On flows in networks with reachability restrictions

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Abstract. This paper we consider graphs with reachability restrictions of three types: barrier, valve and magnetic. As a part of the maximum flow problem in networks with considered restrictions, simple examples are given. These examples demonstrate a significant difference between flows in networks with reachability restrictions and flows in the same networks without any reachability restrictions. In each case under consideration there is a decrease of the maximum flow value. Wherein for magnetic reachability it is shown the possibility to control the network flow, whereas for barrier and valve reachability it is shown the reducing of the maximum flow due to the effect of superimposing of flow to itself, which is never observed in the classical theory of flows in networks.

Keywords: flows in networks, networks with related arcs, networks with non-standard reachability, algorithms on graphs, flow control.

1. Introduction
Graphs with reachability restrictions, which have been introduced into consideration relatively recently (see [1–5]), turned out to be quite interesting objects from the point of view of mathematical modeling. They allow one to take into account various features of real processes (see [6–10]). In particular, reachability restrictions model well communication networks in which there are channels that provide different quality of information transmission (channels with strong attenuation (see, for example, [9,10]).

The main approach for solving problems of shortest paths, random walks and flows in networks on such graphs turned out to be the method of expanders (see [1, 3, 5]). This method consists in constructing an auxiliary graph, which is called an expander of original graph, and transferring the problem under consideration to it. The way of the expander construction depends on the type of reachability restriction, and such construction of the expander itself allows you to ”get away” from the reachability restrictions. It is important that there are no path on the expander that correspond to the invalid by given constraint path on the original graph, and that any valid path on the original graph corresponds to at least one path on the expander.

When constructing an expander, each vertex of the original graph is associated with itself and a set of its twins. Each arc is associated with one or several arcs on the expander. The main problem is to transfer weights of the arcs of the original graph to the corresponding arcs of the expander. For the problems of shortest path and random walks this transfer is trivial, but in the case of flows in networks the problem of transferring is much difficult. Indeed, in the case when one arc of the expander corresponds to one arc of the original graph, their capacities must be equal. However, in the case when one arc of the original graph corresponds to several arcs
of the expander, such a transfer can be impossible, since some flows in the expander cannot be interpreted as a flow on the original graph (examples in following sections will provide additional clarity).

The aforesaid led us to the conclusion that expander in the case of flow problems should be understood as graphs with related arcs (see [1]), that is, on such graphs there are subsets of arcs for which their capacities are not specified. For the arcs of each of these subsets, only total capacity of subset is given. Arcs of such subsets is called related arcs. The presence of related arcs makes the known algorithms for finding the maximum flow in the networks inapplicable.

We have proposed a heuristic breakthrough algorithm which is based on sequentially finding paths on the expander from the source to the sink with their saturation by the flow. This algorithm made it possible to take into account the presence of related arcs. In the case the path contains one of related arcs, it is saturated with the flow and the total throughput of the arcs, which are related to it, decreases by a corresponding value. In the case such a path passes through several arcs of expander, which correspond to the same arc of the original graph, when it is saturated with a flow, it is necessary to take into account the multiplicity of the path passing along such arcs. This case arises, for example, in a network with a barrier reachability.

The main aim of this paper is to construct simple examples that demonstrate the difference between flows in networks with reachability restrictions from flows in the same networks without any reachability restrictions. The most interesting examples, by the authors’ opinion, are described in sections 2 and 3. The decrasing of the maximum flow value in each of these cases occurs due to the effect of ”superimposing” the flow on itself, which is never observed in the classical theory of flows in networks. The examples are so simple that we managed to describe finding the maximum flows without building of corresponding expanders.

2. Networks with barrier reachability

Let us consider graphs with barrier reachability (see [1, 4]). For any such graph it is assumed that there are three subsets of arcs, which are called neutral, increasing and barrier, and it is given a natural number $h$ which is the height of the barrier arcs. What paths are valid on a graph with barrier reachability? At the beginning of the passage along the path, it is assumed that

1. the accumulated energy is equal to zero;
2. the passage along the neutral arc is always allowed and does not change the energy;
3. the passage along the increasing arc is always possible and it decreases the accumulated energy by one;
4. the passage along the barrier arc is possible only if the previously accumulated energy is not less than $h$.

Consider one interesting example in our opinion. The graph $G_1$ in figure 1 is considered as a network with barrier reachability (see [1, 4]). The ”+” sign in the figure 1 marks the increasing arc. The arc which is marked as ”$h = 3$” is a barrier arc of height 3. The rest of the arcs are neutral. The capacity of each arc is 4.

![Network G1 with barrier reachability.](image-url)
It is clear that without any restrictions on reachability, the value of the maximum flow is 4 and its implementation is shown in figure 2. The value of the flow along each arc of network $G_1$ is written out near the arc.

![Figure 2. Maximum flow in network $G_1$ without any reachability restriction.](image)

In the case of barrier reachability, this flow is not valid, since it is implemented by an invalid path in this case.

The maximum flow in network $G_1$ with barrier reachability is shown in figure 3. Its value is equal to 1.

![Figure 3. Maximum flow in network $G_1$ with barrier reachability.](image)

How can we interpret the flow in the figure 3? A unit flow with zero energy passes along the arc leaving the source, and a unit flow with an energy of 3 passes along the barrier arc. The flow of value 4, which passes along the bottom arc of the contour, is a layering of four unit flows with energies which are equal to 0, 1, 2 and 3 respectively. The flows of value 3, which passes along the right and top arcs of the contour, are a layering of three unit flows with energies which are equal to 0, 1 and 2 respectively. The flow of value 3, which passes along the left arc of the contour, is a layering of three unit flows with energies which are equal to 1, 2 and 3 respectively.

In fact, on the expander there is one arc that corresponds to arc leaving the source, one arc that corresponds to the barrier arc of network $G_1$. The bottom arc of the contour of original graph $G_1$ corresponds to four arcs on the expander. Each of the rest of arcs of the contour correspond to three arcs on the expander. A threefold passage of the flow along the contour on the original graph $G_1$ corresponds to a spiral with three turns on the expander. And the entire network $G_1$ corresponds to one simple path on the expander, which leads from the source to the sink. On this path the bottom arc of the contour is represented by four related arcs of the total capacity 4. The rest of arcs of the contour is represented by three related arcs for each them. Therefore, when applying the breakthrough algorithm (see [1]) the capacity of the bottom arc is distributed equally among its four duplicates on the expander, and the capacity of each of the rest of arcs of the contour is distributed equally among its three duplicates on the expander. Thus, any duplicate of the bottom arc of the contour forms the minimum cut of the expander and the capacity of any of them is equal to 1.

**Remark 1.** It is clear that with a higher barrier height, the value of the maximum flow will be even less.
3. Networks with valve reachability

A similar situation is observed for graphs with valve reachability (see [1, 3]). Consider a network $G_2$ with a valve reachability of order 3 in figure 4. In this case we assume the capacity of each arc to 3. Arcs of the sets $U_1$, $U_2$ and $U_3$ are marked with corresponding numbers. The rest of the arcs belong the set $U_0$.

![Image of Network $G_2$ with valve reachability.](image)

**Figure 4.** Network $G_2$ with valve reachability.

For any graph with valve reachability it is assumed that, for forming a path, any arc of the sets $U_0$ can be used without restrictions, and for the rest of the sets the following rule applies: for any natural $i$, to pass along any arc of the set $U_i$ it is necessary to pass along at least one arc of the set $U_{i-1}$. Therefore, in order to pass along the arc to sink on considered graph $G_2$ in figure 4, any valid path have to contain two bottom arcs of the network. And to pass along bottom arcs, any valid path have to contain top arcs. The maximum flow in network $G_2$ is shown in figure 5. Its value is equal to 1.

![Image of Maximum flow in network $G_2$ with valve reachability.](image)

**Figure 5.** Maximum flow in network $G_2$ with valve reachability.

How can we generally interpret the flow in figure 5? At first an unit flow passes along the arc leaving the source, then it passes along the middle arc of the contour (diagonal arc). After this it has to pass along the upper right arc of the contour and along the upper left arc of the contour. That moment considered unit flow turns into an unit flow that has passed along the valve arc of level 1. Further, this unit flow is layered on the already existing unit flow on the middle arc of the countour, and then it passes along the bottom right arc of the countour and along the bottom left arc of the countour. That moment considered unit flow turns into an unit flow that has passed along the valve arc of the 2 level. Then this flow is layered on flows of the middle arc again and passes to sink. A unit flow passes along the arc leading to the sink have to pass first along the valve arc of level 1 (the top left arc of the contour) and then along the valve arc of level 2 (the bottom left arc of the contour). Thus, the flow of value 3, which passes along the middle arc of the contour, is a layering of three unit flows with levels that equal to 0, 1 and 2 respectively.

In fact, as in the previous example for network $G_1$ with barrier reachability, the decreasing of the maximum flow occurs due to the superimposing of the flow on related arcs. Next section,
an example of a network with reachability constraints, when the flow decreases not due to the superimposing of the flow on itself, is given.

4. Networks with mixed reachability

Let’s consider another interesting example. Previous cases of barrier and gate reachability the flow decreased occurred due to the superimpose effect. For the following type of reachability conditions an example, when the maximum flow is significantly affected by the capacities of arcs, for which the maximum flow passing along them is equal to zero in the same network without any restrictions, is constructed.

For any graph with mixed reachability (see [3]) it is assumed that its set of arcs is divided into two sets $U_f$ and $U_a$, which are called "forbidden" and "allowed" arc respectively. Any valid path on such graph does not contain the passages in sequence along two "forbidden" arcs. In other words, for any pair of consecutive arcs of a valid path, at least one of these arcs has to be an "allowed" arc.

Consider network $G_3$ with mixed reachability in figure 6. Here capacities are written next to the arcs. Also the "forbidden" arcs are marked in bold.

![Figure 6. Network $G_3$ with mixed reachability.](image)

It is clear that without any reachability restriction, the value of the maximum flow is 4 and it can be implemented as a flow which is shown in figure 7. The value of the flow along each arc of network is written out near the arc.

![Figure 7. Maximum flow in network $G_3$ without any reachability.](image)

However, in the case of a graph with mixed reachability, this flow is not valid, since it is implemented in an invalid path.

Maximum flow in original network $G_3$ with mixed reachability is shown in figure 8. Its value is equal to 1.

How can we generally interpret the flow in the figure 8? Some flow passes along the arc leaving the source, and since it passes along the forbidden arc, the direct passing to the sink becomes impossible. However, passing along the contour arcs is still possible. Thus, the flow passes along the contour and, since its last arc is an allowed arc, it passes to the sink along the forbidden arc. Since the capacity of this path is equal to 1, therefore, the value of the maximum
flow is also equal to 1. In fact, the decreasing of the maximum flow occurs due to the passage of additional contours with a smaller capacity than on simple paths.

5. Networks with magnetic reachability
Let’s consider an example showing that the flow can be controlled without changing the classical structure of the graph, but only by assigning magnetic reachability restrictions.

For any graph with magnetic reachability of order 1 (see [1,3]) it is assumed that its set of arcs is divided into two sets $U_m$ and $U_0$, which are called “magnetic” and “neutral” arc respectively, and if any initial segment of an arbitrary path contains a magnetic arc and there is a possibility to continue the path along the magnetic arc, then for considered path to be valid the next its arc has to be magnetic. In other words, if the path has passed a magnetic arc, then each of its next arc must be magnetic, if there is such a possibility.

Consider network $G_4$ with magnetic reachability of order 1 (see [1,3]) in figure 9. Here capacities are written next to the arcs. Also the magnetic arc is marked in bold and three arcs $u_1$, $u_2$ and $u_3$ are highlighted.

Note that the value of the maximum flow, both without reachability restrictions and with a magnetic reachability, in this case is equal to 6. However, if in addition the arc $u_1$ becomes magnetic, then the value of the maximum flow in such a network $G_4$ will be equal to 1. This case is shown in figure 10.

Indeed, any path that contains either the arc $u_2$ or the arc $u_3$ does not satisfy the magnetic reachability restriction, since the initial segment of such a path from the source to the sink contains a magnetic arc, therefore it can be continued only along the arc $u_1$.

If the arc $u_3$ becomes magnetic instead of the arc $u_1$, the value of the maximum flow in network $G_4$ is equal to 3. This case is shown in figure 11.

We also note that for the considered network $G_4$ with magnetic reachability, by assigning the magnetic property for arcs of the set $\{u_1, u_2, u_3\}$, it is possible to achieve that the value of the
maximum flow is equal to 1, 2, 3, 4, 5 or 6. In this case, the structure of the original network $G_4$ is such that it is impossible to obtain the maximum flow of another value.

There exist networks for which by assigning the magnetic properties of some arcs it is possible to obtain a cases for which the only valid flow in the network is a flow of zero value, regardless of the capacities of the network arcs.

Consider network $G_5$ with magnetic reachability of order 1 in figure 12. Here magnetic arcs are marked in bold. For definiteness, we assume that the capacities of all arcs of such a network are equal to one.

For the network $G_5$, there is no a valid path from the source to the sink, since the arc leaving the source is magnetic, and two arcs, one of which is magnetic, emerge from the vertex before the sink. Indeed, in the further formation of any path from the source, priority will be given to the magnetic arc (the right arc of the contour).

Remark 2. In fact, in this case the graph $G_5$ in figure 12 is not a network with a source at the left vertex and a drain at the right at all.
Note that there are networks in which any assuming of magnetic reachability restrictions does not affect the maximum flow. We call such networks magnetically uncontrollable. An example of such a network is in figure 13.

![Figure 13. Magnetically uncontrolled network.](image)

At the same time, for any magnetically controlled network without disturbing its structure (in the classical sense), only by assigning the magnetic property to arcs, it is possible to control the flow, changing not only the value of the maximum flow, but also the paths of its passage from the source to the drain.

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