Filters and Functions in Multi-scale Constructions: Extended Abstract

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Abstract

We derive results about geometric means of the Fourier modulus of filters and functions related to refinable distributions with arbitrary dilations and translations. Then we develop multi-scale constructions for dilations by Pisot-Vijayaraghavan numbers and translations in associated quasilattices.

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1 Introduction and Statement of Results

In this paper \( \mathbb{Z}, \mathbb{N} = \{1, 2, 3, \ldots\}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) denote the integer, natural, rational, real, and complex numbers and \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) denotes the circle group.

Let \( \lambda \in \mathbb{R} \setminus [-1, 1], a_1, \ldots, a_m \in \mathbb{C} \setminus \{0\}, a_1 + \cdots + a_m = |\lambda|, \tau_1 < \cdots < \tau_m \in \mathbb{R} \), and

\[
A(y) = |\lambda|^{-1} \sum_{j=1}^{m} a_j e^{2\pi i \tau_j y}, \quad y \in \mathbb{R}.
\]  

(1)

There exists a unique compactly supported distribution \( f \) that satisfies \( \int f(x) dx = 1 \) and the refinement equation

\[
f(x) = \sum_{j=1}^{m} a_j f(\lambda x - \tau_j)
\]  

(2)

The Fourier transform \( \hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x y} dx \) satisfies \( \hat{f}(0) = 1 \) and admits the product expansion

\[
\hat{f}(y) = \prod_{k=1}^{\infty} A \left( y \lambda^{-k} \right).
\]  

(3)

A(\(y\)) and \( \hat{f}(y) \) are the filters and functions of this paper. Equation (3) implies that they are related by the functional equation

\[
\hat{f} \left( y \lambda^k \right) = \hat{f}(y) \prod_{j=1}^{k} A \left( y \lambda^j \right), \quad k \in \mathbb{N}.
\]  

(4)

The Mahler measure \( [25] \) (or height) \( M(P) \) of a polynomial \( P(z_1, \ldots, z_d) \) is defined by

\[
M(P) = \exp \int_{t_1=0}^{1} \cdots \int_{t_d=0}^{1} \ln |P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d})| dt_1 \cdots dt_d.
\]  

(5)

For \( P(z) = c(z-\omega_1) \cdots (z-\omega_q) \) Jensen’s theorem [14] gives \( M(P) = |c| \prod_{j=1}^{q} \max \{ 1, |\omega_j| \} \).

Mahler measure arises in prediction theory of stationary random processes [17], algebraic dynamics [11], and in Lehmer’s open problem in number theory [24].

A(\(y\)) is an almost periodic function in the (uniform) sense of Bohr [3]. Let \( d \) be the
maximal number of linearly independent $\tau_1, ..., \tau_n$ over $\mathbb{Q}$. $A(y)$ is periodic if and only if $d = 1$. There exists a polynomial $P(z_1, ..., z_d)$ and real numbers $r_1, ..., r_d$ such that

$$A(y) = P(e^{2\pi i r_1 y}, ..., e^{2\pi i r_d y}), \quad y \in \mathbb{R}. \quad (6)$$

**Theorem 1** $\ln |A(y)|$ is almost periodic in the (mean squared) sense of Besicovitch [2]. Furthermore

$$\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \ln |A(y)| = \ln M(P). \quad (7)$$

**Definition 1** The Mahler measure of $A$ is $M(A) = M(P)$.

**Theorem 2** Define $\rho(f) = -\ln M(A)/\ln \lambda$.

$$\lim_{L \to \infty} \frac{1}{2L \ln \lambda} \int_{-L}^{L} \ln |\hat{f}(y)| \, dy = -\rho(f). \quad (8)$$

The Hausdorff dimension of a Borel measure is defined in ([30], p. 6) as

$$\dim(\nu) = \inf \{ \dim(B) : B \text{ is a Borel subset of } \mathbb{R} \text{ and } \nu(\mathbb{R}\setminus B) = 0 \} \quad (9)$$

where $\dim(B)$ is the Hausdorff dimension of $B$.

**Conjecture 1** If $f$ is a Borel measure then $\rho(f) = \dim(f)$.

If conjecture [1] is validated we propose to call $\rho(f)$ the Hausdorff dimension of $f$.

An algebraic integer $\lambda$ is a Pisot-Vijayaraghavan (PV) number if the roots $\lambda = \lambda_1, ..., \lambda_n$ of its minimal polynomial $\Lambda(z)$ satisfy $|\lambda_j| < 1$, $j = 2, ..., n$, and $|\lambda_j| = 1$, $j = 1$. They were discovered by Thue [37], rediscovered by Hardy [13] who with Vijayaraghavan studied their Diophantine approximation [7] properties, and studied by Pisot [31] in his dissertation. Examples are:

| $\lambda$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | minimal polynomial |
|-----------|-------------|-------------|-------------|-------------------|
| 1.6180    | -0.6180     |             |             | $z^2 - z - 1$     |
| -1.6180   | 0.6180      |             |             | $z^2 + z - 1$     |
| 3.4142    | 0.5858      |             |             | $z^2 - 4z + 2$    |
| 2.2470    | 0.5550      | -0.8019     |             | $z^3 - 2z - z + 1$|
| 1.3247    | -0.6624 + 0.5623i | -0.6624 - 0.5623i | $z^2 - z - 1$     |
| 1.3803    | -0.8192     | 0.2194 + 0.9145i | 0.2194 - 0.9145i | $z^4 - z^3 - 1$    |

If $\lambda$ is a PV number, $m = 2$, $a_1 = a_2 = |\lambda|/2$, $\tau_1 = 0$, and $\tau_2 = 1$ then $f$ is a measure defined by a Bernoulli convolution and Erdős [10] proved that there exists $\alpha > 0$ and $\gamma \in \mathbb{C}\setminus\{0\}$ such that

$$\lim_{k \to \infty} \frac{1}{k} \hat{f}(\alpha \lambda^k) = \gamma. \quad (10)$$

Erdős’ proof used Equation [1] and the fact that the set of real $\alpha$, for which there exists a sequence of integers $n_k$ with $|\alpha \lambda^k - n_k| \to 0$ exponentially fast, is dense.

An algebraic integer $\lambda$ is a Salem number if the roots $\lambda = \lambda_1, ..., \lambda_n$ of its minimal polynomial satisfy $|\lambda_j| \leq 1$, $j = 2, ..., n$. Like PV numbers, Salem numbers appear in Diophantine approximation and harmonic analysis [27, 35]. The smallest 1.17628 is the Mahler measure of Lehmer’s polynomial $z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 - z + 1$ and is conjectured to be the smallest known Mahler measure > 1 of any polynomial.

Kahane [15] extended Erdős’ result for $\lambda$ a Salem number and Dai, Feng and Wang [8] extended Kahane’s result for $m > 2$ and $\tau_j \in \mathbb{Z}$.

**Theorem 3** If $\lambda$ is a PV number with degree $n$, $\alpha \in \mathbb{Q}(\lambda)$, $\{\tau_1, ..., \tau_m\} \subset \mathbb{Z} \setminus \{1, \lambda, ..., \lambda^{n-1}\}$, and $f(\alpha \lambda^k) \neq 0$ for all $k \in \mathbb{N}$ then the following limit exists

$$\lim_{k \to \infty} \frac{\ln |\hat{f}(\alpha \lambda^k)|}{k \ln |\lambda|} \quad (11)$$
and equals the mean value of $\ln|P_{\text{trig}}|$ over a finite orbit of $C$ in the $n$-dimensional torus group. Here $P_{\text{trig}}$ is the trigonometric polynomial on the torus constructed from the polynomial $P$ by Equation (27) and the polynomial $P$ is related to $A(y)$ by Equation (2). If $\lambda$ is not an integer then $\alpha$ can be chosen such that $\lim_{k \to \infty} f(\alpha \lambda^k) = \gamma$ where $\gamma \in \mathbb{C}\setminus\{0\}$ thus extending the results in [3] to noninteger values of $\tau$.

Section 2 reviews notation and analytical results for Theorems 1 and 2 which are proved in Section 3 and algebraic results for Theorem 3 which is proved in Section 3, and for Lemma 2 which is proved in Section 3. Section 4 constructs multi-scale analyses for spaces of distributions based on translations by points in quasi-lattice subsets of $\mathbb{R}$.

2 Preliminary Results

Let $\mu$ be Lebesgue measure. For a function $K : I \to \mathbb{C}$ on an interval $I \subseteq \mathbb{R}$ let $K'$ be its derivative. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is $n$-dimensional Euclidean vector space with norm $||\cdot||$, $\mathbb{Z}^n$ is a rank $n$ lattice subgroup, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the $n$-dimensional torus group equipped with Haar measure, and $p_n : \mathbb{R}^n \to \mathbb{T}^n$ is the canonical epimorphism. Vectors are represented as column vectors and $T$ denotes transpose. We define $||\cdot|| : \mathbb{T}^n \to \mathbb{R}$ by

$$||g||_{\mathbb{T}^n} = \min\{||x|| : x \in \mathbb{R}^n \text{ and } p_n(x) = g\}, \quad g \in \mathbb{T}^n.$$

Lemma 1 (Titchmarsh) If $h$ is a compactly supported distribution and $[a, b]$ is the smallest interval containing the support of $h$ then

$$H(z) = \int_a^b h(t) e^{2\pi iz} dt \quad (12)$$

is an entire function of exponential type. Moreover the number $n(r)$ of zeros of $H(z)$ in the disk $|z| < r$ satisfies

$$\lim_{r \to \infty} \frac{n(r)}{r} = 2(b - a). \quad (13)$$

Titchmarsh’s proof (Theorem IV, [3]) assumed that $h$ was a Lebesgue integrable function but a simple regularization argument implies that if $h$ is a compactly supported distribution then Equation (13) holds. Therefore there exists $\beta > 0$ such that

$$\text{card}\{y : y \in [-L, L] \text{ and } H(y) = 0\} \leq \beta L, \quad L \geq 1 \quad (14)$$

Lemma 2 Let $I = [a, b]$ and $K : I \to \mathbb{C}$ be differentiable such that the image $K'(I)$ of $I$ under $K'$ is contained in a single quadrant of $\mathbb{C}$. If $u > 0, v > 0$ and $|K| \leq v$ over $I$ and $|K'| \geq u$ over $I$ then

$$\mu(I) \leq \frac{2\sqrt{2}v}{u}. \quad (15)$$

Proof We present the proof that we gave in ([19], Lemma 1). Since $u \leq |K'|$ over $I$ and $\mu(I) = b - a$, the triangle inequality $|K'| \leq |\Re K'| + |\Im K'|$ gives

$$u \mu(I) \leq \int_a^b |K'(y)| dy \leq \int_a^b (|\Re K'(y)| + |\Im K'(y)|) \, dy.$$

Since $K'(I)$ is contained in a single quadrant of $\mathbb{C}$ there exist $c \in \{1, -1\}, d \in \{1, -1\}$ such that $|\Re K'(y)| = c \Re K'(y)$ and $|\Im K'(y)| = d \Im K'(y)$ for all $y \in I$. Therefore

$$\int_a^b (|\Re K'(y)| + |\Im K'(y)|) \, dy = (c \Re K(b) + d \Im K(b)) - (c \Re K(a) + d \Im K(a)).$$

Since $|c| = |d| = 1$ and $|K(y)| \leq v$ over $I$, the quantity on right is bounded above by $2\sqrt{2}v$. Combining these three inequalities completes the proof.
Lemma 3  There exists a sequence $c_2, c_3, ... > 0$ such that for all monic polynomials $P(z)$ with $k \geq 2$ nonzero coefficients

$$
\mu(\{ y \in [0, 1] : |P(e^{2\pi i y})| \leq v \}) \leq c_k v^{1/(k-1)}, \quad v > 0.
$$

(16)

Proof  We proved this in (Theorem 1, [19]) using induction on $k$, lemma 2, and the fact that a polynomial of degree $q$ can have at most $q$ distinct roots.

Remark  We used Lemma 3 and the method we developed in [18] and Boyd developed in [6] to prove a conjecture that he formulated in [6]. The conjecture expresses the Mahler measure of a multidimensional polynomial as a limit of Mahler measures of univariate polynomials and provides an alternative proof of the special case of Lehmer’s conjecture proved in [6]. The conjecture is stated in http://en.wikipedia.org/wiki/Mahler_measure

Lemma 4  There exists a sequence $c_2, c_3, ... > 0$ such that

$$
\mu(\{ y \in [-L, L] : |A(y)| \leq v \}) \leq L c_m v^{1/(m-1)}, \quad L \geq 1, \quad v > 0.
$$

(17)

Proof  We observe that Lemma 4 concerns (not necessarily periodic) trigonometric polynomials. The conclusion is obvious for $m = 2$ so we assume that $m \geq 3$ and proceed by induction on $m$ using a technique similar to the one we used to prove Lemma 3. Since $|e^{-2\pi i A(y)}| = |A(y)|$ we may assume without loss of generality that $0 = \tau_1 < \cdots \tau_m$. Then $A'(y)$ is a trigonometric polynomial with $m - 1$ terms so by induction there exists $c_{m-1} > 0$ such that

$$
\mu(\{ y \in [-L, L] : |A'(y)| \leq u \}) \leq L c_{m-1} u^{1/(m-2)}, \quad L \geq 1, \quad u > 0.
$$

(18)

We observe that $A$ and $A'$, their real and imaginary parts, and their squared moduli are restrictions of functions of exponential type so they satisfy the hypothesis in Lemma 1. Therefore Equation (18) implies that there exists $\beta > 0$ such that for all $u > 0$, $v > 0$ and $L \geq 1$ the subset of $[-L, L]$ where $|A'(y)| \geq u$ and $|A(y)| \leq v$ can be expressed as the union of not more than $\beta L$ closed intervals $I$ such that $A'(I)$ is contained in a single quadrant of $\mathbb{C}$. Therefore Lemma 2 implies that there exists $\beta > 0$ such that

$$
\mu(\{ y \in [-L, L] : |A(y)| \leq v \quad \text{and} \quad |A'(y)| \geq u \}) \leq \frac{L \beta v}{u}, \quad L \geq 1, \quad u > 0, \quad v > 0.
$$

(19)

Combining Equations (18) and (19) gives for every $L \geq 1$, $v > 0$

$$
\mu(\{ y \in [-L, L] : |A(y)| \leq v \}) \leq L \min_{u > 0} \left[ c_{m-1} u^{1/(m-2)} + \frac{\beta v}{u} \right] = L c_m v^{1/(m-1)}
$$

(20)

where $c_m = (1 + \beta(m-2)) \left[ \frac{c_{m-1}}{\beta(m-2)} \right]^{m-2}.$

For $d \in \mathbb{N}$ and $r = [r_1, \ldots, r_d]^T \in \mathbb{R}^d$ we define the homomorphism $\Psi_r : \mathbb{R} \rightarrow \mathbb{T}^d$ by

$$
\Psi_r(y) = p_d(yr), \quad y \in \mathbb{R}
$$

(21)

where $p_d : \mathbb{R}^d \rightarrow \mathbb{T}^d$ is the canonical epimorphism.

Lemma 5  (Bohl–Sierpinski–Weyl)  The image of $\Psi_r$ is a dense subgroup of $\mathbb{T}^d$ if and only if the components $r_1, \ldots, r_d$ of $r$ are linearly independent over $\mathbb{Q}$. If they are independent then for every continuous function $S : \mathbb{T}^d \rightarrow \mathbb{C}$

$$
\lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L S \circ \Psi_r(y) dy = \int_{\mathbb{T}^d} S(y) dy
$$

(22)

where $S \circ \Psi_r : \mathbb{R} \rightarrow \mathbb{C}$ is the composition of $S$ with $\Psi_r$, and $dy$ is Haar measure on $\mathbb{T}^d$.

Proof  Arnold ([1], p. 285–289) discuss the significance and gives Weyl’s proof of this classic result called the Theorem on Averages and asserts it “may be found implicitly in the work of Laplace, Lagrange, and Gauss on celestial mechanics” and “A rigorous proof was given only in 1909 by P. Bohl, V. Sierpinski, and H. Weyl in connection with a problem
of Lagrange on the mean motion of the earth’s perihelion.”

The analytic results above suffice to prove Theorems 1 and 2. The following algebraic results are required to prove Theorem 3 and Lemma 9.

If \( \lambda \) is an algebraic integer with minimal polynomial \( \Lambda(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0 \), \( c_j \in \mathbb{Z} \) that has roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \), then the companion matrix

\[
C = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
-c_0 & -c_1 & \cdots & \cdots & -c_{n-1}
\end{bmatrix},
\]

Vandermonde matrix

\[
V = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{bmatrix},
\]

and diagonal matrix \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) satisfy

\[
C^k V = V D^k, \quad k \in \mathbb{N}.
\]

The matrix \( VV^T \) is invertible since \( \det VV^T = |c_0|^2 \) and it has integer entries since

\[
(VV^T)_{i,j} = \sum_{t=1}^{n} \lambda_t^{i+j-2} = \text{trace } C^{i+j-2}, \quad i, j \in \{1, \ldots, n\}.
\]

Define \( v = [1, \lambda_1, \ldots, \lambda_1^{n-1}]^T \). For \( q \in \mathbb{Q}^n \) let \( u = VV^T - q = [\alpha_1, \ldots, \alpha_n]^T \). Then \( \alpha_j \in \mathbb{Q}(\lambda_j) \), \( j = 1, \ldots, n \) and \( Vu = q \). If \( \alpha \in \mathbb{Q} \) then there exists \( b \in \mathbb{Q}^T \) such that \( \alpha = v^T b \). Then \( u = V^T b = [\alpha_1 \ldots \alpha_n]^T \) satisfies \( \alpha = \alpha_1, \alpha_j \in \mathbb{Q}(\lambda_j) \), \( j = 1, \ldots, n \) and \( q = Vu \in \mathbb{Q}^n \).

Now assume that \( \lambda \) is a PV number. Then Equation (24) implies that

\[
\| C^k q - \alpha \lambda^k v \| \to 0
\]

where the convergence is exponentially fast. Then Equation (25) implies that

\[
\| p_n(C^k q - \alpha \lambda^k v) \|_{\mathbb{T}^n} = \| C^k p_n(q) - p_n(\alpha \lambda^k v) \|_{\mathbb{T}^n} \to 0.
\]

Since \( p_n(\mathbb{Q}^n) \) is the set of preperiodic points of the endomorphism \( C : \mathbb{T}^n \to \mathbb{T}^n \), if \( \alpha \in \mathbb{Q}(\lambda) \) then the sequence \( p_n(\alpha \lambda^k v) \in \mathbb{T}^n \) converges exponentially fast to a periodic orbit of \( C \). Conversely every periodic orbit is the limit of such a sequence.

**Remark 1** We suggest further study of the preperiodic points of \( C \) using results from integral matrices discussed by Newman [29], the open dynamical version of the Manin-Mumford Conjecture solved by Raynaud [26], [27], and connections between Mahler measure and torsion points developed by Le [25]. We also suggest extensions obtained by replacing \( \mathbb{R}^n \) by certain Lie groups (stratified nilpotent groups with rational structure constants) considered in [20].

The following result, proved by Minkowsky in 1896 [28], is the foundation of the geometry of numbers ([12], II.7.2, Theorem 1), ([26], Theorem 4.7)

**Lemma 6** (Minkowski) If \( X \subset \mathbb{R}^n \) is convex, \( X = -X \), and the volume of \( X \) exceeds \( 2^n \) then \( X \) contains a point in \( \mathbb{Z}^n \setminus \{0\} \).
3 Proofs

For every nonzero polynomial \( P(z_1, \ldots, z_d) \) we define the trigonometric polynomial \( P_{\text{trig}} : \mathbb{T}^d \to \mathbb{C} \) by

\[
P_{\text{trig}}(p_d(t_1, \ldots, t_d^T)) = P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d}).
\]

and observe through a simple induction based computation that \( \ln |P_{\text{trig}}| \in L^2(\mathbb{T}^d) \). Henceforth we assume that \( r = [r_1, \ldots, r_d] \in \mathbb{R}^d \) and \( P(z_1, \ldots, z_d) \) are chosen to satisfy Equation [1]. Therefore \( r_1, \ldots, r_d \) are linearly independent over \( \mathbb{Q} \) and moreover

\[
A(y) = P_{\text{trig}} \circ \Psi_r(y), \quad y \in \mathbb{R}.
\]

**Proof of Theorem 1** For all \( v > 0 \) define \( S_v : \mathbb{T}^d \to \mathbb{R} \) by

\[
S_v(g) = \ln \left( \max \{v, |P_{\text{trig}}(g)| \} \right).
\]

Lebesgue’s dominated convergence theorem implies that

\[
\lim_{v \to 0} \int_{\mathbb{T}^d} |S_v(g) - \ln |P_{\text{trig}}(g)||^2 dg = 0.
\]

Lemma [4] implies that

\[
\lim_{v \to 0} \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^L \ln |A(y)| - S_v \circ \Psi_r(y)|^2 dy = 0
\]

and hence \( \ln |A(y)| \) is almost periodic in the sense of Besicovitch. We combine Equations [29] and [30] with Lemma [5] to complete the proof by computing

\[
\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^L \ln |A(y)| dy = \lim_{v \to 0} \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^L S_v \circ \Psi_r(y) dy = \int_{\mathbb{T}^d} \ln |P_{\text{trig}}(g)| dg = M(P).
\]

**Remark** Since the zero set of \( P_{\text{trig}} \) is a real analytic set, an alternative proof based on Lojasiewicz’s structure theorem \([16], [22]\) for real analytic sets may be possible using methods that we developed in \([21]\) to prove the Lagarias-Wang Conjecture.

**Proof of Theorem 2** Let \( L = b|\lambda|^k \) with \( k \in \mathbb{N} \) and \( b \in [|\lambda^{-1}|, 1) \) then use the functional Equation [2] and Theorem [1] to compute

\[
\lim_{k \to \infty} \frac{1}{2k|\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} j |\lambda|^{j+1} \ln \left( \frac{|f(y)\hat{f}(-y)|}{|f(y)|} \right) dy
\]

\[
= \lim_{k \to \infty} \frac{1}{2k|\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} |\lambda|^j \frac{1}{j} \ln \left( \frac{|f(ub|\lambda|^{j})\hat{f}(-ub|\lambda|^{j})|}{|f(y)|} \right) du
\]

\[
= \lim_{k \to \infty} \frac{1}{2k|\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} |\lambda|^j \frac{1}{j} \ln \left( \frac{|f(ub\lambda^{j})\hat{f}(-ub\lambda^{j})|}{|f(y)|} \right) du
\]

\[
\lim_{k \to \infty} \frac{1}{2k|\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} |\lambda|^j \frac{1}{j} \ln \left( \frac{|f(bu\lambda^{j})\hat{f}(-bu\lambda^{j})|}{|f(y)|} + \sum_{i=1}^{j} \ln |A(ub\lambda^{i})A(-ub\lambda^{i})| \right) du
\]

\[
= \lim_{k \to \infty} \frac{(|\lambda|-1) \ln M(A) \ln |\lambda|}{k|\lambda|^k \ln |\lambda|} \sum_{j=1}^{k-1} j |\lambda|^j
\]

\[
= \lim_{k \to \infty} \frac{(|\lambda|-1) \ln M(A) \ln |\lambda|}{k|\lambda|^k \ln |\lambda|} \left( (-|\lambda|(|\lambda|-1)^{-2}(|\lambda|^k - 1) + k|\lambda|^k(|\lambda|-1)^{-1} \right)
\]

\[
= \frac{\ln M(A)}{\ln |\lambda|}.
\]
Applications of Theorem 2 The following computations support Conjecture 1.

The characteristic function $f$ of $[0,1]$ satisfies $f(x) = f(2x) + f(2x - 1)$. Therefore $A(y) = (1 + e^{2\pi iy})/2$ so $\rho(f) = 1$ which equals the Hausdorff dimension of $f$. We observe that $|\hat{f}(y)| = |\sin y/y|$ decays like $|y|^{-1}$.

The uniform measure $f$ on Cantor’s ternary set satisfies $f(x) = \frac{2}{3}f(3x) + \frac{1}{3}f(3x - 2)$. Therefore $A(y) = (1 + e^{6\pi iy})/2$ so $\rho(f) = \ln 2/\ln 3 \approx 0.6309$ which equals the Hausdorff dimension of $f$. This suggests that $\hat{f}(y)$ decays like $|y|^{-\ln 2/\ln 3}$.

The measure $f$ defined by a Bernoulli convolution and PV number $\lambda$ satisfies $f(x) = \frac{1}{\lambda}f(\lambda x) + \frac{1}{\lambda}f(\lambda x - 1)$. Therefore $A(y) = (1 + e^{2\pi iy})/2$ so $\rho(f) = \ln 2/\ln |\lambda|$ which equals the Hausdorff dimension of $f$ computed by Peres, Schlag and Solomyak [30]. This suggests that $\hat{f}(y)$ decays like $|y|^{-\ln 2/\ln |\lambda|}$.

Let $f$ be a nonzero distribution that satisfies $f(2x) = 2f(2x + 2f(2x - 1) - 2f(2x - 3))$. Therefore $A(y) = 1 + e^{2\pi iy} - e^{4\pi iy}$ so $\rho(f) = -\ln((1 + \sqrt{5})/2)/\ln 2 \approx 0.6942$. This suggests that $|\hat{f}(y)|$ grows like $|y|^{-\rho(f)}$.

Proof of Theorem 3 Uses algebraic results in Section 2 with the use of the functional Equation 4 in the proof of Theorem 2. Details will be provided in the full paper.

4 Multiscale Constructions

Let $\lambda$ be a PV number and let $C, D, V, \lambda = \lambda_1, ..., \lambda_n$ be defined as in Section 2. A vector $\sigma \in \mathbb{R}^n$ is called admissible if $\sigma_1 = 0$, $\sigma_j > 0$ for $j \geq 2$, and whenever $2 \leq j < k \leq n$ and $\lambda_j$ and $\lambda_k$ are nonreal complex conjugate pairs then $\sigma_j = \sigma_k$. We denote the set of admissible vectors by $\mathbb{R}^n_\lambda$. It is an open cone and admits the partial order $\sigma \leq \xi$ if and only if $|\sigma_j| \leq |\xi_j|$, $1 \leq j \leq n$.

Definition 2 The quasilattice corresponding to $\sigma \in \mathbb{R}^n_\lambda$ is

$$\mathcal{L}(\sigma) = \{(V^T\ell)_1 : \ell \in \mathbb{Z}^n \text{ and } |(V^T\ell)_j| < \sigma_j \text{ for all } 2 \leq j \leq n\}.$$ 

Lemma 7 If $\sigma, \xi \in \mathbb{R}^n_\lambda$ then

$$0 \in \mathcal{L}(\sigma) \text{ and } \mathcal{L}(\sigma) = -\mathcal{L}(\sigma),$$

$$\mathcal{L}(\sigma) \subseteq \mathcal{L}(\xi) \text{ if and only if } \sigma \leq \xi,$$

$$\mathcal{L}(\sigma) + \mathcal{L}(\xi) = \mathcal{L}(\sigma + \xi).$$

Proof The first assertion is obvious and the second and third follow from the density assertion of Lemma 5.

Lemma 8 Let $\sigma \in \mathbb{R}^n_\lambda$. If $\ell \in \mathcal{L}(\sigma) \setminus \{0\}$ then $|(V^T\ell)_1| \geq \prod_{j=2}^n \sigma_j^{-1}$. The minimal distance between the points in $\mathcal{L}(\sigma)$ is $2^{1-n} \prod_{j=2}^n \sigma_j^{-1}$.

Proof If $\ell \in \mathbb{Z}^n \setminus \{0\}$ then $\prod_{j=1}^n (V^T\ell)_j \in \mathbb{Z} \setminus \{0\}$ is nonzero and a symmetric polynomial in $\mathbb{Z}[\lambda_1, ..., \lambda_n]$ so is a nonzero integer, implying the first assertion. The second assertion follows since lemma 7 implies that the differences between points in $\mathcal{L}(\sigma)$ are in $\mathcal{L}(2\sigma)$.

Lemma 9 If $\sigma \in \mathbb{R}^n_\lambda$ and $L > |\det V| \prod_{j=2}^n \sigma_j^{-1}$ then $\mathcal{L}(\sigma)$ contains a nonzero point in the interval $(-L, L)$.

Proof Let $X = (V^T)^{-1}[(−L, L) \times (−\sigma_1, \sigma_1) \cdots (−\sigma_n, \sigma_n)]^T$. Then $X = −X$ and

$$\text{volume } X = 2^n |\det V|^{-1} L \prod_{j=2}^n \sigma_j > 2^n.$$ 

The result follows from Minkowski’s Lemma 5 since if $\ell \in \mathbb{Z}^n$ then $(V^T\ell)_1 \in (−L, L) \cap \mathcal{L}(\sigma)$ if and only if $\ell \in X$. 

7
Lemma 10 If $\sigma \in \mathbb{R}^n$ then there exists $M(\sigma) > 0$ such that every open interval of length $M(\sigma)$ contains a point in $\Sigma(\sigma)$.

Proof Let $G = \mathbb{R}^n / (VV^T\mathbb{Z}^n)$. Since $VV^T$ is nonsingular and has integer entries $VV^T\mathbb{Z}^n$ is a rank $n$ subgroup of $\mathbb{Z}^n$ so $G$ is isomorphic to a torus group. Let $q_n : \mathbb{R}^n \to G$ and $\varphi_n : G \to T^n$ be the canonical epimorphisms and observe that $p_n((VV^T)^{-1}x) = \varphi_n \circ q_n(x)$, $x \in \mathbb{R}^n$. (31)

Since the entries of $v = [1, \lambda, ..., \lambda^{n-1}]^T$ are linearly independent over $\mathbb{Q}$ the density assertion in Lemma 5 implies that

$$G = q_n \left( (V [\mathbb{R} \times (-\sigma_1, \sigma_1) \times (-\sigma_n, \sigma_n)]^T) \right).$$

Since $G$ is compact there exists $\kappa > 0$ such that

$$G = q_n \left( (V [(-\kappa, \kappa) \times (-\sigma_1, \sigma_1) \times (-\sigma_n, \sigma_n)]^T) \right).$$

Therefore Equation (31) implies that

$$T^n = \varphi_n(G) = p_n \left( (V^T)^{-1} [(-\kappa, \kappa) \times (-\sigma_1, \sigma_1) \times (-\sigma_n, \sigma_n)]^T \right)$$

and the result follows by choosing $M(\sigma) = 2\kappa$.

Lemma 11 If $|c_0| = 1$ then $\lambda \Sigma(\sigma) = \Sigma([0, \lambda_2 \sigma_2, ..., \lambda_n |\sigma_n|^T])$.

Proof Equation 24 gives $\lambda_j(VT\ell)_j = (VT C\ell\ell)_j$, $j = 1, ..., n$ and hence $\lambda \Sigma(\sigma) = \{ (VT C\ell\ell)_1 : \ell \in \mathbb{Z}^n \text{ and } |(VT C\ell\ell)_j| < |\lambda_j| \sigma_j \text{ for } j = 2, ..., n \} = \Sigma([0, \lambda_2 \sigma_2, ..., \lambda_n |\sigma_n|^T])$.

Theorem 4 If $|c_0| = 1$ and $\sigma \in \mathbb{R}^n$ then $\xi = [0, (1 - |\lambda_2|) \sigma_2, ..., (1 - |\lambda_n|) \sigma_n]^T \in \mathbb{R}^n$ and if $f$ is a refinable distribution satisfying Equation 2 and $\tau_j \in \Sigma(\xi)$ then the spaces

$$W_k = \text{span} \{ f(\lambda^k x - \tau) : \tau \in \Sigma(\sigma) \}, \quad k \in \mathbb{Z},$$

satisfy $W_k \subset W_{k+1}$.

Proof If $\tau \in \Sigma(\sigma)$ then Equation 2 implies that

$$f(\lambda^k x - \tau) = \sum_{j=1}^{m} a_j f(\lambda^{k+1} x - \lambda\tau - \tau_j).$$

Since $|c_0| = 1$, Lemma 11 implies that

$$\lambda \tau \in \Sigma([0, \lambda_1 |\sigma_1|, ..., \lambda_n |\sigma_n|^T]),$$

and hence Lemma 2 implies that

$$\lambda \tau + \tau_j \in \Sigma([0, \lambda_1 |\sigma + \xi_1|, ..., \lambda_n |\sigma_n + \xi_n|^T]) = \Sigma(\sigma).$$

Theorem 4 provides the foundation of a multi-scale theory of refinable distributions whose dilations are PV numbers $\lambda$ that are units in the ring of integers in the number field $\mathbb{Q}(\lambda)$ and whose translates are in quasilattices associated to $\lambda$. The union of all the quasilattices equals the set $\mathbb{Z}[1, \lambda, ..., \lambda^{n-1}]$ which is a dense rank $n$ subset of $\mathbb{R}$. It can be shown that the Fourier transform of the sum of unit point measures at each point of a quasilattice is a tempered distribution supported on the countable set $\mathbb{Z}[1, \lambda, ..., \lambda^{n-1}]$. These quasilattices constitute a class of quasicrystals which have been extensively studied. We refer the reader to papers by Bombieri [4, 5], the book by Senechal [36], and references therein.
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