DISCRETE HARMONIC ANALYSIS ASSOCIATED WITH ULTRASPHERICAL EXPANSIONS

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Abstract. In this paper we study discrete harmonic analysis associated with ultraspherical orthogonal functions. We establish weighted $\ell^p$-boundedness properties of maximal operators and Littlewood-Paley $g$-functions defined by Poisson and heat semigroups generated by the difference operator

$$\Delta_\lambda f(n) := a_\lambda^*(n) f(n+1) - 2 f(n) + a_{\lambda-1}^*(n-1), \quad n \in \mathbb{N}, \lambda > 0,$$

where $a_\lambda^* := \{(2\lambda+n)(n+1)/(\lambda(n+1)+\lambda)\}^{1/2}, n \in \mathbb{N},$ and $a_{\lambda-1}^* := 0.$ We also prove weighted $\ell^p$-boundedness properties of transplantation operators associated with the system $\{\varphi_n^\lambda\}_{n \in \mathbb{N}}$ of ultraspherical functions, a family of eigenfunctions of $\Delta_\lambda.$ In order to show our results we previously establish a vector-valued local Calderón-Zygmund theorem in our discrete setting.

1. Introduction

The study of harmonic analysis in discrete settings has attracted considerable attention in the last years. For instance, harmonic analysis on graphs has been studied in [8], [9], [34], [43], and [44], and it has been considered on discrete groups in [18], [19], [21], [23], [33], [38], and [49]. Also, celebrated mathematicians have investigated discrete analogues of Euclidean harmonic analysis problems where the underlying real field $\mathbb{R}$ is replaced by the ring of integers $\mathbb{Z}.$ In this discrete context the exponential sums play the role of oscillatory integrals in the Euclidean setting. Some of these problems are studied in [14], [15], [30], [31], [47], [48] and [49].

As far as we know Titchmarsh ([53]) was the first one who investigated the $\ell^p$-bounded properties of discrete harmonic analysis operators (see also [41]). For every $1 \leq p \leq \infty$ we denote as usual by $\ell^p(\mathbb{Z})$ the space that consists of all those complex sequences $(a_n)_{n \in \mathbb{Z}}$ such that $\| (a_n)_{n \in \mathbb{Z}} \|_{\ell^p(\mathbb{Z})} < \infty,$ where

$$\| (a_n)_{n \in \mathbb{Z}} \|_{\ell^p(\mathbb{Z})} := \begin{cases} \left( \sum_{n \in \mathbb{Z}} |a_n|^p \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{n \in \mathbb{Z}} |a_n|, & p = \infty. \end{cases}$$

The convolution operation on $\mathbb{C}^\mathbb{N}$ associated to the usual group operation on $\mathbb{Z}$ is defined as follows: if $a = (a_n)_{n \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z},$ and $b = (b_n)_{n \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z},$ the convolution $a \ast b \in \mathbb{C}^\mathbb{Z}$ is given by

$$(a \ast b)_n = \sum_{m \in \mathbb{Z}} a_m b_{n-m}, \quad n \in \mathbb{Z},$$

provided that the last sum converges for every $n \in \mathbb{Z}.$ As it is wellknown, the Young’s inequality holds for $\ast$ in $\ell^p(\mathbb{Z})$ spaces. The discrete Hilbert transform $\mathcal{H}_\mathbb{Z}$ was defined in [53] as the $\ast$-convolution operator with the kernel $k = (k(n))_{n \in \mathbb{Z}},$ where $k(n) := (\pi(n+1/2))^{-1}, n \in \mathbb{Z},$ and the convergence of the series is understood as a principal value, i.e., as the limit of partial sums from $-M$ to $M.$ $\mathcal{H}_\mathbb{Z}$ is bounded from $\ell^p(\mathbb{Z})$ into itself, for every $1 < p < \infty.$ As in the continuous case, when $p = 1$ the situation is different. The discrete Hilbert transform is bounded from $\ell^1(\mathbb{Z})$ into $\ell^{1,\infty}(\mathbb{Z})$ where $\ell^{1,\infty}(\mathbb{Z})$ denotes the weak-$\ell^1$ space (see [17] Proposition 5 and Corollary 2]). As the transference results show the $L^p$-boundedness properties of continuous and their discrete analogues operators are closely connected (see, for instance, [7] and [15]).

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The discrete Hilbert transform also adopts other forms. It is usual to consider the operator $\mathcal{H}_Z$ defined by

$\mathcal{H}_Z(f)(n) := \frac{1}{\pi} \sum_{m \in \mathbb{Z}, m \neq n} \frac{f(m)}{n-m}, \quad f \in \ell^p(\mathbb{Z}), \ 1 \leq p < \infty,$

(see [3], [4] and [29]).

The operators $\mathcal{H}_Z$ and $\mathcal{M}_Z$ map $\ell^2(\mathbb{Z})$ into itself with norm 1 ([25] and [53]). Laeng ([53]) has investigated the norms of discrete Hilbert transforms as bounded operators in $\ell^p(\mathbb{Z}), 1 < p < \infty$.

Recently, Ciaurri et al. ([17]) studied discrete harmonic analysis operators related to the discrete Laplacian $\Delta_Z$ defined by

$$(\Delta_Z f)(n) := f(n+1) - 2f(n) + f(n-1), \quad n \in \mathbb{Z},$$

for every $f = (f(n))_{n \in \mathbb{Z}} \in C^2$. By using heat and Poisson semigroups associated with $\Delta_Z$ they defined maximal operators, fractional powers of $\Delta_Z$, Littlewood-Paley $g$-functions and Riesz transforms. The discrete Hilbert transform $\mathcal{H}_Z$ appears as a Riesz transform. $\ell^p$-boundedness properties of those operators are studied in [17] by employing scalar and vector-valued Calderón-Zygmund theory ([42]).

Motivated by [17], in this paper we develop a discrete harmonic analysis associated with ultraspherical expansions.

Assume that $\lambda > 0$. For every $n \in \mathbb{N}$ we consider the $n$-th ultraspherical polynomial of order $\lambda$ defined by ([51], §4.7)]

$$\varphi_n^\lambda(x) := \frac{(-1)^n}{2^n(\lambda + 1/2)n} (1 - x^2)^{\lambda/2} \frac{d^n}{dx^n}(1 - x^2)^{\lambda - 1/2}, \quad x \in [-1, 1].$$

(2)

Here $(a)_n = a(a+1) \cdots (a+n-1)$, for each $a > 0$, $n \in \mathbb{N}$. We have that, for every $n, m \in \mathbb{N}$,

$$\int_{-1}^1 \varphi_n^\lambda(x) \varphi_m^\lambda(x)(1 - x^2)^{\lambda - 1/2}dx = \delta_{n,m} w_\lambda(n),$$

where $\delta_{n,m}$ represents the Kronecker’s delta and

$$w_\lambda(n) := \frac{\Gamma(\lambda)(2\lambda)_n(n + \lambda)}{\sqrt{\pi} \Gamma(\lambda + 1/2)n!}.$$ 

By taking into account that, for certain $C \geq 1$,

$$\frac{1}{C}(m + 1)^{\alpha - 1} \leq \frac{(\alpha)_m}{m!} \leq C(m + 1)^{\alpha - 1}, \quad \alpha > 0 \text{ and } m \in \mathbb{N},$$

we get

$$\frac{1}{C}(n + 1)^{2\lambda} \leq w_\lambda(n) \leq C(n + 1)^{2\lambda}, \quad n \in \mathbb{N}.$$

We consider for each $n \in \mathbb{N}$ the ultraspherical function $\varphi_n^\lambda$ defined by

$$\varphi_n^\lambda(x) := \sqrt{w_\lambda(n)} \varphi_n^\lambda(x)(1 - x^2)^{\lambda - 1/4}, \quad x \in (-1, 1).$$

The sequence $\{\varphi_n^\lambda\}_{n \in \mathbb{N}}$ is an orthonormal basis in $L^2(-1, 1)$.

According to [31] (4.7.17), we have that

$$2x\varphi_n^\lambda(x) = a_n^\lambda \varphi_{n+1}^\lambda(x) + a_{n-1}^\lambda \varphi_n^{\lambda}(x), \quad x \in (-1, 1) \text{ and } n \in \mathbb{N},$$

where

$$a_n^\lambda := \sqrt{(2\lambda + n)(n + 1)} \sqrt{(n + \lambda)(n + 1 + \lambda)}, \quad n \in \mathbb{N}.$$ 

Here and in the sequel $\varphi_{-1}^\lambda := 0$ and $a_{-1}^\lambda := 0$.

We consider the $\lambda$-Laplacian operator given by

$$(\Delta_\lambda f)(n) := a_n^\lambda f(n+1) - 2f(n) + a_{n-1}^\lambda f(n-1), \quad n \in \mathbb{N}, \quad f = (f(n))_{n \in \mathbb{N}} \in C^\mathbb{N}.$$ 

Note that $\Delta_\lambda$ reduces to the discrete Laplacian $\Delta$ on $\mathbb{N}$ in the end point $\lambda = 0$.

For every $1 \leq p \leq \infty$ we denote by $\ell^p(\mathbb{N})$ the space constituted by all those complex sequences $(a_n)_{n \in \mathbb{N}}$ such that $\|(a_n)_{n \in \mathbb{N}}\|_{\ell^p(\mathbb{N})} < \infty$, where $\| \cdot \|_{\ell^p(\mathbb{N})}$ is naturally defined as in [1] with $\mathbb{Z}$ replaced by $\mathbb{N}$.

The operator $\Delta_\lambda$ is selfadjoint in $\ell^2(\mathbb{N})$ and bounded in $\ell^p(\mathbb{N})$, for every $1 \leq p \leq \infty$. 


We define the λ-transform \( \mathcal{F}_\lambda(f) \) of \( f = (f(n))_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}) \), by
\[
\mathcal{F}_\lambda(f) = \sum_{n=0}^{\infty} f(n) \varphi_n^\lambda.
\]

Parseval’s equality says that \( \mathcal{F}_\lambda \) is an isometry from \( \ell^2(\mathbb{N}) \) into \( L^2(-1,1) \).

From (5) we deduce that, for every \( f \in \ell^2(\mathbb{N}) \),
\[
(6) \quad \mathcal{F}_\lambda(\Delta f)(x) = 2(x-1)\mathcal{F}_\lambda(f)(x), \quad x \in (-1,1).
\]

By using again Parseval’s equality we obtain
\[
\sum_{n=0}^{\infty} (\Delta f)(n) f(n) = \int_{-1}^{1} 2(x-1) |\mathcal{F}_\lambda(f)(x)|^2 dx \leq 0, \quad f = (f(n))_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}).
\]

Thus, we show that \( -\Delta \) is a positive operator in \( \ell^2(\mathbb{N}) \).

Since \( \Delta \) is bounded in \( \ell^p(\mathbb{N}) \), \( 1 \leq p \leq \infty \), we have that \( \Delta \) generates a semigroup of operators \( \{W^\lambda_t := e^{t\Delta} \}_{t>0} \) in \( \ell^p(\mathbb{N}) \), \( 1 \leq p \leq \infty \), such that
\[
\lim_{t \to 0^+} e^{t\Delta} f = f, \quad \text{for every } f \in \ell^p(\mathbb{N}).
\]

We can obtain an expression for \( W^\lambda_t \), \( t > 0 \), in terms of a convolution operation \( \#_\lambda \) that is well adapted to our discrete ultraspHERical setting. This \( \#_\lambda \) convolution is a modification of the one introduced in [27].

If \( f = (f(m))_{m \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \) and \( n \in \mathbb{N} \) the \( \lambda \)-translated \( \lambda \tau_n f \) is defined by
\[
(7) \quad (\lambda \tau_n f)(m) := \sum_{k=0}^{\infty} c_\lambda(n,m,k)f(k), \quad m \in \mathbb{N},
\]

where, for every \( m,k \in \mathbb{N} \),
\[
c_\lambda(n,m,k) := \int_{-1}^{1} \phi_n^\lambda(x)\varphi_m^\lambda(x)\varphi_k^\lambda(x)(1-x^2)^{1/4-\lambda/2} dx.
\]

According to [28] (1.1) we have that
\[
(8) \quad c_\lambda(n,m,k) = \sqrt{w_\lambda(n)w_\lambda(m)w_\lambda(k)} \frac{n!m!k!}{(2\lambda)_n(2\lambda)_m(2\lambda)_k} \frac{(\lambda)_{\sigma-n}(\lambda)_{\sigma-m}(\lambda)_{\sigma-k} \Gamma(\sigma+2\lambda)}{(\sigma-n)!(\sigma-m)!(\sigma-k)! \Gamma(\sigma+\lambda+1)}
\]

when \( n,m,k \in \mathbb{N} \), \( |n-m| \leq k \leq n+m \) and \( n+m+k = 2\sigma \), for some \( \sigma \in \mathbb{N} \). Otherwise \( c_\lambda(n,m,k) = 0 \). Note that the series in (7) is actually a finite sum.

Moreover, by using (3) and (4) we have that, there exists \( C > 0 \) such that, for each \( n,m,k \in \mathbb{N} \), such that \( n+m+k = 2\sigma \), with \( \sigma \in \mathbb{N} \), and \( |n-m| \leq k \leq n+m \),
\[
\frac{1}{C} \left( \frac{\sigma(\sigma-n+1)(\sigma-m+1)(\sigma-k+1)}{(n+1)(m+1)(k+1)} \right)^{\lambda-1}
\]

\[
\leq c_\lambda(n,m,k) \leq C \left( \frac{\sigma(\sigma-n+1)(\sigma-m+1)(\sigma-k+1)}{(n+1)(m+1)(k+1)} \right)^{\lambda-1}.
\]

We remark that there is not a group operation \( \circ \) on \( \mathbb{N} \) such that \( (\lambda \tau_n f)(m) = f(n \circ m^{-1}) \), for every \( n,m \in \mathbb{N} \), where \( m^{-1} \) represents the inverse of \( m \) with respect to \( \circ \).

If \( f = (f(n))_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \) and \( g = (g(n))_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \), the \( \#_\lambda \)-convolution \( f \#_\lambda g \) of \( f \) and \( g \) is defined by
\[
(f \#_\lambda g)(n) := \sum_{m=0}^{\infty} f(m)(\lambda \tau_n g)(m), \quad n \in \mathbb{N},
\]

provided that the last series converges.

The triple \( (\mathbb{N}, \mu, \#_\lambda) \), is an hypergroup ([13]), where \( \mu \) is the measure \( \sum_{n \in \mathbb{N}} \delta_n \), and \( \delta_n \), \( n \in \mathbb{N} \), is the point mass probability measure supported on \( n \). From [27] (11) we deduce that
\[
(9) \quad \mathcal{F}_\lambda(f \#_\lambda g)(x) = (1-x^2)^{-\lambda/2+1/4} \mathcal{F}_\lambda(f)(x)\mathcal{F}_\lambda(g)(x), \quad x \in (-1,1),
\]

for every \( f,g \in \mathbb{C}^\mathbb{N} \) such that \( \sqrt{w_\lambda} f \in \ell^1(\mathbb{N}) \) and \( \sqrt{w_\lambda} g \in \ell^1(\mathbb{N}) \).

Analytic continuation and [22] 10.9 (38) allow us to write, for every \( n \in \mathbb{N} \),
\[
(10) \quad \int_{-1}^{1} e^{tx} \varphi_n^\lambda(x)(1-x^2)^{\lambda/2-1/4} dx = 2^\lambda \sqrt{\pi} \Gamma\left( \lambda + \frac{1}{2} \right) \sqrt{w_\lambda(n)} t^{-\lambda} I_{\lambda+n}(t), \quad t > 0.
\]
Here, \( I_\nu \) denotes the modified Bessel function of the first kind and order \( \nu \). By taking into account \( (6) \) and \( (11) \) we obtain that, for every \( t > 0 \),

\[
W^\lambda_t(f) = h^\lambda_t \# f, \quad f \in \ell^p(\mathbb{N}), \ 1 \leq p \leq \infty,
\]

where

\[
h^\lambda_t(n) := \sqrt{\pi} \Gamma \left( \lambda + \frac{1}{2} \right) \sqrt{w_\lambda(n)} e^{-2t^{\lambda} I_{\lambda+n}(2t)}, \quad n \in \mathbb{N}.
\]

By using the subordination formula, the Poisson semigroup \( \{ P^\lambda_t \}_{t > 0} \) associated with \( \Delta_\lambda \) (generated by \( -\Delta_\lambda \)) is defined by

\[
P^\lambda_t(f)(n) := \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-n u} W^\lambda_t(f)(n) du, \quad n \in \mathbb{N}, \ f \in \ell^p(\mathbb{N}), \ 1 \leq p \leq \infty.
\]

The semigroups \( \{ W^\lambda_t \}_{t > 0} \) and \( \{ P^\lambda_t \}_{t > 0} \) are not Markovian, that is, they do not map constants into constants (see Section 3).

From now on, let \( \{ T^\lambda_t \}_{t > 0} \) represents the heat or the Poisson semigroup associated with \( \Delta_\lambda \). If \( k \in \mathbb{N} \setminus \{ 0 \} \), we define the Littlewood-Paley functions \( g^k_{T^\lambda} \) of \( k \)-th order by

\[
g^k_{T^\lambda}(f)(n) := \left( \int_0^\infty |t^k \partial_t^k (T^\lambda_t f)(n)|^2 \frac{dt}{t} \right)^{1/2}, \quad n \in \mathbb{N},
\]

and the maximal operator \( T^\lambda \) by

\[
T^\lambda f := \sup_{t > 0} |T^\lambda_t f|.
\]

Suppose that \( w = (w(n))_{n \in \mathbb{N}} \in (0, \infty)^{\mathbb{N}} \). We say that \( w \in A_p(\mathbb{N}) \) provided that

\[
\sup_{0 \leq m \leq n, n \in \mathbb{N}} \left( \sum_{k=n}^{m} w(k) \right)^p \left( \sum_{k=n}^{m} w(k)^{-1/(p-1)} \right)^{p-1} < \infty, \quad \text{when} \ 1 < p < \infty,
\]

and

\[
\sup_{0 \leq m \leq n, n \in \mathbb{N}} \left( \sum_{k=n}^{m} w(k) \right) \max_{n \leq k \leq m} \frac{1}{w(k)} < \infty, \quad \text{when} \ p = 1.
\]

For every \( 1 \leq p < \infty \) and \( w \in A_p(\mathbb{N}) \) we denote by \( \ell^p(\mathbb{N}, w) \) and \( \ell^\infty(\mathbb{N}, w) \) the usual weighted and weak weighted \( \ell^p \) space, respectively.

By using vector-valued local Calderón-Zygmund theory, we establish the \( \ell^p \)-boundedness properties for our discrete \( g \)-functions and maximal operators.

**Theorem 1.1.** Let \( \lambda > 0 \) and \( k \in \mathbb{N} \setminus \{ 0 \} \). Then,

(i) If \( 1 < p < \infty \) and \( w \in A_p(\mathbb{N}) \), \( g^k_{T^\lambda} \) and \( T^\lambda \) are bounded from \( \ell^p(\mathbb{N}, w) \) into itself.

(ii) If \( w \in A_1(\mathbb{N}) \), \( g^k_{W^\lambda} \), \( g^k_{P^\lambda} \), and \( T^\lambda \) are bounded from \( \ell^1(\mathbb{N}, w) \) into \( \ell^{1,\infty}(\mathbb{N}, w) \).

Let \( 1 \leq p \leq \infty \). Since \( \Delta_\lambda \) is bounded in \( \ell^p(\mathbb{N}) \), for every \( f \in \ell^p(\mathbb{N}) \), \( f = \lim_{t \to 0^+} W^\lambda_t f \) in \( \ell^p(\mathbb{N}) \). Hence, for every \( f \in \ell^p(\mathbb{N}) \),

\[
f(n) = \lim_{t \to 0^+} (W^\lambda_t f)(n), \quad n \in \mathbb{N}.
\]

Subordination formula \( (10) \) allows us to obtain the same convergence properties for the Poisson semigroup \( \{ P^\lambda_t \}_{t > 0} \).

From Theorem 1.1 and by using density arguments we get the following.

**Corollary 1.1.** Let \( \lambda > 0 \), \( 1 \leq p < \infty \) and \( w \in A_p(\mathbb{N}) \). Then, for every \( f \in \ell^p(\mathbb{N}, w) \),

\[
f(n) = \lim_{t \to 0^+} (T^\lambda_t f)(n), \quad n \in \mathbb{N}.
\]

Moreover, if \( 1 < p < \infty \), \( f = \lim_{t \to 0^+} T^\lambda_t f \), in \( \ell^p(\mathbb{N}, w) \).

As a consequence of Theorem 1.1 we can see that the Littlewood-Paley functions \( g^k_{W^\lambda} \) and \( g^k_{P^\lambda} \), \( k \in \mathbb{N} \setminus \{ 0 \} \), define equivalent norms in \( \ell^p(\mathbb{N}, w) \), \( 1 < p < \infty \) and \( w \in A_p(\mathbb{N}) \).

**Proposition 1.1.** Let \( \lambda > 0 \), \( k \in \mathbb{N} \setminus \{ 0 \} \), \( 1 < p < \infty \) and \( w \in A_p(\mathbb{N}) \). Then, there exists \( C > 0 \) such that, for every \( f \in \ell^p(\mathbb{N}, w) \),

\[
\frac{1}{C} \| f \|_{\ell^p(\mathbb{N}, w)} \leq \| g^k_{W^\lambda}(f) \|_{\ell^p(\mathbb{N}, w)} \leq C \| f \|_{\ell^p(\mathbb{N}, w)}.
\]
In [6] Askey and Wainger established a transplantation theorem for ultraspHERical coefficients (see [6] Theorem 1). They obtained weighted $\ell^p$-inequalities with power $A_p(N)$-weights for $1 < p < \infty$. By using discrete local Calderón-Zygmund theory we extend [6] Theorem 1 to general $A_p(N)$-weights for $1 < p < \infty$ and also we get results for $p = 1$.

Suppose that $\lambda, \mu > 0$. We consider the operator $\mathcal{T}_{\lambda, \mu}$ defined by $\mathcal{T}_{\lambda, \mu} = \mathcal{T}_{\lambda}^{-1}$ on $\ell^p(N)$. $\ell^p$-boundedness properties of the operator $\mathcal{T}_{\lambda, \mu}$ are established in the following.

**Theorem 1.2.** Let $\lambda, \mu > 1$. If $1 < p < \infty$ and $w \in A_p(N)$, then there exists $C > 0$ such that

$$
\frac{1}{C} \|f\|_{\ell^p(N, w)} \leq \|\mathcal{T}_{\lambda, \mu}(f)\|_{\ell^p(N, w)} \leq C\|f\|_{\ell^p(N, w)}, \quad f \in \ell^p(N, w).
$$

If $(\mu - 1)/2 < \lambda < \mu$ and $w \in A_1(N)$, the operator $\mathcal{T}_{\lambda, \mu}$ is bounded from $\ell^1(N, w)$ into $\ell^{1, \infty}(N, w)$.

$\mathcal{T}_{\lambda, \mu}$ is really a transplantation operator and Theorem 1.2 extends [6] Theorem 1. Indeed, let $1 < q < \infty$, $v$ a weight in the Muckenhoupt class $A_q(-1, 1)$ and $F \in L^q((-1, 1), v)$. According to [32] Theorem 2 (see also the proof of [2] Proposition 2.2), we have that

$$
\mathcal{S}_\lambda^n(F) = \sum_{k=0}^n c_k^\lambda(F) \varphi_k^\lambda \rightharpoonup F, \quad \text{as } n \to \infty,
$$

in $L^q((-1, 1), v)$. Here, for every $k \in \mathbb{N}$,

$$
c_k^\lambda(F) := \int_{-1}^1 F(x) \varphi_k^\lambda(x) dx.
$$

Since $|\mathcal{P}_k^\lambda|(x) \leq 1$, $k \in \mathbb{N}$ and $\gamma > 0$, (51 Theorem 7.33.1), it follows that

$$
c_m^\lambda(F) = \lim_{n \to \infty} \int_{-1}^1 \mathcal{S}_\lambda^n(F)(x) \varphi_m^\lambda(x) dx = \lim_{n \to \infty} \sum_{k=0}^n c_k^\lambda(F) \int_{-1}^1 \varphi_k^\lambda(x) \varphi_m^\lambda(x) dx = \mathcal{T}_{\lambda, \mu}(f)(m), \quad m \in \mathbb{N},
$$

where $f(k) = c_k^\lambda(F)$, $k \in \mathbb{N}$.

From Theorem 1.2 we deduce the following generalization of [6] Theorem 1.

**Corollary 1.2.** Let $\lambda, \mu > 1$. Assume that $1 < p, q < \infty$, $w \in A_p(N)$ and $v \in A_q(-1, 1)$. Then, there exists $C > 0$ such that, for every $F \in L^q((-1, 1), v)$,

$$
\frac{1}{C} \|\mathcal{S}_\lambda^n(F)\|_{\ell^p(N, w)} \leq \|\mathcal{P}_k^\lambda(F)\|_{\ell^p(N, w)} \leq C \|\mathcal{S}_\lambda^n(F)\|_{\ell^p(N, w)}.
$$

The transplantation operator can also be seen as an extension of Riesz transform operators (see, for instance, [52]).

In Sections 3 and 4 we present proofs of our results. In Section 2 we establish a discrete vector-valued local Calderón-Zygmund theorem that will be very useful to prove Theorems 1.1 and 1.2.

Throughout this paper by $C$ we always denote a positive constant that can change in each occurrence.

2. DISCRETE VECTOR-VALUED LOCAL CALDERÓN-ZYGMUND OPERATORS

Nowak and Stempak [40] developed the so called local Calderón-Zygmund theory that allows us to treat singular integrals on $(0, \infty)$. They used it to obtain $L^p$-boundedness properties for transplantation operators in the Bessel settings ([40] Proposition 4.2]). Banach valued singular integral operators were investigated by Rubio de Francia, Ruiz and Torrea [42] (see also [10]). Recently, Grafakos, Liu and Yang [26] have established a Banach valued version of Calderón-Zygmund theory for singular integrals on spaces of homogeneous type.

In order to show Theorem 1.1 we need to establish the following result that is a local version of [26] Theorem 1.1 for Banach valued Calderón-Zygmund operators on the space $(\mathbb{N}, \mu, | \cdot |)$ of homogeneous type. Here, $\mu$ as above denotes the measure $\mu = \sum_{n \in \mathbb{N}} \delta_n$ on $\mathbb{N}$, and $| \cdot |$ represents the usual metric on $\mathbb{N}$. Next result can also be seen as a discrete version of [40] Proposition 4.2].

Suppose that $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces. By $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ we denote the space of bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$. Assume also that the function

$$
K : (\mathbb{N} \times \mathbb{N}) \setminus D \longrightarrow \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2),
$$

where $D := \{(n, n) : n \in \mathbb{N}\}$, is measurable, and that for certain $C > 0$ the following conditions are satisfied, for each $n, m, \ell \in \mathbb{N}$, $n \neq m$:
(a) the size condition:
\[ ||K(n,m)||_{L^p(B_1,B_2)} \leq \frac{C}{|n-m|}, \]

(b) the regularity properties:
\[ (b1) \quad ||K(n,m) - K(\ell,m)||_{L^p(B_1,B_2)} \leq C \frac{|n-\ell|}{|n-m|^2}, \quad |n-m| > 2|n-\ell|, \quad \frac{m}{2} < n, \ell < \frac{3m}{2}. \]
\[ (b2) \quad ||K(n,m) - K(m,\ell)||_{L^p(B_1,B_2)} \leq C \frac{|n-\ell|}{|n-m|^2}, \quad |n-m| > 2|n-\ell|, \quad \frac{m}{2} < n, \ell < \frac{3m}{2}. \]

We say that $K$ is a local $L^p(B_1,B_2)$-standard kernel when the above conditions are satisfied.

Theorem 2.1. Let $B_1$ and $B_2$ be Banach spaces. Suppose that $T$ is a linear and bounded operator from $\ell^r_{B_1}(\mathbb{N})$ into $\ell^r_{B_2}(\mathbb{N})$, for some $1 < r < \infty$, and such that there exists a local $L^p(B_1,B_2)$-standard kernel $K$ such that, for every sequence $f \in (B_1)^\infty$,
\[ T(f)(n) = \sum_{m \in \mathbb{N}} K(n,m)f(m), \]
for every $n \in \mathbb{N}$ such that $f(n) = 0$. Then,

(i) for every $1 < p < \infty$ and $w \in A_p(\mathbb{N})$ the operator $T$ can be extended from $\ell^r_{B_1}(\mathbb{N}) \cap \ell^r_{B_2}(\mathbb{N},w)$ to $\ell^p_{B_1}(\mathbb{N},w)$ as a bounded operator from $\ell^p_{B_1}(\mathbb{N},w)$ into $\ell^p_{B_2}(\mathbb{N},w)$.

(ii) for every $w \in A_1(\mathbb{N})$, the operator $T$ can be extended from $\ell^r_{B_1}(\mathbb{N}) \cap \ell^r_{B_2}(\mathbb{N},w)$ to $\ell^r_{B_1}(\mathbb{N},w)$ as a bounded operator from $\ell^r_{B_1}(\mathbb{N},w)$ into $\ell^r_{B_2}(\mathbb{N},w)$.

Proof. For every $n \in \mathbb{N}$, we define $W_n := \{ m \in \mathbb{N} : n/2 \leq m \leq 3n/2 \}$ and the operators
\[ T_{glob}(f)(n) := T(\chi_{W_n}f)(n), \quad n \in \mathbb{N}, \]

and
\[ T_{loc}(f) := T(f) - T_{glob}(f), \]
for every $f = (f(n))_{n \in \mathbb{N}} \in (B_1)^\infty$. Since $\chi_{\mathbb{N}\setminus W_n}(n) = 0$, $n \in \mathbb{N}$, we can write
\[ T_{glob}(f)(n) = \sum_{m \in \mathbb{N}\setminus W_n} K(n,m)f(m), \quad n \in \mathbb{N}. \]

According to condition (a) for $K$ we get
\[ ||T_{glob}(f)(n)||_{\ell^r_{B_2}} \leq C \sum_{m \in \mathbb{N}\setminus W_n} \frac{||f(m)||_{B_1}}{|n-m|} \leq C \left( \frac{1}{n} \sum_{m \in \mathbb{N}, m < n/2} ||f(m)||_{B_1} + \sum_{m \in \mathbb{N}, m > 3n/2} \frac{||f(m)||_{B_2}}{m} \right) \]
\[ \leq C \left( H_0(||f||_{B_1})(n) + H_\infty(||f||_{B_2})(n) \right), \quad n \in \mathbb{N}, \]
where $||f||_{B_1} := (||f||_{B_1})_{m \in \mathbb{N}}$, and $H_0$, and $H_\infty$ are the discrete Hardy operators given by
\[ H_0(g)(n) := \frac{1}{n} \sum_{m = 0}^{n} g(m), \quad n \in \mathbb{N} \setminus \{0\}, \]
and
\[ H_\infty(g)(n) := \sum_{m = n}^{\infty} \frac{g(m)}{m}, \quad n \in \mathbb{N} \setminus \{0\}, \]
with $g = (g(m))_{m \in \mathbb{N}} \in \ell^1(\mathbb{N})$. Wellknown $\ell^p$-boundedness properties for discrete Hardy operators allow us to conclude that if $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, $T_{glob}$ can be extended to $\ell^p_{B_1}(\mathbb{N},w)$ as a bounded operator from $\ell^p_{B_1}(\mathbb{N},w)$ into $\ell^p_{B_2}(\mathbb{N},w)$ and if $w \in A_1(\mathbb{N})$, $T_{glob}$ can be extended to $\ell^r_{B_1}(\mathbb{N},w)$ as a bounded operator from $\ell^r_{B_1}(\mathbb{N},w)$ into $\ell^r_{B_2}(\mathbb{N},w)$.

We now study the operator $T_{loc}$. If $f = (f(n))_{n \in \mathbb{N}} \in (B_1)^\infty$, we can write
\[ T_{loc}(f)(n) = \sum_{m \in W_n} K(n,m)f(m), \quad n \in \mathbb{N} \text{ such that } f(n) = 0. \]

We define the function $\tilde{K}$ by
\[ \tilde{K}(n,m) := \chi_{W_n}(m)K(n,m), \quad n, m \in \mathbb{N}, \quad n \neq m. \]
We are going to see that \( \tilde{K} \) satisfies certain Hörmander type conditions that can be seen as discrete analogues of \([10]\) (4.4) and (4.5)].

If \( a, b \in \mathbb{N} \), \( a < b \), and \( I \) is the interval in \( \mathbb{N} \) given by \( I := [a, b] \cap \mathbb{N} \), we denote by \( 2I \) the set
\[
2I := \left[ a - \frac{b - a}{2}, b + \frac{b - a}{2} \right] \cap \mathbb{N}.
\]

By \( \mathcal{M} \) we represent the noncentered Hardy-Littlewood maximal function on \( \mathbb{N} \) defined as follows: for every \( g = (g(m))_{m \in \mathbb{N}} \in \ell^1(\mathbb{N}) \),
\[
\mathcal{M}(g)(n) := \sup_{\text{interval } I} \frac{1}{\#(I)} \sum_{m \in I} g(m), \quad n \in \mathbb{N},
\]
where \( \#(I) \) denotes the cardinal of \( I \).

We assert that the function \( \tilde{K} \) satisfies the following Hörmander conditions: there exists \( C > 0 \) such that, for every interval \( I \) in \( \mathbb{N} \) and \( f = (f(m))_{m \in \mathbb{N}} \in \ell^1(\mathbb{N}) \),
\[
\sum_{m \in \mathbb{N} \setminus 2I} \| \tilde{K}(n, m) - \tilde{K}(\ell, m) \|_{L^\infty(\mathbb{N}^2)} \| f(m) \|_{\ell^1} \leq C \mathcal{M}(\| f \|_{\ell^1})(n), \quad n, \ell \in I,
\]
and
\[
\sum_{m \in \mathbb{N} \setminus 2I} \| \tilde{K}(m, n) - \tilde{K}(m, \ell) \|_{L^\infty(\mathbb{N}^2)} \| f(m) \|_{\ell^1} \leq C \mathcal{M}(\| f \|_{\ell^1})(n), \quad n, \ell \in I.
\]

Since \([16]\) and \([17]\) can be proved similarly, we only show \([16]\). Let \( a, b \in \mathbb{N} \), \( a < b \), \( I := [a, b] \cap \mathbb{N} \) and \( f = (f(m))_{m \in \mathbb{N}} \in \ell^1(\mathbb{N}) \). Suppose that \( n, \ell \in I \), and \( n < \ell \).

First we observe that when \( m \in \mathbb{N} \setminus 2I \), then,
\[
\frac{|m - n|}{3} \leq |m - \ell| \leq 3|m - n|.
\]

To see this estimate, let us write \( 2I = [A, B] \cap \mathbb{N} \) and \( L = (b - a)/2 \), and take \( m \in \mathbb{N} \setminus 2I \). In the case that \( m > B \), we get
\[
m - n > m - \ell > m - b = m - B + L = m - B + \frac{b - a}{3} \geq \frac{m - a}{3} \geq \frac{m - n}{3}.
\]

Similarly, when \( m < A \), we obtain that
\[
\ell - m > n - m > L + A - m = \frac{b - A}{3} + A - m \geq \frac{b - m}{3} \geq \frac{\ell - m}{3},
\]
and \([18]\) is established.

We can decompose the left hand side in \([16]\) as follows:
\[
\sum_{m \in \mathbb{N} \setminus 2I} \| \tilde{K}(n, m) - \tilde{K}(\ell, m) \|_{L^\infty(\mathbb{N}^2)} \| f(m) \|_{\ell^1} = \sum_{m \in \mathbb{N} \setminus 2I} \| K(n, m) - K(\ell, m) \|_{L^\infty(\mathbb{N}^2)} \| f(m) \|_{\ell^1} + \sum_{m \in \mathbb{N} \setminus 2I} \| K(\ell, m) \|_{L^\infty(\mathbb{N}^2)} \| f(m) \|_{\ell^1} + \sum_{m \in \mathbb{N} \setminus 2I} \| K(n, m) \|_{L^\infty(\mathbb{N}^2)} \| f(m) \|_{\ell^1}
\]
\[
:= S_1(n, \ell) + S_2(n, \ell) + S_3(n, \ell).
\]

By taking into account the size condition of \( K \) and \([18]\) we have that
\[
S_2(n, \ell) + S_3(n, \ell) \leq C \left( \sum_{m \in \mathbb{N} \setminus 2I} \| f(m) \|_{\ell^1} \right) \left( \sum_{m \in \mathbb{N} \setminus 2I} \| f(m) \|_{\ell^1} \right).
\]

Assume now that \( 3n < \ell \). In this case we have that \( W_n \cap W_\ell = \emptyset \) and then \( S_1(n, \ell) = 0 \). Also, since for \( m \in \mathbb{N} \setminus 2I \), \( |m - n| > (b - a)/2 \geq (\ell - n)/2 \geq \ell/3 \geq n \), we get
\[
S_2(n, \ell) + S_3(n, \ell) \leq C \left( \frac{1}{n} \sum_{m \in W_n} \| f(m) \|_{\ell^1} + \frac{1}{\ell} \sum_{m \in W_\ell} \| f(m) \|_{\ell^1} \right).
\]

Assume now that \( 3n < \ell \). In this case we have that \( W_n \cap W_\ell = \emptyset \) and then \( S_1(n, \ell) = 0 \). Also, since for \( m \in \mathbb{N} \setminus 2I \), \( |m - n| > (b - a)/2 \geq (\ell - n)/2 \geq \ell/3 > n \), we get
\[
S_2(n, \ell) + S_3(n, \ell) \leq C \left( \frac{1}{n} \sum_{m \in W_n} \| f(m) \|_{\ell^1} + \frac{1}{\ell} \sum_{m \in W_\ell} \| f(m) \|_{\ell^1} \right).
\]
where \( J := [n, 3\ell/2] \cap \mathbb{N} \). By considering that \( 3\ell/2 \geq 3\ell/2 - n > 3\ell/2 - \ell/3 = 7\ell/6 \), we conclude that

\[(20) \quad S_2(n, \ell) + S_3(n, \ell) \leq C.M(\|f\|_{\mathcal{B}_1}(n)).\]

Next we deal with the condition \( \ell \leq 3n \). Now we have that

\[
W_n \cap W_\ell = \left( \frac{\ell}{2}, \frac{3n}{2} \right) \cap \mathbb{N} = \left( \left( \frac{\ell}{2}, \frac{2\ell}{3} \right) \cap \mathbb{N} \right) \cup \left( \left( \frac{2\ell}{3}, \frac{3n}{2} \right) \cap \mathbb{N} \right) =: J_1 \cup J_2,
\]

\[
W_n \setminus W_\ell = \left( \frac{n}{2}, \frac{\ell}{2} \right) \cap \mathbb{N},
\]

and we can write

\[
S_1(n, \ell) + S_2(n, \ell) + S_3(n, \ell) \leq \sum_{m \in J_2 \cap \mathbb{N} \setminus \{2I\}} \|K(n, m) - K(\ell, m)\|_{\mathbf{X}(\mathcal{B}_1, \mathcal{B}_2)} \|f(m)\|_{\mathcal{B}_1} + \left( \sum_{m \in J_2 \cap \mathbb{N} \setminus \{2I\}} + \sum_{m \in J_2 \cap \mathbb{N} \setminus \{2I\}} \right) \|f(m)\|_{\mathcal{B}_1} |n - m| =: T_1(n, \ell) + T_2(n, \ell).
\]

In order to estimate \( T_1(n, \ell) \), we decompose it into two terms as follows:

\[
T_1(n, \ell) = \left( \sum_{m \in J_2 \cap \mathbb{N} \setminus \{2I\}} + \sum_{m \in J_2 \cap \mathbb{N} \setminus \{2I\}} \right) \|K(n, m) - K(\ell, m)\|_{\mathbf{X}(\mathcal{B}_1, \mathcal{B}_2)} \|f(m)\|_{\mathcal{B}_1} |n - m| =: T_{1,1}(n, \ell) + T_{1,2}(n, \ell).
\]

Then, according to the size condition (a) and (18) we get

\[
T_{1,1}(n, \ell) \leq C \sum_{m \in \mathbb{N} \setminus \{2I\}} \frac{|n - \ell|}{|n - m|^2} \|f(m)\|_{\mathcal{B}_1} \leq C \sum_{m \in \mathbb{N} \setminus \{2I\}} \frac{|n - \ell|}{|n - m|^2} \|f(m)\|_{\mathcal{B}_1},
\]

and by taking into account the regularity property (b1) for \( K \) we deduce the same estimate for \( T_{1,2}(n, \ell) \). Hence,

\[
T_1(n, \ell) \leq C \sum_{m \in \mathbb{N} \setminus \{2I\}} \frac{|n - \ell|}{|n - m|^2} \|f(m)\|_{\mathcal{B}_1} \leq C \sum_{k=1}^{\infty} \sum_{m \in 2^{k+1} \setminus 2^{k}} \frac{|n - \ell|}{|n - m|^2} \|f(m)\|_{\mathcal{B}_1} \leq C \sum_{k=1}^{\infty} \sum_{m \in 2^{k+1} \setminus 2^{k}} \|f(m)\|_{\mathcal{B}_1} \leq C.M(\|f\|_{\mathcal{B}_1}(n)).
\]

We analyze now \( T_2(n, \ell) \). It is clear that \( m + n > n/2 \), when \( m > 3n/2 \). We can also establish that if \( m \in \mathbb{N} \setminus \{2I\} \) and \( n/2 \leq m \leq 2\ell/3 \), then \( |m - n| > \ell/3 \). Let \( m \in \mathbb{N} \setminus \{2I\} \) such that \( n/2 \leq m \leq 2\ell/3 \). If \( \ell < 5n/4 \), then, \( |n - m| > 5\ell/6 \), and \( n - m > n/6 \). When \( \ell \geq 5n/4 \), we get that \( |n - m| > (b - a)/2 > (5n/4 - n)/2 = n/8 \) and the result is established.

Then, since \( \ell \leq 3n \), we can write

\[
T_2(n, \ell) \leq C \left( \sum_{m \in \mathbb{N} \setminus \{2n, 2n/2\} \cap \mathbb{N}} + \sum_{m \in \mathbb{N} \setminus \{2n, 3n/2\} \cap \mathbb{N}} \right) \|f(m)\|_{\mathcal{B}_1} \leq C.M(\|f\|_{\mathcal{B}_1}).
\]

By combining (19) and (23) we establish (16).

Since \( T_{\text{glob}}^2 \) and \( T_{\text{glob}}^1 \) are bounded from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}) \) into \( \ell_{\mathcal{B}_2}^1(\mathbb{N}) \), \( T_{\text{loc}} \) is also bounded from \( \ell_{\mathcal{B}_1}^1(\mathbb{N}) \) into \( \ell_{\mathcal{B}_2}^3(\mathbb{N}) \). Hence, according to (26) Theorem 1.1, for every \( 1 < p < \infty \), \( T_{\text{loc}} \) can be extended from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}) \) and \( \ell_{\mathcal{B}_1}^3(\mathbb{N}) \) to \( \ell_{\mathcal{B}_2}^1(\mathbb{N}) \) as a bounded operator from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}) \) into \( \ell_{\mathcal{B}_2}^1(\mathbb{N}) \) and \( T_{\text{loc}} \) can be extended from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}) \) to \( \ell_{\mathcal{B}_2}^1(\mathbb{N}) \) as a bounded operator from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}) \) into \( \ell_{\mathcal{B}_2}^3(\mathbb{N}) \). The same properties are satisfied by \( T \) because \( T_{\text{glob}}^2 \) also verifies them.

Finally, by adapting the arguments in (23) Lemmas 5.15, 7.9 and 7.10 and Theorems 7.11 and 7.12] to vector-valued homogeneous settings, we conclude that \( T_{\text{loc}} \), and then \( T \), can be extended from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}, w) \) to \( \ell_{\mathcal{B}_2}^1(\mathbb{N}, w) \) as a bounded operator from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}, w) \) into \( \ell_{\mathcal{B}_2}^1(\mathbb{N}, w) \), for every \( 1 < p < \infty \) and \( w \in A_p(\mathbb{N}) \), and from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}, w) \) to \( \ell_{\mathcal{B}_1}^3(\mathbb{N}, w) \) as a bounded operator from \( \ell_{\mathcal{B}_1}^2(\mathbb{N}, w) \) into \( \ell_{\mathcal{B}_2}^3(\mathbb{N}, w) \), for every \( w \in A_1(\mathbb{N}) \).

Thus the proof of this theorem is completed. \( \Box \)
3. PROOF OF THEOREM 1.1 FOR MAXIMAL OPERATORS

In this section we prove the boundedness properties for the maximal operators associated with heat and Poisson semigroups stated in Theorem 1.1. From (14) we deduce that \( P_\lambda^f(f) \leq W_\lambda^f(f) \). Hence, the properties in Theorem 1.1 for \( P_\lambda \) can be inferred from the corresponding properties for \( W_\lambda \). Hence, it is sufficient to prove Theorem 1.1 for \( W_\lambda \).

Let \( 1 \leq p \leq \infty \). Since \( \Delta_\lambda \) is a bounded operator from \( \ell^p(\mathbb{N}) \) into itself, \( \Delta_\lambda \) generates the \( C_0 \)-semigroup \( \{ W_\lambda^t := e^{t\Delta_\lambda} \}_{t \geq 0} \) of operators in \( \ell^p(\mathbb{N}) \). We have that

\[
\partial_t W_\lambda^t(f) = \Delta_\lambda W_\lambda^t(f), \quad f \in \ell^p(\mathbb{N}) \quad \text{and} \quad t > 0.
\]

Moreover, \( \{ W_\lambda^t \}_{t \geq 0} \) is not Markovian. Indeed, let \( g = (g(n) = 1)_{n \in \mathbb{N}} \). If \( W_\lambda^t(g) = g, \ t > 0, \) then, by (24) we have that \( \Delta_\lambda g = 0 \), and that is clearly impossible.

According to (10), (11) and (12) we can write, for every \( f = (f(n))_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}) \),

\[
\sum_{n \in \mathbb{N}} |W_\lambda^t(f)(n)|^2 = \int_{-1}^1 |\mathcal{F}_\lambda(W_\lambda^t f)(x)|^2 dx = \int_{-1}^1 |e^{-2(t(1-x))} \mathcal{F}_\lambda(f)(x)|^2 dx
\]

\[
\leq \int_{-1}^1 |\mathcal{F}_\lambda(f)(x)|^2 dx = \sum_{n \in \mathbb{N}} |f(n)|^2, \quad t > 0.
\]

Hence, for every \( t > 0 \), \( W_\lambda^t \) is contractive in \( \ell^2(\mathbb{N}) \). We now prove that the maximal operator \( W_\lambda^* \) is bounded in \( \ell^2(\mathbb{N}) \). By proceeding as in [16] p. 75 we obtain

\[
W_\lambda^t(f) \leq M^\lambda(f) + g_\lambda(f), \quad f \in \ell^2(\mathbb{N}),
\]

where

\[
g_\lambda(f)(n) := \left( \int_0^\infty |t \partial_t(W_\lambda^t f)(n)|^2 dt \right)^{1/2}, \quad n \in \mathbb{N},
\]

and

\[
M^\lambda(f)(n) := \sup_{s > 0} \frac{1}{s} \left| \int_0^s W_\lambda^t(f)(n) dt \right|, \quad n \in \mathbb{N}.
\]

The \( g \)-function \( g_\lambda \) is bounded from \( \ell^2(\mathbb{N}) \) into itself. This can be seen by using spectral arguments (see [16] p. 74]). Since \( W_\lambda^t \) is a contraction in \( \ell^2(\mathbb{N}) \), for every \( t > 0 \), the Hopf-Dunford-Schwartz ergodic theorem (see [15]) allows us to show that the maximal operator \( M^\lambda \) is bounded from \( \ell^2(\mathbb{N}) \) into itself. Hence, \( W_\lambda^* \) is bounded from \( \ell^2(\mathbb{N}) \) into itself.

In order to show \( \ell^p \)-properties of \( W_\lambda \) for \( 1 \leq p < \infty, \ p \neq 2 \), we use Theorem 2.1. We consider the operator

\[
T : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})
\]

\[
f \mapsto T(f)(n; t) := W_\lambda^t(f)(n),
\]

where \( B = L^\infty(0, \infty) \). The operator \( T \) is bounded from \( \ell^2(\mathbb{N}) \) into \( \ell^2(\mathbb{N}) \), because \( W_\lambda^t \) is bounded from \( \ell^2(\mathbb{N}) \) into itself. According to (12), for every \( f \in \mathcal{C}_0^\infty \) we can write

\[
T(f)(n; t) = (h_\lambda^t \# \chi f)(n) = \sum_{m=0}^N f(m) \chi_{\tau_n(h_\lambda^t)}(m), \quad n \in \mathbb{N} \quad \text{and} \quad t > 0,
\]

where \( N \in \mathbb{N} \) is such that \( f(n) = 0, \ n \geq N \).

To simplify we define \( K_\lambda^t(n, m) := \chi_{\tau_n(h_\lambda^t)}(m), \ n, m \in \mathbb{N} \) and \( t > 0 \). By (7) we have that

\[
K_\lambda^t(n, m) \leq \sum_{k=|n-m|}^{n+m} c_{\lambda}(n, m, k) h_\lambda^t(k), \quad m, n \in \mathbb{N},
\]

where \( c_{\lambda}(n, m, k), m, n \in \mathbb{N}, \ k \in \mathbb{N} \), and \( h_\lambda^t, t > 0 \), are given by (8) and (13), respectively.

We are going to see that there exists \( C > 0 \) such that, for every \( n, m, \ell \in \mathbb{N}, \ n \neq m, \)

\[
\|K_\lambda^t(n, m)\|_B \leq \frac{C}{|n - m|};
\]

\[
\|K_\lambda^t(n, m) - K_\lambda^t(\ell, m)\|_B \leq C \frac{|n - \ell|}{|n - m|^2}, \quad |n - m| > 2|n - \ell|, \quad \frac{m}{2} < n, \ell < \frac{3m}{2};
\]

\[
\|K_\lambda^t(n, m) - K_\lambda^t(m, \ell)\|_B \leq C \frac{|n - \ell|}{|n - m|^2}, \quad |n - m| > 2|n - \ell|, \quad \frac{m}{2} < n, \ell < \frac{3m}{2}.
\]
When properties (26), (27) and (28) are established then, from Theorem 2.1 we deduce the \( r \)-boundedness properties stated in Theorem 1.1 for \( W^r_\lambda \). Observe that, since \( K_\lambda^r(n, m) = K_\lambda^r(m, n) \), \( n, m \in \mathbb{N}, t > 0 \), we only have to establish (26) and (27).

**Proof of (26).** We will use the following integral representations for the modified Bessel function \( I_\nu \) ([37] (5.10.22))

\[
I_\nu(z) = \frac{z^\nu}{\sqrt{2\nu \Gamma(\nu + 1/2)}} \int_{-1}^{1} e^{-zs}(1 - s^2)^{\nu - 1/2} ds, \quad z > 0 \text{ and } \nu > \frac{1}{2},
\]

and

\[
I_\nu(z) = -\frac{z^{\nu-1}}{\sqrt{2\nu - 1 \Gamma(\nu - 1/2)}} \int_{-1}^{1} e^{-zs}(1 - s^2)^{\nu - 3/2} ds, \quad z > 0 \text{ and } \nu > \frac{1}{2}.
\]

Note that (30) is obtained by partial integration in (29) (see [17] (34)). We have that

\[
0 \leq e^{-2t} \lambda^{-\lambda} I_{\lambda+k}(2t) \leq \frac{C}{(k+1)^{2\lambda+1}}, \quad k \in \mathbb{N} \text{ and } t > 0,
\]

where \( C > 0 \) does not depend on \( k \in \mathbb{N} \) nor on \( t > 0 \).

To see this estimation we proceed as in [17] Proposition 3. For every \( k \in \mathbb{N} \) and \( t > 0 \), we can write

\[
0 \leq e^{-2t} t^{-\lambda} I_{\lambda+k}(2t) = \frac{t^k}{\sqrt{\pi} \Gamma(\lambda + k + 1/2)} \int_{-1}^{1} e^{-2t(\lambda+1)} (1 - s^2)^{\lambda+k-1/2} ds
\]

\[
= \frac{2t^k}{\sqrt{\pi} \Gamma(\lambda + k + 1/2)} \int_{0}^{t} e^{-2w} \left( \frac{2w}{t} \right)^{\lambda+k-1/2} \left[ 2 \left( 1 - \frac{w}{t} \right) \right]^{\lambda+k-1/2} dw
\]

\[
\leq \frac{2^{2\lambda+k}}{\sqrt{\pi} \Gamma(\lambda + k + 1/2)} \int_{0}^{t} e^{-2w} \left[ \left( \frac{w}{t} \right)^{\lambda+1/2} \left( 1 - \frac{w}{t} \right) \right]^{\lambda+k-1/2} w^{\lambda-1} dw
\]

\[
\leq \frac{2^{2\lambda+k}}{\sqrt{\pi} \Gamma(\lambda + k + 1/2)} \int_{0}^{t} e^{-2w} w^{\lambda-1} dw
\]

\[
\leq \frac{2^{2\lambda+k}}{\sqrt{\pi} \Gamma(\lambda + k + 1/2)} \Gamma(k) \left( \frac{\lambda+1/2}{2\lambda+k} \right)^{\lambda+1/2} \frac{\Gamma(k)}{\Gamma(\lambda + k + 1/2)(2\lambda+k)^{\lambda+1/2}} \leq \frac{C}{(k+1)^{2\lambda+1}}.
\]

We have taken into account that

\[
(1 - r)^{\eta} \gamma \leq \left( \frac{\gamma}{\gamma + \eta} \right)^{\gamma}, \quad 0 < r < 1, \quad \eta > 0 \text{ and } \gamma > 0.
\]

By using (4) and (31) we obtain

\[
0 \leq K_\lambda^r(n, m) \leq C \sum_{k=0}^{\infty} c_\lambda(n, m, k) \frac{1}{(k+1)^{\lambda+1}}, \quad n, m \in \mathbb{N}, n \neq m \text{ and } t > 0.
\]

Since \( c_\lambda(n, m, k) = c_\lambda(m, n, k) \), \( n, m, k \in \mathbb{N} \), Lemma 3b] says that, if \( 0 < \alpha < 1/2 \), then

\[
\sum_{k=0}^{\infty} c_\lambda(n, m, k) \frac{1}{(k+1)^{\lambda+2\alpha+1}} \leq \frac{C}{|n - m|^{\lambda+2\alpha+1}}, \quad n, m \in \mathbb{N}, n \neq m.
\]

The same proof of [5] Lemma 3b] allows us to see that (33) also holds for \( \alpha = 0 \) and \( \alpha = 1/2 \).

Hence,

\[
\|K_\lambda^r(n, m)\|_\infty \leq \frac{C}{|n - m|}, \quad n, m \in \mathbb{N}, n \neq m.
\]

**Proof of (27).** In a first step we show the following estimations:

(A1) \[ \|K_\lambda^r(n + 1, m) - K_\lambda^r(n, m)\|_\infty \leq \frac{C}{|n - m|^2}, \quad n, m \in \mathbb{N}, n > m; \]

(A2) \[ \|K_\lambda^r(n, m) - K_\lambda^r(n - 1, m)\|_\infty \leq \frac{C}{|n - m|^2}, \quad n, m \in \mathbb{N}, 1 \leq n < m \leq 2n. \]
Let \( n, m \in \mathbb{N} \) and \( n > m \). We can write

\[
K_t^\lambda(n+1, m) - K_t^\lambda(n, m) = \sum_{k=n-m}^{n+m+1} c_\lambda(n+1, m, k) h_t^\lambda(k) - \sum_{k=n-m}^{n+m} c_\lambda(n, m, k) h_t^\lambda(k)
\]

\[
= \sum_{k=n-m}^{n+m} \left[ c_\lambda(n+1, m, k+1) h_t^\lambda(k+1) - c_\lambda(n, m, k) h_t^\lambda(k) \right]
\]

\[
= \sum_{k=n-m}^{n+m} c_\lambda(n+1, m, k+1) \left[ h_t^\lambda(k+1) - \frac{\sqrt{w_\lambda(k+1)}}{\sqrt{w_\lambda(k)}} h_t^\lambda(k) \right]
\]

\[
+ \sum_{k=n-m}^{n+m} \left[ \frac{\sqrt{w_\lambda(k+1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n+1, m, k+1) - c_\lambda(n, m, k) \right] h_t^\lambda(k)
\]

(34)

By combining (29) and (30) (see [17] p. 9) we get, for every \( t > 0 \) and \( k \in \mathbb{N} \),

\[
e^{-2t}e^{-\lambda} |I_{\lambda+k+1}(2t) - I_{\lambda+k}(2t)| \leq \frac{2^{2(\lambda+k)+1}}{\sqrt{\pi \Gamma(\lambda + k + 1/2)}} \int_{-1}^{1} e^{-2t} (1 + s) (1 - s)^{\lambda+k-1/2} ds
\]

\[
= \frac{2^{2(\lambda+k)+1}}{\sqrt{\pi \Gamma(\lambda + k + 1/2)}} \int_{0}^{1} e^{-4w} \frac{(w)}{(w+1)} (1 - \frac{w}{\lambda+k+1/2}) \leq C \frac{\Gamma(k+1)}{(2\lambda+1)^{\lambda+1/2}}
\]

(35)

According to (36) and \( |I_2(n, m, t)| \) for \( \alpha = 1/2 \), it follows that

\[
|I_1(n, m, t)| \leq C \sum_{k=n-m}^{n+m} c_\lambda(n, m, k, \lambda^{\alpha+1}) \leq C \frac{C}{|n-m|^2}, \quad t > 0.
\]

We now analyze \( I_2(n, m, t), t > 0 \). Note that, if \( k \in \mathbb{N} \), \( c_\lambda(n, m, k) \neq 0 \) if and only if \( c_\lambda(n+1, m, k+1) \neq 0 \). By using (31) and (31), we can write

\[
|I_2(n, m, t)| \leq \sum_{n-m}^{n+m} c_\lambda(n, m, k) \left[ \sqrt{w_\lambda(k+1)} c_\lambda(n+1, m, k+1) \right] - 1 \left| h_t^\lambda(k) \right|
\]

\[
\leq \sum_{n-m}^{n+m} c_\lambda(n, m, k) \left[ \sqrt{w_\lambda(k+1)} c_\lambda(n+1, m, k+1) \right] - 1 \left| h_t^\lambda(k) \right|
\]

Let \( k \in \mathbb{N}, n - m \leq k \leq n + m \). Assume that \( n + m + k = 2\sigma \), with \( \sigma \in \mathbb{N} \). Straightforward manipulations lead to

\[
\sqrt{w_\lambda(k+1)} c_\lambda(n+1, m, k+1) = \left( n + \frac{1}{n + 2\lambda} \right)^{1/2} \left( n + \lambda + 1 \right)^{1/2} \left( n + \lambda + 1 \sigma - m + \lambda + \sigma + 2\lambda \right) \left( n + \lambda + 1 \right)^{1/2} \left( n + \lambda + 1 \sigma - m + \lambda + \sigma + 2\lambda \right)
\]

To simplify we define

\[
a_1 := \sqrt{\frac{n + 1}{n + 2\lambda}}, \quad a_2 := \sqrt{\frac{n + \lambda + 1}{n + \lambda}}, \quad a_3 := \frac{k + \lambda + 1}{k + \lambda}, \quad a_4 := \frac{\sigma - m + \lambda}{\sigma - m + 1}, \quad a_5 := \frac{\sigma + 2\lambda}{\sigma + \lambda + 1}.
\]

There exists \( C > 0 \) independent on \( n, m \) and \( k \), such that \( 0 \leq a_j \leq C, j = 1, ..., 5 \). Also we get

\[
\left( \prod_{j=1}^{5} a_j \right) - 1 = \sum_{j=1}^{5} (a_j - 1) \prod_{i=j+1}^{5} a_i.
\]
We deduce that
\begin{equation}
|\frac{\sqrt{w_k}(k+1)c_k(n+1,m,k+1)}{\sqrt{w_k}(k)c_k(n,m,k)} - 1| \leq C \sum_{j=1}^{5} |\alpha_j - 1|.
\end{equation}

Since
\[|\alpha_j - 1| \leq \frac{C}{n+1}, \quad j = 1, 2, \quad |\alpha_3 - 1| \leq \frac{C}{k+1}, \quad |\alpha_4 - 1| \leq \frac{C}{\sigma - m}, \quad \text{and} \quad |\alpha_5 - 1| \leq \frac{C}{\sigma},\]
we have that
\[|\alpha_j - 1| \leq \frac{C}{n-m}, \quad j = 1, ..., 5.
\]

By using (4), (31) and (33) (for \(\alpha = 0\)) we obtain
\begin{equation}
|I_2(n,m,t)| \leq \frac{C}{n-m} \sum_{k=n-m}^{n+m} c_k(n,m,k) \frac{(k+1)^{\lambda+1}}{|n-m|^2}, \quad t > 0,
\end{equation}

According to (34), (36) and (38) we deduce property (A1). Next we justify (A2). Suppose that \(1 \leq n < m \leq 2n\). We have that
\[K^\lambda_k(n,m) - K^\lambda_k(n-1,m) = \sum_{m-n}^{m+n} c_k(n,m,k)h^\lambda_k(m) - \sum_{k=m-n+1}^{n+m-1} c_k(n-1,m,k)h^\lambda_k(k), \quad t > 0.
\]

In order to make the estimations easier, we assemble the terms in the sums in a suitable form. We observe that the first sum has two terms more than the second one, so we proceed as follows:
\begin{align*}
K^\lambda_k(n,m) - K^\lambda_k(n-1,m) &= \left(\sum_{k=m-n+1}^{m-n} + \sum_{k=m+2}^{m+n} \right) c_k(n,m,k)h^\lambda_k(k) \\
&+ c_k(n,m,m)h^\lambda_k(m) + c_k(n,m,m+1)h^\lambda_k(m+1) - \sum_{k=m-n+1}^{n+m-1} c_k(n-1,m,k)h^\lambda_k(k) \\
&= \sum_{k=m-n+1}^{m-n} c_k(n,m,k-1)h^\lambda_k(k-1) + \sum_{k=m+1}^{n+m-1} c_k(n,m,k+1)h^\lambda_k(k+1) \\
&+ c_k(n,m,m)h^\lambda_k(m) + c_k(n,m,m+1)h^\lambda_k(m+1) - \sum_{k=m-n+1}^{n+m-1} c_k(n-1,m,k)h^\lambda_k(k) \\
&= \sum_{k=m-n+1}^{m-n} [c_k(n,m,k-1)h^\lambda_k(k-1) - c_k(n-1,m,k)h^\lambda_k(k)] \\
&+ \sum_{k=m+1}^{m+n-1} [c_k(n,m,k+1)h^\lambda_k(k+1) - c_k(n-1,m,k)h^\lambda_k(k)] \\
&+ c_k(n,m,m)h^\lambda_k(m) + c_k(n,m,m+1)h^\lambda_k(m+1) \\
&= J_1(n,m,t) + J_2(n,m,t) + J_3(n,m,t), \quad t > 0.
\end{align*}

We first study \(J_1(n,m,t), t > 0\). Note that \(c_k(n,m,m) \neq 0\) if and only if \(c_k(n,m,m+1) = 0\). Suppose that \(n + 2m = 2\sigma\), with \(\sigma \in \mathbb{N}\). By using (4) and (31) and we obtain
\[c_k(n,m,m)h^\lambda_k(m) \leq C \left(\frac{\sigma(\sigma - n + 1)(\sigma - m + 1)^2}{(n+1)(m+1)^2} \right)^{\lambda-1} \frac{1}{(m+1)^{\lambda+1}} \leq C \frac{[(n+2m)(2m-n)(n+1)]^{\lambda-1}}{(m+1)^{3\lambda-1}}, \quad t > 0.
\]

Since \(n < m \leq 2n\), we have that \(n < 2m - n < 3\), and we get
\[J_1(n,m,t) \leq \frac{C}{(m+1)^2}, \quad t > 0.
\]

In a similar way if \(n + 1 + 2m = 2\sigma\), with \(\sigma \in \mathbb{N}\) then
\[c_k(n,m,m+1)h^\lambda_k(m+1) \leq \frac{C}{(m+1)^2}, \quad t > 0.
\]
Thus, we have obtained that

\begin{equation}
J_3(n, m, t) \leq \frac{C}{(m+1)^2} \leq \frac{C}{|n-m|^2}, \quad t > 0.
\end{equation}

To analyze \(J_1\) we decompose it as follows:

\[
J_1(n, m, t) = \sum_{k=m-n+1}^{m} c_\lambda(n, m, k, 1) \left[ h_\lambda^k(k-1) - \frac{\sqrt{w_\lambda(k-1)}}{\sqrt{w_\lambda(k)}} h_\lambda^k(k) \right] + \sum_{k=m-n+1}^{m} \left[ \frac{\sqrt{w_\lambda(k-1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n, m, k, 1) - c_\lambda(n, m, m, 1) \right] h_\lambda^k(k) =: J_{1,1}(n, m, t) + J_{1,2}(n, m, t), \quad t > 0.
\]

By (4) and (35) and taking into account (33) (for \(\alpha = 1/2\)) we deduce

\[
|J_{1,1}(n, m, t)| \leq \sum_{k=m-n+1}^{m} c_\lambda(n, m, k, 1) \left| \frac{\sqrt{w_\lambda(k-1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n, m, m, 1) - 1 \right| h_\lambda^k(k) \leq C \sum_{k=0}^{\infty} c_\lambda(n, m, k, 1) \leq \frac{C}{|n-m|^2}, \quad t > 0.
\]

On the other hand, by (3) and (31) we can write

\[
|J_{1,2}(n, m, t)| \leq \sum_{m-n+1 \leq k \leq m, n+m+k-1 \text{ even}} c_\lambda(n-1, m, k, 1) \left| \frac{\sqrt{w_\lambda(k-1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n-1, m, 1) - 1 \right| h_\lambda^k(k) \leq C \sum_{k=0}^{\infty} c_\lambda(n-1, m, k, 1) \leq \frac{C}{|n-m|^2}, \quad t > 0.
\]

Let \(k \in \mathbb{N}, m-n+1 \leq k \leq m\) such that \(n+m+k-1 = 2\sigma\), with \(\sigma \in \mathbb{N}\). As before, we can see that

\[
\frac{\sqrt{w_\lambda(k-1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n, m, k, 1) = \frac{n}{n+2\lambda-1} \frac{n+\lambda}{n+\lambda-1} \frac{k+\lambda-1}{k+\lambda} \frac{\sigma-n+1}{\sigma-n+\lambda+1} \frac{\sigma-k+\lambda+1}{\sigma-k+1} = \prod_{j=1}^{5} \beta_j.
\]

There exists \(C > 0\) such that \(|\beta_j| \leq C, j = 1, \ldots, 5,\) and

\[
|\beta_j - 1| \leq \frac{C}{n+1}, \quad j = 1, 2, \quad |\beta_3 - 1| \leq \frac{C}{k+1}, \quad |\beta_4 - 1| \leq \frac{C}{\sigma-n+1}, \quad \text{and} \quad |\beta_5 - 1| \leq \frac{C}{\sigma-k+1}.
\]

Then,

\[
\left| \frac{\sqrt{w_\lambda(k-1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n, m, k, 1) - 1 \right| \leq C \sum_{j=1}^{5} |\beta_j - 1| \leq C \left( \frac{1}{n+1} + \frac{1}{k+1} + \frac{1}{m-n-k+1} \right) \leq C \frac{n+k}{(n+1)(k+1)} \leq \frac{m+1}{(n+1)(k+1)},
\]

and since \(m \leq 2n\) we deduce, by considering (33) (for \(\alpha = 1/2\)), that

\[
|J_{1,2}(n, m, t)| \leq C \sum_{k=0}^{\infty} c_\lambda(n-1, m, k, 1) \leq \frac{C}{|n-m|^2}, \quad t > 0.
\]

Hence, we have established that

\begin{equation}
J_1(n, m, t) \leq \frac{C}{|n-m|^2}, \quad t > 0.
\end{equation}

Finally we deal with \(J_2\). We write

\[
J_2(n, m, t) = \sum_{k=m+1}^{m+n-1} c_\lambda(n, m, k, 1) \left[ h_\lambda^k(k+1) - \frac{\sqrt{w_\lambda(k+1)}}{\sqrt{w_\lambda(k)}} h_\lambda^k(k) \right] + \sum_{k=m+1}^{m+n-1} \left[ \frac{\sqrt{w_\lambda(k+1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n, m, k, 1) - c_\lambda(n, m, m, 1) \right] h_\lambda^k(k) =: J_{2,1}(n, m, t) + J_{2,2}(n, m, t), \quad t > 0.
\]
Again, by taking into account (4), (33) (for $\alpha = 1/2$) and (55) we deduce

$$|J_{2,1}(n, m, t)| \leq \sum_{k=m+1}^{n+m-1} c_{\lambda} (n, m, k) \sqrt{\frac{w_\lambda(k+1)}{(k+1)^{2\lambda+2}}} \leq C \sum_{k=0}^{\infty} c_{\lambda} (n, m, k) \left( \frac{1}{(k+1)^{2\lambda+2}} \right) \leq \frac{C}{|n-m|^2}, \quad t > 0.$$  

Also, according to by (4) and (31), we have that

$$|J_{2,2}(n, m, t)| \leq \sum_{m+1 \leq k \leq n+m-1 \atop n+m+k+1 \text{ even}} c_{\lambda} (n-1, m, k) \left| \frac{\sqrt{w_\lambda(k+1)c_{\lambda} (n-1, m, k)}}{\sqrt{w_\lambda(k)c_{\lambda} (n-1, m, k)}} - 1 \right|, \quad t > 0.$$  

Consider $k \in \mathbb{N}$, $m+1 \leq k \leq m+n$, such that $n+m+k+1 = 2\sigma$, with $\sigma \in \mathbb{N}$. We can write

$$\sqrt{\frac{w_\lambda(k+1)c_{\lambda} (n-1, m, k)}}{\sqrt{w_\lambda(k)c_{\lambda} (n-1, m, k)}} = \sqrt{\frac{n}{n+2\lambda-1}} \sqrt{\frac{n+\lambda}{n+1}} \sqrt{\frac{k+\lambda+1}{k+\lambda}} \sqrt{\frac{\sigma-m+2\lambda-1}{\sigma-m}} \sqrt{\frac{\sigma+2\lambda-1}{\sigma+\lambda}} =: \prod_{j=1}^{5} \gamma_j.$$  

There exists $C > 0$ such that $|\gamma_j| \leq C$, $j = 1, 2, \ldots, 5$, and

$$|\gamma_j - 1| \leq \frac{C}{n+1}, \quad j = 1, 2, \ldots, \gamma_3 - 1 \leq \frac{C}{k+1}, \quad |\gamma_4 - 1| \leq \frac{C}{\sigma - m}, \quad \gamma_5 - 1 \leq \frac{C}{\sigma}.$$  

Since $m \leq 2n$, by proceeding as in the case of $J_{1,2}$, and using again (33) (for $\alpha = 1/2$) it follows that

$$|J_{2,2}(n, m, t)| \leq C \sum_{k=m+1}^{n+m-1} c_{\lambda} (n-1, m, k) \left( \frac{1}{n+1} + \frac{1}{k+1} + \frac{1}{n-m+k+1} + \frac{1}{n+m+k+1} \right) \leq \sum_{k=0}^{\infty} c_{\lambda} (n-1, m, k) \left( \frac{1}{(k+1)^{2\lambda+2}} \right) \leq \frac{C}{|n-m|^2}, \quad t > 0.$$  

Thus,

$$J_2(n, m, t) \leq \frac{C}{|n-m|^2}, \quad t > 0.$$  

Estimations (39), (40), (41) and (42) allow us to conclude property (A2).

We are going now to establish (27). First observe that when $|n-m| > 2|n-\ell|$, $n, m, \ell \in \mathbb{N}$, then

$$\max\{n, m\} > \min\{n, m\} + |n-\ell| \quad \text{and} \quad |n-m| - |n-\ell| > \frac{|n-m|}{2}.$$  

Let $n, m, \ell \in \mathbb{N}$, $n \neq m$, $|n-m| > 2|n-\ell|$ and $m/2 \leq n, \ell \leq 3m/2$. Suppose that $n \leq \ell$. In this case, we have that

$$\|K_{\lambda}^\ell(n, m) - K_{\lambda}^\ell(\ell, m)\|_B \leq \sum_{j=0}^{\ell-n-1} \|K_{\lambda}^\ell(n+j, m) - K_{\lambda}^\ell(n+j+1, m)\|_B.$$  

If $n > m$, we can apply (A1) and obtain

$$\|K_{\lambda}^\ell(n, m) - K_{\lambda}^\ell(\ell, m)\|_B \leq C \sum_{j=0}^{\ell-n-1} \frac{1}{(n+j-m)^2} \leq C \frac{\ell-n}{|n-m|^2}.$$  

When $n < m$, (43) leads to $n+j+1 \leq n+(\ell-n) < m$, $j = 0, \ldots, n-\ell-1$, and we can apply (A2) to get the estimate.

In a similar way if $\ell \leq n$, we write

$$\|K_{\lambda}^\ell(n, m) - K_{\lambda}^\ell(\ell, m)\|_B \leq \sum_{j=0}^{n-\ell-1} \|K_{\lambda}^\ell(n-j, m) - K_{\lambda}^\ell(n-j-1, m)\|_B,$$  

and use (A2) when $n < m$, and (A1) if $n > m$, since in this case by (43) it follows that $n-j-1 \geq \ell = n-(n-\ell) > m$, $j = 0, \ldots, n-\ell-1$.  

\square
By invoking Theorem 2.1, we deduce that, for every $1 \leq p < \infty$ and $w \in A_p(\mathbb{N})$ the operator $T$ can be extended from $\ell^p(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$ to $\ell^p(\mathbb{N}, w)$ as a bounded operator from $\ell^p(\mathbb{N}, w)$ into $\ell^p_B(\mathbb{N}, w)$, when $p \in (1, \infty)$ and from $\ell^1(\mathbb{N}, w)$ into $\ell^1_B(\mathbb{N}, w)$.

Let $p \in [1, \infty)$, $w \in A_p(\mathbb{N})$ and denote by $T$ the extension obtained. We are going to see that, for every $f \in \ell^p(\mathbb{N}, w)$,

$$
\mathbb{T}(f)(n) = (h^p_T \# f)(n), \quad n \in \mathbb{N} \text{ and } t > 0,
$$

and, thus, we prove that the maximal operator $W^\lambda_T$ is bounded from $\ell^p(\mathbb{N}, w)$ into itself.

Let $n \in \mathbb{N}$ and $t > 0$. We consider the operator

$$
P_{t,n} : \ell^p(\mathbb{N}, w) \rightarrow C
$$

$$
f \mapsto P_{t,n}(f) := (h^p_T \# f)(n).
$$

We show that $P_{t,n}$ is a bounded operator. Assume first that $p \in (1, \infty)$. For every $f \in \ell^p(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$, we have that

$$
|w(n)|^{1/p}W^\lambda_T(f)(n) \leq ||W^\lambda_T(f)||_{\ell^p(\mathbb{N}, w)} = ||\mathbb{T}(f)||_{\ell^p(\mathbb{N}, w)} \leq C||f||_{\ell^p(\mathbb{N}, w)}.
$$

Then, for each $f \in \ell^p(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$,

$$
\left| \sum_{m \in \mathbb{N}} f(m)\tau_n(h^p_T)(m) \right| = \left| (h^p_T \# f)(n) \right| = \left| W^\lambda_T(f)(n) \right| \leq C \frac{||f||_{\ell^p(\mathbb{N}, w)}}{|w(n)|^{1/p}}.
$$

Hence, $\lambda^\tau_n h^\lambda_T \in (\ell^p(\mathbb{N}, w))' = \ell^p(\mathbb{N}, w^{-1/(p-1)})$ and we obtain that $P_{t,n}$ is bounded.

If $p = 1$, it is sufficient to establish that the sequence

$$
\frac{\sum_{\lambda \tau_n(h^p_T)}(m)}{w_\lambda(m)}_{m \in \mathbb{N}},
$$

is in $\ell^\infty(\mathbb{N})$.

By taking into account [27] (9) and estimations (4) and (31), we have

$$
0 \leq \lambda^\tau_n(h^p_T)(n) \leq C \sum_{k=0}^\infty c_\lambda(n, n, k) \frac{\sqrt{w_\lambda(k)}}{(k+1)^{2\lambda+1}} \leq C \sum_{k=0}^\infty c_\lambda(n, n, k) w_\lambda(k) = C w_\lambda(n).
$$

On the other hand, size condition (26) says that

$$
\lambda^\tau_n(h^p_T)(m) \leq \frac{C}{|n - m|}, \quad m \neq n,
$$

and, since $w \in A_1(\mathbb{N})$, it follows that, there exists $C > 0$ for which

$$
\frac{1}{w(m)} \leq \max_{n \leq k \leq m} \frac{1}{w(k)} \sum_{k=0}^m \frac{w(k)}{m+1} \left( \sum_{k=0}^m \frac{w(k)}{m+1} \right)^{-1} \leq C(m+1) \left( \sum_{k=0}^m w(k) \right)^{-1} \leq C \frac{m+1}{w(0)}, \quad m \in \mathbb{N}.
$$

Then, we can find $C > 0$ such that

$$
\frac{\lambda^\tau_n(h^p_T)(m)}{w_\lambda(m)} \leq C \frac{m+1}{|n-m|} \leq C, \quad m \neq n,
$$

and we conclude that $P_{t,n}$ is bounded in $\ell^1(\mathbb{N}, w)$.

Let us now consider $f \in \ell^p(\mathbb{N}, w)$ and choose a sequence $(f_k)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$ such that $f_k \rightarrow f$, as $k \rightarrow \infty$, in $\ell^p(\mathbb{N}, w)$. The boundedness of $T$ implies that

$$
T(f_k) = T(f_k) \rightarrow T(f), \quad \text{as } k \rightarrow \infty,
$$

in $\ell^p_B(\mathbb{N}, w)$, when $p \in (1, \infty)$ and in $\ell^1_B(\mathbb{N}, w)$, when $p = 1$.

Suppose $p \in (1, \infty)$. Hence,

$$
T(f_k)(n; t) = P_{t,n}(f_k) \rightarrow [T(f)(n)](t), \quad \text{as } k \rightarrow \infty, \text{ in } C,
$$

and, according to the boundedness of $P_{t,n}$, we can conclude that

$$
[T(f)(n)](t) = P_{t,n}(f) = (f \# h^T_T)(n).
$$

When $p = 1$, we consider $F_k = (F_k(m))_{m \in \mathbb{N}}$, $k \in \mathbb{N}$, and $F = (F(m))_{m \in \mathbb{N}}$, where $F_k(m) = P_{t,m}(f_k)$ and $F(m) = P_{t,m}(f)$, $m, k \in \mathbb{N}$.

Since $P_{t,m}$, $m \in \mathbb{N}$, is a bounded operator, we have that

$$
F_k(m) \rightarrow F(m), \quad \text{as } k \rightarrow \infty, \quad m \in \mathbb{N}.
$$
Also, we have that $F_k = T(\mathcal{F}_k) \rightarrow \mathcal{T}(f)$, as $k \rightarrow \infty$, in $\ell^1_B(N, w)$. Then we can conclude that $\mathcal{T}(f) = F$, that is, (44) is verified when $p = 1$.

4. Proof of Theorem [1.1] for Littlewood-Paley functions

Let us consider along this section the Banach space $\mathbb{B} = L^2((0, \infty), dt/t)$. The following technical lemma will be useful.

Lemma 4.1. Let $k \in \mathbb{N}, a, \alpha, \beta \in \mathbb{R}$ such that $k + 2a > \beta + 1 > a$, $k + \beta + 1 > 0$ and $k + a + 1 > 0$. We have that

$$\int_k^{a, \alpha; \beta} = \left\|e^{2t(1+s)}(1-s)^{k+\alpha}(1+s)^{k+\beta} ds\right\|_B \leq C \frac{\Gamma(k+1)}{(k+1)^{\beta-2a+2}},$$

for certain $C > 0$ which does not depend on $k$.

Proof. We write

$$\int_k^{a, \alpha; \beta} = \int_0^k \left( \int_1^1 \left( 1-s \right)^{k+\alpha}(1+s)^{k+\beta} ds \right) dt$$

$$= \Gamma(2k + 2a) \int_1^{k+\alpha}(1+s)^{k+\beta} ds$$

$$= \frac{\Gamma(2k + 2a)}{\beta-2a} \int_0^1 \left( 1-u \right)^{k+\alpha}(1-v)^{k+\beta} du$$

$$\leq \frac{\Gamma(k+\beta+1)(k+2a-\beta-1)}{\beta-2a} \int_0^1 \left( 1-u \right)^{k+\alpha} du$$

$$= C_{a, \alpha, \beta} \frac{(\Gamma(k+1))^2}{(k+1)^{\beta-2a+1}},$$

for certain $C_{a, \alpha, \beta} > 0$, and then (45) is obtained. \hfill \square

4.1. Proof of Theorem [1.1] for $g_{W^\lambda}^1$.

Since $\Delta_\lambda$ is a bounded operator from $\ell^p(\mathbb{N})$ into itself, $1 \leq p \leq \infty$, it follows that, for every $f \in \ell^p(\mathbb{N})$ and $k \in \mathbb{N}$, $\partial_t W^\lambda_k(f) = W^\lambda_k(\Delta_\lambda f)$, in the sense of derivative in $\ell^p(\mathbb{N})$. Then, for every $f \in \ell^p(\mathbb{N})$ and $n \in \mathbb{N}$, $W^\lambda_k(f)(n)$ is smooth in $(0, \infty)$ and $\partial_t W^\lambda_k(f)(n) = W^\lambda_k(\Delta_\lambda f)(n), k \in \mathbb{N}$.

To prove the $\ell^p$-boundedness properties of the Littlewood-Paley $g_{W^\lambda}^1$, we will apply Theorem [2.1].

Let $G_\lambda$ the operator given by

$$G_\lambda: \ell^2(N, w) \rightarrow \ell^2(N)$$

$$f \rightarrow G_\lambda(f)(n; t) := t \partial_t W^\lambda_k(f)(n).$$

As it was commented in Section [3], $g_{W^\lambda}^1$ is a bounded (sublinear) operator from $\ell^p(\mathbb{N})$ into itself. Then, the operator $G_\lambda$ is bounded. Moreover, by defining $K^\lambda_n(m, n) := \tau_n(t \partial_t h^\lambda_m)(m)$, $m, n \in \mathbb{N}$ and $t > 0$, we can write, for every $f \in \mathcal{C}_0^N$,

$$G_\lambda(f)(n; t) = \sum_{m \in \mathbb{N}} f(m) K^\lambda_n(m, n), \quad n \in \mathbb{N} \text{ and } t > 0.$$
Next, we establish that the kernel function $K_{\ell}(n, m)$, $n, m \in \mathbb{N}$, $t > 0$, satisfies the following properties for a certain $C > 0$ and each $n, m, \ell \in \mathbb{N}$, $n \neq m$:

\begin{align*}
(46) & \quad \|K^\lambda_{\ell}(n, m)\|_B \leq \frac{C}{|n - m|}; \\
(47) & \quad \|K^\lambda_{\ell}(n, m) - K^\lambda_{\ell}(m, n)\|_B \leq C \left|\frac{n - \ell}{|n - m|^2}\right|, \quad |n - m| > 2|n - \ell| \text{ and } \frac{m}{2} \leq n, \ell \leq \frac{3m}{2}; \\
(48) & \quad \|K^\lambda_{\ell}(n, m) - K^\lambda_{\ell}(m, n)\|_B \leq C \left|\frac{n - \ell}{|n - m|^2}\right|, \quad |n - m| > 2|n - \ell| \text{ and } \frac{m}{2} \leq n, \ell \leq \frac{3m}{2}.
\end{align*}

We note that, since $K^\lambda_{\ell}(n, m) = K^\lambda_{\ell}(m, n)$, $n, m \in \mathbb{N}$, $t > 0$, we only have to prove (46) and (47). For that, we use, by making suitable modifications some of the ideas in [17] Section 5.

**Proof of (46).** We denote by $\psi^\lambda_{\ell}(k) := t\partial h^\lambda_{\ell}(k)$, $k \in \mathbb{N}$, $t > 0$. Firstly, we show that, there exists $C > 0$ such that,

\begin{align*}
(49) & \quad \|\psi^\lambda_{\ell}(k)\|_B \leq \frac{C}{(k + 1)^{\lambda + \Gamma}}, \quad k \in \mathbb{N}.
\end{align*}

Since ([37] (5.7.9))

\begin{align*}
2 \frac{d}{dz} I_{\nu}(z) = I_{\nu - 1}(z) + I_{\nu + 1}(z), \quad z > 0 \text{ and } \nu > 0,
\end{align*}

and

\begin{align*}
I_{\nu - 1}(z) - I_{\nu + 1}(z) = \frac{2\nu}{z} I_{\nu}(z), \quad z > 0 \text{ and } \nu > 0,
\end{align*}

we deduce that

\begin{align*}
(50) & \quad \psi^\lambda_{\ell}(k) = \sqrt{\pi} \Gamma\left(\lambda + \frac{1}{2}\right) \sqrt{w_{\lambda}(k)} t \partial_{\lambda} e^{-2t} I_{\lambda + k}(2t) \\
& \quad = \sqrt{\pi} \Gamma\left(\lambda + \frac{1}{2}\right) \sqrt{w_{\lambda}(k)} t e^{-2t} \left[ - \lambda t^{-1} I_{\lambda + k}(2t) - 2I_{\lambda + k}(2t) + 2 \left( \frac{d}{dt} I_{\lambda + k}(2t) \right) \frac{dt}{2t} \right] \\
& \quad = \sqrt{\pi} \Gamma\left(\lambda + \frac{1}{2}\right) \sqrt{w_{\lambda}(k)} \left[ \frac{k}{\lambda + k} I_{\lambda + k - 1}(2t) - 2I_{\lambda + k}(2t) + \frac{2\lambda + k}{\lambda + k} I_{\lambda + k + 1}(2t) \right] \\
& \quad = \frac{k}{\lambda + k} \sqrt{w_{\lambda}(k)} \left[ \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k - 1)}} \left( \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k + 1)}} \right)^{\lambda} \left( I_{\lambda + k}(k - 1) - 2I_{\lambda + k}(k) + \frac{2\lambda + k}{\lambda + k} \sqrt{w_{\lambda}(k)} \right) \right] \\
& \quad = \frac{k}{\lambda + k} \left[ \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k - 1)}} \left( \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k + 1)}} \right)^{\lambda} \left( I_{\lambda + k}(k - 1) - 2I_{\lambda + k}(k) + \frac{2\lambda + k}{\lambda + k} \sqrt{w_{\lambda}(k)} \right) \right] \\
& \quad + \frac{2\lambda + k}{\lambda + k} \left[ \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k + 1)}} \left( I_{\lambda + k}(k - 1) - 2I_{\lambda + k}(k) + \frac{2\lambda + k}{\lambda + k} \sqrt{w_{\lambda}(k)} \right) \right], \quad k \in \mathbb{N}, t > 0,
\end{align*}

where we assume $h^\lambda_{\ell}(1) := 0$, $t > 0$. Then, we can write

\begin{align*}
(51) & \quad \|\psi^\lambda_{\ell}(k)\|_B \leq \frac{k}{\lambda + k} \left[ \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k - 1)}} \left( \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k + 1)}} \right)^{\lambda} \left( I_{\lambda + k}(k - 1) - 2I_{\lambda + k}(k) + \frac{2\lambda + k}{\lambda + k} \sqrt{w_{\lambda}(k)} \right) \right] \\
& \quad + \frac{2\lambda + k}{\lambda + k} \left[ \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k + 1)}} \left( I_{\lambda + k}(k - 1) - 2I_{\lambda + k}(k) + \frac{2\lambda + k}{\lambda + k} \sqrt{w_{\lambda}(k)} \right) \right], \quad k \in \mathbb{N}.
\end{align*}

Now, we are going to establish that

\begin{align*}
(52) & \quad L(k) := \left[ \frac{\sqrt{w_{\lambda}(k)}}{\sqrt{w_{\lambda}(k + 1)}} \left( I_{\lambda + k}(k - 1) - 2I_{\lambda + k}(k) + \frac{2\lambda + k}{\lambda + k} \sqrt{w_{\lambda}(k)} \right) \right] \leq \frac{C}{(k + 1)^{\lambda + \Gamma}}, \quad k > \lambda + \frac{5}{2}.
\end{align*}

By partial integration in (30) we can obtain (17) (35))

\begin{align*}
(53) & \quad I_{\nu}(z) = \frac{z^{-\nu - 2}}{\sqrt{\pi} z^{2\nu - 2 \Gamma(\nu - 3/2)}} \int_{-1}^{1} e^{-z s^{2}} (1 + z s) (1 - s^{2})^{-\nu - 5/2} d s, \quad z > 0 \text{ and } \nu > \frac{3}{2}.
\end{align*}

By using (30) and (54) we have that

\begin{align*}
L(k) & \quad = \sqrt{\pi} \Gamma\left(\lambda + \frac{1}{2}\right) \sqrt{w_{\lambda}(k)} \left[ e^{-2t} t \partial_{\lambda} \left( I_{\lambda + k}(2t) - I_{\lambda + k}(2t) \right) \right] |_{B} \\
& \quad \leq C \sqrt{\Gamma(\lambda + k - 1/2)} \left[ \left| e^{-2t} t \partial_{\lambda} \left( I_{\lambda + k}(2t) - I_{\lambda + k}(2t) \right) \right| \right] |_{B} \\
& \quad \leq C \sqrt{\Gamma(\lambda + k - 1/2)} \int_{-1}^{1} e^{-2t(1 + s)} (1 - s^{2})^{\lambda + k - 3/2} (1 + 2t(1 + s)) d s \leq C \sqrt{\Gamma(\lambda + k - 1/2)} \int_{-1}^{1} e^{-2t(1 + s)} (1 - s^{2})^{\lambda + k - 3/2} (1 + 2t(1 + s)) d s.
\end{align*}
where \( e \) and, for every \( I \)

On the other hand, \([37, (5.16.4) and (5.11.10)]\) say that, for every \( \| \),

By (50) and (57) it follows that

Then, by using (4) and (53), we obtain

Let

From (56) we deduce that, for every \( \nu > 0 \),

and, for every \( n \in \mathbb{N} \),

where \([\nu, r] = (4\nu^2 - 1)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2r - 1)^2) \) \( 2\pi \Gamma(r + 1) \), \( r \in \mathbb{N} \setminus \{0\} \).

From (56) we deduce that, for every \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that

By (50) and (57) it follows that

This estimate, jointly (55), gives (49) with a constant \( C > 0 \) that does not depend on \( k \).

By considering (53) (for \( \alpha = 0 \)) and (49) we can write

and (46) is established.

**Proof of (47).** In order to show (47), we are going to proceed as in the proof of (27). Thus we need to justify the following estimations:

\[
\|K_{\lambda}^\nu(n + 1, m) - K_{\lambda}^\nu(n, m)\|_B \leq \frac{C}{|n - m|^2}, \quad n, m \in \mathbb{N}, n > m;
\]

\[
\|K_{\lambda}^\nu(n, m) - K_{\lambda}^\nu(n, m - 1)\|_B \leq \frac{C}{|n - m|^2}, \quad n, m \in \mathbb{N}, 1 \leq n < m \leq 2n.
\]

Let \( n, m \in \mathbb{N} \) and \( n > m \). As in the proof of (27) we can write

\[
K_{\lambda}^\nu(n + 1, m) - K_{\lambda}^\nu(n, m) = I_1(n, m, t) + I_2(n, m, t), \quad t > 0,
\]

where

\[
I_1(n, m, t) := \sum_{k=n-m}^{n+m} c_{\lambda}(n, m, k + 1) \left[ \psi_{\lambda}^\nu(k+1) - \frac{\sqrt{w_{\lambda}(k+1)}}{\sqrt{w_{\lambda}(k)}} \psi_{\lambda}^\nu(k) \right], \quad t > 0,
\]
and
\[ I_2(n, m, t) := \sum_{k=-n}^{n+m} \left[ \frac{\sqrt{w_\lambda(k+1)}}{\sqrt{w_\lambda(k)}} c_\lambda(n+1, m, k+1) - c_\lambda(n, m, k) \right] \psi^\lambda_t(k), \quad t > 0. \]

By using the estimations \((37), (49)\) and \((33)\) (for \(\alpha = 0\)) we deduce that
\[ ||I_2(n, m, \cdot)||_B \leq \frac{C}{n-m} \sum_{k=-n}^{n+m} c_\lambda(n, m, k) \leq \frac{C}{|n-m|^2}. \]

On the other hand, if we show that
\[ ||\psi_\lambda^\lambda(k+1) - \sqrt{w_\lambda(k+1)} \psi^\lambda_t(k)||_B \leq \frac{C}{(k+1)^{\lambda+\frac{7}{2}}}, \quad k \in \mathbb{N}, \]
we can write, by using also \((33)\) (for \(\alpha = 1/2\)),
\[ ||I_1(n, m, \cdot)||_B \leq \frac{C}{n-m} \sum_{k=-n}^{n+m} c_\lambda(n+1, m, k+1) \leq \frac{C}{|n-m|^2}, \]
which, jointly \((60)\) and \((61)\), gives property \((B1)\).

Let us justify \((62)\). By using \((30)\) and \((51)\) we get
\[
\left\| \psi_\lambda^\lambda(k+1) - \sqrt{w_\lambda(k+1)} \psi^\lambda_t(k) \right\|_B \\
= \sqrt{\pi} \left( \frac{\lambda + 1/2}{2} \right) \sqrt{w_\lambda(k+1)} \left\| t\partial_t [e^{-2t} - \lambda I_{\lambda+k+1}(2t) - I_{\lambda+k}(2t)] \right\|_B \\
\leq C \sqrt{\frac{w_\lambda(k+1)}{\Gamma(\lambda + k - 1/2)}} \left\| t\partial_t \int_{-1}^{1} e^{-2t(1+s)} (1 - s^2)^{\lambda+k-3/2} (1 + 2t(1+s)) ds \right\|_B \\
\leq C \frac{\sqrt{w_\lambda(k+1)}}{\Gamma(\lambda + k - 1/2)} \left( (k-2) k^{-2} \int_{-1}^{1} e^{-2t(1+s)} (1 - s^2)^{\lambda+k-3/2} (1 + 2t(1+s)) ds \right) \\
+ \left\| t^{k-1} \int_{-1}^{1} e^{-2t(1+s)} (1 - s^2)^{\lambda+k-3/2} (1 + 2t(1+s)) ds \right\|_B \\
+ \left\| t^{k-1} \int_{-1}^{1} e^{-2t(1+s)} (1 - s^2)^{\lambda+k-3/2} (1 + s) ds \right\|_B \\
\leq C \frac{\sqrt{w_\lambda(k+1)}}{\Gamma(\lambda + k - 1/2)} \left( (k+1) k^{-2} \int_{-1}^{1} e^{-2t(1+s)} (1 - s^2)^{\lambda+k-3/2} (1 + s) ds \right) \\
+ \left\| (k+1) k^{-1} \int_{-1}^{1} e^{-2t(1+s)} (1 - s^2)^{\lambda+k-3/2} (1 + s)^2 ds \right\|_B \\
+ \left\| k \int_{-1}^{1} e^{-2t(1+s)} (1 - s^2)^{\lambda+k-3/2} (1 + s)^3 ds \right\|_B. \]

Now, by using estimate \((1)\) and Lemma \(4.1\) we obtain
\[
\left\| \psi_\lambda^\lambda(k+1) - \sqrt{w_\lambda(k+1)} \psi^\lambda_t(k) \right\|_B \leq C \frac{(k+1)!}{\Gamma(\lambda + k - 1/2)} \left( \frac{1}{(k+1)^{\lambda+9/2}} + \frac{1}{(k+1)^{\lambda+7/2}} \right) \\
\leq C \frac{k!}{\Gamma(\lambda + k - 1/2)(k+1)^{7/2}} \leq \frac{C}{(k+1)^{\lambda+2}}, \quad \text{when } k > \lambda + 9/2. \]

Also, by taking into account \((49)\), if \(k \in \mathbb{N}, k \leq \lambda + 9/2, \)
\[
\left\| \psi_\lambda^\lambda(k+1) - \sqrt{w_\lambda(k+1)} \psi^\lambda_t(k) \right\|_B \leq C( ||\psi_\lambda^\lambda(k+1)||_B + ||\psi^\lambda_t(k)||_B ) \\
\leq C \frac{C}{(k+1)^{\lambda}} \leq C \frac{C}{(k+1)^{\lambda+2}}, \]
and thus, \((62)\) is established.

Property \((B2)\) can be obtained by proceeding as in the proof of \((A2)\), and using \((49)\) and \((62)\).
Moreover, \((B1)\) and \((B2)\) lead to \((47)\) in the same way as \((27)\) was proved from \((A1)\) and \((A2)\).
According to Theorem 1.1, we deduce that for every $1 \leq p < \infty$ and $w \in A_p(N)$ the operator $G_\lambda$ can be extended from $\ell^p(N) \cap \ell^p(N, w)$ to $\ell^p(N, w)$ as a bounded operator from $\ell^p(N, w)$ into $\ell^p_b(N, w)$, when $1 < p < \infty$ and from $\ell^1(N, w)$ into $\ell^1_b(N, w)$. Let us denote by $T$ to this extension. Our objective now is to show that

$$[T(f)(n)](t) = t\partial_t W_\lambda(f)(n), \quad n \in \mathbb{N} \text{ and } t \in (0, \infty) \setminus E,$$

for certain $E \subset (0, \infty)$ with $|E| = 0$. Thus, we can conclude that the Littlewood-Paley function $g_{W^\lambda}$ is bounded from $\ell^p(N, w)$ into itself, when $1 < p < \infty$ and from $\ell^1(N, w)$ into $\ell^{1,\infty}(N, w)$.

Let $1 < p < \infty$ and $w \in A_p(N)$. Suppose $f \in \ell^p(N, w)$ and that $(f_t)_{t=0}^\infty$ is a sequence in $C_0^\infty$ for which $f_t \to f$, as $t \to \infty$, in $\ell^p(N, w)$. We have that

$$G_\lambda(f_t) \to T(f), \quad \text{as } t \to \infty, \text{ in } \ell^p_b(N, w).$$

Then, for every $n \in \mathbb{N}$,

$$\int_0^\infty |t\partial_t W_\lambda(f_t)(n) - [T(f)(n)](t)|^2 \frac{dt}{t} \to 0, \quad \text{as } t \to \infty.$$  

Let $n \in \mathbb{N}$. There exists an increasing sequence $\phi : \mathbb{N} \to \mathbb{N}$ and a measurable set $E \subset (0, \infty)$ with zero Lebesgue measure such that, for every $t \in (0, \infty) \setminus E$,

$$t\partial_t W_\lambda(f_{\phi(t)})(n) \to [T(f)(n)](t), \quad \text{as } t \to \infty.$$ 

We consider the following maximal operators

$$W_\lambda^-(g)(\ell) := \sup_{t > 0} \left| \sum_{m=0}^{\infty} g(m) \sum_{r=1}^{\infty} c_\lambda(\ell, m, r) h_1^\lambda(r-1) \right|, \quad \ell \in \mathbb{N},$$

and

$$W_\lambda^+(g)(\ell) := \sup_{t > 0} \left| \sum_{m=0}^{\infty} g(m) \sum_{r=1}^{\infty} c_\lambda(\ell, m, r) h_1^\lambda(r+1) \right|, \quad \ell \in \mathbb{N}.$$ 

By proceeding as in Section 3, we can prove that these maximal operators are bounded from $\ell^p(N, w)$ into itself. Also, by (5), (12) and (24) we have that

$$|t\partial_\lambda W_\lambda^1(g)(n)| \leq C t \big[ W_\lambda^1(|g|)(n) + W_\lambda^+1(|g|)(n) + W_\lambda^-1(|g|)(n) \big], \quad t > 0.$$ 

Let $t_0 \in (0, \infty)$. Since $W_\lambda^1, W_\lambda^+1$ and $W_\lambda^-1$ are bounded operators from $\ell^p(N, w)$ into itself, it follows that

$$|t\partial_\lambda W_\lambda^1(f_{\phi(t)})(n)|_{t=t_0} \to |t\partial_\lambda W_\lambda^1(f)(n)|_{t=t_0}, \quad \text{as } t \to \infty.$$ 

Hence, for every $t \in (0, \infty) \setminus E$, $[T(f)(n)](t) = t\partial_\lambda W_\lambda^1(f)(n)$.

In a similar way we can prove that $g_{W^\lambda}$ is bounded from $\ell^1(N, w)$ into $\ell^{1,\infty}(N, w)$.

### 4.2. Proof of Theorem 1.1

for $g_{W^\lambda}$, $k > 1$. In order to show that Littlewood-Paley functions $g_{W^\lambda}, k > 1$, are bounded from $\ell^p(N, w)$ into itself, when $1 < p < \infty$ and $w \in A_p(N)$, we use a reduction argument that we learnt from Professor C. Segovia and J.L. Torrea.

We need the following form of a Krivine’s result (125 Theorem 1.f.14, p. 93).

**Theorem 4.1.** Let $\Omega_i, i = 1, 2,$ be Hilbert spaces and let $(\Omega, c^f_\lambda, \mu)$ be a measure space. Assume that $1 < p < \infty, w \in A_p(N)$ and $T$ is a bounded operator from $\ell^p_{\ell^2_1}(N, w)$ into $\ell^p_{\ell^2_2}(N, w)$. We define the operator $\tilde{T}$ by

$$\tilde{T}(F)(n, \theta) = T(F(., \theta))(n), \quad n \in \mathbb{N} \text{ and } \theta \in \Omega,$$

for every $F : N \times \Omega \to \ell^2$ such that $F(., \theta) \in \ell^p_{\ell^2_1}(N, w)$, for every $\theta \in \Omega$. Then, $\tilde{T}$ is a bounded operator from $\ell^p_{\ell^2_1}(N, w)$ into $\ell^p_{\ell^2_2}(N, w)$.

Let $k \in \mathbb{N}$ and suppose that $g_{W^\lambda, k}^1$ is bounded from $\ell^p(N, w)$ into itself, where $1 < p < \infty$ and $w \in A_p(N)$. We will show that $g_{W^\lambda, k}^2$ is also bounded from $\ell^p(N, w)$ into itself.

For every $m \in \mathbb{N}$ we define the operator

$$G_\lambda^m : \ell^p(N, w) \to \ell^p_b(N, w)$$

$$f \to G_\lambda^m(f)(n, t) := t^m \partial^m_\lambda W_\lambda^1(f)(n).$$
Hence, \( G^1 \) defined by
\[
\widetilde{G}^1(F)(n, \theta; t) = G^1(F(n, \theta))(n, \theta; t), \quad n \in \mathbb{N}, \theta, t > 0.
\]
According to Theorem 4.1 for \( \mathbb{H}_1 = \mathbb{C} \) and \( \mathbb{H}_2 = \mathbb{B} \), the operator \( G^1 \) is bounded from \( \ell^p(\mathbb{N}, w) \) into \( \ell^p(\mathbb{N}, w) \).

A straightforward manipulation (see, for instance, \cite[Proposition 2.5]{12}) leads to
\[
\|G^{k+1}_\lambda f(n; \cdot )\|_\mathbb{B} = \sqrt{2k(2k+1)}\|G^{k}_\lambda |G^{k}_\lambda (f)(n, \cdot ; \cdot )\|_{L^2_2((0, \infty), d\theta / \theta)}, \quad f \in C^\mathbb{N}_0.
\]
Hence, \( G^{k+1}_\lambda \) defines a bounded operator from \( \ell^p(\mathbb{N}, w) \) into \( \ell^p(\mathbb{N}, w) \).

Thus, we have established that \( g^{\ell^2}_k \) is bounded from \( \ell^p(\mathbb{N}, w) \) into \( \ell^p(\mathbb{N}, w) \), for every \( k \in \mathbb{N} \).

**Remark 4.1.** The above argument does not allow us to obtain the weak \((1,1)\) boundedness for the Littlewood-Paley function \( g^{\ell^2}_k \) when \( k > 1 \). In order to get the \( \ell^1 \)-boundedness properties for \( g^{\ell^2}_k \), we need to obtain a treatable expression for \( \partial^k \lambda h^k \), \( t \in (0, \infty) \), for every \( k \in \mathbb{N} \). We cannot get this general form for \( \partial^k \lambda h^k \), \( t \in (0, \infty) \).

### 4.3. Proof of Theorem 1.1 for \( g^{\ell^2}_k \)

Let \( k \in \mathbb{N} \setminus \{0\} \). We consider the operator
\[
Q^{\ell^2}_k : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})
\]
\[
f \mapsto Q^{\ell^2}_k(f)(n; t) := t^k \partial^k \lambda P^1_k(f)(n).
\]
According to \cite{12} and \cite{14} we have that, for every \( f \in C^\mathbb{N}_0 \),
\[
P^1_k(f)(n) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} \frac{W^\lambda}{u} (f)(n) du = \sum_{m=0}^\infty f(m)M^\lambda_k(m, m), \quad n \in \mathbb{N} \text{ and } t > 0,
\]
where
\[
M^\lambda_k(n, m) := \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} \frac{h^\lambda_k(nu^2)}{u^{3/2}}(m) du, \quad n, m \in \mathbb{N} \text{ and } t > 0.
\]
By making a change of variables we can write
\[
M^\lambda_k(n, m) = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}} \frac{\lambda \tau_n(h^\lambda_k)}{u^{3/2}}(m) du = \lambda \tau_n \left[ \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}} h^\lambda_k(u) du \right](m).
\]
For every \( t > 0 \) we define \( \Psi^{\ell^2}_k \) by
\[
\Psi^{\ell^2}_k(f)(n) = \frac{k^k}{2\sqrt{\pi}} \int_0^\infty \partial^k \frac{[te^{-t^2/4u}]}{u^{3/2}} h^\lambda_k(f)(n) du, \quad \ell \in \mathbb{N}.
\]
Then, for every \( f \in C^\mathbb{N}_0 \), we have that
\[
t^k \partial^k \lambda P^1_k(f)(n) = \sum_{m=0}^\infty f(m)Q^{\ell^2}_k(n, m), \quad n \in \mathbb{N} \text{ and } t > 0,
\]
where
\[
Q^{\ell^2}_k(n, m) := \lambda \tau_n(\Psi^{\ell^2}_k(m)), \quad n, m \in \mathbb{N}.
\]
By using spectral theory we can see that \( Q^{\ell^2}_k \) is a bounded operator from \( \ell^2(\mathbb{N}) \) into \( \ell^2(\mathbb{N}) \).

Indeed, let \( f \in \ell^2(\mathbb{N}) \). We have that
\[
\mathcal{F}_\lambda(P^1_k(f))(x) = e^{-\sqrt{2(1-x)}t} \mathcal{F}_\lambda(f)(x), \quad x \in (-1, 1), \ t > 0.
\]
Plancherel equality leads to
\[
\sum_{n=0}^\infty \|Q^{\ell^2}_k(f)(n; \cdot )\|_\mathbb{B}^2 = \int_0^\infty \sum_{n=0}^\infty |Q^{\ell^2}_k(f)(n; t)|^2 dt \leq \int_0^\infty \int_{-1}^1 \left| t \sqrt{2(1-x)} \right|^{2k} e^{-2\sqrt{2(1-x)}t} \left| \mathcal{F}_\lambda(f)(x) \right|^2 dx dt \leq \frac{\Gamma(2k)}{2^{2k}} \int_{-1}^1 \left| \mathcal{F}_\lambda(f)(x) \right|^2 dx = \frac{\Gamma(2k)}{2^{2k}} \sum_{n=0}^\infty |f(n)|^2.
\]
We are going to see that there exists \( C > 0 \) such that, for every \( n, m, \ell \in \mathbb{N} \), \( n \neq m \),
\[
\|Q^{\ell^2}_k(n, m)\|_\mathbb{B} \leq \frac{C}{|n - m|};
\]

\[ (63) \]
\[\|Q^\lambda_k(n, m) - Q^\lambda_k(\ell, m)\|_B \leq C \frac{|n - \ell|}{|n - m|^2}, \quad |n - m| > 2|n - \ell|, \quad \text{and} \quad \frac{m}{2} \leq n, \ell \leq \frac{3m}{2},\]

and
\[\|Q^\lambda_k(n, m) - Q^\lambda_k(\ell, m)\|_B \leq C \frac{|n - \ell|}{|n - m|^2}, \quad |n - m| > 2|n - \ell|, \quad \text{and} \quad \frac{m}{2} \leq n, \ell \leq \frac{3m}{2}.
\]

In order to obtain these estimations we can proceed as in the heat case in Section 4.1. Thus, we only need to establish analogous properties to (49) and (62) for \(\Psi_t^\lambda_k\), that is,
\[\|\Psi_t^\lambda_k(\ell)\|_B \leq \frac{C}{(\ell + 1)^{\lambda+1}}, \quad \ell \in \mathbb{N} \setminus \{0\},\]

and
\[\|\Psi_t^\lambda_k(\ell + 1) - \frac{\sqrt{w_\lambda(\ell + 1)}}{\sqrt{w_\lambda(\ell)}}\Psi_t^\lambda_k(\ell)\|_B \leq \frac{C}{(\ell + 1)^{\lambda+2}}, \quad \ell \in \mathbb{N} \setminus \{0\}.\]

Remark 4.2. Actually, since we assume \(n, m \in \mathbb{N}, n \neq m\), and
\[\lambda \tau_n(\Psi_t^\lambda_k) = \sum_{\ell = [n-m]}^{n+m} c_\lambda(n, m, \ell)\Psi_t^\lambda_k(\ell),\]

we only need to establish properties (66) and (67) for \(\ell \in \mathbb{N} \setminus \{0\}\).

By using Fa di Bruno’s formula ([24, Lemma 4.3, (4.6)]) we can obtain (see [11, Lemma 4])
\[|\partial^k_t [e^{-t^2/(4u)}]| \leq Ce^{-t^2/(8u)}u^{(1-k)/2}, \quad t, u \in (0, \infty).\]

It follows that
\[\|t^k \partial^k_t [e^{-t^2/(4u)}]\|_B \leq Cu^{(1-k)/2} \left(\int_0^\infty t^{2k-1}e^{-t^2/(4u)}dt\right)^{1/2} \leq C\sqrt{u}, \quad u > 0.
\]

Assume \(\ell \in \mathbb{N}, \ell \geq 1\). Let us prove (66). By the Minkowski’s integral’s inequality and (29) we can write
\[\|\Psi_t^\lambda_k(\ell)\|_B \leq C \int_0^\infty \|t^k \partial^k_t [e^{-t^2/(4u)}]\|_B \frac{h_\lambda(t)}{u^{3/2}} du \leq C \sqrt{w_\lambda(\ell)} \int_0^\infty e^{-2u}u^{-\lambda-1}I_{\lambda+\ell}(2u) du
\]

\[\leq C \frac{\sqrt{w_\lambda(\ell)}\Gamma(\ell + \ell + 1/2)}{\Gamma(\ell + 1/2)} \int_0^\infty u^{-\ell-1} \int_{-\frac{1}{2}}^{1} e^{-2u(1+s)(1-s)^{\lambda+\ell-1/2}ds du
\]

\[\leq C \frac{\sqrt{w_\lambda(\ell)}\Gamma(\ell)}{2\Gamma(\ell + 1/2)} \int_{-1}^{1} (1-s^2)^{\lambda+\ell-1/2}(1+s)^{-\ell} ds = C \frac{\sqrt{w_\lambda(\ell)}\Gamma(\ell)}{\Gamma(2\lambda + \ell + 1)} \leq \frac{C}{(\ell + 1)^{\lambda+1}}.
\]

In order to prove (67) we use (29) and (30) and again (66) to get
\[\left|\Psi_t^\lambda_k(\ell + 1) - \frac{\sqrt{w_\lambda(\ell + 1)}}{\sqrt{w_\lambda(\ell)}}\Psi_t^\lambda_k(\ell)\right|_B \leq C \frac{\sqrt{w_\lambda(\ell + 1)}\Gamma(\ell + 1/2)}{\Gamma(\ell + 1/2)} \int_0^\infty e^{-2u}u^{-\lambda-1}I_{\lambda+\ell+1}(2u) - I_{\lambda+\ell}(2u) du
\]

\[\leq C \frac{\sqrt{w_\lambda(\ell + 1)}\Gamma(\ell)}{2\Gamma(\ell + 1/2)} \int_{-1}^{1} (1-s^2)^{\lambda+\ell-1/2}(1+s)^{-\ell} ds
\]

\[= C \frac{\sqrt{w_\lambda(\ell + 1)}\Gamma(\ell)}{\Gamma(2\lambda + \ell + 2)} \leq \frac{C}{(\ell + 1)^{\lambda+2}}.
\]

By proceeding as in the proof of the corresponding property for the heat case in Section 4.1 by using (66) and (67) we can prove (63), (64) and (65).

The proof of the \(\ell^p\)-boundedness properties for \(g^1_{\lambda,k}\) can be finished now similarly to the case of \(g^1_{\lambda}\) (see Section 4.1).
Lemma 5.1. Let $\lambda > 0$, $1 < p < \infty$ and $w \in A_p(N)$. Then, for every $f \in \ell^p(N, w)$ and $g \in \ell^p'(N, v)$, where $1/p + 1/p' = 1$ and $v = w^{-1}/p$,

\begin{equation}
\sum_{n=0}^{\infty} \int_0^\infty t^k \partial_t^k W^\lambda_t(f)(n) \partial_t^k W^\lambda_t(g)(n) \frac{dt}{t} = \frac{\Gamma(2k)}{2^{2k}} \sum_{n=0}^{\infty} f(n)g(n),
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} \int_0^\infty t^k \partial_t^k P^\lambda_t(f)(n) \partial_t^k P^\lambda_t(g)(n) \frac{dt}{t} = \frac{\Gamma(2k)}{2^{2k}} \sum_{n=0}^{\infty} f(n)g(n).
\end{equation}

Proof. Let $f, g \in C_0^\infty$. We can write, by proceeding as in (25),

\begin{equation}
\sum_{n=0}^{\infty} \int_0^\infty t^k \partial_t^k W^\lambda_t(f)(n) \partial_t^k W^\lambda_t(g)(n) \frac{dt}{t} = \int_0^\infty t^{2k-1} \sum_{n=0}^{\infty} \partial_t^k W^\lambda_t(f)(n) \partial_t^k W^\lambda_t(g)(n) dt
\end{equation}

\begin{equation}
= \frac{\Gamma(2k)}{2^{2k}} \int_0^\infty \mathcal{F}_\lambda(f)(x) \mathcal{F}_\lambda(g)(x) e^{-4t(1-x)} (-2(1-x))^{2k} dx dt
\end{equation}

\begin{equation}
= \frac{\Gamma(2k)}{2^{2k}} \int_0^1 \mathcal{F}_\lambda(f)(x) \mathcal{F}_\lambda(g)(x) dx = \frac{\Gamma(2k)}{2^{2k}} \sum_{n=0}^{\infty} f(n)g(n).
\end{equation}

The interchange between the series and the integral is legitimated because $f, g \in C_0^\infty$. Since $C_0^\infty$ is a dense subspace of $\ell^p(N, w)$ and $\ell^p'(N, v)$, by applying Theorem 1.1 and a continuity argument, we can prove the equality (69) for every $f \in \ell^p(N, w)$ and $g \in \ell^p'(N, v)$.

Equality (70) can be established similarly. □

Let $1 < p < \infty$ and $w \in A_p(N)$. By taking into account that $(\ell^p(N, w))' = \ell^p'(N, w^{-1}/p)$, (69) and Theorem 1.1 we get

\begin{equation}
\|f\|_{\ell^p(N, w)} = \sup_{g \in G} \left| \sum_{n=0}^{\infty} f(n)g(n) \right| = \frac{2^{2k}}{\Gamma(2k)} \sup_{g \in G} \left| \int_0^\infty t^k \partial_t^k W^\lambda_t(f)(n) \partial_t^k W^\lambda_t(g)(n) \frac{dt}{t} \right|
\end{equation}

\begin{equation}
\leq C \sup_{g \in G} \|t^k \partial_t^k W^\lambda_t(f)\|_{L^p(N, w)} \|t^k \partial_t^k W^\lambda_t(g)\|_{L^p(N, w^{-1}/p)} \leq C \|t^k \partial_t^k W^\lambda_t(f)\|_{L^p(N, w)},
\end{equation}

where $G$ represents the set of functions $g \in \ell^p(N, w^{-1}/p)$ such that $\|g\|_{\ell^p(N, w^{-1}/p)} \leq 1$. Thus, the left hand side inequality in (15) in the heat case is proved. For the Poisson semigroup $\{P^\lambda_t\}_{t>0}$ we can proceed in a similar way.

6. Proof of Theorem 1.2

Let $\lambda, \mu > 1$. The transplantation operator $\mathcal{R}_\lambda, \mu$ is defined by

\begin{equation}
\mathcal{R}_\lambda, \mu(f) = \mathcal{F}_\mu^{-1}(\mathcal{F}_\lambda f), \quad f \in \ell^2(N).
\end{equation}

Since $\mathcal{F}_\lambda$ is an isometric isomorphism from $\ell^2(N)$ into $L^2(-1, 1)$, $\mathcal{R}_\lambda, \mu$ is an isometry from $\ell^2(N)$ into itself. If $f \in C_0^\infty$, we have that

\begin{equation}
\mathcal{R}_\lambda, \mu(f)(n) = \mathcal{F}_\mu^{-1}\left( \sum_{m=0}^{\infty} f(m)\varphi_m^\lambda \right)(n) = \sum_{m=0}^{\infty} f(m)K_{\lambda, \mu}(m, n), \quad n \in N,
\end{equation}

where

\begin{equation}
K_{\lambda, \mu}(m, n) := \int_{-1}^{1} \varphi_m^\mu(x)\varphi_m^\lambda(x) dx, \quad n, m \in N.
\end{equation}

Moreover, if $f \in \ell^2(N)$, then

\begin{equation}
\mathcal{R}_\lambda, \mu(f)(n) = \lim_{k \to \infty} \sum_{m=0}^{k} f(m)K_{\lambda, \mu}(n, m),
\end{equation}

in the sense of convergence in $\ell^2(N)$ and also pointwisely.
Remark 6.1. Actually, we will also establish that (73) is satisfied for all \( \lambda, \mu > 0 \), when \( n, m \in \mathbb{N} \), \( n \neq m \) and \( m/2 \leq n \leq 3m/2 \), that (72) holds for all \( \lambda > 0 \) and \( \mu > 1 \), and (73), for all \( \lambda > 1 \) and \( \mu > 0 \).

We will use the following two lemmas established in [52, p. 49 and p. 59] (see also [6, p. 400]), which refers to the ultraspherical polynomials \( \mathcal{P}_k^\lambda \) in [2].
Lemma 6.1. Assume that $\gamma > 0$ is not an integer. Then, for every $k, r \in \mathbb{N}$, and $\theta \in (0, \pi)$, \begin{equation} \mathcal{P}_k^\gamma (\cos \theta) = A_{k,r}^\gamma (\theta) + R_{k,r}^\gamma (\theta), \end{equation} where \begin{equation} A_{k,r}^\gamma (\theta) := \sum_{\ell=0}^{r-1} \frac{k!}{\Gamma(k + \ell + \gamma + 1)} \frac{\cos((k + \ell + \gamma)\theta - (\ell + \gamma)\pi/2)}{(2\sin\theta)^{\ell+\gamma}}, \end{equation} with \begin{equation} b_\ell := \frac{2}{\pi} \sin(\gamma\pi) \frac{\Gamma(2\gamma)\Gamma(\ell + \gamma)\Gamma(\ell - \gamma + 1)}{\ell! \Gamma(\gamma)}, \quad \ell = 0, \ldots, r - 1, \end{equation} and \begin{equation} |R_{k,r}^\gamma (\theta)| \leq C(k \sin \theta)^{-(r+\gamma)}, \quad \theta \in (0, \pi), \end{equation} being $C > 0$ independent of $\theta \in (0, \pi)$ and $k, r \in \mathbb{N}$.

Lemma 6.2. Assume that $\gamma \in \mathbb{N}$, $\gamma \geq 1$. Then, for every $k \in \mathbb{N}$ and $\theta \in (0, \pi)$, \begin{equation} \mathcal{P}_k^\gamma (\cos \theta) = \mathcal{A}_k^\gamma (\theta) + \mathcal{R}_k^\gamma (\theta), \end{equation} where \begin{equation} \tilde{b}_\ell := (-1)\ell 2^{2\gamma} \frac{\Gamma(2\gamma)\Gamma(\ell + \gamma)}{\Gamma(\ell)!\Gamma(\gamma - \ell)}, \quad \ell = 0, \ldots, r - 1. \end{equation} Note that if $r, k \in \mathbb{N}$, $r \leq \gamma - 1$, and $\theta \in (0, \pi)$, \begin{equation} \mathcal{P}_k^\gamma (\cos \theta) = \tilde{A}_{k,r}^\gamma (\theta) + \tilde{R}_{k,r}^\gamma (\theta), \end{equation} where \begin{equation} \tilde{A}_{k,r}^\gamma := \sum_{\ell=0}^{r-1} \frac{k!}{\Gamma(k + \ell + \gamma + 1)} \frac{\cos((k + \ell + \gamma)\theta - (\ell + \gamma)\pi/2)}{(2\sin\theta)^{\ell+\gamma}}, \end{equation} and \begin{equation} |\tilde{R}_{k,r}^\gamma (\theta)| \leq C(k \sin \theta)^{-(r+\gamma)}, \quad \theta \in (0, \pi), \end{equation} where $C > 0$ does not depend on $\theta \in (0, \pi)$ nor on $k, r \in \mathbb{N}$, $r \leq \gamma - 1$.

Proof of (74). Let $(\mu - 1)/2 < \lambda < \mu$. When $n, m \in \mathbb{N}$ and $0 \leq n \leq m/2$ or $3m/2 \leq n$, according to [6] pp. 400-401 and since $(\mu - 1)/2 < \lambda < \mu$, we have that \begin{equation} |K_{\lambda,\mu}(n, m)| \leq C \begin{cases} \frac{1}{m}, & 0 \leq n \leq \frac{m}{2}, \\ \frac{1}{n}, & n \geq \frac{3m}{2}, \end{cases} \end{equation} and (74) is established for every $n, m \in \mathbb{N}$, such that $0 \leq n \leq m/2$ or $3m/2 \leq n$.

We now assume that $n, m \in \mathbb{N}$, $m/2 \leq n \leq 3m/2$, $n \neq m$. To analyze this case we use some of the ideas in [6] and, as we can observe, in this case (74) is satisfied for every $\lambda, \mu > 0$.

Suppose that $n$ and $m$ are both even or both odd. We have that \begin{equation} K_{\lambda,\mu}(n, m) = 2 \int_0^{\pi/2} \varphi_n^\lambda (\cos \theta) \varphi_m^\lambda (\cos \theta) \sin \theta d\theta = 2 \left( \int_{0}^{1/2} + \int_{1/2}^{\pi/2} \right) \varphi_n^\lambda (\cos \theta) \varphi_m^\lambda (\cos \theta) \sin \theta d\theta 
= 2 \sqrt{w_n(m)w_m(n)} \left( \int_{0}^{1/2} + \int_{1/2}^{\pi/2} \right) \mathcal{P}_k^\lambda (\cos \theta) \mathcal{P}_m^\lambda (\cos \theta) (\sin \theta)^{-\mu} d\theta \end{equation} \begin{equation} =: K_0(n, m) + K_1(n, m). \end{equation}

By using [4], and since $|\mathcal{P}_k^\lambda (x)| \leq 1$, $x \in (-1, 1)$, $k \in \mathbb{N}$ and $\gamma > 0$, ([74] Theorem 7.33.1)], we get \begin{equation} |K_0(n, m)| \leq Cn^{\lambda+\mu} \int_0^{1/2} \theta^{\lambda+\mu} d\theta \leq \frac{C}{n}, \end{equation} where $C > 0$ does not depend on $n$. 

We use Lemmas 6.1 and 6.2 to estimate $K_1(n, m)$. Assume that $\lambda, \mu$ are not integers. By using \ref{eq:74} with $r = 2$ we get

$$K_1(n, m) = 2 \sqrt{\omega_n(m) \omega_n(n)} \int_{1/n}^{\pi/2} \left[ A_{n,2}^\mu(\theta) A_{n,2}^\lambda(\theta) + A_{n,2}^\mu(\theta) R_{n,2}^\lambda(\theta) + A_{n,2}^\mu(\theta) R_{n,2}^\lambda(\theta) \right] (\sin \theta)^{\lambda + \mu} d\theta.$$ 

We observe that, for every $\gamma > 0, k, r \in \mathbb{N}$, and $\theta \in (0, \pi/2)$,

$$|A_{k,r}(\theta)| \leq C \left( \frac{k!}{(\sin \theta)^\gamma} \sum_{\ell=0}^{r-1} \frac{1}{\Gamma(k + \ell + \gamma + 1)(\sin \theta)^\ell} \right) \leq \frac{C}{(k+1)!} \sum_{\ell=0}^{r-1} \frac{1}{(k+1)^\ell}.$$ 

Then, for every $\theta \in (1/n, \pi/2)$, we have that

$$|A_{n,2}^\mu(\theta) R_{n,2}^\lambda(\theta) + A_{n,2}^\mu(\theta) R_{n,2}^\lambda(\theta) + R_{n,2}^\mu(\theta) R_{n,2}^\lambda(\theta)| \leq C \left( \frac{1}{(n\theta)^{\lambda + \mu + 2}} + \frac{1}{(n\theta)^{\lambda + \mu + 4}} \right) \leq \frac{C}{(n\theta)^{\lambda + \mu + 2}},$$

which jointly with \ref{eq:4}, leads to

$$|K_1(n, m)| \leq C \left( n^{\lambda + \mu} \int_{1/n}^{\pi/2} \left| A_{n,2}^\mu(\theta) A_{n,2}^\lambda(\theta) (\sin \theta)^{\lambda + \mu} d\theta \right| + \frac{1}{n} \int_{1/n}^{\infty} d\theta \right).$$

(79)

Next we estimate the first summand in the last inequality. For every $k \in \mathbb{N}$ and $\gamma > 0$, we consider $\alpha_k^\gamma(\theta) := (k + \gamma)\theta - k\pi/2, \theta \in (0, \pi/2)$. Then, for every $\theta \in (0, \pi/2)$,

$$A_{n,2}^\mu(\theta) A_{n,2}^\lambda(\theta) = \frac{n!m!}{(\sin \theta)^{\gamma + \mu}} \left( b_{k}^\mu b_{k}^\lambda \cos(\alpha_k^\mu(\theta)) \cos(\alpha_k^\lambda(\theta)) + b_{k}^\mu b_{k}^\lambda \sin(\alpha_k^\mu(\theta)) \cos(\alpha_k^\lambda(\theta)) \right) + b_{k}^\mu b_{k}^\lambda \sin(\alpha_k^\mu(\theta)) \cos(\alpha_k^\lambda(\theta)) \sin(\alpha_k^\lambda(\theta) + \theta) + b_{k}^\mu b_{k}^\lambda \sin(\alpha_k^\mu(\theta)) \sin(\alpha_k^\lambda(\theta) + \theta) \sin(\alpha_k^\lambda(\theta) + \theta) \left( \frac{\sin(\alpha_k^\mu(\theta))}{n \sin \theta} + \frac{\sin(\alpha_k^\lambda(\theta) + \theta)}{n \sin \theta} \right).$$

Thus, we obtain

$$n^{\lambda + \mu} \int_{1/n}^{\pi/2} \left| A_{n,2}^\mu(\theta) A_{n,2}^\lambda(\theta) (\sin \theta)^{\lambda + \mu} d\theta \right| \leq C \left( \int_{1/n}^{\pi/2} \cos(\alpha_k^\mu(\theta)) \cos(\alpha_k^\lambda(\theta)) d\theta \right) \leq C \frac{4}{n} \sum_{j=1}^{\infty} I_j(n, m).$$

(80)

We note that

$$I_4(n, m) \leq C \frac{\theta}{n^2} \int_{1/n}^{\pi/2} \frac{\sin(\alpha_k^\mu(\theta)) + \sin(\alpha_k^\lambda(\theta)) + \sin(\alpha_k^\lambda(\theta) + \theta)}{(n \sin \theta)^2} d\theta =: C \frac{1}{n}.$$ 

(81)

For $I_1(n, m)$, we have $I_1(n, m) \leq C$ and, also, when $n + \mu \neq m + \lambda$,

$$I_1(n, m) = \frac{1}{2} \int_{1/n}^{\pi/2} \left( \cos(\alpha_k^\mu(\theta) + \alpha_k^\lambda(\theta)) + \cos(\alpha_k^\mu(\theta) - \alpha_k^\lambda(\theta)) \right) d\theta \leq C \left( \frac{\sin(\alpha_k^\mu(\theta)) + \alpha_k^\lambda(\theta))}{n + m + \lambda + \mu} \right|_{\theta=1/n}^{\pi/2} + \frac{\sin(\alpha_k^\mu(\theta) - \alpha_k^\lambda(\theta))}{n - m + \mu - \lambda} \right|_{\theta=1/n}^{\pi/2} \leq C \left( \frac{1}{n + 1} + \frac{1}{n - m + \mu - \lambda} \right).$$

Then, we get that

$$I_1(n, m) \leq C \frac{1}{n - m} \left( \frac{1}{n - m} \right), \text{ if } |n - m| \leq 2|\mu - \lambda|,$$

and

$$I_1(n, m) \leq C \left( \frac{1}{n} + \frac{1}{|n - m + \mu - \lambda|} \right) \leq C \left( \frac{1}{n} + \frac{1}{|n - m|} \right), \text{ if } |n - m| > 2|\mu - \lambda|.$$

that is,
\[ I_1(n, m) \leq \frac{C}{|n-m|}. \]

On the other hand, we have that
\[ 2\sin(\alpha_n^\mu(\theta) + \theta) \cos(\alpha_m^\mu(\theta)) = \sin(\alpha_n^\mu(\theta) + \alpha_m^\mu(\theta) + \theta) + \sin(\alpha_n^\mu(\theta) - \alpha_m^\mu(\theta) + \theta) \]
\[ = \sin((n + m + \mu + \lambda + 1)\theta) \cos((\lambda + \mu)\pi/2) - \cos((n + m + \mu + \lambda + 1)\theta) \sin((\lambda + \mu)\pi/2) \]
\[ + \sin((n - m + \mu + \lambda + 1)\theta) \cos((\mu - \lambda)\pi/2) - \cos((n - m + \mu + \lambda + 1)\theta) \sin((\mu - \lambda)\pi/2). \]

Let \( A := n + m + \mu + \lambda + 1 \) or \( A := n - m + \mu + \lambda + 1 \). We are going to estimate
\[ J_1 := \left| \int_{1/n}^{\pi/2} \frac{\sin(A\theta)}{\sin \theta} d\theta \right| \quad \text{and} \quad J_2 := \left| \int_{1/n}^{\pi/2} \frac{\cos(A\theta)}{n \sin \theta} d\theta \right|, \]
when \( A \neq 0 \). We can write
\[ J_1 \leq \frac{1}{n} \left( \int_{1/n}^{\pi/2} \left| \frac{1}{\sin \theta} - \frac{1}{\theta} \right| d\theta + \int_{1/n}^{\pi/2} \frac{\sin(\theta) \sin |A|}{\theta} d\theta \right) \leq \frac{1}{n} \left( \int_{1/n}^{\pi/2} \frac{\theta}{\sin \theta} d\theta + \int_{|A|/n}^{\pi/2} \frac{\sin u}{u} du \right) \]
\[ \leq \frac{C}{n} \left( 1 + \int_{|A|/n}^{\pi/2} \frac{\cos u}{u} du \right) \leq \frac{C}{n} \left( 1 + \frac{1}{|A|} \right). \]

In a similar way,
\[ J_2 \leq \frac{C}{n} \left( 1 + \int_{|A|/n}^{\pi/2} \frac{\cos u}{u} du \right) \leq \frac{C}{n} \left( 1 + \frac{1}{|A|} \right). \]

Thus, we can deduce that, when \( n + \mu \neq m + \lambda + 1 \), and \( n + \mu \neq m + \lambda - 1 \),
\[ I_2(n, m) + I_3(n, m) \leq \frac{C}{n} \left( 1 + \frac{1}{|n - m + \mu + \lambda + 1|} + \frac{1}{|n - m + \mu - \lambda + 1|} \right). \]

Then, if \( |n - m| > 2 \max\{|\mu - \lambda + 1|, |\mu - \lambda - 1|\} \),
\[ I_2(n, m) + I_3(n, m) \leq \frac{C}{n} \left( 1 + \frac{1}{|n - m + \mu + \lambda + 1|} + \frac{1}{|n - m + \mu - \lambda - 1|} \right) \leq \frac{C}{n} \left( 1 + \frac{1}{|n - m|} \right). \]

In the case that \( |n - m| \leq 2 \max\{|\mu - \lambda + 1|, |\mu - \lambda - 1|\} \), we write
\[ I_2(n, m) + I_3(n, m) \leq \frac{C}{n} \int_{1/n}^{\pi/2} \frac{d\theta}{\theta} \leq \frac{C}{n} \log \left( \frac{n}{2} + 1 \right) \leq \frac{C}{|n - m|}. \]

Hence,
\[ I_2(n, m) + I_3(n, m) \leq \frac{C}{|n - m|}. \]

By considering estimations \((79)-(83)\) we get
\[ |K_1(n, m)| \leq C \left( \frac{1}{n} + \frac{1}{|n - m|} \right) \leq \frac{C}{|n - m|}, \]
which, jointly \((76)\) and \((77)\), leads to
\[ K_{\lambda,\mu}(n, m) \leq \frac{C}{|n - m|}, \quad \frac{m}{2} \leq n \leq \frac{3m}{2}, \quad n \neq m, \]
provided that \( m/2 \leq n \leq 3m/2 \) and \( \lambda, \mu \) are not integers.

When \( \lambda \) or \( \mu \) is an integer and \( m/2 \leq n \leq 3n/2 \), \((71)\) can be proved in a similar way by using also \((75)\).

\begin{proof}[Proof of \((72)\) for \( K_{\lambda,\mu}^\epsilon \)]
Here we need assume \( \mu > 1 \). Suppose that \( \lambda \) and \( \mu \) are not integers. In other cases we can proceed in a similar way.

It is sufficient to see that, for every \( n, m \in \mathbb{N}, m/2 \leq n \leq 3m/2, \ n \neq m, \)
\[ |K_{\lambda,\mu}^\epsilon(n + 1, m) - K_{\lambda,\mu}^\epsilon(n, m)| \leq \frac{C}{|n - m|^2}. \]

Let \( n, m \in \mathbb{N} \) such that \( n \neq m \) and \( m/2 \leq n \leq 3m/2 \). We can write
\[ K_{\lambda,\mu}^\epsilon(n + 1, m) - K_{\lambda,\mu}^\epsilon(n, m) = 2 \int_0^1 [\varphi_{2n+2}(x) - \varphi_{2n}(x)] \psi_{2n}^\lambda(x) dx \]
Hence, by (71) and Remark 6.1 we deduce that

\[ = 2 \int_0^1 \left[ \varphi_{2n+2}^\Lambda(x) - \frac{\sqrt{w_{\mu}(2n+2)}}{\sqrt{w_{\mu}(2n)}} \varphi_{2n}^\Lambda(x) \right] \varphi_{2n}^\Lambda(x) \, dx \\
+ 2 \left[ \frac{\sqrt{w_{\mu}(2n+2)}}{\sqrt{w_{\mu}(2n)}} - 1 \right] \int_0^1 \varphi_{2n}^\Lambda(x) \varphi_{2n}^\Lambda(x) \, dx \\
= : H_1(n, m) + H_2(n, m). \]

We have that

\[
\left| \frac{w_{\mu}(2n+2)}{w_{\mu}(2n)} - 1 \right| = \left| \frac{w_{\mu}(2n+2)}{w_{\mu}(2n)} + 1 \right| \leq \frac{w_{\mu}(2n+2)}{w_{\mu}(2n)} - 1 \\
= \frac{(2\mu + 2n + 1)(2\mu + 2n)(2n + 2 + \mu)}{(2n + 2)(2n + 1)(2n + \mu)} - 1 \leq \frac{C}{n}.
\]

Hence, by (71) and Remark 6.1 we deduce that

\[ |H_2(n, m)| \leq \frac{C}{n} |K_{\Lambda, \mu}(2n, 2m)| \leq \frac{C}{n|n - m|}. \]

On the other hand, according to [51 (4.7.29)], for every \( x \in (-1, 1) \),

\[
\mathcal{P}_{2n+2}^\mu(x) - \mathcal{P}_{2n}^\mu(x) = \left( \frac{2(n + 2)2n + 1}{(2\mu + 2n + 1)(2\mu + 2n)} - 1 \right) \mathcal{P}_{2n}^\mu(x) + \frac{2(2\mu - 1)(2n + 1 + \mu)}{(2\mu + 2n + 1)(2\mu + 2n)} \mathcal{P}_{2n+2}^\mu(x).
\]

Then, we can decompose \( H_1(n, m) \) as follows:

\[
H_1(n, m) = 2 \sqrt{w_{\mu}(2n+2)}w_{\Lambda}(2m) \int_0^{\pi/2} \left[ \mathcal{P}_{2n+2}^\mu(\cos \theta) - \mathcal{P}_{2n}^\mu(\cos \theta) \right] \mathcal{P}_{2n}^\Lambda(\cos \theta)(\sin \theta)^{\lambda + \mu} d\theta \\
= 2 \sqrt{w_{\mu}(2n+2)}w_{\Lambda}(2m) \\
\times \left[ \left( \frac{2(n + 2)2n + 1}{(2\mu + 2n + 1)(2\mu + 2n)} - 1 \right) \int_0^{\pi/2} \mathcal{P}_{2n}^\mu(\cos \theta) \mathcal{P}_{2n}^\Lambda(\cos \theta)(\sin \theta)^{\lambda + \mu} d\theta \\
+ 2 \frac{(2\mu - 1)(2n + 1 + \mu)}{(2\mu + 2n + 1)(2\mu + 2n)} \int_0^{\pi/2} \mathcal{P}_{2n+2}^{\mu-1}(\cos \theta) \mathcal{P}_{2n+2}^\Lambda(\cos \theta)(\sin \theta)^{\lambda + \mu} d\theta \right] \\
= : H_{1,1}(n, m) + H_{1,2}(n, m).
\]

It is not hard to see that

\[ |H_{1,1}(n, m)| \leq \frac{C}{n} \frac{\sqrt{w_{\mu}(2n+2)}}{w_{\mu}(2n)} |K_{\Lambda, \mu}(2n, 2m)| \leq \frac{C}{n|n - m|}. \]

Then, by using [41, (71)] and Remark 6.1 we obtain that

\[ |H_{1,1}(n, m)| \leq \frac{C}{n} \frac{\sqrt{w_{\mu}(2n+2)}}{w_{\mu}(2n)} |K_{\Lambda, \mu}(2n, 2m)| \leq \frac{C}{n|n - m|}. \]

Also, from [41, 88] (b) and since \(|\mathcal{P}_k^\Lambda(x)| \leq 1, \gamma > 0, k \in \mathbb{N}, x \in (-1, 1)\), we have that

\[
|H_{1,2}(n, m)| \leq Cn^{\lambda + \mu - 1} \left( \int_0^{1/n} + \int_{1/n}^{\pi/2} \right) \mathcal{P}_{2n+2}^{\mu-1}(\cos \theta) \mathcal{P}_{2n}^\Lambda(\cos \theta)(\sin \theta)^{\lambda + \mu} d\theta \\
\leq Cn^{\lambda + \mu - 1} \left( \int_0^{1/n} \theta^{\lambda + \mu} d\theta + \int_{1/n}^{\pi/2} \mathcal{P}_{2n+2}^{\mu-1}(\cos \theta) \mathcal{P}_{2n}^\Lambda(\cos \theta)(\sin \theta)^{\lambda + \mu} d\theta \right) \\
\leq C \left( \frac{1}{n^2} + n^{\lambda + \mu - 1} \int_{1/n}^{\pi/2} \mathcal{P}_{2n+2}^{\mu-1}(\cos \theta) \mathcal{P}_{2n}^\Lambda(\cos \theta)(\sin \theta)^{\lambda + \mu} d\theta \right).
\]

Lemma 6.1 and (78) with \( r = 3 \) leads to

\[
\left| \int_{1/n}^{\pi/2} \mathcal{P}_{2n+2}^{\mu-1}(\cos \theta) \mathcal{P}_{2n}^\Lambda(\cos \theta)(\sin \theta)^{\lambda + \mu} d\theta \right| \leq \int_{1/n}^{\pi/2} A_{2n+2,3}^{\mu-1}(\theta)A_{2n,3}^\Lambda(\theta)(\sin \theta)^{\lambda + \mu} d\theta \\
+ \int_{1/n}^{\pi/2} |A_{2n+2,3}^{\mu-1}(\theta)R_{2n,3}^\Lambda(\theta) + A_{2n,3}^\Lambda(\theta)R_{2n+2,3}^{\mu-1}(\theta) + R_{2n,3}^{\mu-1}(\theta)R_{2n+2,3}^\Lambda(\theta)(\sin \theta)^{\lambda + \mu} d\theta \right|.
\]
Also, we have that

\[ J(\theta) \quad \text{and the method employed to estimate (83) allows us also to obtain that} \]

\[ \sum_{j=1}^{\infty} \frac{\theta^\mu + \lambda}{(n\theta)^{\lambda + \mu + 1}}. \]

To analyze the integral term we consider, as before, \( \alpha_k^\mu(\theta) := (k + \gamma)\theta - \gamma\pi/2, \) \( k \in \mathbb{N}, \gamma > 0 \) and \( \theta \in (0, \pi/2) \).

\[
A_{2n+2,3}^\lambda(\theta)A_{2m,3}^\lambda(\theta) = \frac{(2n + 2)!(2n)!}{(\sin \theta)^{\lambda + \mu + 1}} \left( b_0^{-1} b_0^\lambda \cos(\alpha_{2n+2}^\mu(\theta)) \cos(\alpha_{2m}^\lambda(\theta)) \right.
\]

\[
\left. + b_0^{-1} b_1^\lambda \frac{\cos(\alpha_{2n+2}^\mu(\theta)) \sin(\alpha_{2m}^\lambda(\theta) + \theta)}{\Gamma(2n + 2 + \mu)\Gamma(2m + \lambda + 2)} \sin \theta \right) + b_1^{-1} b_1^\lambda \frac{\sin(\alpha_{2n+2}^\mu(\theta) + \theta) \cos(\alpha_{2m}^\lambda(\theta))}{\Gamma(2n + 3 + \mu)\Gamma(2m + 1 + \lambda + 1)} \sin \theta \right)
\]

\[
- b_1^{-1} b_2^\lambda \frac{\sin(\alpha_{2n+2}^\mu(\theta) + \theta) \cos(\alpha_{2m}^\lambda(\theta) + 2\theta)}{\Gamma(2n + 3 + \mu)\Gamma(2m + 2 + \lambda + 2)} \cos(\alpha_{2m}^\lambda(\theta)) \quad \text{and (83) allows us also to obtain that}
\]

\[ J_2(n, m) + J_4(n, m) \leq \frac{C}{n|n - m|}, \]

and the method employed to estimate (83) allows us also to obtain that

\[ J_3(n, m) + J_5(n, m) + J_7(n, m) \leq \frac{C}{n|n - m|}. \]

Also, we have that

\[ J_6(n, m) + J_8(n, m) + J_9(n, m) \leq C \left( \frac{1}{n^3} \int_{1/n}^1 \frac{d\theta}{\theta^2} + \frac{1}{n^3} \int_{1/n}^1 \frac{d\theta}{\theta^3} \right) \leq \frac{C}{n^2}. \]

Finally, it is not hard to see that

\[
\cos(\alpha_{2n+2}^\mu(\theta)) \cos(\alpha_{2m}^\lambda(\theta)) \sin \theta = - \sin(\alpha_{2n+2}^\mu(\theta)) \cos(\alpha_{2m}^\lambda(\theta)) \sin \theta
\]

\[
= \frac{1}{4} \left( \cos(\alpha_{2n+1}^\mu(\theta) + \alpha_{2m}^\lambda(\theta) + \theta) + \cos(\alpha_{2n+1}^\mu(\theta) - \alpha_{2m}^\lambda(\theta) + \theta)
\]

\[
- \cos(\alpha_{2n+1}^\mu(\theta) + \alpha_{2m}^\lambda(\theta) - \theta) - \cos(\alpha_{2n+1}^\mu(\theta) - \alpha_{2m}^\lambda(\theta) - \theta) \right).
Then, we can write, when $2m + \lambda \neq 2n + 2 + \mu$ and $2m + \lambda \neq 2n + \mu$,
\[
J_1(n, m) = \frac{1}{4} \left[ \frac{\sin(\alpha_{2n+1}^\mu(\theta) + \alpha_{2m}^\lambda(\theta) + \theta)}{2n + 2 + \mu + 2m + \lambda} + \frac{\sin(\alpha_{2n+1}^\mu(\theta) - \alpha_{2m}^\lambda(\theta) + \theta)}{2n + 2 + \mu - 2m - \lambda}\right. \\
- \frac{\sin(\alpha_{2n+1}^\mu(\theta) + \alpha_{2m}^\lambda(\theta) - \theta)}{2n + \mu + 2m + \lambda} - \frac{\sin(\alpha_{2n+1}^\mu(\theta) - \alpha_{2m}^\lambda(\theta) - \theta)}{2n + \mu - 2m - \lambda} \right]_{\theta = \pi/2}^{\theta = 1/n} \\
= \frac{1}{4} \left| F_{n,m}(\theta) \right|^{\theta = \pi/2}_{\theta = 1/n}.
\]
We observe that $F_{n,m}(\pi/2) = 0$ and also, we can write
\[
F_{n,m}(\theta) = \sin(\alpha_{2n+1}^\mu(\theta) + \alpha_{2m}^\lambda(\theta) + \theta) \left[ \frac{1}{2n + 2 + \mu + 2m + \lambda} - \frac{1}{2n + \mu + 2m + \lambda} \right] \\
+ \frac{1}{2n + \mu + 2m + \lambda} \left[ \sin(\alpha_{2n+1}^\mu(\theta) + \alpha_{2m}^\lambda(\theta) + \theta) - \sin(\alpha_{2n+1}^\mu(\theta) + \alpha_{2m}^\lambda(\theta) - \theta) \right] \\
+ \sin(\alpha_{2n+1}^\mu(\theta) - \alpha_{2m}^\lambda(\theta) + \theta) \left[ \frac{1}{2n + 2 + \mu - 2m - \lambda} - \frac{1}{2n + \mu - 2m - \lambda} \right] \\
+ \frac{1}{2n + \mu - 2m - \lambda} \left[ \sin(\alpha_{2n+1}^\mu(\theta) - \alpha_{2m}^\lambda(\theta) + \theta) - \sin(\alpha_{2n+1}^\mu(\theta) - \alpha_{2m}^\lambda(\theta) - \theta) \right].
\]
Hence, when $2m + \lambda \neq 2n + 2 + \mu$ and $2m + \lambda \neq 2n + \mu$, by using mean value theorem we obtain
\[
J_1(n, m) \leq C \left| F_{n,m}(\frac{1}{n}) \right| \\
\leq C \left( \frac{1}{n^2} + \frac{1}{2n + \mu + 2m + \lambda} \right) \\
= C \left( \frac{1}{n^2} + \frac{1}{2n + \mu + 2m + \lambda} + \frac{1}{n(2n + \mu - 2m - \lambda)} \right).
\]
Then, if $|n - m| > \max\{|\mu + 2 - \lambda|, |\mu - \lambda|\}$, we have
\[
J_1(n, m) \leq C \left( \frac{1}{n^2} + \frac{1}{(2|n - m| - |\mu + 2 - \lambda|)(2|n - m| - |\mu - \lambda|)} + \frac{1}{n(2|n - m| - |\mu - \lambda|)} \right) \\
\leq \frac{C}{n^2}. \\
\]
On the other hand, if $|n - m| \leq \max\{|\mu + 2 - \lambda|, |\mu - \lambda|\}$,
\[
J_1(n, m) \leq C \int_{\frac{\pi}{2}}^{1/n} d\theta \leq C \frac{C}{n^2},
\]
and we can deduce that
\[
J_1(n, m) \leq \frac{C}{n^2}.
\]
By joining estimations $[90]-[96]$ it follows that
\[
|H_{1,2}(n, m)| \leq \frac{C}{n^2}. \\
\]
Then, by taking into account $[85]-[87], [89]$ and $[47]$ we conclude property $[84]$. The same procedure allows us to show $[72]$ for $K^\circ_{\lambda,\mu}$.

Proof of $[73]$ for $K^\circ_{\lambda,\mu}$. We observe that
\[
K_{\lambda,\mu}(n, m) = K_{\mu,\lambda}(m, n), \quad n, m \in \mathbb{N}.
\]
Then, since $\lambda > 1$, from $[72]$ we obtain $[73]$.

Suppose that $(\mu - 1)/2 < \lambda < \mu$. According to Theorem $2.1$ the operators $\mathcal{F}^\circ_{\lambda,\mu}$ and $\mathcal{F}^\circ_{\mu,\lambda}$ are bounded
(a) from $\ell^1(N, \omega)$ into $\ell^{1,\infty}(N, \omega)$, for every $\omega \in A_1(N)$;
(b) from $\ell^p(N, \omega)$ into itself, for every $1 < p < \infty$ and $\omega \in A_p(N)$.

Let $1 \leq p < \infty$ and $\omega \in A_p(N)$. We denote by $\mathcal{T}^\circ_{\lambda,\mu}$ and $\mathcal{T}^\circ_{\mu,\lambda}$ the bounded extensions to $\ell^p(N, \omega)$ of $\mathcal{F}^\circ_{\lambda,\mu}$ and $\mathcal{F}^\circ_{\mu,\lambda}$, respectively, and define $\tilde{\omega}(k) := \omega(2k)$ and $\tilde{\omega}(k) := \omega(2k + 1), k \in \mathbb{N}$. Then,
\(\omega, \hat{\omega} \in A_p(\mathbb{N})\). We consider the operator \(T_{\lambda,\mu}\) as follows:

\[
T_{\lambda,\mu}(f)(n) := \begin{cases} 
T_{\lambda,\mu}(\hat{f})(m), & \text{if } n = 2m, \\
T_{\lambda,\mu}(\tilde{f})(m), & \text{if } n = 2m + 1,
\end{cases} \quad f \in \ell^p(\mathbb{N}, \omega).
\]

Note that \(T_{\lambda,\mu}(g) = \mathcal{F}_{\lambda,\mu}(g), g \in \ell^2(\mathbb{N})\).

We have that, if \(1 < p < \infty\) and \(f \in \ell^p(\mathbb{N}, \omega)\),

\[
\sum_{n=0}^{\infty} |T_{\lambda,\mu}(f)(n)|^p \omega(n) = \sum_{n=0}^{\infty} |T_{\lambda,\mu}(f)(2n)|^p \omega(2n) + \sum_{n=0}^{\infty} |T_{\lambda,\mu}(f)(2n+1)|^p \omega(2n+1)
\]

\[
= \sum_{n=0}^{\infty} |T_{\lambda,\mu}(\hat{f})(n)|^p \hat{\omega}(n) + \sum_{n=0}^{\infty} |T_{\lambda,\mu}(\tilde{f})(n)|^p \tilde{\omega}(n)
\]

\[
\leq C \left( \sum_{n=0}^{\infty} |\hat{f}(n)|^p \omega(2n) + \sum_{n=0}^{\infty} |\tilde{f}(n)|^p \omega(2n+1) \right) = C \sum_{n=0}^{\infty} |f(n)|^p \omega(n).
\]

Hence, \(T_{\lambda,\mu}\) is bounded from \(\ell^p(\mathbb{N}, \omega)\) into itself, when \(1 < p < \infty\). In a similar way we can see that \(T_{\lambda,\mu}\) is bounded from \(\ell^1(\mathbb{N}, \omega)\) into \(\ell^{1,\infty}(\mathbb{N}, \omega)\).

Moreover, for every \(f \in \ell^p(\mathbb{N}, \omega)\),

\[
T_{\lambda,\mu}(f)(n) = \lim_{k \to \infty} \sum_{m=0}^{k} K_{\lambda,\mu}(n, m) f(m),
\]

where the convergence of the series can be understood in \(\ell^p(\mathbb{N}, \omega)\), when \(1 < p < \infty\); in \(\ell^{1,\infty}(\mathbb{N}, \omega)\), and also, pointwisely. This property justifies to write \(\mathcal{F}_{\lambda,\mu} = T_{\lambda,\mu}\) in \(\ell^p(\mathbb{N}, \omega), 1 \leq p < \infty\).

Suppose now that \(1 < \lambda < \mu\) and consider \(r \in \mathbb{N}\) such that \(\mu \in (\lambda + r, \lambda + r + 1]\). We have that

\[
\mathcal{F}_{\lambda,\mu} = \mathcal{F}_{\lambda+1,\mu+1} \circ \mathcal{F}_{\lambda+r,\mu+r} \circ \mathcal{F}_{\lambda+r-1,\mu+r-1} \circ \cdots \circ \mathcal{F}_{\lambda+1,\mu+1} \circ \mathcal{F}_{\lambda,\mu+1},
\]

on \(\ell^2(\mathbb{N})\).

Hence, if \(1 < p < \infty\) and \(\omega \in A_p(\mathbb{N})\), the operator \(\mathcal{F}_{\lambda,\mu}\) is bounded from \(\ell^p(\mathbb{N}, \omega)\) into itself.

According to Plancherel equality we can write, for every \(f, g \in \ell^2(\mathbb{N})\),

\[
\sum_{n \in \mathbb{N}} (\mathcal{F}_{\lambda,\mu} f)(n) g(n) = \int_{-1}^{1} \mathcal{F}_{\lambda,\mu}(f)(x) \mathcal{F}_{\lambda,\mu}(g)(x) dx = \sum_{n \in \mathbb{N}} f(n) \mathcal{F}_{\lambda,\mu}(g)(n).
\]

Then, \(\mathcal{F}_{\lambda,\mu}\) is bounded from \(\ell^p(\mathbb{N}, \omega)\) into itself, for every \(1 < p < \infty\) and \(\omega \in A_p(\mathbb{N})\).

Let now \(\lambda, \mu > 1, 1 < p < \infty\) and \(\omega \in A_p(\mathbb{N})\). Since \(\mathcal{F}_{\lambda,\mu} \mathcal{F}_{\lambda,\mu} f = f, f \in \ell^2(\mathbb{N})\), we get

\[
\|f\|_{\ell^p(\mathbb{N}, \omega)} \leq C \|\mathcal{F}_{\lambda,\mu} f\|_{\ell^p(\mathbb{N}, \omega)}, f \in \ell^p(\mathbb{N}, \omega),
\]

because \(\mathcal{F}_{\lambda,\mu}\) and \(\mathcal{F}_{\lambda,\mu}\) are bounded operators from \(\ell^p(\mathbb{N}, \omega)\) into itself.

Thus the proof of this theorem is finished.

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