CONSTRUCTION OF THE VORONOI DIAGRAM AND SECONDARY POLYTOPE

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ABSTRACT

A set $S$ of $n$ points in general position in $\mathbb{R}^d$ defines the unique Voronoi diagram of $S$. Its dual tessellation is the Delaunay triangulation (DT) of $S$. In this paper we consider the parabolic functional on the set of triangulations of $S$ and prove that it attains its minimum at DT in all dimensions. The Delaunay triangulation of $S$ is corresponding to a vertex of the secondary polytope of $S$. We proposed an algorithm for DT’s construction, where the parabolic functional and the secondary polytope are used. Finally, we considered a discrete analog of the Dirichlet functional. DT is optimal for this functional only in two dimensions.

Keywords. Voronoi diagram, Delaunay triangulation, regular triangulation, secondary polytope, flip, incremental algorithm.

1 Introduction

Some of the most well-known names in Computational Geometry are those of two prominent Russian mathematicians: Georgy F. Voronoi and Boris N. Delaunay. Their considerable contribution to the Number Theory and Geometry is well known to the specialists in these fields. Surprisingly, their names (their works remained unread and later re-discovered) became the most popular not among "pure" mathematicians, but among the researchers who used geometric applications. Such terms as "Voronoi diagram" and "Delaunay triangulation" are very important not only for Computational Geometry, but also for Geometric Modeling, Image Processing, CAD, Geographic Information System etc.

The Voronoi diagram is generated by a set of $n$ points $S = \{x_1, ..., x_n\}$ in $\mathbb{R}^d$. The Voronoi diagram is the partition of the $\mathbb{R}^d$ into $n$ convex cells, the Voronoi cells $V_i$, where each $V_i$ contains all points of the $\mathbb{R}^d$ closer to $x_i$ than

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to any other point:

\[ V_i = \{ x|\forall j \neq i, d(x, x_i) \leq d(x, x_j) \}, \]

where \( d(x, y) \) is the Euclidean distance between \( x \) and \( y \).

This concept has independently appeared in various fields of science. The earliest significant use of Voronoi diagrams seems to have occurred in the work of Gauss, Dirichlet and Voronoi in their investigations on the reducibility of positive definite quadratic forms. Different names particular to the respective field have been used, such as medial axis transform in biology and physiology, Wigner-Seitz zones in chemistry and physics, domain of action in crystallography, and Thiessen polygons in meteorology and geography [1].

For generic set of \( n \) points \( S \) in \( \mathbb{R}^d \) the straight-line dual of the Voronoi diagram is triangulation of \( S \), called the Delaunay triangulation and denoted by \( DT(S) \). The \( DT(S) \) is triangulation of the convex hull of \( S \) in \( \mathbb{R}^d \) and set of vertices of \( DT(S) \) is \( S \).

Voronoi [21] was the first to consider the dual structure of the Voronoi diagram, where any two points of \( S \) are connected whose regions have a boundary in common. Later Delaunay [6] obtained the same by defining that two points of \( S \) are connected if and only if they lie on a sphere whose interior contains no points of \( S \). After him, the dual of the Voronoi diagram has been denoted Delaunay tessellation or Delaunay triangulation.

Voronoi diagrams and Delaunay triangulations are used in numerous applications. It is widely used in plane and 3D case. A natural question may arise: why those structures are better than the others. Usually the advantages of planar Delaunay triangulation are rationalized by the max-min angle criterion and other properties [1,2,3,6,16,17,19].

The max-min angle criterion requires that the diagonal of every convex quadrilateral occurring in the triangulation "should be well chosen" [19], in the sense that replacement of the chosen diagonal by the alternative one must not increase the minimum of the six angles in the two triangles making up the quadrilateral. Thus the Delaunay triangulation of a planar point set maximizes the minimum angle in any triangle. More specifically, the sequence of triangle angles, sorted from sharpest to least sharp, is lexicographically maximized over all such sequences constructed from triangulation of \( S \).

In the papers [13–15] we defined several functionals on the set of all triangulations of \( S \) in \( \mathbb{R}^2 \) attaining global minimum on the Delaunay triangulation.

The "mean radius" functional is the mean of circumradii of triangles for planar triangulations. Let \( t \) be a triangulation of \( S \) in the plane. Assume that each triangle \( \Delta_i \) of this triangulation is related to the radius \( R_i \) of its circumscribed circle. Thus every triangulation \( t \) is related to the set \( \{ R_{\Delta_1}, \ldots, R_{\Delta_k} \} \) of circumradii of triangles \( \Delta_i \in t \). The numbers of triangles for any two triangulations of \( S \) are equal, so it is possible to compare sets of radii for different triangulations. In particular, it is possible to compare sums of radii: \( \sum R_{\Delta_i} \) or power sums: \( \sum R_{\Delta_i}^a, a > 0 \). It seems that triangulation having minimal sum of radii is "better", because all its triangles in "average" are nearer to the regular triangles.
The functional $R(t, a) = \sum R_{\Delta_i}^t, \alpha > 0$ attains its minimum iff $t$ is Delaunay triangulation [15].

The harmonic index of triangulation has its origin in the theory of the so called "harmonic maps". For polygon $P$ its harmonic index

$$hrm(P) = \sum a_i^2 / S(P),$$

where $a_1, \ldots, a_n$ are the lengths of sides of $P$ and $S(P)$ is its area. This index is the same for similar polygons. It is easy to prove that harmonic index for triangles achieves its minimum iff triangle is equiangular. For planar triangulation $t$ of a set $S$ let denote by $hrm(t)$ (harmonic index of triangulation $t$) the sum of $hrm$ of its triangles:

$$hrm(t) = \sum_{\Delta_i \in t} hrm(\Delta_i).$$

Harmonic index $hrm(t)$ of triangulation $t$ of $S$ achieves its minimum iff $t$ is the Delaunay triangulation of $S$. The harmonic functional for triangle attains its minimum if this triangle is equiangular. Usually, a triangulation is regarded as "good" for different purposes if its triangles are nearly equiangular. The harmonic index of triangulation $t$ achieves its global minimum if $t$ is a part of a regular triangular lattice. In some sense, this result shows that the Delaunay triangulation is as close as possible to equiangular triangulation.

One of the most popular algorithm for constructing planar Voronoi diagram – Delaunay triangulation is so called flipping algorithm [1,8,10]. (If $t$ is planar triangulation of $S$ and $AC$ is an internal edge of $t$, the two triangles $ACB$ and $ACD$ of $t$ incident with $AC$ in triangulation. If quadrilateral $ABCD$ is convex a new triangulation of $S$ may be obtain by removing the edge $AC$ and inserting the edge $BD$. This operation is called the flip or edge–flip). The proof that sequence of flips does not cycle in the algorithm easily follows from consideration of the mean-radius or harmonic functional. Indeed, using flipping algorithm after each flip the functional decreases until Delaunay triangulation is reached.

Among the various proposed methods for constructing Voronoi diagrams in $R^d$, incremental insertion of points is most intuitive and easy implement. Joe [9], Rajan [16], and Edelsbrunner and Shah [8] generalized of planar incremental and flipping algorithms for higher dimensions.

This paper includes algorithm for construction of the Voronoi diagram. First, in section 1 we consider a so called parabolic functional on a set of all triangulations of $S$ and prove that this functional attains its minimum on the Delaunay triangulation of $S$ in all dimensions. The secondary polytope is the original one due Gel’fand, Kapranov, and Zelevinsky [4,5]. They introduced the secondary polytope $Q$ of $S \subset R^d$ that is a convex polytope in $R^n, n = |S|$, and where the vertices of $Q$ are in one-to-one correspondence with the regular triangulations of $S$. In section 2 we show that parabolic functional is a linear function on $R^n$, and Delaunay triangulation of $S$ is a vertex of $Q$ that gives minimum for parabolic function. The incremental algorithm for construction of the Voronoi diagram i.e.
Delaunay triangulation is given in section 3. The main idea of this algorithm is to construct a sequence of regular triangulations (vertices of $Q$) that decrease parabolic functional, and the last triangulation in this sequence is DT. Finally, in section 4 we consider a discrete analogue of the Dirichlet functional on a set of all triangulations of $S$. For $d = 2$ this functional achieves its minimum for DT. We consider case $n = d + 2$ and prove that optimal triangulation depends on points $S$ configuration only. If $d > 2$ then Delaunay triangulation is not optimal for Dirichlet functional. Thus the problem to finding ”good” triangulations for this functional in higher dimensions is opened and more detailed consideration is necessary.

2 Optimality of Delaunay triangulations for the parabolic functional

Throughout this paper $S = \{x_1, \ldots, x_n\}$ denotes a set of $n$ points in general position in $\mathbb{R}^d$. A triangulation of $S$ is a triangulation of the polytope $P = CH(S)$ (convex hull of $S$) with vertices in $S$.

Let $t$ be a triangulation of the set $S$ in $\mathbb{R}^d$, $\Delta_i$ denotes the $i$-th $d$-simplex of $t$ and $x_{ij} \in \mathbb{R}^d$, $j = 0, 1, \ldots, d$ are its vertices. Let

$$V_r(t) = \sum_i (x_{i0}^2 + \ldots + x_{i0}^d) vol(\Delta_i),$$

where $vol(\Delta_i)$ is volume (area for $d = 2$) of the simplex $\Delta_i$. We call functional $V_r$ parabolic (or Voronoi).

The parabolic functional induces an order on triangulations of the set $S$ by the rule: $t_1 > t_2$ iff $V_r(t_1) > V_r(t_2)$. The value of $V_r$ depends on the choice of the origin. If we move the origin to $x_0$ then this order does not change i.e.

$$V_r(t_1) > V_r(t_2) \text{ iff } V_r(t_1, x_0) > V_r(t_2, x_0).$$

The main result for Voronoi functional $V_r$ is the following:

**Theorem 1.** The parabolic (Voronoi) functional $V_r(t)$ achieves its minimum if and only if $t$ is the Delaunay triangulation.

A simple proof of these theorems follows from paraboloid construction of the DT found by Voronoi [21] and rediscovered only in 1979 for a sphere (K.Brown), and later also for a paraboloid.

Let us consider another functional for triangulations:

$$C_2(t) = \sum_i ||c(\Delta_i)||^2 vol(\Delta_i),$$

Let us consider an arbitrary triangulation $t$ of the set $S$ and ”lift” it onto the paraboloid in $\mathbb{R}^{d+1}$, i.e. let us build a polyhedral surface in $\mathbb{R}^{d+1}$ connecting corresponding vertices on the paraboloid. Note that the functional $V_r$ up to a constant equals to the volume of the solid body below this surface. Thus, the minimum of $V_r$ is attained on the Delaunay triangulation.

Let us consider another functional for triangulations:

$$C_2(t) = \sum_i ||c(\Delta_i)||^2 vol(\Delta_i),$$
where $c(\Delta_i)$ is the center (barycenter) of the $\Delta_i$, $c(\Delta_i) = \sum_j x_{ij}/(d + 1)$ and $\| \cdot \|$ is Euclidean norm.

By direct calculation (it is sufficient to check the formula on a simplex) it is possible to prove that

$$(d + 1)^2 C^2(t) + Vr(t) = (d + 1)(d + 2) \int_{CH(S)} \|x\|^2 dx,$$

where $CH(S)$ is convex hull of set $S$ in $\mathbb{R}^d$.

From Theorem 1 and this formula directly follows that:

**Theorem 2.** The functional $C^2(t)$ on triangulations of the set $S$ achieves its maximum if and only if $t$ is the Delaunay triangulation.

### 3 The secondary polytope, regular triangulations, and flips

The following analytic description of the secondary polytope is the original one due Gel’fand, Kapranov, and Zelevinsky [4,5]. We include it here for completeness. They introduced the secondary polytope $Q = \sum(S)$ of an affine point configuration $S$, where the vertices of $\sum(S)$ are in one-to-one correspondence with the regular triangulations of the "primary polytope" $P = CH(S)$ - convex hull of $S$.

Let $S = \{x_i\}$ be a set of $n$ points in $\mathbb{R}^d$ and $t$ is a triangulation of $S$. We are correspond to triangulation $t$ vector $p(t) = (p_1, p_2, \ldots, p_n)$ in $\mathbb{R}^n$, where $p_i = \sum \text{vol}(\Delta_{ij}, \Delta_{ij})$ denotes the $j$-th $d$-simplex of $t$ that incident to vertex $x_i$, and $\text{vol}(\Delta)$ denotes volume of the $d$-simplex i.e. $p_i$ is volume of star of $i$-th vertex of triangulation $t$.

We have $d + 1$ equations:

$$\sum p_i = (d + 1)\text{vol}(CH(S)), \quad \sum x_i p_i = (d + 1)x_c \text{vol}(CH(S)),$$

where $x_c$ is center (center of mass) of $CH(S)$. Since the right hand sides of these equations not depend of triangulation that in fact $p(t) \in \mathbb{R}^{n-d-1}$. Let

$$Q = \text{convex hull}\{p(t) : t \text{ is a triangulation of } S\},$$

i.e. $Q$ is a convex hull of the set of images in $\mathbb{R}^n$ all triangulations of $S$. The convex polytope $Q$ called secondary polytope and denoted by $\sum(S)$. The dimension of this polytope is $n - d - 1$.

Even for a simple configuration of $S$ the secondary polytope is not simple. When $S$ in the plane consist of $n$ vertices of a convex $n$-gon, then secondary polytope called associahedron [12]. Associahedron is a simple $(n-3)$-dimensional polytope with $n(n - 3)/2$ facets. For example, associahedron of a pentagon is a pentagon, and associahedron of a hexagon is a simple 3-polytope with 14
vertices, 21 edges, and 9 facets. An interesting application of the associahedron to the theoretical Computer Science has been given by Sleator, Tarjan, and Thurston [20]. These authors derive a tight upper bound for the rotation distance between binary trees with \( n \) nodes by proving that the diameter of the associahedron equals \( 2n - 10 \), for large \( n \).

A triangulation \( t \) of \( S \) is said to be regular if there exists a function on \( P \) that is piecewise linear and strictly convex with respect to \( t \). (A convex piecewise linear function over a triangulation \( t \) is said to be strictly convex if it is given by a different linear function on each maximal cell of \( t \)).

Regular triangulations are really just the duals of power diagrams, and Edelsbrunner call them weighted Delaunay triangulations [8].

There are several equivalent ways to define the notion of a regular triangulation of \( S \). For example, Gel’fand, Kapranov, and Zelevinski [5] call regular a following triangulation \( t \):

Choose numbers \( y_1, \ldots, y_n \) and let \( W = \{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{R}^{d+1} \). If \( \Delta = \{(x_{i1}, y_{i1}), \ldots, (x_{i(d+1)}, y_{i(d+1)})\} \) is a facet of \( W \) in the lower hull of \( W \) (i.e., the last component of the outward normal of its supporting hyperplane is negative) then \( \Delta = \{x_{i1}, \ldots, x_{i(d+1)}\} \) is a \( d \)-face of the triangulation \( t \).

Let \( y_i = ||x_i||^2 \). Then we get the Delaunay triangulation of \( S \). Therefore, from this definition the Delaunay triangulation is regular.

Consider a set \( S \) of \( d + 2 \) points in \( \mathbb{R}^d \). From Gale diagram [7] follows (see also Schlegel [18], Lawson [11]) that there are exactly two ways to triangulate \( S \). Indeed, the two ways correspond to the two sides (lower and upper) of the \((d + 1)\)-simplex that is convex hull of corresponding lifted points in \( \mathbb{R}^{d+1} \). A flip is the operation that substitutes one triangulation of \( S \) for the other [8].

There are three types of flips in two dimensions, and we denote a flip by the number of triangles before and after the flip. So the flips in two dimension are of type '1 to 3', '2 to 2', and '3 to 1'. The first type introduces a new point, and the last type removes a point. The flips in three dimensions can be classified as '4 to 1', '3 to 2', '2 to 3', and '1 to 4'.

Now we can give the main results of this section. First theorem belongs to Gel’fand, Kapranov, and Zelevinsky. They used another terminology.

**Theorem 3.** The vertices of the secondary polytope \( Q = \sum(S) \) are in one-to-one correspondence with the regular triangulations of \( S \), and the edges of \( Q \) are corresponding to flips.

From this theorem and Section 1 follow:

**Theorem 4.** There is a sequence of flips that connect any regular triangulation of \( S \) with the Delaunay triangulation of \( S \), and after each flip decreases parabolic functional.

**Proof.** It is easy to see that for any triangulation \( t \) a parabolic functional \( Vr(t) = \sum ||x_1||^2 p_1 \), where \( p_i = \sum \text{vol}(\Delta_{ij}) \) as above. \( p_i \) is \( i \)-th coordinate of \( \mathbb{R}^n \) and therefore \( Vr \)-functional is a linear function on \( \mathbb{R}^n \). Secondary polytope \( Q = \sum(S) \) is a convex polytope in \( \mathbb{R}^n \), and Delaunay triangulation of \( S (DT(S)) \) is a
vertex of $Q$ that gives minimum for function $Vr$. It is clear how to find sequence of neighboring vertices of $Q$ decreasing parabolic functional and connected any regular triangulation (vertex of $Q$) with vertex corresponding to the Delaunay triangulation.

4 Incremental construction of the Voronoi diagram.

A natural idea is to construct the Voronoi diagram by incremental insertion, i.e. to obtain $V(S)$ from $V(S)\setminus \{x\}$ by inserting the point $x$. The insertion process is, maybe, better described, and implemented in the dual environment, for the Delaunay triangulation: construct $DT_i = DT(\{x_1, x_2, \ldots, x_{i-1}, x_i\})$ by inserting the point $x_i$ into $DT_{i-1}$. The advantage over a direct construction of $V(S)$ is that Voronoi vertices that appear in intermediate diagrams but not in the final one need not be constructed and stored.

Several algorithms proposed for Delaunay triangulation are based on the notion of a local transformation henceforth referred to as a flip. Historically the first such algorithm is due to Lawson [10]. Given a finite point set in the plane, the algorithm first construct an arbitrary triangulation of the set. This triangulation is then gradually altered through a sequence of edge-flips until the Delaunay triangulation is obtained. The generalization of this method to $\mathbb{R}^d$, $d > 2$ has difficulties, and it is incorrect if the flips are applied to an arbitrary initial triangulation [8]. Joe [9] shows that if a single point, $x_i$, is added to the Delaunay triangulation $DT_{i-1}$ in $\mathbb{R}^d$ then many different sequences of flips will succeed in constructing the Delaunay triangulation $DT_i$. This can be used as the basis of an incremental algorithm. Rajan [16] considers Delaunay triangulation in arbitrary dimensions, $\mathbb{R}^d$, and argues that a single point can always be added by a sequence of flips. However, he needs a priority queue to find the appropriate sequence, which takes logarithmic time per flip. Edelsbrunner and Shah [8] using ”weighted points” method unifies and extends the algorithmic results of Joe [9] and Rajan [16]. In particular, they show that many different sequences of flips can be used to add a single point to a regular triangulation in $\mathbb{R}^d$. This eliminates the need for a priority queue that sorts the flips. This section we show how to applied parabolic functional and secondary polytope for construction of the Voronoi diagram incrementally.

The algorithm constructs the Delaunay triangulation of a give set $S = \{x_1, x_2, \ldots, x_n\}$ in $\mathbb{R}^d$ incrementally. It is convenient to first construct an artificial $d$-simplex $S_0 = \{x_d, x_{d+1}, \ldots, x_0\}$, so that $S$ is contained in it. The $d + 1$ artificial points can be conveniently chosen at infinity, so that choice of points guarantees that $DT(S)$ is a subcomplex of $DT(S \cup S_0)$ [8]. In fact, $DT(S)$ consist of all simplices of $DT(S \cup S_0)$ that are not incident to any point of $S_0$.

Let $t$ be a triangulation of $S$. Call $t$ locally non-optimal triangulation if there is a convex subcomplex $\sigma \in t$ that after the flip $\sigma$ a new triangulation is
regular and decreases parabolic functional (see section 1.) i.e. \( V_r(t) > V_r(t') \), where \( t' \) is the new triangulation. In accordance with this definition we call \( t \) L.O.T. (locally optimal triangulation) if \( t \) is not locally non–optimal. We denote a locally optimal triangulation of \( S \) as \( LOT(S) \). We will show later that \( LOT(S_i) = DT_i \) for all \( i = 0, 1, ..., n \).

THE INCREMENTAL ALGORITHM FOR CONSTRUCTION OF \( LOT(S) \)

1 Construct \( LOT(S_0) \);
2 for \( i := 1 \) to \( n \) do
3 locate the \( d \)-simplex \( s \) in \( LOT(S_{i-1}) \) that contains \( x_i \)
4 if \( LOT(S_{i-1} \cup \{x_i\}) \) is not L.O.T. then
5 flip \( LOT(S_{i-1} \cup \{x_i\}) \);
6 while there exist non–optimal \( d+2 \)-subcomplex do
7 find a non–optimal \( d+2 \)-subcomplex \( \sigma \);
8 flip \( \sigma \)
9 endwhile
10 endif
11 endfor

This algorithm could fail for two reason. First, it could be that the while loop does not terminate, because it cycles in an infinite loop of flips. Second, if the algorithm would stop before reaching the \( DT(S) \). We show this cannot happen.

The proof that sequence of flips in the algorithm does not cycle easily follows from the consideration of the parabolic functional \( V_r(t) \). Indeed, this functional decreases after each flip in the algorithm.

For each \( i \) in algorithm \( LOT(S_i) \) is locally optimal triangulation of \( S_i \). It means, that \( LOT(S_i) \) is regular triangulation, and there is no a locally non-optimal subcomplex \( \sigma \). Then \( LOT(S_i) \) is a vertex of a secondary polytope \( Q_i = \sum(S_i) \). \( DT_i \) is vertex of \( Q_i \) also, and therefore from Theorem 4 (see section 2.) follows that there is sequence of flips that connect \( LOT(S_i) \) and \( DT_i \). From theorem 1 follows that \( DT_i \) gives minimum for parabolic functional. From other side parabolic functional cannot be decrease for \( LOT(S_i) \). Consequently \( LOT(S_i) \) is \( DT_i \), and therefore

Theorem 5. Constructed in the algorithm a locally optimal triangulation of \( S \) (\( LOT(S) \)) is the Delaunay triangulation of \( S \).

5 The Dirichlet functional on triangulations

Let \( S = \{x_i\} \) be a set of \( n \) points in \( \mathbb{R}^d \), each associated with a real number \( y_i \). Denote by \( Y \) the set of these numbers, i.e. \( Y = (y_1, ..., y_n) \). There are a lot of different problems in Geography, Geology, Topography, CAD/CAM etc., where we need to construct a surface in \( \mathbb{R}^{d+1} \) corresponding to this dataset. The main problem is the following: to find a function \( y = f(x) \), such that \( f(x_i) = y_i \). One of the oldest and the most famous methods is modeling by triangulation. If we
have some triangulation of $S$ then for a set of data $Y$ there is only one method to construct a piecewise linear function (polyhedral surface) on this triangulation. Usually Delaunay triangulation is used for this purpose.

One of the minimum criterion is a discrete analogue of the Dirichlet functional: $\int [||\text{grad } f(x)||^2] \, dx$. For interval $(d = 1)$, a spline of deg = $2k - 1$ is a function $y = f(x)$ such that $f(x_i) = y_i$ and

$$\int_a^b [f^{(k)}(x)]^2 \, dx = \text{min}.$$  

For $k = 1, d > 1$ and piecewise linear function $f$ we get

$$DF(t, Y) = \int_{CH(S)} [||\text{grad } f(x)||^2] \, dx = \sum_i \frac{(\text{vol}(\Delta_i(Y)))^2}{\text{vol}(\Delta_i)} - \text{vol}(CH(S)).$$  

The triangulation of $S$ that minimizing the functional $DF$ can be called the discrete spline triangulation (DST). For the plane DST does not depend upon $Y$ and it is DT. Rippa [11] for $d = 2$ proved that $DF(t, Y)$ achieves its minimum iff $t$ is DT.

For $d > 2$ the Delaunay triangulation of $S$ could be not optimal for this functional. Let

$$x_0 = (0, 0, \cdots, 0); \quad x_i = (0, \cdots, 0, 1, 0, \cdots, 0), i = 1, \cdots, d; \quad x_{d+1} = (a, \cdots, a).$$  

There are exactly two distinct triangulations of $S = (x_0, x_1, \cdots, x_{d+1})$. Denote by $t_1$ a triangulation consist of two simplices: $\Delta_1 = (x_0, x_1, \cdots, x_d); \Delta_2 = (x_{d+1}, x_1, \cdots, x_d)$, and by $t_2$ another one. Then $t_1$ is DT iff $a > 1$. It is easy to show by direct calculation that $t_1$ is DST iff $a > \frac{1}{d-1}$. Therefore, for $\frac{1}{d-1} < a < 1$ $t_1$ is DST, but is not DT.

The proof of Rippa’s theorem directly follows from the fact that the $DF$–functional for triangulation $t$ of a quadrilateral is minimum if $t$ is DT. The proof also follows from some general result that is given below.

Let $S$ be a set of $d + 2$ points $x_1, \cdots, x_{d+2}$ in $\mathbb{R}^d$. Suppose $S$ admits two triangulations $t_1$ and $t_2$, and $Y = (y_1, \cdots, y_{d+2})$ is a set numbers corresponding to $x_1, \cdots, x_{d+2}$ as above. Let $B(Y, S) = DF(t_1, Y) - DF(t_2, Y)$. Note $B(Y, S)$ is a quadratic form depending on $Y$.

**Theorem 6.** The optimal (DST) triangulation of $S$ for $n = d + 2$ does not depend on $Y$.

**Proof.** It is easy to see that for arbitrary set of real numbers $(a_0, a_1, \cdots, a_d)$:

$$B(\hat{Y}, S) = B(Y, S), \quad \text{where} \quad \hat{Y} = (\hat{y}_1, \cdots, \hat{y}_n), \quad \text{and}$$  

$$\hat{y}_i = y_i + a_0 + \sum_{j=1}^d a_j x_{ij}; \quad x_i = (x_{i1}, \cdots, x_{id}) \in S.$$
Then for \( d + 1 \)-dimensional subspace \( \mathcal{R} \subset \mathbb{R}^n \) that is the linear hull of the set of vectors:

\[
(1, \ldots, 1), \quad (x_{11}, \ldots, x_{n1}), \ldots, (x_{1d}, \ldots, x_{nd})
\]

the quadratic form \( B(Y, S) \) is vanished, and \( B(Y + Z, S) = B(Y, S) \) if \( Z \in \mathcal{R} \). Note that \( n = d + 2 \) and \( \dim \mathcal{R} = d + 1 \) therefore \( B(Y, S) \) could be not vanish only on \( 1 \)-dimensional subspace of \( \mathbb{R}^n \) that is orthogonal to \( \mathcal{R} \). Thus \( B(Y, S) = \text{const}(S)L^2(Y) \), where \( L(Y) \) is some linear form on \( Y \in \mathbb{R}^n \). Therefore sign of the \( B(Y, S) = DF(t_1, Y) - DF(t_2, Y) \) does not depend on \( Y \), and if \( t_1 \) is optimal for some \( Y \) (i.e. \( DF(t_1, Y) < DF(t_2, Y) \)) that it is optimal for any \( Y \).

We state an open problem concerning DST:

- Does DST depend on \( Y \), when \( n > d + 2 \) ?

6 Concluding Remarks.

Voronoi diagram and Delaunay triangulation have a fair number of applications, including the generation of grids for point configuration and for surface interpolation. No doubt that in two-dimension these tessellations are the best for these purposes. Indeed, the main motivation for studying the problems solved in this paper is our intention to implement Voronoi and Dirichlet functionals in dimensions beyond \( \mathbb{R}^3 \). The optimal triangulations for these functionals in \( \mathbb{R}^d, d > 2 \) could be not the equal. We do not know is optimal triangulation for Dirichlet functional is regular? If it is regular then for it construction the algorithm in section 3 is suitable. Instead of \( V_r \)-functional there have to be used \( DF \)-functional. It would be interesting to study dual tessellation for DST. In other word, what is analogue of Voronoi diagram for DST?

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