Explosive Synchronization in Inter-pinned Multilayer and Multiplex Networks

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This Letter investigates the nature of synchronization in multilayered and multiplexed populations in which the interlayer interactions are randomly pinned. First, we show that a multilayer network constructed by setting up all-to-all interlayer connections between the two populations leads to explosive synchronization in the two populations successively, leading to the coexistence of coherent and incoherent populations forming chimera states. Second, a multiplex formation of the two populations in which only the mirror nodes are interconnected espouses explosive transitions in the two populations concurrently. The emergence of explosive synchronization is substantiated with rigorous mean-field calculations demonstrating the existence of a bistable region. The random pinning in the interlayer interactions concerns the practical problems where the impact of dynamics of one network on that of other interconnected networks remains elusive, as is the case for many real-world systems.

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Introduction. The dynamical evolution of large-scale complex systems having underlying graph structures has been popularly modeled using coupled Kuramoto oscillators on networks [1, 2]. A multilayer network, which refers to the same sets of nodes having different types of interactions among its units, has brought forward many astonishing phenomena and sheds light on the mechanisms behind emerging behaviors beyond a single layer framework [3–11]. One such behavior is explosive synchronization (ES), which refers to the first-order transition to synchronization [12–20]. In contrast to a smooth transition to synchronization, an abrupt jump to the coherence accompanied by hysteresis is witnessed as the strength of couplings between the interacting units increases. By accommodating coupled Kuramoto oscillators on multilayer networks, it has been shown that the dynamical or structural features in layer(s) characterizing delay, constant phase lag, weight adaptation play a crucial role in governing ES in other or all the layers in multilayer networks [21–26].

Further, a chimera state (CS) refers to the coexistence of coherent and incoherent states, which is an upshot of the partial symmetry breaking of the system [27, 28]. A decade ago, D. M. Abrams et al. [29] reported the breathing chimera states in two groups of identically coupled phase lagged oscillators. Recently, borrowing the same model but composed of non-identical oscillators, the study is extended for a wider parameter space exhibiting various chimera states [30]. Later, a model considering two groups of the phase lagged non-identical oscillators in the presence of adaptively controlled coupling reported the bridging of ES with the chimera state [31].

Strogatz et al. [32, 33] showed that pinning the phases in networked oscillators to random phases leads to the emergence of the ES route. This work introduces a new concept of multilayer inter-pinning, which involves pairs of interconnecting nodes in two populations of non-identical oscillators stuck at independent random phases. Such a scheme is more relevant when a particular impact of the interdependence, i.e., how activities of one network get affected by those of the other networks, is not known or decipherable from the available data, which is the issue for many complex systems (Supplementary Material). Here we show that the inter-pinned multilayer network sports an interesting dynamical feature, the existence of chimera states (CS) during the explosive transition to synchronization and then desynchronization. The multilayer setup leads to the ES transitions in the two populations in succession, i.e., one population stays synchronous while the other stays asynchronous. We also covered multiplex inter-pinning in which the parallel nodes in the two populations are pinned to the same set of random phases. This setup induces ES transitions in the two populations concurrently, i.e., the occurrence of CS is not witnessed. Our investigation creates distinctions between the dynamical characteristics of the multilayer and multiplex inter-pinning.

FIG. 1. (Color online) $r_1-\lambda$ exhibiting ES transitions in GC-GC multilayer network having uniform $\omega_i \in [-0.5, 0.5]$. The green and dark grey solid lines delineate the theoretical predictions discussed later in the study.
A. Dynamics on Multilayer Networks

We begin with considering a multilayer network comprising two interacting non-identical populations of the same number of nodes \( N \). The evolution of phases \( \theta^l_i \) (\( i = 1 \ldots N \)) in either population \( l \in \{1, 2\} \) is governed by

\[
\dot{\theta}^l_i = \omega^l_i + \frac{\lambda}{N} \sum_{j=1}^{N} \sin(\theta^l_j - \theta^l_i) + \frac{D}{N} \sum_{k=1}^{N} \sin(\theta^b_k - \theta^l_i - \alpha^l),
\]

(1)

where the \( \alpha^l \) are independent random phases uniformly distributed on the interval \( \alpha^l \in [0, 2\pi] \). The random pinning phases \( \alpha^l \) corresponding to the mirror nodes \( \{\theta^l_i, \theta^b_i\} \) in the two populations are taken to be the same. Hence, the phase differences \( (\theta^b_k - \theta^l_i) \) of all the \( k \) nodes in population \( l' \) with an interconnected node \( i \) in population \( l \) are pinned at a random phase \( \alpha^l \). For that matter, the third term tends to lead to a static disorderliness among all the interlayer phase differences. This static disorderliness creates frustration among the nodes in either population and hinders the synchronization process in them. Here, the interlayer coupling strength \( D \) also serves as the pinning strength. The second term fosters intra-population coherence as the homogeneous coupling strength \( \lambda = \lambda_l \) strengthens. The frequencies of the nodes in either population follow a uniform or symmetric distribution \( g(\omega^l) \).

Our aim is to comprehend how the behavior of phase synchronization in the two populations is influenced by the random inter-pinning. For that matter, the degree of synchronization in each population is determined by the order parameter defined as

\[
r^l e^{i\psi^l} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta^l_j},
\]

(2)

where \( \psi^l \) is the average phase of population \( l \). A stationary value of \( r(t) = r \approx 1 \) implies coherence whereas \( r(t) \approx 0 \) means complete incoherence. We begin our investigation by constructing a multilayer network of two all-to-all connected (GC) populations, each of size \( N = 1000 \). The interlayer couplings between them are subject to random pinning, as discussed before. Different samples of natural frequencies for the two populations are selected from either uniform \( g(\omega^l) \in [-\Delta, \Delta] \) or unimodal symmetric \( g(\omega^l) \) with mean 0. Distinct samples of phases for the two populations are drawn uniformly randomly on \( [0, 2\pi] \). Phase dynamics of the multilayer network Eq.(1) is evolved using RK4 method with step size \( dt = 0.01 \).

Forward (f) and backward (b) phase transition is then observed by computing order parameters against each adiabatically increasing or decreasing coupling strength \( \lambda \) in the steps of \( d\lambda \), respectively. In Fig.1, the order parameter corresponding to different values of the pinning strength \( D \) is plotted for the forward and backward continuation in \( \lambda \). It unveils that a sufficient pinning strength \( D \) exerts frustration at the interconnected nodes and leads to a discontinuous transition in the two populations, accompanied by hysteresis. It is apparent that two sets of two distinct critical coupling strengths exist, one \( \{\lambda^{f1}_c, \lambda^{f2}_c\} \) for the forward abrupt transitions and the other \( \{\lambda^{b1}_c, \lambda^{b2}_c\} \) for the backward abrupt transitions as shown in Fig.1. At the first forward critical \( \lambda^{f1}_c \), it is the initial condition dependence that one population experiences explosive transition while the other sees complete incoherence \( (r \approx 0) \). The two populations remain in their respective states until second forward \( \lambda^{f2}_c \) is reached, at which the incoherent population also goes through explosive synchronization and traces the other synchronous population. The abrupt desynchronization of the two populations also takes place in a similar fashion. The population which abruptly synchronizes later at the second forward \( \lambda^{f2}_c \) forms a wider hysteresis while the other population sees no hysteresis or negligible hysteresis. An increase in the \( D \) forms even wider hysteresis as it exerts even more frustration among the nodes, in turn entailing even larger values of \( \lambda \) for the onset of an abrupt transition.

**a. Sensitivity to initial condition in \( r \)**

We have already discussed the interesting case of initial condition dependence of the transition route for \( r \) at the critical transition point \( \lambda^{f1}_c \) (see Fig. 1). The sensitivity to initial condition in \( r \) during backward transition is demon-
strated in Fig. 2 for $\omega_1^f = \omega_2^f$ following Lorentzian distribution $\tilde{g}(\omega) = \frac{\Delta}{\pi(\omega^2 + \Delta^2)}$, where the half width at half maximum $\Delta = 1$. It is shown for $D = 2$ that $r_1^b$ forms the hysteresis loop with $r_1^f$, and $r_2^b$ also traces back $r_2^f$ branch from the abrupt transition point $\lambda_b^{11}$. However as shown for $D = 1.5$, $r_2^b$ does not form the hysteresis loop with $r_2^f$ instead $r_1^b$ does, and $r_2^b$ instead traces the $r_1^f$ branch from the transition point $\lambda_b^{11}$. The reason for such random behavior of $r_b^b$ in adopting either of the backward transition routes is its sensitivity to initial condition due to the bistability at $\lambda_b^{11}$. The initial condition dependence of the forward and backward $r_1$ has been witnessed for different values of $D$ and different types of distribution. $r_1 - \lambda$ diagrams exhibiting the sensitivity to initial conditions for Gaussian distribution are shown in Supplementary Material (SM). In Fig. 2, initial transients of the forward $r_1$ are shown for $D = 1.5$ (left panels) and $D = 2$ (right panels) at $\lambda = 2$. Our discussion shows the initial condition dependence of $r_1$ and $r_2$ under the impression of interlayer pinning.

b. Chimeric patterns Here we emphasize upon the occurrence of chimeric state in the two multilayered populations during the forward and backward phase transitions. In the multilayer configuration, the two interlinked populations form the chimeric state in which one population remains coherent while the other dwells in complete incoherence. For instance, in the bottom panels in Fig. 2, the stationary phases $\theta_l^r$ of the two populations are depicted exhibiting chimeric states at $\lambda = 2$ for different values of $D$. The region of chimeric occurrence spans from coupling strength $\lambda_c^{1f}$ to $\lambda_c^{2f}$ during the forward transition. The area of chimera states during the backward transition is stretched out in relatively a narrower region beginning from coupling strength $\lambda_c^{1b}$ to $\lambda_c^{2b}$. Either population, whether it meets the coherence or incoherence in the chimera region depends purely on the sensitivity to the initial condition at the critical point $\lambda_c^{1b}$ in the forward transition and $\lambda_c^{1b}$ in the backward transition. The area of the existing chimera states during either the forward or backward transition gets augmented when the pinning strength is increased as this, in turn, advances the values of $\lambda_c^{1f}$ and $\lambda_c^{1b}$ due to enhanced frustration among the nodes. The same is quite apparent from $r_1 - \lambda$ diagrams in Fig. 1 and Fig. 2.

c. Theoretical predictions In the thermodynamic limit $N \rightarrow \infty$, phases $\theta_l^r$ of the nodes in model (1) are continuous and $2\pi$ periodic function such that $\alpha \rightarrow \theta_1^r$ [33]. Each $\theta_l^r$ is associated with an $\alpha^i$, hence relabelling of each $\theta_l^r$ with its corresponding $\alpha$ allows us to re-express the model (1) as [33]

$$\dot{\theta}_l^r = \omega_l^\alpha + \frac{\lambda}{2\pi} \int_0^{2\pi} \mathrm{d}\alpha' \sin(\theta_l^{\alpha'} - \theta_l^r) + \frac{D}{2\pi} \int_0^{2\pi} \mathrm{d}\alpha' \sin(\theta_l^{\alpha'} - \theta_l^r - \alpha), \tag{3}$$

where $l' \neq l$; $l, l' \in \{1, 2\}$. In the limit $N \rightarrow \infty$, the order parameter (2) for a layer $l$ can be re-written as

$$r_l e^{i\psi_l} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega_l \tilde{g}(\omega_l) \int_0^{2\pi} \mathrm{d}\alpha \ e^{i\theta_l^r} \tag{4}$$

Model (3) can be expressed in terms of mean-field parameters $r_l$ and $\psi_l$

$$\dot{\theta}_l^\alpha = \omega_l^\alpha + \lambda r_l \sin(\psi_l - \theta_l^r) + D r_l v \sin(\psi_l - \theta_l^r - \alpha). \tag{5}$$

Now considering $g(\omega_l)$ such that their mean frequencies $\Omega_l = 0$, then $\psi_l = 0$. The criteria for the synchronous states $\theta_l^s = 0$ in either population, then yields

$$e^{i\theta_l^r} = i \omega_l^\alpha \pm \sqrt{|u + ve^{i\alpha}|^2 - |\omega_l^\alpha|^2} \overline{u + ve^{i\alpha}}, \tag{6}$$

where $u = \lambda r_l$ and $v = Dr_l v$. After substituting $e^{i\theta_l^r}$ from Eq.(6) into Eq.(4), one obtains the following expression for the order parameter

$$r_l = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega_l \tilde{g}(\omega_l) \int_0^{2\pi} \mathrm{d}\alpha \frac{i \omega_l \pm \sqrt{|u + ve^{i\alpha}|^2 - |\omega_l^\alpha|^2}}{u + ve^{i\alpha}}. \tag{7}$$

We theoretically obtain the solutions for synchronous states by considering uniform $\tilde{g}(\omega_l) = \frac{1}{2\gamma}$ for $\omega_l^\alpha \in (-\gamma, \gamma)$ such that $\Omega_l = 0$. For the uniform $\tilde{g}(\omega_l)$, the first part of the integration in the order parameter (7) vanishes:

$$\frac{1}{2\pi} \int_{-\gamma}^{\gamma} \mathrm{d}\omega_l \int_0^{2\pi} \mathrm{d}\alpha \frac{i \omega_l}{u + ve^{i\alpha}} = 0, \tag{8}$$

FIG. 3. (Color online) $r_l$ for GC-GC multiplex network as a function of $\lambda$ simulated for uniform $g(\omega_l)$ with $\Delta = 1$.

FIG. 4. (Color online) Abrupt jump size $|r_1^b - r_1^f|$; $l \in \{1, 2\}$ and hysteresis width in $D - \lambda$ space for GC-GC multiplex network ($N = 1000$) having uniform frequencies with $\Delta = 1$. 

and only the second part accounts for the order parameter. Since we must have $r_1 > 0$, $+$ sign in the second term in Eq. (7) is taken into account and then $r_1$ can be re-expressed in terms of $z = u/v$ as

$$
   r_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\omega_1}{\gamma} \int_0^{2\pi} \frac{d\alpha}{\alpha} \frac{\sqrt{|z + e^{i\alpha}|^2 - |\omega_1/v|^2}}{z + e^{i\alpha}}.
$$

After carrying out some mathematical simplifications, the real part of the order parameter is expressed as

$$
   r_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\omega_1}{\gamma} \int_0^{2\pi} \frac{d\alpha}{\alpha} \frac{\sqrt{z^2 + 2z \cos \alpha + 1 - |\omega_1/v|^2}}{z^2 + 2z \cos \alpha + 1} (z + \cos \alpha).
$$

$r_1 = 0$ is one of the solutions of Eq. (10) for $z = 0$, i.e., $u = 0$ and $v \neq 0$. The bifurcating solutions in the vicinity of $z = 0$ ($z \to 0$) are obtained by the series expansion of Eq. (10) for $0 < z < 1$:

$$
   r_1 = \frac{z^3}{16(1 - \frac{z^2}{\alpha})^{3/2}} + \frac{u}{2\gamma} \arcsin \left( \frac{\gamma}{v} \right) + \mathcal{O}(z^4).
$$

The solutions for the order parameter given by Eq. (10) and Eq. (11) are depicted respectively by green and dark grey solid lines in Fig. 1 for different values of the pinning strength. In Fig. 1, $v$ corresponds to $r_2$ and $r_1$ respectively for $D = 1$ and $D = 2$.

**B. Dynamics on Multiplex Networks**

Next, we treat a multiplex framework of the model given in Eq. (1), which considers interactions only between the mirror adjacent nodes in the two populations. The evolution of phases in the multiplex network possessing ran-dom inter-pinning in the mirror nodes is expressed as

$$
   \dot{\theta}_i^r = \omega_i^r + \frac{\lambda}{N} \sum_{j=1}^{N} \sin(\theta_i^r - \theta_j^r) + D \sin(\theta_i^r - \theta_i^l - \alpha^r).
$$

Synchronization diagrams for such multiplex network consisting of two GC populations are shown in Fig. 3 for different pinning strength. Both the forward critical coupling strength and the hysteresis width increase with the increase in $D$. Also, no initial condition dependence of the order parameter is witnessed for the multiplexed populations unlike we witnessed in the case of the multilayered populations.

**a. Phase plot in $D - \lambda$ space**

To have a complete picture about the nature of transition with change in the pinning strength, we draw phase plot in $D - \lambda$ space for each layer. The color profile in $D - \lambda$ space in Fig. 4 illustrates the abrupt jump size $|r^r_i - r^l_i|$, $l \in [1, 2]$ for a GC-GC multiplexes having natural frequencies drawn from a uniform distribution. A profound distinction in color between hysteresis region and asynchronous or synchronous region can be witnessed for pinning strength $D > 1$. The magnitude of forward and backward critical coupling strength and hysteresis width corresponding to a pinning strength $D$ can also be extracted from the $D - \lambda$ phase plots.

**b. Distribution of mirror phase-difference $|\theta_1^r - \theta_2^r|$**

Here, we investigate the microscopic dynamics of $\Delta \theta^r = |\theta_1^r - \theta_2^r|$, the phase-difference between the mirror nodes. The distribution $P(\Delta \theta^r)$ for any $\lambda < \lambda^r_f$ exhibits two peaks at $\Delta \theta^r = 0$ and $\Delta \theta^r = \pi$ as shown in top left panel of Fig. 5. It implies that for any $\lambda$ belonging to asynchronous state, the $N$ sized population of $\Delta \theta^r$ is segregated notably in two clusters, one at $0$ or $2\pi$, and other at $\pi$, with a few sparsely populated elsewhere. On the other hand, in the synchronous state $\lambda > \lambda^r_f$, the $P(\Delta \theta^r)$ exhibits bimodal peaks with their minima located at either $\pi/2$ or $3\pi/2$ (see top right panel in Fig. 5), hence the two peaks are located at a spread of $\pi/4$ either side of the minima.

Furthermore, we study $\Delta \theta^r$ as a function of $\alpha$ in Fig. 5 for different values of $\lambda$. For any $\lambda < \lambda^f_f$ (see bottom left panel in Fig. 5), the nodes whose initial independent pinning phases are bounded within $\alpha \in [\pi/2, 3\pi/2]$ achieve steady state around $\Delta \theta^r = \pi$, whereas the nodes whose $\alpha \in [0, \pi/2]$ or $\alpha \in [3\pi/2, 2\pi]$ settle on about $\Delta \theta^r = 0, 2\pi$ in the steady state. Nevertheless, for any $\lambda > \lambda^r_f$ (see bottom right panel in Fig. 5), the steady state values of $\Delta \theta^r$ are spread between either $[5\pi/4, 7\pi/4]$ or $[\pi/4, 3\pi/4]$ depending on the value of $\lambda$. The stationary population of $\Delta \theta^r$ in the asynchronous and synchronous state corroborates with the findings for $P(\Delta \theta^r)$. The robustness of the inter-pinning multilayer and multiplex models in respectively exhibiting the successive and simultaneous emergence of the ES for two randomly
connected (Erdős-Rényi topology [34]) populations is demonstrated in Supplementary Material (SM).

Conclusion In summary, we considered multilayer networks in which two populations are randomly inter-pinned. Such an arrangement leads to the explosive synchronization in the two populations in succession. Since the initiation of explosive transition in one population, the multilayer networks stay in the coexisting state of coherent and incoherent populations until the other population also undergoes the explosive transition. Such chimeric pattern in the two populations is witnessed during their explosive transition to synchronization and then desynchronization. Also, the abrupt synchronization and desynchronization transition are found to be initial condition dependent. The analytical predictions for the order parameter are also provided, which fall into good agreement with the numerical estimations. We also explored the phase transition in the multiplex network in which only mirror nodes in the two populations are randomly inter-pinned. In the multiplex network, both the populations espoused explosive routes to synchronization simultaneously. The order parameters of the populations in multiplex formation did not exhibit any chimeric state and initial condition dependence. One can find an analogy of the abrupt onset (explosive transition) to chimeric state and then abrupt return to normalcy in multilayer formation of two populations with the sudden onset and offset of the focal seizures in the brain in which only a part of the brain experiences seizure episode while the other part functions normally [35].

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A. The robustness of inter-pinning prescription against random topology

The random inter-pinning prescription to the multilayer and multiplex networks also successfully applies to the populations’ connectivity manifesting a network topology other than GC-GC as already discussed in the main text. We demonstrate this representing the two populations by ER-ER random networks [1] interconnected in multilayer formation:

$$\dot{\theta}_i^r = \omega_i^r + \lambda \sum_{j=1}^{N} A_{ij}^r \sin(\theta_i^r - \theta_j^r) + D \sum_{k=1}^{N} A_{ik}^f \sin(\theta_i^f - \theta_k^r - \alpha^r),$$

and multiplex formation:

$$\dot{\theta}_i^f = \omega_i^f + \lambda \sum_{j=1}^{N} A_{ij}^f \sin(\theta_i^f - \theta_j^f) + D \sin(\theta_i^r - \theta_i^f - \alpha^f).$$

Here, $A_{ij}; i \in \{1, 2\}$ represent the adjacency matrices for the two ER populations. The multiplex formation Eq. (2) has the interconnections only with the mirror nodes while the interconnections in the multilayer formation Eq. (1) manifest ER network connectivity $A_I$ with average degree $\langle k_i \rangle$. The ER-ER populations inter-pinned in multilayer and multiplex formations also exhibit ES transitions successively (see Fig. S1(a)) and simultaneously (see Fig. S1(b)), respectively, for different values of the pinning strength $D$.

B. The robustness against frequency distribution

Here we show synchronization diagrams $r_1 - \lambda$ for the GC-GC multilayer networks (model (1) in the main text) for $\omega_1 \neq \omega_2$ following Gaussian distribution $g(\omega) = \frac{1}{\sqrt{2\pi} \Delta^{\frac{1}{2}}} \exp(-\omega^2/2\Delta^2); \Delta = 1$ (see Fig. S2(a)). The forward and backward $r_1$ and $r_2$ depending on the initial conditions adopt different routes at forward $\lambda_c^{11}$ and backward $\lambda_c^{11}$ critical transition points. Hence, the proposed model is valid for symmetric unimodal and uniform frequency distributions (as shown in the main text) in giving rise to ES and CS. Since $\omega_1 \neq \omega_2$, therefore one obtains chimera like structures as shown in Fig. S1(b) in which one population is asynchronous while the other is synchronous but with spread in phases.
FIG. S2. (Color online) GC-GC multilayer networks having $\omega_i \neq \omega_j$ drawn from a Gaussian $g(\omega_l)$ with $\Delta = 1$. (a): $r_l - \lambda$ diagrams for different values of $D$, and (b): stationary phases $\theta_{1}^i (i=1\ldots1000)$ and $\theta_{2}^i (i=1001\ldots2000)$ at $\lambda = 2$ for $D = 1.5$ (left) and $D = 2$ (right).

C. The schematic representation of the multilayer and multiplex networks

Fig. S3 illustrates the multilayer and multiplex networks used in model Eq. (1) and model Eq. (12), respectively, in the main text. Interlayer connections in green dashed lines represent the randomly pinned interactions between pairs of the nodes belonging to different populations.

FIG. S3. (Color online) Schematics of two all-to-all connected GC (a) multilayer and (b) multiplex network populations. (a): Each node in one population is interconnected with all the nodes in other population (green dashed lines), for better clarity only one node is shown connecting all the other nodes in other population. (b) Only mirror nodes are interconnected (green dashed lines).

D. Examples of multilayer networks with need for random interlayer pinning model

Strogatz initiated the idea of random pinning for coupled Kuramoto oscillators in 1989 [2], demonstrating pinning led emerging phenomena. Here, the pinning term drives an individual oscillator to stick to a random phase. For multilayer networks, random interlayer accounts for the lack of information about a definite impact of the activities of nodes in one network on those of the other networks. Such a scheme is more relevant when how activities of one layer get affected by those of other layers is not known. In other words, while the existence of different types of relationships among the same set of interacting units can be clearly evident, making the multilayer model more apt, modeling the nature and structure of multiplexing remains an elusive and non-trivial problem. We attempt to explain this using two examples coming from two completely different systems.

a. Glial-Neural Multilayer Networks There have been persistent attempts to understand information processing in the brain under a multilayer network framework consisting of Glial and Neural cells [3] forming different layers. While modeling intralayer interactions for an individual layer is straightforward, for instance, Glial cells interact with other Glial cells through diffusion, and neural cells interact with each other via synaptic connections. There is not enough knowledge of the nature of Glial-neural interactions for modeling "multiplexing" impact [4]. Another such example
is of social multilayer network constructed by different layers representing different social relationships [5]. Here also, the nature of intralayer interactions is rather easy to construe from the available data and hence more easily to model; having such information for multiplexing is non-trivial.

b. Transport Multilayer Network  Another such example is of transport systems with their units (stations) connected with each other with transport routes and a mode defining the corresponding layer. There are definite transport networks based on direct connections between different stations for each mode of transport, for instance, flight and train. Additionally, based on the flight or train frequency, the number of airlines operating, inflow, outflow of the passengers at a particular station and other parameters, one can qualitatively model pattern and the nature of pairwise interactions rather quite accurately. However, similar modeling for multiplexing impact is tricky where one knows for sure that there is an impact of dynamical activities of units and structural patterns of one layer on dynamical activities of units of another layer [6].

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