FANO MANIFOLDS WITH LEFSCHETZ DEFECT 3

C. CASAGRANDE, E. A. ROMANO, AND S. A. SECCI

Abstract. Let $X$ be a smooth, complex Fano variety, and $\delta_X$ its Lefschetz defect. By [Cas12], if $\delta_X \geq 4$, then $X \cong S \times T$, where $\dim T = \dim X - 2$. In this paper we prove a structure theorem for the case where $\delta_X = 3$. We show that there exists a smooth Fano variety $T$ with $\dim T = \dim X - 2$ such that $X$ is obtained from $T$ with two possible explicit constructions; in both cases there is a $\mathbb{P}^2$-bundle $Z$ over $T$ such that $X$ is the blow-up of $Z$ along three pairwise disjoint smooth, irreducible, codimension 2 subvarieties. Then we apply the structure theorem to Fano 4-folds, to the case where $X$ has Picard number 5, and to Fano varieties having an elementary divisorial contraction sending a divisor to a curve. In particular we complete the classification of Fano 4-folds with $\delta_X = 3$, started in [CR22].

This version of the paper incorporates the published article with its corrigendum [CRS22], where a missing case in Prop. 7.1 is added.

1. Introduction

Let $X$ be a smooth, complex Fano variety. The Lefschetz defect $\delta_X$ is an invariant of $X$ which depends on the Picard number of prime divisors in $X$. More precisely, consider the vector space $N_1(X)$ of 1-cycles in $X$, with real coefficients, up to numerical equivalence; its dimension is the Picard number $\rho_X$ of $X$. Given a prime divisor $D$ in $X$, we consider the pushforward $\iota_*: N_1(D) \to N_1(X)$ induced by the inclusion $\iota: D \hookrightarrow X$, and set $N_1(D, X) := \iota_*(N_1(D)) \subseteq N_1(X)$; finally:

$$\delta_X := \max\{\text{codim} N_1(D, X) \mid D \text{ a prime divisor in } X\}.$$ 

The main property of the Lefschetz defect is the following.

Theorem 1.1 ([Cas12], Th. 3.3). Let $X$ be a smooth Fano variety with $\delta_X \geq 4$. Then $X \cong S \times T$, where $S$ is a del Pezzo surface with $\rho_S = \delta_X + 1$.

In this paper we consider Fano varieties $X$ with $\delta_X = 3$. Even if such $X$ does not need to be a product, it turns out that it still has a very rigid and explicit structure. More precisely, we show that there exist a smooth Fano variety $T$ with $\dim T = \dim X - 2$, $\rho_T = \rho_X - 4$, and $\delta_T \leq 3$, and a $\mathbb{P}^2$-bundle $Z$ over $T$, such that $X$ is obtained by blowing-up $Z$ along three pairwise disjoint smooth, irreducible, codimension 2 subvarieties $S_1, S_2, S_3$. The $\mathbb{P}^2$-bundle $\varphi: Z \to T$ is the projectivization of a suitable decomposable vector bundle on $T$, and $S_2$ and $S_3$ are sections of $\varphi$. Instead $\varphi|_{S_1}: S_1 \to T$ is finite of degree 1 or 2.

To state our results, we first describe two explicit ways to obtain $X$ from $T$; the two constructions differ in the degree of $S_1$ over $T$. Then we prove (Th. 1.4) that every

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Fano variety $X$ with Lefschetz defect 3 is obtained by one of these two constructions; this is our main result.

**Proposition 1.2** (Construction A). Let $T$ be a smooth Fano variety with $\delta_T \leq 3$, and $D_1, D_2, D_3$ divisors on $T$ such that $-K_T + D_i - D_j$ is ample for every $i, j \in \{1, 2, 3\}$. Set 

$$Z := \mathbb{P}_T(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3)) \xrightarrow{\varphi} T,$$

and let $S_i \subset Z$ be the section of $\varphi$ corresponding to the projection onto the summand $\mathcal{O}_T(D_i)$, for $i = 1, 2, 3$. Finally let $h: X \to Z$ be the blow-up of $S_1, S_2, S_3$, and $\sigma := \varphi \circ h: X \to T$.

Then $X$ is a smooth Fano variety with $\delta_X = 3$, $\dim X = \dim T + 2$, and $\rho_X = \rho_T + 4$.

**Proposition 1.3** (Construction B). Let $T$ be a smooth Fano variety with $\delta_T \leq 3$, and $N$ a divisor on $T$ such that $N \not\equiv 0$ and both $-K_T + N$ and $-K_T - N$ are ample. Set 

$$Z := \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}) \xrightarrow{\varphi} T,$$

and let $S_2, S_3 \subset Z$ be the sections of $\varphi$ corresponding to the projections onto the summands $\mathcal{O}$. Let $H$ be a tautological divisor of $Z$, and assume that there exists a smooth, codimension 2 subvariety $S_1 \subset Z$ which is a complete intersection of general elements in the linear systems $|H|$ and $|2H|$. Finally let $h: X \to Z$ be the blow-up of $S_1, S_2, S_3$, and $\sigma := \varphi \circ h: X \to T$.

Then $X$ is a smooth Fano variety with $\delta_X = 3$, $\dim X = \dim T + 2$, and $\rho_X = \rho_T + 4$.

Let us note that Constructions A and B give new, explicit ways to construct Fano varieties of any dimension.

**Theorem 1.4** (Structure theorem). Every smooth Fano variety $X$ with $\delta_X = 3$ is obtained with Construction A or B.

When $\dim X = 4$ and $\rho_X = 5$, Th. [1.4] is proved in [CR22], and used to obtain the classification of Fano 4-folds with $\delta_X = 3$ and $\rho_X = 5$. To treat the general case, we apply the same strategy as in [CR22], adapting the proof step by step. Our starting point is the existence of a fibration in del Pezzo surfaces $\sigma: X \to T$ after [Cas12, Th. 3.3]. We give an outline of the proof at the end of the Introduction.

In particular, given a Fano variety $X$ of dimension $n$ with $\delta_X = 3$, by Th. [1.4] there exist a smooth Fano variety $T$ with $\dim T = n - 2$, $\rho_T = \rho_X - 4$, and $\delta_T \leq 3$, and a morphism $\sigma: X \to T$, such that $\sigma$ realizes $X$ as a blow-up of a $\mathbb{P}^2$-bundle $Z$ over $T$, as in Construction A or B. We will denote $X$ by $X_A$ in the former case, and by $X_B$ in the latter.

Let us now describe some applications of the structure theorem.

**Fano 4-folds.** In dimension 4, we complete the classification of Fano 4-folds with Lefschetz defect 3, started in [CR22]; note that all of them are rational, see [MR21 Cor. 1.3]. We treat the last case left open, that is $\delta_X = 3$ and $\rho_X = 6$, in which we show that there are 11 families, among which 8 are toric; we refer the reader to Section 7 for more details and for the description of the three non-toric families and their invariants. Note that the toric case is simpler, as toric Fano 4-folds have been classified by Batyrev [Bat99, Sat00]. The final classification is as follows.

**Proposition 1.5.** Let $X$ be a Fano 4-fold with $\delta_X = 3$. Then $5 \leq \rho_X \leq 8$ and there are 19 families for $X$, among which 14 are toric. More precisely:
- if $\rho_X = 8$, then $X \cong F \times F$, $F$ the blow-up of $\mathbb{P}^2$ at 3 non-collinear points;
- if $\rho_X = 7$, then $X \cong F \times F'$, $F'$ the blow-up of $\mathbb{P}^2$ at 2 points;
- if $\rho_X = 6$, there are 11 families for $X$, among which 8 are toric;
- if $\rho_X = 5$, there are 6 families for $X$, among which 4 are toric.

We also apply our results to the study of conic bundles $\eta: X \to Y$ where $X$ is a Fano 4-fold and $\rho_X - \rho_Y \geq 3$, see Cor. 7.2.

The case $\rho_X = 5$. The second case that we consider in detail is that of Fano varieties with Lefschetz defect 3 and Picard number 5, the minimal one, in Section 8. After Th. 1.1 there are a smooth Fano variety $T$ with $\rho_T = 1$ and a morphism $\sigma: X \to T$ that realizes $X$ as a blow-up of a $\mathbb{P}^2$-bundle over $T$ as in Construction A or B. We show that in this case $T$ and $\sigma$ are uniquely determined, and we list explicitly the different $X$ that are obtained from a given $T$. To state the result, let us fix some notation for the case $\rho_T = 1$: $\mathcal{O}_T(1)$ is the ample generator of $\text{Pic}(T) \cong \mathbb{Z}$, and $\mathcal{O}_T(a) := \mathcal{O}_T(1)^\otimes a$ for every $a \in \mathbb{Z}$. Moreover $\mathcal{O}_T(-K_T) \cong \mathcal{O}_T(i_T)$, where $i_T$ is the index of $T$.

**Proposition 1.6.** Let $X$ be a smooth Fano variety with $\delta_X = 3$ and $\rho_X = 5$. Then $T$ and the morphism $\sigma: X \to T$ as in Construction A or B are uniquely determined; we have $X = X_A$ if $\sigma$ is smooth, and $X = X_B$ if $\sigma$ is not smooth.

Let $X = X_A$. Then there are uniquely determined integers $a, b$ with
\[ b \leq 0, \quad |b| \leq \frac{i_T - 1}{2}, \quad \text{and} \quad |b| \leq a \leq i_T - 1 - |b|, \]
such that $X$ is obtained with Construction A from $T$ with $Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O}(b))$.

Let $X = X_B$. Then there is a uniquely determined integer $a$ with $1 \leq a \leq i_T - 1$ such that $X$ is obtained with Construction B from $T$ with $Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O})$.

**Fano varieties containing a divisor with Picard number 2.** We note that the assumptions $\rho_X = 5$ and $\delta_X = 3$ imply that $X$ contains a prime divisor $D$ with $\dim N_1(D, X) = 2$; in fact it is easy to see that all the varieties as in Prop. 1.6 also contain a prime divisor $D'$ with $\rho_{D'} = 2$. We obtain the following application to Fano varieties containing a prime divisor with $\rho = 2$; an analogous result for the case of a prime divisor with $\rho = 1$ is given in [CD15, Th. 3.8].

**Corollary 1.7.** Let $X$ be a smooth Fano variety containing a prime divisor $D$ with $\rho_D = 2$, or more generally with $\dim N_1(D, X) = 2$. Then either $X \cong S \times T$ where $S$ is a del Pezzo surface and $\rho_T = 1$, or $\rho_X \leq 5$. Moreover, $\rho_X = 5$ if and only if $X$ is as in Prop. 1.6.

**Elementary divisorial contractions.** Corollary 1.7 is also related to the study of Fano varieties having an elementary divisorial contraction $\tau: X \to X'$ where $\tau(\text{Exc}(\tau))$ is a curve, because then automatically $\dim N_1(\text{Exc}(\tau), X) = 2$. It follows from Th. 1.1 that $\rho_X \leq 5$, and Tsukikawa [13su10] has classified the case $\rho_X = 5$ when $\tau$ is the blow-up of a smooth curve in a smooth variety. We generalize this classification to an arbitrary elementary divisorial contraction $\tau$ such that $\dim \tau(\text{Exc}(\tau)) = 1$, as follows.

**Theorem 1.8.** Let $X$ be a smooth Fano variety of dimension $n \geq 4$ with $\rho_X = 5$. Then the following are equivalent:
(i) there is an elementary divisorial contraction \( \tau : X \to X' \) such that \( \tau(\text{Exc}(\tau)) \) is a curve;

(ii) \( X \) is obtained with Construction \( A \) or \( B \) from a smooth Fano variety \( T \) such that 
\[
\dim T = n - 2, \quad \rho_T = 1, \quad i_T > 1,
\]
with \( Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O}) \) and \( a \in \{1, \ldots, i_T - 1\} \).

If these conditions hold we have \( \text{Exc}(\tau) \cong \mathbb{P}^1 \times T, \mathcal{N}_{\text{Exc}(\tau)/X} \cong \pi_{T}^* \mathcal{O}(-1) \otimes \pi_T^* \mathcal{O}(-a), \) and \( \tau(\text{Exc}(\tau)) \cong \mathbb{P}^1 \).

When \( \dim X = 4 \), Cor. 1.7 and Th. 1.8 are already proved in [CR22].

**Strategy of the proof of Theorem 1.4.** Our starting point is the existence, from [Cas12, Th. 3.3], of a flat fibration \( \sigma : X \to T \), where \( T \) is a smooth Fano variety with \( \dim T = \dim X - 2 \) and \( \rho_T = \rho_X - 4 \); moreover \( \sigma \) factors as \( X \to Y \to T \), where the first map is a conic bundle, and the second one is smooth with fiber \( \mathbb{P}^1 \). We collect the properties of \( \sigma \) in Th. 5.3.

We show that the fibration \( \sigma : X \to T \) has a different factorization as \( X \to Z \xrightarrow{\varphi} T \), where \( \varphi \) is a \( \mathbb{P}^2 \)-bundle, and \( X \to Z \) is the blow-up of three pairwise disjoint smooth, irreducible subvarieties \( S_i \subset Z \) of codimension 2, horizontal for \( \varphi \).

Then we prove that \( S_2 \) and \( S_3 \) are always sections of \( \varphi \). When \( S_1 \) is a section as well, we show that \( \sigma : X \to T \) is as in Construction \( A \).

Otherwise, we study the degree \( d > 1 \) of \( S_1 \) and \( T \). Fiberwise, for \( t \in T \) general, we have \( Z_t := \varphi^{-1}(t) \cong \mathbb{P}^2 \), \( X_t := \sigma^{-1}(t) \) a smooth del Pezzo surface, and \( X_t \to Z_t \) the blow-up of the \( d + 2 \) points \( (S_1 \cup S_2 \cup S_3) \cap Z_t \). This is the hardest part of the proof; we restrict to a general curve \( C \subset T \), and construct \( Z := \varphi^{-1}(C) \) a divisor \( D \) which is a \( \mathbb{P}^1 \)-bundle over \( C \) and contains \( S_1 \cap Z \). Therefore, for \( t \in C \) general, the \( d \) points \( S_1 \cap Z_t \) are aligned; since \( X_t \) is del Pezzo, this implies that \( d = 2 \). Finally we show that \( \sigma : X \to T \) is as in Construction \( B \).

**Structure of the paper.** In Section 2 we fix the notation and prove some preliminary results. In Sections 3 and 4 we present Constructions \( A \) and \( B \) respectively, and prove Propositions 1.2 and 1.3 while in Section 5 we prove Th. 1.4.

In the second part of the paper we consider the applications of the structure theorem. In Section 6 we study the del Pezzo fibration \( \sigma : X \to T \); we describe its fibers, the relative cone \( \text{NE}(\sigma) \), the relative contractions, and the different factorizations of \( \sigma \). We also give some conditions on \( T \) in order to perform Constructions \( A \) and \( B \) to obtain Fano varieties \( X \) different from the product \( F \times T \), \( F \) the blow-up of \( \mathbb{P}^2 \) at three non-collinear points. Finally in Section 7 we give the applications to Fano 4-folds, and in Section 8 the applications to Fano varieties with \( \rho_X = 5 \).

**2. Notation and preliminaries**

We work over the field of complex numbers. Let \( X \) be a smooth projective variety of arbitrary dimension.

\( \mathcal{N}_1(X) \) (respectively, \( \mathcal{N}^1(X) \)) is the real vector space of one-cycles (respectively, Cartier divisors) with real coefficients, modulo numerical equivalence, and \( \dim \mathcal{N}_1(X) = \dim \mathcal{N}^1(X) = \rho_X \) is the Picard number of \( X \).

Let \( C \) be a one-cycle of \( X \), and \( D \) a divisor of \( X \). We denote by \([C]\) (respectively, \([D]\)) the numerical equivalence class in \( \mathcal{N}_1(X) \) (respectively, \( \mathcal{N}^1(X) \)). We also denote by \( C \perp \subset \mathcal{N}^1(X) \) (respectively \( D \perp \subset \mathcal{N}_1(X) \)) the orthogonal hyperplanes.
The symbol \( \equiv \) stands for numerical equivalence (for both one-cycles and divisors), and \( \sim \) stands for linear equivalence of divisors.

NE(\( X \)) \( \subset \mathcal{N}_1(\( X \)) \) is the convex cone generated by classes of effective curves. An extremal ray \( R \) is a one-dimensional face of NE(\( X \)). When \( X \) is Fano, the length of \( R \) is \( l(R) = \min \{ -K_X \cdot C | C \text{ a rational curve in } X \} \).

A facet of a convex polyhedral cone \( \mathcal{C} \) is a face of codimension one; moreover \( \mathcal{C} \) is simplicial when it can be generated by linearly independent elements. We denote by \( \varphi \) Notation as in Rem. 2.1 with Lemma 2.2.

When \( Z \) is normal and projective. The corresponding to the two summands of \( \oplus \rho \) \( \mathcal{N}(\( X \)) \) is relatively ample. A contraction is elementary if \( \rho X - \rho Y = 1 \).

A conic bundle \( X \to Y \) is a contraction of fiber type where every fiber is one-dimensional and \(-K_X\) is relatively ample.

We gather here some preliminary results that we need in the sequel.

**Remark 2.1.** Let \( T \) be a smooth projective variety and \( D_1, \ldots, D_r \) divisors on \( T \). Set \( Z := \mathbb{P}(\mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r)) \to T \) and let \( H \) be the tautological divisor. For every \( i = 1, \ldots, r \) set

\[
F_i := \mathbb{P}(\oplus_{j \neq i} \mathcal{O}(D_j)) \hookrightarrow Z
\]

with the embedding given by the projection \( \oplus_j \mathcal{O}(D_j) \to \oplus_{j \neq i} \mathcal{O}(D_j) \). Then:

\[
F_i \sim H - \varphi^* (D_i) \quad \text{for every } i = 1, \ldots, r, \quad \text{and} \quad K_Z - \varphi^* K_T \sim -F_1 - \cdots - F_r.
\]

**Lemma 2.2.** Notation as in Rem. 2.1 with \( r = 3 \). Let \( S_3 \subset Z \) be the section of \( \varphi \) corresponding to the projection \( \mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3) \to \mathcal{O}(D_3) \), and let \( h: X \to Z \) be the blow-up of \( S_3 \), with exceptional divisor \( E_3 \subset X \).

Then \( E_3 \equiv \mathbb{P}(\mathcal{O}(-K_T + D_3 - D_1) \oplus \mathcal{O}(-K_T + D_3 - D_2)) \) and \( (-K_X)|_{E_3} \) is the tautological line bundle.

**Proof.** We have \( \mathcal{N}_{S_3/Z} \cong \mathcal{O}(D_1 - D_3) \oplus \mathcal{O}(D_2 - D_3) \), \( E_3 \equiv \mathbb{P}(\mathcal{N}_{S_3/Z}) \), and \( \mathcal{O}_X(-E_3)|_{E_3} \) is the tautological line bundle.

Let \( E'_i \subset X \) be the transform of \( F_i \subset Z \), for \( i = 1, 2, 3 \); note that \( S_3 = F_1 \cap F_2 \) and \( S_3 \cap F_3 = \emptyset \). Set \( \sigma := \varphi \circ h: X \to T \) and \( \sigma_3 := \sigma|_{E_3}: E_3 \to T \). By Rem. 2.1 we have

\[
-K_Z \sim \varphi^*(-K_T) + F_1 + F_2 + F_3,
\]

so that

\[
-K_X \sim h^*(-K_Z) - E_3 \sim \sigma^*(-K_T) + E'_1 + E'_2 + E'_3 + E_3.
\]

and \( \mathcal{O}_X(-K_X)|_{E_3} \cong \mathcal{O}_{E_3}(\sigma_3^*(-K_T) + G_1 + G_2) \otimes \mathcal{O}_X(E_3)|_{E_3} \)

where \( G_i := E'_i \cap E_3 \) for \( i = 1, 2 \). Note that \( G_1 \) and \( G_2 \) are the sections of \( \sigma_3 \) corresponding to the two summands of \( \mathcal{O}(D_1 - D_3) \oplus \mathcal{O}(D_2 - D_3) \), so by Rem. 2.1 we have \( \mathcal{O}_{E_3}(G_1 + G_2) \cong \mathcal{O}_X(-2E_3 + \sigma^*(2D_3 - D_1 - D_2))|_{E_3} \). Finally we get

\[
\mathcal{O}_X(-K_X)|_{E_3} \cong \mathcal{O}_X(-E_3)|_{E_3} \otimes \mathcal{O}_{E_3}(\sigma_3^*(-K_T + 2D_3 - D_1 - D_2))
\]

which yields the statement.

In a similar way one shows the following.
Remark 2.4. Let $t$ be a tautological line bundle.

Lemma 2.3. Let $h: X \to Z$ be the blow-up of $S_2$ and $S_3$. Let $E'_1 \subset X$ be the transform of $F_1 \subset Z$.

Then $E'_1 \cong \mathbb{P}_T(O(-K_T + D_2 - D_1) \oplus O(-K_T + D_3 - D_1))$ and $(-K_X)|_{E'_1}$ is the tautological line bundle.

We will also need the following properties.

Remark 2.4. Let $X$ be a smooth projective variety. Then $N_1(X)$ is generated, as a vector space, by classes of complete intersections of very ample divisors.

Proof. Let $n$ be the dimension of $X$ and $H$ a very ample divisor. By the hard Lefschetz theorem, the linear map

$$N^1(X) \to N^1(X)$$

$$[D] \mapsto [D \cdot H^{n-2}]$$

is an isomorphism. Since the ample cone has maximal dimension in $N^1(X)$, we can choose very ample divisors $H_1, \ldots, H_{n-1}$ on $X$ such that their classes generate $N^1(X)$; then their images $[H_1 \cdot H^{n-2}], \ldots, [H_{n-1} \cdot H^{n-2}]$ generate $N_1(X)$. ■

Lemma 2.5. Let $Y$ be a smooth Fano variety and $\xi: Y \to T$ a smooth morphism with fiber $\mathbb{P}^1$. Let $A_1, A_2 \subset Y$ be disjoint prime divisors such that $\xi$ is finite on $A_i$ for $i = 1, 2$. Then at least one among $A_1, A_2$ is a section of $\xi$.

Proof. We first show the following claim:

If $[A_1]$ and $[A_2]$ are multiples in $N^1(Y)$, then $Y \cong \mathbb{P}^1 \times T$ and $A_1 = \{pt\} \times T$.

Let $\lambda \in \mathbb{Q}_{>0}$ be such that $A_1 \equiv \lambda A_2$. If $C \subset A_1$ is a curve, then $A_1 \cdot C = \lambda A_2 \cdot C = 0$ because $A_1 \cap A_2 = \emptyset$. This shows that $A_1$ is nef; let $\alpha: Y \to Y_0$ be the contraction such that $NE(\alpha) = A_1^\perp \cap NE(Y)$.

We note that $\alpha(A_1)$ is a point because $\alpha(C) = \{pt\}$ for every curve $C \subset A_1$; on the other hand $[A_1] \in \alpha^*N^1(Y_0)$, so that $\dim Y_0 = 1$ and $Y_0 \cong \mathbb{P}^1$. Moreover $\alpha$ is finite on the fibers of $\xi: Y \to T$, because $\xi$ is finite on $A_1$, hence $A_1 \cdot F > 0$. Then $Y \cong T \times \mathbb{P}^1$ and $\alpha$ is the projection, by Cas09 [Lemma 4.9]. This yields the claim.

Set $d_i := \deg \xi|_{A_i}$ for $i = 1, 2$, so that $d_2 A_1 - d_1 A_2 \equiv \xi^*(N_0)$ for some divisor $N_0$ on $T$.

If $N_0 \equiv 0$, then $d_2 A_1 \equiv d_1 A_2$ and we can apply the claim.

If $N_0 \neq 0$, then $N_0^\perp \subset N_1(T)$ is a hyperplane, and by Rem. 2.4 there exists a curve $C \subset T$, complete intersection of very ample divisors $H_1, \ldots, H_{n-3}$, such that $N_0 \cdot C \neq 0$. We choose $H_i$ general in their linear system.

Set $S := \xi^{-1}(C) \subset Y$ and $C_i := A_i|_S$ for $i = 1, 2$; then $C_1 \cap C_2 = \emptyset$ and by Bertini $C_1, C_2$ are smooth irreducible curves, and $S$ is a smooth ruled surface.

Suppose that $N_0 \cdot C > 0$. Then

$$d_1(C_2)^2 = d_1 A_2 \cdot C_2 = (d_2 A_1 - \xi^*(N_0)) \cdot C_2 = -\xi^*(N_0) \cdot C_2 = -d_2 N_0 \cdot C,$$

so that $(C_2)^2 < 0$. We deduce that $C_2$ is the negative section of the ruled surface $S$, hence $d_2 = 1$. Since $\xi|_{A_2}$ is finite, $A_2$ is a section of $\xi$.

In the case $N_0 \cdot C < 0$, we conclude in a similar way that $A_1$ is a section. ■
Let $n \in \mathbb{Z}_{\geq 2}$ and $T$ a smooth Fano variety of dimension $n - 2$. We consider three divisors $D_1, D_2, D_3$ in $T$ and set

$$Z := \mathbb{P}_T(O(D_1) \oplus O(D_2) \oplus O(D_3)) \rightarrow T,$$

so that $Z$ is a smooth projective variety of dimension $n$ with $\rho_Z = \rho_T + 1$. Let $S_i \subset Z$ be the section of $\varphi$ corresponding to the projection $O_T(D_1) \oplus O_T(D_2) \oplus O_T(D_3) \rightarrow O_T(D_i)$, for $i = 1, 2, 3$. Finally let $h : X \rightarrow Z$ be the blow-up of $S_1, S_2, S_3$, so that $X$ is a smooth projective variety of dimension $n$ with $\rho_X = \rho_T + 4$, and set $\sigma := \varphi \circ h : X \rightarrow T$.

**Lemma 3.1.** The following are equivalent:

(i) $X$ is Fano;

(ii) $-K_T + D_i - D_j$ is ample for every $i, j \in \{1, 2, 3\}$.

**Proof.** For $i = 1, 2, 3$ let $E_i \subset X$ be the exceptional divisor over $S_i$, $F_i \subset Z$ as in Rem. 2.1, and $E'_i \subset X$ its transform.

The implication (i) $\Rightarrow$ (ii) follows directly from Lemma 2.2.

Let us show the converse (ii) $\Rightarrow$ (i). We first show that $-K_X$ is strictly nef, namely that $-K_X \cdot \Gamma > 0$ for every irreducible curve $\Gamma \subset X$.

Again, it follows directly from Lemmas 2.2 and 2.3 that $(-K_X)|E_i$ and $(-K_X)|E'_i$ are ample divisors respectively on $E_i$ and $E'_i$, for every $i = 1, 2, 3$. Therefore if $\Gamma$ is contained in one of these divisors, we have $-K_X \cdot \Gamma > 0$.

Assume that $\sigma(\Gamma)$ is a point. By construction, the sections $S_i$ of $\varphi$ are fibrewise in general linear position, so that every fiber $X_t := \sigma^{-1}(t)$ is a smooth del Pezzo surface. We have $\Gamma \subset X_t$ for some $t \in T$, therefore $-K_X \cdot \Gamma = -K_{X_t} \cdot \Gamma > 0$.

Now assume that $\sigma(\Gamma)$ is not contracted by $\sigma$, and that $\Gamma \not\subset E_i, E'_i$ for $i = 1, 2, 3$. It follows from Rem. 2.1 that:

$$-K_X \sim \sigma^*(-K_T) + \sum_{i=1}^{3} (E_i + E'_i).$$

Then $\sigma(\Gamma)$ is an irreducible curve in $T$, and $-K_X \cdot \Gamma = -K_T \cdot \sigma_*(\Gamma) + \sum_{i=1}^{3} (E_i + E'_i) \cdot \Gamma > 0$.

Now we show that $-K_X$ is big. Consider the divisor $A_Z := F_1 + \varphi^*(-K_T)$. By Rem. 2.1, $F_1 + \varphi^*(D_1)$ is a tautological divisor for $\mathbb{P}_T(O(D_1) \oplus O(D_2) \oplus O(D_3))$, therefore $A_Z = F_1 + \varphi^*(D_1) + \varphi^*(-K_T - D_1)$ is a tautological divisor for $\mathbb{P}_T(O(-K_T) \oplus O(-K_T + D_2 - D_1) \oplus O(-K_T + D_3 - D_1))$, hence $A_Z$ is ample on $Z$.

Therefore there exist non-negative integers $m, c_i$ such that the divisor

$$A_X := mh^*(A_Z) - \sum_{i=1}^{3} c_i E_i$$

$$= m(E'_1 + E_2 + E_3 + \sigma^*(-K_T)) - \sum_{i=1}^{3} c_i E_i$$

$$= m\sigma^*(-K_T) + mE'_1 - c_1 E_1 + (m - c_2)E_2 + (m - c_3)E_3$$
is ample on \(X\). Then
\[
-mK_X \sim m\sigma^*(-K_T) + m\sum_{i=1}^{3}(E_i + E'_i)
\]
\[
= A_X + (c_1 + m)E_1 + c_2E_2 + c_3E_3 + mE'_2 + mE'_3.
\]
We have written a multiple of \(-K_X\) as the sum of an ample divisor and an effective divisor, which implies that \(-K_X\) is big.

Finally, as \(-K_X\) is strictly nef and big, it is ample by the base point free theorem.

**Lemma 3.2.** Suppose that \(X\) is Fano. Then \(\delta_X \geq 3\), and \(\delta_X = 3\) if and only if \(\delta_T \leq 3\).

**Proof.** The divisor \(E_1 \subset X\) is a \(\mathbb{P}^1\)-bundle over \(T\), so that \(\dim \mathcal{N}_1(E_1, X) \leq \rho_{E_1} = \rho_T + 1 = \rho_X - 3\), which implies that \(\delta_X \geq 3\).

If \(\delta_X = 3\), then \(\delta_T \leq \delta_X = 3\) by [Cas12] Rem. 3.3.18. If instead \(\delta_X \geq 4\), then by Th. [1.1] we have \(X \cong S \times X'\) where \(S\) is a del Pezzo surface with \(\rho_S = \delta_X + 1 \geq 5\); moreover \(\sigma\) must be a product morphism (see [Rom19] Lemma 2.10). Since the fiber of \(\sigma\) is a del Pezzo surface with \(\rho = 4\), we must have \(T \cong S \times T'\). Then \(\delta_T \geq \delta_S\) again by [Cas12] Rem. 3.3.18; on the other hand since \(S\) is a surface, it is easy to see that \(\delta_S = \rho_S - 1 = \delta_X\), hence \(\delta_T \geq \delta_X \geq 4\).

Prop. [1.2] follows from Lemmas [3.1] and [3.2]

4. **Construction B**

Let \(n \in \mathbb{Z}_{\geq 2}, T\) a smooth Fano variety of dimension \(n - 2\), and \(N\) a divisor in \(T\) such that \(N \neq 0\). Set
\[
Z := \mathbb{P}_T(\mathcal{O}(N) \otimes \mathcal{O} \otimes \mathcal{O}) \rightarrow T
\]
and let \(H\) be a tautological divisor of \(Z\). Note that \(h^0(Z, H) = h^0(T, \mathcal{O}(N) \otimes \mathcal{O} \otimes \mathcal{O}) \geq 2\), and in fact the constant surjections \(\mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O}\) yield a pencil of divisors \(\mathbb{P}_T(\mathcal{O}(N) \otimes \mathcal{O}) \rightarrow Z\) in the linear system \(|H|\) (see Rem. [2.1]).

We assume that a complete intersection of general elements in the linear systems \(|H|\) and \(|2H|\) is smooth, and we fix such a complete intersection \(S_1 \subset Z\).

We also consider the divisor \(D := \mathbb{P}_T(\mathcal{O} \otimes \mathcal{O}) \rightarrow Z\) given by the projection \(\mathcal{O}(N) \otimes \mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O} \otimes \mathcal{O}\), so that \(D \cong \mathbb{P}^1 \times T\) with \(\mathcal{N}_{D/Z} \cong \pi^*_{\mathcal{O}_T(1)} \otimes \pi^*_{\mathcal{O}_T(-N)}\), and the two sections \(S_2, S_3 \subset D, S_i \equiv \{pt\} \times T \subset D\), again corresponding to the two projections \(\mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O}\).

**Remark 4.1.** There exists a unique divisor \(H_0\) in \(|H|\) that contains \(S_1\); \(H_0\) is smooth, \(H_0 \cong \mathbb{P}_T(\mathcal{O}(N) \otimes \mathcal{O}), H|_{H_0}\) is a tautological divisor, and \(S_1 \sim 2H|_{H_0}\).

**Proof.** Since \(|H|\) contains smooth members, and \(S_1\) is a general complete intersection, there exists \(H_0 \in |H|\) smooth containing \(S_1\), and \(\varphi|_{H_0} : H_0 \rightarrow T\) is a \(\mathbb{P}^1\)-bundle. For general \(t \in T\), \((H_0)|_{Z_t}\) is a line, and \(S_1 \subset 2H|_{H_0}\), so that \(S_1 \cap Z_t\) yields two points which determine uniquely the line \((H_0)|_{Z_t}\).

We note that \(H_0\) must intersect \(D\) in a section \(S_0\) of \(\varphi\), given by a surjection \(\lambda : \mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O}\), so that \(H_0 \cong \mathbb{P}_T(\mathcal{E})\) where \(\mathcal{E}\) is a rank 2 vector bundle on \(T\). We have
commutative diagrams:

\[
\begin{array}{ccc}
Z & \hookrightarrow & H_0 \\
\downarrow & & \downarrow \\
D & \hookrightarrow & S_0
\end{array}
\quad
\begin{array}{ccc}
\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O} & \twoheadrightarrow & \mathcal{E} \\
\pi & & \sigma \\
\mathcal{O} \oplus \mathcal{O} & \overset{\lambda}{\twoheadrightarrow} & \mathcal{O}
\end{array}
\]

Since $\lambda$ and $\pi$ have a section, this yields a section of $\sigma$, which implies that $\mathcal{E} \cong \mathcal{O}(N) \oplus \mathcal{O}$ and gives the rest of the statement. \hfill $\blacksquare$

**Remark 4.2.** We have $H_0 \cap S_2 = H_0 \cap S_3 = D \cap S_1 = \emptyset$; in particular $S_1, S_2, S_3$ are pairwise disjoint.

**Proof.** Every divisor in $|H|$ intersects $D$ in $\{pt\} \times T$, and so it follows that $H_0$ is disjoint from $S_2$ and $S_3$ by generality.

Furthermore, the section $H_0 \cap D$ of $H_0$ has normal bundle $N_{H_0 \cap D/H_0} \cong \mathcal{O}_T(-N)$, and is given by a surjection $\mathcal{O}(N) \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$.

Note that $S_1$ does not intersect any section of $H_0$ with normal bundle $\mathcal{O}_T(-N)$, and it follows that $S_1$ is disjoint from $D$. \hfill $\blacksquare$

**Remark 4.3.** $S_1$ is irreducible and $\varphi_{|S_1}: S_1 \to T$ is finite of degree 2; moreover $2N$ is linearly equivalent to the branch divisor $\Delta \subset T$, $h^0(T, 2N) > 0$, and $H_{|S_1} \sim (\varphi_{|S_1})^*(N) \sim R$ where $R \subset S_1$ is the ramification divisor.

**Proof.** The restriction $\varphi_{|S_1}: S_1 \to T$ is finite of degree 2, because if $\dim(S_1 \cap Z_t) > 0$ for some $t \in T$, we should have $S_1 \cap D \neq \emptyset$, contradicting Rem. 4.2. Let $R \subset S_1$ be the ramification divisor and $\Delta \subset T$ the branch divisor. By adjunction

\[-K_{S_1} = (-K_Z - 3H)_{|S_1} = (\varphi_{|S_1})^*\mathcal{O}_T(-K_T - N),\]

so by the Hurwitz formula $(\varphi_{|S_1})^*(N) \sim R$ and $2N \sim \Delta$. In particular $\varphi_{|S_1}$ is not étale because $N \neq 0$, and we deduce that $S_1$ is irreducible. We also note that $h^0(T, 2N) > 0$ and that $S_1$ is Fano if and only if $-K_T - N$ is ample. Finally we have $D \sim H - \varphi^*(N)$ (see Rem. 2.1) and $D \cap S_1 = \emptyset$ by Rem. 4.2, thus $H_{|S_1} \sim \varphi_{|S_1}^*(N)$. \hfill $\blacksquare$

Let $h: X \to Z$ be the blow-up of $S_1, S_2, S_3$, so that $X$ is a smooth projective variety with $\rho_X = \rho_T + 4$, and set $\sigma := \varphi \circ h: X \to T$.

**Lemma 4.4.** The following are equivalent:

(i) $X$ is Fano;

(ii) $-K_T \pm N$ is ample on $T$.

**Proof.** Let $E_i \subset X$ be the exceptional divisor over $S_i$, and let $\tilde{H}_0$ and $\tilde{D}$ be the transforms of $H_0$ and $D$.

We show $(i) \Rightarrow (ii)$. If $X$ is Fano, by restricting $-K_X$ to $E_2$ (or $E_3$) and using Lemma 2.2 we see that $-K_T - N$ is ample. Then restricting $-K_X$ to $\tilde{D}$ and using Lemma 2.3 we see that $-K_T + N$ is ample.

Let us show the converse $(ii) \Rightarrow (i)$. Again, it follows directly from Lemmas 2.2 and 2.3 that $(-K_X)_{|E_i}$ for $i = 2, 3$ and $(-K_X)_{|\tilde{D}}$ are ample divisors respectively on $E_i$ and $\tilde{D}$. 

We show that \((-K_X)_{\tilde{H}_0}\) is ample on \(\tilde{H}_0\). Since \(\tilde{H}_0 = h^*H_0 - E_1\) by Rem. 4.1 and 4.2, we have \(\mathcal{O}_X(\tilde{H}_0)_{\tilde{H}_0} \cong \mathcal{O}_Z(H_0)_{H_0} \otimes \mathcal{O}_{\tilde{H}_0}(-S_1) \cong \mathcal{O}_{\tilde{H}_0}(-H_0H_{\tilde{H}_0})\) under the isomorphism \(\tilde{H}_0 \cong H_0\). Set \(\varphi_0 := \varphi_{H_0}: H_0 \to T\). Using Rem. 4.1 we get:

\[
\mathcal{O}_X(-K_X)_{\tilde{H}_0} \cong \mathcal{O}_{\tilde{H}_0}(-K_{\tilde{H}_0}) \otimes \mathcal{O}_X(\tilde{H}_0)_{\tilde{H}_0} \cong \mathcal{O}_{\tilde{H}_0}(-K_{\tilde{H}_0} - H_{\tilde{H}_0})
\]

which is the tautological line bundle of \(\mathbb{P}_T(\mathcal{O}(-K_T) \oplus \mathcal{O}(-K_T - N))\), and hence it is ample.

We show that \(-K_X \cdot \Gamma > 0\) for every irreducible curve \(\Gamma \subset X\). If \(\Gamma\) is contained in one of the divisors \(E_2, E_3, \tilde{D}, \tilde{H}_0\), then \(-K_X \cdot \Gamma > 0\) by what precedes.

Now assume that \(\Gamma \not\subset E_2, E_3, \tilde{H}_0, \tilde{D}\). We have \(D \sim H_0 - \varphi^*(N)\) (see Rem. 2.1) and \(-K_Z \sim \varphi^*(-K_T - N) + 3H_0 \sim \varphi^*(-K_T + N) + H_0 + 2\tilde{D}\). Then it follows from Rem. 4.2 that:

\[-K_X \sim h^*(-K_Z) - E_1 - E_2 - E_3 \sim \sigma^*(-K_T + N) + \tilde{H}_0 + 2\tilde{D} + E_2 + E_3.\]

If \(\Gamma\) is not contracted by \(\sigma\), then \(\sigma(\Gamma)\) is an irreducible curve in \(T\), and \(-K_X \cdot \Gamma = (-K_T + N) \cdot \sigma(\Gamma) + (E_2 + E_3 + \tilde{H}_0 + 2\tilde{D}) \cdot \Gamma > 0\).

If instead \(\sigma(\Gamma)\) is a point, then either \(\Gamma\) is a fiber of \(h\), and so \(-K_X \cdot \Gamma = 1\), or \(h(\Gamma)\) is an irreducible curve in a fiber \(Z_t\) of \(\varphi\), for some \(t \in T\). Assume the latter: then \(-K_X \cdot \Gamma = (\tilde{H}_0 + \tilde{D} + h^*(D)) \cdot \Gamma \geq h^*(D) \cdot \Gamma = D \cdot h_0(\Gamma) > 0\), because \(D)_{Z_t}\) is a line in \(Z_t \cong \mathbb{P}^2\).

Now we show that \(-K_X\) is big. Consider the divisor \(A_Z := H_0 + \varphi^*(-K_T)\), which is a tautological divisor for \(\mathbb{P}_T(\mathcal{O}(-K_T + N) \oplus \mathcal{O}(-K_T) \oplus \mathcal{O}(-K_T))\), and hence it is ample on \(Z\). Therefore there exist non-negative integers \(m, c_i\) such that the divisor

\[
A_X := mh^*(A_Z) - \sum_{i=1}^{3} c_i E_i
\]

is ample on \(X\). Finally

\[
-mK_X \sim m(\sigma^*(-K_T) + E_1 + 2\tilde{H}_0 + \tilde{D}) = A_X + m\tilde{H}_0 + m\tilde{D} + \sum_{i=1}^{3} c_i E_i
\]

which implies that \(-K_X\) is big and then ample, as in the proof of Lemma 3.1.

The proof of the following lemma is very similar to the one of Lemma 3.2, thus we omit it.

**Lemma 4.5.** Suppose that \(X\) is Fano. Then \(\delta_X \geq 3\), and \(\delta_X = 3\) if and only if \(\delta_T \leq 3\).

Prop. 1.3 follows from Lemmas 4.4 and 4.5.
5. Proof of Theorem 1.4

We follow [CR22 proof of Th. 1.1], where the case \( \dim X = 4, \rho_X = 5, \) and \( T \cong \mathbb{P}^2 \) is considered, generalizing the proof step by step. We begin by recalling a result on the structure of Fano varieties \( X \) with \( \delta_X = 3. \)

**Theorem 5.1.** Set \( n := \dim X. \) There exists a diagram:

\[
X \xrightarrow{f} X_2 \xrightarrow{\psi} Y \xrightarrow{\xi} T
\]

with the following properties:

(a) all varieties are smooth and projective;
(b) \( Y \) is Fano of dimension \( n - 1 \) and \( T \) is Fano of dimension \( n - 2; \)
(c) \( \psi \) and \( \xi \) are smooth morphisms with fiber \( \mathbb{P}^1; \)
(d) \( \sigma: X \to T \) is a flat contraction of relative dimension 2;
(e) \( \psi \circ f: X \to Y \) is a conic bundle;
(f) \( f \) is the blow-up of two disjoint smooth, irreducible subvarieties \( B_1, B_2 \subset X_2 \) of codimension 2;
(g) for \( i = 1, 2 \) set \( A_i := \psi(B_i) \subset Y \) and \( W_i := \psi^{-1}(A_i) \subset X_2; \) \( A_1 \) and \( A_2 \) are disjoint smooth divisors in \( Y \), and \( B_i \) is a section of \( \psi|_{W_i}: W_i \to A_i \), for \( i = 1, 2; \)
(h) \( \xi \) is finite on \( A_i \) for \( i = 1, 2. \)

**Proof.** See [Cas12, Prop. 3.3.1 and its proof, in particular paragraphs 3.3.15-3.3.17; see also [Rom19, Prop. 3.4 and 3.5] for (g)]. ■

We keep the same notation as in Th. 5.1

**Step 5.2.** By Lemma 2.5, up to switching \( A_1 \) and \( A_2 \), we can assume that \( A_2 \) is a section of \( \xi: Y \to T. \)

**Step 5.3.** We can assume that there exists a commutative diagram:

\[
\begin{array}{ccc}
X_2 & \xrightarrow{\psi} & Y \\
\downarrow{g} & & \downarrow{\xi} \\
Z & \xrightarrow{\varphi} & T
\end{array}
\]

where \( \varphi \) is a \( \mathbb{P}^2 \)-bundle, \( \psi \) and \( \xi \) are \( \mathbb{P}^1 \)-bundles, \( g \) is the blow-up along a section \( S_3 \subset Z \) of \( \varphi \), \( E := \text{Exc}(g) \) is a section of \( \psi \), and \( E \cap (B_1 \cup B_2) = \emptyset. \)

**Proof.** Consider the natural factorization of \( f \) as a sequence of two blow-ups (see Th. 5.1(f)):

\[
X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2
\]

where \( f_2 \) is the blow-up of \( B_2 \) and \( f_1 \) is the blow-up of the transform of \( B_1. \)

Let us consider the morphism \( \zeta := \xi \circ \psi \circ f_2: X_1 \to T. \) Since both \( \psi \) and \( \xi \) are smooth by Th. 5.1(g), the composition \( \xi \circ \psi: X_2 \to T \) is smooth. Moreover, since \( A_2 \subset Y \) is a section of \( \xi \) by Step 5.2 and the center \( B_2 \) of the blow-up \( f_2: X_1 \to X_2 \) is a section over
A_2 (see Th. 5.1[6]), we conclude that B_2 is a section of \( \xi \circ \psi : X_2 \to T \). This implies that \( \zeta : X_1 \to T \) is a smooth contraction of relative Picard number 3.

As in [CR22] proof of Step 2.3 one shows that \(-K_{X_1}\) is \( \zeta \)-ample; every fiber of \( \zeta \) is isomorphic to the blow-up \( F \) of \( \mathbb{P}^2 \) in two points. Moreover, again as in [CR22] proof of Step 2.3 we see that every contraction of the fiber \( F \) extends to a global contraction of \( X_1 \) over \( T \). Therefore the sequence of elementary contractions:

\[
\begin{array}{ccc}
F & \longrightarrow & \mathbb{P}^1 \\
| & & | \\
\mathbb{P}^2 & \longrightarrow & \{ \text{pt} \}
\end{array}
\]

yields a corresponding factorization of \( \zeta \):

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_2} & X'_2 & \xrightarrow{\psi'} & Y' \\
g & & & \downarrow \varepsilon' \\
Z & \xrightarrow{\varphi} & T
\end{array}
\]

We have:

- \( \xi' : Y' \to T \) and \( \psi' : X'_2 \to Y' \) are smooth with fiber \( \mathbb{P}^1 \), and \( \varphi : Z \to T \) is smooth with fiber \( \mathbb{P}^2 \);
- \( g \) is the blow-up of a smooth irreducible subvariety \( S_3 \subset Z \) of codimension 2, which is a section of \( \varphi \);
- \( f'_2 \) is the blow-up of a smooth irreducible subvariety \( B'_2 \subset X'_2 \) of codimension 2, which is a section of \( \varphi \circ g : X'_2 \to T \), and is disjoint from \( E := \text{Exc}(g); \)
- \( A'_2 := \psi'(B'_2) \) is a section of \( \xi' \);
- \( E \) is a section of \( \psi' : X'_2 \to Y' \);
- since \( \varphi, \xi', \) and \( \psi' \) all have a section, they are projectivizations of vector bundles.

Notice that \( \zeta \) is finite on \( f_1(\text{Exc}(f_1)) \), because \( f_2 \) is an isomorphism on it, \( \psi \) is finite on \( B_1 \), and \( \xi \) is finite on \( A_1 \) (see Th. 5.1[6],[8],[11]). In particular \( f_1(\text{Exc}(f_1)) \) cannot contain any \((-1)\)-curve in the fiber \( F \), therefore it cannot meet any such curve, otherwise \( X \) would not be Fano. Hence \( B'_1 := f'_2(f_1(\text{Exc}(f_1))) \) is disjoint from \( B'_2 \) and from \( E \).

We show that properties of Th. 5.1 still hold, so that we can replace the original factorization of \( \zeta \) with the new one. Properties [a],[c],[f] are clear. Since \( \zeta \) is finite on \( f_1(\text{Exc}(f_1)), \xi' \) is finite on \( A'_1 := \psi'(B'_1) \), and we have [h].

The composition \( \psi' \circ f'_2 \circ f_1 : X \to Y' \) is a fiber type contraction with one-dimensional fibers, hence a conic bundle, which yields [e] and [d]. Moreover this conic bundle has only reduced fibers; this implies that \( Y' \) is Fano by [Wis91a Prop. 4.3], and we have [i]. Finally [g] follows from [Rom19 Prop. 3.4 and 3.5].

Set \( S_i := g(B_i) \subset Z \) for \( i = 1, 2 \). Then \( S_1, S_2, \) and \( S_3 \) are pairwise disjoint smooth irreducible subvarieties of codimension 2, and \( X \) is the blow-up of \( Z \) along \( S_1 \cup S_2 \cup S_3 \). We set \( Z_i := \varphi^{-1}(t) \) for every \( t \in T \). Moreover we denote by \( d \) the degree of the finite morphism \( \xi|_{A_1} : A_1 \to T \) (see Th. 5.1[6]).

**Step 5.4.** \( S_2 \) is a section of \( \varphi \), and \( \varphi|_{S_1} : S_1 \to T \) is finite of degree \( d \).
Proof. For \( i = 1, 2 \), since \( B_i \) is a section over \( A_i \) by Th. 5.1, the degree of \( S_i \) over \( T \) is equal to the degree of \( A_i \) over \( T \); in particular \( S_2 \) is a section of \( \varphi \) by Step 5.2. \( \blacksquare \)

Step 5.5. For every \( t \in T \) and for every line \( \ell \subset Z_t \cong \mathbb{P}^2 \), the scheme-theoretic intersection \( \ell \cap (S_1 \cup S_2 \cup S_3) \) has length \( \leq 2 \).

In particular, the points \((S_1 \cup S_2 \cup S_3) \cap Z_t \) (with the reduced structure) are in general linear position in \( Z_t \).

Proof. Let \( a \) be the length of scheme-theoretic intersection \( \ell \cap (S_1 \cup S_2 \cup S_3) \), and let \( \tilde{\ell} \subset X \) be the transform of \( \ell \). Then \( 1 \leq -K_X \cdot \tilde{\ell} = 3 - a \), thus \( a \leq 2 \). \( \blacksquare \)

Step 5.6. If \( d = 1 \), then \( X \to Z \overset{\varphi}{\to} T \) is as in Construction A (Prop. 1.2).

Proof. If \( d = 1 \), then \( \varphi \colon Z \to T \) has three pairwise disjoint sections \( S_i \), which are fiberwise in general linear position, by Steps 5.3, 5.4, and 5.5. This implies that \( Z \cong \mathbb{P}_T(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3)) \) in such a way that the three sections \( S_i \) correspond to the projections onto the summands \( \mathcal{O}(D_i) \). Moreover \(-K_T + D_i - D_j \) is ample for every \( i, j \in \{1, 2, 3\} \) by Lemma 3.1 and \( \delta_T \leq 3 \) by Lemma 3.2. \( \blacksquare \)

From now on we assume that \( d \geq 2 \).

For \( q_1, q_2 \in Z_t \) distinct points, we denote by \( \overline{q_1q_2} \subset Z_t \) the line spanned by \( q_1 \) and \( q_2 \).

Step 5.7. Let \( H_0 \subset Z \) be the relative secant variety of \( S_1 \) in \( Z \), namely the closure in \( Z \) of the set:

\[
\bigcup_{q_1, q_2 \in S_1 \cap Z_t, q_1 \neq q_2, t \in T} \overline{q_1q_2}.
\]

Then \( H_0 \) has pure dimension \( n - 1 \), and for every \( t \in T \), \( H_0 \cap Z_t \) is a union of lines \( \ell \) such that the scheme-theoretic intersection \( \ell \cap S_1 \) has length \( \geq 2 \).

Proof. For \( t \) general we have \( |S_1 \cap Z_t| = d \geq 2 \), so that \( H_0 \) is non-empty.

We first consider the fiber product:

\[
\begin{array}{ccc}
S_1 \times_T S_1 & \longrightarrow & S_1 \\
\downarrow & & \downarrow \\
S_1 & \longrightarrow & T
\end{array}
\]

Since the morphism \( S_1 \to T \) is finite between smooth varieties, it is flat. Therefore also \( S_1 \times_T S_1 \to T \) is finite and flat, so that \( S_1 \times_T S_1 \) has pure dimension \( n - 2 \) and every irreducible component dominates \( T \).

We also note that the morphism \( S_1 \times_T S_1 \to S_1 \) has a natural section, whose image is the diagonal \( \Delta \), which is an irreducible component of the fiber product. We denote by \((S_1 \times_T S_1)_0\) the union of the remaining irreducible components of the fiber product, so that \((S_1 \times_T S_1)_0 \setminus \Delta\) is a dense open subset in \((S_1 \times_T S_1)_0\).

Now we consider the dual projective bundle \( Z^* \to T \). We have a morphism over \( T \):

\[
\alpha \colon (S_1 \times_T S_1)_0 \setminus \Delta \longrightarrow Z^*
\]

given by \( \alpha(q_1, q_2) = [q_1q_2] \). If \( [\ell] \in \text{Im}(\alpha) \), then \( \ell \cap S_1 \) is finite, hence \( \alpha^{-1}([\ell]) \) is finite. Moreover

\[
\text{Im}(\alpha) = \{[q_1q_2] \in Z^*_t \mid q_1, q_2 \in S_1 \cap Z_t, q_1 \neq q_2, t \in T\}.
\]
Let $S \subset Z^*$ be the closure of $\text{Im}(\alpha)$. If $K$ is an irreducible component of $(S_1 \times_T S_1)_0$, set $S_K := \frac{\alpha(K \setminus \Delta)}{\alpha}$. Then $\alpha: K \setminus \Delta \to S_K$ is a dominant morphism between irreducible varieties, hence its image $\alpha(K \setminus \Delta)$ contains an open subset of $S_K$; moreover $\dim S_K = \dim K = n - 2$ and $S_K$ dominates $T$, because $K$ does.

We note that $S$ is the union of $S_K$ when $K$ varies among the finitely many irreducible components of $(S_1 \times_T S_1)_0$, therefore all the irreducible components of $S$ are of type $S_K$.

We conclude that $S$ has pure dimension $n - 2$ and every irreducible component dominates $T$. Moreover every irreducible component of $S$ contains an open subset contained in $\text{Im}(\alpha)$. We also note that, for every $[\ell] \in S$, the scheme-theoretical intersection $\ell \cap S_1$ has length $\geq 2$.

Now we consider the universal family over $T$:

$$
\xymatrix{ U 
\ar[rr]^\pi 
\ar[dr] & & Z 
\ar[dl]_\pi 
}
$$

where $U = \{([\ell], p) \in Z^* \times_T Z \mid p \in \ell\}$.

The inverse image $\hat{\mathcal{S}}$ of $S$ in $U$ has pure dimension $n - 1$; moreover every irreducible component of $\hat{\mathcal{S}}$ is the inverse image of an irreducible component of $S$, and contains an open subset contained in the inverse image $\hat{\mathcal{S}}_0$ of $\text{Im}(\alpha)$. In particular $\hat{\mathcal{S}}_0$ is dense in $\hat{\mathcal{S}}$.

Finally we consider the images $\pi(\hat{\mathcal{S}}_0)$ and $\pi(\hat{\mathcal{S}})$ in $Z$; note that $\pi(\hat{\mathcal{S}}_0)$ is dense in $\pi(\hat{\mathcal{S}})$. By construction we have

$$
\pi(\hat{\mathcal{S}}_0) = \bigcup_{q_1,q_2 \in S_1 \cap Z, q_1 \neq q_2, t \in T} q_1q_2 \subset Z,
$$

so that $H_0 = \overline{\pi(\hat{\mathcal{S}}_0)} = \pi(\hat{\mathcal{S}})$.

Let $W$ be an irreducible component of $\hat{\mathcal{S}}$, and let $W_0$ be an open subset of $W$ contained in $\hat{\mathcal{S}}_0$. Then $\overline{\pi(W_0)} = \overline{\pi(W)}$, so that $\pi_0: W_0 \to \overline{\pi(W)}$ is a dominant morphism between irreducible varieties, and its image contains an open subset $U$ of $\overline{\pi(W)}$. It is clear that $\pi_0^{-1}(p)$ is finite for every $p \in U$, so that $\dim \overline{\pi(W)} = \dim W = n - 1$. This shows that $H_0$ has pure dimension $n - 1$.

Moreover, since $H_0$ is the locus of lines parametrized by $S$, we also have that for every $t \in T$, $H_0 \cap Z_t$ is a union of finitely many lines $\ell$, and the scheme-theoretical intersection $\ell \cap S_1$ has length $\geq 2$.

**Step 5.8.** We have $H_0 \cap (S_2 \cup S_3) = \emptyset$.

**Proof.** Let $\ell$ be a line in $H_0 \cap Z_t$ for some $t \in T$; then $\ell \cap (S_2 \cup S_3) = \emptyset$ by Steps 5.5 and 5.7. Therefore $H_0 \cap (S_2 \cup S_3) = \emptyset$. ◼

Recall that $W_2 = \psi^{-1}(A_2) \subset X_2$ and that $E = \text{Exc}(g) \subset X_2$ (see Th. 5.1 and Step 5.3).

**Step 5.9.** Set $D := g(W_2) \subset Z$. Then $W_2 \cong D \cong \P^1 \times T$, and $D \cap Z_t$ is a line in $Z_t$ for every $t \in T$. Moreover $D$ contains $S_2$ and $S_3$ (as $\{pt\} \times T$), while $D \cap S_1 = \emptyset$.
Proof. We have a commutative diagram:

\[
\begin{array}{ccc}
W_2 & \xrightarrow{\psi_{W_2}} & A_2 \\
\downarrow{g|_{W_2}} & & \downarrow{\xi_{A_2}} \\
D & \xrightarrow{\varphi_D} & T
\end{array}
\]

where the vertical maps are isomorphisms by Steps [5.2 and 5.3] and the horizontal maps are \(\mathbb{P}^1\)-bundles.

We also have \(S_3 = g(E \cap W_2) \subseteq D\); moreover \(B_2 \subseteq W_2\) (see Th. [5.1]), hence \(S_2 = g(B_2) \subseteq D\). By Steps [5.3] and [5.4], \(S_2\) and \(S_3\) are disjoint sections of the \(\mathbb{P}^1\)-bundle \(\varphi_D: D \to T\). Thus we can assume that \(D \cong \mathbb{P}_T(O \oplus M)\) for some \(M \in \text{Pic}(T)\), such that the section \(S_2\) corresponds to the projection \(O \oplus M \to O\), so that \(N_{S_2/D}^Y \cong M\).

Now we have \(H_0 \cap D \neq \emptyset\), because both contain a line in \(Z_t\), so that \(H_0 \cap D\) yields a non-zero effective divisor in \(D\). On the other hand, this divisor is disjoint from both sections \(S_2\) and \(S_3\) by Step [5.8]. Write \(O_D(H_0 \cap D) \cong O_D(mS_2 + (\varphi_D)^*G)\), with \(m \in \mathbb{Z}\) and \(G\) divisor on \(T\).

We have \(O_{S_2} \cong O_D(H_0 \cap D)|_{S_2} \cong O_D((\varphi_D)^*G)|_{S_2}\), which gives \(G \sim 0\) and \(O_D(H_0 \cap D) \cong O_D(mS_2)\). Note that \(m \neq 0\) because \(H_0 \cap D\) is non-zero.

Then \(O_{S_2} \cong O_D(H_0 \cap D)|_{S_2} \cong O_D(mS_2)|_{S_2} \cong M^\otimes (-m)\), thus \(M^\otimes m \cong O_T\). Since \(T\) is a Fano variety, \(\text{Pic}(T)\) is finitely generated and free, so that \(M \cong O_T\) and \(D \cong \mathbb{P}^1 \times T\).

Finally, we note that for every \(t \in T\), \(D \cap Z_t\) is the line spanned by the points \(S_2 \cap Z_t\) and \(S_3 \cap Z_t\), so that \(D \cap S_1 = \emptyset\) by Step [5.5].

Recall from Steps [5.2 and 5.3] that \(\xi: Y \to T\) is a \(\mathbb{P}^1\)-bundle and \(A_2 \subset Y\) is a section. Let \(\mathcal{E}\) be a rank 2 vector bundle on \(T\) such that \(Y = \mathbb{P}_T(\mathcal{E})\); up to tensoring \(\mathcal{E}\) with a line bundle, we can assume that the section \(A_2\) corresponds to a surjection \(\mathcal{E} \to O_T\), and we denote by \(L \in \text{Pic}(Y)\) the tautological line bundle, so that \(L|_{A_2} \cong O_{A_2}\). Moreover let \(N\) be a divisor on \(T\) such that \(O_T(N) \cong N_{A_2/Y}^\vee\) under the isomorphism \(T \cong A_2\), so that we have an exact sequence on \(T\):

\[
0 \to O_T(N) \to \mathcal{E} \to O_T \to 0.
\]

**Step 5.10.** We have \(O_Y(A_1) \cong L^\otimes d\), \(O_Y(A_2 + \xi^*(N)) \cong L\), and \(H^1(Y, L) = 0\).

*Proof.* Both \(L\) and \(O_Y(A_1)\) belong to the kernel of the restriction \(r: N^1(Y) \to N^1(A_2)\); moreover \(r\) is surjective, because for \(G \in \text{Pic}(A_2)\) the line bundle \(\xi^*(((\xi|_{A_2})^{-1})^*G)\) restricts to \(G\), and \(\rho_{A_2} = \rho_T = ry - 1\), so that \(\dim ker r = 1\) and the classes of \(L\) and \(A_1\) must be proportional in \(N^1(Y)\); intersecting with a fiber of \(\xi\) we deduce that \(O_Y(A_1) \cong L^\otimes d\).

Similarly, we have \(O_Y(A_2 + \xi^*(N))|_{A_2} \cong O_{A_2}\), and again intersecting with a fiber we get \(O_Y(A_2 + \xi^*(N)) \cong L\).

Finally we have

\[
-K_Y + L \equiv -K_Y + \frac{1}{d}A_1
\]

where \(-K_Y\) is ample, \(1/d < 1\), and \(A_1\) is a smooth divisor, therefore \(H^1(Y, L) = 0\) by Kawamata-Viehweg vanishing (see [Laz04b, Th. 9.1.18]).
**Step 5.11.** We have $X_2 \cong \mathbb{P}_Y(\mathcal{O} \oplus L)$, and $E$ corresponds (as a section of $\psi$) to the projection $\mathcal{O} \oplus L \to \mathcal{O}$. Moreover $\mathcal{N}_{E/X_2}^\vee \cong L$ and $\mathcal{N}_{S_3/Z}^\vee \cong E$.

**Proof.** By Step 5.3 we have a commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\psi_E} & Y \\
\downarrow & & \downarrow \\
S_3 & \xrightarrow{\psi_{S_3}} & T
\end{array}
\]

where the horizontal maps are isomorphisms, and the vertical maps are $\mathbb{P}^1$-bundles. Since $\psi: X_2 \to Z$ is the blow-up of $S_3$, we get:

\[
\mathbb{P}_{S_3}(\mathcal{N}_{S_3/Z}^\vee) \cong E \cong Y \cong \mathbb{P}_T(\mathcal{E}),
\]

therefore $\mathcal{N}_{S_3/Z}^\vee \cong \mathcal{E} \otimes M$ with $M \in \text{Pic}(T)$. Moreover $\mathcal{N}_{E/X_2}^\vee$ is the tautological line bundle of $\mathbb{P}_T(\mathcal{E} \otimes M)$, which gives $\mathcal{N}_{E/X_2}^\vee \cong L \otimes \xi^*(M)$.

Recall from Step 5.3 that $E \subset X_2$ is a section of the $\mathbb{P}^1$-bundle $\psi: X_2 \to Y$. Let $\mathcal{F}$ be a rank 2 vector bundle on $Y$ such that $X_2 = \mathbb{P}_Y(\mathcal{F})$; up to tensoring $\mathcal{F}$ with a line bundle, we can assume that the section $E$ corresponds to a surjection $\sigma: \mathcal{F} \to \mathcal{O}_Y$, so that $\ker \sigma \cong \mathcal{N}_{E/X_2}^\vee \cong L \otimes \xi^*(M)$. We obtain the following exact sequence over $Y$:

\[
0 \to \ker \sigma \to \mathcal{F} \to \mathcal{O}_Y \to 0.
\]

Now let us consider $A_2 \subset Y$: we have $\ker \sigma|_{A_2} \cong L|_{A_2} \otimes \xi^*(M)|_{A_2} \cong M$, so by restricting to $A_2$ the above exact sequence we get:

\[
0 \to M \to \mathcal{F}|_{A_2} \to \mathcal{O} \to 0.
\]

On the other hand $\mathbb{P}_{A_2}(\mathcal{F}|_{A_2}) = W_2 \cong \mathbb{P}^1 \times T$ by Step 5.3, and we deduce that $M = \mathcal{O}_T$ and $\ker \sigma \cong L$.

Now we have $\text{Ext}^1(\mathcal{O}_Y, \ker \sigma) \cong H^1(Y, L) = 0$ by Step 5.10, hence the sequence \[5.2\] splits, so that $\mathcal{F} \cong \mathcal{O}_Y \oplus L$.

**Step 5.12.** We have $-K_T + N$ ample on $T$, $H^1(T, N) = 0$, $H^0(T, d(d-1)N) \neq 0$, $N \neq 0$, $\mathcal{E} \cong \mathcal{O}_T(T) \oplus \mathcal{O}_T$, and $Y \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O})$.

**Proof.** Set $\xi_1 := \xi|_{A_1}: A_1 \to T$, so that $\xi_1$ is finite of degree $d$. By Step 5.10 we have $L \cong \mathcal{O}_Y(A_2 + \xi^*(N))$ and $\mathcal{O}_Y(A_1) \cong L^\otimes d$, so that $L|_{A_1} \cong \mathcal{O}_{A_1}(\xi_1^*(N))$ and $\mathcal{O}_Y(A_1)|_{A_1} \cong \mathcal{O}_{A_1}(\xi_1^*(dN))$. We have $\det(\mathcal{E}) \cong \mathcal{O}_T(N)$ by \[5.1\], therefore $\mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y(\xi^*(K_T + N)) \otimes L^\otimes (-2)$ and

\[
-K_{Y|_{A_1}} \sim \xi_1^*(-K_T + N).
\]

Since $Y$ is Fano and $\xi_1$ is finite, we deduce that $-K_T + N$ is ample on $T$, so that $H^1(T, N) = H^1(T, K_T - K_T + N) = 0$ by Kodaira vanishing. Moreover we have $\text{Ext}^1(\mathcal{O}_T, \mathcal{O}_T(N)) = 0$, so that the sequence \[5.1\] splits.

Finally:

\[
K_{A_1} \sim (K_Y + A_1)|_{A_1} \sim \xi_1^*(K_T + (d-1)N).
\]

This shows that the ramification divisor $R$ of $\xi_1$ is linearly equivalent to $(d-1)\xi_1^*(N)$; by taking the pushforward we get $(\xi_1)_*(R) \sim d(d-1)N$. On the other hand $R$ is effective and non-zero (because $T$ is simply connected), therefore $H^0(T, d(d-1)N) \neq 0$ and $N \neq 0$.■
Recall from Step 5.9 that $D \cong \mathbb{P}^1 \times T$ and $S_2, S_3 \cong \{\text{pt}\} \times T$.

**Step 5.13.** We have $Z \cong \mathbb{P}_T(O(N) \oplus O \oplus O)$ with the two sections $S_2, S_3$ corresponding to the projections onto the trivial summands; moreover $(N_{D/Z})_{\{\text{pt}\} \times T} \cong \mathcal{O}_T(-N)$.

**Proof.** We know by Steps 5.3, 5.11 and 5.12 that $S_3$ is a section of $\varphi: Z \to T$ with conormal bundle $\mathcal{E} \cong \mathcal{O}_T(N) \oplus \mathcal{O}_T$. As in the proof of Step 5.11, using that $H^1(T, \mathcal{O}_T) = 0$ by Step 5.12, one shows that $Z \cong \mathbb{P}_T(O(N) \oplus O \oplus O)$, with the section $S_3$ corresponding to the projection $p_3$ onto the last summand.

The inclusions $S_2, S_3 \hookrightarrow D \hookrightarrow Z$ yield a diagram:

\[
\begin{array}{ccc}
\mathcal{O}_T(N) \oplus \mathcal{O}_T \oplus \mathcal{O}_T & \xrightarrow{\alpha} & \mathcal{O}_T \oplus \mathcal{O}_T \\
\downarrow{\pi_2} & & \downarrow{\pi_1} \\
\mathcal{O}_T & & \\
\end{array}
\]

where $\pi_i$ are the projections, $\pi_1$ corresponds to $S_2 \subset D$, $\pi_2$ corresponds to $S_3 \subset D$, and $\pi_2 \circ \alpha = p_3$.

By Step 5.12 we have $H^0(T, -N) = 0$, thus $\text{Hom} (\mathcal{O}_T(N), \mathcal{O}_T) = 0$ and $\alpha$ factors through the projection $p_2$: $\mathcal{O}_T(N) \oplus \mathcal{O}_T \oplus \mathcal{O}_T \to \mathcal{O}_T \oplus \mathcal{O}_T$. Then up to changing the isomorphism $Z \cong \mathbb{P}_T(O(N) \oplus O \oplus O)$ we can assume that $\alpha = p_{23}$ and $S_2 \subset Z$ corresponds to the projection $p_2$ onto the second summand.

Since $N \not\equiv 0$ by Step 5.12, as in the proof of Lemma 2.5 we see that there exists a curve $C \subset T$ which is a complete intersection of very ample divisors (general in their linear system) and such that $N \cdot C \neq 0$. Set $S := \xi^{-1}(C) \subset Y$ and $C_1 := A_1|_S$.

**Step 5.14.** We have $\mathcal{O}_S(C_1) = (L|_S)^{\otimes d}$, $\deg(L|_C) > 0$, and $L|_S$ is nef and big in $S$.

**Proof.** Recall that $\mathcal{O}_Y(A_1) \cong L^{\otimes d} \cong \mathcal{O}_Y(d(A_2 + \xi^*(N)))$ by Step 5.10 in particular $\mathcal{O}_S(C_1) = (L|_S)^{\otimes d}$. We have

\[
(C_1)^2 = A_1 \cdot C_1 = d(A_2 + \xi^*(N)) \cdot C_1 = d\xi^*(N) \cdot C_1 = d^2N \cdot C \neq 0.
\]

If $(C_1)^2 < 0$, $C_1$ should be the negative section of the ruled surface $S$, which is impossible because $\xi|_{C_1}$ has degree $d \geq 2$; therefore $(C_1)^2 > 0$, $C_1$ is nef and big in $S$, and $L|_S$ too. Moreover $\deg(L|_C) = C_1 \cdot L|_S > 0$.

Set $\overline{X}_2 := \psi^{-1}(S) \subset X_2$, $\overline{Z} := \varphi^{-1}(C) \subset Z$, and let $\overline{\xi}, \overline{\psi}, \overline{\varphi}, \overline{\eta}$ be the respective restricted morphisms, so that we have a diagram:

\[
\begin{array}{ccc}
\overline{X}_2 & \xrightarrow{\overline{\varphi}} & S \\
\downarrow{\overline{\pi}} & & \downarrow{\overline{\xi}} \\
\overline{Z} & \xrightarrow{\overline{\varphi}} & C
\end{array}
\]

where all varieties are smooth and projective, and $\dim \overline{X}_2 = \dim \overline{Z} = 3$. We also set $\overline{W}_1 := \overline{\psi}^{-1}(C_1) = W_1 \cap \overline{X}_2 \subset \overline{X}_2$ and $\overline{B}_1 := B_1 \cap X_2$, so that $\overline{B}_1$ is a section of $\overline{\psi}|_{\overline{W}_1}: \overline{W}_1 \to C_1$ (see Th. 5.1(g)).

**Step 5.15.** There exists a section $K$ of $\overline{\psi}: \overline{X}_2 \to S$ containing $\overline{B}_1$ and disjoint from $E$. 
We deduce that $\overline{W}_1 = \mathbb{P}_1(\mathcal{O}_{\mathbb{C}_1} \oplus L_{|\mathbb{C}_1})$. Moreover by Steps 5.3 and 5.11 we deduce that $\overline{W}_1 \cap E$ is a section of $\overline{W}_1$, disjoint from $\overline{B}_1$, and corresponding to the projection $\mathcal{O}_{\mathbb{C}_1} \oplus L_{|\mathbb{C}_1} \to \mathcal{O}_{\mathbb{C}_1}$. Then it is not difficult to see that $\overline{B}_1$ corresponds, as a section, to a surjection $\mathcal{O}_{\mathbb{C}_1} \oplus L_{|\mathbb{C}_1} \to L_{|\mathbb{C}_1}$.

Let us consider the restriction $r: \text{Hom}(\mathcal{O}_{\mathbb{S}} \oplus L_{|\mathbb{S}}, L_{|\mathbb{S}}) \to \text{Hom}(\mathcal{O}_{\mathbb{C}_1} \oplus L_{|\mathbb{C}_1}, L_{|\mathbb{C}_1})$. We have

$$\text{Hom}(\mathcal{O}_{\mathbb{C}_1} \oplus L_{|\mathbb{C}_1}, L_{|\mathbb{C}_1}) \cong \text{Hom}(L_{|\mathbb{C}_1} \oplus \mathcal{O}_{\mathbb{C}_1}, \mathcal{O}_{\mathbb{C}_1}) \cong H^0(\mathbb{C}_1, L_{|\mathbb{C}_1}) \oplus H^0(\mathbb{C}_1, \mathcal{O}_{\mathbb{C}_1}),$$
and similarly $\text{Hom}(\mathcal{O}_{\mathbb{S}} \oplus L_{|\mathbb{S}}, L_{|\mathbb{S}}) \cong H^0(\mathbb{S}, L_{|\mathbb{S}}) \oplus H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}})$. Since the restriction $H^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}) \to H^0(\mathbb{C}_1, \mathcal{O}_{\mathbb{C}_1})$ is an isomorphism, $r$ is surjective if the restriction $H^0(\mathbb{S}, L_{|\mathbb{S}}) \to H^0(\mathbb{C}_1, L_{|\mathbb{C}_1})$ is.

We have an exact sequence of sheaves on $\mathbb{S}$:

$$0 \to L_{|\mathbb{S}} \otimes \mathcal{O}_{\mathbb{S}}(-C_1) \to L_{|\mathbb{S}} \to L_{|\mathbb{C}_1} \to 0.$$ Using Step 5.14, Serre duality and Kawamata-Viehweg vanishing:

$$H^1(\mathbb{S}, L_{|\mathbb{S}} \otimes \mathcal{O}_{\mathbb{S}}(-C_1)) = H^1(\mathbb{S}, L_{|\mathbb{S}}^{(1-d)}) = H^1(\mathbb{S}, K_{\mathbb{S}} \otimes L_{|\mathbb{S}}^{(d-1)}) = 0$$

because $d \geq 2$.

We conclude that $r$ is surjective, so that $\tau$ extends to a morphism $\tau_\mathbb{S}: \mathcal{O}_{\mathbb{S}} \oplus L_{|\mathbb{S}} \to L_{|\mathbb{S}}$.

We show that $\tau_\mathbb{S}$ is surjective. Under the isomorphism $\text{Hom}(\mathcal{O}_{\mathbb{S}} \oplus L_{|\mathbb{S}}, L_{|\mathbb{S}}) \cong \text{Hom}(\mathcal{O}_{\mathbb{S}}, L_{|\mathbb{S}}) \oplus \mathbb{C}$, $\tau_\mathbb{S}$ corresponds to $(\alpha, \lambda)$.

If $\lambda = 0$, then $\tau_\mathbb{S}$ factors through the projection $\mathcal{O}_{\mathbb{S}} \oplus L_{|\mathbb{S}} \to \mathcal{O}_{\mathbb{S}}$, and the same happens by restricting to $C_1$. This is impossible, because $\tau: \mathcal{O}_{\mathbb{C}_1} \oplus L_{|\mathbb{C}_1} \to L_{|\mathbb{C}_1}$ is surjective, and $\deg(L_{|\mathbb{C}_1}) > 0$ by Step 5.14.

We conclude that $\lambda \neq 0$ and $\tau_\mathbb{S}$ is surjective, so it yields a section $K \subset X_2$ extending $\overline{B}_1$.

We show that $K \cap E = \emptyset$. Let us consider the projection $\mathcal{O}_{\mathbb{S}} \oplus L_{|\mathbb{S}} \to L_{|\mathbb{S}}$ and the corresponding section $\tilde{K} \subset X_2$. Since $E \cap X_2$ is a section corresponding to the projection onto the other summand, we have $K \cap E = \emptyset$. On the other hand, it is easy to check that $K \sim \tilde{K}$ in $X_2$, hence for every curve $C \subset E \cap X_2$ we have $K \cdot C = 0$. Since $K \neq E \cap X_2$, this implies that $K \cap E = \emptyset$. $lacksquare$

**Step 5.16.** We have $d = 2$.

**Proof.** Recall that $K \cap E = \emptyset$ and $K \supset \overline{B}_1$ by Step 5.15. Consider $\overline{g}(K) \subset Z$, so that $K \cong \overline{g}(K)$ and $\overline{g}(K) \supset S_1 \cap Z$. If $t \in C$ is general, then $g^{-1}(Z_t) \cong \mathbb{P}_1$, and $K \cap g^{-1}(Z_t)$ is a section of $\mathbb{P}_1 \to \mathbb{P}_1$, disjoint from the $(-1)$-curve $E \cap g^{-1}(Z_t)$. Thus $\overline{g}(K) \cap Z_t$ is a line in $Z_t \cong \mathbb{P}^2$, and this line contains the $d$ points $S_t \cap Z_t$. Since these points are in general linear position (see Step 5.5), we conclude that $d = 2$. $lacksquare$

**Step 5.17.** The divisor $H_0$ is a tautological divisor for $Z = \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$.

**Proof.** Since $d = 2$ by Step 5.16, for a general fiber $Z_t$ of $\varphi$ the restriction $(H_0)|_{Z_t}$ is a line (see Step 5.7), therefore $\overline{H}$ is a tautological divisor, we have $H_0 \sim H + \varphi^*(G)$, $G$ a divisor on $T$. Restricting to $S_3$ we have $H_0 \cap S_3 = \emptyset$ by Step 5.8 and $\mathcal{O}_Z(H)|_{S_3} \cong \mathcal{O}_{S_3}$ by Step 5.13, thus $G \sim 0$ and $H_0 \sim H$. $lacksquare$

**Step 5.18.** We have $H^1(Z, H_0) = 0$. 

Proof. By Step 5.18, the divisor $A := \varphi^*(-K_T) + H_0$ is the tautological divisor for $\mathbb{P}_T(\mathcal{O}(-K_T + N) \oplus O(-K_T) \oplus O(K_T))$, hence it is ample by Th. 5.1 and Step 5.12.

Consider the divisors $H'_0, H''_0 \subset Z$ corresponding to the two projections $\mathcal{O}_T(N) \oplus \mathcal{O}_T \to \mathcal{O}_T(N) \oplus \mathcal{O}_T$, such that $H'_0 \cap D = S_2$ and $H''_0 \cap D = S_3$ (see Step 5.13); we note that the divisor $H'_0 + H''_0 + D$ is simple normal crossing in $Z$.

By Step 5.17 and Rem. 2.1 we have $H'_0 \sim H''_0 \sim H_0$ and $D \sim H_0 - \varphi^*(N)$, hence $H'_0 + H''_0 + D \sim 3H_0 - \varphi^*(N)$. Moreover $-K_Z \sim \varphi^*(-K_T - N) + 3H_0$ and $-K_Z + H_0 \sim \varphi^*(-K_T - N) + 4H_0 \sim A + H'_0 + H''_0 + D$.

By Norimatsu’s Lemma [Laz04a, Lemma 4.3.5] we get $H^1(Z, H_0) = H^1(Z, K_Z - K_Z + H_0) = 0$. 

**Step 5.19.** $S_1$ is a complete intersection of $H_0$ and a divisor in $|2H_0|$. 

Proof. We note first of all that $H_0$ is a smooth $\mathbb{P}^1$-bundle over $T$. Indeed, since $d = 2$ by Step 5.16 for every $t \in T$ we have $\mathcal{O}_Z(H_0)|_{Z_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ (see Step 5.7). Note that no fiber $Z_t$ can be contained in $H_0$, because $H_0 \cap S_2 = \emptyset$ by Step 5.8; therefore $(H_0)|_{Z_t}$ is a line for every $t \in T$, and $\varphi|_{H_0} : H_0 \to T$ is a $\mathbb{P}^1$-bundle.

Now we show that $\mathcal{O}_{H_0}(S_1) \cong \mathcal{O}_Z(2H_0)|_{H_0}$. We consider the divisor $H'_0$ as in the proof of Step 5.18 so that $H'_0 \sim H_0$ and $H'_0 \cap D = S_2$.

We have that $D \cap H_0$ is $\{pt\} \times T$ in $D \cong \mathbb{P}^1 \times T$, hence $D \cap H_0 \subset H_0$ is a section of $\varphi$, disjoint from both $S_1$ and $H'_0$, because $D \cap S_1 = \emptyset$ and $D \cap H'_0 \cap H_0 = S_2 \cap H_0 = \emptyset$ (see Steps 5.8 and 5.9). Since $S_1$ has degree 2 over $T$, in $H_0$ we have

$$S_1 \sim H_0 \cdot 2(H'_0)|_{H_0} + (\varphi|_{H_0})^*(G),$$

where $G$ is a divisor in $T$. By restricting to the section $D \cap H_0$, we get $G \sim 0$ and $\mathcal{O}_{H_0}(S_1) \cong \mathcal{O}_Z(2H_0)|_{H_0}$.

Finally we consider the exact sequence in $Z$:

$$0 \to \mathcal{O}_Z(H_0) \to \mathcal{O}_Z(2H_0) \to \mathcal{O}_Z(2H_0)|_{H_0} \to 0.$$ 

By Step 5.18 we have a surjection $H^0(Z, \mathcal{O}_Z(2H_0)) \to H^0(H_0, \mathcal{O}_Z(2H_0)|_{H_0})$, and this yields the statement. 

**Step 5.20.** $X \to Z \xrightarrow{\sigma} T$ is as in Construction B (Prop. 1.3).

Proof. We just note that $-K_T \pm N$ is ample by Lemma 4.4 and $\delta_T \leq 3$ by Lemma 4.5. 

6. **Geometry of the del Pezzo fibration $\sigma : X \to T$**

Let $X$ be a smooth Fano variety with $\delta_X = 3$. By Th. 1.4 there are a smooth Fano variety $T$, and a morphism $\sigma : X \to T$, as in Construction A or B; let us denote $X$ by $X_A$ in the former case, and by $X_B$ in the latter.

In this section we study the geometry of the fibration in del Pezzo surfaces $\sigma$, starting with the description of its fibers in §6.1. Then, in §6.2 and §6.3 we describe the 4-dimensional cone $\text{NE}(\sigma)$ and the associated relative contractions, respectively for $X_A$ and $X_B$. In particular this allows to determine whether the same $\sigma : X \to T$ can be obtained with Construction A or B in different ways, or for different choices of divisors on $T$. 
Finally in §6.4 we show that, if \( X \neq F \times T \) (\( F \) the blow-up of \( \mathbb{P}^2 \) at 3 non-collinear points), then \( T \) must satisfy some non-trivial condition.

We keep the same notation as in Sections 3 and 4; in particular we recall that \( \sigma \) factors as \( X \xrightarrow{h} Z \xrightarrow{\varphi} T \), where \( \varphi \) is a \( \mathbb{P}^2 \)-bundle, and \( h \) is the blow-up along 3 horizontal, pairwise disjoint, codimension 2 smooth subvarieties \( S_1, S_2, S_3 \subset Z \).

6.1. **Fibers of \( \sigma \).** Let \( X \) be as above, and \( X_t \) the fiber of \( \sigma \) over \( t \in T \).

For \( X_A \), the morphism \( \sigma \) is smooth, and \( X_t \) is isomorphic to the blow-up of \( \mathbb{P}^2 \) at non-collinear 3 points for every \( t \in T \).

For \( X_B \), recall that \( \varphi|_{S_1} : S_1 \to T \) is a finite cover of degree 2; let \( \Delta \subset T \) be the branch divisor (note that \( \Delta \neq \emptyset \)). Then \( \sigma \) is smooth over \( T \setminus \Delta \), where \( X_t \) is isomorphic to the blow-up of \( \mathbb{P}^2 \) at 4 points in general linear position. Instead, for \( t \in \Delta \), \( X_t \) is singular and isomorphic to the blow-up of \( \mathbb{P}^2 \) at 3 non-collinear points, one of which is a double point contained in a line (see Rem. 4.1); thus \( X_t \) is irreducible and has one rational double point of type \( A_1 \) (see for instance [EH00, §IV.2.3]).

**Remark 6.1.** Let \( X \) be a smooth Fano variety with \( \delta_X = 3 \). Then \( X \) is toric if and only if \( X \) is obtained using Construction A from a toric Fano variety \( T \).

Indeed, if \( X \) is toric, then \( \sigma : X \to T \) is smooth (so that \( X = X_A \)) and \( T \) is toric.

Conversely, if we apply Construction A to a toric Fano variety \( T \), then \( Z \) is toric, and the three sections \( S_1, S_2, S_3 \subset Z \) are invariant for the torus action, so that \( X \) is toric too.

6.2. **The cone \( \text{NE}(\sigma) \) and relative contractions for \( X_A \).**

**Remark 6.2.** Consider \( X = X_A \), let \( X_t \) be a fiber of \( \sigma \), and \( \iota : X_t \hookrightarrow X \) the inclusion. Since \( \sigma \) is smooth, it follows from [W1891, Prop. 1.3] that the pushforward \( \iota_* : \mathcal{N}_1(X_t) \to \mathcal{N}_1(X) \) yields an isomorphism among \( \mathcal{N}_1(X_t) \) and \( \ker \sigma_* \), and among the cones \( \text{NE}(X_t) \) and \( \text{NE}(\sigma) \). Moreover every relative elementary contraction of \( X/T \) restricts to an elementary contraction of \( X_t \), and viceversa.

Since \( X_t \) is the blow-up of \( \mathbb{P}^2 \) at 3 non-collinear points \( p_1, p_2, p_3 \), the cone \( \text{NE}(X_t) \) is generated by the classes of the six \((-1)\)-curves of \( X_t \), given by the exceptional curve \( e_i \) over \( p_i \), and the transform \( e'_i \) of the line \( p_j p_k \). In \( X \) we have \( e_i = E_i \cap X_t \) and \( e'_i = E'_i \cap X_t \); the exceptional divisors \( E_i, E'_i \) are \( \mathbb{P}^1 \)-bundles over \( T \).

Figure 6.1 shows the 3-dimensional polytope obtained as a hyperplane section of the 4-dimensional cone \( \text{NE}(\sigma) \).

**Lemma 6.3.** Let \( X = X_A \). Then there is a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\hat{h}} & \hat{Z} = \mathbb{P}_T(\mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \oplus \mathcal{O}(-D_3)) \\
\downarrow h & & \downarrow \hat{\varphi} \\
Z & \xrightarrow{\varphi} & T
\end{array}
\]

where \( \text{Exc}(\hat{h}) = E'_1 \cup E'_2 \cup E'_3 \), so that performing Construction A from \( T \) with divisors \( D_1, D_2, D_3 \), or with divisors \( -D_1, -D_2, -D_3 \), yields the same Fano variety \( X \).

**Proof.** By Rem. 6.2, the face of \( \text{NE}(\sigma) \) spanned by the classes of \( e'_1, e'_2, e'_3 \) yields a contraction \( h : X \to \hat{Z} \) such that \( \hat{\varphi} : \hat{Z} \to T \) is a \( \mathbb{P}^2 \)-fibration, and \( \hat{h} \) is the blow-up of three sections that are fibrewise in general linear position. Moreover by Lemmas 2.2
and we have $E_i \cong \mathbb{P}_T(\mathcal{O}(-D_j) \oplus \mathcal{O}(-D_k))$ and $E'_i \cong \mathbb{P}_T(\mathcal{O}(D_j) \oplus \mathcal{O}(D_k))$ for every $i, j, k$ with $\{i, j, k\} = \{1, 2, 3\}$; this implies the statement.

Observe that the birational contractions $h$ and $\hat{h}$ correspond to the two simplicial facets of $\text{NE}(\sigma)$, see Fig. 6.1. Instead the three non-simplicial facets yield conic bundles $X \rightarrow Y$, where $Y$ is a $\mathbb{P}^1$-bundle over $T$.

6.3. The cone $\text{NE}(\sigma)$ and relative contractions for $X_B$.

**Lemma 6.4.** Let $X = X_B$, $X_{t_0}$ a smooth fiber of $\sigma$, and $i: X_{t_0} \hookrightarrow X$ the inclusion. Then every relative elementary contraction of $X/T$ restricts to a non-trivial contraction of $X_{t_0}$, and $i^* \text{NE}(X_{t_0}) = \text{NE}(\sigma)$.

**Proof.** Clearly $i^* \text{NE}(X_{t_0}) \subseteq \text{NE}(\sigma)$. For the converse, let $R$ be an extremal ray of $\text{NE}(\sigma)$, and $c_R: X \rightarrow X_R$ the associated elementary contraction of $X$. We show that $X_{t_0}$ must contain some curve contracted by $c_R$: this implies that $R$ is in the image of $\text{NE}(X_{t_0})$ via $i^*$, so that $i^* \text{NE}(X_{t_0}) = \text{NE}(\sigma)$.

The statement is clear if $c_R$ is of fiber type, so let us assume that $c_R$ is birational. Every fiber $X_t$ of $\sigma$ is irreducible (see §6.1), and $c_R(X_t) \subset X_R$ is the fiber of $X_R \rightarrow t$ over $t$. Since $\dim X_R = n$, we have $\dim c_R(X_t) = 2$ and hence $X_t \not\subseteq \text{Exc}(c_R)$.

On the other hand every fiber of $c_R$ has dimension $\leq 1$. By [Wis91a, Thm. 1.2] we have $\dim \text{Exc}(c_R) = n-1$, therefore $\dim (\text{Exc}(c_R)) = n-2$, $\sigma(\text{Exc}(c_R)) = T$, and $\text{Exc}(c_R)$ meets every fiber $X_t$.

**Remark 6.5.** Let $X = X_B$; we use Lemma 6.4 to describe the cone $\text{NE}(\sigma)$.

Let $X_t$ be a smooth fiber of $\sigma$ and $\{p_1, p'_1, p_2, p_3\} \in Z_t$ be the points blown-up by $h|_{X_t}: X_t \rightarrow Z_t$, where $p_i = S_i \cap Z_t$ for $i = 2, 3$, and $\{p_1, p'_1\} = S_1 \cap Z_t$. The 5-dimensional cone $\text{NE}(X_t)$ is generated by the classes of the ten $(-1)$-curves in $X_t$, given by the exceptional curves and the transforms of the lines through two blown-up points. We denote by $e_i$ (respectively $e'_i$) the exceptional curve over $p_i$ (respectively $p'_i$), and $\ell_{i,j}$ (respectively $\ell_{1,1}$, $\ell_{1',i}$ for $i = 2, 3$) the transform of the line $p_ip'_j$ (respectively $p_1p'_1$, $p'_1p_i$ for $i = 2, 3$).
Consider as above $ι_1 : X_t \hookrightarrow X$ and $ι_* : \mathcal{N}_1(X_t) \to \ker σ_*$. The kernel of $ι_*$ has dimension 1, and is generated by the numerical class in $\mathcal{N}_1(X_t)$ of the 1-cycle $(e_1 - e'_1)$. By looking at the numerical relations in $\mathcal{N}_1(X_t)$, we provide a complete description of the cone $\text{NE}(σ)$: Figure 6.2 shows the 3-dimensional polytope obtained as a hyperplane section of the 4-dimensional cone $\text{NE}(σ)$. We see that $\text{NE}(σ)$ has 7 extremal rays, and the figure shows their generators.

It follows from [Wiś91a, Thm. 1.2] that every relative elementary contraction of $\text{NE}(σ)$ is the blow-up of a smooth variety along a smooth codimension 2 subvariety. The contraction corresponding to $[e_1] = [e'_1]$ (respectively $[e_2], [e_3]$) is the blow-down of $E_1$ (respectively $E_2, E_3$), while the contractions corresponding to $[ℓ_{1,1'}]$ and $[ℓ_{2,3}]$ have respectively exceptional divisors $H_0$ and $D$ (the strict transforms of $H_0$ and $D$ from $Z$). Lastly, for $i = 2, 3$ we denote by $G_i$ the exceptional divisor of the contraction corresponding to $[ℓ_{1,i}] = [ℓ_{1',i}]$.

Observe that $\text{NE}(σ)$ has 4 simplicial facets and 3 non-simplicial facets; $\text{NE}(h)$ is the facet generated by $[e_1], [e_2], [e_3]$.

**Proposition 6.6.** Let $X = X_B$, and consider the facet $⟨[ℓ_{1,2}], [ℓ_{1,1'}], [e_3]⟩$ (respectively $⟨[ℓ_{1,3}], [ℓ_{1,1'}], [e_2]⟩$) of $\text{NE}(σ)$. The associated contraction $\hat{h}$ yields a commutative diagram:

![Diagram](image)

where $\text{Exc}(\hat{h}) = G_2 \cup \tilde{H}_0 \cup E_3$ (respectively $G_3 \cup \tilde{H}_0 \cup E_2$), and $X \xrightarrow{\hat{h}} Z \xrightarrow{\varphi} T$ is as described in Construction B.
Proof. We consider the face $\langle [\ell_{1,2}], [\ell_{1,1'}], [\epsilon_3]\rangle$, the other case being analogous. Let $\hat{h}: X \to \tilde{Z}$ be the associated contraction, and $\tilde{\varphi}: \tilde{Z} \to T$ the map over $T$, so that $\tilde{\varphi}$ is an elementary contraction and $\sigma = \tilde{\varphi} \circ \hat{h}$.

By Rem. 6.5 $\tilde{Z}$ is smooth and $\hat{h}$ is the blow-up of three pairwise disjoint smooth, codimension 2, irreducible subvarieties, with $\text{Exc}(\hat{h}) = G_2 \cup \tilde{H}_0 \cup E_3$.

We show that $\tilde{\varphi}: \tilde{Z} \to T$ is a $\mathbb{P}^2$-bundle. First note that all the fibers $\tilde{Z}_t$ of $\tilde{\varphi}$ are surfaces, and the general one is isomorphic to $\mathbb{P}^2$. Let $A := \hat{h}(E_2)$, and $\tilde{Z}_t$ be general, so that $\mathcal{O}_{\tilde{Z}_t}(A)|_{\tilde{Z}_t} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ and $A \cdot \ell = 1$ for a line $\ell \subset \tilde{Z}_t$. Since $\text{NE}(\tilde{\varphi})$ has dimension 1, the relative Kleiman’s criterion implies that $A$ is $\tilde{\varphi}$-ample, thus there exists an ample divisor $M$ on $T$ such that $A' := A + m\tilde{\varphi}^*(M)$ is ample on $\tilde{Z}$ for $m \gg 0$. Note that $\mathcal{O}_{\tilde{Z}_t}(A'|_{\tilde{Z}_t}) \cong \mathcal{O}_{\mathbb{P}^2}(1)$, so we apply [BS95, Prop. 3.2.1] and deduce that $\tilde{Z} = \mathbb{P}_T(\mathcal{G})$ for some rank 3 vector bundle $\mathcal{G}$ on $T$.

By construction $\hat{S}_3 := \hat{h}(E_3)$ is a section of $\tilde{\varphi}$, and moreover $\mathcal{N}_{\hat{S}_3/\tilde{Z}}^\vee \cong \mathcal{N}_{\hat{S}_3/Z}^\vee \cong \mathcal{O}(N) \oplus \mathcal{O}$. Up to tensoring with some line bundle on $T$, we may assume that the section $\hat{S}_3$ gives an exact sequence on $T$:

$$0 \to \mathcal{O}(N) \oplus \mathcal{O} \to \mathcal{G} \to \mathcal{O} \to 0.$$ 

Since $-K_T + N$ is ample (see Lemma 4.4), we have $h^1(T, N) = 0$ and the above sequence splits, hence $\tilde{Z} \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}) = Z$.

Finally it is not difficult to see that $\hat{h}(\tilde{H}_0)$ is a section of $\varphi$ corresponding to the projection onto $\mathcal{O}_T$, and that $\hat{h}(G_2)$ is a complete intersection of elements in $|H|$ and $|2H|$.

\[ \text{Remark 6.7.} \] Let $X = X_B$, and consider the facet $\langle [\epsilon_2], [\epsilon_3], [\ell_{1,1'}]\rangle$ of $\text{NE}(\sigma)$ (see Fig. 6.2). The associated contraction $h'$ yields a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h'} & W \\
\downarrow h & & \downarrow \alpha \\
Z & \xrightarrow{\varphi} & T
\end{array}
\]

where $\text{Exc}(h') = E_2 \cup E_3 \cup \tilde{H}_0$, $W$ is smooth, and $\alpha: W \to T$ is a quadric bundle, singular over $\Delta \subset T$.

Instead the three non-simplicial facets of $\text{NE}(\sigma)$ yield conic bundles $X \to Y$, where $Y$ is a $\mathbb{P}^1$-bundle over $T$.

We also note that, among the seven exceptional divisors associated to the extremal rays of $\text{NE}(\sigma)$, $E_2$, $E_3$, $\tilde{H}_0$, and $\tilde{D}$ are $\mathbb{P}^1$-bundles over $T$, while $E_1$, $G_2$, and $G_3$ are $\mathbb{P}^1$-bundles over $S_1$, the double cover of $T$ ramified along $\Delta$.

6.4. Conditions on $T$. Given a smooth Fano variety $T$, Constructions A and B give algorithms to construct from $T$ a smooth Fano variety $X$ with $\dim X = \dim T + 2$ and $\rho_X = \rho_T + 4$. For every $T$ one can get $X_A = F \times T$, where $F$ is the blow-up of $\mathbb{P}^2$ at 3 non-collinear points, and $\sigma: X \to T$ the projection; this corresponds to the choice $D_1 \sim D_2 \sim D_3$. In order to get an $X$ different from $F \times T$, the variety $T$ must satisfy some conditions, as follows.
Lemma 6.8. Let $T$ be a smooth Fano variety. Suppose that there exists a Fano variety $X$ obtained from $T$ as in Construction A or B, and such that $\sigma: X \to T$ is not isomorphic to the projection $F \times T \to T$.

Then there exists a hyperplane $\Lambda \subset N_1(T)$ containing all the classes $\{C\}$ where $C \subset T$ is a curve with $-K_T \cdot C = 1$.

In particular, every extremal ray of length 1 of $NE(T)$ is contained in $\Lambda$, and $NE(T)$ has at least one extremal ray $R$ with length $\ell(R) \geq 2$.

Proof. Let $C \subset T$ be a curve with $-K_T \cdot C = 1$. Suppose first that $X = X_A$. By Lemma 3.1, for every $i, j = 1, 2, 3$ we have $1 \leq (-K_T + D_i - D_j) \cdot C = 1 + (D_i - D_j) \cdot C$, so that $D_i \cdot C = D_j \cdot C$. We exclude by assumption the case $D_i \sim D_2 \sim D_3$, hence for some $i \neq j$ we must have $D_i \neq D_j$; set $\Lambda := (D_i - D_j) \perp$, so that $\Lambda$ is a hyperplane in $N_1(T)$, and contains $\{C\}$.

The case where $X = X_B$ is similar, we set $\Lambda := N_1 \perp$. ■

In the next section we apply these conditions to the case where $T$ is a del Pezzo surface.

7. The case of dimension 4

Let $X$ be a Fano 4-fold with $\delta_X = 3$. By Th. 1.4, $X$ is obtained with Construction A or B from a del Pezzo surface $T$ with $\rho_X - \rho_T = 4$ and $\delta_T \leq 3$. Since $T$ is a surface, it is easy to see that $\delta_T = \rho_T - 1$, thus $\rho_T \in \{1, 2, 3, 4\}$ and $\rho_X \in \{5, 6, 7, 8\}$.

If $\rho_T \geq 3$, then $NE(T)$ is generated by classes of $(-1)$-curves, in particular there are no extremal rays of length $\geq 1$. Then it follows from Lemma 6.8 that $X \cong F \times T$, where $F$ is the blow-up of $\mathbb{P}^2$ in 3 non-collinear points.

If instead $\rho_T = 1$, then $\rho_X = 5$; this case is completely classified in [CR22, Th. 1.1], and there are 6 families.

Finally suppose that $\rho_T = 2$, so that $\rho_X = 6$. If $X$ is toric, after the classification of toric Fano 4-folds by Batyrev [Bat99, Sat00], we see that $X$ has combinatorial type $U$ (in the notation of [Bat99]), and there are 8 possibilities for $X$. More precisely, we have $T \cong \mathbb{F}_1$ in the two cases $U_2$ and $U_4 \cong F \times \mathbb{F}_1$, and $T \cong \mathbb{P}_1 \times \mathbb{P}_1$ in the remaining six cases, including $U_5 \cong F \times \mathbb{P}_1 \times \mathbb{P}_1$.

In the non-toric case we get the following result, that together with the previous discussion implies Prop. 1.5.

Proposition 7.1. There are three families of non-toric Fano 4-folds $X$ with $\delta_X = 3$ and $\rho_X = 6$.

Proof. By Th. 1.4 and Rem. 6.1, $X$ is obtained with Construction B from a del Pezzo surface $T$ with $\rho_T = \rho_X - 4 = 2$, namely $T \cong \mathbb{P}_1 \times \mathbb{P}_1$ or $T \cong \mathbb{F}_1$. The divisor $N$ on $T$ is such that the class of $N$ is effective and non-zero, and $-K_T \pm N$ is ample.

It is not difficult to see that there are three choices of $N$ satisfying these conditions: $N \in |\mathcal{O}_{\mathbb{F}_1 \times \mathbb{P}_1}(0, 1)|$, $N \in |\mathcal{O}_{\mathbb{F}_1 \times \mathbb{P}_1}(1, 1)|$, and $N$ the pullback of a general line in $\mathbb{P}^2$ in the case of $\mathbb{F}_1$.

In all cases $N$ is nef, hence the tautological divisor $H$ of $Z = \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$ is nef. On the other hand $Z$ is toric, thus $H$ is globally generated: we conclude that the general complete intersection of elements in $|H|$ and $|2H|$ is smooth, and we can apply Construction B. Finally we get three families of Fano 4-folds, respectively $X_{B_0}$, $X_{B_1}$, and $X_{B_2}$. ■
Description of $X_{B_0}$. We have $X_{B_0} \cong \mathbb{P}^1 \times Y$, where $Y$ is the Fano 3-fold obtained by blowing-up $\mathbb{P}^3$ along a line, a conic disjoint from the line, and two non-trivial fibers of the blow-up of the line. In fact $Y$ is obtained with Construction B from $T_Y = \mathbb{P}^1$ with $N_Y \in |\mathcal{O}_{\mathbb{P}^1}(1)|$.

Description of $X_{B_1}$. Set $Z := \mathbb{P}^1 \times \mathbb{P}^1 \langle \mathcal{O}(1, 1) \oplus \mathcal{O} \rangle \to \mathbb{P}^1 \times \mathbb{P}^1$; the 4-fold $X_{B_1}$ is the blow-up of $Z$ along three pairwise disjoint smooth surfaces $S_1, S_2, S_3$. The surfaces $S_2$ and $S_3$ are sections of $\varphi$, they are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and have normal bundle $\mathcal{O}(−1, −1) \oplus \mathcal{O}$. The blow-up $X_1$ of $Z$ along $S_2$ and $S_3$ is a toric Fano 4-fold of combinatorial type $Q_4$, following [Bat99].

On the other hand the surface $S_1$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with $−K_{S_1} = \varphi^*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|_{S_1}$ (see Rem. 4.3), so that $S_1$ is a del Pezzo surface of degree 4.

Description of $X_{B_2}$. Here $T = F_1$ and $N$ is the pullback of a general line in $\mathbb{P}^2$. Then $Z$ is a $\mathbb{P}^2$-bundle over $F_1$, and $S_2, S_3$ are two sections; the blow-up $X_1$ of $Z$ along $S_2$ and $S_3$ is a toric Fano 4-fold of combinatorial type $Q_2$, following [Bat99]. Moreover $\varphi[S_1] : S_1 \to F_1$ is a double cover with $−K_{S_1} = \varphi^*\mathcal{O}(2N − E)$, where $E \subset F_1$ is the exceptional curve, so that $S_1$ is a del Pezzo surface of degree 6, the blow-up of $\mathbb{P}^2$ at 3 points.

We give in Table 7.1 some numerical invariants of the Fano 4-folds $X_{B_0}, X_{B_1}$, and $X_{B_2}$; they are computed using standard methods, see [CR22] Lemmas 3.2 and 3.3. In the last column $T$ denotes the tangent bundle.

**Table 7.1. Numerical invariants**

| 4-fold | $b_3$ | $h^{2,2}$ | $h^{1,3}$ | $K^4$ | $K^2 \cdot c_2$ | $h^0(-K)$ | $\chi(T)$ |
|--------|-------|----------|----------|------|-----------------|-----------|--------|
| $X_{B_0}$ | 0 | 10 | 0 | 224 | 152 | 51 | 4 |
| $X_{B_1}$ | 0 | 14 | 0 | 222 | 156 | 51 | −2 |
| $X_{B_2}$ | 0 | 12 | 0 | 223 | 154 | 51 | 1 |

Finally we apply our results to the study of conic bundles on Fano 4-folds.

Let $X$ be a Fano 4-fold and $\eta : X \to Y$ a conic bundle such that $\rho_X - \rho_Y \geq 3$. Let us denote by $\Delta := \{ y \in Y | \eta^{-1}(y) \text{ is singular} \}$ the discriminant divisor.

If $X \cong S \times T$ is a product of two del Pezzo surfaces, then it follows easily that $Y \cong \mathbb{P}^1 \times T$ and $\eta$ is induced by a conic bundle $S \to \mathbb{P}^1$ (see for instance [Rom19] Lemma 2.10); in particular all the connected components of $\Delta$ are isomorphic to $T$.

Let us assume that $X$ is not a product of surfaces. Then we have $\rho_X - \rho_Y = \delta_X = 3$ by [Rom19] Th. 4.2(1) and [MR21] Th. 1.1, and $\rho_X \in \{5, 6\}$ by Prop. 1.5, so that the possible $X$ are classified. The case where $\rho_X = 5$ has been studied in [CR22] Cor. 2.18.

As an application of our results, we describe the case $\rho_X = 6$.

**Corollary 7.2.** Let $\eta : X \to Y$ be a conic bundle, where $X$ is a Fano 4-fold with $\rho_X = 6$, and $\rho_X - \rho_Y = 3$. Let $\Delta$ be the discriminant divisor. Then one of the following hold:

(i) $\Delta \cong \mathbb{F}_1 \cup \mathbb{F}_1$ and either $X \cong U_2$ or $X \cong U_4$. 

We show the uniqueness of Proof of Prop. 1.6.

(i) $\Delta \cong \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1$ and $X$ is isomorphic to one of the following varieties: $U_1$, $U_3$, $U_5$, $U_6$, $U_7$, $U_8$, $X_{B_0}$.

(ii) $\Delta \cong \mathbb{P}^1 \times \mathbb{P}^1 \sqcup S_1$ with $S_1$ a del Pezzo surface of degree 4 and $X \cong X_{B_1}$.

(iii) $\Delta \cong \mathbb{P}^1 \times \mathbb{P}^1 \sqcup S_1$ with $S_1$ a del Pezzo surface of degree 6 and $X \cong X_{B_2}$.

Proof. If $X$ is a product of two del Pezzo surfaces, the statement is easy, so let us assume this is not the case; then we have $\delta_X = 3$ by [MR21, Th. 1.1].

Now, replacing $\psi \circ f$ by $\eta$ in Th. 5.1, we check that all the properties $(a)-(h)$ are satisfied. Indeed, by [Rom19 Prop. 3.5 (1)], $\eta$ factors as a composition $X \to X_2 \to Y$ where the first map satisfies $(f)$; and in view of [Rom19 Th. 4.2 (2)] also $(a)$, $(b)$, $(c)$, $(d)$ hold. Finally, $(g)$ follows by the proof of [Rom19 Prop. 3.4], while $(h)$ is shown in Step 2 of the proof of [Rom19 Th. 4.2 (2)]. Therefore, we may run the arguments of the proof of Th. 5.2 with $\eta$ instead of $\psi \circ f$. We keep the notation as in that theorem.

Then $\Delta = A_1 \sqcup A_2$, and by Step 5.3 it follows that $A_i \cong B_i \cong S_i$ for $i = 1, 2$. Moreover, using Step 5.2, we know that $A_2 \cong T$, where in our case either $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $T \cong \mathbb{F}_1$. If $A_1$ is a section of $\xi: Y \to T$ then using Steps 5.4 and 5.6 we get (i) or (ii). Otherwise, by Step 5.20 we deduce that $X$ is obtained as in Construction B, and finally that $X$ is isomorphic to $X_{B_0}$, $X_{B_1}$, or $X_{B_2}$, following the proof of Prop. 7.1.

Remark 7.3. Let $X$ be a Fano 4-fold with $\delta_X = 3$. By Th. 5.1, we know that $\psi \circ f: X \to Y$ is a conic bundle with $\rho_X - \rho_Y = 3$. All the possible targets $Y$ have been classified in [MR21], but in [MR21 Prop. 1.2(b)] there is a missing case, that is $Y \cong \mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(N))$, with $N$ being the pullback of a general line through the blow-up $\mathbb{F}_1 \to \mathbb{P}^2$. In this case, Construction A gives $X = U_2$; while performing Construction B we obtain the non-toric Fano family $X_{B_2}$.

8. The case $\rho_X = 5$

In this section we consider Fano varieties with $\delta_X = 3$ and $\rho_X = 5$, the minimal Picard number, and prove Prop. 1.6. Cor. 1.7, and Th. 1.8.

By Th. 1.4 $X$ is obtained as in Construction A or B from a smooth Fano variety $T$ with $\rho_T = 1$. We denote by $\mathcal{O}_T(1)$ the ample generator of $\text{Pic}(T) \cong \mathbb{Z}$, and $\mathcal{O}_T(a) := \mathcal{O}_T(1)^\otimes a$ for every $a \in \mathbb{Z}$. Recall that $\mathcal{O}_T(-K_T) \cong \mathcal{O}_T(i_T)$, where $i_T$ is the index of $T$.

Proof of Prop. 1.6. We show the uniqueness of $\sigma: X \to T$. Let us assume that there exists another $\tilde{\sigma}: X \to \overline{T}$ as in Construction A or B. We show that $\sigma$ and $\tilde{\sigma}$ coincide up to an isomorphism $T \cong \overline{T}$.

Let $R$ be an extremal ray of the cone $\text{NE}(\sigma)$. By Remarks 6.2, 6.5, and 6.7 the contraction associated to $R$ is the blow-up of a smooth subvariety $S$ which is an irreducible codimension 2 complete intersection, and either $S \cong T$ or $S \cong S_1$. Let $E \subset X$ be the exceptional divisor: since the normal bundle of $S$ is decomposable, we have $E \cong \mathbb{P}_S(\mathcal{O} \oplus L)$ with $L \in \text{Pic}(S)$.

We first assume that $n \geq 5$, where $n := \dim X$. Recall that $S_1$ is a ramified double cover of $T$, and $\rho_T = 1$, so that the ramification divisor is ample; since $\dim S_1 = n - 2 > 2$, [Cor81] yields $\rho_{S_1} = 1$. Therefore in any case we have $\rho_S = 1$ and $\rho_E = 2$. 

Since $\rho_E = 2$, $E$ has at most two elementary contractions, one being the $\mathbb{P}^1$-bundle $E \to S$. If $L \cong \mathcal{O}_S$, then $E \cong \mathbb{P}^1 \times S$, and the second elementary contraction is $E \to \mathbb{P}^1$. Otherwise we can assume that $L$ is ample; in this case $E$ has an elementary divisorial contraction sending a divisor to a point (see for instance [CD15, p. 10768]).

Consider now the restriction $\tilde{\sigma}|_E : E \to \tilde{\sigma}(E) \subseteq \tilde{T}$. The Stein factorization gives

$$E \xrightarrow{\psi} B \xrightarrow{\nu} \tilde{\sigma}(E),$$

where $\psi$ is a contraction of $E$, and $\nu$ is finite. Thus $\dim B = \dim \tilde{\sigma}(E) \leq \dim \tilde{T} = n - 2$; on the other hand every fiber of $\tilde{\sigma}$ is a surface, thus $\dim B \geq \dim E - 2 = n - 3 \geq 2$. By the previous observation on the possible contractions of $E$, we deduce that $B \cong S$ and $\psi$ is the $\mathbb{P}^1$-bundle $E \to S$, so that $\tilde{\sigma}$ contracts the fibers of $E \to S$ and $R \subset \text{NE}(\tilde{\sigma})$.

This holds for every extremal ray $R$ of $\text{NE}(\sigma)$, so that $\text{NE}(\sigma) \subset \text{NE}(\tilde{\sigma})$. Since these two cones are both 4-dimensional faces of $\text{NE}(X)$, we conclude that they are the same, and by [Deb01] Prop. 1.14 $\sigma$ and $\tilde{\sigma}$ coincide up to an isomorphism $T \cong \tilde{T}$.

Suppose now that $n = 4$, so that $T \cong \tilde{T} \cong \mathbb{P}^2$. We repeat the same argument as above with an extremal ray $R$ such that $E$ is a $\mathbb{P}^1$-bundle over $T$. We still have $\dim B \geq \dim E - 2 = 1$, and if $\dim B = 1$, then $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\psi$ the projection onto $\mathbb{P}^1$. Thus the general fiber of $\tilde{\sigma}$ over $\tilde{\sigma}(E)$ is isomorphic to $\mathbb{P}^2$, which is impossible by the description of the fibers in §6.1. We conclude that $\dim B \geq 2$, and as before that $R \subset \text{NE}(\tilde{\sigma})$. If $X = X_A$ we get $\text{NE}(\sigma) \subset \text{NE}(\tilde{\sigma})$ and hence $\text{NE}(\sigma) = \text{NE}(\tilde{\sigma})$. If $X = X_B$, we see from Rem. 6.5 and Fig. 6.2 that the extremal rays $R$ of $\text{NE}(\sigma)$ such that $E$ is a $\mathbb{P}^1$-bundle over $T$ generate a 4-dimensional cone. We conclude that $\dim(\text{NE}(\sigma) \cap \text{NE}(\tilde{\sigma})) = 4 = \dim \text{NE}(\sigma) = \dim \text{NE}(\tilde{\sigma})$, so that again the two cones coincide.

Finally suppose that $n = 3$, so that $T \cong \tilde{T} \cong \mathbb{P}^1$, and repeat the same argument as above. We conclude that either $R \subset \text{NE}(\tilde{\sigma})$, or $E$ has two different $\mathbb{P}^1$-bundle structures, hence $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $O(-1, -1)$. It is not difficult to see that there is at most one extremal ray $R$ with this property, so we conclude that $\text{NE}(\sigma) = \text{NE}(\tilde{\sigma})$ as in the case $n = 4$.

Let $X = X_A$. By Rem. 6.2 and Lemma 6.3 $\sigma : X \to T$ has exactly two different factorizations as in Construction A, through $\mathbb{P}_T(O(D_1) \oplus O(D_2) \oplus O(D_3))$ and $\mathbb{P}_T(O(-D_1) \oplus O(-D_2) \oplus O(-D_3))$. Moreover $O_T(D_1), O_T(D_2), O_T(D_3)$ are determined up to reordering and tensoring with a line bundle.

We may assume that $O_T(D_1) \cong O_T(a), O_T(D_2) \cong O_T(b), O_T(D_3) \cong O_T(b)$, with $a, b \in \mathbb{Z}$. By Lemma 3.1 we see that $(a, b)$ must satisfy the following conditions:

$$(8.1) \quad |a|, |b|, |a - b| \leq i_T - 1.$$

We say that two pairs of integers $(a, b)$ and $(a', b')$ are equivalent (denoted by $\sim$) if they satisfy $(8.1)$ and give isomorphic $X_A$’s. By the previous discussion, we see that all the pairs equivalent to $(a, b)$ can be obtained by the following relations:

$$(a, b) \sim (b, a), (-b, a - b), (-a, -b).$$

Hence up to equivalence we can subsequently assume:

- $a \geq 0$: indeed if $a < 0$, we replace $(a, b)$ with $(-a, -b)$;
- $a \geq b$: if $a < b$, we replace $(a, b)$ with $(b, a)$;
- $b \leq 0$: if $b > 0$, we replace $(a, b)$ with $(a - b, -b)$.
\[ a \geq -b: \text{ if } a < -b, \text{ we replace } (a, b) \text{ with } (-b, -a), \]

so that in the end: \( b \leq 0 \) and \( a \geq |b| \). These conditions, together with (8.1), are equivalent to the conditions in the statement:

\[
(8.2) \quad b \leq 0, \quad |b| \leq \frac{it - 1}{2}, \quad \text{and} \quad |b| \leq a \leq it - 1 - |b|. 
\]

Finally it is not difficult to see that if \( (a, b), (a', b') \) satisfy (8.2) and \( (a, b) \sim (a', b') \), then \( a = a' \) and \( b = b' \).

Let \( X = X_B \). By Rem. 6.3, Prop. 6.6, and Rem. 6.7, \( \sigma: X \rightarrow T \) has exactly three different factorizations as in Construction B, all through \( Z = \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}) \), where \( N \neq 0 \) and \( 2N \) is effective. Therefore \( \mathcal{O}_T(N) \) is ample and isomorphic to \( \mathcal{O}_T(a) \) for some integer \( a \geq 1 \). By Lemma 4.7, we see that a must satisfy \( a \leq it - 1 \).

**Proof of Cor. 1.7.** We have \( \delta_X \geq \operatorname{codim} \mathcal{N}_1(D, X) = \rho_X - 2 \). If \( \rho_X \geq 6 \), then \( \delta_X \geq 4 \), so that by Th. 1.1 we have \( X \cong S \times T \) where \( S \) is a del Pezzo surface with \( \rho_S = \delta_X + 1 \geq 5 \). If \( D_T \subset T \) is a prime divisor then \( \dim \mathcal{N}_1(S \times D_T, X) \geq \rho_S \), therefore \( D \) must dominate \( T \) under the projection \( \pi_T: S \times T \rightarrow T \). Moreover \( \pi_T \) is not finite on \( D \), so that we consider the pushforward \( (\pi_T)_* : \mathcal{N}_1(S \times T) \rightarrow \mathcal{N}_1(T) \), we have \( \ker(\pi_T)_* \cap \mathcal{N}_1(D, X) \neq \{0\} \), and we conclude that \( \rho_T = 1 \).

Suppose that \( \rho_X = 5 \), and note that the assumptions imply that \( \dim X \geq 3 \). If \( \delta_X > \rho_X - 2 \), then \( \delta_X = \rho_X - 1 \), which means that \( X \) contains a prime divisor \( D' \) with \( \dim \mathcal{N}_1(D', X) = 1 \). By [CD15, Lemma 3.1] this would imply that \( \rho_X \leq 3 \), a contradiction. Therefore \( \delta_X = \rho_X - 2 = 3 \), and \( X \) is as in Prop. 1.6.

**Proof of Th. 1.8 (ii) \Rightarrow (i)** In \( Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O}) \) let \( S_2, S_3 \) be the sections of \( Z \rightarrow T \) corresponding to the trivial summands, and \( F_1 := \mathbb{P}_T(\mathcal{O} \oplus \mathcal{O}) \rightarrow Z \) given by the projection \( \mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O} \rightarrow \mathcal{O} \oplus \mathcal{O} \) (this is \( D \) in the notation of Construction B), so that \( S_2, S_3 \subset F_1 \). Then, in both Constructions A and B, \( X \) is the blow-up of \( Z \) along \( S_2, S_3 \), and a third subvariety \( S_1 \) disjoint from \( F_1 \).

If \( E'_1 \subset X \) is the transform of \( F_1 \), we have \( E'_1 \cong \mathbb{P}^1 \times T \) with normal bundle \( \mathcal{N}_{E'_1/X} \cong \pi^{-1}_{E'_1} \mathcal{O}(-1) \oplus \pi^{-1}_T \mathcal{O}(-a) \). Let \( \Gamma_0 \subset T \) be a curve, and let \( \Gamma \subset E'_1 \) be the curve corresponding to \( \{pt\} \times \Gamma_0 \). We show that [\( \Gamma \)] generates an extremal ray of \( \operatorname{NE}(X) \): the associated contraction \( \tau: X \rightarrow X' \) is elementary divisorial with \( \operatorname{Exc}(\tau) = E'_1 \) and \( \tau(\operatorname{Exc}(\tau)) \cong \mathbb{P}^1 \).

Set \( c := \mathcal{O}_T(1) \cdot \Gamma_0 \), and consider \( e'_1 \subset E'_1 \) corresponding to \( \mathbb{P}^1 \times \{\text{pt}\} \). We have:

\[
E'_1 \cdot e'_1 = -1, \quad -K_X \cdot e'_1 = 1, \quad E'_1 \cdot \Gamma = -ac, \quad -K_X \cdot \Gamma = c(it - a).
\]

If \( 2a > it \), consider \( H := a(-K_X) + (it - a)E'_1 \). Then \( H \cdot C \geq 0 \) for every curve \( C \subset X \) not contained in \( E'_1 \). Moreover \( H \cdot e'_1 = 2a - it > 0 \) and \( H \cdot \Gamma = 0 \), so that \( H \) is nef and \( H^\perp \cap \operatorname{NE}(X) = \mathbb{R}_{\geq 0}[\Gamma] \).

If instead \( 2a \leq it \), consider the exceptional divisor \( E_2 \cong \mathbb{P}_T(\mathcal{O} \oplus \mathcal{O}(a)) \) over \( S_2 \), and \( e_2 \subset E_2 \) a fiber of the \( \mathbb{P}^1 \)-bundle \( E_2 \rightarrow T \). We have:

\[
E_2 \cdot e_2 = -1, \quad E_2 \cdot e'_1 = 1, \quad E_2 \cdot \Gamma = 0, \quad -K_X \cdot e_2 = 1, \quad E'_1 \cdot e_2 = 1,
\]

and \( E_2 \cdot R \geq 0 \) for every extremal ray \( R \) of \( \operatorname{NE}(X) \) not containing \([e_2]\). Consider \( H' := a(-K_X) + (it - a)E'_1 + (it - 2a + 1)E_2 \). We have \( H' \cdot e'_1 = 1, \quad H' \cdot \Gamma = 0, \quad H' \cdot e_2 = 2a - 1 > 0 \), and \( H' \cdot R > 0 \) for every extremal ray of \( \operatorname{NE}(X) \) not containing \([e'_1], [\Gamma], [e_2]\). We conclude that \( H' \) is nef and \((H')^\perp \cap \operatorname{NE}(X) = \mathbb{R}_{\geq 0}[\Gamma] \).
(i) ⇒ (ii) If \( n = 4 \) the statement is shown in [CR22, Cor. 1.4], so that we can assume \( n \geq 5 \).

Consider the pushforward \( \tau_*: \mathcal{N}_1(X) \to \mathcal{N}_1(X') \). We have \( \tau_* (\mathcal{N}_1(\text{Exc}(\tau), X)) = \mathbb{R} [\tau(\text{Exc}(\tau))] \), thus \( \dim \mathcal{N}_1(\text{Exc}(\tau), X) = 2 \), and it follows from Cor. 1.7 that \( X \) is as described in Prop. 1.6. We note first of all that \( X \neq F \times T \) (\( F \) the blow-up of \( \mathbb{P}^2 \) at three non-collinear points), as \( F \times T \) has no elementary divisorial contraction sending a divisor to a curve, hence the case \( X = X_A \) with \( a = b = 0 \) is excluded, and \( i_T > 1 \).

Let \( \Gamma \subset X \) be a curve contracted by \( \tau \). We show that at least one of the exceptional divisors of the extremal rays of \( \text{NE}(\sigma) \) has non-zero intersection with \( \Gamma \). We keep the same notation as in Rem. 2.1 and Sections 3 and 4.

By contradiction, suppose otherwise: then the classes of these divisors in \( \mathcal{N}^1(X) \) all belong to the hyperplane \( \Gamma^\perp \). This in turn implies in \( \mathcal{N}^1(Z) \) that, if \( X = X_A \) (respectively \( X = X_B \)), the linear span of the classes of \( F_1, F_3, F_3 \) (respectively the classes of \( D \) and \( H_0 \)) has dimension 1. This is possible only if \( X = X_A \) and \( a = b = 0 \), that we have already excluded.

Hence there exists some extremal ray of \( \text{NE}(\sigma) \) such that the associated exceptional divisor \( E \) satisfies \( E \cdot \Gamma \neq 0 \). We show that \( E = \text{Exc}(\tau) \).

With the same notation as in the proof of Prop. 1.6 \( E \) is a \( \mathbb{P}^1 \)-bundle over \( S \), where \( S \cong T \) or \( S \cong S_1 \), and \( \rho_S = 1 \). If \( T_0 \subset \text{Exc}(\tau) \) is a non-trivial fiber of \( \tau \), we have \( E \cap T_0 \neq \emptyset \) and hence \( \dim(E \cap T_0) \geq n - 3 > 0 \), and the Stein factorization of \( \tau |_E \) induces a non-trivial contraction \( \psi: E \to B \). Since \( E \) meets every non-trivial fiber \( T_0 \) of \( \tau \), \( \psi(\text{Exc}(\psi)) \) is a curve, and by the analysis of the elementary contractions of \( E \) in the proof of Prop. 1.6 we conclude that \( E \cong \mathbb{P}^1 \times S \) and \( \psi \) is the projection onto \( \mathbb{P}^1 \), in the proof of Prop. 1.6 we conclude that \( E \cong \mathbb{P}^1 \times S \) and \( \psi \) is the projection onto \( \mathbb{P}^1 \).

This means that \( \dim \tau(E) = 1 \), hence \( E = \text{Exc}(\tau) \).

If \( X = X_A \), it is not difficult to check that \( \mathbb{P}^1 \times T \) appears among the divisors \( E_1, E_2, E_3, E_1', E_2', E_3' \) if and only if \( b = 0 \), so that \( Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O}) \). Then we have \( E_1 \cong E_1' \cong \mathbb{P}^1 \times T \) with normal bundles \( \mathcal{N}_{E_1/X} \cong \mathcal{O}(-1) \otimes \mathcal{O}(a) \) and \( \mathcal{N}_{E_1'/X} \cong \mathcal{O}(-1) \otimes \mathcal{O}(a) \). Since \( a > 0 \), we have \( \text{Exc}(\tau) = E_1', \) and we get the statement.

If \( X = X_B \), we show that \( S \cong T \). Indeed, if \( S \cong S_1 \), \( E \) should be one of the divisors \( E_1, G_2, G_3 \) (see Rem. 6.5), and by Prop. 6.6 we have \( E_1 \cong G_2 \cong G_3 \cong \mathbb{P}_S(\mathcal{O}(H S_1)) \).

On the other hand \( H | S_1 \) is linearly equivalent to the ramification divisor of the non-trivial double cover \( \varphi_{| S_1} : S_1 \to T \) by Rem. 1.3 so that \( H | S_1 \neq 0 \). Therefore we get \( S \cong T \), and it is not difficult to check that \( \text{Exc}(\tau) = D \).
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CINZIA CASAGRANDE: UNIVERSITÀ DI TORINO, DIPARTIMENTO DI MATEMATICA, VIA CARLO ALBERTO 10, 10123 TORINO - ITALY
Email address: cinzia.casagrande@unito.it

ELEONORA A. ROMANO: UNIVERSITÀ DI GENOVA, DIPARTIMENTO DI MATEMATICA, VIA DODECANESCO 35, 16146 GENOVA - ITALY
Email address: eleonoraanna.romano@unige.it

SAVERIO A. SECCI: UNIVERSITÀ DI TORINO, DIPARTIMENTO DI MATEMATICA, VIA CARLO ALBERTO 10, 10123 TORINO - ITALY
Email address: saverioandrea.secci@unito.it