| 512  | 0.90817(32)| 0.91313(45)| 0.92086(57)| 0.93186(61)|
| 1024 | 0.90777(56)| 0.91280(78)| 0.92117(109)| 0.93191(114)|

| $L$  | $q = 3.25$ | $q = 3.50$ | $q = 3.75$ | $q = 4.00$ |
|------|------------|------------|------------|------------|
| 16   | 0.94453(23)| 0.95554(27)| 0.96872(31)| 0.98129(29)|
| 32   | 0.94431(28)| 0.95731(33)| 0.97163(40)| 0.98913(38)|
| 64   | 0.94428(33)| 0.95874(41)| 0.97494(51)| 0.99544(50)|
| 128  | 0.94522(40)| 0.95967(52)| 0.97789(66)| 0.99884(67)|
| 256  | 0.94465(49)| 0.96087(65)| 0.98047(86)| 1.00431(92)|
| 512  | 0.94516(84)| 0.96028(115)| 0.98054(155)| 1.00374(178)|
| 1024 | 0.94561(160)| 0.95999(229)| 0.98103(329)| 1.00762(386)|
Table 7: Static data for $C_1$, from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, obtained by combining the data for all available $k$ values for each $q$. The quoted error bar corresponds to one standard deviation.
Table 8: Estimates of $\tau_{\text{int},c}$ from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, for $k = 1$. The quoted error bar corresponds to one standard deviation.

| $L$ | $q = 1.25$       | $q = 1.50$       | $q = 1.75$       | $q = 2.00$       |
|-----|------------------|------------------|------------------|------------------|
| 16  | 1.3601 ± 0.0008  | 2.6552 ± 0.0022  | 4.5086 ± 0.0048  | 7.0555 ± 0.0093 |
| 32  | 1.4516 ± 0.0009  | 2.9749 ± 0.0026  | 5.2959 ± 0.0060  | 8.6748 ± 0.0127 |
| 64  | 1.5433 ± 0.0010  | 3.2988 ± 0.0030  | 6.1255 ± 0.0075  | 10.4931 ± 0.0167|
| 128 | 1.6225 ± 0.0011  | 3.6209 ± 0.0034  | 7.0060 ± 0.0093  | 12.4925 ± 0.0217|
| 256 | 1.7014 ± 0.0012  | 3.9284 ± 0.0039  | 7.8884 ± 0.0110  | 14.7173 ± 0.0278|
| 512 | 1.7679 ± 0.0016  | 4.2341 ± 0.0060  | 8.8097 ± 0.0176  | 17.0928 ± 0.0493|
| 1024| 1.8281 ± 0.0028  | 4.5295 ± 0.0109  | 9.8139 ± 0.0340  | 19.8144 ± 0.0982|

| $L$ | $q = 2.25$       | $q = 2.50$       | $q = 2.75$       | $q = 3.00$       |
|-----|------------------|------------------|------------------|------------------|
| 16  | 10.5266 ± 0.0169 | 15.0260 ± 0.0288 | 20.8370 ± 0.0469 | 28.2660 ± 0.0739|
| 32  | 13.4708 ± 0.0243 | 20.0908 ± 0.0443 | 29.2540 ± 0.0778 | 41.3643 ± 0.1309|
| 64  | 17.0098 ± 0.0346 | 26.6206 ± 0.0675 | 40.4635 ± 0.1264 | 60.2637 ± 0.2298|
| 128 | 21.2642 ± 0.0483 | 34.7971 ± 0.1008 | 55.3810 ± 0.2025 | 87.1456 ± 0.3990|
| 256 | 26.1625 ± 0.0657 | 44.9461 ± 0.1479 | 75.4692 ± 0.3214 | 124.4389 ± 0.6808|
| 512 | 32.0642 ± 0.1234 | 57.4369 ± 0.3026 | 101.4174 ± 0.7096 | 175.6975 ± 1.6194|
| 1024| 38.6893 ± 0.2657 | 73.2671 ± 0.6912 | 135.6123 ± 1.7396 | 246.8774 ± 4.2833|

| $L$ | $q = 3.25$       | $q = 3.50$       | $q = 3.75$       | $q = 4.00$       |
|-----|------------------|------------------|------------------|------------------|
| 16  | 37.4020 ± 0.1125 | 48.7655 ± 0.1673 | 62.7027 ± 0.2440 | 79.4340 ± 0.3475|
| 32  | 57.1966 ± 0.2126 | 78.7249 ± 0.3431 | 106.2816 ± 0.5379| 142.0782 ± 0.8310|
| 64  | 87.3178 ± 0.4006 | 126.1715 ± 0.6960| 181.3644 ± 1.1991| 253.4333 ± 1.9791|
| 128 | 134.4255 ± 0.7644| 206.2906 ± 1.4542| 312.1399 ± 2.7062| 462.6885 ± 4.8818|
| 256 | 202.0761 ± 1.4086| 329.0069 ± 2.9261| 534.9698 ± 6.0685| 875.8576 ± 12.7193|
| 512 | 309.5691 ± 3.7861| 532.7875 ± 8.5469| 933.1742 ± 19.8120| 1652.8786 ± 46.7469|
| 1024| 446.2539 ± 10.4070| 822.7567 ± 26.0508| 1581.9426 ± 69.4509| 3049.0566 ± 186.3115|


Table 9: Estimates of $r_{\text{int},N}$ from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, for $k = 1$. The quoted error bar corresponds to one standard deviation.
| $L$  | $q = 1.25$            | $q = 1.50$            | $q = 1.75$            | $q = 2.00$            |
|------|-----------------------|-----------------------|-----------------------|-----------------------|
| 16   | 1.3386 ± 0.0008       | 2.6034 ± 0.0021       | 4.4205 ± 0.0047       | 6.9286 ± 0.0091       |
| 32   | 1.4078 ± 0.0009       | 2.8602 ± 0.0025       | 5.0735 ± 0.0058       | 8.3286 ± 0.0122       |
| 64   | 1.4676 ± 0.0010       | 3.0892 ± 0.0028       | 5.7122 ± 0.0070       | 9.8003 ± 0.0156       |
| 128  | 1.5118 ± 0.0010       | 3.2941 ± 0.0031       | 6.3331 ± 0.0084       | 11.2921 ± 0.0196      |
| 256  | 1.5495 ± 0.0011       | 3.4664 ± 0.0034       | 6.8843 ± 0.0096       | 12.8477 ± 0.0243      |
| 512  | 1.5770 ± 0.0015       | 3.6264 ± 0.0051       | 7.4316 ± 0.0149       | 14.3819 ± 0.0415      |
| 1024 | 1.6009 ± 0.0024       | 3.7573 ± 0.0090       | 7.9652 ± 0.0276       | 16.0737 ± 0.0796      |

| $L$  | $q = 2.25$            | $q = 2.50$            | $q = 2.75$            | $q = 3.00$            |
|------|-----------------------|-----------------------|-----------------------|-----------------------|
| 16   | 10.3488 ± 0.0166      | 14.8081 ± 0.0284      | 20.5896 ± 0.0464      | 28.0138 ± 0.0733      |
| 32   | 12.9917 ± 0.0235      | 19.4472 ± 0.0429      | 28.4240 ± 0.0756      | 40.4259 ± 0.1279      |
| 64   | 15.9573 ± 0.0325      | 25.1767 ± 0.0639      | 38.5615 ± 0.1205      | 57.9953 ± 0.2212      |
| 128  | 19.3490 ± 0.0439      | 32.1053 ± 0.0930      | 51.5394 ± 0.1884      | 82.2076 ± 0.3764      |
| 256  | 23.0132 ± 0.0578      | 40.1938 ± 0.1322      | 68.5937 ± 0.2922      | 115.2704 ± 0.6306     |
| 512  | 27.2615 ± 0.1049      | 49.7110 ± 0.2619      | 89.9306 ± 0.6292      | 159.7078 ± 1.4720     |
| 1024 | 31.7864 ± 0.2183      | 61.4579 ± 0.5798      | 116.7935 ± 1.4982     | 220.6996 ± 3.8291     |

| $L$  | $q = 3.25$            | $q = 3.50$            | $q = 3.75$            | $q = 4.00$            |
|------|-----------------------|-----------------------|-----------------------|-----------------------|
| 16   | 37.2039 ± 0.1119      | 48.6233 ± 0.1669      | 62.6624 ± 0.2439      | 79.6370 ± 0.3484      |
| 32   | 56.2252 ± 0.2090      | 77.8100 ± 0.3391      | 105.4646 ± 0.5338     | 141.5583 ± 0.8279     |
| 64   | 84.5549 ± 0.3879      | 123.3440 ± 0.6804     | 178.5750 ± 1.1806     | 250.9246 ± 1.9595     |
| 128  | 128.5492 ± 0.7310     | 199.5134 ± 1.4064     | 305.0319 ± 2.6446     | 457.0131 ± 4.8219     |
| 256  | 190.1111 ± 1.3252     | 315.2552 ± 2.8038     | 520.8366 ± 5.9082     | 860.9974 ± 12.5035    |
| 512  | 287.5530 ± 3.5168     | 506.8770 ± 8.1313     | 898.2129 ± 19.0697    | 1617.6331 ± 45.7501   |
| 1024 | 408.0224 ± 9.5154     | 777.6178 ± 24.6216    | 1526.7086 ± 67.0260   | 2980.4947 ± 182.1220  |

Table 10: Estimates of $\tau_{\text{int},S_2}$ from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, for $k = 1$. The quoted error bar corresponds to one standard deviation.
Table 11: Estimates of $\tau_{int,F}$ from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, for $k = 1$. The quoted error bar corresponds to one standard deviation.
| $L$ | $q = 1.25$ | $q = 1.50$ | $q = 1.75$ | $q = 2.00$ |
|-----|----------|----------|----------|----------|
| 16  | 1.3101 ± 0.0008 | 2.5278 ± 0.0021 | 4.2775 ± 0.0046 | 6.6989 ± 0.0088 |
| 32  | 1.3708 ± 0.0008 | 2.7571 ± 0.0024 | 4.8684 ± 0.0056 | 7.9857 ± 0.0117 |
| 64  | 1.4224 ± 0.0009 | 2.9585 ± 0.0027 | 5.4412 ± 0.0067 | 9.3266 ± 0.0149 |
| 128 | 1.4596 ± 0.0009 | 3.1373 ± 0.0030 | 5.9956 ± 0.0079 | 10.6786 ± 0.0186 |
| 256 | 1.4918 ± 0.0010 | 3.2868 ± 0.0033 | 6.4566 ± 0.0091 | 12.0813 ± 0.0229 |
| 512 | 1.5155 ± 0.0014 | 3.4287 ± 0.0048 | 6.9711 ± 0.0139 | 13.4617 ± 0.0388 |
| 1024| 1.5364 ± 0.0023 | 3.5411 ± 0.0085 | 7.4506 ± 0.0258 | 14.9718 ± 0.0742 |

| $L$ | $q = 2.25$ | $q = 2.50$ | $q = 2.75$ | $q = 3.00$ |
|-----|----------|----------|----------|----------|
| 16  | 10.0086 ± 0.0161 | 14.3476 ± 0.0275 | 20.0130 ± 0.0451 | 27.2880 ± 0.0714 |
| 32  | 12.4722 ± 0.0225 | 18.7396 ± 0.0414 | 27.4966 ± 0.0731 | 39.3047 ± 0.1244 |
| 64  | 15.2204 ± 0.0310 | 24.0900 ± 0.0611 | 37.1348 ± 0.1160 | 56.1933 ± 0.2143 |
| 128 | 18.3474 ± 0.0416 | 30.5895 ± 0.0886 | 49.4474 ± 0.1808 | 79.3969 ± 0.3635 |
| 256 | 21.6896 ± 0.0544 | 38.1134 ± 0.1254 | 65.5494 ± 0.2792 | 111.0562 ± 0.6076 |
| 512 | 25.5866 ± 0.0984 | 46.9593 ± 0.2474 | 85.7120 ± 0.5997 | 153.1954 ± 1.4120 |
| 1024| 29.7135 ± 0.2041 | 57.6989 ± 0.5443 | 110.7213 ± 1.4203 | 211.5007 ± 3.6695 |

| $L$ | $q = 3.25$ | $q = 3.50$ | $q = 3.75$ | $q = 4.00$ |
|-----|----------|----------|----------|----------|
| 16  | 36.4019 ± 0.1095 | 47.7804 ± 0.1640 | 61.9085 ± 0.2409 | 79.1044 ± 0.3461 |
| 32  | 54.9946 ± 0.2045 | 76.5469 ± 0.3336 | 104.2901 ± 0.5278 | 141.5283 ± 0.8278 |
| 64  | 82.5872 ± 0.3789 | 121.5950 ± 0.6708 | 177.0526 ± 1.1706 | 251.4255 ± 1.9634 |
| 128 | 125.2155 ± 0.7120 | 196.1776 ± 1.3829 | 303.5483 ± 2.6317 | 461.4449 ± 4.8687 |
| 256 | 185.1345 ± 1.2905 | 310.3888 ± 2.7605 | 518.9138 ± 5.8864 | 868.1885 ± 12.6080 |
| 512 | 280.0361 ± 3.4249 | 500.1480 ± 8.0233 | 900.8537 ± 19.1258 | 1637.5816 ± 46.3143 |
| 1024| 395.2831 ± 9.2183 | 762.0676 ± 24.1293 | 1519.4102 ± 66.7056 | 3054.3411 ± 186.6344 |

Table 12: Estimates of $\tau_{\text{int},C_i}$ from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, for $k = 1$. The quoted error bar corresponds to one standard deviation.
error bar corresponds to one standard deviation.

Table 13: Estimates of $\tau_{\text{int},c}$ from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, for $k = 1$. The quoted error bar corresponds to one standard deviation.
Table 14: Estimates of $\tau_{\text{int},c_1}$ from the Monte Carlo simulations at the critical point of the 2-dimensional random cluster model, as a function of $L$ and $q$, for $k = 1$. The quoted error bar corresponds to one standard deviation.

| $L$ | $q = 1.25$       | $q = 1.50$       | $q = 1.75$       | $q = 2.00$       |
|-----|-----------------|-----------------|-----------------|-----------------|
| 16  | 1.0627 ± 0.0007 | 1.9037 ± 0.0015 | 3.0988 ± 0.0033 | 4.7417 ± 0.0063 |
| 32  | 1.0882 ± 0.0007 | 2.0070 ± 0.0017 | 3.3763 ± 0.0039 | 5.3802 ± 0.0079 |
| 64  | 1.1096 ± 0.0007 | 2.0948 ± 0.0019 | 3.6466 ± 0.0045 | 6.0321 ± 0.0096 |
| 128 | 1.1247 ± 0.0007 | 2.1761 ± 0.0021 | 3.9002 ± 0.0051 | 6.6878 ± 0.0116 |
| 256 | 1.1379 ± 0.0008 | 2.2400 ± 0.0022 | 4.1293 ± 0.0058 | 7.3441 ± 0.0139 |
| 512 | 1.1487 ± 0.0011 | 2.3012 ± 0.0032 | 4.3459 ± 0.0087 | 7.9879 ± 0.0230 |
| 1024| 1.1573 ± 0.0018 | 2.3518 ± 0.0056 | 4.5466 ± 0.0158 | 8.6848 ± 0.0430 |

| $L$ | $q = 2.25$       | $q = 2.50$       | $q = 2.75$       | $q = 3.00$       |
|-----|-----------------|-----------------|-----------------|-----------------|
| 16  | 6.9598 ± 0.0112 | 9.8960 ± 0.0190 | 13.7462 ± 0.0310 | 18.6750 ± 0.0488 |
| 32  | 8.2349 ± 0.0149 | 12.2476 ± 0.0270 | 17.8387 ± 0.0474 | 25.5572 ± 0.0809 |
| 64  | 9.6298 ± 0.0196 | 15.0431 ± 0.0382 | 23.1057 ± 0.0722 | 35.0341 ± 0.1336 |
| 128 | 11.1923 ± 0.0254 | 18.4369 ± 0.0534 | 29.6671 ± 0.1085 | 47.6201 ± 0.2180 |
| 256 | 12.8179 ± 0.0322 | 22.2087 ± 0.0731 | 38.1085 ± 0.1623 | 65.1028 ± 0.3562 |
| 512 | 14.7338 ± 0.0567 | 26.6674 ± 0.1405 | 48.6690 ± 0.3405 | 87.6815 ± 0.8081 |
| 1024| 16.7174 ± 0.1148 | 31.8749 ± 0.3007 | 61.5270 ± 0.7892 | 118.9944 ± 2.0645 |

| $L$ | $q = 3.25$       | $q = 3.50$       | $q = 3.75$       | $q = 4.00$       |
|-----|-----------------|-----------------|-----------------|-----------------|
| 16  | 24.9741 ± 0.0751 | 32.8814 ± 0.1128 | 42.7608 ± 0.1664 | 54.9178 ± 0.2403 |
| 32  | 35.7876 ± 0.1330 | 50.1792 ± 0.2187 | 68.8743 ± 0.3486 | 94.4193 ± 0.5522 |
| 64  | 51.7567 ± 0.2374 | 77.0195 ± 0.4249 | 113.3164 ± 0.7492 | 163.7732 ± 1.2789 |
| 128 | 76.0751 ± 0.4326 | 120.8170 ± 0.8516 | 190.4901 ± 1.6515 | 297.1283 ± 3.1350 |
| 256 | 109.5690 ± 0.7638 | 188.0194 ± 1.6722 | 321.0373 ± 3.6417 | 553.7972 ± 8.0423 |
| 512 | 162.8091 ± 1.9912 | 298.8569 ± 4.7942 | 551.3396 ± 11.7054 | 1044.3142 ± 29.5354 |
| 1024| 226.4912 ± 5.2819 | 452.2253 ± 14.3188 | 928.5042 ± 40.7635 | 1950.0423 ± 119.1566 |
Table 15: Fits for $\xi/L$. The quoted error bar corresponds to one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%).
Table 16: Fits of the susceptibility $\chi$ to a pure power law. The quoted error bar corresponds to one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%). The final two columns show the deviation of the estimated $\gamma/\nu$ from the known exact value, in absolute terms and in units of its standard deviation.
| $q$ | $L_{\text{min}}$ | $A$ | $A$ | $\chi^2$ | DF | CL(%) | $B$ | $p$ | $\chi^2$ | DF | CL(%) | $\Delta_1$ (exact) |
|-----|-----------------|-----|-----|--------|-----|-------|-----|-----|--------|-----|-------|---------------------|
| 1.25 | 256             | 1.05017(5) | 2.38 | 2   | 30   | 16   | 1.05013(8) | 0.011(5) | 0.87(19) | 4.72 | 4   | 32   | 1.80702             |
| 1.50 | 128             | 1.06417(6) | 0.70 | 3   | 87   | 16   | 1.06379(53) | 0.003(1) | 0.34(30) | 1.55 | 4   | 82   | 1.63551             |
| 1.75 | 64              | 1.07811(7) | 7.16 | 4   | 13   | —    | —    | —    | —    | —    | —    | —    | 1.47905             |
| 2.00 | 512             | 1.09148(15) | 0.50 | 1   | 48   | [16]  | 1.09195(8) | -0.231(316) | 1.88(50) | 15.63 | 4   | 0.4  | 1.33333             |
| 2.25 | 256             | 1.10545(13) | 2.19 | 2   | 33   | [16]  | 1.10565(15) | -0.078(43) | 1.22(21) | 7.84  | 4   | 10   | 1.19517             |
| 2.50 | 128             | 1.11875(14) | 2.34 | 3   | 51   | 16   | 1.11912(26) | -0.062(22) | 0.94(14) | 3.96  | 4   | 41   | 1.06191             |
| 2.75 | 128             | 1.13159(18) | 0.85 | 3   | 84   | 16   | 1.13192(29) | -0.161(51) | 1.12(12) | 5.80  | 4   | 21   | 0.93110             |
| 3.00 | 256             | 1.14415(25) | 1.71 | 2   | 43   | 16   | 1.14541(49) | -0.086(12) | 0.73(6)  | 0.43  | 4   | 98   | 0.80000             |
| 3.25 | 512             | 1.15559(63) | 0.25 | 1   | 62   | 16   | 1.15656(67) | -0.111(14) | 0.70(6)  | 2.74  | 4   | 60   | 0.66499             |
| 3.50 | 256             | 1.16102(46) | 0.35 | 2   | 84   | 16   | 1.16365(96) | -0.119(15) | 0.65(6)  | 0.99  | 4   | 91   | 0.51991             |
| 3.75 | 256             | 1.15852(61) | 0.42 | 2   | 81   | 16   | 1.16358(199) | -0.080(9) | 0.48(7)  | 0.79  | 4   | 94   | 0.34986             |
| 4.00 | [512]           | 1.07693(124) | 2.97 | 1   | 8    | —    | —    | —    | —    | —    | —    | —    | 0.00000             |

Table 17: Fits for $\chi/L^{\gamma/\nu}$ with $\gamma/\nu$ set to its exact value (1.10). The quoted error bar corresponds to one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%).
| $q$ | $L_{\text{min}}$ | $\beta/\nu$ | $A$ | $\chi^2$ | DF | CL(%) | $\beta/\nu$ (exact) | deviation | dev($\sigma$) |
|-----|-----------------|-------------|-----|---------|----|-------|----------------------|-----------|-----------|
| 1.25 | 256             | 0.11111(6)  | 0.9896(3) | 1.29 | 1   | 26    | 0.11118             | -0.00007  | -1.2      |
| 1.50 | 16              | 0.11681(2)  | 0.9958(1) | 2.01 | 5   | 85    | 0.11678             | 0.00003   | 1.6       |
| 1.75 | 128             | 0.12115(8)  | 1.0004(4) | 2.21 | 2   | 33    | 0.12131             | -0.00016  | -2.1      |
| 2.00 | 128             | 0.12518(7)  | 1.0080(4) | 2.04 | 2   | 36    | 0.12500             | 0.00018   | 2.7       |
| 2.25 | 256             | 0.12787(17) | 1.0118(10) | 1.20 | 1   | 27    | 0.12798             | -0.00011  | -0.6      |
| 2.50 | 128             | 0.13033(13) | 1.0177(7)  | 1.48 | 2   | 48    | 0.13034             | -0.00001  | -0.10     |
| 2.75 | 128             | 0.13230(18) | 1.0237(10) | 1.65 | 2   | 44    | 0.13212             | 0.00018   | 1.0       |
| 3.00 | 64              | 0.13283(11) | 1.0245(5)  | 1.24 | 3   | 74    | 0.13333             | -0.00050  | -4.6      |
| 3.25 | 128             | 0.13366(24) | 1.0295(13) | 0.86 | 2   | 65    | 0.13393             | -0.00027  | -1.1      |
| 3.50 | 64              | 0.13315(19) | 1.0289(9)  | 2.39 | 3   | 50    | 0.13377             | -0.00062  | -3.3      |
| 3.75 | 32              | 0.13174(15) | 1.0263(7)  | 1.74 | 4   | 78    | 0.13242             | -0.00067  | -4.4      |
| 4.00 | 256             | 0.13200(104) | 1.0357(62) | 0.54 | 1   | 46    | 0.12500             | 0.00700   | 6.8       |

Table 18: Fits of the mean size $C_1$ of the largest cluster to a pure power law. The quoted error bar corresponds to one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%). The final two columns show the deviation of the estimated $\beta/\nu$ from the known exact value, in absolute terms and in units of its standard deviation.
Table 19: Fits for the specific heat $C_H$. The quoted error bar corresponds to one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%). The final two columns show the deviation of the estimated $\alpha/\nu$ (from the fit $A L^{\alpha/\nu} + B$) from the known exact value, in absolute terms and in units of its standard deviation.

| $q$ | $L_{\text{min}}$ | $\alpha/\nu$ | $A$ | $\chi^2$ | DF | CL(%) | $L_{\text{min}}$ | $\alpha/\nu$ | $A$ | $B$ | $\chi^2$ | DF | CL(%) | $\alpha/\nu$ (exact) | dev | dev($\sigma$) |
|-----|------------------|---------------|-----|----------|----|-------|------------------|---------------|-----|-----|----------|----|-------|---------------------|-----|----------|
| 2.50 | [256] 0.2778(10) | 1.573(10)     | 9.47 | 1        | 3  | 4.2  | 64               | 0.2025(32)    | 3.421(109) | -3.176(150) | 0.73 | 2     | 69                  | 0.20357 | -0.001  | -0.3 |
| 2.75 | [256] 0.3488(13) | 1.457(11)     | 7.44 | 1        | 3  | 4.2  | 32               | 0.2995(18)    | 2.293(33)  | -2.001(54)  | 0.39 | 3     | 94                  | 0.30168 | -0.002  | -1.2 |
| 3.00 | [256] 0.4254(13) | 1.295(10)     | 0.42 | 1        | 1  | 0.9  | 32               | 0.3947(15)    | 1.681(18)  | -1.296(37)  | 2.14 | 3     | 54                  | 0.40000 | -0.005  | -3.6 |
| 3.25 | [256] 0.5063(17) | 1.120(11)     | 2.94 | 1        | 1  | 0.9  | 32               | 0.4902(16)    | 1.275(14)  | -0.771(38)  | 3.04 | 3     | 39                  | 0.50126 | -0.011  | -6.8 |
| 3.50 | 64 0.5952(7)    | 0.921(3)      | 0.94 | 3        | 4  | 82    | 64               | 0.5945(35)    | 0.925(22)  | -0.020(102) | 0.90 | 2     | 64                  | 0.61007 | -0.016  | -4.5 |
| 3.75 | [256] 0.6920(31) | 0.724(13)     | 0.69 | 1        | 1  | 41    | 128              | 0.7095(86)    | 0.635(38)  | 1.149(369)  | 0.03 | 1     | 86                  | 0.73760 | -0.028  | -3.3 |
| 4.00 | 256 0.7998(35)  | 0.534(11)     | 0.35 | 1        | 55 | 128   | 0.8338(85)       | 0.416(24)     | 2.713(392) | 0.53 | 1     | 47                  | 1 (log$^{-3/2}$) | -0.166  | -19.6 |
Table 20: Ratios \( \tau_{\text{int}, \epsilon'}(k = 1)/\tau_{\text{int}, \epsilon'}(k = 2) \) as a function of \( L \) for \( q = 2 \) and \( q = 3.25 \). Error bars are one standard deviation.

| \( L \) | \( q = 2 \)          | \( q = 3.25 \)     |
|--------|----------------------|--------------------|
| 16     | 2.143(3)             | 2.020(7)           |
| 32     | 2.123(4)             | 2.013(9)           |
| 64     | 2.100(4)             | 1.990(11)          |
| 128    | 2.080(4)             | 2.015(14)          |
| 256    | 2.069(5)             | 1.991(17)          |
| 512    | 2.047(7)             | 2.034(30)          |
| 1024   | 2.051(12)            | 1.953(56)          |

 Fits to \( \tau_{\text{int}, \epsilon'}(1)/\tau_{\text{int}, \epsilon'}(k) = A \)

| \( q \) | \( k \) | \( L_{\text{min}} \) | \( A \)     | \( \chi^2 \) | DF | CL(%) |
|---------|--------|----------------------|-------------|-------------|----|-------|
| 2.00    | 2      | 512                  | 2.048(6)    | 0.09        | 1  | 76    |
| 2.25    | 2      | 256                  | 2.036(5)    | 0.51        | 2  | 77    |
| 2.50    | 2      | 128                  | 2.022(5)    | 0.02        | 3  | 100   |
| 2.75    | 2      | 128                  | 2.006(6)    | 0.87        | 3  | 83    |
| 3.00    | 2      | 32                   | 2.015(5)    | 3.63        | 5  | 60    |
| 3.25    | 2      | 32                   | 2.005(6)    | 5.40        | 5  | 37    |
| 3.50    | 2      | 16                   | 2.015(5)    | 6.74        | 6  | 35    |
| 3.75    | 2      | 16                   | 2.014(6)    | 1.68        | 6  | 95    |
| 4.00    | 2      | 16                   | 2.011(7)    | 2.75        | 6  | 84    |
| 3.00    | 3      | 256                  | 3.011(16)   | 0.66        | 2  | 72    |
| 3.25    | 3      | 64                   | 3.012(11)   | 3.18        | 4  | 53    |
| 3.50    | 3      | 64                   | 3.014(13)   | 3.67        | 4  | 45    |
| 3.75    | 3      | 16                   | 3.034(9)    | 2.32        | 6  | 89    |
| 4.00    | 3      | 64                   | 2.988(19)   | 1.66        | 4  | 80    |
| 4.00    | 4      | 64                   | 4.002(25)   | 3.01        | 4  | 56    |

Table 21: Estimates for the limiting ratios \( \tau_{\text{int}, \epsilon'}(1)/\tau_{\text{int}, \epsilon'}(k) \) as \( L \to \infty \) from fits to a constant \( A \). Error bars are one standard deviation.
Table 22: Estimates for $z_{\text{int}, E'}$ from fits to $\tau_{\text{int}, E'} = AL^z$. Error bars are one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%).
| $q$ | $k$ | $L_{\text{min}}$ | $z_{\text{int}, \varepsilon}'$ | $A$   | $B$   | $\chi^2$ | DF | CL(%) |
|-----|-----|-----------------|-----------------|------|------|--------|----|------|
| 1.25 | 1   | 128             | -0.213(35)      | -1.61(5) | 2.19(8) | 0.49  | 1   | 48    |
| 1.50 | 1   | 32              | -0.034(7)       | -15.86(2.92) | 17.09(2.96) | 0.82  | 3   | 85    |
| 1.75 | 1   | 128             | 0.085(26)       | 9.66(4.42) | -7.56(4.82) | 0.33  | 1   | 57    |
| 2.00 | 1   | 16              | 0.145(4)        | 10.37(51) | -8.44(57) | 2.32  | 4   | 68    |
| 2.00 | 2   | 32              | 0.141(5)        | 5.48(32) | -4.83(36) | 1.32  | 3   | 72    |
| 2.25 | 1   | 32              | 0.235(8)        | 8.92(61) | -6.68(86) | 1.62  | 3   | 65    |
| 2.25 | 2   | [128]           | 0.200(19)       | 6.37(1.31) | -6.45(1.89) | 3.44  | 1   | 6     |
| 2.50 | 1   | 32              | 0.315(8)        | 9.01(59) | -6.75(1.00) | 0.21  | 3   | 98    |
| 2.50 | 2   | 64              | 0.294(11)       | 5.38(48) | -5.23(85) | 0.91  | 2   | 63    |
| 2.75 | 1   | 16              | 0.411(5)        | 8.19(30) | -4.76(57) | 1.06  | 4   | 90    |
| 2.75 | 2   | 128             | 0.348(24)       | 6.87(1.40) | -9.62(3.28) | 0.47  | 1   | 49    |
| 3.00 | 1   | 32              | 0.481(10)       | 9.08(63) | -6.79(1.68) | 0.84  | 3   | 84    |
| 3.00 | 2   | 16              | 0.500(4)        | 4.00(11) | -2.15(25) | 1.35  | 4   | 85    |
| 3.00 | 3   | 16              | 0.497(3)        | 2.72(6) | -1.71(13) | 2.40  | 4   | 66    |
| 3.25 | 1   | 64              | 0.558(21)       | 9.75(1.44) | -11.92(5.96) | 2.93  | 2   | 23    |
| 3.25 | 2   | 32              | 0.577(8)        | 4.24(22) | -2.93(78) | 0.71  | 3   | 87    |
| 3.25 | 3   | 64              | 0.565(12)       | 3.10(26) | -3.51(1.12) | 0.22  | 2   | 90    |
| 3.50 | 1   | 64              | 0.648(25)       | 9.50(1.56) | -14.38(8.87) | 1.15  | 2   | 56    |
| 3.50 | 2   | 16              | 0.691(6)        | 3.57(11) | -0.13(40) | 2.27  | 4   | 69    |
| 3.50 | 3   | 32              | 0.689(8)        | 2.42(11) | -0.66(55) | 0.34  | 3   | 95    |
| 3.75 | 1   | 16              | 0.790(9)        | 6.69(32) | 2.96(1.49) | 0.33  | 4   | 99    |
| 3.75 | 2   | 16              | 0.800(6)        | 3.15(10) | 2.21(50) | 1.62  | 4   | 81    |
| 3.75 | 3   | 16              | 0.795(5)        | 2.15(6) | 1.10(28) | 2.91  | 4   | 57    |
| 4.00 | 1   | 32              | 0.935(18)       | 4.76(48) | 20.52(5.09) | 0.66  | 3   | 88    |
| 4.00 | 2   | 32              | 0.925(13)       | 2.54(18) | 7.49(1.85) | 0.59  | 3   | 90    |
| 4.00 | 3   | 16              | 0.902(6)        | 1.94(6) | 2.60(36) | 2.07  | 4   | 72    |
| 4.00 | 4   | 128             | 0.993(36)       | 0.80(19) | 18.49(6.33) | 0.07  | 1   | 79    |

Table 23: Estimates for $z_{\text{int}, \varepsilon}'$ from fits to $\tau_{\text{int}, \varepsilon}' = AL^z + B$. Error bars are one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%).
Fits to $\tau_{\text{int},E'} = A L^2$

| $q$  | $L_{\text{min}}$ | $z$    | $L_{\text{min}}$ | $z$    | Difference | $\alpha/\nu$ (exact) | $\beta/\nu$ (exact) |
|------|------------------|--------|------------------|--------|------------|---------------------|---------------------|
| 1.25 | —                | —      | 128              | −0.213(35) | —          | −0.35527            | 0.11118             |
| 1.50 | —                | —      | 32               | −0.034(7) | —          | −0.22663            | 0.11678             |
| 1.75 | 256              | 0.158(2)| 128              | 0.085(26)| 0.074      | −0.10929            | 0.12131             |
| 2.00 | 256              | 0.215(3)| 16–32            | 0.143(3) | 0.072      | 0.00000             | 0.12500             |
| 2.25 | 256              | 0.286(4)| 32               | 0.235(8) | 0.051      | 0.10363             | 0.12798             |
| 2.50 | 256              | 0.353(3)| 32–64            | 0.307(7) | 0.046      | 0.20357             | 0.13034             |
| 2.75 | 256              | 0.424(8)| 16–128           | 0.408(5) | 0.016      | 0.30168             | 0.13212             |
| 3.00 | 128–256          | 0.511(3)| 16–32            | 0.497(3) | 0.013      | 0.40000             | 0.13333             |
| 3.25 | 128–256          | 0.587(5)| 32–64            | 0.572(7) | 0.015      | 0.50126             | 0.13393             |
| 3.50 | 16–128           | 0.693(1)| 16–64            | 0.689(4) | 0.004      | 0.61007             | 0.13377             |
| 3.75 | 32–128           | 0.784(2)| 16               | 0.796(4) | −0.012     | 0.73760             | 0.13242             |
| 4.00 | 64–256           | 0.898(4)| 16–128           | 0.910(5) | −0.012     | 1.00000             | 0.12500             |

Table 24: Summary of estimates for $z_{\text{int},E'}$, averaged over all values of $k$ for which the confidence level is $\geq 10\%$. Error bars are one standard deviation.
\[ \tau/C_H = A \log L + B \]

| \( q \) | \( k \) | \( L_{\text{min}} \) | \( A \) | \( B \) | \( \text{CL} \) | \( p \) | \( A \) | \( \text{CL} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1.75 | 1   | 256 | 0.159(8) | 2.28(5) | 9    | 0.049(3) | 2.41(4) | 14   |
| 2.00 | 1   | 128 | 0.224(8) | 2.74(4) | 23   | 0.056(2) | 2.91(3) | 44   |
| 2.00 | 2   | 128 | 0.125(3) | 1.23(1) | 8    | 0.064(1) | 1.34(1) | 30   |
| 2.00 | 2   | 256 | 0.134(5) | 1.18(3) | 59   | 0.067(2) | 1.32(2) | 36   |
| 2.25 | 1   | 128 | 0.337(13) | 3.11(7) | 22   | 0.068(3) | 3.42(5) | 45   |
| 2.25 | 1   | 256 | 0.373(25) | 2.89(14) | 86   | 0.072(5) | 3.33(9) | 69   |
| 2.25 | 2   | 128 | 0.178(5) | 1.45(2) | 30   | 0.073(2) | 1.63(2) | 25   |
| 2.50 | 1   | 64  | 0.410(13) | 3.86(6) | 33   | 0.069(2) | 4.19(4) | 79   |
| 2.50 | 1   | 128 | 0.438(22) | 3.71(12) | 62   | 0.071(3) | 4.13(8) | 85   |
| 2.50 | 2   | 64  | 0.215(5) | 1.85(2) | 15   | 0.073(2) | 2.02(2) | 76   |
| 2.50 | 2   | 256 | 0.244(14) | 1.68(8) | 55   | 0.077(4) | 1.97(5) | 74   |
| 2.75 | 1   | 64  | 0.531(21) | 4.55(10) | 30   | 0.073(3) | 4.99(7) | 65   |
| 2.75 | 1   | 128 | 0.584(35) | 4.26(19) | 89   | 0.078(5) | 4.87(12) | 88   |
| 2.75 | 2   | 64  | 0.286(7) | 2.15(3) | 15   | 0.073(3) | 4.99(7) | 65   |
| 3.00 | 1   | 32  | 0.619(20) | 5.63(9) | 61   | 0.073(2) | 6.05(6) | 89   |
| 3.00 | 1   | 64  | 0.660(32) | 5.42(15) | 100  | 0.075(4) | 5.98(10) | 90   |
| 3.00 | 2   | 64  | 0.354(11) | 2.57(5) | 32   | 0.081(2) | 2.89(4) | 82   |
| 3.00 | 1   | 128 | 0.383(19) | 2.41(10) | 94   | 0.084(4) | 2.83(6) | 98   |
| 3.25 | 1   | 32  | 0.833(30) | 6.30(13) | 22   | 0.083(3) | 6.91(9) | 41   |
| 3.25 | 1   | 64  | 0.907(48) | 5.93(23) | 55   | 0.086(4) | 6.79(15) | 38   |
| 3.25 | 2   | 32  | 0.421(11) | 3.12(4) | 48   | 0.084(2) | 3.43(3) | 99   |
| 3.25 | 2   | 64  | 0.443(17) | 3.01(8) | 81   | 0.084(3) | 3.42(5) | 98   |
| 3.50 | 1   | 32  | 0.387(9) | 2.24(4) | 43   | 0.097(2) | 2.56(2) | 99   |
| 3.50 | 1   | 64  | 0.405(14) | 2.15(7) | 69   | 0.097(3) | 2.57(4) | 98   |
| 3.75 | 1   | 32  | 1.475(67) | 7.73(28) | 81   | 0.103(4) | 9.01(17) | 92   |
| 3.75 | 2   | 32  | 0.767(23) | 3.69(10) | 52   | 0.107(3) | 4.39(6) | 79   |
| 3.75 | 2   | 64  | 0.816(39) | 3.45(18) | 84   | 0.108(5) | 4.38(10) | 64   |
| 3.75 | 3   | 32  | 0.502(13) | 2.49(5) | 0.9  | 0.106(3) | 2.93(3) | 35   |
| 3.75 | 3   | 64  | 0.554(22) | 1.73(10) | 52   | 0.124(5) | 2.41(5) | 63   |

Table 25: Fits for the ratio \( \tau_{\text{int,}E'}/C_H \). The quoted error bar corresponds to one standard deviation.
Table 26: Estimates for \( z_{\text{int}, C_2} \) from fits to \( \tau_{\text{int}, C_2} = AL^z \). Error bars are one standard deviation. \( L_{\text{min}} \) in brackets indicates a poor fit (confidence level < 10%).
| $q$ | $k$ | $L_{\text{min}}$ | $z_{\text{int},c_2}$ | $A$ | $B$ | $\chi^2$ | DF | CL(%) |
|-----|-----|------------------|-----------------|-----|-----|----------|----|------|
| 1.25 | 1   | 64               | -0.189(89)      | -0.19(1) | 1.18(3) | 0.58     | 2   | 75   |
| 1.50 | 1   | 128              | -0.170(97)      | -1.03(4) | 2.51(22) | 6.63     | 1   | 1    |
| 1.75 | 1   | 128              | 0.047(67)       | 3.77(6.81) | -1.19(7.02) | 0.34     | 1   | 56   |
| 2.00 | 1   | 64               | 0.006(26)       | 104.55(437.90) | -102.03(438.13) | 0.85     | 2   | 65   |
| 2.00 | 2   | 64               | 0.019(17)       | 16.72(16.04) | -15.66(16.12) | 1.97     | 2   | 37   |
| 2.25 | 1   | 16               | 0.118(8)        | 8.25(80)   | -5.12(87) | 2.58     | 4   | 63   |
| 2.25 | 2   | 16               | 0.124(5)        | 3.81(24)   | -2.45(26) | 5.01     | 4   | 29   |
| 2.50 | 1   | 16               | 0.216(7)        | 6.24(40)   | -2.55(51) | 1.16     | 4   | 88   |
| 2.50 | 2   | 128              | 0.240(30)       | 2.41(72)   | -0.31(1.17) | 0.07     | 1   | 79   |
| 2.75 | 1   | 16               | 0.326(7)        | 5.04(27)   | -0.43(43) | 0.89     | 4   | 93   |
| 2.75 | 2   | 128              | 0.252(31)       | 5.00(1.47) | -5.15(2.47) | 0.00     | 1   | 98   |
| 3.00 | 1   | 16               | 0.431(8)        | 4.58(23)   | 1.01(45)  | 2.18     | 4   | 70   |
| 3.00 | 2   | 16               | 0.434(5)        | 2.25(8)    | 0.34(15)  | 1.10     | 4   | 89   |
| 3.00 | 3   | 16               | 0.428(4)        | 1.57(4)    | -0.06(9)  | 1.82     | 4   | 77   |
| 3.25 | 1   | 16               | 0.547(8)        | 3.97(19)   | 3.28(50)  | 5.53     | 4   | 24   |
| 3.25 | 2   | 32               | 0.524(9)        | 2.34(14)   | 0.26(43)  | 1.62     | 3   | 65   |
| 3.25 | 3   | 16               | 0.546(5)        | 1.34(4)    | 0.71(10)  | 1.14     | 4   | 89   |
| 3.50 | 1   | 16               | 0.650(9)        | 3.87(19)   | 4.42(64)  | 2.16     | 4   | 72   |
| 3.50 | 2   | 128              | 0.726(39)       | 1.15(31)   | 8.41(3.26) | 0.04     | 1   | 84   |
| 3.50 | 3   | 16               | 0.665(5)        | 1.18(3)    | 1.51(12)  | 0.24     | 4   | 99   |
| 3.75 | 1   | 16               | 0.774(10)       | 3.32(17)   | 7.53(80)  | 0.65     | 4   | 96   |
| 3.75 | 2   | 16               | 0.779(7)        | 1.61(6)    | 3.73(27)  | 0.98     | 4   | 91   |
| 3.75 | 3   | 128              | 0.748(37)       | 1.34(34)   | -1.65(3.86) | 0.28     | 1   | 60   |
| 4.00 | 1   | 32               | 0.940(19)       | 2.23(23)   | 18.11(2.55) | 0.30     | 3   | 96   |
| 4.00 | 2   | 32               | 0.918(13)       | 1.27(10)   | 7.06(95)  | 1.35     | 3   | 72   |
| 4.00 | 3   | 32               | 0.909(11)       | 0.89(5)    | 4.14(53)  | 0.79     | 3   | 85   |
| 4.00 | 4   | 128              | 1.009(37)       | 0.34(8)    | 12.03(2.97) | 0.24     | 1   | 62   |

Table 27: Estimates for $z_{\text{int},c_2}$ from fits to $\tau_{\text{int},c_2} = AL^z + B$. Error bars are one standard deviation. $L_{\text{min}}$ in brackets indicates a poor fit (confidence level < 10%).
Table 28: Summary of estimates for \( z_{\text{int},C_2} \), averaged over all values of \( k \) for which the confidence level is \( \geq 10\% \). Error bars are one standard deviation. The last column of each group shows \( z_{\text{int},C_2} - z_{\text{int},E'} \) from the corresponding fits.

| \( q \) | \( L_{\text{min}} \) | \( z \) | \( z_{C_2} - z_{E'} \) | \( L_{\text{min}} \) | \( z \) | \( z_{C_2} - z_{E'} \) |
|---|---|---|---|---|---|---|
| 1.25 | 256 | 0.010(1) | — | 64 | -0.189(89) | 0.024 |
| 1.50 | — | — | — | — | — | — |
| 1.75 | 128 | 0.062(1) | -0.096 | 128 | 0.047(67) | -0.038 |
| 2.00 | 256 | 0.103(3) | -0.112 | 64 | 0.016(14) | -0.127 |
| 2.25 | 128 | 0.174(3) | -0.112 | 16 | 0.122(4) | -0.113 |
| 2.50 | 128 | 0.247(2) | -0.106 | 16–128 | 0.217(7) | -0.090 |
| 2.75 | 16–256 | 0.333(1) | -0.091 | 16–128 | 0.322(7) | -0.086 |
| 3.00 | 16–64 | 0.429(1) | -0.082 | 16 | 0.431(3) | -0.066 |
| 3.25 | 32–64 | 0.522(1) | -0.065 | 16–32 | 0.543(4) | -0.029 |
| 3.50 | 64–256 | 0.637(4) | -0.056 | 16–128 | 0.662(4) | -0.027 |
| 3.75 | 128 | 0.759(5) | -0.025 | 16–128 | 0.777(5) | -0.019 |
| 4.00 | 128–256 | 0.892(6) | -0.006 | 32–128 | 0.921(7) | 0.011 |
Figure 1: Fits of $\xi/L$ to $x^* + BL^{-q}$. (a) $q = 1.25$, (b) $q = 2$, (c) $q = 2.75$, (d) $q = 3$, (e) $q = 3.25$, (f) $q = 3.5$. For $q = 3.25$ the fit is actually to a constant $x^*$. 

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Figure 2: Fits of $\chi/L^{\gamma/\nu}$ to $A + BL^{-p}$. (a) $q = 1.25$, (b) $q = 2$, (c) $q = 2.25$, (d) $q = 3$. 
Figure 3: Normalized autocorrelation function $\rho_{OO}(t)$ for the observables $O = E', F'$ and $C_2$, in the case $q = 3, k = 3, L = 256$. 
Figure 4: Our best estimates of the dynamic critical exponent $z_{\text{int},E'}$. The purely statistical error bar is indicated in black and is usually about the same size as the symbol; the combined statistical and systematic error bar is indicated in blue. (For $q = 1.25, 1.50$ we are unable to estimate the systematic errors.) The static exponents $\alpha/\nu$ and $\beta/\nu$ are shown for comparison (red and green curves, respectively).
Figure 5: Fits to $\tau_{\text{int}} = A L^{\beta/\nu} + B + C L^{-p}$ for $q = 2$ ($k = 1$). (a) $L_{\text{min}} = 16$, (b) $L_{\text{min}} = 32$. 
Figure 6: Fits to $\tau_{\text{int}, \xi^*} = A L^{\beta/\nu} + B + C L^{-p}$ for $q = 1.75$. (a) $L_{\text{min}} = 32$, (b) $L_{\text{min}} = 64$. 
Figure 7: Fits to $\tau_{\text{int}, \varepsilon'} = A L^{\beta/\nu} + B + C L^{-p}$ for $q = 1.5$. (a) $L_{\text{min}} = 32$, (b) $L_{\text{min}} = 64$. 

(a) $q = 1.5, L_{\text{min}} = 32$

(b) $q = 1.5, L_{\text{min}} = 64$
Figure 8: Fits to $\tau_{\text{int},e'} = AL^{\beta/\nu} + B + CL^{-p}$ for $q = 1.25$, with $L_{\text{min}} = 128$. 