Extensions of classical hypergeometric identities of Bailey and Whipple

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Dedicated to the memory of Professor Plamen Djakov.

Abstract

We obtain extensions of classical hypergeometric identities of Bailey and Whipple that transform nearly-poised and very-well-poised series to Saalschützian series, Saalschützian series to Saalschützian series, and very-well-poised and nearly-poised series to very-well-poised series. We employ a method in which summations and transformations of lower-order series are used to obtain transformations of higher-order series. We also show how a number of other well-known results regarding hypergeometric series follow as special cases of our results.

1 Introduction

Identities among hypergeometric series, both terminating and nonterminating, have been the subject of extensive research. In the 1920s, Bailey and Whipple (see [1, 17, 18, 19, 20, 21]) found a number of identities relating various terminating hypergeometric series that are listed in Chapters 4 and 7 of Bailey’s tract [2]. In this paper, we generalize these terminating hypergeometric identities found by Bailey and Whipple and extend them to higher-order hypergeometric series.

We consider classical identities involving terminating very-well-poised, nearly-poised and Saalschützian series (see Section 2 for the relevant definitions). The highest order classical transformation formula between two terminating very-well-poised series was found by Bailey in [1] and involves two

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terminating very-well-poised 9F6(1) series (see [2, Eq. 4.3.7]). This formula was recently extended to the 11F10 level by Srivastava, Vyas and Fatawat [14, Theorem 3.4] by adding two new numerator and denominator pairs of parameters that differ by one each. In this paper, we further extend Srivastava, Vyas and Fatawat’s result to a relation involving two terminating very-well-poised 13F12(1) series (see Proposition 5.3 in Section 5) by adding two more pairs of numerator and denominator parameters with unit difference.

In Section 4.5 of Bailey’s tract [2], one can find a number of transformation formulas involving terminating nearly-poised series discovered by Whipple and Bailey. The identities [2, Eq. 4.5.1] (originally found by Whipple in [21]) and [2, Eq. 4.5.2] (originally found by Bailey in [1]) relate nearly-poised 4F3(1) and 5F4(1) series, respectively, to Saalschützian 5F4(1) series. In this paper (see Corollary 3.5 in Section 3), we extend these two identities to a single transformation formula between a 5F4(1) series, which is not nearly-poised but in which two pairs of numerator and denominator parameters deviate from “well-poisedness”, and a Saalschützian 6F5(1) series. In addition, this extension is further generalized by Proposition 3.3 in Section 3 to a relation between two terminating 7F6(1) series, one of which is Saalschützian. A special case of Proposition 3.3 given in Corollary 3.4 coincides with a recent special case of a result of Maier [8].

The classical Whipple transformation between a very-well-poised 7F6(1) and a Saalschützian 4F3(1) series (see [18, 20] and [2, Eq. 4.3.4]) was recently generalized by Srivastava, Vyas and Fatawat [14, Theorem 3.2]. In this paper, we provide a very general result in Proposition 3.6 that extends both Srivastava, Vyas and Fatawat’s result [14, Theorem 3.2] and the results described in the paragraph above.

In Section 4 of this paper, we study transformations between two terminating Saalschützian series. The classical result in this area is the Whipple transform (see [18, 19] and [2, Eq. 7.2.1]) involving two terminating Saalschützian 4F3(1) series. In Proposition 4.2 in Section 4, we extend the Whipple transform to a transformation that involves two terminating Saalschützian 6F5(1) series in each of which series two numerator parameters exceed two denominator parameters by one.

In Section 5, in addition to obtaining the above-mentioned relation involving two terminating very-well-poised 13F12(1) series, we also obtain extensions of classical results found by Bailey in [11, Eqs. 8.1, 8.2 and 8.3] and reproduced in [2, Eqs. 4.5.3, 4.5.4 and 4.5.5] that transform terminating nearly-poised 5F4(1) series with parametric excesses ω = 1 and ω = 2 and
a terminating nearly-poised \( _6F_5(1) \) series with parametric excess \( \omega = 1 \) to terminating very-well-poised \( _9F_8(1) \) series. The immediate extension of [2, Eqs. 4.5.3, 4.5.4 and 4.5.5] is given in Corollary 5.5 and a further extension is provided in Proposition 5.4.

To obtain our results, we use a method employed by Bailey in [1] and [2, Chapter 4] that utilizes sums of series of lower order to obtain transformations of series of higher order. The extensions of Bailey’s general formulas [2, Eqs. 4.3.1 and 4.3.6] are provided in Propositions 3.1 and 5.1 respectively. We should point out that Bailey uses the Pfaff–Saalschütz formula (see [2, Eq. 2.2.1]) to obtain [2, Eq. 4.3.1] while we use the extension of the Pfaff–Saalschütz formula to a \( _4F_3(1) \) series given by Rakha and Rathie [11] to obtain the more general Proposition 3.1. Moreover, Bailey uses Dougall’s theorem (see [5, Eq. 6], [6] and [2, Eq. 4.3.5]) to obtain [2, Eq. 4.3.6] and we use the extension of Dougall’s theorem to a \( _9F_8(1) \) series summation provided by Srivastava, Vyas and Fatawat [14, Theorem 3.3] to prove the more general Proposition 5.1. Finally, in this paper, not only do we use the method just described with known summations of series, but we also use it with a known transformation of series and then reverse the order of summation to obtain a new transformation (see Proposition 3.6).

The method described above that we use in this paper is parallel to the Bailey’s transform (see [3], [4] and [13, pp. 58–74]), which is employed by Srivastava, Vyas and Fatawat in [14]. Both methods can be utilized to obtain many of the known transformations of hypergeometric series. We should also note that, according to the Karlsson–Minton summation formula (see [7]), any hypergeometric series in which numerator and denominator parameters differ by positive integers can be written as a finite sum of hypergeometric series of lower order, but we have not used this approach or formula in our paper.

2 Preliminaries

The hypergeometric series of type \( _rF_s \) is defined by

\[
_\alpha F_\beta \left[ \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{n!(b_1)_n(b_2)_n\cdots(b_s)_n} z^n, \quad (2.1)
\]
where $r$ and $s$ are nonnegative integers, $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, z \in \mathbb{C}$, and the rising factorial $(a)_n$ is given by

$$(a)_n = \begin{cases} a(a + 1) \cdots (a + n - 1), & n > 0, \\ 1, & n = 0. \end{cases}$$

In this paper we will be mostly interested in the case where $r = s + 1$. The series of type $s + 1 F_s$ converges absolutely if $|z| < 1$ or if $|z| = 1$ and $\text{Re}(\sum_{i=1}^{s} b_i - \sum_{i=1}^{s+1} a_i) > 0$ (see [2, p. 8]). We assume that no denominator parameter $b_1, b_2, \ldots, b_s$ is a negative integer or zero. If a numerator parameter $a_1, a_2, \ldots, a_{s+1}$ is a negative integer or zero, the series has only finitely many nonzero terms and is said to terminate.

When $z = 1$, we say that the series is of unit argument and of type $s + 1 F_s(1)$. If $\sum_{i=1}^{s} b_i - \sum_{i=1}^{s+1} a_i = 1$, the series is called Saalschützian. If $1 + a_1 = b_1 + a_2 = \cdots = b_s + a_{s+1}$, the series is called well-poised. A well-poised series that satisfies $a_2 = 1 + \frac{1}{2} a_1$ is called very-well-poised. The parametric excess $\omega$ is given by $\omega = \sum_{i=1}^{s} b_i - \sum_{i=1}^{s+1} a_i$. Note that $\omega = 1$ for a Saalschüitzian series.

We will use the following extension of the classical Chu–Vandermonde formula (see [2, Section 1.3]), which extension sums a special terminating $3 F_2(1)$ series where a numerator parameter exceeds a denominator parameter by one:

$$3 F_2 \left( \begin{array}{c} a, p+1, -n \\ b, p \end{array} \right) = \frac{(b-a-1)_n(q+1)_n}{(b)_n(q)_n},$$

where

$$q = \frac{p(b-a-1)}{p-a}. \quad (2.3)$$

The above formula (2.2) appears in [9]. A nonterminating version of the same formula can be found in [10, p. 534, Eq. (10)]. Letting $p \to \infty$ in (2.2) yields the Chu–Vandermonde formula.

We will also use the following extension of the Pfaff–Saalschütz formula (see [2, Eq. 2.2.1]) given by Rakha and Rathie [11] which finds the sum of a special terminating Saalschützian $4 F_3(1)$ series where a numerator parameter exceeds a denominator parameter by one:

$$4 F_3 \left( \begin{array}{c} a, b, p+1, -n \\ c, p, 2 + a + b - c - n \end{array} \right) = \frac{(c-a-1)_n(c-b-1)_n(q+1)_n}{(c)_n(c-a-b-1)_n(q)_n},$$

where

$$q = \frac{p(c-a-1)}{p-c}.$$
where
\[ q = \frac{p(c - a - 1)(c - b - 1)}{a b + p(c - a - b - 1)}. \] (2.5)

Letting \( p = b \) in (2.4) yields the Pfaff–Saalschütz formula, while letting \( b \to \infty \) in (2.4) gives (2.2).

We note that (2.4) can also be written as
\[
\begin{aligned}
\binom{4}{3} F_3 \left( \begin{array}{c}
\begin{array}{c}
(a-b-c, \gamma_1 + 1, a+n, -n
\end{array}
\end{array} \middle| 1 \right)
&= \frac{(b)_n(c)_n(a-p+1)_n(p+1)_n}{(1+a-b)_n(1+a-c)_n(p)(a-p)_n},
\end{aligned}
\] (2.6)

where
\[ \gamma_1 = \frac{p(a-p)(b+c-a)}{bc - p(a-p)}, \] (2.7)

and as
\[
\begin{aligned}
\binom{4}{3} F_3 \left( \begin{array}{c}
\begin{array}{c}
c - a - 1, c - b - 1, \gamma_2 + 1, -n
\end{array}
\end{array} \middle| 1 \right)
&= \frac{(a)_n(b)_n(c)_n(p+1)_n}{(c)_n(1+a+b-c)(p)_n},
\end{aligned}
\] (2.8)

where
\[ \gamma_2 = \frac{p(c-a-1)(c-b-1)}{a b + p(c - a - b - 1)}. \] (2.9)

We will use (2.6) and (2.8) in Sections 3 and 4, respectively, where we study extensions of transformations of nearly-poised and very-well-poised series to Saalschützian series and extensions of transformations of Saalschützian series to Saalschützian series.

In [14, Theorem 3.3], Srivastava, Vyas and Fatawat find a generalization of the classical Dougall’s theorem for the sum of a terminating very-well-poised \( \binom{7}{6} F_6(1) \) series with parametric excess \( \omega = 2 \) (see [5, Eq. 6], [6] and [2, Eq. 4.3.5]). The generalization found by Srivastava, Vyas and Fatawat can be written as
\[
\begin{aligned}
\binom{9}{8} F_8 \left( \begin{array}{c}
\begin{array}{c}
a, 1 + \frac{a}{2}, b, c, d,
\end{array}
\end{array} \middle| 1 \right)
&= \frac{a, 1 + a - b, 1 + a - c, 1 + a - d, 2a - b - c - d + n, a - p + 1, p + 1, -n; 1 + b + c + d - a - n, p, a - p, 1 + a + n; 1}{(1+a-b)_n(a-b-c)_n(a-b-d)_n(a-c-d)_n(\alpha+1)_n}
\end{aligned}
\] (2.10)
where
\[ \alpha = \frac{p(a-p)(a-b-c)(a-b-d)(a-c-d)}{(2a-b-c-d+n)(bcd+p(a-p)(a-b-c-d))}. \] (2.11)

Dougall’s theorem follows from (2.10) by letting \( p = b \). We note that (2.10) can also be written as
\[ 9F_8 \left( \frac{\lambda}{2}, 1 + \frac{\lambda}{2}, \lambda + b - a, \lambda + c - a, \lambda + d - a, \right. \]
\[ \left. a + n, \frac{\lambda}{2} - \gamma + 1, \frac{\lambda}{2} + \gamma + 1, -n; 1 + \lambda - a - n, \frac{\lambda}{2} + \gamma, \frac{\lambda}{2} - \gamma, 1 + \lambda + n; \left| \frac{1}{1} \right. \right) \]
\[ = \frac{(1 + \lambda)_n(b)_n(c)_n(d)_n(a-p+1)_n(p+1)_n}{(a-\lambda)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(p)_n(a-p)_n}, \]
where
\[ \lambda = 2a - b - c - d \] (2.13)
and
\[ \gamma^2 = \frac{\lambda^2}{4} - \frac{p(a-p)(a-b-c)(a-b-d)(a-c-d)}{bcd + p(a-p)(a-b-c-d)}. \] (2.14)

We will use (2.12) in Section 5 where we study extensions of transformations of very-well-poised and nearly-poised series to very-well-poised series.

### 3 Extensions of hypergeometric transformations of nearly-poised and very-well-poised series to Saalschützian series

In this section, we study extensions of the classical identities given in [2] Eqs. 4.5.1, 4.5.2 and 4.3.4. We begin with a general formula that extends [2] Eq. 4.3.1:

**Proposition 3.1.** Let
\[ \gamma = \frac{p(a-p)(b+c-a)}{bc - p(a-p)}. \] (3.1)
Then

\[
\begin{align*}
&\binom{r+6}{s+4} \binom{a, b, c, a - p + 1, p + 1, a_1, \ldots, a_r, -n}{1 + a - b, 1 + a - c, p, a - p, b_1, \ldots, b_s} x \\
&= \sum_{m=0}^{n} \left( \frac{\binom{a}{2} (a+1)_m (a-b-c)_m (\gamma+1)_m (a_1)_m \cdots (a_r)_m (-n)_m (-4x)_m}{m!(1+a-b)_m (1+a-c)_m (\gamma)_m (b_1)_m \cdots (b_s)_m} \right) \\
&\times \binom{r+2}{s} \binom{a + 2m, a_1 + m, \ldots, a_r + m, -n + m}{b_1 + m, \ldots, b_s + m} x. \\
\end{align*}
\]

(3.2)

Proof. Using (2.6), we have

\[
\begin{align*}
&\binom{r+6}{s+4} \binom{a, b, c, a - p + 1, p + 1, a_1, \ldots, a_r, -n}{1 + a - b, 1 + a - c, p, a - p, b_1, \ldots, b_s} x \\
&= \sum_{k=0}^{n} \frac{(a)_k (b)_k (c)_k (a - p + 1)_k (p + 1)_k (a_1)_k \cdots (a_r)_k (-n)_k x^k}{k!(1+a-b)_k (1+a-c)_k (p)_k (a-p)_k (b_1)_k \cdots (b_s)_k} \\
&= \sum_{k=0}^{n} \frac{(a)_k (a_1)_k \cdots (a_r)_k (-n)_k x^k}{k!(b_1)_k \cdots (b_s)_k} \\
&\times \binom{4}{3} \binom{a - b - c, \gamma + 1, a + k, -k}{1 + a - b, 1 + a - c, \gamma} x, \\
\end{align*}
\]

where

\[
\gamma = \frac{p(a - p)(b + c - a)}{bc - p(a - p)}. 
\]

We write the \(4_F3\) series on the right-hand side above as a summation, switch the order of summation in the resulting expression, and then simplify to obtain (3.2).

Remark 3.2. Formula (3.2) is an extension of [2, Eq. 4.3.1]. In fact, [2, Eq. 4.3.1] follows from (3.2) by letting \(x = 1\) and \(p \to \infty\).

We next use Proposition 3.1 to obtain a generalization of [2, Eqs. 4.5.1 and 4.5.2]:

\[
4_F3 \left( \begin{array}{c} a - b - c, \gamma + 1, a + k, -k \\ 1 + a - b, 1 + a - c, \gamma \end{array} \right). 
\]
Proposition 3.3. We have

\[ \begin{align*}
7F_6 & \left( \begin{array}{c}
  a, b, c, a - p + 1, p + 1, q + 1, -n \\
  1 + a - b, 1 + a - c, p, a - p, q, w
\end{array} \right) | 1 \\
= & \frac{(w - a - 1)_n (\alpha + 1)_n}{(w)_n (\alpha)_n} \\
& \times 7F_6 \left( \begin{array}{c}
  1 + a - w, \frac{a + 1}{2}, a - b - c, \beta + 1, \gamma + 1, -n \\
  1 + a - b, 1 + a - c, \frac{2 + a - w - n}{2}, \frac{3 + a - w - n}{2}, \beta, \gamma
\end{array} \right) | 1,
\end{align*} \]

where

\[ \alpha = \frac{q(1 + a - w)}{a - q}, \]

\[ \beta = \frac{q(1 + a - w) + n(a - q)}{1 + 2q - w + n} \]

and

\[ \gamma = \frac{p(a - p)(b + c - a)}{bc - p(a - p)}. \]

Proof. Use \( q + 1, q, w, 1 \) for \( a_1, b_1, b_2, x \), respectively, in (3.2) to obtain

\[ \begin{align*}
7F_6 & \left( \begin{array}{c}
  a, b, c, a - p + 1, p + 1, q + 1, -n \\
  1 + a - b, 1 + a - c, p, a - p, q, w
\end{array} \right) | 1 \\
= & \sum_{m=0}^{n} \frac{(a)_m (a + 1)_m (a - b - c)_m (\gamma + 1)_m (q + 1)_m (q)_m (-n)_m (-4)^m}{m!(1 + a - b)_m (1 + a - c)_m (\gamma)_m (q)_m (w)_m} \\
& \times 3F_2 \left( \begin{array}{c}
  a + 2m, q + 1 + m, -n + m \\
  q + m, w + m
\end{array} \right) | 1,
\end{align*} \]

where

\[ \gamma = \frac{p(a - p)(b + c - a)}{bc - p(a - p)}. \]

Sum the \( 3F_2 \) series on the right-hand side above according to (2.2) and simplify to obtain the result.

We note that the \( 7F_6 \) series on the left-hand side of (3.3) deviates from a well-poised series in two pairs of numerator and denominator parameters while the \( 7F_6 \) series on the right-hand side of (3.3) is Saalschützian.

Letting \( q \to \infty \) in (3.4), we obtain the following special case:
Corollary 3.4. We have

\[
\begin{align*}
_6F_5 & \left( \begin{array}{c}
a, b, c, a - p + 1, p + 1, -n \\
1 + a - b, 1 + a - c, p, a - p, w
\end{array} \right| 1 \right) \\
& = \frac{(w - a)_n}{(w)_n}
\end{align*}
\]

\[
\times _6F_5 \left( \begin{array}{c}
1 + a - w, \frac{a}{2}, \frac{a + 1}{2}, a - b - c, \gamma + 1, -n \\
1 + a - b, 1 + a - c, \frac{1 + a - w - n}{2}, \frac{2 + a - w - n}{2}, \gamma
\end{array} \right| 1 \right),
\]

where

\[
\gamma = \frac{p(a - p)(b + c - a)}{bc - p(a - p)}.
\]

We remark that (3.7) is the special case \(k = 1\) of [8, Theorem 7.1(ii)] as well as [16, Cor. 4].

On the other hand, letting \(p \to \infty\) in (3.3), we obtain the following result:

Corollary 3.5. We have

\[
_5F_4 \left( \begin{array}{c}
a, b, c, q + 1, -n \\
1 + a - b, 1 + a - c, q, w
\end{array} \right| 1 \right) \\
= \frac{(w - a - 1)_n(\alpha + 1)_n}{(w)_n(\alpha)_n}
\]

\[
\times _6F_5 \left( \begin{array}{c}
1 + a - w, \frac{a}{2}, \frac{a + 1}{2}, 1 + a - b - c, \beta + 1, -n \\
1 + a - b, 1 + a - c, \frac{2 + a - w - n}{2}, \frac{3 + a - w - n}{2}, \beta
\end{array} \right| 1 \right),
\]

where

\[
\alpha = \frac{q(1 + a - w)}{a - q}
\]

and

\[
\beta = \frac{q(1 + a - w) + n(a - q)}{1 + 2q - w + n}.
\]

We note that Corollary 3.5 generalizes two well-known results of Whipple and Bailey given in [2, Eqs. 4.5.1 and 4.5.2]. Indeed, we have the following:

(a) Letting \(q \to \infty\) in (3.9) gives [2, Eq. 4.5.1] (originally found by Whipple in [21]).
(b) Letting \( q = a/2 \) in \((3.9)\) gives \[2, Eq. 4.5.2\] (originally found by Bailey in \[1\]).

We next show how Proposition 3.1 along with the result in Corollary 3.5 lead to a formula that generalizes transformations of both nearly-poised and very-well-poised series to Saalschützian series.

**Proposition 3.6.** We have

\[
\pFq{9}{8}{a, b, c, d, e, a - p + 1, p + 1, q + 1, -n}{1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, p, a - p, q, w}{w - a - 1)\(\alpha + 1\)}
\]

\[
= \left( w - a - 1 \right) n(\alpha + 1) \frac{(w - a - 1) n(\alpha + 1) \times \sum_{k=0}^{n} \left( (-n)_k \left( \frac{a}{2} \right)_k \left( \frac{a + 1}{2} \right)_k \frac{(1 + a - w)_k(1 + a - d - e)_k(\beta + 1)_k}{k!(1 + a - d)_k(1 + a - e)_k \left( \frac{2 + a - w - a}{2} \right)_k \left( \frac{3 + a - w - n}{2} \right)_k (\beta)_k \times \pFq{5}{4}{-k, a - b - c, d, e, \gamma + 1}{1 + a - b, 1 + a - c, d + e - a - k, \gamma}{1} \right)}{1 + \frac{q(1 + a - w)}{a - q}} \cdot \frac{q(1 + a - w) + n(a - q)}{1 + 2q - w + n} \cdot \frac{p(a - p)(b + c - a)}{bc - p(a - p)} \right),
\]

\[(3.12)\]

where

\[
\alpha = \frac{q(1 + a - w)}{a - q},
\]

\[(3.13)\]

\[
\beta = \frac{q(1 + a - w) + n(a - q)}{1 + 2q - w + n},
\]

\[(3.14)\]

and

\[
\gamma = \frac{p(a - p)(b + c - a)}{bc - p(a - p)}.
\]

\[(3.15)\]

**Proof.** Use \( d, e, q + 1 \) for \( a_1, a_2, a_3 \), respectively, \( 1 + a - d, 1 + a - e, q, w \) for \( b_1, b_2, b_3, b_4 \), respectively, and \( x = 1 \) in \((3.2)\), and then apply \((3.9)\) to write the \( \pFq{5}{4} \) series on the right-hand side as a Saalschützian \( \pFq{6}{5} \) series. After that reverse the order of summation and simplify. \( \square \)

The formula in Proposition 3.6 is a very general one. It extends both very-well-poised identities as well as nearly-poised identities. In fact, letting \( q \rightarrow a/2 \) first in \( (3.12) \) and then letting \( w \rightarrow 1 + a + n \) in the resulting formula.
yields
\[
\begin{aligned}
\genfrac{}{}{0pt}{}{9F_8}{1}
&= \frac{(1+a)_n(1+a-d-e)_n}{(1+a-d)_n(1+a-e)_n} \\
&\times \genfrac{}{}{0pt}{}{5F_4}{1}
&= \frac{p(a-p)(b+c-a)}{bc-p(a-p)},
\end{aligned}
\]
which is the very-well-poised \(9F_8(1)\) to Saalschützian \(5F_4(1)\) transformation found by Srivastava, Vyas and Fatawat in [14 Theorem 3.2] that generalizes the classical Whipple’s transformation of a very-well-poised \(7F_6(1)\) series to a Saalschützian \(4F_3(1)\) series (see [18], [20] and [2, Eq. 4.3.4]). On the other hand, letting \(b \to \infty\) in (3.12) and then letting \(c \to \infty\) in the resulting formula leads to (3.3), which greatly generalizes the classical nearly-poised to Saalschützian transformations found by Whipple and Bailey (see [2 Eqs. 4.5.1 and 4.5.2]).

4 Extensions of hypergeometric transformations of Saalschützian to Saalschützian series

In this section, we extend the well-known Whipple transform (see [18], [19] and [2, Eq. 7.2.1]) which involves two terminating Saalschützian \(4F_3(1)\) series. We begin with the following general result:

**Proposition 4.1.** Let
\[
\gamma = \frac{p(c-a-1)(c-b-1)}{ab+p(c-a-b-1)},
\]

\[11\]
Then

\[ r+4 F_{s+2} \left( \begin{array}{c} a, b, p + 1, a_1, \ldots, a_r, -n \\ c, p, b_1, \ldots, b_s \end{array} \Big| x \right) \]

\[ = \sum_{m=0}^{n} \frac{(c-a-1)_m(c-b-1)_m(\gamma+1)_m(a_1)_m \cdots (a_r)_m(-n)_m x^m}{m!(c)_m(\gamma)_m(b_1)_m \cdots (b_s)_m} \]

\[ \times r+2 F_s \left( \begin{array}{c} 1 + a + b - c, a_1 + m, \ldots, a_r + m, -n + m \\ b_1 + m, \ldots, b_s + m \end{array} \Big| x \right). \] (4.2)

**Proof.** Using (2.8), we have

\[ r+4 F_{s+2} \left( \begin{array}{c} a, b, p + 1, a_1, \ldots, a_r, -n \\ c, p, b_1, \ldots, b_s \end{array} \Big| x \right) \]

\[ = \sum_{k=0}^{n} \frac{(a)_k(b)_k(p+1)_k(a_1)_k \cdots (a_r)_k(-n)_k x^k}{k!(c)_k(p)_k(b_1)_k \cdots (b_s)_k} \]

\[ = \sum_{k=0}^{n} \frac{(1 + a + b - c)_k(a_1)_k \cdots (a_r)_k(-n)_k x^k}{k!(b_1)_k \cdots (b_s)_k} \]

\[ \times 4 F_3 \left( \begin{array}{c} c - a - 1, c - b - 1, \gamma + 1, -k \\ c, \gamma, c - a - b - k \end{array} \Big| 1 \right), \]

where

\[ \gamma = \frac{p(c-a-1)(c-b-1)}{ab + p(c-a-b-1)}. \]

We write the \( 4 F_3 \) series on the right-hand side above as a summation, switch the order of summation in the resulting expression, and then simplify to obtain (4.2). \( \square \)

The extension of the Whipple transform is given next:

**Proposition 4.2.** We have

\[ _6 F_5 \left( \begin{array}{c} a, b, c, p + 1, q + 1, -n \\ d, e, f, p, q \end{array} \Big| 1 \right) \]

\[ = \frac{(e-c-1)_n(f-c-1)_n(\alpha+1)_n}{(e)_n(f)_n(\alpha)_n} \]

\[ \times _6 F_5 \left( \begin{array}{c} d - a - 1, d - b - 1, c, \gamma + 1, \delta + 1, -n \\ d, 2 + c - e - n, 2 + c - f - n, \gamma, \delta \end{array} \Big| 1 \right). \] (4.3)

12
where
\[ d + e + f - a - b - c + n = 3, \quad (4.4) \]
\[ \alpha = \frac{q(e - c - 1)(f - c - 1)}{(c - q)(d - a - b - 1)}, \quad (4.5) \]
\[ \gamma = \frac{p(d - a - 1)(d - b - 1)}{ab + p(d - a - b - 1)}, \quad (4.6) \]
and
\[ \delta = \frac{q(e - c - 1)(f - c - 1) + n(c - q)(d - a - b - 1)}{(e - c - 1)(f - c - 1) - (c - q)(d - a - b - 1)}. \quad (4.7) \]

**Proof.** Use \( d, c, q+1, e, f = 3+a+b+c-d-e-n, q, 1 \) for \( c, a_1, a_2, b_1, b_2, b_3, x \), respectively, in (4.2), and then sum the Saalschützian \( _4F_3(1) \) series on the right-hand side according to (2.4).

The relation in (4.3) involves two Saalschützian \( _6F_5(1) \) series in each of which series two numerator parameters exceed two denominator parameters by one. This relation is a generalization of the classical Whipple transform (see [18], [19] and [2, Eq. 7.2.1]) involving two terminating Saalschützian \( _4F_3(1) \) series as we show after Corollary 4.4 below.

**Remark 4.3.** Let
\[ \tilde{F}_n(a, b, c; d, e, f; p, q) \]
\[ = (d)_n(e)_n(f)_n(\alpha)_{n6F_5} \left( \begin{array}{c} a, b, c, p + 1, q + 1, -n \\ d, e, f, p, q \end{array} \right| 1 \), \quad (4.8) \]
where
\[ d + e + f - a - b - c + n = 3 \]
and \( \alpha \) is as given in (4.5). Then equation (4.3) implies that
\[ \tilde{F}_n(a, b, c; d, e, f; p, q) \]
\[ = (-1)^n \tilde{F}_n(d - a - 1, d - b - 1, c; d, 2 + c - e - n, 2 + c - f - n; \gamma, \delta), \quad (4.9) \]
where \( \gamma \) and \( \delta \) are as given in (4.6) and (4.7), respectively.
Corollary 4.4. We have
\[
\begin{align*}
\binom{5}{a, b, c, p + 1, -n} & = (e - c)(f - c) \\
\binom{d, e, f, p}{(e)(f)} & = d - a - 1, d - b - 1, c, \gamma + 1, -n \\
\binom{d, 1 + c - e - n, 1 + c - f - n, \gamma}{1}.
\end{align*}
\] (4.10)

where
\[
d + e + f - a - b - c + n = 2,
\] (4.11)

and
\[
\gamma = \frac{p(d - a - 1)(d - b - 1)}{ab + p(d - a - b - 1)}.
\] (4.12)

Proof. Let \(q \rightarrow c\) in (4.3) and then replace \(c + 1\) with \(c\).

The two \(\binom{5}{a, b, c, p + 1, -n}\) series in (4.10) are both Saalschützian and in each one of them a numerator parameter exceeds a denominator parameter by one. Letting \(p = b\) in (4.10) gives the Whipple transform involving two Saalschützian \(\binom{4}{a, b, c, p + 1, -n}\) series.

Corollary 4.5. We have
\[
\begin{align*}
\binom{4}{a, c, p + 1, -n} & = (e - c)(f - c) \\
\binom{d, e, f, p}{(e)(f)} & = d - a - 1, d - b - 1, c, \gamma + 1, -n \\
\binom{d, 1 + c - e - n, 1 + c - f - n, \gamma}{1}.
\end{align*}
\] (4.13)

where
\[
\gamma = \frac{p(d - a - 1)}{p - a}.
\] (4.14)

Proof. In (4.10), fix \(a, c, d, e, p\) and \(n\), and let
\[
f = 2 + a + b + c - d - e - n
\]
derpend on \(b\). Let \(b \rightarrow \infty\) to obtain the result.
We note that (4.13) generalizes the classical relation involving two terminating $3F_2(1)$ series (see Sheppard [12] and Whipple [17] which follow Thomae [15]). Indeed, letting $p \to \infty$ in (4.13) gives
\[
3F_2\left(\begin{array}{c} a, c, -n \\ d, e \end{array} \bigg| 1 \right) = \frac{(e - c)n}{(e)_n} 3F_2\left(\begin{array}{c} d - a, c, -n \\ d, 1 + c - e - n \end{array} \bigg| 1 \right).
\]

Equation (4.13) also extends the Saalschützian $4F_3(1)$ summation (2.4). In fact, (2.4) follows from (4.13) by setting $e = 2 + a + c - d - n$ and then summing the resulting $3F_2(1)$ series on the right-hand side according to (2.2).

5 Extensions of hypergeometric transformations of very-well-poised and nearly-poised series to very-well-poised series

In this section, we extend the relation between two terminating very-well-poised $11F_{10}(1)$ series given by Srivastava, Vyas and Fatawat in [14, Theorem 3.4] (which generalizes Bailey’s $9F_8$ transformation in [2, Eq. 4.3.7]) to a relation between two terminating very-well-poised $13F_{12}(1)$ series. We also extend the formulas found in [2, Eqs. 4.5.3, 4.5.4 and 4.5.5]. We begin with a general formula that extends [2, Eq. 4.3.6]:

**Proposition 5.1.** If
\[
\lambda = 2a - b - c - d,
\]
then
\[
\sum_{m=0}^{n} \frac{\lambda_m (\lambda + b - a)_m (\lambda + c - a)_m (\lambda + d - a)_m (\frac{a}{2})_m (\frac{a + 1}{2})_m}{m! (\frac{1}{2})_m (\frac{1}{2} + \gamma)_m (\frac{1}{2} + \gamma + 1)_m (\frac{1}{2} + \gamma + 2)_m \cdots (\frac{1}{2} + \gamma + m)_m (b_1)_m \cdots (b_s)_m \frac{x^m}{(\frac{1}{2} + \gamma)_m (\frac{1}{2} + \gamma + 1)_m (\frac{1}{2} + \gamma + 2)_m \cdots (\frac{1}{2} + \gamma + m)_m}}
\]
\[
\times \frac{\lambda}{\lambda + 2m} a, c, -n, a_p + 1, p + 1, a_1, \ldots, a_r, -n, 1 + a - b, 1 + a - c, 1 + a - d, p, a - p, b_1, \ldots, b_s \bigg| x \bigg). \]
where
\[ \gamma^2 = \frac{\lambda^2}{4} - \frac{p(a - p)(a - b - c)(a - b - d)(a - c - d)}{bcd + p(a - p)(a - b - c - d)}. \] (5.3)

**Proof.** Using (2.12), we have
\[
\begin{align*}
&\quad \frac{r+7}{9}F_{s+5} \left( \frac{a, b, c, d, a-p+1, p+1}{1+a-b, 1+a-c, 1+a-d, p, a-p, b_1, \ldots, b_s} \bigg| x \right) \\
&= \sum_{k=0}^{n} \frac{(a)_k(b)_k(c)_k(d)_k(a-p+1)_k(p+1)_k}{k!(1+a-b)_k(1+a-c)_k(1+a-d)_k(p)_k(a-p)_k} \\
&\quad \times \frac{(a_1)_k \cdots (a_r)_k(-n)_k x^k}{(b_1)_k \cdots (b_s)_k} \\
&= \sum_{k=0}^{n} \left( \frac{(a)_k(a-\lambda)_k(a_1)_k \cdots (a_r)_k(-n)_k x^k}{k!(1+\lambda)_k(b_1)_k \cdots (b_s)_k} \right) \\
&\quad \times \frac{\lambda, 1+\frac{\lambda}{2}, \lambda+b-a, \lambda+c-a, \lambda+d-a,}{} \\
&\quad \times \frac{a+k, \frac{\lambda}{2}-\gamma+1, \frac{\lambda}{2}+\gamma+1, -k;}{1+\lambda-a-k, \frac{\lambda}{2}+\gamma, \frac{\lambda}{2}-\gamma, 1+\lambda+k;}
\end{align*}
\]

where
\[ \lambda = 2a - b - c - d \]
and
\[ \gamma^2 = \frac{\lambda^2}{4} - \frac{p(a - p)(a - b - c)(a - b - d)(a - c - d)}{bcd + p(a - p)(a - b - c - d)}. \]

We write the \(9F_8\) series on the right-hand side above as a summation, switch the order of summation in the resulting expression, and then simplify to obtain (5.2). \(\square\)

**Remark 5.2.** Formula (5.2) is an extension of [2, Eq. 4.3.6]. In fact, [2, Eq. 4.3.6] follows from (5.2) by letting \(x = 1\) and \(p = d\).

We now obtain the generalization of Srivastava, Vyas and Fatawat’s \(11F_{10}\) transformation given in [14, Theorem 3.4]:

**Proposition 5.3.** Suppose
\[ 3a = b + c + d + e + f + g - n. \] (5.4)
Then

\[
13F_{12} \left( \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f, \right.
\]
\[
g, a - p + 1, p + 1, a - q + 1, q + 1, -n \bigg| 1 \bigg) (5.5)
\]
\[
= \frac{(1 + a)_n(1 + \lambda - e)_n(1 + \lambda - f)_n(1 + \lambda - g)_n}{(1 + \lambda)_n(1 + a - e)_n(1 + a - f)_n(1 + a - g)_n}
\times \frac{(\frac{\mu}{2} - \delta + 1)_n(\frac{\mu}{2} + \delta + 1)_n}{(\frac{\mu}{2} + \delta)_n(\frac{\mu}{2} - \delta)_n}
\times 13F_{12} \left( \frac{\lambda}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + \lambda - e, 1 + \lambda - f, \right.
\]
\[
g, \frac{\lambda}{2} - \gamma + 1, \frac{\lambda}{2} + \gamma + 1, \frac{\lambda}{2} - \epsilon + 1, \frac{\lambda}{2} + \epsilon + 1, -n \bigg| 1 \bigg).
\]

where

\[
\lambda = 2a - b - c - d,
\]
\[
\mu = 2a - e - f - g,
\]
\[
\gamma^2 = \frac{\lambda^2}{4} - \frac{p(a - p)(a - b - c)(a - b - d)(a - c - d)}{bcd + p(a - p)(a - b - c - d)}.
\]
\[
\delta^2 = \frac{\mu^2}{4} - \frac{q(a - q)(a - e - f)(a - e - g)(a - f - g)}{efg + q(a - q)(a - e - f - g)}.
\]

and

\[
\epsilon^2 = \frac{\lambda^2}{4}
\]
\[
- \frac{[q(a - q)(a - e - f)(a - e - g)(a - f - g) + n(\mu + n)(efg + q(a - q)(a - e - f - g))]}{(a - e - f)(a - e - g)(a - f - g) - (\mu + n)(ef + eg + fg + a(a - e - f - g) - q(a - q))}
\]

17
Proof. Use $1 + \frac{a}{2}, e, f, g, a - q + 1, q + 1$ for $a_1, a_2, a_3, a_4, a_5, a_6$, respectively, $\frac{a}{2}, 1 + a - e, 1 + a - f, 1 + a - g, a - q, 1 + a + n$ for $b_1, b_2, b_3, b_4, b_5, b_6, b_7$, respectively, and $x = 1$ in (5.2), and then apply (2.10) to sum the $9F_8(1)$ series on the right-hand side. The result follows after some simplification. \[ \Box \]

Equation (5.5) above involves two terminating very-well-poised $13F_{12}(1)$ series. It generalizes the result of Srivastava, Vyas and Fatawat [14, Theorem 3.4] between two terminating very-well-poised $11F_{10}(1)$ series, which in turn is a generalization of Bailey’s $9F_8$ transformation (see [21 Eq. 4.3.7]). Indeed, [14 Theorem 3.4] follows from our result (5.5) upon setting $q = e$.

We next extend the formulas found in [21 Eqs. 4.5.3, 4.5.4 and 4.5.5]. First, we obtain an even more general result:

**Proposition 5.4.** We have

\[ 8F_7 \left( \begin{array}{c}
  a, b, c, d, a-p+1, p+1, q+1, -n \\
  1 + a - b, 1 + a - c, 1 + a - d, p, a-p, q, w
\end{array} \right) \mid 1 \right) = \frac{(2\lambda - a)_n(\lambda - a)_n(\alpha + 1)_n}{(1 + \lambda)_n(2\lambda - 2a)_n(\alpha)_n} \times
\]

\[ 13F_{12} \left( \begin{array}{c}
  \lambda, 1 + \frac{\lambda}{2}, \frac{a+1}{2}, \lambda + b - a, \lambda + c - a, \lambda + d - a, \\
  1 + a - w, \frac{\lambda}{2} + \gamma + 1, \frac{\lambda}{2} + \gamma + 1, \frac{\lambda}{2} - \delta + 1, \frac{\lambda}{2} + \delta + 1, -n
\end{array} \right) \mid 1 \right),
\]

where

\[ \lambda = 2a - b - c - d, \]

\[ w = 1 + 2a - 2\lambda - n, \]

\[ \alpha = \frac{q(2\lambda - a)}{2q - a}, \]

\[ \gamma^2 = \frac{\lambda^2}{4} - \frac{p(a-p)(a-b-c)(a-b-d)(a-c-d)}{bcd + p(a-p)(a-b-c-d)} \]

and

\[ \delta^2 = \frac{\lambda^2}{4} - \frac{q(2\lambda - a) + n(2q - a)}{2}. \]
Proof. Use \( q + 1, q, w, 1 \) for \( a_1, b_1, b_2, x \), respectively, in (5.2) (where \( w \) is as given in (5.13)) and then apply (2.4) to sum the Saalschützian \( \phi F_3(1) \) series on the right-hand side. The final result follows after some simplification. □

We note that the terminating \( \phi F_7(1) \) series on the left-hand side of (5.11) is Saalschützian (i.e. with parametric excess \( \omega = 1 \)) and deviates from a well-poised series in two pairs of numerator and denominator parameters. The terminating \( \psi F_12(1) \) series on the right-hand side of (5.11) is very-well-poised.

The special case of Proposition 5.4 given in the next corollary is a direct extension of the results found in [2, Eqs. 4.5.3, 4.5.4 and 4.5.5]:

**Corollary 5.5.** We have

\[
\begin{align*}
\phi F_5 \left( \begin{array}{cccc}
0, b, c, d, q + 1, -n \\
1 + a - b, 1 + a - c, 1 + a - d, q, w \\
\end{array} \right) \\
= \frac{(2\lambda - a)_n(\lambda - a)_n(\alpha + 1)_n}{(1 + \lambda)_n(2\lambda - 2a)_n}\times
\phi F_{10} \left( \begin{array}{cccc}
\lambda + 1 + \frac{\lambda}{2}, a + 1, a + \frac{\lambda}{2}, 1 + \lambda \\
-\frac{\lambda}{2}, 1 + \frac{\lambda}{2}, a + \frac{\lambda}{2}, 1 + \lambda \\
\end{array} \right),
\end{align*}
\]

where

\[
\begin{align*}
\lambda &= 1 + 2a - b - c - d, \\
w &= 1 + 2a - 2\lambda - n, \\
\alpha &= \frac{q(2\lambda - a)}{2q - a}
\end{align*}
\]

and

\[
\delta^2 = \frac{\lambda^2}{4} - \frac{q(2\lambda - a) + n(2q - a)}{2}.
\]

**Proof.** Let \( p = b \) in (5.11) and then replace \( b + 1 \) with \( b \). □
Equation (5.17) above expresses a certain terminating Saalschützian (i.e. with parametric excess $\omega = 1$) $6F_5(1)$ series that deviates from a well-poised series in two pairs of numerator and denominator parameters in terms of a terminating very-well-poised $9F_8(1)$ series. This equation is a direct generalization of the classical results found by Bailey in [1, Eqs. 8.1, 8.2 and 8.3] and reproduced in [2, Eqs. 4.5.3, 4.5.4 and 4.5.5] that transform terminating nearly-poised $5F_4(1)$ series with parametric excesses $\omega = 1$ and $\omega = 2$ and a terminating very-well-poised $6F_5(1)$ series with parametric excess $\omega = 1$ in terms of terminating very-well-poised $9F_8(1)$ series. Indeed, we have the following:

(a) Letting $q \to -n$ in (5.17) gives [2, Eq. 4.5.3].

(b) Letting $q \to \frac{2}{3}$ in (5.17) gives [2, Eq. 4.5.4].

(c) Letting $q \to \infty$ in (5.17) gives [2, Eq. 4.5.5].

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