COBORDISM CATEGORY OF MANIFOLDS
WITH BAAS-SULLIVAN SINGULARITIES, PART 2

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Abstract. For a given collection of closed manifolds \( \Sigma_k = (P_1, \ldots, P_k) \), we construct a cobordism category \( \text{Cob}^{\Sigma_k}_d \) of embedded manifolds with Baas-Sullivan singularities of type \( \Sigma_k \). Our main results identify the homotopy type of the classifying spaces \( BC\text{Cob}^{\Sigma_k}_d \) of these cobordism categories with that of the infinite loop-space of certain spectrum \( MT(d)_{\Sigma_k} \). The case of single singularity has been covered by the author’s paper [14].

1. Introduction and Statement of Main Results

1.1. Manifolds with Baas-Sullivan singularities. This paper continues the study we started in [14] where we have determined the homotopy type of the classifying space \( BC\text{Cob}^\Sigma_d \) for a cobordism category of smooth manifolds of dimension \( d \geq 2 \) with Baas-Sullivan singularity \( \Sigma = (P) \). Here we extend those constructions and results for manifolds with multiple Baas-Sullivan singularities. To simplify our presentation we assume for most of the paper that all manifolds are smooth and unoriented.

We fix a sequence \( \Sigma = (P_1, \ldots, P_k, \ldots) \) of smooth closed manifolds and denote \( p_i := \dim P_i \). Then for a positive integer \( k \), we denote by \( \Sigma_k := (P_1, \ldots, P_k) \).

We recall briefly the definition of a manifold with Baas-Sullivan Singularities of type \( \Sigma_k = (P_1, \ldots, P_k) \), postponing details to Section 2. It is convenient to denote by \( P_0 \) the single point.

We say that a compact manifold with corners \( W, \dim W = d \), is a \( \Sigma_k \)-manifold if the boundary \( \partial W \) is given a decomposition

\[
\partial W = \partial_0 W \cup \partial_1 W \cup \cdots \cup \partial_k W
\]

with the property that for each subset \( I \subseteq \{0, 1, \ldots, k\} \), the intersection

\[
\partial_I W := \bigcap_{i \in I} \partial_i W
\]

is a \( (d - |I|) \)-dimensional manifold with corners, where \( |I| \) is the size of \( I \). Furthermore, we suppose that for each \( I \subseteq \{0, 1, \ldots, k\} \), the manifold \( \partial_I W \) is given a factorization,

\[
\partial_I W \cong \beta_I W \times P^I \quad \text{where} \quad P^I = \prod_{i \in I} P_i,
\]

for some \( (d - |I| - \sum_{i \in I} p_i) \)-dimensional manifold with corners \( \beta_I W \).
By definition, the manifold $\beta_I W$ is a $\Sigma_k$-manifold as well for each subset $I \subseteq \{0, 1, \ldots, k\}$. The submanifold $\partial_0 W$ is referred to as the boundary of the $\Sigma_k$-manifold $W$. Closed $\Sigma_k$-manifolds are the ones for which $\partial_0 W$ is empty.

To get actual Baas-Sullivan singularities, we define an equivalence relation on $W$ by declaring two points, $x, y \in W$ to be equivalent if there exists a subset $I \subseteq \{0, 1, \ldots, k\}$ such that $x$ and $y$ both belong to $\partial_I W$ and

$$\text{pr}_I(x) = \text{pr}_I(y) \quad \text{where} \quad \text{pr}_I : (\partial_I W = \beta_I W \times P^I) \to \beta_I W$$

is the projection.

Quotienting-out with respect to this equivalence relation, we obtain a manifold $W_{\Sigma_k}$ with Baas-Sullivan singularities of the type $\Sigma_k$.

**Remark 1.1.** It is more convenient to work with $\Sigma_k$-manifolds. We take special care to ensure that all of our constructions factor through the above equivalence relation.

We denote by $\Omega_*$ the graded bordism group of unoriented manifolds, and by $\Omega_*^{\Sigma_k}$ a graded bordism group of manifolds with Baas-Sullivan Singularities of type $\Sigma_k$. We denote for now, $\Omega_*^{\Sigma_0} := \Omega_*$. For varying integer $k$, the groups $\Omega_*^{\Sigma_k}$ are related to each other by the well-known Bockstein-Sullivan exact couple:

$$\begin{array}{ccc}
\Omega_*^{\Sigma_{k-1}} & \xrightarrow{\times P_k} & \Omega_*^{\Sigma_k} \\
\downarrow & & \uparrow \\
\Omega_*^{\Sigma_k} & \xrightarrow{\beta_k} & \Omega_*^{\Sigma_{k-1}}
\end{array}$$

The map $\times P_k$ is given by multiplication by the manifold $P_k$ and is of degree $p_k$, $i_k$ is the map given by inclusion, and $\beta_k$ is the degree $-(p_k + 1)$ map given by sending a $\Sigma_k$-manifold $W$ to the $\Sigma_{k-1}$-manifold $\beta_k W$, see definitions in Section 2.

The classifying spectra for these cobordism theories, which we will denote by $\text{MO}_{\Sigma_k}$, can be constructed inductively as cofibres of maps between spectra starting with $\text{MO} := \text{MO}_{\Sigma_0}$. The homomorphism $\times P_1 : \Omega_* \to \Omega_*$ of degree $p_1 = \dim P_1$ defines a map of spectra

$$\mu_{P_1} : \Sigma^{p_1} \text{MO} \to \text{MO}.$$ 

We then set

$$\text{MO}_{\Sigma_1} := \text{Cofibre}(\mu_{P_1} : \Sigma^{p_1} \text{MO} \to \text{MO}).$$

Then, assuming that $\text{MO}_{\Sigma_{k-1}}$ is defined, we obtain a map $\mu_{P_k} : \Sigma^{p_k} \text{MO}_{\Sigma_{k-1}} \to \text{MO}_{\Sigma_{k-1}}$ given by the homomorphism $\times P_k : \Omega_*^{\Sigma_{k-1}} \to \Omega_*^{\Sigma_{k-1}}$ of degree $p_k = \dim P_k$. We define by induction:

$$\text{MO}_{\Sigma_k} := \text{Cofibre}(\mu_{P_k} : \Sigma^{p_k} \text{MO}_{\Sigma_{k-1}} \to \text{MO}_{\Sigma_{k-1}}).$$

The exact couple (1) is induced by the resulting cofibre sequence,

$$\Sigma^{p_k} \text{MO}_{\Sigma_{k-1}} \to \text{MO}_{\Sigma_{k-1}} \to \text{MO}_{\Sigma_k} \to \Sigma^{p_k+1} \text{MO}_{\Sigma_{k-1}} \to \cdots$$

For constructions and calculations relating to this exact couple, see [3]. An important example of the above is the case when $k = 1$ and $P_1$ is equal to the set of $n$ discrete points. In this case $\Sigma_1$-manifolds are sometimes referred to as $\mathbb{Z}/n$-manifolds.
1.2. Cobordism categories. Motivated by the ideas in [2], we will construct a cobordism category of manifolds with Baas-Sullivan singularities and determine the homotopy-type of its classifying space.

First, we recall that in [3], the authors construct a cobordism category $\text{Cob}_d$ whose morphisms are $d$-dimensional submanifolds $W \subseteq [a,b] \times \mathbb{R}^{d-\infty}$ that intersect the walls $\{a,b\} \times \mathbb{R}^{d-\infty}$ orthogonally in $\partial W$. This category is topologically enriched in such a way so that there are homotopy equivalences,

$$\text{Ob}(\text{Cob}_d) \sim \bigsqcup_M \text{Diff}(M), \quad \text{Mor}(\text{Cob}_d) = \bigsqcup_W \text{Diff}(W; \partial_{\text{in}}, \partial_{\text{out}})$$

where $M$ varies over diffeomorphism classes of $(d-1)$-dimensional closed manifolds and $W$ varies over diffeomorphism classes of $d$-dimensional cobordisms. Here $\text{Diff}(M)$ is the group of diffeomorphisms of $M$ and $\text{Diff}(W; \partial_{\text{in}}, \partial_{\text{out}})$ is the group of diffeomorphisms of $W$ that restrict to diffeomorphisms of the incoming and outgoing boundaries. In [3], the authors determine the homotopy type of the classifying space of $\text{Cob}_d$. In particular, they prove a weak homotopy equivalence

$$(2) \quad B\text{Cob}_d \sim \Omega^{\infty-1} \text{MT}(d),$$

where $\text{MT}(d)$ is a certain Thom-spectrum. This deep result, could be interpreted as a parametrized version of the classical Thom-Pontryagin construction.

Following this work on cobordism categories, we construct a cobordism category of $\Sigma_k$-manifolds. Roughly, the construction goes as follows (for more details, see Section 2).

We denote $\mathbb{R}^+_k := [0, \infty)^k$ and $\mathbb{R}^+_k(I) := \{(t_1, \ldots, t_k) \in \mathbb{R}^+_k \mid t_i = 0 \text{ if } i \notin I\}$ for $I$ a subset of $\{1, \ldots, k\}$. We then fix once and for all smooth embeddings

$$(3) \quad \phi_i : P_i \hookrightarrow \mathbb{R}^{p_i+m_i} \quad \text{for each } P_i \text{ from } \Sigma \text{ with } m_i > p_i = \dim P_i,$$

for each $k \geq 0$. Now we are ready to construct a topologically enriched category $\text{Cob}_d^{\Sigma_k}$. The morphisms are given by embedded $d$-dimensional $\Sigma_k$-cobordisms,

$$W \subset \left([a, b] \times \mathbb{R}^+_k \times \mathbb{R}^{d-1+\infty} \times \prod_{1 \leq i \leq k} \mathbb{R}^{p_i+m_i}\right)$$

such that for each subset $I \subseteq \{1, \ldots, k\},$

$$W \cap \left([a, b] \times \mathbb{R}^+_k(I^c) \times \mathbb{R}^{d-1+\infty} \times \prod_{1 \leq i \leq k} \mathbb{R}^{p_i+m_i}\right) = \partial_I W$$

where $I^c$ is the compliment of $I$ in $\{1, \ldots, k\}$. It is required that the submanifold $\partial_I W$ has factorization,

$$\partial_I W = \beta_I W \times P^I$$

where,

$$\beta_I W \subset \left([a, b] \times \mathbb{R}^+_k(I^c) \times \mathbb{R}^{d-1+\infty} \times \prod_{i \in I^c} \mathbb{R}^{p_i+m_i}\right)$$
and the submanifold

\[ P^I \subset \prod_{i \in I} \mathbb{R}^{p_i + m_i} \]

is given by a product of the embeddings from (3). Furthermore, we have

\[ \partial_0 W \subset \left( \{a, b\} \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+\infty} \times \prod_{1 \leq i \leq k} \mathbb{R}^{p_i + m_i} \right). \]

All intersections of \( W \) with the boundary of \([a, b] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+\infty} \times \prod_{1 \leq i \leq k} \mathbb{R}^{p_i + m_i}\) are required to be orthogonal. This category is topologized in such a way similar to as in [6] so that there are homotopy equivalences,

\[ Ob(\text{Cob}^\Sigma_k) \sim \bigcup_M \text{B Diff}(M)^{\Sigma_k}, \quad \text{Mor}(\text{Cob}^\Sigma_k) \sim \bigcup_W \text{B Diff}(W)^{\Sigma_k} \]

where \( M \) varies over diffeomorphism classes of \((d-1)\)-dimensional closed \((\partial_0 M = \emptyset) \Sigma_k\)-manifolds and \( W \) varies over diffeomorphism classes of \(d\)-dimensional \(\Sigma_k\)-manifolds with boundary. The diffeomorphism groups \(\text{Diff}(W)^{\Sigma_k}\) are defined to be the space of all smooth maps \(g : W \to W\) that are diffeomorphisms of \( W \) as a manifold with corners, i.e. \(g(\partial_I W) = \partial_I W\) for all \(I\), with the additional property that for each subset \(I \subset \{0, \ldots, k\}\), the restriction of \(g\) to \(\partial_I W\) has the factorization

\[ g |_{\partial_I W} = g_{\beta_I} \times Id_{P^I} \]

where \(g_{\beta_I} : \beta_I W \to \beta_I W\) is a diffeomorphism of a manifold with corners. Such diffeomorphisms descend to unique homeomorphisms of the singular manifold \(W^{\Sigma_k}\) obtained when quotienting-out by the equivalence relation described earlier. The construction of this category depends on our choice of embeddings from [3]. However, we will latter see that for any two choices of collections of embeddings of the manifolds from the list \(\Sigma\), the resulting categories will be isomorphic, provided the dimension of the ambient space is large enough so as to make the embeddings isotopic.

**Remark 1.1.** If \(k = 0\), then the category \(\text{Cob}^\Sigma_0\) is the same cobordism category of smooth manifolds studied in [6]. The case of a single singularity type, i.e. \(k = 1\), is covered in [14].

1.3. **First results.** To state our first results, we need more definitions. We will need to compare the categories \(\text{Cob}^\Sigma_k\) to other similar cobordism categories of manifolds with singularity sets that differ slightly from \(\Sigma\). For a non negative integer \(\ell \leq k\), we denote by \(\Sigma^\ell_k\) the length \(k\) list \((P_1, \ldots, P_{k-\ell}, *, \ldots, *)\) obtained by swapping the last \(\ell\) entries of \(\Sigma_k\) with single points. Using these singularity sets we define categories \(\text{Cob}^\Sigma_{k-1}\) in the same way as before. With this definition in place, for any pair \(\ell \leq k\) the category \(\text{Cob}^\Sigma_{k-1}\) can be identified with the
pull-back of the diagram,

\[
\begin{array}{ccc}
\text{Cob}^{\Sigma^k_d} & \xrightarrow{\partial_{k-\ell+1}} & \text{Cob}^{\Sigma^k_d} \\
\downarrow & & \downarrow \\
\text{Cob}^{\Sigma_{k-1}^{\ell-1}}_{d-p_{k-\ell+1}} & \times P_{k-\ell+1} & \to & \text{Cob}^{\Sigma_{k-1}^{\ell-1}}_{d-1}
\end{array}
\]

where \(\partial_{k-\ell+1}\) is the functor defined by sending a morphism \(W\) to \(\partial_{k-\ell+1}W\), which in this case equals \(\beta_{k-\ell+1}W\). The map \(\times P_{k-\ell+1}\) is the functor defined by sending a morphism \(W\) to \(W \times P_{k-\ell+1}\). Taking the pull-back of this diagram has the effect of adding one singularity type, in this case singularities of type \(P_{k-\ell+1}\), to the morphisms and objects in \(\text{Cob}^{\Sigma^k_d}\).

From this identification, we see that the category \(\text{Cob}^{\Sigma^k_d}\) can be defined inductively starting with \(\text{Cob}^{\Sigma^k_d}_{k}\), by iterating this pull-back construction for each \(\ell\) from \(k\) down to 0. Applying the classifying space functor, we obtain a cartesian diagram

\[
\begin{array}{ccc}
\text{BCob}^{\Sigma^k_d}_{d-1} & \xrightarrow{B(\times P_{k-\ell+1})} & \text{BCob}^{\Sigma^k_d}_{d-1} \\
\downarrow & & \downarrow \\
\text{BCob}^{\Sigma^k_d}_{k-1} & \xrightarrow{B\beta_{k-\ell+1}} & \text{BCob}^{\Sigma^k_d}_{k-1}
\end{array}
\]

This brings us to the statement of our first results:

**Theorem 1.1.** The above diagram \(\text{(5)}\) is homotopy-cartesian for all \(\ell\) and \(k\) with \(0 < \ell \leq k\).

We then with some work identify the homotopy fibre of the vertical maps of the above homotopy-cartesian square with the space \(\text{BCob}^{\Sigma^k_d}_{d-1}\). For the case that \(\ell = 1\) this yields:

**Theorem 1.2.** For all \(k\) there is a homotopy fibre-sequence,

\[
\begin{array}{ccc}
\text{BCob}^{\Sigma^k_d}_{d-1} & \longrightarrow & \text{BCob}^{\Sigma^k_d}_{d} \\
& & \downarrow \text{B}\beta_{k} \\
& & \text{BCob}^{\Sigma^k_d}_{d-1-\ell}
\end{array}
\]

The fact that \(\text{(5)}\) is homotopy cartesian implies that the homotopy type of the classifying space \(\text{BCob}^{\Sigma^k_d}_{d}\) can be determined inductively starting with \(\text{BCob}^{\Sigma^k_d}_{d}\), by iterating the the homotopy-pull-back construction with respect to the maps \(\text{B}(\beta_{k-\ell+1})\) and \(\text{B}(\times P_{k-\ell+1})\) for each \(\ell\) from \(k\) down to 0. For the case \(\ell = k\), the category \(\text{Cob}^{\Sigma^k_d}\) is the cobordism category of manifolds with corners. This category has been studied in [7]. There, the author identifies the homotopy type of the classifying space of this category with that of the infinite loop-space of a homotopy-colimit of a certain diagram comprised of the spectra \(MT(n)\). We will identify the homotopy type of \(\text{BCob}^{\Sigma^k_d}_{d}\) with the infinite loop-space of a similar homotopy colimit. Below we give a rough description of the construction.
1.4. Cubes of spaces and spectra. To state our results which determines the weak homotopy type of the classifying spaces $\text{B}C\text{ob}_{\Sigma_k}^d$, we have to consider $k$-dimensional cubical diagrams of spaces and spectra. We denote by $\text{Top}$ or $\text{Spec}$ the corresponding categories. A $k$-dimensional cubical diagram of spaces or spectra (which we will sometimes refer to as $k$-cubes of spaces) is a contravariant functor from the lattice of subsets of a finite set $\{1, \ldots, k\}$, which we denote by $2^{(k)}$, to the categories $\text{Top}$ or $\text{Spec}$. Let $X_\bullet: 2^{(k)} \to \text{Top}$, be such a functor. For instance, for $k = 1, 2, 3$, the functor $X_\bullet$ produces the corresponding diagrams:

$$
\begin{array}{cccc}
X_0 & \leftarrow & X_1 \\
\downarrow & & \downarrow \\
X_0 & \leftarrow & X_1 \\
\end{array}
$$

Now, given a functor $X_\bullet: 2^{(k)} \to \text{Top}$, we associate a space called the total homotopy cofibre of $X_\bullet$, which we denote by $t\text{Cofibre} X_\bullet$. The total homotopy cofibre is defined inductively as follows. First if $k = 1$, we set $t\text{Cofibre} X_\bullet := \text{Cofibre} X_\bullet$.

This makes sense because in this case $\text{Cofibre} X_\bullet = \text{Cofibre}(X_1 \to X_0)$.

Now assume that it is defined for all $(k - 1)$-cubes. We denote by $X_{\cdot,k}$ the $(k - 1)$-cube obtained by restricting $X_\bullet$ to the sub-lattice of all subsets containing the element $k \in \{1, \ldots, k\}$. We denote by $X_{\cdot,k}$ the $(k - 1)$-cube obtained by restricting to the sub-lattice of all subsets disjoint from $k$.

For instance, if $k = 3$, the top square gives $X_{\cdot,3}$, and the bottom one gives $X_{\cdot,3}$, see (6).

Clearly, there is a map of $(k - 1)$-cubes $X_{\cdot,k} \to X_{\cdot,k}$ which induces a map

$$t\text{Cofibre} X_{\cdot,k} \to t\text{Cofibre} X_{\cdot,k}.$$ 

We use this map and the induction hypothesis to define

$$t\text{Cofibre} X_\bullet := \text{Cofibre}(t\text{Cofibre} X_{\cdot,k} \to t\text{Cofibre} X_{\cdot,k}).$$

1.5. Weak homotopy type of $\text{B}C\text{ob}_{\Sigma_k}^d$. Next, we would like to construct a $k$-cube of spectra $\text{M}T_\bullet(d)_{\Sigma_k}$. First, recall that the spectrum $\text{M}T(d)$ from the main theorem in [6] is defined as follows. The $(n + d)$-th space of this spectrum is given by the Thom space $\text{M}T(d)_{n+d} := \text{Th}(U^\perp_{d,n})$ where $U^\perp_{d,n} \to G(d, n)$ is the orthogonal compliment to the canonical
bundle $U_{d,n} \rightarrow G(d,n)$ over the Grassmanian manifold $G(d,n)$ consisting of $d$-dimensional vector subspaces of the Euclidean space $\mathbb{R}^{d+n}$.

The bundle $U_{d,n+1}^+ \rightarrow G(d,n+1)$ restricts over $G(d,n)$ to the direct sum $U_{d,n}^+ \oplus \epsilon^1$, where $\epsilon^1$ is a trivial line bundle. Then the maps

$$S^1 \wedge \text{Th}(U_{d,n}^+) \rightarrow \text{Th}(U_{d,n+1}^+)$$

are the structure maps $\Sigma^1 \text{MT}(d)_{n+d} \rightarrow \text{MT}(d)_{n+1+d}$ of the spectrum $\text{MT}(d)$.

Recall that our manifolds $P_i$ come together with embeddings $\phi_i : P_i \hookrightarrow \mathbb{R}^{p_i+m_i}$, see (??). For a subset $I \subseteq \{1, \ldots, k\}$, we denote by $p_I$ and $m_I$ the sums $\sum_{i \in I} p_i$ and $\sum_{i \in I} m_i$, and by $P^I$ the product of manifolds $P_i$, $i \in I$. We notice that the embeddings $\phi_i$ induce the product embedding

$$\phi_I : P^I \hookrightarrow \mathbb{R}^{p_I+m_I}.$$ 

Let $I \subseteq J \subseteq \{1, \ldots, k\}$. Then we have the corresponding embedding

$$\phi_{I \setminus J} : P^{J \setminus I} \hookrightarrow \mathbb{R}^{p_{J \setminus I}+m_{J \setminus I}}.$$ 

We denote by $N_{J \setminus I}$ the normal bundle of the embedding $\phi_{I \setminus J}$ which comes together with a Gauss map

$$N_{J \setminus I} \rightarrow U_{p_{J \setminus I},m_{J \setminus I}}^+$$

of the embedding $\phi_{I \setminus J}$. We obtain a corresponding Thom-Pontryagin map

$$(7) \quad \rho_{J \setminus I} : S^{p_{J \setminus I}+m_{J \setminus I}} \rightarrow \text{Th}(U_{p_{J \setminus I},m_{J \setminus I}}^+).$$

Now we define a $k$-cube of spectra, $\text{MT}_\bullet(d)_{\Sigma_k}$ defined by setting

$$\text{MT}_J(d)_{\Sigma_k} := \Sigma^{-|J|} \text{MT}(d - p_J - |J|) \quad \text{for each } J \subseteq \{1, \ldots, k\}.$$ 

Thus the spectrum $\text{MT}_J(d)_{\Sigma_k}$ is the vertex in our cubical diagram corresponding to $J$.

![Figure 1](image_url)  

**Figure 1.** The cubical diagram of spectra $\text{MT}_\bullet(d)_{\Sigma_k}$ for the case that $k = 3$. 

Then for $I \subseteq J \subseteq \{1, \ldots, k\}$, we will denote the edge in the cubical diagram connecting the vertices associated to $I$ and $J$ by

$$\kappa_{J,I} : \Sigma^{-|J|} \text{MT}(d - p_J - |J|) \rightarrow \Sigma^{-|I|} \text{MT}(d - p_I - |I|).$$

The construction of $\kappa_{J,I}$ goes as follows. There is a natural map of the Grassmanians

$$\mu_{I,J} : G(d - p_J - |J|, n - m_J) \times G(p_J \setminus I, m_J \setminus I) \rightarrow G(d - p_I - |I|, n - m_I)$$

which sends a pair of vector subspaces

$$(\Pi, \Pi') \in G(d - p_J - |J|, n - m_J) \times G(p_J \setminus I, m_J \setminus I)$$

to their product

$$\Pi \times \Pi' \subset \mathbb{R}^{d-p_J-|J|+n-m_J} \times \mathbb{R}^{p_J \setminus I + m_J \setminus I}.$$ 

The map $\mu_{I,J}$ gives the standard pairing map of the Thom spaces:

$$\hat{\mu}_{J,I} : \text{Th}(U_{d-p_J-|J|,n-m_J}) \wedge \text{Th}(U_{p_J \setminus I, m_J \setminus I}) \rightarrow \text{Th}(U_{d-p_I-|I|,n-m_I})$$

Then the maps $\hat{\mu}_{I,J}$ and $\rho_{I,J}$ yield a map $\tau_{I,J}$ given by the composition:

$$\text{Th}(U_{d-p_J-|J|,n-m_J}) \wedge S^{p_J \setminus I \wedge m_J \setminus I} \wedge S^{p_I \setminus m_I} \xrightarrow{\tau_{I,J}} \text{Th}(U_{d-p_J-|J|,n-m_J}) \wedge \text{Th}(U_{p_J \setminus I, m_J \setminus I}) \wedge S^{p_I \setminus m_I}$$

which in turn induces a map of spectra:

$$\tau_{J,I} : \Sigma^{-|J|} \text{MT}(d - p_J - |J|) \rightarrow \Sigma^{-|I|} \text{MT}(d - p_I - |I|).$$

Now let

$$\text{Th}(U_{d-p_I-|I|,n-m_I}) \rightarrow \text{Th}(U_{d-p_I-|I|,n-m_I})$$

be the map induced by the natural embedding

$$G(d - p_I - |I|, n - m_I) \hookrightarrow G(d - p_I - |I|, n - m_I),$$

which is given by sending a $(d - p_I - |I|)$-dimensional subspace $\Pi$ to the product $\Pi \times \mathbb{R}^{|J|-|I|}$. This induces a map of spectra

$$j_{J,I} : \Sigma^{-|J|} \text{MT}(d - p_I - |J|) \rightarrow \Sigma^{-|I|} \text{MT}(d - p_I - |I|).$$

We then set $\kappa_{J,I} := j_{J,I} \circ \tau_{J,I}$ to be the edge in our cubical diagram of spectra connecting the $J$-th vertex to the $I$-th vertex. With some care, this construction can be carried out so that the cubical diagram obtained by putting together the maps $\kappa_{J,I}$ commutes on the nose (though it is relatively easy prove that the diagram commutes up to homotopy). Thus our functor $\text{MT}_*(d)_{\Sigma_k} : 2^{(k)} \rightarrow \text{Spec}$ is well defined. We denote,

$$\text{MT}(d)_{\Sigma_k} := \text{tCofibre } \text{MT}_*(d)_{\Sigma_k}.$$
This leads to us to the statement of our main theorem:

**Theorem 1.3.** There is a weak homotopy equivalence,

\[ \text{BCob}_d^{\Sigma_k} \sim \Omega^\infty-1 \text{MT}(d)\Sigma_k. \]

The above equivalence is constructed using a relative-parametric version of the classical Thom-Pontryagin construction. This theorem for the case where \( k = 1 \) was proven in [14].

Now, recall the Thom-Pontryagin maps

\[ \rho_{P,J}: S^{p,J+1+m,J/I} \to \text{Th}(U_{p,J/I,m,J/I}^J) \]

used in the construction of the cube \( \text{MT}_*(d)\Sigma_k \). By standard Thom-Pontryagin theory, the homotopy classes of these maps depend only on the cobordism classes of the manifold \( P_{J/I} \).

From this observation we have:

**Corollary 1.4.** Let \( \widehat{\Sigma} \) be a sequence of manifolds obtained by replacing each \( P_i \) from the sequence \( \Sigma \) by a manifold \( \widehat{P}_i \) cobordant to \( P_i \). Then the classifying space \( \text{BCob}_d^{\Sigma_k} \) of the resulting cobordism category \( \text{Cob}_d^{\Sigma_k} \), is weakly homotopy equivalent to \( \text{BCob}_d^{\Sigma_k} \), for all \( k \geq 0 \).

1.6. **Plan of the paper.** The paper is structured as follows. Section 2 is devoted to carefully constructing the category \( \text{BCob}_d^{\Sigma_k} \). Along the way we define the appropriate spaces of embeddings, diffeomorphism groups, and moduli-spaces of \( \Sigma_k \)-manifolds. In Sections 3 and 4 we construct sheaves \( C_d^{\Sigma_k} \), \( D_d^{\Sigma_k} \), defined on the category of smooth manifolds, which reduce to the sheaves \( C_d \) and \( D_d \) from [6] in the case \( k = 0 \). The representing spaces of these sheaves will be seen to weakly homotopy equivalent to \( \text{Cob}_d^{\Sigma_k} \) and \( \text{BCob}_d^{\Sigma_k} \) respectively. In Section 5 we prove Theorems 1.1 and 1.2. Section 6 is devoted to basic constructions and results regarding cubical diagrams. In section 7 we give a construction of the spectrum \( \text{MT}(d)\Sigma_k \) and in section 8 we give a proof of Theorem 1.3.

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# 2. The Cobordism Category

We start with the same sequence of closed smooth manifolds \( \Sigma = (P_1, \ldots, P_n, \ldots) \) specified in the introduction. We denote \( p_i := \dim(P_i) \). We fix once and for all a positive integer \( k \). We denote by \( \Sigma_k \) the truncated list of manifolds \( \Sigma_k = (P_1, P_2, \ldots, P_k) \). We will have to work with contracted or augmented lists \( \Sigma_{k-j} \) or \( \Sigma_{k+j} \), but in order to keep all of our constructions consistent, the integer \( k \) will be of the same value throughout the whole paper.

For \( 0 \leq \ell \leq k \) we denote by \( \Sigma_k^\ell \) the list \( (P_1, \ldots, P_{k-\ell}, *, \ldots, *) \) of the length \( k \).
As it was outlined in the introduction we construct a cobordism category of embedded $\Sigma^\ell_k$-manifolds. That is, for each positive integer $n$ and $0 \leq \ell \leq k$, we construct a cobordism category $\text{Cob}_{d,n}^\Sigma_{\ell,k}$ with morphisms given by $d$-dimensional $\Sigma^\ell_k$-cobordisms embedded in $(d+n)$-dimensional Euclidean space. Objects will be given by closed $(d-1)$-dimensional $\Sigma^\ell_k$-manifolds embedded in $(d+n-1)$-dimensional Euclidean space.

2.1. The Objects. In this section we construct the space of objects of our cobordism categories $\text{Cob}_{d,n}^\Sigma_{\ell,k}$.

For each positive integer $i$, we fix once and for all integers $m_i$ with $m_i > p_i = \dim(P_i)$, and smooth embeddings

$$\phi_i : P_i \hookrightarrow \mathbb{R}^{p_i+m_i}.$$  

For each subset $I \subseteq \{1, \ldots, k\}$, we denote the product embedding,

$$\phi_I := \prod_{i \in I} \phi_i : P_I \hookrightarrow \prod_{i \in I} \mathbb{R}^{p_i+m_i}.$$  

We denote by $\phi$ the collection of embeddings $\{\phi_1, \ldots, \phi_k\}$. We now introduce some notation that will be used throughout the paper.

Notational Convention 2.1. Denote by $\langle k \rangle$ the set $\{1, \ldots, k\}$. We define,

$$\bar{p} = p_1 + \cdots + p_k, \quad \bar{m} = m_1 + \cdots + m_k,$$

and for any subset $I \subseteq \langle k \rangle$,

$$p_I = \sum_{i \in I} p_i, \quad m_I = \sum_{i \in I} m_i.$$  

For any integer $n$ with $n \geq k + \bar{p} + \bar{m}$, we denote

$$\hat{n} := n - k + \bar{p} + \bar{m}.$$  

These conventions will become important when we define the space of embeddings of a $\Sigma^\ell_k$-manifold.

For each subset $I \subseteq \langle k \rangle$ we define,

$$\mathbb{R}^{\bar{p}+\bar{m}}_{\langle I \rangle} := \{(x_1, \ldots, x_k) \in \mathbb{R}^{p_1+m_1} \times \cdots \times \mathbb{R}^{p_k+m_k} | x_i = 0 \text{ if } i \notin I \}.$$  

We will also use the spaces,

$$\mathbb{R}^k_{\langle I \rangle} := \{(t_1, \ldots, t_k) \in \mathbb{R}^k | t_i = 0 \text{ if } i \in I \},$$  

$$\mathbb{R}^k_{+\langle I \rangle} := \{(t_1, \ldots, t_k) \in \mathbb{R}^k_{\langle I \rangle} | t_i \geq 0 \text{ for all } 1 \leq i \leq k \}.$$  

For a subset $I \subseteq \langle k \rangle$ we set

$$\{0, 1\}^k_{\langle I \rangle} := \{(t_1, \ldots, t_k) \in [0, 1)^k | t_i = 0 \text{ if } i \notin I \}.$$
For a list of positive constants \( \bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_k) \), we will denote,
\[
[0, \bar{\epsilon})_{(I)} := \{ (t_1, \ldots, t_k) \in [0, 1)^k \mid t_i < \epsilon_i \text{ for each } i \}.
\]
If \( I \subseteq \langle k \rangle \), we denote by \( I^c \) the compliment of \( I \) in \( \langle k \rangle \). We will frequently use the following identifications:
\[
[0, 1)^k_{\langle J \setminus I \rangle} \times [0, 1)^k_{\langle J \rangle} = [0, 1)^k_{\langle J \rangle}, \quad \mathbb{R}^{p+\bar{m}}_{\langle I \rangle} \times \mathbb{R}^{p+\bar{m}}_{\langle J \setminus I \rangle} = \mathbb{R}^{p+\bar{m}}_{\langle J \rangle}, \quad \text{and} \quad \mathbb{R}^k_{\langle+I \rangle} \times \mathbb{R}^k_{\langle+J \setminus I \rangle} = \mathbb{R}^k_{\langle J \rangle},
\]
for any subsets \( I \subseteq J \subseteq \langle k \rangle \). We will be using these notational conventions throughout the paper.

We now give a definition of a closed \( \Sigma^\ell_k \)-manifold.

**Definition 2.1.** Let \( M \) be a compact \((d-1)\)-dimensional manifold with \( k \)-order corners. We say that \( M \) is a closed \( \Sigma^\ell_k \)-manifold if it satisfies the following conditions:

i. The boundary \( \partial M \) is given a decomposition
\[
\partial M = \partial_1 M \cup \cdots \cup \partial_k M
\]
of the boundary \( \partial M \) into a union of \((d-2)\)-dimensional manifolds with \((k-1)\)-order corners such that for all subsets \( I \subseteq \langle k \rangle \), the intersection
\[
\partial_I M := \bigcap_{i \in I} \partial_i M
\]
is a manifold with \((k - |I|)\)-order corners with its boundary given by
\[
\partial(\partial_I M) = \bigcup_{J \subseteq \langle k \rangle} (\partial_J M \cap \partial_I M).
\]

(ii) The faces \( \partial_I M \) are given compatible collar embeddings, i.e., for all \( I \subseteq \langle k \rangle \), there exist compatible collar embeddings
\[
h_I : \partial_I M \times [0, 1)^k_{(I)} \to M.
\]
Compatibility here means the following. Let \( I \subseteq J \subseteq \langle k \rangle \) be subsets. We require that the embedding \( h_I : \partial_I M \times [0, 1)^k_{(I)} \to M \) maps the subspace
\[
\partial_J M \times [0, 1)^k_{(J \setminus I)} \subseteq \partial_J M \times [0, 1)^k_{(J)}
\]
into \( \partial_I M \times [0, 1)^k_{(I)} \). We denote this restriction by \( h_I|_{J \setminus I} \), and require that \( h_I \) factors through the composition:
\[
\partial_J M \times [0, 1)^k_{(J)}, \quad \partial_J M \times [0, 1)^k_{(J \setminus I)} \times [0, 1)^k_{(I)} \xrightarrow{h_I|_{J \setminus I} \times Id} \partial_I M \times [0, 1)^k_{(I)} \xrightarrow{h_I} M
\]
where we are identifying \( \partial_J M \times [0, 1)^k_{(J \setminus I)} \times [0, 1)^k_{(I)} \) with \( \partial_I M \times [0, 1)^k_{(I)} \), see (??).
(iii) For each subset $I \subseteq \{1, \ldots, k - \ell\}$ there are diffeomorphisms,

$$\psi_I : \partial_I M \xrightarrow{\cong} \beta_I M \times P^I$$

where $\beta_I M$ is a $(d - 1 - |I| - p_I)$-dimensional manifold with corners which satisfies the conditions (i) and (ii), and $P^I = \prod_{i \in I} P_i$ is as in the above notational convention. Furthermore these diffeomorphisms must be compatible in the following sense: if $I \subset J$ and $\iota_{J,I} : \partial_J M \hookrightarrow \partial_I M$ is the corresponding inclusion, then the map

$$\psi_I \circ \iota_{J,I} \circ \psi_J^{-1} : \beta_J M \times P^J \longrightarrow \beta_I M \times P^I$$

is the identity on the direct factor of the $P^I$.

It follows directly from the above definition that for $i \in \{1, \ldots, k - \ell\}$ $\beta_i M$ is a $\Sigma_{k - 1}^{\ell}$-manifold of dimension $d - 2 - p_i$ and that for $j \in \{k - \ell + 1, \ldots, k\}$, $\partial_j W$ is a closed $\Sigma_{k - 1}^{\ell}$ manifold of dimension $d - 2$. We note also that a closed $\Sigma_{k}^{\ell}$-manifold is automatically a $\Sigma_{k}^{\ell}$ manifold for all $j \geq \ell$.

We now define the space of diffeomorphisms of a closed $\Sigma_{k}^{\ell}$ manifold.

**Definition 2.2.** Let $M$ a $(d - 1)$-dimensional closed $\Sigma_{k}^{\ell}$-manifold as above. We again denote by $h_I$ and $\psi_I$ the collar embeddings and product structures given in the previous definition. Let $\epsilon_i$ for $i \in \langle k \rangle$, be positive constants and denote $\overline{\epsilon} = (\epsilon_1, \ldots, \epsilon_k)$. We define

$$\Diff(M)_{\Sigma_{k}^{\ell}}$$

to be the space of diffeomorphisms $g : M \longrightarrow M$, of a manifold with corners, subject to the following conditions:

i. For each subset $I \subseteq \langle k \rangle$, we have $g(\partial_I M) = \partial_I M$.

ii. The map $g$ respects collars about each face $\partial_i M$ of widths $\epsilon_i$ in the following way. For each $I \subseteq \langle k \rangle$, we require that

$$g \circ h_I (w, t_1, \ldots, t_k) = h_I (g|_{\partial_I M}(w), t_1, \ldots, t_k)$$

for $w \in \partial_I M$, where $(t_1, \ldots, t_k) \in [0, \overline{\epsilon})^k_{\langle I \rangle}$.

iii. For each $I \subseteq \{1, \ldots, k - \ell\}$, the restrictions $g|_{\partial_I M}$ have the factorizations,

$$g|_{\partial_I M} = \psi_I^{-1} \circ (g_{\beta_I M} \times Id_{P^I}) \circ \psi_I$$

with $g_{\beta_I M} : \beta_I M \longrightarrow \beta_I M$ a diffeomorphism of $\beta_I M$ satisfying the conditions i. and ii.

The space $\Diff(M)_{\Sigma_{k}^{\ell}}$ is topologized using the $C^\infty$-Whitney topology.

To eliminate dependence on $\epsilon$, we take the direct limit,

$$\Diff(M)_{\Sigma_{k}^{\ell}} := \colim_{\epsilon \to 0} \Diff(M)_{\Sigma_{k}^{\ell}}.$$
Recall that we have smooth embeddings $\phi_i : P_i \hookrightarrow \mathbb{R}^{p_i + m_i}$ for each $i$. We denote by $\phi$ the collection of embeddings $\{\phi_1, \ldots, \phi_k\}$, and

$$
\phi_I := \prod_{i \in I} \phi_i : P^I \hookrightarrow \prod_{i \in I} \mathbb{R}^{p_i + m_i} = \mathbb{R}_{(I)}^{\tilde{p} + \tilde{m}}.
$$

Below we use the notation introduced in Notational Convention 2.1. For what follows let $M$ a $d - 1$-dimensional closed $\Sigma_k^f$-manifold.

**Definition 2.3.** For each $i \in \langle k \rangle$ let $\epsilon_i$ be a positive constant. Denote $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_k)$. We define $\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}})_{\Sigma_k^f, \phi}$ to be the space of smooth embeddings of manifolds with corners,

$$
f : M \longrightarrow \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}},
$$

subject to the following conditions:

i. For each $I \subseteq \langle k \rangle$,

$$
f(\partial_I M) \subseteq \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}}.
$$

ii. The map $f$ respects collars of widths $\epsilon_i$ in the following way. Let $i_I : [0, \bar{\epsilon})^k_{(I)} \hookrightarrow \mathbb{R}^k_+(I)$ be the standard inclusion. Then for all $I \subseteq \langle k \rangle$, the following diagram commutes

$$
\begin{array}{ccc}
\partial_I M \times [0, \bar{\epsilon})^k_{(I)} & \xrightarrow{f \circ h_I} & \mathbb{R}^k_+(I) \times (\mathbb{R}_+^{k_{(I')}} \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}}) \\
\downarrow \text{proj} & & \downarrow \text{proj} \\
[0, \bar{\epsilon})^k_{(I)} & \xrightarrow{i_I} & \mathbb{R}_+^k(I)
\end{array}
$$

where in the upper-right corner we are using the identification,

$$
\mathbb{R}_+^k(I) \times \mathbb{R}_+^{k_{(I')}} \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}} \cong \mathbb{R}_+^k(I) \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}}.
$$

iii. For each $I \subseteq \{1, \ldots, k - \ell\}$ there is a factorization:

$$
f \mid_{\partial_I M} = f_{\beta_I M} \times \phi_{p_I}
$$

where the $\phi_{p_I}$ are the embeddings specified in (10) and

$$
f_{\beta_I M} : \beta_I M \longrightarrow [a, b] \times \mathbb{R}_+^k_{(I')} \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}}
$$

is an embedding which satisfies conditions i. and ii. given above.

The space $\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}})_{\Sigma_k^f, \phi}$ is topologized using the $C^\infty$-Whitney topology. To eliminate dependence on $\bar{\epsilon}$, we take the direct limit,

$$
(12) \quad \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}})_{\Sigma_k^f, \phi} := \colim_{\bar{\epsilon} \to 0} \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}})_{\Sigma_k^f, \phi}.
$$
Let $M$ be a $\Sigma_{k-1}$-manifold. Since $M$ is then a $\Sigma_k'$-manifold as well, the space

$$\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k'}$$

is defined. For what follows, to save space we temporarily denote

$$H_{k,\ell}^\natural := \mathbb{R}_+^k \times \mathbb{R}^{(k-\ell+1)+}(\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}).$$

The space $\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k',\phi}$ can be identified with the pull-back of the diagram,

$$\begin{array}{c}
\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k',\phi} \\
\downarrow \partial_{k-\ell+1}
\end{array}$$

$$\begin{array}{c}
\text{Emb}(\beta_{k-\ell+1}M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_{k-1}',\phi} \\
\downarrow \beta_{k-\ell+1}
\end{array}$$

where the right vertical map sends an embedding $g : M \to \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}$ to its restriction $g |_{\partial_{k-\ell+1}M}$ and the bottom horizontal map sends an embedding $f : \beta_{k-\ell+1}M \to \mathbb{R}_+^k \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})$ to the product embedding

$$f \times \phi_{\beta_{k-\ell+1}M} : \beta_{k-\ell+1}M \times P_{k-\ell+1} \to (\mathbb{R}_+^k \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})) \times (\mathbb{R}^{\bar{p}+\bar{m}}).$$

**Remark 2.1.** The spaces of diffeomorphisms $\text{Diff}(M)^{\Sigma_{k+1}}$ can be realized as the pull-back of

$$\begin{array}{c}
\text{Diff}(M)^{\Sigma_{k+1}} \\
\downarrow \partial_{k-\ell+1}
\end{array}$$

$$\begin{array}{c}
\text{Diff}(\beta_{k-\ell+1}M)^{\Sigma_{k-1}} \\
\downarrow \beta_{k-\ell+1}
\end{array}$$

in a similar way.

The following lemma will prove to be quite useful.

**Lemma 2.1.** For each all $k$ and $l$, the map

$$\partial_{k-\ell+1} : \text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k',\phi} \to \text{Emb}(\partial_{k-\ell+1}M, \mathbb{R}_+^k \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_{k-1}',\phi}$$

is a Serre-fibration.

**Proof.** The proof follows from Theorem [A.1] given in Appendix [A].

$\square$
This lemma implies that the space $\text{Emb}(M, \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\hat{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma^{f-1}_k,\phi}$ is actually the homotopy pull-back of the diagram (13).

In the case that $\ell = k$, the space $\text{Emb}(M, \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\hat{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k^f,\phi}$ is simply the space of embeddings of a manifold with corners, the product structure on faces $\partial_1M$ are forgotten.

We take the direct limit to define,

\begin{equation}
\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k^f,\phi} = \colim_{n \to \infty} \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\hat{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k^f,\phi}.
\end{equation}

This brings us to the following result:

**Theorem 2.2.** Let $M$ be a $\Sigma^j_k$-manifold. Then $\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k^f,\phi}$ is weakly contractible for all $\ell \geq j$. In particular, the space

$$\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k^f,\phi},$$

which corresponds to the case where $j = 0$, is weakly contractible for any $\Sigma_k$-manifold $M$.

**Proof.** We prove this result by induction on the difference $k - \ell$. Now, by Theorem 2.7 of [7], for any $k \geq 0$ the space $\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma_k^f,\phi}$ is weakly contractible. This space is the space of neat embeddings of a manifold with corners. Now suppose that the theorem holds for all $\Sigma^s_r$ manifolds where $r - s \leq k - \ell$. We will show that this implies that

$$\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma^{f-1}_k,\phi}$$

is weakly contractible. Lemma 2.1 together with (13) implies that the diagram,

\begin{equation}
\begin{array}{ccc}
\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma^{f-1}_k,\phi} & \xrightarrow{i} & \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma^f_k,\phi} \\
\downarrow{\beta_{k-\ell+1}} & & \downarrow{\partial_{k-\ell+1}} \\
\text{Emb}(\beta_{k-\ell+1}M, H^\infty_{k,\ell})^{\Sigma^{f-1}_k,\phi} & \xrightarrow{\phi_{k-\ell+1}} & \text{Emb}(\partial_{k-\ell+1}M, \mathbb{R}^k_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma^{f-1}_k,\phi}
\end{array}
\end{equation}

is homotopy-cartesian where $H^\infty_{k,\ell}$ in the diagram above has the same meaning as in (13). By the induction hypothesis, the lower-left, lower-right, and upper-right spaces are weakly contractible. This together with the fact that the diagram is homotopy-cartesian implies that the space

$$\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma^{f-1}_k,\phi}$$

is weakly contractible as well. This proves the theorem. \[ \square \]

The proof of the above theorem demonstrates the utility of the pull-back construction (13). This method of proof will be used throughout this paper.

The above construction depends on the choice of embeddings $\phi_i : P_i \hookrightarrow \mathbb{R}^{\bar{p}+\bar{m}}$. However, the homeomorphism type of the spaces $\text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+\hat{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})^{\Sigma^{f-1}_k,\phi}$ do not.
**Theorem 2.3.** For each $\ell$, the homeomorphism type of the space

$$\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi}$$

does not depend on our choice of embeddings $\phi$.

**Proof.** Proof is given in Appendix A \(\square\)

For each choice of $\phi$ and positive integer $n$, the group $\text{Diff}(M)_{\Sigma_k}$ acts freely (and smoothly) on the space $\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi}$, by pre-composition,

$$(g, f) \mapsto f \circ g$$

for $g$ a diffeomorphism and $f$ an embedding. We denote by

$$\mathcal{B}_n(M)_{\Sigma_k} := \frac{\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi}}{\text{Diff}(M)_{\Sigma_k}}$$

the quotient space induced by the group action. Notice that the underlying set of $\mathcal{B}_n(M)_{\Sigma_k}$ is the set of all $\Sigma_k$-submanifolds of $\mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m}$ diffeomorphic to $M$. Indeed the action of $\text{Diff}(M)_{\Sigma_k}$ identifies any two embeddings with the same image. We have the following:

**Theorem 2.4.** For each $k$ and $\ell$, the quotient map

$$q : \text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi} \longrightarrow \mathcal{B}_n(M)_{\Sigma_k}$$

is a locally trivial principal fibre-bundle with structure group $\text{Diff}(M)_{\Sigma_k}$.

**Proof.** Proof of this theorem is essentially the same as Lemma A.1 from [14]. We refer the reader there. \(\square\)

The local-triviality of $q$ along with the smooth structures on

$$\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi} \quad \text{and} \quad \text{Diff}(M)_{\Sigma_k}$$

make the spaces $\mathcal{B}_n(M)_{\Sigma_k}$ into (infinite dimensional) manifolds. The inclusion maps

$$\text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi} \hookrightarrow \text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n+1} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi}$$

are compatible with the action of $\text{Diff}(M)_{\Sigma_k}^{-1}$ and so we can take the direct limit to define

$$\mathcal{B}_\infty(M)_{\Sigma_k} = \colim_{n \to \infty} \mathcal{B}_n(M)_{\Sigma_k}. \tag{16}$$

**Theorem 2.4** combined with **Theorem 2.2** implies that the projection map

$$q : \text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\beta+m})_{\Sigma_k, \phi} \longrightarrow \mathcal{B}_\infty(M)_{\Sigma_k}$$

is the universal $\text{Diff}(M)_{\Sigma_k}$-principal bundle and that it is homotopy equivalent to the classifying space $B\text{Diff}(M)_{\Sigma_k}$. 
Using the well-known *Borel construction* we define,

\[(17) \quad E_n(M)^\Sigma^k := \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})^\Sigma^k \times \text{Diff}(M)^\Sigma^k M.\]

We obtain a fibre-bundle:

\[E_n(M)^\Sigma^k \hookrightarrow B_n(M)^\Sigma^k\]

with fibre \(M\) and structure group \(\text{Diff}(M)^\Sigma_{k-1}\). This fibre bundle comes with a natural embedding \(E_n(M)^\Sigma^k \subset (B_n(M)^\Sigma^k \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})\).

2.2. Morphisms. We now must define a \(\Sigma^l_k\)-bordism between two closed \(\Sigma^l_k\)-manifolds. Let \(M_a\) and \(M_b\) be closed \(\Sigma^l_k\)-manifolds of dimension \(d - 1\). For \(\nu = a, b\), let \(h^\nu_I\) and \(\psi^\nu_I\) be the collar embeddings and product structure maps associated to \(M_a\) and \(M_b\) from Definition 2.1.

**Definition 2.4.** A \(d\)-dimensional manifold with \(k+1\)-order corners is said to be a \(\Sigma^l_k\)-bordism from \(M_a\) to \(M_b\) if it satisfies the following conditions:

i. The boundary \(\partial W\) is given the decomposition,

\[\partial W = (M_a \sqcup M_b) \cup \partial_1 W \cup \cdots \cup \partial_k W\]

such that for each subset \(I \subset \langle k \rangle\), the intersection \(\partial_I W := \cap_{i \in I} W_i\) is a \(d - |I|\)-dimensional manifold with \(k - |I| + 1\)-order corners just as in condition i. of definition 2.1. Furthermore, for each \(I\) we have,

\[\partial_I W \cap M_\nu = \partial_I M_\nu \quad \text{for } \nu = a, b.\]

ii. There are collar embeddings

\[h_I : \partial_I W \times [0, 1]^k_I \longrightarrow W\]

which satisfy the same compatibility conditions given in condition ii. of Definition 2.1. Furthermore, for each \(I\) there are embeddings,

\[j^\nu_I : \partial_I M_\nu \times [0, 1) \longrightarrow \partial_I W\]

which make the following diagram commute,

\[
\begin{array}{ccc}
\partial_I M_\nu \times [0, 1)^k_{\langle I \rangle} \times [0, 1) & \cong & (\partial_I M_\nu \times [0, 1)) \times [0, 1)^k_{\langle I \rangle} \\
\downarrow h^\nu_I \times 1d & & \downarrow j^\nu_I \\
M_\nu \times [0, 1) & \longrightarrow & \beta_I W \times P^I
\end{array}
\]

where above it is understood that \(\partial_0 M_\nu = M_\nu\) and \(j^\nu := j^\nu_0\).

iii. For each subset \(I \subseteq \{1, \ldots, k - l\}\), there are diffeomorphisms,

\[\psi_I : \partial_I W \cong \beta_I W \times P^I\]
which satisfy the same compatibility conditions as given in condition iii. of Definition 2.1. Furthermore, for each $I$, the composition,

$$\beta_I M_\nu \times P^I \xrightarrow{(\psi_I^\nu)^{-1}} \partial_I M_\nu = \partial_I W \cap M_\nu \xrightarrow{\psi_I|_{\partial_I W \cap M_\nu}} \beta_I W \times P^I$$

is factors as a product $i_{I,\nu} \times Id_{P^I}$:

$$\beta_I M_\nu \times P^I \rightarrow \beta_I W \times P^I,$$

where $i_{I,\nu}$ is an embedding.

In the above definition we will refer to $M_a \sqcup M_b$ as the boundary of the $\Sigma^l_k$-bordism $W$. We will sometimes denote this boundary by $\partial W$ or by $\partial_0 W$ as we did in the introduction.

**Remark 2.2.** It follows directly from the definition that for $i \in \{1, \ldots, k - l\}$, $\beta_i W$ is a $\Sigma^l_{k-1}$-bordism from $\beta_i M_a$ to $\beta_i M_b$ and that for $j \in \{k - l + 1, \ldots, k\}$, $\partial j W$ is a $\Sigma^l_{k-1}$-bordism from $\partial j M_a$ to $\partial j M_b$.

![Figure 2](image.png)

**Figure 2.** Above is a $\Sigma^l_k$-bordism in the case that $k = l = 1$.

We now proceed to define the spaces of diffeomorphisms and embeddings of a $\Sigma^l_k$-bordism. The definitions will be similar to the spaces of diffeomorphisms and embeddings of closed $\Sigma^l_k$-manifolds.

Let $M_a$ and $M_b$ closed $d-1$-dimensional $\Sigma^l_k$ manifolds with collar embeddings and product structures given by $h_I^\nu$ and $\psi_I^\nu$ for $\nu = a, b$. Let $W$ be a $d$-dimensional $\Sigma^l_k$-bordism from $M_a$ to $M_b$ with collar embeddings $h_I$, $j_I^\nu$ and product structures $\psi_I$ given as in the previous definition.

**Definition 2.5.** Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_k$ be positive constants and denote $\bar{\epsilon} := (\epsilon_1, \ldots, \epsilon_k)$ (notice that we exclude $\epsilon_0$ from this list). We define $\text{Diff}(W; \delta)^{\Sigma^l_k}_{\epsilon, \epsilon_0}$ to be the space of diffeomorphisms $g : W \rightarrow W$, of a manifold with corners, subject to the following conditions:

i. For each subset $I \subseteq \langle k \rangle$ we have, $g(\partial_I W) = \partial_I W$. Furthermore, it is required that the restrictions of $g$ to $M_a$ and $M_b$ are both elements of the spaces $\text{Diff}(M_a)^{\Sigma^l_k}_{\bar{\epsilon}, \epsilon_0}$ and $\text{Diff}(M_b)^{\Sigma^l_k}_{\bar{\epsilon}, \epsilon_0}$. 
ii. The map \( g \) respects collars about each face \( \partial_i W \) of widths \( \epsilon_i \) for \( i \in \langle k \rangle \) and collars about \( M_\nu \) of width \( \epsilon_0 \) for \( \nu = a, b \) in the following way. For each \( I \subseteq \langle k \rangle \), it is required that
\[
g \circ h_I(w, t_1, \ldots, t_k) = h_I(g|_{\partial_i W}(w, t_1, \ldots, t_k)
\]
for \( w \in \partial_i W \), where \( (t_1, \ldots, t_k) \in [0, \bar{\epsilon})^{\langle I \rangle} \), and \( t_i < \epsilon_i \) for all \( i = 1, \ldots, k \). Furthermore, it is required that
\[
g \circ j_i^\nu(m, t) = j_i^\nu((g|_{\partial_i M_\nu}(m), t)
\]
for \( m \in M_\nu \), \( t \in [0, \epsilon_0) \).

iii. For each \( I \subseteq \{1, \ldots, k - \ell \} \), the restrictions \( g|_{\partial_i W} \) have the factorizations,
\[
g|_{\partial_i W} = \psi_I^{-1} \circ (g_{\beta_i W} \times Id_{\partial t}) \circ \psi_I
\]
with \( g_{\beta_i W} : \beta_i W \to \beta_i W \) a diffeomorphism of \( \beta_i W \) satisfying the conditions i. and ii.

The space \( \text{Diff}(W; \delta)^{\Sigma^\ell_k} \) is topologized using the \( C^\infty \)-Whitney topology.

As before we eliminate the dependence on \( \bar{\epsilon} \) by taking the direct limit,
\[
\text{Diff}(W; \delta)^{\Sigma^\ell_k} = \colim_{\bar{\epsilon} \to 0} \text{Diff}(W; \delta)^{\Sigma^\ell_k}_{\bar{\epsilon}, \epsilon_0}.
\]

Now let \( \phi \) the same collection of embeddings \( \phi_i : P_i \hookrightarrow \mathbb{R}^{n_i + m_i} \) used in the previous section.

**Definition 2.6.** Let \( \epsilon_0, \epsilon_1, \ldots, \epsilon_k \) be positive constants and set \( \bar{\epsilon} := (\epsilon_1, \ldots, \epsilon_k) \). We define
\[
\text{Emb}(W, [0, 1] \times \mathbb{R}^{k} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}})_{\bar{\epsilon}, \epsilon_0}^{\Sigma^\ell_k, \phi}
\]
to be the space of smooth embeddings of a manifold with corners
\[
f : W \to [0, 1] \times \mathbb{R}^{k} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}},
\]
subject to the following conditions:

i. For each subset \( I \subseteq \langle k \rangle \) it is required that,
\[
f(\partial_i W) \subset [0, 1] \times \mathbb{R}^{k}_{+,(I^c)} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}},
\]
f(\( M_\nu \)) \subset \{0\} \times \mathbb{R}^{k}_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}} \) and \( f(\{1\}) \subset \mathbb{R}^{k}_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}} \).

Furthermore the restriction of \( f \) to \( M_\nu \), for \( \nu = a, b \), is an element of the space
\[
\text{Emb}(M_\nu, \mathbb{R}^{k}_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}})_{\bar{\epsilon}}^{\Sigma^\ell_k}.
\]

ii. For each \( i \in \langle k \rangle \) the map \( f \) respects collars of widths \( \epsilon_i \) about the faces \( \partial_i W \) in the same way as in Definition 2.3. Furthermore, the restriction of \( f \circ j^a \) to \( M_a \times [0, \epsilon_0) \) is given by the formula,
\[
(m, t) \mapsto (t, f(m)) \in [0, 1] \times (\mathbb{R}^{k}_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}}) \text{ for } m \in M_a \text{ and } t \in [0, \epsilon_0)
\]
and the restriction of \( f \circ j^b \) to \( M_b \times [0, \epsilon_0) \) is given by the formula,
\[
(m, t) \mapsto (1 - t, f(m)) \in [0, 1] \times (\mathbb{R}^{k}_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p} + \bar{m}}) \text{ for } m \in M_b \text{ and } t \in [0, \epsilon_0).
iii. For each $I \subseteq \{1, \ldots, k - \ell\}$ there is a factorization:

$$f \big|_{\partial_M} = f_{\beta,M} \times \phi_{p}$$

just as in condition iii. of Definition 2.3

We eliminate dependence on $\epsilon$ and $\epsilon_0$ by taking the direct limit,

\begin{equation}
\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi} := \colim_{\ell \to 0} \text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{\ell,\phi}.
\end{equation}

We also may take the direct limit as $n \to \infty$ to define,

\begin{equation}
\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\beta+m})_{k,\phi} = \colim_{n \to \infty} \text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi}.
\end{equation}

Now, we obtain direct analogues of Lemma 2.1 and Theorems 2.2 and 2.4 from the previous section. Specifically the topology of the spaces $\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi}$ is independent of our choice of embeddings $\phi$ and the space

$$\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\beta+m})_{k,\phi}$$

is weakly contractible for all $0 \leq \ell \leq k$. The proofs go through in exactly the same way as before. Furthermore, the space $\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi}$ can be identified with the pull-back of the diagram,

\begin{equation}
\begin{array}{ccc}
\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k & \times & \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi} \\
\downarrow & & \downarrow \\
\text{Emb}(\beta_{k-\ell+1}W; [0, 1] & \times & \mathbb{R}_+^k \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{\beta+m})_{k,\phi}
\end{array}
\end{equation}

where as before, $H^n_{k,\ell} := \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m}|_{(k-\ell+1)^{\eta}}$.

For $W$ a $\Sigma^f_k$-bordism from $M_a$ to $M_b$, the space $\text{Diff}(W; \delta)_{k}^{\Sigma^f_k}$ acts freely and smoothly on the space $\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi}$ and so we may for each $n$ define the spaces $\text{B}_n(W; \delta)^{\Sigma^f_k}$ in exactly the same way as in the previous section. Furthermore, the quotient map

$$\text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi} \to \text{B}_n(W; \delta)^{\Sigma^f_k}$$

is a principle $\text{Diff}(W; \delta)_{k}^{\Sigma^f_k}$-fibre-bundle. We take the direct limit as $n \to \infty$ to define,

\begin{equation}
\text{B}_\infty(W; \delta)^{\Sigma^f_k} = \colim_{n \to \infty} \text{B}_n(W; \delta)^{\Sigma^f_k}.
\end{equation}

The space $\text{B}_\infty(W; \delta)^{\Sigma^f_k}$ is a model for the classifying space $\text{B} \text{Diff}(W; \delta)^{\Sigma^f_k}$.

Using the well-known Borel construction we define,

\begin{equation}
\text{E}_n(W; \delta)^{\Sigma^f_k} := \text{Emb}(W; [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\beta+m})_{k,\phi} \times_{\text{Diff}(W; \delta)^{\Sigma^f_k}} W.
\end{equation}
We obtain a fibre-bundle:
\[ E_n(W; \delta)^{\Sigma_k} \longrightarrow B_n(W; \delta)^{\Sigma_k} \]
with fibre \( M \) and structure group \( \text{Diff}(W; \delta)^{\Sigma_k-1} \). This fibre bundle comes with a natural embedding
\[ E_n(W; \delta)^{\Sigma_k} \subset (B_n(W; \delta)^{\Sigma_k} \times [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}}). \]

**Remark 2.3.** We emphasize the similarity with the corresponding construction from [6]. This embedding makes the bundle \( E_n(W; \delta)^{\Sigma_k} \longrightarrow B_n(W; \delta)^{\Sigma_k} \) universal in the following sense. If \( f : X \longrightarrow B_n(W; \delta)^{\Sigma_k} \) is a smooth map from a smooth manifold \( X \) of dimension \( j \), then the pull-back
\[ f^*(B_n(W; \delta)^{\Sigma_k}) = \{(x, v) \in X \times [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \mid (f(x), v) \in B_n(E)^{\Sigma_k}\} \]
is a smooth \((j + d)\)-dimensional \( \Sigma_k \)-submanifold
\[ E \subset X \times [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \]
such that the projection onto \( X \) is a fibre-bundle with fibre \( W \) and structure group \( \text{Diff}(W; \delta)^{\Sigma_k} \). Furthermore, \( E \) is a \( \Sigma_k \)-manifold and it can be easily verified that for each \( x \in X \), the inclusion map of the fibre \( E_x \) over \( x \) into \( \{x\} \times [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \) satisfies the conditions of Definition 2.3. Any such embedded fibre-bundle
\[ E \subset X \times [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \]
with the property that for each \( x \in X \) the inclusion of the fibre
\[ E_x \hookrightarrow \{x\} \times [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \]
satisfies all conditions of Definition 2.3 is induced by a unique smooth map
\[ f : X \longrightarrow B_n(W; \delta)^{\Sigma_k}. \]

Similar remarks apply to the bundle
\[ E_n(M)^{\Sigma_k} \longrightarrow B_n(M)^{\Sigma_k} \]
for a closed \( \Sigma_k \)-manifold \( M \).

### 2.3. The Category \( \text{Cob}_{d,n}^{\Sigma_k} \)
We are now ready to define the category \( \text{Cob}_{d,n}^{\Sigma_k} \). An object of \( \text{Cob}_{d,n}^{\Sigma_k} \) is a pair \((M, a)\) with \( a \in \mathbb{R} \) and \( M \subseteq \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \) a closed \((d-1)\)-dimensional \( \Sigma_k \)-submanifold with \( \partial_0 M = \emptyset \) such that the inclusion map of \( M \) into \( \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \) is an element of the space \( \text{Emb}(M, \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}})^{\Sigma_k, \phi} \).

A non-identity morphism of \( \text{Cob}_{d,n}^{\Sigma_k} \) from \((M, a)\) to \((M_b, b)\) is a triple \((W, a, b)\) with \( a < b \) and
\[ W \subset [a, b] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}} \]
a \( d \)-dimensional \( \Sigma_k \) submanifold such that the inclusion map of \( W \) is an element of the space \( \text{Emb}(W, [0, 1] \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}})^{\Sigma_k, \phi} \).
after a linear rescaling in the first coordinate. Two morphisms \((W, a, b)\) and \((V, c, d)\) can be composed if \(b = c\) and the submanifolds

\[
W \cap (\{b\} \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\ell+\bar{m}}) \quad \text{and} \quad (V \cap \{c\} \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\ell+\bar{m}})
\]

are equal. In this case, the composition of \((W, a, b)\) of \((V, c, d)\) is given by \((W \cup V, a, d)\).

Condition iii. of Definition 2.3 (the condition requiring embeddings of \(\Sigma^k\)-manifolds to respect collars) ensures that the union \(W \cup V\) is a smooth manifold with corners and so the composition is well defined. It is easy to check that this composition rule is associative.

We want to make \(\text{Cob}^{\Sigma^k_{d,n}}\) into a topological category. Observe that as sets we have isomorphisms,

\[
\text{Ob}(\text{Cob}^{\Sigma^k_{d,n}}_d) \cong \bigsqcup_M \mathcal{B}_n(M)^{\Sigma^k_{d,n}} \times \mathbb{R},
\]

(24)

\[
\text{Mor}(\text{Cob}^{\Sigma^k_{d,n}}_d) \cong \bigsqcup_W \mathcal{B}_n(W; \delta)^{\Sigma^k_{d,n}} \times \{(a, b) \in \mathbb{R}^2 \mid a < b\} \sqcup \text{Ob}(\text{Cob}^{\Sigma^k_{d,n}}_d),
\]

where \(M\) varies over diffeomorphism classes of closed \(d-1\)-dimensional \(\Sigma^k\) manifolds and \(W\) varies over diffeomorphism classes of \(d\)-dimensional \(\Sigma^k\)-bordisms. These isomorphisms as sets follow from the fact that for each \(W\), as a set \(\mathcal{B}_n(W; \delta)^{\Sigma^k_{d,n}}\) is precisely equal to all \(d\)-dimensional embedded \(\Sigma^k\)-bordisms of the space \([0, 1] \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\ell+\bar{m}}\); diffeomorphic to \(W\) as a \(\Sigma^k\)-bordism. We then take (24) to be the definition of the category \(\text{Cob}^{\Sigma^k_{d,n}}\). Defined in this way we see that composition, and the target and source maps are all continuous.

We take the direct limit as \(n \to \infty\) to define,

(25)

\[
\text{Cob}^{\Sigma^k_d} = \text{colim}_{n \to \infty} \text{Cob}^{\Sigma^k_{d,n}}.
\]

Now, in a way similar to (13), we can identify \(\text{Cob}^{\Sigma^k_{d-1,n-1}}\) with the pull-back of the diagram

\[
\text{Cob}^{\Sigma^k_{d,n}}_{d-\ell+1,1,n-m_{\ell-1}} \times P_{k-\ell+1} \xrightarrow{\partial_{k-\ell+1}} \text{Cob}^{\Sigma^k_{d-1,n}}_{d-\ell+1,1,n-m_{\ell-1}}.
\]

The bottom-horizontal map is the functor defined by sending a morphism

\[
W \subset [a, b] \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\ell+\bar{m}},
\]

to the product

\[
W \times P_{k-\ell+1} \subset [a, b] \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\ell+\bar{m}} \times \mathbb{R}^{\ell+\bar{m}} \times \mathbb{R}^{\ell+\bar{m}}\]

The right-vertical map is the functor which sends a morphism

\[
W \subset [a, b] \times \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\ell+\bar{m}}
\]
to
\[ \partial_{k-\ell+1}(W) \subset [a, b] \times \mathbb{R}^k_{+(k-\ell+1)} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}. \]

Notice that the definition of the functors \( \times P_{k-\ell+1} \) depend on our choice of the embeddings \( \phi \) from \([10]\).

The category \( \text{Cob}^\Sigma_{d,n} \) is the cobordism category of manifolds with corners studied in \([7]\).

The category of interest, the one appearing in the statement of the main theorem from the introduction is \( \text{Cob}^\Sigma_{d,n} \), which we will often denote by \( \text{Cob}^\Sigma_{d,n} \). Using \((26)\), we can realize \( \text{Cob}^\Sigma_{d,n} \) inductively starting with \( \text{Cob}^\Sigma_{k,d,n} \), by iterating the above pull-back construction from \( \ell = k \) down to 0. Since geometric realization of simplicial spaces preserves finite limits, applying \( B(\cdot) \), the classifying space functor to diagram \((26)\) yields a cartesian square,

\[
\begin{array}{ccc}
\text{BCob}^{\Sigma_{d,n}}_{d-1} & \overset{B\text{Cob}^{\Sigma_{d,n}}_{d-1}}{\longrightarrow} & \text{BCob}^{\Sigma_{d,n}}_d \\
\downarrow & & \downarrow \\
\text{BCob}^{\Sigma_{d-1,n}}_{d-1} \times P_{k-\ell-1} & \overset{\partial_{k-\ell+1}}{\longrightarrow} & \text{BCob}^{\Sigma_{d-1,n}}_{d-1}
\end{array}
\]

Theorem \([1,1]\) states that this cartesian square is in fact homotopy-cartesian.

**Remark 2.4.** The construction of this category does depend on our choice of collection of embeddings \( \phi \). In order for all of our constructions relating to this category to be well defined and consistent, we will need to stick with this choice throughout the paper. However, it follows from Theorem \([A,2]\) and \((24)\) that the isomorphism type of this category, as a topologically enriched category, does not depend on \( \phi \).

### 3. A Sheaf Model for \( \text{Cob}^{\Sigma_{d,n}}_d \)

In order to determine the homotopy type of the classifying space of the topological category \( \text{Cob}^{\Sigma_{d,n}}_d \), we will need to study certain sheaves defined on the category of smooth manifolds (without boundary) that are modeled on these spaces \( \text{Cob}^{\Sigma_{d,n}}_d \) and \( \text{BCob}^{\Sigma_{d,n}}_d \).

#### 3.1. A Recollection of Sheaves.

Let \( \mathcal{X} \) denote the category with objects given by smooth manifolds without boundary with morphisms given by smooth maps.

**Definition 3.1.** By a sheaf (set valued) on \( \mathcal{X} \) we mean a contravariant functor \( \mathcal{F} \) from \( \mathcal{X} \) to \( \text{Sets} \) which satisfies the following condition. For any good open covering \( \{U_i \mid i \in \Lambda\} \) of some \( X \in \text{Ob}(\mathcal{X}) \), and every collection \( s_i \in \mathcal{F}(U_i) \) satisfying \( s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \) for all \( i, j \in \Lambda \), there is a unique \( s \in \mathcal{F}(X) \) such that \( s|_{U_i} = s_i \) for all \( i \in \Lambda \).

This is the same definition used in \([13]\).
Definition 3.2. Let $\mathcal{F}$ be a sheaf on $\mathcal{X}$. Two elements $s_0$ and $s_1$ of $\mathcal{F}(X)$ are said to be concordant if there exists $s \in \mathcal{F}(X \times \mathbb{R})$ that agrees with $pr^*(s_0)$ in an open neighborhood of $X \times (-\infty, 0]$ and agrees with $pr^*(s_1)$ in an open neighborhood of $X \times [1, \infty)$, where $pr : X \times \mathbb{R} \to X$ is the projection on the first factor.

We denote the set of concordance classes of $\mathcal{F}(X)$ by $\mathcal{F}[X]$. The correspondence $X \mapsto \mathcal{F}[X]$ is clearly functorial in $X$.

Definition 3.3. For a sheaf $\mathcal{F}$ we define the representing space, denoted by $|\mathcal{F}|$, to be the geometric realization of the simplicial set given by the formula $k \mapsto \mathcal{F}(\Delta^k_e)$ where

$$\Delta^k_e := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum x_i = 1\}$$

is the standard extended $k$-simplex.

From this definition it is easy to see that any map of sheaves $\mathcal{F} \to \mathcal{G}$ induces a map between the representing spaces $|\mathcal{F}| \to |\mathcal{G}|$.

Definition 3.4. Let $\mathcal{F}$ be a sheaf on $\mathcal{X}$. Assume $A \subset X$ is a closed subset, and let $s$ be a germ near $A$, i.e., $s \in \text{colim}_U \mathcal{F}(U)$ with $U$ ranging over all open sets containing $A$ in $X$. Then we define $\mathcal{F}(X,A;s)$ to be the set of all $t \in \mathcal{F}(X)$ whose germ near $A$ coincides with $s$. Then two elements $t_0$ and $t_1$ are concordant relative to $A$ and $s$ if they are related by a concordance whose germ near $A$ is the constant concordance equal to $s$. The set of such relative concordance classes is denoted by $\mathcal{F}[X,A;s]$.

Now, any element $z \in \mathcal{F}(\ast)$ determines a point in $|\mathcal{F}|$ which we also denote by $z$. Also, for any $X \in \text{Ob}(\mathcal{X})$, such an element $z$ determines an element, which we give the same name, $z \in \mathcal{F}(X)$ by pulling back by the constant map. In [13, 2.4.3] it is proven that there is a natural bijection of sets

$$(28) \quad [(X,A), (|\mathcal{F}|, z)] \cong \mathcal{F}[X,A;z].$$

Here the set on the left hand side is the set of homotopy classes of maps of pairs. The non-relative case of this isomorphism with $A$ the empty set holds as well.

Using these observations we define the homotopy groups of a sheaf by setting

$$(29) \quad \pi_n(\mathcal{F}, z) := \mathcal{F}[^n\ast, \ast; z].$$

By (28) we get $\pi_n(\mathcal{F}, z) \cong \pi_n(|\mathcal{F}|, z)$ for any choice of $z \in \mathcal{F}(\ast)$.

Definition 3.5. A map of sheaves $\mathcal{F} \to \mathcal{G}$ is said to be a weak equivalence if it induces a homotopy equivalence $|\mathcal{F}| \sim |\mathcal{G}|$ of representing spaces.

The following result will give us a useful way to determine when a map of sheaves is a weak equivalence.
Proposition 3.1 (Relative surjectivity criterion). Let $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ be a map of sheaves. Suppose that $\alpha$ induces a surjective map
\[ F[X, A; s] \rightarrow G[X, A; \alpha(s)] \]
for every $X \in \mathcal{X}$ with a closed subset $A \subset X$ and any germ $s \in \text{colim}_U \mathcal{F}(U)$ where $U$ ranges over the neighborhoods of $A$ in $X$. Then $\alpha$ is a weak equivalence.

Proof. See [13].

In addition to set-valued sheaves, we will also have to consider sheaves on $\mathcal{X}$ which take values in the category of small categories, which we denote by $\text{CAT}$. A $\text{CAT}$-valued sheaf on $\mathcal{X}$ is a contravariant functor from $\mathcal{X}$ to $\text{CAT}$ satisfying the same sheaf condition with respect to good open covers given in Definition 3.1. Notice that for a $\text{CAT}$-valued sheaf $\mathcal{F}$, for each positive integer $k$, one has set-valued sheaves $\mathcal{N}_k \mathcal{F}$ defined by sending $X \in \text{Ob}(\mathcal{X})$ to the $k$-th nerve set of the category $\mathcal{F}(X)$.

In the case that $\mathcal{F}$ is a $\text{CAT}$-valued sheaf, the rule,
\[ k \mapsto \mathcal{F}(\Delta^k) \]
defines a simplicial category. One can define a topological category $|\mathcal{F}|$ by setting
\[ \text{Ob}(|\mathcal{F}|) := |\mathcal{N}_0 \mathcal{F}|, \quad \text{Mor}(|\mathcal{F}|) := |\mathcal{N}_1 \mathcal{F}|. \]

One can construct the classifying space of the topological category $|\mathcal{F}|$ by taking the geometric-realization of the diagonal bi-simplicial space given by
\[ n \mapsto \mathcal{N}_n \mathcal{F}(\Delta^n). \]

We have
\[ B|\mathcal{F}| \cong |k \mapsto \mathcal{N}_k \mathcal{F}(\Delta^k)|. \]

For details see [13, 4.1]. A map of $\text{CAT}$-valued sheaves $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ induces a map of spaces
\[ B|\alpha| : B|\mathcal{F}| \rightarrow B|\mathcal{G}|. \]

This induced map $B|\alpha|$ is a weak homotopy equivalence if for each non-negative integer $k$, the induced maps
\[ \mathcal{N}_k(\alpha) : \mathcal{N}_k \mathcal{F} \rightarrow \mathcal{N}_k \mathcal{G}, \]
are weak equivalences of set-valued sheaves.

3.2. The Sheaf Model. We define a $\text{CAT}$-valued sheaf on $\mathcal{X}$ whose representing space is homotopy equivalent to $\text{Cob}^\Sigma_k$.

Remark 3.1. So far in this paper we have defined closed-$\Sigma^l_k$-manifolds and $\Sigma^l_k$-bordisms. In this section and throughout the rest of this paper we will encounter manifolds with corners which satisfy all conditions of definition 2.1 the definition closed $\Sigma^l_k$-manifold, with the one
exception that they may fail to be compact. We will refer to such spaces as simply as $\Sigma_k^l$-manifolds. These spaces, by themselves, will not be the objects or morphisms of any cobordism category.

**Notational Convention 3.1.** As before $k$ will be fixed throughout. For an integer $n$ with $n \geq k + \bar{p} + \bar{m}$, we set, as before,

$$\hat{n} := n - k - \bar{p} - \bar{m}.$$ 

Let $X \in \mathcal{Ob}(\mathcal{X})$. For $1 \leq i \leq k$, let

$$\bar{\epsilon} := (\epsilon_1, \ldots, \epsilon_k) : X \longrightarrow (0, \infty)^k$$

be a smooth function (the $\epsilon_i$ are coordinate functions). For $I \subseteq \langle k \rangle$, we set

$$X \times [0, \bar{\epsilon}]_{(I)}^k := \{(x, t_1, \ldots, t_k) \in X \times \mathbb{R}_+^k | 0 \leq t_i \leq \epsilon_i(x) \forall i \text{ and } t_i = 0 \text{ if } i \notin I\}.$$ 

For a map $\pi : W \longrightarrow X$ we set,

$$W \times [0, \bar{\epsilon}]_{(I)}^k := \{(x, t_1, \ldots, t_k) \in W \times \mathbb{R}_+^k | 0 \leq t_i \leq \epsilon_i(\pi(x)) \forall i \text{ and } t_i = 0 \text{ if } i \notin I\}.$$ 

Similiarly for smooth functions $a, b : X \longrightarrow \mathbb{R}$ with $a(x) < b(x)$ for all $x \in X$ we denote,

$$X \times (a, b) = \{(x, t) \in X \times \mathbb{R} | a(x) < t < b(x) \text{ for all } x \in X \}.$$ 

We will often identify $X \times (a, b) \times [0, \bar{\epsilon}]_{(I)}^k \times \mathbb{R}_+^{k \langle I^c \rangle}$ as a subspace of $X \times (a, b) \times \mathbb{R}_+^k$ where $\bar{\epsilon}$ and $a, b$ are functions as above.

**Definition 3.6.** Let $X \in \mathcal{Ob}(\mathcal{X})$ and $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_k) : X \longrightarrow (0, \infty)^k$, $\epsilon_0 : X \longrightarrow (0, \infty)$ be smooth functions and let $a, b : X \longrightarrow \mathbb{R}$ smooth functions with $a(x) \leq b(x)$ for all $x \in X$. We define $C_{d, n, k}^\Sigma_l(X; a, b, \epsilon_0, \bar{\epsilon})$ to be the set of embedded, $d + \dim(X)$-dimensional $\Sigma_k^l$-submanifolds

$$(\pi, f, j) : W \hookrightarrow X \times (a - \epsilon_0, b + \epsilon_0) \times (\mathbb{R}_+^k \times \mathbb{R}_+^{d-1+n} \times \mathbb{R}^\bar{p} + \bar{m}),$$

subject to the following conditions:

i. For each $I \subseteq \langle k \rangle$, the face $\partial_I W$, is embedded with a collar parametrized by $X$. By this we mean that there is equality,

$$W \cap (X \times (a - \epsilon_0, b + \epsilon_0) \times [0, \bar{\epsilon}]_{(I)}^k \times \mathbb{R}_+^{k \langle I^c \rangle} \times \mathbb{R}_+^{d-1+n} \times \mathbb{R}^\bar{p} + \bar{m}) = \partial_I W \times [0, \bar{\epsilon}]_{(I)}^k.$$ 

It is also required that for $\nu = a, b$, the intersection

$$W \cap (X \times (\nu - \epsilon_0, \nu + \epsilon_0) \times [0, \bar{\epsilon}]_{(I)}^k \times \mathbb{R}_+^{k \langle I^c \rangle} \times \mathbb{R}_+^{d-1+n} \times \mathbb{R}^\bar{p} + \bar{m}),$$

be equal to the fibred product

$$(\pi, f)^{-1}(X \times \{\nu\}) \times (\nu - \epsilon_0, \nu + \epsilon_0)$$

as subspaces of $X \times (\nu - \epsilon_0, \nu + \epsilon_0) \times [0, \bar{\epsilon}]_{(I)} \times \mathbb{R}_+^{k \langle I^c \rangle} \times \mathbb{R}_+^{d-1+n} \times \mathbb{R}^\bar{p} + \bar{m}$. 

ii. For each $I \subseteq \{1, \ldots, k-\ell\}$, the $I$th face $\partial_I W$ has the factorization $\partial_I W = \beta_I W \times P^I$ where

$$\beta_I W \subset X \times (a - \epsilon_0, b + \epsilon_0) \times \mathbb{R}^k_{+I'} \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^\beta_{\beta I'}$$

is a $(\text{dim}(X) + d - p_I - |I|)$-dimensional $\Sigma_k$-submanifold and $P^I \subset \mathbb{R}^\beta_{\beta I'}$ is the closed submanifold given in (10).

iii. For each $I \subseteq \langle k \rangle$, the map $\pi$ and its restrictions $\pi \mid_{\partial_I W}$ are submersions such that the restriction of $\pi$ to the collar $\partial_I W \times [0, \bar{\epsilon}]_{\langle I \rangle}$ has the factorization,

$$\partial_I W \times [0, \bar{\epsilon}]_{\langle I \rangle} \xrightarrow{\text{proj}} \partial_I W \xrightarrow{\pi_{\partial_I W}} X.$$

Furthermore, if $I \subseteq \{1, \ldots, k-\ell\}$, the restriction $\pi \mid_{\partial_I W}$ has the further factorization,

$$\partial_I W = \beta_I W \times P^I \xrightarrow{\text{proj}} \beta_I W \xrightarrow{\pi_{\beta_I W}} X$$

where the map $\pi_{\beta_I W}$ is a submersion as well.

iv. The map $(\pi, f)$ is a proper map. For each $I \subset \langle k \rangle$, the restriction of $f$ to the collar $\partial_I W \times [0, \bar{\epsilon}]_{\langle I \rangle}$ has the factorization,

$$\partial_I W \times [0, \bar{\epsilon}]_{\langle I \rangle} \xrightarrow{\text{proj}} \partial_I W \xrightarrow{f_{\mid_{\partial_I W}}} \mathbb{R}.$$

If $I \subseteq \{1, \ldots, k-\ell\}$ the restriction $f_{\mid_{\partial_I W}}$ has the further factorization,

$$\partial_I W = \beta_I W \times P^I \xrightarrow{\text{proj}} \beta_I W \xrightarrow{f_{\mid_{\beta_I W}}} \mathbb{R}$$

where $(\pi_{\beta_I W}, f_{\mid_{\beta_I W}})$ is a proper map as well.

v. For each $I \subseteq \langle k \rangle$, the restrictions of $(\pi \mid_{\partial_I W}, f \mid_{\partial_I W})$ to the pre-images

$$(\pi_{\mid_{\partial_I W}}, f_{\mid_{\partial_I W}})^{-1}(X \times (\nu - \epsilon_0, \nu + \epsilon_0)) \quad \text{for } \nu = a, b$$

are both submersions.

**Remark 3.2.** The factorization from condition iii. of the above definition implies that the fibres $\pi^{-1}(x)$ are all $\Sigma^\ell_k$-manifolds of dimension $d$. Condition iv. implies that for any submanifold $Y \subset X \times (a - \epsilon_0, b + \epsilon_0)$ that is transverse to the restriction $(\pi, f) \mid_{\partial_I W}$ for all $I \subseteq \langle k \rangle$, the inverse image $(\pi, f)^{-1}(Y)$ is a $\Sigma^\ell_k$ manifold as well. For details on the proof of these statements, see [7]. There the author proves similar statements regarding transversality for manifolds with corners. The situation for $\Sigma^\ell_k$-manifolds is similar. These points will be used in proofs to come.

Condition v. of the above definition implies that for $(W, \pi, f) \in C^\Sigma^\ell_k(X; a, b, \epsilon_0, \bar{\epsilon})$, the restrictions $f \mid_{\pi^{-1}(x)}$ are transverse to both $a(x)$ and $b(x)$ for all $x \in X$. This implies that the pre-image $(\pi, f)^{-1}(X \times [a, b])$ is a $\Sigma^\ell_k$-bordism from

$$M_a := (\pi, f)^{-1}(X \times \{a\}) \quad \text{to} \quad M_b := (\pi, f)^{-1}(X \times \{b\}).$$
This condition also implies that the pre-images \((\pi, f)^{-1}(X \times (\nu - \epsilon_0, \nu + \epsilon_0))\) are elements of the sets \(C^\Sigma_{d,n}(X; \nu, \nu, \bar{\epsilon}, \epsilon_0)\) for \(\nu = a, b\). From these observations we see that for each \(X\), the set
\[
\bigsqcup_{a \leq b} C^\Sigma_{d,n}(X; a, b, \bar{\epsilon}, \epsilon_0)
\]
has the structure of a category. Morphisms,
\[(W, \pi_W, f_W) \in C^\Sigma_{d,n}(X; a, b, \bar{\epsilon}, \epsilon_0) \quad \text{and} \quad (V, \pi_V, f_V) \in C^\Sigma_{d,n}(X; c, d, \bar{\epsilon}, \epsilon_0)\]
can be composed if \(b = c\) and if the pre-images \((\pi_W, f_W)^{-1}(X \times \{b\})\) and \((\pi_V, f_V)^{-1}(X \times \{b\})\) agree. We now eliminate dependence on \(\epsilon_0\) and \(\bar{\epsilon}\) by setting
\[(31) \quad C_{d,n}^{\Sigma_k}(X; a, b) = \colim_{\epsilon_0 \to 0, \bar{\epsilon} \to 0} C_{d,n}^{\Sigma_k}(X; a, b, \epsilon_0, \bar{\epsilon}).\]

**Definition 3.7.** For \(X \in \mathcal{O}b(\mathcal{X})\), we set,
\[C_{d,n}^{\Sigma_k}(X) = \bigsqcup_{a \leq b} C_{d,n}^{\Sigma_k}(X; a, b)\]
with union ranging over all smooth functions \(a, b : X \to \mathbb{R}\) with \(a \leq b\) and
\[\{x \in X \mid a(x) = b(x)\}\]
an open subset of \(X\).

The assignment \(X \mapsto C_{d,n}^{\Sigma_k}(X)\) for \(X \in \mathcal{X}\) is a contravariant functor. Indeed, if \(g : X \to Y\) is a smooth map then for any embedded \(\Sigma_k\)-manifold
\[(\pi, f, j) : W \hookrightarrow Y \times (a - \epsilon_0, b + \epsilon_0) \times (\mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^p + \bar{m})\]
representing an element of \(C_{d,n}^{\Sigma_k}(Y)\), it follows from condition iii. of definition 3.6 that the space \(g^*(W)\) defined by the pull-back,
\[
g^*(W) \xrightarrow{g^*} W \\
\downarrow \hat{\pi} \\
X \xrightarrow{g} Y
\]
is a \(\Sigma_k\)-manifold, see remark 3.2. Furthermore it has natural embedding
\[(\hat{\pi}, \hat{f}, \hat{j}) : g^*(W) \hookrightarrow X \times (a - \epsilon_0, b + \epsilon_0) \times (\mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^p + \bar{m})\]
induced by \((\pi, f, j)\), which satisfies all conditions of definition 3.6, thus it yields an element of \(C_{d,n}^{\Sigma_k}(X)\). One can check also that \(C_{d,n}^{\Sigma_k}\) satisfies the sheaf condition on \(\mathcal{X}\). Furthermore for each \(X \in \mathcal{O}b(\mathcal{X})\), by the above discussion, \(C_{d,n}^{\Sigma_k}(X)\) has the structure of a category, it follows that \(C_{d,n}^{\Sigma_k}\) is a \(\text{CAT}\)-valued sheaf on \(\mathcal{X}\).
Using maps induced by inclusion, $C_{d,n}^{\Sigma_k,\ell,h} \hookrightarrow C_{d,n+1}^{\Sigma_k,\ell,h}$, we set

$$C_{d}^{\Sigma_k,\ell,h} = \colim_{n \to \infty}^* C_{d,n}^{\Sigma_k,\ell,h},$$

where by $\colim^*$ we mean colimit in the category of sheaves which is taken by sheafifying the colimit taken in the category of presheaves. There is a homotopy equivalence

$$|C_{d}^{\Sigma_k,\ell,h}| \sim \colim_{n \to \infty} |C_{d,n}^{\Sigma_k,\ell,h}|,$$

see [14, 5.2].

**Definition 3.8.** Let

$$C_{d,n}^{\Sigma_k,\ell,h}(X; a, b, \epsilon_0, \bar{\epsilon}) \subseteq C_{d,n}^{\Sigma_k,\ell,h}(X; a, b, \epsilon_0, \bar{\epsilon})$$

be the subset satisfying the further condition:

vi. For $x \in X$ let $J_a(x)$ be the interval $((a - \epsilon_0)(x), (a + \epsilon_0)(x)) \subseteq \mathbb{R}$ and let

$$V_a = (\pi, f)^{-1}(\{x\} \times J_a(x)) \subseteq \{x\} \times J_a(x) \times \mathbb{R}^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}.$$

Then

$$V_a = \{x\} \times J_a \times M \subseteq \{x\} \times J_a(x) \times \mathbb{R}^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}$$

for some $(d-1)$-dimensional $\Sigma_k$-submanifold $M$. The same condition must hold for the function $b$.

We then proceed to define

$$C_{d,n}^{\Sigma_k,\ell,h}(X; a, b) := \colim_{\epsilon_0 \to 0, \bar{\epsilon} \to 0} C_{d,n}^{\Sigma_k,\ell,h}(X; a, b, \epsilon_0, \bar{\epsilon}),$$

and

$$C_{d,n}^{\Sigma_k,\ell,h}(X) := \bigsqcup_{a \leq b} C_{d,n}^{\Sigma_k,\ell,h}(X; a, b).$$

As with $C_{d,n}^{\Sigma_k,\ell,h}$, the contravariant functor $X \mapsto C_{d,n}^{\Sigma_k,\ell,h}(X)$ is a sheaf on $X$. We also set

$$C_{d,n}^{\Sigma_k,\ell,h} = \colim_{n \to \infty}^* C_{d,n}^{\Sigma_k,\ell,h}$$

to get rid of dependence on $n$.

This added condition from definition 3.8 implies that for $(W, \pi, f; a, b) \in C_{d,n}^{\Sigma_k,\ell,h}(X; a, b, \epsilon_0, \bar{\epsilon})(X)$, for all $x \in X$, the inclusion map of the pre-mage $(\pi, f)^{-1}(\{x\} \times [a(x), b(x)])$ into

$$[a(x), b(x)] \times \mathbb{R}^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}$$

is an element of the space $\text{Emb}(W, [a(x), b(x)] \times \mathbb{R}^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}) \Sigma_k,\phi$. This implies that the fibre-bundle $\pi : W \to X$ is classified by a smooth map $g : X \to \text{Mor}(\text{Cob}_{d,n}^{\Sigma_k})$, see [24]. From this observation we have:
Proposition 3.2. For all $n$, the sheaf $C_{d,n}^{\Sigma_k}$ is isomorphic to the category valued sheaf given by

$$X \mapsto C^\infty(X, \text{Mor}(\text{Cob}_{d,n}^{\Sigma_k})).$$

The category structure on $C^\infty(X, \text{Mor}(\text{Cob}_{d,n}^{\Sigma_k}))$ is given by point-wise multiplication.

Proof. Proof follows from the above discussion. □

Proposition 3.3. For each $n$ there are weak homotopy equivalences

$$B|C_{d,n}^{\Sigma_k}| \sim B\text{Cob}_{d,n}^{\Sigma_k}.$$ 

Proof. Proof is the same as Proposition 2.9 of [6]. □

For each $n$ there is a map of sheaves $i_n : C_{d,n}^{\Sigma_k} \longrightarrow C_{d,n}^{\Sigma_k,\mathcal{C}}$ induced by inclusion.

Proposition 3.4. For each $n$, the above map $i_n$ induces a weak homotopy equivalence,

$$B|C_{d,n}^{\Sigma_k}| \sim B|C_{d,n}^{\Sigma_k,\mathcal{C}}|.$$ 

Proof. The proof is almost the same as proof of Proposition 4.4 of [6]. The only potential complication arrises in dealing with transversality in the context of $\Sigma_k$-manifolds. We provide a sketch of the proof.

We show that $i_n$ induces a weak equivalence of set-valued sheaves

$$\mathcal{N}_k C_{d,n}^{\Sigma_k} \sim \mathcal{N}_k C_{d,n}^{\Sigma_k,\mathcal{C}}$$ 

for each $k = 0, 1, 2, \ldots$ by the relative surjectivity criterion.

Let $X \in \text{Ob}(\mathcal{X})$. Fix a smooth function $\psi : \mathbb{R} \rightarrow [0, 1]$ which is 0 near $(-\infty, \frac{1}{3}]$ and is 1 near $[\frac{2}{3}, \infty)$, such that $\psi' \leq 0$ everywhere and $\psi' > 0$ on $\psi^{-1}(0, 1)$. Given smooth functions $a \leq b : X \longrightarrow \mathbb{R}$ with $(a - b)^{-1}(0) \subseteq X$ an open subset, we define $\phi : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ by the formulas

$$\phi(x, u) = (x, \phi_x(u)),$$

$$\phi_x(u) = \begin{cases} a(x) - (b(x) - a(x)) \cdot \psi\left(\frac{u - a(x)}{b(x) - a(x)}\right) & \text{if } a(x) < b(x) \\ a(x) & \text{if } a(x) = b(x) \end{cases}$$

Suppose that $W \in C_{d,n}^{\Sigma_k,\mathcal{C}}(X; a, b)$ with $a \leq b$. Condition v. of Definition 3.6 implies that the maps $(\pi, f)$ and $\phi$ are transverse, i.e. the product map

$$(\pi, f) \times \phi : W \times X \times \mathbb{R} \longrightarrow (X \times \mathbb{R}) \times (X \times \mathbb{R})$$
is transverse to the diagonal $\triangle$. Furthermore, condition v. of Definition [3.6] implies that for all $I \subset \langle k \rangle$, $(\pi, f) |_{\partial_j W}$ and $\phi$ are transverse and if $I \subseteq \{1, \ldots, k-l\}$, then $(\pi_{\beta_j W}, f_{\beta_j W})$ and $\phi$ are transverse. These transversality conditions imply that the space

$$W_\phi := \phi^* W = \{(x, u, z) \mid \pi(z) = x, f(z) = \phi_x(u)\}$$

is a $\Sigma^l_k$-submanifold of $X \times \mathbb{R} \times W$. Using the embedding

$$(\pi, f, j) : W \hookrightarrow X \times \mathbb{R} \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m},$$

we can rewrite $W_\phi$ as

$$W_\phi = \{(x, u, r) \mid (x, \phi_x(u), r) \in W\} \subseteq X \times \mathbb{R} \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}.$$ 

By inspection of the formula for $\phi$, it follows that

$$W_\phi \cap (X \times (-\infty, a+\epsilon) \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}) = M_a \times (-\infty, a+\epsilon)$$

$$W_\phi \cap (X \times (b-\epsilon, \infty) \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}) = M_b \times (b-\epsilon, \infty)$$

for closed $\Sigma^l_k$-manifolds $M_a$ and $M_b$ where $\epsilon = 1$ on $(b-a)^{-1}(0)$ and $\epsilon = \frac{1}{3}(b-a)$ otherwise. Thus $W_\phi$ determines an element of $C^{n,\Sigma^l_k}_{d,n}(X; a, b)$. We now show that $W_\phi$ is actually concordant to $W$ in $C^{n,\Sigma^l_k}_{d,n}(X; a, b)$. Define

$$\psi_s(u) = \rho(s)\psi(u) + (1-\rho(s))u$$

with $\rho$ any smooth function from $\mathbb{R}$ to $[0,1]$ for which $\rho = 0$ near $(-\infty,0]$ and $\rho = 1$ near $[1,\infty)$. Define $\Phi : X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}$ as $\Phi(x, s, u) = (x, \Phi_x(s, u))$ where

$$\Phi_x(s, u) = \begin{cases} a(x) + (b(x) - a(x))\psi_s\left(\frac{u-a(x)}{b(x)-a(x)}\right) & \text{if } a(x) < b(x), \\ \rho(s) \cdot a(x) + (1-\rho(s))u & \text{if } a(x) = b(x). \end{cases}$$

Like before, $\Phi$ is transverse to $(\pi |_{\partial_j W}, f |_{\partial_j W})$ for all $I \subseteq \langle k \rangle$ and to $(\pi_{\beta_j W}, f_{\beta_j W})$ for all $I \subseteq \{1, \ldots, k-l\}$ and so

$$W_\Phi = \{((x, s), u, r) \mid (x, \phi_x(s, u), r) \in W\} \subset (X \times \mathbb{R}) \times \mathbb{R} \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}$$

is a $\Sigma^l_k$-submanifold and defines the required concordance in $C^{n,\Sigma^l_k}_{d,n}(X \times \mathbb{R})$ from $W$ to $W_\phi$. This proves that $i_n$ induces surjections

$$\mathcal{N}_0 C^{\Sigma^l_k}_{d,n}[X] \to \mathcal{N}_0 C^{n,\Sigma^l_k}_{d,n}[X] \quad \text{and} \quad \mathcal{N}_1 C^{\Sigma^l_k}_{d,n}[X] \to \mathcal{N}_1 C^{n,\Sigma^l_k}_{d,n}[X].$$

The argument needed to prove relative surjectivity is similar. This proves that

$$\mathcal{N}_k C^{\Sigma^l_k}_{d,n} \xrightarrow{\sim} \mathcal{N}_k C^{n,\Sigma^l_k}_{d,n}$$

is a weak equivalence for $k = 0, 1$. The case for general $k$ is similar. □
4. The Sheaf $D_{d,n}^{\Sigma_k}$

4.1. The Sheaf. In this section we define sheaves $D_{d,n}^{\Sigma_k}$ who’s representing spaces are weakly homotopy equivalent to the classifying spaces $BCob_{d,n}^{\Sigma_k}$.

**Definition 4.1.** Let $X \in Ob(\mathcal{X})$, and $\bar{\epsilon} := (\epsilon_1, \ldots, \epsilon_k) : X \to (0, \infty)^k$ be a smooth function. We define $D_{d,n,\bar{\epsilon}}^{\Sigma_k}(X)$ to be the set of embedded, $(dim(X) + d)$-dimensional $\Sigma_k^l$-submanifolds $$(\pi, f, j) : W \to X \times \mathbb{R} \times (\mathbb{R}^{\bar{k}}_+ \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})$$ subject to the following conditions:

i. For each $I \subseteq (k)$, the face $\partial_I W$, is embedded with a collar parametrized by $X$. By this we mean that there is equality,

$$W \cap (X \times \mathbb{R} \times [0, \bar{\epsilon}]_{(I)} \times \mathbb{R}^k_{+\langle I \rangle} \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}}) = \partial_I W \times [0, \bar{\epsilon}]_{(I)}.$$ 

Here, we are identifying $W$ with its image under $(\pi, f, j)$.

ii. For each $I \subseteq \{1, \ldots, k-\ell\}$, the $I$th face $\partial_I W$ has the factorization $\partial_I W = \beta_I W \times P^I$ where

$$\beta_I W \subset X \times \mathbb{R} \times \mathbb{R}^k_{+\langle I \rangle} \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}}$$

is a $(dim(X) + d - p_I - |I|)$-dimensional $\Sigma_k^l$-submanifold and $P^I \subset \mathbb{R}^{\bar{p}+\bar{m}}$ is the closed submanifold given in [10].

iii. The map $\pi$ and the restrictions $\pi |_{\partial_I W}$ are submersions. Furthermore they satisfy exactly the same factorization conditions as in condition iii. of Definition 3.6.

iv. The map $(\pi, f)$ is a proper map. Furthermore $(\pi, f)$ and the restrictions $(\pi |_{\partial_I W}, f |_{\partial_I W})$ satisfy exactly the same factorization conditions as in condition iv. of Definition 3.6.

**Remark 4.1.** The above definition looks quite similar to the definition of $C_{d,n,\bar{\epsilon}}^{\Sigma_k}(X)$. However condition v. of the definition of $C_{d,n,\bar{\epsilon}}^{\Sigma_k}(X)$ implies that the map $\pi : W \to X$ is a proper-submersion and hence a fibre-bundle for an element $(W, \pi, f) \in C_{d,n,\bar{\epsilon}}^{\Sigma_k}(X; a, b)$. Since the above definition does not include this condition we cannot assume that the submersion $\pi : W \to X$ a fibre bundle for $(W, \pi, f) \in D_{d,n,\bar{\epsilon}}^{\Sigma_k}(X)$. The map $\pi$ will not be proper in general and the diffeomorphism types of the fibres $\pi^{-1}(x)$ may change with $x$.

We now take the direct limit over all functions $\bar{\epsilon} : X \to (0, \infty)^k$ to define

$$D_{d,n}^{\Sigma_k}(X) := \colim_{\bar{\epsilon} \to \emptyset} D_{d,n,\bar{\epsilon}}^{\Sigma_k}(X).$$

The assignment $X \mapsto D_{d,n}^{\Sigma_k}(X)$ is a contravariant functor in the same way as described for $C_{d,n,\bar{\epsilon}}^{\Sigma_k}$ in the previous section. It can be verified easily that $D_{d,n}^{\Sigma_k}$ satisfies the sheaf condition with respect to locally finite open covers on elements of $Ob(\mathcal{X})$. 

Taking the direct limit over the direct system given by the maps \( D_{d,n}^{\Sigma_k} \to D_{d,n+1}^{\Sigma_k} \) induced by inclusion, we define,

\[
D_{d,n}^{\Sigma_k} = \text{colim}_{n \to \infty} D_{d,n}^{\Sigma_k}
\]

where as before, the \( \text{colim}^* \) is the direct limit in the category of sheaves. As in \([14]\) we have a homotopy equivalence \( |D_{d,n}^{\Sigma_k}| \sim \text{colim}_{n \to \infty} |D_{d,n}^{\Sigma_k}| \).

The definition of \( D_{d,n}^{\Sigma_k} \) depends on the choice embeddings \( \phi \) from \((10)\). It will latter be seen that the homotopy type of \( |D_{d,n}^{\Sigma_k}| \) does not depend on these choices.

4.2. A Fibre Sequence. We now specialize to the case where \( \ell = 0 \). We denote, \( D_{d,n}^{\Sigma_k} := D_{d,n}^{\Sigma_0} \).

Consider the element \((W, \pi, f) \in D_{d,n}^{\Sigma_k}(X)\) with \( X \) arbitrary. Note that \( \beta_k W \) is a \( \Sigma_{k-1} \)-submanifold of \( X \times \mathbb{R} \times \mathbb{R}^k_{+(\{k\})^c} \times \mathbb{R}^{d-1+n \bar{\pi}} \times \mathbb{R}^{p+\bar{m}} \)

and so \((\beta_k W, \pi, f)\) is naturally an element of \( D_{d-1-p_k,n-m_k}^{\Sigma_{k-1}}(X)\). As a result, there is a map of sheaves,

\[
\beta_k : D_{d,n}^{\Sigma_k} \to D_{d-1-p_k,n-m_k}^{\Sigma_{k-1}}
\]

given by sending an element \( W \in D_{d,n}^{\Sigma_k}(X) \) to \( \beta_k W \in D_{d-1-p_k,n-m_k}^{\Sigma_{k-1}}(X) \).

**Theorem 4.1.** For all \( n > 0 \), there is a homotopy fibre sequence

\[
|D_{d,n}^{\Sigma_{k-1}}| \to |D_{d,n}^{\Sigma_k}| \xrightarrow{\beta_k} |D_{d-1-p_k,n-m_k}^{\Sigma_{k-1}}|.
\]

In order to prove this theorem we will need to use some results regarding the concordance theory of sheaves on \( X \). We recall now from \([13], 4.1.5\) the definition of a certain property for sheaves analogous to the homotopy lifting property for spaces.

**Definition 4.2.** A map of sheaves \( \alpha : \mathcal{F} \to \mathcal{G} \) is said to have the **Concordance Lifting Property** if for any \( X \in \mathcal{X} \), given \( s \in \mathcal{F}(X) \) and \( h \in \mathcal{G}(X \times \mathbb{R}) \) such that there exists \( \epsilon > 0 \) with

\[
\text{pr}^*(\alpha(s))|_{X \times (-\infty, \epsilon)} = h|_{X \times (-\infty, \epsilon)},
\]

where \( \text{pr} : X \times \mathbb{R} \to X \) is the projection, then there exists \( \widehat{h} \in \mathcal{F}(X \times \mathbb{R}) \) such that

\[
\widehat{h}|_{X \times (-\infty, \bar{\epsilon})} = \text{pr}^*(\alpha(s))|_{X \times (-\infty, \epsilon)}\quad \text{and} \quad \alpha(\widehat{h}) = h
\]

where \( \bar{\epsilon} \) is some positive real number, possibly different than \( \epsilon \).
Any element $z \in \mathcal{G}(\ast)$ gives rise to an element (which we give the same name) $z \in \mathcal{G}(X)$ for $X \in \mathcal{O}b(X)$ by pulling back over the constant map.

**Definition 4.3.** Let $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ and $z \in \mathcal{G}(\ast)$ be as above. The fibre of the map $\alpha$ over $z$ is the sheaf $\mathcal{F}^\alpha_z$ defined by

\[ \mathcal{F}^\alpha_z(X) = \{ s \in \mathcal{F}(X) \mid \alpha(s) = z \} . \]

In [13, A.2.4] the authors prove the following theorem:

**Proposition 4.2.** Suppose given sheaves $\mathcal{E}, \mathcal{F}, \mathcal{G}$ on $\mathcal{X}$ and maps of sheaves $u : \mathcal{E} \rightarrow \mathcal{G}$ and $v : \mathcal{F} \rightarrow \mathcal{G}$.

Let $\mathcal{E} \times_\mathcal{G} \mathcal{F}$ be the fibred product of $u$ and $v$. If $u$ has the concordance lifting property, then the projection $\mathcal{E} \times_\mathcal{G} \mathcal{F} \rightarrow \mathcal{F}$ has the concordance lifting property and the following square is homotopy cartesian:

\[
\begin{array}{ccc}
|\mathcal{E} \times_\mathcal{G} \mathcal{F}| & \longrightarrow & |\mathcal{F}| \\
\downarrow & & \downarrow |v| \\
|\mathcal{E}| & \longrightarrow & |\mathcal{G}|.
\end{array}
\]

For this we get an immediate corollary as a special case:

**Corollary 4.3.** If $v : \mathcal{F} \rightarrow \mathcal{G}$ has the concordance lifting property then there is a homotopy fibre sequence,

\[ |\mathcal{F}^\alpha_z| \rightarrow |\mathcal{F}| \rightarrow |\mathcal{G}| \]

for any choice of $z \in \mathcal{G}(\ast)$.

We now apply these results to the map $\beta_k : \mathbf{D}^{\Sigma_k}_{d,n} \rightarrow \mathbf{D}^{\Sigma_{k-1}}_{d-p_k-1,n-m_k}$.

For $0 < \ell \leq k$, there are maps,

\[ \partial_{k-\ell+1} : \mathbf{D}^{\Sigma_k}_{d,n} \rightarrow \mathbf{D}^{\Sigma_{k-1}}_{d-\ell,n} , \]

\[ \beta_{k-\ell+1} : \mathbf{D}^{\Sigma_{k-1}}_{d,n} \rightarrow \mathbf{D}^{\Sigma_{k-1}}_{d-p_{k-\ell+1}-1,n-m_{k-\ell+1}} , \]

given by,

\[ W \mapsto \partial_{k-\ell+1}W \quad \text{and} \quad W \mapsto \beta_{k-\ell+1}W. \]

There is also an inclusion map

\[ i : \mathbf{D}^{\Sigma_{\ell-1}}_{d,n} \rightarrow \mathbf{D}^{\Sigma_k}_{d,n} \]

given by simply forgetting the product structure $\partial_{k-\ell+1}W = \beta_{k-\ell+1}W \times P_{k-\ell+1}$ on the $k-\ell+1$-th face of the boundary of $W$. Now in a way similar to the the situation with the spaces of
embeddings of $\Sigma_k^\ell$-manifolds from (13), the following diagram is cartesian

\[ D_{\Sigma_k^\ell_{d,n}} \xrightarrow{i} D_{\Sigma_k^\ell_{d,n}} \]
\[ \downarrow \beta_{k-\ell+1} \quad \downarrow \partial_{k-\ell+1} \]
\[ D_{\Sigma_k^{\ell-1}_{d-p_k-\ell+1-1,n-m_k-\ell+1}} \xrightarrow{\times P_{k-\ell+1}} D_{\Sigma_k^{\ell-1}_{d-1,n}}. \]

We will need the following technical lemma:

**Lemma 4.4.** For any $k$ and $\ell > 0$, the map

\[ \partial_{k-\ell+1} : D_{\Sigma_k^\ell_{d,n}} \longrightarrow D_{\Sigma_k^{\ell-1}_{d-1,n}} \]

has the concordance lifting property.

**Proof.** Fix an element $X \in \mathcal{O}b(\mathcal{X})$ and let $W \in D_{\Sigma_k^\ell_{d-1,n}}(X \times \mathbb{R})$ be a concordance. Let $V \in D_{\Sigma_k^\ell_{d,n}}(X)$ be an element which satisfies the following condition: there exists $\epsilon > 0$ such that

\[ \text{pr}^*(\partial_{k-\ell+1}V) |_{X \times (-\infty, \epsilon)} = W |_{X \times (-\infty, \epsilon)} \]

where $\text{pr} : X \times \mathbb{R} \longrightarrow X$ is the projection. We will construct a concordance $\tilde{V} \in D_{\Sigma_k^\ell_{d,n}}(X \times \mathbb{R})$ such that

\[ \partial_{k-\ell+1}\tilde{V} = W \quad \text{and} \quad \tilde{V} |_{X \times (-\infty, \hat{\epsilon})} = \text{pr}^*(V) |_{X \times (-\infty, \hat{\epsilon})} \]

for some $\hat{\epsilon} > 0$, possibly different from $\epsilon$. This will prove the lemma.

First we make the observation that equation (36) implies that the restriction $W |_{X \times (-\infty, \epsilon)}$ is cylindrical. This means that

\[ W |_{X \times (-\infty, \epsilon)} = \pi^{-1}(X \times \{t\}) \times (-\infty, \epsilon) \quad \text{for any} \ t < \epsilon \]

where $\pi : W \longrightarrow X \times \mathbb{R}$ is the projection that comes with $W$ from the definition as an element of $D_{\Sigma_k^\ell_{d,n}}(X \times \mathbb{R})$. This is a fact which we will exploit in our construction of the concordance $\tilde{V}$.

Recall that by definition of $D_{\Sigma_k^\ell_{d,n}}(X)$, the face $\partial_{k-\ell+1}V$ is embedded into the space

\[ X \times \mathbb{R} \times \mathbb{R}_+^{d-1+n} \times \mathbb{R}^{p+m} \]

with a collar whose width is parametrized by $X$. Specifically, there exists a smooth function

\[ \epsilon_{k-\ell+1} : X \longrightarrow (0, \infty), \]

such that

\[ V \cap (X \times \mathbb{R} \times \mathbb{R}_+^{\{k-\ell+1\}^c} \times [0, \epsilon_{k-\ell+1}) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}) = \partial_{k-\ell+1}V \times [0, \epsilon_{k-\ell+1}). \]
In the above equation $\partial_{k-\ell+1}V \times [0, \epsilon_{k-\ell+1})$ is given by the set
$$\{(v, s) \in \partial_{k-\ell+1}V \times [0, \infty) \mid 0 \leq s < \epsilon_{k-\ell+1}(\pi(v))\},$$
where $\pi$ is the projection onto $X$. The space $X \times \mathbb{R} \times \mathbb{R}^k_{\{k-\ell+1\}^c} \times [0, \epsilon_{k-\ell+1}) \times \mathbb{R}^{d-1-n} \times \mathbb{R}^{p+m}$
is given by the set of all tuples $(x, r, (t_1, \ldots, t_k), y)$ such that
$$(x, r) \in X \times \mathbb{R}, \quad 0 \leq t_{k-\ell+1} < \epsilon_{k-\ell+1}(x), \quad \text{and} \quad y \in \mathbb{R}^{d-1-n} \times \mathbb{R}^{p+m}.$$

Let $\epsilon > 0$ be as in (36). Let $\rho : \mathbb{R} \times [0, 1) \longrightarrow \mathbb{R} \times [0, 1)$ be a smooth function which satisfies
the following conditions:

i. The image of $\rho$ is contained in the subspace $\mathbb{R} \times [0, \frac{2}{3}) \cup [0, \frac{2\epsilon}{3}) \times [0, 1)$.
ii. When restricted to the subspace $\mathbb{R} \times [0, \frac{1}{3}) \cup [0, \frac{5}{3}) \times [0, 1), \rho$ is equal to the identity function. Specifically, $\rho(t, s) = (t, s)$ if $t \leq \frac{\epsilon}{3}$ or if $s \leq \frac{1}{3}$.

Figure 3. Above is a schematic for the function $\rho$.

Let $\lambda : \mathbb{R} \times [0, 1) \longrightarrow \mathbb{R}$ be the projection $(t, s) \mapsto t$, for $(t, s) \in \mathbb{R} \times [0, 1)$. Using these two functions $\rho$ and $\lambda$, we define,
$$\hat{\rho} : X \times \mathbb{R} \times [0, \epsilon_{k-\ell+1}) \longrightarrow X \times \mathbb{R}$$
by the formula
$$(x, t, s) \mapsto (x, \lambda \circ \rho(t, \frac{s}{\epsilon_{k-\ell+1}(x)})).$$

It follows directly from the definition of $\rho$ and $\lambda$ that for all $x \in X$,

(a) if $s \leq \frac{\epsilon_{k-\ell+1}(x)}{3}$ or $t \leq \frac{\epsilon}{3}$, then $\hat{\rho}(x, t, s) = (x, t),$
(b) if $s \geq \frac{2\epsilon_{k-\ell+1}(x)}{3}$, then $\hat{\rho}(x, t, s) = (x, \frac{\epsilon}{2})$ for all $x$ and $t$. 

We now form the pull-back,

\[
\begin{array}{ccc}
\hat{\rho}^*(W) & \xrightarrow{\hat{\pi}} & W \\
X \times \mathbb{R} \times [0, \epsilon_{k-\ell+1}] & \xrightarrow{\hat{\rho}} & X \times \mathbb{R}
\end{array}
\]

where \(\pi : W \rightarrow X \times \mathbb{R}\) is the projection onto \(X \times \mathbb{R}\). By definition of \(D^{\Sigma_{d-1,n}}_{d-1,n}(X \times \mathbb{R})\), \(\pi\) is a submersion and for each \(I \subseteq \langle k \rangle\), the restriction \(\pi |_{\partial W}\) is a submersion as well. It follows from the factorization in condition iii. of Definition 4.1 that \(\hat{\rho}^*(W)\) is a \(\Sigma^f_k\)-manifold, see remark 3.2. Furthermore, the induced map \(\hat{\pi}\) is a submersion as well. The space \(\hat{\rho}^*(W)\) is by definition equal to the space

\[
((x, t, s), w) \in X \times \mathbb{R} \times [0, \epsilon_{k-\ell+1}) \times W \mid \hat{\rho}(x, t, s) = \pi(w)
\]

and so there is a natural embedding

\[
i : \hat{\rho}^*(W) \hookrightarrow (X \times \mathbb{R}) \times \mathbb{R} \times k_{\langle \{k-\ell+1\}\rangle} \times [0, \epsilon_{k-\ell+1}] \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}
\]

induced by the inclusion of \(W\) into \((X \times \mathbb{R}) \times \mathbb{R} \times k_{\langle \{k-\ell+1\}\rangle} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}\). From now on we refer to \(\hat{\rho}^*(W)\) as the submanifold determined by the above embedding.

It can be verified directly using condition (b) and (39) that the intersections

\[
\hat{\rho}^*(W) \cap \left[ (X \times \mathbb{R}) \times \mathbb{R} \times k_{\langle \{k-\ell+1\}\rangle} \times \left( \frac{2\epsilon_{k-\ell+1}}{3}, \epsilon_{k-\ell+1} \right) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m} \right],
\]

\[
\text{pr}^*V \cap \left[ (X \times \mathbb{R}) \times \mathbb{R} \times k_{\langle \{k-\ell+1\}\rangle} \times \left( \frac{2\epsilon_{k-\ell+1}}{3}, \epsilon_{k-\ell+1} \right) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m} \right],
\]

are equal. Denoting by \(\tilde{V}\) the second of the above intersections, this implies that the union

\[\tilde{V} \cup \hat{\rho}^*(W)\]

is \(\Sigma^f_k\)-submanifold of \((X \times \mathbb{R}) \times \mathbb{R} \times k_{\langle \{k-\ell+1\}\rangle} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}\). We set

\[\tilde{V} := \tilde{V} \cup \hat{\rho}^*(W)\]

We claim that \(\tilde{V}\) is the desired lift of the concordance \(W\) that we seek. It is easy to check that the projection onto \((X \times \mathbb{R})\) is a submersion and the projection onto \((X \times \mathbb{R}) \times \mathbb{R}\) is proper. Furthermore, it follows from condition (a) that the intersection

\[\tilde{V} \cap \left[ (X \times \mathbb{R}) \times \mathbb{R} \times k_{\langle \{k-\ell+1\}\rangle} \times [0, \frac{\epsilon_{k-\ell+1}}{3}) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m} \right],\]

is equal to the product \(\partial_{k-\ell+1} W \times [0, \frac{\epsilon_{k-\ell+1}}{3})\) and so \(\partial_{k-\ell+1} W\) is embedded with a parametrized collar. Thus the manifold \(\tilde{V}\) does determine an element of \(D^{\Sigma^f_{d,n}}_{d,n}(X \times \mathbb{R})\). We have from (a), that \(\partial_{k-\ell+1} \tilde{V}\), which is given by the intersection

\[
\tilde{V} \cap \left[ (X \times \mathbb{R}) \times \mathbb{R} \times k_{\langle \{k-\ell+1\}\rangle} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m} \right],
\]
is equal to $\partial_{k-\ell+1}W$. One also sees that the intersection
\[ \tilde{V} \cap \left( (X \times (-\infty, \frac{\epsilon}{3})) \times \mathbb{R} \times \mathbb{R}_k^+ \times \mathbb{R}_d^{d-1+n} \times \mathbb{R}^{p+m} \right), \]
is equal to $\text{pr}^*V |_{X \times (-\infty, \frac{\epsilon}{3})}$. This implies that the element of $D_{d,n}^{\Sigma^k} (X \times \mathbb{R})$ determined by $\tilde{V}$ is the desired lift of $W$ which extends $\text{pr}^*V |_{X \times (-\infty, \frac{\epsilon}{3})}$. This proves the lemma. □

Applying this lemma together to the pull-back diagram (35), Proposition 4.2 implies the following theorem:

**Theorem 4.5.** The cartesian diagram

\[
\begin{diagram}
\node{\text{D}^{\Sigma^k}_{d,n}} \arrow{s,l}{|}\arrow{e,l}{\text{D}^{\Sigma^k}_{d,n}} \arrow{s,l}{|\beta_{k-\ell+1}} \arrow{e,l}{\text{D}^{\Sigma^k}_{d-1,n}} \arrow{s,l}{|\partial_{k-\ell+1}} \arrow{e,l}{\text{D}^{\Sigma^k}_{d-1,n}}
\end{diagram}
\]

obtained from (35) is homotopy cartesian.

**Proof.** Follows from the previous lemma. □

We now identify the homotopy-fibres of the vertical maps in the above diagram. Since the diagram is homotopy-cartesian, the homotopy fibres of both columns will be the same.

The element $\emptyset \in D_{d-1,n}^{\Sigma^k-1}(\star)$, given by the empty set, determines an element $\emptyset \in D_{d-1,n}^{\Sigma^k-1}(X)$ for all $X \in \text{Ob}(\mathcal{X})$. Denote by $F_{\emptyset}^{\partial_{k-\ell+1}}$ the fibre-sheaf over $\emptyset$ of the map

\[ \partial_{k-\ell+1} : D_{d,n}^{\Sigma^k} \rightarrow D_{d-1,n}^{\Sigma^k-1}. \]

This is defined by

\[ F_{\emptyset}^{\partial_{k-\ell+1}} (X) = \{ W \in D_{d,n}^{\Sigma^k} (X) | \partial_{k-\ell+1}(W) = \emptyset \}. \]

**Lemma 4.6.** The fibre sheaf $F_{\emptyset}^{\partial_{k-\ell+1}}$ is isomorphic to the sheaf $D_{d,n}^{\Sigma^k-1}$

**Proof.** Elements of $D_{d,n}^{\Sigma^k-1}(X)$ are given by $\Sigma^{k-1}_{k-1}$-submanifolds of

\[ X \times \mathbb{R} \times \mathbb{R}_k^{k-1} \times (-\infty, \infty) \times \mathbb{R}_d^{d-1+n} \times \mathbb{R}^{p+m}. \]

For $X \in \text{Ob}(\mathcal{X})$, an element of $F_{\emptyset}^{\partial_{k-\ell+1}} (X)$ is given by a $\Sigma^\ell_k$-submanifold

\[ W \subset X \times \mathbb{R} \times \mathbb{R}_k^+ \times \mathbb{R}_d^{d-1+n} \times \mathbb{R}^{p+m}. \]
satisfying all conditions of the definition of $\mathbf{D}_{d,n}^{\Sigma_k^l}(X)$ with the added property that
\[
W \cap (X \times \mathbb{R} \times \mathbb{R}_+^{k_{(k,c)}} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}) = \partial_{k-\ell+1}W = \emptyset.
\]
So $F_0^{\partial_{k-\ell+1}}(X)$ can be characterized as the subset of $\mathbf{D}_{d,n}^{\Sigma_{k-1}^l}(X)$ consisting of all manifolds that lie in
\[
X \times \mathbb{R} \times \mathbb{R}_+^{k_{-1}} \times (0, \infty) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}.
\]
Choosing a diffeomorphism,
\[
\Phi : X \times \mathbb{R} \times \mathbb{R}_+^{k_{-1}} \times (-\infty, \infty) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}} \longrightarrow X \times \mathbb{R} \times \mathbb{R}_+^{k_{-1}} \times (0, \infty) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}
\]
that is identical on the factors of $X$, $\mathbb{R}_+^{k_{-1}}$, $\mathbb{R}^{d-1+n}$, and $\mathbb{R}^{\bar{p}+\bar{m}}$, we can define a map
\[
\mathbf{D}_{d,n}^{\Sigma_{k-1}^l} \longrightarrow F_0^{\partial_{k-\ell+1}}
\]
by sending $W \in \mathbf{D}_{d,n}^{\Sigma_{k-1}^l}(X)$ to $\Phi(W)$, which is an element of $F_0^{\partial_{k-\ell+1}}(X)$. This map is clearly natural in $X$ since the map $\Phi$ is identical on the $X$-factor. Since $\Phi$ is a diffeomorphism, this map of sheaves is invertible, and thus is a natural isomorphism. □

The previous lemmas implies that for all $0 < \ell \leq k$ there is a homotopy-fibre sequence
\[
|\mathbf{D}_{d,n}^{\Sigma_{k-1}^l}| \longrightarrow |\mathbf{D}_{d,n}^{\Sigma_k^l}| \longrightarrow |\partial_{k-\ell+1}^{\ell-1}| \longrightarrow |\mathbf{D}_{d,n}^{\Sigma_{k-1}^l}|
\]
This combined with the fact that diagram (40) is homotopy cartesian implies that for all $0 < \ell \leq k$ there is a homotopy fibre sequence,
\[
|\mathbf{D}_{d,n}^{\Sigma_{k-1}^l}| \longrightarrow |\mathbf{D}_{d,n}^{\Sigma_{k-1}^l}| \longrightarrow |\partial_{k-\ell+1}^{\ell-1}| \longrightarrow |\mathbf{D}_{d,n}^{\Sigma_{k-1}^l}|
\]
Applying this theorem to the case when $\ell = 1$, the homotopy-fibre sequence (41) implies that there is a homotopy fibre sequence,
\[
|\mathbf{D}_{d,n}^{\Sigma_{k-1}^l}| \longrightarrow |\mathbf{D}_{d,n}^{\Sigma_k^l}| \longrightarrow |\mathbf{D}_{d-1-p_k,n-m_k}^{\Sigma_k^l}|
\]
This proves Theorem 4.1

5. The Classifying Space of $\mathbf{Cob}_{d,n}^{\Sigma_k^l}$

In this section we show that for each $n$ there is a weak homotopy equivalence
\[
\mathbf{BCob}_d^{\Sigma_k^l} \sim |\mathbf{D}_{d,n}^{\Sigma_k^l}|
\]
Combining this with the results of Section 8 will prove Theorem 1.3 stated in the introduction. We define a category-valued sheaf $\mathbf{D}_{d,n}^{\Sigma_k^l}$ which we can map to both $\mathbf{D}_{d,n}^{\Sigma_k^l}$ and $\mathbf{C}_{d,n}^{\Sigma_k^l}$. In this section we will need a bit more technical machinery relating to $\mathbf{CAT}$-valued sheaves. We will need to consider Co-cycle Sheaves.
5.1. Cocycle Sheaves. Here we will review the definition of Co-cycle sheaves and state the main results that we need to use. For more details on Co-cycle sheaves, we refer the reader to [13].

**Definition 5.1.** Let \( F \) be any \( \text{CAT} \)-valued sheaf on \( X \). The is an associated set valued sheaf \( \beta F \) on \( X \). Choose once and for all an uncountable set \( J \). An element of \( \beta F(X) \) is a pair \((U, \Phi)\) where \( U = \{ U_j \mid j \in J \} \) is a locally finite open cover of \( X \), indexed by \( J \), and \( \Phi \) is a collection of morphisms, \( \phi_{RS} \in \mathcal{N}_1 F(U_S) \) indexed by pairs \( R \subset S \) of non-empty finite subsets of \( J \) and \( U_S = \cap_{i \in S} U_i \), subject to the conditions:

i. \( \phi_{RR} = \text{Id}_{c_R} \) for an object \( c_R \in \mathcal{N}_0(U_R) \),
ii. For each non-empty finite \( R \subset S \), \( \phi_{RS} \) is a morphism from \( c_S \) to \( c_R \mid U_S \) (where by condition i. \( \text{Id}_{c_S} = \phi_{SS} \) and \( \text{Id}_{c_R} = \phi_{RR} \)),
iii. For all triples \( R \subseteq S \subseteq T \) of finite non-empty subsets of \( J \), we have \( \phi_{RT} = (\phi_{RS} \mid U_T) \circ \phi_{ST} \).

**Theorem 5.1.** There is a weak homotopy equivalence

\[ |\beta F| \sim B|F| \]

**Proof.** See Theorem 4.1.2 of [13]. \( \square \)

**Remark 5.1.** The above homotopy equivalence is natural in the following sense. The assignments

\[ (42) \quad F \mapsto B|F| \quad \text{and} \quad F \mapsto |\beta F| \]

are functors from the category of \( \text{CAT} \)-valued sheaves on \( X \) to the category of Topological Spaces. In [13 A.3], the authors connect these two functors (42) through a zig-zag of natural transformations of functors sending \( \text{CAT} \)-valued sheaves to Spaces. The authors then show that these natural transformations induce equivalences upon passing to the homotopy category.

**Remark 5.2.** A set valued sheaf \( F \) can be considered a category-valued sheaf by defining \( F(X) \) to be the object set of a category with only identity morphisms. In this way it makes sense to define \( \beta F \). In this case with \( F \) a set valued sheaf, there is a forgetful projection \( \beta F \to F \) which is is a weak equivalence. For proof of this see Section 4.2 of [13].

5.2. The Classifying Space. We now define a Poset-valued sheaf \( D_{d,n}^{\Sigma_k,h} \) which we can map to both \( C_{d,n}^{\Sigma_k,h} \) and \( D_{d,n}^{\Sigma_k} \).

**Definition 5.2.** For \( X \in \mathcal{X} \), we define \( D_{d,n}^{\Sigma_k,h}(X) \) to be the set of quadruples \((W; \pi, f, a)\) subject to the following conditions:

i. \((W; \pi, f) \in D_{d,n}^{\Sigma_k}(X)\),
ii. \( a : X \to \mathbb{R} \) is a smooth function,
iii. the function $f : W \rightarrow \mathbb{R}$ and all restrictions $f |_{\partial I W}$ are fibrewise transverse to $a$ with respect to the submersion $\pi$.

By fibrewise transverse we mean that for each $x \in X$ the restriction of $f$ (and $f |_{\partial I W}$ for each $I$) to $\pi^{-1}(x)$ is transverse to the point $a(x)$. For each $X \in \mathcal{X}$, the set $D^{\Sigma_k, e}\Sigma_k(X)$ actually has the structure of a category, namely a partially ordered set. The objects are the elements $(W; \pi, f, a)$ as described above. A morphism is given by a quintuple $(W; \pi, f, a, b)$ where $(W; \pi, f, a)$ and $(W; \pi, f, b)$ are elements of $D^{\Sigma_k, e}\Sigma_k(X)$ and $a(x) \leq b(x)$ for all $x \in X$. In this way $D^{\Sigma_k, e}\Sigma_k(X)$ has the structure of a partially ordered set. Notice that there are no morphisms connecting elements $(W; \pi, f, a)$ and $(V; \pi', f', b)$ if $W \neq V$.

For each positive integer $n$, there is a map of sheaves

\[ \alpha_n : D^{\Sigma_k, e}\Sigma_k(X) \rightarrow C^{\Sigma_k, e}\Sigma_k(X) \]

defined as follows. Let $(W; \pi, f, a)$ be an object of $D^{\Sigma_k, e}\Sigma_k(X)$. There exists a smooth function $\epsilon : X \rightarrow (0, \infty)$ such that $f$ is fibre-wise transverse to all other smooth functions $g : X \rightarrow \mathbb{R}$ such that $a - \epsilon \leq g \leq a + \epsilon$. The properness of $(\pi, f)$ then implies that the restriction of $(\pi, f)$ to the open subset

\[ W_\epsilon := (\pi, f)^{-1}(X \times (a - \epsilon, a + \epsilon)), \]

is a proper submersion $W_\epsilon \rightarrow X \times (a - \epsilon, a + \epsilon)$. Thus the class $[W_\epsilon]$, as $\epsilon \rightarrow 0$, is a well defined element of $C^{\Sigma_k, e}\Sigma_k(X; a, a)$ hence an element of $\mathcal{O}(C^{\Sigma_k, e}\Sigma_k(X))$. We define,

\[ \alpha_n(W; \pi, f, a) := ([W_\epsilon], \pi, f, a, a). \]

This defines $\alpha_n$ on the level of objects, it is defined similarly on the level of morphisms thus giving a map of $\textbf{CAT}$-valued sheaves.

We now state:

**Proposition 5.2.** The map $\alpha : D^{\Sigma_k, e}\Sigma_k \rightarrow C^{\Sigma_k, e}\Sigma_k$ defined above induces a weak homotopy equivalence

\[ B|D^{\Sigma_k, e}\Sigma_k| \sim B|C^{\Sigma_k, e}\Sigma_k|. \]

**Proof.** This proposition is proven by showing that $\alpha_n$ induces homotopy equivalences

\[ |N_k(D^{\Sigma_k, e}\Sigma_k)| \sim |N_k(C^{\Sigma_k, e}\Sigma_k)| \]

for all $k \geq 0$. This can be done in exactly the same way as in the proof of Proposition 4.3 from [6].

Now we consider the map $h : D^{\Sigma_k, e}\Sigma_k \rightarrow D^{\Sigma_k}$ defined by forgetting the Partially ordered set structure on $D^{\Sigma_k, e}\Sigma_k(X)$. Applying the cocycle construction induces a map given by the
composition
\[
\beta D_{d,n}^{\Sigma_k,\ell} \xrightarrow{\beta h} \beta D_{d,n}^{\Sigma_k} \xrightarrow{\sim} D_{d,n}^{\Sigma_k} \]
where the second map is the forgetful projection which is a weak equivalence, see remark 5.2

**Proposition 5.3.** The above map \(\beta D_{d,n}^{\Sigma_k,\ell} \rightarrow D_{d,n}^{\Sigma_k}\) is a weak equivalence of sheaves. Thus there is a weak homotopy equivalence \(B|D_{d,n}^{\Sigma_k,\ell}| \sim |D_{d,n}^{\Sigma_k}|\).

**Proof.** The proof is essentially the same as the proof of [6, 4.2]. A proof is also written explicitly in [7] for the case \(\ell = k\), for manifolds with corners.

The results from this section yield a zig-zag of weak homotopy equivalences
\[
\begin{align*}
B|C_{d,n}^{\Sigma_k,\ell}| & \xrightarrow{\sim} B|D_{d,n}^{\Sigma_k}| \xrightarrow{\sim} |D_{d,n}^{\Sigma_k}|.
\end{align*}
\]

Theorems 3.4 and 3.3 yield a zig zag,
\[
\begin{align*}
BCob_{d,n}^{\Sigma_k} & \xrightarrow{\sim} B|C_{d,n}^{\Sigma_k}| \xrightarrow{\sim} B|C_{d,n}^{\Sigma_k,\ell}|.
\end{align*}
\]
Combining the above yields a weak homotopy equivalence,
\[
BCob_{d,n}^{\Sigma_k} \sim |D_{d,n}^{\Sigma_k}|.
\]

### 5.3. Proof of Theorems 1.2 and 1.1
Recall the functor
\[
\beta_k : \text{Cob}_{d,n}^{\Sigma_k} \rightarrow \text{Cob}_{d-p_k-m,n-m_k}^{\Sigma_k-1}
\]
defined by sending a \(\Sigma_k\)-cobordism \(W\) to \(\beta_k W\) and the map of sheaves by the same name
\[
\beta_k : D_{d,n}^{\Sigma_k} \rightarrow D_{d-p_k-1,n-m_k}^{\Sigma_k-1}
\]
from Section 4.2. Since the weak homotopy equivalences from the above zig-zag diagrams are all induced by natural transformations of sheaves, the diagram
\[
\begin{align*}
\xymatrix{ BCob_{d,n}^{\Sigma_k} \ar[r] & |D_{d,n}^{\Sigma_k}| \ar[d]^{B\beta_k} \ar[r] & |D_{d,n}^{\Sigma_k}| \ar[d]^{|\beta_k|} \\
BCob_{d-p_k-1,n-m_k}^{\Sigma_k-1} \ar[r] & |D_{d-p_k-1,n-m_k}^{\Sigma_k-1}| &
}
\end{align*}
\]
commutes, where the horizontal arrows come from the zig-zags of weak equivalences from (44) and (45). Commutativity of this diagram implies that the vertical maps have weakly equivalent homotopy-fibres. Applying Theorem 4.1, we see that the homotopy-fibre of
\[
B(\beta_k) : BCob_{d,n}^{\Sigma_k} \rightarrow BCob_{d-p_k-m,n-m_k}^{\Sigma_k-1}
\]
is weakly equivalent to \( |D_{d,n}^{\Sigma_{k-1}}| \) and thus by (46), weakly equivalent to \( BCob_{d,n}^{\Sigma_{k-1}} \). This holds for all \( n \). Taking the direct limit as \( n \to \infty \) yields the homotopy fibre sequence,

\[
BCob_{d}^{\Sigma_{k-1}} \to BCob_{d}^{\Sigma_{k}} \to BCob_{d-p_k-m_k}^{\Sigma_{k-1}}.
\]

This proves Theorem 1.2 stated in the introduction. We in fact just proved the stronger result that the above is a fibre sequence prior to letting \( n \) run to \( \infty \).

The above zig-zags from (44) and (45), since they are induced by natural transformations, yield a commutative diagram,

where the front and back squares are the cartesian squares (27) and (40) respectively. By Theorem 4.5, the back square is homotopy cartesian. Since the diagonal zig-zags are weak equivalences, the front square is homotopy cartesian as well. This proves Theorem 1.1 from the introduction.

6. Cubical Diagrams

6.1. \( k \)-Cubic Spaces. As in previous sections, let \( \langle k \rangle \) denote the set \( \{1, \cdots, k\} \) of \( k \)-elements. We denote by \( 2^{\langle k \rangle} \) the Power Set of \( \langle k \rangle \) made into a category with objects the subsets of \( \langle k \rangle \) and morphisms given by the inclusion maps. We call a contravariant functor from \( 2^{\langle k \rangle} \) to \( \text{Top} \) (the category of topological spaces), a \( k \)-cubic space. In order to define a \( k \)-cubic space one needs to associate to each subset \( J \subseteq \langle k \rangle \) a space \( X_J \) and to any pair of subsets \( I \subset J \), a map

\[ f_{J,I} : X_J \to X_I \]

such that for any triple \( K \subset I \subset J \), the equation \( f_{J,K} = f_{I,K} \circ f_{J,I} \) holds. In this case we will generally denote the resulting \( k \)-cubic space by \( X_\bullet \). The spaces \( X_J \) and maps \( f_{J,I} \) fit together to form a \( k \)-dimensional cubical commutative diagram. We will often times refer to the spaces \( X_J \) as vertices of the cube and the maps \( f_{J,I} \) as edges.
We define a map of $k$-cubic spaces, $F : X \rightarrow Y$, to be a natural transformation of the functors $X$ and $Y$. Or in other words a collection of maps $F_J : X_J \rightarrow Y_J$ such that for any pair of subsets $I \subseteq J$, the diagram

$$
\begin{array}{ccc}
X_I & \xrightarrow{F_I} & Y_I \\
\uparrow & & \uparrow \\
X_J & \xrightarrow{F_J} & Y_J
\end{array}
$$

commutes where the vertical maps are the edges in the cubes $X$ and $Y$.

We denote the set of all such $k$-cubic space maps by $C^0_{(k)}(X, Y)$.

This set is topologized naturally as a subset of the product $\prod_{J \subseteq (k)} C^0(X_J, Y_J)$ where $C^0(X_J, Y_J)$ is the space of continuous maps from $X_J$ to $Y_J$ with the compact-open topology.

It is easy to see that a $\Sigma_k$-manifold $W$ determines a $k$-cubic space by the correspondence $I \mapsto \partial I W$ for $I \subseteq \langle k \rangle$. Also notice that the spaces $\mathbb{R}_{+}^k$ and $\mathbb{R}_{p}^{p+m}$ from the previous sections determine $k$-cubic spaces via the correspondences,

$$
I \mapsto \mathbb{R}_{+}^{k(I)} = \{(t_1, \ldots, t_k) \mid t_i = 0 \text{ if } i \notin I\}
$$

$$
I \mapsto \mathbb{R}_{p}^{p+m(I)} = \{(x_1, \ldots, x_k) \in \mathbb{R}_{p_1+m_1} \times \cdots \times \mathbb{R}_{p_k+m_k} \mid x_i = 0 \text{ if } i \notin I\}.
$$

6.2. $k$-Cubic Spectra. In addition to $k$-cubic spaces we will also have to consider $k$-cubic spectra. A $k$-cubic spectrum is a functor

$$
\hat{X} : 2^{(k)} \rightarrow Spe
$$

where $Spe$ is the category of spectra. It is required that for each $I \subset J \subseteq \langle k \rangle$ the edge (in the cubical diagram),

$$
\hat{X}_J \rightarrow \hat{X}_I
$$

is defined to be a map of spectra of degree 0. Let $\hat{X}$ be a $k$-cubic spectrum. Then for each integer $n$, there is a $k$-cubic space $(\hat{X})_n$ defined by sending each $J \subseteq \langle k \rangle$ to the $n$th space of the spectrum $\hat{X}_J$. The operations of suspending, $\hat{X} \mapsto \Sigma \hat{X}$, and de-suspending $\hat{X} \mapsto \Sigma^{-1} \hat{X}$ still make sense in the context of $k$-cubic spectra.

6.3. The Total Homotopy Cofibre of a $k$-Cubic Space. We define an important type of homotopy colimit associated to a $k$-cubic space called the Total Homotopy Cofibre. This will be defined inductively on $k$. We first introduce some new notation to enable us to carry out the induction. Let $\hat{X}$ be a $k$-cubic space. For $j \in \langle k \rangle$ we define $X_{\cdot j}$ to be the $k-1$-cubic space defined by restricting the functor

$$
X : 2^{(k)} \rightarrow Top
$$
to the full subcategory of $2^{(k)}$ on all subsets of $\langle k \rangle$ that contain $j$. Similarly, we define $X_{\bullet,j}$ to be the restriction of $X_{\bullet}$ to the full subcategory of $2^{(k)}$ on all subsets disjoint from $j$. There is a natural map of $k-1$-cubic spaces
\begin{equation}
\iota^{(k)}_j : X_{\bullet,j} \to X_{\bullet,j}
\end{equation}
induced by these inclusions.

**Definition 6.1.** Let $X_{\bullet}$ be a $k$-cubic space. We define the Total Homotopy Cofibre of $X_{\bullet}$, which we denote by
\[ t\text{Cofibre}_{(k)} X_{\bullet}, \]
inductively on $k$ as follows. If $k = 1$, we define
\[ t\text{Cofibre}_{(k)} X_{\bullet} := \text{Cofibre}_{(k)} X_{\bullet} \]
where by Cofibre we technically mean the homotopy-cofibre. In this case the total homotopy cofibre is simply the normal homotopy cofibre since when $k = 1$, a $k$-cubic space consists of just one map. Now to define for arbitrary $k$. Assume that the total homotopy cofibre is defined for all $k-1$-cubic spaces. The map $\iota^{(k)}_j$ induces a map
\[ t\text{Cofibre}_{(k-1)} X_{\bullet,k} \to t\text{Cofibre}_{(k-1)} X_{\bullet,k}. \]
We set,
\begin{equation}
t\text{Cofibre}_{(k)} X_{\bullet} := \text{Cofibre}_{(k)} \left( t\text{Cofibre}_{(k-1)} X_{\bullet,k} \to t\text{Cofibre}_{(k-1)} X_{\bullet,k} \right). \end{equation}
For a $k$-cubic spectrum $\hat{X}_{\bullet}$, we define $t\text{Cofibre}_{(k)} \hat{X}_{\bullet}$ similarly. Since the operation of taking suspension commutes with taking homotopy cofibres, we can define $t\text{Cofibre}_{(k)} \hat{X}_{\bullet}$ by setting the $n$-th space
\[ (t\text{Cofibre}_{(k)} \hat{X}_{\bullet})_n := t\text{Cofibre}_{(k)} (\hat{X}_n). \]

6.4. **Loop Spaces.** In this section we describe a useful stratification of the infinite loop-space of the total homotopy cofibre of a $k$-cubic spectrum.

**Definition 6.2.** For $n \geq 0$, let $D_{\bullet,I}^{n,(k)}$ denote the $k$-cubic space defined by
\[ D_{\bullet,I}^{n,(k)} := (\mathbb{R}^k_{+(J^c)} \times \mathbb{R}^{n-k})^c, \]
where the super-script of $c$ denotes one-point compactification. The maps $D_{\bullet,I}^{n,(k)} \to D_{\bullet,I}^{n,(k)}$ for $I \subseteq J$ are given by inclusion.

Let $X_{\bullet}$ be a $k$-cubic space. We denote,
\begin{equation}
\Omega_{(k)}^n X_{\bullet} := C^0_{(k)} (D_{\bullet,I}^{n,(k)}, X_{\bullet}),
\end{equation}
where the space on the right hand side is the space of $k$-cubic space maps topologized as a subspace of the product

$$\prod_{i \in \langle k \rangle} C^0(D^{n,(k)}_{\langle i \rangle}, X_{\langle i \rangle})$$

as described in the previous section. Now let $\hat{X}_\bullet$ be a $k$-cubic spectrum. For integers $m, n$ with $n > 0$, there are maps,

$$\Omega_{(k)}^n(\hat{X}_\bullet)_{n+m} \to \Omega_{(k)}^{n+1}(\hat{X}_\bullet)_{n+m+1}$$

induced by the structure maps of the spectrum. Using these maps we define,

$$\Omega_{(k)}^{\infty-m} \hat{X}_\bullet := \text{colim}_{n \to \infty} \Omega_{(k)}^n(\hat{X}_\bullet)_{n+m}.$$ (50)

For each $n$ and $m$, the natural maps

$$(\hat{X}_{J})_{n+m} \to (\text{tCofibre} \hat{X}_\bullet)_{n+m} \text{ for each } J \subset \langle k \rangle$$

induce,

$$\Omega_{(k)}^n(\hat{X}_\bullet)_{n+m} \to \Omega_{(k)}^n(\text{tCofibre} \hat{X}_\bullet)_{n+m}$$

and a map

$$\Omega_{(k)}^{\infty-m} \hat{X}_\bullet \to \Omega_{(k)}^{\infty-m} \text{tCofibre} \hat{X}_\bullet.$$ (51)

in the direct limit where the spaces on the right are the normal loop-spaces. This brings us to the useful result,

**Theorem 6.1.** For all $m$, the map from (51) yields a homotopy equivalence,

$$\Omega_{(k)}^{\infty-m} \hat{X}_\bullet \sim \Omega_{(k)}^{\infty-m} \text{tCofibre} \hat{X}_\bullet.$$ 

**Proof.** We prove this by induction on $k$. For $k = 0$, this is trivial. The two spaces in question are clearly the same space. Now suppose the statement holds for all $k - 1$-cubic spectra. Let $X$ be $k$-cubic spectrum. Recall the notation $\hat{X}_\bullet$ and $\hat{X}_\bullet$ from Section 6.3. These are both $k - 1$-cubic spectra. The induction hypothesis implies that we have homotopy equivalences,

$$\Omega_{(k-1)}^{\infty-m} \hat{X}_{\bullet,k} \sim \Omega_{(k-1)}^{\infty-m} \text{tCofibre} \hat{X}_{\bullet,k}$$

$$\Omega_{(k-1)}^{\infty-m} \hat{X}_{\bullet,k} \sim \Omega_{(k-1)}^{\infty-m} \text{tCofibre} \hat{X}_{\bullet,k},$$

induced by the map from (51). Now for all positive integers $n$, there is a map,

$$r_n^{(k)} : \Omega_{(k)}^n(\hat{X}_\bullet)_{n+m} \to \Omega_{(k-1)}^{n-1}(\hat{X}_{\bullet,k})_{n+m} := \Omega_{(k-1)}^{n-1}(\Sigma \hat{X}_{\bullet,k})_{n+m-1}$$ (52)

given by restricting a $k$-cubic space map $f_\bullet : D^{(k),n}_{\bullet} \to (\hat{X}_\bullet)_{n+m}$ to the $k - 1$-cubic space $D^{(k),n}_{\bullet,k}$. Notice that the $k - 1$-cubic space $D^{(k),n}_{\bullet,k}$ is equal to $D^{(k-1),n-1}_{\bullet,k}$. Now this map $r_n^{(k)}$...
is a Serre-fibration and the fibre over the constant map can be identified with the space 
\( \Omega_{(k-1)}(\tilde{X}_{\ast,k})_{n+m} \) and so we have a fibre-sequence
\[
\Omega_{(k-1)}^n(\tilde{X}_{\ast,k})_{n+m} \longrightarrow \Omega_{(k)}(\tilde{X}_{\ast})_{n+m} \longrightarrow \Omega_{(k-1)}^{n-1}(\Sigma \tilde{X}_{\ast,k})_{n+m-1}.
\]
Taking the direct limits as \( n \to \infty \) yields a homotopy-fibre sequence,
\[
(53) \quad \Omega_{(k-1)}^\infty \tilde{X}_{\ast,k} \longrightarrow \Omega_{(k)}^\infty \tilde{X}_{\ast} \longrightarrow \Omega_{(k-1)}^{\infty-1} \Sigma \tilde{X}_{\ast,k}.
\]
Now applying \( \Omega_{(k-1)}^\infty \) to the cofibre-sequence,
\[
tCofibre \tilde{X}_{\ast,k} \longrightarrow tCofibre \tilde{X}_{\ast,k} \longrightarrow tCofibre \tilde{X}_{\ast} \longrightarrow (tCofibre \Sigma \tilde{X}_{\ast,k} \sim \Sigma tCofibre \tilde{X}_{\ast,k}),
\]
yields a homotopy-fibre sequence
\[
(54) \quad \Omega_{(k-1)}^\infty tCofibre \tilde{X}_{\ast,k} \longrightarrow \Omega_{(k)}^\infty tCofibre \tilde{X}_{\ast} \longrightarrow \Omega_{(k-1)}^{\infty-1} tCofibre \Sigma \tilde{X}_{\ast,k}.
\]
Above we used the identification, up to homotopy, \( tCofibre \Sigma \tilde{X}_{\ast,k} \sim \Sigma tCofibre \tilde{X}_{\ast,k} \). The maps (51) yield the map of homotopy-fibre sequences,
\[
\begin{align*}
\Omega_{(k-1)}^\infty \tilde{X}_{\ast,k} & \sim \Omega_{(k-1)}^\infty tCofibre_{(k-1)} \tilde{X}_{\ast,k} \\
\Omega_{(k)}^\infty \tilde{X}_{\ast} & \sim \Omega_{(k)}^\infty tCofibre_{(k)} \tilde{X}_{\ast} \\
\Omega_{(k-1)}^{\infty-1} \Sigma \tilde{X}_{\ast,k} & \sim \Omega_{(k-1)}^{\infty-1} tCofibre_{(k-1)} \Sigma \tilde{X}_{\ast,k}.
\end{align*}
\]
The top and bottom horizontal maps are homotopy equivalences by the induction hypothesis. It follows that the middle-horizontal map is a homotopy equivalence by an application of the five lemma on the long exact of homotopy groups induced by these fibre-sequences. □

7. A Homotopy Colimit of Thom Spectra

7.1. A Cubical Diagram of Thom-Spectra. We now construct a particular \( k \)-cubic spectrum, which we denote by \( MT_\ast(d)_{\Sigma_k} \) with the property that for each \( J \subseteq \langle k \rangle \), \( MT_J(d)_{\Sigma_k} \) is homotopy equivalent (as a spectrum) to the spectrum \( \Sigma^{-|J|} MT(d - p_J - |J|) \) from [6]. The total homotopy cofibre of the \( k \)-cubic spectrum, which we denote by \( MT(d)_{\Sigma_k} \), is the spectrum that appears in the statement of our main theorem. In order to carry out the construction of \( MT_\ast(d)_{\Sigma_k} \), we will have to first make some auxiliary definitions.

For each manifold \( P_i \in \Sigma \), denote by \( N_{P_i} \longrightarrow P_i \) the normal bundle for \( P_i \) induced by the embedding \( \phi_i : P_i \hookrightarrow \mathbb{R}^{p_i+m_i} \). For each subset \( I \subseteq \langle k \rangle \), we denote by \( N_{P_I} \longrightarrow P_I \) the normal
bundle for the product embedding $\phi_I : P^I \hookrightarrow \prod_{(I)} \mathbb{R}^{\hat{p}+\hat{m}_I}$. Clearly there is a factorization

$$N_{P^I} = \prod_{i \in I} N_{P_i}.$$ 

Now, we fix once and for all tubular neighborhood embeddings

$$e_{P_i} : N_{P_i} \hookrightarrow \mathbb{R}^{p_i+m_i} := \mathbb{R}^{\hat{p}+\hat{m}}_{(\{i\})}.$$ 

For each $J \subseteq \langle k \rangle$, the products,

$$e_{P_J} := \prod_{j \in J} e_{P_j} : N_{P_J} \hookrightarrow \mathbb{R}^{\hat{p}+\hat{m}}_{(J)}$$

give tubular neighborhood embeddings for the normal bundles $N_{P_J} \hookrightarrow P^I$. These tubular neighborhoods induce collapsing maps

$$c_{P_J} : (\mathbb{R}^{\hat{p}+\hat{m}}_{(J)})^c \longrightarrow \text{Th}(N_{P_J})$$

defined by the formula,

$$c_{P_J}(x) = \begin{cases} 
 e_{P_J}^{-1}(x) & \text{if } x \in e_{P_J}(N_{P_J}) \\
 \infty & \text{if } x \notin e_{P_J}(N_{P_J}) 
\end{cases}$$

where $\infty$ means the point at infinity in the Thom-space $\text{Th}(N_{P_J})$. The tubular neighborhood embeddings (55) are unique up to homotopy. As a result of this uniqueness it follows that our constructions will not depend on these choices made. We will be using these maps throughout the rest of this paper.

**Definition 7.1.** For $d - |J| - p_J \geq 0$, we set $G(\Sigma_k, d, n, J)$ to be the Grassmanian manifold of all $(d - |J| - p_J)$-dimensional vector subspaces of $\mathbb{R} \times \mathbb{R}^k_{\langle J \rangle} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}}_{\langle J \rangle}$. If $d - |J| - p_I < 0$ then $G(\Sigma_k, d, n, J)$ is defined to be a single point.

**Remark 7.1.** The space $G(\Sigma_k, d, n, J)$, is the Grassmanian manifold of $(d - |J| - p_J)$-dimensional vector subspaces of $(n - m_J + d - p_J - |J|)$-dimensional Euclidean space. In [6] and in [14], this space is denoted by $G(d - |J| - p_J, n - m_J)$.

For each $J \subseteq \langle k \rangle$, the space $G(\Sigma_k, d, n, J)$, being a Grassmanian manifold, is equipped with the canonical vector bundle, which we denote by

$$U_{\Sigma_k,d,n,J} \longrightarrow G(\Sigma_k, d, n, J).$$

The orthogonal compliment to $U_{\Sigma_k,d,n,J}$ we denote by

$$U_{\Sigma_k,d,n,J}^\perp \longrightarrow G(\Sigma_k, d, n, J).$$

Now, let

$$i_{J,n} : G(\Sigma_k, d, n, J) \longrightarrow G(\Sigma_k, d, n + 1, J)$$

be the embedding induced by the inclusion map

$$\mathbb{R} \times \mathbb{R}^k_{\langle J \rangle} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\hat{p}+\hat{m}}_{\langle J \rangle} \hookrightarrow \mathbb{R} \times \mathbb{R}^k_{\langle J \rangle} \times \mathbb{R}^{d-1+n+1} \times \mathbb{R}^{\hat{p}+\hat{m}}_{\langle J \rangle}.$$
This yields a bundle map
\[
\begin{array}{ccc}
e^1 \oplus U^\perp_{\Sigma_k,d,n,J} & \xrightarrow{i^*_J,n} & U^\perp_{\Sigma_k,d,n+1,J} \\
\downarrow & & \downarrow \\
G(\Sigma_k,d,n,J) & \xrightarrow{i_J,n} & G(\Sigma_k,d,n+1,J)
\end{array}
\]
which in-turn induces a map of Thom-Spaces,
\[
\text{Th}(i^*_J,n) : \text{Th}(e^1 \oplus U^\perp_{\Sigma_k,d,n,J}) \cong S^1 \wedge \text{Th}(U^\perp_{\Sigma_k,d,n,J}) \longrightarrow \text{Th}(U^\perp_{\Sigma_k,d,n+1,J}).
\]

**Definition 7.2.** For \( J \subseteq \langle k \rangle \), \( MT_J(d)\Sigma_k \) is the spectrum defined by setting,
\[
(\text{MT}_J(d)\Sigma_k)_{d+n} := \text{Th}(U^\perp_{\Sigma_k,d,n,J}) \wedge (R^{p+m})^c.
\]
The structure maps
\[
\sigma^J_n : (\text{MT}_J(d)\Sigma_k)_{d+n} \longrightarrow (\text{MT}_J(d)\Sigma_k)_{d+(n+1)}
\]
are given by smashing \( \text{Th}(i^*_J,n) \) from (57) with the identity on \((R^{p+m})^c\).

**Remark 7.2.** The space \( \text{Th}(U^\perp_{\Sigma_k,d,n,J}) \) is the \((d+n-|J|-p_J-m_J)\) the space of the spectrum \( MT(d-p_J-|J|) \) given in [6]. The space \((R^{p+m})^c\) is homeomorphic to \( S^{p_J+m_J} \) and so for each \( J \), the spectrum \( MT_J(d)\Sigma_k \) is homotopy equivalent to the spectrum \( \Sigma^{-|J|}MT(d-p_J-|J|) \). We use the notation \((R^{p+m})^c\) instead of \( S^{p_J+m_J} \) because in this way it will be easier to describe how these spectra fit together to form a \( k \)-dimensional cube of spectra.

We now assemble the spectra \( MT_J(d)\Sigma_k \) together into the \( k \)-cubic spectrum \( MT_\bullet(d)\Sigma_k \). For \( I \subseteq J \), we construct the edge
\[
\kappa_{J,I} : MT_J(d)\Sigma_k \longrightarrow MT_I(d)\Sigma_k
\]
connecting the vertices that correspond to \( J \) and \( I \) as follows.

For each \( J \subseteq \langle k \rangle \), we denote by \( G(p,m,J) \) the Grassmanian manifold consisting of \( p_j \)-dimensional vector subspaces of \( R^{p+m} \). We denote by
\[
U_{\bar{p},m,J} \longrightarrow G(\bar{p},m,J),
\]
\[
U_{\bar{p},m,J}^\perp \longrightarrow G(\bar{p},m,J),
\]
the canonical vector bundle and its orthogonal compliment. Now for each \( I \subseteq J \) and \( n \geq 0 \) there are maps,
\[
\tau_{J,I} : G(\Sigma_k,d,n,J) \times G(\bar{p},m,J \setminus I) \longrightarrow G(\Sigma_k,d,n,I)
\]
given by sending a pair of subspaces \((W,V)\) with
\[
W \subset R^{k}_{\langle J \rangle} \times R^{d-1+n} \times R^{p+m}_{\langle J' \rangle} \quad \text{and} \quad V \subset R^{p+m}_{\langle J \rangle}
\]
to the subspace
\[
(R^{k}_{\langle J \rangle} \times W \times V) \subset R^{k}_{\langle J \rangle} \times R^{d-1+n} \times R^{p+m}_{\langle J' \rangle}.
\]
(Notice that the product $W \times V$ is a subspace of $\mathbb{R}^k_{(J^c)} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}$ and so $\mathbb{R}^k_{(J \setminus I)} \times W \times V$ is a subspace of $\mathbb{R}^k_{(J^c)} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}$ via the identification of $\mathbb{R}^k_{(J \setminus I)} \times \mathbb{R}^k_{(J^c)}$ with $\mathbb{R}^k_{(J^c)}$.)

The maps $\tau_{J,I}$ are covered by bundle maps

\[
\begin{align*}
U^\perp_{\Sigma_k,d,n,I} \times U^\perp_{\bar{p},\bar{m},J \setminus I} & \xrightarrow{\tau^*_{J,I}} U^\perp_{\Sigma_k,d,n,I} \\
G(\Sigma_k, d, n, J) \times G(\bar{p}, \bar{m}, J \setminus I) & \xrightarrow{\tau_{J,I}} G(\Sigma_k, d, n, I).
\end{align*}
\]

(Implied here is that the bundle $U^\perp_{\Sigma_k,d,n,J} \times U^\perp_{\bar{p},\bar{m},J \setminus I}$ is isomorphic to the pull-back $\tau^*_{J,I}(U^\perp_{\Sigma_k,d,n,I})$.)

For each $J \subseteq \langle k \rangle$, the normal bundles $N_{P,J} \to P^J$ admit Gauss maps,

\[
\begin{align*}
N_{P,J} & \xrightarrow{\gamma_{P,J}^*} U^\perp_{\bar{p},\bar{m},J} \\
P^J & \xrightarrow{\gamma_{P,J}} G(\bar{p}, \bar{m}, J)
\end{align*}
\]

which induce

\[
\Th(\gamma_{P,J}^*) : \Th(N_{P,J}) \longrightarrow \Th(U^\perp_{\bar{p},\bar{m},J}).
\]

We define

\[
\kappa_{n+d}^{n+d}_{J,I} : (\MT_J(d)_{\Sigma_k})_{n+d} \longrightarrow (\MT_I(d)_{\Sigma_k})_{n+d}
\]

by the composition,

\[
\begin{align*}
\Th(U^\perp_{\Sigma_k,d,n,J}) \wedge (\mathbb{R}^{\bar{p}+\bar{m}})_{(J \setminus I)}^c \wedge (\mathbb{R}^{\bar{p}+\bar{m}})_{(I)}^c & \xrightarrow{(1)} \Th(U^\perp_{\Sigma_k,d,n,J}) \wedge \Th(N_{P,J \setminus I}) \wedge (\mathbb{R}^{\bar{p}+\bar{m}})_{(I)}^c \\
(\mathbb{R}^{\bar{p}+\bar{m}})_{(J \setminus I)}^c \wedge (\mathbb{R}^{\bar{p}+\bar{m}})_{(I)}^c & \xrightarrow{(2)} \Th(U^\perp_{\Sigma_k,d,n,J}) \wedge (\mathbb{R}^{\bar{p}+\bar{m}})_{(I)}^c.
\end{align*}
\]

The first map is given by $Id \wedge (c_{P,J \setminus I}) \wedge Id$ where $c_{P,J \setminus I}$ the collapsing map from (56). The second map is given by $Id \wedge \Th(\gamma_{P,J \setminus I}^*) \wedge Id$ where $\Th(\gamma_{P,J \setminus I}^*)$ is the map from (61). The third map is given by $\Th(\tau^*_{J,I}) \wedge Id$ where $\tau^*_{J,I}$ is the map from (59). Above, we are using the identification

\[
(\mathbb{R}^{\bar{p}+\bar{m}})_{(J \setminus I)}^c \wedge (\mathbb{R}^{\bar{p}+\bar{m}})_{(I)}^c = (\mathbb{R}^{\bar{p}+\bar{m}})_{(J \setminus I)}^c.
\]

It is easy to check that the $\kappa_{n+d}^{n+d}$ are compatible, namely for subsets $K \subseteq I \subset J \subseteq \langle k \rangle$, we have

\[
\kappa_{n+d}^{n+d}_{I,K} \circ \kappa_{n+d}^{n+d}_{J,I} = \kappa_{n+d}^{n+d}_{J,K}.
\]

From this compatibility, the maps $\kappa_{n+d}^{n+d}$ make $(\MT_\bullet(d)_{\Sigma_k})_{d+n}$ into a $k$-cubic space.
It can be checked that for each \( n \), and \( I \subseteq J \), the diagram
\[
\begin{array}{c}
\vspace{5pt}
S^1 \wedge (\text{MT}_J(d)\Sigma_k)_{n+d} \\ \downarrow \sigma^n \\
\text{MT}_J(d)\Sigma_k \\
\end{array}
\begin{array}{c}
\vspace{5pt}
\kappa_{J,I}^n \\
\downarrow \sigma^n \\
\text{MT}_I(d)\Sigma_k \\
\end{array}
\begin{array}{c}
\vspace{5pt}
\kappa_{J,I}^{n+1} \\
\downarrow \sigma^n \\
\text{MT}_I(d)\Sigma_k \\
\end{array}
\end{array}
\]
commutes, thus the \( \kappa_{J,I}^n \) piece together to define maps of spectra
\[
(63) 
\kappa_{J,I} : \text{MT}_J(d)\Sigma_k \to \text{MT}_I(d)\Sigma_k
\]
such that \( \kappa_{I,K} \circ \kappa_{J,I} = \kappa_{J,K} \) for any \( K \subseteq I \subseteq J \). thus they piece together to define the \( k \)-cubic spectrum which we denote by \( \text{MT}_\ast(d)\Sigma_k \).

We denote,
\[
(64) \quad \text{MT}(d)\Sigma_k := t\text{Cofibre}_{\langle k \rangle} \text{MT}_{\ast}(d)\Sigma_k.
\]
Recall that by definition,
\[
t\text{Cofibre}_{\langle k \rangle} \text{MT}_{\ast}(d)\Sigma_k = \text{Cofibre}_{\langle k-1 \rangle} \text{MT}_{\ast,k}(d)\Sigma_k \to t\text{Cofibre}_{\langle k-1 \rangle} \text{MT}_{\ast,\bar{k}}(d)\Sigma_k.
\]
The \( (k-1) \)-cubic spectra \( \text{MT}_{\ast,k}(d)\Sigma_k \) and \( \text{MT}_{\ast,\bar{k}}(d)\Sigma_k \) were defined to be the cubic spectra obtained by restricting \( \text{MT}_{\ast}(d)\Sigma_k \) to the sub-lattice of subsets of \( \langle k \rangle \) containing \( k \) and to the sub-lattice of subsets disjoint from \( k \) respectively. By inspection, one sees that
\[
\text{MT}_{\ast,k}(d)\Sigma_k = \Sigma^{-1}\text{MT}_{\ast}(d-p_k-1)\Sigma_{k-1} \quad \text{and} \quad \text{MT}_{\ast,\bar{k}}(d)\Sigma_k = \text{MT}_{\ast}(d)\Sigma_{k-1}.
\]
Thus we have a cofibre sequence,
\[
\Sigma^{-1}\text{MT}(d-p_k-1)\Sigma_{k-1} \to \text{MT}(d)\Sigma_{k-1} \to \text{MT}(d)\Sigma_k.
\]
Continuing this cofibre sequence one term to the right and then applying \( \Omega^\infty(\cdot) \) yields a homotopy-fibre sequence,
\[
\Omega^\infty\text{MT}(d)\Sigma_{k-1} \to \Omega^\infty\text{MT}(d)\Sigma_k \to \Omega^\infty\text{MT}(d-p_k-1)\Sigma_{k-1}.
\]
For the case that \( k = 1 \) and so \( \Sigma_k = (P) \) for some closed manifold \( P \), the above fibre sequence is the fibre sequence studied in \cite{14}.

We now make one observation about the construction of \( \text{MT}_\ast(d)\Sigma_k \). The edge maps
\[
\kappa_{J,I} : \text{MT}_J(d)\Sigma_k \to \text{MT}_I(d)\Sigma_k
\]
were constructed using Thom-Pontryagin maps for the embeddings \( \phi^i : P_i \hookrightarrow \mathbb{R}^{p_i+m_i} \). Specifically these Thom-Pontryagin map for this embedding is the composition
\[
(\mathbb{R}^{p_i+m_i})^c \overset{c_{P_i}}{\longrightarrow} \mathbb{T}(N_{P_i}) \overset{\mathbb{T}(\gamma_{P_i})}{\longrightarrow} \mathbb{T}(U_{\bar{P}_i,m_{\bar{P}_i},J}^1)
\]
with $c_{P_i}$ the collapse map from (56) and $\text{Th}(\gamma_P)$ the map from (61). It is well known by classic cobordism theory that the homotopy-type of this map is determined by the cobordism class of the manifold $P_i$. This observation leads us to:

**Proposition 7.1.** Let $\hat{\Sigma}_k$ be a list of closed manifolds $(\hat{P}_1, \ldots, \hat{P}_k)$ such that for each $0 \leq i \leq k$, the manifold $\hat{P}_i$ is cobordant to the manifold $P_i$. Then the spectra $MT(d)\Sigma_k$ and $MT(d)\hat{\Sigma}_k$ are homotopy equivalent.

**Proof.** Follows from the above discussion. □

Combining this result with Theorem 1.3 stated in the introduction yields Corollary 1.4 from the introduction.

8. The Main Theorem

In this section we identify the homotopy type of the space $|D_{d}^{\Sigma_k}|$. Let the list $\Sigma_k = (P_1, \ldots, P_k)$ and embeddings

$$\phi_i : P_i \hookrightarrow \mathbb{R}^{\bar{p} \langle i \rangle}$$

be the same as in previous sections used in the definitions of $D_d^{\Sigma_k}$ and of $MT\cdot(d)\Sigma_k$. We have

**Theorem 8.1.** There is a homotopy equivalence

$$|D_d^{\Sigma_k}| \sim \Omega^{\infty-1}MT(d)\Sigma_k.$$ 

Combining this with the weak homotopy equivalence $|D_d^{\Sigma_k}| \sim BCob_d^{\Sigma_k}$ proved in Section 5 yields the weak homotopy equivalence

$$BCob_d^{\Sigma_k} \sim \Omega^{\infty-1}MT(d)\Sigma_k,$$

our main result.

By Theorem 6.1 it will suffice to prove the homotopy equivalence

$$|D_d^{\Sigma_k}| \sim \Omega^{\infty-1}MT\cdot(d)\Sigma_k.$$ 

We do this by constructing maps

$$T_{d,k}^{d,n} : |D_d^{\Sigma_k}| \longrightarrow \Omega^{n+d-1}(MT\cdot(d)\Sigma_k)_{n+d}$$

for each $n$ via a relative-parametric Thom-Pontryagin construction which will be seen to induce a homotopy equivalence

$$T_{\Sigma_k}^d : |D_d^{\Sigma_k}| \sim \Omega^{\infty-1}MT\cdot(d)\Sigma_k$$

in the limit as $n \rightarrow \infty$.

To begin, we define a sheaf $\mathcal{Z}_{d,n}^{\Sigma_k}$ on $\mathcal{X}$ by setting

$$\mathcal{Z}_{d,n}^{\Sigma_k}(X) = Maps(X \times \mathbb{R}, \Omega^{d+n-1}(MT\cdot(d)\Sigma_k)_{d+n}).$$
This is a set valued sheaf; by Maps(·, ·) we mean the set of continuous maps, free of any topology. We then define \( \mathcal{Z}^{\Sigma_k}_d \) by setting,

\[
\mathcal{Z}^{\Sigma_k}_d = \text{Maps}(X \times \mathbb{R}, \Omega^{\infty-1}\text{MT}_*(d)\Sigma_k).
\]

Recall the restriction map

\[
(r^{n+d}_{\{k\}} : \Omega^{d+n-1}(\text{MT}_*(d)\Sigma_k)_{d+n} \to \Omega^{d+n-2}(\text{MT}_*(d)\Sigma_k)_{d+n}).
\]

This is a Serre-fibration with fibre over the constant map homeomorphic to \( \Omega^{d+n-1}(\text{MT}_*(d)\Sigma_k)_{d+n} \), see section 6.4. By inspection one sees that

\[
\text{MT}_*(d)\Sigma_k + \text{MT}_*(d-1)\Sigma_{k-1} \wedge S^{p_k+m_k}.
\]

For what follows, in order to save space we denote \( j_k = p_k + m_k \). We define another sheaf on \( \mathcal{X} \) denoted by \( \mathcal{Z}^{\Sigma_{k-1}}_{d-p_k-1,n} \) by setting

\[
\mathcal{Z}^{\Sigma_{k-1}}_{d-p_k-1,n}(X) := \text{Maps}\left(X \times \mathbb{R}, \Omega^{d+n-2}_{\{k-1\}}\left[\text{MT}_*(d-p_k-1)\Sigma_{k-1}(d+n-j_k-1) \wedge S^j_k\right]\right).
\]

There is a suspension map

\[
\mathcal{S}_n : \Omega^{d+n-2-j_k}_{\{k-1\}}\text{MT}_*(d-p_k-1)\Sigma_{k-1}(d+n-j_k-1) \to \Omega^{d+n-2}_{\{k-1\}}\left[\text{MT}_*(d-p_k-1)\Sigma_{k-1}(d+n-j_k-1) \wedge S^j_k\right]
\]

defined by sending a map a \( k-1 \)-cubic space map

\[
f_* : D^{(k-1),d+n-2-j_k} \to \text{MT}_*(d-p_k-1)\Sigma_{k-1}(d+n-j_k-1)
\]
to its \( j_k \)-fold suspension,

\[
f_* \wedge \text{Id}_{\mathcal{S}_n} : D^{(k-1),d+n-2} \to \text{MT}_*(d-p_k-1)\Sigma_{k-1}(d+n-j_k-1) \wedge S^j_k,
\]

where we are using the fact that \( D^{(k-1),d+n-2-j_k} \wedge S^j_k = D^{(k-1),d+n-2} \) as \( k-1 \)-cubic spaces. This induces the map of sheaves,

\[
(\mathcal{S}_n)_* : \mathcal{Z}^{\Sigma_{k-1}}_{d-p_k-1,n} \to \mathcal{Z}^{\Sigma_{k-1}}_{d-p_k-1,n}.
\]

**Lemma 8.2.** In the limit as \( n \to \infty \), the induced map

\[
\mathcal{S} : \Omega^{\infty-1}_{\{k-1\}}\text{MT}_*(d-p_k-1)\Sigma_{k-1} \to \Omega^{\infty-j_k}_{\{k-1\}}\left[\text{MT}_*(d-p_k-1)\Sigma_{k-1} \wedge S^j_k\right]
\]

is a homotopy equivalence.
and suppose that \( p \). We now define a modified version of the sheaf \( D \) consists of a smooth vector bundle \( q \), there is a homotopy-fibre sequence

Since \( r \) of \( r \) the fibre-sheaf of \( (\cdot) \) induces a map of sheaves, \( S : \Omega_{(k-1)}^{\infty-1}(\cdot) \to \Omega_{(k-1)}^{\infty+pk+m-1}(\cdot) \) in the limit \( n \to \infty \), is a homotopy equivalence. □

It follows that the induced map of sheaves

| \( S_* : Z_{d-pk-1}^{\Sigma_{k-1}} \to \hat{Z}_{d-pk-1}^{\Sigma_{k-1}} \) |
|---|
| is a weak equivalence. |

The restriction map \( r_{d+n}^{(k)} \) from (67) induces a map of sheaves,

| \( (r_{d+n}^{(k)})_* : Z_{d,n}^{\Sigma_{k}} \to \hat{Z}_{d-pk-1,n}^{\Sigma_{k-1}} \) |
|---|
| Since \( r_{d+n}^{(k)} \) is a Serre-fibration and thus has the homotopy lifting property with respect to CW complexes, it follows that \( (r_{d+n}^{(k)})_* \) then has the concordance lifting property. Since the fibre of \( r_{d+n}^{(k)} \) over the constant map is homeomorphic to

| \( \Omega_{(k-1)}^{d+n-1}((\cdot))_{d+n} = \Omega_{(k-1)}^{d+n-1}(\cdot)_{d+n} \) |
|---|
| it follows that the fibre-sheaf of \( (r_{d+n}^{(k)})_* \) over the constant map is isomorphic to \( Z_{d,n}^{\Sigma_{k-1}} \). Thus, there is a homotopy-fibre sequence

| \( Z_{d,n}^{\Sigma_{k-1}} \to Z_{d,n}^{\Sigma_{k}} \to \hat{Z}_{d-pk-1,n}^{\Sigma_{k-1}} \) |
|---|

We now define a modified version of the sheaf \( D_{d,n}^{\Sigma_{k}} \).

**Definition 8.1.** Let \( p : Y \to X \) be a submersion. Let \( i_C : C \hookrightarrow Y \) be a smooth submanifold and suppose that \( p \mid_C \) is still a submersion. A vertical tubular neighborhood for \( C \) in \( Y \) consists of a smooth vector bundle \( q : N \to C \) (which in our case will always be the normal bundle of \( C \)) with zero section \( s \), along with an open embedding \( e : N \to Y \) such that \( e \circ s = i_C \), and \( p \circ e = p \circ i_C \circ q \).
**Definition 8.2.** For $X \in \mathcal{X}$ we define $D_{d,n}^{k,\nu}(X)$ to be the set of quadruples $(W, \pi, f, e)$ such that

i. $(W, \pi, f) \in D_{d,n}^{k}(X)$,

ii. The map $e : N_W \hookrightarrow X \times \mathbb{R} \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}}$, where $N_W$ is the normal bundle of $W$, is a *vertical tubular neighborhood* of $W$ with respect to the submersion $\pi$. It is required that for each $I \subseteq \langle k \rangle$, the restriction $e|_{\partial I W}$ has the factorization,

$$e|_{\partial IW} = e_{N_{\beta I W}} \times e_{P I}$$

where

$$e_{N_{\beta I W}} : N_{\beta I W} \hookrightarrow X \times \mathbb{R} \times \mathbb{R}_+^{\langle I \rangle} \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}},$$

is a vertical tubular neighborhood for $\beta I W$, where $N_{\beta I W}$ is normal bundle, and

$e_{P I} : N_{P I} \hookrightarrow \mathbb{R}^{\tilde{p}+\tilde{m}}$

is the tubular neighborhood for $N_{P I}$ specified in (55) and used in the construction of $\text{MT}_{\bullet}(d)\Sigma_k$.

Clearly, $D_{d,n}^{k,\nu}$ satisfies the sheaf condition. There is a forgetful map

$$D_{d,n}^{k,\nu} \rightarrow D_{d,n}^{k}$$

defined by sending the quadruple $(W, \pi, f, e) \in D_{d,n}^{k,\nu}(X)$ to the tripple $(W, \pi, f) \in D_{d,n}^{k}(X)$.

**Proposition 8.3.** The forgetful map $D_{d,n}^{k,\nu} \rightarrow D_{d,n}^{k}$ defined above is a weak equivalence.

**Proof.** This forgetful map satisfies the relative surjectivity criterion, see 3.1. One needs to show that for any element $(W, \pi, f) \in D_{d,n}^{k}(X)$, the normal bundle of the manifold $W$ admits a vertical tubular neighborhood with respect to the submersion $\pi$. The existence of such a neighborhood follows from the usual construction of tubular neighborhoods, see [9], using the exponential map induced by a chosen metric on the ambient space. In this case, one just needs to use a metric on $X \times \mathbb{R} \times \mathbb{R}_+^k \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}}$ under which $X$ and $\mathbb{R}_+^k \times \mathbb{R}^{d-1+\tilde{n}} \times \mathbb{R}^{\tilde{p}+\tilde{m}}$ are orthogonal, since the submersion in question is just projection onto $X$. □

We now proceed to define a map of sheaves

$$T_{\Sigma_k} : D_{d,n}^{k,\nu} \rightarrow Z_{d,n}^{\Sigma_k}$$

such that the following diagram commutes,

$$\begin{array}{ccc}
D_{d,n}^{k,\nu} & \xrightarrow{i} & D_{d,n}^{k,\nu} \\
\downarrow T_{d,n}^{\Sigma_k} & & \downarrow S_n \circ [T_{d-pk-1,n-mk}^{\Sigma_{k-1}}] \\
Z_{d,n}^{\Sigma_{k-1}} & \xrightarrow{i} & Z_{d,n}^{\Sigma_k} \\
\downarrow T_{d,n}^{\Sigma_k} & & \downarrow S_n \circ [T_{d-pk-1,n-mk}^{\Sigma_{k-1}}] \\
Z_{d,n}^{\Sigma_{k-1}} & \xrightarrow{i} & Z_{d,n}^{\Sigma_k} \\
\end{array}$$

(68)
Notice that the right-vertical map is the composition

\[ T_{d-p_k-1,n-m_k}^\Sigma \colon D_{d-p_k-1,n-m_k}^{\Sigma_k-1} \to Z_{d-p_k-1,n-m_k}^{\Sigma_k-1} \to Z_{d}^{\Sigma_k} \to \hat{Z}_{d-p_k-1,n}^{\Sigma_k} \]

Theorem 4.1 implies that (68) induces a map of homotopy fibre sequences,

\[
\begin{array}{ccc}
|D_{d,n}^{\Sigma_k-1,\nu}| & \longrightarrow & |D_{d,n}^{\Sigma_k,\nu}| \longrightarrow |D_{d-p_k,n-m_k}^{\Sigma_k-1,\nu}| \\
\downarrow & & \downarrow \\
|Z_{d,n}^{\Sigma_k-1}| & \longrightarrow & |Z_{d,n}^{\Sigma_k}| \longrightarrow |\hat{Z}_{d-p_k-1,n}^{\Sigma_k}|.
\end{array}
\]

This map \( T_{d,n}^{\Sigma_k} \) will be constructed using a relative-parametric Thom-Pontryagin Construction. With \( T_{d,n}^{\Sigma_k} \) defined for all \( k, d, \) and \( n \) making (68) commutative, we can then prove Theorem 8.1 by induction on \( k \) using the the above homotopy fibre sequence.

Assuming that \( T_{d,n}^{\Sigma_k} \) is defined in such a way so as to make diagram (68) commutative, we have:

**Proof of 8.1.** In the limit \( n \to \infty \), diagram (69) induces a map of homotopy fibre-sequences,

\[
\begin{array}{ccc}
|D_{d}^{\Sigma_k-1,\nu}| & \longrightarrow & |D_{d}^{\Sigma_k,\nu}| \longrightarrow |D_{d-p_k}^{\Sigma_k-1,\nu}| \\
\downarrow & & \downarrow \\
|Z_{d}^{\Sigma_k-1}| & \longrightarrow & |Z_{d}^{\Sigma_k}| \longrightarrow |\hat{Z}_{d-p_k-1}^{\Sigma_k}|.
\end{array}
\]

where we have replaced \( |Z_{d-p_k-1}^{\Sigma_k-1}| \) with the homotopy equivalent space \( |\hat{Z}_{d-p_k-1}^{\Sigma_k-1}| \) (see Lemma 8.2). Now in the case \( k = 0 \), the map

\[ |T_{d}^{\Sigma_k}| : |D_{d}^{\Sigma_k-1,\nu}| \longrightarrow |Z_{d}^{\Sigma_k}| \sim \Omega_{(k)}^{\Sigma_k-1} MT_\bullet (d) \Sigma_k \]

is a homotopy equivalence. This is precisely the map used in the main theorem of [6]. In this case for \( k = 0 \), \( |D_{d}^{\Sigma_k-1,\nu}| \sim |D_{d}| \) and \( \Omega_{(k)}^{\Sigma_k-1} MT_\bullet (d) \sim \Omega_{(k)}^{\Sigma_k-1} MT_\bullet (d) \Sigma_k \). Now assume that

\[ |T_{d}^{\Sigma_k-1}| : |D_{d}^{\Sigma_k-1,\nu}| \longrightarrow |Z_{d}^{\Sigma_k-1}| \sim \Omega_{(k-1)}^{\Sigma_k-1} MT_\bullet (d) \Sigma_{k-1} \]

is a homotopy equivalence for all \( d \). Then, the first and third columns of (70) our are homotopy equivalences. The result follows from an application of the five lemma on the map of long-exact sequences in homotopy groups induced by the fibre-sequences (70). \( \square \)

8.1. **A Parametrized Thom Pontryagin Construction.** Now, it is left to construct

\[ T_{d,n}^{\Sigma_k} : D_{d,n}^{\Sigma_k,\nu} \longrightarrow Z_{d,n}^{\Sigma_k} \]
The map $T^{\Sigma_k}_{d,n}$ will be defined by first sending an element $(W, \pi, f, e) \in D^{\Sigma_k,v}_{d,n}(X)$ to a $k$-cubic space map

$$\tilde{T}^{\Sigma_k}_{d+n}(W)_* : X \times \mathbb{R} \times D^{(k),d-1+n}_{\bullet} \to (MT_\bullet(d)_{\Sigma_k})_{n+d}.$$ (71)

Applying the adjunction

$$C^0_{(k)} \left( X \times \mathbb{R} \times D^{(k),d-1+n}_{\bullet}, (MT_\bullet(d)_{\Sigma_k})_{n+d} \right) \xrightarrow{\cong} C^0 \left( X \times \mathbb{R}, \Omega^{d-1+n}_{(k)}(MT_\bullet(d)_{\Sigma_k})_{n+d} \right),$$

will then yield an element $T^{\Sigma_k}_{d,n}((W, \pi, f, e))$ of $\mathcal{Z}^{\Sigma_k}_{d,n}(X)$. This construction will be natural in the variable $X$, so as to give us an actual map of sheaves.

**Remark 8.1.** The map $T^{\Sigma_k}_{d,n}$ will be constructed in such a way so that in the case that $k = 1$, it will be equal to the parametrized Thom-Pontryagin map constructed in [14, 7.2], part 1 of this project. Here, $T^{\Sigma_k}_{d,n}$ will be defined in a similar way. The only added difficulty here is that for arbitrary $k$ we have an entire cube of spaces to keep track of. The amount of data in the cube grows exponentially with $k$.

Let $X \in \mathcal{O}b(X)$ and let $(W, \pi, f, e)$ be an element of $D^{\Sigma_k,v}_{d,n}(X)$. For each subset $J \subseteq \langle k \rangle$ we denote by

$$N_{\partial_j W} \to \partial_j W$$

the normal bundle for $\partial_j W$ as a submanifold of $\mathbb{R} \times \mathbb{R}^k_{i+(Jc)} \times \mathbb{R}^{d-1+\hat{n}} \times \mathbb{R}^{p+m}$. Clearly, for each $I \subseteq J$ we have

$$N_{\partial_j W} |_{\partial_j W} = N_{\partial_i W}.$$ 

Furthermore, for each for each $J$, there is a factorization $N_{\partial_j W} = N_{\partial_j W} \times N_{P^J}$ where $N_{\partial_j W}$ is the normal bundle for $\partial_j W$ as a submanifold of $\mathbb{R} \times \mathbb{R}^k_{i+(Jc)} \times \mathbb{R}^{d-1+\hat{n}} \times \mathbb{R}^{p+m}$ and $N_{P^J}$ is the normal bundle for $P^J \subset \mathbb{R}^{p+m}_{(Jc)}$. The submersion $\pi : W \to X$ determines a bundle given by the kernel of the differential of $\pi$, which we denote by $T^\pi W \to W$. This bundle has fibre-dimension $d$. For each $J \subseteq \langle k \rangle$, the restriction of $T^\pi W$ to $\partial_j W$ has the factorization,

$$T^\pi W |_{\partial_j W} = T^\pi \beta_j W \times TP^J \times \mathbb{R}^k_{(Jc)}$$

where $T^\pi \beta_j W$ is the kernel bundle of the differential of the submersion $\pi_{\beta_j W} : \beta_j W \to X$. This factorization follows directly from condition iii. of definition 4.1.

By the above discussion, for each $J \subseteq \langle k \rangle$ the Gauss map for $N_{\partial_j W}$ factors as the product,

$$N_{\partial_j W} = N_{\partial_j W} \times N_{P^J} \xrightarrow{\gamma_{\partial_j W} \times \gamma_{P^J}} U_{d+n,J}^{\Sigma_k,d,n,J} \times U_{\hat{p},m,J}^{\hat{p},m,J}$$

$$\partial_j W = \beta_j W \times TP^J \xrightarrow{\gamma_{\partial_j W} \times \gamma_{P^J}} G(\Sigma_k, d, n, J) \times G(\hat{p}, m, J).$$
We note that the map $\gamma_\beta$ above is also the Gauss-map corresponding to the kernel bundle $T^\pi J W$. It is covered by a bundle map

$$T^\pi J W \to U_{\Sigma, d, n, J},$$

which is the orthogonal complement to $\gamma_\beta^* : N_{\beta J} W \to U_{\Sigma, d, n, J}^\perp$ from the above diagram. These bundle maps $\gamma^*_\beta \times \gamma^*_p, J$ induce maps of Thom-spaces,

$$\tilde{\gamma}_\beta W \wedge \tilde{\gamma}_p J : \text{Th}(N_{\beta J} W) \wedge \text{Th}(N_{p, J}) \to \text{Th}(U_{\Sigma, d, n, J}^\perp) \wedge \text{Th}(U_{\beta J}^\perp, n, J).$$

Now, the vertical tubular neighborhood embeddings

$$e_{\theta J} W = e_{\beta J} W \times e_{p, J} : N_{\beta J} W \times N_{p, J} \to (X \times \mathbb{R} \times \mathbb{R}^k_{+<J_{\rho}}) \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}) \times (\mathbb{R}^{\bar{p}+\bar{m}})$$

induce collapse maps

$$\tilde{e}_{\theta J} W = \tilde{e}_{\beta J} W \times \tilde{e}_{p, J} : (X \times \mathbb{R} \times \mathbb{R}^k_{+<J_{\rho}}) \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}) \times (\mathbb{R}^{\bar{p}+\bar{m}}) \to \text{Th}(N_{\beta J} W) \times \text{Th}(N_{p, J})$$

(notice that the domain is not the one-point compactification). These collapse maps are defined by the formula,

$$\tilde{e}_{\theta J} W(x, t, z) = \begin{cases} \epsilon_{\theta J} W(x, t, z) & \text{if } (x, t, z) \in e_{\theta J} W(N_{\theta J} W) \\ \infty & \text{if } (x, t, z) \notin e_{\theta J} W(N_{\theta J} W). \end{cases}$$

From this formula it is clear that the factorization $\tilde{e}_{\theta J} = \tilde{e}_{\beta J} W \times \tilde{e}_{p, J}$ from (74) follows from the factorization $e_{\theta J} W = e_{\beta J} W \times e_{p, J}$. Also note that $\tilde{e}_{p, J}$ is equal to the restriction of $e_{p, J}$ from (56) to $(\mathbb{R}^{\bar{p}+\bar{m}})^c \backslash \infty = \mathbb{R}^{\bar{p}+\bar{m}}$.

**Proposition 8.4.** The maps $\tilde{e}_{\theta J} W$ and $\tilde{e}_{\beta J} W$ given above have well defined extensions

$$c_{\theta J} W : X \times \mathbb{R} \times (\mathbb{R}^k_{+<J_{\rho}}) \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^c \to \text{Th}(N_{\theta J} W),$$

$$c_{\beta J} W : X \times \mathbb{R} \times (\mathbb{R}^k_{+<J_{\rho}}) \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^c \to \text{Th}(N_{\beta J} W)$$

defined at $X \times \mathbb{R} \times \{\infty\}$ by sending $(x, t, \infty)$ to $\infty \in \text{Th}(N_{\theta J} W)$ for all $(x, t) \in X \times \mathbb{R}$.

**Proof.** Properness of the map $(\pi, f)$ implies that for each $(x, t) \in X \times \mathbb{R}$, the pre-image $(\pi, f)^{-1}(x, t)$ is compact. Therefore for each $(x, t) \in X \times \mathbb{R}$, there is a positive real number, which we denote by $\lambda(x, t)$, with the property that if $z \in \mathbb{R}^k_{+<J_{\rho}} \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})$ is such that $|z| > \lambda(x, t)$, then $z \notin W$. We can choose $\lambda(x, t)$ so that $z \notin e_{\theta J} W(N_{\theta J} W)$ when $|z| > \lambda(x, t)$. We may also assume that $(x, t) \mapsto \lambda(x, t)$ is a continuous function. In this case it is clear that $\tilde{e}_{\theta J} W(x, t, z) = \infty \in \text{Th}(N_{\theta J} W)$ whenever $|z| > \lambda(x, t)$, thus the extension $c_{\theta J} W$ is well defined. Same holds for $c_{\beta J} W$. \qed

Now, for $J \subseteq \langle k \rangle$, fixing $(x, t) \in X \times \mathbb{R}$, the collapse map $c_{\theta J} W$ yields a map

$$(\mathbb{R}^k_{+<J_{\rho}}) \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^c \to \text{Th}(N_{\theta J} W),$$
given by $y \mapsto c_{\theta J} W(x, t, y)$.

Using the identifications $\text{Th}(N_{\beta J} W) \wedge \text{Th}(N_{p, J}) \simeq \text{Th}(N_{\beta J} W)$ and

$$(\mathbb{R}^k_{+<J_{\rho}}) \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^c \wedge (\mathbb{R}^k_{<J_{\rho}})^c = (\mathbb{R}^k_{+<J_{\rho}}) \times (\mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})^c,$$
the factorization \( \bar{\partial}_{JW} = \bar{\partial}_{JW} \times \bar{\partial}_{PJ} \) implies
\[
\bar{\partial}_{JW}(x, t, y) = (\partial_{\beta I} W \times P^K)_{\partial_{\beta I} W = 0},
\]
where \( y_1 \in (\mathbb{R}^k_{+}(J_c) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}_c) \) and \( y_2 \in (\mathbb{R}^{p+n+m}_k) \).

Using the collapse map \( c_{\beta J} W \) together with \( \hat{\gamma}_{\beta J} W \) from (3), we define:

**Definition 8.3.** For \((W, \pi, f, e) \in D_{d+n}^{\Sigma k}(X)\) and \( J \subseteq \langle k \rangle \) we define \( \hat{T}_{d+n}^{\Sigma k}(W)_J \) to be the map
\[
X \times \mathbb{R} \times \left[ (\mathbb{R}^k_{+}(J_c) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}_c) \right. \left. \cap (\mathbb{R}^{p+n+m}_k) \right] \rightarrow \Sigma (U^1_{\Sigma k,d,n,I} \times (\mathbb{R}^{p+m}_c)
\]
defined by the formula
\[
(x, t, y) \mapsto \left( \hat{\gamma}_{\beta J} W \circ c_{\beta J} W(x, t, y_1), y_2 \right).
\]

This formula is well defined because of equation (76).

We set \( \hat{T}_{d+n}^{\Sigma k}(W)_J \) to be the \( J \)th map in our sought after \( k \)-cubic space map
\[
T_{d+n}^{\Sigma k}(W)_J : X \times \mathbb{R} \times D_{d+n}^{\Sigma k}(X) \rightarrow (MT_j(d)_{\Sigma k})_{n+d}
\]
from (71). This map and hence \( \hat{T}_{d+n}^{\Sigma k}(W)_J \) will be defined once we show that the maps \( \hat{T}_{d+n}^{\Sigma k}(W)_J \) for all \( J \subseteq \langle k \rangle \) are compatible.

**Proposition 8.5.** For each \( I \subseteq J \subseteq \langle k \rangle \) the diagram
\[
\begin{array}{ccc}
X \times \mathbb{R} \times (\mathbb{R}^k_{+}(J_c) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}_c) & \xrightarrow{T_{d+n}^{\Sigma k}(W)_J} & \Theta(U^1_{\Sigma k,d,n,I} \times (\mathbb{R}^{p+m}_c) \\
\downarrow & & \downarrow \kappa_{J,I}^{n+d} \\
X \times \mathbb{R} \times (\mathbb{R}^k_{+}(J_c) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}_c) & \xrightarrow{T_{d+n}^{\Sigma k}(W)_J} & \Theta(U^1_{\Sigma k,d,n,I} \times (\mathbb{R}^{p+m}_c)
\end{array}
\]

commutes, where the left vertical map is the inclusion and the maps \( \kappa_{J,I}^{n+d} \) are the edges in the \( k \)-cubic space \( (MT_j(d)_{\Sigma k})_{n+d} \). Thus, the \( T_{d+n}^{\Sigma k}(W)_J \) piece together to yield the \( k \)-cubic space map from (71).

**Proof.** Commutativity follows from several observations. First recall from Section ?? that \( \beta_I W \) is a \( \Sigma_k \)-manifold with boundary faces given by
\[
\partial_{\Sigma k}(\beta_I W) = \begin{cases} \beta_{K \cup I} W \times P^K & \text{if } K \cap I = \emptyset \\ \emptyset & \text{if } K \cap I \neq \emptyset. \end{cases}
\]

This implies that \( \partial_{J \setminus I}(\beta_I W) = \beta_J W \times P^{J \setminus I} \). Also recall that by condition i. of the definition of \( D_{d,n}^{\Sigma k}(X) \) (see 4.1), we have
\[
\partial_{J \setminus I}(\beta_I W) = \beta_J W \times P^{J \setminus I} = \beta_I W \cap \left[ (X \times \mathbb{R} \times (\mathbb{R}^k_{+}(J_c) \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m}_c) \times (\mathbb{R}^{p+m}_c) \right].
\]
This implies that the restriction of the normal bundle $N_{\beta_j W}$ to $\partial_j \beta_j W$ has the factorization,

$$N_{\beta_j W} |_{\partial_j \beta_j W} = N_{\beta_j W} \times N_{P_{\beta_j I}}.$$  

From this it follows that the restriction of the Gauss map $\langle \gamma^*_{\beta_j W}, \gamma_{\beta_j W} \rangle$ to the bundle

$$N_{\beta_j W} \times N_{P_{\beta_j I}} \longrightarrow \beta_j W \times P_{\beta_j I},$$

has the following factorization,

$$N_{\beta_j W} \times N_{P_{\beta_j I}} \xrightarrow{\gamma_{\beta_j W} \times \gamma^*_{\beta_j W}} U_{\Sigma_k, d, n, I}^\perp \times U_{\bar{\beta}, m, \bar{J}, I}^\perp \xrightarrow{\tau_{\beta j I}} U_{\Sigma_k, d, n, I}^\perp$$

$$\beta_j W \times P_{\beta_j I} \xrightarrow{\gamma_{\beta_j W} \times \gamma^*_{\beta_j W}} G(\Sigma_k, n, J, d) \times G(\bar{p}, \bar{m}, \bar{J}, I) \xrightarrow{\tau_{\beta j I}} G(\Sigma_k, n, I, d)$$

where $\tau_{\beta j I}$ is the map from (69) defined by sending a pair $(W, V)$ of subspaces to the product $\mathbb{R}_{(J, I)}^k \times W \times V$. Thus the restriction of $\tilde{\gamma}_{\beta_j W}$ from (73) to $\mathcal{TH}(N_{\beta_j W} \times N_{P_{\beta_j I}}) \subset \mathcal{TH}(N_{\beta_j W})$

has the factorization,

$$\mathcal{TH}(N_{\beta_j W}) \wedge \mathcal{TH}(N_{P_{\beta_j I}}) \xrightarrow{\tilde{\gamma}_{\beta_j W} \wedge \tilde{\gamma}^*_{\beta_j W}} \mathcal{TH}(U_{\Sigma_k, d, n, I}^\perp) \wedge \mathcal{TH}(U_{\bar{\beta}, m, \bar{J}, I}^\perp) \mathcal{TH}(\tau_{\beta j I}) \xrightarrow{\mathcal{TH}(\tau_{\beta j I})} \mathcal{TH}(U_{\Sigma_k, d, n, I}^\perp).$$

The factorization $N_{\beta_j W} |_{\partial_j \beta_j W} = N_{\beta_j W} \times N_{P_{\beta_j I}}$ implies that the restriction of $c_{\partial_j W}$ to

$$X \times \mathbb{R} \times (\mathbb{R}_{+,(J, C)}^k \times \mathbb{R}_{(J, C)}^{d-1+n} \times \mathbb{R}_{(J, C)}^{\bar{p}+\bar{m}})^c := X \times \mathbb{R} \times (\mathbb{R}_{+,(J, C)}^k \times \mathbb{R}_{(J, C)}^{d-1+n} \times \mathbb{R}_{(J, C)}^{\bar{p}+\bar{m}} \times \mathbb{R}_{(I)}^{\bar{p}+\bar{m}})^c,$$

has the formula,

$$(x, t, y_1, y_2, y_3) \mapsto (c_{\beta_j W}(x, t, y_1), c_{P_{\beta_j I}}(y_2), c_{P_{I}}(y_3))$$

where $(x, t) \in X \times \mathbb{R}$, $y_1 \in \mathbb{R}^k_{+,(J, C)} \times \mathbb{R}_{(J, C)}^{d-1+n} \times \mathbb{R}_{(J, C)}^{\bar{p}+\bar{m}}$ and $y_2 \in \mathbb{R}^k_{(J, I)}$, and $y_3 \in \mathbb{R}_{(I)}^{\bar{p}+\bar{m}}$.

It follows that the restriction of $\tilde{T}_{d+n}^\Sigma(W)_I$ to $X \times \mathbb{R} \times (\mathbb{R}_{+,(J, C)}^k \times \mathbb{R}_{(J, C)}^{d-1+n} \times \mathbb{R}_{(J, C)}^{\bar{p}+\bar{m}})^c$ has the formula,

$$(x, t, y_1, y_2, z) \mapsto \left(\tilde{\gamma}_{\beta_j W} \circ c_{\beta_j W}(x, t, y_1), \tilde{\gamma}^*_{P_{\beta_j I}} \circ c_{P_{\beta_j I}}(y_2), z\right)$$

where $\tilde{\gamma}_{\beta_j W} := \mathcal{TH}(\gamma_{\beta_j W})$, $(x, t) \in X \times \mathbb{R}$, $y_1 \in \mathbb{R}^k_{+,(J, C)} \times \mathbb{R}_{(J, C)}^{d-1+n} \times \mathbb{R}_{(J, C)}^{\bar{p}+\bar{m}}$, $y_2 \in \mathbb{R}^k_{(J, I)} \times \mathbb{R}_{(J, I)}^{\bar{p}+\bar{m}}$ as before, and $z \in \mathbb{R}_{(\bar{I})}^{\bar{p}+\bar{m}}$.

Now, recall that

$$\kappa_{J, I}^{d+n} : \mathcal{TH}(U_{\Sigma_k, d, n, I}^\perp) \wedge (\mathbb{R}_{(J, I)}^{\bar{p}+\bar{m}})^c \longrightarrow \mathcal{TH}(U_{\Sigma_k, d, n, I}^\perp) \wedge (\mathbb{R}_{(I)}^{\bar{p}+\bar{m}})^c,$$

from (62) is given by the formula

$$[\tilde{\gamma}_{J, I} \circ (Id_{\mathcal{TH}(U_{\Sigma_k, d, n, I}^\perp)} \wedge c_{P_{\beta_j I}})] \wedge Id_{(\mathbb{R}_{(I)}^{\bar{p}+\bar{m}})^c}.$$

Comparing this formula to that of (80), one sees that the restriction of $\tilde{T}_{d+n}^\Sigma(W)_I$ to $X \times \mathbb{R} \times (\mathbb{R}_{+,(J, C)}^k \times \mathbb{R}_{(J, C)}^{d-1+n} \times \mathbb{R}_{(J, C)}^{\bar{p}+\bar{m}})^c$ is equal to $\kappa_{J, I}^{d+n} \circ \tilde{T}_{d+n}^\Sigma(W)_I$ and so diagram (78) is commutative. \qed
The above proposition implies that the maps
\[ \tilde{T}^{\Sigma_k}_{d+n}(W)_I : X \times \mathbb{R} \times D^{(k),d+n-1}_{I} \longrightarrow (\mathcal{M}T(d)\Sigma_k)_{d+n} \]
are compatible and thus piece together to define the \( k \)-cubic space map \( \tilde{T}^{\Sigma_k}_{d+n}(W)_* \) from (71). We then set \( T^{\Sigma_k}_{d,n}(W,\pi,f,e) \) equal to the image of \( \tilde{T}^{\Sigma_k}_{d+n}(W)_* \) under the adjunction from (72).

It is easy to check that in the case that \( k = 0 \), \( T^{\Sigma_k}_{d,n} \) is precisely the map of sheaves inducing the isomorphism of concordance sets in Theorem 3.4 of [6]. In the case that \( k = 1 \), this is the same map constructed as in 7.2 of [14]. This construction concludes the proof of Theorem 8.1.

**Appendix A.**

In this section we prove a result that implies Lemma 2.1. It is essentially the same as the main result from [11] adapted for \( \Sigma_k \)-manifolds.

**Theorem A.1.** Let \( W \) be a \( \Sigma_k \)-manifold. Let \( \phi_i : P_i \hookrightarrow \mathbb{R}^{p_i+m_i} \) be the embeddings used throughout the paper. Let \( \bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_k) \) be a list of positive constants. Then for any \( I \subseteq \langle k \rangle \), the restriction map
\[ r_I : \text{Emb}(M, \mathbb{R}^{k}_+ \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})_{\Sigma_k,\phi} \longrightarrow \text{Emb}(\beta_I M, \mathbb{R}^{k}_+ \langle I^c \rangle \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})_{\Sigma_k,\phi} \]
defined by sending an embedding of \( M \) to its restriction on \( \beta_I M \) is a locally trivial fibre-bundle.

**Proof.** Let \( f \in \text{Emb}(\beta_I M, \mathbb{R}^{k}_+ \langle I^c \rangle \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})_{\Sigma_k,\phi} \). We denote by
\[ \text{Diff}(\mathbb{R}^{k}_+ \langle I^c \rangle \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})_{\bar{\epsilon}} \]
the space of compactly supported diffeomorphisms of \( \mathbb{R}^{k}_+ \langle I^c \rangle \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}} \), as a manifold with corners, respecting collars of lengths \( \epsilon_i \). We construct a continuous map
\[ \eta : U \longrightarrow \text{Diff}(\mathbb{R}^{k}_+ \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})_{\bar{\epsilon}} \]
with \( U \) a suitably chosen neighborhood about \( f \), satisfying the following conditions:

i. The element \( \eta(f) \in \text{Diff}(\mathbb{R}^{k}_+ \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}})_{\bar{\epsilon}} \) is the identity map.
ii. For any \( g \in U \) and \( \hat{f} \in r_{\beta_i}^{-1}(f) \), we have \( r_I(\eta(g) \circ \hat{f}) = g \).

With such a map we can define a trivialization of \( r_I \) over \( U \),
\[ U \times r_I^{-1}(f) \overset{\cong}{\longrightarrow} r_I^{-1}(U) \]
via the formula,
\[ (g, \hat{f}) \mapsto (\eta(g) \circ f) \]
thus proving the theorem. The conditions above imply that this map is actually a diffeomorphism which commutes with projection onto \( U \) thus making it a local trivialization.

Now to construct \( \eta \). Choose \( \delta < \min\{\epsilon_i | i = 0, \ldots, k\} \). Let \( N_\delta \) be the tubular neighborhood about \( f(\beta_I W) \) in \( \mathbb{R}^{k}_+ \langle I^c \rangle \times \mathbb{R}^{d-1+\bar{n}} \times \mathbb{R}^{\bar{p}+\bar{m}} \) of length \( \delta \). Let \( \pi : N_\delta \rightarrow f(\beta_I W) \) be the projection.
map. Let \( \tilde{U} \) be an open neighborhood of \( f \) in \( \text{Emb}(\beta_t M, \mathbb{R}^k_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}) \) with the property that \( g(\beta_t W) \subset N_{\frac{\delta}{2}} \) for all \( g \in \tilde{U} \).

Let \( \lambda : [0, \infty) \rightarrow [0, 1] \) be a non-increasing smooth function such that \( \lambda(t) = 1 \) for \( t \leq \frac{\delta}{2} \) and \( \lambda(t) = 0 \) for \( t \geq \frac{3\delta}{4} \). We define a map

\[
\tilde{\eta} : \tilde{U} \rightarrow C^\infty(\mathbb{R}^k_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})
\]

via the formula

\[
\tilde{\eta}(g)(\bar{t}_{I^c}, x) = (\bar{t}_{I^c}, x) + \lambda(\text{dist}[(\bar{t}_{I^c}, x), f(\beta_t W)]) \cdot \left( g \circ f^{-1} \circ \pi(\bar{t}_{I^c}, x) - \pi(\bar{t}_{I^c}, x) \right)
\]

where

\[
\bar{t}_{I^c} \in \mathbb{R}^k_{+} \quad \text{and} \quad x \in \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}},
\]

\( \text{dist}(\cdot, \cdot) \) Euclidean distance, and the addition and subtraction symbols denote coordinate-wise sum and difference of vectors in Euclidean space. We note that this formula is very similar to the formula used in the proof of the main result of [11].

Since both \( g, f \), and \( \pi \) respect collars of lengths \( \epsilon_i \) and since \( \delta < \min\{\epsilon_i | i = 0, \ldots, k \} \), it follows from a simple calculation that \( \tilde{\eta}(g)(\bar{t}_{I^c}, x) \) is actually in \( \mathbb{R}^k_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}} \), (i.e. each coordinate of \( \bar{t}_{I^c} \) is actually greater than or equal to 0) for all \( g \in \tilde{U} \) and all \( (\bar{t}_{I^c}, x) \). It is also clear that \( \tilde{\eta} \) is continuous in \( g \). Now, observe that \( \tilde{\eta}(f) \) is the identity function. Since the space of diffeomorphisms is open in the space of smooth maps, we can choose a smaller neighborhood \( U \subset \tilde{U} \) about \( f \) such that \( \tilde{\eta} \) maps \( U \) to \( \text{Diff}(\mathbb{R}^k_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}) \). Now for each \( i \in I \), define smooth non-increasing functions \( \alpha_i : [0, \infty) \rightarrow [0, 1] \) with the property that \( \alpha_i(t) = 1 \) for \( t_i \leq \epsilon_i \) and \( \alpha_i(t) = 0 \) for \( t \geq 2\epsilon_i \). Then define \( \alpha : \mathbb{R}^k_{+} \rightarrow [0, 1] \) by setting \( \alpha(t_{i_1}, \ldots, t_{i_j}) = \alpha_{i_1}(t_{i_1}) \cdots \alpha_{i_j}(t_{i_j}) \). We now define a map

\[
\eta : U \times \mathbb{R}^k_{+} \rightarrow \text{Diff}(\mathbb{R}^k_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})
\]

by

\[
\eta(g)(\bar{t}_I, \bar{t}_{I^c}, x) = (\bar{t}_I, x) + \alpha(\bar{t}_I) \cdot \lambda(\text{dist}[(\bar{t}_I, x), f(\beta_t W)]) \cdot \left( g \circ f^{-1} \circ \pi(\bar{t}_I, x) - \pi(\bar{t}_I, x) \right),
\]

where

\[
\bar{t}_I \in \mathbb{R}^k_{+}, \quad \bar{t}_{I^c} \in \mathbb{R}^k_{+}, \quad \text{and} \quad x \in \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}
\]

and the addition and subtraction in the above formula is the addition and subtraction of vectors in Euclidean space. Finally, we define our map

\[
\eta : U \rightarrow \text{Diff}(\mathbb{R}^k_{+} \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}})
\]

by the formula

\[
\eta(g)(\bar{t}_I, \bar{t}_{I^c}, x, y) = (\bar{t}_I, \tilde{\eta}(g)(\bar{t}_{I^c}, x, \bar{t}_I^c), y)
\]

where

\[
\bar{t}_I \in \mathbb{R}^k_{+}, \quad \bar{t}_{I^c} \in \mathbb{R}^k_{+}, \quad x \in \mathbb{R}^{d-1+n} \times \mathbb{R}^{\bar{p}+\bar{m}}, \quad y \in \mathbb{R}^{\bar{p}+\bar{m}}
\]
using the identification
\[ R^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m} = R^k_+ \times R^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m} \times \mathbb{R}^{p+m}. \]

This function \( \eta \) has the desired properties, namely that \( r_I(\eta(g) \circ f) = g \) for any \( g \in U \) and \( f \in r^{-1}_I(f) \). Using \( \eta \) we can define a trivialization of \( r_I \) over \( U \) as described above. This proves the theorem. \( \square \)

**Theorem A.2.** For each \( I \), the homeomorphism type of the space
\[ \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\Sigma_k \phi} \]
does not depend on our choice of embeddings \( \phi \).

**Proof.** The proof of this theorem is a simple application of the Isotopy Extension Theorem, [9]. Since \( m_i > p_i \) for all \( i \), there exists isotopies connecting the embeddings \( \phi_i^1 \) and \( \phi_i^2 \) for all \( i \). Let \( \Phi_i : P_i \times [0, 1] \to \mathbb{R}^{p_i+m_i} \) be such isotopies so that \( \Phi_i \) equals \( \phi_i^1 \) at time 0 and \( \phi_i^2 \) at time 1. By the Isotopy Extension Theorem, for each \( i \) there exists diffeotopies
\[ \hat{\Phi}_i : \mathbb{R}^{p_i+m_i} \times [0, 1] \to \mathbb{R}^{p_i+m_i}, \]
such that
\[ \hat{\Phi}_i(x, 0) = Id_{\mathbb{R}^{p_i+m_i}} \quad \text{for all } x \in \mathbb{R}^{p_i+m_i}, \]
\[ \hat{\Phi}_i(\phi_i^1(p), 1) = \phi_i^2(p) \quad \text{for all } p \in P_i \text{ and } \]
\[ \hat{\Phi}_i(\Phi_i(p, t), t) = \Phi_i(p, t) \quad \text{for all } (p, t) \in P_i \times I. \]

Using these diffeotopies, we define a map
\[ \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\Sigma_k \phi^1} \to \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\Sigma_k \phi^2}, \]
by sending an embedding \( g \in \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\Sigma_k \phi^1} \), to the embedding given by
\[ \left( Id_{\mathbb{R}^k_+ \times \mathbb{R}^{d-1+n}} \times \prod_{i \in \{k\}} \hat{\Phi}_i \mid_{\mathbb{R}^{p_i+m_i} \times \{1\}} \right) \circ g \]
which is an element of \( \text{Emb}(M, \mathbb{R}^k_+ \times \mathbb{R}^{d-1+n} \times \mathbb{R}^{p+m})^{\Sigma_k \phi^2} \). This map is clearly a homeomorphism because the maps \( \hat{\Phi}_i \mid_{\mathbb{R}^{p_i+m_i} \times \{1\}} \) are diffeomorphisms. \( \square \)

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