Hidden constructions in abstract algebra, Krull Dimension, Going Up, Going Down

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Abstract

We present constructive versions of Krull’s dimension theory for commutative rings and distributive lattices. The foundations of these constructive versions are due to Joyal, Espanöl and the authors. We show that this gives a constructive version of basic classical theorems (dimension of finitely presented algebras, Going up and Going down theorem, . . . ), and hence that we get an explicit computational content where these abstract results are used to show the existence of concrete elements. This can be seen as a partial realisation of Hilbert’s program for classical abstract commutative algebra.

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## Contents

Introduction

1 Constructive definition of Krull dimension of commutative rings
   1.1 Idealistic chains .................................................. 5
   1.2 Simultaneous collapse ........................................... 7
   1.3 Pseudo regular sequences and Krull dimension ................. 10
   1.4 Krull dimension and local-global principle .................... 13

2 Distributive lattice, Entailment relations and Krull dimension .... 15
   2.1 Distributive lattices, filters and spectrum .................... 15
   2.2 Distributive lattices and entailment relations ............... 19
   2.3 Krull dimension of distributive lattices ....................... 20

3 Zariski and Krull lattice associated to a commutative ring ....... 27

4 Going Up and Going Down
   4.1 Relative Krull dimension ........................................ 30
   4.2 Going Up ................................................................ 34
   4.3 Going Down ................................................................ 36

Bibliographie

Annex: The compactness theorem and LLPO ............................. 41
   A.1 Theories and models ................................................. 41
   A.2 Completeness theorem ............................................. 41
   A.3 Compactness theorem .............................................. 41
   A.4 LPO and LLPO .......................................................... 42
   A.5 Geometric formulae and theories .................................. 42
Introduction

We present constructive versions of Krull’s dimension theory for commutative rings and distributive lattices. The foundations of these constructive versions are due to Joyal, Espanòl and the authors. We show that this gives a constructive version of basic classical theorems (dimension of finitely presented algebras, Going up and Going down theorem, . . .), and hence that we get an explicit computational content when these abstract results are used to show the existence of concrete elements. This can be seen as a partial realisation of Hilbert’s program for classical abstract commutative algebra.

Our presentation follows Bishop’s style (cf. in algebra [16]). As much as possible, we kept minimum any explicit mention to logical notions. When we say that we have a constructive version of an abstract algebraic theorem, this means that we have a theorem the proof of which is constructive, which has a clear computational content, and from which we can recover the usual version of the abstract theorem by an immediate application of a well classified non constructive principle. An abstract classical theorem can have several distinct interesting constructive versions.

In the case of abstract theorem in commutative algebra, such a non constructive principle is the compactness theorem, which claims the existence of a model of a formally consistent propositional theory\(^1\). When this is used for algebraic structures of enumerable presentation (in a suitable sense) this principle is nothing else than a reformulation of Bishop LLPO (a real number is $\geq 0$ or $\leq 0$).

To avoid the use of compactness theorem is not motivated by philosophical but by practical considerations. The use of this principle leads indeed to replace quite direct (but usually hidden) arguments by indirect ones which are nothing else than a double contraposition of the direct proofs, with a corresponding lack of computational content. For instance [1] the abstract proof of 17\(^{th}\) Hilbert’s problem claims : if the polynomial \(P\) is not a sum of rational fractions there is a field \(K\) in which one can find an absurdity by reading the (constructive) proof that the polynomial is everywhere positive or zero. The direct version of this abstract proof is: from the (constructive) proof that the polynomial is everywhere positive or zero, one can show (using arguments of the abstract proofs) that any attempt to build \(K\) will fail. This gives explicitly the sum of squares we are looking for. In the meantime, one has to replace the abstract result: “any real field can be ordered” by the constructive theorem: “in a field in which any attempt to build an ordering fails $-1$ is a sum of squares”. One can go from this explicit version to the abstract one by compactness theorem, while the proof of the explicit version is hidden in the algebraic manipulations that appear in the usual classical proof of the abstract version.

Here is the content of the paper.

Pseudo regular sequences and Krull dimension of commutative rings

In section 1 we give “more readable” proofs for some of the results contained in [10], which were there proved using the notion of dynamical structures [1]. The central notion which is used is the one of partial specification of a chain of ideal primes. Abstract constructions on chains of prime ideals are then expressed constructively in the form of simultaneous collapsing theorems [11] (theorem 1.10). We present the notion of pseudo-regular sequence (a weakened form of the notion of regular sequence), which allows us to define constructively the Krull dimension of a ring. We show in this way that the notion of Krull dimension has an explicit computational content in the form of existence (or non existence) of some algebraic identities. This confirms

\(^1\)Mathematically, this result can be seen as stating the compactness of product spaces $\{0,1\}^V$; thus it can be seen as a special case of Tychonov’s theorem.
the feeling that commutative algebra can be seen computationally as a machinery producing
algebraic identities (the most famous of which being called Nullstellensatz). Finally we give a
constructive version of the theorem which says that the Krull dimension of a ring is the upper
bound of the Krull dimension of its localizations along maximal ideals.

Distributive lattices
In section 2 we develop the theory of Krull dimension of distributive lattices, first in the style of
section 1, and then in the style of the theory of Joyal. We then show the connections between
these two presentations. An important simplification of proofs and computations is obtained
via the systematic use of the notion of entailment relation, which has its origin in the cut rule
in Gentzen’s sequent calculus, with the fundamental theorem 2.10.

Zariski and Krull lattice In section 3 we define the Zariski lattice of a commutative ring
(whose elements are radicals of finitely generated ideals), which is the constructive counterpart
of Zariski spectrum: the points of Zariski spectrum are the prime ideals of Zariski lattice,
and the constructible subsets of Zariski spectrum are the elements of the Boolean algebra
generated by the Zariski lattice. Joyal’s idea is to define Krull dimension of a commutative ring
as the dimension of its Zariski lattice. This avoids any mention of prime ideals. We show the
equivalence between this (constructive) point of view of the (constructive) presentation given
in section 1.

Going Up and Going Down
Section 4 presents the famous Going up theorem for integral extensions, and the Going down
theorem for integral extensions of integrally closed domains and for flat extensions. We show
that these theorems, which seem at first quite abstract (since they claim the existence of
some prime ideals) have quite concrete meaning as constructions of algebraic identities. These
constructive versions may seem at first a little strange, but they are directly suggested by this
process of making explicit the abstract arguments of these classical results.

Conclusion
This article confirms the actual realisation of Hilbert’s program for a large part of abstract
commutative algebra. (cf. [1, 3, 6, 7, 8, 9, 10, 11, 12, 13]). The general idea is to replace ideal
abstract structures by partial specifications of these structures. The very short elegant abstract
proof which uses these ideal objects has then a corresponding computational version at the level
of the partial specifications of these objects. Most of classical results in abstract commutative
algebras, the proof of which seem to require in an essential way excluded middle and Zorn’s
lemma, seem to have in this way a corresponding constructive version. Most importantly, the
abstract proof of the classical theorem always contains, more or less implicitly, the constructive
proof of the corresponding constructive version.

Finally one should note that the constructive theorems which concern the Krull dimension
of polynomial rings and of finitely presented algebra over a field, the Going up and Going down
are new (they could not be obtained in the framework of Joyal’s theory as long as one only
looks at Zariski lattice without explicitating the computations and algebraic identities involved
there).
1 Constructive definition of Krull dimension of commutative rings

Let \( R \) be a commutative ring. We write \( \langle J \rangle \) or explicitly \( \langle J \rangle_R \) the ideal of \( R \) generated by the subset \( J \subseteq R \). We write \( \mathcal{M}(U) \) the monoid \(^2\) generated by the subset \( U \subseteq R \).

1.1 Idealistic chains

Definition 1.1 In a commutative ring \( R \)
- A partial specification for a prime ideal (in abbreviated form idealistic prime) is a couple \( \mathcal{P} = (J, U) \) of subsets of \( R \).
- An idealistic prime \( \mathcal{P} = (J, U) \) is said to be complete if \( J \) is an ideal \( U \) is a monoid and \( J + U = U \).
- Let \( \mathcal{P}_1 = (J_1, U_1) \) and \( \mathcal{P}_2 = (J_2, U_2) \) be two idealistic primes. We say that \( \mathcal{P}_1 \) is contained into \( \mathcal{P}_2 \) written \( \mathcal{P}_1 \subseteq \mathcal{P}_2 \) if \( J_1 \subseteq J_2 \) and \( U_2 \subseteq U_1 \).
- We say that \( \mathcal{P}_2 \) refines \( \mathcal{P}_1 \) and we write \( \mathcal{P}_1 \leq \mathcal{P}_2 \) if \( J_1 \subseteq J_2 \) and \( U_1 \subseteq U_2 \).
- A partial specification of a chain of prime ideals (in abbreviated form idealistic chain) is defined as follows. An idealistic chain of length \( \ell \) is a list of \( \ell + 1 \) idealistic primes: \( \mathcal{C} = (\mathcal{P}_0, \ldots, \mathcal{P}_\ell) \) \( (\mathcal{P}_i = (J_i, U_i)) \). We shall write \( \mathcal{C}(i) \) for \( \mathcal{P}_i \). The idealistic chain will be said finite iff all sets \( J_i \) and \( U_i \) are finite.
- An idealistic chain \( \mathcal{C} = (\mathcal{P}_0, \ldots, \mathcal{P}_\ell) \) is said to be complete iff the idealistic primes \( \mathcal{P}_i \) are complete and if we have \( \mathcal{P}_i \subseteq \mathcal{P}_{i+1} \) \( (i = 0, \ldots, \ell - 1) \).
- Let \( \mathcal{C} = (\mathcal{P}_0, \ldots, \mathcal{P}_\ell) \) be two idealistic chains of length \( \ell \) and \( \mathcal{C}' = (\mathcal{P}_0', \ldots, \mathcal{P}_\ell') \). We say that \( \mathcal{C}' \) is a refinement of \( \mathcal{C} \) and we write \( \mathcal{C} \leq \mathcal{C}' \) if \( \mathcal{P}_i \leq \mathcal{P}_i' \) for \( i = 0, \ldots, \ell \).

We can think of an idealistic chain \( \mathcal{C} \) of length \( \ell \) in \( A \) as a partial specification of an increasing chain of prime ideals (in the usual sense) \( P_0, \ldots, P_\ell \) such that \( \mathcal{C} \leq (Q_0, \ldots, Q_\ell) \), where \( Q_i = (P_i, A \setminus P_i) \) \( (i = 0, \ldots, \ell) \).

Fact 1.2 Any idealistic chain \( \mathcal{C} = ((J_0, U_0), \ldots, (J_\ell, U_\ell)) \) generates a complete minimal idealistic chain
\[ \mathcal{C}' = ((I_0, V_0), \ldots, (I_\ell, V_\ell)) \]
defined by \( I_0 = \langle J_0 \rangle \), \( I_1 = \langle J_0 \cup J_1 \rangle, \ldots, I_\ell = \langle J_0 \cup \cdots \cup J_\ell \rangle \), \( U_0' = \mathcal{M}(U_0) \) \( (i = 0, \ldots, \ell) \), \( V_0 = U_0' + I_0, V_\ell = U_\ell' + I_\ell, V_{\ell-1} = U_{\ell-1}'V_\ell + I_{\ell-1}, \ldots, V_0 = U_0'V_1 + I_0 = U_0'(U_1'(\cdots(U_\ell'(I_\ell) + \cdots) + I_1) + I_0) \).
Furthermore any element of \( U_0 \) can be rewritten as
\[ u_0 \cdot (u_1 \cdot \cdots (u_\ell + j_\ell) + \cdots) + j_1) + j_0 = u_0 \cdots u_\ell + u_0 \cdots u_{\ell-1} \cdot j_\ell + \cdots + u_0 \cdot j_1 + j_0 \]
with \( j_i \in \langle J_i \rangle \) and \( u_i \in \mathcal{M}(U_i) \).

Definition 1.3 An ideal \( I \) and a monoid \( S \) are said to be conjugate if we have:
\[ (s \cdot a \in I, \ s \in S) \implies a \in I \]
\[ a^n \in I \implies a \in I \quad (n \in \mathbb{N}, \ n > 0) \]
\[ (j \in I, \ s \in S) \implies s + j \in S \]
\[ s_1 \cdot s_2 \in S \implies s_1 \in S \]

In this case we say also that the idealistic prime \( (I, S) \) is saturated.

\(^2\) A monoid will always be multiplicative.
For instance a detachable prime ideal \(^3\) and the complementary monoid are conjugate. When an ideal \(I\) and a monoid \(S\) are conjugate we have

\[
1 \in I \iff 0 \in S \iff (I, S) = (A, A)
\]

**Definition 1.4 (Collapsus)** Let \(C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))\) be an idealistic chain and \(C' = ((I_0, V_0), \ldots, (I_\ell, V_\ell))\) the complete idealistic chain it generates.

- We say that the idealistic chain \(C\) collapses iff we have \(0 \in V_0\). An alternative definition is: there exists \(j_i \in \langle J_i \rangle\), \(u_i \in M(U_i)\), \((i = 0, \ldots, \ell)\) satisfying the equation

\[
u_0 \cdot (u_1 \cdot (\cdots (u_\ell + j_\ell) + \cdots) + j_1) + j_0 = 0\]

Such a relation is called a collapsus of the idealistic chain \(C\).

- An idealistic chain is said to be saturated iff it is complete and the idealistic primes \((J_i, U_i)\) are saturated.

- The idealistic chain \(((A, A), \ldots, (A, A))\) is said to be trivial: a saturated chain that collapses is trivial.

Notice that the idealistic prime \((0, 1)\) collapses if and only if \(1 = A 0\).

The following lemma is direct.

**Fact 1.5** An idealistic chain \(C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))\) with \(U_h \cap J_h \neq \emptyset\) for some \(h\) collapses. More generally if an idealistic chain \(C'\) extracted from an idealistic chain \(C\) collapses then \(C\) collapses. Similarly if \(C\) collapses, any refinement of \(C\) collapses.

The proofs of the following properties are instructive.

**Fact 1.6** Let \(C_1 = (P_0, \ldots, P_\ell)\) and \(C_2 = (P_{\ell+1}, \ldots, P_{\ell+r})\) be two idealistic chains on a ring \(A\). Let \(C = C_1 \bullet C_2 = (P_0, \ldots, P_{\ell+r})\).

1. Suppose that \(C_1\) is saturated. Then \(C\) collapses in \(A\) if, and only if, \(P_\ell \bullet C_2\) collapses in \(A\) if, and only if, \(C_2\) collapses in the quotient \(A/I_\ell\) (\(P_\ell = (I_\ell, U_\ell)\)).

2. Suppose that \(C_2\) is complete. Then \(C\) collapses in \(A\) if, and only if, \(C_1 \bullet P_{\ell+1}\) collapses in \(A\) if, and only if, \(C_2\) collapses in the localisation \(A_{U_{\ell+1}}\) (\(P_{\ell+1} = (I_{\ell+1}, U_{\ell+1})\)).

3. Suppose that \(C_1\) is saturated and \(C_2\) is complete. Then \(C\) collapses in \(A\) if, and only if, \((P_\ell, P_{\ell+1})\) collapses in \(A\) if, and only if, \(I_\ell \cap U_{\ell+1} \neq \emptyset\).

**Proof.**

Left to the reader. \(\Box\)

\(^3\) A subset of a set is said to be detachable if we can decide membership to this subset. For example the finitely generated ideals of a polynomial ring with integer coefficients are detachable.
1.2 Simultaneous collapse

Notation 1.7 In the sequel we will use the following notations for an idealistic prime or an idealistic chain obtained by refinement. If $P = (J, U), C = (P_1, \ldots, P_n)$, we write

- $(J; x; U)$ or still $P \& \{x \in P\}$ for $(J \cup \{x\}, U)$
- $(J; x, U)$ or still $P \& \{x \notin P\}$ for $(J, U \cup \{x\})$
- $P \& \{I \subseteq P\}$ for $(J \cup I, U)$
- $P \& \{V \subseteq A \setminus P\}$ for $(J, U \cup V)$
- $C \& \{x \in C(i)\}$ for $(P_0, \ldots, P_i \& \{x \in P_i\}, \ldots, P_n)$
- $C \& \{x \notin C(i)\}$ for $(P_0, \ldots, P_i \& \{x \notin P_i\}, \ldots, P_n)$
- etc.

Theorem 1.8 (Simultaneous collapse for an idealistic prime)

Let $P = (J, U)$ be an idealistic prime in a commutative ring $R$.

1. Let $x$ be an element of $R$. Suppose that the idealistic primes $P \& \{x \in P\}$ and $P \& \{x \notin P\}$ both collapse, then so does $P$.

2. The idealistic prime $P$ generates a minimum saturated idealistic prime. We get it by adding in $U$ (resp. $J$) any element $x \in R$ such that the idealistic prime $P \& \{x \in P\}$ (resp. $P \& \{x \notin P\}$) collapses.

Proof.
The proof of point (1) is Rabinovitch trick. From the two equalities $u_1 + j_1 + ax = 0$ and $u_2 x^m + j_2 = 0$ (with $u_i \in M(U), j_i \in \langle J \rangle, a \in R, n \in \mathbb{N}$) we build a third one, $u_3 + j_3 = 0$ by eliminating $x$: we get $u_2(u_1 + j_1)^m + (-a)^m j_2 = 0$, with $u_3 = u_2 u_1^m$.

The point (2) is seen to be a consequence of (1) as follows. Let $P' = (\langle J \rangle, \langle J \rangle + M(U))$ be the complete idealistic prime generated by $P$. Let $P'' = (I', S')$ be a saturated idealistic prime which refines $P$. Let $P_1 = (K, S)$ the idealistic prime described in (2). It is easily seen that we have $P \leq P' \leq P_1 \leq P''$. Thus we are left to check that $P_1$ is a saturated idealistic prime. We shall see that this results from (1) without having to do any computations. Let us show for instance that $K + K \subseteq K$. Let $x$ and $y$ be in $K$ i.e., such that $(I; x, U)$ and $(I; y, U)$ both collapse. We have to show that $(I; x + y, U)$ also collapses. For this, by (1), it is enough to show that $P_2 = (I; x; x + y, U)$ and $P_3 = (I; x + y, x, U)$ collapse. For $P_3$ it is by hypothesis. If we complete $P_2$, we get $y = x + y - x$ in the monoid, hence this is a refinement of $(I; y, U)$ which collapses by hypothesis.

The other verifications are direct, by similar arguments. \hfill \Box

Notice that the saturation of $(0, 1)$ is $(N, A^\times)$ where $N$ is the nilradical of $R$ and $A^\times$ the group of units.

Corollary 1.9 (Krull’s theorem or formal Hilbert Nullstellensatz )

Let $P = (J, U)$ be an idealistic prime in a commutative ring $R$. Compactness theorem implies that the following properties are equivalent:

- For all $j \in \langle J \rangle$, $u \in M(U)$, we have $u \neq j$. 

• There exists a detachable prime ideal $Q$ such that $\mathcal{P} \leq Q$, i.e., such that $J \subseteq Q$ and $U \cap Q = \emptyset$.

• There exists an homomorphism $\psi$ from $R$ to an entire ring $S$ such that $\psi(J) = 0$ and $0 \notin \psi(U)$.

Furthermore if the saturation of $\mathcal{P}$ is $(I, V)$, compactness theorem implies that $I$ is the intersection of all prime ideals containing $J$ and disjoint from $U$, whereas $V$ is the union of all complement of these prime ideals.

Proof.
See the proof of theorem 1.13 (which is more general).

\begin{proof}

Theorem 1.10 (Simultaneous collapse for the idealistic chains)
Let $\mathcal{C} = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$ be an idealistic chain in a commutative ring $R$

1. Suppose $x \in R$ and $i \in \{0, \ldots, \ell\}$. Suppose that the idealistic chains $\mathcal{C} \& \{ x \in \mathcal{C}^{(i)} \}$ and $\mathcal{C} \& \{ x \notin \mathcal{C}^{(i)} \}$ both collapse, then so does $\mathcal{C}$.

2. The idealistic chain $\mathcal{C}$ generates a minimum saturated idealistic chain. We get it by adding in $U_i$ (resp. $J_i$) all element $x \in R$ such that the idealistic chain $\mathcal{C} \& \{ x \in \mathcal{C}^{(i)} \}$ (resp. $\mathcal{C} \& \{ x \notin \mathcal{C}^{(i)} \}$) collapses.

Proof.
Let us write (1)$_\ell$ and (2)$_\ell$ the statements for a fixed $\ell$. Notice that (1)$_0$ and (2)$_0$ are proved in theorem 1.8. We shall reason then by induction on $\ell$. We can suppose that the idealistic chain $\mathcal{C}$ is complete (since a chain collapses iff its completion does). The fact that (1)$_\ell$ $\Rightarrow$ (2)$_\ell$ is direct, and can be proved by similar arguments as in the proof of theorem 1.8.

We are left to show ((1)$_{\ell-1}$ and (2)$_{\ell-1}$) $\Rightarrow$ (1)$_\ell$ (for $\ell > 0$).
If $i < \ell$ we use fact 1.6: we have then idealistic chains of length $i$ in the localisation $A_{U_i+1}$ and we can apply the induction hypothesis.
If $i = \ell$ we consider the idealistic chain of length $\ell - 1$ $((K_0, S_0), \ldots, (K_{\ell-1}, S_{\ell-1}))$ that we get by saturating $((J_0, U_0), \ldots, (J_{\ell-1}, U_{\ell-1}))$ (we use (2)$_{\ell-1}$). For arbitrary $j_i \in J_i$ and $u_i \in \mathcal{M}(U_i)$ ($0 \leq i \leq \ell$), let us consider the following assertions:

\begin{align*}
{u}_0 \cdot \left( {u}_1 \cdot \left( \cdots \left( {u}_{\ell-1} \cdot \left( {u}_\ell + {j}_\ell \right) + {j}_{\ell-1} \right) + \cdots \right) + {j}_1 \right) + {j}_0 &= 0 \quad \text{(\alpha)} \\
({u}_\ell + {j}_\ell) &\in K_{\ell-1} \quad \text{(\beta)}
\end{align*}

\[ \exists n \in \mathbb{N} \quad {u}_0 \cdot \left( {u}_1 \cdot \left( \cdots \left( {u}_{\ell-1} \cdot \left( {u}_\ell + {j}_\ell \right)^n + {j}_{\ell-1} \right) + \cdots \right) + {j}_1 \right) + {j}_0 = 0 \quad \text{(\gamma)} \]

We have \((\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)\). Hence the following properties are equivalent. Primo: the idealistic chain $\mathcal{C}$ collapses in $R$ (this is certified by an equality of type (\alpha) or (\gamma)). Secundo: the idealistic prime $(J_\ell, U_\ell)$ collapses in $R/K_{\ell-1}$ (which is certified by an equality of type (\beta)). We are then reduced to the case (1)$_0$ in the ring $R/K_{\ell-1}$, and this case has been dealt with in theorem 1.8.

\end{proof}

Notice that we have made no use in the end of this argument of fact 1.6, which is not powerful enough in this situation. The following facts are simple corollaries of theorem 1.10. Notice that the second point allows an improved utilisation of fact 1.6.

Fact 1.11
An idealistic chain $C$ collapses if, and only if, any saturated idealistic chain which refines $C$ is trivial.

One does not change the collapsus of an idealistic chain $C$ if we replace a subchain by its saturation.

Suppose $x_1, \ldots, x_k \in R$ and that the idealistic chains $((J_0, U_0), \ldots, (J_i, U_i))$ collapse for all complement pair $(H, H')$ of $\{1, \ldots, k\}$, then $C$ collapses.

Definition 1.12

Two idealistic chains that generate the same saturated idealistic chain are said to be equivalent.

An idealistic chain of finite type is an idealistic chain which is equivalent to a finite idealistic chain.

An idealistic chain is strict iff $V_i \cap I_{i+1} \neq \emptyset$ $(i = 0, \ldots, \ell - 1)$ in the corresponding generated saturated idealistic chain.

A saturated idealistic chain $C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$ is frozen if it does not collapse and if for all $i = 0, \ldots, \ell$, $J_i \cup U_i = R$. An idealistic chain is frozen iff its saturation is frozen. To give a frozen idealistic chain is equivalent to give an increasing chain of detachable prime ideals.

We think that theorem 1.10 reveals a computational content which is “hidden” in usual classical proofs about increasing chains of prime ideals. We illustrate this point in the following theorem, which, in classical terms, gives a concrete characterisation of idealistic chains which are incomplete specifications of increasing chains of prime ideals.

Theorem 1.13 (formal Nullstellensatz for chain of prime ideals) Let $R$ be a ring and $((J_0, U_0), \ldots, (J_\ell, U_\ell))$ an idealistic chain in $R$. The compactness theorem implies that the following are equivalent:

(a) There exist $\ell + 1$ detachable prime ideals $P_0 \subseteq \cdots \subseteq P_\ell$ such that $J_i \subseteq P_i$, $U_i \cap P_i = \emptyset$, $(i = 0, \ldots, \ell)$.

(b) For all $j_i \in \langle J_i \rangle$ and $u_i \in \mathcal{M}(U_i)$, $(i = 0, \ldots, \ell)$

$$u_0 \cdot (u_1 \cdot (\cdots (u_\ell + j_\ell) + \cdots ) + j_i) + j_0 \neq 0.$$

Proof.

Only $(b) \Rightarrow (a)$ is not direct. Let us start by a proof that uses not the compactness theorem but excluded middle principle and Zorn’s lemma. We consider an idealistic chain $C_1 = ((P_0, S_0), \ldots, (P_\ell, S_\ell))$ maximal (w.r.t. the refinement relation) among all idealistic chain which refine $C$ and that are not collapsing. It is first clear that $C_1$ is complete, since collapsus is not changed by completion. If this was not a chain of prime ideals with complement, we would have for some index $i : S_i \cup P_i \neq R$. In this case let $x \in A \setminus (S_i \cup P_i)$. Then $((P_0, S_0), \ldots, (P_i \cup \{x\}, S_i), \ldots, (P_\ell, S_\ell))$ has to collapse (by maximality). The same holds for $((P_0, S_0), \ldots, (P_i, S_i \cup \{x\}), \ldots, (P_\ell, S_\ell))$. By theorem 1.10 the idealistic chain $((P_0, S_0), \ldots, (P_\ell, S_\ell))$ collapses, which is absurd.
Let us present next a proof which uses only the compactness theorem, hence with a restricted use of excluded middle principle.

We consider the syntactical propositional theory which describes an increasing chain of prime ideals of length $\ell$ in $R$. In this theory we have atomic proposition for $x \in P_i$ and the axioms state that each $P_i$ defines a proper prime ideal, that is

- $\neg(1 \in P_i)$
- $0 \in P_i$
- $a \in P_i \land b \in P_i \rightarrow (a + b) \in P_i$
- $a \in P_i \rightarrow ab \in P_i$
- $ab \in P_i \rightarrow (a \in P_i \lor b \in P_i)$

and we have furthermore $a \in P_i \rightarrow a \in P_{i+1}$ and $a \in P_i$ for $a \in J_i$ and $\neg(b \in P_i)$ for $b \in U_i$.

By theorem 1.10 this theory is consistent. By the compactness theorem, it has a model. This model gives us $\ell + 1$ prime ideals as desired. \hfill \Box

Notice also that theorem 1.13 implies (in two lines) the simultaneous collapsus theorem 1.10. This last result can thus be considered to be the constructive version of the first. A possible corollary of theorem 1.13 would be a characterisation of the saturated idealistic chain generated by an idealistic chain $C$ via the family of chains of prime ideals that are refinements of $C$ like in the last claim of theorem 1.8.

1.3 Pseudo regular sequences and Krull dimension

In a constructive framework, it is sometimes better to consider an inequality relation $x \neq 0$ which is not simply the negation of $x = 0$. For instance a real number is said to be $\neq 0$ iff it is invertible, i.e., apart from 0. Whenever we mention an inequality relation $x \neq 0$ in a ring, we always suppose implicitly that this relation has been defined first in the ring. We require that this relation is a standard apartness relation, that is we ask that, modulo the use of the excluded middle, it can be shown to be equivalent to $\neg(x = 0)$. We ask also the conditions $(x \neq 0, y = 0) \Rightarrow x + y \neq 0$, $xy \neq 0 \Rightarrow x \neq 0$, and $\neg(0 \neq 0)$. Finally $x \neq y$ is defined as $x - y \neq 0$. Without any further precisions on $x \neq 0$ one can always consider that it is $\neg(x = 0)$. When the ring is a discrete set, that is when there is an equality test, we always chose the inequality $\neg(x = 0)$. Nevertheless it would be a misguided conception to believe that algebra should limit itself to discrete sets.

Definition 1.14 Let $(x_1, \ldots, x_\ell)$ be a sequence of length $\ell$ in a commutative ring $R$.

- The idealistic chain $((0, x_1), (x_1, x_2), \ldots, (x_{\ell-1}, x_\ell), (x_\ell, 1))$ is said to be an elementary idealistic chain. It is said to be associate to the sequence $(x_1, \ldots, x_\ell)$. We write it $(x_1, \ldots, x_\ell)$.

- The sequence $(x_1, \ldots, x_\ell)$ is said to be pseudo singular when the associate elementary idealistic chain $(x_1, \ldots, x_\ell)$ collapses. This means that there exist $a_1, \ldots, a_\ell \in R$ and $m_1, \ldots, m_\ell \in \mathbb{N}$ such that

\[
x_1^{m_1}(x_2^{m_2}(x_\ell^{m_\ell}(1 + a_\ell x_\ell) + \cdots + a_2 x_2) + a_1 x_1) = 0
\]
The sequence \((x_1, \ldots, x_\ell)\) is pseudo regular when the corresponding elementary idealistic chain does not collapse. This means that for all \(a_1, \ldots, a_\ell \in R\) and all \(m_1, \ldots, m_\ell \in \mathbb{N}\), we have
\[
x_1^{m_1}(x_2^{m_2}\cdots(x_\ell^{m_\ell}(1 + a_\ell x_\ell)) + \cdots + a_2 x_2) + a_1 x_1) \neq 0
\]
Notice that the length of the elementary idealistic chain associated to a sequence is the same as the length of this sequence.

The connection with the usual notion of regular sequence is given by the following proposition, which is direct.

**Proposition 1.15** In a commutative ring \(R\) any regular sequence is pseudo regular.

The following lemma is sometimes useful.

**Lemma 1.16** Let \((x_1, \ldots, x_\ell)\) and \((y_1, \ldots, y_\ell)\) be two sequences in a commutative ring \(R\). Suppose that for each \(j\), \(x_j\) divides a power of \(y_j\) and that \(y_j\) divides a power of \(x_j\). The sequence \((x_1, \ldots, x_\ell)\) is then pseudo singular if, and only if, the sequence \((y_1, \ldots, y_\ell)\) is pseudo singular.

**Proof.** Indeed, if \(x\) divides a power of \(y\) and \(y\) divides a power of \(x\) we have the following refinement relations
\[
(a; x)(x; b) \leq (a; x, y)(x, y; b) \leq \text{the saturation of the sequence } (a; x)(x; b)
\]
one add the first \(y\) by the relation \(yc = x^k\) (\(y\) is hence in the saturation of the monoid generated by \(x\)), one add the second by the relation \(y^m = dx\) (\(y\) is hence in the radical of the ideal generated by \(x\)). We deduce by symmetry that \((a; x)(x; b)\) and \((a; y)(y; b)\) have the same saturation. \(\square\)

An immediate corollary of theorem 1.13 is the following theorem 1.17.

**Theorem 1.17** (pseudo regular sequences and increasing chain of prime ideals) The compactness theorem implies the following result. In a ring \(R\) a sequence \((x_1, \ldots, x_\ell)\) is pseudo regular if, and only if, there exist \(\ell + 1\) prime ideals \(P_0 \subseteq \cdots \subseteq P_\ell\) with \(x_1 \in P_1 \setminus P_0, x_2 \in P_2 \setminus P_1, \ldots, x_\ell \in P_\ell \setminus P_{\ell - 1}\).

This leads to the following definition, which gives an explicit constructive content to the notion of Krull dimension of a ring.

**Definition 1.18** (Krull dimension of a ring)

- A ring \(R\) is of dimension \(-1\) if, and only if \(1 =_A 0\). It is of dimension \(\geq 0\) if, and only if, \(1 \neq_A 0, > -1\) if, and only if, \(\neg(1 =_A 0)\) and \(< 0\) if, and only if, \(\neg(1 \neq_A 0)\).

Let us now suppose \(\ell \geq 1\).

- A ring is of dimension \(\leq \ell - 1\) if, and only if, all elementary idealistic chains of length \(\ell\) collapse.
- A ring is of dimension \(\geq \ell\) if, and only if, there exists a pseudo regular sequence de longueur \(\ell\).
- A ring is of dimension \(\ell\) if, and only if, it is both of dimension \(\geq \ell\) and \(\leq \ell\).
A ring is of dimension \(<\ell\) if, and only if, it is impossible for it to be of dimension \(\geq \ell\).

A ring is of dimension \(>\ell\) if, and only if, it is impossible for it to be of dimension \(\leq \ell\). (\(^4\)).

A ring is thus of (Krull) dimension \(\leq \ell - 1\) if for all sequence \((x_1, \ldots, x_\ell)\) in \(R\), one can find \(a_1, \ldots, a_\ell \in R\) and \(m_1, \ldots, m_\ell \in \mathbb{N}\) such that

\[
x_1^{m_1} \cdots (x_\ell^{m_\ell} (1 + a_\ell x_\ell) + \cdots) + a_1 x_1 = 0
\]

In particular a ring is of dimension \(\leq 0\) if, and only if, for all \(x \in R\) there exists \(n \in \mathbb{N}\) and \(a \in R\) such that \(x^n = ax^{n+1}\). And it is of dimension \(< 1\) if, and only if, it is absurd to find \(x \in R\) such that, for all \(n \in \mathbb{N}\) and all \(a \in R\), \(x^n \neq ax^{n+1}\).

Notice that the ring of real numbers is a local ring of dimension \(< 1\), but it cannot be proved constructively to be of dimension \(\leq 0\).

Notice also that a local ring is of dimension \(\leq 0\) if, and only if,

\[
\forall x \in A \quad x \text{ is invertible or nilpotent}
\]

Krull dimension of a polynomial ring over a discrete field

First we have.

**Proposition 1.19** Let \(K\) be a discrete field, \(R\) a commutative \(K\)-algebra, and \(x_1, \ldots, x_\ell\) in \(R\) algebraically dependent over \(K\). The sequence \((x_1, \ldots, x_\ell)\) is pseudo singular.

**Proof.**
Let \(Q(x_1, \ldots, x_\ell) = 0\) be a algebraic dependence relation over \(K\). Let us order the non zero monomials of \(Q\) along the lexicographic ordering. We can suppose that the coefficient of the first monomial is 1. Let \(x_1^{m_1} x_2^{m_2} \cdots x_\ell^{m_\ell}\) be this momial, it is clear that \(Q\) can be written on the form

\[
Q = x_1^{m_1} \cdots x_\ell^{m_\ell} + x_1^{m_1} \cdots x_\ell^{1+m_\ell} R_\ell + x_1^{m_1} \cdots x_{\ell-1}^{1+m_{\ell-1}} R_{\ell-1} + \cdots + x_1^{m_1} x_2^{1+m_2} R_2 + x_1^{1+m_1} R_1
\]

and this is the desired collapsus. \(\square\)

It follows that we have:

**Theorem 1.20** Let \(K\) be a discrete field. The Krull dimension of the ring \(K[X_1, \ldots, X_\ell]\) is equal to \(\ell\).

**Proof.**
Given proposition 1.19 it is enough to check that the sequence \((X_1, \ldots, X_\ell)\) is pseudo regular. But this sequence is regular. \(\square\)

Notice that we got this basic result quite directly, and that our argument is of course also valid classically (with the usual definition of Krull dimension). This contradicts the current opinion that constructive arguments are necessarily more involved than classical proofs.

\(^4\) Actually, there exists one and only one elementary idealistic chain of length 0: \((0, 1)\), hence there was no need to begin with a particular definition of ring of dimension \(-1\). In this framework, we recover the distinction between being of dimension \(\geq 0\) and of dimension \(> -1\), as well as the distinction between being of dimension \(\leq -1\) and dimension \(< 0\).
1.4 Krull dimension and local-global principle

Comaximal monoids

Definition 1.21

(1) The monoids $S_1, \ldots, S_n$ of a ring $R$ are comaximal if, and only if, an ideal of $R$ that meets each $S_i$ contains 1, i.e.,

$$\forall s_1 \in S_1 \cdots \forall s_n \in S_n \exists a_1, \ldots, a_n \in A \ \sum_{i=1}^{n} a_is_i = 1.$$ 

(2) The monoids $S_1, \ldots, S_n$ of the ring $R$ cover the monoid $S$ if $S$ is a subset of each $S_i$ and if any ideal of $R$ which meets each of the $S_i$ meets also $S$, i.e.,

$$\forall s_1 \in S_1 \cdots \forall s_n \in S_n \exists a_1, \ldots, a_n \in A \ \sum_{i=1}^{n} a_is_i \in S.$$ 

Notation 1.22 If $(I; U)$ is an idealistic prime of $R$, we write $S(I; U)$ the monoid $M(U) + \langle I \rangle$ of the idealistic prime obtained by completing $(I; U)$.

The fundamental example of comaximal monoids is the following: when $s_1, \ldots, s_n \in R$ are such that $\langle s_1, \ldots, s_n \rangle = \langle 1 \rangle$, the monoids $M(s_i)$ are comaximal.

The following two lemmas are also quite useful to build comaximal monoids.

Lemma 1.23 (easy computations)

(1) (associativity) If the monoids $S_1, \ldots, S_n$ of the ring $R$ cover the monoid $S$ and if each $S_\ell$ is covered by the monoids $S_{\ell,1}, \ldots, S_{\ell,m_\ell}$ then the monoids $S_{\ell,1} \cdots S_{\ell,m_\ell}$ cover $S$.

(2) (transitivity) Let $S$ be a monoid of the ring $R$ and $S_1, \ldots, S_n$ be comaximal monoids of the localization $R_\Sigma$. For $\ell = 1, \ldots, n$ let $V_\ell$ be the monoid of $R$ which consist of the numerators of the elements of $S_\ell$. The monoids $V_1, \ldots, V_n$ cover $S$.

Lemma 1.24 Let $U$ and $I$ be two subsets of the ring $R$ and $a \in R$, then the monoids $S(I; a; U)$ and $S(I; a; U)$ cover the monoid $S(I; U)$.

Proof.

For $x \in S(I; U, a)$ and $y \in S(I; a; U)$ we must find a linear combination $x_1x + y_1y \in S(I; U)$ $(x_1, y_1 \in R)$. We write $x = u_1a^k + j_1$, $y = (u_2 + j_2) - (az) + i_1$, $u_2, u_3, u_4, j_1, j_4 \in I(I)$, $z \in R$. The fundamental identity $c^k - d^k = (c - d) \cdots$ gives $y_2 \in R$ such that $y_2y = (u_2 + j_2)^k - (az)^k = (u_3 + j_3) - (az)^k$ and we write $c^kx + y_1y_2y = u_1u_3 + u_1j_3 + j_1z^k = u_4 + j_4$. □

Corollary 1.25 Let $u_1, \ldots, u_n \in R$. Let $S_k = S((u_i)_{i \leq k}; u_k)$ ($k = 1, \ldots, n$), $S_0 = S((u_i)_{i=1,\ldots,n}; 1)$. Then the monoids $S_0, S_1, \ldots, S_n$ are comaximal.

The comaximal monoids are a constructive tool which allows in general to replace abstract local-global arguments by explicit computations. If $S_1, \ldots, S_n$ are comaximal monoids of the ring $R$, the product of all localisation $R_{S_i}$ is a faithfully flat $R$-algebra. Hence a lot of properties are true for $R$ if, and only if, they hold for each of the $R_{S_i}$.

In the next paragraph this will be illustrated on the example of Krull dimension.
Local character of Krull dimension

The following proposition is direct.

**Proposition 1.26** Let $R$ be a ring. Its Krull dimension is always greater or equal to any of its quotient or localisation. More precisely, any elementary idealistic chain which collapses in $R$ collapses in any quotient and localisation of $R$ and any elementary idealistic chain in a localisation of $R$ is equivalent to an elementary idealistic chain of $R$. Finally, if an elementary idealistic chain $(a_1, \ldots, a_\ell)$ of $R$ collapses in a localisation $R_S$, there exists $m$ in $S$ such that $(a_1, \ldots, a_\ell)$ collapses in $R[1/m]$.

**Proposition 1.27** Let $S_1, \ldots, S_n$ be comaximal monoids of the ring $R$ and $C$ be an idealistic chain of $R$. Then $C$ collapses in $R$ if, and only if, it collapses in each of the $R_{S_i}$. In particular the Krull dimension of $R$ is $\leq \ell$ if, and only if, the Krull dimension of each of the $R_{S_i}$ is $\leq \ell$.

**Proof.**

We have to show that an idealistic chain $C$ collapses in $R$ if it collapses in each of the $R_{S_i}$. To simplify let us take a chain of length 2: $((J_0, U_0), (J_1, U_1), (J_2, U_2))$ with ideals $J_k$ and monoids $U_k$. In each $R_{S_i}$ we have an equality

$$u_{0,i} u_{1,i} u_{2,i} + u_{0,i} u_{1,i} J_{2,i} + u_{0,i} j_{1,i} + j_{0,i} = 0$$

with $u_{k,i} \in U_k$ and $j_{k,i} \in J_k R_{S_i}$. This implies an equality in $R$ of the form

$$s_i u_{0,i} u_{1,i} u_{2,i} + u_{0,i} u_{1,i} J_{2,i} + u_{0,i} j_{1,i} + j_{0,i} = 0$$

with $s_i \in S_i$, $u_{k,i} \in U_k$ and $j_{k,i} \in J_k$. We take $u_k = \prod_i u_{k,i}$. By multiplying the previous equation by a suitable product we get an equality

$$s_i u_{0} u_{1} u_{2} + u_{0} u_{1} J_{2,i} + u_{0} j_{1,i} + j_{0,i} = 0$$

(E$_i$)

with $s_i \in S_i$, $u_k \in U_k$ and $j_{k,i} \in J_k$. We now write $\sum_i a_is_i = 1$, we multiply the each equality (E$_i$) by $a_i$ and we sum all these equalities. \qquad \square

**An application**

In classical mathematics the Krull dimension of a ring is the upper bound of the Krull dimension of the localisation in each maximal ideals. This follows easily (classically) from propositions 1.26 and 1.27.

Proposition 1.27 should have the same concrete consequences (that we can obtain non constructively by using the classical property above) even we don’t have access to the maximal ideals.

We will limit ourselves here to describe a simple example, where we do have access to the maximal ideals. Suppose that we have a simple constructive argument showing that the Krull dimension of $\mathbb{Z}[p][x_1, \ldots, x_\ell]$ is $\leq \ell + 1$ (p being an arbitrary prime number, and $\mathbb{Z}[p]$ the localisation of $\mathbb{Z}$ in $p\mathbb{Z}$). We can then deduce that the same holds for $R = \mathbb{Z}[x_1, \ldots, x_\ell]$ using the local-global principle above.

Indeed, consider a sequence $(a_1, \ldots, a_{\ell+2})$ in $R$. The collapsus of the elementary idealistic chain $(a_1, \ldots, a_{\ell+2})$ in $\mathbb{Z}(p)[x_1, \ldots, x_\ell]$ can be read as a collapsus in $\mathbb{Z}[1/m_0][x_1, \ldots, x_\ell]$ for some odd $m_0$. For each of the prime divisor $p_i$ of $m$ ($i = 1, \ldots, k$), the collapsus of the elementary idealistic chain $(a_1, \ldots, a_{\ell+2})$ in $\mathbb{Z}(p_i)$ can be read as a collapsus in $\mathbb{Z}[1/m_i][x_1, \ldots, x_\ell]$ for some $m_i$ relatively prime to $p_i$. The integers $m_i$ ($i = 0, \ldots, k$) generate the ideal $\langle 1 \rangle$, hence the monoids $M(m_i)$ are comaximal and we can apply proposition 1.27.
2 Distributive lattice, Entailment relations and Krull dimension

2.1 Distributive lattices, filters and spectrum

A distributive lattice is an ordered set with finite sups and infs, a minimum element (written 0) and a maximum element (written 1). The operations sup and inf are supposed to be distributive w.r.t. the other. We write these operations $\lor$ and $\land$. The relation $a \leq b$ can then be defined by $a \lor b = b$. The theory of distributive lattices is then purely equational. It makes sense then to talk of distributive lattices defined by generators and relations.

A quite important rule, the cut rule, is the following

$$(((x \land a) \leq b) \land (a \leq (x \lor b))) \implies a \leq b.$$  

In order to prove this, write $x \land a \land b = x \land a$ and $a = a \land (x \lor b)$ hence

$$a = (a \land x) \lor (a \land b) = (a \land x \land b) \lor (a \land b) = a \land b.$$  

A totally ordered set is a distributive lattice as soon as it has a maximum and a minimum element. We write $\mathbb{n}$ the totally ordered set with $n$ elements (this is a distributive lattice for $n \neq 0$.) A product of distributive lattices is a distributive lattice. Natural numbers with the divisibility relation form a distributive lattice (with minimum element 1 and maximum element 0). If $T$ and $T'$ are two distributive lattices, the set Hom$(T, T')$ of all morphisms (i.e., maps preserving sup, inf, 0 and 1) from $T$ to $T'$ has a natural order given by

$$\varphi \leq \psi \iff \forall x \in T \varphi(x) \leq \psi(x).$$  

A map between two totally ordered distributive lattices $T$ and $S$ is a morphism if, and only if, it is nondecreasing and $0_T$ and $1_T$ are mapped into $0_S$ and $1_S$.

The following proposition is direct.

Proposition 2.1 Let $T$ be a distributive lattice and $J$ a subset of $T$. We consider the distributive lattice $T'$ generated by $T$ and the relations $x = 0$ for $x \in J$ ($T'$ is a quotient of $T$). Then

- the equivalence class of 0 is the set of $a$ such that for some finite subset $J_0$ of $J$:
  $$a \leq \bigvee_{x \in J_0} x \text{ in } T$$

- the equivalence class of 1 is the set of $b$ such that for some finite subset $J_0$ of $J$:
  $$1 = \left( b \lor \bigvee_{x \in J_0} x \right) \text{ in } T$$

- More generally $a \leq_T b$ if, and only if, for some finite subset $J_0$ of $J$:
  $$a \leq \left( b \lor \bigvee_{x \in J_0} x \right)$$
In the previous proposition, the equivalence class of 0 is called an *ideal* of the lattice; it is the ideal generated by $J$. We write it $\langle J \rangle_T$. We can easily check that an ideal $I$ is a subset such that:

\[
\begin{align*}
0 & \in I \\
x, y \in I & \implies x \lor y \in I \\
x \in I, z \in T & \implies x \land z \in I
\end{align*}
\]

(the last condition can be written $(x \in I, y \leq x) \implies y \in I$).

Furthermore, for any morphism $\varphi : T_1 \to T_2$, $\varphi^{-1}(0)$ is an ideal of $T_1$.

A *principal ideal* is an ideal generated by one element $a$. We have $\langle a \rangle_T = \{ x \in T ; x \leq a \}$.

Any finitely generated ideal is principal.

The dual notion of ideal is the one of *filter*. A filter $F$ is the inverse image of 1 by a morphism. This is a subset such that:

\[
\begin{align*}
1 & \in F \\
x, y \in F & \implies x \land y \in F \\
x \in F, z \in T & \implies x \lor z \in F
\end{align*}
\]

**Notation 2.2** We write $P_f(X)$ the set of all finite subsets of the set $X$. If $A$ is a finite subset of a distributive lattice $T$

\[
\bigvee A := \bigvee_{x \in A} x \quad \text{and} \quad \bigwedge A := \bigwedge_{x \in A} x
\]

We write $A \vdash B$ or $A \vdash_T B$ the relation defined on the set $P_f(T)$:

\[
A \vdash B \iff A \leq \bigvee B
\]

Note the relation $A \vdash B$ is well defined on finite subsets because of associativity commutativity and idempotence of the operations $\land$ and $\lor$. Note also $\emptyset \vdash \{x\} \implies x = 1$ and $\{y\} \vdash \emptyset \implies y = 0$. This relation satisfies the following axioms, where we write $x$ for $\{x\}$ and $A, B$ for $A \cup B$.

\[
\begin{align*}
a \vdash a & \quad (R) \\
A \vdash B & \implies A, A' \vdash B, B' \quad (M) \\
(A, x \vdash B) & \& (A \vdash B, x) \implies A \vdash B \quad (T)
\end{align*}
\]

we say that the relation is reflexive, monotone and transitive. The last rule is also called emphcut rule. Let us also mention the two following rules of “distributivity”:

\[
\begin{align*}
(A, x \vdash B) & \& (A, y \vdash B) \iff A, x \lor y \vdash B \\
(A \vdash B, x) & \& (A \vdash B, y) \iff A \vdash B, x \land y
\end{align*}
\]

The following is a corollary of proposition 2.1.

**Proposition 2.3** Let $T$ be a distributive lattice and $(J, U)$ a couple of subsets of $T$. We consider the distributive lattice $T'$ generated by $T$ and by the relations $x = 0$ for $x \in J$ and $y = 1$ for $y \in U$ ($T'$ is a quotient of $T$). We have that:

- the equivalence class of 0 is the set of elements $a$ such that:

  \[
  \exists J_0 \in P_f(J), U_0 \in P_f(U) \quad a, U_0 \vdash_T J_0
  \]
• the equivalence class of 1 is the set of elements \( b \) such that: vérifiant:

\[ \exists J_0 \in P_1(J), U_0 \in P_1(U) \quad U_0 \vdash_T b, J_0 \]

• More generally \( a \leq_T b \) if, and only if, there exists a finite subset \( J_0 \) of \( J \) and a finite subset \( U_0 \) of \( U \) such that, in \( T \):

\[ a, U_0 \vdash_T b, J_0 \]

We shall write \( T/(J = 0, U = 1) \) the quotient lattice \( T' \) described in proposition 2.3. Let \( \psi : T \to T' \) be the canonical surjection. If \( I \) is the ideal \( \psi^{-1}(0) \) and \( F \) the filter \( \psi^{-1}(1) \), we say that the ideal \( I \) and the filter \( F \) are conjugate. By the previous proposition, an ideal \( I \) and a filter \( F \) are conjugate if, and only if, we have:

\[
[x \in T, I_0 \in P_1(I), F_0 \in P_1(F), (x, F_0 \vdash I_0)] \implies x \in I \quad \text{and} \quad [x \in T, I_0 \in P_1(I), F_0 \in P_1(F), (F_0 \vdash x, I_0)] \implies x \in F.
\]

This can also be formulated as follows:

\[
(f \in F, x \land f \in I) \implies x \in I \quad \text{and} \quad (j \in I, x \lor j \in F) \implies x \in F.
\]

When an ideal \( I \) and a filter \( F \) are conjugate, we have

\[
1 \in I \iff 0 \in F \iff (I, F) = (T, T).
\]

We shall also write \( T' = T/(J = 0, U = 1) \) as \( T/(I, F) \). By proposition 2.3, an homomorphism \( \varphi \) from \( T \) to another lattice \( T_1 \) satisfying \( \varphi(J) = \{0\} \) and \( \varphi(U) = \{1\} \) can be factorised in an unique way through the quotient \( T' \).

As shown by the example of totally ordered sets a quotient of distributive lattices is not in general characterised by the equivalence classes of 0 and 1.

Classically a prime ideal \( I \) of a lattice is an ideal whose complement \( F \) is a filter (which is then a prime filter). This can be expressed by

\[
1 \notin I \quad \text{and} \quad (x \land y) \in I \implies (x \in I \lor y \in I)
\]

which can also be expressed by saying that \( I \) is the kernel of a morphism from \( T \) into the lattice with two elements written \( 2 \). Constructively it seems natural to take the definition (\( \ast \)), where “or” is used constructively. The notion of prime filter is then defined in a dual way.

The spectrum of the lattice \( T \), written \( \text{Spec}(T) \) is defined as the set \( \text{Hom}(T, 2) \). It is isomorphic to the ordered set of all detachables prime ideals. The order relation is then reverse inclusion. We have \( \text{Spec}(2) \simeq 1, \text{Spec}(3) \simeq 2, \text{Spec}(4) \simeq 3 \), etc…

**Definition 2.4** Let \( T \) be a distributive lattice.

- An idealistic prime in \( T \) is given by a pair \((J, U)\) of subsets of \( T \). We consider this as an incomplete specification for a prime ideal \( P \) satisfying \( J \subseteq P \) and \( U \cap P = \emptyset \). It is finite iff \( J \) and \( U \) are finite, and trivial iff \( J = U = T \).

- An idealistic prime \((J, U)\) is saturated iff \( J \) is an ideal, \( U \) a filter and \( J \) and \( U \) are conjugate. Any idealistic prime generates a saturated idealistic prime \((I, F)\) as described in proposition 2.3.
We say that the idealistic prime \((J, U)\) collapses iff the saturated idealistic prime \((I, F)\) it generates is trivial. This means that the quotient lattice \(T' = T/(J = 0, U = 1)\) is a singleton i.e., \(1 \leq T' 0\), which means that there is a finite subset \(J_0\) of \(J\) and a finite subset \(U_0\) of \(U\) such that
\[
U_0 \vdash J_0.
\]

We have the following theorem, similar to theorem 1.8.

**Theorem 2.5** (Simultaneous collapse for idealistic primes) Let \((J, U)\) be an idealistic prime for a lattice \(T\) and \(x\) be an element of \(T\).

1. If the idealistic primes \((J \cup \{x\}, U)\) and \((J, U \cup \{x\})\) collapse, then so does \((J, U)\).
2. The idealistic prime \((J, U)\) generates a minimum saturated idealistic prime. We get it by adding in \(U\) (resp. \(J\)) any \(x \in A\) such that the idealistic prime \((J \cup \{x\}, U)\) (resp. \((J, U \cup \{x\})\)) collapses.

**Proof.**
Let us prove (1). We have two finite subsets \(J_0, J_1\) of \(J\) and two finite subsets \(U_0, U_1\) of \(U\) such that
\[
x, U_0 \vdash J_0 \quad \text{and} \quad U_1 \vdash x, J_1
\]
donc
\[
x, U_0, U_1 \vdash J_0, J_1 \quad \text{and} \quad U_0, U_1 \vdash x, J_0, J_1
\]
Hence by the cut rule
\[
U_0, U_1 \vdash J_0, J_1
\]
The point (2) has already been proved (in a slightly different formulation) in proposition 2.3. \(\square\)

Notice the crucial role of the cut rule.
We deduce the following proposition.

**Proposition 2.6** The compactness theorem implies the following result. If \((J, U)\) is an idealistic prime which does not collapse then there exists \(\varphi \in \text{Spec}(T)\) such that \(J \subseteq \varphi^{-1}(0)\) and \(U \subseteq \varphi^{-1}(1)\). In particular if \(a \not\leq b\), there exists \(\varphi \in \text{Spec}(T)\) such that \(\varphi(a) = 1\) and \(\varphi(b) = 0\). Also, if \(T \neq 1\), \(\text{Spec}(T)\) is non empty.

A corollary is the following representation theorem (Birkhoff theorem)

**Theorem 2.7** (Representation theorem) The compactness theorem implies the following result. The map \(\theta_T : T \to P(\text{Spec}(T))\) defined by \(a \mapsto \{\varphi \in \text{Spec}(T) ; \varphi(a) = 1\}\) is an injective map of distributive lattice. This means that any distributive lattice can be represented as a lattice of subsets of a set.

Another corollary is the following proposition.

**Proposition 2.8** The compactness theorem implies the following result. Let \(\varphi : T \to T'\) a map of distributive lattices; \(\varphi\) is injective if, and only if, \(\text{Spec}(\varphi) : \text{Spec}(T') \to \text{Spec}(T)\) is surjective.
Proof.
We have the equivalence

\[ a \neq b \iff a \land b \neq a \lor b \iff a \lor b \not\leq a \land b \]

Assume that \( \text{Spec}(\varphi) \) is surjective. If \( a \neq b \) in \( T \), take \( a' = \varphi(a) \), \( b' = \varphi(b) \) and let \( \psi \in \text{Spec}(T) \) be such that \( \psi(a \lor b) = 1 \) and \( \psi(a \land b) = 0 \). Since \( \text{Spec}(\varphi) \) is surjective there exists \( \psi' \in \text{Spec}(T') \) such that \( \psi = \psi' \varphi \) hence \( \psi'(a' \lor b') = 1 \) is \( \psi'(a' \land b') = 0 \), hence \( a' \lor b' \not\leq a' \land b' \) and \( a' \neq b' \).

Suppose that \( \varphi \) is injective. We identify \( T \) to a sublattice of \( T' \). If \( \psi \in \text{Spec}(T) \), take \( I = \psi^{-1}(0) \) and \( F = \psi^{-1}(1) \). Then \( (I, F) \) cannot collapse in \( T' \) since it would then collapse in dans \( T \). Hence there exists \( \psi' \in \text{Spec}(T') \) such that \( \psi'(I) = 0 \) and \( \psi'(F) = 1 \), which means \( \psi = \psi' \varphi \).

Of course, these three last results are hard to interpret in a computational way. An intuitive interpretation is that we can proceed “as if” any distributive lattice is a lattice of subsets of a set. The goal of Hilbert’s program is to give a precise meaning to this sentence, and explain what is meant by “as if” there.

2.2 Distributive lattices and entailment relations

An interesting way to analyse the description of distributive lattices defined by generators and relations is to consider the relation \( A \vdash B \) defined on the set \( P_f(T) \) of finite subsets of a lattice \( T \). Indeed if \( S \subseteq T \) generates the lattice \( T \), then the relation \( \vdash \) on \( P_f(S) \) is enough to characterise the lattice \( T \), because any formula on \( S \) can be rewritten, in normal conjunctive form (inf of sups in \( S \)) and normal disjunctive form (sup of infs in \( S \)). Hence if we want to compare two elements of the lattice generated by \( S \) we write the first in normal disjunctive form, the second in normal conjunctive form, and we notice that

\[
\bigvee_{i \in I} \left( \bigwedge A_i \right) \leq \bigwedge_{j \in J} \left( \bigvee B_j \right) \iff \&_{(i,j) \in I \times J} (A_i \vdash B_j)
\]

Definition 2.9 For an arbitrary set \( S \), a relation over \( P_f(S) \) which is reflexive, monotone and transitive (see page 16) is called an entailment relation.

The notion of entailment relations goes back to Gentzen sequent calculus, where the rule \( (T) \) (the cut rule) is first explicitly stated, and plays a key role. The connection with distributive lattices has been emphasized in [2, 3]. The following result (cf. [2]) is fundamental. It says that the three properties of entailment relations are exactly the ones needed in order to have a faithful interpretation in distributive lattices.

Theorem 2.10 (fundamental theorem of entailment relations) Let \( S \) be a set with an entailment relation \( \vdash_S \) over \( P_f(S) \). Let \( T \) be the lattice defined by generators and relations as follows: the generators are the elements of \( S \) and the relations are

\[ A \vdash_T B \]

whenever \( A \vdash_S B \). For any finite subsets \( A \) and \( B \) of \( S \) we have

\[ A \vdash_T B \iff A \vdash_S B. \]
Proof.
We give an explicit possible description of the lattice $T$. The elements of $T$ are represented by
finite sets of finite sets of elements of $S$

$$X = \{A_1, \ldots, A_n\}$$

(intuitively $X$ represents $\bigwedge A_1 \lor \cdots \lor \bigwedge A_n$). We define then inductively the relation $A \prec Y$ with $A \in P_1(S)$ and $Y \in T$ (intuitively $\bigwedge A \leq \bigvee_{C \in Y} (\bigwedge C)$)

- if $B \in Y$ and $B \subseteq A$ then $A \prec Y$
- if $A \vdash_S y_1, \ldots, y_m$ and $A, y_j \prec Y$ for $j = 1, \ldots, m$ then $A \prec Y$

It is easy to show that if $A \prec Y$ and $A \subseteq A'$ then we have also $A' \prec Y$. It follows that $A \prec Z$ holds whenever $A \prec Y$ and $B \prec Z$ for all $B \in Y$. We can then define $X \leq Y$ by $A \prec Y$ for all $A \in X$ and one can then check that $T$ is a distributive lattice\(^5\) for the operations

$$0 = \emptyset, \quad 1 = \{\emptyset\}, \quad X \lor Y = X \cup Y, \quad X \land Y = \{A \cup B \mid A \in X, B \in Y\}.$$ 

For establishing this one first show that if $C \prec X$ and $C \prec Y$ we have $C \prec X \land Y$ by induction on the proofs of $C \prec X$ and $C \prec Y$. We notice then that if $A \vdash_S y_1, \ldots, y_m$ and $A, y_j \vdash_S B$ for all $j$ then $A \vdash_T B$ using $m$ times the cut rule. It follows that if we have $A \vdash_T B$, i.e., $A \prec \{\{b\} \mid b \in B\}$, then we have also $A \vdash_T B$. \(\square\)

As a first application, we give the description of the Boolean algebra generated by a distributive lattice. A Boolean algebra can be seen as a distributive lattice with a complement operation $x \mapsto \overline{x}$ such that $x \land \overline{x} = 0$ and $x \lor \overline{x} = 1$. The application $x \mapsto \overline{x}$ is then a map from the lattice to its dual.

**Proposition 2.11** Let $T$ be a distributive lattice. There exists a free Boolean algebra generated by $T$. It can be described as the distributive lattice generated by the set $T_1 = T \cup \overline{T}$ (\(^6\)) with the entailment relation $\vdash_{T_1}$ defined as follows: if $A, B, A', B'$ are finite subsets of $T$ we have

$$A, \overline{B} \vdash_{T_1} A', \overline{B'} \iff A, B' \vdash A', B \quad \text{in } T$$

If we write $T_{\text{Bool}}$ this lattice (which is a Boolean algebra), there is a natural embedding of $T_1$ in $T_{\text{Bool}}$ and the entailment relation of $T_{\text{Bool}}$ induces on $T_1$ the relation $\vdash_{T_1}$.

**Proof.**
See [2]. \(\square\)

Notice that by theorem 2.10 we have $x \vdash_T y$ if, and only if, $x \vdash_{T_1} y$ hence the canonical map $T \to T_1$ is one-to-one and $T$ can be identified to a subset of $T_1$.

### 2.3 Krull dimension of distributive lattices

To develop a suitable constructive theory of the Krull dimension of a distributive lattice we have to find a constructive counterpart of the notion of increasing chains of prime ideals.

One can do it along the same lines as what has been done for commutative rings in section 1, or else use an idea due to Joyal. It consists in building an universal lattice $\text{Kr}_\ell(T)$ associated to $T$ such that the points of $\text{Spec}(\text{Kr}_\ell(T))$ are (in a natural way) the chains of prime ideals of length $\ell$. We shall present the two descriptions and establish their equivalence.

\(^5\) $T$ is actually the quotient of $P_1(P_1(S))$ by the equivalence relation: $X \leq Y$ and $Y \leq X$.
\(^6\) $\overline{T}$ is a disjoint copy of $T$. 
Partially specified chains of prime ideals

Definition 2.12 In a distributive lattice $T$

- A partial specification for a chain of prime ideals (that we shall call idealistic chain) is defined as follows. An idealistic chain of length $\ell$ is a list of $\ell + 1$ idealistic primes of $T$: $C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$. The idealistic chain is finite iff all the subsets are finite. An idealistic chain of length 0 is nothing but an idealistic prime.

- An idealistic chain is saturated if, and only if, all the $J_i$ and $U_i$ are conjugate, and if we have furthermore $J_i \subseteq J_{i+1}$, $U_{i+1} \subseteq U_i$ ($i = 0, \ldots, \ell$).

- An idealistic chain $C'$ is a refinement of the idealistic chain $C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$ if, and only if, $J_k \subseteq J'_k$, $U_k \subseteq U'_k$.

- An idealistic chain $C$ collapses if, and only if, the only saturated idealistic chain that refines $C$ is the trivial idealistic chain $(T, T, \ldots, (T, T))$.

Lemma 2.13 An idealistic chain $C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$ in which we have $U'_h \vdash J'_h$ with $U'_h \in P_t(U_h)$ and $J'_h \in P_t(J_h)$ (in particular if $U_h \cap J_h \neq \emptyset$) for some index $h$ collapses.

Proof. Let $((J_0, F_0), \ldots, (I_\ell, F_\ell))$ be a saturated idealistic chain which is a refinement of $C$. Since the idealistic prime $(I_h, F_h)$ collapses and since $I_h$ and $F_h$ are conjugate, we have $1 \in I_h$ and $0 \in F_h$. For all index $j > h$ we thus have $1 \in I_j$ and hence $0 \in F_j$. Similarly for all index $j < h$ we have $0 \in F_j$ and hence $1 \in I_j$. \qed

In the following theorem the points (3) and (2) correspond to the point (1) and (2) in theorem 1.10.

Theorem 2.14 (Simultaneous collapse for the idealistic chains in distributive lattices) Let $C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$ be an idealistic chain in a distributive lattice $T$.

(1) The idealistic chain $C$ collapses if, and only if, there exists $x_1, \ldots, x_\ell \in T$ and a finite idealistic chain $C' = ((J'_0, U'_0), \ldots, (J'_\ell, U'_\ell))$ of which $C$ is a refinement, with the following relations in $T$ (where $\vdash$ is the entailment relation of $T$):

$$
\begin{align*}
  x_1, U'_0 & \vdash J'_0 \\
  x_2, U'_1 & \vdash J'_1, x_1 \\
  & \vdots \\
  x_\ell, U'_{\ell-1} & \vdash J'_{\ell-1}, x_{\ell-1} \\
  U'_\ell & \vdash J'_\ell, x_\ell
\end{align*}
$$

(2) The idealistic chain $C$ generates a minimum saturated idealistic chain. We get it by adding to $U_i$ (resp. $J_i$) each element $a \in A$ such that the idealistic chain $((J_0, U_0), \ldots, (J_i \cup \{a\}, U_i), \ldots, (I_\ell, U_\ell))$ (resp. $((J_0, U_0), \ldots, (J_i, U_i \cup \{a\}), \ldots, (I_\ell, U_\ell))$) collapses.

(3) Take $x \in T$. Suppose that the idealistic chains $((J_0, U_0), \ldots, (J_i \cup \{x\}, U_i), \ldots, (I_\ell, U_\ell))$ and $((J_0, U_0), \ldots, (J_i, U_i \cup \{x\}), \ldots, (I_\ell, U_\ell))$ both collapse, then so does $C$.  

21
Proof.
Let us begin with the two first points. We can always suppose the idealistic chain $C$ to be finite, for one can always deduce the general case from this one by looking as the given idealistic chain as an inductive limit of all the finite idealistic chain of which it is a refinement. In the case where $C$ is finite, we can systematically replace $U_j'$ by $U_i$ and $J_i'$ par $J_i$. Let $C_1 = ((I_0, F_0), \ldots, (I_\ell, F_\ell))$ be the idealistic chain defined in (2). We shall show that

(α) If $C$ satisfies the relations (1) any saturated idealistic chain which refines $C$ is trivial (i.e., $C$ collapses).

(β) The idealistic chain $C_1$ is saturated.

(γ) Any saturated idealistic chain which refines $C$ also refines $C_1$.

(δ) If $C_1$ is trivial, $C$ satisfies the relations (1).

This will establish (1) and (2). (α) Let $((I_0', F_0'), \ldots, (I_\ell', F_\ell'))$ be a saturated idealistic chain which refines $C$. We consider the relations (1)

\[
x_1, U_0 \vdash J_0 \\
x_2, U_1 \vdash J_1, x_1 \\
\vdots \quad \vdots \\
x_\ell, U_{\ell-1} \vdash J_{\ell-1}, x_{\ell-1} \\
U_\ell \vdash J_\ell, x_\ell
\]

Since $I_0'$ and $F_0'$ are conjugate, the first of these relations gives $x_1 \in I_0'$. Hence $x_1 \in I_1'$, and the second relation gives $x_2 \in I_1'$. Going on in this way we get for the last relation $U_\ell \vdash J_\ell, x_\ell$. We have by hypothesis some $x_i$'s and $y_i$'s satisfying the following relations

\[
x_1, U_0 \vdash J_0 \\
x_2, U_1, x \vdash J_1, x_1 \\
x_3, U_2 \vdash J_2, x_2 \\
U_3 \vdash J_3, x_3 \\
y_1, U_0 \vdash J_0 \\
y_2, U_1, y \vdash J_1, y_1 \\
y_3, U_2 \vdash J_2, y_2 \\
U_3 \vdash J_3, y_3
\]

Using distributivity, we get

\[
(x_1 \lor y_1), U_0 \vdash J_0 \\
(x_2 \land y_2), U_1, (x \lor y) \vdash J_1, (x_1 \lor y_1) \\
(x_3 \land y_3), U_2 \vdash J_2, (x_2 \land y_2) \\
U_3 \vdash J_3, (x_3 \land y_3)
\]

Let us show now that the corresponding ideals and filters are conjugate, for instance that $I_1$ and $F_1$ are conjugate. We assume $x \land y \in I_1$, $y \in F_1$ and we show $x \in I_1$. We have by hypothesis some $x_i$'s and $y_i$'s satisfying the following relations

\[
x_1, U_0 \vdash J_0 \\
x_2, U_1, (x \land y) \vdash J_1, x_1 \\
x_3, U_2 \vdash J_2, x_2 \\
U_3 \vdash J_3, x_3 \\
y_1, U_0 \vdash J_0 \\
y_2, U_1 \vdash J_1, y_1, y \\
y_3, U_2 \vdash J_2, y_2 \\
U_3 \vdash J_3, y_3
\]
Using distributivity, we get

\[(x_1 \lor y_1), U_0 \vdash J_0\]
\[(x_2 \land y_2), U_1, x, y \vdash J_1, (x_1 \lor y_1)\]
\[(x_2 \land y_2), U_1, x \vdash J_1, (x_1 \lor y_1), y\]
\[(x_3 \land y_3), U_2 \vdash J_2, (x_2 \land y_2)\]
\[U_3 \vdash J_3, (x_3 \land y_3)\]

The relations n°2 and 3 give by cut

\[(x_2 \land y_2), U_1, x \vdash J_1, (x_1 \lor y_1)\]

The proof is finished.

(γ) We give the proof for \(\ell = 3\). Let \(((I'_0, F'_0), \ldots, (I'_3, F'_3))\) be a saturated idealistic chain which refines \(C\). Let us show \(I_1 \subseteq I'_1\). Take \(x \in I_1\), we have

\[x_1, U_0 \vdash J_0\]
\[x, x_2, U_1 \vdash J_1, x_1\]
\[x_3, U_2 \vdash J_2, x_2\]
\[U_3 \vdash J_3, x_3\]

We deduce from this successively \(x_1 \in I'_0 \subseteq I'_1\), \(x_3 \in F'_3 \subseteq F'_2\), \(x_2 \in F'_2 \subseteq F'_1\), and finally \(x \in I'_1\). Notice that the proof of the point (α) can be seen as a particular case of the proof of the point (γ).

(δ) is direct.

Finally we prove (3). We have \(x \in I_i\) and \(x \in F_i\), and hence \(C_i\) collapses (lemma 2.13). Hence \(C\) collapses.

\[\square\]

**Definition 2.15**

- Two idealistic chains that generate the same saturated idealistic chain are equivalent.
- An idealistic chain of finite type is one which is equivalent to a finite idealistic chain.
- An idealistic chain is strict if, and only if, we have \(V_i \cap I_{i+1} \neq \emptyset\) \((i = 0, \ldots, \ell - 1)\) in its generated saturated idealistic chain.
- A saturated idealistic chain \(C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))\) is frozen if, and only if, it does not collapse and if we have \(J_i \cup U_i = T\) for \(i = 0, \ldots, \ell\). An idealistic chain is frozen iff its saturation is. To give a strict frozen idealistic chain is the same as to give a strictly increasing chain of detachable prime ideals.

We think of an idealistic chain of length \(\ell\) as a partial specification of an increasing chains of prime ideals \(P_0, \ldots, P_\ell\) such that \(J_i \subseteq P_i\), \(U_i \cap P_i = \emptyset\), \((i = 0, \ldots, \ell)\).

From the simultaneous collapse theorem we deduce the following result which justifies this idea of partial specification.

**Theorem 2.16** (formal Nullstellensatz for chains of prime ideals) The compactness theorem implies the following result. Let \(T\) be a distributive lattice and \(((J_0, U_0), \ldots, (J_\ell, U_\ell))\) be an idealistic chain in \(T\). The following properties are equivalent:

(a) There exist \(\ell + 1\) prime ideals \(P_0 \subseteq \cdots \subseteq P_\ell\) such that \(J_i \subseteq P_i\), \(U_i \cap P_i = \emptyset\), \((i = 0, \ldots, \ell)\).

(b) The idealistic chain does not collapse.

The proof is the same as the proof of theorem 1.13.
Joyal’s Theory

The idea of Joyal is to introduce a lattice $K_T(T)$ associated to $T$ by an universal condition such that the points of $\text{Spec}(K_T(T))$ are (in a natural way) the chains of prime ideals of length $\ell$. To give such a chain is equivalent to give morphisms $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_\ell$ from $T$ to $2$. If we have a distributive lattice $K$ and $\ell + 1$ homomorphisms $\varphi_0 \geq \varphi_1 \geq \cdots \geq \varphi_\ell$ from $T$ to $K$ such that for all lattices $T'$ and all $\psi_0 \geq \psi_1 \geq \cdots \geq \psi_\ell \in \text{Hom}(T,T')$ we have a unique homomorphism $\eta : K \to T'$ such that $\eta \varphi_0 = \psi_0$, $\eta \varphi_1 = \psi_1$, \ldots, $\eta \varphi_\ell = \psi_\ell$, then the elements of $\text{Spec}(K)$ can be identified canonically with chains of prime ideals of length $\ell$ in $T$.

The advantage is that $K$ is an object that we can build effectively from $T$, in opposition to the chain of prime ideals or points in the spectrum.

The fact that such an universal object $K_T(T)$ always exists and is unique follows, constructively, from general abstract algebra arguments.

The explicit description of $K_T(T)$ is simplified by the notion of entailment relations ([2]). More precisely we have the following result.

\textbf{Theorem 2.17} Let $T$ be a distributive lattice. We consider the following universal problem, called here “Krull problem”: to find a distributive lattice $K$ and $\ell + 1$ homomorphisms $\varphi_0 \geq \varphi_1 \geq \cdots \geq \varphi_\ell$ from $T$ to $K$ such that, for any lattice $T'$ and any morphism $\psi_0 \geq \psi_1 \geq \cdots \geq \psi_\ell \in \text{Hom}(T,T')$ we have one and only one morphism $\eta : K \to T'$ such that $\eta \varphi_0 = \psi_0$, $\eta \varphi_1 = \psi_1$, \ldots, $\eta \varphi_\ell = \psi_\ell$. This universal problem admits a unique solution (up to isomorphism). We write $K_T(T)$ the corresponding distributive lattice. It can be described as the lattice generated by the disjoint union $S$ of $\ell + 1$ copies of $T$ $(\text{we shall write } \varphi \text{ the bijection between } T \text{ and the ith copy})$ with the entailment relation $\vdash_S$ defined as follows. If $U_i$ and $J_i$ $(i = 0, \ldots, \ell)$ are finite subsets of $T$ we have

$$\varphi_0(U_0), \ldots, \varphi_\ell(U_\ell) \vdash_S \varphi_0(J_0), \ldots, \varphi_\ell(J_\ell)$$

if, and only if, there exist $x_0, \ldots, x_\ell \in T$ such that (where $\vdash$ is the entailment relation of $T$):

\begin{align*}
x_0, U_0 & \vdash J_0 \\
x_1, U_1 & \vdash J_1, x_1 \\
& \vdots \\
x_\ell, U_\ell & \vdash J_{\ell-1}, x_{\ell-1} \\
U_\ell & \vdash J_\ell, x_\ell
\end{align*}

\textbf{Proof.}

First we show that the relation $\vdash_S$ on $P(T)$ described in the statement of the theorem is indeed an entailment relation. The only point that needs explanation is the cut rule. To simplify notations, we take $\ell = 3$. We have then 3 possible cases, and we analyse only one case, where $X, \varphi_1(z) \vdash_S Y$ and $X \vdash_S Y, \varphi_1(z)$, the other cases being similar. By hypothesis we have $x_0, x_1, x_2, y_1, y_2, y_3$ such that

\begin{align*}
x_0, U_0 & \vdash J_0 \\
x_2, U_1, z & \vdash J_1, x_1 \\
x_3, U_2 & \vdash J_2, x_2 \\
U_3 & \vdash J_3, x_3
\end{align*}

\begin{align*}
y_1, U_0 & \vdash J_0 \\
y_2, U_1 & \vdash J_1, y_1, z \\
y_3, U_2 & \vdash J_2, y_2 \\
U_3 & \vdash J_3, y_3
\end{align*}

The two entailment relations on the second line give

$$x_2, y_2, U_1, z \vdash J_1, x_1, y_1 \quad x_2, y_2, U_1 \vdash J_1, x_1, y_1, z$$

$$x_1, U_0 \vdash J_0 \quad y_1, U_0 \vdash J_0$$

24
hence by cut

\[ x_2, y_2, U_1 \vdash J_1, x_1, y_1 \]

i.e.,

\[ x_2 \land y_2, U_1 \vdash J_1, x_1 \lor y_1 \]

Finally, using distributivity

\[
\begin{align*}
(x_1 \lor y_1), U_0 & \vdash J_0 \\
(x_2 \land y_2), U_1 & \vdash J_1, (x_1 \lor y_1) \\
(x_3 \land y_3), U_2 & \vdash J_2, (x_2 \land y_2) \\
U_3 & \vdash J_3, (x_3 \land y_3)
\end{align*}
\]

and hence \( \varphi_0(U_0), \ldots, \varphi_3(U_3) \vdash_S \varphi_0(J_0), \ldots, \varphi_3(J_3) \).

It is left to show that the lattice \( \text{Kr}_\ell(T) \) defined from \( (S, \vdash_S) \) satisfied the desired universal condition. For this it is enough to notice that the entailment relation we have defined is clearly the least possible relation ensuring the \( \varphi_i \) to form an increasing chain. \( \square \)

Notice that the morphisms \( \varphi_i \) are injective: it is easily seen that for \( a, b \in T \) the relation \( \varphi_i(a) \vdash_S \varphi_i(b) \) implies \( a \vdash b \), and hence that \( \varphi_i(a) = \varphi_i(b) \) implies \( a = b \).

**Comparing the two approaches**

The analogy between the proofs of theorems 2.14 and 2.17 is striking. Actually these two theorems show together that an idealistic chain \( C = ((J_0, U_0), \ldots, (J_\ell, U_\ell)) \) collapses in \( T \) if, and only if, the idealistic prime \( P = (\varphi_0(J_0), \ldots, \varphi_\ell(J_\ell); \varphi_0(U_0), \ldots, \varphi_\ell(U_\ell)) \) collapses in \( \text{Kr}_\ell(T) \).

This is not a coincidence: given the universal property that defines \( \text{Kr}_\ell(T) \) to give a detachable prime ideal of \( \text{Kr}_\ell(T) \) is the same as to give an increasing chain of detachable prime ideals of \( T \) (of length \( \ell \)). One could then think that one of the two proofs is superfluous.

Classically, one could organize things as follows. One would define first a priori the collapsus of an idealistic prime (resp. an idealistic chain) as meaning that it is impossible to refine this idealistic prime in a prime ideal (resp. to refine this idealistic chain in an increasing chain of prime ideals). The simultaneous collapsus theorems (theorems 2.5 (1) and 2.14 (3)) are direct with such definitions. Furthermore, the algebraic characterisation of the collapsus of an idealistic prime \( (J, U) \) (i.e., \( U_0 \vdash J_0 \) for some finite subsets \( U_0 \subseteq U \) and \( J_0 \subseteq J \)) are also easily established. The description of \( \text{Kr}_\ell(T) \) given in theorem 2.17 implies then (taking into account the algebraic characterisation of the collapsus of an idealistic prime) the algebraic characterisation of the collapsus of an idealistic chain, i.e., the point (1) of theorem 2.14.

Constructively, we have defined the collapsus of an idealistic prime (resp. of an idealistic chain) as meaning the impossibility of a refinement of this idealistic prime into a saturated non trivial idealistic prime \(^7\) (resp. of this idealistic chain in a saturated non trivial idealistic chain). To deduce the algebraic characterisation of the collapsus of an idealistic chain from the algebraic characterisation of the collapsus of an idealistic prime and of the description of \( \text{Kr}_\ell(T) \) (which would avoid the “superfluous proof”) it is enough to explain how to derive from a saturated idealistic chain \( ((I_0, F_0), \ldots, (I_\ell, F_\ell)) \) an increasing chain of morphisms \( (\psi_0, \ldots, \psi_\ell) \) from \( T \) in a distributive lattice with \( \psi_k^{-1}(0) = I_k \) and \( \psi_k^{-1}(1) = F_k \) (\( k = 0, \ldots, \ell \)). For this it is enough to apply the following lemma.

\(^7\) More precisely the double negation (…impossibility … non trivial) has to be taken, of course, in the form of an explicit affirmation.
Lemma 2.18 Let $C = ((I_0,F_0),\ldots,(I_\ell,F_\ell))$ be a saturated idealistic chain in a distributive lattice $T$. Let $T_C$ be the quotient distributive lattice of $Kr_\ell(T)$ by $\varphi_0(I_0) = \cdots = \varphi_\ell(I_\ell) = 0$, $\varphi_0(F_0) = \cdots = \varphi_\ell(F_\ell) = 1$. Let $\pi$ be the canonical projection from $Kr_\ell(T)$ onto $T_C$ and $\psi_k = \pi \circ \varphi_k$. Then $\psi_k^{-1}(0) = I_k$ and $\psi_k^{-1}(1) = F_k$ ($k = 0,\ldots,\ell$).

Proof.
For instance $\psi_k^{-1}(0) = \{x \in T; \varphi_k(x) =_{T_C} 0\}$ is equal to, by proposition 2.3
\[
\{x \in T ; \varphi_k(x),\varphi_0(F_0),\ldots,\varphi_\ell(F_\ell) \vdash_{Kr_\ell(T)} \varphi_0(I_0),\ldots,\varphi_\ell(I_\ell)\}
\]
i.e., the set $x$ such that there exist $x_1,\ldots,x_\ell$ such that (where $\vdash$ is the entailment relation of $T$)
\[
\begin{align*}
x_1, F_0 & \vdash I_0 \\
x_2, F_1 & \vdash I_1, x_1 \\
& \vdots \\
x, x_{k+1}, F_k & \vdash I_k, x_k \\
& \vdots \\
x_\ell, F_{\ell-1} & \vdash I_{\ell-1}, x_{\ell-1} \\
F_\ell & \vdash I_\ell, x_\ell
\end{align*}
\]
Since the idealistic chain $C$ is saturated one has successively $x_1 \in I_1 \subseteq I_2$, $x_2 \in I_2, \ldots x_k \in I_k$, and $x_\ell \in F_\ell$, ..., $x_{k+1} \in F_{k+1} \subseteq F_k$, hence $x \in I_k$. \hfill $\blacksquare$

Constructive definition of the Krull dimension of a distributive lattice

Since an idealistic chain $C = ((J_0,U_0),\ldots,(J_\ell,U_\ell))$ collapses in $T$ if, and only if, the idealistic prime $P = (\varphi_0(J_0),\ldots,\varphi_\ell(J_\ell);\varphi_0(U_0),\ldots,\varphi_\ell(U_\ell))$ collapses in $Kr_\ell(T)$, the two variations in the definition below of the dimension of a distributive lattice are equivalent.

Definition 2.19

1) An elementary idealistic chain in a distributive lattice $T$ is an idealistic chain of the form
\[
((0,x_1),(x_1,x_2),\ldots,(x_\ell,1))
\]
(with $x_i$ in $T$).

2) A distributive lattice $T$ is of dimension $\leq \ell - 1$ iff it satisfies one of the equivalent conditions
- Any elementary idealistic chain of length $\ell$ collapses.
- For any sequence $x_1,\ldots,x_\ell \in T$ we have
\[
\varphi_0(x_1),\ldots,\varphi_{\ell-1}(x_\ell) \vdash \varphi_1(x_1),\ldots,\varphi_{\ell}(x_\ell)
\]
in $Kr_\ell(T)$.

The condition in (2) is that: $\forall x_1,\ldots,x_\ell \in T \exists a_1,\ldots,a_\ell \in T$ such that
\[
\begin{align*}
a_1, x_1 & \vdash 0 \\
a_2, x_2 & \vdash a_1, x_1 \\
& \vdots \\
a_\ell, x_\ell & \vdash a_{\ell-1}, x_{\ell-1} \\
1 & \vdash a_\ell, x_\ell
\end{align*}
\]
In particular the distributive lattice $T$ is of dimension $\leq -1$ if, and only if, $1 = 0$ in $T$, and it is of dimension $\leq 0$ if, and only if, $T$ is a Boolean algebra (any element has a complement).

We shall not give for distributive lattices neither the definition of $\dim(T) < \ell$ nor the one of $\dim(T) \geq \ell$ and we limit ourselves to mention that $\dim(T) > \ell$ means that $\dim(T) \leq \ell$ is impossible. One could refine as in section 1.3 these definitions when one has a primitive inequality relation in $T$.

The second variant in the definition is useful for deriving easily the simpler following characterisation.

**Lemma 2.20** A distributive lattice $T$ generated by a set $S$ is of dimension $\leq \ell - 1$ if, and only if, for any sequence $x_1, \ldots, x_\ell \in S$

$$\varphi_0(x_1), \ldots, \varphi_{\ell-1}(x_\ell) \vdash \varphi_1(x_1), \ldots, \varphi_\ell(x_\ell)$$

in $\text{Kr}_\ell(T)$.

Indeed using distributivity, one can deduce

$$a \lor a', A \vdash b \lor b', B$$

from $a, A \vdash b, B$ and $a', A \vdash b', B$. Furthermore any element of $T$ is an inf of sups of elements of $S$.

Notice the analogy between the formulation of this condition and the definition of pseudo regular sequence 1.14.

**Connections with Joyal’s definition**

Let $T$ be a distributive lattice, Joyal [4] gives the following definition of $\dim(T) \leq \ell - 1$. Let

$$\varphi^\ell_1 : T \to \text{Kr}_\ell(T)$$

be the $\ell + 1$ universal morphisms. By universality of $\text{Kr}_{\ell+1}(T)$, we have $\ell + 1$ morphisms $\sigma_i : \text{Kr}_{\ell+1}(T) \to \text{Kr}_\ell(T)$ such that $\sigma_i \circ \varphi^{\ell+1}_j = \varphi^\ell_j$ if $j < i$ and $\sigma_i \circ \varphi^{\ell+1}_j = \varphi^\ell_{j-1}$ if $j > i$. Joyal defines then $\dim(T) \leq \ell$ to mean that $(\sigma_0, \ldots, \sigma_\ell) : \text{Kr}_{\ell+1}(T) \to \text{Kr}_\ell(T)^{\ell+1}$ is injective. This definition can be motivated by proposition 2.8: the elements in the image of de $\text{Sp}(\sigma_i)$ are the chains of prime ideals $(a_0, \ldots, a_\ell)$ with $a_i = a_{i+1}$, and $\text{Sp}(\sigma_0, \ldots, \sigma_\ell)$ is surjective if, and only if, for any chain $(a_0, \ldots, a_\ell)$ there exists $i < \ell$ such that $a_i = a_{i+1}$. This means exactly that there is no non trivial chain of prime ideals of length $l + 1$. Using compactness theorem, one can then see the equivalence with definition 2.19. One could check directly this equivalence using a constructive metalanguage, but for lack of space, we shall not present here this argument. Similarly, it would be possible to establish the equivalence of our definition with the one of Espanól [4] (here also, this connection is clear via compactness theorem).

3 Zariski and Krull lattice associated to a commutative ring

**Zariski lattice**

Given a commutative ring $R$ the Zariski lattice $\text{Zar}(R)$ has for elements the radical ideals (the order relation being inclusion). It is well defined as a lattice. Indeed $\sqrt{I_1} = \sqrt{J_1}$ and $\sqrt{I_2} = \sqrt{J_2}$ imply $\sqrt{I_1I_2} = \sqrt{J_1J_2}$ (which defines $\sqrt{I_1} \land \sqrt{I_2}$) and $\sqrt{I_1 + I_2} = \sqrt{J_1 + J_2}$ (which defines $\sqrt{I_1} \lor \sqrt{I_2}$). The Zariski lattice of $R$ is always distributive, but may not be decidable. Nevertheless an inclusion $\sqrt{I_1} \subseteq \sqrt{I_2}$ can always be certified in a finite way if the ring $R$ is
discrete. This lattice contains all the informations necessary for a constructive development of the abstract theory of the Zariski spectrum.

We shall write $\overline{a}$ for $\sqrt{(a)}$. Given a subset $S$ of $A$ we write $\overline{S}$ the subset of $\text{Zar}(R)$ the elements of which are $\overline{s}$ for $s \in S$. We have $\overline{a_1} \vee \cdots \vee \overline{a_m} = \sqrt{(a_1, \ldots, a_m)}$ and $\overline{a_1} \wedge \cdots \wedge \overline{a_m} = a_1 \cdots a_m$

Let $U$ and $J$ be two finite subsets of $R$, we have

$$\overline{U} \vdash_{\text{Zar}(R)} \overline{J} \iff \prod_{u \in \overline{U}} u \in \sqrt{(\overline{J})} \iff \mathcal{M}(U) \cap \langle I \rangle \neq \emptyset$$

i.e.,

$$(J, U) \text{ collapses in } R \iff (\overline{J}, \overline{U}) \text{ collapses in } \text{Zar}(R)$$

This describes completely the lattice $\text{Zar}(R)$. More precisely we have:

**Proposition 3.1** The lattice $\text{Zar}(R)$ of a commutative ring $R$ is (up to isomorphsim) the lattice generated by $(R, \vdash)$ where $\vdash$ is the least entailment relation over $R$ such that

$$0_A \vdash x, y \vdash xy$$

$$\vdash 1_A, xy \vdash x, x + y \vdash x, y$$

**Proof.**

It is clear that the relation $U \vdash J$ defined by “$\mathcal{M}(U)$ meets $\langle J \rangle$” satisfies these axioms. It is also clear that the entailment relation generated by these axioms contains this relation. Let us show that this relation is an entailment relation. Only the cut rule is not obvious. Assume that $\mathcal{M}(U, a)$ meets $\langle J \rangle$ and that $\mathcal{M}(U)$ meets $\langle J, a \rangle$. There exist then $m_1, m_2 \in \mathcal{M}(U)$ and $k, x$ such that $a^k m_1 \in \langle J \rangle$, $m_2 + ax \in \langle J \rangle$. Eliminating $a$ this implies that $\mathcal{M}(U)$ intersects $\langle J \rangle$. \qed

We have $\overline{a} = \overline{b}$ if, and only if, $a$ divides a power of $b$ and $b$ divides a power of $a$.

**Proposition 3.2** In a commutative ring $R$ to give an ideal of the lattice $\text{Zar}(R)$ is the same as to give a radical ideal of $R$. If $I$ is a radical ideal of $R$ one associates the ideal

$$\mathcal{I} = \{ J \in \text{Zar}(R) \mid J \subseteq I \}$$

of $\text{Zar}(R)$. Conversely if $\mathcal{I}$ is an ideal of $\text{Zar}(R)$ one can associate the ideal

$$I = \bigcup_{J \in \mathcal{I}} J = \{ x \in A \mid \overline{x} \in \mathcal{I} \},$$

which is a radical ideal of $R$. In this bijection the prime ideals of the ring correspond to the prime ideals of the Zariski lattice.

**Proof.**

We only prove the last assertion. If $I$ is a prime ideal of $R$, if $J, J' \in \text{Zar}(R)$ and $J \wedge J' \in \mathcal{I}$, let $a_1, \ldots, a_n \in R$ be some “generators” of $J$ (i.e., $J = \sqrt{(a_1, \ldots, a_n)}$) and let $b_1, \ldots, b_m \in A$ be some generators of $J'$. We have then $a_i b_j \in I$ and hence $a_i \in I$ or $b_j \in I$ for all $i, j$. It follows from this (constructively) that we have $a_i \in I$ for all $i$ or $b_j \in I$ for all $j$. Hence $J \in \mathcal{I}$ or $J' \in \mathcal{I}$ and $\mathcal{I}$ is a prime ideal of $\text{Zar}(R)$.

Conversely if $\mathcal{I}$ is a prime ideal of $\text{Zar}(R)$ and if we have $\overline{x} \overline{y} \in \mathcal{I}$ then $\overline{x} \wedge \overline{y} \in \mathcal{I}$ and hence $\overline{x} \in \mathcal{I}$ or $\overline{y} \in \mathcal{I}$. This shows that $\{ x \in A \mid \overline{x} \in \mathcal{I} \}$ is a prime ideal of $R$. \qed

**Definition 3.3** We define $\text{Kru}_\ell(R) := \text{Kr}_\ell(\text{Zar}(R))$. This is called the Krull lattice of order $\ell$ of the ring $R$. 28
Theorem 3.4 Let $C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$ be an idealistic chain in a commutative ring $R$. It collapses if, and only if, the idealistic chain $((\widetilde{J}_0, \widetilde{U}_0), \ldots, (\widetilde{J}_\ell, \widetilde{U}_\ell))$ collapses in $\operatorname{Zar}(R)$. For instance if $C$ is finite, the following properties are equivalent:

1. there exist $j_i \in \langle J_i \rangle$, $u_i \in \mathcal{M}(U_i)$, $(i = 0, \ldots, \ell)$, such that
   $$u_0 \cdot (u_1 \cdot (\cdots (u_\ell + j_\ell) + \cdots) + j_1) + j_0 = 0$$

2. there exist $x_1, \ldots, x_\ell \in \operatorname{Zar}(R)$ such that in $\operatorname{Zar}(R)$:
   $$\begin{align*}
x_1, \widetilde{U}_0 &\vdash \widetilde{J}_0 \\
x_2, \widetilde{U}_1 &\vdash \widetilde{J}_1, x_1 \\
\vdots &\vdots \\
x_\ell, \widetilde{U}_{\ell-1} &\vdash \widetilde{J}_{\ell-1}, x_{\ell-1} \\
\widetilde{U}_\ell &\vdash \widetilde{J}_\ell, x_\ell
\end{align*}$$

3. same thing but with $x_1, \ldots, x_\ell \in \widetilde{A}$

Proof.
It is clear that 1 entails 3: simply take
$$v_\ell = u_\ell + j_\ell, \ v_{\ell-1} = v_\ell u_{\ell-1} + j_{\ell-1}, \ldots, \ v_0 = v_1 u_0 + j_0 \quad \text{and} \quad x_i = \widetilde{v}_i$$
and that 3 entails 2. The fact that 1 follows from 2 can be seen by reformulating 2 in the following way. We consider the idealistic chain $C_1 = ((K_0, V_0), \ldots, (K_\ell, V_\ell))$ obtained by saturating the idealistic chain $C$. We define the $\ell + 1$ radical ideals $I_0, \ldots, I_\ell$ of $R$

- $I_0 = \{x \in A \mid \mathcal{M}(x, U_0) \cap \langle J_0 \rangle \neq \emptyset\}$
- $I_1 = \{x \in A \mid \mathcal{M}(x, U_1) \cap (\langle J_1 \rangle + I_0) \neq \emptyset\}$
- $\vdots$
- $I_{\ell-1} = \{x \in A \mid \mathcal{M}(x, U_{\ell-1}) \cap (\langle J_{\ell-1} \rangle + I_{\ell-2}) \neq \emptyset\}$
- $I_\ell = \langle J_\ell \rangle + I_{\ell-1}$

It is clear that $I_i \subseteq K_i$ $(i = 0, \ldots, \ell)$. In the correspondance given in 3.2 these ideals correspond to the following ideals of $\operatorname{Zar}(R)$

- $\mathcal{I}_0 = \{u \in \operatorname{Zar}(R) \mid u, \widetilde{U}_0 \vdash \widetilde{J}_0\}$
- $\mathcal{I}_1 = \{u \in \operatorname{Zar}(R) \mid (\exists v \in \mathcal{I}_0) u, \widetilde{U}_1 \vdash \widetilde{J}_1, v\}$
- $\vdots$
- $\mathcal{I}_{\ell-1} = \{u \in \operatorname{Zar}(R) \mid (\exists v \in \mathcal{I}_0) u, \widetilde{U}_{\ell-1} \vdash \widetilde{J}_{\ell-1}, v\}$
The condition 2 becomes then \( \tilde{U}_\ell \vdash \tilde{J}_i, v \) for some \( v \in \mathcal{I}_{\ell-1} \). This means that \( \mathcal{M}(U) \) intersects \( I_\ell \), or \( I_\ell \subseteq K_\ell \). Hence \( C_1 \) collapses, and hence \( C \) collapses.

Let us give another direct proof that (2) implies (3). We rewrite the entailment relations of (2) as follows. Each \( \tilde{U}_i \) can be replaced by a \( \tilde{u}_i \) with \( u_i \in R \), each \( \tilde{J}_i \) can be replaced by a radical of finitely generated ideal \( I_i \) of \( R \), and we write \( L_i \) instead of \( x_i \) to indicate that this is a radical of a finitely generated ideal. We get:

\[
L_1, \tilde{u}_0 \vdash I_0 \\
L_2, \tilde{u}_1 \vdash I_1, L_1 \\
L_3, \tilde{u}_2 \vdash I_2, L_2 \\
\tilde{u}_3 \vdash I_3, L_3
\]

The last line means that \( \mathcal{M}(u_3) \) intersects \( I_3 + L_3 \) and hence \( I_3 + \langle y_3 \rangle \) for some element \( y_3 \) of \( L_3 \). Hence we have \( \tilde{u}_3 \vdash I_3, \tilde{y}_3 \). Since \( \tilde{y}_3 \leq L_3 \) in \( \text{Zar}(R) \) we have \( \tilde{y}_3, \tilde{u}_2 \vdash I_2, L_2 \). We have then replaced \( L_3 \) by \( \tilde{y}_3 \). Reasoning as previously one sees that one can replace as well \( L_2 \) by a suitable \( \tilde{y}_2 \), and then \( L_1 \) by a suitable \( \tilde{y}_1 \). One gets then (3). \( \square \)

**Corollary 3.5** The Krull dimension of a commutative ring \( R \) is \( \leq \ell \) if, and only if, the Krull dimension of its Zariski lattice \( \text{Zar}(R) \) is \( \leq \ell \).

**Proof.**
By the previous theorem and lemma 2.20. \( \square \)

This would be a natural place to relate decidability properties of \( R \) and of \( \text{Kr}_n(\text{Zar}(R)) \). For instance, it can be shown that if \( R \) is coherent, noetherian and strongly discrete then each of the \( \text{Kr}_n(\text{Zar}(R)) \) is decidable. Due to lack of space, we shall not present these results here.

## 4 Going Up and Going Down

### 4.1 Relative Krull dimension

**General remarks about relative Krull dimension**

We shall develop here a constructive counterpart of the notion of increasing chain of prime ideals which all lie over the same prime ideal of a given subring. This paragraph can apply as well to the case of an arbitrary distributive lattice (here it is \( \text{Zar}(R) \)) with evident modifications. There is no real computations going on, just some simple combinatorics.

**Definition 4.1** Let \( R \subseteq S \) be two commutative rings and \( C = ((J_0, U_0), \ldots, (J_\ell, U_\ell)) \) an idealistic chain in \( S \).

- The idealistic chain \( C \) collapses above \( R \) if, and only if, there exist \( a_1, \ldots, a_k \in R \) such that for all couple of complementary subsets \( (H, H') \) of \( \{1, \ldots, k\} \), the idealistic chain
  \[
  (\{a_h\}_{h \in H} \cup J_0, U_0), (J_1, U_1) \ldots, (J_\ell, U_\ell \cup \{a_h\}_{h \in H'})
  \]
  collapses.

- The (relative) Krull dimension of the extension \( S/R \) is \( \leq \ell - 1 \) if, and only if, any elementary idealistic chain \( ((0, x_1), (x_1, x_2), \ldots, (x_\ell, 1)) \) collapses above \( R \).
• The (relative) Krull dimension of the extension $S/R$ is $\geq \ell$ if, and only if, there exist $x_0, \ldots, x_\ell$ in $S$ such that the elementary idealistic chain $((0, x_1), (x_1, x_2), \ldots, (x_\ell, 1))$ does not collapse\(^8\) above $R$.

• The (relative) Krull dimension of the extension $S/R$ is $< \ell$ if, and only if, it is impossible that it is $\geq \ell$.

• The (relative) Krull dimension of the extension $S/R$ is $> \ell$ if, and only if, it is impossible that it is $\leq \ell$.

One can consider a more general case of a ring extension: a map $R \to S$ non necessarily injective. It is possible to adapt the previous definition by replacing $R$ by its image in $S$.

One has a relative simultaneous collapse theorem.

**Theorem 4.2** (Relative simultaneous collapse for the idealistic chains) Let $R \subseteq S$ be two commutative rings and $C$ an idealistic chain of length $\ell$ in $S$.

1. Take $x \in S$ and $i \in \{0, \ldots, \ell\}$. Suppose that the idealistic chains $C$ & $\{x \in C(i)\}$ and $C \& \{x \notin C(i)\}$ both collapse above $R$, then so does $C$.

2. Take $x \in R$. Suppose that the idealistic chains $C$ & $\{x \in C(0)\}$ and $C \& \{x \notin C(0)\}$ both collapse above $R$, then so does $C$.

This is an easy consequence of the (non relative) theorem 1.10, that is left to the reader. From this, we deduce (classically) a characterisation of the idealistic chains which collapse relatively.

**Theorem 4.3** (Formal Nullstellensatz for the chains of prime ideals in a ring extension) The compactness theorem implies the following result. Let $R \subseteq S$ be commutative rings and $C = ((J_0, U_0), \ldots, (J_\ell, U_\ell))$ an idealistic chain in $S$. The following properties are equivalent:

(a) There exists a detachable prime ideal $P$ of $R$ and $\ell + 1$ detachable prime ideals $P_0 \subseteq \cdots \subseteq P_\ell$ of $S$ such that $J_i \subseteq P_i$, $U_i \cap P_i = \emptyset$ and $P_i \cap A = P$ $(i = 0, \ldots, \ell)$.

(b) The idealistic chain $C$ does not collapse above $S$.

**Proof.**

We have clearly (a) $\Rightarrow$ (b). For proving (b) $\Rightarrow$ (a) we do the (easier) proof which relies on excluded middle principle and Zorn’s lemma. We consider a maximal idealistic chain $C_1 = ((P_0, S_0), \ldots, (P_\ell, S_\ell))$ (for the extension relation) among all the idealistic chains which refines $C$ and that do not collapse above $R$. Given the relative simultaneous collapse theorem, the same proof that in theorem 1.13 shows that it is an increasing chain of prime ideals (with their complements). It is left to show that all $P_i \cap A$ are equals, which is equivalent to $S_0 \cap P_\ell \cap A = \emptyset$. If this was not so we would have $x \in S_0 \cap P_\ell \cap A$. Then $((P_0 \cup \{x\}, S_0), \ldots, (P_\ell, S_\ell))$ and $((P_0, S_0), \ldots, (P_\ell, S_\ell \cup \{x\}))$ collapses (absolutely) and hence $C_1$ collapses above $A$ (with the finite subset $\{x\}$). This is absurd. □

Constructively we have the following result. We omit the proof for reason of space.

---

\(^8\) More precisely, constructively, we have to say: for any $k$ and any $a_1, \ldots, a_k \in R$ there exist a pair of complementary subsets $(H, H')$ of $\{1, \ldots, k\}$, such that the idealistic chain $((\{a_h\}_{h \in H}); x_1), (x_1, x_2), \ldots, (x_\ell; \{a_h\}_{h \in H'})$ “does not collapse” with the meaning of the inequality relation defined over $R$ (cf. the explanation in the beginning of section 1.3 page 10).
Theorem 4.4 Let $R \subseteq S$ be commutative rings.

(1) Suppose that the Krull dimension of $R$ is $\leq m$ and that the relative Krull dimension of the extension $S/R$ is $\leq n$, then the Krull dimension of $B$ is $\leq (m + 1)(n + 1) - 1$.

(2) Suppose that $R$ and $S$ have an inequality $\neq 0$ defined as the negation of $= 0$. Suppose that the Krull dimension of the extension $S/R$ is $\leq n$ and that the collapse of elementary idealistic chains in $R$ is decidable. Given a pseudo regular sequence of length $(m+1)(n+1)$ in $B$, one can build a pseudo regular sequence of length $m+1$ in $R$.

Case of integral extensions

In the following proposition (1) is the constructive version of the “incompatibility theorem” (theorem 13.33 in the book of Sharp citeSha).

Proposition 4.5 Let $R \subseteq S$ be commutative rings.

(1) If $S$ is integral over $R$ the relative Krull dimension of the extension $S/R$ is 0.

(2) More generally we have the same result if any element of $S$ is a zero of a polynomial in $R[X]$ which has a coefficient equal to 1. For instance if $R$ is a Prüfer domain, this applies to any overring of $R$ in its quotient field.

(3) In particular, using theorem 4.4 if $\dim(R) \leq n$ then $\dim(S) \leq n$.

Proof.
We show (2). We have to show that for any $x \in S$ the idealistic chain $((0,x),(x,1))$ collapses above $R$. The finite list in $R$ is the one given by the coefficients of the polynomial of which $x$ is a zero. Suppose that $x^k = \sum_{i \neq k, i \leq r} a_i x^i$. Let $G, G'$ be two complementary subsets of $\{a_i; i \neq k\}$. The collapse of $((G,x),(x,G'))$ is of the form $x^m(g'bx) = g$ with $g \in (G)_S, g' \in \mathcal{M}(G')$, $b \in S$. Actually we take $g \in G[x]$ and $b \in R[x]$. If $G'$ is empty we take $m = k$, $g' = 1$. Otherwise let $h$ be the smallest index $\ell$ such that $a_\ell \in G'$. All $a_j$ with $j < h$ are in $G$. If $h < k$ we take $m = h$, $g' = a_h$. If $h > k$ we take $m = k$, $g' = 1$.

NB: notice that the disjunction has only $r$ cases and not $2^r$:

- $a_0 \in G'$, or
- $a_0 \in G, a_1 \in G'$, or
- $a_0, a_1 \in G, a_2 \in G'$, or
- $\vdots$

Relative Krull dimension with polynomial rings

We give a constructive version of the classical theorem on the relative Krull dimension of an extension $A[x_1, \ldots, x_n]/A$.

We shall need the following elementary lemma from linear algebra.

Lemma 4.6 Let $V_1, \ldots, V_{n+1}$ be vectors in $R^n$. 32
• If $R$ is a discrete field, there exists an index $k \in \{1, \ldots, n+1\}$ such that $V_k$ is a linear combination of the following vectors (if $k = n+1$ this means $V_{n+1} = 0$).

• If $R$ is a commutative ring, write $V$ the matrix the columns of which are the $V_i$. Let $\mu_1, \ldots, \mu_\ell$ (with $\ell = 2^n - 1$) be the list of all minors of $V$ extracted on the $n$ or $n-1$ or ... or 1 last columns, and ranked by decreasing size. Take $\mu_{\ell+1} = 1$ (the corresponding minor for the empty extracted matrix). For each $k \in \{1, \ldots, \ell+1\}$ we take $I_k = \langle (\mu_i)_{i<k} \rangle$ and $S_k = S/(I_k; \mu_k)$. If the minor $\mu_k$ is of order $j$, the vector $V_{n+1-j}$ is, in the ring $(R/I_k)S_k$, equal to a linear combination of the following vectors.

**Proof.**
For the second point, one uses Cramer formulas.

**Proposition 4.7** Let $S = R[X_1, \ldots, X_n]$ be a polynomial ring. The relative Krull dimension of the extension $S/R$ is equal to $n$. Hence if the Krull dimension of $R$ if $\leq r$ the one of $S$ is $\leq r+n$. Furthermore if the Krull dimension of $S$ is $\leq r+n$ the one of $R$ is $\leq r$.

**Proof.**
The last assertion follows from the fact that if the sequence $(a_1, \ldots, a_r, X_1, \ldots, X_n)$ is pseudo singular in $S$ then the sequence $(a_1, \ldots, a_r)$ is pseudo singular in $R$: we have indeed, considering only the case $m = n = 2$, an equality in $S$ of the form:

$$a_1^m a_2^m X_1^{n} X_2^{m} + a_1^m a_2^m X_1^{n} X_2^{m+1} R_4 + a_1^m a_2^m X_1^{m+1} R_3 + a_1^m a_2^m X_1^{n+1} R_2 + a_1^m + 1 R_1 = 0$$

Looking the coefficient of $X_1^m X_2^{m+1}$ in the polynomial of the left hand-side we get

$$a_1^m a_2^m + a_1^m a_2^{m+1} r_2 + a_1^{m+1} r_1 = 0$$

which gives the collapsus of $(a_1, a_2)$ in $R$.

The second assertion follows from the first (cf. theorem 1.20(1)).

The proof of the first assertion in classical mathematics relies directly on considering the case of fields. We give a constructive proof which follows the same pattern, and consider the case of (discrete) fields. We analyse the proof of proposition 4.19 and we substitute everywhere the field $K$ by a ring $R$, which will allow us to use the definition of collapsus above $R$. Take $(y_1, \ldots, y_{n+1})$ in $R[X_1, \ldots, X_n]$. We can write as a proof in linear algebra a proof that the $y_i$ are algebraically dependent over $R$, assuming first that $R$ is a discrete field. For instance if the $y_i$ are polynomials of degree $\leq d$ the polynomials $y_i^m_1 \cdots y^m_{n+1}$ with $\sum_i m_i \leq m$ are in the vector space of polynomials of degree $\leq dm$ which is of dimension $\leq \binom{dn+n}{n}$, and there are $\binom{m+n+1}{n+1}$ of them. For an explicit value of $m$ we have $\binom{m+n+1}{n+1} > \binom{dn+n}{n}$ (since one term is a polynomial of degree $n+1$ and the other a polynomial of degree $n$). We fix $m$ to this value. We order the corresponding "vectors" $y_1^m \cdots y^m_{n+1}$ (such that $\sum_i m_i \leq m$) along the lexicographic order for $(m_1, \ldots, m_{n+1})$. We can limit ourselves to consider $\binom{dn+n}{n} + 1$ vectors. Using lemma 4.6, we get in each rings $(R/I_k)S_k$ a vector $y_1^m \cdots y^m_{n+1}$ which is a linear combination of the following vectors. This gives, like in the proof of proposition 1.19 a collapsus, but this time we have to add at the beginning and at the end of the elementary idealistic chain $(y_1, \ldots, y_{n+1})$ the "additional hypothesis": the idealistic chain

$$(\langle \mu_i \rangle)_{i<k}, (y_1; y_2), (y_2; y_3), \ldots, (y_{n-1}, (y_n), (y_n; y_{n+1}, \mu_k)$$

collapses (for each $k$). Indeed, for showing the collapsus of an idealistic chain one can always quotient by the first of the ideals (or by a smaller ideal) and localised along the last of the monoids (or along a smaller monoid).

Lastly, all these collapsus give us the collapsus of $(y_1, \ldots, y_{n+1})$ above $R$ using the finite set of the $\mu_i$. □
4.2 Going Up

If \( R \) is a subring of \( S \), an idealistic chain of \( R \) which collapses in \( R \) collapses in \( S \) and the trace on \( R \) of a saturated idealistic prime of \( S \) is a saturated idealistic prime of \( R \). Among other things, we shall establish in this section that for integral extensions we have also the converse of these assertions.

**Lemma 4.8** Let \( R \subseteq S \) be two commutative rings where \( S \) is integral over \( R \). Let \( I \) be an ideal of \( R \) and take \( x \in R \). Then

\[ x \in \sqrt{I} \iff x \in \sqrt{IB} \]

**Proof.**

Suppose \( x \in \sqrt{IB} \), i.e., \( x^n = \sum j_i b_i \), with \( j_i \in I \), \( b_i \in B \). The \( b_i \) and 1 generate a sub \( R \)-module faithful and finitely generated \( M \) of \( S \) and \( x^n \) can be written as a linear combination with coefficients in \( I \) over a system of generators of \( M \). The characteristic polynomial of the matrix of the multiplication by \( x^n \) (expressed using this system of generators) have then all its coefficients (except maybe the leading one) in \( I \).

\[ \blacksquare \]

**Definition 4.9** Let \( R \subseteq S \) two commutative rings. Take \( P = (J,V) \) an idealistic prime of \( S \) and \( C = (P_1,\ldots,P_n) \) an idealistic chain de \( S \). We say that \((J \cap R,V \cap R)\) is the trace of \( P \) on \( R \). We write \( P|_R \) this idealistic prime of \( R \). We say that \((P_1|_R,\ldots,P_n|_R)\) is the trace of \( C \) on \( R \). We write \( C|_R \) this idealistic chain of \( R \).

It is clear that the trace of a complete (resp. saturated) idealistic chain is complete (resp. saturated).

**Corollary 4.10** (Lying over) Let \( R \subseteq S \) be commutative rings with \( S \) integral over \( R \).

- Let \( P \) be an idealistic prime in \( R \).
  
  (1) If \( P \) collapses in \( S \), it collapses in \( R \).
  
  (2) If \( Q \) is the saturation of \( P \) in \( S \), then \( Q|_R \) is the saturation of \( P \) in \( R \).

- The compactness theorem implies the following result. Any prime ideal of \( R \) is the trace on \( A \) of a prime ideal of \( S \).

**Proof.**

We show (1), the other points are then easy consequences. If \( P = (I,U) \) collapses in \( S \) an element of \( M(U) \) is in the radical of \( \langle I \rangle_S = \langle I \rangle_R S \) and hence, by the previous lemma, is in the radical of \( \langle I \rangle_R \).

\[ \blacksquare \]

**Theorem 4.11** (Going Up) Let \( R \subseteq S \) be commutative rings with \( S \) integral over \( R \). Let \( C_1 \) be a saturated idealistic chain of \( S \) and \( C_2 \) an idealistic chain of \( R \).

(1) The idealistic chain \( C = C_1 \bullet C_2 \) collapses in \( S \) if, and only if, the idealistic chain \( C_1|_R \bullet C_2 \) collapses in \( R \).

(2) Let \( C' \) be the saturation of \( C \) in \( S \). The trace of \( C' \) on \( R \) is the saturation of \( C_1|_R \bullet C_2 \) in \( R \).

In particular any idealistic chain of \( R \) that collapses in \( S \) collapses in \( R \), and the trace on \( R \) of the saturation in \( S \) of an idealistic chain of \( R \) is equal to its saturation in \( R \).
Proof.
Take \( C_1 = ((J_1, V_1), \ldots, (J_\ell, V_\ell)) \) in \( S \), \( C_1|_R = ((I_1, U_1), \ldots, (I_\ell, U_\ell)) \) its trace on \( R \) and \( C_2 = ((I_{\ell+1}, U_{\ell+1}), \ldots, (I_{\ell+r}, U_{\ell+r})) \). Write (1)\(_{\ell,r}\) and (2)\(_{\ell,r}\) the assertions for given \( \ell \) and \( r \). These two statements are in fact directly equivalent, given the characterisation of the saturation of an idealistic chain in term of collapsus. Notice also that (1)\(_{0,1}\) and (2)\(_{0,1}\) state the lying over. We show (2)\(_{0,r}\) \(\Rightarrow\) (1)\(_{\ell,r}\). The idealistic chain \( C_1|_R \) is saturated in \( R \). Consider the quotient ring \( R'/R = R/I_\ell \subseteq S'/S = S/I_\ell \). The ring \( S' \) is still integral over \( R' \). (2)\(_{0,r}\) applied to these quotients gives (1)\(_{1,r}\) using fact 1.6 (1).

It is enough now to show (2)\(_{1,r}\) \(\Rightarrow\) (1)\(_{0,r+1}\). Let \( (P_1, \ldots, P_{r+1}) \) be an idealistic chain in \( R \) which collapses in \( S \). Let \( Q_1 \) be the saturation of \( P_1 \) in \( S \). The trace of \( Q_1 \) on \( R \) is \( P_1 \) by the lying over. We can then apply (2)\(_{1,r}\) with \( C_1 = Q_1 \) and \( C_2 = (P_2, \ldots, P_{r+1}) \).

\[ \square \]

Corollary 4.12 (Going Up, classical version) The compactness theorem implies the following result. Let \( R \subseteq S \) be commutative rings with \( S \) integral over \( R \). Let \( Q_1 \subseteq \cdots \subseteq Q_\ell \) be prime ideals in \( S \), \( P_1 = Q_1 \cap R \) (\( i = 1, \ldots, \ell \)), and \( P_{\ell+1} \subseteq \cdots \subseteq P_{\ell+r} \) prime ideals in \( R \) with \( P_\ell \subseteq P_{\ell+1} \).

There exist \( Q_{\ell+1}, \ldots, Q_{\ell+r} \), prime ideals in \( S \) which satisfy \( Q_\ell \subseteq Q_{\ell+1} \subseteq \cdots \subseteq Q_{\ell+r} \) and \( Q_{\ell+j} \cap A = P_{\ell+j} \) for \( j = 1, \ldots, r \).

Proof.
We consider the idealistic chains \( C_1 \) (in \( S \)) and \( C_2 \) (in \( R \)) associate to the given chains. by hypothesis, \( C_1|_R \cdot C_2 \) does not collapse in \( R \) and hence, by constructive Going up, \( C = C_1 \cdot C_2 \) does not collapse in \( S \). We can then consider a chain of prime ideals of \( S \) that refines the idealistic chain \( C \) (theorem 1.13). Since \( C_1 \) is frozen it does not change in this process (otherwise it would collapse). The last part of the chain \( Q_{\ell+1}, \ldots, Q_{\ell+r} \), has its trace on \( R \) frozen, and hence it has to be equal to \( P_{\ell+1}, \ldots, P_{\ell+r} \) (otherwise it would collapse).

\[ \square \]

Notice that it seems difficult to show directly, in classical mathematics, theorem 4.11 from corollary 4.12, even using theorem 1.13 which connects the idealistic chains and the chains of prime ideals.

Corollary 4.13 (Krull dimension of an integral extension) Let \( R \subseteq S \) be commutative rings with \( S \) integral over \( R \).

1. The Krull dimension of \( R \) is \( \leq n \) if, and only if, the Krull dimension of \( S \) is \( \leq n \).

2. A pseudo regular sequence of \( R \) is pseudo regular in \( S \).

3. If the collapse of elementary idealistic chains in \( R \) is decidable, from a pseudoregular sequence in \( S \) one can build a pseudo regular sequence of same length in \( R \).

\[ \text{NB: for the points (2) and 3 the rings are supposed to be equipped with the apartness relation } \neg(x = 0). \]

Proof.
(1) Proposition 4.5 (3) gives \( \dim(R) \leq n \Rightarrow \dim(S) \leq n \), and the last part of theorem 4.11 gives the converse.

(2) follows by contraposition from the last part of theorem 4.11.

(3) Since the relative Krull of \( S \) on \( R \) is 0, we can apply theorem 4.4 (2).

\[ \square \]

A corollary of the previous result and of theorem 1.20 is the following theorem, which says that the Krull dimension of a finitely presented algebra over a discrete field is the one given by Noether’s normalisation lemma.
Theorem 4.14 Let $K$ be a discrete field, $I$ a finitely generated ideal of the ring $K[X_1, \ldots, X_r]$ and $A$ the quotient algebra. Noether’s normalisation lemma applied to the ideal $I$ gives an integer $r$ and elements $y_1, \ldots, y_r$ of $A$ that are algebraically independent over $K$ and such that $A$ is a finitely generated module over $K[y_1, \ldots, y_r]$. The Krull dimension of $A$ is equal to $r$.

4.3 Going Down

This section is copied over the section on Going Down given in Sharp’s book. Since one cannot in general compute the minimal polynomial of an element algebraic over a field, the first lemma is a little less simple than the corresponding lemma in classical mathematics and the proof of the Going Down actually explicits an algorithm that searches a polynomial “which may be seen as the minimal polynomial for the ongoing computation”.

Lemma 4.15 Let $R \subseteq S$ be two entire rings with $S$ integral over $R$ and $R$ integrally closed. Let $I$ be a radical ideal of $R$ and $x \in IS$. There exists a monic polynomial $M(X)$ whose all non leading coefficients are in $I$ and such that $M(x) = 0$. Let $P$ be another polynomial in $R[X]$ such that $P(x) = 0$. Let $K$ be the quotient field of $R$ and $Q$ the monic gcd of $P$ and $M$ in $K[X]$. Then $Q$ has its non leading coefficients in $I$, is such that $Q(x) = 0$ and divides $M$ and $P$ in $R[X]$.

Proof. The existence of $M$ is easy. We write $x = \sum a_kb_k$ with $a_k$ in $I$ and $b_k$ in $B$, and consider the sub $R$-algebra $T$ of $S$ generated by the $b_k$. This is a faithful finitely generated $R$-module. We write the matrix of the multiplication by $x$ over a system of generators of the $R$-module $C$, which has all its coefficients in $I$, and we take its characteristic polynomial. Let $L$ be the quotient field of $S$. Given the Bézout relation $UP + VM = Q$, we have $Q(x) = 0$ in $K[x] \subseteq L$ and hence in $S$. Given the relation $QS_1 = S$, and since $R$ is integrally closed, $Q$ and $S_1$ have their non leading coefficients in $\sqrt{I} = I$. Finally the quotient $P_1 = P/Q$ can be computed in $R[X]$ by Euclidean division. \hfill \Box

NB: this lemma can be seen as the computational content of the lemma in classical mathematics that the monic minimal polynomial of $x$ has all its non leading coefficients in $I$.

Proposition 4.16 (Going Down in one step) Let $R \subseteq S$ be two integral rings with $S$ entire over $R$ and $R$ integrally closed. Let $Q_1$ be a saturated idealistic prime in $S$, $P_1 = (I_1, U_1)$ its trace on $R$ and $P_0 = (I_0, U_0)$ a saturated idealistic prime of $R$ with $I_0 \subseteq I_1$. If the idealistic chain $(P_0, Q_1)$ collapses in $S$, then the idealistic prime $Q_1$ collapses in $S$ (and a fortiori the idealistic prime $P_1$ collapses in $R$).

Proof. Take $Q_1 = (J_1, V_1)$. Since $Q_1$ is complete, we can write the collapsus on the form

$$u_0v_1 = j_0 \quad \text{avec } u_0 \in U_0, \; v_1 \in V_1, \; j_0 \in I_0B$$

We know that $j_0$ cancels a monic polynomials $A$ with non leading coefficients in $I_0$

$$A(X) = X^k + \sum_{i<k} a_iX^i \quad \text{avec } a_i \in I_0$$
hence \( v_1 \) cancels \( M(u_0 X) \)

\[
0 = R(u_0 v_1) = u_0^k v_1^k + \sum_{i<k} (a_i u^i) v_1^i
\]

We know also that \( v_1 \) cancels a monic polynomial \( B \) with coefficients in \( R \) of degree \( d \).

**First case:** \( u_0^d S(X) = A(u_0 X) \). Then the non leading coefficients of \( B \), the \( b_i = a_i u_0^{k-i} \) are in \( I_0 \), since \( \mathcal{P}_0 \) is saturated in \( R \). Hence \( v_1^k \in I_0 S \), so \( v_1 \in \sqrt{I_0 S} \subseteq \sqrt{I_1 S} \subseteq J_1 \) (\( I_0 \subseteq I_1 \subseteq J_1 \) and \( J_1 \) is a radical ideal of \( S \)). Hence \( v_1 \in V_1 \cap J_1 \).

**Second case:** We don’t have \( u_0^d S(X) = A(u_0 X) \). We shall reduce this case to the first one. We apply the previous lemma with \( v_1 \) and the ideal \( R \). We get that \( v_1 \) cancels the monic \( B_1 \) monic gcd of \( A(u_0 X) \) and \( B(X) \), and with coefficients in \( R \). Let \( d_1 \) be the degree of \( S_1 \). We consider the following monic polynomial with coefficients in \( R \) \( A_1(X) = u_0^{d_1} S_1(X/u_0) \). We have \( A_1(j_0) = 0 \), \( A_1(u_0 X) = u_0^{d_1} B_1(X) \), \( A_1(X) \) is the monic gcd of \( R(X) \) and \( u_0^{d_1} B(X/u_0) \). Applying the previous lemma with \( j_0 \) and the ideal \( I_0 \) we get that \( A_1 \) has its non leading coefficients in \( I_0 \). We are thus back, with \( A_1 \) and \( B_1 \), to the first case.

Notice that the beginning of the proof (before the analysis of the second case, that Sharp avoids by considering the minimal polynomial of \( j_0 \)) is almost the same as the proof in Sharp’s book. Sharp does not use the word “collapsus”, and has for hypothesis that \( \mathcal{Q}_1 \) is given as a prime ideal, and shows that it would be absurd to have an equality \( u_0 v_1 = j_0 \) since this would imply \( v_1 \in V_1 \cap J_1 \) which is contradictory with his hypothesis. This is a good case showing that our work consists essentially to explicitate algorithms that are only implicit in classical arguments. This illustrates also well a systematic feature of classical proofs, which inverses positive and negative by the introduction of abstract objects. The collapsus, which is a concrete fact, is seen as a negative fact (“this would be absurd since we have a prime ideal”) while the absence of collapsus, which requires a priori an infinity of verifications, and hence is by essence negative, is felt as a positive fact, which ensures the existence of abstract object. The price to pay for the apparent comfort provided by these abstract objects is to transform constructive arguments by non constructive one, changing the (constructive) direct proof of \( P \Rightarrow Q \) by a proof by contradiction of \( \neg Q \Rightarrow \neg P \), i.e., a proof of \( \neg \neg P \Rightarrow \neg Q \).

**Theorem 4.17** (Going Down) Let \( R \subseteq S \) be two entire rings with \( S \) integral over \( R \) and \( R \) integrally closed. Let \( C_1 \) be a saturated idealistic chain of \( R \) and \( C_2 \) a saturated idealistic chain of \( S \), non empty. Let \( I_\ell \) be the last of the ideals in the idealistic chain \( C_1 \) and \( I_{\ell+1} \) the first of the ideals in the idealistic chain \( C_2 | R \). We assume \( I_\ell \subseteq I_{\ell+1} \). If the idealistic chain \( C_1 \bullet C_2 \) collapses in \( S \), then the idealistic chain \( C_2 \) collapses in \( S \).

**Proof.**

If \( \ell \geq 1 \) and \( r \geq 1 \) are the numbers of idealistic primes in the idealistic chains \( C_1 \) and \( C_2 \), write \( GD_{\ell,r} \) the property that we want to establish. We know already \( GD_{1,1} \) from the Going Down in one step.

Since the idealistic chains \( C_2 \) and \( C_2 | R \) are saturated, and by fact 1.6 (2) only the first idealistic primes of the second idealistic chainmatters for the collapsus. Thus it is enough to show \( GD_{\ell,1} \), i.e., the case where \( C_2 \) contains only one idealistic prime. We proceed by induction on \( \ell \). Take \( C_1 = (P_1, \ldots, P_{\ell}) \) a saturated idealistic chain in \( R \), \( (P_k = (I_k, U_k)) \) and \( Q \) a saturated idealistic prime in \( S \). Assume that \( C_1 \bullet Q \) collapses in \( S \). Let \( C = (P_2, \ldots, P_\ell, Q) \) and \( C' \) its saturation in \( S \). If \( Q_2 = (J_2, V_2) \) is the first of the idealistic primes in \( C' \) we have \( I_1 \subseteq I_2 \subseteq (J_2 \cap R) \), and we can apply the Going Down in one step (or more precisely \( GD_{1,1} \)): \( C' \) collapses in \( S \). Hence \( C \) collapses in \( S \). One can then apply the induction hypothesis. \( \square \)
Corollary 4.18 (Going Down, classical version) The compactness theorem implies the following result. Let \( R \subseteq S \) be two entire rings with \( S \) integral over \( R \) and \( R \) integrally closed. Let \( Q_{\ell+1} \subseteq \cdots \subseteq Q_\ell \) be prime ideals in \( S \), \( P_i = Q_i \cap A \) (\( i = \ell + 1, \ldots, \ell + r \)), and \( P_1 \subseteq \cdots \subseteq P_\ell \) be prime ideals in \( R \) with \( P_\ell \subseteq P_{\ell+1} \). There exist \( Q_1, \ldots, Q_\ell \), prime ideals in \( S \) which satisfy \( Q_1 \subseteq \cdots \subseteq Q_\ell \subseteq Q_{\ell+1} \) and \( Q_j \cap A = P_j \) pour \( j = 1, \ldots, \ell \).

Proof.
Like in the proof of corollary 4.12.

We finish with a Going Down theorem for flat extensions. (cf. [15]).

Theorem 4.19 (Going Down for flat extensions) Let \( R \subseteq S \) be two commutative rings with \( S \) flat over \( A \).

(1) Let \( Q_1 = (J_1, V_1) \) be a saturated idealistic prime of \( S \), \( P_1 = (I_1, U_1) \) its trace on \( R \) and \( P_0 = (I_0, U_0) \) a saturated idealistic prime of \( A \) with \( I_0 \subseteq I_1 \). It the idealistic chain \( (P_0, Q_1) \) collapses in \( S \), then the idealistic prime \( Q_1 \) collapses in \( S \) (and a fortiori the idealistic prime \( P_1 \) collapses in \( R \)).

(2) Let \( C_1 \) be a saturated idealistic chain of \( R \) and \( C_2 \) a saturated idealistic chain of \( S \), non empty. Let \( I_\ell \) be the last of the ideals in the idealistic chain \( C_1 \) and \( I_{\ell+1} \) the first of the ideals in the idealistic chain \( C_2|_R \). Assume \( I_\ell \subseteq I_{\ell+1} \). If the idealistic chain \( C_1 \bullet C_2 \) collapses in \( S \), then the idealistic chain \( C_2 \) collapses in \( S \).

Proof.
It is enough to show (1), because we can then show (2) like in the proof of theorem 4.17. Take \( J_0 = I_0 B \). If \( (P_0, Q_1) \) collapses in \( S \), we have \( v_1 \in V_1 \), \( u_0 \in U_0 \) and \( j_0 \in J_0 \) with \( v_1 u_0 + j_0 = 0 \). We write \( j_0 = i_1 b_1 + \cdots + i_r b_r \) with the \( i_k \in I_0 \) and \( b_k \in B \). We get that the elements \( v_1, b_1, \ldots, b_r \) of \( S \) are linearly dependent over \( R \).

\[
(u_0, i_1, \ldots, i_r) \begin{pmatrix} v_1 \\ b_1 \\ \vdots \\ b_r \end{pmatrix} = 0
\]

Since \( S \) is flat over \( R \) this relation can be decomposed to

\[
(u_0, i_1, \ldots, i_r) M = (0, \ldots, 0) \quad \text{and} \quad \begin{pmatrix} v_1 \\ b_1 \\ \vdots \\ b_r \end{pmatrix} = M \begin{pmatrix} b_1' \\ \vdots \\ b_r' \end{pmatrix}
\]

where \( M = (m_{k,\ell}) \in R^{s \times (r+1)} \) and the \( b_k' \) are in \( S \). Each relation

\[
\left. u_0 m_{0,\ell} + i_1 m_{1,\ell} + \cdots + i_r m_{1,\ell} \right. = 0
\]

implies that \( m_{0,\ell} \in I_0 \) since \( P_0 \) is saturated. A fortiori \( m_{0,\ell} \in J_1 \). Hence the relation

\[
v_1 = m_{0,1}b_1' + \cdots + m_{0,s}b_s'
\]

is a collapsus \( Q_1 \) of \( S \).
For this proof, that was literally forced upon us, we did not try to analyse the quite abstract argument given by Matsumara for the following corollary.

**Corollary 4.20** (Going Down for flat extension, classical version) *The compactness theorem implies the following result. Let $R \subseteq S$ be two commutative rings with $S$ flat over $R$. Let $Q_{\ell+1} \subseteq \cdots \subseteq Q_{\ell + r}$ be prime ideals in $S$, $P_i = Q_i \cap R$ ($i = \ell + 1, \ldots, \ell + r$), and $P_1 \subseteq \cdots \subseteq P_\ell$ be prime ideals in $R$ with $P_\ell \subseteq P_{\ell + 1}$. There exist $Q_1, \ldots, Q_\ell$, prime ideals in $S$ such that $Q_1 \subseteq \cdots \subseteq Q_\ell \subseteq Q_{\ell + 1}$ and $Q_j \cap R = P_j$ for $j = 1, \ldots, \ell$.***
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Annex: Completeness, compactness theorem, LLPO and geometric theories

A.1 Theories and models

We fix a set $V$ of atomic propositions or propositional letters. A proposition $\phi, \psi, \ldots$ is a syntactical object built from the atoms $p, q, r \in V$ with the usual logical connectives

$$0, 1, \phi \land \psi, \phi \lor \psi, \phi \rightarrow \psi, \neg \phi$$

We let $P_V$ be the set of all propositions. Let $F_2$ be the Boolean algebra with two elements. A valuation $v \in F_2^V$ that assigns a truth value to any of the atomic propositions. Such a valuation can be extended to a map $P_V \rightarrow \{0, 1\}$, $\phi \mapsto v(\phi)$ in the expected way. A theory $T$ is a subset of $P_V$. A model of $T$ is a valuation $v$ such that $v(\phi) = 1$ for all $\phi \in T$.

More generally given a Boolean algebra $B$ we can define $B$-valuation to be a function $v \in B^V$. This can be extended as well to a map $P_V \rightarrow B$, $\phi \mapsto v(\phi)$. A $B$-model of $T$ is a valuation $v$ such that $v(\phi) = 1$ for all $\phi \in T$. The usual notion of model is a direct special case, taking for $B$ the Boolean algebra $F_2$. For any theory there exists always a free Boolean algebra over which $T$ is a model, the Lindenbaum algebra of $T$, which can be also be defined as the Boolean algebra generated by $T$, thinking of the elements of $V$ as generators and the elements of $T$ as relations. The theory $T$ is formally consistent if, and only if, its Lindenbaum algebra is not trivial.

A.2 Completeness theorem

**Theorem A.1** (Completeness theorem) Let $T$ be a theory. If $T$ is formally consistent then $T$ has a model.

This theorem is the completeness theorem for propositional logic. Such a theorem is strongly related to Hilbert’s program, which can be seen as an attempt to replace the question of existence of model of a theory by the formal fact that this theory is not contradictory.

Let $B$ the Lindenbaum algebra of $T$. To prove completeness, it is enough to find a morphism $B \rightarrow F_2$ assuming that $B$ is not trivial, which is the same as finding a prime ideal (which is then automatically maximal) in $B$. Thus the completeness theorem is a consequence of the existence of prime ideal in nontrivial Boolean algebra. Notice that this existence is clear in the case where $B$ is finite, hence that the completeness theorem is direct for finite theories.

A.3 Compactness theorem

The completeness theorem for an arbitrary theory can be seen as a corollary of the following fundamental result.

**Theorem A.2** (Compactness theorem) Let $T$ be a theory. If all finite subsets of $T$ have a model then so does $T$.

Suppose indeed that the compactness theorem holds, and let $T$ be a formally consistent theory. Then an arbitrary finite subset $T_0$ of $T$ is also formally consistent. Furthermore, we have seen that this implies the existence of a model for $T_0$. It follows then from the compactness theorem that $T$ itself has a model.
Conversely, it is clear that the compactness theorem follows from the completeness theorem, since a theory is formally consistent as soon as all its finite subsets are.

A simple general proof of the compactness theorem is to consider the product topology on \( \{0, 1\}^V \) and to notice that the set of models of a given subset of \( T \) is a closed subset. The theorem is then a corollary of the compactness of the space \( W := \{0, 1\}^V \) when compactness is expressed (in classical mathematics) as: if a family of closed subsets of \( W \) has non-void finite intersections, then its intersection is non-void.

### A.4 LPO and LLPO

If \( V \) is countable (i.e., discrete and enumerable) we have the following alternative argument. One writes \( V = \{p_0, p_1, \ldots\} \) and builds by induction a partial valuation \( v_n \) on \( \{p_i \mid i < n\} \) such that any finite subset of \( T \) has a model which extends \( v_n \), and \( v_{n+1} \) extends \( v_n \). To define \( v_{n+1} \) one first tries \( v_{n+1}(p_n) = 0 \). If this does not work, there is a finite subset of \( T \) such that any of its model \( v \) that extends \( v_n \) satisfies \( v(p_n) = 1 \) and one can take \( v_{n+1}(p_n) = 1 \).

The non-effective part of this argument is contained in the choice of \( v_{n+1}(p_n) \), which demands to give a global answer to an infinite set of (elementary) questions.

Now let us assume also that we can enumerate the infinite set \( T \). We can then build a sequence of finite subsets of \( T \) in a nondecreasing way \( K_0 \subseteq K_1 \subseteq \ldots \) such that any finite subset of \( T \) is a subset of some \( K_n \). Assuming we have construct \( v_n \) such that all \( K_j \)'s have a model extending \( v_n \), in order to define \( v_{n+1}(p_n) \) we have to give a global answer to the questions: do all \( K_j \)'s have a model extending \( v_{n+1} \) when we choose \( v_{n+1}(p_n) = 1 \) ? For each \( j \) this is an elementary question, having a clear answer. More precisely let us define \( g_n : N \to \{0, 1\} \) in the following way: \( g_n(j) = 1 \) if there is a model \( v_{n,j} \) of \( K_j \) extending \( v_n \) with \( v_{n,j}(p_n) = 1 \), else \( g_n(j) = 0 \). By induction hypothesis if \( g_n(j) = 0 \) then all \( K_\ell \) have a model \( v_{n,\ell} \) extending \( v_n \) with \( v_{n,\ell}(p_n) = 1 \), and all models \( v_{n,\ell} \) of \( K_\ell \) extending \( v_n \) satisfy \( v_{n,\ell}(p_n) = 1 \) if \( \ell \geq j \). So we can “construct” inductively the infinite sequence of partial models \( v_n \) by using at each step the non-constructive Bishop’s principle LPO (Least Principle of Omniscience): given a function \( f : N \to \{0, 1\} \), either \( f = 1 \) or \( \exists j \in N \ f(j) \neq 1 \). This principle is applied at step \( n \) to the function \( g_n \).

In fact we can slightly modify the argument and use only a combination of Dependant Choice and of Bishop’s principle LLPO (Lesser Limited Principle of Omniscience), which is known to be strictly weaker than LPO: given two non-increasing functions \( g, h : N \to \{0, 1\} \) such that, for all \( j \)

\[
g(j) = 1 \vee h(j) = 1
\]

then we have \( g = 1 \) or \( h = 1 \). Indeed let us define \( h_n : N \to \{0, 1\} \) in a symmetric way: \( h_n(j) = 1 \) if there is a model \( v_{n,j} \) of \( K_j \) extending \( v_n \) with \( v_{n,j}(p_n) = 0 \), else \( h_n(j) = 0 \). Clearly \( g_n \) and \( h_n \) are non-increasing functions. By induction hypothesis, we have for all \( j \) \( g_n(j) = 1 \vee h_n(j) = 1 \). So, applying LLPO, we can define \( v_{n+1}(p_n) = 1 \) if \( g_n = 1 \) and \( v_{n+1}(p_n) = 1 \) if \( h_n = 1 \). Nevertheless, we have to use dependant choice in order to make this choice infinitely often since the answer “\( g = 1 \) or \( h = 1 \)” given by the oracle LLPO may be ambiguous.

In a reverse way it is easy to see that the compactness theorem restricted to the countable case implies LLPO.

### A.5 Geometric formulae and theories

*What would have happened if topologies without points had been discovered before topologies with points, or if Grothendieck had known the theory of distributive lattices?* (G. C. Rota [17]).
A formula is *geometric* if, and only if, it is built only with the connectives \(0, 1, \phi \land \psi, \phi \lor \psi\) from the propositional letters in \(V\). A theory if a (propositional) *geometric* theory iff all the formula in \(T\) are of the form \(\phi \rightarrow \psi\) where \(\phi\) and \(\psi\) are geometric formulae.

It is clear that the formulae of a geometric theory \(T\) can be seen as relations for generating a distributive lattice \(L_T\) and that the Lindenbaum algebra of \(T\) is nothing else but the free Boolean algebra generated by the lattice \(L_T\). It follows from Proposition 2.11 that \(T\) is formally consistent if, and only if, \(L_T\) is nontrivial. Also, a model of \(T\) is nothing else but an element of \(\text{Spec}(L_T)\).

**Theorem A.3 (Completeness theorem for geometric theories)** Let \(T\) be a geometric theory. If \(T\) generates a nontrivial distributive lattice, then \(T\) has a model.

The general notion of geometric formula allows also existential quantification, but we restrict ourselves here to the propositional case. Even in this restricted form, the notion of geometric theory is fundamental. For instance, if \(R\) is a commutative ring, we can consider the theory with atomic propositions \(D(x)\) for each \(x \in R\) and with axioms

- \(D(0_R) \rightarrow 0\)
- \(1 \rightarrow D(1_R)\)
- \(D(x) \land D(y) \rightarrow D(xy)\)
- \(D(xy) \rightarrow D(x)\)
- \(D(x + y) \rightarrow D(x) \lor D(y)\)

This is a geometric theory \(T\). The model of this theory are clearly the complement of the prime ideals. What is remarkable is that, while the existence of models of this theory is a nontrivial fact which may be dependent on set theoretic axioms (such as dependent axiom of choices) its formal consistency is completely elementary (as explained in the beginning of the section 3). This geometric theory, or the distributive lattice it generates, can be seen as a point-free description of the Zariski spectrum of the ring. The distributive lattice generated by this theory (called in this paper the Zariski lattice of \(R\)) is isomorphic to the lattice of compact open of the Zariski spectrum of \(R\), while the Boolean algebra generated by this theory is isomorphic to the algebra of the constructible sets.