On the Positive Moments of Ranks of Partitions

William Y.C. Chen¹, Kathy Q. Ji² and Erin Y.Y. Shen³

¹,²,³Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

¹Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

¹chen@nankai.edu.cn, ²ji@nankai.edu.cn, ³shenyiying@mail.nankai.edu.cn

Abstract. By introducing $k$-marked Durfee symbols, Andrews found a combinatorial interpretation of $2^k$-th symmetrized moment $\eta_{2^k}(n)$ of ranks of partitions of $n$ in terms of $(k+1)$-marked Durfee symbols of $n$. In this paper, we consider the $k$-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks of partitions of $n$ which is defined as the truncated sum over positive ranks of partitions of $n$. As combinatorial interpretations of $\bar{\eta}_{2^k}(n)$ and $\bar{\eta}_{2^k-1}(n)$, we show that for fixed $k$ and $i$ with $1 \leq i \leq k + 1$, $\bar{\eta}_{2^k-1}(n)$ equals the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$-th rank being zero and $\bar{\eta}_{2^k}(n)$ equals the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$-th rank being positive. The interpretations of $\bar{\eta}_{2^k-1}(n)$ and $\bar{\eta}_{2^k}(n)$ also imply the interpretation of $\eta_{2^k}(n)$ given by Andrews since $\eta_{2^k}(n)$ equals $\bar{\eta}_{2^k-1}(n)$ plus twice of $\bar{\eta}_{2^k}(n)$. Moreover, we obtain the generating functions of $\bar{\eta}_{2^k}(n)$ and $\bar{\eta}_{2^k-1}(n)$.

Keywords: rank of a partition, $k$-marked Durfee symbol, moment of ranks

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1 Introduction

This paper is concerned with a combinatorial study of the symmetrized positive moments of ranks of partitions. The notion of symmetrized moments was introduced by Andrews [1]. The odd symmetrized moments are zero due to the symmetry of ranks. For even symmetrized moments, Andrews found a combinatorial interpretation by introducing $k$-marked Durfee symbols. It is natural to investigate the combinatorial interpretation of the odd symmetrized moments which are truncated sum over positive ranks of partitions of $n$. We give combinatorial interpretations of the even and odd positive moments in terms of $k$-marked Durfee symbols, which also lead to the combinatorial interpretation of the even symmetrized moments of ranks given by Andrews.

The rank of a partition $\lambda$ introduced by Dyson [6] is defined as the largest part minus the number of parts. Let $N(m, n)$ denote the number of partitions of $n$ with rank $m$. The generating function of $N(m, n)$ is given by
Theorem 1.1 (Dyson-Atkin-Swinnerton-Dyer [3]). For fixed integer $m$, we have
\[
\sum_{n=0}^{+\infty} N(m, n)q^n = \frac{1}{(q; q)_\infty} \sum_{n=1}^{+\infty} (-1)^{n-1}q^{n^{(3n-1)/2+|m|n}}(1 - q^n).
\] (1.1)

Recently, Andrews [1] introduced the $k$-th symmetrized moment $\eta_k(n)$ of ranks of partitions of $n$ as given by
\[
\eta_k(n) = \sum_{m=-\infty}^{+\infty} \left( m + \left\lfloor \frac{k-1}{2} \right\rfloor \right) N(m, n).
\] (1.2)

It can be easily seen that for given $k$, $\eta_k(n)$ is a linear combination of the moments $N_j(n)$ of ranks given by Atkin and Garvan [4]
\[
N_j(n) = \sum_{m=-\infty}^{+\infty} mv^j N(m, n).
\]

For example,
\[
\eta_6(n) = \frac{1}{720}N_6(n) - \frac{1}{144}N_4(n) + \frac{1}{180}N_2(n).
\]

In view of the symmetry $N(-m, n) = N(m, n)$, we have $\eta_{2k+1}(n) = 0$. As for the even symmetrized moments $\eta_{2k}(n)$, Andrews gave the following combinatorial interpretation by introducing $k$-marked Durfee symbols. For the definition of $k$-marked Durfee symbols, see Section 2.

Theorem 1.2 (Andrews [1]). For fixed $k \geq 1$, $\eta_{2k}(n)$ is equal to the number of $(k + 1)$-marked Durfee symbols of $n$.

Andrews [1] proved the above theorem by using the $k$-fold generalization of Watson’s $q$-analog of Whipple’s theorem. Ji [8] gave a combinatorial proof of Theorem 1.2 by establishing a map from $k$-marked Durfee symbols to ordinary partitions. Kursungoz [9] provided another proof of Theorem 1.2 by using an alternative representation of $k$-marked Durfee symbols.

In this paper, we introduce the $k$-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks as given by
\[
\bar{\eta}_k(n) = \sum_{m=1}^{+\infty} \left( m + \left\lfloor \frac{k-1}{2} \right\rfloor \right) N(m, n),
\]
or equivalently,
\[
\bar{\eta}_{2k-1}(n) = \sum_{m=1}^{+\infty} \left( m + k - 1 \right) 2k - 1 N(m, n) \] (1.3)
and
\[ \bar{\eta}_{2k}(n) = \sum_{m=1}^{\infty} \left( \frac{m + k - 1}{2k} \right) N(m, n). \] (1.4)

Furthermore, it is easy to see that for given \( k \), \( \bar{\eta}_k(n) \) is a linear combination of the positive moments \( \bar{N}_j(n) \) of ranks introduced by Andrews, Chan and Kim \cite{2} as given by
\[ \bar{N}_j(n) = \sum_{m=1}^{\infty} m^j N(m, n). \]

For example,
\[ \bar{\eta}_4(n) = \frac{1}{24} \bar{N}_4(n) - \frac{1}{12} \bar{N}_3(n) - \frac{1}{24} \bar{N}_2(n) + \frac{1}{12} \bar{N}_1(n), \]
\[ \bar{\eta}_5(n) = \frac{1}{120} \bar{N}_5(n) - \frac{1}{24} \bar{N}_3(n) + \frac{1}{30} \bar{N}_1(n). \]

By the symmetry \( N(-m, n) = N(m, n) \), it is readily seen that
\[ \eta_{2k}(n) = 2\bar{\eta}_{2k}(n) + \bar{\eta}_{2k-1}(n). \] (1.5)

The main objective of this paper is to give combinatorial interpretations of \( \bar{\eta}_{2k}(n) \) and \( \bar{\eta}_{2k-1}(n) \). We show that for given \( k \) and \( i \) with \( 1 \leq i \leq k + 1 \), \( \bar{\eta}_{2k-1}(n) \) equals the number of \((k + 1)\)-marked Durfee symbols of \( n \) with the \( i \)-th rank being zero and \( \bar{\eta}_{2k}(n) \) equals the number of \((k + 1)\)-marked Durfee symbols of \( n \) with the \( i \)-th rank being positive. It should be noted that \( \bar{\eta}_{2k-1}(n) \) and \( \bar{\eta}_{2k}(n) \) are independent of \( i \) since the ranks of \( k \)-marked Durfee symbols are symmetric, see Andrews \cite[Corollary 12]{1}.

With the aid of Theorem 2.1 and Theorem 2.2 together with the generating function (1.1) of \( N(m, n) \), we obtain the generating functions of \( \bar{\eta}_{2k}(n) \) and \( \bar{\eta}_{2k-1}(n) \).

## 2 Combinatorial interpretations

In this section, we give combinatorial interpretations of \( \bar{\eta}_{2k-1}(n) \) and \( \bar{\eta}_{2k}(n) \) in terms of the \( k \)-marked Durfee symbols. For a partition \( \lambda \), we write \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \), so that \( \lambda_1 \) is the largest part and \( \lambda_s \) is the smallest part of \( \lambda \). Recall that a \( k \)-marked Durfee symbol of \( n \) introduced by Andrews \cite{1} is a two-line array composed of \( k \) pairs \((\alpha^i, \beta^i)\) of partitions along with a positive integer \( D \) which is represented in the following form:
\[ \tau = \left( \begin{array}{ccccc} \alpha^k & \alpha^{k-1} & \ldots & \alpha^1 \\ \beta^k & \beta^{k-1} & \ldots & \beta^1 \end{array} \right)_D, \]
where the partitions \( \alpha^i \) and \( \beta^i \) satisfy the following four conditions:
(1) The partitions $\alpha^i$ ($1 \leq i < k$) are nonempty, while $\alpha^k$ and $\beta^i$ ($1 \leq i \leq k$) are allowed to be empty;

(2) $\beta^{i-1}_1 \leq \alpha^{i-1}_1 \leq \min\{\alpha^i_s, \beta^i_s\}$ for $2 \leq i \leq k$;

(3) $\alpha^k_1, \beta^k_1 \leq D$;

(4) $\sum_{i=1}^k (|\alpha^i| + |\beta^i|) + D^2 = n$.

Let $\tau = \left( \begin{array}{cccc} \alpha^k_1 & \alpha^{k-1}_1 & \ldots & \alpha^1_1 \\ \beta^k_1 & \beta^{k-1}_1 & \ldots & \beta^1_1 \end{array} \right)_D$

be a $k$-marked Durfee symbol. The pair $(\alpha^i, \beta^i)$ of partitions is called the $i$-th vector of $\tau$. Andrews defined the $i$-th rank $\rho_i(\tau)$ of $\tau$ as follows

$$\rho_i(\tau) = \begin{cases} \ell(\alpha^i) - \ell(\beta^i) - 1, & \text{for } 1 \leq i < k, \\ \ell(\alpha^k) - \ell(\beta^k). & \text{for } i = k. \end{cases}$$

For example, consider the following 3-marked Durfee symbol $\tau$.

$$\tau = \left( \begin{array}{cccc} \alpha^2_1 & \alpha^2_2 & \alpha^2_3 \\ 3_3 & 4_2, 3_2 & 3_2 & 2_2 \\ \beta^3_1 & 3_2, 2_2, 2_2 & 2_1, 2_1 \end{array} \right)_5.$$

We have $\rho_1(\tau) = -2$, $\rho_2(\tau) = 0$, and $\rho_3(\tau) = 1$.

For odd symmetrized moments $\bar{\eta}_{2k-1}(n)$, we have the following combinatorial interpretation.

**Theorem 2.1.** For fixed positive integers $k$ and $i$ with $1 \leq i \leq k+1$, $\bar{\eta}_{2k-1}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$-th rank equal to zero.

For the even case, we have the following interpretation.

**Theorem 2.2.** For fixed positive integers $k$ and $i$ with $1 \leq i \leq k+1$, $\bar{\eta}_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$-th rank being positive.

The proofs of the above two interpretations are based on the following partition identity given by Ji [3]. We shall adopt the notation $D_k(m_1, m_2, \ldots, m_k; n)$ as used by Andrews [11] to denote the number of $k$-marked Durfee symbols of $n$ with $i$-th rank equal to $m_i$.

**Theorem 2.3.** Given $k \geq 2$ and $n \geq 1$, we have

$$D_k(m_1, m_2, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1} = 0}^{\infty} N \left( \sum_{i=1}^k |m_i| + 2 \sum_{i=1}^{k-1} t_i + k - 1, n \right). \quad (2.1)$$
To prove the above two interpretations, we also need the following symmetric property
given by Andrews [1]. Boulet and Kursungoz [5] found a combinatorial proof of this fact.

**Theorem 2.4.** For \( k \geq 2 \) and \( n \geq 1 \), \( D_k(m_1, \ldots, m_k; n) \) is symmetric in \( m_1, m_2, \ldots, m_k \).

We are now in a position to prove Theorem 2.1 and Theorem 2.2 with the aid of
Theorem 2.3 and Theorem 2.4.

**Proof of Theorem 2.1.** By Theorem 2.4, it suffices to show that
\[
\sum_{m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n) = \bar{\eta}_{2k-1}(n). \tag{2.2}
\]
Using Theorem 2.3, we get
\[
\sum_{m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n)
= \sum_{m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{t_1, \ldots, t_k = 0}^{\infty} N \left( \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k, n \right). \tag{2.3}
\]

Given \( k \) and \( n \), let \( c_k(n) \) denote the number of integer solutions to the equation
\[
|m_2| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = n,
\]
where the variables \( m_i \) are integers and the variables \( t_i \) are nonnegative integers. It is
easy to see that the generating function of \( c_k(n) \) is equal to
\[
\sum_{n=0}^{\infty} c_k(n) q^n = (1 + 2q + 2q^2 + 2q^3 + \cdots)^k (1 + q^2 + q^4 + q^6 + \cdots)^k
= \left( \frac{1 + q}{1 - q} \right)^k \left( \frac{1}{1 - q^2} \right)^k
= \frac{1}{(1 - q)^{2k}}
= \sum_{n=0}^{\infty} \binom{n + 2k - 1}{2k - 1} q^n. \tag{2.4}
\]
Equating the coefficients of \( q^n \) on the both sides of (2.4), we get
\[
c_k(n) = \binom{n + 2k - 1}{2k - 1},
\]
that is,
\[
c_k(m - k) = \binom{m + k - 1}{2k - 1}.
\]
Thus, \((2.3)\) can be written as
\[
\sum_{m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n)
= \sum_{m=1}^{\infty} \binom{m + k - 1}{2k - 1} N(m, n)
\]
which is equal to \(\bar{\eta}_{2k-1}(n)\). This completes the proof.

**Proof of Theorem 2.2.** Similarly, by Theorem 2.4 it is enough to show that
\[
\sum_{m_1 > 0, m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n) = \bar{\eta}_{2k}(n). \tag{2.5}
\]

Using Theorem 2.3 we get
\[
\sum_{m_1 > 0, m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n)
= \sum_{m_1 > 0}^{\infty} \sum_{m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} N \left( m_1 + \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k, n \right). \tag{2.6}
\]

Given \(k\) and \(n\), let \(\bar{c}_k(n)\) denote the number of integer solutions to the equation
\[
m_1 + |m_2| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = n,
\]
where the variable \(m_1\) is a positive integer, the variables \(m_i\) \((2 \leq i \leq k + 1)\) are integers and the variables \(t_i\) are nonnegative integers. An easy computation shows that
\[
\sum_{n=0}^{\infty} \bar{c}_k(n)q^n = \frac{q}{(1 - q)^{2k+1}}, \tag{2.7}
\]
so that
\[
\bar{c}_k(n) = \binom{n + 2k - 1}{2k}.
\]

We write
\[
\bar{c}_k(m - k) = \binom{m + k - 1}{2k}.
\]
It follows that
\[
\sum_{m_1 > 0, m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n)
= \sum_{m=1}^{\infty} \binom{m + k - 1}{2k} N(m, n),
\]
which equals \(\bar{\eta}_{2k}(n)\), as required.

Note that the number \(D_k(m_1, \ldots, m_k; n)\) has the mirror symmetry with respect to each \(m_i\), that is, for \(1 \leq i \leq k\), we have
\[
D_k(m_1, \ldots, m_i, \ldots, m_k; n) = D_k(m_1, \ldots, -m_i, \ldots, m_k; n).
\]

Using this mirror symmetry, Theorem 2.2 can be restated as follows.

**Theorem 2.5.** For fixed positive integers \(k\) and \(i\) with \(1 \leq i \leq k + 1\), \(\bar{\eta}_{2k}(n)\) is also equal to the number of \((k + 1)\)-marked Durfee symbols of \(n\) with the \(i\)-th rank being negative.

| \(\bar{\eta}_1(5)\) | \(\bar{\eta}_2(5)\) | \(\bar{\eta}_2(5)\) |
|---------------------|---------------------|---------------------|
| \(\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}_1\) |
| \(\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 & 1 \end{pmatrix}_1\) |
| \(\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 & 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 & 1 \end{pmatrix}_1\) |
| \(\begin{pmatrix} 1 & 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 \end{pmatrix}_1\) |
| \(\begin{pmatrix} 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 \end{pmatrix}_1\) | \(\begin{pmatrix} 1 \end{pmatrix}_1\) |

Table 2.1: 2-Marked Durfee Symbols of 5.
For example, for $n = 5$, $k = 1$ and $i = 1$, there are twenty-one 2-marked Durfee symbols of 5 as listed in Table 2.1. The first column in Table 2.1 gives seven 2-marked Durfee symbols $\tau$ with $\rho_1(\tau) = 0$, the second column contains seven 2-marked Durfee symbols $\tau$ with $\rho_1(\tau) > 0$ and the third column contains seven 2-marked Durfee symbols $\tau$ with $\rho_1(\tau) < 0$. It can be verified that $\bar{\eta}_1(5) = 7$, $\bar{\eta}_2(5) = 7$ and $\eta_2(5) = \bar{\eta}_1(5) + 2\bar{\eta}_2(5) = 21$.

### 3 The generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$

In this section, we obtain the generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ with the aid of Theorem 2.1 and Theorem 2.2. In doing so, we use the generating function of $N(m, n)$ to derive the generating functions of $D_{k+1}(0, m_2, \ldots, m_{k+1}; n)$ and $D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n)$ ($m_1 > 0$).

**Theorem 3.1.** For $k \geq 1$, we have

$$\sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{n=0}^\infty D_{k+1}(0, m_2, \ldots, m_{k+1}; n)x_1^{m_2} \cdots x_k^{m_{k+1}} q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^\infty (-1)^{n-1} q^{(3n-1)/2+kn} \prod_{j=1}^k (1 - x_j q^n)(1 - x_j^{-1} q^n). \quad (3.1)$$

**Proof.** Let

$$G_k(x_1, \ldots, x_k; q) = \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{n=0}^\infty D_{k+1}(0, m_2, \ldots, m_{k+1}; n)x_1^{m_2} \cdots x_k^{m_{k+1}} q^n.$$ 

By Theorem 2.3, we have

$$G_k(x_1, \ldots, x_k; q) = \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} x_1^{m_2} \cdots x_k^{m_{k+1}} \sum_{n=0}^\infty N \left( \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^k t_i + k, n \right) q^n. \quad (3.2)$$

Using (1.1) with $m$ replaced by $\sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^k t_i + k$, we get

$$\sum_{n=0}^\infty N \left( \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^k t_i + k, n \right) q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^\infty (-1)^{n-1} q^{(3n-1)/2+kn(\sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^k t_i + k)} (1 - q^n).$$
Therefore (3.2) becomes

\[
G_k(x_1, \ldots, x_k; q) = \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{t_1, \ldots, t_k = 0}^{\infty} x_1^{m_2} \cdots x_k^{m_{k+1}} \\
\times \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+n(\sum_{i=2}^{k+1} |m_i|+2 \sum_{i=1}^{k} t_i+k)}(1 - q^n).
\]

Write (3.3) in the following form

\[
G_k(x_1, \ldots, x_k; q) = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn}(1 - q^n) \\
\times \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{t_1, \ldots, t_k = 0}^{\infty} x_1^{m_2} \cdots x_k^{m_{k+1}} q^{n|\sum_{i=2}^{k+1} |m_i|+2 \sum_{i=1}^{k} t_i|}.
\]

(3.4)

Notice that

\[
\sum_{a=\infty}^{+\infty} \sum_{b=0}^{+\infty} x^a q^{n(|a|+2b)} = \frac{1}{(1 - xq^n)(1 - x^{-1} q^n)}.
\]

(3.5)

Applying the above formula repeatedly to (3.4), we deduce that

\[
G_k(x_1, \ldots, x_k; q) = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} \frac{(1 - q^n)}{\prod_{j=1}^{k} (1 - x_j q^n)(1 - x_j^{-1} q^n)},
\]

as required.

Setting \(x_j = 1\) for \(1 \leq j \leq k\) in Theorem 3.1 and using Theorem 2.1, we obtain the following generating function of \(\tilde{\eta}_{2k-1}(n)\).

**Corollary 3.2.** For \(k \geq 1\), we have

\[
\sum_{n=1}^{\infty} \tilde{\eta}_{2k-1}(n) q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} \frac{1}{(1 - q^n)^{2k-1}}.
\]

(3.6)

Taking \(k = 1\) in (3.6) and observing that \(\tilde{\eta}_1(n) = \overline{N}_1(n)\), we are led to the generating function for \(\overline{N}_1(n)\) as given by Andrews, Chan and Kim in [2, Theorem 1].

The following generating function can be shown by using the same reasoning as in the proof of Theorem 3.1.
Theorem 3.3. For $k \geq 1$, we have
\[
\sum_{m_1 > 0} \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n)x_1^{m_1} \cdots x_{k+1}^{m_{k+1}} q^n
\]
\[
= \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n+1)/2+kn} \frac{x_1(1-q^n)}{(1-x_1 q^n) \prod_{j=2}^{k+1} (1-x_j q^n)(1-x_j^{-1} q^n)}. \quad (3.7)
\]

Setting $x_j = 1$ for $1 \leq j \leq k+1$ in Theorem 3.3 and using Theorem 2.2 we arrive at the following generating function of $\tilde{\eta}_{2k}(n)$.

**Corollary 3.4.** For $k \geq 1$, we have
\[
\sum_{n=1}^{\infty} \tilde{\eta}_{2k}(n) q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n+1)/2+kn} \frac{1}{(1-q^n)^{2k}}. \quad (3.8)
\]

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