Bootstrapping a Five-Loop Amplitude Using Steinmann Relations

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The analytic structure of scattering amplitudes is restricted by Steinmann relations, which enforce the vanishing of certain discontinuities of discontinuities. We show that these relations dramatically simplify the function space for the hexagon function bootstrap in planar maximally supersymmetric Yang-Mills theory. Armed with this simplification, along with the constraints of dual conformal symmetry and Regge exponentiation, we obtain the complete five-loop six-particle amplitude.

INTRODUCTION

To “bootstrap” generally refers to solving a problem via an ansatz constrained by symmetries and physical principles. This is naturally most successful in very special theories such as low-dimensional integrable models, but it has also proved powerful for conformal field theories in arbitrary dimensions. The hexagon function bootstrap [1, 2] is a perturbative version aimed at solving a scattering problem in a four-dimensional quantum field theory: the planar limit of $\mathcal{N} = 4$ SYM, the six-point amplitude, which is a transcendental function of these three variables, lives in a restricted space of “hexagon” functions [1]. These are iterated integrals with singularities generated by logarithms of the nine letters [1].

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad w = \frac{s_{34}s_{61}}{s_{345}s_{234}},$$

where $s_{i...k} = (p_i + \cdots + p_k)^2$ are Mandelstam invariants. The same symmetry forces the four- and five-particle amplitudes to be essentially trivial, which is why we concentrate on six particles. It has been conjectured that the amplitude, which is a transcendental function of these three variables, lives in a restricted space of “hexagon” functions [1].

$$S = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\},$$

where

$$y_u = \frac{1+u-v-w-\sqrt{\Delta}}{1+u-v-w+\sqrt{\Delta}}, \quad \Delta = (1-u-v-w)^2 - 4uvw,$$

and cyclic rotations act as

$$C: \quad u \to v \to w \to u, \quad y_u \to 1/y_v \to y_w \to 1/y_u,$$

while parity acts as $u_i \to u_i$, $y_i \to 1/y_i$. These letters arise naturally as projectively invariant combinations of momentum twistors [12], variables that make manifest the dual conformal symmetry. Multiple zeta values $\zeta_{q_1,q_2,...}$ with positive indices $q_i$ also appear.

Branch cuts for massless scattering amplitudes start only at vanishing values of the Mandelstam invariants, $s_{i...k} = 0$. Consequently, there is a canonical Riemann

HEXAGON STEINMANN FUNCTIONS

We consider the scattering amplitude for six gluons (or other partons) in the planar limit of $\mathcal{N} = 4$ SYM. A priori, such an amplitude can depend, in four spacetime dimensions, on 8 Mandelstam invariants. Dual conformal symmetry of this model restricts the nontrivial dependence to be on 3 cross-ratios [1] [10]

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad w = \frac{s_{34}s_{61}}{s_{345}s_{234}},$$

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The naming convention will be explained shortly. Already, the Steinmann relations’ impact is noticeable: without it there would be three additional functions, \( \log^2 u, \log^2 v \) and \( \log^2 w \), which do not satisfy eq. (6).

**The Steinmann Basis to Weight 4**

A complete basis of 88 hexagon functions at transcendental weight 4 was originally constructed in ref. [16]. The Steinmann relations imply that only a subspace is physically relevant, a subspace sufficiently small that it can be described in this Letter. We begin with weight 1, where the first entry condition allows only elementary logarithms: \( \log u, \log v, \log w \). To build the higher weight basis, we use the fact that all derivatives of a Steinmann function also obey the Steinmann relations.

The derivative of a weight-\( k \) hexagon function \( F \) has the form

\[
dF = \sum_{i=1}^{9} F^i \ d\ln S_i, \tag{7}
\]

where \( F^i \) are weight-(\( k-1 \)) hexagon functions and \( S_i \in S \) in eq. (2). We thus make an ansatz (7) for the derivatives of \( F \) where the \( F^i \) are Steinmann functions. For the ansatz to represent a function, the partial derivatives must commute (“integrability condition”). Once this condition is solved, the analyticity and Steinmann properties simplify dramatically. It suffices to impose the following constraints, which serve only to fix a few coefficients of zeta-values of weight \((k-1)\) and \((k-2)\):

- \( F^{1-u}, F^{v_w} \) and \( F^{w_v} \) must vanish at \((u,v,w) = (1,0,0) \) [2, 17].
- The \( s_{234} \)-discontinuity of \( F^u + F^{1-u} + F^w + F^{1-w} \) must vanish at \((u,v,w) = (+\infty,0,-\infty)\).

Cyclic rotations of these conditions are implied. The first condition enforces the absence of unwanted discontinuities [13] at function level; the second condition does the same for the Steinmann condition [5].

Following this procedure, at weight 2 we find 7 elements: the constant \( \zeta_2 \) and two cyclic orbits containing

\[
K^u_{1,1} \equiv \text{Li}_2(1-1/u), \quad L^a_2 \equiv \frac{1}{2} \left[ \log^2(u) + \log^2(v/w) \right]. \tag{8}
\]

The focus of this Letter is the Steinmann relations, which state that an amplitude \( A \) can have no double discontinuities in overlapping channels [7]. Using the correspondence between discontinuities and cut diagrams via the Cutkosky rules [14], overlapping channels correspond to cut lines that intersect. Thus for example the channels \( s_{345} \) and \( s_{234} \) overlap, which leads, schematically, to:

\[
\text{fig. 1. Illustration of the channels } s_{345} \text{ and } s_{234} \text{ for } 3 \rightarrow 3 \text{ kinematics. The discontinuity in one channel should not know about the discontinuity in the other channel.}
\]

Steinmann relation: \( \text{Disc}_{s_{345}}(\text{Disc}_{s_{234}} A) = 0, \) \( \tag{4} \)

illustrated in figure [1]

We focus on three-particle invariants \( s_{ijk} \) because these can change sign along fairly generic codimension-1 surfaces in the space of external momenta. The relation can therefore be probed with real external momenta. (In contrast, massless thresholds in two-particle invariants \( s_{ij} \) occur at phase space boundaries where other invariants may change sign; it is unclear to the authors how to extract putative constraints from these thresholds beyond the Regge limit [15].) For functions of the cross-ratios \( u, v, w \), the discontinuity with respect to \( s_{234} \) can be computed by rotating \( v, w \) by a common phase, as follows from eq. (1). The general Steinmann relation (4) thus implies — for the special case of dual-conformally invariant functions — that the following combination is analytic in a neighborhood of \( r = \infty \):

\[
0 = \text{Disc}_{r=\infty} \left[ A(ru, ve^{i\pi}, re^{i\pi}) - A(ru, ve^{-i\pi}, re^{-i\pi}) \right], \tag{5}
\]

where \( u, v > 0 \) (and \( r > 0 \) before taking the discontinuity). The reason why \( r = \infty \) appears is that the three-particle invariants appear in the denominators of eq. (1).

Focusing on the region where all three cross-ratios are large and combining this condition with its permutations, we obtain an equivalent but more practical statement: the amplitude must be expressible as a sum of terms with singularities in only one three-particle channel:

\[
A = \sum_k \left[ a_k^u \log^k \left( \frac{ru}{vw} \right) + a_k^v \log^k \left( \frac{v}{wu} \right) + a_k^w \log^k \left( \frac{wu}{rv} \right) \right], \tag{6}
\]

with the \( a_k^{uv,w} \) analytic around \( u = v = w = \infty \).
At weight 3, the basis contains 17 elements, the 5 cyclic 3-orbits of

\[ K_3^u ≡ \frac{1}{3!} \log^3(1/u) + \frac{1}{2} \log(1/u) \log^2(v/w), \]
\[ K_{2,1}^u ≡ \text{Li}_2(1/u) \log(1/u) - 2\text{Li}_3(1/u) + 2\zeta_3, \]
\[ K_{1,1,1}^u ≡ \zeta_2 \log(1/u), \]
\[ K_{1,1,1,1}^u ≡ \zeta_2 \log(1/u), \] (9)

the constant \( \zeta_3 \), and a single parity-odd element: the six-dimensional scalar hexagon integral \( \Phi_6 \) \[17\] \[18\].

At this stage we see that the functions in eqs. (8)-(9) depend nontrivially on only \( u \), apart from simple powers of \( \log(v/w) \). We can construct \( 3 \times 2^{k-1} \) similar elements at weight \( k \), as follows. We start from “seeds” which trivially satisfy eq. (6):

\[ \begin{align*}
K_k^u(u, \frac{v}{w}) &\equiv \frac{1}{2 \cdot k!} \left[ \log^k \left( \frac{v}{uw} \right) - \log^k \left( \frac{uw}{w} \right) \right], \\
L_k^u(u, \frac{v}{w}) &\equiv \frac{1}{2 \cdot k!} \left[ \log^k \left( \frac{v}{uw} \right) + \log^k \left( \frac{uw}{w} \right) \right].
\end{align*} \] (10)

We then construct nontrivial functions as a simple generalization of harmonic polylogarithms (HPLs) \[19\] with argument \( x = 1/u \), by integrating the seeds from the base point \( u = \infty \). Using this base point automatically maintains the Steinmann relations. The constraint of analyticity for \( u > 0 \) is enforced by recursively removing values at \( u = 1 \):

\[ K_{k,...}^u(u, \frac{v}{w}) \equiv \sum_j c_j L_{j}^u + \int_0^{1/u} \frac{dx}{1-x} \left( \log^{1-j} \left( \frac{v}{uw} \right) K_{j}^u(\frac{1}{x}, \frac{v}{w}) \right), \] (11)

where the zeta-valued coefficients \( c_j \) are chosen uniquely to make the total vanish at \( u = 1 \). Without the \( c_j \), the recursive definition would be identical to that of HPLs with argument \( x = 1/u \), which makes it straightforward to express the \( K^u \) as combinations of HPLs. At weights 2 and 3, this definition agrees with the examples given.

Defining \( K^v, K^w, L^v \) and \( L^w \) as cyclic images of \( K^u \), \( L^u \), the \( K \) functions with positive indices do generate \( 3 \times 2^{k-1} \) linearly independent elements. There is one exception: the three \( K_{k,v,w}^u \) for even weight \( k \) are linearly dependent, so for even \( k \) we use \( L_{k,v,w}^u \) instead.

At weight 4, the Steinmann basis contains the 8 3-orbits generated by:

\[ \begin{align*}
L_4^u, & \quad K_3^{1,3}, \quad K_2^{2,2}, \quad K_1^{1,1}, \quad K_1^{1,1,1,1}, \quad K_2^{1,1,1,1}.
\end{align*} \]

The iterative construction also generates 5 “non-\( K \)” functions: 3 parity-even functions — the integral \( \Omega^{(2)} \) \[16\] \[17\] and its cyclic permutations — plus 2 parity-odd functions. Ten more functions come from multiplying \( \zeta_2 \), \( \zeta_3 \) and \( \zeta_4 \) by the lower-weight Steinmann functions listed earlier. In summary, at weight 4 there are 39 physically relevant Steinmann functions, to be contrasted with 88 in the original hexagon function space.

### Table I. Free parameters remaining after applying each constraint, for the 6-point (MHV,NMHV) amplitude at \( L \) loops.

| Constraint | \( L = 1 \) | \( L = 2 \) | \( L = 3 \) | \( L = 4 \) | \( L = 5 \) |
|------------|-------------|-------------|-------------|-------------|-------------|
| 0. Functions | (10,10) | (82,88) | (639,761) | (5153,6916) | (???????) |
| 1. Steinmann | (7,7) | (37,39) | (174,190) | (758,839) | (3105,3434) |
| 2. Symmetry | (3,5) | (11,24) | (44,106) | (174,451) | (???????) |
| 3. Final-entry | (2,2) | (5,5) | (19,12) | (72,32) | (272,83) |
| 4. Collinear | (0,0) | (0,0) | (1,1) | (3,5) | (9,15) |
| 5. Regge | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) |

This gap increases rapidly with higher weights, as evidenced by the first two lines of Table I, which was generated by implementing the construction iteratively. The paucity of Steinmann functions is because the space is not a ring: the product of two Steinmann functions is generically not an allowed function.

### Application to Two Loops

Before using the Steinmann basis to help bootstrap the hexagon amplitude, we comment on the subtraction of its infrared divergences. A particularly convenient scheme for removing infrared divergences in the SYM model is to divide by the so-called BDS ansatz \[20\]. This soaks up the dual conformal anomaly, leaving a remainder which depends only on the cross-ratios \( u, v, w \), and furthermore vanishes in soft and collinear limits \[10\] \[11\].

However, in order to preserve the Steinmann relation \[4\], it is critical to divide only by quantities which are free of three-particle discontinuities. This singles out the so-called BDS-like ansatz \[2\] \[22\] \( R_6' \):

\[ R_6' \equiv M_6^{bare} / M_6^{BDS-like} \] (12)

In fact, the amplitude is a function of the helicity of all 6 particles, in a way which can be neatly encoded in so-called \( R \)-invariants \[12\] \[23\]. In this Letter we thus deal with bosonic functions \( E, E \) and \( \tilde{E} \) which encode all the information and correspond to suitable components of the MHV and NMHV BDS-like remainders. Schematically, \( R_6' \simeq E \oplus \tilde{E} \oplus \bar{E} \). The relations to the more conventional BDS MHV remainder (\( R_6 \)) and NMHV ratio function (\( V, \tilde{V} \)), defined for example in ref. \[2\] (to which...
we refer for further details), are:

\[ e^{\mathcal{R}_6} \equiv e^{-\frac{\Lambda}{24} \Gamma_{\text{cusp}} \mathcal{E}^{(1)}}, \quad \mathcal{V} = E/\mathcal{E}, \quad \tilde{V} = \tilde{E}/\mathcal{E}, \quad (13) \]

where \( \frac{1}{2} \Gamma_{\text{cusp}} = g^2 - 2\zeta_2 g^4 + \ldots \) is the cusp anomalous dimension, known exactly as a function of the coupling \( g^2 \equiv \frac{\alpha_s N}{16\pi^2} \) [24]. We stress that while \( \mathcal{E}, E \) and \( \tilde{E} \) obey the Steinmann relations, \( \mathcal{R}_6,\mathcal{V} \) and \( \tilde{V} \) do not: the space of Steinmann functions is not a ring.

Let us describe a concrete example, the bootstrap of \( \mathcal{E} \) at two loops. We begin by applying the following:

1. \( \mathcal{E} \) is a hexagon Steinmann function
2. \( \mathcal{E} \) is parity-even and dihedrally symmetric
3. The collinear limit to leading power is universal:

\[ \lim_{\nu \to 0} \mathcal{E} = e^{-\frac{1}{4} \Gamma_{\text{cusp}} (L_2^2 + 2\zeta_2)} + O(\sqrt{\nu} \ln^{L-1} \nu). \]

In the weight 4 Steinmann space, no linear combination vanishes in all three collinear limits. Therefore the two-loop MHV amplitude is fully determined by just the above three conditions! Loop expanding using \( \mathcal{E} = \mathcal{E}^{(0)} + g^2 \mathcal{E}^{(1)} + g^4 \mathcal{E}^{(2)} + \ldots \), the result at tree level is \( \mathcal{E}^{(0)} = 1 \), at one loop

\[ \mathcal{E}^{(1)} = K_{1,1}^u + K_{1,1}^v + K_{1,1}^w, \quad (14) \]

and at two loops

\[ \mathcal{E}^{(2)} = (1 + C + C^2) \left[ (2) \sum K_{1,1,1}^u - 4K_{1,1,1}^u \right] + 8\zeta_4, \quad (15) \]

where the cyclic rotation \( C \) is defined in eq. (3). This result agrees completely with refs. [11, 16].

For MHV at higher loops, and for NMHV, we imposed an additional “final-entry” condition, obtained by considering the action of the \( \tilde{Q} \) generator of dual superconformal transformations [8]. The MHV final-entry condition is simply \( \tilde{E}^{1-u} = -E^u \), plus the cyclic relations. Similarly, the differential of the NMHV BDS-like remainder is spanned by the 18 elements listed in eq. (3.10) of ref. [2]. These conditions almost completely determine the higher-loop amplitudes; we need information from only one more limit.

**REGGE EXPONENTIATION AND BOOTSTRAP**

In the multi-Regge limit of \( 2 \to 4 \) gluon scattering, the four outgoing gluons are strongly ordered in rapidity. The cross-ratios have the limits \( u \to 1, v, w \to 0 \), but on an analytically continued Riemann sheet which ensures nontrivial Lorentzian kinematics. This limit has been thoroughly analyzed for both MHV and NMHV amplitudes [15, 25, 29]. Amplitudes exponentiate in terms of Fourier-Mellin variables \( \nu, m \) which are conjugate to the transverse plane coordinates, schematically:

\[ \mathcal{E}(\nu, m, vw) \xrightarrow{\text{Regge}} \Phi(\nu, m) \times (-1/\sqrt{vw})^{\omega(\nu, m)} \quad (16) \]

where the Regge trajectory \( \omega \) vanishes at tree level and \( \Phi \) is an “impact factor”. Exponentiation implies that terms with \( \log^2(vw) \) or higher powers of the large logarithm are predicted by the multi-Regge limit at lower loops.

Remarkably, through five loops such terms suffice to fix all remaining parameters and uniquely determine \( \mathcal{E}, E, \) and \( \tilde{E} \). Terms with \( \log(vw) \) or lower were not needed, but rather led to predictions for the next loop order, enabling a pure bootstrap with no external information. The constraints are summarized in table I.

With \( \mathcal{E}, E, \) and \( \tilde{E} \) fixed through five loops we can evaluate them numerically on a variety of lines in cross-ratio space. Figure 2 shows the remainder function on the line \( u = v = w \). We have also used “hedgehog” variables [30] to generate multiple polylog representations of these functions in one bulk region [31].

Past implementations of the hexagon function bootstrap employed a variety of other constraints, which the Steinmann relations render unnecessary, or relegate to cross checks. For NMHV, the representation in terms of \( R \)-invariants has poles at kinematically spurious points that must cancel between different permutations of \( E \) and \( \tilde{E} \) [16]. Now, after imposing the collinear constraint in table I, the spurious poles cancel automatically. Similarly, for MHV and NMHV the \( \tilde{Q} \) equation predicts not only final entries, but next-to-final entries; however, again these constraints are satisfied automatically.

For both MHV and NMHV, the pentagon operator product expansion (POPE) [8, 9] served previously as a powerful bootstrap constraint [17, 29]. Now Regge exponentiation is enough to obtain a unique result. Nonetheless, we do check our results against the POPE predictions. We find complete agreement through five loops, to each order in the OPE we have computed (\( T^1 \) and \( T^2 F^2 \) for MHV [8, 22, 30] and \( T^1 \) for the (6134) component of NMHV [8, 32]).
In ref. [29], two of the authors conjectured a relationship between the $L$-loop MHV amplitude and the $(L-1)$-loop NMHV amplitude. Our five-loop MHV amplitude allows us to verify this relation at one more loop order. Expressed in terms of the functions defined in eq. (13), it reads (using the coproduct notation [29])

$$g^2 (2E - \mathcal{E}) = \mathcal{E}^{y_u+y_v} + \mathcal{E}^{y_u+y_w} - 3\mathcal{E}^{y_u+y_v} - \mathcal{E}^{v,v} - \mathcal{E}^{1-v,v} + 2(\mathcal{E}^{y_u+y_v} + \mathcal{E}^{y_u+y_w}) - \mathcal{E}^{y_u+y_v} - \mathcal{E}^{y_u+y_w}. \quad (17)$$

This relation calls out for explanation.

Remarkably, the space of Steinmann functions appears to be “not much larger” than required to contain $\mathcal{E}$, $E$ and $\tilde{E}$, if we include all derivatives of higher loop amplitudes. Up to at least weight 6, the complete space is needed, apart from certain unexpected restrictions on zeta values. For example, the weight 2 functions found by taking 8 derivatives of $\mathcal{E}^{(5)}$, $E^{(5)}$ and $\tilde{E}^{(5)}$ span a 6 dimensional subspace of the 7 dimensional Steinmann space: $K^{(1)} + L^2 + 2C_2$, plus cyclic; $C_2$ is not an independent element. In an ancillary file, we provide a coproduct representation of this trimmed basis, which suffices to describe $\mathcal{E}$, $E$ and $\tilde{E}$ through five loops. We also give HPL expressions for these functions on the lines $(1,v,v)$ and $(u,1,1)$ [31].

**CONCLUSION**

Leveraging the power of the Steinmann relations, we have bootstrapped six-point scattering amplitudes in planar $\mathcal{N} = 4$ SYM, the Steinmann relations apply in general quantum field theories. Their strength here suggests that these often-neglected constraints may have broader applicability, perhaps making similar bootstrap techniques viable in other theories, such as QCD.

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