AdS$_3 \times \mathbb{R}$ AS A TARGET SPACE FOR THE (2,1) STRING THEORY

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Abstract

We study a target space geometry of the form $AdS_3 \times \mathbb{R}$ for the (2,1) heterotic string. This target space arises as the near horizon limit of a solitonic configuration in $2 + 2$ dimensions. We investigate the null isometries of this space and discuss the reduction to $1 + 1$ dimensions of the target space geometry arising from the consistent gauging of one of these isometries.

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1 Introduction

The $(2, 1)$ string theory is a string theory that has an ostensibly four dimensional target space of signature $(- - + +)$. This four manifold in turn is the world-volume of an extended object with a twelve dimensional target space of signature $(2, 10)$ \[1\]. In addition to the equations of motion for the four dimensional geometry, the $(2, 1)$ string theory requires that the four manifold $\mathcal{M}$ possess an isometry and that in the language of the $(2, 1)$ sigma model this isometry must be gauged \[2\]. This paper presents a particular non-trivial solution to the target space equations of motion that possesses null isometries. We discuss the meaning of the gauged isometries of the sigma model in the context of the action for the $2 + 2$ dimensional world volume (which we shall heretofore refer to as the M-brane). In some respects the construction reflects recent work in string theory relating supergravity in anti-de-Sitter space-time to D-brane world-volume theories. This similarity arises as the metric that we consider is the metric on $AdS_3 \times \mathbb{R}$ and the gauging of the isometry reminds one of the way in which the boundary of the $AdS$ space-time is related to the D-brane world-volume.

In this paper we will restrict ourselves to the null isometries of $\mathcal{M}$. The requirement of the null isometry arises as the world-sheet of the $(2, 1)$ string possess a $U(1)$ (on the $N = 1$ side) that is geometrically realized in the target space of the string. The $U(1)$ current must be gauged for consistency of the world-sheet theory, and this requires that the target space possess an isometry \[2\]. The requirement of freedom from gauge anomalies on the world sheet additionally requires that the $U(1)$ corresponds to a null isometry of the target space. The target space of this string theory has in addition to the $2 + 2$ space-time co-ordinates eight chiral scalar fields on the left-moving $N = 1$ side. As the $U(1)$ is in the left moving sector of the supersymmetry algebra, the null isometry can either lie completely in the $2 + 2$ target space, or it can include a component in one of the additional eight directions. From the conformal field theory analysis of the $(2, 1)$ world sheet one finds that in the former case after gauging the isometry one finds a D-string world volume as the target space (which is the only case that we will consider here), and in the latter one finds the D2-brane world volume \[3\].

The second section of the paper is devoted to an overview of the $(2, 1)$ string theory, and its relevance to recent discussions of M-theory and the unification of string theories. In this section we also review the $(2, 1)$-sigma model geometry. This has been discussed in detail in a series of papers by Hull and Abou-Zeid, and we refer to their papers for a more complete discussion \[3\]. In the third section we introduce the space-time of interest that solves the low energy effective field theory of the $(2, 1)$ string theory. In the fourth section we look at the null isometries of this space-time and show that they satisfy the additional consistency condition presented in section one. In the final section we discuss the mechanics of the reduction process in the context of the world-volume action of the M-brane.
2 AN OVERVIEW OF THE (2, 1) STRING THEORY

The (2, 1) string theory provides a potentially unifying framework for the various known string theories. In this section we firstly present an overview of this string theory and in particular the origin of the null $U(1)$ current - for more details see the Cargese lectures of Martinec [4]. We then review the features of the (2, 1) sigma model relevant to the gauging of this $U(1)$ current.

2.1 The (2, 1) string and M-theory

The world-sheet theory of the (2, 1) string is analogous in structure to that of the heterotic (1, 0) string theory in the manner in which the left and right moving sectors are combined. The $N = 2$ string theory, has a target space critical dimension of four with a complex structure and a signature that respects this structure, and therefore for a lorentzian signature we need a metric of signature $(- - + +)$. For the $N = 1$ fields we find a standard fermionic string living in $9 + 1$ dimensions. To put the left and right moving fields together, thus giving us a $2 + 2$ target space, we need to extend the chiral $9 + 1$ dimensional target space (related to the $N = 1$ superalgebra) to a $10 + 2$ dimensional target space, leaving over an internal 8 bosonic fields which for reasons of modular invariance must be compactified on an $E_8$ lattice, (analogous to the situation for the sixteen chiral bosonic fields of the usual heterotic string theory). However, now we of course find that the conformal anomaly is no longer saturated and to repair this defect one must supplement the $N = 1$ superconformal algebra by a $U(1)$ current algebra. This is in fact fortuitous as the $N = 2$ supersymmetry of the right moving sector demands that corresponding to the $U(1)$ of the $N = 2$ algebra, the $N = 1$ left-moving SUSY algebra needs to have an anomaly free $U(1)$ current.

The bosonic fields on the worldsheet are,

$$x^i_r, \ i = 0 \ldots 3; x^m_l, \ m = 0 \ldots 11,$$  \hspace{1cm} (2.1)

and their fermionic partners are,

$$\psi^i_r; \psi^m_l,$$  \hspace{1cm} (2.2)

where the fields with subscript $r(l)$ are right(left) moving fields on the worldsheet. Putting the $x^i_{r(l)}$ together, we find fields $x^i$ which are co-ordinates in a four dimensional target space with signature $(- - + +)$. The gauged $U(1)$ supercurrent is $J_l = v_m \partial x^m_l$, $\Psi_l = v_m \psi^m_l$ and for anomaly freedom of this current we need $v$ to be a null vector, $v^2 = 0$. The effect of this gauged current is a reduction of the naively $2+2$ dimensional target space to a $1+1$ or $2+1$ dimensional space depending on whether the null vector $v$ lies entirely in the $2+2$ dimensional part of the target, or if it contains a component in one of the time-like directions and the other component in one of the eight internal dimensions of the left moving fields. For example if $v = (1, 0, 1, 0, 0, \ldots, 0)$ then the target is a
1-brane with bosonic co-ordinates \((x^1, x^3)\), fermionic partners \((\psi^1, \psi^3)\) and it has the world-volume field structure of the D-string of type IIB string theory, i.e., it carries eight bosonic co-ordinates and a gauge field \([1]\). If on the other hand \(v = (1, 0, 0, \ldots, 0, 1)\) then the target is \(2 + 1\) dimensional and has the action and field content of the D2-brane world-volume theory, with seven bosonic co-ordinates and a gauge field \([1]\). For the details of this construction, including the explicit vertex operators, and the conditions arising from the null vector and the Virasoro constraints see \([1]\).

From the world-sheet theory of the \((2, 1)\) string we can also understand something about the space-time in which the \(2 + 2\) brane (or more properly its appropriate reduction via \(v\)), is embedded. The vertex operators of the string theory are the fields of the \(2 + 2\) brane and thus are the co-ordinates of the space-time in which it is embedded. In this way one discovers the above relationship between the choice of \(v\), D1 or D2 brane world volumes, and furthermore we learn something about the ten dimensional target spaces of these D-branes in the corresponding string theories.

One can move on from this point to also find physics of other critical ten dimensional theories with less supersymmetry, using the basic idea of the construction of Horava-Witten, wherein the heterotic and type IA string theories arise from M-theory compactified on \(S^1/Z_2\). In the context of the \((2, 1)\) string, this construction is carried out by performing an orientifold projection. One then finds, depending on the choice of the null vector \(v\), either a configuration involving a D-string stretched between two D7-branes in a type IIB configuration of the variety that one finds from F-theory, or one finds a D2-brane stretched between a pair of D8-branes. The orientifold projection that is used to construct these configurations is such that it acts on one time and one space direction in the M-brane world-volume. In the above, the null direction was chosen to always include the time-like direction on which an orientifold projection has been performed. Alternatively one can choose the other time-like direction, in which case one finds a configuration that has a Dirichlet boundary condition in a time-like direction. This gives rise to a Euclidean D-brane, or E-brane as it is referred to in recent work \([3]\). In the language of that paper, we find then that simply a rotation of \(v\) interchanges D-branes and E-branes.

It is important to emphasise the nature of the construction of D-branes from the target space of the \((2, 1)\) string. In all of these situations one begins the construction with the string theory world-sheet, the actual ten dimensional critical string target space is two steps away as it is the target space of the M-brane. To fully understand the various ten dimensional string theories from this perspective really requires a string field theory of the \((2, 1)\) string. The purpose of the remainder of this paper is to continue the program initiated in \([3, 6]\) of studying directly the M-brane world-volume theory. In particular we wish to focus on the issue of the gauging of the \(U(1)\) of the string theory and its relationship to null isometries of the M-brane geometry. We introduce a non-trivial target space that possesses such isometries and show that these isometries satisfy the
consistency conditions for their gauging [3]. We then look at the reduction of the action under these isometries and the geometry of the D-brane world-volume that remains after the gauging.

2.2 The gauged (2, 1) sigma model

We first review the low energy effective field theory of the (2, 1) string theory. The action in (1, 1) superspace for the bosonic part of the target space is,

\[ S = \frac{1}{4i} \int d^2\sigma d^2\theta (g_{ij}(x) + b_{ij}(x)) D_1^i x^i D_1^- x^j. \] \hspace{1cm} (2.3)

The \( x^i \) are co-ordinates in the target space \( \mathcal{M} \).

Additionally the action will have (2, 1) supersymmetry if \( \mathcal{M} \) is even dimensional with a complex structure satisfying

\[ J^i_j J^j_k = -\delta^i_k \]
\[ N_{ij}^k = 0 \] \hspace{1cm} (2.4)

which is covariantly conserved with respect to the generalised connection,

\[ \Gamma^{(+)}_{jk} = \begin{cases} i \\ j \\ k \end{cases} + g^d H_{jkl} \] \hspace{1cm} (2.5)

where \( H_{ijk} = (db)_{ijk} \) and \( \begin{cases} i \\ j \end{cases} \) is the Christoffel connection. The complex structure must also be such that,

\[ g_{ij} J^i_k J^j_l = g_{kl}. \] \hspace{1cm} (2.6)

The action is furthermore conformally invariant and thus a valid starting point for the corresponding string theory if the \( \beta \)-function equations following from the effective action

\[ S = \int d^D x e^{-2\Phi} \sqrt{-g} (R - \frac{1}{3} H^2 + 4(\nabla \Phi)^2) \] \hspace{1cm} (2.7)

are satisfied and \( D = 4[7, 8] \).

From now on we will use complex co-ordinates on a 2 + 2 dimensional target manifold with co-ordinates, \( z^\alpha, \bar{z}^\beta, \alpha, \beta = 1, 2 \). We will also consider only metrics for which the line element is of the form, \( ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \) and the complex structure is \( J_{\alpha\bar{\beta}} = ig_{\alpha\bar{\beta}} \).

The above conditions on the metric, torsion and complex structure of the target space imply that the geometry is determined by a vector field, \( k_\alpha \) in the following manner,

\[ g_{\alpha\bar{\beta}} = \partial_\alpha k_{\bar{\beta}} + \partial_{\bar{\beta}} k_\alpha \]
\[ b_{\alpha\bar{\beta}} = \partial_\alpha k_{\bar{\beta}} - \partial_{\bar{\beta}} k_\alpha \]
\[ H_{\alpha\beta\gamma} = \frac{1}{2} \partial_\gamma (\partial_\alpha k_{\beta} - \partial_\beta k_\alpha) \] \hspace{1cm} (2.8)
It was shown by Hull [9] that a sufficient condition for the satisfaction of the beta function equations (following from the above effective action with \( D = 4 \)) is,

\[
\Gamma^{(+)}_i = \Gamma^{(+)}_{jk} J^k_i = 0, \tag{2.9}
\]

and for this solution the dilaton field is given algebraically by the metric through,

\[
\Phi = -\log |\det g_{\alpha\beta}|. \tag{2.10}
\]

It is in fact easy to show that for a complex manifold with anti-symmetric tensor field related to the metric as above, and for which the metric is conformally flat, the vanishing of \( \Gamma^{(+)}_{jk} J^k_i \) follows immediately. We now turn to a discussion of a particular solution to these equations which we believe is an illustrative example for the discussion of theories with two time directions. Notice that a four manifold that satisfies all of the above conditions and in addition the conditions that we present in section 4 following from the gauging of the null isometry [3] is likely to be quite restricted, and thus one may suspect that there are no other solutions apart from the one we present here.

As we see from the above discussion, the equations for the target space of the \((2,1)\) string can be reduced to the equation \( \Gamma^{(+)}_i = 0 \), which is then an equation to be solved for the vector field \( k_i \). This equation follows from the effective action [6, 10],

\[
S = \int d^4 x \sqrt{-\det g_{\alpha\beta}}. \tag{2.11}
\]

### 3 \( AdS_3 \times \mathbb{R} \)

We want to consider a non-compact version of the Hopf manifold, \( S^3 \times S^1 \) [11]. The metric is the product metric on \( AdS_3 \times \mathbb{R} \) with a complex structure such that in complex co-ordinates we have,

\[
ds^2 = \frac{dz_1 d\bar{z}_1 - dz_2 d\bar{z}_2}{z_1 \bar{z}_1 - z_2 \bar{z}_2} = \frac{\eta_{\alpha\beta} dz^\alpha d\bar{z}^\beta}{\eta_{\alpha\beta} z^\alpha \bar{z}^\beta}. \tag{3.1}
\]

It is easy to see that this is the standard product metric on \( AdS_3 \times \mathbb{R} \) by noticing that at fixed \( \rho = \eta_{\alpha\beta} z^\alpha \bar{z}^\beta \) we find the metric on a flat \( 2 + 2 \) dimensional space-time restricted to the hyperboloid \( \rho = \text{constant} \) (the standard construction of the metric on \( AdS_3 \)), and then in the radial (\( \rho \)) direction the metric is a flat metric in logarithmic co-ordinates thus the flat metric on \( \mathbb{R} \).

In the euclidean case, this manifold has been discussed in [12], as a four dimensional string theory target space with the metric of \( S^3 \times \mathbb{R} \) and a linear dilaton in the flat direction. The corresponding conformal field theory is then an \( SU(2) \times U(1) \) WZW
model. The $AdS_3$ configuration has recently been analyzed as a non-compact WZW model in the context of the AdS – conformal field theory correspondence [13]. In our case the conformal field theory will be $SU(1,1) \times U(1)$ and deserves further study in the context of $(2,1)$ string theory.

In fact, this solution is really the near horizon limit of a solitonic solution that interpolates between flat space and a non-trivial configuration as a function of $\rho$. This geometry is given by the metric,

$$ds^2 = (dz_1d\bar{z}_1 - dz_2d\bar{z}_2)(1 + k\frac{z_1}{z_1 - z_2}). \quad (3.2)$$

We now see that the $\rho \to 0$ “near-horizon” limit or alternatively the $k \to \infty$ large charge limit gives rise to the geometry of equation (3.1), and therefore we see here a 2 + 2 dimensional version of the constructions that originally led Maldacena to his conjecture relating supergravity to SUSY Yang-Mills theory. In [14], the near horizon geometry of a solitonic D-brane configuration was related to the SUSY Yang-Mills theory that lives on the D-brane world-volume. In our case we should find a relationship between the $AdS_3 \times \mathbb{R}$ self-dual gravity of the $(2,1)$ string target space with null isometry, and the D-string world volume theory that has been shown in [1] to arise after gauging a null $U(1)$ that lies entirely within $\mathcal{M}$.

The vector field $k_\alpha$ from which one can derive the metric and antisymmetric tensor field of (4.1) is,

$$k_1 = \frac{1}{4z^1} \log \frac{\rho}{\sqrt{z^2}} \quad (3.3)$$

$$k_2 = \frac{1}{4z^2} \log \frac{\rho}{\sqrt{z^1}} \quad (3.4)$$

where the antisymmetric tensor field strength is,

$$H_{\alpha\beta\bar{\gamma}} = \frac{1}{2}(g_{\alpha\beta,\bar{\gamma}} - g_{\beta\bar{\gamma},\alpha})$$

$$= \frac{1}{4\rho^2} (\eta_{\beta\bar{\gamma}} z_\alpha - \eta_{\alpha\bar{\gamma}} z_\beta), \quad (3.5)$$

and is identical (up to a factor of $k$ - the charge of the soliton) for the $AdS_3 \times \mathbb{R}$ configuration (3.1) and the interpolating soliton configuration (3.2). As this soliton carries a non-trivial anti-symmetric tensor field, the configuration is appropriately interpreted as the solitonic form of the fundamental $(2,1)$ string. Summarising the above, we have argued that the near horizon/large charge geometry of the solitonic fundamental $(2,1)$ string configuration in the 2+2 dimensional target space of the $(2,1)$ string, may in turn be thought of as the 1+1 dimensional world-volume of the type IIB D-string embedded in the 1+9 dimensional space-time of the IIB string theory.

As explained at the end of the previous section, this manifold (and also the interpolating soliton) will satisfy the equations of motion that arise from the $(2,1)$ string theory as a
consequence of the fact that it is conformally flat and that the geometry can be encoded entirely in the vector field $k_a$. From (3.8) we see immediately that the dilaton in the M-brane world-volume is proportional to $\log \rho$ and thus on each $AdS_3$ represented by an hyperboloid in $R^{2,2}$ the dilaton field is a constant, once more as happens in the non-singular solitonic D-branes of ten and eleven dimensional supergravities where Maldacena’s conjecture [14] works best - M2-brane, M5-brane, D1+D5-brane and D3-brane.

The isometry group is $SO(2,2) \times \mathbb{R}$ and is made up of the isometries of $AdS_3$ and translations in the additional flat direction. The condition of a gauged $U(1)$ in the world sheet theory requires a null isometry in this target space. The invariance of the complex structure under the isometry additionally requires that the isometry be holomorphic. The additional conditions on the null isometry for a consistent gauging of the sigma model will be discussed in the next section. We will now turn to the Killing vectors of our metric.

The Killing vectors belonging to $SO(2,2)$ in real coordinates $x^i$ are $L_{ij} = x_i \partial_j - x_j \partial_i$, where $z^1 = x^1 + ix^2$ and $z^2 = x^3 + ix^4$, where we are for the moment raising and lowering indices with the flat metric $\eta_{ij}$. It is convenient for us to consider the following linear combinations of these generators, corresponding to the isomorphism between $SO(2,2)$ and $SU(1,1) \times SU(1,1)$,

$$
\begin{align*}
\xi_{(1)} &= \frac{1}{2} (L_{23} - L_{14}) \\
\eta_{(1)} &= \frac{1}{2} (L_{23} + L_{14}) \\
\xi_{(2)} &= \frac{1}{2} (L_{31} - L_{24}) \\
\eta_{(2)} &= \frac{1}{2} (L_{31} + L_{24}) \\
\xi_{(3)} &= \frac{1}{2} (L_{12} + L_{34}) \\
\eta_{(3)} &= \frac{1}{2} (L_{12} - L_{34}).
\end{align*}
$$

In the complex basis these vectors have the following components,

$$
\begin{align*}
\xi_{(1)} &= i(z^2, -z^1) \\
\eta_{(1)} &= i(\bar{z}^2, \bar{z}^1) \\
\xi_{(2)} &= (-z^2, -z^1) \\
\eta_{(2)} &= (-\bar{z}^2, -\bar{z}^1) \\
\xi_{(3)} &= i(z^1, -z^2) \\
\eta_{(3)} &= i(z^1, z^2)
\end{align*}
$$

The remaining $U(1)$ isometry is generated by global scale transformations and has components,

$$
S = (z^1, z^2).
$$

Of these isometries, clearly $\eta_{(1)}$ and $\eta_{(2)}$ are not holomorphic and thus do not preserve the complex structure. Of the remaining vectors we have several linear combinations that are also null. A representative selection of these vectors are $\xi_{(1,2)} \pm S$ and $\xi_{(1,2)} \pm \xi_{(3)}$. 

7
4 Null Isometries

The conditions for the gauging of a null isometry $\zeta$ in the target space of the $(2,1)$ supersymmetric sigma model was analyzed in [3]. It is shown in that work that the isometry must be holomorphic, that there must be a vector field $u_\alpha$ solving the equation,

$$\partial[\alpha u_\beta] = \zeta^\gamma H_{\alpha\beta\gamma}, \quad (4.1)$$

and there must exist a complex scalar potential $iX + Y$ such that,

$$\zeta_\alpha + u_\alpha = \partial_\alpha(iX + Y). \quad (4.2)$$

Note from (4.1) that the solution for $u_\alpha$ has a shift symmetry by the gradient of a scalar, and thus up to global considerations one can remove the function $Y$ by absorbing it into $u_\alpha$. These fields must satisfy the following additional conditions (see [15, 3] respectively),

$$\mathcal{L}_\zeta X = 0$$

$$\zeta^\dagger u_\iota = 0. \quad (4.3)$$

In words the first of these follows from the equivariance of $u_\alpha$ under the action of the $U(1)$ isometry group and the second from the consistency of the WZW term in the $(2,1)$ sigma model.

For our particular solution there are two distinct classes of null isometries both of which satisfy the above conditions as we will now show explicitly. In one class we have the null isometries of the $AdS_3$ space-time and the other class involves the sum of an isometry of the $AdS_3$ and a translation in $\mathcal{R}$.

The first example that we will consider is the case in which the isometry lies entirely within the $AdS_3$ space. The holomorphic null vector, $\zeta^{(1)}_\alpha = (\xi^{(3)} - \xi^{(2)})^\alpha$ which in complex co-ordinates is $(iz^1 + z^2, -iz^2 + z^1)$. In this case one finds,

$$u_\alpha = \frac{1}{2\rho} (iz^1 - \bar{z}^2, iz^2 + \bar{z}^1). \quad (4.4)$$

It is easy to see then that after lowering the indices on $\zeta^{(1)}$ we have $\zeta^{(1)} + u = 0$ implying $X + iY = \text{const.}$ and (4.3) are satisfied.

As a second example consider the null isometry $\zeta^{(2)}_\alpha = (S - \xi^{(1)})^\alpha$, which in complex co-ordinates is $(z^1 - iz^2, z^2 + iz^1)$. For this choice of $\zeta^{(2)}$ $u$ is,

$$u_\alpha = -\frac{i}{2\rho} (z^2, \bar{z}^1) \quad (4.5)$$

and

$$(\zeta^{(2)} + u)_\alpha = \frac{1}{2\rho} (\bar{z}^1, -\bar{z}^2) = \frac{1}{2} \partial_\alpha \log \rho. \quad (4.6)$$
\[ \zeta + u \text{ is therefore the gradient of} \]
\[ iX + Y = \frac{1}{2} \log \rho + \text{const}. \quad (4.7) \]

Making use of the gauge freedom that allows us to absorb \( Y \) into \( u \) we can shift \( u \) by \(-\partial_{\alpha} Y\) so that \( \zeta = -u \), \( X = \text{const.} \) and the consistency conditions (5.3) are again clearly satisfied.

5 Reduction along null isometries

In this section we make a preliminary investigation of the null reduction from the point of view of the world-volume theory of the M-brane. We want to look at the bosonic part of the target space action and investigate how the null reduction works. In particular we will find that the mechanism has some features in common with the study of singleton fields in AdS spaces and also appears as an interesting lower dimensional version of the relationship between gauge theories and supergravity in AdS space-times [14]. In this paper we will discuss the situation for null vectors living in the M-brane world volume. Null vectors that include some of the internal left-moving co-ordinates will be left for future work.

We claim that the reduction proceeds by considering the above background, in the context of the complete 2 + 2 dimensional Dirac-Born-Infeld type lagrangian derived for this model by Kutasov and Martinec [6]. Imposing the condition that the fields of the model have vanishing Lie derivative along the null isometry, we see that this effectively reduces the world-volume action from 2 + 2 to 1 + 1 dimensions. A direct comparison can then be made between the terms in this action and the terms in the action for the D-string world-volume. The M-brane action is the effective action we introduced at the end of section 2, but now we include also the contributions from the eight target-space scalars [6],

\[ S = \int d^4x \sqrt{-\det(g_{\alpha\bar{\beta}} + \partial_\alpha \phi^a \partial_{\bar{\beta}} \phi^a)} \quad (5.1) \]

where \( a = 1, \ldots, 8 \) and \( \alpha = 1, 2 \). The additional eight scalar fields arise as fields in the target space from the \( N = 1 \) side of the supersymmetry algebra, just as the \( N = 0 \) sector of the \( (1,0) \) heterotic string gives rise to target space gauge fields. We will expand this action about the above background so that we can study the kinetic term only.

The expansion of the M-brane action to second order in \( \phi \) is,

\[ S = \int d^4x \left( \sqrt{-\det(g_{\alpha\bar{\beta}})} + \frac{(g_{11}A_{22} + g_{22}A_{11} - g_{12}A_{21} - g_{21}A_{12})}{2 \sqrt{-\det(g_{\alpha\bar{\beta}})}} + \cdots \right) \quad (5.2) \]

where \( A_{\alpha\bar{\beta}} = \partial_\alpha \phi^a \partial_{\bar{\beta}} \phi^a \). In terms of the flat metric \( \eta_{\alpha\bar{\beta}} \) the kinetic term is proportional to \( \eta^\alpha_{\bar{\beta}} A_{\alpha\bar{\beta}} \). Note that this term, quadratic in \( \phi \), is conformally invariant, a property
which one would expect for a standard kinetic term in two dimensions. This is a consequence of the non-standard factor of \((-\det g_{\alpha \beta})^{1/2} = (-\det g)^{1/4}\) in the M-brane action and is consistent with the fact that the dimensional reduction of the M-brane action to 1 + 1 dimensions gives rise to a standard Dirac Born-Infeld action for the type IIB D-string.

To understand more fully what happens in the reduction procedure, let us first consider the case for a target space that is 2 + 2 dimensional flat space. We will consider the kinetic term for the eight scalar fields that exist in the target space of the (2, 1) string, and appear in the square root of the action. For the flat target space with all other fields turned off we find the standard kinetic term for the flat metric \(\eta_{ij}\) of signature \((- - + +)\),

\[
S_{\text{kin}} = - \int d^4 x \partial_i \phi^a \partial_j \phi^a \eta^{ij} \quad (5.3)
\]

We consider the null isometry of \(\mathcal{M}\) corresponding to the vector, \(v = (1, 0, 0, 1)\). The vanishing of the Lie derivative of the scalar field in this direction reduces the kinetic term from a manifestly four dimensional form to a two dimensional form,

\[
S_{\text{kin}} = \int d^4 x \left( - \partial_2 \phi^a \partial_2 \phi^a + \partial_3 \phi^a \partial_3 \phi^a \right). \quad (5.4)
\]

Notice in particular that the null direction orthogonal to \(v\) has also disappeared from the action. The meaning of this absence is the new freedom to add any function of \(x_− = x^1 - x^4\) to \(\phi\). This represents a gauge invariance of the same form that arises in the discussion of singleton fields in \(AdS\) space \([16]\). This type of invariance is basically a condition that the fields of the theory are chiral in the sense that they are functions only of \(x_-, x^2\) and \(x^3\). However, we can easily see that the action does not actually govern the \(x_-\) behaviour of the fields and we effectively end up with a theory on a flat 1 + 1 dimensional space-time.

Returning now to our non-trivial metric we will first change co-ordinates in the kinetic term using the null Killing vectors. Consider two pairs of real light-cone co-ordinates, \(U = x^1 + x^4\), \(V = x^1 - x^4\), \(X = x^2 + x^3\) and \(Y = x^2 - x^3\) for which the metric takes the form,

\[
ds^2 = \frac{dU dV + dX dY}{UV + XY} \quad (5.5)
\]

It turns out to be convenient to consider the additional change of variables, \(U = R \cos \theta\), and \(Y = R \sin \theta\). The Killing vectors that we will consider are then,

\[
\zeta_1^i \partial_i = (\xi_3 - \xi_2) i^i \partial_i = U \partial_X - Y \partial_Y,
\]

\[
\zeta_2^i \partial_i = (S - \xi_1) i^i \partial_i = U \partial_U + Y \partial_Y = R \partial_R. \quad (5.6)
\]

In terms of these Killing vectors the kinetic term in the world-volume action takes the form,

\[
\int d^4 x \left( \frac{\partial Y \phi^a}{U} \zeta_1^i \partial_i \phi^a + \frac{\partial Y \phi^a}{U} \zeta_2^i \partial_i \phi^a \right). \quad (5.7)
\]
When we consider configurations for which \( \zeta^i \partial_i \phi = 0 \) then we find that the kinetic term has only a single off diagonal component,

\[
\frac{R}{U} \partial_R \phi^a \partial_V \phi^a = \frac{\partial_R \phi^a \partial_V \phi^a}{\cos \theta}.
\] (5.8)

The vanishing of the Lie derivative of \( \phi \) in the \( \zeta \) direction implies that \( \cos \theta \partial_X \phi - \sin \theta \partial_Y \phi = 0 \) which implies that \( \phi \) depends on \( X \) and \( V \) only in the combination \( y = X \sin \theta + V \cos \theta \). Changing variables in the surviving part of the kinetic term from \( V \) to \( y \), the \( \cos \theta \) in the denominator drops out and including the factor of \( R \) coming from the change to polar co-ordinates in the measure we get the final simple diagonal form, \( R \partial_y \phi^a \partial_R \phi^a \).

A further change of variables \( R = R_0 e^\xi \), results in a flat space kinetic term with 1 + 1 dimensional Lorentz invariance in almost precise analogy with the reduction discussed above for a flat target space. We also see that \( \phi \) may have, independent of the fixed \( \theta \) dependence of \( y \), an additional undetermined \( \theta \) dependence analogous to the undetermined \( x_- \) dependence of \( \phi \) in the flat space example.

Considering a general solution to \( \zeta^i \partial_i \phi = R \partial_R \phi = 0 \) the kinetic term in the action (using also \( \partial_Y = \sin \theta \partial_R + \frac{1}{R} \cos \theta \partial_\theta \)) is,

\[
S_{kin} = \int dV dX dR d\theta (\frac{\cos \theta \partial_X \phi^a - \sin \theta \partial_Y \phi^a}{\cos \theta} (\sin \theta \partial_R \phi^a + \frac{1}{R} \cos \theta \partial_\theta \phi^a)).
\] (5.9)

Unlike for the gauging of \( \zeta \) we cannot further simplify this kinetic term since the first term cannot be written as a derivative along some co-ordinate direction. On the other hand the metric that we deduce from this kinetic term has eigenvalues 0, \( \pm \frac{1}{2} \) and thus we have again ended with a 1 + 1 dimensional theory.

In the above examples of “dimensional reduction” we have ignored the behaviour of the field in the direction in \( M \) complementary to the direction of the null isometry of interest, for example in the discussion of the reduction for the flat target space we ignored the undetermined dependence of \( \phi \) on \( x_- \). To extend the analysis we need to look at the constrained dynamics of a chiral bosonic field of the form \( \phi(x_-, x_2, x_3) \) (in the notation used for the flat space example) in 2 + 2 dimensions, or the similar co-ordinate dependence of \( \phi \) for the non-trivial metric of this paper.

For the “gauging” of the direction \( \zeta \) a more suggestive approach to the undetermined behaviour of \( \phi \) in the direction orthogonal to the gauged null direction may be the following. In our metric the orthogonal null direction to \( \zeta \) is actually, \( \eta^i \partial_i = V \partial_Y + X \partial_X \). If the Lie derivatives in this direction are also zero (akin to removing the \( x_- \) dependence of \( \phi \) in the previous paragraph) then additionally the Lie derivatives in the direction \( \eta + \zeta = S \) must vanish, and thus \( \phi \) is in particular independent of \( \rho \). Using a \( \phi \) of this form in the kinetic term results in (for a fixed value of \( \rho \)),

\[
\partial_X \phi^a \partial_Y \phi^a (\frac{\rho}{\rho - XY}).
\] (5.10)
Furthermore as $\phi$ is independent of $\rho$ we may take $\rho \to \infty$ to get a $1 + 1$ dimensional kinetic term with a flat Minkowski metric. This is precisely the way in which the singleton fields are found as $(d - 2) + 1$ dimensional fields which live on the boundary at $\rho \to \infty$ of the $AdS_d$ space-time, (see [17] for a recent review). In this case also, the reduction has involved an additional assumption which is in a similar spirit to the holography assumption invoked in the recent work on $AdS$ spaces initiated in [14].

The null reduction including fermion fields does not appear to be as straightforward since the action is only first order in derivatives. Thus the removal of the direction complementary (in the above sense) to the direction of the null isometry does not appear to happen in as simple a manner as for the bosonic fields described above. In the reduction from $2 + 2$ to $1 + 1$ dimensions one does have the freedom to now restrict the two dimensional chirality of the spinors that one obtains from the four dimensional spinors. This may be sufficient to allow the null reduction to take us all the way to two dimensions. Related to this subtlety arising in the incorporation of fermions in the above construction is the fact that in the current formulation of the M-brane world-volume action the correct form for the fermionic terms is not currently known [1]. Turning this discussion around then, it is possible that formulating the dimensional reduction including fermions appropriately will lead to some new insight into the construction of the full supersymmetric low energy effective action for the $(2,1)$ string theory.

Acknowledgements

The authors would like to thank Matthias Blau for useful conversations during the course of this work and C. Vafa for a useful email exchange. This work was supported in part by the EC under the TMR contract ERBFMRX-CT96-0090.

References

[1] D. Kutasov, E. Martinec, “New principles for string/membrane unification”, Nucl. Phys. B477 (1996), 652, hep-th/9602049. D. Kutasov, E. Martinec and M. O’Loughlin, “Vacua of M-theory and $N = 2$ strings”, Nucl.Phys. B477 (1996), 675, hep-th/9603119.

[2] H. Ooguri and C. Vafa, “$N = 2$ heterotic strings”, Nucl. Phys. 367 (1991), 83.

[3] M. Abou-Zeid and C.M. Hull, “The gauged $(2,1)$ heterotic sigma model”, Nucl. Phys. B513 (1998), 490, hep-th/9708047. “Geometry, isometries and gauging of $(2,1)$ heterotic sigma models”, Phys. Lett. B398 (1997), 291, hep-th/9612208.

[4] E.J. Martinec, “M-theory and $N = 2$ strings”, Talk given at the NATO Advanced Study Institute on Strings, Branes and Dualities, Cargese, France, 1997, hep-th/9710122.

[5] C.M. Hull, “Timelike T-duality, de-Sitter space, Large N gauge theories and topological field theory”, hep-th/9806146.
[6] D. Kutasov and E. Martinec, “M-branes and $N = 2$ strings”, Class. Quant. Grav. 14 (1997), 2483, hep-th/9612102.

[7] C. Callan, D. Friedan, E. Martinec and M. Perry, “Strings in background fields”, Nucl. Phys. B262 (1985), 593.

[8] C.M. Hull and P. Townsend, “Finiteness and conformal invariance in non-linear sigma models”, Nucl. Phys. B274 (1986), 349.

[9] C.M. Hull, “Compactifications of the Heterotic Superstring”, Phys. Lett. 178B (1986), 357, “Sigma model beta functions and string compactifications”, Nucl. Phys. B267 (1986), 266.

[10] C. M. Hull, “Actions for (2, 1) sigma-models and strings”, Nucl. Phys. B509 (1998), 252.

[11] H. Hopf, Courant anniversary volume, (1948), 168; see also E. Calabi and B. Eckmann, Ann. of Math. 58 (1953), 494.

[12] C. Callan, J. Harvey and A. Strominger, “Supersymmetric string solitons”, Trieste 1991, Proceedings, String theory and quantum gravity 1991, 208, hep-th/9112030.

[13] J.M. Evans, M.R. Gaberdiel and M.J. Perry, “The no-ghost theorem for $AdS_3$ and the string exclusion principle”, hep-th/9806024 A. Giveon, D. Kutasov and N. Seiberg, “Comments on string theory on $AdS(3)$”, hep-th/9806194 K. Behrndt, I. Brunner and I. Gaida, “$AdS(3)$ gravity and conformal field theories”, hep-th/9806195.

[14] J. Maldacena, “The large $N$ limit of superconformal field theories and supergravity”, hep-th/9711200 E. Witten, “Anti-de Sitter space and holography”, hep-th/9802150 S. Gubser, I Klebanov and A. Polyakov, “Gauge theory correlators from non-critical string theory”, hep-th/9802109.

[15] C. Hull, G. Papadopoulos and B. Spence, “Gauge symmetries for the $(P,Q)$ supersymmetric sigma models”, Nucl. Phys. B363 (1991), 593.

[16] A.M. Harun ar Rashid, C. Fronsdal, M. Flato, “Three D singletons and 2-D C.F.T.”, Int. J. Mod. Phys. A7 (1992), 2193.

[17] S. Ferrara and C. Fronsdal, “Conformal Maxwell theory as a singleton field theory on $AdS(5)$, IIB three-branes and duality”, hep-th/9712239.