An anti-Schwarzschild solution: wormholes and scalar-tensor solutions

José P Mimoso* and Francisco S N Lobo
Centro de Astronomia e Astrofísica da Universidade de Lisboa, Avenida Professor Gama Pinto 2, P-1649-003 Lisbon, Portugal
E-mail: jpmimoso@cii.fc.ul.pt, flobo@cii.fc.ul.pt

Abstract. We investigate a static solution with an hyperbolic nature, characterised by a pseudo-spherical foliation of space. This space-time metric can be perceived as an anti-Schwarzschild solution, and exhibits repulsive features. It belongs to the class of static vacuum solutions termed “a degenerate static solution of class A” (see [1]). In the present work we review its fundamental features, discuss the existence of generalised wormholes, and derive its extension to scalar-tensor gravity theories in general.

1. Introduction
We consider a largely ignored metric which belongs to a class of vacuum solutions referred to as degenerate solutions of class A [1] given by

\[ ds^2 = -e^{\mu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (du^2 + \sinh^2 u dv^2) , \]  

(1)

which are axisymmetric solutions [2], but where the usual 2-dimensional spheres are replaced by pseudo-spheres, \( dr^2 = du^2 + \sinh^2 u dv^2 \), i.e., by surfaces of negative, constant curvature. These are still surfaces of revolution around an axis, and \( v \) represents the corresponding rotation angle. For the vacuum case we get

\[ e^{\mu(r)} = e^{-\lambda(r)} = \left( \frac{2\mu}{r} - 1 \right) , \]  

(2)

where \( \mu \) is a constant [1, 2]. We immediately see that the static solution holds for \( r < 2\mu \) and that there is a coordinate singularity at \( r = 2\mu \) (note that \( |g| \) neither vanishes nor becomes \( \infty \) at \( r = 2\mu \))[3]. This is the complementary domain of the exterior Schwarzschild solution. In the region \( r > 2\mu \), as in the latter solution, the \( g_\mu \) and \( g_\tau \) metric coefficients swap signs. Defining \( \tau = r \) and \( \rho = t \), we obtain \( ds^2 = -d\tilde{\tau}^2 + A^2(\tilde{\tau}) d\rho^2 + B^2(\tilde{\tau}) (du^2 + \sinh^2 u dv^2) \), with the following parametric definitions \( \tilde{\tau} = -\tau + 2\mu \ln |\tau - 2\mu| \), \( A^2 = 2\mu/\tau - 1 \) and \( B^2(\tau) = \tau^2 \), which is a particular case of a Bianchi III axisymmetric universe.

Using pseudo-spherical coordinates \( \{ x = r \sinh u \cos v, y = r \sinh u \sin v, z = r \cosh u, w = b(r) \} \), the spatial part of the metric (1) can be related to the hyperboloid \( w^2 + x^2 + y^2 - z^2 = (b^2/r^2 - 1) r^2 \) embedded in a 4-dimensional flat space. We then have

\[ dw^2 + dx^2 + dy^2 - dz^2 = \left[ (b'(r))^2 - 1 \right] dr^2 + r^2 \left( du^2 + \sinh^2 u dv^2 \right) . \]  

(3)
where the prime stands for differentiation with respect to \( r \), and \( b(r) = \mp \sqrt{2\mu / \bar{r}} \). We can recast metric (1) into the following
\[
ds^2 = - \tan^2 \left[ \ln (\bar{r}) + \frac{2\mu}{\bar{r}} \right] \, \dd \tau^2 + \left( \frac{2\mu}{\bar{r}} \right)^2 \cos^4 \left[ \ln (\bar{r}) + \frac{2\mu}{\bar{r}} \right] \left[ \dd^2 \bar{r}^2 + \dd^2 \left( \sinh^2 \bar{u} \, \dd \bar{v}^2 \right) \right],
\]
which is the analogue of the isotropic form of the Schwarzschild solution. The spatial surfaces are conformally flat, but the flat metric is not euclidean. Indeed, the 3–dim spatial metric \( \dd \sigma^2 = \dd^2 \bar{r}^2 + \dd^2 \left( \sinh^2 \bar{u} \, \dd \bar{v}^2 \right) \) is foliated by 2-dimensional surfaces of negative curvature, since \( R^2 = - \sinh^2 \bar{u} \), and it corresponds to \( \dd \sigma^2 = \dd x^2 + \dd y^2 - \dd z^2 \). We thus cannot recover the usual Newtonian weak-field limit.

Analysing the “radial” motion of test particles, we have the following equation \( \dd^2 \bar{r} + \left( \frac{2\mu}{\bar{r}} \right) \left( 1 + \frac{ar{h}^2}{\bar{r}^2 \sinh^2 \bar{u}} \right) = \epsilon \) where \( \epsilon \) and \( \bar{h} \) are constants of motion defined by \( \epsilon = (2\mu / \bar{r}^2 - 1) \bar{r} = \text{const} \) and \( \bar{h}^2 = r^2 \sinh^2 \bar{u} \, \bar{v} = \text{const} \), for fixed \( \bar{u} = u_* \). The former and latter constants represent the energy and angular momentum per unit mass, respectively. We thus define the potential \( 2V(r) = \left( \frac{2\mu}{\bar{r}} - 1 \right) \left( 1 + \frac{ar{h}^2}{\bar{r}^2 \sinh^2 u_*} \right) \). This potential is manifestly repulsive, crosses the \( r \)-axis at \( r = 2\mu \), and for sufficiently high values of \( \bar{h} \) it has a minimum at \( r_{\pm} = (\bar{h}^2 \mp \sqrt{\bar{h}^4 - 12\mu^2 \bar{h}^2})/(2\mu) \). However this minimum falls outside the \( r = 2\mu \) divide. So a test particle is subject to a repulsive potential forcing it to inevitably cross the event horizon at \( r = 2\mu \) attracted either by some mass at the minimum or by masses at infinity. In [2] it is hinted that the non-existence of a clear Newtonian analogue is related to the existence of mass sources at \( \infty \), but no definite conclusions were drawn.

2. Alter-ego of Morris-Thorne wormholes

A natural extension of the solution (2) would be to add exotic matter to obtain static and pseudo-spherically symmetric traversable wormhole solutions [4]. Consider the metric (1) given by \( \mu(r) = 2\Phi(r) \) and \( \lambda(r) = - \ln[1 - b(r)/r] \). The coordinate \( r \) decreases from a constant value \( r_0 \) to a minimum value \( r_\text{th} \), representing the location of the throat of the wormhole, where \( b(r_\text{th}) = r_\text{th} \), and then it increases from \( r_\text{th} \) back to the value \( \mu \). The condition \( (b/r - 1) \geq 0 \) imposes that \( b(r) \geq r_\text{th} \), contrary to the Morris-Thorne counterpart [5].

The solution provides the following stress-energy scenario
\[
\rho(r) = -\frac{1}{8\pi} \frac{b'}{r^2}, \quad p_r(r) = \frac{1}{8\pi} \left[ \frac{b}{r^3} + 2 \left( \frac{b}{r} - 1 \right) \frac{\Phi'}{r} \right],
\]
\[
p_t(r) = \frac{1}{8\pi} \left( \frac{b}{r} - 1 \right) \left[ \Phi'' + \left( \Phi' \right)^2 + \frac{b'r + b - r}{2r(b - r)} \Phi' + \frac{b'r - b}{2r^2(b - r)} \right],
\]
in which \( \rho(r) \) is the energy density, \( p_r(r) \) is the radial pressure, \( p_t(r) \) is the pressure measured in the tangential directions. Note that the radial pressure is always positive at the throat, i.e, \( p_r = 1/(8\pi r^2 \Phi) \), contrary to the Morris-Thorne wormhole, where a radial tension at the throat is needed to sustain the wormhole. In addition to this, the mathematics of embedding imposes that \( b'(r_\text{th}) = 1 \) at the throat, which implies a negative energy density at the throat (see [4] for more details). This condition is another significant difference to the Morris-Thorne wormhole, where the existence of negative energy densities at the throat is not a necessary condition. Several interesting equations of state were considered in [4], and we refer the reader to this work for more details.

3. Pseudo-spherical scalar-tensor solution

A theorem by Buchdahl [6] establishes the reciprocity between any static solution of Einstein’s vacuum field equations and a one-parameter family of solutions of Einstein’s equations with a
(massless) scalar field. In the conformally transformed Einstein frame, note that scalar-tensor gravity theories are described by

\[ S = \int \sqrt{-\tilde{g}} \left[ \tilde{R} - \frac{1}{2} (\nabla \varphi)^2 + (16\pi G_N/\Phi^{-2}(\varphi)) L_{\text{Matter}} \right]. \]

In this representation we have GR plus a massless scalar field which is now coupled to the matter fields. Different scalar-tensor theories correspond to different couplings. In the absence of matter we can use Buchdahl’s theorem. So, given the metric (1), we derive the corresponding scalar-tensor solution

\[ ds^2 = -\left(\frac{2\mu}{r} - 1\right)^B dt^2 + \left(\frac{2\mu}{r} - 1\right)^{-B} dr^2 + \left(\frac{2\mu}{r} - 1\right)^{1-B} r^2 (du^2 + \sinh^2 u dv^2), \tag{7} \]

\[ \varphi(r) = \sqrt{\frac{C^2(2\omega + 3)}{16\pi}} \varphi_0 \ln \left(\frac{2\mu}{r} - 1\right), \tag{8} \]

where \( C^2 = (1 - B^2)/(2\omega + 3) \) and \(-1 \leq B \leq 1\). This clearly reduces to our anti-Schwarzschild metric (1) in the GR limit when \( B = 1 \), and hence \( C = 0 \) implying that \( G = \Phi^{-1} \) is constant. Reverting \( \varphi = \int \sqrt{\Phi_0(2\omega + 3)/(16\pi)} d\ln(\Phi/\Phi_0) \), and the conformal transformation, \( \tilde{g}_{ab} = (2\mu/r - 1)^{-C} g_{ab} \), we can recast this solution in the original frame in which the scalar-field is coupled to the geometry and the content is vacuum, the so-called Jordan frame. The \( r = 2\mu \) limit is no longer just a coordinate singularity, but rather a true singularity as it can be verified from the analysis of the curvature invariants. In the Einstein frame this occurs because the energy density of the scalar field diverges likewise in the Schwarzschild case[7]. Of paramount importance is that, once again, the ST-solution has no Newtonian limit (as its GR limit does not have one). This implies that the usual Parametrized Post-Newtonian formalism that assesses the departures of modified gravity theories from GR does not hold for this class of metrics (see [8]).

4. Discussion

We have outlined the exotic features of the vacuum static solution with a pseudo-spherical foliation of space. We have revealed the existence of generalised wormholes, and derived its extension to scalar-tensor gravity theories. A fundamental feature of these solutions is the absence of a Newtonian weak field limit, which reminds us of a quotation from John Barrow [9]

\[ \text{The miracle of general relativity is that a purely mathematical assembly of second-rank tensors should have anything to do with Newtonian gravity in any limit.} \]

Acknowledgments

The authors are grateful to Raül Vera and Guillermo A. González for helpful discussions. JPM also acknowledges the LOC members, Ruth, Raül, Jesús and José for a very enjoyable conference.

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