SPECTRA OF QUANTUM KDV HAMILTONIANS, LANGLANDS DUALITY, AND AFFINE OPERS

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Abstract. We prove a system of relations in the Grothendieck ring of the category \( \mathcal{O} \) of representations of the Borel subalgebra of an untwisted quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) introduced in [HJ]. This system was discovered, under the name \( \hat{Q}\hat{Q} \)-system, in [MRV1, MRV2], where it was shown that solutions of this system can be attached to certain \( L\hat{\mathfrak{g}} \)-affine opers, introduced in [FF5], where \( L\hat{\mathfrak{g}} \) is the Langlands dual affine Kac–Moody algebra of \( \hat{\mathfrak{g}} \). Together with the results of [BLZ3, BHK] which enable one to associate quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians to representations from the category \( \mathcal{O} \), this provides strong evidence for the conjecture of [FF5] linking the spectra of quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians and \( L\hat{\mathfrak{g}} \)-affine opers. As a bonus, we obtain a direct and uniform proof of the Bethe Ansatz equations for a large class of quantum integrable models associated to arbitrary untwisted quantum affine algebras, under a mild genericity condition. We also conjecture analogues of these results for the twisted quantum affine algebras and elucidate the notion of opers for twisted affine algebras, making a connection to twisted opers introduced in [FF6].

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1. Introduction

The purpose of this paper is two-fold. First, it is to elucidate the link, proposed in [FF5], between (i) the spectra of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians acting on highest weight representations of \( \mathcal{W} \)-algebras, and (ii) affine opers – differential operators in one variable associated to \( L\hat{\mathfrak{g}} \), the affine Kac–Moody algebra that is Langlands dual to \( \hat{\mathfrak{g}} \). And second, it is to prove a system of relations in the Grothendieck ring \( K_0(\mathcal{O}) \) of the category \( \mathcal{O} \) of representations of the Borel subalgebra of the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \), introduced in [HJ]. These relations generalize the “quantum Wronskian relation” in the case of \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_2 \).
described in [BLZ3]. As far as we know, for general affine algebras \( \hat{\mathfrak{g}} \) these relations, viewed as relations in \( K_0(O) \), are new. Among other things, these relations enable one to quickly derive the Bethe Ansatz equations in a uniform fashion and under minimal assumptions.

Remarkably, these relations were discovered on the other side of the above KdV-oper correspondence – namely, in the domain of affine opers – in the striking recent papers [MRV1, MRV2] by Masoero, Raimondo, and Valeri, which provided a catalyst for the present work. Following [MRV1, MRV2], we call these relations the \( \tilde{Q}\tilde{Q} \)-system. According to [MRV1, MRV2], solutions of the \( \tilde{Q}\tilde{Q} \)-system can be attached to affine \( \hat{\mathfrak{g}} \)-opers of special kind – precisely the simplest \( \hat{\mathfrak{g}} \)-affine opers proposed in [FF5] to describe the spectra of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians. Now let’s look on the KdV side: following the construction of Bazhanov, e.a. [BLZ2, BLZ3, BHK], one can attach non-local quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians to elements of \( K_0(O) \). Therefore, our results (in the present paper) imply that the spectra of these Hamiltonians should also yield solutions of this \( \tilde{Q}\tilde{Q} \)-system. Thus, we obtain strong evidence for the conjecture of [FF5] linking the spectra of these Hamiltonians to \( \hat{\mathfrak{g}} \)-opers:

\[
\begin{array}{c}
\text{spectra of quantum} \\
\hat{\mathfrak{g}} \text{-KdV Hamiltonians} \\
\text{solutions of} \\
\text{the } \tilde{Q}\tilde{Q} \text{-system} \\
\hat{\mathfrak{g}} \text{-affine opers}
\end{array}
\]

We now explain the interrelations between these topics in more detail.

In the case of \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_2 \) the link between the spectra of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians and ordinary differential operators (the precursors of affine opers of [FF5]) was discovered and investigated in [DT1, BLZ4, BLZ5]. It was further generalized and studied in the case of \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_3 \) in [BHK], and \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_r \) in [DDT1, DMST].

Motivated by those works, Feigin and one of the authors of the present paper interpreted in [FF5] the quantum \( \hat{\mathfrak{g}} \)-KdV integrable system as a generalization of the Gaudin model, in which a simple Lie algebra \( \mathfrak{g} \) is replaced by the affine Kac–Moody algebra \( \hat{\mathfrak{g}} \). It is known [FFR, F1, FFT] that the spectra of the Hamiltonians of the Gaudin model associated to \( \mathfrak{g} \) can be encoded by differential operators known as \( \mathcal{L} \)-opers, where \( \mathcal{L} \mathfrak{g} \) is the Langlands dual Lie algebra of \( \mathfrak{g} \). (This follows from an isomorphism between the center of the completed enveloping algebra of \( \hat{\mathfrak{g}} \) at the critical level and the algebra of functions on \( \mathcal{L} \hat{\mathfrak{g}} \)-opers on the formal disc [FF3, F2].) Therefore, by analogy with the simple Lie algebra case, it was conjectured in [FF5] that the spectra of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians should be encoded by what was called in [FF5] the affine opers associated to the Langlands dual affine algebra \( \mathcal{L} \hat{\mathfrak{g}} \), or \( \hat{\mathfrak{g}} \)-affine opers (\( \hat{\mathfrak{g}} \)-opers for short). Moreover, it was shown in [FF5] that this

\footnote{This \( \tilde{Q}\tilde{Q} \)-system should not be confused with the \( Q \)-system satisfied by (ordinary) characters of Kirillov–Reshetikhin modules, see [H1] and references therein.}
proposal is consistent with the results obtained in the papers cited above for \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_r = L\hat{\mathfrak{g}}_r \).

In the present paper we give more details on the structure of the relevant \( L\hat{\mathfrak{g}}_\text{-opers} \) in the case that \( L\hat{\mathfrak{g}} \) is a twisted affine algebra, making a connection to the twisted opers introduced in [FG] (Section 8.6).

Already in the pioneering works [DT1, BLZ4, DT2, DDT1, BHK] various systems of functional equations were constructed for \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_r \) with the property that its solution could be attached to an \( r \)th order differential operator of a special kind. From the perspective of [FF5], those differential operators are the same as \( \hat{\mathfrak{sl}}_r \)-affine opers corresponding to the highest weight vectors for \( r > 2 \), and to all eigenvectors for \( r = 2 \). On the other hand, the eigenvalues of certain non-local quantum \( \hat{\mathfrak{sl}}_r \)-KdV Hamiltonians were shown to satisfy the same relations; namely, for \( \hat{\mathfrak{sl}}_2 \), in [BLZ4, BLZ5]; for \( \hat{\mathfrak{sl}}_3 \), in [BHK]; and for \( \hat{\mathfrak{sl}}_r \), in [Ko]. Therefore, in the case of \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_r \) these relations provided a link between the spectra of quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians and \( L\hat{\mathfrak{g}}_\text{-affine opers} \) (because \( L\hat{\mathfrak{sl}}_r = \hat{\mathfrak{sl}}_r \)).

Unfortunately, these systems were either analogues of the Baxter’s TQ-relation – and so they involved the classes of finite-dimensional representations of \( U_q(\hat{\mathfrak{g}}) \), with the number of terms in the relation growing as the dimensions of those representations – or Wronskian-type relations with the number of terms growing as \( r! \). Analogues of such systems were unknown for a general affine algebra \( \hat{\mathfrak{g}} \), and this impeded further progress in understanding the link between the spectra of quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians and \( L\hat{\mathfrak{g}}_\text{-affine opers} \), beyond the case of \( \hat{\mathfrak{sl}}_r \).

That’s why an elegant and uniform \( Q\tilde{Q} \)-system proposed for an arbitrary untwisted affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) in the papers [MRV1, MRV2] is such an important development. However, the \( Q\tilde{Q} \)-system was constructed in [MRV1, MRV2] only on the affine oper side of the KdV-oper correspondence. In fact, it was shown in [MRV1, MRV2] that solutions of the \( Q\tilde{Q} \)-system can be attached to the simplest \( L\hat{\mathfrak{g}}_\text{-affine opers} \) of the kind proposed in [FF5] (those are in fact the \( L\hat{\mathfrak{g}}_\text{-affine opers} \) that are supposed to encode the eigenvalues of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians on a highest weight vector of a representation of the \( \mathcal{W} \)-algebra). We note that some partial results in this direction were obtained earlier in [S].

If the solutions of this \( Q\tilde{Q} \)-system could also be constructed using the methods of [MRV1, MRV2] for more general \( L\hat{\mathfrak{g}}_\text{-affine opers} \) from [FF5] and the present paper (which are supposed to encode other eigenvalues of the \( \hat{\mathfrak{g}} \)-KdV Hamiltonians on representations of \( \mathcal{W} \)-algebras), this would open the possibility of using this \( Q\tilde{Q} \)-system to establish the link between the spectra of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians and \( L\hat{\mathfrak{g}}_\text{-affine opers} \) proposed in [FF5] (analogously to the link established in [BLZ5] in the case of \( \hat{\mathfrak{sl}}_2 \)).

However, in order to do that, we first need to understand the meaning of the \( Q\tilde{Q} \)-system on the side of the quantum KdV Hamiltonians. In other words, we need to answer the following question: Can a solution of the \( Q\tilde{Q} \)-system from [MRV1, MRV2] be attached to each joint eigenvector of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians?

In this paper we show that in fact the \( Q\tilde{Q} \)-system of [MRV1, MRV2] is a universal system of relations in the (commutative) Grothendieck ring \( K_0(\mathcal{O}) \) of the category \( \mathcal{O} \) of representations of the Borel subalgebra of the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) introduced by Jimbo and one of the authors in [HI]. Since one can attach non-local quantum \( \hat{\mathfrak{g}} \)-KdV
Hamiltonians to elements of $K_0(\mathcal{O})$ using the construction of $[\text{BLZ}_2, \text{BLZ}_3, \text{BHK}]$ (modulo some convergence issues discussed in Section 7.2 below), this gives us a way to attach a solution of the $\tilde{Q}\tilde{Q}$-system to each joint eigenvector of the quantum $\hat{g}$-KdV Hamiltonians in a representation of the corresponding $\mathcal{W}$-algebra.

An interesting aspect of the $\tilde{Q}\tilde{Q}$-system is that it involves two sets of variables, denoted in $[\text{MRV}_1, \text{MRV}_2]$ by $Q_i$ and $\tilde{Q}_i$, where $i$ runs over the set of simple roots of $g$. The challenge is then to find the corresponding two sets of representations of $U_q(\hat{g})$ from the category $\mathcal{O}$ whose classes satisfy the $\tilde{Q}\tilde{Q}$-system.

We find these representations in the present paper. Namely, the first set of representations, corresponding to the $Q_i$, are the representations denoted by $L_{i,a}^+$ and called prefunamental in $[\text{FH}]$. They were first constructed for $\hat{g} = \hat{\text{sl}}_2$ in $[\text{BLZ}_2, \text{BLZ}_3]$, for $\hat{g} = \hat{\text{sl}}_3$ in $[\text{BHK}]$, and for $\hat{g} = \hat{\text{sl}}_{n+1}$ with $i = 1$ in $[\text{Ko}]$. For general $\hat{g}$, these representations were introduced and studied in $[\text{FH}]$, and they were further investigated in $[\text{FH}]$.

The representations of the second set, corresponding to the $\tilde{Q}_i$, have not been previously studied for general affine algebras, as far as we know. We denote these representations by $X_{i,a}$ in Section 3. If $L_{i,a}^+$ in some sense corresponds to the $i$th fundamental weight $\omega_i$ of $g$, then $X_{i,a}$ corresponds to the weight $\omega_i - \alpha_i$. We prove that together, $Q_i = [L_{i,a}^+]$ and $\tilde{Q}_i = [X_{i,a}]$ satisfy the $\tilde{Q}\tilde{Q}$-system of $[\text{MRV}_1, \text{MRV}_2]$, up to some scalar multiples (which are inessential normalization constants from the point of view of the eigenvalues of the quantum KdV Hamiltonians).

Thus, we prove that the $\tilde{Q}\tilde{Q}$-system of $[\text{MRV}_1, \text{MRV}_2]$ appears naturally in representation theory of the Borel subalgebra of the quantum affine algebra $U_q(\hat{g})$ for an arbitrary untwisted affine Kac–Moody algebra $\hat{g}$. Furthermore, we conjecture an analogous $\tilde{Q}\tilde{Q}$-system in the Grothendieck ring $K_0(\mathcal{O})$ for an arbitrary twisted affine Kac–Moody algebra $\hat{g}$ (Conjecture 3.3), as well as an analogue of the result of $[\text{MRV}_1, \text{MRV}_2]$ that solutions of this system can be attached to $L_\mathcal{O}$-opers (Conjecture 8.1).

It follows from our results that the $\tilde{Q}\tilde{Q}$-system arises whenever there is an action of the Grothendieck ring $K_0(\mathcal{O})$ on a vector space, as the relation between the joint eigenvalues of the commuting operators corresponding to the classes of the representations $L_{i,a}^+$ and $X_{i,a}$ in $K_0(\mathcal{O})$. For instance, let $V$ be the tensor product a finite number of irreducible finite-dimensional representation of $U_q(\hat{g})$. The well-known transfer-matrix construction yields an action of $K_0(\mathcal{O})$ on $V$ (see $[\text{FH}]$). Therefore we obtain that the joint eigenvalues of the transfer-matrices corresponding to $L_{i,a}^+$ and $X_{i,a}$ satisfy the $\tilde{Q}\tilde{Q}$-system. There is also a similar relation in the dual category $\mathcal{O}^*$, in which the role of $L_{i,a}^+$ is played by its dual representation denoted by $R_{i,a}^+$ in $[\text{FH}]$. Furthermore, in $[\text{FH}]$ it was proved that every eigenvalue of the transfer-matrix of $R_{i,a}^+$ on such a $V$ is equal, up to a common scalar factor that depends only on $V$ and $i$, to a polynomial in the spectral parameter (this was originally conjectured in $[\text{FR}]$). These polynomials are the generalizations of the celebrated Baxter’s polynomials (see $[\text{FH}]$ for more details and references).

As an immediate consequence of this $\tilde{Q}\tilde{Q}$-system, one can derive a system of equations on the roots of these polynomials, which are nothing but the Bethe Ansatz equations for
the quantum integrable systems associated to $U_q(\hat{\mathfrak{g}})$. Previously, for a general $\hat{\mathfrak{g}}$ these equations were essentially guessed from the relations between these eigenvalues and the eigenvalues of the transfer-matrices corresponding to finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$, as explained in [FR, FH]; these relations are generalizations of Baxter’s $TQ$-relation (see, e.g., [FH]). This argument, which goes back to Reshetikhin’s analytic Bethe Ansatz method [R1, R2, R3] (see also [BR, KS]), gives strong evidence for the Bethe Ansatz equations, but short of a proof. On the other hand, the $Q\tilde{Q}$-system gives us a direct proof (albeit under a genericity assumption) of the Bethe Ansatz equations for the roots of the generalized Baxter polynomials arising from every joint eigenvector of the transfer-matrices on $V$. This is explained in Section 5 below, following [MRV1, MRV2].

The non-local quantum $\hat{\mathfrak{g}}$-KdV Hamiltonians give us another example of an action of the Grothendieck ring $K_0(O)$ via the construction of [BLZ2, BLZ3, BHK] – in this case, on representations of the corresponding $W$-algebra (modulo the convergence issues discussed in Section 7.2). Thus, for each common eigenvector of the quantum $\hat{\mathfrak{g}}$-KdV Hamiltonians we also obtain a solution of the above $Q\tilde{Q}$-system. In this case, the $Q_i$ and $\tilde{Q}_i$, viewed as functions of the spectral parameter, are expected to be entire functions on the complex plane with a particular asymptotic behavior at infinity [BLZ2, BLZ3, BLZ4, BLZ5, BHK]. The corresponding Bethe Ansatz equations are then the equations on the positions of the zeros of the functions $Q_i$. In the case of $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_r$ these equations are equivalent to the ones that have been previously considered in the literature.

In a similar way, solutions of the $Q\tilde{Q}$-system can be attached to joint eigenvalues of the non-local quantum Hamiltonians of the “shift of argument” affine Gaudin model introduced in [FF4, Section 3]. The zeros of the corresponding functions $Q_i$ satisfy the same Bethe Ansatz equations, but these functions have analytic properties different from the analytic properties of the functions $Q_i$ corresponding to the joint eigenvalues of the non-local quantum $\hat{\mathfrak{g}}$-KdV Hamiltonians.

We also want to note that Baxter’s $Q$-operators $Q_i$ arise in other important quantum integrable models, such as the ones studied in [NPS] that appear in the $\Omega$-deformations of the five-dimensional supersymmetric quiver gauge theories. It is natural to expect that the construction of these $Q$-operators can be extended to the entire Grothendieck ring $K_0(O)$. Then the operators $\tilde{Q}_i$ can be constructed in those models as well, so that together with the Baxter’s operators $Q_i$ they satisfy the $Q\tilde{Q}$-system. The corresponding Bethe Ansatz equations may then be derived from the $Q\tilde{Q}$-system.

As we mentioned above, according to [MRV1, MRV2], a solution of the $Q\tilde{Q}$-system can be obtained from special $L\hat{\mathfrak{g}}$-affine opers; namely, the ones that correspond to highest weight vectors of the representations of the $W$-algebra (they are automatically eigenvectors of the quantum $\hat{\mathfrak{g}}$-KdV Hamiltonians because the Virasoro operator $L_0$ is one of these Hamiltonians). Thus, the $Q\tilde{Q}$-system links a $L\hat{\mathfrak{g}}$-affine oper of this kind and the joint eigenvalues of the $\hat{\mathfrak{g}}$-KdV Hamiltonians on the highest weight vector. (This generalizes the earlier results for $\mathfrak{sl}_r$ [DT1, BLZ4, DT2, DDT1, BHK, BLZ3].)

Davide Masoero and Andrea Raimondo informed us that they expect that their construction can be generalized to the more general $L\hat{\mathfrak{g}}$-affine opers from [FF5] (see also Section 8).
below) that were conjectured to correspond to other eigenvectors of the quantum $\hat{g}$-KdV Hamiltonians. If this is indeed the case, then the $QQ$-system will provide a link between the $L\hat{g}$-affine opers and eigenvalues of the quantum $\hat{g}$-KdV Hamiltonians (similarly to the case of $\hat{\mathfrak{sl}}_2$ in [BLZ3]).

Finally, we note that an analogue of the $QQ$-system also exists for the quantum $\mathfrak{gl}_1$ toroidal algebra. We present it in Section 3.5. We also note that for quantum affine algebras another system of relations in $K_0(\mathcal{O})$ was established in [HL2]. It arises naturally from a cluster algebra structure introduced in [HL2], as the first step of the Fomin–Zelevinsky mutation relations. We call it the $QQ^*$-system and discuss it in Section 3.4. This system gives rise to the same Bethe Ansatz equations as the $QQ$-system (see Section 5). Furthermore, a system analogous to the $QQ^*$-system has been recently defined in [FJMM] for the quantum $\mathfrak{gl}_1$ toroidal algebra. The corresponding Bethe Ansatz equations were also proved in [FJMM] (unconditionally). We show in Section 5 that the same Bethe Ansatz equations also follow from the $QQ$-system of Section 3.5 (under a genericity assumption).

The paper is organized as follows. In Section 2, we present the necessary definitions and results concerning quantum affine algebras and the category $\mathcal{O}$. In Section 3 we introduce the irreducible representations $X_{i,a}$ and state the $QQ$-system (Theorem 3.2). We describe the examples of the $QQ$-system for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{sl}_3$, connecting the $QQ$-system in these cases to relations found in earlier works [BLZ3, BHK]. We also conjecture an analogue of the $QQ$-system for twisted affine algebras (Section 3.3), state the $QQ^*$-system for quantum affine algebras (Section 3.4) and an analogue of the $QQ$-system for the quantum $\mathfrak{gl}_1$ toroidal algebra (Section 3.5). In Section 4 we prove the $QQ$-system for untwisted affine algebras using the theory of $q$-characters. We then derive the Bethe Ansatz equations from the $QQ$-system in Section 5 and discuss applications of the Bethe Ansatz equations in various situations.

After that, we shift our focus to the KdV system. In Section 6 we recall the definition of the classical KdV system and the corresponding spaces of opers for both twisted and untwisted affine algebras. Then we discuss the quantization of the KdV Hamiltonians in Section 7. We explain the construction of [BLZ1, BLZ2, BLZ3, BHK] assigning non-local quantum KdV Hamiltonians to elements of $K_0(\mathcal{O})$. In Section 7.4 we state Conjecture 7.2 that the quantum $\hat{g}$- and $L\hat{g}$-KdV Hamiltonians commute with each other. If true, this should yield a somewhat surprising correspondence between solutions of the $QQ$-systems (as well as other equations stemming from $K_0(\mathcal{O})$ such as the $QQ^*$-system) for $U_q(\hat{\mathfrak{g}})$ and $U_\tilde{q}(L\hat{\mathfrak{g}})$, where $q = e^{\pi i/\beta^2}$ and $\tilde{q} = e^{\pi i \gamma/\beta^2}$.

In Section 8 we discuss and give more details on the conjecture of [FF5] linking the spectra of quantum $\hat{g}$-KdV Hamiltonians and $L\hat{g}$-opers of a certain kind, elucidating a number of points, including a more precise interpretation of the $L\hat{g}$-opers in the non-simply laced case. We also discuss the results of [MRV1, MRV2] associating a solution of the

\footnote{Vladimir Bazhanov drew out attention to a system of relations in the case of $\mathfrak{sl}_{n+1}$ which were introduced in [BLMS, Equation (1.3)]. Moreover, a referee gave us an explicit comparison between this system and the $QQ$-system, see Remark 3.4 below for more details.}
Q\(\tilde{Q}\) system (for \(U_q(\mathfrak{g})\)) to the simplest \(L\hat{\mathfrak{g}}\)-opers of this kind and conjecture a generalization of these results to the case of twisted affine algebras \(\hat{\mathfrak{g}}\) (Section 8.7). Finally, we discuss a conjectural duality between \(\mathfrak{g}\)- and \(L\hat{\mathfrak{g}}\)-affine opers (Section 8.8) and comment on the appearance of opers associated to two Langlands dual Lie algebras in the classical and the quantum pictures, which still largely remains a mystery (Section 8.9).

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2. Background on Quantum Affine Algebras

In this section we collect some definitions and results on quantum affine algebras and their representations. We refer the reader to [CP] for a canonical introduction, and to [CH L] for more recent surveys on this topic. We also discuss representations of the Borel subalgebra of a quantum affine algebra, see [HJ F] for more details.

2.1. Quantum affine algebras and Borel algebras. Let \(C = (C_{i,j})_{0 \leq i,j \leq n}\) be an indecomposable Cartan matrix of untwisted affine type. We denote by \(\hat{\mathfrak{g}}\) the Kac–Moody Lie algebra associated with \(C\). Set \(I = \{1, \ldots, n\}\), and denote by \(\mathfrak{g}\) the finite-dimensional simple Lie algebra associated with the Cartan matrix \((C_{i,j})_{i,j \in I}\). Let \(\{\alpha_i\}_{i \in I}, \{\alpha_i'\}_{i \in I}, \{\omega_i\}_{i \in I}, \{\omega_i'\}_{i \in I}\), and \(\mathfrak{h}\) be the simple roots, the simple coroots, the fundamental weights, the fundamental coweights, and the Cartan subalgebra of \(\mathfrak{g}\), respectively. We set \(Q = \oplus_{i \in I} \mathbb{Z} \alpha_i\), \(Q^+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i\), \(P = \oplus_{i \in I} \mathbb{Z} \omega_i\). Let \(D = \text{diag}(d_0, \ldots, d_n)\) be the unique diagonal matrix such that \(B = DC\) is symmetric and \(d_i\)'s are relatively prime positive integers. We will also use \(P_\mathbb{Q} = P \otimes \mathbb{Q}\) with its partial ordering defined by \(\omega \leq \omega'\) if and only if \(\omega' - \omega \in Q^+\). We denote by \((\ , \ ) : Q \times Q \to \mathbb{Z}\) the invariant symmetric bilinear form such that \((\alpha_i, \alpha_i) = 2d_i\).

We use the numbering of the Dynkin diagram as in [Ka]. Let \(a_0, \ldots, a_n\) stand for the Kac labels ([Ka], pp.55-56). We have \(a_0 = 1\) and we set \(a_0 = -(a_1a_1 + a_2a_2 + \cdots + a_na_n)\). We set

\[i \sim j \text{ if } C_{i,j} < 0.\]

Throughout this paper, we fix a non-zero complex number \(q\) which is not a root of unity. We set \(q_i = q^{d_i}\). We also set \(h \in \mathbb{C}\) such that \(q = e^h\), so that \(q^r\) is well-defined for any \(r \in \mathbb{Q}\). We will use the standard symbols for \(q\)-integers

\[[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \prod_{j=1}^m [j]_z, \quad \left[ \frac{s}{r} \right]_z = \frac{[s]_z!}{[r]_z! [s-r]_z!}.
\]
The quantum loop algebra \( U_q(\hat{\mathfrak{g}}) \) is the \( \mathbb{C} \)-algebra defined by generators \( e_i, f_i, k_i^{\pm 1} \) \((0 \leq i \leq n)\) and the following relations for \( 0 \leq i, j \leq n \).

\[
k_i k_j = k_j k_i, \quad k_0^{a_1} k_1^{a_2} \cdots k_n^{a_n} = 1, \quad k_i e_j k_i^{-1} = q_i^{C_{i,j}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-C_{i,j}} f_j,
\]

\[
[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},
\]

\[
\sum_{r=0}^{1-C_{i,j}} (-1)^r e_i^{(1-C_{i,j}-r)} e_j e_i^{(r)} = 0 \quad (i \neq j), \quad \sum_{r=0}^{1-C_{i,j}} (-1)^r f_i^{(1-C_{i,j}-r)} f_j f_i^{(r)} = 0 \quad (i \neq j).
\]

Here we have set \( x_i^{(r)} = x_i^r / [r]_q! \) \((x_i = e_i, f_i)\). The algebra \( U_q(\hat{\mathfrak{g}}) \) has a Hopf algebra structure such that

\[
\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i,
\]

where \( i = 0, \ldots, n \).

The algebra \( U_q(\hat{\mathfrak{g}}) \) can also be presented in terms of the Drinfeld generators \( \mathbf{Dr} \) \( \mathbf{Be} \)

\[
x_i^{\pm} \quad (i \in I, r \in \mathbb{Z}), \quad \phi_{i,m}^{\pm} \quad (i \in I, m \geq 0), \quad k_i^{\pm 1} \quad (i \in I).
\]

We will use the generating series \((i \in I)\):

\[
\phi_{i,m}^{\pm} (z) = \sum_{m \geq 0} \phi_{i,m}^{\pm} z^m = k_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{m > 0} h_{i,m} z^m \right).
\]

We also set \( \phi_{i,m}^{\pm} = 0 \) for \( m < 0, i \in I \).

The algebra \( U_q(\hat{\mathfrak{g}}) \) has a \( \mathbb{Z} \)-grading defined by \( \deg(e_i) = \deg(f_i) = \deg(k_i^{\pm 1}) = 0 \) for \( i \in I \) and \( \deg(e_0) = - \deg(f_0) = 1 \). It satisfies \( \deg(x_{i,m}^{\pm}) = \deg(\phi_{i,m}^{\pm}) = m \) for \( i \in I, m \in \mathbb{Z} \). For \( a \in \mathbb{C}^\times \), there is a corresponding automorphism

\[
\tau_a : U_q(\hat{\mathfrak{g}}) \rightarrow U_q(\hat{\mathfrak{g}})
\]

such that an element \( g \) of degree \( m \in \mathbb{Z} \) satisfies \( \tau_a(g) = a^m g \).

**Definition 2.1.** The Borel algebra \( U_q(\mathfrak{b}) \) is the subalgebra of \( U_q(\hat{\mathfrak{g}}) \) generated by \( e_i \) and \( k_i^{\pm 1} \) with \( 0 \leq i \leq n \).

This is a Hopf subalgebra of \( U_q(\hat{\mathfrak{g}}) \). The algebra \( U_q(\mathfrak{b}) \) contains the Drinfeld generators \( x_{i,m}^{\pm}, x_{i,r}^{\pm}, k_i^{\pm 1}, \phi_{i,r}^{\pm} \) where \( i \in I, m \geq 0 \) and \( r > 0 \). When \( \mathfrak{g} = \mathfrak{sl}_2 \), these elements generate \( U_q(\mathfrak{b}) \).

Denote \( \mathfrak{t} \subset U_q(\mathfrak{b}) \) the subalgebra generated by \( \{k_i^{\pm 1}\}_{i \in I} \). Set \( \mathfrak{t}^\times = (\mathbb{C}^\times)^I \), and endow it with a group structure by pointwise multiplication. We define a group morphism \( \theta : P_{\mathfrak{q}} \rightarrow \mathfrak{t}^\times \) by setting \( \theta_i(j) = q_i^{\delta_{i,j}} \). We shall use the standard partial ordering on \( \mathfrak{t}^\times \):

\[
(2.1) \quad \omega \leq \omega' \quad \text{if} \quad \omega \omega'^{-1} \quad \text{is a product of} \quad \{\alpha_i^{-1}\}_{i \in I}.
\]
2.2. Category $\mathcal{O}$ for representations of Borel algebras. For a $U_q(b)$-module $V$ and $\omega \in t^\times$, we set
\begin{equation}
V_\omega = \{v \in V \mid k_i v = \omega(i) v \ (\forall i \in I)\},
\end{equation}
and call it the weight space of weight $\omega$. For any $i \in I$, $r \in \mathbb{Z}$ we have $\phi^\pm_{i,r}(V_\omega) \subset V_\omega$ and $x^\pm_{i,r}(V_\omega) \subset V_{\omega \pm 1}$. We say that $V$ is Cartan-diagonalizable if $V = \bigoplus_{\omega \in t^\times} V_\omega$.

**Definition 2.2.** A series $\Psi = (\Psi_{i,m})_{i \in I, m \geq 0}$ of complex numbers such that $\Psi_{i,0} \neq 0$ for all $i \in I$ is called an $\ell$-weight.

We denote by $t^\times_\ell$ the set of $\ell$-weights. Identifying $(\Psi_{i,m})_{m \geq 0}$ with its generating series we shall write
$$\Psi = (\Psi_i(z))_{i \in I}, \quad \Psi_i(z) = \sum_{m \geq 0} \Psi_{i,m} z^m.$$ Since each $\Psi_i(z)$ is an invertible formal power series, $t^\times_\ell$ has a natural group structure. We have a surjective morphism of groups $\varpi : P_\ell \to P_\mathbb{Q}$ given by $\Psi_i(0) = q_i^{\varpi(\Psi_i(\alpha_i^\vee))}$.

**Definition 2.3.** A $U_q(b)$-module $V$ is said to be of highest $\ell$-weight $\Psi \in t^\times_\ell$ if there is $v \in V$ such that $V = U_q(b)v$ and the following hold:
$$e_i v = 0 \quad (i \in I), \quad \phi^+_i v = \Psi_{i,m} v \quad (i \in I, \ m \geq 0).$$

The $\ell$-weight $\Psi \in t^\times_\ell$ is uniquely determined by $V$. It is called the highest $\ell$-weight of $V$. The vector $v$ is said to be a highest $\ell$-weight vector of $V$.

**Proposition 2.4.** For any $\Psi \in t^\times_\ell$, there exists a simple highest $\ell$-weight module $L(\Psi)$ of highest $\ell$-weight $\Psi$. This module is unique up to isomorphism.

The submodule of $L(\Psi) \otimes L(\Psi')$ generated by the tensor product of the highest $\ell$-weight vectors is of highest $\ell$-weight $\Psi \Psi'$. In particular, $L(\Psi \Psi')$ is a subquotient of $L(\Psi) \otimes L(\Psi')$.

**Definition 2.5.** \cite{H] For $i \in I$ and $a \in \mathbb{C}^\times$, let
\begin{equation}
L^\pm_{i,a} = L(\Psi_{i,a}) \text{ where } (\Psi_{i,a})_j(z) = \begin{cases} (1 - za)^{\pm 1} & (j = i), \\ 1 & (j \neq i). \end{cases}
\end{equation}

We call $L^+_i$ (resp. $L^-_i$) a positive (resp. negative) prefundamental representation in the category $\mathcal{O}$.

**Definition 2.6.** \cite{H] For $\omega \in t^\times$, let
$$[\omega] = L(\Psi_\omega) \text{ where } (\Psi_\omega)_i(z) = \omega(i) \quad (i \in I).$$

Note that the representation $[\omega]$ is 1-dimensional with a trivial action of $e_0, \cdots, e_n$. For $\lambda \in P$, we will simply use the notation $[\lambda]$ for the representation $[\vec{\lambda}]$.

For $a \in \mathbb{C}^\times$, the subalgebra $U_q(b)$ is stable under $\tau_a$. Denote its restriction to $U_q(b)$ by the same letter. Then the pullbacks of the $U_q(b)$-modules $L^\pm_{i,a}$ by $\tau_a$ is $L^\pm_{i,a b}$.

For $\lambda \in t^\times$, we set $D(\lambda) = \{\omega \in t^\times \mid \omega \leq \lambda\}$. The following category $\mathcal{O}$ is introduced in \cite{H].
Definition 2.7. A \( U_q(b) \)-module \( V \) is said to be in category \( \mathcal{O} \) if:

i) \( V \) is Cartan-diagonalizable,

ii) for all \( \omega \in \mathfrak{t}^\times \) we have \( \dim(V_\omega) < \infty \),

iii) there exist a finite number of elements \( \lambda_1, \ldots, \lambda_s \in \mathfrak{t}^\times \) such that the weights of \( V \) are in \( \bigcup_{j=1, \ldots, s} D(\lambda_j) \).

The category \( \mathcal{O} \) is a monoidal category.

Let \( \tau \) be the subgroup of \( \mathfrak{t}^\times \times \ell \) consisting of \( \Psi \) such that \( \Psi_i(z) \) is rational for any \( i \in I \).

Theorem 2.8. [HJ] Let \( \Psi \in \mathfrak{t}^\times \). The simple module \( L(\Psi) \) is in category \( \mathcal{O} \) if and only if \( \Psi \in \tau \).

Let \( \mathcal{E} \) be the additive group of maps \( c : P_\mathcal{Q} \to \mathbb{Z} \) whose support

\[
\text{supp}(c) = \{ \omega \in P_\mathcal{Q}, c(\omega) \neq 0 \}
\]

is contained in a finite union of sets of the form \( D(\mu) \). For \( \omega \in P_\mathcal{Q} \), we define \( [\omega] = \delta_{\omega,} \in \mathcal{E} \).

For \( V \) in the category \( \mathcal{O} \) we define the character of \( V \) to be an element of \( \mathcal{E} \)

\[
\chi(V) = \sum_{\omega \in \mathfrak{t}^\times} \dim(V_\omega)[\omega].
\]

As for the category \( \mathcal{O} \) of a classical Kac–Moody algebra, the multiplicity of a simple module in a module of our category \( \mathcal{O} \) is well-defined (see [Ka, Section 9.6]) and we have the corresponding Grothendieck ring \( K_0(\mathcal{O}) \) (see also [HL2, Section 3.2]). Its elements are the formal sums

\[
\chi = \sum_{\Psi \in \tau} \lambda_{\Psi} [L(\Psi)]
\]

where the \( \lambda_{\Psi} \in \mathbb{Z} \) are set so that \( \sum_{\Psi \in \tau, \omega \in P_\mathcal{Q}} [\lambda_{\Psi} \dim((L(\Psi))_\omega)][\omega] \) is in \( \mathcal{E} \).

We naturally identify \( \mathcal{E} \) with the Grothendieck ring of the category of representations of \( \mathcal{O} \) with constant \( \ell \)-weights, the simple objects of which are the \( [\omega], \omega \in P_\mathcal{Q} \). Thus as in [Ka, Section 9.7] we will regard elements of \( \mathcal{E} \) as formal sums

\[
c = \sum_{\omega \in \text{supp}(c)} c(\omega)[\omega].
\]

The multiplication is given by \( [\omega][\omega'] = [\omega + \omega'] \) and \( \mathcal{E} \) is regarded as a subring of \( K_0(\mathcal{O}) \).

The character defines a ring morphism \( \chi : K_0(\mathcal{O}) \to \mathcal{E} \) which is not injective.

2.3. Monomials and finite-dimensional representations. Following [FR], consider the ring of Laurent polynomials \( \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times} \) in the indeterminates \( \{Y_{i,a}\}_{i \in I, a \in \mathbb{C}^\times} \). Let \( \mathcal{M} \) be the group of monomials of \( \mathcal{Y} \). For example, for \( i \in I, a \in \mathbb{C}^\times \), define \( A_{i,a} \in \mathcal{M} \) to be

\[
Y_{i,aq^{-1}} Y_{i,aq} \left( \prod_{j \in I, C_{j,i} = -1} Y_{j,a} \prod_{j \in I, C_{j,i} = -2} Y_{j,aq^{-1}} Y_{j,aq} \prod_{j \in I, C_{j,i} = -3} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \right)^{-1}.
\]
For a monomial \( m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}} \), we consider its ‘evaluation on \( \phi^+(z) \)’. By definition it is an element \( m(\phi(z)) \in \mathfrak{r} \) given by

\[
m(\phi(z)) = \prod_{i \in I, a \in \mathbb{C}^\times} (Y_{i,a}(\phi(z)))^{u_{i,a}} \quad \text{where} \quad \left( Y_{i,a}(\phi(z)) \right)_j = \begin{cases} 
\frac{1 - aq_i^{-1}z}{1 - aq_iz} & (j = i), \\
1 & (j \neq i).
\end{cases}
\]

This defines an injective group morphism \( M \to \mathfrak{r} \). We identify a monomial \( m \in M \) with its image in \( \mathfrak{r} \). Note that \( \varpi(\bar{Y}_{i,a}) = \omega_i \).

Let \( \mathcal{C} \) be the category of (type 1) finite-dimensional representations of \( U_q(\hat{\mathfrak{g}}) \).

A monomial \( M \in \mathcal{M} \) is said to be dominant if \( M \in \mathbb{Z}[Y_{i,a}]_{i \in I, a \in \mathbb{C}^\times} \). Then \( L(M) \) is finite-dimensional. Moreover, the action of \( U_q(\mathfrak{b}) \) can be uniquely extended to an action of the full quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \), and any simple object in the category \( \mathcal{C} \) is of this form. By [CP] and [FH, Remark 3.11], for \( L(\Psi) \) a finite-dimensional module in the category \( \mathcal{O} \), there is \( M \) as above and \( \omega \in \mathfrak{t}^\times \) such that

\[
L(\Psi) \simeq L(M) \otimes [\omega].
\]

Note that if \( \Psi \) is a monomial in the variables

\[
\bar{Y}_{i,a} = [-\omega_i] Y_{i,a},
\]

then \( L(\Psi) \) is finite-dimensional. We will also use in the following notation:

\[
\bar{A}_{i,a} = \Psi_{i,aq_i^{-1}} \Psi_{i,aq_i}^{-1} \prod_{j \sim i, r_j > 1} \Psi_{j,aq_j}^{-1} \Psi_{j,aq_j} \prod_{j \sim i, r_j = 1} \Psi_{j,aq_j}^{-1} \Psi_{j,aq_j} = [-\alpha_i] A_{i,a}.
\]

For \( i \in I, a \in \mathbb{C}^\times \) and \( k \geq 0 \), we have the Kirillov–Reshetikhin (KR) module

\[
W^{(i)}_{k,a} = L(Y_{i,a} Y_{i,aq_i^2} \cdots Y_{i,aq_i^{2(k-1)}}).
\]

The representations \( W^{(i)}_{k,a} = L(Y_{i,a}) \) are called fundamental representations. The simple tensor product of a KR-module by a one-dimensional representation \([\omega], \omega \in P\), will also be called a KR-module.

2.4. Example. The prefundamental representations have a relatively simple structure in comparison to the general simple representations in the category \( \mathcal{O} \).

As an example, let us consider the case \( \mathfrak{g} = B_2 \) and the representation \( L^+_{2,1} \). From [H1] we know the action of a large number of generators of the Borel algebra on this representation. For \( r > 0, \phi^+_{1,r}, \phi^+_{2,r+1}, x^+_{1,r}, x^+_{2,r}, x^-_{1,r}, x^-_{2,r+1} \) act by 0 on this representation; \( k_2^{-1} \phi^+_{2,1} \) is the operator \(-\text{Id}; (r)\); the root operators \( E_{-\alpha_1 - \alpha_2/(r + 1)} \delta, E_{-2\alpha_1 - \alpha_2/(r + 1)} \delta \) act by 0.

As this representation \( L^+_{2,1} \) is constructed in [H1] as a limit of finite-dimensional Kirillov-Reshetikhin modules whose structure is well-known (see [HH1] and references therein), it has a basis

\[
L^+_{2,1} = \bigoplus_{T \in \mathcal{T}} \mathbb{C} v_T
\]

of weight vectors parametrized by the semi-infinite tableaux \( T = (T_{i,j})_{i=1,2,j \geq 0} \) with coefficients in the ordered set

\[
1 < 2 < 0 < \overline{T} < \overline{2}.
\]
satisfying $T_{i,j} \preceq T_{i,j+1}$, $(T_{i,j}, T_{i,j+1}) \neq (0, 0)$ and $(T_{1,j} < T_{2,j}$ or $(T_{1,j}, T_{2,j}) = (0, 0)$). It is required in addition that $T_{i,j} = i$ for $j$ is large enough.

The weight of $v_T$ is

$$\omega_T = -\sum_{j \geq 0} \alpha(T_{1,j}) + \beta(T_{2,j})$$

where $\alpha(1, 2, 0, 1) = (0, \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2)$ and $\beta(2, 0, 1, 2) = (0, 2\alpha_2, 2\alpha_2 + \alpha_1)$. In particular, we have the explicit character formula

$$\chi_2 = \chi(L_{2,1}^+) = \sum_{T \in \mathcal{T}} [\omega_T].$$

3. The $Q\tilde{Q}$-system

In this section we prove the $Q\tilde{Q}$-system of relations in $\mathcal{K}_0(O)$ (Theorem 3.2). This is one of the main results of this paper.

3.1. Statement. Let $\mathfrak{g}$ be an arbitrary simple Lie algebra. We start by introducing the following representations.

**Definition 3.1.** For $i \in I$ and $a \in \mathbb{C}^\times$, we define the representation

$$X_{i,a} = L(\Psi_{i,a})$$

where

$$\tilde{\Psi}_{i,a} = \Psi_{i,a}^{-1} \left( \prod_{J_{C_{i,j}} = -1} \Psi_{j,aq} \right) \left( \prod_{J_{C_{i,j}} = -2} \Psi_{j,aq^2} \right) \left( \prod_{J_{C_{i,j}} = -3} \Psi_{j,aq^{-1}} \Psi_{j,aq} \Psi_{j,aq^3} \right).$$

Note that if we think of $\Psi_{i,a}$ as an analogue of the fundamental weight $\omega_i$, then $\tilde{\Psi}_{i,a}$ is an analogue of the weight $\omega_i - \alpha_i$.

**Remark 3.1.** (i) If $\mathfrak{g}$ is simply-laced, then

$$X_{i,a} = L(\Psi_{i,a}^{-1} \prod_{j \sim i} \Psi_{j,aq}).$$

(ii) The $\ell$-weight $\Psi_{i,a}$ may be written as an infinite product

$$\Psi_{i,a} = \tilde{\Psi}_{i,a} \tilde{Y}_{i,aq} \tilde{Y}_{i,aq^2} \tilde{Y}_{i,aq^3} \cdots,$$

where $\tilde{Y}_{i,a}$ as in Section 2.3 is the analogue of a fundamental weight. Similarly, one may write $\tilde{\Psi}_{i,a}$ as an infinite product involving the $\tilde{A}_{i,a}$ as in Section 2.3 which is the analogue of a simple root. Indeed,

$$\tilde{A}_{i,aq^2} \tilde{Y}_{i,aq} \tilde{Y}_{i,aq^2} \tilde{Y}_{i,aq^3} \cdots = \Psi_{i,a}^{-1} \prod_{K \geq 1} \left( \prod_{j \sim i, r_j > 1} \Psi_{j,aq^2 K q_j^{-1}} \Psi_{j,aq^{-1} K q_j} \right) \left( \prod_{j \sim i, r_j = 1} \Psi_{j,aq^2 K q_j^{-1}} \Psi_{j,aq^{-1} K q_j} \right).$$


\[= \Psi_{i,a}^{-1} \left( \prod_{j=1}^{r_i} \Psi_{j, q_a^2 q_j - r_j} \Psi_{j, q_a^2 q_j + 2r_i} \cdots \Psi_{j, q_a^2 q_j - 2r_i} \right) \left( \prod_{j=1}^{r_i} \Psi_{j, a^2 q_j} \right) = \tilde{\Psi}_{i,a}. \]

So \(\tilde{\Psi}_{i,a}\) may be indeed be viewed as an analogue of the difference \(\omega_i - \alpha_i\).

We define the \(Q\) and \(\bar{Q}\) variables as follows:

\[Q_{i,a} = [L_{i,a}^+] \quad \text{and} \quad \bar{Q}_{i,a} = [X_{i,a q_i^{-2}}] \chi_i^{-1} \left[ -\frac{\alpha_i}{2} \right],\]

where

\[\chi_i = \chi(L_{i,a}^+) \in \mathcal{E}\]

does not depend on \(a\) and is seen as an element of \(K_0(\mathcal{O})\).

Now we state the \(Q\bar{Q}\)-system which is one of the main results of this paper.

**Theorem 3.2.** For any \(i \in I, a \in \mathbb{C}^\times\) we have the following \(Q\bar{Q}\)-system:

\[(3.6) \quad \left[\frac{\alpha_i}{2}\right] Q_{i,a q_i^{-1}} \bar{Q}_{i,a q_i} - \left[\frac{\alpha_i}{2}\right] Q_{i,a q_i} \bar{Q}_{i,a q_i^{-1}} = \left( \prod_{j | C_{i,j} = -1} Q_{j,a} \right) \left( \prod_{j | C_{i,j} = -2} Q_{j,a q_i^{-2}} \right) \left( \prod_{j | C_{i,j} = -3} Q_{j,a q_i^{-2}} \right).

**Remark 3.2.** (i) This \(Q\bar{Q}\)-system matches \([\text{MRV2}]\) Formula (5.4), with \(E\) replaced by \(a\) and \(\Omega\) replaced by \(q^{-2}\).

(ii) In this simply-laced case, the \(Q\bar{Q}\)-system specializes to the following:

\[\left[\frac{\alpha_i}{2}\right] Q_{i,a q_i^{-1}} \bar{Q}_{i,a q_i} - \left[\frac{\alpha_i}{2}\right] Q_{i,a q_i} \bar{Q}_{i,a q_i^{-1}} = \prod_{j=1}^{r_i} Q_{j,a}.

This matches \([\text{MRV1}]\) Formula (4.6).

(iii) An analogous system may be written for the category \(\mathcal{O}^*\) dual to the category \(\mathcal{O}\), which was defined in [HJ] Section 3.6. It suffices to set \(Q_{i,a} = [R_{i,a}^+]\) and \(\bar{Q}_{i,a} = [X_{i,a q_i^{-2}}] \chi_i^{-1} \left[\frac{\alpha_i}{2}\right]\) and to replace \(\pm \frac{\alpha_i}{2}\) by \([\mp \frac{\alpha_i}{2}]\). Here \(R_{i,a}^+\) and \(X_{i,a}^*\) are defined by \((R_{i,a}^+)^* \simeq L_{i,a}^+, (X_{i,a}^*)^* \simeq X_{i,a}\) and \(\chi_i' = \chi(R_{i,a}^+).\) Indeed the same proof as for formula (3.8) gives us a relation in the category \(\mathcal{O}\) of the opposite Borel as considered in [FH] Section 3.5] with the representations \(L_{i,a}^+, X_{i,a}\) replaced by the simple representations in \(\mathcal{O}\) with the same highest \(\ell\)-weight. Then it suffices to twist by the involutive automorphism \(\hat{\omega}\) as in [FH].

\[3.2. \text{First examples. Let } g = s\mathfrak{l}_2. \text{ We have the following relation in } K_0(\mathcal{O}):\]

\([L_a^+][L_a^-] - [-\alpha][L_{aq_i^2}][L_{aq_i^{-2}}] = \chi\]

where as above

\(\chi = \chi(L_a^+) = \sum_{r \geq 0} [-r \alpha] \in K_0(\mathcal{O}).\)
does not depend on \( a \). By setting

\[ Q_a = [L^+_a] \quad \text{and} \quad \tilde{Q}_a = [L^-_{aq-2}] \chi^{-1} \left[-\frac{\alpha}{2}\right] \]

we get the \( Q\tilde{Q} \)-relation:

\[ \left[\frac{\alpha}{2}\right] Q_{aq^{-1}} \tilde{Q}_{aq} - \left[-\frac{\alpha}{2}\right] Q_{aq} \tilde{Q}_{aq^{-1}} = 1 \]

which is essentially the “quantum Wronskian relation” \cite[Formula (9)]{BLZ5} (see also \cite[BLZ2, BLZ3]{BLZ3}).

Let \( g = \mathfrak{sl}_3 \). We consider 6 families of representations as in \cite[BHK]{BHK}: the prefundamental representations \( L^+_{1,a}, L^+_2, L^-_{1,a}, L^-_{2,a} \) and the new representations

\[ X_{1,a} = L(\Psi^{-1}_{1,a} \Psi_{2,a}q), \quad X_{2,a} = L(\Psi^{-1}_{2,a} \Psi_{1,a}q). \]

Let

\[ \chi_1 = \chi(L^+_{1,a}) = \sum_{0 \leq r \leq s} [-r \alpha_2 - s \alpha_1] = \quad \text{and} \quad \chi_2 = \chi(L^+_{2,a}) = \sum_{0 \leq r \leq s} [-r \alpha_1 - s \alpha_2]. \]

Remark 3.3. The representations \( X_{1,a} \) and \( X_{2,a} \) have the same character in this case (this is not true for general \( g \)):

\[ \chi(X_{1,a}) = \chi(X_{2,a}) = \sum_{\lambda, \mu \geq 0} (1 + \min(\lambda, \mu))[-\lambda \alpha_1 - \mu \alpha_2], \]

which is the character of the Verma module of \( \mathfrak{sl}_3 \) of highest weight 0. However, the representations \( X_{1,a}, X_{2,a} \) are not evaluation modules of this Verma module as the action of \( U_q(\mathfrak{b}) \) cannot be extended to the full quantum affine algebra.

We view \( \chi_1, \chi_2 \) as elements of \( K_{q}(0) \). Now we can define the \( Q \) and \( \tilde{Q} \) variables:

\[ Q_{1,a} = [L^+_{1,a}], \quad \tilde{Q}_{1,a} = [X_{1,a}q^{-2}] \chi^{-1}_1 \left[-\frac{\alpha_1}{2}\right], \]

\[ Q_{2,a} = [L^+_{2,a}], \quad \tilde{Q}_{2,a} = [X_{2,a}q^{-2}] \chi^{-1}_2 \left[-\frac{\alpha_2}{2}\right]. \]

Then get the \( Q\tilde{Q} \)-system in \( \text{Frac}(K_0(0)) \) as in \cite[BHK]{BHK} Formulas (5.4), (5.5):

\[ \left[\frac{\alpha_1}{2}\right] Q_{1,a} \tilde{Q}_{1,a} - \left[-\frac{\alpha_1}{2}\right] Q_{1,a} \tilde{Q}_{1,a}^{-1} = Q_{2,a}. \]

\[ \left[\frac{\alpha_2}{2}\right] Q_{2,a} \tilde{Q}_{2,a} - \left[-\frac{\alpha_2}{2}\right] Q_{2,a} \tilde{Q}_{2,a}^{-1} = Q_{1,a}. \]

Let \( g = B_2 \). We have given in section \cite{24} an explicit formula for \( \chi_2 \) and a description of the representation \( L^+_{2,a} \). The corresponding \( Q\tilde{Q} \)-relation is

\[ \left[\frac{\alpha_2}{2}\right] Q_{2,a} \tilde{Q}_{2,a} - \left[-\frac{\alpha_2}{2}\right] Q_{2,a} \tilde{Q}_{2,a}^{-1} = Q_{1,a} \]

where the \( Q_{1,a} \) are classes of prefundamental representations and \( \tilde{Q}_{2,a} = [X_{2,a}q^{-2}] \chi^{-1}_2 \left[-\frac{\alpha_2}{2}\right] \).
Remark 3.4. Vladimir Bazhanov drew out attention to a system of relations in the case of \( \mathfrak{sl}_{n+1} \) which were introduced in [BFLMS] Equation (1.3) in the context of finite-dimensional representations of the corresponding Yangians, which is equivalent to ours (for example, the analogues of the prefundamental representations for the Yangians of \( \mathfrak{sl}_{n+1} \) are described in [BFLMS] and called there “partonic” representations). Moreover, an explicit comparison between this system and the \( \widetilde{Q}\overline{Q} \)-system was kindly given to us by the referee which we now present (it is worth mentioning that in this case it coincides with the Hirota equations, according to [BFLMS]). This is not immediately clear as the system of [BFLMS] involves a seemingly different set of variables \( \overline{Q}_{j_1, \ldots, j_i} \) with \( j_1, \ldots, j_i \in \{1, \ldots, n\} \). Let us pick what can be called a path in the Hasse diagram
\[
\mathcal{P} : \emptyset \subset \{t_1\} \subset \{t_1, t_2\} \subset \cdots \subset \{t_1, \ldots, t_n\} = \{1, 2, \ldots, n\}.
\]
Then [BFLMS] Equation (1.3)] with \( I = \emptyset \) and \( a = t_1, b = t_2 \) is the type A \( \widetilde{Q}\overline{Q} \)-system with \( i = 1 \) after the identification of \( Q_1, Q_2, \overline{Q}_1 \) with \( Q_{t_1}, Q_{t_1, t_2}, Q_{t_1, t_2} \setminus \{t_1\} \) (up to a multiplication by a weight). More generaly, if we consider a second path
\[
\mathcal{P} : \emptyset \subset \{\hat{t}_1\} = \{t_1, t_2\} \setminus \{t_1\} \subset \{\hat{t}_1, \hat{t}_2\} = \{t_1, t_2, t_3\} \setminus \{t_2\} \subset \cdots \subset \{1, \ldots, n\},
\]
then [BFLMS] Equation (1.3)] with \( I = \{t_1, \ldots, t_{i-1}\} \) and \( a = t_i, b = t_{i+1} \) is the \( \widetilde{Q}\overline{Q} \)-system at \( i \) after the identification of \( Q_{j}, \overline{Q}_{j} \) with \( Q_{t_1, \ldots, t_j}, Q_{\hat{t}_1, \ldots, \hat{t}_j} \) (up to a multiplication by a weight).

3.3. \( \widetilde{Q}\overline{Q} \)-system for the twisted quantum affine algebras. There is also a \( \widetilde{Q}\overline{Q} \)-system for the twisted quantum affine algebras. To explain this, we use the notation of [HL] Section 2.4 (except that the Lie algebra denoted by \( g \) in [HL] will now be denoted by \( g' \), and \( I \) will be denoted by \( I' \)). Let \( \sigma \) be an automorphism of the Dynkin diagram of a simply-laced simple finite-dimensional Lie algebra \( g' \); that is, a bijection \( \sigma : I' \to I' \) of the set \( I' \) of nodes of the Dynkin diagram of \( g \) such that \( C_{\sigma(i),\sigma(j)} = C_{i,j} \) for any \( i, j \in I' \), where \( C \) is the Cartan matrix of \( g' \). Let \( r \) be the order of \( \sigma \). Consider the twisted case, so that \( r \in \{2, 3\} \) (in fact, \( g \) must be of type \( A_n \) \((n \geq 2)\), \( D_n \) \((n \geq 4)\), or \( E_6 \)). Let \( I_\sigma \) denote the set of orbits of \( \sigma \) and for \( i \in I' \) we denote by \( \overline{r} \) in \( I_\sigma \) the orbit of \( i \).

Using the Cartan generators of \( g' \), we obtain an automorphism of \( g' \) of the same order, which we also denote by \( \sigma \). The Lie algebra \( g' \) decomposes into a direct sum of eigenspaces of \( \sigma \):
\[
g' = \bigoplus_{\overline{r} \in \mathbb{Z}/r\mathbb{Z}} g'_{\overline{r}},
\]
where \( g'_{\overline{r}} \) is the simple Lie algebra corresponding to the quotient of the Dynkin diagram of \( g' \) by the action of the automorphism. The twisted affine Kac--Moody algebra \( \widehat{g'} \) is defined as the universal central extension of the twisted loop algebra
\[
\mathcal{L}_\sigma g = \bigoplus_{n \in \mathbb{Z}} g'_{\overline{r}n} \otimes z^n
\]
Note that its constant part is the simple Lie algebra \( g'_{\overline{0}} \) the \( \sigma \)-invariants of \( g' \). The nodes of the Dynkin diagram of \( g'_{\overline{0}} \) are naturally parametrized by \( I_\sigma \).

There is a quantum affine algebra \( U_q(\widehat{g'}) \) attached to the twisted affine algebra \( \widehat{g} \), whose finite-dimensional representations were studied by several authors, see [HL] and references.
therein. This algebra has a Borel subalgebra, and one can define the corresponding category \( \mathcal{O} \) in the same way as in [11] (see Section 2.2). Though this category has not been studied in the twisted case, it is natural to conjecture that it contains analogues of the representations \( L_{i,a}^+ \) and \( X_{i,a} \) defined in [11] and the present paper, respectively, in the untwisted case.

More precisely, for each \( \mathfrak{g} \) will be discussed in another paper. We fix such a choice and identify \( L_{i,a}^+ \) for the simply-laced Lie algebra \( \mathfrak{g} \). It arises naturally in the context of cluster algebras, as the first step of the Fomin–Zelevinsky mutation relations. Namely, for \( \mathfrak{g} \) of type \( A_n \), it is well-defined for any \( i, j \in I_\sigma \). Then we expect that in the category \( \mathcal{O} \) in the twisted case there are representations \( L_{i,a}^+ \) and \( X_{i,a} \) for all \( i \in I_\sigma \) and \( a \in \mathbb{C}^\times \).

Now let us define \( Q_{i,a} \) and \( \tilde{Q}_{i,a} \) by formulas (3.6), and for each \( i \in I_\sigma \), set \( q_i = q^{d_i} \) where

\[
d_i = \begin{cases} r & \text{if } C_{i,\sigma(i)} = 2, \\ 1 & \text{if } C_{i,\sigma(i)} = 0, \\ 1/2 & \text{if } C_{i,\sigma(i)} = -1. \end{cases}
\]

**Conjecture 3.3.** The variables \( Q_{i,a}, \tilde{Q}_{i,a}, i \in I_\sigma, a \in \mathbb{C}^\times \) satisfy the following \( QQ \)-system:

\[
\left[ \frac{\alpha_i}{2} \right] Q_{i,a} q_i^{-1} \tilde{Q}_{i,a} q_i - \left[ \frac{\alpha_i}{2} \right] Q_{i,a} q_i \tilde{Q}_{i,a} q_i^{-1} \quad = \quad \begin{cases} \left( \prod_{r \sim i, d_r = r} Q_{j,a} \right) \left( \prod_{r \sim i, d_r \neq r \text{ and } b,b' = a} Q_{j,a} \right) & \text{if } d_i = r, \\
\left( \prod_{r \sim i, d_r = r} \tilde{Q}_{j,a} \right) \left( \prod_{r \sim i, d_r \neq r} Q_{j,a} \right) & \text{if } d_i = 1, \\
Q_{i,-a} \times \left( \prod_{r \sim i} Q_{j,a} \right) & \text{if } d_i = 1/2, \end{cases}
\]

(3.9)

where the products run on the \( j \in I_\sigma \), \( j \sim i \) means \( C_{i,j} < 0 \) as above and \( \alpha_i \) is a simple root of \( \mathfrak{g}' \).

We note that we have obtained the system (3.9) by a kind of “folding” of the \( QQ \)-system for the simply-laced Lie algebra \( \mathfrak{g}' \) (analogously to how the \( T \)-system in the twisted case was written in [KS2] and established in [HL4] by “folding” the \( T \)-system in the untwisted case).

We expect that the proof of this conjecture can be obtained along the same lines as our proof in Section 6.1 of the \( QQ \)-system in the case of untwisted quantum affine algebras. This will be discussed in another paper.

### 3.4. \( QQ^* \)-system.

Another system of relations in \( K_0(\mathcal{O}) \) was established in [HL2, Section 6.1.3]. It arises naturally in the context of cluster algebras, as the first step of the Fomin–Zelevinsky mutation relations. Namely, for \( i \in I \) and \( a \in \mathbb{C}^\times \) we have

\[
Q_{i,a} Q_{i,a}^* = \prod_{j, C_{j,i} \neq 0} Q_{j,a} q_{j,C_{j,i}} - \left[ -\alpha_i \right] \prod_{j, C_{j,i} \neq 0} Q_{j,a} q_{j,C_{j,i}}
\]

where

\[
Q_{i,a}^* = [L(\Psi_{i,a}^{-1} \prod_{j, C_{j,i} \neq 0} \Psi_{j,a}^{-1} q_{j,C_{j,i}})] ; \quad Q_{i,a} = [L_{i,a}^+].
\]
This is the relation \([HL2\) Equation 6.14]. Note that \(Q^*_i\) is the mutated cluster variable obtained from \(Q_{i,a}^*\) in a certain subring of \(K_0(\mathcal{O})\), which was introduced and proved to be a cluster algebra in \([HL2\).

To distinguish them from the relations of the \(QQ^*\)-system, we call these relations the \(QQ^\vee\)-system.

An analogous system of relations was also established in \([HL2\) Example 7.8] in terms of the negative fundamental representations. Taking the duals in those relations, we get the \(QQ^*\)-system in the Grothendieck ring of the dual category \(\mathcal{O}^*\), with the variables 

\[Q_{i,a}^* = [R_{i,a}^{-1}], \quad Q_{i,a} = L(\Psi_{i,a}^{-1}\prod_{j,C_{j,i} \neq 0} \Psi_{j,a-1q^a_jC_{j,i}})\]

and with \([-\alpha_i]\) replaced by \([\alpha_i]\).

### 3.5. \(QQ^\vee\)-system for the quantum \(\mathfrak{gl}_1\) toroidal algebra

An analogue of the \(QQ^\vee\)-system holds for the quantum \(\mathfrak{gl}_1\) toroidal algebra \(\mathcal{E}\) as well. The quantum parameters of \(\mathcal{E}\) are \(q_1, q_2, q_3\) satisfying \((q_1q_3)^{-1} = q_2 = q^2\) and we use the notations of \([FJMM]\) except that we replace their spectral parameter \(z\) by \(z^{-1}\) for consistency. For \(a \in \mathbb{C}^\times\) we introduce the representation \(L_a\) of highest \(\ell\)-weight \((1 - az)^{-1}(1 - q_1^{-1}az)(1 - q_3^{-1}az)\). \(L_a^+\) denotes the prefundamental representation of highest \(\ell\)-weight \((1 - az)\).

If we set as above \(Q_a = [L_a^+]\) and \(\tilde{Q}_a = [X_{aq-2}][\frac{-\alpha}{2}]{\chi}^{-1}\), then we obtain the following \(QQ^\vee\)-system:

\[\left[\frac{\alpha}{2}\right]Q_{aq^{-1}}\tilde{Q}_aq - \left[-\frac{\alpha}{2}\right]Q_{aq}\tilde{Q}_aq^{-1} = Q_{aq^{-1}q_1^{-1}}Q_{aq^{-1}q_3^{-1}},\]

where \([\pm \frac{\alpha}{2}]\) is the class of the one-dimensional representation corresponding to \(t^{\pm \frac{1}{2}}\).

The proof is analogous to the proof of the \(QQ^\vee\)-system presented in the next section. One needs to use the theory of \(q\)-characters of the category \(\mathcal{O}\) of the quantum \(\mathfrak{gl}_1\) toroidal algebra, which has recently been developed in \([FJMM]\) in parallel with the theory for the quantum affine algebras \([FR]\ [HL] [T1]\).

Note that the \(QQ^*\)-system for quantum affine algebras discussed in Section 3.3 also has an analogue for the \(\mathfrak{gl}_1\) quantum toroidal algebra. This is the system of relations obtained in \([FJMM]\) Formula (4.24)].

**Remark 3.5.** We were informed by Michio Jimbo that the above \(QQ^\vee\)-system for the quantum \(\mathfrak{gl}_1\) toroidal algebra was known to the authors of \([FJMM]\). \(\square\)

### 4. Proof of the \(QQ^\vee\)-system

In this section we prove the \(QQ^\vee\)-system \([HL]\) stated in Theorem 3.2. One of the crucial tools used in the proof is the theory of \(q\)-characters.

**4.1. \(q\)-characters.** For a \(U_q(\mathfrak{b})\)-module \(V\) and \(\Psi \in \mathfrak{t}^*_i\), the linear subspace

\[V_\Psi = \{v \in V \mid \exists p \geq 0, \forall i \in I, \forall m \geq 0, (\phi_{i,m}^+ - \Psi_{i,m})^p v = 0\}\]

is called the \(\ell\)-weight space of \(V\) of \(\ell\)-weight \(\Psi\).

**Theorem 4.1. \([HL]\)** For \(V\) in category \(\mathcal{O}\), \(V_\Psi \neq 0\) implies \(\Psi \in \mathfrak{t}\).
Given a map \( c : \mathfrak{r} \to \mathbb{Z} \), consider its support

\[
\text{supp}(c) = \{ \Psi \in \mathfrak{r} \mid c(\Psi) \neq 0 \}.
\]

Let \( \mathcal{E}_\ell \) be the additive group of maps \( c : \mathfrak{r} \to \mathbb{Z} \) such that \( \varpi(\text{supp}(c)) \) is contained in a finite union of sets of the form \( D(\mu) \), and such that for every \( \omega \in \mathcal{P}_Q \), the set \( \text{supp}(c) \cap \varpi^{-1}(\{ \omega \}) \) is finite. The map \( \varpi \) is naturally extended to a surjective homomorphism \( \varpi : \mathcal{E}_\ell \to \mathcal{E} \).

For \( \Psi \in \mathfrak{r} \), we define \( \{ \Psi \} = \delta_{\Psi_\omega} \in \mathcal{E}_\ell \).

Let \( V \) be a \( U_q(\mathfrak{b}) \)-module in category \( \mathcal{O} \). We define [FR] the \( q \)-character of \( V \) as

\[
\chi_q(V) = \sum_{\Psi \in \mathfrak{r}} \dim(V_\Psi)[\Psi] \in \mathcal{E}_\ell.
\]

**Example 4.2.** For \( \omega \in \mathfrak{t}^\times \), the \( q \)-character of the 1-dimensional representation \( [\omega] \) is just its \( \ell \)-highest weight \( \chi_q([\omega]) = [\omega] \). That is why the use of the same notation \( [\omega] \) will not lead to confusion.

Note that we have \( \chi(V) = \varpi(\chi_q(V)) \) for \( V \) a representation in the category \( \mathcal{O} \).

By [FR] Theorem 3 and [FJ] Proposition 3.12, we have the following.

**Proposition 4.3.** The \( q \)-character morphism

\[
\chi_q : \text{Rep}(U_q(\mathfrak{b})) \to \mathcal{E}_\ell, \quad [V] \mapsto \chi_q(V),
\]

is an injective ring morphism.

It is proved in [FR] that a finite-dimensional \( U_q(\mathfrak{g}) \)-module \( V \) satisfies \( V = \bigoplus_{m \in \mathbb{N}} V_m(\phi(z)) \). In particular, \( \chi_q(V) \) can be viewed as an element of \( \mathfrak{y} \). It is proved in [FR] that if moreover \( V = L(m) \) is simple, then

\[
\chi_q(L(m)) \in m(1 + \mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}).
\]

**Theorem 4.4.** (i) For any \( a \in \mathbb{C}^\times \), \( i \in I \) we have

\[
\chi_q(L_{i,a}^+) = [\Psi_{i,a}] \chi(L_{i,a}^+) = [\Psi_{i,a}] \chi(L_{i,a}).
\]

(ii) For any \( a \in \mathbb{C}^\times \), \( i \in I \) we have

\[
\chi_q(L_{i,a}^-) \in [\Psi_{i,a}^{-1}] (1 + A_{i,a}^{-1} \mathbb{Z}[A_{j,b}^{-1}]_{j \in I, b \in \mathbb{C}^\times}).
\]

**Remark 4.1.** (i) The statement (i) is proved in [FJ] [FH].

(ii) The statement (ii) is proved in [FJ] indeed it is established there that \( [\Psi_{i,a}] \chi_q(L_{-1,a}^-) \) is a certain limit of \( q \)-characters of KR modules as a formal power series in the \( A_{j,b}^{-1} \). It is of the form written in the Theorem by [FJ] Lemma 4.4.

(iii) As a consequence the \( \chi_i \in \mathcal{E} \) defined in formula (3.7) is equal to

\[
\chi_i = \chi(L_{i,a}^+) = \chi(L_{i,a}^-) = [\Psi_{i,a}^{-1}] \chi_q(L_{i,a}^+).
\]

**Example 4.5.** In the case \( \mathfrak{g} = \mathfrak{sl}_2 \), we have:

\[
\chi_q(L_{1,a}^+) = \left(1 - za\right) \sum_{r \geq 0} [−2r \omega_1], \quad \chi_q(L_{1,a}^-) = \left(\frac{1}{1 - za}\right) \sum_{r \geq 0} A_{1,a}^{-1} A_{1,aq^{-2}} \cdots A_{1,aq^{-2(r-1)}}.
\]
Example 4.6. In the case of $\mathfrak{g} = B_2$ (see section 2.4), we have:

$$\chi_q(L_+^{2,a}) = [(1 - za)] \sum_{T \in \mathcal{T}} \chi(T),$$

$$\chi_q(L_-^{2,a}) = \left[\frac{1}{(1 - za)}\right] \sum_{T \in \mathcal{T}, j \geq 0} \prod_{T_j} \left(A_j(T_1,j)B_j(T_2,j)\right)^{-1},$$

where $A_j(1,2,0,T) = (1, A_1,aq^{-4j+4}, A_2,aq^{-4j+2}A_2,aq^{-4j+2}A_2,aq^{-4j+4}A_2,aq^{-4j+2})$ and $B_j(2,0,T) = (1, A_2,aq^{-4j}, A_2,aq^{-4j}, A_2,aq^{-4j}, A_2,aq^{-4j}, A_2,aq^{-4j}A_1,aq^{-4j}).$

Theorem 4.7. [FH] Any tensor product of positive (resp. negative) fundamental representations $L_{i,a}^+$ (resp. $L_{i,a}^-$) is simple.

4.2. Examples. Let us explain the examples from Section 3.2 in terms of $q$-characters.

For $\mathfrak{g} = \mathfrak{sl}_2$, the relations follow directly from the $q$-character explicit formulas given in Example 4.5.

For $\mathfrak{g} = \mathfrak{sl}_3$, we can prove the following explicit $q$-character formula (see the general result in Proposition 4.8):

$$\chi_q(X_{1,a}) = \left[\Psi^{-1}_1 \Psi_2 \right] \sum_{r \geq 0} (A_1,aq^{-2} \cdots A_1,aq^{-2(r-1)})^{-1}$$

and an analog formula for $\chi_q(X_{2,a})$.

For $\mathfrak{g} = B_2$, we can prove the following explicit $q$-character formula:

$$\chi_q(X_{2,a}) = \left[\Psi_2^{-1} \Psi_1 \right] \sum_{r \geq 0} (A_2,aq^{-2} \cdots A_2,aq^{-2(r-1)})^{-1}.$$ 

This gives some insights on the structure of the representation $X_{2,a}$: it has a basis of the $\ell$-weight vectors

$$X_{2,a} = \bigoplus_{T \in \mathcal{T}, r \geq 0} \mathbb{C} v_{T,r},$$

where $v_{T,r}$ has $\ell$-weight

$$\left[\Psi_2^{-1} \Psi_1 \right] \sum_{r \geq 0} (A_2,aq^{-2} \cdots A_2,aq^{-2(r-1)})^{-1}.$$ 

4.3. A $q$-character formula. Our Theorem 3.2 is a consequence of the following

Proposition 4.8. For any $i \in I$, $a \in \mathbb{C}^\times$, we have

$$\chi_q(X_{i,a}) = \left[\Psi_i \right] \chi_i a \prod_{j \neq i} \chi_j^{-C_{i,j}},$$

where

$$\chi_i a = \sum_{r \geq 0} (A_i,aq^{-2} \cdots A_i,aq^{-2(r-1)})^{-1}.$$ 

Remark 4.2. This explicit $q$-character formula implies

$$\chi(X_{i,a}) = \chi_i a \prod_{j \neq i} \chi_j^{-C_{i,j}}.$$ 

□
By using the automorphism $\tau_a$, it suffices to prove the formula for $a = 1$. For $m \in \mathbb{Z}$ we denote by $[m]$ its integer part. For $N \leq 0 < M$ let us set

$$
\widetilde{\Psi}_i^{(N,M)} = \widetilde{\Psi}_i \Psi_{i,q_i^{-2N}} \left( \prod_{j \mid C_i, j = -1} \Psi^{-1}_{j,q_i^{i+2r_j}+2r_j[1+(M-r_i)/(2r_j)]} \right) \cdot 
$$

$$
\cdot \left( \prod_{j \mid C_i, j = -2} \Psi^{-1}_{j,q_i^{4[1+M/4]}} \Psi^{-1}_{j,q_i^{6+4(M-2)/4}} \right) \cdot 
$$

$$
\cdot \left( \prod_{j \mid C_i, j = -3} \Psi^{-1}_{j,q_i^{4[1+6(M+1)/6]}} \Psi^{-1}_{j,q_i^{7+6(M-1)/6}} \Psi^{-1}_{j,q_i^{8+6(M-3)/6}} \right) \cdot 
$$

$$
= \left( \tilde{Y}_{i,q_i^{-1}} \tilde{Y}_{i,q_i^{-3}} \cdots \tilde{Y}_{i,q_i^{2N}} \right) \left( \prod_{j \mid C_i, j = -1} \tilde{Y}_{j,q_i^{r_i+r_j}} \tilde{Y}_{j,q_i^{r_i+3r_j}} \cdots \tilde{Y}_{j,q_i^{r_i+2r_j[(M-r_i)/(2r_j)]+r_j}} \right) \cdot 
$$

$$
\cdot \left( \prod_{j \mid C_i, j = -2} \left( \tilde{Y}_{j,q_i^{2}} \tilde{Y}_{j,q_i^{4}[4(M+1)/6]} \cdots \tilde{Y}_{j,q_i^{4}} \tilde{Y}_{j,q_i^{8}} \cdots \tilde{Y}_{j,q_i^{8+4(M-2)/4}} \right) \right) \cdot 
$$

$$
\cdot \left( \prod_{j \mid C_i, j = -3} \left( \tilde{Y}_{j,q_i^{6}} \tilde{Y}_{j,q_i^{12}} \cdots \tilde{Y}_{j,q_i^{6[1+(M-3)/6]}} \right) \right) \cdot 
$$

As $\widetilde{\Psi}_i^{(N,M)}$ is expressed as a product of variables $\tilde{Y}_{j,b}$, the representation $L(\widetilde{\Psi}_i^{(N,M)})$ is finite-dimensional (see Section 2.3). We will also consider the $\ell$-weight $\widetilde{\Psi}_i^{(M)}$ obtained from $\widetilde{\Psi}_i^{(N,M)}$ by removing the factors depending on $N$, that is

$$
\widetilde{\Psi}_i^{(M)} = \widetilde{\Psi}_i^{(N,M)} \Psi_{i,q_i^{-2N}}^{-1}. 
$$

As discussed in Section 4.1, we have

$$
\chi_q(L(\widetilde{\Psi}_i^{(N,M)})) \in [\tilde{\Psi}_i^{(N,M)}] \mathbb{Z}[A_{j,q_i}^{-1}]_{j \in I, r \in \mathbb{Z}}. 
$$

As moreover it follows from Theorem 4.4 that

$$
\chi_q(\Psi_{i,q_i^{-2N}}^{-1}) \in [\Psi_{i,q_i^{-2N}}^{-1}] \mathbb{Z}[A_{j,q_i}^{-1}]_{j \in I, r \in \mathbb{Z}}, 
$$

we get

$$
\chi_q(L(\widetilde{\Psi}_i^{(M)})) \in [\tilde{\Psi}_i^{(M)}] \mathbb{Z}[A_{j,q_i}^{-1}]_{j \in I, r \in \mathbb{Z}}. 
$$

Following [HL1], let us consider the truncated $q$-characters

$$
\chi_q^{<M}(L(\widetilde{\Psi}_i^{(N,M)})) \in \mathcal{E}_\ell \text{ and } \chi_q^{<M}(L(\widetilde{\Psi}_i^{(M)})) \in \mathcal{E}_\ell. 
$$
Remark 4.3. The first formula proves a particular case of [HL2, Conjecture 7.15].

Lemma 4.4. \[\sum \text{ is also simple (we can argue as in [H1, Proposition 5.3]). The n it is proved in [H1, Lemma 4.9].} \]

Proof. For the first formula, note that \(m(\Psi_i^{(N,M)})^{-1} \in \mathbb{Z}[A_{i,r}]_{i \in I, r < M}.\)

Lemma 4.9. We have:
\[
\chi^<_{q} (L(\Psi_i^{(N,M)})) = \left[\Psi_i^{(N,M)}\right] \sum_{0 \leq r \leq -N+1} (A_{i,1}A_{i,q_i}^{-2} \cdots A_{i,-2(r-1)})^{-1},
\]

\[
\chi^<_{q} (L(\Psi_i^{(M)})) = [\Psi_i^{(M)}]_{\chi_{i,1}}.
\]

Remark 4.3. The first formula proves a particular case of [HL2, Conjecture 7.15].

Proof. For the first formula, note that \(L(\Psi_i^{(N,M)})\) is a subquotient of
\[
L(\Psi_i^{(N,0)}) \otimes L(\Psi_i^{(0,M)}),
\]
where we set
\[
\Psi_i^{(0,M)} = \Psi_i^{(N,M)}(\Psi_i^{(N,0)})^{-1}.
\]
Here \(L(\Psi_i^{(N,0)})\) is a KR-module and \(L(\Psi_i^{(0,M)})\) is a tensor product of KR-modules which is also simple (we can argue as in [HL1 Proposition 5.3]). Then it is proved in [HL2 Lemma 4.4] that \(\Psi_i^{(0,M)}\) is the only \(\ell\)-weight in \(\chi_{q}(L(\Psi_i^{(0,M)}))\) which may occur in \(\chi^<_{q}(L(\Psi_i^{(0,M)}))\); that is,
\[
\chi^<_{q}(L(\Psi_i^{(0,M)})) = [\Psi_i^{(0,M)}].
\]
Consequently the \(\ell\)-weights occurring in \(\chi^<_{q}(L(\Psi_i^{(N,M)}))\) are of the form
\[
\Psi_i^{(N,M)} \Psi
\]
where \((\Psi_i^{(0,M)})^{-1}\) is an \(\ell\)-weight occurring in \(\chi_{q}(L(\Psi_i^{(N,0)}))\). As \(L(\Psi_i^{(N,0)})\) is a KR module, it is proved in [HL2, Section 4] that its \(q\)-character can be computed by using the algorithm introduced in [FM Section 5.5]. We also have precise information on the monomials occurring in its \(q\)-character in [HL2 Lemma 5.5]. In particular, we have the following: suppose that \(\Psi\) is not of the form
\[
(A_{i,1}A_{i,q_i}^{-2} \cdots A_{i,-2(r-1)})^{-1} \quad \text{for some } 0 \leq r \leq -N+1,
\]
that is it is not in the set denoted by \(B'\) in [HL2 Lemma 5.5]. Then there is an \(\ell\)-weight \(\Psi'\) occurring in \(\chi^<_{q}(L(\Psi_i^{(N,M)}))\) whose weight is of the form \(-r\alpha_i - \alpha_j\) for some \(0 \leq r \leq -N+1\) and some \(j \sim i\) in [HL2 Lemma 5.5] this is stated with \(j \neq i\), but the Frenkel-Mukhin algorithm mentioned above gives immediately that necessarily \(j \sim i\). Let \(r\) minimal with this property. There is \(\alpha \in \mathbb{Z}\) such that
\[
\Psi_i^{(N,M)} (A_{i,1}A_{i,q_i}^{-2} \cdots A_{i,-2(r-1)})^{-1} A_{j,q_0}^{-1}
\]
occurs as an \(\ell\)-weight in \(\chi_{q}(L(\Psi_i^{(N,M)}))\). But such an \(\ell\)-weight satisfies exactly the hypothesis of [H2 Theorem 5.1] which gives sufficient conditions so that a monomial do not
occur in the \( q \)-character of a simple module (here the \( i \) in \[12\] Theorem 5.1 is \( j, m \) is 
\( \tilde{\Psi}_i^{(N,M)}(A_{i,1}A_{i,q_2} \cdots A_{i,q_{2(r-1)}})^{-1} A_{i,q_1}^{-1} \) and \( M \) is \( mA_{j,q} \) up to a constant \( \ell \)-weight multiple). Hence we get a contradiction.

For the second formula, it follows from \[11\] Theorem 6.1] generalized in \[12\] Theorem 7.6] that we can take the limit \( N \to -\infty \), that is \( \tilde{\Psi}_i^{(N,M)} \) converges to \( (\tilde{\Psi}_i^{(M)})^{-1} \chi_q^M(L(\tilde{\Psi}_i^{(M)})) \) as a formal power series in the \( A_{j,b}^{-1} \).

\( \square \)

Consider the partial ordering \( \leq \) is defined on \( \mathcal{E}_\ell \) so that \( \chi \leq \chi' \) if the coefficients of \( \chi \) are lower than those of \( \chi' \).

**Lemma 4.10.** We have \( \chi_q(X_{i,1}) \preceq [\tilde{\Psi}_{i,1}]_1 i,1] \prod_{j \neq i} \chi_j^{-C_{i,j}}. \)

**Proof.** \( X_{i,1} \) is a subquotient of 
\[ L(\tilde{\Psi}_i^{(M)}) \otimes L(\tilde{\Psi}_{i,1})(\tilde{\Psi}_i^{(M)})^{-1}. \]

By Theorem \[17\] \( L(\tilde{\Psi}_{i,1})(\tilde{\Psi}_i^{(M)})^{-1} \) is a simple tensor product of positive prefundamental representations and 
\[ \chi_q(L(\tilde{\Psi}_{i,1})(\tilde{\Psi}_i^{(M)})^{-1}) = [\tilde{\Psi}_{i,1}(\tilde{\Psi}_i^{(M)})^{-1}] \prod_{j \neq i} \chi_j^{-C_{i,j}}. \]

This implies 
\[ [\tilde{\Psi}_{i,1}]_1 \chi_q(X_{i,1}) \leq [\tilde{\Psi}_{i,1}]_1 \chi_q(L(\tilde{\Psi}_i^{(M)}) \chi_q(L(\tilde{\Psi}_{i,1})(\tilde{\Psi}_i^{(M)})^{-1})). \]

This is true for any \( M > 0 \). For each \( \ell \)-weight \( \Psi \) in the left term, there is \( M \) such that no \( A_{j,q}^{-1} \) with \( r \geq M \) occurs as a factor in \( \Psi \). So this \( \ell \)-weight \( \Psi \) occurs only in the product with the truncated \( q \)-character 
\[ [(\tilde{\Psi}_i^{(M)})^{-1}] \chi_q^M(L(\tilde{\Psi}_i^{(M)})) \prod_{j \neq i} \chi_j^{-C_{i,j}} = \chi_{i,1} \prod_{j \neq i} \chi_j^{-C_{i,j}}. \]

\( \square \)

To conclude, we prove

**Lemma 4.11.** We have \( \chi_q(X_{i,1}) \preceq [\tilde{\Psi}_{i,1}]_1 \chi_q(L(\tilde{\Psi}_i^{(M)})) \).

**Proof.** Consider the representation 
\[ X_{i,1} \otimes L(\tilde{\Psi}_{i,1}^{-1}(\tilde{\Psi}_i^{-1})^{-1} \).

It admits \( L(\tilde{\Psi}_i^{-1}) \) as a simple constituent. By Theorem \[17\] \( L(\tilde{\Psi}_{i,1}) \) is a tensor product of negative prefundamental representations. Let \( \Psi' \) be an \( \ell \)-weight occurring in 
\[ \chi_{i,1} \preceq \Psi_{i,1} \chi_q(L(\tilde{\Psi}_i^{-1})). \]
Hence by (ii) in Theorem 4.4, $\Psi'$ is a product $M \overline{\Psi}_{i,1}^{-1}$ where $M$ is an $\ell$-weight of $\chi_q(X_{i,1})$. We get

$$\chi_q(X_{i,1}) \geq [\overline{\Psi}_{i,1}]\chi_{i,1}.$$ 

Note that for $\Psi'$ an $\ell$-weight in $\chi_{i,1}$, the product $\Psi_{i,1} \overline{\Psi}_{i,1} \Psi'$ is a monomial in the $\Psi_{j,a}$. So by Theorem 4.7 and (i) in Theorem 4.4, $L(\Psi_{i,1} \overline{\Psi}_{i,1} \Psi')$ is a simple tensor products of positive prefundamental representations and

$$\chi_q(L(\Psi_{i,1} \overline{\Psi}_{i,1} \Psi')) = [\Psi_{i,1} \overline{\Psi}_{i,1} \Psi'] \chi_i \prod_{j \neq i} \chi^{-C_{i,j}}_j.$$ 

Hence these simple modules are simple constituents of

$$[X_{i,1} \otimes L_{i,1}^+]$$

that is

$$[\Psi_{i,1}]\chi_i \chi_q(X_{i,1}) = \chi_q(X_{i,1}) \chi_q(L_{i,1}^+) \geq \chi_{i,1} [\Psi_{i,1} \overline{\Psi}_{i,1}] \chi_i \prod_{j \neq i} \chi^{-C_{i,j}}_j,$$

which implies the result. \hfill $\square$

4.4. Completion of the proof of Theorem 3.2. We can now complete the proof of Theorem 3.2.

Note that $C_{i,j} < -1$ implies $r_i = 1$. It suffices to prove that

$$\left( \prod_{j \mid C_{i,j} = -1} \Psi_{j,a}^{-1} \right) \left( \prod_{j \mid C_{i,j} = -2} \Psi_{j,aq^{-1}} \Psi_{j,a} \chi_i \prod_{j \mid C_{i,j} = -3} \Psi_{j,aq^{-2}} \Psi_{j,aq^2} \right) = [-\alpha_i] \Psi_{i,aq_i} \Psi_{i,aq_i}^{-1} \chi_i \prod_{j \mid C_{i,j} = -3} \Psi_{j,aq^{-2}} \Psi_{j,aq^2} \chi_i \prod_{j \mid C_{i,j} = -2} \Psi_{j,aq^{-3}} \Psi_{j,aq^3} \chi_i \prod_{j \mid C_{i,j} = -1} \Psi_{j,a}^{-1} \Psi_{j,aq^{-2}} \chi_i \prod_{j \mid C_{i,j} = -2} \Psi_{j,aq^{-1}} \Psi_{j,a} \chi_i \prod_{j \mid C_{i,j} = -3} \Psi_{j,aq^{-2}} \Psi_{j,aq^2} \chi_i \prod_{j \mid C_{i,j} = -1} \Psi_{j,a}^{-1},$$

that is

$$\chi_i \Psi_{i,aq_i}^{-1} = 1 + [-\alpha_i] \Psi_{i,aq_i} \Psi_{i,aq_i}^{-1} \chi_i \Psi_{i,aq_i}^{-3}$$

(4.14) $\times \left( \prod_{j \mid C_{i,j} = -1} \Psi_{j,a}^{-1} \Psi_{j,aq^{-2}} \right) \left( \prod_{j \mid C_{i,j} = -2} \Psi_{j,aq^{-1}} \Psi_{j,a} \right) \left( \prod_{j \mid C_{i,j} = -3} \Psi_{j,aq^{-2}} \Psi_{j,aq^2} \right) \left( \prod_{j \mid C_{i,j} = -1} \Psi_{j,aq^{-1}} \Psi_{j,a} \right) \left( \prod_{j \mid C_{i,j} = -2} \Psi_{j,aq^{-2}} \Psi_{j,aq^2} \right)$.

Note that $A_{i,aq_i}^{-1} [-\alpha_i] \Psi_{i,aq_i}^{-1} \Psi_{i,aq_i}^{-3}$ is equal to

$$\left( \prod_{j \mid C_{i,j} = -1} \Psi_{j,aq^{-1}} \Psi_{j,a} \Psi_{j,aq^{-2}} \right) \left( \prod_{j \mid C_{i,j} = -2} \Psi_{j,aq^{-1}} \Psi_{j,a} \right).$$
This is exactly the last factor in Equation (5.11):

- if \( r_j = 1 \), then \( C_{i,j} = -1 \) and both factors are equal to \( \Psi_{j,a}^{-1} \Psi_{j,aq^2}^{-1} \).
- if \( r_j = 2 \) and \( r_i = 1 \), then \( C_{i,j} = -2 \), \( C_{j,i} = -1 \) and both factors are equal to \( \Psi_{j,a}^{-1} \Psi_{j,aq}^{-3} \).
- if \( r_j = r_i = 2 \), then \( C_{i,j} = C_{j,i} = -1 \) and both factors are equal to \( \Psi_{j,a}^{-1} \Psi_{j,aq^{-4}} \).
- if \( r_j = 3 \) and \( r_i = 1 \), then \( C_{i,j} = -3 \), \( C_{j,i} = -1 \) and both factors are equal to \( \Psi_{j,a}^{-1} \Psi_{j,aq^{-4}} \).

We get the desired result because

\[
\chi_{i,aq_i^{-1}} = 1 + A^{-1}_{i,aq_i^{-1}} \chi_{i,aq_i^{-3}}.
\]

\[
\square
\]

5. BETHE ANSATZ

We now derive the Bethe Ansatz equations from the \( Q\tilde{Q} \)-system [3.8], following [MRV1, MRV2]. We focus of the case of untwisted affine algebras, but one can obtain the Bethe Ansatz equations for the twisted affine algebras from the \( Q\tilde{Q} \)-system (3.9) in a similar way.

Suppose that we have an action of the commutative algebra \( K_0(\mathcal{O}) \) on a vector space \( V \), and let \( v \) be one of its joint eigenvectors. Then we obtain an algebra homomorphism from \( K_0(\mathcal{O}) \) to \( \mathbb{C} \). Let us denote the values of the elements \( Q_{i,u} \) and \( \tilde{Q}_{i,u} \) of \( K_0(\mathcal{O}) \) under this homomorphism by \( Q_i(u) \) and \( \tilde{Q}_i(u) \), respectively. Depending on the space \( V \), these functions will have different analytic properties.

In addition, under any homomorphism from \( K_0(\mathcal{O}) \) to \( \mathbb{C} \), we have

\[
\left[ \frac{\pm \Omega_i}{2} \right] \mapsto v_i^{\pm 1}
\]

for some \( v_i \in \mathbb{C}^\times \), for all \( i \in I \). (Note that what we denoted by \( v_i \) in [FH] corresponds to \( v_i^2 \) here; however, that \( v_i \) was a formal variable in [FH], whereas here it is a non-zero complex number.)

The relations in (3.22) then give rise to algebraic relations between these functions:

\[
(5.15) \quad v_i Q_i(uq_i^{-1})\tilde{Q}_i(uq_i) - v_i^{-1} Q_i(uq_i)\tilde{Q}_i(uq_i^{-1})
= \left( \prod_{j|C_{i,j} = -1} Q_j(u) \right) \left( \prod_{j|C_{i,j} = -2} Q_j(uq^{-1})Q_j(uq) \right) \left( \prod_{j|C_{i,j} = -3} Q_j(uq^{-2})Q_j(u)Q_j(uq^2) \right).
\]

Now suppose that \( w \) is a zero of \( Q_i(u) \) that is not a zero of \( \tilde{Q}_i(u) \) and that the terms in formula (5.15) have no poles when \( u = wq_i^{\pm 1} \) (we will refer to this as a genericity condition). Substituting \( u = wq_i^{\pm 1} \) into (5.15) and taking the ratio of the resulting equations, we obtain:

\[
(5.16) \quad v_i^{-2} \prod_{j \in I} \frac{Q_j(wq^{B_{ij}})}{Q_j(wq^{-B_{ij}})} = -1,
\]

where \( (B_{ij}) \) is the symmetrized Cartan matrix, \( B_{ij} = (\alpha_i, \alpha_j) \). These are the Bethe Ansatz equations.
Thus, under the genericity condition, the zeros of $Q_i(u)$ must satisfy the Bethe Ansatz equations (5.16).

In Section 5.6 of [FH] (see also Section 6 of [FR]) we obtained these equations in the case that $V$ is the tensor product of irreducible finite-dimensional representations of $U_q(\hat{g})$ and the action of $K_0(\mathcal{O})$ on $V$ is obtained using the standard transfer-matrix construction.

If we switch to the dual category $\mathcal{O}^*$, so that $Q_{i,u}$ becomes $[R_{i,u}^+]$, as explained in Remark 3.2(iii), then the corresponding Bethe Ansatz equation (5.16) is equivalent to formula (5.8) of [FH] (note that an overall minus sign is missing in that formula). This case is special in that for a given $V$, any eigenvalue of the transfer-matrix of $R_{i,u}^+$ on $V$ has the form $Q_i(u) = f_i(u)Q_i(u)$, where $f_i(u)$ is a universal factor that depends only on $V$ and $i$, and $Q_i(u)$ is a polynomial (this is an analogue of the Baxter polynomial). Thus, the analytic behavior of $Q_i(u)$ has a very special form in this case.

Though we did not prove it in [FH], we do expect that the eigenvalues of the transfer-matrix of $L_{i,u}^+$ have the same general form as those of the transfer-matrix of $R_{i,u}^+$. If this is indeed the case, then equations (5.8) of [FH] may also be viewed as the equations on the zeros of the generalized Baxter polynomials occurring in the eigenvalues of the transfer-matrix of $L_{i,u}^+$ on $V$.

However, the derivation of the Bethe Ansatz equations presented in [FH] (following the analytic Bethe Ansatz method [R1, R2, R3, BR, KS1]) is much less direct than the derivation presented in this section. Indeed, the argument of [FH] started with the formula expressing the eigenvalues of the transfer-matrix of a finite-dimensional representation $W$ of $U_q(\hat{g})$ in terms of the eigenvalues of the transfer-matrices of $R_{i,u}^+$ (or $L_{i,u}^+$), see Theorem 5.11 of [FH] (these are the analogues of Baxter’s $TQ$-relation). If we make a specific assumption about how poles get canceled in this formula (namely, that the cancellation happens between the terms in the formula corresponding to the monomials $M$ and $MA_i^{-1}q_i$ from the $q$-character), then we obtain the above Bethe Ansatz equations (5.16), see Section 5.8 of [FH] for details. In contrast, in our present argument we immediately get the Bethe Ansatz equations under a mild genericity condition.

The non-local quantum KdV Hamiltonians give us (conjecturally, see Section 7.2 below) another way to construct an action of $K_0(\mathcal{O})$, as explained in Section 7. In this case, $V$ is a graded component in a representation of a $W$-algebra associated to $\hat{g}$; for example, a Fock representation. Under the same genericity condition, the zeros of the corresponding eigenvalues $\tilde{Q}_i(u)$ satisfy Bethe Ansatz equations (5.16) (with specific values of $v_i$). Note that for $\hat{sl}_2$ this was shown in [BLZ5] using the quantum Wronskian relation, to which the $Q\tilde{Q}$-system reduces in the case of $\hat{sl}_2$ (see Section 3.2). In this case, the function $Q_1(u)$ is expected to be an entire function of $u$, see [BLZ4, BLZ5].

We close this section with two remarks. First, the Bethe Ansatz equations (5.16) can be derived, in a similar fashion, from the $QQ^*$-system of [HL2] (see Section 3.3), under an assumption that is similar to the above genericity condition.

Second, for the quantum $g_l$ toroidal algebra, a system of relations in $K_0(\mathcal{O})$ was established in [FJMM]. It could be viewed as an analogue of the $QQ^*$-system of [HL2] (for the analogue of the $Q\tilde{Q}$-system, see Section 3.5 above). In [FJMM], Bethe Ansatz equations were derived from that system in a similar fashion. However, the authors of [FJMM] went
a step further: they proved that the analogue of the above genericity assumption is in fact not necessary. This gives us hope that the genericity assumption can be dropped in the affine case as well, for both the $QQ^*$-system and the $\tilde{Q}\tilde{Q}$-system.

Finally, we can derive the Bethe Ansatz equations of [FJMM] from the $\tilde{Q}\tilde{Q}$-system for the quantum $\hat{g}_1$ toroidal algebra from Section 3.5 under the genericity assumption. Namely, if $w$ is a zero of $Q(z)$ which is not a zero of $\tilde{Q}(z)$, then we get the following Bethe Ansatz equation:

$$Q(wq_1)Q(wq_2)Q(wq_3) + [\alpha]Q(wq_1^{-1})Q(wq_2^{-1})Q(wq_3^{-1}) = 0.$$  

It coincides with the Bethe Ansatz equation obtained in a different way in [FJMM].

6. Classical KdV system

In the rest of this paper, we discuss the affine opers that should encode the eigenvalues of the quantum $\hat{g}$-KdV Hamiltonians according to the conjecture of [FF5]. We start by recalling the definition of the classical KdV systems and opers.

6.1. Drinfeld–Sokolov reduction and opers. The phase space of the classical $\hat{g}$-Kdv system is obtained from a certain space of first order differential operators by Hamiltonian reduction, which is called the Drinfeld–Sokolov reduction [DS]. We will first discuss the case of an untwisted affine algebra $\hat{g}$, by which we mean the universal central extension of the formal loop algebra $g((t))$:  

$$0 \rightarrow \mathbb{C}1 \rightarrow \hat{g} \rightarrow g((t)) \rightarrow 0$$

(we are slightly abusing notation here, because in our discussion of the quantum affine algebras $\hat{g}$ stands for the Laurent polynomial version). The commutation relations read:

$$[1, A(t)] = 0 \text{ and } \quad [A(t), B(t)] = [A(t), B(t)] - \text{Res}_{t=0} \kappa_0(A(t), dB(t)),$$

where $\kappa_0$ is the invariant inner product on $g$ normalized in the standard way, so that the square length of the maximal root is equal to 2.

Consider the space of differential operators

$$\partial_t + A(t), \quad A(t) \in g((t)),$$

The inner product

$$\langle A(t), B(t) \rangle = \text{Res}_{t=0} \kappa_0(A(t), B(t)) dt$$

enables us to identify $g((t))$ with its dual space. It is known (see, e.g., [FB], Ch. 16.4) that under this identification, the space of differential operators (6.17) may be identified with a hyperplane in the dual space to $\hat{g}$ that consists of all linear functionals on $\hat{g}$ taking value 1 on the central element 1. The standard Kirillov–Kostant Poisson structure on the dual space to $\hat{g}$ restricts to a Poisson structure on the hyperplane. So do the coadjoint actions of the group $G((t))$ and its Lie algebra $g((t))$, and when written in terms of the operators (6.17), they become the gauge actions of $G((t))$ and $g((t))$, respectively.

Fix the Cartan decomposition

$$g = n_+ \oplus \mathfrak{h} \oplus n_-,$$
where $n_+$ and $n_-$ are the upper and lower nilpotent subalgebras of $g$, respectively, and $\mathfrak{h}$ is the Cartan subalgebra. The above inner product on $g((t))$ identifies the dual space to $n_+(\langle t \rangle)$ with $n_-(\langle t \rangle)$. Let $f_i, i = 1, \ldots, n$, be generators of $n_-$ corresponding to negative simple roots of $g$. Consider the Hamiltonian reduction of the space of the operators (6.17) with respect to the gauge (that is, coadjoint, hence Poisson) action of the Lie algebra $n_+(\langle t \rangle)$ and its character (that is, a one-point coadjoint orbit in $n_-(\langle t \rangle)$) corresponding to the element

$$p_{-1} = \sum_{i=1}^{n} f_i \in n_- \subset n_-(\langle t \rangle) = n_+(\langle t \rangle)^*.$$  

This is the Drinfeld–Sokolov reduction [DS].

The reduced phase space of the Drinfeld–Sokolov reduction is therefore the quotient of the space $\tilde{M}(\hat{g})$ of operators of the form

$$\partial_t + p_{-1} + v(t), \quad v(t) \in b_+(\langle t \rangle),$$

where $b_+ = h \oplus n_+$ is the Borel subalgebra of $g$, under the gauge action of the loop group $N_+(\langle t \rangle)$.

According to [DS], the action of $N_+(\langle t \rangle)$ on $\tilde{M}(\hat{g})$ is free. The resulting quotient space

$$M(g) = \tilde{M}(\hat{g})/N_+(\langle t \rangle)$$

is called the space of $g$-opers on the punctured disc $D^\times = \text{Spec} \mathbb{C}((t))$ (for a general curve, the space of $g$-opers has also been defined by Beilinson and Drinfeld [BD1, BD2]). The Poisson algebra of local functionals on $M(g)$ is known as the classical $W$-algebra. We denote it by $\mathcal{W}(g)$.

For example, for $g = sl_2$ we have

$$p_{-1} = f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and so $M(sl_2)$ is the quotient of the space of operators of the form

$$\partial_t + \begin{pmatrix} a(t) & b(t) \\ 1 & -a(t) \end{pmatrix}, \quad a(t), b(t) \in \mathbb{C}((t)),$$

by the upper triangular gauge transformations depending on $t$. It is easy to see that each gauge equivalence class contains a unique operator of the form

$$\partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}, \quad v(t) \in \mathbb{C}((t)),$$

and hence we may identify $M(sl_2)$ with the space of such operators, or, equivalently, with the space of second order differential operators

$$\partial_t^2 - v(t), \quad v(t) \in \mathbb{C}((t)).$$

Likewise, the space $M(sl_r)$ may be identified with the space of $n$th order differential operators

$$\partial_t^n - v_1(t)\partial_t^{n-2} + \ldots + (-1)^{r} v_{r-2}(t)\partial_t - (-1)^{r} v_{r-1}(t).$$

In a similar way, for Lie algebras of types $B$ and $C$ one can identify $g$-opers with self-adjoint and anti-self adjoint scalar differential operators, and for type $D$, pseudo-differential
operators of a special kind \([DS]\). However, there is no such uniform identification for a general Lie algebra \(g\). The best we can do in general is to choose special representatives in the \(N_+((t))\)-gauge equivalence classes on the space of operators of the form \([6.19]\) in the following way.

Recall the element \(\overline{p}_{-1} \in n_-\) given by formula \([6.18]\). There exists a unique element of \(n_+\) of the form

\[
\overline{p}_1 = \sum_{i=1}^n c_i e_i, \quad c_i \in \mathbb{C},
\]

where \(e_i, i = 1, \ldots, n\), are generators of \(n_+\), such that \(\overline{p}_1, \overline{p}_{-1}, \) and \(\overline{p}_0 = [\overline{p}_1, \overline{p}_{-1}]\) form an \(\mathfrak{sl}_2\) triple. The element \(\frac{1}{2}\overline{p}_0 \in \mathfrak{h}\) then defines the principal grading on \(g\) such that \(\deg \overline{p}_1 = 1, \deg \overline{p}_{-1} = 1\). Let

\[
V_{\text{can}} = \bigoplus_{i \in E} V_{\text{can}, i}
\]

be the space of \(\text{ad} \overline{p}_1\)-invariants in \(n_+\), decomposed according to the principal grading. Here

\[
E = \{d_1, \ldots, d_n\}
\]

is the set of exponents of \(g\). Then \(\overline{p}_1\) spans \(V_{\text{can}, 1}\). Choose a linear generator \(\overline{p}_j\) of \(V_{\text{can}, d_j}\) (if the multiplicity of \(d_j\) is greater than one, which happens only in the case \(\hat{g} = D_{2n}^{(1)}, d_j = 2n\), then we choose linearly independent vectors in \(V_{\text{can}, d_j}\) ). The following result is due to Drinfeld and Sokolov \([DS]\) (see also \([BD1, BD2]\)).

**Lemma 6.1.** The gauge action of \(N_+((t))\) on the space \(\widetilde{M}(g)\) is free, and each gauge equivalence class contains a unique operator of the form \(\partial_t + \overline{p}_{-1} + v(t)\), where \(v(t) \in V_{\text{can}}((t))\), so that we can write

\[
v(t) = \sum_{j=1}^n v_j(t) \cdot \overline{p}_j, \quad v_j(t) \in \mathbb{C}((t)).
\]

Thus, each point of the reduced phase space \(M(g)\) of the Drinfeld–Sokolov reduction is canonically represented by an operator \(\partial_t + \overline{p}_{-1} + v(t)\), where \(v(t)\) is of the form \([6.20]\).

**6.2. Spectral parameter.** Now we insert the “spectral parameter” \(z\) into our operators. This means that we go from \(g\) to \(g((z^{-1}))\), and from \(g((t))\) to \(g((z^{-1}))((t))\). Let

\[
f_0 = e_\theta z \in g((z^{-1})), \quad \text{where} \quad e_\theta \in n_+ \subset g
\]

is a non-zero element in the one-dimensional weight subspace of the nilpotent subalgebra \(n_+\) of \(g\) corresponding to the maximal root (this is a highest weight vector in the adjoint representation of \(g\)). For instance, if \(g = \mathfrak{sl}_r\), we can take as \(e_\theta\) the matrix with 1 in the upper right corner and 0 in all other places.

Note that \(f_i, i = 0, \ldots, n\), are the generators of the lower nilpotent subalgebra \(\tilde{n}_-\) of \(g((z^{-1}))\), which consists of all elements of \(g[z]\) whose value at \(z = 0\) is in \(n_- \subset g\). Therefore,

\[
p_{-1} = \sum_{i=0}^n f_i
\]

may be viewed as a “principal nilpotent element” of \(\tilde{n}_-\).
Now we shift the operators \(6.17\) by \(f_0 = e_\theta z\). We then obtain the following operators:
\[
\partial_t + A(t) + e_\theta z, \quad A(t) \in \mathfrak{g}(t),
\]
Since \(f_0 = e_\theta z\) is stable under the action of \(N_+(t)\), this shift does not change the gauge action of \(N_+(t)\). Therefore, we can identify the reduced phase space \(M(\mathfrak{g})\) of the Drinfeld–Sokolov reduction with the quotient of the space \(\hat{M}(\mathfrak{g})\) of operators of the form
\[
\partial_t + p_{-1} + v(t), \quad v(t) \in \mathfrak{b}_+(t),
\]
by the gauge action of the loop group \(N_+(t)\).

According to Lemma \(6.1\), each gauge equivalence class contains a unique operator of the form
\[
\partial_t + p_{-1} + v(t), \quad v(t) \in \mathcal{V}_{\text{can}}(t).
\]
We will denote this quotient by \(M(\hat{\mathfrak{g}})\). It is isomorphic to the space \(M(\mathfrak{g})\), but its elements are differential operators with "spectral parameter" \(z\) that we need to construct the \(\hat{\mathfrak{g}}\)-KdV Hamiltonians.

For example, \(M(\hat{\mathfrak{sl}}_2)\) is the quotient of the space of operators of the form
\[
\partial_t + \begin{pmatrix} a(t) & b(t) + z \\ 1 & -a(t) \end{pmatrix}, \quad a(t), b(t) \in \mathbb{C}(t),
\]
by the upper triangular gauge transformations depending on \(t\) (but not on \(z\)). Each gauge equivalence class contains a unique operator of the form
\[
\partial_t + \begin{pmatrix} 0 & v(t) + z \\ 1 & 0 \end{pmatrix}, \quad v(t) \in \mathbb{C}(t),
\]
and hence we may identify \(M(\hat{\mathfrak{sl}}_2)\) with the space of such operators, or, equivalently, with the space of second order differential operators with spectral parameter
\[
\partial_t^2 - v(t) - z, \quad v(t) \in \mathbb{C}(t).
\]
Likewise, the space \(M(\hat{\mathfrak{sl}}_r)\) may be identified with the space of \(n\)th order differential operators with spectral parameter
\[
\partial_t^r - v_1(t)\partial_t^{r-2} + \ldots + (-1)^r v_{n-2}(t)\partial_t - (-1)^r v_{r-1}(t) - (-1)^r z.
\]

The reason why inserting the spectral parameter is important is that after we do that we can define the \(\hat{\mathfrak{g}}\)-KdV Hamiltonians. These are Poisson commuting functions (more properly, functionals) on the reduced phase space \(M(\hat{\mathfrak{g}}) = M(\mathfrak{g})\). There are two types of \(\hat{\mathfrak{g}}\)-KdV Hamiltonians: local and non-local, and they are both constructed using the formal monodromy matrix of the operators \(6.19\) specialized to different representations of \(\mathfrak{g}\). Because our operators now depend on \(z\), the monodromy matrix depends on \(z\) as well, and this enables us to take the coefficients of its expansion.

More precisely, let \(M_V(z) \in G\) be the monodromy matrix of the operator \(6.21\) specialized to an irreducible representation \(V\) of \(\mathfrak{g}\) (see Section 3.2 of [FF5] for the precise definition). For any \(\mathfrak{g}\)-invariant function \(\varphi\) on \(V\) the corresponding function \(\hat{H}_\varphi(z) = \varphi(M_V(z))\) on \(\hat{M}(\mathfrak{g})\) is invariant under the gauge action of \(N_+(t)\) and hence gives rise to a well-defined function \(\hat{H}_\varphi(z)\) on the quotient \(\hat{M}(\mathfrak{g})\).
The asymptotic expansion of $H_\varphi(z)$ at $z = \infty$ yields the local $\hat{g}$-KdV Hamiltonians, which generate the $\hat{g}$-KdV hierarchy of commuting Hamiltonian flows on the Poisson manifold $\mathcal{M}(\hat{g})$ (see [DS]). These Hamiltonians have the form

\begin{equation}
H_s = \int P_s(v_j(t), v'_j(t), \ldots) dt, \quad s = d_i + Nh, \quad N \in \mathbb{Z}_+,
\end{equation}

where $d_i \in E$ is an exponent of $\hat{g}$ and $h$ is the Coxeter number. The integrand $P_s$ is a differential polynomial of degree $s + 1$, where we set $\text{deg} v^{(m)}_j = d_j + m + 1$.

On the other hand, the $z$-expansion of $H_\varphi(z)$ at $z = 0$ yields Poisson commuting non-local $\hat{g}$-KdV Hamiltonians. Poisson commutativity of these Hamiltonians is easily proved from the commutativity of the functions $H_\varphi(z)$ (see, for example, [RS1, RSF, RS2]).

6.3. Miura transformation. A convenient way to compute the higher order terms in the $z$-expansion of $\varphi(M_V(z))$ is to realize the variables of the KdV hierarchy in terms of the variables of the modified KdV (mKdV) hierarchy. This provides a kind of “free field realization,” also known as the Miura transformation, for the commuting Hamiltonians.

Consider the space $\overline{\mathcal{M}(\hat{g})}$ of operators of the form

\begin{equation}
\partial_t + p_{-1} + u(t), \quad u(t) \in \mathfrak{h}(\mathfrak{t})).
\end{equation}

The natural map $\overline{\mathcal{M}(\hat{g})} \to \mathcal{M}(\hat{g})$ given by the composition of the inclusion $\overline{\mathcal{M}(\hat{g})} \to \hat{\mathcal{M}(\hat{g})}$ and the projection $\mathcal{M}(\hat{g}) \to \hat{\mathcal{M}(\hat{g})}$ is called the Miura transformation (see [DS]). It is a Poisson map with respect to the Heisenberg–Poisson structure on $\overline{\mathcal{M}(\hat{g})}$ and the Drinfeld–Sokolov Poisson structure on $\mathcal{M}(\hat{g})$. Therefore it gives rise to an embedding of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ into the Heisenberg–Poisson algebra of functions on $\overline{\mathcal{M}(\hat{g})}$.

Because the operator (6.25) has such a simple structure, it is easier to compute the monodromy matrix $M_V(z)$, and hence the functions $\varphi(M_V(z))$, for it rather than for the operators of the form (6.22). The coefficients of the asymptotic expansion of the function $\varphi(M_V(z))$ are the local Hamiltonians of the modified KdV (or mKdV) hierarchy associated to $\hat{g}$. They are connected to the above $\hat{g}$-KdV Hamiltonians by the Miura transformation. On the other hand, the coefficients in the $z$-expansion of $\varphi(M_V(z))$ are the non-local $\hat{g}$-mKdV Hamiltonians.

For example, in the case when $\mathfrak{g} = \mathfrak{sl}_2$ the operator (6.25) has the form

$$
\partial_t + \begin{pmatrix}
  u(t) & z \\
  1 & -u(t)
\end{pmatrix}.
$$

The coefficients in the $z$-expansion of the trace of the monodromy of this operator are written down explicitly in [BLZ1]. They are given by multiple integrals of $\exp(\pm 2\phi(t))$, where $\phi(t)$ is the anti-derivative of $u(t)$, that is, $u(t) = \phi'(t)$ (these are classical screening operators, see [FF4]).

Similar formulas may be obtained for other affine Kac–Moody algebras.

6.4. Twisted affine algebras. Finally, we consider the case of a twisted affine Kac–Moody algebra $\hat{\mathfrak{g}}$. We recall that it is constructed from a finite-dimensional simple Lie algebra $\mathfrak{g'}$ whose Dynkin diagram has an automorphism of order $r = 2$ or 3. Using the Cartan
generators of \( g' \), we obtain an automorphism of \( g' \) of the same order denoted by \( \sigma \). The Lie algebra \( g' \) decomposes into a direct sum of eigenspaces of \( \sigma \):

\[
g' = \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} g'_i,
\]

where \( g'_0 \) is the simple Lie algebra corresponding to the quotient of the Dynkin diagram of \( g' \) by the action of the automorphism. The twisted affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) is defined as the universal central extension of the twisted loop algebra \( \mathcal{L}_\sigma \mathfrak{g} \), which is the completion of the Lie algebra

\[
\bigoplus_{n \in \mathbb{Z}} g'_n \otimes z^n
\]
in \( g'((z^{-1})) \).

Let \( f_i, i = 1, \ldots, n \), be the generators of the lower nilpotent subalgebra of \( g'_n \), and \( f_0 = e_{\theta_0} z \), where \( e_{\theta_0} \) is a non-zero generator of the one-dimensional highest weight subspace of the \( g_0 \)-module \( \mathfrak{g} \) (this is the twisted affine algebra analogue of the element \( e_{\theta} z \)). We denote its weight (from the point of view of the Cartan subalgebra of \( g_0 \)) by \( \theta_0 \). The element

\[
p_{-1} = \sum_{i=0}^{n} f_i
\]
is then the “principal nilpotent element” of the twisted affine Kac–Moody algebra \( \hat{\mathfrak{g}} \). Therefore the analogues of the operators \((6.22)\) are the operators of the form

\[
\partial_t + p_{-1} + \mathbf{v}(t), \quad \mathbf{v}(t) \in \mathfrak{b}_{\mathfrak{n}} \cap \mathfrak{g}_{\mathfrak{l}}
\]
where \( \mathfrak{b}_{\mathfrak{n}} \cap \mathfrak{g}_{\mathfrak{l}} \) is the Borel subalgebra of \( g'_n \). We denote the space of operators \((6.26)\) by \( \tilde{\mathcal{M}}(\hat{\mathfrak{g}}) \).

Next, we take the quotient of the space \( \tilde{\mathcal{M}}(\hat{\mathfrak{g}}) \) by the gauge action of the loop group \( N_{\mathfrak{n}}((t)) \), where \( N_{\mathfrak{n}} \) is the unipotent Lie group corresponding to the Lie algebra \( \mathfrak{n}_{\mathfrak{n}} = \mathfrak{n}_{\mathfrak{n}} + \mathfrak{g}_{\mathfrak{l}} \).

Since \( f_0 \) is invariant under the action of \( N_{\mathfrak{n}}((t)) \), we can remove \( f_0 \) from \((6.26)\) (in the same way as in the untwisted case); that is, we can replace \( p_{-1} \) by the element \( \sum_{i=1}^{n} f_i \) of \( g'_n \). The resulting space consists of the operators of the form

\[
\partial_t + \sum_{i=1}^{n} f_i + \mathbf{v}(t), \quad \mathbf{v}(t) \in \mathfrak{b}_{\mathfrak{n}} \cap \mathfrak{g}_{\mathfrak{l}}
\]
and hence coincides with the space \( \tilde{\mathcal{M}}(g'_n) \) that we considered in Section \( 6.1 \) when we discussed the untwisted case. Therefore the quotient of \( \tilde{\mathcal{M}}(\hat{\mathfrak{g}}) \) by the gauge action of \( N_{\mathfrak{n}}((t)) \) is nothing but the reduced space \( \mathcal{M}(g'_n) \) arising in the Drinfeld–Sokolov reduction of the untwisted affine algebra \( \hat{\mathfrak{g}} \) which is the central extension of the loop algebra \( g'_n((z^{-1})) \). The reduced phase space is therefore the same as the one which we get in the case of the untwisted affine algebra \( \hat{\mathfrak{g}} \), and so the corresponding Poisson algebra is nothing but the classical \( \mathcal{W} \)-algebra \( \mathcal{W}(g'_n) \).

However, we now insert the spectral parameter differently, by shifting our operators by \( e_{\theta_0} z \) (rather than \( e_{\theta} z \)). Then, using Lemma \( 6.1 \) we can realize the space \( \tilde{\mathcal{M}}(g'_n) \) as the space
of differential operators of the form
\begin{equation}
\partial_t + \mathcal{P}_{-1} + e_{\theta_0}z + \mathbf{v}(t), \quad \mathbf{v}(t) \in V_{\text{can}}((t)).
\end{equation}
That is to say, we “insert” the element \( f_0 = e_{\theta_0}z \in \mathfrak{g}'_1z \) of the twisted loop algebra \( \mathcal{L}_a \mathfrak{g}' \) rather than the element \( e_{\theta}z \in \mathfrak{g}'_0(z^{-1}) \) (see formula (6.23)).

We then construct the invariants of the monodromy matrices for these operators in the same way as in the untwisted case. Note, however, that these monodromy matrices are different from the monodromy matrices for the operators (6.23) corresponding to the untwisted affine algebra \( \hat{\mathfrak{g}}' \), even though both may be viewed as functionals on the same reduced phase space \( \mathcal{M}(\mathfrak{g}'_0) \). The expansions of the invariants of the monodromy matrices at \( z = \infty \) and \( z = 0 \) give rise to the local and non-local classical Hamiltonians, respectively, for the KdV system corresponding to the twisted affine algebra \( \hat{\mathfrak{g}} \) we started with. These Hamiltonians are different from those defined for the operators (6.23) corresponding to the untwisted affine algebra \( \mathfrak{g}'_0(z^{-1}) \) even though in both cases they are functionals on the same reduced phase space \( \mathcal{M}(\mathfrak{g}'_0) \).

7. Quantum KdV System

7.1. Local Hamiltonians. KdV Hamiltonians can be quantized. First, let’s consider the problem of quantization of the classical local \( \hat{\mathfrak{g}} \)-KdV Hamiltonians. They are Poisson commuting elements of the classical \( \mathcal{W} \)-algebra \( \mathcal{W}(\mathfrak{g}) \); namely, the Poisson algebra of local functionals on the space \( \mathcal{M}(\hat{\mathfrak{g}}) \) discussed in the previous section. It has been shown in [FF4] that \( \mathcal{W}(\mathfrak{g}) \) can be quantized; that is, there exists a one-parameter associative algebra \( \mathcal{W}_\beta(\mathfrak{g}) \) whose limit as \( \beta \to 0 \) is a commutative algebra with a natural Poisson structure that is isomorphic to \( \mathcal{W}(\mathfrak{g}) \). For example, \( \mathcal{W}_\beta(\mathfrak{sl}_2) \) is a completed enveloping algebra of the Virasoro algebra.

More precisely, in [FF4] it was shown that \( \mathcal{W}_\beta(\mathfrak{g}) \) may be defined as a subalgebra in a Heisenberg algebra. Here by “Heisenberg algebra” we mean a completion of the universal enveloping algebra of the Heisenberg Lie subalgebra \( \mathfrak{h} \) of \( \hat{\mathfrak{g}} \) (central extension of the formal loop algebra \( \mathfrak{h}((t)) \)) in which the central element is identified with the identity. The \( \mathcal{W} \)-algebra \( \mathcal{W}_\beta(\mathfrak{g}) \) is defined for generic \( \beta \) (i.e., such that \( \beta^2 \) is not a rational number) as the intersection of the kernels of the so-called screening operators, which depend on \( \beta \), associated to the simple roots of \( \mathfrak{g} \) (see [FF4, Section 4.6] or [FB, Section 15.4.11]). The resulting algebra is then extended to all complex values of \( \beta \). This implies, in particular, that each Fock representation \( \pi_\mu \) of \( \mathfrak{h} \) (where \( \mu \in \mathfrak{h}^* \) is the highest weight) is naturally a module over \( \mathcal{W}_\beta(\mathfrak{g}) \). In fact, \( \mathcal{W}_\beta(\mathfrak{g}) \) is first defined as a vertex subalgebra of the Heisenberg vertex algebra \( \pi_0 \), and then as an associative algebra corresponding to this vertex algebra (see [FF4] and [FB, Chapter 15]).

In the limit \( \beta \to 0 \) we obtain an embedding of \( \mathcal{W}(\mathfrak{g}) \) (viewed as a Heisenberg–Poisson algebra) into the Heisenberg–Poisson algebra of functions on the space \( \overline{\mathcal{M}(\hat{\mathfrak{g}})} \) discussed in the previous section. This embedding is induced by the Miura transformation \( \overline{\mathcal{M}(\hat{\mathfrak{g}})} \to \mathcal{M}(\hat{\mathfrak{g}}) \) discussed in the previous section. Thus, the embedding of the quantum \( \mathcal{W} \)-algebra \( \mathcal{W}_\beta(\mathfrak{g}) \) into the Heisenberg algebra may be viewed as a quantization of the Miura transformation.
Alternatively, the quantum $W$-algebra $W_\beta(g)$ may be defined via the quantum Drinfeld–Sokolov reduction, see [FF2, FF3] and [FB, Chapter 15] (the equivalence between the two constructions of $W_\beta(g)$ is discussed in [FB, Chapter 15.4]).

Now, by a quantization of local $\hat{g}$-KdV Hamiltonians we understand a commutative subalgebra of $W_\beta(g)$ whose limit as $\beta \to 0$ is the Poisson commutative subalgebra of $W(g)$ generated by the classical local KdV Hamiltonians (6.24). (If $\hat{g}$ is a twisted affine algebra, then by $g$ we mean here the Lie algebra $g_0$ as in the previous section.) The existence of this quantization is a non-trivial statement, which has been proved in [FF4]. Local quantum $\hat{g}$-KdV Hamiltonians are elements $\hat{H}_s$ of this commutative subalgebra which are quantizations of the $H_s$ given by formula (6.24) in the sense that $\hat{H}_s$ tends to $H_s$ when $\beta \to 0$, if we rescale the generators of $W_\beta(g)$ in the appropriate way (see [FF4] for details).

If we apply the quantum Miura transformation to the local quantum $\hat{g}$-KdV Hamiltonians $\hat{H}_s$, we obtain the corresponding local quantum $\hat{g}$-mKdV Hamiltonians. Those may also be viewed as quantum integrals of motion of the affine Toda field theory associated to $\hat{g}$. The latter is equivalent to saying that they commute with the screening operators associated to the simple roots of $\hat{g}$ (see [FF4] for details).

The commutative algebra of local quantum $\hat{g}$-KdV Hamiltonians acts on any module over $W_\beta(g)$ on which the eigenvalues of the Virasoro operator $L_0$ are “bounded from below” (that is, the set of these eigenvalues is the union of the sets of the form $\gamma + Z_+, \gamma \in \mathbb{C}$). We will refer to such modules as highest weight modules. In particular, we can consider their action on the Fock representations $\pi_\mu$, $\mu \in \mathfrak{h}^*$. After passing to the periodic coordinate $\varphi$ such that $z = e^{i\varphi}$, we obtain a commuting algebra of quantum Hamiltonians which includes the Virasoro operator $L_0$. Thus, all local quantum Hamiltonians are homogeneous of degree 0 with respect to the grading defined by $L_0$, and so they preserve the homogeneous components of highest weight modules. In the case of Fock representations $\pi_\mu$ these components are finite-dimensional. It is natural to ask what are the spectra of the local quantum $\hat{g}$-KdV Hamiltonians on these components. This problem naturally arises in the study of deformations of conformal field theories with $W$-algebra symmetry [Z, EY, KM, FF4]. However, it proved to be elusive for general $\beta$, because we do not have much structure on the commutative algebra generated by the local quantum KdV Hamiltonians.

7.2. Non-local Hamiltonians. A breakthrough in the study of the quantum KdV system was made by Bazhanov, Lukyanov, and Zamolodchikov in a series of papers starting with [BLZ1], in which a procedure for quantization of the non-local classical KdV Hamiltonians was developed. We recall from the previous section that those can be obtained from the traces of the monodromy matrix of the first order differential operators (6.25) (taken in the Miura form) with spectral parameter $z$, expanded as a power series near $z = 0$.

The quantum non-local $\hat{g}$-KdV Hamiltonians were introduced in [BLZ1] in the case of $\hat{g} = \hat{sl}_2$ and in [BHK] in the case of $g = \hat{sl}_3$. As noted in [BHK], the latter construction generalizes in a straightforward way to other affine algebras $\hat{g}$.

These non-local Hamiltonians are defined as elements of the completed Heisenberg algebra acting on the Fock representations $\pi_\mu$ for $\mu \in \mathfrak{h}^*$. More precisely, according to the construction of [BLZ1] [BHK], for each finite-dimensional representation $V$ of $U_q(\hat{g})$, one
defines a generating series of commuting non-local Hamiltonians $T_{V,n}$ by the formula

$$T_{V}(z) = \sum_{n \geq 0} T_{V,n} z^n = \text{Tr}_{V(z)}(e^{\pi i P \cdot h} \mathcal{L}).$$

Here $\mathcal{L}$ is the operator obtained from the reduced universal $R$-matrix of $U_q(\widehat{g})$, which is an element of a completion of the tensor product $U_q(\widehat{n}_+) \otimes U_q(\widehat{n}_-)$, by mapping the first factor to $\text{End} \; V(z)$ and the second factor to $\text{End} \oplus \pi_\mu$ using the screening operators corresponding to the simple roots of $\widehat{g}$ (they satisfy the Serre relations of $U_q(\widehat{n}_-)$), and

$$P \cdot h = \sum_{j=1}^{n} P^j h^j.$$ 

Here $\{h_j\}$ is a basis in $\mathfrak{g}$ and $\{P^j\}$ is the dual basis in $\mathfrak{g}$ with respect to the inner product on $\mathfrak{h}$ obtained by restricting the normalized invariant inner product $\kappa_0$ on $q$ (see Section 6.1). The elements $h_j$ are defined so that $K_j = q^{h_j}$ are the standard Drinfeld-Jimbo generators of $U_q(\widehat{g}) \subset U_q(\widehat{\mathfrak{g}})$, where

$$q = e^{\pi i \beta^2}.$$ 

The $h_j$'s act on $V$. On the other hand, the elements $P^j$ are elements of the constant Cartan subalgebra $\mathfrak{h}$ in the Heisenberg algebra $\widehat{\mathfrak{h}}$. They act on the Fock representation $\pi_\mu$ according to the formula $P^j \mapsto \mu(P^j) \text{Id}$. 

Note that this $T_{V}(z)$ may be viewed as a quantum analogue of the monodromy matrix $M_V(z)$, see Section 6.2.

The claim of [BLZ1, BHK] is that the formal power series $T_{V}(z)$ commute with each other:

$$[T_{V}(z), T_{W}(w)] = 0$$

for any finite-dimensional representations $V$ and $W$ of $U_q(\widehat{g})$. They also commute with the local quantum $\widehat{g}$-KdV Hamiltonians. Technically, this is proved in [BLZ1, BHK] for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{sl}_3$, but the proof generalizes to other simple Lie algebras. Alternatively, these statements can be proved [FFS] using the methods of [FF4] (see Sects. 3.3 and 5.3 of [FF5] for an outline).

Furthermore, it follows from the construction that

$$T_{V \oplus W}(z) = T_{V}(z) + T_{W}(z), \quad T_{V \otimes W}(z) = T_{V}(z)T_{W}(z).$$

Thus, we obtain an action of the commutative algebra $K_0(\mathfrak{c})$, where $\mathfrak{c}$ is the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ (see Section 2.3), by endomorphisms of the Fock representation $\pi_\mu$ for any $\mu \in \mathfrak{h}^*$. All of the non-local Hamiltonians $T_{V,n}$ are homogeneous endomorphisms of $\pi_\mu$ of degree 0 with respect to the grading by the Virasoro operator $L_0$ (which is in fact the simplest local quantum KdV Hamiltonian). Therefore we obtain an action of $K_0(\mathfrak{c})$ on each finite-dimensional graded component of $\pi_\mu$. This action of non-local quantum $\widehat{g}$-KdV Hamiltonians commutes with the action of the local quantum $\widehat{g}$-KdV Hamiltonians, and so these Hamiltonians have common eigenvectors. Furthermore, according to [BLZ1, BHK, BLZ5], the eigenvalues of the local Hamiltonians can be recovered from the eigenvalues of the non-local ones as coefficients of their asymptotic expansions. Therefore it makes sense to replace...
the question of describing the spectra of the local $\hat{g}$-KdV Hamiltonians with the question of describing the joint eigenvalues of the non-local Hamiltonians. We now have a lot of additional structure because, by construction, these Hamiltonians correspond to elements of the algebra $K_0(\mathfrak{O})$ and therefore they must satisfy the relations in this algebra, such as the $T$-system satisfied by the classes of the KR modules (see [HI] and references therein). Bazhanov, Lukyanov, and Zamolodchikov then made an additional step of generalizing this construction to infinite-dimensional representations of $U_q(\mathfrak{b}_+).$ Indeed, formula (7.29) only requires $V$ to be a representation of $U_q(\mathfrak{b}_+),$ and hence we can take as $V$ a representation from the category $\mathfrak{O}$ (see Section 2.2). For instance, we can take $V = L^+_i.$ Then, in the case $\mathfrak{g} = \mathfrak{sl}_2,$ if we denote $T_V(z)$ by $Q(z),$ we obtain the Baxter relation linking $Q(z)$ and $T_W(z),$ where $W$ is the two-dimensional fundamental representation. In fact, it is in this context that the representations $L^+_i$ were discovered in [BLZ2, BLZ3] and [BHK] for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{sl}_3$ (in $K_0$ this construction was generalized to the case of $\mathfrak{sl}_\nu.$) However, if $V$ is infinite-dimensional, then we need to address the convergence of the trace over $V.$ A priori, it is not clear that the trace over $V,$ and hence $T_V(z)$ given by formula (7.29), is well-defined. One way to approach this question is to use the term $e^{\pi i P h}$ in formula (7.29), which acts on $\pi_\mu$ as $\exp(\pi i \sum_j \mu(P^j) h_j).$ If we denote $e^{\pi i u(P)}$ by $u_j,$ we can express the trace as a power series in the $u_j.$ Then the trace will make sense as a formal power series in the $u_j, j = 1, \ldots, n.$ This is similar to the approach taken in [FH]. However, in applications to quantum field theories (such as conformal field theories and their deformations), one needs to consider the $u_j$ as specific numbers. Then the convergence of the trace will become problematic. Nevertheless, it is natural to conjecture that the series will converge for generic values of $u_j$ — that is, for generic $\mu \in \mathfrak{h}^*.$ We formulate this as the following

**Conjecture 7.1.** For generic $\mu \in \mathfrak{h}^*,$ there is an action of the commutative algebra $K_0(\mathfrak{O})$ on the Fock representation $\pi_\mu$ of $\mathfrak{h}$ which commutes with the local quantum $\hat{g}$-KdV Hamiltonians (including the operator $L_0$).

7.3. **Connection to the $Q\hat{Q}$-system.** Conjecture 7.1 means that for any relation in $K_0(\mathfrak{O}),$ the joint eigenvalues of the corresponding non-local $\hat{g}$-KdV Hamiltonians will satisfy this relation for any joint eigenvector in $\pi_\mu$ for generic $\mu \in \mathfrak{h}^*.$ In particular, each joint eigenvector of the non-local $\hat{g}$-KdV Hamiltonians should give rise to a solution of the $Q\hat{Q}$-system (5.15) with $q = e^{\pi i \beta^2}$ and the corresponding Bethe Ansatz equations (5.16) (provided that the genericity assumption of Section 5 holds).

In the case $\mathfrak{g} = \mathfrak{sl}_2,$ the $\hat{Q}\hat{Q}$-system reduces to the quantum Wronskian relation of [BLZ2, BLZ3, BLZ5], and in the case $\mathfrak{g} = \mathfrak{sl}_3$ the $\hat{Q}\hat{Q}$-system reduces to relations considered in [BHK] (see Section 6.2 for more details). However, for general simple Lie algebras the $Q\hat{Q}$-system has not previously been considered as a relation on the spectra of the non-local $\hat{g}$-KdV Hamiltonians. Using the $Q\hat{Q}$-system in the study of non-local $\hat{g}$-KdV Hamiltonians has the important advantage that we can use the results of [MRV1, MRV2], where it was shown that solutions of the same $Q\hat{Q}$-system naturally arises from the $\hat{g}$-opers that were
proposed in [FF5] as the parameters for the spectra of the non-local $\hat{g}$-KdV Hamiltonians. We will discuss these affine opers in detail in the next section.

7.4. Langlands duality of the spectra of quantum KdV Hamiltonians. We recall the Langlands duality of quantum $\mathcal{W}$-algebras established in [FF3] (see [FF4, Section 4.8.1] or [FB, Section 15.4.15] for an exposition):

$$\mathcal{W}_\beta(g) \simeq \mathcal{W}_{\hat{\beta}}(Lg),$$

where $Lg$ is the Langlands dual Lie algebra to $g$ and

$$\hat{\beta} = -(\hat{r})^{1/2}/\beta,$$

$\hat{r}$ being the maximal number of edges connecting the vertices of the Dynkin diagram of $g$ (so $\hat{r} = 1$ for simply-laced $g$; $\hat{r} = 2$ for $g$ of types $B_n, C_n, \text{ and } F_4$; and $\hat{r} = 3$ for $g = G_2$). The proof follows from the fact that for generic $\beta$ the kernels of the screening operators corresponding to simple roots of $g$ and parameter $\beta$ are equal to the kernels of the screening operators corresponding to the simple roots of $\hat{g}$ and parameter $\hat{\beta}$. This implies the above isomorphism for all $\beta$.

Since the local quantum $\hat{g}$-KdV Hamiltonians are defined as elements in the intersection of the kernels of the screening operators corresponding to the simple roots of $g$ and parameter $\beta$, by using the same argument, we identify for generic $\beta$ the commutative algebra of local quantum $\hat{g}$-KdV Hamiltonians and the commutative algebra of local quantum $L\hat{g}$-KdV Hamiltonians (see the end of [FF4, Section 4.8.1]).

Furthermore, recall from Section 7.2 that the non-local quantum $\hat{g}$-KdV Hamiltonians are defined using the the screening operators corresponding to the simple roots of $\hat{g}$ and parameter $\hat{\beta}$. Since the latter essentially commute with the screening operators corresponding to the simple roots of $L\hat{g}$ and parameter $\hat{\beta}$, it is natural to expect that the non-local quantum $\hat{g}$-KdV Hamiltonians commute with the non-local quantum $L\hat{g}$-KdV Hamiltonians (we use the normalized inner product on $\mathfrak{h}$ to identify $\mathfrak{h}^*$ with $(\mathfrak{h}^*)^* = \mathfrak{h}$, so that the $\hat{g}$- and $L\hat{g}$-KdV Hamiltonians act on the same Fock representations). This has been stated in [BLZ2] in the case $\hat{g} = \hat{sl}_2$ (see formula (2.26)). We formulate this as a conjecture in general. (Note that since the local quantum $\hat{g}$- and $L\hat{g}$-KdV Hamiltonians coincide, they automatically commute with the non-local quantum $\hat{g}$- and $L\hat{g}$-KdV Hamiltonians.)

**Conjecture 7.2.** The action of the non-local quantum $\hat{g}$-KdV Hamiltonians on a Fock representation $\pi_\mu, \mu \in \mathfrak{h}^*$, commutes with the action of the non-local quantum $L\hat{g}$-KdV Hamiltonians.

This implies that using joint eigenvectors of the non-local quantum $\hat{g}$- and $L\hat{g}$-KdV Hamiltonians in $\pi_\mu, \mu \in \mathfrak{h}^*$, we obtain a surprising correspondence between solutions of the $QQ$-systems (as well as other equations stemming from $K_0(\mathcal{O})$ such as the $Q\mathcal{Q}^*$-system of [HL2], see Section 3.4) for $U_q(\hat{g})$ and $U_q(L\hat{g})$, where

$$q = e^{\pi i \beta^2}, \quad \hat{q} = e^{\pi i \hat{\beta}^2} = e^{\pi i \hat{r}/\beta^2}.$$ 

This correspondence (or duality) deserves further study.
8. Spectra of the Quantum KdV Hamiltonians

In [FF5], it was conjectured that the spectra of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians can be parametrized by \( L_{\hat{\mathfrak{g}}} \)-affine opers of special kind. This conjecture was motivated by the results of [BLZ5] in the case of \( \hat{\mathfrak{sl}}_2 \) and an analogy between the quantum \( \hat{\mathfrak{g}} \)-KdV system and the Gaudin model associated to a simple Lie algebra \( \mathfrak{g} \), in which case the joint eigenvalues of the commuting Hamiltonians are known to be parametrized by \( L_{\mathfrak{g}} \)-opers. We refer the reader to [FF5] (especially, Sections 4.4, 5.4, and 5.5) for the explanation of this analogy.

The general definition of a \( \hat{\mathfrak{g}} \)-affine oper was given in Section 4.1 of [FF5], inspired by the work of Drinfeld and Sokolov [DS] and Beilinson and Drinfeld [BD1, BD2]. Here we review the \( \hat{\mathfrak{g}} \)-opers that are related to the joint eigenvalues of the \( \hat{\mathfrak{g}} \)-KdV Hamiltonians according to the conjecture of [FF5].

8.1. The case of \( \hat{\mathfrak{sl}}_2 \). The \( \hat{\mathfrak{sl}}_2 \)-opers that appear here are gauge equivalence classes of the first order differential operators on \( \mathbb{P}^1 \), equipped with a coordinate \( z \), with values in the Lie algebra \( \mathbb{C} d \ltimes \mathfrak{sl}_2((\lambda)) \), where \( d = \lambda \partial_\lambda \), of the form

\[
\partial_z + \begin{pmatrix} a(z) & b(z) + \lambda \\ 1 & -a(z) \end{pmatrix} + \frac{k}{z} \mathbf{d}, \quad k \in \mathbb{C},
\]

(where \( a(z) \) and \( b(z) \) are rational functions in \( z \)) under the action of the group \( N_+ \)-valued rational functions in \( z \), where \( N_+ \) is the upper unipotent subgroup of \( SL_2 \).

Each gauge equivalence class contains a unique operator of the form

\[
\partial_z + \begin{pmatrix} 0 & v(z) + \lambda \\ 1 & 0 \end{pmatrix} + \frac{k}{z} \mathbf{d},
\]

and the operators relevant to the spectra of the quantum KdV Hamiltonians are the ones with

\[
v(z) = \frac{r(r+1)}{z^2} + \frac{1}{z} \left( 1 - \sum_{j=1}^{m} \frac{k}{w_j} \right) + \sum_{j=1}^{m} \frac{2}{(z-w_j)^2} + \sum_{j=1}^{m} \frac{k}{w_j(z-w_j)},
\]

such that the coefficients \( v_{j,k} \) in the expansion of \( v(z) \) in \( z-w_j \) satisfy the equations

\[
\frac{1}{4} \left( \frac{k}{w_j} \right)^3 - \frac{k}{w_j} v_{j,0} + v_{j,1} = 0, \quad j = 1, \ldots, m.
\]

As explained in Section 4.4 of [FF5], this is the condition that the solutions of the differential equation corresponding to the operator (8.32) have no monodromy around the singular points \( w_j \).

As in [FF5], applying gauge transformation by \( z^{kd} \), we can eliminate the term \( \frac{k}{z} \mathbf{d} \) from (8.32) obtaining the operator

\[
\partial_z + \begin{pmatrix} 0 & v(z) + \lambda z^k \\ 1 & 0 \end{pmatrix},
\]
which is equivalent to the following second order differential operator with spectral parameter:

\[
\frac{\partial^2}{\partial z^2} - \frac{1}{z} \left(1 - \sum_{j=1}^{m} \frac{k}{w_j}\right) - \frac{r(r+1)}{z^2} - \sum_{j=1}^{m} \frac{2}{(z-w_j)^2} - \sum_{j=1}^{m} \frac{1}{w_j z - w_j} - \lambda z^k.
\]

Again, the equations (8.34) are equivalent to the condition that this operator has no monodromy around \(w_j, j = 1, \ldots, m\), and therefore no monodromy on \(\mathbb{P}^1\), except around the points 0 and \(\infty\), for all values of \(\lambda\). This operator also has regular singularity at \(z = 0\) and the mildest possible irregular singularity at \(z = \infty\) (indeed, the restriction of (8.36) to the punctured disc at the point \(z = \infty\) has the form \(\partial^2_s - \tilde{v}(s)\), where \(\tilde{v}(s) = 1/s^3 + \ldots\) with respect to the local coordinate \(s = z^{-1}\) at \(\infty\)).

According to the proposal of [FF5], for generic \(r\) and \(k\) the differential operators (8.36) should encode the common eigenvalues of the quantum KdV Hamiltonians on the irreducible highest weight module over the Virasoro algebra with the central charge

\[
c_k = 1 - \frac{6(k+1)^2}{k+2}
\]

and highest weight (with respect to the operator \(L_0\))

\[
\Delta_{r,k} = \frac{(2r+1)^2 - (k+1)^2}{4(k+2)}
\]

This “numerology” is explained as follows: the Virasoro algebra can be obtained by the Drinfeld–Sokolov reduction (with respect to \(n_-(t)\)) from the affine Kac–Moody algebra of level \(k\) has central charge \(c_k\) (see [FF1]). Further, the Drinfeld–Sokolov reduction of the irreducible \(\widehat{sl}_2\)-module with highest weight \(\lambda = 2r\) (“spin” \(r\)) and level \(k\) is the irreducible module over the Virasoro algebra with the highest weight \(\Delta_{r,k}\) (provided that \(k - \lambda \notin \mathbb{Z}_+\)); see [FF1]. The number \(m\) of poles of the \(\widehat{sl}_2\)-oper on \(\mathbb{P}^1\setminus\{0, \infty\}\) should be equal to the \(L_0\)-degree of the corresponding eigenvector. By that we mean that it should occur in the subspace in the irreducible module with highest weight \(\Delta_{r,k}\) on which \(L_0\) acts by \(\Delta_{r,k} + m\).

Note that in general, some of the poles \(w_j\) may coalesce, see Section 5.5 of [FF5].

8.2. Change of variables. The advantage of formula (8.36) is that it is clearly linked, via a gauge transformation, to an affine oper (8.33). However, the disadvantage of (8.36) is that the spectral parameter \(\lambda\) appears not by itself but multiplied with \(z^k\). To fix that, we make a change of variables \(z \mapsto x\), where

\[
z = \frac{x^{2\alpha + 2}}{(2\alpha + 2)^2}
\]

and

\[
\alpha = -\frac{k + 1}{k + 2}
\]

(we assume that \(k \neq -2\)). We note that this \(\alpha\) is related to the parameter \(\beta\) discussed in Section 7 by the formula

\[
\alpha + 1 = \frac{1}{\beta^2}.
\]
Recall (see, e.g., [FB], Ch. 8.2) that the general transformation formula for a second order operator (also known as a projective connection) of the form
\[ \partial_z^2 - v(z) : \Omega^{-1/2} \to \Omega^{3/2} \]
(we need to consider our second order operators as acting from \( \Omega^{-1/2} \) to \( \Omega^{3/2} \) to ensure that their property of having the principal symbol 1 and subprincipal symbol 0 is coordinate-independent) under the change of variables \( z = \varphi(x) \) is
\[ v(z) \mapsto v(\varphi(x)) \left( \varphi' \right)^2 - \frac{1}{2} \{ \varphi, x \}, \tag{8.40} \]
where
\[ \{ \varphi, x \} = \frac{\varphi'''}{\varphi'} - \frac{3}{2} \left( \frac{\varphi''}{\varphi'} \right)^2 \tag{8.41} \]
is the Schwarzian derivative of \( \varphi \).

Applying the change of variables \( z \mapsto x \) to the operator (8.36), we obtain the operator
\[ \partial_x^2 - \frac{\ell(\ell + 1)}{x^2} - x^{2\alpha} + 2 \frac{d^2}{dx^2} \sum_{j=1}^{m} \log(x^{2\alpha+2} - z_j) + E, \tag{8.42} \]
where
\[ \ell(\ell + 1) = 4(\alpha + 1)^2 r(r + 1) + \alpha^2 + 2\alpha + \frac{3}{4} = 4(\alpha + 1) \Delta_{r,k} + \alpha^2 - \frac{1}{4}, \]
so that we have
\[ \Delta_{r,k} = \Delta(\ell, \alpha) = \frac{(2\ell + 1)^2 - 4\alpha^2}{16(\alpha + 1)}, \]
and
\[ z_j = (2\alpha + 2)^2 w_j, \]
\[ E = -(2\alpha + 2)^2 \frac{\alpha}{\alpha + 1} \lambda. \]
The operator (8.42) has the spectral parameter \( \lambda \) (which is obtained by rescaling \( \lambda \)) without any additional factors, so that (8.42) looks like a typical Schrödinger operator with a spectral parameter (the price to pay for this is that this operator is multivalued with respect to the coordinate \( x \)).

As shown in [FF5], the operators (8.42) coincide with the Schrödinger operators in [BLZ5] (formula (1)) parametrizing the spectra of the quantum KdV Hamiltonians on the irreducible module over the Virasoro algebra with highest weight \( \Delta(\alpha, \ell) \) and central charge
\[ c(\alpha) = c_k = 1 - 6 \frac{\alpha^2}{\alpha + 1}. \]
Moreover, as explained above, our condition (8.34) means that the operator (8.42) has no monodromy around the points \( z_j \) for all \( E \). This condition is equivalent to the algebraic equations given by formula (3) in [BLZ5].

\[ ^3 \text{Note that what we denote by } \ell \text{ here coincides with } \ell \text{ of [BLZ5] but was denoted by } \bar{\ell} \text{ in [FF5], whereas what we denote by } r \text{ here was denoted by } \ell \text{ in [FF5].} \]
In particular, the oper associated to the highest weight vector (corresponding to \( m = 0 \)) is given by the formula

\[
\partial_x^2 - \frac{\ell(\ell + 1)}{x^2} - x^{2\alpha} + E,
\]

or in the matrix form

\[
\partial_x + \begin{pmatrix}
\frac{\ell}{x} & x^{2\alpha} & E \\
1 & -\frac{\ell}{x} \\
\end{pmatrix}.
\]

We note that for generic \( \ell \) and \( k \) the above irreducible module over the Virasoro algebra is isomorphic to the Verma module and the Fock representation \( \pi_{\mu} \) with \( \mu = (2\ell + 1)(\alpha + 1)/4 \).

8.3. The case of \( \hat{sl}_r \). Next, we consider the case of \( \hat{g} = \hat{sl}_r \). In this case the Langlands dual Lie algebra \( \hat{g}^* \) is also \( \hat{sl}_r \). We consider \( \hat{sl}_r \)-opers on \( \mathbb{P}^1 \) which are the gauge equivalence classes of differential operators with values in \( C\mathfrak{d} \rtimes \mathfrak{sl}_r((\lambda)) \), where \( \mathfrak{d} = \lambda \partial_\lambda \), of the form

\[
\partial_z + \begin{pmatrix}
* & * & * & \cdots & * + \lambda \\
1 & * & * & \cdots & * \\
0 & 1 & * & \cdots & * \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & 1 & * \\
\end{pmatrix} + \frac{k}{z} \mathfrak{d}
\]

(8.43)

(where each * stands for a rational function on \( \mathbb{P}^1 \)) under the action of the group of \( N_+ \)-valued rational functions on \( \mathbb{P}^1 \) where \( N_+ \) is the upper unipotent subgroup of \( SL_r \).

Each gauge equivalence class contains a unique operator of the form

\[
\partial_z + \begin{pmatrix}
0 & v_1(z) & v_2(z) & \cdots & v_{r-1}(z) + \lambda \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix} + \frac{k}{z} \mathfrak{d}.
\]

After the gauge transformation by \( z^{kd} \), we can eliminate the last term at the cost of multiplying \( \lambda \) by \( z^k \):

\[
\partial_z + \begin{pmatrix}
0 & v_1(z) & v_2(z) & \cdots & v_{r-1}(z) + z^k \lambda \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}.
\]

(8.44)

The last operator is equivalent to the \( r \)th order scalar differential operator with spectral parameter:

\[
\partial_z^r - v_1(z)\partial_z^{r-2} + \ldots + (-1)^r v_{r-2}(z)\partial_z - (-1)^r v_{r-1}(z) + (-1)^r z^k \lambda,
\]

acting from \( \Omega^{-(r-1)/2} \) to \( \Omega^{(r+1)/2} \). This determines the transformation properties of these operators under the changes of coordinate \( z \); in particular, this ensures that the property
that their principal symbol is 1 and the subprincipal symbol is 0 is preserved by the changes of coordinate.

As conjectured in [FF5], the operators \( (8.45) \) that are relevant to the spectra of the quantum \( \hat{sl}_r \)-KdV Hamiltonians are those in which the functions \( v_i(z) \) is a rational function in \( z \) with poles at \( z = 0, \infty, \) and finitely many other points \( w_j, j = 1, \ldots, m. \) They satisfy the following properties:

1. At \( z = \infty, \) the operator \( (8.45) \) has the mildest possible irregular singularity; namely, we have
   \[
   \tilde{v}_i(s) \sim \frac{c_i}{s^{i+1}} + \ldots, \quad r = 1, \ldots, r - 2;
   \]
   \[
   \tilde{v}_{r-1}(s) \sim \frac{(-1)^r}{s^{r+1}} + \ldots,
   \]
   where \( \tilde{v}_i(s) \) are the coefficients of the operator obtained from \( (8.45) \) by the change of variables \( z \mapsto s = z^{-1}. \) By rescaling the coordinate \( z \) we can make the leading coefficient of \( \tilde{v}_{r-1}(s) \) to be equal to any non-zero number; we choose that number to be \( (-1)^r. \)

2. At \( z = 0, \) the operator \( (8.45) \) has regular singularity, that is
   \[
   v_i(z) \sim \frac{c_i(\nu)}{z^{i+1}} + \ldots,
   \]
   where the coefficients \( c_i(\nu) \) are determined by the highest weight \( \nu \) of the \( \hat{sl}_r \)-module \( L_{\nu,k}. \)

   Namely, representing \( \nu \) as \( (\nu_1, \ldots, \nu_r) \), where \( \nu_i \in \mathbb{C} \) and \( \sum_{i=1}^r \nu_i = 0, \) we find the \( c_i(\nu) \)'s from the following formula:
   \[
   (8.46) \quad \partial_z^r + \sum_{i=1}^{r-1} (-1)^i \frac{c_i(\nu)}{z^{i+1}} \partial_z^{r-i-1} = \left( \partial_z - \frac{\nu_1}{z} \right) \ldots \left( \partial_z - \frac{\nu_r}{z} \right).
   \]

3. At the points \( w_j, \) the operator \( (8.45) \) has regular singularity,
   \[
   v_i(z) \sim \frac{c_i(\theta)}{(z - w_j)^{i+1}} + \ldots,
   \]
   where \( \theta = (1, 0, \ldots, 0, -1) \) is the maximal root of \( sl_r, \) which is the highest weight of its adjoint representation. In addition, we require that the operator \( (8.45) \) has trivial monodromy around the point \( w_j \) for each \( j = 1, \ldots, m, \) and all \( \lambda. \)

The proposal of [FF5] is that the \( r \)th order differential operators of this kind should correspond to the common eigenvalues of the quantum \( \hat{sl}_r \)-KdV Hamiltonians on the subspace of \( L_g \)-degree \( m \) in the generic irreducible module over the \( W \)-algebra obtained by the quantum Drinfeld–Sokolov reduction (with respect to \( n_-(t) \)) of the irreducible \( \hat{sl}_r \)-module \( L_{\nu,k} \) of generic highest weight \( \nu \in \mathfrak{h}^* \) and level \( k. \) The central charge of this module is
\[
\text{c}_k = r - 1 - r(r^2 - 1) \frac{(k + r - 1)^2}{k + r}.
\]
Note that for generic \( \nu \) and \( k \) this irreducible module over the \( W \)-algebra is isomorphic to the Verma module and the Fock representation \( \pi_\mu \) with appropriate \( \mu \in \mathfrak{h}^*. \)
In order to obtain a stand-alone spectral parameter, we apply the change of variables $z \mapsto x$, where

$$z = \frac{x^{r\alpha + r}}{(r\alpha + r)^r}$$

and

$$\alpha = -\frac{k + r - 1}{k + r}.$$

The dependence of the central charge on $\alpha$ is

$$c(\alpha) = c_k = r - 1 - r(r^2 - 1)\frac{\alpha^2}{\alpha + 1}.$$

It is easy to see that under this change of variables the irregular term $z^{-r+1}$ in the last summand $v_{r-1}$ of our differential operator becomes the term $x^{r\alpha}$, and the term $\lambda x^k$ gives rise to the new spectral parameter term $-E$, where

$$E = -(r\alpha + r)^{\frac{r}{r+1}}\lambda,$$

which is independent of $x$ (the sign is just a matter of normalization). The poles of the new operator will be at the points $x = 0, \infty$ and $x^{r\alpha + r} = z_j, j = 1, \ldots, m$, where

$$z_j = (r\alpha + r)^r w_j.$$

In particular, the $\widehat{\mathfrak{sl}}_r$-oper corresponding to the highest weight vector may be written in the following way:

$$\partial_x + \mathfrak{p}_{-1} + \frac{\nu}{x} + (x^{r\alpha} - E)e_\theta,$$

where $\mathfrak{p}_{-1} = \sum_{i=1}^{r-1} f_i$, the sum of the generators of the lower nilpotent subalgebra of $\mathfrak{sl}_r$ ($f_i$ is the matrix having 1 in the $i$th place below the diagonal, and 0 everywhere else), and $e_\theta$ is a generator of the maximal root subspace (the matrix having 1 in the upper right corner and 0 everywhere else).

If we re-write these opers as $r$th order differential operators, we obtain the differential operators of $[\text{BHK}, \text{DDT1}, \text{DMST}]$. They correspond to the highest weight vectors of the representations of $\mathcal{W}$-algebras.

8.4. General simply-laced case. Let $\mathfrak{g}$ be a simply-laced simple Lie algebra (that is, of $ADE$ type). Then $L\mathfrak{g} = \mathfrak{g}$ and $L\mathfrak{g} = \mathfrak{g}$. If $\mathfrak{g}$ is of classical type ($A$ or $D$), then we can realize the corresponding affine opers as scalar (pseudo)differential operators, following $[\text{DS}]$. It is therefore possible to describe those of them that encode the spectra of the quantum Hamiltonians of the $\mathfrak{g}$-KdV system in a way that is similar to the case of $\mathfrak{sl}_r$ (see the Section 8.3). And in fact, in the special case that there are no singular points other than 0 and $\infty$ (this is the case of the highest weight vector), after a change of variables one obtains the (pseudo)differential operators proposed in $[\text{BHK}, \text{DDT1}, \text{DMST}]$ (see $[\text{DDT2}]$ for a survey). We can generalize these operators by including extra singular points $w_j$ on $\mathbb{P}^1$, as in the case of $\mathfrak{sl}_r$ (see above).
However, in order to describe these affine opers for an arbitrary affine Kac–Moody algebra \( \hat{g} \) (including the case of non-simply laced \( \hat{g} \) discussed below), we need to define \( \hat{g} \)-affine opers as gauge equivalence classes of first order \( \hat{g} \)-valued differential operators (as in Section 4.1 of [FF5]).

Recall a basis \( \{ \mathcal{P}_1, \ldots, \mathcal{P}_n \} \) of the canonical slice \( V_{\text{can}} \subset n_+ \subset g \), as introduced in Section 6.1 and Lemma 6.1. We will also use the elements \( \mathcal{P}_{-1} = \sum_{i=1}^n f_i \) and \( f_0 = e_\theta \lambda \) of \( \hat{g} \). These are the same elements as those introduced in 6.4 except that now we use the variable \( \lambda \) (\( z \) is reserved for the spectral parameter of the KdV system and appears here as the coordinate on \( P^1 \) on which the affine opers are defined), and we take the completion in Laurent power series in \( \lambda \) rather than in \( z^{-1} \) (the reason for this is explained in Section 4.1 of [FF5]).

The \( \hat{g} \)-opers (or equivalently, \( L\hat{g} \)-opers) that parametrize the common eigenvalues of the \( \hat{g} \)-KdV Hamiltonians are \( \mathbb{C}d \ltimes \hat{g} \)-valued differential operators of the form

\[
\partial_z + \mathcal{P}_{-1} + (z^{-h^\vee} + \lambda)e_\theta + \sum_{i=1}^n v_i(z) \cdot \mathcal{P}_i + \frac{k}{z^d},
\]

where each \( v_i(z) \) is a rational function on \( P^1 \) with poles at \( z = 0, \infty \), and finitely many other points \( w_j, j = 1, \ldots, m \).

Here we denote by \( h^\vee \) the dual Coxeter number of \( \hat{g} \), which coincides with the Coxeter number of \( \hat{g} \) if \( g \) is simply-laced.

As before, applying the gauge transformation by \( z^k d \) to the operator (8.48), we eliminate the last term at the cost of multiplying \( \lambda \) by \( z^k \):

\[
\partial_z + \mathcal{P}_{-1} + (z^{-h^\vee} + \lambda z^k)e_\theta + \sum_{i=1}^n v_i(z) \cdot \mathcal{P}_i.
\]

Recall that the quantum \( \hat{g} \)-KdV Hamiltonians act on the irreducible module over the \( W \)-algebra associated to \( g \) obtained by the quantum Drinfeld–Sokolov reduction (with respect to \( n_-((t)) \)) from the irreducible module \( L_{\nu,k} \) over \( \hat{g} \) with highest weight \( \nu \) and level \( k \) (such a module is isomorphic to the Fock representation \( \pi_\mu \) with appropriate \( \mu \in h^* \)). The proposal of [FF5] is that for generic \( \nu \) and \( k \), the common eigenvalues of these Hamiltonians are encoded by the \( \hat{g} \)-opers (equivalently, \( L\hat{g} \)-opers) (8.49) satisfying the following properties:

1. At \( z = \infty \), the operator (8.49) has the mildest possible irregular singularity. The terms \( v_i(z) \) are regular:

\[
\tilde{v}_i(s) \sim \frac{c_i}{s^{d_i+1}} + \ldots, \quad i = 1, \ldots, \ell,
\]

where \( \tilde{v}_i(s) \) are the coefficients of the operator obtained from (8.49) by the change of variables \( z \mapsto s = z^{-1} \). But the term \( z^{-h^\vee}e_\theta \) creates an irregular singularity term \( (-1)^{h^\vee} s^{-h^\vee-1}e_\theta \).

2. At \( z = 0 \), the operator (8.49) has regular singularity, that is

\[
v_i(z) \sim \frac{c_i(\nu)}{z^{d_i+1}} + \ldots,
\]
where \( \nu \in h^* \simeq h \) and the \( c_i(\nu) \) are determined by the following rule: the element

\[
\bar{p}_{-1} + \sum_{i=1}^{n} \left( c_i(\nu) + \frac{1}{4} \delta_i,1 \right) \bar{p}_i
\]

is the unique element in the Kostant slice of regular elements of \( g \),

\[
\bar{p}_{-1} + \nu, \quad \nu \in b,
\]

which is conjugate to \( \bar{p}_{-1} - \nu \) (see [F2], Section 9.1).

Here we use the identification of \( h^* \) and \( h \) (so \( \nu \in h^* \) becomes an element of \( h \subset b \)) provided by the restriction of the invariant inner product on \( g \) normalized so that the square length of each root is equal to 2.

(3) At the points \( w_j \), the operator \( (8.49) \) has regular singularity, and for all \( i = 1, \ldots, \ell \),

\[
v_i(z) \sim \frac{c_i(\theta)}{(z-w_j)^{d_i+1} + \ldots},
\]

where \( \theta \) is the maximal root of \( g \). In addition, we require trivial monodromy around the point \( w_j \) for each \( j = 1, \ldots, m \), and all \( \lambda \).

As before, the number \( m \) should correspond to the \( L_0 \)-degree of the corresponding eigenvector.

### 8.5. Change of variables: general case.

Next, we make a change of variables so as to make the spectral parameter appear independently of the coordinate:

\[
(8.50) \quad z = \frac{x^{h^\vee(\alpha+1)}}{(h^\vee(\alpha + 1))^{h^\vee}}
\]

where

\[
\alpha = -\frac{k + h^\vee - 1}{k + h^\vee}. \quad \text{When we make this change, we find that } \bar{p}_{-1}, \bar{p}_i, \text{ and } e_\theta \text{ all get multiplied by}
\]

\[
\frac{dz}{dx} = \frac{x^{h^\vee(\alpha+1)-1}}{(h^\vee(\alpha + 1))^{h^\vee-1}}.
\]

In order to bring the operator to the oper form \( \partial_z + \bar{p}_{-1} + \ldots \), we then need to apply a gauge transformation by an element of the Cartan subgroup \( H \).

Recall the weight \( \rho \in h^* \), the half-sum of positive roots. We have

\[
\langle \rho, \alpha_i^\vee \rangle = 1, \quad i = 1, \ldots, n.
\]

Here we view \( \rho \) as a coweight of \( h \) identified with \( h^* \) as above, via a normalized invariant inner product. Therefore, \( \rho \) gives rise to a homomorphism (one-parameter subgroup) \( C^\times \to H \), the Cartan subgroup of the simply-connected Lie group \( G \) with the Lie algebra \( g \). We will denote this one-parameter subgroup by \( \rho \). It acts as follows:

\[
\rho(a) \cdot \bar{p}_{-1} = a^{-1}\bar{p}_{-1}, \quad \rho(a) \cdot \bar{p}_i = a^{d_i}\bar{p}_i, \quad i = 1, \ldots, n,
\]

and

\[
\rho(a) \cdot e_\theta = a^{h^\vee-1}e_\theta.
\]
Let us apply the gauge transformation by

\[ \rho(x)^{h^\vee (\alpha + 1) - 1} \]

As the result we get back \( \bar{p}_{-1} \), and the coefficient \( z^k \lambda e_\theta \) becomes \(-E e_\theta\), where

\[ E = -(h^\vee (\alpha + 1))^{h^\vee (\alpha + 1) - 1} \]

which is a pure spectral parameter, independent of \( x \) (due to the identity \( kh^\vee (\alpha + 1) + h^\vee (h^\vee (\alpha + 1) - 1) = 0 \)). In addition, the term \( z^{-h^\vee + 1} \) becomes \( x^{h^\vee (\alpha + 1)} \) (that was the reason for including the coefficient \( (h^\vee (\alpha + 1))^{-h^\vee} \) in the coordinate change from \( z \) to \( x \)).

The singularities at \( z = w_j \) become singularities at \( x^{h^\vee (\alpha + 1)} = z_j \), where

\[ z_j = (h^\vee (\alpha + 1))^{h^\vee w_j} \]

The oper (8.49) now takes the form

\[ \partial_z + \bar{p}_{-1} + (x^{h^\vee (\alpha + 1)} - E) e_\theta + \sum_{i=1}^{n} v_i(x) \cdot \bar{p}_i, \]

where

\[ v_1(x) = v_1(\varphi(x))(\varphi')^2 - \frac{1}{2} \{ \varphi, x \}, \]

\[ v_i(x) = v_i(\varphi(x))(\varphi')^{d_i+1}, \quad i > 1, \]

and \( \varphi(x) \) is the function on the right hand side of formula (8.50) (see [F2], Section 4.2.4).

In the case \( m = 0 \), the \( L\hat{g} \)-oper corresponding to the highest weight vector may be written in the form

\[ \partial_x + \bar{p}_{-1} + \nu x + (x^{h^\vee (\alpha + 1)} - E) e_\theta, \]

where \( \nu \in \mathfrak{h}^\ast = L\mathfrak{h} \).

8.6. The non-simply laced case. Let us recall that to each affine Dynkin diagram we can associate an affine Kac–Moody algebra. We are interested in its quotient by the central element. In the untwisted case, this is the semi-direct product of \( d = \lambda \partial_\lambda \) and the universal central extension of \( g(\lambda) \), where \( g \) is a simple Lie algebra. In the twisted case, it is the semi-direct product of \( d = \lambda \partial_\lambda \) and the universal central extension of the twisted loop algebra \( L_g' \), defined as in Section 6.4, except that the loop variable is now denoted by \( \lambda \) rather than \( z \), and we take the completion in formal power series in \( \lambda \) rather than in \( z^{-1} \).

Given an affine Kac–Moody algebra \( \hat{g} \) associated to a Dynkin diagram \( \hat{\Gamma} \), we define its Langlands dual as the affine Kac–Moody algebra \( L\hat{g} \) associated to the dual Dynkin diagram \( L\hat{\Gamma} \) which is obtained from \( \hat{\Gamma} \) by reversing all arrows. The dual of \( L\hat{g} \) is of course \( \hat{g} \) itself.

Note that if we remove the 0th nodes from the Dynkin diagrams \( \hat{\Gamma} \) and \( L\hat{\Gamma} \), we obtain the Dynkin diagrams of two simple Lie algebras that are Langlands dual to each other. These are the degree zero Lie subalgebras of \( \hat{g} \) and \( L\hat{g} \) with respect to the grading defined by \( d \). We denote them by \( g \) and \( Lg \).

If \( \hat{g} \) is the untwisted Kac–Moody algebra associated to a simply-laced simple Lie algebra \( g \), then \( L\hat{g} = \hat{g} \). This is the case we have just discussed. However, if \( \hat{g} \) is the untwisted
Kac–Moody algebra associated to a non-simply laced simple Lie algebra \( \mathfrak{g} \), then \( L\hat{\mathfrak{g}} \) is a twisted affine Kac–Moody algebra. In this case the Dynkin diagram of \( L\hat{\mathfrak{g}} \) is the quotient of the Dynkin diagram of a simply-laced Lie algebra \( \mathfrak{g}' \) by an automorphism of order \( r^\vee \) equal to 2 or 3, and \( L\hat{\mathfrak{g}} \) is the corresponding twisted affine algebra of type \( \mathfrak{g}'^{(r^\vee)} \) (see Section 6.4). Note that in this case \( L\hat{\mathfrak{g}} = \mathfrak{g}'_0 \), the \( \sigma \)-invariant part of \( \mathfrak{g}' \), where \( \sigma \) is an outer automorphism of \( \mathfrak{g}' \) of order \( r^\vee \) corresponding to the automorphism of the Dynkin diagram of \( \mathfrak{g}' \).

Here we consider the case that \( \mathfrak{g} \) is non-simply laced, and so \( L\hat{\mathfrak{g}} \) is a twisted affine Kac–Moody algebra. The conjecture of [FF5] is that the joint eigenvalues of the quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians are encoded by the \( L\hat{\mathfrak{g}} \)-affine opers, and so the relevant finite-dimensional Lie algebra is \( L\hat{\mathfrak{g}} \). Hence we consider the basis \( \{\mathcal{P}_1, \ldots, \mathcal{P}_n\} \) of the canonical slice \( V_{can} \) of \( L\hat{\mathfrak{g}} \) rather than \( \mathfrak{g} \) (see Section 6.4 and Lemma 6.1). We will also use the elements \( \mathcal{P}_{-1} = \sum_{i=1}^n f_i \) and \( f_0 \) of \( L\hat{\mathfrak{g}} \). The latter is now equal to \( f_0 = \epsilon_{\theta_0} \lambda \), where \( \epsilon_{\theta_0} \) is a highest weight vector of the \( L\hat{\mathfrak{g}} \)-module \( \mathfrak{g}'_1 \), the eigenspace of \( \sigma \) with the eigenvalue \( e^{2\pi i/r^\vee} \) (see Section 6.4).

It is natural to generalize formula (8.48) to this case as follows:

\[
\partial_z + \mathcal{P}_{-1} + (z^{-r^\vee+1} + \lambda)\epsilon_{\theta_0} + \nu(z) + \frac{k}{z}d.
\]

However, as D. Masoero and A. Raimondo pointed out to us, this formula requires a further justification since the operator (8.53) does not, on the face of it, take values in \( \hat{\mathfrak{g}} \) if \( \mathfrak{g} \) is non-simply laced. Indeed, according to the definition of the twisted affine Kac–Moody algebras (see Section 6.4), the \( \lambda \)-part of an element of \( L\hat{\mathfrak{g}} \) should be in \( L\hat{\mathfrak{g}} = \mathfrak{g}'_0 \), and the \( \lambda^1 \)-part should be in \( \mathfrak{g}'_1 \). But in formula (8.53), we have the \( \lambda \)-term \( z^{-r^\vee+1}\epsilon_{\theta_0} \), which is in \( \mathfrak{g}'_1 \).

In order to make sense of formula (8.53) in the non-simply laced case, we recall the notion of twisted opers introduced by B. Gross and one of the authors in [FG], Section 5.1.

The idea is to make the automorphism \( \sigma \) act on both the Lie algebra \( \mathfrak{g}' \) and the space on which our differential operators are defined; that is, the projective line with the coordinate \( z \). We will view this \( \mathbb{P}^1 = \mathbb{P}_1^1 \) as an \( r^\vee \)-sheeted cover of another projective line \( \mathbb{P}_1^1 \) with the coordinate \( t \) such that \( z^{r^\vee} = t \). We define an automorphism \( \tilde{\sigma} \) of \( \mathbb{P}_1^1 \times \mathfrak{g}' \) by the formula

\[
(z, g) \mapsto (ze^{-2\pi i/r^\vee}, \sigma(g)).
\]

We then have a natural notion of a \( \sigma \)-twisted connection on \( \mathbb{P}_1^1 \): namely, a \( \tilde{\sigma} \)-invariant (meromorphic) connection on the trivial \( \mathfrak{g}' \)-bundle on \( \mathbb{P}_1^1 \) (it is automatically flat, since \( \mathbb{P}_1^1 \) is one-dimensional as a complex manifold, so the flatness condition is vacuous),

\[
\nabla = d + A(z)dz,
\]

where \( A(z)dz \) is a \( \tilde{\sigma} \)-invariant \( \mathfrak{g}' \)-valued one-form on \( \mathbb{P}_1^1 \) (and \( d \) is the de Rham differential on \( \mathbb{P}_1^1 \)).

Furthermore, in [FG] the notion of a twisted oper was introduced, as a \( \tilde{\sigma} \)-invariant connection (8.54) satisfying a natural generalization of the oper condition. Here’s an example of a twisted oper constructed in [FG] (we use the notation of the present paper):

\[
d + (\mathcal{P}_{-1} + z\epsilon_{\theta_0}) \frac{dz}{z}.
\]
It is clear that the restriction of this operator to the formal disc around the point \( z = 0 \) can be viewed as an element of the dual space to the twisted affine Kac–Moody algebra \( L\hat{g} \) (with respect to the coordinate \( z \)).

The difference between the twisted opers introduced in [FG], such as (8.55), and the \( L\hat{g} \)-affine opers that we need here is that we have to incorporate the additional (formal) variable \( \lambda \). We do that as follows.

Given \( g' \) and \( L\hat{g} \) as above, we define a \( L\hat{g} \)-affine oper as an operator

\[
(8.56) \quad \nabla = d + A(z, \lambda)dz, \quad A(z, \lambda) \in \left( g' \otimes C(z)(\langle \lambda \rangle) \right) \oplus \left( d \otimes C(z) \right)
\]

(note that we denote the element of the affine algebra \( \lambda \partial \) by \( d \) to avoid any confusion with the de Rham differential \( d \)), which are invariant under the action of the automorphism \( \tilde{\sigma} \):

\[
(z, \lambda, g) \mapsto (ze^{-2\pi i/r^\vee}, \lambda e^{-2\pi i/r^\vee}, \sigma(g)), \quad g \in g'.
\]

Thus, in our definition of \( L\hat{g} \)-affine opers, the automorphism is given not only as \( \sigma \) on \( g' \) and the map \( \lambda \mapsto \lambda e^{-2\pi i/r^\vee} \), but also the map \( z \mapsto ze^{-2\pi i/r^\vee} \) acting on the coordinate \( z \) of \( \mathbb{P}^1_z \), on which the oper is defined.

Now we interpret our operator (8.53) as a \( L\hat{g} \)-affine oper in the sense of this definition. Let us recall the element \( \rho \in \mathfrak{h}^* \) defined in Section 8.3. Since we have a canonical isomorphism \( \mathfrak{h}^* \simeq L\mathfrak{h} \), we view \( \rho \) as an element of \( L\mathfrak{h} \). Therefore, \( \rho \) gives rise to a homomorphism \( \mathbb{C}^\times \to L\mathbb{G} \), the Cartan subgroup of the simply-connected Lie group \( L\mathbb{G} \) with the Lie algebra \( L\hat{g} \), and we have

\[
\rho(a) \cdot \mathfrak{p}_{-1} = a^{-1}\mathfrak{p}_{-1}, \quad \rho(a) \cdot e_{\theta_0} = a^{h^\vee -1}e_{\theta_0}.
\]

Applying the gauge transformation by \( \rho(z) \) to the operator (8.53) (and multiplying it by \( dz \) in order to make it into a connection operator of the form (8.56)), we obtain

\[
(8.57) \quad d + \left( \mathfrak{p}_{-1} + (z + \lambda z^{h^\vee})e_{\theta_0} + \nabla(z) \right) \frac{dz}{z} + kd \frac{dz}{z},
\]

where \( \nabla(z) = z\rho(z) \cdot \nabla(z) - \rho \). Finally, we apply the gauge transformation by \( z^{h^\vee}d \) to bring it to the following form:

\[
(8.58) \quad d + \left( \mathfrak{p}_{-1} + (z + \lambda)e_{\theta_0} + \nabla(z) \right) \frac{dz}{z} + (k + h^\vee)d \frac{dz}{z}
\]

Note that the resulting shift of \( k \) by the dual Coxeter number \( h^\vee \) is akin to the shift of \( \nabla(z) \) by \(-\rho\); in a sense, it properly centers the weights around the affine version of \( \rho \) (note that the shift of \( \nabla(z) \) is here by \(-\rho\), rather than \( \rho \), because of our convention that the generator \( f_0 \) is \( e_{\theta_0} \lambda \), rather than \( e_{\theta_0} \lambda^{-1} \) which leads to the multiplication of the weights by \(-1 \), so that the “central” weight is \( \rho \) rather than the more traditional \(-\rho\)).

The connection (8.58) is \( \tilde{\sigma} \)-invariant, and is indeed a \( L\hat{g} \)-oper, provided that

\[
(8.59) \quad \nabla(z) = \nabla(ze^{-2\pi i/r^\vee})
\]

(in other words, \( \nabla(z) \) really depends on \( t = z^{r^\vee} \)).

Note that formula (8.58) makes perfect sense for any affine Kac–Moody algebra \( L\hat{g} \) and can be taken as an alternative definition of the \( L\hat{g} \)-opers in the simply-laced case (in this case, of course, \( \sigma \) is the identity, \( r^\vee = 1 \), \( t = z \), and we set \( \theta_0 = \theta \)).
As before, we can eliminate the last term in (8.58) by applying the gauge transformation by $z^{(k + h^\vee)}d$. Then we obtain
\[(8.60)\]
\[d + \left(\bar{p}_{-1} + (z + \lambda z^{k + h^\vee})e_{\theta_0} + \nabla(z)\right) \frac{dz}{z}.\]

By using an $L^N$-valued gauge transformation, we can bring $\nabla(z)$ to the form
\[\nabla(z) = \sum_{i=1}^{\ell} \pi_i(z)\bar{p}_i.\]

Now we list the conditions that the $L^\g$-opers (8.60) should satisfy in order to encode joint eigenvalues of the quantum KdV Hamiltonians. Note that these conditions are slightly different from (and simpler than) the conditions listed in Section 8.4. This is in part because in formula (8.60) we have the overall factor $1/z$ that creates a more convenient “gauge” for our opers.

(0) $\nabla(z)$ should satisfy (8.59), and hence each $\pi_i(z)$ should satisfy the same invariance property.

(1) At $z = \infty$ (if we set $\lambda = 0$), the operator (8.49) should have the mildest possible irregular singularity. The terms $\pi_i(z)$ are regular:
\[\pi_i(s) \sim -\pi_i(\nu + \rho) + \ldots, \quad i = 1, \ldots, \ell,\]
but the term $e_{\theta_0}dz$ creates an irregular singularity term $-s^{-2}e_{\theta_0}ds$ (here, as above, $s = z^{-1}$).

(2) At $z = 0$, the operator (8.49) should have regular singularity, that is
\[\pi_i(z) \sim \pi_i(\nu + \rho) + \ldots,\]
for some $\nu \in \mathfrak{h}^* = L\mathfrak{h}$, where $\pi_i(\mu)$ is determined by the following rule:
The element
\[\bar{p}_{-1} + \sum_{i=1}^{\ell} \pi_i(\mu)\bar{p}_i\]
is the unique element in the Kostant slice of regular elements of $\g$, which is conjugate to $\bar{p}_{-1} - \mu$ (see [F2], Section 9.1).

(3) The $\pi_i(z)$ are allowed to have regular singularities at $m \cdot r^\vee$ points on $\mathbb{C}^\times \subset \mathbb{P}^1$, $w_j^{(p)} = w_j e^{2\pi ip/r^\vee}, \quad j = 1, \ldots, m, \quad p = 0, 1, r^\vee - 1,$ so that we have the following expansions in $z - w_j^{(p)}$:
\[\pi_i(z) \sim \pi_i(\theta_0) + \ldots\]
In addition, the connections (8.58) and (8.60) should have trivial monodromy around each of the points $w_j^{(p)}$ for all $\lambda$.

Thus, we see that the set of singular points (other than 0 and $\infty$) forms a union of $r^\vee$ families – orbits of the cyclic group $\mathbb{Z}_{r^\vee}$ naturally acting on $\mathbb{P}^1$; that’s because we need $\nabla(z)$ to be invariant under the action of this group. As before, the number $m$ should correspond to the $L_0$-degree of the corresponding eigenvector of the $\hat{\g}$-KdV Hamiltonians.
We can make a change of variables (8.50) so as to make the spectral parameter appear independently of the coordinate. All calculations of Section 8.5 apply in the same way as in the simply-laced case.

8.7. **QQ-system from affine opers.** Let \( \hat{\mathfrak{g}} \) be an untwisted quantum affine algebra (simply-laced or non-simply laced). In the papers [MRV1, MRV2] a solution of the \( Q\tilde{Q} \)-system (5.15) is assigned to the affine \( L\hat{\mathfrak{g}} \)-oper (8.52) for generic \( \nu \in \mathfrak{h}^* \) (note that our \( \alpha \) corresponds to \( M \) in [MRV1, MRV2], and our \( q \) corresponds to \( \Omega^{-1/2} \) in [MRV1, MRV2]). This construction generalizes earlier results [DT1, BLZ4, DT2, DDT1, BHK, S].

More precisely, for an untwisted affine Kac–Moody algebra \( \hat{\mathfrak{g}} \), the authors of [MRV1, MRV2] define \( n \) evaluation representations \( V^{(i)} \), \( i = 1, \ldots, n \), of the affine algebra \( L\hat{\mathfrak{g}} \) (which is twisted if \( \mathfrak{g} \) is not simply-laced) and study the first order linear differential equations obtained from the operator (8.52) specialized in the representations \( V^{(i)} \). They show that each of these equations has a unique (properly normalized) solution that goes to 0 most rapidly as \( x \to +\infty \). Then they define \( Q_i \) and \( \tilde{Q}_i \) as the leading coefficients appearing in the expansion of the above solutions near \( x = 0 \) (these functions may be viewed as generalizations of the spectral determinants). Finally, they show that for generic \( \nu \in \mathfrak{h}^* \) these functions are entire functions of \( E \) which satisfy the \( Q\tilde{Q} \)-system (5.15) (with \( E \) instead of \( u \) and particular values of \( v_i, i = 1, \ldots, n \)). They obtain this system from a system of equations satisfied by the above solutions (which they call the \( \Psi \)-system).

Now let us consider the case that \( \hat{\mathfrak{g}} \) is a twisted affine algebra. In this case \( L\hat{\mathfrak{g}} \) is untwisted (unless \( \hat{\mathfrak{g}} = A_{2n}^{(2)} \), in which case \( L\hat{\mathfrak{g}} = A_{2n}^{(2)} \) as well). Note that this case was not considered in [MRV1, MRV2]. However, it is natural to expect that using the construction of [MRV1, MRV2], one can attach to the \( L\hat{\mathfrak{g}} \)-affine oper (8.52) with generic \( \nu \) one can attach a solution of the \( U_q(\hat{\mathfrak{g}}) Q\tilde{Q} \)-system (3.9), with \( \pm 2\frac{\alpha_\nu}{\alpha} \) mapping to some \( v^{\pm 1}_i \in \mathbb{C}^* \) depending on \( \alpha \) and \( \nu \).

In this section, we have discussed the conjecture of [FF5] linking the eigenvectors (or, equivalently, the spectra) of the \( \mathfrak{g} \)-KdV Hamiltonians to \( L\hat{\mathfrak{g}} \)-affine opers on \( \mathbb{P}^1 \) with special analytic behavior. Note that this conjecture was not based on a direct construction, but rather on an analogy with the Gaudin model (the fact that the spectra of the \( \mathfrak{g} \)-Gaudin model can be encoded by \( L\mathfrak{g} \)-opers on \( \mathbb{P}^1 \) with special analytic behavior, see [FF5], especially Sections 4.4, 5.4, and 5.5, for more details).

However, if the results of [MRV1, MRV2] (as well as Conjecture 8.1) could be generalized to other \( L\hat{\mathfrak{g}} \)-opers discussed in this section, given by formulas (8.49) and (8.60), that we expect to correspond to the excited states in the highest weight representations of \( \mathcal{W} \)-algebra, then, as we discussed in the Introduction, this would indeed open the possibility of establishing a direct link between the spectra of quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians and \( L\hat{\mathfrak{g}} \)-opers.

In [FF5, Section 4], another quantum integrable system was discussed: the “shift of argument” \( \hat{\mathfrak{g}} \)-Gaudin model, and its spectrum was also conjectured to be encoded by \( L\hat{\mathfrak{g}} \)-affine opers on \( \mathbb{P}^1 \). These opers differ from the \( L\hat{\mathfrak{g}} \)-affine opers arising in the quantum \( \hat{\mathfrak{g}} \)-KdV
system in the way they behave near \( \infty \in \mathbb{P}^1 \). Nevertheless, we expect that solutions of the \( U_q(\hat{\mathfrak{g}}) \) \( \mathbb{Q}Q \)-system can be attached to these \( L^\mathfrak{g} \)-affine opers as well. At the same time, the joint eigenvalues of the quantum Hamiltonians of this \( \hat{\mathfrak{g}} \)-Gaudin model should satisfy the same \( \mathbb{Q}Q \)-system for the same reason as in the quantum KdV case. Therefore, we again expect that \( \mathbb{Q}Q \)-system would provide a link between between these joint eigenvalues and the \( L^\mathfrak{g} \)-affine opers described in [FF5, Section 4].

8.8. **Duality of affine opers.** Let us recall that Conjecture 7.2 implies that there is a correspondence between solutions of the \( \mathbb{Q}Q \)-systems (as well as other equations stemming from \( K_0(\mathcal{O}) \) such as the \( QQ^* \)-system of \[11L2\], see Section 3.3) for \( U_q(\hat{\mathfrak{g}}) \) and \( U_qL^\hat{\mathfrak{g}} \), where

\[
q = e^{\pi i \beta^2}, \quad \bar{q} = e^{\pi i \bar{v}/\beta^2}.
\]

If that is true, then this should also hold on the side of affine opers. This means that there should be a correspondence (or duality) between the \( L^\mathfrak{g} \)-opers of the form discussed in this section with the parameter \( \alpha \), and \( \hat{\mathfrak{g}} \)-opers of the same form but with the parameter \( \bar{\alpha} \), where

\[
\bar{\alpha} + 1 = \frac{1}{\bar{r}(\alpha + 1)}
\]

(see formulas (7.30) and (8.39)). For \( \hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_r \), this duality was discussed in [DDT1].

8.9. **Two appearances of opers.** At first glance, it may appear that the conjecture of [FF5] discussed in this section is not so surprising: after all, the phase space of the classical KdV system is the space of opers. Why should we then be surprised that the spectra of the quantum KdV Hamiltonians would be linked to opers as well? However, it is important to realize that the two spaces of opers appearing here are quite different.

The phase space of the classical \( \mathfrak{g} \)-KdV system is the space of \( \mathfrak{g} \)-opers (not \( \hat{\mathfrak{g}} \)-opers!) on a circle, or a punctured disc (with coordinate \( t \), see Section 6. We inserted a spectral parameter \( z \) into these \( \mathfrak{g} \)-opers in order to construct the Poisson commuting KdV Hamiltonians.

On the other hand, the spectra of the corresponding algebra of quantum \( \hat{\mathfrak{g}} \)-KdV Hamiltonians are conjecturally encoded by \( L^\mathfrak{g} \)-affine opers on the projective line \( \mathbb{P}^1 \) with coordinate \( z \) (it is the same \( z \) as the spectral parameter in the classical story). So opers appear again, but these are affine opers, and they are associated to the Langlands dual affine algebra \( L^\hat{\mathfrak{g}} \). Therefore, \textit{a priori} they have nothing to do with the \( \mathfrak{g} \)-opers appearing in the definition of the classical KdV system (other than the fact that a coordinate on the space on which the \( L^\mathfrak{g} \)-affine opers “live” is the spectral parameter of the \( \mathfrak{g} \)-opers).

For instance, in the case of \( \mathfrak{sl}_2 \), the points of the phase space of the KdV system are \( \mathfrak{sl}_2 \)-opers with spectral parameter (see Section 6.2)

\[
\partial_t^2 - v(t) - z,
\]

where \( t \) is a coordinate on a circle, or a punctured disc, and \( z \) is the spectral parameter. The classical KdV Hamiltonians are constructed by expanding the monodromy matrix of this operator, considered as function of \( z \), near \( z = 0 \) (non-local) or \( z = \infty \) (local).

On the other hand, the \( \hat{\mathfrak{sl}}_2 \)-opers that encode the eigenvalues of the quantum KdV Hamiltonians have the form (see Section 8.1)

\[
\partial_z^2 - v(z) - \lambda z^k,
\]
where \( z \) is the spectral parameter of the classical KdV system, which is now viewed as a coordinate on \( \mathbb{P}^1 \), and \( \tau(z) \) is a meromorphic function on this \( \mathbb{P}^1 \) with poles at \( z = 0, \infty \), and finitely many other points (see formula (8.36)). There is another spectral parameter \( \lambda \). When we make a change of variables \( z \mapsto x \), we obtain the differential operators of the form (8.42) with the spectral parameter \( E \).

For non-simply laced \( \mathfrak{g} \), the difference between the two spaces is even more drastic because of the appearance of the Langlands dual affine algebra \( \hat{\mathfrak{g}} \).

As argued in [FF5], the “quantum KdV – affine opers” duality may be viewed as a generalization of the duality observed in the generalized Gaudin quantum integrable systems, in which the spectra of the quantum Hamiltonians in a model associated to a simple Lie algebra \( \mathfrak{g} \) turn out to be encoded by \( L\mathfrak{g} \)-opers [FFR, F1, FFT]. The latter is explained by the isomorphism between the center of the completed enveloping algebra of \( \hat{\mathfrak{g}} \) at the critical level and the algebra of functions on \( L\mathfrak{g} \)-opers on the formal disc [FF3] (see [F2] for an exposition). In other words, in order to understand the duality between spectra of the generalized \( \mathfrak{g} \)-Gaudin systems and \( L\mathfrak{g} \)-opers, we need to use the affinization of the Lie algebra \( \mathfrak{g} \). The “master algebra” lurking behind the generalized Gaudin quantum integral systems is the center of the completed enveloping algebra of \( \hat{\mathfrak{g}} \) at the critical level, and the fact that it is isomorphic to the algebra of functions on \( L\mathfrak{g} \)-opers on the formal disc gives rise to identifications of the spectra of the \( \mathfrak{g} \)-Gaudin Hamiltonians with \( L\mathfrak{g} \)-opers of particular kind.

Therefore it is natural to expect that in order to understand the “quantum KdV – affine opers” duality we need to study the affinization of \( \hat{\mathfrak{g}} \); that is to say, a toroidal algebra of \( \mathfrak{g} \), but the big open problem here is to figure out what is the analogue of the “critical level” of \( \hat{\mathfrak{g}} \) and the corresponding center (see Section 7 of [FF5] for a discussion of this point). It may well be that to do so, one needs to study the “gerbal representations” of the toroidal algebra introduced in [FZ].

From this point of view, the “quantum KdV – affine opers” duality offers us glimpses into the mysterious “critical level” structures arising in toroidal algebras.

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