The Onset of Chaos with a Quadrupole–Quadrupole Interaction (*)

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Abstract

The transition from order to chaos in atomic nuclei has been studied analytically and numerically using a quadrupole-quadrupole residual interaction. This interaction leads to chaotic behaviour, but the critical energy $E_C \simeq 12.6 \text{ MeV}$, corresponding to the onset of chaos, is higher than that of the experimental one.
1. Introduction

As is well known, in many–body systems states of different "complexity\(^1\)" generally coexist [1,2,3]. In particular, in atomic nuclei the "hierarchy" of complexity can be represented by a simple picture. In zero order approximation the relevant elementary excitations (single–particle, rotations, vibrations) may regarded as independent modes and then the interaction between these elementary modes can be considered [4]. At the end of the chain of complexity there are the so called "stochastic" or "chaotic states", whereas a single mode can be considered "regular".

The experimental data of nuclear spectroscopy (see, e.g. references [5,6]) suggest regular states near the ground state (0–3 MeV) and chaotic states near the neutron emission threshold (6–8 MeV). On the other hand, in the framework of the so called "rotational damping", the authors of references [7] have shown that the surface delta interaction produces chaotic behaviour while a pairing plus quadrupole force is unable to do so. The aim of this paper is to analyze in detail the quadrupole–quadrupole residual interaction showing that it is able to reproduce the transition order–chaos in the atomic nuclei, but at an energy (\(\sim 12.6\) MeV) higher that that of the experimental one (\(\sim 8\) MeV).

The structure of the article is as follows: in section 2 we present a schematic nuclear model that describes two nucleons in a mean–field interacting with a separable quadrupole–quadrupole residual interaction; in

\(^1\)In this paper the terms "complexity" and "hierarchy" have been used in a general sense; for precise definitions, see [17].
section 3 we apply the curvature of potential energy criterion to calculate
the energy $E_C$ of the onset of chaos and in section 4 we plot the Poincarè
sections of the model for bound energies. Finally in section 5 we diagonalize
the quantum hamiltonian and calculate the distribution $P(S)$ of spacings
between adjacent levels.

2. The schematic quadrupole–quadrupole interaction

In our model nucleons move independently in an oscillator potential under
the influence of a separable quadrupole–quadrupole effective residual inter-
action. The hamiltonian of the system is given by [8]:

$$H = \sum_{i=1}^{A} \frac{p_i^2}{2m} + V_0 \beta^2 (A - 1) \sum_{i=1}^{A} r_i^2 -$$

$$- \frac{16\pi V_0}{15} \beta^4 \sum_{i<j}^{A} \sum_{m=-2}^{2} (-1)^m \{r_i^2 Y_{2m}(\theta_i, \phi_i)\} \{r_j^2 Y_{2-m}(\theta_j, \phi_j)\}, \quad (1)$$

where $R_0 = 1/\beta$ is the range of the interaction and $A$ the number of nucleons.

We consider only 2 nucleons and suppose that they move in one dimension,
so that $Y_{lm} = \sqrt{\frac{(2l+1)}{4\pi}} \delta_{m,0}$, and we have:

$$H = \frac{1}{2m} (p_1^2 + p_2^2) + V_0 \beta^2 (r_1^2 + r_2^2) - \frac{4}{3} V_0 \beta^4 r_1^2 r_2^2. \quad (2)$$

To simplify the problem we perform the transformation:

$$p_k \rightarrow \sqrt{\frac{1}{2m\hbar\omega}} p_k, \quad q_k \rightarrow \sqrt{\frac{m\omega}{2\hbar}} q_k \quad (3)$$

where $m\omega^2/2 = V_0/R_0^2$. So the hamiltonian becomes:

$$H = \epsilon(p_1 + p_2 + q_1 + q_2) - \chi q_1^2 q_2^2, \quad (4)$$
with ϵ = ℏω and χ = 8/3(ℏ²/mR₀²).

In nuclei ϵ = ℏω ≃ 41 A⁻¹/³ MeV is the single particle energy for the
harmonic oscillator potential [8] and R₀ = 1/β ≃ 1.2A¹/³ fm is the nuclear
radius, and so we have χ ≃ 74A⁻²/³ MeV.

3. The onset of chaos with the curvature criterion

We apply the criterion of curvature [9] to the hamiltonian (4) in order to
calculate the critical energy of the transition to chaos, which will be indicated
by E_C. The criterion of curvature is based on the estimation of the rate of
separation of neighbouring trajectories in the phase space. To calculate the
time evolution of the dynamical system with hamiltonian:

\[ H = \frac{1}{2m}(p_1 + p_2) + V(r_1, r_2) \] (5)

the following equations have to be solved:

\[ \frac{d}{dt} \vec{r} = \frac{\partial H}{\partial \vec{p}}, \quad \frac{d}{dt} \vec{p} = -\frac{\partial H}{\partial \vec{r}}, \] (6)

where \( \vec{r} = (r_1, r_2) \) and \( \vec{p} = (p_1, p_2) \). The deviations from the two initially
neighbouring trajectories \((\hat{r}(t), \hat{p}(t))\) are given by:

\[ \delta \vec{r}(t) = \hat{r}(t) - \vec{r}(t), \quad \delta \vec{p}(t) = \hat{p}(t) - \vec{p}(t) \] (7)

and the linearized equations of motion for the deviations are:

\[ \frac{d}{dt} \delta \vec{p}(t) = M^{-1} \delta \vec{p}(t), \quad \frac{d}{dt} \delta \vec{r}(t) = -S(t) \delta \vec{r}(t) \] (8)

where \( M^{-1}_{ij} = \delta_{ij}m^{-1} \), and:

\[ S_{ij}(t) = \left. \frac{\partial^2 V}{\partial r_i \partial r_j} \right|_{\vec{r} = \vec{r}(t)}. \] (9)
The stability of the dynamical system is then determined by the eigenvalues of the $4 \times 4$ matrix:

$$\Gamma(t) = \begin{pmatrix} 0 & M^{-1} \\ -S(t) & 0 \end{pmatrix}. \quad (10)$$

If at least one of the eigenvalues $\lambda_i(t)$ is real, then the separation of the trajectories grows exponentially, and the motion is unstable. Imaginary eigenvalues correspond to stable motion. To diagonalize the matrix $\Gamma(t)$, we must first solve the equations of motion of the differences (8). The problem can be significantly simplified by assuming that the time dependence can be eliminated, i.e. $\Gamma(\vec{r}(t)) = \Gamma(\vec{r})$. The eigenvalues then are:

$$\lambda_{1,2,3,4} = \pm[-b \pm \sqrt{b^2 - 4c}]^{\frac{1}{2}}, \quad (11)$$

where:

$$b = m^{-1}[\frac{\partial^2 V}{\partial r_1^2} + \frac{\partial^2 V}{\partial r_2^2}], \quad (12)$$

$$c = m^2[\frac{\partial^2 V}{\partial r_1^2} \frac{\partial^2 V}{\partial r_2^2} - (\frac{\partial^2 V}{\partial r_1 \partial r_2})^2]. \quad (13)$$

Now, if $b > 0$ then with $c \geq 0$ the eigenvalues are purely imaginary and the motion is stable, meanwhile with $c < 0$ the pair of eigenvalues become real, and this leads to exponential separation of neighbouring trajectories, i.e. chaotic motion. The parameter $c$ has the same sign as the Gaussian curvature $K(r_1, r_2)$ of the potential–energy surface:

$$K(r_1, r_2) = \frac{\frac{\partial^2 V}{\partial r_1^2} \frac{\partial^2 V}{\partial r_2^2} - (\frac{\partial^2 V}{\partial r_1 \partial r_2})^2}{[1 + (\frac{\partial^2 V}{\partial r_1^2})^2 + (\frac{\partial^2 V}{\partial r_2^2})^2]^2}. \quad (14)$$

Let us now return to our nuclear problem. The potential energy is:

$$V(r_1, r_2) = \epsilon(r_1^2 + r_2^2) - \chi r_1^2 r_2; \quad (15)$$

it has one minimum for \( r_1 = r_2 = 0 \) with \( V = 0 \) MeV, and four saddle points for \( r_1 = \pm \sqrt{\frac{2}{\chi}}, \ r_2 = \pm \sqrt{\frac{2}{\chi}} \), and \( r_1 = \pm \sqrt{\frac{2}{\chi}}, \ r_2 = \mp \sqrt{\frac{2}{\chi}} \), with \( V = \frac{\epsilon^2}{\chi} \approx 23.3 \) MeV. These are the points for which the curvature criterion is exact (\( \nabla H = 0 \)); the origin is a stable elliptic point with \( \lambda_{1,2,3,4} = \pm \sqrt{(2)}i \) and the four saddle points are unstable hyperbolic points with \( \lambda_{1,2} = \pm 2 \) and \( \lambda_{3,4} = \pm 2i \).

The equipotentials (curves of constant potential \( V \)) are shown in figure 1, and we see that \textit{unbounded motion} occurs if \( E > V(\sqrt{\frac{2}{\chi}}, \sqrt{\frac{2}{\chi}}) = \frac{\epsilon^2}{\chi} \) and \textit{bounded motion} for \( E < \frac{\epsilon^2}{\chi} \). We concentrate here upon the region \( 0 \leq E \leq \frac{\epsilon^2}{\chi} \).

At low positive energies, the motion near the minimum of the potential energy, where the curvature is positive, is periodic or quasi–periodic and is separated from the region of instability by a line of zero curvature. If the energy is increased, the system will, for certain initial conditions, be in a region of negative curvature where the motion is chaotic. In accordance with this scenario, the energy of order–chaos transition \( E_C \) is equal to the minimum value of the line of zero gaussian curvature \( K(r_1, r_2) \) of the potential–energy surface of the system. For our potential, the gaussian curvature vanishes at the points that satisfy the equation:

\[
4\epsilon(1 - \frac{\chi}{\epsilon} r_1^2 - \frac{\chi}{\epsilon} r_2^2 - 3\frac{\chi^2}{\epsilon^2} r_1^2 r_2^2) = 0.
\]

(16)

The energy on the zero–curvature line is determined by the expression:

\[
V(K = 0, r_2) = \frac{\epsilon \chi^2 r_2^4}{3\chi^2 r_2^2 + \epsilon \chi} - \frac{2\epsilon^2 \chi r_2^2}{3\chi^2 r_2^2 + \epsilon \chi} + \frac{\chi^3}{3\chi^2 r_2^2 + \epsilon \chi} + \epsilon r_2^2,
\]

(17)

(see also fig. 2). It is easy to show that the minimal energy on the zero–curvature line is given by:

\[
V_{\text{min}}(K = 0, \bar{r}_1) = \frac{5\epsilon^2}{9\chi},
\]

(18)
and occurs at $\bar{r}_1 = \pm \frac{\sqrt{2}}{2} \beta$. This is the critical energy of the transition to chaos $E_C$ of the model, and we have $E_C \simeq 12.62$ MeV. We observe that the energy of the onset of chaos ($E_C \simeq 12.6$ MeV) is much higher than that required for the atomic nuclei one (chaotic behaviour $\simeq 8$ MeV) [5,6].

In figure 3 we show the rate of regular points of configuration space obtained with the curvature criterion; the transition to chaos seems to be quite smooth.

4. Classical calculations

The curvature criterion is only able to characterize the local behaviour of the system (for example local instability) and gives a rough signature of the global properties (e.g. global instability) [10]. As is well known, global properties are more appropriately studied using Poincaré sections [12,13].

The classical equations of motion obtained from the hamiltonian (4) are:

$$\dot{r}_1 = 2\epsilon p_1, \quad \dot{r}_2 = 2\epsilon p_2$$

$$\dot{p}_1 = -2\epsilon r_1 + 2\chi r_1 r_2^2, \quad \dot{p}_2 = -2\epsilon r_2 + 2\chi r_1^2 r_2. \quad (19)$$

We used a fourth order Runge–Kutta method [11] to compute the classical trajectories. Poincaré sections [12,13] have been produced for a variety of energies, showing the transition from almost totally regular motion at low energies to almost totally irregular motion at high energies. Conservation of energy restricts any trajectory in four–dimensional phase space to a three–dimensional energy shell. At a particular energy, therefore, the restriction $r_1 = 0$ defines a two–dimensional surface in phase space. Each time a particular trajectory passes through the surface, i.e. each time it crosses the $r_2$
axis, a point is plotted at the position of intersection \((r_2, p_2)\). We employ a first-order interpolation process to reduce inaccuracies due to the use of a finite step length [13].

Regular regions on the surface of section plots are characterised by sets of invariant intersection points. The surface of section pictures (fig. 4) show that motion is almost wholly regular for \(E = 15\) MeV. However, for \(E = 21\) MeV we see evidence of bifurcation: one can count at least 6 new elliptic points and some points are distributed irregularly. For \(E = 23\) MeV the number of irregularly distributed points increases but many regular regions remain.

These numerical results are in good agreement with the smooth transition to chaos predicted by the curvature criterion.

5. Quantum calculations

In quantum mechanics one cannot apply classical concepts and methods directly since the notion of trajectory is absent. Nevertheless, many efforts have been made to establish the features of quantum systems which reflect the qualitative difference in the behaviour of their classical counterparts [2]. Many schematic models [2,3] have shown that this difference reveals itself in the properties of fluctuations in eigenvalue sequences. The spectral statistics for the systems with underlying chaotic behaviour agree with the predictions of the random matrix theory. By contrast, quantum analogs of classically integrable systems display the characteristics of Poisson distribution.

To obtain the quantum mechanical energy levels of the system we can
write the Hamiltonian (4) with the creation and destruction operators:

\[ a_k = (r_k + ip_k), \quad a_k^+ = (r_k - ip_k), \]  

(20)

where \( k = 1, 2 \), so we have:

\[ H = \epsilon(a_1^+a_1 + a_2^+a_2 + 1) - \frac{\chi}{16}(a_1^+ + a_1^2)(a_2 + a_2^+)^2. \]  

(21)

The eigenvalues can be calculated by diagonalizing the Hamiltonian in the basis \(|n_1n_2>\) of the occupation numbers of the two harmonic oscillators. The matrix elements can be written:

\[
< n_1'n_2'|H|n_1n_2 > = \epsilon(n_1 + n_2 + 1)\delta_{n_1'n_1}\delta_{n_2'n_2} + \\
\frac{\chi}{16}[\sqrt{n_1(n_1-1)}\delta_{n_1'n_1-2} + \sqrt{(n_1+1)(n_1+2)}\delta_{n_1'n_1+2} + (2n_1+1)\delta_{n_1'n_1}]	imes \\
\times [\sqrt{n_2(n_2-1)}\delta_{n_2'n_2-2} + \sqrt{(n_2+1)(n_2+2)}\delta_{n_2'n_2+2} + (2n_2+1)\delta_{n_2'n_2}] 
\]

(22)

and each submatrix can be labelled by the parity of the occupation numbers \( n_1, n_2 \). By performing the unfolding procedure described in detail in reference [14], each spectrum has been mapped into one with quasi-uniform level density.

The distribution \( P(S) \) of spacings between adjacent levels has been calculated and compared to the Brody [15] distribution:

\[ P(S) = \alpha(q + 1)S^q exp(-\alpha S^{q+1}), \]  

(23)

with:

\[ \alpha = (\Gamma(q + 2)/q + 1)^{q+1}, \quad 0 \leq q \leq 1. \]  

(24)
The distribution interpolates between the Poisson distribution \((q = 0)\) of integrable systems and the Wigner–Dyson distribution \((q = 1)\) of chaotic ones. In figure 5 (a) the distribution \(P(S)\) obtained with all the four classes is plotted for \(0 \leq E \leq 12\) MeV with \(q = 0\), and in (b) for \(12 \leq E \leq 23\) MeV with \(q = 0.15\).

6. Conclusions

In this paper we have shown that the quadrupole–quadrupole residual interaction leads to the transition from order to chaos. However the energy of the onset of chaos \((E_C \simeq 12.6\) MeV\) is much higher than the experimental one of atomic nuclei (chaotic behaviour \(\simeq 8\) MeV) [5,6]. Therefore the quadrupole–quadrupole force is unable to reproduce experimental nuclear data and higher multipole components should be added [16].

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Figure Captions

Figure 1: The potential energy of the model ($A = 200$).

Figure 2: The contour plot of the potential energy and the region of negative curvature (marked by ·).

Figure 3: Ratio of stable points in phase space obtained from the curvature criterion.

Figure 4: The Poincaré sections of the model: (a) $E = 7$ MeV, (b) $E = 15$ MeV, (c) $E = 23$ MeV.

Figure 5: The distribution $P(S)$ of spacings $S$ between adjacent levels: (a) $0 \leq E \leq 12$ MeV ($q = 0$), (b) $12 \leq E \leq 23$ MeV ($q=0.15$).
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