1 Introduction

The librationist system now named \( \mathcal{L} \) (libra) is detailed some in (Bjørdal, 2011) and (Bjørdal, 2012), but we go beyond those accounts here and inter alia focus more incisively upon aspects related to how it gives rise to new perspectives concerning inferential principles, and the nature of connectives. These are matters which are central to our understanding of thinking and rationality, and we will in the end e.g. point out that unlike other foundational systems that are on offer librationism is in a very precise sense a disconnectionist point of view as \( \mathcal{L} \) has theses which are disconnected. \( \mathcal{L} \) is not a non-classical system as it extends classical logic soberly. Unlike paraconsistent systems engendered by adding naive comprehension to paraconsistent logics suggested in the literature for dealing with the paradoxes, \( \mathcal{L} \) is neither inconsistent nor contradictory. As divulged in Bjørdal (2013) we show how \( \mathcal{L} \) is able to soberly interpret \( \mathbf{ZF} \). As divulged in Bjørdal (2014) we show that \( \mathcal{L} \) also makes it possible to isolate the definable real numbers and so prepare a space for definable analysis. In section 21 we show that \( \mathcal{L} \) has an interpretation of \( \mathbf{ZFC} \) given what we name the Skolem-Fraenkel Postulation (Postulation 21.20).

We use the term thesis for a formula which is librationistically valid, and theorem is used as expected for results concerning \( \mathcal{L} \). Unlike in (Bjørdal, 2012) and in earlier superseded accounts of the librationist approach we use prescription where one would expect axiom schema and regulation where one would expect inference rule; moreover, prescript is used where one would expect (proper) axiom and regula should be used if one wants to denote a specific instance of a regulation. A prescribe is a fundamental thesis which cannot be universally generalized. Just prescribes, prescripts and prescriptions are posits. The terms postulate and postulation are for statements that one thinks should be provable as theorems though for reasons connected with Gödel’s second incompleteness theorem only by means much stronger than those used to isolate the notion of being a theorem of \( \mathcal{L} \).

This and further terminology is introduced in order to make distinctions called for by new concepts, and also to underline the fact that \( \mathcal{L} \) is not an axiomatic or formal system in the common sense. \( \mathcal{L} \) is super-formal, or semi-formal according to what the author takes as unfortunate contemporary terminology. We suggest to also consider \( \mathcal{L} \) a contentual system as it is categorical and so rather more unambiguous with respect to content than merely formal systems. Nevertheless, one may as much as \( \omega \)-logic is partly denoted by the term “logic” also understand \( \mathcal{L} \) as a logic, especially as Frege in the opening phrases of Der Gedanke nails down logic’s subject matter to be truth itself. For truth is indeed a most central theme of \( \mathcal{L} \), and it should be compared with axiomatic theories of truth as overviewed in Halbach and Leigh (2014); \( \mathcal{L} \) much extends such formal relatives, and provides iself its sentences with names.

Our considerations concerning librationism are inherently semantical, and soundness and completeness considerations are irrelevant. As we focus upon one intended model considerations concerning compactness are extraneous. The posits we isolate are only examples of formula schemas which hold librationistically and the regulations we display are likewise just examples of principles which tell us when thesishood is regulated from
that of others. Nevertheless, the prescriptions and precepts and regulations we do
isolate are quite informative and comprehensive, and they moreover provide liberation
justifications for important axiomatic systems.

2 Fundamentals of £

As in (Bjørnland, 2012) we continue to consider £ a theory of sorts and thence also a
t and properties. However, in the final analysis the language of £ may be taken as
just that of the ordinary language of set theory without identity including set brackets
plus two special extra sort constants for truth and enumeration. We take sorts or
sets that are designated by terms (pronomina) that are formed (in a sense made more
precise below) without using the alethizor or the enumerator to be sets. Hence £ is
also an alternative theory of sets, and it contains paradoxical sets as e.g. Russell’s set
\( \{x | x \notin x \} \). So pace Gödel and others this author now thinks that there always
will be and always were set theoretical paradoxes.

The motivation for the following pedantic policies pertaining to the formal language
of £ is stated in Remark 3.3 of Section 3, and these policies enable our statement of
semantical principle P5 in Section 6 and the articulation of the truth prescription in
Section 7.

The austere Polish language \( \mathcal{L}(£) \) of £ is its alphabet \( \mathcal{A}(£) \) plus its formation rules
\( \mathcal{F}(£) \). \( \mathcal{A}(£) \) is the two signs . (dot) and | (bar). \( \mathcal{F}(£) \) is \( \mathcal{F}S(0) - \mathcal{F}S(3), \mathcal{F}F(0) - \mathcal{F}F(9),
\mathcal{F}E(0) - \mathcal{F}E(9) \) plus \( \mathcal{F}V(0) - \mathcal{F}V(4) \) as in the following. Minuscule letters from the beginning
of the Latin alphabet range over terms and their capital counterparts range over formulas.
We let \( u, v, w, x, y, z, u' \) and so on stand for arbitrary noemata (see the formation
rules right below). We take \( m \) and \( n \) to range over natural numbers. For any numeral
\( n \) we let \( \underline{n} \) be bar followed by \( n \) dots so that \( \underline{0} \) is | and \( \underline{n+1} \) is \( \underline{n} \); when just the latter
convention is used in presenting we say that the presentation is in bare form, and when
the convention is suppressed so that only bars and dots are used we say that it is in
austere form.

\[
\begin{align*}
\mathcal{F}S(0) : & \quad \underline{0} \text{ is the sortifier.} \\
\mathcal{F}S(1) : & \quad \underline{1} \text{ is the universalizor.} \\
\mathcal{F}S(2) : & \quad \underline{2} \text{ is the norifyer.} \\
\mathcal{F}S(3) : & \quad \text{Just the sortifier, the universalizor and the norifyer are syncategoremata.} \\
\mathcal{F}F(0) : & \quad \underline{3} \text{ is the alethizor.} \\
\mathcal{F}F(1) : & \quad \underline{4} \text{ is the enumerator.} \\
\mathcal{F}F(2) : & \quad \underline{5} \text{ is a noema.} \\
\mathcal{F}F(3) : & \quad \text{If } \underline{n} \text{ is a noema then } \underline{n+1} \text{ is a noema.} \\
\mathcal{F}F(4) : & \quad \text{Nothing else is a noema.} \\
\mathcal{F}F(5) : & \quad \text{Just the alethizor, the enumerator and noemata are praeomina.} \\
\mathcal{F}F(6) : & \quad \text{Just syncategoremata and praeomina are symbols.} \\
\mathcal{F}F(7) : & \quad \text{If } p \text{ is a symbol then } p \text{ is a formation.} \\
\mathcal{F}F(8) : & \quad \text{If } p \text{ and } q \text{ are formations then } pq \text{ is a formation.}
\end{align*}
\]
FE(9): Nothing else is a formation.
FE(0): Praenomina are terms.
FE(1): If $a$ and $b$ are terms then $ba$ is a formula.
FE(2): If $A$ and $B$ are formulas then $\lfloor AB \rfloor$ is a formula.
FE(3): If $a$ and $b$ are terms then $\lfloor ba \rfloor$ is a term.
FE(4): If $A$ is a formula and $y$ is a noema then $\lfloor yA \rfloor$ is a formula.
FE(5): If $A$ is a formula and $y$ is a noema then $\lceil yA \rceil$ is a term.
FE(6): Nothing else is a term or a formula.
FE(7): Just terms and formulas are expressions.
FE(8): Just formulas are sentences.
FE(9): Just terms are nomina and sort constants.

FV(0): If $A$ is a formula and $y$ is a noema then all occurrences of $y$ in $\lfloor yA \rfloor$ are variables.
FV(1): If $A$ is a formula and $y$ is a noema then all occurrences of $y$ in $\lceil yA \rceil$ are variables.
FV(2): The first occurrence of $y$ in $\lfloor yA \rfloor$, or $\lceil yA \rceil$, is the binding variable.
FV(3): All occurrences of $y$ in $\lfloor yA \rfloor$ and in $\lceil yA \rceil$ are bound variables.
FV(4): Nothing else is a variable.

The semantical principles of $\mathcal{L}$ in the final analysis compel us to the unusual policies embodied by FE(8) and FE(9) (cfr. section 11 and 12 for the deeper motivation). The resulting nominist view supported by $\mathcal{L}$ is accounted for in more detail in section 12, and the related and necessitated nominist turn is explained in section 13 as a consequence has no free variables. We have instead opted for using the expressions noema (singular) and noemata (plural) where one would expect free variable. This is inter alia justified by the fact that one meaning of the word noema as listed in the Oxford English Dictionary is: A figure of speech whereby something stated obscurely is nevertheless intended to be understood or worked out. Also, the Greek letter $\nu$ in the original Greek word $\nuonema$ typographically very much resembles lower case $v$. We also use variants of Latin nomen for terms.

It may be that our knowledge of proto Indo European does not entirely justify us in thinking that there is an etymological relatedness between Greek $\nuonema$ (noema) and Greek $\nuonoma$ (onoma), but the terms are surely conceptually related enough e.g. in as far as we often come to know things by their names. Quite possibly, ancient Greek philosophers did not make a connection between these notions. However these matters may turn out, in $\mathcal{L}$ we take noemata to also be nomina.

We follow the convention that the expression “$X \equiv Y$” stands for the idea that $X$ is defined as $Y$, and “$\equiv$” is accordingly pronounced as “is defined as” or its likes.

Let us define:

D0: $\hat{y}A \equiv \lfloor yA \rfloor$
D1: $\forall yA \equiv \lfloor yA \rfloor$
D2: $\downarrow AB \equiv \lfloor AB \rfloor$
D3: $\downarrow ab \equiv \lfloor ab \rfloor$

6
With these notions we use Hebrew Н (Nun) in our definition of the set of noemata of expressions, where we as also in what follows further on use square brackets for sets as used in the metalanguage and η for the element relation between such sets:

**Definition 2.1**

\[ |
(\text{u}) \triangleq [\text{u}] \\
(\text{T}) \triangleq [] \\
(\text{E}) \triangleq [] \\
(\text{ba}) \triangleq |(\text{a}) \cup |(\text{b}) \\
(\text{\textae}A) \triangleq |(\text{A}) \setminus [\text{u}] \\
(\forall \text{u}A) \triangleq |(\text{A}) \setminus [\text{u}] \\
(|\text{AB}) \triangleq |(\text{A}) \cup |(\text{B}) \\
(|\text{ab}) \triangleq |(\text{a}) \cup |(\text{b}) \]

A noema \text{u} is present in an expression \text{A} iff \text{u} ∈ |(\text{A}). A formula \text{A} is a proposition iff no noema is present in \text{A}. A formula \text{A} is atomic iff \text{A} is of the form \text{ba} with terms \text{a} and \text{b}. With this terminology, all propositions are sentences, so we do not presuppose that propositions are extralinguistic entities in our framework. In \£ it turns out that all formulas are parivalent (see section 13 below) with a proposition.

A term \text{a} is a cognomen iff no noema is present in \text{a}. Some terms are neither cognomina nor praenomina. A term is a pronomen iff it is a cognomen and neither \text{T} (the alethizor) nor \text{E} (the enumerator) occurs in it. In \£ we take a sort to be a set iff \£ shows that it is identical with a pronomen.

For any term \text{a} and noema \text{u}, \text{(a/\text{u})} is a substitution function from expressions to expressions:

\[
(\text{a/\text{u}})w \triangleq \text{a} \text{ if } w = w, \text{ otherwise } (\text{a/\text{u}})w \triangleq w \\
(\text{a/\text{u}})\text{cb} \triangleq (\text{a/\text{u}})c(\text{a/\text{u}})b \\
(\text{a/\text{u}})\text{\textae}\text{cb} \triangleq (\text{a/\text{u}})c(\text{a/\text{u}})b \\
(\text{a/\text{u}})\text{T} \triangleq \text{T} \\
(\text{a/\text{u}})E \triangleq E \\
(\text{a/\text{u}})\text{\textae}yA \triangleq \text{\textae}y(a/\text{u})A \text{ if } u \neq y, \text{ else } (\text{a/\text{u}})\text{\textae}yA \triangleq \text{\textae}yA \\
(\text{a/\text{u}})(\forall y)A \triangleq (\forall y)(\text{a/\text{u}})A \text{ if } u \neq y, \text{ else } (\text{a/\text{u}})(\forall y)A \triangleq (\forall y)A \\
(\text{a/\text{u}})\text{\textae}AB \triangleq (\text{a/\text{u}})A(\text{a/\text{u}})B
\]

We will make use of a suffix notation and write \text{A(a/\text{u})} for (\text{a/\text{u}})A. Iterated uses of the substitution function like \text{A(a/\text{u})(b/\text{w})(c/\text{y})} should be written \text{A(a/\text{u},b/\text{w},c/\text{y})}.
We define the notion ‘a is substitutable for u in . . . ’ by the recursion: a is substitutable for u in y; a is substitutable for u in cb iff a is substitutable for u in b and in c; a is substitutable for u in \( \downarrow bc \) iff a is substitutable for u in b and in c; a is substitutable for u in \( \downarrow AB \) iff a is substitutable for u in A and in B; a is substitutable for u in \( \check{y} A \) iff either y is not a noema in a or u is not a noema in A and a is substitutable for u in A; a is substitutable for u in \( \forall y A \) iff either y is not a noema in a or u is not a noema in A and a is substitutable for u in A.

In the metalinguistic account we also presuppose the following definitions:

D7: \( \neg A \equiv \downarrow AA \)
D8: \( A \lor B \equiv \downarrow \downarrow AB \downarrow AB \)
D9: \( A \land B \equiv \downarrow \downarrow AA \downarrow BB \)
D10: \( A \rightarrow B \equiv \downarrow \downarrow \downarrow AAB \downarrow \downarrow AAB \)
D11: \( A \leftrightarrow B \equiv (A \rightarrow B) \land (B \rightarrow A) \)
D12: \( \overline{a} \equiv \downarrow aa \)
D13: \( a \cup b \equiv \downarrow \downarrow ab \downarrow ab \)
D14: \( a \cap b \equiv \downarrow \overline{\overline{ab}} \)
D15: \( a \setminus b \equiv a \cap \overline{b} \)
D16: \( \exists y A \equiv \neg \forall y \neg A \)
D17: \( a \in b \equiv ba \)
D18: \( \{y|A\} \equiv \check{y} A \)
D19: \( T A \equiv (\exists y)(y \in \{y|A\}) \), where y is the smallest noema not in \( \mathcal{J}(A) \)

The metalinguistic connectives have the precedence order \( \neg, \land, \lor, \rightarrow \) and then \( \leftrightarrow \). Parentheses are suppressed accordingly, and also associativity taken into account and outer parentheses omitted in order to better the legibility of the metalinguistic presentation. We call the \( \downarrow \) the joint, and the joint is a connective when operating on formulas and a juncture when operating on terms; just junctures and connectives defined in terms of the joint are concourses. As already stated, we use \( u, v, w, x, y, z, u' \) etc. to range over noemata and variables in the metalinguistic account when availing ourselves of D(0)-D(19). When needed we use subscripts such as in \( x_0 \) and \( x_1 \). \( A(x, y, z) \) signifies that \( x, y \) and \( z \) are noemata present in \( A \). We use \( \overline{x} \) in \( A(\overline{x}) \) for \( A(x_0, \ldots, x_{n-1}) \) where \( n \) is unspecified for vectors of arbitrary dimension; the quantified expression \( \forall \overline{x} A \) is to be interpreted accordingly. Sentences and nomina that are presented (more or less) in line with these metalinguistic conventions are said to be in (more or less) presentable form.
3 Formations as Numbers, Mathematicalism and Coding

Our semantic constructions may be thought of as taking place in Gödel’s $L$. We adopt the policy of taking a formation of $L$ to be the natural number (finite von Neumann ordinal) of $L$ denoted by the binary numeral which is gotten by replacing every occurrence of “|” in its austere expression with “1” and every occurrence of “.” in it austere expression with “0”. Alternatively put, if one in the austere expression takes “|” as the numeral “1” and “.” as the numeral “0” one may take the formation as the natural number denoted by it when taken as a binary numeral. The number corresponding to any symbol $e$ is $2^n$. The length of a natural number $n$ in the binary system is $l(n) = \mu_y(2^y > n)$. Define, for $m \cdot n > 0$, $m \cdot n = (m \cdot 2^{l(n)} + n)$. $l(m \cdot n) = l(m) + l(n)$ can be shown to hold by induction, and associativity then follows easily. For formations $e$ and $e'$ of $L$, $ee'$ is taken as $e \sim e'$.

Our approach to formations as natural numbers is one which can be taken in general to formal and super (semi) formal systems. As an upshot the sets of theses of these coding function as $e$ is be, from a mathematicalist point of view as suggested here the consequence relation of than its theses or consequences from the empty set of assumptions. However that may induction, and associativity then follows easily. For formations $e$ and $e'$ of $L$, $ee'$ is taken as $e \sim e'$.

Our approach to formations as natural numbers is one which can be taken in general to formal and super (semi) formal systems. As an upshot the sets of theses of these systems may be understood as real numbers, i.e. sets of natural numbers. This supports a mathematicalist point of view which holds that mathematics is more fundamental than logic. One may want to take the consequence relation of a logic as more fundamental than its theses or consequences from the empty set of assumptions. However that may be, from a mathematicalist point of view as suggested here the consequence relation of any given logic just is a function from real numbers to real numbers.

Let $N$ be the natural numbers taken as the finite von Neumann ordinals of $L$. The coding function $\sim': N \to N$ is given by the following recursion wherein $\triangle$ is taken to abbreviate "|"; the numeral "1" and "." as the binary number system denoting finite von Neumann ordinals of $L$, so that we take expressions of $L$ to be natural numbers of $L$.

Remark 3.1 Recall that we adopt the numerical policy of taking “|” and “.” as the numerals “1” and “0” of the binary number system denoting finite von Neumann ordinals of $L$, so that we take expressions of $L$ to be natural numbers of $L$.

Remark 3.2 $\sim'$ is stated in bare form and $\sim' n + 1$ in bare form modulo $\sim' n$; it is left as an exercise to work these out in more presentable forms given definitions D0–D19 above.

Remark 3.3 Our coding function and numerical policy are so devised that for any formation $f$ of $L$ a natural number is $f$ according to $L$ iff it is $\sim' f$ according to $L$. In light of our policy laid down above concerning expressions of $L$ as numbers of $L$, this means that whenever $ef$ is a formation of $L$, then $\sim' ef$ is $\sim' e \sim' f$; with the numerical policy in mind, the following definition is useful.

Definition 3.4 $\sim' e \sim' f = \sim' e \cdot 2^{l(f)} + f$.
Remark 3.5 The central idea for the base case is to find the smallest number of $N$ of $L$ that denotes 0 according to $L$ without invoking noemata. In more presentable form $^\ast$, corresponds with $\{v_0\vdash\forall v_1(v_0 \in v_1 \rightarrow v_0 \in v_1)\}$, but notice that the bare or austere form is strictly speaking needed to be exact and that the choice of variables is not arbitrary; to the last point, cfr. remark [3.3] just below. If my count is right $^\ast$ is about the size of $2^{111}$. Work the opposite way for the successor case so that the definens of $^\ast n+1$ in more presentable form is $\{v_0|v_0 \in \forall v_1(v_0 \in v_1 \rightarrow \exists n \in v_1)\}$. Obviously, one could presuppose other choices for zero and successor.

Remark 3.6 In the definiens of $^\ast n+1$ and of $^\ast 0$ the Leibnizian-Russellian theory of identity justified in section 4 of [Bjordal, 2012] is presupposed.

Remark 3.7 Notice that $^\ast$ is a different number from $\mathbb{I}_{\mathbb{S}}^\ast \mathbb{I}_{\mathbb{R}}^\ast$ in $L$, but $\mathbb{I}_{\mathbb{M}}^\ast = \mathbb{I}_{\mathbb{S}}^\ast \mathbb{I}_{\mathbb{R}}^\ast$ is a maxim of $L$ given our policy on alphabetological variants (see Section 4 on this notion) as per principle P6 in the semantic recursion in Section 4 below. So the notion of identity of $L$ does not coincide with that of $L$.

4 Substitution

We define a substitution function on triples of natural numbers by a course of value recursion:

$\text{sub}(^\ast v_1, i, y) = y$

$\text{sub}(^\ast v_2, j, y) = ^\ast v_2$ if $i \neq j$

$\text{sub}(^\ast a b, i, y) = \text{sub}(^\ast a, i, y) \rightarrow \text{sub}(^\ast b, i, y)$ for atomic formula $a b$

$\text{sub}(^\ast \forall v_1 A, i, y) = \forall ^\ast v_1 A$

$\text{sub}(^\ast \exists v_1 A, i, y) = \exists ^\ast v_1 A$

$\text{sub}(\forall v_1 A, i, y) = \forall ^\ast v_1 A$ if $i \neq j$

$\text{sub}(\exists v_1 A, i, y) = \exists ^\ast v_1 A$ if $i \neq j$

$\text{sub}(x, i, y) = 0$ if $x$ is not of one of the above forms.

We see that for any term $t$ and formula $A(v_1)$, $\text{sub}(^\ast A(v_1) i, ^\ast t) = ^\ast A(t)$. Compare here and in the following [Smorynski, 1977], p. 837.

We define $\text{Sub}(x, y) = \text{sub}(x, i, y)$ whenever $x$ is the code of a formula with a noema and $i$ is the least number such that noema $v_i$ occurs in the formula $x$ is a code of, and $\text{Sub}(x, y) = x$ if $x$ is not the code of a formula with a noema.

We define $\text{Sub}(x, y)$ and abbreviate $\text{sub}(^\ast A(x), y)$ by $^\ast A(y)$). Iterated uses are as expected, and if the vector’s dimensionality is unspecified we write $^\ast A(\vec{x})$.
5 Alphabetological Variants and their Enumeration

Let $\mathfrak{S}(A \leftrightarrow B)$ hold in $L$ iff $A$ and $B$ are propositions (“closed sentence”) according to classical logic in the language of $L$ which are provably equivalent in classical logic, i.e. they are Lindenbaum-Tarski congruent. Proposition $A$ is an alphabetological variant of proposition $B$ iff $\mathfrak{S}(A \leftrightarrow B)$ or there are cognomina (i.e. terms wherein no noema occurs) $a$ and $b$ which are alphabetological variants of each other and such that $A$ is an alphabetological variant of $B(a/b)$. Cognomina $a$ and $b$ are alphabetological variants of each other iff $a = b = T(|_3)$ or $a = b = \epsilon (= |_4)$, or there are cognomina $c$ and $d$ and $e$ and $f$ such that $a = \downarrow cd$ and $b = \downarrow ef$ and either $c$ is an alphabetological variant of $e$ and $d$ is an alphabetological variant of $f$ or $c$ is an alphabetological variant of $f$ and $d$ is an alphabetological variant of $e$, or there are formulas $A$ and $B$ so that $a = \{x : A(x)\}$ and $b = \{y : B(y)\}$ and $(\exists x) A(x)$ is an alphabetological variant of $(\exists y) B(y)$.

Cognomina of $L$, as other formations of $L$, have a natural order according to their sizes as natural numbers or finite von Neumann ordinals of $L$. We presuppose a Kuratowskian definition of ordered pairs here and use the following notation to distinguish object language and meta language statements involving ordered pairs: $(a, b)$ is an object language ordered pair of $L$ and $\langle a, b \rangle$ is an ordered pair of $L$. We define:

**Definition 5.1** $/\alpha, b \setminus \eta \equiv n \eta N$ and $b$ a cognomen and an alphabetological variant $c$ of $b$ is the smallest cognomen such that for no natural number $m < n$ is $/\alpha, m \setminus c \setminus \eta e$.

6 The Semantics of $L$

In stating the semantical principles below we use almost presentable form. In defining the semi inductive Herzbergerian style semantic process (cfr. Gupta (1982) and descendent literature for the related revisionary approach) we use square brackets for sets assumed at the meta level and $\eta$ is used for elementhood. $\setminus$ is the arithmetical formula predicate. We avail ourselves of the class of all von Neumann ordinals in the transfinite recursion, and let minuscule Greek letters except $\eta$ and including the archaic $\Omega$ (koppa) denote ordinals. We let $m$ and $n$ range over natural numbers. In P5 and P6 we presuppose the Leibnizian-Russellian definition shown adequate in (Bjordal 2012) section 4, so $a = b \equiv \forall u (a \in u \leftrightarrow b \in u)$, $\vdash$ is a function from ordinals to real numbers taken as sets of natural numbers of $L$, and we write $\alpha \vdash A$ for $A \eta = (\alpha)$. We presuppose that formulahood is fulfilled. In P7 $\mathbb{N}$ is $\{x | \forall y (\emptyset \in y \wedge \forall z (z \in y \rightarrow z' \in y) \rightarrow x \in y)\}$ where $\emptyset$ is $\{x | x \neq x\}$ and $z'$ is $\{x | x \in z \vee x = z\}$.

P1: $\alpha \vdash a \in \{u | A\}$ iff $a$ is free for $u$ in $A$ and $\exists \beta (\beta < \alpha \& \forall \gamma (\beta \leq \gamma < \alpha \rightarrow \gamma \vdash A(u/a)))$

P2: $\alpha \vdash A B$ iff neither $\alpha \vdash A$ nor $\alpha \vdash B$

P3: $\alpha \vdash a \in b \wedge c$ iff neither $\alpha \vdash a \in b$ nor $\alpha \vdash a \in c$

P4: $\alpha \vdash y A$ iff for all $a$ substitutable for $y$ in $A$, $\alpha \vdash A(a/y)$

P5: $\alpha \vdash Tu$ iff for some formula $A$, $\alpha \vdash u = 'A' \wedge T A$

P6: Given $\alpha, \eta, \eta e$ and $n \geq 0$: If $\forall m (m < n \Rightarrow \alpha \vdash \eta = v_m \neq a)$ then $\alpha \vdash v_n = a$, else $\alpha \vdash v_n = v_{n-1}$.
\(\alpha \models u \in \mathbb{E}\) iff for some set \(a\) and some natural number \(n\) and some noema with corresponding numeral \(\mathfrak{a}\), \(\alpha \models u = \langle n, a \rangle \) \(\land n \in \mathbb{N} \land n = v_n\).

Define

\[
\begin{align*}
IN(\models) & \triangleq \left[ A : \forall \exists \forall \gamma (\beta \leq \gamma \rightarrow \gamma \models A) \right] \\
OUT(\models) & \triangleq \left[ A : \exists \forall \forall \gamma (\beta \leq \gamma \rightarrow \gamma \not\models A) \right] \\
STAB(\models) & \triangleq \left[ A : \exists \forall \forall \gamma (\beta \leq \gamma \rightarrow \gamma \models A) \right] \cup UNSTAB(\models) \\
\models [\gamma] & \triangleq \{ A : \gamma \models T \}
\end{align*}
\]

Define

\[
\begin{align*}
\text{Limit } \kappa \text{ covers } \models & \triangleq \text{ for every } \gamma \geq \kappa, \text{ } IN(\models) \subset \models [\gamma] \\
\text{and } \models [\gamma] \subset IN(\models) \cup UNSTAB(\models). \\
\text{Limit } \sigma \text{ stabilizes } \models & \triangleq \sigma \text{ covers } \models \text{ and } \models [\sigma] \subset IN(\models).
\end{align*}
\]

**Theorem 6.1 (Herzberger)**

(i) There is an ordinal \(\kappa\) which covers \(\models\).

(ii) There is an ordinal \(\sigma\) which stabilizes \(\models\).

**Proof.** See §2 of Bjørdal (2012), and relatedly also Cantini (1996) and Herzberger (1982) and elsewhere. The semantic set up we provide here in some respects deviates from and simplifies the arguments given earlier. Notice also that we now presuppose an extension of the formal language which also gives us the real complement of sets, so that at the initial ordinal 0 for all terms \(b\) of \(\mathcal{L}\), \(0 \models \forall x (x \notin b)\) or \(0 \models \forall x (x \in b)\).

We let \(\wp\) - archaic Greek Koppa - be the closure ordinal, i.e. the smallest stabilizing ordinal. We have that \(\wp \models T A\) (see D19 on page [S]) iff for all \(\beta \geq \wp\), \(\beta \models A\).

We make the crucial *librationist twist* to isolate the intended model of librationism, and shift our attention to those formulas (sentences) \(A\) which are such that \(\wp \models \neg T \neg A\). So our official definition of what we take as the roadstyle sign is \(\models A \equiv \wp \models \neg T \neg A\).

All ordinals \(\alpha\) are *maximally progression-consistent* in the sense that \(\alpha \models B\) iff \(\alpha \not\models \neg B\), and *progression closed* in the sense that \(\alpha \models A\) and \(\alpha \models A \rightarrow B\) only if \(\alpha \models B\). Suppose \(\models \neg A\), i.e. not \(\models A\). It follows that \(\wp \models T \neg A\) as \(\wp\) is maximally progression-consistent. But \(\wp \models T \neg A \rightarrow \neg T A\) on account of \(\mathcal{L}_M^2\) below. As \(\wp\) is progression closed, it follows that \(\wp \models \neg T A\), i.e. \(\models \neg A\). So \(\models A\) or \(\models \neg A\), i.e. \(\mathcal{L}\) is negjunction (“negation”) complete.

Our definition of the roadstyle supports the following precise definitions of maxims (signified with capital M as subscript) and minors (signified with minuscule m as subscript): \(\models_M A \equiv \wp \models T A\) and \(\models_m A \equiv \models A \& \models \neg A\). As \(\wp\) is maximally progression-consistent we also have that \(\models_M A\) iff both \(\models A\) and \(\models \neg A\). As \(\mathcal{L}\) is negjunction complete, \(\models_M A\) iff \(\models \neg A\). The maxims are the unparadoxical theses while the minors are the paradoxical ones.
7 Prescriptions, Prescrits and Prescribes of \( \mathcal{L} \)

We give a partial list of posits of \( \mathcal{L} \). All prescriptions that follow hold with all generalizations, so that generalization is not a primitive regulation. We can show, however, by an inductive argument going back to Tarski, that generalization holds as a derived regulation relative to theses which follow from the prescriptions presupposed with all generalizations. Maximal prescriptions are those which only have maxims as instances, and they are marked with a subscripted capital \( M \) in their appellation. All instances of minor prescriptions (which are marked with a subscripted minuscule \( m \) in their appellation) are theses, but some of their instances are minors.

\[
\begin{align*}
\mathcal{L}^1_M & \quad A \rightarrow (B \rightarrow A) \\
\mathcal{L}^2_M & \quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\
\mathcal{L}^3_M & \quad (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B) \\
\mathcal{L}^4_M & \quad A \rightarrow \forall x A, \text{ provided } x \text{ is not a noema in } A \\
\mathcal{L}^5_M & \quad \forall x (A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B) \\
\mathcal{L}^6_M & \quad \forall x A \rightarrow A(a/x), \text{ if } a \text{ is substitutable for } x \text{ in } A \\
\mathcal{L}^7_M & \quad T(A \rightarrow B) \rightarrow (TA \rightarrow TB) \\
\mathcal{L}^8_M & \quad TA \rightarrow \neg T\neg A \\
\mathcal{L}^9_M & \quad TB \lor T\neg B \lor (T\neg T\neg A \rightarrow TA) \\
\mathcal{L}^{10}_M & \quad TB \lor T\neg B \lor (TA \rightarrow TTA) \\
\mathcal{L}^{11}_M & \quad T(TA \rightarrow A) \rightarrow TA \lor T\neg A \\
\mathcal{L}^{12}_M & \quad \exists x TA \rightarrow T\exists x A \\
\mathcal{L}^{13}_M & \quad T\forall x A \rightarrow \forall x TA \\
\mathcal{L}^{14}_M & \quad TA \rightarrow A \\
\mathcal{L}^{15}_M & \quad A \rightarrow TA \\
\mathcal{L}^{16}_M & \quad \forall x TA \rightarrow T\forall x A \\
\mathcal{L}^{17}_M & \quad T\exists x A \rightarrow \exists x TA
\end{align*}
\]

The alethic comprehension prescription:
\[
A^C_M \quad \forall x \in \{y \mid A\} \leftrightarrow TA(x/y), \text{ if } x \text{ is substitutable for } y \text{ in } A
\]

The truth prescription:
\[
TT_M \quad \forall \overline{x}(TA(\overline{x}) \iff TA(\overline{x}))
\]

Definition 7.1 \( KIND(a) \equiv \forall x (Tx \in a \lor Tx \notin a) \)
The enumeration precepts:

\[\mathcal{E}_1^M: \forall x \exists y (y \in \mathbb{N} \land \langle y, x \rangle \in \mathcal{E})\]

\[\mathcal{E}_2^M: \forall x (x \in \mathcal{E} \rightarrow \exists y, z (\langle x, y, z \rangle \in \mathcal{E} \land \langle y, z \rangle \in \mathcal{E} \land \langle y, z \rangle \in \mathcal{E} \rightarrow y = z)\]

\[\mathcal{E}_3^M: \forall x, y, z (\langle x, y, z \rangle \in \mathcal{E} \land \langle y, z \rangle \in \mathcal{E} \land \langle y, z \rangle \in \mathcal{E} \rightarrow y = z)\]

\[\mathcal{E}_4^M: \forall x, y, z (\langle x, y, z \rangle \in \mathcal{E} \land \langle y, z \rangle \in \mathcal{E} \land \langle y, z \rangle \in \mathcal{E} \rightarrow x = y)\]

\[\mathcal{E}_5^M: \text{KIND}(\mathcal{E})\]

The prescribes prescription:

\[\langle n, v_n \rangle \in \mathcal{E}, \text{ for } n \in \mathbb{N} \text{ and corresponding } v_n \in \mathbb{N} \cdot\]

The disunion prescript:

\[\downarrow M: \forall x, y, z (x \in \downarrow yz \leftrightarrow x \notin y \land x \notin z)\]

The complement prescript:

\[\overline{M}: \forall x, y (x \in \overline{y} \leftrightarrow x \notin y)\]

The relative complement prescript:

\[\setminus M: \forall x, y, z (x \in y \setminus z \leftrightarrow x \in y \land x \notin z)\]

The union prescript:

\[\cup M: \forall x, y, z (x \in y \cup z \leftrightarrow x \in y \lor x \in z)\]

The intersection prescript:

\[\cap M: \forall x, y, z (x \in y \cap z \leftrightarrow x \in y \land x \in z)\]

**Definition 7.2** Let \( F \equiv \exists \mathcal{E} \setminus T \)

The bivalence prescript:

\[\text{BV}_M: \forall x (\exists x \rightarrow Fx \lor Tx)\]

I leave it as an exercise here to verify the prescriptions and precepts above. But see (Bjørdal, 2012) for accounts of a number of prescriptions, some of which are lifted from Cantini (1996).

There are infinitely many distinct prescribes which contain at least one noema. Importantly, prescribes are not instances of precepts and cannot be universally generalized upon. Nevertheless, by the prescribes prescription fundamental prescribes are instances of a prescription. As \( a = b \equiv \forall u (a \in u \rightarrow b \in u), v_\exists T = T \) is an example of a prescribe relative to the presupposed enumeration \( \mathcal{E} \) as \( T \) is the first cognomen in \( \mathcal{E} \).

Librationist comprehension is constituted by those principles which are engendered by alethic comprehension in cooperation with all posits and regulations of £.
8 Regulations of £

The regulations valid in librationism are sensitive as to whether the initial or consequential theses are maxims or minors. In the following partial list of some salient regulations we as above use subscripts $M$ and $m$ to indicate whether a thesis is supposed to be a maxim or a minor. If there is no subscript on the roadstyle this signifies that it is left open whether what follows the roadstyle is a maxim or a minor.

\[
\begin{align*}
R_1 & \models_M A \land \models_M (A \rightarrow B) \Rightarrow \models_M B & \text{modus maximus} \\
R_2 & \models_m A \land \models_M (A \rightarrow B) \Rightarrow \models_B & \text{modus subiunctionis} \\
R_3 & \models_M A \land \models_m (A \rightarrow B) \Rightarrow \models_mB & \text{modus antecedentiae} \\
R_4 & \models_M A \Rightarrow \models_M TA & \text{modus ascendens maximus} \\
R_5 & \models_m A \Rightarrow \models_m TA & \text{modus ascendens minor} \\
R_6 & \models_M TA \Rightarrow \models_M A & \text{modus descendens maximus} \\
R_7 & \models_m TA \Rightarrow \models_m A & \text{modus descendens minor} \\
R_8 & \models_M \neg TA \Rightarrow \models_m TA & \text{modus scandens maximus} \\
R_9 & \models_m \neg TA \Rightarrow \models_m TA & \text{modus scandens minor} \\
R_{10} & \models_M \forall x TA \Rightarrow \models_m T \forall x A & \text{modus Barcanicus} \\
R_{11} & \models_T \exists x A \Rightarrow \models_m \exists x TA & \text{modus attestans generalis} \\
R_{12} & \models_m T \exists x A \Rightarrow \models_m \exists x TA & \text{modus attestans minor} \\
R_{13} & \models_m A \land \models_m B \Rightarrow \models_m \neg T \neg A \land \neg T \neg B & \text{modus minor} \\
R_Z & \models_M A(x/y) \text{ for all noemata } x \Rightarrow \models_M \forall x A(x/y) & \text{The Z-regulation}
\end{align*}
\]

It is reminded that this list of regulations is not complete, as librationism is not recursively axiomatizable and no such list can be safeguarded as complete. Moreover, we have aimed at providing a fairly comprehensive list instead of circumscribing a list of independent regulations. The Z-regulation is named in analogy with the $\omega$-rule by using the last letter of the Latin alphabet.

Notice that on account of $R_1$ conjoined with $\models_M \neg TA$ and the fact that all generalizations of the latter are presupposed as prescriptions, £ is super-classical. We take our regulation modus maximus above to be the reasonable interpretation of modus ponens as classically intended, so that the novelty of our regulations does not as we see it constitute a weakening of but an extension of classical principles. However, the facial value of modus ponens only appeals to thisishd so we use the novel term to distinguish.

We verify regulation $R_3$. Suppose $\models_M A$ and $\models_m A \rightarrow B$. We then have that $\models_T TA$, $\models_T \neg (A \rightarrow B)$ and $\models_T (A \rightarrow B)$. That $\models_T \neg (A \rightarrow B)$ means that $A \rightarrow B$ is unbounded under $\models_T$. That $\models_T TA$ means that $A$ holds as from some ordinal below $\models_T$. As all ordinals below $\models_T$ are progression closed, $B$ is unbounded under $\models_T$, i.e. $\models_T \neg \neg B$. That $\models_T \neg (A \rightarrow B)$ means that $A \land \neg B$ is unbounded under $\models_T$. But so a fortiori also $\neg B$ is unbounded under $\models_T$, i.e. $\models_T \neg TB$. So $\models_B$ and $\models \neg B$, i.e. $\models_mB$. The other regulations are left at exercises, but (Bjørdal, 2012) can be consulted.
9 Sobriety of Extensions and Interpretations

A formula $A$ is considered an anti thesis of a system iff its negjunction $\neg A$ is a thesis of the system. Let us agree that a theory $T$ extends theory $S$ soberly iff the set of theses of $S$ is a proper subset of the set of theses of $T$ and no thesis of $T$ is an anti thesis of $S$. $T$ soberly interprets $S$ iff $T$ interprets $S$ and for no interpretans $A'$ in $T$ of the interpretandum $A$ of $S$ does $T$ have $A'$ as a thesis and an anti thesis. As it is, $\mathcal{L}$ soberly extends classical logic and impredicative mathematics in the sense of reverse mathematics. By (Bjørdal, 2013) below, $\mathcal{L}$ soberly interprets ZF if ZF plus “there are omega inaccessibles” is consistent. By section 21 below, $\mathcal{L}$ interprets ZFC if ZF is consistent under the Skolem-Fraenkel Postulation.

10 $\mathcal{E}$, Countability, Order and Kind Choice

Given the enumeration prescripts of section 6, $\mathcal{E}$ is a bijection from $\mathbb{N}$ to a full universe as $\{x|x=x\}$; the non-extensionality of $\mathcal{L}$ prohibits us from talking about the full universe. As stressed and made precise in Bjørdal (2012), in librationism all sets or sorts are countable. Let $<$ be the usual order on $\mathbb{N}$:

**Definition 10.1**

1) $a \triangleright b \equiv \exists x, y (x \in \mathbb{N} \land y \in \mathbb{N} \land < x, a > \in \mathcal{E} \land < y, b > \in \mathcal{E} \land x < y)$

2) $a \trianglerightleq b \equiv a \triangleright b \lor a = b$

The following exercise illustrates the paradoxicality of the power set operation and of sets given by the comprehension condition invoked in Cantorian arguments. Sections 7 and 8 of Bjørdal (2012) should be compared.

**Exercise 10.2** Show that if $\vdash a \notin \{x|x=x\}$ then $C = \{x|x \in a \land \forall y (< x, y > \in \mathcal{E} \rightarrow x \notin y)\}$, $f = \{< x, y > | < x, y > \in \mathcal{E} \land y \subset a\}$ and $P(a) = \{x|x \subset a\}$ are paradoxical.

**Theorem 10.3 (The Kind Choice Theorem)**

If $a$ is a kind with only kind members, $\{x|\exists y(y \in a \land x \in y \land \forall z(z \in y \rightarrow x \notin z)\}$ is a kind with precisely one member from each member of $a$.

We leave the proof of the Kind Choice Theorem as an exercise.

11 Curries

Let $c^F \equiv \{x : x \in x \rightarrow F\}$. We are interested in $c^F$ for various sentences $F$, but here consider the Curry situation quite generally. By $A^C_M: c^F \in c^F \leftrightarrow T(c^F \in c^F \rightarrow F)$ is a maxim. $\mathcal{L}$ and sentence logic give us $\vdash c^F \in c^F \rightarrow (c^F \in c^F \rightarrow F)$. But $\vdash_M (c^F \in c^F \rightarrow (c^F \in c^F \rightarrow F)) \rightarrow (c^F \in c^F \rightarrow F)$, so by modus subUNCTIONIS or modus maximus
it follows that \( \vdash (c^F \in c^F \rightarrow F) \). By modus ascendens we get \( \vdash T(c^F \in c^F \rightarrow F) \), and so next \( \vdash c^F \in c^F \). By alethic comprehension by modus subiunctionis or modus maximus. So we have that both \( \vdash c^F \in c^F \) and \( \vdash c^F \in c^F \rightarrow F \) for any arbitrary sentence \( F \). Clearly, then, modus ponens as classically stated cannot hold for the roadstyle as otherwise \( \mathcal{L} \) would have been trivial.

Now, \( \vdash c^F \in c^F \) being a thesis, it must either be a minor or a maxim. If we have \( \vdash_M c^F \in c^F \), we easily derive that \( F \) is a maxim, i.e. \( \vdash_M F \). If \( c^F \in c^F \) is a minor it follows that \( \neg F \) is a thesis; for then also \( \vdash c^F \notin c^F \) so by alethic comprehension and modus subiunctionis \( \vdash -T(c^F \in c^F \rightarrow F) \) and thence by modus ascendens and modus descendens, \( \vdash c^F \in c^F \land \neg F \), so that by tautologies and modus subiunctionis, \( \vdash \neg F \). It follows, by parity of reasoning, that for any sentence \( F \), either \( F \) is a maxim, \( F \) is a minor, or \( \neg F \) is a maxim. But in our contentual framework this is as we knew it to be, for we observed in the penultimate paragraph of section 6 that \( \mathcal{L} \) is negjunction complete.

If we consider \( c^1 \equiv \{ x \mid x \in x \rightarrow 1 \} \), where \( 1 \) is falsum or a contradiction, we realize that it behaves semantically as Russell’s paradoxical set \( \{ x \mid x \notin x \} \). Much more complicated paradoxicalities can be constructed when \( D \in c^D \) is itself is paradoxical.

12 Nominism: Nominalism Released, Platonism Restrained

As pointed out by Kripke it is common policy in logic to presuppose a generality-interpretation of variables, or what we instead take as noemata. Such a policy cannot be sustained in librationism. We need only consider e.g. \( c^\{13=96\} \) to see this. By the previous section, and as \( \mathcal{L} \) is negjunction complete, \( v_{13} = v_{96} \) is a thesis or \( v_{13} \neq v_{96} \) is a thesis. As identity is kind (see (Bjørdal, 2012)) either \( v_{13} = v_{96} \) is a maxim of \( \mathcal{L} \) or \( v_{13} \neq v_{96} \) is a maxim of \( \mathcal{L} \). Had one adopted the generality-interpretation for \( \mathcal{L} \) one would have to conclude that there is only one object according to \( \mathcal{L} \) lest \( \mathcal{L} \) be trivial. But a theory that is trivial or only postulates the existence of one object is not interesting and so we do not adopt the generality-interpretation. Instead we adopt a nominality-interpretation where what is usually taken as free variables are considered noemata and nomina, i.e. names. In our set up \( v_{13} \neq v_{96} \) is indeed a maxim, but we have no regulation or regula that allows generalizing universally.

We take nominism to be the view that all mathematical objects have a name while it nonetheless upholds the platonist view that mathematical objects are abstract. Nominism is supported by librationism by the latters avoidance of Cantor’s conclusion that there are uncountable infinites and insistence instead that there are only denumerably many objects; for the reader’s convenience we repeat that it is not claimed anywhere that Cantor’s arguments are invalid (cfr. (Bjørdal, 2012) and (Bjørdal, 2011) for more on this). As there, according to \( \mathcal{L} \), are only a denumerable infinity of objects, we have enough names to name all mathematical objects and we have invoked a nominality policy with that as objective in our semantics.

Notice that the term “platonism” in the context of set theory is sometimes taken to stand for the Cantorian view that the endless hierarchy of alephs exists in a non-relative
sense. Here we instead abide by what we take as a more plausible usage of the term in the philosophy of mathematics where it denotes the view that takes mathematical objects to be abstract objects. The nominism we propound of course rejects the Cantorian version of platonism.

13 Contradiction, Contravalence and Complementarity

In this section we develop new ideas which pertain to how the connectives should be understood in the librationist framework.

As we have seen in section 6 the semantics of $\mathcal{L}$ comes about by elaborations, and a librationist twist, upon the semi inductive type of approach employed by (Herzberger, 1982) to analyse the Liar’s paradox in a way which in some important respects improved upon that of (Kripke, 1975). However, there is not a simple duality between the librationist semantics and an envisioned Herzbergerian style approach as $\mathcal{L}$ has noemata and no free variables. $v_{13} \neq v_{16}$ may e.g. be a maxim of $\mathcal{L}$ but $v_{13} = v_{16}$ is not unbounded in all Herzbergerian semantic processes. One important aspect of the librationist elaboration upon the Herzberger style approach is that we focus upon one semantic process in order to gain categoricity; this we dub the nominist turn, and it is with the nominist turn accompanied by the librationist twist that $\mathcal{L}$ becomes a contentual system.

Induced principles for truth and abstraction are such as to e.g. make $r \in r$ a thesis of $\mathcal{L}$ and also the negjunction $r \notin r$ a thesis of $\mathcal{L}$ when $r$ is Russell’s set $\{x : x \notin x\}$ of all and only those sets which are not members of themselves. However, $r \in r \land r \notin r$ is not a thesis of $\mathcal{L}$; notice that if it had been this would have contradicted that $\mathcal{L}$ is a sober extension of classical logic. Pay also heed here to the fact that the induced regulations of $\mathcal{L}$ are all novel and in particular that modus ponens as classically stated is not generally truth preserving in $\mathcal{L}$.

Librationism indeed comes very close to being a so-called paraconsistent system. However, I shall on the basis of its semantics argue that $\mathcal{L}$ is neither paraconsistent nor inconsistent nor contradictory. This will involve concepts and a terminological policy from a crestal (meta) level.

A sentence’s valency is the set of von Neumann ordinals of $\mathcal{L}$ in the semantical process where it holds. The valor of a sentence is the least upper bound of its valency, and thus the union of its valency as we presuppose von Neumann ordinals in the semantic set up. A sentence has at least one of two values, viz. it can be true or it can be false. A sentence is true iff its valor is the closure ordinal $\wp$ and a formula is false iff its negjunction is true.

We let $\mathcal{V}(A)$ be the valency of $A$ and $\mathcal{V}(A)$ be the valor of $A$. If $\mathcal{V}(A) < \wp$, $A$ is false and not true and we say from our external perspective that $A$ is pseudic; if $\mathcal{V}(\neg A) < \wp$, $A$ is true and not false and we say that $A$ is veridic. It turns out that a sentence $A$ is veridic (pseudic) iff it is a maxim $T A$ ($T \neg A$), and the latter holds iff $A$ ($\neg A$) is a maxim. There are no special challenges with accounting for paradoxical conditions that involve the notions of being pseudical or veridical, for $\mathcal{L}$ has no access to the external perspective on $\mathcal{L}$ needed to talk about veridicity and pseudicity.
Two sentences $A$ and $B$ are *parivalent* iff the valency of $A$, $\forall(A)$, is identical with $\forall(B)$; two sentences are *altervalent* iff they are not parivalent; two sentences $A$ and $B$ are *contravalent* iff $\forall(A) \not\equiv \forall(B)$; the *ambivalence* of $A$ and $B$ is the intersection $\forall(A) \cap \forall(B)$; the *valence* of $A$ and $B$ is the union $\forall(A) \cup \forall(B)$; the *subvalence* of $A$ under $B$ is $\forall(\neg A) \cup \forall(B)$ and the *homovalence* of $A$ and $B$ is $(\forall(A) \cap \forall(B)) \cup (\forall(\neg A) \cap \forall(\neg B))$.

The connectives do not work truth-functionally in librationism, but they work *valency-functionally* and by following the classical interdefinability connections as in any Boolean algebra. The valency of the negjunction of $A$, $\forall(\neg A)$, is the contravalence of $A$; the valency of the *adjunction* (conjunction) of $A$ and $B$, $\forall(A \land B)$, is the ambovalence of $A$ and $B$; the valency of the *veljunction* (disjunction) of $A$ and $B$, $\forall(A \lor B)$, is the homovalence of $A$ and $B$; the valency of the *subjunction* (material implication) $A \rightarrow B$, $\forall(A \rightarrow B)$, is the subvalence of $A$ under $B$ and the valency of the *equivjunction* (material equivalence) $A \leftrightarrow B$, $\forall(A \leftrightarrow B)$, is the homovalence of $A$ and $B$.

We use slightly non-standard names for the connectives in part to forestall irrelevant objections which appeal to something like the one and only true meaning of them in ordinary language. One should appreciate that the connectives do behave truth-functionally for maxims, and so in this essential respect they do have their classical meaning in ordinary discourse.

Two sentences $A$ and $B$ are *incompatible* iff their ambovalence is empty and, consequently, $A \land B$ fails to be a thesis. Contravalent sentences are thence also incompatible. All sentences are contravalent with their negjunction; moreover, if $A$ and $B$ are contravalent then $A$ and $\neg B$ are parivalent.

What does “What does it say?” say? A variety of things, to be sure; but we lay down the convention that a sentence *dictates* its valor, and that a sentence *presents* its valency. Given this convention, we take contravalent sentences to be *contrapresentive* and parivalent sentences to be *paripresentive*. We on occasion say that the sentence’s valency is the *way* the sentence dictates its valor.

Two sentences are *paridictive* iff they dictate the same valor, and otherwise they are *alterdictive*. Two sentences are contradictive iff they are contravalent and alterdictive. A contradiction is the adjunction of two contradictive sentences. In consequence precisely one of two contradictive sentences is true, so it can by our light never be correct, *pace* certain paraconsistentists, to maintain contradictions.

Two sentence are *complementary* iff they are contravalent and paridictive. Two complementary sentences are thence both true, and they dictate the same (viz. 9) in *opposite* ways. In particular, for Russell’s set $r = \{x \mid x \notin x\}$, $r \in r$ and $r \notin r$ are complementary. Thence $r \in r$ and $r \notin r$ dictate the same in opposite ways and so do not by our standards contradict each other.

We cannot strictly speaking say “sentences $r \in r$ and $r \notin r$ are both true” in one mouthful as $r \in r$ and $r \notin r$ are incompatible. Librationism is committed to a peculiar discretist or sententialist, though nonetheless holistic, point of view.

We regard a theory as inconsistent iff it has theses of the form $A \land \neg A$, and as contradictory iff it has contradictions as theses; consequently, we hold that a theory is inconsistent iff it is contradictory. Given the foregoing it is our considered opinion that $\mathcal{L}$ is consistent.
and not contradictory. As a consequence librationism should not be considered to be a dialetheist point of view as() dialetheism is canonically characterized as a view which takes some contradictions to be true. Instead we take librationism to offer a bialethic point of view.

We take our considerations in this section to meet a challenge that remains one for those who as some dialetheists believe that a sentence and its contradiction can both be true. Such paraconsistentists owe others an explanation as to what they in dialethec cases think it is a true sentence p says which its true contradictory ¬p contradicts.

14 Disconnectionism, Contrapresentationism, Parasistency

A sentence A is taken as a contrapresentive thesis of a theory iff A is both a thesis and an anti thesis of the theory. Let us agree that a theory is contrapresentive iff it has contrapresentive theses. A theory is trivial iff all sentences of its language are theses. Trivial systems and inconsistent theories with simplification or adjunction elimination are contrapresentive. £ is contrapresentive, but neither trivial nor inconsistent. Contrapresentationism is the view that a contrapresentive theory, such as £, is true.

We say that two formulas A and B of a theory T are connected iff A and B are theses of T only if also the conjunction A ∨ B is a thesis of T. Two sentences disconnect with each other iff they do not connect with each other. A sentence is disconnected iff it disconnects with some sentence. A set b is disconnected iff for some set a the sentence a ∈ b is disconnected. Paradoxical theses of £ are disconnected theses of £, and vice versa. It is straightforward that if A is not a thesis then A connects with all sentences, and further that all maxims connect with all sentences. All sentences are self-connected and the relation connects with is also symmetric, but not transitive. A theory is disconnected iff it has disconnected theses. A topic is disconnected iff a true theory about it is disconnected and disconnectionism is the view that there are sound disconnected theories

Some paraconsistent logics, such as the ones following the approach by Jaskowski, are non-adjunctive. But such logics do not in and of themselves have disconnected theses, though extensions of such logics with suitable comprehension principles or semantic principles may be disconnected if not trivial.

Notice also that £ is distinct from paraconsistent systems in that the latter, including Jaskowski’s system, do not have ex falso quodlibet, (A ∧ ¬A → B), as a theorem whereas £ has it as a maxim. We emphasize this obvious corollary for the reason that the failure of ex falso quodlibet is very arguably a defining feature of paraconsistent approaches.

We correctly hold it against someone if she utters A and thence not A when she is meant to elucidate a connected topic. However, we take our discussion to have revealed that paradoxes are essentially disconnected. I therefore contend that £ should be regarded as fulfilling a very important adequacy requirement by being a disconnected theory about the disconnected and quasi incoherent and absurd topic of paradoxicality.

The librationist theory £ may, as far as its dealing with paradoxical phenomena is concerned, be thought of as one accompanied with many perspectives which we shift
between in our reasoning in those contexts. One may employ the term parasistency for this idea that £ lets us stand beyond, so to speak, and shift between perspectives. We have an external overarching crestal (meta) perspective which helps us illuminate many issues, as e.g. illustrated here. Besides, for each true sentence its valency may be thought of as associated with a partial perspective we may take upon mathematical and semantical reality. There are, then, oscillating switches, as it were, between different perspectives we may have upon paradoxical sentences, and this justifies our adoption of librationism’s neologist name. Moreover, such oscillating shifts between perspectives correspond well with our contemplative experiences in connection with the paradoxes, and so librationism on this count appropriately fulfills an important desideratum, arguably an adequacy condition, by incorporating them.

It is of course important in all of this that the right balance is struck between various desiderata. As it is, £ seems well suited to strike precisely such a balance as it soberly extends classical logic and even soberly interprets classical mathematics in as far as the latter is consistent. Moreover, theses are only disconnected theses of £ if they are minor and hence paradoxical theses of £. Our isolated notions of complementarity and valency functionality appropriately alleviate the loss of intuitiveness brought upon us by the paradoxical phenomenon of disconnectedness.

15 Manifestation Points with Parameters

We have earlier indicated that we follow the standard practice of writing $A(x, y)$ to indicate that the noemata $x$ and $y$ are present in the formula $A$. We extend this practice so that we e.g. write $c(x, z^2)$ to indicate that the noema $x$ and the noemata from the vector $z^2$ occur as parameters in the term $c$; as we now work metalinguistically it makes sense to think as if we invoke noemata as parameters.

We rehearse the definition of manifestation point of section 6 of Biordal (2012) for convenience and to show that we can make use of instances with parameters.

**Theorem 15.1** If $A(x, y, z)$ is a formula with the noemata indicated we can find a term $h(z)$ such that $\vdash_M \forall w \exists z (w \in h(z) \leftrightarrow \text{TTA}(w, h(z), z))$.

**Proof.** Let $d = \{\{x, g\}|A(x, \{u|\langle u, g \rangle \in g\}, z)\}$ and $h(z) = \{g|\langle y, d \rangle \in d\}$. If we spell out, we have: $h(z) = \{y|\langle y, \{\{x, g\}|A(x, \{u|\langle u, g \rangle \in g\}, z)\}\} \in \{\{x, g\}|A(x, \{u|\langle u, g \rangle \in g\}, z)\}$. By $A^C_M$ we have that $\vdash_M \forall w \exists h(z) \leftrightarrow \text{T}(w, \{\{x, g\}|A(x, \{u|\langle u, g \rangle \in g\}, z)\}) \in \{\{x, g\}|A(x, \{u|\langle u, g \rangle \in g\}, z)\}$, and so by $A^C_M$ again it follows that $\vdash_M \forall w \exists h(z) \leftrightarrow \text{TTA}(w, \{u|\{x, g\}|A(x, \{u|\langle u, g \rangle \in g\}, z)\}) \in \{\{x, g\}|A(x, \{u|\langle u, g \rangle \in g\}, z)\}, z)$. But this is $\vdash_M \forall w \exists h(z) \leftrightarrow \text{TTA}(w, h(z), z)$, as wanted. As we only invoked generalizable prescriptions also $\vdash_M \forall w \exists z (w \in h(z) \leftrightarrow \text{TTA}(w, h(z), z))$. Obviously, we can have more than one parameter or a vector of parameters in $A$. ■
16 Defying an Orthodox Identity

Theorem 5 of [Bjørdal, 2012] implies that $\mathcal{L}$ is highly non-extensional in that for any kind $a$ there are infinitely many mutually distinct kinds coextensional with $a$. The main result of this section has the same as consequence for non-empty kinds, but is a more general result and based upon different considerations.

Lemma 16.1 $\models_M a = \{ x | x \in a \}$ only if $a$ is kind.

Proof. Suppose $\models_M a = \{ x | x \in a \}$. By substitution of identicals and alethic comprehension $A^C_M$, $\models_M \forall x T(x \in a \leftrightarrow x \in a)$, so $\models_M \forall x T(x \in a \rightarrow x \in a)$. By $L^C_M \models_M \forall x (T(x \in a \rightarrow x \in a) \rightarrow (T x \in a \lor \lnot T x \in a))$, so by classical logic $\models_M \forall x (T x \in a \lor \lnot T x \in a)$ which by the terminology of [Bjørdal, 2012] means that $a$ is kind. ■

Lemma 16.2 If $\alpha = \beta = 0$ and $\alpha > 0$ then $\alpha \models T a = b$

Proof. As for Lemma 1 of Bjørdal (2012), and using the librationist definition of identity and the fact that $\alpha \models a \in \{ x | x = a \}$.

Lemma 16.3 $\models \alpha = b$ and $\beta < \alpha$ only if $\models \beta = a = b$

Proof. Suppose $\models \alpha = b$ and $\beta < \alpha$ for some ordinal $\beta$. It follows that $\alpha > 0$, so that by Lemma 16.2 then $\alpha \models T a = b$. It follows that for some $\delta < \alpha$, for all $\alpha > \gamma \geq \delta$, $\gamma \models a = b$; in particular $\delta \models a = b$. By the wellfoundedness of ordinals one can iterate the process to any ordinal $\beta$ below $\alpha$.

Lemma 16.4 If $\models_M a = \{ x | x \in a \}$ and $\beta < \varphi$ then $\models \forall x (x \notin a)$

Proof. Suppose to the contrary that for some $b$ and $\beta$: $\models_M a = \{ x | x \in a \}$, $\beta < \varphi$ and $\beta = b \in a$. By Lemma 16.3 and the theory of identity, $\models b \in \{ x | x \in a \}$; by alethic comprehension $A^C_M$, $\models T b \in a$; so for some $\delta < \beta$, $\delta = b \in a$. But this could be iterated so that we get $0 \models b \in a$, so that $1 \models a \in \{ x | b \in x \}$. But by Lemma 16.3 again and our theory of identity, then also $1 \models \{ x | x \in a \} \in \{ x | b \in x \}$, so that $0 \models b \in \{ x | x \in a \}$; but the latter statement is impossible on account of principle $P(1)$ of our semantics as 0 is the smallest ordinal and $\{ x | x \in a \}$ is of caliber zero (see Definition 21.18).

Lemma 16.5 If for all $\beta < \varphi$, $\beta \models \forall x (x \notin a)$, then $\models_M a = \{ x | x \in a \}$

Proof. Suppose to the contrary. By the logic of identity for all $\beta < \varphi$, $\beta \models \forall x (x \notin a)$ and $\models_M a \neq \{ x | x \in a \}$. If for all $\beta < \varphi$, $\beta \models \forall x (x \notin a)$ also for all $\beta < \varphi$, $\beta \models \forall x (x \notin \{ x | x \in a \})$; this we can infer without invoking the identity between $a$ and $\{ x | x \in a \}$, $0 \models a = \{ x | x \in a \}$, as $0 \models b = c$ for all terms $b$ and $c$. As $\models_M a \neq \{ x | x \in a \}$ there must be a first ordinal $\gamma$ such that $\gamma \models a \neq \{ x | x \in a \}$. Let $\delta$ be any ordinal smaller than $\gamma$. From the information we have $\delta \models b \in a \leftrightarrow b \in \{ x | x \in a \}$ and $\delta \models a \in b \leftrightarrow \{ x | x \in a \} \in b$ for all terms $b$; moreover, such biconditionals also hold within any number of applications of the defined truth operator $T$. Consequently, there cannot be anything below $\gamma$ which justifies a sudden change to non-identity between $a$ and $\{ x | x \in a \}$. ■
Lemma 16.6  If for all $\beta < \varphi$, $\beta \models \forall x (x \notin a)$, then $\models_M a = \emptyset$

Proof. Here $\emptyset = \{x| x \neq x\}$. The argument is as for Lemma 16.5  ■

Theorem 16.7  $\models_M a = \{x| x \in a\}$ iff $\models_M a = \emptyset$

Proof. The argument is by combining the preceding lemmas.  ■

The non-extensionality results of [Bjørdal, 2012] and preceding results in earlier literature referred to there are surprising in the sense that they force us to revise the prejudice of extensionality in type free contexts by means of explicit conditions that convince us. Theorem 16.7 is different in that it surprises us in its statement of the very wide prevalence of non-extensionality in cases where one should not expect it at all. One may hold against this that some of these features are consequences of a perhaps arbitrary decision of having all terms of the form $\{x| A(x)\}$ empty at the ordinal 0; but other choices would have had their own blend of arbitrary seeming consequences, and the author thinks that there are so good aesthetical reasons concerned with uniformity to support the semantical set up presupposed that it is not arbitrary.

17 Capturing Collection, Replacement, Specification and Choice

We introduce ordinary capture by the following definition, where $\alpha(x, y)$ is any first order condition on $x$ and $y$ as standardly conceived in classical set theory.

Definition 17.1 (Ordinary capture)

\[
\forall v \exists w \forall x (x \in w \leftrightarrow \exists y (y \in v \land \alpha(y, x) \land (\forall z)(\alpha(y, z) \rightarrow x = z)))
\]

Let $W$ be Zermelo set theory $\mathcal{Z}$ minus the axiom schema of specification.

Theorem 17.2  $ZF = W + \text{ordinary capture}$

Proof. The condition $\beta(y, x) = \alpha(y, x) \land (\forall z)(\alpha(y, z) \rightarrow x = z)$ is functional, so capture follows from replacement. If $\alpha(y, x)$ is already functional the clause requiring only one set captured is redundant, so capture entails replacement. Specification follows from capture by letting $\alpha(y, x) = \beta(y) \land y = x$.  ■

We introduce librationist capture by the following definition, where $\alpha$ is as above and the order relation $\downarrow$ as in Definition 10.1.

Definition 17.3 (Librationist capture)

If $a$ is a term, so is $\{x| \exists y (y \in a \land \alpha(y, x) \land (\forall z)(\alpha(y, z) \rightarrow x \downarrow z))\}$. 23
Proposition 17.4 (Librationist capture of choice)

If $c$ is a kind with only nonempty kinds whose intersections are pairwise empty for members not coextensional, then \{x|\exists y(x \in y \land (\forall z)(z \in y \rightarrow x \triangleleft z))\} is a kind that, modulo extensionality, contains precisely one kind from each kind of $c$. So librationist capture entails full choice in those contexts it is operative.

We will see in section 21 that librationist capture in some contexts where extensionality fails is stronger than ordinary capture also in that the former provides collection and the latter not. Our considerations in this section are of a general nature, but the relationships pointed out between capture, replacement, specification and choice carry over to the more intricate constructions carried out in the next section and in section 21.

Our use of capture is not merely motivated by its pleasing aesthetic qualities such as that it is a biconditional or that we avoid special restrictions on the invoked first order condition, but also by the fact that it is a more manageable closure principle which is more useful for our definitional purposes below.

18 Varieties of Conditions

We explore some uniform maximal closure conditions for $H$ beyond the Jensen rudimentary functions amounting to bounded separation discussed in (Bjørdal, 2012).

Recall definition 2.1. We make an analogous object language definition:

**Definition 18.1**

\[ \forall (v_i) \equiv \{v_i\} \]
\[ \forall (T) \equiv \{T\} \]
\[ \forall (\epsilon) \equiv \{\epsilon\} \]
\[ \forall (\forall v_i) \equiv \forall (A) \setminus \{v_i\} \]
\[ \forall (\forall v_i A) \equiv \forall (A) \setminus \{v_i\} \]
\[ \forall (\forall \forall v_i A) \equiv \forall (A) \setminus \{v_i\} \]
\[ \forall (\forall A B) \equiv \forall (A) \cup \forall (B) \]
\[ \forall (\forall ab) \equiv \forall (a) \cup \forall (b) \]

We define conditions upon noemata.

**Definition 18.2** $\text{Cnd}(x, y) \equiv x$ is the code of a formula that does not contain $\hat{\ }$, $T$ or $\epsilon$ and contains the joint only as connective and not as a juncture and $\forall z(z \in 'x \rightarrow z \in y)$.

We will e.g. have $\text{Cnd}(\alpha, \{v_0\})$ just if $\alpha$ is a first-order condition upon $v_0$ precisely as in classical set theory without the identity sign as primitive.

For $X$ a possibly more restricted set of (codes of) formulas we define:
Definition 18.3 \( X \text{-Cnd}(x, y) \doteq \text{Cnd}(x, y) \) and \( x \in X \).

In the following we presuppose the full set of first order conditions. Recall our substitution function \( \text{SUB} \) of Section \[3\] above.

Definition 18.4 \( z \) is a Capture without parameters of \( a \) via \( x \)

\[
\text{Cpt}(z, a, x) \doteq \text{Cnd}(x, \{v_0^x, v_1^x\}) \land z = \{u|\exists v (v \in a \land \text{TSUB}(\text{SUB}(x, v), u) \land \forall w (\text{TSUB}(\text{SUB}(x, v), w) \rightarrow u = w)\})
\]

Definition 18.5 \( z \) is a Capture without parameters of \( a \)

\[
\text{Cpr}(z, a) \doteq \exists x \text{Cpt}(z, a, x)
\]

19 Domination

For some gain in effectiveness, we state the domination requirements in terms of capture.

Definition 19.1 (Domination requirement with parameter of parameters)

\[
D(x, y, a, z) \doteq \exists t (\text{Cnd}(t, z) \land {}^t v_0^x \in {}^t \Gamma(t) \land {}^t v_1^x \in {}^t \Gamma(t) \land 
\]

\[
\land s = \text{SUB}({}^t \Gamma(t) \land {}^t \Gamma(t); v_1 = v_2^x, y) \land
\]

\[
x = \{u|\exists v (v \in a \land \text{TSUB}(\text{SUB}(s, v), u) \land \forall w (\text{TSUB}(\text{SUB}(s, v), w) \rightarrow u = w)\})
\]

Definition 19.2 (Domination requirement without parameters)

\[
D^-(x, y, a, z) \doteq \exists t (\text{Cnd}(t, \{v_0^x, v_1^x, v_2^x\}) \land {}^t v_0^x \in {}^t \Gamma(t) \land {}^t v_1^x \in {}^t \Gamma(t) \land 
\]

\[
\land {}^t v_2^x \in {}^t \Gamma(t) \land s = \text{SUB}({}^t \Gamma(t) \land {}^t \Gamma(t); v_1 = v_2^x, y) \land
\]

\[
x = \{u|\exists v (v \in a \land \text{TSUB}(\text{SUB}(s, v), u) \land \forall w (\text{TSUB}(\text{SUB}(s, v), w) \rightarrow u = w)\})
\]

Domination without parameters can now be defined by manifestation point with one parameter as described in Section \[13\] above:

Definition 19.3 \( \vdash_M \forall x (x \in D^- (h) \leftrightarrow \text{TTD}^-(x, D^-(h), h)) \)

Lemma 19.4 \( \vdash_M \forall x (\text{KIND}(D^- (x))) \)

Proof. As \( \mathcal{L} \) soberly extends the theory of identity and Peano arithmetic. \( \blacksquare \)

Corollary 19.5 \( \vdash_M \forall x (x \in D^- (h) \leftrightarrow D^-(x, D^-(h), h)) \)

Definition 19.6 (Ordinary specification, short hand)

\[
\forall (x \in a | \alpha (x) ^o) \doteq \{ x | \exists y (y \in a \land T^r \alpha (y) \land \dot{y} = \dot{x} \land \forall z (T^r \alpha (y) \land \dot{y} = \dot{z} \rightarrow z = x) \}
\]

25
Lemma 19.7 (The Impredicativity Lemma)

If \( z = \{ u | u \in D^- (h) \land \alpha (u) \} \), \( Cnd('\alpha', \{ 'u' \}) \) and kind \( h \) then a kind \( w \) coextensional with \( \cup z \) is a member of \( D^- (h) \).

**Proof.** \( D^- (h) \) is impredicative as \( \forall x \in h (\exists u (u \in D^- (h) \land \alpha (u) \land x \in u)) \in D^- (h) \). In desired kind contexts the latter is coextensional with \( \cup \{ u | u \in D^- (h) \land \alpha (u) \} \).

20 The Postulate, Closures and Definable Real Numbers

As in [Bjørdal (2012)](Bjørdal2012) we take a sort to be good iff it is hereditarily kind and in \( \mathbb{H} \).

**Postulate 20.1 (The Postulate)**

If \( c \) is a good and \( Cpr(z, c) \) (with possible parameters from \( \mathbb{H} \)), then so is \( z \). More precisely, \( \models_{\mathcal{M}} c \in \mathbb{H} \land Cpr(z, c) \rightarrow z \in \mathbb{H} \).

Notice that the Postulate is not an original posit, and it amounts to a leap of faith that it is a theorem, as it were, or a presupposition on a par with the credo in the ZF tradition that replacement or collection with first order conditions does not lead to inconsistency. Given Gödel’s second incompleteness theorem, we cannot expect to prove that the postulate holds in the liberationist semantic set up unless with means that surpasses the resources used in that set up. We state the two critical adjoint theorems conditional upon the Postulate:

**Theorem 20.2** \( \models_{\mathcal{M}} (\forall x)(x \in \mathbb{H} \rightarrow D^- (x) \in \mathbb{H}) \)

**Proof.** The proof depends on the Postulate.

**Theorem 20.3** \( \models_{\mathcal{M}} (\forall x)(\forall y)(x \in \mathbb{H} \land Cpr(y, x) \rightarrow y \in \mathbb{H}) \)

**Proof.** The proof depends on the Postulate.

Let \( \mathbb{Q} \) be the set of rational numbers and \( < \) their usual order. We use a formula with precisely the noemata \( x \) and \( y \) to appropriately capture Dedekind cuts when defining the set \( \mathbb{D} \) of definable real numbers:

**Definition 20.4 (The Definable Real Numbers)**

\[
\mathbb{D} \triangleq \{ x | x \in D^- (\mathbb{Q}) \exists u (u \in x) \land \exists u (u \in \mathbb{Q} \land u \notin x) \land \forall u (u \in x \rightarrow \\
\forall v (v \in \mathbb{Q} \land v < u \rightarrow v \in x)) \land \forall u (u \in x \rightarrow \exists v (v \in \mathbb{Q} \land u < v \land v \in x)) \}
\]

**Theorem 20.5** If \( z \) is a definable sort of real numbers from \( \mathbb{D} \) with an upper bound in \( \mathbb{D} \) then \( z \) has a least upper bound in \( \mathbb{D} \).

**Proof.** The hint is to invoke the Impredicativity Lemma[19.7].
21 The Definable Echelon

Definition 21.1 (Conditions restricted to $u$)

$Cn(x,u) ≜ x$ is the code of a formula that does not contain $\sim$, $T$ or $∈$ and contains the joint only as connective and not as a juncture and $∀ z (z ∈ T x → z ∈ u)$ and all the quantifiers of the formula $x$ is a code of are restricted to $u$.

Definition 21.2 ($w$ is an ordinary capture from $z$ of $a$ via $x$)

$Ct(w,z,a,x) ≜ Cn(x,z) \land \forall v [z ∈ z \land T ≤ SUB(x,v), w) \land \forall y (T ≤ SUB(x,v), y) → w = y)]$

Definition 21.3 ($w$ is a librationist capture from $z$ of $a$)

$C(w,z,a) ≜ (\exists x)(Cn(x,z) \land \forall v [z ∈ z \land w = \{ w | \exists v (v ∈ a \land T ≤ SUB(x,v), w) \land \forall y (T ≤ SUB(x,v), y) → w = y)]$

Definition 21.4 (Librationist specification, short hand)

$\{ x ∈ a | α(x) \} ≜ \{ x | \exists y (y ∈ a \land T ≤ T ≤ α(y) \land \dot{y} = \dot{x} \land \forall z (T ≤ α(y) \land \dot{y} = \dot{z} \rightarrow z = x)\}$

Definition 21.5 (Librationist semantic specification, short hand)

$\{ u ∈ a | T ≤ SUB(t,u) \} ≜ \{ x | \exists y (y ∈ a \land T ≤ T ≤ T ≤ \dot{y} = \dot{x} \land \forall z (T ≤ α(y) \land \dot{y} = \dot{z} \rightarrow z = x)\}$

Definition 21.6 (Domination requirement with parameters from $z$ relative to $u$)

$D(x,y,h,z) ≜ \exists s,t (Cn(t,z) \land \forall v [z ∈ z \land s = T ≤ SUB(t,y) \land x = \{ u | T ≤ SUB(s,u)\})$

By manifestation point:

Definition 21.7 (Domination with parameters from $z$)

$\vDash_M \forall x (x ∈ D(h,z) ← T T D(x,D(h,z),h,z))$

Definition 21.8

i) $a ≜ b$ if $a ∈ u \land b ∈ u \land \forall v (v ∈ u → (a ∈ v → b ∈ v))$

ii) $\{ a, b \}^u_≤ \{ x | x ≜ a \lor x ≜ b \}$

iii) $\{ a \}^u_≤ \{ a, a \}^u$

iv) $S(a,u) ≜ \{ x | x ∈ u \land (x ∈ a \lor x ≜ a)\}$

v) $\emptyset ≜ \{ x | x ∈ u \land x \not= x \}$

vi) $Ω(a,u) ≜ \{ x | x ∈ u \land (\forall y)(\emptyset ∈ y \land \forall z (z ∈ y → S(z,u) ∈ y) → x ∈ y)\}$
Lemma 21.14

For ii) also Lemma 21.13 is invoked, noting that $\vdash \emptyset \subseteq \Omega(\emptyset, y, u) \to x \in y$

Definition 21.10 (u is a Fraenkel of x) $F(x, u) \equiv \forall y(S(\emptyset, y, u) \to x \in y)$

Definition 21.11 (The definable echelon by manifestation point)

$\vdash_M \forall x(x \in \hat{D} \leftrightarrow TTF(x, \hat{D}))$

Lemma 21.12

i) $\vdash_M F(\emptyset, \hat{D})$

ii) $\vdash_M \forall x(F(x, \hat{D}) \to F(\Omega(x, \hat{D}), \hat{D}))$

iii) $\vdash_M \forall x(F(x, \hat{D}) \to F(D(x, \hat{D}), \hat{D}))$

iv) $\vdash_M \forall x, y(F(x, \hat{D}) \land C(y, \hat{D}, w) \to F(y, \hat{D}))$

v) $\vdash_M \forall x, y(F(x, \hat{D}) \land F(y, \hat{D}) \to F([x, y]^{D}, \hat{D}))$

vi) $\vdash_M \forall x(F(x, \hat{D}) \to F([x, x], \hat{D}))$

The proof of Lemma 21.12 is by invoking classical predicate logical tautologies. For the next lemma, recall Definition 21.10.

Lemma 21.13 $\vdash_M KIND(\Omega(\emptyset, \hat{D}))$.

Proof. Adapt the proofs of Theorem 3 (i)-(iii) of Bjørdal (2012). ■

Lemma 21.14

i) $\vdash_M \emptyset \in \{x|F(x, \hat{D})\}$

ii) $\vdash_M \forall x(x \in \{x|F(x, \hat{D})\} \to \Omega(x, \hat{D}) \in \{x|F(x, \hat{D})\})$

iii) $\vdash_M \forall x(x \in \{x|F(x, \hat{D})\} \to D(x, \hat{D}) \in \{x|F(x, \hat{D})\})$

iv) $\vdash_M \forall x, y(x \in \{x|F(x, \hat{D})\} \land C(y, \hat{D}, x) \to y \in \{x|F(x, \hat{D})\})$

v) $\vdash_M \forall x, y(x \in \{x|F(x, \hat{D})\} \land y \in \{x|F(x, \hat{D})\} \to \{x, y\}^{D} \in \{x|F(x, \hat{D})\})$

vi) $\vdash_M \forall x(x \in \{x|F(x, \hat{D})\} \to \cup x \in \{x|F(x, \hat{D})\})$

Proof. The proof of i) is by invoking modus ascendens and alethic comprehension ($A^C_{M}$) on Lemma 21.12 i). For iii)-vii), invoke also $\mathcal{L}_{M}$ (LO1$_{M}$ of p. 339 of Bjørdal (2012)), and for ii) also Lemma 21.13 is invoked, noting that $\vdash_M \forall x(x \in \Omega(\emptyset, \hat{D}) \to T x \in \Omega(\emptyset, \hat{D}))$. ■
Lemma 21.15 \(\vdash_M \text{KIND}(\{x|F(x, \dot{Z})\})\)

Proof. Adapt the proof of Theorem 3 iii) of Bjørdal (2012) and invoke Lemma 21.14.

Lemma 21.16 \(\vdash_M \text{KIND}(\dot{D})\)

Proof. A consequence of 21.15 given the definition of \(\dot{D}\).

Definition 21.17 Let \(H\) be defined by manifestation point as in section 9 of Bjørdal (2012), so that we can show that \(\vdash_M \forall x(x \in H \leftrightarrow \text{KIND}(x) \land x \subset H)\).

Notice that we here and in the following have “\(H\)” for “\(H\)” to emphasize.

Definition 21.18 The caliber of sorts

A cognomen \(a\) has caliber 0 \(\doteq\ a\) is \(T\) or \(a\) is \(\in\) or for some \(A\) and \(x\), \(a\) is \(\sim A\).

A cognomen \(a\) has caliber \(n + 1\) \(\doteq\ a\) is \(\downarrow bc\) and the maximal caliber of \(b\) and \(c\) is \(n\).

Theorem 21.19 The Regularity Rule for \(H\)

If \(\vdash_M a \in H\) then \(\vdash_M (\exists x)(x \in a) \rightarrow (\exists x)(x \in a \land \forall y(y \in a \rightarrow y \notin x))\)

Proof. We need to extend the argument of Bjørdal (2012), and therefore first quote from its page 353:

“We wrote that \(H\) is a sort of iterative sorts. This holds in the following sense of a regularity rule:

If \(\vdash_M b \in H\) then \(\vdash_M \exists x(x \in b) \supset \exists x(x \in b \land \forall y(y \in b \supset y \notin x))\)

We can justify the regularity rule briefly as follows: Suppose instead that \(\vdash_M b \in H\) and \(\vdash \exists x(x \in b) \land \forall x(x \in b \supset \exists y(y \in b \land y \in x))\). As \(b\) is hereditarily kind it follows that \(\vdash_M \exists x(x \in b) \land \forall x(x \in b \supset \exists y(y \in b \land y \in x))\). But the latter can only be satisfied if \(b\) is circular, a cycle or has an infinitely descending chain. Given the nature of \(H\), it would follow that \(X(0) \vdash \exists x(x \in H)\), which is contrary to our minimalist stipulations. Hence, \(H\) only contains well-founded sorts as maximal members.”

The proof of Bjørdal (2012) only covers sets with caliber 0. Suppose \(n + 1\) is the smallest caliber of a set \(b = \downarrow cd\) such that \(\vdash_M b \in H\) and \(b\) is not regular, i.e. well-founded. But this is impossible as the caliber 0 set \(\{x : x \notin c \land x \notin d\}\) is then also good, i.e. \(\vdash_M \{x : x \notin c \land x \notin d\} \in H\); clearly, however, \(\vdash_M b \subset \{x : x \notin c \land x \notin d\}\), so \(b\) is regular as \(\{x : x \notin c \land x \notin d\}\) is regular.

Postulation 21.20 (The Skolem-Fraenkel Postulation)

If \(b\) is good and \(C(w, \dot{D}, b)\), \(w\) is good. More precisely, \(\vdash_M b \in H \land C(w, \dot{D}, b) \rightarrow w \in H\).

We write “SFP” to alert that the proof depends on the Skolem-Fraenkel Postulation.
Lemma 21.21  Given Postulate 20.1, Postulation 21.20 holds iff $\vdash M \dot{D} \in H$.

Proof. Exercise. □

Remark 21.22  If one should want “higher infinities” in a Skolem relative sense in our framework one will need adequate postulations, in analogy with the content of Lemma 21.21 which we intuit are then equivalent under Postulate 20.1 with the goodness of such a Skolem relative “higher” infinity.

Lemma 21.23  $\vdash M \forall x, y(x \in H \land C(y, \dot{D}, x) \rightarrow y \in H)$

Proof. Exercise. SFP. □

Lemma 21.24  $\vdash M \forall x(x \in H \rightarrow \dot{D}(x, \dot{D}) \in H)$

Proof. Exercise. SFP. □

Theorem 21.25  $\vdash M \dot{D} \in H$

Proof. Exercise. SFP. □

Theorem 21.26  If $A(x)$ is any formula with $x$ free and only $\in$ as its non-logical symbol then $\vdash M a \equiv b \rightarrow (\dot{A}(a) \rightarrow \dot{A}(b))$.

Proof. Exercise. SFP. □

Let the following be an axiomatization of ZFC, where the identity sign of A7 abbreviates the consequent of A8:

\[
\begin{align*}
A1 & \quad (\forall a)(\forall b)(\exists x)(a \in x \land b \in x) \\
A2 & \quad (\forall a)(\exists x)(\forall y)(y \in x \leftrightarrow (\exists z)(z \in a \land y \in z)) \\
A3 & \quad (\exists x)(\forall y)(y \in x \leftrightarrow (\forall z)(z \in x \land y \in z)) \\
A4 & \quad (\forall a)(\exists x)(\forall y)(y \in x \leftrightarrow (y \in a \land A(y))) \\
A5 & \quad (\exists x)((\forall y)(y \in x \rightarrow A(y)) \rightarrow A(x)) \rightarrow (\forall x)A(x) \\
A6 & \quad (\forall a)((\forall x)(x \in a \leftrightarrow (\exists y)(A(x, y))) \rightarrow (\exists z)(\forall x)(x \in a \leftrightarrow (\exists y)(y \in z \land A(x, y)))) \\
A7 & \quad (\forall a)((\forall b)(\forall c)(b \in a \land c \in a \rightarrow ((\exists x)(x \in b \land x \in c) \leftrightarrow b = c))) \\
& \quad \rightarrow (\exists b)(\forall c)(c \in a \rightarrow (\exists x)(\forall y)(y \in x \leftrightarrow y \in c \land y \in b)) \\
A8 & \quad (\forall x)(x \in a \leftrightarrow x \in b) \rightarrow (\forall a)(a \in u \leftrightarrow b \in u) \\
A9 & \quad (\forall a)(\exists y)(\forall x)(x \in y \leftrightarrow x \subset a)
\end{align*}
\]

Lemma 21.27  If $A$ is one of $A1$ to $A7$ then $\vdash M \dot{A}$.
Theorem 21.32

Lemma 21.28 (Domination potency)

Proof.

Our strategy will here be to extend the interpretation invoked in the proof of Corollary 21.30 (Weak power) in Friedman (1973) let $A_1$.

Exercise 21.29 Hint: let $z = \{u \in x | u \in y \}.

Corollary 21.30 (Weak power)

Theorem 21.31 $L$ interprets ZF.

Proof. By Corollary 21.30 and Lemma 21.28 $L$ interprets system $S$ of Friedman (1973) which by its Theorem 1 interprets ZF.

Theorem 21.32 $L$ interprets ZFC.

Proof. Our strategy will here be to extend the interpretation invoked in the proof of the previous theorem. Whereas Friedman (1973) uses the identity sign to abbreviate coextensionality, use e.g. $\sim$ if needed in unravelling to interpret. Adapting Definition 3 in Friedman (1973) let $EQR(a)$ abbreviate

$\forall x, y((x, y) \in a \rightarrow (\forall z \in x)(\exists w \in y)((z, w) \in a) \wedge (\forall w \in x)(\exists z \in y)((z, w) \in a))$.

$a \sim b$ abbreviate $(\exists x)(x \in b \wedge EQR(x) \wedge a \in EQR(x))$.

$a_1 \sim b$ abbreviate $\exists x(a \sim a \wedge x \in b)$.
In the following we establish the standard Zermelian version, dubbed “the multiplicative axiom” by Russell and others, for \( e^* \) of the axiom of choice which is equivalent with our \( A7 \) above under the other principles of \( ZF \): We assume that a set \( c \) is such that \( \vdash c \in \check{D} \land \forall a(a \in e^* c \rightarrow \exists x(x \in e^* a)) \) and \( \vdash c \in \check{D} \land \forall b(b \in e^* c \rightarrow \exists x(x \in e^* a \land x \in e^* b)) \). As \( \vdash_M \check{D} \in \check{H} \), then \( \vdash_M c \in \check{D} \land \forall a(a \in e^* c \rightarrow \exists x(x \in e^* a)) \) and \( \vdash_M c \in \check{D} \land \forall a(b \in e^* a \land b \in e^* c \land \exists x(x \in e^* a \land x \in e^* b) \rightarrow (\forall y)(y \in \check{D} \rightarrow (a \in e^* y \leftrightarrow b \in e^* y)) \). Obviously \( a \in e^* c \) entails \( a \in e^* c \), and obviously \( \exists x(x \in e^* a) \) entails \( \exists x(x \in a) \), so \( \vdash_M \forall a(a \in e^* c \rightarrow \exists x(x \in a)) \). Clearly \( \vdash_M a \in c \land b \in c \land \exists x(x \in a \land x \in b) \) entails \( \vdash_M a \in e^* c \land b \in e^* c \land \exists x(x \in e^* a \land x \in e^* b) \), and we observe that \( \vdash_M (\forall y)(y \in \check{D} \rightarrow (a \in e^* y \leftrightarrow b \in e^* y)) \) entails \( \vdash_M (\forall y)(y \in \check{D} \rightarrow (a \in y \leftrightarrow b \in y)) \). This establishes that \( \vdash_M \forall a(a \in e^* c \rightarrow \exists x(x \in a)) \) and \( \vdash_M \forall a \exists b(a \in e^* c \land b \in e^* c \land \exists x(x \in e^* a \land x \in e^* b)) \). By the multiplicative axiom for \( e^* \), i.e. also \( A7 \), \( \vdash_M \exists d(d \in \check{D} \land \forall a(a \in e^* c \rightarrow \exists x(x \in a \land x \in d)) \land \forall a(a \in e \rightarrow \forall x \forall y(x \in a \land x \in d \land y \in a \land y \in d \rightarrow x \in y)) \). Let \( e \) be a witness and consider first \( \vdash_M \forall a(a \in e \rightarrow \exists x(x \in a \land x \in e)) \). Suppose \( \vdash_M b \in e^* c \). Then for some \( a \), \( a \sim b \), \( \vdash_M a \in e^* c \) and thus \( \vdash_M \forall a(a \in e^* c \rightarrow \exists x(x \in a \land x \in e)) \). By Lemma 14 of Friedman (1973) \( \vdash_M a \sim b \leftrightarrow (\forall x)(x \in e^* a \leftrightarrow x \in e^* b) \). Hence \( \vdash_M (\forall x)(x \in e^* b \land x \in e^* c) \). So \( \vdash_M \forall a(a \in e^* c \rightarrow \exists x(x \in e^* a \land x \in e^* e)) \). Consider next \( \vdash_M \forall a(a \in e \rightarrow \forall x \forall y(x \in a \land x \in e \land y \in a \land y \in e \rightarrow x \in y)) \). Suppose \( \vdash_M b \in e^* c \). Then for some \( a \), \( \vdash_M a \sim b \) and \( \vdash_M a \in e^* c \). So \( \vdash_M (\forall x)(\forall y)(x \in a \land x \in e \land y \in a \land y \in e \rightarrow x \in y)) \). Suppose \( \vdash_M f \in e^* a \land f \in e^* e \land g \in e^* a \land g \in e^* e \). Then for some \( h \) and \( i \), \( \vdash_M h \sim f \sim h \in a \land h \in e \) and \( \vdash_M i \sim g \sim i \in a \land i \in e \). Then \( \vdash_M h = i \). Given Theorem 21.26 \( \vdash_M f \sim g \). By adapting Lemma 19 of Friedman (1973), \( \vdash_M f \notin g \). So \( \vdash_M \forall u(u \in \check{D} \rightarrow (f \in e^* u \leftrightarrow g \in e^* u)) \). Consequently, \( \vdash_M \forall a(a \in e^* c \rightarrow \forall x \forall y(x \in e^* a \land x \in e^* e \land y \in a \land y \in e \rightarrow \forall u(u \in \check{D} \rightarrow (f \in e^* u \rightarrow g \in e^* u))) \). As \( \vdash_M c \in \check{D} \), \( \vdash_M \{x|\exists y(y \in c \land T^x \in y \land \forall z(T^x \in y \rightarrow z \in x))\} \in \check{D} \). By adapting Proposition 11.3 \( \{x|\exists y(y \in c \land T^x \in y \land \forall z(T^x \in y \rightarrow z \in x))\} \) can stand in for \( e \) so that \( \vdash_M (\exists d(d \in \check{D} \land \forall a(a \in e^* c \rightarrow \exists x(x \in e^* a \land x \in e^* d)) \land \forall a(a \in e^* c \rightarrow \forall x \forall y(x \in e^* a \land x \in e^* d \land y \in e^* a \land y \in e^* d \rightarrow \forall u(u \in \check{D} \rightarrow (f \in e^* u \leftrightarrow g \in e^* u)))) \). SFP. ■

### 22 Conclusion

We suggest that \( \mathcal{L} \) justifies useful set theoretic and mathematical principles more appropriately than alternative accounts. The definable echelon \( \check{D} \) may serve as an attractive arena for definable mathematical analysis. We are interested in how far \( \mathcal{L} \) can account for higher infinities, in a relative sense complying with that which was revealed by my compatriot Thoralf Skolem; this would inter alia be by means of the manifestation point construction going back via the set theoretic or property theoretic presentation of Cantini (1996) to its semantical origin (?) in Visser (1983). Moreover, we speculate on whether and if so how category theory may be thought of as librationistically. We are also concerned with a more general quest as to how we best can think of concrete crowds (properties), queues (instances of relations), crowds of queues (relations) and individuals with an extension of \( \mathcal{L} \) that lets us think about them in a type free, adicity liberal
and order eased manner. There are, I think, other fundamental questions that may be
dressed advantageously if librationistically.

Bibliography

Bjørdal, F. (2011). Considerations Contra Cantorianism. In Pelis, M. and Puncochar,
V., editors, The LOGICA Yearbook 2010, pages 43–52. College Publications, London
2011.

Bjørdal, F. (2012). Librationist Closures of the Paradoxes. Logic and Logical Philosophy,
21(4):323–361.

Bjørdal, F. (2013). A Sober Librationist Interpretation of ZF. Lecture for the Non-
Classical Mathematics workshop at the 4th World Congress and School on Uni-
versal Logic in Rio de Janeiro, Brasil, March 29 - April 7, 2013.

Bjørdal, F. (2014). Definable real numbers with librationist capture and domination.
Lecture for the Encontro Brasileiro de Lógica in Petrópolis, Brazil, April 8th 2014.

Cantini, A. (1996). Logical Frameworks for Truth and Abstraction. Elsevier.

Friedman, H. (1973). The Consistency of Classical Set Theory Relative to a Set Theory
with Intuitionistic Logic. The Journal of Symbolic Logic, 38(2):315–319.

Gupta, A. (1982). Truth and paradox. Journal of Philosophical Logic, 11(1):1–60.

Halbach, V. and Leigh, G. E. (2014). Axiomatic theories of truth. In Zalta, E. N., editor,
The Stanford Encyclopedia of Philosophy. http://plato.stanford.edu/archives/sum2014/entries/truth-axiomatic/
summer 2014 edition.

Herzberger, H. (1982). Notes on Naive Semantics. Journal of Philosophical Logic,
11(1):61–102.

Kripke, S. (1975). Outline of a Theory of Truth. Journal of Philosophy, 72(19):690–716.

Kunen, K. (1980). Set theory - An Introduction to Independence Proofs. Elsevier.

Scott, D. (1966). More on the Axiom of Extensionality. In Y. Bar Hillel, e. a., edi-
tor, Essays on the Foundations of Mathematics, pages 115–131. Hebrew University,
Jerusalem.

Smorynsky, C. (1977). The Incompleteness Theorems. In Barwise, J., editor, Handbook
of Philosophical Logic, pages 821–865. North Holland.

Visser, A. (1989). Semantics and the liar paradox. In Handbook of Philosophical Logic,
volume 4, pages 617–706.