Gravitational Waves from Rotating Black Strings

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Abstract

In the Randall-Sundrum two-brane model (RS1), a Kerr black hole on the brane can be naturally identified with a section of rotating black string. To estimate Kaluza-Klein (KK) corrections on gravitational waves emitted by perturbed rotating black strings, we give the effective Teukolsky equation on the brane which is separable equation and hence numerically manageable. In this process, we derive the master equation for the electric part of the Weyl tensor $E_{\mu\nu}$ which would be also useful to discuss the transition from black strings to localized black holes triggered by Gregory-Laflamme instability.

1 Introduction

The recent progress of superstring theory has provided a new picture of our universe, the so-called braneworld. The evidence of the extra dimensions in this scenario should be explored in the early stage of the universe or the black hole. In particular, gravitational waves are key probes because they can propagate into the bulk freely. Cosmology in this scenario has been investigated intensively. While, gravitational waves from black holes have been less studied so far. In this paper, we shall take a step toward this direction.

Here, we will concentrate on a two-brane model which is proposed by Randall and Sundrum as a simple and phenomenologically interesting model [1]. In this RS1 model, the large black hole on the brane is expected to be black string. Hence, it would be important to clarify how the gravitational waves are generated in the perturbed black string system and how the effects of the extra dimensions come into the observed signal of the gravitational waves. It is desired to have a basic formalism for analyzing gravitational waves generated by perturbed rotating black string.

It is well known in general relativity that the perturbation around Kerr black hole is elegantly treated in the Newman-Penrose formalism. Indeed, Teukolsky derived a separable master equation for the gravitational waves in the Kerr black hole background [2]. The main purpose of this paper is to extend the Teukolsky formalism to the braneworld context and derive the effective Teukolsky equation.

The organization of this paper is as follows. In sec.II, we present the model and demonstrate the necessity of solving $E_{\mu\nu}$ in deriving the effective Teukolsky equation. In sec.III, a perturbed equation and the junction conditions for $E_{\mu\nu}$ are obtained. In sec.IV, we give the explicit solution for $E_{\mu\nu}$ using the gradient expansion method. Then, the effective Teukolsky equation is presented. The final section is devoted to the conclusion.

2 Teukolsky Equation on the Brane

Based on the Newmann-Penrose (NP) null-tetrad formalism, in which the tetrad components of the curvature tensor are the fundamental variables, a master equation for the curvature perturbation was developed by Teukolsky for a Kerr black hole with source. The master equation is called the Teukolsky equation, and it is a wave equation for a null-tetrad component of the Weyl tensor $\Psi_0 = -C_{pqrs}^{\ell p \ell q \ell r \ell s}$ or $\Psi_4 = -C_{pqrs}n^p \bar{m}^q \bar{m}^r \bar{m}^s$, where $C_{pqrs}$ is the Weyl tensor and $\ell, n, m, \bar{m}$ are null basis in the NP formalism. All information about the gravitational radiation flux at infinity and at the event horizon can be extracted from $\Psi_0$ and $\Psi_4$. The Teukolsky equation is constructed by combining the Bianchi identity with the Einstein equations. The Riemann tensor in the Bianchi identity is written in terms of the Weyl
tensor and the Ricci tensor. The Ricci tensor is replaced by the matter fields using the Einstein equations. In this way, the Bianchi identity becomes no longer identity and one can get a master equation in which the curvature tensor is the fundamental variable.

It is of interest to consider the four-dimensional effective Teukolsky equation in braneworld to investigate gravitational waves from perturbed rotating black strings. We consider an $S_1/Z_2$ orbifold space-time with the two branes as the fixed points. In this RS1 model, the two 3-branes are embedded in AdS$_5$ with the curvature radius $\ell$ and the brane tensions given by $\sigma_\parallel = 6/(\kappa^2 \ell)$ and $\sigma_\parallel = -6/(\kappa^2 \ell)$. Our system is described by the action

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( \mathcal{R} + \frac{12}{\ell^2} \right) - \sum_{i=\parallel,\parallel} \sigma_i \int d^4x \sqrt{-g_{^\parallel\text{-brane}}} + \sum_{i=\parallel,\parallel} \int d^4x \sqrt{-g_{^\parallel\text{-brane}}} \mathcal{L}_i^{\text{matter}} ,$$  

where $(g_{\mu\nu}, \mathcal{R}, g_{^{\text{brane}}}^{\mu\nu}$, and $\kappa^2$ are the 5-dimensional metric, the 5-dimensional scalar curvature, the induced metric on the $i$-brane, and the 5-dimensional gravitational constant, respectively.

Since the Bianchi identity is independent of dimensions, what we need is the projected Einstein equation

$$\mathcal{L}_i^{\text{Einstein}} = \frac{1}{2} (\Delta + 3 \gamma - \gamma^* + 4 \mu + \mu^*) (D + 4 \rho - \delta^* - \tau^* + \beta^* + 3 \alpha + 4 \pi) (\delta - \tau + 4 \beta) - 3 \Psi_2 \partial_\Psi_4$$

where $\left( D + 4 \rho - \frac{4}{\ell^2} \right) \delta E_{\mu\nu}$ is described as a source term in the projected Teukolsky equation. It should be stressed that the projected Teukolsky equation on the brane derived by Shiromizu, Maeda and Sasaki \[3\]. The first order perturbation of the projected Einstein equation is

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \delta E_{\mu\nu} ,$$

where $8\pi G = \kappa^2 / \ell$. If we replace the Ricci curvature in the Bianchi identity to the matter fields and $\delta E_{\mu\nu}$ using Eq. \[2\], then the projected Teukolsky equation on the brane is written in the following form,

$$[(\Delta + 3 \gamma - \gamma^* + 4 \mu + \mu^*) (D + 4 \rho - \delta^* - \tau^* + \beta^* + 3 \alpha + 4 \pi) (\delta - \tau + 4 \beta) - 3 \Psi_2] \delta \Psi_4$$

Here our notation follows that of \[2\]. We see the effects of a fifth dimension, $\delta E_{\mu\nu}$, is described as a source term in the projected Teukolsky equation. It should be stressed that the projected Teukolsky equation on the brane Eq. \[3\] is not a closed system yet. One must solve the gravitational field in the bulk to obtain $\delta E_{\mu\nu}$.

### 3 Master Equation for $\delta E_{\mu\nu}$

Starting with the 5-dimensional Bianchi identities, we can derive the perturbed equation for $\delta E_{\mu\nu}$. The background we consider is a Ricci flat string without source ($T_{\mu\nu} = 0$) whose metric is written as

$$ds^2 = dy^2 + e^{-2\pi} g_{\mu\nu}(x^\mu) dx^\mu dx^\nu ,$$

where $g_{\mu\nu}(x^\mu)$ is supposed to be the Ricci flat metric. The equation of motion for $\delta E_{\mu\nu}$ in the bulk is found as

$$\left( \partial_y^2 - \frac{4}{\ell} \partial_y + \frac{4}{\ell^2} \right) \delta E_{\mu\nu} = -e^{2\pi} \tilde{\mathcal{L}}_{\mu\nu}^{\alpha\beta} \delta E_{\alpha\beta} \equiv -e^{2\pi} \tilde{\mathcal{L}} \delta E_{\mu\nu} ,$$

where $\tilde{\mathcal{L}}_{\mu\nu}^{\alpha\beta}$ stands for the Lichnerowicz operator,

$$\tilde{\mathcal{L}}_{\mu\nu}^{\alpha\beta} = \Box \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + 2 R_{\mu}^{\alpha} \nu^{\beta} .$$

Here, the covariant derivative and the Riemann tensor are constructed from $g_{\mu\nu}(x)$. 
On the other hand, the junction conditions on each branes become

\[ e^{2\hat{y}} \left[ e^{-2\hat{y}} \delta E_{\mu\nu} \right]_{y=0} = -\frac{\kappa^2}{6} T_{\mu\nu} - \frac{\kappa^2}{2} \hat{L}_{\mu\nu}^{\alpha\beta} \left( T_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} T \right), \]

\[ e^{2\hat{y}} \left[ e^{-2\hat{y}} \delta E_{\mu\nu} \right]_{y=d} = \frac{\kappa^2}{6} T_{\mu\nu} + \frac{\kappa^2}{2} \hat{L}_{\mu\nu}^{\alpha\beta} \left( T_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} T \right). \]

Let \( \phi(x) \) and \( \hat{\phi}(x) \) be the scalar fields on each branes which satisfy

\[ \square \phi = \frac{\kappa^2}{6} T, \quad \square \hat{\phi} = \frac{\kappa^2}{6} T, \]

respectively. In the Ricci flat space-time, the identity

\[ (\square \phi)_{|\mu\nu} = \hat{L}_{\mu\nu}^{\alpha\beta} (\phi_{|\alpha\beta}) \]

holds. Thus, Eqs. (7) and (8) can be rewritten as

\[ e^{2\hat{y}} \left[ e^{-2\hat{y}} \delta E_{\mu\nu} \right]_{y=0} = -\hat{L} \left( \phi_{|\mu\nu} - g_{\mu\nu} \square \phi \right) - \frac{\kappa^2}{2} \hat{L} T_{\mu\nu} \equiv \hat{L} S_{\mu\nu} , \]

\[ e^{2\hat{y}} \left[ e^{-2\hat{y}} \delta E_{\mu\nu} \right]_{y=d} = \hat{L} \left( \hat{\phi}_{|\alpha\beta} - g_{\mu\nu} \square \hat{\phi} \right) + \frac{\kappa^2}{2} \hat{L} T_{\mu\nu} \equiv -\hat{L} S_{\mu\nu} . \]

The scalar fields \( \phi \) and \( \hat{\phi} \) can be interpreted as the brane fluctuation modes.

4 Effective Teukolsky Equation

Now, we solve Eq. (5) under the boundary conditions (11) and (12) using the gradient expansion method.

4.1 Gradient Expansion Method

It is known that the Gregory-Laflamme instability occurs if the curvature length scale of the black hole \( L \) is less than the Compton wavelength of KK modes \( \sim \ell \exp(d/\ell) \). As we are interested in the stable rotating black string, we assume

\[ \epsilon = \left( \frac{\ell}{T} \right)^2 \ll 1 . \]

This means that the curvature on the brane can be neglected compared with the derivative with respect to \( y \). Our iteration scheme consists in writing the Weyl tensor \( E_{\mu\nu} \) in the order of \( \epsilon \) [4]. Hence, we will seek the Weyl tensor as a perturbative series

\[ \delta E_{\mu\nu}(y, x^\mu) = \delta E_{\mu\nu}(y, x^\mu) + \delta E_{\mu\nu}(y, x^\mu) + \delta E_{\mu\nu}(y, x^\mu) + \cdots . \]

Substituting this Eq. (14) into Eq. (5), we can obtain \( \delta E_{\mu\nu} \) perturbatively.

4.2 Effective Teukolsky Equation

As a result, Teukolsky equation takes the following form

\[ \hat{P} \delta \Psi_4 = \hat{Q} (8\pi G T_{nm} - \delta E_{nm}) + \cdots , \]

where \( \hat{P} \) and \( \hat{Q} \) are operators defined in [4]. What we needed is \( \delta E_{\mu\nu} \) in the above equation. We can write down \( \delta E_{\mu\nu} \) on the brane up to the second order as [4]

\[ \delta E_{\mu\nu}|_{y=0} = \frac{2}{\ell} \left[ \hat{S}_{\mu\nu} + \Omega^2 \hat{S}_{\mu\nu} \right] + \left[ \frac{1}{2} + \frac{d}{\ell} \frac{1}{\Omega^2 - 1} \right] \frac{\ell}{1 - \Omega^2} \left[ \hat{L} \hat{S}_{\mu\nu} + \Omega^4 \hat{L} \hat{S}_{\mu\nu} \right] + \frac{\ell}{4} \left[ \hat{L} \hat{S}_{\mu\nu} + \Omega^2 \hat{L} \hat{S}_{\mu\nu} \right] , \]

(16)
where

\[
\begin{align*}
\tilde{S}_{\mu\nu} &= -\phi_{\mu\nu} + g_{\mu\nu}\Box\phi - \frac{\kappa^2}{2}T_{\mu\nu}, \\
\tilde{S}'_{\mu\nu} &= -\phi_{\mu\nu} + g_{\mu\nu}\Box\phi - \frac{\kappa^2}{2}T_{\mu\nu}.
\end{align*}
\]

Substituting this \( \delta E_{\mu\nu} \) into (4), we get the effective Teukolsky equation on the brane. From Eq. (17), we see KK corrections give extra sources to Teukolsky equation. To obtain quantitative results, we must resort to numerical calculations. It should be stressed that the effective Teukolsky equation is separable like as the conventional Teukolsky equation. Therefore, it is suitable for numerical treatment.

Notice that our result in this section assume only Ricci flatness. If we do not care about separability, we can study the gravitational waves in the general Ricci flat background. Moreover, we can analyze other types of waves using \( \delta E_{\mu\nu} \). As \( \delta E_{\mu\nu} \) has 5 degrees of freedom which corresponds to the degrees of freedom of the bulk gravitational waves, one expect the scalar gravitational waves and vector gravitational waves. Without KK effects, no vector gravitational waves exist and the scalar gravitational waves can be described as the Brans-Dicke scalar waves. However, KK effects produce new effects which might be observable.

5 Conclusion

We formulated the perturbative formalism around the Ricci flat two-brane system. In particular, the master equation for \( \delta E_{\mu\nu} \) is derived. The gradient expansion method is utilized to get a series solution. This gives the closed system of equations which we call the effective Teukolsky equations in the case of type D induced metric on the brane. This can be used for estimating the Kaluza-Klein corrections on the gravitational waves emitted from the perturbed rotating black string. Our effective Teukolsky equation is completely separable, hence the numerical scheme can be developed in a similar manner as was done in the case of 4-dimensional Teukolsky equation.

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