Non-Gaussianity in the inflating curvaton

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Inflating curvaton can create curvature perturbation when the curvaton density is slowly varying. Using the delta-N formalism, we discuss the evolution of the curvature perturbation during curvaton inflation and find analytic formulation of the non-Gaussianity parameter. We first consider the inflating curvaton with sufficiently long inflationary expansion. Then we compare the result with short curvaton inflation.

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I. INTRODUCTION

The primordial curvature perturbation $\zeta(k)$ that exists on cosmological scales just before they start to enter the horizon is usually related to the perturbations generated during inflation. Recent observation suggests that $\zeta(k)$ is strongly constrained and provides a window on the very early universe [1].

The mechanism of generating $\zeta$ can be diverse, but begins presumably during inflation. There are many proposals for generating the curvature perturbation from the field perturbations, which use one or more fields for the mechanism.

The paradigm of the multi-field inflation scenario has been widely investigated, but it has usually been supposed that $\zeta(x,t)$ evaluated at an epoch $t_{\text{end}}$ just after the end of inflation is to be identified with the observed quantity. In this respect, a lot of papers consider the calculation of the spectrum of $\zeta$ just at the end of inflation [2–9].

On the other hand, we know that multi-field inflation may lead to multi-component Universe, in which the curvature perturbation may evolve significantly. The typical example is the curvaton mechanism, in which the mixed state of the radiation and the matter causes significant evolution of the curvature perturbation [10–12]. Recently, the curvaton mechanism has been extended to include the slowly varying component [13, 14]. The idea of the inflating curvaton has been mentioned earlier in many papers [15]. There has been numerical calculation for the hilltop curvaton, in which the curvaton inflation may take place [16]. We hope that the reader will be able to compare our analytic calculation with these preceding works.

Let us first summarize the motivations of this paper. In the conventional curvaton model there will be a significant non-Gaussianity when the curvaton decays before it dominates the energy density. In that way one usually has the formula $f_{NL} \sim r_{\sigma}^{-1}$, where “$r_{\sigma}$” is comparable to the density ratio. One might suspect that similar mechanism does not work in the inflating curvaton scenario, simply because the inflating curvaton is already dominating the density. This is an important problem. We must examine the details of the mechanism to answer that question. We are basically using the non-linear formalism. The non-linear formalism is reviewed in the appendix.

If one is going to apply the non-linear formalism into the inflating curvaton, one must be careful about the end boundary of the curvaton mechanisms. The usual non-linear calculation assumes uniform density hypersurfaces at the end, where the oscillating curvaton decays. For the inflating curvaton, the end boundary is where the inflation ends. The important point is that the end of inflation may not coincide with the uniform density hypersurfaces even though there is no other light scalar field (moduli). The mismatch (modulation) is due to the conventional isocurvature perturbations of the curvaton field, not from perturbations of extra light field (moduli). The mismatch (modulation) is due to the conventional isocurvature perturbations of the curvaton field, not from perturbations of extra light field [14, 15, 16]. Although the isocurvature perturbations of the curvaton field dies away (or converted into curvature perturbations) during the curvaton inflation, it could not be negligible when the curvaton inflation is short. Therefore, after establishing the non-linear formalism of the inflating curvaton, we have to go further to deal with the modula-

1 Here, we should remember that the conventional quadratic potential (without additional vacuum energy) is usually not suitable for the inflating curvaton mechanism in the sense that the second inflation (curvaton inflation) may not create enough perturbation via the curvaton mechanism.
tion at the end.\(^2\) Here the modulation is defined as the discrepancy between the transition hypersurfaces and the uniform density hypersurfaces.\(^3\)

We basically use the \(\delta N\) formalism for the calculation. In this formalism, \(\zeta\) is defined by smoothing the energy density \(\rho\) on a super-horizon scale shorter than any scale of interest. Then the local energy continuity equation is given by

\[
\frac{\partial \rho(x, t)}{\partial t} = -\frac{3}{a(x, t)} \frac{\partial a(x, t)}{\partial t} (\rho(x, t) + p(x, t)),
\]

where \(t\) is time along a comoving thread of spacetime and \(a\) is the scale factor.

During inflation, the vacuum fluctuation of a light scalar field \(\phi_i\) is converted at horizon exit to a nearby Gaussian classical perturbation with spectrum \(H/2\pi\). Here \(H \equiv \dot{a}(t)/a(t)\) is the Hubble parameter defined in the unperturbed universe. The \(\delta N\) formalism gives

\[
\zeta = \delta \ln \left[ a(x, t) / t \right] \equiv \delta N,
\]

where taking \(t_*\) to be an epoch during inflation after relevant scales leave the horizon, one can assume \(N(x, t_*) = N_i(x)\) so that

\[
\zeta(x, t) = N_i \delta \phi_i(x, t_*) + \frac{1}{2} N_{i j} \delta \phi_i(x, t_*) \delta \phi_j(x, t_*) + \ldots,
\]

where a subscript \(i\) denotes the derivative with respect to \(\phi_i\), which is evaluated on the unperturbed trajectory.

In the curvaton calculation\(\text{[10]}\) one usually assumes that these expressions are dominated by the single “curvaton” field \(\sigma\), which starts oscillation in the radiation dominated Universe when \(\sigma\) has the negligible contribution to the curvature perturbation. Then the non-Gaussianity parameter is given by\(\text{[23, 25]}\)

\[
f_{NL} \simeq \frac{5}{4r_\sigma} \left( 1 + \frac{g''(g)}{g'^2} \right) - \frac{5}{3} - \frac{5}{6} r_\sigma,
\]

where \(g(\sigma_*)\) is the initial amplitude of the oscillation as a function of the curvaton field at horizon exit\(\text{[23, 24]}\), and \(r_\sigma\) will be defined later in this paper. The above result is obtained for the two-component Universe, in which one component behaves like matter while the other behaves like radiation. Our calculation is aimed to generalize and extend the above result to include a slowly varying component. Any kind of components (e.g., cosmological defects\(\text{[26]}\)) can be included in the same way.

\(\text{\footnote{Typical examples of the usual modulation will be the modulated reheating\(\text{[15, 21]}\), modulated end of inflation\(\text{[22]}\) or the inhomogeneous phase transition\(\text{[15, 22]}\). We are not arguing the modulation of this kind, in which additional light scalar field causes modulation.}}\)

\(\text{\footnote{Usually, the curvaton mechanism assumes sudden-decay approximation. We are assuming abrupt change of the density scaling relation at the ‘transition’. In the usual curvaton mechanism, modulation at the beginning of the curvaton mechanism has been calculated as the deviation in the function \(g(\sigma_*)\), which has been introduced by D.H. Lyth in Ref.\(\text{[23]}\).}}\)

A. Non-linear formalism (brief review)

Let us first review the basic idea of the non-linear formalism. (See also the appendix.) In this paper we consider the non-linear formalism defined in Ref.\(\text{[27, 28]}\);

\[
\zeta_i = \delta N + \int_{\rho_i}^{\rho_i^*} \frac{d\rho_i}{3(1 + w_i) \rho_i} - \frac{1}{3} \ln \left( \frac{\rho_i}{\rho_i^*} \right)
\]

\[
\sim \delta N + \frac{1}{3(1 + w_i)} \delta \rho_i^{iso} / \rho_i,
\]

where \(\delta \rho_i^{iso}\) will be defined in Eq.(7). Here \(w_i = 1/3\) for the radiation fluid and \(w_i = 0\) for the matter fluid. A bar is for the homogeneous quantity, and \(\rho_i\) is defined on the uniform density hypersurfaces. Due to the isocurvature perturbations, \(\bar{\rho}_i(t) \neq \rho_i(x, t)\) is possible in the multi-component Universe. The curvature perturbation of the total fluid should be discriminated from the component curvature perturbation \(\zeta_i\). The standard definition of the adiabatic perturbation is given by

\[
\delta N = -M H \delta \rho_i^{adi} / \rho_i,
\]

where \(\delta \rho_i^{adi} \equiv \sum \delta \rho_i^{iso}\) must be evaluated on the spatially flat hypersurfaces. In contrast to \(\delta \rho_i^{iso}\), the isocurvature quantity \(\delta \rho_i^{iso}\) is related to the fraction perturbation defined on the uniform density hypersurfaces. Using the homogeneous density \(\bar{\rho}_i(t)\), the isocurvature density perturbation is defined on the uniform density hypersurfaces as

\[
\delta \rho_i^{iso} (x, t) \equiv \rho_i(x, t) - \bar{\rho}_i(t),
\]

which satisfies \(\sum \delta \rho_i^{iso} = 0\). In the multi-component Universe, the hypersurface defining uniform “density” \((\rho \equiv \sum \rho_i)\) is usually different from the one defining uniform “component density” \((\rho_i)\).

We find from the second line of Eq.(5);

\[
\rho_i = \bar{\rho}_i e^{3(1 + w_i)(\zeta_1 - \delta N)}.
\]

Using the above equation for the two-component Universe, the definition of the total energy density \(\rho^{total} = \rho_1 + \rho_2\) on the uniform density hypersurfaces leads to

\[
f_1 e^{3(1 + w_1)(\zeta_1 - \delta N)} + (1 - f_1) e^{3(1 + w_2)(\zeta_2 - \delta N)} = 1,
\]

where the fraction of the energy density is defined by

\[
f_1 \equiv \frac{\bar{\rho}_1}{\bar{\rho}_1 + \bar{\rho}_2}.
\]

Expanding Eq.(9) and solving the equation for \(\delta N\), we find at first order

\[
\delta N = r_1 \zeta_1 + (1 - r_1) \zeta_2
\]

\[
= r_1 (\zeta_1^{iso} + (1 - r_1) \zeta_2^{iso}) + \zeta_1^{adi},
\]
where \( \zeta_i \equiv \zeta - \zeta_{\text{adi}} \) is introduced in the last line. \( r_1 \) is defined by
\[
r_1 \equiv \frac{3(1 + w_1) \rho_1}{3(1 + w_1) \rho_1 + 3(1 + w_2) \rho_2},
\]
where \( w_1 = 0 \) and \( w_2 = 1/3 \) gives for the usual curvaton;
\[
r_1 \equiv \frac{3 \rho_1}{3 \rho_1 + 4 \rho_2}.
\]
Note that \( r_1 \) and \( f_1 \) are comparable in the usual curvaton scenario; however in the inflating curvaton the discrepancy between these quantities is significant. For the slowly varying curvaton density, we consider \( 1 + w_1 \ll 1 \). In that case, Eq. (12) can give \( r_1 \ll 1 \) even if the curvaton density \( (\rho_1) \) is already dominating the Universe.

Before discussing the evolution of the curvature perturbation for the slowly varying curvaton density, let us remember why the “evolution” is possible in the curvaton mechanism. Note first that \( \zeta_{\text{adi}} \) is identical to \( \delta N \) at the time when the initial quantities are evaluated. Usual curvaton scenario assumes \( \delta N = \zeta_{\text{adi}} = 0 \) at the beginning of the oscillation and it defines the initial condition. Since \( w_1 = 0 \) and \( w_2 = 1/3 \) are constant in the usual curvaton mechanism, \( \zeta_i \) and \( \zeta_d \) are constant during the evolution \( [27] \). Hence, \( \delta N \) is time-dependent only when \( r_1(t) \) is changing, which defines the evolution of the curvature perturbation in the usual curvaton mechanism. The definition given by Eq. (11) can be used anytime and for any long-wavelength perturbations. However, because we are formulating the evolution of the adiabatic perturbation that is caused by the adiabatic-isocurvature mixings, we need first to define the “starting point” at an epoch and then we can discuss the “evolution” thereafter. In this paper, the quantities evaluated at the starting point are defined in the function \( g(\phi_1) \), which will be defined later in Eq. (25).

The slowly varying curvaton density is realized by \( 1 + w_1 \ll 1 \) in the above formalism. The perturbation of the function \( g(\phi_1) \equiv (1 + w_1) \) is essential. This contribution is important, since it can change the sign of \( f_{NL} \).

Besides the curvaton mechanism, which we have discussed above, there could be some important contributions from the boundaries. In the non-linear formalism the evolution “before” the starting point is totally included in the function \( g \). On the other hand, the transition “at the end of the curvaton mechanism” is usually supposed to occur on the uniform density hypersurfaces; however the inflating curvaton may break the rule. In that sense, these two sources from the boundaries must be unified in the formalism \([18]\).

II. MODELS

A. Inflating Curvaton 1 (Slow-roll and sufficient inflation)

In this paper \( \sigma \) is the curvaton and \( \phi \) is the inflaton of the primordial inflation. The subscripts “\( \sigma \)” and “\( \phi \)” are used to define the related quantities. About the densities, \( \rho_\sigma \) is the curvaton density and \( \rho_r \) is the radiation density that is generated after the primordial inflation. For simplicity, we assume that \( \phi \) decays directly into radiation just after inflation, and \( \sigma \) decays into radiation just after the curvaton inflation. (The direct-decay approximations.) We assume no mixing between \( \phi \) and \( \sigma \). \( H_I \) denotes the Hubble parameter during the primordial inflation. In the direct-decay approximation, the sinusoidal \( \sigma \)-oscillation of the curvaton field starts at \( H_{\text{osc}} \) and then it decays instantly at \( H_{\text{dec}} \approx H_{\text{osc}} \).

Then, there are two phases (A, C) characterized by the parameter \( w_\sigma \), which are separated by \( H = H_{\text{osc}} \approx H_{\text{dec}} \):

(A) \( \rho_\sigma \) is slowly varying,
\[
\rho_r \text{ is the radiation,}
\]
\[
1 + w_\sigma \equiv \epsilon_\sigma \ll 1 \text{ and } w_r = 1/3.
\]

(B) This phase is skipped in the direct-decay approximation.
\[
\sigma \text{ is oscillating,}
\rho_\sigma \text{ is the radiation,}
\]
\[
w_\sigma = 0 \text{ and } w_r = 1/3
\]

(C) Both \( \rho_\sigma \) and \( \rho_r \) are the radiation,
\[
(w_\sigma = w_r = 1/3).
\]

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4 This is the reason why the non-linear formulation is useful for the curvaton mechanism. Alternatively, the constancy of the scaling exponents can be used.

5 There are many papers in which numerical calculation has been used for the curvaton mechanism. See for instance Ref. \([16]\).
In contrast to $\zeta$, $\zeta_r$ is always invariant during the evolution. (See also Fig.1)

The evolution of the curvature perturbations in the phase (A) may not be simple. We thus need further simplifications of the scenario:

1. As far as $\rho_r$ is dominating the Universe, we always find $r_\sigma \sim \epsilon_w \frac{a(t)}{H(t)} < \epsilon_w \ll 1$. Our first assumption for the analytic calculation is that the curvaton mechanism is negligible before the curvaton inflation.

2. Our second assumption is that $\zeta$ can behave like constant from $t = t_{ini}$, somewhat after the beginning of the curvaton inflation, and can remain approximately a constant when the significant conversion of the curvaton mechanism is working.

At the end, one will find $r_\sigma \sim 1$ when the curvaton inflation is “sufficient”; while “short” inflation may lead to $r_\sigma \ll 1$. In this section we are considering the former scenario.

At the beginning of the curvaton inflation, we define the initial perturbation $\zeta_{\sigma, ini}$. Using the same argument as in Eq. (11), we find just before the end of the curvaton inflation:

$$\delta N_- \equiv r_\sigma - \zeta_{\sigma} + (1 - r_\sigma)\zeta_r - \zeta_{\sigma, ini} + (1 - r_\sigma)\zeta_{r, ini}.$$  \hspace{1cm} (14)

where

$$r_\sigma \equiv \frac{3\epsilon_w \rho_{\sigma}}{3\epsilon_w \rho_{\sigma} + 4\rho_r}.$$  \hspace{1cm} (15)

Here the minus sign denotes the quantities just before the end of the curvaton inflation. These quantities satisfy

$\bar{\rho}_{\sigma} - \rho_{\sigma, ini} \approx \rho_{\sigma, ini}$ (because $\rho_\sigma$ is slowly varying) and $\rho_{\sigma, ini} \approx \rho_r, ini \gg \bar{\rho}_r$.

Then, we find the curvature perturbation created by the evolution

$$\delta N_{long} \sim r_\sigma - \zeta_{\sigma, ini} \approx \frac{\delta \rho_\sigma}{3\epsilon_w \bar{\rho}_\sigma}_{ini}.$$  \hspace{1cm} (16)

where $r_\sigma \sim 1$ is assumed for the sufficient inflation.

For the curvaton inflation, we find that the number of e-foldings required for $r_\sigma \sim 0.9$ (we consider 0.9 just for instance) is given by

$$\tilde{N}_e = \frac{1}{4} \ln \left( \frac{\rho_{\sigma} - \bar{\rho}_{\sigma}}{\rho_r - \bar{\rho}_r} \right)^{1/4} = -\frac{1}{4} \left[ \ln \epsilon_w + \ln \left( \frac{1 - r_\sigma}{r_\sigma} \right) - \ln \frac{4}{3} \right] \approx -\frac{1}{4} \left[ \ln \epsilon_w + \ln \left( \frac{1}{12} \right) \right],$$  \hspace{1cm} (17)

where $\tilde{N}_e$ is the number of e-foldings elapsed during the curvaton inflation. When $\epsilon_w \ll 1$, it is obvious that significant inflation is needed to achieve $r_\sigma \sim 1$ before the end.

B. Inflating curvaton 2 (Slow-roll but not sufficient inflation)

Practically this model requires fine-tuning of the initial conditions since we are considering short inflation in the slow-roll limit. The benefit of the model is that it gives a quite suggestive consequence in the well-defined limit.

We find from Eq. (13):

$$\bar{\rho}_{\sigma -} = \frac{4r_\sigma - 3\epsilon_w (1 - r_\sigma)}{\rho_r - \bar{\rho}_r}.$$  \hspace{1cm} (18)

If the curvaton inflation is short (ends with $r_\sigma \ll 1$), we find $3\epsilon_w - \rho_{\sigma -} + 4\bar{\rho}_r \approx 4\bar{\rho}_r$. Then, the curvature perturbation created by the evolution is given by

$$\delta N_{short} \sim r_\sigma - \zeta_{\sigma, ini} \approx \frac{\delta \rho_{\sigma, ini}}{4 \bar{\rho}_r} \approx \frac{r_\sigma - \delta \rho_{\sigma, ini}}{3\epsilon_w \bar{\rho}_{\sigma, ini}}.$$  \hspace{1cm} (19)

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6 The standard assumptions of the curvaton mechanism are $\zeta_r \ll \zeta_\sigma$ and $\zeta_{\sigma, ini} \approx 0$. The above result is obtained using these assumptions.
Therefore, \( \Delta N^{\text{short}} \ll \Delta N^{\text{long}} \) is obvious, which shows that longer inflation (curvaton inflation) is efficient for the curvaton mechanism.

The above result is valid when the end of the curvaton inflation coincides with the uniform density hypersurfaces; otherwise the modulation at the end may not be negligible. This condition is not obvious and highly model-dependent. The most obvious counter-example is the hybrid curvaton, which will be discussed in Sec. IV

### III. NON-GAUSSIANITY

In this section we are basically following the non-linear calculation in Ref.[23]. We are extending the calculation to include the slowly varying curvaton density and the perturbations related to \( \epsilon_w(\sigma) \). Lengthy equations are omitted, since the calculation itself is quite simple and straight. More details of the expansions will be shown in the appendix. Writing down the expansions in the non-linear formalism, one will automatically find \( f_{NL} \).

Generically, one can expand

\[
\sigma = \bar{\sigma} + \sum_{k=1}^{\infty} \frac{1}{k!} \delta^{(k)} \sigma,
\]

where \( \delta^{(1)} \) is a Gaussian random field. In that way, the primordial perturbation can be expanded as

\[
\zeta = \zeta^{(1)} + \sum_{k=2}^{\infty} \frac{1}{k!} \delta^{(k)} \sigma,
\]

where \( \zeta^{(1)} \) is Gaussian. Non-linearity parameters are defined for the adiabatic perturbation \( \zeta \):

\[
\zeta = \zeta^{(1)} + \frac{3}{5} f_{NL} (\zeta^{(1)})^2 + \frac{9}{25} g_{NL} (\zeta^{(1)})^3 + ...\]

Assume that the curvaton potential during the curvaton inflation is given by the quadratic potential:

\[
\rho_\sigma = V_0 \pm \frac{1}{2} m^2 \sigma^2,
\]

where the effective mass term may have either positive or negative signs.

Using the Gaussian quantum fluctuations at the horizon exit \( \delta \sigma_* \), we can write\[23\]

\[
\sigma_* = \bar{\sigma} + \delta \sigma_*.
\]

In that case we write

\[
\sigma_{\text{ini}} = g(\sigma_*)
\]

and expand \[23\]

\[
\sigma_{\text{ini}} = \bar{g} + \sum_{k=1}^{\infty} \frac{1}{k!} g^{(k)} \left( \frac{\bar{g}}{g' / \sigma} \right)^k,
\]

where we wrote \( g^{(n)} \equiv \partial^n g / \partial \sigma^n \). Assuming that \( \delta N \ll \zeta_\sigma \) for the starting-point perturbations, we find

\[
\rho_{\sigma, \text{ini}} \simeq \rho_{\sigma, \text{ini}} e^{3 \epsilon_w \zeta_{\sigma, \text{ini}}}.
\]

Although \( \epsilon_w \) is approximately constant during evolution, one cannot ignore its perturbations. We find for \( \epsilon_w \propto \sigma^l \):

\[
\epsilon_w = \epsilon_w^{(0)} + l \epsilon_w \frac{\delta \sigma}{\sigma} + ...\]

We find \( l = 2 \) for the quadratic hilltop potential. For simplicity, we will use \( \epsilon_w \) instead of \( \epsilon_w^{(0)} \) when there is no confusion.

In the calculation below we will omit the subscript “ini” when it is apparent. Substituting Eq.(26) into Eq.(27) we obtain for the inflating curvaton;

\[
e^{3 \epsilon_w \zeta_\sigma} = \frac{V_0 \pm \frac{1}{2} m^2 g' \delta \sigma}{V_0 \pm \frac{1}{2} m^2 g'^2},
\]

where \( \epsilon_w \ll 1 \) is assumed for the inflating curvaton. Order by order, we have for the expansion \( \rho_\sigma = \bar{\rho}_\sigma + \sum_{k=1}^{\infty} \frac{1}{k} \delta^{(k)} \rho_\sigma \):

\[
\delta^{(1)} \rho_\sigma = m^2 g \delta \sigma
\]

\[
\delta^{(2)} \rho_\sigma = m^2 \left( 1 + \frac{g g''}{g'^2} \right) (\delta \sigma)^2
\]

Defining the ratio \( R \equiv \frac{1}{2} m^2 \bar{\sigma}^2 / \bar{\rho}_\sigma \), we find for \( \zeta_\sigma \equiv \zeta_\sigma^{(1)} + \frac{2}{3} \zeta_\sigma^{(2)} + \frac{4}{3} \zeta_\sigma^{(3)} + ... \):

\[
\zeta_\sigma^{(1)} \simeq \frac{2R \delta \sigma}{3 \epsilon_w \bar{\sigma}}
\]

\[
\zeta_\sigma^{(2)} \simeq \frac{3 \epsilon_w}{2R} \left( 1 - 2R + \frac{g g''}{g'^2} - 4 \right) (\zeta_\sigma^{(1)})^2.
\]

In the last line, \(-4\) appears from the expansion of \( \epsilon_w \). In the slowly-varying density phase \( \rho_\sigma \), we find

\[
f_\sigma e^{3 \epsilon_w (\zeta_\sigma - \delta N)} + (1 - f_\sigma) e^{4 (\zeta_\sigma - \delta N)} = 1,
\]

where \( f_\sigma \equiv \frac{\bar{\rho}_\sigma}{\bar{\rho}_\sigma + \bar{\rho}_\sigma} \). Assuming \( \zeta_\sigma \simeq 0 \) and expanding the equation, we obtain at first order

\[
f_\sigma 3 \epsilon_w \left[ \epsilon_\sigma^{(1)} + (1 - f_\sigma) \left[ -4 \delta N^{(1)} \right] = 0,
\]

where the expansion is given by \( \delta N \equiv \delta N^{(1)} + \frac{1}{2} \delta N^{(2)} + \frac{1}{3} \delta N^{(3)} + ... \). We can solve this equation as

\[
\delta N^{(1)} \simeq r_\sigma \zeta_\sigma^{(1)}.
\]
At second order, we find

\[-4(1 - f_\sigma)\delta N^{(2)} + 16(1 - f_\sigma)(\delta N^{(1)})^2\]
\[+ 3\epsilon_w f_\sigma \left(\epsilon^{(2)}_w - \delta N^{(2)}\right) + 9\epsilon_w f_\sigma \left(\epsilon^{(1)}_w - \delta N^{(1)}\right)^2\]
\[+ 6\epsilon^{(1)}_w f \left(\epsilon^{(1)}_w - \delta N^{(1)}\right) = 0,\]

where the last line includes \(\epsilon_w\)-expansion. We thus find that

\[\frac{\delta N^{(2)}}{(\delta N^{(1)})^2} = \frac{1}{r_\sigma} \left[\frac{3\epsilon_w}{2R} \left\{-4r_\sigma - 2R + 1 + \frac{gg''}{g^2}\right\}\right.\]
\[\left.\quad + 3(1 - r_\sigma)^2\epsilon_w + 4r_\sigma(1 - r_\sigma)\right].\]

The above result does NOT reproduce the standard curvaton even if one substitutes \(\epsilon_w = 1\) and \(R = 1\), since our result has been obtained after expanding \(\epsilon_w \propto \sigma^2\). More details about the expansion and the explicit relation between the conventional curvaton mechanism will be discussed in the appendix.

We thus find the non-Gaussianity parameter for the slowly-varying curvaton density:\(^8\)

\[f_{NL} = \frac{1}{r_\sigma} \frac{5\epsilon_w}{4R} \left\{1 + \frac{gg''}{g^2}\right\}\]
\[+ \frac{5}{3} \epsilon_w - \frac{10}{3} + \left(\frac{5}{2} \epsilon_w - \frac{10}{3}\right) r_\sigma - \frac{5\epsilon_w}{R}.\]

In contrast to the standard curvaton, a minus sign \((R < 0)\) is possible for the hilltop potential \(^{29}\). For the quadratic potential we find \(\epsilon_w / R \sim \frac{2}{3}\eta_\sigma\) in the slow-roll limit, where \(\eta_\sigma \equiv m^2 / 3H^2\) is the conventional slow-roll parameter during the curvaton inflation.

Fast-roll

For the fast-roll scenario, \(\epsilon_w\) will have a different \(\eta\)-dependence. For the fast-roll field one will find

\[\dot{c}H\dot{\sigma} \simeq V'\]

with the coefficient defined by

\[\dot{c} = \frac{3 + \sqrt{9 - 12\eta_\sigma}}{2}.\]

We thus find

\[-H\frac{\delta \sigma}{\sigma} \simeq \frac{\dot{c}}{3\eta_\sigma} \frac{\delta \sigma}{\sigma}.\]

Our definition of \(\epsilon_w (\dot{\rho} = -3H\epsilon_w\rho)\) can be combined with \(\dot{\rho} \simeq V'\dot{\sigma}\) to give another expression

\[-H\frac{\delta \sigma}{\sigma} \simeq \frac{2}{3} \frac{R}{\epsilon_w} \frac{\delta \sigma}{\sigma}.\]

One may choose either \(\dot{c}\) or \(\epsilon_w\) for the calculation. The choice of the definition cannot cause any discrepancy in the result.

Estimation of the function \(g\)

For the practical estimation of the curvature perturbations one has to calculate \(g\). \(g\) is trivial in the slow-roll limit (since in that limit the motion is negligible), while for the fast-rolling one has to calculate the model-dependent evolution before the curvaton mechanism.\(^9\)

Quadratic potential is an exception, for which \(\delta \sigma\) and \(\sigma\) obey identical equation of motion (which is a linear differential equation) and their combination \(\delta \sigma / \sigma\) behaves like constant. Moreover, with the quadratic assumption we have

\[g' = \frac{dg}{d\sigma} = \frac{g}{\sigma},\]
\[g'' = -g + \frac{g' \sigma}{\sigma^2} = 0,\]

which is the result obtained in Ref.\(^{[14]}\) and suggests that there is no \(g\)-dependence in \(f_{NL}\) when the quadratic assumption is valid. See Ref.\(^{[14]}\) for more details.

Spectral index and the slow-roll parameters

The spectral index requires \(|\eta_\sigma| \ll 1\) during primordial inflation, which must be explained in the specific inflationary model that creates the perturbation \(\delta \sigma\).\(^{10}\)

Isocurvature perturbation?

In the usual curvaton mechanism, \(f_{NL} \gg 1\) may lead to significant isocurvature perturbation \(^{32}\). To avoid the unwanted isocurvature perturbation, the baryon number asymmetry (BAU) must be created after the curvaton mechanism. This is not the case in the inflating curvaton scenario, since the energy density is already dominated by the curvaton even if \(r_\sigma\) is smaller than unity. This is a unique character of the inflating scenario of the curvaton mechanism.

About \(g\)

The evaluation of the function \(g\) is very important when one estimates the non-linearity caused by the evolution of the curvaton field. Note first that the method required for the evaluation of \(g\) is compatible among different curvaton scenarios (the oscillating and the inflating curvatons). To begin with, we summarize the past discussions about the function \(g\) in the usual (oscillating) curvaton model.

1. In the (oscillating) curvaton model, the curvature perturbation is determined by the quantity \(\frac{\delta \sigma}{\sigma}\).

\(^8\) This is partly different from earlier simple estimations \(^{[14]}\).

\(^9\) Just to avoid the complexity of the calculations, we have ignored \(g\) in \(\epsilon_w\); note that \(g\)-dependence appears from \(\epsilon^{(2)}_w\), which does not appear in the calculation of \(f_{NL}\).

\(^{10}\) According to Ref. \(^{31}\), one can expect \(|\eta_\sigma| \ll 1\) during the radiation domination epoch. The inflating curvaton without mass protection can cause significant scale-dependence of the perturbations and it may cause generation of the primordial black holes \(^{14}\).
Useful point is that when the potential is quadratic the above combination behaves like constant before the oscillation. Hence, one usually finds simple estimation without knowing the explicit form of the function $g$.\footnote{33}

2. The simple calculation does not apply when the potential deviates from the quadratic. A detailed study can be found in \cite{33,34}. Quantum corrections are discussed in \cite{35}.

In the name of the “curvaton”, the above points apply to the inflating curvaton, since the evolution before the curvaton mechanism (i.e, the method required for the estimation of $g$, which is defined at the beginning of the curvaton mechanism) is compatible among those curvaton scenarios.

A. More extension (higher-order potential)

In the above calculation we have considered a simple quadratic potential. The quadratic potential is convenient for the analytic estimation of the function $g$; however the curvaton potential could be different from the quadratic one. Here we will find the basic formalism for the calculation. Analytic estimation of the function $g$ is quite difficult when the slow-roll conditions are violated. We consider the potential given by a polynomial:

$$
\rho_\sigma = V_0 + \frac{\lambda}{n M^{n-4}} \sigma^n,
$$

(44)

and $\epsilon_w$ could have non-trivial $\sigma$-dependence. In that case, assuming the instant and direct decay after the slowly varying density phase (A), we find for the inflating curvaton

$$
\bar{\rho}_\sigma = V_0 + \frac{\lambda}{n M^{n-4}} \bar{\sigma}^n,
$$

(45)

$$
\rho_\sigma = V_0 + \frac{\lambda}{n M^{n-4}} \left[ \bar{\sigma} + \frac{1}{k!} g(k) \left( \frac{\delta \sigma}{\sigma} \right)^k \right]^n.
$$

(46)

Defining the ratio

$$
R = \frac{\lambda}{n M^{n-4}} \frac{\bar{\rho}_n}{\rho_\sigma},
$$

(47)

and assuming $\epsilon_w \propto \sigma^4$, we find order by order;

$$
\zeta_\sigma^{(1)} = \frac{n R \delta \sigma}{3 \epsilon_w \sigma} \simeq -H \frac{\delta \sigma}{\sigma},
$$

(48)

$$
\frac{\zeta_\sigma^{(2)}}{\zeta_\sigma^{(1)}} = \frac{3 \epsilon_w}{n R} \left[ n - n R + \frac{g g''}{g'^2} - 2 \right]
$$

(49)

In the slowly varying density phase (A), the non-linear formalism gives Eq. \cite{35}. Expanding the equation, we obtain Eq. \cite{34}. At second order, we find

$$
\frac{\delta N^{(2)}}{(\delta N^{(1)})^2} = \frac{1}{r_\sigma} \left[ \frac{3 \epsilon_w}{n R} \left\{ -2 l r_\sigma - n R + (n - 1) + \frac{g g''}{g'^2} \right\} \right.
$$

$$
+ 3(1 - r_\sigma)^2 \epsilon_w + 4 r_\sigma (1 - r_\sigma) \bigg]\bigg). \tag{50}
$$

We thus find the non-Gaussianity parameter

$$
f_{NL} = \frac{5}{6} \left[ \frac{3 \epsilon_w}{n R r_\sigma} \left\{ -2 l r_\sigma - n R + (n - 1) + \frac{g g''}{g'^2} \right\} \right.
$$

$$
+ 3 \epsilon_w \left( \frac{1}{r_\sigma} - 2 + r_\sigma \right) + 4(1 - r_\sigma) \bigg]. \tag{51}
$$

The standard curvaton corresponds to $n = 2$, $l = 0$, $R = 1$ and $\epsilon_w = 1$, which gives

$$
f_{NL} = \frac{5}{6} \left[ \frac{3}{2r_\sigma} \left\{ -2 + 1 + \frac{g^2}{g'^2} \right\} \right.
$$

$$
+ 3 \left( \frac{1}{r_\sigma} - 2 + r_\sigma \right) + 4(1 - r_\sigma) \bigg]. \tag{52}
$$

IV. MODULATION AT THE END OF THE INFLATING CURVATON

A. Modulation in the non-linear formalism

The modulation at the end (at the transition $w_{\sigma-} \to w_{\sigma+}$) can be implemented in the curvaton mechanism. Assuming that the transition occurs at the density $\rho_x$, the non-linear formalism evaluated at the end can be separated as

$$
\zeta_\sigma = \delta N + \int_{\rho_x}^{\rho_{\sigma-}} H \frac{d \rho_\sigma}{3(1 + w_{\sigma+}) \rho_\sigma} + \int_{\rho_x}^{\rho_{\sigma-}} H \frac{d \rho_\sigma}{3(1 + w_{\sigma-}) \rho_\sigma}
$$

$$
= \delta N + \frac{1}{3(1 + w_{\sigma-})} \ln \left( \frac{\rho_x}{\rho_{\sigma-}} \right)
$$

$$
+ \frac{1}{3(1 + w_{\sigma-})} \ln \left( \frac{\rho_x}{\rho_{\sigma-}} \right). \tag{53}
$$

Here $\rho_\sigma$ ($\rho_{\sigma-} > \rho_x > \rho_{\sigma+}$) is placed after the transition. Allowing oscillation after curvaton inflation (i.e, phase (B) is not ignored), the above quantities are defining the initial condition for the usual curvaton mechanism during the phase (B).

First take the limit $\rho_{\sigma-} \to \rho_x$ (i.e, the transition hypersurface is identical to the uniform density hypersurface). In that case $\zeta_\sigma$ at the end of the inflating curvaton is given by

$$
\zeta_{\sigma,e} = \delta N + \frac{1}{3} \ln \left( \frac{\rho_x}{\rho_{\sigma-}} \right)_e. \tag{54}
$$

Eq. \cite{34} gives the initial condition for the oscillating curvaton that may work after inflation. Obviously, $\delta \rho_{\sigma,e}$
\(\rho_{\sigma,e} - \bar{\rho}_{\sigma,e}\) is negligible after sufficient inflation, while it could be significant if the curvaton inflation is short.

With regard to the boundary at the end, the opposite limit \(\rho_x \rightarrow \bar{\rho}_{\sigma}\) is needed for the hybrid curvaton \([36]\), in which the waterfall begins presumably at \(\rho_x = \bar{\rho}_{\sigma} = V(\sigma_e)\). In that case we find (in contrast to the conventional calculation):

\[
\zeta_{\sigma,e} = \delta N + \frac{1}{3\epsilon_w} \ln \left( \frac{\rho_{\sigma}}{\bar{\rho}_{\sigma}} \right), \quad (55)
\]

which is enhanced when \(\epsilon_w \ll 1\). The isocurvature density perturbation \((\delta \rho_{\sigma,e})\) is indeed significant when the curvaton inflation is short. Of course the waterfall stage of the hybrid curvaton may start \textit{without curvaton inflation}. In that case the isocurvature perturbation is obviously significant.

Note that the enhancement at the end is not due to the “evolution” (the curvaton mechanism), but caused by the “modulation”. Unlike the usual modulation, the above “modulation” is simply due to the isocurvature perturbation of the curvaton field. There is no additional moduli that causes extra modulation.

V. CONCLUSIONS AND DISCUSSIONS

In this paper we have calculated the curvature perturbation caused by the inflating curvaton mechanism. To explain the mechanism, a typical two-component Universe has been considered, in which one component (curvaton) is slowly varying. In this paper the modulation at the end of the curvaton mechanism is specifically defined as the “inhomogeneous transition in the curvaton sector that does not require additional light field”. The modulation caused by an additional light field has been discussed in Ref. \([18, 22]\).

Generically, \(f_{N1}\) derived from the non-linear formalism depends on the model-dependent function \(g\). In that way the analytic estimation of the non-Gaussianity parameter is possible only when \(g\) is obvious. In the slow-roll limit, \(g\) is trivial and the analytic estimation is possible. For the fast-roll curvaton, estimation is possible when the potential is quadratic. Otherwise the function \(g\) requires more assumptions or highly model-dependent analyses. Exact calculation of the model-dependent function \(g\) has been separated from the present work. The relation between the curvaton mechanism and the modulation at the end is clear in the non-linear formalism. This is the first exact calculation of the non-Gaussianity created by the slowly varying curvaton density.

For our purposes, we took some simple set-ups for the calculation. We thus need further study, such as

1. Instant phase transition has been considered in this paper; however the transition should be more complicated depending on the details of the model parameters \([37]\). In the standard curvaton scenario the sudden-decay approximation gives a good intuitive derivation, however in practice the curvaton density is continually decaying into radiation.

2. In this paper, we were avoiding the interaction between components \([38]\). If the interaction is significant the dissipation may appear, which (in the most extreme case) may lead to warm curvaton scenario \([39]\).

3. In this paper the second component is always the radiation. This assumption is useful in capturing the essential of the scenario: however the assumption may not be valid in practice. We need to discuss multi-component Universe that could be a mixture of the slow-varying density, defects, matter and radiation.

4. The curvaton evolution and the modulation might be significant after N-flation, while the model considered in this paper is based on two-component Universe. We thus need some statistical argument for the evolution of the perturbations after N-flation \([40]\).

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Appendix A: Non-Linear formalism for the conventional curvaton

In this section we are going to review the basics of the non-linear formalism applied to the oscillating curvaton mechanism. The key point in this calculation is the constancy of the component perturbations.

First remember the non-linear formalism

\[
\zeta_{\sigma} = \delta N + \frac{1}{3} \ln \left( \frac{\rho_{\sigma}}{\bar{\rho}_{\sigma}} \right); \quad (A1)
\]

\[
\zeta_{r} = \delta N + \frac{1}{4} \ln \left( \frac{\rho_{r}}{\bar{\rho}_{r}} \right). \quad (A2)
\]

Here \(\delta N\) is the perturbation of \(N\) between two hypersurfaces; one is the flat hypersurface, and the other is usually a uniform density hypersurface. Besides \(\delta N\), we have to define the other quantities \((\rho_{\sigma}, \rho_{r})\) and \((\bar{\rho}_{\sigma}, \bar{\rho}_{r})\). Following the above definition of \(\delta N\), we define those quantities on the uniform density hypersurface on which \(\delta N\) is defined.
we find at first order
\[ \delta N_o = r_{\sigma,o}\zeta_{\sigma,o} + (1 - r_{\sigma,o})\zeta_{r,o} \]
\[ = r_{\sigma,o}\left[ \delta N_o + \frac{1}{3} \ln \left( \frac{\rho_{\sigma,o}}{\rho_{\sigma,o}} \right) \right] 
+ (1 - r_{\sigma,o}) \left[ \delta N_o + \frac{1}{4} \ln \left( \frac{\rho_{r,o}}{\rho_{r,o}} \right) \right] \]
\[ = \delta N_o + \frac{r_o}{3} \ln \left( \frac{\rho_{\sigma,o}}{\rho_{\sigma,o}} \right) + \frac{1 - r_o}{4} \ln \left( \frac{\rho_{r,o}}{\rho_{r,o}} \right) . \]  

The trivial identity is
\[ \frac{r_o}{3} \ln \left( \frac{\rho_{\sigma,o}}{\rho_{\sigma,o}} \right) + \frac{1 - r_o}{4} \ln \left( \frac{\rho_{r,o}}{\rho_{r,o}} \right) = 0. \]  

Equivalently, for the expansion \( \delta \rho_i = \rho_i - \bar{\rho}_i \) the above equation gives
\[ \delta \rho_{\sigma,o} + \delta \rho_{r,o} = 0, \]  

where the trivial expansion used above is
\[ \ln(1 + X) = X - \frac{1}{2} X^2 + ... \]  

**Here, the point is that the component perturbations are constant during the curvaton mechanism.**

One can evaluate the non-linear formalism away from \( H = H_o \). Choosing another hypersurface at \( H = H_d \), one can evaluate
\[ \delta N_d = r_{\sigma,d}\zeta_{\sigma,d} + (1 - r_{\sigma,d})\zeta_{r,d} \]
\[ = r_{\sigma,d}\left[ \delta N_d + \frac{1}{3} \ln \left( \frac{\rho_{\sigma,d}}{\rho_{\sigma,d}} \right) \right] 
+ (1 - r_{\sigma,d}) \left[ \delta N_d + \frac{1}{4} \ln \left( \frac{\rho_{r,d}}{\rho_{r,d}} \right) \right] \]
\[ = \delta N_d + \frac{r_d}{3} \ln \left( \frac{\rho_{\sigma,d}}{\rho_{\sigma,d}} \right) + \frac{1 - r_d}{4} \ln \left( \frac{\rho_{r,d}}{\rho_{r,d}} \right) + \frac{r_d}{3} \ln \left( \frac{\rho_{\sigma,d}}{\rho_{\sigma,d}} \right) + \frac{1 - r_d}{4} \ln \left( \frac{\rho_{r,d}}{\rho_{r,d}} \right) . \]  

Note that \( \delta N_o \) disappears from the result because of the obvious cancellation.

If one chooses \( H_o = H_{osc} \) at the beginning of the curvaton oscillation and \( H_d = H_{dec} \) at the decay, the above result gives the evolution of the curvature perturbation in the conventional curvaton mechanism. If one assumes \( \delta N_{inf} \simeq \delta N_{osc} \), which explains the curvature perturbation generated during primordial inflation, one will find
\[ \delta N_{dec} \equiv \delta N_{inf} + \delta N_{curv}. \]
Appendix B: More about the expansion

When we calculate the expansion of the component perturbation $\zeta$, we expand the relation $\rho_\sigma e^{3e_w\zeta} = \rho_\sigma$, which gives

$$
V_0 + \frac{\lambda}{n M^{n-4}} [g + \sum_{k=1}^{n} \frac{1}{k!} \delta^{(k)} \left( \frac{\delta_1 \sigma}{\sigma} \right)^k]^n
$$

where $e_w$ is expanded for $e_w \propto \sigma^l$;

$$
e_w = \epsilon^{(0)} + \epsilon^{(1)} + \ldots$$

The left-hand side of (B1) has

$$
\frac{\lambda}{n M^{n-4}} [g + \sum_{k=1}^{n} \frac{1}{k!} \delta^{(k)} \left( \frac{\delta_1 \sigma}{\sigma} \right)^k]^n
$$

when the subscripts 0th, 1st and 2nd are added to show clearly the expansion. The right-hand side of Eq. (B1) has

$$
\frac{\lambda}{n M^{n-4}} [g + \sum_{k=1}^{n} \frac{1}{k!} \delta^{(k)} \left( \frac{\delta_1 \sigma}{\sigma} \right)^k]^n
$$

Defining $\bar{V} = \left[ V_0 + \frac{\lambda}{n M^{n-4}} \right]$, the 0-th order terms on both sides give the relation

$$
\bar{\rho} = \bar{\rho},
$$

which is trivial.

The first order relation is

$$
\bar{\rho}_\sigma \times 3 \epsilon^{(0)} \zeta^{(1)} = \frac{\lambda \gamma^n}{n M^{n-4}} \left[ n \frac{\delta_1 \sigma}{\sigma} \right],
$$

which leads to

$$
\zeta^{iso(1)} = \frac{nR \delta_1 \sigma}{3 \epsilon^{(0)} \zeta^{(1)}},
$$

where we have introduced the ratio $R = \frac{\lambda \gamma^n}{n \rho_\sigma}$.

The second order equation is

$$
\left[ 3 \epsilon^{(0)} \zeta^{(1)} + \frac{3}{2} \epsilon^{(0)} \zeta^{(2)} + \frac{1}{2} (3 \epsilon^{(0)} \zeta^{(1)})^2 \right] = R \left[ \frac{n(n-1)}{2} + \frac{n g g'}{2 g^2} \right] \left[ \frac{\delta_1 \sigma}{\sigma} \right]^2.
$$

We find for $\epsilon^{(0)} \propto \sigma^l$;

$$
3 \epsilon^{(1)} \zeta^{(1)} = 3 \epsilon^{(0)} \delta_1 \sigma \zeta^{(1)}
$$

$$
\left[ \frac{g}{n} \right] \left[ \frac{\epsilon^{(0)}}{R} \right]^2 \left( \frac{\zeta^{(1)}}{\sigma} \right)^2,
$$

where the last equation uses (B7). As far as there will be no confusion in the calculations, we are going to replace $\epsilon^{(0)} \rightarrow e_w$ for simplicity. We find for the second order relation;

$$
\left[ \frac{g}{n} \right] \left[ \frac{\epsilon^{(0)}}{R} \right]^2 \left( \frac{\zeta^{(1)}}{\sigma} \right)^2
$$

where the last equation uses (B7). Solving the above equation for $\zeta^{(2)}$, we find

$$
\zeta^{(2)} = \frac{3 \epsilon^{(0)} \zeta^{(1)}}{nR} \left[ -2 + n \frac{n(n-1)}{2} + n \frac{g g''}{2 g^2} \right] \left( \frac{\zeta^{(1)}}{\sigma} \right)^2.
$$

The standard curvatton scenario corresponds to $e_w = 1$ ($l = 0$), $n = 2$ (quadratic potential) and $R = 1 (V_0 = 0)$, which gives the conventional curvatton result [25] for the component perturbation

$$
\zeta^{(2)} = \frac{3}{2} \left[ 1 + n \frac{g g''}{2 g^2} \right] \left( \frac{\zeta^{(1)}}{\sigma} \right)^2.
$$

We thus finished the expansion of the component perturbation $\zeta$. In order to calculate the curvature perturbation $\zeta$ from the component perturbation $\zeta$, we use the equation

$$
f_\sigma e^{3e_w(\zeta - \delta N)} + (1 - f_\sigma) e^{-4\delta N} = 1,
$$

where $\zeta$ has been neglected. Expanding $e^X = 1 + X + \frac{X^2}{2} + \ldots$, for $X = 3\epsilon^{(0)}(\zeta - \delta N)$ and $X = -4\delta N$, we obtain

$$
f_\sigma \left[ 1 + 3 \left( \epsilon^{(0)} + \epsilon^{(1)} + \frac{1}{2} \epsilon^{(2)} + \ldots \right) \right] \zeta^{(1)} + \frac{3}{2} \left( \epsilon^{(0)} + \epsilon^{(1)} + \frac{1}{2} \epsilon^{(2)} + \ldots \right)^2 \zeta^{(2)} + \ldots
$$

$$
+ (1 - f_\sigma) \left[ 1 - 4 \left( \delta N^{(1)} + \frac{1}{2} \delta N^{(2)} + \ldots \right) \right] \zeta^{(1)} + \frac{4}{2} \left( \delta N^{(1)} + \frac{1}{2} \delta N^{(2)} + \ldots \right)^2 \zeta^{(2)} + \ldots
$$

$$
= 1.
$$
At 0-th order we find
\[ f_\sigma + (1 - f_\sigma) = 1, \tag{B15} \]
which is trivial. At the first order we find
\[ 3f_\sigma \epsilon_w^{(0)} \left[ \zeta^{(1)}_\sigma - \delta N^{(1)} \right] + (1 - f_\sigma) \left[ -4\delta N^{(1)} \right] = 0. \tag{B16} \]
Introducing \( r_\sigma \equiv \frac{3\epsilon_w^{(0)} + 3\epsilon_w^{(0)} f_\sigma}{4(1-f_\sigma)+3\epsilon_w^{(0)} f_\sigma} \), we can solve this equation as
\[ \delta N^{(1)} = r_\sigma \zeta^{(1)}_\sigma. \tag{B17} \]
At second order, we find
\[ 3f_\sigma \epsilon_w^{(0)} \left( \zeta^{(1)}_\sigma - \delta N^{(1)} \right) + \frac{3}{2} f_\sigma \epsilon_w^{(0)} \left( \zeta^{(2)}_\sigma - \delta N^{(2)} \right) + \frac{9}{2} f_\sigma \left( \epsilon_w^{(0)} \right)^2 \left( \zeta^{(1)}_\sigma - \delta N^{(1)} \right)^2 \]
\[ -2(1 - f_\sigma) \delta N^{(2)} + 8(1 - f_\sigma) \left( \delta N^{(1)} \right)^2 = 0. \tag{B18} \]
We have to solve this equation for \( \delta N^{(2)} \). We are going to use the relations
\[ \epsilon_w^{(1)} \equiv \frac{\delta \sigma}{\sigma} = \frac{3\epsilon_w^{(0)} f_\sigma}{nR \zeta^{(1)}_\sigma} \tag{B19} \]
\[ \zeta^{(1)}_\sigma - \delta N^{(1)} = (1 - r_\sigma) \zeta^{(1)}_\sigma \tag{B20} \]
\[ \frac{\zeta^{(2)}_\sigma}{(\zeta^{(1)}_\sigma)^2} = \frac{3\epsilon_w}{nR} \left[ -2l + nR + \frac{2}{n} \left[ \frac{n(n-1)}{2} + \frac{n gg''}{g''} \right] \right]. \tag{B21} \]
Therefore, the second order equation can be rewritten as
\[ \frac{1}{2} \left[ 3f_\sigma \epsilon_w^{(0)} + 4(1 - f_\sigma) \right] \delta N^{(2)} = \frac{\left( \delta N^{(1)} \right)^2}{r_\sigma^2} \left[ \frac{9f_\sigma \epsilon_w^{(0)} \left( \epsilon_w^{(0)} \right)^2}{nR} (1 - r_\sigma) \right] \]
\[ + \frac{9}{2} f_\sigma \left( \epsilon_w^{(0)} \right)^2 \left[ -2l + nR + \frac{2}{n} \left[ \frac{n(n-1)}{2} + \frac{n gg''}{g''} \right] \right] \]
\[ + \frac{9}{2} f_\sigma (1 - r_\sigma)^2 \left( \epsilon_w^{(0)} \right)^2 + 8r_\sigma^2 (1 - f_\sigma). \tag{B22} \]
Using the equation
\[ \frac{1 - f_\sigma}{f_\sigma} = \frac{3\epsilon_w^{(0)}}{4r_\sigma} (1 - r_\sigma), \tag{B23} \]
we can write
\[ \frac{3f_\sigma \epsilon_w}{2r_\sigma} \delta N^{(2)} = \frac{\left( \delta N^{(1)} \right)^2}{r_\sigma^2} \left[ \frac{9f_\sigma \epsilon_w^{(0)} \left( \epsilon_w^{(0)} \right)^2}{nR} (1 - r_\sigma) \right] \]
\[ + \frac{9}{2} f_\sigma \left( (1 - r_\sigma)^2 \left( \epsilon_w^{(0)} \right)^2 + 8r_\sigma^2 (1 - f_\sigma) \right]. \tag{B24} \]
Finally, we find
\[ \frac{\left( \delta N^{(1)} \right)^2}{r_\sigma^2} = \frac{2}{3f_\sigma \epsilon_w r_\sigma} \left[ \frac{9f_\sigma \epsilon_w^{(0)} \left( \epsilon_w^{(0)} \right)^2}{2nR} \right] \]
\[ + \frac{9}{2} f_\sigma (1 - r_\sigma)^2 \left( \epsilon_w^{(0)} \right)^2 \left[ -2l + nR + \frac{2}{n} \left[ \frac{n(n-1)}{2} + \frac{n gg''}{g''} \right] \right] \]
\[ + \frac{9}{2} f_\sigma (1 - r_\sigma)^2 \left( \epsilon_w^{(0)} \right)^2 + 8r_\sigma^2 (1 - f_\sigma). \tag{B25} \]
Therefore, the non-linear parameter \( f_{NL} \) is
\[ f_{NL} = \frac{5}{6} \left[ \frac{3\epsilon_w}{nRR_\sigma} \left[ -2l_\sigma + nR + (n-1) + \frac{gg''}{g''} \right] \right] \]
\[ + 3\epsilon_w \left[ \frac{1}{r_\sigma} - 2 + r_\sigma \right] + 4(1 - r_\sigma) \]
\[ = \frac{5}{6} \left[ \frac{3\epsilon_w}{nRR_\sigma} \left[ -2l_\sigma + (n-1) + \frac{gg''}{g''} \right] \right] \]
\[ + 3\epsilon_w \left[ -2 + r_\sigma \right] + 4(1 - r_\sigma). \tag{B26} \]
The standard curvaton scenario corresponds to \( \epsilon_w = 1 \) \((l = 0, n = 2\) (quadratic potential) and \( R = 1 \) \((V_0 = 0\)), which give the conventional curvaton result \[25\].

**Appendix C: How to compare Eq. (33) and Eq. (51) with conventional curvaton Eq. (4)**

In this section we are going to compare our result of the generalized curvaton calculation with the standard curvaton result Eq. (4), to show clearly how one can obtain the conventional curvaton from the generalized formulation.

Our first calculation gives Eq. (33): however if one needs to compare the result with Eq. (4), one will soon find that the equation contains many new parameters that are making the equation rather confusing, and what is worse, one is required to remove the contribution coming from the expansion with respect to the parameter \( \epsilon_w \), since in the original curvaton scenario \( \epsilon_w \) is a constant that does not contribute to the expansion. To remove
the expansion of $\epsilon_w$, it would be useful to use the more
generalized formula \((51)\), in which $l = 0$ corresponds to
a “constant $\epsilon_w$.” Then the other parameters are; $n = 2$
gives the quadratic potential for the curvaton, $R = 1$
responds to $V_0 = 0$, and finally $\epsilon_w = 1$ means $\rho_w \propto a^{-3}$.
Substituting those quantities into Eq.\((51)\), one will find Eq.\((52)\).

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