Quantum corrections in mirror symmetry for a 2-dimesional Lagrangian submanifold with an elliptic umbilic

G. Marelli

Department of Mathematics, Kyoto University,
Kitashirakawa, Saky-o-ku, Kyoto 606-8502, Japan

Abstract. Given the Lagrangian fibration $T^4 \to T^2$ and a Lagrangian submanifold, exhibiting an elliptic umbilic and supporting a flat line bundle, we study, in the context of mirror symmetry, the “quantum” corrections necessary to solve the monodromy of the holomorphic structure of the mirror bundle on the dual fibration.

2000 Mathematics Subject Classification: 14J32, 37G25, 51P05, 53D12, 70K60, 81T30
E-Mail address: marelli@kusm.kyoto-u.ac.jp
1 Introduction

The first steps in the study of mirror symmetry, assuming the existence of dual torus fibrations $X$ and $\hat{X}$, has been undertaken in papers such as [6], [1], [14], [2] and [3]: under certain hypotheses, it is provided a transform, defined on some subcategory of the Fukaya category of $X$, which maps pairs formed by a Lagrangian submanifold $L$ and a $U(1)$-flat connection $\nabla$, to holomorphic bundles $\hat{E}$ over $\hat{X}$. The caustic $K$ of $L$ is always assumed to be empty. The purpose of this paper is to start understanding how to remove this hypothesis. We focus our attention on the Lagrangian torus fibration $T^4 \to T^2$ and consider a Lagrangian map $f : L \hookrightarrow T^4 \to T^2$. Since all our reasonings take place in neighbourhoods of critical points, we can confine ourselves to the fibration $\mathbb{R}^4 \to \mathbb{R}^2$. Generically, $f$ exhibits only folds and cusps, which are singularities of codimension 1 and 2 respectively. If we restrict the fibrations and $L$ to the subset $\mathbb{R}^2 \setminus K$, then the Lagrangian map $f$ has no singular points, and so we can try to apply the constructions contained in the papers mentioned before, and hope to get a holomorphic bundle $\hat{E}$ on the dual fibration restricted to $\mathbb{R}^2 \setminus K$, and whose holomorphic structure can be extended to the whole fibration over $\mathbb{R}^2$. However this hope is in general vain (we consider the elliptic umbilic in chapter 3, but see also the same example described in paragraph 5.4 of [6]): what may happen, as in the case we are going to study, is that $K$ is a compact curve and in the non-compact subset of $\mathbb{R}^2$ determined by $K$ the holomorphic structure of $\hat{E}$ presents a monodromy when going around the caustic $\hat{K}$, which hinders us from extending the mirror bundle to $K$ and glueing it to the mirror bundle constructed inside $K$, and so producing a holomorphic bundle $\hat{E}$ on the whole dual fibration. Some kind of quantum corrections is thus required in order to obtain a holomorphic bundle defined on the whole dual fibration $\hat{\mathbb{R}}^4 \to \mathbb{R}^2$. The idea, outlined in [5], is that quantum corrections are provided by the instanton effect, that is, by counting pseudoholomorphic strips in $\mathbb{R}^4$ which bound $L$ and the fibre $F_x$ of the fibration. As proposed in [5] as a general idea, holding beyond the specific case considered there, the fibre over $x \in \mathbb{R}^2 \setminus K$ of the mirror bundle $\hat{E}$ on $\hat{\mathbb{R}}^4$ is constructed as the Lagrangian intersection Floer homology of $L$ and of the Lagrangian fibre of $\mathbb{R}^4$ over $x$. It is interesting as well as useful for drawing information about how performing in general quantum corrections, to assume that $K$ contains just one singular point and that such point is an elliptic umbilic. We know that
in dimension 2 this singularity is neither stable nor generic, however, from 
[15] and [16], we know how the caustic $K$ and the bifurcation locus $B$ change when $f$ is slightly perturbed. The idea is that $f$ is Hamiltonian equivalent to some small perturbation $\tilde{f}$ of it, so that $f$ and $\tilde{f}$ define the same object in the Fukaya category. According to a conjecture proposed by K. Fukaya in [6] (see paragraph 3.5), near $K$, Lagrangian intersection Floer homology is equivalent to Morse homology defined by means of the generating function $f$ of $L$, which is a Morse function far from $K$ and $B$: this conjecture allows us to switch from Floer homology to Morse homology. This conjecture has been proved in [7] for the case of the cotangent bundle and its purpose is just to provide a way to simplify the computations involved in working with pseudoholomorphic discs. Quantum corrections are then defined, that is, rules to glue the holomorphic Morse homology bundle $\hat{\tilde{E}}$, relative in our case to $\tilde{f}$, across folds which are not limit points of bifurcation lines, and across bifurcation lines far from their intersections. We check that in this way the holomorphic structure of $\hat{\tilde{E}}$ can be extended to the codimension 2 subset of $\mathbb{R}^2$, containing the remaining points of $\tilde{K}$ and $\tilde{B}$: the intersection points of bifurcation lines, folds which are limit points of bifurcation lines and cusps. We realize however that these corrections are not yet enough to extend the holomorphic structure of $\hat{\tilde{E}}$ to cusps. A correction of different kind is thus required: it is related to the possibility of defining a spin structure on $\tilde{L}$, or, better, a relative spin structure. This has to do with the orientation problem in Floer homology theory (see [8]), and, probably, to the possibility of orienting a family of Morse homologies. In this way, also the monodromy around the caustic is cancelled, and so the mirror bundle $\hat{\tilde{E}}$ can be endowed with a holomorphic structure defined on the whole dual fibration.

Acknowledgements. I wish to thank K. Fukaya, whose suggestions and help were decisive for the achievement of the results here expounded.

2 The mirror bundle

We briefly remind the idea of how the mirror bundle should be constructed. In [14], [2] and [3] it is defined by means of a kind of Fourier-Mukai transform associating to a pair formed by a Lagrangian submanifold $L$, in the given Lagrangian fibration, and a local system $\nabla$ on it, a vector bundle $\tilde{E}$ on the dual fibration, endowed with a connection $\hat{\nabla}$: it is verified that its curvature
$\hat{F}$ satisfies $\hat{F}^{0,2} = 0$ and so it induces a holomorphic structure on $\hat{E}$. This is achieved under certain hypothesis, among which that $L$ has no caustic. On the other hand, in [5], of the mirror bundle $\hat{E}$ it is defined its fibre over the point $(x, w)$ of the dual fibration ($x$ is a coordinate on the base and $w$ on the fibre) as the Lagrangian intersection Floer homology of $L$ and $F_x$:

$$\hat{E}(x, w) = HF((L, \nabla), (F_x, w))$$

where $w$, belonging to $\hat{E}_x$, defines a flat connection on $F_x$ (in the specific case of affine Lagrangian submanifolds considered in [5], $HF^k$ does not vanish only when $k$ equals the dimension of the fibre). A holomorphic frame is then defined on $\hat{E}$. These two construction are equivalent in the cases considered in the mentioned papers, so when assuming at least that the fibration has no singular fibres and that $L$ has no caustic.

In this work we are going to follow mainly the second construction (though sometimes also the Fourier-Mukai construction will be used), since this approach seems to be more suitable if quantum correction are provided by pseudoholomorphic discs. However, as explained in the introduction, using a conjecture by K. Fukaya, presented in paragraph 3.5 of [6], near the caustic we switch from Floer homology and pseudoholomorphic discs to Morse homology and gradient lines. In order to not introduce notation which we are not going to use, we report the conjecture in an informal way and we refer to [6] for the precise formulation.

**Conjecture 2.1.** The moduli space of gradient lines (see few lines below for their definition) is isotopic to the moduli space of pseudo-holomorphic discs in a neighbourhood of a point of the caustic.

Partial results towards a proof of this conjecture are due to A. Floer in [4] and to K. Fukaya and Y.G. Oh in [7] (proof in the case of the cotangent bundle).

The transfer to Morse homology is then performed as follows. Consider the trivial Lagrangian fibration $\mathbb{R}^{2n} \to \mathbb{R}^n$. To $L$ it is associated a (local) generating function $f : \mathbb{R}^n \to \mathbb{R}$ (though in this paper we will consider a specific case, however this is expected to be the general idea rephrasing the construction in [5] using families of Floer homologies). We define the family of function $f_x : \mathbb{R}^n \to \mathbb{R}$, where $x$ is a point in the base of the fibration, as

$$f_x(y) = f(y) - x \cdot y$$
and consider the gradient system
\[ \nabla f_x(y) = \frac{dy}{dt} \tag{1} \]
whose solutions are called gradient lines. Let \( K \) be the caustic of \( L \), that is, the subset of critical values of the projection of \( L \) onto the base of the fibration or, equivalently, the subset of points \( x \) where the gradient field \( \nabla f_x \) exhibits a degenerate critical points, and let \( B \) be the bifurcation locus of \( L \), the subset of points \( x \) where \( f_x \) is a Morse function but \( \nabla f_x \) is not Morse-Smale, that is, where the phase portrait of \( \nabla f_x \) features a saddle-to-saddle separatrix (\( K \) and \( B \) are described in more details in [15]). If \( x \notin K \cup B \), with some further hypothesis on \( f \) (see [18]), the Morse complex is defined over \( x \): the space of \( k \)-chains is the free \( \mathbb{C} \)-module generated by critical points of Morse index \( k \) and the differential is defined counting gradient lines, that is, the solutions of the gradient systems (1), joining two critical points whose Morse indexes differs by 1. The fibre of the mirror bundle is defined as the Morse homology of the Morse complex over \( x \):
\[ \hat{E}_{(x,w)} = HM(f_x) \]
and a holomorphic frame is constructed in a similar way as proposed in [5] and [6], namely, writing \( \nabla = d + A \), a section \( e(x) \) of \( \hat{E} \) turns out to be holomorphic and descends on the torus fibres when multiplied by the weight
\[ \exp \left[ 2\pi \left( \frac{h(x)}{2} - \frac{A(x)}{4\pi} + i \frac{\partial h}{\partial x} \cdot w \right) \right] \]
where \( h \) is a multi-valued function on the base such that each sheet of \( L \) is locally the graph of \( dh \): in other words, \( h \) is a set of local generating functions, defined in the coordinates of the base, one for each sheet of \( L \). The problem is to glue this bundle along the caustic \( K \) and the bifurcation locus \( B \).

3 The monodromy of the elliptic umbilic

Consider the trivial Lagrangian torus fibration \( T^4 \to T^2 \) and a Lagrangian submanifold \( L \) whose caustic \( K \) contains an elliptic umbilic \( q \). We know that, in a neighbourhood of \( q \), we can choose symplectic coordinates \((y_1, y_2, x_1, x_2)\),
where \( y_1 \) and \( y_2 \) are coordinates on the fibres, \( x_1 \) and \( x_2 \) on the base of the fibration, such that \( L \) is given by the generating function \( f : \mathbb{R}^2 \to \mathbb{R} \)

\[
f(y_1, y_2) = \frac{1}{3} y_1^3 - 2y_1 y_2^2
\]

(2)

Since all the considerations we are going to do are in a neighbourhood of \( q \), we will work with the local coordinates just introduced: this means to consider the Lagrangian fibration \( \mathbb{R}^4 \to \mathbb{R}^2 \) and the Lagrangian submanifold \( L \) defined by the generating function \( f \). Associated to \( f \), we have the caustic \( K \) and the bifurcation locus \( B \): by hypothesis \( K = \{ (0, 0) \} \), while \( B \), in \([16]\), is shown to be given by three half-lines from \((0, 0)\), defined by \( t \to te^{i\alpha} \), for \( \alpha = 0, 2\pi/3, 4\pi/3 \), and \( t > 0 \).

Consider a line bundle \( E \) over \( L \) with a flat \( U(1) \)-connection \( \nabla \). The pair \((L, \nabla)\) defines an object in the Fukaya category of the symplectic manifold \( \mathbb{R}^4 \). On \( \mathbb{R}^2 \setminus K \) the generating function \( f \) has no critical points, so we are in a position of applying the results of \([3]\) or of \([5]\), thus producing a bundle \( \hat{E} \) of rank 2 over the total space of the dual fibration, restricted to \( \mathbb{R}^2 \setminus K \). On \( \hat{E} \) a hermitian connection \( \hat{\nabla} \) can be defined thus inducing a holomorphic structure on \( \hat{E} \): note that \( L \) is a 2-sheets cover of \( \mathbb{R}^2 \setminus K \), so if, for \( x \in \mathbb{R}^2 \setminus K \), \( p_1(x) \) and \( p_2(x) \) denote the elements of \( L \cap F_x \), where \( F_x \) is the fibre of the Lagrangian fibration \( \mathbb{R}^4 \to \mathbb{R}^2 \) over \( x \), and if \( z_1 \) and \( z_2 \) are coordinates along the fibres of the dual fibration, then the connection \( \hat{\nabla} \) can be written as \( d + \hat{A} \), with

\[
\hat{A}(x) = i(p_1(x)dz_1 + p_2(x)dz_2)
\]

(3)

However, let \( \Gamma \in \pi_1(\mathbb{R}^2 \setminus K) \), \( \Gamma : [0, 1] \to \mathbb{R}^2 \), and consider the continuous maps

\[
M^i_\Gamma : [0, 1] \to \mathbb{R}^4
\]

\[
M^i_\Gamma(t) = p_i(\Gamma(t))
\]

(4)

with \( i = 1, 2 \). Let \( M^i_\Gamma(t)_F \) be the projection onto \( F_{\Gamma(t)} \cong \mathbb{R}^2 \) of \( M^i_\Gamma(t) \).

**Definition 3.1.** The monodromy of the holomorphic structure of \( \hat{E} \) is the map

\[
\mathcal{M} : \pi_1(\mathbb{R}^2 \setminus K) \to \text{End}(\mathbb{R}^2)
\]

\[
\mathcal{M}(\Gamma)(M^i_\Gamma(0)_F) = M^i_\Gamma(1)_F
\]

(5)

Note that, since \( \Gamma(0) = \Gamma(1) \), \( M^i_\Gamma(0) \) and \( M^i_\Gamma(1) \) belong to the same fibre. Moreover, the endomorphism \( \mathcal{M}(\Gamma) \) is well defined, as \( \{M^i_\Gamma(t)\} \), for \( i = 1, 2 \), is a basis of \( F_{\Gamma(t)} \).
Lemma 3.2. If $\Gamma$ is a non-trivial simple loop in $\pi_1(\mathbb{R}^2 \setminus K)$, then the monodromy $M$ of the holomorphic structure $\hat{E}$ on $\Gamma$ can be represented by the matrix

$$M(\Gamma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Proof. It is a consequence of the fact that the points $p_1(x)$ and $p_2(x)$ exchange when going around the origin: in fact, since $L$ has equation

$$\begin{cases} x_1 = y_1^2 - y_2^2 \\ x_2 = -2y_1y_2 \end{cases}$$

writing $z = x_1 + ix_2$ and $w = y_1 + iy_2$, the equation of $L$ becomes $z = \bar{w}^2$. □

This lemma shows that $\hat{E}$ can not be extended to a holomorphic bundle on the whole dual fibration over $\mathbb{R}^2$. To reach this purpose, some "quantum correction" must be added (see also paragraph 5.4 in [5]).

4 Perturbations of the elliptic umbilic

Consider now a small perturbation $\tilde{f}$ of $f$. The caustic $\tilde{K}$ and the bifurcation locus $\tilde{B}$ of $\tilde{f}$ were studied in [15] and [16] respectively: more precisely, $\tilde{K}$ was shown to be diffeomorphic to a tricuspoid; as to $\tilde{B}$, outside a disc containing $\tilde{K}$, it looks as the bifurcation locus of the unperturbed $f$, while, inside this disc, its structure can be highly complicated and bifurcation lines can intersect (we refer to [16] for the pictures of the several admissible diagrams representing the reciprocal postions of $\tilde{K}$ and $\tilde{B}$ inside the disc). At first we restrict our attention to the subset $\mathbb{R}^2 \setminus \tilde{K}$. Given a flat connection $\tilde{\nabla}$ on the Lagrangian submanifold $\tilde{L}$ defined by $\tilde{f}$, we construct a holomorphic bundle $\tilde{\hat{E}}$ on each of the two connected components of $\mathbb{R}^2 \setminus \tilde{K}$, as explained in [3] or in [5]. As done in section 3 for $\hat{E}$, we can define the monodromy $\tilde{M}$ of the holomorphic structure of $\tilde{\hat{E}}$ and prove the following lemma:

Lemma 4.1. If $\Gamma$ is a non-trivial simple loop in $\pi_1(\mathbb{R}^2 \setminus K)$, then the monodromy $\tilde{M}$ of the holomorphic structure of $\tilde{\hat{E}}$ on $\Gamma$ can be represented by the matrix

$$\tilde{M}(\Gamma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
Proof. Since $f$ is perturbed on a compact subset $D$ containing the origin, it follows that $\tilde{f}$ coincides with $f$ outside $D$ and that $\tilde{K} \subset D$. So $\tilde{\mathcal{M}}(\Gamma) = \mathcal{M}(\Gamma)$.

Therefore, outside the caustic, also the holomorphic structure of $\tilde{E}$ exhibits a monodromy.

5 Quantum corrections to perturbations of the elliptic umbilic

The problem is to solve the monodromy and extend the holomorphic structure of $\tilde{E}$ across the caustic $\tilde{K}$, glueing it with the holomorphic structure inside $\tilde{K}$. The way to achieve this is to construct $\tilde{E}$ with its holomorphic structure on $\mathbb{R}^2 \setminus (\tilde{K} \cup \tilde{B})$, define morphism glueing this structure across $\tilde{K}$ and $\tilde{B}$ and check if the monodromy is solved. This is what we mean by quantum corrections. We are going to define quantum corrections on sections of $\tilde{E}$, then, since a holomorphic section is obtained, as explained in section 2 multiplying a section of $\tilde{E}$ by a suitable weight, we will obtain quantum corrections for holomorphic sections of $\tilde{E}$; so, if a section can be extended to $\tilde{K} \cup \tilde{B}$, the same will hold for a holomorphic section. The features of the set $\mathbb{R}^2 \setminus (\tilde{K} \cup \tilde{B})$, namely, the possible mutual position of $\tilde{K}$ and $\tilde{B}$, are described in theorem 4.14 of [16].

We explain now how the construction of the mirror bundle, described in chapter 2 far from $\tilde{K} \cup \tilde{B}$ is carried out in this case. Observe first that the function $\tilde{f}_x$, defined by $\tilde{f}_x(y) = \tilde{f}(y) - x \cdot y$, is a Morse function for every $x \in \mathbb{R}^2 \setminus (\tilde{K} \cup \tilde{B})$. As computed in [16], if $x$ lies inside the caustic, $\tilde{f}_x$ has four critical points, more precisely, three saddles $s_i(x)$ (the points with Morse index 1) and an unstable node $n(x)$ (the point with Morse index 2), thus the Morse complex is

$$0 \leftarrow 0 \leftarrow \bigoplus_{i=1}^{3} \mathbb{C}[s_i(x)] \leftarrow \partial_x \mathbb{C}[n(x)] \leftarrow 0 \leftarrow \ldots$$  \hspace{1cm} (6)

where $\mathbb{C}[s_i(x)]$ and $\mathbb{C}[n(x)]$ denote the free modules over $\mathbb{C}$ generated by $s_i(x)$ and $n(x)$ respectively. The differential $\partial$ can be defined after an orientation is chosen on the moduli space of gradient lines from $n$ to $s_i$ (see [18] or [17] for a more detailed construction of Morse homology): in our case, $\partial_x$ can be
defined, for example, as $\partial x_n(x) = s_1(x) + s_2(x) + s_3(x)$ (anyway, having the Morse complex only two non trivial terms, $\partial$ automatically satisfy $\partial^2 = 0$); we fix this choice of orientation of gradient lines.

If $x$ lies outside the caustic, $f_x$ has two saddles as critical points, so the Morse complex is simply given by

$$0 \leftarrow 0 \leftarrow \mathbb{C}[s_i(x)] \oplus \mathbb{C}[s_j(x)] \leftarrow 0 \leftarrow ...$$ (7)

**Definition 5.1.** The fibre $\widehat{E}_x$ of $\widehat{E}$ over $x \in \mathbb{R}^2 \setminus (\tilde{K} \cup \tilde{B})$ is defined to be the Morse homology of the Morse complex (6) or (7) respectively if $x$ lies inside or outside the caustic.

In our case, Morse homology has only one non-trivial term, so for $x$ inside the caustic

$$\widehat{E}_x = \bigoplus_{i=1}^{3} \mathbb{C}[s_i(x)] / \partial_x(\mathbb{C}[n(x)])$$

while for $x$ outside the caustic

$$\widehat{E}_x = \mathbb{C}[s_i(x)] \oplus \mathbb{C}[s_j(x)]$$

**Definition 5.2.** On each $U_i$ we define $\widehat{E}$ as the trivial bundle whose fibre at $x \in U_i$ is given by definition 5.1.

We define now morphisms glueing the holomorphic bundle $\widehat{E}$ along $\tilde{K}$ and $\tilde{B}$. We start by considering the subset $\tilde{K}_F$ of $\tilde{K}$ consisting of folds which are not limit points of bifurcation lines. It is a codimension 1 subset of $\mathbb{R}^2$. Suppose $U$ and $V$ are two connected components of $\mathbb{R}^2 \setminus (\tilde{K} \cup \tilde{B})$, lying respectively outside and inside the caustic, such that $\partial U \cap \partial V \neq \emptyset$, and let $\tilde{K}_i \subset \partial U \cap \partial V \cap \tilde{K}_F$ be a connected component of $\tilde{K}_F$. For simplicity, suppose that $V$ is inside the caustic and $U$ outside, so that along $\tilde{K}_i$ the node $n$ and the saddle $s_i$ in $V$ glue together and disappear in $U$ ($(n,s_i)$ is also called a birth/death pair).

**Definition 5.3.** The isomorphism $\widehat{E}(U) \cong \widehat{E}(V)$ glueing $\widehat{E}$ along $\tilde{K}_i$ is defined as the one induced in homology by the inclusion

$$\mathbb{C}[s_j] \oplus \mathbb{C}[s_k] \hookrightarrow \bigoplus_{l=1}^{3} \mathbb{C}[s_l(x)]$$

for $j,k \neq i$. 8
It is a good definition since the inclusion preserves kernel and image of the differential of the Morse complex.

The second group of definitions is concerned instead with glueing along the subset \( \tilde{B}_1 \) of \( \tilde{B} \) consisting of points which are not intersection of bifurcation lines. It is a codimension 1 subset of \( \mathbb{R}^2 \).

**Definition 5.4.** For each \( x \in \mathbb{R}^2 \setminus (\tilde{K} \cup \tilde{B}) \) lying inside the caustic we define the incidence matrix \( I(x) = (I(x)_i) \in \text{Mat}(3, 1) \) such that \( I(x)_i = 0 \) if there is no gradient line from \( n(x) \) to \( s_i(x) \), and \( I(x)_i = 1 \) otherwise.

**Remark 5.5.** Similar definitions, though in a different setting, appear in [9], [10] and [11], highlighting the relations between Morse theory and algebraic K-theory. The definition of incidence matrix also resembles that of transition matrix given by H. Kokubu in [13].

Note that the incidence matrix at \( x \) gives information about the phase portrait of the gradient vector field \( \nabla \tilde{f}_x \) and is related to the Morse differential simply as follows:

\[
\partial_x n(x) = I(x)_1 s_1(x) + I(x)_2 s_2(x) + I(x)_3 s_3(x)
\]

Observe also that the incidence matrix is constant on each connected components of \( \mathbb{R}^2 \setminus (\tilde{K} \cup \tilde{B}) \): indeed, the gradient vector fields \( \nabla \tilde{f}_x \) are orbit equivalent for all \( x \) in the same connected component, and so the Morse complexes are isomorphic. Let \( U \) and \( V \) be two such components, lying inside the caustic, such that \( \partial U \cap \partial V \neq \emptyset \), with incidence matrix \( I(U) \) and \( I(V) \) respectively. For \( \tau \in \{1, 0, -1\} \), let \( E_{ij}(\tau) \in \text{Mat}(3, 3) \) be the triangular matrix whose \((k, l)\)-entry is 1 if \( k = l \), \( \tau \) if \( k = i \) and \( l = j \), and 0 otherwise. Observe that from results in [16], crossing a bifurcation line can change at most only one of the entries of the incidence matrix, so we have that

- either \( I(U) \neq I(V) \): in this case there exists only one \( k \in \{1, 2, 3\} \) such that \( I(U)_k \neq I(V)_k \);
- or \( I(U) = I(V) \)

**Definition 5.6.** The transformation matrix from \( U \) to \( V \) associated to points in \( \partial U \cap \partial V \cap \tilde{B}_1 \) of a bifurcation line of \( \tilde{B} \), characterized by the appearance of a non-generic gradient line from \( s_i \) to \( s_j \), is a matrix of the form \( E_{ij}(\tau) \), such that \( E_{ij}(\tau)I(U) = I(V) \).
Note that when $I(U) \neq I(V)$ it follows that $\tau = 1$ if $I(U)_j = 0$, and $\tau = -1$ if $I(U)_j = 1$; when instead $I(U) = I(V)$, there is an ambiguity in the choice of $\tau$ which will be discussed below in example 5.8.

We make two examples to clarify the previous definition:

**Example 5.7.** Suppose the phase portrait of $\nabla f_x$ for $x \in U$ and for $x \in V$ is represented by the incidence matrix $I(U) = (1, 1, 1)$ and $I(V) = (1, 1, 0)$ respectively. There are two possible bifurcations from $U$ to $V$ (see [15] and [16] for further explanations and some pictures): either the non-generic gradient line $\gamma_{s_1 s_3}$ or the non-generic gradient line $\gamma_{s_2 s_3}$ appears in the phase portrait of $\nabla f_x$ when $x$ is the bifurcation point. The first bifurcation corresponds to the transformation matrix $E_{31}(-1)$, while the second corresponds to $E_{32}(-1)$. Instead, if crossing from $V$ to $U$, the same bifurcations give the transformation matrices $E_{31}(1)$ and $E_{32}(1)$ respectively.

**Example 5.8.** $I(U) = I(V)$ occurs only in case (c) analyzed in proposition 5.11 and shown in figure 5.3 (refer to this for the notation), along the bifurcation line between $\delta$ and $\epsilon$. The phase portraits in $\delta$ and $\epsilon$, which are represented respectively in figure 4.20 and 4.19 of [10], can be resumed here as follows: the separatrixes which connected $s_1$ and $s_3$ to $n$ in $\alpha$ (the phase portrait over $\alpha$ is shown in figure 4.17 of [10]), can form a saddle-to-saddle separatrix in $\epsilon$, but this can not occur in $\delta$. This can provide a criterion for the choice of $\tau$, which can not be justified further on here, considering only the specific example of the perturbed elliptic umbilic. The matrix $M(w_3)$ in the proof of proposition 5.11 is the transformation matrix from $\epsilon$ to $\delta$: there the choice of $\tau$ is the one which solves the monodromy.

Suppose now that $U$ and $V$ lie outside the caustic $\tilde{K}$ and $\partial U \cap \partial V \cap \tilde{B}_1$ is a subset of $\tilde{B}_j$, one of the three bifurcation lines forming the bifurcation diagram $\tilde{B}$, and assume $\tilde{B}_j$ enters into $\tilde{K}$ at a point $p$, through the side $l_j$ of $\tilde{K}$, where $n$ and $s_j$ form a birth/death pair. Since we are working in a neighbourhood of $\tilde{K}$, we can assume that $p \in \partial U \cap \partial V$. To $\tilde{B}_j$, inside the caustic and in a neighbourhood of $p$, we can associate a transformation matrix $E_{ik}(\tau)$ according to definition 5.6.

**Definition 5.9.** If $U$ and $V$ lie outside $\tilde{K}$ and are as described above, the transformation matrix from $U$ to $V$, associated to points in $\partial U \cap \partial V \cap \tilde{B}_1$ of the bifurcation line $\tilde{B}_j$, is the matrix $E_{ik}(\tau) \in \text{Mat}(2, 2)$, obtained from $E_{ik}(\tau) \in \text{Mat}(3, 3)$ above, by deleting the $j$-row and the $j$-column.
The transformation matrix associated to a bifurcation line $\tilde{B}_j$ from $U$ to $V$ defines a morphism between the Morse complexes of $U$ and $V$.

**Definition 5.10.** The isomorphism $\tilde{E}(U) \cong \tilde{E}(V)$ gluing $\tilde{E}$ along $\tilde{B}_j$ is the one induced by the transformation matrix of definition 5.6 or 5.9 associated to the bifurcation line $\tilde{B}_j$.

We have now to check that we can extend $\tilde{E}$ through the codimension 2 subset given by intersection points of bifurcation lines, limit points of bifurcation lines on the caustic and the three cusps.

We start by considering intersection points of bifurcation lines. In [16] we analyzed under which conditions two bifurcation lines can intersect themselves.

**Proposition 5.11.** The holomorphic bundle $\tilde{E}$ can be extended through intersection points of bifurcation lines.

**Proof.** We check that, for all possible cases of intersection of bifurcation lines, described in [16], chosen a loop $\Gamma$ around the intersection point $p$, the composition of the transformation matrices of bifurcation lines, at intersection points with $\Gamma$, is the identity. From [16] we know there are three cases:

![Diagram](image)

*Fig. 5.1: Intersection of bifurcation lines: case (a)*
The pictures of phase portraits in the subsets determined by bifurcation lines and of bifurcations in cases (a), (b) and (c) are represented in [16], and precisely in figures 4.7, 4.8, 4.9 for (a), 4.11, 4.12, 4.13, 4.14 for (b), and 4.17, 4.18, 4.19, 4.20, 4.21, 4.22 for (c).

In case (a), represented in figure 5.1, we know that the two bifurcation lines are characterized by the appearance of the same saddle-to-saddle separatrix, obtained by glueing the same pair of separatrices: so, chosen a simple loop $\Gamma$ around $p$, intersecting for simplicity the bifurcation lines into four points $w_i$, $i = 1, \ldots, 4$, and associated to each $w_i$ a transition matrix $M(w_i)$ according to definition 5.6 we have $M(w_1) = M(w_3) = M(w_2)^{-1} = M(w_4)^{-1}$, and thus $M(w_4)M(w_3)M(w_2)M(w_1) = \text{Id}$. This implies that there is no monodromy around $p$ and so the holomorphic bundle $\widehat{E}$ can be extended across $p$.

As to case (b), represented in figure 5.2, we chose again a simple loop $\Gamma$ around $p$, intersecting the bifurcation lines for simplicity into four points $w_i$, $i = 1, \ldots, 4$: suppose $w_1$ belongs to the bifurcation line from $\alpha$ to $\beta$, $w_2$ to the
bifurcation line from $\beta$ to $\delta$, $w_3$ to the bifurcation line from from $\delta$ to $\gamma$ and $w_4$ to the bifurcation line from from $\gamma$ to $\alpha$. The transformation matrices according to definition 5.6 associated to the bifurcation lines, at each $w_i$, in the chosen order, are given by:

\[
M(w_1) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M(w_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}
\]

\[
M(w_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M(w_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\]

Then $M(w_4)M(w_3)M(w_2)M(w_1) = Id$, and so the holomorphic bundle $\tilde{E}$ can be extended across $p$.

As to case (c), represented in figure 5.3, chosen a simple loop $\Gamma$ around $p$, which intersects the bifurcation lines for simplicity into five points $w_i$, $i = 1, \ldots, 5$, starting from the bifurcation line from $\alpha$ to $\beta$ and then proceeding anti-clockwise, the transformation matrices according to definition 5.6 are:

\[
M(w_1) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M(w_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}
\]

\[
M(w_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M(w_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\]

Then $M(w_5)M(w_4)M(w_3)M(w_2)M(w_1) = Id$, and so the holomorphic bundle $\tilde{E}$ can be extended across $p$.\qed

We analyze now the behaviour of $\tilde{E}$ around limit points of bifurcation lines belonging to the caustic.

**Proposition 5.12.** The holomorphic bundle $\tilde{E}$ can be extended through limit points of bifurcation lines belonging to the caustic, when they are not not cusps.
Proof. From [16] we know there are two cases: generically, either (a) the bifurcation line \( \tilde{B} \) enters into the caustic \( \tilde{K} \) at a fold or (b) it is an half-line with origin at a fold (and the bifurcation line \( \tilde{B} \), near its origin, lies inside \( \tilde{K} \)). In both cases, let us denote this fold by \( p \).

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
\hline
\( \tilde{K} \) & \( p \) & \( \alpha \) \\
\hline
\( \tilde{B} \) & & \\
\hline
\end{tabular}
\end{figure}

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
\hline
\( \tilde{K} \) & \( p \) & \( \alpha \) \\
\hline
\( \tilde{B} \) & & \\
\hline
\end{tabular}
\end{figure}

\textbf{Fig. 5.4: Mutual positions of bifurcation lines and caustic}

As to case (a), since \( p \) is not a cusp, at each point of the caustic \( \tilde{K} \) near \( p \), the node \( n \) glues with a saddle, which we suppose for simplicity to be \( s_1 \). Suppose also that the half-line \( \tilde{B} \) has his extreme on the side of the caustic where \( n \) glues with \( s_2 \) and that for \( x \in \alpha \), where \( \alpha \) is the region highlighted in figure 5.4, the phase portrait of \( \nabla \tilde{f}_x \) contains all the gradient lines \( \gamma_{ns_i} \). Choose a simple loop \( \Gamma \) around \( p \), intersecting for simplicity \( \tilde{B} \) into two points \( w_1 \) and \( w_3 \), and \( \tilde{K} \) into two points \( w_2 \) and \( w_4 \). Suppose \( w_1 \) lies inside the caustic and \( w_4 \) outside. We write the transition matrices at \( w_1 \) and \( w_3 \), according respectively to definition 5.6 and 5.9:

\[ M(w_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad M(w_3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

Consider an element \( h \in \tilde{E}_x \), for \( x \in \alpha \). Since \( \tilde{E}_x = \frac{\partial^{3}_{i=1}s_i(s_i(x))}{\partial_s(x[0])} \), we write \( h \) as an equivalence class \([\{h_1, h_2, h_3\}]\) on the basis \((s_1, s_2, s_3)\) of \( \mathbb{C}^3 \), where \((h_1, h_2, h_3) \sim (h_1 + c, h_2 + c, h_3 + c)\) for every \( c \in \mathbb{C} \). Moving along \( \Gamma \) from \( \alpha \) into \( \beta \), crossing \( \tilde{B} \) in \( w_1 \), \( h \) is transformed by \( M(w_1) \). In \( \beta \) we have
\[(h_1, h_2, h_3) \sim (h_1 + c, h_2 + c, h_3)\] for every \(c \in \mathbb{C}\), so we can write

\[M(w_1)h = [(h_1, h_2, -h_2 + h_3)] = [(0, h_2 - h_1, -h_2 + h_3)]\]

According to definition 5.3, when crossing \(\tilde{K}\) at \(w_2\), we have the glueing isomorphism:

\[[(0, h_2 - h_1, -h_2 + h_3)] \cong (h_2 - h_1, -h_2 + h_3)\]

crossing now \(\tilde{B}\) along \(\Gamma\) at \(w_3\)

\[M(w_3)(h_2 - h_1, -h_2 + h_3) = (h_2 - h_1, h_3 - h_1)\]

crossing \(\tilde{K}\) at \(w_4\) and using the glueing isomorphism of definition 5.3 we obtain:

\[(h_2 - h_1, h_3 - h_1) \cong [(0, h_2 - h_1, h_3 - h_1)] = [(h_1, h_2, h_3)]\]

This shows that there is no monodromy and so \(\tilde{E}\) can be extended through \(p\).

As to case (b), suppose for simplicity that: at \(p\) the node \(n\) and the saddle \(s_1\) form the birth/death pair, and that \(\tilde{B}\) intersects further \(\tilde{K}\) into another point where \(n\) and \(s_2\) form the birth/death pair; for \(x \in \alpha\) the phase portrait of \(\nabla f_x\) contains all the gradient lines \(\gamma_{n,s_i}\). Choose a simple loop \(\Gamma\) around \(p\), intersecting for simplicity \(\tilde{B}\) into the point \(w_1\), and \(\tilde{K}\) into two points \(w_2\) and \(w_3\). We know \(w_1\) lies inside the caustic. The transformation matrix according to definition 5.6 at \(w_1\) is

\[
M(w_1) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

Consider an element \(h \in \tilde{E}_x\), for \(x \in \alpha\), which we write as an equivalence class \([(h_1, h_2, h_3)]\) on the basis \((s_1, s_2, s_3)\) of \(\mathbb{C}[s_1(x)]\), where \((h_1, h_2, h_3) \sim (h_1 + c, h_2 + c, h_3 + c)\) for every \(c \in \mathbb{C}\). Going along \(\Gamma\) into \(\beta\), crossing \(\tilde{B}\) in \(w_1\), \(h\) is transformed by \(M(w_1)\). In \(\beta\) we have \((h_1, h_2, h_3) \sim (h_1 + c, h_2 + c, h_3)\) for every \(c \in \mathbb{C}\), so we can write

\[M(w_1)h = [(h_1, h_2, -h_1 + h_3)] = [(0, h_2 - h_1, -h_1 + h_3)]\]
Now, crossing $\tilde{K}$ at $w_2$ and using the glueing isomorphism of definition 5.3:

$$[(0, h_2 - h_1, -h_1 + h_3)] \cong (h_2 - h_1, -h_1 + h_3)$$

finally, entering into $\tilde{K}$ through $w_3$ and using again the glueing isomorphism, we obtain in (a):

$$(h_2 - h_1, -h_1 + h_3) \cong [(0, h_2 - h_1, -h_1 + h_3)] = [(h_1, h_2, h_3)]$$

This shows that there is no monodromy and so $\tilde{E}$ can be extended through $p$.  

Now we check if $\tilde{E}$ can be extended to cusps. To start suppose that at a cusp $c$ the node $n$ glues with the saddles $s_2$ and $s_3$. According to [16] there are two cases: either (a) for $x$ in a neighbourhood of $c$, inside the caustic, the phase portrait of $∇f_x$ contains all the gradient lines $γ_{ns_1}$, or (b) it contains only $γ_{ns_2}$ and $γ_{ns_3}$. In both cases a monodromy appears around the cusp.

**Lemma 5.13.** In case (a), if $Γ$ is a non-trivial simple loop around $c$, the monodromy of the holomorphic structure of $\tilde{E}$ along $Γ$ is represented by the matrix

$$M = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

(8)

**Proof.** For $x$ outside the caustic, since $\tilde{E}_x = \mathbb{C}[s_1] \oplus \mathbb{C}[s_j]$, we write an element $h ∈ \tilde{E}_x$ as $(h_1, h_j)$: on the branch $l_k$ of the caustic, with $k ∈ \{2, 3\}$, where $n$ glues with $s_k$, the glueing isomorphism of definition 5.3 identifies $s_j$ with the saddle different from $s_k$ and $s_1$. So, entering into the caustic through $l_2$ we have

$$(h_1, h_j) \cong [(h_1, h_j, 0)] = [(h_1 - h_j, 0, -h_j)]$$

now exiting from the caustic through $l_3$ we have

$$[(h_1 - h_j, 0, -h_j)] \cong (h_1 - h_j, -h_j)$$

which gives the expected monodromy.  

**Lemma 5.14.** In case (b), if $Γ$ is a non-trivial simple loop around $c$, the monodromy of the holomorphic structure of $\tilde{E}$ along $Γ$ is represented by the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(9)
Proof. Using the notation in the proof of the previous lemma, we have, entering into the caustic through $l_2$

$$(h_1, h_j) \cong [(h_1, h_j, 0)] = [(h_1, 0, -h_j)]$$

and exiting from caustic through $l_3$

$$[(h_1, 0, -h_j)] \cong (h_1, -h_j)$$

which gives the expected monodromy. \hfill \square

Observe that in both cases, the matrix $M$ is invertible. This means that both to $\Gamma$ and to its opposite $\Gamma^{-1}$ in $\pi_1(L \setminus \{c\})$ the same monodromy is associated.

If now at $c$ the node $n$ glues with the saddles $s_1$ and $s_2$ we have a similar result:

**Lemma 5.15.** If $\Gamma$ is a non-trivial simple loop around $c$, the monodromy of the holomorphic structure of $\tilde{E}$ along $\Gamma$ is represented, in case (a), by the matrix

$$M = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

in case (b), by the matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Proof.** The proof is analogous to that of lemma 5.13 and 5.14. \hfill \square

Again observe that the matrix $M$ is invertible, meaning that $\Gamma$ and $\Gamma^{-1}$ provide the same monodromy.

Lastly, if at $c$ the node $n$ glues with the saddles $s_1$ and $s_3$ we obtain the following result:

**Lemma 5.16.** If $\Gamma$ is a non-trivial simple loop around $c$, the monodromy of the holomorphic structure of $\tilde{E}$ along $\Gamma$ is represented, in case (a), by the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

17
or by its inverse

$$M^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

(13)

while in case (b), by the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(14)

or by its inverse

$$M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(15)

Proof. The proof is similar to that of lemma 5.13 and 5.14.

Observe that, in both cases, if $\Gamma$ is associated, for example, to $M$, then $\Gamma^{-1}$ is associated to $M^{-1}$.

To solve the monodromy around the cusps it is necessary to add a new kind of correction. It is related to the possibility of defining a spin structure on $\tilde{L}$ and probably to the problem of orientation in Lagrangian intersection Floer homology (in fact from [8] we know that the existence of a relative spin structure on $\tilde{L}$ is a condition for the orientability of the moduli space of pseudo-holomorphic discs) or to the problem of orientation for families of Morse homologies. This is suggested intuitively by what follows: consider the composition $\pi \circ i : \tilde{L} \hookrightarrow T^4 \to T^2$, where $\pi$ is the projection of the fibration and $i$ is the Lagrangian immersion, and note that a spin structure can be induced at least on the subset of $\tilde{L}$ where $d\pi$ is invertible, that is, on $\tilde{L} \setminus \pi^{-1}(\tilde{K})$; this means that the caustic or a subset of it represents an obstruction to the existence of a spin structure on $\tilde{L}$.

The following result shows that the set of cusps is actually the obstruction to the existence of a spin structure on a Lagrangian submanifold $L$ with generating function $f$: it proves, in fact, that the second Stiefel-Whitney class $w_2(L) \in H^2(L, \mathbb{Z}_2)$ of $L$, which represents the obstruction to the existence of spin structures on $L$, has the set of cusps as Poincaré dual in $H_0(L, \mathbb{Z}_2)$.

Lemma 5.17.

$$PD(w_2(L)) = A_3(f)$$

where $A_3(f)$ is the set of singular points of $f$ of type $A_3$, that is, the set of cusps.
The main tool in proving this equality is represented by Thom polynomials of Lagrangian singularities. The proof is essentially given in [12] by M. Kazarian, where it follows from other major results given there: it is first demonstrated that the cohomology class \( PD(\Omega(f)) \), the Poincaré dual to the locus \( \Omega(f) \) of singularities of \( f \) of class \( \Omega \), is equal to the Thom polynomial \( P_\Omega \) associated to \( \Omega \); then Thom polynomials are computed (see also [19]), and in particular, when \( \Omega = A_3 \), it is shown that \( P_\Omega = w_2(T^*L) = w_2(TL) \).

Let \( A \) be an immersed 1-dimensional submanifold of \( \mathbb{R}^2 \) with three non-intersecting connected components, each of which being an half-line with vertex at one of the three cusps of the caustic. To solve the monodromy around the cusps it is enough to glue, for example along \( A \), the holomorphic structure in such a way to cancel the monodromy. The problem is to justify, if anything, this procedure, which for the moment is just an \textit{ad hoc} correction.

As said, the idea, coming from the orientation problem of Lagrangian intersection Floer homology, and confirmed by lemma 5.17, is that the possibility to define a spin structure on some flat bundle on \( \tilde{L} \) should provide, in some way, such correction. We make the following natural definition:

\textbf{Definition 5.18.} Along each half-line forming the submanifold \( A \), depending on which cusp the half-line has as vertex, we glue the holomorphic bundle \( \tilde{\mathcal{E}} \) using the inverse of morphism (8) or (10) or (12) or (13) in case (a), and (9) or (11) or (14) or (15) in case (b).

This correction is called \textit{orientation twist} in [6].

\textbf{Proposition 5.19.} If \( \tilde{\mathcal{E}} \) is glued along \( A \) according to definition 5.18, then its holomorphic structure can be extended across the cusp.

\textit{Proof.} The proof is a direct consequence of lemma 5.13, 5.14, 5.15 and 5.16, since the corrections applied are just the inverse of what we want to cancel.

We try now to justify definition 5.18, though, in this paper, it will be done only in a heuristic way. Before considering the case of a perturbed elliptic umbilic, let us examine for simplicity a Lagrangian submanifold \( L \) exhibiting a cusp \( c \): in this case, \( A \) is an half-line with vertex in \( c \). Consider a ball \( U \) containing \( c \). Since \( U \) is contractible, \( L \) owns a spin structure over \( U \). On the other hand, over the complement of \( U \), \( d\pi \) is invertible and so it induces
a spin structure on $L$. Since, by lemma 5.17, $w_2(L)$ does not vanish because of $c$, it follows that the non-existence of a spin structure on $L$ comes from the glueing of $TL$ along the boundary of $U$. The purpose now is to show how $A$ can provide both a “correction” to $TL$, by defining a new bundle carrying a spin structure, and a “correction” to to the flat $U(1)$-line bundle $L$ on $L$, yielding the glueing which cancels the monodromy. Consider representations

$$\rho : \pi_1(\mathbb{R}^2 \setminus \{c\}) \to \{1, -1\} = O(1) \subset U(1)$$

defining two representations $\rho^{O(1)}$ and $\rho^{U(1)}$. According to this choice we have, respectively, a flat $O(1)$-bundle $L^{O(1)}_\rho$ or a flat $U(1)$-bundle $L^{U(1)}_\rho$ on $\mathbb{R}^2 \setminus \{c\}$. There are two possibilities for $\rho$, that is, it is either the trivial or the non-trivial group homomorphism $\mathbb{Z} \to \{1, -1\}$. Particularly, when $\rho$ is the non-trivial representation, its values on a path $\Gamma \in \pi_1(\mathbb{R}^2 \setminus \{c\})$ are given by the intersection number of $\Gamma$ and $A$. $L^{O(1)}_\rho$ is the trivial bundle when $\rho$ is trivial, while it is a Möbius strip when $\rho$ is non-trivial. The same holds for $L^{U(1)}_\rho$: in particular, the bundle $L^{U(1)}_\rho$ restricted on a generator $\Gamma \cong S^1$ of $\pi_1(\mathbb{R}^2 \setminus \{c\})$, is the flat line bundle on the torus $T^1 = S^1$ with factor of automorphy equal to either 1 or -1, according to which $\rho$ is trivial or not. In other words, we may think of a section of $L^{U(1)}_\rho$ over $\Gamma$ as multiplied by respectively 1 or -1 at $\Gamma \cap A$ (the factor of automorphy for $U(1)$-line bundles on tori and the induced connection on the mirror bundle are treated and exposed in [2] and [3]). The projection of the fibration $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ and the composition $\pi \circ i$, where $i : L \hookrightarrow \mathbb{R}^4$ is the Lagrangian immersion, defines, respectively, bundles $L^{\mathbb{R}^4}_\rho = i^*L^\mathbb{R^4}_\rho$ on $\mathbb{R}^4$ and $L^L_\rho = (\pi \circ i)^*L^\rho$ on $L$, away, respectively, from $\pi^{-1}(c)$ and $(\pi \circ i)^{-1}(c)$, where $\rho$ can be either $\rho^{O(1)}$ or $\rho^{U(1)}$.

If $\rho^{O(1)}$ is the non-trivial representation, since a Möbius strip has $w_1 = 1$, then, setting $M = L^{\mathbb{R}^4}_{\rho^{O(1)}} \oplus L^{\mathbb{R}^4}_{\rho^{O(1)}}$, we have $w_1(M) = 2w_1(L^{\mathbb{R}^4}_{\rho^{O(1)}}) = 0$ and $w_2(M) = 2w_2(L^{\mathbb{R}^4}_{\rho^{O(1)}}) + w_1(L^{\mathbb{R}^4}_{\rho^{O(1)}})w_1(L^{\mathbb{R}^4}_{\rho^{O(1)}}) = 1$. This implies that the bundle $TL \oplus M|_L$ over $L$ carries a spin structure: in fact, since $L^L_\rho = i^*(\pi^*L^\rho) = i^*(L^{\mathbb{R}^4}_\rho)$ and so $w_2(L^L_\rho) = i^*w_2(L^{\mathbb{R}^4}_\rho)$, we have that $w_2(TL \oplus M|_L) = w_2(TL) + w_1(TL)w_1(M|_L) + w_2(M|_L) = 0$ in $H^2(L; \mathbb{Z}_2)$. This, together with the fact that $L$ has dimension 2 and that $M$ is a real orientable vector bundle on $\mathbb{R}^4$, implies, by definition, that $L$ is relative spin.

Now, consider the flat line bundle $L \otimes L^{L}_\rho$ over $L$, carrying the connection $\nabla_\rho = \nabla \otimes \nabla^{L}_\rho$, where $(L, \nabla)$ is the given flat line bundle over $L$.
\[ \nabla_{\rho_U} L \]

is the flat connection of \( L_{\rho_U} \) defined by \( \rho_U \), and consider the effect given by the connection \( \hat{\nabla}_\rho \) on the transformed bundle \( \hat{E} \): it induces a non-trivial glueing along \( A \), given by multiplication by -1, which cancels the monodromy along \( c \), given also by a multiplication by -1. In fact, if \( s_1 \) and \( s_2 \) are the saddles and \( l_1 \) and \( l_2 \) are the sides of the caustic where the node \( n \) glues together with \( s_1 \) and \( s_2 \) respectively, we have that along \( l_1 \), according to definition 5.3

\[(h) \cong [(h, 0)]\]

in Morse homology we have the equality

\[[h, 0] = [(0, -h)]\]

along \( l_2 \), according to definition 5.3, we have

\[[0, -h] \cong (-h)\]

and, finally, along \( A \), the connection \( \hat{\nabla}_\rho \) gives the glueing

\[(-h) \cong (h)\]

Consider now our case of a perturbed elliptic umbilic. Take a suitable ball \( U \) containing a cusp \( c \) of \( \tilde{L} \) such that \( \tilde{L} \cap \pi^{-1}(U) \) has two connected components. For simplicity, suppose that \( c \) is the cusp of the caustic where \( n, s_2 \) and \( s_3 \) glue together. Identifying critical points of the gradient system over \( x \) and points of \( \tilde{L} \) over \( x \), we have that, of the two components of \( \tilde{L} \cap \pi^{-1}(U) \), one contains \( s_1 \) and the other \( s_2 \) and \( s_3 \). Note that \( T\tilde{L} \) carries a spin structure over the first component but not over the second, where we find the same situation described above for the cusp. So choose \( \rho \) in such a way that \( L_{\rho_U}^L \) and \( L_{\rho_U}^L \) are the trivial flat line bundles over the component containing \( s_1 \) and the non-trivial one over the component containing \( s_2 \) and \( s_3 \). As described above, setting \( M = \mathcal{L}_{\rho_U}^L \oplus \mathcal{L}_{\rho_U}^L \), \( T\tilde{L} \oplus M_L \) carries a spin structure on both the components. Moreover, the connection \( \tilde{\nabla}_\rho \) on the mirror bundle \( \tilde{E} \), induced by the connection \( \tilde{\nabla}_\rho = \tilde{\nabla} \oplus \tilde{\nabla}_{\rho_U}^L \), cancels the monodromy of lemma 5.13 and 5.14 as we will explain now. Consider first case (b) described by lemma 5.14: as no gradient line exists from \( n \) to \( s_1 \), it can be treated as done above for the cusp, obtaining that the flat connection gives a glueing along \( A \) which is a multiplication by 1 on chains generated by \( s_1 \) and a multiplication by -1 on chains generated by \( s_2 \) or \( s_3 \);
this cancels in homology the monodromy of lemma 5.14. Consider now case (a) described in lemma 5.13. The gluing provided by \( \nabla_\rho \) must commute with the equivalence among cycles in Morse homology in order to define a gluing in homology, and this is not automatic as in case (b) because of the gradient line from \( n \) to \( s_1 \). In fact, the connection \( \nabla_\rho \) induces a connection on \( \partial(n) = < s_1 + s_2 + s_3 > \) characterized by a gluing which is a multiplication by -1. On the other hand, the connection on \( \sum_{i=1}^{3} \mathbb{C}[s_i] \) has factor of automorphy -1 on the chains \( s_2 \) and \( s_3 \) and 1 on \( s_1 \); this means that it does not commute with the action on cycles determined by the differential \( \partial \).

Thus, to induce a connection in homology, that is, on the quotient \( \sum_{i=1}^{3} \mathbb{C}[s_i] / \partial(n) \), the connection at the chains level, that is, on \( \sum_{i=1}^{3} \mathbb{C}[s_i] \), must be split into two parts, one of which, commuting with that on \( \partial(n) \), will induce a connection in homology. The problem is the choice of a splitting of the connection at the chains level. This is performed as follows: the gluing

\[(h_1, h_2, h_3) \cong (h_1, -h_2, -h_3)\]

is split as

\[(h_1, -h_2, -h_3) = (h_1 - h_2 - h_3, -h_2, -h_3) + (h_2 + h_3, 0, 0)\]

and on the quotient it is induced the gluing given by

\[ [(h_1, -h_2, -h_3)] = [(h_1 - h_2 - h_3, -h_2, -h_3)].\]

Note, indeed, that it commutes with the Morse differential:

\[(h_1, h_2, h_3) \cong (h_1+g, h_2+g, h_3+g) \cong (h_1+g-h_2-g-h_3-g, -h_2-g, -h_3-g) =
\[(h_1 - h_2 - h_3 - g, -h_2 - g, -h_3 - g)\]

where the first equivalence is that among cycles in Morse homology and the second is the gluing, and

\[(h_1, h_2, h_3) \cong (h_1 - h_2 - h_3, -h_2, -h_3) \cong (h_1 - h_2 - h_3 - g, -h_2 - g, -h_3 - g)\]

where now the first equivalence is the gluing and the second is that among cycles in Morse homology. The splitting we chose corresponds to a gluing, at the chains level, given by a multiplication by 1 on the generator \( s_1 \), while, on the generators \( s_2 \) and \( s_3 \), by a multiplication by -1, followed by a projection,
parallel to $s_1$, onto the line generated by $s_3$ and $s_2$ respectively. A better justification for this choice requires, perhaps, the consideration of a more general situation than that of a perturbed elliptic umbilic. Anyway, this solves the monodromy: indeed, as in the proof of lemma 5.13 we have along $l_2$

$$(h_1, h_j) \cong [(h_1, h_j, 0)] = [(h_1 - h_j, 0, -h_j)]$$

the connection $\nabla_\rho$ gives the glueing

$$[(h_1 - h_j, 0, -h_j)] \cong [(h_1 - h_j + h_j, 0, h_j)] = [(h_1, 0, h_j)]$$

and along $l_3$ we have

$$[(h_1, 0, h_j)] \cong (h_1, h_j).$$

What remains to do now is to check that there is no monodromy in the holomorphic structure of $\tilde{E}$ when going along a loop $\Gamma$ such that the caustic lies in the compact region of $\mathbb{R}^2$ determined by $\Gamma$, as described in lemma 4.1.

**Theorem 5.20.** The monodromy of lemma 4.1 is solved if the following corrections are applied: $\tilde{E}$ is glued by means of the morphisms of definition 5.3 along the caustic $\tilde{K}$, of definition 5.10 along the bifurcation locus $\tilde{B}$, and of definition 5.18 along the relative cycle $A$.

**Proof.** The theorem follows from propositions 5.11, 5.12 and 5.19.  

As an example, we write the transformation matrices associated to bifurcation lines and to half-lines forming the relative cycles $A$, which a loop $\Gamma$ as described above meets, and show that their composition is the identity, implying that the expected monodromy is cancelled. Consider, for instance, the following bifurcation diagram:
Fig. 5: An allowed bifurcation diagram together with the half-cycle $A$ and the loop $\Gamma$.

Assumed for simplicity that $\Gamma$ is directed counter-clockwise, set $a_i = A_i \cap \Gamma$ and $b_i = \tilde{B}_i \cap \Gamma$, where $A_i$ are the half-lines forming the relative cycle $A$ and $\tilde{B}_i$ are the bifurcation lines, with $i = 1, 2, 3$, then the matrices corresponding to glueing morphisms at points $a_i$ and $b_i$ are:

\[
M(b_1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad M(a_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
M(b_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M(a_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
M(a_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad M(b_3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

Observe now that $M(b_3)M(a_3)M(a_2)M(b_2)M(a_1)M(b_1) = Id$, which implies that the monodromy is solved.

With such corrections, the mirror bundle $\tilde{E}$ is endowed with a holomorphic structure which can be extended along the caustic and the bifurcation locus.
References

[1] D. Arinkin, A. Polishchuk, *Fukaya category and Fourier transform*, math.AG/9811023.

[2] U. Bruzzo, G. Marelli, F. Pioli, *A Fourier transform for sheaves on real tori: Part I: the equivalence Sky(T) $\cong$ Loc(T)*, J. Geo. Phys. 39 (2001), 174-182.

[3] U. Bruzzo, G. Marelli, F. Pioli, *A Fourier transform for sheaves on real tori: Part II: Relative theory*, J. Geo. Phys. 41 (2002), 312-329.

[4] A. Floer, *Morse theory for Lagrangian intersections*, J. Diff. Geom. 28 (1988), 513-547.

[5] K. Fukaya, *Mirror symmetry of Abelian varieties and multi-theta functions*, (2000). Available from the web page http://www.math.kyoto-u.ac.jp/~fukaya/abelrev.pdf

[6] K. Fukaya, *Multivalued Morse theory, asymptotics analysis and mirror symmetry*, (2002). Available from the web page http://www.math.kyoto-u.ac.jp/~fukaya/fukayagrapat.dvi

[7] K. Fukaya, Y.G. Oh, *Zero-loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math. 1 (1997), 99-180.

[8] K. Fukaya, Y.G. Oh, H. Ohta, K. Ono, *Lagrangian intersection Floer Theory - anomaly and obstruction -*, (2000). Available from the web page http://www.math.kyoto-u.ac.jp/~fukaya/fooo.dvi

[9] K. Igusa, *Higher Franz-Reidmeister torsion*, AMS/IP Studies in Advanced Mathematics, (2002).

[10] K. Igusa, *The Borel regulator map on pictures, I: a dilogarithm formula*, K-Theory 7 (1993), 201-224.

[11] K. Igusa, J. Klein, *The Borel regulator map on pictures, II: an example from Morse theory*, K-Theory 7 (1993), 225-267.

[12] M. Kazarian, *Thom polynomials for Lagrange, Legendre, and critical point function singularities*, Proc. LMS (3) 86 (2003), 707-734.

[13] H. Kokubu, *On transition matrices*, EQUADIFF99, Proceedings of the International Conference on Differential Equations, Berlin, Germany 1-7 August 1999, World Scientific, 2000, pp.219-224.
[14] N.C. Leung, S.-T. Yau, E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, math.DG/0005118.

[15] G. Marelli, *Two-dimensional Lagrangian singularities and bifurcations of gradient lines I*, J. Geo. Phys. 56/9 (2006), 1688-1708.

[16] G. Marelli, *Two-dimensional Lagrangian singularities and bifurcations of gradient lines II*, J. Geo. Phys. 56/9 (2006), 1875-1892.

[17] D. McDuff, D. Salamon *J-holomorphic curves and symplectic topology*, American Mathematical Society, Colloquium Publications, vol. 52.

[18] M. Schwarz, *Morse homology*, Birkäuser, Basel-Boston-Berlin, (1993).

[19] V.A. Vassilyev, *Lagrange and Legendre characteristic classes*, Gordon and Breach Science Publishers, New York (1988).