OCTONIONS AND THE TWO RATIONAL TIGHT PROJECTIVE 5-DESIGNS

BENJAMIN NASMITH

ABSTRACT. There are precisely two unique tight projective 5-designs with rational angle sets, one constructed from the short vectors of the Leech lattice and the other corresponding to a generalized hexagon structure in the octonion projective plane. This paper describes a new connection between these two projective 5-designs—a common construction using octonions. A set of well chosen octonion involutionary matrices acts on a three-dimensional octonion vector space to produce the first 5-design and these same matrices act on the octonion projective plane to produce the second 5-design. This result draws on the octonion construction of the Leech lattice due to Robert Wilson and provides a new link between the generalized hexagon GH(2,8) and the Leech lattice.

1. INTRODUCTION

A spherical $t$-design is a finite subset of points on the unit sphere in a real vector space with the following special property: the average value of any polynomial of degree at most $t$ over the sphere is equal to the average value of the polynomial evaluated at the points of the $t$-design. A projective $t$-design generalizes this concept from spheres in $\mathbb{R}^d$ to projective spaces. Projective $t$-designs are interesting objects in part because of their connections to certain real, complex, and quaternionic reflection groups, as well as other sporadic simple groups.

Tight $t$-designs (whether spherical or projective) meet an absolute bound. While $t$-designs are common, tight $t$-designs are rare. It turns out that there are precisely two tight projective 5-designs with rational angle sets—one constructed from the short vectors of the Leech lattice and the other constructed from the lines of the unique generalized hexagon GH(2,8) [Hog89]. A third tight projective 5-design exists on the complex projective line, but it has an irrational angle set [Lyu09]. Some authors have conjectured that the two rational tight projective 5-designs are closely related [Hog82]. This paper provides a common construction of both using well-chosen octonion involutionary matrices. These matrices can act on the octonion vector space $\mathbb{O}^3$ and also on the octonion projective plane $\mathbb{O}\mathbb{P}^2$ to produce the two unique tight projective 5-designs with rational angle sets. This common construction serves as a new connection between the generalized hexagon GH(2,8) and the Leech lattice.

2. TWO UNIQUE TIGHT PROJECTIVE 5-DESIGNS WITH RATIONAL ANGLE SET

This section reviews tight projective $t$-designs with rational angle set and an important uniqueness theorem. We denote a projective space by $\mathbb{F}P^{d-1}$ where $\mathbb{F}$

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is one of \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) or \( \mathbb{O} \) (with \( d = 2, 3 \) in the case of \( \mathbb{F} = \mathbb{O} \)). As described in [Hog82 234], there are two ways to model a projective space. The first model applies for \( \mathbb{F} \neq \mathbb{O} \). In the first model, a point \( [x] \) in \( \mathbb{FP}^{d-1} \) (with \( x \in \mathbb{F}^d \)) is represented by the set,

\[
[x] = \{ \lambda x \mid 0 \neq \lambda \in \mathbb{F} \}, \quad \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}.
\]

The second model applies whenever the first model does, and also when \( \mathbb{F} = \mathbb{O} \). In the second model, a point \( X \) in \( \mathbb{FP}^{d-1} \) is an idempotent \( d \times d \) Hermitian matrix over \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \). The Euclidean inner product is defined in terms of the matrix product as \( \langle X, Y \rangle = \frac{1}{2} \text{Tr}(XY + YX) \). This inner product is a real-valued positive definite symmetric bilinear form, and we can define an angle between any two idempotent matrices \( X, Y \) via \( \cos^2 \theta = \langle X, Y \rangle \). Taking \( x \) to be a row vector in \( \mathbb{F}^d \), with \( \mathbb{F} \neq \mathbb{O} \), we can convert from the first model to the second via the map \( [x] \to x^t x/(xx^t) = X \), where \( x^t \) is the complex conjugate transpose of \( x \).

To define projective \( t \)-designs, we first need the define the polynomials \( Q_k(x) \) of degree \( k \) [CD07 54.33]:

\[
Q_k(x) = \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} (N + 2k - 1) \frac{(N)_{k+r-1}}{(m)_r k!} x^r,
\]

where \( m = \frac{1}{2}[\mathbb{F} : \mathbb{R}] \), \( N = md \), and \( (a)_i = a(a+1)(a+2)\cdots(a+i-1) \). A projective \( t \)-design is a subset \( T \subset \mathbb{FP}^{d-1} \) that satisfies,

\[
\sum_{a \in T} \sum_{b \in T} Q_k((a,b)) = 0, \quad k = 1, 2, \ldots, t.
\]

We denote by \( A \) the angle set of possible inner products \( \langle a, b \rangle = \cos^2 \theta_{a,b} \) for \( a \neq b \) in \( T \). We can constrain the size of \( |T| \) based on properties of \( A \). When the constraint depends on \( |A| \) only, it is called an absolute bound. When it depends on the values in \( A \), it is called a special bound. A spherical or projective \( t \)-design is called tight when it reaches the absolute bound [CD07 54.17]. In general, a projective \( t \)-design satisfies \( t \leq 2l + \varepsilon \) and \( |T| \geq Q_0(1) + Q_1(1) + \ldots + Q_l(1) \) where we have [Hog82 88],

\[
l = |A\setminus\{0\}|, \quad \varepsilon = \begin{cases} 1, & \text{if } 0 \in |A| \\ 0, & \text{otherwise} \end{cases}
\]

A projective \( t \)-design is tight when it satisfies \( t = 2l + \varepsilon \). An equivalent condition is that \( |T| = Q_0(1) + Q_1(1) + \ldots + Q_l(1) \) for a tight projective \( t \)-design.

**Theorem 2.1** ([Hog89]). A tight projective \( t \)-design in \( \mathbb{FP}^{d-1} \neq \mathbb{RP}^1 \) always satisfies \( t \leq 5 \). When \( t = 5 \), there are just two examples with a rational angle set. The only two tight projective 5-designs with rational angle sets are the unique system of 98280 points in \( \mathbb{RP}^2 \) defined by the short vectors of the Leech lattice and the unique system of 819 points in \( \mathbb{OP}^2 \) corresponding to the lines of the generalized hexagon \( \text{GH}(2,8) \).

**Proof.** A version of this theorem is the main result of [Hog89], which draws on [BS81] for the uniqueness of the tight 5-design in \( \mathbb{RP}^2 \) and on [CT85] for the uniqueness of the tight 5-design in \( \mathbb{OP}^2 \) via the uniqueness of the generalized hexagon \( \text{GH}(2,8) \). In [Hog89], the two tight 5-designs mentioned above are taken to be the only tight projective 5-designs, on the assumption that tight \( t \)-designs always have rational angle sets. However, [Lyu09] identifies a third tight 5-design with irrational
angle set, consisting of 12 points in \( \mathbb{CP}^1 \) related to the symmetries of an icosahedron. This counter-example requires that the theorem in [Hog89] be weakened to claim that there are only two unique tight projective 5-designs with rational angle sets.

\[ \square \]

**Remark 2.2.** The counter-example in [Lyu09] to the original theorem in [Hog89] is a tight projective 5-design with an irrational angle set, a system of 12 points in \( \mathbb{CP}^1 \) constructed from an orbit of the binary icosahedral subgroup of \( \text{SU}(2) \). For \( F = \mathbb{R}, \mathbb{C}, \mathbb{H} \), this is the only counter-example to the theorem in [Hog89]. It is unclear from [Lyu09] whether room remains for another counter-example with \( F = \mathbb{O} \).

In addition to the tight projective 5-design in \( \mathbb{RP}^{23} \), there are other important projective 5-designs constructed from the Leech lattice. These include the designs in \( \mathbb{CP}^{11} \) and \( \mathbb{HP}^2 \) that exhibit the symmetries of sporadic simple groups \( \text{Suz} \) and \( \text{HJ} \) respectively, shown in Fig. 1. The remarks in [Hog82] include a conjecture that the tight projective 5-design in \( \mathbb{OP}^2 \), which exhibits \( 3\text{D}_4(2) \) symmetries, is also constructed in terms of the Leech lattice, which exhibits \( \text{Co}_1 \) symmetries. The justification offered is that \( 196560 = 819 \cdot 240 \), the number of points in the 5-design multiplied by the number of roots in the \( E_8 \) lattice equals the number of short vectors in the Leech lattice. Against this possible connection, one referee of [Hog82] highlights certain facts about the representation theory of the groups \( \text{Co}_1 \) and \( 3\text{D}_4(2) \). However, the uniqueness of these two tight projective 5-designs with rational angle sets, established in [Hog89] and [Lyu09], provides some further motivation to find a common construction of both.

| \( F \) | \( d \) | \( v \) | \( A \) | \( t \) | Isometries |
|-------|------|-----|------|-----|----------|
| \( \mathbb{R} \) | 24   | 98280 | \( \{0, \frac{1}{4}, \frac{1}{2} \} \) | 5   | \( \text{Co}_1 \) |
| \( \mathbb{C} \) | 12   | 32760 | \( \{0, \frac{1}{12}, \frac{1}{4}, \frac{1}{3} \} \) | 5   | \( \text{Suz} \) |
| \( \mathbb{H} \) | 3    | 315  | \( \{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2} (3 \pm \sqrt{5}) \} \) | 5   | \( \text{HJ} \) |
| \( \mathbb{O} \) | 3    | 819  | \( \{0, \frac{1}{2}, \frac{1}{3} \} \) | 5   | \( 3\text{D}_4(2) \) |

**Figure 1.** Some projective 5-designs related to the Leech lattice.

This paper will identify a common construction for the two unique projective 5-designs with rational angle sets via the action of the same set of well-chosen octonion involutory matrices acting on (a) vectors in \( \mathbb{O}^3 \cong \mathbb{R}^4 \) and (b) idempotents in the octonion projective plane \( \mathbb{OP}^2 \). The same involutory matrices, acting on different objects, yield the two unique tight projective 5-designs with rational angle sets. This also establishes a previously elusive connection between the Leech lattice and the \( \text{GH}(2, 8) \) 5-design on the octonion projective plane.

### 3. Octonions and the Albert Algebra

This section reviews the octonion algebra, the Albert algebra, and the octonion projective plane. The *octonion algebra* \( \mathbb{O} \) is a real algebra with a basis indexed by the projective line over \( \mathbb{F}_7 \), namely \( \{i_t \mid t \in \text{PL}(7) = \{\infty\} \cup \mathbb{F}_7\} \). The identity element \( i_{\infty} = 1 \) spans the center of \( \mathbb{O} \) and this center is the real subalgebra \( \mathbb{R} \cong \mathbb{R} i_{\infty} \). The remaining basis vectors are square-roots of \(-1\), so that \( i_t^2 = -1 \) for all \( t \in \mathbb{F}_7 \).
The remaining octonion products are given by,
\[ i_t = i_{t+1}i_{t+3} = -i_{t+3}i_{t+1} = i_{t+2}i_{t+6} = -i_{t+6}i_{t+2} = i_{t+4}i_{t+5} = -i_{t+5}i_{t+4}, \quad t \in \mathbb{F}_7. \]

Fig. 2 provides a memory-aid to assist with octonion multiplication, which is invariant under the maps \( i_t \mapsto i_{t+1} \) and \( i_t \mapsto i_{2t} \). The map \( i_t \mapsto i_{t+1} \) corresponds to rotating the diagram labels and the map \( i_t \mapsto i_{2t} \) corresponds to mapping the red arrow to the green arrow to the blue arrow in a cycle.

Many details about the unique properties of the octonion algebra are available in [Bae02] and [CS03]. It is important to remember that the octonion algebra is not associative, but that it contains important associative subalgebras such as the real numbers \( \mathbb{R} \), the complex numbers \( \mathbb{C} \), and Hamilton’s quaternion algebra \( \mathbb{H} \).

**Lemma 3.1.** The subalgebra of \( \mathbb{O} \) generated by a single octonion is commutative (isomorphic to \( \mathbb{R} \) or \( \mathbb{C} \)) and the subalgebra generated by any two octonions is associative (isomorphic to \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{H} \)).

**Proof.** For details see [CS03 76]. \( \square \)

Just like the complex numbers and the quaternions, the octonion algebra \( \mathbb{O} \) is equipped with a **conjugation map** \( X \mapsto \overline{X} \) sending \( i_\infty \mapsto i_\infty \) and \( i_t \mapsto -i_t \) for \( t \in \mathbb{F}_7 \). A Hermitian octonion matrix is a matrix with octonion entries equal to its own octonion conjugate transpose, i.e. \( \mathcal{X} = \mathcal{X}^\dagger \) where \( \dagger \) denotes conjugate transpose. The **Albert algebra** \( \text{Herm}(3, \mathbb{O}) \) is defined as the algebra of \( 3 \times 3 \) Hermitian octonion matrices,
\[
(d, e, f \mid D, E, F) = \begin{pmatrix}
d & F & \overline{E} \\
F & e & D \\
\overline{E} & D & f
\end{pmatrix}, \quad d, e, f \in \mathbb{R}, \ D, E, F \in \mathbb{O},
\]
with a product \( \circ \) defined in terms of standard matrix multiplication as,
\[
\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}), \quad \mathcal{X}, \mathcal{Y} \in \text{Herm}(3, \mathbb{O}).
\]
The trace is defined as the sum of the diagonal entries, and \( \langle \mathcal{X}, \mathcal{Y} \rangle = \text{Tr}(\mathcal{X} \circ \mathcal{Y}) \) is a real positive-definite symmetric bilinear form on the Albert algebra. The Albert
algebra is a real 27-dimensional commutative but nonassociative algebra with the exceptional Lie group $F_4$ for its automorphism group.

The octonion projective plane $\mathbb{O}P^2$ is the manifold of primitive idempotents in the Albert algebra $\text{Herm}(3, \mathbb{O})$. A primitive idempotent is a matrix $X$ in the Albert algebra that satisfies $X \circ X = X$ and $\text{Tr}(X) = 1$. Equivalently, a primitive idempotent is an idempotent that cannot be written as the sum of two other nonzero orthogonal idempotents. To illustrate the axioms of the projective plane, we need to introduce a second product on $\text{Herm}(3, \mathbb{O})$. That is, we can define a symmetric cross product $[SV00, 122]$, $X \times Y = X \circ Y + \frac{1}{2} (\text{Tr}(X) \text{Tr}(Y) - \text{Tr}(X \circ Y)) - \frac{1}{2} (\text{Tr}(X)Y + \text{Tr}(Y)X)$

This cross product can be used to identify primitive idempotents.

**Lemma 3.2.** $[SV00, 141]$ Let $X \in \text{Herm}(3, \mathbb{O})$ with $\text{Tr}(X) \neq 0$. Then $X \times X = 0$ if and only if $X$ is a scalar multiple of a primitive idempotent.

Now we can describe the projective plane geometry using the cross product. First, $X$ is a point in the octonion projective plane if and only if $\text{Tr}(X) \neq 0$ and $X \times X = 0$. Two points $X$ and $Y$ are equivalent points if they are real scalar multiples. We can always represent a point by normalizing so that $\text{Tr}(\lambda X) = 1$, as we do when we select a primitive idempotent to represent the point. Two points $X$ and $Y$ are joined by the line consisting of all the points orthogonal to $X \times Y$. When two lines are represented by points $M$ and $N$, they intersect in the common point $M \times N$.

**Lemma 3.3.** Let $X = (d,e,f | D,E,F)$ be a primitive idempotent representing a point in the octonion projective plane $\mathbb{O}P^2$. Then $D, E, F$ belong to the same quaternion subalgebra of $\mathbb{O}$ and $X = x^\dagger x$ for some row vector $x$ in $\mathbb{H}^3$.

**Proof.** For details see $[DM15, 119-125]$. $\square$

More details about the Albert algebra and octonion projective plane are available in $[SV00]$, $[Bae02]$, and $[DM15]$. 

### 4. An Octonion Leech Lattice

This section reviews Wilson’s construction of the Leech lattice using octonion vectors in $\mathbb{O}^3$, rather than $\mathbb{R}^{24}$ $[Wh09a]$, $[Wh09b]$. We adopt his notation and specific Leech lattice construction in what follows.

First we need to construct subsets of $\mathbb{O}$ isometric to a (possibly scaled) $E_8$ lattice. Let $B$ denote the $E_8$ lattice with the following natural choice of 240 roots, for all $t \in \mathbb{F}_7$:

$$\pm 1, \pm i_t, \frac{1}{2} (\pm 1 \pm i_t \pm i_{t+1} \pm i_{t+3}), \frac{1}{2} (\pm i_{t+2} \pm i_{t+4} \pm i_{t+5} \pm i_{t+6}).$$

We call this $E_8$ lattice Kirmse’s octonions. The $E_8$ lattice defined by these roots is not closed under octonion multiplication. However, we can construct 7 distinct octonion $E_8$ lattices that do close under multiplication as follows:

$$A_t = \frac{1}{2} (1 - i_t) B (1 - i_t), \quad t \in \mathbb{F}_7.$$
The $A_t$ octonion rings are each known as Coxeter-Dickson integral octonions. Finally, we can recover the standard coordinates for the $E_8$ lattice in both the left-handed $L$ and right-handed $R$ form as follows:

$$L = (1 + i_l)A_l = B(1 - i_l), \quad R = A_l(1 + i_l) = (1 - i_l)B.$$ 

The intersection $L \cap R$ is a standard copy of the $D_8$ lattice, namely the span of all 112 $D_8$ roots $\pm i_r \pm i_t$ for $r, t \in \text{PL}(7)$. The remaining roots in $L$ are the 128 vectors of the form $\frac{1}{2}(\pm 1 \pm i_0 \pm i_1 \cdots \pm i_6)$ with an odd number of minus signs. The remaining roots in $R$ are the 128 vectors of this form with an even number of minus signs. So the vector $s = \frac{1}{2}(-1 + i_0 + i_1 + i_2 + i_3 + i_4 + i_5 + i_6)$ is in $L$ while the vector $\overline{s}$ is in $R$.

The Moufang laws that octonions satisfy yield the following simple relations between $L$, $R$, and $B$:

$$LR = 2B, \quad BL = L, \quad RB = R.$$ 

Wilson uses these relations to construct the following octonionic definition of the Leech lattice as the row vectors $(x, y, z)$ in $O^3$ that satisfy,

1. $x, y, z \in L$
2. $x + y, y + z, x + z \in L$\hspace{1em}$L\overline{s}$
3. $x + y + z \in Ls$.

In this construction the Leech lattice norm is given by,

$$N(x, y, z) = \frac{1}{2}(x, y, z)(x, y, z)^\dagger = \frac{1}{2}(x\overline{x} + y\overline{y} + z\overline{z}).$$ 

We will call this set of octonion vectors in $O^3$ Wilson’s Leech lattice. The short vectors (of norm 4) of Wilson’s Leech lattice are listed in Fig. 3 for $\lambda$ any root in $L$, for all $j, k$ in $\{\pm i_t \mid t \in \{\infty\} \cup F_7\}$, and all permutations of the coordinates.

| Shape | Count | Total |
|-------|-------|-------|
| $(2\lambda, 0, 0)$ | $3 \times 240$ | 720 |
| $(\lambda\overline{s}, (\lambda\overline{s})j, 0)$ | $3 \times 240 \times 16$ | 11520 |
| $((\lambda s)_j, \lambda k, (\lambda j)k)$ | $3 \times 240 \times 16 \times 16$ | 184320 |
| | | 196560 |

Figure 3. The octonion Leech lattice short vectors

Wilson shows how to construct $2 \cdot \text{Co}_1$, the automorphism group of the Leech lattice, using $3 \times 3$ octonion matrices acting on the right of the Leech lattice row vectors $(x, y, z)$. In particular, Wilson shows that the following actions preserve the Leech lattice. First, if we write,

$$R_x = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix},$$ 

then the following maps preserve the Leech lattice,

$$(x, y, z) \mapsto \frac{1}{2}((x, y, z)R_{1-i_0})R_{1+i_1}.$$
In fact, these nested matrix actions generate a $2 \cdot \text{A}_8$ action on the Leech lattice. If we adjoin right multiplication by the following matrices,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & i^t & 0 \\ 0 & 0 & i^t \end{pmatrix}, \quad t \in \mathbb{F}_7,$$

Then we obtain the action of the maximal subgroup $2^{3+12}\langle \text{A}_8 \times S_3 \rangle$ of the Leech lattice automorphism group $2 \cdot \text{Co}_1$. To obtain the full $2 \cdot \text{Co}_1$ action, we need to adjoin right multiplication by,

$$-\frac{1}{2} \begin{pmatrix} 0 & \pi & \bar{\pi} \\ s & -1 & 1 \\ s & 1 & -1 \end{pmatrix}.$$

Wilson uses subsets of these actions, and closely related ones, to exhibit the groups of the Suzuki chain of $\text{Co}_1$ subgroups acting on this octonion Leech lattice:

$$S_3 < S_4 < \text{PSL}_2(7) < \text{PSU}_3(3) < HJ < G_2(4) < 3 \cdot \text{Suz}.$$

In this paper we identify a number of different octonion involutionary matrices that also generate the Conway group $\text{Co}_1$ and use them to link the two unique projective 5-designs.

5. Constructing the Unique Real Projective Tight 5-Design

We now construct our first tight projective 5-design using vectors in $\mathbb{O}^3$ and provide octonion involutionary matrices that generate the actions of Suzuki chain subgroups on this 5-design. The unique tight projective 5-design in $\mathbb{R}^{24} \cong \mathbb{O}^3$ is defined by the 98280 pairs of opposite Leech lattice short vectors, projected onto $\mathbb{RP}^{23}$. In what follows we will always identify opposite octonion Leech lattice vectors. The action of the Leech lattice automorphisms on opposite pairs of short vectors is the simple group $\text{Co}_1$ acting on 98280 points, the smallest permutation representation.

Let’s begin with the Suzuki chain subgroup $S_3 < \text{Co}_1$. We already know that any permutation of the three components of the octonion triples $(x, y, z)$ in Wilson’s Leech lattice is a Leech lattice automorphism, and therefore an element of $\text{Co}_1$. We can model this group action using reflection matrices. First consider the short vectors in Wilson’s Leech lattice of Fig. 3 with the form $(\lambda s, (\lambda s)j, 0)$ and $\lambda = \pm s$ and $j = \mp i_\infty = \mp 1$. There are six vectors of this form, e.g. $(2, -2, 0)$ and all permutations of the coordinates. These form a scaled $\text{A}_2$ system of roots. We can construct a Weyl reflection matrix from a row vector $r$ as,

$$W(r) = I - 2rr^\dagger/(rr^\dagger).$$

To reflect any $(x, y, z) \in \text{A}_2$ by any $r$ in $\text{A}_2$ we compute,

$$(x, y, z) \mapsto (x, y, z)W(r).$$

In fact, for any $(x, y, z)$ in Wilson’s Leech lattice and $r$ in $\text{A}_2$ this map returns another member of Wilson’s Leech lattice. That is, for $r$ in $\text{A}_2$, right multiplication by $W(r)$ is a Leech lattice automorphism and an element of $\text{Co}_1$. This is not true for any $r$ in general, but for certain well chosen vectors $r$, right multiplication by $W(r)$ is in fact a Leech lattice automorphism. Restricting to $r$ in $\text{A}_2$ for the moment, these maps generate a $S_3 = W(\text{A}_2) < \text{Co}_1$ subgroup of Leech lattice automorphisms, the first and smallest in the Suzuki chain. By using other careful choices of reflection
vector \( r \), we can generalize this procedure to recover the action of all the remaining Suzuki chain subgroups acting on the Leech lattice short vector opposite pairs.

To see which choices of \( r \) are suitable for exhibiting Leech lattice symmetries, note that there is a correspondence between idempotent matrices in \( \mathbb{O} \mathbb{P}^2 \) and involutive matrices in the Albert algebra \( \text{Herm}(3, \mathbb{O}) \). That is, for any idempotent matrix \( \lambda \) in \( \mathbb{O} \mathbb{P}^2 \), the element \( I - 2\lambda \) squares to \( I \) (with \( I \) the identity matrix):

\[
(I - 2\lambda)^2 = I^2 - 4I\lambda + 4\lambda^2 = I.
\]

Suppose that \( v = (x, y, z) \) is a vector in \( \mathbb{O}^3 \) with the three components contained in a common associative subalgebra of \( \mathbb{O} \) (i.e., a common quaternion, complex, or real subalgebra). Then \( \lambda = v^tv/(vv^t) \) is an idempotent matrix in \( \mathbb{O} \mathbb{P}^2 \). So for any \( v = (x, y, z) \) in \( \mathbb{O}^3 \) with \( x, y, z \) in a common quaternion subalgebra, we can define the reflection matrix of \( v \) as,

\[
\mathcal{W}(v) = I - 2v^tv/(vv^t).
\]

This matrix has the involution property \( \mathcal{W}(v)^2 = I \).

Next, let \( v_1 \) and \( v_2 \) be vectors in \( \mathbb{O}^3 \) that admit the construction of idempotents \( v_1^tv_1/(v_1v_1^t) \) (i.e., such that the components of \( v_i \) belong to a common quaternion subalgebra of \( \mathbb{O} \)) so that \( \mathcal{W}(v_1) \) and \( \mathcal{W}(v_2) \) are well defined. These matrices act on row vectors of the form \( (x, y, z) \) through nesting—the composition of right matrix multiplication maps—so that we never actually compute the matrix product \( \mathcal{W}(v_1)\mathcal{W}(v_2) \). That is, the successive application of \( \mathcal{W}(v_1) \) and \( \mathcal{W}(v_2) \) has the form,

\[
(x, y, z) \mapsto [(x, y, z)\mathcal{W}(v_1)]\mathcal{W}(v_2).
\]

Provided that \( \mathcal{W}(v_1) \) and \( \mathcal{W}(v_2) \) each permute Leech lattice short vectors, the composition of these two maps will also permute Leech lattice vectors. But in general, this composition is not represented by the matrix product \( \mathcal{W}(v_1)\mathcal{W}(v_2) \), due to the non-associativity of the octonion algebra.

In what follows we will restrict ourselves to reflection matrices \( \mathcal{W}(v) \) of vectors \( v = (x, y, z) \) with \( x, y, z \) in a common complex subalgebra of \( \mathbb{O} \). There are \( 2 \times 1260 \) short vectors in Wilson’s Leech lattice for which the three coordinates belong to a common complex subalgebra of the octonions. We’ve already seen that the six \( \mathbb{A}_2 \) vectors identified above are among these, and that their reflection matrices generate a \( S_4 = W(\mathbb{A}_2) \) action on the Leech lattice short vector pairs.

Next, consider the following vectors among the short vectors of Wilson’s Leech lattice with all components in a common complex subalgebra of \( \mathbb{O} \):

\[
V_i = \{ \sigma(\pm 2, \pm 2i, 0), \ \sigma \in \text{Sym}(3) \}, \quad t \in \{ \infty \} \cup F_7.
\]

We obtain these by taking the short vectors of the form \( (\lambda\bar{s}, (\lambda\bar{s})j, 0) \) in Fig. \[3\] and setting \( \lambda = s \). We see that \( V_\infty \) contains 6 pairs of opposite Leech lattice short vectors. In fact, \( V_\infty \) is a scaled \( \mathbb{A}_3 \) system of roots. Each \( V_i \) for \( t \in F_7 \) contains 12 pairs of opposite Leech lattice short vectors.

Each \( v \) in each \( V_i \) defines an octonion involutionary matrix \( \mathcal{W}(v) \) corresponding to an involutionary automorphism of the Leech lattice. In particular, the \( V_\infty \) vectors yield involutions generating an \( S_4 = W(\mathbb{A}_3) < \text{Co}_1 \) action on the Leech lattice short vector pairs.
Lemma 5.1. Let $W(v) = I - 2v^1v/(vv^1)$ for $v$ any vector in $V_t$, for any $t \in \text{PL}(7)$. Then the following map is a Leech lattice automorphism:

$$(x, y, z) \mapsto (x, y, z)W(v).$$

The action on opposite pairs is an automorphism of the real projective tight 5-design and an involution in $\text{Co}_1$. The groups $G$ generated by these involutions are described in terms of their order and centralizer in $\text{Co}_1$ in the following table (with $t \neq t' \in \mathbb{F}_7$):

| Vectors defining $W(v)$ | $|G|$ | $\text{Cent}_{\text{Co}_1}(G)$ |
|--------------------------|------|------------------------|
| $V_1$ ⊂ $V_\infty$     | 6 = $|S_3|$ | $A_0$ |
| $A_3 = V_\infty$       | 24 = $|S_4|$ | $A_6$ |
| $V_t$                   | 96   | $A_6$ |
| $V_t \cup V_{t'}$      | 384  | $A_5$ |
| $V_t \cup V_{t+1} \cup V_{t+3}$ | 1536 | $A_4$ |
| $V_t \cup V_{t+5} \cup V_{t+6} \cup V_{t+7}$ | 6144 | $A_3$ |
| $V_\infty \cup V_0 \cup V_1 \cup \cdot \cdot \cdot \cup V_6$ | 98304 | 1 |

Proof. These facts are confirmed using GAP computations acting on the pairs of opposite short vectors in Wilson’s Leech lattice. □

We now provide a set of generators for a $\text{PSL}_2(7)$ subgroup in $\text{Co}_1$. Notice that the octonion algebra $\mathbb{O}$ has the following cyclic automorphism:

$$\alpha : i_t \mapsto i_{t+1}.$$  

The action of $\alpha$ on the coefficients of Wilson’s Leech lattice short vectors induces a permutation of the short vector pairs. This permutation is an element of order 7 of $\text{Co}_1$. We may also denote this element of order 7 in $\text{Co}_1$ by $\alpha$. The centralizer of $\alpha$ in $\text{Co}_1$ has the form $C_7 \times \text{PSL}_2(7)$. In fact, $\alpha$ fixes exactly 21 opposite pairs of short vectors in Wilson’s Leech lattice, and these 21 vectors all have components within the same complex subalgebra $\mathbb{R}(s)$ of $\mathbb{O}$. In fact, $s$ has the form $\frac{1}{2}(-1 + \sqrt{-7})$ in $\mathbb{R}(s) \cong \mathbb{C}$. We denote by $S$ the $2 \times 21$ short vectors fixed by $\alpha$.

To obtain the vectors in $S$, we take all the short vectors in Fig. 3 corresponding to $\lambda = s$ and $j, k = \pm 1$. This yields six short vectors of the form $(\lambda, 0, 0)$, 12 short vectors of the form $(\lambda s, (\lambda s)j, 0)$, and 24 short vectors of the form $((\lambda s)j, \lambda k, (\lambda j)k)$.

Lemma 5.2. For each short vector $v$ in $S$, i.e. the short vectors in Wilson’s Leech lattice fixed under $i_t \mapsto i_{t+1}$, the map $(x, y, z) \mapsto (x, y, z)W(v)$ is a Leech lattice involutionary automorphism. There are 21 such involutions corresponding to fixed vectors. The group generated by these 21 involutions is the group $\text{PSL}_2(7)$ with centralizer $A_7$ in $\text{Co}_1$. This $\text{PSL}_2(7)$ is also the centralizer of $A_7$ in $\text{Co}_1$.

Proof. These facts have been confirmed by computation using GAP. □

We can now use the generators corresponding to the vectors in the sets $S$ and $V_t$, with $t \in \mathbb{F}_7$, to generate the Suzuki chain subgroups of $\text{Co}_1$.

Lemma 5.3. The following Suzuki chain subgroups can be generated via the right action,

$$(x, y, z) \mapsto (x, y, z)W(v),$$

where $W(v)$ are octonion involution matrices constructed from the following vectors:
Vectors defining $W(v)$ & $G$ & Cent$_{Co_1}(G)$

| $v$ | $G$ | $A$ |
|-----|-----|-----|
| $a_2 \subset V_\infty$ & $S_3$ & $A_3$
| $a_3 = V_\infty$ & $S_4$ & $A_8$
| $S$ & PSL$_2(7)$ & $A_7$
| $S \cup V_t$ & PSU$_3(3)$ & $A_6$
| $S \cup V_t \cup V_t'$ & HJ & $A_5$
| $S \cup V_t \cup V_{t+1} \cup V_{t+3}$ & $G_2(4)$ & $A_4$
| $S \cup V_{t+2} \cup V_{t+5} \cup V_{t+6} \cup V_{t+7}$ & $3 \cdot$ Suz & $A_3$
| $S \cup V_\infty \cup V_0 \cup V_1 \cup \cdots \cup V_6$ & Co$_1$ & 1 |

Proof. These facts have been confirmed by computation in GAP [noa20] and making use of the automorphism $i_t \mapsto i_{t+1}$. □

The groups $G_2(4)$, $3 \cdot$ Suz, and Co$_1$ are each transitive on the Leech lattice short vector pairs. This means that the octonion reflection matrices corresponding to the vectors listed for each of these three groups in Lemma 5.3 are sufficient to construct the unique tight real projective 5-design in $\mathbb{RP}^{23}$. Minimally, we can construct the unique tight real projective 5-design in $\mathbb{RP}^{23}$ using only a $G_2(4)$ subgroup of Leech lattice automorphisms. That is, the vectors of $S \cup V_t \cup V_t' \cup V_{t+1} \cup V_{t+3}$ for $t \in F_7$ define the reflection matrices $W(v)$ needed to construct the unique tight projective 5-design in $\mathbb{RP}^{23}$. The design itself is independent of our choice of $t \in F_7$.

Remark 5.4. We can make better sense of the choices of vectors in Lemmas 5.1 and 5.3 by returning to Fig. 2 and replacing the labels $i_0, i_1, \ldots, i_6$ with $V_0, V_1, \ldots, V_6$. In particular, the generators of PSU$_3(3)$ correspond to any point on the edge of the diagram, those of HJ to any pair of points or an arc, those of $G_2(4)$ to a triangle constructed from an arc and its product (i.e. 1, 3, 7), those of $3 \cdot$ Suz to any pair of arcs, and those of Co$_1$ to all seven points.

6. Action on the Octonion Projective Plane

In this section we use the same sets of generating vectors $v$ in $S$ and $V_t$, to both (a) define points in the octonion projective plane of the form $U(v) = v^\dagger v/\langle vv^\dagger \rangle$ and (b) act on these points in $\mathbb{OP}^2$ via the group action generated involutions of the form,

$$X \mapsto W(v')^\dagger X W(v').$$

By using the generating vectors $v$ of various different subgroups in the Suzuki chain, described in Lemma 5.3, we can construct a number of interesting projective $t$-designs in $\mathbb{OP}^2$. Most notably, the generating vectors of both HJ and $G_2(4)$ (when acting on $\mathbb{O}^3$ as described above) both yield the same projective $t$-design—the octonionic projective tight 5-design with 819 points in $\mathbb{OP}^2$.

The first task is to confirm that the action $U(v) \mapsto W(v')^\dagger U(v) W(v')$ is well-defined given the non-associativity of the octonion algebra.

Lemma 6.1 ([DM15]). Suppose that the components of $W(v')$ belong to the same complex subalgebra of $\mathbb{O}$. Then we have,

$$W(v')^\dagger (U(v) W(v')) = (W(v')^\dagger U(v)) W(v'),$$

for all $U(v)$ in the octonion projective plane $\mathbb{OP}^2$. 

Proof: The Hermitian octonion matrix $U(v)$ has the form $(d, e, f | D, E, F)$, being an element in the Albert algebra $\text{Herm}(3, \mathbb{O})$. This means that we can write this matrix as a sum of terms of the form $(d, 0, 0 | 0, 0, 0), (0, e, 0 | 0, 0, 0), \ldots, (0, 0, 0 | 0, 0, F)$. Let $\mathcal{X}$ be one of these six matrices. The coordinates of $\mathcal{X}$ and of $\mathcal{W}(v')$ belong to a common quaternion subalgebra of $\mathbb{O}$, since $\mathcal{X}$ only contains one independent octonionic coordinate and this coordinate combines with the common complex subalgebra of $\mathbb{O}$ given by the coordinates of $\mathcal{W}(v')$ to define an associative (at most quaternion) subalgebra of $\mathbb{O}$. This means that we have in each case,

$$\mathcal{W}(v')^\dagger (\mathcal{X} \mathcal{W}(v')) = (\mathcal{W}(v')^\dagger \mathcal{X}) \mathcal{W}(v').$$

Since each term associates, the sum also associates. 

We see then that the action,

$$U(v) \mapsto \mathcal{W}(v')^\dagger U(v) \mathcal{W}(v'),$$

is well defined for our choice of $v$ in $S$ or $V_t$ for some $t \in \mathbb{F}_7$. It turns out that this action defines an involutory automorphism of the Albert algebra $\text{Herm}(3, \mathbb{O})$, contained in the Lie group $F_4$. As before, we compose two involutions corresponding to $v_1$ and $v_2$ by nesting:

$$U \mapsto \mathcal{W}(v_2)^\dagger (\mathcal{W}(v_1)^\dagger U \mathcal{W}(v_1)) \mathcal{W}(v_2).$$

We do not need to evaluate $\mathcal{W}(v_1) \mathcal{W}(v_2)$ and in general this matrix will not represent the composition of the involutions given by the two factors, due to the non-associativity of $\mathbb{O}$.

We now have a method to construct subgroups of $F_4$ acting on the octonion projective plane $\mathbb{OP}^2$ generated by involutions corresponding to a set of suitable octonion involutory matrices $\mathcal{W}(v)$. The orbit of each $U(v) \in \mathbb{OP}^2$, with $v$ a generating vector, under the group action generated by the involutions defined above can form a projective $t$-design.

**Lemma 6.2.** For $v, v'$ generating vectors in the subsets of $S \cup V_0 \cup V_1 \cup \cdots \cup V_6$ given below, we can construct the corresponding projective $t$-designs as the orbits of all $\mathcal{X} = U(v)$ under the group action on $\mathbb{OP}^2$ generated by the involutions,

$$\mathcal{X} \mapsto \mathcal{W}(v')^\dagger \mathcal{X} \mathcal{W}(v').$$

| Vectors $v, v'$ | Group Action | Orbit |
|-----------------|--------------|-------|
| $A_3 = V_\infty$ | $S_4$        | 1-design on 6 points in $\mathbb{OP}^2$ |
| $S$             | $\text{PSL}_2(7)$ | 2-design on 21 points in $\mathbb{CP}^2$ |
| $S \cup V_t$    | $\text{PSU}_3(3)$ | 3-design on 63 points in $\mathbb{HP}^2$ |
| $S \cup V_t \cup V_{t'}$ | $3D_4(2)$ | 5-design on 819 points in $\mathbb{OP}^2$ |
| $S \cup V_t \cup V_{t+1} \cup V_{t+3}$ | $3D_4(2)$ | 5-design on 819 points in $\mathbb{OP}^2$ |

Proof: These facts have been confirmed by computation in GAP [noa20] and making use of the automorphism $i_t \mapsto i_{t+1}$. 

The 5-design in $\mathbb{OP}^2$ described in Lemma 6.2 is the unique tight octonion projective 5-design with rational angle set. The construction via generators corresponding to the vectors in $S \cup V_t \cup V_{t+1} \cup V_{t+3}$ yields seven intersecting 5-designs, one for each value of $t \in \mathbb{F}_7$. The intersection of these 5-designs is the 2-design generated by the vectors in $S$ alone.
7. Conclusion

We have seen that the vectors in \( v \in S \cup V_t \cup V_{t+1} \cup V_{t+3} \) with \( t \in \mathbb{F}_7 \) define octonion reflection matrices \( W(v) \) that exhibit a \( G_2(4) \) action on octonion row vectors in \( \mathbb{O}^3 \) and a corresponding \( 3D_4(2) \) action on points in the octonion projective plane \( \mathbb{O}P^2 \). In the first case, the \( G_2(4) \) action yields a single orbit of 98280 pairs of opposite vectors in \( \mathbb{O}^3 \approx \mathbb{R}^{24} \), which form a tight projective 5-design in \( \mathbb{R}P^{23} \). The corresponding \( 3D_4(2) \) action yields a single orbit of 819 points in the octonion projective plane \( \mathbb{O}P^2 \), which form another tight projective 5-design. These are the only two tight projective 5-designs with rational angle sets and we can now see that they share a common construction.

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Current address: Department of Physics and Space Science, Royal Military College of Canada, Kingston, ON

E-mail address: ben.nasmith@gmail.com