Fundamental formulation of electrodynamics revisited, and the precision of quantum electrodynamics

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Abstract

It was shown recently that unambiguous description of electromagnetic environments requires electromagnetic potentials; knowledge only of electric and magnetic fields is insufficient and can lead to error. Consequences of that demonstration are here applied to propagating fields, such as laser fields. Gauge invariance is replaced by symmetry preservation. This alteration makes it possible to understand how the known failure of the convergence of perturbation expansions in quantum electrodynamics (QED) follows from the fact that QED is incomplete; it does not contain its strong-field limit. Inherent in that demonstration are the strong-field coupling constant and the strong-field alteration of the mass shell of a charged particle. A variety of physically important consequences ensue, including the loss of guidance from Feynman diagrams. The meaning of tests for the precision of QED is questioned since such evaluations apply only to perturbative QED, but not to extensions required for complete QED.

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I. INTRODUCTION

Gauge transformations are designed to find alternative sets of electromagnetic potentials associated with a specific configuration of electric and magnetic fields. The conventional belief has long been that electric and magnetic fields define the electromagnetic environment, so that the existence of alternative sets of potential functions identifies them as auxiliary quantities. That situation is inverted in Ref. [1], where gauge transformations are shown to alter the fundamental symmetries associated with a physical system. That opens the possibility of finding spurious solutions in Maxwellian electrodynamics and Newtonian mechanics. It is necessary to select an appropriate set of potentials to define fully the physical problem. This identifies potentials as more fundamental than electric and magnetic fields. It is also shown in Ref. [1] that gauge transformations are not, in general, unitary transformations.

The symmetries appropriate to a physical environment are determined by properties of the Lagrangian describing the system [2]. The quantum theory of fields is based on the properties of Lagrangians, and so that discipline is free of the ambiguities explored here and in Ref. [1]. Gauge invariance is a foundation principle when employed in the quantum theory of fields, but it has important limitations in the context of how electric and magnetic fields are related to potential functions.

The conclusions of Ref. [1] are applied to the important case where the electromagnetic environment includes a propagating field (also known as a transverse field, a plane-wave field, a sourceless field, or a photon field). For example, these results apply to all laser-field problems.

The usual meaning of quantum electrodynamics (QED) is in the sense of Feynman-Dyson perturbation theory, with the primary computational method being the use of Feynman diagrams. The subjects examined here become very important when strong fields exist, with the accompanying concepts of intensity-dependent coupling constant and mass shell for charged particles. These concepts are not part of standard QED. It is known that Feynman-Dyson perturbation theory is not convergent. That lack is ascribed to the incompleteness of QED, which does not include strong-field phenomena.

The significance of studies of the precision of QED is questioned, since such studies focus on the value of the fine-structure constant. That has meaning only for perturbative QED, but not if the meaning of QED is extended to include strong-fields. Laboratory experiments with
strong fields interacting with charged particles have for many years exceeded perturbative limits. Importantly different conditions for extended QED are proposed.

The background for results presented here spans the entire history of strong-field physics. It combines recent results, such as Ref. [1], with early work on strong fields from the 1950s and 1960s, whose significance could not be fully appreciated at the time.

Gaussian units are used for electromagnetic quantities.

II. GAUGE INVARIANCE

In the quantum theory of fields, gauge invariance has profound importance, but that theory is entirely in terms of potentials. See, for example, Ref. [3]. The sense in which gauge invariance is discussed here is in the more mundane connection between potentials and electric and magnetic fields. This semiclassical connection has not had a detailed examination equivalent to that for quantized fields.

The principle of gauge invariance in its classical sense has, as its origin, the notion that electric and magnetic fields determine all dynamical consequences in electromagnetism, and that scalar and vector potentials have only an auxiliary function. This type of gauge invariance is upset as a basic principle when potentials are shown to be necessary to define an electromagnetic environment. The propagation property constitutes a symmetry that has not been adequately considered, but it is fundamental in showing that there exist nominally valid gauge transformations that violate that symmetry. Gauge invariance is replaced by symmetry preservation as a requirement for equivalence.

A. Equations of quantum mechanics

Acting upon the long-standing assumption that fields are basic and potentials are secondary, many investigators attempted to express the Schrödinger equation directly in terms of fields [4–9]. All such attempts resulted in making the Schrödinger equation nonlocal and thus unsatisfactory. This conclusion applies to relativistic equations of motion (Klein-Gordon, Dirac, Proca) in addition to the nonrelativistic Schrödinger and Pauli equations. The implication, not realized at the time, is that potentials are more fundamental than fields.
B. Aharonov-Bohm effect

The Aharonov-Bohm effect [10, 11] is a direct demonstration that potentials are more fundamental than fields in that the deflection of an electron beam passing over a solenoid takes place in a region that is free of fields, but has a potential that explains the deflection. This is a quantum effect, and it is discussed in textbooks as being exclusively a quantum anomaly that represents a departure from the notion that fields are primary and potentials are secondary. Even that limited role has been questioned [12].

The Aharonov-Bohm effect involves a magnetic field, so it has no direct significance for the study of propagating fields.

C. Altered symmetries

Reference [1] demonstrates that a gauge transformation can alter the basic symmetries that characterize a problem in electromagnetism. Symmetries determine conservation properties [2], so that a change in symmetries represents a change to a different problem in electrodynamics. This finding is quite general, and it occurs in both classical and quantum physics. Different gauges for a given field configuration need not be equivalent, and potential functions are required to define the appropriate electromagnetic environment.

A practical example presented in Ref. [1] has important implications. A propagating electromagnetic field, such as a laser field, must satisfy the Einstein Principle [13] that the speed of light in vacuum is the same in all inertial frames of reference. The formal statement of this principle is that the spacetime 4-vector $x^\mu$ can occur only as a scalar product with the propagation 4-vector $k^\mu$. That scalar product:

$$\varphi \equiv k^\mu x_\mu = \omega t - k \cdot r,$$

is the phase of a propagating field. This will be referred as the propagation condition. When the field is a propagating field, the 4-vector potential must be expressible as $A^\mu(\varphi)$.

A gauge transformation in electrodynamics is expressed as

$$A^\mu \rightarrow \tilde{A}^\mu = A^\mu + \partial^\mu \Lambda,$$

where $\Lambda$, the generating function for the transformation, is a scalar function that satisfies
the homogeneous wave equation:

$$\partial^\mu \partial_\mu \Lambda = 0.$$  \hspace{1cm} (3)

When those conditions are satisfied, the electric and magnetic fields are unchanged by the transformation.

A valid gauge transformation that produces an invalid 4-vector potential for a transverse field has the generating function \[1, 14\]

$$\Lambda = -A^\mu x_\mu.$$  \hspace{1cm} (4)

This is a scalar function that satisfies the condition (3) required of a generating function and it is also stated covariantly. However, it is clear that this will introduce a violation of the propagation condition because \(x^\mu\) appears in isolation from \(k^\mu\). The gauge-transformed 4-vector potential is

$$\tilde{A}^\mu = -k^\mu (x^\nu A'_\nu), \text{ where } A'_\nu \equiv \frac{d}{d\varphi} A_\nu (\varphi).$$  \hspace{1cm} (5)

The propagation condition is violated, but the electric and magnetic fields are unchanged by this gauge transformation. This violation of a basic property of a propagating field leads to the general conclusion that potentials are more fundamental than fields, since preservation of the fields can nevertheless lead to an invalid representation of a propagating wave.

An alternative form of the gauge-transformed potential in Eq. (5) is \[1, 14\]

$$\tilde{A}^\mu = -\left(\frac{k^\mu}{\omega/c}\right) \mathbf{r} \cdot \mathbf{E}(\varphi).$$  \hspace{1cm} (6)

Although this gauge is unacceptable, it has found some favor \[15\] because of its resemblance to the ubiquitous length-gauge potential \(−\mathbf{r} \cdot \mathbf{E}(t)\).

D. Gauge invariance versus symmetry preservation

The historically accepted conditions required to maintain electric and magnetic fields in a gauge transformation are insufficient to characterize an electromagnetic environment. For maintenance of the propagation property of a plane-wave field, a necessary additional requirement is preservation of the propagation condition of Eq. (1).

When an electromagnetic environment has both a propagating field and a scalar field, then only the radiation gauge (also known as Coulomb gauge) can be compatible with an
origin of coordinates, typifying a scalar potential such as a Coulomb field and, simultaneously, the absence of an origin of coordinates necessary to describe a propagating field such as a laser field. In the radiation gauge, the time component of the 4-vector potential represents the scalar field, and the 3-vector component represents the propagating field. When only the propagating field is present, the Lorenz condition $\partial^\mu A_\mu = 0$ reduces to $\nabla \cdot A = 0$. The 3-vector gradient condition is often stated to be the defining condition for the radiation gauge.

The general expression for a gauge transformation is given by Eq. (2). The condition that $A^\mu$ must be a function only of $\varphi$ imposes the same constraint on $\Lambda$, leading to

$$\tilde{A}^\mu = A^\mu + k^\mu \Lambda',$$

where $\Lambda' = d\Lambda/d\varphi$. That is, the only alteration of the potential that is possible differs from $A^\mu$ by a component that lies on the light cone. An important consequence is that

$$\tilde{A}^\mu \tilde{A}_\mu = A^\mu A_\mu.$$  

This follows from the transversality condition $k^\mu A_\mu = 0$ as well as the fact that a 4-vector on the light cone is self-orthogonal: $k^\mu k_\mu = 0$.

Gauge invariance employed in classical and semiclassical electrodynamics as a general principle cannot be correct since a gauge change will normally alter symmetry conditions, meaning that the physical problem is changed. For radiation fields in interaction with matter, gauge invariance is replaced by symmetry preservation.

There is some flexibility possible even when the propagation condition is enforced. That possibility arises when the generator of the gauge transformation itself satisfies the propagation condition, as expressed in Eq. (7). A further differentiation gives

$$\partial_\mu \tilde{A}^\mu = \partial_\mu A^\mu + k_\mu k^\mu \Lambda'' = \partial_\mu A^\mu,$$

where $\Lambda'' = d^2\Lambda/d^2\varphi$ and $k_\mu k^\mu = 0$. The Lorenz condition $\partial_\mu A^\mu = 0$ and the condition of Eq. (3) are automatically satisfied.

### E. Length gauge aberration

Of the several gauges in use in strong-field physics, there is one that stands out for the insupportable claims made for it.
The “length gauge” is a name used to refer to the $-\mathbf{r} \cdot \mathbf{E}(t)$ scalar function to represent the interaction of radiation fields with matter. There is an influential body of literature devoted to the claim that the length gauge is the only proper gauge to be used, and if a different gauge is to be employed, then it must always carry with it the gauge transformation factor. This hypothesis is here termed an “aberration” since it is not possible for a scalar potential to be fundamental for treatment of a vector field like a laser field.

Two of the most ambitious efforts advocating the primacy of the length gauge are cited here: [17, 18]. Both of these papers assume that the interaction Hamiltonian behaves unitarily in a gauge transformation. This is untrue [1]. It is plainly a contradiction since $U (\mathbf{r} \cdot \mathbf{E}) U^{-1}$ remains just $\mathbf{r} \cdot \mathbf{E}$ in any attempted gauge transformation because that interaction Hamiltonian contains no operators. This means that, if a gauge transformation is made into the length gauge, then it is impossible to do the inverse transformation back to the initial gauge. This is untenable.

III. QUANTUM ELECTRODYNAMICS IS INCOMPLETE

In 1952, Dyson showed [19] that the perturbation expansion of QED is not convergent. This continues to pose a dilemma since increasing accuracy in both experiments and computation have failed to show any discrepancies. Dyson conjectured that QED was somehow incomplete.

That Dyson’s conjecture is correct was demonstrated long ago [20, 21]. Standard QED has the basic defect that it does not contain its strong-field limit. That is, QED exists as a perturbation expansion without knowledge of the complete theory to which it is an approximation.

The identification of the defect followed from an effort to find the convergence properties of relativistic quantum mechanics (RQM) as obtained from a problem using the Volkov solution [22, 23]. This is an exact solution of the Klein-Gordon equation found by Gordon [22] for a scalar charged particle in a plane-wave field, and an exact solution of the Dirac equation found by Volkov [23] for a spin-$\frac{1}{2}$ particle in a plane-wave field. It has become conventional to refer to both solutions as the Volkov solution. In the context of the application to Breit-Wheeler pair production [24] for arbitrarily high intensities, an analysis showed that RQM possesses a convergent perturbation expansion with the radius of convergence...
limited by intensity-dependent singularities in the complex coupling-constant plane. Two very important and unexpected features arose in the investigation: an altered mass shell, and an intensity-dependent coupling constant.

A. Strong-field mass shell

The quantity referred to as the “mass shell” is the expression

\[ p^\mu p_\mu = (mc)^2, \]  

(10)

where \( p^\mu \) is the 4-momentum vector of the particle of mass \( m \). It was discovered independently by Sengupta [25] and by the present author [20, 21] that, in strong fields, the mass shell is altered to

\[ p^\mu p_\mu = (mc)^2 + 2mU_p, \]  

(11)

where \( U_p \) is the ponderomotive potential of a particle of charge \( q \) in a transverse field, defined as

\[ U_p = \frac{q^2}{2mc^2} \langle |A^\mu A_\mu| \rangle. \]  

(12)

The ponderomotive potential is plainly Lorentz invariant and, from Eq. (8) it is also gauge-invariant for a propagating field. The absolute value \( |A^\mu A_\mu| \) is employed because \( A^\mu \) is a spacelike 4-vector and it is best to use a positive number as a basic measure. The angle brackets in Eq. (12) refer to an average over a full cycle of the field. That is employed because, within any cycle of oscillation in a periodic field, it is known [26] that there is a continuous exchange between kinetic and potential energies even though, in any complete cycle, there can be no net energy transfer between a transverse field and a free charged particle.

The difference between the mass shell expressions in Eqs. (11) and (12) is minor in experiments with low-power laser beams, but it is a major factor with modern high-power pulsed laser beams.

B. Strong-field coupling constant

The coupling constant in standard QED between a plane-wave field and a particle of charge \( e \) is the fine-structure constant \( \alpha = e^2/\hbar c \). In the convergence investigation of Refs.
It was found that the coupling parameter of strong-field physics is given by the dimensionless intensity-dependent quantity

\[ z_f = \frac{2U_p}{mc^2}. \]  

(13)

Expressed in terms of \( z_f \), the mass shell of Eq. (11) is

\[ p^\mu p_\mu = (mc)^2 (1 + z_f), \]  

(14)

so that \( z_f \) is a direct measure of the distinction between strong-field and standard electrodynamics.

There is another way to express \( z_f \) that is very informative. If \( \alpha \) is extracted as a multiplier, then \( z_f \) can be written in the form

\[ z_f = \alpha \rho V = \alpha \rho \left( \frac{2\lambda_C^2}{c} \lambda \right), \]  

(15)

where \( \rho \) is the density of photons and \( V = 2\lambda_C^2 \lambda \) is the volume that supplies photons to a strong-field process. This volume is essentially a cylinder of radius \( \lambda_C \) and length \( \lambda \), where \( \lambda_C \) is the Compton wavelength and \( \lambda \) is the wavelength of the propagating field. The Compton wavelength is the usual measure of the interaction radius of a charged particle in a propagating field. The wavelength \( \lambda \) is a macroscopic quantity. That is, all the photons in this cylinder contribute to the interaction even though \( \lambda \) might be many orders of magnitude larger than the size of a target that is subjected to the strong field.

It is the presence of a macroscopic quantity that can be said to characterize a strong field.

C. Dressed electrons

When an electron (or any charged particle) is immersed in a strong propagating field, it possesses a field-caused potential energy of \( U_p \). This energy comes from the propagating field, which is a relativistic phenomenon, so that the electron must also acquire the momentum \( U_p/c \) of the “dressing” field. Photons from the background field have a 4-momentum on the light cone. That is, the electron must acquire the 4-momentum

\[ U^\mu = U_p \left( \frac{k^\mu}{\omega/c} \right). \]  

(16)
The dressed electron can be regarded as a free particle with the 4-momentum \( p^\mu + U^\mu/c \), and satisfy the usual mass shell condition of Eq. (10), which becomes

\[
\left( p^\mu + \frac{1}{c} U^\mu \right) \left( p_\mu + \frac{1}{c} U_\mu \right) = p^\mu p_\mu + \frac{2}{c^2} p^\mu U_\mu,
\]

(17)
since \( U^\mu U_\mu \sim k^\mu k_\mu = 0 \). The Lorentz invariant quantity on right-hand side of this equation can be evaluated in the rest frame of the electron, so that Eq. (17) becomes

\[
\left( p^\mu + \frac{1}{c} U^\mu \right) \left( p_\mu + \frac{1}{c} U_\mu \right) = (mc)^2 + 2mU_p,
\]

(18)
which is exactly Eq. (11). That is, Eq. (11) can be regarded as the mass shell of a free electron dressed by the propagating field.

D. Summed Feynman diagrams

Fried and Eberly [29] showed that it was possible to sum the Feynman diagrams of QED to all orders in a Compton scattering problem in which the spinor electron is replaced by a scalar particle. With one important revision, the result they found exactly duplicates what is obtained by using a Volkov solution. The only difference is that \( z_f \) does not appear, and the mass shell of Eq. (10) is obtained.

The Fried and Eberly calculation verifies that QED is incomplete.

A subsequent investigation by Eberly and Reiss [30] examined a class of diagrams, each of which is divergent, that was omitted from the Fried and Eberly calculation on the grounds that the divergent diagrams are unphysical. The Eberly and Reiss paper showed that these diagrams can be summed exactly, and the sum is finite. The importance of this demonstration is that the summed divergent diagrams introduce exactly the strong-field mass-shell expression of Eq. (11) or (14).

The Eberly and Reiss calculation shows the manner in which QED fails to be complete.

E. Radius of convergence of perturbation theory and the failure of Feynman diagrams

The convergence investigation of Refs. [20] and [21] shows that an extreme upper limit for perturbation theory is marked by the first channel closing that can occur. A feature of
strong-field processes is that the field must supply the basic energy required for a transition as well as the potential energy $U_p$ of any created charged particle immersed in a strong field. For example, in the strong-field Breit-Wheeler pair production process, multiple photons are needed to supply the rest energy $2mc^2$ of the pair produced, but also the ponderomotive energy of the two charged particles created. As the field intensity increases, the lowest-order process that can occur must index upward to supply the required ponderomotive energy of the electron pair. That indexing is referred to as a “channel closing”, and perturbation theory will fail at or before the intensity for the first channel closing.

Reference [30] shows that an infinite number of Feynman diagrams must be summed in order to explain the strong-field mass shell. The strong-field mass shell is therefore a nonperturbative effect. When the intensity is high enough for perturbation theory to fail, then Feynman diagrams lose all meaning since they provide a pictorial representation only of perturbative processes.

Figures 7 and 8 of Ref. [32] show graphically the change from perturbative to nonperturbative behavior. At low field intensity the lowest allowed order of interaction is the sole contributor to a transition. At high field intensity the superposition of many different photon orders is necessary to describe a quantum transition.

F. Green’s function of the Volkov solution

The properties of the Green’s function of the Volkov solution provide a clear picture of the differences between a standard propagator in QED and that for strong fields. It is conventional to examine the behavior of the Green’s function in a complex $p^0$ space (i.e. complex energy space). This is instructive because the path employed for an integration in this space establishes whether the Green’s function represents an advanced solution, a retarded solution, or a Feynman solution in which positive energy solutions propagate forward in time and negative energy solutions propagate backward.

In QED, the mass shell of Eq. (10) applies, and the only poles in the complex energy space occur at $p^0 = \pm \sqrt{p^2 + mc^2}$. In the Volkov Green’s function, the mass shell of Eq. (11) applies, and families of sideband singularities appear in addition to the two QED poles. Each QED pole has its own set of sideband states, but they do not overlap. See Ref. [31] for details.
These Green's function properties make explicit the differences found between the QED calculation of Ref. [29] and the strong field calculation of Ref. [30]. QED has but two singularities on the complex $p^0$ space, whereas the strong-field case has infinite families of sidebands. This phenomenon illustrates the failure of Feynman diagrams, which cannot represent an infinity of singularities.

IV. STRONG-FIELD INFERENCE

Two important strong-field matters will be mentioned here, in addition to the strong-field features discussed in preceding Sections.

A. Fixed origin for energy measure

The ponderomotive energy $U_p$ is the fundamental measure applicable to a charged particle in a strong field. This is reflected in the essential properties of $z_f$ as the coupling constant for charged particle interactions with strong fields, as well as its role in the strong-field altered mass shell. When interactions of the field with bound-state particles are considered, another dimensionless intensity parameter arises, which is the ratio of the ponderomotive potential to the binding energy $E_B$ \cite{27, 32}:

$$z_1 = \frac{2U_p}{E_B}. \hspace{1cm} (19)$$

A third dimensionless parameter is

$$z = \frac{U_p}{\hbar \omega}, \hspace{1cm} (20)$$

which is of universal applicability in strong-field problems since it is a measure of the minimum number of photons that enter into a strong-field-induced interaction. As $U_p$ increases, this minimum indexes upward, illustrating channel closing. As mentioned above, the first such channel closing identifies an upper limit for the convergence of perturbation theory \cite{20, 21, 27, 32}.

A novel feature that follows from the basic importance of $U_p$ is that it fixes an absolute origin for energy measures. As Eq. (12) shows, the zero of energy is established by the zero of the 4-vector $A^\mu$. In the dipole approximation as employed in atomic physics, the zero of energy can be arbitrary as long as it is applied universally.
This fixed origin of energy measure is an important feature distinguishing the two varieties of the Strong-Field Approximation (SFA).

B. Ambiguity in the SFA

The SFA is regarded as the standard analytical approximation for the interaction of strong laser fields with matter. There is an existential problem with this appraisal in that the SFA exists in two incompatible forms.

1. SEFA

The dipole approximation as used in atomic physics neglects entirely the magnetic component of laser fields. When the dipole approximation is imposed from the outset, the SFA is a theory of oscillatory electric fields. It is not a theory of propagating fields like those of lasers. Despite the similarities in some ranges of parameters, the differences are fundamental, and of major importance in other ranges of parameters.

The first strong-field analytical approximation employed for laser-induced processes is that of Keldysh [33], who employs the dipole approximation. This and subsequent dipole-approximation methods will be termed the Strong Electric-Field approximation (SEFA).

The dipole approximation is also employed in numerical solution of the time-dependent Schrödinger equation (TDSE), so that it is also of the SEFA character. It is not an exact calculation of laser-induced transitions, as is often claimed.

2. SPFA

A different analytical approximation follows from taking the nonrelativistic limit of a theory based on propagating fields, which will be referred to as the Strong Propagating-Field Approximation (SPFA). The genesis of the SFA of Ref. [32] from relativistic origins is demonstrated in Refs. [34, 35]. When a laser field is very strong, it is the dominant influence in interactions of the field with matter. Laser fields propagate at the speed of light, so that
a relativistic treatment is necessary. When such a theory is reduced to its nonrelativistic long-wavelength form [32], its provenance from a relativistic formalism remains important even though the general appearance of the SPFA resembles that of the SEFA.

3. **SEFA/SPFA differences**

When field frequencies are relatively high, SEFA and SPFA theories coalesce. This is shown in detail in Ref. [36], for example. The authors do numerical integration of the time-dependent Schrödinger equation, employing the dipole approximation, so it corresponds to a SEFA. The method they cite as SFA is the SPFA.

When field frequencies are low, then SEFA and SPFA predictions become profoundly different [37, 38]. The SEFA trends toward what has been labeled as the "asymptotic limit", where the field becomes a constant electric field. By contrast, as the field frequency decreases, the SPFA increasingly manifests the effects of the magnetic component of a laser field, trending towards relativistic behavior. The location of the transition from high and low frequency domains has yet to be established, but it corresponds approximately to wavelengths in the few-µm range.

A very important matter is that the SPFA of Refs. [32, 34, 35] is the only strong-field approximation method that can be categorized as SPFA. Everything else, including TDSE, is SEFA.

V. **VARIETIES OF ELECTRODYNAMICS**

Electrodynamics can be viewed from the standpoint of quantum field theory (QFT), or of relativistic quantum mechanics, or as a purely classical phenomenon. Each of these viewpoints overlaps the adjacent one, generating a unified view of electrodynamics.

A. **Electrodynamics as a quantum field theory**

QFT has become a highly developed formalism based on symmetry principles, leading to the modern “Standard Model”. Electrodynamics has a place in this scheme as the gauge particle of the electromagnetic field. For purposes of this article, the discipline that earned a
Nobel prize for Feynman, Schwinger, and Tomonaga is sufficient. The salient point in QFT is the existence of a number operator whose eigenvalues count the number of photons that participate in an interaction.

The practical application of quantum electrodynamics to laboratory phenomena is accomplished according to the graphic means of the Feynman diagrams that follow from perturbation theory.

B. Electrodynamics in relativistic quantum mechanics

In RQM, the field is not quantized. That is, there is no number operator. Nevertheless, RQM employs what is called the Floquet property, in which the periodicity of an electromagnetic plane wave leads to transfer of energy in integer packets of $\hbar \omega$. Using standard S-matrix methods in a relativistic formulation, a set of computational rules can be evolved that are identical to the Feynman rules of QFT. A clear representation of the equality of the Feynman rules in QED and in RQM is given by the two textbooks of Bjorken and Drell. The first volume uses RQM methods to produce the Feynman rules, followed by the QFT volume that produces exactly the same rules.

The novel feature of RQM is the existence of the Volkov solution, which makes possible the construction of a nonperturbative domain of electrodynamics. The essential distinctions between the QFT of electrodynamics and electrodynamics within RQM is elucidated by Refs. 29 and 30.

C. Classical electrodynamics

Classical electrodynamics does not employ a quantized version of the electromagnetic field, but it is nevertheless possible to define the photon density of a monochromatic field by using the classical energy density of a plane-wave field divided by $\hbar \omega$. This is the method used in Eq. (15) to evaluate photon density.

The application of classical electrodynamics to such practical matters as antenna theory or the properties of transmission lines seems to have no correspondences with RQM or QFT, but there is nevertheless an important connection with nonperturbative RQM. This connection arises through the wavelength dependence of the coupling constant of RQM in
the form given by Eq. (15). The salient question arises from the possibility that $\lambda$ can be such a large quantity that its connection to the microscopic world of quantum mechanics becomes difficult to understand.

Some insight into this question comes from a situation in which classical phenomena at extreme wavelengths is also difficult to understand.

There is a practical application of extremely long wavelengths to the problem of communication with deeply submerged submarines. Several countries have devised systems operating at very long wavelengths to take advantage of the fact that the skin depth of a conducting medium such as seawater varies as $\frac{\lambda^{1/2}}{2}$. The system employed by the U.S. Navy \([41]\) operated at a frequency of 76 Hz, corresponding to a wavelength of $4 \times 10^6 m$, which is almost equal to the Earth’s radius of $6.4 \times 10^6 m$. The receiving antenna can be regarded as the length of the submarine, of the order of $10^2 m$, or one part in 40,000 of the wavelength of the radio signal. On the scale of the submarine, the electric and magnetic fields of the radio wave are constant. Nevertheless, the radio wave carries a coded signal that is intelligible.

This is related to the problem of constant crossed fields. The two relativistic invariants of a propagating electromagnetic field have zero value:

$$E^2 - B^2 = 0, \quad E \cdot B = 0.$$  \hspace{1cm} (21)

It is also possible to generate constant $E$ and $B$ fields that satisfy the conditions (21). If the electric and magnetic fields are the governing quantities that identify fields, then constant crossed fields and propagating fields of very long wavelengths should be equivalent. They are not. Transverse fields propagate in vacuum at the speed of light. Constant crossed fields “propagate” at zero speed. When identified by potentials, constant crossed fields and propagating fields are unrelated. This is direct proof of the primacy of potentials over fields.

The problem of how an atom can respond to a propagating field many orders of magnitude greater than the size of the atom is analogous to the problem of how a submarine can decipher a radio signal with a wavelength 40,000 times the length of the submarine. In each case the target of the plane wave can respond to the information carried by the potential functions of the propagating field. Properties of the $E$ and $B$ fields are secondary.

The critical strong-field parameter $z_f$ varies as the square of the wavelength. It is possible to achieve $z_f = O(1)$ with commercial radio-frequency equipment. That is, familiar classical environments can exhibit certain strong-field effects of powerful lasers.
D. Summary of the varieties of electrodynamics

Descriptions of the effects of transverse fields are equivalent within QFT and RQM within the domain of the validity of perturbation theory of QFT. RQM makes available an analytical continuation of the effects of the transverse field into a domain where QFT fails. Classical electromagnetism as applied to macroscopic problems has no relevance to the microscopic world of quantum systems, but nonperturbative RQM shares some of the important behavior of macroscopic transverse fields at very long wavelengths.

VI. PRECISION OF QED

Appraisals of the precision of QED are based on the accuracy to which the value of \( \alpha \) can be determined [42]. That is, the premise is accepted that \( \alpha \) measures the coupling of transverse fields to charged particles. One intent of the present article is to show that Feynman-Dyson perturbation theory applies only to a subset of electrodynamics. Strong fields are neglected, and the coupling parameter of strong fields is \( z_f \), not \( \alpha \).

The proposition is now made that the failure of QED to be convergent is governed by the inability of QED to explain strong field phenomena. QED is not a subset of strong-field physics, but rather it is an approximation to strong-field physics. The importance of strong fields is measured by \( z_f = 2U_p/mc^2 \), and perturbation theory in the context of strong fields has an absolute limit \( z < 1 \), in terms of the intensity parameter \( z \) of Eq. (20). That is QED is subject to the limit

\[
z_f = \frac{2U_p}{mc^2}, \quad z = \frac{U_p}{\hbar \omega} < 1, \quad \text{so that} \quad z_f < \frac{2\hbar \omega}{mc^2}.
\]

(22)

For a typical laser photon energy of 1.5eV, the limit given in Eq. (22) is \( z_f < 6 \times 10^{-6} \). This is an extreme upper limit, and nonperturbative behavior is known to exist at much smaller \( z_f \) values. The best-known manifestation of nonperturbative behavior is the above-threshold ionization (ATI) effect, first observed by Agostini, et al. [43]. The first successful match of a nonperturbative theory to experiment, reported in Ref. [44], was for a case where the peak
$z_f$ was $z_f = 8 \times 10^{-6}$, and this was clearly well into the nonperturbative domain.

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