Lyapunov exponents in Minkowskian U(1) gauge theory

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U(1) gauge fields are decomposed into a monopole and photon part across the phase transition from the confinement to the Coulomb phase. We analyze the leading Lyapunov exponents of such gauge field configurations on the lattice which are initialized by quantum Monte Carlo simulations. We observe that the monopole field carries the same Lyapunov exponent as the original U(1) field. Evidence is found that monopole fields stay chaotic in the continuum whereas the photon fields are regular.

1. Monopole and photon part of U(1)

We begin with a 4d U(1) gauge theory described by the Euclidean action

\[ S\{U_l\} = \beta \sum_p (1 - \cos \theta_p) \]

where \( U_l = U_{x,\mu} = \exp(i\theta_{x,\mu}) \) and \( \theta_p = \theta_{x,\mu} + \theta_{x+\mu,\nu} - \theta_{x,\nu} \). We are interested in the relationship between monopoles and classical chaos across the phase transition at \( \beta_c \approx 1.01 \). Following Ref. [1], we have factorized our gauge configurations into monopole and photon fields. The U(1) plaquette angles \( \theta_{x,\mu\nu} \) are decomposed into the “physical” electromagnetic flux through the plaquette \( \bar{\theta}_{x,\mu\nu} \) and a number \( m_{x,\mu\nu} \) of Dirac strings through the plaquette

\[ \theta_{x,\mu\nu} = \bar{\theta}_{x,\mu\nu} + 2\pi m_{x,\mu\nu} \],

where \( \bar{\theta}_{x,\mu\nu} \in (-\pi, +\pi] \) and \( m_{x,\mu\nu} \neq 0 \) is called a Dirac plaquette.

2. Classical chaotic dynamics from quantum Monte Carlo initial states

Chaotic dynamics in general is characterized by the spectrum of Lyapunov exponents. These exponents, if they are positive, reflect an exponential divergence of initially adjacent configurations. In case of symmetries inherent in the Hamiltonian of the system there are corresponding zero values of these exponents. Finally negative exponents belong to irrelevant directions in the phase space; perturbation components in these directions die out exponentially. Pure gauge fields on the lattice show a characteristic Lyapunov spectrum consisting of one third of each kind of exponents [2]. Assuming this general structure of the Lyapunov spectrum we investigate presently its magnitude only, namely the maximal value of the Lyapunov exponent, \( L_{\text{max}} \).

The general definition of the Lyapunov exponent is based on a distance measure \( d(t) \) in phase space,

\[ L := \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{1}{t} \ln \frac{d(t)}{d(0)}. \]  

In case of conservative dynamics the sum of all Lyapunov exponents is zero according to Liouville’s theorem, \( \sum \lambda_i = 0 \). We utilize the gauge invariant distance measure consisting of the local differences of energy densities between two 3d field configurations on the lattice:

\[ d := \frac{1}{N_P} \sum_P |\text{tr} U_P - \text{tr} U_P'|. \]

Here the symbol \( \sum_P \) stands for the sum over all \( N_P \) plaquettes, so this distance is bound in the interval \( (0, 2N) \) for the group SU(N). \( U_P \) and \( U_P' \) are the familiar plaquette variables, constructed from the basic link variables \( U_{x,i} \),

\[ U_{x,i} = \exp \left( aA^c_{x,i} T^c \right), \]

located on lattice links pointing from the position \( x = (x_1, x_2, x_3) \) to \( x + a e_i \). The generators of the group are \( T^c = -ig \tau^c/2 \) with \( \tau^c \) being the Pauli matrices in case of SU(2) and \( A^c_{x,i} \) is the vector potential. The elementary

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plaquette variable is constructed for a plaquette with a corner at $x$ and lying in the $ij$-plane as $U_{x,ij} = U_{x,i}U_{x+i,j}U_{x+j,i}^\dagger U_{x,j}^\dagger$. It is related to the magnetic field strength $B_{x,k}^c$:

$$U_{x,ij} = \exp(\varepsilon_{ijk}aB_{x,k}^cT_c).$$

The electric field strength $E_{x,i}^c$ is related to the canonically conjugate momentum $P_{x,i} = \dot{U}_{x,i}$ via

$$E_{x,i}^c = \frac{2a}{g^3}\text{tr}\left(T_c\dot{U}_{x,i}U_{x,i}^\dagger\right).$$

The Hamiltonian of the lattice gauge field system can be casted into the form

$$H = \sum \left[ \frac{1}{2}\langle P, P \rangle + 1 - \frac{1}{4}\langle U, V \rangle \right].$$

Here the scalar product stands for $\langle A, B \rangle = \frac{1}{2}\text{tr}(AB^\dagger)$. The staple variable $V$ is a sum of triple products of elementary link variables closing a plaquette with the chosen link $U$. This way the Hamiltonian is formally written as a sum over link contributions and $V$ plays the role of the classical force acting on the link variable $U$.

We prepare the initial field configurations from a standard four dimensional Euclidean Monte Carlo program on a $12^3 \times 4$ lattice varying the inverse gauge coupling $\beta$ [3]. We relate such four dimensional Euclidean lattice field configurations to Minkowskian momenta and fields for the three dimensional Hamiltonian simulation by identifying a fixed time slice of the four dimensional lattice.

3. Chaos, confinement and continuum

We start the presentation of our results with a characteristic example of the time evolution of the distance between initially adjacent configurations. An initial state prepared by a standard four dimensional Monte Carlo simulation is evolved according to the classical Hamiltonian dynamics in real time. Afterwards this initial state is rotated locally by group elements which are chosen randomly near to the unity. The time evolution of this slightly rotated configuration is then pursued and finally the distance between these two evolutions is calculated at the corresponding times. A typical exponential rise of this distance followed by a saturation can be inspected in Fig. 1 from an example of $U(1)$ gauge theory in the confinement phase and in the Coulomb phase. While the saturation is an artifact of the compact distance measure of the lattice, the exponential rise (the linear rise of the logarithm) can be used for the determination of the leading Lyapunov exponent. The left plot exhibits that in the confinement phase the original field and its monopole part have similar Lyapunov exponents whereas the photon part has a smaller $L_{max}$. The right plot in the Coulomb phase suggests that all slopes and consequently the Lyapunov exponents of all fields decrease substantially.

The main result of the present study is the dependence of the leading Lyapunov exponent $L_{max}$ on the inverse coupling strength $\beta$, displayed in
Fig. 2. As expected the strong coupling phase is more chaotic. The transition reflects the critical coupling to the Coulomb phase. The plot shows that the monopole fields carry Lyapunov exponents of nearly the same size as the full U(1) fields. The photon fields yield a non-vanishing value in the confinement ascending toward $\beta = 0$ for randomized fields which indicates that the decomposition works well only around the phase transition.

An interesting result concerning the continuum limit can be viewed from Fig. 3 which shows the energy dependence of the Lyapunov exponents for the U(1) theory and its components. One observes an approximately linear relation for the monopole part while a quadratic relation is suggested for the photon part in the weak coupling regime. From scaling arguments one expects a functional relationship between the Lyapunov exponent and the energy [2,4] 

$$L(a) \propto a^{k-1} E^k(a),$$

with the exponent $k$ being crucial for the continuum limit of the classical field theory. A value of $k < 1$ leads to a divergent Lyapunov exponent, while $k > 1$ yields a vanishing $L$ in the continuum. The case $k = 1$ is special leading to a finite non-zero Lyapunov exponent. Our analysis of the scaling relation (8) gives evidence, that the classical compact U(1) lattice gauge theory and especially the photon field have $k \approx 2$ and

REFERENCES

1. J.D. Stack and R.J. Wensley, Nucl. Phys. B371 (1992) 597; T. Suzuki, S. Kitahara, T. Okude, F. Shoji, K. Moroda, and O. Miyamura, Nucl. Phys. B (Proc. Suppl.) 47 (1996) 374.

2. T.S. Biró, S.G. Matinyan, and B. Müller: Chaos and Gauge Field Theory, World Scientific, Singapore, 1995.

3. T.S. Biró, M. Feurstein, and H. Markum, APH Heavy Ion Physics 7 (1998) 235; T.S. Biró, N. Hörmann, H. Markum, and R. Pullirsch, Nucl. Phys. B (Proc. Suppl.) 86 (2000) 403; H. Markum, R. Pullirsch, and W. Sakuler, hep-lat/0201001; hep-lat/0205003.

4. L. Casetti, R. Gatto, and M. Pettini, J. Phys. A32 (1999) 3055; H.B. Nielsen, H.H. Rugh, and S.E. Rugh, ICHEP96 1603, hep-th/9611128; B. Müller, chao-dyn/9607001; H.B. Nielsen, H.H. Rugh, and S.E. Rugh, chao-dyn/9605013.