A modified sinusoidal perturbation on a two distinguishable spin—$\frac{1}{2}$ particles system

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Abstract. We analyze a quantum system of two distinguishable spin—$\frac{1}{2}$ particles subjected to modified sinusoidal perturbation of the form $V(t) = (V_0 + V_1 x + V_2 y + V_3 z) e^{-\gamma t / \gamma t} \cos(\omega t)$, $0 < t < \tau$; where $\gamma$ is a real positive number; $V_0, V_1, V_2, V_3$ are real constants and have very small values; $x = r \sin \theta \cos \phi$; $y = r \sin \theta \sin \phi$; $z = r \cos \theta$; $\sigma_1$ is the Pauli matrix for the first particle, and $\sigma$ is the three-dimensional position operator in spherical coordinates; $V(t)$ is zero when $t \leq 0$ and $t \geq \tau$. We use the time-dependent perturbation theory to solve this problem. Initially, we define the initial system state before being subjected to perturbation, namely $|00\rangle$, and the final state of the form $|nlm;SM\rangle = |nlm\rangle \otimes |SM\rangle$. The total spin state $|SM\rangle$ actually contains the linear combination of the states of each particle. Next, we calculate the expected value of $(nlm;SM|V(t)|00;000)000$ is used in calculating the probability amplitude $C_\alpha(\tau)$, and then this form is used to analyze the conditions required for the transition to occur. From the results of the analysis, we present various conditions which include the conditions for $n, l, m, M, S$; constants $V_1, V_2$ and $V_3$; and the relationship between $r$ and $\gamma$; so that the transition process can occur. We have also presented the energy emission and absorption processes that this type of perturbation phenomenon might produce in the system under study.

1. Introduction

The perturbation theory is a theory that plays a very important role in physics, starting from the subatomic level [1–3] to large-scale levels such as cosmology [4–7]. When a physical system that has a certain Hamiltonian is subjected to perturbation with a certain Hamiltonian, the system can be studied using this theory. There are two types of perturbation theory, namely the time-independent perturbation theory (which deals with time-independent potentials) and the time-dependent perturbation theory (if perturbation changes with time). This theory is an approximation theory that is very useful for analyzing systems that cannot be solved exactly, but of course with various limitations as stated above.

In the context of quantum mechanics, there are many problems that require us to apply this theory, for example in the case of the Stark effect [8–10], the Ziemann effect [9–11], fine structure [11–13], hyperfine structure [11], and so on [2,14], also for cases such as the quantum system of two distinguishable particles spins—$\frac{1}{2}$. Studies of this system with various types of perturbation have been carried out by many researchers, both time-independent and time-dependent perturbation. Especially for time-dependent perturbation, the perturbation can be in the form of harmonic or anharmonic perturbations [9,10,15].
In this paper, we analyze a quantum system of two distinguishable spin-$\frac{1}{2}$ fermionic particles subjected to time-dependent perturbation. This type of perturbation is sinusoidal which is modified by the presence of the damping term, and there are also perturbation terms simultaneously in all three cartesian coordinate bases, but then it is studied in spherical coordinates. Not only that, but this perturbation modification also involves multiplying by the scalar product of the Pauli spin matrix of one of the particles with a three-dimensional position operator in spherical coordinates. This potential form of modification is of great interest to study because it provides possible more general solutions that can cover standard harmonic perturbation states. The solution obtained is a transition amplitude that has a non-standard form, but we can choose some special conditions so that the form of this general solution can be reduced to some standard forms.

2. Methods
The study in this paper uses the Time-Dependent Perturbation method. This method is described in two parts namely.

2.1. Dyson series
Suppose a quantum system has a state $|\alpha\rangle$ at time $t = t_0$, then at time $t$ then the system is in the state $|\alpha, t_0; t\rangle$, where the form $|\ldots\rangle$ represents the state in the interaction picture. For a quantum system that is subject to time-dependent perturbation, in the interaction picture, the state equation using the time evolution operator $U(t, t_0)$ can be expressed in the form [9].

$$|\alpha, t_0; t\rangle = U(t, t_0)|\alpha, t_0\rangle$$  \quad (1)

where $U(t, t_0) = e^{-\frac{iH_0t}{\hbar}}U(t, t_0)e^{-\frac{-iH_0t_0}{\hbar}}; U(t, t_0)$ is the time evolution operator in the Shrodinger picture; and $H_0$ is the time-independent Hamiltonian. The time evolution operator in equation (1) satisfies the equation

$$i\hbar \frac{dU(t, t_0)}{dt} = V(t)U(t, t_0),$$  \quad (2)

where $V(t)$ is a potential of interaction picture that has a form $V(t) = e^{\frac{iH_0t}{\hbar}}V(t)e^{-\frac{-iH_0t}{\hbar}}$. Next, equation (2) becomes

$$\frac{1}{i\hbar} \int_{U(t, t_0)|_{t_0}}^{U(t, t_0)|_{t}} dU(t', t_0) = \frac{1}{i\hbar} \int_{t_0}^{t} V(t')U(t', t_0)(t')dt'. \quad (3)$$

By using the initial condition $U(t, t_0)|_{t=t_0} = 1$, the integral result of equation (3) is obtained, namely

$$U(t, t_0) = 1 + \frac{1}{i\hbar} \int_{t_0}^{t} V(t')U(t', t_0)dt'. \quad (4)$$

Furthermore, assuming that $V(t)$ is small, then the solution of equation (4) can be approximated successively. For the first-order approximation, entering $U(t', t_0) = 1$, we get

$$U(t, t_0)^{(1)} = 1 + \frac{1}{i\hbar} \int_{t_0}^{t} V(t')dt'. \quad (5)$$

For the second order approximation, we use $U(t', t_0) = U(t, t_0)^{(1)}(t', t_0)$ and then inserting this form and equation (5) into equation (4), we obtain

$$U(t, t_0)^{(2)} = 1 + \frac{1}{i\hbar} \int_{t_0}^{t} V(t')dt' + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} \int_{t_0}^{t} V(t')dt' \int_{t_0}^{t} V(t'')dt''. \quad (6)$$

The higher orders can be obtained using the same method. Therefore, the general form of this approximation can be expressed in terms of
\[ U(t, t_0) = 1 + \frac{1}{\hbar} \int_{t_0}^{t} V_j(t') dt' + \left( \frac{1}{\hbar} \right)^2 \int_{t_0}^{t} V_j(t') dt' \int_{t_0}^{t} V_j(t'') dt'' + \cdots \]
\[ + \left( \frac{1}{\hbar} \right)^n \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \cdots \int_{t_0}^{t} dt^{(n)} \int_{t_0}^{t} dt^{(n)} V_j(t') V_j(t'') \cdots V_j(t^{(n)}) + \cdots \]  
(7)

This form is known as the Dyson Series which is used to calculate the state vector up to the required perturbation order.

2.2. Transition amplitude

The transition amplitude of the quantum system with the initial unperturbed \( |i\rangle \) to \( |n\rangle \) state is

\[ C_n = C_n^{(0)} + C_n^{(1)} + C_n^{(2)} + \cdots = \langle n | U(t, t_0) | i \rangle \]

which has a relationship with transition probability \( P(i \rightarrow n) = |\langle n | U(t, t_0) | i \rangle|^2 = |C_n|^2 \), where \( n \neq i \). By using the forms of \( V_j(t) \); \( C_n \); and equation (7), we obtain

\[ C_n^{(0)} = \delta_{ni} \quad (\text{independent of } t) \]

\[ C_n^{(1)} = \frac{1}{\hbar} \int_{t_0}^{t} \langle n | V_j(t') | i \rangle dt' \]

\[ C_n^{(2)} = \frac{1}{\hbar^2} \sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' e^{i\omega m t'} \langle n | V_j(t') | m \rangle e^{i\omega m t''} \langle m | V_j(t'') | i \rangle, \]
(11)

where \( e^{i\omega m t} \equiv e^{i(\epsilon_n - E_i)t/\hbar} \). Equation (8) is a statement of the amplitude of a quantum state transition to a certain order in \( V_j(t) \). However, values in high orders generally have a very small contribution to the value of the transition amplitude in low orders, especially for first-order. Therefore, most of the transition amplitude analyzes are only carried out in first-order because this order alone is sufficient to represent the physical state of a quantum system such as the problem of atoms and nuclear physics

3. Results and discussion

This paper examines the two distinguishable spin-\( \frac{1}{2} \) particles system. This type of system can be found in particle systems such as muonium, protonium, positronium, pionium, tauronium, and others. For such systems, the Hamiltonian has a form \[\hat{H} = \frac{\hbar^2}{2m_1} \hat{V}_1^2 - \frac{\hbar^2}{2m_2} \hat{V}_2^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \omega^2 (\hat{r}_1^2 - \hat{r}_2^2)^2 + g \hat{\sigma}_1 \cdot \hat{\sigma}_2 \]
(12)

where \( g \ll \hbar \omega \); \( g \) is the Lande factor; \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \) are the Pauli spin matrix for the first particle and the second particle, respectively; \( m_1 \) and \( m_2 \) are the masses of the first and second particles, respectively.

In general, the Hamiltonian form in equation (12) is converted into the Hamiltonian form with the coordinates of the center of mass, taking into account \( \hat{R} = \frac{m_1 \hat{r}_1 + m_2 \hat{r}_2}{m_1 + m_2} \) and \( \hat{r} = \hat{r}_1 - \hat{r}_2 \), so that equation (12) becomes

\[ \hat{H} = \frac{\hbar^2}{2M} \hat{V}_R^2 - \frac{\hbar^2}{2\mu} \hat{V}_R^2 + \frac{\mu}{2} \omega^2 r^2 + 2g \left( S(S+1) - \frac{3}{2} \right), \]
(13)

where \( S = \hat{s}_1 + \hat{s}_2 = \frac{1}{2} (\hat{\sigma}_1 + \hat{\sigma}_2) \) and \( M = m_1 + m_2; \mu = \frac{m_1 m_2}{m_1 + m_2} \). From equation (13), the first term of the right side is the motion of the system as a whole; the second and third terms are Hamiltonian terms for an isotropic harmonic oscillator without spin; and the last term in the right-hand side arises from the product of the Pauli spin matrix, which uses the basis \( |s_1 s_2 ; SM \rangle \), where the indices correspond to the Pauli spin matrix, and which are associated with the first and second particles. Meanwhile, the bases of each particle are \( |s_1 m_1 \rangle \) and \( |s_2 m_2 \rangle \), respectively.

Now we analyze a two distinguishable spin-\( \frac{1}{2} \) particles system when subjected to a modified sinusoidal perturbation of the form

\[ V(t) = (V_0 + V_1 x + V_2 y + V_3 z)e^{-\gamma t/\tau} \cos(\omega t) \hat{\sigma}_1 \cdot \hat{\sigma}_2 \]; \( 0 < t < \tau \),
(14)
where \( \gamma \) is a real positive number; \( V_0, V_1, V_2, V_3 \) are real constants; \( x = r \sin \theta \cos \phi \); \( y = r \sin \theta \sin \phi \); \( z = r \cos \theta \); \( \hat{r}_1 \) is the Pauli matrix for the first particle, and \( \hat{\mathbf{x}} \) is the three-dimensional position operator in spherical coordinates; \( V_1, V_2, V_3 \) have very small values; \( \frac{\alpha}{\hbar} \ll 1 \); \( V(t) \) is zero when \( t < 0 \) and \( t > \tau \). By doing some mathematical steps and using the inverse form of Euler’s formula \( e^{\pm i \phi} = \cos \phi \pm i \sin \phi \), equation (14) can also be written

\[
V(t) = \left( V_0 + \frac{V}{2} r \sin \theta e^{-i \phi} + \frac{V^*}{2} r \sin \theta e^{i \phi} + V_3 r \cos \theta \right) e^{-r \tau \cos(\omega t)} \hat{r}_1 \cdot \hat{\mathbf{x}} \quad 0 < t < \tau ,
\]

where \( V \equiv V_1 + i V_2 \) and \( V^* \equiv V_1 - i V_2 \).

In the initial state, the system is in the ground state, and we define this state in the form \( |000; 00\rangle \), and the final state has the form \( |nlm; SM\rangle \), that appears from the direct product \( |nlm; SM\rangle = |nlm\rangle \otimes |SM\rangle \). Since both particles are fermions with each spin \(-\frac{1}{2}\), the total spin state \( |SM\rangle \) of two distinguishable systems contain the base state of each particle, namely [9, 11]

\[
|11\rangle = |s_1(1)s_2(1)\rangle; |10\rangle = \frac{1}{\sqrt{2}} (|s_1(1)s_2(1)\rangle + |s_2(1)s_1(1)\rangle); |1-\rangle = |s_1(1)s_2(1)\rangle; \nonumber
\]

\[
|00\rangle = \frac{1}{\sqrt{2}} (|s_1(1)s_2(1)\rangle - |s_1(1)s_2(1)\rangle).
\]

(16)

To facilitate calculations, we start by calculating the form \( \hat{\sigma}_1 \cdot \hat{\mathbf{x}} \) in spherical coordinates, i.e.

\[
\hat{\sigma}_1 \cdot \hat{\mathbf{x}} = \sin \theta \cos \phi \sigma_{1x} + \sin \theta \sin \phi \sigma_{1y} + \cos \theta \sigma_{1z}
\]

(17)

where \( \hat{\mathbf{x}} = \hat{x} \hat{\mathbf{r}} + \hat{y} \hat{\mathbf{r}} + \hat{z} \hat{\mathbf{r}} \); with \( x = r \sin \theta \cos \phi \); \( y = r \sin \theta \sin \phi \); \( z = r \cos \theta \); and \( \hat{\sigma}_1 = \sigma_{1x} \hat{x} + \sigma_{1y} \hat{y} + \sigma_{1z} \hat{z} \). Then, the operator in equation (17) is worked on state \( |00\rangle \), and then we get the form

\[
(\hat{\sigma}_1 \cdot \hat{\mathbf{x}}) |00\rangle = \frac{1}{\sqrt{2}} \left( \sin \theta \cos \phi \sigma_{1x} |s_1(1)s_2(1)\rangle - \sin \theta \sin \phi \sigma_{1y} |s_1(1)s_2(1)\rangle + \sin \theta \sin \phi \sigma_{1y} |s_1(1)s_2(1)\rangle - \sin \theta \sin \phi \sigma_{1x} |s_1(1)s_2(1)\rangle \right)
\]

(18)

By using the forms \( \sigma_{1x} |s_1(1)\rangle = |s_1(1)\rangle; \sigma_{1y} |s_1(1)\rangle = |s_1(1)\rangle; \sigma_{1y} |s_1(1)\rangle = -i |s_1(1)\rangle; \) and \( \sigma_{1z} |s_1(1)\rangle = |s_1(1)\rangle; \sigma_{1z} |s_1(1)\rangle = -|s_1(1)\rangle \); and taking advantage of equations (16), then equation (18) becomes

\[
(\hat{\sigma}_1 \cdot \hat{\mathbf{x}}) |00\rangle = \frac{1}{\sqrt{2}} \sin \theta e^{i \phi} |1-\rangle + \cos \theta |10\rangle - \frac{1}{\sqrt{2}} \sin \theta e^{-i \phi} |11\rangle.
\]

(19)

First, we calculate the expectation value which contains \( \hat{\sigma}_1 \cdot \hat{\mathbf{x}} \), and by performing several calculation steps, we obtain

\[
\langle nlm; SM | \hat{\sigma}_1 \cdot \hat{\mathbf{x}} | 00; 00 \rangle = \langle nlm; SM | r \left( \sqrt{\frac{4\pi}{3}} Y_1^0 |1-\rangle - \sqrt{\frac{4\pi}{3}} Y_{10}^0 |10\rangle - \sqrt{\frac{4\pi}{3}} Y_{11}^0 |11\rangle \right) Y_0^0 \otimes |0\rangle
\]

where \( Y_l^m \) is the spherical harmonic function. Then, the form of equation (20) can be written

\[
\langle nlm; SM | \hat{\sigma}_1 \cdot \hat{\mathbf{x}} | 00; 00 \rangle = \frac{1}{\sqrt{2}} \delta_{n0} \delta_{l0} \delta_{s1} \left( \delta_{m0} \delta_{M0} - \delta_{m1} \delta_{M-1} - \delta_{m1} \delta_{M1} \right) \equiv 0,
\]

(21)

because \( l = 0 \) for \( n = 0 \).

Second, we calculate the expectation value which contains \( r \sin \theta e^{-i \phi} \hat{\sigma}_1 \cdot \hat{\mathbf{x}} \), and by performing several calculation steps, we obtain

\[
\langle nlm; SM | r \sin \theta e^{-i \phi} \hat{\sigma}_1 \cdot \hat{\mathbf{x}} | 00; 00 \rangle
\]

\[
= \langle nlm; SM | r \left( \frac{2}{3\sqrt{2}} |1-\rangle - \frac{1}{3} \sqrt{\frac{8\pi}{5}} Y_2^{-1} |11\rangle - \sqrt{\frac{16\pi}{15}} Y_{2}^{-2} |11\rangle + \sqrt{\frac{8\pi}{15}} Y_{2}^{-1} |10\rangle \right) Y_0^0 \otimes |0\rangle
\]

\[
\otimes |0\rangle
\]

(22)

Then, the equation (22) can be written

\[
\langle nlm; SM | r \sin \theta e^{-i \phi} \hat{\sigma}_1 \cdot \hat{\mathbf{x}} | 00; 00 \rangle = \frac{2}{3\sqrt{2}} (\langle nl | 00 \rangle \delta_{ls} \delta_{s1} \delta_{m0} \delta_{M-1} - \langle nl | 00 \rangle \delta_{l1} \delta_{s1} \left( \frac{2}{15} \delta_{m-1} \delta_{M0} - \frac{4}{\sqrt{15}} \delta_{m-2} \delta_{M1} - \frac{1}{3} \sqrt{2} \delta_{m0} \delta_{M-1} \right) \right)
\]

(23)

where \( \langle nl | 00 \rangle = \int_0^\infty R_{nl} R_{00} r^3 \, dr \). For \( l = 2 \), \( \langle n2 | 00 \rangle = \int_0^\infty R_{n2} R_{00} r^3 \, dr \).
Third, we calculate the expectation value which contains $r \sin \theta \ e^{i \phi} \hat{\sigma}_z \cdot \hat{x}$, and by performing several calculation steps, we obtain

$$
\langle nlm;SM | r \sin \theta \ e^{i \phi} \hat{\sigma}_z \cdot \hat{x} | 00;00 \rangle = \langle nlm;SM | (\frac{16 \pi}{15} Y_{1}^{0} | 1 - 1) - \frac{8 \pi}{15} Y_{1}^{0} | 10) + \frac{1}{3} \left( \frac{8 \pi}{5} Y_{2}^{0} | 11 \right) - \frac{2}{3 \sqrt{2}} | 11 \rangle Y_{0}^{0} \otimes | 0 \rangle.
$$

(24)

Then, the equation (24) can be written

$$
\langle nlm;SM | r \sin \theta \ e^{i \phi} \hat{\sigma}_z \cdot \hat{x} | 00;00 \rangle
= \frac{2}{3 \sqrt{2}} (n|00) \delta_{i,0} \delta_{S,1} \left( \frac{4}{15} \delta_{m,0} - \frac{1}{3} \frac{2}{3 \sqrt{2}} \delta_{m,0} + \frac{1}{\sqrt{15}} \delta_{m,-1} \delta_{M,1} - \frac{1}{\sqrt{15}} \delta_{m,1} \delta_{M,0} \right)
$$

(25)

Fourth, we calculate the expectation value which contains $r \cos \theta \ \hat{\sigma}_z \cdot \hat{x}$, we have (by performing several calculation steps and by making use of the properties of the spherical harmonic function)

$$
\langle nlm;SM | r \cos \theta \ \hat{\sigma}_z \cdot \hat{x} | 00;00 \rangle
= \langle nlm;SM | (\frac{16 \pi}{15} Y_{1}^{0} | 1 - 1 - \frac{4 \pi}{15} Y_{1}^{0} | 11) + \frac{1}{3} \left( \frac{16 \pi}{5} Y_{2}^{0} | 10 \right) + \frac{1}{3} (10)) Y_{0}^{0} \otimes | 0 \rangle.
$$

(26)

The equation above can also be expressed in the form

$$
\langle nlm;SM | r \cos \theta \ \hat{\sigma}_z \cdot \hat{x} | 00;00 \rangle
= \frac{1}{\sqrt{15}} \left( \frac{2}{3 \sqrt{5}} \delta_{m,0} - \frac{1}{\sqrt{15}} \delta_{m,1} \delta_{M,1} - \frac{1}{\sqrt{15}} \delta_{m,-1} \delta_{M,0} \right)
$$

(27)

Next, we calculate $C_n (\tau)$ (please note that we can also write it with $C_n (\omega)$ because after time $\tau$ is infinite, there is no more perturbation to the system so they both mean the same), namely

$$
C_n (\tau) = \frac{1}{i \hbar} \int_{0}^{\tau} e^{i \omega t} \langle nlm;SM | V(t) | 00;00 \rangle d \tau,
$$

(28)

$$
C_n (\tau) = \frac{1}{i \hbar} \langle nlm;SM \left( \frac{V}{2 \tau} \sin \theta e^{-i \phi} + \frac{V}{2 \tau} \sin \theta e^{i \phi} + V r \cos \theta \right) \hat{\sigma}_z \cdot \hat{x} | 00;00 \rangle \int_{0}^{\tau} e^{i \omega t} e^{-\gamma t \cos(\omega t)} d \tau,
$$

(29)

because $\langle nlm;SM \hat{\sigma}_z \cdot \hat{x} | 00;00 \rangle = 0$, so it doesn’t contribute to $C_n (\tau)$. By using Euler’s formula, which is the inversion of the form $e^{i \theta} = \cos \theta + i \sin \theta$, the integral result is

$$
\int_{0}^{\tau} e^{i \omega t} e^{-\gamma t \cos(\omega t)} d \tau = \frac{1}{i(n+1)\omega - \frac{V}{\tau}} \left[ e^{i(n+1)\omega t} - 1 \right] + \frac{1}{i(n-1)\omega - \frac{V}{\tau}} \left[ e^{i(n-1)\omega t} - 1 \right].
$$

(30)

From equations (25) and (30), equation (29) can be written

$$
C_n (\tau) = \frac{(nl|00)}{4i \hbar} \delta_{S,1} \left( \frac{2V}{3 \sqrt{2}} \delta_{m,0,2,1} \delta_{M,0} - \frac{1}{3} \frac{2}{3 \sqrt{2}} \delta_{m,0,1} \delta_{M,1} - \frac{1}{3 \sqrt{2}} \delta_{m,0,0} \delta_{M,0} \right) + \frac{V}{2} \delta_{i,0} \delta_{m,0} \delta_{M,0} + \frac{V}{2} \delta_{i,0} \delta_{M,1} - \frac{1}{3} \frac{2}{3 \sqrt{2}} \delta_{m,0} \delta_{M,1} + \frac{V}{2} \delta_{i,0} \delta_{m,0} \delta_{M,1} - \frac{1}{3} \frac{2}{3 \sqrt{2}} \delta_{m,0} \delta_{M,1} + \frac{V}{2} \delta_{i,0} \delta_{m,0} \delta_{M,1} - \frac{1}{3} \frac{2}{3 \sqrt{2}} \delta_{m,0} \delta_{M,1}
$$

(31)

For three-dimensional harmonic oscillators, $n = l + 2n_r$, with $l = 2$, then $n = 2 + 2n_r = 2(1 + n_r)$ is even. Therefore, we get

$$
C_{2n} (\tau) = 0,
$$

(32)
where \( l = 2; \) \( \eta = 1, 2, 3, \ldots; \) \( (2\eta; 2)^r|00\rangle = \int_0^\infty \hat{R}(2\eta, 2) \hat{R}_0 r^3 dr; \) and

\begin{align*}
A_1 &= \frac{2}{15} \sqrt{\delta_{m,-1}\delta_{M,0} - \frac{4}{15} \sqrt{\delta_{m,-2}\delta_{M,1} - \frac{1}{3} \sqrt{\delta_{m,0}\delta_{M,-1}}}, \tag{34} \\
A_2 &= \frac{4}{15} \sqrt{\delta_{m,2}\delta_{M,-1} + \frac{2}{3} \sqrt{\delta_{m,0}\delta_{M,1} - \frac{2}{15} \sqrt{\delta_{m,1}\delta_{M,0}}}, \tag{35} \\
A_3 &= 2 \left( \frac{2}{3\sqrt{5}} \delta_{m,0}\delta_{M,0} - \frac{1}{\sqrt{15}} \delta_{m,1}\delta_{M,-1} - \frac{1}{\sqrt{15}} \delta_{m,-1}\delta_{M,1} \right). \tag{36}
\end{align*}

Equation (33) states the transition amplitude for this system from the initial state \(|000; 00\rangle\) to the final state \(|nlm; SM\rangle\) in the time range \(0 < t < \tau\) when the system is subjected to the perturbation. It appears that this quantum state transition depends on a variety of factors including the radial term, angular momentum, constants, and sinusoidal terms of the perturbation, even the damping factor. For more details, we discuss them one by one.

First, we examine the factors containing the radial term \((2\eta; 2)^r|00\rangle\). For this factor, the transition is only allowed at \( l = 2\), and \( \eta \) is a positive integer. Since there is a multiplier of 2, it is only possible to transition for even \( n\), while for odd \( n\), it is absolutely impossible, as shown by equation (32). Note that \( n = 2\eta\).

Second, we examine the term \(\delta_{S,1}(VA_1 + V^\dagger A_2 + V_3 A_3)\). From this form, it is very clear that a transition is only possible if \( S = 1 \). Meanwhile, for the terms contained in brackets, we divide the transition conditions into two parts, namely for the constant \( V_k (k = 1, 2, 3)\) and \( A_k (K = 1, 2, 3)\). The following describes these two conditions, namely: a) Condition for \( V_k \): keep in mind that \( V \) and \( V^\dagger \) contain \( V_1 \) and \( V_2 \). Therefore, for this form, we can select the requirements for \( V_1, V_2, \) and \( V_3 \) only. If we describe this term explicitly, then in fact the three terms contained in the brackets will become five terms in which there is \( V_k \). Because there are five of these terms, then based on the relationship \( V = V_1 + iV_2 \) and \( V^\dagger = V_1 - iV_2 \), we can obtain seven conditions that allow a transition, namely the constants \( V_1, V_2, \) and \( V_3 \) working simultaneously (this is the most common case); \( V_1 \) only; \( V_2 \) only; \( V_3 \) only, \( V_1 \) and \( V_2 \) only; \( V_1 \) and \( V_3 \) only; and the last one is \( V_2 \) and \( V_3 \) only. Of course it should be emphasized here that, this does not involve the presence of the factor \( A_k \); b) The condition for \( A_k \): specifically for this part, \( A_1 \) requires that a transition occurred when \( m = -2, -1, 0; M = 0, \pm 1 \). For \( A_2 \), a transition is possible if \( m = 0, 1, 2; M = 0, \pm 1 \). Meanwhile, \( A_3 \) requires \( m = 0, \pm 1; M = 0, \pm 1 \). But it needs to be emphasized again, the conditions presented for all \( A_k \) are in order that all terms in each equation (34)-(36) are not zero. Of course, for even more special circumstances, the three terms for each \( A_k \) do not have to exist simultaneously. For example, for \( A_1 \), transitions can occur with several possible terms such as, 1) \( m = -1 \) and \( M = 0 \); 2) \( m = -2 \) and \( M = 1 \); 3) \( m = 0 \) and \( M = -1 \); or it could be by choosing two non-zero terms, of course, with the appropriate conditions based on equation (34). This also applies to \( A_2 \) and \( A_3 \). Furthermore, after these two parts, namely \( V_k \) and \( A_k \) have been described in detail, and because of the five terms in brackets there is always a multiplication between one of the elements \( V_\kappa \) and \( A_\kappa \), the terms of “mutually independent” between these two factors in the description above are reduced.

More specifically, we present the five explicit terms referred to, namely \( V_1 A_1 + iV_2 A_2 + V_3 A_2 - iV_2 A_2 + V_3 A_3 \). From this form, we can see that the requirements we described earlier can no longer stand-alone, but the elements become interrelated. For example, if we select only \( V_1 \) which is non-zero from \( V_k \), then only \( V_1 A_1 + V_3 A_2 \) remains. That is, the requirements selected must involve the requirements \( A_1 \) and \( A_2 \) as described above. This also applies to other circumstances, of course, it must comply with the conditions previously described.

Third, we analyze the last two terms containing \( e^{i(2\eta+1)\omega t - \gamma} \) and \( e^{i(2\eta-1)\omega t - \gamma} \), where the form is \( (2\eta \pm 1)\omega \approx \omega_{2\eta} \pm \omega \). This first term corresponds to \( \omega_{2\eta} + \omega \approx 0 \), which is a state when the energy is \( E_{2\eta} = E_0 - \hbar \omega \), or physically it is a state of stimulated emission, which means that when the system is subjected to perturbation, the system undergoes a transition by releasing energy as much as \( \hbar \omega \).
Meanwhile, for the second term, this corresponds to $\omega_{2\eta} - \omega \approx 0$, which is the state when the energy is $E_{2\eta} = E_\eta + \hbar \omega$, or physically it is an absorption state, which means that when the system is subjected to perturbation, the system receives energy equal to $\hbar \omega$ of the perturbation potential. When we study separately between absorption and emission states, of course, it can be reduced to the case of constant perturbation as well. However, one thing that is important here is the presence of the factor $e^{-\gamma}$ which of course reduces the "transition amplitude oscillations" exponentially. From equation (33), it can be seen that if $\tau$ is very large ($\tau \gg \gamma$), then the transition amplitude will be reduced to being like ordinary harmonic perturbation case [9,17]. However, if $\tau \to 0$, then there is no oscillation and only a constant if $\gamma$ doesn’t go infinite. The third possibility, if $i(2\eta \pm 1)\omega \tau - \gamma \to 0$, then the numerator of the last two terms of equation (33) become $i(2\eta \pm 1)\omega \tau - \gamma$. This means that the last two terms of the right-hand side of equation (33) become $2\tau$, so the transition amplitude is proportional to $\tau$. In this state, $\gamma$ does not contribute to the amplitude of the transition. However, this is not physically intuititionally acceptable because it shows that the transition probability of obtaining the state $|nlm;SM\rangle$ after time $\tau$ is quadratic, not linear, over the duration of time when the perturbation $V(t)$ operates on the system. This is clearly intuitively unreasonable.

From the above description, in order for a transition to occur, the three general requirements of the terms discussed based on equation (33) must be met. For further study, especially about the emission state and absorption state of the system like this is beyond the study of this paper. This paper does not discuss until the calculation of the transition probability and even the transition rate (Fermi Golden Rule), but from the formulation $P_n(\tau) = \left| \mathcal{C}_n^{(1)}(\tau) \right|^2$, we can see that the description of the conditions the transitions of this system that have been carried out are very representative. To calculate the respective states, namely emission or absorption, what needs to be done next is to review one by one the terms of the last two terms in equation (33). This means that when calculating the emission state, we only need to consider the term $e^{i(2\eta+1)\omega \tau - \gamma}$, while the term containing $e^{i(2\eta-1)\omega \tau - \gamma}$ is ignored. On the other hand, when calculating absorption, only the term containing $e^{i(2\eta-1)\omega \tau - \gamma}$ is considered.

4. Conclusion
We have performed an analysis to calculate the transition amplitude of a quantum system which initially has an unperturbed Hamiltonian as shown by equation (12), and which is then subjected to a modified sinusoidal perturbation, which involves the spin matrix Pauli factor (its explicit form can be seen in the equation (14)). We obtain the general form of the transition amplitudes as shown in equations (32) and (33). It can be seen that the general transition is only possible for $n$ even positive integers, $l = 2$, and $S = 1$, whereas there is no transition at odd $n$. Specific requirements, as described, include those for potential constants $V_1, V_2, V_3$, and quantum numbers $m$ and $M$. This perturbation can also produce energy absorption and emission phenomena, but in the presence of the factor $e^{-\gamma}$, the transition amplitude oscillations attenuate exponentially, of course, according to the conditions already described. However, it does not always mean that the presence of the factor $\gamma$ always results in attenuation; when $\tau \gg \gamma$, then the transition amplitude will be reduced to being like ordinary harmonic perturbation; when $\tau \to 0$, then there is no oscillation and only a constant if $\gamma$ doesn’t go infinite. Further analysis that can be done based on the results of this study is to examine the transition probability, the transition rate of the quantum state, even to obtain the band state.

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