Some Versions of Supercyclicity for a Set of Operators

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Abstract. Let $X$ be a complex topological vector space and $L(X)$ the set of all continuous linear operators on $X$. An operator $T \in L(X)$ is supercyclic if there is $x \in X$ such that; $COrb(T,x) = \{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\}$, is dense in $X$. In this paper, we extend this notion from a single operator $T \in L(X)$ to a subset of operators $\Gamma \subseteq L(X)$. We prove that most of related proprieties to supercyclicity in the case of a single operator $T$ remains true for subset of operators $\Gamma$. This leads us to obtain some results for $C$-regularized groups of operators.

1. Introduction and Preliminary

Let $X$ be a complex topological vector space and $L(X)$ the space of all continuous linear operators on $X$. By an operator, we always mean a continuous linear operator.

The most important and studied notion in the linear dynamics is that of hypercyclicity: an operator $T$ acting on $X$ is said to be hypercyclic if there exists some vector $x$ whose orbit under $T$;

$$Orb(T,x) := \{T^n x : n \geq 0\},$$

is dense in $X$. Such a vector $x$ is called a hypercyclic vector for $T$, and the set of all hypercyclic vectors for $T$ is denoted by $HC(T)$. The first examples of hypercyclic operators on a Banach space were given by Rolewicz in 1969 in [13]. He proved that if $B$ is a backward shift on the Banach space $\ell^p(N); 1 \leq p < \infty$, then $\lambda B$ is hypercyclic for any complex number $\lambda$ such that $|\lambda| > 1$.

Another important notion in the linear dynamics is that of supercyclicity: we say that $T \in L(X)$ is a supercyclic operator if there is some vector $x \in X$ such that the cone generated by $Orb(T,x)$;

$$COrb(T,x) = \{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\},$$

is dense in $X$. Such a vector $x$ is called a supercyclic vector for $T$, and the set of all supercyclic vectors for $T$ is denoted by $SC(T)$, see [11]. In the context of separable Banach spaces, Feldman [9] proved that an operator $T$ is supercyclic if and only if it is supercyclic transitive, that is; for each pair $(U, V)$ of nonempty open subsets of $X$ there exist $\alpha \in \mathbb{C}$ and $n \geq 0$ such that

$$\alpha T^n(U) \cap V \neq \emptyset.$$ 

Another important notion that implies the supercyclicity is the supercyclicity criterion [14]. It provides several sufficient conditions that ensure supercyclicity. We say that an operator $T \in L(X)$ satisfies the
supercyclicity criterion if there exist an increasing sequence of integers \((n_k)\), a sequence \((\alpha_m)\) of nonzero complex numbers, two dense sets \(X_0, Y_0 \subset X\) and a sequence of maps \(S_m : Y_0 \to X\) such that:

- \(\alpha_m T^n x \to 0\) for any \(x \in X_0\);
- \(\alpha_m^{-1} S_m y \to 0\) for any \(y \in Y_0\);
- \(T^n S_m y \to y\) for any \(y \in Y_0\).

For a general overview of the hypercyclicity and supercyclicity see [6, 10].

An operator \(T \in L(X)\) is called quasi-conjugate or quasi-similar to an operator \(S \in L(Y)\) if there exists an operator \(\phi : X \to Y\) with dense range such that \(\phi \circ T = S \circ \phi\). If \(\phi\) can be chosen to be a homeomorphism, then \(T\) and \(S\) are called conjugate or similar, see [10, Definition 1.5]. A property \(P\) is said to be preserved under quasi-similarity if the following holds: if an operator \(T \in L(X)\) has property \(P\), then every operator \(S \in L(Y)\) that is quasi-similar to \(T\) has also property \(P\), see [10, Definition 1.7].

A set \(\Gamma\) of operators is called hypercyclic if there exists a vector \(x\) in \(X\) such that its orbit under \(\Gamma\):

\[
\text{Orb}(\Gamma, x) = \{T x : T \in \Gamma\},
\]

is dense in \(X\). Such a vector \(x\) is called a hypercyclic vector for \(\Gamma\). The set of all hypercyclic vectors for \(\Gamma\) is denoted by \(HC(\Gamma)\), see [2, 4]. If the space generated by \(\text{Orb}(\Gamma, x)\):

\[
\text{span}([\text{Orb}(\Gamma, x)]) = \text{span}([\{T x : T \in \Gamma\}]),
\]

is dense in \(X\) for some vector \(x\), then \(\Gamma\) is cyclic. The vector \(x\) is called a cyclic vector for \(\Gamma\). The set of all cyclic vector for \(\Gamma\) is denoted by \(C(\Gamma)\), see [1, 5].

In this work, we introduce and study the supercyclicity for a set of operators.

In Section 2, we introduce the notion of supercyclicity for a subset \(\Gamma \subseteq L(X)\) and we prove most of related results to supercyclicity for \(\Gamma\). We show that the set of supercyclic vectors for a set \(\Gamma\) is \(G_0\) type and that the supercyclicity for a set \(\Gamma\) is preserved under quasi-similarity.

In Section 3, we introduce the notions of supercyclic transitivity, strictly supertransitivity, supertransitivity and the supercyclicity criterion for a set \(\Gamma\) of operators. Also, we give the relationship between these notions and the supercyclicity.

In Sections 4, we apply previous results to prove some results for \(C\)-regularized groups of operators.

2. Supercyclic sets of operators

In the following definition, we introduce the notion of the supercyclicity of a set of operators. This definition generalizes the notion of the supercyclicity of a single operator.

**Definition 2.1.** Let \(\Gamma \subseteq L(X)\). We say that \(\Gamma\) is a supercyclic set of operators if there exists \(x \in X\) such that the cone generated by \(\text{Orb}(\Gamma, x)\):

\[
\text{COrb}(\Gamma, x) := \{\alpha T x : \alpha \in \mathbb{C}, T \in \Gamma\},
\]

is dense in \(X\). Such a vector \(x\) is called a supercyclic vector for \(\Gamma\). The set of all supercyclic vectors for \(\Gamma\) is denoted by \(SC(\Gamma)\).

The following example shows the existence of supercyclic sets of operators on the field of complex numbers.

**Example 2.2.** Let \(X = \mathbb{C}\) and \(T\) be a nonzero operator on \(\mathbb{C}\), then there exists \(x \in \mathbb{C}\) such that \(Tx \neq 0\). Let \(\Gamma = \{T\}\), then

\[
\text{COrb}(\Gamma, x) = \mathbb{C}[T x] = \mathbb{C}.
\]

This means that \(\Gamma\) is supercyclic and \(x\) is a supercyclic vector for \(\Gamma\).
Remark 2.3. Let \( \Gamma \) be a subset of \( L(X) \). Since for all \( x \in X \) we have
\[
\text{Orb}(\Gamma, x) \subset \text{COrb}(\Gamma, x),
\]
if \( \Gamma \) is hypercyclic, then it is supercyclic. The converse does not hold in general. Indeed, let \( \Gamma \) be the set defined as in Example 2.2, then \( \Gamma \) is supercyclic, but it is not hypercyclic.

It has been shown in [11] that \( X \) supports supercyclic operators if and only if \( \dim(X) = 1 \) or \( \dim(X) = +\infty \). This result does not hold in general in the case of a set of operators. Moreover, the supercyclicity of a set of operators exists in each topological vector space \( X \).

Example 2.4. Let \( f \) be a nonzero linear form on a locally convex space \( X \) and \( D \) be a subset of \( X \) such that the set
\[
\mathcal{C}D := \{ax : \alpha \in \mathbb{C}, x \in D\}
\]
is a dense subset of \( X \). For all \( x \in X \), let \( T_x \) be an operator defined by:
\[
T_x : X \rightarrow X,
\]
\[
y \mapsto f(y)x.
\]
We consider \( \Gamma_f = \{T_x : x \in D\} \) and let \( y \) be a vector of \( X \) such that \( f(y) \neq 0 \). Then
\[
\text{COrb}(\Gamma_f, y) = \{\alpha T_x y : x \in D, \alpha \in \mathbb{C}\} = \{\alpha f(y)x : x \in D, \alpha \in \mathbb{C}\} = \mathcal{C}D.
\]
Hence, \( \text{COrb}(\Gamma_f, y) = X \), which means that \( \Gamma_f \) is supercyclic and \( y \) is a supercyclic vector for \( \Gamma_f \).

Remark 2.5. Let \( T \) be an operator acting on a complex separable Banach space \( X \) such that \( \dim(X) \geq 1 \). By Ansari’s theorem, if \( T \) is supercyclic, then for any \( n \geq 2 \), the operator \( T^n \) is supercyclic. Moreover, \( T \) and \( T^n \) share the same supercyclic vectors, see [3]. This result does not hold in general in the case of a set of operators. Indeed, let \( \Gamma_f \) be the set of operators defined as in Example 2.4, then \( \Gamma_f \) is supercyclic. However, every single operator \( T_x \) is not supercyclic since supercyclic operators are of dense range, see [7].

Let \( \Gamma \subset L(X) \). We denote by \( \{\Gamma\}' \) the set of all elements of \( L(X) \) which commute with every element of \( \Gamma \).

Proposition 2.6. Let \( T \) be an operator with dense range. If \( T \in \{\Gamma\}' \), then \( Tx \in \text{SC}(\Gamma) \), for all \( x \in \text{SC}(\Gamma) \).

Proof. Let \( \Omega \) be a nonempty and open subset of \( X \). Since \( T \) is of dense range, \( T^{-1}(\Omega) \) is nonempty and open. Let \( x \in \text{SC}(\Gamma) \), then there exist \( \alpha \in \mathbb{C} \) and \( S \in \Gamma \) such that \( \alpha Sx \in T^{-1}(\Omega) \), that is \( \alpha T(Sx) \in \Omega \). Since \( T \in \{\Gamma\}' \), it follows that \( \alpha S(Tx) = \alpha T(Sx) \in \Omega \). Hence, \( Tx \in \text{SC}(\Gamma) \).

Corollary 2.7. Let \( \Gamma \) be a supercyclic set of operators. If \( x \in \text{SC}(\Gamma) \), then \( \alpha x \in \text{SC}(\Gamma) \), for all \( \alpha \in \mathbb{C} \setminus \{0\} \).

Let \( X \) and \( Y \) be topological vector spaces and let \( \Gamma \subset L(X) \) and \( \Gamma_1 \subset L(Y) \). Recall from [1], that \( \Gamma \) and \( \Gamma_1 \) are called quasi-similar if there exists an operator \( \phi : X \rightarrow Y \) with dense range such that for all \( T \in \Gamma \), there exists \( S \in \Gamma_1 \) satisfying \( S \circ \phi = \phi \circ T \). If \( \phi \) is a homeomorphism, then \( \Gamma \) and \( \Gamma_1 \) are called similar.

It has been shown in [10] that the supercyclicity of a single operator is stable under quasi-similarity. In the following, we prove that the same result holds for sets of operators.

Proposition 2.8. If \( \Gamma \subset L(X) \) and \( \Gamma_1 \subset L(Y) \) are quasi-similar, then \( \Gamma \) is supercyclic in \( X \) implies that \( \Gamma_1 \) is supercyclic in \( Y \). Moreover, \( \phi(\text{SC}(\Gamma)) \subset \text{SC}(\Gamma_1) \).

Proof. Let \( \Omega \) be a nonempty open subset of \( Y \), then \( \phi^{-1}(\Omega) \) is a nonempty open subset of \( X \). If \( x \in \text{SC}(\Gamma) \), then there exist \( \alpha \in \mathbb{C} \) and \( T \in \Gamma \) such that \( \alpha Tx \in \phi^{-1}(\Omega) \), that is \( \alpha \phi(Tx) \in \Omega \). Let \( S \in \Gamma_1 \) such that \( S \circ \phi = \phi \circ T \). Hence, \( \alpha S(\phi(Tx)) = \alpha \phi(Tx) \in \Omega \). Hence \( \phi(Tx) \in \text{SC}(\Gamma_1) \).

Corollary 2.9. Assume that \( \Gamma \subset L(X) \) and \( \Gamma_1 \subset L(Y) \) are similar. Then \( \Gamma \) is supercyclic in \( X \) if and only if \( \Gamma_1 \) is supercyclic in \( Y \). Moreover,
\[
\phi(\text{SC}(\Gamma)) = \text{SC}(\Gamma_1).
\]
Proof. This is since \( C \) is supercyclic if and only if \( \Gamma_1 := \{ \alpha T : T \in \Gamma \} \) is supercyclic. Moreover, \( \Gamma \) and \( \Gamma_1 \) share the same supercyclic vectors.

**Proposition 2.11.** Let \( \{ X_i \}_{i=1}^n \) be a family of complex topological vector spaces and \( \Gamma_i \) be a subset of \( L(X_i) \), for all \( 1 \leq i \leq n \). If \( \Gamma_i \) is a supercyclic set in \( \Gamma_i \), then \( \Gamma_i \) is a supercyclic set in \( X_i \), for all \( 1 \leq i \leq n \). Moreover, if \( (x_1, x_2, \ldots, x_n) \in SC(\Gamma_i) \), then \( x_i \in SC(\Gamma_i) \), for all \( 1 \leq i \leq n \). That is \( SC(\Gamma_i) \subset SC(\Gamma_i) \).

Proof. Let \( (x_1, x_2, \ldots, x_n) \in SC(\Gamma_i) \). For all \( 1 \leq i \leq n \), let \( O_i \) be a nonempty open subset of \( X_i \), then \( O_1 \times O_2 \times \cdots \times O_n \) is a nonempty open subset of \( \Gamma_i \). Since \( Orb(\Gamma_i, \Gamma_i) \) is dense in \( \Gamma_i \), it follows that there exist \( a \in C \) and \( T_i \in \Gamma_i \); \( 1 \leq i \leq n \) such that

\[
(\alpha T_1 x_1, \alpha T_2 x_2, \ldots, \alpha T_n x_n) = a(T_1 \times T_2 \times \cdots \times T_n)(x_1, x_2, \ldots, x_n) \in O_1 \times O_2 \times \cdots \times O_n,
\]

that is \( \alpha T_i x_i \in O_i \), for all \( 1 \leq i \leq n \). Hence, \( \Gamma_i \) is a supercyclic set in \( X_i \) and \( x_i \in SC(\Gamma_i) \), for all \( 1 \leq i \leq n \).

A subset of \( X \) is said to be \( G_\delta \) type if it is an intersection of a countable collection of open sets.

Using a countable basis of the topology of \( X \), we can prove that the set of all supercyclic vectors for a set \( \Gamma \) is \( G_\delta \) type as shows the next proposition.

**Proposition 2.12.** Let \( X \) be a second countable topological vector space and \( \Gamma \subset L(X) \) a supercyclic set. Then,

\[
SC(\Gamma) = \bigcap_{n \geq 1} \left( \bigcup_{\beta \in C \setminus \{0\}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n) \right),
\]

where \( (U_n)_{n \geq 1} \) is a countable basis of the topology of \( X \). As a consequence, \( SC(\Gamma) \) is a \( G_\delta \) type set.

Proof. Suppose that \( \Gamma \) is a supercyclic set. Then, \( x \in SC(\Gamma) \) if and only if \( COrb(\Gamma, x) = X \). Equivalently, for all \( n \geq 1 \) we have \( U_n \cap COrb(\Gamma, x) \neq \emptyset \). That is, for all \( n \geq 1 \) there exist \( \alpha \in C \) and \( T \in \Gamma \) such that \( x \in aT^{-1}(U_n) \).

This is equivalent to the fact that \( x \in \bigcap_{n \geq 1} \left( \bigcup_{\beta \in C \setminus \{0\}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n) \right) \). Hence, \( SC(\Gamma) = \bigcap_{n \geq 1} \left( \bigcup_{\beta \in C \setminus \{0\}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n) \right) \).

Since \( \bigcup_{\beta \in C \setminus \{0\}} \bigcup_{T \in \Gamma} T^{-1}(\beta U_n) \) is an open subset of \( X \), for all \( n \geq 1 \), it follows that \( SC(\Gamma) \) is a \( G_\delta \) type.

3. Density and Transitivity of Sets of Operators

The supercyclic transitivity of a single operator was introduced in [9]. In the following definition, we extend this notion to sets of operators.

**Definition 3.1.** We say that \( \Gamma \) is a supercyclic transitive set of operators if for each pair of nonempty open subsets \( (U, V) \) of \( X \), there exist \( \alpha \in C \setminus \{0\} \) and \( T \in \Gamma \) such that

\[
T(\alpha U) \cap V \neq \emptyset.
\]

The following example shows that each topological vector space \( X \) supports supercyclic transitive sets of operators.

**Example 3.2.** Assume that \( X \) is a locally convex space. Let \( x, y \in X \) and let \( f_y \) be a linear form on \( X \) such that \( f_y(y) \neq 0 \). We define an operator \( T_{f_y, x} \) by

\[
T_{f_y, x} : X \rightarrow X, \quad z \mapsto f_y(z)x.
\]
Let $\Gamma$ be a set of operators on $X$ defined by $\Gamma = \{T_{f, x} : x, y \in X \text{ such that } f(y) \neq 0\}$. Then $\Gamma$ is a supercyclic transitive set of operators. Indeed, let $U$ and $V$ be two nonempty open subsets of $X$. There exist $x, y \in X$ such that $x \in U$ and $y \in V$. We have

\[ T_{f, x}(y) = f(y)x. \]

Since $f(y) \neq 0$, it follows that $x = \frac{1}{f(y)} T_{f, x}(y)$. Hence $x \in U$ and $x \in \frac{1}{f(y)} T_{f, x}(V)$, which implies that $U \cap \frac{1}{f(y)} T_{f, x}(V) \neq \emptyset$. Thus $\Gamma$ is a supercyclic transitive set of operators.

The supercyclicity of a single operators is preserved under quasi-similarity, see [10, Proposition 1.13]. The following proposition proves that this result holds for sets of operators.

**Proposition 3.3.** Assume that $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are quasi-similar. If $\Gamma$ is supercyclic transitive in $Y$, then $\Gamma_1$ is supercyclic transitive in $Y$.

**Proof.** Let $U, V$ be nonempty open and subsets of $Y$. Since $\phi$ is of dense range, $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty and open subsets of $X$. Since $\Gamma$ is supercyclic transitive in $X$, there exist $y \in \phi^{-1}(U)$ and $x \in X$. There exist $x, y \in X$ such that $x \in U$ and $y \in V$. Hence $x \in U$ and $y \in V$.

**Corollary 3.4.** Assume that $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar. Then, $\Gamma$ is supercyclic transitive in $X$ if and only if $\Gamma_1$ is supercyclic transitive in $Y$.

In the following result, we give necessary and sufficient conditions for a set of operators to be supercyclic transitive.

**Theorem 3.5.** Let $X$ be a normed space and $\Gamma \subset L(X)$. The following assertions are equivalent:

(i) $\Gamma$ is supercyclic transitive;

(ii) For each $x, y \in X$, there exist sequences $\{k\}$ in $\mathbb{N}$, $\{x_k\}$ in $X$, $\{a_k\}$ in $\mathbb{C}$ and $\{T_k\}$ in $\Gamma$ such that $x_k \to x$ and $T_k(a_k x_k) \to y$;

(iii) For each $x, y \in X$ and for $W$ a neighborhood of zero, there exist $z \in X$, $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $x - z \in W$ and $T(\alpha z) - y \in W$.

**Proof.** (i) $\Rightarrow$ (ii) Let $x, y \in X$. For all $k \geq 1$, let $U_k = B(x, \frac{1}{k})$ and $V_k = B(y, \frac{1}{k})$. Then $U_k$ and $V_k$ are nonempty open subsets of $X$. Since $\Gamma$ is supercyclic transitive, there exist $a_k \in \mathbb{C}$ and $T_k \in \Gamma$ such that $T_k(a_k x_k) \in V_k$.

For all $k \geq 1$, let $x_k \in U_k$ such that $T_k(a_k x_k) \in V_k$, then $||x_k - x|| < \frac{1}{k}$ and $||T_k(a_k x_k) - y|| < \frac{1}{k}$ which implies that $x_k \to x$ and $T_k(a_k x_k) \to y$.

(ii) $\Rightarrow$ (iii) Clear;

(iii) $\Rightarrow$ (i) Let $U$ and $V$ be two nonempty open subsets of $X$. There exist $x, y \in X$ such that $x \in U$ and $y \in V$. Since for all $k \geq 1$, $W_k = B(0, \frac{1}{k})$ is a neighborhood of $0$, there exist $z_k \in X$, $\alpha_k \in \mathbb{C}$ and $T_k \in \Gamma$ such that $||x - z_k|| < \frac{1}{k}$ and $||T_k(\alpha_k z_k) - y|| < \frac{1}{k}$. This implies that $z_k \to x$ and $T_k(\alpha_k z_k) \to y$. There exists $N \in \mathbb{N}$ such that $z_k \in U$ and $T_k(\alpha_k z_k) \in V$, for all $k \geq N$. This implies that $\Gamma$ is supercyclic transitive.

**Theorem 3.6.** Let $X$ be a second countable Baire topological vector space and $\Gamma$ a subset of $L(X)$. The following assertions are equivalent:

(i) $\text{SC}(\Gamma)$ is dense in $X$;

(ii) $\Gamma$ is supercyclic transitive.

As a consequence, a supercyclic transitive set is supercyclic.
Proof. Since $X$ is a second countable topological vector space, we can consider $(U_m)_{m=1}^\infty$ a countable basis of the topology of $X$.

(i) $\Rightarrow$ (ii): Assume that $SC(\Gamma) = \bigcap_{n\in\mathbb{N}} \bigcup_{\beta\in C[0]} \bigcup_{T\in \text{Orb}} T^{-1}(\beta U_n)$ is dense in $X$. Hence, for all $n \geq 1$ the set $A_n = \bigcup_{\beta\in C[0]} \bigcup_{T\in \text{Orb}} T^{-1}(\beta U_n)$ is dense in $X$. Thus, for all $n, m \geq 1$, we have $A_n \cap U_m \neq \emptyset$ which implies that for all $n, m \geq 1$ there exist $\beta \in C \setminus \{0\}$ and $T \in \Gamma$, such that $T(\beta U_m) \cap U_n \neq \emptyset$, which implies that $\Gamma$ is supercyclic transitive.

(ii) $\Rightarrow$ (i): Let $n, m \geq 1$, then there exist $\beta \in C \setminus \{0\}$ and $T \in \Gamma$ such that $T(\beta U_m) \cap U_n \neq \emptyset$ which implies that $T^{-1}(\beta U_n) \cap U_m \neq \emptyset$. Hence, for all $n \geq 1$ the set $\bigcup_{\beta\in C[0]} \bigcup_{T\in \text{Orb}} T^{-1}(\beta U_n)$ is dense in $X$. □

In the following, we prove that the converse of Theorem 3.6 holds with some additional assumptions.

**Theorem 3.7.** Let $\Gamma \subset L(X)$ such that for all $T, S \in \Gamma$ with $T \neq S$, there exists $A \in \Gamma$ such that $T = AS$. Then $\Gamma$ is supercyclic implies that $\Gamma$ is supercyclic transitive.

Proof. Since $\Gamma$ is supercyclic, there exists $x \in X$ such that $C\text{Orb}(\Gamma, x)$ is a dense subset of $X$. Let $U, V$ be nonempty and open subsets of $X$, then there exist $\alpha, \beta \in C \setminus \{0\}$, and $T, S \in \Gamma$ such that

$$\alpha Tx \in U \quad \text{and} \quad \beta Sx \in V. \quad (1)$$

There exists $A \in \Gamma$ such that $T = AS$. By (1), we have $\alpha A(Sx) \in U$ and $\beta A(Sx) \in V$ which implies that $U \cap A(\frac{\beta}{\alpha} V) \neq \emptyset$. Hence, $\Gamma$ is supercyclic transitive. □

**Remark 3.8.** Let $\Gamma$ be a set of mutually commuting operators, that is; for each $T$ and $S$ in $\Gamma$, we have $TS = ST$. Assume that each operator of $\Gamma$ is of dense range. Then $\Gamma$ is supercyclic implies that $\Gamma$ is supercyclic transitive.

**Definition 3.9.** We say that $\Gamma \subset L(X)$ is strictly supertransitive if for each pair of nonzero elements $x, y$ in $X$, there exist $\alpha \in C$ and $T \in \Gamma$ such that $\alpha Tx = y$.

**Example 3.10.** Let $X$ be a locally convex space and $f$ a nonzero linear form on $X$. Let $D$ be a subset of $X$ such

$$CD := \{ax : \alpha \in C, x \in D\}$$

is dense in $X$. Let $\Gamma_f$ be the set of operators defined as in Example 2.4. Let $x$ and $y$ be two elements of $X$, then

$$T_x(y) = f(y)x = ax.$$

Hence $\Gamma_f$ is strictly supertransitive.

**Proposition 3.11.** A strictly supertransitive set is supercyclic transitive. As a consequence, a strictly supertransitive set is supercyclic.

Proof. Let $\Gamma \subset L(X)$ be a strictly supertransitive set. If $U$ and $V$ are two nonempty open subsets of $X$, there exist $x, y \in X$ such that $x \in U$ and $y \in V$. Since $\Gamma$ is strictly supertransitive, there exist $\alpha \in C$ and $T \in \Gamma$ such that $\alpha Tx = y$. Hence, $\alpha Tx \in \alpha T(U)$ and $\alphaTx \in V$. Thus, $\alpha T(U) \cap V \neq \emptyset$, which implies that $\Gamma$ is supercyclic transitive. □

**Proposition 3.12.** Assume $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar. Then $\Gamma$ is strictly supertransitive in $X$ if and only $\Gamma_1$ is strictly supertransitive in $Y$.

Proof. Let $x, y \in Y$. There exist $a, b \in X$ such that $\phi(a) = x$ and $\phi(b) = y$. Since $\Gamma$ is strictly supertransitive in $X$, there exist $\alpha \in C$ and $T \in \Gamma$ such that $\alpha Ta = b$. Let $S \in \Gamma_1$ such that $S \circ \phi = \phi \circ T$, this implies that $\alpha Sx = y$. Hence $\Gamma_1$ is strictly supertransitive in $Y$. □
The strong operator topology (SOT for short) on $L(X)$ is the topology for which a neighborhood of $T \in L(X)$ is given by

$$
\Omega = \{ S \in L(X) : S e_i - T e_i \in U, \, i = 1, 2, \ldots, k \},
$$

where $k \in \mathbb{N}$, $e_1, e_2, \ldots, e_k \in X$ are linearly independent and $U$ is a neighborhood of zero in $X$.

In the following theorem, the proof is true for norm-density if $X$ is assumed to be a normed linear space.

**Theorem 3.13.** Let $X$ be a topological vector space. Then for each pair of nonzero linearly independent vectors $x, y \in X$ there exists a SOT-dense set $\Gamma_{xy} \subset L(X)$ which is not strictly supertransitive. Furthermore, $\Gamma \subset L(X)$ is a dense nonstrictly transitive set if and only if $\Gamma$ is a dense subset of $\Gamma_{xy}$ for some $x, y \in X$.

**Proof.** Fix nonzero linearly independent vectors $x, y \in X$ and put

$$
\Gamma_{xy} = \{ T \in L(X) : y \text{ and } Tx \text{ are linearly independent} \}.
$$

It is clear that $\Gamma_{xy}$ is not strictly supertransitive. Let $\Omega$ be a nonempty open subset of $L(X)$ and $S \in \Omega$. If $Sx$ and $y$ are linearly independent, then $S \in \Omega \cap \Gamma_{xy}$. Otherwise, putting $S_n = S + \frac{1}{n}I$, we see that $S_n \in \Omega$ for some $k$, but $S_n x$ and $y$ are linearly independent. Hence, $\Omega \cap \Gamma_{xy} \neq \emptyset$ and the proof of the first part is complete.

We prove the second part of the theorem. Suppose that $\Gamma$ is a dense subset of $L(X)$ that is not strictly supertransitive. Then there exist nonzero vectors $x, y \in X$ such that $Tx$ and $y$ are linearly independent for all $T \in \Gamma$ and hence $\Gamma \subset \Gamma_{xy}$. To show that $\Gamma$ is dense in $\Gamma_{xy}$, assume that $\Omega_0$ is an open subset of $\Gamma_{xy}$. Thus, $\Omega_0 = \Gamma_{xy} \cap \Omega$ for some open set $\Omega$ in $L(X)$. Then $\Gamma \cap \Omega_0 = \Gamma \cap \Omega \neq \emptyset$.

For the converse, let $\Gamma$ be a dense subset of $\Gamma_{xy}$ for some $x, y \in X$. Then $\Gamma$ is not strictly supertransitive. Also, since $\Gamma_{xy}$ is a dense open subset of $L(X)$, we conclude that $\Gamma$ is also dense in $L(X)$. Indeed, if $\Omega$ is any open set in $L(X)$ then $\Omega \cap \Gamma_{xy} \neq \emptyset$ since $\Gamma_{xy}$ is dense in $L(X)$. On the other hand, $\Omega \cap \Gamma_{xy}$ is open in $\Gamma_{xy}$ and so it must intersect $\Gamma$ since $\Gamma$ is dense in $\Gamma_{xy}$. Thus, $\Omega \cap \Gamma \neq \emptyset$ and so $\Gamma$ is dense in $L(X)$. □

**Corollary 3.14.** Let $X$ be a topological vector space and $\Gamma$ be a SOT-dense subset of $L(X)$. Then there is a subset $\Gamma_1$ of $\Gamma$ such that $\Gamma_1 \supseteq L(X)$ and $\Gamma_1$ is not strictly supertransitive.

**Proof.** For nonzero linearly independent vectors $x, y$ put $\Gamma_1 = \Gamma \cap \Gamma_{xy}$. □

**Definition 3.15.** A set $\Gamma \subset L(X)$ is said to be supertransitive if $SC(\Gamma) = X \setminus \{0\}$.

**Remarks 3.16.** Let $T \in L(X)$.

(i) If $T$ is supertransitive, then it is injective of dense range.

(ii) $T$ is supertransitive if and only if $T^p$ is supertransitive, for all $p \geq 2$.

The next proposition shows that supertransitivity implies supercyclic transitivity.

**Proposition 3.17.** If $\Gamma$ is supertransitive, then it is supercyclic transitive.

**Proof.** Let $U, V$ be nonempty and open subsets of $X$. There exists $x \in X \setminus \{0\}$ such that $x \in U$. Since $\Gamma$ is supertransitive, there exist $a \in \mathbb{C}$ and $T \in \Gamma$ such that $aTx \in V$. Thus, it follows that $aT(U) \cap V \neq \emptyset$. Hence, $\Gamma$ is supercyclic transitive. □

**Proposition 3.18.** Assume that $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar. Then, $\Gamma$ is supertransitive on $X$ if and only if $\Gamma_1$ is supertransitive on $Y$.

**Proof.** It suffices to use Proposition 2.8 and verify that $\phi(X \setminus \{0\}) = X \setminus \{0\}$. □

The following result shows that the SOT-closure of $\Gamma$ is not large enough to have more supercyclic vectors than $\Gamma$. 

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Proposition 3.19. Let $\Gamma \subset L(X)$. Then $SC(\Gamma) = SC\left(\Gamma^{\text{SOT}}\right)$.

Proof. Let $x \in SC\left(\Gamma^{\text{SOT}}\right)$. If $U$ is an open set of $X$, then there is some $a \in C$ and $T \in \Gamma^{\text{SOT}}$ such that $aTx \in U$. Since $\Omega = \{S \in L(X) : aSx \in U\}$ is a SOT-neighborhood of $T$, there is some $S \in \Gamma$ such that $aSx \in U$ and this shows that $x \in SC(\Gamma)$. $\blacksquare$

Corollary 3.20. Let $\Gamma \subset L(X)$. Then, $\Gamma$ is supertransitive if and only if $\Gamma^{\text{SOT}}$ is supertransitive.

In the following definition, we introduce the notion of the supercyclicity criterion for a set of operators.

Definition 3.21. We say that $\Gamma$ satisfies the criterion of supercyclicity if there exist two dense subsets $X_0, Y_0$ in $X$, and sequences $\{k\} \subset \mathbb{N}$, $\{\alpha_k\} \subset C \setminus \{0\}$, $\{T_k\} \subset \Gamma$, and maps $S_k : Y_0 \to X$ such that:

1. $\alpha_kT_kx \to 0$ for all $x \in X_0$;
2. $\alpha_k^{-1}S_ky \to 0$ for all $y \in Y_0$;
3. $T_kS_ky \to y$ for all $y \in Y_0$.

Theorem 3.22. Let $X$ be a second countable Baire topological vector space and $\Gamma$ a subset of $L(X)$. If $\Gamma$ satisfies the criterion of supercyclicity, then it is supercyclic.

Proof. Let $U, V$ be nonempty open subsets of $X$. There exist $x_0, y_0$ in $X$ such that $x_0 \in X_0 \cap U$ and $y_0 \in Y_0 \cap V$. For all $k \geq 1$, let $z_k = x_0 + \alpha_k^{-1}S_ky$. It follows that $z_k \to x_0$, and $\alpha_kT_kz_k \to y_0$. Hence, there exists $k$ such that $\alpha_kT_k(U) \cap V \neq \emptyset$. $\blacksquare$

4. Supercyclicity of $C$-Regularized Groups

In this section, we study the particular case where $\Gamma$ stands for a $C$-regularized group. Recall from [8], that an entire $C$-regularized group is an operator family $(S(z))_{z \in C}$ on $L(X)$ that satisfies:

1. $S(0) = C$;
2. $S(z + w)C = C\{S(z)w\}$ for every $z, w \in C$,
3. The mapping $z \mapsto S(z)x$, with $z \in C$, is entire for every $x \in X$.

Example 4.1. Let $X = C$. For all $z \in C$, let $S(z)x = \exp(z)x$, for all $x \in C$. $(S(z))_{z \in C}$ is a $C$-regularized group of operators and we have $\overline{\text{COB}}((S(z))_{z \in C}, x) = C$, for all $x \in C \setminus \{0\}$. Hence $(S(z))_{z \in C}$ is supercyclic and $SC((S(z))_{z \in C}) = C \setminus \{0\}$.

By Theorem 3.6, every supercyclic transitive $C$-regularized group is supercyclic. In the following, we prove that the converse holds.

Theorem 4.2. Assume that $C$ is of dense range. If $(S(z))_{z \in C}$ is supercyclic, then it is supercyclic transitive.

Proof. Let $x \in SC((S(z))_{z \in C})$. If $U$ and $V$ are two nonempty open subsets of $X$, then there exist $\alpha, \beta, z_1, z_2 \in C$ such that $\alpha S(z_1)x \in C^{-1}(U)$ and $\beta S(z_2)x \in V$. Let $z_3 = z_1 - z_2$, then $U \cap \frac{\alpha}{\beta}S(z_3)(V) \neq \emptyset$. Hence, $(S(z))_{z \in C}$ is a supercyclic $C$-regularized group. $\blacksquare$

Theorem 4.3. Let $(S(z))_{z \in C}$ be a supercyclic $C$-regularized group on a Banach infinite-dimensional space $X$. Assume that $C$ is of dense range. If $x \in X$ is a supercyclic vector of $(S(z))_{z \in C}$, then the following assertions hold:

1. $S(z)x \neq 0$ for all $z \in C$;
2. The set $\{aS(z)x : \alpha, z \in C, |z| > |\omega_0|\}$ is dense in $X$ for all $\omega_0 \in C$. 

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Proof. (1) Clear.

(2) Let $\omega_0 \in \mathbb{C}$ such that the set $A := \{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| > |\omega_0|\}$ is not dense in $X$. Hence there exists a bounded open set $U$ such that $U \cap \overline{A} = \emptyset$. Therefore we have

$$ U \subset \{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| \leq |\omega_0|\} $$

by using the relation

$$ X = \{\alpha S(z)x : \alpha, z \in \mathbb{C}\} = \{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| > |\omega_0|\} \cup \{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| \leq |\omega_0|\}. $$

Since $S(z)x$ is continuous with $z$ and $S(z)x \neq 0$ holds for all $z \in \mathbb{C}$ by (1), there exist $m_1, m_2 > 0$ such that $0 < m_1 \leq \|S(z)x\| < m_2$ for $z \in \mathbb{C}$ with $|z| \leq |\omega_0|$. There exists $M > 0$ such that $\|y\| \leq M$ for any $y \in U$ because $U$ is bounded. So we have

$$ U \subset \{\alpha S(z)x : |z| \leq |\omega_0|, |\alpha| \leq M, |z| \leq \frac{M}{m_1}\}, $$

which means that $\overline{U}$ is compact. Hence $X$ is finite dimensional, which contradicts that $X$ is infinite dimensional. $\square$

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