On the Geometry of Certain Isospectral Sets in the Full Kostant-Toda Lattice

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Abstract: We use momentum mappings on generalized flag manifolds and their momentum polytopes to study the geometry of the level sets of the 1-chop integrals of the full Kostant-Toda lattice in certain isospectral submanifolds of the phase space. We derive expressions for these integrals in terms of Plücker coordinates on the flag manifold in the case that all eigenvalues are zero and compare the geometry of the base locus of their level set varieties with the corresponding geometry for distinct eigenvalues. Finally, we illustrate and extend our results in the context of the full sl(3, C) and sl(4, C) Kostant-Toda lattices.

1 Introduction

The system of differential equations for the full Kostant-Toda lattice is written in Lax form as
\[ \dot{X}(t) = [X(t), \Pi_{N-}X(t)], \]
in which \( X \) belongs to the space of matrices
\[ \epsilon + B_{-} = \left\{ \begin{pmatrix} * & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ \cdots & \cdots & \cdots & \cdots & * \end{pmatrix} : \text{tr}X = 0 \right\}, \]
where \( B_{-} \) is the Lie algebra of lower triangular matrices with trace zero, \( \epsilon \) is the matrix with 1’s on the superdiagonal and zeros elsewhere, and \( \Pi_{N-}X \) is the strictly lower triangular part of \( X \). The space \( \epsilon + B_{-} \) is not a symplectic manifold. It is a Poisson manifold which is foliated by symplectic leaves of different dimensions. The full Kostant-Toda lattice is known to be completely integrable ([?, ?, ?]) on its generic symplectic leaves; moreover, it is a system for which there exist distinct maximal involutive families of integrals.

In [?] the authors consider a particular family of integrals known as the “k-chop integrals” for \( k = 0, \ldots, \lfloor \frac{n}{2} \rfloor \), which we describe in Section 2. The 0-chop integrals are simply the eigenvalues of \( X \in \epsilon + B_{-} \). Indeed, the form of the
solution readily shows that these are constants of motion. To solve the system (1) with initial condition \( X_0 \), we factor the exponential \( e^{tX_0} \) as
\[
e^{tX_0} = n(t)b(t),
\]
where \( n(t) \) is lower unipotent and \( b(t) \) is upper triangular. Conjugating \( X_0 \) by \( n(t) \) yields the solution, \( X(t) \) ([7, 8]):
\[
X(t) = n^{-1}(t) X_0 n(t),
\]
which clearly preserves the spectrum. We point out here that the k-chop integrals for \( k > 0 \) are rational functions in the entries of \( X \) and that typically the flows associated to these integrals exit the phase space in finite (complex) time. In particular, if this factorization cannot be done for some \( t = t_0 \), then \( X(t) \) has a pole at \( t_0 \).

The geometry of the flag manifold \( SL(n, \mathbb{C})/B \), where \( B \) is the subgroup of upper triangular matrices, is used in [8] to study the restriction of (1) to an isospectral submanifold of \( \mathbb{C}^+ + B^- \) in which the eigenvalues are distinct. Using a factorization result of Kostant, one can define an embedding of this submanifold into \( SL(n, \mathbb{C})/B \) as an open dense set. Under this embedding, the Toda flows associated to the 0-chop integrals generate the action of the diagonal complex torus \( (\mathbb{C}^*)^{n-1} \) on the flag manifold so that the compactified level sets of the constants of motion are unions of torus orbits. This embedding has therefore become known as the “torus embedding.”

The fact that these compactified level sets are invariant under the action of the diagonal torus is the basis of the study of the full Kostant-Toda lattice in [7], which brings in the geometry of certain convex polytopes associated to these torus orbits through the momentum mapping, extending some of the results in [8] and providing geometric descriptions of certain distinctive features of the system including the symplectic leaf stratification, the relationship between level sets cut out by different involutive families of integrals, and monodromy near certain “degenerate” level sets.

This paper continues in the same spirit, appealing to the structure of momentum polytopes to illuminate the geometry of the full Kostant-Toda lattice. In Section 4 we describe in a more general setting the momentum mapping on the flag manifold associated to the action of the diagonal compact torus, \((S^1)^{n-1}\), and a convexity theorem which describes the convex polytopes arising as the images under this mapping of the closures of the orbits of \((\mathbb{C}^*)^{n-1}\).

It is shown in [7] that under the torus embedding, the 1-chop integrals have simple expressions in terms of certain homogeneous coordinates \([\pi_i] \times [\pi^*_i]\) on \( P^n \times (P^n)^* \). For example, when \( n = 3 \), the single 1-chop integral has the form
\[
\frac{\lambda_2 \lambda_3 \pi_1 \pi^*_1 + \lambda_1 \lambda_3 \pi_2 \pi^*_2 + \lambda_1 \lambda_2 \pi_3 \pi^*_3}{\lambda_1 \pi_1 \pi^*_1 + \lambda_2 \pi_2 \pi^*_2 + \lambda_3 \pi_3 \pi^*_3},
\]
which clearly shows its invariance under the torus \( (\mathbb{C}^*)^2 \) since the coordinates \( \pi_i \) and \( \pi^*_i \) are scaled under the flow in such a way that the products \( \pi_i \pi^*_i \) remain
unchanged. Moreover, the 1-chop integrals for general \( n \) depend only on the projection of the flag manifold to the manifold of partial flags \( \{V^1 \subset V^{n-1} \subset \mathbb{C}^n\} \subset P^n \times (P^n)^* \), which we denote by \( Sl(n, \mathbb{C})/P_1 \). Clearing the denominators in the expressions of these integrals defines a collection of subvarieties in this partial flag manifold. We show that the intersection of these varieties, their base locus, has a simple description in terms of the momentum mapping on \( Sl(n, \mathbb{C})/P_1 \) and its momentum polytope. Indeed, this base locus is precisely the inverse image under the momentum mapping of the boundary of the momentum polytope.

Our discussion so far and the treatments of the full Kostant-Toda lattice in [?] and [?] concern only the situation in which the eigenvalues of \( X \in \epsilon + B_- \) are distinct. One of the reasons for this is that it is only in this case that the torus embedding is defined, producing compactified level sets of the constants of motion which are unions of orbits of the diagonal torus. There is, however, a different mapping which embeds an arbitrary isospectral submanifold of \( \epsilon + B_- \) into the flag manifold. In this case the Toda flows for the 0-chop integrals generate the action (by left multiplication) of the exponential of the abelian nilpotent algebra whose elements have the form

\[
\begin{pmatrix}
0 & a_1 & \cdots & a_{n-1} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & a_3 \\
0 & \cdots & \cdots & 0
\end{pmatrix},
\]

where the \( a_i \) belong to \( \mathbb{C} \), on the flag manifold. This embedding is called the “companion embedding” since the role of the diagonal matrix in the torus embedding is replaced by the companion matrix of \( X \in \epsilon + B_- \).

Computing the expressions for the 1-chop integrals in terms of the Plücker coordinates on the flag manifold under the companion embedding, we discover that as in the case of the torus embedding, they involve only the coordinates \( [\pi_i] \times [\pi_i^*] \) on the partial flag manifold \( Sl(n, \mathbb{C})/P_1 \). So as in the case of the torus embedding when the eigenvalues are distinct, we obtain again a collection of subvarieties of \( Sl(n, \mathbb{C})/P_1 \).

However, since the varieties in \( Sl(n, \mathbb{C})/P_1 \) defined by the 1-chop integrals in the case of the companion embedding are not preserved by the diagonal torus, the level sets of the constants of motion are not unions of orbits of this torus and we cannot apply the convexity theorem directly to study their geometry. Neither does the common intersection (base locus) of these varieties for an arbitrary choice of isospectral level set lend itself to a geometric description in terms of momentum polytopes. However, in the case of the isospectral submanifold \( (\epsilon + B_-)_0 \), in which all the eigenvalues coincide and are equal to zero, the base locus of the varieties determined by the 1-chop integrals is invariant under the action of the diagonal torus. Here the companion matrix is simply the nilpotent algebra.
matrix with 1’s on the superdiagonal and 0’s elsewhere, and the expressions of
the 1-chop integrals become much simpler. When \( n = 3 \), for example, the single
1-chop integral is
\[
\frac{\pi_3\pi_1^*}{\pi_2\pi_1^* + \pi_3\pi_2^*}.
\]

In this paper we refer to the situation in which an isospectral submanifold
of \( \epsilon + B_- \) with distinct eigenvalues is embedded into the flag manifold by the
torus embedding as Case A and to the companion embedding of \((\epsilon + B_-)_0\)
into \( Sl(n, \mathbb{C})/B \) as Case B. We show that in Case B the base locus of the
varieties in \( Sl(n, \mathbb{C})/P_1 \) determined by the 1-chop integrals is not the complete
inverse image under the momentum mapping of the boundary of the momentum
polytope, in contrast to what we find in Case A. Here the base locus corresponds
to only certain faces of this polytope, one associated to each of the fundamental
weights of \( sl(n, \mathbb{C}) \).

It is interesting to note that first, it is in the two extreme cases, the generic
case in which all eigenvalues are distinct and the most degenerate case in which
all eigenvalues coincide, that the base locus of the varieties in \( Sl(n, \mathbb{C})/P_1 \)
corresponding to the 1-chop integrals (under the appropriate embedding) have
simple descriptions in terms of the momentum polytope, and secondly that the
two embeddings that reveal this geometry in the two cases are different.

After first developing our results for Cases A and B in Section 4, we consider
the expressions of the 1-chop integrals for an arbitrary isospectral level set under
the consistent use of the companion embedding. The expressions we obtain
provide some insight into how the 1-chop integrals under this embedding simplify
as a level set with distinct eigenvalues degenerates to the extreme case in which
all eigenvalues coincide (at zero).

We devote the final two sections to the special cases \( n = 3 \) and \( n = 4 \). For
\( n = 3 \) we consider in more detail the geometry of the level sets of the single
1-chop integral (a Casimir) in the flag manifold in both Case A and Case B
and show how the common intersection of the level set varieties in each of these
cases precisely encodes the stratification of the relevant isospectral submanifold
induced by the symplectic stratification of \( \epsilon + B_- \). In our example with \( n = 4 \),
after illustrating our results for the 1-chop integrals, we consider a different
maximal involutive family of integrals whose geometry is studied in detail in
[?]. We obtain for this family of integrals statements which are completely
analogous to those for the 1-chop integrals, the partial flag manifold \( Sl(4, \mathbb{C})/P_1 \)
being replaced by the Grassmannian \( G(2, 4) \) of two-dimensional subspaces of \( \mathbb{C}^4 \).
2 Background

To describe the Poisson structure on $\epsilon + B_-$ we consider the decomposition of the Lie algebra $sl(n, \mathbb{C})$ into its upper triangular and lower nilpotent subalgebras:

$$sl(n, \mathbb{C}) = B_+ \oplus N_-.$$  

Using the nondegenerate Killing form $\langle X, Y \rangle = 2n \cdot tr(XY)$ on $sl(n, \mathbb{C})$, we identify the dual, $B^*_+$, of $B_+$ with $N_{\perp}^-$, which is $B_-$. This gives an identification of $\epsilon + B_- \cong B^*_-$ with the dual of a Lie algebra:

$$\epsilon + B_- \cong B^*_- \cong B_{-}^*.$$ 

The space $\epsilon + B_-$ acquires its Poisson structure through this identification from the Lie-Poisson structure on $B_+^*$, whose symplectic leaves are the coadjoint orbits. Indeed, the differential equations generated by the Hamiltonian $H = \frac{1}{2} trX^2$ are precisely the Toda equations (1).

From the solution (2), we know that the functions $I_{rk} = \frac{1}{k} trX^k$ for $k = 2, \ldots , n$ are constants of motion. To obtain a completely integrable system on a generic symplectic leaf, one must find Casimirs and sufficiently many additional integrals. The full Kostant-Toda lattice is an integrable system for which there exist distinct maximal families of constants of motion in involution, where the constants in different families do not necessarily commute with each other. Using a method introduced by Thimm [?] and applied to the Toda lattice in [?], such involutive families may be found by considering different nested chains of parabolic subalgebras of $sl(n, \mathbb{C})$. [?] deals primarily with a particular family of integrals.

**Proposition 2.1** ([?, ?]) For $k = 0, \ldots , \left[ \frac{n-1}{2} \right]$, denote by $(X - \lambda Id)_{(k)}$ the result of removing the first $k$ rows and last $k$ columns from $X - \lambda Id$, and let $\lambda_{rk}$, $r = 1, \ldots , n-2k$, denote the roots of

$$\hat{Q}_k(X, \lambda) = \det(X - \lambda Id)_{(k)} = E_{0k}\lambda^{n-2k} + \cdots + E_{n-2k,k}.$$ 

The $\lambda_{rk}$ are constants of motion for the full Kostant-Toda lattice. The coefficients, $I_{rk}$, of the monic polynomial

$$Q_k(X, \lambda) = \frac{\det(X - \lambda Id)_{(k)}}{E_{0k}} = \lambda^{n-2k} + I_{1k}\lambda^{n-2k-1} + \cdots + I_{n-2k,k}$$

are constants of motion equivalent to the $\lambda_{rk}$. They are called the $k$-chop integrals. (The $I_{10}$ are the coefficients of the characteristic polynomial of $X$.) The functions $I_{1k} = \sum_r \lambda_{rk}$ are Casimirs on $\epsilon + B_-$, and the $I_{rk}$ for $r > 1$ constitute a complete involutive family of integrals for the generic symplectic leaves of $\epsilon + B_-$ cut out by the Casimirs $I_{1k}$.

In [?], the authors present a method of computing the $k$-chop integrals.
Proposition 2.2 Choose $X \in \epsilon + B_-$, and break it into blocks of the indicated sizes,

$$
X = \begin{bmatrix}
\begin{array}{ccc}
k & n-2k & k \\

k & \begin{pmatrix}
X_1 & X_2 & X_3 \\
X_4 & X_5 & X_6 \\
X_7 & X_8 & X_9
\end{pmatrix}
\end{array}
\end{bmatrix},
$$

where $k$ is an integer, $0 \leq k \leq \left[\frac{(n-1)}{2}\right]$. If $\det X_7 \neq 0$, define the matrix $\phi_k(X)$ by

$$
\phi_k(X) = X_5 - X_4 X_7^{-1} X_8 \in \text{Gl}(n-2k, \mathbb{C}), \quad k \neq 0,
\phi_0(X) = X.
$$

Then the $I_{rk}$ are the coefficients of the polynomial $\det(\lambda - \phi_k(X))$.

The “generic” level sets of these constants of motion are studied in [?] in terms of the geometry of generalized flag manifolds. This is made possible through the following result of Kostant.

Proposition 2.3 Let $X \in \epsilon + B_-$, and let

$$
C = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \vdots & 1 \\
s_n & \cdots & \cdots & s_2 & 0
\end{pmatrix}
$$

be the companion matrix. Then there is a unique lower unipotent matrix $L$ such that $X = LCL^{-1}$.

Now fix the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $X \in \epsilon + B_-$. Let $\Lambda$ be the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ for some fixed ordering of the $\lambda_i$, and denote by $(\epsilon + B_-)_\Lambda$ the isospectral submanifold of $\epsilon + B_-$ with these eigenvalues. Because of the uniqueness of $L$, there is an embedding of $(\epsilon + B_-)_\Lambda$ into the flag manifold $\text{Sl}(n, \mathbb{C})/B$, where $B$ is the subgroup of upper triangular matrices, defined by

$$
\Phi_\Lambda : (\epsilon + B_-)_\Lambda \to \text{Sl}(n, \mathbb{C})/B
$$

$$
X \mapsto L^{-1} \mod B.
$$

This is known as the companion embedding.

Now suppose that the eigenvalues $\lambda_i$ are distinct. Then $C = VAV^{-1}$, where $V$ is a Vandermonde matrix, and the theorem implies that for each $X$
in \((\epsilon + B_-)_A\), there is a unique lower unipotent matrix \(L\) such that \(X = LVAV^{-1}L^{-1}\). Again, the uniqueness of \(L\) implies that the mapping

\[
\Psi_A : (\epsilon + B_-)_A \to SL(n, \mathbb{C})/B
\]

\[
X \mapsto V^{-1}L^{-1} \mod B
\]

is also an embedding.

Since the flows for the integrals \(\frac{1}{k}trX^k\) generate the action of the diagonal complex torus \((\mathbb{C}^*)^n\) on the flag manifold under this embedding (see [?]), it is called the *torus embedding*. Under this mapping, each compactified level set of the constants of motion is a union of torus orbits in the flag manifold. By an application of a more general convexity theorem of Atiyah in [?], we can associate to the closure of each torus orbit a convex polytope by means of the momentum mapping, as described in the next section. The geometry of generalized flag manifolds and their momentum polytopes is the basis of the study of the full Kostant-Toda lattice in [?], in which the eigenvalues are taken to be distinct. In this paper we extend some of the work in [?] to the case in which all the eigenvalues are equal to zero and compare our results to the corresponding statements in the case of distinct eigenvalues.

### 3 The Momentum Mapping on \(G/P\)

In this section we first introduce the momentum mapping and discuss the convexity theorem in the more general setting in which \(SL(n, \mathbb{C})\) is replaced by an arbitrary complex semisimple Lie group. We then specialize our discussion to the case of \(SL(n, \mathbb{C})\) and establish some notation.

Let \(G\) be a complex semisimple Lie group, \(H\) a Cartan subgroup of \(G\), and \(B\) a Borel subgroup containing \(H\). We denote by \(\mathcal{H}\) the Lie algebra of \(H\) and by \(\mathcal{H}^*\) the dual of \(\mathcal{H}\). Let \(P\) be a parabolic subgroup of \(G\) containing \(B\). The homogeneous space \(G/P\) can be realized as the orbit of \(G\) through a projectivized highest weight vector in the projectivization, \(P(V)\), of an irreducible representation \(\rho : G \to GL(V)\) (see [?], for example).

Because \(G/P\) is a projective algebraic variety, it is Kähler and is therefore also a symplectic manifold. The action of the compact torus \(T\) in \(H\) preserves the symplectic structure. This gives rise to a momentum mapping \(\mu : G/P \to \tau^*\), where \(\tau^*\) is the dual of the Lie algebra of \(T\). (\(\tau^*\) is the real part of \(\mathcal{H}^*\).) To describe this mapping, let \(\mathcal{A}\) be the set of weights of \(\rho\) taken with multiplicity, and choose a basis of weight vectors \(\{v_{\alpha} : \alpha \in \mathcal{A}\}\) for \(V\). A point \([X] \in G/P \subset P(V)\), represented by a vector \(X \in V\), determines uniquely up to a scalar the collection of numbers \(\pi_\alpha(X)\), where \(X = \sum_{\alpha \in \mathcal{A}} \pi_\alpha(X)v_{\alpha}\). We refer to the homogeneous coordinates \([\pi_\alpha(X)]\) on \(G/P\) as *Plücker coordinates*. 
The momentum mapping for the action of $T$ on $G/P$ is defined by

$$\mu : G/P \to \tau^*$$

$$[X] \mapsto \sum_{\alpha \in A} |\pi_{\alpha}(X)|^2 \alpha / \sum_{\alpha \in A} |\pi_{\alpha}(X)|^2.$$

Let $H \cdot [X]$ be the orbit of $H$ through $[X]$, and let $\overline{H \cdot [X]}$ be its closure in $G/P$. To describe the image of $\overline{H \cdot [X]}$ under the momentum mapping, we consider the orbit $W \cdot \alpha^V$ of the Weyl group $W$ of $G$ through the highest weight vector $\alpha^V$ of $V$ determined by our choice of $B$. It is shown in ?? using the more general results in ?? that $\mu(\overline{H \cdot [X]})$ is the convex polytope whose vertices are the weights $\{\alpha \in W \cdot \alpha^V : \pi_{\alpha}(X) \neq 0\}$. In particular, the image of $G/P$ under the momentum mapping is the convex hull of the orbit of the Weyl group through $\alpha^V$. This image is therefore the weight polytope of the irreducible representation $V$; we refer to it also as the moment polytope of $G/P$. The vertices of this polytope are the images under $\mu$ of the fixed points, $[v_\alpha]$ for $\alpha \in W \cdot \alpha^V$, of the torus action; $\mu([v_\alpha]) = \alpha$.

The momentum mapping induces a partition of $G/P$ into equivalence classes called strata (see ??), where each stratum is the union of all torus orbits in $G/P$ whose closures have as their images under the momentum mapping the same convex polytope. The generic stratum is the union of all torus orbits whose closures have the full momentum polytope as their common image under $\mu$. These are the generic torus orbits, on which no Plücker coordinate vanishes.

Now let $G$ be the group $SL(n, \mathbb{C})$, $H$ its diagonal subgroup, and $V$ the adjoint representation of $sl(n, \mathbb{C})$. We denote by $L_i$ the linear function in $H^*$ which sends an element of $H$ to its $i$th diagonal entry. Then $H^*$ is the quotient space

$$H^* = \{\sum_{i=1}^n c_i L_i : c_i \in \mathbb{C}\} / \langle \sum_{i=1}^n L_i \rangle \cong \mathbb{C}^{n-1},$$

and the roots of the adjoint representation are $L_i - L_j, i \neq j$. These roots are the vertices of the weight polytope of $V$, which we denote by $\Delta_n$; they constitute the orbit of the Weyl group of $SL(n, \mathbb{C})$, the symmetric group on $n$ elements, through the highest weight $L_1 - L_n$. The adjoint representation of $sl(n, \mathbb{C})$ is the $(n^2 - 1)$-dimensional irreducible component in the representation $\mathbb{C}^n \otimes \wedge^{n-1} \mathbb{C}^n$, where $\mathbb{C}^n$ is the standard representation of $sl(n, \mathbb{C})$.

Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{C}^n$. We identify $\wedge^{n-1} \mathbb{C}^n$ with the dual of $\mathbb{C}^n$ by requiring that $\omega(\beta)$ for $\omega \in \wedge^{n-1} \mathbb{C}^n$ and $\beta \in \mathbb{C}^n$ satisfy $\beta \wedge \omega = \omega(\beta)e_1 \wedge \ldots \wedge e_n$. Under this identification, the dual basis of $\{e_i\}_{i=1}^n$ is $\{e_i^*\}_{i=1}^n$, where $e_i^* = (-1)^{i+1} e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \ldots \wedge e_n$. Our Plücker coordinates on $G[\{e_i \otimes e_i^*\}] \subset P(V)$ are defined with respect to the weight vectors $v_{L_1 - L_2} = e_i \otimes e_i^*$ for the root spaces of $sl(n, \mathbb{C})$ and the basis $\{e_i \otimes e_i^* - e_i+1 \otimes e_{i+1}^*\}_{i=1}^{n-1}$ of $H$. (The vectors $e_i \otimes e_i^*, i = 1, \ldots, n$ span the $n$-dimensional zero weight space in the tensor product $\mathbb{C}^n \otimes \wedge^{n-1} \mathbb{C}^n$.)
The orbit $G \cdot [e_1 \otimes e_n^*]$ is realized as the homogeneous space $G/P_1$, where $P_1$ is the stabilizer of $[e_1 \otimes e_n^*]$ in $P(V)$. It consists of the elements $[v \otimes \sigma] \in P(C^n \otimes \wedge^{n-1} C^n)$ for which $v \wedge \sigma = 0$. Its obvious embedding into $P(C^n) \times P(\wedge^{n-1} C^n)$ defines coordinates $[\pi_1 : \cdots : \pi_n] \times [\pi_1^* : \cdots : \pi_n^*]$ on $G/P_1$, where $[\pi_i]$ are the Plücker coordinates on $P(C^n)$ with respect to the standard basis and $[\pi_i^*]$ are the Plücker coordinates on $P(\wedge^{n-1} C^n)$ with respect to the dual basis $\{e_i^*\}$. (If we represent $[X] \in G/P_1$ by $M$ mod $P_1$ with $M \in G$, then $[\pi_i] = [M_{i1}]$, and $[\pi_i^*] = [(-1)^{i+1}M_{i\bar{n}}]$, where $M_{i\bar{n}}$ is the determinant of the matrix obtained by deleting the $i$th row and the $n$th column of $M$.) In these coordinates, the variety $G/P_1 \subset P(C^n) \times P(\wedge^{n-1} C^n)$ is cut out by the equation $\sum_{i=1}^n \pi_i \pi_i^* = 0$.

It will be convenient to use both the Plücker coordinates $[\pi_o]$ for the embedding $G/P_1 \to P(V)$ and the coordinates $[\pi_i] \times [\pi_i^*]$ for the embedding $G/P_1 \to P(C^n) \times P(\wedge^{n-1} C^n)$. For $[X] \in G/P_1$, $X$ has the form

$$X = \sum_{i \neq j} \pi_{L_i - L_j}(X)e_i \otimes e_j^* + \sum_{i=1}^n a_i(X)e_i \otimes e_i^*,$$

where $\sum_{i=1}^n a_i(X) = 0$. In terms of the coordinates $[\pi_i] \times [\pi_i^*]$ on $[X]$, the Plücker coordinates $\pi_{L_i - L_j}(X)$ are projectively equal to the products $\pi_i \pi_j^*$ for $i \neq j$:

$$[\pi_{L_i - L_j}(X)]_{i \neq j} = [\pi_i \pi_j^*]_{i \neq j},$$

and the $a_i(X)$ are projectively equal to the products $\pi_i \pi_i^*$:

$$[a_i(X)]_{i=1}^n = [\pi_i \pi_i^*]_{i=1}^n.$$

Throughout the paper, when we refer to a torus orbit, we mean an orbit of the diagonal complex torus $(C^*)^{(n-1)}$, and reference to a face or edge of a polytope means its closure.

4 Results for General $Sl(n, C)$

We consider two types of isospectral submanifolds of $\epsilon + B_-$, those with distinct eigenvalues and the one whose eigenvalues are all equal to zero, which we denote by $(\epsilon + B_-)_0$. In the case that the eigenvalues are distinct, we take the torus embedding of $(\epsilon + B_-)_\Lambda$ into $G/B$; we embed $(\epsilon + B_-)_0$ into the flag manifold by the companion embedding. We refer to these as Case A and Case B, respectively.

4.1 The two extreme cases

It is shown in [?] that under the torus embedding of $(\epsilon + B_-)_\Lambda$ (with distinct eigenvalues) into $G/B$, the 1-chop integrals have the expressions

$$I_{r_1} = \frac{\sum_{i=1}^n \sigma_{r+1}(\hat{i}) \pi_i \pi_i^*}{\sum_{i=1}^n \sigma_1(\hat{i}) \pi_i \pi_i^*},$$
where $\sigma_j(\hat{i})$ is the $j^{th}$ symmetric polynomial on the $n - 1$ eigenvalues different from $\lambda_i$. Observe that these functions on $G/B$ involve only the Plücker coordinates $[\pi_i] \times [\pi_i^*]$ and are therefore defined in terms of the projection of the flag manifold to $G/P_1$. The situation in Case B is similar.

**Proposition 4.1** Under the companion embedding of $(\epsilon + B_-)_0$ into the flag manifold, the 1-chop integrals have the expressions

$$I_{r1} = \frac{\sum_{i=1}^{n-r-1} \pi_{i+r+1} \pi_i^*}{\sum_{i=1}^{n-1} \pi_{i+1} \pi_i^*},$$

which depend only on the projection of $G/B$ to $G/P_1$. In terms of the Plücker coordinates $\pi_\alpha$ on $G/P_1$, these expressions are

$$I_{r1} = \frac{\sum_{\text{ht}(\alpha) = -1-r} \pi_\alpha}{\sum_{\text{ht}(\alpha) = -1} \pi_\alpha},$$

where $\text{ht}(\alpha)$ is the height of the weight $\alpha$.

**Proof.** Let $C_0$ denote the nilpotent matrix with 1’s on the superdiagonal and zeros elsewhere, and for $M \in \text{Sl}(n, \mathbb{C})$, denote by $M_{ij}$ the determinant of the matrix obtained by deleting row $i$ and column $j$ from $M$. Then for $X \in (\epsilon + B_-)_0$,

$$\det(X - \lambda I)_{(1)} = (L(C_0 - \lambda I)L^{-1})_{\hat{i}\hat{n}}$$

$$= \sum_{i,j=1}^{n} L_{ij}(C_0 - \lambda I)_{\hat{i}\hat{j}} L_{\hat{i}\hat{n}}^{-1} L_{\hat{j}\hat{n}}^{-1}$$

$$= \sum_{i,j=1}^{n} (-1)^{i+j} L_{ij}^{-1} L_{\hat{i}\hat{n}}^{-1} (C_0 - \lambda I)_{\hat{i}\hat{j}}$$

$$= \sum_{i,j=1}^{n} (-1)^{i+j} \pi_i \pi_j^* (C_0 - \lambda I)_{\hat{i}\hat{j}}$$

$$= \sum_{i=1}^{n} \sum_{1 \leq j \leq i} \pi_i \pi_j^* \lambda^{n-1-(i-j)},$$

where $\pi_i$ and $\pi_i^*$ denote $\pi_i(\Phi_0(X))$ and $\pi_i^*(\Phi_0(X))$, respectively. By Proposition 2.1, $I_{r1}$ is the coefficient of $\lambda^{n-r-2}$ in this polynomial divided by the coefficient of $\lambda^{n-2}$. (The coefficient of $\lambda^{n-1}$ is the Plücker relation $\sum_{i=1}^{n} \pi_i \pi_i^*$, which vanishes.)

**Remark:** Observe that in Case A, the expressions for the $I_{r1}$ involve only the products $\pi_i \pi_i^*$, which correspond to the Plücker coordinates for the zero weight space in $P(\mathbb{C}^n \times \land^{n-1} \mathbb{C}^n)$, whereas in Case B, the $I_{r1}$ depend only on

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the Plücker coordinates $\pi_i\pi_j^*$ with $i > j$, corresponding to the negative roots of $sl(n, \mathbb{C})$. We point out, however, that we are using different embeddings in the two cases. Later we show that using the companion embedding for a level set with distinct eigenvalues, none of which is zero, the expressions for the 1-chop integrals involve all the weights of $sl(n, \mathbb{C})$; this, however, obscures the geometry of the base locus defined below, which is revealed clearly by the torus embedding.

**Definition 4.1** Let $\mathcal{F}_{r,\alpha}^0$ for $\alpha \in \mathbb{C}$ denote the variety in $G/P_1$ cut out by the homogeneous polynomial

$$I_{r1} \left( \sum_{i=1}^{n-1} \pi_{i+1} \pi_i^* \right) - \sum_{i=1}^{n-r-1} \pi_{i+r+1} \pi_i^* = 0,$$

where $I_{r1} = \alpha$, and let $\mathcal{F}_{r,\infty}^0$ denote the variety

$$\sum_{i=1}^{n-1} \pi_{i+1} \pi_i^* = 0.$$

The base locus of these varieties is the intersection

$$Z_0 = \bigcap_{r=1}^{n-2} \bigcap_{\alpha \in \mathbb{P}^1} \mathcal{F}_{r,\alpha}^0.$$

This base locus corresponds to the subset of $(\epsilon + \mathcal{B}_-)_0$ on which all the $I_{r1}$ are undefined (and not equal to infinity). It consists of the $n - 1$ components

$$\pi_1^* = \cdots = \pi_k^* = \pi_{k+1}^* = \cdots = \pi_n = 0$$

for $k = 1, \ldots, n - 1$; we denote them by $Z_0^k$. Each of these components is the closure of a single torus orbit in $G/P_1$ of complex dimension $n - 2$. To see this, observe that any two generic points in $Z_0^k$ have Plücker coordinates of the form

$$[\pi_1 : \cdots : \pi_k : 0 : \cdots : 0] \times [0 : \cdots : 0 : \pi_{k+1}^* : \cdots : \pi_n^*]$$

and

$$[\pi_1^* : \cdots : \pi_k^* : 0 : \cdots : 0] \times [0 : \cdots : 0 : \pi_{k+1} : \cdots : \pi_n],$$

where $\pi_1 \cdots \pi_k \pi_{k+1}^* \cdots \pi_n^* \neq 0$ and $\pi_1^* \cdots \pi_k^* \pi_{k+1} \cdots \pi_n^* \neq 0$. Since the element in the torus whose diagonal entries are $h_i = \pi_i^*$ for $i = 1, \ldots, k$ and $h_i = \pi_i^*$ for $i = k+1, \ldots, n$ takes the first into the second, these points belong to the same torus orbit; its dimension is $n - 2$.

The geometry of $Z_0$ has a simple description in terms of the momentum mapping on $G/P_1$ and the momentum polytope $\Delta_n$.

**Definition 4.2** Let $Q$ be the weight polytope of an irreducible representation of $SL(n, \mathbb{C})$ with highest weight $w$. We denote by $\text{Star}(Q, w)$ the union of all faces which have $w$ as a vertex and are perpendicular to a fundamental weight of $sl(n, \mathbb{C})$ with respect to the Killing form on $\mathfrak{h}^*$. We denote the fundamental weight $\sum_{i=1}^k L_i$ by $w_k$ for $k = 1, \ldots, n - 1$. 
Remarks: 1. The face of $\text{Star}(Q, w)$ perpendicular to $w_k$ intersects the positive ray through $w_k$ perpendicularly at its center.

2. For an arbitrary irreducible representation, $\text{Star}(Q, w)$ is the union of at most $n - 1$ faces of dimension $n - 2$. In the case of $\text{Star}(\Delta_n, L_1 - L_n)$, this number is equal to $n - 1$.

Definition 4.3 Let $S_k$ denote the set $\{1, \ldots, k\}$, and for $R \subset S_n$ with $\lvert R \rvert = k$, denote by $R'$ the complement $S_n \setminus R$. For each $R \subset S_n$, we define $G_R$ to be the subalgebra of $sl(n, \mathbb{C})$ generated by the roots $L_i - L_j$ for $i, j \in R$ ($G_R \cong sl(\lvert R \rvert, \mathbb{C})$). The subalgebras $G_R \oplus G_{R'}$ for $R \subset S_n$ with $1 \leq \lvert R \rvert \leq n - 1$ have rank $n - 2$. We will refer to these as the rank $n - 2$ principal subalgebras of $sl(n, \mathbb{C})$. For $R = S_k$, $k = 1, \ldots, n - 1$, the subalgebras $G_{S_k} \oplus G_{S'_k} \cong sl(k, \mathbb{C}) \oplus sl(n - k, \mathbb{C})$ will be called the rank $n - 2$ principal block subalgebras of $sl(n, \mathbb{C})$.

Proposition 4.2 Let $\mu$ be the momentum mapping on $G/P_1$. Then $Z_0 = \mu^{-1}(\text{Star}(\Delta_n, L_1 - L_n))$. The momentum mapping induces a one-to-one correspondence between the $n - 1$ components of $Z_0$, each of which is the closure of a single torus orbit of complex dimension $n - 2$, and the set of $(n - 2)$-dimensional faces of $\text{Star}(\Delta_n, L_1 - L_n)$; $\mu(Z_0^k)$ is the face of $\text{Star}(\Delta_n, L_1 - L_n)$ perpendicular to the fundamental weight $w_k$.

Proof. We have already shown that $Z_0^k$ is the closure of a single torus orbit in $G/P_1$ of complex dimension $n - 2$. Its image under the momentum mapping is therefore an $(n - 2)$-dimensional face of $\Delta_n$ whose vertices are the images of the fixed points of the torus action contained in it. These fixed points are those whose single nonvanishing product of the form $\pi_i \pi_j^*$ is such that $i \in S_k$ and $j \in S'_k$; their images under $\mu$ are the (positive) roots $L_i - L_j$ for $i \in S_k$ and $j \in S'_k$. These are precisely the weights of the representation $V_{S_k} \otimes V_{S'_k}^*$ of the subalgebra $G_{S_k} \oplus G_{S'_k}$, where $V_{S_k}$ and $V_{S'_k}^*$ denote the standard representation of $G_{S_k}$ and the dual of the standard representation of $G_{S'_k}$, respectively. This representation is the irreducible component containing $e_1 \otimes e_n^*$ in the decomposition of the adjoint representation of $sl(n, \mathbb{C})$ under the action of this subalgebra. Since the root lattice of $G_{S_k} \oplus G_{S'_k}$ is generated by the simple roots $L_i - L_{i+1}$ for $i \neq k$, which are all perpendicular to the fundamental weight $w_k$ with respect to the Killing form, $\mu(Z_0^k)$ is the face of $\text{Star}(\Delta_n, L_1 - L_n)$ perpendicular to $w_k$.

Remark: $Z_0^k$, being the closure of a single torus orbit in $G/P_1$, is a toric variety. The geometry of such varieties in homogeneous spaces $G/P$ has been studied in detail in [?]; see also the corrections in [?]. The polytope $\mu(Z_0^k)$, which is the weight polytope of the representation $V_{S_k} \otimes V_{S'_k}^*$ of $G_{S_k} \oplus G_{S'_k}$, contains significant information about the geometry of this variety in $G/P_1$.

For Case A, we have a result similar to Proposition 4.2.
Definition 4.4 Let $\mathcal{V}_{r,\alpha}$ for $\alpha \in \mathbb{C}$ denote the variety in $G/P_1$ cut out by the homogeneous polynomial

$$I_{r}(\sum_{i=1}^{n} \sigma_1(i) \pi_i \pi_i^*) - (\sum_{i=1}^{n} \sigma_{r+1}(i) \pi_i \pi_i^*) = 0,$$

where $I_{r} = \alpha$, and let $\mathcal{V}_{r,\infty}$ denote the variety

$$\sum_{i=1}^{n} \sigma_1(i) \pi_i \pi_i^* = 0.$$

The base locus of these varieties is the intersection

$$Z_\Lambda = \bigcap_{r=1}^{n-2} \bigcap_{\alpha \in \mathbb{P}^1} \mathcal{V}_{r,\alpha}.$$

$Z_\Lambda$ corresponds (under the torus embedding) to the subset of $(\epsilon + B_\alpha)_\Lambda$ on which all the 1-chop integrals are undefined (and not equal to infinity). Since the common vanishing set of equations (6) and (7) coincides with the intersections of the varieties $\pi_i \pi_i^* = 0$ for $i = 1, \ldots, n$, $Z_\Lambda$ is the union of the $\sum_{k=1}^{n-1} \binom{n}{k}$ components

$$\pi_{\sigma(1)}^* = \cdots = \pi_{\sigma(k)}^* = \pi_{\sigma(k+1)} = \cdots = \pi_{\sigma(n)} = 0$$

for $k = 1, \ldots, n-1$ and $\sigma \in \Sigma_n$, the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$. We denote the component (6) by $Z_{\Lambda}^{k,\sigma}$. (Note that the permutation $\sigma$ is not unique for $n \geq 3$.)

In this case, as in Case B, one can easily show that each of these components is the closure of a single torus orbit in $G/P_1$ of complex dimension $n - 2$. Its image under the momentum mapping is therefore an $(n-2)$-dimensional face of $\Delta_n$. In fact, all such faces in the boundary of $\Delta_n$ are obtained in this way.

Proposition 4.3 $Z_\Lambda = \mu^{-1}(\partial \Delta_n)$. The momentum mapping induces a one-to-one correspondence between the components of $Z_\Lambda$, each of which is the closure of a single torus orbit of complex dimension $n - 2$, and the set of all $(n-2)$-dimensional faces of $\Delta_n$; $\mu(Z_{\Lambda}^{k,\sigma})$ is the face of $\Delta_n$ which intersects the positive ray through the weight $\sigma(w_k) = \sum_{i=1}^{k} L_{\sigma(i)}$ perpendicularly.

Proof. By Proposition 4.2, $\mu(Z_{\Lambda}^{k,\sigma})$ is the face of $\Delta_n$ which intersects the positive ray through $w_k$ perpendicularly at its center. Its vertices are $L_i - L_j$ for $i \in S_k, j \in S_k'$. The action of $\sigma \in \Sigma_n$ on the roots of $\mathfrak{sl}(n, \mathbb{C})$ takes this face to the face of $\Delta_n$ whose vertices are $L_i - L_j$ for $i \in \sigma(S_k)$ and $j \in \sigma(S_k')$; it intersects the positive ray through the weight $\sigma(w_k) = \sum_{i=1}^{k} L_{\sigma(i)}$ perpendicularly. But these vertices are the images of the fixed points of the torus action in $G/P_1$ for which the nonvanishing product $\pi_i \pi_j^*$ is such that $i \in \sigma(S_k)$ and $j \in \sigma(S_k')$. 

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which are precisely the fixed points contained in the component $Z^k,\sigma_\Lambda$ of $Z_\Lambda$. The image $\mu(Z^k,\sigma_\Lambda)$ is therefore the $(n-2)$-dimensional face of $\triangle_n$ which intersects the positive ray through $\sigma(w_k)$ perpendicularly.

To complete the proof, it remains to show that every $(n-2)$-dimensional face of $\triangle_n$ is perpendicular to a weight of a fundamental representation of $\mathfrak{sl}(n,\mathbb{C})$. To see this, observe that the vertices of the face which intersects the positive ray through the weight $\sigma(w_k)$ perpendicularly are the weights of the representation $V_{\sigma(S_k)} \otimes V^*_{\sigma(S'_k)}$ of the principal rank $n-2$ subalgebra $G_{\sigma(S_k)} \oplus G_{\sigma(S'_k)}$ of $\mathfrak{sl}(n,\mathbb{C})$, where $V_{\sigma(S_k)}$ and $V^*_{\sigma(S'_k)}$ are the standard representation of $G_{\sigma(S_k)}$ and the dual of the standard representation of $G_{\sigma(S'_k)}$, respectively. If we denote the weight polytopes of $V_{\sigma(S_k)}$ and $V^*_{\sigma(S'_k)}$ by $Q_{\sigma(S_k)}$ and $Q^*_{\sigma(S'_k)}$, respectively, then this face is the polytope $Q_{\sigma(S_k)} \times Q^*_{\sigma(S'_k)}$, the Cartesian product of a $k$-simplex and an $(n-k)$-simplex. One can easily write down an arbitrary $(n-3)$-dimensional face of this polytope and verify that it is also a face of another such $(n-2)$-dimensional face of $\triangle_n$.

Remark: In Case A, the image $\mu(Z_\Lambda)$ of the base locus $Z_\Lambda$ contains all the roots of $\mathfrak{sl}(n,\mathbb{C})$, whereas in Case B, $\mu(Z_0)$ contains precisely the vertices of $\triangle_n$ corresponding to the positive roots. Compare this with the remark following Proposition 4.1.

4.2 The companion embedding for an arbitrary isospectral level set

Let $C$ be the companion matrix defined in Proposition 2.3 for an arbitrary choice of spectrum. From the proof of Proposition 1.1,

$$\det(L(C - \lambda I)L^{-1})_{(1)} = \sum_{i,j=1}^{n} (-1)^{i+j} \pi_i \pi_j^*(C - \lambda I)_{ij}. \tag{9}$$

Recall that in Case B, in which all the eigenvalues are zero, all the symmetric functions vanish, and the coefficients in this polynomial involve only the Plücker coordinates corresponding to the negative roots of $\mathfrak{sl}(n,\mathbb{C})$. However, on an isospectral level set for which not all eigenvalues are zero, the symmetric functions $s_2, \ldots, s_n$ do not all vanish, and the coefficients of the polynomial (9) involve additional terms which depend on the values of these functions. Rewriting (9) by collecting the coefficients of the $s_i$, one obtains
\((-1)^{n+1} \det(L(C - \lambda I)L^{-1})_{(1)}\) = \sum_{i=1}^{n-1} \left( \lambda^{n-1-i} \sum_{ht(\alpha) = -i} \pi_\alpha \right) \\
+ \sum_{p=2}^{n} s_p \left[ \sum_{i=1}^{p-1} \left( \lambda^{n-1-i} \sum_{ht(\alpha) = p-i} \pi_\alpha \right) \right] \\
- \lambda^{n-1-p} \sum_{i=1}^{n-p} \pi_i \pi_i^* \\
- \sum_{i=1}^{n-1-p} \left( \lambda^{n-1-p-i} \sum_{ht(\alpha) = -i} \pi_\alpha \right) \right],

where \(\Gamma(\mathcal{G}_R)\) denotes the sublattice of the root lattice of \(sl(n, \mathbb{C})\) determined by the subalgebra \(\mathcal{G}_R\) (see definition 13). This expression is calculated easily in low-dimensional examples and is verified in general by induction on \(n\). It is the sum of \(n\) polynomials of degree \(n-2\); the first is the polynomial \(\det(L(C_0 - \lambda I)L^{-1})_{(1)}\) of Case B, and each of the others is multiplied by a symmetric polynomial \(s_i\).

Observe that the coefficients of the polynomial containing the factor \(s_p\) correspond to the subalgebra \(\mathcal{G}_{S_{n-p}} \oplus \mathcal{G}_{S_{n-p}'}\) of \(sl(n, \mathbb{C})\). The coefficients of \(\lambda^{n-2}, \ldots, \lambda^{n-p}\) are the sums of the Plücker coordinates for the (positive) roots of heights \(p-1, \ldots, 1\), respectively, of the rank \(p-1\) subalgebra \(\mathcal{G}_{S_{n-p}'}\); the coefficient of \(\lambda^{n-p-1}\) is (minus) the sum of the Plücker coordinates for the zero weight space of \(\mathcal{C}^{n-p} \otimes (\mathcal{C}^{n-p})^*\), and the coefficients of \(\lambda^{n-p-2}, \ldots, \lambda^0\) are (minus) the sums of the Plücker coordinates for the (negative) roots of heights \(-1, \ldots, -(n-p-1)\), respectively, of the rank \(n-p-1\) subalgebra \(\mathcal{G}_{S_{n-p}}\).

For purposes of illustration, consider the case \(n = 4\):

\[-\det(L(C - \lambda I)L^{-1})_{(1)}\] = \lambda^2 (\pi_2 \pi_1^* + \pi_3 \pi_2^* + \pi_4 \pi_3^* + \lambda (\pi_3 \pi_1^* + \pi_4 \pi_2^*) + \pi_4 \pi_1^* \\
+ s_2 [\lambda^2 \pi_3 \pi_1^* - \lambda (\pi_1 \pi_2^* + \pi_2 \pi_1^*) - \pi_2 \pi_1^*] \\
+ s_3 [\lambda^2 \pi_2 \pi_1^* + \lambda (\pi_2 \pi_3^* + \pi_3 \pi_2^*) - \pi_1 \pi_1^*] \\
+ s_4 [\lambda^2 \pi_1 \pi_4^* + \lambda (\pi_1 \pi_5^* + \pi_5 \pi_1^*) + (\pi_1 \pi_2^* + \pi_2 \pi_1^* + \pi_3 \pi_4^*)].

When \(s_2, s_3,\) and \(s_4\) all vanish, this reduces to the polynomial of Case B, whose coefficients depend only on the Plücker coordinates for the negative roots of \(sl(4, \mathbb{C})\). The coefficients of the polynomial containing the factor \(s_2\) correspond to the subalgebra \(\mathcal{G}_{(1,2)} \oplus \mathcal{G}_{(3,4)}\). The constant term is (minus) the Plücker coordinate for the negative root of \(\mathcal{G}_{(1,2)}\) (of height -1), the coefficient of \(\lambda\) is (minus) the sum of the Plücker coordinates for the zero weight space of \(\mathcal{C}^2 \otimes (\mathcal{C}^2)^*\),
and the coefficient of \( \lambda^2 \) is the Plücker coordinate for the positive root of \( G_{(3,4)} \) (of height 1). This pattern repeats; the coefficients of the polynomial containing the factor \( s_3 \) correspond to the subalgebra \( G_{(1)} \oplus G_{(2,3,4)} \) in the same way. The constant term is (minus) the Plücker coordinate for the zero weight space of \( C \otimes C^* \), and the coefficients of \( \lambda \) and \( \lambda^2 \) are the sum of the roots of \( G_{(2,3,4)} \) of height 1 and the root of \( G_{(3,4)} \) of height 2, respectively. Finally, the coefficients of \( \lambda^2 \), \( \lambda \), and \( \lambda^0 \) in the polynomial corresponding to \( s_4 \) are simply the sums of the Plücker coordinates for the positive roots of \( \mathfrak{sl}(3,C) \) (= \( G_{(1,2,3,4)} \)) of heights 3, 2, and 1, respectively.

Writing the polynomial \( \det(L(C - \lambda I)L^{-1})_{(1)} \) in this way allows one to see how the expressions of the 1-chop integrals in the case of the companion embedding simplify as a generic isospectral level set approaches the extreme case in which all eigenvalues are zero through a sequence of degenerations in which one additional eigenvalue is set equal to zero in each step. Indeed, the \( m + 1 \) symmetric functions \( s_{n-m}, \ldots, s_n \) vanish simultaneously if and only if zero is an eigenvalue of multiplicity at least \( m + 1 \). In the case that all the \( \lambda_i \) are nonzero, \( s_n \neq 0 \), and the coefficients of the 1-chop polynomial depend on the Plücker coordinates for all the roots of \( \mathfrak{sl}(n,C) \). With each additional eigenvalue that is set equal to zero, the 1-chop polynomial loses its dependence on the Plücker coordinates for the roots of the greatest height that occurs in the previous step so that when \( m \) eigenvalues vanish, the 1-chop integrals do not depend on the roots of height greater than or equal to \( n - m \).

5 The Special Case \( n = 3 \)

In this example, we take \( G = SL(3,C) \). The elements of \( \epsilon + B_- \) have the form

\[
X = \begin{pmatrix} f_1 & 1 & 0 \\ g_1 & f_2 & 1 \\ h & g_2 & f_3 \end{pmatrix}, \quad \sum_{i=1}^{3} f_i = 0,
\]

and there is a single 1-chop integral, \( C = f_2 - \frac{g_1 h}{f_3} \), which is a Casimir. In this case \( G/P_1 \) coincides with the flag manifold \( G/B \), and the momentum polytope is the regular hexagon shown in Figure 1.

In Case A, the Casimir has the expression

\[
C = \frac{\lambda_2 \lambda_3 \pi_1 \pi_1^* + \lambda_1 \lambda_3 \pi_2 \pi_2^* + \lambda_1 \lambda_2 \pi_3 \pi_3^*}{\lambda_1 \pi_1 \pi_1^* + \lambda_2 \pi_2 \pi_2^* + \lambda_3 \pi_3 \pi_3^*}.
\]  

(10)

The base locus \( Z_\Lambda \) is the common intersection of the varieties \( \mathcal{F}_C^\Lambda \) defined by

\[
C(\sum_{i=1}^{3} \sigma_1(i)\pi_i \pi_i^*) - \sum_{i=1}^{3} \sigma_2(i)\pi_i \pi_i^* = 0
\]
for \( C \in \mathbb{C} \) and by the vanishing of the denominator in (11) for \( C = \infty \). It is the union of the strata in the flag manifold whose images under the momentum mapping lie on the boundary of the momentum polytope. Each of these strata consists of a unique torus orbit; there are six one-dimensional complex orbits corresponding to the edges of the hexagon and six fixed points whose images are the vertices.

This base locus and its image under the momentum mapping precisely encode the structure of the intersection of the isospectral submanifold \((\epsilon + B_-)_\Lambda\) with the set of symplectic leaves in \(\epsilon + B_-\) of complex dimension strictly less than four, the dimension of the generic leaves. Each of the six 1-dimensional orbits in \(B_\Lambda\) corresponds under the torus embedding to the intersection of \((\epsilon + B_-)_\Lambda\) with a symplectic leaf of complex dimension two; the six fixed points of the torus action are the images of the six 0-dimensional leaves contained in \((\epsilon + B_-)_\Lambda\).

In terms of the momentum polytope, the intersections of the lower-dimensional leaves with \((\epsilon + B_-)_0\) correspond to the six edges and the six vertices of the hexagon. This is explained in detail and illustrated in Chapter Three of [?].

The expression for the Casimir in Case B is

\[
C = \frac{\pi_3 \pi_1^*}{\pi_2 \pi_1^* + \pi_3 \pi_2^*}.
\]  

(11)

The base locus \( Z_0 \) is the union of the closures of two 1-dimensional complex torus orbits in \( G/B \), one satisfying \( \pi_1^* = \pi_2 = \pi_3 = 0 \) and the other satisfying \( \pi_1^* = \pi_2^* = \pi_3 = 0 \). Their images under the momentum mapping are the two edges of \( \text{Star}(\Delta_3, L_1 - L_3) \), shown in Figure 2.

As in Case A, the intersection of the isospectral submanifold \((\epsilon + B_-)_0\) with the set of symplectic leaves of dimension less than four corresponds to the base locus \( Z_0 \). \((\epsilon + B_-)_0\) has nontrivial intersections with only two of the two-dimensional leaves; these intersections have the forms

\[
\begin{pmatrix}
  a & 1 & 0 \\
  b & -a & 1 \\
  0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & a & 1 \\
  0 & b & -a
\end{pmatrix}
\]

with \( a^2 - b = 0 \), which correspond under the companion embedding to the two 1-dimensional complex torus orbits in \( Z_0 \). The unique 0-dimensional leaf in \((\epsilon + B_-)_0\) is the companion matrix. Its image under the companion embedding is the fixed point of the torus action whose image under the momentum mapping is the highest weight \( L_1 - L_3 \), in which the two edges of \( \text{Star}(\Delta_3, L_1 - L_3) \) intersect.

Consider now the variety \( F^0_\mathcal{C} \) defined by

\[
C(\pi_2 \pi_1^* + \pi_3 \pi_2^*) - \pi_3 \pi_1^* = 0
\]

for \( C \in \mathbb{C} \) and by the vanishing of the denominator in (11) for \( C = \infty \). For \( C = 0 \), \( F^0_\mathcal{C} \) is the union of the two components \( \pi_3 = 0 \) and \( \pi_1^* = 0 \). It is easy to
show that each of these components is the closure of a single nongeneric two-dimensional complex torus orbit in $G/P$. Their images under the momentum mapping are the two half-hexagons shown in Figure 3. For a nonzero value of the Casimir, the variety $F_0^C$ is not invariant under the torus action. The image of $F_0^C$ under the momentum mapping for $C \notin \{0, \infty\}$ contains exactly two points on the boundary of the momentum polytope which are not in the base locus. These are the points $\frac{1}{|C|} [(L_2 - L_1) + |C|^2(L_3 - L_1)]$ and $\frac{1}{|C|} [(L_3 - L_2) + |C|^2(L_3 - L_1)]$, which lie on the two edges opposite the edges of $\text{Star}(\triangle_3, L_1 - L_3)$. They are the images of $[0 : 1 : C] \times [1 : 0 : 0]$ and $[0 : 0 : 1] \times [C : 1 : 0]$, respectively, in $F_0^C$. Observe that each of these points lies at the same distance from the common vertex $L_3 - L_1$ of these two edges and that this distance depends only on the modulus of $C$. As $|C|^2$ increases from 0 to $\infty$, the two points move monotonically along the two edges, starting at the simple negative roots $L_2 - L_1$ and $L_3 - L_2$ when $C = 0$ and coinciding in the limit at $L_3 - L_1$ when $C = \infty$.

Furthermore, each of the two edges opposite $\text{Star}(\triangle_3, L_1 - L_3)$ is the image under $\mu$ of the closure of a unique 1-dimensional complex torus orbit in $G/P$, which is a copy of $\mathbf{P}^1$. These two copies intersect in the fixed point for which $\pi_3 \pi_1^* \neq 0$; they are parametrized as $[0 : z : w] \times [1 : 0 : 0]$ and $[0 : 0 : 1] \times [z : w : 0]$ for $[z : w] \in \mathbf{P}^1$. Since each of these copies of the projective line intersects $F_0^C$ in a unique point for every $C \in \mathbf{P}^1$, it can be considered as the parameter space for the varieties $F_0^C$; the two fixed points of the torus action contained in it correspond to $C = 0$ and $C = \infty$. Thus, the modulus of $C$ determines the image of each point in this parameter space under the momentum mapping so that each orbit of the compact torus $S^1$ in $\mathbf{P}^1$ has the same image under $\mu$.

**Remark:** The geometry of the base locus $Z_0$ and the reducible variety $F_0^C$ is seen in the Bruhat decomposition of the flag manifold,

$$G/B = \bigcup_{w \in \Sigma_3} BX_wB/B,$$

where $X_w$ is a matrix representing the permutation $w$. The single zero-dimensional cell is the fixed point in $Z_0$ whose image under $\mu$ is the highest weight $L_1 - L_3$, corresponding to the unique zero-dimensional leaf in $(\epsilon + B_-)_0$. The two one-dimensional complex cells are the torus orbits in $Z_0$ corresponding to the two edges of $\text{Star}(\triangle_3, L_1 - L_3)$. The two two-dimensional complex cells are the components of the reducible variety $F_0^C$, and the big cell contains the union of the generic level set varieties $F_0^C$ (minus the base locus) for $C \in \mathbf{P}^1 \setminus \{0\}$.

For a geometrical study of the varieties $F_0^C$ in Case A, we refer to Chapter Three of [?], in which it is shown that the geometry of the base locus $Z_0$, the three reducible varieties $F_{\lambda_i}^A$, and the irreducible varieties $F_{\lambda_i}^A$ for $\lambda_i \neq \lambda_i$, is precisely encoded in the partitioning of the flag manifold into strata (see [?]). This stratification of $G/B$ is finer than the Bruhat decomposition which, as we have shown, describes the analogous geometry in Case B.
6 The Special Case $n = 4$

In the full $\text{Sl}(4, \mathbb{C})$ Kostant-Toda lattice, there are two 1-chop integrals, $I_{11}$, which is a Casimir, and the constant of motion $I_{21}$. In Case A, these integrals have the expressions

$$I_{11} = \frac{\sum_{i=1}^{4} \sigma_2(\hat{i}) \pi_i \pi_i^*}{\lambda_1 \pi_1 \pi_1^* + \lambda_2 \pi_2 \pi_2^* + \lambda_3 \pi_3 \pi_3^* + \lambda_4 \pi_4 \pi_4^*},$$

$$I_{21} = -\frac{\lambda_2 \lambda_4 \pi_1 \pi_1^* + \lambda_1 \lambda_3 \lambda_4 \pi_2 \pi_2^* + \lambda_1 \lambda_2 \lambda_4 \pi_1 \pi_3^* + \lambda_1 \lambda_2 \lambda_3 \pi_4 \pi_4^*}{\lambda_1 \pi_1 \pi_1^* + \lambda_2 \pi_2 \pi_2^* + \lambda_3 \pi_3 \pi_3^* + \lambda_4 \pi_4 \pi_4^*}.$$

The momentum polytope $\triangle_4$ of $G/P_1$ is the cuboctahedron, shown in Figure 4; it has fourteen two-dimensional faces. The base locus $\mathcal{Z}_\lambda$ is the union of fourteen components, each of which is the closure of the unique two-dimensional complex torus orbit in $G/P_1$ whose image under $\mu$ is a particular two-dimensional face of the momentum polytope.
In Case B the 1-chop integrals are expressed as
\[ I_{11} = \frac{\pi_3 \pi_1^* + \pi_4 \pi_2^*}{\pi_2 \pi_1^* + \pi_3 \pi_2^* + \pi_4 \pi_3^*}, \]
\[ I_{21} = \frac{\pi_4 \pi_1^*}{\pi_2 \pi_1^* + \pi_3 \pi_2^* + \pi_4 \pi_3^*}. \]

The base locus, \( Z_0 \), of the corresponding varieties in \( G/P_1 \) is the union of the closures of three two-dimensional complex torus orbits. Its image under the momentum mapping is \( \text{Star}(\triangle_4, L_1 - L_4) \), which consists of the two triangular faces perpendicular to the fundamental weights \( L_1 \) and \( -L_4 \) and the square face perpendicular to the fundamental weight \( L_1 + L_2 \), as illustrated in Figure 5.

The isomorphism \( \rho : \mathfrak{sl}(4, \mathbb{C}) \to \mathfrak{so}(6, \mathbb{C}) \) gives rise to another constant of motion, \( J \), which is in involution with the integrals \( I_{1k} \) for \( k = 2, 3, 4 \), but not with \( I_{21} \). It is obtained by a chopping construction on the matrix \( \rho(X) - \lambda I \) similar to the one on \( X - \lambda I \) described in Proposition 2.2 which gives the 1-chop integrals \( I_{r1} \). The restriction of \( J \) to the generic symplectic leaf on which the Casimir \( I_{11} \) is equal to zero exhibits a geometry very similar to that of the level sets of the 1-chop integrals in each of the two types of isospectral submanifolds we have been considering.

We start with Case A. Let \( \mathcal{V}_\Lambda \) be the subvariety of the flag manifold defined by
\[ \sum_{i=1}^{4} \sigma_2(i) \pi_i \pi_i^* = 0, \]
on which \( I_{11} \) vanishes. In [?] it is found that the function \( J \), when restricted to \( \mathcal{V}_\Lambda \), depends only on its projection to the Grassmannian \( G(2, 4) \) of two-dimensional subspaces of \( \mathbb{C}^4 \). \( G(2, 4) \) is the orbit \( SL(4, \mathbb{C}) : [e_1 \wedge e_2] \subset P(\wedge^2 \mathbb{C}^4) \).

The weights of the irreducible representation \( \wedge^2 \mathbb{C}^4 \) of \( \mathfrak{sl}(4, \mathbb{C}) \) are \( L_i + L_j \) for \( i \neq j \). We will take the Plücker coordinates on \( G(2, 4) \) with respect to the weight basis \( \{ e_i \wedge e_j \}_{i<j} \) and denote \( \pi_{L_i + L_j} \) by \( \pi_{ij} \). In terms of these coordinates, the expression for the restriction of \( J \) to \( \mathcal{V}_\Lambda \) as given in [?] is
\[ J|_{\mathcal{V}_\Lambda} = \frac{A_A A_A' \pi_{12} \pi_{34} - B_\Lambda B_\Lambda' \pi_{13} \pi_{24}}{A_A \pi_{12} \pi_{34} - B_\Lambda \pi_{13} \pi_{24}}, \]
where
\[ A_A = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_4), \]
\[ A_A' = (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_4), \]
\[ B_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4), \]
\[ B_\Lambda' = (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4). \]

This is simply a linear fractional transformation of the cross-ratio \( c \) defined by
\[ c = \frac{\pi_{12} \pi_{34}}{\pi_{13} \pi_{24}}. \]
Let $\hat{Z}_\Lambda$ denote the common intersection of the varieties
\[ c \pi_{13}\pi_{24} - \pi_{12}\pi_{34} = 0 \]
in $G(2, 4)$ for $c \in C$, and let $\mu_{G(2, 4)}$ be the momentum mapping on the Grassmannian. The momentum polytope, which we denote by $\Box$, is shown in Figure 6. Our statement in this situation is parallel to Proposition 4.3 for the integrals $I_{r1}$ in Case A. Its proof is found in [?].

**Proposition 6.1** $\hat{Z}_\Lambda = \mu_{G(2, 4)}^{-1}(\partial\Box)$. The momentum mapping induces a one-to-one correspondence between the eight components of $\hat{Z}_\Lambda$, each of which is the closure of a single torus orbit of complex dimension two, and the set of two-dimensional faces of $\Box$.

Consider now Case B, in which we take the companion embedding of $(\epsilon + B_-)_0$ into $G/B$, and let $\mathcal{V}_0$ be the subvariety of $G/B$ defined by
\[ \pi_3\pi_1^* + \pi_4\pi_2^* = 0, \]
on which $I_{11}$ vanishes. The expression for $J$ restricted to $\mathcal{V}_0$ is
\[ J|_{\mathcal{V}_0} = \frac{\pi_{34}^2}{\pi_{34}(\pi_{14} + \pi_{23}) - \pi_{24}^2}. \] (12)
(To see this, it suffices to show that it holds on the matrices $X(I_{21}, u)$ defined by equation (5.9) in [?] with all eigenvalues set equal to zero since $J$ is invariant under the torus action and $u$ parametrizes the generic torus orbits in each generic level set of $I_{21}$ in $(\epsilon + B_-)_0$. Indeed, evaluating our formula (12) on the image under the companion embedding of $X(I_{21}, u)$ with the eigenvalues equal to zero gives precisely the expression for $J$ in equation (6.4) of [?] with all eigenvalues zero.)

Let $\hat{Z}_0$ denote the common intersection of the varieties
\[ J|_{\mathcal{V}_0}(\pi_{34}(\pi_{14} + \pi_{23}) - \pi_{24}^2) - \pi_{34}^2 = 0 \]
in $G(2, 4)$ for $J|_{\mathcal{V}_0} \in C$. Since $G(2, 4)$ is the vanishing set of the polynomial
\[ \pi_{12}\pi_{34} - \pi_{13}\pi_{24} + \pi_{14}\pi_{23} = 0 \] (13)
in $P(\wedge^2C^4)$, this base locus consists of two components, $\pi_{34} = \pi_{24} = \pi_{14} = 0$ and $\pi_{34} = \pi_{24} = \pi_{23} = 0$. It is easily seen that each of these components is the closure of a single torus orbit of complex dimension two. Taking their images under the momentum mapping, we obtain a statement similar to Proposition 4.2 for the base locus $Z_0$.

**Proposition 6.2** $\hat{Z}_0 = \mu_{G(2, 4)}^{-1}(\text{Star}(\Box, L_1 + L_2))$. Each of the two components of $\hat{Z}_0$ is the closure of a single torus orbit of complex dimension two; their images under the momentum mapping are the two faces of $\text{Star}(\Box, L_1 + L_2)$.

This is illustrated in Figure 7.