Self-organized instability in graded-index multimode fibres

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Multimode fibres (MMFs) are attracting interest in the study of spatiotemporal dynamics as well as in the context of ultrafast fibre sources, imaging and telecommunications. This interest stems from three differences compared with single-mode fibre structures: their spatiotemporal complexity (information capacity), the role of disorder, and their complex intermodal interactions. To date, MMFs have been studied in limiting cases in which one or more of these properties can be neglected. Here, we study a regime in which all these elements are integral. We observe a spatial beam-cleaning phenomenon that precedes spatiotemporal modulation instability. We provide evidence that the origin of these processes is a universal unstable attractor in graded-index MMFs. The self-organization and instability of the attractor are both caused by intermodal interactions characterized by cooperating disorder, nonlinearity and dissipation. Disorder-enhanced nonlinear processes in MMFs have important implications for future telecommunications, and the multifaceted nature of the considered dynamics showcases MMFs as potential laboratories for a variety of topics in complexity science.

Multimode fibres (MMFs) are now at the core of many developments in optics. For imaging and telecommunications, MMFs offer unprecedented information density. For telecommunications, this facilitates an increased bandwidth for internet traffic1–7. For imaging8–10, this means a variety of high-resolution optical imaging modalities may be performed through a robust and flexible fibre endoscope with a diameter of ~100–1,000 μm. To access these features, researchers have tamed the spatiotemporal complexity of MMFs by a complete measurement of the fibre’s transmission (or transfer) matrix and its principal modes4–10. For spatiotemporally complex propagation in MMFs, these tools recover underlying order in the linear coupling between the modes, which allows spatial and temporal control of the linear propagation.

MMFs have also been studied in the nonlinear regime11–35. Nonlinear coupling between modes causes a wide range of novel effects, many of which may be useful for high-power, ultrashort-pulsed fibre sources. A cubic nonlinearity couples up to four distinct waves. As a result, breaking down multimode (MM) nonlinear dynamics into a mode-coupling picture requires four-dimensional (4D) tensors11, which describe the nonlinear coupling between spatial eigenmodes. The dimension and nonlinearity restrict the amount of insight that can be directly inferred from the structure of these tensors compared with the insight from the linear transmission and group delay matrices, but this approach provides an efficient description that is useful in many situations.

This coupled-mode approach is not unique to optics. Today, research on many complex systems focuses primarily on the coupling of the systems’ distinct elements. This is the key insight of complexity science—in many complex systems, important behaviours result from the characteristic features of the interactions between distinct elements of the system. Of particular interest is the topology of the interactions, also referred to as the complex network36,37. As opposed to optics, however, for many systems of interest in complexity science, including human social interactions, financial markets and the brain, experiments are not straightforward. Often researchers rely, instead, on computer simulations or natural experiments. Such difficulties have led to controversy38,39. For this reason, an experimental system such as a MMF should be of broad scientific interest: it is highly controllable and measurable, and yet it supports complex phenomena. It is essentially a large network of interacting dynamic systems. The topology of this network, and the nature of the connections, can be readily controlled by the mode structure of the fibre, the properties of the exciting field, the initial distribution of light within the modes, the presence of disorder or active media and the length and orientation of the fibre.

Here we study the self-organization of nonlinear waves in normal-dispersion, multimode graded-index (GRIN) fibres, in a regime in which intermodal interactions are mediated by disorder, nonlinearity and dissipation. In this regime, we observe a spatial transformation of arbitrary input fields to a consistent attractor (that is, a ‘beam clean-up’), which is the fundamental mode with a weak background of higher-order modes. This attractor is unstable and, consequently, once it is reached we observe spatiotemporal modulation instability (STMI), which in turn causes the field to evolve towards a spatiotemporally complex steady state. We provide a simplified theoretical model to understand the origin of this self-organized instability in terms of cooperative intermodal interactions.

Our study integrates a wide range of MM phenomena, and many of these individual physical processes have been studied to some extent before, including four-wave mixing (FWM)15,16,18,20,24,25, stimulated Raman scattering (SRS)12,13,32–34 and simultaneous multiple nonlinear processes12,13,18,21–23,30,31,34. Theoretical studies explored the limit of strong fibre disorder in the context of nonlinear pulse propagation36–39. More broadly, many studies investigated strong disorder in optics40–45, motivated in particular by Anderson localization. Research in this area has explored a variety of phenomena defined by disorder and dissipation, of which a key example is the random laser45. Nonlinear interactions have also been studied in disordered systems—they underlie a variety of interesting effects, as well as important open questions (see refs 41–44, and references therein).

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nonlinearity, MMFs are valuable because these properties (as well as dispersion) can be easily and precisely adjusted over a broad range. Furthermore, because of the important applications mentioned above, understanding MMFs has a clear value beyond basic science.

**Results**

We launch nanosecond-duration pulses in the normal-dispersion regime of long, coiled GRIN MMFs, which excite a large number of transverse modes, and then monitor the spatial and spectral profiles of the output field. Figure 1 shows the results from a typical experiment. The variation of spectral features (Fig. 1a) and spatial profiles (Fig. 1b,c) of the output beam with increasing energy are shown for a fixed 100 m fibre length. Figure 1c shows the near-field beam profiles for increasing pulse energy. At low energy, the beam undergoes a dramatic transformation of self-focusing. Remarkably, this transformation is observed when imaging the full-spectrum beam (using achromatic lenses), rather than only a particular Stokes wave. By roughly 1.5 µJ, the field has reached a clean Gaussian beam—the critical state of the attractor. With the attractor reached, the field is acutely unstable and spectral sidebands appear. These sidebands result from multiple FWM processes with spatiotemporal phase-matching.

Accurate modelling of these observations using standard field-envelope propagation equations is prohibitively difficult. Disorder, dissipation and Raman and Kerr nonlinearities are all important, and the field occupies 55 modes with a time-bandwidth product that exceeds $2 \times 10^2$ (that is, $1 \text{ ns} \times (200 \text{ THz})$). Many complex systems exhibit winner-take-all and self-organized critical dynamics, and complex network analysis has proved useful when more exact complete models are extremely time-consuming or so complicated that they fail to provide much insight. Hence, in what follows we describe the multifaceted nature of the mode coupling—the complex network—in a GRIN fibre to find a qualitative understanding of the experimental observations.

**Theoretical coupling model.** In general, the coupling between two of the fibre’s eigenmodes $\phi_l$ and $\phi_m$ caused by a general local index perturbation $\Delta n(x,y,z)$, is of the form:

$$C_{lm}(z) \propto \int \int \int n(x,y)\phi_l^*(x,y)\phi_m(x,y)\Delta n(x,y,z)\,dx\,dy$$

where $n(x,y)$ is the ideal refractive index profile of the fibre. If we factor $\Delta n(x,y,z)$ into its transverse and longitudinal components, $\Delta n(x,y,z) = \Delta n(x,y)\delta(z)$, and we neglect the influence of all other modes, the amplitude of mode $l$ at a given position is proportional to $F(K = \Delta K)$. $F(K)$ is the Fourier transform of $f(z)$, evaluated at the propagation constant mismatch $\Delta K = k_l - k_l$ between the two modes. In general, the energy exchanged between $\phi_l$ and $\phi_m$ will be higher when $C_{lm}(z)$ is larger. Hence, the mode coupling depends on the spatial overlap between the modes and the perturbation, as well as on the longitudinal evolution $f(z)$ of the perturbation. With two modes and $f(z) = 1$, energy is periodically exchanged. With many modes and/or with more-complex $f(z)$, energy transferred from $\phi_m$ to $\phi_l$ may never return to $\phi_m$ (that is, it may stay in $\phi_l$ or be coupled to another mode, $\phi_l'$).

**Nonlinearity.** For the Kerr nonlinearity we have that $\Delta n(x, y, z) = n_2I(x, y, z)$, where:

$$I(x, y, z) = \sum_{m=1}^{N} a_m(z)\phi_m^*(x,y)e^{-\phi_m^2}$$

where $n_2$ is the Kerr nonlinearity of the fibre.

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**Figure 1 | Experimental measurements of self-organized instability in normal-dispersion GRIN fibre.** a. The spectrum of the field that exits the fibre as a function of increasing launched energy. b. The spatial size (mode field diameter, MFD) of the field over the same range. c. Output near-field intensity spatial profile for increasing energy launched into the fibre. As energy increases, the field organizes until it reaches the unstable attractor (about 1.5 µJ), the fundamental mode on a background of higher-order modes. Increasing the energy from this point leads to spatiotemporal modulation instability and spatiotemporal complexity. Scale bars, 11 µm. The evolution is shown in Supplementary Movie 1. For improved graphical clarity, the spectrum for each energy value is normalized to the peak spectral intensity at that energy.
However, as the sole member of its group, the fundamental mode is with LOMs, the bias and energy in LOMs both grow monotonically. Of the bias of linear dynamics are complicated. However, on average, energy is many neighbouring modes with mode-coupling structure: HOMs have complex symmetries and are longitudinally aperiodic, so disorder is still small, it is much stronger for our experiments at irregularities lead to scattering, which increases rapidly with dopants, as well as small manufacturing errors and environmental fluctuations of the glass and dopants, as well as small manufacturing errors and environmental effects such as bending, twisting and core ellipticity. These small irregularities lead to scattering, which increases rapidly with decreasing wavelength. For this reason, although the net effect of disorder is still small, it is much stronger for our experiments at 532 nm than at longer wavelengths. These random perturbations are longitudinally aperiodic, so \( F(K) \) is centred around \( K=0 \). As a result, disorder causes coupling between the modes with similar propagation constants, and primarily within mode groups. As most sources of disorder are asymmetric, \( C_{ii}(z) \) is usually large when \( \Delta n(x,y) \) corresponds to a symmetry between the two modes. As the perturbations are small, \( C_{ii}(z) \) is also large for similarly sized modes only (Supplementary Figs 2, 5 and 6). As a consequence of these characteristics, the fundamental mode is the least affected by disorder, whereas HOMs, which have reduced symmetries and many neighbours, are the most affected (Fig. 2c and Supplementary Figs 2 and 6).

**Dissipation.** Linear and nonlinear losses occur during propagation. Linear loss that results from the coupling to cladding modes most strongly affects HOMs, because their propagation constants are the closest to those of cladding modes. SRS is a nonlinear dissipative process, and it transfers energy to red-shifted Stokes waves in proportion with the overlap between the pump and the Stokes field. If the pump occupies a single pure mode, the Stokes field will be dominated by the same mode. However, for a multimode pump, the Stokes modes will differ from the pump modes because of gain competition. As the average intensity of a MM field in a parabolic fibre overlaps most with the fundamental mode, for most pump fields the fundamental mode dominates.

**Total coupling effects.** In our experiments, mode coupling is the combination of the couplings caused by nonlinearity, disorder and dissipation. Considering Raman coupling alone, Raman beam clean-up is only the most probable outcome of Raman-gain competition in a multimode GRIN fibre. Disorder alone is diffusive: it causes additional dissipation of HOMs, but otherwise leads to energy equipartition. In the absence of other effects, the Kerr nonlinearity alone can lead to the observed attractor for some initial conditions. However, our experiments include substantial disorder and dissipation. Moreover, the attractor is observed for virtually all coupling conditions that are accessible with a near-Gaussian beam, provided that sufficient energy (typically 2–3 \( \mu \)) is coupled into the fibre.

The interaction between coupling mechanisms is important. Through its influence on dissipation and nonlinear coupling, weak disorder enhances the attractor (Supplementary Figs 7–14). Disorder couples HOMs to cladding modes, and thereby increases their attenuation. This increases the energy in LOMs relative to HOMs. As a result, the overlap of the nonlinear coupling integral with those modes is increased, and the bias of \( C_{ii}(z) \) is enhanced. Disorder-induced coupling within mode groups also interacts with nonlinearity. If disorder overwhelms nonlinearity, it can completely suppress intermodal (or intergroup) energy exchange.

Disorder-induced coupling within mode groups also interacts with nonlinearity. If disorder overwhelms nonlinearity, it can completely suppress intermodal (or intergroup) energy exchange. However, for the weaker coupling considered here, random coupling enhances the attractor. By averaging over HOMs, Raman beam clean-up is more likely to occur (as was suggested in refs 33 and 34).
Figure 3 | Spatiotemporal modulation instability. a, Theoretical spatial profiles, integrated over one spatial dimension, for increasing orders of MI (R is the fibre core radius). b, Spectrum of the entire field. c, Experimentally measured spatial profiles at the indicated frequencies. The comparison with a is not direct because a Raman cascade also leads to a spatiotemporal structure and because multiple pumps lead to MI sidebands. Rather, it is meant to illustrate the overall similar hyperbolic shape. d, Average position of measured MI sidebands (for five different initial conditions) compared with theory from ref. 15. The theoretical curves are shown for the 532 nm pump and two Stokes waves (545 nm, 559 nm).

By seeding perturbations, disorder lowers the power threshold for nonlinear HOM instability (Supplementary Figs 11–14). As the fundamental mode is spatially stable and is the least affected by disorder, this is an important process that underlies the attractor for the ~1 kW peak powers considered here.

Spatiotemporal instability. The fundamental origin of the observed spectral sidebands is STMI, which consists of multiple FWM processes with spatiotemporal phase-matching—spatial (modal) dispersion compensates for chromatic dispersion, as in STMI in free space. Figure 3 shows the theoretical calculations along with the experimental measurements of STMI. As STMI sidebands overlap with the red-most edge of the Raman cascade, the Raman peaks blur together. Furthermore, as there are several intense Stokes waves, the STMI sidebands may originate from several different pump wavelengths. With the attractor as the pump, the higher-order sidebands correspond to increasingly higher-order mode groups (Fig. 3a). Figure 3b shows the measured spectrum, and Fig. 3c the spatial profiles measured at the indicated frequencies, which exhibit the theoretical spatiotemporal trend. Figure 3d shows the typical locations of the modulation instability (MI) sidebands (for the fibre used in Fig. 1) along with the theory.

Despite its 2D stability, the attractor is the maximally unstable state of the field when its full 3D nature is considered. Figure 4, Supplementary Movies 2 and 3 and Supplementary Figs 15–20 demonstrate the maximal instability of the attractor through numerical simulations. The fact that the nonlinear attractor studied here is the most-unstable (or critical) state is related to the broader concept of self-organized criticality.

Understanding the instability of the attractor requires consideration of time-domain processes. The attractor is driven by spatial processes and is primarily a 2D narrowband process; however, the field is 3D. For sufficiently intense and/or broadband fields, spatiotemporal instabilities become important. Intense fields may become susceptible to spatiotemporal instabilities before reaching the attractor, but they are particularly relevant once the attractor is reached, because the attractor is the maximally unstable state. Rigorous understanding of the coupling between different frequencies and modes requires a generalization of the monochromatic approach used so far. Nonetheless, we may simply extend the existing approach to include chromatic dispersion, letting \( \Delta \beta = \beta_m(\omega_m) - \beta_l(\omega_l) \) to obtain an intuitive understanding of the observed phenomena. The qualities that make the attractor two-dimensionally stable make it three-dimensionally unstable—it has the highest intensity and, being the sole member of its group, it is unaffected by the intergroup-coupling suppression that arises from nonlinear and disorder-induced intragroup coupling. In our experiments, the instability develops from the attractor, for which...
periodic oscillations are small. Therefore, $F(K)$ is large only near $K = 0$, and coupling between different modes can only take place if they have different frequencies. This differs from geometric parametric instability (GPI)\textsuperscript{16,18}. In terms of our qualitative model for spatiotemporal coupling, GPI results from strong periodic oscillations that create high-amplitude harmonics in $F(K)$, which allow for the coupling between the same modes at different frequencies. We therefore primarily observe STMI, which is evident from the hyperbolic spatiotemporal profile shown in Fig. 3.

Discussion

Although the unstable attractor we studied here is of particular interest because of its multifaceted origin, its basic features appear to be quite universal and manifest themselves across a broad range of parameters. Here we observe self-focusing towards a maximally unstable attractor, followed by rapid development of STMI. This occurs with kilowatt peak power, nanosecond-duration pulses in 100 m long normal-dispersion fibres with 55 modes. Qualitatively similar behaviour has been observed in shorter fibres with many more modes (~100–300), with a higher power and normal dispersion\textsuperscript{16–18}, and with up to megawatt peak power, ~100 fs long pulses in short anomalous-dispersion fibres\textsuperscript{12,13}. This universality stems from the common characteristics of the mode-coupling network.

Mode coupling in GRIN fibre is primarily local, that is, it takes place between modes with similar propagation constants (intr grou p coupling), but also comes with some intergroup coupling. This is a small-world network\textsuperscript{16} in which nodes are primarily connected locally to their near neighbours, with some small number of ‘shortcut’ connections between neighbourhoods. Small-world networks exhibit a phase transition to global interconnectivity when the strength of the shortcut coupling exceeds a given threshold. Considering disorder only, intergroup coupling is crucial to observe a phase transition from dispersive to diffusive propagation\textsuperscript{7} and for the suppression of FWM\textsuperscript{9}. With nonlinearity, intergroup coupling arises from the periodic component of $f(z)$ and from broadband FWM processes such as STMI. In the attractor, this suggests a crucial role for the lowest-order modes in each mode group. These modes, as can be seen in Fig. 2a, are strongly coupled to one another, which makes them intergroup coupling ‘hubs’. Shortcut coupling transitions may underlie the 2D attractor, the growth of spatiotemporal complexity that follows STMI and the emergence of ultra-broadband spatiotemporal coherence through the GPI\textsuperscript{16–18}.

The confluence of different intermodal coupling processes and emergent behaviour of many modes observed here is a compelling example that highlights MM waveguides as an experimental test bed for complexity science. This use could be similar to that of single-mode fibres (SMFs)\textsuperscript{30}, but MMFs have many more degrees of freedom and control, and therefore offer a way to connect to a much broader class of complex dynamical systems. Furthermore, this capability also makes MM waveguides suitable platforms for the realization of particular applications, such as neural network computers, which have experienced a resurgence of attention recently.

For other applications, MM fibres have significant potential, but it is still not clear if nonlinear processes can be controlled in a useful and reliable manner. Beam clean-up appears to provide one route to spatially coherent MM sources, but the instability of the attractor may undermine this potential. The worldwide demand for increased telecommunications bandwid th at decreasing cost poses an obstacle that SMF-based systems are fundamentally unable to avoid. Indications are that by 2020 SMF transmission capacity will be two orders of magnitude less than the anticipated needs\textsuperscript{3}. The performance limits of MMF-based transmission are still the subject of research, and complex intermodal interactions are a potential obstacle. Here we have observed collective dynamics because of the cooperation of disorder and dissipation with nonlinearity. We used relatively low peak powers (~1 kW), and our experiments roughly compare to a loss-managed line of 150 km (Supplementary Table 1). This highlights the potential complexity of MMF transmission impairment, and the importance of understanding not only nonlinear interference effects between modes, but also the possibility of collective dynamics, which involves many channels. Ultimately, realistic multimode telecommunications scenarios will involve disorder, dissipation and nonlinearity. Understanding their individual and collective effects is an important area for future research.

We have shown how a given initial field in a GRIN MMF self-organizes into a state that is the most unstable through the cooperation of nonlinearity, dissipation and disorder. The subsequent evolution from this state causes the field to develop a spatiotemporally complex nature. The results showcase the wide-ranging complex phenomena that can be investigated effectively in MMFs and raise many questions and opportunities. Applications include powerful and flexible fibre lasers, space-division multiplexing and novel computing platforms.

Methods

Methods and any associated references are available in the online version of the paper.

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Author contributions
L.G.W. performed simulations and experiments, with assistance provided by Z.L. D.A.N. and M.-J.L. made and provided small-core GRIN fibres. L.G.W. and F.W.W. wrote the final version.

Competing financial interests
D.A.N. and M.-J.L. are employed by Corning Incorporated, which manufactures optical fibres for applications including telecommunications.
Methods

Main experiments. We conducted experiments by launching pulses -1 ns long from a Q-switched, frequency-doubled neodymium yttrium aluminium garnet laser into GRIN fibres of length 50–100 m. The fibres have a numerical aperture of 0.137, a radius of 13 µm and are nearly parabolic (α = 1.95). The small core translates into enhanced nonlinearity and to relatively few modes, and thereby increases the validity of the simulations. The initial spatial condition (that is, the combination of modes that are excited) was varied by adjusting the position of the polarizer. For the majority of initial conditions, we observed results similar to those shown in Figs 1 and 3. In exceptional cases, STMI may develop before the ultimate state of the attractor (see, for example, Supplementary Figs 23 and 24). However, in these cases self-organized instability and evolution towards the attractor are still observed.

Mode-coupling matrices. For Fig. 2a,b, which show the nonlinear mode-coupling matrix, we computed the coefficients from equation (1) at a particular z value (that is, \( C_{0f(z')} \)) for 500 random fields. These fields are each generated as the coherent sum of random combinations of the fibre’s 55 modes, that is:

\[
I(x,y) = \sum_{i=1}^{55} c_i \psi_i(x,y)^2
\]

where \( c_i = a_i + ib_i \) and \( a_i \) and \( b_i \) are random numbers from a uniform distribution from −1 to 1. This is, therefore, the average mode-coupling matrix for a random field in the fibre at a given position \( z' \) along the fibre. This illustrates the typical characteristics of the nonlinear mode coupling.

For the disorder-induced mode-coupling matrix (Fig. 2c and Supplementary Figs 1 and 2), we computed the coefficients from equation (1) for 100 random bends of the fibre, described by \( \Delta n(x,y) = (\Delta_n x/R) + (\Delta_n y/R) \) for \((x^2 + y^2)^{0.5} \leq R\) and equal to 0 otherwise. The values \( \Delta_n \) were obtained from a uniform distribution that ranged from \( \Delta_n = -0.025\Delta \) to \( 0.025\Delta \), where \( \Delta \) is the difference between the centre and cladding index of the fibre, \( 0.0064 \).

Spatiospectral measurement. We inserted a cylindrical lens at the end of the 4-f telescope used to image the near-field beam profile, so that the focus of the cylindrical lens coincided with the imaging plane of the telescope. A bare fibre was scanned through the beam along the dimension unfocused by the cylindrical lens. To observe the clearest hyperbolic spatiotemporal shape, it is helpful to have a large size difference between the modes. Therefore, we chose a fibre with a larger core size (−20 µm radius) but similar numerical aperture as the one used for the experiment in Fig. 1. Similar measurements as those shown in Fig. 3 for the fibre used in Fig. 1 are presented in Supplementary Figs 21 and 22. Furthermore, for comparison with the theory in Fig. 3a, we used parameters of a typical GRIN fibre with a 62.5 µm core diameter, which has more modes than the fibres studied, so that the plot could be extended to high mode groups to show the trend. The modes in the plot are integrated over one spatial dimension to facilitate comparison with the experiment.

Instability simulations. We conducted the simulations using the generalized multimode nonlinear Schrödinger equation. We considered the first 30 linearly polarized modes of the fibre, which were calculated numerically. To ensure that spatiotemporal instabilities were excited and that the attractor was not important for the evolution, and to conduct the simulations in a timely fashion, we launched high-power (560 kW) pulses. This corresponds to \( 0.45P_{\text{crit}} \) at 532 nm. The launched pulses were 3.2 µJ, 5 ps long Gaussians, with the initial spatial profiles shown in Fig. 4. The sideband energies were measured after 0.6 cm, which was sufficient to see a substantial gain for most initial conditions. To make sure the results were not merely a result of the large peak power, several selected initial conditions were tested at lower power (0.03\( P_{\text{crit}} \)) and longer propagation lengths (>30 cm). These simulations yielded the same conclusion as the high-power ones (Supplementary Figs 15–20).