ORIGINAL RESEARCH PAPER

Quasi-bipartite synchronisation of multiple inertial signed delayed neural networks under distributed event-triggered impulsive control strategy

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Abstract

The central concern of this paper is to study leader-following quasi-bipartite synchronisation of a multiple inertial signed neural networks with varying time-delay by utilising distributed event-triggered impulsive control scheme, where connections between adjacent nodes of the neural networks either positive or negative. The second-order neural networks, called inertial neural networks, can be transformed into differential equations of first-order by implementing suitable variable substitution. Under certain hypothesis about the node dynamics, signed graph theory and balanced topology of networks, some conditions are derived in terms of lower-dimensional linear matrix inequalities (LMIs) to achieve leader-following quasi-bipartite synchronisation. In addition, a basic algebraic condition is derived to estimate the theoretical upper bound for the error node. Finally, some numerical simulations are provided to illustrate the correctness of the theoretical results.

Funding information

UAE University (UAE), Grant/Award Number: 126005-UPAR-5-2020

1 | INTRODUCTION

A neural network (NN) is a technology to imitate the human brain into artificial intelligence which helps to solve many technological problems and can lead to the discovery of new technologies. The complex neuronal interconnections in the brain transmit the information in an unimaginable manner that allows unexpected brain responses. Examining these interconnections through mathematical modelling is the secret to manipulating brain functions and imitating the same process in man-made technologies such as robotics. In this regard, both theoretical and practical researches are necessary to accomplish the developments on dynamical NNs. As we know, conventional NNs are typically described by differential equations of the first order. In [1], firstly the inertial neural networks (INNs) is proposed by introducing inductors to the NNs circuit that could be described as second-order differential equations [2–5]. It should be noted that the INNs are different from conventional NNs and their dynamic behaviours are much harder to analyse. Besides that, INNs has actual background in biological science, for example in [8], by means of $M$-matrix method and matrix decomposition theory, the bipartite synchronisation of coupled inertial neural networks(CINNs) was studied with non-reduced order method. In [5], the researchers studied bipartite synchronisation criteria for memristor-based coupled inertial neural networks for both competitive and cooperative interactions. This paper deals with the problem of quasi-bipartite synchronisation of signed INNs with time-delays which is distinguished from most existing work in literature; see [6, 7].

Moreover, according to various topological structures, the NNs system has different dynamical behaviours such as the stability, synchronisation and bifurcations. As one of the classical phenomenon of NNs, synchronisation analysis has been extensively studied in [2–4, 12–15]. It is noticeable that the results of all synchronisation or anti-synchronisation has been mentioned in the works concerned with co-operative networks. Due to NNs learning and high approximation capabilities, multiple neural networks (MNNs) became an essential form of multi-agent systems (MASs). In particular, the synchronisation of MNNs is very close to the consensus of MASs and it received much

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attention in the literature. Besides, in NNs system, collaborative as well as antagonistic relationships coexist between nodes of NNs. Most of previous research have focused on using neural networks for unsigned graphs (or graphs consisting of only positive links). Accordingly, signed graphs are getting very common, specially with the rise in popularity of online social media. Naturally, this raises the issue of whether unsigned NNs can be used on signed networks. For this reason, during the past few decades, substantial efforts have been given in the research of synchronisation or anti-synchronisation of signed NNs which is described by a signed graph. In signed graph the sign of a path is the product of the signs of its edges. As a result, a path is only positive if it has an equal number of negative edges (where zero is even). A signed graph is balanced when every cycle is positive, according to Frank Harary’s mathematical balance theory. For delayed NNs, each node is considered as a delayed nonlinear device and, therefore, it is imperative to tackle some complex bipartisan synchronisation problems. In fact, node dynamics and node delay on bipartite network synchronisation have been extensively studied in the literature; see [8, 16–28]. The node and/or edge based adaptive control strategy was designed and bipartite synchronisation of MNNs with antagonistic interactions are studied in [17]. Using Halanay’s differential inequality and algebraic graph theory, the quasi-bipartite synchronisation of signed NNs is investigated under impulsive control strategy in [21].

In recent technological trends the choice of control strategy has an important effect on system output. In the existing results, state-feedback control strategies for synchronisation can be roughly divided into two categories. One form is continuous or piecewise continuous, which necessarily involves the controller to adjust as the state error changes for example sampled data control [29, 30], optimal control [31], sliding mode control [32]. On the other hand, continuously upgrading the controller, would result in significant energy consumption. The other is discontinuous, and can help with control issues like impulses. Impulsive control technique has attracted attention of many researchers, due to its applications in finance, biological models and medicine [33–35]. It is a discontinuous control method and many advantages compared to continuous control (state-feedback control) techniques such as high reliability, efficiency, easy installation, and low cost maintenance. Generally speaking, sampling period of the time-triggered control cannot be chosen too large to guarantee the synchronisation of delayed networks. Nevertheless, if delayed networks works in perfect state to be reliable, a shorter sampling period may result in unnecessary waste of resources. In addition, the time-triggered scheme in a network channel with a small bandwidth significantly increases the probability of packet waiting and collision. Hence, researchers have developed a control which is triggered by events (event-triggered) to resolve the limitations caused by other time-triggered control (TTC) methods. The event based control event-triggered control (ETC) strategy is a kind of control application technique inserted between the sampler and the controller by a pre-designed event-triggering (ET) strategy. Based on these merits, many interesting results have been proposed, for example a new adaptive memory ETC strategy for network based T-S fuzzy systems is discussed in [36] with asynchronous premise constraints and in [37] the same control strategy is considered for networked control systems under deception attacks. Recently, event-triggering impulsive control scheme (ETICS) consisting of both impulsive control and event based control are discussed in [38]. The main principle of ETICS is to implement impulsive control only at the moment of the event. Therefore, ETICS findings have much research attention and has been widely studied by many researchers, see for example [38, 39]. Besides that, observer-based ETICS of nonlinear systems is designed and the input to state stability criteria has been extensively studied in [40]. Utilising distributed ETICS, the consensus of leader-following multiagent systems was investigated in [38].

In the present paper, a distributed event-triggered impulsive control design is proposed to achieve quasi-bipartite synchronisation in a network of inertial neural networks with time varying delay. Event-triggered impulsive control strategy is a suitable one and the control approach will effectively boost the use of resources for bandwidth-limited systems. Therefore, to improve the performance of systems under event triggered control, the strategy must be improved based on event triggered condition. By using certain hypothesis about the node dynamics and signed graph, main results are derived in terms of lower-dimensional linear matrix inequalities to achieve leader-following quasi-bipartite synchronisation inertial neural networks with time-delays. In addition, a basic algebraic condition is derived to estimate the theoretical upper bound for the error node. The authors believe that the underlying study of quasi-bipartite synchronisation of signed inertial neural networks and it is different from most existing works reported in the literature. This study provides new results in quasi-bipartite synchronisation of inertial delayed neural networks with unsigned graphs. The remaining parts of this paper are organized as: Section 2 describes the system architecture and provides some required preliminaries for the main results. Section 3 provides the ETICS and derives the theoretical results of the main problem. A numerical simulation result and the conclusion of the main result is given in Section 4 and Section 5 respectively.

2 PROBLEM DESCRIPTION AND PRELIMINARIES

In this section, some basic definitions and lemmas are given that are used in the rest of this paper.

2.1 Signed-graph theory

Let $G = \{U, T, W\}$ be a signed directed graph, where $U$ represents the set of all nodes, $T$ represents the edge set of the signed digraph $G$ and $W = (a_{pq})_{N \times N}$ denotes the adjacency matrix of signed graph $G$ and the elements of $W$ satisfies $a_{pq} \neq 0$ when $p \neq q$ this yields that there is a directed edge form the node $p$ and $q$, otherwise $a_{pq} = 0$. Assume $a_{pq} = 0$ for any node $p \in U$.

\footnote{The signed graph’s row sum is not always zero and is entirely different from the non-negative graph.}
the Laplacian matrix of the signed graph $G$ is represented as $L = \{p,q\} = D - W$, where $D = \text{diag}(d_1, d_2, ..., d_N)$ such that $d_p = \sum_{q=1}^{N} |a_{pq}|$, $N_p = \{q \in U | a_{pq} \in T\}$. For graph $G = \{U, T, W\}$, suppose that the node set $U$ is separated into two sets that are disjoint named as $U_1$ and $U_2$ such that $a_{pq} \geq 0$ hold for $p,q \in U_1$, $i = 1,2$ and $a_{pq} \leq 0$ hold for $p \in U_1$, $q \in U_2$, $r \neq s, (r,s) \in 1,2$, then signed directed graph $G$ is structurally balanced. In this current work, we assume the network topology is structurally balanced.

### 2.2 Network model

Consider the family of inertial neural networks with time-varying delay consisting of $N$ followers and one leader. The $p$th neuron of the inertial neural networks is represented as follows:

$$
\frac{d^2 \psi_p(t)}{dt^2} = -\sum_{q=1}^{N_p} |a_{pq}| \frac{d\psi_q(t)}{dt} + B\psi_p(t) + Cf(\psi_p(t)) + Df(\psi_p(t) - \tau(t)) + I(t) + r_p(t) \tag{1}
$$

where the second derivative $d^2 \psi_p(t)/dt^2$ is known as the inertial term of the neural networks model (1). $\psi_p(t) = (\psi_{p1}, ..., \psi_{pN})^T$, $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times n}$ are positive definite diagonal matrices. $C = (c_{iu})_{n \times n} \in \mathbb{R}^{n \times n}$, $D = (d_{iu})_{n \times n} \in \mathbb{R}^{n \times n}$ are connection weight matrices, $\tau(t)$ is the time varying delay. The time-delay $\tau(t)$ satisfies $0 \leq \tau(t) \leq \tau$, $\tau(t)$ is the external input. $r_p(t) \in \mathbb{R}^n$ is the control input of the $p$th neural networks to be defined later. The initial values of model (1) is represented as follows:

$$
\psi_p(0) = \xi(0), \quad \frac{d\psi_p(t)}{dt} = \xi(t), \quad -\tau \leq t \leq 0.
$$

where, $\xi(t), \xi(t)$ are the continuous real valued functions on $t \in [-\tau, 0]$. Let us transform the suitable variable transformation

$$
\left\{
\begin{align*}
\psi_p(t) & = \xi_p(t) \\
\psi_q(t) & = \xi_q(t)
\end{align*}
\right.
$$

From (2), the neural networks model (1) can be transformed into first order ODE as:

$$
\left\{
\begin{align*}
\frac{d\xi_p(t)}{dt} & = -\sum_{q=1}^{N_p} |a_{pq}| \frac{d\xi_q(t)}{dt} + B\xi_p(t) + C
\end{align*}
\right.
$$

where, $p = 1, 2, ..., N$, $\psi_p(t) = (\psi_{p1}(t), ..., \psi_{pN}(t))^T \in \mathbb{R}^n$, $I_p(t) = (I_{p1}(t), ..., I_{pN}(t))^T \in \mathbb{R}^n$, $\gamma = \text{diag}(\gamma_1, ..., \gamma_n)\{A = \text{diag}(a_1 - \gamma_1), ..., (a_n - \gamma_n), B = \text{diag}(b_1, ..., b_n + \gamma_n\gamma_n - a_n), I(t) = \text{diag}(I_1(t), ..., I_n(t))\}$. $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))^T$ denotes the disturbance signal, where $\gamma_p(t) \in \mathbb{R}^n$ satisfies $||\gamma(t)||^2 \leq \nu, \nu$ is positive constant.

The leader node of (3) is defined as follows:

$$
\left\{
\begin{align*}
\frac{d\chi(t)}{dt} & = -\gamma \chi(t) + \gamma(t), \\
\frac{dy(t)}{dt} & = -Af(t) - B\chi(t) + C\chi(t) + Df(\chi(t) - \tau(t)) + I(t)
\end{align*}
\right.
$$

where $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))^T, \chi(t) = (\chi_1(t), ..., \chi_n(t))^T$ and all the other parameters are consistent with model (3). The impulsive effect in model (3) is described as follows:

$$
\left\{
\begin{align*}
\Delta \xi_p(t_k^p) & = \bar{P}_p(t_k^p) + \bar{b} \sum_{j=1}^{N} \rho_{pj} \xi_p(t_k^j), \\
\Delta \xi_p(t_k^p) & = \bar{P}_p(t_k^p) + \bar{b} \sum_{j=1}^{N} \rho_{pj} \xi_p(t_k^j)
\end{align*}
\right.
$$

where, $\bar{P}, \bar{P}$ are the impulsive strength of the $p$th node of (3), $\bar{b}, \bar{b}$ are the disturbance weight. The impulsive sequence $\{t_k^p\}$ satisfies $0 < t_k^p < t_k^p < ... < t_k^p <$, $\lim_{k \to \infty} t_k^p = \infty$. $\Delta \xi_p(t_k^p) = \xi_p(t_k^p) - \xi_p(t_k^-), \xi_p(t_k^p) = \xi_p(t_k^-) = \lim_{k \to \infty} \xi_p(t_k^p - b), \Delta I_p(t_k^p) = I_p(t_k^p) - I_p(t_k^p) = I_p(t_k^-) = \lim_{k \to \infty} I_p(t_k^p - b), \quad \bar{P}_p(t_k^p) = \sum_{j \in \mathcal{N}_p} a_{pj} (\xi_j(t_k^p) - \xi_p(t_k^p)) + b_p (\xi_j(t_k^p) - \xi_p(t_k^p)), \bar{P}_p(t_k^p) = \sum_{j \in \mathcal{N}_p} a_{pj} (\xi_j(t_k^p) - 0) + b_p (\xi_j(t_k^p) - I_p(t_k^-)), \text{where } t_k^p \text{ be the triggering time sequence of } p\text{th node and defined as } t_{k+1}^p = \inf \{t : t > t_k^p, f_p(t) \geq 0\}$.
Assumption 1. For the signed inertial neural network (1), the activation functions are odd functions i.e. \( f_q(z_q(t)), f_p(z_p(t – \tau(t))) \) satisfies the following condition

\[
f_q(-z_q(t)) = -f_q(z_q(t)),
\]

\[
f_p(-z_p(t – \tau(t))) = -f_p(z_p(t – \tau(t)));
\]

\[
\| f_p(z_p(t) – f_p(z_p(t)) \| \leq L_1 \| z_p - z_p \|, \]

\[
\| f_p(z_p(t – \tau(t))) – f_p(z_p(t – \tau(t))) \| \leq L_2 \| z_p - z_p \|,
\]

where \( z_p, z_p \in \mathbb{R}^n, z_p \neq z_p \) are positive constants.

Now, model (3) can be written as follows:

\[
\begin{aligned}
&\frac{d\xi_p(t)}{dt} = -\gamma \xi_p(t) + I_p(t), \\
&\frac{dI_p(t)}{dt} = -A\rho I_p(t) - B\xi_p(t) + Cf(z_p(t)) + Df(z_p(t – \tau(t))) + I(t), \\
&\Delta \xi_p(t) = \gamma \xi_p(t) + \sum_{q=1}^{N} \rho_{pq} \xi_p(t), \\
&\Delta I_p(t) = \gamma I_p(t) + \sum_{q=1}^{N} \rho_{pq} I_p(t).
\end{aligned}
\]

Now, one can get a matrix \( \rho = \operatorname{diag}\{\rho_1, ..., \rho_N\}, \rho_p \in \{-1, 1\} \) for a structurally balanced graph \( G \) such that \( \bar{W} = \left(\bar{W}_{pq}\right)_{N \times N} = \rho W \rho, \) where \( \bar{W}_{pq} = \rho_{pq} \rho \rho = |\bar{W}_{pq}|. \) Let \( \xi_p(t) = \rho_p \xi_p(t), I_p(t) = \rho_p I_p(t), \) where \( \rho_p \in \{-1, 1\}. \) Then, \( \xi_p(t) = \rho_p \xi_p(t), I_p(t) = \rho_p I_p(t). \)

Remark 3. The word inertia is considered a vital tool for generating bifurcation and instability in a nonlinear system, and neural networks with a term inertial are represented by the second order differential equations called inertial neural networks. It is worth noting that in [2–4, 8], the authors study the inertial neural networks under reduced order method and non reduced order method. The author’s proposed new inertial neural network stability conditions by developing the novel L-K functionals which laid the foundations for future research into inertial neural networks stability and synchronisation. On that basis, this paper deals with quasi-bipartite synchronisation problem of signed inertial neural networks via distributed event-triggered impulsive control by reduced order method.

Remark 4. In event-based control, sampling is only achieved if the state-dependent error reaches a tolerable limit. Compared to time-triggered process, event-based method has a lower probability of redundant transmission of information. The event-based control law that determines when the sampling is performed quickly or slowly and whether the sampling information needs to be transmitted for control updates. The approach was introduced to networks, and distributed event-triggered control techniques which were proposed in [13, 38, 40–42, 46]. In [38] the controller is only upgraded for each agent if those state-dependent errors reach a tolerable limit and the control inputs are performed by the actor only at event triggering impulsive instants and in [46], the controller updates are event-based for each agent and will only be triggered at their own event times. In [13], by the combination of delayed impulsive control with the event-triggering mechanism, a the novel event-triggered delayed impulsive control is formulated using the quadratic Lyapunov function. Between them, event-triggering instants are produced when the event-based impulsive control strategy is violate, and delayed impulsive control is only applied at event-triggering instants. Therefore, to improve the performance of systems under event triggered control, the strategy must be improved based on event triggered condition. From application point of view, event-triggered impulsive control strategy is a suitable one and the control approach will effectively boost the use of resources for bandwidth-limited systems. In this paper, the distributed event-triggered impulsive control strategy is applied and in addition, disturbance is considered in original model. Because of disturbance between the adjacent nodes, quasi-bipartite synchronisation could not succeed exactly. But, under certain mild conditions, it can reach quasi-bipartite synchronisation.

Definition 1. [21] Based on the Assumption 1, the inertial signed neural networks (1) achieves quasi bipartite synchronisation if the following condition holds:

\[
\lim_{t \to \infty} \| \xi_p(t) - \rho \xi(t) \| \leq w_1
\]

where \( \rho = 1 \) if \( p \in U_1, \rho = -1 \) if \( p \in U_2, p = 1, 2, ..., n, w_1 \) is a nonnegative constant.

Definition 2. [21] The average impulsive interval of the sequence \( r = \{t_1, t_2, ..., t_n\} \) is equal to \( T_n \) then

\[
\frac{T - t}{T_n} - H_0 \leq H_0 \leq \frac{T - t}{T_n} + H_0, \forall T \geq 0.
\]

holds if there is a positive integer \( H_0 \), where \( H_0(T, t) \) is the number which is the occurrence of impulse in the time interval \( (t, T] \)

Then, rewrite the model (6) with impulsive effects

\[
\begin{aligned}
&\frac{d\xi_p(t)}{dt} = -\gamma \xi_p(t) + I_p(t), \\
&\frac{dI_p(t)}{dt} = -A\rho I_p(t) - B\xi_p(t) + Cf(z_p(t)) + Df(z_p(t – \tau(t))) + I(t), \\
&\Delta \xi_p(t) = \gamma \xi_p(t) + \sum_{q=1}^{N} \rho_{pq} \xi_p(t), \\
&\Delta I_p(t) = \gamma I_p(t) + \sum_{q=1}^{N} \rho_{pq} I_p(t).
\end{aligned}
\]
where,

\[
\hat{\mathbf{P}}_p(t) = \sum_{q \in \mathcal{N}_p} \tilde{a}_{pq} (\tilde{\mathbf{z}}_q(t) - \tilde{\mathbf{z}}_p(t)) + \tilde{b}_p (\tilde{\mathbf{z}}_0(t) - \tilde{\mathbf{z}}_p(t)),
\]

\[
\check{\mathbf{P}}_p(t) = \sum_{q \in \mathcal{N}_p} \check{a}_{pq} (\check{\mathbf{z}}_q(t) - \check{\mathbf{z}}_p(t)) + \check{b}_p (\check{\mathbf{z}}_0(t) - \check{\mathbf{z}}_p(t)),
\]

\[
\tilde{\rho}_{pq} = \rho_{pq} \tilde{\rho}_{pq} \tilde{B} = (\tilde{b}_p)_{\mathcal{N} \times \mathcal{N}} = \rho B \rho
\]

The error vector is then defined as \( \bar{e}_p(t) = \tilde{e}_p(t) - x(t) \), \( \bar{e}_p^{(2)}(t) = \check{e}_p(t) - y(t) \). Then, one can obtain the error system as

\[
\begin{align*}
\frac{d \bar{e}_p^{(1)}(t)}{dt} &= -\gamma \bar{e}_p^{(1)}(t) + \bar{e}_p^{(2)}(t) \\
\frac{d \bar{e}_p^{(2)}(t)}{dt} &= -A \bar{e}_p^{(2)}(t) - B \bar{e}_p^{(1)}(t) + C f \left( \bar{e}_p^{(1)}(t) \right) + D f (\bar{e}_p^{(1)}(t - \tau(t))) \\
t &\geq 0, t \neq t_k, k \in \mathcal{N}
\end{align*}
\]

(12)

\[
\begin{align*}
\bar{e}_p^{(1)}(t_k^+) &= \gamma \check{P}_p^{(1)}(t_k^{(1)}) + \sum_{q = 1}^{\mathcal{N}_p} \tilde{\rho}_{pq} \check{e}_p^{(1)}(t_k^{(2)}) + \bar{e}_p^{(1)}(t_k^{-}) \\
\bar{e}_p^{(2)}(t_k^+) &= \gamma \check{P}_p^{(2)}(t_k^{(1)}) + \sum_{q = 1}^{\mathcal{N}_p} \tilde{\rho}_{pq} \check{e}_p^{(2)}(t_k^{(2)}) + \bar{e}_p^{(2)}(t_k^{-})
\end{align*}
\]

where,

\[
\check{P}_p^{(1)}(t) = \sum_{q = 1}^{\mathcal{N}_p} \check{a}_{pq} \left( \check{e}_q^{(1)}(t) - \check{e}_p^{(1)}(t) \right) + \check{b}_p \left( \check{e}_0^{(1)}(t) - \check{e}_p^{(1)}(t) \right),
\]

\[
\check{P}_p^{(2)}(t) = \sum_{q = 1}^{\mathcal{N}_p} \check{a}_{pq} \left( \check{e}_q^{(2)}(t) - \check{e}_p^{(2)}(t) \right) + \check{b}_p \left( \check{e}_0^{(2)}(t) - \check{e}_p^{(2)}(t) \right),
\]

\[
f \left( \bar{e}_p^{(1)}(t) \right) = f \left( \bar{e}_p^{(1)}(t) + \tilde{x}(t) \right) - f(x(t)),
\]

\[
f \left( \bar{e}_p^{(2)}(t) \right) = f \left( \bar{e}_p^{(2)}(t) + \check{y}(t) \right) - f(y(t)).
\]

It remains to prove the main results according to (12), the following compact form is utilised to represent the error dynamics at impulsive instant \( t = t_k \),

\[
\bar{e}_p^{(1)}(t_k^+) = \bar{e}_p^{(1)}(t_k^-) + \gamma \sum_{q = 1}^{\mathcal{N}_p} \check{a}_{pq} \left( \check{e}_q^{(1)}(t_k) - \check{e}_p^{(1)}(t_k) \right)
\]

\[
+ \check{b}_p \left( \check{e}_0^{(1)}(t_k) - \check{e}_p^{(1)}(t_k) \right) + \tilde{\rho}_{pq} \check{e}_p^{(1)}(t_k) + \check{b}_p \left( \check{e}_0^{(1)}(t_k) - \check{e}_p^{(1)}(t_k) \right)
\]

\[
= -\gamma \sigma_p (k) \sum_{q = 1}^{\mathcal{N}_p} \tilde{a}_{pq} \left( \tilde{e}_q^{(1)}(t_k) - \tilde{e}_p^{(1)}(t_k) \right)
\]

\[
+ \tilde{b}_p \left( \tilde{e}_0^{(1)}(t_k) - \tilde{e}_p^{(1)}(t_k) \right) + \tilde{\rho}_{pq} \tilde{e}_p^{(1)}(t_k) + \tilde{b}_p \left( \tilde{e}_0^{(1)}(t_k) - \tilde{e}_p^{(1)}(t_k) \right)
\]

(13)

where,

\[
H = L + B, L = D - W, W = (a_{pq})_{\mathcal{N} \times \mathcal{N}}, d_p = \sum_{q = 1}^{\mathcal{N}_p} a_{pq}
\]

\[
B = \text{diag} \{ b_1, \ldots, b_{\mathcal{N}} \}, D = \text{diag} \{ d_1, d_2, \ldots, d_{\mathcal{N}} \},
\]

\[
\sigma(K) = \begin{pmatrix}
\sigma_1(k) & \sigma_2(k) & \ldots & \sigma_{\mathcal{N}}(k) \\
\sigma_1(k) & \sigma_2(k) & \ldots & \sigma_{\mathcal{N}}(k) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_1(k) & \sigma_2(k) & \ldots & \sigma_{\mathcal{N}}(k)
\end{pmatrix}
\]

At sampling instant \( t_k, \sigma_p (k) = 1 \) if and only if the event occurs for \( p \), otherwise \( \sigma_p (k) = 0 \). Then, the error system (12) can be
further transformed in the following compact form as

\[
\begin{align*}
\frac{d\mathcal{E}^{(1)}(t)}{dt} &= -(I_N \otimes \gamma)\mathcal{E}^{(1)}(t) + \mathcal{E}^{(2)}(t) \\
\frac{d\mathcal{E}^{(2)}(t)}{dt} &= -(I_N \otimes A)\mathcal{E}^{(2)}(t) - (I_N \otimes B)\mathcal{E}^{(1)}(t) \\
&\quad + (I_N \otimes C) \phi(\mathcal{E}^{(1)}(t)) + (I_N \otimes D) \phi(\mathcal{E}^{(1)}(t) - \tau(t)) \\
\end{align*}
\]

Remark 5. In previous work [5], bipartite synchronisation for inertial memristor based neural networks were investigated. In particular, the multiple neural networks synchronisation issue similar to multi agent systems consensus problem was studied by many authors [2, 17, 38, 43, 44]. They investigated synchronisation of multiple neural networks through impulsive control [44, 47], respectively. Wang et al., in [42, 43], have investigated the synchronisation problem of multiple memristive neural networks using cyber-physical attacks and switching topologies with parameter mismatches respectively. In the above mentioned works the event-triggering conditions is utilised for the error system convergence. In [48], robust synchronisation of multiple memristive neural networks with uncertain parameters through nonlinear coupling is discussed and global synchronisation of multiple recurrent neural networks through impulsive interactions have been investigated in [49].

### 3 MAIN RESULTS

In this section, we investigate the quasi-bipartite synchronisation criterion of the multiple inertial neural networks with time varying delays with event triggered impulsive control.

**Theorem 1.** If the Assumption 1 holds, for any positive scalars $\theta, \theta_1$ and $2\varepsilon > b > 0$ and

\[
\begin{pmatrix}
\psi_1 & \psi_2 \\
\psi_3 & \psi_4 \\
\psi_5 & \psi_6
\end{pmatrix}
\begin{pmatrix}
1_{N-a} - (I_N \otimes B) \\
1_{N-a} \\
(I_N \otimes C) \\
(I_N \otimes D) \\
(L \otimes C) \\
(L \otimes B)
\end{pmatrix} < 0
\]

where,

\[
\begin{align*}
\psi_1 &= 2\varepsilon I_{N-a} - 2(I_N \otimes \gamma) - (L_1 \otimes A), \\
\psi_2 &= -(I_N \otimes \gamma), \\
\psi_3 &= -2(I_N \otimes A) + 2\varepsilon I_{N-a}, \\
\psi_4 &= -(I_N \otimes A), \\
\psi_5 &= -(L_1 \otimes B) - bI_{N-a},
\end{align*}
\]

then the inertial signed neural network (1) is globally exponentially converge to the set $\xi$. 

**Proof.** Consider the Lyapunov function as follows:

\[
\mathcal{V}(t) = \mathcal{E}^{(1)}(t)^T \mathcal{E}^{(1)}(t) + \mathcal{E}^{(2)}(t)^T \mathcal{E}^{(2)}(t)
\]

Then, taking the time derivative of $\mathcal{V}(t)$ along (15), we have

\[
\begin{align*}
D^+ \mathcal{V}(t) &\leq 2\varepsilon \mathcal{E}^{(1)}(t)^T \mathcal{E}^{(1)}(t) + 2\varepsilon \mathcal{E}^{(2)}(t)^T \mathcal{E}^{(2)}(t) \\
D^+ \mathcal{V}(t) &\leq 2\varepsilon \mathcal{E}^{(1)}(t)^T \mathcal{E}^{(1)}(t) + 2\varepsilon \mathcal{E}^{(2)}(t)^T \mathcal{E}^{(2)}(t)
\end{align*}
\]

(20)
On the other hand, the following qualities hold:

\[
2[\tilde{\tau}^{(1)}(t) + \tilde{\tau}^{(2)}(t)] - (I_N \otimes \gamma)\tilde{\tau}^{(1)}(t) + \tilde{\tau}^{(2)}(t) - \tilde{\tau}^{(1)}(t) = 0
\]

and

\[
2[\tilde{\tau}^{(2)}(t) + \tilde{\tau}^{(2)}(t)] - \tilde{\tau}^{(2)}(t) - (I_N \otimes A)\tilde{\tau}^{(2)}(t)
- (I_N \otimes B)\tilde{\tau}^{(1)}(t) + (I_N \otimes C)\tilde{\tau}^{(1)}(t)
+ (I_N \otimes D)\tilde{\tau}^{(1)}(t - \tau(t)) = 0.
\]

From Assumption 1, for any positive diagonal matrix \(A\) and \(B\), then we get the inequalities as follows:

\[
\Delta_1^T \begin{bmatrix} -(\mathcal{L} \otimes A) & (\mathcal{L} \otimes A) \\ * & -(I_N \otimes A) \end{bmatrix} \Delta_1 \geq 0
\]

and

\[
\Delta_2^T \begin{bmatrix} -(\mathcal{L} \otimes B) & (\mathcal{L} \otimes B) \\ * & -(I_N \otimes B) \end{bmatrix} \Delta_2 \geq 0,
\]

where

\[
\Delta_1 = (\tilde{\tau}^{(1)}(t))(f(\tilde{\tau}^{(1)}(t)))^T, \Delta_2 = (\tilde{\tau}^{(2)}(t - \tau(t)))f(\tilde{\tau}^{(2)}(t - \tau(t)))^T.
\]

Therefore, from (19)–(24) we get

\[
D^+ \mathcal{V}(t) \leq \eta^T(t)\Xi \eta(t) + \tilde{\nu}(t)\tilde{\tau}^{(1)}(t - \tau(t)) - 2\mathcal{V}(t),
\]

\[
D^+ \mathcal{V}(t) \leq \eta^T(t)\Xi \eta(t) + \tilde{\nu}(t)\tilde{\tau}^{(2)}(t - \tau(t)) - 2\mathcal{V}(t),
\]

where

\[
\Xi = \begin{bmatrix} \psi_1 & \psi_2 & (\mathcal{L} \otimes A) & 0 \\ * & -2I_n & 0 & 0 \\ * & * & -(I_N \otimes A) & 0 \\ * & * & * & -(I_N \otimes B) \\ * & * & * & * \\ * & * & * & * \\ \end{bmatrix}
\]

\[
\psi_1 = 2\epsilon I_{N_n} - 2(I_N \otimes \gamma) - (\mathcal{L} \otimes A), \psi_2 = -(I_N \otimes \gamma),
\]

\[
\psi_3 = -2(I_N \otimes A) + 2\epsilon I_{N_n}, \psi_4 = -(I_N \otimes A), \psi_5 = -(\mathcal{L} \otimes B).
\]

Then

\[
D^+ \mathcal{V}(t) + 2\epsilon \mathcal{V}(t) \leq \eta^T(t)\Xi \eta(t),
\]

\[
D^+ \mathcal{V}(t) \leq -2\tilde{\nu}(t) + \eta^T(t)\Xi \eta(t),
\]

\[
\mathcal{V}(t_k) = [I_N - \tilde{\mathcal{H}}\mathcal{S}(K)] \otimes I_n \tilde{\tau}^{(1)}(t_k) + \tilde{\mathcal{S}}(\tilde{\mathcal{H}} \mathcal{S}(K)) \otimes I_n \tilde{\tau}^{(2)}(t_k)
\]

It follows from (13) and (14)

\[
\mathcal{V}(t_k) = [I_N - \tilde{\mathcal{H}}\mathcal{S}(K)] \otimes I_n \tilde{\tau}^{(1)}(t_k) + \tilde{\mathcal{S}}(\tilde{\mathcal{H}} \mathcal{S}(K)) \otimes I_n \tilde{\tau}^{(2)}(t_k)
\]

From (28) we have

\[
\tilde{\tau}^{(1)}(t_k) [I_N - \tilde{\mathcal{H}}\mathcal{S}(K)] \otimes I_n
\]

\[
\times [I_N - \tilde{\mathcal{H}}\mathcal{S}(K) ]\otimes I_n \tilde{\tau}^{(1)}(t_k) + \tilde{\mathcal{S}}(\tilde{\mathcal{H}} \mathcal{S}(K)) \otimes I_n \tilde{\tau}^{(2)}(t_k)
\]

\[
\leq \tilde{\delta}^2 \tilde{\tau}^{(1)}(t_k) [\tilde{\mathcal{H}}^T \tilde{\mathcal{H}}] \otimes I_n \tilde{\tau}^{(1)}(t_k)
\]

\[
\leq \tilde{\delta}^2 \tilde{\tau}^{(1)}(t_k) [\tilde{\mathcal{H}}^T \tilde{\mathcal{H}}] \otimes I_n \tilde{\tau}^{(1)}(t_k)
\]
According to Lemma 1, the following condition is obtained for \( t \in (t_i, t_{i+1}) \):

\[
\mathcal{V}(t) \leq b_1^k e^{-\tilde{\mu}(t-h_i)}\mathcal{V}(t_i) + b_2^k e^{-\tilde{\mu}(t-h_i)} + b_2^k e^{-\tilde{\mu}(t-h_i)} + \ldots + b_2^k e^{-\tilde{\mu}(t-h_i)}.
\]

From the Definition 2, we can write (34) as:

\[
\mathcal{V}(t) \leq b_1^k e^{-\tilde{\mu}(t-h_i)}\mathcal{V}(t_i) + \frac{b_2^k e^{-\tilde{\mu}(t-h_i)} e^{\ln(t_{i+1}-h_i)}}{1 - e^{\ln(t_{i+1}-h_i)}}.
\]

Hence the convergence of \( \mathcal{V}(t) \) is depending on the value of \( b_1 \). We then get two cases: First one is when the value of \( b_1 \) is \( 0 < b_1 \leq 1 \) it follows that \( \lim_{t \to +\infty} ||e(t)||^2 \) \( \leq \lim_{t \to +\infty} \mathcal{V}(t) \leq \frac{b_2^k e^{-\tilde{\mu}(t-h_i)} e^{\ln(t_{i+1}-h_i)}}{1 - e^{\ln(t_{i+1}-h_i)}} \). Second one is when \( b_1 > 0 \), \( \lim_{t \to +\infty} ||e(t)||^2 \leq \lim_{t \to +\infty} \mathcal{V}(t) \leq \frac{b_2^k e^{-\tilde{\mu}(t-h_i)} e^{\ln(t_{i+1}-h_i)}}{1 - e^{\ln(t_{i+1}-h_i)}} \). Therefore, there is no changes when the value of \( 0 < b_1 \leq 1, b_1 > 1 \) and the following inequality holds:

\[
\lim_{t \to +\infty} ||e(t)||^2 \leq \lim_{t \to +\infty} \mathcal{V}(t) \leq \frac{b_2^k e^{-\tilde{\mu}(t-h_i)} e^{\ln(t_{i+1})}}{1 - e^{\ln(t_{i+1})}}.
\]

Then, the convergence set of \( e(t) \) is defined as

\[
\xi = \left\{ e(t) ||e(t)||^2 \leq \frac{b_2^k e^{-\tilde{\mu}(t-h_i)} e^{\ln(t_{i+1})}}{1 - e^{\ln(t_{i+1})}} \right\}.
\]

Hence the proof is complete and the error of inertial signed neural network (12) cannot necessarily achieve leader-following bipartite synchronisation because of the presence of disruption between the neighbouring nodes. But under certain mild conditions it can reach quasi-bipartite synchronisation. Equation (37) gives clear expression of the error bound, which relates primarily to impulsive strength and corresponding impulsive disturbance.

**Remark 6.** In [50], the bipartite quasi synchronisation of signed fractional-order neural networks with antagonistic interactions in coupled sense through the impulse effects is considered. The impulse control is designed depending on impulse functions and the related fractional order. This paper, an exhaustive derivation process in Lemma 1 is proposed. If we want to extend the results to fractional-order signed neural networks, it is necessary to extend the Lemma 1 in [21] for fractional-order version.

**Corollary 1.** Suppose that Assumption 1 holds and impulsive sequence \( r = \{t_1, t_2, t_3, \ldots, t_i\} \) is in \( R(T) \). Then, the inertial signed neural
networks (1) can reach the bipartite synchronisation under distributed event-triggered impulsive control without any external disturbances with the leader node (4), such that the condition (18) satisfies for any positive scalars $\bar{\theta}, \bar{\theta}_1$ and $2e > b > 0$.

**Proof.** Suppose we have to rewrite the model (6) with the distributed event-triggered impulsive control [38] without any external disturbances. Then, (9) can be modelled as follows:

\[
\begin{aligned}
\frac{d\tilde{y}_p(t)}{dt} &= -\gamma \tilde{y}_p(t) + \tilde{I}_p(t) \\
\frac{d\tilde{\xi}_p(t)}{dt} &= -A\tilde{y}_p(t) - B\tilde{y}_p(t) + C_f(\tilde{y}_p(t)) + Df(\tilde{y}_p(t) - \tau(t)) + I(t) \\
\Delta \tilde{y}_p(t_k^+) &= \hat{P}_p(t_k^+) \Delta \tilde{y}_p(t_k^-) = \hat{P}_p(t_k^+)
\end{aligned}
\]

where, $\tilde{P}_p(t_k), \hat{P}_p(t_k)$ is defined in (10) and (11) respectively. Then, the error system is designed between the leader node (4) and the corresponding followers (38), then we can obtain the error system (12) without the disturbances $\tilde{u}_{\tilde{\xi}} \tilde{y}_p(t_k^+), \tilde{u}_{\tilde{\xi}} \tilde{y}_p(t_k^-)$. Then, the following compact form is presented to represent the error dynamics at impulsive instant $t = t_k, \tilde{y}_p^{{(1)}}(t_k^+) = [I_N - \tilde{y} \tilde{H}(\sigma(K)) \otimes \hat{\xi}^{{(1)}}(t_k)] \hat{\xi}^{{(1)}}(t_k), \tilde{y}_p^{{(2)}}(t_k^+) = [I_N - \tilde{y} \tilde{H}(\sigma(K)) \otimes \hat{\xi}^{{(2)}}(t_k)]$.

Compared to the proof in Theorem 1, consider the following Lyapunov functional candidate:

\[
\Psi(t) = \tilde{z}^{(1)T}(t)(R_1 \otimes Q_1)\tilde{z}^{(1)}(t) + \tilde{z}^{(2)T}(t)(R_2 \otimes Q_2)\tilde{z}^{(2)}(t)
\]

we can easily get

\[
D^+ \Psi(t) + 2\tilde{e}\tilde{v}(t) \leq \eta^{(1)T}(t)\hat{\xi}(t) + \tilde{I}_p^{(1)T}(t - \tau(t)) (R_1 \otimes Q_1)
\]

\[
D^+ \Psi(t) \leq -2\tilde{e}\tilde{v}(t) + h \tilde{V}(t),
\]

which holds for $t_k < t \leq t_{k+1}, k = 1, 2, \ldots, n$ and

\[
\hat{\xi} = \begin{bmatrix}
\psi_1 & \psi_2 & (L \otimes A) & 0 \\
* & -2I_N & 0 & 0 \\
* & * & -(I_N \otimes A) & 0 \\
* & * & * & -(I_N \otimes B) \\
* & * & * & * \\
* & * & * & *
\end{bmatrix}
\]

where,

\[
\begin{align*}
\psi_1 &= 2\tilde{e}(R_1 \otimes Q_1) - 2(I_N \otimes \gamma) - (L \otimes A), \\
\psi_2 &= (R_1 \otimes Q_1) - I_N - (I_N \otimes \gamma), \\
\psi_3 &= -2(I_N \otimes A) + 2\tilde{e}(R_2 \otimes Q_2), \\
\psi_4 &= (R_2 \otimes Q_2) - I_N - (I_N \otimes A), \\
\psi_5 &= -(L \otimes B) - b(R_1 \otimes Q_1).
\end{align*}
\]

When $t = t_k$, according to (13) and (14) the disturbances are neglected, then one can obtain

\[
\Psi(t_k^+) = \tilde{z}^{(1)T}(t_k^-)[I_N - \tilde{y} \tilde{H}(\sigma(K)) \otimes \hat{\xi}^{(1)}(t_k^-)] + \tilde{z}^{(2)T}(t_k^-)[I_N - \tilde{y} \tilde{H}(\sigma(K)) \otimes \hat{\xi}^{(2)}(t_k^-)]
\]

\[
\leq k_{\max}^2[I_N - \tilde{y} \tilde{H}(\sigma(K))]\bar{V}(t_k^-)
\]

which implies that

\[
\Psi(t_k^+) \leq \tilde{h}_1 \Psi(t_k^-)
\]

where $\tilde{h}_1 = \max[k_{\max}^2|I_N - \tilde{y} \tilde{H}(\sigma(K))|, k_{\max}^2|I_N - \tilde{y} \tilde{H}(\sigma(K))|]$. According to Lemma 1, in (35) we have to apply $b_2 = 0$, then the following condition is obtained for $t \in (t_k, t_{k+1})$.

\[
\Psi(t) \leq k_{\max}^2 e^{-\tilde{h}_1(t-t_k)} \bar{V}(t_k^-).
\]

From (8) we have

\[
\Psi(t) \leq e^{-\left(\frac{\tilde{h}_1}{\tilde{h}_0}+\tilde{h}_0\right)(t-t_0)} \bar{V}(t_0)\max\left\{\tilde{h}_0, \tilde{h}_1\right\} \bar{V}(t_0).
\]
Then, we get
\[
\lim_{t \to +\infty} ||\xi(t)||^2 = \lim_{t \to +\infty} V(t) \\
\leq \lim_{t \to +\infty} \hat{V}(\eta_0) e^{-\xi(t-\tau_0)} \max \left\{ \gamma_{11}, \gamma_{22} \right\},
\]
(44)
where \( \xi = (-\frac{\ln h}{\tau_0} + \bar{\omega}) > 0 \). Therefore, \( \lim_{t \to +\infty} ||\xi(t)||^2 = 0 \). Then, the inertial signed neural networks (1) can reach the bipartite leader-following synchronisation under distributed event-triggered impulsive control without any external disturbances with the leader node (4).

Remark 7. From Theorem 1, we conclude the following:

1. Under the term of (33), the impulsive strength \( \gamma, \bar{\gamma} \) can be arbitrary. That plays a significant role in neural network synchronisation. It shows that, the influence of the impulsive strength \( \gamma, \bar{\gamma} \) is positive if \( \gamma, \bar{\gamma} > 0 \), and the influence of the impulsive strength \( \gamma, \bar{\gamma} \) is negative if \( \gamma, \bar{\gamma} < 0 \).
2. From (33) and (37), we can see that both the terms \( b_2 \) and average impulsive interval of the impulsive sequence \( \{\tau_1, \tau_2, ..., \tau_n\} \), that is \( T_a \) can be effective elements on network synchronisation. To improve the error system convergence rate, the average impulsive interval \( T_a \) and impulsive strength \( \gamma, \bar{\gamma} \) selection should be the smallest possible. In addition, the synchronisation effect does not depends on \( b_1 \), when \( 0 < b_1 \leq 1, \), \( b_1 > 1 \). Because of disturbance between the adjacent nodes, the multiple inertial neural networks (15), could not succeed the bipartite synchronisation exactly. Under certain mild conditions, it can reach quasi-bipartite synchronisation.

4 | NUMERICAL EXAMPLE

In this section, a simulation example is given to demonstrate the theoretical results of this paper. Consider the multiple inertial signed neural networks with distributed event-triggered impulsive control (1) composed of \( N = 12 \) nodes (See Figure 1), with time-delay \( \tau(t) = 4t \), \( \xi(t) = (\xi_0^T(t), \bar{\xi}_0^T(t))^T \) is the state neuron of the \( j \)-th inertial neural network and the network is represented by
\[
\frac{d^2 \xi(t)}{dt^2} = -a \frac{d\xi(t)}{dt} - b_1 \xi(t) + Cf(\xi(t)) \\\n+ Df(\xi(t - \tau(t)))+I + r_p(t),
\]
(45)
where \( p \in N, f(\xi(t)) \) and \( f(\xi(t - \tau(t)) \) are the activation functions without and with time-delay respectively.

Figure 2 shows the state trajectories of the given signed inertial neural networks (45) for the values of \( \epsilon = 0.1, \gamma = 1.1, \eta = 0.01, \psi = 26.3 \). Then, we get the impulsive strength \( b_1 = 0.1694, \) and the impulsive disturbance \( b_2 = 0.6457 \). By using \( \epsilon = 1.265, \eta = 1.086, \) the solution of \( m - 2\epsilon + b \gamma^\omega = 0 \) is \( \bar{\omega} = 0.1023 \) for \( \eta = 2.07 \). The convergence rate of signed inertial neural network (1) is calculated using \( \xi = (-\frac{\ln h}{\tau_0} + \bar{\omega}) \) and \( b_2(1 - a_{11}) \) is the state neuron of the \( j \)-th inertial neural network and the network is represented by
\[
\frac{d^2 \xi(t)}{dt^2} = -a \frac{d\xi(t)}{dt} - b_1 \xi(t) + Cf(\xi(t)) \\\n+ Df(\xi(t - \tau(t)))+I + r_p(t),
\]
(45)
where \( p \in N, f(\xi(t)) \) and \( f(\xi(t - \tau(t)) \) are the activation functions without and with time-delay respectively.

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\[
\frac{d^2 \xi(t)}{dt^2} = -a \frac{d\xi(t)}{dt} - b_1 \xi(t) + Cf(\xi(t)) \\\n+ Df(\xi(t - \tau(t)))+I + r_p(t),
\]
(45)
where \( p \in N, f(\xi(t)) \) and \( f(\xi(t - \tau(t)) \) are the activation functions without and with time-delay respectively.

Remark 8. In [21], quasi bipartite synchronisation problem for the signed neural networks is discussed. In which the nodes are interacted both cooperatively and antagonistically. Moreover, disturbance occurs at impulsive instants in communication networks between some adjacent agents. In the present paper, the same problem is considered for inertial neural networks with reduced order approach. Comparing with the results of [21], for the inertial neural network (1), we achieve quasi-bipartite synchronisation results for the values of \( \epsilon = 0.1, \gamma = 1.1, \eta = 0.01, \psi = 26.3 \). Then, we get the impulsive strength \( b_1 = 1.0694, \) and impulsive disturbance \( b_2 = 0.6457 \). It is observed that the upper bound of error states of inertial neural network (1) becomes larger, compared with [21].
5 | CONCLUSION

In this paper, we have investigated the problem of quasi-bipartite synchronisation of multiple inertial signed neural networks under distributed event triggered impulsive control strategy, where the nodes of the networks have cooperative as well as antagonistic interactions. First the second-order neural networks can be transformed into differential equations of first-order by implementing suitable variable substitution. Simultaneously, a leader node has been introduced, which can be viewed as a reference signal. In addition, we have designed a distributed event triggered impulsive control for each network node to achieve the quasi-bipartite synchronisation on a reference signal. Secondly, based on structurally balanced network topology and extended Halanay differential inequality, some criteria in terms of LMIs for quasi-bipartite synchronisation of multiple inertial signed neural networks have been obtained. Moreover, a basic algebraic condition is derived to estimate the theoretical upper bound for the error node. Finally, one numerical simulation result have been provided to illustrate the correctness of the obtained results. Future work will focus on solving finite/fixed-time quasi-bipartite synchronisation problem of fractional-order discontinuous coupled neural network with switching topologies.
ACKNOWLEDGMENT
The authors would like to thank the editor and reviewers for their valuable and constructive comments which improved the quality of this manuscript. This work was funded by the project of fund # 12S005-UPAR -5-2020, UAE University (UAE).

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FIGURE 4 The state trajectories of error states without disturbance
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How to cite this article: Udhayakumar K, Rihan FA, Li X, Rakkiyappan R. Quasi-bipartite synchronisation of multiple inertial signed delayed neural networks under distributed event-triggered impulsive control strategy. IET Control Theory Appl., 2021;15, 1615–1627. https://doi.org/10.1049/cth2.12146