Weighted Simultaneous Approximation of the Linear Combinations of Baskakov Operators

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In this paper, the approximation property of the linear combinations of Baskakov operators is investigated by using a Jacobi weight function. Moreover, both the positive and inverse theorems for the weighted simultaneous approximation of the linear combinations of Baskakov operators are proved.

1. Introduction

The functional analysis methods are widely used to study the approximation theory since the 20th century. Due to the combination of functional analysis methods and classical analysis techniques, such approximation theories developed quickly and had formed a theoretical system.

It is well known that the positive and inverse theories are the most significant problems in the operator approximation theorem. In 1972, some pioneering works had been done by H. Berens and G. Lorentz [1], which lead to a hot topic in related research fields. In the meantime, many kinds of approximation tools had been proposed and widely used in practice, such as smooth modulus and K-functional. In particular, the smooth modulus introduced by Ditzian [2] in 1994 contains the related results by using the classic smooth modulus and the Ditzian-Totik modulus. Moreover, such kinds of smooth modulus are usually used to construct an optimal polynomial to approximate a complicated function [3–5].

For any \( f \in C[0, +\infty) \), the corresponding Baskakov operator [6] is defined as

\[
V_n(f, x) = \sum_{k=0}^\infty f \left( \frac{k}{n} \right) V_{nk}(x),
\]

where \( V_{nk}(x) = \binom{n+k-1}{k} x^k (1 + x)^{-n-k}, k = 1, 2, \ldots, n. \)

Linear combination of the Baskakov operator [7] is defined as follows:

\[
V_{nk}(f, r, x) = \sum_{i=0}^{r-1} c_i(n)V_{ni}(f, x),
\]

where \( n_i \) and \( c_i(n) \) are related constants which satisfy the following conditions:

\[
\begin{align*}
(a) & \quad n = n_0 < n_1 < \ldots < n_{r-1} \leq Cn, \\
(b) & \quad \sum_{i=0}^{r-1} |c_i(n)| \leq C, \\
(c) & \quad \sum_{i=0}^{r-1} c_i(n) = 1, \\
(d) & \quad \sum_{i=0}^{r-1} c_i(n)n_i^{-\rho} = 0, \quad \rho = 1, 2, \ldots, r-1.
\end{align*}
\]

Let \( \phi(x) = \sqrt{x(1 + x)}, 0 < \lambda < 1, \delta_\lambda(x) = \phi(x) + 1/\sqrt{n}, \omega(x) = x^a (1 + x)^{-b} (0 < a < 1, b \geq 0), \| \cdot \| \) be the supremum or infimum norm. \( \| f \|_w = \| w f \|_{\mathcal{C}} + | f(0)|, \)

\[
\omega_{\phi^\lambda}(f, t) = \sup_{0 < h \leq t} \| \omega(x) \Delta_{\phi^\lambda}(x)f(x) \|,
\]

where \( V_{nk}(x) = \binom{n+k-1}{k} x^k (1 + x)^{-n-k}, k = 1, 2, \ldots, n. \)
where
\[
\Delta_{n,r}^{\omega}(f,x) = \begin{cases} \sum_{k=0}^{r} \left( \frac{r}{k} \right) f(x + \frac{r-k}{2}h_\omega(x)), x \geq \frac{rh_\omega(x)}{2} \\ 0, & 0 < x < \frac{rh_\omega(x)}{2} \end{cases}.
\]

Hence, the weighted K-functional can be defined as follows:
\[
K_{r,\omega}(f,t) = \inf \left\{ \|w(f-g)\| + t\|w_\omega^{\frac{1}{2}}g^{(r)}\| : g^{(r-1)} \in AC.loc \right\},
\]
and the corrected weighted K-functional is defined as
\[
\tilde{K}_{r,\omega}(f,t) = \inf \left\{ \|w(f-g)\| + t\|w_\omega^{\frac{1}{2}}g^{(r)}\| + t^{r(1-\alpha)/2}\|w_\omega^{\frac{1}{2}}g^{(r)}\| : g^{(r-1)} \in AC.loc \right\}.
\]

According to [8], one can obtain the relationship between the smooth modulus and K-functional
\[
\omega_\rho^r(f,t) \sim K_{r,\omega}(f,t) \sim \omega_{\rho}^r(f,t).
\]
Note that \(f^{(s)} \in C[0, \infty)\) and \(s \in N\), and we have
\[
V_{n,s}^{(s)}(f,x) = \left( \frac{n+s-1}{(n-1)!} \right) \sum_{k=0}^{\infty} \left( \frac{1}{n} \right)^{s} \left( \frac{1}{n} \right)^{s} \ldots \left( \frac{1}{n} \right)^{s} f(x)
\]
\[
\cdot \frac{k}{n} + \sum_{i=1}^{s} u_i \right) du_1 \ldots du_s v_{n,s}(x).
\]
Let \(g \in C[0, \infty)\) and \(s \in N\), and introducing an auxiliary operator
\[
V_{n,s}(g,x) = \left( \frac{n+s-1}{(n-1)!} \right) \sum_{k=0}^{\infty} \left( \frac{1}{n} \right)^{s} \left( \frac{1}{n} \right)^{s} \ldots \left( \frac{1}{n} \right)^{s} g(x)
\]
\[
\cdot \frac{k}{n} + \sum_{i=1}^{s} u_i \right) du_1 \ldots du_s v_{n,s}(x),
\]
then the linear combination of the auxiliary operator can be defined as follows:
\[
V_{n,s}(g,x) = \sum_{i=0}^{r-1} C_i(x)V_{n,s}(g,x),
\]
where \(n_i\) and \(C_i(x)\) are related constants which satisfy (3).
Note that \(V_{n,s}(1, x) = 0\). Assuming \(f^{(s)} \in C[0, \infty)\), we have
\[
V_{n,s}^{(s)}(f^{(s)}, x) = V_{n}^{(s)}(f, x),
\]
\[
V_{n}^{(s)}(f, x) = V_{n,s}^{(s)}(f^{(s)}, x).
\]
The Baskakov operator has been studied in many research studies by using a lot of deep methods [9–13]. In previous work [14, 15], some pointwise results of the Baskakov operator with weighted approximate were obtained.

From the literature [16], some preexisting results are given as follows.

**Theorem 1.** Let \(s \in N, 0 \leq \lambda \leq 1, f^{(s)} \in [0, \infty)\), then we have
\[
\left| V_n^{(s)}(f, x) - f^{(s)} \right| = O\left( \omega_\rho^r\left( f^{(s)}, n^{-1/2}\delta_n^{-1}(x) \right) \right).
\]

**Theorem 2.** Let \(s \in N, 0 \leq \lambda \leq 1, f^{(s)} \in [0, \infty)\), \(s \leq \alpha \leq s + r\), then we have
\[
\left| V_n^{(s)}(f, x) - f^{(s)} \right| = O\left( \left( n^{-1/2}\delta_n^{-1}(x) \right)^{\alpha-s} \right) \iff \omega_\rho^r\left( f^{(s)}, t \right) = O(t^{\alpha-s}).
\]

According to [17], the following theorems hold.

**Theorem 3.** Let \(s \in N, 0 \leq \lambda \leq 1, f^{(s)} \in [0, \infty)\), then we obtain
\[
\omega(x)\left| V_n^{(s)}(f, x) - f^{(s)} \right| = O\left( \omega_\rho^r\left( f^{(s)}, n^{-1/2}\delta_n^{-1}(x) \right) \right).
\]

**Theorem 4.** Let \(s \in N, 0 \leq \lambda \leq 1, f^{(s)} \in [0, \infty)\), \(s \leq \alpha \leq s + r\), then we have
\[
\omega(x)\left| V_n^{(s)}(f, x) - f^{(s)} \right| = O\left( \left( n^{-1/2}\delta_n^{-1}(x) \right)^{\alpha-s} \right) \iff \omega_\rho^r\left( f^{(s)}, t \right) = O(t^{\alpha-s}).
\]

According to [18], we get the following theorems.

**Theorem 5.** Let \(r, s \in N, 0 \leq \lambda \leq 1, f^{(s)} \in [0, \infty)\), then we have
\[
\left| V_n^{(s)}(f, r, x) - f^{(s)} \right| = O\left( \omega_\rho^r\left( f^{(s)}, n^{-1/2}\delta_n^{-1}(x) \right) \right).
\]

**Theorem 6.** Let \(r, s \in N, 0 \leq \lambda \leq 1, f^{(s)} \in [0, \infty)\), \(s \leq \alpha \leq s + r\), then we get
\[
\left| V_n^{(s)}(f, r, x) - f^{(s)} \right| = O\left( \left( n^{-1/2}\delta_n^{-1}(x) \right)^{\alpha-s} \right) \iff \omega_\rho^r\left( f^{(s)}, t \right) = O(t^{\alpha-s}).
\]

Although there are many research studies about Baskakov operator approximation [19], we can further improve these theories. On the basis of the aforementioned research literature, we firstly applied the equivalence relation between the weighted smooth modulus and K-functional to explore the simultaneous approximation of the linear combination of the Baskakov operator with Jacobi weight. And we obtained both the positive and inverse theorems for the weighted simultaneous approximation of the linear combinations of Baskakov operators. Thereby, we unite and expand the results about the existing smooth modulus \(\omega^2(f, t)\), \(\omega^2(f, t)\), \(\omega^2(f, t)\), and \(s = 0\), where \(C\) is a constant.
2. Concepts and Properties

The weight function \( w(x) \) that we used has the following properties.

**Property 1.** Let \( 0 < a < 1, 0 < b, x \leq 0, t \leq u \leq x, \) then we have

\[
\begin{align*}
w^{-1}(u) &\leq C(t^a(1 + x)^b + x^a(1 + t)^b), \\
\Omega_{X}^\gamma(f, t)_{w} &\leq \sup_{0 \leq h \leq t} \|w \Delta^\gamma_h f\|_{\|C(\gamma^{(1-\beta)}_{\infty, \infty})\|}, \quad 0 < \lambda < 1, \\
\Omega_{X}^\gamma(f, t)_{w} &\leq \sup_{0 \leq h \leq t} \|w \Delta^\gamma_h f\|_{\|C(\gamma^{(1-\beta)}_{\infty, \infty})\|}, \quad 0 < \lambda < 1.
\end{align*}
\]

Moreover, weighted smooth modulus and weighted main part smooth modulus have the following connection.

**Property 2.** \( w^\gamma(f, t^{1/(1-\beta)}) w \leq Cw^\gamma(f, t) w, \beta < 1. \)

\[
\begin{align*}
C^{-1} \Omega_{X}^\gamma(f, t)_{w} &\leq \sup_{0 \leq h \leq t} \|w \Delta^\gamma_h f\|_{\|C(\gamma^{(1-\beta)}_{\infty, \infty})\|} \\
\Omega_{X}^\gamma(f, t)_{w} &\leq C \left( \int_{t}^{r} \frac{\Omega_{X}^\gamma(f, \tau) w}{\tau^{r+1}} d\tau + \|w f\| \right).
\end{align*}
\]

In order to prove the cons-theorem, we need to introduce a new K-functional and put some symbols firstly. Let \( 0 < \lambda < 1, 0 < a < \infty, \) then

\[
\begin{align*}
\|f\|_{\lambda} &\leq \sup_{x \in [0, \infty]} \|w(x)\varphi^{(a-\lambda)(\lambda-1)}(x) f(x)\|, \\
\|f\|_{\lambda} &\leq \sup_{x \in [0, \infty]} \|w(x)\varphi^{(a-\lambda)(\lambda-1)}(x) f^{(x)}(x)\|, \\
C_{0} &\equiv \{ f \mid f(x) \in C[0, \infty), \|f\|_{\lambda}, \|f^{(1)}\|_{\lambda} \leq +\infty \}, \\
C_{0} &\equiv \{ f \mid f(x) \in C[0, \infty), \|f\|_{\lambda} \leq +\infty \}.
\end{align*}
\]

A new K-functional is defined:

\[
K_{\lambda}^{\gamma} = \inf_{\frac{\varphi}{\varphi}^{(a-\lambda)(\lambda-1)}(x)} \| \Delta^\gamma_h f \|_{\|C(\gamma^{(1-\beta)}_{\infty, \infty})\|} + \|f\|_{1/2},
\]

(23)

and a new smooth modulus is defined:

\[
\omega_{\varphi}^{\gamma}(f, t)_{a} = \sup_{0 \leq h \leq t} \left\{ \|w \Delta^\gamma_h f\|_{\|C(\gamma^{(1-\beta)}_{\infty, \infty})\|} \right\}.
\]

(24)

3. Pros-Theorem of Simultaneous Approximation of the Linear Combination of the Baskakov Operator

3.1. Auxiliary Lemmas

**Lemma 1** (see [18]). Let \( s \in N, r \in n, m \geq n \), then we have

\[
V_{n,s}(t - x)^m, r, x) = 0, \quad m = 1, 2, \ldots, r - 1,
\]

(25)

\[
V_{n,s}(t - x)^m, x) \leq Cn^{-m}x^{2m}(x).
\]

(26)

**Lemma 2** (see [15]). If \( a \geq 0, b \in R \), then

\[
\sum_{k=1}^{\infty} v_{n,k}(x) \left( \frac{k}{n} \right)^a \left( 1 + \frac{k}{n} \right)^b \leq Cx^a (1 + x)^b.
\]

(27)

According to Lemma 2, we obtain the following corollaries.

**Corollary 1.** When \( a \geq 0, b \in R \), we have

\[
V_{n,s}(t - x)^m, x) \leq Cx^a,
\]

(28)

\[
V_{n,s}(1 + x)^b, x) \leq C(1 + x)^b.
\]

(29)

**Lemma 3** (see [17]). If \( f \in C[0, \infty), s \in N \), then

\[
\|V_{n,s}(f, x)\|_{w} \leq C\|f\|_{w}.
\]

(30)

**Corollary 2.** If \( f \in C[0, \infty), s \in N \), then

\[
\|V_{n,s}(f, r, x)\|_{w} \leq C\|f\|_{w}.
\]

(31)

**Proof.** It follows from (30) that we get

\[
|w(x)V_{n,s}(f, r, x)| \leq \|w(x)\|_{w} \sum_{i=0}^{r-1} C_{i} \|V_{n,s}(f, x)\|_{w} \leq \|f\|_{w}. \]

(32)

\[
\sum_{i=0}^{r-1} C_{i} \|w(x)\|_{w} \|V_{n,s}(f, x)\|_{w} \leq C\|f\|_{w}.
\]

(33)

**Lemma 4**

\[
(1) \quad \text{If } t \leq u \leq x, \ x \in E_{n}^{C}[0, 1/n], \quad \left| t - u \right|^{r-1} \leq \left| t - x \right|^{r-1} \left( \frac{1}{\delta_{n}^{(r)}(u)} + \frac{1}{(1 + x)^{r/2}} + \frac{1}{(1 + t)^{r/2}} \right).
\]

(33)

\[
(2) \quad \text{If } t \leq u \leq x, \quad \left| t - u \right|^{r-1} \leq \left| t - x \right|^{r-1} \left( \frac{1}{x^{r/2}} + \frac{1}{(1 + x)^{r/2}} + \frac{1}{(1 + t)^{r/2}} \right).
\]

(34)

**Proof.**

(1) When \( x < t \), the conclusion is established obviously.

For the case of \( x > t \), let \( f(x) = x/\delta_{n}^{(r)}(x) \), then we have
\[ f'(x) = \frac{\delta_n^2(x) - n\delta_n(x)(\sqrt{x+1}/x + \sqrt{x}(x+1))}{\delta_n^2(x)} \]
\[ = \frac{1}{\sqrt{n} - x\sqrt{x}(x+1)} \frac{\delta_n(x)}{\delta_n^2(x)} \]

(35)

For \( x < 1/n \), \( f'(x) > 0 \), so when \( x \) in \((0, 1/n)\), \( f \) increases. Hence, we have \( u/\delta_n^2(u) < x/\delta_n^2(x) \); for \( x > u \), we obtain \( u^2/\delta_n^2(u) < x^2/\delta_n^2(x) \), \( u^{-2}/\delta_n^{-2}(u) < x^{-2}/\delta_n^{-2}(x) \), and then \( u^{-1}/\delta_n^2(u) < x^{-1}/\delta_n^2(x) \), combined with inequality \( |t - u||x| < |t - x||u| \), then we get

\[ \frac{|t - u|^{-1}}{\delta_n^3(u)} < \frac{|t - x|^{-1}}{\delta_n^3(x)}. \]

(36)

(2) Due to \( 1/(1 + u)^{r/2} \leq (1/(1 + x)^{r/2} + 1/(1 + t)^{r/2}) \), thereby we have

\[ \frac{u^{r/2}}{q'(u)} \leq \frac{x^{r/2}}{x^{r/2} + 1/(1 + t)^{r/2}}. \]

(37)

and combined with inequality \( |t - u||x| < |t - x||u| \), then we get

\[ \frac{|t - u|^{r/2}}{q'(u)} \leq \frac{|t - x|^{r/2}}{x^{r/2}} \left( \frac{1}{(1 + x)^{r/2}} + \frac{1}{(1 + t)^{r/2}} \right). \]

(38)

and then combined with inequality \( |t - u| < |t - x| \), the proposition is established.

\[ \square \]

Lemma 5. Let \( g \in C[0, \infty) \), \( 0 \leq \lambda \leq 1 \), \( R_r(g, t, x) = \int_x^t (t - u)^{-r} g^\lambda(u)du \), \( r \in N \), then we have

\[ w(x)V_{n,r}(R_r(g, t, x)) \leq C\left( n^{-r/2} (1 - 1/3r \lambda) \right) \left\| g^{\lambda \lambda} \left\|_{w} \right\| \right\|_{w} \]

(39)

Proof

(1) If \( t \leq u \leq x \), and \( x \in E_n^{1/2}(0, 1/n) \) using Lemma 4, we get

\[ w(x)\left| V_{n,r}(R_r(g, t, x)) \right| \leq Cw(x) \left( \int_x^t (t - u)^{-1} g^{\lambda}(u)du, x \right) \]

\[ \leq Cw(x) \left( \delta_n^{1/2} g^{\lambda}(x) \right) \left( \int_x^t (t - u)^{-1} \delta_n^{1/2} g^{\lambda}(u)du, x \right) \]

\[ \leq Cw(x) \left( \delta_n^{1/2} g^{\lambda}(x) \right) \left( \int_x^t (t - u)^{-1} w^{1/2}(u)du, x \right) \]

\[ \leq Cw(x) \left( \delta_n^{1/2} g^{\lambda}(x) \right) \left( \int_x^t (t - u)^{-1} w^{1/2}(u)du, x \right) \]

\[ \leq Cw(x) \left( \delta_n^{1/2} g^{\lambda}(x) \right) \left( \left| t - x \right|^{1/2} \left( 1 + x \right)^{b}, x \right) \]

\[ \leq Cn^{-r/2}(1 - 1/3r \lambda) \left( \delta_n^{1/2} g^{\lambda}(x) \right) \left( \left| t - x \right|^{1/2} \left( 1 + x \right)^{b}, x \right) \]

(40)

In the proof process above, we used the Hölder inequality and the conclusion of formula (26). i.e.,

\[ V_{n,a}(\left| t - x \right|^{1/2} \left( 1 + x \right)^{b}, x) \]

\[ \leq C \left( V_{n,a}(\left| t - x \right|^{2}, x) \right)^{1/2} \left( V_{n,a}(\left( 1 + x \right)^{b}, x) \right)^{1/2} \]

\[ \leq C \left( V_{n,a}(\left| t - x \right|^{2}, x) \right)^{1/2} \left( V_{n,a}(\left| t - x \right|^{2}, x) \right)^{1/2} \]

\[ \leq Cn^{-r/2}(1 - 1/3r \lambda) \left( \left| t - x \right|^{1/2} \left( 1 + x \right)^{b}, x \right) \]

(41)

Similarly, we obtain

\[ V_{n,a}(\left| t - x \right|^{1/2} \left( 1 + x \right)^{b}, x) \leq Cn^{-r/2}(1 - 1/3r \lambda) \left( \left| t - x \right|^{1/2} \left( 1 + x \right)^{b}, x \right) \]

(42)

(2) When \( x \in (1/n, \infty) \), by using the Hölder inequality, (26), (28), and (29) and the conclusion of [15], we can get
\[ w(x)\mid V_{n,s}(R, (g, t, x), x) \mid \]
\[ = w(x)\left( \int_{x}^{t} (t-u)^{-1} g^{(r)}(u)du, x \right) \]
\[ \leq Cw(x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w} V_{n,s}\left( \int_{x}^{t} |t-u|^{-1} \delta_{n}^{\lambda} (u)w^{-1}(u)du, x \right) \]
\[ \leq Cw(x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w} V_{n,s}\left( \frac{|x-t|}{1 + x^{\lambda/2}} \left( \frac{1}{(1 + t)^{\lambda/2}} + \frac{1}{(1 + t)^{\lambda/2}} \right) (t^{-a}(1 + x)^{b} + x^{-a}(1 + t)^{b}), x \right) \]
\[ \leq Cw(x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w} V_{n,s}\left( \frac{|x-t|}{(1 + t)^{\lambda/2}} (t^{-a}(1 + x)^{b} + x^{-a}(1 + t)^{b}), x \right) \]
\[ + V_{n,s}\left( \frac{|x-t|}{(1 + t)^{\lambda/2}} (t^{-a}(1 + x)^{b} + x^{-a}(1 + t)^{b}), x \right) \]
\[ \leq Cn^{-r/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w}. \]

Therefore,
\[ Cn^{-r/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w} \]
\[ \leq Cn^{-r/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \left( \left\| \phi_{\lambda}^{\lambda} g^{(r)} \right\|_{w} \right) \]
\[ + n^{-\lambda/2} \left\| g^{(r)} \right\|_{w} \]
\[ \leq C\left( n^{-r/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \left\| \phi_{\lambda}^{\lambda} g^{(r)} \right\|_{w} \right) \]
\[ + \left( n^{-\lambda/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \right)^{r/(1-\lambda/2)} \left\| g^{(r)} \right\|_{w}. \]

and the proposition is established. □

### 3.2. Positive Theorem

**Theorem 7.** Let \( s \in N, 0 \leq \lambda \leq 1, f^{(s)} \in C[0, \infty), s \leq \alpha \leq s + r, \) then we get
\[ w(x)\mid V_{n,s}^{(s)}(f, r, x) - f^{(s)}(x) \mid \]
\[ = O\left( w_{\phi}^{\phi}(f^{(s)}, n^{-1/2} \delta_{n}^{\lambda(1-\lambda)} (x))u \right). \]

**Proof.** For \( g^{(r-1)} \in C[0, \infty), \) let \( g(t) \) Taylor expand in x,
\[ g(t) = g(x) + (t-x)g^{(1)}(x) + \cdots \]
\[ + \frac{(t-x)^{r-1}}{(r-1)!} g^{(r-1)}(x) + \frac{1}{(r-1)!} \int_{x}^{t} (t-u)^{r-1} g^{(r)}(u)du. \]

It follows from (25) that we obtain
\[ w(x)\mid V_{n,s}^{(s)}(f, r, x) - f^{(s)}(x) \mid \]
\[ = w(x)\left( \int_{x}^{t} (t-u)^{-1} g^{(r)}(u)du, x \right) \]
\[ \leq Cw(x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w} V_{n,s}\left( \int_{x}^{t} |t-u|^{-1} \delta_{n}^{\lambda} (u)w^{-1}(u)du, x \right) \]
\[ \leq Cw(x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w} V_{n,s}\left( \frac{|x-t|}{1 + x^{\lambda/2}} \left( \frac{1}{(1 + t)^{\lambda/2}} + \frac{1}{(1 + t)^{\lambda/2}} \right) (t^{-a}(1 + x)^{b} + x^{-a}(1 + t)^{b}), x \right) \]
\[ \leq Cw(x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w} V_{n,s}\left( \frac{|x-t|}{(1 + t)^{\lambda/2}} (t^{-a}(1 + x)^{b} + x^{-a}(1 + t)^{b}), x \right) \]
\[ + V_{n,s}\left( \frac{|x-t|}{(1 + t)^{\lambda/2}} (t^{-a}(1 + x)^{b} + x^{-a}(1 + t)^{b}), x \right) \]
\[ \leq Cn^{-r/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \left\| \delta_{n}^{\lambda} g^{(r)} \right\|_{w}. \]

Then, for \( f^{(s)} \in C[0, \infty) \) and all of \( g^{(r-1)} \in A.C.\text{loc}, \) we have
\[ w(x)\mid V_{n,s}^{(s)}(f, r, x) - f^{(s)}(x) \mid \]
\[ \leq w(x)\mid V_{n,s}^{(s)}(f - g, r, x) \mid + w(x)\mid f^{(s)}(x) - g(x) \mid + w(x)\mid V_{n,s}(f, r, x) - g(x) \mid \]
\[ \leq C\left( \left\| f^{(s)}(x) - g(x) \right\|_{w} + C(n^{-r/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \left\| \phi_{\lambda}^{\lambda} g^{(r)} \right\|_{w} \right) \]
\[ + \left( n^{-\lambda/2} \delta_{n}^{\lambda(1-\lambda)r} (x) \right)^{r/(1-\lambda/2)} \left\| g^{(r)} \right\|_{w}. \]

**Theorem 8.** Let \( s \in N, 0 \leq \lambda \leq 1, x \in [0, \infty), f^{(s)} \in C[0, \infty), s \leq \alpha \leq s + r, \) and if \( w_{\phi}^{\phi}(f^{(s)}, t)u = O(t^{\alpha-\epsilon}), \) then we have
\[ w(x)\mid V_{n,s}^{(s)}(f, r, x) - f^{(s)}(x) \mid = O\left( \left( n^{-1/2} \delta_{n}^{\lambda(1-\lambda)} (x) \right)^{\alpha-\epsilon} \right). \]
Proof. From Theorem 7, we obtain
\[
\|w(x)\|_{\mathcal{V}_{n,s}(f,r,x) - f^{(q)}(x)} = O\left(\omega^{(q)}(f^{(q)}, n^{-1/2} s^{1/2} (x))\right)
\]
\[
= O\left(\left(n^{-1/2} s^{1/2} (x)\right)^{a-s}\right).
\]
(50)

\[\square\]

4. Cons-Theorem of Simultaneous Approximation of the Linear Combination of the Baskakov Operator

4.1. Base Lemmas

Lemma 6. If \(0 \leq \lambda \leq 1, r \in N, s \leq \alpha \leq s + r, f \in C^\alpha\), we have
\[
\|\mathcal{V}_{n,s}(f,r,x)\|_r \leq C n^{r/2} f_0.
\]
(51)

Proof

(1) When \(x \in [0,1/n]\), we have \(\phi(x) \leq 1/\sqrt{n}\), and then
\[
\|\mathcal{V}_{n,s}^{(r)}(f,x)\| \leq C n^{r/2} \sum_{k=0}^\infty \mathcal{V}_{n+r,k}(x) \int_0^{1/n} \ldots \int_0^{1/n} \sum_{j=0}^r (-1)^j \binom{r}{j} f\left(\frac{k+r-j}{n} + \sum_{i=1}^s u_i\right) du_1 \ldots du_r.
\]
(52)

Firstly, considering the first term of
\[
\sum_{j=0}^r (-1)^j \binom{r}{j} f\left(\frac{k+r-j}{n} + \sum_{i=1}^s u_i\right) du_1 \ldots du_r,
\]
the following part is similar, and by using the methods of [17] and the H"older inequality, we get
\[
\|\mathcal{V}_{n,s}^{(r)}(f,x)\| \leq C n^{r/2} \sum_{k=0}^\infty \mathcal{V}_{n+r,k}(x) \int_0^{1/n} \ldots \int_0^{1/n} \sum_{j=0}^r (-1)^j \binom{r}{j} f\left(\frac{k+r-j}{n} + \sum_{i=1}^s u_i\right) du_1 \ldots du_r
\]
\[
\leq C \|f\| \|n^{r/2} \cdot \|\mathcal{V}_{n+r,k}(x)\| \left(\sum_{k=0}^\infty \binom{r}{j} f\left(\frac{k+r-j}{n} + \sum_{i=1}^s u_i\right) du_1 \ldots du_r\right).
\]
(53)

Thus, we obtain \(\|\mathcal{V}_{n,s}(f,r,x)\|_r \leq C n^{r/2} f_0\). Therefore,
\[
\|\mathcal{V}_{n,s}(f,r,x)\|_r \leq C n^{r/2} f_0.
\]
(54)

(2) When \(x \in [1/n, \infty)\), similar to the proof of [8], we have
\[
\|\mathcal{V}_{n,s}^{(r)}(f,x)\| \leq C \|f\| \|n^{r/2} \cdot \|\mathcal{V}_{n+r,k}(x)\| \left(\sum_{k=0}^\infty \binom{r}{j} f\left(\frac{k+r-j}{n} + \sum_{i=1}^s u_i\right) du_1 \ldots du_r\right).
\]
(55)

where \(I_1\) is the situation when \(k = 0\), and \(I_2\) is the last situations. Let \(p \in N, s \leq \alpha \leq \alpha - (1-\lambda)/2\), by the H"older inequality, then we have
\[
I_2 \leq C \|f\| \|n^{r/2} \cdot \|\mathcal{V}_{n+r,k}(x)\| \left(\sum_{k=0}^\infty \binom{r}{j} f\left(\frac{k+r-j}{n} + \sum_{i=1}^s u_i\right) du_1 \ldots du_r\right).
\]
(56)

Then, we obtain \(\|\mathcal{V}_{n,s}(f,r,x)\|_r \leq C n^{r/2} f_0\); therefore,
\[
\|\mathcal{V}_{n,s}(f,r,x)\|_r \leq C n^{r/2} f_0.
\]
(57)

Lemma 7. Supposing that \(0 \leq \lambda \leq 1, r \in N, s \leq \alpha \leq s + r, \), we get
\[
\|\mathcal{V}_{n,s}(f,r,x)\|_r \leq C \|f\|_r.
\]
(58)
Proof. Supposing that \( F(x) \) satisfies \( F^s(x) = f(x) \), and then by reference [8], we have
\[
\left| w(x)\varphi^{(r(a-s))(1-\lambda)}(x)V_{n,s}^{(r)}(f,x) \right|
\leq w(x)\varphi^{(r(a-s))(1-\lambda)}(x) \frac{n^r(n-s-1)}{(n-s-1)!} V_{n,s}^{(r+s)}(F,x)
\leq Cn^{r+s}w(x)\varphi^{(r(a-s))(1-\lambda)}(x) \sum_{k=0}^{\infty} \Delta_{1/n}^{r+s} \frac{k}{n} v_{n,r+s,k}(x),
\]
where \( \Delta_{1/n}^{r+s} F(k/n) = \sum_{j=0}^{r+s} (-1)^j \binom{r+s}{j} F((k+r+s-j)/n) \), and when \( k \geq 1 \), we get
\[
n^{r+s} \left| \Delta_{1/n}^{r+s} F(k/n) \right| \leq \sup_{k/n \in [k/(k+r+s)]/n} \left| V_{n,s}^{(r+s)}(x) \right|
\leq C \frac{k}{n} \lambda^{-((r(a-s))(1-\lambda))/2} \| f \|_{r,s}.
\]

When \( k = 0 \), using the method of [8], we can obtain
\[
n^{r+s} \left| \Delta_{1/n}^{r+s} F(0) \right| \leq Cw^{(-1)}(x)n^{-((r(a-s))(1-\lambda))/2} \| f \|_{r,s}.
\]

So, using the Hölder inequality and Lemma 2, we get
\[
\left| w(x)\varphi^{(r(a-s))(1-\lambda)}(x)V_{n,s}^{(r)}(f,x) \right|
\leq Cw(x)\varphi^{(r(a-s))(1-\lambda)}(x) \| f \|_r \| \sum_{k=0}^{\infty} \frac{k}{n} \lambda^{-((r(a-s))(1-\lambda))/2} \| v_{n,r+s,k}(x) \|
\leq C \| f \|_r.
\]

But \( \| V_{n,s}(f,r,x) \| \leq C \| V_{n,s}(f,s,x) \| \leq C \| f \|_r \), and thereby the conclusion is established.

\[ \square \]

Lemma 8 (see [20]). Supposing \( r \in N, 0 \leq \lambda \leq 1, 0 < h < 1/(2r), x > rh^{\frac{1}{2}}(x)/2 \), then we have
\[
\begin{align*}
\omega^{(r)}(x) \varphi^{(r(a-s))(1-\lambda)}(x) & = \mathcal{O} \left( \left( n^{-1/2} \delta_n^{\lambda-1}(x) \right)^{a-s} \right) \\
\iff \omega^{(r)}(f,t) & = \mathcal{O}(r^{a-s}).
\end{align*}
\]

4.2. Cons-Theorem

Theorem 9. Supposing \( s \in N, 0 \leq \lambda \leq 1, x \in [0,\infty), f^{(i)} \in [0,\infty), s \leq a \leq s + r, \) then we have
\[
\omega^{(r)}(x) V_{n,s}^{(r)}(f,r,x) - f^{(i)} = \mathcal{O} \left( \left( n^{-1/2} \delta_n^{\lambda-1}(x) \right)^{a-s} \right).
\]

Proof. By the definition of \( K_{\lambda}^{(a-s)}(f,t)_w \), selecting a suitable \( g \), which satisfies the following formula
\[
\left\| f^{(i)} - g \right\|_0 + n^{-r/2} \| g \|_r \leq 2K_{\lambda}^{(a-s)}(f,n^{-r/2})_w,
\]
by the known condition
\[
\omega^{(r)}(x) V_{n,s}^{(r)}(f,r,x) - f^{(i)} = \mathcal{O} \left( \left( n^{-1/2} \delta_n^{\lambda-1}(x) \right)^{a-s} \right),
\]
we have
\[
\omega^{(r)}(x) \varphi^{(r(a-s))(1-\lambda)}(x) \| V_{n,s}(f,r,x) - f^{(i)}(x) \| \leq Cn^{-(a-s)/2}
\implies \| V_{n,s}^{(r)}(f,r,x) - f^{(i)}(x) \|_0 \leq Cn^{-(a-s)/2}.
\]

By using Lemmas 6 and 7, we obtain
\[
K_{\lambda}^{(a-s)}(f^{(i)},t)_w \leq \left\| V_{n,s}^{(r)}(f^{(i)},r,x) - f^{(i)}(x) \right\|_0 + t' \left\| V_{n,s}(f^{(i)},r,x) \right\|_r
\leq Cn^{-(a-s)/2} + t' \left( \| V_{n,s}(f^{(i)} - g,r,x) \|_r + \left\| V_{n,s}(g,r,x) \right\|_r \right)
\leq C \left( n^{-(a-s)/2} + t' n^{-r/2} \right) \| f^{(i)} - g \|_0 + t' \| g \|_r
\leq C \left( n^{-(a-s)/2} + t' n^{-r/2} \right) K_{\lambda}^{(a-s)}(f,n^{-r/2})_w.
\]

By using the Berens-Lorentz lemma [8], we get
\[
K_{\lambda}^{(a-s)}(f^{(i)},t)_w \leq C \| f \|_w. \quad \text{Especially, when } \lambda = 1, \text{ we can obtain } K_{\lambda}^{(a-s)}(f^{(i)},t)_w = K_{\varphi}(f^{(i)},t)_w \leq C \| f \|_w, \quad \text{i.e.,}
\omega^{(r)}(f^{(i)},t) = \mathcal{O}(r^{a-s}). \quad \text{By contrast, when } 0 \leq \lambda \leq 1, x \geq rh^{\frac{1}{2}}(x)/2, x \in [ (rh)^{1/(1-\lambda)}, \infty), \text{ for } f \in C_{\lambda,w},
\]
by using Lemma 8, we get
\[
\omega^{(r)}(f^{(i)},t) = \mathcal{O}(r^{a-s}).
\]
Conflicts of Interest
and there are no experimental data.

\textbf{Theorem 10.} Equivalence Theorem

5 Complexity

\begin{equation}
\begin{aligned}
w(x)\left|\Delta_{h(x)}^{\alpha}(x) - f(x)\right| & \\
& \leq w(x)\left|\Delta_{h(x)}^{\alpha}(x) - g(x)\right| + w(x)\left|\Delta_{h(x)}^{\alpha}(x)/g(x)\right| \\
& = w(x)\left(\sum_{k=0}^{r} (-1)^{k} r_{\Delta_{h(x)}^{\alpha}(x)} \left(f^{(k)}(x) - g(x)\right) + \frac{x + (r - k) h(x)}{x + 2/k} \right) \\
& + \left(\sum_{k=0}^{r} (-1)^{k} r_{\Delta_{h(x)}^{\alpha}(x)} \left|\Delta_{h(x)}^{\alpha}(x)\right| \right) \\
& \leq C \Phi_{\alpha}(\lambda, 1-\lambda) \left(\|f - g\|_0 + \|g\|_{\beta'} \Phi_{\alpha}(\lambda, 1-\lambda)\right) \\
& \leq C \Phi_{\alpha}(\lambda, 1-\lambda) \left(\frac{h_{\beta'}(\lambda, 1-\lambda)}{\Phi_{\alpha}(\lambda, 1-\lambda)}\right) \\
& \leq C h_{\alpha-s}.
\end{aligned}
\end{equation}

Then, \( \Omega_{\phi}^{r}(f^{(s)}, t) \), and thereby

\begin{equation}
\omega_{\phi}^{r}(f^{(s)}, t) \leq C \int_{0}^{t} \frac{\Omega_{\phi}^{r}(f^{(s)}, t)}{\tau} d\tau
\end{equation}

\begin{equation}
\leq C \int_{0}^{t} \tau^{\alpha-s-1} d\tau \leq C t^{\alpha-s}.
\end{equation}

\textbf{5. Equivalence Theorem}

\textbf{Theorem 10.} Supposing \( s \in N, 0 \leq \lambda \leq 1, x \in [0, \infty), f^{(s)} \in C[0, \infty), s \leq \alpha \leq s + r, \) then the following propositions are equivalent:

\begin{enumerate}
\item \( w(x)\left|V_{n}^{(s)}(f, r, x) - f(x)\right| = O\left((n^{-1/2})^{\lambda} \Phi_{\alpha}(\lambda, 1-\lambda)\right)\) \hspace{1cm} (71)
\item \( \omega_{\phi}^{r}(f^{(s)}, t) \leq O\left(t^{\alpha-s}\right)\) \hspace{1cm} (72)
\item \( \omega_{\phi}^{(r)}(f^{(s)}, t) \leq O\left(t^{\alpha}\right)\) \hspace{1cm} (73)
\end{enumerate}

\textbf{Proof.} It follows from Theorem 8, and we know (71) \( \Longrightarrow \) (72) is established. And from Theorem 9, (72) \( \Longrightarrow \) (71) is established. Finally, by reference [8], we know (72) \( \Longrightarrow \) (73) is established, too.

\textbf{Data Availability}

Mathematical deduction is the main method in this paper, and there are no experimental data.

\textbf{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

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