On the spectra of coupled harmonic oscillators

Francisco M. Fernández*
INIFTA, DQT, Sucursal 4, C.C 16,
1900 La Plata, Argentina

Abstract

We discuss the diagonalization of a general Hamiltonian operator for a set of coupled harmonic oscillators and determine the conditions for the existence of bound states. We consider the particular cases of two and three oscillators studied previously and show the conditions for bound states in the latter example that have been omitted in an earlier treatment of this model.

1 Introduction

Models of coupled harmonic oscillators (CHO) have been extensively used to approximate and illustrate a wide variety of physical problems [1] (and references therein). They appear, for example, in the analysis of small oscillations in classical mechanics [2] and in the theory of molecular vibrations [3]. There has recently been great interest in the analysis of the symmetries of CHO and the two-mode squeezed states [4,5]. The model proved to be a pedagogical illustrative example of Feynman’s rest of the universe [6] and suitable for the study of entanglement in quantum mechanics [7–9]. The starting point of these studies consists of rewriting the Hamiltonian in diagonal form by means of two canonical
transformations of the coordinates and their conjugate momenta [4–9] but it seems that the results in some of the papers are not correct [8].

The parameters in the Hamiltonians for those CHO should be chosen with care in order to have bound states. The conditions have been completely specified in the case of some two-dimensional models [4,5], only partially specified in some cases [7] and omitted in others [6,8]. In the only treatment of three CHO the parameter conditions for bound states were completely ignored, most probably because the second canonical transformation, based on the SU(3) group, far from solving the problem leads to six transcendental equations that the authors never solved [9]. The one-step algorithm based on the diagonalization of two symmetric matrices [2,3] appears to be simpler and more straightforward than the one just mentioned [4–9] but it seems to have been overlooked in the latter treatments of the CHO. A pedagogical geometrical interpretation of this one-step algorithm in the case of two oscillators looks rather confusing because it resorts to more than one transformation [10].

The purpose of this paper is the application of the one-step algorithm [2,3] to the particular cases of two and three oscillators studied recently [4–9] with the purpose of determining the conditions that the coefficients of the Hamiltonian for the three-oscillator model [9] should satisfy so that there are bound states.

In section 2 we develop the approach for a quantum-mechanical CHO model instead of using the results for the classical version considered earlier [2,3,10]. Although the frequencies of the normal modes of both the classical and quantum-mechanical CHO are exactly the same it is worth developing the approach for the latter case because it does not appear to be so widely discussed [2,3,10]. In section 3 we apply the general results to the particular cases of two and three CHO already mentioned above [4–9]. In section 4 we summarize the main results and draw conclusions and at the end of this paper the reader will find the Appendix A with the necessary and sufficient conditions for bound states in the cases of four and five coupled harmonic oscillators.
2 Diagonalization of the model Hamiltonian

We consider a quantum-mechanical system with \( N \) coordinates \( x_i \) and conjugate momenta \( p_j \) that satisfy the well known commutation relations \([x_i, p_j] = i\hbar\delta_{ij},\) \( i, j = 1, 2, \ldots, N\). The Hamiltonian is a quadratic function of these dynamical variables

\[
H = \frac{1}{2} (p'^T T p + x'^T V x),
\]

(1)

where \( p' = (p_1, p_2, \ldots, p_N), x' = (x_1, x_2, \ldots, x_N) \) (t stands for transpose) and \( T, V \) are \( N \times N \) real symmetric matrices.

We carry out the canonical transformation

\[
x = C x', \quad p = (C'^{-1}) p',
\]

(2)

so that the new momenta \( p'' = (p'_1, p'_2, \ldots, p'_N) \) and coordinates \( x'' = (x'_1, x'_2, \ldots, x'_N) \) satisfy \([x''_i, p''_j] = i\hbar\delta_{ij}, i, j = 1, 2, \ldots, N\). We choose the \( N \times N \) matrix \( C \) so that

\[
C^{-1} T (C'^{-1}) = I, \quad C'^T V C = \Lambda,
\]

(3)

where \( I \) is the \( N \times N \) identity matrix and \( \Lambda \) is a diagonal matrix with elements \( \lambda_i, i = 1, 2, \ldots, N \). Therefore, the Hamiltonian operator (1) becomes

\[
H = \frac{1}{2} (p''^T p' + x''^T \Lambda x').
\]

(4)

Clearly there will be bound states provided that \( \lambda_i > 0, i = 1, 2, \ldots, N \). Because of the commutation relations between the new coordinates and momenta the eigenvalues are given by

\[
E_{\{n\}} = \hbar \sum_{i=1}^{N} \sqrt{n_i} \left( n_i + \frac{1}{2} \right), \quad \{n\} = \{n_1, n_2, \ldots, n_N\}, \quad n_i = 0, 1, \ldots
\]

(5)

It follows from equations (3) that

\[
C^{-1} T V C = \Lambda,
\]

(6)

so that the whole problem reduces to the diagonalization of the non-symmetric matrix \( A = TV \). This result is well known in molecular spectroscopy where it
has proved suitable for the study of molecular vibrations in terms of generalized coordinates, although it was derived in the realm of classical mechanics [3]. A slightly different, though entirely equivalent, equation has also been derived in the study of small oscillations in classical mechanics [2].

There are alternative ways of obtaining $H$ in diagonal form. If we prefer diagonalizing symmetric matrices we can define $C = T^{1/2}U$ provided $T$ is positive definite. In this case equation (6) becomes

$$U^{-1}T^{1/2}VT^{1/2}U = \Lambda.$$  

(7)

Since $S = T^{1/2}VT^{1/2}$ is symmetric then $U$ is orthogonal ($U^{-1} = U^t$) and we can use well known efficient diagonalization routines. The calculation of $T^{1/2}$ is particularly straightforward when $T$ is diagonal (as in the examples mentioned above [4–9]). Any $N \times N$ orthogonal matrix has only $N(N-1)/2$ independent matrix elements. Therefore, for $N = 2$ and $N = 3$ we can write $U$ in terms of two and tree independent quantities (angles, for example), respectively [4–9].

Notice that $x'_i$ and $p'_i$ do not longer have units of length and momentum, respectively, because $C$ has units of mass$^{-1/2}$ (assuming that $T$ has units of mass$^{-1}$). However, we obtain the correct eigenvalues because the transformed dynamical variables satisfy the standard canonical commutation relations. But if we want the dynamical variables to keep their standard physical units we simply change the conditions (3) to

$$C^{-1}TC^{t} = \frac{1}{m}I, \quad C^tV = K,$$

(8)

where $m$ is an arbitrary mass and $K$ a diagonal matrix. In this case $C$ is dimensionless, the diagonalization equation becomes

$$C^{-1}TVC = \frac{1}{m}K = \Lambda,$$

(9)

and the resulting Hamiltonian reads

$$H = \frac{1}{2m}p'^{2} + \frac{1}{2}x'^{2}Kx'.$$

(10)

It is clear that its eigenvalues are exactly those given above in equation (5) and, consequently, independent of the arbitrary mass $m$. This fact may appear to be
strange at first sight but one has to take into consideration that the transformation $C^{-1}T(C^t)^{-1}$ is merely a normalization condition for the eigenvectors of $A$ that are the columns of the matrix $C$. If one feels uncomfortable about having an arbitrary mass in the intermediate equations one may set it to be, for example, the geometric mean $m = (m_1 m_2 \ldots m_N)^{1/N}$ (when $T_{ij} = \delta_{ij}/m_i$, $i, j = 1, 2, \ldots, N$) as in earlier studies of the particular cases $N = 2 \ [4-8]$ and $N = 3 \ [9]$.

The symmetric matrix $S$ is particularly useful for determining the values of the model parameters that are compatible with positive eigenvalues $\lambda_i$ and, consequently, bound-state solutions. It is well known that a symmetric matrix is positive definite if and only if each of its leading principal minors is positive $[11]$. This theorem will prove useful in the analysis of the examples below.

## 3 Examples

We first consider the particular case of $N = 2$ coupled harmonic oscillators $[4-8]$

$$H = \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 + \frac{1}{2} (C_1 x_1^2 + C_2 x_2^2 + C_3 x_1 x_2).$$  \(11\)

In this case $T$ is positive-definite and diagonal which renders the calculation of $T^{1/2}$ trivial.

The matrices

$$A = \begin{pmatrix} C_1/m_1 & C_3/m_1 \\ C_2/m_2 & C_3/m_2 \end{pmatrix},$$

$$S = \begin{pmatrix} C_1/m_1 & \sqrt{m_1/m_2} C_3 \\ \sqrt{m_1/m_2} C_2/m_2 \end{pmatrix},$$  \(12\)

have the characteristic polynomial

$$\lambda^2 - \frac{\lambda (m_2 C_1 + m_1 C_2)}{m_1 m_2} + \frac{4 C_1 C_2 - C_3^2}{4 m_1 m_2} = 0,$$  \(13\)

which will have two real and positive roots provided that

$$m_2 C_1 + m_1 C_2 > 0, \ 4 C_1 C_2 - C_3^2 > 0.$$  \(14\)
It follows from these two conditions that \( C_1, C_2 > 0 \), already mentioned in some treatments of this model \[4, 5\]. Notice that it is only necessary to specify two conditions instead of three and that some of the conditions are omitted in some earlier treatments of this model \[6–8\]. The two principal minors of \( S \) are positive provided that \( C_1 > 0 \) and \( 4C_1C_2 - C_3^2 > 0 \) which are the necessary and sufficient conditions for positive definiteness and, consequently, positive eigenvalues \( \lambda_i \). They are equivalent to those discussed above.

The eigenvalues of \( A \) and \( S \) are

\[
\lambda_1 = \frac{m_1C_2 + m_2C_1 - R}{2m_1m_2}, \quad \lambda_2 = \frac{m_1C_2 + m_2C_1 + R}{2m_1m_2},
\]

\[
R = \sqrt{(m_2C_1 - m_1C_2)^2 + m_1m_2C_3^2}. \tag{15}
\]

The second particular example is given by the three coupled oscillators \[9\]

\[
H = \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 + \frac{1}{2m_3}p_3^2
+ \frac{1}{2} \left( m_1\omega_1^2x_1^2 + m_2\omega_2^2x_2^2 + m_3\omega_3^2x_3^2 + D_{12}x_1x_2 + D_{13}x_1x_3 + D_{23}x_2x_3 \right). \tag{16}
\]

In this case the matrix \( T \) is also positive-definite and diagonal. The matrices that are relevant for the diagonalization of this Hamiltonian operator are

\[
A = \begin{pmatrix}
\omega_1^2 & D_{12}/2m_1 & D_{13}/2m_1 \\
D_{12}/2m_2 & \omega_2^2 & D_{23}/2m_2 \\
D_{13}/2m_3 & D_{23}/2m_3 & \omega_3^2
\end{pmatrix},
\]

\[
S = \begin{pmatrix}
\omega_1^2 & D_{12}/2\sqrt{m_1m_2} & D_{13}/2\sqrt{m_1m_3} \\
D_{12}/2\sqrt{m_1m_2} & \omega_2^2 & D_{23}/2\sqrt{m_2m_3} \\
D_{13}/2\sqrt{m_1m_3} & D_{23}/2\sqrt{m_2m_3} & \omega_3^2
\end{pmatrix}. \tag{17}
\]

where the symmetric matrix \( S \) is identical to the matrix \( R \) derived by Merdaci and Jellal \[9\]. These authors claimed to have solved this problem exactly but they merely derived six transcendental equations for the six independent elements of their matrix \( R \) in terms of its three eigenvalues \( \Sigma_i^2 \) and three angles that define the matrix elements of the transformation matrix \( M \) (identical to present orthogonal matrix \( U \)).
The characteristic polynomial of any of those matrices (multiplied by \(-1\)) is

$$\lambda^3 - a\lambda^2 + b\lambda - c = 0,$$

$$a = (\omega_1^2 + \omega_2^2 + \omega_3^2),$$

$$b = \omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2 \left( \frac{D_{12}^2}{4m_1m_2} - \frac{D_{13}^2}{4m_1m_3} - \frac{D_{23}^2}{4m_2m_3} \right),$$

$$c = \omega_1^2\omega_2^2\omega_3^2 \left( \frac{D_{12}^2\omega_3^2}{4m_1m_2} + \frac{D_{13}^2\omega_2^2}{4m_1m_3} + \frac{D_{23}^2\omega_1^2}{4m_2m_3} - \frac{D_{12}D_{13}D_{23}}{4m_1m_2m_3} \right). \quad (18)$$

If the three roots are real and positive, then \(b > 0\) and \(c > 0\). These conditions are necessary but not sufficient because they are also compatible with one positive root and two complex-conjugate ones with positive real part. In order to remove the latter possibility we add the discriminant \([12]\) of the characteristic polynomial

$$\Delta = (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2 = a^2b^2 - 4a^3c + 18abc - 4b^3 - 27c^2 \geq 0 \quad (19)$$

Merdaci and Jellal \([9]\) did not derive any conditions for bound states probably because they did not solve their six transcendental equations which are too complicated for such an analysis.

We can derive two remarkably simpler necessary and sufficient conditions for the existence of bound states from two of the three leading principal minors of the matrix \(S\):

$$4m_1m_2\omega_1^2\omega_2^2 - D_{12}^2 > 0,$$

$$4m_1m_2m_3\omega_1^2\omega_2^2\omega_3^2 + D_{12}D_{13}D_{23} - m_1\omega_1^2D_{23}^2 - m_2\omega_2^2D_{13}^2 - m_3\omega_3^2D_{12}^2 > 0. \quad (20)$$

Notice that one of the conditions has been omitted because it is trivial in this case \((\omega_1^2 > 0)\). It is worth mentioning that all the results about entanglement discussed by Merdaci and Jellal \([9]\) are not valid unless the model parameters satisfy the two conditions \((20)\).

Merdaci and Jellal \([9]\) tested their unsolved equations by uncoupling one of the oscillators and restricting the problem to just two coupled oscillators. This
particular case can be achieved by choosing $D_{13} = D_{23} = 0$. If we do exactly the same we recover the results of two coupled oscillators discussed above (plus, of course an eigenvalue $\lambda_3 = \omega_2^2$ coming from the uncoupled oscillator). The two conditions for bound states reduce to just the first one.

The analytical expressions for the eigenvalues $\lambda_i$ and the transformation matrix $C$ are quite cumbersome in the general case (probably the reason why Merdaci and Jellal [9] did not attempt to solve their equations (12-17)). However, the particular case of three identical oscillators is remarkably simple and most useful for testing the general theoretical results given above.

If we set $m_i = m$, $\omega_i = \omega$, $D_{ij} = D$, $i, j = 1, 2, 3$, we have

$$A = S = \frac{1}{2m} \begin{pmatrix} m\omega^2 & D & D \\ D & m\omega^2 & D \\ D & D & m\omega^2 \end{pmatrix}, \quad (21)$$

with eigenvalues

$$\lambda_1 = \lambda_2 = \omega^2 - \frac{D}{2m}, \quad \lambda_3 = \omega^2 + \frac{D}{m}. \quad (22)$$

This problem is particularly simple because $TV = VT$ which explains why $A = S$. From the eigenvalues we conclude that there are bound states only when $-m\omega^2 < D < 2m\omega^2$. On the other hand, from the three leading principal minors we obtain $m\omega^2 > 0$ (trivial) and

$$4m^2\omega^4 - D^2 > 0, \quad m\omega^2 + D > 0, \quad (23)$$

that lead to exactly the same conditions derived from the eigenvalues.

The calculation of the eigenvectors of the matrix $S$ is also extremely simple and we obtain the transformation matrix

$$C = U = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \end{pmatrix}, \quad (24)$$

so that $x = Ux'$ and $p = Up'$. 

8
4 Conclusions

In order to transform a general Hamiltonian for a set of coupled oscillators into a diagonal form it is only necessary to obtain the eigenvalues and eigenvectors of either the nonsymmetric matrix $A$ or the symmetric matrix $S$ as shown in equations (6) and (7), respectively. This procedure is more general than the one based on two canonical transformations that is suitable for the particular case of a diagonal matrix $T$ \(^4\) \(^9\). Besides, the application of the algebraic method proposed by Merdaci and Jellal \(^9\) appears to become increasingly cumbersome as $N$ increases (they were unable to solve the resulting equations even for the second simplest case $N = 3$). On the other hand, the expressions shown in section 2 are valid for all $N$. Notice that it was quite easy to obtain the necessary and sufficient conditions for the existence of bound states in the two simplest cases $N = 2$ and $N = 3$, the latter of which have not been taken into account before \(^9\). Besides, it has been argued that the parameters of the resulting diagonal Hamiltonian operator have not been derived correctly even in the simplest case $N = 2$ \(^8\). The approach sketched here (known since long ago for the classical model \(^2\)\(^3\)) can be straightforwardly applied to a wider variety of oscillators with more general couplings than those based on a diagonal matrix $T$. In particular, the analysis of the matrix $S$ in terms of its principal minors is one of the simplest ways of determining the conditions for bound states.

A Necessary and sufficient conditions for bound states in the cases $N = 4$ and $N = 5$

In the case of $N = 4$ we should add
\[ D_{12}^2 D_{34}^2 - 4D_{12}^2 m_3 m_4 \omega_3^2 \omega_4^2 + 4D_{12} D_{13} D_{23} m_4 \omega_4^2 - 2D_{12} D_{13} D_{24} D_{34} \\
-2D_{12} D_{14} D_{23} D_{34} + 4D_{12} D_{14} D_{24} m_3 \omega_3^2 \\
+D_{13}^2 D_{24}^2 - 4D_{13}^2 m_2 m_4 \omega_2^2 \omega_4^2 - 2D_{13} D_{14} D_{23} D_{24} + 4D_{13} D_{14} D_{34} m_2 \omega_2^2 \\
+D_{14}^2 D_{23}^2 - 4D_{14}^2 m_2 m_3 \omega_2^2 \omega_3^2 - 4D_{23}^2 m_1 m_4 \omega_1^2 \omega_4^2 \\
+4D_{23} D_{24} D_{34} m_1 \omega_1^2 - 4D_{24}^2 m_1 m_3 \omega_3^2 \omega_4^2 - 4D_{34}^2 m_1 m_2 \omega_1^2 \omega_2^2 \\
+16 m_1 m_2 m_3 m_4 \omega_1^2 \omega_2^2 \omega_3^2 \omega_4^2 > 0, \quad (A.1) \]
to the two conditions shown above for $N = 3$. For $N = 5$ we also have

$$D_{12}^2 D_{34}^2 m_5 \omega_5^2 - D_{12}^2 D_{34} D_{35} D_{45} + D_{12}^2 D_{35}^2 m_4 \omega_4^2$$
$$+ D_{12}^2 D_{35}^2 m_3 \omega_3^2 - 4 D_{12}^2 m_3 m_4 m_5 \omega_3^2 \omega_4^2 \omega_5^2$$
$$- D_{12} D_{13} D_{23} D_{35}^2 + 4 D_{12} D_{13} D_{23} m_5 \omega_3^2 \omega_5^2$$
$$- 2 D_{12} D_{13} D_{24} D_{34} m_5 \omega_5^2 + D_{12} D_{13} D_{24} D_{35} D_{45}$$
$$+ D_{12} D_{13} D_{25} D_{34} D_{45} - 2 D_{12} D_{13} D_{25} D_{35} m_4 \omega_4^2$$
$$- 2 D_{12} D_{14} D_{23} D_{34} m_5 \omega_5^2 + D_{12} D_{14} D_{23} D_{35} D_{45}$$
$$- D_{12} D_{14} D_{24} D_{35}^2 + 4 D_{12} D_{14} D_{24} m_3 m_5 \omega_3^2 \omega_5^2 + D_{12} D_{14} D_{25} D_{34} D_{35}$$
$$- 2 D_{12} D_{14} D_{25} D_{45} m_3 \omega_3^2 + D_{12} D_{15} D_{23} D_{34} D_{45} - 2 D_{12} D_{15} D_{23} D_{35} m_4 \omega_4^2$$
$$+ D_{12} D_{15} D_{24} D_{34} D_{35} - 2 D_{12} D_{15} D_{24} D_{45} m_3 \omega_3^2$$
$$- D_{12} D_{15} D_{25} D_{34}^2 + 4 D_{12} D_{15} D_{25} m_3 m_4 \omega_3^2 \omega_4^2$$
$$+ D_{12}^2 D_{24} m_5 \omega_5^2 - D_{12}^2 D_{24} D_{25} D_{45} + D_{12}^2 D_{25} m_4 \omega_4^2$$
$$+ D_{12}^2 D_{25} m_2 \omega_2^2 - 4 D_{12}^2 m_2 m_4 m_5 \omega_2^2 \omega_4^2 \omega_5^2 - 2 D_{12} D_{13} D_{23} D_{24} m_5 \omega_5^2$$
$$+ D_{12} D_{13} D_{23} D_{25} D_{45} + D_{13} D_{14} D_{24} D_{25} D_{35} - D_{13} D_{14} D_{23} D_{34}^2$$
$$+ 4 D_{13} D_{14} D_{34} m_2 m_5 \omega_5^2 \omega_2^2 - 2 D_{13} D_{14} D_{35} D_{45} m_2 \omega_2^2$$
$$+ D_{13} D_{15} D_{23} D_{24} D_{45} - 2 D_{13} D_{15} D_{23} D_{25} m_4 \omega_4^2 - D_{13} D_{15} D_{24} D_{25} D_{45}$$
$$+ D_{13} D_{15} D_{24} D_{25} D_{34} - 2 D_{13} D_{15} D_{24} D_{34} m_2 \omega_2^2 + 4 D_{13} D_{15} D_{35} m_2 m_4 \omega_2^2 \omega_4^2$$
$$+ D_{13}^2 D_{23} m_5 \omega_5^2 - D_{13}^2 D_{23} D_{25} D_{35} + D_{13}^2 D_{25} m_3 \omega_3^2 + D_{13}^2 D_{35} m_2 \omega_2^2$$
$$- 4 D_{13}^2 m_2 m_3 m_5 \omega_3^2 \omega_2^2 \omega_5^2 - 2 D_{13}^2 D_{24} D_{35} D_{45} + D_{13} D_{15} D_{23} D_{24} D_{35}$$
$$+ D_{13} D_{15} D_{23} D_{25} D_{34} - 2 D_{13} D_{15} D_{24} D_{25} m_3 \omega_3^2 - 2 D_{13} D_{15} D_{34} D_{35} m_2 \omega_2^2$$
$$+ 4 D_{14} D_{15} D_{23} D_{24} D_{25} m_3 \omega_3^2 - 2 D_{14} D_{15} D_{34} D_{35} m_2 \omega_2^2$$
$$+ D_{14} D_{15} D_{24} D_{25} D_{34} - 2 D_{14} D_{15} D_{24} D_{34} m_2 \omega_2^2 + 4 D_{14} D_{15} D_{35} m_2 m_4 \omega_2^2 \omega_4^2$$
$$+ D_{15}^2 D_{24} m_3 \omega_3^2 + D_{15}^2 D_{24} m_2 \omega_2^2 - 4 D_{15}^2 m_2 m_3 m_4 \omega_2^2 \omega_4^2 \omega_5^2$$
$$+ D_{24}^2 D_{35} m_1 \omega_1^2 - 4 D_{24}^2 m_1 m_4 m_5 \omega_1^2 \omega_4^2 \omega_5^2$$
$$+ 4 D_{24} D_{25} D_{34} m_1 m_5 \omega_5^2 \omega_2^2 - 2 D_{24} D_{25} D_{34} D_{45} m_1 \omega_1^2 - 2 D_{24} D_{25} D_{35} D_{45} m_1 \omega_1^2$$
$$+ 4 D_{24} D_{25} D_{35} m_1 m_4 \omega_1^2 \omega_4^2 + D_{24}^2 D_{35}^2 m_1 \omega_1^2$$
$$- 4 D_{24}^2 m_1 m_3 m_5 \omega_5^2 \omega_1^2 \omega_2^2 - 2 D_{24} D_{25} D_{34} D_{35} m_1 \omega_1^2$$
$$+ 4 D_{24} D_{25} D_{45} m_1 m_4 \omega_1^2 \omega_4^2 + D_{25}^2 D_{35}^2 m_1 \omega_1^2$$
$$- 4 D_{25}^2 m_1 m_3 m_4 \omega_1^2 \omega_2^2 \omega_4^2 - 4 D_{25}^2 m_1 m_2 m_5 \omega_1^2 \omega_2^2 \omega_5^2$$
$$+ 4 D_{34} D_{35} D_{45} m_1 m_4 \omega_1^2 \omega_2^2 - 4 D_{35}^2 m_1 m_2 m_4 \omega_1^2 \omega_2^2 \omega_4^2$$
$$- 4 D_{45}^2 m_1 m_3 m_4 \omega_1^2 \omega_2^2 \omega_5^2 + 16 m_1 m_2 m_3 m_4 m_5 \omega_1^2 \omega_2^2 \omega_3^2 \omega_4^2 \omega_5^2 > 0 \quad (A.2)
in addition to the three conditions indicated above.

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