STRONG COUPLING LIMIT OF BETHE ANSATZ SOLUTIONS IN MASSIVE THIRRING MODEL

T. Fujita, T. Kake, and H. Takahashi
Department of Physics, Faculty of Science and Technology
Nihon University, Tokyo, Japan

ABSTRACT

We study the strong coupling limit of the Bethe ansatz solutions in the massive Thirring model. We find analytical expressions for the energy eigenvalues for the vacuum state as well as $n$–particle $n$– hole states. This formula is compared with the numerical results and is found to achieve a very good agreement.

Also, it is found that the 2–particle 2– hole and higher particle–hole states describe $n$– free bosons states in this limit. The behaviors of the strong coupling limit of the boson mass for various model calculations are examined. We discuss an ambiguity of the coupling constant normalization due to the current regularization.

1e-mail: fujita@phys.cst.nihon-u.ac.jp
2e-mail: htaka@phys.cst.nihon-u.ac.jp
1 Introduction

Recent calculations for the massive Thirring model have presented a debate over the energy spectrum of the bound states [1-4]. Several different methods give different results on the spectrum of the bound state. For a long time, people have believed that the semiclassical results by Dashen et al.[5] are exact in spite of the fact that they took into account only the lowest order quantum fluctuations in the path integral method. However, the recent calculation by the light cone procedure shows that there is only one bound state, and the spectrum of the bound state energy as the function of the coupling constant is different from the semiclassical result [1-3]. Further, the recent calculations based on the Bethe ansatz solutions [4] present a numerical proof that there is only one bound state, and the spectrum seems to be consistent with the light cone results.

In this paper, we present analytical calculations of the strong coupling limit of the Bethe ansatz solutions for the massive Thirring model and show that the analytical expressions obtained here agree very well with those calculated by numerically solving the Periodic Boundary Condition (PBC) equations of the Bethe ansatz solutions [6,7]. Here, we obtain the energy eigenvalues of the vacuum, $1p-1h$ states (symmetric and asymmetric cases ) and $2p-2h$ and higher particle hole states (symmetric case). The analytical formula shows that the $n$ particle—$n$ hole state is just $n$ times $1p-1h$ state energy (a boson mass). Therefore, there is only one boson state in the massive Thirring model. This shows that the $n$ particle—$n$ hole states are all scattering states.

Further, we show the behaviors of the strong coupling limit of the boson mass for various model calculations. It turns out that the analytical expression of the boson mass at the strong coupling limit with the Bethe ansatz solutions is different from the light cone prediction. This may indicate that the normalization ambiguity of the coupling constant due to the fermion current regularization in the massive Thirring model is different from the massless Thirring model. For the massless Thirring model, Klaiber [8] proves that the coupling constant has a normalization ambiguity which arises from the fermion current regularization. In the case of the massive Thirring model, it is expected that the same type of the coupling constant ambiguity may well appear. However, we will see later that the normalization ambiguity of the coupling constant is more complicated than expected for the massive Thirring model.

In the next section, we briefly describe the Bethe ansatz method which is applied to solving the massive Thirring model. In section III, we discuss analytical expressions for the energy eigenvalues with the strong coupling expansion. Then, section IV treats numerical calculations of the PBC equations in the strong coupling region and we compare them with the analytical expressions. In section V, we examine the boson mass at the strong coupling limit. Finally, section VI summarizes what we have understood in this paper.
2 Massive Thirring model and Bethe ansatz solutions

The massive Thirring model is a 1+1 dimensional field theory with current current interactions \([4,11]\), and can be solved by the Bethe ansatz method \([4,6,7]\). In this case, the eigenvalue problem is reduced to solving the following PBC equations,

\[
m_0 \sinh \beta_i = \frac{2\pi n_i}{L} - \sum_j \frac{2}{L} \tan^{-1} \left[ \frac{g_B}{2} \tanh \frac{1}{2} (\beta_i - \beta_j) \right] \tag{2.1}
\]

where \(\beta_i\) denotes the rapidity of the particle and \(n_i\)'s are integer. \(m_0, L\) and \(g_B\) denote the bare mass, the box length and the coupling constant, respectively.

2.1 Vacuum state

The PBC equations for the vacuum which is filled with negative energy particles (\(\beta_i = i\pi - \alpha_i\)) can be written as

\[
\sinh \alpha_i = \frac{2\pi i}{L_0} - \frac{2}{L_0} \sum_{j \neq i} \tan^{-1} \left[ \frac{1}{2} g_B \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right], \tag{2.2}
\]

\((i = 0, \pm 1, \ldots, \pm N_0)\),

where \(L_0\) is defined as \(L_0 = m_0 L\) and \(N_0 = (N - 1)/2\). In this case, the vacuum energy \(E_v\) can be written as

\[
E_v = - \sum_{i = -N_0}^{N_0} m_0 \cosh \alpha_i. \tag{2.3}
\]

2.2 1p – 1h state

For one particle-one hole (1p – 1h) states, we take out one negative energy particle (\(i_0\)-th particle) and put it into a positive energy state. In this case, the PBC equations become

\[
\begin{align*}
i \neq i_0 \\
\sinh \alpha_i &= \frac{2\pi i}{L_0} - \frac{2}{L_0} \sum_{j \neq i, i_0} \tan^{-1} \left[ \frac{g_B}{2} \coth \frac{1}{2} (\alpha_i + \beta_{i_0}) \right] \\
&\quad - \frac{2}{L_0} \sum_{j \neq i, i_0} \tan^{-1} \left[ \frac{g_B}{2} \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right] \tag{2.4a}
\end{align*}
\]

\[
\begin{align*}
i = i_0 \\
\sinh \beta_{i_0} &= \frac{2\pi i_0}{L_0} + \frac{2}{L_0} \sum_{j \neq i_0} \tan^{-1} \left[ \frac{g_B}{2} \coth \frac{1}{2} (\beta_{i_0} + \alpha_j) \right] \tag{2.4b}
\end{align*}
\]
where $\beta_{i0}$ can be a complex variable as long as it can satisfy eqs.(2.10). These PBC equations determine the energy of the one particle-one hole states which we denote by $E^{(i_0)}_{1p1h}$,

$$E^{(i_0)}_{1p1h} = m_0 \cosh \beta_{i0} - \sum_{i=1}^{N_0} m_0 \cosh \alpha_i. \quad (2.5)$$

### 2.3 $2p - 2h$ states

For two particle-two hole ($2p - 2h$) states, we take out the $i_1$–th and the $i_2$–th particles and put them into positive energy states. The PBC equations for the two particle-two hole states become

\[ i \neq i_1, i_2 \]

\[
\sinh \alpha_i = \frac{2\pi i}{L_0} - \frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_B \coth \frac{1}{2} (\alpha_i + \beta_{i1}) \right] \\
- \frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_B \coth \frac{1}{2} (\alpha_i + \beta_{i2}) \right] \\
- \frac{2}{L_0} \sum_{j \neq i, i_1, i_2} \tan^{-1} \left[ \frac{1}{2} g_B \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right] \quad (2.6a)
\]

\[ i = i_1 \]

\[
\sinh \beta_{i1} = \frac{2\pi i}{L_0} + \frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_B \tanh \frac{1}{2} (\beta_{i1} + \beta_{i2}) \right] \\
+ \frac{2}{L_0} \sum_{j \neq i, i_1, i_2} \tan^{-1} \left[ \frac{1}{2} g_B \coth \frac{1}{2} (\beta_{i1} - \alpha_j) \right] \quad (2.6b)
\]

\[ i = i_2 \]

\[
\sinh \beta_{i2} = \frac{2\pi i}{L_0} + \frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_B \tanh \frac{1}{2} (\beta_{i2} + \beta_{i1}) \right] \\
+ \frac{2}{L_0} \sum_{j \neq i, i_1, i_2} \tan^{-1} \left[ \frac{1}{2} g_B \coth \frac{1}{2} (\beta_{i2} - \alpha_j) \right]. \quad (2.6c)
\]

In this case, the energy of the $2p - 2h$ states $E^{(i_1,i_2)}_{2p2h}$ becomes

$$E^{(i_1,i_2)}_{2p2h} = m_0 \cosh \beta_{i1} + m_0 \cosh \beta_{i2} - \sum_{i=-N_0}^{N_0} m_0 \cosh \alpha_i. \quad (2.7)$$

Here, we note that the symmetric case ($i_1 = -i_2$) always gains the energy and therefore is lower than other asymmetric cases of $2p - 2h$ states. Higher particle-hole states are constructed just in the same way as above.
3 Strong coupling expansion

Here, we present the strong coupling expansion of the PBC equations. We take the limit of \( g_B \to \infty \) and obtain the energy eigenvalues analytically. This limit of \( g_B \to \infty \) can be taken since there is no coupling constant renormalization in the massive Thirring model [15].

3.1 Vacuum state

First, we consider the strong coupling limit for the vacuum [16]. We assume that \( g_B \) is much larger than any of the rapidity \( \alpha_i \), namely,

\[
\sqrt{g_B} \gg \alpha_i.
\]

In this case, eq.(2.2) becomes in terms of \( b_i = \sqrt{g_B L_0} \alpha_i \),

\[
b_i = 8 \sum_{j \neq i} \frac{1}{b_i - b_j}.
\]

Further, the vacuum energy is written as

\[
E_v = -m_0 \sum_{i=-N_0}^{N_0} \cosh \frac{b_i}{\sqrt{g_B L_0}} \approx -m_0 - 2m_0 \sum_{i=1}^{N_0} \left( 1 + \frac{b_i^2}{2g_B L_0} \right).
\]

(3.3)

Since \( b_i \)'s have the symmetry of \( b_i = -b_{-i} \), we can rewrite eq.(3.2) as

\[
b_i^2 = 16 \sum_{j=1}^{N_0} \frac{b_i^2}{b_i^2 - b_j^2} + 12 \ (i = 1, \ldots, N_0).
\]

(3.4)

From this equation, we can easily obtain

\[
\sum_{i=1}^{N_0} b_i^2 = 8N_0(N_0 - 1) + 12N_0.
\]

(3.5)

Therefore, the vacuum energy can be explicitly written up to \( 1/g_B \) order,

\[
E_v = -(2N_0 + 1)m_0 - \frac{1}{g_B L} \left[ 8N_0(N_0 - 1) + 12N_0 \right].
\]

(3.6)
3.2 1p − 1h state (symmetric)

Next, we treat the 1p − 1h states [15]. We assign the positive energy particle by $i_0$. In the strong coupling limit, eqs. (2.10) become

$$\sinh \alpha_i = \frac{2\pi i}{L_0} - \frac{\pi}{L_0} \varepsilon(\alpha_i + \beta_{i_0})$$
$$- \frac{2}{L_0} \sum_{j \neq i, i_0} \left[ \frac{\pi}{2} \epsilon(\alpha_i - \alpha_j) - \frac{4}{g_B(\alpha_i - \alpha_j)} + \ldots \right]$$  \hspace{1cm} (3.7a)

$$\sinh \beta_{i_0} = \frac{2\pi i_0}{L_0}$$
$$+ \frac{2}{L_0} \sum_{j \neq i_0} \left[ \frac{\pi}{2} \epsilon(\beta_{i_0} + \alpha_j) - \frac{4}{g_B(\beta_{i_0} + \alpha_j)} + \ldots \right].$$  \hspace{1cm} (3.7b)

where $\epsilon(\alpha)$ denote the step function. These equations have two solutions, the symmetric and the asymmetric solutions. For the symmetric case, one easily sees, since $\alpha_i = -\alpha_{-i}$

$$\beta_{i_0} = 0.$$  \hspace{1cm} (3.8)

Also, for other $\alpha_i$'s, the equations can be rewritten using $b_i$'s,

$$b_i^2 = 16 \sum_{j=1}^{N_0} \frac{b_i^2}{b_i^2 - b_j^2} + 4 \quad (i = 1, \ldots, N_0).$$  \hspace{1cm} (3.9)

In this case, we can evaluate the energy of 1p − 1h symmetric case in the same way as the vacuum and obtain

$$E^{(0)}_{1p1h} = -(2N_0 - 1)m_0 - \frac{1}{g_BL_0} [8N_0(N_0 - 1) + 4N_0].$$  \hspace{1cm} (3.10)

Therefore, the 1p − 1h energy for the $i_0 = 0$ case with respect to the vacuum becomes

$$\Delta E^{(0)}_{1p1h} = 2m_0 + \frac{8N_0}{g_BL}.$$  \hspace{1cm} (3.11)

3.3 1p − 1h states (asymmetric)

Now, we discuss the asymmetric solutions. In this case, we obtain the following coupled equations up to $1/g_B$.

$$\sinh \beta_{i_0} = \frac{2\pi i_0}{L_0} (N_0 + i_0) - \frac{4}{g_BL_0} \sum \tanh \frac{1}{2}(\beta_{i_0} + \alpha_j)$$  \hspace{1cm} (3.12a)

$$\sinh \alpha_i = \frac{4}{g_BL_0} \left[ \tanh \frac{1}{2}(\beta_{i_0} + \alpha_i) + \sum_{j \neq i_0, i} \frac{1}{\tanh \frac{1}{2}(\alpha_i - \alpha_j)} \right]$$
for $(N_0 \geq i > i_0)$  \hspace{1cm} (3.12b)

$$\sinh \alpha_i = -\frac{2\pi i}{L_0} + \frac{4}{g_BL_0} \left[ \tanh \frac{1}{2}(\beta_{i_0} + \alpha_i) + \sum_{j \neq i_0, i} \frac{1}{\tanh \frac{1}{2}(\alpha_i - \alpha_j)} \right]$$
for $(i_0 - 1 \geq i > -N_0).$  \hspace{1cm} (3.12c)
From the numerical analysis, we can put

$$|\beta_{i_0}| \gg |\alpha_i|.$$  

Therefore, the above PBC equations are reduced to

$$\sinh \beta_{i_0} = \frac{2\pi}{L_0} (N_0 + i_0) - \frac{8N_0}{g_BL_0} \tanh \frac{\beta_{i_0}}{2}$$  

(3.13a)

$$\alpha_i = \frac{4}{g_BL_0} \left[ \frac{\beta_{i_0}}{2} + \sum_{j \neq i, i_0} \frac{2}{\alpha_i - \alpha_j} \right]$$  

for $$N_0 \geq i > i_0$$ (3.13b)

$$\alpha_i = -\frac{2\pi}{L_0} + \frac{4}{g_BL_0} \left[ \frac{\beta_{i_0}}{2} + \sum_{j \neq i, i_0} \frac{2}{\alpha_i - \alpha_j} \right]$$  

for $$(i_0 - 1 \geq i \geq -N_0)$$. (3.13c)

The energy of the one particle-one hole states $$E_{1p1h}^{(i_0)}$$ becomes

$$E_{1p1h}^{(i_0)} \simeq m_0 \cosh \beta_{i_0} - 2m_0N_0 - \frac{m_0}{2} \sum_{i=-N_0}^{N_0} \alpha_i^2.$$  

(3.14)

Therefore, we should calculate the $$\cosh \beta_{i_0}$$ and the sum of the $$\alpha_i^2$$. First we evaluate the $$\beta_{i_0}$$ from eq.(3.13a), which can be reduced to a cubic equation with $$x \equiv \tanh(\beta_{i_0}/2),$$

$$x^3 - \frac{\pi g_B}{4N_0} (N_0 + i_0)x^2 - \frac{g_BL_0 + 4N_0}{4N_0} x + \frac{\pi g_B}{4N_0} (N_0 + i_0) = 0.$$  

(3.15)

It is easy to solve the above equation, and we obtain

$$\cosh \beta_{i_0} \simeq \frac{1}{2} - \frac{2}{g_Bm_0} \left( 4 - \pi g_B \frac{N_0 + i_0}{N_0} \right) \rho.$$  

(3.16)

Next, we calculate the sum of the $$\alpha_i^2$$. From eqs.(3.13b) and (3.13c), we obtain

$$\sum_{i=-N_0}^{N_0} \alpha_i^2 = \sum_{i<i_0} \alpha_i^2 + \sum_{i>i_0} \alpha_i^2$$

$$= \left( \frac{2\pi}{L_0} \right)^2 (N_0 + i_0) - 16\pi g_BL_0^2 (N_0 + i_0) \tanh \frac{\beta_{i_0}}{2} - \frac{16\pi}{g_BL_0^2} \sum_{i<i_0} \sum_{j>i_0} \frac{2}{\alpha_i - \alpha_j}.$$  

(3.17)

Also, we have from eqs.(3.13b) and (3.13c) when $$i > i_0$$ and $$j < i_0$$,

$$\alpha_i - \alpha_j = 2\frac{\pi}{L_0} + O\left( \frac{1}{g_BL_0} \right).$$  

(3.18)
Therefore, we find

\[
\sum_{i=0}^{N_0} \alpha_i^2 = \left(\frac{2\pi}{L_0}\right)^2 (N_0 + i_0) - \frac{16\pi}{g_BL_0^2} (N_0 + i_0) \tanh \frac{\beta_{i_0}}{2} + \frac{16\pi}{g_BL_0^2} (N_0^2 - i_0^2). \tag{3.19}
\]

Finally, we obtain for the \(E_{1p1h}^{(i_0)}\),

\[
E_{1p1h}^{(i_0)} = m_0 \cosh \beta_{i_0} - 2m_0N_0 - \frac{2\pi^2}{m_0L} \left(\rho + \frac{i_0}{L}\right) + \frac{8\pi}{g_BL_0} \left(\rho + \frac{i_0}{L}\right) \tanh \frac{\beta_{i_0}}{2} - \frac{8}{g_BL_0} (N_0^2 - i_0^2). \tag{3.20}
\]

### 3.4 2p−2h and higher particle–hole states

Now, we consider 2p−2h and higher particle–hole states. In this case, the symmetric solutions always gain the energy. Therefore, we only treat the symmetric solution here. Due to the symmetry, we can easily find

\[
\alpha_0 = 0. \tag{3.21}
\]

Let us first consider the 2p−2h configuration. We assume that the \(i_0\)–th and the \(-i_0\)–th particles are in the positive energy state. In this case, eqs.(2.12) reduce to

\[
b_i^2 = 16 \sum_{j=1}^{N_0-1} \frac{b_i^2 - b_j^2}{b_i^2 - b_j^2} + 12 \quad (i \neq i_0), \tag{3.22a}
\]

\[
b_i = -\frac{4}{b_i} \quad (i = \pm i_0). \tag{3.22b}
\]

Note that this leads to the string–like configurations since the solution for \(b_{i_0}\) becomes pure imaginary. That is,

\[
b_{\pm i_0} = \pm 2i. \tag{3.23}
\]

In terms of \(\beta_{i_0}\), this becomes

\[
\beta_{\pm i_0} = \pm \frac{2}{\sqrt{g_B}L_0} i. \tag{3.24}
\]

Therefore, the rapidity interval \(\Delta\) of the string becomes

\[
\Delta = \frac{4}{\sqrt{g_B}L_0} i + O \left(\frac{i}{\frac{g_B}{4}}\right), \tag{3.25}
\]

where we explicitly write the behavior of the next order of \(1/g_B\) expansion.

On the other hand, Bergknoff and Thacker [6] assume that the rapidity interval of the string behaves for large \(g_B\) as

\[
\Delta_{BT} = \frac{4}{g_B} i. \tag{3.26}
\]
This behavior as the function of $g_B$ does not agree with eq. (3.25) which is a solution of the PBC equation. Therefore, the string configurations taken by Bergknoff and Thacker are not consistent with the string-like solution that satisfies the PBC equations. But this is not at all surprising if one considers the way of obtaining the string configurations by Bergknoff and Thacker. They assume that the wave functions of the particles should not diverge at $x_i = -\infty$ as a sufficient condition. However, this cannot happen due to the two reasons. The first reason is that one constructs the field theory in the box of $0 \leq x_i \leq L$. Therefore, the boundary is always periodic, that is, the wave functions at $x_i = L$ and $x_i = 0$ are the same. The second reason is more physical. The interactions between particles considered here are always repulsive. Therefore, the wave functions cannot diverge at any points of the space, since they are in the scattering states as bare particles.

To avoid the confusions, we clarify the string picture. In the nonlinear Schrödinger model, the string corresponds to the bound states of the particles since they make bound states due to the attractive $\delta-$ type interaction. However, this is only possible for the bosonic particles. For fermions, there is neither two particle bound state nor three or higher particle bound state due to the Pauli principle with the $\delta-$ type interaction. In the massive Thirring model, therefore, we should not consider the string configuration which simulates the many particle bound states.

Now, in the same way as the vacuum case, we obtain the energy for the $2p - 2h$ state

$$E_{2p-2h}^{(0)} = -(2N_0 - 3)m_0 - \frac{1}{g_B L} [8N_0(N_0 - 1) - 4N_0],$$

(3.27)

where we have ignored those terms which vanish when $L \to \infty$ with $N_0/L$ finite. Therefore, the $2p - 2h$ energy with respect to the vacuum becomes

$$\Delta E_{2p-2h}^{(0)} = 4m_0 + \frac{16N_0}{g_B L}.$$  

(3.28)

For higher $p - h$ states, we can evaluate the energy just in the same way as the $2p - 2h$ state case. For $n$-particle-$n$-hole states, the energy with respect to the vacuum can be written as

$$\Delta E_{np-nh}^{(0)} = 2nm_0 + \frac{8nN_0}{g_B L}.$$  

(3.29)

It is important to find that the $np - nh$ state energy is just $n$ times as large as that of $1p - 1h$ state energy, that is,

$$\Delta E_{np-nh}^{(0)} = n\Delta E_{1p-1h}^{(0)}.$$  

(3.30)

This shows that the $n$-particle-$n$-hole states are composed of $n$ free bosons in this limit. This result is consistent with the numerical calculations presented in ref.[4].
4 Numerical method

We solve the PBC equations by the Newton method. The type of equation we want to solve can be schematically written as

\[ G(f) = 0 \] (4.1)

where \( f = (f_1, f_2, ..., f_N) \) are the \( N \) variables that should be determined. \( G \) is some function. First, we denote some initial solution by \( f_0 \). We expand eq.(4.1) near \( f_0 \) as

\[ f = f_0 + \delta x \] (4.2)

\[ G(f_0) + \frac{\partial G(f_0)}{\partial f_0} \delta x = 0. \] (4.3)

We solve this equation for \( \delta x \) and put them into eq.(4.1). This leads to a new set of \( f \), and we consider \( f \) as a new \( f_0 \) and repeat the same procedure until we get some convergent results for \( f \).

This method has a great advantage over the iteration method proposed in ref.[4], namely it gives a good convergence even for the strong coupling region. However, the matrix diagonalization can be possible only for a few thousand of matrix dimensions if we have to know all of the eigenvalues. It is found that the present calculations agree perfectly with those calculated in ref.[4].

It should also be interesting to check the accuracy of eqs.(3.6), (3.10) and (3.20). In Table 1, we show the comparison of the vacuum energies, the \( 1p - 1h \) (symmetric) and \( 1p - 1h \) (asymmetric) energies between the analytical expressions and the numerical calculations with the number of particles of \( N = 1601 \). As can be seen from the table 1, we find quite a good agreement between the predictions of the strong coupling expansion and the exact numerical calculations. This indicates that the strong coupling expansion is indeed a good approximate scheme.

|        | Numerical | Analytical |
|--------|-----------|------------|
| \( E_v \) | -1807.44  | -1809.08   |
| \( E_{1p1h}^{(0)} \) | -1805.20  | -1806.81   |
| \( E_{1p1h}^{(1)} \) | -1767.40  | -1760.44   |

The predictions of \( 1/g_B \) expansion are compared with the computer calculations. Table 1 shows the vacuum energy \( E_v \), \( 1p - 1h \) energies \( E_{1p-1h}^{(0)} \) with the symmetric state and \( E_{1p-1h}^{(1)} \) with the asymmetric state for \( g_B = 245 \) with \( N = 1601 \) and \( L_0 = 100 \).


5 Bound states at the strong coupling limit

5.1 Bound state

Now, we calculate the bound state of the massive Thirring model at the strong coupling limit. In ref.[4], Fujita et al. showed that there is one isolated boson state and all the other states are continuum states. Therefore, the 1p-1h continuum energy should start from the free fermion antifermion mass, that is twice the physical fermion mass.

\[ m = \frac{1}{2} \left( E_{1p1h}^{(1)} - E_v \right). \]  \hfill (5.1)

Thus, the bound state mass \( \mathcal{M} \) can be defined as

\[ \mathcal{M} = 2m \lim_{\rho \to \infty} \left( \frac{\Delta E_{1p1h}^{(0)}}{\Delta E_{1p1h}^{(1)}} \right). \]  \hfill (5.2)

The \( \Delta E_{1p1h}^{(1)} \) is given for the large \( \rho \)

\[
\Delta E_{1p1h}^{(1)} = E_{1p1h}^{(1)} - E_v \\
= \frac{3}{2} m_0 + \left[ 2\pi - \frac{4}{g_B} + \left( \frac{8\pi}{g_B m_0} - \frac{2\pi^2}{m_0} \right) \frac{1}{L} \right] \rho
\]  \hfill (5.3)

Thus, at the large \( L \) and \( \rho \) limit, \( \Delta E_{1p1h}^{(1)} \) becomes

\[ \Delta E_{1p1h}^{(1)} = \frac{3}{2} m_0 + \left( 2\pi - \frac{4}{g_B} \right) \rho. \]  \hfill (5.4)

Finally, from eq.(5.2) and eq.(5.4), we obtain the bound state mass as

\[
\mathcal{M} = 2m \lim_{\rho \to \infty} \frac{2m_0 + \frac{8}{g_B} \rho}{\frac{3}{2} m_0 + \left( 2\pi - \frac{4}{g_B} \right) \rho} \\
\simeq \frac{8}{\pi} \frac{m}{g_B}.
\]  \hfill (5.5)

This result can be compared with the prediction of Fujita and Ogura in the infinite momentum frame calculation.

\[ \mathcal{M}_{FO} \simeq 2\sqrt{2\pi} \frac{m}{g_0}, \]  \hfill (5.6)

where \( g_0 \) is the Schwinger type coupling constant [8,9]. As can be seen, they are different from each other if we assume \( g_B = g_0 \). In ref.[4], it was shown that the Bethe ansatz solutions for several cases of the coupling constant
are consistent with those of infinite momentum frame calculation by Fujita
and Ogura [1] with the identification of $g_B = g_0$. However, it became also
apparent that the boson mass calculated by the Bethe ansatz solution starts
to deviate from the light cone result in the strong coupling region. This
difference may well be related to the normalization ambiguity of the coupling
constant in the massive Thirring model.

5.2 Coupling constant ambiguity

As Klaiber pointed out long time ago, there is an ambiguity of the coupling
constant in the massless Thirring model. He proves that the coupling con-
stant is related to each other depending on the regularization. We briefly
review it in the appendix A.

The results of the boson mass at the strong coupling limit for the light
cone and the Bethe ansatz method indicate that the coupling constant am-
biguity may well be different from eq.(A.8) for the massive Thirring model.
From the comparison of the numerical calculations and analytical evaluations
between the light cone and the Bethe ansatz solutions, we can anticipate the
following relation between $g_0$ and the Bethe ansatz coupling constant $g_B$

$$ g_B = g_0 \left( \frac{2\sqrt{2}}{\pi^2} + \frac{B \left( 1 - \frac{2\sqrt{2}}{\pi^2} \right)}{B + \frac{2\sqrt{2}}{\pi^2}} \right), $$

(5.7)

where $B$ is a free parameter. In this case, the boson mass of the Bethe ansatz
solutions at the strong coupling becomes identical to the light cone result.
If we take $B \sim 10$, then the agreement between the light cone result and
the Bethe ansatz solutions become very good for whole range of the coupling
constant. From eq.(5.7), $g_B$ becomes for the small value of $g_0$

$$ g_B \simeq g_0. $$

(5.8)

For the large value of $g_0$, we obtain

$$ g_B \simeq g_0 \frac{2\sqrt{2}}{\pi^2}. $$

(5.9a)

This can be derived if we assume the following relation for $s$ and $t$

$$ t = \frac{1}{2} \left( \frac{1}{X} - 1 \right) (s - t), $$

(5.9b)

where $X$ is given as the solution of the following equation,

$$ \left[ \frac{g_B}{\pi} (B - 1) + B - \frac{2\sqrt{2}}{\pi^2} \right] X^2 + \left( \frac{g_B}{\pi} - B + \frac{4\sqrt{2}}{\pi^2} \right) X - \frac{2\sqrt{2}}{\pi^2} = 0. $$

(5.10)

Up to now, we do not know any physically simple meaning of choosing the
fermion current regularization which corresponds to eq.(5.10). Further stud-
ies of the normalization ambiguity of the coupling constant arising from the
fermion current regularization would be very interesting since we believe
that it may well be related to some symmetry which is hidden in the massive
Thirring model.

12
6 Conclusions

We have presented numerical calculations as well as the analytical expressions of the energy eigenvalues of the vacuum and $n$ particle $n$ hole states in the strong coupling regions. It is found that the analytical expressions agree very well with the numerical values of the vacuum and $1p−1h$ state energies for the large values of the coupling constant.

From the analytical expressions, we find that the $2p−2h$ and higher particle hole states appear as free boson states in the strong coupling limit. This is consistent with the recent proof [13] that the $S$-matrix factorization assumed by Zamolodchikov and Zamolodchikov [14] is violated at the quantum level, and therefore, the spectrum predicted by the $S$-matrix factorization is only semiclassical.

We have also obtained the boson mass $M$ at the strong coupling limit analytically. To compare the present result with other calculations, we write here the expressions of the boson mass at the strong coupling limit for various calculations.

\[
M_{\text{DHN}} \simeq \frac{\pi^2 m}{2 g_0}, \tag{6.1a}
\]
\[
M_{\text{FO}} \simeq 2\sqrt{2}\pi m g_0, \tag{6.1b}
\]
\[
M_{\text{BA}} \simeq \frac{8 m}{\pi g_B}, \tag{6.1c}
\]

where $M_{\text{DHN}}$ denotes the result of the semiclassical calculation by Dashen et al., $M_{\text{FO}}$ is the prediction of the light cone calculation by Fujita and Ogura, and the present result is denoted by $M_{\text{BA}}$. If we identify $g_B = g_0$, then $M_{\text{BA}}$ is different from the light cone calculation.

At the present stage, we believe that the coupling constant normalization arising from the fermion current regularization in the massive Thirring model is slightly different from the massless Thirring model. It may well be that the regularization ambiguity is related to some hidden symmetry which is not clearly understood up to now.

We would like to thank M. Hiramoto and A. Ogura for helpful discussions and comments.
References

[1] T. Fujita and A. Ogura, Prog. Theor. Phys. 89 (1993), 23.
[2] A. Ogura, T. Tomachi and T. Fujita, Ann. Phys. 237 (1995), 12.
[3] M. Cavicchi, Int. J. Mod. Phys. A10 (1995), 167.
[4] T. Fujita, Y. Sekiguchi and K. Yamamoto, Ann. Phys. 255 (1997), 204.
[5] R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D11 (1975), 3432.
[6] H. Bergknoff and H.B. Thacker, Phys. Rev. Lett. 42 (1979), 135.
[7] H.B. Thacker, Rev. Mod. Phys. 53 (1981), 253.
[8] B. Klaiber, in Lectures in Theoretical Physics, 1967, edited by A. Barut and W. Britten (Gordon and Breach, NY, 1968).
[9] J. Schwinger, Phys. Rev. Lett. 3 (1959), 296.
[10] K. Johnson, Nuovo Cimento, 20 (1961), 773.
[11] W. Thirring, Ann. Phys. (N.Y) 3 (1958), 91.
[12] A. Ogura, Ph. D thesis, Nihon University (1994).
[13] T. Fujita and M. Hiramoto, Phys. Rev. D58 (1998) 125019 − 1 − 10.
[14] A.B.Zamolodchikov and A.B.Zamolodchikov, Ann. Phys. 120 (1979), 253.
[15] M. Gomes and J.H. Lowenstein, Nucl. Phys. B45 (1972), 252.
[16] T. Fujita, C. Itoi and H. Mukaida, unpublished.
[17] N.Nakanishi, Prog. Theor. Phys. 57 (1977), 580.
A Coupling constant ambiguity of the massless Thirring model

Here, we briefly review the normalization ambiguity of the coupling constant in the massless Thirring model \[8,12,17\].

For the right mover fermion field \( \psi_R \), we can express it by the massless boson fields \( \phi_R \) and \( \phi_L \) as

\[
\psi_R \sim e^{is\phi_R - it\phi_L}
\]  \hspace{1cm} (A.1)

where \( s \) and \( t \) are free parameters which satisfy the following constraint

\[
s^2 - t^2 = 4\pi.
\]  \hspace{1cm} (A.2)

The \( s \) and \( t \) can be expressed in terms of the boson coupling constant \( \beta \) as

\[
s = \frac{1}{2} \left( \frac{4\pi}{\beta} + \beta \right), \quad t = \frac{1}{2} \left( \frac{4\pi}{\beta} - \beta \right). \hspace{1cm} (A.3)
\]

Now, the fermion current regularization gives another constraint. For example, Schwinger’s regularization \( (g_0) \) which makes the fermion current regularization only in terms of the space coordinate point splitting implies that

\[
t = \frac{g_0}{2\pi} (s - t). \hspace{1cm} (A.4)
\]

In this case, one obtains the following equation,

\[
\frac{\beta^2}{4\pi} = \frac{1}{1 + \frac{g_0}{\pi}}. \hspace{1cm} (A.5)
\]

On the other hand, Johnson’s regularization \( (g) \) which makes the fermion current regularization in terms of the space-time coordinate point splitting in a symmetric fashion implies that

\[
t = \frac{g}{2\pi} s. \hspace{1cm} (A.6)
\]

In this case, one obtains the following equation,

\[
\frac{\beta^2}{4\pi} = \frac{2 - \frac{g}{\pi}}{2 + \frac{g}{\pi}}. \hspace{1cm} (A.7)
\]

Therefore, one obtains the relation between \( g_0 \) and \( g \) as written

\[
g_0 = \frac{2g}{2 - \frac{g}{\pi}}. \hspace{1cm} (A.8)
\]