Point Enclosure Problem for Homothetic Polygons

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Abstract

In this paper, we investigate the homothetic point enclosure problem: given a set \( S \) of \( n \) triangles with sides parallel to three fixed directions, find a data structure for \( S \) that can report all the triangles of \( S \) that contain a query point efficiently. The problem is “inverse” of the homothetic range search problem. We present an \( O(n \log n) \) space solution that supports the queries in \( O(\log n + k) \) time, where \( k \) is the output size. The preprocessing time is \( O(n \log n) \). The same results also hold for homothetic polygons.

1 Introduction

Point enclosure is one of the fundamental problems that has been well studied in Computational Geometry \cite{1, 4, 8, 10}. The problems of this type, typically, are formulated as follows:

Preprocess a set \( S \) of input geometrical objects so that given a query point \( q \), the objects of \( S \) containing the query point \( q \) can be reported or counted efficiently.

In the counting version of the problem, the goal is to report the number of objects in \( S \) containing a query point instead of the objects themselves. The interval overlapping problem \cite{1, 2} and inverse range reporting problem \cite{2, 10} are extensively studied problems of this type.

In this paper, we consider the following point enclosure problem. Given a set \( S \) of \( n \) triangles of the same shape in the plane, preprocess \( S \) so that a given query point \( q \) in the plane, all the triangles \( T \) of \( S \) with \( T \cap q \neq \emptyset \) can be reported.

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efficiently. The trivial way of doing this is to sequentially process each triangle of $S$ and check whether it contains the query point or not, which will take linear time. Note that this solution takes $\Omega(n)$ time in the worst case even when no triangle contains the query point. We achieve a solution with $O(\log n + k)$ query time, with $O(n \log n)$ preprocessing time and $O(n \log n)$ space.

Chazelle et al.\cite{3} considered its dual or inverse, namely, homothetic range search problem. The point enclosure problems with orthogonal input objects have been well studied\cite{1, 2, 10}. The point enclosure problem with input triangles has been studied with different constraints—Katz\cite{7} and Gupta\cite{5} provided solutions for the problem when input triangles are fat triangles. Sharir\cite{9} proposed a randomized solution for a more general problem. We are not aware of any work which deals with homothetic input triangles and provides a deterministic solution. Guting\cite{6} gave an optimal time solution for its counting version.

2 A solution for Right-angled Isosceles Triangles

First, we consider a simpler variant of the problem in which all the input triangles are right-angled isosceles triangles with equal sides parallel to the axes. We present a $O(n \log n)$ space solution for the variant that supports query in $O(\log n + k)$ time, where $k$ is the output size. We can obtain a solution for the original homothetic point enclosure problem for triangles, by using suitable linear transformation(s).

To obtain a solution for the simpler variant, consider a point $p$ and a right-angled isosceles triangle $T$ with equal sides parallel to the axes of the plane. The point $p$ can not be contained by the triangle $T$ if their horizontal or vertical projections on the axes are disjoint. We also observe that a triangle whose horizontal projection contains the $x$-coordinate of point $p$ will contain the point $p$ if and only if the point $p$ lies between its horizontal side and hypotenuse. We use these observations to design our solution.

We call the orthogonal projection of a triangle $T$ on the $X$-axis its $x$-interval, denoted by $T_h$. The $y$-interval of a triangle $T$ is analogously defined which is denoted by $T_v$. We denote the output size by $k(Q)$, where $Q$ is a query point. We omit $Q$ where the context is clear.

A right-angled triangle is called an axis-parallel right-angled triangle if its sides incident at the right angle are parallel to axes.

A triangle $T$ is said to be horizontally closer to a point $p$, if $T_h \cap p_x \neq \phi$. The vertical closeness of a triangle for a point is analogously defined.

Let $S$ be a set of $n$ homothetic right-angled isosceles triangles. Without loss of generality, we assume that the triangles in the set $S$ are axis-parallel triangles and present in the first quadrant of the plane. A point $q = (q_x, q_y)$ lies in a triangle $T$ of $S$ if and only if $q$ is horizontally closer to $T$ and $q_y$ lies on the line segment of the line $x = q_x$ inside the triangle $T$.

Based on the $x$-intervals of the triangles of $S$, we construct a one-dimensional segment tree\cite{1}, denoted by $T$. Each node $v$ of the segment tree $T$ corresponds
to a closed vertical slab \( H(v) \). We say that a triangle \( T \) of \( S \) crosses \( H(v) \) if \( T \) intersects \( H(v) \) and none of its vertices lie in the interior of \( H(v) \). Let \( S(v) \) be the subset of triangles of \( S \) that cross \( H(v) \) but do not cross \( H(u) \), where \( u \) is the parent of \( v \) in \( T \). There are \( O(\log n) \) nodes \( v \) such that \( T \in S(v) \), for a triangle \( T \) in \( S \). Moreover, any triangle is included in \( O(\log n) \) parts, one in each slab. The size of the segment tree \( T \) is \( O(n \log n) \).

Consider a triangle \( T \in S(v) \), for some node \( v \). The portion of \( T \) (trapezoid) lying inside the slab \( H(v) \) can be partitioned into a right-angled triangle and a (possibly empty) rectangle. Formally, for each node \( v \) and each triangle \( T \) of \( S(v) \), we define the trimmed triangle for \((T, v)\) as the right-angle triangle in \( H(v) \cap T \) that has the points of intersections of the hypotenuse of \( T \) with the vertical sides of the slab \( H(v) \) as the endpoints of its hypotenuse. The remaining portion of \( H(v) \cap T \) is a (possibly empty) rectangle lying below the trimmed triangle. We call it the trimmed rectangle for \((T, v)\).

Observe that

**Lemma:** Let \( T \) be a triangle in \( S(v) \), for some node \( v \) in the segment tree \( T \). The trimmed triangle for \((T, v)\) is always a right-angle isosceles triangle and all the trimmed triangles corresponding to a node \( v \) are congruent.

For each node \( v \) in the segment tree \( T \), we associate two structures. One would return those trimmed triangles of \( v \) that contain the query point while the other one would return the trimmed rectangles containing the query point. The data structure is built as follows.

1. Build a segment tree, denoted by \( T \), for the \( x \)-intervals of the triangles present in \( S \).
2. If triangle \( T \in S(v) \), then compute the trimmed triangle and rectangle for \((T, v)\). For each trimmed triangle and rectangle, we maintain a pointer to the defining triangle.
3. For each node \( v \) in \( T \):
   
   (a) The trimmed triangles are stored in a list sorted by the ordinates of their horizontal sides. We denote the sorted list by \( L(v) \).
   (b) Preprocess the set of the trimmed rectangles corresponding to \( S(v) \) into an interval overlapping structure\(^2\), denoted by \( I(v) \).

Queries are processed as follows. Let \( q = (q_x, q_y) \) be a query point.

1. Find the search path for \( q_x \) in the segment tree \( T \).
2. For each node \( v \) on the search path
2.1 Find the trimmed rectangles (and hence the corresponding triangles) of $S(v)$ that contains the point $q$ by querying the structure $I(v)$ with $q_y$ using the method of Chazelle\[2\].

2.2 Using the binary search in the sorted list, find the position of $q_y$ in the sorted list $L(v)$.

2.3 While moving to the left in the list $L(v)$, report all the trimmed triangles (and hence corresponding original triangle) until we get a trimmed triangle for which does not contain the query point.

The data structure can be constructed in $O(n \log n)$ space and time. The query time of the algorithm is $O(\log^2 n + k)$. Using the fractional cascading technique\[4\], we can improve the query time by a log factor. Hence, we have the following result.

**Theorem:** We can compute a data structure for a given set of $n$ right-angled isosceles triangles with the same orientation so that, for a given query point, it can report all the triangles of the input set containing the query point in $O(\log n + k)$ time. The structure can be constructed in $O(n \log n)$ space and time.

Using suitable transformations, we can easily obtain a solution for the problem with homothetic input triangles. The bounds will remain intact.

By triangulating input polygons, the result can be extended to the point enclosure problem for homothetic polygons.
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