Mean-field dynamo in a turbulence with shear and kinetic helicity fluctuations

Nathan Klecorin and Igor Rogachevskii

Department of Mechanical Engineering, The Ben-Gurion University of the Negev, POB 653, Beer-Sheva 84105, Israel

(Dated: March 7, 2008)

We study effects of kinetic helicity fluctuations in a turbulence with large-scale shear using two different approaches: the spectral tau-approximation and the second order correlation approximation (or first-order smoothing approximation). These two approaches demonstrate that homogeneous kinetic helicity fluctuations alone with zero mean value in a sheared homogeneous turbulence cannot cause large-scale dynamo. Mean-field dynamo can be possible when kinetic helicity fluctuations are inhomogeneous which cause a nonzero mean alpha effect in a sheared turbulence. On the other hand, shear-current effect can generate large-scale magnetic field even in a homogeneous nonhelical turbulence with large-scale shear. This effect was investigated previously for large hydrodynamic and magnetic Reynolds numbers. In this study we examine the threshold required for the shear-current dynamo versus Reynolds number. We demonstrate that there is no need for a developed inertial range in order to maintain the shear-current dynamo (e.g., the threshold in the Reynolds number is of the order of 1).

PACS numbers: 47.65.Md

I. INTRODUCTION

It has been widely recognized that astrophysical large-scale magnetic fields originate due to the mean-field dynamo (see, e.g., [1, 2, 3, 4, 5, 6, 7]). Such dynamo can be driven by the joint action of the mean kinetic helicity of turbulence and large-scale differential rotation. On the other hand, recently performed numerical experiments [8, 9, 10] have demonstrated existence of a nonhelical large-scale dynamo in a turbulence with a large-scale shear whereby mean alpha effect vanishes. Note that a sheared turbulence is a universal feature in astrophysical [7, 11] and laboratory [12] flows.

One of the possible mechanism of the nonhelical large-scale dynamo in a homogeneous sheared turbulence is a shear-current effect that has been extensively studied during recent years (see [12, 14, 15, 16]). In particular, the deformations of the original nonuniform magnetic field lines are caused by upward and downward turbulent eddies. In a sheared turbulence the inhomogeneity of the original mean magnetic field breaks a symmetry between the influence of the upward and downward turbulent eddies on the mean magnetic field. This creates the mean electric current along the mean magnetic field and results in the nonhelical shear-current dynamo. Indeed, the large-scale velocity shear creates anisotropy of turbulence that produces a contribution to the electromotive force, \( \mathbf{W} \times \mathbf{J} \), caused by the shear, where \( \mathbf{W} \) is the background large-scale vorticity due to the shear and \( \mathbf{J} \) is the large-scale electric current. Joint effects of the electromotive force \( \mathbf{W} \times \mathbf{J} \) and stretching of the mean magnetic field due to the large-scale shear motions cause the mean-field dynamo instability. Note also that the numerical experiment with Taylor-Green forcing [17] seems to be another example of a mean-field dynamo produced by a combined effect of a nonhelical turbulence and a complicated large-scale flow.

Other effect that might explain the nonhelical large-scale dynamo is related to kinetic helicity fluctuations in a sheared turbulence. A problem associated with dynamics of large-scale magnetic field in the presence of kinetic helicity fluctuations in a shear-free turbulence has been formulated for the first time by Kraichnan [18] (see also [1]). In particular, he assumed that in small scales \( l_\nu \ll l_{\text{turb}} \ll l_0 \) (and \( \tau_\nu \ll \tau_{\text{turb}} \ll \tau_0 \) there is a small-scale turbulence generated by forcing \( \mathbf{F}(\alpha) \). On the other hand, in the scales \( l_0 \ll l \ll l_\chi \) (and \( \tau_0 \ll \tau \ll \tau_\chi \) there are kinetic helicity fluctuations (or \( \alpha \) fluctuations) with a zero mean generated by forcing \( \mathbf{F}(\chi) \). The mean-field effects occur at large scales \( L \gg l_\chi \) (and times \( \tau_\chi \gg \tau_\chi \)) where the large-scale helicity is zero. The large-scale quantities are determined by double averaging over velocity fluctuations and over kinetic helicity fluctuations. It was found in [18] (see also [1]) that kinetic helicity fluctuations in a shear-free turbulence cause both, a reduction of turbulent magnetic diffusion and a large-scale drift velocity, \( \mathbf{V}^{(\alpha)} \propto \nabla(\alpha^2) \), of the mean magnetic field.

In recent time the dynamo problem related to kinetic helicity fluctuations has been extensively studied. In particular, various mathematical aspects of this problem has been discussed in [19] (see also [20]). Some numerical experiments which examine effects of kinetic helicity fluctuations have been performed in [10, 21]. In particular, numerical simulations of the magnetic field evolution in accretion discs in [21] have demonstrated that kinetic helicity fluctuations with a zero mean can result in generation of large-scale magnetic field.

It has been pointed out in [22] that inhomogeneous kinetic helicity fluctuations in a sheared turbulence can produce a mean-field dynamo. In particular, a joint action of a large-scale shear and a nonzero mean alpha ef-
fect caused by the inhomogeneous kinetic helicity fluctuations can result in generation of a large-scale magnetic field, where the mean alpha effect in a sheared turbulence is proportional to $\nabla \langle \hat{\alpha}^2 \rangle$. This mean-field dynamo is similar to the $\alpha \Omega$-dynamo. On the other hand, it has been suggested in [23] using phenomenological arguments that homogeneous kinetic helicity fluctuations in a homogeneous turbulence with shear may generate a large-scale magnetic field.

The main goal of this study is to examine a possibility for a nonhelical large-scale dynamo associated with homogeneous kinetic helicity fluctuations with zero mean in a homogeneous turbulence with a large-scale shear. In a study of a sheared turbulence we use two different approaches, namely, the spectral tau-approximation [14] and second order correlation approximation (SOCA), sometimes refers in astrophysical literature as first-order smoothing approximation (FOSA), see, e.g., [24, 25]. We also investigate dynamo effects associated with inhomogeneous kinetic helicity fluctuations in a sheared homogeneous turbulence.

This paper is organized as follows. In Sec. II we formulate the governing equations and outline the procedure of derivation based on the $\tau$-approach, that allows us to determine a contribution to the mean electromotive force caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations. In Sec. III we study the effects of kinetic helicity fluctuations in a sheared turbulence using the $\tau$-approach. In Sec. IV we investigate similar effects using the SOCA-approach. In Sec. V we discuss the threshold required for the shear-current dynamo versus Reynolds number. In Sec. VI we draw concluding remarks. Finally, the detailed derivations of the effects of kinetic helicity fluctuations in a sheared turbulence using the $\tau$-approach and the SOCA-approach have been performed in Appendixes A and B, respectively.

II. GOVERNING EQUATIONS AND THE $\tau$-APPROACH

In order to study mean-field dynamo in a turbulence with kinetic helicity fluctuations and large-scale shear we use a procedure which is similar to that applied in [14] for an investigation of a sheared turbulence. In particular, we use the following equations for fluctuations of velocity, $u$, and magnetic field, $b$, in order to determine the effect of shear on a turbulence:

$$\frac{\partial u}{\partial t} = -(U \cdot \nabla) u - (u \cdot \nabla) U - \nabla \left( \frac{\rho}{\rho} \right) + \frac{1}{\rho} \{ (b \cdot \nabla) B + (B \cdot \nabla) b \} + \nu \Delta u + u^N + F(u) + F(x),$$

$$\frac{\partial b}{\partial t} = (B \cdot \nabla) u - (u \cdot \nabla) B + (b \cdot \nabla) U - (U \cdot \nabla) b + \eta \Delta b + b^N.$$  \hspace{1cm} (1)  \hspace{1cm} (2)

The velocity field is assumed to be incompressible. Here $B = \langle b' \rangle$, $b' = b + B$ is total magnetic field, the angular brackets $\langle \rangle$ denote averaging over ensemble of turbulent velocity field, the velocity $U = \langle u' \rangle = U^{(S)} + V$ includes an imposed large-scale sheared velocity $U^{(S)}$, $u' = u + U$ is total velocity field, $\nu$ is the kinematic viscosity, $\eta$ is the magnetic diffusion due to electrical conductivity of the fluid, $\rho$ is the fluid density, $p$ are the fluctuations of total (hydrodynamic and magnetic) pressure, the magnetic permeability of the fluid is included in the definition of the magnetic field, $v^N$ and $b^N$ are the nonlinear terms, $F(u)$ and $F(x)$ are the stirring forces for velocity and kinetic helicity fluctuations, respectively.

Using Eqs. (1) and (2) written in a Fourier space we derive equations for the instantaneous two-point second-order correlation functions of the velocity fluctuations $\langle u_i u_j \rangle$, the magnetic fluctuations $\langle b_i b_j \rangle$, and the cross-helicity tensor $\langle b_i u_j \rangle$. The equations for these correlation functions are given by Eqs. (A2)-(A4) in Appendix A. We split the tensor of magnetic fluctuations into nonhelical, $h_{ij} = \langle b_i b_j \rangle$, and helical, $h_{ij}^{(H)}$, parts. The helical part $h_{ij}^{(H)}$ depends on the magnetic helicity, and it is determined by the dynamic equation that follows from the magnetic helicity conservation arguments (see, e.g., [26, 27, 28, 29, 30, 31, 32, 33, 34, 35], and a review [7]).

The second-moment equations include the first-order spatial differential operators $\hat{N}$ applied to the third-order moments $M^{(11)}$. A problem arises how to close the system, i.e., how to express the set of the third-order terms $\hat{N} M^{(11)}$ through the lower moments $M^{(II)}$ (see, e.g., [34, 35, 36]). We use the spectral $\tau$-closure-approximation which postulates that the deviations of the third-moment terms, $\hat{N} M^{(11)}(k)$, from the contributions to these terms afforded by the background turbulence, $\hat{N} M^{(11 \cdot 0)}(k)$, are expressed through the similar deviations of the second moments, $M^{(II)}(k) - M^{(II \cdot 0)}(k)$:

$$\hat{N} M^{(11)}(k) - \hat{N} M^{(11 \cdot 0)}(k) = -\frac{1}{\tau_r(k)} \left[ M^{(II)}(k) - M^{(II \cdot 0)}(k) \right],$$

(3)  \hspace{1cm} (3)

(see [12, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]), where $\tau_r(k)$ is the scale-dependent relaxation time, which can be identified with the correlation time of the turbulent velocity field for large hydrodynamic and magnetic Reynolds numbers. The quantities with the superscript $\langle 0 \rangle$ correspond to the background shear-free turbulence. We apply the spectral $\tau$ approximation only for the nonhelical part $h_{ij}$ of the tensor of magnetic fluctuations. Note that a justification of the $\tau$ approximation for different situations has been performed in numerical simulations and analytical studies in [7, 40, 41, 42, 43, 44, 45, 46] (see also detailed discussion in [46], Sec. 6).

We assume that the characteristic time of variation of the magnetic field $B$ is substantially larger than the correlation time $\tau(k)$ for all turbulence scales. This allows us to get a stationary solution for the equations for the second-order moments, $M^{(II)}$. We split all second-order correlation functions, $M^{(II)}$, into sym-
metric \( h^{(\epsilon)}_{ij} = \frac{[h_{ij}(k) + h_{ij}(-k)]}{2} \) and antisymmetric \( h^{(\alpha)}_{ij} = \frac{[h_{ij}(k) - h_{ij}(-k)]}{2} \) parts with respect to the wave vector \( k \). For the integration in \( k \)-space we have to specify a model for the background shear-free turbulence. A non-helical part of the homogeneous background turbulence is given by the following equations

\[
\langle u_i u_j \rangle^{(0)}(k) = \langle u^2 \rangle \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{E(k)}{8\pi k^2},
\]

\[
\langle b_i b_j \rangle^{(0)}(k) = \langle b^2 \rangle \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{E(k)}{8\pi k^2},
\]

where \( \delta_{ij} \) is the Kronecker tensor, the energy spectrum function is \( E(k) = k_0^{-1}(q-1)(k/k_0)^{-\gamma} \), the wave number \( k_0 = 1/l_0 \), the length \( l_0 \) is the maximum scale of turbulent motions. The turbulent correlation time is \( \tau(k) = C \tau_0 (k/k_0)^{-\mu} \), where the coefficient \( C(q, \mu) = (q - 1 + \mu)/(q - 1) \). This value of the coefficient \( C \) corresponds to the standard form of the turbulent diffusion coefficient in the isotropic case, i.e., \( \eta_r = \int \tau(k) \langle u^2 \rangle E(k) \, dk = \tau_0 \langle u^2 \rangle/3 \). Here the time \( \tau_0 = l_0/\sqrt{\langle u^2 \rangle} \) and \( \sqrt{\langle u^2 \rangle} \) is the characteristic turbulent velocity in the scale \( l_0 \). For the Kolmogorov’s type background turbulence (i.e., for a turbulence with a constant energy flux over the spectrum), the exponent \( \mu = q - 1 \) and the coefficient \( C = 2 \). In the case of a turbulence with a scale-independent correlation time, the exponent \( \mu = 0 \) and the coefficient \( C = 1 \).

On the other hand, a helical part of the background turbulent velocity field is given by the following equation:

\[
\langle u_i u_j \rangle^{(0)}(k) = i \chi^v \varepsilon_{jin} k_n \frac{E(k)}{8\pi k^4},
\]

where \( \varepsilon_{ijk} \) is the fully antisymmetric Levi-Civita tensor, \( \chi^v = (u, (\nabla \times u)) \) is the kinetic helicity, the spectrum function is \( E^v(k) = k_0^{-1} C_x (k/k_0)^{-\gamma} \) and \( C_x = q - 1 \) for large hydrodynamic Reynolds numbers. In the scales \( l_0 \ll l \ll l_x \) there are fluctuations of kinetic helicity \( \chi^v \) (see Sec. 3).

Using the solution of the derived second-moment equations, we determine the contributions to the electromotive force, \( \mathcal{E}^{(S,\alpha)}_i = \varepsilon_{imn} \int \langle b_m u_n \rangle^{(S,\alpha)} \, dk \), caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations (see Appendix A).

### III. EFFECTS OF KINETIC HELICITY FLUCTUATIONS IN A SHEARED TURBULENCE: \( \tau \)-APPROACH

The procedure described in Sec. II allows us to determine the contributions to the electromotive force (in particular, to the \( \tilde{\alpha} \) tensor) caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations (for details see Appendix A). This procedure yields the equation for the evolution of the magnetic field \( \mathbf{B} \):

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \tilde{\alpha} \mathbf{B} + \mathbf{U}^{(S)} \times \mathbf{B} - \eta_r \mathbf{J} \right) + \mathbf{B}^N,
\]

where \( \eta_r \) is the turbulent magnetic diffusion coefficient, \( \mathbf{J} = \nabla \times \mathbf{B} \) is the electric current, \( \mathbf{B}^N \) are the nonlinear terms, \( \mathbf{U}^{(S)} \) is the imposed background sheared velocity and we assume for simplicity that \( \mathbf{V} = 0 \). Here the total \( \tilde{\alpha} \) tensor is given by

\[
\tilde{\alpha}_{ij} = \tilde{\alpha}_{ij} + \alpha^{(S,\alpha)}_{ij},
\]

where \( \tilde{\alpha}_{ij} \) determines a contribution to the total \( \tilde{\alpha} \) tensor caused by a shear-free turbulence, while \( \alpha^{(S,\alpha)}_{ij} \) describes a contribution to the \( \tilde{\alpha} \) tensor caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations. The tensor \( \alpha^{(S,\alpha)}_{ij} \) reads

\[
\alpha^{(S,\alpha)}_{ij} = -\tilde{\alpha}_{ij} \left[ C_1 (\tilde{\partial} U^{(S)}_{ij} + C_2 \varepsilon_{ijn} W_n^{(S)}) \right],
\]

where \( (\partial U)^{(S)}_{ij} = (\nabla_i U^{(S)}_j + \nabla_j U^{(S)}_i)/2 \), \( W^{(S)} = \nabla \times U^{(S)} \), the coefficients \( C_1 = (3I/5)(3-2\mu) \), \( C_2 = I/2 \), and the parameter \( I \) is given by

\[
I = \tau_0^{-2} \int \tau^2(k) E(k) \, dk = \frac{(q - 1 + \mu)^2}{(q - 1 + 2\mu)(q - 1)}.
\]

For the Kolmogorov’s type background turbulence (i.e., for a turbulence with a constant energy flux over the spectrum), the exponent \( \mu = q - 1 \) and the coefficients \( C_1 = (4/5)(3-2\mu) \), \( C_2 = 2/3 \). The tensor \( \alpha^{(S,\alpha)}_{ij} \) has been derived in Appendix A.

Using Eq. (7) we derive equation for the correlation function \( \langle \alpha_{ij} B_p \rangle^{(\alpha)} \):

\[
\frac{\partial}{\partial t} \langle \alpha_{ij} B_p \rangle^{(\alpha)} = \varepsilon_{pmn} \left[ \left( \nabla_m \tilde{B}_k \right) \langle \alpha_{ij} \alpha_{nk} \rangle^{(\alpha)} + \tilde{B}_k \left( \langle \alpha_{ij} \nabla_m \alpha_{nk} \rangle^{(\alpha)} + \langle \alpha_{ij} B_n \rangle^{(\alpha)} \nabla_n U^{(S)} \right) + \nabla \langle \alpha_{ij} B_p \rangle^{(\alpha)} \right],
\]

where \( \tilde{B} = \langle B^{(\alpha)} \rangle \), the brackets \( \langle \ldots \rangle^{(\alpha)} \) denote an averaging over random \( \tilde{\alpha} \) fluctuations, \( \mathbf{J} = \nabla \times \mathbf{B} \) and \( \nabla \langle \alpha_{ij} B_p \rangle^{(\alpha)} \) determines the third-order moments caused by the nonlinear terms, which include also the turbulent diffusion term. In Eq. (11) we use the spectral \( \tau \) approximation [3], whereby the relaxation time is of the order of the time \( \tau_x \). We also take into account that the characteristic time of variation of the mean magnetic field \( \mathbf{B} \) is substantially larger than the relaxation time \( \tau_x \). Then the steady state solution of Eq. (11) allows us to determine the correlation function \( \langle \alpha_{ij} B_p \rangle^{(\alpha)} \), that is given by Eq. (A6) in Appendix A.

Now let us consider for simplicity a linear mean velocity shear \( \mathbf{U}^{(S)} = (0, Sx, 0) \) and \( \mathbf{W}^{(S)} = (0, 0, S) \) with \( S \ll 1 \). We also consider the mean magnetic field \( \mathbf{B} \) in a most simple form \( \mathbf{B} = (\tilde{B}_z(z), B_y(z), 0) \). Therefore, the correlation function \( \langle \alpha_{ij} B_j \rangle^{(\alpha)} \) is given by

\[
\langle \alpha_{ij} B_j \rangle^{(\alpha)} = \frac{1}{2} \tau_x \left[ S (\tau_x + 2C_2 \tau_0) \left[ 2J_z - (\mathbf{B} \times \nabla)_z \right] + 2J_y - (\mathbf{B} \times \nabla)_y \right] \langle \tilde{\alpha}^2 \rangle^{(\alpha)},
\]

where we used Eq. (A6) given in Appendix A.
A. Inhomogeneous kinetic helicity fluctuations

Let us first analyze inhomogeneous kinetic helicity fluctuations. The last term in Eq. (12) describes a large-scale drift velocity of the mean magnetic field:

$$V^{(a)} = \frac{\tau_x}{2} \nabla (\hat{\alpha}^2)^{(a)} ,$$

(13)

where $\tau_x = \frac{l_x^2}{\eta_r}$. The third term in Eq. (12) determines a negative contribution to the turbulent magnetic diffusion of the mean magnetic field:

$$\eta_r^{(a)} = -\tau_x (\hat{\alpha}^2)^{(a)} .$$

(14)

Note that the total turbulent magnetic diffusion coefficient, $\eta_r + \eta_r^{(a)}$, should be positive. The reduction of the turbulent magnetic diffusion and the large-scale drift velocity, $V^{(a)} \propto \nabla (\hat{\alpha}^2)$, of the mean magnetic field caused by kinetic helicity fluctuations have been obtained previously in [15] (see also [1]) using the SOCA approach. The second term in Eq. (12) describes a mean $\alpha$ effect:

$$\hat{\alpha}^{(S, a)} = -\frac{\tau_x}{2} (\tau_x + 2C_2 \tau_0) \nabla \hat{\alpha}^{(S, a)} ,$$

(15)

caused by a combined action of a large-scale shear and inhomogeneous kinetic helicity fluctuations. This effect can result in a mean-field dynamo (see [22]) that acts as the $\alpha \Omega$-dynamo. The first term in Eq. (12), that is proportional to $\bar{J}_x$, contributes to the coefficient $\sigma_\alpha$ determined by Eq. (20) below.

B. Homogeneous kinetic helicity fluctuations

Now let us consider homogeneous kinetic helicity fluctuations ($\nabla (\hat{\alpha}^2)^{(a)} = 0$) when the mean $\alpha$ effect vanishes. The total contribution to the mean electromotive force caused by the sheared turbulence and the kinetic helicity fluctuations is $E^{(S, a)} = (\alpha_{ij} B_j)^{(a)} + \hat{b}_{ijk} \nabla_k B_j$. Here the tensor $\hat{b}_{ijk}^{(S)}$ determines the shear-current effect and is given by

$$\hat{b}_{ijk}^{(S)} = \frac{l_0^2}{3} \left[ C_3 \varepsilon_{i+k} (\partial U)^{(S)}_{n j} + C_4 \delta_{ij} W_k^{(S)} \right] ,$$

(16)

(see [14]), where $C_3 = I [1 - 2 \mu + \epsilon (9 + 10 \mu)]/30$, $C_4 = I [3 - 2 \mu - \epsilon (5 + 2 \mu)]/60$, the parameter $\epsilon = E_m/E_e$, $E_m$ and $E_e$ are the magnetic and kinetic energies per unit mass in the background turbulence, and $I$ is determined by Eq. (10). Magnetic fluctuations in the background turbulence are caused by a small-scale dynamo (see, e.g., [17, 18, 49]).

The $y$-component of the mean electromotive force caused by the sheared turbulence and the kinetic helicity fluctuations reads

$$E^{(S, a)}_y = \frac{l_0^2}{3} S (\nabla \hat{\alpha} \hat{\alpha}^{(a)} B_y) \sigma_s - \eta_{2}^{(a)} \bar{J}_y ,$$

(17)

where

$$\sigma_s = \sigma_B - \frac{\tau_x^2}{\tau_0} \frac{(\hat{\alpha}^2)^{(a)}}{u_0^2} \sigma_\alpha ,$$

(18)

$$\sigma_B = \frac{1}{2} C_3 + C_4 = \frac{I}{15} [1 - \mu + \epsilon (1 + 2 \mu)] ,$$

(19)

$$\sigma_\alpha = 1 + 2C_2 \tau_0 \tau_x = 1 + 4 \frac{\tau_0}{3 \tau_x} ,$$

(20)

and $u_0 = l_0/\tau_0$. The parameter $\sigma_s$ describes the shear-current effect, while the parameter $\sigma_\alpha$ determines the combined effect of the kinetic helicity fluctuations and the sheared turbulence. Equation (19) for the parameter $\sigma_B$ has been derived in [14] for the case of large hydrodynamic and magnetic Reynolds numbers. In Sec. V we determine the parameter $\sigma_B$ for the case when hydrodynamic and magnetic Reynolds numbers are not large.

C. Mean-field dynamo

The equation for the evolution of the mean magnetic field, $\bar{B} = (\bar{B}_x(z), \bar{B}_y(z), 0)$, reads

$$\frac{\partial \bar{B}_x}{\partial t} = -\sigma_s S l_0^2 \bar{B}_y'' + \eta_\tau \bar{B}_x'' ,$$

(21)

$$\frac{\partial \bar{B}_y}{\partial t} = S \bar{B}_x + \eta_\tau \bar{B}_y'' ,$$

(22)

where $\eta_\tau = \eta_r + \eta_r^{(a)}$ and $\bar{B}_i'' = \partial^2 \bar{B}_i/\partial z^2$. Here we neglect small contributions to the coefficient of turbulent magnetic diffusion caused by the shear motions because $S \tau_0 \ll 1$. The solution of equations (21) and (22) we seek for is given by $\gamma_s \bar{B}_y + \bar{J}_y$, where $\gamma_s$, of the mean magnetic field is given by

$$\gamma_s = S l_0 \eta_r \tau_0 K_z - \eta_\tau K_z^2 .$$

(23)

The necessary condition for the magnetic dynamo instability is $\gamma_s > 0$. The parameter $\sigma_\alpha > 0$ (see Eqs. (18) and (20)). This implies that homogeneous kinetic helicity fluctuations in a homogeneous turbulence with shear cause a negative contribution to the parameter $\sigma_s$. Therefore, homogeneous kinetic helicity fluctuations in a sheared turbulence act against mean-field dynamo (see Eqs. (13) and (23)), while the shear-current effect may cause the generation of the large-scale magnetic field when the parameter $\sigma_B > 0$ (see also Sec. V).

Note that two effects determined by the parameters $\sigma_B$ and $\sigma_\alpha$, can be interpreted as the off-diagonal terms in the tensor of turbulent magnetic diffusion. The kinetic helicity fluctuations cause a negative contribution to the diagonal components $\propto \eta_{2}^{(a)}$ of turbulent magnetic diffusion of the mean magnetic field (see Eq. (14)). On the other hand, the kinetic helicity fluctuations also results in a negative contribution to the off-diagonal term $\propto \sigma_s \propto -\sigma_\alpha$ in the tensor of turbulent magnetic diffusion.
In order to determine the threshold required for the excitation of the mean-field dynamo instability, we consider the solution of Eqs. (21) and (22) with the following boundary conditions: \( B(t, |z| = L) = 0 \) for a layer of the thickness \( 2L \) in the \( z \) direction. The solution for the mean magnetic field is determined by

\[
\begin{align*}
B_y(t, z) &= B_0 \exp(\gamma_s t) \cos(K_z z + \varphi), \\
B_z(t, z) &= l_0 K_z \sqrt{\varrho_s} B_y(t, z).
\end{align*}
\]

For the symmetric mode the angle \( \varphi = \pi n \) and the large-scale wave number \( K_z = (\pi/2)(2m + 1)L^{-1} \), where \( n, m = 0, 1, 2, \ldots \). For this mode the mean magnetic field is symmetric relative to the middle plane \( z = 0 \). Let us introduce the dynamo number \( D = (l_0 S/\nu)^2 \sigma_s \), where \( \sigma_s = S L^2/\eta \) is the dimensionless shear number. For the symmetric mode the mean magnetic field is generated due to the shear-current effect when the dynamo number \( D > D_c = (\pi^2/4)(2m + 1)^2 \). For the antisymmetric mode the angle \( \varphi = (\pi/2)(2m + 1) \) with \( n, m = 0, 1, 2, \ldots \), the wave number \( K_z = \pi m L^{-1} \) and the magnetic field is generated when the dynamo number \( D > D_c = \pi^2 m^2 L^{-1} \) and the maximum growth rate of the mean magnetic field, \( \gamma_{\text{max}} = S^2 l_0^2 \sigma_s/4 \eta \), is attained at \( K_z = S l_0 \sqrt{\varrho_s} /2 \eta \). Therefore, the characteristic scale of the mean magnetic field variations \( L_B = 2\pi/K_z = 4 u_0/(S \sqrt{\varrho_s}) \).

IV. EFFECTS OF KINETIC HELICITY FLUCTUATIONS IN A SHEARED TURBULENCE: THE SOCA APPROACH

Now we study the effects of kinetic helicity fluctuations in a sheared turbulence using a second order correlation approximation (SOCA). This approximation is valid only for small hydrodynamic Reynolds numbers. Even in a highly conductivity limit (large magnetic Reynolds numbers), SOCA can be valid only for small Strouhal numbers (i.e., for very short correlation time).

The procedure of the derivation of the electromotive force in a homogeneous turbulence with shear and kinetic helicity fluctuations is as follows (for details see Appendix B). We use Eqs. (11) and (12) for fluctuations of velocity and magnetic fields, exclude the pressure term from the equation of motion (11) by calculation \( \nabla \times (\nabla \times \mathbf{u}) \). We rewrite the obtained equation and Eq. (2) in a Fourier space and apply the two-scale approach (i.e., we use large-scale and small-scale variables). We neglect nonlinear terms but keep molecular dissipative terms (i.e., we use the tensor approximation). We seek for a solution for fluctuations of velocity and magnetic fields as an expansion for weak velocity shear.

This procedure allows us to determine the contributions to the electromotive force caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations. In particular, the tensor \( \alpha_{ij}^{(S, \alpha)} \) caused by the kinetic helicity fluctuations in a sheared turbulence is given by

\[
\alpha_{ij}^{(S, \alpha)} = -\tilde{\alpha} \tau_0 \left[ C_1 (\partial U_{ij})^{(S)} + \tilde{C}_2 \varepsilon_{ijn} W_{n}^{(S)} \right],
\]

for details see Appendix B), where the coefficients \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are

\[
\tilde{C}_1 = -\frac{C^*}{10} (11 \text{Re} + 3 \text{Rm}) , \quad \tilde{C}_2 = \frac{C^*}{8} (\text{Rm} - 2 \text{Re}),
\]

and \( \text{Re} = u_0 l_0 / \nu \) is the hydrodynamic Reynolds number, \( \text{Rm} = u_0 l_0 / \eta \) is the magnetic Reynolds number. Here we take into account that

\[
\tilde{\alpha}_{ij} \equiv i \epsilon_{ijn} \int k_j G_\eta f_{rn}^{(0, \chi)} dk \, d\omega = \tilde{\alpha} \delta_{ij},
\]

where

\[
\tilde{\alpha} = -\frac{C_0}{3} \tau_0 \chi^* \text{Rm}, \quad C_0 = \frac{\tilde{C}_1}{q + 1} \left[ 1 - \left( \frac{l_0}{l_*} \right)^{q+1} \right],
\]

the function \( G_\eta(k, \omega) = (\eta k^2 - i \omega)^{-1} \) and \( f_{rn}^{(0, \chi)}(k, \omega) \equiv \langle u_i u_j \rangle^{(0, \chi)}(k, \omega) = -i \chi^* \varepsilon_{ijn} k_n \tilde{E}_\chi(k, \omega)/(\pi^2 k \nu) \). To integrate in \( k \) and \( \omega \) space we used the following model for the spectrum function \( \tilde{E}_\chi(k, \omega) = \tilde{C}_1 k_{\nu}^{-1} (k/\nu)^{-q} \delta(\omega) \), where \( \tilde{C}_1 = (q-1) \left[ 1 - (l_0/\nu)^q \right]^{-1} \) and the wave number \( k \) varies in the interval from \( l_0^{-1} \) to \( l_*^{-1} \). Since SOCA is valid for small hydrodynamic Reynolds numbers the scale \( l_* \) is not related to Kolmogorov (viscous) scale \( l_\nu \).

In the scales \( l_0 \ll l \ll l_* \) there are fluctuations of kinetic helicity \( \chi^* \) (or fluctuations of \( \tilde{\alpha} \)). In order to determine the correlation function \( \langle \alpha_{ij} B_{ij} \rangle^{(a)} \) we use Eq. (17) in which \( \eta_\nu \) is replaced by \( \eta + \eta_\nu \) and we neglect the nonlinear terms. Solving this equation in a Fourier space we determine the magnetic field \( B_\chi(\mathbf{K}, \Omega) \), where the wave vector \( \mathbf{K} \) and the frequency \( \Omega \) are in the spatial scales \( l_0 \ll l \ll l_* \) and the time scales \( \tau_0 \ll \tau \ll \tau_\chi \).

Multiplying the magnetic field \( B_\chi(\mathbf{K}, \Omega) \) by the tensor \( \alpha_{ij} = \tilde{\alpha} \delta_{ij} + \alpha_{ij}^{(S, \alpha)} \) and averaging over kinetic helicity fluctuations we determine the correlation function \( \langle \alpha_{ij} B_{ij} \rangle^{(a)} \). It is given by Eq. (19) in Appendix B.

Now consider a linear mean velocity shear \( U^{(S)} = (0, Sx, 0) \) and assume that the mean magnetic field \( \mathbf{B} \) has the form \( \mathbf{B} = (B_x(z), B_y(z), 0) \). Therefore, Eq. (19) yields the correlation function \( \langle \alpha_{xy} B_{y} \rangle^{(a)} \):

\[
\langle \alpha_{xy} B_{y} \rangle^{(a)} = \frac{1}{2} \left[ G_T \left[ S (G_T + 2 \tilde{C}_2 \tau_0) [2 \tilde{I}_x - (\mathbf{B} \times \nabla)_x] + 2 \tilde{J}_y - (\mathbf{B} \times \nabla) y \right] \langle \tilde{\alpha}^2 \rangle^{(a)} + \tilde{K} \right] d\Omega,
\]

where

\[
G_T(\tilde{K}, \Omega) = \left| (\eta + \eta_\nu) \tilde{K}^2 - i \Omega \right|^{-1}.
\]
A. Homogeneous kinetic helicity fluctuations

Let us consider homogeneous kinetic helicity fluctuations. The $y$-component of the mean electromagnetic force caused by the sheared turbulence and the kinetic helicity fluctuations is given by Eq. (17), the parameter $\sigma_\alpha$ is determined by Eq. (18) whereby the time $\tau_\chi$ is replaced by the time $\bar{\tau}_\chi = l_\chi^2/(\eta + \eta_r)$, and the parameter $\sigma_\alpha$ is given by

$$\sigma_\alpha = \frac{1}{\bar{\tau}_\chi^2 \langle \tilde{\alpha}^2 \rangle} \left( G_T + 2\tilde{C}_2 \tau_0 \right) \langle \tilde{\alpha}^2 \rangle_{K} dK d\Omega$$

$$= \frac{q - 1}{q + 3} + \frac{C_s \tilde{C}_0}{4} \left( \frac{\tau_0}{\bar{\tau}_\chi} \right) \left( \frac{Rm - 2Re}{l_\chi} \right) .$$

Here the coefficient $\tilde{C}_0 = (q - 1)/(q + 1)$, the function $\langle \tilde{\alpha}^2 \rangle_{K} = \langle \tilde{\alpha}^2 \rangle \tilde{E}_\chi(\tilde{K}, \Omega)$ and we use the following model for the spectrum function $\tilde{E}_\chi(\tilde{K}, \Omega) = k_\chi^{-1} (q - 1) (K/k_\chi)^{-\delta} \delta(\Omega)$, where the wave number $k_\chi = 1/l_\chi$ and we take into account that $l_0 \ll l_\chi$.

As follows from Eq. (32), the parameter $\sigma_\alpha > 0$. This implies that homogeneous kinetic helicity fluctuations in a homogeneous turbulence with shear act against mean-field dynamo (see Eqs. (18) and (23)). For small hydrodynamic and magnetic Reynolds numbers (i.e., for the range of validity of SOCA) the parameter $\sigma_\alpha$ is negative and the shear-current effect cannot generate the large-scale magnetic field (see Eqs. 24 and 25). This result is in agreement with [4] (see also Sec. V).

B. Inhomogeneous kinetic helicity fluctuations

Now let us consider inhomogeneous kinetic helicity fluctuations. The last term in Eq. (31) determines a large-scale drift velocity of the mean magnetic field:

$$\mathbf{V}^{(\alpha)} = \frac{1}{2} \nabla \int G_T \langle \tilde{\alpha}^2 \rangle_{K} dK d\Omega = \frac{\tilde{C}_0}{2} \bar{\tau}_\chi \nabla \langle \tilde{\alpha}^2 \rangle_{(\alpha)} ,$$

and the third term in Eq. (31) describes a negative contribution to the turbulent magnetic diffusion of the mean magnetic field [18] (see also [4]):

$$\eta_x^{(\alpha)} = - \int G_T \langle \tilde{\alpha}^2 \rangle_{K} dK d\Omega = -\tilde{C}_0 \bar{\tau}_\chi \langle \tilde{\alpha}^2 \rangle_{(\alpha)} .$$

The second term in Eq. (31) determines the mean $\alpha$ effect caused by a combined action of a large-scale shear and inhomogeneous kinetic helicity fluctuations:

$$\tilde{\alpha}^{(S, \alpha)} = -\frac{S}{2} \nabla_z \int G_T \left( G_T + 2\tilde{C}_2 \tau_0 \right) \langle \tilde{\alpha}^2 \rangle_{K} dK d\Omega$$

$$= -\sigma_\alpha \frac{\bar{\tau}_\chi^2}{2} \nabla_z \langle \tilde{\alpha}^2 \rangle_{(\alpha)} .$$

The first term in Eq. (31) that is proportional to $\tilde{J}_x$, contributes to the coefficient $\sigma_\alpha$ determined by Eq. (32).

V. THRESHOLD FOR SHEAR-CURRENT DYNAMO VERSUS REYNOLDS NUMBER

In Sections III and IV we have shown that homogeneous kinetic helicity fluctuations with zero mean in a sheared turbulence act against mean-field dynamo. On the other hand, shear-current effect can generate large-scale magnetic field even in a homogeneous nonhelical turbulence with large-scale shear. The shear-current dynamo has been studied in [13, 14, 15] for large hydrodynamic and magnetic Reynolds numbers. In this Section we demonstrate that hydrodynamic and magnetic Reynolds numbers can be not large in order to maintain the shear-current dynamo. To this end we examine the threshold required for the generation of a large-scale magnetic field by the shear-current dynamo.

Let us neglect the effect of kinetic helicity fluctuations discussed in previous sections (i.e., consider the case when $\langle \tilde{\alpha}^2 \rangle_{(\alpha)} \ll u_0^2$). A general form of the parameter $\sigma_\alpha$ entering in Eqs. (18) and (24) and defining the shear-current effect is given by

$$\sigma_\alpha = \frac{1}{15 \tau_0} \int \left( 1 + \frac{k (d\tau_r / dk)}{\tau_r(k)} \right) \tau_r^2(k) E(k) dk .$$

Equation (36) has been derived in [15] using the $\tau$ approach. Here $E(k)$ is the energy spectrum function, $\tau_r(k)$ is the relaxation time of the cross-helicity tensor that determines the mean electromagnetic force. For large hydrodynamic and magnetic Reynolds numbers the relaxation time $\tau_r(k)$ of the cross-helicity tensor is of the order of the correlation time of turbulent velocity field $\tau(k)$. For simplicity we consider the case $\epsilon = 0$.

A. Developed turbulence at low magnetic Prandtl numbers

Let us first consider a developed turbulence at low magnetic Prandtl numbers. In this case the relaxation time $\tau_r(k)$ of the cross-helicity tensor which takes into account the magnetic diffusion $\eta$ due to electrical conductivity of the fluid, is determined by equation:

$$\tau_r^{-1}(k) = \eta k^2 + (C \tau_0)^{-1} \left( \frac{k}{\tau_0} \right)^\mu ,$$

where we take into account that $\nu \ll \eta$. For example, the Kolmogorov scaling corresponds to $\mu = 2/3$, 

i.e., \( \tau_r(k) \propto k^{-2/3} \) (large hydrodynamic and magnetic Reynolds numbers), while for small magnetic Reynolds numbers, the time \( \tau_r(k) \propto 1/(\eta k^2) \), i.e., \( \tau_r(k) \propto k^{-2} \). Using Eqs. (38) and (37) we determine the parameter \( \sigma_B \) defining the shear-current effect versus the magnetic Reynolds number:

\[
\sigma_B = \frac{16}{135} \left[ 1 + \frac{15 \pi}{\sqrt{2} \mathrm{Rm}^{3/2}} \left( 1 - \frac{2}{\pi} \arctan \sqrt{2/\mathrm{Rm}} \right) \right. \\
\left. - \frac{6}{\mathrm{Rm}} \left( 2 + \frac{1}{2 + \mathrm{Rm}} \right) - \frac{3}{4} \left( \frac{\mathrm{Rm}}{2 + \mathrm{Rm}} \right)^2 \right]. \tag{38}
\]

The asymptotic formulas for the parameter \( \sigma_B \) are as follows. When \( \mathrm{Rm} \ll 1 \) the parameter \( \sigma_B \) reads

\[
\sigma_B = -\frac{2}{105} \mathrm{Rm}^2.
\]

This implies that there is no shear-current dynamo for \( \mathrm{Rm} \ll 1 \) in a developed turbulence at low magnetic Prandtl numbers. When \( \mathrm{Re} \gg \mathrm{Rm} \gg 1 \) the parameter \( \sigma_B \) is

\[
\sigma_B = \frac{4}{135} \left( 1 - \frac{36}{\mathrm{Rm}} \right).
\]

The coefficient \( \sigma_B \) defining the shear-current effect versus the magnetic Reynolds number is shown in Fig. 1. This figure demonstrates that in a developed turbulence at low magnetic Prandtl numbers the threshold in the magnetic Reynolds number \( \mathrm{Rm}_{cr} \) required for the shear-current dynamo is \( \mathrm{Rm}_{cr} \approx 10 \).

**B. A random flow with a scale-independent correlation time**

Let us consider a random flow with a scale-independent correlation time. In this case the exponent \( \mu = 0 \), the coefficient \( C = 1 \) and the relaxation time \( \tau_r(k) \) of the cross-helicity tensor which takes into account kinematic viscosity \( \nu \) and the magnetic diffusion \( \eta \) due to electrical conductivity of the fluid, is determined by equation:

\[
\tau_r^{-1}(k) = (\nu + \eta) k^2 + \tau_0^{-1}.
\]

In this case the turbulent energy spectrum function is

\[
E(k) = k_0^{-1} (q - 1) \left( 1 - \frac{\mathrm{Re}(1 - q)/2}{(1 + a \mathrm{Re})^2} \right)^{-1} \left( \frac{k}{k_0} \right)^{-q}. \tag{40}
\]

Note that in a random flow with a scale-independent correlation time the viscous scale is \( l_v = l_0/\sqrt{\mathrm{Re}} \), while the viscous scale of the Kolmogorov turbulence is \( l_\sigma = l_0/\mathrm{Re}^{3/4} \). Here the hydrodynamic Reynolds number \( \mathrm{Re} > 1 \).

When the exponent \( q = 0 \), the parameter \( \sigma_B \) defining the shear-current effect reads

\[
\sigma_B = \frac{1}{60 (\sqrt{\mathrm{Re}} - 1)} \left[ \sqrt{\mathrm{Re}} (3 + a \mathrm{Re}) - \frac{3 + a}{(1 + a)^2} \right.
\]

\[
+ \frac{1}{\sqrt{a}} \left[ \arctan \sqrt{a \mathrm{Re}} - \arctan \sqrt{a} \right], \tag{41}
\]

where \( a = \mathrm{Rm}^{-1} + \mathrm{Re}^{-1} \). The asymptotic formulas for the parameter \( \sigma_B \) are as follows. When \( \mathrm{Rm} \ll 1 \) and \( \mathrm{Pr}_m \ll 1 \), the parameter \( \sigma_B \) is

\[
\sigma_B = -\frac{\mathrm{Rm}^2}{30 \mathrm{Re}^{3/2}} (\mathrm{Re} + \sqrt{\mathrm{Re}} + 1),
\]

where \( \mathrm{Pr}_m = \nu/\eta \) is the magnetic Prandtl number. For \( \mathrm{Rm} \gg \mathrm{Re} \gg 1 \) the parameter \( \sigma_B \approx 1/38 \), while for \( \mathrm{Re} \gg \mathrm{Rm} \gg 1 \) the parameter \( \sigma_B \approx 1/34 \).

When \( q = 2 \) the parameter \( \sigma_B \) defining the shear-current effect reads

\[
\sigma_B = \frac{1}{15 (\sqrt{\mathrm{Re}} - 1)} \left\{ \sqrt{\mathrm{Re}} \left[ 1 + \frac{a (7 + 5 a)}{4 (1 + a)^2} \right.ight.
\]

\[
+ \frac{9 \sqrt{a}}{4} \left[ \arctan \sqrt{a \mathrm{Re}} - \arctan \sqrt{a} \right],
\]

\[
- \frac{a \mathrm{Re} (7 + 5 a \mathrm{Re})}{4 (1 + a \mathrm{Re})^2} - 1 \right\}. \tag{42}
\]

When \( \mathrm{Rm} \ll 1 \) and \( \mathrm{Pr}_m \ll 1 \), the parameter \( \sigma_B \) is

\[
\sigma_B = -\frac{3}{20 (\sqrt{\mathrm{Re}} - 1)},
\]

while for \( \mathrm{Rm} \gg \mathrm{Re} \gg 1 \) the parameter \( \sigma_B \approx 1/15 \). For \( \mathrm{Re} \gg \mathrm{Rm} \gg 1 \) the parameter \( \sigma_B \) is

\[
\sigma_B = \frac{1}{15} \left[ 1 - \frac{9 \pi \sqrt{\mathrm{Re}}}{8 \mathrm{Rm}} \right].
\]

The coefficient \( \sigma_B \) defining the shear-current effect versus the magnetic Reynolds number for a random flow with a
with a scale-independent correlation time and
versus the magnetic Reynolds number \( R_m \) for a random flow
independent correlation time and
dynamic Reynolds numbers \( R_e \) for a random flow with a scale-
R_m
FIG. 3: The threshold in the magnetic Reynolds number
for the shear-current dynamo versus the hydrodynamic Reynolds numbers \( R_e \) is
Reynolds number \( R_m \) (dashed-dotted line). The dashed line is \( Pr_m = \nu/\eta = 1 \).

scale-independent correlation time is shown in Fig. 2 (for
\( q = 0 \)). The function \( \sigma_q(R_m) \) for the exponent \( q = 2 \) is
similar to that for \( q = 0 \). The threshold in the magnetic
Reynolds number \( R_m \) required for the shear-current dyna-
mos the evolution of small-scale magnetic helicity, is of
a great importance due to the conservation law for the
total (large and small scales) magnetic helicity in turbu-
ence with very large magnetic Reynolds numbers (see,
e.g., [26, 27, 28, 29, 30, 31, 32, 33], and a review [7]).
In particular, the mean-field dynamo is essentially non-
linear due to the evolution of the small-scale magnetic
helicity ([28]). Even for very small mean magnetic field
the magnetic \( \alpha \) effect that is related to the small-scale
magnetic helicity, is not small.

The nonlinear mean-field dynamo due to a shear-
current effect has been studied in [15], whereby the trans-
port of magnetic helicity as a dynamical nonlinearity has
been taken into account. It has been demonstrated in [15]
that the magnetic helicity flux strongly affects the mag-
netic field dynamics in the nonlinear stage of the she-
current dynamo. Numerical solutions of the nonlinear
mean-field dynamo equations which take into account the
shear-current effect (see [15]), show that if the magnetic
helicity flux is not small, the saturated level of the mean
magnetic field is of the order of the equipartition field de-
termined by the turbulent kinetic energy. The results of
this study are in a good agreement with numerical sim-
ulations performed in [8, 53, 51]. Note that a non-zero
magnetic helicity flux is related to open boundary con-
ditions (see [30, 31], and a review [7]). Finally, we point
out an important issue related to the gauge invariance
formulation for the magnetic helicity that is described in
details in [7, 52].

VI. DISCUSSION

We study effects of kinetic helicity fluctuations in a
homogeneous turbulence with large-scale shear using the
spectral tau-approximation and the second order corre-
lation approximation (SOCA). We show that homogeneous
kinetic helicity fluctuations alone with a zero mean can-
ot cause a large-scale dynamo in a sheared turbulence.
This negative result based on SOCA and tau approach,
is a quantitative one: the sign of a certain coefficient in
the mean electromotive force (\( \propto -\sigma_\alpha \)) turns out to be
unfavorable for the mean-field dynamo. This result is
in a contradiction to that suggested in [23] using phe-
nomenological arguments. In order to compare with the
results obtained in [22], we rewrite Eqs. (21) and (22) in
the following form:

\[
\frac{\partial A}{\partial t} = \sigma_s S I_0^2 B_y + \bar{\eta}_y A'',
\]

\[
\frac{\partial B_y}{\partial t} = -S A' + \bar{\eta}_y B_y',
\]

where the mean magnetic field, \( B = \bar{B}_y(z) e_y + \nabla \times [\bar{A}(z) e_y] \). Equation (43) is similar to Eq. (8) derived
in [22], whereby \( \Omega' \) is replaced by \(-S\), while Eq. (44) is
similar to Eq. (9) derived in [22].

The first term in the right hand side of Eq. (43) de-
determines two different effects, namely the shear-current

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The coefficient \( \sigma_q \) defining the shear-current effect versus the magnetic Reynolds number \( R_m \) for a random flow with a scale-independent correlation time and \( q = 0 \). The different curves corresponds to the following values of the hydrodynamic Reynolds numbers \( R_e \): \( R_e = 10 \) (solid); \( R_e = 6 \) (dotted); \( R_e = 3 \) (dashed); \( R_e = 1.5 \) (dashed-dotted).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{The threshold in the magnetic Reynolds number \( R_m \) for the shear-current dynamo versus the hydrodynamic Reynolds numbers \( R_e \) for a random flow with a scale-independent correlation time and \( q = 0 \). The dashed line is \( Pr_m = \nu/\eta = 1 \).}
\end{figure}
effect $\propto \sigma_n S \tilde{B}'_n$ and the effect $\propto -\sigma_n S \tilde{B}'_n$ caused by the homogeneous kinetic helicity fluctuations in a sheared turbulence. The shear-current effect results in the mean-field dynamo, while the second effect (\(x - \sigma_n\)) acts against the mean-field dynamo (see Eqs. (13)-(20) and (32)). These two effects can be interpreted as the off-diagonal terms in the tensor of turbulent magnetic diffusion, and they cannot be reduced to the standard $\alpha$ effect.

Note that the first term in the right hand side of Eq. (13) that determines the effect of homogeneous kinetic helicity fluctuations in a sheared turbulence, is similar to the first term in the right hand side of Eq. (9) derived in [23] except for it has opposite sign. The latter is crucial for the mean-field dynamo. In this comparison we have not taken into account the shear-current effect determined by the coefficient $\sigma_n$, because this effect has not been considered in [23]. The results obtained in the present study are derived using the rigorous mean field theory based on SOCA (see Sec. IV). These results are also in a good agreement with those obtained by the tau-approach (see Sec. III). On the other hand, the results of derivation performed in [23] by phenomenological arguments using ad hoc mean-field equations are in a disagreement with our results.

The shear-current effect causes the mean-field dynamo in a homogeneous nonhelical turbulence with imposed large-scale shear. This effect has been studied previously (see [13, 14, 15]) only for large hydrodynamic and magnetic Reynolds numbers. In the present study we determine the threshold required for the shear-current dynamo as a function of Reynolds number and demonstrate that threshold value of the Reynolds number is of the order of 1. This implies that there is no need for a developed inertial range in order to maintain the shear-current dynamo.

In the present study we also recover the results obtained in [18] (see also [1]) for a shear-free turbulence, whereby a negative contribution of kinetic helicity fluctuations to the turbulent magnetic diffusion, $\eta^\alpha = -\tau^2 \tilde{\chi} (\tilde{\alpha}^2)^\alpha$, and a large-scale drift velocity of the mean magnetic field, $V^\alpha \propto \tau^\alpha \nabla \tilde{\chi} (\tilde{\alpha}^2)^\alpha$, have been found.

On the other hand, we have demonstrated that inhomogeneous kinetic helicity fluctuations in a sheared turbulence cause a nonzero mean alpha effect, $\tilde{\alpha}^{(S, \alpha)} \propto -\tau^2 \tilde{\chi} \nabla z \tilde{\alpha}^{(2)}$, where the mean vorticity due to the large-scale shear is $W^{(S)} = S e_2$. The mean alpha effect $\tilde{\alpha}^{(S, \alpha)}$ is formed by a combined action of a large-scale shear in turbulent flow and inhomogeneous kinetic helicity fluctuations even when $\tilde{\alpha}$ is zero. The large-scale shear and the mean alpha effect can cause a mean-field dynamo (see [22]) that is similar to the $\alpha\Omega$-dynamo.

The discussed effects in this study might be important in astrophysics (e.g., accretion discs, colliding protogalactic clouds, merging protostellar clouds [16]) and laboratory dynamo experiments. In particular, non-symmetrical explosions of supernova may produce fluctuations of kinetic helicity located in larger scales than small-scale turbulence existing in convective zones inside stars. On the other hand, the shear-current dynamo acts together with the $\alpha$-shear dynamo which is similar to the $\alpha\Omega$ dynamo. The shear-current effect does not quench (see [14, 15]) contrary to the quenching of the nonlinear $\alpha$ effect, the turbulent magnetic diffusion, the effective drift velocity. This implies that the shear-current effect might be the only surviving effect, which can explain the origin of large-scale magnetic fields in sheared astrophysical turbulence.

**Acknowledgments**

We have benefited from stimulating discussions with Alexander Schekochihin and Dmitry Sokoloff.

**APPENDIX A: DERIVATION OF Eqs. (11) AND (12) USING THE $\tau$-APPROACH**

In order to study the effect of kinetic helicity fluctuations in a sheared turbulence we use a procedure applied in [14] for a sheared turbulence. Let us derive equations for the second moments. To exclude the pressure term from the equation of motion (11) we calculate $\nabla \times (\nabla \times \mathbf{u})$. Then we rewrite the obtained equation and Eq. (2) in a Fourier space. We also apply the two-scale approach, e.g., we use large scale $\mathbf{R} = (x + y)/2$, $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$ and small scale $\mathbf{r} = x - y$, $\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2)/2$ variables (see, e.g., [23]). We derive equations for the following correlation functions:

$$
\begin{align*}
    f_{ij}(\mathbf{k}) &= \hat{L}(u_i; u_j) , \quad h_{ij}(\mathbf{k}) = \hat{L}(b_i; b_j) , \\
    y_{ij}(\mathbf{k}) &= \hat{L}(b_i; u_j) , \\
\end{align*}
$$

where

$$
\hat{L}(a; c) = \int \langle a(\mathbf{k} + \mathbf{K}/2)c(-\mathbf{k} + \mathbf{K}/2) \rangle \times \exp(i\mathbf{K}\cdot\mathbf{R}) d\mathbf{K} ,
$$

where $\langle \cdot \rangle$ denotes averaging over ensemble of turbulent velocity field. The equations for these correlation functions are given by

$$
\begin{align*}
    \frac{\partial f_{ij}(\mathbf{k})}{\partial t} &= i(\mathbf{k} \cdot \mathbf{B}) \Phi_{ij} + I_{ij} + I_{ijmn}^S(U)f_{mn} \\
    &\quad + \hat{N} f_{ij} + F_{ij} , \\
    \frac{\partial h_{ij}(\mathbf{k})}{\partial t} &= -i(\mathbf{k} \cdot \mathbf{B}) \Phi_{ij} + I^h_{ij} + E^S_{ijmn}(U)h_{mn} + \hat{N} h_{ij} , \\
    \frac{\partial y_{ij}(\mathbf{k})}{\partial t} &= i(\mathbf{k} \cdot \mathbf{B}) [f_{ij}(\mathbf{k}) - h_{ij}(\mathbf{k}) - h_{ij}^H] + I^g_{ij} \\
    &\quad + J^S_{ijmn}(U)y_{mn} + \hat{N} y_{ij} ,
\end{align*}
$$

(see [14]), where hereafter we omit argument $t$ and $\mathbf{R}$ in the correlation functions and neglect small terms $\sim$
$O(\nabla^2)$. Here $F_{ij}$ is related to the forcing terms and $\nabla = \partial / \partial \mathbf{R}$. In Eqs. (A2)-(A4), $\Phi_i(\mathbf{k}) = g_{ij}(\mathbf{k}) - g_{ij}(-\mathbf{k})$, and $\mathcal{N}f_{ij}$, $\mathcal{N}f_{ij}$, $\mathcal{N}g_{ij}$, are the third-order moment terms appearing due to the nonlinear terms which include also molecular dissipation terms. The tensors $I_{ijmn}^S(\mathbf{U})$, $E_{ijmn}^S(\mathbf{U})$ and $J_{ijmn}^S(\mathbf{U})$ are given by

\begin{align*}
I_{ijmn}^S(\mathbf{U}) &= \left[2k_iq_j\delta i_p \delta j_m + 2k_jq_i\delta m_p \delta i_m - \delta i_m \delta j_p \delta m_p \right. \\
&\quad - \delta m_q \delta j_p \delta m_p + \delta i_m \delta j_p \delta m_p \left. \right] \nabla_p U_q , \\
E_{ijmn}^S(\mathbf{U}) &= \left[\delta i_m \delta j_p \delta m_p + \delta j_m \delta i_p \delta m_p \right. \\
&\quad + \delta i_m \delta j_p \delta m_p \left. \right] \nabla_p U_q , \\
J_{ijmn}^S(\mathbf{U}) &= \left[2k_iq_j\delta i_p \delta j_m - \delta i_m \delta j_p \delta m_p + \delta j_m \delta j_p \delta m_p \right. \\
&\quad + \delta i_m \delta j_p \delta m_p \left. \right] \nabla_p U_q ,
\end{align*}

where $k_{ij} = k_ik_j/k^2$. The source terms $I_{ij}^S$, $I_{ij}^b$ and $I_{ij}^g$ which contain the large-scale spatial derivatives of the magnetic field $\mathbf{B}$, are given in [14]. Next, in Eqs. (A2)-(A4) we split the tensor for magnetic fluctuations into nonhelical, $h_{ij}$, and helical, $h_{ij}(h)$, parts. The helical part of the tensor of magnetic fluctuations $h_{ij}(h)$ depends on the magnetic helicity and it follows from the magnetic helicity conservation arguments (see, e.g., [20, 27, 28, 29, 30, 31]). We also use the spectral $\tau$ approximation which postulates that the deviations of the third-moment terms, $\mathcal{N}M^{(II)}(\mathbf{k})$, from the contributions to these terms afforded by the background turbulence, $\mathcal{N}M^{(II,0)}(\mathbf{k})$, are expressed through the similar deviations of the second moments, $M^{(II)}(\mathbf{k}) - M^{(II,0)}(\mathbf{k})$ [see Eq. (3)].

We take into account that the characteristic time of variation of the magnetic field $\mathbf{B}$ is substantially larger than the correlation time $\tau(k)$ for all turbulence scales. This allows us to get a stationary solution for Eqs. (A2)-(A4) for the second-order moments, $M^{(II)}(\mathbf{k})$, which are the sum of contributions caused by a shear-free turbulence, a sheared turbulence and kinetic helicity fluctuations. The contributions to the mean electromotive force caused by a shear-free turbulence and the sheared turbulence without kinetic helicity fluctuations are given in [14]. On the other hand, the contributions to the electromotive force caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations, are given by $E_{in}^{(S,\alpha)} = \epsilon_{mji} \int d\mathbf{k} g_{ij}^{(S,\alpha)}(\mathbf{k})$, where the corresponding contributions to the cross-helicity tensor $g_{ij}^{(S,\alpha)}$ in the kinematic approximation, are given by

\begin{equation}
g_{ij}^{(S,\alpha)}(\mathbf{k}) = i\tau \left[J_{ijmn}^S \tau (\mathbf{k} \cdot \mathbf{B}) + \tau (\mathbf{k} \cdot \mathbf{B}) I_{ijmn}^S \right] f_{mn}^{(0,\chi)} ,
\end{equation}

where $f_{mn}^{(0,\chi)} = \langle u_i u_j \rangle^{(0,\chi)} = -i \chi^\varepsilon_{ijm} k_n E_\chi(k)/(8\pi k^4)$ is the helical part of the velocity fluctuations. Straightforward calculations using Eq. (A5) yields the contributions $\alpha_{ij}^{(S,\alpha)}$ to the $\alpha$ tensor caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations, that is determined by Eq. (9). The total $\alpha$ tensor is given by $\alpha_{ij} = \alpha_{ij} + \alpha_{ij}^{(S,\alpha)}$, where $\alpha_{ij}$ determines a shear-free turbulence contribution. This procedure allows us to derive the equation for the evolution of the magnetic field $\mathbf{B}$ that is given by Eq. (7).

Using Eq. (7) we derive equation for the correlation function $\langle \alpha_{ij} B_p \rangle$ that is given by Eq. (11), where $\langle \ldots \rangle^{(A)}$ denote an averaging over random $\alpha$ fluctuations. Equation (11) include the third-order moments caused by the nonlinear terms. In Eq. (11) we use the spectral $\tau$ approximation [3], where the large-scale relaxation time is of the order of $\tau_\chi$. We also take into account that the characteristic time of variation of the mean magnetic field $\mathbf{B} = (\mathbf{B})^{(\alpha)}$ is substantially larger than the relaxation time $\tau_\chi$. This yields the correlation function $\langle \alpha_{ij} B_j \rangle^{(\alpha)}$:

\begin{align}
\langle \alpha_{ij} B_j \rangle^{(\alpha)} &= \frac{\tau_\chi}{2} \left\{2\epsilon_{jk}\int d\mathbf{k} \left[ B_n \left( \delta_{ij} \langle \partial k \alpha^{mn}_{(S,\alpha)} \rangle^{(\alpha)} \right) \right. \\
&\quad + \delta_{mn} \langle \alpha^{(S,\alpha)}_{ij} \nabla_k \alpha_{n} \rangle^{(\alpha)} \right) + \left( \nabla_k B_n \right) \left( \delta_{ij} \langle \alpha^{(S,\alpha)}_{mn} \rangle^{(\alpha)} \right) + \delta_{mn} \langle \alpha^{(S,\alpha)}_{ij} \alpha_{n} \rangle^{(\alpha)} \right\} \langle 2J_j \rangle \\
&\quad - \langle \mathbf{B} \times \nabla \alpha \rangle^2 \right\} ,
\end{align}

where $\mathbf{J} = \nabla \times \mathbf{B}$. We consider for simplicity a linear mean velocity shear $\mathbf{U}^{(S)} = (0, Sx, 0)$ and $\mathbf{W}^{(S)} = (0, 0, S)$, where $S \tau_\chi \ll 1$. The mean magnetic field $\mathbf{B}$ in a most simple form is $\mathbf{B} = (B_x(z), B_y(z), 0)$. Equation (A6) allows us to determine the correlation function $\langle \alpha_{ij} B_j \rangle^{(\alpha)}$ that is given by Eq. (12).

\section*{APPENDIX B: DERIVATION OF EQ. (31) USING THE SOCA-APPROACH}

In order to study the effect of kinetic helicity fluctuations in a turbulence with large-scale shear we use a second order correlation approximation (SOCA) applied in [24] for a sheared turbulence. To exclude the pressure term from the equation of motion we calculate $\nabla \times (\nabla \times \mathbf{u})$, then we rewrite the obtained equation and Eq. (2) in a Fourier space, apply the two-scale approach (i.e., we use large-scale and small-scale variables), and we neglect nonlinear terms in Eqs. (1) and (2). On the other hand, we keep molecular dissipative terms in these equations. We seek for a solution for fluctuations of velocity and magnetic fields as an expansion for weak velocity shear:

\begin{align}
\mathbf{u} &= \mathbf{u}^{(0)} + \mathbf{u}^{(1)} + \ldots , \\
\mathbf{b} &= \mathbf{b}^{(0)} + \mathbf{b}^{(1)} + \ldots ,
\end{align}

where
where

\[
b_i^{(0)}(k, \omega) = G_\eta(k, \omega) \left[ i (k \cdot \mathbf{B}) \delta_{ij} - \left( \delta_{ij} k_m \frac{\partial}{\partial k_n} + \delta_{im} \delta_{jn} \right) \left( \nabla_n B_m \right) \right] u_j^{(0)}(k, \omega),
\]

(\text{B3})

\[
u_i^{(1)}(k, \omega) = G_{\nu}(k, \omega) \left[ 2 k_i \delta_{jp} + \delta_{ij} k_q \frac{\partial}{\partial k_p} - \delta_{iq} \delta_{jp} \right] \left( \nabla_p U_q \right) u_j^{(0)}(k, \omega),
\]

(\text{B4})

\[
b_i^{(1)}(k, \omega) = G_\eta(k, \omega) \left\{ i (k \cdot \mathbf{B}) \delta_{ij} - \left( \delta_{ij} k_m \frac{\partial}{\partial k_n} + \delta_{im} \delta_{jn} \right) \left( \nabla_n B_m \right) u_j^{(1)}(k, \omega) + \left[ \delta_{ij} k_q \frac{\partial}{\partial k_p} + \delta_{iq} \delta_{jp} \right] \left( \nabla_p U_q \right) b_j^{(0)}(k, \omega) \right\},
\]

(\text{B5})

(for details see [24]). Here \( G_{\nu}(k, \omega) = (\nu k^2 - i \omega)^{-1} \) and \( G_\eta(k, \omega) = (\eta k^2 - i \omega)^{-1} \).

Equations (\text{B3})–(\text{B5}) allow us to determine the cross-helicity tensor \( g_i^{(1)} = (b_i^{(0)} u_j^{(0)}) + (b_i^{(1)} u_j^{(0)}) + (u_i^{(1)} b_j^{(0)}) + (u_i^{(0)} b_j^{(0)}/2) \). This procedure yields the contributions, \( \varepsilon_{mij} = \varepsilon_{mij} \int g_i^{(1,S,\alpha)}(k, \omega) \, dk \, d\omega \), to the electromotive force caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations. In particular, the \( \alpha \) tensor caused by the kinetic helicity fluctuations in a sheared turbulence reads:

\[
\alpha_{ij}^{(S,\alpha)} = D_1 \left( \partial U \right)^{(S)}_{ij} + D_2 \varepsilon_{ijm} W^{(S)}_{mn},
\]

(\text{B6})

where

\[
D_1 = \frac{1}{60} \int \left[ G_\eta (11 G_\nu + 11 G^{*}_\nu + 5 G_\eta) + G^{*}_\eta (G_\nu + 11 G^{*}_\nu + 5 G_\eta) + 4k \left( G_\eta G^{*}_\nu + G^{*}_\eta (G_\nu + G^{*}_\nu) + G^{*}_\nu G^{*}_\nu \right) \right] \tilde{E}_\chi(k, \omega) \, dk \, d\omega,
\]

(\text{B7})

\[
D_2 = \frac{1}{24} \int \left[ G_\eta (G_\eta - G_\nu - G^{*}_\nu) + G^{*}_\eta (G_\nu - G^*_\nu) \right] \tilde{E}_\chi(k, \omega) \, dk \, d\omega.
\]

(\text{B8})

Here \( E(k, \omega) = \tilde{E}(k, \omega)/8\pi k^2 \) and we integrate over the angles in \( k \) space. The total \( \alpha \) tensor is given by \( \alpha_{ij} = \tilde{\alpha} \delta_{ij} + \alpha_{ij}^{(S,\alpha)} \), where \( \tilde{\alpha} \delta_{ij} \) determines a shear-free turbulence contribution. Let us consider the following model for the spectrum function \( \tilde{E}_\chi(k, \omega) = \tilde{C}_\chi k^{-1}(k/k_0)^{-a} \delta(\omega) \). Then integration over \( k \) and \( \omega \) in Eqs. (\text{B7}) and (\text{B8}) yields the contributions to the \( \alpha \) tensor caused by a combined action of the sheared turbulence and the kinetic helicity fluctuations. In particular, the tensor \( \alpha_{ij}^{(S,\alpha)} \) is given by Eq. (\text{B6}) in Fourier space we determine the magnetic field \( B_j(\mathbf{K}, \Omega) \), where the wave number \( K \) and the frequency \( \Omega \) are in the spatial scales \( l_0 \ll l \ll l_\lambda \) and the time scales \( \tau_0 \ll \tau \ll \tau_\lambda \). Multiplying the magnetic field \( B_j(\mathbf{K}, \Omega) \) by the tensor \( \alpha_{ij} = \tilde{\alpha} \delta_{ij} + \alpha_{ij}^{(S,\alpha)} \) and averaging over kinetic helicity fluctuations we determine the correlation function \( \langle \alpha_{ij} B_j \rangle^{(\alpha)} \):

\[
\langle \alpha_{ij} B_j \rangle^{(\alpha)} = \frac{1}{2} \int G_T \left\{ 2 \varepsilon_{jkm} \left[ \tilde{B}_n \left( \delta_{ij} \tilde{\alpha} \nabla_k (S^{(S,\alpha)}) \right) \right. \right.

\[
+ \delta_{mn} \left( \delta_{ij} \tilde{\alpha} (S^{(S,\alpha)} \nabla_k \tilde{\alpha})^{(\alpha)} \right)

\[
\left. + \delta_{mn} \left( \delta_{ij} \tilde{\alpha} (S^{(S,\alpha)} \nabla_k \tilde{\alpha})^{(\alpha)} \right) \right] + \left[ \delta_{ij} + \left( \nabla_p U_q^{(S)} \right) \left( \delta_{jq} \delta_{ip} \right) \right. \right.

\[
\left. + \delta_{ij} \tilde{K}_q \frac{\partial}{\partial K_p} \right] G_T \left[ 2 J_j - (\mathbf{B} \times \nabla)_j \right] (\tilde{\alpha}^{(\alpha)}_2 R) \left. \right] dK \, d\Omega,
\]

(\text{B9})

where \( G_T(\mathbf{K}, \Omega) = (\eta \tilde{K}^2 - i \Omega)^{-1} \). We consider a linear mean velocity shear \( \mathbf{U}^{(S)} = (0, 0, 0) \) and assume that the mean magnetic field \( \mathbf{B} \) has a form: \( \mathbf{B} = (\tilde{B}_x(z), \tilde{B}_y(z), 0) \). This allows us to simplify Eq. (\text{B9}) and to determine the correlation function \( \langle \alpha_{ij} B_j \rangle^{(\alpha)} \) (see Eq. (\text{B11})).

[1] H. K. Moffatt, Magnetic Field Generation in Electrically Conducting Fluids (Cambridge University Press, New York, 1978).

[2] E. Parker, Cosmical Magnetic Fields (Oxford University Press, New York, 1979).

[3] F. Krause, and K. H. Rädler, Mean-Field Magnetohydrodynamics and Dynamo Theory (Pergamon, Oxford, 1980).

[4] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, Magnetic Fields in Astrophysics (Gordon and Breach, New York, 1983).

[5] A. Ruzmaikin, A. M. Shukurov, and D. D. Sokoloff, Magnetic Fields of Galaxies (Kluwer Academic, Dordrecht, 1988).

[6] M. Ossendrijver, Astron. Astrophys. Rev. 11, 287 (2003).

[7] A. Brandenburg and K. Subramanian, Phys. Rept. 417, 1 (2005).

[8] A. Brandenburg, Astrophys. J. 625, 539 (2005).

[9] T. A. Yousef, T. Heinemann, A. A. Schekochihin, N. Kleeorin, I. Rogachevskii, A. B. Iskakov, S. C. Cowley and J. C. McWilliams, Phys. Rev. Lett., submitted, ArXiv: 0710.3359 (2007).
