APPLICATIONS OF THE ALGEBRAIC GEOMETRY OF THE
PUTMAN-WIELAND CONJECTURE

AARON LANDESMAN AND DANIEL LITT

ABSTRACT. We give two applications of our prior work toward the Putman-Wieland conjecture. First, we deduce a strengthening of a result of Marković-Tošić on virtual mapping class group actions on the homology of covers. Second, let $\Sigma_{g',n'} \to \Sigma_{g,n}$ be a finite $H$-cover of topological surfaces. We show the virtual action of the mapping class group of $\Sigma_{g,n+1}$ on an $H$-isotypic component of $H^1(\Sigma_{g'})$ has non-unitary image.

1. INTRODUCTION

1.1. Review of the Putman-Wieland conjecture. We aim to explain applications of our prior work \cite{LL22a} toward the Putman-Wieland conjecture, and so we begin by reviewing the statement of the Putman-Wieland conjecture.

Let $\Sigma_{g,n}$ denote an orientable topological surface of genus $g$ with $n$ punctures. Let $H$ be a finite group. Given a finite unramified $H$-cover of topological surfaces $\Sigma_{g',n'} \to \Sigma_{g,n}$, there is an action of a finite index subgroup $\Gamma$ of the mapping class group $\text{Mod}_{g,n+1}$ of $\Sigma_{g,n+1}$ on $H_1(\Sigma_{g'},C)$, as we now explain. The group $\text{Mod}_{g,n+1}$ acts on $\pi_1(\Sigma_{g,n},x)$ for some basepoint $x$, and we can take $\Gamma$ to be the stabilizer of the surjection $\phi: \pi_1(\Sigma_{g,n},x) \to H$, where $\phi$ corresponds to the cover $\Sigma_{g',n'} \to \Sigma_{g,n}$. Then, for $x' \in \Sigma_{g',n'}$ mapping to $x$, $\Gamma$ acts on $\pi_1(\Sigma_{g',n'},x') = \ker \phi$, preserving the conjugacy classes of the loops around the punctures in $\Sigma_{g',n'}$, and hence acts on $H_1(\Sigma_{g'},C)$.

Conjecture 1.2 (Putman-Wieland, \cite{PW13} Conjecture 1.2). Fix $g \geq 2$ and $n \geq 0$. For any unramified cover $\Sigma_{g',n'} \to \Sigma_{g,n}$, the vector space $H_1(\Sigma_{g'},C)$ has no nonzero vectors with finite orbit under the action of $\Gamma$.

Note that the Putman-Wieland conjecture is false when $g = 2$, see \cite{Mar22}. We next describe applications of our prior work relating to the Putman-Wieland conjecture.
1.3. The results of Marković-Tošić. In [MT Theorem 1.5], Marković-Tošić verified some new cases of the Putman-Wieland conjecture. We are able to recover their results from ours. In fact, we are able to deduce a slight generalization of their results, as explained in [Remark 1.7].

We now formally define what it means for \( f : X \to Y \) to furnish a counterexample to Putman-Wieland.

**Definition 1.4.** Suppose \( \Sigma_{g',n'} \to \Sigma_{g,n} \) is a finite covering of topological surfaces. Let \( \Gamma \subset \text{Mod}_{g,n+1} \) denote the finite index subgroup preserving \( h \). The action of \( \Gamma \) on \( \pi_1(\Sigma_{g',n'}, \mathbb{C}) \) induces an action on \( H_1(\Sigma_{g',n'}, \mathbb{C}) \) which preserves the subspace spanned by homology classes of loops around punctures, and hence also induces an action on \( H_1(\Sigma_{g}, \mathbb{C}) \) via Poincaré duality. We say \( h \) furnishes a counterexample to Putman-Wieland, if there is some nonzero \( v \in H_1(\Sigma_{g'}, \mathbb{C}) \) with finite orbit under \( \Gamma \).

Let \( Y \) be a compact Riemann surface of genus \( g \) with \( n \) marked points \( p_1, \ldots, p_n \). Upon identifying \( \pi_1(Y - \{ p_1, \ldots, p_n \}) \cong \pi_1(\Sigma_{g,n}) \), let \( f : X \to Y \) be the covering of compact Riemann surfaces, ramified at most over \( p_1, \ldots, p_n \), corresponding to the topological cover \( h \). If \( h \) furnishes a counterexample to Putman-Wieland, we also say \( f : X \to Y \) furnishes a counterexample to Putman-Wieland.

Suppose \( h \) is Galois with Galois group \( H \), and \( \rho \) is an irreducible \( H \)-representation. Note that \( H \) and \( \Gamma \) simultaneously act on \( H_1(\Sigma_{g'}, \mathbb{C}) \). We say a counterexample to Putman-Wieland is \( \rho \)-isotypic if every element of the \( \rho \)-isotypic subspace \( H_1(\Sigma_{g'}, \mathbb{C})^\rho \) has finite orbit under \( \Gamma \). Here if \( V \) is an \( H \)-representation, \( V^\rho \) denotes the \( \rho \)-isotypic subspace of \( V \), i.e. the image of the natural evaluation map

\[
\text{Hom}_H(\rho, V) \otimes \rho \to V.
\]

**Remark 1.5.** Even though the Putman-Wieland conjecture assumes \( g \geq 2 \), we still say \( f : X \to Y \) furnishes a counterexample to Putman-Wieland when \( Y \) has genus \( g \leq 1 \).

Here is the slight improvement on [MT Theorem 1.5], which we will prove in §5.10. We let \( \lambda_1(X) \) denote the smallest nonzero eigenvalue of the Laplacian acting on \( L^2 \) functions on \( X \).

**Theorem 1.6.** Suppose \( Y \) has genus \( g \geq 2 \) and \( f : X \to Y \) furnishes a counterexample to Putman-Wieland. Then \( \frac{1}{g-1} \geq 2\lambda_1(X) \). If \( f \) is Galois, then \( \frac{1}{g-1} > 2\lambda_1(X) \).

**Remark 1.7.** [Theorem 1.6] is slightly stronger than [MT Theorem 1.5] as we now explain. In their result, they first choose \( \epsilon \) and then choose \( g \) subject
to the inequality $g \geq (1 + \epsilon)/\epsilon$, or equivalently $\epsilon \geq \frac{1}{g-1}$. They conclude $\epsilon \geq \lambda_1(X)$. (Though their argument in fact gives $\epsilon \geq 2\lambda_1(X)$, as they accidentally omitted a factor of 2.) In general, we also find this same inequality. However, when the cover is moreover Galois, using \textbf{Theorem 1.6} we find $\epsilon \geq \frac{1}{g-1} > 2\lambda_1(X)$, so we obtain a strict inequality instead of a weak inequality.

Recall that the gonality of a curve $X$ is the smallest degree of a non-constant map $f : X \to \mathbb{P}^1$. We use $\text{gon}(X)$ to denote the gonality of $X$. In order to prove \textbf{Theorem 1.6} the first step is the same as in \cite{MT}, as we reduce to a statement about the gonality of $X$ using the Li-Yau inequality and then apply the following proposition, which may be of independent interest.

\textbf{Proposition 1.8.} Suppose $Y$ has genus $g$ and $f : X \to Y$ is a cover furnishing a counterexample to Putman-Wieland. Then, $\text{gon}(X) \leq \deg f$. When $f$ is additionally Galois, we have $\text{gon}(X) \leq \deg f - (g - 1)$.

We prove this in \S 5.9. The non-Galois case of the above proposition is proven via different methods in \cite{MT}.

\textbf{1.9. Prill’s problem.} Along the way to our proof of \textbf{Theorem 1.6}, we make a brief digression regarding Prill’s problem \cite[Chapter VI, Exercise D]{ACGH85}. There, we explain why a general genus 2 curve has a cover such that every fiber moves in a pencil. As detailed in \textbf{Remark 5.6} we prove a stronger result in the shorter and less technical article \cite{LL22c}, but include these observations here as it is a simple consequence of the tools we develop to recover the results of Marković-Tošić described above.

\textbf{1.10. Isotypicity.} We next prove a result ruling out $\rho$-isotypic counterexamples to Putman-Wieland in genus at least 2.

\textbf{Theorem 1.11.} Suppose $f : X \to Y$ is a Galois $H$-cover furnishing a counterexample to Putman-Wieland which is $\rho$-isotypic in the sense of \textbf{Definition 1.4}. Then $Y$ has genus at most 1.

Equivalently, if the genus of $Y$ is at least 2, for each irreducible $H$-representation $\rho$, there exists an element of $H^1(X, \mathbb{C})^\rho$ with infinite orbit under the virtual action of the mapping class group of $Y$.

This will follow from a stronger statement:
Theorem 1.12. Let $X \to Y$ be a Galois $H$-cover, where $Y$ has genus at least 2. Let $\rho$ be an irreducible complex $H$-representation. Then the virtual action of the mapping class group of $Y$ on $H^1(X, \mathbb{C})^\rho$ is not unitary.

Theorem 1.11 is immediate, as representations with finite image are unitary.

We prove Theorem 1.12 in §4.2. The idea is to investigate a certain bilinear pairing, arising from the derivative of a period map and studied earlier in [LL22a].

A recent paper of Boggi-Looijenga [BL21], which has since been retracted by the authors, claims to show that in some settings the Jacobian of the generic $H$-curve has $\mathbb{Q}$-endomorphism algebra $\mathbb{Q}[H]$. While their argument is incomplete, we show the claimed result would imply the Putman-Wieland conjecture in Corollary 4.3.

In [Mar22], Marković gives a counterexample to Putman-Wieland $f : X \to Y$, where $Y$ has genus 2. Theorem 1.11 immediately implies the following.

Corollary 1.13. Let $f' : X' \to Y$ be the Galois closure of the counterexample to Putman-Wieland $f : X \to Y$ from [Mar22]. For any irreducible $H$-representation $\rho$, $f'$ is not a $\rho$-isotypic counterexample to Putman-Wieland.

Remark 1.14. Corollary 1.13 is especially interesting, as all the counterexamples where $Y$ has genus 0 or 1 which we know of are $\rho$-isotypic for some $\rho$. This example and other examples of covers which factor through this one are the only ones we know of which are not $\rho$-isotypic. It would be quite interesting to produce other counterexamples to Putman-Wieland which are not $\rho$-isotypic for any $\rho$.

1.15. Overview. In this paper, we give two algebro-geometric consequences of our work on the Putman-Wieland conjecture. In §2, we recall background on parabolic bundles and notation for versal families. In §3, we connect non-generically globally generated vector bundles to the Putman-Wieland conjecture; this connection is a straightforward consequences of our past work, but we spell it out for completeness. In §4, we discuss a certain isotypicity property which would imply the Putman-Wieland conjecture, and we prove Theorem 1.11. Finally, in §5 we give a proof of a slight improvement of the main result of [MT], Theorem 1.6.

1.16. Acknowledgements. We would like to thank an anonymous referee, Marco Boggi, Anand Deopurkar, Joe Harris, Neithalath Mohan Kumar, Eric
Larson, Rob Lazarsfeld, Eduard Looijenga, Vladimir Marković, Anand Patel, Andy Putman, Will Sawin, Ravi Vakil, and Isabel Vogt for helpful discussions related to this paper. Landesman was supported by the National Science Foundation under Award No. DMS-2102955. Litt was supported by NSF Grant DMS-2001196. This material is based upon work supported by the Swedish Research Council under grant no. 2016-06596 while the authors were in residence at Institut Mittag-Leffler in Djursholm, Sweden during the fall of 2021.

2. Background and notation

Throughout, we work over the complex numbers, unless otherwise stated. For a pointed finite-type scheme or Deligne-Mumford stack \((X, x)\) over \(\mathbb{C}\), we will use \(\pi_1(X, x)\) to denote the topological fundamental group of the associated complex-analytic space or analytic stack. In the remainder of this section, we recall notation for parabolic bundles, so that we can state a relevant proposition on vector bundles which are not generically globally generated. For more detail, we recommend the reader consult [LL22b, §2]. We also review notation for versal families, which we also used in [LL22a].

2.1. A lightning review of parabolic bundles. Fix a smooth proper connected curve \(C\) over an arbitrary field and let \(D = x_1 + \cdots + x_n\) be a reduced divisor on \(C\). Recall that a parabolic bundle on \((C, D)\) is a vector bundle \(E\) on \(C\), a decreasing filtration \(E|_D = E^1_j \supseteq E^2_j \supseteq \cdots \supseteq E^{m+1}_j = 0\) for each \(1 \leq j \leq n\), and an increasing sequence of real numbers \(0 \leq \alpha^1_j < \alpha^2_j < \cdots < \alpha^n_j < 1\) for each \(1 \leq j \leq n\) referred to as weights. We use \(E_* = (E, \{E^i_j\}, \{\alpha^i_j\})\) to denote the data of a parabolic bundle.

Given a parabolic bundle \(E_* = (E, \{E^i_j\}, \{\alpha^i_j\})\), let \(J \subset \{1, \ldots, n\}\) denote the set of integers \(j \in \{1, \ldots, n\}\) for which \(\alpha^1_j = 0\), and define

\[
\hat{E}_0 := \ker(E \to \bigoplus_{j \in J} E_{x_j}/E^2_j).
\]

(This is a special case of more general notation used for coparabolic bundles as in [LL22b, 2.2.8] or the equivalent [BY96, Definition 2.3], but is all we will need for this paper.) In particular, \(\hat{E}_0 \subset E\) is a subsheaf.

We next make sense of a relative variant of the above notions. Namely, let \(\mathcal{C} \to \mathcal{B}\) be a relative smooth proper curve with geometrically connected fibers and let \(\mathcal{D} \subset \mathcal{C} \to \mathcal{B}\) be a relative étale Cartier divisor. Then, a relative parabolic bundle on \((\mathcal{C}, \mathcal{D})\) is a vector bundle \(\mathcal{E}\) on \(\mathcal{C}\), a decreasing filtration \(\mathcal{E}|_{\mathcal{D}} = \mathcal{E}^1 \supseteq \mathcal{E}^2 \supseteq \cdots \supseteq \mathcal{E}^{m+1} = 0\), and an increasing sequence of
real numbers $0 \leq \alpha^1 < \alpha^2 < \cdots < \alpha^m < 1$ referred to as weights. Above, the $\mathcal{E}^i$ are sheaves supported on $\mathcal{D}$. We use $\mathcal{E}_* = (\mathcal{E}, \{\mathcal{E}^i\}, \{\alpha^i\})$ to denote the data of a relative parabolic bundle.

**Remark 2.2.** Note that if $(\mathcal{E}, \{\mathcal{E}^i\}, \{\alpha^i\})$ is a relative parabolic bundle on $(\mathcal{C}, \mathcal{D})$, and $(C, D)$ is the fiber over a point $b \in \mathcal{B}$, with $D = x_1 + \cdots + x_n$, then we can recover the data of a parabolic bundle $(\mathcal{E}|_C, \{\mathcal{E}^i|_C\}, \{\alpha^i|_C\})$ over a field, in the above sense, as follows. Take $\mathcal{E}|_C$ as the underlying vector bundle. Take the filtration at $x_j$ to be the restriction of the filtration $\mathcal{E}^i$ to $x_j$, with repetitions removed. Finally, take $\alpha^i|_C : \max_k \{\alpha^k : \mathcal{E}^k|_{x_j} = \mathcal{E}^i|_{x_j}\}$.

To conclude our treatment of the relative setting, we define

$$
\mathcal{E}_0 := \begin{cases}
\ker(\mathcal{E} \to \mathcal{E}|_\emptyset / \mathcal{E}^2) & \text{if } \alpha^1 = 0, \\
\mathcal{E} & \text{otherwise.}
\end{cases}
$$

(2.2)

Returning to the absolute setting, parabolic bundles admit a notion of parabolic stability, analogous to the usual notion of stability for vector bundles, which we next recall. First, the parabolic degree of a parabolic bundle $E_*$ is

$$\text{par-deg}(E_*) := \deg(E) + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \alpha^i_j \dim(\mathcal{E}^i_j/\mathcal{E}^{i+1}_j).$$

Then, the parabolic slope is defined by $\mu_*(E_*) := \text{par-deg}(E_*) / \text{rk}(E_*)$. Any subbundle $F \subset E$ has an induced parabolic structure $F_* \subset E_*$ defined as follows: the filtration over $x_j$ on $F$ is obtained from the filtration

$$F_{x_j} = E^1_j \cap F_{x_j} \supset F^2_j \cap F_{x_j} \supset \cdots \supset E^{n_{x_j}+1}_j \cap F_{x_j} = 0$$

by removing redundancies. For the weight associated to $F^i_j \subset F_{x_j}$ one takes

$$\max_{k, 1 \leq k \leq n_j} \{\alpha^k_j : F^i_j = E^k_j \cap F_{x_j}\}.$$

A parabolic bundle $E_*$ is parabolically semi-stable if for every nonzero subbundle $F \subset E$ with induced parabolic structure $F_*$, we have $\mu_*(F_*) \leq \mu_*(E_*)$.

Mehta and Seshadri [MS80, Theorem 4.1, Remark 4.3], give a correspondence between parabolically stable parabolic bundles of parabolic degree zero on $(\mathcal{C}, D)$ and irreducible unitary local systems on $C \setminus D$. This bijection sends a local system $\mathcal{V}$ to the Deligne canonical extension of $(\mathcal{V} \otimes \mathcal{E}, \text{id} \otimes d)$ with the parabolic structure induced by the connection (as in [LL22b, Definition 3.3.1]).
For our later results, it will also be useful to have a lower bound on the rank of certain vector bundles. This follows from the following general result about non-generically generated vector bundles, whose proof ultimately relies on Clifford’s theorem for vector bundles. The reader may take this as a black box. For a less technical variant of this statement, see [LL22b, Proposition 6.3.1].

**Proposition 2.3** ([LL22b, Proposition 6.3.6]). Suppose $C$ is a smooth proper connected genus $g$ curve and $E_\star = (E, \{ E_i \}, \{ \alpha_i \})$ is a nonzero parabolic bundle $C$ with respect to $D = x_1 + \cdots + x_n$. Suppose $E_\star$ is parabolically semistable. Let $U \subset \hat{E}_0$ be a (non-parabolic) subbundle with $c := \text{rk } E - \text{rk } U$ and $\delta := h^0(C, \hat{E}_0) - h^0(C, U)$.

(I) If $\mu_\star(E_\star) > 2g - 2 + n$, then $\text{rk } E > gc - \delta$.

(II) If $\mu_\star(E_\star) = 2g - 2 + n$, then $\text{rk } E \geq gc - \delta$.

In particular, if $\hat{E}_0$ fails to be generically globally generated, and $\mu_\star(E_\star) \geq 2g - 2 + n$, $\text{rk } E \geq g$.

**Remark 2.4.** The statement of Proposition 2.3 is equivalent to [LL22b, Proposition 6.3.6], but differs slightly in that we write “$E_\star$ is parabolically stable” in place of “$\hat{E}_\star$ is coparabolically stable” and $\mu_\star(E_\star)$ in place of $\mu_\star(\hat{E}_\star)$. However, by definition $E_\star$ is parabolically stable if and only if $\hat{E}_\star$ is coparabolically stable and $\mu_\star(E_\star) = \mu_\star(\hat{E}_\star)$ [LL22b, Definitions 2.2.9 and 2.4.2]. Finally, the final “In particular,…” statement is an immediate consequence of (II).

We next introduce the notation $E_\rho^\star$ as the parabolic bundle corresponding to a representation $\rho$.

**Notation 2.5.** Let $Y$ be a curve and $D \subset Y$ a divisor. Recall that under the Mehta-Seshadri correspondence [MS80], there is a bijection between reducible representations of $\pi_1(Y - D)$ and parabolic degree 0 stable parabolic vector bundles on $Y$, with parabolic structure along $D$. Given an irreducible $H$ representation $\rho$, we use $E_\rho^\star$ to denote the parabolic bundle corresponding to the representation $\pi_1(Y - D) \simeq \pi_1(\Sigma_{g, n}) \xrightarrow{\phi} H \xrightarrow{\rho} \text{GL}_{\dim \rho}(C)$.

### 2.6. Notation for versal families.

We set some notation to describe families of covers of curves.

**Notation 2.7.** We fix non-negative integers $(g, n)$ so that $n \geq 1$ if $g = 1$ and $n \geq 3$ if $g = 0$, i.e., $\Sigma_{g, n}$ is hyperbolic. Let $\mathcal{M}$ be a connected complex variety. A family of $n$-pointed curves of genus $g$ over $\mathcal{M}$ is a smooth proper morphism.
\[ \pi : \mathcal{C} \to \mathcal{M} \] of relative dimension one, with geometrically connected genus \( g \) fibers, equipped with \( n \) sections \( s_1, \ldots, s_n : \mathcal{M} \to \mathcal{C} \) with disjoint images. Call such a family \textit{versal} if the induced map \( \mathcal{M} \to \mathcal{M}_{g,n} \) is dominant and étale. Here \( \mathcal{M}_{g,n} \) denotes the Deligne-Mumford moduli stack of \( n \)-pointed genus \( g \) smooth proper curves with geometrically connected fibers.

If \( \pi : \mathcal{C} \to \mathcal{M} \) is a family of \( n \)-pointed curves, we let \( \mathcal{D} := \bigsqcup_{i=1}^{n} \text{im}(s_i) \) denote the union of the images of the sections, which is finite étale of degree \( n \) over \( \mathcal{M} \). Also let \( \mathcal{C}^0 := \mathcal{C} \setminus \bigcup_i \text{im}(s_i) \), let \( j : \mathcal{C}^0 \to \mathcal{C} \) be the natural inclusion, and let \( \pi^0 := \pi \circ j : \mathcal{C}^0 \to \mathcal{M} \) denote the composition. We will refer to \( \pi^0 : \mathcal{C}^0 \to \mathcal{M} \) as the \textit{associated family of punctured curves}. If \( \pi^0 \) arises as the family of punctured curves associated to a versal family of \( n \)-pointed curves, we will call it a \textit{punctured versal family}. We will frequently use \( m \in \mathcal{M} \) as a basepoint, and \( c \in \mathcal{C}^0 \) as a basepoint with \( \pi^0(c) = m \).

We also will need the notion of a versal family of \( \phi \)-covers.

**Definition 2.8.** Specify a surjection \( \phi : \pi_1(\Sigma_{g,n}, v_0) \to H \), where \( H \) is a finite group. A \textit{versal family of \( \phi \)-covers} is the data of

1. a dominant étale morphism \( \mathcal{M} \to \mathcal{M}_{g,n} \), with \( \pi^0 : \mathcal{C}^0 \to \mathcal{M} \) the associated punctured versal family,
2. a point \( c \in \mathcal{C}^0, m = \pi^0(c) \), and an identification \( i : \pi_1(\Sigma_{g,0,n}, v_0) \simeq \pi_1(\mathcal{C}^0_m, c) \), and
3. a finite étale Galois \( H \)-cover \( f : \mathcal{X}^0 \to \mathcal{C}^0 \) inducing a map
   \[ \pi_1(\mathcal{C}^0_m, c) \to \pi_1(\mathcal{C}^0, c) \to \pi_1(\mathcal{X}^0, x) \simeq H \]
   agreeing with the surjection \( \phi \) under the identification of (2).

We use \( \mathcal{X} \) to denote the normalization of \( \mathcal{C} \) in the function field of \( \mathcal{X}^0 \), so \( \mathcal{X} \) is the relative smooth proper curve compactifying \( \mathcal{X}^0 \).

### 3. Connection to Vector Bundles Which Are Not Generically Globally Generated

We next set up notation to relate the Putman-Wieland conjecture to a certain period map.
Notation 3.1. Fix a surjection $\phi : \pi_1(\Sigma_{g,n},v_0) \rightarrow H$, where $H$ is a finite group. Let $\mathcal{C} \rightarrow \mathcal{M}$, $\mathcal{X} \rightarrow \mathcal{C}$ be a versal family of $\phi$-covers, as in Definition 2.8.

Let $\rho$ be an irreducible $H$-representation and let $U^\rho$ denote the local system on $\mathcal{C}$ with monodromy representation given by $\pi_1(\mathcal{C}) \rightarrow H \xrightarrow{\rho} \text{GL}_{\dim \rho}(\mathbb{C})$, where the first map is induced by the $H$-cover $\mathcal{X} \rightarrow \mathcal{C}$.

Let $(E^\rho, \nabla^\rho)$ denote the Deligne canonical extension with its natural parabolic structure (the Deligne canonical extension is defined in [LL22b, Definition 4.1.2]; the parabolic structure is defined in [LL22b, Definition 3.3.1] for curves, but the definition there generalizes naturally to families) of $(U^\rho \otimes \mathcal{O}_C, \text{id} \otimes d)$ to $\mathcal{C}$. Let $W^\rho := R^1\pi_*^{\rho}(U^\rho)$ and let $(\mathcal{H}^\rho, \nabla^{\rho}_{GM}) := (W^\rho \otimes \mathcal{O}_\mathcal{M}, \text{id} \otimes d)$.

Because $U^\rho$ is unitary, the Hodge-de Rham spectral sequence for $R^1\pi_*^{\rho}(U^\rho)$ degenerates [Tim87, Theorem 7.1(a)] (see also [Sai88, Théorème 5.3.1] for a much more general result). Hence there is a 2-step Hodge filtration of $\mathcal{H}^\rho = W^\rho \otimes \mathcal{O}_\mathcal{M}$ satisfying

$$F^1\mathcal{H}^\rho \simeq \pi_*(\mathcal{E}^\rho \otimes \Omega^1_{\mathcal{C}/\mathcal{M}}(\log \mathcal{D}))$$
$$\mathcal{H}^\rho / F^1\mathcal{H}^\rho \simeq R^1\pi_*\mathcal{E}^\rho.$$  

(3.1)

In this paper, we will primarily be concerned with the weight 1 part of cohomology (essentially ignoring the part related to the punctures) and so we now repeat the above construction for the weight 1 part. See, for example, [Del71] for background on the weight and hodge filtration, [Tim87] for further background in the unitary case, and [LL22a, §4] for a concise description of the weight and hodge filtration in the case of curves.

We now let $W^\rho$ be the weight 1 part of $W^\rho$, which we can explicitly identify as $R^1\pi_*(j_*U^\rho)$ by [LL22a, Theorem 4.1.1]. Let

$$(\mathcal{G}^\rho, \nabla^{\rho}_{GM}) := (W^\rho \otimes \mathcal{O}_\mathcal{M}, \text{id} \otimes d).$$

To understand how the Hodge filtration of $\mathcal{G}^\rho$ interacts with the weight filtration, the following lemma is key. The proof is a matter of unwinding definitions.

Lemma 3.2. Suppose $C$ is a smooth proper connected curve, $D \subset C$ is a divisor, and $\mathcal{V}$ is a local system on $C^\circ := C - D$ with unitary monodromy. Let $(E_\ast, \nabla)$ be the Deligne canonical extension of $(\mathcal{V} \otimes \mathcal{O}_{C^\circ}, \text{id} \otimes d)$ to $C$ as in, for example, [LL22b, Definition 3.3.1]. Under the natural isomorphism $F^1(H^1(C^\circ, \mathcal{V})) \simeq H^0(C, \mathcal{E} \otimes \omega_C(D))$, the weight 1 part is given by

$$(W^1 \cap F^1)(H^1(C^\circ, \mathcal{V})) \simeq H^0(C, \hat{\mathcal{E}}_0 \otimes \omega_C(D)) \subset H^0(C, E \otimes \omega_C(D)).$$
Proof. First, we claim \((W^1 \cap F^1)(H^1(C^\circ, V))\) is identified with the subspace of \(H^0(C, E \otimes \omega_C(D))\) vanishing under the residue map \(H^0(C, E \otimes \omega_C(D)) \to i_*(\mathcal{O}_D \otimes E|_D)\), for \(i : D \to C\) the inclusion. Indeed, this follows from the degeneration of the Hodge-de Rham spectral sequence for unitary local systems [Tim87, Theorem 7.1(a)] and the definition of the weight and Hodge filtrations as in [Tim87], culminating in [Tim87, Definition 6.1].

It only remains to identify \(H^0(C, \hat{E}_0 \otimes \omega_C(D)) = \ker(H^0(C, E \otimes \mathcal{O}_X \omega_C(D)) \to i_*(\mathcal{O}_D \otimes E|_D))\). We will explain why this equality is a matter of unwinding definitions. Specifically, we must unwind the definition of the Deligne canonical extension which yields the parabolic structure on \(E\), as in [Del70], Remarques 5.5(i)], (see also [LL22b, Definition 3.3.1] for a summary in the relevant case) and the definition of \(\hat{E}_0\) in (2.1). Indeed, using notation as in [§2.1] we recall \(J\) denotes the set of integers \(j \in \{1, \ldots, n\}\) for which \(\alpha_j = 0\). For any \(j \in J\), by definition of the Deligne canonical extension, \(E^j\) is the sum of the generalized eigenspaces of the residue map at \(x_j\) with nonzero eigenvalue, as described in [LL22b, Definition 3.3.1]. Because the 0-generalized eigenspace at \(x_j\) is semisimple by assumption that the monodromy of \(V\) is unitary, it is identically 0. Hence, \(\ker(E \to E_{x,j}/E^j_0)\) is the kernel of the residue map at \(x_j\). It follows that \(\hat{E}_0 := \ker(E \to \bigoplus_{j \in J} E_{x,j}/E^j_0)\) is the kernel of the residue map along \(D\), and the analogous statement holds for \(E\) replaced by \(E \otimes \omega_C(D)\). \(\square\)

Consequently we obtain a simple description of the graded parts of the Hodge filtration of \(\mathcal{G}^\rho\). Since the weight 1 part of the Hodge filtration surjects onto the second graded piece of the weight filtration, Lemma 3.2 yields the following description of the Hodge filtration of \(\mathcal{G}^\rho\), where we use notation as in (2.2):

\[
F^1\mathcal{G}^\rho \simeq \pi_*(\mathcal{E}^\rho_0 \otimes \Omega^1_M(\log \mathcal{D}))
\]

\[
\mathcal{G}^\rho / F^1\mathcal{G}^\rho \simeq R^1\pi_*\mathcal{E}^\rho.
\]

For \(m \in \mathcal{M}\), let \(\nabla^\rho\) denote the fiber over \(m\) of the map

\[
\nabla^\rho : F^1\mathcal{G}^\rho \to \mathcal{G}^\rho \to \mathcal{G}^\rho \otimes \Omega^1_M \to \mathcal{G}^\rho / F^1\mathcal{G}^\rho \otimes \Omega^1_M
\]

where the first map is the natural inclusion and the last is the natural quotient map.
Using a version of the theorem of the fixed part, we now show how counterexamples to the Putman-Wieland conjecture yield nonzero elements in the kernel of $\nabla_m^\rho$ (compare to [LL22a Lemma 6.1.1]).

**Lemma 3.3.** Suppose $f : X \to Y$ furnishes a counterexample to Putman-Wieland. Let $\mathcal{X} \xleftarrow{\tilde{f}} \mathcal{C} \xrightarrow{\pi} \mathcal{M}$ denote a versal family of $\phi$-covers, as in Definition 2.8, whose fiber over some $m \in \mathcal{M}$ is $f^\circ : X^\circ \to Y^\circ$ with regular compactification given by $f : X \to Y$. Then there is some irreducible complex representation $\rho$ of $H$ such that $\nabla_m^\rho$, as defined in (3.3), has a nontrivial kernel. Moreover, if $f$ is $\rho$-isotypic in the sense of Definition 1.4, then $\nabla_m^\rho$ vanishes.

**Proof.** To begin, let us set up some notation. By assumption $f$ corresponds to a covering of topological surfaces $h : \Sigma_{g',n'} \to \Sigma_{g,n}$ which furnishes a counterexample to Putman-Wieland. Then, there is some finite index $\Gamma' \subset \Gamma$ fixing a nonzero vector in $H^1(\Sigma_{g',C})$. Because this action is tensored up from an action defined over the rational numbers, $\Gamma'$ also fixes a nonzero vector $v \in H^1(\Sigma_{g',Q})$. After replacing $\mathcal{M}$ by a finite étale cover, we may assume the action of $\pi_1(\mathcal{M})$ on $H^1(\Sigma_{g',Q})$ factors through $\Gamma'$. Let $V = H^0(\mathcal{M}, R^1(\tilde{f}^\circ \circ \pi^\circ)_*Q)$; by the theorem of the fixed part [PS08, 14.52], $V$ has a natural mixed Hodge structure compatible with its natural embedding $V \hookrightarrow (R^1(\tilde{f}^\circ \circ \pi^\circ)_*Q)_x$, for any $x \in \mathcal{M}$. By assumption $V$ is nonzero, and moreover intersects the weight 1 part of $R^1(\tilde{f}^\circ \circ \pi^\circ)_*Q$, because our chosen vector fixed by $\Gamma'$ lies in the weight 1 part $H^1(\Sigma_{g',C}) = W^1(H^1(\Sigma_{g',n'}, C))$ of the cohomology.

We next show that for some $\rho$, $\nabla_m^\rho$ has a nontrivial kernel. Let $j : Y^\circ \to Y$ denote the inclusion. As $W^1V$ is a nonzero rational sub-mixed Hodge structure of the mixed Hodge structure on $H^1(Y^\circ, f_*C) = (R^1(\tilde{f}^\circ \circ \pi^\circ)_*Q)_m$, $W^1V$ has non-trivial intersection with $(F^1 \cap W^1)H^1(Y^\circ, f_*C)$, i.e. $(F^1 \cap W^1)V$ is nonzero. Any element of $V = H^0(\mathcal{M}, R^1(\tilde{f}^\circ \circ \pi^\circ)_*Q)$ lies in the kernel of the Gauss-Manin connection by definition. Hence any element of $(F^1 \cap W^1)V$ lies in the kernel of

$$\nabla_m : F^1H^1(Y, j_*f_*C) \to H^1(Y, j_*f_*C)/F^1H^1(Y, j_*f_*C) \otimes T^\vee_{\mathcal{M},m}.$$

Decomposing into $\rho$-isotypic pieces yields the claim that for some $\rho$, $\nabla_m^\rho$ has a nontrivial kernel: indeed, $R^1(\tilde{f}^\circ \circ \pi^\circ)_*C$ is a direct sum of copies of $W^\rho$, as defined in Notation 3.1 by the Leray spectral sequence associated to the composition $\mathcal{X} \xleftarrow{\tilde{f}} \mathcal{C} \xrightarrow{\pi} \mathcal{M}$. Since this decomposition respects
the weight filtration, we similarly find \( W^1(R^1(\tilde{f} \circ \pi \circ \rho)^*_\ast) \) is a direct sum of copies of \( V^\rho \).

Finally, if \( f \) is \( \rho \)-isotypic, this means that \( V \) contains all of \( W^1(R^1(\tilde{f} \circ \pi \circ \rho)^*_\ast)Q^\rho \), and hence \( \nabla^m \) vanishes. \( \square \)

For the next statement, recall that a vector bundle \( V \) on an integral variety \( X \) is generically globally generated if the evaluation map \( H^0(X, V) \otimes O_X \rightarrow V \) is a surjection over the generic point of \( X \). Recall also that for \( \rho \) an \( H \)-representation and \( X \rightarrow Y \) a Galois \( H \)-cover, the associated map \( \pi_1(Y) \rightarrow H \rightarrow \text{GL}_{\text{dim}_{\rho}(C)} \) yields a parabolic vector bundle \( E^\rho \) on \( Y \) with underlying vector bundle \( E^\rho := E^\rho_0 \) under the Mehta-Seshadri correspondence, as in Notation 2.5.

**Proposition 3.4.** Suppose \( f : X \rightarrow Y \) furnishes a counterexample to Putman-Wieland. Then there exists an irreducible \( H \)-representation \( \rho \) so that the vector bundle \( (E^\rho)^\vee \otimes \omega_Y \) on \( Y \) is not generically globally generated.

**Proof.** By Lemma 3.3 (with notation as in the statement of Lemma 3.3), there exists \( \rho \) such that \( \nabla^m_{\rho} \) has a nontrivial kernel for any \( m \in M \). Fixing \( m \in M \), let \( Y := \mathcal{C}_m, E^\rho := (\mathcal{C}_m^\rho)_m \) and \( v \in \ker \nabla^m_{\rho} \) nonzero. Let \( D \subset Y \) denote the branch divisor of \( f : X \rightarrow Y \). We can identify

\[
v \in F^1\mathcal{C}^\rho \simeq H^0(Y, (\hat{E}^\rho)_0 \otimes \omega_Y(D)) \subset H^0(Y, E^\rho \otimes \omega_Y(D))
\]

and think of \( v \) as a nonzero map \( \mu_v : (E^\rho)^\vee \otimes \omega_Y \rightarrow \omega^2_Y(D) \). By [LL22a, Theorem 5.1.6], the vanishing of \( \nabla^m_{\rho}(v) \) is equivalent to the vanishing of the induced map \( H^0(Y, (E^\rho)^\vee \otimes \omega_Y) \rightarrow H^0(Y, \omega^2_Y(D)) \). In other words, the map \( \mu_v \) defined above induces the 0 map on global sections. Therefore, all global sections of \( (E^\rho)^\vee \otimes \omega_Y \) factor through the proper subbundle \( \ker \mu_v \), and so \( (E^\rho)^\vee \otimes \omega_Y \) is not generically globally generated. \( \square \)

4. ISOTYPICITY AND NON-UNITARITY

The main result of this section is Theorem 1.11, which states that counterexamples to Putman-Wieland in genus \( \geq 2 \) cannot be isotypic, i.e., there exists an element of \( H^1(\Sigma_g', C)^\rho \) with infinite orbit under the action of a finite index subgroup of the mapping class group. We show more, namely Theorem 1.12, if \( X \rightarrow Y \) is an \( H \)-cover, where \( Y \) has genus at least 2, the virtual action of the mapping class group of \( Y \) on an \( H \)-isotypic component of the cohomology of \( X \) is non-unitary.
In Corollary 4.3 we use this to show how a result from the retracted paper of Boggi-Looijenga [BL21] would imply the Putman-Wieland conjecture.

Our main tool for proving this is a natural bilinear pairing, which we next introduce. Let \( C \) be a smooth proper connected curve of genus \( g \), \( D \subset C \) a reduced divisor, \( E_* \) a parabolic vector bundle on \((C, D)\), and \( E := E_0 \). As described in [LL22a (5.5)], there is a nondegenerate bilinear pairing

\[
(4.1) \quad B_E : (E \otimes \omega_C(D)) \times (E^\vee \otimes \omega_C) \rightarrow \omega_C^2(D)
\]

given as the composition

\[
B_E : (E \otimes \omega_C(D)) \times (E^\vee \otimes \omega_C) \xrightarrow{\otimes} (E \otimes E^\vee) \otimes \omega_C^2(D) \xrightarrow{\text{tr} \otimes \text{id}} \omega_C^2(D),
\]

where \( \text{tr} \) denotes the trace pairing \( E \otimes E^\vee \rightarrow \mathcal{O}_C \). Note this pairing is nondegenerate, since on the fiber over \( x \) it is obtained by the pairing between the vector spaces \( E_x \) and \( E^\vee_x \). By restriction to \( H^0(C, \mathcal{E}_0 \otimes \omega_C(D)) \subset H^0(C, E \otimes \omega_C(D)) \), we also obtain an induced pairing

\[
H^0(C, \mathcal{E}_0 \otimes \omega_C(D)) \times H^0(C, E^\vee \otimes \omega_C) \rightarrow H^0(C, \omega_C^2(D)).
\]

**Theorem 4.1.** Let \( E_* \) be a semistable parabolic bundle on \((C, D)\) of parabolic degree zero, with underlying vector bundle \( E := E_0 \). Suppose \( g \geq 2 \). Then the pairing \( H^0(\mathcal{E}_0 \otimes \omega_C(D)) \otimes H^0(E^\vee \otimes \omega_C) \rightarrow H^0(C, \omega_C^2(D)) \) cannot vanish.

**Proof.** First, we may assume \( \text{rk} \ E > 1 \) by [LL22a, Proposition 5.2.3], as \( g \geq 2 \).

If \( F_* \) is a parabolic sheaf, we call the image of \( H^0(C, F_0) \otimes \mathcal{O}_C \rightarrow F_0 \subset F_* \) the globally generated subsheaf of \( F_* \), which is by definition also a subsheaf of \( F_0 \). Let \( U \subset \mathcal{E}_0 \otimes \omega_C(D) \) denote the globally generated subsheaf of \( \mathcal{E}_0 \otimes \omega_C(D) \) and \( V \) denote the globally generated subsheaf of \( (E^\vee) \otimes \omega_C(D) \).

Under the identifications

\[
\left( \mathcal{E}_0 \otimes \omega_C(D) \right)_0 = \mathcal{E}_0 \otimes \omega_C(D)
\]

\[
((E^\vee) \otimes \omega_C(D))_0 = E^\vee \otimes \omega_C,
\]

(see [LL22b, Definition 2.6.1] for the notion of a dual of a parabolic bundle) \( U \) and \( V \) are also the globally generated subsheaves of \( \mathcal{E}_0 \otimes \omega_C(D) \) and \( E^\vee \otimes \omega_C \) respectively.

Let \( c_V := \text{rk} \ E - \text{rk} \ V \) and \( c_U := \text{rk} \ E - \text{rk} \ U \). Using Proposition 2.3 applied to the bundles \( E_* \otimes \omega_C(D) \) and \( E^\vee_* \otimes \omega_C(D) \), we know \( \text{rk} \ E \geq gc_V \) and \( \text{rk} \ E \geq gc_U \).
On the other hand, we claim $\text{rk } U + \text{rk } V \leq \text{rk } E$. Granting this claim, we find $c_V + c_U \geq \text{rk } E$. Since $\text{rk } E \geq gc_V$ and $\text{rk } E \geq gc_U$, adding these gives
\begin{equation}
2 \text{rk } E \geq g(c_V + c_U) \geq g \text{rk } E,
\end{equation}
implying $2 \geq g$.

In the case $g > 2$, it remains to show $\text{rk } U + \text{rk } V \leq \text{rk } E$. We will argue this using the fact that an isotropic subsheaf for a non-degenerate quadratic form on a vector bundle of rank $2 \text{rk } E$ has rank at most $\text{rk } E$. Indeed, consider the quadratic form $q_E$ on $E \otimes \omega_C(D) \oplus E^\vee \otimes \omega_C$ associated to the nondegenerate bilinear form $B_E$ of (4.1):
\[
q_E : E \otimes \omega_C(D) \oplus E^\vee \otimes \omega_C \to \omega_C^{\otimes 2}(D) \quad (v, w) \mapsto B_E(v, w).
\]
Any vector bundle subsheaf of $E \otimes \omega_C(D) \oplus E^\vee \otimes \omega_C$ isotropic for this quadratic form has rank at most $\text{rk } E$, as may be verified on the generic fiber using that isotropic subspaces of a rank $2 \text{rk } E$ non-degenerate quadratic space have dimension at most $\text{rk } E$. Therefore, it is enough to show $U \oplus V$ is killed under $q_E$. Using $q_E(U \oplus V) = B_E(U \times V)$, it is enough to show $B_E(U \times V) = 0$. We have a commutative diagram
\begin{equation}
\begin{array}{ccc}
(H^0(C, U) \otimes \mathcal{O}_C) \times (H^0(C, V) \otimes \mathcal{O}_C) & \longrightarrow & H^0(C, \omega_C^{\otimes 2}(D)) \otimes \mathcal{O}_C \\
\downarrow & & \downarrow \\
U \times V & \stackrel{B_E}{\longrightarrow} & \omega_C^{\otimes 2}(D)
\end{array}
\end{equation}
where the top horizontal map vanishes by assumption. Since the vertical maps are surjective, the bottom horizontal map satisfies $B_E(U \times V) = 0$, as desired.

To conclude, we also rule out the case $g = 2$. If $g = 2$, we must have equality in (4.2), which forces $\text{rk } E = 2c_U$. This means we have equality in Proposition 2.3(II), which is proved in [LL22b Proposition 6.3.6]. If equality holds, we also have equality in [LL22b Lemma 6.2.3], which means $\mu(\mathcal{E}_0 \otimes \omega_C(D)) = 2g - 2$ (where we are taking the bundle named $V$ in [LL22b Lemma 6.2.3] to be $\mathcal{E}_0 \otimes \omega_C(D)$). This means $E_s$ has trivial parabolic structure at each parabolic point, so $\mathcal{E}_0 \otimes \omega_C(D) = E \otimes \omega_C$. If we had $\text{rk } E = 2c_U$, we would have $H^0(C, E \otimes \omega_C) = H^0(C, U)$, which means $h^0(C, U) = 2 \text{rk } U$ and so $h^1(C, U) = \text{rk } U$. By Clifford’s theorem for vector bundles, as in [LL22b Lemma 6.2.3], this can only happen when $\mu(U) = 2g - 2$, in which case we would have $U = E \otimes \omega_C$. This equality contradicts the assumption that $\text{rk } E = 2c_U$, i.e. that $\text{rk } E = 2 \text{rk } U$. □
4.2. We now deduce our desired isotypicity consequence, Theorem 1.11 and Theorem 1.12, for the Putman-Wieland conjecture. With setup as in Conjecture 1.2, we have a cover \( \Sigma_{g,n'} \to \Sigma_{g,n} \) and an action of a finite index subgroup \( \Gamma \subset \text{Mod}_{g,n+1} \) on \( H^1(\Sigma_{g,n'},\mathbb{C}) \). We aim to show that if \( \rho \) is any irreducible \( H \)-representation so that every element of the characteristic subspace \( H^1(\Sigma_{g',n'},\mathbb{C})^\rho \subset H^1(\Sigma_{g,n'},\mathbb{C}) \) has finite orbit under \( \Gamma \), then \( g \leq 2 \).

Proof of Theorem 1.11 and Theorem 1.12. We first prove Theorem 1.12. Assume to the contrary that the virtual action of the mapping class group on \( H^1(\Sigma_{g,n},\mathbb{C})^\rho \) is unitary. Let \( \mathcal{X} \to \mathcal{C} \to \mathcal{M} \) be a versal family of \( H \)-covers, and \( \mathcal{X} \to \mathcal{C} \to \mathcal{M} \) the associated families of proper curves. Let \( m \in \mathcal{M} \) be a point, and \( X \to Y \) the fiber over \( \mathcal{M} \). Let \( E^\rho \) be the parabolic bundle on \( Y \) corresponding to the representation \( \rho \) as in Notation 2.5.

We claim the map \( \nabla^0_{\rho} \) vanishes identically, as \( W^1 R^1 \pi_*^c \rho \), the variation of Hodge structure on \( \mathcal{M} \) associated to \( H^1(X,\mathbb{C})^\rho \), is unitary by assumption. Indeed, by [Del87, 1.13], we may write

\[
W^1 R^1 \pi_*^c \rho = \bigoplus_i V_i \otimes W_i,
\]

where the \( V_i \) are complex variations of Hodge structure with irreducible monodromy and the \( W_i \) are constant variations. The \( V_i \) carry a unique structure of a \( \mathbb{C} \)-VHS, up to renumbering. As \( W^1 R^1 \pi_*^c \rho \) is unitary by assumption, the same is true of each \( V_i \), and so the Hodge filtration of each \( V_i \) has length at most one. This proves the claim that \( \nabla^0_{\rho} = 0 \).

Note that \( \nabla^0_{\rho} \) is the weight 1 part of the map adjoint to the multiplication map

\[
H^0(Y, E^\rho_0 \otimes \omega_Y(D)) \otimes H^0(Y, (E^\rho_0)^* \otimes \omega_Y) \to H^0(Y, \omega_Y^{\otimes 2}(D))
\]

by [LL22a, Theorem 5.1.6]. Since the subspace of \( H^0(Y, E^\rho_0 \otimes \omega_Y(D)) \) corresponding to \( W^1 \cap F^1 \) is \( H^0(Y, E^\rho_0 \otimes \omega_Y(D)) \) by Lemma 3.2, we obtain that the multiplication map

\[
H^0(Y, E^\rho_0 \otimes \omega_Y(D)) \otimes H^0(Y, (E^\rho_0)^* \otimes \omega_Y) \to H^0(Y, \omega_Y^{\otimes 2}(D))
\]

also vanishes. Using Theorem 4.1, this implies \( g \leq 2 \).

Theorem 1.11 is immediate, as representations with finite image are unitary. \( \square \)
As a consequence, we show how a claimed result from a paper of Boggi-Looijenga (which has since been retracted by the authors) implies the Putman-Wieland conjecture.

**Corollary 4.3.** Suppose [BL21, Theorem B(i)] were true. Then the Putman-Wieland conjecture, Conjecture 1.2, would hold for all $g \geq 3$.

**Remark 4.4.** We note that, unfortunately, there is a fatal error in the proof of [BL21, Theorem B(i)], appearing in [BL21, Lemma 1.6] (which is used in the proof of [BL21, Theorem 1.1], and hence of [BL21, Theorem B(i)]). The error comes in the penultimate sentence of the proof of [BL21, Lemma 1.6], where it is claimed that “It follows…” but in fact no argument is given for this claim. It is for this reason that the authors of [BL21] retracted that paper.

**Proof.** Suppose $g \geq 3$, and we are given a finite étale $H$-cover of Riemann surfaces $f : \Sigma_{g',n'} \rightarrow \Sigma_{g,n}$ furnishing a counterexample to Putman-Wieland. This means some subrepresentation $\chi \subset H_1(X, C)$ has finite orbit under the action of the mapping class group $\text{Mod}_{g,n+1}$. The isotypicity statement of [BL21, Theorem B(i)] implies that if $\chi \subset H_1(X, C)$ has finite orbit under the action of the mapping class group $\text{Mod}_{g,n+1}$, every element of the $\chi$-isotypic component $H_1(X, C)^\chi$ has finite orbit under the action of $\text{Mod}_{g,n+1}$. By definition, this means $f : \Sigma_{g',n'} \rightarrow \Sigma_{g,n}$ is $\chi$-isotypic in the sense of Definition 1.4, which contradicts Theorem 1.11. 

5. **Deducing a Result of Marković-Tošić**

Our goal in this section is to show how our results toward the Putman-Wieland conjecture yield an alternate proof of [MT Theorem 1.5], which gives constraints on any counterexample to Putman-Wieland. Specifically, we will prove Theorem 1.6, which is slightly stronger than [MT Theorem 1.5], as explained in Remark 1.7.

We now briefly outline the structure of the proof. We will deduce Theorem 1.6 from Proposition 1.8 using the Li-Yau inequality, following the idea in [MT]. Hence, we focus on explaining Proposition 1.8 which shows that any counterexample to Putman-Wieland has low gonality. To verify this claim about gonality, we recall from Proposition 3.4 that any counterexample to Putman-Wieland $f : X \rightarrow Y$ yields a representation $\rho$ so that $(E^p)^Y \otimes \omega_Y$ is not generically globally generated. With this in mind, we give some equivalent criteria for this non-generic global generation in Lemma 5.4. Using Lemma 5.4, we obtain cohomological bounds in Lemma 5.8 which then easily implies the desired bound for gonality of Proposition 1.8.
Hence, our first goal is to describe equivalent conditions for generic global generation of \((E^\rho)^Y \otimes \omega_Y\) in Lemma 5.4. For this, it will be useful to reduce to the case \(f : X \to Y\) is Galois, in which the following decomposition of \(f_*\mathcal{O}_X\) will be useful. This result is well known to experts, and we include it for completeness.

**Lemma 5.1.** Let \(f : X \to Y\) be a Galois \(H\)-cover, let \(\rho\) be an irreducible \(H\)-representation, and let \(E^\rho\) denote the parabolic bundle associated to \(\rho\) as in Notation 2.5, and let \(E^\rho = E^\rho_0\). Then \(f_*\mathcal{O}_X \simeq \bigoplus H\text{-irreps} \rho (E^\rho) \oplus \dim \rho\).

*Proof.* In Lemma 5.2 below, we show \(f_*\mathcal{O}_X \simeq E^\rho_{reg}\). Therefore, it suffices to show \(E^\rho_{reg} \simeq \bigoplus H\text{-irreps} \rho (E^\rho) \oplus \dim \rho\). Since the regular representation of \(H\) decomposes as \(\text{reg} \simeq \bigoplus H\text{-irreps} \rho \rho \oplus \dim \rho\), the Mehta-Seshadri correspondence [MS80] gives the desired isomorphism \(E^\rho_{reg} \simeq \bigoplus H\text{-irreps} \rho (E^\rho) \oplus \dim \rho\). □

**Lemma 5.2.** Let \(f : X \to Y\) be a Galois \(H\)-cover. Let \(E^\rho_{reg}\) denote the parabolic bundle on \(Y\) associated to the regular \(H\) representation under the Mehta-Seshadri correspondence. Then, \(E^\rho_{reg} \simeq f_*\mathcal{O}_X\).

*Proof.* Let \(Y^\circ \subset Y\) be the locus on which \(f\) is étale, let \(f^\circ : X^\circ := f^{-1}(Y^\circ) \to Y^\circ\) be the restriction of \(f\) to \(Y^\circ\), and let \(j : Y^\circ \to Y\) be the inclusion. We find

\[E^\text{reg}_{Y^\circ} \simeq E^\rho_{reg, X^\circ} \simeq f^\circ_* E^{\text{triv}, X^\circ} \simeq f^\circ_* \mathcal{O}_{X^\circ}^\text{reg}.\]

By adjunction, this gives us a map

\[E^\text{reg}_{Y^\circ} \to j_* j^* E^\text{reg}_{Y^\circ} \simeq j_* E^\text{reg}_{Y^\circ} \simeq j_* f^\circ_* \mathcal{O}_{X^\circ} \simeq f_* \mathcal{O}_X.\]

Since this is an isomorphism over \(Y^\circ\), it is enough to verify it is an isomorphism in a neighborhood of each point of \(Y - Y^\circ\), which follows from a local computation. □

Using the above decomposition in the Galois case, we now give several reformulations of non-generic global generation of \(E^\rho \otimes \omega_Y\). For the first, we recall a definition of Prill exceptional covers, named after their relation to Prill’s problem, as discussed in [LL22c], see also Remark 5.6 and Remark 5.7

**Definition 5.3.** A finite cover \(f : X \to Y\) of smooth proper geometrically connected curves is **Prill exceptional** if \(h^0(X, \mathcal{O}_X(f^{-1}(y))) \geq 2\) for every point \(y \in Y\).

**Lemma 5.4.** Let \(f : X \to Y\) be a finite cover of smooth proper connected curves whose Galois closure has Galois group \(H\). The following are equivalent:

1. The map \(f\) is Prill exceptional.
(2) There is some irreducible nontrivial $H$-representation $\rho$ for which the associated vector bundle $E^\rho := E^\rho_0$ as in [Notation 2.3] is a summand of $f_*\mathcal{O}_X$ and $h^0(Y, E^\rho(p)) > 0$ for a general $p \in Y$.

(3) For the same $\rho$ as in the previous part, $(E^\rho)^\vee \otimes \omega_Y$ is not generically globally generated.

**Proof.** We first show the equivalence of (1) and (2). In the case $f$ is a Galois cover, we can apply the decomposition $f_*\mathcal{O}_X \cong \text{⊕}_H\text{-irrep}\rho(E^\rho)^\oplus_{\dim \rho}$ from Lemma 5.1. We find

$$h^0(X, \mathcal{O}_X(f^{-1}(p))) = h^0(Y, (f_*\mathcal{O}_X)(p)) = \text{⊕}_H\text{-irrep}\rho h^0(Y, (E^\rho)^\oplus_{\dim \rho}(p)).$$

Therefore, in the Galois case, $f$ is Prill exceptional if and only if there is some $E^\rho$, a summand of $f_*\mathcal{O}_X$, with $h^0(Y, E^\rho(p)) > 0$ for a general $p \in Y$.

We next verify the equivalence of (1) and (2) in the case that $f$ is not Galois. Let $X' \xrightarrow{h} X \xrightarrow{f} Y$ denote its Galois closure, so that $f \circ h$ has Galois group $H$. Since we are working in characteristic 0, $\mathcal{O}_{X'}$ is a summand of $h_*\mathcal{O}_X$, and therefore $f_*\mathcal{O}_X$ shows up as a summand in $(f \circ h)_*\mathcal{O}_{X'}$. Therefore, we can decompose $f_*\mathcal{O}_X$ as a sum $f_*\mathcal{O}_X \cong \text{⊕}_H\text{-irrep}\rho(E^\rho)^\oplus a_\rho$ where $0 \leq a_\rho \leq \dim \rho$. As in the Galois case, we again find $f$ is Prill exceptional if and only if there is some $E^\rho$ with $h^0(Y, E^\rho(p)) > 0$ for a general $p \in Y$.

We conclude by demonstrating the equivalence of (2) and (3). Let $F^\rho := (E^\rho)^\vee \otimes \omega_Y$. We wish to show $F^\rho$ is not generically globally generated if and only if $h^0(Y, E^\rho(p)) > 0$. Note that $F^\rho$ is not generically globally generated if and only if, for a general $p \in Y$, we have an exact sequence

$$0 \longrightarrow H^0(Y, F^\rho(-p)) \longrightarrow H^0(Y, F^\rho) \longrightarrow H^0(Y, F^\rho|_p)$$

which is not right exact. Since $F^\rho|_p$ is supported at $p$, $h^0(Y, F^\rho|_p) = \text{rk} F^\rho$, and so failure of right exactness is equivalent to $h^0(Y, F^\rho(-p)) > h^0(Y, F^\rho) - \text{rk} F^\rho$. Note that because $\rho$ is nontrivial, $0 = h^0(E^\rho) = h^1(Y, F^\rho)$. Therefore, using Riemann-Roch, $h^0(Y, F^\rho(-p)) > h^0(Y, F^\rho) - \text{rk} F^\rho$ is equivalent to $h^1(Y, F^\rho(-p)) > h^1(Y, F^\rho) = 0$. By Serre duality, this is equivalent to $h^0(Y, E^\rho(p)) > 0$, as we wanted to show.

Combining Proposition 3.4 and Lemma 5.4, we easily deduce that counterexamples to Putman-Wieland yield Prill exceptional covers.

**Lemma 5.5.** If $X$ has genus $g'$, $Y$ has genus $g$, and $f : X \to Y$ is a cover furnishing a counterexample to Putman-Wieland, then $f : X \to Y$ is Prill exceptional.
Proof. Suppose $f : X \to Y$ is a Galois $H$-cover which furnishes a counterexample to Putman-Wieland. We saw in Proposition 3.4 that there is some irreducible $H$-representation $\rho$ so that $(E^\rho) \otimes \omega$ is not generically globally generated. By Lemma 5.4, $f$ is Prill exceptional. □

The following remarks will not be needed in what follows. We include them as a pleasant, immediate application of the work we have done so far.

Remark 5.6. Prill’s problem, [ACGH85, p. 268, Chapter VI, Exercise D], asks whether any curve of genus $g \geq 2$ has a Prill exceptional cover. By Lemma 5.5, any counterexample to Putman-Wieland yields a Prill exceptional cover. Since [Mar22, Theorem 1.3] gives $f : X \to Y$ which is a counterexample to Putman-Wieland in genus 2, we find that a general curve of genus 2 has a Prill exceptional cover.

Remark 5.7. In a companion paper, [LL22c], we prove a stronger result than that mentioned in Remark 5.6. Namely, we show that any smooth proper connected curve of genus 2 over the complex numbers has a Prill exceptional cover, as opposed to just a general curve of genus 2. We include Remark 5.6 in the present paper as it is an immediate consequence of the work we do here to obtain an alternate proof of the result of Marković-Tošić. However, we chose to write [LL22c] as a separate paper in order to give a mostly self-contained exposition of Prill’s problem, which is easier to follow than the above construction of Prill exceptional covers. In particular, it is not mired in the notation of parabolic bundles.

Another consequence of the above results is the following bound on sections associated to the divisor given by a fiber of $f$, which will be the key to verifying a gonality estimate to deduce the result of Marković-Tošić, Theorem 1.6.

Lemma 5.8. Any Prill exceptional cover $f : X \to Y$ satisfies $h^0(X, \mathcal{O}_X(f^{-1}(p))) \geq 2$. If $f$ is Galois, we moreover have $h^0(X, \mathcal{O}_X(f^{-1}(p))) \geq g + 1$.

Proof. The non-Galois case follows from the definition of a Prill exceptional cover.

We next show, in the case that $f$ is Galois, that $\dim \rho \geq g$. Let $D$ denote the branch divisor of $f$, and let $n := \deg D$. We wish to apply Proposition 2.3 to the bundle $E_\otimes = (E^\rho)^\vee \otimes \omega_Y(D)$ and so we next verify its hypotheses. If $f$ is Galois, $(E^\rho)^\vee \otimes \omega_Y$ is not generically globally generated by Lemma 5.4. By [Yok95 (3.1)] we have $\tilde{E}_0^\rho(D) \simeq (E_0^\rho)^\vee$. This implies $\tilde{E}_0^\rho \otimes \omega_Y(D) \simeq$
\((E_0^\vee \otimes \omega_Y). \) Hence, the bundle \(E_\ast^\vee \otimes \omega_Y(D)\) is not generically globally generated. We also have

\[
\mu_\ast(E_\ast^\vee \otimes \omega_Y(D)) = \mu_\ast(E_\ast^\vee) + \deg \omega_Y(D) = 0 + 2g - 2 + n = 2g - 2 + n.
\]

Therefore, we may apply the final statement of Proposition 2.3 to the parabolic bundle \(E_\ast = E_\ast^\vee \otimes \omega_Y(D)\) to conclude \(\dim \rho \geq g\).

Combining the above observations, we have

\[
h^0(X, \mathcal{O}_X(f^{-1}(p))) = h^0(Y, (f_\ast \mathcal{O}_X)(p)) 
\geq h^0(Y, E^\text{triv}_0(p) \oplus (E_0^\vee) \oplus \dim \rho(p)) 
\geq h^0(Y, E^\text{triv}_0(p) \oplus (E_0^\vee) \oplus g(p)) 
\geq 1 + g.
\]

\[\square\]

5.9. We next prove Proposition 1.8, regarding the gonality of counterexamples to Putman-Wieland.

Proof of Proposition 1.8. We first want to show that if \(f : X \to Y\) furnishes a counterexample to Putman-Wieland, then \(\text{gon}(X) \leq \deg f\). In the case \(f : X \to Y\) is Galois, we wish to show \(X\) has gonality at most \(\deg f - (g - 1)\).

Note first that \(f : X \to Y\) is Prill exceptional by Lemma 5.5. Hence, we may apply Lemma 5.8, which implies that for a general point \(p \in Y\), we have \(h^0(X, \mathcal{O}(f^{-1}(p))) \geq 2\), and moreover \(h^0(X, \mathcal{O}(f^{-1}(p))) \geq g + 1\) when \(f\) is Galois. In the general case, \(\mathcal{O}(f^{-1}(p))\) is a line bundle of degree \(\deg f\) with 2 global sections, meaning \(X\) has gonality at most \(\deg f\).

We now assume \(f\) is Galois and aim to show \(\text{gon}(X) \leq \deg f - (g - 1)\). Choose a generic degree \(g - 1\) divisor \(S\) on \(X\). Then

\[
h^0(X, \mathcal{O}_X(f^{-1}(p) - S)) \geq h^0(X, \mathcal{O}_X(f^{-1}(p))) - \deg S = 1 + g - (g - 1) = 2,
\]

and so \(f^{-1}(p) - S\) is a divisor of degree \(\deg f - (g - 1)\) which still has a 2-dimensional space of global sections. Therefore, \(X\) has gonality at most \(\deg f - (g - 1)\), as claimed.

\[\square\]

To conclude the paper, we deduce Theorem 1.6 from Proposition 1.8. For this, we will connect nonzero eigenvalues of the Laplacian of a curve \(X\) to the gonality of \(X\), following the idea of [MT]. The key input in [MT, Theorem 1.5] is their verification in the end of [MT, §8] that if \(f : X \to Y\) furnishes a counterexample to Putman-Wieland, then the gonality of \(X\) is at most \(\deg f\). We prove prove a slightly stronger statement, using different
methods. Following the idea of [MT], we now explain why the gonality estimate of Proposition 1.8 implies Theorem 1.6 using the Li-Yau inequality.

5.10. Proof of Theorem 1.6. Let $X$ be a curve of genus $g'$, $Y$ a curve of genus $g$ and $f : X \to Y$ a cover furnishing a counterexample to Putman-Wieland. Recall we are trying to show that if $\lambda_1(X)$ denotes the smallest nonzero eigenvalue of the Laplacian acting on $X$, then $\frac{1}{g-1} \geq 2\lambda_1(X)$, and moreover there is a strict inequality when $f$ is Galois.

It follows from the Li-Yau inequality [LY82, Theorem 1], as explained in [EHK12 (11)], that $\text{gon}(X) \geq 2\lambda_1(X)(g'-1)$. The statement of Proposition 1.8 tells us that any counterexample to Putman-Wieland, $f : X \to Y$, satisfies $\deg f \geq \text{gon}(X)$. Combining this with the above consequence of the Li-Yau inequality and Riemann-Hurwitz yields

$$\deg f \geq \text{gon}(X) \geq 2\lambda_1(g'-1) \geq 2\lambda_1(X) \deg f(g-1).$$

Dividing both sides by $\deg f(g-1)$ gives $\frac{1}{g-1} \geq 2\lambda_1(X)$.

In the case $f$ is Galois, since $g \geq 2$, we get a strict inequality $\deg f > \text{gon}(X)$, from Proposition 1.8 which similarly implies $\frac{1}{g-1} > 2\lambda_1(X)$. □

References

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. Geometry of algebraic curves. Vol. I, volume 267 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985.

[BL21] Marco Boggi and Eduard Looijenga. Curves with prescribed symmetry and associated representations of mapping class groups. Math. Ann., 381(3-4):1511–1535, 2021.

[BY96] Hans U. Boden and Kôji Yokogawa. Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves. I. Internat. J. Math., 7(5):573–598, 1996.

[Del70] Pierre Deligne. Équations différentielles à points singuliers réguliers. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[Del71] Pierre Deligne. Théorie de Hodge. II. Inst. Hautes Études Sci. Publ. Math., (40):5–57, 1971.

[Del87] Pierre Deligne. Un théorème de finitude pour la monodromie. In Discrete groups in geometry and analysis, pages 1–19. Springer, 1987.

[EHK12] Jordan S. Ellenberg, Chris Hall, and Emmanuel Kowalski. Expander graphs, gonality, and variation of Galois representations. Duke Math. J., 161(7):1233–1275, 2012.

[LL22a] Aaron Landesman and Daniel Litt. Canonical representations of surface groups. arXiv preprint arXiv:2205.15352v2, 2022.
Aaron Landesman and Daniel Litt. Geometric local systems on very general curves and isomonodromy. \textit{arXiv preprint arXiv:2202.00039v2}, 2022.

Aaron Landesman and Daniel Litt. Prill's problem. \textit{arXiv preprint arXiv:2209.12958v1}, 2022.

Peter Li and Shing Tung Yau. A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. \textit{Invent. Math.}, 69(2):269–291, 1982.

Vladimir Marković. Unramified correspondences and virtual properties of mapping class groups. \textit{Bull. Lond. Math. Soc.}, 54(6):2324–2337, 2022.

Vikram Bhagvandas Mehta and Conjeevaram Srinivasa Seshadri. Moduli of vector bundles on curves with parabolic structures. \textit{Mathematische Annalen}, 248(3):205–239, 1980.

Vladimir Marković and Ognjen Tošić. The second variation of the Hodge norm and higher Prym representations. \texttt{https://people.maths.ox.ac.uk/~markovic/MT-hodge.pdf}. Accessed: August 25, 2022.

Chris A. M. Peters and Joseph H. M. Steenbrink. \textit{Mixed Hodge structures}, volume 52 of \textit{Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]}. Springer-Verlag, Berlin, 2008.

Andrew Putman and Ben Wieland. Abelian quotients of subgroups of the mappings class group and higher Prym representations. \textit{J. Lond. Math. Soc. (2)}, 88(1):79–96, 2013.

Morihiko Saito. Modules de Hodge polarisables. \textit{Publ. Res. Inst. Math. Sci.}, 24(6):849–995 (1989), 1988.

Klaus Timmerscheidt. Mixed Hodge theory for unitary local systems. \textit{J. Reine Angew. Math.}, 379:152–171, 1987.

Kōji Yokogawa. Infinitesimal deformation of parabolic Higgs sheaves. \textit{Internat. J. Math.}, 6(1):125–148, 1995.