Delay’s propagation of *citrus tristeza virus* on citrus plant

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**Abstract.** Citrus plant is one of the biggest production fruit plants in the world as well as in Indonesia. Citrus plant can be infected with a disease by virus, such as *Citrus Tristeza Virus* (CTV) propagated by *Toxoptera citricida*. Based on the well-known SIR-SI model, it was derived the CTV propagation model of the citrus plant. The obtained model involves citrus plant which consists of subpopulation susceptible, infected, and recovered. While the vector’s population were subpopulation susceptible and infected insects. Then two equilibrium points were determined for the model with and without delay, i.e. the disease–free equilibrium and the endemic equilibrium points. Hence the obtained model, which were analysed around two equilibrium points with and without delay, were asymptotic stable. Moreover, the basic reproduction number (\(\rho\)) was determined for obtaining the parameter value \(m\). The numerical simulation with and without delay were demonstrated and the responses of the model were asymptotic stable. By applying time delays, it is concluded that the spread of CTV on citrus plant can be inhibited by accelerating the harvest before it is infected to the virus, so that citrus fruits can be harvested more better than without giving any delay.

1. Introduction

Plants have a very important role for life as a producer of oxygen, producer of food and the fulfillment of all kinds of needs of living things. One of the largest fruit production plants in the world is citrus. In 2014 citrus plantation production occupies the position of the soil with a total of 70.86 million tons [1]. Citrus plants may develop viral illnesses, such as Citrus Tristeza (CTV) [2].

Healthy citrus plants can be infected by the virus because it is infested by insects which previously infected citrus plants that have been infected. One of the insects that spread the virus is aphids (*Toxoptera citricida*) [3]. *Toxoptera citricida* proliferates depending on the temperature between 20° \(\mathcal{C}\) – 30° \(\mathcal{C}\) and the capacity increases to 0.4 when the temperature is 25° \(\mathcal{C}\). Female *Toxoptera citricida* can produce a population of over 4,400 eggs within 3 weeks without any natural predators [4].

It has been interested to study the model of the spread of plant virus that describes the interaction between citrus plants and virus-carrying insects, and one was inspired by an article [5]. The purpose of this study is to discuss CTV dispersion model with and without delay, by assuming the virus would be transmitted by contact between citrus and insect plants using Holling Type II. Then the study demonstrates also the stability and simulations of CTV deployment model in citrus plants with and without delay. It is necessary to obtain an alternative to control the spread of the virus in citrus plants. Such alternatives can be taken into consideration for appropriate action in handling any problem posed by the CTV deployment.

2. Mathematical Modelling for The Propagation of CTV on Citrus Plant
Epidemic model is utilized to determine the spread of a disease in certain areas. The SIR epidemic model was first introduced by [6]. The mathematical model was applied in the spread of the virus in citrus plants involves the population of citrus plant $K$ and insect population $N$. Based on the well-known SIR-SI model, it was derived the CTV propagation model of the citrus plant. The obtained model involved citrus plant population which consists of subpopulation susceptible $S(t)$, infected $I(t)$, and recovered $R(t)$, respectively. While the vector’s population were subpopulation susceptible insect $S_X(t)$ and infected insect $I_Y(t)$, respectively. The interactions between citrus plants and insects influence the spread of viruses in citrus and insects. These interactions implement Holling Type II through two ways, namely virus transmission from insects to citrus plants and transmission of virus from citrus plants to insects. Thus, a compartment diagram of the propagation of CTV on citrus plants is depicted as follows:

$$\begin{align*}
\frac{dS(t)}{dt} &= \Gamma - \mu S(t) - \frac{\beta I_Y(t)}{1 + \alpha I_Y(t)} S(t) \\
\frac{dI(t)}{dt} &= \beta I_Y(t) S(t) - vI(t) - \mu I(t) - \gamma I(t) \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t) \\
\frac{dS_X(t)}{dt} &= \Lambda - \frac{\beta I_Y(t)}{1 + \alpha I_Y(t)} S_X(t) - mS_X(t) \\
\frac{dI_Y(t)}{dt} &= \frac{\beta I_Y(t)}{1 + \alpha I_Y(t)} S_X(t) - mI_Y(t)
\end{align*}$$

(1)

Meanwhile, the propagation of CTV on citrus plant is relating to some delays. Assuming that delay time is only occurred on susceptible citrus plants, infected and infected insects because with these delays the occurred CTV may inhibit its spreading. The model is formulated as follows:

$$\begin{align*}
\frac{dS(t)}{dt} &= \Gamma - \mu S(t) - \frac{\beta I_Y(t-\tau_1)}{1 + \alpha I_Y(t-\tau_1)} S(t-\tau_1) \\
\frac{dI(t)}{dt} &= \beta I_Y(t-\tau_1) S(t-\tau_1) - vI(t) - \omega I(t) \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t) \\
\frac{dS_X(t)}{dt} &= \Lambda - \frac{\beta I_Y(t)}{1 + \alpha I_Y(t)} S_X(t) - mS_X(t) \\
\frac{dI_Y(t)}{dt} &= \frac{\beta I_Y(t-\tau_2)}{1 + \alpha I_Y(t-\tau_2)} \left( \frac{\Lambda}{m} - I_Y(t-\tau_2) \right) - mI_Y(t)
\end{align*}$$

(2)

where, $\Gamma = \mu K + vI(t)$; $\Lambda = mX + mY$; $K = \text{total population of citrus plants}$; $N = \text{total population of insects}$; $S(t) = \text{population of susceptible citrus plant}$; $I(t) = \text{population of citrus plants infected by CTV}$; $R(t) = \text{population of recovered citrus plant}$; $S_X(t) = \text{population of susceptible insect}$; $I_Y(t) = \text{population of insects infected by CTV}$; $\beta = \text{infection rate of citrus plants caused by insects}$; $\beta_1$ rate of insect infections caused by citrus plants; $\alpha = \text{constant saturation of citrus plants caused by insects}$; $\alpha_1 = \text{constant saturation of insects caused by citrus plants}$; $\mu = \text{natural mortality rate in citrus plant}$; $m = \text{natural rate of death on insects}$; $\gamma = \text{recovered rate of citrus plant}$; $\Lambda = \text{rate of population increase of citrus plant}$.
insects (birth or immigration; $\Gamma = \text{rate of population growth of citrus plants}$; and $v = \text{death rate of infected citrus plants due to the virus}$.

\subsection*{2.1 Stability of Equilibrium}

\textbf{Definition 1.}

Given first-order differential equations $\dot{x} = f(x, t)$, and $x(t, x_0)$ are system solutions with initial values $x(0) = x_0$. A vector $\vec{x}$ that satisfies $f(\vec{x}) = 0$ is called the equilibrium point. The equilibrium point $\vec{x}$ is stable if for $\epsilon > 0$ there is $\delta > 0$, so for $\|x_0 - \vec{x}\| < \delta$ apply $\|x(t, x_0) - \vec{x}\| < \epsilon$, for $t > 0$. An equilibrium point $\vec{x}$ is said to be asymptotically stable if $\vec{x}$ is stable and there is $\delta_1 > 0$, so for $\|x_0 - \vec{x}\| < \delta_1$ imply $\lim_{t \to \infty} ||x(t, x_0) - \vec{x}|| = 0$. In addition the stability of the equilibrium point of the system can be determined by considering the real part of eigen values of the characteristic equation [7].

\section*{3. Result and Discussion}

\subsection*{3.1 The Analysis of Equilibrium Point}

The parameter values used in the propagation of CTV on citrus plants are $(K; N; \beta; \beta_1; \alpha; a_1; \mu; \gamma; \Lambda; v) = (1000; 40; 0.001; 0.001; 0.2; 0.1; 0.01; 0.01, 20, 0.2)$. The analysis of equilibrium point of the propagation of CTV on citrus plant is by making the right-hand side of each equation (2) equal to zero. Thus, the equilibrium point is $\vec{E} = (\vec{S}, \vec{I}, \vec{R}, \vec{S}_X, \vec{I}_Y)$, where $\vec{S} = \frac{\Gamma(1+\alpha t)}{\mu(1+\gamma t)}$, $\vec{I} = 0$ at or $\vec{I} = \frac{\beta\beta_1S_X}{m\omega a_1+\alpha\beta_1\omega S_X}$; $\vec{R} = \frac{\gamma t}{\mu}$; $\vec{S}_X = \frac{\lambda}{(\beta_1+1+\alpha t)^m}$; $\vec{I}_Y = \frac{\beta_1+1+\alpha t)}{m(1+\gamma t)}$.

By assuming two conditions, namely $\vec{I} = 0$ or $\vec{I} = \frac{\beta\beta_1S_X}{m\omega a_1+\alpha\beta_1\omega S_X}$, two equilibrium points are obtained as follows.

1. The disease–free equilibrium point is $E^0 = (S^0, I^0, R^0, S_X^0, I_Y^0)$.

Where $S^0 = \frac{\Gamma(1+\alpha t)}{\mu(1+\gamma t)}$; $I^0 = 0$; $R^0 = \frac{\gamma t}{\mu}$; $S_X^0 = \frac{\lambda}{(\beta_1+1+\alpha t)^m}$; $I_Y^0 = \frac{\beta_1+1+\alpha t)}{m(1+\gamma t)}$.

Note that $\Gamma = \mu K + v(I(t))$. Substituting all parameter values, it is obtained that $E^0 = (S^0, I^0, R^0, S_X^0, I_Y^0) = (K; 0; 0; N; 0)$

2. Assuming $\vec{S} \neq 0, \vec{I} \neq 0, \vec{R} \neq 0, \vec{S}_X \neq 0, \vec{I}_Y \neq 0$. The endemic equilibrium point is $E^* = (S^*, I^*, R^*, S_X^*, I_Y^*)$

Where $S^* = \frac{\Gamma(1+\alpha t)}{\mu(1+\gamma t)}$; $I^* = \frac{\beta\beta_1S_X}{m\omega a_1+\alpha\beta_1\omega S_X}$; $R^* = \frac{\gamma t}{\mu}$; $S_X^* = \frac{\lambda}{(\beta_1+1+\alpha t)^m}$; $I_Y^* = \frac{\beta_1+1+\alpha t)}{m(1+\gamma t)}$.

Obviously, the endemic equilibrium point depends on $I^*$, and it is assumed that $I^* = 2.66$, substituting the parameter values, then the endemic equilibrium point is $E^* = (S^*, I^*, R^*, S_X^*, I_Y^*) = (994.68; 2.66; 2.66; 39.23; 0.33)$.

\subsection*{3.2 Basic Reproduction Number}

A basic reproduction numbers ($\rho$) is used to determine the extent of disease spread. The value $\rho$ is determined by using the next generation matrix $F$. The matrix $F$ consists of a matrix consisting of $F_i$, indicating a new infection rate in different compartments and $V$ consisting of $V_i$ showing the rate of the rate of displacement from one compartment to another.

$$F = \begin{pmatrix} 0 & \frac{\beta\beta_1S_X^0}{m(1+\gamma t)} \\ \frac{\beta_1+1}{m(1+\gamma t)} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \omega & 0 \\ 0 & m \end{pmatrix}$$

The basic reproduction numbers is as follows: $\rho = \sqrt{\frac{\beta_1+1}{m^2}K}$.
Substituting the parameter values to the basic reproduction number, then it is obtained that if \( \rho < 1 \), then the virus does not spread for \( m > 0.30 \). Otherwise, an endemic state will occur whenever \( m < 0.30 \).

3.3 Linearization Model Without Delays

1. The Jacobian matrix of the linear model around disease free equilibrium point is

\[
J = \begin{pmatrix}
-\mu - \frac{\beta I^0}{1 + \alpha I^0} & v & 0 & 0 & -\frac{\beta S^0}{(1 + \alpha I^0)^2} \\
n & -\omega & 0 & 0 & -\frac{\beta S^0}{(1 + \alpha I^0)^2} \\
0 & \gamma & -\mu & 0 & 0 \\
\frac{\beta_1 S^0}{(1 + \alpha I^0)^2} & 0 & 0 & -\frac{\beta I^0}{1 + \alpha I^0} - m & 0 \\
0 & \frac{\beta_1 S^0}{(1 + \alpha I^0)^2} & 0 & \frac{\beta I^0}{1 + \alpha I^0} & -m
\end{pmatrix}
\]

The linear model around disease free equilibrium point \( E^0 = (S^0, I^0, R^0, S_X^0, I_Y^0) \) is therefore given as follows:

\[
\begin{align*}
\frac{dS}{dt} &= -\mu S + v I - \frac{\beta S}{(1 + \alpha I)^2} I_Y \\
\frac{dI}{dt} &= \mu S - \omega I + \frac{\beta I}{(1 + \alpha I^0)^2} I_Y \\
\frac{dR}{dt} &= \gamma I - \mu R \\
\frac{dS_X}{dt} &= -\frac{\beta_1 I}{(1 + \alpha I)^2} I - \left( \frac{\beta I^0}{1 + \alpha I^0} + m \right) S_X \\
\frac{dI_Y}{dt} &= \frac{\beta I^0}{(1 + \alpha I^0)^2} I + \frac{\beta I^0}{1 + \alpha I^0} S_X - m I_Y
\end{align*}
\]

2. The Jacobian matrix of the linear model around endemic equilibrium point is

\[
J = \begin{pmatrix}
-\mu - \frac{\beta I^0}{1 + \alpha I^0} & v & 0 & 0 & -\frac{\beta S^*}{(1 + \alpha I^0)^2} \\
n & -\omega & 0 & 0 & -\frac{\beta S^*}{(1 + \alpha I^0)^2} \\
0 & \gamma & -\mu & 0 & 0 \\
\frac{\beta_1 S^0}{(1 + \alpha I^0)^2} & 0 & 0 & -\frac{\beta I^0}{1 + \alpha I^0} - m & 0 \\
0 & \frac{\beta_1 S^0}{(1 + \alpha I^0)^2} & 0 & \frac{\beta I^0}{1 + \alpha I^0} & -m
\end{pmatrix}
\]

The linear model around the equilibrium endemic point \( E^* = (S^*, I^*; R^*; S_X^*; I_Y^*) \) is given as follows:

\[
\begin{align*}
\frac{dS}{dt} &= -\mu S + v I - \frac{\beta S^*}{(1 + \alpha I^0)^2} I_Y \\
\frac{dI}{dt} &= \mu S - \omega I + \frac{\beta S^*}{(1 + \alpha I^0)^2} I_Y \\
\frac{dR}{dt} &= \gamma I - \mu R \\
\frac{dS_X}{dt} &= -\frac{\beta_1 I^*}{(1 + \alpha I^0)^2} I - \left( \frac{\beta I^0}{1 + \alpha I^0} + m \right) S_X \\
\frac{dI_Y}{dt} &= \frac{\beta I^0}{(1 + \alpha I^0)^2} I + \frac{\beta I^0}{1 + \alpha I^0} S_X - m I_Y
\end{align*}
\]

3.4 Simulation the model with and without delays

Consider the equation (1), our numerically simulations are demonstrated with two different values \( m = 0.5 \) and \( m = 0.25 \). Furthermore, the simulations are shown as follows.

1. Simulation of the model, with and without delays, around the disease-free equilibrium point.
Consider the disease-free equilibrium point $E^0 = (1000; 0; 40; 0)$ with $K = 1000$, $N = 40$, and the initial value $(945; 55; 0; 35; 5)$. $\tau_1 = 40$, $\tau_2 = 18$ and $m = 0.5$. The responses are given as follows in figure 2.

![Figure 2](image)

**Figure 2.** (a) Susceptible Citrus Plants, (b) Infected Citrus Plants, (c) Infected Citrus Plants, (d) Susceptible Insects, and (e) Insects Infected with $m = 0.5$

2. Simulation of the model, with and without delays, around the endemic equilibrium point.

Consider the endemic equilibrium point $E^* = (994.68; 2.66; 2.66; 39.23; 0; 33)$ with $K = 1000$, $N = 40$, the initial value $(945; 55; 0; 35; 5)$. $\tau_1 = 40$, $\tau_2 = 18$ and $m = 0.25$. The responses of the model are given as follows.
Figure 3. (a) Susceptible Citrus Plants, (b) Infected Citrus Plants, (c) Infected Citrus Plants, (d) Susceptible Insects, and (e) Insects Infected with \( m = 0.25 \).

Based on Figure 2 and Figure 3, it demonstrates the model responses with and without delays blue and red responses, respectively. The red graph of the citrus plant population with some delays could provide more advantageous than without delays (blue graph). Therefore, it can be concluded that the CTV deployment of citrus crops can be inhibited by accelerating the harvest before citrus plants are infected with the virus, so that the harvesting of citrus fruit can be more better than without delays.

3.5 Stability Analysis

1. Stability of the model around the disease–free equilibrium point \( E^0 = (S^0; I^0; R^0; S^*_X; I^*_Y) \).

   Based on the matrix \( J \) in the equation (4) and by substituting the parameter value, it is obtained that \( \lambda_1 = -0.01, \lambda_2 = -0.01, \lambda_3 = -0.50, \lambda_4 = -0.12 \) and \( \lambda_5 = -0.60 \). Hence, the model is asymptotically stable.

2. Stability of the model around endemic equilibrium point \( E^* = (S^*; I^*; R^*; S^*_X; I^*_Y) \).

   Based on the matrix \( J \) in the equation (5) and by substituting the parameter value, it is obtained that \( \lambda_1 = -0.01, \lambda_2 = -0.38, \lambda_3 = -0.09, \lambda_4 = -0.02 \), and \( \lambda_5 = -0.25 \). Therefore the model is also asymptotically stable.

4. Conclusion

The CTV propagation models with and without delays had been well-derived based on the model SIR – SXLV. Two obtained equilibrium points, i.e. the disease-free equilibrium point \( E^0 = (1000, 0, 0, 40, 0) \) and the endemic equilibrium point \( E^* = (994.68, 2.66, 2.66, 39.23, 0.77) \), are utilized to linearize and analyze the model. The stability of the model without any delay addressed that the model is asymptotically stable, as well as the model with some delays. In addition, the basic reproduction number \( (\rho) \) is significantly involved to determine the parameter value \( m \), and it is obtained that \( m \geq 0.30 \) and \( m < 0.30 \) for the disease-free and endemic conditions, respectively. The simulation of models with and without delays, with \( m = 0.5 \) for disease free and \( m = 0.25 \) for endemic state, emphasized that both populations are asymptotically stable. By applying some delays, it can be concluded that the CTV deployment of citrus plants can be inhibited by accelerating the harvest before the citrus plants are infected with the virus, so that the citrus fruits harvesting can be better than without delays.

5. References

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