DYNAMICS OF NEMATIC LIQUID CRYSTAL FLOWS: 
THE QUASILINEAR APPROACH 

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Abstract. Consider the (simplified) Leslie-Erickson model for the flow of nematic liquid crystals in a bounded domain \( \Omega \subset \mathbb{R}^n \) for \( n > 1 \). This article develops a complete dynamic theory for these equations, analyzing the system as a quasilinear parabolic evolution equation in an \( L_p - L_q \)-setting. First, the existence of a unique local strong solution is proved. This solution extends to a global strong solution, provided the initial data are close to an equilibrium or the solution is eventually bounded in the natural norm of the underlying state space. In this case the solution converges exponentially to an equilibrium. Moreover, the solution is shown to be real analytic, jointly in time and space. 

1. Introduction 

We consider the following system modeling the flow of nematic liquid crystal materials in a bounded domain \( \Omega \subset \mathbb{R}^n \) 

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla \pi &= -\lambda \text{div}([\nabla d]^T \nabla d) \quad \text{in } (0, T) \times \Omega, \\
\partial_t d + (u \cdot \nabla)d &= \gamma (\Delta d + |\nabla d|^2 d) \quad \text{in } (0, T) \times \Omega, \\
\text{div } u &= 0 \quad \text{in } (0, T) \times \Omega, \\
(u, d)|_{t=0} &= (u_0, d_0) \quad \text{in } \Omega.
\end{align*}
\]

Here, the function \( u : (0, \infty) \times \Omega \to \mathbb{R}^n \) describes the velocity field, \( \pi : (0, \infty) \times \Omega \to \mathbb{R} \) the pressure, and \( d : (0, \infty) \times \Omega \to \mathbb{R}^n \) represents the macroscopic molecular orientation of the liquid crystal. Due to the physical interpretation of \( d \) it is natural to impose the condition 

\[
|d| = 1 \quad \text{in } (0, T) \times \Omega. 
\]

We will show in the following that this condition is indeed preserved by the above system; see Proposition 4.3 below for details. 

The constants \( \nu > 0, \lambda > 0, \) and \( \gamma > 0 \) represent viscosity, the competition between kinetic energy and potential energy and the microscopic elastic relaxation time for the molecular orientation field, respectively. For simplicity, we set \( \nu = \lambda = \gamma = 1 \), which does not change our analysis. 

The continuum theory of liquid crystals was developed by Ericksen and Leslie during the 1950's and 1960's in \cite{9, 17}. The Ericksen-Leslie theory is widely used as a model for the flow of liquid crystals, see for example the survey articles by Leslie in \cite{10} and also \cite{3, 6, 13, 20}. 

The set of equations (1.1) was considered first in \cite{21}, however for the situation where in the second equation of (1.1) the term \( |\nabla d|^2 d \) is replaced by \( f(d) = \nabla F(d) \), i.e. 

\[
d_t + (u \cdot \nabla)d = \gamma (\Delta d - f(d)),
\]

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where $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth, bounded function. Note that in this situation, the condition (1.2) cannot be preserved in general. Thus, this condition was replaced in \cite{20} and \cite{21} by the Ginzburg-Landau energy functional, i.e. $f$ is assumed to satisfy $f(d) = \nabla F(d) = \nabla \frac{1}{2}(|d|^2 - 1)^2$. In 1995, Lin and Liu \cite{21} proved the existence of global weak solutions to (1.1) in dimension 2 or 3 under the assumptions that $u_0 \in L_2(\Omega)$, $d_0 \in H^1(\Omega)$, and $d_0 \in H^{3/2}(\partial \Omega)$. Existence and uniqueness of global classical solutions was also obtained by them in dimension 2 provided $u_0 \in H^1(\Omega)$, $d_0 \in H^2(\Omega)$, and provided the viscosity $\nu$ is large in dimension 3. For regularity results of weak solutions in the spirit of Caffarelli-Kohn-Nirenberg we refer to \cite{22}.

Hu and Wang \cite{14} considered in 2010 the case of $f(d) = 0$ and proved existence and uniqueness of a global strong solution for small initial data in this case. They proved moreover that whenever a strong solutions exist, all global weak solutions as constructed in \cite{21} must be equal to this strong solution. The idea of their approach was to consider the above system (1.1) as a semilinear equation with a forcing term $\lambda \text{div}([\nabla d]^T \nabla d)$ on the right-hand side.

The full system (1.1) with $f(d) = |\nabla d|^2 d$ was revisited by Lin, Lin, and Wang in 2010. They proved in \cite{19} interior and boundary regularity theorems under smallness condition in dimension 2 and established the existence of global weak solutions on bounded smooth domains $\Omega \subset \mathbb{R}^2$ that are smooth away from a finite set. Furthermore, Wang proved in \cite{31} global well-posedness for this system for initial data being small in $BMO^{-1} \times BMO$ in the case of a whole space, i.e. $\Omega = \mathbb{R}^n$, by combining techniques of Koch and Tataru with methods from harmonic maps to certain Riemannian manifolds.

For results on the compressible case we refer to \cite{15, 23}. Here, local existence of strong solutions is proved. The latter turn out to be even local classical solution.

Summarizing, we observe that in particular results for local as well as global, strong solutions in the three dimensional setting for the full system (1.1), obeying also the condition (1.2), do not seem to exist so far.

Recently, Li and Wang claimed in \cite{18} such a result. More precisely, they claimed the existence and uniqueness of a strong solutions to (1.1) in bounded, smooth domains (however, not satisfying (1.2)). Their idea was to rewrite (1.1) as a semilinear equation for the Stokes equation coupled to the heat equation with a right hand side of the form

$$\tilde{F}(u,d) := \left( -(u \cdot \nabla)u - \text{div}([\nabla d]^T \nabla d), - (u \cdot \nabla) d + |\nabla d|^2 d \right).$$

Unfortunately, their approach and their main result \cite{18} Theorem 2.1 relies on an incorrect regularity property for the solution of the heat equation \cite{18} Theorem 3.1. This result would imply further regularity properties for $d$ and hence for $\tilde{F}(u,d)$, which however are not true. Note that the (incorrect) assertion of \cite{18} Theorem 3.1 is crucial for their approach. Thus, the theory for local as well as for global strong solutions to (1.1), also satisfying (1.2), needs clarification.

It is the aim of this paper to present a complete theory for global strong solutions to (1.1) satisfying (1.2) as well as for their dynamical behaviour in the $n$-dimensional setting, where $n > 1$.

Our main idea is to consider (1.1) not as a semilinear equation as done in all of the previous approaches but as a quasilinear evolution equation. We thus incorporate the term $\text{div}([\nabla d]^T \nabla d)$ into the quasilinear operator $A$ given by

$$A(d) = \begin{bmatrix} A & \mathbb{P} \mathbb{B}(d) \\ 0 & \mathcal{D} \end{bmatrix},$$

where $A$ denotes the Stokes operator, $\mathcal{D}$ the Neumann Laplacian, and $\mathbb{B}$ is given by

$$[\mathbb{B}(d)h]_i := \partial_i d_l \Delta h_l + \partial_k d_l \partial_k \partial_i h_l,$$

for which we employ the sum convention. Note that $\mathbb{B}(d)d = \text{div}([\nabla d]^T \nabla d)$.

We then develop a complete dynamic theory for (1.1)–(1.2). In fact, first by local existence theory for abstract quasilinear parabolic problems, we prove the existence and uniqueness of a strong solution
to (1.1)—(1.2) on a maximal time interval. Thus, (1.1)—(1.2) give rise to a local semi-flow in the natural state space.

Furthermore, the equilibria \( \mathcal{E} \) of (1.1)—(1.2) are determined to be
\[
\mathcal{E} = \{(0, d_*) : d_* \in \mathbb{R}^n, |d_*| = 1\},
\]
and the well-known energy functional
\[
E = \frac{1}{2} \int_{\Omega} [ |u|^2 + |\nabla d|^2 ] dx
\]
for (1.1)—(1.2) is shown to be a strict Lyapunov-functional. In addition, the equilibria are shown to be normally stable, i.e. for an initial value close to \( \mathcal{E} \), the solution of (1.1)—(1.2) exists globally and the solution converges exponentially to an equilibrium. More generally, a solution, eventually bounded on its maximal interval of existence, exists globally and converges to an equilibrium exponentially fast.

Due to the polynomial character of the nonlinearities, we can even show that the solution of (1.1)—(1.2) is real analytic, jointly in time and space.

Our approach is based on the theory of quasilinear parabolic problems and relies in particular on the maximal \( L_p \)-regularity property for the heat and the Stokes equation. In particular, we refer here to [1, 2, 7, 5, 16, 25, 26, 28].

The plan for this paper is as follows. We begin by collecting general results from the theory of quasilinear parabolic evolution equations. Then, in Section 3 we introduce our formulation of (1.1). Section 4 deals with local well-posedness and regularity of solutions to (1.1)—(1.2); in particular we see that the solution is real analytic. The generalized principle of linearized stability yields the stability of equilibria and convergence of solutions is proved in Section 5. Moreover, by means of the associated energy functional, we prove convergence of a solution to an equilibrium, whenever the solution is eventually bounded in the natural state space.

2. Quasilinear Evolution Equations

Let \( X_0 \) and \( X_1 \) be Banach spaces such that \( X_1 \xleftarrow{d} X_0 \), i.e. \( X_1 \) is continuously and densely embedded in \( X_0 \). Let \( J = [0, a] \) for an \( a > 0 \). By a *quasilinear autonomous parabolic evolution equation* we understand an equation of the form
\[
(\text{QL}) \quad \dot{z}(t) + A(z(t))z(t) = F(z(t)), \quad t \in J, \quad z(0) = z_0,
\]
where \( A \) is a mapping from a real interpolation space \( X_{\gamma,\mu} \) with suitable weights between \( X_0 \) and \( X_1 \) into \( \mathcal{L}(X_0, X_1) \). Our approach relies on the maximal \( L_p \)-regularity of \( A(v) \) for \( v \in X_{\gamma,\mu} \). For details we refer e.g. to [7].

The equation (QL) is investigated in spaces of the form \( L_p(J; X_0) \) with temporal weights. More precisely, for \( p \in (1, \infty) \) and \( \mu \in (1/p, 1] \), the spaces \( L_{p,\mu} \) and \( H^1_{p,\mu} \) are defined by
\[
L_{p,\mu}(J; X_1) := \{ z : J \to X_1 : t^{1-\mu}z \in L_p(J; X_1) \},
\]
\[
H^1_{p,\mu}(J; X_0) := \{ z \in L_{p,\mu}(J; X_0) \cap W^1_1(J; X_0) : \dot{z} \in L_{p,\mu}(J; X_0) \}.
\]

It is clear, that
\[
L_p(J; X) \hookrightarrow L_{p,\mu}(J; X) \quad \text{and} \quad L_p([0, a]; X) \hookrightarrow L_{p,\mu}([\tau, a]; X),
\]
for all Banach spaces \( X \) and \( \tau \in (0, a) \). It has been shown in [26, Theorem 2.4] that \( L_p \)-maximal regularity implies also \( L_{p,\mu} \)-maximal regularity, provided \( p \in (1, \infty) \) and \( \mu \in (1/p, 1] \). The trace space of the maximal regularity class containing temporal weights,
\[
z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)
\]
Proposition 2.3. Let $u$ only that $A$.

The following result on global existence and stability of was proved in [28, Theorem 2.1] assuming $\sup_{v \in X_{\gamma,\mu}} \mu$ was shown by Clément-Li [5] in the case $\mu = 1$ and by Kühne-Prüss-Wilke [16] Theorem 2.1, Corollary 2.2] for the case $\mu \in (1/p, 1]$.

Proposition 2.1. Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $z_0 \in X_{\gamma,\mu}$, and suppose that the assumptions (A) and (F) are satisfied. Then, there exists $a > 0$, such that (QL) admits a unique solution $z$ on $J = [0, a]$ in the regularity class $z \in H^1_{p,\mu}(J; X_0) \cap L^{p,\mu}(J; X_1) \hookrightarrow C(J; X_{\gamma,\mu}) \cap C((0, a]; X_\gamma)$. The solution depends continuously on $z_0$, and can be extended to a maximal interval of existence $J(z_0) = [0, t^*(z_0))$.

Parabolic problems allow for additional smoothing effects. In this respect, a method due to Angenent [3] is well known. We only state here a variant of it which is adapted to (QL); see [25] Theorem 5.1] for the case $\mu = 1$. By a slight adjustment of its proof to the situation of temporal weights, this result remains true also for maximal regularity classes using this type of weights.

Proposition 2.2. Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $a > 0$, and assume that (A) and (F) hold. Let $z \in H^1_{p,\mu}(J; X_0) \cap L^{p,\mu}(J; X_1)$ be a solution of (QL) on $J = [0, a]$. Then

$$t^k \frac{d}{dt}^k z \in H^1_{p,\mu}(J; X_0) \cap L^{p,\mu}(J; X_1), \quad k \in \mathbb{N}.$$  

Furthermore, $z$ is real analytic with values in $X_1$ on $(0, a)$.

We denote the set of equilibria of (QL) by

$$\mathcal{E} = \{z_* \in X_1: A(z_*) z_* = F(z_*)\}.$$  

The following result on global existence and stability of was proved in [28] Theorem 2.1] assuming only that $A$ and $F$ are of class $C^1$.

Proposition 2.3. Let $1 < p < \infty$ and assume that assumptions (A) and (F) hold. Let $A_0$ be the linearization of (QL), i.e. let

$$A_0 w = A(z_*) w + (A'(z_*) w) u_* - F'(u_*) w, \quad w \in X_1.$$  

Suppose that $u_* \in \mathcal{E}$ is normally stable equilibrium, i.e.

(i) near $u_*$ the set of equilibria $\mathcal{E} \subset X_1$ is a $C^1$-manifold in $X_1$ of dimension $m \in \mathbb{N}_0$,

(ii) the tangent space of $\mathcal{E}$ at $u_*$ is given by $N(A_0)$,

(iii) $0$ is semi-simple eigenvalue of $A_0$, i.e. $N(A_0) \oplus R(A_0) = X_0$,

(iv) $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{x \in \mathbb{C}: \text{Re} x > 0\}$.

Then $u_*$ is stable in $X_{\gamma}$. Further, there exists a number $\rho > 0$ such that the unique solution $z$ of (QL) with initial value $z \in B_{X_\gamma}(0, \rho)$ exists on $\mathbb{R}_+$ and converges at an exponential rate to some $u_\infty \in \mathcal{E}$ in $X_{\gamma}$ as $t \to \infty$.

We finish the section with another result on global existence result for (QL); see [16] Theorem 3.1].
Proposition 2.4. Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $z_0 \in X_{\gamma, \mu}$ and let $J = [0, a]$ or $J = \mathbb{R}_+$. Suppose that assumptions (A) and (F) are satisfied and that the embedding $X_{\gamma, \mu} \hookrightarrow X_\gamma$ is compact. Assume furthermore that the solution $z$ of (QL) is eventually bounded in $X_\gamma$ on its maximal interval of existence, i.e. that $z$ satisfies
\[ z \in BC((\tau, t^+(z_0)); X_\gamma) \]
for some $\tau \in (0, t^+(z_0))$. Then the solution $z$ exists globally and for each $\delta > 0$, the orbit $\{z(t)\}_{t \geq \delta}$ is relatively compact in $X_\gamma$. If in addition $z_0 \in X_\gamma$, then $\{z(t)\}_{t \geq 0}$ is relatively compact in $X_\gamma$.

3. Nematic liquid crystals as quasilinear evolution equations

We now reformulate (1.1) equivalently as a quasilinear parabolic evolution equation for the unknown $z = (u, d)$. To this end, for $1 < q < \infty$ define the Banach spaces $X_0$ by
\[ X_0 := L_{q, \sigma}(\Omega) \times L_q(\Omega)^n, \]
where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial \Omega \subset C^2$. The subscript $\sigma$ in $L_{q, \sigma}(\Omega)$ means as usual the subspace of $L_q(\Omega)^n$ consisting of solenoidal vector fields.

The Neumann-Laplacian $D_q$ in $L_q(\Omega)$ is defined by $D_q = -\Delta$ with domain
\[ D(D_q) := \{d \in H^2_q(\Omega)^n : \partial_\nu d = 0 \text{ on } \partial \Omega\}. \]
It is well-known that $D_q$ has the property of $L_p$-maximal regularity; see [7, Theorem 8.2].

Let $\mathbb{P} : L_q(\Omega)^n \to L_{q, \sigma}(\Omega)$ denote the Helmholtz projection. We then define the Stokes Operator $A_q = -\mathbb{P}\Delta$ in $L_{q, \sigma}(\Omega)$ with domain
\[ D(A_q) = \{u \in H^2_q(\Omega)^n : \text{div } u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial \Omega\}. \]
It is also well-known that $A_q$ has the property of $L_p$-maximal regularity; see e.g. [28, 12, 11].

Next, we define the space $X_1$ by
\[ X_1 := D(A_q) \times D(D_q), \]
equipped with its canonical norms. Then $X_1 \hookrightarrow X_0$ densely.

The quasilinear part $A(z)$ of (QL) is given by the tri-diagonal matrix
\[ A(z) = \begin{bmatrix} A_q & \mathbb{P} B_q(d) \\ 0 & D_q \end{bmatrix}, \]
where the operator $B_q$ is given by
\[ [B_q(d)h]_i := \partial_i d_i \Delta h_i + \partial_k d_i \partial_k \partial_i h_i, \]
for which we employed the sum convention. Note that
\[ B_q(d)d = \text{div}([\nabla d]^T \nabla d). \]

Obviously, $B(d) : X_1 \to X_0$ is bounded for each $d \in C^1(\overline{\Omega})^n$ and the map $d \mapsto \mathbb{P} B_q(d)$ is polynomial, hence real analytic. By the tri-diagonal structure of $A(z)$ and by the regularity of $B$ one can easily see that $A(z)$ also has the property of $L_p$-maximal regularity, for each $z \in C^1(\overline{\Omega})^{2n}$. Indeed, for a fixed right-hand side $(f_u, f_d) \in L_p(0, a; X_{\gamma, \mu})$ and initial values $(u_0, d_0) \in X_{\gamma, \mu}$, we may use the maximal regularity of $D_q$ to obtain a solution $d$ of the heat equation with Neumann boundary condition in the right maximal regularity class. By setting
\[ \tilde{f}_u := f_u - \mathbb{P} B_q(d)\tilde{d} \]
as right-hand side for the Stokes equation, we obtain a solution $\tilde{u}$ in the right maximal regularity class due to the fact that $B_q(d)$ is linear and bounded.

The right-hand side $F(z)$ of (QL) is defined by
\[ F(z) = (-\mathbb{P} u \cdot \nabla u, -u \cdot \nabla d + |\nabla d|^2d), \]
which is also polynomial, hence a real analytic mapping from \( C^1(\overline{\Omega})^{2n} \) into \( X_0 \).

Note that (A) and (F) hold, as soon as we have the embedding
\[
X_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega})^{2n}.
\]
The space \( X_\gamma \) is given by
\[
X_\gamma := (X_0, X_1)_{1-1/p, p} = D_{A_\gamma}(1-1/p, p) \times D_{D_\gamma}(1-1/p, p);
\]
see \[18\]. As explained in Section 2 we consider \( L_p \)-spaces with temporal weights. The trace space of the class
\[
z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)
\]
now reads
\[
X_{\gamma,\mu} = (X_0, X_1)_{1-1/p, p} = D_{A_\gamma}(\mu-1/p, p) \times D_{D_\gamma}(\mu-1/p, p),
\]
provided \( p \in (1, \infty) \) and \( \mu \in (1/p, 1] \); see [21, Theorem 4.12].

In order to obtain the embeddings \( X_\gamma \hookrightarrow C^1(\overline{\Omega})^{2n} \) and more generally \( X_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega})^{2n} \) we impose on \( p, q \in (1, \infty) \) now the conditions
\[
\frac{2}{p} + \frac{n}{q} < 1, \quad \frac{1}{2} + \frac{1}{p} + \frac{n}{2q} < \mu \leq 1.
\]
Standard Sobolev embedding theorems can then be applied.

Further, we recall from [39, Theorem 4.3.3] and [2] Theorem 3.4], respectively, the following characterizations of the interpolation spaces involved,
\[
d \in D_{D_\gamma}(\mu-1/p, p) \iff d \in B^{2\mu-2/p}(\Omega)^n, \quad \partial_\nu d = 0 \text{ on } \partial \Omega,
\]
and
\[
u \in D_{A_\gamma}(\mu-1/p, p) \iff u \in B^{2\mu-2/p}(\Omega)^n \cap L_{q,\sigma}(\Omega), \quad u = 0 \text{ on } \partial \Omega.
\]
Observe that both of these characterizations make sense, since the condition (3.1) guarantees the existence of the trace.

4. Existence, uniqueness, and regularity of solutions

We start this section by applying Proposition 2.1 to obtain the following result on local well-posedness of (1.1).

**Theorem 4.1.** Let \( p, q, \mu \) be subject to (3.1), and assume \( z_0 = (u_0, d_0) \in X_{\gamma,\mu} \), which means that \( u_0, d_0 \in B^{2\mu-2/p}(\Omega)^n \) satisfy the compatibility conditions
\[
\text{div } u_0 = 0 \text{ in } \Omega, \quad u_0 = \partial_\nu d_0 = 0 \text{ on } \partial \Omega.
\]
Then for some \( a = a(z_0) > 0 \), there is a unique solution
\[
z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1), \quad J = [0, a],
\]
of (1.1) on \( J \). Moreover,
\[
z \in C([0, a]; X_{\gamma,\mu}) \cap C((0, a]; X_\gamma),
\]
i.e. the solution regularizes instantly in time. It depends continuously on \( z_0 \) and exists on a maximal time interval \( J(z_0) = [0, t^+(z_0)) \). Therefore problem (1.1), i.e. (QL), generates a local semi-flow in its natural state space \( X_{\gamma,\mu} \).

**Remark 4.2.** Assuming that \( 2/p + n/q < 1 \), for \( \varepsilon > 0 \) we may choose \( \mu \) subject to (3.1) such that
\[
H_q^{1+\frac{\varepsilon}{q} + \varepsilon}(\Omega)^n \hookrightarrow B^{2\mu-2/p}(\Omega)^n \hookrightarrow H_q^{1+\frac{\varepsilon}{q} - \varepsilon}(\Omega)^n
\]
due to Sobolev embeddings [39, Theorem 4.6.1]. Furthermore, we can choose \( p, q \) large with
\[
C^{1+\varepsilon}(\Omega)^n \hookrightarrow B^{2\mu-2/p}(\Omega)^n.
\]
Employing different time weights for $u$ and $d$, an inspection of the above proofs shows that the assertion of the above theorem remains true provided $u_0 \in C^\omega(\Omega)$.

The following result tells that the condition \( (1.2) \) is preserved by \( (1.1) \).

**Proposition 4.3.** Suppose that $\mu, p, q$ are satisfying \( (3.1) \) and let $z_0 = (u_0, d_0) \in X_{\gamma, \mu}$ with $|d_0| \equiv 1$, $a > 0$. Let
\[
  z \in H^1_{p, \mu}(J; X_0) \cap L_{p, \mu}(J; X_1)
\]
be a solution of \( (1.1) \) on the interval $J = [0, a]$. Then $|d(t)| \equiv 1$ holds for all $t \in [0, a]$.

**Proof.** Setting $\varphi = |d|^2 - 1$ the elementary identities,
\[
  \partial_t |d|^2 = 2d \cdot \partial_t d, \quad \Delta |d|^2 = 2\Delta d \cdot d + 2|\nabla d|^2, \quad \nabla |d|^2 = 2d \cdot \nabla d,
\]
and multiplication with $d$ of the second line in \( (1.1) \) yields the problem
\[
  \begin{cases}
    \partial_t \varphi + u \cdot \nabla \varphi = \Delta \varphi + 2|\nabla d|^2 \varphi & \text{in } \Omega \\
    \partial_n \varphi = 0 & \text{on } \partial \Omega, \\
    \varphi(0) = 0 & \text{in } \Omega,
  \end{cases}
\]
provided $|d_0| \equiv 1$. Uniqueness of this parabolic convection-reaction diffusion equations yields $\varphi \equiv 0$, i.e. $|d| \equiv 1$. \qed

As the nonlinearities $A$ and $F$ are real analytic we may employ Angenent’s method (Proposition 2.2) to obtain further regularity of the solutions of \( (1.1) \).

**Proposition 4.4.** Suppose that $\mu, p, q$ satisfy \( (3.1) \), $z_0 \in X_{\gamma, \mu}$, and $a > 0$ and let
\[
  z \in H^1_{p, \mu}(J; X_0) \cap L_{p, \mu}(J; X_1)
\]
be a solution of \( (1.1) \) on the interval $J = [0, a]$. Then for each $k \in \mathbb{N}$,
\[
  t^k \frac{d}{dt}^k z \in H^1_{p, \mu}(J; X_0) \cap L_{p, \mu}(J; X_1).
\]

Moreover, $z \in C^\omega((0, a); X_1)$.

We will employ Proposition 4.4 in the following to justify the regularity of time derivatives of the energy functional.

**Remark 4.5.** Employing scaling techniques jointly in time and space, it is possible to show via maximal regularity and the implicit function theorem that $u, \pi, d$ are analytic in $(0, t^+(z_0)) \times \Omega$; see [25, Section 5] for parabolic problems, and specifically for a Navier-Stokes problem [27]. As we will not use this result below we omit the details, here.

5. **Stability and Convergence to Equilibria**

We consider the set $\mathcal{E}_0 = \{0\} \times \mathbb{R}^n$, which are obviously equilibria of \( (1.1) \). This set forms a $n$-dimensional subspace of $X_1$, hence a $C^1$-manifold with tangent space $\{0\} \times \mathbb{R}^n$ at each point $(0, d_\ast) \in \mathcal{E}_0$. The linearization of \( (1.1) \) at $z_\ast \in \mathcal{E}_0$ is given by the linear evolution equation
\[
  \dot{z} + A_\ast z = f, \quad z(0) = z_0,
\]
in $X_0$, where
\[
  A_\ast = \text{diag}(A_q, D_q), \quad D(A_\ast) = X_1.
\]
As $\Omega$ is bounded, the spectrum $\sigma(A_q)$ consists only of positive eigenvalues and $0 \notin \sigma(A_q)$. On the other hand, $D_q$ has 0 as an eigenvalue, which is semi-simple and the remaining part of $\sigma(D_q)$ consist only of positive eigenvalues. Thus $\sigma(A_\ast) \setminus \{0\} \subset [\delta, \infty)$ for some $\delta > 0$ and the kernel of $A_\ast$ is given by
\[
  N(A_\ast) = \{0\} \times \mathbb{R}^n,
exists an open set $V$ contained in a manifold $E_\Gamma (5.1)$

As a result we see that the equilibrium is normally stable.

But, as the lemma below shows, this implies $(5.3)$

Therefore $E(t)$ is non-increasing along solutions. But $E$ is also a strict Lyapunov functional, i.e. strictly decreasing along non-constant solutions. In fact, if $dE(t)/dt = 0$ at some time instant, then by the energy equality we have $\nabla u = 0$ and $\Delta d + |\nabla d|^2d = 0$ in $\Omega$. Therefore $u = 0$ by the no-slip condition on $\partial \Omega$, and $d$ satisfies the nonlinear eigenvalue problem

\[ \begin{aligned}
\Delta d + |\nabla d|^2d &= 0 \quad \text{in } \Omega, \\
|d|^2 &= 1 \quad \text{in } \Omega, \\
\partial_d d &= 0 \quad \text{on } \partial \Omega.
\end{aligned} \] (5.3)

But, as the lemma below shows, this implies $\nabla d = 0$ in $\Omega$, hence $d = d_\ast$ is constant and $z_\ast := (0, d_\ast)$, $|d_\ast| = 1$ is an equilibrium of the problem.
Lemma 5.2. Suppose that \( d \in H^2_0(\Omega; \mathbb{R}^n) \) satisfies (5.3). Then \( d \) is constant in \( \Omega \).

Proof. The idea is to reduce inductively the dimension \( N = n \) of the vector \( d \). This can be achieved by introducing polar coordinates according to

\[
d_1 = c_1 \cos \theta, \quad d_2 = c_1 \sin \theta, \quad d_j = c_{j-1}, \quad j \geq 3.
\]

Simple computations yield

\[
1 = |d|^2 = |c|^2, \quad |\nabla d|^2 = |\nabla c|^2 + c_1^2 |\nabla \theta|^2,
\]

and

\[
\Delta c_j + |\nabla c|^2 + c_1^2 |\nabla \theta|^2 c_j = 0 \quad \text{in} \ \Omega,
\]

as well as \( \partial \nu c_j = 0 \) on \( \partial \Omega \) for \( j = 2, \ldots, n-1 \). Moreover, by an easy calculations we further obtain

\[
\Delta c_1 + c_1 |\nabla \theta|^2 = |\nabla c|^2 + c_1^2 |\nabla \theta|^2 c_1 \quad \text{in} \ \Omega,
\]

and

\[
c_1 \Delta \theta + 2 \nabla c_1 \cdot \nabla \theta = 0 \quad \text{in} \ \Omega,
\]

as well as

\[
\partial \nu c_1 = c_1 \partial \nu \theta = 0 \quad \text{on} \ \partial \Omega.
\]

Multiplying the former equation by \( c_1 \theta \) and integrating over \( \Omega \) we deduce

\[
0 = \int_\Omega [c_1 \Delta \theta + 2 \nabla c_1 \cdot \nabla \theta] c_1 \theta dx = \int_\Omega \text{div}[c_1^2 \nabla \theta] \theta dx = - \int_\Omega c_1^2 |\nabla \theta|^2 dx,
\]

hence \( c_1 \nabla \theta = 0 \). This implies that \( c \) satisfies the problem (5.3) where the vector \( c \) has dimension \( N-1 \). Inductively, we arrive at dimension \( N = 1 \) and if \( d \) is a solution of (5.3) with dimension 1, then \( d = 1 \) or \( d = -1 \) by connectedness of \( \Omega \).

\[\square\]

Note that the side condition \( |d| = 1 \) is important at this point. Summarizing we proved the following result.

Proposition 5.3. The energy functional \( E \) defined on \( X_\gamma \) is a strict Ljapunov function for system (1.1)–(1.2). The equilibria of this system are given by the set

\[
E = \{ z_\ast = (u_\ast, d_\ast) : u_\ast = 0, \quad d_\ast \in \mathbb{R}^n, \ |d_\ast| = 1 \},
\]

which forms a manifold of dimension \( n-1 \). The corresponding pressures \( p_\ast \) are constant as well.

Suppose finally that \( z \) is a solution of (1.1)–(1.2) which is eventually bounded in \( X_\gamma \) on its maximal interval of existence. Then, by Proposition 2.3 this solution is global and \( z((\delta, \infty)) \subset X_\gamma \) is relatively compact. Therefore its limit set

\[
\omega(z_0) = \{ v \in X_\gamma : \exists t_n \uparrow \infty \text{ s.t. } z(t_n; z_0) \to v \text{ in } X_\gamma \}
\]

is nonempty. As \( E \) is a strict Ljapunov functional for (1.1)–(1.2), we obtain \( \text{dist}(z(t, z_0), \omega(z_0)) \to 0 \) in \( X_\gamma \) for \( t \to \infty \) and \( \omega(z_0) \subset E \subset X_1 \). Now Theorem 5.4 applies and we may conclude that \( z(t) \to z_\infty \in E \) in \( X_\gamma \) as \( t \to \infty \). In summary we proved the following result.

Theorem 5.4. Let \( \mu, p, q \) satisfy (1.1). Let \( z_0 = (u_0, d_0) \in X_{\gamma, \mu} \) with \( |d_0| = 1 \) and suppose that the solution \( z(t) \) of (1.1) is eventually bounded in \( X_\gamma \) on its maximal interval of existence, i.e.

\[
z \in BC([\tau, t^+(z_0)]; X_\gamma)
\]

for some \( \tau \in (0, t^+(z_0)) \). Then \( z(t) \) exists globally and converges to an equilibrium \( z_\infty \in E \) in \( X_\gamma \), as \( t \to \infty \). The converse is also true.
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DYNAMICS OF NEMATIC LIQUID CRYSTAL FLOWS

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