PHASE TRANSITIONS IN NJL AND SUPER-NJL MODELS

Zygmunt Lalak\textsuperscript{a,b,1}, Jacek Pawelczyk\textsuperscript{a,1} and Stefan Pokorski\textsuperscript{a,c,1}

\textsuperscript{a} Institute of Theoretical Physics
University of Warsaw
Hoża 69, Warsaw, PL 00-681

\textsuperscript{b} Physics Department
Technische Universität München
D-85748 Garching

\textsuperscript{c} Max-Planck-Institut für Physik
Werner-Heisenberg-Institut
P.O.Box 401212 München

ABSTRACT

An elementary method of determination of the character of the hot phase transition in 4d four-fermion NJL-type models is applied to non-supersymmetric and supersymmetric versions of simple NJL model. We find that in the non-susy case the transition is usually of the second order. It is weakly first order only in the region of parameters which correspond to fermion masses comparable to the cut-off. In the supersymmetric case both kinds of phase transitions are possible. For sufficiently strong coupling and sufficiently large susy-breaking scale the transition is always of the first order.

\textsuperscript{1}lalak,pawelc,pokorski@fuw.edu.pl

\textsuperscript{*}Work partially supported by Polish Commite for Scientific Research.
1. Introduction. Recently a lot of attention has been devoted to the problem of phase transitions in a variety of particle physics models. A sound motivation for this activity is given by cosmology. In particular, it is believed that first order phase transitions imply several interesting and potentially observable effects. One well known and widely considered example is the possibility of baryon number generation during first-order electroweak phase transition [1]. Other examples include gravitational wave production during bubble collisions [2] and modifications of relic dark matter particle abundancies [3].

As the symmetry breaking in the standard model is achieved by means of a scalar field with a nonzero vacuum expectation value, it is the temperature phase structure of the bosonic models, for example $\phi^4$ theory, which attracts at present attention of the public and is subject to a careful investigation. At the same time similar problems in simplest fermionic models, like the Nambu-Jona Lasinio (NJL) one, remain rather poorly known in four dimensions. It should be stressed that the fermionic models, although generically nonrenormalizable in 4d, are very useful as phenomenological effective Lagrangians. They facilitate quantitative studies of phenomena for which fundamental theories are not known and provide a tractable description of dynamical symmetry breaking. On more practical side, a specific interest in this type of theories may be justified by composite Higgs models of BHL-type [4],[5] and by recent revival of studies on NJL-type QCD condensates [6].

In this note we shall discuss the temperature phase structure of the NJL and super-NJL (SNJL) models to leading order in large $N$ expansion. An elementary but effective algorithm is used to determine the type of hot phase transitions in those models. We study parameter space of the nonsupersymmetric and supersymmetric four-fermion theories with particular regard to regions interesting for phenomenological applications.

The structure of the paper is as follows. In section 2 we describe our approach to hot phase transitions in NJL models, in section 3 we apply the algorithm to the nonsupersymmetric model which allows for a fully analytical treatment, and in section 4 we elaborate on the more complex supersymmetric NJL case.
2. *NJL models of fermion condensation at finite temperature.* We set out to investigate the 4d four-fermion models to the leading order in large $N$ expansion. Let us recapitulate briefly main steps of the standard procedure which converts the original Lagrangians into the form suitable for $1/N$ expansion. The goal is achieved through the introduction of auxiliary scalar fields [7],[8]. The original Lagrangian of the NJL model then reads:

\[
L_{NJL} = i\bar{\lambda} \gamma^\mu \partial_\mu \lambda + \frac{g^2}{4} [ (\bar{\lambda} \lambda)^2 - (\bar{\lambda} \gamma_5 \lambda)^2 ]
\]

where $\lambda_+, \lambda_-$ are chiral Weyl components of $\lambda$ and $B$ is an auxiliary scalar field, non-propagating at tree level. In order to apply the $1/N$ expansion to this lagrangian one considers $N$ generations of fermions $\lambda$ and rescales $B \rightarrow \sqrt{N} B$, $g \rightarrow g/\sqrt{N}$. Integration over fermions gives an effective action for the auxiliary field $B$. The vacuum expectation value of that field is related to the fermion condensate through the relation $<\bar{\lambda} \lambda> = \frac{2B}{g}$ valid to the leading order in $1/N$. Hence, the object of interest to us is the potential for $B$. To determine this potential it is sufficient to consider real valued, homogeneous $B$. Denoting $m \equiv gB$ one gets the effective action in the form

\[
S_{NJL}[m] = N \int d^4x \left( \frac{m^2}{g^2} - 2 \text{tr} \log(-\partial^2 + m^2) \right)
\]

where $S$ is written in the Euclidean space. We use finite temperature imaginary time formalism due to Matsubara i.e. we compactify the euclidean time on a circle with circumference $\beta \equiv 1/T$. The fermionic functional determinant in (2) is calculated with anti-periodic boundary conditions in the compact direction.

The supersymmetric generalization of NJL model was constructed in [9]. The Lagrangian with auxiliary superfields has the form

\[
L_{SNJL} = \int d^4\theta [(\bar{\Phi}_+ \Phi_+ + \bar{\Phi}_- \Phi_-)(1 - \Delta^2 \theta \bar{\theta} \theta \bar{\theta}) + \bar{\Phi}_2 \Phi_2]
+ \left[ \int d^2\theta [\Phi_1 \Phi_2 - g\Phi_1 \Phi_+ \Phi_-] + c.c. \right].
\]

\footnote{We use notation of the book [13].}
All superfields in (3) are chiral, $\Phi_{\pm}$ contain fermions $\lambda_{+}, \lambda_{-}$, $\Phi_{1,2}$ are auxiliary superfields and $\Delta$ is a susy-breaking scale. The first component of $\Phi_1$, which we denote by $A_1$, is proportional to $<\bar{\lambda}\lambda>$ and $m \equiv gA_1$ is, as before, the mass of the fermion $\lambda$. Functional integration over all matter fields leads to the following effective action for the parameter $m$

\[ S_{SNJL}[m] = N \left( \int d^4x \frac{m^2}{g^2} - 2 \text{tr} \log(-\partial^2 + m^2) + 2 \text{tr} \log(-\partial^2 + \Delta^2 + m^2) \right) \]  

(4)

Performing direct evaluation of (2) and (4) one obtains finite temperature effective potentials $V_{NJL}$ and $V_{SNJL}$ respectively

\[ V_{NJL}(m, T) = \frac{m^2}{g^2} - \frac{1}{16\pi^2} \left\{ m^2\Lambda^2 - m^4 \log(1 + \frac{\Lambda^2}{m^2}) \right\} - \frac{4}{\beta} \int \frac{d^3k}{(2\pi)^3} \log(1 + e^{-\beta\sqrt{m^2+k^2}}) \]  

and

\[ V_{SNJL}(m, T, \Delta) = \frac{m^2}{g^2} - \frac{1}{16\pi^2} \left\{ m^2\Lambda^2 - m^4 \log(1 + \frac{\Lambda^2}{m^2}) \right\} - \frac{4}{\beta} \int \frac{d^3k}{(2\pi)^3} \log(1 + e^{-\beta\sqrt{m^2+k^2}}) \]  

\[ - \left\{ (m^2 + \Delta^2) \log(1 + \frac{\Lambda^2}{m^2 + \Delta^2}) \right\} - \frac{4}{\beta} \int \frac{d^3k}{(2\pi)^3} \log(1 + e^{-\beta\sqrt{m^2+k^2}}) \]  

\[ + \frac{4}{\beta} \int \frac{d^3k}{(2\pi)^3} \log(1 - e^{-\beta\sqrt{m^2 + \Delta^2 + k^2}}) \]  

(5)

where the divergencies have been regularized by a simple momentum cut-off at the scale $\Lambda$. One should note at this point that all the formulae we have written till now are well defined for all values of parameters. As discussed in the literature ([10],[11],[12]) the form of $V_{NJL}$ and $V_{SNJL}$ as functions of the temperature determines the character of the phase transition between localized vacuum configurations. Positions of the extrema of the potential are given by the gap equation which is just the condition for vanishing of its first derivatives with respect to $m$.

Our purpose is to determine the type of the phase transition as a function of the parameters of the theory. First of all, let us make the following observation. By direct
inspection of eqs. (3) and (4) one concludes that the derivatives of both effective potentials with respect to \( m \) consist of two parts: the first part is temperature independent, the second temperature dependent. It is easy to check that the temperature independent part is an increasing function of \( m \) while the temperature dependent one is a decreasing function of \( m \), with monotonically decreasing (increasing) their first derivatives (i.e. the second derivatives of the respective terms in the potential), respectively. Moreover, the second derivatives of these functions (third derivatives of the respective terms in the potential) never cross each other. All together, this implies that each gap equation, which is the summ of the two terms, has at most two different from zero roots. The greater of them always corresponds to a minimum because the potentials are increasing functions of \( m \) for large enough \( m \). Hence these effective potentials have at most one nontrivial minimum, at any temperature.

There is another observation which simplifies the reasoning -the effective potentials of both models under consideration are analytic functions of \( m^2 \) around \( m = 0 \) for nonzero \( T \) and \( \Delta \). This is due to the additional mass gap which appears for fermions compactified on a circle with antiperiodic boundary conditions. Hence one can expand the finite temperature effective potentials in the vicinity of \( m = 0 \) in the Taylor series in the variable \( m^2 \).

We define the critical temperature \( T_c \) in the standard way, demanding \( \frac{\partial^2 V(T_c)}{\partial m^2} |_{m=0} = 0 = \frac{\partial V(T_c)}{\partial (m^2)} |_{m=0} \). Of course, below \( T_c \) the \( \frac{\partial V}{\partial m^2} |_{m=0} \) is negative and above \( T_c \) – positive. At this critical temperature the term in the expansion which is proportional to \( m^4 \) can be negative or positive. In the former case, at \( T = T_c \) the potential has to have a minimum for \( m \neq 0 \). The phase transition which may take place at \( T \neq T_c \) is of the first order. This follows from the fact that if one slightly increases the temperature above \( T_c \), one gets \( \frac{\partial V}{\partial (m^2)} |_{m=0} > 0 \) i.e. a "hill" arises between the local minimum at \( m = 0 \) and the global minimum which exists for \( m > 0 \) due to \( \frac{\partial^2 V}{\partial (m^2)^2} |_{m=0, T=T_c} < 0 \). In the second case the phase transition takes place at \( T = T_c \) and is of the second order.

\(^2\)We disregard the case when it vanishes as the one which requires a fine tuning of parameters.
Indeed, for $T > T_c$ the potential has a minimum at $m = 0$ and slightly below $T_c$ one has $\frac{\partial V}{\partial (m^2)}|_{m=0} < 0$ which implies that a “valley” of an infinitesimal depth arises next to $m = 0$. There cannot be any other non trivial minimum for $m \neq 0$ as this would contradict the observation that the potential can have only one nontrivial minimum.

In conclusion, we can determine the order of the phase transition by studying the behaviour of the effective potential in the vicinity of the point $m = 0$.

In the next two sections we shall analyse phase transitions in NJL and SNJL models according to the prescription presented in this section.

3. Nonsupersymmetric NJL model at finite temperature. Let us warm-up with the simpler, nonsupersymmetric model. Since we are operating in the vicinity of $m = 0$ and the effective potential is an analytic function at that point we can Taylor expand it in $m^2$, as explained in section 2.

$$V_{NJL}(m, T) = V_{NJL}(m = 0, T) + m^2 \left( \frac{1}{12 \beta^2} + \frac{1}{g^2} - \frac{\Lambda^2}{8 \pi^2} \right) + m^4 \frac{1 + 2 c_f + 2 \log(\beta^2 \Lambda^2)}{32 \pi^2} + o(m^6 \beta^2) \tag{7}$$

where $\beta = 1/T$, $c_f = 2\gamma - 3/2 - 2 \log \pi \approx -2.64$.

According to the program outlined in section 2, we find the critical temperature $T_c$ demanding that the $m^2$ term in (7) vanishes at $T_c$. We get

$$T_c^2 = \frac{3 \Lambda^2}{2 \pi^2} (1 - c) \tag{8}$$

where $c = \frac{8 \pi^2}{g^2 \Lambda^2}$.

Next, the order of the phase transition is determined by the sign of the $m^4$ term at $T = T_c$. The analysis of the different physical situations is straightforward thus we simply summarize the results. For $c > 1$ - there is no phase transition as there is no (chiral) symmetry breaking at $T = 0$. For $1 - \frac{2 \pi^2}{3} e^{c_f + 0.5} < c < 1$ - the phase transition is of the second order and it take place at $T_c$. For $0 < c < 1 - \frac{2 \pi^2}{3} e^{c_f + 0.5} \approx 0.22$ - the phase transition is of the first order and it may take place at $T > T_c$. One should note
that the squared temperature of the phase transition is greater than \( \Lambda^2 e^{c_f+0.5} \approx \Lambda^2 / 10 \) in this case.

Looking at the \( T = 0 \) gap equation one can see that the last scenario corresponds to fermion masses comparable to the cut-off \( \Lambda \). This may be relevant for QCD, [6], but it is definitely not the range of fermion masses suitable for top-mode standard model of BHL [4], where to have the top mass of the order of \( 10^2 \) GeV one needs \( \frac{8\pi^2}{g^2m^2} \approx 1 - O(10^{-26}) \). The top-mode scenario falls then safely into the region of the second-order phase transition (to the leading order in large \( N \)), which confirms earlier results [14].

4. Susy-NJL model at finite temperature. The construction of the supersymmetric extension of the simplest version of the NJL model has been discussed carefully in [9]. It turns out that in exactly supersymmetric theory fermion condensation is impossible for any choice of parameters. However, if one introduces a soft susy-breaking scalar mass \( \Delta \), then the fermion condensate may appear and chiral symmetry breaking becomes possible. Following the same procedure as in the non-susy case we obtain to the leading order in large \( N \), Taylor expanding in \( m^2 \), the supersymmetric counterpart of (7)

\[
V_{SNJL}(m, T, \Delta) - V_{SNJL}(0, T, \Delta) =
\frac{1}{12\beta^2} + \frac{1}{g^2} - \frac{\Delta^2}{8\pi^2} \log(1 + \Lambda^2/\Delta^2) + \frac{1}{\beta^2\pi^2} I_2(\beta^2\Delta^2)
\]

\[
+ \frac{1}{32\pi^2} + \frac{\Lambda^2}{16\pi^2(\Lambda^2 + \Delta^2)}
\]

\[
+ \frac{1}{16\pi^2} \log\left( \frac{\beta^2 \Lambda^2 \Delta^2}{\Lambda^2 + \Delta^2} \right) - \frac{1}{4\pi^2} I_0(\beta^2\Delta^2) + o(m^6\beta^2)
\]

(9)

where \( I_n(x) = \int_0^\infty dy y^n(y^2 + x)^{-1/2} / (\exp(\sqrt{y^2 + x} - 1)) \).

Following the procedure applied already in section 3 we find the equation for the critical temperature demanding that the \( m^2 \) term in formula (8) vanishes:

\[
\frac{1}{12\beta^2} + \frac{1}{g^2} - \frac{\Delta^2}{8\pi^2} \log(1 + \Lambda^2/\Delta^2) + \frac{1}{\beta^2\pi^2} I_2(\beta^2\Delta^2) = 0
\]

(10)

The resulting equation cannot be solved algebraically due to the presence of the func-
tion $I_2$. In order to give a flavour of the dependence of $T_c$ on parameters of the theory we can find a rough approximate solution noting that values of the function $I_2$ range from $\pi^2/6$ to 0 at $\infty$. We arrive at the formula

$$\frac{T_c^2}{\Lambda^2} \approx \frac{3}{2\pi^2} \left( \frac{\Lambda^2}{\Delta^2} \log\left( \frac{\Lambda^2}{\Delta^2} + 1 \right) - c \right). \quad (11)$$

We also see that for $\Delta <\!\!<\!\!\Lambda$ the term proportional to $m^4$ in eq.(9) depends only on the product $\beta \Delta$. The condition for vanishing of this term can be easily solved. We get that if $T_c < \Delta/2.16$ the phase transition is of the second order. For $T_c > \Delta/2.16$ the phase transition is of the first order. The characteristic scale which determines the critical temperature is now the susy-breaking scale.

The analytic results one can get using the above approximations can be summarized as follows. If $c > 1$ - there cannot be any phase transition – the symmetry remains unbroken. If $0.07 < c < 1$ - for small susy-breaking scale, $0 < \Delta < \Delta_1$, again no phase transition is possible; for $\Delta > \Delta_1$ there is second order phase transition. Its temperature is given by eq.(11). The boundary value of $\Delta$ i.e. $\Delta_1$ is given by $\frac{\partial V_{SNJL}}{\partial m^2}(T = 0, \Delta_1) = 0$ (eq.(10)). If $0 < c < 0.07$ - for $0 < \Delta < \Delta_1$ no phase transition is possible ($\Delta_1$ is as above); for $\Delta_1 < \Delta < \Delta_2$ there is a second order phase transition; for $\Delta > \Delta_2$ there is a first order phase transition. The value of $\Delta_2$ is given by the condition that both terms on the r.h.s. of (10) vanish.

The exact results of the numerical study are presented in the figure 1. All values in this figure are scaled by the cut-off $\Lambda$. The curve “$d1$” represents the dependence of $d1 = \Delta_1/\Lambda$ on $c$ (“$d1$” is the solution to the eq.(11) at $T = 0$). The curve “$d2$” represents $d2 \equiv \Delta_2/\Lambda$. Thus, above the curve “$d1$” no phase transition is possible, between “$d1$” and “$d2$” only the second order phase transition occurs, below “$d2$” the first order phase transition occurs. The remaining curves are solutions to the equation (10) for $T/\Lambda = 0.1$, $0.2$, $0.25$ respectively. These are contours of the fixed critical temperature in the plane $(d,c)$.

The phase transition in the super-top-mode scenario, [5], described in the leading order of the large N expansion of the SNJL model, can be of the first order as long
as one expects that the critical temperature for the electroweak breaking is close to 200 GeV and the supersymmetry breaking scale \( \Delta \) lies in the TeV range. There \( T_c > \Delta/2.16 \) and in addition for fermion mass \( O(200) \) GeV we get \( \Delta_2 < \Delta \).

5. Summary. In summary, a simple method has been used to determine the character of the hot phase transition in 4d four-fermion NJL model, both in its non-supersymmetric and supersymmetric version. We conclude that in the non-susy case the transition may be of the first order, however in the region of parameters which correspond to fermion masses comparable to the cut-off. In the supersymmetric case also both kinds of phase transitions are possible. For sufficiently strong coupling and sufficiently large susy-breaking scale the transition is always of the first order. We applied these results to models under current investigation. We have found that the super-NJL scenario of the top quark condensation can exhibit the first order phase transition in the physically interesting range of parameters.

Authors thank J. Wosiek and K. Zalewski for useful discussion.
Z.L. was supported by A. von Humboldt Foundation, also partially supported by Deutsche Forschungsgemeinschaft and EC grant SC1-CT91-0729.
References

[1] V.Kuzmin,V.Rubakov,M.Shaposhnikov, *Phys. Lett.* **155B**, 36 (1985).

[2] A.Kosowsky,M.Turner, Fermilab-Pub-92-295-A, (1992).

[3] S.Dimopoulos,R.Esmailzadeh,L.Hall,N.Tetradis, *Phys. Lett.* **247B**, 601 (1990).

[4] W.Bardeen,C.Hill,M.Lindner, *Phys. Rev.* **D41**, 1647 (1990).

[5] T.Clark,S.Love,W.Bardeen, *Phys. Lett.* **B237**, 235 (1990), M.Carena, T.Clark, C.E.M.Wagner,W.Bardeen,K.Sasaki, *Nucl. Phys.* **B369**, 33 (1992).

[6] G.Ripka, ”Introduction to Nambu Jona-Lasinio Models Applied to Low Energy Hadronic Matter”, in *Hadrons and Hadronic Matter*, eds. D.Vautherin, F.Lenz and J.W.Negele, Plenum Press, New York, 1990; M.Bander,H.Rubinstein, *Phys. Lett.* **B289B**, 385 (1992).

[7] A.Polyakov, *Gauge Fields and Strings*, Harwood Academic Publishers, 1987.

[8] Z.Lalak, *Phys. Lett.* **B278B**, 284 (1992).

[9] W.Buchmuller,U.Ellwanger, *Nucl. Phys.* **B245**, 237 (1984).

[10] M.Sher, *Phys. Rep.* **179**, 273 (1989).

[11] R.Rivers, *Path Integral Methods in Quantum Field Theory*, Cambridge University Press, 1987.

[12] L.Dolan,R.Jackiw, *Phys. Rev.* **D9**, 3320 (1974).

[13] J.Wess and J.Bagger, *Supersymmetry and Supegravity*, Princeton University Press, Princeton, NJ, 1983.

[14] S.Kawati,H.Miyata, *Phys. Rev.* **D23**, 3010 (1981).

T.Hatsuda, T.Kunihiro, *Phys. Lett.* **B198B**, 126 (1987).

S.Klimt,M.Lutz,W.Weise, *Phys. Lett.* **B249B**, 386 (1990).
Figure caption

Figure 1. Exact results of the numerical study of the expression (9). Label d denotes the ratio $\Delta/\Lambda$, c represents $\frac{8\pi^2}{g^2\Lambda^2}$. The curve “$d1$” represents dependence of $d1 = \Delta_1/\Lambda$ on $c$ being the solution to the eq. (10) at $T = 0$. The curve “$d2$” represents $d2 \equiv \Delta_2/\Lambda$. The remaining curves are solutions of the equation (10) for $T/\Lambda = 0.1$, 0.2, 0.25 respectively. These are contours of the fixed critical temperature in the plane $(d,c)$. 