RUELLE OPERATOR FOR INFINITE CONFORMAL IFS

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Abstract

Let \((X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)\) \((2 \leq m < \infty)\) be a contractive iterated function system (IFS), where \(X\) is a compact subset of \(\mathbb{R}^d\). It is well known that there exists a unique nonempty compact set \(K\) such that \(K = \bigcup_{j=1}^m w_j(K)\). Moreover, the Ruelle operator on \(C(K)\) determined by the IFS \((X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)\) \((2 \leq m < \infty)\) has been introduced in [FL]. In the present paper, the Ruelle operators determined by the infinite conformal IFSs are discussed. Some separation properties for the infinite conformal IFSs are investigated by using the Ruelle operator.

1. Introduction

The (finite) conformal iterated function systems (IFSs) have been studied in various ways. Fan and Lau [FL] considered the eigenmeasure by using the Ruelle operator. Fan et al. ([FL], [L]) extended Schief’s idea [S] to prove the equivalence of the OSC and SOSC. Peres et al. [P] showed that for conformal IFSs satisfying the Hölder condition with the invariant set \(K\), both OSC and SOSC are equivalent to \(0 < H^s(K) < \infty\), where \(s\) is the zero point of the pressure function \(P(\cdot)\). A simple proof of the result was also given by Ye [Y].

The infinite iterated function systems (IFSs) of similarity maps were developed by Moran [M]. Moran [M] extended the classical results for finite systems of similitudes satisfying the open set condition to the infinite case. Mauldin and Urbański [MU] considered the dimension and Hausdorff measure of the limit sets of the infinite conformal IFSs. They found a way to determine the Hausdorff dimension of the limit sets under the condition that the basic open sets were connected. In this paper, we continue the

* The corresponding author: the Research was partially supported by SRF for ROCS, SEM.
study of the infinite conformal IFSs. We only assume that the space \( X \) is locally connected. This is similar to the finite case [Y].

Let \((X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)\) be an IFS, where \(w_j\)'s are contractive self-maps on a compact subset \( X \subset \mathbb{R}^d \) and \(p_j\)'s are positive Dini functions on \( X \) [FL]. Then there exists a unique compact set \( K \) invariant under IFS, i.e.,

\[
K = \bigcup_{j=1}^m w_j(K).
\]

The Ruelle operator on \( C(K) \) is defined by

\[
T(f)(x) = \sum_{j=1}^m p_j(w_j(x))f(w_j(x)), \quad f \in C(K),
\]

where \( C(K) \) is the space of all continuous functions on \( X \). For the importance of studying the Ruelle operators can be found in [LY]. We could generalize the Ruelle operators to the infinite conformal IFSs. Then we obtain the PF-property (Theorem 1.1) using the Ruelle operator we define.

The separation properties (OSC and SOSC) are useful for studying the (finite) conformal IFSs. However, for the infinite conformal IFSs, Moran [M] showed that self-similar set generated by a countable system of similitudes may not be s-set even if the open set condition (OSC) is satisfies. Szarek and Wedrychowicz [SW] showed that for infinite IFSs, the OSC does not imply the SOSC. So it is necessary to consider the weak separation properties for the infinite conformal IFSs. Let \( \{s_j\}_{j=1}^\infty \) be an infinite conformal IFS on an open set \( U_0 \). We call the infinite conformal IFS \( \{s_j\}_{j=1}^\infty \) satisfies the finite open set condition if for any integer \( n \), there is a nonempty open set \( U_n \subset U_0 \) such that \( s_i(U_n) \subset U_n \) for any \( i \leq n \) and \( s_i(U_n) \cap s_j(U_n) = \emptyset \) for any \( i, j \leq n, i \neq j \). Such weak separation property has been used to study the Hausdorff dimension of invariant set [M]. We give a sufficient condition for the infinite conformal IFS \( \{s_j\}_{j=1}^\infty \) satisfies the finite (strong) open condition in Section 4. Our basic results are:

**Theorem 1.1.** Let \((X, \{s_j\}_{j=1}^\infty, \{p_j\}_{j=1}^\infty)\) be an infinite conformal uniformly Dini IFS, and let \( J \) be the limit set of the IFS \( \{s_j\}_{j=1}^\infty \). Then there exists a unique \( \theta < h \in C(J) \) and a unique probability measure \( \mu \in M(J) \) such that

\[
T^* \mu = \theta \mu, \quad \langle \mu, h \rangle = 1.
\]

Moreover, for every \( f \in C(J) \), \( \theta^{-n} T^n f \) converges to \( \langle \mu, f \rangle \) in the supremum norm, and for every \( \xi \in M(J) \), \( \theta^{-n} T^n \xi \) converges weakly to \( \langle \xi, h \rangle \mu \).

**Theorem 1.2.** Let \( \{s_j\}_{j=1}^\infty \) be an infinite conformal IFS, and let \( J \) be the limit set of the IFS. If Hausdorff measure \( H^a(J) = H^a(J) > 0 \), then the system \( \{s_j\}_{j=1}^\infty \) satisfies the finite strong open set condition.
Theorem 1.1 generalizes THEOREM 1.1 of the paper [FL]. Theorem 1.2 extends the result of the paper [Y]. Observe that the system is infinite, to prove Theorem 1.1 we need to assume that the potential functions are uniformly Dini continuous. Under the assumption that there exists a number \(a\) determined by the Ruelle operator, we can prove Theorem 1.2 by making use of Schief’s idea [S].

The paper is organized as follows. In Section 2, we present some concepts for infinite conformal IFSs and some elementary facts about the Ruelle operators determined by infinite conformal IFSs. In Section 3, we study the Ruelle operators defined by the infinite conformal IFSs. In Section 4, we investigate the separation properties of the infinite conformal IFSs.

2. Preliminaries

In the paper, we always assume that \(X\) is a locally connected compact subset of \(\mathbb{R}^d\). Let \(w\) be a self-map on compact set \(X \subset \mathbb{R}^d\). We call \(w\) a conformal map if there exists an open set \(X \subset U_0\) such that \(w\) is continuous differentiable on \(U_0\) and for each \(x \in U_0\), \(w'(x)\) is a self-similar matrix. We call the \(\{s_j\}_{j=1}^\infty\) an infinite conformal iterated function system (IFS), if each \(s_j\) is self-conformal map. The family of functions \(\{p_j\}_{j=1}^\infty\) is said to be uniformly Dini continuous on \(U_0\) provided that \(\int_0^1 \frac{\alpha(t)}{t} \, dt < +\infty\), where

\[
\alpha(t) := \sup_{j \geq 1} \sup_{x,y \in U_0} \frac{\log p_j(x) - \log p_j(y)}{|x-y|}
\]

and \(\sum_{j=1}^{\infty} p_j(x) < \infty\) on \(U_0\). Under this assumption, we call the triple \((X, \{s_j\}_{j=1}^\infty, \{p_j\}_{j=1}^\infty)\) an infinite conformal uniformly Dini IFS.

Throughout the paper, we always assume that the infinite conformal IFS \(\{s_j\}_{j=1}^\infty\) satisfies the following conditions:

(i) each \(s_j : X \to X\) is one-to-one and the system \(\{s_j\}_{j=1}^\infty\) is uniformly contractive, i.e., there exists 0 < \(s < 1\) such that for any 1 \(\leq j < \infty\), \(|s_j(x) - s_j(y)| \leq s|x-y|\), \(\forall x, y \in X\);

(ii) each \(s_j\) is conformal on \(U_0(\supset X)\) with \(0 < \inf_{x \in U_0} |s'_j(x)| \leq \sup_{x \in U_0} |s'_j(x)| < 1\);

(iii) \(\{|s'_j|\}_{j=1}^\infty\) is uniformly Dini continuous on \(U_0\).

Let the \(\{s_j\}_{j=1}^\infty\) be an infinite conformal IFS. Denote \(I = \{1, 2, 3, \cdots\}\),

\[
I^n = \{u = u_1u_2\cdots u_n : u_j \in I, \forall 1 \leq j \leq n\},
\]

\[
I^\infty = \{v = v_1v_2\cdots : v_j \in I, \forall 1 \leq j < \infty\},
\]

and let \(I^* = \bigcup_{n \geq 1} I^n\). For arbitrary \(w = w_1w_2\cdots w_k \in I^k\), we let \(|w| = k\) and \(s_w = s_{w_1} \circ s_{w_2} \cdots \circ s_{w_k}\).
For any \( w \in I^* \cup I^\infty \) and integer \( n \), if \( n \leq |w| \), we denote by \( w|_n \) the word \( w_1 w_2 \cdots w_n \). By the contractiveness of the system \( \{s_j\}_{j=1}^\infty \), we have
\[
\lim_{n \to \infty} \text{diam}\{s_{w|_n}(X)\} \to 0, \quad \forall w \in I^\infty.
\]
This implies the set \( \bigcap_{n=1}^\infty s_{w|_n}(X) \) is a singleton. Hence we can define a map \( \pi : I^\infty \to X \) by
\[
\pi(w) := \bigcap_{n=1}^\infty s_{w|_n}(X).
\]
We endow the \( I^\infty \) with the metric \( d(w, u) = e^{-n(w, u)} \), where \( n(w, u) \) is the largest integer \( n \) such that \( w|_n = u|_n \). We can prove that the map \( \pi \) is continuous.

Define
\[
J = \pi(I^\infty) = \bigcup_{w \in I^\infty} \bigcap_{n=1}^\infty s_{w|_n}(X).
\]
We can check that the set \( J \) satisfies the following equation:
\[
J = \bigcup_{j \in I} s_j(J).
\]
The set \( J \) is said to be the limit set of the system \([MU]\). For an infinite conformal uniformaly Dini IFS \((X, \{s_i\}_{i=1}^\infty, \{p_j\}_{j=1}^\infty)\), we define an operator \( T : C(J) \to C(J) \) by
\[
Tf(x) = \sum_{j=1}^\infty p_j(s_j(x))f(s_j(x)).
\]
\( T \) is called the Ruelle operator of the system. The dual operator \( T^* \) on the measure space \( M(J) \) is given by
\[
T^* \mu(E) = \sum_{j=1}^\infty \int_{s_j^{-1}(E)} p_j(x) \, d\mu(x) \quad \text{for any Borel set } E \subseteq J.
\]
(For the finite see e.g. \([B]\)).

For \( j = j_1 j_2 \cdots j_n \), we let
\[
p_{s_j}(x) = p_{j_1}(s_{j_2} \circ s_{j_3} \circ \cdots \circ s_{j_n}(x)) \cdots p_{j_{n-1}}(s_{j_n}(x)) p_{j_n}(x).
\]
By induction we have for any integer \( n \),
\[
T^n f(x) = \sum_{|j| = n} p_{s_j(x)} f(s_j(x)).
\]
Let \( \rho = \rho(T) \) be the spectral radius of \( T \). Since \( T \) is a positive operator, we have \( \|T^n 1\| = \|T^n\| \) and
\[
\rho = \lim_{n \to \infty} \|T^n\|^\frac{1}{n} = \lim_{n \to \infty} \|T^n 1\|^\frac{1}{n}.
\]
For the sake of convenience, we can assume that

\[ \text{diam}(U_0) = \sup\{|x - y| : x, y \in U_0\} \leq 1. \]

Let \( J \) be the limit set of the system \( \{s_j\}_{j=1}^\infty \). For any \( r > 0 \), denote \( B(J, r) = \bigcup_{x \in J} B(x, r) \). Choose \( \kappa > 0 \) such that

\[ (2.4) \quad X_0 := \bigcup_{x \in X} \overline{B(x, \kappa)} \subset U_0. \]

From the contractiveness of \( s_j \)'s, we can show that \( s_j(X_0) \subset X_0 \) for any \( j \in I \). Hence we may assume that

\[ B(J, \kappa) \subset X. \]

We call the infinite conformal IFS \( \{s_j\}_{j=1}^\infty \) satisfies the open set condition (OSC) if there exists a nonempty bounded open set \( U \subset U_0 \) such that

\[ s_j(U) \subset U \quad \text{and} \quad s_i(U) \cap s_j(U) = \emptyset \quad \text{for any} \ i, j \in I \text{ and} \ i \neq j. \]

Such a open set \( U \) is called a basic open set for the system \( \{s_j\}_{j=1}^\infty \). If moreover \( U \cap J \neq \emptyset \), then the system \( \{s_j\}_{j=1}^\infty \) is said to satisfy the strong open set condition (SOSC).

To study the separation properties of the infinite conformal IFS, we introduce the finite open set condition as follows.

**Definition 2.1.** We call the infinite conformal IFS \( \{s_j\}_{j=1}^\infty \) satisfies the finite open set condition if for any integer \( n \), there is a nonempty open set \( U_n \subset U_0 \) such that

\[ s_j(U_n) \subset U_n \quad \text{for each} \ 1 \leq j \leq n \quad \text{and} \quad s_i(U_n) \cap s_j(U_n) = \emptyset \quad \text{for each} \ 1 \leq i, j \leq n \quad \text{and} \ i \neq j. \]

The finite strong open set condition holds if furthermore \( U_n \cap J \neq \emptyset \).

Obviously the finite open set condition is weaker than the OSC.

3. **RUELLE OPERATOR**

**Proposition 3.1.** Let \( \{s_j\}_{j=1}^\infty \) be an infinite conformal IFS, and let \( J \) be the limit set of the system \( \{s_j\}_{j=1}^\infty \). Then

\[ \overline{\{s_u(x) : u \in I^*\}} = \overline{J}, \quad \forall x \in J. \]

**Proof.** For any \( x \in J \) and \( u \in I^* \), we have

\[ s_u(x) \in s_u(J) \subset \overline{s_u(J)} \subset \overline{J}. \]

This implies that \( \overline{\{s_u(x) : u \in I^*\}} \subset \overline{J} \).
For any \( y \in J \), by noticing that (2.1) and the contractiveness of the system \( \{ s_j \}_{j=1}^{\infty} \), there exists \( w \in I^* \) such that
\[
\lim_{n \to \infty} \{ s_{w/n}(x) \} = y, \quad \forall x \in \mathcal{J}.
\]
This implies that \( J \subset \{ s_u(x) : u \in I^* \} \). We have
\[
\mathcal{J} \subset \{ s_u(x) : u \in I^* \}.
\]
Thus we conclude the assertion. \( \square \)

**Proposition 3.2.** Let the operator \( T \) be defined as in (2.2). Then
(i) \( \min_{x \in \mathcal{J}} \varrho^{-n}T^n1(x) \leq 1 \leq \max_{x \in \mathcal{J}} \varrho^{-n}T^n1(x), \quad \forall n > 0. \)
(ii) if there exist \( \lambda > 0 \) and \( 0 < h \in C(\mathcal{J}) \) such that \( Th = \lambda h \), then \( \lambda = \varrho \) and there exist \( A, B > 0 \) such that
\[
A \leq \varrho^{-n}T^n1(x) \leq B, \quad \forall n > 0.
\]
**Proof.** (i) We will prove the second inequality of (i), the first inequality is similar. Suppose it is not true, then there exists a \( k \) such that
\[
\varrho = (\varrho(T^k))^\frac{1}{k} \leq \|T^k1\|^\frac{1}{k} = \|T^k1\|^\frac{1}{k} < \varrho,
\]
which it is a contradiction.
(ii) Let \( a_1 = \min_{x \in \mathcal{J}} h(x) \), \( a_2 = \max_{x \in \mathcal{J}} h(x) \). Then
\[
0 < \frac{a_1}{a_2} \leq \frac{h(x)}{a_2} = \frac{\lambda^{-n}T^n h(x)}{a_2} \leq \lambda^{-n}T^n1(x) \tag{3.1}
\]
Since \( T^n \) is a positive operator, we have
\[
\frac{a_1}{a_2} \leq \lambda^{-n}\|T^n\|.
\]
Similarly we can show that
\[
\lambda^{-n}T^n1(x) \leq \frac{a_2}{a_1} \tag{3.2}
\]
This implies that
\[
\lambda^{-n}\|T^n\| \leq \frac{a_2}{a_1}.
\]
Hence \( \varrho = \lim_{n \to \infty} \|T^n\|^\frac{1}{n} = \lambda \). This, together with (3.1) and (3.2), implies that
\[
A := \frac{a_1}{a_2} \leq \varrho^{-n}T^n1(x) \leq B := \frac{a_2}{a_1}, \quad \forall n > 0. \tag*{\square}
\]

We call the operator \( T : C(\mathcal{J}) \to C(\mathcal{J}) \) irreducible if for any non-trivial, non-negative \( f \in C(\mathcal{J}) \) and for any \( x \in \mathcal{J} \), there exists \( n > 0 \) such that \( T^n f(x) > 0 \).
Proposition 3.3. Let \((X, \{s_i\}_{i=1}^{\infty}, \{p_j\}_{j=1}^{\infty})\) be an infinite conformal uniformly Dini IFS. Then the Ruelle operator \(T\) is irreducible and
\[
\dim \{h \in C(\overline{J}) : Th = \rho h, \ h \geq 0\} \leq 1;
\]
if \(h \geq 0\) is a \(\rho\)-eigenfunction of \(T\), then \(h > 0\).

Proof. For any given \(f \in C(\overline{J})\) with \(f \geq 0\) and \(f \neq 0\), let \(V = \{x \in \overline{J} : f(x) > 0\}\). For any \(x \in \overline{J}\), by Proposition 3.1 there exists a \(u_0\) such that \(s_{u_0}(x) \in V\). Let \(n_0 = |u_0|\), then
\[
T^{n_0}f(x) = \sum_{|u|=n_0} p_{s_u}(x)f(s_u(x)) \geq p_{s_{u_0}}(x)f(s_{u_0}(x)) > 0.
\]
This implies that \(T\) is irreducible.

For the dimension of the eigensubspace, we suppose that there exist two independent strictly positive \(\rho\)-eigenfunctions \(h_1, h_2 \in C(\overline{J})\). Without loss of generality we assume that \(0 < h_1 \leq h_2\) and \(h_1(x_0) = h_2(x_0)\) for some \(x_0 \in \overline{J}\). Then \(h = h_2 - h_1(\geq 0)\) is a \(\rho\)-eigenfunction of \(T\) and \(h(x_0) = 0\). It follows that \(T^n h(x_0) = \rho^n h(x_0) = 0\), which contradicts to the irreducibility of \(T\). Hence the dimension of the \(\rho\)-eigensubspace is at most 1.

The strict positivity of \(h\) follows directly from the irreducibility of \(T\). □

Proposition 3.4. Let \((X, \{s_i\}_{i=1}^{\infty}, \{p_j\}_{j=1}^{\infty})\) be an infinite conformal uniformly Dini IFS and \(\varrho\) be the spectral radius of \(T\). Then
\begin{enumerate}[(i)]
\item there exists \(0 < h \in C(\overline{J})\) such that \(Th = \varrho h\);
\item there exist \(A, B > 0\) such that \(A \leq \varrho^{-n}T^n 1(x) \leq B, \ \forall n > 0\).
\end{enumerate}

Proof. (i) Let
\[
\Phi(t) = \sum_{j=0}^{\infty} \alpha(s^j t),
\]
where \(\alpha(t) = \sup_{j \geq 1} \max_{|x-y| \leq 1} |\log p_j(x) - \log p_j(y)|\). From the uniformly Dini continuity of \(\{p_j\}_{j=1}^{\infty}\), it follows that \(\Phi(\cdot)\) is well-defined and \(\Phi(t) < +\infty\).

Let \(C^+(\overline{J}) := \{f \in C(\overline{J}) : f > 0\}\), and set
\[
F := \{f \in C^+(\overline{J}) : f(x) \leq f(y)e^{\Phi(|x-y|)}\}.
\]
For any \(f \in F\) and any \(x, y \in \overline{J}\), we have
\[
T f(x) = \sum_{j=1}^{\infty} p_j(s_j(x)) f(s_j(x)) \leq \sum_{j=1}^{\infty} p_j(s_j(y)) f(s_j(y)) e^{\alpha(|x-y|) + \Phi(|x-y|)} = T f(y) e^{\alpha(|x-y|) + \Phi(|x-y|)} \leq T f(y) e^{\Phi(|x-y|)}.
\]
We define $L : F \rightarrow C(\mathcal{J})$ by

$$Lf(x) := \frac{Tf(x)}{\|Tf\|}.$$  

By (3.4) we deduce

$$Lf(x) \leq Lf(y)e^{\Phi(|x-y|)}.$$  

Let

$$F_0 = F \cap \{e^{-\Phi(1)} \leq f \leq 1\}.$$  

It is easy to show that $F_0$ is a convex compact subset of $C(\mathcal{J})$, and $L(F_0) \subset F_0$. The Schauder fixed point yields an $h \in F_0$ such that $Lh = h$. This implies that $Th = \|Th\|h$. By the Proposition 3.2, we have $\|Th\| = \rho$. Hence $Th = \rho h$.

(ii) The proof comes from Proposition 3.2 immediately.  

We are now ready to prove our first result.

**Proof of Theorem 1.1**: The proof is modified from the paper [LY]. We include the details here for the sake of completeness. By Proposition 3.4, there exists $0 < h \in C(\mathcal{J})$ and constants $B \geq A > 0$ such that $Th = \rho h$ and

$$A \leq \rho^{-n}T^n1(x) \leq B, \quad \forall \ n > 0.$$  

Let $C^+(\mathcal{J}) := \{f \in C(\mathcal{J}) : f > 0\}$, and let

$$D = \{f \in C^+(\mathcal{J}) : \text{there exists } c > 0 \text{ such that } f(x) \leq f(y)e^{c|x-y|}\}.$$  

Then $D$ is dense in $C^+(\mathcal{J})$. Let

$$D_k := \{f \in C^+(\mathcal{J}) : f(x) \leq f(y)e^{k\Phi(|x-y|)}\},$$  

where $\Phi$ is given in (3.3). It is easy to see that $D \subseteq \tilde{D} := \bigcup_{k=1}^\infty D_k$, hence $\tilde{D}$ is dense in $C^+(\mathcal{J})$.

For any $f \in D_k$ and $x, y \in \mathcal{J}$, we have

$$Tf(x) = \sum_{j=1}^\infty p_j(s_j(x))f(s_j(x))$$

$$\leq \sum_{j=1}^\infty p_j(s_j(y))f(s_j(y))e^{k\alpha(|x-y|)+k\Phi(s|x-y|)}$$

$$= Tf(y)e^{k\alpha(|x-y|)+k\Phi(s|x-y|)}$$

$$= Tf(y)e^{k\Phi(|x-y|)}.$$  

It follows that $TD_k \subset D_k$.  

□
For any \( g \in C^+(\overline{J}) \), \( f \in D_k \) and \( n > 0 \), we have
\[
|\varrho^{-n}T^n g(x) - \varrho^{-n}T^n g(y)|
\leq \|\varrho^{-n}T^n f\| \left(1 - \frac{T^n f(y)}{T^n f(x)}\right) + 2\|\varrho^{-n}T^n\| \|f - g\|
\leq B\|f\|(e^{\kappa \Phi(|x-y|)} - 1) + 2\|f - g\|
\]
for any \( x, y \in \overline{J} \). By the assumptions on \( D \) and \( \Phi \), we can deduce that for any \( g \in C^+(\overline{J}) \), \( \{\varrho^{-n}T^n g\}_{n=1}^{\infty} \) is a bounded equicontinuous sequence.

For any \( f \in C(\overline{J}) \), we can choose \( a > 0 \) such that \( f + a > 0 \). Then from the above prove process, it follows that the sequences \( \{\varrho^{-n}T^n (f+a)\}_{n=1}^{\infty} \) and \( \{\varrho^{-n}T^n a\}_{n=1}^{\infty} \) are bounded equicontinuous, hence the sequence \( \{\varrho^{-n}T^n f\}_{n=1}^{\infty} \) is also bounded equicontinuous. We let
\[
q_j(x) = \frac{p_j(s_j(x))h(s_j(x))}{gh(x)}
\]
and define an operator \( P : C(\overline{J}) \to C(\overline{J}) \) by
\[
P f(x) = \sum_{j=1}^{\infty} q_j(x)f(s_j(x)).
\]

For any \( f \in C(\overline{J}) \), we have \( T^n f = \varrho^n h P^n f \cdot h^{-1} \). This implies that \( \{P^n f\}_{n=1}^{\infty} \) is a bounded equicontinuous sequence in \( C(\overline{J}) \). We know from the Arzelà-Ascoli theorem that there exists \( \tilde{f} \in C(\overline{J}) \) and a subsequence \( \{P^n f\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} \|P^n f - \tilde{f}\| = 0 \).

We claim that \( \tilde{f} \) is a constant function and \( \lim_{n \to \infty} \|P^n f - \tilde{f}\| = 0 \). For this we let \( \tau(g) = \min_{x \in \overline{J}} g(x) \). Since \( \sum_{j=1}^{\infty} q_j(x) = 1 \), it is easy to see that \( \tau(\tilde{f}) \leq \tau(P \tilde{f}) \) and
\[
\tau(f) \leq \tau(P \tilde{f}) \leq \cdots \leq \tau(\tilde{f}).
\]
By taking the limit, we have \( \tau(P \tilde{f}) \leq \tau(\tilde{f}) \), hence \( \tau(P \tilde{f}) = \tau(\tilde{f}) \). For any \( n > 0 \), we select \( x_n \in \overline{J} \) such that \( P^n \tilde{f}(x_n) = \tau(P^n \tilde{f}) \). Then \( \sum_{|j|=n} q_j(x_n) = 1 \) implies that
\[
\tilde{f}(s_j(x_n)) = \tau(\tilde{f}), \quad \forall j \in I^*, |j| = n.
\]
Similarly there exists \( y_n \in \overline{J} \) such that
\[
\tilde{f}(s_j(y_n)) = \eta(\tilde{f}) := \max_{x \in \overline{J}} \tilde{f}(x), \quad \forall j \in I^*, |j| = n.
\]
We assume \( j_n = 11 \cdots 1, |j_n| = n \). Then
\[
z := \lim_{n \to \infty} s_{j_n}(x_n) = \lim_{n \to \infty} s_{j_n}(y_n) \in \overline{J}.
\]
Hence
\[
\tau(\tilde{f}) = \lim_{n \to \infty} \tilde{f}(s_{j_n}(x_n)) = \tilde{f}(z) = \lim_{n \to \infty} \tilde{f}(s_{j_n}(y_n)) = \eta(\tilde{f}).
\]
We deduce \( \tilde{f}(x) \equiv \tau(\tilde{f}) \) is constant function. By (3.6) and the dual version for \( \eta(\tilde{f}) \), we have \( \lim_{n \to \infty} \|P^n f - \tilde{f}\| = 0. \)

In particular, by taking \( f = h^{-1} \), we see that \( P^n(h^{-1}) \) converges uniformly, then \( \varphi^{-n}T^n1 \) converges uniformly.

To prove that

\[
\lim_{n \to \infty} \|\varphi^{-n}T^n1 - h\| = 0,
\]

we let

\[
f_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi^{-i}T^i1(x).
\]

From (3.5), it follows that \( \{f_n\}_{n=1}^\infty \) is bounded by \( A \) and \( B \) and is an equicontinuous subset of \( C(\bar{J}) \). By Arzelà-Ascoli theorem, we assume without loss of generality that there exists a \( \tilde{h} \in C(\bar{J}) \) such that \( \lim_{n \to \infty} \|f_n - \tilde{h}\| = 0 \). We have

\[
\|T\tilde{h} - \varphi\tilde{h}\| = \lim_{n \to \infty} \|Tf_n - \varphi f_n\| \leq \lim_{n \to \infty} \frac{\varphi}{n} \|1 - \varphi^{-n}T^n1\| \leq \lim_{n \to \infty} \frac{\varphi}{n} \left(1 + B\right) = 0,
\]

i.e., \( T\tilde{h} = \varphi\tilde{h} \) and also \( \tilde{h} \geq A > 0 \). Proposition 3.3 implies that \( \tilde{h} = ch \) for some \( c > 0 \). Without loss of generality, we assume that \( h = \tilde{h} \). This implies that \( \lim_{n \to \infty} \|\varphi^{-n}T^n1 - h\| = 0 \). So \( \lim_{n \to \infty} \|P^n(h^{-1}) - 1\| = 0. \)

Now we define a function \( \upsilon : C(\bar{J}) \to \mathbb{R}, \langle \upsilon, f \rangle = \tau(\tilde{f})(= \tilde{f}(x), x \in \bar{J}) \). Then \( \upsilon \) is a bounded linear functional on \( C(\bar{J}) \), and \( \langle \upsilon, 1 \rangle = 1, \langle \upsilon, h^{-1} \rangle = 1 \). From

\[
\langle \upsilon, Pf \rangle = \tau(P\tilde{f}) = \tau(\tilde{f}),
\]

we have \( P^*\upsilon = \upsilon \). Let \( \mu : C(\bar{J}) \to \mathbb{R} \) be defined by \( \langle \mu, f \rangle = \langle \upsilon, fh^{-1} \rangle \). Then \( \langle \mu, 1 \rangle = \langle \upsilon, h^{-1} \rangle = 1 \) and \( \mu \) is a probability measure. It is easy to see that \( T^*\mu = \varphi\mu \) and \( \langle \mu, h \rangle = \langle \upsilon, 1 \rangle = 1 \). Hence for any \( f \in C(\bar{J}) \), \( \varphi^{-n}T^n f \) converges to \( \langle \mu, f \rangle h \) in the supremum norm. Also it follows that for every \( \xi \in M(\bar{J}), \varphi^{-n}T^n\xi \) converges weakly to \( \langle \xi, h \rangle \mu \).

By Proposition 3.3, we conclude that the eigen-function \( h \) is unique. For the uniqueness of the eigen-measure, we note that if \( \sigma \in M(\bar{J}) \) satisfies \( T^*\sigma = \varphi\sigma \) and \( \langle \sigma, h \rangle = 1 \), then for every \( f \in C(\bar{J}) \),

\[
\langle \sigma, f \rangle = \lim_{n \to \infty} \langle \varphi^{-n}T^n\sigma, f \rangle = \lim_{n \to \infty} \langle \sigma, \varphi^{-n}T^n f \rangle = \langle \sigma, \langle \mu, f \rangle h \rangle = \langle \sigma, f \rangle.
\]

This implies that \( \sigma = \mu. \)

\[\square\]

4. SEPARATION PROPERTIES

For any \( i \in I^* \), let

\[
J_i = s_i(J), \quad r_i = \inf_{x \in U_0} |s_i(x)|, \quad R_i = \sup_{x \in U_0} |s_i(x)|.
\]
Lemma 4.1. Let $X$ and $\{s_j\}_{j=1}^\infty$ be defined as above. Then
(i) there exists $c_1 > 1$ such that
\begin{equation}
R_j \leq c_1 r_j, \quad \forall j \in I^*; \tag{4.1}
\end{equation}
\begin{equation}
c_1^{-1} r_i r_j \leq r_{ij} \leq c_1 r_i r_j, \quad \forall i, j \in I^*; \tag{4.2}
\end{equation}
(ii) there exist $c_2 \geq c_1$ and $\delta > 0$ such that for any $x, y \in X$ with $|x - y| \leq \delta$,
\begin{equation}
c_2^{-1} r_j \leq \frac{|s_1(x) - s_1(y)|}{|x - y|} \leq c_2 r_j, \quad \forall j \in I^*; \tag{4.3}
\end{equation}
(iii) there exist $c_3 \geq c_2$ and $k_0$ such that for any $x, y \in X$,
\begin{equation}
|s_j(x) - s_j(y)| \leq c_3 r_j |x - y|, \quad \forall j \in I^*, |j| > k_0. \tag{4.4}
\end{equation}

Proof. (i) For each $j \in I$, let $p_j(\cdot) = |s_j'(\cdot)|$ and still denote (3.3) by $\Phi(t)$. For any $x, y \in U_0$, $j \in I^*$, $|j| = n$, we have
\begin{align*}
\log |s_j'(y)| - \log |s_j'(x)| & \leq \sum_{i=1}^n \log |s_j'(y_{i+1})| - \log |s_j'(x_{i+1})| \\
& \leq \sum_{i=1}^n \alpha(s^{n-i}) \leq \Phi(1) < \infty.
\end{align*}

where $y_i = s_{j_i} \cdots s_{j_n}(y), y_{n+1} = y, y \in U_0$. Consequently, we deduce (4.1). And the chain rule yields (4.2).

(ii) For any $x \in X$, there exist $\delta_x > 0$ such that $B(x, \delta_x) \subset U_0$. Since $X$ is locally connected, we can assume that $B(x, \delta_x) \cap X$ is connected. Let $\delta$ be the Lebesgue number. Then for any $x, y \in X$, if $|x - y| \leq \delta$, there exists $x' \in X$ such that $x, y \in B(x', \delta_x')$. For such $x$ and $y$, we have $s_j(x), s_j(y) \in B(y', \delta_x')$ for some $y' \in X$. The self-similar property of $s_j$ implies that
\begin{equation*}
|s_j(x) - s_j(y)| \leq R_j |x - y| \leq c_1 r_j |x - y|.
\end{equation*}

On the other hand, let $u_j(x) := s_j^{-1}(x), \forall x \in B(y', \delta_x') \cap s_j(B(x', \delta_x'))$. Then
\begin{equation*}
R_j^{-1} \leq |u_j'(x)| \leq r_j^{-1}, \quad \forall x \in B(y', \delta_x') \cap s_j(B(x', \delta_x')).
\end{equation*}
Since $B(y', \delta_x') \cap s_j(B(x', \delta_x'))$ is convex connected, similarly, by the mean value theorem, we have
\begin{equation*}
|u_j(s_j(x)) - u_j(s_j(y))| \leq r_j^{-1} |s_j(x) - s_j(y)|.
\end{equation*}
Consequently, we have
\begin{equation*}
r_j |x - y| \leq |s_j(x) - s_j(y)| \leq c_1 r_j |x - y|.
\end{equation*}
(iii) By the contractiveness of the infinite conformal IFS, we can select integer $k_0$ such that
\[ |s_j(x) - s_j(y)| \leq \delta, \quad \forall |j| > k_0. \]
The choice of the $\delta$ (the Lebesgue number) and the self-similar property of $s_j$ implies that
\[ |s_j(x) - s_j(y)| \leq R_j |x - y| \leq c_3 r_j |x - y|. \]
\[ \square \]

Let $\kappa$ be as given by (2.4). We take $0 < \varepsilon < c_2^{-1} \cdot \min\{\kappa, \delta\}$. By the assumption on $X$, we have
\[ (4.5) \quad B(J, c_2 \varepsilon) \subset X. \]
For $j \in I^*$, let $G_j = s_j(B(J, \varepsilon))$. By (4.3) and (4.5), we have for any $x \in J$,
\[ (4.6) \quad B(s_j(x), c_2^{-1} \varepsilon r_j) \subset s_j(B(x, \varepsilon)) \subset B(s_j(x), c_2 \varepsilon r_j). \]
It follows that
\[ (4.7) \quad B(J, c_2^{-1} \varepsilon r_j) = \bigcup_{x \in J} B(s_j(x), c_2^{-1} \varepsilon r_j) \subset G_j = s_j(\bigcup_{x \in J} B(x, \varepsilon)) \subset B(s_j(x), c_2 \varepsilon r_j) = B(J, c_2 \varepsilon r_j). \]

For any two compact subsets $E, F$ of $\mathbb{R}^d$, we define
\[ |E| = \sup\{|x - y| : x, y \in E\}; \]
\[ D(E, F) = \inf\{|x - y| : x \in E, y \in F\}; \]
\[ d(E, F) = \inf\{\varepsilon : E \subset B(F, \varepsilon), F \subset B(E, \varepsilon)\}. \]
We say $u, v \in I^*$ are comparable if there exists $w \in I^*$ such that $u = vw$ or $v = uw$.

Denote $F_n = \{1, 2, \ldots, n\}$. Let $F^*_n = \bigcup_{k \geq 1} F^*_n$, and let
\[ F^*_n = \{u = u_1 u_2 \cdots u_k : u_i \in F_n, 1 \leq i \leq k\}. \]
For any $n \in I$ and $0 < r \leq 1$, we let
\[ Q^n(r) = \{v = v_1 v_2 \cdots v_m \in F^*_n : s_v < r \leq s_{v_1 v_2 \cdots v_{m-1}}\}. \]
Let $k_0$ be as given by Lemma 41(iii). For any $w$ with $|w| = k_0 + 1$, we define
\[ I^n(w) = \{v \in Q^n(|G_w|) : J_v \bigcap G_w \neq \emptyset\}. \]
Suppose $I^n(w)$ is defined, and then for any $1 \leq j \leq n$, we define
\[ I^n(jw) = M \bigcup N, \]
where $M = \{jv : v \in I^n(w)\}$ and
\[ N = \{v \in Q^n(|G_{jw}|) : v_1 \neq j, J_v \bigcap G_{jw} \neq \emptyset\}. \]
Lemma 4.2. For any fixed $n$, there exist constants $k_n$ and $c_4^{(n)} > 0$ such that

$$\frac{1}{c_4^{(n)}} \leq \frac{r_u}{r_v} \leq c_4^{(n)}, \quad \forall u \in I^n(v), \ |v| \geq k_n.$$ 

Proof. Let $k_0$ be as given in Lemma 4.1(iii). For any integer $n$, we select integer $k_n \geq k_0$ such that

$$\min \{|u| : u \in I^n(v) \text{ and } |v| \geq k_n\} > k_0.$$ 

We consider the following two cases:

(1) If $v_1 \neq u_1$, by the construction of $N$, we have $u \in Q^n(|G_v|)$. Then

$$r_u < |G_v| \leq r_{u_1u_2\cdots u_{n-1}} \leq c_1 (r^{(n)})^{-1} r_u,$$

where $r^{(n)} = \min_{1 \leq j \leq n} \{r_j\}$. It follows from (4.3) that

$$c_2^{-1} \varepsilon r_v \leq |G_v|.$$ 

Hence

$$c_2^{-1} \varepsilon r_v \leq |G_v| \leq c_1 (r^{(n)})^{-1} r_u.$$ 

Note that

$$r_u < |G_v| \leq c_2 \varepsilon r_v + |J_v| \leq c_3 (2 \varepsilon + |J|) r_v.$$ 

From the arguments above, we conclude that there exists $a > 0$ such that

$$\frac{1}{a} \leq \frac{r_u}{r_v} \leq a.$$ 

(2) If $v_1 = u_1$, we write

$$v = v_1 v_2 \cdots v_l v_{l+1} \cdots v_n := v_1 v_2 \cdots v_l v',$$

$$u = v_1 v_2 \cdots v_l u_{l+1} \cdots u_n := v_1 v_2 \cdots v_l u'.$$

where $v_{l+1} \neq u_{l+1}$. From the construction of $M$, it follows that $u' \in I^n(v')$. Similarly to the case of (1), we can deduce that

$$a^{-1} \leq \frac{r'_u}{r'_v} \leq a.$$ 

Together with (4.2), this implies that

$$(ac_1^2)^{-1} \leq \frac{r_u}{r_v} \leq ac_1^2.$$ 

Let $c_4^{(n)} = ac_1^2$. Then the result follows. \hfill \Box

Proposition 4.3. Let $\{s_j\}_{j=1}^{\infty}$ be an infinite conformal IFS.

(i) The system $\{s_j\}_{j=1}^{\infty}$ satisfies the finite strong condition if it satisfies the open set condition;

(ii) $\sum_{i=1}^{\infty} R_i^d \leq c_4^d$ if the system $\{s_j\}_{j=1}^{\infty}$ satisfies finite open set condition.
Proof. (i) For any $n$, from the assumption that the system $\{s_j\}_{j=1}^n$ satisfies open set condition, the result of Schief [S] implies that the system $\{s_j\}_{j=1}^n$ satisfies finite strong open set condition. Hence there exists a nonempty bounded open set $U_n$ such that $s_i(U_n) \subset U_n, \forall i \leq n$, and
\[
s_i(U_n) \cap s_j(U_n) = \emptyset, \quad \forall i, j \leq n, i \neq j.
\]
Moreover $U_n \cap J_{F_n} \neq \emptyset$, where $J_{F_n}$ is the invariant set of the system $\{s_i\}_{i=1}^n$. It is easy to see that $J_{F_n} \subset J$. Hence $U_n \cap J \neq \emptyset$.

(ii) Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}^d$. For any $n$, by definition of the finite open set condition, there exists a nonempty bounded open set $U_n$ such that $s_i(U_n) \subset U_n, \forall i \leq n$, and
\[
s_i(U_n) \cap s_j(U_n) = \emptyset, \quad \forall i, j \leq n, i \neq j.
\]
It follows that
\[
\lambda(U_n) \geq \sum_{i=1}^{n} \lambda(s_i(U_n)) = \sum_{i=1}^{n} \int_{U_n} |s_i'(x)|^d \lambda(x) \geq c_1^{-d} \sum_{i=1}^{n} R_i^d \lambda(U_n).
\]
Since $\lambda(U_n) > 0$, we have
\[
\sum_{i=1}^{n} R_i^d \leq c_1^d.
\]
Hence $\sum_{i=1}^{\infty} R_i^d \leq c_1^d$. \hfill \Box

For any $t \geq 0$, let $\psi(t) = \sum_{i \in I} R_i^d$, and let $\theta = \inf\{t : \psi(t) < \infty\}$. For any $s > \theta$, we define the Ruelle operator $T_s : C(\overline{J}) \to C(\overline{J})$ by
\[
T_s f(x) = \sum_{j=1}^{\infty} |s_j'(x)|^s \cdot f(s_j(x)).
\]
Let $\rho(T_s)$ denote the spectral radius of $T_s$.

Proposition 4.4. Assume $\psi(\theta) = \lim_{t \to 0} \psi(t) > 1$ and $\psi(\infty) = \lim_{t \to \infty} \psi(t) < 1$, then there exists a unique $a > \theta$ such that $\rho(T_a) = 1$.

Proof. We claim that the function
\[
\rho(T_a) := \lim_{n \to \infty} \|T_a^n 1\|^{\frac{1}{n}}
\]
is continuous and strictly decreasing on $(\theta, +\infty)$. Indeed, it is easy to see that
\[
T_a^n 1(x) = \sum_{|j|=n} |s_j'(x)|, \quad \forall x \in \overline{J}.
\]
This implies that
\[
\|T_a^n 1\| = \sup_{x \in \overline{J}} \sum_{|j|=n} |s_j'(x)|^s.
\]
By Lemma 4.1, we have
\[
\sum_{|j|=n} r_j^s \leq \sum_{|j|=n} |s_j'(x)|^s \leq \sum_{|j|=n} R_j^s \leq c_1 \sum_{|j|=n} r_j^s.
\]
It follows that
\[
\lim_{n \to \infty} \|T_n^1\|^\frac{s}{n} = \lim_{n \to \infty} \left( \sum_{|j|=n} r_j^s \right)^{\frac{1}{n}}.
\]
Let
\[
f_n(s) := \frac{1}{n} \log \sum_{|j|=n} r_j^s.
\]
For any \(0 \leq \lambda \leq 1\), \(s_1, s_2 \in (\theta, +\infty)\), by the Hölder’s inequality, we have
\[
f_n(\lambda s_1 + (1 - \lambda)s_2) = \frac{1}{n} \log \left( \sum_{|j|=n} r_j^{\lambda s_1 + (1 - \lambda)s_2} \right)
\leq \frac{1}{n} \log \left( \sum_{|j|=n} r_j^{\lambda s_1} r_j^{(1 - \lambda)s_2} \right)
\leq \frac{1}{n} \log \left( \sum_{|j|=n} r_j^{s_1} \right)^\lambda \left( \sum_{|j|=n} r_j^{s_2} \right)^{(1 - \lambda)}
\leq \lambda f_n(s_1) + (1 - \lambda)f_n(s_2).
\]
We conclude that \(f_n(\cdot)\) is convex. This implies that \(\lim_{n \to \infty} \frac{1}{n} \log \sum_{|j|=n} r_j^s\) is convex. Since the convex function on \((\theta, +\infty)\) is continuous, we confirm that \(p(T_s)\) is continuous.

If \(s \in (\theta, +\infty)\), then there exists \(i_0 \in I\) such that \(r_{i_0}^s \geq r_j^s\) for every \(j \in I\). We have
\[
\sum_{|j|=n} r_j^{s+t} \leq \sum_{|j|=n} r_j^s r_j^t \leq \sum_{|j|=n} r_j^s c_1 r_{i_0}^{nt},
\]
for \(t > 0\). We can check that \(p(T_s)\) is strictly decreasing by using (4.8). Combining the assumption, we see that there exists a unique \(a > \theta\) such that \(p(T_a) = 1\). \(\square\)

In the following we always let \(a\) be the constant such that \(p(T_a) = 1\). From [MU] we know that \(a\) is the Hausdorff dimension of \(J\).

**Example 4.5.** Let \(X = \left[\frac{1}{4}, \frac{1}{2}\right]\) and let \(\{s_j\}_{j=1}^\infty = \{s_j : X \to X : j \geq 1\}\) be an infinite conformal IFS consisting of similarities \(s_j(x) = 2^{-2j}x + 2^{-j} - 2^{-2j}\). Thus \(|s_j'(\cdot)| = 2^{-2j}\) and the system satisfies the OSC. It is easy to see that \(J\) is compact and \(a = \frac{1}{2}\). So from Theorem [1.1] we know that there exist \(h \in C(J)\) and \(\mu \in M(J)\) such that
\[
Th = h, \quad T^*\mu = \mu, \quad \mu, h > 1.
\]
Proposition 4.6. $H^a(J) \leq H^a(\overline{J}) < \infty$.

Proof. For any fixed $z \in \overline{J}$, by (4.4), there exist $k_0$ and $|j| > k_0$ such that $|J_j| \leq c|s'_j(z)|$, so we have
\[
\sum_{j=n} |J_j|^a \leq c^a \sum_{j=n} |s'_j(z)|^a.
\]
From Theorem 1.1, we know that there exists a unique $h(z) \in C(\overline{J})$ such that
\[
\lim_{n \to \infty} \sum_{j=n} |s'_j(z)|^a = \lim_{n \to \infty} T^a_1 = h(z).
\]
This implies $H^a(J) \leq H^a(\overline{J}) < \infty$. \(\square\)

Lemma 4.7. [FL] Let $a > 0$ be a constant. Let $w$ be conformal and invertible and $D$ be a Borel subset in the domain of $w$ and $0 < H^a(D) < \infty$. Then we have the following change of variable formula:
\[
H^a(w(D)) = \int_D |w'(x)|^a dH^a(x).
\]

Lemma 4.8. Suppose $0 < H^a(J) < \infty$, then
\[
H^a(J \cup J_v) = 0 \quad \text{for any incomparable } \; u, v \in I^*.
\]

Proof. Note that $\{|s'_j(\cdot)|\}_{j=1}^\infty$ is uniformly Dini continuous. Since $T_a$ has spectral radius 1, by Theorem 1.1 there exists $0 < h \in C(\overline{J})$ such that
\[
h(x) = \sum_{j=1}^\infty |s'_j(x)|^a h(s_j(x)), \quad \forall x \in \overline{J}.
\]
Let $\tilde{h} := h|_J$. Then $0 < \tilde{h} \in C(J)$ and $\tilde{h}(x) = \sum_{j=1}^\infty |s'_j(x)|^a \tilde{h}(s_j(x))$. Hence
\[
\sum_{j=1}^\infty \int_{J_j} \tilde{h}(x)dH^a(x) \geq \int_J \tilde{h}(x)dH^a(x)
\]
\[
= \int_J \sum_{j=1}^\infty |s'_j(x)|^a \tilde{h}(s_j(x))dH^a(x)
\]
\[
= \sum_{j=1}^\infty \int_{J_j} \tilde{h}(x)dH^a(x).
\]
The last equality follows Lemma 4.7. This implies that \( H^a(J_i \cap J_j) = 0 \) for any \( i \neq j \). It follows immediately that \( H^a(J_u \cap J_v) = 0 \) for any incomparable \( u, v \in I^* \).

**Proof of Theorem 1.2:** Let \( c_1, c_2, c_3 \) and \( \delta \) be as given in Lemma 4.1. And let \( \iota > 0 \) satisfy the condition: \( 2^{-1}c_1^{-\alpha} > \iota \). There exists an open covering \( V_1, \ldots, V_n \) of \( \mathcal{J} \) such that

\[
(4.9) \quad V := \bigcup_{i=1}^n V_i \supset \mathcal{J}, \quad \delta' := D(\mathcal{J}, V^c) < \delta
\]

and

\[
H^a(V) \leq \sum_{i=1}^n |V_i|^a \leq (1 + \iota)H^a(\mathcal{J}) \leq (1 + \iota)H^a(J).
\]

For any given \( u, v \in I^n(w) \), we let \( \mathcal{J}_u = s_u(J) \) and \( \mathcal{J}_v = s_v(J) \). We assume that \( H^a(J_u) \leq H^a(J_v) \). Then for any given \( \varepsilon > 0 \) satisfying \( c_1^2 \iota < \varepsilon < 1 \), we have \( \varepsilon H^a(J_u) \leq H^a(J_v) \). We claim that \( d(\mathcal{J}_u, \mathcal{J}_v) \geq c_2^{-1}\delta' r_u \). Otherwise \( d(\mathcal{J}_u, \mathcal{J}_v) < c_2^{-1}\delta' r_u \), by (4.3) and (4.9), we have

\[
D(\mathcal{J}_u, s_u(V)^c) \geq c_2^{-1}\delta' r_u.
\]

This implies that \( J_v \subset \mathcal{J}_v \subset s_u(V) \). By Lemma 4.8 we have

\[
(1 + \varepsilon)H^a(J_u) < H^a(J_u) + H^a(J_v) = H^a(J_u \cup J_v) \leq H^a(s_u(V)).
\]

Thus

\[
\varepsilon r_u H^a(J) \leq \varepsilon H^a(J_u) < H^a(s_u(V \setminus J)) \\
\leq (c_1 r_u)^a H^a(V \setminus J) < (c_1 r_u)^a \iota H^a(J).
\]

This implies that \( \varepsilon < c_1^2 \iota \), and it contradicts to the choice of \( \varepsilon \). Hence, there exists \( \delta_0 > 0 \) such that for any \( w \in I^* \),

\[
(4.10) \quad d(\mathcal{J}_u, \mathcal{J}_v) \geq \delta_0 r_w, \quad \forall \ u, v \in I^n(w), u \neq v.
\]

Let \( \delta_1 = (3c_3c_4^{(n)})^{-1}\delta_0 \). We can find a finite set \( Z \subset \mathcal{J} \) such that

\[
\mathcal{J} \subset \bigcup_{x \in Z} B(x, \delta_1).
\]

For any \( w \in I^* \) with \( |w| > k_n \), by (4.10) there exists \( x \in \mathcal{J} \) such that

\[
|s_u(x) - s_v(x)| \geq \delta_0 r_w, \quad \forall \ u, v \in I^n(w), u \neq v.
\]

For such \( x \) there exists a \( z \) such that \( |x - z| < \delta_1 \). By (4.4) and the choice of \( k_n \) in Lemma 4.2 we have

\[
|s_u(x) - s_u(z)| \leq \frac{1}{3}\delta_0 r_w, \quad |s_v(x) - s_v(z)| \leq \frac{1}{3}\delta_0 r_w.
\]

This implies that

\[
(4.11) \quad |s_u(z) - s_v(z)| \geq \delta_0 r_w/3.
\]
For each $z \in Z$, let
\[ T^n_z = \{ u \in I^n(w) : \exists v \in I^n(w) \text{ such that (4.11) holds} \}. \]
Together with (4.11), it implies that
\[ I^n(w) = \bigcup_{z \in Z} T^n_z. \]

For each $z$, the sets
\[ \{ B(s_u(z), \delta_0 r_w/6) : u \in T^n_z \} \]
are disjoint and contained in $B(G_w, |J_u| + \delta_0 r_w/6)$. By (4.4) and Lemma 4.2, there exist $x \in J$ and $c > 0$ such that
\[ B(G_w, |J_u| + \delta_0 r_w/6) \subset B(x, cr_w). \]
We deduce that there exists $k$ such that $\max_{z \in Z} \sharp T^n_z \leq k$, then we have
\[ \sharp I^n(w) \leq \sharp Z \cdot \max_{z \in Z} \sharp T^n_z \leq \sharp Z \cdot k. \]

If $w \in I^*$ with $|w| > k_n$ and
\[ \sharp I^n(w) = \sup_{|w| \geq k_n} \sharp I^n(w), \]
we prove that
\[ (4.12) \quad I^n(iw) = \{ ij : j \in I^n(w) \} \quad \forall i \in F^*_n. \]

By the maximality of $w$, we only prove the following:
\[ \{ il : l \in I^n(w) \} \subset I^n(iw). \]
The definition of $I^n(w)$ implies that
\[ I^n(jw) \ni \{ jv : v \in I^n(w) \}, \quad j = 1, 2, \ldots, n. \]
We conclude that
\[ \{ il : l \in I^n(w) \} \subset I^n(iw). \]

For any fixed $1 \leq l \leq n$, $v = v_1 \cdots v_n \in I^*$, $v_1 \neq l$, we consider the family
\[ \mathfrak{J}_l = \{ J_1 : l \in Q^n(|v|w), l_1 = l \}. \]
Then $\mathfrak{J}_l$ is a cover of $J_l$. Since $v_1 \neq l$, then $l \notin I^n(vw)$. By the construction of $N$, we have $J_l \cap G_{vw} = \emptyset$. Hence by (4.7) we have
\[ D(J_l, J_{vw}) \geq c_2^{-1} \varepsilon r_{vw}, \]
which implies
\[ (4.13) \quad D(J_l, J_{vw}) \geq c_2^{-1} \varepsilon r_{vw}, \quad l \neq v_1. \]
Now we let \( U_n = \bigcup_{u \in F_n^*} G_{uw}^* \), where \( G_{uw}^* = s_u(B(J, 2^{-1}c_2^{-2}\varepsilon)) \). We claim that the set \( U_n \) satisfies the condition of finite SOSC. Indeed, \( U_n \) is an open set and for each \( i \in F_n \),

\[
s_i(U_n) = \bigcup_{u \in F_n^*} s_i(G_{uw}^*) = \bigcup_{u \in F_n^*} G_{uw}^* \subset U_n.
\]

We claim that for each \( i, j \in F_n, i \neq j \), \( s_i(U_n) \cap s_j(U_n) = \emptyset \). Otherwise, there exist \( u, v \in F_n^* \) such that \( G_{iuw}^* \cap G_{jvw}^* \neq \emptyset \). We assume that

\[
r_{iuw} \geq r_{jvw}.
\]

Let \( y \) be in the intersection, then there exist \( y_1 \in J_{iuw} \) and \( y_2 \in J_{jvw} \) such that

\[
d(y, y_1) < c_2 \cdot \frac{1}{2c_2^2} \varepsilon \cdot r_{iuw} \leq \frac{c_2^{-1} \varepsilon}{2} r_{iuw},
\]

\[
d(y, y_2) < c_2 \cdot \frac{1}{2c_2^2} \varepsilon \cdot r_{jvw} \leq \frac{c_2^{-1} \varepsilon}{2} r_{jvw}.
\]

Hence

\[
D(J_{iuw}, J_j) < c_2^{-1} \varepsilon r_{iuw},
\]

which contradicts to (4.13). And it is easy to see that \( U_n \cap J \neq \emptyset \). This completes the proof. \( \square \)

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