A characterization of normal forms for control systems

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1 Introduction

The study of the behavior of solutions of ODEs often benefits from deciding on a convenient choice of coordinates. This choice of coordinates may be used to “simplify” the functional expressions that appear in the vector field in order that the essential features of the flow of the ODE near a critical point become more evident. In the case of the analysis of an ordinary differential equation in the neighborhood of an equilibrium point, this naturally leads to the consideration of the possibility to remove the maximum number of terms in the Taylor expansion of the vector field up to a given order. This idea was introduced by H. Poincaré in [25] and the “simplified” system is called normal form. There have been several applications of the method of normal forms particularly in the context of bifurcation theory where one combines between the method of normal forms and the center manifold theorem in order to classify bifurcations [9]. This approach was extended to control systems in continuous-time by Kang and Krener ([17], see also [18] for a survey) and Tall and Respondek ([23], see [24] for a survey), and by Barbot et al. [2] and Hamzi et al. in discrete-time [15, 16]. The center manifold theorem was extended to control systems by Hamzi et al. [10, 11] and combined with the normal forms approach to analyze and stabilize systems with bifurcations in continuous and discrete-time [12, 13, 14].

On another side, even though in many textbook treatments (see eg [9]) the emphasis is on the reduction of the number of monomials in the Taylor expansion, one of the main reasons for the success of normal forms lies in the fact that it allows to analyze a dynamical system based on a simpler form and a simpler form doesn’t necessarily mean to remove the maximum number of terms in the Taylor series expansion. This observation, led to introduce the so-called “inner-product normal forms” in [3,19,7]. They are based on properly choosing an inner product that allows to simplify the computations. This inner-product will characterize the space over which one performs the Taylor series expansion. The elements in this space are the ones that characterize the normal form. Our goal in this paper is to generalize such an approach to control systems.

In section §2, we review some results about normal forms. In section §3, we develop a new method for deriving normal forms for control systems.

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2 Normal forms near equilibria of ODEs

In this section we briefly review some results on normal forms near equilibria of nonlinear ODEs.

Consider the nonlinear ODE in $\mathbb{R}^n$

$$\dot{x} = Ax + f(x), \quad (2.1)$$

with $f \in C^{r+1}(\mathbb{R}^n; \mathbb{R}^n)$, $f(0) = 0$ and $A = \frac{\partial f}{\partial x}|_{x=0}$ is in real or complex Jordan form. Without loss of generality the latter condition can be met by application of a linear coordinate transformation.

The goal is to find a change of coordinates

$$x = \xi(y), \quad (2.2)$$

with $\xi \in C^r(\mathbb{R}^n; \mathbb{R}^n)$ in a neighborhood of the origin, such that the Taylor expansion of (2.1) is simple, making essential features of the flow of (2.1) near the equilibrium $x = 0$ more evident. The desired simplification of (2.1) will be obtained, up to terms of a specified order, by constructing a near identity coordinate transformation from a sequence of compositions of coordinate transformations of the form (2.2) with

$$\xi(y) = \exp(\xi[y])(y) = y + \xi[y](y) + O(|y|^{k+1}), \quad (2.3)$$

where $y \in \mathbb{R}^n$ is close to zero, $\xi[y] \in H^k_n$ ($k \geq 2$), the vector space of homogeneous polynomials of degree $k$ in $n$ variables with values in $\mathbb{R}^n$, and $\exp(\xi[y])$ denotes the time-one flow of the ODE $\dot{y} = \xi[y](y)$. We consider a formal power series expansion of $f$ in (2.1) and write

$$f(x) = f^{[2]}(x) + f^{[3]}(x) + \ldots, \quad (2.4)$$

with $f^{[k]} \in H^k_n$. From (2.3) we obtain

$$\xi^{-1}(y) = y - \xi[y](y) + O(|y|^{2k}). \quad (2.5)$$

Substituting (2.2), (2.3) and (2.5) in (2.1), we get

$$\dot{y} = Ay + \cdots + f^{[k-1]}(y) + f^{[k]}(y) - (L_A \xi^{[k]})(y) + O(|y|^{k+1}), \quad (2.6)$$

with the Lie derivative $L_A$ defined on vector fields $f$ as

$$(L_A f)(y) := \frac{\partial f(y)}{\partial y} Ay - Af(y). \quad (2.7)$$

In the present context $L_A$ is also known as the homological operator.

The Lie derivative leaves $H^k_n$ invariant, $L_A : H^k_n \to H^k_n$. We denote its range in $H^k_n$ as $\mathcal{R}^k$ and let $\mathcal{C}^k$ denote a complement of $\mathcal{R}^k$ in $H^k_n$

$$H^k_n = \mathcal{R}^k \oplus \mathcal{C}^k, \quad k \geq 2. \quad (2.8)$$

We define a normal form of $f$ of order $r$ as a Taylor expansion of the vector field with linear part and terms $f^{[k]} \in \mathcal{C}^k$ for $2 \leq k \leq r$. We may associate the choice of complement $\mathcal{C}^k$ to an inner product on $H^k_n$, for which it is the orthogonal complement of $\mathcal{R}^k$ in $H^k_n$, i.e. $\mathcal{C}^k := (\mathcal{R}^k)\perp$. \newpage
A convenient choice of inner product was introduced by Belitskii [3], Meyer [19] and Elphick et al. [7], enabling the characterization of expression of $C_k$ as the kernel of the Lie derivative of $A^*$ (the adjoint of linear part $A$ of the vector field at the equilibrium). Denoting monomials in shorthand notation as $x^\ell := x_1^{\ell_1} \cdots x_n^{\ell_n}$ with $\ell! := \ell_1! \cdots \ell_n!$, we define an inner product on polynomials

$$p(x) = \sum \ell \ p_x^\ell, \ q(x) = \sum \ q_m x^m, \ \text{as} \ \langle p, q \rangle = \sum \ m! p_m q_m. \quad (2.9)$$

For vector polynomials we define the corresponding inner product as the sum of the inner products between the polynomials of corresponding vector components. The inner product $(2.9)$ with $T \in gl(n, \mathbb{R})$ and $T^*$ denoting its adjoint (with respect to the standard inner product on $\mathbb{R}^n$) satisfies

$$\langle p \circ T, q \rangle = \langle p, q \circ T^* \rangle. \quad (2.10)$$

Accordingly, one obtains that the adjoint of $L_A$ on $H_n^k$ with the above defined inner product satisfies the following relation $[3, 7]$

$$(L_A)^* = L_{A^*}. \quad (2.11)$$

By application of the Fredholm alternative, it follows that $(\mathcal{R}^k)^\perp = \ker(L_{A^*}|_{H^k_n})$. In combination with $(2.11)$, this leads us to

$$C_k = \ker(L_{A^*}|_{H^k_n}),$$

as a result of which nonlinear elements of the normal form $g$ satisfy the linear PDE

$$L_{A^*} g = 0. \quad (2.12)$$

This PDE can be solved explicitly using the method of characteristics (for more details on this method, see for example [6]).

We recall that since $L_{A^*}$ is a Lie derivative, it follows that the nonlinear elements of the normal form commute with the group

$$G = \{ \exp(A^* t) \mid t \in \mathbb{R} \}. \quad (2.13)$$

We finally note that

$$\ker(L_{A^*}^k) = \ker(L_{A^*_s}^k) \cap \ker(L_{A^*_n}^k), \quad (2.14)$$

where $A = A_s + A_n$ is the Jordan-Chevalley decomposition of $A$ in its (mutually commuting) semi-simple and nilpotent parts. As $A^*$ commutes with $A_s$ but not with $A_n$ (if nonzero), if $A$ is not semi-simple, only a subgroup of $G$ (as defined above) is a symmetry group of the normal form. In general, with the above choices made, the normal form is equivariant with respect to the group

$$G_s = \{ \exp(A^*_s t) \mid t \in \mathbb{R} \}. \quad (2.15)$$

The appearance of this symmetry group is an important feature.

3 Normal Forms of Nonlinear Control Systems

The object of this section is to extend the normal form theory set out above to nonlinear control systems. We consider the nonlinear control system

$$\dot{x} = f(\dot{x}), \quad (3.16)$$
With $\tilde{x} = (x, u)^T \in \mathbb{R}^n \times \mathbb{R}^m$ and $u \in \mathbb{R}^m$ representing the control. In lowest linear order Taylor expansion, the control system takes the form

$$\dot{x} = A\tilde{x} + O(|\tilde{x}|^2),$$

with $A := \begin{pmatrix} A & B \end{pmatrix}$ and $\tilde{x} := \begin{pmatrix} x_\epsilon & u \end{pmatrix}$.

We consider the effect of coordinate transformations of the form (3.17) takes precisely the form of the Lie derivative viewpoint as if the coordinate transformation would be for the ODE $(\dot{p})_{\text{using coordinate transformations of the form}}$

$$L f = \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} \cdot \frac{\partial \pi}{\partial x_i}(\tilde{x}) \cdot \frac{\partial \pi}{\partial x}$$. 

We may thus choose the complement of the range of $L A_0$ with $x_{n+i} = u_i$, $i = 1, \ldots, m$, i.e.

$$C^k := \{ q \in H^k_{m+n} \mid \langle q, L A_0 p \rangle = 0, \forall p \in S^k_{n,m} \}. \tag{3.21}$$

By the Fredholm alternative we have

$$\langle q, L A_0 p \rangle = \langle L A_0^* q, p \rangle, \tag{3.22}$$

$$\langle q, L A_0 p \rangle = \langle L A_0^* q, p \rangle,$$
so that this complement takes the form
\[ C^k := \{ q \in H^{k}_{m+n} \mid L_{A^*_0} q \in (S^k_{n,m})^\perp \} . \] (3.23)

By the definition of the inner product (2.9),
\[ (S^k_{n,m})^\perp = \{ q \in H^{k}_{m+n} \mid q(x,0) = 0 \}, \]

i.e. the subset of vector polynomials in \( H^{k}_{m+n} \) for which each constituting monomial contains a factor \( u_i, \ i = 1, \ldots, m \). The complement to the range of \( \mathcal{L}_A \) characterising the corresponding normal form is \( \pi C^k \). By writing out the relevant operators, the following result follows immediately.

**Theorem 3.1 (Control normal form)** Consider a finite order in Taylor expansion of the vector field defining the control system (3.16),
\[ f(\tilde{x}) = A\tilde{x} + \sum_{k=2}^{N} f^{[k]}(\tilde{x}) + O(|\tilde{x}|^{k+1}), \text{ with } f^{[k]} \in H^{k}_{m+n,n}. \]

By a choice of coordinates, the nonlinear parts \( f^{[k]} \) can be made to satisfy
\[ \hat{\mathcal{L}}_A^* f^{[k]}(x,0) = 0 \] (3.24)
where
\[ \hat{\mathcal{L}}_A^* f^{[k]}(\tilde{x}) := D_{\tilde{x}} f^{[k]}(\tilde{x}) A^* x - A^* f^{[k]}(\tilde{x}), \] (3.25)
and \( \tilde{x} = (x, u) \).

**Remark.** We note that by restricting first to coordinate transformations that do not involve \( u \), we can achieve \( G_s \)-equivariance of the control system to any desired order. Then we can refine the normalization further using \( G_s \)-equivariant coordinate transformations that preserve this equivariance.

\[ \therefore \]

4 **Illustrations**

4.1 **Linearly Controllable Case**

To illustrate this method, consider the nonlinear control system \( \Sigma \) in (3.16) with one input, i.e. \( m = 1 \), and assume that its linearization is controllable. From linear control theory we know that there exists a linear change of coordinates and feedback that allows to transform the linear part in the Brunovský form, i.e.

\[ A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \] (4.26)
In this case, the PDE (3.25) becomes
\[
\begin{align*}
\begin{bmatrix}
\frac{x_1}{\partial x_2} & + & \cdots & + & \frac{x_n}{\partial x_n} & + & x_n & \frac{\partial p_1}{\partial u} & = & 0 \\
\frac{x_1}{\partial x_2} & + & \cdots & + & \frac{x_n}{\partial x_n} & + & x_n & \frac{\partial p_2}{\partial u} & - & p_1 & = & 0 \\
\vdots & & & & & & & & & & & & \\
\frac{x_1}{\partial x_2} & + & \cdots & + & \frac{x_n}{\partial x_n} & + & x_n & \frac{\partial p_n}{\partial u} & - & p_{n-1} & = & 0 \\
\end{bmatrix}
\end{align*}
\]

(4.27)

that we’ll solve using the method of characteristics.

**Theorem 4.1** Consider the nonlinear control system \( \Sigma \) given by (3.16). There exist a change of coordinates and feedback (3.17) such that \( \Sigma \) writes as
\[
\begin{align*}
\dot{x}_1 & = x_2 + \Phi_1(\ell_1, \cdots, \ell_{r+1}), \\
\dot{x}_2 & = x_3 + \Phi_2(\ell_1, \cdots, \ell_{r+1}) + \int p_1(x, u) \frac{dx_2}{x_1}, \\
& \vdots \\
\dot{x}_n & = u + \Phi_n(\ell_1, \cdots, \ell_{r+1}) + \int p_{n-1}(x, u) \frac{dx_2}{x_1},
\end{align*}
\]

(4.28)

with \( \Phi_i(\ell_1, \cdots, \ell_n) \) are functions satisfying
\[
\left. \frac{\Phi_i(\ell_1, \cdots, \ell_n)}{x_1^p} \right|_{x_1=0} = 0 \quad \text{for} \quad p = 0, \cdots, n-i,
\]

(4.29)

(4.30)

and \( \ell_1(x) = x_1, \ell_2(x) = \frac{x_2^2}{2} - x_1 x_3, \cdots, \ell_i(x) = \frac{1}{2} x_i^2 + \sum_{k=1}^{n-p} (-1)^k x_{i-k} x_{i+k} \) for \( i = 2, \cdots, r+1 \), and \( r \) is such that \( r = n/2 \) if \( n \) is even, and \( r = (n-1)/2 \) if \( n \) is odd (here, \( x_0 = 0 \) and \( x_{n+1} = u \)).

**Proof.**

In the \( n \)-dimensional space of the variables \( x_1, x_2, \cdots, x_n \) we determine the curves \( x_i = x_i(s) \) in terms of a parameter \( s \) by means of the system of ordinary differential equations that represent the characteristic curves
\[
\begin{align*}
\begin{bmatrix}
\frac{dx_1}{ds} & = & 0 \\
\frac{dx_2}{ds} & = & x_1 \\
& \vdots \\
\frac{dx_n}{ds} & = & x_{n-1} \\
\frac{du}{ds} & = & x_n
\end{bmatrix}
\end{align*}
\]

(4.31)
Along the characteristic curves and using the chain rule, the systems of PDEs (3.25) writes as

\[
\frac{dp_1}{ds} = \frac{dx_1}{ds} \frac{\partial p_1}{\partial x_1} + \frac{dx_2}{ds} \frac{\partial p_1}{\partial x_2} + \cdots + \frac{dx_n}{ds} \frac{\partial p_1}{\partial x_n} + \frac{du}{ds} \frac{\partial p_1}{\partial u} = 0
\]

\[
\frac{dp_2}{ds} = \frac{dx_1}{ds} \frac{\partial p_1}{\partial x_1} + \frac{dx_2}{ds} \frac{\partial p_1}{\partial x_2} + \cdots + \frac{dx_n}{ds} \frac{\partial p_1}{\partial x_n} + \frac{du}{ds} \frac{\partial p_1}{\partial u} = p_1
\]

\[
\vdots
\]

\[
\frac{dp_n}{ds} = \frac{dx_1}{ds} \frac{\partial p_1}{\partial x_1} + \frac{dx_2}{ds} \frac{\partial p_1}{\partial x_2} + \cdots + \frac{dx_n}{ds} \frac{\partial p_1}{\partial x_n} + \frac{du}{ds} \frac{\partial p_1}{\partial u} = p_{n-1}
\]

Hence, along the characteristic curves defined by (4.31), the systems of PDEs (3.25) transforms into a set of ODEs

\[
\frac{dp_1}{ds} = 0
\]

\[
\frac{dp_2}{ds} = p_1
\]

\[
\vdots
\]

\[
\frac{dp_n}{ds} = p_{n-1}
\]

This system of ODEs can be solved explicitly

\[
p_1(s) = c_1
\]

\[
p_2(s) = c_2 + \int p_1(s) ds
\]

\[
\vdots
\]

\[
p_n(s) = c_n + \int p_{n-1}(s) ds
\]

The “constants of integration”, \(c_i\), are the constants along the characteristic curves which are the trivial first integrals of the system (4.31). One can check that they are given by \(\ell_1(x) = x_1\), \(\ell_2(x) = \frac{x_1^2}{2} - x_1 x_3, \ldots, \ell_i(x) = \frac{1}{2}x_i^2 + \sum_{k=1}^{n-p} (-1)^k x_{i-k} x_{i+k}\) for \(i = 2, \ldots, r+1\), and \(r\) is such that \(r = n/2\) if \(n\) is even, and \(r = (n-1)/2\) if \(n\) is odd (for notation convenience, \(x_0 = 0\) and \(x_{n+1} = u\)). From (4.31), we have\(^1\) \(ds = \frac{dx_2}{x_1}\), and the solution of (4.33) is given by

\[
p_1(x, u) = \Phi_1(\ell_1, \cdots, \ell_{r+1})
\]

\[
p_2(x, u) = \Phi_2(\ell_1, \cdots, \ell_{r+1}) + \int p_1(x, u) \frac{dx_2}{x_1}
\]

\[
\vdots
\]

\[
p_n(x, u) = \Phi_n(\ell_1, \cdots, \ell_{r+1}) + \int p_{n-1}(x, u) \frac{dx_2}{x_1}
\]

where \(\Phi_i(\ell_1, \cdots, \ell_n), i = 1, \cdots, n\), are functions of the variables \(\ell_1, \cdots, \ell_n\) and are thus constants along the characteristic curves defined in (4.31). Since \(\Phi_i(\ell_1, \cdots, \ell_n)\) and \(\tilde{q}_i(x, u)\) satisfy the conditions

\[
\left.\frac{\Phi_i(\ell_1, \cdots, \ell_n)}{x_1^p}\right|_{x_1=0} = 0 \quad \text{for} \quad p = 0, \cdots, n-i,
\]

\(^1\)We can also use \(ds = \frac{dx_{i+1}}{x_i}\) and in this case the normal form will be parametrized by \(x_{i+1}\). We can also parameterize each component with a different parameterization.
thus \( p_1, \ldots , p_{n-1} \) are divisible by \( x_1 \). Hence \( p_1(x,u), \ldots , p_n(x,u) \) in (4.34) are polynomials.

### 4.1.1 Example

Consider a two dimensional system with controllable linearization. In this case, the linear part of (3.16) writes as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u
\end{align*}
\]

and the PDE (3.25) writes as

\[
\begin{align*}
x_1 \frac{\partial p_1}{\partial x_2} + x_2 \frac{\partial p_1}{\partial u} &= 0, \\
x_1 \frac{\partial p_2}{\partial x_2} + x_2 \frac{\partial p_2}{\partial u} &= 0
\end{align*}
\]

Hence, we get

\[
\begin{align*}
\frac{dp_1}{ds} &= 0, \\
\frac{dp_2}{ds} - p_1 &= 0
\end{align*}
\]

with

\[
\begin{align*}
\frac{dx_1}{ds} &= 0, \\
\frac{dx_2}{ds} &= x_1, \\
\frac{du}{ds} &= x_2, \\
\frac{dp_1}{ds} &= 0, \\
\frac{dp_2}{ds} - p_1 &= 0
\end{align*}
\]

We thus deduce the following parametrization of the solution

\[
\begin{align*}
x_1 &= x_{1,0}, \\
x_2 &= x_{1,0} s + x_{2,0}, \\
u &= \frac{x_{1,0}}{2} s^2 + x_{2,0} s + x_{3,0}
\end{align*}
\]

The first integrals are \( \ell_1(x,u) = x_1 \) and \( \ell_2(x,u) = 2x_1 u - x_2^2 \). From (4.39d)-(4.39e) we deduce that

\[
p_1(x,u) = \Phi_1(x_1, 2x_1 u - x_2^2)
\]
\[ p_2(x, u) = \int p_1(t) dt + \Phi_2(x_1, 2x_1 u - x_2^2) \]  

(4.42)

We can use either (4.39b) or (4.39c) to express the normal form as a function of \( x_2 \) or \( u \). For example, using (4.39b) we deduce that

\[ dt = dx_2 \]

Moreover, using (4.29), we obtain conditions on \( \Phi_i(\ell_1, \ell_2) \) and \( \tilde{q}_i, i = 1, 2 \),

\[ \Phi_i(\ell_1, \ell_2)|_{x_1=0} = 0, \]

At the quadratic level these conditions imply that

\[ \Phi_1(\ell_1, \ell_2)|_{x_1=0} = \phi_{11} x_1^2 + O(x, u)^3 \]  

(4.43a)

\[ \Phi_2(\ell_1, \ell_2)|_{x_1=0} = \tilde{\phi}_{11} x_1^2 + \tilde{\phi}_{12}(2x_1 u - x_2^2) + O(x, u)^3 \]  

(4.43b)

Hence

\[ p_1(x, u) = \phi_{11} x_1^2 + O(x, u)^3 \]  

(4.44a)

\[ p_2(x, u) = \phi_{11} x_1 x_2 + \tilde{\phi}_{11} x_1^2 + \tilde{\phi}_{12}(2x_1 u - x_2^2) + O(x, u)^3 \]  

(4.44b)

Hence the normal form has the form

\[ \dot{x}_1 = x_2 + \phi_{11} x_1^2 + O(x, u)^3 \]  

(4.45a)

\[ \dot{x}_2 = u + \phi_{11} x_1 x_2 + \tilde{\phi}_{11} x_1^2 + \tilde{\phi}_{12}(2x_1 u - x_2^2) + O(x, u)^3 \]  

(4.45b)

### 4.2 Systems with Uncontrollable Linearization

Now, consider the nonlinear control system \( \Sigma \) in (3.16) with one input, i.e. \( m = 1 \), and assume that the system has \( r \) uncontrollable modes. From linear control theory we know that there exists a linear change of coordinates and feedback that allows to write the linear part as

\[ \dot{z} = A_1 z + O(z, x, u)^2, \]

\[ \dot{x} = A_2 x + B_2 u + O(z, x, u)^2 \]

where \( z \in \mathbb{R}^{r\times 1}, x \in \mathbb{R}^{(n-r)\times 1}, A_1 \in \mathbb{R}^{r\times r}, \) and \( (A_2, B_2) \in \mathbb{R}^{(n-r)\times (n-r)} \times \mathbb{R}^{(n-r)\times 1} \) are in the Brunovský form.

In this case, \( A = \begin{pmatrix} A_1 & 0 \\ A_2 & B_2 \end{pmatrix} \), \( \tilde{x} = (z, x, u)^T \) in the PDE (3.25). Let’s note that when \( r = 0 \) we recover the case in the preceding section and we can find a general explicit solution. However, when \( r \neq 0 \) a general solution is not as easily found and depends on \( A_0 \). We’ll illustrate the method through an example.
4.2.1 Example

Consider the system whose linear part writes as

\[
\begin{align*}
\dot{z} &= O(z, x, u)^2, \\
\dot{x}_1 &= x_2 + O(z, x, u)^2 \\
\dot{x}_2 &= u + O(z, x, u)^2
\end{align*}
\] (4.48)

This system has uncontrollable linearization and the uncontrollable dynamics corresponds to the \( z \)-dynamics.

The elements of the normal form satisfy the PDE

\[
\begin{align*}
3 \frac{\partial p_1}{\partial x_2} + x_2 \frac{\partial p_1}{\partial u} &= 0 \\
3 \frac{\partial p_2}{\partial x_2} + x_2 \frac{\partial p_2}{\partial u} &= 0 \\
3 \frac{\partial p_3}{\partial x_2} + x_2 \frac{\partial p_3}{\partial u} - p_2 &= 0
\end{align*}
\] (4.49)

The equation of the characteristics is

\[
\begin{align*}
\frac{dp_1}{ds} &= 0 \\
\frac{dp_2}{ds} &= 0 \\
\frac{dp_3}{ds} &= p_2
\end{align*}
\] (4.50)

The characteristic equations are

\[
\begin{align*}
z &= c_1, \\
x_1 &= c_2, \\
x_2 &= c_1 s + c_3, \\
u &= \frac{c_1}{2} s^2 + c_3 s + c_4.
\end{align*}
\]

We can either parametrize by \( x_2 \) or \( u \) by writing \( ds = \frac{dx_2}{2} \) or \( ds = \frac{du}{x_2} \).

The solution of the system of PDEs (4.49) is

\[
\begin{align*}
p_1 &= \Psi_0(z, x_1, x_2^2 - 2zu) \\
p_2 &= \Psi_1(z, x_1, x_2^2 - 2zu) \\
p_3 &= \Psi_2(z, x_1, x_2^2 - 2zu) + \int p_2(z, x, u) \frac{dx_2}{2}
\end{align*}
\] (4.51)

The normal form is thus given by

\[
\begin{align*}
\dot{z} &= \Psi_0(z, x_1, x_2^2 - 2zu) \\
\dot{x}_1 &= x_2 + \Psi_1(z, x_1, x_2^2 - 2zu) \\
\dot{x}_2 &= u + \Psi_2(z, x_1, x_2^2 - 2zu) + \int p_2(z, x, u) \frac{dx_2}{2}
\end{align*}
\] (4.52)

5 Concluding remark and future extensions:

Given the preceding, one could think about hyper normal forms where instead of normalizing with respect to the linear term, one normalizes the quadratic term with respect to the linear term, then normalize the cubic term with respect to the sum of the linear and quadratic terms, and so forth. This direction has been fruitful for systems without control \[21, 22\] and its extension to the control case is the object of future research. Several other extensions are possible for this work. One could think about characterizing completely the normal form in the case of systems with uncontrollable
linearization, developing the Hamiltonian case, and computing the coefficients in the normal form directly from the original system.

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References

[1] Arnold, V.I. (1983). Geometrical Methods in the Theory of Ordinary Differential Equations. Springer.

[2] Barbot, J.-P., S. Monaco and D. Normand-Cyrot (1997). Quadratic forms and approximated feedback linearization in discrete time, in Int. Journal of Control, 67, 4, pp 567-587.

[3] Belitskii, G. R. (1979). Invariant Normal Forms and Formal Series. Functional Analysis and Applications, 13, 59-60.

[4] Belitskii, G. R. (2002). $C^\infty$-normal forms of local vector fields, Acta Appl. Math. 70, 23-41.

[5] Chow, S.-N., C. Li, D. Wang (1994). Normal Forms and Bifurcation of Planar Vector Fields. Cambridge University Press.

[6] Courant, R. and D. Hilbert (1961). Methods of Mathematical Physics, vol. II. Interscience Publishers.

[7] Elphick, C., E. Tirapegui, M.E. Brachet, P. Coullet and G. Iooss (1987). A Simple Global Characterization for Normal Forms of Singular Vector Fields. Physica D, 29, 95-127.

[8] Elphick, C. (1988). Global Aspects of Hamiltonian Normal Forms. Physics Letters A, 127, 418-424.

[9] Guckenheimer, J. and P. Holmes. (1983). Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer.

[10] Hamzi, B., A. J. Krener and W. Kang, The Controlled Center Dynamics of Discrete-Time Control Bifurcations, Systems and Control Letters, 55, 7, 585-596, 2006.

[11] Hamzi, B., W. Kang and A. J. Krener, The Controlled Center Dynamics, SIAM J. on Multiscale Modeling and Simulation, 3, 4, 838-852, 2005.

[12] Hamzi, B., W. Kang and J.-P. Barbot, Analysis and Control of Hopf Bifurcations, SIAM J. on Control and Optimization, 42, 6, 2200-2220, 2004.

[13] Hamzi, B., J.-P. Barbot, S. Monaco, and D. Normand-Cyrot, Nonlinear Discrete-Time Control of Systems with a Naimark-Sacker Bifurcation, Systems and Control Letters, 44, pp. 245-258, 2001.

[14] Hamzi, B. (2001). Quadratic Stabilization of Nonlinear Control Systems with a Double-Zero Control Bifurcation, Proc. of the 5th IFAC symposium on Nonlinear Control Systems (NOLCOS’2001), pp. 161-166, 2001.
[15] Hamzi, B., J.-P. Barbot and W. Kang, *Normal Forms for Discrete-Time Parameterized Systems with Uncontrollable Linearization*, Proc. of the 38th IEEE Conference on Decision and Control, pp. 2035–2039, 1999.

[16] Hamzi, B. and I. A. Tall, Normal Forms for Discrete-Time Control Systems, Proc. of the 42nd IEEE Conference on Decision and Control, 2, 1357 - 1361, 2003.

[17] W. Kang and A. J. Krener, Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems, SIAM J. Control and Optimization, 30 (1992), 1319-1337.

[18] Kang, W., and A. J. Krener (2006). Normal Forms of Nonlinear Control Systems, in Chaos in Automatic Control, W. Perruquet and J-P. Barbot (Eds.), pp. 345-376.

[19] Meyer, K. R. (1984). Normal Forms for the General Equilibrium, Funkcialaj Ekvacioj, 27, pp. 261-271.

[20] Meyer, K. R., G. R. Hall, and D. Offin (2009). *Introduction to Hamiltonian dynamical systems and the N-body problem*. Springer.

[21] Murdock, J. (2003). *Normal Forms and Unfoldings for Local Dynamical Systems*. Springer.

[22] Murdock, J. (2004). Hypernormal form theory: foundations and algorithms, *Journal of Differential Equations*, 205, 424-465.

[23] Tall, I.A. and W. Respondek (2003), ”Feedback Classification of Nonlinear Single-Input Control Systems with Controllable Linearization: Normal Forms, Canonical Forms, and Invariants”, in SIAM Journal on Control and Optimization, 41(5), pp. 1498-1531

[24] Tall, I.A. and W. Respondek (2006). Feedback Equivalence of Nonlinear Control Systems: A Survey on Formal Approach, in Chaos in Automatic Control, W. Perruquet and J-P. Barbot (Eds.), pp. 137-262.

[25] Poincaré, H. (1885). Mémoire sur les courbes définies par une équation différentielle, J. Maths Pures Appl., 4, 1, pp. 167-244.