A Factor Matching of Optimal Tail Between Poisson Processes

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Abstract
Consider two independent Poisson point processes of unit intensity in the Euclidean space of dimension \( d \) at least 3. We construct a perfect matching between the two point sets that is a factor (i.e., a measurable function of the point configurations that commutes with translations), and with the property that the distance between two matched configuration points has a tail distribution that decays as fast as possible in magnitude, namely, as \( b \exp(-cr^d) \) with suitable constants \( b, c > 0 \). This settles the most difficult version of such matching problems: bicolored (versus unicolored) and deterministic (versus randomized). Our proof relies on two earlier results: an allocation (“land-division”) rule of similar tail for a Poisson point process by Markó and the author, and a recent breakthrough result of Bowen, Kun and Sabok that enables one to obtain perfect matchings from fractional perfect matchings under suitable conditions.

Keywords  Factor matching · Poisson point process · Allocation rule · Land division

Mathematics Subject Classification  05C70 · 60D05 · 60G55

Let \( \omega_1 \) and \( \omega_2 \) be two independent random sets of points in the Euclidean space \( \mathbb{R}^d \), given by Poisson point processes of unit intensity. We are interested in measurable perfect matchings ( bijections) between \( \omega_1 \) and \( \omega_2 \). We want the perfect matching to commute with translations of \( \mathbb{R}^d \), in other words, to be invariant under translations. One may allow the use of extra randomness in a version of this problem, as in [12],
but in our setting the matching has to be a deterministic, equivariant function of the point configurations. Such a matching is called a factor matching.

Our goal is to make matched points “as close as possible”, that is, we want \( \mathbb{P}(|m(0)| > r | 0 \in \omega_1) \) to decay as fast as possible when \( r \to \infty \), where \( m(0) \) is the point matched to 0. Note that conditioning on the zero-probability event that \( 0 \in \omega_1 \) is a standard and natural construction, called the Palm version, and in case of a Poisson point process it is the same as adding a configuration point to 0; see e.g. [18] for general background.

A trivial lower bound for the possible tail is given by the closest point of \( \omega_2 \) to 0, whose tail distribution is of order \( \exp(-cr^d) \). Our main result is that a decay of this magnitude is attainable (up to the value of the constant) when the dimension is at least 3. While Holroyd et al. [12] solved the problem with a similar tail for a factor matching of a single Poisson point process or a randomized matching between two Poisson point processes, the optimal behavior for the 2-color factor matching remained unknown until now.

**Theorem 1** Let \( \omega_1 \) and \( \omega_2 \) be two independent Poisson point processes of intensity 1 in the Euclidean space \( \mathbb{R}^d \) of dimension \( d \geq 3 \). There exists a factor perfect matching \( m \) between \( \omega_1 \) and \( \omega_2 \) with the property that

\[
\mathbb{P}(|m(0)| > r | 0 \in \omega_1) < b \exp(-cr^d)
\]

with some \( b, c > 0 \), where \( m(0) \) is the point matched to 0.

There are several reasons why the requirement of no extra randomness in a factor matching is of importance. It is certainly natural: each configuration point \( x \) has to find its image \( y = m(x) \) using only the actual local information (up to arbitrarily small error) and no “central planning”. Analogous problems for graphs have shown striking difference when optimized over all invariant or over all equivariant objects, the most famous example possibly being the independent set of transitive graphs, where the maximal factor (of iid) independent set may have lower density than the maximal invariant independent set [5]. A similar difference can be observed for certain functions from point processes. For example, the fastest possible decay of a perfect matching over a single Poisson point process in dimension 1 is very different depending on whether we allow extra randomness or not, with possible exponential tail for the former versus infinite first moment for any factor (Theorem 3 in [12]). The requirement of no extra randomness usually makes the problem more difficult, as illustrated by optimal allocations for a Poisson point process (which are related to matchings, as also seen below). A (randomized) allocation of optimal decay with the use of extra randomness was found in [13], while the construction of factor allocations needed significant further efforts, including the analysis of gravitational allocation [7], and later a factor allocation of optimal tail [16]. Similarly, for matchings between two Poisson point processes, an optimal one using extra randomness was found in [12] by Holroyd, Pemantle, Peres and Schramm, while the same paper only contains a factor matching of worse than polynomial decay. This bound was improved in a preprint [20] by the present author, where a matching scheme of decay \( \exp(-cr^{d-2-\epsilon}) \) was constructed. (That paper was dealing with the essentially equivalent problem of flipping a fair coin
for each vertex of \( \mathbb{Z}^d, d \geq 3 \), and trying to find a factor matching of optimal tail between vertices with a head and vertices with a tail.) The case of \( d = 1, 2 \) shows a different behavior, with \( c'r^{-d/2} \) being the optimal decay, as proved in [17] \((d = 1)\) and the simpler part of [20] \((d = 2)\). Various related versions of the problem have been studied, such as optimal tail behavior of matchings on a unique Poisson point process (allowing extra randomness or factor matchings) [12], minimal planar matchings [8], tail behavior of the so-called stable matching [9, 10], or multicolor matchings [3]. Let us mention that matching two Poisson point processes is significantly more difficult than the problem of finding a perfect matching factor for a single Poisson point process, essentially because of the discrepancy in the number of configuration points from the two processes within any fixed box.

**Definition 1** Consider a translation invariant point process \( \omega \) of intensity 1 in the Euclidean space \( \mathbb{R}^d \). A (factor) allocation is a measurable map \( \alpha_\omega \) from \( \mathbb{R}^d \) to \( \omega \) such that for every \( x \in \omega \), \( \alpha_\omega^{-1}(x) \) has unit volume, and \( \alpha_\omega \) is equivariant (with regard to translations) and measurable. We will assume for simplicity that \( \alpha_\omega(x) = x \) for every \( x \in \omega \). The sets \( \alpha_\omega^{-1}(x) (x \in \omega) \) will be called cells. Given an allocation \( \alpha_\omega \), for a point \( p \in \mathbb{R}^d \) let \( A(p) \) denote the cell containing \( p \), i.e. \( A(p) = \alpha_\omega^{-1}(\alpha_\omega(p)) \).

The proof will rely on two earlier results. We denote the Euclidean ball of radius \( r \) around 0 by \( B(r) \).

**Theorem 2** (Markó–Timár, [16]) Let \( \omega \) be a Poisson point process of intensity 1 in \( \mathbb{R}^d, d \geq 3 \). Then there is an allocation rule with the property that

\[
P(\text{diam}(A(0)) > r \mid 0 \in \omega) < b_0 \exp(-c_0 r^d),
\]

with some \( b_0, c_0 > 0 \). The allocation has the further property that any bounded subset of \( \mathbb{R}^d \) intersects only finitely many allocation cells.

The last assertion of the theorem is not stated explicitly in [16] but it follows from the arguments there. Indeed, in the construction of [16] the allocation cell of every configuration point \( \xi \in \omega \) is contained in the support of a function \( f^{\omega, \xi} \) that was assigned to that point earlier (Lemma 3.8 of [16]). So it is enough to verify that an arbitrary bounded subset of \( \mathbb{R}^d \) intersects only finitely many of these \( \text{supp}(f^{\omega, \xi}) \)'s that intersect a unit cube \( z + [0, 1]^d \) is almost surely finite for every \( z \), so our claim follows.

**Definition 2** A perfect fractional matching on a graph \( G = (V(G), E(G)) \) is a map \( f : E(G) \to [0, \infty) \) such that \( \sum_{v \in e, e \in E(G)} f(e) = 1 \) for every \( v \in V(G) \).

**Definition 3** Given a Borel \( \sigma \)-algebra \( (\Omega, B) \), a graph \( G \) on \( \Omega \) is a Borel graph if its edge set is Borel in the product \( \sigma \)-algebra.

Consider a probability measure \( \mu \) on \( (\Omega, B) \). \( G \) is a graphing if

\[
\int_A \deg_B(x) d\mu(x) = \int_B \deg_A(x) d\mu(x)
\]

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for every $A, B \in \mathcal{B}$.

A graphing is \textit{hyperfinite} if for any $\varepsilon > 0$ there is $U \subset \Omega$ with $\mu(U) < \varepsilon$ such that the components of $G \setminus U$ are uniformly bounded almost surely. A graphing is \textit{one-ended} if almost every component is one-ended.

We included the definitions for completeness, but refer the reader to [15] for more background on graphings. We also omit the connection between unimodular random graphs and graphings and the description of the Mass Transport Principle; these can be found in [2] for example.

**Theorem 3** (Bowen–Kun–Sabok, [6]) Suppose that a hyperfinite one-ended bipartite graphing $G$ has some measurable perfect fractional matching which is almost everywhere positive. Then $G$ has a measurable perfect matching almost everywhere.

**Proof of Theorem 1** The idea is that for both point configurations we take the corresponding allocations given by Theorem 2. Then to the edge $xy$ (where $x \in \omega_1$ and $y \in \omega_2$) we assign the measure of the intersection of the cells of $x$ and $y$. This defines a fractional perfect matching between the configurations, and hence Theorem 3 can be applied, resulting in the desired matching $m$.

First we construct a graph $G(\omega_1, \omega_2)$ on vertex set $\omega_1 \cup \omega_2$, with $\omega_1$ and $\omega_2$ as classes of bipartition, and furthermore show that there exists an almost everywhere positive measurable perfect fractional matching on $G(\omega_1, \omega_2)$. Let $\alpha_1$ and $\alpha_2$ be the allocation functions for $\omega_1$ and $\omega_2$ respectively, as given by Theorem 2, and let $A_1 = \alpha_1^{-1} \alpha_1$ (respectively, $A_2 = \alpha_2^{-1} \alpha_2$) be the map assigning its cell to every point of $\mathbb{R}^d$. Let $x \in \omega_1$ and $y \in \omega_2$ be adjacent in $G(\omega_1, \omega_2)$ if the closures of their cells, $\overline{A}_1(x)$ and $\overline{A}_2(y)$, have nonempty intersection. Define the perfect fractional matching by assigning the Lebesgue measure of $A_1(x) \cap A_2(y)$ to the edge $\{x, y\} \in E(G(\omega_1, \omega_2))$. (This graph and fractional matching applied for the construction of a matching from an allocation, already appears in [1], in a finite setting.) This will indeed be a fractional perfect matching, by definition of an allocation. Now we can obtain a graphing by picking $\omega_1$ and $\omega_2$ as independent Poisson point processes of unit intensity conditioned on $0 \in \omega_1 \cup \omega_2$, and calling $G(\omega_1, \omega_2)$ and $G(\omega', \omega')$ adjacent if there is a translation $\phi$ of $\mathbb{R}^d$ with $\phi(\omega_1) = \omega'$, $\phi(\omega_2) = \omega'$, that takes one of the neighbors of 0 in $G(\omega_1, \omega_2)$ to 0. The component of the so-defined graphing that contains $G(\omega_1, \omega_2)$ is isomorphic to the graph $G(\omega_1, \omega_2)$, as we can find every vertex of this component by repeatedly shifting the configuration so that a neighbor of 0 in $\omega_1 \cup \omega_2$ is mapped to 0.

To see that this is a graphing we have to check (3), which is standard by the invariance of the point process and the equivariance of the construction, e.g. combine Example 9.5 and 9.9 in [2]. It is bipartite by definition, every vertex has finite degree by the last assertion of Theorem 2, and $G = G(\omega_1, \omega_2)$ is one-ended almost everywhere, as explained in the rest of this paragraph. Proving by contradiction, suppose there is a finite $U \subset V(G)$ such $G \setminus U$ has at least two infinite components. By switching to neighborhoods if necessary, we may assume that every point of $U$ is in the same class of bipartition, say $\omega_1$. By the bound in Theorem 2, every cell is bounded almost surely, thus $\bigcup_{x \in U} \overline{A}_1(x)$ is bounded. Hence $\mathbb{R}^d \setminus \bigcup_{x \in U} \overline{A}_1(x)$ has only one unbounded component; call it $I$. Consider two points $u, v \in I \cap (\omega_1 \cup \omega_2)$. We claim that then
Let $u$ and $v$ be in the same component of $G(\omega_1, \omega_2) \setminus U$. To see this, let $\gamma$ be a curve between $u$ and $v$ in $I$. (A connected open subset of $\mathbb{R}^d$ is always path-connected, hence such a curve exists.) There is a path between $u$ and $v$ in $G(\omega_1, \omega_2) \setminus U$, because every point of $\gamma$ is contained in the closure of at least one cell of $\{A_1(x) : x \in \omega_1 \setminus U\}$ and one of $\{A_2(x) : x \in \omega_2\}$, and there are finitely many cells from these sets that intersect $\gamma$. We conclude that $G(\omega_1, \omega_2) \setminus U$ can have only one unbounded component. We obtain that $G(\omega_1, \omega_2)$ has one end.

So it only remains to prove that $G$ is hyperfinite, that is, to find for any $\epsilon > 0$ a subset of the configuration points as a factor (measurable equivariant function), such that the density of the chosen points in $V(G)$ is less than $\epsilon$ and their removal breaks $G$ into all finite components. (By “density” we mean the probability that the “root” $0$ of $G$ is among the chosen points, where we condition on $0 \in \omega_1 \cup \omega_2$.)

Given $\epsilon > 0$, choose $r$ such that $\mathbb{P}(\max(|x| : (0, x) \in E(G)) > r | 0 \in \omega_1 \cup \omega_2) < \epsilon / 2$. Let $U_1$ be the set of vertices $u$ in $V(G)$ such that $\max(|u - x| : (u, x) \in E(G)) > r$. Then $\mathbb{P}(0 \in U_1 | 0 \in \omega_1 \cup \omega_2) < \epsilon / 2$. Now choose $N$ large enough so that $dr / N < \epsilon / 2$.

We will take a partition $\mathcal{P}$ of $\mathbb{R}^d$ into convex polyhedra as a factor of $\omega_1 \cup \omega_2$, in such a way that every polyhedron in $\mathcal{P}$ contains a ball of radius $N$. The volume of $P \in \mathcal{P}$ divided by its surface area is at least $N / d$ (see Lemma 3.3 in [19] for the simple proof), which implies that $\text{Vol}(\partial P) / \text{Vol}(P) \leq r \text{Area}(\partial P) / \text{Vol}(P) \leq rd / N$, with $\partial P$ denoting the $r$-neighborhood of the boundary of $P$ inside $P$. This implies $\mathbb{P}(0 \in \cup_{P \in \mathcal{P}} \partial P) < rd / N < \epsilon / 2$ (see Lemma 3.5 of [19]). Hence, denoting by $U_2$ the set of vertices in $V(G) \setminus U_1$ whose cell intersects the boundary of some $P \in \mathcal{P}$, we have $\mathbb{P}(0 \in U_2 | 0 \in \omega_1) < \epsilon / 2$. The removal of $U_1 \cup U_2$ splits $G$ into only finite components, because points of $\omega_1 \cup \omega_2$ that are in different elements of $\mathcal{P}$ cannot be in the same component of $G \setminus (U_1 \cup U_2)$. Note that splitting into finite components (instead of bounded-size components) is enough to show hyperfiniteness; see Section 21.1 of [15] for this equivalence. Furthermore, $\mathbb{P}(0 \in U_1 \cup U_2 | 0 \in \omega_1 \cup \omega_2) < \epsilon$. So we deduce hyperfiniteness of $G$ as soon as we construct the $\mathcal{P}$. To do so, it is enough to find a factor subset $X$ of $\omega_1 \cup \omega_2$ consisting of points whose pairwise distance in $\mathbb{R}^d$ is at least $2N$, and then take the Voronoi tessellation for $X$ to be $\mathcal{P}$. One way to do this is to consider the points of $\omega_1 \cup \omega_2$ whose distance from every other configuration point is at least $2N$. This is a nonempty set, since the configuration is from a Poisson point process.

Now consider the graphing $G$ and let $m$ be a perfect matching that bijectively assigns to every $x \in \omega_1$ a point $m(x) \in \omega_2$, as provided by Theorem 3. We only have to show (1). Conditional on $0 \in \omega_1$, the random marked rooted graph $(G, 0; A)$ is unimodular, where by $A$ we denote the assignment of the number $\text{diam}(A_i(x))$ (with $i = 1$ if $x \in \omega_1$ and $i = 2$ otherwise) to each vertex $x$. Define the following simple mass transport: let $y \in \omega_2$ send mass 1 to $x \in \omega_1$ if $m(x) = y$ and $\text{diam}(A_2(y)) > r$. By the Mass Transport Principle,

$$
\mathbb{P}(\text{diam}(A_2(m(0))) > r | 0 \in \omega_1) = \mathbb{E}(\text{“mass received”}) = \mathbb{E}(\text{“mass sent out”}) = \mathbb{P}(\text{diam}(A_2(0)) > r | 0 \in \omega_2).
$$

We conclude that

$$
\mathbb{P}(|m(0)| > 2r | 0 \in \omega_1) < \mathbb{P}(\text{diam}(A_1(0)) > r | 0 \in \omega_1).
$$
\[ + \mathbb{P}(\text{diam}(A_2(m(0))) > r \mid 0 \in \omega_1) < 2b_0 \exp(-c_0r^d) \]

by Theorem 2.

The fact that the underlying point processes are Poisson were only used in the proof at two places: at the point when Theorem 2 is applied, and at the choice of the subset \( X \). The latter can be replaced by a more refined argument that would work for any point processes. Namely, define a graph \( H \) on \( \omega_1 \cup \omega_2 \) where two points are adjacent if their distance is at most \( 2N \). We need to find a nonempty independent set that is a factor in this graph. To this end choose \( D \) so that there exist vertices with degree less than \( D \), then delete points whose degree is at least \( D \). Now use a measurable Brooks theorem to find a proper coloring of the resulting induced graph by finitely many colors (see [14] for the measurable Brooks theorem, which is applied a few times in [6] in a similar way as here). Choose one of the color classes to be \( X \). The rest of the proof generalizes right away, to show the following theorem.

**Theorem 4** Let \( \omega_1 \) and \( \omega_2 \) be invariant point processes of unit intensity. Suppose that for \( i = 1, 2 \) there are functions \( f_i \) with \( f_i(r) \to 0 \) as \( r \to \infty \), and there is an allocation rule with the property that the cell \( A_i(0) \) satisfies

\[ \mathbb{P}(\text{diam}(A_i(0)) > r \mid 0 \in \omega_i) < f_i(r), \tag{4} \]

and such that any bounded subset of \( \mathbb{R}^d \) intersects only finitely many allocation cells. Then there is a factor perfect matching \( m \) between \( \omega_1 \) and \( \omega_2 \) with the property that

\[ \mathbb{P}(|m(0)| > r \mid 0 \in \omega_1) < f_1(r/2) + f_2(r/2). \tag{5} \]

**Remark 5** In Theorem 1 we construct a matching that is equivariant with respect to translations. However, the same construction would work for all isometries, if the allocation considered from Theorem 2 were isometry-equivariant. While the proof in [16] is elaborated only for translations, a bit of extra work would make it isometry-equivariant (see Remark 3.9 in [16]), giving rise to an isometry-equivariant matching in Theorem 1 of the current paper.

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