Team Decision Problems with Convex Quadratic Constraints
AtherGattami

Abstract—In this paper, we consider linear quadratic team problems with an arbitrary number of quadratic constraints in both stochastic and deterministic settings. The team consists of players with different measurements about the state of nature. The objective of the team is to minimize a quadratic cost subject to additional finite number of quadratic constraints. We first consider the problem of countably infinite number of players in the team for a bounded state of nature with a Gaussian distribution and show that linear decisions are optimal. Then, we consider the problem of team decision problems with additional convex quadratic constraints and show that linear decisions are optimal for both the finite and infinite number of players in the team. For the finite player case, the optimal linear decisions can be found by solving a semidefinite program. Finally, we consider the problem of minimizing a quadratic objective for the worst case scenario, subject to an arbitrary number of deterministic quadratic constraints. We show that linear decisions are optimal and can be found by solving a semidefinite program. Finally, we apply the developed theory on dynamic team decision problems in linear quadratic settings.

Index Terms—Team Decision Theory, Stochastic, Deterministic, Game Theory, Quadratic Constraints, Convex Functional Optimization.

I. INTRODUCTION

We consider the problem of distributed decision making with information constraints and quadratic constraints under linear quadratic settings. For instance, information constraints appear naturally when making decisions over networks. Quadratic constraints appear due to the power limited controllers in practice for instance. These problems can be formulated as team problems. The team problem is an optimization problem with several decision makers possessing different information aiming to optimize a common objective. Early results in [1] considered static team theory in stochastic settings and a more general framework was introduced by Radner [2], where existence and uniqueness of solutions where shown. Generalization to dynamic team problems for control purposes where introduced in [3]. In [4], the deterministic team problem with two team members was solved. The solution can’t be easily extended to more than two players since it uses the fact that the two members have common information; a property that doesn’t necessarily hold for more than two players. Also, a nonlinear team problem with two team members was considered in [5], where one of the team members is assumed to have full information whereas the other member has only access to partial information about the state of the world. Related team problems with exponential cost criterion were considered in [6]. Optimizing team problems with respect to affine decisions in a minimax quadratic cost was shown to be equivalent to stochastic team problems with exponential cost, see [7]. The connection is not clear when the optimization is carried out over nonlinear decision functions. The deterministic version (minimizing the worst case scenario) of the linear quadratic team decision problem was solved in [8]. The problem of countably infinite number of players under the power semi-norm was solved in [9] under certain assumptions.

In this paper, we will consider both Gaussian and deterministic settings (worst case scenario) for team decision problems under additional quadratic constraints. It’s well-known that additional constraints, although convex, could give rise to complex optimization problems if the optimized variables are functions (as opposed to real numbers). For instance, linear functions (that is functions of the form \( \mu(x) = Kx \) where \( K \) is a real matrix) are no longer optimal. We will illustrate this fact by the following example.

Example 1: For \( x \in \mathbb{R} \), we want to minimize the objective function

\[
|u|^2
\]

subject to

\[
|x - u|^2 \leq \gamma
\]

Some Hilbert space theory shows that the optimal \( u \) is given by

\[
u = \mu(x) = (|x| - \sqrt{\gamma})x/|x| \quad \text{if} \quad |x|^2 > \gamma,\]

and

\[
u = \mu(x) = 0 \quad \text{otherwise.}\]

Obviously, the optimal \( u \) is a nonlinear function of \( x \).

Increasing the dimension of \( x \), and adding constraints on the structure of \( u \), for instance \( x \in \mathbb{R}^N \) and \( u = \mu(x) = (\mu(x_1), \ldots, \mu(x_N)) \), certainly makes the constrained optimization more complicated. The example above shows that, in spite of having a convex optimization carried out over a Hilbert space, the optimal decision function is nonlinear. However, we show in the upcoming sections that problems multiple quadratic constraints behave nicely when considering the expected values of the objectives in the Gaussian case, in the sense that linear decisions are optimal. We also extend the results to the case of infinite number of players in the team. For the deterministic counterpart which is not an optimization problem over a Hilbert space, we show that linear decisions are optimal and we show how to find the linear optimal decisions by semidefinite programming. Finally, we apply the developed theory on dynamic team decision problems in linear quadratic settings.
II. NOTATION

The following table gives a list of the notation we are going to use throughout the paper:

| $\mathbb{R}_+$ | The set of nonnegative real numbers. |
| $\mathbb{N}$ | The set of positive integers. |
| $S_n^+$ | The set of $n \times n$ symmetric positive semidefinite matrices. |
| $S_n^{++}$ | The set of $n \times n$ symmetric positive definite matrices. |
| $\mathcal{M}$ | The set of measurable functions. |
| $\mathcal{C}$ | The set of functions $\mu : \mathbb{R}^p \rightarrow \mathbb{R}^m$ with $\mu(y) = (\mu_1(y_1), \mu_2(y_2), \ldots, \mu_N(y_N))^*$, $\mu_i : \mathbb{R}^p \rightarrow \mathbb{R}^{m_i}$, $\sum m_i = m$, $\sum p_i = p$. |
| $A_i$ | The element of $A$ in row position $i$. |
| $\succeq$ | $A \succeq B$ if $A - B \in S_n^+$. |
| $\succ$ | $A \succ B$ if $A - B \in S_n^{++}$. |
| $K$ | $K = \{K | K = \text{diag}(K_1, \ldots, K_N), K_i \in \mathbb{R}^{m_i \times m_i}\}$. |
| $\Tr$ | $\Tr(A)$ is the trace of the matrix $A$. |
| $\mathcal{N}(m, X)$ | The set of Gaussian variables with mean $m$ and covariance $X$. |

III. LINEAR QUADRATIC GAUSSIAN TEAM THEORY

In this section, we will review a classical result in stochastic team theory for a finite number of decision variables and present an extension to the case of infinite number of decision variables.

In the static team decision problem, one would like to solve

$$\begin{align*}
\min_{\mu} \quad & \mathbb{E} \left[ \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right] \\
\text{subject to} \quad & y_i = C_i x + v_i \\
& u_i = \mu_i(y_i) \\
& \text{for } i = 1, \ldots, N.
\end{align*}$$

(1)

Here, $x$ and $v$ are independent Gaussian variables taking values in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, with $x \sim \mathcal{N}(0, V_{xx})$ and $v \sim \mathcal{N}(0, V_{vv})$. Also, $y_i$ and $u_i$ will be stochastic variables taking values in $\mathbb{R}^n$, $\mathbb{R}^{m_i}$, respectively, and $p_1 + \ldots + p_N = p$. We assume that

$$\begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \in \mathbb{S}^{m+n},$$

(2)

and $Q_{uu} \in S_n^{++}$, $m = m_1 + \ldots + m_N$.

If full state information about $x$ is available to each decision maker $u_i$, the minimizing $u$ can be found easily by completion of squares. It is given by $u = L x$, where $L$ is the solution to

$$Q_{uu} L = -Q_{ux}.$$

Then, the cost function in (1) can be rewritten as

$$J(x, u) = \mathbb{E}\{x^T (Q_{xx} - L^T Q_{uu} L) x\} + \mathbb{E}\{(u - Lx)^T Q_{uu} (u - Lx)\}.$$  

(3)

Minimizing the cost function $J(x, u)$, is equivalent to minimizing

$$\mathbb{E}\{(u - Lx)^T Q_{uu} (u - Lx)\},$$

since nothing can be done about $\mathbb{E}\{x^T (Q_{xx} - L^T Q_{uu} L) x\}$ (the cost when $u$ has full information).

The next result is due to Radner [2], showing that linear decision are optimal for the finite-dimensional static team problem

**Proposition 1 (Radner):** Let $x$ and $v_i$ be Gaussian variables with zero mean, taking values in $\mathbb{R}^n$ and $\mathbb{R}^{p_i}$, respectively, with $p_1 + \ldots + p_N = p$. Also, let $u_i$ be a stochastic variable taking values in $\mathbb{R}^{m_i}$, $Q_{uu} \in S_n^{++}$, $m = m_1 + \ldots + m_N$, $L \in \mathbb{R}^{m \times n}$, $C_i \in \mathbb{R}^{p_i \times n}$, for $i = 1, \ldots, N$. Then, the optimal decision $\mu$ to the optimization problem

$$\begin{align*}
\min_{\mu} \quad & \mathbb{E}\{(u - Lx)^* Q_{uu} (u - Lx)\} \\
\text{subject to} \quad & y_i = C_i x + v_i \\
& u_i = \mu_i(y_i) \\
& \text{for } i = 1, \ldots, N.
\end{align*}$$

(4)

is unique and linear in $y$.

**Proof:** For a proof, see [2].

It’s not clear how to extend the result above to the case of infinite number of state and decision variables, that is $N = \infty$. This is an important case to approach dynamic team problems, where the decision variables are over space and time, and the time horizon that goes to infinity.

The next theorem establishes a generalization of the above proposition for the infinite-dimensional case.

**Theorem 1:** Let $x = (x_1^* x_2^* \ldots)^*$ and $v = (v_1^* v_2^* \ldots)^*$ be infinite-dimensional vectors where $x_i$ is Gaussian taking values in $\mathbb{R}^{m_i}$ and $v_i$ is Gaussian taking values in $\mathbb{R}^{p_i}$. Suppose that $\mathbb{E}\{x^* x\}, \mathbb{E}\{v^* v\} < \infty$. Also, let $u = (u_1^* \ u_2^* \ldots)^*$ be a random infinite-dimensional vector with $u_i$ taking values in $\mathbb{R}^{m_i} \ \mathbb{R}^{p_i}$ an infinite dimensional, bounded, positive definite, self-adjoint linear operator, $L$ a bounded linear operator, and $C_i \in \mathbb{R}^{p_i \times n}$ a bounded linear operator, for all $i \in \mathbb{N}$. Then, the optimal decision $\mu$ to the optimization problem

$$\begin{align*}
\min_{\mu} \quad & \mathbb{E}\{(u - Lx)^* Q_{uu} (u - Lx)\} \\
\text{subject to} \quad & y_i = C_i x + v_i \\
& u_i = \mu_i(y_i) \\
& \text{for } i \in \mathbb{N}.
\end{align*}$$

(5)

is unique and linear in $y$.

**Proof:** Note first that $y_i$ is bounded since $C_i$, $x$, and $v_i$ are bounded.

Let $Z$ be the linear space of functions such that $z \in Z$ if $z_i$ is a linear transformation of $y_i$, that is $z_i = A_i y_i$ for some real matrix $A_i \in \mathbb{R}^{m_i \times p_i}$. Since $Q_{uu}$ is a bounded symmetric positive definite linear operator, $Z$ is a linear space under the inner product

$$\langle g, h \rangle = \mathbb{E}\{g^* Q_{uu} h\},$$

and norm

$$\|g\|^2 = \mathbb{E}\{g^* Q_{uu} g\}.$$

The optimization problem in (5) where we search for the linear optimal decision can be written as

$$\min_{u \in Z} \|u - Lx\|^2$$

(6)
Finding the best linear optimal decision \( u_* \in \mathcal{Z} \) to the above problem is equivalent to finding the shortest distance from the subspace \( \mathcal{Z} \) to the element \( Lx \) (\( Lx \) is bounded since \( L \) and \( x \) are bounded), where the minimizing \( u_* \) is the projection of \( Lx \) on \( \mathcal{Z} \), and hence unique. Also, since \( u_* \) is the projection, we have

\[
0 = \langle u_* - Lx, u \rangle = E\{ (u_* - Lx)^* Q_{uu} u \},
\]

for all \( u \in \mathcal{Z} \). In particular, for \( f_i = (0, 0, \ldots, z_i, 0, 0, \ldots) \in \mathcal{Z} \), we have

\[
E\{ (u_* - Lx)^* Q_{uu} f_i \} = E\{ (u_* - Lx)^* Q_{uu} z_i \} = 0.
\]

The Gaussian assumption implies that

\[
[u_* - Lx]^* Q_{uu} \text{ is independent of } z_i = A_i y_i, \text{ for all linear transformations } A_i.
\]

This gives in turn that \( [u_* - Lx]^* Q_{uu} \) is independent of \( y_i \). Hence, for any decision \( \mu \in \mathcal{M} \cap \mathcal{C} \), linear or nonlinear, we have that

\[
E(\mu(y) - Lx)^* Q_{uu}(\mu(y) - Lx) = E(\mu(y) - Lx)^* Q_{uu}(\mu(y) - Lx)
\]

\[
= E(u_* - Lx)^* Q_{uu}(u_* - Lx)
\]

\[
+ 2E(\mu(y) - Lx)^* Q_{uu}(\mu(y) - Lx)
\]

\[
\geq E(u_* - Lx)^* Q_{uu}(u_* - Lx)
\]

with equality if and only if \( \mu(y) = u_* \). This concludes the proof.

IV. TEAM DECISION PROBLEMS WITH POWER CONSTRAINTS

In practice, we often have power constraints on the control variables of the form \( \gamma_i \geq E|\mu_i(y_i)|^2 \). The question is whether linear decisions are optimal and if there is a practical algorithm that can obtain optimal decisions. The introductory example clearly showed that linear decisions are not optimal for the case of point-wise optimization with a power constraint. Thus, there is no reason to expect that linear decisions are optimal for the stochastic (average) case. This will be addressed in this section.

Consider the modified version of the optimization problem [1]:

\[
\min_{\mu} E \left[ x^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right]
\]

subject to \( y_i = C_i x \)

\( u_i = \mu_i(y_i) \)

\( \gamma_i \geq E|\mu_i(y_i)|^2 \)

for \( i = 1, \ldots, N \).

The difference from Radner’s original formulation is that we have added power constraints to the decision functions, \( \gamma_i \geq E|\mu_i(y_i)|^2 \). Note that additional constraints in functional optimization could give rise to complex nonlinear optimal solution as was shown in Example [1] in the introduction.

In the sequel, we will prove a more general theorem, where we consider power constraints on a set of quadratic forms in both the state \( x \) and the decision function \( \mu \).

**Theorem 2:** Let \( x \) be a Gaussian variable with zero mean and given covariance matrix \( \Sigma \), taking values in \( \mathbb{R}^n \). Also, let \( Q \in \mathbb{S}_+^{m+n} \), \( R_0 \in \mathbb{S}_+^n \), \( \left[ \begin{array}{cc} Q_j & S_j \\
 & R_j \end{array} \right] \in \mathbb{S}_+^{m+n} \) and \( R_j \in \mathbb{S}_+^n \), for \( j = 1, \ldots, M \). Assume that the optimization problem

\[
\min_{\mu \in \mathcal{C}} E \left[ \begin{bmatrix} x \\ \mu(x) \end{bmatrix}^* \begin{bmatrix} Q_0 & S_0 \\
S_0^* & R_0 \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \right] + \sum_{j=1}^M \lambda_j \left( E \left[ \begin{bmatrix} x \\ \mu(x) \end{bmatrix}^* \begin{bmatrix} Q_j & S_j \\
 & R_j \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \right] \right) \leq \gamma_j
\]

is feasible. Then, linear decisions \( \mu \) given by \( \mu(x) = K(X)x \), with \( K(X) \in \mathbb{K} \), are optimal.

**Proof:** Consider the expression

\[
E \left[ \begin{bmatrix} x \\ \mu(x) \end{bmatrix}^* \begin{bmatrix} Q_0 & S_0 \\
S_0^* & R_0 \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \right] + \sum_{j=1}^M \lambda_j \left( E \left[ \begin{bmatrix} x \\ \mu(x) \end{bmatrix}^* \begin{bmatrix} Q_j & S_j \\
 & R_j \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \right] \right)
\]

Take the expectation of a quadratic form with index \( j \) to be larger than \( \gamma_j \). Then, \( \lambda_j \to \infty \) makes the value of the expression above infinite. On the other hand, if the expectation of a quadratic form with index \( j \) is smaller than \( \gamma_j \), then the maximizer \( \lambda_j \) is optimal for \( \gamma_j = 0 \).

Now let \( p_* \) be the optimal value of the optimization problem [8], and consider the objective function

\[
\begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} Q_0 & S_0 \\
S_0^* & R_0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x^* (Q_0 - S_0 R_0^{-1} S_0^*) x \\ (u + R_0^{-1} S_0 x)^* R_0 (u + R_0^{-1} S_0 x) \end{bmatrix}\]

We have that \( Q_0 - S_0 R_0^{-1} S_0^* \geq 0 \), since it’s the Schur complement of \( R_0 \) in the positive semi-definite matrix \( \begin{bmatrix} Q_0 & S_0 \\
S_0^* & R_0 \end{bmatrix} \). Since \( R_0 \succ 0 \), a necessary condition for the objective function to be zero is that \( u = -R_0^{-1} S_0^* x \), and so \( u \) must be linear (in order for \( u \) to have the structure given by \( C, R_0^{-1} S_0^* \) must be in \( \mathbb{K} \), to satisfy the information constraints).
Now assume that $p_\star > 0$. We have

\[
\begin{align*}
p_\star &= \min_{\mu \in \mathbb{C}} \max_{\lambda_j \in \mathbb{R}_+} \mathbb{E} \left[ x \mu(x) \right]^* \begin{bmatrix} Q_0 & S_0 & R_0 \end{bmatrix} \begin{bmatrix} x \mu(x) \end{bmatrix} \\
&\quad + \sum_{j=1}^M \lambda_j \left( \mathbb{E} \begin{bmatrix} x \mu(x) \end{bmatrix} \begin{bmatrix} Q_j^* & S_j & R_j \end{bmatrix} \begin{bmatrix} x \mu(x) \end{bmatrix} \right) - \gamma_j \\
&= \min_{\mu \in \mathbb{C}} \max_{\lambda_j \in \mathbb{R}_+} \mathbb{E} \left[ x \mu(x) \right]^* \left( \begin{bmatrix} Q_0^* & S_0 & R_0 \end{bmatrix} + \sum_{j=1}^M \lambda_j \begin{bmatrix} Q_j^* & S_j & R_j \end{bmatrix} \right) \begin{bmatrix} x \mu(x) \end{bmatrix} \\
&\quad - \sum_{j=1}^M \lambda_j \gamma_j.
\end{align*}
\]

Now introduce $\lambda_0$ and the matrix

\[
\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} = \sum_{j=0}^M \lambda_j \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix},
\]

and consider the minimax problem

\[
\begin{align*}
p_0 &= \min_{\mu \in \mathbb{C}} \max_{\lambda_j \geq 0} \sum_{j=0}^M \lambda_j \mathbb{E} \left[ x \mu(x) \right]^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \mu(x) \end{bmatrix} \\
&\quad - \sum_{j=1}^M \lambda_j \gamma_j.
\end{align*}
\]

Note that a maximizing $\lambda_0$ must be positive, since $\lambda_0 = 0$ implies that $p_0 \leq 0$, while $\lambda_0 > 0$ gives $p_0 > 0$. We can always recover the optimal solutions of (9) from that of (10) by dividing all variables by $\lambda_0$, that is $p_\star = p_0/\lambda_0$, $\lambda_j \mapsto \lambda_j/\lambda_0$, and $\mu \mapsto \mu/\lambda_0$. Now we have the obvious inequality

\[
\min \max \{ \cdot \} \geq \max \min \{ \cdot \}
\]

where the equality is obtained by applying Proposition 2 in the Appendix, the second inequality follows from the fact that the set of linear decisions $K \mathbb{C}$, $K \in \mathbb{R}$, is a subset of $\mathbb{C}$, and the second equality follows from the definition of $p_0$. Hence, linear decisions are optimal, and the proof is complete.

**Remark:** Although Theorem 1 is stated and proved for $y = x$ and $u = \mu(y) = \mu(x)$, it extends easily to the case $y = Cx$ for any matrix $C$, which is often the case in applications. Note also that we may set $N = \infty$ and the result would still hold by using Theorem 2 in the proof.

V. **Computation of the Optimal Team Decisions**

The optimization problem that we would like to solve when assuming linear decisions is

\[
\begin{align*}
\min_{\gamma_0, K \in \mathbb{K}} \gamma_0 \\
\text{subject to } & \mathbb{E} \left[ x_{KC} \right] \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x_{KC} \end{bmatrix} \leq \gamma_j, \\
& j = 0, \ldots, M, \\
& x \sim \mathcal{N}(0, H^2).
\end{align*}
\]

where $K$ is unique. Thus,
with $H \succeq 0$. Note that we can write the constraints as

$$
E \left[ x^T KCx \right] \succeq 0
$$

subject to

$$
\begin{align*}
0 & \leq P_j - HQ_j H - HS_j KCH - HC^* K^* S_j^* H - HC^* K^* R_j^* KCH \\
& \leq P_j - HQ_j H - HS_j KCH - HC^* K^* S_j^* H - HC^* K^* R_j^* KCH.
\end{align*}
$$

Hence, our optimization problem to be solved is given by

$$
\begin{align*}
\min_{\gamma_j, K \in \mathbb{R}} \quad & \gamma_0 \\
\text{subject to} \quad & \text{Tr} P_j \leq \gamma_j \\
& 0 \leq P_j - HQ_j H - HS_j KCH - HC^* K^* S_j^* H - HC^* K^* R_j^* KCH.
\end{align*}
$$

VI. Deterministic Team Problems with Quadratic Constraints

We considered the problem of static stochastic team decision in the previous sections. This section treats an analogous version for the deterministic (or worst case) problem. For the dynamic setting with partially nested information (which will be discussed in the next section), this corresponds to the $H_\infty$ control problem. Although the problem formulation is very similar, the ideas of the solution are considerably different, and in a sense more difficult.

The deterministic problem considered is a quadratic game between a team of players and nature. Each player has limited information that could be different from the other players in the team. This game is formulated as a minimax problem, where the team is the minimizer and nature is the maximizer.

Consider the following team decision problem

$$
\begin{align*}
\min_{\mu} & \quad J(x, u) \\
\text{subject to} & \quad y_i = C_i x \\
& \quad u_i = \mu_i(y_i) \\
& \quad i = 1, \ldots, N
\end{align*}
$$

where $u_i \in \mathbb{R}^m$, $m = m_1 + \cdots + m_N$, $C_i \in \mathbb{R}^{p \times n}$.

$J(x, u)$ is a quadratic cost given by

$$
J(x, u) = \sup_{x \leq 1} x^* \left[ Q_{xx} \quad Q_{xu} \right] \left[ \begin{array}{c} x \\ u \end{array} \right],
$$

where

$$
\left[ Q_{xx} \quad Q_{xu} \quad Q_{uu} \right] \in \mathbb{R}^{m+n}.
$$

We will be interested in the case $Q_{uu} \succeq 0$. The players $u_1, \ldots, u_N$ make up a team, which plays against nature represented by the vector $x$, using $\mu \in \mathcal{S}$, that is

$$
\mu(Cx) = \left[ \begin{array}{c} \mu_1(C_1 x) \\ \vdots \\ \mu_N(C_N x) \end{array} \right].
$$

Now consider the team problem (16) and note that an equivalent condition for the existence of a decision function $\mu_* \in \mathcal{C}$ that achieves the value of the game $\gamma_*$ is that

$$
\begin{align*}
\min_{\mu_*(Cx)} \quad & \left[ \begin{array}{c} x \\ \mu_*(Cx) \end{array} \right] \left[ \begin{array}{cc} Q & S \\ S^* & R \end{array} \right] \left[ \begin{array}{c} x \\ \mu_*(Cx) \end{array} \right] \leq \gamma_* |x|^2
\end{align*}
$$

for all $x$. This is equivalent to

$$
\begin{align*}
\min_{\mu_*(Cx)} \quad & \left[ \begin{array}{c} x \\ \mu_*(Cx) \end{array} \right] \left[ \begin{array}{cc} Q - \gamma_* & S \\ S^* & R \end{array} \right] \left[ \begin{array}{c} x \\ \mu_*(Cx) \end{array} \right] \leq 0
\end{align*}
$$

for all $x$. This is an example of a quadratic constraint. We could also have a set of quadratic constraints that have to be mutually satisfied. For instance, in addition to the minimization of the worst case quadratic cost, we could have constraints on the induced norms of the decision functions

$$
\frac{1}{|x|^2} \leq \gamma_i \quad \text{for all } x \neq 0, \quad i = 1, \ldots, M,
$$

or equivalently given by the quadratic inequalities

$$
\begin{align*}
\end{align*}
$$
\[
|\mu_i(C(x))|^2 - \gamma_i|\mu|^2 \leq 0 \quad \text{for all } x, \quad i = 1, \ldots, M.
\]

Also, the team members could share a common power source, and the power is proportional to the squared norm of the decisions \(\mu_i\):
\[
\sum_{i=1}^{M} |\mu_i(C(x))|^2 - c|x|^2 \leq 0 \quad \text{for all } x,
\]
for some positive real number \(c\).

It's not clear whether linear decisions are optimal, since the example given at the introduction indicates that, in deterministic settings, nonlinear decisions are optimal pointwise.

**Theorem 4:** Let
\[
\begin{bmatrix}
Q_j & S_j \\
S_j^* & R_j
\end{bmatrix} \in \mathbb{S}^{m+n}
\]
for \(j = 0, 1, \ldots, M,\) \(R_0 \in \mathbb{S}^{m+m}, R_j \in \mathbb{S}^n_m\) for \(j = 1, \ldots, M.\)

Suppose that there exists a decision function \(\mu \in \mathcal{C}\) such that
\[
\sup_{x \in \mathbb{R}^n} x^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \leq 0, \quad j = 0, \ldots, M.
\]

Then, there exists a linear decision \(\mu(x) = Kx, K \in \mathbb{K}\), such that (17) is satisfied.

**Proof:** Suppose there exists a decision function \(\mu \in \mathcal{C}\) such that (17) is satisfied. Then, for any Gaussian variable \(x \sim \mathcal{N}(0, X)\), we have that
\[
\mathbf{E} \left[ x^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \right] \leq 0.
\]
Equivalently, for a given \(x \sim \mathcal{N}(0, X)\), the optimal value \(s\) of the optimization problem
\[
\min_{\mu \in \mathcal{C}} \mathbf{E} \left[ x^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \right],
\]
subject to \(\mathbf{E} \left[ x^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \\ \mu(x) \end{bmatrix} \right] \leq 0 \quad (18)\)

\(j = 1, \ldots, M\)

must be nonpositive, \(s \leq 0.\) But Theorem 2 gives that the decision function \(\mu(x) = K(X)x\) is optimal, with \(K(X) \in \mathbb{K}.\)

Since \(\mathbf{E}xx^* = X\), we get the inequalities
\[
0 \geq \mathbf{E} \left[ x^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \\ K(X)x \end{bmatrix} \right] = \mathbf{E} \left[ x^* K(X)^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} I \\ K(X) \end{bmatrix} \right] = \mathbf{Tr} \left[ x^* K(X)^* \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} I \\ K(X) \end{bmatrix} \right] X \quad (19)
\]
for all \(j.\) Now let \(\lambda_i \geq 0, i = 0, \ldots, M,\) and
\[
\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} = \sum_{j=0}^{M} \lambda_j \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix}.
\]

Introduce the set
\[
\mathcal{X} = \{ X : X \succeq 0, \mathbf{Tr} X = 1 \}.
\]
The fact that for every covariance matrix \(X\) there is a matrix \(K(X)\) such that (19) holds implies
\[
\max_{\lambda_i \geq 0, X \in \mathcal{X}} \min_{K \in \mathbb{K}} \mathbf{Tr} \left[ \begin{bmatrix} I \\ K \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \right] X \leq 0.
\]
For every fixed \(X\), we have a max-min problem which is convex in \(K\) and linear in \(\lambda_i\), so we can switch the order of the minimization and maximization to get the max-min-max inequality
\[
\max_{X \in \mathcal{X}} \min_{K \in \mathbb{K}} \min_{\lambda_i \geq 0} \mathbf{Tr} \left[ \begin{bmatrix} I \\ K \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \right] X \leq 0.
\]

which is equivalent to the existence of a matrix \(K \in \mathbb{K}\) such that
\[
\begin{bmatrix} I \\ K \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \leq 0, \quad (20)
\]
for every set of \(\lambda_i \geq 0, i = 0, \ldots, M.\) This implies that there must exist a matrix \(K \in \mathbb{K}\) such that
\[
\begin{bmatrix} I \\ K \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \leq 0, \quad (20)
\]
for all \(j\). Finally, (20) implies that
\[
x^* \begin{bmatrix} I \\ K \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} x \leq 0 \quad \text{for all } x, \quad j = 0, \ldots, M, \quad (21)
\]
and the proof is complete.

**Theorem 5:** Let \(C_i \in \mathbb{R}^{p_i \times n}\), for \(i = 1, \ldots, N.\) Let
\[
\begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \in \mathbb{S}^{m+n}\quad \text{for } j = 0, \ldots, M,\) and \(R_j \in \mathbb{S}^n_m\) for \(j = 1, \ldots, M.\) Then, the set of quadratic matrix inequalities
\[
\begin{bmatrix} x \\ KCx \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \\ KCx \end{bmatrix} \leq 0 \quad \forall x, \quad j = 0, \ldots, M, \quad (22)
\]
equivalent to
\[
Q_j + S_jKC + C^*K^*S_j^* + C^*K^*R_jC \leq 0, \quad i = 0, \ldots, M. \quad (23)
\]

**Proof:** We have the following chain of inequalities:
\[
\begin{bmatrix} x \\ KCx \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} x \\ KCx \end{bmatrix} \leq 0
\]
\[
\begin{bmatrix} I \\ KC \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \begin{bmatrix} I \\ KC \end{bmatrix} \leq 0
\]
\[
Q_j + S_jKC + C^*K^*S_j^* + C^*K^*R_jC \leq 0
\]
\[
A = \begin{bmatrix} Q_j + S_jKC + C^*K^*S_j^* & C^*K^*R_j \\ R_jKC & -R_j \end{bmatrix} \leq 0,
\]
Complement of between four systems. The arrow from node 2 to node 1 indicates that system 1 affects the dynamics of system 2 directly.

The observation of system *i* at time *k* is given by

\[ y_i(k) = C_i x_i(k), \]

where

\[ C_i = \begin{bmatrix} C_{i1} & 0 & 0 & 0 \\ 0 & C_{i2} & 0 & 0 \\ 0 & 0 & C_{i3} & 0 \\ 0 & 0 & 0 & C_{i4} \end{bmatrix}. \] (26)

Here, *C*<sub>ij</sub> = 0 if system *i* does not have access to *y*<sub>j</sub>(*k*). The subsystems could exchange information about their outputs. Every subsystem receives the information with some time delay, that is reflected by the interconnection structure. Let \( \mathbb{I}_k^j \) denote the set of observations \( y_j(n) \) and control signals \( u_j(n) \) available to node *j* up to time *k*, \( n \leq k, j = 1, ..., N \).

Consider the following (general) dynamic team decision problem with additional quadratic constraints:

\[
\begin{align*}
\inf_{\mu} & \ J(u, w) \\
\text{subject to} & \ x(k+1) = Ax(k) + Bu(k) + w(k) \\
& \ y_i(k) = C_i x_i(k), \\
& \ u_i(k) = \mu_i : \mathbb{I}_k^i \rightarrow \mathbb{R}^{p_i} \\
& \text{for } i = 1, ..., N \\
& \sum_{k=0}^{T-1} \mathbb{E} \left[ x(k) \right]^T Q_j \left[ x(k) \right] \leq \gamma_j \\
& \text{for } j = 1, ..., M
\end{align*}
\]

where

\[
J(u, w) = \mathbb{E} x^*(T) Q_f x^*(T) \\
+ \sum_{k=0}^{T-1} \mathbb{E} \left[ x(k) \right]^T Q \left[ x(k) \right] u(k). \]

Now write \( x(k) \) and \( y(k) \) as

\[
\begin{align*}
x(k) &= A^t x(k-t) + \sum_{n=0}^{t-1} A^n B u(k-n-1) \\
&+ \sum_{n=0}^{t-1} A^n w(k-n-1), \\
y_i(k) &= C_i A^t x(k-t) + \sum_{n=0}^{t-1} C_i A^n B u(k-n-1) \\
&+ \sum_{n=0}^{t-1} C_i A^n w(k-n-1).
\end{align*}
\]

Note that the summation over *n* is defined to be zero when *t* = 0. The next result is an extension of [3] for the case of optimal control with additional quadratic constraints where it presents a condition on the information structure for which a dynamic problem can be transformed to a static team problem. The condition is known as the partially nested information structure.
Let given by equation (27) reduces to the following static team problem in the form (30) if
\[
y_j(k) \in I_i^k \Rightarrow u_j(k-n-1) \in I_i^k \text{ for } [C_i A^n B]_{ij} \neq 0
\]
for all \( n \) such that \( 0 \leq n < t \), \( t = 0, 1, 2, \ldots \). In particular, the optimal solution to the optimization problem given by (27) is linear in the observations \( I^t_i \) if condition (30) is satisfied.

**Proof:** Introduce
\[
\bar{x} = \begin{bmatrix} w(0) \\ w(1) \\ \vdots \end{bmatrix}, \quad \bar{u}_i = \begin{bmatrix} u_i(0) \\ u_i(1) \\ \vdots \end{bmatrix},
\]
Then, we can write the cost function \( J(x, u) \) as
\[
E \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right]^* Q \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right].
\]
for some symmetric positive definite matrix \( Q \). Similarly, the quadratic constraints
\[
\sum_{k=0}^{T-1} E \left[ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right]^* Q_j \left[ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right] \leq \gamma_j
\]
may be written as
\[
E \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right]^* Q_j \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right] \leq \gamma_j.
\]
for some symmetric positive definite matrices \( Q_j \), \( j = 1, \ldots, M \). Consider the expansion given by (29). The problem here is that \( y_i(k) \) depends on previous values of the control signals \( u(n) \) for \( n = 0, \ldots, k-1 \). The components \( u_j(k-n-1) \) that \( y_i(k) \) depends on are completely determined by the structure of the matrix \( [C_i A^n B]_{ij} \).

Now if condition (30) is satisfied, node \( i \) has the information of \( u_j(k-n-1) \) available at time \( k \) if the element \( [C_i A^n B]_{ij} \neq 0 \). Then, node \( i \) can form the new output measurement
\[
\tilde{y}_i(k) = y_i(k) - \sum_{n=0}^{k-1} C_i A^n B u(k-n-1)
\]
\[
= A^k x(0) + \sum_{n=0}^{k-1} C_i A^n w(k-n-1).
\]
Let
\[
\tilde{y}_i(k) = \begin{bmatrix} \tilde{y}_i(0) \\ \tilde{y}_i(1) \\ \vdots \end{bmatrix}.
\]
With these new variables introduced, the optimization problem given by equation (27) reduces to the following static team decision problem:
\[
\inf_{\mu} \ E \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right]^* Q \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right]
\]
subject to \( u_j(k) = \mu_j(y_i(k)) \)
for \( i = 1, 2, \ldots \) (32)
\[
E \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right]^* Q_j \left[ \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right] \leq \gamma_j
\]
for \( j = 1, 2, \ldots, M \)
and the optimal solution \( \bar{u} \) is linear according to Theorem 2. This completes the proof.

**VIII. CONCLUSIONS**

We have studied multi-objective linear quadratic optimization of team decisions in both stochastic and deterministic settings. Constrained decision problems tend to have nonlinear optimal solutions. We have shown that for the Gaussian and worst case scenario settings, respectively, linear decisions are in fact optimal, and we can find the respective linear optimal solutions by solving a semidefinite program. We also showed that linear decision are optimal when the number of players in the time is infinite. Future work will consider an S-procedure sort of a result, where we want to find decision function \( \mu \) such that the inequality \( J_0(\mu(x), x) \leq 0 \) is satisfied if \( J_j(\mu(x), x) \leq 0 \), where \( J_0, J_j \) are some quadratic forms in \( \mu \) and \( x \). However, this is a much harder problem since the search for linear a function \( \mu(x) \) is not a convex problem, and it’s not clear if it can be convexified.

**IX. ACKNOWLEDGEMENTS**

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APPENDIX

Let \( J = J(u, w) \) be a functional defined on a product vector space \( U \times W \), to be minimized by \( u \in U \subset U \) and maximized by \( w \in W \subset W \), where \( U \) and \( W \) are the constrained sets. This defines a zero-sum game, with kernel \( J \), in connection with which we can introduce two values, the upper value

\[
\bar{J} := \inf_{u \in U} \sup_{w \in W} J(u, w),
\]

and the lower value

\[
\underline{J} := \sup_{w \in W} \inf_{u \in U} J(u, w).
\]

Obviously, we have the inequality \( \bar{J} \geq \underline{J} \). If \( \bar{J} = \underline{J} = J^* \), then \( J^* \) is called the value of the zero-sum game. Furthermore, if there exists a pair \( (u^*, w^* \in U, w^* \in W) \) such that

\[
J(u^*, w^*) = J^*,
\]

then the pair \( (u^*, w^*) \) is called a (pure-strategy) saddle-point solution. In this case, we say that the game admits a saddle-point (in pure strategies). Such a saddle-point solution will equivalently satisfy the so-called pair of saddle-point inequalities:

\[
J(u^*, w) \leq J(u^*, w^*) \leq J(u, w^*), \quad \forall u \in U, \forall w \in W.
\]

**Proposition 2:** Consider a two-person zero-sum game on convex finite dimensional action sets \( U_1 \times U_2 \), defined by the continuous kernel \( J(u_1, u_2) \). Suppose that \( J(u_1, u_2) \) is strictly convex in \( u_1 \) and strictly concave in \( u_2 \). Suppose that either

\( i \) \( U_1 \) and \( U_2 \) are closed and bounded, or

\( ii \) \( U_i \subseteq \mathbb{R}^{m_i}, i = 1, 2, \) and \( J(u_1, u_2) \to \infty \) as \( \|u_1\| \to \infty \), and \( J(u_1, u_2) \to -\infty \) as \( \|u_2\| \to \infty \).

Then, the game admits a unique pure-strategy saddle-point equilibrium.

**Proof:** See [11], pp. 177.