Calculation of coverage intervals for repeated measurements (Bayesian inference)

A. Chunovkina, A. Stepanov
VNIIM, Russia
E-mail: A.G.Chunovkina@vniim.ru

Abstract. The aims of the work are to derive coverage factor $K_{0.95}$ for confidence level of 95% using a Bayesian approach to uncertainty analysis for a linear measurement model with two input quantities. The calculations are made for the cases when the first quantity is presented by repeated indications from the normal or uniform law, and the distribution of the second belongs to an exponential family.

1. Introduction
The work deals with a measurement model with two input quantities and (consequently) two uncertainty sources evaluated by types A and B. The coverage factor expressions are given below for coverage probability level of 95%. Normal and uniform distributions of the indications and exponential distributions family for the measuring instrument systematic error are considered.

The work discusses evaluation of coverage interval as a product of a coverage factor and a standard measurement uncertainty. The expressions for the coverage factor are given as a functions of a number of repeated indications and a ratio of uncertainties associated with the above mentioned sources.

The work presents the further development of the results presented in the article [1].

2. Measurement model
Consider the following simple measurement model: $Y = X + B$, here $X$ is a measurand value, $Y$ is an indication of measuring instrument, and $B$ is a systematic measurement error. Quantity $Y$ is presented by repeated indications $y_1, \ldots, y_n \in Y$, $n > 3$; $y_i$ are drawn from normal or uniform distribution: $Y \in \mathcal{N}(X + B, \sigma)$ or $Y \in \mathcal{U}(X + B - \theta, X + B + \theta)$. Distribution of $B$ belongs to the following parametrized class of exponential distributions:

$$p(b) = \frac{\alpha}{2\lambda u_B \Gamma(1/\alpha)} \exp\left\{ -\frac{b}{\lambda u_B} \right\} \Gamma\left(\frac{1}{\alpha}\right), \quad \lambda = \sqrt{\frac{\Gamma(1/\alpha)}{\Gamma(3/\alpha)}}, \quad \alpha > 0; \quad (1)$$

the family contains the normal ($\alpha = 2$) and the uniform ($\alpha = +\infty$) distributions.

3. Bayesian inference
At a first stage, we calculate pdf for the quantity $X$, and then the coverage factor is estimated as a ratio of the coverage interval length and the standard uncertainty.
Prior pdfs for \( \sigma, \theta, X \) are: \( p(s) = s^{-1} \) \( (s = \sigma, \theta) \), \( p(X) \sim \text{Const} \), and joint pdf is given by the formula:

\[
p(x, b, s \mid y_1, \ldots, y_n) \sim L(y_1, \ldots, y_n \mid x, b, s) p(b) s^{-1},
\]

where \( L \) is a likelihood function. So the posterior pdf for \( X \) is

\[
p(x \mid y_1, \ldots, y_n) \sim \int_{-\infty}^{+\infty} db \int_{0}^{+\infty} L(y_1, \ldots, y_n \mid x, b, s) p(b) s^{-1} ds.
\]

Coverage interval \((-\alpha_{0.95}, \alpha_{0.95})\) is given by:

\[
\int_{-\alpha_{0.95}}^{\alpha_{0.95}} p(x) dx = 0.95,
\]

and the coverage factor could be obtained by:

\[
K_{0.95} = \frac{\alpha_{0.95}}{u(x)} = \frac{u_B \sqrt{n}}{S}, \quad n > 3,
\]

where \((-\alpha_{0.95}, \alpha_{0.95})\) is a coverage interval for \( p(x) \sim \int_{-\infty}^{+\infty} \left( \frac{1}{n-1} (\gamma z + \tilde{x})^2 + 1 \right)^{-\frac{n}{2}} \exp\left\{ -\frac{z^2}{\lambda} \right\} dz.\)

Asymptotics for \( K_{0.95} \) look as follows:

\[
K_{0.95}(\gamma) \xrightarrow{\gamma \to 0} \sqrt{\frac{n-3}{n-1}} t_{0.95}(n-1), \quad K_{0.95}(\gamma) \xrightarrow{\gamma \to +\infty} e^{(1)}_{0.95},
\]

\[
e^{(1)}_{0.95} = -\frac{\ln 0.05}{\sqrt{2}} \approx 2.118, \quad e^{(2)}_{0.95} \approx 1.96, \quad e^{(\infty)}_{0.95} = 0.95 \sqrt{3} \approx 1.645, \quad \text{etc.,}
\]

here \( e^{(\alpha)}_{0.95} \) is a 95\% coverage factor corresponding to the distribution family \((1)\) (see Fig. 3); and \( t_{0.95} \) is a quantile of Student’s distribution. Some plots for \( K_{0.95}(\gamma) \) \((n = 4, 7)\) are given on Fig. 1, 2.

![Figure 1](image-url)

**Figure 1.** \( K_{0.95}(\gamma), \gamma = \frac{u_B \sqrt{n}}{S}, n = 4. \)
3.2. Uniform distribution for the repeated indications

In case of uniform distribution for $y_i$ the coverage factor could be calculated as

$$K_{0.95}(\mu) = \frac{\tilde{\alpha}_{0.95}}{\sqrt{\frac{2(n-2)(n-3)}{n} + \mu^2}}, \quad \mu = \frac{u_B}{\sqrt{r}}, \quad n > 3,$$

where $(-\tilde{\alpha}_{0.95}, \tilde{\alpha}_{0.95})$ is a coverage interval for

$$p(\tilde{x}) \sim \int_{-\infty}^{+\infty} \left( \frac{2}{\sqrt{n}} |\mu z + \tilde{x}| + 1 \right)^{-n} \exp \left\{ - \left| \frac{z}{\lambda} \right|^{\alpha} \right\} dz,$$

Asymptotics:

$$K_{0.95}(\mu) \xrightarrow{\mu \to 0} \sqrt{\frac{(n-2)(n-3)}{2} \left( \frac{n}{\sqrt{20}} - 1 \right)};$$

for $\mu \to \infty$ the asymptotics are the same as in the previous case. Some plots for $K_{0.95}(\mu)$ ($n = 4, 7$) are given on Fig. 4, 5.
4. Conclusions
The coverage interval for a sum of two quantities (measuring instrument indication and systematic measurement error) was obtained using the Bayesian approach; various distributions were considered. The following conclusions, regarding the coverage factor, could be made:

- A single common value of $K_{0.95}$ can not be recommended for all the cases examined, the dependence on $n$ and $\gamma(\mu)$ is very significant.
- $K_{0.95}$ exceeds 2 for some values of the parameter $\alpha$ of the family (1).
- The value of $K_{0.95}$ changes rapidly and significantly when the parameters $\gamma$ and $\mu$ vary in a relatively narrow range of values near zero.
- Starting with a certain value of $\gamma$, $\mu$ ($\sim 4...5$), the values of $K_{0.95}$ for each $n$ can be regarded as constants with a sufficiently high degree of reliability.

References
[1] Stepanov A., Chunovkina A., Burnistrova N. Calculation of coverage intervals: some study cases. In: Series on Advances Mathematics for Applied Sciences, Vol. 86. World Scientific Publishing, 2015.