A FPTAS for the Subset Sum Problem with Real Numbers

MARIUS COSTANDIN, General Digits

In this paper we study the subset sum problem with real numbers. Starting from the given problem, we formulate a quadratic maximization problem over a polytope which is eventually written as a distance maximization to a fixed point. For solving this, we provide a polynomial algorithm which maximizes the distance to a fixed point over a certain convex set. This convex set is obtained by intersecting the unit hypercube with two relevant half spaces. We show that in case the subset sum problem has a solution, our algorithm gives the correct maximum distance up to an arbitrary chosen precision. In such a case, we show that the obtained maximizer is a solution to the subset sum problem. Therefore, we compute the maximizer and upon analyzing it we can assert the feasibility of the subset sum problem.

Additional Key Words and Phrases: computational geometry, subset sum problem, quadratic optimization, 0-1 integer programming

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1 INTRODUCTION
We begin this work with the following subsections:

1.1 Notations and definitions
For \( n, m \in \mathbb{N} \) and \( D \subseteq \mathbb{R} \) we denote by
\[
\mathcal{D}^{n \times m} = \left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \mid a_{i,j} \in D, \, 1 \leq i \leq n, 1 \leq j \leq m \right\}
\]
(1)

We denote by \( I_n \) the unit matrix in \( \mathbb{R}^{n \times n} \), with \( 1_{n \times m} \) the matrix of appropriate size where each entry is 1, and with \( 0_{n \times m} \) the matrix of appropriate size where each entry is 0.

For \( x \in \mathbb{R}^{n \times 1} \) and \( r > 0 \), we denote by \( B(x, r) \) the open ball centered in \( x \) and of radius \( r \), and with \( \bar{B}(x, r) \) the closed ball centered at \( x \) and of radius \( r \), i.e
\[
B(x, r) = \{ y \in \mathbb{R}^{n \times 1} \mid \| y - x \| < r \} \quad \bar{B}(x, r) = \{ y \in \mathbb{R}^{n \times 1} \mid \| y - x \| \leq r \}
\]
(2)

Definition 1.1 (An outer approximation of a polytope). Given \( \epsilon \in \mathbb{R} \), \( \epsilon > 0 \) and compact polytope \( \mathcal{P} \subseteq \mathbb{R}^{n \times 1} \) we say that the compact set \( Q \subseteq \mathbb{R}^{n \times 1} \) is an outer approximation of \( \mathcal{P} \) of precision \( \epsilon \) if
\[
\mathcal{P} \subseteq Q \subseteq \bigcup_{x \in \mathcal{P}} B(x, \epsilon)
\]
(3)

Author’s address: Marius Costandin, costandinmarius@gmail.com, General Digits.

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For $D \subseteq \mathbb{R}^{n \times 1}$ we denote by $\partial D$ the frontier of $D$.

1.2 Problem definition
Let us consider the real subset sum problem (RSSP): for a given $S \in \mathbb{R}^{n \times 1}$ we ask if exists $x \in \{0, 1\}^{n \times 1}$ such that $S^T \cdot x = 0$.

1.3 Approaches from literature
We will present some algorithms for the literature for solving the Subset Sum problem where the numbers are positive integers. These FPTAS use a very different approach to our work. One of them, is presented here, [1]. This algorithm first sorts the numbers in the list that we will call $S$ (i.e what we defined as a vector and left unsorted, this will now become a list, which can be sorted). Then it defines merging of two such ordered lists. Finally they provide a solution to the exact Subset Sum problem in terms of merging of lists. This proves to be an exponential algorithm. In order to improve on this, they define a trimming operation: remove an item in the list if it can be approximated (a proper definition of approximation is given) by another element. Then applying the previous algorithm with merging of lists together with trimming they provide a FPTAS to the subset sum problem. For more information see [1]

Another classical approach from the literature uses the tools from dynamic programming as follows: the list of integers is ordered and for simplicity of presentation it is assumed to be composed out of positive integers. Let $M$ be the sum of all the integers in the list. It is obvious that it is not possible to have a subset which adds up to a negative number, zero or a positive number (strictly) greater than $M$. It is also obvious that we have a subset which adds up to $M$. Let us ask if there is a subset of $S$ which adds up to $1 \leq T \leq M - 1$. Applying the dynamic programming based algorithm we (parasitically) obtain an answer to all the possible values of $T$. The algorithm works as follows: form a table with as many rows as numbers are in $S$ and as many columns as numbers are between $1$ and $M - 1$ i.e $M - 1$. Assuming that the numbers in $S$ are ordered, proceed as follows to fill the table. The table will be filled with the logical values $1$ – true or $0$ – false. At the end, if the column corresponding to $T$ has a $1$ then there is a subset of $S$ which adds up to $T$ otherwise there is none.

So the table is filled as follows: the first line is filled with zeros, except for the column equal to the first number in $S$, where we put $1$. All the other rows are filled with the values of the previous row until we meet the column equal to the number in $S$ the row corresponds to, where we put $1$. Continuing on the same row, we fill each remaining column with the value from the previous row on the column obtained by subtracting from the current one the value of the number in $S$ corresponding to the row. For instance, assume that $S = \{x_1, \ldots, x_N\}$. Then we will have $N$ lines. Let’s denote the table by $A$. The line $k$ is filled as follows: $A(k, j) = A(k - 1, j)$ for all $1 \leq j < x_k$. Then $A(k, x_k) = 1$ and $A(k, x_k + p) = A(k - 1, p)$ for all $1 \leq p \leq M - 1 - x_k$. It can be proven that this approach gives the correct results. For more advanced algorithm using Dynamic Programming related tools see [2], [3], [4], [5].

Finally, another approach is based on writing an associated quadratic optimization problem to the subset sum problem, see [6]. Then a quadratic function is ought to be maximized over a convex set. We therefore mention here that such algorithms (maximization of quadratic functions over convex domains) are able to provide an approximate answer to the subset sum problem. Some references in this direction are: [8], [9], [10], [11].

Overall a good presentation of the state of the art approaches can be found here [26].
1.4 Our proposed approach

For some $\hat{\beta} > 0$ we formulate the well known optimization problem associated to the subset sum problem:

$$\max \sum_{k=1}^{n} x_k \cdot (x_k - 1) + \hat{\beta} \cdot \sum_{k=1}^{n} x_k \cdot s_k$$

s.t.

$$S^T \cdot x \leq 0$$

$$0 \leq x_k \leq 1$$

$$1_{n_{x=1}}^T \cdot x - \frac{1}{2} \geq 0$$

(4)

where $x = [x_1 \ldots x_n]^T \in \mathbb{R}^{n_{x=1}}$ and $S = [s_1 \ldots s_n]^T \in \mathbb{R}^{n_{x=1}}$. Please note that due to the last constraint, the origin $0_{n_{x=1}}$ does not belong to the search space.

As presented here [6] note that upon solving (4) the answer to the problem is positive if and only if the maximum is zero. Indeed, on the search space, the objective function is less than or equal to zero and reaches its maximum value of zero on $x^* \in \mathbb{R}^{n_{x=1}}$ if and only if $x^* \in \{0, 1\}^{n_{x=1}}$ and $S^T \cdot x^* = 0$. We write the function in (4) as follows

$$\sum_{k=1}^{n} x_k \cdot (x_k - 1) + \hat{\beta} \cdot \sum_{k=1}^{n} x_k \cdot s_k = x^T \cdot x + (\hat{\beta} \cdot S - 1_{n_{x=1}})^T \cdot x =$$

$$= x^T \cdot x + 2 \cdot x^T \cdot \frac{\hat{\beta} \cdot S - 1_{n_{x=1}}}{2} + \frac{1}{4} \|\hat{\beta} \cdot S - 1_{n_{x=1}}\|^2 - \frac{1}{4} \|\hat{\beta} \cdot S - 1_{n_{x=1}}\|^2$$

$$= \|x - \frac{1}{2} (1_{n_{x=1}} - \hat{\beta} \cdot S)\|^2 - \frac{1}{4} \|\hat{\beta} \cdot S - 1_{n_{x=1}}\|^2$$

(5)

For a given $\frac{1}{4} \geq \epsilon > 0$, being able to assert the existence of an $x$ in the feasible set such that

$$-\epsilon \leq x^T \cdot (x - 1_{n_{x=1}}) + \hat{\beta} \cdot S^T \cdot x \leq 0$$

(6)

means that we can assert the existence of an $x$ such that

$$-\epsilon \leq x_k \cdot (x_k - 1) \leq 0$$

$$-\epsilon \leq \hat{\beta} \cdot S^T \cdot x \leq 0$$

(7)

which finally means $x_k \in \left\{ \frac{1 + \sqrt{1 - 4 \delta}}{2} \right\}$ for all $k \in \{1, \ldots, n\}$. Please note that for $\epsilon \rightarrow 0$ we get $x_k \in \{0, 1\}$, and for $\hat{\beta} \geq 1$ we get $S^T \cdot x \rightarrow 0$. We will be satisfied with being able to assert the above, i.e to solve (4) to an arbitrary given precision determined by the strictly positive $\epsilon > 0$ and for any $\hat{\beta} \geq 1$.

It is obvious that we can solve (4) for a given $\hat{\beta}$ if we solve:

$$\max \left\| x - \frac{1}{2} \left( 1_{n_{x=1}} - \frac{\beta}{\|S\|^2} \cdot S \right) \right\|^2$$

s.t.

$$S^T \cdot x \leq 0$$

$$0 \leq x_k \leq 1$$

$$1_{n_{x=1}}^T \cdot x - \frac{1}{2} \geq 0$$

(8)

for $\beta = \|S\| \cdot \hat{\beta}$. We take $\beta \geq \|S\|$. In the following we focus on solving the problem (8).

Having a closer look at (8) let’s denote

$$C = \frac{1}{2} \left( 1_{n_{x=1}} - \frac{\beta}{\|S\|^2} \cdot S \right) \quad \mathcal{P} = \{ x \in \mathbb{R}^{n_{x=1}} | S^T \cdot x \leq 0, 0 \leq x_k \leq 1, 1_{n_{x=1}}^T \cdot x - \frac{1}{2} \geq 0 \}$$

(9)
where $e_k$ is the $k'$th column of the unit matrix in $\mathbb{R}^{n \times n}$. Then (8) becomes

$$\max_{x \in \mathcal{P}} \|x - C\|^2$$

i.e we have to find the furthest point in the polytope $\mathcal{P} \subseteq \mathbb{R}^{n \times 1}$ to the fixed point $C \in \mathbb{R}^{n \times 1}$

**Remark 1.** Let

$$x^* \in \arg\max_{x \in \mathcal{P}} \|x - C\|^2$$

then if $\|x^* - C\| = \|C\|$ the maximum of (4) is zero, so we have an affirmative answer to the RSSP problem, otherwise the answer is negative.

**Remark 2.** Since $\mathcal{P}$ is compact, it is obvious that $x^* \in \partial \mathcal{P}$.

We attack the RSSP problem on two ways:

1. Given the polytope $\mathcal{P}$, we will find an outer-approximation $\mathcal{Q}$ of $\mathcal{P}$ such that:
   - (a) if exists $x^* \in \partial \mathcal{P}$ for which $\|x^* - C\| = \|C\|$ then $x^* \in \partial \mathcal{Q}$.
   - (b) if exists $y^* \in \partial \mathcal{Q}$ for which $\|y^* - C\| = \|C\|$ then $y^* \in \partial \mathcal{P}$.
   - (c) One has $\|y - C\| \leq \|C\|$ for all $y \in \mathcal{Q}$

2. Find

$$x^*_1 \in \arg\max_{x \in \mathcal{Q}} \|x - C\|^2$$

Indeed, if $\|x^*_1 - C\| = \|C\|$ then the answer to RSSP is affirmative otherwise it is negative.

In the section below, we construct $\mathcal{Q}$ and present an algorithm which ends in polynomial time. This algorithm is guaranteed to find a solution to RSSP if one exists. It searches for the furthest point to $C$ in $\mathcal{Q}$. If a solution to the RSSP exists then it will find it, otherwise will find something else, not necessarily relevant. However, because it stops in polynomial time and presents a point, we can assert if a solution exists or not by analyzing what was found: if the point is a solution to the RSSP then the RSSP has a solution, otherwise, if the point is not a solution to the RSSP, then the RSSP does not have a solution.

2 MAIN RESULTS: WALK-THROUGH

2.1 Construction and analysis of a suitable intersection of balls

Please note that $\mathcal{P}$ is formed by intersecting the unit hypercube with two additional half-spaces $\{x \in \mathbb{R}^{n \times 1} | S^T \cdot x \leq 0 \}$ and $\{x \in \mathbb{R}^{n \times 1} | 1^T_{n \times 1} \cdot x - \frac{1}{2} \geq 0 \}$

We will construct a set according to the following instructions: let’s consider the ball $\mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$. This ball, obviously includes the unit hypercube, hence it includes $\mathcal{P}$. The only points of the unit hypercube on the boundary of this ball are the corners. Therefore any point of $\mathcal{P}$ on the boundary of this ball has to be a corner of the unit hypercube, hence we are interested to preserve them in our construction. Moreover, if there is a corner of the hypercube on the hyperplane $\{x \in \mathbb{R}^{n \times 1} | S^T \cdot x = 0 \}$ this means it is on the intersection of the hyperplane with the boundary of the ball $\mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$. According to the above reasoning, a general rule for our construction arises: *we should preserve the intersection of each hyperplane (forming $\mathcal{P}$) with the boundary of the ball $\mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$.* For each halfspace we construct a ball. We consider that each halfspace is generated by a hyperplane that we know. Therefore, for the construction of the ball, we have to answer two questions:

1. Where is the center?
2. How large is the radius?
We provide the following answers:

1. The center of the ball is
   (a) in the halfspace,
   (b) on an axis going through the point \( \frac{1}{2} \cdot 1_{n \times 1} \), parallel to the director vector of the hyperplane
2. The radius is such that the ball leaves the same imprint on the boundary of \( B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \) as the hyper-plane.

The following construction leaves free a unidimensional parameter: the distance from the center of the ball to the point \( \frac{1}{2} \cdot 1_{n \times 1} \). We will take this to be equal for all the balls we construct. We therefore obtained for each halfspace determining the polytope \( \mathcal{P} \) a ball. Intersecting all these balls we obtain a set. Associated to each ball we had one parameter (which we took equal for all of the balls) hence there is a unidimensional parameter associated with the intersection of the balls. Our choice of taking this parameter equal for all of the balls is based purely on the desire to keep the notations and the presentation simple. Let’s denote this parameter with \( \rho \) and the intersection of balls with \( \mathcal{Q}_\rho \).

One can already imagine that increasing or decreasing \( \rho \) will affect the shape of \( \mathcal{Q}_\rho \). One can intuitively think that increasing \( \rho \) will make \( \mathcal{Q}_\rho \) shrink while decreasing \( \rho \) will make \( \mathcal{Q}_\rho \) bloat. For instance we would like to think that (sometimes breaking the rule (1.a) for imagination purposes only) for \( \rho = 0 \) one has \( \mathcal{Q}_\rho = B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \) while for \( \rho \rightarrow \infty \) one has \( \mathcal{Q}_\rho \rightarrow \mathcal{P} \). Although not used in this algorithm, but interesting to note, one sees that \( \mathcal{Q}_{\rho_1} \subseteq \mathcal{Q}_{\rho_2} \) for \( \rho_2 \leq \rho_1 \). All of these will eventually be rigorously proven later. This presentation here was done only to infer the reader the correct intuition about the reason and the style of the construction. We will prove:

**Lemma 2.1.** Given \( \delta > 0 \) fixed, if \( \exists B(x, \delta) \subseteq \mathcal{P} \) then exists \( \rho_\delta > 0 \) such that for all \( \rho \geq \rho_\delta \) one has

\[
\mathcal{P} \subseteq Q_\rho \subseteq \bigcup_{x \in \mathcal{P}} B(x, \delta)
\]

i.e we prove that \( \mathcal{Q}_\rho \) is an outer approximation of \( \mathcal{P} \).

### 2.2 Deciding the inclusion of an intersection of balls in another ball

As explained in the introduction, we can solve the RSSP if we are able to maximize the distance to a fixed point \( C \in \mathbb{R}^{n \times 1} \) over an intersection of balls. For this, we consider w.l.o.g the following problem:

\[
\max_{x \in \mathcal{Q}_\rho} \|x\|
\]

for some fixed \( \rho > 0 \). It is obvious that this problem can be easily solved if for every \( R > 0 \) one can answer the question: \( \mathcal{Q}_\rho \subseteq B(0_{n \times 1}, R) \). In order to answer this question, we assume that \( \mathcal{Q}_\rho = \bigcap_{k=1}^{m} B(C_k, r_k) \). Let’s define

\[
h_\rho(x) = \max_{k \in \{1, \ldots, m\}} \|x - C_k\|^2 - r_k^2 = f(x) = \|x\|^2
\]

then \( \mathcal{Q}_\rho = \{ x \mid h_\rho(x) \leq 0 \} \). We have:

\[
(h_\rho - f)(x) = -\|x\|^2 + \max_{k \in \{1, \ldots, m\}} \|x - C_k\|^2 - r_k^2 = \max_{k \in \{1, \ldots, m\}} -2 \cdot C_k^T \cdot x + \|C_k\|^2 - r_k^2
\]

Observing that the above function, \( (h_\rho - f)(x) \) contains linear terms, we define the polytope:

\[
\mathcal{P}_{\rho, R^2} = \left\{ x \in \mathbb{R}^{n \times 1} \mid (h_\rho - f)(x) \leq -R^2 \right\}
\]

for the given \( R \).
Let us consider the following convex optimization problem:

$$X^*_p = \arg\min_{h_p(x) \leq 1} (h_p - f)(x)$$

We choose the search space \( \{x|h_p(x) \leq 1\} \) just to have a compact, hence assuring the fact that the result is finite, larger set than \( \{x|h_p(x) \leq 0\} \). We assume in this paper that \( X^*_p = \{x^*_p\} \) i.e contains a single point! Or, put otherwise, the algorithm we develop is for these situations! There are two possibilities now:

1. \( X^*_p \cap \{x|h_p(x) = 0\} \neq \emptyset \)
2. \( X^*_p \cap \{x|h_p(x) = 0\} = \emptyset \) which becomes
   a. \( X^*_p \subseteq \{x|h_p(x) < 0\} \)
   b. \( X^*_p \subseteq \{x|h_p(x) > 0\} \)

We have the following remarks about the polytope \( \mathcal{P}_{p,R_0} \):

**Remark 3.** Please note that for \( R = 0 \) we have that \( \mathcal{Q}_p = \{x|h_p(x) \leq 0\} \subseteq \mathcal{P}_{p,R=0} \). Indeed, let \( x \in \{x|h_p(x) \leq 0\} \) then \( \max_{x}\|x - C_k\|^2 - r_k^2 \leq 0 \). On the other hand, \( x \in \mathcal{P}_{p,0} \) if \( (h_p - f)(x) \leq 0 \). Since \( f(x) = \|x\|^2 \geq 0 \), one can see that this easily happens.

According to the above possibilities we prove the following phenomena will happen if \( R \) is increased from 0.

1. \( X^*_p \cap \{x|h_p(x) = 0\} \neq \emptyset \) Then let \( R^*_p = \|x^*_p\| \)
2. \( X^*_p \cap \{x|h_p(x) = 0\} = \emptyset \) Then if
   a. \( X^*_p \subseteq \{x|h_p(x) < 0\} \) exists the smallest \( R^*_p \) such that \( \mathcal{P}_{p,R^*_p} \subseteq \mathcal{Q}_p \) for all \( R \geq R^*_p \). That is, the polytope \( \mathcal{P}_{p,R^*_p} \) enters the set \( \mathcal{Q}_p \) as \( R \) increases. Finding this \( R^*_p \) is hard in general!
   b. \( X^*_p \subseteq \{x|h_p(x) > 0\} \) exists the smallest \( R^*_p \) such that \( \mathcal{P}_{p,R^*_p} \cap \mathcal{Q}_p = \emptyset \) for all \( R \geq R^*_p \). That is, the polytope \( \mathcal{P}_{p,R^*_p} \) separates from the set \( \mathcal{Q}_p \) as \( R \) increases. Since \( \mathcal{Q}_p \) and \( \mathcal{P}_{p,R^*_p} \) are convex sets, testing the non-emptiness of their intersection is a convex optimization problem and can be done in \( P \) time hence finding \( R^*_p \) in this case is not hard!

We will prove the main result of this section in the following theorem:

**Theorem 2.2.**

$$R^*_p = \max_{x \in \mathcal{Q}_p} \|x\|$$

It follows that once \( R^*_p \) is found, one actually solved the optimization problem \( \max_{x \in \mathcal{Q}_p} \|x\| \). However, as stated above, finding \( R^*_p \) for the case (2.a) is difficult. In the following we present a method for doing just that in one particular case: \( X^*_p \subseteq \mathcal{P} \subseteq \mathcal{Q}_p \). While this might be seen as another limitation of the presented results, we will add the fact that if \( X^*_p \subseteq \mathcal{Q}_p \setminus \mathcal{P} \) then since for \( \rho \to \infty \) one has \( \mathcal{Q}_p \to \mathcal{P} \) one should just increases \( ho \). However, this scenario will not be further analyzed in this paper.

Finding \( R^*_p \) for the case (2.a) boils down to being able to tell if the polytope \( \mathcal{P}_{p,R^*_p} \) is included in an intersection of balls, i.e \( \mathcal{Q}_p \). However, this seems initially more difficult than asserting the inclusion of a polytope in one ball (this was actually the initial problem: assert if \( \mathcal{P} \) is included in \( \mathcal{B}(C,R) \)). Please recall the fact that \( \mathcal{Q}_p \) is an outer approximation of the polytope \( \mathcal{P} \). We can assert in \( P \) time if \( \mathcal{P}_{p,R^*_p} \subseteq \mathcal{P} \), since this is actually testing polytope containment in another polytope. We show that if the RSSP has a solution then \( \mathcal{P}_{p,R^*_p} \subseteq \mathcal{P} \iff \mathcal{P}_{p,R^*_p} \subseteq \mathcal{Q}_p \) for sufficiently large values of \( ho \). We give a computable lower bound. Also, we will always be able to find the "last" point of \( \mathcal{P}_{p,R^*_p} \) to enter the set \( \mathcal{P} \), therefore eventually our algorithm not only is able to assert in \( P \) time the existence or non existence of solutions to the RSSP, but in case of existence it will deliver a solution too.
3 MAIN RESULTS: PROOFS

In this section we reiterate with proofs the assertions made in the previous one.

3.1 The Construction of $Q_{\rho}$

For approximating the unit hypercube, we consider the following balls $B(C_{k+}, r_{k+})$ and $B(C_{k-}, r_{k-})$ where

$$C_{k\pm} = \frac{1}{2} \cdot 1_{n \times 1} \pm q_k \cdot e_k$$  \hspace{1cm} (20)

where $e_k$ is the $k$'th column of the unit matrix $I_n$ and $q_k$ and $r_k$ are to be determined such that

$$\left\| \frac{1}{2} \cdot 1_{n \times 1} - \frac{1}{2} \cdot e_k - C_{k+} \right\|^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2 = r_{k+}^2$$

$$\left\| \frac{1}{2} \cdot 1_{n \times 1} + \frac{1}{2} \cdot e_k - C_{k-} \right\|^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2 = r_{k-}^2$$  \hspace{1cm} (21)

These conditions assure that each corner of the unit hypercube belongs to the boundary of $\bigcap_{k=1}^n (B(C_{k+}, r_{k+}) \cap B(C_{k-}, r_{k-}))$. Indeed, w.l.o.g let us consider the corners of the unit hypercube which belong to the facet $x_1 = 1$. These will be in the form $\left[ \frac{1}{2} \right]_y$ where $y \in \{0, 1\}^{n-1}$. Then we have

$$\left\| \left[ \frac{1}{2} \right]_y - \left( \frac{1}{2} \cdot 1_{n \times 1} + \frac{1}{2} \cdot e_1 \right) \right\|^2 = \left\| y - \frac{1}{2} \cdot 1_{n \times 1} \right\|^2 = \left( \frac{\sqrt{n} - 1}{2} \right)^2$$  \hspace{1cm} (22)

Finally, we show that $\left[ \frac{1}{2} \right]_y \in B(C_{k-}, r_{k-})$.

$$\left\| \left[ \frac{1}{2} \right]_y - C_{k-} \right\|^2 = \left\| \left[ \frac{1}{2} \right]_y - \left( \frac{1}{2} \cdot 1_{n \times 1} + \frac{1}{2} \cdot e_1 \right) + \left( \frac{1}{2} \cdot 1_{n \times 1} + \frac{1}{2} \cdot e_1 \right) \right\|^2 - C_{k-} \right\|^2 = \left\| y - \frac{1}{2} \cdot 1_{n \times 1} \right\|^2 + \left\| \frac{1}{2} \cdot 1_{n \times 1} + \frac{1}{2} \cdot e_1 - C_{k-} \right\|^2 = r_{k-}^2$$  \hspace{1cm} (23)

because $\frac{1}{2} \cdot 1_{n \times 1} + \frac{1}{2} \cdot e_1 - C_{k-} = e_1 \cdot (\frac{1}{2} + q_k)$ which is orthogonal on $\left[ \frac{1}{2} \right]_y$. Therefore the unit hypercube is included in the intersection of $n$-disks and has the corners touching the boundary of the intersection.

Next we approximate the half space $\{x | S^T \cdot x \leq 0\}$ with the closed ball $\bar{B}(C_s, r_s)$ where we compute $C_s$ and $r_s$ as follows. Let $P_s = \frac{1}{2} \cdot 1_{n \times 1} + \frac{S}{\|S\|} \cdot S$ such that $S^T \cdot P_s = 0$ hence $S^T \cdot 1_{n \times 1} + \frac{S}{\|S\|} \cdot \|S\|^2 = 0$ and consequently

$$P_s = \frac{1}{2} \cdot 1_{n \times 1} - \frac{S^T \cdot 1_{n \times 1}}{\|S\|^2} \cdot S.$$  \hspace{1cm} Let

$$C_s = P_s - \frac{\tilde{q}_s}{\|S\|} \cdot S = \frac{1}{2} \cdot 1_{n \times 1} - \left( \frac{S^T \cdot 1_{n \times 1}}{\|S\|^2} + \tilde{q}_s \right) \cdot \frac{S}{\|S\|}$$

and

$$\tilde{r}_s^2 = \frac{n}{4} - \left\| \frac{1}{2} \cdot 1_{n \times 1} - P_s \right\|^2 = \frac{n}{4} - \left( \frac{S^T \cdot 1_{n \times 1}}{\|S\|} \right)^2$$  \hspace{1cm} (24)
and we require the following relation to hold:

\[ \|P_s - C_s\|^2 + r_s^2 = r_s^2 \]

This is need to ensure that if a corner of the unit hypercube belongs to the hyperplane \( \{x | S^T \cdot x = 0\} \) then it will also belong to the hyper sphere \( \{x | \|x - C_s\| = r_s\} \). Indeed, let \( x \in \{0, 1\}^n \) such that \( x^T \cdot S = 0 \), then we show that \( \|C_s - x\| = r_s \).

\[
\|C_s - x\|^2 = \|C_s - P_s + P_s - x\|^2 = \|C_s - P_s\|^2 + \|P_s - x\|^2 + 2 \cdot (C_s - P_s)^T \cdot (P_s - x)
\]

\[
= r_s^2 - r_s^2 + \left| x - \frac{1}{2} \cdot 1_n \right|^2 + \left| \frac{1}{2} \cdot 1_n - P_s \right|^2 - 2 \cdot \bar{q}_s \cdot \frac{S^T}{\|S\|} \cdot (P_s - x)
\]

\[
= r_s^2 - r_s^2 + \frac{n}{4} - \left| \frac{1}{2} \cdot 1_n - P_s \right|^2 = r_s^2
\]

(27)

because \( S^T \cdot P_s = 0 = S^T \cdot x \) and \( \|P_s - x\|^2 + \left| \frac{1}{2} \cdot 1_n - P_s \right|^2 = \left| x - \frac{1}{2} \cdot 1_n \right|^2 \) by applying Pitagora’s theorem in the right triangle \( \triangle x \frac{1}{2} P_s \). Finally please note that \( \|x - \frac{1}{2} 1_n\| = \frac{\sqrt{n}}{2} \) for all \( x \in \{0, 1\}^n \).

The last constraint is the half space \( \{x | \frac{1}{2} \cdot 1_n \cdot x - \frac{1}{2} \geq 0\} \) which will be “approximated” with the closed n-disk \( \bar{B}(C_h, r_h) \) with \( C_h \) and \( r_h \) to be calculated below. Let \( P_h = \frac{1}{2} \cdot 1_n + t \cdot \frac{1}{\|1_n\|} \) such that \( \frac{1}{2} P_h - \frac{1}{2} = 0 \). It follows that \( \frac{1}{2} \cdot 1_n \cdot (\frac{1}{2} \cdot 1_n + t \cdot \frac{1}{\|1_n\|}) - \frac{1}{2} = 0 \) hence \( P_h = \frac{1}{2} \cdot 1_n + \frac{1 - n}{2 \sqrt{n}} \cdot \frac{1}{\|1_n\|} = \frac{1}{2} \cdot 1_n \) then

\[
C_h = P_h + \bar{q}_h \cdot \frac{1}{\|1_n\|} = \frac{1}{2} \cdot 1_n + \left( \frac{1 - n}{2 \sqrt{n}} \cdot \bar{q}_h \right) \cdot \frac{1}{\|1_n\|}
\]

(28)

Let

\[
r_h^2 = n \frac{4}{4} - \left| \frac{1}{2} \cdot 1_n + P_h \right|^2 = n \frac{4}{4} - \left( \frac{1}{2} - \frac{1}{2} \cdot n \right)^2 \cdot n = \ldots = \frac{1}{2} - \frac{1}{4} \cdot n
\]

(29)

As above, we require \( q_h \) and \( r_h \) to meet the following constraint:

\[
\|C_h - P_h\|^2 + r_h^2 = r_h^2
\]

(30)

This is enough to ensure that \( \{x | \|x - \frac{1}{2} \cdot 1_n\| = \frac{\sqrt{n}}{2}, \frac{1}{2} \cdot 1_n \cdot x - \frac{1}{2} \geq 0\} \subseteq \bar{B}(C_h, r_h) \) as we will see in the next subsection.

Finally, we choose

\[
q_{kh} = q_h = q_s = \rho
\]

(31)

and define:

\[
Q_\rho = \bigcap_{k=1}^{n} (\bar{B}(C_{k+}, r_{k+}) \cap \bar{B}(C_{k-}, r_{k-})) \cap \bar{B}(C_s, r_s) \cap \bar{B}(C_h, r_h)
\]

(32)

Please note that there is a one to one relation between \( \rho \) and the radii of the n-disks. If the radii are constrained to be greater than some fixed value, then this is achievable by choosing a sufficiently large \( \rho \).
3.2 The Analysis of $Q_\rho$

We begin this subsection with a very useful lemma:

**Lemma 3.1.** Let us consider the following n-disks

$$
D_1 = \{ x \in \mathbb{R}^{n \times 1} \mid q_1 \cdot e_1 - x \mid \leq r_1 \} \\
D_2 = \{ x \in \mathbb{R}^{n \times 1} \mid q_2 \cdot e_1 - x \mid \leq r_2 \} \\
D_3 = \{ x \in \mathbb{R}^{n \times 1} \mid q_3 \cdot e_1 - x \mid \leq r_3 \}
$$

(33)

with $q_1, q_2 > 0$ and $r_1 > r_2 > r_3 \geq 0$ such that exists

$$
0 < a^2 = r_1^2 - q_1^2 = r_2^2 - q_2^2 = r_3^2 - q_3^2
$$

(34)

that is the n-disks share a common n-1 sphere $\{ x | e_1^T \cdot x = 0, \|x\| = a \}$. Let $\mathcal{H} = \{ x \in \mathbb{R}^{n \times 1} | e_1^T \cdot x \leq 0 \}$ and $\mathcal{G} = \{ x \in \mathbb{R}^{n \times 1} | e_1^T \cdot x \geq 0 \} = \mathbb{R}^{n \times 1} \setminus \text{int}(\mathcal{H})$. Then the following inclusions are true

1. $\mathcal{H} \cap D_3 \subseteq D_1 \cap D_3 \subseteq D_2 \cap D_3 \subseteq D_2$

(2)

$$
\mathcal{G} \cap D_1 \subseteq \mathcal{G} \cap D_2 \subseteq \mathcal{G} \cap D_3
$$

(35)

**Proof.** see Appendix \(\square\)

**Remark 4.** Using the above Lemma 3.1 and the construction of the set $Q_\rho$ the following can be proven:

1. Exists $\bar{\rho} > 0$ such that $\min\{ r_{k+}, r_\rho, r_{h} \} \geq \sqrt{n} \frac{\sqrt{n}}{2}$, $C_2 \cdot S < 0$ and $C_0^T \cdot 1_{n \times 1} > 0$ for all $\rho \geq \bar{\rho}$. This is easy to prove using the construction of the n-disks.

2. If $\rho > \bar{\rho}$ then one has

$$
\mathcal{P} \subseteq Q_\rho
$$

(36)

Indeed, since we know that $\mathcal{P} \subseteq \mathcal{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{n} \frac{\sqrt{n}}{2} \right)$ in order to prove that $\mathcal{P} \subseteq Q_\rho$ is enough to prove that any half space composing $\mathcal{P}$ intersected with $\mathcal{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{n} \frac{\sqrt{n}}{2} \right)$ is included in an n-disk composing $Q_\rho$ intersected with $\mathcal{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{n} \frac{\sqrt{n}}{2} \right)$. W.l.o.g take in the above Lemma 3.1 $\mathcal{H} = \{ x \mid S^T \cdot x \leq 0 \}$, $D_3 = \mathcal{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{n} \frac{\sqrt{n}}{2} \right)$ and $D_1 = \mathcal{B}(C_\rho, r_\rho)$ to obtain $\mathcal{H} \cap D_3 \subseteq D_1 \cap D_3$

3. For $\rho > \bar{\rho}$ one also has

$$
Q_\rho \subseteq \mathcal{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{n} \frac{\sqrt{n}}{2} \right)
$$

(37)

Indeed

$$
Q_\rho \subseteq \bigcap_{k=1}^{n} (\mathcal{B}(C_{k+}, r_{k+}) \cap \mathcal{B}(C_{k-}, r_{k-})) = \check{Q}_\rho
$$

(38)

W.l.o.g let us analyze one of the intersecting balls:

$$
\mathcal{B}(C_{k+}, r_{k+}) \subseteq \{ x \mid e_k^T \cdot x \geq 0 \} \cup \{ x \mid e_k^T \cdot x \leq 0 \} \cap \mathcal{B}(C_{k+}, r_{k+})
$$

(39)
It can be proven using the above Lemma 3.1 and the construction of the ball \( \overline{B}(C_k, r_k) \) that 
\[
\left\{ x | e_k^T \cdot x \leq 0 \right\} \cap \overline{B}(C_k, r_k) \subseteq \left( \left\{ x | e_k^T \cdot x \leq 0 \right\} \cap \overline{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \right)
\]

hence we obtained
\[
\overline{B}(C_k, r_k) \subseteq \left\{ x | e_k^T \cdot x \geq 0 \right\} \cup \overline{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)
\]

Finally considering the following property regarding a finite intersection of sets for \( A_k, B \subseteq \mathbb{R}^{n \times 1} \)
\[
\bigcap_{k=1}^{n} A_k \subseteq B \Rightarrow \bigcap_{k=1}^{n} (A_k \cup B) \subseteq B
\]

Therefore, since the unit hypercube \( \bigcap_{k=1}^{n} \left( \left\{ x | e_k^T \cdot x \geq 0 \right\} \cap \left\{ x | e_k^T \cdot x \leq 1 \right\} \right) \subseteq \overline{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \) one obtains
\[
\tilde{Q}_p \subseteq \overline{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right).
\]

Finally, we give the result:

**Lemma 3.2.** Given \( \delta > 0 \) fixed, if \( \exists B(x, \delta) \subseteq \mathcal{P} \) then exists \( \rho_\delta > 0 \) such that for all \( \rho \geq \rho_\delta \) one has
\[
\mathcal{P} \subseteq Q_\rho \subseteq \bigcup_{x \in \mathcal{P}} B(x, \delta)
\]

**Proof.** See Appendix \( \square \)

What the above lemma says is that by using large enough radii for the balls involved (please note that the construction presented above allows the radii of the n-disks to be chosen arbitrarily large) one can assure a fixed desired level of approximation of \( \mathcal{P} \).

### 3.3 Deciding the inclusion of an intersection of balls in another ball

Here we prove some results related to (18). Please recall:

\[
X^*_p = \arg\min_{h_p(x) \leq 1} (h_p - f)(x)
\]

and

\[
\mathcal{P}_{\rho,R^2} = \left\{ x \in \mathbb{R}^{n \times 1} | (h_p - f)(x) \leq -R^2 \right\}
\]

One can see \( \mathcal{P}_{\rho,R^2} \) as a parameterized family of level sets (of \( h_p - f \)).

We assume that \( X^*_p = \{ x^*_p \} \) therefore letting \( -R^2 = (h_p - f)(x^*_p) < 0 \) follows that \( \mathcal{P}_{\rho,R^2} = \{ x^*_p \} \). It is now obvious that

1. If \( x^*_p \in Q_\rho \) then since, as already said, for \( R^2 = 0 \) we have \( Q_\rho \subseteq \mathcal{P}_{\rho,R^2=0} \) and for \( R^2 = R^2_\rho \) one has \( \mathcal{P}_{\rho,R^2=R^2_\rho} = \{ x^*_p \} \subseteq Q_\rho \) follows that exists the smallest \( R^*_p \) such that \( \mathcal{P}_{\rho,R^2} \subseteq Q_\rho \) for all \( R \geq R^*_p \). Hence let
\[
R^*_p = \inf \{ R > 0 \} \mathcal{P}_{\rho,R^2} \subseteq Q_\rho \}
\]
(2) If \( x^*_p \notin Q_p \), then, as above, since for \( R^2 = 0 \) we have \( Q_p \subseteq \mathcal{P}_{R^2=0} \) and for \( R^2 = R^2_p \) one has \( \mathcal{P}_{R^2=R^2_p} \cap Q_p = \emptyset \) follows that exists the smallest \( R^*_p \) such that \( \mathcal{P}_{R=R^*_p} \cap Q_p = \emptyset \) for all \( R \geq R^*_p \).

We now give a fundamental result of this paper:

**Theorem 3.3.**

\[
R^*_p = \max_{x \in Q_p} \|x\| \tag{47}
\]

**Proof.** The problem (47) is maximizing a convex function over a convex domain, hence the maximizer will be on the boundary. We give the proof for the following two cases:

1. \( \{x^*_p\} = X^*_p \subseteq \text{int}(Q_p) \) i.e \( h_p(x^*_p) < 0 \): For some \( 0 < R < R^*_p \) we shall prove that \( \{x|h_p(x) = 0 \} \setminus \mathcal{B}(0_{nx1}, R) \subseteq \mathcal{P}_{R,R^2} \), which means that the points on the boundary of \( Q_p \) of magnitude greater than \( R \), are inside \( \mathcal{P}_{R,R^2} \).

Indeed, this is easy to verify: let \( h_p(x) = 0 \) and \( \|x\| > R \) then

\[
h_p(x) + R^2 - \|x\|^2 \leq 0 \iff (h_p - f)(x) \leq -R^2 \iff x \in \mathcal{P}_{R,R^2} \tag{48}
\]

Since \( \mathcal{P}_{R,R^2} \cap \partial Q_p = \emptyset \) for all \( R > R^*_p \) (by the definition of \( R^*_p \)), follows that \( \exists x_2 \) with \( h_p(x_2) = 0 \), i.e. on the boundary of \( Q_p \), and \( \|x_2\| > R^*_p \). Indeed, assuming that \( \exists \|x_2\| = R_2 > R^*_p \) follows that \( x_2 \in \mathcal{P}_{R,R^2} \) since \( h_p(x_2) - \|x_2\|^2 = -R^2 < -R^*_p \). Also, since \( h_p(x_2) = 0 \) follows that \( x_2 \in \partial Q_p \) a contradiction arises with the definition of \( R^*_p \) because now \( x_2 \in \mathcal{P}_{R,R^2} \cap \partial Q_p \) and \( R > R^*_p \).

2. \( \{x^*_p\} = X^*_p \cap Q_p = \emptyset \) i.e \( h_p(x^*_p) > 0 \): In this case we also have that \( \mathcal{P}_{R,R^2} \cap \partial Q_p = \emptyset \) for all \( R > R^*_p \) (by the definition of \( R^*_p \)), but this time the reason is that instead of \( \mathcal{P}_{R,R^2} \) being strictly inside \( Q_p \), for all \( R > R^*_p \) as above, we have that \( \mathcal{P}_{R,R^2} \cap Q_p = \emptyset \). Hence the same reasoning, as in the previous case, stays valid here: \( \exists x_2 \in \partial Q_p \) with \( \|x_2\| = R_2 > R^*_p \) then \( x_2 \in \mathcal{P}_{R,R^2} \cap \partial Q_p \) which is a contradiction.

3. \( h_p(x^*_p) = 0 \) In this case we argue that \( R^*_p = \|x^*_p\| = \max_{x \in Q_p} \|x\| \). Indeed, simply because \( \mathcal{P}_{R,R^2} = \emptyset \) for all \( R > R^*_p \) follows that \( \mathcal{P}_{R,R^2} \cap \partial Q_p = \emptyset \) for all \( R > R^*_p \). Hence the same reasoning, as in the previous cases, stays valid here: \( \exists x_2 \in \partial Q_p \) with \( \|x_2\| = R_2 > R^*_p \) then \( x_2 \in \mathcal{P}_{R,R^2} \cap \partial Q_p \) which is a contradiction.

\[\square\]

### 3.4 Solving the RSSP

Let us define

\[
C = \frac{1}{2} \cdot 1_{nx1} - \frac{\beta}{2} \cdot \frac{S}{\|S\|} \tag{49}
\]

and consider the problem

\[
\max \|x - C\| \quad \text{s.t} \quad x \in Q_p \tag{50}
\]

for some \( \rho > \hat{\rho} \) with \( \hat{\rho} \) being given by Remark 4. We continue the subsection with a very important lemma:

**Lemma 3.4.** For \( \hat{\rho} - \frac{\hat{\beta}}{2} < q_s = \rho \) let

\[
x^*_p \in \mathcal{U}^*_p = \operatorname{argmax}_{x \in Q_p} \|x - C\|^2 \tag{51}
\]

then

1. \( \|x^*_p - \frac{1}{2} \cdot 1_{nx1}\| \leq \sqrt{\frac{\beta}{2}} \)
2. \( \|x^*_p - \frac{1}{2} \cdot 1_{nx1}\| = \sqrt{\frac{\beta}{2}} \) iff \( x^*_p \in \{0, 1\}^{nx1} \)
3. If exists \( 0_{nx1} \neq x_1 \in \{0, 1\}^{nx1} \) with \( S^T \cdot x_1 = 0 \) then \( x_1 \in \mathcal{U}^*_p \)
(4) If exists $0_{n \times 1} \neq x_1 \in \{0, 1\}^{n \times 1}$ with $S^T \cdot x_1 = 0$ then for all $x_p^* \in U_p^*$ one has $x_p^* \in \{0, 1\}^{n \times 1}$ and $S^T \cdot x_p^* = 0$

Proof. Since $\rho > \bar{\rho}$, from Remark 4 follows $Q_0 \subseteq B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$. Then:

(1) The first claim follows easily from the fact that $x_p^* \in Q_0 \subseteq B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)

(2) For the second claim it is easy to verify the reverse implication. We focus on the direct one. Let $\|x_p^* - \frac{1}{2} \cdot 1_{n \times 1}\| = \frac{\sqrt{n}}{2}$. Recall $\bar{Q}_0$ from (38). Since

\[ x_p^* \in Q_0 \cap \partial B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \subseteq \bar{Q}_0 \cap B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \]  

(52)

follows that $x_p^* \in \{0, 1\}^{n \times 1}$ since it can be proven that for $\rho > \bar{\rho}$ one has $\bar{Q}_0 \cap B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \subseteq \{0, 1\}^{n \times 1}$.

From (38) we have that $\bar{Q}_0$ is basically an approximation of the unit hypercube and we will not make here further efforts to prove that it only shares its corners with the ball centered in $\frac{1}{2} \cdot 1_{n \times 1}$.

(3) Let $x_1 \neq 0_{n \times 1}, x_1 \in \{0, 1\}^{n \times 1}$ with $S^T \cdot x_1 = 0$ then by construction we get $x_1 \in Q_0 \cap \{x|S^T \cdot x = 0\}$. Hence

\[ \|x_1 - \frac{1}{2} \cdot 1_{n \times 1}\| = \frac{\sqrt{n}}{2} \]  

and $\|C_1 - x_1\| = r_s$. We shall prove that $\|x^* - C\| \leq \|x_1 - C\|$. We prove this by showing that $\|x - C\| \leq \|x_1 - C\|$ for all $x \in Q_0$. This would imply that $x_1 \in U_p^*$. Please note that the points $\frac{1}{2} \cdot 1_{n \times 1}, C, \bar{C}$ are on the same axis, therefore, we can use the Lemma 3.1 as follows.

Since $q_s > \frac{\beta}{2} > \bar{\rho}$ we can choose in Lemma 3.1 $D_1 = \bar{B}(C_s, r_s), D_2 = B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$ and $D_2 = \bar{B}(C, \|C - x_1\|)$ to assert that $D_1 \cap D_2 \subseteq D_2$. Furthermore, because $Q_0 \subseteq D_1 \cap D_2$, i.e. $Q_0 \subseteq \bar{B}(C, r_s) \cap B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$ follows that for all $x \in Q_0$ one has $x \in D_2$ hence $\|x - C\| \leq \|C - x_1\|$.

(4) It is known from the previous point that $\|C - x_p^*\| = \|C - x_1\|$. Then it follows that $\|C - x_p^*\| = r_s$ and $\|x_p^* - \frac{1}{2} \cdot 1_{n \times 1}\| = \frac{\sqrt{n}}{2}$. Indeed, assume that $\|C - x_p^*\| < r_s$ and/or $\|x_p^* - \frac{1}{2} \cdot 1_{n \times 1}\| < \frac{\sqrt{n}}{2}$. Then by the use of Lemma 3.1, similarly to previous point, one obtains $\|C - x_p^*\| < \|C - x_1\|$, which is false. Since $\|x_p^* - \frac{1}{2} \cdot 1_{n \times 1}\| = \frac{\sqrt{n}}{2}$ using the second point in the lemma, we get that $x_p^* \in \{0, 1\}^{n \times 1}$. Finally because $\|x_p^* - \frac{1}{2} \cdot 1_{n \times 1}\| = \frac{\sqrt{n}}{2}$ and $\|C - x_p^*\| = r_s$ we can prove through a simple calculation that $S^T \cdot x_p^* = 0$.

Let us consider the problem:

\[ V_p^* = \arg\max_{x \in p} \|x - C\|^2 \]  

(53)

For $\bar{\rho} < \frac{\beta}{2} < q_s = \rho$ let

\[ U_{p,C}^* = \arg\max_{x \in Q_p} \|x - C\|^2 \]  

(54)

We reason as follows: if $V_p^* \subseteq \{0, 1\}^{n \times 1} \cap \{x|S^T \cdot x = 0\}$ then from Lemma 3.4 point 4, follows immediately that $V_p^* = U_{p,C}^*$ and every point in $U_{p,C}^*$ has that property. In the following we show that if a point has this property we will find it.

In order to solve (54) as described in Theorem 3.3 we form the function

\[ h_{p}(x) = \max_{p \in \{k, l, s, h\}} \|x - C_p\|^2 - r_p^2 \]  

(55)
where obviously $Q_p = \{x|h_p(x) \leq 0\}$. Let us define
\[ \hat{x}^*_{p,c} = \operatorname{argmin}_{h_p(x) \leq 1} h_p(x) - \|x - C\|^2 \] (56)

We assume that $X^*_{p,c}$ has only one element, therefore $X^*_{p,c} = \{x^*_{p,c}\}$. We apply Theorem 3.3 to solve (54). The value of $h_p(x^*_{p,c})$ determines what case will be used in the Theorem 3.3. The most exciting case (and likely to meet) is when $h_p(x^*_{p,c}) < 0$. This is what we treat in the following. All the other cases are actually convex optimization problems.

Remark 5. The assumption that $X^*_{p,c} = \{x^*_{p,c}\}$ actually limits the number of RSSP problems we can solve since this might not be true for every $S$, therefore this algorithm might not be applicable for any $S \in \mathbb{R}^{n \times 1}$.

We analyze the case $h_p(x^*_{p,c}) < 0$. We give a solution for the situation where $\exists B(x^*_{p,c}, \epsilon > 0) \subseteq P \subseteq Q_p$ i.e $x^*_{p,c} \in \text{int}(P)$.

Let us form the family of polytopes for $R \geq 0$
\[ P_{p,R;C} = \{x|h_p(x) - \|x - C\|^2 \leq -R^2\} \] (57)

Remark 6. Please note that the definition of polytope $P_{p,R;C}$ accounts for the point $C \in \mathbb{R}^{n \times 1}$ to which we want to find the furthest in $Q_p$. According to Theorem 3.3 in order to solve (54) we should increase $R$ until $P_{p,R;C} \subseteq Q_p$. The smallest $R$ for which this happens shall be denoted $R^*_{p,c}$
\[ R^*_{p,c} = \inf\{R > 0|P_{p,R;C} \subseteq Q_p\} \] (58)

It follows (from Theorem 3.3) that $U^*_{p,c} = \partial P_{p,(R^*_{p,c})^2;C} \cap \partial Q_p$.

The difficulty here is, off course, asserting if a polytope (a.k.a $P_{p,R;C}$) is included in an intersection of n-disks (a.k.a $Q_p$).

Remark 7. The crucial observation here is: it is easy to assert if $P_{p,R;C} \subseteq P$.

We will continue this section as a sequence of questions and answers which reflect the reasoning process. Our main interest area is obtaining a method for asserting if $P_{p,R;C}$ is included in $Q_p$ for a given $p$ and $R$.

We already know that $P \subseteq Q_p$, so having $P_{p,R;C} \subseteq P$ is sufficient to say that $P_{p,R;C} \subseteq Q_p$. We formulate the following question:

Is $R^*_{p,c}$ the smallest $R$ for which $P_{p,R;C} \subseteq P$?

Since $P \subseteq Q_p \subseteq P_{p,c}$ and $x^*_{p,c} \in P$ follows that exists $\hat{R}^*_{p,c}$ such that $\hat{R}^*_{p,c}$ is the smallest for which $R_{p,R;C} \subseteq P$. It is obvious that $\hat{R}^*_{p,c} \geq R^*_{p,c}$ i.e $P_{p,R;C}$ enters $P$ after it entered $Q_p$, assuming that $R$ increases with time. It is clear that we can compute $\hat{R}^*_{p,c}$. We investigate the relationship between $\hat{R}^*_{p,c}$ and $R^*_{p,c}$. The next question is:

For $R = R^*_{p,c}$ exists $x \in P_{p,R^2;C} \setminus P$?

That is, we ask if by the time $P_{p,R^2;C}$ entered $Q_p$ there are still some points of it outside of $P$. If the answer to the above question is NO then it follows that $\hat{R}^*_{p,c} = R^*_{p,c}$. We shall prove that this is the case if the RSSP has a solution. Let us assume that exists
\[ x_0 \in P_{p,(R^*_{p,c})^2;C} \setminus P \] (59)
and exists $\delta_0 > 0$ such that $B(x_0, \delta_0) \subseteq P_{p,(R^*_{p,c})^2;C} \setminus P$. In order to obtain a contradiction we proceed as follows: starting from $Q_p$ we form another intersection of balls by letting $\hat{\rho} = \alpha \cdot \rho$ for some $\alpha > 1$ large enough such that
the new intersection of balls, \( Q_\beta \), is much “tighter” around \( P \) to the extend that \( x_0 \not\in Q_\beta \). Due to the construction of \( Q_\beta \) this still includes \( P \) and has all the other properties of \( Q_\beta \).

Associated with this new \( \hat{\beta} \) we construct a new point \( \hat{C} \) and we are now considering the problem:

\[
\max_{x \in Q_\hat{\beta}} \| x - \hat{C} \|
\]

(60)

For solving this we resort again to Theorem 3.3. We can define the polytope \( P_{\hat{\beta}, R, \hat{C}} \) and show that for a given \( R > 0 \) exists \( \hat{R} > 0 \) such that \( P_{\hat{\beta}, R, \hat{C}} = P_{\beta, R^*, C} \). This is a crucial result! This will be a consequence of the construction method of \( \hat{C} \). Therefore, in Theorem 3.3 we use the same cases for solving the problem (60) as for the problem (50).

Therefore, associated with the polytope \( P_{\hat{\beta}, R, \hat{C}} \) we obtain \( R^*_{\hat{\beta}, \hat{C}} \) as the smallest value of \( R \) for which \( P_{\hat{\beta}, R, \hat{C}} \subseteq Q_\hat{\beta} \). We will also show that if the RSSP has a solution, then \( P_{\hat{\beta}, (R^*_{\hat{\beta}, \hat{C}})^2, \hat{C}} = P_{\beta, (R^*_{\beta, C})^2, C} \). This finally generates a contradiction to (59) because \( x_0 \not\in Q_\beta \) and from \( P_{\beta, (R^*_{\beta, C})^2, \hat{C}} \subseteq Q_\beta \) follows that \( x_0 \not\in P_{\beta, (R^*_{\beta, C})^2, \hat{C}} \) while it was assumed that \( x_0 \in P_{\beta, (R^*_{\beta, C})^2, \hat{C}} \). Once (59) is refuted, this means that for the cases when the RSSP has a solution one has \( R^*_{\beta, C} = R^*_{\beta, \hat{C}} \).

Finally, in order to check if the RSSP has a solution, while computing \( R^*_{\beta, C} \) we can also obtain a point which corresponds to \( R^*_{\beta, C} \) and check if it is a solution to the RSSP. This is sufficient to assert the existence of solutions to the RSSP using the Remark 1.

So, starting from \( Q_\beta \) we form another intersection of \( n \)-disks as follows:

\[
Q_{\hat{\beta}} = \bigcap_{k=1}^{n} \left( B(\hat{C}_{k-}, \hat{r}_{k-}) \cap B(\hat{C}_{k+}, \hat{r}_{k+}) \right) \cap B(\hat{C}_{\hat{s}}, \hat{r}_{\hat{s}}) \cap B(\hat{C}_{\hat{h}}, \hat{r}_{\hat{h}})
\]

(61)

where for \( \alpha > 1 \) we have:

\[
\begin{align*}
\hat{C}_{k \pm} &= \frac{1}{2} \cdot 1_{n \times 1} \pm \alpha \cdot q_k \cdot e_k \\
\hat{C}_{\hat{s}} &= \frac{1}{2} \cdot 1_{n \times 1} - \alpha \cdot q_s \cdot \frac{S}{\|S\|} \\
\hat{C}_{\hat{h}} &= \frac{1}{2} \cdot 1_{n \times 1} + \alpha \cdot q_h \cdot \frac{1_{n \times 1}}{\|1_{n \times 1}\|}
\end{align*}
\]

(62)

with the constraints on the radii

\[
\begin{align*}
\| \frac{1}{2} \cdot 1_{n \times 1} - \frac{1}{2} \cdot e_k - \hat{C}_{k+} \|^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2 &= \hat{r}_{k+}^2 \\
\| \frac{1}{2} \cdot 1_{n \times 1} + \frac{1}{2} \cdot e_k - \hat{C}_{k-} \|^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2 &= \hat{r}_{k-}^2 \\
\| \hat{C}_{\hat{s}} - P_s \|^2 + \hat{r}_{\hat{s}}^2 &= \hat{r}_{\hat{s}}^2 \\
\| \hat{C}_{\hat{h}} - P_h \|^2 + \hat{r}_{\hat{h}}^2 &= \hat{r}_{\hat{h}}^2
\end{align*}
\]

(63)

with \( \hat{r}_{\hat{s}} \) and \( \hat{r}_{\hat{h}} \) are the same as in (25) and (29) respectively.
Given the fact that $\exists B(x^*_\rho, \epsilon) \subseteq P$ we can apply Lemma 3.2 with $\delta = \min \left\{ \epsilon, \frac{\delta_0}{2} \right\}$ to assure the existence of $\rho_\delta$ such that if $\hat{\rho} \geq \rho_\delta$ then

$$P \subseteq Q_{\hat{\rho}} \subseteq \bigcup_{x \in P} B \left( x, \frac{\delta_0}{2} \right)$$

Please note that we can achieve $\hat{\rho} = \alpha \cdot \rho \geq \rho_\delta$ by increasing $\alpha$ in (62 and 63).

Remark 8. From the equation (59) and (64) one has $x_0 \notin Q_{\hat{\rho}}$

Finally we define inspired from (49) and in accordance to (62) and (63)

$$\hat{C} = \frac{1}{2} \cdot 1_{n \times 1} - \alpha \cdot \frac{\beta}{2} \cdot \frac{S}{\|S\|}$$

Remark 9. Please note that the modifications in $\hat{C}$ are proportional to the modifications $Q_{\hat{\rho}}$, i.e the point is moved further, on the same axis, by a factor of $\alpha > 1$. This will later create desired congruence like properties.

Consider the problem:

$$U_{\hat{\rho}, \hat{C}}^* = \argmax_{x \in Q_{\hat{\rho}}} \| x - \hat{C} \|^2$$

In order to solve this problem, as above, we define:

$$h_{\hat{\rho}}(x) = \max_{p \in \{k_s, k_h\}} \| x - \hat{C}_p \|^2 - \hat{r}_p^2$$

with the obvious remark that $Q_{\hat{\rho}} = \{ x | h_{\hat{\rho}}(x) \leq 0 \}$. Then finally define

$$X_{\hat{\rho}, \hat{C}}^* = \argmin_{h_{\hat{\rho}}(x) \leq 1} h_{\hat{\rho}}(x) - \| x - \hat{C} \|^2$$

The last main contribution of this paper is the following lemma:

Lemma 3.5. For $R \geq R^*$ exists $\hat{R}$ such that

$$P_{\rho, R^2, C} = P_{\hat{\rho}, \hat{R}^2, \hat{C}}$$

where

$$P_{\rho, R^2, C} = \{ x | h_{\rho}(x) - \| x - C \|^2 \leq -R^2 \}$$

Proof. See proof for Lemma C.1 in Appendix.

Using the above Lemma 3.5 it is easy to prove that $X_{\hat{\rho}, \hat{C}}^* \in \text{int}(P) \subseteq Q_{\hat{\rho}}$ because if for some $\hat{R}_{\rho, C} \geq R^*_{\rho, C}$ one has $P_{\rho, \hat{R}^2_{\rho, C}, C} = X_{\hat{\rho}, \hat{C}}^* = \{ x^*_{\rho, C} \} \subseteq P$ then it follows that exists $\hat{R}_{\rho, C}$ such that $P_{\rho, \hat{R}^2_{\rho, C}, C} = P_{\hat{\rho}, \hat{R}^2_{\rho, C}, \hat{C}} = X_{\hat{\rho}, \hat{C}}^* = \{ x^*_{\rho, C} \} \subseteq P$

hence we can define

$$R^*_{\rho, C} = \inf \{ R > 0 \mid P_{\rho, R^2, C} \subseteq Q_{\rho} \}$$

Corollary 3.6. If $\exists x \in P \cap \{ 0, 1 \}^{n \times 1} \cap \{ x | S^T \cdot x = 0 \}$ then

$$P_{\rho, (R^*_{\rho, C})^2, C} = P_{\hat{\rho}, (R^*_{\rho, C})^2, \hat{C}}$$
Proof. If \( \exists x \in P \cap \{0, 1\}^{nx_1} \cap \{x \mid S^T \cdot x = 0\} \) then from Lemma 3.4 applied for \( Q_p \) \( \exists x \in Q_p \cap \{0, 1\}^{nx_1} \cap \{x \mid S^T \cdot x = 0\} \) and follows that \( U^*_{\hat{\rho}, \hat{C}} \subseteq \{0, 1\}^{nx_1} \cap \{x \mid S^T \cdot x = 0\} \). It is also known that \( \forall x \in U^*_{\hat{\rho}, \hat{C}} \) one has \( x \in \partial P_{\hat{\rho}, \hat{R}^*_p, \hat{C}} \) because \( U^*_{\hat{\rho}, \hat{C}} \in \partial Q_p. \) Since for all \( R_1 < R_2 \) one has \( P_{\hat{\rho}, \hat{R}_2^*, \hat{C}} \cap P_{\hat{\rho}, \hat{R}_1^*, \hat{C}} = \emptyset \) (because the level sets are strictly monotone) it is easy to prove that \( P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} = P_{\hat{\rho}, \hat{R}_1^*, \hat{C}}. \)

Indeed, for \( R = \hat{R}_1^* \) let \( \hat{R}_1 \) be given by the Lemma 3.5 such that

\[
P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} = P_{\hat{\rho}, \hat{R}_1^*, \hat{C}}.
\]

It is known that \( \forall x \in \{0, 1\}^{nx_1} \cap \{x \mid S^T \cdot x = 0\} \cap P \) one has \( x \in \partial P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} = \partial P_{\hat{\rho}, \hat{R}_1^*, \hat{C}}. \) Assuming \( \hat{R}_1 < R^*_{\hat{\rho}, \hat{C}} \) follows \( \partial P_{\hat{\rho}, \hat{R}_1^*, \hat{C}} \subseteq \partial P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} \) which is a contradiction with Lemma 3.4 point (4) (applied for the problem (60)) and the definition of \( R^*_{\hat{\rho}, \hat{C}}. \) That is, for this case the level set \( P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} \) is too inside the set \( Q_p. \) Its boundary is no longer touching the boundary of \( Q_p. \)

On the other hand, assuming \( \hat{R}_1 > R^*_{\hat{\rho}, \hat{C}} \), follows that \( P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} \cap P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} = \emptyset \) hence \( x \notin \partial P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} \) which is again a contradiction with Lemma 3.4 point (4) (applied for the problem (60)) and the definition of \( R^*_{\hat{\rho}, \hat{C}}. \) Because for all \( x \in \{0, 1\}^{nx_1} \cap \{x \mid S^T \cdot x = 0\} \cap P \) one should have \( x \in \partial P_{\hat{\rho}, (\hat{R}^*_p)^2, \hat{C}} \)

It follows that \( \hat{R}_1 = R^*_{\hat{\rho}, \hat{C}}. \)

\[\Box\]

3.5 Brief overview on the complexity

Given the RSSP problem, choose \( \rho \geq \hat{\rho} \) and under the hypothesis that in (56) one gets that \( \lambda^*_{\rho, \hat{\rho}, \hat{C}} \) contains a single element which happens to be in the interior of \( P \) then the complexity of the RSSP problem is given by the complexity of finding the smallest \( R \) such that \( P_{\rho, R, \hat{C}} \subseteq P. \) For this one solves the convex optimization problem:

\[
\min_{h_p(x) \leq 1} \ h_p(x) - \|x - C\|^2
\]

(74)

Since we know that for \( R = 0 \) \( P \subseteq Q_p \subseteq P_{\rho, R, \hat{C}} \) and for \( R = \hat{R}_{\rho, \hat{C}} \) one has \( P_{\rho, R, \hat{C}} \subseteq P \) we apply a bisection on \( R. \)

This bisection will allow the finding of \( R^*_{\rho, \hat{C}} \) with the desired precision \( \epsilon > 0 \) in \( O \left( \log \left( \frac{\hat{R}_{\rho, \hat{C}}}{\epsilon} \right) \right) \) steps. Each step requires asserting polytope containment in another polytope. In our case, this can be done by solving \( 2 \cdot n + 2 \) linear programs. This is known to be of polynomial complexity, hence we finally assert that the presented method has complexity \( O \left( \log \left( \frac{\hat{R}_{\rho, \hat{C}}}{\epsilon} \right) \cdot \text{poly}(n) \right) \). Linear programs can be solved exactly only for rational coefficients in \( P \) time, but \( P \) time approximations exist to any desired precision. Also, one may be interested in a bound for the parameter \( \hat{R}_{\rho, \hat{C}}. \) We leave this for a future work. In order to perform the calculations we will assume a theoretical BSS computation device. It is worth noting however, that in case the coefficients of the polytopes involved are integers the Polytope containment in another polytope is solved exactly in polynomial time. In such cases, as a future work one can define a lower bound on epsilon which would guarantee that having the solution to the optimization problem computed with that precision is sufficient to assert the feasibility of the subset problem exactly. We will not pursue that goal in this paper.

4 CONCLUSION AND FUTURE WORK

We provide a solution to a well known and studied problem in a more general case: the subset problem for real numbers. Our approach is to solve a classic optimization problem associated to the subset sum problem, i.e maximization of a quadratic function over a polytope. We rewrite this problem as a maximization of the distance to a fixed point over the polytope, and show that the subset sum problem has a solution if and only
if the maximum distance has a certain easily computable value. Finally, we give an algorithm which requires polynomial space and ends in polynomial time, which in case the subset problem has a solution delivers the correct maximum and a maximizer which is proved to be a solution to the subset sum problem. Otherwise, if the subset sum problem does not have a solution, the algorithm returns something not necessarily relevant. We argue that applying this method one can always obtain a candidate for a solution to the subset sum problem.

**Therefore, if the returned candidate is a solution to the subset sum, then we assert that the subset sum has a solution otherwise we assert that it does not!**

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A INTERSECTION OF LARGE AND SMALL SPHERES

Let us first give a fundamental lemma:

**Lemma A.1.** Let us consider the following $n$-disks
\[
D_1 = \{x \in \mathbb{R}^{n \times 1} \mid \| - q_1 \cdot e_1 - x \| \leq r_1 \},
\]
\[
D_2 = \{x \in \mathbb{R}^{n \times 1} \mid \| - q_2 \cdot e_1 - x \| \leq r_2 \}.
\]
with $q_1 > 0$ and $r_1 > r_2 \geq 0$ such that exists
\[
0 < a^2 = r_1^2 - q_1^2 = r_2^2 - q_2^2
\]
that is the $n$-disks share a common $n-1$ sphere i.e $\{x \mid e_1^T \cdot x = 0, \|x\| = a \}$. Let $\mathcal{H} = \{x \in \mathbb{R}^{n \times 1} \mid e_1^T \cdot x \leq 0 \}$ and $\mathcal{G} = \{x \in \mathbb{R}^{n \times 1} \mid e_1^T \cdot x \geq 0 \} = \mathbb{R}^{n \times 1} \setminus \text{int}(\mathcal{H})$. Then the following inclusions are true
\[
(1) \quad \mathcal{H} \cap D_2 \subseteq \mathcal{H} \cap D_1
\]
\[
(2) \quad \mathcal{G} \cap D_1 \subseteq \mathcal{G} \cap D_2
\]

**Proof.** Let $x = q \cdot e_1 + v$ with $e_1^T \cdot v = 0$ and assume that $x \in \mathcal{H} \cap D_2$, i.e $q \leq 0$ and $\|q \cdot e_1 + v - (-q_2 \cdot e_1)\| \leq r_2$. It follows that $(q + q_2)^2 + \|v\|^2 \leq r_2^2$. We want to check if $x \in D_1$, i.e $\|q \cdot e_1 + v + q_1 \cdot e_1\| \leq r_1$. In $r_1^2$ we only prove that
\[
(q + q_1)^2 + \|v\|^2 \leq (q + q_1)^2 + r_1^2 = (q + q_2)^2 \iff (q + q_1)^2 - (q + q_2)^2 \leq r_1^2 - r_2^2
\]
that is
\[
(q_1 - q_2) \cdot (2 \cdot q + (q_1 + q_2)) = 2 \cdot (q_1 - q_2) \cdot q + q_1^2 - q_2^2 \leq r_1^2 - r_2^2
\]
But from (76) one has $r_1^2 - r_2^2 = q_1^2 - q_2^2$ hence (81) is equivalent to
\[
2 \cdot (q_1 - q_2) \cdot q \leq 0
\]
But since $|q_1| > |q_2|$ and $q_1 > 0$ follows that $q_1 - q_2 > 0$. Finally because $q \leq 0$ (82) is true and so is the claim (77).

The claim in (78) is easily proved in a similar fashion.

Next we have

**Lemma A.2.** Let us consider the following $n$-disks
\[
D_1 = \{x \in \mathbb{R}^{n \times 1} \mid \| - q_1 \cdot e_1 - x \| \leq r_1 \},
\]
\[
D_2 = \{x \in \mathbb{R}^{n \times 1} \mid \| - q_2 \cdot e_1 - x \| \leq r_2 \},
\]
\[
D_3 = \{x \in \mathbb{R}^{n \times 1} \mid \| - q_3 \cdot e_1 - x \| \leq r_3 \}
\]
with $q_1, q_2 > 0$ and $r_1 > r_2 > r_3 \geq 0$ such that exists
\[
0 < a^2 = r_1^2 - q_1^2 = r_2^2 - q_2^2 = r_3^2 - q_3^2
\]
that is the n-disks share a common n-sphere i.e \( \{ x \mid e_i^T \cdot x = 0, \| x \| = a \} \). Let \( \mathcal{H} = \{ x \in \mathbb{R}^{n \times 1} \mid e_i^T \cdot x \leq 0 \} \) and \( \mathcal{G} = \{ x \in \mathbb{R}^{n \times 1} \mid e_i^T \cdot x \geq 0 \} = \mathbb{R}^{n \times 1} \setminus \text{int}(H) \). Then the following inclusions are true

(1)
\[
\mathcal{H} \cap D_3 \subseteq D_1 \cap D_3 \subseteq D_2 \cap D_3 \subseteq D_2
\] (85)

(2)
\[
\mathcal{G} \cap D_1 \subseteq \mathcal{G} \cap D_2 \subseteq \mathcal{G} \cap D_3
\] (86)

**Proof.** For (86) once can successively apply Lemma A.1 claim (78) to obtain the desired result.

For (85) the last inclusion is obvious. For the first inclusion, let \( x \in \mathcal{H} \cap D_3 \). Then we already have from Lemma A.1 that \( x \in \mathcal{H} \cap D_3 \subseteq D_1 \cap D_3 \). For the second inclusion we prove the following:

\[
D_1 \cap D_3 \cap \mathcal{H} \subseteq D_2 \cap D_3 \cap \mathcal{H} \quad \text{and} \quad D_1 \cap D_3 \cap \mathcal{G} \subseteq D_2 \cap D_3 \cap \mathcal{G}
\] (87)

Indeed, in the above equation for the first inclusion let \( x \in D_1 \cap D_3 \cap \mathcal{H} \). We have from Lemma A.1 that \( D_3 \cap \mathcal{H} \subseteq D_2 \cap \mathcal{H} \) hence (since \( x \in D_3 \))

\[
x \in D_1 \cap D_2 \cap \mathcal{H} \cap D_3 \subseteq D_2 \cap D_3 \cap \mathcal{H}
\] (88)

Finally, let \( x \in D_1 \cap D_3 \cap \mathcal{G} \). Applying Lemma A.1 we obtain that \( D_1 \cap \mathcal{G} \subseteq D_2 \cap \mathcal{G} \). It follows

\[
x \in D_3 \cap D_2 \cap \mathcal{G}
\] (89) \[\square\]

**B APPROXIMATING POLYTOPES WITH A FINITE INTERSECTION OF N-DISKS**

**Lemma B.1.** Given \( \delta > 0 \) fixed, if \( \exists B(x, \delta) \subseteq P \) then exists \( \rho_\delta > 0 \) such that for all \( \rho \geq \rho_\delta \) one has

\[
P \subseteq Q_\rho \subseteq \bigcup_{x \in P} B(x, \delta)
\] (90)

**Proof.** The first inclusion is true for \( \rho \geq \rho_\delta \) from Remark 4. For the second inclusion one has the following. From Figure 1 one has

\[
x = r_1 - q_1 = \sqrt{q_1^2 + a^2} - q_1 = \frac{a^2}{q_1 + \sqrt{q_1^2 + a^2}}
\] (91)

For a fixed \( a \), letting \( q_1 \) be large enough one can see that \( x \leq \delta \) will eventually occur. Therefore for any point \( u \) in the intersection of the large disk with the right half-space exists a point \( v \) in the intersection of the hyper-plane with the small disk such that \( u \in B(v, \delta) \).

Consider w.l.o.g the set \( \mathcal{H}_v = \{ x \mid S^T \cdot x \leq 0 \} \) i.e a facet of \( P \). Since \( C_{\rho_\delta} \subseteq \hat{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{\frac{v}{2}} \right) \) we want to prove that

\[
\hat{B}(C_v, r_v) \cap \hat{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{v}}{2} \right) \subseteq \bigcup_{x \in \mathcal{H}_v \cap \hat{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{\frac{v}{2}} \right)} B(x, \delta)
\] (92)

Regarding the above equation, it is easy to see that

\[
\mathcal{H}_v \cap \hat{B}(C_v, r_v) \cap \hat{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{v}}{2} \right) \subseteq \mathcal{H}_v \cap \hat{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{\frac{v}{2}} \right)
\]

\[
\subseteq \bigcup_{x \in \mathcal{H}_v \cap \hat{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \sqrt{\frac{v}{2}} \right)} B(x, \delta)
\] (93)
Therefore we will now focus only on the elements of
\[ G_s \cap \bar{B}(C_s, r_s) \cap \bar{B}\left(\frac{1}{2} \cdot 1_{n, 1, \frac{\sqrt{n}}{2}}\right) \] (94)
where \( G_s = \{ x \mid S^T \cdot x \geq 0 \} \). For this in Figure 1, take the large disk \( \bar{B}(C_s, r_s) \) and the small disk \( \bar{B}(P_s, \tilde{r}_s) \). Here consider \( \|C_s - P_s\| = q_1 \) and \( a = \tilde{r}_s \). Then, as shown, given \( \delta > 0 \) exists \( q_\delta \) such that for any \( q_1 \geq q_\delta \) one has
\[ \bar{B}(C_s, r_s) \cap G_s \subseteq \bigcup_{x \in H_s \cap G_s \cap \bar{B}(P_s, \tilde{r}_s)} B(x, \delta) \] (95)
However, please note that due to construction one has
\[ H_s \cap G_s \cap \bar{B}(P_s, \tilde{r}_s) = H_s \cap G_s \cap \bar{B}\left(\frac{1}{2} \cdot 1_{n, 1, \frac{\sqrt{n}}{2}}\right) \subseteq H_s \cap \bar{B}\left(\frac{1}{2} \cdot 1_{n, 1, \frac{\sqrt{n}}{2}}\right) \] (96)
hence from (95) and (96) it is obtained:
\[ G_s \cap \bar{B}(C_s, r_s) \cap \bar{B}\left(\frac{1}{2} \cdot 1_{n, 1, \frac{\sqrt{n}}{2}}\right) \subseteq G_s \cap \bar{B}(C_s, r_s) \subseteq \bigcup_{x \in H_s \cap G_s \cap \bar{B}\left(\frac{1}{2} \cdot 1_{n, 1, \frac{\sqrt{n}}{2}}\right)} B(x, \delta) \subseteq \bigcup_{x \in H_s \cap \bar{B}\left(\frac{1}{2} \cdot 1_{n, 1, \frac{\sqrt{n}}{2}}\right)} B(x, \delta) \] (97)
Finally, since \( \|P_s - \frac{1}{2} \cdot 1_{n, 1}\| \) is fixed, form the existence of \( q_\delta \) the existence of \( \rho_\delta \) easily follows. \( \square \)
C Equivalent Polytopes

Lemma C.1. For \( R \geq R^* \) and \( \hat{\rho}, \rho \geq \hat{\rho} \) exists \( \hat{R} \) such that

\[
P_{R^*} = \hat{P}_{\hat{R}^*}
\]

where

\[
P_{R^*} = \{ x | h_p(x) - \| x - C \|^2 \leq -R^2 \}
\]

\[
\hat{P}_{\hat{R}^*} = \{ x | h_{\hat{\rho}}(x) - \| x - \hat{C} \|^2 \leq -\hat{R}^2 \}
\]

with \( C, \hat{C} \) being given by (49) and (65) respectively and \( h, h_{\hat{\rho}} \) being given by

\[
h_p = \max_{p \in \{k, s, h\}} \| x - C_p \|^2 \quad h_{\hat{\rho}} = \max_{p \in \{k, s, h\}} \| x - \hat{C}_p \|^2
\]

while \( C_p \) and \( \hat{C}_p \) are given by (20, 24, 28) and (62) respectively for all \( p \in \{k, s, h\} \) and \( k \in \{1, \ldots, n\} \). The value of \( \hat{\rho} \) is taken from Remark 4.

Proof. Let \( R \geq R^* \) and consider the inequalities:

\[
\| x - C_{k+} \|^2 - r_{k+}^2 - \| x - C \|^2 + R^2 \leq 0
\]

\[
\| x - \hat{C}_{k+} \|^2 - \hat{r}_{k+}^2 - \| x - \hat{C} \|^2 + \hat{R}^2 \leq 0
\]

We wonder if exists \( \hat{R} \) such that the above inequalities are actually only one for all \( k \). We have for the first inequality

\[
\left\| x - \left( \frac{1}{2} \cdot 1_{n \times 1} + q_k \cdot e_k \right) \right\|^2 - \left\| x - \left( \frac{1}{2} \cdot 1_{n \times 1} - \frac{\beta}{2} \cdot \frac{S}{\| S \|} \right) \right\|^2 - r_{k+}^2 + R^2 \leq 0
\]

which becomes the linear inequality

\[
\left( \left( x - \left( \frac{1}{2} \cdot 1_{n \times 1} + q_k \cdot e_k \right) \right) - \left( x - \left( \frac{1}{2} \cdot 1_{n \times 1} - \frac{\beta}{2} \cdot \frac{S}{\| S \|} \right) \right) \right)^T \cdot \left( \left( x - \left( \frac{1}{2} \cdot 1_{n \times 1} + q_k \cdot e_k \right) \right) + \left( x - \left( \frac{1}{2} \cdot 1_{n \times 1} - \frac{\beta}{2} \cdot \frac{S}{\| S \|} \right) \right) \right) - r_{k+}^2 + R^2 \leq 0
\]

then

\[
\left( -q_k \cdot e_k - \frac{\beta}{2} \cdot \frac{S}{\| S \|} \right)^T \cdot \left( 2 \cdot x - 1_{n \times 1} - q_k \cdot e_k + \frac{\beta}{2} \cdot \frac{S}{\| S \|} \right) - r_{k+}^2 + R^2 \leq 0
\]

After similar calculations, the second inequality in (101) becomes

\[
\left( -\alpha \cdot q_k \cdot e_k - \frac{\alpha \cdot \beta}{2} \cdot \frac{S}{\| S \|} \right)^T \cdot \left( 2 \cdot x - 1_{n \times 1} - \alpha \cdot q_k \cdot e_k + \frac{\alpha \cdot \beta}{2} \cdot \frac{S}{\| S \|} \right) - \hat{r}_{k+}^2 + \hat{R}^2 \leq 0
\]

We would want to prove that exists \( \hat{R} \) not depending on \( k \) such that the inequality in (104) multiplied by \( \alpha \) becomes the inequality from (105).

About \( r_{k+} \) and \( \hat{r}_{k+} \) we know from (21) and (63) that

\[
\left\| \frac{1}{2} \cdot 1_{n \times 1} - \frac{1}{2} \cdot e_k - \left( \frac{1}{2} \cdot 1_{n \times 1} + q_k \cdot e_k \right) \right\|^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2 = r_{k+}^2
\]

\[
\left\| \frac{1}{2} \cdot 1_{n \times 1} - \frac{1}{2} \cdot e_k - \left( \frac{1}{2} \cdot 1_{n \times 1} + \alpha \cdot q_k \cdot e_k \right) \right\|^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2 = \hat{r}_{k+}^2
\]
The first equality in (106) becomes:
\[
\left( \frac{1}{2} + q_k \right)^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2 = r_{k+}^2
\]
while the second one becomes:
\[
r_{k+}^2 = \left( \frac{1}{2} + \alpha \cdot q_k \right)^2 + \left( \frac{\sqrt{n} - 1}{2} \right)^2
\]

We write (105) as follows
\[
\alpha \cdot 
\left( -q_k \cdot e_k - \frac{\beta}{2} \cdot \frac{S}{||S||} \right)^T \cdot 
\left( 2 \cdot x - 1_{n \times 1} - q_k \cdot e_k + \frac{\beta}{2} \cdot \frac{S}{||S||} \right) - \alpha \cdot (r_{k+}^2 - R^2) +
\]
\[
\alpha \cdot 
\left( -q_k \cdot e_k - \frac{\beta}{2} \cdot \frac{S}{||S||} \right)^T \cdot \left( \alpha - 1 \right) \cdot 
\left( -q_k \cdot e_k + \frac{\beta}{2} \cdot \frac{S}{||S||} \right) + \alpha \cdot (r_{k+}^2 - R^2) +
\]
\[
- \hat{r}_{k+}^2 + \hat{R}^2 \leq 0
\]

Let
\[
\theta_{k+} =
\]
\[
\alpha \cdot 
\left( -q_k \cdot e_k - \frac{\beta}{2} \cdot \frac{S}{||S||} \right)^T \cdot \left( \alpha - 1 \right) \cdot 
\left( -q_k \cdot e_k + \frac{\beta}{2} \cdot \frac{S}{||S||} \right) + \alpha \cdot (r_{k+}^2 - R^2) - \hat{r}_{k+}^2 + \hat{R}^2
\]

We want to prove that exists \( \hat{R} \) not depending on \( k \) such that \( \theta_{k+} = 0 \). Indeed, we rewrite \( \theta_{k+} \) as follows
\[
\alpha \cdot (\alpha - 1) \cdot 
\left( q_k^2 - \frac{\beta^2}{4} \right) + \alpha \cdot r_{k+}^2 - \hat{r}_{k+}^2 - \alpha \cdot R^2 + \hat{R}^2
\]

Finally we show that \( \alpha \cdot (\alpha - 1) \cdot q_k^2 + \alpha \cdot r_{k+}^2 - \hat{r}_{k+}^2 \) does not depend on \( k \). Using (107) and (108) we obtain
\[
\alpha \cdot r_{k+}^2 - \hat{r}_{k+}^2 = \alpha \cdot 
\left( \frac{1}{2} + q_k \right)^2 + \frac{n - 1}{4} - 
\left( \frac{1}{2} + \alpha \cdot q_k \right)^2 + \frac{n - 1}{4}
\]
\[
= \alpha \cdot \left( \frac{n}{4} + q_k^2 \right) - \left( \frac{n}{4} + \alpha \cdot q_k + \alpha^2 \cdot q_k^2 \right)
\]
\[
= (\alpha - 1) \cdot \frac{n}{4} + \alpha \cdot (1 - \alpha) \cdot q_k^2
\]

hence
\[
\theta_{k+} = (\alpha - 1) \cdot \frac{n}{4} + \alpha \cdot (1 - \alpha) \cdot \frac{-\beta^2}{4} - \alpha \cdot R^2 + \hat{R}^2
\]

We will not make the calculations for what would be \( \theta_{k-} \) and simply assume (because of the high similarity) that \( \theta_{k-} = \theta_{k+} \).

Let us now consider the inequalities
\[
||x - C_k||^2 - r_k^2 - ||x - C||^2 + R^2 \leq 0
\]
\[
||x - \hat{C}_k||^2 - r_{k+}^2 - ||x - \hat{C}||^2 + \hat{R}^2 \leq 0
\]

The first of them becomes:
\[
\left\| x - \left( \frac{1}{2} \cdot 1_{n \times 1} + q_k \cdot \frac{1_{n \times 1}}{||1_{n \times 1}||} \right) \right\|^2 - \left\| x - \left( \frac{1}{2} \cdot 1_{n \times 1} - \frac{\beta}{2} \cdot \frac{S}{||S||} \right) \right\|^2 - r_{k+}^2 + R^2 \leq 0
\]

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We rewrite (117) as follows
\[
\left( -\frac{\beta}{2} \frac{S}{||S||} - q_h \cdot \frac{1_{nx1}}{1_{nx1}} \right)^T \cdot \left( 2 \cdot x - 1_{nx1} - \frac{1_{nx1}}{||1_{nx1}||} \right) - r_h^2 + R^2 \leq 0 \tag{116}
\]

Similarly, the second inequality becomes:
\[
\alpha \cdot \left( -\frac{\beta}{2} \frac{S}{||S||} - q_h \cdot \frac{1_{nx1}}{1_{nx1}} \right)^T \cdot \left( 2 \cdot x - 1_{nx1} - \alpha q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} + \frac{\alpha \beta}{2} \frac{S}{||S||} \right) - \hat{r}_h^2 + \hat{R}^2 \leq 0 \tag{117}
\]

From (30) and (??) we obtain:
\[
||C_h - P_h||^2 + \hat{r}_h^2 = r_h^2 = \left\| \frac{1}{2} \cdot 1_{nx1} + q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} - P_h \right\|^2 + \hat{r}_h^2
\]
\[
||\hat{C}_h - P_h||^2 + \hat{r}_h^2 = \hat{r}_h^2 = \left\| \frac{1}{2} \cdot 1_{nx1} + \alpha \cdot q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} - P_h \right\|^2 + \hat{r}_h^2 \tag{118}
\]

We rewrite (117) as follows
\[
\alpha \cdot \left( -\frac{\beta}{2} \frac{S}{||S||} - q_h \cdot \frac{1_{nx1}}{1_{nx1}} \right)^T \cdot \left( 2 \cdot x - 1_{nx1} - q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} + \beta \frac{S}{||S||} \right) + \alpha (-\hat{r}_h^2 + \hat{R}^2) + \alpha \cdot \left( -\frac{\beta}{2} \frac{S}{||S||} - q_h \cdot \frac{1_{nx1}}{1_{nx1}} \right)^T \cdot (\alpha - 1) \left( -q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} + \beta \frac{S}{||S||} \right) - \hat{r}_h^2 + \hat{R}^2 \leq 0 \tag{119}
\]

Let us denote
\[
\theta_h = \alpha \cdot (\alpha - 1) \cdot \left( q_h^2 - \frac{\beta^2}{4} \right) + \alpha \cdot \hat{r}_h^2 + \hat{r}_h^2 - \alpha \cdot R^2 + \hat{R}^2 \tag{120}
\]

Here we evaluate:
\[
\alpha \cdot \hat{r}_h^2 - \hat{r}_h^2 = \alpha \cdot \left( \left\| \frac{1}{2} \cdot 1_{nx1} + q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} - P_h \right\|^2 + \hat{r}_h^2 \right) - \left( \left\| \frac{1}{2} \cdot 1_{nx1} + \alpha \cdot q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} - P_h \right\|^2 + \hat{r}_h^2 \right)
\]
\[
= \alpha \cdot \left( \left\| \frac{1}{2} \cdot 1_{nx1} + q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} - P_h \right\|^2 + \hat{r}_h^2 \right) - \left( \left\| \frac{1}{2} \cdot 1_{nx1} - P_h \right\|^2 + \hat{r}_h^2 + \alpha \cdot q_h \cdot \frac{1_{nx1}}{||1_{nx1}||} \right)^2 + \hat{r}_h^2
\]
\[
= \alpha \cdot \left( \left\| \frac{1}{2} \cdot 1_{nx1} - P_h \right\|^2 + \hat{r}_h^2 + q_h^2 + 2 \cdot q_h \cdot \left( \frac{1}{2} \cdot 1_{nx1} - P_h \right)^T \cdot \frac{1_{nx1}}{||1_{nx1}||} \right) - \left( \left\| \frac{1}{2} \cdot 1_{nx1} - P_h \right\|^2 + \hat{r}_h^2 + \alpha \cdot q_h^2 + 2 \cdot \alpha \cdot q_h \cdot \left( \frac{1}{2} \cdot 1_{nx1} - P_h \right)^T \cdot \frac{1_{nx1}}{||1_{nx1}||} \right) \tag{121}
\]
But from (29) one has \( \frac{d}{dt} = \frac{1}{2} \cdot 1_{n \times 1} - P_h \) hence

\[
\alpha \cdot r_h^2 - \hat{r}_h^2 = (\alpha - 1) \cdot \frac{n}{4} + \alpha \cdot q_h^2 - \alpha^2 \cdot q_h^2
\]

and

\[
\theta_h = \alpha \cdot (\alpha - 1) \cdot \frac{-\beta^2}{4} + (\alpha - 1) \cdot \frac{n}{4} - \alpha \cdot R^2 + \hat{R}^2
\]

Finally, we consider the inequalities:

\[
\|x - C_x\|^2 - r_s^2 - \|x - C\|^2 + R^2 \leq 0
\]

\[
\|x - \hat{C}_x\|^2 - \hat{r}_s^2 - \|x - \hat{C}\|^2 + \hat{R}^2 \leq 0
\]

The first inequality becomes:

\[
\left\| x - \left( \frac{1}{2} \cdot 1_{n \times 1} - q_s \cdot \frac{S}{\|S\|} \right) \right\|^2 - \left\| x - \left( \frac{1}{2} \cdot 1_{n \times 1} - \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right) \right\|^2 r_s^2 + R^2 \leq 0
\]

then

\[
\left( q_s \frac{S}{\|S\|} - \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right)^T \cdot \left( 2 \cdot x - 1_{n \times 1} + q_s \frac{S}{\|S\|} + \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right) - r_s^2 + R^2 \leq 0
\]

and similarly the second inequality becomes:

\[
\alpha \cdot \left( q_s \frac{S}{\|S\|} - \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right)^T \cdot \left( 2 \cdot x - 1_{n \times 1} \right) + \alpha q_s \frac{S}{\|S\|} - \alpha \cdot \frac{\beta S}{2 \|S\|} - r_s^2 + \hat{R}^2 \leq 0
\]

From (26) and (63) we obtain:

\[
\|C_x - P_s\|^2 + \hat{r}_s^2 = r_s^2 = \left\| \frac{1}{2} \cdot 1_{n \times 1} - q_s \cdot \frac{S}{\|S\|} - P_s \right\|^2 + \hat{r}_s^2
\]

\[
\|\hat{C}_x - P_s\|^2 + \hat{r}_s^2 = r_s^2 = \left\| \frac{1}{2} \cdot 1_{n \times 1} - \alpha \cdot q_s \cdot \frac{S}{\|S\|} - P_s \right\|^2 + \hat{r}_s^2
\]

As we did above with the other so we do here. We focus on (127) and rewrite it as

\[
\alpha \cdot \left( q_s \frac{S}{\|S\|} - \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right)^T \cdot \left( 2 \cdot x - 1_{n \times 1} \right) + q_s \frac{S}{\|S\|} + \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right) - \alpha \cdot (-r_s^2 + R^2) +
\]

\[
\alpha \cdot \left( q_s \frac{S}{\|S\|} - \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right)^T \cdot \left( \alpha - 1 \right) \cdot \left( q_s \frac{S}{\|S\|} + \frac{\beta}{2} \cdot \frac{S}{\|S\|} \right) - \alpha \cdot (-r_s^2 + R^2) -
\]

\[
- \hat{r}_s^2 + \hat{R}^2 \leq 0
\]

Let

\[
\theta_s = \alpha \cdot (\alpha - 1) \cdot \left( q_s^2 - \frac{\beta^2}{4} \right) + \alpha \cdot r_s^2 - \hat{r}_s^2 - \alpha \cdot R^2 + \hat{R}^2
\]
and we evaluate
\[
\alpha \cdot r_s^2 - \hat{r}_s^2 = \alpha \cdot \left( \left\| \frac{1}{2} \cdot 1_{n \times 1} - q_s \cdot \frac{S}{\|S\|} - P_s \right\|^2 + \hat{r}_s^2 \right) - \\
\left( \left\| \frac{1}{2} \cdot 1_{n \times 1} - \alpha \cdot q_s \cdot \frac{S}{\|S\|} - P_s \right\|^2 + \hat{r}_s^2 \right)
\]
\[
= \alpha \cdot \left( \left\| \frac{1}{2} \cdot 1_{n \times 1} - P_s \right\|^2 - q_s \cdot \frac{S}{\|S\|} \right) - \\
\left( \left\| \frac{1}{2} \cdot 1_{n \times 1} - P_s \right\|^2 - \alpha \cdot q_s \cdot \frac{S}{\|S\|} \right) - \\
\left( \alpha \cdot \left( \frac{1}{2} \cdot 1_{n \times 1} - P_s \right)^T \cdot \frac{S}{\|S\|} \right) - \\
\left( \alpha \cdot \left( \frac{1}{2} \cdot 1_{n \times 1} - P_s \right)^T \cdot \frac{S}{\|S\|} \right)
\]
But from (25) one has \( \frac{n}{4} = \left\| \frac{1}{2} \cdot 1_{n \times 1} - P_s \right\|^2 + \hat{r}_s^2 \) hence
\[
\alpha \cdot r_s^2 - \hat{r}_s^2 = (\alpha - 1) \cdot \frac{n}{4} + \alpha \cdot q_s^2 - \alpha^2 \cdot q_s^2
\] (132)
and
\[
\theta_s = \alpha \cdot (\alpha - 1) \cdot \frac{-b^2}{4} + (\alpha - 1) \cdot \frac{n}{4} - \alpha \cdot R^2 + \hat{R}^2
\] (133)
For \( R = R^* \), from (113), (123) and (133) in order to have \( \theta_{k+} = \theta_{k-} = \theta_{h} = \theta_{s} = 0 \) we take
\[
\hat{R}^2 = \alpha \cdot R^2 + \alpha \cdot (\alpha - 1) \cdot \frac{b^2}{4} - (\alpha - 1) \cdot \frac{n}{4}
\] (134)
\]