EVOLUTION OF FIXED-END STRINGS AND THE OFF-SHELL DISK AMPLITUDE

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Abstract

An exact integral expression is found for the amplitude of a Bosonic string with ends separated by a fixed distance \( R \) evolving over a time \( T \) between arbitrary initial and final configurations. It is impossible to make a covariant subtraction of a divergent quantity which would render the amplitude non-zero. It is suggested that this fact (and not the tachyon) is responsible for the lack of a continuum limit of regularized random-surface models with target-space dimension greater than one. It appears consistent, however, to remove this quantity by hand. The static potential of Alvarez and Arvis \( V(R) \), is recovered from the resulting finite amplitude for \( R > R_c = \pi \sqrt{\frac{d-2}{3} \alpha'} \). For \( R < R_c \), we find \( V(R) = -\infty \), instead of the usual tachyonic result. A rotation-invariant expression is proposed for special cases of the off-shell disk amplitude. None of the finite amplitudes discussed are Nambu or Polyakov functional integrals, except through an unphysical analytic continuation. We argue that the Liouville field does not decouple in off-shell amplitudes, even when the space-time dimension is twenty-six.

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1 Introduction

In this paper, we study amplitudes for string world sheets with fixed boundaries. Our motivation is to better understand off-shell amplitudes for string theory. Knowledge of these amplitudes yields the effective action of string theory at lowest nontrivial order. This in turn may lead to a simpler covariant formalism of string field theory and perhaps yield insight into QCD as well.

Direct computation of off-shell amplitudes with Polyakov’s path integral is difficult, since the ghost degrees of freedom are more complicated with Dirichlet rather than with Neumann boundary conditions. The boundary conditions on the ghost fields themselves was discussed long ago [1]. Instead of working directly with the Polyakov path integral as has been done earlier [2], [3] we will study the energy operator for a string with fixed endpoints in a covariant gauge, evolving over a finite time interval. Unfortunately, this places some restrictions on the boundary data. On the other hand, the method does yield a definite expression.

Our first result is that the world-sheet-regularized form of these amplitudes is ill-defined. In general, the expression contains a divergent piece which cannot be removed by a covariant subtraction. We believe that this divergence is related to the inability of lattice and dynamically triangulated strings to have a proper continuum limit for \( d > 1 \) [4]. The form of this divergent piece is not universal, however. Therefore, it seems that it may be sensibly set to zero. When this is done, the amplitude decays exponentially with area, with the standard semiclassical corrections of Lüscher, Symanzik and Weisz [5] (who used a loop-wave equation. See reference [6] for a discussion of the Polyakov action). In fact, the well-known static potential of Alvarez [7] and Arvis [8] is easily recovered for \( R \) greater than the so called critical radius. Instead of becoming imaginary below this radius, the potential becomes negative infinite. This does not mean that the usual conclusion is wrong, for our method of continuing below the critical radius is different mathematically from that in references [7], [8]. This phenomenon can be understood as as a divergence of the string amplitude.

The expression obtained for the amplitude is not rotation invariant. If the world-sheet boundaries lie on a rectangle of size \( R \times T \), this expression changes upon interchange of \( R \) and \( T \). This is not surprising; in the appendix, we show that a similar expression constructed for relativistic particles is not Lorentz invariant. We suggest the corrected expression for the disk amplitude, by analogy with the particle case.

The Liouville field does not decouple in off-shell amplitudes even in the critical dimension. The correct interpretation of this fact is most likely that if the string degrees of freedom \( X^\mu \) obey Dirichlet boundary conditions, the central charge of the world-sheet Fadeev-Popov ghosts is \(-2\) instead of \(-26\).

The particular geometries studied here are fairly special. They are such that Cartesian coordinates \( X^\mu \) have the property that the boundary values of \( X^0 \) and \( X^1 \) describe a rectangle. The remaining coordinates, \( X^\perp = X^2, \ldots, X^{d-1} \) are left arbitrary on the boundary. For such boundaries, an exact expression for the amplitude is found. Setting
$X^\perp = 0$ and letting one dimension of the rectangle go to infinity may be used to obtain the Bosonic-string static potential \[7, 8\]. We discuss in Section 7 how the amplitude for slightly more general boundary geometries may be determined.

## 2 Off-shell amplitudes

The action of the Bosonic string is

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\mu ,$$

where $\xi^a$ are the coordinates of the world-sheet, $a, b = 0, 1$, $\partial_a = \partial/\partial \xi^a$, and $\mu, \nu = 1, \ldots d$. The functional integral of the disk with fixed boundary data $X(\varphi)$ with boundary coordinate $\varphi \in [0, 2\pi)$

$$\Phi[X] = \int_{X(\varphi)} DX e^{-S} .$$

If this expression exists, it has the expansion:

$$\Phi[X] \approx \frac{1}{\sqrt{-\det \delta^2 S}} e^{-S_0} ,$$

where $S_0$ is the classical action of a solution to the equations of motion and the prefactor is the reciprocal of the square root of the determinant of the fluctuation operator. We shall argue that (2.2) is not well-defined as it stands, but that it is possible to formally define a string Green’s function $\Phi[X]$ anyway. Nonetheless, it is interesting to determine $S_0$ from the boundary data, as was done in reference \[9\]. Though the on-shell tree amplitudes for string theory obtained from $S_0$ are correct \[9\], we will show that the off-shell amplitudes are not.

## 3 Determination of the classical action

It is possible to obtain $S_0$ for Dirichlet data $X^\mu(\varphi)$ on the boundary by solving the equations of motion \[9\]. We review this procedure here (which is not generally known) because the result will be important in our discussion of the complete solution for the off-shell disk amplitude.

Let us suppose we know the solution in the interior of the disk $X^\mu(r, \varphi)$, $g_{ab}(r, \varphi)$. These quantities satisfy the equations of motion

$$\frac{1}{\sqrt{g}} \partial_a \sqrt{g} g^{ab} \partial_b X^\mu = 0 ,$$

where
and

$$
\partial_a X^\mu \partial_b X^\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X^\mu = 0 .
$$

(3.2)

Once any solution is obtained, by making a conformal transformation on the metric, it is possible to chose \( r \in [0, a) \), \( g_{rr} = e^\phi \), \( g_{\phi'\phi'} = r^2 e^\phi \) and \( g_{r\phi'} = 0 \). The prime in \( \phi' \) is present because \( \phi \) is mapped to \( \phi' \) by the conformal transformation. Then the equation of motion is Laplace’s equation (3.1) which has the solution:

$$
X^\mu = B^\mu (re^{i\phi'}) + \bar{B}^\mu (re^{i\phi'}) ,
$$

where \( B^\mu(z) \) is an analytic function

$$
B^\mu(z) = \sum_{n=0}^{\infty} b_n^\mu z^n
$$

satisfying \( B^\mu(ae^{i\phi'}) + \bar{B}^\mu(ae^{i\phi'}) = X^\mu(\phi') \). The values of \( b_n^\mu \) are determined by the boundary data \( X^\mu(\phi') = X^\mu(a, \phi') \):

$$
b_n^\mu = \frac{1}{2\pi a} \int_0^{2\pi} d\phi' e^{-in\phi'} X^\mu(\phi') .
$$

(3.3)

In terms of the coefficients \( b_n^\mu \), the action (2.1) is, after integration by parts and using the equation of motion, a pure surface term:

$$
S = S_0 = \frac{1}{4\pi \alpha'} \int_0^a dr \int_0^{2\pi} d\phi' \left[ r (\partial_r X)^2 + \frac{1}{r} (\partial_{\phi'} X)^2 \right]
$$

$$
= \frac{1}{4\pi \alpha'} \int_0^a dr \int_0^{2\pi} d\phi' \left[ \partial_r (r X^\mu \partial_r X^\mu) + \frac{1}{r} \partial_{\phi'} (X^\mu \partial_{\phi'} X^\mu) \right] ,
$$

(3.4)

where \( a \) is the disk diameter (the actual value of which is irrelevant). Only the first term in (3.4) is not zero. The general solution for \( X \) into (3.4) yields

$$
S_0 = \frac{1}{2\alpha'} \sum_{n=1}^{\infty} n b_n^\mu b_n^\mu ,
$$

and substituting (3.3) gives

$$
S_0 = -\frac{1}{8\pi \alpha'} \int_0^{2\pi} d\phi'_1 \int_0^{2\pi} d\phi'_2 \sum_{n=1}^{\infty} n e^{in(\phi'_1-\phi'_2)} \left\{ [X(\phi'_1) - X(\phi'_2)]^2 + X(\phi'_1)^2 - X(\phi'_2)^2 \right\} .
$$
The last two terms are zero. Summation over $n$ is straightforward, giving the result:

$$e^{-S_0} = \exp - \frac{1}{16\pi\alpha'} \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \frac{[X(\varphi_1) - X(\varphi_2)]^2}{\sin \frac{1}{2}(\varphi_1 - \varphi_2)}.$$  

This expression is invariant under diffeomorphisms of $\varphi'$, mapping the interval $[0, 2\pi)$ to itself. We are therefore free to map $\varphi'$ back to $\varphi$, obtaining

$$e^{-S_0} = \exp - \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \sum_{j=-\infty}^{\infty} \frac{[X(\varphi_1) - X(\varphi_2)]^2}{(\varphi_1 - \varphi_2 - 2\pi j)^2}.$$  

(3.5)

Since $S_0$ is the value of an extremum of the Polyakov action, it is the value of an extremum of the Nambu action as well. Therefore $S_0$ is the minimal area of a surface whose boundary is $X^\mu(\varphi)$.

The expression (3.3) is invariant under diffeomorphisms of the boundary, $\varphi \rightarrow F(\varphi)$. In the form of this result presented by Polyakov, the range of $\varphi$ is extended to the entire real axis, so that diffeomorphisms become $\text{SL}(2,\mathbb{R})$ transformations [9]. One finds that the off-shell expression (3.3) becomes

$$e^{-S_0} = \exp - \frac{1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\varphi_1 \int_{-\infty}^{\infty} d\varphi_2 \frac{[X(\varphi_1) - X(\varphi_2)]^2}{(\varphi_1 - \varphi_2)^2}.$$  

This expression yields the open-string Koba-Nielsen amplitudes when extended to the mass shell [10]. Naively, one expects the Liouville field to decouple when $d = 26$ and that (3.5) should be the final expression for $Z[X]$. We will see that this is not the case.

4 The string energy operator and subtractions

In this section, the string energy operator, which governs time development in target space, is critically reexamined. Certain conventions are different from Arvis' [8]. We will quantize in the Schrödinger representation.

We change the notation for the coordinates, specifically $\xi^0 = \tau$ and $\xi^1 = \sigma$, and write the string degrees of freedom as $X^\mu(\sigma, \tau)$. The coordinate $\sigma$ lies in the interval $[0, \pi]$, while we treat the coordinate $\tau$ as the world-sheet time on world sheets of Minkowski signature. Target space will also be assumed to have Minkowski signature (in the next section, we Wick-rotate to Euclidean signature). Unlike the case of the on-shell string, the world-sheet time $\tau$ is not identified with target-space time.

The boundary conditions taken in most of this paper are $X^\perp(0, \tau) = 0$, $X^\perp(\pi, \tau) = 0$, $X^\perp(0, \tau) = 0$, $X^\perp(\pi, \tau) = R$, $\partial_\sigma X^0(0, \tau) = 0$ and $\partial_\sigma X^0(\pi, \tau) = 0$. We will discuss slightly more general boundary conditions in Section 7.
In these coordinates the action (2.1) is that of a free field theory

\[ S = -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \eta_{\mu\nu} [(\partial_\tau X^\mu)^2 - (\partial_\sigma X^\mu)^2], \]

where \( \eta \) has signature \((+,-,\ldots,-)\). This action leads to the momentum density, conjugate to \( X^\mu \)

\[ P_\mu(\sigma, \tau) = -\eta_{\mu\nu} \partial_\tau X^\nu(\sigma, \tau). \]

The energy of the string is not the canonical Hamiltonian, but rather the integral of \( P_0 \):

\[ E = -\int_0^\pi d\sigma P_0(\sigma, \tau) = \int_0^\pi d\sigma \partial_\tau X^0(\sigma, \tau). \]

(4.1)

The boundary conditions are consistent with the normal-mode expansions:

\[ X^\perp(\sigma, \tau) = 2\sqrt{\alpha'} \sum_{n=1}^{\infty} \sqrt{n} \sin n\sigma \ X^\perp_n(\tau), \]

\[ X^1(\sigma, \tau) = \frac{R}{\pi} \sigma + 2\sqrt{\alpha'} \sum_{n=1}^{\infty} \sqrt{n} \sin n\sigma \ X^1_n(\tau), \]

\[ X^0(\sigma, \tau) = X^0_0(\tau) + 2\sqrt{\alpha'} \sum_{n=1}^{\infty} \sqrt{n} \cos n\sigma \ X^0_n(\tau). \]

The action (2.1) in conformal coordinates \( \sigma, \tau \), after Wick rotating to Minkowski signatures and written in term of normal modes is

\[ S = -\frac{1}{4\pi\alpha'} \int d\tau \left\{ \pi(\partial_\tau X^0_0)^2 + \sum_{n=1}^{\infty} [(\partial_\tau X^0_n)^2 - (\partial_\tau X^1_n)^2 - (\partial_\tau X^\perp_n)^2] \right. \]

\[ \left. - n^2(X^0_n)^2 + n^2(X^1_n)^2 + n^2(X^\perp_n)^2 \right\}. \]

The canonical momenta of \( X^0_0, X^0_n, X^1_n \) and \( X^\perp_n \) \((n > 0)\) are

\[ P^0_0 = -\frac{1}{2\alpha'} \partial_\tau X^0_0, \quad P^0_n = -\frac{1}{2\pi\alpha'} \partial_\tau X^0_n, \quad P^1_n = \frac{1}{2\pi\alpha'} \partial_\tau X^1_n, \quad P^\perp_n = \frac{1}{2\pi\alpha'} \partial_\tau X^\perp_n, \]

respectively. It is clear that \( P^0_0 \) is equal to the total energy \( E \), defined in (4.1). The canonical Hamiltonian is

\[ H = -\alpha'E^2 - \pi\alpha' \sum_{n=1}^{\infty} [(P^0_n)^2 - (P^1_n)^2 - (P^\perp_n)^2] \]
It is not hard to show that the Poisson bracket of $E(\tau)$ with the Hamiltonian (4.2) vanishes. Therefore $E$ is independent of $\tau$.

The energy-momentum conditions (3.2), after rotating to Minkowski signature are

$$(\partial_{\sigma} X^0 \pm \partial_{\sigma} X^0)^2 - (\partial_{\sigma} X^1 \pm \partial_{\sigma} X^1)^2 - (\partial_{\sigma} X^\perp \pm \partial_{\sigma} X^\perp)^2 = 0.$$  

In terms of normal modes and their conjugate momenta, these conditions are

$$L_n \equiv \sum_{j+k=n} n^\mu_\alpha j^\nu_\beta \alpha_j^\mu \alpha_k^\nu = 0 \quad (4.3)$$

where $j, k, n$ are (positive, zero or negative) integers and

$$\alpha_0^0 = \sqrt{2} \alpha' E \quad \alpha_0^1 = \frac{R}{\sqrt{2\pi}} \quad \alpha_0^\perp = 0,$$

$$\alpha_n^0 = \sqrt{2\pi} \alpha' P^0_n - \frac{in}{2\sqrt{\pi}} X^0_n \quad \alpha_{-n}^0 = \sqrt{2\pi} \alpha' P^0_n + \frac{in}{2\sqrt{\pi}} X^0_n, \quad (n \geq 1)$$

$$\alpha_n^1 = -i\sqrt{2\pi} \alpha' P^1_n + \frac{n}{2\sqrt{\pi}} X^1_n \quad \alpha_{-n}^1 = i\sqrt{2\pi} \alpha' P^1_n - \frac{n}{2\sqrt{\pi}} X^1_n, \quad (n \geq 1)$$

$$\alpha_n^\perp = -i\sqrt{2\pi} \alpha' P^\perp_n + \frac{n}{2\sqrt{\pi}} X^\perp_n \quad \alpha_{-n}^\perp = i\sqrt{2\pi} \alpha' P^\perp_n - \frac{n}{2\sqrt{\pi}} X^\perp_n, \quad (n \geq 1).$$

These conditions imply that the canonical Hamiltonian (4.2) vanishes. It is possible to write $X^\mu(\sigma, \tau)$ directly in terms of the variables $\alpha^\mu_n$:

$$X^0(\sigma, \tau) = X^0_0(\tau) + i\sqrt{2} \sum_{n=-\infty}^{\infty} \frac{\cos n\sigma}{n} \alpha^0_n(\tau),$$

$$X^1(\sigma, \tau) = \frac{R\sigma}{\pi} + i\sqrt{2} \sum_{n=-\infty}^{\infty} \frac{\sin n\sigma}{n} \alpha^1_n(\tau),$$

$$X^\perp(\sigma, \tau) = i\sqrt{2} \sum_{n=-\infty}^{\infty} \frac{\sin n\sigma}{n} \alpha^\perp_n(\tau). \quad (4.4)$$
Invariance under pseudo-conformal transformations $\sigma \rightarrow \sigma + f(\tau + \sigma) - f(\tau - \sigma)$, $\tau \rightarrow \tau + f(\tau + \sigma) + f(\tau - \sigma)$, allows for further gauge fixing:

$$\partial_\tau X^1 + \partial_\sigma X^0 = 0 \ , \ \partial_\tau X^0 + \partial_\sigma X^1 = 2\alpha' E + \frac{R}{\pi} \ , \ X^0_{\sigma} = 0 . \tag{4.5}$$

Substituting the expansions (4.4) into (4.5) gives

$$\partial_\tau \alpha_{n}^0 - i n \alpha_{n}^0 = 0 \ , \ \alpha_{n}^0 + \alpha_{n}^1 = 0 \ (n \geq 1) . \tag{4.6}$$

Substituting (4.6) into (4.3) for $n \neq 0$ yields

$$2\sqrt{2}\alpha' \left( E + \frac{R}{2\pi \alpha'} \right) \alpha_{n}^0 - \sum_{j+k=n} \alpha_{j}^\perp \cdot \alpha_{k}^\perp = 0 \ , \ n \neq 0 ,$$

or

$$\alpha_{n}^0 = -\alpha_{n}^1 = \frac{\sqrt{2}}{2\alpha'E + \frac{R}{\pi}} \sum_{j+k=n} \alpha_{j}^\perp \cdot \alpha_{k}^\perp , \ n \neq 0 . \tag{4.7}$$

The conditions (4.7) reduce the total number of degrees of freedom, but do not affect the spectrum of the energy operator and will not be discussed further. Substituting the algebraic conditions (4.6) into (4.3) for $n = 0$ gives

$$2\alpha'E^2 - \frac{R^2}{2\pi^2} - \sum_{j=-\infty \atop j \neq 0}^{\infty} \alpha_{j}^\perp \cdot \alpha_{-j}^\perp = 0 , \tag{4.8}$$

or

$$E = \left( \frac{R^2}{4\pi^2\alpha'^2} + \frac{1}{2\alpha'^2} \sum_{j=-\infty \atop j \neq 0}^{\infty} \alpha_{j}^\perp \cdot \alpha_{-j}^\perp \right)^{\frac{1}{2}} .$$

Instead of quantizing yet, as was done in reference [8], we will first write the energy $E$ in terms of variables which are functions of $\sigma$. Reintroducing the field $X^\perp(\sigma, \tau)$ and its conjugate momentum $P^\perp(\sigma)$, the energy operator is

$$E = \left\{ \frac{R^2}{4\pi^2\alpha'^2} + \int_0^{\pi} d\sigma \left[ \pi (P_{\perp})^2 + \frac{1}{4\pi\alpha'^2} (\partial_{\sigma} X_{\perp})^2 \right] \right\}^{\frac{1}{2}} .$$

We see that $E^2$ contains a piece resembling the Hamiltonian of a free massless field theory. Upon quantization, we therefore regard the energy operator as

$$E = \left\{ \frac{R^2}{4\pi^2\alpha'^2} + \int_0^{\pi} d\sigma \left[ \frac{\delta^2}{\delta X^\perp(\sigma) \cdot \delta X^\perp(\sigma)} + \frac{1}{4\pi\alpha'^2} (\partial_{\sigma} X_{\perp})^2 \right] \right\}^{\frac{1}{2}} . \tag{4.9}$$
Notice that this operator, defined with any sensible regularization is positive definite.

Rewriting (4.9) in terms of annihilation operators $a_n^\dagger \perp = \frac{i}{\sqrt{n} \alpha_{\perp n}}$ and creation operators $a_n^\perp = -\frac{i}{\sqrt{n} \alpha_{\perp n}}$, where $n \geq 1$:

$$E = \left( \frac{R^2}{4\pi^2 \alpha'^2} + \frac{1}{\alpha'} \sum_{j=1}^{\infty} na_j^\dagger \perp \cdot a_j^\perp + \frac{d-2}{2\alpha'} \sum_{j=1}^{\infty} j \right)^{\frac{1}{2}}. \quad (4.10)$$

Unfortunately, this expression presents us with a problem, because $E^2$ is positive definite. We would like to make a subtraction $C_0$ from $E^2$:

$$E = \left( \frac{R^2}{4\pi^2 \alpha'^2} + \frac{1}{\alpha'} \sum_{j=1}^{\infty} na_j^\dagger \perp \cdot a_j^\perp - C_0 + \frac{d-2}{2\alpha'} \sum_{j=1}^{\infty} j \right)^{\frac{1}{2}}. \quad (4.11)$$

so that

$$-\frac{2\alpha'}{2-d} C_0 + \sum_{j=1}^{\infty} j = \zeta(-1) = -\frac{1}{12}, \quad (4.12)$$

where $\zeta(s)$ is the Riemann zeta function. Equivalently, the zeta function is analytically continued to $s = -1$. This would be in accord with Lorentz invariance in the critical dimension $D = 26$. The same choice follows from requiring the smallest eigenvalue of $E^2$ to be the “Casimir energy”; the expression is in quotes, because $E^2$ is not actually an energy. The smallest eigenvalue of $E$ could then be read off from (4.11) to give the static potential [7, 8]:

$$V(R) = \left( \frac{R^2}{4\pi^2 \alpha'^2} - \frac{d-2}{12 \alpha'} \right) \frac{1}{2} = \frac{1}{2\pi \alpha'} \sqrt{R^2 - R_c^2}. \quad (4.13)$$

For large values of $R$, this has the form [6]

$$V(R) = \frac{R}{2\pi \alpha'} - \frac{\pi (d-2)}{24R} + \ldots,$$

where the leading correction to the linear potential is universal [11]. Clearly $C_0$ is an infinite constant. The subtraction procedure can only be physically meaningful if (once the world-sheet is suitably regularized) it can be made by introducing a local counterterm in (2.1) [1]. According to our analysis in the next section, there is no such procedure. This does not mean that amplitudes with the constant $C_0$ satisfying (4.12) introduced are meaningless; only that these amplitudes are not directly obtainable from a quantum action principle, such as (2.2).

1We are not disputing that it is mathematically sensible to define the zeta function for negative arguments by analytic continuation. The point is that there is no physical justification for such a procedure without introducing an explicit dimensionful cut-off, and a sensible procedure for making the subtraction. Such a subtraction is impossible for the problem we are considering, though it is possible in other situations, e.g. the light-cone gauge string [12].
5 String evolution and the divergence of string amplitudes

In this section we begin by calculating the amplitude for a string to begin in one eigenstate of $X^\perp$ at time $X^0 = 0$ and evolve to another eigenstate of $X^\perp$ after a time $X^0 = T$. This is not quite the same thing as the disk amplitude with the same boundary conditions; we propose what the correct form of the latter should be on the basis of symmetry considerations in Section 7. To illustrate the distinction between the two quantities, we consider an analogous situation, namely the difference between the relativistic particle amplitude and the propagator using similar technology in the appendix.

For any positive operator $D$, the following integral transform will be useful:

$$e^{-T\sqrt{D}} = \frac{T}{\sqrt{\pi}} \int_0^\infty \frac{du}{u^2} e^{-\frac{u^2}{4\alpha'^2} - u^2 D}.$$ (5.1)

Hence we expect the amplitude for a string to begin in an eigenstate of the operators $X^\perp_i(\sigma)$ with eigenvalues $X^\perp_i(\sigma)$ and finish in an eigenstate with eigenvalues $X^\perp_f(\sigma)$ over a time duration $T$ is

$$\langle X^\perp_f(\sigma) | e^{-TE} | X^\perp_i(\sigma) \rangle = \frac{T}{\sqrt{\pi}} \int_0^\infty \frac{du}{u^2} e^{-\frac{u^2}{4\alpha'^2}} \langle X^\perp_f(\sigma) | e^{-u^2E^2} | X^\perp_i(\sigma) \rangle,$$ (5.2)

where $E$ is the operator (4.9) or (4.10). If $E^2$ is not positive, the integral fails to converge and this formula breaks down.

To evaluate the matrix element on the right-hand side of (5.2), we use the familiar expression for the simple-harmonic-oscillator kernel:

$$\langle q_f | e^{-\frac{\beta}{2} \left( -\frac{q^2}{4\alpha'^2} + q^2 \right)} | q_i \rangle = \sqrt{\frac{1}{2 \pi \sinh T}} e^{-A(q_f, q_i; \beta)},$$

where

$$A(q_f, q_i; \beta) = \frac{1}{2} (q_f^2 + q_i^2) \coth \beta - \frac{q_f q_i}{\sinh \beta} = \frac{1}{2} \int_0^\beta dt \left( q_{\text{class}}^2 + q_{\text{class}}^2 \right)$$

is the Wick-rotated action of the harmonic oscillator for the classical solution $q_{\text{class}}(t)$ for which $q_{\text{class}}(0) = q_i$ and $q_{\text{class}}(\beta) = q_f$. We thereby obtain

$$\langle X^\perp_f(\sigma) | e^{-TE} | X^\perp_i(\sigma) \rangle = \frac{T}{\sqrt{\pi}} \int_0^\infty \frac{du}{u^2} e^{-\frac{u^2}{4\alpha'^2} - \frac{u^2}{4\alpha'^2} + \frac{u^2}{4\alpha'^2}}$$

$$\times \left( \prod_{n=1}^\infty \pi e^{4\alpha'^2} \left( 1 - e^{-\frac{2nu^2}{\alpha'^2}} \right) \right)^{-\frac{4\alpha'^2}{u^2}} e^{-S_0[u; X^\perp_f, X^\perp_i]},$$ (5.3)
where

\[ S_0[u; X^\perp_f, X^\perp_i] = \frac{1}{4\pi\alpha'} \sum_{n=1}^{\infty} \int_0^\pi d\sigma_1 \int_0^\pi d\sigma_2 n \sin n\sigma_1 \sin n\sigma_2 \]

\[ \times \left\{ [X^\perp_f(\sigma_1)^2 + X^\perp_i(\sigma_2)^2] \coth \frac{nu^2}{\alpha'} - \frac{2X^\perp_f(\sigma_1) \cdot X^\perp_i(\sigma_2)}{\sinh \frac{nu^2}{\alpha'}} \right\}. \quad (5.4) \]

Recall that the Dedekind eta function of the complex number \( \tau \) is

\[ \eta(\tau) = e^{-\pi i \tau} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau}). \quad (5.5) \]

and under an inversion of \( \tau \) through the unit circle, followed by a reflection through the imaginary axis, transforms as follows:

\[ \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau). \quad (5.6) \]

Notice that this function resembles the infinite product in (5.3). We write this product as

\[ \left[ \prod_{n=1}^{\infty} \pi e^{\frac{u^2}{\alpha'} (1 - e^{-2\pi in\tau})} \right]^{-\frac{d-2}{2}} = \eta\left(\frac{iu^2}{\pi\alpha'}\right)^{-\frac{d-2}{2}} e^{-\frac{d-2}{2} \sum_{n=1}^{\infty} \ln \pi e^{-\frac{(d-2)u^2}{2\alpha'}} (\sum_{n=1}^{\infty} n + \frac{1}{12})}. \]

With these manipulations

\[ \left\langle X^\perp_f(\sigma) e^{-TE} | X^\perp_i(\sigma) \right\rangle = Z_0 T \int_0^{\infty} \frac{du}{u^2} \eta\left(\frac{iu^2}{\pi\alpha'}\right)^{-\frac{d-2}{2}} e^{-\frac{u^2}{4\alpha'} - \frac{\alpha^2}{4\pi^2\alpha'} e^{-S_0[u; X^\perp_f, X^\perp_i]}} \]

\[ \times \exp \left[ -u^2 \frac{d-2}{2\alpha'} \left( \sum_{n=1}^{\infty} n + \frac{1}{12} \right) \right], \quad (5.7) \]

where \( Z_0 \) is an infinitesimal constant (which can be interpreted as a string-wavefunction renormalization).

The last factor of (5.7):

\[ \exp -u^2 \left[ \frac{d-2}{2\alpha'} \left( \sum_{n=1}^{\infty} n + \frac{1}{12} \right) \right], \quad (5.8) \]

gives a nonsensical result for the amplitude, presenting us with a problem. We would like to introduce a subtraction inside the integral on the right-hand side so that (5.8) can be set equal to unity. Such a procedure mutilates the theory we began with. If
we multiply this integral by a factor $e^{C_0 T}$ we can only make a subtraction from the energy operator. This is obviously useless for removing the unwanted factor, even if the limits of integration in $u$ are cut off. We therefore seem to be in a bind; there is no counterterm which removes this factor. The integral vanishes at large $u$, even if the world-sheet is regularized. The reason is that any physical regularization of the sum $\sum_{n=1}^\infty n + \frac{1}{12}$ is greater than $R^2/4\pi^2\alpha'^2$ for sufficiently large cut-off (though there are formal techniques which obscure the situation). We believe that this divergence is a fact, and the amplitude

$$\langle X_f^\perp(\sigma) \lvert e^{-TE} \rvert X_i^\perp(\sigma) \rangle$$

is simply meaningless from a mathematically careful standpoint.

What we are saying should not be surprising in the light of analytic and numerical studies of random surface models \[1\]. No attempt to define strings with dimension greater than one has been successful in such studies. As far as we can determine, the difficulty is not related to the tachyon in standard quantization, but simply the impossibility of making a subtraction.

### 6 Formally defined string evolution and the static potential

At the end of the last section, we showed that removing the factor (5.8) leads to an expression which does not come from the theory of Bosonic random surfaces. But let’s remove this factor anyway! The answer is an amplitude consistent with the usual results for the static potential, at least for $R > R_c$. If, as we expect, but do not prove here, the on-shell extrapolation of such amplitudes are Veneziano amplitudes, they are still worth investigating. With the unwanted factor removed from the integrand, the amplitude (5.7) becomes

$$\langle X_f^\perp(\sigma) \lvert e^{-TE} \rvert X_i^\perp(\sigma) \rangle = Z_0 T \int_0^\infty \frac{du}{u^2} \left( \frac{iu^2}{\pi\alpha'} \right)^{-\frac{d-2}{4}} e^{-\frac{u^2}{4\pi\sigma^2} - \frac{4u^2\alpha'^2}{4\pi\sigma^2} e^{-S_0[u,X^\perp_i,X^\perp_f]} } . \quad (6.1)$$

The quotation marks mean that this quantity is not really the transition amplitude for a theory of random surfaces, but instead the expression for the amplitude with (5.8) removed.

At this point, we shall show that the mode series (5.4) for $S_0$ may be evaluated. The quantity $S_0$ is the Wick-rotated classical action

$$S_0[u;X^\perp_f,X^\perp_i] = \frac{1}{4\pi\alpha'} \int_0^\pi ds \int_0^\pi d\sigma \left[ (\partial_\sigma X^\perp)^2 + (\partial_s X^\perp)^2 \right] . \quad (6.2)$$

of the field $X^\perp(\sigma,s)$ with “time” $s$ the time and initial and final conditions

$$X^\perp(\sigma,0) = X_i^\perp(\sigma), \quad X^\perp(\sigma,u^2/\alpha') = X_f^\perp(\sigma) ,$$
respectively and the further boundary conditions
\[ X^\perp(0, s) = X^\perp(\pi, s) = 0. \]

In Section 3 we found the solution (3.5) for the classical action with arbitrary Dirichlet boundary data. It follows from this result that
\[ S_0[u; X^\perp_f, X^\perp_i] = A[u, X^\perp_f, X^\perp_i] + A[u, X^\perp_f, X^\perp_i] - B[u, X^\perp_i, X^\perp_f], \tag{6.3} \]
where \( A[u, X^\perp, Y^\perp] \) and \( B[u, X^\perp, Y^\perp] \) are the quadratic forms
\[ A[u, X^\perp, Y^\perp] = \frac{\pi \alpha'}{16(\pi \alpha' + u^2)^2} \int_0^\pi d\sigma_1 \int_0^\pi d\sigma_2 \frac{[X^\perp(\sigma_1) - Y^\perp(\sigma_2)]^2}{\sin^2 \frac{\alpha'(\sigma_1 - \sigma_2)}{2(\pi \alpha' + u^2)}}, \tag{6.4} \]
\[ B[u, X^\perp, Y^\perp] = \frac{\pi \alpha'}{8(\pi \alpha' + u^2)^2} \int_0^\pi d\sigma_1 \int_0^\pi d\sigma_2 \frac{[X^\perp(\sigma_1) - Y^\perp(\sigma_2)]^2}{\sin^2 \frac{u^2 - \alpha'(\sigma_1 + \sigma_2)}{2(\pi \alpha' + u^2)}}. \tag{6.5} \]

It is simple to recover the static potential \( V(R) \) in (1.13) from (6.1) for \( R > R_c \). If we take \( T \) very large, then the integral is dominated by large \( u \). For very large \( u \), the quadratic forms (6.4), (6.5) may be neglected. The infinite product formula for the eta function (5.9) implies that
\[ \left( \langle X^\perp_f(\sigma) \mid e^{-TE} \mid X^\perp_i(\sigma) \rangle \right) \rightarrow Z_0 T \int_0^\infty \frac{du}{u^2} e^{(\frac{d-4}{12\alpha'})u^2} e^{-\frac{u^4}{4\pi^2} - \frac{R^2}{4\pi^2}} = \sqrt{\pi} Z_0 e^{-TV(R)}. \tag{6.6} \]
This implies that the bulk contribution to the free energy of the world sheet is \( TV(R) \), so that the energy of the string in its ground state is \( V(R) \). If \( R < R_c \) we do not find the usual imaginary result for \( V(R) \). In this case the energy operator has a negative eigenvalue. The integral over \( u \) simply fails to converge. This means that
\[ V(R) = \begin{cases} \frac{1}{2\pi \alpha'} \sqrt{R^2 - R_c^2}, & R > R_c \\ -\infty, & R < R_c \end{cases} \tag{6.7} \]
We can regularize the integral by cutting off the integration variable \( u \). As the cut-off is removed, a negative infinite value for \( V(R) \) is unavoidable. We are not claiming that references [7], [8] are incorrect. Our result is a consequence of defining amplitudes (5.1). Under circumstances when the square of the energy is not positive, the logarithm of this transform does not become imaginary but infinite.

The amplitude (6.1) is not rotation invariant. For simplicity, assume \( X^\perp \) on the boundary is zero. Then \( S_0 = 0 \). If we take a new integration variable \( \pi \alpha' / u \) and use (5.10), (6.4) becomes
\[ \left( \langle X^\perp_f(\sigma) \mid e^{-TE} \mid X^\perp_i(\sigma) \rangle \right) = Z_0 T \int_0^\infty \frac{du}{u^2} (\pi \alpha')^{\frac{d-6}{4}} u \frac{du^2}{2} \eta \left( \frac{iu^2}{\pi \alpha'} \right)^{-\frac{d-2}{4}} e^{-\frac{u^2}{4\pi^2} - \frac{R^2}{4\pi^2}}. \]
Rotating by $90^\circ$ would yield the same expression, except for the quantity $(\pi\alpha')^{\frac{4u}{3} + \frac{4u^2}{3}}$ appearing in the integrand. The lack of rotation invariance is not really so strange. We are calculating the amplitude for a string of a specified shape at one time to become a string with another specified shape at another time. This is analogous to the amplitude for a relativistic particle to travel from one space-time point to another space-time point; such an amplitude is not Lorentz invariant, as shown in the appendix. This particle amplitude and the (Lorentz-invariant) propagator have similar integral transforms of type (5.1), but with different powers of the integration variable in the integrand. The kernel also has a prefactor proportional to the time separation, but the propagator does not.

7 The off-shell disk amplitude

In the light of the above considerations, the only sensible form of the off-shell disk amplitude in 26 dimensions contains an extra factor of $u^{-5}$ and no overall factor of $T$:

$$
\Phi[T, R; X_0^+, X_1^+] = Z \int_{0}^{\infty} \frac{du}{u^7} \eta\left(\frac{iu^2}{\pi\alpha'}\right)^{-12} e^{-\frac{u^2}{4\pi\alpha'} - \frac{u^2}{4\pi\alpha'}} e^{-S_0[u, X_0^+, X_1^+]},
$$

(7.1)

where $Z$ is presumably not the same as $Z_0$ and $S_0$ is given as before by (5.3), (5.4), (5.5). It is easily checked that when $X^\perp$ vanishes on the boundary the right-hand side of (7.1) is invariant under rotations by $90^\circ$. Furthermore, the large $T$ behavior (6.6) is not affected.

The integral expression on the right-hand side of (7.1) suggests a guess for the amplitude with $X^\perp$ arbitrary on the boundary (but with $X^0$ and $X^1$ in the shape of a rectangle of dimensions $T$ and $R$). We use the classical action $S_0$ given in (6.2), but without specifying $X^\perp = 0$ at $\sigma = 0, \pi$. The boundary data on the rectangle of dimensions $T \times R$ must be mapped to boundary data on a rectangle of dimensions $u^2/\alpha' \times \pi$, whose perimeter is $2\pi + 2\frac{u^2}{\alpha'}$. We convert boundary data

$$
X_i^\perp(\sigma), \ X_{\text{left}}^\perp(X^0) = X_i^\perp(\sigma = \pi, X^0), \ X_f^\perp(\sigma), \ X_{\text{right}}^\perp(X^0) = X_i^\perp(\sigma = 0, X^0),
$$

into $X^\perp(s)$ with $s \in [0, 2\pi + 2\frac{u^2}{\alpha'}]$ by

$$
X^\perp(s) = \begin{cases} 
X_i^\perp(\sigma), & s = \sigma, \quad s \in [0, \pi], \\
X^\perp(s) = X_{\text{left}}^\perp(X^0), & s = \pi + \frac{u^2}{\alpha'T}, & s \in [\pi, \pi + \frac{u^2}{\alpha'}], \\
X^\perp(s) = X_f^\perp(\sigma), & s = 2\pi + \frac{u^2}{\alpha'} - \sigma, & s \in [\pi + \frac{u^2}{\alpha'}, 2\pi], \\
X^\perp(s) = X_{\text{right}}^\perp(X^0), & s = 2\pi + 2\frac{u^2}{\alpha'} - \frac{u^2}{\alpha'T}, & s \in [2\pi, 2\pi + 2\frac{u^2}{\alpha'}].
\end{cases}
$$

(7.2)
This boundary function $X^\perp(s)$ can then be used to determine $S_0$:

$$S_0[u, X^\perp] = \frac{\pi \alpha'}{16(\pi \alpha' + u^2)^2} \oint ds_1 \oint ds_2 \frac{[X^\perp(s_1) - X^\perp(s_2)]^2}{\sin^2 \frac{\alpha'(s_1 - s_2)}{2(\pi \alpha' + u^2)}},$$  \hspace{1cm} (7.3)

Since (7.1) is clearly not the same as (3.5). The static potential obtained by taking $T$ large as in (6.6) is (4.13). The classical result (3.5), however, is a simple area law and therefore leads to a purely linear potential $V(R) = R/(2\pi \alpha')$. The implication of the failure of the classical result is clear; the quantum fluctuations around the classical solution are important. This undoubtably means that the Liouville field does not decouple in twenty-six dimensions unless Neumann boundary conditions are taken.

8 Discussion

There are several issues raised in this paper which need to be studied further. The first concerns the question of what the term “off-shell Bosonic string amplitude” actually means. We have argued that such amplitudes are not mathematically equivalent to those of quantized strings with a Nambu or Polyakov action principle. The question is not simply a formal one; we would like to know how strings arise effectively in gauge theories. Somehow, these effective strings are free of the bulk contribution to the square of the energy operator. World-sheet supersymmetry may cure this contribution.

Another issue is the determination of off-shell amplitudes for more general boundary conditions. At present it is not clear how to obtain these directly. It may be possible to determine string amplitudes of quasi-static strings using our methods. These amplitudes differ from the static string amplitudes in that the length $R$, of the string is dependent on the time $X^0$. This may not solve the problem, since the resulting amplitudes will not be rotation invariant; but perhaps it is possible to guess the answer which generalizes (7.1).

We remark that it is possible to formally remove the divergence of (7.1) when one dimension $R$ or $T$ is made sufficiently small. This can be done by simply cutting off the integration on $u$:

$$\Phi[T, R; X_f^\perp, X_i^\perp] = Z \int_{\pi \alpha'}^{-\lambda} \frac{du}{u^2} \eta \left( \frac{iu^2}{\pi \alpha'} \right)^{-12} e^{-\frac{T^2}{4u^2} - \frac{R_2u^2}{4\pi^2 \alpha'}} e^{-S_0[u, X^\perp_i, X^\perp_f]},$$  \hspace{1cm} (8.1)

where $S_0$ is given by (7.2), (7.3). This expression is invariant upon interchange of $R$ and $T$. The static potential $V(R)$ is no longer negative-infinite below the critical distance $R < R_c = \pi \sqrt{\frac{(d-2)\alpha'}{3}}$, but is instead a finitely-deep potential well. An interesting question is whether the static potential obtained from (8.1) has conceptual or phenomenological usefulness for QCD.

In both (5.1) and (7.1), there is a phase transition at sufficiently small $T$, analogous to the Hagedorn transition with periodic time. This transition differs from the usual
Hagedorn transition in that it is not associated with windings of the string around a cylinder. The Hagedorn transition is associated with vortex condensation in the two-dimensional $XY$-model \[14\]. The transition of the world-sheet with rectangular boundary conditions is instead associated with that of the two-dimensional restricted gaussian model, with partition function

$$Z = \left[ \prod_j \int_{-\beta/2}^{\beta/2} dx_j \right] \exp \left\{ \frac{-1}{2} \sum_{j,k} (x_j - x_k)^2 \right\},$$

where $j$ and $k$ denote sites on a two-dimensional flat lattice and $< j, k >$ means that $j$ and $k$ are nearest neighbors. This lattice model behaves like a massless field theory for sufficiently large $\beta$. It also has a well-defined high-temperature expansion around $\beta = 0$ however, and therefore has a strongly-coupled phase with exponentially-decaying correlations for small $\beta$. The transition is driven by the condensation of loops on the lattice where $x = \pm \beta/2$, rather than by vortices.

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**Appendix: Integral transforms for particle evolution operators and propagators**

As an application of the formula (5.1), we will calculate the wave function of position $x$ and time $t$, $\Psi(x, t) = K(x, t; x_i, t_i)$ for a relativistic particle which has been restricted to be at initial position $x_i$ at initial time $t_i$ (we assume $t > t_i$). This wave function, the relativistic Schrödinger kernel, is closely related to the propagator, but they are not the same, and we discuss the connection between them. Though such matters are elementary, they are neither trivial nor discussed in textbooks (at least to our knowledge), and clarifying them will facilitate our discussion of the disk amplitude. The particle amplitude to start from some fixed configuration at one time and finish in a fixed configuration at a later time is not the two-point Green’s function.

Imagine a relativistic particle scatters isotropically from a small target of size $(\Delta x)_{\text{target}}$ at $x_i$ at time $t_i$. We measure the probability for the particle to be at $x_f$ at time $t_f > t_i$ with a small detector of size $(\Delta x)_{\text{detect}}$ located at $x_f$. This probability is

$$(\Delta x)_{\text{target}}^3 (\Delta x)_{\text{detect}}^3 |K(x_f, t_f; x_i, t_i)|^2.$$

The wave function is the kernel

$$K(x_f, t_f; x_i, t_i) = \left\langle x_f \left| \exp \left[ i(t_f - t_i + i\varepsilon)\sqrt{-\nabla^2 + m^2} \right] \right| x_i \right\rangle,$$  

(A.1)
where we have set the speed of light and Planck’s constant equal to one and \( m \) is the particle mass.

To evaluate (A.1) explicitly, we use (5.1) to obtain

\[
K(x_f, t_f; x_i, t_i) = \frac{1}{8\sqrt{\pi}} \int_0^\infty \frac{du}{u^5} e^{\frac{(t_f-t_i)^2 + i\varepsilon}{4u^2}} \left( e^{-u^2\sqrt{-\nabla^2} + m^2} x_i \right)
\]

\[
= \frac{1}{8\sqrt{\pi}} \int_0^\infty \frac{du}{u^5} \exp \left[ \frac{(t_f-t_i)^2 - (x_f-x_i)^2 + i\varepsilon}{4u^2} - m^2u^2 \right]. \tag{A.2}
\]

This integral may be evaluated for \(|(t_f - t_i)^2 - (x_f - x_i)^2| \gg m^2\), with the result

\[
K(x_f, t_f; x_i, t_i) \approx i \frac{1}{8\sqrt{\pi}} \int_0^\infty \frac{du}{u^5} \exp \left[ \frac{(t_f-t_i)^2 - (x_i-x_f)^2 + i\varepsilon}{4u^2} - m^2u^2 \right]
\]

\[
\times \int_0^\infty \frac{du}{u^5} \exp \left[ \frac{(t_f-t_i)^2 - (x_i-x_f)^2 + i\varepsilon}{4u^2} - m^2u^2 \right]
\]

\[
= \frac{i(t_f-t_i)}{8\pi[(t_f-t_i)^2 - (x_i-x_f)^2 + i\varepsilon]^{3/2}} \exp \frac{im\sqrt{(t_f-t_i)^2 - (x_i-x_f)^2 + i\varepsilon}}. \tag{A.3}
\]

If the proper-time interval is space-like, i.e. \( t_f - t_i < |x_f - x_i| \), the result (A.3) shows that the wave function \( K(x_f, t_f; x_i, t_i) \) is not zero, but exponentially decaying. This establishes that in relativistic quantum mechanics the amplitude for a particle to travel outside the light cone does not vanish. This fact is almost obvious from the uncertainty principle, but it is satisfying to understand its origins clearly. Feynman showed this result as well, not by an explicit calculation, but using a theorem from harmonic analysis \cite{13}. Since the interval between events is space-like, there exists a Lorentz transformation reversing the temporal order of the the scattering and detection of a particle, which implies the existence of antiparticles (and explains the PCT theorem). One can even add spin to describe electrons (rather than scalar particles). To eliminate superluminal communication, it is necessary to second quantize (for details see Feynman’s article).

As interesting as (A.3) may be, this wave function is not the same as the propagator. The reason the two are different is that the the function \( K \) is not invariant under a Lorentz transformation. This can be seen by inspecting our result (A.3). Yet a source for a scalar field \( j(x) \) is itself a scalar, i.e. is Lorentz invariant; therefore the propagator must be Lorentz invariant.

The retarded Green’s function is

\[
S_{\text{ret}}(x_f, t_f; x_i, t_i) = \left\langle x_f \left\vert \frac{\theta(t_f-t_i)}{2\sqrt{-\nabla^2 + m^2}} e^{i(t_f-t_i)\sqrt{-\nabla^2 + m^2}} \right\vert x_i \right\rangle,
\]
and the advanced Green’s function is
\[
S_{\text{adv}}(x_f, t_f; x_i, t_i) = \left\langle x_f \left| \frac{\theta(-t_f + t_i)}{2\sqrt{-\nabla^2 + m^2}} e^{-i(t_f - t_i)\sqrt{-\nabla^2 + m^2}} \right| x_i \right\rangle,
\]
where \( \theta(t) = 0 \) for \( t < 0 \) and \( \theta(t) = 1 \) for \( t \geq 0 \). The average of the retarded and advanced Green’s functions is the scalar-field propagator:
\[
S(x_f, t_f; x_i, t_i) = \frac{1}{2} S_{\text{ret}}(x_f, t_f; x_i, t_i) + \frac{1}{2} S_{\text{adv}}(x_f, t_f; x_i, t_i),
\]
or
\[
S(x_f, t_f; x_i, t_i) = \left\langle x_f \left| \frac{1}{4\sqrt{-\nabla^2 + m^2}} e^{i[t_f - t_i]\sqrt{-\nabla^2 + m^2}} \right| x_i \right\rangle.
\]

The integral transform for the propagator is not (A.2), but
\[
S(x_f, t_f; x_i, t_i) = -\frac{1}{16\sqrt{\pi}} \int_0^\infty \frac{du}{u^3} \exp \left[ \frac{(t_f - t_i)^2 - (x_f - x_i)^2 + i\varepsilon}{4u^2} - m^2 u^2 \right], \quad (A.4)
\]
which can be checked by differentiating (A.2) with respect to \( t_f \). The right-hand side of equation (A.4) is Lorentz invariant. Notice that the power of \( u \) in the integral in (A.4) is different from that in (A.2). It is argued in Section 6 that the integral expression for the one-closed-string vacuum expectation value is similar to that for the open-string evolution operator, but has a different power of \( u \) in the integrand.

All of the discussion of this appendix has been in Minkowski space. We could have just as easily done the analysis in Euclidean space, as we do for strings in the main text of this article.

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