Quantum corrections to spinning superstrings in $AdS_3 \times S^3 \times M^4$: determining the dressing phase

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Abstract

We study the leading quantum string correction to the dressing phase in the asymptotic Bethe Ansatz system for superstring in $AdS_3 \times S^3 \times T^4$ supported by RR flux. We find that the phase should be different from the phase appearing in the $AdS_3 \times S^5$ case. We use the simplest example of a rigid circular string with two equal spins in $S^3$ and also consider the general approach based on the algebraic curve description. We also discuss the case of the $AdS_3 \times S^3 \times S^3 \times S^1$ theory and find the dependence of the 1-loop correction to the effective string tension function $h(\lambda)$ (expected to enter the magnon dispersion relation) on the parameters $\alpha$ related to the ratio of the two 3-sphere radii. This correction vanishes in the $AdS_3 \times S^3 \times T^4$ case.

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1 Introduction and summary

Recent remarkable progress in uncovering integrable structure behind the spectrum of quantum strings in \( AdS_5 \times S^5 \) [1] which was much aided by duality to \( N = 4 \) supersymmetric gauge theory raises the question about applying similar integrability-based methods (algebraic curve description of finite-gap solutions, its discretisation and magnon scattering S-matrix as guides towards asymptotic Bethe ansatz (ABA), its TBA generalisation, etc.) also in the similar but less supersymmetric “low-dimensional” cases of superstring in \( AdS_3 \times S^3 \times M^4 \) and \( AdS_2 \times S^2 \times M^6 \) supported by R-R fluxes. In these cases the dual conformal theories are poorly understood and thus one has less data in trying to fix the structure of the corresponding Bethe ansatz.

The first important step was made in [2] where the set of ABA equations was proposed for the spectrum of strings on \( AdS_3 \times S^3 \times T^4 \) and \( AdS_3 \times S^3 \times S^3 \times S^3 \) described by the GS superstring action on the supercosets \( PSU(1,1|2) \times PSU(1,1|2)/SU(1,1) \times SU(2) \) and \( D(2,1;\alpha) \times D(2,1;\alpha)/SU(1,1) \times SU(2) \times SU(2) \). The first model may be viewed as a special case of the second: if the radius of \( AdS_3 \) is set to 1, then the radii of the two 3-spheres can be parametrized as \( 1^{-\alpha}, R_2 = (1-\alpha)^{-1} \), i.e. the \( AdS_3 \times S^3 \times T^4 \) model with \( R_2 = \infty \) corresponds to \( \alpha = 1 \).

The starting point was the classical integrable supercoset sigma model and the discretisation of the corresponding finite-gap equations following closely the analogy with the \( AdS_5 \times S^5 \) case [4] (see [5]). It was conjectured in [2] that the corresponding dressing phase should be the same BES phase [6] as in the \( AdS_5 \times S^5 \) case.

Further elaborations of the proposed ABA system appeared in [7, 8, 9, 10]. In particular, it was noted in [10] that due to different algebraic structure here one cannot fix the dressing scalar factors in the magnon S-matrix using crossing symmetry constraints as was done [11, 12, 13] in the \( AdS_5 \times S^5 \) case, but until very recently it was assumed that the original conjecture of [2] that the phase should be given by the BES expression should be correct.\(^2\)

The aim of the present paper is to suggest a proposal for the leading quantum string correction to the “classical” AFS phase in the ABA system of [2, 8] following the same first-principles approach as originally used in \( AdS_5 \times S^5 \) case [15, 16, 17, 18], i.e. by comparing the ABA predictions to the quantum string and algebraic curve computations of the 1-loop corrections to semiclassical string energies.\(^3\) We will study the simplest example of rigid circular string with two equal spins in \( S^3 \) [23] (and also closely related, via an analytic continuation, case of \( (S,J) \) folded long string [24]) and also consider more general algebraic curve approach. Our conclusion is that the phase in the ABA of [2, 8] requires a modification from the standard

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\(^1\)The meaning of the relation \( 1 = R_+^2 + R_-^2 \) between the three radii can be easily understood as follows. The \( AdS_3 \times S^3 \times S^3 \times S^3 \) background supported by the R-R 3-form background is a type IIB supergravity solution related by S-duality to the same metric supported by the NS-NS 3-form flux (see, e.g. [3]). The corresponding string sigma model is simply \( SL(2,R) \times SU(2) \times SU(2) \times SO(2) \) WZW model (with world-sheet supersymmetry if treated in NSR approach). The dilaton equation of motion (with constant dilaton) is then the total central charge condition relating the three (shifted) levels, i.e. \( -\frac{3}{k_{s(2)}} + \frac{3}{k_{su(2)+}} + \frac{3}{k_{su(2)-}} = 0 \). Since the levels are proportional to the radii, the above relation follows.

\(^2\)While the present paper was in preparation, there appeared a preprint [14] where it is claimed that there should be several scalar phase factors and they may differ from the BES expression.

\(^3\)Some semiclassical computations for superstrings in \( AdS_3 \times S^3 \times M^4 \) appeared earlier in [19, 20, 21, 22].
AdS$_5 \times S^5$ form of [15, 16], i.e. the ABA for the AdS$_3 \times S^3 \times M^4$ theory can not have the standard BES phase.

In more details, the phase for the scattering of two magnons with momenta $p_j$ and $p_k$ in the AdS$_5 \times S^5$ theory can be written as [15, 25]

$$\vartheta(p_j, p_k) = 2 \sum_{r=2}^{\infty} \sum_{s \geq r+1}^{s \text{ odd}} c_{r,s}(\lambda) \left( \frac{\lambda}{16 \pi^2} \right)^{r+s-1} \left[ q_r(p_j) q_s(p_k) - q_s(p_j) q_r(p_k) \right].$$

(1.1)

Here, $q_n(p)$ is the elementary magnon $n$-th charge. The strong coupling expansion of the coefficient functions $c_{r,s}(\lambda)$ is

$$c_{r,s}(\lambda) = c_{r,s}^{(0)} + \frac{1}{\sqrt{\lambda}} c_{r,s}^{(1)} + \ldots,$$

(1.2)

where $c_{r,s}^{(0)} = \delta_{r+1,s}$ is the AFS contribution [4] and the one-loop correction is the HL phase found for $r = 2, s = 3$ in [15] and then in general in [16]. It is non-vanishing for odd $r + s$,

$$c_{r,s}^{(1)} = -8 \frac{(r-1)(s-1)}{(r+s-2)(s-r)},$$

(1.3)

and reproduces the “non-analytic” part of the 1-loop correction to SU(2) circular string energy [15]

$$\delta E_1^{\text{AdS}_5} = \frac{1}{\sqrt{J^2 + m^2}} \left( m^2 + 2 J^2 \log \frac{J^2}{J^2 + m^2} - \frac{J^2 - m^2}{2} \log \frac{J^2 - m^2}{J^2 + m^2} \right).$$

(1.4)

Our analysis of the same SU(2) circular string with two equal spins in $R_t \times S^3 \subset AdS_3 \times S^3 \times T^4$ suggests that the corresponding non-analytic term that should be reproduced by the dressing contribution is instead

$$\delta E_1^{\text{AdS}_3} = \frac{1}{\sqrt{J^2 + m^2}} \left( m^2 + J^2 \log \frac{J^2}{m^2 + J^2} \right).$$

(1.5)

This expression is indeed found when the $c_{r,s}^{(1)}$ coefficients for the LL or RR scattering take the following new form

$$c_{r,s}^{(1)} = 2 \frac{s - r}{r + s - 2},$$

(1.6)

provided also that the summation in (1.1) now starts from $r = 1$. This constitutes our proposal for the 1-loop dressing phase coefficients.

Below we show that the coefficients (1.6) are consistent with the string prediction for the SU(2) circular string energy. We arrive at (1.6) using the semiclassical algebraic curve approach to the derivation of the dressing phase [18]. The circular string case serves as a guide to how

\[\footnote{L and R stand for the left and right moving sectors [14].}\]
to resolve the regularization ambiguity of the algebraic curve approach. The expression (1.6) follows after requiring the antisymmetry of the coefficients $c^{(1)}_{r,s}$, which is shown to be consistent with the string result (1.5).

Although the example of the circular string solution does not test the mixed LR or RL scattering, we also propose that the scattering phase between the opposite chirality magnons takes also the above general form (1.1) (again with summation in (1.1) starting from $r = 1$), but with the coefficients

$$
\pi^{(1)}_{r,s} = -2 \frac{r + s - 2}{s - r}. \tag{1.7}
$$

Additional tests of these expressions for $c^{(1)}_{r,s}$ and $\bar{c}^{(1)}_{r,s}$ would certainly be important.

We also consider the more general $\alpha$-dependent integrable model based on $AdS_3 \times S^3 \times S^3 \times S^1$. Here we do not attempt to fix the dressing phase in ABA in a systematic way but compute the non-analytic (i.e. dressing-related) contribution to the one-loop energy for the corresponding generalized $SU(2)$ circular string. Remarkably, this correction turns out to be independent of $\alpha$ when written in terms of the effective string tension $h(\lambda)$ that has the following strong coupling expansion

$$
h(\lambda) = \frac{\sqrt{\lambda}}{4 \pi} + a + \mathcal{O} \left( \frac{1}{\sqrt{\lambda}} \right), \quad a_{AdS_3 \times S^3 \times S^3 \times S^1} = \frac{\alpha \log \alpha + (1 - \alpha) \log (1 - \alpha)}{4 \pi}. \tag{1.8}
$$

This function is expected to enter the corresponding magnon dispersion relation and thus appear in the Bethe Ansatz. Notice that the 1-loop shift in (1.8) vanishes at $\alpha = 0, 1$ when we go back to the $AdS_3 \times S^3 \times T^4$ case.

This paper is organized as follows. We shall start in section 2 with a brief review of the ABA equations of [2, 8]. We shall then consider the constraints on the leading quantum string correction to the dressing phase for the $AdS_5 \times S^5 \times T^4$ case that follow from the expression for the non-analytic term in the 1-loop quantum correction to the $SU(2)$ circular string energy as an input (section 3).

Next, in section 4 we shall discuss the algebraic curve setup [18], finding a non-antisymmetric expression for the coefficients $c_{r,s}$ in the phase, apparently contradicting the requirement following from the discrete form of the Bethe Ansatz. In section 5 we shall compare the present computation with the one in the $AdS_5 \times S^5$ case [15, 16] and point out a mismatch between the standard string and the algebraic curve regularizations. In section 6 we shall show that requiring the antisymmetry of the $c_{r,s}$ coefficients resolves disagreement between the algebraic curve approach and string computation in section 3 and leads to our proposal for the coefficients in (1.6).

In section 7 we shall derive the non-analytic (dressing) part of the 1-loop energy for the $SU(2)$ circular string case in the $AdS_3 \times S^3 \times S^3 \times S^1$ case, pointing out the role of the 1-loop shift in (1.8) and discuss the $\alpha \to 1$ limit.

In Appendix A we shall consider the $(S, J)$ folded string with large spins and determine the corresponding coefficients $c_{1,s}$ in the phase that agree with the ones in the $SU(2)$ case.
2 Asymptotic Bethe Ansatz equations for $AdS_3 \times S^3 \times T^4$ model

As discussed in [2], type IIB GS superstring theory on $AdS_3 \times S^3 \times S^3 \times S^1$ space with RR 3-form flux reduces, in a particular $\kappa$-symmetry gauge, to a supercoset sigma model which is classically integrable. The string theory on the $AdS_3 \times S^3 \times T^4$ background with RR flux can be formally treated as the limiting case ($\alpha = 1$) of the $AdS_3 \times S^3 \times S^3 \times S^1$ supercoset model.

From the general classical integrability structure of $\mathbb{Z}_4$ symmetric (super)cosets (see [5]), one can derive the finite gap equations which may be written entirely in terms of the group-theory data. Discretization of these finite gap equations leading to the associated quantum Bethe equations was proposed in [2] for the symmetric point ($\alpha = \frac{1}{2}$) where the radii of the two 3-spheres are equal and also for the limiting case ($\alpha = 1$) of $AdS_3 \times S^3 \times T^4$. It should be mentioned that from the point of view of the integrability structure the limit $\alpha \rightarrow 1$ is a non-trivial one [9].

In order to fix the notation, here we shall briefly review the form of the quantum Bethe equations for the case of $AdS_3 \times S^3 \times T^4$ that we will be mostly considering here. The starting point is the Dynkin diagram of the $PSU(1,1|2) \times PSU(1,1|2)/(SU(1,1) \times SU(2))$ supercoset. It contains $3+3$ nodes associated to the left/right moving sectors. The quantum Bethe equations are written in terms of the Bethe roots in the spectral plane $x_{i,\ell}$ where $i = 1,2,3,\bar{1},\bar{2},\bar{3}$, $\ell = 1,\ldots,K_i$ and the parameters $x^\pm$ defined by the Zhukovsky map

$$x^\pm + \frac{1}{x^\pm} = x + \frac{1}{x} \pm \frac{i}{2h(\lambda)}.$$

Here the function $h(\lambda)$ cannot be determined by the integrability alone. The asymptotic Bethe equations are given by

$$1 = \prod_{k} \frac{x_{1,j} - x_{2,k}^+}{x_{1,j} - x_{2,k}} \prod_{k} \frac{1 - \frac{1}{x_{1,j}x_{2,k}}}{1 - \frac{1}{x_{1,j}x_{2,k}}},$$

$$\left(\frac{x_{2,j}^+}{x_{2,j}^-}\right)^L = \prod_{k \neq j} \frac{x_{2,j}^+ - x_{2,k}^-}{x_{2,j}^- - x_{2,k}} \frac{1 - \frac{1}{x_{2,j}x_{2,k}}}{1 - \frac{1}{x_{2,j}x_{2,k}}} \sigma^2(x_{2,j},x_{2,k}) \prod_{k} \frac{x_{3,j} - x_{1,k}^-}{x_{3,j} - x_{1,k}} \frac{x_{3,j}^+ - x_{3,k}}{x_{3,j}^+ - x_{3,k}} \frac{1 - \frac{1}{x_{2,j}x_{3,k}}}{1 - \frac{1}{x_{2,j}x_{3,k}}} \prod_{k} \gamma(x_{2,j},x_{3,k})$$

$$1 = \prod_{k} \frac{x_{3,j} - x_{2,k}^+}{x_{3,j} - x_{2,k}} \frac{1 - \frac{1}{x_{3,j}x_{2,k}}}{1 - \frac{1}{x_{3,j}x_{2,k}}},$$

together with other three equations with $(1,2,3) \leftrightarrow (\bar{1},\bar{2},\bar{3})$ which are are identical in stucture to the above apart from the l.h.s of the middle equation that is reversed, i.e. is $\left(\frac{x_{2,j}^-}{x_{2,j}^+}\right)^L$.

The dressing phase factor $\sigma^2 = e^{i\vartheta}$ has the AFS [4] limit at leading order in strong coupling

$$\sigma_{\text{AFS}}(x_j, x_k) = \frac{1 - \frac{1}{x_{j}x_{k}}}{1 - \frac{1}{x_{j}^+x_{k}^-}} \left(\frac{x_{j}x_{j}^--1 - x_{j}^+x_{j}^-1}{x_{j}x_{j}^--1 - x_{j}^+x_{j}^-1}\right)^i h(\lambda + \frac{1}{x_{j}x_{j}^--1 - x_{j}^+x_{j}^-1})$$

$$= \frac{1 - \frac{1}{x_{j}x_{k}}}{1 - \frac{1}{x_{j}x_{k}}},$$
as required in order to match the classical finite-gap equations. Apart from this constraint and unitarity, nothing is known a priori about the dressing phase: it should be determined by the dynamics of the integrable system, i.e. its specific form is not fixed just by the symmetry structure.

The expression for the quantum string energy and the momentum constraint are

$$E = 2h(\lambda) \sum_{i=2,2}^{K_i} \left( \frac{1}{x_{i,\ell}^+} - \frac{1}{x_{i,\ell}^-} \right), \quad \prod_{\ell=1}^{K_1} x_{2,\ell}^+ \prod_{\ell=1}^{K_\sigma} x_{2,\ell}^- = 1. \quad (2.6)$$

These above equations describe bound states of $4_B + 4_F$ massive magnon modes with mass 1. From comparison with semiclassical string theory it follows that the relation between $h$ and $\lambda$ at strong coupling is

$$h(\lambda) = \frac{\sqrt{\lambda}}{4\pi} + O(1), \quad \lambda \gg 1. \quad (2.7)$$

As we shall find in the next section, the $O(1)$ 1-loop correction here vanishes in the $AdS_3 \times S^3 \times T^4$ case.

3 One-loop correction to energy of circular string with two equal spins in $S^3$: fixing leading quantum term in the dressing phase

Here we shall present the calculation of the one-loop correction to energy for the rigid circular string with two equal spins $J_1 = J_2$ in $S^3$ part of $AdS_3 \times S^3 \times T^4$. Similar computation in the $AdS_5 \times S^5$ case can be found in [23, 15]. Following [15], we shall extract the so-called non-analytic part of the 1-loop correction that should be arising from the dressing phase in the ABA and thus fix the subleading strong-coupling part of the coefficients $c_{r,s}$ in the phase.

3.1 Non-analytic term in one-loop string energy

The classical solution we consider here is exactly the same as in the $AdS_5 \times S^5$ case: the motion in the $S^3$ is described by

$$X_1 + iX_2 = \frac{1}{\sqrt{2}} e^{iJ_0 + im\sigma}, \quad X_3 + iX_4 = \frac{1}{\sqrt{2}} e^{iJ_0 - im\sigma} \quad (3.1)$$

where $X_k$ are the embedding coordinates on $S^3$ and the $AdS_3$ part of the solution is

$$Y_3 + iY_0 = e^{im\sigma}, \quad Y_1 = Y_2 = 0, \quad -Y_0^2 + Y_1^2 + Y_2^2 - Y_3^2 = -1. \quad (3.2)$$

Here the spins are $J_1 = J_2 = \frac{1}{2}\sqrt{\lambda}J$, $m$ is the winding number and the classical energy of this string is

$$E_0 = \sqrt{\lambda} \kappa, \quad \kappa = \sqrt{J^2 + m^2}. \quad (3.3)$$
The 1-loop correction to the energy is given by the sum of fluctuation frequencies:

\[ E_1 = \frac{1}{2\kappa} \sum_{n \in \mathbb{Z}} \left( \omega_n^B - \omega_n^F \right). \] (3.4)

The individual frequencies in the \( \text{AdS}_5 \times S_5 \) case were given in [26]. The 1-loop correction in the \( \text{AdS}_3 \times S^3 \times T^4 \) case is obtained from the \( \text{AdS}_5 \times S_5 \) one by a simple truncation – we remove two bosonic frequencies that correspond to fluctuations in the transverse directions of \( S^5 \), and halve the AdS and fermionic contributions. There will also be four bosonic and four fermionic massless modes coming from the \( T^4 \); we will not write them out explicitly as their contributions cancel each other. We are then left with two bosonic frequencies that come from the \( S^3 \) in which the string is rotating,

\[ \omega_n^B = \left[ n^2 + 2\kappa^2 - 2m^2 \pm 2\sqrt{(\kappa^2 - m^2)^2 + \kappa^2 n^2} \right]^{1/2}, \] (3.5)

two frequencies from the AdS part

\[ \omega_n^B = \sqrt{n^2 + \kappa^2}, \] (3.6)

and four fermionic frequencies

\[ \omega_n^F = \sqrt{n^2 + \kappa^2 - m^2} \pm \text{const}. \] (3.7)

The additive constant shifts are irrelevant as they will cancel in the result for (3.4) which is

\[ E_1 = \sum_{n \in \mathbb{Z}} e(n), \] (3.8)

\[ e(n) = \sqrt{1 + \frac{(n + \sqrt{n^2 - 4m^2})^2}{4(J^2 + m^2)}} + \sqrt{1 + \frac{n^2}{J^2 + m^2}} - 2\sqrt{1 + \frac{n^2 - m^2}{J^2 + m^2}}. \] (3.9)

It is straightforward to check that this sum is UV finite.

In the \( \text{AdS}_5 \times S^5 \) case, the computation of the large \( J \) expansion of the 1-loop energy made it possible to partially fix the coefficients of the leading quantum correction to the AFS dressing phase. To study the large \( J \) expansion in the present case we will use the same method as in [15]. When expanding \( e(n) \) at large \( J \) one gets terms with divergent sums over \( n \); separating out the convergent (i.e. regular) and divergent (i.e. singular) parts we can write \( e(n) = e^{\text{reg}}(n) + e^{\text{sing}}(n) \). To deal with the singular part we define \( e^{\text{int}}(x) = e(Jx) \) and expand it for large \( J \) at fixed \( x \), getting \( e^{\text{int}}(x) = e^{\text{reg}}(x) + e^{\text{sing}}(x) \) where \( e^{\text{sing}} \) is the part whose integral is divergent at \( x = 0 \). The regular part in one regime is in fact equal to the singular part in the other regime (as in \( \text{AdS}_5 \times S^5 \) case this can be checked order by order in the large \( J \) expansion)

\[ e^{\text{int}}(x) = e^{\text{reg}}(Jx), \quad e^{\text{sing}}(n) = e^{\text{reg}}(n/J), \] (3.10)

so that (3.8) takes the form

\[ E_1 = E_1^{\text{analytic}} + E_1^{\text{non-analytic}}, \] (3.11)
\[ E_{\text{analytic}}^{1} = \sum_{n \in \mathbb{Z}} e_{\text{reg}}^{\text{sum}}(n), \quad E_{\text{non-analytic}}^{1} \equiv \delta E_{1} = \int_{-\infty}^{\infty} J \, dx_{\text{reg}}^{\text{int}}(x). \]  

\((3.12)\)

\(E_{\text{analytic}}^{1}\) gives the “analytic” part of the 1-loop correction: its large \(J\) expansion contains only \textit{even} powers of \(J\) which translate to \textit{integer} powers of the coupling \(\lambda\) if we rewrite the result in terms of the total angular momentum \(J = \sqrt{\lambda}J\). The integral term, \(\delta E_{1}\), gives the “non-analytic” contribution: it contains \textit{odd} powers of \(J\) and thus leads to \textit{half-integer} powers of \(\lambda\) when expressed in terms of \(J\). This non-analytic part was responsible in the AdS\(_{5} \times S^{5}\) case for the famous “3-loop discrepancy”. In the present AdS\(_{3} \times S^{3} \times T^{4}\) case we find from (3.8),(3.9)

\[ \delta E_{1}^{\text{AdS}_{3}} = \frac{m^{4}}{2J^{3}} - \frac{7m^{6}}{12J^{5}} + \frac{29m^{8}}{48J^{7}} - \frac{97m^{10}}{160J^{9}} + \frac{2309m^{12}}{3840J^{11}} + \ldots . \]  

\((3.13)\)

This expansion is reproduced by the following closed expression

\[ \delta E_{1}^{\text{AdS}_{3}} = \frac{1}{\sqrt{J^{2} + m^{2}}} \left( m^{2} + J^{2} \log \frac{J^{2}}{m^{2} + J^{2}} \right), \]  

\((3.14)\)

that we found using the same method as used in [27], i.e. by rewriting the sum as a contour integral in the \(n\) plane. This method also shows that the analytic part is the same as it was for AdS\(_{5} \times S^{5}\) case because it is essentially determined by the \(S^{3}\) frequencies (3.5) only. On the contrary, for the non-analytic part, the contributions of \textit{all} frequencies are important.

For comparison, let us recall the corresponding non-analytic contribution in the AdS\(_{5} \times S^{5}\) case [15, 28]

\[ \delta E_{1}^{\text{AdS}_{5}} = \frac{1}{\sqrt{J^{2} + m^{2}}} \left( m^{2} + 2J^{2} \log \frac{J^{2}}{J^{2} + m^{2}} - \frac{J^{2} - m^{2}}{2} \log \frac{J^{2} - m^{2}}{J^{2} + m^{2}} \right) \]  

\[ = - \frac{m^{6}}{3J^{5}} + \frac{m^{8}}{3J^{7}} - \frac{49m^{10}}{120J^{9}} + \frac{2m^{12}}{5J^{11}} + \ldots . \]  

\((3.15)\)

### 3.2 Constraining the dressing phase in SU(2) sector

The above result for the non-analytic part of the one-loop energy is expected to originate from the dressing phase in the ABA equations. The Bethe ansatz equations in the \(su(2)\) sector, corresponding to strings with nontrivial motion only in the \(S^{3}\) part of the background, are the same as in the AdS\(_{5} \times S^{5}\) case:

\[ \left( \frac{x_{j}^{2}}{x_{j}^{2}} \right)^{L} = \prod_{k \neq j} x_{j}^{2} - x_{k}^{2} \frac{1 - \frac{1}{x_{j}^{2}x_{k}^{2}}}{1 - \frac{1}{x_{j}^{2}x_{k}^{2}}} \sigma^{2}(x_{j}, x_{k}). \]  

\((3.16)\)

They are obtained from the full set of ABA equations in section 2 by considering states with only \(x_{2}\) Bethe roots excited.

From these equations we can compute the ABA prediction for the non-analytic part of the 1-loop string energy similarly to how this was done in the AdS\(_{5} \times S^{5}\) case. Let us first assume that the dressing phase has the form (1.1) with summation starting from \(r = 2\) but keep
coefficients $c_{r,s}^{(1)}$ unfixed. Such structure of the phase would be in agreement with the proposal of [25] which is expected to apply to a large class of integrable systems. Following the same method as used in [16] we find that the Bethe ansatz then predicts that the non-analytic part of the 1-loop energy should be

$$\delta E_1 = \frac{m^6}{16J^5} c_{2,3}^{(1)} + \frac{m^8}{64J^7} (-5c_{2,3}^{(1)} - 2c_{2,5}^{(1)} + c_{3,4}^{(1)}) + \ldots . \tag{3.17}$$

This prediction is, however, in structural disagreement with the AdS$_3 \times S^3 \times T^4$ string result in (3.13) as the latter starts with a $1/J^3$ term, i.e. one order earlier than the expansion (3.15) obtained for $AdS_5 \times S^5$ case. This difference may be attributed to the reduced amount of supersymmetry (and thus supersymmetry protection) in the AdS$_3$ case compared to AdS$_5$ one.

This forces us to modify the structure of (1.1): we propose to include also the $c_{r,s}^{(1)}$ coefficients, i.e. to assume that the summation in the phase (1.1) should start from $r = 1$. Then the Bethe ansatz prediction becomes

$$\delta E_1 = \frac{m^4}{4J^3} c_{1,2}^{(1)} - \frac{m^6}{16J^5} (4c_{1,2}^{(1)} + c_{1,4}^{(1)} - c_{2,3}^{(1)}) + \frac{m^8}{64J^7} (15c_{1,2}^{(1)} + 5c_{1,4}^{(1)} + 2c_{1,6} - 5c_{2,3}^{(1)} - 2c_{2,5}^{(1)} + c_{3,4}^{(1)}) + \ldots , \tag{3.18}$$

which reduces to (3.17) in the $AdS_5 \times S^5$ case where one has $c_{1,s}^{(1)} = 0$. Comparing to (3.13), we find that in the $AdS_3 \times S^3 \times T^4$ case

$$c_{1,2}^{(1)} = 2 , \tag{3.19}$$

$$c_{1,4}^{(1)} = \frac{4}{3} + c_{2,3}^{(1)} , \tag{3.20}$$

$$c_{1,6}^{(1)} = \frac{1}{2} (2c_{2,5}^{(1)} - c_{3,4}^{(1)} + 2) . \tag{3.21}$$

Our proposed coefficients $c_{r,s}^{(1)}$ in (1.6) are consistent with these relations.

In the above derivation we assumed that the 1-loop correction to $h(\lambda)$ is zero, i.e.

$$h(\lambda) = \frac{\sqrt{\lambda}}{4\pi} + a + O\left(\frac{1}{\sqrt{\lambda}}\right) , \quad a_{AdS_3 \times S^3 \times T^4} = 0 . \tag{3.22}$$

Having a nonzero would produce (after replacing $\sqrt{\lambda}$ by $4\pi h$ in the classical energy (3.3)) an extra $1/J$ term in $\delta E_1^{AdS_3}$ which is absent in (3.13).

Let us note that the analytic part of the energy, which, as we discussed in Section 3, is the same in $AdS_3 \times S^3 \times T^4$ and $AdS_5 \times S^5$ cases, is correctly reproduced from the ABA (3.16), since it is only sensitive to the AFS part of the phase.

The dressing phase should be universal, i.e. the same phase should be possible to extract also from the study of other classical solutions. Indeed, as we shall find in Appendix A, the same relations (3.19)–(3.21) follow also from the expression for the non-analytic part of the 1-loop energy of the large spin $(S,J)$ folded string solution in the $SL(2)$ sector. This is not totally
surprising as the two solutions are related by an analytic continuation [29]; nevertheless, this is a nontrivial check, since on the Bethe ansatz side the two calculations are quite different. Moreover, as we shall explain in detail in sections 4-6, the same 1-loop phase can be found for a generic semiclassical solution using the algebraic curve method used in $AdS_5 \times S^5$ case in [18].

4 Semiclassical dressing phase from the algebraic curve approach

Considering the strong-coupling (string semiclassical) expansion in the Bethe equations, the leading term is given by the integral equations parametrized by an algebraic curve which represents a generic finite gap string solution. Starting with an algebraic curve description of such string solution one may compute the 1-loop correction by summing up the corresponding fluctuation frequencies and then extract the dressing phase contribution. This powerful approach has been developed in [18, 30] for the $AdS_5 \times S^5$ case (see also [31] where the algebraic curve method is reviewed). Here we will use the same method in the $AdS_3 \times S^3 \times T^4$ case.

4.1 Scaling limit of the Bethe equations and finite-gap equations

Let us introduce the function

$$\mathring{\alpha}(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1}, \quad (4.1)$$

and define the discrete resolvents

$$G_a(x) = \sum_{k=1}^{K_a} \frac{\mathring{\alpha}(x_{a,k})}{x - x_{a,k}}, \quad H_a(x) = \sum_{k=1}^{K_a} \frac{\mathring{\alpha}(x)}{x - x_{a,k}}, \quad \overline{G}(x) = G(1/x), \quad \overline{H}(x) = H(1/x). \quad (4.2)$$

Expand the quantum Bethe equations equations at large $\hbar \sim \sqrt{\lambda}$ we find for first three equations in (2.2)–(2.4)

$$x \in \mathcal{C}_1, \quad 2\pi n_1 = -G_2 - \overline{H}_2 - \frac{G_T(0) + G'_T(0)}{x^2 - 1} x,$$

$$x \in \mathcal{C}_2, \quad 2\pi n_2 + \frac{4\pi \mathcal{J} x}{x^2 - 1} = 2H_2 - H_1 - H_3 + 2\overline{H}_2 - \overline{H}_1 - \overline{H}_3 + 2 \frac{G_T(0) - G_2(0)}{x^2 - 1} x,$$

$$x \in \mathcal{C}_3, \quad 2\pi n_3 = -G_2 - \overline{H}_2 - \frac{G_T(0) + G'_T(0)}{x^2 - 1} x. \quad (4.3)$$

These match the finite-gap equations in Eqs.(7.39)-(7.41) of [2] upon use of the identity

$$G(x) = H(x) - \frac{G(0) + G'(0)}{x^2 - 1} x. \quad (4.4)$$

We use $\mathring{\alpha}(x)$ notation instead of the standard $\alpha(x)$ to avoid confusion with the background parameter $\alpha$ used in other sections.
4.2 Quasi-momenta and algebraic curve

The finite-gap equations can be written as (the equation with $n_{\ell}$ is evaluated at $x \in \mathcal{C}_{\ell}$)

$$2\pi n_1 = -H_2 - \bar{\mathcal{T}}_\pi + \frac{G_2(0) + xG'_2(0)}{x^2 - 1} - \frac{G_\pi(0) + xG'_\pi(0)}{x^2 - 1}, \quad (4.5)$$

$$2\pi n_2 + \frac{4\pi \mathcal{J} x}{x^2 - 1} = 2H_2 - H_1 - H_3 + 2\bar{\mathcal{T}}_2 - \bar{\mathcal{T}}_\pi - 2 + 2\frac{G_\pi(0) - G_2(0)}{x^2 - 1}, \quad (4.6)$$

$$2\pi n_3 = -H_2 - \bar{\mathcal{T}}_\pi + \frac{G_2(0) + xG'_2(0)}{x^2 - 1} - \frac{G_\pi(0) + xG'_\pi(0)}{x^2 - 1}, \quad (4.7)$$

$$2\pi n_\pi = -H_2 - \bar{\mathcal{T}}_\pi - \frac{G_2(0) + xG'_2(0)}{x^2 - 1} + \frac{G_\pi(0) + xG'_\pi(0)}{x^2 - 1}, \quad (4.8)$$

$$2\pi n_\pi - \frac{4\pi \mathcal{J} x}{x^2 - 1} = 2H_2 - H_1 - H_3 + 2\bar{\mathcal{T}}_2 - \bar{\mathcal{T}}_\pi - 2 + 2\frac{G_\pi(0) - G_2(0)}{x^2 - 1}, \quad (4.9)$$

$$2\pi n_\pi = -H_2 - \bar{\mathcal{T}}_\pi - \frac{G_2(0) + xG'_2(0)}{x^2 - 1} + \frac{G_\pi(0) + xG'_\pi(0)}{x^2 - 1}. \quad (4.10)$$

Building on the work of [32], we set \(^6^)

$$p_1 = -p_4 = -\frac{1}{2}H_1 - \frac{1}{2}\bar{\mathcal{T}}_1 - \frac{1}{2}H_3 - \frac{1}{2}\bar{\mathcal{T}}_3 - \frac{2\pi \mathcal{J} x}{x^2 - 1} + \frac{x}{x^2 - 1}[G_2(0) - G'_2(0)], \quad (4.11)$$

$$p_2 = -p_3 = H_2 + \bar{\mathcal{T}}_2 - \frac{1}{2}H_1 - \frac{1}{2}\bar{\mathcal{T}}_1 - \frac{1}{2}H_3 - \frac{1}{2}\bar{\mathcal{T}}_3 - \frac{2\pi \mathcal{J} x}{x^2 - 1}, \quad (4.12)$$

$$p_\pi = -p_\pi = -\frac{1}{2}H_1 - \frac{1}{2}\bar{\mathcal{T}}_1 - \frac{1}{2}H_3 - \frac{1}{2}\bar{\mathcal{T}}_3 + \frac{2\pi \mathcal{J} x}{x^2 - 1} + \frac{x}{x^2 - 1}[G_\pi(0) - G'_\pi(0)], \quad (4.13)$$

$$p_\pi = -p_\pi = H_2 + \bar{\mathcal{T}}_2 - \frac{1}{2}H_1 - \frac{1}{2}\bar{\mathcal{T}}_1 - \frac{1}{2}H_3 - \frac{1}{2}\bar{\mathcal{T}}_3 + \frac{2\pi \mathcal{J} x}{x^2 - 1}. \quad (4.14)$$

Up to winding contributions, we have

$$p_{1,2,3,4}(x) = p_{\pi,2,3,\pi}(1/x). \quad (4.15)$$

The above finite-gap equations are obtained with $p_i - p_j = 2\pi n_{ij}$ where

$$(i, j) = (1, 2), (2, 3), (3, 4), \quad (\bar{1}, \bar{2}), (\bar{2}, \bar{3}), (\bar{3}, \bar{4}). \quad (4.16)$$

Note that here the algebraic curve is a connected sum of two pieces interchanged by the $x \to 1/x$ transformation, while in the $AdS_5 \times S^5$ case the curve is a single connected invariant piece.

4.3 Semiclassical one-loop dressing factor

The semiclassical one-loop dressing factor is built according to the prescription in [18]. Fig.1 gives the picture of the Dynkin nodes, algebraic curve sheets and physical fluctuations for the unbarred $PSU(1,1|2)$ factor.

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\(^6^\)For bosonic classical solutions in $PSU(1,1|2)$ we can set $p_1 + p_4 = p_2 + p_3 = 0$ and similarly for the barred quasi-momenta.
Figure 1: Setup for computation of $\mathcal{V}$.

For each quasi-momentum $p_I$ (with $I \in \{1, 2, 3, 4, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$), we are to compute the corresponding correction $V_I$ which is the following sum over all polarizations (i.e. from both $PSU(1,1|2)$ symmetry factors)

$$V_I(x) = \frac{1}{2} \sum_{ij} (-1)^{F_{ij}} \int \frac{dy}{2\pi} \left( p'_i - p'_j \right) \left[ \delta H_{ij} \frac{\hat{\alpha}(x)}{x-y} + \delta \tilde{H}_{ij} \frac{\hat{\alpha}(1/x)}{1/x-y} \right].$$  \hspace{1cm} (4.17)

Here, for each polarization $(ij)$, we determine the Bethe roots $B_{ij} \subset \{u_{1,2,3}, u_{1,2,\bar{3},\bar{3}}\}$ that lie between the sheets $i$ and $j$ and evaluate the total variation of the functions $H$ and $\tilde{H}$ appearing in $p_I$ due to the addition of one root for each element of $B_{ij}$. The total phase corrections to the Bethe equations are obtained by evaluating $V_I - V_J$. The result for the middle node 2 equation is given by

$$\mathcal{V}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[ \left( G_{2}(y) + \tilde{G}_{2}(y) \right) \frac{\hat{\alpha}(x)}{x-y} + \left( \tilde{G}_{2}(y) + G_{2}(y) \right) \frac{\hat{\alpha}(1/x)}{1/x-y} \right].$$  \hspace{1cm} (4.18)

The equations for the nodes 1, 3 are not corrected. The potential $\mathcal{V}$ contains the correction to the two dressing contributions. If we excite only the node 2, it reduces to

$$\mathcal{V}_2(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[ G'_{2}(y) \frac{\hat{\alpha}(x)}{x-y} + G'_{2}(y) \frac{\hat{\alpha}(1/x)}{1/x-y} \right].$$  \hspace{1cm} (4.19)

where the notation is $\int_{-1}^{1} = \frac{1}{2} \int_{C^+} + \frac{1}{2} \int_{C^-}$ and the half circumferences $C^\pm$ (and their orientation) are defined in the caption of figure 4 of [18].

The next step is to evaluate $\mathcal{V}_2(x)$ for large $x$. This is done by factoring out $\hat{\alpha}(x)$ in the integrand of (4.19) and expanding at large $x$. Using the relation between $G_2$ and the charges $Q_n$,

$$G_2(y) = - \sum_{n=0}^{\infty} Q_{n+1} y^n,$$  \hspace{1cm} (4.20)
the resulting function of \( y \) is not singular anywhere on the circle \( |y| = 1 \) and the integration is trivial. The result is

\[
\mathcal{V}_2(x) = \frac{\tilde{\alpha}(x)}{2\pi} \sum_{r=2}^{\infty} \sum_{s=1}^{\infty} \tilde{c}_{r,s} \frac{Q_r}{x^s},
\]

(4.21)

\[
\tilde{c}_{r,s} = -4 \left(1 - \frac{1}{2} \delta_{s,1}\right) \frac{1 - (-1)^{r+s}}{2} \frac{r - 1}{r + s - 2}.
\]

(4.22)

Here \( \tilde{c}_{r,s} \) are the (naive) prediction of the algebraic curve method for the values of the \( c_{r,s}^{(1)} \) coefficients which parametrize the phase according to (1.2). We notice that the coefficients \( \tilde{c}_{r,s} \) are not antisymmetric. This is a serious problem since the antisymmetry of the coefficients \( c_{r,s} \) in the phase is an important consistency requirement (see section 4.4 below).

Indeed, in section 5 we will show on the example of the SU(2) sector circular string that the part that breaks the antisymmetry induces a mismatch with the string theory result (3.14) for the non-analytic term in the one-loop energy. This disagreement turns out to be due to a regularization ambiguity in the sum over fluctuation frequencies. Once this regularization problem is fixed, the algebraic curve approach result agrees with the string theory result and the the antisymmetry of \( c_{r,s} \) is recovered.

### 4.4 On consistency condition on the phase

Let us make a comment concerning the orig of antisymmetry of the phase coefficients \( c_{r,s} \) and why this antisymmetry is not obvious in the algebraic curve approach in the present \( AdS_3 \) case.

If we consider the \( sl(2) \) Bethe equations written in the form

\[
\left(\frac{x_i^+}{x_i^-}\right)^J = \prod_{J \neq i} \frac{x_i^- - x_j^+}{x_i^+ - x_j^-} \frac{1 - \frac{1}{x_i^+ x_j^-}}{1 - \frac{1}{x_i^- x_j^+}} e^{i \vartheta_{ij}}
\]

(4.23)

and take the product over \( i \), we find

\[
\sum_{i,j} \vartheta_{ij} = 0.
\]

(4.24)

This is automatic if the coefficients \( c_{r,s} \) which define the phase \( \vartheta_{ij} \) are antisymmetric.

While this constraint is thus a direct consequence of the discrete form of ABA equations, it is not automatic in the thermodynamic (“semiclassical”) limit due to infinite summation and thus regularization issues involved. The relation (4.24) implies the following condition for the potential (4.18) that determines the dressing phase:

\[
\sum_k \mathcal{V}(x_k) = 0.
\]

(4.25)

In the \( AdS_5 \times S^5 \) case we have

\[
\mathcal{V}^{AdS_5}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[ G(y) - G(1/y) \right]' \left[ \frac{\tilde{\alpha}(x)}{x-y} - \frac{\tilde{\alpha}(1/x)}{1/x-y} \right].
\]

(4.26)
Using the relations
\[ \sum_k \hat{\alpha}(x_k) = -G(y), \quad \sum_k \frac{\hat{\alpha}(1/x_k)}{1/x_k - y} = G(0) - G(1/y), \] (4.27)
we find indeed that
\[ \sum_k \mathcal{V}^{AdS_5}(x_k) = \int_{-1}^{1} \frac{dy}{2\pi} \left[ -\frac{1}{2} \left( G(y) - G(1/y) \right)^2 - G(0) \left( G(y) - G(1/y) \right) \right] = 0. \] (4.28)

Instead, in the $AdS_3 \times S^3 \times T^4$ case we have
\[ \mathcal{V}^{AdS_3}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[ G'(y) \frac{\hat{\alpha}(x)}{x - y} + G'(1/y) \frac{\hat{\alpha}(1/x)}{1/x - y} \right], \] (4.29)
and thus
\[ \sum_k \mathcal{V}^{AdS_3}(x_k) = \int_{-1}^{1} \frac{dy}{2\pi} \left\{ -\frac{1}{2} \left[ G^2(y) + G^2(1/y) \right] + G(0)G(1/y) \right\} = 0. \] (4.30)

does not vanish automatically. Even assuming that $G(0) = 0$ we get an a priori non-vanishing term
\[ \sum_k \mathcal{V}^{AdS_3}(x_k) = -\frac{1}{2\pi} \left[ G^2(1) - G^2(-1) \right], \] (4.31)
implying that the dressing phase coefficients $c_{r,s}$ coming from this potential will not be automatically antisymmetric. This is, indeed, what we have found above in (4.22).

However, as we shall explain below, it is possible to adjust the regularization involved in the definition of the “semiclassical” limit (subtracting from the potential a regularization-related part) so that to ensure the vanishing of (4.31) and thus the antisymmetry of the $c_{r,s}$ coefficients in the $AdS_3 \times S^3 \times T^4$ theory.

5 Algebraic curve approach applied to $SU(2)$ circular string case: regularization ambiguity

Let us now apply the discussion of the previous section to the circular string example discussed from the string theory perspective in section 3.

5.1 Non-analytic part of one-loop energy

The expansion of $\mathcal{V}_2(x)$ in (4.19),(4.21) can be analysed using the strategy developed in [16]. One considers a dressing contribution which is the usual combination (1.1) of charges with some unknown coefficients $c_{r,s}$. Then one perturbs the quadratic equation for the resolvent associated to a given solution of the finite-gap integral equation. The result is a compact expression for
the dressing correction to the string energy. Applied to the $SU(2)$ circular string case the ingredients in this expression are the classical charges $Q_n(m, J)$ defined by

$$ - \sum_{n=0}^{\infty} Q_{n+1} x^n = 2 \pi m + \frac{\sqrt{1 + (m/J)^2} - \sqrt{1 + (4\pi m x)^2}}{2 \left( x - \frac{1}{(4\pi J)^2} \right)}. \quad (5.1) $$

The energy correction for the perturbation associated with $\mathcal{V}_2$ computed in the algebraic curve approach is then

$$ \delta E_{1AC} = - \frac{1}{\pi \left[ 1 + \frac{2Q_2}{(4\pi J)^2} \right]} \sum_{s \geq 1} \left( \frac{1}{4\pi J} \right)^{r+s} \tilde{c}_{r,s} Q_{s+1} Q_r. \quad (5.2) $$

where the coefficients $\tilde{c}_{r,s}$ were defined in (4.22). Expanding this correction at large $J$ we get

$$ \delta E_{1AC} = \frac{m^4}{2J^3} - \frac{17m^6}{24J^5} + \frac{41m^8}{48J^7} - \frac{623m^{10}}{640J^9} + \frac{4139m^{12}}{3840J^{11}} - \frac{126079m^{14}}{107520J^{13}} + \frac{18069m^{16}}{14336J^{15}} + \ldots \quad (5.3) $$

This is nothing but the expansion of

$$ \delta E_{1AC} = \frac{1}{\sqrt{m^2 + J^2} \left( m^2 + J^2 \log \frac{J^2}{m^2 + J^2} \right)} - \frac{m^2 \left( 2J \left( J - \sqrt{J^2 + m^2} \right) + m^2 \right)}{2 \left( J^2 + m^2 \right)^{3/2}}, \quad (5.4) $$

where $\delta E_{1AdS}$ is the string-theory result in (3.14), while $\Delta E_1$ is a discrepancy. As we shall explain below, the latter is related to an implicit choice of regularization in the algebraic curve approach.\(^7\)

### 5.2 Regularization origin of the mismatch

Let us rederive the result (5.4) directly. First, let us relabel the quasi momenta for the two $PSU(1,1|2)$ factors as follows

$$ \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \quad PSU(1,1|2)_+ \quad \tilde{p}_2, \tilde{p}_1, \tilde{p}_4, \tilde{p}_3, \quad PSU(1,1|2)_- \quad (5.5) $$

The physical $4_B + 4_F$ (massive) polarizations are organized as

$$ B : \quad (\tilde{2} \ 3), \quad (\tilde{1} \ 4), \quad (\tilde{1} \ 3), \quad (2 \ 3) \quad (5.6) $$

$$ F : \quad (\tilde{1} \ 3), \quad (\tilde{2} \ 4), \quad (\tilde{2} \ 3), \quad (\tilde{1} \ 3). \quad (5.7) $$

The explicit quasi momenta are ($\kappa = \sqrt{J^2 + m^2}$)

$$ \tilde{p}_1 = \tilde{p}_2 = -\tilde{p}_3 = -\tilde{p}_4 = \frac{2 \pi \kappa x}{x^2 - 1}, \quad (5.8) $$

\(^7\)For the $AdS_4 \times CP^3$ background, related regularization issues were discussed in [33] for the folded string and in [34] for giant magnon solutions.
\[ \tilde{p}_2 = -\tilde{p}_3 = 2\pi \frac{x}{x^2 - 1} \sqrt{m^2 x^2 + J^2} - 2\pi m, \quad (5.9) \]

\[ \tilde{p}_1 = -\tilde{p}_4 = 2\pi \frac{x}{x^2 - 1} \sqrt{m^2 / x^2 + J^2}. \quad (5.10) \]

Computing the off-shell frequencies \( \Omega^{ij} \) for each polarization and defining

\[ \Omega^{ij}_n = \Omega^{ij}(x^{ij}_n), \quad (5.11) \]

where \( x^{ij}_n \) is the solution of \( p_i - p_j = 2\pi n \), we find

\[ \Omega^{2\bar{3}}_n = \sqrt{\frac{2J^2 + M^2 - 2\sqrt{J^4 + m^2 M^2 + J^2 M^2}}{J^2 + m^2}}, \quad M = n + 2m, \quad (5.12) \]

\[ \Omega^{1\bar{4}}_n = \sqrt{\frac{J^2 + m^2 + n^2}{J^2 + m^2} - 1}, \quad (5.13) \]

\[ \Omega^{1\bar{3}}_n = \Omega^{2\bar{4}}_n = \sqrt{\frac{J^2 + M^2}{J^2 + m^2} - 1}, \quad M = n + m, \quad (5.14) \]

\[ \Omega^{1\bar{4}}_n = -\frac{2J}{\kappa} + \sqrt{\frac{2J^2 + M^2 + 2\sqrt{J^4 + m^2 M^2 + J^2 M^2}}{J^2 + m^2}}, \quad M = n, \quad (5.15) \]

\[ \Omega^{2\bar{3}}_n = \sqrt{\frac{J^2 + m^2 + n^2}{J^2 + m^2} - 1}, \quad (5.16) \]

\[ \Omega^{2\bar{4}}_n = \Omega^{1\bar{3}}_n = -\frac{J}{\kappa} + \sqrt{\frac{J^2 + M^2}{J^2 + m^2} - 1}, \quad M = n. \quad (5.17) \]

Notice that for the evaluation of the non-analytic part of the 1-loop correction to the energy, any finite set of modes is irrelevant since each mode contribution is separately analytic in the large \( J \) expansion. The non-analytic contribution arises from the infinite summation. Note also that, with our choice of quasimomenta, the first set of four frequencies have a shift \( M \neq n \). In terms of \( M \), these frequencies are the same as in the string world-sheet calculation in section 3.

The one-loop energy can then be computed as usual as the sum over polarizations or in terms of the integral representation (see, e.g., [18, 30, 35])

\[ E_1 = \sum_{ij} (-1)^{F_{ij}} \oint \frac{dx}{2\pi i} \Omega^{ij}(x) \partial_x \log \sin \frac{p_i - p_j}{2\pi}, \quad (5.18) \]

where the integration encircles the points \( x^{ij}_n \). This contour can be transformed in the unit circumference as usual up to cut terms that do not contribute to the non-analytic part. Neglecting exponentially suppressed contributions in the large \( J \) limit, we checked numerically that from (5.18) we obtain precisely the result (5.4), i.e. the sum of \( \delta E_1^{AdS_3} \) plus an extra term \( \Delta E_1 \).

Let now show that the origin of the extra term \( \Delta E_1 \) is due to a particular choice of regularization used in the algebraic curve approach. When we evaluate the unit circumference
contribution, we consider a contour like that shown in Fig. (2) and it is the same for all polarizations.

![Figure 2: Contour defining the AC regularization](image)

The crosses are the poles $x_{n}^{ij}$ and the small part of circle $\gamma$ around $x = 1$ determines a cut-off on $n$ that depends on the polarization $(ij)$. To see this, one can expand at small $\varepsilon$ the differences

$$M_{ij} = \frac{1}{2\pi} [p_{i}(x) - p_{j}(x)]$$

(5.19)

after setting $^{8}$

$$x = 1 + \kappa \varepsilon + \frac{\kappa^{2}}{2} \varepsilon^{2}.$$  

(5.20)

The explicit results are

$$M_{\tilde{2} \tilde{3}} = \frac{1}{\varepsilon} + \left(\frac{m^{2}}{\sqrt{m^{2} + \mathcal{J}^{2}}} - 2m\right) + \left(\frac{m^{4}}{4m^{2} + 4\mathcal{J}^{2}} + \frac{2\mathcal{J}^{2}m^{2}}{4m^{2} + 4\mathcal{J}^{2}}\right) \varepsilon + O\left(\varepsilon^{2}\right),$$

(5.21)

$$M_{\tilde{1} \tilde{4}} = \frac{1}{\varepsilon} + O\left(\varepsilon^{2}\right),$$

(5.22)

$$M_{\tilde{1} \tilde{3}} = \frac{1}{\varepsilon} + \left(\frac{m^{2}}{2\sqrt{m^{2} + \mathcal{J}^{2}}} - m\right) + \left(\frac{m^{4}}{4m^{2} + 4\mathcal{J}^{2}} + \frac{2\mathcal{J}^{2}m^{2}}{4m^{2} + 4\mathcal{J}^{2}}\right) \varepsilon + O\left(\varepsilon^{2}\right),$$

(5.23)

$$M_{\tilde{2} \tilde{4}} = \frac{1}{\varepsilon} + \left(\frac{m^{2}}{2\sqrt{m^{2} + \mathcal{J}^{2}}} - m\right) + \left(\frac{m^{4}}{4m^{2} + 4\mathcal{J}^{2}} + \frac{2\mathcal{J}^{2}m^{2}}{4m^{2} + 4\mathcal{J}^{2}}\right) \varepsilon + O\left(\varepsilon^{2}\right),$$

(5.24)

$$M_{\tilde{1} \tilde{4}} = \frac{1}{\varepsilon} + \left(\frac{m^{2}}{\sqrt{m^{2} + \mathcal{J}^{2}}} + \left(\frac{m^{4}}{2m^{2} + 2\mathcal{J}^{2}} + \frac{2\mathcal{J}^{2}m^{2}}{2m^{2} + 2\mathcal{J}^{2}}\right) \varepsilon + O\left(\varepsilon^{2}\right),$$

(5.25)

$$M_{\tilde{2} \tilde{3}} = \frac{1}{\varepsilon} + O\left(\varepsilon^{2}\right),$$

(5.26)

$^{8}$Here the expression for $x$ in terms of $\varepsilon$ is not an expansion, but just a convenient quadratic parametrization $x(\varepsilon)$ with the property that $x(\varepsilon) \to 1$ when $\varepsilon \to 0$. 

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\[ M_{\tilde{2}\tilde{4}} = \frac{1}{\epsilon} - \frac{m^2}{2\sqrt{m^2 + J^2}} + \left( \frac{m^4}{4m^2 + 4J^2} + \frac{2J^2m^2}{4m^2 + 4J^2} \right) \epsilon + O(\epsilon^2) \]  
(5.27)

\[ M_{\tilde{1}\tilde{3}} = \frac{1}{\epsilon} - \frac{m^2}{2\sqrt{m^2 + J^2}} + \left( \frac{m^4}{4m^2 + 4J^2} + \frac{2J^2m^2}{4m^2 + 4J^2} \right) \epsilon + O(\epsilon^2). \]  
(5.28)

At this point, it is convenient to introduce the “offsets”

\[ \Delta_{ij} = (-2m, 0, -m, -m, 0, 0, 0, 0). \]  
(5.29)

These are such that the sum over the frequencies with a common (i.e. universal) cutoff \( M \leq \frac{1}{\epsilon} + \Delta_{ij} \) reproduces by construction the string-theory result \( \delta E_{AdS_3}^{1} \) in the \( \epsilon \to 0 \) limit. This may be called the standard regularization for the non-analytic (dressing) contribution, as compared to the AC regularization which is \( M \leq M_{ij} \) with \( \epsilon \to 0 \).

To evaluate the difference between the results for the two regularization prescriptions we have to evaluate the additional terms in the sum from \( M = \frac{1}{\epsilon} + \Delta_{ij} \) to \( M = M_{ij} \). Using the Maclaurin summation formula we get

\[ \Delta E_1 = \lim_{\epsilon \to 0} \sum_{ij} (-1)^{F_{ij}} \left[ \frac{1}{2} \Omega^{ij} \left( \frac{1}{\epsilon} + \Delta_{ij} \right) + \int_{\frac{1}{\epsilon} + \Delta_{ij}}^{M_{ij}} dM \Omega^{ij}(M) \right]. \]  
(5.30)

The integral here can be done by replacing the integrand by its \( O(x) \) and \( O(x^0) \) terms in the large \( x \) expansion up to terms that vanish as \( \epsilon \to 0 \). The computation gives exactly the expression in (5.4)

\[ \Delta E_1 = -\frac{m^2 \left[ 2J (J - \sqrt{J^2 + m^2} + m^2) \right]}{2 \left( J^2 + m^2 \right)^{3/2}}, \]  
(5.31)

explaining the regularization origin of the discrepancy.

### 6 Proposal for the 1-loop dressing phase coefficients from the algebraic curve approach

As was noted in section 4, one should expect to find a set of antisymmetric coefficients \( c_{r,s} \) in \( \psi \) in (1.1) as this is a consequence of the antisymmetry of the elementary magnon scattering phases. The regularization ambiguity that we have discussed in the previous section should be fixed to ensure this antisymmetry. Here we will show that enforcing the antisymmetry of \( c_{r,s} \) selects, for the \( SU(2) \) circular string case discussed above, the standard regularization, removing the mismatch with string-theory result.

Motivated by a discussion in [7] let us try to enforce the antisymmetry by integrating by parts. We start with the potential correcting the Bethe equation for the left moving sector middle node (4.18), i.e. (here \( f' = \frac{\partial}{\partial y} f \))

\[ \mathcal{V}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[ \left( G_2(y) + \overline{G}_2(y) \right)' \frac{\tilde{\alpha}(x)}{x-y} + \left( \overline{G}_2(y) + G_2(y) \right)' \frac{\tilde{\alpha}(1/x)}{1/x-y} \right]. \]  
(6.1)
Let us integrate by parts and define
\[
\tilde{\mathcal{V}}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[ \left( G_2(y) + \overline{G_7}(y) \right) \frac{\hat{\alpha}(x)}{x-y} - \left( \overline{G_2(y)} + G_7(y) \right) \left( \frac{\hat{\alpha}(1/x)}{1/x-y} \right) \right].
\] (6.2)

The difference is
\[
\mathcal{V}(x) - \tilde{\mathcal{V}}(x) = \frac{\hat{\alpha}(1/x)}{2\pi} \frac{G_2(1) + G_7(1)}{1/x - 1} - \frac{G_2(-1) + G_7(-1)}{1/x + 1}.
\] (6.3)

The large $x$ expansion of $\tilde{\mathcal{V}}(x)$ is
\[
\tilde{\mathcal{V}}(x) = \frac{\hat{\alpha}(x)}{2\pi} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} c_{r,s}^{(1)} \frac{Q_r - \overline{Q}_{r,s}}{x^s},
\] (6.4)

where the expansion coefficients are now \textit{antisymmetric}:
\[
c_{r,s}^{(1)} = 2 \frac{1 - (-1)^{r+s}}{2} \frac{s-r}{r+s-2}, \quad c_{r,s}^{(1)} = -2 \frac{1 - (-1)^{r+s}}{2} \frac{r+s-2}{s-r}.
\] (6.5)

These are the expressions for the phase coefficients that we announced in the Introduction, see eqs. (1.3) and (1.6). These coefficients (only $c_{r,s}^{(1)}$ is actually contributing) now lead precisely to the string theory expression $\delta E_1^{AdS_3}$ that we got in (3.14) for the non-analytic term in the circular $SU(2)$ string case, i.e. (cf. (5.2))
\[
\delta E_1^{AdS_3} = \frac{1}{\pi} \left[ 1 + \frac{2Q_2}{(4\pi J)^2} \right] \sum_{r \geq 1} \sum_{s \geq r+1} \left( \frac{1}{4\pi J} \right)^{r+s} \frac{s-r}{r+s-2} \left( Q_{r+1} Q_s - Q_r Q_{s+1} \right),
\] (6.6)

This suggests that (6.3) is responsible for the regularization mismatch term $\Delta E_1$ in (5.31). A hint in this direction is that the $G$ functions in (6.3) are evaluated at $y = 1$ and this is the large $n$ region where regularization issues are relevant.

We can now compute the scattering phases between magnons as in the final part of section 3.2 of [18]. To this aim, we identify the dressing phase contribution in (2.3) as
\[
e^{i \tilde{\mathcal{V}}_{x_2,i}} = \prod_{k \neq j}^{K_2} \sigma^2(x_{2,j}, x_{2,k}) \prod_{k}^{K_7} \sigma^{-2}(x_{2,j}, x_{7,k}) = \prod_{k \neq j}^{K_2} e^{\vartheta(x_{2,j}, x_{2,k})} \prod_{k}^{K_7} e^{-i \vartheta(x_{2,j}, x_{7,k})}.
\] (6.7)

Using the discrete definition (4.2) of the function $G$ and integrating over $y$ in (6.2) we obtain
\[
\vartheta(x, y) = -\frac{\hat{\alpha}(x) \hat{\alpha}(y)}{2\pi (x-y)^2} \left[ 2 \log \left( \frac{x+1}{x-1} \frac{y-1}{y+1} \right) + 2 \frac{(x-y)(x^2+y^2-2)}{(x^2-1)(y^2-1)} \right],
\] (6.8)

\[
\overline{\vartheta}(x, y) = -\frac{\hat{\alpha}(x) \hat{\alpha}(y)}{2\pi (1-x-y)^2} \left[ 2 \log \left( \frac{x+1}{x-1} \frac{y-1}{y+1} \right) - 2 \frac{(x-y)(x^2+y^2-1)}{(x^2-1)(y^2-1)} \right].
\] (6.9)
We notice that
\[ \vartheta(x, y) + \tilde{\vartheta}(x, y) = \vartheta_{\text{AdS}_5}(x, y), \]  
(6.10)
where \( \vartheta_{\text{AdS}_5} \) is the corresponding \( \text{AdS}_5 \times S^5 \) expression for the 1-loop phase (see eq.(28) in [18])
\[ \vartheta_{\text{AdS}_5} = -\frac{\hat{\alpha}(x)\hat{\alpha}(y)}{\pi} \left[ \left( \frac{1}{(x - y)^2} + \frac{1}{(xy - 1)^2} \right) \log \left( \frac{x + 1}{x - 1} + \frac{1}{y} \right) + \frac{2}{(x - y)(xy - 1)} \right]. \]  
(6.11)

7 Non-analytic term in the 1-loop energy of \( SU(2) \) circular string in \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \)

In this section we shall compute the 1-loop energy correction for the \( SU(2) \) circular string moving in the string model based on the \( \alpha \)-dependent \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) background. We shall discuss the structure of the non-analytic contribution for generic \( \alpha \) as well as in the \( \alpha \to 1 \) limit corresponding to \( \text{AdS}_3 \times S^3 \times T^4 \) case.

7.1 The classical solution

We write the metric of \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) as
\[ ds^2 = ds^2_{\text{AdS}_3} + \frac{1}{\alpha} ds^2_{S^3} + \frac{1}{1 - \alpha} ds^2_{S^3} + d\psi^2, \]  
(7.1)
\[ ds^2_{\text{AdS}_3} = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\phi^2, \quad ds^2_{S^3} = d\gamma^2 + \sin^2 \gamma d\varphi_1^2 + \cos^2 \gamma d\varphi_2^2. \]

The embedding coordinates are
\[ Y_3 + i Y_0 = \cosh \rho e^{i t}, \quad Y_1 + i Y_2 = \sinh \rho e^{i \phi}, \]  
(7.2)
\[ X_1, \pm + i X_2, \pm = \frac{1}{\sqrt{\alpha}} \sin \gamma \pm e^{i \varphi_1, \pm}, \quad X_3, \pm + i X_4, \pm = \frac{1}{\sqrt{\alpha}} \cos \gamma \pm e^{i \varphi_2, \pm}. \]  
(7.3)

We choose the following classical solution
\[ \rho = 0, \quad t = \kappa \tau, \quad \gamma = \frac{\pi}{4}, \quad \varphi_{1, \pm} = w_\pm \tau + m_\pm \sigma, \quad \varphi_{2, \pm} = w_\pm \tau - m_\pm \sigma, \quad \psi = 0. \]  
(7.4)

The Virasoro constraints give
\[ \kappa^2 = \frac{w_+^2 + m_+^2}{\alpha} + \frac{w_-^2 + m_-^2}{1 - \alpha}. \]  
(7.5)

and the classical energy is \( E_0 = \sqrt{\lambda} \kappa \). We first specialize to the following case of
\[ w_+ = \alpha J, \quad w_- = (1 - \alpha) J. \]  
(7.6)

Remarkably, a similar relation was found for the corresponding Pohlmeyer-reduced theories: a product of the two phase factors which appear in the S-matrix of the reduced \( \text{AdS}_3 \times S^3 \) theory gives the phase factor of the reduced \( \text{AdS}_5 \times S^5 \) theory [36]. The presence of the two phase factors in the \( \text{AdS}_3 \times S^3 \) case is connected with the non-simple (product) structure of the supergroup defining the corresponding supercoset. We are grateful to B. Hoare for a related discussion.
The relation $w_+/w_- = \alpha/(1 - \alpha)$ is due to the requirement that in the point-like $m_\pm = 0$ limit this configuration should reduce to the supersymmetric massless BMN-like geodesic discussed in [2]. The solution then has two equal spins in each of the two spheres:

\begin{align*}
J_1 &= J_2 = \sqrt{\frac{\lambda}{2\alpha}} \frac{w_+}{2(1 - \alpha)} = \frac{1}{2} \sqrt{\lambda} J \quad \text{on} \quad S^3_+ ,
J_1 &= J_2 = \sqrt{\frac{\lambda}{2}} \frac{w_-}{2} = \frac{1}{2} \sqrt{\lambda} J \quad \text{on} \quad S^3_.
\end{align*}

Choosing further\(^\text{10}\)

\begin{equation}
m_+ = m, \quad m_- = 0 ,
\end{equation}
we get in $S^3_-$ a single orbital momentum instead of two spins (by an $SO(4)$ rotation the solution on $S^3_-$ can be transformed into geodesic along big circle). Since we have two spins in $S^3_+$, we may refer to this case as an $SU(2)$ solution.

### 7.2 Fluctuation frequencies

Let us start with bosonic fluctuations. We find one massless and two massive fluctuations in $AdS_3$

\begin{equation}
\omega^{AdS_3 (1)} = n, \quad \omega^{AdS_3 (2,3)} = \sqrt{n^2 + \kappa^2} .
\end{equation}

From the point of view of bosonic fluctuations, the two 3-spheres are decoupled. The characteristic equation for the $S^3_\pm$ frequencies is

\begin{equation}
\begin{vmatrix}
\omega^2 - n^2 & i(w_\omega - mn) & i(-w_\omega - mn) \\
-2i(w_\omega - mn) & \omega^2 - n^2 & 0 \\
2i(w_\omega + mn) & 0 & \omega^2 - n^2
\end{vmatrix} = 0
\end{equation}

giving as in section 3 a massless and two massive modes

\begin{align*}
\omega^{S^3_+ (1)} &= n , \quad \omega^{S^3_+ (2,3)} = \sqrt{n^2 + 2w_\pm^2 \pm 2\sqrt{n^2 (w_\pm^2 + m_\pm^2) + w_\pm^4} .
\end{align*}

Finally, there is also a massless mode from $S^1$

\begin{equation}
\omega^{S^1} = n .
\end{equation}

The discussion of the fermionic fluctuations is similar, e.g., to the one in [20]. The quadratic part of the GS Lagrangian reads

\begin{equation}
L_{GS} = i \left( \sqrt{-h} h^{ab} \delta^{I,J} - \epsilon^{ab} \sigma^I_3 J \right) \bar{\theta} \rho_a D^{JK} \theta^K ,
\end{equation}

\(^{10}\text{Below we shall also consider the case with non-zero } m_- .\)
where \( \rho_a = \partial_a X^\mu \, E^A_\mu \Gamma_A \), and

\[
D_b^{JK} \theta^K = \delta^{JK} \left( \partial_b + \frac{1}{4} \omega^{AB}_\mu \partial_b X^\mu \Gamma_{AB} \right) \theta^K + \frac{1}{24} F_{MNP} \Gamma^{MNP} \rho_b \sigma_a^{JK} \theta^K. \tag{7.14}
\]

The RR 3-form flux term here is

\[
F_{MNP} \Gamma^{MNP} = 6 \left( \Gamma^{012} + \sqrt{\alpha} \Gamma^{345} + \sqrt{1-\alpha} \Gamma^{678} \right) \equiv 6 \Gamma, \tag{7.15}
\]

where the 012, 345, 678, and 9 coordinates refer to the factors in \( AdS_3 \times S_3^+ \times S_3^- \times S_1^1 \), i.e.

\[
\begin{array}{cccccccccc}
\mu & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
X^\mu & t & \rho & \phi & \gamma_+ & \varphi_{1,+} & \varphi_{2,+} & \gamma_- & \varphi_{1,-} & \varphi_{2,-} & \psi
\end{array}
\tag{7.16}
\]

Fixing \( \kappa \)-symmetry by \( \theta^1 = \theta^2 = \theta \), we end with

\[
L_{GS} = -2i \tilde{\theta} \left( -\rho^a D_a - \frac{1}{4} \rho^a \Gamma_a \right) \theta = -2i \tilde{\theta} D_F \theta, \tag{7.17}
\]

\[
D_a = \partial_a + \frac{1}{4} \omega^{AB}_\mu \partial_a X^\mu \Gamma_{AB}. \tag{7.18}
\]

The fermionic frequencies are the zeroes of the determinant of the fermionic operator. Let us define the two polynomials

\[
P_1(\omega) = \frac{1}{16 \alpha^2 \kappa} \left[ \kappa \left( m^4 + 2am^2 \left((3-4\alpha)\mathcal{J}^2 + (4-8\alpha)n^2 \right) \right) \\
+ \alpha^2 \left( (5-8\alpha)\mathcal{J}^4 + 16n^4 + 8\mathcal{J}^2n^2 \right) \right] \\
-4(\alpha - 1)\mathcal{J} \left( \alpha \mathcal{J}^2 + m^2 \right) \left( m^2 + \alpha \left( \mathcal{J}^2 + 4n^2 \right) \right) \right] \\
+ \omega \left( (2\alpha - 1)\kappa \mathcal{J}^2 + \frac{(\alpha - 1)\mathcal{J} \left( m^2 + \alpha \left( \mathcal{J}^2 + 4n^2 \right) \right)}{2\alpha} \right) \\
+ \frac{\omega^2 \left( m^2(\kappa - 2\alpha \mathcal{J} + \mathcal{J}) - \alpha(\mathcal{J} - \kappa)(3(2\alpha - 1)\mathcal{J}^2 + 4n^2) \right)}{2\alpha(\mathcal{J} - \kappa)} + \omega^3(2\mathcal{J} - 2\alpha \mathcal{J}) + \omega^4, \tag{7.19}
\]

\[
P_2(\omega) = \frac{1}{16 \alpha^2 \kappa} \left[ 4(\alpha - 1)\mathcal{J} \left( \alpha \mathcal{J}^2 + m^2 \right) \left( m^2 + \alpha \left( \mathcal{J}^2 + 4n^2 \right) \right) \right] \\
+ \kappa \left( m^4 + 2am^2 \left( (3-4\alpha)\mathcal{J}^2 + (4-8\alpha)n^2 \right) + \alpha^2 \left( (5-8\alpha)\mathcal{J}^4 + 16n^4 + 8\mathcal{J}^2n^2 \right) \right) \right] \\
+ \omega \left( (2\alpha - 1)\kappa \mathcal{J}^2 - \frac{(\alpha - 1)\mathcal{J} \left( m^2 + \alpha \left( \mathcal{J}^2 + 4n^2 \right) \right)}{2\alpha} \right) \\
- \frac{\omega^2 \left( m^2(\kappa + (2\alpha - 1)\mathcal{J}) + \alpha(\kappa + \mathcal{J})(3(2\alpha - 1)\mathcal{J}^2 + 4n^2) \right)}{2(\alpha(\kappa + \mathcal{J}))} + \omega^3(2\alpha \mathcal{J} - 2\mathcal{J}) + \omega^4. \tag{7.20}
\]

We can prove that the roots of \( P_{1,2}(\pm \omega) = 0 \) are the distinct roots of

\[
\det \left( -\rho^a D_a - \frac{1}{4} \rho^a \Gamma_a \right) = 0. \tag{7.21}
\]
Choosing the signs of $\omega_n^{F(1,2,3,4,5,6,7,8)}$ such that for large $n$ we have $\omega_n^{F(i)} = n + O(1)$, we can check that\footnote{The two zero modes come from one in $AdS_3$, one in $S^4_+$, one in $S^4_-$, one from $S^1$ minus two conformal-gauge ghosts.} 

$$
\tilde{e}(n) = \sum_{i=1}^{2} \omega_n^{AdS_3(i)} + \sum_{i=2}^{3} \omega_n^{S^4_+(i)} + \sum_{i=2}^{3} \omega_n^{S^4_-(i)} + 2|n| - \sum_{i=1}^{8} \omega_n^{F(i)} = O\left(\frac{1}{n^2}\right),
$$

ensuring UV finiteness.

### 7.3 Non-analytic part of $E_1$

The one-loop energy is

$$
E_1 = \sum_{n \in \mathbb{Z}} e(n), \quad e(n) = \frac{\tilde{e}(n)}{2\kappa}, \quad \kappa = \sqrt{J^2 + \frac{m^2}{\alpha}}. \tag{7.23}
$$

The non-analytic part can be found by as discussed in (3.11)

$$
\delta E_1 = \frac{J}{2\kappa} \int_{-\infty}^{\infty} dx \tilde{e}(Jx). \tag{7.24}
$$

From the above expressions one finds

$$
\tilde{e}(Jx) = \frac{m^2}{\alpha J} \left[ \frac{\sqrt{2\alpha^2 - 2\alpha \sqrt{\alpha^2 + x^2} + x^2} - \sqrt{2\alpha^2 + 2\alpha \sqrt{\alpha^2 + x^2} + x^2}}{2\sqrt{\alpha^2 + x^2}} + \frac{\alpha - 1}{\sqrt{(\alpha - 1)^2 + x^2}} + \frac{1}{\sqrt{x^2 + 1}} \right] + \ldots. \tag{7.25}
$$

Computing the integral, we find

$$
\delta E_1 = \frac{m^2}{\alpha J} \left[ \alpha \log \alpha + (1 - \alpha) \log(1 - \alpha) \right] + \ldots. \tag{7.26}
$$

Going to the next order, and setting

$$
L \equiv \alpha \log \alpha + (1 - \alpha) \log(1 - \alpha), \tag{7.27}
$$

we find that for $0 < \alpha < 1$

$$
\delta E_1 = \frac{m^2 L}{\alpha J} + \frac{m^4}{\alpha^2 J^3} \left[ \frac{1}{4} - \frac{L}{2} + \frac{1 - \alpha}{8} \log \alpha + \frac{(\alpha - 1)(\alpha + 3)}{8 \alpha} \log(1 - \alpha) \right] + \ldots, \tag{7.28}
$$

while for $\alpha = 1$

$$
\delta E_1 = \frac{m^4}{2 J^3} + \ldots. \tag{7.29}
$$

The $\alpha = 1$ case is in agreement with (3.14). Actually, the limit $\alpha \to 1$ is discontinuous with a jump that is due to the extra massless modes that appear when $\alpha = 1$. Notice that the $O(1/J^3)$ correction is not symmetric under $\alpha \to 1 - \alpha$ since we have set $m_- = 0$.  

\footnote{The two zero modes come from one in $AdS_3$, one in $S^4_+$, one in $S^4_-$, one from $S^1$ minus two conformal-gauge ghosts.}
7.4 Case of $\alpha \to 1 - \alpha$ symmetric solution and “renormalization” of string tension

It is interesting to consider the case with manifest symmetry under $\alpha \to 1 - \alpha$. To this end we repeat the 1-loop calculation assuming that instead of (7.6), (7.8) our classical solution now has

\[ w_+ = \alpha J, \quad w_- = (1 - \alpha)J, \quad m_+ = \alpha m, \quad m_- = (1 - \alpha) m. \quad (7.30) \]

The calculation is completely similar, but the result is much simpler:

\[ \delta E_1 = \begin{cases} \frac{m^2 L}{J} + \frac{m^4}{J^3} \left( \frac{1}{4} - \frac{L}{2} \right) + \frac{m^6}{24 J^5} \left( -7 + 9 L \right) + \ldots, & 0 < \alpha < 1 \\ \frac{m^4}{2 J^3} - \frac{7 m^6}{12 J^5} + \ldots, & \alpha = 1 \end{cases} \quad (7.31) \]

Let us now show that the $L = L(\alpha)$ dependent terms in (7.31) can be removed by a coupling redefinition. Recall that in the above expressions we used $J = \frac{L}{\sqrt{\lambda}}$. Let us now introduce $h(\lambda)$ such that at strong coupling

\[ h(\lambda) = \frac{\sqrt{\lambda}}{4\pi} + a + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad (7.32) \]

and define

\[ J_h = \frac{J}{4\pi h(\lambda)}. \quad (7.33) \]

The classical plus one-loop energy corresponding to the case of $m_+ = \alpha m$ and $m_- = (1 - \alpha) m$

\[ E = \sqrt{\lambda} \sqrt{J^2 + m^2} + \left[ \frac{m^2 L}{J} + \frac{m^4}{J^3} \left( \frac{1}{4} - \frac{L}{2} \right) + \frac{m^6}{24 J^5} \left( -7 + 9 L \right) + \ldots \right] + O\left(\frac{1}{h}\right) \quad (7.34) \]

can be expressed in terms of $J_h$ and then expanded at large $h$. The choice of

\[ a_{AdS_3 \times S^3 \times S^3 \times S^1} = \frac{L}{4\pi} \quad (7.35) \]

removes all the $L$-dependent (i.e. $\alpha$-dependent, with $0 < \alpha < 1$) terms in the 1-loop energy

\[ E = 4\pi h \sqrt{J_h^2 + m^2} + \left( \frac{m^4}{4 J_h^3} - \frac{7 m^6}{24 J_h^5} + \ldots \right) + O\left(\frac{1}{h}\right), \quad (7.36) \]

\[ ^{12}\text{We need then to assume that } m_\pm \text{ are integers (which imposes a restriction on } \alpha \text{) but this is not important for the present computation.} \]

\[ ^{13}\text{In (7.32) we defined } h(\lambda) \text{ so that it has } \lambda^{3/4} \text{ as the leading term at strong coupling by analogy with the } AdS_5 \times S^5 \text{ and } AdS_3 \times S^3 \times S^3 \times S^1 \text{ cases. Had we chosen it to be twice this value, i.e. } h(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + a + O\left(\frac{1}{\sqrt{\lambda}}\right) \text{ as in [8], we would get, instead of (7.35), the relation } a = \frac{L}{2\pi} \text{ which is consistent with what was found in [37] for the giant magnon case.} \]
Note that for the non-symmetric solution (7.8) this redifinition of the tension also removes the $L$-dependent part of the 1-loop energy (7.28) (although it does not eliminate all of the dependence on $\alpha$).

It is natural to expect that this effective string tension $h(\lambda)$ should be identified with the interpolating coupling in the corresponding Bethe Ansatz.

**Note added:**
While this paper was in preparation there appeared ref.[37] which also discussed 1-loop corrections to some semiclassical string configurations in $AdS_3 \times S^3 \times S^3 \times S^1$.

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**A One-loop correction to energy of long folded spinning string in $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$**

In this section we will study the 1-loop correction to energy of long folded spinning string in the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ backgrounds. In both cases the solution has the same form as in $AdS_5 \times S^5$ and describes a folded string carrying large AdS spin $S$ and and angular momentum $J$ (for more details see [29, 20]). We will consider the long string limit,

$$S \gg J, \quad x = \frac{\sqrt{\lambda}}{\pi J} \log S = \text{fixed},$$

in which the solution takes simple "homogeneous" form.

Below we will show that for the $AdS_3 \times S^3 \times T^4$ theory, matching with the one-loop energy of the folded string provides the same constraints on coefficients of the dressing phase in the ABA as the matching with the circular string discussed in section 3. We will also discuss the folded solution in the $AdS_3 \times S^3 \times S^3 \times S^1$ background, obtaining a closed expression for the one-loop energy which will allow us to analyze the structure of its non-analytic part.

**A.1 Bethe ansatz calculation in $AdS_3 \times S^3 \times T^4$ case**

The energy of the long folded string can be written in the form

$$E = E_0 + E_1 = S + J\sqrt{1+x^2} + \frac{J}{\sqrt{\lambda}}F(x) + \ldots,$$
where the first two terms are the classical energy $E_0$ and the third term is the one-loop correction $E_1$. This correction was computed for the $AdS_3 \times S^3 \times T^4$ case in \[20\],

$$E_1 = \frac{\mathcal{J}}{u} \left[ - (1 - u^2) + \sqrt{1 - u^2} - u^2 \log u - (2 - u^2) \log \left( 1 + \sqrt{1 - u^2} \right) \right], \quad (A.3)$$

where

$$\mathcal{J} \equiv \frac{J}{\sqrt{\lambda}}, \quad u \equiv \frac{1}{\sqrt{1 + x^2}}. \quad (A.4)$$

The “non-analytic” terms here, which in the $AdS_5 \times S^5$ case were captured by the dressing phase in the ABA, are the terms with even powers of $x$ in the small $x$ expansion

$$\frac{1}{\mathcal{J}} E_1 = -\frac{4x^3}{3} + \frac{x^4}{2} + \frac{4x^5}{5} - \frac{5x^6}{12} - \frac{64x^7}{105} + \frac{17x^8}{48} + \frac{32x^9}{63} - \frac{149x^{10}}{480} + \ldots. \quad (A.5)$$

Comparing with the $AdS_5 \times S^5$ result \[29\],

$$\frac{1}{\mathcal{J}} E_1 = -\frac{4x^3}{3} + \frac{4x^5}{5} + \frac{x^6}{3} - \frac{64x^7}{105} - \frac{2x^8}{3} + \frac{32x^9}{63} + \frac{43x^{10}}{40} + \ldots, \quad (A.6)$$

we see that for $AdS_3 \times S^3 \times T^4$ the non-analytic part starts one order earlier than in $AdS_5 \times S^5$ case, just as it happened for the circular string discussed in section 3.

The one-loop energy of the long folded string in $AdS_5 \times S^5$ was reproduced from the ABA in \[38\]. The ABA equations for the $sl(2)$ sector of the $AdS_3 \times S^3 \times S^3 \times S^1$ theory are the same as in $AdS_5 \times S^5$ up to the dressing phase, so the ABA prediction for the analytic part of the energy (sum of terms with even powers of $x$ in the small $x$ expansion) does not change, since it is only sensitive to the classical (AFS) part of the phase. In agreement with this prediction, the analytic part of the string result (A.5) is the same as in the $AdS_5 \times S^5$ case (A.6). For the non-analytic part the calculation of \[38\] is straightforward to adapt to our case, and the dressing phase we propose leads to the following expression:

$$\frac{1}{\mathcal{J}} \delta E_1 = \frac{1}{4} x^4 c_{1,2}^{(1)} + \frac{1}{16} x^6 (-4c_{1,2}^{(1)} + c_{1,4}^{(1)} - c_{2,3}^{(1)})$$

$$+ \frac{1}{64} x^8 (15c_{1,2}^{(1)} - 7c_{1,4}^{(1)} + 2c_{1,6}^{(1)} + 7c_{2,3}^{(1)} - 2c_{2,5}^{(1)} + c_{3,4}^{(1)}) + \ldots \quad (A.7)$$

Matching it with the even powers of $x$ in the string result (A.5) we find exactly the same relations (3.19)-(3.21) for coefficients $c_{r,s}^{(1)}$ as for the circular string!

### A.2 One-loop correction to energy in $AdS_3 \times S^3 \times S^3 \times S^1$

For the $AdS_3 \times S^3 \times S^3 \times S^1$ background with generic $\alpha$ the fluctuation frequencies around the folded string solution in the limit (A.1) were computed in \[20\], and the 1-loop energy was obtained in closed form for special cases\[14\] $\alpha = 0$ and $\alpha = \frac{1}{2}$. Here we will compute it for generic

\[14\] Note that physical quantities, e.g. the one-loop energy, are symmetric under $\alpha \to 1 - \alpha$ as this is a symmetry of the background.
\( \alpha \), which will allow us to explore, in particular, the dependence on \( \alpha \) of the non-analytic part of the energy.

The one-loop correction is defined by

\[
E_1^\alpha = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[ \sum_{i=1}^{8} \omega_i^B(n) - \sum_{i=1}^{8} \omega_i^F(n) \right],
\]

where in the regime we consider \( \kappa = \frac{J}{\sqrt{\lambda}} \sqrt{1 + x^2} \gg 1 \) and the frequencies are given in [20]

\[
\omega_{1,2}^B(n) = n, \quad \omega_{3,4}^B(n) = \sqrt{n^2 + \alpha^2 \mathcal{J}^2}, \quad \omega_{5,6}^B(n) = \sqrt{n^2 + (1 - \alpha)^2 \mathcal{J}^2},
\]

\[
\omega_{7,8}^B(n) = \sqrt{n^2 + 2\kappa^2} \pm 2\sqrt{n^2 \mathcal{J}^2 + \kappa^4},
\]

\[
\omega_{1,2}^F(n) = \pm \frac{\mathcal{J}}{2} + n, \quad \omega_{3,4}^F(n) = \pm \frac{\mathcal{J}}{2} + \sqrt{n^2 + \kappa^2}.
\]

The four other fermionic frequencies are given by the roots of two quartic equations

\[
\left[ (\omega_i^F)^2 - n^2 - \left( \frac{1}{2} - \alpha \right)^2 \mathcal{J}^2 \right]^2 = \kappa^2 \left[ \omega_i^F + s\left( \frac{1}{2} - \alpha \right) \mathcal{J} \right]^2 - (\kappa^2 - \mathcal{J}^2)n^2,
\]

where \( s = \pm 1 \). Note that the equation with \( s = -1 \) is obtained from the one with \( s = +1 \) by replacing \( \omega \rightarrow -\omega \). At \( \alpha = 0 \) the roots of this equation reduce to

\[
\omega_{5,6}^F(n) = \pm \frac{\mathcal{J}}{2} + n, \quad \omega_{7,8}^F(n) = \pm \frac{\mathcal{J}}{2} + \sqrt{n^2 + \kappa^2}
\]

in agreement with discussion of the spectrum in section C.4 of [20]. It is straightforward to check that the resulting 1-loop correction is UV finite.

To find the 1-loop correction in the limit \( \kappa \gg 1 \) one has to evaluate the integral

\[
\frac{1}{2\kappa} \int_{-\infty}^{\infty} dn \left[ \sum_{i=1}^{8} \omega_i^B(n) - \sum_{i=1}^{8} \omega_i^F(n) \right],
\]

which is nontrivial. The complication here is with the four fermionic frequencies that are solutions of the quartic equation – they can be found as explicit but very involved functions of \( n \). However, let us make use of the fact that this quartic equation can be solved explicitly for \( n(\omega) \) instead of \( \omega(n) \). Then the trick is to use integration by parts: introducing a cutoff \( \Lambda \) we get for these frequencies

\[
\int_{-\Lambda}^{\Lambda} dn \omega_i(n) = 2 \int_{0}^{\Lambda} dn \omega_i(n) = 2 (\omega_i n) \bigg|_{n=0}^{n=\Lambda} - 2 \int_{\omega_i(0)}^{\omega_i(\Lambda)} d\omega_i n(\omega_i).
\]

\( \text{The corresponding equation in [20] contains a typo in the sign in front of } (\frac{1}{2} - \alpha)^2 \mathcal{J}^2 \text{ in the l.h.s.} \)

\( \text{Since } n \text{ enters the equation only as } n^2 \text{ the frequencies are also even functions of } n. \)
After several changes of variables the integral over $\omega_i$ can be evaluated in elementary functions.\footnote{An important fact which turns out to reduce complexity of the integrand is that two of the roots $\omega$ of the quartic equation coincide when $n=0$.} The frequencies $\omega_i(0)$ can be found explicitly, and $\omega_i(\Lambda)$ are straightforward to find as an expansion at large $\Lambda$.

As a result, $E_1^{\alpha}$ is obtained in closed form:

$$\frac{1}{J} E_1^{\alpha} = \frac{1}{2u} \sqrt{1-u^2} \sqrt{1-(1-2\alpha)^2u^2} + \frac{u^2-1}{2u} + \frac{\sqrt{1-u^2}}{u}$$

$$+ \frac{(2(1-\alpha)au^2-1)}{u} \log 2 + \frac{(u^2-2)\log(\sqrt{1-u^2}+1)}{u}$$

$$- u \left[ \alpha^2 \log(\alpha) + (1-\alpha)^2 \log(1-\alpha) + (2(\alpha-1)\alpha + 1) \log u \right]$$

$$- \frac{(2(\alpha-1)au^2+1)}{4u} \log(1-u^2) + \sum_{i=1}^{2} \left[ f_i(\alpha) + f_i(1-\alpha) \right],$$

(A.16)

where

$$f_1(\alpha) = \frac{(\alpha-1)(\alpha u^2-1)}{2u} \log \left( (1-2\alpha)u^2 + \sqrt{(1-u^2)(1-(1-2\alpha)^2u^2)} + 1 \right),$$

$$f_2(\alpha) = \frac{(\alpha-1)(\alpha u^2-1)}{2u} \log \left( (1-u^2) \sqrt{1-(1-2\alpha)^2u^2} + \sqrt{1-u^2} ((1-2\alpha)u^2 + 1) \right).$$

This expression respects $\alpha \rightarrow 1-\alpha$ symmetry and reduces at special values $\alpha = 0$ and $\alpha = \frac{1}{2}$ to the expressions found in [20].

An important outcome of this result is the expression for the non-analytic part of the energy obtained from the small $x$ expansion of (A.16):

$$\frac{1}{J} \delta E_1^{\alpha} = L x^2 + \frac{1}{4} (2L+1) x^4 + \frac{1}{24} (9L-5) x^6 + \frac{1}{96} (-30L+17) x^8 + \ldots$$

(A.17)

where $L$ is the same quantity that appeared for the circular string in (7.27). In complete analogy with the circular string case, all dependence on $\alpha$ is again removed by the same shift in the tension (7.32). The reason for this is the relation

$$\delta E_1^{\alpha} = \frac{1}{2} \delta E_1^{\alpha=0} + L J x \frac{d}{dx} \sqrt{1+x^2},$$

(A.18)

which also shows that, again, as $\alpha \rightarrow 0$ the non-analytic part experiences a jump by a factor of two. If we shift the tension in (A.2) as

$$\sqrt{x} \rightarrow \sqrt{x} - 4\pi a$$

(A.19)

while holding the charges $S$ and $J$ fixed, then the one-loop energy will get a contribution coming from the classical part $E_0$. The latter depends on the tension only through the variable $x$ (see (A.2)), and we find that (A.2) becomes

$$E = S + J \sqrt{1+x^2} + \frac{J}{2\sqrt{x}} \left( F(x) - 4\pi a x \frac{d}{dx} \sqrt{1+x^2} \right) + \ldots,$$

(A.20)
where the expression in the round brackets is the modified one-loop energy. Then choosing
\[ a = \frac{L}{4\pi} \] (A.21)
we see that due to (A.18) all terms with \( L \) are removed. The shift (A.19) is equivalent to rewriting the string result in terms of the same interpolating coupling that was discussed in (7.32) for the circular string:

\[ h(\lambda) = \frac{\sqrt{\lambda}}{4\pi} + \frac{L}{4\pi} + O\left(\frac{1}{\sqrt{\lambda}}\right). \] (A.22)

We can also compute the analytic part of the energy in the small \( x \) expansion:

\[ \frac{1}{J} E_1^{\text{analytic}} = \frac{4}{3}(R - 1)x^3 + \left( -\frac{14R}{15} + \frac{1}{30R} + \frac{4}{5} \right)x^5 
+ \left( -\frac{1}{1120R^3} + \frac{157R}{210} - \frac{13}{420R} - \frac{64}{105R} \right)x^7 + \cdots , \] (A.23)

where
\[ R \equiv \sqrt{\alpha(1-\alpha)}. \] (A.24)

This result is valid for all \( \alpha \) except \( \alpha = 0 \) or \( \alpha = 1 \) where it becomes singular. At these special values the analytic part can be found from (A.5).

\section*{A.2.1 Large \( \ell \) expansion}

Let us also discuss the expansion in terms of \( \ell \) defined by
\[ \ell = \frac{\pi J}{\log S} = \frac{1}{x} \] (A.25)
and the corresponding re-expansion at weak coupling (the corresponding expansion for the \( AdS_5 \times S^5 \) case is described in, e.g., [39]). The energy has the form

\[ E = S + \frac{\sqrt{\lambda}}{\pi} f(\ell, \sqrt{\lambda}) \log S + \cdots , \quad f(\ell, \sqrt{\lambda}) = f_0(\ell) + \frac{1}{\sqrt{\lambda}} f_1(\ell) + \cdots \] (A.26)

where the classical part is the same as in \( AdS_5 \times S^5 \) case
\[ f_0(\ell) = \sqrt{1 + \ell^2} , \] (A.27)
while the 1-loop part can be found from (A.16). With the aim of making a re-expansion at weak coupling (as in [39]) we rewrite the energy as

\[ E = S + f(\lambda, \ell) \ln S + \cdots , \quad f = \frac{f(\ell, \sqrt{\lambda})}{\ell} j , \quad j = \frac{J}{\log S} = \frac{\sqrt{\lambda}}{\pi} \ell . \] (A.28)
In the $AdS_5 \times S^5$ case one obtains then what looks like a weak-coupling gauge theory expansion:

\[ f(\lambda, \ell) = \left( j + \frac{\lambda}{2\pi^2 j} - \frac{\lambda^2}{8\pi^4 j^3} + \frac{\lambda^3}{16\pi^6 j^5} + \cdots \right) \]

\[ + \left( - \frac{4\lambda}{3\pi^5 j^2} + \frac{4\lambda^2}{5\pi^5 j^4} + \frac{\lambda^2\sqrt{\lambda}}{3\pi^6 j^5} + \cdots \right), \quad (A.29) \]

where terms in the first line come from $f_0$, and in the second line from $f_1$. These two parts mix only at the order $\frac{1}{j^5}$ where $f_1$ provides a non-analytic contribution $\propto \lambda^{5/2}$, so that in the expansion

\[ f(\lambda, \ell) = j + \frac{c_{10}\lambda}{j} + \frac{c_{11}\lambda}{j^2} + \frac{c_{12}\lambda + c_{20}\lambda^2}{j^3} + \frac{c_{13}\lambda + c_{21}\lambda^2}{j^4} + \frac{p_5(\lambda)}{j^5} + \cdots \quad (A.30) \]

the coefficients $c_{ij}$ of lower-order terms should be protected and independent of $\lambda$, while $p_5(\lambda)$ should be a nontrivial interpolating function.

In $AdS_3 \times S^3 \times S^3 \times S^1$ the tree-level part $f_0$ is the same, while the 1-loop part is different, and we find

\[ f(\lambda, \ell) = \left( j + \frac{\lambda}{2\pi^2 j} - \frac{\lambda^2}{8\pi^4 j^3} + \frac{\lambda^3}{16\pi^6 j^5} + \cdots \right) \]

\[ + \left( \frac{\lambda^{1/2} L}{\pi^2 j} + \frac{4\lambda(R + 1)}{3\pi^3 j^2} - \frac{\lambda^{3/2}(2L - 1)}{4\pi^4 j^3} + \frac{\lambda^2(28R + \frac{1}{4R} + 24)}{30\pi^5 j^4} + \frac{\lambda^{5/2}(9L - 5)}{24\pi^6 j^5} + \cdots \right) \quad (A.31) \]

The first non-analytic term $\frac{\lambda^{1/2} L}{\pi^2 j}$ here already appears at order $\frac{1}{j}$ which is two orders earlier than in $AdS_5 \times S^5$. This suggests that in this case there should be no protected coefficients at all, at least in the part with odd powers of $\frac{1}{j}$.

This conclusion, however, changes if we shift the tension as in (A.19) above. Then an extra contribution comes from $\frac{\sqrt{\lambda}}{\pi} f_0(\ell) \log S$, the terms with $L$ in (A.31) cancel and we get

\[ f(\lambda, \ell) = \left( j + \frac{\lambda}{2\pi^2 j} - \frac{\lambda^2}{8\pi^4 j^3} + \frac{\lambda^3}{16\pi^6 j^5} + \cdots \right) \]

\[ + \left( \frac{4\lambda(R + 1)}{3\pi^3 j^2} + \frac{\lambda^{3/2}}{4\pi^4 j^3} + \frac{\lambda^2(28R + \frac{1}{4R} + 24)}{30\pi^5 j^4} - \frac{5\lambda^{5/2}}{24\pi^6 j^5} + \cdots \right). \quad (A.32) \]

Now the first nontrivial term in $f(\lambda, \ell)$, i.e. the $\frac{1}{j}$ term, appears to be protected.

### A.2.2 Subleading corrections in large $\kappa$

Finally, we can also study subleading terms in the large $\kappa$ expansion of $E_1$, taking $J \to 0$. In the $AdS_5 \times S^5$ we have

\[ E_1^{(0)} = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[ \sqrt{n^2 + 4\kappa^2} + 2\sqrt{n^2 + 2\kappa^2} + 5\sqrt{n^2} - 8\sqrt{n^2 + \kappa^2} \right], \quad (A.33) \]
which gives [24, 27, 40]

\[ E_1^{(0)} = -3 \log 2 \kappa - \frac{5}{12\kappa} + \ldots , \quad (A.34) \]

where dots denote exponentially suppressed terms. In the $AdS_3 \times S^3 \times S^3 \times S^1$ case setting $J = 0$ (i.e. $J = 0$) restricts solution to $AdS_3$, the fluctuations frequencies become independent of $\alpha$ and we find

\[ E_1^{(0)} = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[ \sqrt{n^2 + 4\kappa^2} + 3\sqrt{n^2 - 4\kappa^2} + \kappa^2 \right] . \quad (A.35) \]

Then using the Euler-Maclaurin formula we get

\[ E_1^{(0)} = -2 \log 2 \kappa - \frac{3}{12\kappa} + \ldots \quad (A.36) \]

Terms of the type $\frac{k}{12\kappa}$ in expansion of $E_1^{(0)}$ come from the $k$ massless modes in the sum over $n$. In $AdS_5 \times S^5$ the five bosonic massless modes give $\frac{5}{12\kappa}$, while in $AdS_3 \times S^3 \times S^3 \times S^1$ the three massless modes (7 bosonic minus 4 fermionic) give $\frac{3}{12\kappa}$.

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