Positive explicit and implicit computational techniques for reaction–diffusion epidemic model of dengue disease dynamics

Nauman Ahmed¹,², Muhammad Rafiq³, Dumitru Baleanu⁴,⁵,⁶, Ali Saleh Alshomrani⁷ and Muhammad Aziz-ur Rehman¹

¹Correspondence: dumitru@cankaya.edu.tr
²Department of Mathematics, Cankaya University, Ankara, Turkey
³Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
Full list of author information is available at the end of the article

Abstract
The aim of this work is to develop a novel explicit unconditionally positivity preserving finite difference (FD) scheme and an implicit positive FD scheme for the numerical solution of dengue epidemic reaction–diffusion model with incubation period of virus. The proposed schemes are unconditionally stable and preserve all the essential properties of the solution of the dengue reaction diffusion model. This proposed FD schemes are unconditionally dynamically consistent with positivity property and converge to the true equilibrium points of dengue epidemic reaction diffusion system. Comparison of the proposed scheme with the well-known existing techniques is also presented. The time efficiency of both the proposed schemes is also compared, with the two widely used techniques.

Keywords: Structure preserving methods; Finite difference schemes; Dengue model; Diffusion epidemic system; Numerical simulations

1 Introduction
Dengue fever is a mosquito born infection which causes flu-like illness, fever and severe pain in the body. Dengue virus is transmitted by Aedes mosquito bite. It is a lethal disease that starts with painful fever. Some people have non-febrile illness with rash, headache, pain behind eyes and joint pains. Dengue hemorrhagic fever is highly complicated which can cause high fever, hemorrhage and enlargement of liver and circulatory failure. Dengue is an epidemic disease which can be prevented by awareness programs against it. There is no vaccine and specific medication to treat it. In order to get a good understanding of the nature and dynamics of the transmission of dengue epidemics, various epidemic models of dengue disease dynamics are discussed in the literature [1–7]. Most of the epidemic models of infection disease dynamics are based on ODE systems. Recently, the researchers were developing integer order models in the setup of fractional calculus. Since the fractional calculus is an extension of the classical calculus, its scope is wider than that of its counterpart. Epidemic models, in the framework of fractional calculus, address more parameters which reduces errors. Also, by including these parameters, the models express a...
very close behavior to the actual physical problem. Several authors used various fractional operators for developing epidemic models, for detail, see the references [8–10]. These models provide a motivation to the young researchers.

In order to get appropriate perception of dispersal and control of dengue transmission, it is important to consider the dengue epidemic model with diffusion because individuals do not mix homogenously.

Let us consider a dengue epidemic model proposed by Rafiq et al. [11]:

\[
\frac{dS}{dt} = \mu_h - \beta_h SI_v \left( \frac{C}{\mu_v} \right) - \mu_h S, \quad (1.1)
\]

\[
\frac{dX}{dt} = - (\alpha_h + \mu_h) X + \beta_h SI_v \left( \frac{C}{\mu_v} \right), \quad (1.2)
\]

\[
\frac{dI}{dt} = - (r + \mu_h) I + \alpha_h X, \quad (1.3)
\]

\[
\frac{dX_v}{dt} = - (\alpha_v + \mu_v) X_v + \beta_v IN_T (1 - X_v - I_v), \quad (1.4)
\]

\[
\frac{dI_v}{dt} = - \mu_v I_v + \alpha_v X_v, \quad (1.5)
\]

and the conditions

\[
S + X + I + R = 1 \quad \text{and} \quad S_v + X_v + I_v = 1. \quad (1.6)
\]

Note that (1.1)–(1.5) is a normalized system discussed by M. Rafiq et al. [11].

Let \(d_S, d_X, d_I, d_{X_v}, \) and \(d_{I_v}\) be the diffusive constants of \(S, X, I, \) \(X_v, \) and \(I_v, \) respectively. Then system (1.1)–(1.5) with diffusion can be written as

\[
\frac{\partial S}{\partial t} = d_S \frac{\partial^2 S}{\partial x^2} + \mu_h - \beta_h SI_v \left( \frac{C}{\mu_v} \right) - \mu_h S, \quad (1.7)
\]

\[
\frac{\partial X}{\partial t} = d_X \frac{\partial^2 X}{\partial x^2} + \beta_h SI_v \left( \frac{C}{\mu_v} \right) - \mu_h X, \quad (1.8)
\]

\[
\frac{\partial I}{\partial t} = d_I \frac{\partial^2 I}{\partial x^2} + \alpha_h X - rI - \mu_h I, \quad (1.9)
\]

\[
\frac{\partial X_v}{\partial t} = d_{X_v} \frac{\partial^2 X_v}{\partial x^2} + \beta_v IN_T (1 - X_v - I_v) - \alpha_v X_v - \mu_v X_v, \quad (1.10)
\]

\[
\frac{\partial I_v}{\partial t} = d_{I_v} \frac{\partial^2 I_v}{\partial x^2} + \alpha_v X_v - \mu_v I_v, \quad (1.11)
\]

where \(S(x, t), X(x, t)\) and \(I(x, t)\) are population sizes of susceptible, exposed, and infectious humans, while \(X_v(x, t)\) and \(I_v(x, t)\) are population sizes of exposed and infectious vectors, respectively, at location \(x\) and time \(t,\) where \(\mu_h, \beta_h, \alpha_h, r, \) and \(N_T\) are the rate of death of humans population, infection rate from vector population to human population, rate at which infected human population becomes infectious, recovery rate for human population and total human population, respectively. The rates \(\beta_v, \alpha_v, \mu_v, \) and \(C\) are infection rate from human population to vector population, rate at which infected vector population becomes infectious, death rate of vector population and recruitment rate of vector population, respectively.
The main theme of this work is to provide numerical techniques which are consistent to the continuous diffusive epidemic model. The epidemic models describe the unknown quantities as population sizes therefore negative solutions of epidemic models are meaningless. In order to find the solution of dengue epidemic model, the method of solution should preserve the positivity of the solution. Also proposed epidemic models of dengue dynamics have two stable equilibrium points, the numerical technique must show convergence towards these equilibrium points. Normally, classical and well-known numerical techniques have flaws in their construction, therefore these techniques cannot preserve most of the properties possessed by the continuous epidemic models. The proposed techniques not only preserve the positive solution but also converge towards the true equilibrium points of the continuous system.

2 Equilibrium points
Dengue epidemic model (1.7)–(1.11) describes two possible equilibrium points, disease free equilibrium (DFE) and endemic equilibrium (EE), namely

**DFE**

$$E_0(1,0,0,0,0),$$

**EE**

$$E_1(S^*, X^*, I^*, X^*_v, I^*_v),$$

where

$$S^* = \frac{(\alpha_h + \mu_v)(MN\mu^2_h\mu_v + \alpha_h\gamma_v\mu_h)}{\alpha_h\gamma_v[\mu_h(\alpha_v + \mu_v) + \alpha_v\gamma_h]},$$

$$X^* = \frac{M\mu^2_h\mu_v(\alpha_v + \mu_v)(R_0 - 1)}{\alpha_h[\mu_h(\alpha_v + \mu_v) + \alpha_v\gamma_h]},$$

$$I^* = \frac{\mu_h\mu_v(\alpha_v + \mu_v)(R_0 - 1)}{\alpha_h[\mu_h(\alpha_v + \mu_v) + \alpha_v\gamma_h]},$$

$$X^*_v = \frac{\mu_v(MN\mu^2_h\mu_v)(R_0 - 1)}{\gamma_v[\mu_h(\alpha_v + \mu_v) + MN\mu^2_h\mu_v]},$$

$$I^*_v = \frac{MN\mu^2_h\mu_v}{\gamma_v[\mu_h(\alpha_v + \mu_v) + MN\mu^2_h\mu_v]}(R_0 - 1),$$

and where

$$R_0 = \frac{\alpha_h\alpha_v\gamma_v\gamma_h}{(r + \mu_h)(\alpha_h + \mu_h)\mu_v(\alpha_v + \mu_v)},$$

when $d_S = d_X = d_I = d_{X_v} = d_{I_v} = 0,$ is the reproductive number.

Also

$$\gamma_h = \beta_h\left(\frac{C}{\mu_v}\right), \quad \gamma_v = \beta_vN_T, \quad M = \frac{r + \mu_h}{\mu_h} \quad \text{and} \quad N = \frac{\alpha_h + \mu_h}{\mu_h}.$$  

The reproductive number $R_0$ decides the outcome, namely if $R_0 < 1,$ the disease is eliminated from the given population, and if $R_0 > 1,$ the disease persists in the population.
3 Computational method

A nonstandard FD method, introduced by Mickens [12], is an efficient tool to solve epidemic models because this method is structure-preserving and consistent with the solution of epidemic models. Many researchers used structure-preserving methods to solve epidemic models with a system of ordinary and partial differential equations [11, 13–22]. For the reaction–diffusion epidemic model and some of the work concerning positivity-preserving finite difference schemes, we refer to the literature [23–26].

In the present study, we design two structure-preserving finite difference schemes [27–30] for dengue epidemic model with diffusion. The proposed schemes are convergent towards all steady states of the continuous system and preserve positivity property that highlights the significance and efficacy of the proposed schemes.

In this section, we rewrite system (1.7)–(1.11) as

\[
\frac{\partial S}{\partial t} = d S \frac{\partial^2 S}{\partial x^2} + \mu_h - \beta_h SI_v(C/\mu_v) - \mu_h S, \quad (3.1)
\]

\[
\frac{\partial X}{\partial t} = d X \frac{\partial^2 X}{\partial x^2} + \beta_h SI_v(C/\mu_v) - \mu_h X - \mu_h X, \quad (3.2)
\]

\[
\frac{\partial I}{\partial t} = d I \frac{\partial^2 I}{\partial x^2} + \alpha_h X - r I - \mu_h I, \quad (3.3)
\]

\[
\frac{\partial X_v}{\partial t} = d X_v \frac{\partial^2 X_v}{\partial x^2} + \beta_v I N_T(1 - X_v - I_v) - \alpha_v X_v - \mu_v X_v, \quad (3.4)
\]

\[
\frac{\partial I_v}{\partial t} = d I_v \frac{\partial^2 I_v}{\partial x^2} + \alpha_v X_v - \mu_v I_v, \quad (3.5)
\]

for all \( t \geq 0, x \in [0, L] \) and the initial conditions are:

\[
S(x, 0) = \delta_1(x) \geq 0, \quad X(x, 0) = \delta_2(x) \geq 0, \quad I(x, 0) = \delta_3(x) \geq 0, \quad (3.6)
\]

\[
X_v(x, 0) = \delta_4(x) \geq 0 \quad \text{and} \quad I_v(x, 0) = \delta_5(x) \geq 0, \quad (3.7)
\]

while the boundary conditions are:

\[
\frac{\partial S(0, t)}{\partial x} = \frac{\partial X(0, t)}{\partial x} = \frac{\partial I(0, t)}{\partial x} = \frac{\partial X_v(0, t)}{\partial x} = \frac{\partial I_v(0, t)}{\partial x} = 0, \quad t > 0, \quad (3.8)
\]

\[
\frac{\partial S(L, t)}{\partial x} = \frac{\partial X(L, t)}{\partial x} = \frac{\partial I(L, t)}{\partial x} = \frac{\partial X_v(L, t)}{\partial x} = \frac{\partial I_v(L, t)}{\partial x} = 0, \quad t > 0. \quad (3.9)
\]

Divide \([0, L] \times [0, T]\) using \( M \times N \) mesh points having time and space step sizes \( h = \frac{L}{M} \) and \( \tau = \frac{T}{N} \).

The nodal points then are

\[
x_i = i h, \quad i = 0, 1, 2, \ldots, M, \]

\[
t_n = n \tau, \quad n = 0, 1, 2, \ldots, N,
\]

\( S^n_i, X^n_i, I^n_i, X_v^n_i \) and \( I_v^n_i \) are denoted as FD values of \( S(ih, n\tau), X(ih, n\tau), I(ih, n\tau), X_v(ih, n\tau) \) and \( I_v(ih, n\tau) \), respectively.

Four FD schemes are used to solve system (3.1)–(3.5) numerically: forward Euler explicit FD scheme, Crank–Nicolson implicit FD scheme and the proposed FD scheme. Forward
Euler FD scheme for system (3.1)–(3.5) is:

\[
S_{i+1}^{n+1} = S_i^n + R_1(S_{i-1}^n + S_{i+1}^n) + \tau \mu_h - \tau \beta_h(C/\mu_h)S_i^n I_{i+1}^n - \tau \mu_h S_i^n, \\
X_{i+1}^{n+1} = X_i^n + R_2(X_{i-1}^n - 2X_i^n + X_{i+1}^n) + \tau \beta_h S_i^n I_{i+1}^n \left(\frac{C}{\mu_h}\right) - \tau (\mu_h + \alpha_h) X_i^n, \\
I_{i+1}^{n+1} = I_i^n + R_1(I_{i-1}^n - 2I_i^n + I_{i+1}^n) + \tau \alpha_h X_i^n - \tau (r + \mu_h) I_i^n, \\
X_{i+1}^{n+1} = X_i^n + R_4(X_{i-1}^n - 2X_i^n + X_{i+1}^n) + \tau \beta_h N_i I_{i+1}^n(1 - X_i^n - I_i^n) - \tau (\alpha_r + \mu_i) X_i^n, \\
I_{i+1}^{n+1} = I_i^n + R_5(I_{i-1}^n - 2I_i^n + I_{i+1}^n) + \tau \alpha_r X_i^n - \tau \mu_r I_i^n.
\]

The stability range of Forward Euler explicit scheme is \(R_1 \leq \frac{2 - \tau \mu_h}{4}, R_2 \leq \frac{2 - \tau (\mu_h + \alpha_h)}{4}, R_3 \leq \frac{2 - \tau (r + \mu_h)}{4}\) and \(R_5 \leq \frac{2 - \tau \mu_r}{4}\).

Crank–Nicolson FD scheme for system (3.1)–(3.5) is:

\[
(1 + R_1)S_i^{n+1} - \frac{R_1}{2}(S_{i-1}^{n+1} + S_{i+1}^{n+1}) \\
= (1 - R_1)S_i^n + \frac{R_1}{2}(S_{i-1}^n + S_{i+1}^n) + \tau \mu_h - \tau \beta_h(C/\mu_h)S_i^n I_{i+1}^n - \tau \mu_h S_i^n, \\
(1 + R_2)X_i^{n+1} - \frac{R_2}{2}(X_{i-1}^{n+1} + X_{i+1}^{n+1}) \\
= (1 - R_2)X_i^n + \frac{R_2}{2}(X_{i-1}^n + X_{i+1}^n) + \tau \beta_h S_i^n I_{i+1}^n \left(\frac{C}{\mu_h}\right) - \tau (\mu_h + \alpha_h) X_i^n, \\
(1 + R_3)I_i^{n+1} - \frac{R_3}{2}(I_{i-1}^{n+1} + I_{i+1}^{n+1}) = (1 - R_3)I_i^n + \frac{R_3}{2}(I_{i-1}^n + I_{i+1}^n) + \tau \alpha_h X_i^n - \tau (r + \mu_h) I_i^n, \\
(1 + R_4)X_i^{n+1} - \frac{R_4}{2}(X_{i-1}^{n+1} + X_{i+1}^{n+1}) \\
= (1 - R_4)X_i^n + \frac{R_4}{2}(X_{i-1}^n + X_{i+1}^n) + \tau \beta_h N_i I_{i+1}^n(1 - X_i^n - I_i^n) - \tau (\alpha_r + \mu_i) X_i^n, \\
(1 + R_5)I_i^{n+1} - \frac{R_5}{2}(I_{i-1}^{n+1} + I_{i+1}^{n+1}) = (1 - R_5)I_i^n + \frac{R_5}{2}(I_{i-1}^n + I_{i+1}^n) + \tau \alpha_r X_i^n - \tau \mu_r I_i^n.
\]

Crank–Nicolson scheme is unconditionally stable.

Now the proposed explicit positive FD scheme is developed [27–30] with the help of rules defined by Mickens [12] as follows:

\[
S_i^{n+1} = S_i^n + R_1(S_{i-1}^n + S_{i+1}^n) - 2R_1S_i^{n+1} + \tau \mu_h - \tau \beta_h(C/\mu_h)S_i^n I_{i+1}^n - \tau \mu_h S_i^{n+1}, \\
S_i^{n+1} + \tau \mu_h I_{i+1} \left(\frac{C}{\mu_h}\right)S_i^n + \tau \mu_h S_i^{n+1} + 2R_1S_i^{n+1} = S_i^n + R_1(S_{i-1}^n + S_{i+1}^n) + \tau \mu_h, \\
\left(1 + \tau \beta_h I_{i+1} \left(\frac{C}{\mu_h}\right) + \tau \mu_h + 2R_1\right)\frac{S_i^{n+1}}{1 + \tau \beta_h I_{i+1} \left(\frac{C}{\mu_h}\right) + \tau \mu_h + 2R_1} = S_i^n + R_1(S_{i-1}^n + S_{i+1}^n) + \tau \mu_h, \\
\frac{S_i^{n+1}}{1 + \tau \mu_h + \tau \alpha_h + 2R_2} = S_i^n + R_1(S_{i-1}^n + S_{i+1}^n) + \tau \mu_h.
\]

A similar process is used and we get

\[
X_i^{n+1} = \frac{X_i^n + R_2(X_{i-1}^n + X_{i+1}^n) + \tau \beta_h S_i^n I_{i+1}^n \left(\frac{C}{\mu_h}\right)}{1 + \tau \mu_h + \tau \alpha_h + 2R_2},
\]

\[
I_i^{n+1} = \frac{I_i^n + R_3(I_{i-1}^n + I_{i+1}^n) + \tau \alpha_r X_i^n}{1 + \tau \mu_r + 2R_5},
\]

\[
X_i^{n+1} = \frac{X_i^n + R_4(X_{i-1}^n + X_{i+1}^n) + \tau \beta_h N_i I_{i+1}^n}{1 + \tau \mu_r + 2R_5}.
\]
Here, a similar process is used for $X_{n+1}$:

$$X_{n+1}^{(2)} = R_4 (X_{n+1}^{(1)} + X_{n+1}^{(4)}) + \tau \beta_3 N_i I_t^n (1 - I_t^n),$$

$$I_{n+1}^{(2)} = I_{n+1}^{(1)} + R_5 (I_{n+1}^{(1)} + I_{n+1}^{(4)}) + \tau \alpha \nu X_{n+1}^{(4)},$$

(3.14) (3.15)

Now we design the proposed positive implicit scheme for the given model (3.1)-(3.5) as:

$$S_{n+1}^{(2)} - R_1 (S_{n+1}^{(1)} + S_{n+1}^{(4)}) = S_n^n + \tau \mu - \tau \beta_3 (C/\mu_b) N_i I_t^n - \tau \mu S_{n+1}^{(2)},$$

(3.16)

$$X_{n+1}^{(2)} - R_3 (X_{n+1}^{(1)} + X_{n+1}^{(4)}) = X_n^n + \tau \beta_3 N_i I_t^n \left( \frac{C}{\mu_b} \right) - \tau (\mu_b + \alpha) X_{n+1}^{(2)},$$

(3.17)

$$I_{n+1}^{(2)} - R_5 (I_{n+1}^{(1)} + I_{n+1}^{(4)}) = I_n^n + \tau \alpha \nu X_{n+1}^{(4)} - \tau \mu I_{n+1}^{(2)}.$$

(3.18) (3.19) (3.20)

Here,

$$R_1 = d_s \frac{\tau}{h^2}, \quad R_2 = d_x \frac{\tau}{h^2}, \quad R_3 = d_j \frac{\tau}{h^2}, \quad R_4 = d_x \tau \frac{\omega}{\Delta t} \quad \text{and} \quad R_5 = d_j \frac{\tau}{h^2}.$$

### 3.1 Stability

For the stability analysis, we apply the von Neumann method to (3.10). Substituting $S_i^n$ with $\xi(t)e^{j\omega t}$ and linearizing, we have

$$\xi(t + \Delta t)e^{j\omega t} = \xi(t)e^{j\omega t} + R_1 (e^{j\omega(x-h\Delta x)} + e^{j\omega(x+h\Delta x)})\xi(t)$$

$$\quad - 2R_1 \xi(t + \Delta t)e^{j\omega t} - \tau \mu \xi(t + \Delta t)e^{j\omega t}.$$

After simplification, we have

$$\frac{\xi(t + \Delta t)}{\xi(t)} = \left| \frac{1 + 2R_1 - 4R_1 \sin^2(\omega \Delta z/2)}{1 + 2R_1 + \tau \mu \mu_b} \right| \leq \frac{1 - 2R_1}{1 + 2R_1 + \tau \mu \mu_b} < 1.$$  

(3.21)

A similar process is used for $X_i^n, I_t^{(2)}, X_{n+1}^{(2)}$ and $I_{n+1}^{(2)}$, so we have:

$$\frac{\theta(t + \Delta t)}{\theta(t)} = \left| \frac{1 + 2R_2 - 4R_2 \sin^2(\omega \Delta z/2)}{1 + 2R_2 + \tau (\mu_b + \mu_b)} \right| \leq \frac{1 - 2R_2}{1 + 2R_2 + \tau (\mu_b + \mu_b)} < 1,$$

(3.22)

$$\frac{\theta(t + \Delta t)}{\theta(t)} = \left| \frac{1 + 2R_3 - 4R_3 \sin^2(\omega \Delta z/2)}{1 + 2R_3 + \tau (r + \mu_b)} \right| \leq \frac{1 - 2R_3}{1 + 2R_3 + \tau (r + \mu_b)} < 1,$$

(3.23)

$$\frac{\theta(t + \Delta t)}{\theta(t)} = \left| \frac{1 + 2R_4 - 4R_4 \sin^2(\omega \Delta z/2)}{1 + 2R_4 + \tau (\mu_b + \nu)} \right| \leq \frac{1 - 2R_4}{1 + 2R_4 + \tau (\mu_b + \nu)} < 1,$$

(3.24)

$$\frac{\theta(t + \Delta t)}{\theta(t)} = \left| \frac{1 + 2R_5 - 4R_5 \sin^2(\omega \Delta z/2)}{1 + 2R_5 + \tau \mu_v} \right| \leq \frac{1 - 2R_5}{1 + 2R_5 + \tau \mu_v} < 1.$$  

(3.25)

From (3.21)-(3.25), it is clear that the proposed FD scheme is unconditionally stable.
In a similar fashion, the stability of the proposed implicit scheme (3.16)–(3.20) can be verified [21].

3.2 Consistency
To check the consistency of the proposed FD scheme, we use Taylor series expansion of \( S_{i+1}^n, S_{i-1}^n \) and \( S_i^n \):

\[
S_{i+1}^n = S_i^n + \tau \frac{\partial S}{\partial t} + \frac{\tau^2}{2!} \frac{\partial^2 S}{\partial t^2} + \frac{\tau^3}{3!} \frac{\partial^3 S}{\partial t^3} + \cdots,
\]

\[
S_{i-1}^n = S_i^n - h \frac{\partial S}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 S}{\partial x^2} + \frac{h^3}{3!} \frac{\partial^3 S}{\partial x^3} + \cdots,
\]

\[
S_i^n = S_i^n - h \frac{\partial S}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 S}{\partial x^2} - \frac{h^3}{3!} \frac{\partial^3 S}{\partial x^3} + \cdots.
\]

The proposed scheme for equation (3.1) is

\[
S_{i+1}^n = S_i^n + R_1(S_{i+1}^n + S_{i-1}^n) - 2R_1S_{i-1}^n + \tau \mu_h - \tau \beta_h S_i^n \left( \frac{C}{\mu_h} \right) S_i^n - \tau \mu_h S_i^n + \tau \beta_h S_i^n. \quad (3.26)
\]

Putting the values of \( S_{i+1}^n, S_{i-1}^n \) and \( S_i^n \) in (3.26) and simplifying, we get

\[
\left( \frac{\partial S}{\partial t} + \frac{\tau}{2!} \frac{\partial^2 S}{\partial t^2} + \frac{\tau^2}{3!} \frac{\partial^3 S}{\partial t^3} + \cdots \right) \left( 1 + 2dS \frac{\tau}{h^2} + \tau \beta_h S_i^n \left( \frac{C}{\mu_h} \right) + \tau \mu_h \right)
\]

\[
= \mu_h + 2dS \left( \frac{\tau}{2!} \frac{\partial^2 S}{\partial x^2} + \frac{h^2}{4!} \frac{\partial^4 S}{\partial x^4} + \cdots \right) - S_i^n \left( \beta_h S_i^n \left( \frac{C}{\mu_h} \right) + \mu_h \right). \quad (3.27)
\]

Putting \( \tau = h^2 \) and letting \( h \to 0 \), equation (3.27) becomes (3.1) [27, 28].

In a similar way, by using Taylor series expansion of \( X_{i+1}^n, X_{i-1}^n \) and \( X_i^n \) in (3.12) and simplifying, we get

\[
\left( \frac{\partial X}{\partial t} + \frac{\tau}{2!} \frac{\partial^2 X}{\partial t^2} + \frac{\tau^2}{3!} \frac{\partial^3 X}{\partial t^3} + \cdots \right) \left( 1 + 2dX \frac{\tau}{h^2} + \tau (\alpha_h + \mu_h) \right)
\]

\[
= 2dX \left( \frac{\tau}{2!} \frac{\partial^2 X}{\partial x^2} + \frac{h^2}{4!} \frac{\partial^4 X}{\partial x^4} + \cdots \right) + \beta_h S_i^n \left( \frac{C}{\mu_h} \right) + X_i^n \left( - (\alpha_h + \mu_h) \right). \quad (3.28)
\]

Putting \( \tau = h^2 \) and letting \( h \to 0 \), equation (3.28) becomes (3.2).

Substituting the Taylor series expansion of \( I_{i+1}^n, I_{i-1}^n \) and \( I_i^n \) in (3.13) and simplifying, we get

\[
\left( \frac{\partial I}{\partial t} + \frac{\tau}{2!} \frac{\partial^2 I}{\partial t^2} + \frac{\tau^2}{3!} \frac{\partial^3 I}{\partial t^3} + \cdots \right) \left( 1 + 2dI \frac{\tau}{h^2} + \tau (r + \mu_h) \right)
\]

\[
= 2dI \left( \frac{\tau}{2!} \frac{\partial^2 I}{\partial x^2} + \frac{h^2}{4!} \frac{\partial^4 I}{\partial x^4} + \cdots \right) + \alpha_h X_i^n - I_i^n (r + \mu_h). \quad (3.29)
\]

Putting \( \tau = h^2 \) and letting \( h \to 0 \), equation (3.29) becomes (3.3).

Putting the values of \( X_{i+1}^n, X_{i-1}^n \) and \( X_i^n \) in (3.14) and simplifying, we get

\[
\left( \frac{\partial X}{\partial t} + \frac{\tau}{2!} \frac{\partial^2 X}{\partial t^2} + \frac{\tau^2}{3!} \frac{\partial^3 X}{\partial t^3} + \cdots \right) \left( 1 + 2dX \frac{\tau}{h^2} + \tau (\alpha_v + \mu_v) \right)
\]
Putting $\tau = h^3$ and letting $h \to 0$, equation (3.30) becomes (3.4).

Again substituting the Taylor series expansion of $I_{n+1}^{\nu_i}$, $I_{n+1}^{\mu_i}$ and $I_{n-1}^{\mu_i}$, in (3.15) and simplifying, we get

\[
= 2d_{\nu}( \frac{1}{2!} \frac{\partial^2 X_{\nu}}{\partial t^2} + \frac{h^2}{3!} \frac{\partial^3 X_{\nu}}{\partial t^3} + \cdots ) + \beta_\nu N_{\nu} I_{n}^{\nu} \left( 1 - I_{n}^{\nu} \right) \\
- X_{\nu}^{\nu} (\beta_\nu N_{\nu} I_{n}^{\nu} + (\alpha_{\nu} + \mu_{\nu})).
\]  

(3.30)

Putting $\tau = h^3$ and letting $h \to 0$, equation (3.31) becomes (3.5).

The consistency of the proposed implicit scheme (3.16)–(3.20) by applying Taylor series expansion and after simplification is given as:

\[
E_x = (1 + 2R_1) S_{i+1}^{\nu_1} - R_1 ( S_{i+1}^{\nu_1} + S_{i+1}^{\mu_1} ) - S_i^{\mu} - \tau \mu h + \tau \beta h \left( \frac{C}{\mu h} \right) S_{i}^{\nu_1} I_{n}^{\nu} + \tau \mu h S_{i}^{\nu_1} \\
= \tau \left( \frac{dS}{dt} + \frac{\tau h^3}{2!} \frac{\partial^2 S}{\partial t^2} + \frac{\tau h^4}{3!} \frac{\partial^3 S}{\partial t^3} + \frac{\tau h^5}{4!} \frac{\partial^4 S}{\partial t^4} + \cdots \right) \left( 1 + 2R_1 - R_1 + \tau \beta h \left( \frac{C}{\mu h} \right) I_{n}^{\nu} + \tau \mu h \right) \\
- h^2 \left( \frac{dS}{12} \frac{\partial^4 S}{\partial t^4} + \cdots \right) \\
\to 0, \quad \text{as} \quad h \to 0, \tau \to 0,
\]

\[
E_X = (1 + 2R_2) X_{i+1}^{\nu_1} - R_2 ( X_{i+1}^{\nu_1} + X_{i+1}^{\mu_1} ) - X_i^{\mu} - \tau \mu h S_{i}^{\nu_1} \left( \frac{C}{\mu h} \right) + \tau (\mu h + \alpha h) X_{i+1}^{\nu_1} \\
= \tau \left( \frac{dX}{dt} + \frac{\tau h^3}{2!} \frac{\partial^2 X}{\partial t^2} + \frac{\tau h^4}{3!} \frac{\partial^3 X}{\partial t^3} + \frac{\tau h^5}{4!} \frac{\partial^4 X}{\partial t^4} + \cdots \right) \left( 1 + 2R_2 - R_2 + \tau (\mu h + \alpha h) \right) \\
- h^2 \left( \frac{dX}{12} \frac{\partial^4 X}{\partial t^4} + \cdots \right) \\
\to 0, \quad \text{as} \quad h \to 0, \tau \to 0,
\]

\[
E_I = (1 + 2R_3) I_{i+1}^{\nu_1} - R_3 ( I_{i+1}^{\nu_1} + I_{i+1}^{\mu_1} ) - I_i^{\mu} - \tau \mu h X_{i}^{\mu} + \tau (r + \mu h) I_{i+1}^{\mu_1} \\
= \tau \left( \frac{dI}{dt} + \frac{\tau h^3}{2!} \frac{\partial^2 I}{\partial t^2} + \frac{\tau h^4}{3!} \frac{\partial^3 I}{\partial t^3} + \frac{\tau h^5}{4!} \frac{\partial^4 I}{\partial t^4} + \cdots \right) \left( 1 + 2R_3 - R_3 + \tau (r + \mu h) \right) \\
- h^2 \left( \frac{dI}{12} \frac{\partial^4 I}{\partial t^4} + \cdots \right) \\
\to 0, \quad \text{as} \quad h \to 0, \tau \to 0,
\]

\[
E_{XV} = (1 + 2R_4) X_{i+1}^{\nu_1} - R_4 ( X_{i+1}^{\nu_1} + X_{i+1}^{\mu_1} ) - X_i^{\mu} - \tau \mu h X_{i}^{\nu_1} \left( 1 - X_{i+1}^{\nu_1} - I_{i}^{\nu} \right) \\
+ \tau (\alpha h + \mu h) X_{i+1}^{\nu_1} \\
= \tau \left( \frac{dX}{dt} + \frac{\tau h^3}{2!} \frac{\partial^2 X}{\partial t^2} + \frac{\tau h^4}{3!} \frac{\partial^3 X}{\partial t^3} + \frac{\tau h^5}{4!} \frac{\partial^4 X}{\partial t^4} + \cdots \right) \\
\times \left( 1 + 2R_4 - R_4 + \tau \beta h N_{\nu} I_{n}^{\nu} + \tau (\alpha_{\nu} + \mu_{\nu}) \right) - h^2 \left( \frac{dX}{12} \frac{\partial^4 X}{\partial t^4} + \cdots \right)
\]
\[ \rightarrow 0, \text{ as } h \rightarrow 0, \tau \rightarrow 0, \]
\[ L_{Iv} = (1 + 2R_5)I_{v+1}^{n+1} - R_5(I_{v-1}^{n+1} + I_{v+1}^n) - I_{v+1}^n - \tau \alpha_v X_v^n + \tau \mu_v I_{v+1}^{n+1} \]
\[ = \tau \left( \frac{\partial I_{v+1}}{\partial \tau} + \frac{\tau^2}{2!} \frac{\partial^2 I_{v+1}}{\partial \tau^2} + \frac{\tau^3}{3!} \frac{\partial^3 I_{v+1}}{\partial \tau^3} + \cdots \right) (1 + 2R_5 - R_5 + \tau \mu_v) \]
\[ - h^2 \left( \frac{\partial^4 I_{v+1}}{\partial \tau^4} + \cdots \right) \]
\[ \rightarrow 0, \text{ as } h \rightarrow 0, \tau \rightarrow 0. \]

Hence the given implicit scheme is consistent.

The accuracy depends upon the numerical design. In the present scenario, the proposed implicit method has an order of accuracy \( O(h^2 + \tau) \) and the stability is unconditional. It is independent of the step size but implicit in nature. On the other hand, the proposed explicit scheme is a modification of the forward Euler technique. The consistency of the scheme can be observed when \( \tau = h^3 \). The forward Euler design also has the order of accuracy \( O(h^2 + \tau) \).

### 3.3 Positivity

This section is devoted to the positivity analysis of both proposed techniques.

**Lemma 1** Expressions (3.11)–(3.15) have the non-negativity property associated with auxiliary data.

**Proof** The proof smoothly follows from the non-negativity properties that appear on the right-hand side of expressions (3.11)–(3.15). This fact, along with the non-negative initial conditions, yields the required proof. \( \square \)

Next we furnish the matrix representation of the designed implicit scheme (3.16)–(3.20). System (3.16)–(3.20) can be arranged as

\[ AS^{n+1} = L, \quad (3.32) \]
\[ BX^{n+1} = M, \quad (3.33) \]
\[ CX^{n+1} = N, \quad (3.34) \]
\[ DX^{n+1} = O, \quad (3.35) \]
\[ EX^{n+1} = P. \quad (3.36) \]

Here \( A, B, C, D, E \) are square matrices of dimension \((N + 1) \times (N + 1)\); \( L, M, N, O, P \) are block matrices:

\[
A = \begin{pmatrix}
  a_3 & a_1 & 0 & 0 & 0 \\
  a_2 & a_3 & a_2 & \cdots & 0 & 0 & 0 \\
  0 & a_2 & a_3 & 0 & 0 & 0 & 0 \\
  \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & a_3 & a_2 & 0 \\
  0 & 0 & 0 & \cdots & a_2 & a_3 & a_2 \\
  0 & 0 & 0 & 0 & a_1 & a_3 \\
\end{pmatrix},
\]
In these matrices the entries are

\[ a_1 = -2R_1, \quad a_2 = R_1, \quad a_3 = (1 + 2R_1 + \tau \beta h \left( \frac{C}{\alpha h} \right) I_{\alpha h} + \tau \mu h), \]

\[ b_1 = -2R_2, \quad b_2 = -R_2, \quad b_3 = 1 + 2R_2 + \tau (\mu h + \alpha h), \]

\[ c_1 = -2R_3, \quad c_2 = -R_3, \quad c_3 = 1 + 2R_3 + \tau (r + \mu h), \]

\[ d_1 = -2R_4, \quad d_2 = -R_4, \quad d_3 = 1 + 2R_4 + \tau (\alpha_v + \mu_v), \]

\[ e_1 = 2R_5, \quad e_2 = -R_5, \quad e_3 = 1 + 2R_5 + \tau \mu v. \]

**Definition 1** A real matrix is called an $M$-matrix, if it is

- A square matrix with strictly dominant diagonal;
- Diagonal entries are positive;
- Off-diagonal entries are non-positive.

**Lemma 2** The matrices $A, B, C, D$ and $E$ possess all the properties of $M$-matrices.

**Proof** At the initial stage, observe that all $R_1, R_2, R_3, R_4$ and $R_5$ are positive. It may be noted further that $A, B, C, D$ and $E$ are strictly diagonally-dominant matrices. The other properties of $M$-matrices are also fulfilled by the entries of the matrices $A, B, C, D$ and

\[
B = \begin{pmatrix}
 b_3 & b_1 & 0 & 0 & 0 \\
 b_2 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\
 0 & b_2 & b_3 & \cdots & 0 & 0 & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 0 & 0 & 0 & b_3 & b_2 & 0 \\
 0 & 0 & 0 & \cdots & b_2 & b_3 & a_2 \\
 0 & 0 & 0 & 0 & b_1 & b_3
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
 c_3 & c_1 & 0 & 0 & 0 \\
 c_2 & c_3 & c_2 & \cdots & 0 & 0 & 0 \\
 0 & c_2 & c_3 & \cdots & 0 & 0 & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 0 & 0 & 0 & c_3 & c_2 & 0 \\
 0 & 0 & 0 & \cdots & c_2 & c_3 & c_2 \\
 0 & 0 & 0 & 0 & c_1 & c_3
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
 d_3 & d_1 & 0 & 0 & 0 \\
 d_2 & d_3 & d_2 & \cdots & 0 & 0 & 0 \\
 0 & d_2 & d_3 & \cdots & 0 & 0 & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 0 & 0 & 0 & d_3 & d_2 & 0 \\
 0 & 0 & 0 & \cdots & d_2 & d_3 & a_2 \\
 0 & 0 & 0 & 0 & d_1 & d_3
\end{pmatrix},
\]

\[
E = \begin{pmatrix}
 e_3 & e_1 & 0 & 0 & 0 \\
 e_2 & e_3 & e_2 & \cdots & 0 & 0 & 0 \\
 0 & e_2 & e_3 & \cdots & 0 & 0 & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 0 & 0 & 0 & e_3 & e_2 & 0 \\
 0 & 0 & 0 & \cdots & e_2 & e_3 & e_2 \\
 0 & 0 & 0 & 0 & e_1 & e_3
\end{pmatrix}.
\]
$E$ as their diagonal and off-diagonal entries satisfy the definition of the $M$-matrix. Thus ultimately, we arrive at the logical result.

\[\square\]

**Theorem 1** ([21]) For every positive $h$ and $\tau$, the system has positive solution, i.e. $S^n$, $X^n$, $I^n$, $X_v^n$ and $I_v^n$ are positive for all $n \in \{0, 1, 2, \ldots\}$.

For detailed proof of the above theorem, we refer to [21].

4 Experiment and simulations

The parametric values [1, 11] used in this experiment are presented in Table 1.

4.1 Experiment

In the experiment, we take the following initial conditions:

- $S(x, 0) = 0.1$,
- $X(x, 0) = 0.0001$,
- $I(x, 0) = 0.0001$,
- $X_v(x, 0) = 0.001$,
- $I_v(x, 0) = 0.001$.

The values of the diffusion coefficients in this experiment are $d_S = d_X = d_I = d_{X_v} = d_{I_v} = 0.0001$.

4.1.1 Disease-free equilibrium

Now we present simulations for DFE (disease-free equilibrium) using all methods. For the DFE, we use $C = 3$ ($R_0 < 1$). DFE graph for the forward Euler explicit FD scheme is presented in Fig. 1, and DFE graph for Crank–Nicolson implicit FD scheme is presented in Fig. 2.

Figures 1–2 represent the graphs of exposed population by using the forward Euler and Crank–Nicolson methods. The graphs clearly show that both methods fail to retain positive solution, which is the main feature of the continuous model as we deal with the population dynamics. Now we present the graphs of DFE using the proposed implicit scheme in Figs. 3–7.

Figures 3–7 clearly show that the proposed implicit scheme converges to disease-free equilibrium point $E_0(1, 0, 0, 0, 0)$ and preserves the positivity property.

Next the simulations by using the proposed explicit positive scheme are presented at same equilibrium point as above.

| Table 1 | Parametric values |
|---------|-------------------|
| Parameters | DFE values | Endemic values |
| $N_T$ | 5000 | 5000 |
| $\alpha_T$ | $1/5$ | $1/5$ |
| $\beta_T$ | 0.00005 | 0.00005 |
| $\mu_{T}$ | 0.0000391 | 0.0000391 |
| $\alpha_v$ | $1/10$ | $1/10$ |
| $\mu_v$ | $1/14$ | $1/14$ |
| $r$ | $1/14$ | $1/14$ |
| $C$ | 3 | 300 |
Figure 1  Graph representing the exposed humans for DFE implementing forward Euler approach with $h = 0.1$, $R_2 = 0.0800007$

Figure 2  Graph representing the exposed humans for DFE using Crank–Nicolson approach with $h = 0.1$, $R_2 = 0.0800007$

Figure 3  Graph representing the infected humans for DFE using the proposed implicit approach with $h = 0.1$, $R_3 = 0.0800007$
Figure 4 Graph representing the exposed vectors for DFE using the proposed implicit approach with $h = 0.1$, $R_4 = 0.0800007$

Figure 5 Graph representing the infected vectors for DFE using the proposed implicit approach with $h = 0.1$, $R_5 = 0.0800007$

Figure 6 Graph representing the exposed vectors for DFE using the proposed implicit approach with $h = 0.1$, $R_4 = 0.0800007$
Figures 7–12 represent the graphs using the proposed implicit approach. These graphs show the disease-free equilibrium. Graphs clearly show that the proposed FD scheme converges to the disease-free equilibrium point $E_0(1, 0, 0, 0, 0)$ and preserves the positivity property.

**4.1.2 Endemic equilibrium**

The graphs of EE (endemic equilibrium) are presented using four finite difference schemes: forward Euler FD scheme, Crank–Nicolson FD scheme, the proposed implicit scheme and the proposed positive explicit FD scheme.

For the endemic equilibrium, we use $C = 300$ ($R_0 > 1$).

Figures 13–14 represent the graphical behavior of susceptible individuals at EE for the forward Euler and Crank–Nicolson methods. Graphs clearly show that both schemes demonstrate nonphysical behavior and do not converge to the EE.

Figures 15–24 represent the graphs of the endemic point by implementing the proposed implicit numerical method and the proposed FD scheme. Graphs clearly show that the
Figure 9  Graph representing the exposed humans for DFE using the proposed FD scheme with $h = 0.1$, $R_2 = 0.0800007$

Figure 10  Graph representing the infected humans for DFE using the proposed FD scheme with $h = 0.1$, $R_3 = 0.0800007$

Figure 11  Graph representing the exposed vectors for DFE using the proposed explicit approach with $h = 0.1$, $R_4 = 0.0800007$
Figure 12  Graph representing the infected vectors for DFE using the proposed explicit approach with $h = 0.1$, $R_0 = 0.0800007$

Figure 13  Graph representing the susceptible humans for EE implementing the forward Euler approach with $h = 0.1$, $R_1 = 0.0800007$

Figure 14  Graph representing the susceptible humans for EE using Crank–Nicolson approach with $h = 0.1$, $R_1 = 0.0800007$
Figure 15  Graph representing the infected humans for EE using the proposed implicit approach with $h = 0.1$, $R_3 = 0.0800007$

Figure 16  Graph representing the exposed vectors for EE using the proposed implicit approach with $h = 0.1$, $R_4 = 0.0800007$

Figure 17  Graph representing the infected vectors for EE using the proposed implicit approach with $h = 0.1$, $R_5 = 0.0800007$
Figure 18  Graph representing the exposed vectors for EE using the proposed implicit approach with $h = 0.1$, $R_4 = 0.0800007$

Figure 19  Graph representing the infected vectors for EE using the proposed implicit approach with $h = 0.1$, $R_5 = 0.0800007$

Figure 20  Graph representing the susceptible humans for EE using the proposed explicit approach with $h = 0.1$, $R_1 = 0.0800007$
Figure 21  Graph representing the exposed humans for EE using the proposed explicit approach with $h = 0.1$, $R_2 = 0.0800007$

Figure 22  Graph representing the infected humans for EE using the proposed explicit approach with $h = 0.1$, $R_3 = 0.0800007$

Figure 23  Graph representing the exposed vectors for EE using the proposed explicit approach with $h = 0.1$, $R_4 = 0.0800007$
proposed schemes converge to the endemic equilibrium point $E_1(S^*, X^*, I^*, X_v^*, I_v^*)$ and sustain the positive solution.

Now we present the time efficiency at disease-free equilibrium point which is discussed in Table 2.

Table 2 demonstrates that the positive explicit method is time efficient as compared to the positive implicit method. The implicit method takes more than double execution time than the explicit method. This efficiency is shown in the reaction–diffusion model in one space dimension. As far as the complicated situation of two and three space dimensions is concerned, the implication of the proposed implicit scheme is very difficult. We have to observe the long-term behaviour of such a model, therefore time efficiency is crucial for the numerical scheme in a multidimensional space; for the details, see [29].

5 Conclusion

In this paper, we proposed two positive FD schemes to solve a reaction–diffusion dengue epidemic model with incubation period of the virus. We used four FD schemes to solve numerically the reaction–diffusion dengue epidemic model. These schemes were the forward Euler FD scheme, Crank–Nicolson scheme, the proposed implicit FD and the proposed explicit FD schemes. Both existing schemes fail to preserve the positivity property, show nonphysical behavior and converge to false steady states, whereas the proposed FD schemes converge towards the true steady states of the continuous model. The proposed FD schemes are unconditionally dynamically consistent with the positivity property, which is necessary as negative values of a subpopulation are meaningless. Simulations of a test problem were presented in this paper. These simulations show that the proposed implicit and explicit FD schemes converge to all the steady states of the system and preserve
the positivity property. The proposed explicit method is time efficient as compared to the implicit method. In the future the proposed explicit scheme will be an important tool to solve many other infectious disease reaction–diffusion mathematical models in multiple space dimensions because of its time efficiency. Our future plans include spatio-temporal numerical analysis of a stochastic dengue epidemic model [31] and fractional order dynamical systems [32, 33].

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Author details
1 Department of Mathematics, University of Management and Technology, Lahore, Pakistan. 2 Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan. 3 Faculty of Engineering, University of Central Punjab, Lahore, Pakistan. 4 Department of Mathematics, Cankaya University, Ankara, Turkey. 5 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan. 6 Institute of Space Sciences, Magurele-Bucharest, Romania. 7 Faculty of Science, Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia.

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