Abelian symmetry and the Palatini variation

James T. Wheeler*

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Abstract

Independent variation of the metric and connection in the Einstein-Hilbert action, called the Palatini variation, is generally taken to be equivalent to the usual formulation of general relativity in which only the metric is varied. However, when an abelian symmetry is allowed for the connection, the Palatini variation leads to an integrable Weyl geometry, not Riemannian. We derive this result using two possible metric/connection pairs: (1) the metric and general coordinate connection and (2) the solder form and local Lorentz spin connection of Poincaré gauge theory. Both lead to the same conclusion. Finally, we relate our work to other treatments in the literature.

1 The Palatini variation

General relativity describes spacetimes, \((\mathcal{M}, g)\), where \(\mathcal{M}\) is a Riemannian manifold and \(g\) is a Lorentzian metric. The field equation follows by metric variation of the action functional

\[
S_{GR}[g] = \int R\sqrt{-g}d^4x
\]

where \(R\) is the scalar curvature computed from the metric compatible Christoffel connection. Sources are included by adding the action for any generally coordinate invariant matter action to \(S_{GR}\),

\[
S = S_{GR} + S_{\text{Matter}}
\]

The beginning of an alternative variation dates back to a 1919 paper by [1] and was brought to its current formulation by Einstein [2] (see [3] for the interesting history leading Einstein to the connection variation, and Appendix I for Einstein’s calculation). The alternative formulation showed that the assumption of the metric compatible connection could be replaced by varying the metric and connection independently, using the action,

\[
S_P[g, \hat{\Gamma}] = \int \hat{R}\sqrt{-g}d^4x
\]  

(1)

Here \(\hat{\Gamma}\) is any symmetric connection, \(\hat{\Gamma}^\alpha{}_{\mu\nu} = \hat{\Gamma}^\alpha{}_{\nu\mu}\). This symmetry condition is preserved by changes of coordinates because the inhomogeneous term from a general coordinate transformation is symmetric.

Notice that in treating the connection independently, we consider spacetime to be a triple, \((\mathcal{M}, g, \hat{\Gamma})\).

Although the independent variable \(\hat{\Gamma}^\alpha{}_{\mu\nu}\) is assumed to be a general symmetric connection, this is not the form taken by the connection for an abelian symmetry, which carries a weight factor and can apply nontrivially to scalars as well as vectors. For the connection variation to be complete, a more general expression is required. In the remainder of this Section, we carry out the usual Palatini variation of \(S_P\), then show the altered effect of an abelian covariance to the derivation. We find that including an abelian term in the connection results in an integrable Weyl geometry.

*Utah State University, Logan, UT 84322, jim.wheeler@usu.edu
In Section 2 we study the Palatini variation in Poincaré gauge theory, where the independent variables are the solder form and the spin connection instead of the metric \( g_{\mu\nu} \) and \( \hat{\Gamma}^\alpha_{\mu\nu} \). The details differ in interesting ways but the end result is the same—when the possibility of an abelian symmetry is included in the variation of the spin connection, we obtain an integrable Weyl geometry.

As in our investigation, Einstein’s original development of the Palatini variation leads to the introduction of an additional vector field. In the final Section we discuss the relationship between this vector field and our inclusion of abelian symmetry. We conclude by noting the differences between ours and some standard treatments of the Palatini variation.

Throughout these notes, in order to distinguish coordinate and orthonormal frames, Greek indices refer to any coordinate basis, and Latin indices to any orthonormal basis. We do not use coordinate-free tensor notation since this would unnecessarily complicate the notation.

1.1 The standard Palatini variation

With the Palatini variation of Eq.(1), the metric variation becomes much simpler. Writing the metric dependence explicitly and varying

\[
\delta g_S P [g, \hat{\Gamma}] = \delta g \int \hat{R}_{\alpha\beta} g^{\alpha\beta} \sqrt{-g} d^4 x
\]

\[
= \delta g \int \left( \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R} \right) \delta g^{\alpha\beta} \sqrt{-g} d^4 x
\]

gives the Einstein tensor,

\[
\hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R} = 0
\] (2)

Here \( \hat{R}_{\alpha\beta} \) is the Ricci tensor computed from \( \hat{\Gamma} \), but it is only after varying \( \hat{\Gamma} \) that we know what connection to use.

The connection variation gives

\[
\delta_{\Gamma} g_S P [g, \hat{\Gamma}] = \delta_{\Gamma} \int \hat{R}_{\alpha\beta} g^{\alpha\beta} \sqrt{-g} d^4 x
\]

\[
= \int \left( \hat{\nabla}_\mu \left( \delta \hat{\Gamma}^\alpha_{\mu\beta} \right) - \hat{\nabla}_\beta \left( \delta \hat{\Gamma}^\mu_{\alpha\mu} \right) \right) g^{\alpha\beta} \sqrt{-g} d^4 x
\] (3)

At this point it is useful to write the covariant derivative as the sum of a metric compatible piece and an additional, non-compatible tensor.

\[
\hat{\nabla}_\mu v^\alpha = \partial_\mu v^\alpha + v^\beta \hat{\Gamma}^\alpha_{\beta\mu} = \nabla_\mu v^\alpha + v^\beta C^\alpha_{\beta\mu}
\]

so the connection variation becomes variation of the non-metric piece, \( \delta \hat{\Gamma}^\mu_{\alpha\beta} = \delta C^\mu_{\alpha\beta} \). Here \( \nabla_\mu g_{\alpha\beta} = 0 \) by definition, implying the usual Christoffel/Levi-Civita connection for \( \nabla_\mu \). The remaining tensor \( C^\alpha_{\beta\mu} \) is intended to characterize any further properties of the connection. We carry this out in considerable detail for subsequent reference.

For the variation we need

\[
\hat{\nabla}_\mu \left( \delta C^\nu_{\alpha\beta} \right) = \nabla_\mu \left( \delta C^\nu_{\alpha\beta} \right) + \left( \delta C^\rho_{\alpha\beta} \right) C^\nu_{\rho\mu} - \left( \delta C^\nu_{\rho\beta} \right) C^\rho_{\alpha\mu} - \left( \delta C^\nu_{\alpha\rho} \right) C^\rho_{\beta\mu}
\]

Taking the required contractions, the variation becomes

\[
\delta g_S P \left[ g, \hat{\Gamma} \right] = \int \nabla_\mu \left( \delta C^\mu_{\alpha\beta} \right) g^{\alpha\beta} \sqrt{-g} d^4 x
\]

\[
+ \int \left( \delta C^\rho_{\alpha\beta} C^\mu_{\rho\mu} - \delta C^\mu_{\rho\beta} C^\rho_{\alpha\mu} - \delta C^\nu_{\alpha\rho} C^\rho_{\beta\mu} \right) g^{\alpha\beta} \sqrt{-g} d^4 x
\]

\[
+ \int \delta C^\nu_{\rho\mu} C^\rho_{\alpha\beta} g^{\alpha\beta} \sqrt{-g} d^4 x
\]
The compatible part of each derivative in the variation is integrated by parts and vanishes by metric-compatibility

\[
\int \nabla_{\mu} \left( \delta C^\mu_{\alpha\beta} \right) g^{\alpha\beta} \sqrt{-g} d^4x = -\int \delta C^\mu_{\alpha\beta} \nabla_{\mu} (g^{\alpha\beta} \sqrt{-g}) d^4x = 0
\]

\[
-\int \nabla_{\beta} (\delta C^\mu_{\alpha\mu}) g^{\alpha\beta} \sqrt{-g} d^4x = \int \delta C^\mu_{\alpha\mu} \nabla_{\beta} (g^{\alpha\beta} \sqrt{-g}) d^4x = 0
\]

Collecting the remaining terms and setting the whole to zero,

\[0 = \int \delta C^{\rho\lambda} (\delta_{\sigma\mu} C^\mu_{\rho\mu} - \delta_{\beta\rho} C^\sigma_{\alpha\rho} - \delta_{\alpha} C^\lambda_{\beta\rho} + \delta_{\lambda} C^\sigma_{\alpha\beta}) g^{\alpha\beta} \sqrt{-g} d^4x\]

from which we conclude

\[g^{\alpha\beta} (\delta_{\sigma\mu} C^\mu_{\rho\mu} - \delta_{\beta\rho} C^\sigma_{\alpha\rho} - \delta_{\alpha} C^\lambda_{\beta\rho} + \delta_{\lambda} C^\sigma_{\alpha\beta}) = 0\]

Carrying out the contractions

\[g^{\sigma\lambda} C^\mu_{\rho\mu} - C^\sigma_{\rho\lambda} - C^\lambda_{\rho\sigma} + \delta_{\rho} C^\sigma_{\lambda\beta} = 0\]

This is easily solved. Recalling the symmetry of the connection \(C^\mu_{\alpha\beta} = C^\mu_{\beta\alpha}\), there are only two independent contractions. From the \(\sigma\lambda\) and \(\lambda\rho\) contractions, the two traces must satisfy both

\[2C^\sigma_{\lambda\rho} + C^\beta_{\rho\beta} = 0\]

\[3C^{\sigma\beta}_{\beta\beta} = 0\]

and therefore both vanish. Substituting into the full field equation and lowering indices we have

\[C^\sigma_{\lambda\rho} + C^\lambda_{\sigma\rho} = 0\]

Finally, this succumbs to the usual technique of cycling the indices, then adding the first two permutations and subtracting the third. The result is the vanishing of the non-metric part of the connection and we appear to have established metric compatibility.

### 1.2 The connection of an abelian symmetry

In addition to assuming symmetry of the connection, there is a further hidden assumption. We noted above that \(C^\mu_{\alpha\beta}\) is intended to account for all characteristics beyond metric compatibility, but it fails to include the possibility of an abelian symmetry.

By contracting a connection such as \(\hat{\Gamma}^\mu_{\alpha\beta}\) above with a small displacement \(dx^\beta\) we see that

\[M^\mu_{\alpha} = \hat{\Gamma}^\mu_{\alpha\beta} dx^\beta\]

has the form of a linear transformation. The transformation characterizes the relationship between components of a tangent vector in tangent spaces separated by \(dx^\beta\). By correcting for this change of tangent basis in moving about a manifold, the covariant derivative is able to separate the change in a physical vector field from the arbitrariness of the coordinates. Parallel transport around a closed loop therefore gives intrinsic geometric information—the curvature.

A linear transformation such as \(M^\mu_{\alpha}\) is appropriate for any non-abelian group of transformations. Linear representations of non-abelian groups act on real or complex vector spaces, and must be characterized by matrix transformations of dimension \(n \geq 2\). To accomplish the Leibnitz rule for products of fields, we include \(k\) linear such transformations on tensors of rank \(k\). For example, the covariant derivative of the rank-2 metric is

\[\hat{D}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - g_{\rho\beta} \hat{\Gamma}^\rho_{\alpha\mu} - g_{\alpha\rho} \hat{\Gamma}^\rho_{\beta\mu}\]

(4)
For an abelian group, the transformation is simple multiplication, so that even scalars may provide nontrivial linear representations. As a result, the connection for an abelian transformation takes a different form, acting nontrivially on weighted scalars.

For example, a complex wave function under a $U(1)$ transformation will transform as $\psi \to e^{i\alpha} \psi$, so the connection required to make the $U(1)$ symmetry local acts on $\psi$ as

$$\hat{D}_\mu \psi = \partial_\mu \psi - iA_\mu \psi$$

and it follows that for a field transforming as $\chi(k) \to (e^{ik})^k \chi(k)$ (e.g., $(\psi)^k$) the derivative must include a weight $k$

$$\hat{D}_\mu \chi(k) = \partial_\mu \chi(k) - ikA_\mu \chi(k)$$

The derivative $\hat{D}_\mu$ of $\chi(k)$ is then covariant under the combined transformation

$$\chi(k) \to e^{ik\varphi} \chi(k)$$

$$A_\alpha \to A_\alpha + \partial_\alpha \varphi$$

Derivations are transformations which are both linear and Leibnitz. The weight is necessary in order to satisfy the Leibnitz rule. Thus, for fields $\chi(k)$ and $\psi(m)$ of weights $k$ and $m$ the weights are additive,

$$D_\alpha (\chi(k)\psi(m)) = \partial_\alpha (\chi(k)\psi(m)) - (k + m)W_\alpha (\chi(k)\psi(m))$$

$$= D_\alpha \chi(k)\psi(m) + \chi(k) (D_\alpha \psi(m))$$

These considerations apply to both scalars and vectors. For weighted, vector-valued fields $v_\beta^{(k)}$ the covariant derivative is

$$D_\alpha v_\beta^{(k)} = \nabla_\alpha v_\beta^{(k)} + e_\mu^{(k)} \hat{\Gamma}^{\beta}_{\mu \alpha} - k v_\beta^{(k)} W_\alpha$$

where $\hat{\Gamma}^{\beta}_{\mu \alpha}$ provides covariance under non-abelian transformations and $kW_\alpha$ under abelian transformations. Notice that a linear transformation $\hat{\Gamma}^{\beta}_{\mu \alpha} dx^\alpha$ of dimension $n \geq 2$ cannot be restricted to act on scalars.

Dilatations provide another example of an abelian symmetry. A dilatation will rescale a dimensionful field such as the volume element $\sqrt{-g} \to e^{4\varphi} \sqrt{-g}$ when the metric scales as $g_{\alpha\beta} \to e^{2\varphi} g_{\alpha\beta}$. The metric is said to be of conformal weight 2, and the volume form of weight 4, so the scale-covariant derivative of the volume form is

$$\hat{D}_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - 4W_\mu \sqrt{-g}$$

where $W_\mu$ is the Weyl vector. Because the metric has nonzero conformal weight, the general form of the combined general coordinate and scale covariant derivative is

$$\hat{D}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - g_{\rho\beta} \hat{\Gamma}^{\rho}_{\alpha \mu} - g_{\alpha\rho} \hat{\Gamma}^{\rho}_{\beta \mu} - 2g_{\alpha\beta} W_\mu \sqrt{-g} \tag{5}$$

This is tensorial under general coordinate transformations with the usual inhomogeneous transformation of $\hat{\Gamma}^{\mu}_{\alpha \beta}$, and also under the combined conformal transformation

$$g_{\alpha\beta} \to e^{2\varphi} g_{\alpha\beta}$$

$$W_\mu \to W_\mu + \partial_\mu \varphi$$

It is natural to include the possibility of a Weyl geometry when considering the differential geometry of spacetime. Indeed, other systematic approaches to the underlying geometry of spacetime also lead to Weyl geometry. Recent work by Trautman, Matveev, and Scholz [5, 6] puts fresh rigor to the physically insightful work of Ehlers, Pirani, and Schild [7]. These studies show that agreement of the projective structure of timelike geodesics and the conformal structure of lightlike geodesics in the lightlike limit leads to an integrable Weyl geometry. Thus, since ultimately we measure only paths of particles, we should expect the world to be described by a Weyl geometry. For agreement with experiment it is important that within
strong experimental limits this should be an integrable Weyl geometry, in which the Weyl vector takes the pure gauge form \( W_\mu = \partial_\mu \phi \). There then exists a gauge in which the Weyl vector vanishes, and transport of physical objects around closed paths does not lead to measurable relative size change.

Whatever abelian symmetry we envision, when varying the connection the most general ansatz for the covariant derivative of a weighted vector is

\[
\hat{D}_\mu v^\alpha_{(k)} = \partial_\mu v^\alpha_{(k)} + v_\beta \hat{\Gamma}^\alpha_{\beta \mu} - kW_\mu v^\alpha_{(k)}
\]

Our central point is this: If the metric has nonzero weight, then use of Eq. (5) instead of Eq. (4) is necessary and will change the result of the Palatini variation.

While our discussion applies to any abelian symmetry, our results apply when the abelian symmetry affects the metric \( (w_g \neq 0) \). Given this, it does not matter whether the symmetry is interpreted as scale covariance or some other physical symmetry. The resulting structure is always that of a Weyl geometry.

### 1.3 The Palatini variation again

We now repeat the argument of 1.1 using the fully general form given in Eq. (5) for the connection. The curvature experienced by a weight zero field is the of the usual form in terms of \( \hat{\Gamma}^\alpha_{\beta \mu} \) alone, so the connection variation still takes the form given in Eq. (5).

\[
\delta T S P \left[ g, \hat{\Gamma} \right] = \int \left( \hat{D}_\mu \left( \delta \hat{\Gamma}^\mu_{\alpha \beta} \right) - \hat{D}_\beta \left( \delta \hat{\Gamma}^\mu_{\alpha \mu} \right) \right) g^{\alpha \beta} \sqrt{-g} d^4 x
\]

The only difference is the addition of a possible abelian term in the derivative of the connection variation,

\[
\hat{D}_\mu \left( \delta \hat{\Gamma}^\nu_{\alpha \beta} \right) = \nabla_\mu \left( \delta \hat{\Gamma}^\nu_{\alpha \beta} \right) + \left( \delta \hat{\Gamma}^\nu_{\rho \beta} \right) C^\rho_{\mu \alpha} - \left( \delta \hat{\Gamma}^\nu_{\rho \alpha} \right) C^\rho_{\beta \mu} - \delta \hat{\Gamma}^\nu_{\alpha \beta} W_\mu \left( \delta \hat{\Gamma}^\nu_{\alpha \beta} \right)
\]

where \( w_\Gamma \) is the weight of \( \delta \hat{\Gamma}^\nu_{\alpha \beta} \) and we again separate out the metric compatible \( \nabla_\mu \). Taking the required contractions and substituting yields

\[
\delta T S P \left[ g, \hat{\Gamma} \right] = \int \left( \nabla_\mu \left( \delta \hat{\Gamma}^\mu_{\alpha \beta} \right) - \nabla_\beta \left( \delta \hat{\Gamma}^\nu_{\alpha \nu} \right) \right) g^{\alpha \beta} \sqrt{-g} d^4 x + \int \left( \delta \hat{\Gamma}^\rho_{\alpha \beta} C^\rho_{\mu \mu} - \delta \hat{\Gamma}^\rho_{\rho \beta} C^\rho_{\alpha \mu} - \delta \hat{\Gamma}^\rho_{\rho \alpha} C^\rho_{\beta \mu} - w_\Gamma W_\mu \delta \hat{\Gamma}^\mu_{\alpha \beta} \right) g^{\alpha \beta} \sqrt{-g} d^4 x + \int \left( \left( \delta \hat{\Gamma}^\nu_{\rho \nu} \right) C^\rho_{\alpha \beta} + w_\Gamma \left( W_\beta \delta \hat{\Gamma}^\nu_{\alpha \nu} \right) \right) g^{\alpha \beta} \sqrt{-g} d^4 x
\]

Integration by parts of the metric compatible derivative gives zero acting on \( g^{\alpha \beta} \sqrt{-g} \), and we are once again left with an algebraic condition for the non-metric piece. Factoring out the variation,

\[
0 = \int \left( \delta \hat{\Gamma}^\rho_{\sigma \lambda} \left( \delta \hat{\Gamma}^\rho_{\sigma \delta} C^\rho_{\mu \mu} - \delta \hat{\Gamma}^\rho_{\rho \mu} C^\rho_{\alpha \rho} - \delta \hat{\Gamma}^\rho_{\rho \alpha} C^\rho_{\beta \mu} - w_\Gamma W_\mu \delta \hat{\Gamma}^\mu_{\alpha \beta} \right) \right) g^{\alpha \beta} \sqrt{-g} d^4 x + \int \delta \hat{\Gamma}^\rho_{\sigma \lambda} \left( \delta \hat{\Gamma}^\rho_{\sigma \delta} C^\rho_{\alpha \beta} + w_\Gamma W_\beta \delta \hat{\Gamma}^\rho_{\alpha \rho} \right) g^{\alpha \beta} \sqrt{-g} d^4 x
\]

and carrying out the contractions with the metric the field equation becomes

\[
0 = g^{\sigma \lambda} C^\mu_{\rho \mu} - C^{\sigma \lambda}_{\rho} - C^{\lambda \sigma}_{\rho} - w_\Gamma W_\rho g^{\sigma \lambda} + \delta \hat{\Gamma}^\rho_{\sigma \beta} C^\rho_{\sigma \beta} + w_\Gamma W^\sigma \delta \hat{\Gamma}^\rho_{\sigma \beta}
\]

(6)

Now the \( \sigma \lambda \) contraction becomes

\[
0 = 2C^\mu_{\rho \mu} + C^\rho_{\beta} \beta - 3w_\Gamma W_\rho
\]
The $\sigma \rho$ trace vanishes identically, while contracting $\lambda \rho$ gives

$$0 = 3C^{\sigma \beta} + 3w_T W^\sigma$$

and therefore, solving we have

$$C^{\sigma \beta} = -w_T W^\sigma$$

$$C^\mu_{\rho \mu} = 2w_T W^\rho$$

Substituting the contractions back into Eq.(6),

$$0 = 2w_T W^\rho g^{\sigma \lambda} - C^{\sigma \lambda} - C^{\lambda \sigma} - w_T W^\rho g^{\sigma \lambda} - \delta^\lambda_\rho w_T W^\sigma + w_T W^\sigma \delta^\lambda_\rho$$

Lowering the indices, we permute indices and combine in the usual way to isolate $C_{\lambda \sigma \rho}$.

$$C_{\sigma \lambda \rho} + C_{\lambda \sigma \rho} + C_{\lambda \rho \sigma} - C_{\rho \sigma \lambda} - C_{\sigma \rho \lambda} = w_T W^\rho g_{\sigma \lambda} + w_T W^\sigma g_{\lambda \rho} - w_T W^\rho g_{\sigma \rho}$$

Restoring the first index to its natural position

$$C^\lambda_{\sigma \rho} = \frac{1}{2} w_T (\delta^\lambda_{\rho} W^\rho + \delta^\lambda_{\sigma} W^\sigma - W^\lambda g_{\sigma \rho})$$

We can choose the weight $w_T$ to insure metric compatibility. With this expression for $C^\lambda_{\sigma \rho}$, the covariant derivative of a weight $w_y$ metric is

$$\hat{D}_\rho g_{\alpha \beta} = g_{\alpha \beta, \rho} - g_{\lambda \beta} \tilde{\Gamma}_{\alpha \rho}^\lambda - g_{\alpha \lambda} \tilde{\Gamma}_{\beta \rho}^\lambda - w_y W^\rho g_{\alpha \beta}$$

$$= g_{\alpha \beta, \rho} - g_{\lambda \beta} \left( \frac{1}{2} g^{\lambda \mu} (g_{\mu \alpha, \rho} + g_{\mu \rho, \alpha} - g_{\alpha \rho, \mu}) + \frac{1}{2} w_T g^{\lambda \mu} (g_{\mu \alpha} W^\rho + g_{\mu \rho} W^\alpha - g_{\alpha \rho} W^\mu) \right)$$

$$- g_{\alpha \lambda} \left( \frac{1}{2} g^{\lambda \mu} (g_{\mu \beta, \rho} + g_{\mu \rho, \beta} - g_{\beta \rho, \mu}) + \frac{1}{2} w_T g^{\lambda \mu} (g_{\mu \beta} W^\rho + g_{\mu \rho} W^\beta - g_{\beta \rho} W^\mu) \right) - w_y W^\rho g_{\alpha \beta}$$

$$- \frac{1}{2} w_g g_{\alpha \beta} W^\rho - \frac{1}{2} w_g g_{\beta \alpha} W^\rho + \frac{1}{2} w_g g_{\rho \alpha} W^\beta - \frac{1}{2} w_g g_{\rho \beta} W^\alpha - \frac{1}{2} w_g g_{\rho \beta} W^\alpha - 2W^\rho g_{\alpha \beta}$$

so we set $w_T = -w_g$. Then despite the non-vanishing of $C^\lambda_{\sigma \rho}$ we have metric compatibility. The full connection is

$$\hat{\Gamma}^{\nu}_{\alpha \beta} = \frac{1}{2} g^{\nu \mu} (g_{\mu \lambda, \alpha} + g_{\mu \alpha, \lambda} - g_{\alpha \lambda, \mu}) - \frac{w_g}{2} g^{\nu \mu} (g_{\mu \alpha} W^\rho + g_{\mu \rho} W^\alpha - g_{\alpha \rho} W^\mu)$$

$$= \frac{1}{2} g^{\nu \mu} (\mathcal{D}_\rho g_{\mu \alpha} + \mathcal{D}_\alpha g_{\mu \rho} - \mathcal{D}_\rho g_{\mu \alpha})$$

(7)

where

$$\mathcal{D}_\mu g_{\alpha \beta} = g_{\alpha \beta, \mu} - w_y g_{\alpha \beta} W^\mu$$

is the abelian-covariant derivative of the metric. This makes the full connection invariant under the abelian transformations. The $W_\alpha$ terms in Eq.(7) represent decoupling of the abelian and non-abelian parts of the derivative.

Equation(7) is the connection of a Weyl geometry. In this sense, the Palatini variation leads to a Weyl geometry.
1.4 Integrability of the Weyl geometry

When the weight of the metric is nonzero, \( w_g \neq 0 \), the defining vector of a Weyl geometry, \( W_\mu \), is called the Weyl vector. Through its coupling to the metric it affects lengths. Suppose \( s^\alpha \) is a constant, weight zero vector associated with a physical object so that in a flat geometry (\( \Gamma^\alpha_{\mu\nu} = 0 \)) we have

\[
D_\alpha s^\beta = \partial_\alpha s^\beta - 0 \cdot W_\alpha s^\alpha = 0
\]

Then the covariant derivative of \( s^2 = \eta_{\alpha\beta} s^\alpha s^\beta \) is

\[
D_\mu s^2 = D_\mu (\eta_{\alpha\beta} s^\alpha s^\beta) = (D_\mu \eta_{\alpha\beta}) s^\alpha s^\beta = -w_g W_\mu s^\alpha
\]

If two identical such rods are carried along different paths and brought back together forming a closed curve \( C \), their lengths no longer match but differ by

\[
\Delta s^2 = -w_g \int_C W_\mu dx^\mu = -w_g \int_S (\partial_\mu W_\nu - \partial_\nu W_\mu) dS^\mu\nu
\]

This constitutes a measurable change in physical size unless the curl of the Weyl vector vanishes. Even on small scales such an effect would drastically spread atomic, nuclear, and particle spectral lines and resonances, in conflict with experiment. Therefore, it is important that unless the coupling is immeasurably small, the curl of the Weyl vector must vanish. When this is the case, the Weyl connection describes an integrable Weyl geometry. In an integrable Weyl geometry \( W_\mu \) is a gradient, and there exists a rescaling such that \( \tilde{W}_\mu = 0 \), returning the appearance of the geometry to Riemannian.

With this background in mind, we examine the form of the Weyl vector by looking in detail at the derivative of the volume form, \( g = \det (g_{\alpha\beta}) \).

For the abelian symmetry we know that

\[
\hat{D}_\mu g = \partial_\mu g - 4w_g W_\mu
\]

Expanding the determinant in terms of the metric,

\[
g = \det g_{\alpha\beta} = \frac{1}{4!} \varepsilon^{\alpha\beta\mu\nu} \varepsilon^{\rho\sigma\lambda\tau} g_{\alpha\rho} g_{\beta\sigma} g_{\mu\lambda} g_{\nu\tau}
\]

we may express \( \hat{D}_\mu g \) in terms of the metric derivative,

\[
\hat{D}_\mu g = \frac{1}{3!} \varepsilon^{\alpha\beta\mu\nu} \varepsilon^{\rho\sigma\lambda\tau} \left( \hat{D}_\mu g_{\alpha\rho} \right) g_{\beta\sigma} g_{\rho\lambda} g_{\nu\tau} = \Sigma^{\alpha\rho} \left( \hat{D}_\mu g_{\alpha\rho} \right)
\]

We define

\[
\Sigma^{\alpha\rho} = \frac{1}{3!} \varepsilon^{\alpha\beta\mu\nu} \varepsilon^{\rho\sigma\lambda\tau} g_{\beta\sigma} g_{\rho\lambda} g_{\nu\tau}
\]

\[
= -\frac{1}{3!} \varepsilon^{\alpha\beta\mu\nu} \varepsilon^{\rho\sigma\lambda\tau} g_{\beta\sigma} g_{\rho\lambda} g_{\nu\tau}
\]

\[\text{Even in the Riemannian gauge with } W_\mu = 0, \text{ there is still a difference between a Weyl geometry and a Riemannian geometry, since in the former a rescaling will restore a nonzero Weyl vector while keeping physical scalars unchanged, while the same rescaling will substantially change physical predictions in a Riemannian geometry.}\]
where \( e^{\alpha\beta\varphi\nu} = \frac{1}{\sqrt{-g}} \varepsilon^{\alpha\beta\varphi\nu} \) is the Levi-Civita tensor. Then contracting with another copy of the metric, we lower the indices on the second Levi-Civita tensor,

\[
\Sigma^{\alpha\rho} g_{\rho\theta} = -\frac{1}{3!} \varepsilon^{\alpha\beta\varphi\nu} g_{\alpha\beta} g_{\varphi\nu} g^{\rho\sigma\lambda\tau} g_{\rho\sigma} g_{\varphi\lambda} g_{\nu\tau} = -\frac{1}{3!} \varepsilon^{\alpha\beta\varphi\nu} \epsilon_{\theta\beta\varphi\nu} = g_{\delta\theta}^\rho
\]

This shows that

\[
\Sigma^{\alpha\rho} = gg^{\alpha\rho}
\]

since the inverse metric is unique and the volume element nonvanishing. Therefore, returning to Eq. (9),

\[
\hat{D}_\mu g = gg^{\alpha\rho} \hat{D}_\mu g_{\alpha\rho}
\]

The same argument shows that the partial derivative of the metric determinant is

\[
\partial_\mu g = gg^{\alpha\rho} \partial_\mu g_{\alpha\rho}
\]  (10)

With the covariant derivative of the metric given by

\[
\hat{D}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - g_{\nu\beta} \hat{\Gamma}_\nu^\alpha_{\alpha\mu} - g_{\alpha\alpha} \hat{\Gamma}_\nu^\nu_{\beta\mu} - w_g W_\mu g_{\alpha\beta}
\]

the covariant derivative of the volume form becomes

\[
\hat{D}_\mu g = gg^{\alpha\beta} \hat{D}_\mu g_{\alpha\beta} = gg^{\alpha\beta} \partial_\mu g_{\alpha\beta} - 2g \hat{\Gamma}_\alpha^\alpha_{\alpha\mu} - 4w_g g W_\mu
\]  (11)

Substituting Eqs. (10) into (11) and equating to the expression in Eq. (8),

\[
\partial_\mu g - 4w_g g W_\mu = \partial_\mu g - 2g \hat{\Gamma}_\alpha^\alpha_{\alpha\mu} - 4w_g g W_\mu
\]

and therefore

\[
\hat{\Gamma}_\alpha^\alpha_{\alpha\mu} = 0
\]

From the form of the Weyl connection Eq. (7) this implies

\[
\hat{\Gamma}^\alpha_{\alpha\beta} = \frac{1}{2} g^{\alpha\mu} (g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) - \frac{w_g}{2} g^{\alpha\mu} (g_{\mu\alpha} W_\beta + g_{\mu\beta} W_\alpha - g_{\beta\alpha} W_\mu)
\]

0 = \frac{1}{2} g^{\alpha\mu} g_{\mu\alpha,\beta} - \frac{w_g}{2} (4W_\beta + W_\beta - W_\beta)

0 = \frac{1}{2} g^{\alpha\mu} g_{\mu\alpha,\beta} - 2w_g W_\beta

and therefore

\[
W_\beta = \frac{1}{4w_g} g^{\alpha\mu} g_{\mu\alpha,\beta} = \frac{1}{4w_g} \partial_\mu (\ln g)
\]

The Weyl vector is therefore a gradient and the Palatini variation leads to an integrable Weyl geometry. This is in good agreement with experiment and with the conclusions about measurability by Matveev and Trautman [5].
The result is striking in two ways. First, the integrability of the Weyl vector means that there exists a choice of gauge in which the field equation takes the usual form from general relativity. In this sense, the usual Palatini conclusion holds: the Palatini action \( S[g, \hat{\Gamma}, W] \) leads to the usual Einstein equation together with the Christoffel connection, \textit{but only in a particular conformal gauge.}

The second striking feature is that we are led by the Palatini variation to an integrable Weyl geometry. In this sense, the usual conclusion is wrong. We do not get only the Christoffel connection. The physical arguments of [7, 5] are supported by the free variation of the connection.

Before concluding, we must ask whether the presence of sources will make the Weyl vector non-integrable. To high precision this would conflict with observation. However, the Einstein tensor of a Weyl geometry is given by [8]

\[
G_{ab} = R_{ab} - \frac{1}{2}R\eta_{ab} + 2W_{a;b} + 2W_a W_b + (W^2 - 2W^c_{\ c}) \eta_{ab}
\]

and this will equal the energy tensor of the source, which in turn arises from the metric variation of the matter action. Since the metric is symmetric, this always yields a symmetric source tensor,

\[
\mathcal{E}_{ab} = \kappa T_{ab}
\]

Taking the antisymmetric part of this expression leaves only one term,

\[
0 = \mathcal{E}_{[ab]} = W_{[a;b]}
\]

and this is the condition for the Weyl vector to be pure gauge. Therefore, even including sources, Palatini variation leads to an integrable Weyl geometry and is therefore gauge-equivalent to general relativity.

It is amusing to note that this conclusion agrees with the result of Einstein in his original formulation of the Palatini variation [2]. In [2] the metric is replaced by an asymmetric tensor density and the connection is fully general. The variation leads to the introduction of a vector field in addition to the usual metric and connection, even when the metric is symmetric. Einstein’s full argument is presented in the Appendix.

### 2 Different independent variables

When we write general relativity as a Poincaré gauge theory the form of the metric and connection are altered to give a local Lorentz fiber bundle. The change of variables begins by replacing coordinate differentials by an orthonormal 1-form basis. Whereas the metric in a general coordinate basis is related to coordinate 1-form basis by

\[
\langle dx^\alpha, dx^\beta \rangle = g^{\alpha\beta}
\]

the solder form is an orthonormal linear combination \( e^a = e^a_\alpha dx^\alpha \) such that

\[
\langle e^a, e^b \rangle = \eta^{ab}
\]

where \( \eta_{ab} \) is the Minkowski metric. Preserving the orthonormality of the frame field reduces the symmetry from local general linear to local Lorentz while maintaining complete generality of the geometry. The change replaces the coordinate metric and connection \( (g, \hat{\Gamma}) \) of the Palatini action with the solder form and spin connection, \( (e^a, \omega^a_{\ b}) \).

The Cartan structure equations take the form

\[
d\omega^a_{\ b} = \omega^c_{\ b} \wedge \omega^a_{\ c} + \mathcal{R}^a_{\ b}
\]

\[
de^a = e^b \wedge \omega^a_{\ b} + T^a
\]
where $\mathcal{R}^a_{\ b}$ is the curvature 2-form and $T^a$ is the torsion 2-form. These require integrability conditions (Bianchi identities) similar to general relativity,

$$DT^a = e^b \wedge \mathcal{R}^a_{\ b}$$
$$D\mathcal{R}^a_{\ b} = 0$$

but the first Bianchi identity now involves the torsion.

To achieve the Riemannian geometry of general relativity directly we would set the torsion to zero. This eliminates torsion dependence of the curvature, $\mathcal{R}^a_{\ b} \rightarrow R^a_{\ b}$. Then, along with the correspondingly reduced Bianchi identity $0 = e^b \wedge R^a_{\ b}$, the reduced structure equations

$$d\omega^a_{\ b} = \omega^c_{\ b} \wedge \omega^a_{\ c} + R^a_{\ b}$$
$$de^a = e^b \wedge \omega^a_{\ b}$$

describe a Riemannian geometry. Solder form or metric variation of the Einstein-Hilbert action

$$S_{EH}[e^a] = \frac{1}{2} \int \mathcal{R}^{ab} \wedge e^c \wedge e^d e_{abcd}$$
gives the Einstein equation. Varying the solder form alone, the torsion makes no appearance in the field equations, so setting torsion to zero is consistent throughout.

When torsion is not set to zero by hand, the variation of the Einstein-Hilbert action with respect to the solder form or metric leads to the Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity. While vacuum ECSK theory still leads to vanishing torsion, torsion can be nonzero in the presence of spinor sources.

By contrast, the Palatini variation introduces a second field equation directly dependent on the torsion. While some variants of ECSK theory vary the metric and torsion, in the gauge theory formulation it is natural to take the Cartan connection 1-forms as the independent variables. For Poincaré gauge theory these are the solder form $e^a$ and the spin connection $\omega^a_{\ b}$.

We study whether this change in the choice of independent variables affects our conclusions regarding the Palatini variation.

Retaining the Einstein-Hilbert action, we write it as

$$S_P[e^a, \omega^a_{\ b}] = \frac{1}{2} \int \mathcal{R}^{ab} \wedge e^c \wedge e^d e_{abcd}$$

with the curvature and connection given by Eqs. (12) and (13). Working within the rigid context of these Cartan structure equations, there is no freedom to modify the connection as in the previous Section.

### 2.1 Solder form variation

There are no surprises when we vary the solder form.

$$\delta_e S_P = \int \mathcal{R}^{ab} \wedge e^c \wedge e^d e_{abcd}$$

Defining a volume form as the dual of one, $\Phi = \ast 1 = \frac{1}{4!} \varepsilon^{abcd} e^a \wedge e^b \wedge e^c \wedge e^d$, so that

$$e^a \wedge e^b \wedge e^c \wedge e^d = -e_{abcd} \Phi$$

the field equation becomes

$$0 = \mathcal{R}^{ab} \wedge e^c \wedge e^d e_{abcd}$$
$$= -\frac{1}{2} \mathcal{R}^{ab} f^c_{\ f} g^{de} e_{abcd} \Phi$$
and reducing the doubled Levi-Civita tensor \( e^{fgde}_{abcd} = -\frac{1}{2} \delta^f_g \delta^d_c \delta^e_b \) we have the Einstein equation

\[
R_{ab} - \frac{1}{2} R \eta_{ab} = 0
\]

The only difference is that the curvature is that of an Einstein-Cartan geometry, hence dependent upon the torsion.

### 2.2 Varying the spin connection

Varying the spin connection, some features emerge as before and some are different. All of the structure is determined by the Cartan equations, Eqs. (12) and (13). Using Eq. (12) we have

\[
\delta_\omega S_P = \frac{1}{2} \int D (\delta_\omega^{ab}) \wedge e^c \wedge e^d e_{abcd}
\]

where

\[
D \delta_\omega^{ab} = d (\delta_\omega^{ab}) + \omega^a \wedge \delta_\omega^b - \omega^b \wedge \delta_\omega^a
\]

Integrating by parts we need to exercise caution because \( \frac{1}{2} \int D (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}) \) is not necessarily just a surface term. From the Leibnitz rule we must have

\[
\delta_\omega S_P = \frac{1}{2} \int D (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}) + \frac{1}{2} \int \delta_\omega^{ab} \wedge D (e^c \wedge e^d e_{abcd})
\]

but if there is a non-abelian symmetry the action of \( D \) on a scalar is not just the exterior derivative. Rather,

\[
D (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}) = d (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}) - w_\Sigma \omega \wedge (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd})
\]

where \( w_\Sigma \) is the weight of the scalar 3-form

\[
\Sigma = \delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}
\]

For the action to have an abelian symmetry, its weight should be zero. Given the weight \( w_g \) of the metric, it is straightforward to determine the weights of the remaining fields. This is carried out for all relevant fields in Appendix II to show that

\[
\begin{align*}
w (e^a) &= \frac{1}{4} w_g \quad w (\omega^a b) &= 0 \\
w (\Phi) &= 2 w_g \quad w (\omega^a bc) &= -\frac{1}{2} w_g \\
w (e_{abcd}) &= 0 \quad w (T^a) &= \frac{1}{2} w_g \\
w (\eta_{ab}) &= 0 \quad w_\Sigma &= w_g
\end{align*}
\]

In particular we have \( w_\Sigma = w_g \).

Returning to the variation

\[
\delta_\omega S_P = \frac{1}{2} \int D (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}) + \frac{1}{2} \int \delta_\omega^{ab} \wedge D (e^c \wedge e^d e_{abcd})
\]

\[
= \frac{1}{2} \int d (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}) - \frac{1}{2} \int w_\Sigma \omega \wedge (\delta_\omega^{ab} \wedge e^c \wedge e^d e_{abcd}) + \frac{1}{2} \int \delta_\omega^{ab} \wedge D (e^c \wedge e^d e_{abcd})
\]

Discarding the surface term, writing \( \delta_\omega^{ab} = \delta_\omega^{ab} e^c \), then setting the variation to zero, the field equation becomes

\[
0 = \frac{1}{2} e^c \wedge D (e^c \wedge e^d e_{abcd}) + \frac{1}{2} w_\Sigma e^c \wedge \omega \wedge e^c \wedge e^d e_{abcd}
\]

\[
= \frac{1}{2} e^c \wedge (De^c \wedge e^d e_{abcd} - e^c \wedge De^d e_{abcd} + e^c \wedge e^d De^d e_{abcd}) + \frac{1}{2} w_g W_f e^c \wedge e^f \wedge e^c \wedge e^d e_{abcd}
\]

\[
= T^c \wedge e^c \wedge e^d e_{abcd} + \frac{1}{2} e^c \wedge De^d e_{abcd} + \frac{1}{2} w_g W_f e^d e_{abcd} \Phi
\]

\[
= \frac{1}{2} w_g W_f e^c \wedge e^d e_{abcd} - \frac{1}{2} w_g W_f e^d e_{abcd} \Phi
\]
and since \( w(\epsilon_{abcd}) = 0 \),
\[
\mathbf{D}e_{abcd} = \text{d}e_{abcd} = 0
\]

We are left with
\[
\begin{align*}
0 &= \frac{1}{2} T^{c}_{f g} e^{f} \wedge e^{g} \wedge e^{e} \wedge e^{d} e_{abcd} + w_{g} W_{f} \left( \delta_{a}^{c} \delta_{b}^{f} - \delta_{a}^{f} \delta_{b}^{c} \right) \Phi \\
0 &= -T^{c}_{f g} e^{c} e^{g} e^{d} e_{abcd} + w_{g} \left( \delta_{a}^{c} W_{b} - \delta_{a}^{d} W_{b} \right)
\end{align*}
\]

Resolving the double Levi-Civita, the field equation becomes
\[
0 = T^{c}_{ab} + T^{d}_{da} \delta^{c}_{b} - T^{d}_{db} \delta^{c}_{a} + w_{g} \left( \delta^{c}_{a} W_{b} - \delta^{d}_{a} W_{b} \right)
\]

The \( ea \) trace of the field equation shows that
\[
T^{a}_{ab} = \frac{3}{2} w_{g} W_{b}
\]

so that
\[
T^{e}_{ab} = \frac{1}{2} w_{g} \left( \delta^{e}_{b} W_{b} - \delta^{e}_{a} W_{a} \right)
\]

Writing the torsion as a 2-form,
\[
T^{a} = \frac{1}{2} w_{g} e^{a} \wedge \omega
\]

As a Riemannian geometry, the spacetime has torsion. However, substituting into the Cartan structure equations yields
\[
\begin{align*}
\text{d} \omega^{a} = \omega^{c}_{b} \wedge \omega^{a}_{c} + R^{a}_{b} \\
\text{d} e^{a} = e^{b} \wedge \omega^{a}_{b} + \frac{1}{2} w_{e} e^{a} \wedge \omega
\end{align*}
\]

These are the structure equations of Weyl geometry [8]. Since \( w_{e} = \frac{1}{2} w_{g} \) this shows that the Weyl covariant derivative of the solder form vanishes,
\[
\mathbf{D} e^{a} = \text{d} e^{a} - e^{b} \wedge \omega^{a}_{b} - w_{e} e^{a} \wedge \omega = 0
\]

Taking Eqs. (14) as the structure equations of a Weyl geometry, the Weyl connection is metric compatible and torsion free.

### 2.3 Integibility of the Weyl geometry

With the structure equations now in the form
\[
\begin{align*}
\text{d} \omega^{a} &= \omega^{c}_{b} \wedge \omega^{a}_{c} + R^{a}_{b} \\
\text{d} e^{a} &= e^{b} \wedge \omega^{a}_{b} + w_{e} e^{a} \wedge \omega
\end{align*}
\]

we may find the contribution of \( \omega \) to the curvature. Observing that the solution to Eq. (16) for the spin connection must be the Riemannian spin connection plus a term linear in \( W_{a} \), we set
\[
\omega^{a}_{b} = \alpha^{a}_{b} + \beta^{a}_{b}
\]

where
\[
\text{d} e^{a} = e^{b} \wedge \alpha^{a}_{b}
\]
defines the Christoffel spin connection and we let
\[ \beta^{ab} = \alpha \eta^{ab} e^a W + \beta e^a W_b \]

Antisymmetry on \( ab \) requires \( \alpha = -\beta \). Substituting into Eq. (16) leaves \( \beta e^a \wedge \omega = e^a \wedge \omega \) so \( \beta = 1 \). Therefore, the spin connection is
\[ \omega^{ab} = \alpha a^b + e^a W_b - \eta^{ab} e^e W^e \quad (17) \]

The curvature follows by substituting (17) into Eq. (15). After collecting terms and setting \( de^a = e^b \wedge \alpha^{ab} \),
\[ R^a_{\ b} = d\omega^a_b - \omega^e_b \wedge \omega^a_c = \left( \frac{1}{2} R^a_{bde} (\alpha) - (\delta_d^a \delta_b^f - \eta_{bd} \eta^{af}) \left( D_e W_f - W_e W_f + \frac{1}{2} \eta_{ef} W^2 \right) \right) e^d \wedge e^e \]

Antisymmetrizing \( de \) to remove the basis and contracting, the Ricci tensor becomes
\[ R_{be} = R^a_{bae} = R_{be} (\alpha) - 2 D_e W_b - \eta_{be} D_d W_d + 2 W_e W_b - 2 \eta_{be} W^2 \]

Because the connection is no longer simply \( \alpha^{ab} \), the Ricci tensor acquires an antisymmetric part,
\[ R_{[be]} = -2 W_{[be]} \]

In vacuum this must vanish independently. Even when we consider the full Einstein equation including sources, the symmetry of the energy tensor implies
\[ W_{[be]} = W_{[b,e]} = 0 \]

Therefore, \( W_a = \partial_a \phi \) for some \( \phi \) and the geometry is integrable Weyl. The vector \( W_a \) may be removed by a conformal scaling of the solder form.

### 3 Summary and discussion

We have presented both general coordinate and Poincaré gauge theory demonstrations that when the Palatini variation includes a possible abelian symmetry, the result is scale covariant general relativity in an integrable Weyl geometry.

The difference between the usual claims and these results hinges on the different form of the covariant derivative for abelian symmetries. For non-abelian symmetries, the covariant derivative takes the form
\[ D_\alpha v^\beta = \nabla_\alpha v^\beta + v^\mu \Gamma^\beta_{\mu \alpha} \]

However, fields which transform covariantly under an abelian symmetry, \( \chi \to e^{k \phi} \chi \) are assigned a weight \( k \), \( \chi^{(k)} \) and the covariant derivative reflects this,
\[ D_\alpha \chi^{(k)} = \partial_\alpha \chi^{(k)} - kW_\alpha \chi^{(k)} \]

or for weighted, vector-valued fields \( v^\beta_{(k)} \),
\[ D_\alpha v^\beta_{(k)} = \nabla_\alpha v^\beta_{(k)} + v^\mu_{(k)} \hat{\Gamma}^\beta_{\mu \alpha} - k v^\beta_{(k)} W_\alpha \]

where \( \hat{\Gamma}^\beta_{\mu \alpha} \) provides covariance under non-abelian transformations and \( kW_\alpha \) under abelian transformations.

These considerations affect the Palatini variation whenever the metric has nonzero weight \( w_g \), since the covariant derivatives of the metric and metric determinant become
\[ D_\mu g_{\alpha \beta} = \partial_\alpha g_{\mu \beta} - g_{\beta \mu} \hat{\Gamma}^\beta_{\mu \alpha} - g_{\mu \beta} \hat{\Gamma}^\beta_{\nu \alpha} - w_g W_\alpha g_{\mu \nu} \quad (18) \]
\[ D_\mu g = \partial_\alpha g - 4 w_g W_\alpha g_{\mu \nu} \quad (19) \]
respectively. We have shown that the solution for $\hat{\Gamma}^\beta_{\mu\alpha}$ is

$$\hat{\Gamma}^\beta_{\mu\alpha} = \frac{1}{2}g^{\beta\nu} \left[ (g_{\nu\mu,\alpha} + w_g g_{\nu\mu} W_\alpha) + (g_{\nu\alpha,\mu} + w_g g_{\nu\alpha} W_\mu) - (g_{\mu\alpha,\nu} + w_g g_{\mu\alpha} W_\nu) \right]$$  \hspace{1cm} (20)

The Christoffel term has been augmented in each metric derivative by a the abelian connection vector $w_g W_\alpha$ so that the combination gives the non-abelian connection weight zero and $\hat{\Gamma}^\beta_{\mu\alpha}$ is the connection of a Weyl geometry.

Expressing the covariant derivative of the determinant of the metric in terms of the derivative of the metric, we showed that $W_\alpha$ must be a gradient, so the Palatini variation yields an integrable Weyl geometry.

We have shown that these same conclusions apply to Poincaré gauge theory, even though the independent fields become the solder form and the spin connection, and the Cartan structure equations leave no room to modify the fields. By systematic determination of the weights of all relevant fields in terms of the weight $w_g$ of the metric, and careful treatment of the integration by parts, we show that the torsion acquires a nonvanishing piece

$$T^a = \frac{1}{2}w_g e^a \wedge \omega$$  \hspace{1cm} (21)

This changes the Cartan structure equation to that of a Weyl geometry.

It is interesting to note that when we use $g_{\alpha\beta}$ and $\Gamma^{\alpha}_{\beta\gamma}$ as independent variables the torsion vanishes by the symmetry of the connection and the Weyl vector emerges as a limited form of non-metricity. By contrast, using the orthonormal variables $(e^a, \omega^a_b)$ leads to vanishing non-metricity from the antisymmetry of the spin connection while the Weyl vector emerges as a limited form of the torsion. Thus, the Weyl connection is exactly that generalization of the connection that may be interpreted as either non-metricity or torsion, or equivalently the Weyl connection is the intersection between non-metricity and torsion.

Notice that the specification of a symmetric connection has a certain ambiguity, because a Riemannian geometry with torsion in the form of Eq. (21) is equivalent to a Weyl geometry without torsion. The final Weyl connection is symmetric and metric compatible.

The Weyl covariant derivative includes the Weyl vector $W_\alpha$ in two distinct ways. First, a weight $k$ vector $v^\alpha_{(k)}$, the covariant derivative is

$$D_\alpha v^\beta_{(k)} = \partial_\alpha v^\beta_{(k)} + v^\mu_{(k)} \hat{\Gamma}^\beta_{\mu\alpha} + kW_\alpha v^\beta_{(k)}$$

Second, $\hat{\Gamma}^\beta_{\mu\alpha}$ itself includes additional terms as in Eq. (20) to make the non-abelian part of the connection invariant under abelian transformations. The weights of these two occurrences of the Weyl vector are generally different. The $w_g$ in $\hat{\Gamma}^\beta_{\mu\alpha}$ is the weight of the metric, while the $w_k$ is the weight of the field $v^\beta_{(k)}$.

A third set of independent variables was studied by Einstein in 1925. In [2], the metric and the connection are both generalized to asymmetric fields $g^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$. This calculation is reproduced in the Appendix, where it is again seen to lead to Weyl geometry when $g^{\mu\nu}$ is symmetric or its antisymmetric part may be neglected. It is significant to note that the fully general ansatz for the independent variables leads to a vector field in addition to the metric and Christoffel connection.

Finally, we compare our treatment with that of two standard references: the comprehensive text by Misner, Thorne, and Wheeler, and R. M. Wald’s excellent modern approach.

The argument by Misner, Thorne, and Wheeler MTW begins with what the authors stress is a definition

$$D_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - \sqrt{-g} \Gamma^{\alpha}_{\mu\alpha}$$  \hspace{1cm} (22)

However, we show how this determinant may be written in terms of the covariant derivative of the metric and it follows that

$$D_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - \sqrt{-g} \hat{\Gamma}^{\beta}_{\beta\mu} - 2w_g W_\mu \sqrt{-g}$$

Therefore, the definition of [9] requires the vanishing of abelian part of the connection, the Weyl vector. This allows them to conclude from their variational equation, equivalent to

$$D_\alpha (g^{\beta\rho} \sqrt{-g}) = 0$$
that the covariant derivative of the metric vanishes, \( D_\mu g_{\alpha\beta} = 0 \).

Understanding of the difference between our result for the Palatini variation and the usual conclusion of metric compatibility as due to abelian symmetry was fostered by studying the proof given in [10]. Here, the generalized derivative is written as the sum of the metric compatible derivative, \( \nabla_\alpha \),

\[
\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - g_{\beta\nu} \Gamma^\beta_{\mu\alpha} - g_{\mu\beta} \Gamma^\beta_{\nu\alpha} = 0
\]

(23)

and an additional symmetric tensor \( C^{\alpha}_{\mu\nu} \), then varying the additional tensor. In this case, the derivative of a vector is written as

\[
D_\alpha v^\beta = \nabla_\alpha v^\beta + v^\mu C^{\beta}_{\mu\alpha}
\]

(24)

and the metric compatibility condition determines \( \Gamma^\beta_{\mu\alpha} \) to be the Christoffel connection. Eq.(24) is the general form of covariant derivative for a nonabelian group, and should be modified to

\[
D_\alpha v^\beta(\mathbf{r}) = \partial_\alpha v^\beta(\mathbf{r}) + v^\mu(\mathbf{r}) \Gamma^\beta_{\mu\alpha} - kW_{\alpha}v^\beta(\mathbf{r})
\]

for a weight \( k \) vector field. The difference in conclusion follows by replacing Eq.(23) by the more general form of \( D_\alpha \) in Eq.(18).

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Appendix I: Einstein’s Original Palatini variation

In 1925, Einstein proposed a unified theory of Gravitation and Electricity [2]. This was one of many attempts to geometrically unify the two known interactions of the era. The resulting model represents a third set of independent variables beyond \((g_{\mu\nu}, \Gamma^\alpha_{\mu\nu})\) and \((e^a, \omega^a_{\beta})\) because the metric and connection are both generalized to asymmetric fields \( g^{\mu\nu} \) and \( \Gamma^\alpha_{\mu\nu} \). It was in this paper that Einstein carried out the full Palatini variation, treating \( g^{\mu\nu} \) and \( \Gamma^\alpha_{\mu\nu} \) independently. What interests us here is that he also found a Weyl geometry.

The basic premise of this model was the Palatini variation, which, after introducing the connection \( \Gamma^\beta_{\alpha\beta} \) and the curvature

\[
R^\alpha_{\mu\nu\beta} = -\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\beta} + \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\mu\beta} + \frac{\partial \Gamma^\alpha_{\mu\beta}}{\partial x_\nu} - \Gamma^\sigma_{\mu\nu} \Gamma^\alpha_{\sigma\beta}
\]

built from it, Einstein describes as follows:\(^2\)

Independently of this affine connection we introduce a contravariant tensor density \( g^{\mu\nu} \) whose symmetry we leave undetermined. From both we build the scalar

\[
\mathcal{S} = g^{\mu\nu} R_{\mu\nu}
\]

and postulate that simultaneous variation of the integral

\[
\mathcal{J} = \int \mathcal{S} dx_1 dx_2 dx_3 dx_4
\]

with respect to \( g^{\mu\nu} \) and \( \Gamma^\alpha_{\mu\nu} \) as independent variables (and not varied on the boundary) vanishes.

\(^2\)Any errors in translation are my own. The notation is preserved from the original, possibly with different indices where the the photocopy of the original is too blurry.
Below we reproduce the subsequent calculation of [2] with a few additional comments. A small amount of newer notation is introduced for clarity.

The variation of \( J \) with respect to \( g^{\mu \nu} \) yields the 16 equations

\[
R_{\mu \nu} = 0,
\]

the variation of \( \Gamma^\alpha_{\mu \nu} \) next gives 64 equations,

\[
0 = g^{\alpha \nu},_\alpha - g^{\nu \alpha},_\alpha \tag{26}
\]

\[
0 = -3 \left( g^{\alpha \mu},_\alpha + g^{\alpha \beta} \Gamma^\mu_{\mu \beta} \right) - g^{\mu \alpha} \left( \Gamma^\beta_{\alpha \beta} - \Gamma^\beta_{\beta \alpha} \right) \tag{27}
\]

Next, the inverse is defined with reversed index order \( g_{\alpha \beta} \), so that

\[
g_{\alpha \nu} g^{\alpha \beta} = \delta^\nu_{\beta} = g_{\nu \alpha} g^{\beta \alpha}
\]

Using this, the third contraction of the field equation Eq.(25) with \( g_{\mu \nu} \) gives

\[
0 = g_{\mu \nu} g^{\mu \nu},_\alpha + \Gamma^\beta_{\beta \alpha} - 3 \Gamma^\beta_{\alpha \beta} - g_{\mu \alpha} \left( g^{\mu \beta},_\beta + g^{\rho \beta} \Gamma^\mu_{\rho \beta} \right) \tag{28}
\]

Also using Eq.(27) Einstein defines a vector density

\[
\tilde{f}^\mu = \frac{1}{3} g^{\mu \alpha} \left( \Gamma^\beta_{\alpha \beta} - \Gamma^\beta_{\beta \alpha} \right) = - \left( g^{\mu \alpha},_\alpha + g^{\alpha \beta} \Gamma^\mu_{\alpha \beta} \right) \tag{29}
\]

Notice that the vector density \( \tilde{f}^\mu \) is proportional to the trace of the torsion and vanishes for a symmetric Riemannian connection.

Using the determinant relation Eq.(28) and Eq.(27), write the third contraction as

\[
0 = g_{\mu \nu} g^{\mu \nu},_\alpha + \Gamma^\beta_{\beta \alpha} - 3 \Gamma^\beta_{\alpha \beta} - g_{\mu \alpha} \left( g^{\mu \beta},_\beta + g^{\rho \beta} \Gamma^\mu_{\rho \beta} \right)
= -2 \left( \partial_\alpha \ln \sqrt{-g} + \Gamma^\beta_{\alpha \beta} \right) + \left( \Gamma^\beta_{\beta \alpha} - \Gamma^\beta_{\alpha \beta} \right) + g_{\mu \alpha} \tilde{f}^\mu
\]

Raising the index and substituting the vector density for the trace of the torsion, we find a third expression for the vector density.

\[
\tilde{f}^\mu = -g^{\mu \alpha} \left( \partial_\alpha \ln \sqrt{-g} + \Gamma^\beta_{\alpha \beta} \right) \tag{30}
\]

or, lowering an index, \( g_{\mu \alpha} \tilde{f}^\mu = - \left( \partial_\alpha \ln \sqrt{-g} + \Gamma^\beta_{\alpha \beta} \right). \)

Now using [29] the full equation takes the form

\[
0 = g^{\mu \nu},_\alpha + g^{\beta \nu} \Gamma^\mu_{\beta \alpha} + g^{\alpha \beta} \Gamma^\nu_{\alpha \beta} - g^{\mu \nu} \Gamma^\beta_{\alpha \beta} + \delta^\nu_{\alpha} \tilde{f}^\mu
\]
This is Eq.(10) in [2]. We also still have

\[ 0 = g^{\alpha\nu,\alpha} - g^{\nu\alpha,\alpha} \]

Finally, we convert the tensor densities to tensors by defining

\[ g_{\alpha\beta} = \frac{g_{\alpha\beta}}{\sqrt{-g}} \]

It follows that \( g = \frac{1}{g} \) and therefore

\[ g_{\alpha\beta} = g_{\alpha\beta} \sqrt{-g} \]

Lower indices of the full equation by contracting with \( g^{\mu\rho}g^{\sigma\nu} \)

\[ 0 = -g_{\sigma\rho,\alpha} + g_{\mu\rho}\Gamma^{\mu}_{\sigma\alpha} + g_{\nu\sigma}\Gamma^{\nu}_{\alpha\rho} - g_{\sigma\rho}\Gamma^{\beta\alpha}_{\alpha\beta} + g_{\sigma\alpha}g_{\mu\rho}\Gamma^{\mu}_{\sigma\nu} \]

Now we substitute to eliminate the densities. Using Eq.(30) and dividing out the determinant yields the final form of the field equation

\[ 0 = -g_{\sigma\rho,\alpha} + g_{\mu\rho}\Gamma^{\mu}_{\sigma\alpha} + g_{\nu\sigma}\Gamma^{\nu}_{\alpha\rho} + g_{\sigma\rho}\phi_{\alpha} + g_{\sigma\alpha} \sqrt{-g} \]

where we define the vector

\[ \phi_{\rho} \equiv -g_{\mu\rho} \sqrt{-g}\Gamma^{\mu}_{\sigma\nu} \]

We also still have

\[ 0 = g^{\alpha\nu,\alpha} - g^{\nu\alpha,\alpha} \]

Along with 16 equations from the variation of \( g^{\alpha\nu} \), and 64 from the connection variation, \( \phi_{\alpha} \) must already be determined. In fact, it is the trace of the torsion.

The remaining parts of [2] consider special cases. First, Einstein shows that if \( \phi_{\alpha} = 0 \) and \( g^{\alpha\beta} \) is symmetric we arrive at general relativity. The paper concludes with a perturbative study focussing on the antisymmetric part of \( g^{\alpha\beta} \).

Here we digress, noting that if we take \( g^{\alpha\beta} \) symmetric but do not set \( \phi_{\alpha} \) to zero, the connection acquires an antisymmetric piece. Solving Eq.(31) with \( g^{\alpha\beta} \) symmetric, by cycling indices and combining in the usual way yields pure-trace torsion

\[ T_{\sigma\alpha\rho} = \Gamma_{\sigma\alpha\rho} - \Gamma_{\sigma\rho\alpha} = g_{\sigma\alpha}\phi_{\rho} - g_{\rho\sigma}\phi_{\alpha} \]

We have seen that this leads to Weyl geometry, with its symmetric, metric compatible connection.

### Appendix II: Weyl Weights

An action functional with an abelian symmetry will have weight zero. From this and the weight \( w_g \) of the metric, the weights of all other fields follow. Only the fields have weight; coordinates do not. The purely numerical arrays \( \eta_{\mu\nu} \) and \( \varepsilon_{\mu\nu\sigma} \) have zero weight.

Since we know the relationship between the metric and the components of the solder form, we have

\[ w[g_{\mu\nu}] = w[e_{\mu}^a e_{\nu}^b \eta_{ab}] \]

\[ w_g = 2w_e + w_\eta \]

and since \( w_\eta = 0 \), the weight of the component matrix of the solder form is \( w_e = \frac{1}{2}w_g \). Since the coordinates have zero weight,

\[ w(e^\alpha) = w_e + w(dx^\mu) = \frac{1}{2}w_g \]

\(^3\text{If not, suppose } w(\eta_{ab}) = w_\eta \neq 0. \text{ Then we may always define a weight zero flat metric by setting } \tilde{\eta}_{ab} = (-\det \eta_{ab})^{-1/4} \eta_{ab}.\]
The volume form is
\[
\Phi = \ast 1 = \frac{1}{4!} e^a \wedge e^b \wedge e^c \wedge e^d e_{abcd}
\]
\[
= \frac{1}{4!} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \sqrt{-g} \varepsilon_{\mu \nu \alpha \beta}
\]
(32)
so its weight is \(2w_g\). We also need the weight of the orthonormal Levi-Civita tensor. We have
\[
e^a \wedge e^b \wedge e^c \wedge e^d e_{abcd} = dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \sqrt{-g} \varepsilon_{\mu \nu \alpha \beta}
\]
where \(\varepsilon_{\mu \nu \alpha \beta}\) is a purely numerical array and the coordinates are of weight zero. With \(w(\sqrt{-g}) = 2w_g\) we have
\[
w(e^a \wedge e^b \wedge e^c \wedge e^d e_{abcd}) = 2w_g
\]
\[
4 \cdot \frac{w_g}{2} + w(e_{abcd}) = 2w_g
\]
and therefore \(w(e_{abcd}) = 0\).

To find the weight of the spin connection, consider the structure equation
\[
d e^a = e^b \wedge \omega^a_{\ b} + T^a
\]
The weight of \(d e^a\) is
\[
w(d e^a) = w(dx^\mu \partial_\mu e^a) = w(e^a)
\]
while
\[
w(e^b \wedge \omega^a_{\ b}) = w(e^b) + w(\omega^a_{\ b})
\]
Equating these gives
\[
w(\omega^a_{\ b}) = 0
\]
\[
w(T^a) = w(e^a) = \frac{1}{2} w_g
\]
For the components of the spin connection,
\[
0 = w(\omega^a_{\ bc})
\]
\[
= w(\omega^a_{\ bc} e^c)
\]
\[
= w(\omega^a_{\ bc}) + w(e^c)
\]
so that \(w(\omega^a_{\ bc}) = -w(e^c)\).

Then for
\[
\Sigma = \delta \omega^a_{\ bc} e^c \wedge e^e \wedge e^d e_{abcd}
\]
we wedge with a solder form and extract the volume form \(\Phi\),
\[
e^f \wedge \Sigma = \delta \omega^a_{\ bc} e^f e^c \wedge e^e \wedge e^d e_{abcd}
\]
\[
= -\delta \omega^a_{\ bc} e^f e_{eabcd} \Phi
\]
\[
= 2 \delta \omega^a_{\ bc} \left( \delta^f_{\ a} \delta^e_{\ b} - \delta^e_{\ a} \delta^f_{\ b} \right) \Phi
\]
\[
= 4 \delta \omega^a_{\ bc} \Phi
\]
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Taking the weight of both sides,

\[ w(e' \Lambda \Sigma) = w(\delta \omega^f_{\epsilon} \Phi) \]
\[ w(e') + w(\Sigma) = w(\delta \omega^f_{\epsilon}) + w(\Phi) \]
\[ w(\Sigma) = -2w(e') + w(\Phi) \]
\[ = -w_g + 2w_g \]

and we conclude that \( w_\Sigma = w_g \).

Collecting these results:

\[ w(e^a) = \frac{1}{2}w_g \]
\[ w(\Phi) = 2w_g \]
\[ w(e_{abcd}) = 0 \]
\[ w(\eta_{ab}) = \frac{1}{2}w_g \]
\[ w(\Sigma) = w_g \]

The final equality, \( w_\Sigma = w_g \), is important for carrying out integration by parts with a covariant derivative.

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