Manifest supersymmetry and the ADHM construction of instantons

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Abstract
We present the (0, 4) superspace version of Witten’s sigma model construction for ADHM instantons. We use the harmonic superspace formalism, which exploits the three complex structures common to both (0, 4) supersymmetry and self-dual Yang-Mills theory. A novel feature of the superspace formulation is the manifest interplay between the ADHM construction and its twistor counterpart.
1 Introduction

In a recent very interesting paper [1] Witten constructed a $(0, 4)$ supersymmetric linear sigma model in two dimensions with a potential term with surprising properties. Under certain assumptions about the structure of the multiplets involved, the Yukawa couplings of the model were found to satisfy the ADHM equations of the instantons construction [2]. In the infrared regime the massless chiral fermions in the model turned out to be coupled to the instanton gauge field, obtained exactly in accord with the ADHM prescription. The fact that $(0, 4)$ supersymmetry implies self-duality of the target-space gauge fields is not new (see, for instance, [3]). However, it is a very unexpected feature of Witten’s model that it requires couplings to instantons, i.e., self-dual gauge fields with finite action.

Non-linear $(0, 4)$ sigma models with potentials had been discussed before (see, e.g., [4],[5]) and the restrictions on the potential imposed by supersymmetry had been found. However, this had been done in the traditional way of [6]. There one starts with one supersymmetry and then tries to add three more. This approach breaks the intrinsic automorphism $SO(4) \sim SU(2) \times SU(2)'$ of $(0, 4)$ supersymmetry to $SO(3)$. Witten carried out the analysis in a fully $SO(4)$ covariant way, which greatly facilitated not only finding the conditions on the potential but also solving them. In addition, Witten required that the potential be invariant under one of the two $SU(2)$ automorphism groups. All of this allowed him to discover the remarkable relation between this type of sigma models and the ADHM construction.

In [1] the sigma model was given in terms of component fields and on shell, i.e., without manifest $(0, 4)$ supersymmetry. In this paper we present the superspace version of Witten’s model. The natural setting for $N = 4$ off-shell supersymmetry is $SU(2)$ harmonic superspace [7]. On the one hand, it turns out that one of the multiplets used by Witten (the chiral fermion one) can, in general, only exist off shell with an infinite set of auxiliary fields. Harmonic superfields are just the right objects of this type. On the other hand, the harmonic approach is best adapted to exploit the three complex structures common for self-dual Yang-Mills (SDYM) fields and $(0, 4)$ supersymmetry. In this way we find a natural bridge between the ADHM construction for instantons and its twistor counterpart given by Ward [8].

In section 2 we first recall some basic facts about $SU(2)$ harmonics and show how they are used [2] to reformulate the twistor approach of Ward [8] to SDYM theory. We also give a brief summary of the ADHM construction in a notation adapted to field theory. In section 3 we give the harmonic superspace description of the relevant $(0, 4)$ off-shell multiplets, namely, the scalar, chiral fermion and “twisted” scalar multiplet and construct their coupling of potential type. It turns out that the “twisted” scalar multiplet of [1] is described by an abelian gauge odd $(0, 4)$ superfield (its gauge invariance is an artifact of supersymmetry and is not related to the Yang-Mills group). It is precisely this gauge invariance that severely restricts the possible couplings and leads to the ADHM conditions. The component version of the “twisted” multiplet is obtained in a Wess-Zumino (not manifestly supersymmetric) gauge for this superfield. However, there also exists a manifestly supersymmetric gauge. In it part of the chiral fermion superfields are gauged away, another part are absorbed by the “twisted” multiplet superfields, which become massive. The remaining massless chiral fermion superfields interact with a target-space
SDYM field, which naturally arises in its twistor (harmonic) form. Thus the superspace sigma model allows us to see the relation between the ADHM construction and its twistor counterpart. It also makes it possible to generalize the linear sigma model to non-linear ones, i.e., to switch on a hyper-Kähler background, again given in its twistor version.

In Appendix A we explain why the chiral fermion multiplet requires, in general, infinitely many auxiliary fields off shell. In Appendix B we show how the (0, 4) superfields used in this paper can be obtained from $\mathcal{N} = 2, D = 4$ ones by trivial dimensional reduction to (4, 4) supersymmetry in two dimensions and further truncation down to (0, 4). The set of $N = 2, D = 4$ harmonic superfields is very simple, it just consists of matter (hypermultiplet) and abelian gauge ones [7].

2 Self-duality, (0, 4) supersymmetry and $SU(2)$ harmonics

In this section we shall recall the basic concepts of the method of harmonic space [7]. We shall do this in two different contexts, that of SDYM theory and of (0, 4) supersymmetry. One of our tasks will be to convince the reader that the harmonic method allows one to exhibit very clearly the remarkable similarity between these two theories. The basic reason for this similarity is the existence of three complex structures in both cases.

2.1 Self-dual Yang-Mills theory

Let us start with self-duality. SDYM fields can be defined in a Euclidean space of dimension $4k$, $R^{4k}$. This space has, in general, $SO(4k)$ as a symmetry group. In the context of self-duality it is convenient to reduce this symmetry to $Sp(1) \times Sp(k)$ and to introduce coordinates $X^{AY}$. Here $A = 1, 2$ is an $SU(2) \sim Sp(1)$ index and $Y = 1, \ldots, 2k$ is an $Sp(k)$ index. The coordinates are real in the sense $X^{AY} = \epsilon_{AB}X^{BZ}Y$, where $\epsilon$ are the antisymmetric constant tensors of the two symplectic groups.

The Yang-Mills fields for some internal symmetry group are introduced via the covariant derivatives $\nabla_{AY} = \partial_{AY} + A_{AY}(X)$. Their commutator defines the field strength tensor, which is decomposed under $Sp(1) \times Sp(k)$ as follows:

$$[\nabla_{AY}, \nabla_{BZ}] = \epsilon_{AB}F_{AY} + F_{(AB)[YZ]} .$$

Here $F_{(AB)[YZ]}$ is symmetric in $A, B$ and antisymmetric in $Y, Z$. Self-duality means to impose the condition

$$F_{(AB)[YZ]} = 0 .$$

These first-order equations imply the standard Yang-Mills equations $\nabla^{AY}F_{AYBZ} = 0$, but unlike in the latter case, one is able to find exact solutions to eqs. [2].

\[\text{We use the following convention to raise/lower symplectic indices: } X_A = \epsilon_{AB}X_B; X^A = \epsilon^{AB}X_B; \epsilon^{AB}\epsilon_{BC} = \delta^A_C. \text{ We always assume summation over repeated indices.}\]
2.2 Harmonic variables

The way to the exact solutions of the self-duality equations \((2)\) goes, in one way or another, through finding an interpretation of \((2)\) as an integrability condition. We shall do that in the harmonic framework \([9]\). We must point out that the harmonic treatment of self-duality as integrability is a modification of the twistor one given by Ward \([8]\). We believe that the harmonic approach might be somewhat closer to physicists with a field-theoretical background.

The harmonic variables are defined as \(SU(2)\) matrices:

\[ u^\pm A \in SU(2) : \quad u^+ A u^- = 1, \quad u^- A = u^+ A. \] (3)

In what follows we shall not use any particular parametrization of \(SU(2)\), but instead we shall use the matrices \((3)\) themselves as (constrained) variables. One can write down three differential operators compatible with the definition \((3)\):

\[ D^{++} = u^+ A \frac{\partial}{\partial u^- A}, \quad D^{--} = D^{++}, \quad D^0 = u^+ A \frac{\partial}{\partial u^+ A} - u^- A \frac{\partial}{\partial u^- A}. \] (4)

The third one \(D^0\) clearly counts the \(U(1)\) charge \(\pm\) of the variables \(u^\pm A\). We shall define harmonic functions as eigenfunctions of this charge operator,

\[ D^0 f^q(u) = q f^q(u), \quad q = 0, \pm 1, \pm 2, \ldots, \] (5)

that have a harmonic expansion

\[ f^q(u) = \sum_{n=0}^{\infty} f^{A_1 \ldots A_{n+q} B_1 \ldots B_n} u^+_{(A_1} \ldots u^+_{A_{n+q}} u^-_{B_1} \ldots u^-_{B_n}) \] (6)

(for \(q \geq 0\); for \(q < 0\) the expression is analogous). In fact, this is nothing but the spherical harmonic expansion of a (square integrable) function (or tensor, if \(q \neq 0\)) on the sphere \(S^2 \sim SU(2)/U(1)\). The coset \(SU(2)/U(1)\) is obtained via the homogeneity condition \((3)\).

We want to emphasize that this way of describing the sphere (as opposed to using, e.g., a complex parametrization) is global, we don’t have to worry about the analytic properties of our functions in different coordinate patches.

We should mention that for even values of the \(U(1)\) charge \(q\) the harmonic functions \((3)\) can be made real with respect to the following special conjugation:

\[ f^{AB \ldots} = f^{AB \ldots}, \quad u^\pm A = u^\pm A, \quad u^- A = -u^+ A. \] (7)

In fact this is just a combination of usual complex conjugation and of the antipodal map on \(S^2\), realized in terms of \(u^\pm\) (see \([9]\) for details). This conjugation will systematically be used below.

To complete our short introduction into the harmonic calculus, here are two formulas which we shall use in what follows. A harmonic integral is defined by the simple rule

\[ \int du \ 1 = 1, \quad \int du \ u^+_{(A_1} \ldots u^+_{A_p} u^-_{B_1} \ldots u^-_{B_q)} = 0 \text{ for } p \text{ and/or } q > 0. \] (8)
In other words, the harmonic integral projects out the singlet part in the expansion of the integrand. The above integration rule is compatible with integration by parts for the derivatives (4). In addition, one can prove the following identity

\[ D_{1}^{++} \frac{1}{u_{1}^{+} u_{2}^{+}} = \delta^{+-}(u_{1}, u_{2}) \, , \]  

where \( u_{1}^{+} u_{2}^{+} \equiv u_{1}^{+A} u_{2}^{+A} \) and \( \delta^{+-}(u_{1}, u_{2}) \) is a harmonic delta function. \(^2\)

Now let us come back to the self-duality condition (2). Multiplying eq. (1) by \( u_{1}^{+A} u_{2}^{+B} \) and introducing the notation

\[ \nabla_{Y}^{+} = u_{1}^{+A} \nabla_{AY} \, , \]  

we can rewrite (2) as follows

\[ [\nabla_{Y}^{+}, \nabla_{Z}^{+}] = 0 \, . \]  

Note also that from the definition (10) and from the properties of the harmonic derivative \( D_{1}^{++} \) immediately follows that

\[ [D_{1}^{++}, \nabla_{Y}^{+}] = 0 \, . \]  

The important point is that the two new conditions (11) and (12) are in fact equivalent to the original self-duality one (2). Indeed, let the connection \( A_{Y}^{+}(x, u) \) in \( \nabla_{Y}^{+} \) be a harmonic function satisfying (12). Inspecting the harmonic expansion (6), we easily see that the solution of (12) must have the form (10). Then, using the fact that \( u_{1}^{+}, u_{2}^{+} \) are arbitrary commuting doublets, from (11) we derive (2).

Equation (11) provides the desired integrability interpretation of the self-duality condition. Namely, consider a matter field \( \phi(X, u) \) satisfying the following analyticity condition

\[ \nabla_{Y}^{+}\phi(X, u) = 0 \]  

in a Yang-Mills background. Then eq. (11) is obviously the integrability condition for the existence of such fields. Further, eq. (11) has the following (local) general solution of “pure gauge” type

\[ A_{Y}^{+} = h^{-1} \partial_{Y}^{+} h \, , \]  

where \( h(X, u) \) is some harmonic dependent element of the gauge group. This new object allows us to rewrite the analyticity condition (13) in a much simpler way

\[ \partial_{Y}^{+}\Phi = 0 \, , \quad \Phi \equiv h\phi \, . \]  

which has solutions in the form of functions holomorphic with respect to the projected variable \( X^{+Y} = u_{1}^{+A} X^{AY} \):

\[ \partial_{Y}^{+}\Phi = 0 \quad \Rightarrow \quad \Phi = \Phi(X^{+}, u) \, . \]  

In other words, the “pure gauge” transformation (14), which generates the solutions to (11), allows us to define a new gauge frame, in which holomorphicity (16) becomes

\(^2\)This identity is equivalent to \( \partial/\partial \bar{z}(1/z) = \pi \delta(z) \). Indeed, using a complex parametrization of \( S^2 \), one can show that \( D^{++} \sim \partial/\partial \bar{z} \) and \( u_{1}^{+} u_{2}^{+} \sim z_{1} - z_{2} \).
manifest. It is clear that this procedure can only be non-trivial because of the harmonic
dependence of the fields.

The last point about the harmonic interpretation of self-duality concerns the harmonic
derivative $D^{++}$. The change of gauge frame $\phi \rightarrow h\phi$ made the connection in the derivative
$\nabla_Y^+$ disappear. However, at the same time the previously "short" derivative $D^{++}$ acquires
now a connection:

$$D^{++} \rightarrow \nabla^{++} = D^{++} + V^{++}.$$  \hfill (17)

where

$$V^{++} = hD^{++}h^{-1}.$$  \hfill (18)

In these new terms the second self-duality constraint (12) becomes

$$[\nabla^{++}, \partial_Y^+] = 0 \Rightarrow V^{++} = V^{++}(X^+, u).$$  \hfill (19)

The conclusion is that the solutions of the SDYM equations can be parametrized (locally)
by the holomorphic harmonic connection $V^{++}$ from (17), (19). We should mention here
that the harmonic connection $V^{++}$ is in a sense the analog of the "twistor transform" of
the SDYM field in the approach of Ward [8]. Given an (almost) arbitrary $V^{++}(X^+, u)$,
one is in principle able to find the "pure gauge" transformation matrix $h(X, u)$ from (18)
and then reconstruct the Yang-Mills connection $A$ from (14). In practice, however, this
amounts to solving the non-linear differential equation (18) on $S^2$, which is not an easy
task. Moreover, in this framework it is not obvious which $V^{++}$ would generate solutions
of the SDYM equations with finite action (instantons).

2.3 The ADHM construction

An alternative way of obtaining (or, rather, parametrizing) the instanton solutions is
the ADHM construction, which is the main subject of this article. One of the results
of this paper will be to establish a one-to-one correspondence (modulo gauge freedom)
between the instanton-generating connections $V^{++}$ and the building blocks of the ADHM
construction (see section 3.3). It turns out that the superspace realization of Witten’s
sigma model makes this correspondence transparent. It should be mentioned that a
similar relation between $V^{++}$ and the ADHM construction has already been presented in
a different context in [11].

For the reader’s convenience we give here a short summary of the ADHM construction.
More details can be found in the original instanton literature ([2],[11],[12]). Here we
shall treat only the case of an $SO(n)$ gauge group. The principal difference from the
standard presentation of the ADHM construction will be that we do not use the traditional
quaternionic notation for the various matrices. The obvious reason for this is that we want
to construct a field theory (following Witten), which reproduces the ADHM construction.
Using quaternion-valued fields in an action is rather inconvenient.

The starting point of the ADHM construction for the case $SO(n)$ is the rectangular
matrix $\Delta_{aY'}^+$ with indices $a = 1, \ldots, n+4k'$, $Y' = 1, \ldots, 2k'$ ($n$ and $k'$ are any two positive
integers); $A$ is the same $Sp(1)$ index as above. This matrix is required to satisfy several
conditions:
• It must be real in the sense
\[ \Delta_{AY}^{a} = \epsilon^{AB} \epsilon^{Y'Z'} \Delta_{BZ'}^{a}. \]  
(20)

• It must be linear in \( X \),
\[ \Delta_{AY}^{a} = \alpha_{AY}^{a} + \beta_{AY}^{a} X^{Y}_{A} \]  
(21)
with constant matrices \( \alpha, \beta \).

• It must satisfy the algebraic constraint
\[ \Delta_{AY}^{a} \Delta_{BZ'}^{a} = \epsilon_{AB} R_{Y'Z'}. \]  
(22)
where \( R_{Y'Z'} \) is an invertible antisymmetric \( 2k' \times 2k' \) matrix.

• The matrices \( \alpha, \beta \) must have maximal rank.

To obtain the SDYM field one needs another real rectangular matrix \( v_{i}^{a} \), where the index \( i = 1, \ldots, n \) is an \( SO(n) \) one. It is defined to be orthogonal to the matrix \( \Delta \),
\[ v_{i}^{a} \Delta_{AY}^{a} = 0 \]  
(23)
and orthonormal,
\[ v_{i}^{a} v_{j}^{a} = \delta_{ij}. \]  
(24)
Then the \( SO(n) \) gauge field is given by the simple expression
\[ (A_{AY})_{ij} = v_{i}^{a} \partial_{AY} v_{j}^{a}. \]  
(25)

It is not hard to check that (23) is indeed a self-dual field in the sense (2). In the process one essentially uses the linear dependence of \( \Delta \) on \( X \). It is not much more difficult to show that (23) corresponds to an instanton solution with finite action and instanton number \( k' \). What is really hard is the proof that this construction gives all instanton solutions. For more details see [2],[11],[12].

From the procedure described above it is clear that the matrices \( \Delta \) and \( v \) contain a considerable arbitrariness. Indeed, the resulting gauge field (25) is not affected by global \( SO(n + 4k') \) transformations on the index \( a \). Further, the index \( Y' \) can be multiplied by global \( GL(2k',C) \) matrices, compatible with the reality condition (20), which reduces them to \( GL(k',Q) \) ones. Of course, local \( SO(n) \) transformations on the index \( i \) correspond to the usual Yang-Mills freedom in the gauge field. The \( SO(n + 4k') \times GL(k',Q) \) arbitrariness is essential when one counts the number of independent parameters of the instanton solution. In particular, one is able to fix a very simple canonical form for the coefficient matrices \( \alpha, \beta \) (21). In the orthogonal case and for \( R^{k} \) \( (k = 1) \) it is [11]:
\[ \alpha_{AY}^{a} \rightarrow \left( \begin{array}{c} b_{4k' \times n} \\ B_{4k' \times 4k'} \end{array} \right), \quad \beta_{AY}^{a} \rightarrow \left( \begin{array}{c} 0_{4k' \times n} \\ 1_{4k' \times 4k'} \end{array} \right). \]  
(26)
The remaining non-trivial matrices \( b, B \) satisfy algebraic constraints following from (22). This, together with the rest of the symmetry transformations leads to the correct number of independent parameters in the instanton solutions [11],[12].
The point about the freedom of redefinitions above is rather important. As we shall see in section 3, in the sigma model construction of Witten one is forced to interpret $Y'$ as an $Sp(k')$ index. The consequences of this will be discussed later on.

Finally, for those who are more familiar with the traditional quaternionic notation in the instanton literature, we indicate the correspondence. In the four-dimensional case $(k = 1)$ $X^{A'}$ is a $2 \times 2$ matrix, $X^{A'} = X^\mu (\sigma_\mu)^{A'}$. The matrix $\Delta$ can be rewritten as a quaternionic one if one splits the index $Y' \rightarrow A'y'$, where $A'$ is a new $Sp(1)$ index and now $y' = 1, \ldots, k'$. Then the two $Sp(1)$ indices $AA'$ can be saturated with Pauli matrices and $\Delta$ becomes $\Delta_y$ with quaternionic entries. The analog of condition (22) is

\[ (\Delta_y)\epsilon_1^a \Delta_y^a = \text{Re}(R_{y'y'z'}q) , \]  

valid for an arbitrary quaternion $q$.

2.4 (0, 4) supersymmetry

The harmonic variables are very useful in application to (0, 4) supersymmetry on a two-dimensional world sheet. The basic reason is the presence of three complex structures in this supersymmetry.

The (0, 4) world sheet can be regarded as a superspace with coordinates $x_{++}, x_{--}, \theta_{A'A'}$. Here $\pm$ indicate the Lorentz ($SO(1,1)$) weights (to avoid confusion with the harmonic $U(1)$ charges we shall always write the weights as lower and the charges as upper indices). The Grassmann variables $\theta_{A'A'}$ carry doublet indices of the $SO(4) \sim SU(2) \times SU(2)'$ automorphism group of (0, 4) supersymmetry and satisfy the reality condition $\theta_{A'A'} = \epsilon_{AB} \epsilon_{A'B'} \theta_{B'B'}$. The spinor covariant derivatives (the counterparts of the supersymmetry generators)

\[ D_{-AA'} = \frac{\partial}{\partial \theta_{-AA'}} + i \theta_{+AA'} \frac{\partial}{\partial x_{++}} \]  

satisfy the (0, 4) supersymmetry algebra

\[ \{D_{-AA'}, D_{-BB'}\} = 2i \epsilon_{AB} \epsilon_{A'B'} \partial_{--} . \]  

Comparing eq. (29) with (1) and (2), we see the first similarity between (0, 4) supersymmetry and SDYM theory in $R^{4k}$. This suggests to apply the same trick of projecting the $SU(2)$ indices $A, B$ with harmonic variables, as we did in (10). Defining

\[ D_{-A'} \equiv u^{AA'} D_{-AA'} , \]  

we can rewrite (29) as follows

\[ \{D_{-A'}, D_{-B'}\} = 0 . \]  

Note that after this projection the torsion term from the right-hand side of (29) disappeared, just like the curvature term in (11). This analogy goes even further. In the Yang-Mills case we interpreted eq. (11) as the integrability condition for the existence

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3 We are grateful to F. Delduc for helping us establish this correspondence.
4 An analogous treatment of the case $k > 1$ has been given in [13].
of analytic fields defined by (13). Here we can define Grassmann analytic superfields satisfying
\[ D_{-A'}^+ \Phi(x, \theta, u) = 0 , \tag{32} \]
with eq. (31) as the integrability condition. Further, in subsection 2.2 we proceeded by changing the gauge frame as in (14),(15), in order to “shorten” the covariant derivative \( \nabla_+^x \). Here the analogous operation is the introduction of a new analytic basis in harmonic (i.e., extended by adding the harmonic variables) superspace
\[ \hat{x}_{++} = x_{++} + i \theta^{AA'} \theta^B_{+A'} u^+_A u^-_{(A} u_{B)} , \quad x_{--}, \quad \theta^\pm_{+A'} = u^\pm_A \theta^{AA'} , \quad u^\pm . \tag{33} \]
Its existence is guaranteed by the integrability conditions (31). In it the derivative \( D_{-A'}^+ \) is just the partial one: \( D_{-A'}^+ = \partial / \partial \theta^{--} \). Then the Grassmann analyticity condition (32) can be solved in the form
\[ D_{-A'}^+ \Phi(x, \theta, u) = 0 \Rightarrow \Phi = \Phi(\hat{x}_{++}, x_{--}, \theta^+_+, u) . \tag{34} \]
At the same time the harmonic derivative \( D^{++} \), which acquired a connection in the new frame in subsection 2.2, here gets a vielbein term
\[ D^{++} = u^{+A} \frac{\partial}{\partial u^{-A}} + i \theta^+_{+A'} \theta^{++}_{+A'} \frac{\partial}{\partial \hat{x}_{++}} . \tag{35} \]
To complete the parallel between the two pictures, note that the analog of (12) here is
\[ [D^{++}, D_{-A'}^-] = 0 , \tag{36} \]
which is basis-independent.

The analytic superfields defined in (34) can have a non-vanishing \( U(1) \) harmonic charge, \( \Phi^q(x, \theta^+, u) \). They have a very short Grassmann expansion,
\[ \Phi^q(x, \theta^+, u) = \phi^q(x, u) + \theta^+_{+A'} \xi_{-A'}^{q-1}(x, u) + (\theta^+_+)^2 f_{q-2}^--(x, u) , \tag{37} \]
where \( (\theta^+_+)^2 \equiv \theta^+_{+A'} \theta^{++}_{+A'} \). The coefficients in (37) are harmonic-dependent fields (note that the overall \( U(1) \) charge \( q \) is conserved in all the terms in (37)). We should also mention that in certain cases the harmonic analytic superfields (37) can be made real in the sense of the special conjugation (7).

The careful reader may note that we have “harmonized” just one \( SU(2) \) subgroup of the \( SO(4) \) automorphism group, but haven’t touched the other one, \( SU(2)' \). It is this fact that makes harmonic superfields very useful in CFT. There, \( SU(2)' \) is a part of the \( (0,4) \) superconformal group, while \( SU(2) \) is not. In the model considered below, the action is \( SU(2)' \) invariant and not \( SU(2) \) invariant. So, it has the right symmetries to flow in the infrared to an \( (0,4) \) CFT.

\[ ^5 \]To simplify the notation we shall not write explicitly \( \hat{x}_{++} \) when it is clear that we work in the analytic basis (33).
3 The (0, 4) sigma model

In this section we give the superspace formulation of the sigma model used by Witten in [1] to show the relation between (0, 4) supersymmetry and the ADHM constructions of instantons. One of the superfields needed turns out to have an abelian gauge invariance (which has nothing to do with the gauge symmetry of the Yang-Mills theory). It is this invariance which, to a large extent, is responsible for the peculiar coupling leading to the ADHM construction. We show that the manifestly supersymmetric sigma model action gives rise to the twistor transform of the instanton gauge field. We also point out a subtlety concerning the exact relationship between the (0, 4) sigma model and the ADHM construction.

3.1 The free supermultiplets

Witten makes use of three types of (0, 4) supermultiplets. Here we shall give their superspace counterparts. Their origin is perhaps more clear if they are regarded as obtained by trivial dimensional reduction and truncation from the two basic \( N = 2, D = 4 \) superfields, namely, the matter (hypermultiplet) and the abelian gauge multiplet. We shall explain this in Appendix B.

The first multiplet includes the coordinates \( X^{AY} \) of the Euclidean target space \( \mathbb{R}^4_k \), in which the Yang-Mills fields will be defined. In harmonic superspace it is described by the analytic superfields (cf. (37))

\[
X^+(x, \theta^+, u) = X^+(x, u) + i\theta^+A\psi^A(x, u) + (\theta^+)^2 f^A(x, u) .
\]

These superfields can be chosen real in the sense of the conjugation defined in (3):

\[
\tilde{X}^+ = \epsilon_{YZ}X^+ .
\]

Each field in (38) is harmonic, i.e., has an infinite harmonic expansion on \( S^2 \). However, the multiplet we are looking for should have a finite number of physical fields. It turns out that we can truncate the harmonic expansions in (38) by imposing the following harmonic irreducibility condition

\[
D^{++}X^{+Y} = 0 .
\]

It is not hard to obtain the component solution of this constraint. Using the analytic basis form of \( D^{++} \) (35), for \( X^{+Y}(x, u) \) we find (\( \partial^{++} \) denotes the purely harmonic part of (35))

\[
\partial^{++}X^{+Y}(x, u) = 0 \Rightarrow X^{+Y}(x, u) = u^+_AX^{AY}(x) ,
\]

as follows from the harmonic expansion (3) for \( q = +1 \). Similarly, for the other two fields in (38) we get

\[
\psi^A(x, u) = \psi^A(x), \quad f^A(x, u) = -iu_\theta^A\partial^+X^{AY}(x) .
\]

The component fields are real because of (39), \( \bar{X}^{AY} = \epsilon_{AB}\epsilon_{YZ}X^{BZ} \), \( \bar{\psi}^A = \epsilon_{AB'}\epsilon_{YZ}\psi^{B'Z} \). So, as a result of the constraint (40) the harmonic dependence of the fields in (38) was...
reduced to a linear one. Since the constraint (40) is manifestly supersymmetric and does not involve equations of motion for the component fields, we conclude that the fields in (41), (42) form an off-shell \((0,4)\) supersymmetry multiplet. It coincides with the one given in [1].

The action for the above multiplet is given as an integral over the analytic superspace:

\[
S_X = i \int d^2x d^2\theta^+ X^{+Y} \partial^{++} X^+_Y. \tag{43}
\]

Note that in constructing (43) we took special care to match the \(U(1)\) charge \(-2\) and Lorentz weight \(-2\) of the Grassmann measure with those of the integrand, so that the action is both an \(SU(2)\) and Lorentz singlet. To obtain the component content of (43) we substitute the short form (41), (42) of the expansion (38) in (43) and perform two more steps. First, we do the Grassmann integral, i.e., we pick out only the \((\theta^+)^2\) terms. Then we do the harmonic integral according to the rules (8), which amounts to extracting the \(SU(2)\) singlet part. The result is

\[
S_X = \int d^2x \left( X^{AY} \partial^{++} \partial^{--} X^{AY} + \frac{i}{2} \psi^{AY}_+ \partial^{++} \psi^{AY}_- \right). \tag{44}
\]

This is the action for \(4k\) real scalars \(X^{AY}\) and their chiral spinor superpartners \(\psi^{AY}_\pm\). The multiplet in (44) is clearly free and massless, which corresponds to a flat target space \(R^{4k}\).

The way to endow the target space with a non-flat (necessarily hyper-Kähler) metric or even a torsion has been studied in [14], [15]. One simply replaces the flat irreducibility condition (40) by

\[
D^{++} X^{+Y} = \mathcal{L}^{+3Y} (X^+, u), \tag{45}
\]

with an arbitrary function of \(U(1)\) charge +3 (the hyper-Kähler case, i.e., no torsion, corresponds to choosing \(\mathcal{L}^{+3Y} = (\partial/\partial X^+_Y) \mathcal{L}^{+4}\), see [14] for more details). In what follows we shall not be interested in such generalizations, we prefer to keep (40) flat.

The second type of \((0, 4)\) supermultiplet used by Witten involves only chiral fermions (at least on shell) of chirality, opposite to that in (44). Its harmonic superspace description is very simple indeed. Take the following anticommuting and real (in the sense of (7)) superfields

\[
\Lambda^a_+ (x, \theta^+, u) = \lambda^a_+ (x, u) + \theta^+_+ \lambda^a_+ (x, u) + i (\theta^+_+)^2 \sigma^{--}_a (x, u). \tag{46}
\]

The action for them is

\[
S_\Lambda = \frac{1}{2} \int d^2x d^2u d^2\theta^+ \Lambda^a_+ D^{++} \Lambda^a_+. \tag{47}
\]

From here it is clear that we can regard the external index \(a = 1, \ldots, n + 4k'\) as an \(SO(n + 4k')\) one: integrating \(D^{++}\) in (47) by parts changes the sign, but the superfields \(\Lambda^a_+\) anticommute, so the trace \(aa\) is symmetric.

Obtaining the component content of (47) is as easy as in the case of (43). First we do the Grassmann integral and find

\[
S_\Lambda = \int d^2x d^2u \left( \frac{i}{2} \lambda^a_+ \partial^{--} \lambda^a_+ + i \sigma^{--}_a \partial^{++} \lambda^a_+ + \frac{1}{4} g^{-aA'} \partial^{++} g^{-aA'} \right). \tag{48}
\]
Note that the space-time derivative $\partial$ in (48) originates from the harmonic derivative (35). The field $\sigma^{-a}$ serves as a Lagrange multiplier for the harmonic condition $\partial^{++} \lambda^a(x,u) = 0$ which makes $\lambda^a$ harmonic independent. The harmonic-dependent field $g^{-aA}(x,u)$ is clearly auxiliary too. Eliminating both auxiliary harmonic fields, we obtain simply the action for $n + 4k'$ free chiral fermions

$$S_A = \frac{i}{2} \int d^2x \, \lambda^a(x) \partial_{--} \lambda^a(x) \, .$$

(49)

This represents an on-shell supermultiplet, in which supersymmetry acts in a trivial way. However, and we want to stress on this, the off-shell version of the multiplet (48) requires an infinite number of auxiliary fields. Assuming a finite number of auxiliary fields, one can show by the standard “no-go” counting arguments of [16] that the number of chiral fermions must be an integer multiple of four. Moreover, for the multiplets with a finite number of auxiliary fields, the natural $SO(n + 4k')$ symmetry of the free action (49) is necessarily broken (see Appendix A). Naively, one might conclude that it is impossible to have an arbitrary number of chiral fermions in an off-shell representation of $(0, 4)$ supersymmetry [17]. However, the harmonic formalism avoids this by using an infinity of auxiliary fields. This is an analog of the $N = 2$ hypermultiplet [1] and $N = 3$ supersymmetric Yang-Mills multiplet [18] in four dimensions, for which there exist no finite off-shell multiplets.

Finally, Witten utilizes the so-called “twisted” scalar multiplet, in which the $SU(2)$ indices carried by the bosons and fermions are interchanged (as compared to the standard multiplet (44)). Its superspace description turns out to be quite unusual. This time we need a set of anticommuting abelian gauge superfields

$$\Phi_+^{Y'}(x, \theta^+, u) = \rho_+^{Y'}(x,u) + \theta_+^{X'} \phi_{X'}(x,u) + i(\theta_+^+)^2 \chi^{-Y'}(x,u) \, ,$$

(50)

satisfying the reality condition $\Phi_+^{Y'} = \epsilon_{Y''Z'} \Phi_+^{Z'}$ (see (1)). Here $Y' = 1, \ldots, 2k'$ is an index of some new symplectic group $Sp(k')$, as we shall see below (it turns out that $k'$ is just the number of instantons in the corresponding ADHM construction). These superfields undergo the following abelian gauge transformations

$$\delta \Phi_+^{Y'} = D^{++} \omega_+^{-Y'}$$

(51)

with analytic parameters $\omega_+^{-Y'}(x, \theta^+, u)$. We want to stress that this gauge invariance has nothing to do with the target space gauge group. It is an artifact of the superspace description of the multiplet.

The rôle of the gauge invariance (51) may become more transparent if we make comparison with an abelian gauge field in four dimensions, $A_\mu(x)$. It is known that this field contains two representations of the Poincaré group with spins 1 and 0. There are two ways to eliminate the spin 0 part. One is to impose an irreducibility condition, e.g., $\partial^\mu A_\mu = 0$, the other is to submit $A_\mu$ to gauge transformations $\delta A_\mu = \partial_\mu \omega(x)$ and write down a gauge invariant action. Something similar we observe here. The analog of the irreducibility condition $\partial^\mu A_\mu = 0$ is eq. (40) for the superfield $X^+$. The effect of this condition is that the superfield becomes short in the harmonic sense, which means the elimination of an (infinite number) of extra degrees of freedom. The analog of the second,
gauge mechanism for $A_\mu$ is given by (51). To understand what happens we should look at the expansion of the parameter

$$\omega_+^{-Y'}(x, \theta^+, u) = \tau_+^{-Y'}(x, u) + \theta_+^{A'} l_+^{-Y'}(x, u) + i(\theta_+^+)^2 \mu_-^{-3Y'}(x, u). \quad (52)$$

From (50) and (52) and using (33), we get, for instance, $\delta \rho_+^+(x, u) = \partial^{++} \tau_-(x, u)$. Comparing the harmonic expansions (3) of $\rho^+$ and $\tau^-$, we easily see that each term in the expansion of $\rho^+$ has its counterpart in that of $\tau^-$. So, the component $\rho^+$ can be completely gauged away. Similar arguments show that the expansions of the parameters $l^-$ and $\mu^{-3}$ are a little “shorter” than those of the fields $\phi$ and $\chi^-$, correspondingly. What is missing is just the lowest order term, the smallest $SU(2)$ representations in each expansion. Thus, the fields $\phi_A^Y(x, u)$ and $\chi^{-Y'}(x, u)$ can also be gauged away, except for the first terms in their harmonic expansions, the fields $\phi_A^Y(x)$ and $u^- A Y^A(x)$. The net result of all this is the following “short”, i.e., irreducible harmonic superfield in the Wess-Zumino-type gauge

$$\Phi_+^{+Y'}(x, \theta^+, u) = \theta_+^{A'} \phi_A^Y(x) + i(\theta_+^+)^2 u^- A Y^A(x). \quad (53)$$

This is precisely the content of the “twisted” multiplet of Witten. We note that this multiplet is off shell, just like the one described by the superfield $X^+$. In fact, in terms of component fields the difference between the two multiplets is very small, just the two automorphism groups $SU(2)$ are flipped. However, we have seen that their superspace descriptions are quite different, and the reason for this is clear: we have harmonic variables for one of the $SU(2)$ groups only. We would also like to stress that in the Wess-Zumino gauge (53) one loses manifest supersymmetry.

Next, we have to find a gauge invariant action for the superfield $\Phi_+^+$. It is modeled after the action for a gauge superfield for $N = 2, D = 4$ supersymmetry [7] and has the form

$$S_\Phi = i \int d^2xd^4\theta_+ du_1du_2 \frac{1}{u_1^+ u_2^+} \Phi_+^{+Y'}(1) \partial_{++} \Phi_+^{+Y'}(2). \quad (54)$$

The notation $\Phi_+^+(1)$ means that the analytic superfield is written down in an analytic basis (33) defined by the harmonic variable $u_1$; similarly, $\Phi_+^+(2)$ is given in another basis, defined by $u_2$. Since both superfields appear in the same integral, they should be written down in the same non-analytic basis $x_{\pm\pm}, \theta^{AA'}, u$. This explains why the Grassmann integral in (54) is taken over the full superspace and not over an analytic subspace, as in (13) or (17). Note that, as always, we take care of matching $U(1)$ charges and Lorentz weights (this accounts for the presence of $\partial_{++}$ in (54)). Note also that the contraction $\chi^{-Y'} \equiv Y' \epsilon_{Y'Z}^{Z'}$ must be of the type $Sp(k')$. Indeed, interchanging 1 and 2 involves integration by parts of $\partial_{++}$, flipping the odd superfields $\Phi(1), \Phi(2)$ and using $u_1^+ u_2^+ = -u_1^+ u_2^+$, so the fourth antisymmetric factor $\epsilon_{Y'Z}^{Z'}$ restores the symmetry required by the double harmonic integral. The issue of the symmetry of this kinetic term is important and we shall come back to it later on.

The reason for the exotic form of (54) is the gauge invariance (51). It works in the following way. Varying in (54) with respect to, e.g., $\Phi_+^+(1)$ and according to the transformation law (51) makes the harmonic derivative $D_1^{++}$ appear under the integral. Integrating it by parts, we see that it only acts upon the harmonic distribution $(u_1^+ u_2^+)^{-1}$ (since $\Phi_+^+(2)$ depends on the second harmonic variable $u_2$). Then we use the formula (3)
and obtain a harmonic delta function, which removes one of the harmonic integrals. After this both superfields $\Phi^+_{Y^A}$ become analytic with respect to the same basis, i.e., they depend only on the two Grassmann variables $\theta^+_A$. However, the Grassmann integral in (54) is over $d^4\theta$, so it gives 0. This establishes the gauge invariance of the action (54).

The action (54) may seem strange because of its harmonic non-locality, but it is only apparent. Indeed, in the Wess-Zumino gauge (53) all the fields have short harmonic expansions. Working out the Grassmann integral (not forgetting the different arguments of $\Phi(1,2)$), we obtain factors of $(u^+u^+_2)$ in the numerator which cancel out the singular denominator. Then the double harmonic integral becomes trivial and we find

$$S_\Phi = \int d^2x \left( \phi^{Y^A} \partial_{++} \phi_{Y^A} + \frac{i}{2} \chi_{-Y^A} \partial_{++} \chi_{-Y^A} \right), \quad (55)$$

which is the “twisted” multiplet action used by Witten.

### 3.2 Interaction and the ADHM construction

The main question now is how to couple the above multiplets. Following the idea of Witten, we are only interested in couplings of the potential type, i.e., without space-time derivatives. The coupling should be controlled by a parameter $m$ with the dimension of mass. The aim is to examine the resulting theory in the limit $m \to \infty$ and show that it flows into a CFT. The latter are known to have an unbroken $SU(2)$ invariance, and this is put as a requirement in Witten’s construction. In our harmonic superspace description this corresponds to preserving the symmetry $SU(2)'$ (indices $A'$).

The interaction terms of the action can be integrals over either the full (0,4) harmonic superspace $\int d^2xdud^4\theta$ or the analytic superspace $\int d^2xdud^2\theta^+_+$; they should involve a positive power of $m$. To see how this can be arranged, let us first examine the dimensions of our superfields $X^+, \Lambda^+_a$ and $\Phi^+_{Y^A}$. The position of the physical spinors (dimension 1/2) $\psi_-$ in (33), $\lambda_+$ in (46) and $\chi_-$ in (50) fixes the dimensions $[X] = 0$, $[\Lambda] = 1/2$, $[\Phi] = -1/2$. It is easy to see that the full superspace integrals are ruled out. Indeed, the full measure has dimension 0 and Lorentz weight 0. The presence of the mass in $m \int d^2x d^4\theta$ requires at least a pair of $\Phi^+_{Y^A}$ superfields ($[\Phi^2] = -1$), but this is not consistent with the Lorentz invariance. Thus, we are left with analytic superspace interaction terms only. The analytic measure $\int d^2xdud^2\theta^+_+$ has dimension 1, Lorentz weight −2 and $U(1)$ charge −2. There are only two possible coupling terms in the action, in which the dimensions, charges and weights are matched (we omit the indices):

$$S_{\Phi\Lambda} = m \int d^2x dud^2\theta^+_+ \Phi^+_+ \Lambda_+ (X^+, u), \quad (56)$$

$$S_{\Phi\Lambda} = m^2 \int d^2x dud^2\theta^+_+ \Phi^+_+ t(X^+, u). \quad (57)$$

Here $v^+$ and $t$ are arbitrary functions of the dimensionless and weightless superfields $X^+$ and of the harmonic variables $u$.

Let us first investigate the term (56). To get an idea how it can be constructed, we first take a simplified case, in which $n = 0$ and the index $a$ can be written as a pair of
symplectic indices, \(a = AY'\). Then the charged object \(v^+\) can be the harmonic variable \(u^+_A\) itself. Thus we come to the following coupling term

\[
S_{\Phi \Lambda} = m \int d^2x d^2\theta^+ \Phi^{+Y'} u^+_A \Lambda_{+AY'} .
\]  

(58)

The most important point now is to make sure that the coupling (58) is invariant under the gauge transformation (51). It is very easy to see that to this end the superfield \(\Lambda_{+AY'}\) must also transform as follows

\[
\delta \Lambda_{+AY'} = m u^+_A \omega_{+Y'} .
\]

(59)

Indeed, the variation with respect to \(\Lambda\) in the kinetic term (47) compensates for that of \(\Phi\) in (58), whereas \(\delta \Lambda\) in (58) is annihilated by the harmonic \(u^+ (u^+_A u^+_A = 0)\).

The second possibility (57) to construct a coupling term is now clearly ruled out by the gauge invariance (51) (however, soon we shall see a term of the type (57) appearing in a fixed gauge).

Surprisingly enough, in spite of the presence of gauge invariance, the coupling (58) is nothing but a mass term. There are two ways to see this. One is to examine the component Lagrangian, the other is to do it directly in terms of superfields. We postpone the former until we come to the general interaction and do the latter, which is in fact much easier. Let us decompose the index \(A\) of \(\Lambda_{+AY'}\) in the \(u^\pm\) basis:

\[
\Lambda_{+AY'} = u^+_A \Lambda_{+A Y'} + u^-_A \Lambda_{-A Y'} , \quad \Lambda_{-A Y'} \equiv -u^-_A \Lambda_{+A Y'} , \quad \Lambda_{+A Y'} \equiv u^+_A \Lambda_{+A Y'} .
\]

(60)

Then from (59) we see that \(\delta \Lambda_{-A Y'} = m u^+_A \omega_{-Y'}\), which allows us to fix the following supersymmetric gauge (as opposed to the non-supersymmetric Wess-Zumino gauge (53))

\[
\Lambda_{+Y'} = 0 .
\]

(61)

Substituting this into (58) and the kinetic term (47), we obtain two action terms involving the remaining projection \(\Lambda^+_{+Y'}\):

\[
\int d^2x d^2\theta^+_+ \left( -\frac{1}{2} \Lambda^+_{+Y'} \Lambda^+_{+Y'} + m \Phi^{+Y'} \Lambda^+_{+Y'} \right) .
\]

(62)

Clearly, \(\Lambda^+_{+Y'}\) can be eliminated from (62) by means of its algebraic field equation \(\Lambda^+_{+Y'} = m \Phi^{+Y'}\). Then we are left only with the superfield \(\Phi\) which has the action (recall (54))

\[
S_{\Phi} = i \int d^2x d^4\theta_+ du_1 du_2 \frac{1}{u_1 u_2} \Phi^{+Y'} (1) \partial_+ \Phi^{+Y'} (2) + \frac{m^2}{2} \int d^2x d^2\theta^+_+ \Phi^{+Y'} \Phi^{+Y'} .
\]

(63)

The very appearance of (63) shows that it is the action for \(2k'\) massive superfields. We conclude that when the number of the multiplets \(\Lambda^a\) equals \(4k'\), the coupling term (58) is in fact a mass term for the \(4k'\) real bosons \(\phi^{AY'}\) contained in \(\Phi\) and for the \(4k'\) pairs of right-handed fermions \(\chi_{+Y'}\) from \(\Phi\) and left-handed ones \(\lambda_{+Y'}\) from \(\Lambda\).

Let us now come back to the general case when \(n \neq 0\). We can try to generalize the coupling (58) by replacing \(u^+_A\) by a matrix \(u^+_A (X^+, u)\), satisfying the reality condition
\( \widetilde{v}_Y^+ v^+_Z = \epsilon^{YZ} v^+_Z \). It cannot depend on the superfields \( \Phi \) or \( \Lambda \) because of Lorentz invariance, but it can still be a function of the weightless superfields \( X^Y \) and of the harmonic variables. So, we write down

\[
S_{\text{int}} = m \int d^2xdud^2\theta^+ \Phi^+ \Phi^+ v^+_{Y'}(X^+, u)\Lambda^a_+ .
\]  

(64)

The main question is how to make (64) compatible with the gauge invariance (51). Like in (59), we can try

\[
\delta \Lambda^a_+ = mv^{+a}_Y(X^+, u)\omega_+^{-Y'}.
\]  

(65)

It is not hard to see that this transformation compensates for \( \delta \Phi \) in (64), provided the following two conditions hold:

\[
v^{+a}_Y v^+_Z = 0 ,
\]  

(66)

\[
D^{++} v^{+a}_Y(X^+, u) = 0 .
\]  

(67)

Condition (67) is very restrictive. Indeed, assume that the function \( v^{+a}_Y(X^+, u) \) is regular, i.e., can be expanded in a Taylor series in \( X^+ \):

\[
v^{+a}_Y(X^+, u) = \sum_{p=0}^{\infty} v^{(1-p)a}_{Y_{Y_1...Y_p}}(u)X^{Y_1}...X^{Y_p} .
\]  

(68)

Since \( X^+ \) satisfies the irreducibility condition (40), (67) implies

\[
D^{++} v^{(1-p)a}_{Y_{Y_1...Y_p}}(u) = 0,
\]  

where \( 1 - p \) is the \( U(1) \) charge. The only non-vanishing solution to this is (see the harmonic expansion (4))

\[
v^{+a}_Y(X^+, u) = u^{+A} \alpha^a_{AY'} + \beta^a_{Y'Y}X^{+Y'},
\]  

(69)

where the matrices \( \alpha, \beta \) are constant. In other words, the matrix \( v^{+a}_Y(X^+, u) \) can at most depend linearly on \( X^+ \). If we put \( \theta^+ = 0 \), i.e., consider only the lowest components in the superfield \( X^+ \) (see (41)), then (68) becomes

\[
v^{+a}_Y(X^+, u)|_{\theta=0} = u^{+A} (\alpha^a_{AY'} + \beta^a_{Y'Y}X^Y_A) \equiv u^{+A} \Delta^a_{AY'} .
\]  

(70)

The other condition (66) is purely algebraic. Putting (70) in it and removing the harmonic variables \( u^{+A}u^{+B} \), we get

\[
\Delta^a_{(AY')\Delta^a_{BZ'}} = 0 , \quad \text{i.e.,} \quad \Delta^a_{AY'}\Delta^a_{BZ'} = \epsilon_{AB} R_{Y'Z'} .
\]  

(71)

What we have obtained are precisely the two defining conditions (21), (22) on the matrix \( \Delta \), used as a starting point in the ADHM construction for instantons. As we mentioned in subsection 2.3, in the ADHM construction one requires that the matrices \( \alpha, \beta \) have maximal rank (in the present case \( 4k' \)). In the sigma model context [1] this implies that all the \( 4k' \) right-handed chiral fermions in \( \Phi^+ Y' \) are paired with a subset of \( 4k' \) left-handed ones from \( \Lambda^a_+ \) and become massive. The sigma model described in this section and first proposed by Witten [1] corresponds to \( k' \) instantons in \( R^{4k} \) with gauge group \( SO(n) \). The somewhat subtle point about the allowed freedom of redefinition of the matrix \( \Delta \) will be discussed at the end of subsection 3.3.
It should be pointed out that throughout this subsection we always preserved the $SU(2)'$ (indices $A'$) invariance of the free theory. In fact, this invariance is not required by supersymmetry. Indeed, we can construct another potential term, similar to (64), but with broken $SU(2)'$. To this end we introduce manifest $\theta^+$ dependence:

$$m\int d^2xdud^2\theta^+_+ A' \theta^{+A'} M^{+a}(X^+, u) \Lambda^a_+.$$  \hspace{1cm} (72)

One might think that such a term would break supersymmetry. However, the variation $\delta \theta^{+A'} = \epsilon^{+A'}$ can be compensated by a suitable transformation of the superfield $\Lambda$,

$$\delta \Lambda^a_+ = me^{-A'} M^{+a}_+.$$  \hspace{1cm} (73)

Here we used $D^{++} \epsilon^{+A'} = \epsilon^{+A'}$. Note that after this $\Lambda^a_+$ ceases to be a superfield. In order for this trick to work the matrix $M^{+a}_+(X^+, u)$ must satisfy conditions similar to (66), (67). These constraints imply a structure of $M^{+a}_+(X^+, u)$ like in (69), but this time involving constant matrices with $SU(2)'$ indices. In a sense, the term (72) corresponds to introducing a constant superfield $\Phi_+^{+A'} = \theta^{+A'}$ in (64). Yet another way to say this is that we have replaced the field $\phi$ in (53) by a unit matrix (and put $\chi_- = 0$). Note that the matrix $M^{+a}_+(X^+, u)$ explicitly breaks $SU(2)'$. By analyzing the general restrictions on the potential term following by (0, 4) supersymmetry, Witten [1] has also found such terms. He argued that they should not be allowed if one wants to obtain a sigma model which makes contact with CFT.

To summarize this subsection, we have shown how (0, 4) supersymmetry can efficiently determine the kind of interaction for the linear sigma model and put it in correspondence with the starting point of the ADHM constructions. We emphasize the rôle of the abelian gauge invariance (51) in fixing this interaction. Of course, the model we have found is just the superspace version of the one of Witten.

### 3.3 The instanton gauge field

In the ADHM constructions one is able, on the basis of the matrix $\Delta$ above, to give an explicit expression for the instanton gauge field. This can also be done in our supersymmetric sigma model. One way is to follow Witten and extract the gauge field from the component sigma model action. We shall do this at the end of this subsection, but first we prefer to continue with our manifestly supersymmetric approach. As we shall see, it will lead us directly to the twistor counterpart of the SDYM field introduced in subsection 2.2.

As we have shown in the beginning of subsection 3.2 when $n = 0$ and the coupling is simply given by eq. (58), then the model describes $4k' + 4k'$ free massive bosons and fermions. To a certain extent this remains true even for the interaction Lagrangian with (58) replaced by (64). The more precise statement is that among the $n + 4k'$ left-handed fermions $\lambda_+$ contained in $\Lambda^a_+$ there is a subset of $4k'$ which are paired with the right-handed fermions in $\Phi$ and become massive (together with the bosons from $\Phi$). The remaining

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6The other half of the automorphism group $SO(4)$ of (0, 4) supersymmetry (the group $SU(2)$, indices $A$) is broken by the presence of the constant matrices $\alpha_{A'}^{a'}$, in (69).
chiral fermions stay massless. The question is how to separate the massive from the massless modes, i.e., how to diagonalize the action. We shall try a diagonalization which resembles \((60)\).

As the first step let us complete the \(2k' \times (n+4k')\) matrix \(v^+_{\hat a} (X^+, u)\) to a full orthogonal matrix \(v^\hat a_a (X^+, u)\), where the \(n+4k'\) dimensional index \(\hat a = (+Y', -Y', i)\) and \(i = 1, \ldots, n\) is an index of the group \(SO(n)\). Orthogonality means

\[
v^\hat a_a v^\hat b_b = \delta^{\hat a \hat b},
\]

where \(\delta^{+Y',-Z'} - \delta^{-Y',+Z'} = \epsilon Y' Z', \quad \delta^{+Y',+Z'} - \delta^{-Y',-Z'} = \delta^{\pm Y', i} = 0\). In particular, for \(\hat a = +Y'\) and \(\hat b = +Z'\) we obtain just condition \((66)\). Since \(v^+_{\hat a} \) is a function of \(X^+\) and \(u^\pm\), we expect the other blocks of \(v^\hat a\), namely \(v^{-Y'a}\) and \(v^a\) to be such functions too. Of course, the fact that \(v^+_{\hat a}\) must be a linear function of \(X^+\) (see \((60)\)) does not imply that the rest of \(v^\hat a\) are linear as well. It is also clear that the new matrix blocks are not completely fixed by \((74)\). The freedom consists of the subset of \(SO(n+4k')\) transformations acting on the index \(\hat a\) with analytic (i.e., functions of \(X^+\) and \(u^\pm\)) parameters, which leave \(v^+_{\hat a}\) invariant.

With the help of the newly introduced matrix we can make a change of variables from the superfield \(\Lambda^a_+\) to \(\Lambda^a_\hat = v^\hat a \Lambda^a_+\). Then the gauge transformation \((53)\) can be translated into

\[
\delta \Lambda^\hat a_+ Y' = m \omega_d Y', \quad \delta \Lambda^a_+ Y' = \delta \Lambda^a_+ = 0.
\]

As in the flat case of subsection 3.2 eq. \((75)\) allows us to fix the supersymmetric gauge (cf. \((61)\))

\[
\Lambda^a_+ Y' = 0.
\]

Now we can switch to the new superfields \(\Lambda^\hat a_+\) in the kinetic term for \(\Lambda\) \((47)\) and the coupling term \((64)\). It is useful to introduce the following notation

\[
(V^+)\hat a b = v^\hat a \delta^{++} v^\hat b\tag{77}
\]

Then the terms of the Lagrangian containing \(\Lambda\) become (in the gauge \((76)\))

\[
L_{++}^+(\Lambda) = \frac{1}{2} \Lambda^i_+ [\delta^{ij} D^{++} + (V^+) ij] \Lambda^j_+ + \Lambda^{\hat a}_+ Y' [\frac{1}{2} (V)_{Y'Z'} \Lambda^{++} Z' + (V^+) \Lambda^a_+ (V^{-1})_{Y'} \Lambda^a_+ + m \Phi^+_{Y'Z'}],
\]

where we have denoted \(V_{Y'Z'} = (V^+)_{Y'Z'} (78)\). We see that \(\Lambda^a Y'\) enters without derivatives, so it can be eliminated from \((78)\). The result is

\[
L_{++}^+(\Lambda) = \frac{1}{2} \Lambda^i_+ [\delta^{ij} D^{++} + (V^+) ij] \Lambda^j_+ - \frac{1}{2} \left( (V^+) \Lambda^a_+ + m \Phi^+_{Y'Z'} \right) \left( (V^{-1})_{Y'Z'} \Lambda^a_+ + m \Phi^+_{Y'Z'} \right).
\]

The complete action of the model is obtained by adding the kinetic terms for \(X^\pm\) \((43)\) and \(\Phi^\pm\) \((74)\).

To make contact with the free case \((63)\) we can put the superfields \(\Lambda^a_+\) to 0 and \(V_{Y'Z'} = \epsilon_{Y'Z'}\). What remains is just the mass term for the superfields \(\Phi\) (cf. \((63)\)).
This explains why the subset of superfields $\Lambda^+_+(X)$ (and, consequently, the chiral fermions therein) correspond to massive modes. The more interesting simplification occurs when we let the mass $m$ go to infinity. This corresponds to suppressing the kinetic term for $\Phi$, after which the second term in (74) becomes auxiliary and we can drop it. After that we are left with the simple result

$$L_{++}^+(\Lambda)|_{m \to \infty} = \Lambda^i_+ [\delta^{ij} D^{++} + (V^{++})^{ij}] \Lambda^j_+,$$

where

$$(V^{++})^{ij} = v^{ia}(X^+, u) D^{++} v^{ja}(X^+, u).$$

Now, let us compare this with the harmonic treatment of the SDYM equations, in particular, with the covariant derivative $\nabla^{++}$ [18], [19]. We realize that $V^{++}$ in (81) is the counterpart of the holomorphic harmonic connection for the gauge group $SO(n)$ and $\Lambda^i_+$ is the analog of the matter field. As explained in subsection 2.2, starting from a given $V^{++}(X^+, u)$ one can reconstruct a solutions of the SDYM equations by first finding the “pure gauge” transformation $h(X, u)$ from the equation $hD^{++}h^{-1} = V^{++}$ and then substituting it in (14). It is important to realize that the matrices $h^{ij}(X^{AY}, u^\pm)$ belong to $SO(n)$ and are not holomorphic (i.e., they depend on both $X^\pm$). In [14], [81] we have found another representation of $V^{++}(X^+, u)$, this time in terms of the holomorphic, but bigger ($SO(n + 4k')$) matrices $v^{\hat{a}a}(X^+, u)$. More precisely, $V^{++}(X^+, u)$ is given in terms of the rectangular matrix $v^{ia}$, which is in turn determined by $v^{+a} \sim \Delta$. O. Ogievetsky [14] has found another, direct expression for $V^{++}$ in terms of $\Delta$ (at least for $R^4$ and the gauge group $SU(2) \sim Sp(1)$). Starting from $\Delta$, he was able to find the matrix $h^{ij}(X^{AY}, u^\pm)$ and from it to compute $V^{++}$. The result is remarkably simple. If we use the canonical form of $\Delta$ (24), then

$$(V^{++})^{ij} = b^{+i}_{Y'}(B^{-2})^{Y'Z'} b^{+j}_{Z'},$$

with $B_{Y'Z'} = u^+ A u^- (B_{AY'} B_{Z'} + X_{AB} e_{Y'Z'})$. It would be very interesting to find out the correspondence between this expression and ours (up to gauge transformations, of course).

To summarize, we have seen that the manifestly supersymmetric formulation of our sigma model naturally makes contact with the harmonic (or twistor) interpretation of self-duality. On the other hand, we have an alternative way to extract the SDYM field encoded in the action term (14). It consists in obtaining the part of the component action in which the massless chiral fermions $\lambda_+$ are coupled to a composite gauge field. To this end we have to use the non-supersymmetric Wess-Zumino gauge (23) rather than the manifestly supersymmetric one (26). To simplify our task we shall keep only the relevant fields, i.e., the fermions in $\Phi^+_+$ and $\Lambda^a_+$ and the bosons in $X^+$. Since $\chi^{-AY'}$ is accompanied by $(\theta^-)^2$ in (53), the other superfields in (64) contribute with their lowest order components only, i.e., $\Delta^a_{AY'}$ from (70) and $\Lambda^a_+$ from (46). This, together with the kinetic term for $\Lambda^a_+$ (48), gives

$$S = \int d^2 x du \left( \frac{i}{2} \lambda^a_+ \partial^+ \lambda^a_+ + i \sigma^{-a} \partial^+ \lambda^a_+ + m u^+ \chi^{A Y'} u^+ B_{AY'} \Delta^a_{Y'}(X) \lambda^a_+ \right).$$

The important point here is that the Lagrange multiplier term with $\sigma^{-a}$ has not changed, so we still have the field equation $\partial^+ \lambda^a_+(x, u) = 0 \rightarrow \lambda^a_+(x) = \lambda^a_+(x)$, like in the free case.
Then the harmonic integral in (83) becomes trivial and we obtain the action
\[ S = \int d^2 x \left( \frac{i}{2} \lambda^a_+ \partial_- \lambda^a_+ - \frac{m}{2} \chi^a_+ \Delta^a_{AY}(X) \lambda^a_+ \right). \] (84)

The subsequent steps are described by Witten and we only sketch them. One introduces an \( n \times (n + 4k') \) matrix \( v^a_i(X) \), orthogonal to \( \Delta \), \( \Delta^a_{AY} v^a_i = 0 \) and orthonormalized, \( v^a_i v^a_j = \delta_{ij} \) (cf. subsection 2.3). In a sense, this step is similar to the introduction of the matrix \( \hat{v}^{aa} (74) \). The aim is to diagonalize (84), i.e., to separate the massive fermions from the massless ones. The massless fermions are just \( \lambda^i_+ = v^i a \lambda^a_+ \). After that one puts all massive fields to 0 (or, equivalently, \( m \to \infty \)) and obtains
\[ S = \frac{i}{2} \int d^2 x \lambda^i_+ (\delta^{ij} \partial_- + \partial_- X^{BY} A^{ij}_{BY}) \lambda^j_+, \] (85)

where
\[ A^{ij}_{BY} = v^i a \frac{\partial v^j a}{\partial X^{BY}}. \] (86)

This is precisely the expression for the instanton field in the ADHM construction, see [23].

In the context of comparing the sigma model to the ADHM construction we would like to come back to the issue of counting the number of instanton parameters. As explained in subsection 2.3, the ADHM construction has another important ingredient, namely the freedom to redefine the matrices \( \Delta \) and still obtain the same instanton solution. There one uses \( GL(k', Q) \) transformations of the index \( Y' \) in the defining constraints (22), (23). In the sigma model above the kinetic term (54) is a quadratic form invariant under \( Sp(k') \) only. The consequence is that there exist equivalence classes of sigma models (obtained from one another by transformations from \( GL(k', Q)/Sp(k') \)), which produce the same SDYM field. However, the problem may not be so serious, as we only clearly see the SDYM field in the limit \( m \to \infty \), where the kinetic term (54) is suppressed and, effectively, the \( GL(k', Q) \) invariance of the ADHM construction is restored.

In conclusion we can say that the supersymmetric sigma model described in this section exhibits the following features. First, supersymmetry and the gauge invariance (51) lead to the very special interaction (64). In it we found the matrix \( \Delta (70) \), which is the starting point in the ADHM construction. Then, proceeding in a manifestly supersymmetric way, we were able to separate the massless from the massive modes in the model. In the massless part of the Lagrangian (80) we found the harmonic connection \( V^{++} \), which is the “twistor transform” of the SDYM field. Alternatively, a non-supersymmetric gauge allowed us to arrive at the harmonic-free action (84). Separating the massless and massive modes in it lead to the ADHM instanton field, instead of its “twistor transform”.

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Appendix A

Here we shall prove a “no-go” theorem about the off-shell content of the (0, 4) chiral fermion supermultiplet. In this multiplet the only propagating fields are chiral fermions $\lambda^a_\pm(x)$, with the action

$$S_\lambda = \frac{i}{2} \int d^2x \lambda^a_+ \partial^- \lambda^a_+,$$  \hspace{1cm} (A.1)

where $a = 1, 2, \ldots, n$, and $n$ is a positive integer. Note that this action possesses an $SO(n)$ symmetry with $\lambda^a_+(x)$ transforming as $SO(n)$ vector.

Since on shell $\partial^- \lambda^a_+ = 0$, the chiral fermions are inert under (0, 4) supersymmetry. An off-shell multiplet of chiral fermions must also include a number of auxiliary fields, since the number of fermionic fields should match the number of bosonic fields. The auxiliary fields vanish on shell.

We assume the following:

- (0, 4) supersymmetry is realized linearly;
- the multiplet fields form representations of $SO(n)$ and of the supersymmetry automorphism group $SU(2) \times SU(2)'$;
- the action is quadratic in the fields and invariant under $SO(n) \times SU(2) \times SU(2)'$.

Given this, there are no off-shell chiral fermion multiplets with a finite number of auxiliary fields, except for $n = 4$, where $SO(4)$ must be identified with $SU(2) \times SU(2)'$.

To prove this, let us first consider an off-shell multiplet describing one real chiral fermion, $n = 1$. Since the dimensions of $\lambda^a_+(x)$ and of the supersymmetry parameter $\epsilon^{AA'}$ are half-integer, the auxiliary bosonic fields should have integer dimensions $0, \pm 1, \pm 2, \ldots$, and the auxiliary fermions should have half-integer dimensions. Therefore, the auxiliary fermions must enter the Lagrangian (of dimension 2) in pairs, hence the total number of fermionic fields would be odd. Further, the bosonic fields always occur in multiples of four: the supersymmetry transformation implies that the bosons carry odd number of vector indices of $SO(4) = SU(2) \times SU(2)'$ (decomposing such a representation into irreducible ones, we find that all of them are multiples of four). Hence the fermionic and bosonic degrees of freedom do not match, and the off-shell multiplet does not exist.

The above argument is immediately generalized to $n > 1$. There the auxiliary fields carry the same vector index $a$ of $SO(n)$ as the chiral fermions. This clearly leads to the same mismatch of degrees of freedom. The only exception is the case $n = 4$, where it is possible to identify $SO(4)$ with $SU(2) \times SU(2)'$. \footnote{Now the four chiral fermions can be matched with four auxiliary bosons \cite{17}. Note, however, that in such a formulation the $SO(4)$ symmetry of (A.1) is tied to the supersymmetry automorphism group. As a consequence, it cannot be gauged independently, as required in the sigma model of Witten.\label{footnote1}}

For $n = 3$ $SO(3)$ can be identified with either $SU(2)$ or $SU(2)'$, but this case is ruled out because the total number of fermions is odd.
We want to stress that the above argument is a counting one, i.e., based on the assumption of finite numbers of auxiliary fields. As we have shown in the text, off-shell representations with infinitely many auxiliary fields exist for any number of chiral fermions.

Appendix B

The origin of the three types of \((0,4)\) superfields used in this paper becomes more clear if we regard them as obtained by trivial dimensional reduction and truncation from the two basic \(N = 2, D = 4\) supermultiplets, matter and gauge.

The adequate superspace for \(N = 2, D = 4\) supersymmetry is again harmonic [7]. The Grassmann variables in \(N = 2, D = 4\) supersymmetry are two-component \(SL(2, \mathbb{C})\) spinors \(\theta^A_\alpha\) and their complex conjugates \(\bar{\theta}^A_\dot{\alpha}\) with an additional \(SU(2)\) automorphism index \(A\). The spinor covariant derivatives

\[
D_{A\alpha} = \frac{\partial}{\partial \theta^A_\alpha} + i\bar{\theta}^A_\alpha \frac{\partial}{\partial \theta^A_{\dot{\alpha}}} \quad (B.1)
\]
satisfy the \(N = 2, D = 4\) supersymmetry algebra

\[
\{D^A_\alpha, D^B_\beta\} = \{\bar{D}^A_\dot{\alpha}, \bar{D}^B_\dot{\beta}\} = 0, \quad \{D^A_\alpha, \bar{D}^B_\dot{\beta}\} = 2i\epsilon_{AB}\partial^{\alpha\dot{\beta}}. \quad (B.2)
\]

Projecting the \(SU(2)\) indices \(A, B\) with harmonic variables, \(D^+\equiv u^A D^A_\alpha, \quad \bar{D}^+\equiv u^A \bar{D}^A_{\dot{\alpha}}\), we can rewrite (B.2) as follows

\[
\{D^+\alpha, D^+\beta\} = \{\bar{D}^+\dot{\alpha}, \bar{D}^+\dot{\beta}\} = \{D^+\alpha, \bar{D}^+\dot{\beta}\} = 0. \quad (B.3)
\]

Grassmann analytic superfields are defined by the constraint

\[
D^+\alpha \Phi(x, \theta, u) = \bar{D}^+\dot{\alpha} \Phi(x, \theta, u) = 0, \quad (B.4)
\]
that is explicitly solved in the analytic basis in harmonic superspace

\[
x^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + 2i\theta^A_\alpha \bar{\theta}^B_{\dot{\beta}} u^+_{(A} u^-_{B)}, \quad \theta^{\pm\alpha} = u^A_\alpha \theta^A_{\alpha}, \quad \bar{\theta}^{\pm\dot{\alpha}} = u^A_{\dot{\alpha}} \bar{\theta}^A_{\dot{\alpha}}, \quad u^\pm \quad (B.5)
\]
in the form \(\Phi = \Phi(x, \theta^+, \bar{\theta}^+, u)\) (we shall not write \(\hat{x}\) explicitly). In this basis the spinor derivatives \(D^+\) are just partial derivatives, but the harmonic derivative \(D^{++}\) becomes covariant:

\[
D^{++} = \partial^{++} + 2i\theta^{+\alpha} \bar{\theta}^{+\dot{\alpha}} \partial^{\alpha\dot{\alpha}}. \quad (B.6)
\]

In \(N = 2, D = 4\) rigid supersymmetry there are two basic supermultiplets, the matter (hyper)multiplet and the gauge one (we do not discuss local supersymmetry, i.e., supergravity). Both of them can be described by appropriate analytic harmonic superfields [7]. Matter is described in terms of a set of commuting superfields \(q^{+a}(x, \theta^+, \bar{\theta}^+, u)\) with the action

\[
S_q = \frac{1}{2} \int d^4x d^4u \theta^+ q^{+a} D^{++} q^{+a}, \quad (B.7)
\]
where the index $a$ corresponds to the $2n$ representation of $Sp(n)$. In addition, these superfields satisfy the reality condition $\tilde{q}^{+a} = \epsilon_{ab}q^{+b}$. It should be emphasized that off shell the superfields $q^+$ contain an infinite number of auxiliary fields, in accord with the “no-go” theorem of ref. [10]. Only on shell the equation of motion $D^{++}q^+ = 0$ eliminates the harmonic dependence and one obtains short multiplets consisting of $4n$ real bosons and $4n$ fermions. Here one should recall the similar equation (40), which was a purely kinematical (off-shell) constraint in $N = (0,4), D = 2$ supersymmetry.

The (abelian) gauge multiplet is described by a real commuting superfield of the type $A^{++}$ undergoing the gauge transformation $\delta A^{++} = D^{++}\Omega$ with an analytic superfield parameter $\Omega$. In the Wess-Zumino gauge the superfield $A^{++}$ has the short harmonic expansion

\[
A^{++} = \left[(\theta^+)^2M(x) + c.c.] + i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}A_{\alpha\dot{\alpha}}(x) + [(\theta^+)^2\theta^{+\alpha}u^-\chi^\alpha(x) + c.c.]
+ (\theta^+)2(\theta^+)2u^-_Au^-_BD^{AB}(x) \right].
\] (B.8)

Here $M$ is the physical scalar (complex), $A$ is the gauge field, $\chi$ are the physical spinors and $D$ are auxiliary fields. Note that this gauge multiplet, unlike the matter one above, does not need infinitely many auxiliary fields off shell.

The action for $A^{++}$ resembles (54):

\[
S_A = \int d^4x d^8\theta du_1 du_2 \frac{1}{(u_1^+u_2^+)^2}A^{++}(1)A^{++}(2).
\] (B.9)

This time we had to use the harmonic distribution $(u_1^+u_2^+)^{-2}$ in order to match the $U(1)$ charges. Instead of (H), it satisfies the identity

\[
D^{++}_1 \frac{1}{(u_1^+u_2^+)^2} = D^{--}_1 \delta^{2-2}(u_1,u_2),
\] (B.10)

which is still sufficient to prove the gauge invariance of (B.9).

Now we dimensionally reduce the above superfields to $D = 2$. The resulting (4,4) superfields depend on the left- and right-handed Grassmann variables $\theta^+_\pm A'$. Further, they can be truncated to (0,4) superfields. This simply means that we decompose the (4,4) superfields in the right-handed $\theta^+_\pm$ and then discard some of the resulting (0,4) superfields. Thus, the abelian gauge superfield has the decomposition

\[
A^{++} = B^{++} + \theta^+_\pm A'^+ + (\theta^+_\pm)^2C^{++}.
\] (B.11)

Each of the coefficients here is a (0,4) superfield, $B$ and $C$ are even, whereas $\Phi$ is odd. It is quite obvious that $\Phi^{+A'}_+\Phi^{+A'}_{+A'}$ is the model for the odd abelian gauge superfield (II). The decomposition and truncation of the gauge transformation law for $A^{++}$ explain (P). A similar procedure leads from the gauge invariant action (B.9) to its (0,4) counterpart (54) (a factor of $u^+_1u^+_2$ appears in the numerator due to the integration over $\theta^+_\pm$).

The other two multiplets used in section 3 both have their origin in an $N = 2, D = 4$ hypermultiplet, they just correspond to different truncations. Thus, from the decomposition of $q^{+a}$ we take only the odd term

\[
q^{+a} = \theta^+_A A'^a_a.
\] (B.12)
and from that of $X^+Y$ only the even terms

$$X^+Y = X^+Y + (\theta^+)^2 P^-Y.$$ (B.13)

In both cases the corresponding actions (17) and (13) are obtained from the hypermultiplet action (3.7) (the superfield $P^-Y$ serves as a Lagrange multiplier in the action for $X^+$, which produces the condition (10)).

We should point out that the above truncation gives the correct type of (0,4) superfields, but does not completely reproduce the symmetry of the (0,4) sigma model. For instance, the superfield $\Lambda^t_a$ in (3.12) describes the chiral fermions in multiples of 4, but we know this does not have be the case in the (0,4) sigma model. The reason is that (0,4) supersymmetry is less restrictive than the (4,4) one.
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