Traveling Salesperson Problems for a double integrator

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Abstract

In this paper we propose some novel path planning strategies for a double integrator with bounded velocity and bounded control inputs. First, we study the following version of the Traveling Salesperson Problem (TSP): given a set of points in $\mathbb{R}^d$, find the fastest tour over the point set for a double integrator. We first give asymptotic bounds on the time taken to complete such a tour in the worst-case. Then, we study a stochastic version of the TSP for double integrator where the points are randomly sampled from a uniform distribution in a compact environment in $\mathbb{R}^2$ and $\mathbb{R}^3$. We propose novel algorithms that perform within a constant factor of the optimal strategy with high probability. Lastly, we study a dynamic TSP: given a stochastic process that generates targets, is there a policy which guarantees that the number of unvisited targets does not diverge over time? If such stable policies exist, what is the minimum wait for a target? We propose novel stabilizing receding-horizon algorithms whose performances are within a constant factor from the optimum with high probability, in $\mathbb{R}^2$ as well as $\mathbb{R}^3$. We also argue that these algorithms give identical performances for a particular nonholonomic vehicle, Dubins vehicle.

I. INTRODUCTION

The Traveling Salesperson Problem (TSP) with its variations is one of the most widely known combinatorial optimization problems. While extensively studied in the literature, these problems continue to attract great interest from a wide range of fields, including Operations Research, Mathematics and Computer Science. The Euclidean TSP (ETSP) [1], [2] is formulated as follows: given a finite point set $P$ in $\mathbb{R}^d$ for $d \in \mathbb{N}$, find the minimum-length closed path through all points.

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in $P$. It is quite natural to formulate this problem in the context of other dynamical vehicles. The focus of this paper is the analysis of the TSP for a vehicle with double integrator dynamics or simply a double integrator; we shall refer to it as DITSP. Specifically, DITSP will involve finding the *fastest* tour for a double integrator through a set of points.

Exact algorithms, heuristics and polynomial-time constant factor approximation algorithms are available for the Euclidean TSP, see [3], [4], [5]. However, unlike most other variations of the TSP, it is believed that the DITSP cannot be formulated as a problem on a finite-dimensional graph, thus preventing the use of well-established tools in combinatorial optimization. On the other hand, it is reasonable to expect that exploiting the geometric structure of feasible paths for a double integrator, one can gain insight into the nature of the solution, and possibly provide polynomial-time approximation algorithms.

The motivation to study the DITSP arises in robotics and uninhabited aerial vehicles (UAVs) applications. In particular, we envision applying our algorithm to the setting of an UAV monitoring a collection of spatially distributed points of interest. Additionally, from a purely scientific viewpoint, it is of general interest to bring together the work on dynamical vehicles and that on TSP. UAV applications also motivate us to study the Dynamic Traveling Repairperson Problem (DTRP), in which the aerial vehicle is required to visit a dynamically generated set of targets. This problem was introduced by Bertsimas and van Ryzin in [6] and then decentralized policies achieving the same performances were proposed in [7]. Variants of these problems have attracted much attention recently [7], [8], [9], [10], [11]. However, as with the TSP, the study of DTRP in conjunction with vehicle dynamics has eluded attention from the research community.

The contributions of this paper are threefold. First, we analyze the minimum time to traverse DITSP in $\mathbb{R}^d$ for $d \in \mathbb{N}$. We show that the minimum time to traverse DITSP belongs to $O(n^{1-\frac{1}{2d}})$ and in the worst case, it also belongs\(^1\) to $\Omega(n^{1-\frac{1}{2d}})$. Second, we study the *stochastic* DITSP, i.e., the problem of finding the fastest tour through a set of target points that are uniformly randomly generated. We show that the minimum time to traverse the tour for the stochastic DITSP belongs to $\Omega(n^{2/3})$ in $\mathbb{R}^2$ and $\Omega(n^{4/5})$ in $\mathbb{R}^3$. Drawing inspiration from our earlier work [12], we propose two novel algorithms for the stochastic DITSP: the *Recursive Bead-Tiling Algorithm*

\(^1\)For $f, g : \mathbb{N} \rightarrow \mathbb{R}$, we say that $f \in O(g)$ (respectively, $f \in \Omega(g)$) if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_+$ such that $|f(N)| \leq k|g(N)|$ for all $N \geq N_0$ (respectively, $|f(N)| \geq k|g(N)|$ for all $N \geq N_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$. 

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for \( \mathbb{R}^2 \) and the \textbf{Recursive Cylinder-Covering Algorithm} for \( \mathbb{R}^3 \). We prove that these algorithms provide a constant-factor approximation to the optimal DITSP solution with high probability. Third, we propose two algorithms for the DTRP in the heavy load case based on the fixed-resolution versions of the corresponding algorithms for stochastic DITSP. We show that the performance guarantees for the stochastic DITSP translate into stability guarantees for the average performance of the DTRP problem for a double integrator. Specifically, the performances of the algorithms for the DTRP are within a constant factor of the optimal policies. We contend that the successful application to the DTRP problem does indeed demonstrate the significance of the DITSP problem from a control viewpoint. As a final minor contribution, we also show that the results obtained for stochastic DITSP carry over to the stochastic TSP for the Dubins vehicle, i.e., for a nonholonomic vehicle moving along paths with bounded curvature, without reversing direction.

This work completes the generalization of the known combinatorial results on the ETSP and DTRP (applicable to systems with single integrator dynamics) to double integrators and Dubins vehicle models. It is interesting to compare our results with the setting where the vehicle is modeled by a single integrator; this setting corresponds to the so-called Euclidean case in combinatorial optimization. The results are summarized as follows:

|                     | Single integrator | Double integrator | Dubins vehicle |
|---------------------|-------------------|-------------------|----------------|
| Min. time for       | \( \Theta(n^{1-\frac{1}{d}}) \) [2] | \( \Omega(n^{1-\frac{1}{d}}), O(n^{1-\frac{1}{2d}}) \) | \( \Theta(n) \) [13] |
| TSP tour            |                   |                   | \( (d = 2, 3) \) |
| (worst-case)        |                   |                   |                |
| Exp. min. time      | \( \Theta(n^{1-\frac{1}{d}}) \) [2] | \( \Theta(n^{1-\frac{1}{2d-1}}) \) w.h.p. | \( \Theta(n^{1-\frac{1}{2d-1}}) \) w.h.p. |
| for TSP tour        |                   |                   | \( (d = 2, 3) \) |
| (stochastic)        |                   |                   |                |
| System time         | \( \Theta(\lambda^{d-1}) \) [6] \( (d = 1) \) | \( \Theta(\lambda^{2(d-1)}) \) \( (d = 2, 3) \) | \( \Theta(\lambda^{2(d-1)}) \) \( (d = 2, 3) \) |

for the DTRP problem we propose
novel policies and show their stability for a uniform target-generation process with intensity $\lambda$. It is clear from the table that motion constraints make the system much more sensitive to increases in the target generation rate $\lambda$.

II. SETUP AND WORST-CASE DITSP

For $d \in \mathbb{N}$, consider a vehicle with double integrator dynamics:

$$\ddot{p}(t) = u(t), \quad \|u(t)\| \leq r_{\text{ctr}}, \quad \|\dot{p}(t)\| \leq r_{\text{vel}},$$

where $p \in \mathbb{R}^d$ and $u \in \mathbb{R}^d$ are the position and control input of the vehicle, $r_{\text{vel}} \in \mathbb{R}_+$ and $r_{\text{ctr}} \in \mathbb{R}_+$ are the bounds on the attainable speed and control inputs. Let $Q \subset \mathbb{R}^d$ be a unit hypercube. Let $P = \{q_1, \ldots, q_n\}$ be a set of $n$ points in $Q$ and $\mathcal{P}_n$ be the collection of all point sets $P \subset Q$ with cardinality $n$. Let $\text{ETSP}(P)$ denote the cost of the Euclidean TSP over $P$ and let $\text{DITSP}(P)$ denote the cost of the TSP for double integrator over $P$, i.e., the time taken to traverse the fastest closed path for a double integrator through all points in $P$. We assume $r_{\text{vel}}$ and $r_{\text{ctr}}$ to be constant and we study the dependence of $\text{DITSP}: \mathcal{P}_n \rightarrow \mathbb{R}_+$ on $n$. Without loss of generality, we assume the vehicle starts traversing the TSP tour at $t = 0$ with initial position $q_1$.

**Lemma 2.1: (Worst-case Lower Bound on the TSP for Double Integrator)** For $r_{\text{vel}}, r_{\text{ctr}} \in \mathbb{R}_+$ and $d \in \mathbb{N}$, there exists a point set $P \in \mathcal{P}_n$ in $Q \subset \mathbb{R}^d$ such that $\text{DITSP}(P)$ belongs to $\Omega(n^{1-\frac{1}{d}})$.

**Proof:** We consider the class of point sets that give rise to the worst case scenario for the ETSP; we refer the reader to [2]. It suffice to note that, for such a point set of cardinality $n$ in $\mathbb{R}^d$, the minimum distance between any two points belongs to $\Omega(n^{-\frac{1}{d}})$. The minimum time required for a double integrator with initial speed $\tilde{v}$ to go from one point to another at a distance $\tilde{\delta}$ is lower bounded by $\sqrt{(\tilde{v}/r_{\text{ctr}})^2 + 2(\tilde{\delta}/r_{\text{ctr}}) - \tilde{v}/r_{\text{ctr}}}$. However, $\tilde{v} \leq r_{\text{vel}}$ and for the point set under consideration, $\tilde{\delta}$ belongs to $\Omega(n^{-\frac{1}{d}})$. This implies that the minimum time required for a double integrator to travel between two points of the given point set belongs to $\Omega(n^{-\frac{1}{d}})$. Hence, the minimum time required for the vehicle to complete the tour over this point set belongs to $n\Omega(n^{-\frac{1}{d}})$, i.e., $\Omega(n^{1-\frac{1}{d}})$.

We now propose a simple strategy for the DITSP and analyze its performance. The STOP-GO-STOp strategy can be described as follows: The vehicle visits the points in the same order as in the optimal ETSP tour over the same set of points. Between any pair of points, the vehicle
starts at the initial point at rest, follows the shortest-time path to reach the final point with zero velocity.

**Theorem 2.2: (Upper Bound on the TSP for Double Integrator)** For any point set \( P \subseteq \mathcal{P}_n \) in \( Q \subseteq \mathbb{R}^d \) and \( r_{\text{ctr}} > 0, r_{\text{vel}} > 0 \) and \( d \in \mathbb{N} \), \( \text{DITSP}(P) \) belongs to \( O(n^{1 - \frac{1}{2d}}) \).

**Proof:** Without any loss of generality, let \((q_1, \ldots, q_n, q_1)\) be the optimal order of points for the Euclidean TSP over \( P \). For \( 1 \leq i \leq n - 1 \), let \( \delta_i = \|q_i - q_{i+1}\| \) and \( \delta_n = \|q_n - q_1\| \). If \( \delta_i \) is the distance between a set of points, then the time \( t_i \) required to traverse that distance by a double integrator following the STOP-GO-STOP strategy is given by:

\[
  t_i = \begin{cases} 
    2 \sqrt{\frac{\delta_i}{r_{\text{ctr}}}}, & \text{if } \delta_i \leq \frac{r_{\text{vel}}^2}{r_{\text{ctr}}} \\
    \frac{r_{\text{vel}}}{r_{\text{ctr}}} + \frac{\delta_i}{r_{\text{vel}}}, & \text{otherwise} \end{cases}
\]

Let \( I = \{1 \leq i \leq n \mid \delta_i \leq \frac{r_{\text{vel}}^2}{r_{\text{ctr}}}\} \) and \( I^c = \{1, \ldots, n\} \setminus I \). Also, let \( n_I \) be the cardinality of the set \( I \) and let \( n_{I^c} = n - n_I \). Therefore, an upper bound on the minimum time taken to complete the tour as obtained from this strategy is

\[
  \text{DITSP}(P) \leq \sum_{i=1}^{n} t_i = \sum_{i \in I} t_i + \sum_{i \in I^c} t_i = 2 \sqrt{\frac{r_{\text{ctr}}}{r_{\text{vel}}}} \sum_{i \in I} \sqrt{\delta_i} + n_{I^c} \frac{r_{\text{vel}}}{r_{\text{ctr}}} + \frac{1}{r_{\text{vel}}} \sum_{i \in I^c} \delta_i \leq 2 \sqrt{\frac{r_{\text{ctr}}}{r_{\text{vel}}}} \sum_{i \in I} \sqrt{\delta_i} + n_{I^c} \left(\frac{r_{\text{vel}}}{r_{\text{ctr}}} + \frac{\text{diam}(Q)}{r_{\text{vel}}}\right),
\]

where \( \text{diam}(Q) \) is the length of the largest segment lying completely inside \( Q \). From the well known upper bound [2] on the tour length of optimal ETSP, there exists a constant \( \beta(Q) \) such that \( \sum_{i \in I} \delta_i \leq \sum_{i=1}^{n} \delta_i \leq \beta(Q) n^{1 - \frac{1}{d}} \). Hence an upper bound on the term \( \sum_{i \in I} \sqrt{\delta_i} \) in eqn. \( 2 \) can be obtained by solving the following optimization problem:

\[
  \text{maximize } \sum_{i \in I} \sqrt{\delta_i}, \quad \text{ subj. to } \sum_{i \in I} \delta_i \leq \beta(Q) n^{1 - \frac{1}{d}}.
\]

By employing the method of Lagrange multipliers, one can see that the maximum is achieved when \( \delta_i = \beta(Q) \frac{n^{1 - \frac{1}{d}}}{n_I} \forall i \in I \). Hence \( \sum_{i \in I} \sqrt{\delta_i} \leq \sqrt{\beta(Q)} \sqrt{n_I n^{1 - \frac{1}{d}}} \). Substituting this in eqn. \( 2 \), we get that

\[
  \text{DITSP}(P) \leq 2 \sqrt{\frac{\beta(Q)}{r_{\text{ctr}}}} \sqrt{n_I n^{\frac{1}{2} - \frac{1}{d}}} + n_{I^c} \left(\frac{r_{\text{vel}}}{r_{\text{ctr}}} + \frac{\text{diam}(Q)}{r_{\text{vel}}}\right).
\]

However, \( n_I \leq n \) and Lemma 2.3 implies that \( n_{I^c} \) belongs to \( O(n^{1 - \frac{1}{d}}) \). Incorporating these facts into eqn. \( 3 \), one arrives at the final result.

\[\square\]
The above theorem relies on the following key result.

**Lemma 2.3:** Given any point set \( P \in \mathcal{P}_n \) in \( Q \subset \mathbb{R}^d \), if \((q_1, q_2, \ldots, q_n, q_1)\) is the order of points for the optimal ETSP tour over \( P \), then for any \( \eta \in \mathbb{R}_+ \), the cardinality of the set \( \{q_i \in P \mid \|q_i - q_{i+1}\| > \eta\} \) belongs to \( O(n^{1-\frac{1}{d}}) \).

**Proof:** By contradiction, assume there exists \( \tilde{\eta} \in \mathbb{R}_+ \) such that the cardinality of \( \{p_i \in P \mid \|q_i - q_{i+1}\| > \tilde{\eta}\} \) belongs to \( \Omega(n^{1-\frac{4}{d}+\epsilon}) \) for some \( \epsilon > 0 \). This implies that \( \text{ETSP}(P) \) belongs to \( \tilde{\eta} \times \Omega(n^{1-\frac{4}{d}+\epsilon}) = \Omega(n^{1-\frac{4}{d}+\epsilon}) \). However, we know from [2] that \( \text{ETSP}(P) \in O(n^{1-\frac{1}{d}}) \).

### III. The Stochastic DITSP

The results in the previous section showed that based on a simple strategy, the STOP-GO-STOP strategy, we are already guaranteed to have sublinear cost for the DITSP when the point sets are considered on an individual basis. However, it is reasonable to argue that there might be better algorithms when one is concerned with average performance. In particular, one can expect that when \( n \) target points are stochastically generated in \( Q \) according to a uniform probability distribution function, the cost of DITSP should be lower than the one given by the STOP-GO-STOP strategy. We shall refer to the problem of studying the average performance of DITSP over this class of point sets as stochastic DITSP. In this section, we present novel algorithms for stochastic DITSP and then establish bounds on their performances.

We make the following assumptions: in \( \mathbb{R}^2 \), \( Q \) is a rectangle of width \( W \) and height \( H \) with \( W \geq H \); in \( \mathbb{R}^3 \), \( Q \) is a rectangular box of width \( W \), height \( H \) and depth \( D \) with \( W \geq H \geq D \). Different choices for the shape of \( Q \) affect our conclusions only by a constant. The axes of the reference frame are parallel to the sides of \( Q \). The points \( P = (p_1, \ldots, p_n) \) are randomly generated according to a uniform distribution in \( Q \).

#### A. Lower bounds

First we provide lower bounds on the expected length of stochastic DITSP for the 2 and 3 dimensional case.

**Theorem 3.1:** (Lower bounds on stochastic DITSP) For all \( r_{vel} > 0 \) and \( r_{ctr} > 0 \), the expected cost of a stochastic DITSP visiting a set of \( n \) uniformly-randomly-generated points satisfies the
following inequalities:

\[
\lim_{n \to +\infty} \frac{\text{E}[\text{DITSP}(P \subset Q \subset \mathbb{R}^2)]}{n^{2/3}} \geq \frac{3}{4} \left( \frac{6WH}{r_{\text{vel}}r_{\text{ctr}}} \right)^{1/3} \quad \text{and}
\]

\[
\lim_{n \to +\infty} \frac{\text{E}[\text{DITSP}(P \subset Q \subset \mathbb{R}^3)]}{n^{4/5}} \geq \frac{5}{6} \left( \frac{20WHD}{\pi r_{\text{vel}}^2 r_{\text{ctr}}^2} \right)^{1/5}.
\]

**Proof:** We first prove the first inequality. Choose a random point \( q_i \in P \) as the initial position and \( v_i \) as the initial speed of the vehicle on the tour, and choose the heading randomly. We would like to compute a bound on the expected time to the closest next point in the tour; let us call such a time \( t^* \). To this purpose, consider the set \( R_t \) of points that are reachable by a second order vehicle within time \( t \). It can be verified that the area of such a set can be bounded, as \( t \to 0^+ \), by

\[
\text{Area}(R_t) \leq \frac{r_{\text{ctr}}v_it^3}{6} + o(t^3) \leq \frac{r_{\text{ctr}}v_it^3}{6} + o(t^3).
\]

(4)

Given time \( t \), the probability that \( t^* > t \) is no less than the probability that there is no other target reachable within a time at most \( t \); in other words,

\[
\text{Pr}[t^* > t] \geq 1 - n \frac{\text{Area}(R_t)}{\text{Area}(Q)} \geq 1 - n \frac{r_{\text{ctr}}v_it^3}{6WH} - o(t^3).
\]

In terms of expectation, defining \( c = \frac{n r_{\text{ctr}}v_i}{6WH} \),

\[
\text{E}[t^*] = \int_0^{+\infty} \text{Pr}(t^* > \xi) \, d\xi \\
\geq \int_0^{+\infty} \text{max} \left( 0, 1 - \frac{n r_{\text{ctr}}v_i}{6WH} \xi^3 - o(\xi^3) \right) \, d\xi \\
\geq \int_0^{c^{-1/3}} (1 - c\xi^3) \, d\xi - n \int_0^{c^{-1/3}} o(\xi^3) \, d\xi \\
= \frac{3}{4} \left( \frac{6WH}{r_{\text{vel}}r_{\text{ctr}}} \right)^{1/3} - o(n^{-1/3}).
\]

The expected total tour time will be no smaller than \( n \) times the expected shortest time between two points, i.e.,

\[
\text{E}[\text{DITSP}(P)r_{\text{vel}}, r_{\text{ctr}}2] \geq \frac{3}{4} \left( \frac{6n^2WH}{r_{\text{vel}}r_{\text{ctr}}} \right)^{1/3} - o(n^{2/3}).
\]

Dividing both sides by \( n^{2/3} \) and taking the limit as \( n \to +\infty \), we get the first result.

We now prove the second inequality. Choose a random point \( q_i \in P \) as the initial position and \( v_i \) as the initial speed of the vehicle on the tour, and choose the heading randomly. We would like to compute a bound on the expected time to the closest next point in the tour; let us call
such a time $t^*$. To this purpose, consider the set $R_t$ of points that are reachable by a second order vehicle within time $t$. It can be verified that the volume of such a set can be bounded, as $t \to 0^+$, by

$$\text{Volume}(R_t) \leq \frac{\pi r_{\text{cr}}^2 v_t t^5}{20} + o(t^5).$$

(5)

Given time $t$, the probability that $t^* > t$ is no less than the probability that there is no other target reachable within a time at most $t$; in other words,

$$\Pr[t^* > t] \geq 1 - n \frac{\text{Volume}(R_t)}{\text{Volume}(Q)} \geq 1 - n \frac{\pi r_{\text{cr}}^2 r_{\text{vel}} t^5}{20WHD} - o(t^5).$$

In terms of expectation, defining $c = \frac{n \pi r_{\text{cr}}^2 r_{\text{vel}}}{20WHD}$,

$$E[t^*] = \int_0^{+\infty} \Pr(t^* > \xi) \, d\xi$$

$$\geq \int_0^{+\infty} \max \left\{ 0, 1 - n \frac{\pi r_{\text{cr}}^2 r_{\text{vel}} \xi^5}{20WHD} - o(\xi^5) \right\} \, d\xi$$

$$\geq \int_0^{c^{-1/5}} (1 - c\xi^5) \, d\xi - n \int_0^{c^{-1/5}} o(\xi^5) \, d\xi$$

$$= \frac{5}{6} \left( \frac{20WHD}{r_{\text{vel}} r_{\text{cr}}^2 n} \right)^{1/5} - o(n^{-1/5}).$$

The expected total tour time will be no smaller than $n$ times the expected shortest time between two points, i.e.,

$$E[\text{DITSP}(P) r_{\text{vel}}, r_{\text{cr}}^3] \geq \frac{5}{6} \left( \frac{20n^4WHD}{r_{\text{vel}} r_{\text{cr}}^2 n} \right)^{1/5} - o(n^{4/5}).$$

Dividing both sides by $n^{4/5}$ and taking the limit as $n \to +\infty$, we get the second result.

B. Constructive upper bounds

In this section, we first recall the Recursive Bead-Tiling Algorithm from our earlier work [14] on Dubins vehicle and use it to propose novel algorithms for the stochastic DITSP: the Recursive Bead-Tiling Algorithm for $\mathbb{R}^2$ and Recursive Cylinder-Covering Algorithm for $\mathbb{R}^3$. The performances of these algorithms will be shown to be within a constant factor of the optimal with high probability.

In [14], we studied stochastic versions of TSP for Dubins vehicle. Though conventionally Dubins vehicle is restricted to be a planar vehicle, one can easily generalize the model even
for the three (and higher) dimensional case. Correspondingly, Dubins vehicle can be defined as a vehicle that is constrained to move with a constant speed along paths of bounded curvature, without reversing direction. Accordingly, a *feasible curve for Dubins vehicle* or a *Dubins path* is defined as a curve that is twice differentiable almost everywhere, and such that the magnitude of its curvature is bounded above by $1/\rho$, where $\rho > 0$ is the minimum turn radius. Based on this, one can immediately come up with the following analogy between feasible curves for Dubins vehicle and a double integrator.

**Lemma 3.2: (Trajectories of Dubins and double integrators)** A feasible curve for Dubins vehicle with minimum turn radius $\rho > 0$ is a feasible curve for a double integrator (modeled in equation (1)) moving with a constant speed $\sqrt{\rho r_{ct}}$. Conversely, a feasible curve for a double integrator moving with a constant speed $s \leq r_{vel}$ is a feasible curve for Dubins vehicle with minimum turn radius $\frac{s^2}{r_{ct}}$.

In [12], we proposed a novel algorithm, the **Recursiv BEad-TILING Algorithm** for the stochastic version of the Dubins TSP (DTSP) in $\mathbb{R}^2$; we showed that this algorithm performed within a constant factor of the optimal with high probability. In this paper, taking inspiration from those ideas, we propose algorithms to compute feasible curves for a double integrator moving with a constant speed. Note that moving at the maximum speed $r_{vel}$ is not necessarily the best strategy since it restricts the maneuvering capability of the vehicle. Nonetheless, this strategy leads to efficient algorithms. In what follows we assume that the double integrator is moving with some constant speed $s \leq r_{vel}$. Next, we proceed towards devising strategies which perform within a constant factor of the optimal for stochastic DITSP in $\mathbb{R}^2$ as well as $\mathbb{R}^3$, both with high probability.

1) **The basic geometric construction:** Here we define useful geometric objects and study their properties. Given the constant speed $s$ for the double integrator let $\rho = \frac{s^2}{r_{ct}}$; from Lemma 3.2, this constant corresponds to the minimum turning radius of the analogous Dubins vehicle. Consider two points $p_-$ and $p_+$ on the plane, with $\ell = \|p_+ - p_-\|_2 \leq 4\rho$, and construct the bead $B_\rho(\ell)$ as detailed in Figure 1.

Associated with the bead is also the rectangle $efgh$. Rotating this rectangle about the line passing through $p_-$ and $p_+$ gives rise to a cylinder $C_\rho(\ell)$. The regions $B_\rho(\ell)$ and $C_\rho(\ell)$ enjoy the following asymptotic properties as $(\ell/\rho) \to 0^+$:
(P1) The maximum “thickness” of $B_\rho(\ell)$ is equal to

$$w(\ell) = 4\rho \left( 1 - \sqrt{1 - \frac{\ell^2}{16\rho^2}} \right) = \frac{\ell^2}{8\rho} + \rho \cdot o\left(\frac{\ell^3}{\rho^3}\right).$$

The radius of cross-section of $C_\rho(\ell)$ is $w(\ell)/4$ and the length of $C_\rho(\ell)$ is $\ell$.

(P2) The area of $B_\rho(\ell)$ is equal to

$$\text{Area}(B_\rho(\ell)) = \frac{\ell w(\ell)}{2} = \frac{\ell^3}{16\rho} + \rho^2 \cdot o\left(\frac{\ell^4}{\rho^4}\right).$$

The volume of $C_\rho(\ell)$ is equal to

$$\text{Volume}[C_\rho(\ell)] = \pi \left(\frac{w(\ell)}{4}\right)^2 \frac{\ell}{2} = \frac{\pi \ell^5}{2048 \rho^2} + \rho^3 \cdot o\left(\frac{\ell^6}{\rho^6}\right).$$

(P3) For any $p \in B_\rho$, there is at least one feasible curve $\gamma_p$ through the points $\{p_-, p, p_+\}$, entirely contained within $B_\rho$. The length of any such path is at most

$$\text{Length}(\gamma_p) \leq 4\rho \arcsin\left(\frac{\ell}{4\rho}\right) = \ell + \rho \cdot o\left(\frac{\ell^3}{\rho^3}\right).$$

Analogously, for any $\tilde{p} \in C_\rho$, there is at least one feasible curve $\gamma_{\tilde{p}}$ through the points $\{p_-, \tilde{p}, p_+\}$, entirely contained within the region obtained by rotating $B_\rho(\ell)$ about the line.
passing through $p_-$ and $p_+$. The length of $\gamma_{\bar{p}}$ satisfies the same upper bound as the one established for $\gamma_p$.

The geometric shapes introduced above can be used to cover $\mathbb{R}^2$ and $\mathbb{R}^3$ in an organized way. The plane can be periodically tiled by identical copies of $B_\rho(\ell)$, for any $\ell \in [0, 4\rho]$. The cylinder, however, does not enjoy any such special property. For our purpose, we consider a particular covering of $\mathbb{R}^3$ by cylinders described as follows.

A row of cylinders is formed by joining cylinders end to end along their length. A layer of cylinders is formed by placing rows of cylinders parallel and on top of each other as shown in Figure 2. For covering $\mathbb{R}^3$, these layers are arranged next to each other and with offsets as shown in Figure 3(a), where the cross section of this arrangement is shown. We refer to this construction as the covering of $\mathbb{R}^3$.

2) The 2D case: The Recursive Bead-Tiling Algorithm (RecBTA): Consider a tiling of the plane such that $\text{Area}[B_\rho(\ell)] = \text{Area}[Q \subset \mathbb{R}^2]/(2n) = WH/(2n)$; to obtain this equality we assume $\ell$ to be a decreasing function of $n$ such that $\ell(n) \leq 4\rho$. Furthermore, we assume the tiling is chosen to be aligned with the sides of $Q \subset \mathbb{R}^2$, see Figure 4.

A tiling of the plane is a collection of sets whose intersection has measure zero and whose union covers the plane.
The proposed algorithm consists of a sequence of phases; during each of these phases, a feasible curve will be constructed that “sweeps” the set $Q$. In the first phase, a feasible curve is constructed with the following properties:

(i) it visits all non-empty beads once,
(ii) it visits all rows\(^3\) in sequence top-to-down, alternating between left-to-right and right-to-left passes, and visiting all non-empty beads in a row,
(iii) when visiting a non-empty bead, it services at least one target in it.

In order to visit the outstanding targets, a new phase is initiated. In this phase, instead of considering single beads, we will consider “meta-beads” composed of two beads each, as shown in Figure 4 and proceed in a similar way as the first phase, i.e., a feasible curve is constructed with the following properties:

(i) the curve visits all non-empty meta-beads once,
(ii) it visits all (meta-bead) rows in sequence top-to-down, alternating between left-to-right and right-to-left passes, and visiting all non-empty meta-beads in a row,
(iii) when visiting a non-empty meta-bead, it services at least one target in it.

\(^3\)A row is a maximal string of beads with non-empty intersection with $Q$. 

Fig. 3. (a): Cross section of the arrangement of the layers of cylinders used for covering $Q \subset \mathbb{R}^3$, (b): The relative position of the bigger cylinder relative to smaller ones of the prior phase during the phase transition.
This process is iterated at most \( \log_2 n + 1 \) times, and at each phase meta-beads composed of two neighboring meta-beads from the previous phase are considered; in other words, the meta-beads at the \( i \)-th phase are composed of \( 2^{i-1} \) neighboring beads. After the last phase, the leftover targets will be visited using, for example, a greedy strategy.

Fig. 4. Sketch of “meta-beads” at successive phases in the recursive bead tiling algorithm.

The following result is related to a similar result in [15].

**Theorem 3.3 (Targets remaining after recursive phases):** Let \( P \in \mathcal{P}_n \) be uniformly randomly generated in \( Q \subseteq \mathbb{R}^2 \). The number of unvisited targets after the last recursive phase of the REC BTA is less than \( 24 \log_2 n \) with high probability, i.e., with probability approaching one as \( n \to +\infty \).

**Proof:** Associate a unique identifier to each bead, let \( b(t) \) be the identifier of the bead in which the \( t \)-th target is sampled, and let \( h(t) \in \mathbb{N} \) be the phase at which the \( t \)-th target is visited. Without loss of generality, assume that targets within a single bead are visited in the same order in which they are generated, i.e., if \( b(t_1) = b(t_2) \) and \( t_1 < t_2 \), then \( h(t_1) < h(t_2) \). Let \( v_i(t) \) be the number of beads that contain unvisited targets at the inception of the \( i \)-th phase, computed after the insertion of the \( t \)-th target. Furthermore, let \( m_i \) be the number of \( i \)-th phase meta-beads (i.e., meta-beads containing \( 2^{i-1} \) neighboring beads) with a non-empty intersection with \( Q \). Clearly, \( v_i(t) \leq v_i(n) \), \( m_i \leq 2m_{i+1} \), and \( v_1(n) \leq n \leq m_1/2 \) with certainty. The \( t \)-th target will not be visited during the first phase if it is sampled in a bead that already contains other targets. In other words,

\[
\Pr [h(t) \geq 2 \mid v_1(t)] = \frac{v_1(t)}{m_1} \leq \frac{v_1(n)}{2n} \leq \frac{1}{2}.
\]

Similarly, the \( t \)-th target will not be visited during the \( i \)-th phase if (i) it has not been visited before the \( i \)-th pass, and (ii) it belongs to a meta-bead that already contains other targets not visited
before the \( i \)th phase:

\[
\Pr [h(t) \geq i + 1 \mid (v_{i}(t - 1), v_{i-1}(t - 1), v_{1}(t - 1))] = \Pr [h(t) \geq i \mid v_{i}(t - 1)] \cdot \Pr [h(t) \geq i \mid (v_{i-1}(t - 1), \ldots, v_{1}(t - 1))]
\]

\[
\leq \frac{v_{i}(t - 1)}{m_{i}} \Pr [h(t) \geq i \mid (v_{i-1}(t - 1), \ldots, v_{1}(t - 1))]
\]

\[
= \prod_{j=1}^{i} \frac{v_{j}(t - 1)}{m_{j}} \leq \prod_{j=1}^{i} \frac{2^{j-1}v_{j}(n)}{2n} = \left(\frac{2^{2-i}}{n}\right)^{i} \prod_{j=1}^{i} v_{j}(n).
\]

Given a sequence \( \{\beta_{i}\}_{i \in \mathbb{N}} \subset \mathbb{R}_{+} \) and given a fixed \( i \geq 1 \), define a sequence of binary random variables

\[
Y_{t} = \begin{cases} 
1, & \text{if } h(t) \geq i + 1 \text{ and } v_{i}(t - 1) \leq \beta_{i}n, \\
0, & \text{otherwise.}
\end{cases}
\]

In other words, \( Y_{t} = 1 \) if the \( t \)th target is not visited during the first \( i \) phases even though the number of beads still containing unvisited targets at the inception of the \( i \)th phase is less than \( \beta_{i}n \). Even though the random variable \( Y_{t} \) depends on the targets generated before the \( t \)th target, the probability that it takes the value 1 is bounded by

\[
\Pr [Y_{t} = 1 \mid b(1), b(2), \ldots, b(t - 1)] \leq 2^{\frac{i(i-3)}{2}} \prod_{j=1}^{i} \beta_{j} =: q_{i},
\]

regardless of the actual values of \( b(1), \ldots, b(t - 1) \). It is known [15] that if the random variables \( Y_{t} \) satisfy such a condition, the sum \( \sum_{t} Y_{t} \) is stochastically dominated by a binomially distributed random variable, namely,

\[
\Pr \left[ \sum_{t=1}^{n} Y_{t} > k \right] \leq \Pr [B(n, q_{i}) > k].
\]

In particular,

\[
\Pr \left[ \sum_{t=1}^{n} Y_{t} > 2nq_{i} \right] \leq \Pr [B(n, q_{i}) > 2np_{i}] < 2^{-nq_{i}/3}, \tag{6}
\]

where the last inequality follows from Chernoff’s Bound [16]. Now, it is convenient to define \( \{\beta_{i}\}_{i \in \mathbb{N}} \) by

\[
\beta_{1} = 1, \quad \beta_{i+1} = 2q_{i} = 2^{\frac{i(i-3)}{2}+1} \prod_{j=1}^{i} \beta_{j} = 2^{i-2} \beta_{i}^{2},
\]
which leads to $\beta_i = 2^{1-i}$. In turn, this implies that equation (6) can be rewritten as

$$\Pr \left[ \sum_{t=1}^n Y_t > \beta_{i+1} n \right] < 2^{-\beta_{i+1} n/6} = 2^{-\frac{n}{3^2}} ,$$

which is less than $1/n^2$ for $i \leq i^* (n) := \lfloor \log_2 n - \log_2 \log_2 n - \log_2 6 \rfloor \leq \log_2 n$. Note that $\beta_i \leq 12 \frac{\log_2 n}{n}$, for all $i > i^* (n)$.

Let $E_i$ be the event that $v_i (n) \leq \beta_i n$. Note that if $E_i$ is true, then $v_{i+1} (n) \leq \sum_{t=1}^n Y_t$: the right hand side represents the number of targets that will be visited after the $i$th phase, whereas the left hand side counts the number of beads containing such targets. We have, for all $i \leq i^* (n)$:

$$\Pr \left[ v_{i+1} > \beta_{i+1} n \mid E_i \right] \cdot \Pr [E_i] \leq \Pr \left[ \sum_{t=1}^n Y_t > \beta_{i+1} n \right] \leq \frac{1}{n^2} ,$$

that is, $\Pr [\neg E_{i+1} \mid E_i] \leq \frac{1}{n^2 \Pr [E_i]}$, and thus (recall that $E_1$ is true with certainty):

$$\Pr [\neg E_{i+1}] \leq \frac{1}{n^2} + \Pr [\neg E_i] \leq \frac{i}{n^2} .$$

In other words, for all $i \leq i^* (n)$, $v_i (n) \leq \beta_i n$ with high probability.

Let us now turn our attention to the phases such that $i > i^* (n)$. The total number of targets visited after the $(i^*)$th phase is dominated by a binomial variable $B(n, 12 \log_2 n/n)$; in particular,

$$\Pr \left[ v_{i^*+1} > 24 \log_2 n \mid E_{i^*} \right] \cdot \Pr [E_{i^*}] \leq \Pr \left[ \sum_{t=1}^n Y_t > 24 \log_2 n \right] \leq \Pr \left[ B(n, 12 \log_2 n/n) > 24 \log_2 n \right] \leq 2^{-12 \log_2 n} .$$

Dealing with conditioning as before, we obtain

$$\Pr [v_{i^*+1} > 24 \log_2 n] \leq \frac{1}{n^{12}} + \Pr [\neg E_{i^*}] \leq \frac{1}{n^{12}} + \frac{\log_2 n}{n^2} .$$

In other words, the number of targets that are left unvisited after the $(i^*)$th phase is bounded by a logarithmic function of $n$ with high probability.

In summary, Theorem 3.3 says that after a sufficiently large number of phases, almost all targets will be visited, with high probability. The second key point is to recognize that (i) the length of the first phase is of order $n^{2/3}$ and (ii) the length of each phase is decreasing at such a rate that the sum of the lengths of the $\lfloor \log_2 n \rfloor$ recursive phases remains bounded and proportional to the length of the first phase. (Since we are considering the asymptotic case in
which the number of targets is very large, the length of the beads will be very small; in the
remainder of this section we will tacitly consider the asymptotic behavior as $\ell/\rho \to 0^+$.

**Lemma 3.4 (Path length for the first phase):** Consider a tiling of the plane with beads of
length $\ell$. For any $\rho > 0$ and for any set of target points, the length $L_1$ of a path visiting
once and only once each bead with a non-empty intersection with a rectangle $Q$ of width $W$
and length $H$ satisfies

$$L_1 \leq \frac{16\rho WH}{\ell^2} \left(1 + \frac{7}{3} \pi \rho \frac{\rho}{W}\right) + \rho \cdot o\left(\frac{\rho}{\ell}\right).$$

**Proof:** A path visiting each bead once can be constructed by a sequence of passes, during
which all beads in a row are visited in a left-to-right or right-to-left order. In each row, there are
at most $\lceil W/\ell \rceil + 1$ beads with a non-empty intersection with $Q$. Hence, the cost of each pass
is at most:

$$L_1^{\text{pass}} \leq W + 2\ell + \rho \cdot o\left(\frac{\ell^2}{\rho^2}\right).$$

Two passes are connected by a U-turn maneuver, in which the direction of travel is reversed,
and the path moves to the next row, at distance equal to one half the width of a bead. The length
of the shortest path to reverse the heading of the vehicle with co-located initial and final points
is $(7/3)\pi \rho$, the length of the U-turn satisfies

$$L_1^{U\text{-turn}} \leq \frac{7}{3} \pi \rho + \frac{1}{2} w(\ell) + \frac{\ell^2}{16\rho} + \rho \cdot o\left(\frac{\ell^3}{\rho^3}\right).$$

The total number of passes, i.e., the total number of rows of beads with non-empty intersection
with $Q$, satisfies

$$N_1^{\text{pass}} \leq \left\lceil \frac{2H}{w(\ell)} \right\rceil + 1 \leq \frac{16\rho H}{\ell^2} + 2 + o\left(\frac{\rho}{\ell}\right).$$

A simple upper bound on the cost of closing the tour is given by

$$L_1^{\text{close}} \leq (W + 2\ell) + (H + 2w(\ell)) + 2\pi \rho = W + H + 2\pi \rho + 2\ell + \rho \cdot o(\ell/\rho).$$

In summary, the total length of the path followed during the first phase is

$$L_1 \leq N_1^{\text{pass}} \left(L_1^{\text{pass}} + L_1^{U\text{-turn}}\right) + L_1^{\text{close}}$$

$$\leq \left(\frac{16\rho H}{\ell^2} + 2 + o\left(\frac{\rho}{\ell}\right)\right) \left(W + 2\ell + \frac{7}{3} \pi \rho + \frac{\ell^2}{16\rho} + \rho \cdot o\left(\frac{\ell^2}{\rho^2}\right)\right) + W + H + 2\pi \rho + 2\ell + \rho \cdot o(\ell/\rho)$$

$$\leq \frac{16\rho WH}{\ell^2} \left(1 + \frac{7}{3} \pi \rho \frac{\rho}{W}\right) + \rho \cdot o\left(\frac{\rho}{\ell}\right).$$
Based on the calculation for the first phase, we can estimate the length of the paths in generic phases of the algorithm. Since the total number of phases in the algorithm depends on the number of targets \( n \), as does the length of the beads \( \ell \), we will retain explicitly the dependency on the phase number.

**Lemma 3.5 (Path length at odd-numbered phases):** Consider a tiling of the plane with beads of length \( \ell \). For any \( \rho > 0 \) and for any set of target points, the length \( L_{2j-1} \) of a path visiting once and only once each meta-bead with a non-empty intersection with a rectangle \( Q \) of width \( W \) and length \( H \) at phase number \( (2j-1) \), \( j \in \mathbb{N} \) satisfies

\[
L_{2j-1} \leq 2^{5-j} \left[ \frac{\rho WH}{\ell^2} \left( 1 + \frac{7 \pi \rho}{3 W} \right) + \rho \cdot o \left( \frac{\rho}{\ell} \right) \right] + 32 \frac{\rho H}{\ell} + \rho \cdot o \left( \frac{\rho}{\ell} \right) + 2^j \left[ 3\ell + \rho \cdot o \left( \frac{\ell}{\rho} \right) \right].
\]

**Proof:** During odd-numbered phases, the number of beads in a meta-bead is a perfect square and the considerations made in the proof of Lemma 3.4 can be readily adapted. The length of each pass satisfies

\[
L_{2j-1}^{\text{pass}} \leq (W + 2^j \ell) \left[ 1 + o \left( \frac{\ell}{\rho} \right) \right].
\]

The length of each U-turn maneuver is bounded as

\[
L_{2j-1}^{\text{U-turn}} \leq \frac{7}{3} \pi \rho + 2^j - 2w(\ell) \leq \frac{7}{3} \pi \rho + 2^j - 2 \left[ \frac{\ell^2}{8\rho} + \rho \cdot o \left( \frac{\ell^3}{\rho^2} \right) \right],
\]

from which

\[
L_{2j-1}^{\text{pass}} + L_{2j-1}^{\text{U-turn}} = W + \frac{7}{3} \pi \rho + o \left( \frac{\ell}{\rho} \right) + 2^j \left[ \ell + \rho \cdot o \left( \frac{\ell}{\rho} \right) \right].
\]

The number of passes satisfies:

\[
N_{2j-1}^{\text{pass}} \leq 2^{5-j} \left[ \frac{\rho H}{\ell^2} + o \left( \frac{\rho}{\ell} \right) \right] + 2.
\]

Finally, the cost of closing the tour is bounded by

\[
L_{2j-1}^{\text{close}} \leq W + H + 2\pi \rho + 2^j \left[ \ell + \rho \cdot o(\ell/\rho) \right].
\]

Therefore, a bound on the total length of the path is

\[
L_{2j-1} = N_{2j-1}^{\text{pass}}(L_{2j-1}^{\text{pass}} + L_{2j-1}^{\text{U-turn}}) + L_{2j-1}^{\text{close}}
\]

\[
\leq 2^{5-j} \left[ \frac{\rho WH}{\ell^2} \left( 1 + \frac{7 \pi \rho}{3 W} \right) + \rho \cdot o \left( \frac{\rho}{\ell} \right) \right] + 32 \frac{\rho H}{\ell} + \rho \cdot o \left( \frac{\rho}{\ell} \right) + 2^j \left[ 3\ell + \rho \cdot o \left( \frac{\ell}{\rho} \right) \right].
\]
Lemma 3.6 (Path length at even-numbered phases): Consider a tiling of the plane with beads of length \( \ell \). For any \( \rho > 0 \), a rectangle \( Q \) of width \( W \) and length \( H \) and any set of target points, paths in each phase of the \textsc{Bead-Tiling Algorithm} can be chosen such that \( L_{2j} \leq 2L_{2j+1} \), for all \( j \in \mathbb{N} \).

Proof: Consider a generic meta-bead \( B_{2j+1} \) traversed in the \((2j+1)\)th phase, and let \( l_3 \) be the length of the path segment within \( B_{2j+1} \). The same meta-bead is traversed at most twice during the \((2j)\)th phase; let \( l_1, l_2 \) be the lengths of the two path segments of the \((2j)\)th phase within \( B_{2j+1} \). By convention, for \( i \in \{1, 2, 3\} \), we let \( l_i = 0 \) if the \( i \)th path does not intersect \( B_{2j+1} \). Without loss of generality, the order of target points can be chosen in such a way that \( l_1 \leq l_2 \leq l_3 \), and hence \( l_1 + l_2 \leq 2l_3 \). Repeating the same argument for all non-empty meta-beads, we prove the claim.

Finally, we can summarize these intermediate bounds into the main result of this section. We let \( L_{\text{RBT A}, \rho}(P) \) denote the length of the path computed by the \textsc{Recursive Bead-Tiling Algorithm} for a point set \( P \).

Theorem 3.7 (Path length for the \textsc{Recursive Bead-Tiling Algorithm}): Let \( P \in \mathcal{P}_n \) be uniformly randomly generated in the rectangle of width \( W \) and height \( H \). For any \( \rho > 0 \), with high probability

\[
\lim_{n \to +\infty} \frac{\text{DTSP}_\rho(P)}{n^{2/3}} \leq \lim_{n \to +\infty} \frac{L_{\text{RBT A}, \rho}(P)}{n^{2/3}} \leq 24\sqrt{\rho WH} \left( 1 + \frac{7}{3\pi} \frac{\rho}{W} \right).
\]

Proof: For simplicity we let \( L_{\text{RBT A}, \rho}(P) = L_{\text{RBT A}} \). Clearly, \( L_{\text{RBT A}} = L'_{\text{RBT A}} + L''_{\text{RBT A}} \), where \( L'_{\text{RBT A}} \) is the path length of the first \( \lfloor \log_2 n \rfloor \) phases of the \textsc{Recursive Bead-Tiling Algorithm} and \( L''_{\text{RBT A}} \) is the length of the path required to visit all remaining targets. An immediate consequence of Lemma 3.6 is that

\[
L'_{\text{RBT A}} = \sum_{i=1}^{\lfloor \log_2(n) \rfloor} L_i \leq 3 \sum_{j=1}^{\lfloor \log_2(n)/2 \rfloor} L_{2j-1}.
\]

The summation on the right hand side of this equation can be expanded using Lemma 3.8.
yielding

\[
L'_{\text{RBTA}} \leq 3 \left\{ \frac{\rho WH}{\ell^2} \left( 1 + \frac{7\pi\rho}{3W} \right) + \rho \cdot o \left( \frac{\rho^2}{\ell^2} \right) \sum_{j=1}^{\log_2(n)/2} 2^{5-j} + \left( 32\frac{\rho H}{\ell} + \rho \cdot o \left( \frac{\rho}{\ell} \right) \right) \left[ \log_2 n \right] + [3\ell + \rho \cdot o(\ell/\rho)] \sum_{j=1}^{\log_2(n)/2} 2^j \right\}.
\]

Since \( \sum_{j=1}^{k} 2^{-j} \leq \sum_{j=1}^{+\infty} 2^{-j} = 1 \), and \( \sum_{j=1}^{k} 2^j = 2^{k+1} - 2 \leq 2^{k+1} \), the previous equation can be simplified to

\[
L'_{\text{RBTA}} \leq 3 \left\{ 32 \left[ \frac{\rho WH}{\ell^2} \left( 1 + \frac{7\pi\rho}{3W} \right) + \rho \cdot o \left( \frac{\rho}{\ell} \right) \right] + \left( 32\frac{\rho H}{\ell} + \rho \cdot o \left( \frac{\ell}{\rho} \right) \right) \left[ \log_2 n \right] + [3\ell + \rho \cdot o(\ell/\rho)] \cdot (4\sqrt{n}) \right\}.
\]

Recalling that \( \ell = 2(\rho WH/n)^{1/3} + o(n^{-1/3}) \) for large \( n \), the above can be rewritten as

\[
L'_{\text{RBTA}} \leq 24\sqrt{\rho WH n^2} \left( 1 + \frac{7}{3} \frac{\rho}{W} \right) + o(n^{2/3}).
\]

Now it suffices to show that \( L''_{\text{RBTA}} \) is negligible with respect to \( L'_{\text{RBTA}} \) for large \( n \) with high probability. From Theorem 3.3, we know that with high probability there will be at most \( 24 \log_2 n \) unvisited targets after the \( \lceil \log_2 n \rceil \) recursive phases. From [13] we know that, with high probability, the length of a alternating algorithm tour through these points satisfies

\[
L''_{\text{RBTA}} \leq \kappa \left[ 12 \log_2 n \right] \pi \rho + o(\log_2 n).
\]

In order to obtain an upper bound on the DITSP(\( P \)), we derive the expression for time taken, \( T_{\text{RecBT A}} \), by the recbta to execute the path of length \( L_{\text{RBTA}, \rho}(P) \) and then optimize it with respect to \( \rho \). Based on this calculation, we get the following result.

**Theorem 3.8:** (Upper bound on the total time in \( \mathbb{R}^2 \)) Let \( P \in \mathcal{P}_n \) be uniformly randomly generated in the rectangle of width \( W \) and height \( H \). For any double integrator \( \Xi \), with high probability

\[
\lim_{n \to +\infty} \frac{T_{\text{RecBT A}}}{n^{2/3}} \leq 24 \left( \frac{WH}{r_{\text{vel}}r_{\text{ctr}}} \right)^{1/3} \left( 1 + \frac{7\pi r_{\text{vel}}^2}{3W} \right).
\]

**Remark 3.9:** Theorems 3.1 and 3.8 imply that, with high probability, the recbta is a \( \frac{32}{\sqrt{6}} \left( 1 + \frac{7\pi r_{\text{vel}}^2}{3\text{cei}W} \right) \)-factor approximation (with respect to \( n \)) to the optimal stochastic DITSP in \( \mathbb{R}^2 \) and that \( \mathbb{E}[\text{DITSP}(P \subset Q \subset \mathbb{R}^2)] \) belongs to \( \Theta(n^{2/3}) \).
3) The 3D case: The Recursive Cylinder-Covering Algorithm (RecCCA): Consider a covering of $Q \in \mathbb{R}^3$ by cylinders such that $\text{Volume}[C_\rho(\ell)] = \frac{\text{Volume}[Q \subset \mathbb{R}^3]}{4n} = \frac{WHD}{4n}$ (Again implying that $n$ is sufficiently large). Furthermore, the covering is chosen in such a way that it is aligned with the sides of $Q \subset \mathbb{R}^3$.

The proposed algorithm will consist of a sequence of phases; each phase will consist of five sub-phases, all similar in nature. For the first sub-phase of the first phase, a feasible curve is constructed with the following properties:

(i) it visits all non-empty cylinders once,
(ii) it visits all rows of cylinders in a layer in sequence top-to-down in a layer, alternating between left-to-right and right-to-left passes, and visiting all non-empty cylinders in a row,
(iii) it visits all layers in sequence from one end of the region to the other,
(iv) when visiting a non-empty cylinder, it services at least one target in it.

In subsequent sub-phases, instead of considering single cylinders, we will consider “meta-cylinders” composed of 2, 4, 8 and 16 beads each for the remaining four sub-phases, as shown in Figure 5 and proceed in a similar way as the first sub-phase, i.e., a feasible curve is constructed with the following properties:

(i) the curve visits all non-empty meta-cylinders once,
(ii) it visits all (meta-cylinder) rows in sequence top-to-down in a (meta-cylinder) layer, alternating between left-to-right and right-to-left passes, and visiting all non-empty meta-cylinders in a row,

Fig. 5. Starting from top left in the left-to-right, top-to-bottom direction, sketch of projection of “meta-cylinders” on the corresponding side of $Q \subset \mathbb{R}^3$ at second, third, fourth and fifth sub-phases of a phase in the recursive cylinder covering algorithm.
(iii) it visits all (meta-cylinder) layers in sequence from one end of the region to the other,
(iv) when visiting a non-empty meta-cylinder, it services at least one target in it.

A meta-cylinder at the end of the fifth sub-phase, and hence at the end of the first phase will
consist of 16 nearby cylinders. After this phase, the transitioning to the next phase will involve
enlarging the cylinder to 32 times its current size by increasing the radius of its cross section by
a factor of 4 and doubling its length as outlined in Figure 3(b). It is easy to see that this bigger
cylinder will contain the union of 32 nearby smaller cylinders. In other words, we are forming
the object $C_\rho(2\ell)$ using a conglomeration of 32 $C_\rho(\ell)$ objects. This whole process is repeated at
most $\log_2 n + 2$ times. After the last phase, the leftover targets will be visited using, for example,
a greedy strategy.

We have the following results, which are similar to the one for the RECURSIVE BEAD-TILING
ALGORITHM.

**Theorem 3.10 (Targets remaining after recursive phases):** Let $P \in \mathcal{P}_n$ be uniformly randomly
generated in $Q \subset \mathbb{R}^3$. The number of unvisited targets after the last recursive phase of the
RECURSIVE CYLINDER-COVERING ALGORITHM over $P$ is less than $24 \log_2 n$ with high prob-
ability, i.e., with probability approaching one as $n \to +\infty$.

**Lemma 3.11 (Path length for the first sub phase):** Consider a covering of the space with cylin-
ders $C_\rho(\ell)$. For any $\rho > 0$ and for any set of target points, the length $L_I$ of a path executing the
first sub-phase of the RECURSIVE CYLINDER-COVERING ALGORITHM in a rectangular box $Q$
of width $W$, height $H$ and depth $D$ satisfies

$$L_I \leq \frac{1024 \rho^2 WHD}{\ell^4} \left(1 + \frac{7}{3} \pi \frac{\rho}{W}\right) + \rho \cdot o \left(\frac{\rho^3}{\ell^3}\right).$$

**Proof:** A path visiting each cylinder once can be constructed by a sequence of passes,
during which all cylinders in a row are visited by making left-to-right and then right-to-left
passes. This is done for all the rows of cylinders. In each row, there are at most $\lceil W/\ell \rceil + 1$
cylinders encountered in one pass. Hence, the cost of each pass is at most:

$$L_I^{\text{pass}} \leq W + 2\ell + \rho \cdot o \left(\frac{\ell^2}{\rho^2}\right).$$

In order to visit all cylinders in a row, the vehicle needs to make two passes through that row
and the paths for these two passes are connected by a u-turn path whose length is $\frac{7}{3} \pi \rho + \frac{\ell}{2}$. 
Therefore the length of the path required to visit all cylinders in one row is:

\[ L_{I}^{\text{row}} \leq 2W + \frac{9}{2} \ell + \frac{7}{3} \pi \rho + \rho \cdot o \left( \frac{\ell^2}{\rho^2} \right). \]

During the transition from one row to another, the vehicle needs to make a U-turn maneuver, in which the direction of travel is reversed, and the path moves to the next row, at distance equal to the diameter of the cylinder. Since the length of the shortest path to reverse the heading of the vehicle with co-located initial and final points is \((7/3)\pi \rho\), the length of the U-turn satisfies

\[ L_{I}^{\text{U-turn}} \leq \frac{7}{3} \pi \rho + \frac{1}{2} w(\ell) \leq \frac{7}{3} \pi \rho + \frac{\ell^2}{16 \rho} + \rho \cdot o \left( \frac{\ell^3}{\rho^3} \right). \tag{7} \]

The total number of rows, i.e., the total number of rows of cylinders with non-empty intersection with \(Q\), satisfies

\[ N_{I}^{\text{row}} \leq \left\lceil \frac{2H}{w(\ell)} \right\rceil + 1 \leq \frac{16 \rho H}{\ell^2} + o \left( \frac{\rho}{\ell} \right). \]

During the transition from one row to another, the vehicle needs to make a U-turn maneuver whose length satisfies the same bound as in Eq. (7). The total number of layers of cylinders satisfies

\[ N_{I}^{\text{layer}} \leq \left\lceil \frac{4D}{w(\ell)} \right\rceil + 1 \leq \frac{32 \rho D}{\ell^2} + o \left( \frac{\rho}{\ell} \right). \]

A simple upper bound on the cost of closing the tour is given by

\[ L_{I}^{\text{close}} \leq (W + 2\ell) + (H + 2w(\ell)) + (D + w(\ell)) + 2\pi \rho = W + H + D + 2\pi \rho + 2\ell + \rho \cdot o(\ell/\rho). \]

In summary, the total length of the path followed during the first sub-phase is

\[ L_{I} \leq N_{I}^{\text{layer}} \left( N_{I}^{\text{row}} \left( L_{I}^{\text{row}} + L_{I}^{\text{U-turn}} \right) + L_{I}^{\text{U-turn}} \right) + L_{I}^{\text{close}} \leq \frac{1024 \rho^2 W H D}{\ell^4} \left( 1 + \frac{7}{3} \pi \frac{\rho}{W} \right) + \rho \cdot o \left( \frac{\rho^3}{\ell^3} \right). \]

Based on this calculation, we can estimate the length of the paths in subsequent sub-phases.
The length of path to execute the first phase is then the sum of the path lengths for these five sub-phases.

**Lemma 3.12 (Path length at the first phase):** Consider a covering of the space with cylinders $C_{\rho}(\ell)$. For any $\rho > 0$ and for any set of target points, the length $L_1$ of a path visiting once and only once each cylinder with a non-empty intersection with a rectangular box $Q$ of width $W$, height $H$ and depth $D$ satisfies

$$L_1 \leq \frac{3328 \rho^2 WHD}{\ell^4} \left(1 + \frac{7}{3} \pi \frac{\rho}{W} + \rho \cdot o\left(\frac{\rho^3}{\ell^3}\right)\right).$$

Since we increase the length of cylinders by a factor of two while doing the phase transition from one phase to the another, the length of path for the subsequent $i^{th}$ phase is given by:

$$L_i \leq \frac{3328 \rho^2 WHD}{16 \ell^4} \left(1 + \frac{7}{3} \pi \frac{\rho}{W} + \rho \cdot o\left(\frac{\rho^3}{\ell^3}\right)\right).$$

Finally, we can summarize these intermediate bounds into the main result of this section. We let $L_{\text{RCFA},\rho}(P)$ denote the length of the path computed by the RECURSIVE CYLINDER-COVERING ALGORITHM for a point set $P$.

**Theorem 3.13 (Path length for the RECURSIVE CYLINDER-COVERING ALGORITHM):** Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangle of width $W$, height $H$ and depth $D$. For any $\rho > 0$, with high probability

$$\lim_{n \to +\infty} \frac{\text{DITSP}(P \subset Q \subset \mathbb{R}^3)}{n^{4/5}} \leq \lim_{n \to +\infty} \frac{L_{\text{RCFA},\rho}(P)}{n^{4/5}} \leq \frac{3328}{15} \left(\frac{\pi}{16}\right)^{4/5} (\rho^2 WHD)^{1/5}.$$

**Proof:** Clearly,

$$L_{\text{RCFA}} = \sum_{i=1}^{\lfloor \log_2(2n+7) \rfloor} \left(\frac{3328 \rho^2 WHD}{16 \ell^4} \left(1 + \frac{7}{3} \pi \frac{\rho}{W} + \rho \cdot o\left(\frac{\rho^3}{\ell^3}\right)\right)\right)$$

$$\leq \frac{53248 \rho^2 WHD}{15 \ell^4} \left(1 + \frac{7}{3} \pi \frac{\rho}{W} + \rho \cdot o\left(\frac{\rho^3}{\ell^3}\right)\right).$$

Recalling that $\ell = 2 \left(\frac{16 \rho^2 WHD}{\pi n}\right)^{1/5} + o(n^{-1/5})$ for large $n$, the above can be rewritten as

$$L_{\text{RCFA}} \leq \frac{3328}{15} \left(\frac{\pi}{16}\right)^{4/5} (\rho^2 WHD)^{1/5} \left(1 + \frac{7}{3} \pi \frac{\rho}{W}\right) n^{4/5} + o(n^{4/5}).$$

**Theorem 3.14: (Upper bound on the total time in $\mathbb{R}^3$)** Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangular box of width $W$, height $H$ and depth $D$. For any double integrator
with high probability

\[
\lim_{n \to +\infty} \frac{T_{\text{RecCCA}}}{n^{4/5}} \leq 61 \left( \frac{WHD}{r_{\text{ctr}}^2 r_{\text{vel}}} \right)^{1/5} \left( 1 + \frac{7\pi r_{\text{vel}}^2}{3W r_{\text{ctr}}} \right).
\]

**Remark 3.15:** Theorems 3.1 and 3.14 imply that, with high probability, the RecCCA is a 50 \( \left( 1 + \frac{7\pi r_{\text{vel}}^2}{3W r_{\text{ctr}}} \right) \)-factor approximation (with respect to \( n \)) to the optimal stochastic DITSP in \( \mathbb{R}^3 \) and that \( \mathbb{E}[\text{DITSP}(P \subset Q \subset \mathbb{R}^3)] \) belongs to \( \Theta(n^{4/5}) \).

**IV. The DTRP for Double Integrator**

We now turn our attention to the Dynamic Traveling Repairperson Problem (DTRP) that was introduced in [6] and that we here tackle for a double integrator.

**A. Model and problem statement**

In the DTRP the double integrator is required to visit a dynamically growing set of targets, generated by some stochastic process. We assume that the double integrator has unlimited range and target-servicing capacity and that it moves at a unit speed with minimum turning radius \( \rho > 0 \).

Information about the outstanding targets representing the demand at time \( t \) is described by a finite set \( n(t) \) of positions \( \mathcal{D}(t) \). Targets are generated, and inserted into \( \mathcal{D} \), according to a time-invariant spatio-temporal Poisson process, with time intensity \( \lambda > 0 \), and uniform spatial density inside the region \( Q \), which we continue to assume to be a rectangle for two dimensions and a rectangular box for three dimensions. Servicing of a target and its removal from the set \( \mathcal{D} \), is achieved when the double integrator moves to the target position. A control policy \( \Phi \) for the DTRP assigns a control input to the vehicle as a function of its configuration and of the current outstanding targets. The policy \( \Phi \) is a stable policy for the DTRP if, under its action

\[
n_{\Phi} = \lim_{t \to +\infty} \mathbb{E}[n(t)|\dot{p} = \Phi(p, \mathcal{D})] < +\infty,
\]

that is, if the double integrator is able to service targets at a rate that is, on average, at least as fast as the rate at which new targets are generated.

Let \( T_j \) be the time elapsed from the time the \( j^{\text{th}} \) target is generated to the time it is serviced and let \( T_{\Phi} := \lim_{j \to +\infty} \mathbb{E}[T_j] \) be the steady-state system time for the DTRP under the policy \( \Phi \). (Note that if the system is stable, then it is known [17] that \( n_{\Phi} = \lambda T_{\Phi} \).) Clearly, our objective is to design a policy \( \Phi \) with minimal system time \( T_{\Phi} \).
B. Lower and constructive upper bounds

In what follows, we design control policies that provide constant-factor approximation of the optimal achievable performance. Consistently with the theme of the paper, we consider the case of heavy load, i.e., the problem as the time intensity \( \lambda \to +\infty \). We first provide lower bounds for the system time, and then present novel approximation algorithms providing upper bound on the performance.

**Theorem 4.1 (Lower bound on the DTRP system time):** For a double integrator, the system time \( T_{DTRP,2} \) and \( T_{DTRP,3} \) for the DTRP in two and three dimensions satisfy

\[
\lim_{\lambda \to \infty} \frac{T_{DTRP,2}}{\lambda^2} \geq \frac{81}{32} \frac{WH}{r_{vel} r_{ctr}}, \quad \lim_{\lambda \to \infty} \frac{T_{DTRP,3}}{\lambda^4} \geq \frac{7813 WHD}{972 r_{vel}^2 r_{ctr}^2}.
\]

**Proof:** We prove the lower bound on \( T_{DTRP,2} \); the bound on \( T_{DTRP,3} \) follows on similar lines. Let us assume that a stabilizing policy is available. In such a case, the number of outstanding targets approaches a finite steady-state value, \( n^* \), related to the system time by Little’s formula, i.e., \( n^* = \lambda T_{DTRP,2} \). In order for the policy to be stabilizing, the time needed, on average, to service \( m \) targets must be no greater than the average time interval in which \( m \) new targets are generated. The average time needed by the double integrator to service one target is no greater than the expected minimum time from an arbitrarily placed vehicle to the closest target; in other words, we can write the stability condition \( E[t^*(n^*)] \leq 1/\lambda \). A bound on the expected value of \( t^* \) has been computed in the proof of Theorem 3.1, yielding

\[
\frac{3}{4} \left( \frac{6WH}{r_{vel} r_{ctr} n} \right)^{1/3} \leq E[t^*(n^*)] \leq 1/\lambda.
\]

Using Little’s formula \( n^* = \lambda T_{DTRP,2} \), and rearranging, we get the desired result.

We now propose simple strategies, the BEAD TILING ALGORITHM (for \( \mathbb{R}^2 \)) and the CYLINDER COVERING ALGORITHM (for \( \mathbb{R}^3 \)), based on the concepts introduced in the previous section. The BEAD TILING ALGORITHM (BTA) strategy consists of the following steps:

(i) Tile the plane with beads of length \( \ell := \min\{C_{BTA}/\lambda, 4\rho\} \), where

\[
C_{BTA} = 0.5241 r_{vel} \left( 1 + \frac{7\pi \rho}{3W} \right)^{-1}.
\]

(ii) Traverse all non-empty beads once, visiting one target per non-empty bead. Repeat this step.
The Cylinder Covering Algorithm (CCA) strategy is akin to the BTA, where the region is covered with cylinders constructed from beads of length \( \ell := \min\{C_{CFA}/\lambda, 4\rho\} \), where

\[
C_{CCA} = 0.1615 r_{vel} \left( 1 + \frac{7\pi\rho}{3W} \right)^{-1}.
\]

The policy is then to traverse all non-empty cylinders once, visiting one target per non-empty cylinder. The following result characterizes the system time for the closed loop system induced by these algorithms and is based on the bounds derived to arrive at Theorems 3.8 and 3.14.

**Theorem 4.2 (Upper bound on the DTRP system time):** For a double integrator (1) and \( \lambda > 0 \), the BTA and the CCA are stable policies for the DTRP and the resulting system times \( T_{BTA} \) and \( T_{CFA} \) satisfy:

\[
\lim_{\lambda \to \infty} T_{DTRP,2} = \lim_{\lambda \to \infty} \frac{T_{BTA}}{\lambda^2} \leq 70.5 \frac{WH}{r_{vel} r_{ctr}} \left( 1 + \frac{7\pi r_{vel}^2}{3W r_{ctr}} \right)^3.
\]

\[
\lim_{\lambda \to \infty} T_{DTRP,3} = \lim_{\lambda \to \infty} \frac{T_{CFA}}{\lambda^4} \leq 2 \cdot 10^7 \frac{WH \rho}{r_{vel}^2 r_{ctr}} \left( 1 + \frac{7\pi \rho}{3W} \right)^5.
\]

**Proof:** We prove the upper bound on \( T_{DTRP,2} \); the upper bound on \( T_{DTRP,3} \) follows on similar lines. Consider a generic bead \( B \), with non-empty intersection with \( Q \). Target points within \( B \) will be generated according to a Poisson process with rate \( \lambda_B \) satisfying

\[
\lambda_B = \frac{\lambda \text{Area}(B \cap Q)}{WH} \leq \lambda \frac{\text{Area}(B)}{WH} = \frac{C_{BTA}^3}{16\rho WH \lambda^2} + o \left( \frac{1}{\lambda^2} \right).
\]

The vehicle will visit \( B \) at least once every \( T_{REC,BTA,1} \) time units, where \( T_{REC,BTA,1} \) is the bound on the time required to traverse a path of length \( L_1 \), as computed in Lemma 3.4. As a consequence, targets in \( B \) will be visited at a rate no smaller than

\[
\mu_B = \frac{C_{BTA}^2 r_{vel}}{16\rho WH \lambda^2} \left( 1 + \frac{7\pi \rho}{3W} \right)^{-1} + o \left( \frac{1}{\lambda^2} \right).
\]

In summary, the expected time \( T_B \) between the appearance of a target in \( B \) and its servicing by the vehicle is no more than the system time in a queue with Poisson arrivals at rate \( \lambda_B \), and deterministic service rate \( \mu_B \). Such a queue is called a \( M/D/1 \) queue in the literature [17], and its system time is known to be

\[
T_{M/D/1} = \frac{1}{\mu_B} \left( 1 + \frac{1}{2} \frac{\lambda_B}{\mu_B - \lambda_B} \right).
\]

Using the computed bounds on \( \lambda_B \) and \( \mu_B \), and taking the limit as \( \lambda \to +\infty \), we obtain

\[
\lim_{\lambda \to +\infty} \frac{T_B}{\lambda^2} \leq \lim_{\lambda \to +\infty} \frac{T_{M/D/1}}{\lambda^2} \leq \frac{16\rho WH}{C_{BTA}^2 r_{vel} \left( 1 + \frac{7\pi \rho}{3W} \right)} \left( 1 + \frac{1}{2} \frac{C_{BTA}}{r_{vel} \left( 1 + \frac{7\pi \rho}{3W} \right)} \right).
\]
Since equation (9) holds for any bead intersecting \(Q\), the bound derived for \(T_B\) holds for all targets and is therefore a bound on \(T_{DTRP,2}\). The expression on the right hand side of (9) is a constant that depends on problem parameters \(\rho\), \(W\), and \(H\), and on the design parameter \(C_{BTA}\), as defined in equation (8). Stability of the queue is established by noting that \(C_{BTA} < r_{vel}(1 + 7/3 \pi \rho/W)^{-1}\). Additionally, the choice of \(C_{BTA}\) in equation (8) minimizes the right hand side of (9) yielding the numerical bound in the statement. We then substitute \(\rho = r_{vel}^2/r_{ctr}\) to yield the final result.

Remark 4.3: Note that the achievable performances of the BTA and the CCA provide a constant-factor approximation to the lower bounds established in Theorem 4.1.

V. EXTENSION TO THE TSPs FOR THE DUBINS VEHICLE

In our earlier works [13], [18], [12], we have studied the TSP for the Dubins vehicle in the planar case. In [13], we proved that in the worst case, the time taken to complete a TSP tour by the Dubins vehicle will belong to \(\Theta(n)\). One could shown that this result holds true even in \(\mathbb{R}^3\). In [12], the first known algorithm with strictly sublinear asymptotic minimum time for tour traversal was proposed for the stochastic DTSP in \(\mathbb{R}^2\). This algorithm was modified in [12] to give a constant factor approximation to the optimal with high probability. This naturally lead to a stable policy for the DTRP problem for the Dubins vehicle in \(\mathbb{R}^2\) which also performed within a constant factor of the optimal with high probability. The RECCA developed in this paper can naturally be extended to apply to the stochastic DTSP in \(\mathbb{R}^3\). It follows directly from Lemma 3.2 that in order to use the RECCA for a Dubins vehicle with minimum turning radius \(\rho\), one has to simply compute feasible curves for double integrator moving with a constant speed \(\sqrt{\rho r_{ctr}}\). Hence the results stated in Theorem 3.14 and Theorem 4.2 also hold true for the Dubins vehicle.

This equivalence between trajectories makes the RECCA the first known strategy with a strictly sublinear asymptotic minimum time for tour traversal for stochastic DTSP in \(\mathbb{R}^3\). The fact that it performs within a constant factor of the optimal with high probability and that it gives rise to a constant factor approximation and stabilizing policy for DTRP for Dubins vehicle in \(\mathbb{R}^3\) is also novel.
VI. CONCLUSIONS

In this paper we have proposed novel algorithms for various TSP problems for vehicles with double integrator dynamics. Future directions of research include extensive simulations to support the results obtained in this paper, study of centralized and decentralized versions of the DTRP, and more general task assignment and surveillance problems for vehicles with nonlinear dynamics.

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