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SPECTRAL TRIPLES FOR FINITELY PRESENTED GROUPS, INDEX 1

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ABSTRACT. Using a Cayley complex (generalizing the Cayley graph) and Clifford algebras, we are able to give, for a large class of finitely presented groups, a uniform construction of spectral triples with $D_+$ of index 1.

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Warning 0.1. This paper is just a first draft, it contains very few proofs. It is possible that some propositions are false, or that some proofs are incomplete or trivially false.

1. Introduction

In this paper, we define even $\theta$-summable spectral triples for a large class of finitely presented groups such that $D_+$ is index 1. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory as a Cayley complex (generalizing the Cayley graph).

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2. Basic definitions

Definition 2.1. A spectral triple $(\mathcal{A}, H, D)$ is given by a unital $\star$-algebra $\mathcal{A}$ represented on the Hilbert space $H$, and an unbounded operator $D$, called the Dirac operator, such that:

1. $D$ is self-adjoint.
2. $(D^2 + I)^{-1}$ is compact.
3. $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ is dense in $\mathcal{A}$.

See the article [6] of G. Skandalis, dedicated to A. Connes and spectral triple.

Definition 2.2. A group $\Gamma$ is finitely presented if it exists a finite generating set $S$ and a finite set of relations $R$ such that $\Gamma = \langle S \mid R \rangle$. We always take $S$ equals to $S - 1$ and the identity element $e \notin S$ (see [3] for more details).

3. Geometric construction

Definition 3.1. Let $\Gamma_n$ be the set of irreducible $n$-blocks, defined by induction:

- $\Gamma_0 = \Gamma$.
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

An $(n + 2)$-block is a finite set $a$ of $(n + 1)$-blocks such that:

$\forall b \in a, \forall c \in b, \exists b' \in a$ such that $b \cap b' = \{c\}$.

Let $a, a'$ be $n$-blocks then the commutative and associative composition:

$a \cdot a' := a \Delta a' = (a \cup a') \setminus a \cap a'$

gives also an $n$-block if it’s non empty (we take $n \neq 0$).

Let $n > 1$, an $n$-block $a''$ is called irreducible if $\forall a, a'$ $n$-blocks:

1. $a'' = a \cdot a' \Rightarrow \text{card}(a) \text{ or } \text{card}(a') \geq \text{card}(a'')$
2. $\forall b \in a''$, $b$ is an irreducible $(n - 1)$-block.

- $\Gamma_{n+2}$ is the set of irreducible $(n + 2)$-blocks.

Note that if $b \in \Gamma_n$, we call $n$ the dimension of $b$.

Definition 3.2. An $n$-block is called admissible if it decomposes into irreducibles.

Example 3.3. Let $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$ then $a = \{e, s^{10}\}$ is an admissible 1-block because $a = \{e, s\}, \{s, s^2\}, \ldots, \{s^9, s^{10}\}$; but, $b = \{e, s\}, \{e, s^{-1}\}, \{s^{-1}, s\}$ is a non-admissible 2-block, because there is no irreducible 2-block in this case.

Remark 3.4. The graph with vertices $\Gamma_0$ and edges $\Gamma_1$ is the Cayley graph $\mathcal{G}$.

Remark 3.5. Let $a$ be an $n$-block then $a \cdot a = \emptyset$ and if $a = \{b_1, \ldots, b_r\}$ then $b_i = b_1, b_2, \ldots, b_{i-1}, b_{i+1} \ldots b_r$ and $b_1, b_2, \ldots, b_r = \emptyset$.

Remark 3.6. $\Gamma_{n+1} \neq \emptyset$ iff $\exists r > 1; a_1, \ldots, a_r \in \Gamma_n$ all distincts with $a_1 \ldots a_r = \emptyset$. 
Remark 3.7. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\exists N$ such that $\Gamma_N \neq \emptyset$ and $\forall n > N$, $\Gamma_n = \emptyset$. In fact $N \leq \text{card}(S)$.

Examples 3.8. For $\mathbb{F}_r = \langle s_{1}^{\pm 1}, \ldots, s_{r}^{\pm 1} \mid \rangle$, we have $N = 1$.

For $\mathbb{Z}^{r} = \langle s_{1}^{\pm 1}, \ldots, s_{r}^{\pm 1} \mid s_{i}s_{j}s_{i}^{-1}s_{j}^{-1}, i, j = 1, \ldots, r \rangle$, we have $N = r$.

Here an $n$-block $(n \leq r)$ is just an $n$-dimensional hypercube.

Definition 3.9. We define the action of $\Gamma$ on $\Gamma_n$ recursively:

- $\Gamma$ acts on $\Gamma_0 = \Gamma$ as: $u_g : h \rightarrow g.h$ with $g, h \in \Gamma$.
- Action on $\Gamma_{n+1}$: $u_g : a \rightarrow g.a = \{g.b \mid b \in a\}$ with $g \in \Gamma$, $a \in \Gamma_{n+1}$.

Note that the action is well-defined: $g.\Gamma_n = \Gamma_n$, $\forall g \in \Gamma$.

Definition 3.10. Let $a$ and $b$ be blocks, then we say that $b \in a$ if $b = a$ or if $b \subseteq a$ or if $\exists c \in a$ such that $b \subseteq c$ (recursive definition).

Definition 3.11. Let $n > 1$ then an $n$-block $c$ is connected if $\forall b \subseteq c$: $b$ is an $n$-block $\Rightarrow b = c$.

Definition 3.12. An $n$-block $b$ is called maximal if there is no $(n+1)$-block $c$ with $b \subseteq c$. We note $\Gamma_{\text{max}}$ the set of maximal irreducible blocks.

Example 3.13. Let $\Gamma = \mathbb{Z}^2 \times \mathbb{Z} = \langle s_{1}^{\pm 1}, s_{2}^{\pm 1}, s_{3}^{\pm 1} \mid s_{1}s_{2}s_{1}^{-1}s_{2}^{-1} \rangle$, then $\{e, s_{3}\}$ is a maximal $1$-block, $\{\{e, s_{1}\}, \{s_{1}, s_{1}s_{2}\}, \{s_{1}s_{2}, s_{2}\}, \{s_{2}, e\}\}$ is a maximal $2$-block.

Definition 3.14. We define the block length $\ell(.)$ as follows: let $b$ be a block, then $\ell(b)$ is the minimal number of irreducible blocks decomposing a connected admissible block $c$ with $e \subseteq c$ and, $b \subseteq c$ or $b \cap c \neq \emptyset$.

Definition 3.15. Let $b$ be a block, then a sequence $(c_{1}, \ldots, c_{\ell(b)})$ with $b \subseteq c_{1}$, $e \subseteq c_{\ell(b)}$, $c_{i}$ irreducible and $c_{i} \cap c_{i+1} \neq \emptyset$ is called a geodesic block-path, from $b$ to $e$ beginning with $c_{1}$.

Definition 3.16. Let $\Upsilon_{b}$ be the set of irreducible blocks of minimal dimension beginning a geodesic block-path from $b$ to $e$.

Remark 3.17. In general, $\Upsilon_{b}$ is not of cardinal one. It is for CAT(0) groups, but not for the Baumslag-Solitar group $BS(1, 2) = \langle a^{\pm 1}, b^{\pm 1} \mid bab^{-1} = a^{2} \rangle$.

Remark 3.18. Consider the group $\Gamma$ and its finite presentation $\langle S \mid R \rangle$, then we can complete the presentation as follows: let $T$ be a finite subset of $\Gamma$ with $T \cap S = \emptyset$, $T = T^{-1}$ and $e \not\in T$, let $S' = T \cup S$ an amplified generating set and $R' = R \cup \{t = \bar{t} \mid t \in T\}$ where $\bar{t}$ is $t$ considered as a generator.

Then $\Gamma = \langle S' \mid R' \rangle$.

Lemma 3.19. We can choose $T$ such that if we build the blocks with the completed presentation $\langle S' \mid R' \rangle$, then every irreducible blocks are triangular, i.e. $\forall b \in \Gamma_n$, $\text{card}(b) = n + 1$. We call $\langle S' \mid R' \rangle$ a triangularized presentation.
Example 3.20. The complete triangularization: let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\Gamma$ acts on $\Gamma_{\max}$ (def. 3.3, 3.12); there are only finitely many orbits $O_1, \ldots, O_r$; choose $b_i \in O_i$; let $E_i = \{ g \in \Gamma \mid g \in b_i \}$; let $T_i = \{ gh^{-1} \mid g, h \in E_i, gh^{-1} \notin S \cup \{ e \} \}$. Then amplifying the generating set with $T = \bigcup T_i$, we obtain obviously a triangularization called the complete triangularization. Note that this process increases the maximal dimension of the blocks. Note that $\text{card}(T)$ is finite because the group is finitely presented.

4. Clifford algebra

We first quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

Definition 4.1. For $V$ a $n$-dimensional Hilbert space, define the exterior algebra $\Lambda(V)$ equals to $\bigoplus_{k=0}^n \Lambda^k(V)$ with $\Lambda^0(V) = \mathbb{C} \Omega$. We called $\Omega$ the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$.

Note that $\dim(\Lambda^k(V)) = C^n_k$ and $\dim(\Lambda(V)) = 2^n$.

Definition 4.2. Let $\alpha_v$ be the creation operator on $\Lambda(V)$ defined by:

$$\alpha_v(v_1 \wedge \ldots \wedge v_r) = v \wedge v_1 \wedge \ldots \wedge v_r$$

and $\alpha_v(\Omega) = v$

Reminder 4.3. The dual $\alpha_v^*$ is called the annihilation operator, then:

$$\alpha^*_v(v_1 \wedge \ldots \wedge v_r) = \sum_{i=0}^r (-1)^{i+1}(v, v_i)v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_r$$

and $\alpha^*_v(\Omega) = 0$

Reminder 4.4. Let $\gamma_v = \alpha_v + \alpha_v^*$, then $\gamma_v = \gamma_v^*$ and $\gamma_v \gamma_w + \gamma_w \gamma_v = 2(v, w)I$.

Definition 4.5. The operators $\gamma_v$ generate the Clifford algebra $\text{Cliff}(V)$. Note that the operators $\gamma_v$ are bounded and that $\text{Cliff}(V).\Omega = \Lambda(V)$.

Remark 4.6. $V$ admits the orthonormal basis $(v_a)_{a \in I}$.

We will write $\gamma_a$ instead of $\gamma_{v_a}$, so that $[\gamma_a, \gamma_{a'}]_+ = 2\delta_{aa'}I$.

Let $\Gamma$ be a finitely presented group, with a triangularized presentation $\langle S \mid R \rangle$.

Definition 4.7. For any irreducible block $c$, let $\Delta_c = \{ b \in \bigcup \Gamma_n \mid c \in \Upsilon_b \}$, with $\Upsilon_b$ defined on definition 3.14.

Remark 4.8. If $\Delta_c \neq \emptyset$ then $c \in \Delta_c$.

$\bigcup \Gamma_n = \bigcup \Delta_c$ (it’s not a partition in general)

If $\Delta_c \cap \Delta_{c'} \neq \emptyset$ with $c \neq c'$ then $\dim(c) = \dim(c')$ and $c, c'$ is connected.

Definition 4.9. Let $\bigcup_{a \in J} \mathcal{P}_a$ be the minimal partition generated by $\bigcup \Delta_c$, ie $\bigcup_{a \in J} \mathcal{P}_a = \bigcup \Gamma_n$, $\forall \alpha \in J$, $\mathcal{P}_a \neq \emptyset$ and $\exists n > 0$, $\exists c_1, \ldots, c_n$ irreducibles such that $\mathcal{P}_a = \Delta_{c_1} \cap \ldots \cap \Delta_{c_n}$. 

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Lemma 4.10. For any \( \alpha \in \mathcal{J} \), \( P_\alpha \) admits a unique element \( c_\alpha \) (resp. \( c'_\alpha \)) of minimal dimension \( m \) (resp. of maximal dimension \( M \)). Denote by \( I_\alpha \) the set of blocks of dimension \( m + 1 \) in \( P_\alpha \), then \( P_\alpha \) is in one-to-one correspondence with the power set \( \mathcal{P}(I_\alpha) \); in particular, the cardinality of \( P_\alpha \) is \( 2^M-m \).

Definition 4.11. We naturally identify \( \ell^2(P_\alpha) \) with the exterior algebra \( \Lambda(\ell^2(I_\alpha)) \) on which operates the Clifford algebra \( \text{Cliff}(\ell^2(I_\alpha)) \) generated by \( \gamma_a, a \in I_\alpha \).

5. Dirac operator

Definition 5.1. We define the \( n \)-block length \( \ell_n(.) \) as follows: let \( b \) be a block, then \( \ell_n(b) \) is the minimal number of irreducible blocks decomposing a connected admissible \( n \)-dimensional block \( c \) with \( e \in c \) and, \( b \in c \) or \( b \cap c \neq \emptyset \).

Definition 5.2. Let \( b \) be a block, then a sequence \((c_1, ..., c_{\ell_n(b)}) \) with \( e \in c_1 \), \( c_i \in \Gamma_n \) and \( c_i \cap c_{i+1} \neq \emptyset \) is called a geodesic \( n \)-block-path, from \( b \) to \( e \) beginning with \( c_1 \).

Definition 5.3. For any \( \alpha \in \mathcal{J} \), let \( n = \text{dim}(c_\alpha) + 1 \); for any \( a \in I_\alpha \) define \( p_a(\alpha) \) the number of geodesic \( n \)-block path from \( c_\alpha \) to \( e \) beginning with \( a \); let \( p(\alpha) = \sum_{a \in I_\alpha} p_a(\alpha) \); let \( \lambda_a = \frac{p_a(\alpha)}{p(\alpha)} \ell_n(c_\alpha) \).

Definition 5.4. On \( \ell^2(P_\alpha) = \Lambda(\ell^2(I_\alpha)) \), define the Dirac operator \( D_\alpha \) by:

\[
D_\alpha = \sum_{a \in I_\alpha} \lambda_a \cdot \gamma_a
\]

Remark 5.5. \( P_e := \Delta_e = \{e\} \), \( \ell^2(P_e) = C e_1 \), \( I_e = \emptyset \) and \( D_e = 0 \).

Definition 5.6. Consider then the Hilbert space:

\[
\mathcal{H} = \bigoplus_n \ell^2(\Gamma_n) = \bigoplus_c \ell^2(\Delta_c) = \bigoplus_{a \in \mathcal{J}} \ell^2(\Delta_a) = \bigoplus_{\alpha \in \mathcal{J}} \Lambda(\ell^2(I_\alpha))
\]

\( \mathbb{Z}_2 \)-graded by the decomposition into even and odd dimensional blocks:

\[
\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-
\]

Define the unbounded selfadjoint operator \( \mathcal{D} = \sum_{\alpha \in \mathcal{J}} D_\alpha \).

Lemma 5.7. \( \mathcal{D}^2 = \sum_\alpha \lambda_\alpha^2 \cdot p_\alpha \),

with \( \lambda_\alpha^2 = \sum_{a \in I_\alpha} \lambda_a^2 \) and \( p_\alpha \) the projection on \( \ell^2(\Delta_a) \).

Proof. We use the orthonormal decomposition and the Clifford relations. \( \square \)

Proposition 5.8. \( \mathcal{D}_+ : \mathcal{H}^+ \rightarrow \mathcal{H}^- \) is a Fredholm operator of index 1.

Proposition 5.9. \( (\mathcal{D}^2 + I)^{-1} \) is compact.

For \( t > 0 \), the operator \( e^{-t\mathcal{D}_2} \) is trass-class.
Definition 5.10. For any $g \in \Gamma$ and for any $s \in S$ define $p_s(g)$ the number of geodesic 1-block path from $g$ to $e$ beginning with $\{g, gs\}$; let $p(g) = \sum_{s \in S} p_s(g)$.

Definition 5.11. Let $C$ be the class of finitely presented groups $\Gamma = \langle S \mid R \rangle$ such that $\forall g \in \Gamma, \exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h, gh \neq e$):
$$\left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| \leq \frac{K_g}{\ell_1(h)}$$

Examples 5.12. The class $C$ is stable by direct or free product, it contains $\mathbb{Z}^n$, $\mathbb{F}_n$, the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see [2]).

Proposition 5.13. Let $\Gamma$ of class $C$, $\mathcal{A} = C_r^*(\Gamma)$ and $\mathcal{D}$ as previously then: $\{a \in \mathcal{A} \mid [\mathcal{D}, a] \in B(\mathcal{H})\}$ is dense in $\mathcal{A}$.

Theorem 5.14. $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even $\theta$-summable spectral triple and $\mathcal{D}^+$ is index 1. It then gives a non-trivial element for the $K$-homology of $\mathcal{A}$.

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