HOLOMORPHIC CURVES WHOSE DOMAINS ARE RIEMANN SURFACES

XIANJING DONG

Abstract. We establish a defect relation of holomorphic curves from a general open Riemann surface into a normal complex projective variety, with Zariski-dense image intersecting effective Cartier divisors.

1. Introduction

Value distribution of holomorphic curves has grown into a very rich branch in Nevanlinna theory \cite{14, 15} since H. Cartan \cite{5} established his Second Main Theorem of holomorphic curve from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ intersecting hyperplanes in general position. Many well-known results were obtained, referred to Ahlfors \cite{1}, Nochka \cite{11, 12}, Noguchi-Winkelmann \cite{13, 14}, Ru \cite{15, 16, 17, 18, 19}, Shabat \cite{20}, Tiba \cite{21} and Yamanoi \cite{22}, etc. In the paper, we would further develop the well-known Ru’s result of holomorphic curves by generalizing the source space $\mathbb{C}$ to a general open Riemann surface through Brownian motion initiated by Carne \cite{6} and developed by Atsuji \cite{2, 3}.

Let $S$ be an open Riemann surface. By uniformization theorem, one could equip $S$ with a complete Hermitian metric $ds^2 = 2g dz d\bar{z}$ such that the Gauss curvature $K_S \leq 0$ associated to $g$, here $K_S$ is defined by

$$K_S = -\frac{1}{4} \Delta_S \log g = -\frac{1}{g} \frac{\partial^2 \log g}{\partial z \partial \bar{z}}.$$ 

Obviously, $(S, g)$ is a complete Kähler manifold with associated Kähler form $\alpha = g \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$. Set

$$\kappa(t) = \min \left\{ K_S(x) : x \in \overline{D(t)} \right\}$$

which is a non-positive and decreasing continuous function defined on $[0, \infty)$.

Fix $o \in S$ as a reference point. Denoted by $D(r)$ the geodesic disc centered at $o$ with radius $r$, and by $\partial D(r)$ the boundary of $D(r)$. By Sard’s theorem,
\( \partial D(r) \) is a submanifold of \( S \) for almost all \( r > 0 \). Also, we denote by \( g_r(o,x) \) the Green function of \( \Delta_S/2 \) with Dirichlet boundary condition and a pole at \( o \), and by \( d\pi^r_o(x) \) the harmonic measure on \( \partial D(r) \) with respect to \( o \).

Let 
\[ f : S \to X \]
be a holomorphic curve, where \( X \) is a complex projective variety. Let us first introduce Nevanlinna’s functions on Riemann surfaces which are extensions of the classical ones on \( \mathbb{C} \). Let \( L \to X \) be an ample holomorphic line bundle equipped with Hermitian metric \( h \). We define the characteristic function of \( f \) with respect to \( L \) by
\[
T_{f,L}(r) = \pi \int_{D(r)} g_r(o,x) f^* c_1(L, h) = -\frac{1}{4} \int_{D(r)} g_r(o,x) \Delta_S \log h \circ f(x) dV(x),
\]
where \( dV(x) \) is the Riemannian volume measure of \( S \). It can be easily known that \( T_{f,L}(r) \) is independent of the choices of metrics on \( L \), up to a bounded term. Since a holomorphic line bundle on \( X \) can be written as the difference of two ample holomorphic line bundles, the definition of \( T_{f,L}(r) \) can extend to an arbitrary holomorphic line bundle. For a convenience, we use \( T_{f,D}(r) \) to replace \( T_{f,L}(r) \) for an effective Cartier divisor \( D \) on \( X \). Given an ample effective Cartier divisor \( D \) on \( X \), the Weil function of \( D \) is well defined by
\[
\lambda_D(x) = -\log \| s_D(x) \|
\]
up to a bounded term, and here \( s_D \) is the canonical section associated to \( D \). Note also that an effective Cartier divisor can be written as the difference of two ample effective Cartier divisors, and so the definition of Weil functions can extend to an arbitrary effective Cartier divisor. We define the proximity function of \( f \) with respect to \( D \) by
\[
m_f(r,D) = \int_{\partial D(x)} \lambda_D \circ f(x) d\pi^r_o(x).
\]
Now write \( s_D = \tilde{s}_D e \) locally, where \( e \) is a local holomorphic frame of \( (L_D, h) \). The counting function of \( f \) with respect to \( D \) is defined by
\[
N_f(r,D) = \pi \sum_{x \in f^* D \cap D(r)} g_r(o,x)
= \pi \int_{D(r)} g_r(o,x) dd^c \left[ \log |\tilde{s}_D \circ f(x)|^2 \right]
= \frac{1}{4} \int_{D(r)} g_r(o,x) \Delta_S \log |\tilde{s}_D \circ f(x)|^2 dV(x)
\]
in the sense of currents.
Remark. When \( S = \mathbb{C} \), the Green function is \((\log \frac{r}{|z|})/\pi\) and the harmonic measure is \(d\theta/2\pi\). By integration by part, we observe that it agrees with the classical ones.

We introduce the concept of Nevanlinna constant proposed by Ru.

**Definition 1.1** ([17] [18]). Let \( L \) be a holomorphic line bundle over \( X \), and \( D \) be an effective Cartier divisor on \( X \). If \( X \) is normal, then we define

\[
\text{Nev}(L, D) = \inf_{k, V, \mu} \frac{\dim_{\mathbb{C}} V}{\mu},
\]

where “inf” is taken over all triples \((k, V, \mu)\) such that \( V \subseteq H^0(X, kL) \) is a linear subspace with \( \dim_{\mathbb{C}} V > 1 \), and \( \mu > 0 \) is a number with the property: for each \( x \in \text{Supp}D \), there exists a basis \( B_x \) of \( V \) such that

\[
\sum_{s \in B_x} \text{ord}_E(s) \geq \mu \text{ord}_E(kD)
\]

for all irreducible components \( E \) of \( D \) passing through \( x \). If there exists no such triples \((k, V, \mu)\), one defines \( \text{Nev}(L, D) = \infty \). If \( X \) is not normal, then \( \text{Nev}(L, D) \) is defined by pulling back to the normalization of \( X \).

The main purpose of this paper is to explore the value distribution theory of holomorphic curves into complex projective varieties by extending source space \( \mathbb{C} \) to a general open Riemann surface. We prove the following theorem

**Theorem 1.2.** Let \( L \) be a holomorphic line bundle over a normal complex projective variety \( X \) with \( \dim_{\mathbb{C}} H^0(X, kL) \geq 1 \) for some \( k > 0 \). Let \( D \) be an effective Cartier divisor on \( X \). Let \( f : S \to X \) be a holomorphic curve with Zariski-dense image. Then

\[
m_f(r, D) \leq \text{Nev}(L, D)T_{f, L}(r) + o(T_{f, L}(r)) + O\left(-\kappa(r)r^2 + \log + \log r\right),
\]

where \( \kappa \) is defined by (1), and “\( \| \)" means that the inequality holds for \( r > 1 \) outside a set of finite Lebesgue measure.

The term \( \kappa(r)r^2 \) in the above theorem appears from the bending of metric of \( S \). In particular, when \( S = \mathbb{C} \), it deduces \( \kappa(r) \equiv 0 \) and \( T_{f, L}(r) \geq O(\log r) \) for a holomorphic curve \( f \) with Zariski-dense image in \( X \). As a consequence, we recover a result of Ru:

**Corollary 1.3** ([17]). The same conditions are assumed as in Theorem 1.2. Let \( f : \mathbb{C} \to X \) be a holomorphic curve with Zariski-dense image. Then

\[
m_f(r, D) \leq \text{Nev}(L, D)T_{f, L}(r) + o(T_{f, L}(r))\|
\]

Theorem 1.2 implies a defect relation.
Corollary 1.4. The same conditions are assumed as in Theorem 1.2. Let $f : S \to X$ be a holomorphic curve with Zariski-dense image satisfying
\[ \liminf_{r \to \infty} \frac{\kappa(r)r^2}{T_{f,L}(r)} = 0. \]
Then
\[ \delta_f(D) \leq \text{Nev}(L,D). \]

2. First Main Theorem

2.1. Stochastic formulas.

We would use the stochastic method to study value distribution theory for Riemann surfaces. To start with, we introduce Brownian motion and Dynkin formula \cite{8,10}. It is known that the Dynkin formula plays a similar role as Green-Jensen formula \cite{15}. Indeed, the co-area formula is also introduced.

Let $(M,g)$ be a Riemannian manifold with Laplace-Beltrami operator $\Delta_M$ associated to $g$. For $x \in M$, we denote by $B_x(r)$ the geodesic ball centered at $x$ with radius $r$, and denote by $S_x(r)$ the geodesic sphere centered at $x$ with radius $r$. By Sard’s theorem, $S_x(r)$ is a submanifold of $M$ for almost every $r > 0$. A Brownian motion $X_t$ in $M$ is a heat diffusion process generated by $\frac{1}{2}\Delta_M$ with transition density function $p(t,x,y)$ which is the minimal positive fundamental solution of the heat equation
\[ \frac{\partial}{\partial t} u(t,x) - \frac{1}{2} \Delta_M u(t,x) = 0. \]
We denote by $\mathbb{P}_x$ the law of $X_t$ started at $x \in M$ and by $\mathbb{E}_x$ the corresponding expectation with respect to $\mathbb{P}_x$.

Let $D$ be a bounded domain with smooth boundary $\partial D$ in $M$. Fix $x \in D$, we use $d\pi^D_x$ to denote the harmonic measure on $\partial D$ with respect to $x$. This measure is a probability measure. Set
\[ \tau_D := \inf \{ t > 0 : X_t \notin D \} \]
which is a stopping time. Denoted by $g_D(x,y)$ the Green function of $\Delta_M/2$ for $D$ with a pole at $x$ and Dirichlet boundary condition, namely
\[ -\frac{1}{2} \Delta_M g_D(x,y) = \delta_x(y), \quad y \in D; \quad g_D(x,y) = 0, \quad y \in \partial D, \]
where $\delta_x$ is the Dirac function. For $\phi \in \mathcal{C}_b(D)$ (space of bounded continuous functions on $D$), the co-area formula \cite{4} asserts that
\[ \mathbb{E}_x \left[ \int_0^{\tau_D} \phi(X_t) dt \right] = \int_D g_D(x,y)\phi(y) dV(y). \]
From Proposition 2.8 in [4], we also have the relation of harmonic measures and hitting times that
\[ (2) \quad \mathbb{E}_x [\psi(X_{\tau_D})] = \int_{\partial D} \psi(y) d\pi^D_x (y) \]
for any \( \psi \in \mathcal{C}(\mathbb{D}) \).

Let \( u \in \mathcal{C}^2(M) \) (space of bounded \( \mathcal{C}^2 \)-class functions on \( M \)), we have the famous \( \text{Itô formula} \) (see [2, 8, 9, 10])
\[ u(X_t) - u(x) = B \left( \int_0^t \| \nabla_M u \|^2 (X_s) ds \right) + \frac{1}{2} \int_0^t \Delta_M u(X_s) dt, \quad \mathbb{P}_x - \text{a.s.} \]
where \( B_t \) is the standard Brownian motion in \( \mathbb{R} \) and \( \nabla_M \) is gradient operator on \( M \). Take expectation of both sides of the above formula, it follows \( \text{Dynkin formula} \) (see [2, 10])
\[ \mathbb{E}_x [u(X_T)] - u(x) = \frac{1}{2} \mathbb{E}_x \left[ \int_0^T \Delta_M u(X_t) dt \right] \]
for a stopping time \( T \) such that each term makes sense.

**Remark.** Thanks to expectation “\( \mathbb{E}_x \)”, the Dynkin formula, co-area formula and (2) still work when \( u, \phi \) or \( \psi \) has a pluripolar set of singularities.

### 2.2. First Main Theorem.

Let \( S \) be a complete open Riemann surface with Kähler form \( \alpha \) associated to Hermitian metric \( g \). Fix \( o \in S \), we let \( X_t \) be the Brownian motion with generator \( \Delta_S/2 \) started at \( o \in S \). Moreover, we set a stopping time
\[ \tau_r = \inf \{ t > 0 : X_t \notin D(r) \} \]
Let \( f : S \to X \) be a holomorphic curve into a complex projective variety \( X \). Let \( L \to X \) be an ample holomorphic line bundle equipped with Hermitian metric \( h \). Apply co-area formula, we have
\[ T_{f,L}(r) = -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \log h \circ f(X_t) dt \right]. \]
A relation of harmonic measures and hitting times implies that
\[ m_f(r,D) = \mathbb{E}_o [\lambda_D \circ f(X_{\tau_r})]. \]

We here give the First Main Theorem of a holomorphic curve \( f : S \to M \) such that \( f(o) \notin \text{Supp}D \), where \( D \) is an effective Cartier divisor on \( X \). Apply Dynkin formula to \( \lambda_D \circ f(x) \),
\[ \mathbb{E}_o [\lambda_D \circ f(X_{\tau_r})] - \lambda_D \circ f(o) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \lambda_D \circ f(X_t) dt \right]. \]
The first term on the left hand side of the above equality is equal to $m_f(r, D)$, and the term on the right hand side equals

$$\frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \lambda_D \circ f(X_t) dt \right] = \frac{1}{2} \int_{D(r)} g_r(o, x) \Delta_S \log \frac{1}{\|s_D \circ f(x)\|} dV(x)$$

due to co-area formula. Since $\|s_D\|^2 = h|\tilde{s}_D|^2$, where $h$ is a Hermitian metric on $L_D$, we get

$$\frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \lambda_D \circ f(X_t) dt \right] = -\frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log h \circ f(x) dV(x)$$

$$-\frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log |\tilde{s}_D \circ f(x)|^2 dV(x)$$

$$= T_{f,D}(r) - N_f(r, D).$$

Therefore, we obtain

$$F. M. T. \quad T_{f,D}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

**Remark.** $N_f(r, D)$ is of a probabilistic expression

$$N_f(r, D) = \lim_{\lambda \to \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq t \leq \tau_r} \log \frac{1}{\|s_D \circ f(X_t)\|} > \lambda \right).$$

### 3. Logarithmic Derivative Lemma

Let $(S, g)$ be a simply-connected and complete open Riemann surface with Gauss curvature $K_S \leq 0$ associated to $g$. By uniformization theorem, there exists a nowhere-vanishing holomorphic vector field $X$ on $S$.

#### 3.1. Calculus Lemma.

Let $\kappa$ be defined by (11). As is noted before, $\kappa$ is a non-positive, decreasing continuous function on $[0, \infty)$. Associate the ordinary differential equation

$$G''(t) + \kappa(t) G(t) = 0; \quad G(0) = 0, \quad G'(0) = 1.$$

We compare (3) with $y''(t) + \kappa(0) y(t) = 0$ under the same initial conditions, $G$ can be easily estimated as

$$G(t) = t \quad \text{for} \quad \kappa \equiv 0; \quad G(t) \geq t \quad \text{for} \quad \kappa \neq 0.$$

This implies that

$$G(r) \geq r \quad \text{for} \quad r \geq 0; \quad \int_1^r \frac{dt}{G(t)} \leq \log r \quad \text{for} \quad r \geq 1.$$

On the other hand, we rewrite (3) as the form

$$\log' G(t) \cdot \log' G'(t) = -\kappa(t).$$
Since $G(t) \geq t$ is increasing, then the decrease and non-positivity of $\kappa$ imply that for each fixed $t$, $G$ must satisfy one of the following two inequalities

$$\log' G(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t > 0; \quad \log' G(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t \geq 0.$$

By virtue of $G(t) \to 0$ as $t \to 0$, by integration, $G$ is bounded from above by

(5) $$G(r) \leq r \exp \left(r \sqrt{-\kappa(r)}\right) \quad \text{for } r \geq 0.$$

The main result of this subsection is the following

**Theorem 3.1 (Calculus Lemma).** Let $k \geq 0$ be a locally integrable function on $S$ such that it is locally bounded at $o \in S$. Then for any $\delta > 0$, there exists a constant $C > 0$ independent of $k, \delta$, and a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure such that

$$E_o[k(X_t)] \leq \frac{F(\tilde{k}, \kappa, \delta)e^{\sqrt{-\kappa(r)}} \log r}{2\pi C} \int_0^{r} k(X_t) dt$$

holds for $r > 1$ outside $E_\delta$, where $\kappa$ is defined by (1) and $F$ is defined by

$$F(\tilde{k}, \kappa, \delta) = \left\{ \log^+ \tilde{k}(r) \cdot \log^+ \left(r e^{\sqrt{-\kappa(r)} \tilde{k}(r)} \left\{ \log^+ \tilde{k}(r) \right\}^{1+\delta} \right) \right\}^{1+\delta}$$

with

$$\tilde{k}(r) = \frac{\log r}{C} \int_0^r k(X_t) dt.$$

Moreover, we have the estimate

$$\log F(\tilde{k}, \kappa, \delta) \leq O \left( \log^+ \log E_o \left[ \int_0^r k(X_t) dt \right] + \log^+ r \sqrt{-\kappa(r)} + \log^+ \log r \right).$$

To prove theorem 3.1 we need some lemmas.

**Lemma 3.2 ([3]).** Let $\eta > 0$ be a constant. Then there is a constant $C > 0$ such that for $r > \eta$ and $x \in B_o(r) \setminus B_o(\eta)$

$$g_r(o, x) \int_\eta^r \frac{dt}{G(t)} \geq C \int_{r(x)}^r \frac{dt}{G(t)}$$

holds, where $G$ be defined by (3).

**Lemma 3.3 ([15]).** Let $T$ be a strictly positive nondecreasing function of $C^1$-class on $(0, \infty)$. Let $\gamma > 0$ be a number such that $T(\gamma) \geq e$, and $\phi$ be a strictly positive nondecreasing function such that

$$c_\phi = \int_\gamma^\infty \frac{1}{t \phi(t)} dt < \infty.$$

Then, the inequality $T'(r) \leq T(r) \phi(T(r))$ holds for all $r \geq \gamma$ outside a set of Lebesgue measure not exceeding $c_\phi$. In particular, if taking $\phi(t) = \log^{1+\delta} t$ for a number $\delta > 0$, then we have $T'(r) \leq T(r) \log^{1+\delta} T(r)$ holds for all $r > 0$ outside a set $E_\delta \subset (0, \infty)$ of finite Lebesgue measure.
Proof of Theorem 3.1

Proof. The argument refers to Atsuji [3]. The simple-connectedness and the non-positivity of Gauss curvature of $S$ imply the following relation (see [7])

$$d\pi_0^r(x) \leq \frac{1}{2\pi r} d\sigma_r(x),$$

where $d\sigma_r(x)$ is the induced volume measure on $\partial D(r)$. By Lemma 3.2 and (4), we have

$$E_o \left[ \int_0^r k(X_t) dt \right] = \int_{D(r)} g_r(o, x) k(x) dV(x)$$

$$= \int_0^r dt \int_{\partial D(t)} g_r(o, x) k(x) d\sigma_t(x)$$

$$\geq C \int_0^r \frac{\int_r^s G^{-1}(s) ds}{\int_1^s G^{-1}(s) ds} dt \int_{\partial D(t)} k(x) d\sigma_t(x)$$

$$\geq \frac{C}{\log r} \int_0^r dt \int_t^r \frac{ds}{G(s)} \int_{\partial D(t)} k(x) d\sigma_t(x),$$

$$E_o [k(X_{r \tau})] = \int_{\partial D(r)} k(x) d\pi_0^r(x) \leq \frac{1}{2\pi r} \int_{\partial D(r)} k(x) d\sigma_r(x).$$

Hence,

$$E_o \left[ \int_0^r k(X_t) dt \right] \geq \frac{C}{\log r} \int_0^r dt \int_t^r \frac{ds}{G(s)} \int_{\partial D(o, t)} k(x) d\sigma_t(x),$$

$$E_o [k(X_{r \tau})] \leq \frac{1}{2\pi r} \int_{\partial D(r)} k(x) d\sigma_r(x).$$

Set

$$\Lambda(r) = \int_0^r dt \int_t^r \frac{ds}{G(s)} \int_{\partial D(t)} k(x) d\sigma_t(x).$$

We conclude that

$$\Lambda(r) \leq \frac{\log r}{C} E_o \left[ \int_0^r k(X_t) dt \right] = \tilde{k}(r).$$

Since

$$\Lambda'(r) = \frac{1}{G(r)} \int_0^r dt \int_{\partial D(t)} k(x) d\sigma_t(x),$$

then it yields from (6) that

$$E_o [k(X_{r \tau})] \leq \frac{1}{2\pi r} \frac{d}{dr} \left( \Lambda'(r) G(r) \right).$$
Using Lemma 3.3 twice with (5), then for any $\delta > 0$
\[
\frac{d}{dr}(\Lambda'(r)G(r))
\leq G(r)\left\{ \log^+ \Lambda(r) \cdot \log^+ \left( G(r) \Lambda(\{ \log^+ \Lambda(r) \}^{1+\delta}) \right) \right\}^{1+\delta} \Lambda(r)
\leq re^{r\sqrt{-\kappa(r)}}\left\{ \log^+ \tilde{k}(r) \cdot \log^+ \left( re^{r\sqrt{-\kappa(r)}}\tilde{k}(r) \left\{ \log^+ \tilde{k}(r) \right\}^{1+\delta} \right) \right\}^{1+\delta} \tilde{k}(r)
= \frac{F(\tilde{k},\kappa,\delta)}{C}re^{r\sqrt{-\kappa(r)}}\log r \sum_{0}^{\tau_{r}} k(X_{t})dt
\]
holds outside a set $E_{\delta} \subset (1, \infty)$ of finite Lebesgue measure. Thus,
\[
E_{o}\left[ k(X_{\tau_{r}}) \right] \leq \frac{F(\tilde{k},\kappa,\delta)}{2\pi C}re^{r\sqrt{-\kappa(r)}}\log r \sum_{0}^{\tau_{r}} k(X_{t})dt.
\]
Hence, we get the desired inequality. Indeed, for $r > 1$ we compute that
\[
\log F(\tilde{k},\kappa,\delta) \leq O\left( \log^+ \log^+ \tilde{k}(r) + \log^+ r\sqrt{-\kappa(r)} + \log^+ \log r \right)
\]
and
\[
\log^+ \tilde{k}(r) \leq \log E_{o}\left[ k(X_{\tau_{r}}) \right] + \log^+ \log r + O(1).
\]
We have arrived at the required estimate. \qed

3.2. Logarithmic Derivative Lemma.

Let $\psi$ be a meromorphic function on $(S, g)$. The norm of the gradient of $\psi$ is defined by
\[
\|\nabla_{S}\psi\|^{2} = \frac{1}{g} \left| \frac{\partial \psi}{\partial z} \right|^{2}
\]
in a local coordinate $z$. Locally, we write $\psi = \psi_{1}/\psi_{0}$, where $\psi_{0}, \psi_{1}$ are local holomorphic functions without common zeros. Regard $\psi$ as a holomorphic mapping into $\mathbb{P}^{1}(\mathbb{C})$ by $x \mapsto [\psi_{0}(x) : \psi_{1}(x)]$. We define
\[
T_{\psi}(r) = \frac{1}{4} \int_{D(r)} g_{r}(o,x)\Delta_{S} \log \left( |\psi_{0}(x)|^{2} + |\psi_{1}(x)|^{2} \right)dV(x)
\]
and $T(r, \psi) := m(r, \psi) + N(r, \psi)$ with
\[
m(r, \psi) = \int_{\partial D(r)} \log^+ |\psi(x)|d\pi_{o}^{r}(x),
\]
\[
N(r, \psi) = \pi \sum_{x \in \psi^{-1}(\infty) \cap D(r)} g_{r}(o, x).
\]
Let $i : \mathbb{C} \hookrightarrow \mathbb{P}^{1}(\mathbb{C})$ be an inclusion defined by $z \mapsto [1 : z]$. Via the pull-back by $i$, we have a $(1,1)$-form $i^{*}\omega_{FS} = dd^{c} \log(1 + |\zeta|^{2})$ on $\mathbb{C}$, where $\zeta := w_{1}/w_{0}$ and
$[w_0 : w_1]$ is the homogeneous coordinate system of $\mathbb{P}^1(\mathbb{C})$. The characteristic function of $\psi$ with respect to $i^*\omega_{FS}$ is defined by

$$\hat{T}_\psi(r) = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log(1 + |\psi(x)|^2) dV(x).$$

Clearly, $\hat{T}_\psi(r) \leq T_\psi(r)$. We adopt the spherical distance $\| \cdot, \cdot \|$ on $\mathbb{P}^1(\mathbb{C})$, the proximity function of $\psi$ with respect to $a \in \mathbb{P}^1(\mathbb{C})$ is defined by

$$\hat{m}_\psi(r, a) = \int_{\partial D(r)} \log \frac{1}{\|\psi(x), a\|} d\pi_o(x).$$

Then $\hat{T}_\psi(r) = \hat{m}_\psi(r, a) + \hat{N}_\psi(r, a) + O(1)$. Note that $\hat{m}(r, \psi) = \hat{m}_\psi(r, \infty) + O(1)$, which implies that

$$T(r, \psi) = \hat{T}_\psi(r) + O(1), \quad T\left(r, \frac{1}{\psi - a}\right) = T(r, \psi) + O(1).$$

Hence, we arrive at

$$(7) \quad T(r, \psi) + O(1) = \hat{T}_\psi(r) \leq T_\psi(r) + O(1).$$

We establish the following Logarithmic Derivative Lemma (LDL):

**Theorem 3.4 (LDL).** Let $\psi$ be a nonconstant meromorphic function on $S$. Let $\mathcal{X}$ be a nowhere-vanishing holomorphic vector field on $S$. Then

$$m\left(r, \frac{\mathcal{X}^k(\psi)}{\psi}\right) \leq \frac{3k}{2} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) - \kappa(r) r^2 + \log^+ \log r\right),$$

where $\kappa$ is defined by (1).

On $\mathbb{P}^1(\mathbb{C})$, we take a singular metric

$$\Phi = \frac{1}{|\zeta|^2(1 + \log^2 |\zeta|)} \frac{\sqrt{-1}}{4\pi^2} d\zeta \wedge d\overline{\zeta}.$$

A direct computation gives that

$$\int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 2\pi \psi^* \Phi = \frac{\|\nabla_S \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)} \alpha.$$

Set

$$T_\psi(r, \Phi) = \frac{1}{2\pi} \int_{D(r)} g_r(o, x) \frac{\|\nabla_S \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(x) dV(x).$$
By Fubini’s theorem
\[
T_\psi(r, \Phi) = \int_{D(r)} g_r(o, x) \frac{\psi^* \Phi}{\alpha} dV(x)
= \pi \int_{C^1(\mathbb{C})} \Phi \sum_{x \in \psi^{-1}(\zeta) \cap D(r)} g_r(o, x)
= \int_{C^1(\mathbb{C})} N_\psi(r, \zeta) \Phi \leq T(r, \psi) + O(1).
\]

We get
(8) \quad T_\psi(r, \Phi) \leq T(r, \psi) + O(1).

**Lemma 3.5.** Assume that \( \psi(x) \neq 0 \). Then for any \( \delta > 0 \), there exists a constant \( C > 0 \) independent of \( \psi \), and a set \( E_\delta \subset (1, \infty) \) of finite Lebesgue measure such that
\[
\frac{1}{2} \text{E}_o \left[ \log^{+} \frac{\| \nabla S \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_{r_0}) \right] \leq \frac{1}{2} \log T(r, \psi) + O \left( \log^+ T(r, \psi) + r\sqrt{-\kappa(r)} + \log^+ \log r \right),
\]
where \( \kappa \) is defined by (1).

**Proof.** By Jensen’s inequality, it is clear that
\[
\text{E}_o \left[ \log^{+} \frac{\| \nabla S \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_{r_0}) \right] \leq \text{E}_o \left[ \log \left( 1 + \frac{\| \nabla S \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_{r_0}) \right) \right] \leq \log^+ \text{E}_o \left[ \frac{\| \nabla S \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_{r_0}) \right] + O(1).
\]

By Lemma 3.1 and (8)
\[
\log^+ \text{E}_o \left[ \frac{\| \nabla S \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_{r_0}) \right] \leq \log^+ \text{E}_o \left[ \int_0^{r_f} \frac{\| \nabla M \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_t) dt \right] + \log \frac{F(\tilde{k}, \kappa, \delta) e^{r \sqrt{-\kappa(r)}} \log r}{2\pi C} \leq \log T(r, \Phi) + \log F(\tilde{k}, \kappa, \delta) + r\sqrt{-\kappa(r)} + \log^+ \log r + O(1) \leq \log T(r, \psi) + O \left( \log^+ \log^+ \tilde{k}(r) + r\sqrt{-\kappa(r)} + \log^+ \log r \right),
\]
where
\[
\tilde{k}(r) = \frac{\log r}{C} \text{E}_o \left[ \int_0^{r_f} \frac{\| \nabla S \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)} (X_t) dt \right].
\]
Indeed, we note that
\[
\tilde{k}(r) = \frac{2\pi \log r}{C} T_\psi(r, \Phi) \leq \frac{2\pi \log r}{C} T(r, \psi).
\]
Then we have the desired inequality. □

We first prove LDL for the first-order derivative:

**Theorem 3.6 (LDL).** Let \( \psi \) be a nonconstant meromorphic function on \( S \). Let \( \mathcal{X} \) be a nowhere-vanishing holomorphic vector field on \( S \). Then

\[
m\left( r, \frac{\mathcal{X}(\psi)}{\psi} \right) \leq \frac{3}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right),
\]

where \( \kappa \) is defined by (1).

**Proof.** Write \( \mathcal{X} = a \frac{\partial}{\partial z} \), then \( \|\mathcal{X}\|^2 = g|a|^2 \). We have

\[
m\left( r, \frac{\mathcal{X}(\psi)}{\psi} \right) = \int_{\partial D(r)} \log \left| \frac{\mathcal{X}(\psi)}{\psi} \right|^2 (x)d\pi^r_\psi(x)
\] 
\[
\leq \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{|\mathcal{X}(\psi)|^2}{\|\mathcal{X}\|^2|\psi|^2(1 + \log^2 |\psi|)}(x)d\pi^r_\psi(x)
\] 
\[
+ \frac{1}{2} \int_{\partial D(r)} \log(1 + \log^2 |\psi(x)|)|d\pi^r_\psi(x) + \frac{1}{2} \int_{\partial D(r)} \log^+ \|\mathcal{X}_x\|^2 d\pi^r_\psi(x)
\] 
\[
:= A + B + C.
\]

We next handle \( A, B, C \) respectively. For \( A \), it yields from Lemma 3.5 that

\[
A = \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{|a|^2 |\frac{\partial \psi}{\partial z}|^2}{g|a|^2|\psi|^2(1 + \log^2 |\psi|)}(x)d\pi^r_\psi(x)
\] 
\[
= \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}(x)d\pi^r_\psi(x)
\] 
\[
\leq \frac{1}{2} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) + r\sqrt{-\kappa(r)} + \log^+ \log r \right).
\]

For \( B \), the Jensen’s inequality implies that

\[
B \leq \int_{\partial D(r)} \log \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right)d\pi^r_\psi(x)
\] 
\[
\leq \log \int_{\partial D(r)} \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right)d\pi^r_\psi(x)
\] 
\[
\leq \log T(r, \psi) + O(1).
\]

Finally, we estimate \( C \). By the condition, \( \|\mathcal{X}\| > 0 \). Since \( S \) is non-positively curved and \( a \) is holomorphic, then \( \log \|\mathcal{X}\| \) is subharmonic, i.e., \( \Delta_S \log \|\mathcal{X}\| \geq 0 \). Clearly, we have

\[
\Delta_S \log^+ \|\mathcal{X}\| \leq \Delta_S \log \|\mathcal{X}\|
\]
for $x \in S$ satisfying $\|X_x\| \neq 1$. Notice that $\log^+ \|X_x\| = 0$ for $x \in S$ satisfying $\|X_x\| \leq 1$. Thus, by Dynkin formula we have

$$C = \frac{1}{2} \mathbb{E}_o \left[ \log^+ \|X(X_r)\|^2 \right]$$

$$\leq \frac{1}{4} \mathbb{E}_o \left[ \int_0^r \Delta S \log \|X(X_t)\|^2 dt \right] + O(1)$$

$$= \frac{1}{4} \mathbb{E}_o \left[ \int_0^r \Delta S \log g(X_t) dt \right] + \frac{1}{4} \mathbb{E}_o \left[ \int_0^r \Delta S \log |a(X_t)|^2 dt \right] + O(1)$$

$$= -\mathbb{E}_o \left[ \int_0^r K_S(X_t) dt \right] + O(1)$$

$$\leq -\kappa(r) \mathbb{E}_o [\tau_r] + O(1),$$

where we use the fact $K_S = -\left(\frac{\Delta S \log g}{4}\right)$. Thus, we prove the theorem by using $\mathbb{E}_o [\tau_r] \leq r^2 / 2$ which is due to Lemma 3.7 below.

**Lemma 3.7.** Let $X_t$ be a Brownian motion in a simply-connected complete Riemann surface $S$ of non-positive Gauss curvature. Then

$$\mathbb{E}_o [\tau_r] \leq \frac{r^2}{2}.$$

**Proof.** We refer to arguments of Atsuji [3]. Apply Itô formula to $r(x)$

$$r(X_t) = B_t - L_t + \frac{1}{2} \int_0^t \Delta S r(X_s) ds,$$

where $B_t$ is the standard Brownian motion in $\mathbb{R}$, $L_t$ is the local time on cut locus of $o$, an increasing process which increases only at cut loci of $o$. Since $S$ is simply connected and non-positively curved, then

$$\Delta S r(x) \geq \frac{1}{r(x)}, \quad L_t \equiv 0.$$

By (10), we arrive at

$$r(X_t) \geq B_t + \frac{1}{2} \int_0^t \frac{ds}{r(X_s)}.$$

Associate the stochastic differential equation

$$dW_t = dB_t + \frac{1}{2} \frac{dt}{W_t}, \quad W_0 = 0,$$

where $B_t$ is the standard Brownian motion in $\mathbb{R}$, and $W_t$ is the 2-dimensional Bessel process defined as the Euclidean norm of Brownian motion in $\mathbb{R}^2$. By the standard comparison arguments of stochastic differential equations, one gets that

$$W_t \leq r(X_t)$$
almost surely. Set
\[ \tau_r = \inf \{ t > 0 : W_t \geq r \}, \]
which is a stopping time. From (11), we can verify that \( \tau_r \geq \tau_r \).
This implies
\[ E_o[\tau_r] \geq E_o[\tau_r]. \]
Since \( W_t \) is the Euclidean norm of the Brownian motion in \( \mathbb{R}^2 \) starting from
the origin, then applying Dynkin formula to \( W_t^2 \) we have
\[ E_o[W_t^2] = \frac{1}{2} E_o \left[ \int_0^{\tau_r} \Delta_{\mathbb{R}^2} W_t^2 dt \right] = 2E_o[\tau_r], \]
where \( \Delta_{\mathbb{R}^2} \) is the Laplace operator on \( \mathbb{R}^2 \). Using (11) and (12), we conclude
that \( r^2 = E_o[r^2] = 2E_o[\tau_r] \geq 2E_o[\tau_r] \).
This certifies the assertion. \( \Box \)

Proof of Theorem 3.4

Proof. Note that
\[ m(r, \mathcal{X}^k(\psi)) \leq \sum_{j=1}^{k} m(r, \mathcal{X}^j(\psi)), \]
Therefore, we finish the proof by using Lemma 3.8 below. \( \Box \)

Lemma 3.8. We have
\[ m(r, \mathcal{X}^{k+1}(\psi)) \leq \frac{3}{2} \log T(r, \psi) + O \left( \log^+ \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right), \]
where \( \kappa \) is defined by (11).

Proof. We first claim that
\[ T(r, \mathcal{X}^k(\psi)) \leq 2^k T(r, \psi) + O \left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right). \]
By virtue of Theorem 3.6 when \( k = 1 \)
\[ T(r, \mathcal{X}(\psi)) = m(r, \mathcal{X}(\psi)) + N(r, \mathcal{X}(\psi)) \]
\[ \leq m(r, \psi) + 2N(r, \psi) + m(r, \mathcal{X}(\psi)) \]
\[ \leq 2T(r, \psi) + m(r, \mathcal{X}(\psi)) \]
\[ \leq 2T(r, \psi) + O \left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right) \]
holds for \( r > 1 \) outside a set of finite Lebesgue measure. Assuming now that
the claim holds for \( k \leq n - 1 \). By induction, we only need to prove the claim.
in the case when \( k = n \). By the claim for \( k = 1 \) proved above and Theorem \ref{3.6} repeatedly, we have
\[
T(r, \mathcal{X}^n(\psi)) \leq 2T(r, \mathcal{X}^{n-1}(\psi)) + O\left(\log T(r, \mathcal{X}^{n-1}(\psi)) - \kappa(r)r^2 + \log^+ \log r\right)
\]
\[
\leq 2^n T(r, \psi) + O\left(\log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r\right)
\]
\[
+ O\left(\log T(r, \mathcal{X}^{n-1}(\psi)) - \kappa(r)r^2 + \log^+ \log r\right)
\]
\[
\leq 2^n T(r, \psi) + O\left(\log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r\right)
\]
\[
\cdots \cdots \cdots
\]
\[
\leq 2^n T(r, \psi) + O\left(\log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r\right)\|.
\]

The claim \eqref{13} is proved. Using Theorem \ref{3.6} and \eqref{13} to get
\[
m \left( r, \frac{\mathcal{X}^{k+1}(\psi)}{\mathcal{X}^k(\psi)} \right)
\]
\[
\leq \frac{3}{2} \log T(r, \mathcal{X}^k(\psi)) - \frac{\kappa(r)r^2}{2} + \frac{1}{2} \log \frac{G(r)}{r}
\]
\[
+ O\left(\log^+ \log T(r, \mathcal{X}^k(\psi)) + \log^+ \log G(r)\right)
\]
\[
\leq \frac{3}{2} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r\right)\|.
\]

\[\square\]

4. Second Main Theorem

4.1. Wronskian determinants.

Let \( S \) be an open Riemann surface with a nowhere-vanishing holomorphic vector field \( \mathcal{X} \) (it always exists), which is equipped with a complete Hermitian metric \( h \) such that the Gauss curvature \( K_S \leq 0 \). Let
\[
f : S \to \mathbb{P}^n(\mathbb{C})
\]
be a holomorphic curve into complex projective space with the Fubini-Study form \( \omega_{FS} \). Locally, we may write \( f = [f_0 : \cdots : f_n] \), a reduced representation, i.e., \( f_0 = w_0 \circ f, \cdots \) are local holomorphic functions without common zeros, where \( w = [w_0 : \cdots : w_n] \) denotes homogenous coordinate system of \( \mathbb{P}^n(\mathbb{C}) \). Set \( \|f\|^2 = |f_0|^2 + \cdots + |f_n|^2 \). Notice that \( \Delta_S \log \|f\|^2 \) is independent of the choices of representations of \( f \), so it is well defined on \( S \). The height function of \( f \) is defined by
\[
T_f(r) = \pi \int_{D(r)} g_r(o, x) f^* \omega_{FS} = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log \|f(x)\|^2 dV(x).
\]
Let $H$ be a hyperplane of $\mathbb{P}^n(\mathbb{C})$ with defining function $\hat{H}(w) = a_0w_0 + \cdots + a_nw_n$. Set $\|\hat{H}\|^2 = |a_0|^2 + \cdots + |a_n|^2$. The counting function of $f$ with respect to $H$ is defined by

$$N_f(r, H) = \pi \int_{D(r)} g_r(o, x)d\ell [\log |\hat{H} \circ f|^2]$$

$$= \frac{1}{4} \int_{D(r)} g_r(o, x)\Delta_S \log |\hat{H} \circ f|^2dV(x)$$

in the sense of currents. We define the proximity function of $f$ with respect to $H$ by

$$m_f(r, H) = \int_{\partial D(r)} \log \frac{\|\hat{H} \||f(x)||}{|\hat{H} \circ f(x)|}d\pi_0^\circ(x).$$

**Lemma 4.1.** Assume that $f_k \neq 0$ for some $k$. We have

$$\max_{0 \leq j \leq n} T\left(r, \frac{f_j}{f_k}\right) \leq T_f(r) + O(1).$$

**Proof.** By (7), we arrive at

$$T\left(r, \frac{f_j}{f_k}\right) \leq T_{f_j/f_k}(r) + O(1)$$

$$\leq \frac{1}{4} \int_{D(r)} g_r(o, x)\Delta_S \log \left(\sum_{j=0}^n |f_j(x)|^2\right)dV(x) + O(1)$$

$$= T_f(r) + O(1).$$

$\square$

Let $H_1, \cdots, H_q$ be $q$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with defining functions given by

$$\hat{H}_j(w) = \sum_{k=0}^n a_{jk}w_k, \quad 1 \leq j \leq q.$$ 

One defines Wronskian determinant and logarithmic Wronskian determinant of $f$ with respect to $X$ respectively by

$$W_X(f_0, \cdots, f_n) = \begin{vmatrix} f_0 & \cdots & f_n \\ X(f_0) & \cdots & X(f_n) \\ \vdots & \vdots & \vdots \\ X^n(f_0) & \cdots & X^n(f_n) \end{vmatrix}, \quad \Delta_X(f_0, \cdots, f_n) = \begin{vmatrix} 1 & \cdots & 1 \\ \frac{X(f_0)}{f_0} & \cdots & \frac{X(f_n)}{f_n} \\ \vdots & \vdots & \vdots \\ \frac{X^n(f_0)}{f_0} & \cdots & \frac{X^n(f_n)}{f_n} \end{vmatrix}. $$
For a \((n+1) \times (n+1)\)-matrix \(A\) and a nonzero meromorphic function \(\phi\) on \(S\), we can check the following basic properties:

\[
\begin{align*}
\Delta_X(\phi f_0, \cdots, \phi f_n) &= \Delta_X(f_0, \cdots, f_n), \\
W_X(\phi f_0, \cdots, \phi f_n) &= \phi^{n+1}W_X(f_0, \cdots, f_n), \\
W_X((f_0, \cdots, f_n)A) &= \det(A)W_X(f_0, \cdots, f_n), \\
W_X(f_0, \cdots, f_n) &= \left(\prod_{j=0}^{n} f_j\right)\Delta_X(f_0, \cdots, f_n).
\end{align*}
\]

Obviously, \(\Delta_X(f_0, \cdots, f_n)\) is globally well defined on \(S\).

**Lemma 4.2.** Let \(Q \subseteq \{1, \cdots, q\}\) with \(|Q| = n+1\). If \(S\) is simply connected, then we have

\[
m\left(r, \Delta_X(\hat{H}_k \circ f, k \in Q)\right) \leq O\left(\log T_f(r) - \kappa(r)r^2 + \log^+ \log r\right),
\]

where \(\kappa\) is defined by (1).

**Proof.** We write \(Q = \{j_0, \cdots, j_n\}\) and suppose that \(\hat{H}_{j_0} \circ f \neq 0\) without loss of generality. The property of logarithmic Wronskian determinant implies

\[
\begin{align*}
\Delta_X(\hat{H}_{j_0} \circ f, \cdots, \hat{H}_{j_n} \circ f) &= \Delta_X\left(1, \frac{\hat{H}_{j_1} \circ f}{\hat{H}_{j_0} \circ f}, \cdots, \frac{\hat{H}_{j_n} \circ f}{\hat{H}_{j_0} \circ f}\right).
\end{align*}
\]

Since \(\hat{H}_{j_0} \circ f, \cdots, \hat{H}_{j_n} \circ f\) are linear forms of \(f_0, \cdots, f_n\), by Theorem 3.4 and Lemma 4.1 we have

\[
m\left(r, \Delta_X(\hat{H}_k \circ f, k \in Q)\right) \leq O\left(\log T_f(r) - \kappa(r)r^2 + \log^+ \log r\right).
\]

We have arrived at the desired inequality. \(\square\)

**Lemma 4.3.** Let \(H_1, \cdots, H_q\) be hyperplanes of \(\mathbb{P}^n(\mathbb{C})\). Let \(f : S \to \mathbb{P}^n(\mathbb{C})\) be a linearly non-degenerate holomorphic curve. Then

\[
\begin{align*}
\int_{\partial D(r)} \max_Q \sum_{k \in Q} \log \left|\frac{\hat{H}_k\|f(x)\|}{\hat{H}_k \circ f(x)}\right|d\pi_0^r(x) &
\leq (n+1)T_f(r) + O\left(\log T_f(r) - \kappa(r)r^2 + \log^+ \log r\right),
\end{align*}
\]

where \(Q\) ranges over all subsets of \(\{1, \cdots, q\}\) such that \(\{\hat{H}_k\}_{k \in Q}\) are linearly independent.
Proof. Without loss of generality, we assume that \( q \geq n + 1 \) and \( H_1 \cdots, H_q \) are in general position. Then
\[
\int_{\partial D(r)} \max_{Q \subseteq \mathbb{Q}} \log \frac{\| \hat{H}_k \| \| f(x) \|}{\| \hat{H}_k \circ f(x) \|} \, d\pi_o^r(x)
\]
\[
= \int_{\partial D(r)} \max_{|Q|=n+1} \log \prod_{k \in Q} \frac{\| \hat{H}_k \| \| f(x) \|}{\| \hat{H}_k \circ f(x) \|} \, d\pi_o^r(x)
\]
\[
\leq \int_{\partial D(r)} \max_{|Q|=n+1} \log \frac{\| f(x) \|^{n+1}}{\prod_{k \in Q} \| \hat{H}_k \circ f(x) \|} \, d\pi_o^r(x) + O(1)
\]
\[
= \int_{\partial D(r)} \max_{|Q|=n+1} \log \frac{\Delta_{\chi}(\hat{H}_k \circ f(x), k \in Q)}{W_{\chi}(\hat{H}_k \circ f(x), k \in Q)} \| f(x) \|^{n+1} \, d\pi_o^r(x) + O(1).
\]
By \( W_{\chi}(\hat{H}_k \circ f, k \in Q) = b_Q W_{\chi}(f_0, \ldots, f_n) \) (with \( |Q| = n + 1 \)) for a nonzero constant \( b_Q \) depending on \( Q \), we further conclude that
\[
\int_{\partial D(r)} \max_{Q \subseteq \mathbb{Q}} \log \frac{\| \hat{H}_k \| \| f(x) \|}{\| \hat{H}_k \circ f(x) \|} \, d\pi_o^r(x)
\]
\[
\leq \int_{\partial D(r)} \max_{|Q|=n+1} \log \frac{\Delta_{\chi}(\hat{H}_k \circ f(x), k \in Q)}{W_{\chi}(f_0(x), \cdots, f_n(x))} \| f(x) \|^{n+1} \, d\pi_o^r(x) + O(1)
\]
\[
:= A + B + O(1).
\]
We next handle the terms \( A \) and \( B \). By Lemma 4.2
\[
A \leq \int_{\partial D(r)} \log \sum_{|Q|=n+1} \frac{\Delta_{\chi}(\hat{H}_k \circ f(x), k \in Q)}{d\pi_o^r(x)} \, d\pi_o^r(x)
\]
\[
\leq \sum_{|Q|=n+1} \left( r, \Delta_{\chi}(\hat{H}_k \circ f, k \in Q) \right) + O(1)
\]
\[
\leq O \left( \log T_f(r) - \kappa(r) r^2 + \log^+ \log r \right).
\]
Apply Dynkin formula to \( B \),
\[
B = \frac{1}{2} \int_{D(r)} g_r(o, x) \Delta_s \log \frac{\| f(x) \|^{n+1}}{W_{\chi}(f_0(x), \cdots, f_n(x))} \, dV(x) + O(1)
\]
\[
= (n + 1) T_f(r) - N_{W_{\chi}}(r) + O(1)
\]
\[
\leq (n + 1) T_f(r) + O(1).
\]
Putting together the above, we have the desired inequality. \( \square \)
Combining (14) with \( \lambda T \) (17) \( L \) In further, we have (16) \( \pi \) projection morphisms \( \lambda \) (14) \( f \) \( \pi \) \( s \) on \( M \) where \( \lambda \) Set Proof. \( M \). variety

**Theorem 4.4.** Let \( D \) be an effective Cartier divisor on a complex projective variety \( M \). Let \( s_1, \cdots, s_q \) be nonzero elements of a nonzero linear subspace \( V \subseteq H^0(M, L_D) \). Let \( f : S \to M \) be a holomorphic curve with Zariski dense image. Then

\[
\int_{\partial D(r)} \max_Q \sum_{k \in Q} \lambda_{s_k} \circ f(x) d\pi^r(x) \\
\leq \dim \mathbb{C} VT_{f,D}(r) + O\left( \log T_{f,D}(r) - \kappa(r) r^2 + \log^+ \log r \right),
\]

where \( \lambda_{s_k} \) denotes the Weil function of \( (s_k) \), and \( Q \) ranges over all subsets of \( \{ 1, \cdots, q \} \) such that \( \{ s_k \}_{k \in Q} \) are linearly independent.

**Proof.** Set \( d = \dim \mathbb{C} V \). If \( d = 1 \), then \( |Q| = 1 \). Hence, for \( 1 \leq j \leq q \), we have \( s_k = b_k s_D \) for some constant \( b_k \neq 0 \). By the First Main Theorem

\[
\int_{\partial D(r)} \max_Q \sum_{k \in Q} \lambda_{s_k} \circ f(x) d\pi^r(x) \leq \int_{\partial D(r)} \lambda_D \circ f(x) d\pi^r(x) + O(1) \\
\leq T_{f,D}(r) + O(1).
\]

If \( d > 1 \), we treat the projective space \( P(V) \) of \( V \) that can be regarded as \( \mathbb{P}^{d-1}(\mathbb{C}) \). Let \( M' \) be the closure of graph of \( f \), then there are the canonical projection morphisms \( \pi : M' \to M \) and \( \phi : M' \to \mathbb{P}^{d-1}(\mathbb{C}) \). We lift \( f \) to \( \tilde{f} : S \to M' \). Note that (see [18]) there exists an effective Cartier divisor \( B \) on \( M' \) such that for each \( s \in V \), we can choose a hyperplane \( H_s \) (depending on \( s \)) of \( \mathbb{P}^{d-1}(\mathbb{C}) \) which satisfies \( \pi^*(s) - B = \phi^*H_s \) (more precisely, \( \phi^*G(1) \cong L_{\pi^*D-B} \)). For \( 1 \leq j \leq q \), one chooses hyperplanes \( H_j \) of \( \mathbb{P}^{d-1}(\mathbb{C}) \) such that \( \pi^*(s_j) - B = \phi^*H_j \). Since \( M \) is compact, then we have

\[
\lambda_{\pi^*(s_j)} = \lambda_{\phi^*H_j} + \lambda_B + O(1).
\]

In further, we have

\[
N_{\tilde{f}}(r, \pi^*(s_j)) = N_{\tilde{f}}(r, \phi^*H_j) + N_{\tilde{f}}(r, B),
\]

\[
m_{\tilde{f}}(r, \pi^*(s_j)) = m_{\tilde{f}}(r, \phi^*H_j) + m_{\tilde{f}}(r, B) + O(1).
\]

Note that \( \phi \circ \tilde{f} : S \to \mathbb{P}^{d-1}(\mathbb{C}) \) is a holomorphic curve, using the First Main Theorem, it yields that \( T_{\phi \circ f}(r) = m_{\phi \circ f}(r, H_j) + N_{\phi \circ f}(r, H_j) + O(1) \). Indeed, \( L_{(s_j)} \cong L_D \) and \( f = \pi \circ \tilde{f} \) are noted. By (15) and (16), we arrive at

\[
T_{f,D}(r) = T_{\phi \circ f}(r) + T_{f,B}(r) + O(1).
\]

Combining (14) with \( \lambda_{s_j} \circ f = \lambda_{\pi^*(s_j)} \circ \tilde{f} + O(1) \), it suffices to show that

\[
\int_{\partial D(r)} \max_Q \sum_{k \in Q} \left( \lambda_{H_k} \circ \phi \circ \tilde{f}(x) + \lambda_B \circ \tilde{f}(x) \right) d\pi^r(x) \\
\leq dT_{f,D}(r) + O\left( \log T_{f,D}(r) - \kappa(r) r^2 + \log^+ \log r \right).
\]
In fact, by Lemma 4.3 and (17) we have
\[
\int_{\partial D(r)} \max_Q \sum_{k \in Q} \lambda_{H_k} \circ \phi \circ \tilde{f}(x) d\pi_o^r(x)
= \int_{\partial D(r)} \max_Q \sum_{k \in Q} \log \frac{\|\hat{H}_k\| \|\phi \circ \tilde{f}(x)\|}{\|H_k \circ \phi \circ \tilde{f}(x)\|} d\pi_o^r(x) + O(1)
\leq d T_{\phi \circ \tilde{f}}(r) + O \left( \log T_{\phi \circ \tilde{f}}(r) - \kappa(r) r^2 + \log \log r \right)
\leq d \left( T_{f,D}(r) - T_{\tilde{f},B}(r) \right) + O \left( \log T_{f,D}(r) - \kappa(r) r^2 + \log \log r \right).
\]
Since \(|Q| \leq d\), the First Main Theorem implies that
\[
\int_{\partial D(r)} \max_Q \sum_{k \in Q} \lambda_B \circ \tilde{f}(x) d\pi_o^r(x) \leq d T_{\tilde{f},B}(r) + O(1).
\]
Combining the above, we conclude the proof. \(\Box\)

4.2. Second Main Theorem.

In this subsection, we aim to prove the main theorem of the paper, namely, the Second Main Theorem (Theorem 1.2).

Let \(S\) be a complete open Riemann surface with Gauss curvature \(K_S \leq 0\). We here consider the universal covering \(\pi : \tilde{S} \to S\). By the pull-back of \(\pi\), \(\tilde{S}\) could be equipped with the induced metric from the metric of \(S\). In such case, \(\tilde{S}\) is a simply-connected and complete open Riemann surface of non-positive Gauss curvature. Take a diffusion process \(\tilde{X}_t\) in \(\tilde{S}\) so that \(X_t = \pi(\tilde{X}_t)\), then \(\tilde{X}_t\) becomes a Brownian motion with generator \(\Delta_{\tilde{S}}/2\) which is induced from the pull-back metric. Let \(\tilde{X}_t\) start from \(\tilde{o} \in \tilde{S}\) with \(o = \pi(\tilde{o})\), then we have
\[
\mathbb{E}_\tilde{o}[\phi(X_t)] = \mathbb{E}_\tilde{o}[\phi \circ \pi(\tilde{X}_t)]
\]
for \(\phi \in \mathcal{C}_b(S)\). Set
\[
\tilde{\tau}_r = \inf \{ t > 0 : \tilde{X}_t \notin \tilde{D}(r) \},
\]
where \(\tilde{D}(r)\) is a geodesic disc centered at \(\tilde{o}\) with radius \(r\) in \(\tilde{S}\). If necessary, one can extend the filtration in probability space where \((X_t, \mathbb{P}_o)\) are defined so that \(\tilde{\tau}_r\) is a stopping time with respect to a filtration where the stochastic calculus of \(X_t\) works. By the above arguments, we would assume \(\tilde{S}\) is simply connected by lifting \(f\) to the covering.

Proof of Theorem 1.2

Proof. Let \(\mathfrak{P}\) be the set of all prime divisors occurring in \(D\), then
\[
D = \sum_{E \in \mathfrak{P}} \operatorname{ord}_E(D) \cdot E.
\]
Set \( \Lambda = \{ \sigma \subseteq \mathcal{P} : \cap_{E \in \sigma} E \neq \emptyset \} \) which is a finite set. For any \( \sigma \in \Lambda \), we write

\[
D = D_{\sigma,1} + D_{\sigma,2},
\]

where

\[
D_{\sigma,1} = \sum_{E \in \sigma} \text{ord}_E(D) \cdot E, \quad D_{\sigma,2} = \sum_{E \notin \sigma} \text{ord}_E(D) \cdot E.
\]

From the definition of \( \text{Nev}(L, D) \), for each \( \sigma \in \Lambda \), there exists a basis \( \mathcal{B}_\sigma \) of a linear subspace \( V_k \subseteq H^0(X, kL) \) with \( \dim_C V_k > 1 \) (for some \( k \)) such that

\[
\frac{1}{\mu_k} \sum_{s \in \mathcal{B}_\sigma} \text{ord}_E(s) \geq \text{ord}_E(kD)
\]

at some (and hence all) points \( x \in \cap_{E \in \sigma} E \). For each \( E \in \sigma \), we have

\[
\text{(18)} \quad \frac{1}{\mu_k} \sum_{s \in \mathcal{B}_\sigma} \text{ord}_E(s) \cdot \lambda_E \geq \text{ord}_E(kD) \cdot \lambda_E.
\]

Note that (refer to the proof of Proposition 3.1 in [17]) there exists a number \( B > 0 \) such that for each \( x \in X \), one can pick \( \sigma_x \in \Lambda \) (depending on \( x \)) such that

\[
\lambda_{D_{\sigma_x,1}}(x) \leq B, \quad \text{here } B \text{ is independent of } x.
\]

Thus,

\[
\text{(19)} \quad \lambda_D(\sigma_x, 1)(x) + O(1).
\]

By properties of Weil functions, we have from [19] and [18] that

\[
\lambda_{kD}(x) \leq \frac{1}{\mu_k} \max_{\sigma \in \Lambda} \sum_{s \in \mathcal{B}_\sigma} \lambda_s(x) + O(1),
\]

where \( \lambda_s(x) \) is the Weil function of \( (s) \). Taking the expectation to get

\[
km_f(r, D) \leq \frac{1}{\mu_k} \int_{\partial D(r)} \max_{\sigma \in \Lambda} \sum_{s \in \mathcal{B}_\sigma} \lambda_s(x) d\pi_{r_0}(x) + O(1).
\]

Making use of Theorem 4.4, we obtain

\[
km_f(r, D) \leq \frac{\dim_C V_k}{\mu_k} T_f(kL(r)) + o(T_f(kL(r))) + O\left( -\kappa(r)r^2 + \log^+ \log r \right).
\]

This proves Theorem 1.2. \( \square \)

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Email address: xjdong@amss.ac.cn