The Schrödinger Functional — a Renormalizable Probe for Non-Abelian Gauge Theories

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Abstract

Following Symanzik we argue that the Schrödinger functional in lattice gauge theories without matter fields has a well-defined continuum limit. Due to gauge invariance no extra counter terms are required. The Schrödinger functional is, moreover, accessible to numerical simulations. It may hence be used to study the scaling properties of the theory and in particular the evolution of the renormalized gauge coupling from low to high energies. A concrete proposition along this line is made and the necessary perturbative analysis of the Schrödinger functional is carried through to 1-loop order.
1. Introduction

The Schrödinger functional is the propagation kernel for going from some field configuration at time $x^0 = 0$ to some other configuration at $x^0 = T$. In euclidean space-time it can be written as a functional integral over all fields with the specified initial and final values. The renormalization of the Schrödinger functional has been discussed by Symanzik [1] some time ago in the course of his proof of the existence of the Schrödinger picture in renormalizable quantum field theories (for an introduction to this paper see ref.[2]). His result is that the Schrödinger functional can be renormalized by adding the usual counterterms to the action plus possibly a set of further terms that are integrals of local polynomials in the field and its derivatives over the $x^0 = 0$ and $x^0 = T$ hyper-planes. In the case of the $\phi^4$ theory, for example, two polynomials, $\phi^2$ and $\phi\partial_0\phi$, are needed. Pure gauge theories are simpler in this respect, because no extra counterterms are required here, as we shall argue in sect. 2. If we choose a lattice to regularize the theory, this statement simply means that the Schrödinger functional converges in the continuum limit, provided, of course, the bare coupling is scaled in the usual way.

Our motivation to consider the Schrödinger functional is that we would like to apply the finite size scaling technique of ref.[3] to gauge theories. The ultimate goal of this programme is to compute the running coupling in say the minimal subtraction (MS) scheme of dimensional regularization at short distances given in units of the low energy scales of the theory (the string tension, for example, or the mass of the lightest glueball). In other words, our aim is to connect the non-perturbative infrared behaviour of the theory with the high energy regime, where the coupling is logarithmically decreasing according to the renormalization group.

To explain in which way the Schrödinger functional enters this calculation, we need to recall the basic strategy of ref.[3]. One begins by putting the theory in a finite spatial volume with linear extension $L$ and periodic boundary conditions in all directions. Next one introduces some renormalized coupling $\bar{g}^2(L)$ which does not depend on any scale other than $L$ and which can hence be considered a running coupling. This coupling is then computed over a range of $L$ through numerical simulation of the lattice theory, using a recursive procedure which allows one to go from large values of $L$ (where contact is made with the non-perturbative scales) to the perturbative small $L$ domain. At these distances $\bar{g}^2(L)$ can be analytically related to other more commonly used couplings such as the coupling in the MS scheme of dimensional regularization.
Note that the latter are usually defined in infinite volume. All reference to a finite volume thus disappears from the final result.

The precise definition of the running coupling $\bar{g}^2(L)$ is not of principal importance. The coupling should however be accurately computable through numerical simulation and its scaling properties should not be strongly influenced by the presence of a non-zero lattice spacing. It is our experience that these requirements are not easy to fulfil. In particular, extracting a running coupling from correlation functions of Wilson loops is difficult, because large loops give a poor signal while small loops are affected by lattice artefacts. As we shall explain in sect. 2, running couplings can be straightforwardly defined through the Schrödinger functional. With carefully chosen boundary values for the gauge field, these couplings have, moreover, the desired technical properties which make our finite size scaling study feasible.

The idea to probe gauge theories through external fields is not new of course. In particular, the background field method [4–8] has long proved to be an efficient tool for perturbative calculations, in a wide range of theories and contexts (see e.g. refs.[9–12]). Our work can also be understood as a continuation of ref.[13]. Most of the basic concepts that we now exploit already appear there, in a language oriented towards statistical mechanics.

In the present paper the Schrödinger functional in Yang-Mills theories is discussed in detail. Numerical results on the scaling behaviour of the couplings derived from it will be published elsewhere. After going through the formal definition of the functional in sect. 2, we verify explicitly, using dimensional regularization, that indeed there are no additional counterterms at one-loop order of perturbation theory (sect. 3). We then discuss the definition and cutoff dependence of the Schrödinger functional on the lattice (sect. 4). In our numerical work, we take constant Abelian fields for the boundary values of the gauge field at $x^0 = 0$ and $x^0 = T$, and so, in sects. 5 and 6, we set up the perturbation expansion for this case. In particular, the stability of the gauge field configuration around which one expands must be established and the gauge fixing must be done with care to obtain all boundary terms correctly. As an application we calculate our running coupling to one-loop order, for the theory with gauge group SU(2) (sect. 7).
2. A first look at the Schrödinger functional

The aim in this section is to give an introduction to the Schrödinger functional in Yang-Mills theories and to establish the basic notation. We shall not at this point worry about the mathematical status of the quantities considered. Later on when the theory is regularized either dimensionally or by passing to the lattice formulation, we shall be able to deal with the Schrödinger functional on a more rigorous level.

For definiteness we assume that the gauge group is SU($N$). Our conventions regarding indices, group generators etc. are listed in appendix A.

2.1 Formal definition of the Schrödinger functional

Our starting point is the Hamiltonian formulation of the theory in the temporal gauge. In this framework the theory is specified at a fixed time, say $x^0 = 0$, by assuming canonical commutation relations among the basic field variables and by giving the Hamilton operator.

As explained in the introduction, the Schrödinger functional will be used to study the scaling properties of the theory in finite volume, and so we take space to be an $L \times L \times L$ box with periodic boundary conditions. SU($N$) gauge fields are accordingly represented by periodic vector potentials $A_k(x)$ on $\mathbb{R}^3$ with values in the Lie algebra of SU($N$)†. To preserve periodicity under gauge transformations

$$A_k(x) \to A_k^\Lambda(x) = \Lambda(x)A_k(x)\Lambda(x)^{-1} + \Lambda(x)\partial_k\Lambda(x)^{-1}, \quad (2.1)$$

only periodic gauge functions $\Lambda(x)$ will be admitted. $\Lambda$ can thus be regarded as a mapping from a 3-dimensional torus to SU($N$). Continuous functions of this kind are topologically non-trivial in general. More precisely, they fall into disconnected topological classes labelled by an integer winding number

$$n = \frac{1}{24\pi^2} \int_0^L d^3x \epsilon_{klj} \text{tr} \left\{ (\Lambda\partial_k\Lambda^{-1}) (\Lambda\partial_l\Lambda^{-1}) (\Lambda\partial_j\Lambda^{-1}) \right\}. \quad (2.2)$$

Gauge transformations with non-zero winding number are explicitly allowed and will play an important rôle later on.

† We do not consider the possibility of twisted periodic boundary conditions in this paper.
In the Schrödinger representation the quantum mechanical states of the theory are wave functionals \( \psi[A] \), where \( A \) runs over all gauge fields as described above. A scalar product is formally given by

\[
\langle \psi | \chi \rangle = \int D[A] \psi[A]^\ast \chi[A], \quad D[A] = \prod_{x,k,a} dA^a_k(x). \tag{2.3}
\]

Only gauge invariant states \( \psi[A] \), i.e. those satisfying

\[
\psi[A^\Lambda] = \psi[A] \tag{2.4}
\]

for all gauge transformations \( \Lambda \), are physical. In particular, we choose the vacuum angle \( \theta \) [15,16] to vanish. Any given wave functional \( \psi[A] \) can be projected on the physical subspace through

\[
\psi[A] \rightarrow \mathbf{P} \psi[A] = \int D[A] \psi[A^\Lambda], \quad D[A] = \prod_x d\Lambda(x), \tag{2.5}
\]

where \( dU, U \in \text{SU}(N) \), denotes the normalized invariant measure on \( \text{SU}(N) \).

The gauge field \( A^a_k(x) \) can in the obvious way be interpreted as an operator field acting on wave functionals \( \psi[A] \). The canonically conjugate field is the colour electric field

\[
F^a_{0k}(x) = \frac{1}{i} \frac{\delta}{\delta A^a_k(x)}. \tag{2.6}
\]

The magnetic components of the colour field tensor are

\[
F^a_{kl}(x) = \partial_k A^a_l(x) - \partial_l A^a_k(x) + f^{abc} A^b_k(x) A^c_l(x), \tag{2.7}
\]

and the Hamilton operator \( \mathbf{H} \) is given by

\[
\mathbf{H} = \int_0^L d^3x \left\{ \frac{g_0^2}{2} F^a_{0k}(x) F^a_{0k}(x) + \frac{1}{4g_0^2} F^a_{kl}(x) F^a_{kl}(x) \right\} \tag{2.8}
\]

with \( g_0 \) being the (bare) gauge coupling.

For any smooth classical gauge field \( C_k(x) \), a state \( |C\rangle \) may be introduced such that

\[
\langle C|\psi = \psi[C] \tag{2.9}
\]
for all wave functionals $\psi[A]$. $|C\rangle$ is not gauge invariant, of course, but it may be made so by applying the projector $\mathbf{P}$. The (euclidean) Schrödinger functional $\mathcal{Z}[C',C]$ is now defined by

$$\mathcal{Z}[C',C] = \langle C'| e^{-\mathbf{H}T} \mathbf{P} | C \rangle. \quad (2.10)$$

We shall always assume $T > 0$ and do not explicitly indicate the dependence of the Schrödinger functional on this parameter. If we insert an orthonormal basis $|\psi_n\rangle$, $n = 0,1,2,\ldots$, of gauge invariant energy eigenstates, the spectral representation

$$\mathcal{Z}[C',C] = \sum_{n=0}^{\infty} e^{-E_n T} \psi_n[C'] \psi_n[C]^* \quad (2.11)$$

is obtained, where $E_n$ are the energy eigenvalues (the spectrum is discrete in finite volume). Since only physical intermediate states contribute, it is evident that $\mathcal{Z}[C',C]$ is invariant under arbitrary gauge transformations of the boundary fields $C$ and $C'$.

### 2.2 Functional integral representation

The matrix elements of the euclidean time evolution operator $e^{-\mathbf{H}T}$ between gauge invariant states can be expressed through a functional integral over all gauge field configurations $A_\mu(x)$ in four dimensions with $0 \leq x^0 \leq T$ and periodic boundary conditions in the spatial directions. For the matrix element (2.10), the appropriate initial and final values of the gauge field are

$$A_k(x) = \begin{cases} C_k^A(x) & \text{at } x^0 = 0, \\ C_k'(x) & \text{at } x^0 = T, \end{cases} \quad (2.12)$$

and an integration over all gauge transformations $\Lambda$ is required to account for the projector $\mathbf{P}$ [cf. eq.(2.5)]. The functional integral representation of the Schrödinger functional thus reads

$$\mathcal{Z}[C',C] = \int \mathbf{D}[A] \int \mathbf{D}[A] e^{-S[A]}, \quad (2.13)$$

where the measure $\mathbf{D}[A]$ now stands for an integration over all components of the euclidean field. The euclidean action is given by

$$S[A] = -\frac{1}{2g_0^2} \int d^4x \tr \{ F_{\mu\nu} F_{\mu\nu} \}, \quad (2.14)$$
with
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \] 
\[(2.15)\]

Finally, an overall normalization factor independent of the boundary values \(C\) and \(C'\) has been dropped in eq.(2.13). In the following we do not keep track of such factors, because we are ultimately only interested in the field dependence of the Schrödinger functional.

The reader may wonder at this point how it comes that the time component of the gauge field appears in the functional integral, while in the Hamiltonian formulation it did not. The reason is that the integral (2.13) and the boundary conditions (2.12) are invariant under the gauge transformation
\[ A_\mu(x) \rightarrow \Omega(x) A_\mu(x) \Omega(x)^{-1} + \Omega(x) \partial_\mu \Omega(x)^{-1}, \] 
\[ \Lambda(x) \rightarrow \Omega(x)|_{x^0=0} \Lambda(x), \] 
\[(2.16, 2.17)\]
provided the gauge function \(\Omega(x)\) satisfies
\[ \Omega(x) = 1 \text{ at } x^0 = T. \] 
\[(2.18)\]

An admissible gauge fixing condition for this symmetry is the temporal gauge \(A_0 = 0\). The associated Faddeev-Popov determinant is field independent and may be included in the overall normalization of the integral. The time component of the gauge field can thus be eliminated and it is now straightforward to make contact with the Hamiltonian expression for the Schrödinger functional.

For the perturbative calculations to be discussed later on, other gauge choices will be more convenient so that in the following we keep the time component of the gauge field as an integration variable.

We may however use the gauge invariance of the functional integral to reduce the integration over the gauge transformation \(\Lambda\) to a sum over topological classes. To this end, first note that after the inner integral in eq.(2.13) has been performed, one is left with some function of \(\Lambda\) to be integrated over. This function depends only on the winding number of \(\Lambda\), because any two gauge functions in the same topological class may be connected through a gauge transformation \(\Omega\) as described above [eqs.(2.16–18)]. So if we choose, for each integer \(n\), some fixed representative \(\Lambda_n(x)\) in the class of gauge functions with winding number \(n\), the Schrödinger functional becomes
\[ Z[C', C] = \sum_{n=-\infty}^{\infty} \int D[A] e^{-S[A]}, \] 
\[(2.19)\]
where now we require that

\[ A_k(x) = \begin{cases} 
    C_k^{\Lambda_n}(x) & \text{at } x^0 = 0, \\
    C_k'(x) & \text{at } x^0 = T.
\end{cases} \tag{2.20} \]

It is convenient to set \( \Lambda_0 = 1 \), but an explicit choice of the non-trivial \( \Lambda_n \)'s would not be useful in what follows. That a sum over topological classes is needed on top of the functional integration is perhaps not too surprising, if we recall that this is the normal situation in Yang-Mills theories on compact manifolds without boundary.

### 2.3 Instanton bound

It is well-known [14] that the gauge field action is bounded by

\[ S[A] \geq \frac{8\pi^2}{g_0^2} |Q[A]|, \tag{2.21} \]

where \( Q[A] = -\frac{1}{16\pi^2} \int d^4x \text{tr} \{ F_{\mu\nu}^* F_{\mu\nu} \} \) (2.22) denotes the topological charge of \( A \) and

\[ F_{\mu\nu}^* = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{2.23} \]

the dual of the field tensor (2.15). \( Q[A] \) may be expressed through the boundary values \( C \) and \( C' \) and the winding number \( n \). To see this note that

\[ \frac{1}{2} \text{tr} \{ F_{\mu\nu}^* F_{\mu\nu} \} = \epsilon_{\mu\nu\rho\sigma} \partial_\mu \text{tr} \{ A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \}. \tag{2.24} \]

Taking the boundary conditions (2.20) into account and integrating by parts then yields (after some algebra)

\[ Q[A] = S_{CS}[C'] - S_{CS}[C] - n, \tag{2.25} \]

where

\[ S_{CS}[C] = -\frac{1}{8\pi^2} \int d^3x \epsilon_{klj} \text{tr} \{ C_k \partial_l C_j + \frac{2}{3} C_k C_l C_j \} \tag{2.26} \]

is the Chern-Simons action of the boundary field \( C \).
2.4 Induced background field

For small couplings $g_0$, the functional integral (2.19) is dominated by the fields around the absolute minima of the action. From the instanton bound it follows that we only need to inspect a small number of topological sectors to find the fields with least action (and the specified boundary values). In general there are several gauge inequivalent minimal action configurations and these can even occur in different winding number sectors.

For simplicity we shall from now on restrict attention to boundary values $C$ and $C'$, where the absolute minimum is attained in the $n = 0$ sector and where, furthermore, the minimal action configuration $B_\mu(x)$ is unique up to gauge transformations. This is the typical situation if $C$ and $C'$ are small. In the following $B$ will be referred to as the induced background field.

For given boundary values $C$ and $C'$, it is usually impossible to obtain the induced background field in closed analytical form. We may, however, invert the procedure and start from any known solution $B$ of the field equations. If we define $C$ and $C'$ through

$$C_k(x) = B_k(x)|_{x^0=0}, \quad C'_k(x) = B_k(x)|_{x^0=T},$$

(2.27)

the boundary conditions will be trivially satisfied. We are then left with the problem to prove that the chosen field $B$ is the unique minimal action configuration with these boundary values.

Little work is needed to show this, if the field tensor

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

(2.28)

is self-dual,

$$G_{\mu\nu} = {}^*G_{\mu\nu},$$

(2.29)

and if, furthermore, the bound

$$S_{CS}[C'] - S_{CS}[C] < \frac{1}{2}$$

(2.30)

holds. The point is that the instanton bound (2.21) is saturated in this case. All fields $A$ in the $n = 0$ sector hence have an action greater or equal to

$$S[A] = \frac{8\pi^2}{g_0^2} \{S_{CS}[C'] - S_{CS}[C]\}.$$  

(2.31)

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And the same is true for all other sectors, as one may quickly show by combining eqs. (2.21), (2.25), (2.31) with the bound (2.30).

We have thus proved that under the conditions stated above, $B$ is an absolute minimum of the action. It is in fact the only one, up to gauge transformations, because any other minimizing configuration would have to be self-dual and satisfy the same boundary conditions. Since the self-duality equation is first order in the time derivatives, the solution in the temporal gauge is uniquely determined by the initial values at $x^0 = 0$. In a general gauge, this means that any two solutions are related by a gauge transformation.

A simple example of a self-dual field is obtained by making the ansatz

$$B_0(x) = 0, \quad B_k(x) = b(x^0)I_k. \quad (2.32)$$

Here, $b$ is a real function and the group generators $I_k$ are chosen such that they form an irreducible representation of the angular momentum algebra

$$[I_k, I_l] = \epsilon_{klj}I_j. \quad (2.33)$$

Up to unitary transformations, there exists only one irreducible representation of this algebra with dimension $N$. In particular, the square of the angular momentum is given by

$$I_kI_k = -\frac{1}{4}(N^2 - 1). \quad (2.34)$$

Note that the action of the field (2.32) is finite if $b(x^0)$ is smooth and if $L < \infty$.

With the ansatz (2.32) the self-duality equation (2.29) reduces to

$$\partial_0 b = b^2, \quad (2.35)$$

so that

$$b(x^0) = (\tau - x^0)^{-1}. \quad (2.36)$$

To ensure the regularity of the solution in the interval $0 \leq x^0 \leq T$, the integration constant $\tau$ must either be negative or greater than $T$. We leave it to the reader to work out the condition (2.30), but it is obviously satisfied for sufficiently large values of $\tau$. We have thus found a one-parameter family of globally stable background fields.
2.5 Renormalization of the Schrödinger functional

In the weak coupling domain, the Schrödinger functional can be computed by performing a saddle point expansion of the functional integral (2.19) about the induced background field $B$. For the effective action

$$\Gamma[B] = -\ln Z[C', C],$$

an asymptotic series of the form

$$\Gamma[B] = g_0^{-2}\Gamma_0[B] + \Gamma_1[B] + g_0^2\Gamma_2[B] + \ldots$$

(2.38)
is thus obtained †. The leading term is given by

$$\Gamma_0[B] = g_0^2 S[B],$$

(2.39)

while the higher order contributions are sums of vacuum bubble Feynman diagrams with an increasing number of loops. The Feynman rules involve vertices and propagators that depend on the background field. To make the diagrams well-defined an ultra-violet regularization will be needed at this point. We shall either use dimensional regularization or introduce a space-time lattice. In both cases the gauge invariance of the theory is preserved, which is crucial for the renormalization to work out in the way described below.

Initially the saddle point expansion of the Schrödinger functional is performed in the regularized bare theory. That is, we integrate over the bare gauge field $A$, use the bare action to generate the Feynman rules and impose the boundary conditions (2.20) on the bare field. Since the regularization respects the gauge symmetry, the effective action $\Gamma[B]$ does not depend on the gauge fixing condition employed. Moreover, it is invariant under gauge transformations

$$B_\mu(x) \rightarrow B'_\mu(x) = \Omega(x)B_\mu(x)\Omega(x)^{-1} + \Omega(x)\partial_\mu\Omega(x)^{-1}$$

(2.40)
of the background field for arbitrary (periodic) gauge functions $\Omega$.

Ultra-violet divergences now appear in each order of the bare coupling $g_0$ when we try to remove the regularization. We certainly expect that some of

† As already mentioned earlier, we are only interested in the field dependence of the Schrödinger functional. Any additive contributions to the effective action independent of $B$ are hence dropped without further notice.
the divergent terms are cancelled by the usual renormalization of the coupling constant. As we shall show in detail in the following sections, the one-loop effective action actually becomes finite after the coupling is renormalized. In other words, what is suggested is that the (bare) Schrödinger functional does not need any renormalization besides the coupling constant renormalization.

A rigorous proof of this conjecture to all orders of perturbation theory is beyond the scope of this paper. We may, however, make our proposition more plausible by recalling Symanzik’s work on the Schrödinger representation in quantum field theory [1]. Symanzik studied the massless $\phi^4$ theory and showed that all divergences in the Schrödinger functional can be cancelled by renormalizing the coupling constant and by including the boundary counterterms

$$\int_{x^0=T} d^3x \left\{ Z_1 \phi^2 + Z_2 \phi \partial_0 \phi \right\} + \int_{x^0=0} d^3x \left\{ Z_1 \phi^2 - Z_2 \phi \partial_0 \phi \right\}$$

in the bare action. The terms proportional to $\phi^2$ do not influence the propagators and vertices and just add to the effective action. Because the renormalization constant $Z_1$ is linearly divergent, they are not needed if one employs dimensional regularization. The other counterterms are equivalent to a rescaling of the boundary values of the field [1]. They are logarithmically divergent and must be taken into account when deriving the renormalization group equation for the Schrödinger functional.

Symanzik also expressed the expectation that in a general renormalizable field theory, the Schrödinger functional can be made finite by renormalizing the coupling and mass parameters and by adding a few boundary counterterms to the action. These are proportional to some local composite fields of dimension less than or equal to 3 integrated over the $x^0 = 0$ and $x^0 = T$ hyper-planes.

Coming back to Yang-Mills theories, the obvious question is whether there are any candidates for such counterterms. Since the effective action is independent of the gauge fixing condition employed, we should be able to write them in a gauge invariant form not involving the Faddeev-Popov ghosts. We now note, however, that any non-trivial gauge invariant polynomial in the gauge potential and its derivatives has dimension greater than 3. A counterterm proportional to the Chern-Simons action (2.26) is also excluded, because the latter is odd under parity. The upshot then is that divergent boundary terms cannot occur in this theory and that consequently the (bare) Schrödinger functional is finite after the coupling has been renormalized.
2.6 Running couplings in finite volume

As explained in sect. 1, the finite size scaling study that we propose to apply to Yang-Mills theories is based on the idea of a renormalized coupling $\bar{g}^2(L)$, which runs with the box size $L$. Starting from the Schrödinger functional, there are many ways to introduce such a coupling. For example, we may choose some background field $B$ which depends on a dimensionless parameter $\eta$. From the above we then infer that

$$\Gamma'[B] = \frac{\partial}{\partial \eta} \Gamma[B]$$

(2.42)

is a renormalization group invariant. A renormalized coupling may hence be defined through

$$\bar{g}^2 = \frac{\Gamma'_0[B]}{\Gamma'[B]}$$

(2.43)

[cf. eq.(2.38)]. Note that $\Gamma'_0[B]$ is just a normalizing factor, which guarantees that $\bar{g}^2$ coincides with $g_0^2$ to leading order of perturbation theory.

In general the chosen background field depends on several external scales and the corresponding renormalized coupling must be regarded as a function of all these parameters. In addition it depends on the box size $L$ and the propagation time $T$. To obtain a coupling which runs with $L$, we simply scale all dimensionful parameters with fixed proportions relative to $L$. For example, we may set $T = L$ and take the self-dual configuration (2.32),(2.36) with

$$\tau = -L/\eta$$

(2.44)

as the background field.

There are, of course, other choices of background fields and a corresponding manifold of renormalized couplings. In particular, a simple alternative are spatially constant Abelian fields, which, for reasons given later, are actually more suitable for our numerical work.
3. Perturbation Expansion

We now discuss the saddle point expansion of the Schrödinger functional around the induced background field using dimensional regularization. In particular, we would like to show that to one-loop order the renormalization of the Schrödinger functional works out in the way described above. Our calculations rely on well-established techniques, which have previously been applied in the context of instantons and the semi-classical approximation [17–22]. There is also a formal similarity with the so-called background field method [4–8], although here we do not expand in powers of the background field.

As before we shall assume that the absolute minimum of the action occurs in the \( n = 0 \) sector and that the minimal action configuration \( B \) is unique up to gauge transformations. To avoid some technicalities when fixing the gauge, we shall in addition require that the gauge group acts freely on the boundary values \( C \) and \( C' \). This means that the only spatial gauge transformations \( \Lambda \) which leave \( C \) or \( C' \) invariant are constant and equal to a central element of \( SU(N) \). The self-dual background field considered earlier has this property. Abelian fields however do not and the gauge fixing procedure must consequently be reexamined in this case (see sect. 6).

3.1 Dimensional regularization

The familiar techniques of dimensional regularization apply to Feynman diagrams in the momentum space representation [23–26]. A different approach is required in the presence of background fields, because the propagators and vertices do not in general have their standard form. One possibility is to stay in position space and to insert the heat kernel representation for the propagators. The Feynman integrals then become well-defined analytic functions of the space-time dimension. In the following we shall proceed along these lines and refer the reader to ref.[27] for an introduction to the method.

Dimensional regularization starts by extending space-time to a \( D \) dimensional manifold, where the extra \( D - 4 \) dimensions are here assumed to be spatial with the usual periodic boundary conditions. The bare action of a gauge field \( A_\mu(x) \) on this manifold is again given by eq.(2.14), except that we now integrate over all \( D \) coordinates of \( x \) and that the Lorentz indices run from 0 to \( D - 1 \) (cf. appendix A). The boundary values \( C \) and \( C' \) and the background field \( B \) are taken to be independent of the extra coordinates.
Their components in these directions are set to zero. In particular,

\[ S[B] = L^{D-4} \{ S[B] \}_{D=4}, \tag{3.1} \]

and the same volume factor also appears in all higher order contributions to the effective action, because the translation symmetry in the added dimensions is not affected by the background field.

The renormalization of the gauge coupling is performed as usual. We choose the minimal subtraction (MS) scheme \([28]\) and accordingly denote the renormalized coupling by \(g_{\text{MS}}\). The relation between the bare and renormalized couplings reads

\[ g_0^2 = \mu^{2\varepsilon} g_{\text{MS}}^2 \left\{ 1 + \sum_{l=1}^{\infty} z_l(\varepsilon) g_{\text{MS}}^{2l} \right\}, \quad D = 4 - 2\varepsilon, \tag{3.2} \]

where \(\mu\) is the normalization mass. The singular coefficients

\[ z_l(\varepsilon) = \sum_{k=1}^{l} z_{lk} \varepsilon^{-k} \tag{3.3} \]

coincide with the counterterms calculated in ordinary perturbation theory. In particular, the one-loop coefficient is given by

\[ z_1(\varepsilon) = -\frac{11}{3\varepsilon} \frac{N}{16\pi^2}, \tag{3.4} \]

and the two- and three-loop coefficients can be found in refs.\([29–31]\).

### 3.2 Gauge fixing

The action \(S[A]\) and the a priori measure \(D[A]\) are invariant under arbitrary gauge transformations \(A \rightarrow A^\Omega\) [cf. eq.\((2.40)\)]. We are interested in evaluating the functional integral for fixed boundary values \(C\) and \(C'\), and so, in this subsection, restrict attention to the group \(\hat{G}\) of transformations \(\Omega\) which leave the boundary values intact. Our assumptions on the boundary values imply that a gauge function \(\Omega\) belongs to \(\hat{G}\) if and only if

\[ \Omega(x) = \begin{cases} e^{i2\pi m/N} & \text{at } x^0 = 0, \\ e^{i2\pi m'/N} & \text{at } x^0 = T, \end{cases} \tag{3.5} \]
for some integers $m$ and $m'$. $\hat{G}$ accordingly decomposes into $N^2$ disconnected components.

The center $Z_N$ of $\hat{G}$ consists of all gauge functions $\Omega = e^{i2\pi m/N}$. These transformations act trivially on gauge fields and so it is really the factor group $G = \hat{G}/Z_N$ with which we are concerned when fixing the gauge. $G$ may be identified with the $m' = 0$ component of $\hat{G}$. It acts freely on the space of gauge fields, i.e. $A^\Omega = A$ for some $A$ implies $\Omega = 1$.

The gauge fixing procedure is the usual one: we add a gauge fixing term to the action and include an integration over Faddeev-Popov ghost fields $c$ and $\bar{c}$ with the appropriate action. To write down the gauge fixing term, the gauge field $A$ integrated over is decomposed according to

$$A_\mu(x) = B_\mu(x) + g_0 q_\mu(x). \quad (3.6)$$

The “quantum” field $q$ introduced here is the new integration variable, while $B$ will be kept fixed. In perturbation theory, only the winding number zero sector contributes. The boundary conditions (2.20) thus become

$$q_k(x) = 0 \quad \text{at } x^0 = 0 \text{ and } x^0 = T. \quad (3.7)$$

Note that the time component $q_0$ remains unrestricted at this point.

In terms of the quantum field $q$, a gauge transformation $A \to A^\Omega$ amounts to the substitution

$$q_\mu(x) \to \Omega(x)q_\mu(x)\Omega(x)^{-1} + g_0^{-1} \left[ B^\Omega_\mu(x) - B_\mu(x) \right]. \quad (3.8)$$

In particular, if we set

$$\Omega(x) = 1 - g_0 \omega(x) + O(g_0^2), \quad (3.9)$$

the transformation becomes

$$q_\mu(x) \to q_\mu(x) + D_\mu \omega(x) + O(g_0), \quad D_\mu = \partial_\mu + \text{Ad } B_\mu. \quad (3.10)$$

This suggests that we take

$$S_{gf}[B, q] = -\lambda_0 \int \! d^D x \, \text{tr} \{ D_\mu q_\mu D_\nu q_\nu \} \quad (3.11)$$
as the gauge fixing term, where $\lambda_0$ denotes the bare gauge fixing parameter. This gauge is referred to as the background gauge. The associated Faddeev-Popov action reads

$$S_{\text{FP}}[B, q, c, \bar{c}] = 2 \int d^D x \ tr \{ \bar{c} D_\mu (D_\mu + g_0 A q_\mu) c \}, \quad (3.12)$$

and so we end up with

$$e^{-\Gamma[B]} = \int D[q] \int D[c] D[\bar{c}] e^{-S_{\text{total}}[B, q, c, \bar{c}]}, \quad (3.13)$$

where

$$S_{\text{total}}[B, q, c, \bar{c}] = S[B + g_0 q] + S_{\text{gf}}[B, q] + S_{\text{FP}}[B, q, c, \bar{c}] \quad (3.14)$$

is the total action.

We have already noted above that the spatial components of $q$ must vanish at $x^0 = 0$ and $x^0 = T$. The boundary conditions on $q_0$ and the Faddeev-Popov fields $c$ and $\bar{c}$ are determined by the gauge fixing procedure. In particular, they depend on the choice of the gauge fixing term and on the gauge transformation properties of the boundary values $C$ and $C'$. The influence of the latter is subtle and will only become clear after a while.

In the background gauge and for boundary values as specified at the beginning of this section, it turns out that

$$D_0 q_0(x) = c(x) = \bar{c}(x) = 0 \quad \text{at } x^0 = 0 \text{ and } x^0 = T \quad (3.15)$$

are the correct boundary conditions. This is not easy to prove, because the functional integral is not really well-defined before we fix the gauge. A rigorous derivation can (and will) be given after we pass to the lattice formulation of the theory (cf. sect. 6).

At this point it is still possible to explain, on a heuristic level, how the boundary conditions (3.15) arise. Let us first consider the Faddeev-Popov fields. When going through the gauge fixing procedure, one notes that the Faddeev-Popov operator

$$\Delta_{\text{FP}} = -D_\mu (D_\mu + g_0 A q_\mu) \quad (3.16)$$

acts in the linear space of infinitesimal gauge transformations. So if $t$ is an anti-commuting parameter, the transformation $\Omega(x) = \exp[tc(x)]$ must be an
element of $G$, and the same is true for $\bar{c}$. From eq. (3.5) we now infer that the ghost fields vanish at $x^0 = 0$ and $x^0 = T$.

The boundary conditions on $q_0$ are harder to justify. An obvious case is the background field Landau gauge $D_\mu q_\mu = 0$, where the vanishing of $D_0 q_0$ at the boundaries is an immediate consequence of the boundary conditions on $q_k$. For general values of the gauge fixing parameter, we may resort to the slightly obscure argument that the gauge fixing function $D_\mu q_\mu$ should be a mapping from the space of gauge fields to the space of infinitesimal gauge transformations. Since the latter vanish at $x^0 = 0$ and $x^0 = T$, we again conclude that $q_0$ satisfies the boundary conditions (3.15).

3.3 One-loop effective action

It is now straightforward to expand the total action and the gauge fixed functional integral (3.13) in powers of $g_0$. To second order we have

$$S_{\text{total}}[B, q, c, \bar{c}] = S[B] - 2 \int d^D x \text{tr} \left\{ \frac{1}{2} q_\mu \hat{\Delta}_1 q_\mu + \bar{c} \hat{\Delta}_0 c \right\} + O(g_0), \quad (3.17)$$

where the operators $\hat{\Delta}_1$ and $\hat{\Delta}_0$ are defined through

$$\hat{\Delta}_1 q_\mu = -D_\nu D_\nu q_\mu + (1 - \lambda_0) D_\mu D_\nu q_\nu - 2 [G_{\mu
u}, q_\nu], \quad (3.18)$$

$$\hat{\Delta}_0 c = -D_\nu D_\nu c. \quad (3.19)$$

$\hat{\Delta}_1$ and $\hat{\Delta}_0$ act on fields in $D$ dimensions and are to be distinguished from

$$\Delta_1 = \hat{\Delta}_1 \bigg|_{D=4} \quad \text{and} \quad \Delta_0 = \hat{\Delta}_0 \bigg|_{D=4}. \quad (3.20)$$

Both are hermitean operators relative to the obvious scalar products in the spaces of fields satisfying the boundary conditions (3.7) and (3.15). They are also elliptic (if $\lambda_0 > 0$) and bounded from below. In particular, there exists a complete set of eigenfunctions with a discrete spectrum of real eigenvalues.

From the definition (3.19) it follows that all eigenvalues of $\hat{\Delta}_0$ are strictly positive. Negative eigenvalues of $\hat{\Delta}_1$ are excluded too, because $B$ is a minimal action configuration. There could be a finite number of zero modes, but this is an unlikely case which we shall not discuss any further, i.e. our assumptions on $B$ are supplemented accordingly.

If we now insert eq. (3.17) in the functional integral and expand the integrand, we obtain the series (2.38) for the effective action. The leading term
is given by eq.(2.39) as expected. At one-loop order we only need to perform the Gaussian integrals over the quantum field and the Faddeev-Popov ghosts and so end up with

\[ \Gamma_1[B] = \frac{1}{2} \ln \det \hat{\Delta}_1 - \ln \det \hat{\Delta}_0. \] (3.21)

The determinants occurring here are defined according to the rules of dimensional regularization (cf. sect. 3.3 of ref.[27]).

In the following it is convenient to set \( \lambda_0 = 1 \). We are free to make this choice, because \( \Gamma_1[B] \) is independent of the gauge fixing parameter. A rigorous proof of this statement could actually be given at this point, but since the argument is a bit lengthy (and the result is expected anyway), we do not present it here.

A more explicit expression for the determinants, exhibiting the dependence on the space-time dimensionality, can be given in terms of the heat kernels of \( \hat{\Delta}_1 \) and \( \hat{\Delta}_0 \). Heat kernels have simple factorization properties, and if we take these into account, one gets [27]

\[ \ln \det \hat{\Delta}_1 = - \int_0^\infty \frac{dt}{t} \left( \text{Tr} e^{-t\Delta_1} \right)^{-2\varepsilon} \left( \text{Tr} e^{-t\Delta_1} - 2\varepsilon \text{Tr} e^{-t\Delta_0} \right), \] (3.22)
\[ \ln \det \hat{\Delta}_0 = - \int_0^\infty \frac{dt}{t} \left( \text{Tr} e^{-t\Delta_1} \right)^{-2\varepsilon} \text{Tr} e^{-t\Delta_0}. \] (3.23)

The first factor in these integrals arises from the added dimensions. It involves the Laplacian \( \Delta_1 \) on the circle and is explicitly given by

\[ \text{Tr} e^{-t\Delta_1} = \sum_{n=-\infty}^{\infty} e^{-t(2\pi n/L)^2}. \] (3.24)

The other factors, \( \text{Tr} e^{-t\Delta_1} \) and \( \text{Tr} e^{-t\Delta_0} \), are defined in four dimensions and are thus independent of \( \varepsilon \). Finally, the term proportional to \( \varepsilon \) in eq.(3.22) must be included because the quantum field \( q \) has \( D \) vector components rather than 4.

An important point to note is that the integrals over the proper time \( t \) are convergent for \( \text{Re} \, \varepsilon > 2 \). At the upper end of the integration range, convergence is guaranteed by the exponential fall-off of the integrand (\( \Delta_1 \) and \( \Delta_0 \) are positive). For \( t \to 0 \), on the other hand, the asymptotic behaviour
of the heat kernels is computable. This goes under the name of the Seeley-
DeWitt expansion and will be discussed below. The result is that the functions
integrated over are proportional to $t^{-3+\varepsilon}$ at small $t$ and so are integrable for
$\text{Re} \varepsilon > 2$.

3.4 Seeley-DeWitt expansion

To evaluate the one-loop effective action near four dimensions, the integrals
(3.22) and (3.23) must be analytically continued from $\text{Re} \varepsilon > 2$ to a region
including the origin $\varepsilon = 0$. The key result which enables us to do so is the
Seeley-DeWitt expansion. There exists an extensive literature on the subject
(an incomplete list of references is [32–41]) and we shall, therefore, be rather
brief.

Let $\Delta$ stand for one of the operators $\Delta_1$ or $\Delta_0$. The Seeley-DeWitt
expansion states that

$$
\text{Tr} e^{-t\Delta} \sim \alpha_2(\Delta) t^{-2} + \alpha_{3/2}(\Delta) t^{-3/2} + \alpha_1(\Delta) t^{-1} + \ldots ,
$$

(3.25)

where the coefficients $\alpha_{j/2}(\Delta)$ are algebraically computable. There are two
different kinds of contributions, the volume and the boundary terms. Vol-
ume terms only occur for even $j$ and are equal to a local polynomial in the
background field $B$ and its derivatives integrated over the space-time mani-
fold. The boundary terms are constructed similarly, except that the integral
is taken over the hyper-planes at $x^0 = 0$ and $x^0 = T$ (with possibly a sign
difference between the two). The dimension of the polynomials must be equal
to $4 - j$ and $3 - j$, respectively.

A crucial observation now is that the spectrum of $\Delta_1$ is invariant under
the substitution $B \to B^\Omega$, for any gauge function $\Omega$. We simply transform the
eigenfunctions covariantly, viz.

$$
q_\mu(x) \to \Omega(x)q_\mu(x)\Omega(x)^{-1},
$$

(3.26)

and then note that the eigenvalue equation and the boundary conditions are
preserved. The same remark also applies to $\Delta_0$ and so it is clear that the
coefficients $\alpha_{j/2}(\Delta)$ must be gauge invariant.

The gauge symmetry just described is not accidental. The complete gauge
fixed functional integral is actually invariant if $B$ is gauge transformed and the
fields integrated over are rotated covariantly. In particular, as we have noted
earlier, the effective action satisfies
\[ \Gamma[B^\Omega] = \Gamma[B]. \]  
(3.27)

We emphasize that this property does not depend on our choice of gauge fixing term. The effective action is independent of the latter. The advantage of the background gauge only is that every individual contribution to the effective action is already invariant, while in a general gauge this would not be the case.

Besides the trivial polynomial (the constant) there is no gauge invariant polynomial in the background field and its derivatives with dimension less than 4. The polynomials with dimension 4 are linear combinations of \( \text{tr} \{ G_{\mu\nu} G_{\rho\sigma} \} \). Taking this into account it is clear that all coefficients \( \alpha_{j/2}(\Delta) \) with \( j > 0 \) are independent of the background field. Furthermore, using one of the known computational techniques (ref.[39] for example), one finds that
\[ \alpha_0(\Delta_0) = -\frac{1}{20} \alpha_0(\Delta_1) = \frac{N}{96\pi^2} \int d^4x \text{tr} \{ G_{\mu\nu} G_{\mu\nu} \}. \]  
(3.28)

It has certainly not escaped the reader’s notice that our argumentation here parallels the discussion of the renormalization of the Schrödinger functional in subsect. 2.5. In particular, it is because of gauge invariance that we are able to exclude the presence of boundary terms in the coefficients \( \alpha_0(\Delta) \). As we shall see below, this is intimately related to the fact that the renormalization of the effective action does not require any extra counterterms.

### 3.5 Removal of the ultra-violet cutoff

To perform the analytic continuation of the integrals (3.22) and (3.23) towards \( \varepsilon = 0 \), it is useful to break up the integration range at \( t = 1 \). Since the integrands fall off exponentially, it is clear that the integrals from 1 to infinity are entire analytic functions of \( \varepsilon \). So we only need to care about the integration from 0 to 1. In that range the integrands may be split into singular and regular parts according to
\[ \text{Tr} e^{-t\Delta} = \sum_{j=0}^4 \alpha_{j/2}(\Delta) t^{-j/2} + R(t|\Delta). \]  
(3.29)

From the Seeley-DeWitt expansion we know that \( R(t|\Delta) \) is of order \( t^{1/2} \) at small \( t \). Furthermore, using the Poisson summation formula, one may show
that
\[ \text{Tr} \, e^{-t\Delta^{s_1}} = \frac{L}{(4\pi t)^{1/2}} \sum_{n=-\infty}^{\infty} e^{-n^2L^2/4t}, \tag{3.30} \]
i.e. this factor is proportional to \( t^{-1/2} \). The contributions of the regular parts to the integrals are hence analytic in the region \( \text{Re}\, \epsilon > -\frac{1}{2} \). The remaining terms are easy to work out, since the integrands are explicitly known.

To be able to write the result of these computations concisely, we introduce the zeta function
\[ \zeta(s|\Delta) = \text{Tr} \, \Delta^{-s}. \tag{3.31} \]
The trace converges for \( \text{Re}\, s > 2 \), but after passing to the heat kernel representation, one may show, following the steps taken above, that \( \zeta(s|\Delta) \) extends to a meromorphic function in the whole complex plane. In particular, there is no singularity at \( s = 0 \) and we may define
\[ \zeta'(0|\Delta) = \left. \frac{d}{ds} \zeta(s|\Delta) \right|_{s=0}. \tag{3.32} \]
The determinants of \( \hat{\Delta}_1 \) and \( \hat{\Delta}_0 \) are now given by
\[ \ln \det \hat{\Delta}_1 = -\left[ \frac{1}{\epsilon} + \ln 4\pi - \gamma_E + \frac{1}{16} \right] L^{-2\epsilon} \alpha_0(\Delta_1) - \zeta'(0|\Delta_1), \tag{3.33} \]
\[ \ln \det \hat{\Delta}_0 = -\left[ \frac{1}{\epsilon} + \ln 4\pi - \gamma_E \right] L^{-2\epsilon} \alpha_0(\Delta_0) - \zeta'(0|\Delta_0), \tag{3.34} \]
where \( \gamma_E = 0.577... \) denotes Euler’s constant and all terms vanishing at \( \epsilon = 0 \) have been dropped.

From these results we immediately deduce that
\[ \Gamma_1[B] = \left. \frac{11}{3\epsilon} \frac{N}{16\pi^2} \Gamma_0[B] \right|_{\epsilon \to 0} + O(1). \tag{3.35} \]
This singularity is exactly cancelled by the coupling constant renormalization. Indeed, from eq.(3.2) we get
\[ \Gamma[B] = \mu^{-2\epsilon} \left\{ \frac{1}{g_{\text{MS}}^2} - z_1(\epsilon) \right\} \Gamma_0[B] + \Gamma_1[B] + O(g_{\text{MS}}^2). \tag{3.36} \]
and after inserting eq.(3.4), we are left with

\[
\{\Gamma[B]\}_{D=4} = \left\{ \frac{1}{g_{\text{MS}}} - \frac{11}{3} \frac{N}{16\pi^2} \left[ \ln 4\pi\mu^2 - \gamma_E + \frac{1}{11} \right] \right\} \Gamma_0[B]
\]

(3.37)

\[
- \frac{1}{2} \zeta'(0|\Delta_1) + \zeta'(0|\Delta_0) + O(g_{\text{MS}}^2).
\]

We have thus shown that to one-loop order the Schrödinger functional renormalizes in the expected way.

The zeta functions appearing in eq.(3.37) are complicated functionals of the background field, which cannot normally be reduced to simpler expressions. For spatially constant background fields they can be worked out numerically, essentially by listing all eigenvalues of \(\Delta_1 \) and \(\Delta_0 \) up to a certain size (cf. sect. 7).

4. Lattice Formulation

A non-perturbatively meaningful definition of the Schrödinger functional can be given in the framework of lattice gauge theories. The lattice regularization is not unique — a choice of lattice action must be made and some further arbitrariness is involved when the Schrödinger functional is introduced. We expect that these details do not matter in the continuum limit and what follows should therefore be regarded as one possible approach to the problem.

4.1 Gauge fields

We choose to set up the lattice theory in euclidean space and thus imagine that a regular hypercubic lattice is superimposed on the space-time manifold. \(T\) is assumed to be an integer multiple of the lattice spacing \(a\) so that the possible values of the time coordinate \(x^0\) of a lattice point \(x\) are \(0, a, 2a, \ldots, T\).

A lattice gauge field \(U\) is an assignment of a link variable \(U(x, \mu) \in \text{SU}(N)\) to every pair \((x, x + a\hat{\mu})\) of lattice points (\(\hat{\mu}\) denotes the unit vector in the \(\mu\)-direction). In particular, the temporal link variables \(U(x, 0)\) are defined for all lattice points with \(0 \leq x^0 < T\). Gauge functions \(\Omega(x)\) live on the sites \(x\) of the lattice and take values in \(\text{SU}(N)\). They act on the link variables according to

\[
U(x, \mu) \rightarrow U^\Omega(x, \mu) = \Omega(x)U(x, \mu)\Omega(x + a\hat{\mu})^{-1}.
\]

(4.1)
As in the continuum we require that gauge fields and gauge transformation functions are periodic in spatial directions with period \( L \) (which must also be an integer multiple of \( a \)).

The lattice regularized Schrödinger functional will be defined later on as an integral over all lattice gauge fields with prescribed boundary values

\[
W(x, k) = U(x, k)|_{x^0 = 0} \quad \text{and} \quad W'(x, k) = U(x, k)|_{x^0 = T}.
\]

To make contact with the continuum definition of the Schrödinger functional, \( W \) and \( W' \) must be related to the continuum boundary values \( C \) and \( C' \).

A natural relationship is suggested if we recall that \( U(x, \mu) \) is the parallel transporter for colour vectors from \( x + a\hat{\mu} \) to \( x \) along the straight line connecting these two points. To achieve the best possible matching of lattice and continuum boundary values, we should thus identify \( W(x, k) \) with the corresponding parallel transporter determined by the field \( C_k(x) \). In other words, we set

\[
W(x, k) = \mathcal{P} \exp \left\{ a \int_0^1 dt C_k(x + a\hat{k} - ta\hat{k}) \right\},
\]

and \( W'(x, k) \) is similarly given by the field \( C'_k(x) \) [in eq.(4.3) the symbol \( \mathcal{P} \) implies a path ordered exponential such that the fields at the larger values of the integration variable \( t \) come first]. Note that this construction is gauge covariant: if we perform a gauge transformation \( C \rightarrow C^\Lambda \), the associated boundary field \( W' \) transforms as a lattice gauge field should.

### 4.2 Lattice action and the Schrödinger functional

Following Wilson [42] the action \( S[U] \) of a lattice gauge field \( U \) is taken to be

\[
S[U] = \frac{1}{g_0^2} \sum_p w(p) \text{tr} \left\{ 1 - U(p) \right\},
\]

where the sum runs over all oriented plaquettes \( p \) on the lattice and \( U(p) \) denotes the parallel transporter around \( p \). The weight \( w(p) \) is equal to 1 in all cases except for the spatial plaquettes at \( x^0 = 0 \) and \( x^0 = T \) which are given the weight \( \frac{1}{2} \). The significance of this weight factor will be discussed later on.

The lattice regularized Schrödinger functional is now defined through

\[
\mathcal{Z}[C', C] = \int D[U] e^{-S[U]} , \quad D[U] = \prod_{x, \mu} dU(x, \mu),
\]

23
where one integrates over all lattice gauge fields \( U \) with fixed boundary values as described above [recall that \( dU \) denotes the normalized invariant measure on \( SU(N) \); the product in eq.(4.5) is over all lattice points \( x \) and directions \( \mu \) such that the link \( (x, x+a\hat{\mu}) \) is in the interior of the lattice]. We emphasize that the lattice Schrödinger functional is regarded as a functional of the continuous fields \( C \) and \( C' \) rather than the boundary values of the link field. The latter are determined through eq.(4.3) and its primed relative. In the continuum limit, \( C \) and \( C' \) are kept fixed, while the lattice spacing \( a \) is sent to zero and the bare coupling \( g_0 \) is scaled according to the renormalization group.

Compared to the formal continuum expression (2.19) we seem to miss a sum over topological classes here. Such an average is not needed on the lattice, because the functional integral (4.5) is already invariant under arbitrary gauge transformations of the boundary fields. To see this, recall that a gauge transformation \( C \to C^\Lambda \) induces a corresponding transformation \( W \to W^\Lambda \) of the lattice boundary field at \( x^0 = 0 \). In the functional integral, such a change of the boundary values can be compensated by a substitution \( U \to U^\Omega \) of the integration variables, where \( \Omega \) is an arbitrary gauge transformation function with

\[
\Omega(x) = \begin{cases} 
\Lambda(x) & \text{at } x^0 = 0, \\
1 & \text{at } x^0 = T.
\end{cases}
\]  

(4.6)

The crucial point to note is that gauge functions with these boundary values exist on the lattice, independently of whether \( \Lambda \) is topologically trivial or not.

A quantum mechanical interpretation of the lattice Schrödinger functional (4.5) can be given through the well-known transfer matrix construction \([43–47]\). The transfer matrix \( T_T \) in the temporal gauge \( U(x, 0) = 1 \) is a hermitean operator which acts on Schrödinger wave functions \( \psi[W] \) at time \( x^0 = 0 \). It may be regarded as the lattice expression for the step evolution operator \( e^{-i\hat{H}a} \). As in the continuum theory, physical wave functions \( \psi[W] \) must be gauge invariant. The associated projector is again denoted by \( \mathbf{P} \) and we may also introduce a state \( |W\rangle \) in a way analogous to \( |C\rangle \) [cf. eq.(2.9)].

The point we wish to make is that our definition of the lattice Schrödinger functional is precisely such that the identity

\[
Z[C', C] = \langle W' | (T_T)^{T/a} \mathbf{P} | W \rangle
\]

(4.7)

In some of the papers quoted above it is the gauge projected operator \( T = T_T \mathbf{P} \) which is referred to as the transfer matrix. There is no difference between \( T \) and \( T_T \) on the physical subspace. We here stick to \( T_T \) to match with the notation employed in sect. 2.
holds. This formula is entirely analogous to the corresponding continuum expression (2.10). In particular, a spectral representation of the type (2.11) may be obtained, for all values of the lattice spacing.

4.3 Background field

For given boundary values $C$ and $C'$ (with properties as specified earlier) and sufficiently small lattice spacings, we expect that the absolute minimum of the lattice action $S[U]$ is non-degenerate up to gauge transformations. The minimizing configuration $V$ should moreover converge to the induced background field $B$ of the continuum theory,

$$V(x, \mu) = 1 + aB_\mu(x) + O(a^2),$$  \hspace{1cm} (4.8)

provided that both fields are transformed to a definite gauge (the temporal gauge, for example). In perturbation theory, the lattice background field $V$ plays the same rôle as $B$ did in the continuum. It is, therefore, often necessary to find an analytical or at least an accurate numerical representation of $V$.

To illustrate these remarks, let us consider the self-dual background field which was introduced in sect. 2 [eqs.(2.32)–(2.36)]. Since $C$ and $C'$ are spatially constant in this case, the lattice boundary fields are simply given by

$$W(x, k) = \exp\{ab(0)I_k\} \quad \text{and} \quad W'(x, k) = \exp\{ab(T)I_k\}. \hspace{1cm} (4.9)$$

The associated minimal action configuration $V$ must be a solution of the field equations which one derives from the Wilson action $S[U]$. To be able to write them concisely, we introduce the plaquette field

$$P(x, \mu, \nu) = V(x, \mu)V(x + a\hat{\mu}, \nu)V(x + a\hat{\nu}, \mu)^{-1}V(x, \nu)^{-1} \hspace{1cm} (4.10)$$

and its covariant divergence

$$d^*P(x, \mu) = \sum_{\nu=0}^3 \{P(x, \mu, \nu) - V(x - a\hat{\nu}, \nu)^{-1}P(x - a\hat{\nu}, \mu, \nu)V(x - a\hat{\nu}, \nu)\}. \hspace{1cm} (4.11)$$

One may now show that the lattice action is stationary if and only if the traceless anti-hermitean part of $d^*P(x, \mu)$ vanishes,

$$d^*P(x, \mu) - d^*P(x, \mu)^\dagger - \frac{1}{N} \text{tr} \{d^*P(x, \mu) - d^*P(x, \mu)^\dagger\} = 0, \hspace{1cm} (4.12)$$
for all links \((x, x + a\hat{\mu})\) in the interior of the lattice.

The form of the continuum background field suggests that we try to solve the lattice field equations (4.12) with the ansatz

\[
V(x, 0) = 1, \quad V(x, k) = \exp\{av(x^0)I_k\}, \tag{4.13}
\]

where \(v\) is some real function to be determined. In the case of the SU(2) theory, a solution is in fact obtained in this way, if \(v\) satisfies

\[
\partial^* \left[ \frac{1}{a^2} \sin \left( \frac{1}{2} a^2 \partial v \right) \right] = 8 \cos \left( \frac{1}{2} av \right) \left[ \frac{1}{a} \sin \left( \frac{1}{2} av \right) \right]^3, \quad 0 < x^0 < T. \tag{4.14}
\]

The operator \(\partial\) in this equation denotes the forward lattice derivative,

\[
\partial f(x^0) = \frac{1}{a} \left[ f(x^0 + a) - f(x^0) \right], \tag{4.15}
\]

and \(\partial^*\) the backward derivative. The ansatz (4.13) also works out for SU(3), but for \(N \geq 4\) we suspect that a more complicated expression is needed, with at least two unknown functions.

In the practically relevant range of \(T/a\), say \(T/a = 10, \ldots, 100\), the solution of eq.(4.14) with the required boundary values (and the lowest action) can be determined numerically to any desired precision. Alternatively, if we assume that an expansion of the form

\[
v(x^0) = v_0(x^0) + av_1(x^0) + a^2v_2(x^0) + \ldots, \tag{4.16}
\]

holds, where \(v_k(x^0)\) are smooth functions independent of \(a\), the resulting tower of equations,

\[
\partial_0^2 v_0 = 2v_0^3, \tag{4.17}
\]

\[
\partial_0^2 v_1 = 6v_0^2 v_1, \tag{4.18}
\]

etc., can be solved iteratively. We expect, of course, that the leading term \(v_0\) coincides with the continuum solution \(b\). This is consistent with the lowest order equation (4.17), and from eq.(4.18) we now infer that \(v_1\) is a linear combination of \(b^2\) and \(b^{-3}\). If we take into account that \(v_1\) must vanish at \(x^0 = 0\) and \(x^0 = T\), a little calculation shows that \(v_1\) is in fact equal to zero everywhere, and so we conclude that

\[
v(x^0) = b(x^0) + O(a^2). \tag{4.19}
\]
We have also worked out the order $a^2$ correction and compared our result with the numerical solution of eq.(4.14). Complete agreement was found, and the approach of the lattice solution to the continuum background field is, therefore, well under control.

One point, however, that remains to be checked is that the lattice background field so constructed is indeed a configuration with least action. To convince oneself that there are no other lower minima, one may run a relaxation program, on a range of lattices, starting from various initial configurations. In the case of most interest to us, the constant Abelian fields, such a numerical “proof” of stability is fortunately not needed, because the minimum property of these fields can be established by analytical means (cf. sect. 5).

4.4 Continuum limit in perturbation theory

The effective action on the lattice, $\Gamma[B]$, is defined through eq.(2.37) and is considered to be a functional of the continuum background field. We are certainly free to do so, because the Schrödinger functional (4.5) depends on the boundary values $C$ and $C'$ and these are in one-to-one correspondence with $B$. The notation is also in accord with our understanding of the lattice as a device to regularize the theory which is to be removed at fixed $B$.

The perturbation expansion (2.38) of the lattice effective action may be derived following the steps taken in sect. 3 for the case of dimensional regularization. In particular, the leading term $\Gamma_0[B]$ is equal to $g_0^2$ times the action of the lattice background field $V$. An interesting new aspect is that the gauge fixing procedure can be carried out rigorously. This will be discussed in sect. 6 and an explicit one-loop calculation is presented in sect. 7. Our aim here is to describe how the continuum limit is reached in perturbation theory.

Symanzik’s analysis [48,49] of the cutoff dependence of Feynman diagrams on the lattice suggests that the $l$–loop contribution to the effective action may be expanded in an asymptotic series of the form

$$\Gamma_l[B] \sim \sum_{m=0}^{\infty} \sum_{n=0}^{l} \Gamma_{lmn}[B] a^m (\ln a)^n. \quad (4.20)$$

Close to the continuum limit, all terms with $m > 0$ can be neglected. We moreover expect that the logarithmically divergent terms cancel after the gauge coupling $g_0$ is renormalized, i.e. after we substitute

$$g_0^2 = g_{\text{lat}}^2 + z_1(a\mu)g_{\text{lat}}^4 + \ldots, \quad z_1(a\mu) = \frac{22}{3} \frac{N}{16\pi^2} \ln(a\mu), \quad (4.21)$$
where $g_{\text{lat}}$ is a renormalized coupling and $\mu$ some normalization mass. This is entirely analogous to what happened in the case of the dimensionally regularized effective action. In particular, the resulting renormalized expansion in powers of $g_{\text{lat}}^2$ must coincide with eq.(3.37), if the gauge couplings $g_{\text{lat}}$ and $g_{\text{MS}}$ are properly matched.

To one-loop order and for the background fields considered, all these statements will be confirmed explicitly, and so we are confident that the dependence of the effective action on the lattice spacing is indeed as described.

### 4.5 $O(a)$ improved action

Numerical simulations of the functional integral (4.5) are limited to lattice spacings $a$ that are not too small compared to the scales set by the background field. The cutoff dependent contributions to the effective action (the terms of order $a$ and higher) may hence have a non-negligible influence on the outcome of such calculations. In that instance it may be desirable to choose an improved lattice action so as to cancel the dominant cutoff effects.

This idea has previously been worked out for scalar theories [50–52] and pure gauge theories on lattices without boundaries [53–58]. The general principle is that the cutoff effects of order $a(\ln a)^n$, $a^2(\ln a)^n$, etc. can be successively removed by adding a linear combination of local counterterms to the action, with increasing dimensions and properly adjusted coefficients. This may be viewed as an extension of the renormalization procedure to the level of irrelevant operators. We thus expect that the counterterms needed to improve the Schrödinger functional come in two forms, the boundary and the volume terms (cf. sect. 2).

At order $a$ there are two possible counterterms. They are obtained by summing any local lattice expression for the fields

$$a^4 \text{tr} \{F_{0k}F_{0k}\} \quad \text{and} \quad a^4 \text{tr} \{F_{kl}F_{kl}\} \quad (4.22)$$

over the $x^0 = 0$ and $x^0 = T$ hyper-planes. A simple $O(a)$ improved lattice action is thus given by eq.(4.4), where

$$w(p) = \begin{cases} 
\frac{4}{3}c_s(g_0) & \text{if } p \text{ is a spatial plaquette at } x^0 = 0 \text{ or } x^0 = T, \\
ct(g_0) & \text{if } p \text{ is a time-like plaquette attached to a boundary plane,} 
\end{cases} \quad (4.23)$$

and $w(p) = 1$ in all other cases. The coefficients $c_s(g_0)$ and $ct(g_0)$ multiplying the boundary plaquettes allow us to monitor the size of the counterterms.
They are to be adjusted so as to achieve the desired improvement of the theory.

Ideally one would like to determine $c_s(g_0)$ and $c_t(g_0)$ through numerical simulations of the Schrödinger functional. While this is not an impossible task, it is surely demanding, since one needs precise data on a range of lattices to be able to isolate the cutoff effects properly. In perturbation theory we have

\begin{align}
  c_s(g_0) &= c_s^{(0)} + c_s^{(1)} g_0^2 + \ldots, \quad (4.24) \\
  c_t(g_0) &= c_t^{(0)} + c_t^{(1)} g_0^2 + \ldots, \quad (4.25)
\end{align}

where the coefficients $c_s^{(1)}$ and $c_t^{(1)}$ can be extracted from the $l$–loop contribution to the effective action. In particular, at tree level we only need to work out the order $a$ term in the small $a$ expansion of $\Gamma_0[B]$ for two independent choices of $B$, say a constant Abelian field and a self-dual field. As a result one obtains

\begin{equation}
  c_s^{(0)} = c_t^{(0)} = 1 \quad \text{for all } N. \quad (4.26)
\end{equation}

On the basis of our calculations to one-loop order (sect. 7), we have moreover been able show that

\begin{equation}
  c_s^{(1)} = -0.166(1) \quad \text{and} \quad c_t^{(1)} = -0.0543(5) \quad \text{for } N = 2. \quad (4.27)
\end{equation}

Note that in the SU(2) theory the crossover from strong to weak coupling behaviour occurs around $g_0^2 = 2$ and simulations nowadays are performed at couplings close to 1.5. The one-loop correction (4.27) is thus reasonably small.

We finally mention that the improved action coincides with the Wilson action to lowest order of $g_0^2$. The latter is hence $O(a)$ improved at tree level, and this implies that all coefficients $\Gamma_{lmn}[B]$ in the expansion (4.20) with $m = 1$ and $n = l$ vanish.
5. Abelian Background Fields

As explained in sect. 2, a large variety of running couplings can be defined by differentiating the effective action $\Gamma[B]$ with respect to some parameter of the background field $B$. A careful choice of $B$ is however necessary, if one attempts to study the scaling properties of the coupling through numerical simulations of the lattice Schrödinger functional. In particular, since one cannot afford to make the lattice spacing arbitrarily small, a background field is required for which the lattice corrections to the effective action are tolerable. It is our experience that constant Abelian fields are optimal in this respect and so we discuss their properties here.

In the following we allow the space-time dimensionality $D$ to assume any integer value greater or equal to 2. For the lattice action we take Wilson’s action (subsect. 4.2) multiplied with $a^{D-4}$.

5.1 Definition

The boundary values of the background fields considered in this section are spatially constant and diagonal. In other words, we have

$$C_k = \frac{i}{L} \begin{pmatrix} \phi_{k1} & 0 & \ldots & 0 \\ 0 & \phi_{k2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi_{kN} \end{pmatrix},$$  \hspace{1cm} (5.1)

where the angles $\phi_{k\alpha}$ satisfy

$$\sum_{\alpha=1}^{N} \phi_{k\alpha} = 0 \quad \text{for all } k.$$  \hspace{1cm} (5.2)

The other boundary field $C'$ is defined similarly, with $\phi_{k\alpha}$ replaced by a second set of angles $\phi'_{k\alpha}$.

Gauge fields of this type have previously occurred in the literature and are referred to as “torons” (see e.g. refs.[59,60]). Locally they are pure gauge configurations, i.e. the only gauge invariant information is contained in the parallel transporters for closed curves winding “around the world”. In the field $C$ these are products of the matrices $\exp \{LC_k \}$ and their inverses. In particular, they only depend on the phase factors $e^{i\phi_{k\alpha}}$ and the angular character of the parameters $\phi_{k\alpha}$ thus becomes apparent.
An obvious solution of the field equations with the boundary values specified above is
\[ B_0 = 0, \quad B_k = \left[ x^0 C_k' + (T - x^0) C_k \right] / T. \] (5.3)

For the associated field tensor one obtains
\[ G_{0k} = \left[ C_k' - C_k \right] / T, \quad G_{kl} = 0, \] (5.4)
and the action is given by
\[ S[B] = \frac{L^{D-3}}{g_0 T} \sum_{k=1}^{D-1} \sum_{\alpha=1}^{N} \left( \phi_{k\alpha}' - \phi_{k\alpha} \right)^2. \] (5.5)

This solution — a constant colour electric field — is not the only solution of the Yang-Mills field equations with the required boundary values. It is in fact quite obvious that other solutions must exist, because the action (5.5) does not reflect the expected periodicity in the angles \( \phi_{k\alpha} \). The stability of the field (5.3) thus needs to be discussed and we shall come back to this problem in subsect. 5.2.

The link field
\[ V(x, \mu) = \exp \{ a B_\mu(x) \} \] (5.6)
(with \( B \) as above) is a candidate for the induced background field on the lattice. \( V \) has the required boundary values and one may easily verify that it is a solution of the lattice field equations (4.12). Note that the action
\[ S[V] = \frac{T L^{D-1}}{g_0^2} \sum_{k=1}^{D-1} \sum_{\alpha=1}^{N} \left\{ \frac{2}{a^2} \sin \left[ \frac{a^2}{2TL} \left( \phi_{k\alpha}' - \phi_{k\alpha} \right) \right] \right\}^2 \] (5.7)

coincides with the continuum action (5.5) up to terms of order \( a^4 \) (rather than \( a^2 \)). Constant Abelian fields thus show their quality of leading to minimal cutoff effects already at the classical level.

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5.2 Stability theorem

Before the lattice field (5.6) can be accepted as the induced background field for the chosen boundary values, we must show that it is a configuration with least action and that any other field with the same action is gauge equivalent to $V$. Such a proof can in fact be given if $\varphi_{k\alpha}$ and $\varphi'_{k\alpha}$, are restricted to a certain bounded domain.

To specify this region it is useful to arrange $\varphi_{k\alpha}$ (and similarly $\varphi'_{k\alpha}$) into $D - 1$ colour vectors $\Phi_k$ in the obvious way. We say that a vector $\Phi = (\varphi_1, \ldots, \varphi_N)$ is in the fundamental domain if

$$\varphi_1 < \varphi_2 < \ldots < \varphi_N, \quad |\varphi_\alpha - \varphi_\beta| < 2\pi, \quad \sum_{\alpha=1}^{N} \varphi_\alpha = 0. \quad (5.8)$$

This is a bounded convex set, which has a certain group theoretical significance. Namely, if $u \in SU(N)$ has pairwise different eigenvalues $\lambda_\alpha$, there exists a unique ordering of the eigenvalues and a unique vector $\Phi$ in the fundamental domain, such that $\lambda_\alpha = e^{i\phi_\alpha}$.

Our result on the stability of the background fields considered in this section is now summarized by

**Theorem 1.** Suppose the angle vectors $\Phi_k$ and $\Phi'_k$ are in the fundamental domain for all $k = 1, \ldots, D - 1$. Let $V$ be the associated background field (5.6) on a lattice with

$$TL/a^2 > (N-1)\pi^2 \max\{1, N/16\}. \quad (5.9)$$

The action of any other lattice gauge field $U$ with the same boundary values then satisfies $S[U] \geq S[V]$, where the equality holds if and only if $U$ is gauge equivalent to $V$.

The proof of the theorem is quite involved and is deferred to appendix B. The condition (5.9) is of a technical nature and is perhaps not really needed for the validity of the theorem. It is, in any case, insignificant for $N \leq 3$ and the lattices of interest.

There is, of course, a continuum version of the theorem, provided one is willing to restrict attention to the category of differentiable fields. The minimum property of the solution (5.3) is in fact a simple corollary of theorem 1, since any differentiable gauge field can be approximated arbitrarily well by lattice fields.
5.3 Definition of a running coupling for the SU(2) theory

We are now ready to introduce the running coupling \( \bar{g}^2(L) \) which will be used in our finite size scaling study of the SU(2) theory in four dimensions and which will, therefore, be in the focus of interest for the rest of this paper.

Following the general scheme described in subsect. 2.6, we choose the solution (5.3) for the background field \( B \) and set \( T = L \). \( B \) depends on altogether 6 parameters, say \( \phi_{k1} \) and \( \phi'_{k1} \). A symmetrical one-parameter submanifold is

\[
\phi_{k1} = -\eta, \quad \phi'_{k1} = -\pi + \eta, \tag{5.10}
\]

and the coupling is now given by eq.(2.43).

We should of course make sure that the premises of theorem 1 are satisfied so that the stability of the chosen background field is guaranteed. This is the case if

\[
0 < \eta < \pi \quad \text{and} \quad L/a \geq 4. \tag{5.11}
\]

In the numerical work we take \( \eta = \pi/4 \), which is half-way between the zero action point \( \eta = \pi/2 \) and the boundary of the stability interval. All these details are quite arbitrary, but we emphasize (once more) that they are in no way of fundamental importance for the scaling analysis we have in mind.

6. Fixing the Gauge in the Lattice Theory

The problem of gauge fixing has already been addressed in sect. 3 on a somewhat formal level. In particular, a truly convincing derivation of the boundary conditions on the Faddeev-Popov ghosts and the time component of the gauge field could not be given and so we feel that it is worthwhile to go through the gauge fixing procedure once more, in the mathematically rigorous framework of the lattice theory. We closely follow the general scheme discussed in subsect. 3.5 of ref.[47], where further details and a proof of the basic lemma can be found.

In this section the background field is assumed to be an Abelian field as described in sect. 5, with angle vectors \( \Phi_k \) and \( \Phi'_k \) in the fundamental domain.
6.1 Group of gauge transformations

The lattice Schrödinger functional (4.5) is invariant under all gauge transformations \( U \rightarrow U^\Omega \) which leave the boundary fields \( W \) and \( W' \) intact. According to lemma 1 of appendix B, this condition is satisfied if and only if the boundary values of the gauge function \( \Omega(x) \) at \( x^0 = 0 \) and \( x^0 = T \) are constant and diagonal. The group of all these transformation functions is denoted by \( \hat{G} \).

The constant diagonal gauge functions \( \Omega(x) \) form a subgroup of \( \hat{G} \) isomorphic to the Cartan subgroup \( C_N \) of SU\( (N) \). Such transformations act trivially on the background field and there are actually no further transformations with this property (cf. appendix B). The gauge directions in the space of infinitesimal deformations of the background field are thus generated by

\[
\mathcal{G} = \hat{G}/C_N,
\]

and so it is this set of transformations which needs to be fixed. The group \( C_N \) then survives as a global symmetry of the theory.

In the following it is convenient to identify \( G \) with the group of all transformations \( \Omega \in \hat{G} \) that are equal to 1 at \( x^0 = T \). From a purely mathematical point of view, we then have the canonical situation, with a compact gauge group acting freely on the space of fields integrated over. The discussion in ref.[47] thus carries over literally.

6.2 Spaces of infinitesimal fields

The Lie algebra \( \mathcal{L} \) of \( \mathcal{G} \) consists of all fields \( \omega(x) \) such that the infinitesimal transformation

\[
\Omega(x) = 1 - g_0 \omega(x) + O(g_0^2) \tag{6.2}
\]

belongs to \( \mathcal{G} \). In other words, \( \omega \) must be a spatially periodic lattice field which takes values in the Lie algebra of SU\( (N) \) and which satisfies the boundary conditions

\[
\omega(x)|_{x^0=0} = \kappa, \quad \omega(x)|_{x^0=T} = 0, \tag{6.3}
\]

where \( \kappa \) is constant and diagonal.

A linear space \( \mathcal{H} \) of lattice vector fields \( q_\mu(x) \) may be introduced similarly. We here require that

\[
U(x, \mu) = \left\{ 1 + g_0 aq_\mu(x) + O(g_0^2) \right\} V(x, \mu) \tag{6.4}
\]

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is a valid infinitesimal deformation of the background field. In particular, to
guarantee that $U$ has the same boundary values as $V$, the spatial components
$q_k(x)$ must be equal to zero at $x^0 = 0$ and $x^0 = T$.

We shall later find it useful to have a scalar product on these spaces at
our disposal. The obvious choice for the product of two vector fields is

$$(q, r) = -2a^4 \sum_{x, \mu} \text{tr} \{q_\mu(x)r_\mu(x)\}, \quad (6.5)$$

and the product on $L$ is defined similarly.

Another notational item are the covariant lattice derivatives $D_\mu$ and $D^*_\mu$. These operators act on functions $f(x)$ with values in the Lie algebra of SU($N$) and are given by

$$D_\mu f(x) = \frac{1}{a} [V(x, \mu)f(x + a\hat{\mu})V(x, \mu)^{-1} - f(x)], \quad (6.6)$$

$$D^*_\mu f(x) = \frac{1}{a} [f(x) - V(x - a\hat{\mu}, \mu)^{-1}f(x - a\hat{\mu})V(x - a\hat{\mu}, \mu)]. \quad (6.7)$$

They only make sense if $f$ is defined at the lattice points referred to and should thus be used with care.

### 6.3 Gauge fixing function

The gauge fixing term to be added to the Wilson action is the square of
a suitable gauge fixing function $F$. As explained in ref.[47], $F$ must be a
regular mapping from the space of fields integrated over to the Lie algebra $L$.
For perturbation theory it is actually sufficient to specify $F$ on an arbitrarily
small but finite neighborhood of the background field $V$. The fields $U$ in such
a neighborhood may be parametrized by

$$U(x, \mu) = \exp \{g_0aq_\mu(x)\} \ V(x, \mu), \quad (6.8)$$

where $q \in \mathcal{H}$ and say $\|q\| < \varepsilon$. The simplest possibility then is to choose the
gauge fixing function to be a linear mapping from $\mathcal{H}$ to $\mathcal{L}$.

To be able to write it in a compact form we introduce the operator

$$d : \mathcal{L} \mapsto \mathcal{H}, \quad (d\omega)_\mu(x) = D_\mu \omega(x). \quad (6.9)$$

The covariant derivative occurring here is well-defined, for all relevant lattice
points $x$ and directions $\mu$, and it is also easy to verify that $d\omega$ has the proper
boundary values. The significance of $d$ becomes apparent if we note that the gauge directions in the space of infinitesimal deformations of the background field are precisely the modes of the form $d\omega$. As already mentioned, the group $G$ acts freely on the space of gauge fields, and so it is not surprising that the kernel of $d$ turns out to be trivial, i.e. $d\omega = 0$ implies $\omega = 0$.

The basic property the gauge fixing function should have is that it does not vanish on the gauge modes $d\omega$. It is quite obvious now that this requirement (and all the other conditions listed in ref.[47]) will be fulfilled if we choose

$$F(U) = d^*q,$$ (6.10)

where $d^*$ is minus the adjoint of $d$. That is, $d^*$ maps any vector field $q \in \mathcal{H}$ onto an element of $\mathcal{L}$ such that

$$(d^*q, \omega) = -(q, d\omega) \quad \text{for all } \omega \in \mathcal{L}.$$ (6.11)

This property implies

$$d^*q(x) = D^*_\mu q_\mu(x) \quad \text{if } 0 < x^0 < T,$$ (6.12)

and for the boundary values at $x^0 = 0$ and $x^0 = T$ one obtains

$$[d^*q(x)]_{\alpha\beta} = \begin{cases} (a^2/L^3) \sum_y [q_0(0,y)]_{\alpha\beta} & \text{if } \alpha = \beta \text{ and } x^0 = 0, \\ 0 & \text{otherwise.} \end{cases}$$ (6.13)

Note that these are indeed as required for a function contained in $\mathcal{L}$.

**6.4 Gauge fixed functional integral**

Now that we have specified the gauge fixing function $F$, the gauge fixed form of the Schrödinger functional (4.5) is obtained almost mechanically, following the steps described in ref.[47]. There is no new element involved here and so we merely introduce the necessary notations and state the final result.

Our choice of gauge fixing function corresponds to the gauge fixing term

$$S_{gf}[B, q] = \frac{\lambda_0}{2}(d^*q, d^*q).$$ (6.14)

In view of eq.(6.12), this may be regarded as a lattice version of the background gauge. The associated Faddeev-Popov ghosts $c$ and $\bar{c}$ are in all respects like infinitesimal gauge transformations except that they are fermi fields. That is,
if we choose a complete set of functions in $L$, the ghost fields are equal to the general linear combination of these basis elements, with anti-commuting coefficients that generate a Grassmann algebra.

The action of the Faddeev-Popov fields is

$$S_{FP}[B, q, c, \bar{c}] = -(\bar{c}, d^* \delta_c q),$$

where $\delta_c q$ denotes the first order variation of $q$ under the gauge transformation generated by $c$. To second order in $g_0$ we have

$$\delta_c q_\mu = D_\mu c + g_0 \text{Ad} q_\mu c$$

$$+ \left[ \frac{1}{2} g_0 a \text{Ad} q_\mu + \frac{1}{12} (g_0 a \text{Ad} q_\mu)^2 + \ldots \right] D_\mu c$$

(no sum over $\mu$ is implied here). Note that $\delta_c q$ is a vector field with the correct boundary values. The action of $d^*$ in eq.(6.15) is therefore well-defined.

The gauge fixed form of the Schrödinger functional is now given by

$$e^{-\Gamma[B]} = \int D[U] \int D[c] D[\bar{c}] e^{-S_{total}[B, q, c, \bar{c}]},$$

$$S_{total}[B, q, c, \bar{c}] = S[U] + S_{gf}[B, q] + S_{FP}[B, q, c, \bar{c}],$$

where it is understood that $U$ and $q$ are related by eq.(6.8). The first integral in eq.(6.17) is restricted to a small neighborhood of the background field $V$, i.e. we are neglecting terms that are exponentially small in the coupling. The integration over the ghost fields is the usual one, resulting in the determinant of the Faddeev-Popov operator.

As in the continuum theory, the gauge fixed functional integral is the starting point from which the perturbation expansion of the effective action is obtained. Noting

$$D[U] = D[q] \left\{ 1 + \mathcal{O}(g_0^2) \right\},$$

$$S_{total}[B, q, c, \bar{c}] = S[V] + \frac{1}{2} (q, \Delta_1 q) + (\bar{c}, \Delta_0 c) + \mathcal{O}(g_0),$$

we get, for the first two terms in eq.(2.38),

$$\Gamma_0[B] = g_0^2 S[V],$$

$$\Gamma_1[B] = \frac{1}{2} \ln \det \Delta_1 - \ln \det \Delta_0.$$
$\Delta_0$ and $\Delta_1$ are symmetric operators acting on functions in $L$ and $H$ respectively. We have already used the symbols $\Delta_0$ and $\Delta_1$ for the corresponding operators in the continuum theory, but this will not lead any confusion in the following.

From the above we infer that

$$\Delta_0 = -d^*d,$$

$$\Delta_1 = \Delta'_1 - \lambda_0 dd^*, \quad (6.23)$$

where $\Delta'_1$ is obtained by expanding the Wilson action $S[U]$ to second order in $q$. With the help of the $\star$ product notation introduced in appendix A, the result of that computation may be written in the compact form

$$\Delta'_1 q_\mu(x) = \sum_{\nu \neq \mu} \left\{ \cosh (a^2 G_{\mu\nu}) \star \left[ -D^*_\nu D_\nu q_\mu(x) + D^*_\nu D_\nu q_\mu(x) \right] + a^{-2} \sinh (a^2 G_{\mu\nu}) \star \left[ 2q_\nu(x) + a (D^*_\nu + D_\nu) q_\mu(x) + a^2 D^*_\nu D_\nu q_\mu(x) \right] \right\}. \quad (6.25)$$

It should be emphasized that this formula is only valid for constant Abelian background fields, the case of immediate interest to us. We have also assumed that the link $(x, x + a\hat{\mu})$ is contained in the interior of the lattice, viz.

$$0 \leq x^0 < T \text{ if } \mu = 0 \text{ and } 0 < x^0 < T \text{ if } \mu > 0. \quad (6.26)$$

All degrees of freedom of the fields in $H$ reside on these links and the operator $\Delta'_1$ is thus completely specified by eq.(6.25).

6.5 Boundary conditions

The lattice quantum field $q$ and the ghost fields $c$ and $\bar{c}$ do not satisfy any boundary conditions beyond those implicit in the definition of $H$ and $L$. In particular, the time component $q_0(x)$ is defined at all points $x$ with $0 \leq x^0 < T$ and is unconstrained.

However, after passing to the continuum limit, the situation regarding boundary conditions may be quite different. In a free scalar theory, for example, the propagator on a lattice with free boundary conditions converges to a Green function which satisfies Neumann boundary conditions. We should thus be prepared to find that the eigenfunctions of the lattice operators $\Delta_0$ and $\Delta_1$ satisfy additional boundary conditions in the limit $a \to 0$. 

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To be absolutely clear, we are not suggesting that the lattice theory should be amended in some way. Our aim rather is to determine which boundary conditions must be chosen when one attempts to formulate the theory directly in the continuum, using dimensional regularization for example. By comparing with the lattice theory this question can now be decided, because all cutoff prescriptions should yield identical results at tree level.

In the case of the lattice operator $\Delta_1$ it is not immediately obvious how to pass to the continuum limit, because the definition of the gauge term $dd^*$ is slightly non-uniform at the boundary [cf. eqs.(6.12) and (6.13)]. This difficulty can be removed by choosing a better notation. To this end we extend the time component $q_0(x)$ of the lattice field to all points with $x^0 = -a$ and $x^0 = T$. Its values there are chosen such that

$$D_0^*q_0(x) = d^*q(x) \quad \text{at} \quad x^0 = 0 \text{ and } x^0 = T.$$  \hspace{1cm} (6.27)

No additional degrees of freedom are thus introduced and the extension should be regarded purely as a matter of convention.

The operator $dd^*$ is now given by the simple expression

$$dd^*q_\mu(x) = D_\mu D_\nu^*q_\nu(x),$$  \hspace{1cm} (6.28)

for all $x$ and $\mu$ in the range (6.26). Furthermore, we may interpret eq.(6.27) as a boundary condition on $q_0$. At $x^0 = T$, for example, the requirement simply is that $D_0^*q_0(x) = 0$. At the other end of the lattice $q_0$ is again required to satisfy Neumann boundary conditions with the exception of the spatially constant diagonal modes, which must vanish at $x^0 = -a$ and thus satisfy Dirichlet boundary conditions. This latter complication depends on our choice of background field and is absent for the irreducible background fields considered in sect. 3.

In the present formulation one may straightforwardly pass to the continuum limit. In particular, taking $a \rightarrow 0$ in eqs.(6.25) and (6.28) leads to the continuum expression (3.18) for $\Delta_1$ and the boundary conditions stated above carry over literally. To remove all doubts about this procedure, we have verified in a number of cases, by exact numerical computation, that the eigenfunctions of the lattice operator converge to smooth functions in the continuum limit and that these indeed satisfy the expected boundary conditions.

We thus conclude that the proper boundary conditions on the quantum field $q$ in the continuum theory are a mixture of Dirichlet and Neumann bound-
ary conditions as described above. A similar analysis applies to the Faddeev-Popov fields. The result here is that besides the boundary conditions already present on the lattice, the spatially constant diagonal modes satisfy Neumann boundary conditions at $x^0 = 0$. This is again special to our choice of background field in this section, i.e. for irreducible boundary values $C$ and $C'$ one would end up with Dirichlet boundary conditions for all modes.

7. First Order Correction to the Running Coupling [SU(2) Theory]

We now set $N = 2$ and consider the one-parameter family of background fields defined in subsect. 5.3. Our aim is to compute the associated running coupling $\bar{g}^2(L)$ to one-loop order of perturbation theory. The calculation will be performed in the framework of the lattice theory, but as a check we have also worked out the relevant determinants in the continuum theory, using dimensional regularization. Some of the details involved in that calculation will be sketched at the end of this section.

7.1 Preliminaries

In the following we choose lattice units and thus set $a = 1$. From the perturbation expansion of the effective action we deduce that

$$\bar{g}^2(L) = g_0^2 + m_1(L)g_0^4 + m_2(L)g_0^6 + \ldots \quad (7.1)$$

The one-loop coefficient is a sum of two contributions,

$$m_1 = h_0 - \frac{1}{2} h_1, \quad (7.2)$$

one from the ghost and the other from the vector fields. Explicitly, they are given by

$$h_s = \frac{\partial}{\partial \eta} \ln \det \Delta_s / \frac{\partial}{\partial \eta} \Gamma_0[B], \quad s = 0, 1. \quad (7.3)$$

By differentiating the classical action (5.7) one obtains

$$\frac{\partial}{\partial \eta} \Gamma_0[B] = -24L^2 \sin \left[ \frac{1}{L^2}(\pi - 2\eta) \right], \quad (7.4)$$
and we are thus left with the problem to compute the determinants of $\Delta_0$ and $\Delta_1$ for the background fields of interest. We do not expect that this can be done analytically, but as will be shown in the following subsections, it is possible to evaluate the determinants numerically, for a useful range of $L$ and to essentially any desired precision.

7.2 Symmetries

$\Delta_0$ is a symmetric operator which acts in a real vector space. In particular, there exists a complete set of eigenfunctions and the determinant of $\Delta_0$ is the product of all eigenvalues.

In the following we consider $\Delta_0$ to be a linear operator in the complex space $\mathcal{L} \oplus i\mathcal{L}$. The eigenfunctions and eigenvalues are not affected by this extension and so it is clear that also the determinant is the same as before. The reason for going to the complexified space is that now we can more easily pass to a basis which is adapted to the symmetries of the problem.

To exploit the invariance of $\Delta_0$ under constant diagonal gauge transformations, we choose the standard basis $T^a$ of SU(2) generators [eq.(A.4)] and define $T^\pm = T^1 \pm iT^2$. We then consider the decomposition

$$\mathcal{L} \oplus i\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_- \oplus \mathcal{L}_+ \quad (7.5)$$

where $\mathcal{L}_0$ and $\mathcal{L}_\pm$ are the subspaces of all fields that are proportional to $T^3$ and $T^\pm$ respectively. Constant diagonal gauge transformations reduce to a multiplication by a phase factor on $\mathcal{L}_\sigma$ and so we expect that these spaces are invariant under the action of $\Delta_0$.

To show this explicitly, we introduce the corresponding subspaces $\mathcal{H}_0$ and $\mathcal{H}_\pm$ of complex vector fields and note that the operator $d$ maps any function $\omega \in \mathcal{L}_\sigma$ onto an element of $\mathcal{H}_\sigma$. More precisely, we have

$$D_\mu \omega(x) = \begin{cases} 
\omega(x + \hat{\mu}) - \omega(x) & \text{if } \mu = 0, \\
e^{i\sigma \beta(x^0)} \omega(x + \hat{\mu}) - \omega(x) & \text{if } \mu > 0,
\end{cases} \quad (7.6)$$

where the phase $\beta$ is given by

$$\beta(t) = -(2/L^2) [\eta L + (\pi - 2\eta)t] \quad (7.7)$$

It is quite obvious now that $d^*$ maps $\mathcal{H}_\sigma$ into $\mathcal{L}_\sigma$ and so we conclude that $\Delta_0$ leaves $\mathcal{L}_\sigma$ invariant.
The operator $\Delta_1$ is similarly reduced by the subspaces $\mathcal{H}_\sigma$. This is an immediate consequence of the identities

$$\cosh G_{0k} \star T^a = \cos(\gamma/2)T^a, \quad \gamma = (2/L^2)(\pi - 2\eta), \quad (7.8)$$

$$\sinh G_{0k} \star T^3 = 0, \quad (7.9)$$

$$\sinh G_{0k} \star T^\pm = \mp i \sin(\gamma/2)T^\pm, \quad (7.10)$$

and the discussion above of the covariant derivatives. In particular, we have

$$\det \Delta_1 = \det \Delta_1|_{\mathcal{H}_0} \det \Delta_1|_{\mathcal{H}_-} \det \Delta_1|_{\mathcal{H}_+}, \quad (7.11)$$

and the analogous factorization holds for the determinant of $\Delta_0$. Note that the last two factors in eq.(7.11) are equal.

A further factorization of the determinants is obtained if we take into account that $\Delta_0$ and $\Delta_1$ are invariant under spatial translations. The eigenfunctions of this group of symmetries are the plane waves $e^{ipx}$ where

$$p = 2\pi n/L, \quad n_k \in \mathbb{Z}, \quad -L/2 < n_k \leq L/2. \quad (7.12)$$

We are thus led to introduce the subspaces $\mathcal{L}_\sigma(p)$ and $\mathcal{H}_\sigma(p)$ of all functions which are proportional to $e^{ipx}$ and have no other dependence on $x$. Since the momentum $p$ is conserved under the action of $\Delta_1$, it follows that

$$\det \Delta_1|_{\mathcal{H}_\sigma} = \prod_p \det \Delta_1|_{\mathcal{H}_\sigma(p)}, \quad (7.13)$$

and the same factorization holds in the case of the operator $\Delta_0$.

The simplification which has thus been achieved is substantial. In each symmetry sector the spatial dependence of the wave functions and also the SU(2) degrees of freedom are completely fixed. The problem is, therefore, effectively reduced to computing the determinants of a set of finite difference operators in one dimension.
7.3 Computation of $h_0$

From the discussion above it is evident that $\Delta_0$ is independent of the background field on the subspace $L_0$. The contribution from this sector hence drops out and we are left with

$$\frac{\partial}{\partial \eta} \ln \det \Delta_0 = 2 \sum_p \frac{\partial}{\partial \eta} \ln \det \Delta_0|_{L_+(p)}.$$  \hspace{1cm} (7.14)

So let us consider the action of $\Delta_0$ in the subspace $L_+(p)$ in some more detail. The functions in this symmetry sector are of the form

$$\omega(x) = \psi(x^0) e^{ipx^+ T^+},$$  \hspace{1cm} (7.15)

where $\psi(t), 0 \leq t \leq T$, is a complex function with

$$\psi(0) = \psi(T) = 0$$  \hspace{1cm} (7.16)

[cf. eq.(6.3); the constant $\kappa$ vanishes because $T^+$ is off-diagonal].

On this set of functions $\Delta_0$ reduces to an ordinary second order difference operator,

$$\Delta_0 \psi(t) = A(t) \psi(t + 1) + B(t) \psi(t) + C(t) \psi(t - 1),$$  \hspace{1cm} (7.17)

with coefficients given by

$$A(t) = C(t) = -1,$$  \hspace{1cm} (7.18)

$$B(t) = 8 - 2 \sum_{k=1}^{3} \cos [p_k + \beta(t)].$$  \hspace{1cm} (7.19)

The determinant of such operators can be computed by solving a simple recursion relation. This is a known trick which is discussed in some generality in appendix C. In the present case we need to solve

$$\Delta_0 \psi(t) = 0, \quad 0 < t < T,$$  \hspace{1cm} (7.20)

starting from the initial values

$$\psi(0) = 0, \quad \psi(1) = 1.$$  \hspace{1cm} (7.21)
The determinant is then given by

\[ \det \Delta_0 |_{\mathcal{L}_+(p)} = \psi(T). \] (7.22)

Note that the calculational effort to solve eq.(7.20) is proportional to \( T \), while one usually needs of order \( T^3 \) arithmetic operations to evaluate the determinant of a \( T \times T \) matrix.

To compute the derivative of the determinant with respect to \( \eta \), the best way to proceed is to derive a recursion for \( \partial \psi(t) / \partial \eta \) by differentiating eq.(7.20). Both recursion relations are trivial to program, and so one is able to compute the coefficient \( h_0 \) for say \( L \leq 32 \) with a negligible amount of computer time.

### 7.4 Computation of \( h_1 \)

For notational convenience we always assume that the field \( q_0(x) \) is extended one step beyond the lattice to all points with \( x^0 = -1 \) and \( x^0 = T \), as discussed in subsect. 6.5. The boundary conditions on \( q_\mu \) then are a mixture of Dirichlet and Neumann boundary conditions and the action of \( \Delta_1 \) is given by eqs.(6.24),(6.25) and (6.28).

We first consider the contribution of the sector \( \mathcal{H}_0(p) \). The fields in this subspace may be written as

\[ q_0(x) = \psi_0(x^0)(-i)e^{ipx} T^3, \] (7.23)

\[ q_k(x) = \psi_k(x^0)e^{i2px}e^{ipx} T^3, \] (7.24)

where \( \psi_\mu(t) \) is some complex vector field. The extra phase factors in these equations have no particular significance except that they lead to simpler expressions later on.

The action of the operator \( \Delta_1 \) on wave functions of the above type is of the general form

\[ \Delta_1 \psi_\mu(t) = A_{\mu\nu}(t)\psi_\nu(t+1) + B_{\mu\nu}(t)\psi_\nu(t) + C_{\mu\nu}(t)\psi_\nu(t-1). \] (7.25)

It is not difficult to work out the coefficient matrices \( A, B \) and \( C \) explicitly [cf. appendix D].

For all momenta \( p \neq 0 \) the boundary conditions on \( \psi_\mu(t) \) are

\[ \partial^* \psi_0(t) = \psi_k(t) = 0 \quad \text{at } t = 0 \text{ and } t = T. \] (7.26)
\( \partial^* \) here denotes the backward difference, defined in the usual way. If \( p = 0 \) it is \( \psi_0(-1) \) instead of \( \partial^* \psi_0(0) \) which is required to vanish, while all other boundary conditions are as above.

The determinant of \( \Delta_1 \) in this latter sector can actually be computed immediately. The matrices \( A, B \) and \( C \) assume a particularly simple form in this case, leading to a decoupling of the \( \psi_0 \) and \( \psi_k \) components. Furthermore, only the spatial modes make a field dependent contribution and one quickly finds that

\[
\frac{\partial}{\partial \eta} \ln \det \Delta_1|_{\mathcal{H}_0(0)} = 3(T - 1) \frac{\partial}{\partial \eta} \ln \cos \frac{\gamma}{2}.
\]  

(7.27)

This is nice because now we are left with the \( p \neq 0 \) subspaces, where the boundary conditions are simply given by eq. (7.26).

To compute the associated determinants, we proceed essentially as in the case of the Faddeev-Popov operator. We first construct all solutions of

\[
\Delta_1 \psi_0(t) = 0, \quad 0 \leq t < T, \\
\Delta_1 \psi_k(t) = 0, \quad 0 < t < T,
\]

with \( \partial^* \psi_0(0) = \psi_k(0) = 0 \). For specified initial values, \( \psi_0(0) \) and \( \partial \psi_k(0) \), there exists exactly one solution. It is obtained recursively, first by solving eq. (7.28) at \( t = 0 \), then eq. (7.29) at \( t = 1 \), and so on.

It is useful to group the initial data in a vector \( v_\mu \) with components

\[
v_0 = \psi_0(0), \quad v_k = \partial \psi_k(0).
\]  

(7.30)

After completing the recursion we may compute the final values

\[
w_0 = \partial^* \psi_0(T), \quad w_k = \psi_k(T).
\]  

(7.31)

It is evident that these are related linearly to the initial values, i.e. there exists a matrix \( M \) such that

\[
w_\mu = M_{\mu\nu} v_\nu.
\]  

(7.32)

The determinant of \( \Delta_1 \) is then given by

\[
\det \Delta_1|_{\mathcal{H}_0(p)} = \lambda_0^T \left( \cos \frac{\gamma}{2} \right)^{3T-3} \det M,
\]  

(7.33)

as one may show by adapting the arguments of appendix C to the case at hand.
Let us now turn to the subspaces $H_+(p)$. The wave functions here are of the form

$$q_0(x) = \psi_0(x^0)(-i)e^{ipx} T^+, \quad (7.34)$$

$$q_k(x) = \psi_k(x^0)e^{\frac{i}{2} [p_k + \beta(x^0)]} e^{ipx} T^+, \quad (7.35)$$

and the action of $\Delta_1$ is again given by eq.(7.25), with coefficients $A$, $B$ and $C$ listed in appendix D. Furthermore, the boundary conditions are as before [eq.(7.26)], and so it is clear that the determinant of $\Delta_1$ in this sector can be computed following the lines discussed above. In particular, if $M$ is the matrix connecting initial and final values, we have

$$\det \Delta_1|_{H_+(p)} = \chi_0^T \det M, \quad (7.36)$$

for all momenta $p$.

### 7.5 Results

We now set $\eta = \pi/4$ and compute the one-loop correction to the running coupling using the techniques described above. Some of our results are listed in table 1. The calculations were done so as to guarantee a final precision of at least 20 decimal places. In particular, all digits quoted in table 1 are significant.

In lattice units the continuum limit is approached by taking $L$ to infinity. As discussed in subsects. 4.4 and 4.5 we expect that an asymptotic expansion

| $L$   | $m_1(L)$     | $\varepsilon(L)$ | $L$   | $m_1(L)$     | $\varepsilon(L)$ |
|-------|--------------|-------------------|-------|--------------|-------------------|
| 6     | 0.35422030   | 0.00352           | 20    | 0.47541031   | 0.00024           |
| 8     | 0.38370429   | 0.00177           | 22    | 0.48471069   | 0.00020           |
| 10    | 0.40644167   | 0.00107           | 24    | 0.49316916   | 0.00017           |
| 12    | 0.42483107   | 0.00072           | 26    | 0.50092488   | 0.00014           |
| 14    | 0.44023578   | 0.00052           | 28    | 0.50808526   | 0.00012           |
| 16    | 0.45347745   | 0.00039           | 30    | 0.51473490   | 0.00010           |
| 18    | 0.46508332   | 0.00031           | 32    | 0.52094162   | 0.00009           |
of the form
\[ m_1(L) \sim \sum_{n=0}^{\infty} \left( r_n + s_n \ln L \right) / L^n \] (7.37)
holds, with
\[ s_0 = \frac{11}{12\pi^2} \quad \text{and} \quad s_1 = 0. \] (7.38)
The first few coefficients \( r_n \) and \( s_n \) may be extracted from the data listed in table 1 by any suitable extrapolation procedure. In particular, using the blocking method of ref.[58], we were able to confirm, to four significant decimal places, that \( s_0 \) is given by eq.(7.38). This again shows that all ultra-violet divergences in the Schrödinger functional cancel after renormalizing the coupling in the usual way.

If we now assume that \( s_0 \) is precisely given by eq.(7.38), the extrapolation of our data yields
\[ r_0 = 0.202349(3), \] (7.39)
\[ r_1 = -0.1084(11). \] (7.40)
As shown by table 1, the remainder
\[ \varepsilon(L) = m_1(L) - s_0 \ln L - r_0 - r_1 / L, \] (7.41)
with coefficients as given above, is rapidly decreasing over the range of \( L \) covered. The higher order cutoff effects are thus comfortably small for our choice of background field. It must be emphasized, however, that other background fields, such as the constant self-dual field considered in sect. 2, do not fare as well in this respect. Partly this is because the strength of the Abelian field is constant while in the other cases the lattice must be fine enough to resolve the often appreciable time dependence of the field.

After eliminating \( g_0 \) in favour of the renormalized coupling \( g_{\text{lat}} \) introduced in subsect. 4.4, the continuum limit can be taken and one ends up with the expansion
\[ \bar{g}^2(L) = g_{\text{lat}}^2 + \left[ \frac{11}{12\pi^2} \ln(\mu L) + r_0 \right] g_{\text{lat}}^4 + \ldots \] (7.42)
At this point we may pass to any other renormalization scheme defined in the continuum theory. In particular, if we choose the running coupling \( \bar{g}_{\text{MS}}^2(L) \) in
the MS scheme of dimensional regularization as our new expansion parameter and use the known relation between \( g_{\text{lat}} \) and \( g_{\text{MS}} \) [62,9,63], we obtain

\[
\bar{g}^2(L) = \bar{g}_{\text{MS}}^2(L) + c_1 \bar{g}_{\text{MS}}^4(L) + \ldots, \tag{7.43}
\]

where the coefficient \( c_1 \) is given by

\[
c_1 = \frac{11}{24\pi^2} \left[ \ln 4\pi - \gamma_E - 1.61638(8) \right]. \tag{7.44}
\]

For the corresponding \( \Lambda \) parameter ratio one finds

\[
\frac{\Lambda}{\Lambda_{\text{MS}}} = 1.18378(5), \tag{7.45}
\]

which shows that our coupling is very nearly equal to \( \bar{g}_{\text{MS}}^2(L) \) in the perturbative domain.

The term proportional to \( r_1 \) in the expansion (7.37) is an order \( a \) cutoff effect. As discussed in subsect. 4.5 it can be removed by adding a combination of two boundary counterterms to the Wilson action, with coefficients \( c_s \) and \( c_t \). Our result above, eq.(7.40), allows us to compute \( c_t \) to one-loop order [cf. eq.(4.27)].

To also determine \( c_s \), the effective action of a further background field, with non-vanishing spatial field strength, must be computed. The constant self-dual fields are a convenient choice for this calculation, because the determinants of \( \Delta_0 \) and \( \Delta_1 \) again factorize after going to momentum space. With some additional work, we have thus been able to obtain the result quoted in subsect. 4.5.

7.6 Computation of \( \bar{g}^2(L) \) using dimensional regularization

It is possible to derive the expansion (7.43) directly in the continuum theory, starting from the dimensionally regularized Schrödinger functional. This provides an important check on our calculations on the lattice and gives us additional confidence that the Schrödinger functional is indeed a universal amplitude.

In sect. 3 we have already obtained the one-loop effective action for a general irreducible background field, using dimensional regularization. The final formula, eq.(3.37), is also valid for Abelian background fields, if \( \Delta_0 \) and \( \Delta_1 \) are considered to be operators in the proper function spaces [cf. subsect. 6.5]. The non-trivial remaining problem then is to compute \( \zeta'(0|\Delta_0) \) and \( \zeta'(0|\Delta_1) \) or rather their derivatives with respect to \( \eta \).
In the following we adopt the notation of sect. 3. In particular, the symbol \( \Delta \) denotes any one of the operators \( \Delta_0 \) or \( \Delta_1 \). From the definition of the zeta function and the Seeley-DeWitt expansion we infer that

\[
\zeta'(0|\Delta) = \gamma_E \alpha_0(\Delta) + \int_0^\infty \frac{dt}{t} \left\{ \text{Tr} e^{-t\Delta} - \sum_{j=1}^{4} \alpha_j/2(\Delta) t^{-j/2} - \alpha_0(\Delta) \theta(1-t) \right\}.
\]

This integral is absolutely convergent and the lower end of the integration range may, therefore, be replaced by some positive cutoff \( \delta \) which is subsequently sent to zero. It is then straightforward to show that

\[
\frac{\partial}{\partial \eta} \zeta'(0|\Delta) = \lim_{\delta \to 0} \left\{ (\gamma_E + \ln \delta) \frac{\partial}{\partial \eta} \alpha_0(\Delta) - \sum_{n=0}^{\infty} e^{-\delta E_n} \frac{\partial}{\partial \eta} \ln E_n \right\},
\]

where \( E_n, n = 0, 1, 2, \ldots \), are the eigenvalues of \( \Delta \).

Our computational strategy now is as follows. We first calculate all eigenvalues \( E_n \) and their derivatives \( \partial E_n/\partial \eta \) up to a certain level. For values of \( \delta \) that are not too small, the bracket in eq.(7.47) is then accurately given by restricting the spectral sum to these eigenvalues. Finally, the result is extrapolated to the limit \( \delta = 0 \), taking into account that the bracket has an asymptotic expansion in powers of \( \delta^{1/2} \).

The success of the method of course depends on our ability to compute sufficiently many eigenvalues of \( \Delta \). This goal is easy to reach. We first fix the symmetry properties of the wave functions as before, by choosing an appropriate basis of group generators and by going to momentum space. The range of momenta \( p \) is here unbounded, but since the eigenvalues are growing essentially like \( p^2 \), only a finite subset of momenta needs to be considered.

In every symmetry sector the problem is then reduced to finding the eigenvalues of a certain second order ordinary differential operator acting on functions with Dirichlet or Neumann boundary conditions. The reduced operators are actually so simple that the eigenfunctions can be given explicitly in terms of hypergeometric functions with a parameter to be adjusted. Alternatively, one may use exact numerical methods (Runge-Kutta integration and root finding algorithms) or one may set up a variational calculation with a suitable basis of plane wave states.
Proceeding either way we have been able, with a small amount of computer time, to calculate the bracket in eq.(7.47) for values of \( \delta \) in the range between say \( 0.01 \times L^2 \) and \( 0.001 \times L^2 \). From there an extrapolation to \( \delta = 0 \) is safely possible and our result for the coefficient \( c_1 \) was found to be in agreement with eq.(7.44) (the estimated precision after the extrapolation was about 4 significant digits).

8. Concluding Remarks

The renormalizability of the Schrödinger functional opens the door to a new generation of scaling studies of pure gauge theories. We have in this paper prepared the ground for such an investigation. Numerical simulations of the SU(2) theory are now underway and first results are expected soon.

It would be quite trivial to extend our calculations in sect. 7 of the running coupling to the physically more interesting case of the SU(3) theory. Before that one however needs to fix the parameters of the background field. In particular, some experience must be accumulated to determine which fields lead to an acceptably large signal in numerical simulations, while keeping the cutoff effects low.

We also believe that an extension of our computations to the two-loop level is feasible, although this is clearly going to be hard work.

Appendix A

Our notational conventions are as follows. Lorentz indices \( \mu, \nu, \ldots \) normally run from 0 to 3. In the context of dimensional regularization they run up to \( D - 1 \), the dimension of space. Since the metric is euclidean, it does not matter in which position these indices appear. The totally anti-symmetric symbol \( \epsilon_{\mu \nu \rho \sigma} \) is normalized such that \( \epsilon_{0123} = 1 \). Latin indices \( k, l, \ldots \) run from 1 to 3 (or \( D - 1 \)) and are used to label spatial vector components. Colour vectors in the fundamental representation of SU(\( N \)) carry indices \( \alpha, \beta, \ldots \) ranging from 1 to \( N \), while for vectors in the adjoint representation, latin indices \( a, b, \ldots \)
running from 1 to $N^2 - 1$ are employed. Repeated indices are automatically summed over unless stated otherwise.

The Lie algebra $\mathfrak{su}(N)$ of $\text{SU}(N)$ can be identified with the space of complex $N \times N$ matrices $X_{\alpha \beta}$ which satisfy

$$X^\dagger = -X, \quad \text{tr}\{X\} = 0,$$

where $X^\dagger$ denotes the adjoint matrix of $X$ and $\text{tr}\{X\} = X_{\alpha \alpha}$ is the trace of $X$. This is a real vector space of dimension $N^2 - 1$. A natural inner product is given by

$$(X, Y) = -2 \text{tr}\{XY\},$$

and we may thus choose a basis $T^a$, $a = 1, 2, \ldots, N^2 - 1$, such that

$$(T^a, T^b) = \delta^{ab}.$$

For $N = 2$, the standard basis is

$$T^a = \frac{\tau^a}{2i}, \quad a = 1, 2, 3,$$

where $\tau^a$ denote the Pauli matrices. With these conventions, the structure constants $f^{abc}$, defined through

$$[T^a, T^b] = f^{abc} T^c,$$

are real and totally anti-symmetric under permutations of the indices.

An $\text{SU}(N)$ gauge potential is a vector field $A_\mu(x)$ with values in the Lie algebra $\mathfrak{su}(N)$. With respect to a basis $T^a$ of group generators, we have

$$A_\mu(x) = A^a_\mu(x) T^a,$$

where the component fields $A^a_\mu(x)$ are real. The components of other fields such as the gauge field tensor or the Faddeev-Popov ghost field are defined in the same way.

The representation space of the adjoint representation of $\mathfrak{su}(N)$ is the Lie algebra itself, i.e. the elements $X$ of $\mathfrak{su}(N)$ are represented by linear transformations

$$\text{Ad} X : \quad \mathfrak{su}(N) \mapsto \mathfrak{su}(N).$$
Explicitly, \( \text{Ad} X \) is defined through

\[
\text{Ad} X(Y) = [X, Y] \quad \text{for all} \ Y \in \text{su}(N). \tag{A.8}
\]

With respect to a basis \( T^a \), the associated matrix \( \text{(Ad} X)^{ab} \) representing the transformation is given by

\[
\text{Ad} X(T^b) = T^a(\text{Ad} X)^{ab}, \tag{A.9}
\]

which is equivalent to

\[
(\text{Ad} X)^{ab} = -f^{abc} X^c, \quad (\text{Ad} X(Y))^a = f^{abc} X^b Y^c, \tag{A.10}
\]
in terms of the structure constants.

We finally define the \( \star \) product of two \( N \times N \) matrices \( M \) and \( X \) through

\[
M \star X = \frac{1}{2} (MX + XM^\dagger) - \frac{1}{2N} \text{tr} (MX + XM^\dagger). \tag{A.11}
\]

\( M \star X \) is contained in \( \text{su}(N) \), for all \( X \in \text{su}(N) \) and arbitrary \( M \).

**Appendix B**

The proof of theorem 1 is elementary from the mathematical point of view, but it is lengthy and is therefore broken up into several steps. Our first goal will be to show that the theorem holds in all dimensions if it is true in two dimensions. After that we shall work out the consequences of the lattice field equations in two dimensions. We shall find that there is a discrete set of solutions and we are then left to prove that among these the configuration with least action is the background field \( V \).

Without further notice, we assume in the following that the angle vectors \( \Phi_k \) and \( \Phi'_k \) are in the fundamental domain (5.8) and that the bound (5.9) holds. The discussion will be solely concerned with the lattice theory and the boundary values of the gauge fields considered are always as specified in subsect. 5.1. Furthermore, the letter \( V \) is strictly reserved for the background field defined through eqs.(5.3) and (5.6).
B.1 Remarks on gauge transformations

The boundary fields $W$ and $W'$ have an important stability property under gauge transformations which is expressed by

Lemma 1. If $\Lambda$ is a spatial gauge transformation which leaves $W$ or $W'$ invariant, it must be equal to a constant diagonal matrix.

Proof: Let us assume that it is $W$ which is invariant under the action of $\Lambda$, viz.

$$\Lambda(x)W(x,k)\Lambda(x+a\hat{k})^{-1} = W(x,k). \quad (B.1)$$

The parallel transporter along any lattice curve at $x^0 = 0$ is an ordered products of the link variables and so must be invariant too. In particular, we may consider a closed curve starting at some point $x$ and winding once around the world in the $k$–direction. Since $\Lambda$ is periodic, we then deduce that

$$\Lambda(x) \exp \{LC_k\} \Lambda(x)^{-1} = \exp \{LC_k\}. \quad (B.2)$$

The parallel transporter $\exp \{LC_k\}$ is a diagonal matrix with eigenvalues $\lambda_\alpha = e^{i\phi_{k\alpha}}$. These are all distinct (the angle vector $\Phi_k$ is in the fundamental domain) and eq.(B.2) thus implies that $\Lambda(x)$ must be diagonal. That it must also be constant, is now an immediate consequence of eq.(B.1).

A simple corollary of this lemma is that the background field $V$ is invariant under a space-time gauge transformation $\Omega$ if and only if $\Omega$ is constant and diagonal.

For any given lattice gauge field $U$ we define

$$\Theta(x) = U(y,0)U(y+a\hat{0},0)U(y+2a\hat{0},0)\ldots U(x-a\hat{0},0), \quad y = (0,x), \quad (B.3)$$

which is just the parallel transporter along the time axis from $x$ to the “base point” $y$ at the boundary. $\Theta$ is periodic in the space directions and may be regarded as a gauge transformation function.

Lemma 2. If $U$ is known to be gauge equivalent to $V$, we have $V = U^\Theta$.

Proof: Let us define $\bar{U} = U^\Theta$. From the definition (B.3) of $\Theta$ we infer that

$$\bar{U}(x,0) = V(x,0) = 1 \quad \text{for all } x, \quad (B.4)$$

$$\bar{U}(x,k) = V(x,k) = W(x,k) \quad \text{for all } k \text{ and all } x \text{ with } x^0 = 0. \quad (B.5)$$
Furthermore, since $U$ and $V$ are gauge equivalent, there exists a gauge transformation $\Omega$ such that $\tilde{U} = V^\Omega$.

Eq.(B.4) now implies that $\Omega$ is time independent, and if we combine eq.(B.5) with lemma 1, one concludes that $\Omega$ must be constant and diagonal. It follows that $V^\Omega = V$ and so we have proved the lemma. ■

B.2 Reduction to two dimensions

A significant simplification is now achieved by

**Lemma 3.** Let us assume that theorem 1 holds in two dimensions. Then it is true for all $D \geq 2$.

**Proof:** The action of any field $U$ in $D$ dimensions may be written in the form

$$S[U] = S_E[U] + S_M[U], \quad \text{(B.6)}$$

where $S_E$ and $S_M$ are the contributions of the time-like and spatial plaquettes, respectively. The magnetic part $S_M$ is always greater or equal to zero and so we conclude that

$$S[U] \geq S_E[U]. \quad \text{(B.7)}$$

The electric part $S_E$ may be split up according to

$$S_E[U] = a^{D-2} \sum_P S[U|_P], \quad \text{(B.8)}$$

The sum here extends over all two-dimensional time-like planes $P$ on the lattice and $U|_P$ is the two-dimensional gauge field that one obtains by restricting $U$ to $P$.

We now make use of the premise of the lemma to show that

$$S_E[U] \geq a^{D-2} \sum_P S[V|_P] = S_E[V] = S[V]. \quad \text{(B.9)}$$

When combined with eq.(B.7), this proves that $S[U] \geq S[V]$ for all fields $U$.

We still need to show that $U$ is gauge equivalent to $V$ if $S[U] = S[V]$. So let us assume that $U$ is a minimal action configuration. From the above we then infer that the two-dimensional fields $U|_P$ and $V|_P$ are gauge equivalent, for all planes $P$. This does not immediately imply that $U$ is gauge equivalent to $V$, because the gauge transformations which map $U|_P$ onto $V|_P$ need not coincide at the intersections of different planes.
At this point lemma 2 comes to rescue. It is valid in any dimension and may be applied to $U|_{P}$ and $V|_{P}$ (in place of $U$ and $V$). We then conclude that

$$[U|_{P}]^{\Theta} = V|_{P}, \quad (B.10)$$

where $\Theta$ is the transformation defined by eq.(B.3). Since $\Theta(x)$ is independent of the plane $P$ passing through $x$, we have thus shown that $U^{\Theta} = V$. 

**B.3 Group theoretical lemmas**

To be able to work out the consequences of the lattice field equations later on, some technical preparation is needed.

The eigenvalues $\lambda_{\alpha}$ of any matrix $u \in \text{SU}(N)$ lie on the unit circle and their product is equal to 1. We say that $u$ belongs to the set $S$, if

$$\lambda_{\alpha} = e^{i\chi_{\alpha}}, \quad \text{where} \quad |\chi_{\alpha}| < \pi/2 \quad \text{and} \quad \sum_{\alpha=1}^{N} \chi_{\alpha} = 0, \quad (B.11)$$

It is obvious that $S$ is an open neighborhood of the group identity. A useful criterion for a matrix $u$ to be an element of $S$ is

**Lemma 4.** Any matrix $u \in \text{SU}(N)$ with $\text{Re} \; \text{tr}(1 - u) < \min\{1, 16/N\}$ is contained in $S$.

**Proof:** The eigenvalues $\lambda_{\alpha}$ of $u$ may always be parametrized by a set of angles $\chi_{\alpha}$ such that

$$\lambda_{\alpha} = e^{i\chi_{\alpha}}, \quad -\pi < \chi_{\alpha} \leq \pi, \quad \sum_{\alpha=1}^{N} \chi_{\alpha} = 2\pi n, \quad (B.12)$$

where $n$ is an integer. From the bound

$$\text{Re} \; \text{tr}(1 - u) = \sum_{\alpha=1}^{N} (1 - \cos \chi_{\alpha}) < 1 \quad (B.13)$$

we conclude that $|\chi_{\alpha}| < \pi/2$. To prove that the sum of the angles $\chi_{\alpha}$ vanishes, we first note that

$$1 - \cos \chi \geq (2\chi/\pi)^{2} \quad \text{if} \quad |\chi| < \pi/2. \quad (B.14)$$
The sequence of inequalities

$$\left| \sum_{\alpha=1}^{N} \chi_{\alpha} \right|^2 \leq N \sum_{\alpha=1}^{N} \chi_{\alpha}^2 \leq \frac{N \pi^2}{4} \text{Re} \text{tr}(1-u) < 4\pi^2 \quad (B.15)$$

is then easily established. It follows from these that \(|n| < 1\), and since \(n\) must be an integer, we are left with \(n = 0\) as the only possible value. \( \Box \)

The significance of the set \(\mathcal{S}\) is elucidated by Lemma 5. Any two matrices \(u, v \in \mathcal{S}\) with the same traceless anti-hermitean parts, viz.

$$u - u^\dagger - \frac{1}{N} \text{tr} (u - u^\dagger) = v - v^\dagger - \frac{1}{N} \text{tr} (v - v^\dagger), \quad (B.16)$$

are equal.

Proof: Let \(\lambda_\alpha = e^{i\chi_\alpha}\) be the eigenvalues of \(u\) as in eq.(B.11). The associated eigenvectors are also eigenvectors of \(u - u^\dagger\), with eigenvalues \(2i \sin \chi_\alpha\). Since \(\chi_\alpha \neq \chi_\beta\) implies \(\sin \chi_\alpha \neq \sin \chi_\beta\), the eigenspaces of \(u\) and \(u - u^\dagger\) must in fact coincide. The same statement applies to the other pair of matrices, \(v\) and \(v - v^\dagger\). So if we take eq.(B.16) into account, it follows that all four matrices can be simultaneously diagonalized.

Let us now choose a basis of simultaneous eigenvectors. The eigenvalues of \(v\) may be parametrized in the same way as those of \(u\) through a set of angles \(\psi_\alpha\). Eq.(B.16) then reduces to

$$\sin \chi_\alpha - \sin \psi_\alpha = t, \quad (B.17)$$

where \(t\) is independent of \(\alpha\). In particular, if \(t > 0\) one infers that \(\chi_\alpha > \psi_\alpha\) for all \(\alpha\) (recall that all angles are between \(-\pi/2\) and \(\pi/2\)). This is impossible, however, because both sets of angles must sum up to zero. The same argument excludes \(t < 0\) and so we are left with \(\chi_\alpha = \psi_\alpha\) as the only acceptable solution of eq.(B.17). The matrices \(u\) and \(v\) are hence equal. \( \Box \)

B.4 Proof of theorem 1 for \(D = 2\)

From now on the discussion is restricted to the two-dimensional theory. To simplify the notation we set \(\phi_{1\alpha} = \phi_\alpha\) and \(\phi'_{1\alpha} = \phi'_\alpha\). Since the set of all lattice gauge fields is a compact manifold, the infimum of the action is attained by at least one configuration \(U\). In the following we assume that \(U\) is such a field. We then need to show that \(U\) is gauge equivalent to \(V\).
Lemma 6. For all $x$ the plaquette matrix

$$P(x) = U(x,0)U(x+a\hat{0},1)U(x+a\hat{1},0)^{-1}U(x,1)^{-1} \quad (B.18)$$

is contained in the set $S$.

Proof: Since $U$ is a minimal action configuration, we have

$$S[U] \leq S[V] \leq (g_0^2TL)^{-1} \sum_{\alpha=1}^{N} (\phi'_\alpha - \phi_\alpha)^2. \quad (B.19)$$

If we remember that $\Phi$ and $\Phi'$ are in the fundamental domain, it is easy to prove that the angle differences $\psi_\alpha = \phi'_\alpha - \phi_\alpha$ satisfy

$$|\psi_\alpha - \psi_\beta| < 2\pi, \quad \sum_{\alpha=1}^{N} \psi_\alpha = 0. \quad (B.20)$$

Using these properties, the estimate

$$\sum_{\alpha=1}^{N} \psi_\alpha^2 = \frac{1}{2N} \sum_{\alpha,\beta=1}^{N} (\psi_\alpha - \psi_\beta)^2 < 2(N - 1)\pi^2 \quad (B.21)$$

may be derived.

The action $S[U]$ is a sum of non-negative contributions, one from each (unoriented) plaquette. Any one of these must be smaller than the right hand side in eq.(B.19) and thus, taking eq.(B.21) and the bound (5.9) into account, one deduces that

$$\text{Retr} (1 - P(x)) < \min \{1, 16/N\} \quad (B.22)$$

for all $x$. Lemma 4 now tells us that $P(x)$ is contained in $S$. □

Lemma 7. The plaquette field $P(x)$ is covariantly constant, i.e. it satisfies

$$P(x) = U(x,\mu)P(x+a\hat{\mu})U(x,\mu)^{-1} \quad (B.23)$$

for all $x$ and directions $\mu$.

Proof: Being a configuration with least action, $U$ must be a solution of the lattice field equations (cf. subsect. 4.3). In two dimensions these are equivalent to the requirement that the unitary matrices

$$u = P(x) \quad \text{and} \quad v = U(x,\mu)P(x+a\hat{\mu})U(x,\mu)^{-1} \quad (B.24)$$
have the same traceless anti-hermitean parts, for all $x$ and directions $\mu$. From the above we know that $u$ and $v$ are contained in the set $S$ and lemma 5 thus implies that $u = v$, as was to be shown. □

The plaquette field $P(x)$ may be regarded as a particular gauge transformation function. Eq.(B.23) then simply says that $U$ is invariant under this transformation. In particular, the boundary field $W$ is left invariant and so, by lemma 1, we conclude that $P(x)$ must be constant and diagonal at $x^0 = 0$. In other words, we have

$$P(x)|_{x^0=0} = \begin{pmatrix} e^{i\chi_1} & 0 & \ldots & 0 \\
0 & e^{i\chi_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & e^{i\chi_N} \end{pmatrix}, \quad (B.25)$$

where the angles $\chi_\alpha$ may be chosen such that

$$|\chi_\alpha| < \pi/2 \quad \text{and} \quad \sum_{\alpha=1}^N \chi_\alpha = 0 \quad (B.26)$$

(as a consequence of lemma 6).

Another important implication of eq.(B.23) is that the eigenvalues of $P(x)$ are the same for all $x$. In particular, the action

$$S[U] = \frac{2TL}{g_0^2a^4} \sum_{\alpha=1}^N \left(1 - \cos \chi_\alpha \right) \quad (B.27)$$

is determined by the angles $\chi_\alpha$.

**Lemma 8.** There exists a permutation $\sigma$ and a set of integers $n_\alpha$ such that

$$\chi_\alpha = \frac{a^2}{TL} \left( \phi'_{\sigma(\alpha)} + 2\pi n_\alpha - \phi_\alpha \right). \quad (B.28)$$

In the special case where $\sigma$ is the identity and $n_\alpha = 0$ for all $\alpha$, the configuration $U$ is gauge equivalent to $V$.

**Proof:** We first pass to the temporal gauge by applying the gauge transformation $\Theta$ [eq.(B.3)]. The transformed field $\tilde{U} = U\Theta$ and the associated plaquette field $\tilde{P}$ satisfy

$$\tilde{U}(x, 0) = 1 \quad \text{for all} \ x, \quad (B.29)$$
\[ \tilde{U}(x, 1) = W(x, 1), \quad \tilde{P}(x) = P(x) \quad \text{at} \quad x^0 = 0. \quad (B.30) \]

Since \( \tilde{P}(x) \) is covariantly constant, we conclude that it must be equal to \( P(0) \) for all \( x \).

Next we note that
\[ \tilde{P}(x) = \tilde{U}(x + a\bar{0}, 1)\tilde{U}(x, 1)^{-1}, \quad (B.31) \]
which immediately leads to
\[ \tilde{U}(x, 1) = [P(0)]^{x^0/a} W(x, 1). \quad (B.32) \]

The matrices on the right hand side of this equation are diagonal, i.e. we have shown that \( \tilde{U} \) is an Abelian solution of the field equations similar to \( V \).

Let us now consider the parallel transporter for a loop winding once around the world at \( x^0 = T \). In the temporal gauge it may be computed directly using eq.(B.32), or we may note that it is just the gauge transform of the parallel transporter determined by the original field \( U \). We thus obtain the relation
\[ \Theta(x) \exp \{LC'_1\} \Theta(x)^{-1} = [P(0)]^{T/L/a^2} \exp \{LC_1\}. \quad (B.33) \]

The eigenvalues of the matrix on the left are the phase factors \( e^{i\delta_\alpha} \). Up to a possible reordering, they must be equal to the diagonal elements of the matrix on the other side of the equation and so we conclude that eq.(B.28) holds for some permutation \( \sigma \) and some integers \( n_\alpha \). Furthermore, if \( \sigma \) happens to be the identity and \( n_\alpha = 0 \) for all \( \alpha \), it is immediately clear from eq.(B.32) that \( \tilde{U} = V \), i.e. \( U \) and \( V \) are gauge equivalent in this case.

The upshot then is that the possible minimal action configurations \( U \) are labelled by a permutation \( \sigma \) and a set of integers \( n_\alpha \). Not all configurations \( (\sigma, n) \) can occur, but only those for which the angles (B.28) satisfy the constraints (B.26). These will be called admissible in the following. In particular, the trivial configuration, where \( \sigma \) is the identity and \( n_\alpha = 0 \) for all \( \alpha \), is admissible. Note that the set of admissible configurations \( (\sigma, n) \) is finite.

Taking eq.(B.27) and lemma 8 into account, the proof of theorem 1 is now completed by
Lemma 9. On the set of admissible configurations \((\sigma, n)\), the minimum of the function
\[
s(\sigma, n) = \sum_{\alpha=1}^{N} (1 - \cos \chi_{\alpha})
\] (B.34)
is not degenerate and the unique minimizing configuration is the trivial one.

Proof: Let us assume that \((\sigma, n)\) is a non-trivial admissible configuration. We then show that there exists another admissible configuration \((\tilde{\sigma}, \tilde{n})\), such that
\[s(\sigma, n) > s(\tilde{\sigma}, \tilde{n}).\]

We first consider a configuration \((\sigma, n)\) where not all \(n_\alpha\)'s are equal to zero. Since \((\sigma, n)\) is admissible, the integers \(n_\alpha\) must add up to zero. It is, therefore, possible to find two indices \(\alpha\) and \(\beta\) such that
\[n_\alpha \geq 1 \quad \text{and} \quad n_\beta \leq -1.\] (B.35)

Let us now define a new configuration \((\tilde{\sigma}, \tilde{n})\) through
\[
\tilde{\sigma}(\gamma) = \begin{cases} 
\sigma(\beta) & \text{if } \gamma = \alpha, \\
\sigma(\alpha) & \text{if } \gamma = \beta, \\
\sigma(\gamma) & \text{otherwise},
\end{cases}
\] (B.36)
and
\[
\tilde{n}_\gamma = \begin{cases} 
n_\beta + 1 & \text{if } \gamma = \alpha, \\
n_\alpha - 1 & \text{if } \gamma = \beta, \\
n_\gamma & \text{otherwise}.
\end{cases}
\] (B.37)
The associated angles \(\tilde{\chi}_\gamma\) are equal to \(\chi_\gamma\) with the exception of
\[
\tilde{\chi}_\alpha = \frac{a^2}{TL} \left[ \phi'_{\sigma(\beta)} + 2\pi (n_\beta + 1) - \phi_\alpha \right],
\] (B.38)
\[
\tilde{\chi}_\beta = \frac{a^2}{TL} \left[ \phi'_{\sigma(\alpha)} + 2\pi (n_\alpha - 1) - \phi_\beta \right].
\] (B.39)

Note that \(\tilde{\sigma}\) is just a transposition of \(\alpha\) and \(\beta\) followed by \(\sigma\).

It is not difficult to show that \((\tilde{\sigma}, \tilde{n})\) is an admissible configuration. Furthermore, a little algebra yields
\[
s(\sigma, n) - s(\tilde{\sigma}, \tilde{n}) =
\]
\[
4 \cos \left[ \frac{1}{4} (\chi_\alpha + \chi_\beta + \tilde{\chi}_\alpha + \tilde{\chi}_\beta) \right] \sin \left[ \frac{1}{2} (\chi_\alpha - \tilde{\chi}_\alpha) \right] \sin \left[ \frac{1}{2} (\chi_\alpha - \tilde{\chi}_\beta) \right].
\]
The arguments of all trigonometric functions in this expression are less than \( \pi/2 \) in magnitude. In particular, the first factor is positive. Concerning the other two factors, we note that their arguments

\[
\chi_\alpha - \widetilde{\chi}_\alpha = \frac{a^2}{TL} \left[ \phi'_{\sigma(\alpha)} - \phi'_{\sigma(\beta)} + 2\pi (n_\alpha - n_\beta - 1) \right], \\
\chi_\alpha - \widetilde{\chi}_\beta = \frac{a^2}{TL} \left[ 2\pi + \phi_\beta - \phi_\alpha \right],
\]

are positive, because the angle vectors \( \Phi \) and \( \Phi' \) are in the fundamental domain and because the integers \( n_\alpha \) and \( n_\beta \) satisfy the bounds (B.35). So we conclude that the action of the new configuration \((\widetilde{\sigma}, \widetilde{n})\) is strictly lower than \( s(\sigma, n) \).

Let us now consider a configuration \((\sigma, n)\), where \( \sigma \) is non-trivial but where all \( n_\alpha \)'s vanish. All configurations of this type are admissible. Since \( \sigma \) is not the identity, there are two indices \( \alpha \) and \( \beta \) such that

\[
\alpha < \beta \quad \text{and} \quad \sigma(\alpha) > \sigma(\beta).
\]

The new configuration \((\widetilde{\sigma}, \widetilde{n})\) in the present case is given by eq.(B.36) and \( \widetilde{n}_\gamma = 0 \) for all \( \gamma \). Eq.(B.40) is then still valid and in view of

\[
\chi_\alpha - \widetilde{\chi}_\alpha = \frac{a^2}{TL} \left[ \phi'_{\sigma(\alpha)} - \phi'_{\sigma(\beta)} \right], \\
\chi_\alpha - \widetilde{\chi}_\beta = \frac{a^2}{TL} \left[ \phi_\beta - \phi_\alpha \right],
\]

all factors on the right hand side are positive. It follows that \( s(\sigma, n) > s(\widetilde{\sigma}, \widetilde{n}) \) and we have thus proved the lemma. \( \square \)

**Appendix C**

In this appendix we derive a useful expression for the determinant of a general second order difference operator acting on a space of wave functions with Dirichlet boundary conditions. This result is needed in sect. 7 to compute the determinants of \( \Delta_0 \) and \( \Delta_1 \) in a constant Abelian background field. The basic idea is borrowed from Coleman’s Erice lecture on the uses of instantons...
where he treats the analogous case of a second order differential operator (see ref.[61], p.340).

C.1 Definitions

The general second order difference operator $\Delta$ acts on complex wave functions $\psi(t)$ with $n$ components, defined at integer values of $t$. Explicitly, $\Delta$ is given by

$$\Delta \psi(t) = A(t)\psi(t+1) + B(t)\psi(t) + C(t)\psi(t-1),$$  \hspace{1cm} (C.1)

where $A$, $B$ and $C$ are some complex $n \times n$ matrices depending on $t$.

The wave functions which are defined for $t = 0, 1, 2, \ldots, T$ and satisfy Dirichlet boundary conditions,

$$\psi(t) = 0 \quad \text{at} \quad t = 0 \text{ and } t = T,$$  \hspace{1cm} (C.2)

form a vector space $\mathcal{F}$ of dimension $d_{\mathcal{F}} = n(T-1)$. If $\psi(t)$ is an element of this space, eq.(C.1) is meaningful for $0 < t < T$ and $\Delta$ may thus be regarded as a linear operator in $\mathcal{F}$.

In the following we assume that $A(t)$ is invertible for all $t$ and that $\Delta$ is hermitean relative to some scalar product on $\mathcal{F}$. In particular, we take it for granted that there exists a complete set of eigenfunctions.

C.2 Statement of result

For any complex $\lambda$ the equation

$$(\Delta - \lambda)\psi(t) = 0, \quad t > 0,$$  \hspace{1cm} (C.3)

has a unique solution with $\psi(0) = 0$ and prescribed initial value at $t = 1$. It can be computed recursively by first solving eq.(C.3) at $t = 1$, then at $t = 2$, and so on. After $T - 1$ steps one obtains $\psi(T)$ which is in general not equal to zero. It is obvious, however, that $\psi(T)$ depends linearly on the initial value $\psi(1)$, i.e. there exists an $n \times n$ matrix $M(\lambda)$ such that

$$\psi(T) = M(\lambda)\psi(1).$$  \hspace{1cm} (C.4)

This matrix is evidently determined through the coefficients $A$, $B$ and $C$. It is comparatively easy to evaluate, requiring a computational effort proportional to $n^3T$. 

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The formula alluded to above now reads

\[
\det (\Delta - \lambda) = \det M(\lambda) \prod_{t=1}^{T-1} \det [-A(t)], \tag{C.5}
\]

where \(\Delta\) is here considered to be an operator in \(\mathcal{F}\), as described above. In particular, the determinant of \(\Delta\) is obtained if we set \(\lambda = 0\).

C.3 Proof of eq.(C.5)

It is easy to show that the matrix \(M(\lambda)\) is a polynomial in \(\lambda\) of degree \(T - 1\). The leading term is

\[
M(\lambda) = \lambda^{T-1} \{A(1)A(2)\ldots A(T-1)\}^{-1} + O(\lambda^{T-2}). \tag{C.6}
\]

It follows from these remarks that

\[
P(\lambda) = \det M(\lambda) \prod_{t=1}^{T-1} \det [-A(t)] \tag{C.7}
\]

is a polynomial in \(\lambda\) too, with leading term equal to \((-\lambda)^{d_F}\).

Let us now assume that \(\mu\) is an eigenvalue of \(\Delta\) with multiplicity \(k\). If \(\psi \in \mathcal{F}\) is one of the associated eigenfunctions, we have

\[
0 = \psi(T) = M(\mu)\psi(1). \tag{C.8}
\]

The matrix \(M(\mu)\) thus has a zero mode and we conclude that \(P(\mu) = 0\). Actually, since there are \(k\) linearly independent eigenfunctions, we may choose a basis such that the first \(k\) columns of \(M(\mu)\) vanish. The multiplicity of the zero of \(P(\lambda)\) at \(\lambda = \mu\) is, therefore, greater or equal to \(k\).

The total number of eigenvalues of \(\Delta\), including multiplicities, is equal to \(d_F\). Since this is also the degree of \(P(\lambda)\), it follows that this polynomial cannot have any further zeros besides the eigenvalues of \(\Delta\) and that, moreover, the corresponding multiplicities must coincide.

We have thus shown that \(P(\lambda)\) is the characteristic polynomial of the operator \(\Delta\), which is precisely the content of eq.(C.5).
Appendix D

In this appendix the matrices $A_{\mu\nu}$, $B_{\mu\nu}$ and $C_{\mu\nu}$ introduced in subsect. 7.4 are given explicitly. They are all real and satisfy

\[ B_{\mu\nu}(t) = B_{\nu\mu}(t), \quad C_{\mu\nu}(t) = A_{\nu\mu}(t - 1), \quad (D.1) \]

as required for a symmetric operator. In the following we shall use the abbreviations

\[ \hat{p}_k = 2 \sin \left[ \frac{1}{2} p_k \right], \quad (D.2) \]
\[ s_k(t) = 2 \sin \left[ \frac{1}{2} (p_k + \beta(t)) \right]. \quad (D.3) \]

We now list the independent elements of the matrices which describe the action of $\Delta_1$ in the $\mathcal{H}_0(p)$ sector.

\[ A_{00}(t) = -\lambda_0, \quad (D.4) \]
\[ A_{kl}(t) = -\cos \frac{\gamma}{2} \delta_{kl}, \quad (D.5) \]
\[ A_{0k}(t) = - \left[ \cos \frac{\gamma}{2} - \lambda_0 \right] \hat{p}_k, \quad (D.6) \]
\[ A_{k0}(t) = 0, \quad (D.7) \]
\[ B_{00}(t) = 2\lambda_0 + \cos \frac{\gamma}{2} \hat{p}_j \hat{p}_j, \quad (D.8) \]
\[ B_{kl}(t) = \left[ 2 \cos \frac{\gamma}{2} + \hat{p}_j \hat{p}_j \right] \delta_{kl} + (\lambda_0 - 1)\hat{p}_k \hat{p}_l, \quad (D.9) \]
\[ B_{0k}(t) = \left[ \cos \frac{\gamma}{2} - \lambda_0 \right] \hat{p}_k, \quad (D.10) \]

The matrices describing the action of $\Delta_1$ in the $\mathcal{H}_+(p)$ subspace are as follows.

\[ A_{00}(t) = -\lambda_0, \quad (D.11) \]
\[ A_{kl}(t) = -\delta_{kl}, \quad (D.12) \]
\[ A_{0k}(t) = \lambda_0 s_k(t + 1) - s_k(t), \quad (D.13) \]
\[ A_{k0}(t) = 0, \quad (D.14) \]
\[ B_{00}(t) = 2\lambda_0 + s_j(t)s_j(t + 1), \quad (D.15) \]

\[ B_{kl}(t) = \left[ 2\cos\frac{\gamma}{2} + s_j(t)s_j(t) \right] \delta_{kl} + (\lambda_0 - 1)s_k(t)s_l(t), \quad (D.16) \]

\[ B_{0k}(t) = s_k(t + 1) - \lambda_0 s_k(t). \quad (D.17) \]

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