Origin of hydrodynamic instability from noise: from laboratory flow to accretion disk

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We attempt to address the old problem of plane shear flows: the origin of turbulence and hence transport of angular momentum in accretion flows as well as laboratory flows, such as plane Couette flow. We undertake the problem by introducing an extra force in Orr-Sommerfeld and Squire equations along with the Coriolis force mimicking the local region of the accretion disk. For plane Couette flow, the Coriolis term drops. Subsequently we solve the equations by WKB approximation method. We investigate the dispersion relation for the Keplerian flow and plane Couette flow for all possible combinations of wave vectors. Due to the very presence of extra force, we show that both the flows are unstable for a certain range of wave vectors. However, the nature of instability between the flows is different. We also study the Argand diagrams of the perturbation eigenmodes. It helps us to compare the different time scales corresponding to the perturbations as well as accretion. We ultimately conclude with this formalism that fluid gets enough time to be unstable and hence plausibly turbulent particularly in the local regime of the Keplerian accretion disks. Repetition of the analysis throughout the disk explains the transport of angular momentum and matter along outward and inward direction respectively.

I. INTRODUCTION

A long-standing mismatch between theory and experiment regarding the transition from laminar to turbulent flows for laboratory fluids, e.g. plane Couette flow and plane Poiseuille flow, is there in literatures. The linear theory of perturbation says that plane Poiseuille flow becomes unstable beyond Reynolds number (Re) 5772.22 [1] and, on the other hand, plane Couette flow is stable for any Re [2]. However, according to experiments/simulations, beyond Re ∼ 1000 [3,4] and Re ∼ 350 [5,6] the laminar flow becomes turbulent in case of plane Poiseuille flow and plane Couette flow respectively. The similar kind of mismatch is there in the context of astrophysics particularly in case of accretion disks. Accretion disks are astrophysical objects formed around a denser object mainly in the form of a disk. Nevertheless, the accretion disk involves very sophisticated (or rich) physics behind the formation and evolution of its various parts depending on the nature of the central objects (black holes, white dwarfs, neutron stars, main sequence stars, etc) around which the matter accretes in the form of a disk. The physics also involves with the nature of mass supply (e.g. mass supplied from evolved stars, from interstellar medium, molecular cloud, etc) that helps accretion around the central object. However, in this work, we shall be discussing a geometrically thin and optically thick disk, where the accreting matter almost follows Kepler’s law, i.e. the fluid particle in the corresponding flow revolves around the central object at a particular radius due to the almost balance between inward gravitational force and outward centrifugal force. The flow therefore is called Keplerian flow. The change in the angular momentum per unit mass of the fluid particle, therefore, occurs in increasing proportion to the square root of the radial distance of the particle. Due to the very nature of the Keplerian rotation, the perturbation of the fluid particle decays down and eventually the particle returns to its initial position. This is called Rayleigh stability. The Keplerian flow, therefore, is Rayleigh stable.

However, due to the Keplerian rotation, two fluid layers across the radial direction in the disk will have different angular velocities. Since the flow has differential velocity across the radial direction, molecular viscosity comes into picture. However, observational evidences, e.g. temperature, luminosity, etc. from the Keplerian accretion disk do not support the molecular viscosity as the origin of matter transport. The molecular viscosity is so weak that it cannot transport the angular momentum outward and matter inward and hence cannot explain the observables [7]. The belief is that it is the turbulent viscosity which is behind the transport. The idea was put forward by Shakura & Sunyaev [8] and Lynden-Bell & Pringle [9] without explicitly revealing the reason behind the turbulence. In 1991, Balbus & Hawley [10] came up with an idea of instability mechanism due to the interplay between weak magnetic field and the rotation of the fluid parcel, naming magneto-rotational instability (MRI), following the idea of Velikhov [11] and Chandrasekhar [12]. In spite of the overwhelming success of MRI in explaining the origin of turbulence, it is not out of caveats. In the colder systems, e.g. protoplanetary disk [13,14], cataclysmic variables in their low states [15,16], the outer part of active galactic nucleus (AGN) disks and the underlying dead zone [17], where the ionization is very small such that matter cannot be coupled with the magnetic field, MRI gets suppressed. It is not only the low ionization that challenges MRI, there are, in fact, a lot of other examples too. Nath & Mukhopadhyay [15] argued that it is the magnetic transient growth that brings nonlinearity and hence plausible turbulence in the system beyond Reynolds number (Re) 10⁹, since their growth rate is faster than MRI in that regime. Usually,
$Re$ in accretion disks \cite{19} is larger than this value, hence the relevance of MRI in large $Re$ systems is questionable. As a general interest, the transient energy growth in the case of magnetohydrodynamical shear flows (with viscosity and resistivity included) was studied further by Bhatia & Mukhopadhyay \cite{20}. They showed that even transient energy growth ceased to occur beyond certain magnetic field. In addition to this, Pessah & Psaltis \cite{21} and Das et al. \cite{22}, using local and global analysis respectively, showed the stabilization of the axisymmetric MRI above a certain magnitude of a toroidal component of the magnetic field for compressible and differentially rotating flows. It is, therefore, of great concern whether there is any instability in the system from hydrodynamical origin.

However, in the literature \cite{23–34}, there is a long standing debate regarding the stability of Rayleigh stable flows, particularly in the context of accretion disks. Approximating the local hot accretion flow to be shearing sheet, people \cite{35–36} attempted, analytically and with simulation, to resolve the issue without considering viscosity. They concluded that the sustained turbulence and hence outward transport of angular momentum were not possible in the Keplerian flow if hydrodynamics was considered only. However, Lesur & Longaretti \cite{37} with shearing sheet approximation and considering viscosity strongly disagreed with the aforementioned authors and claimed that the absence of turbulence in the simulation in the above mentioned works was resolution issue. Although they agreed that there was lack of computer resources to resolve the Keplerian regime, their extrapolated numerical data could not produce astrophysically sufficient subcritical turbulent transport in the Keplerian flow. Pumir \cite{38} claimed for sustained turbulence if the mean flow is plane Couette typed. However, they did not consider rotational effects. Fromang and Papaloizou \cite{39}, though did magnetohydrodynamical (MHD) simulation, argued for considering explicit diffusion coefficients: both resistive and viscous, whose effect is stronger than numerical dissipation effect, before making any conclusion based on MHD simulation. Therefore, we notice that in all of these works some important physics are missing, i.e., viscosity \cite{35–36}, resolution of the Keplerian region \cite{37}, the Coriolis force \cite{38}, explicit diffusion coefficients (both viscous and resistive) \cite{39} are not adequately considered. Even if we have well-resolved simulations \cite{19–22}, the previously mentioned facts or parameter regions exist, where MRI is inapplicable/insufficient as an instability mechanism. Nevertheless, the authors argued for plausible emergence of hydrodynamics instability and hence further turbulence by experiment (e.g. \cite{13}), simulations in the context accretion disks (e.g. \cite{14}), transient growth in the case of otherwise linearly stable flows (e.g. \cite{33, 45, 47}).

We, therefore, search for a hydrodynamical origin of nonlinearity and hence plausible turbulence in the accretion disk. We, in particular, consider an extra force in this work and the force has stochastic origin. The existence and consequences of the stochastic force in the hydrodynamical systems were initiated by Mukhopadhyay & Chattopadhyay \cite{44} inspired by the idea of Nelson & Foster \cite{48} and DeDominicis & Martin \cite{49}. They showed that the presence of the stochastic force in the rotating shear flows in a narrow gap limit reveals large correlation of energy growth of the perturbation. Later, Nath & Mukhopadhyay \cite{50} obtained the dispersion relation of the linear perturbations considering stochastic force in the Orr-Sommerfeld and Squire equations, describing the fluid flow in a small radial patch of accretion disk. However, they considered plane wave perturbation with constant amplitude as the trial solution of the Orr-Sommerfeld and Squire equations. In the present work, we consider three-dimensional perturbations and WKB approximation to obtain the solutions for Orr-Sommerfeld and Squire equations. While qualitatively we obtain similar result as Nath & Mukhopadhyay \cite{50}, it brings new quantitative insight which is useful to infer observed data and/or experimental results based on our model. We also obtain the Argand diagrams corresponding to the perturbations and these are necessary to compare the timescales corresponding to the growth with that of oscillation of the perturbations. In addition to this, we also confirm whether the fluid parcel inside the shearing box within a small patch of accretion disk gets enough time to enter into the nonlinear regime and hence becomes turbulent within the timescale it came across the box. However, for plane Couette flow, we do not need to worry about any such time scale, as there is no radial infall.

The plan of the paper is the following. In \hspace{1em}§IV \hspace{1em}we describe the governing equations which are Orr-Sommerfeld and Squire equations in the presence of Coriolis force and noise for linearly perturbed flow inside a shearing box at a smaller patch of accretion disk. We then write them in the Fourier space to obtain a general dispersion relation. In \hspace{1em}§V \hspace{1em}the dispersion relation is studied extensively for the Keplerian and plane Couette flows. The Argand diagrams corresponding to the linear perturbations in the case of Keplerian flow are studied in \hspace{1em}§VI \hspace{1em}for various parameters. In the end, we discuss about the plausibility of occurrence of instability which could further lead to nonlinearity and hence turbulence in the context of accretion disks and laboratory flows, e.g. plane Couette flows in \hspace{1em}§V. We finally conclude in \hspace{1em}§VI \hspace{1em}that our model is able to explain the origin of instability and hence turbulence in the context of accretion disk as well as plane Couette flow.

\section{Formalism}

The detailed description of the local formulation can be found in Mukhopadhyay et al. \cite{45} and also in Bhatia & Mukhopadhyay \cite{20}. The schematic diagram of the background flow inside the shearing box is shown in \hspace{1em}§VI \hspace{1em}. As the fluid is in the local region, we assume the fluid to
be incompressible \[\text{[18] [40]}\). There we recast the Navier-Stokes equation in Orr-Sommerfeld and Squire equations in the presence of Coriolis force and extra force, by eliminating the pressure term from different components of the Navier-Stokes equation and utilizing the continuity equation for incompressible flow \[\text{[50]}\]. The ensemble averaged Orr-Sommerfeld and Squire equations in the presence of Coriolis force and extra force are given by

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right) \nabla^2 u - \frac{\partial^2 U}{\partial x^2} \frac{\partial u}{\partial y} + \frac{2 \partial \zeta}{q} = \frac{1}{Re} \nabla^4 u + \eta_1, \tag{1} \]

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right) \zeta - \frac{\partial U}{\partial x} + \frac{2 \partial \zeta}{q} \frac{\partial u}{\partial z} = \frac{1}{Re} \nabla^2 \zeta + \eta_2, \tag{2} \]

where \(U = -x\) is the \(y\)-component of background velocity. The other components of background velocity are zero; \(u, \zeta\) are \(x\)-components of velocity and vorticity perturbations respectively; \(q\) is the rotation parameter which describes the radial dependence of the angular frequency of fluid element around the central object, given by \(\Omega \propto q\). The ensemble averaged frequency of fluid element around the central object, given

\[
\frac{\partial}{\partial t} u = (\alpha + \beta) u + \gamma \zeta + \frac{i}{Re} \nabla^2 u + \delta(k) \delta(\omega), \tag{4} \]

where the Fourier transform of \(\eta_i\) is \(m_i \delta(k) \delta(\omega)\) with \(m_i\) being the constant mean corresponding to \(\eta_i\). The traveling wave solutions for equations \([1]\) and \([2]\) are assumed to be

\[
\zeta = (x) e^{i(\alpha r - \beta t)}, \tag{8} \]

where the wave vector, \(\alpha\), is given by \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\), and \(\beta\) is the frequency. Usually \(\beta\) is a complex quantity and, according to our convention, if the imaginary part of \(\beta\), i.e. \(Im(\beta)\), is positive, then the perturbation grows with time. To obtain the dispersion relation, we transform equation \([3]\) in the Fourier space (see Appendix \([A]\)) and substitute them in the equations \([6]\) and \([7]\) and then we integrate with respect to \(\omega\) and \(k\). See Appendix \([A]\) for details. We further use WKB approximation to obtain the solution. Therefore, we neglect second and higher order derivatives, as they are varying slowly over the length \(1/\alpha_1\). The dispersion relations from equations \([6]\) and \([7]\) are then

\[
\left( i \beta \alpha^2 - \frac{\alpha^4}{Re} \right) u(0) + 2i \alpha_1 \left( \frac{2 \alpha^2}{Re} - i \beta \right) u'(0) + 2i \alpha_3 \left( 1 - \frac{2 \alpha}{q} \right) u(0) + \left( i \beta - \frac{\alpha^2}{Re} \right) \zeta(0) + 2i \alpha_1 \left( \frac{2 \alpha^2}{Re} - q \right) \zeta'(0) = 0, \tag{9} \]

Here, \(u(0)\) and \(u'(0)\) are respectively values of \(u(x)\) and \(u'(x)\) at \(x = 0\). We also consider the first order derivatives to be

\[
u(0) = \gamma u(0), \quad \zeta(0) = \gamma \zeta(0), \quad u'(0) = \gamma u'(0), \quad \zeta'(0) = \gamma \zeta'(0), \]

and the same strength for the extra forces, i.e. \(m_1 = m_2 = m\). Now if we eliminate \(\zeta\) with all the assumptions from equations \([6]\) and \([7]\), we obtain the dispersion relation, which is given by

\[
m \left( 2 \alpha_3 + \beta q + \frac{4i \alpha_2 q}{Re} + \frac{4\alpha_1 \gamma q}{Re} \right) u_0 + \left( 2 \alpha_3 + \beta q + \frac{4i \alpha_2 q}{Re} + \frac{4\alpha_1 \gamma q}{Re} \right) u_0 + \left( 2 \alpha_3 + \beta q + \frac{4i \alpha_2 q}{Re} + \frac{4\alpha_1 \gamma q}{Re} \right) u_0.
\tag{10} \]

For clarity, we consider \(\gamma = \pm 1, \pm \alpha_1, \pm \alpha_3\). However, only \(\gamma = i \alpha_1\) gives \(Im(\beta) < 0\) for any \(Re\) without extra force is considered, i.e. \(m = 0\), which is physical. We, therefore, stick to \(\gamma = i \alpha_1\) throughout the paper. For the computational purpose, we consider the components of wave vectors along \(y\)-direction to be zero, i.e. \(\alpha_2^2 = \alpha_1^2 + \alpha_3^2\). However, if we make \(\alpha_3 = 0\) and \(\alpha_2^2 = \alpha_1^2 + \alpha_2^2\), from equation \([6]\) it is clear that the problem will become qualitatively plane Couette flow.
III. DISPERSION RELATION

A. Keplerian flow

Here we shall study the solutions of equation (10) for different parameters. Equation (10) is a quadratic equation of \( \beta \) with complex coefficients. Among the two solutions of \( \beta \), the one which we are interested in is

\[
\beta = -\frac{0.5i}{3\alpha_1^2 q + \alpha_3^2} \left[ mq + \frac{14\alpha_1^2 q \alpha_3^2}{m/u_0} + \frac{12\alpha_0^2 \alpha_1^2 q^2}{m/u_0} \right] \\
+ \frac{2\alpha_3^4 q}{m/u_0} - \frac{1}{\frac{m}{u_0} - \frac{3\alpha_1^2 \alpha_3^2 \alpha_4^2 q^2}{u_0} + \frac{8\alpha_0^2 \alpha_1^2 q^2}{u_0}} - \frac{8\alpha_0^2 \alpha_1^2 q^2}{u_0} + \frac{16\alpha_1^4 q^2 + 24\alpha_0^2 \alpha_1^2 q^2}{u_0} \\
+ \frac{8\alpha_0^4 q^2 - 48\alpha_0^2 \alpha_1^2 q^2}{u_0} - \frac{16\alpha_1^4 q^2}{u_0} \right]^{\frac{1}{2}}. \tag{11}
\]

The other solution of \( \beta \) is always stable irrespective of extra force. However, equation (11) expectedly provides negative \( \text{Im}(\beta) \) for \( m = 0 \) irrespective of \( \text{Re} \). Interestingly, equation (11) also provides positive \( \text{Im}(\beta) \) within a particular window of \( \alpha_1 \) and \( \alpha_3 \) beyond certain \( m \) depending on \( \text{Re} \) for a fixed \( q \). Here we observe the dispersion relations, i.e. the variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) for different \( \text{Re} \) and \( m/u_0 \) for the Keplerian flow. FIGs. 1 and 2 show the variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) for the Keplerian and plane Couette flow (see III B) respectively for \( m/u_0 = 0 \). From linear stability analysis, we know these two flows are stable for any \( \text{Re} \) and this is confirmed in the FIGs. 1 and 2. The kinks in FIG. 1 around \( \alpha_3 = 0 \) are there for \( q < 2 \) and hence their presence is due to the rotation in the system.

The color codes that we use for the contour plots for FIG. 1 to FIG. 6 are the same. We use bluish and reddish colors to indicate \( \text{Im}(\beta) \)'s negativity and positivity respectively. We further use white color to indicate the transition from the negative to positive of \( \text{Im}(\beta) \).

As we introduce the extra force, i.e. \( m \neq 0 \), \( \text{Im}(\beta) \) becomes positive for a particular range of \( \alpha_1 \) and \( \alpha_3 \). Throughout the paper, we use \( \text{Im}(\beta)_{\text{max}} \) and \( \text{Re}(\beta)_{\text{max}} \) to indicate the maximum value of \( \text{Im}(\beta) \) and at which \( \text{Re}(\beta) \), it occurs, respectively. FIGs. 3 and 4 show the variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) for \( m/u_0 = 10 \) for the Keplerian flow but for \( \text{Re} = 10^2 \) and \( 10^4 \) respectively. FIGs. 5 and 6 show the variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) for \( \text{Re} = 10^4 \) for the Keplerian flow but for \( m/u_0 = 10 \) and \( 10^2 \) respectively. These two figures depict that the increment of \( m/u_0 \) increases \( \text{Im}(\beta) \) value for a fixed \( \text{Re} \). Note that the bounds on the axes of FIGs. 4 and 6 are different than that of FIGs. 1 and 2. The reason is described later in this section itself.

Now if we fix \( m/u_0 \) and increase \( \text{Re} \), it is expected that the value of \( \text{Im}(\beta) \) increases. FIGs. 7, 8, 9 and 10 depict the same. These four figures show the variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) in three dimensions for \( \text{Re} = 10, 10^2, 10^3 \) and \( 10^4 \) for \( m/u_0 = 10 \) in case of the Keplerian flow. \( \text{Im}(\beta)_{\text{max}} \) is given in the caption corresponding to each figure to compare one with other. We make three dimensional plots for these cases to capture \( \text{Im}(\beta)_{\text{max}} \), as it is not obvious from the contour plots, particularly from FIGs. 3 and 4. This fact becomes clear once we compare between FIGs. 2 and 10. From these four three dimensional figures and also from FIGs. 3, 4 and 5, it is clear that the increment of \( \text{Re} \) for a fixed \( m/u_0 \) also increases the range of \( \alpha_1 \) and \( \alpha_3 \) which could give rise to positive \( \text{Im}(\beta) \) and hence instability in the system. To capture this particular fact, we zoom out the axes of the FIGs. 3 and 5 as these two figures look almost similar if the bound on the axes is chosen from -10 to 10. Similarly, FIGs. 9 and 10 may apparently look same, however they are not. If we check the fact that at which value of \( \text{Im}(\beta) \), the surfaces of \( \text{Im}(\beta) \) corresponding to these two figures cut the \( \text{Im}(\beta) \) axis at \( \alpha_1 = -10 \), then we can be sure that they are not same. Apart from this, FIG. 3 shows that at \( \alpha_1 = -10 \), the surface of \( \text{Im}(\beta) \) is downwards while the same for FIG. 10 is almost flat.

However, \( \text{Im}(\beta)_{\text{max}} \) does not increase beyond 0.91, even if we increase \( \text{Re} \) for \( m/u_0 = 10 \) for the Keplerian flow. It, therefore, looks like \( \text{Im}(\beta)_{\text{max}} \) gets saturated at 0.91 at \( \text{Re} = 10^4 \) and any further increment in \( \text{Re} \) increases only the range of \( \alpha_1 \) and \( \alpha_3 \) that makes \( \text{Im}(\beta) \) positive. This saturation of \( \text{Im}(\beta) \) depends on \( m/u_0 \). FIG. 6 shows the variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) for \( \text{Re} = 10^4 \) and \( m/u_0 = 10^2 \) in the case of Keplerian flow. In this case, \( \text{Im}(\beta)_{\text{max}} \) is 2.12. Increment of \( m/u_0 \), therefore, increases the saturation in \( \text{Im}(\beta)_{\text{max}} \). This situation is well-depicted in FIG. 11 which shows the variation of \( \text{Im}(\beta)_{\text{max}} \) as a function of \( \text{Re} \) for \( m/u_0 = 10 \) and \( m/u_0 = 100 \) for the Keplerian flow. In addition, the same figure also shows the saturation of \( \text{Im}(\beta)_{\text{max}} \) for a fixed \( m/u_0 \). If we consider \( \alpha_1 = 0 \) in equation (11), we obtain the dispersion relations as shown by Nath & Mukhopadhyay [30] in Figure 2.

B. Plane Couette Flow

For plane Couette flow, equation (11) becomes

\[
\beta = -\frac{0.5i}{3\alpha_1^2 q + \alpha_3^2} \left[ mq + \frac{14\alpha_1^2 q \alpha_3^2}{m/u_0} + \frac{12\alpha_0^2 \alpha_1^2 q^2}{m/u_0} \right] \\
+ \frac{2\alpha_3^4 q}{m/u_0} - \frac{1}{\frac{m}{u_0} - \frac{3\alpha_1^2 \alpha_3^2 \alpha_4^2 q^2}{u_0} + \frac{8\alpha_0^2 \alpha_1^2 q^2}{u_0}} - \frac{8\alpha_0^2 \alpha_1^2 q^2}{u_0} + \frac{16\alpha_1^4 q^2}{u_0} \right]^{\frac{1}{2}}. \tag{12}
\]

It is quite obvious that \( \beta \) is an imaginary quantity for plane Couette flow. To have instability therefore, the quantity within the square bracket must be negative and this leads to the condition

\[
\frac{m}{u_0} < -\frac{45}{\text{Re} \left( 9\alpha_1^4 + 6\alpha_1^2 \alpha_3^2 + \alpha_3^4 \right) \left( \alpha_1^4 + 1.867\alpha_1^6 \alpha_3^2 + 1.111\alpha_1^4 \alpha_3^4 + 0.267\alpha_1^2 \alpha_3^6 + 0.022\alpha_3^8 \right)}.
\tag{13}
\]
FIG. 1: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $Re = 10^{10}$ and $m/u_0 = 0$ for the Keplerian flow.

$\text{Im}(\beta)[\alpha_1,\alpha_3]$

$\alpha_3$

$\alpha_1$

$-10 -5 0 5 10$

$-10$

$-5$

$0$

$5$

$10$

$-10 -5 0 5 10$

$-10$

$-5$

$0$

$5$

$10$

$\alpha_1$

$\alpha_3$

FIG. 2: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $Re = 10^{10}$ and $m/u_0 = 0$ for plane Couette flow.

$\text{Im}(\beta)[\alpha_1,\alpha_3]$

$\alpha_3$

$\alpha_1$

$-20 -10 0 10 20$

$-20$

$-10$

$0$

$10$

$20$

$-20 -10 0 10 20$

$-20$

$-10$

$0$

$10$

$20$

$\alpha_1$

$\alpha_3$

FIG. 3: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $Re = 10^2$ and $m/u_0 = 10$ for the Keplerian flow.

$\text{Im}(\beta)[\alpha_1,\alpha_3]$

$\alpha_3$

$\alpha_1$

$-20 -10 0 10 20$

$-20$

$-10$

$0$

$10$

$20$

$-20 -10 0 10 20$

$-20$

$-10$

$0$

$10$

$20$

$\alpha_1$

$\alpha_3$

FIG. 4: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $Re = 10^{10}$ and $m/u_0 = 10$ for the Keplerian flow.

$\text{Im}(\beta)$ has positive value, i.e. flow is unstable.

We use the same color codes for the contour plots in FIGs. 1, 2, 3, 4 as used in §III A. However, we use grayish color to indicate the transition from the positive to negative of $\text{Im}(\beta)$. As $\alpha_1, \alpha_3 \to 0$, $\text{Im}(\beta) \to \infty$. The region where $\alpha_1, \alpha_3 \to 0$, therefore, cannot be captured in the contour plots. This region, therefore, is covered with white color by default. However, to avoid any confusion, we mention ‘Infinity’ inside this region whenever possible, otherwise we mention it in the correspond-

$m/u_0$, therefore, has to be negative to have instability in plane Couette flow. If we make $\alpha_1 = 0$, the condition in equation (13) becomes

$$\frac{m}{u_0} < -\frac{\alpha_3^4}{Re},$$

which was obtained by Nath & Mukhopadhyay (2016) for vertical perturbation.

From equation (12), it is obvious that $\text{Im}(\beta)$ blows up at $\alpha_1 = \alpha_3 = 0$. The color bars in the contour plots corresponding to plane Couette flow, therefore, have different meaning than indicating the value of $\text{Im}(\beta)$. They, rather, indicate the range of $\alpha_1$ and $\alpha_3$ within which

$\text{Im}(\beta)$ has positive value, i.e. flow is unstable.

We use the same color codes for the contour plots in FIGs. 1, 2, 3, 4 as used in §III A. However, we use grayish color to indicate the transition from the positive to negative of $\text{Im}(\beta)$. As $\alpha_1, \alpha_3 \to 0$, $\text{Im}(\beta) \to \infty$. The region where $\alpha_1, \alpha_3 \to 0$, therefore, cannot be captured in the contour plots. This region, therefore, is covered with white color by default. However, to avoid any confusion, we mention ‘Infinity’ inside this region whenever possible, otherwise we mention it in the correspond-

$\text{Im}(\beta)$ has positive value, i.e. flow is unstable.
FIG. 5: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $\text{Re} = 10^4$ and $m/u_0 = 10$ for the Keplerian flow.

FIG. 6: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $\text{Re} = 10^4$ and $m/u_0 = 10^2$ for the Keplerian flow.

FIG. 7: Variation of $\text{Im}(\beta)$ in three dimensions as a function of $\alpha_1$ and $\alpha_3$ for $\text{Re} = 10$ and $m/u_0 = 10$ for the Keplerian flow. $\text{Im}(\beta)_{\text{max}} = 0.33$.

FIG. 8: Variation of $\text{Im}(\beta)$ in three dimensions as a function of $\alpha_1$ and $\alpha_3$ for $\text{Re} = 10^2$ and $m/u_0 = 10$ for the Keplerian flow. $\text{Im}(\beta)_{\text{max}} = 0.822$.

The system unstable. FIGs. 12 and 13 make this point even clearer. These two figures represent the variation of $\text{Im}(\beta)$ ($\geq 0$) in three dimensions as a function of $\alpha_1$ and $\alpha_3$ for $\text{Re} = 10^2$ and $10^{10}$ respectively for $m/u_0 = -10^{-2}$ for plane Couette flow. However, if we compare carefully FIG. 12 (or FIG. 13) with FIG. 16 (or FIG. 17), we see that FIG. 16 (or FIG. 17) has a larger range of $\alpha_1$ and $\alpha_3$ to give rise to positive $\text{Im}(\beta)$. It is also expected that if we increase the magnitude of $m/u_0$, the system becomes more unstable as in the case of Keplerian flow. This phenomenon also happens here but in different way. FIGs. 16 and 17 show the variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $\text{Re} = 10^2$ and $10^{10}$ respectively and for $m/u_0 = -10$ for plane Couette flow. However, if we compare carefully FIG. 12 (or FIG. 13) with FIG. 16 (or FIG. 17), we see that FIG. 16 (or FIG. 17) has a larger range of $\alpha_1$ and $\alpha_3$ to give rise to positive $\text{Im}(\beta)$. 
On the other hand, \( \alpha \) varies in a certain range of \( \beta \), therefore, 

\[
\text{Re}(u), \text{Re}(\zeta) \sim \cos(\text{Re}(\beta)t)e^{\text{Im}(\beta)t}.
\]  

Here, we observe the variation of \( \text{Im}(\beta) \) as a function of \( \text{Re}(\beta) \). FIG. 18 shows Argand diagrams for \( \text{Re} = 10 \), \( 10^2 \), \( 10^3 \) for fixed \( \alpha_1 = 1 \) and \( m/u_0 = 10 \) by varying \( \alpha_3 \) in case of the Keplerian flow. We observe that \( \text{Im}(\beta)_{\text{max}} \), i.e., the maximum growth rate increases as we increase \( \text{Re} \) for a fixed \( m/u_0 \). FIG. 19 shows the Argand diagrams for \( \text{Re} = 10^4 \), \( m/u_0 = 10 \) and for \( \alpha_1 = 1, 5, 10 \) for the Keplerian flow, where for each \( \alpha_1 \), we vary \( \alpha_3 \) from \(-2000\) to \(2000\). From FIG. 19 it is clear that as we decrease \( \alpha_1 \), \( \text{Im}(\beta)_{\text{max}} \) increases. For smaller \( \alpha_1 \), therefore, the system becomes unstable at smaller time and plausibly becomes turbulent for those \( \alpha_1 \) first.

The phenomenon of increment in the maximum growth rate with decreasing \( \alpha_3 \) is described through the energy of perturbations in FIG. 20. Here, \( (\text{Re}(u))^2 \) represents the temporal evolution of energy corresponding to the \( x \)-component of the perturbed velocity field for \( \text{Im}(\beta)_{\text{max}} \) and \( \text{Re}(\beta)_{\text{max}} \) corresponding to three different \( \alpha_1 \), \( \text{Re} = 10^4 \) for \( m/u_0 = 10 \) in the case of the Keplerian flow. In FIG. 20 the maximum value along vertical axis is \( 10^4 \). We consider this value to be the limit of linearity following [15]. We notice that the higher \( \text{Im}(\beta)_{\text{max}} \) has higher \( \text{Re}(\beta)_{\text{max}} \), i.e., the higher growth rates have the higher frequency.

It is always interesting to check what happens to the \( \text{Im}(\beta)_{\text{max}} \) if \( m/u_0 \) increases for the same \( \text{Re} \). FIG. 21 shows the variation of \( \text{Im}(\beta)_{\text{max}} \) as a function of \( m/u_0 \), for \( \alpha_1 = 1, 5, 10 \) for \( \text{Re} = 10^4 \) for the Keplerian flow. Here we notice that \( \text{Im}(\beta)_{\text{max}} \) increases as we increase \( m/u_0 \). However, at larger \( m/u_0 \), the \( \text{Im}(\beta)_{\text{max}} \) becomes almost independent on \( \alpha_1 \). At higher \( m/u_0 \), the extra
FIG. 12: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $Re = 10^2$ and $m/u_0 = -10^{-2}$ for plane Couette flow. At $\alpha_1 = \alpha_3 = 0$, $\text{Im}(\beta) \to \infty$. It is indicated by white point at the center of the plot.

FIG. 13: Variation of $\text{Im}(\beta)$ as a function of $\alpha_1$ and $\alpha_3$ for $Re = 10^{10}$ and $m/u_0 = -10^{-2}$ for plane Couette flow.

force and $Re$ almost completely take control of the system of a fixed $q$. This phenomenon can be explained from the equation (11). At large $m/u_0$, the equation (11) becomes

$$\beta \sim -\frac{0.5i}{3\alpha_1^2 q + \alpha_3^2 q} \left[ \frac{mq}{u_0} - \frac{1}{Re} \left( \frac{24i\alpha_3q^2mRe^2}{u_0} + \frac{8i\alpha_3^3 mqRe^2}{u_0} + \frac{m^2 q^2 Re^2}{u_0^2} - \frac{8\alpha_3^4 mq^2 Re}{u_0} \right) \right].$$

(17)

We obtain equation (17) from equation (11) by retaining the terms that involve with $m/u_0$ as the magnitude of other terms become negligible compared to those involving with $m/u_0$. From equation (17), it is evident that as $m/u_0$ increases, the effect of $\alpha_1$ on $\beta$ and, hence, $\text{Im}(\beta)$ decreases.

To make the study complete, we should have enough comparison among Argand diagrams like FIG. 19 but with different $m/u_0$ and $Re$. FIG. 22 represents the Argand diagrams for $Re = 10^4$ and $m/u_0 = 10^2$ for three values of $\alpha_3$ mentioned in the figure where $\alpha_3$ is varied from $-2000$ to $2000$ for each value of $\alpha_1$. Here we see that, $\text{Im}(\beta)_{\text{max}}$ and $\text{Re}(\beta)_{\text{max}}$ for three different $\alpha_3$ are greater than those for $m/u_0 = 10$. We, therefore, confirm that as $m/u_0$ increases the value of $\text{Im}(\beta)_{\text{max}}$ and $\text{Re}(\beta)_{\text{max}}$ also increase. FIGs. 23 and 24 show the Argand diagrams for $Re = 10^{10}$ but for $m/u_0 = 10$ and $10^2$ respectively for three different $\alpha_1$ as shown in the
FIG. 16: Variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) for \( Re = 10^2 \) and \( m/u_0 = -10 \) for plane Couette flow.

FIG. 17: Variation of \( \text{Im}(\beta) \) as a function of \( \alpha_1 \) and \( \alpha_3 \) for \( Re = 10^3 \) and \( m/u_0 = -10 \) for plane Couette flow.

V. COMPARISON OF VARIOUS TIMESCALES

In §III, we obtain the dispersion relation for the linear perturbation in the presence of Coriolis force and extra force for the Keplerian flow as well as plane Couette flow. It shows that there is a range of wave vectors in which \( \text{Im}(\beta) \) is positive. On the other hand, we also see the presence of temporal oscillation in the linear perturbation due to the presence of \( Re(\beta) \) in §IV. It, therefore, is important to compare the time period of the tempo-

corresponding figures and for each \( \alpha_1 \), we vary \( \alpha_3 \) from \(-100000\) to \(100000\). If we compare between FIGs. 19 and 23 (and also between FIGs. 22 and 24), we notice that \( \text{Im}(\beta)_{\text{max}} \) and \( Re(\beta)_{\text{max}} \) do not change as we increase \( Re \) for a fixed \( m/u_0 = 10 \), but the range of \( \alpha_1 \), that gives rise to positive \( \text{Im}(\beta) \), does increase, as the positive area under the curve increases with increasing \( Re \).
To calculate the infall time scale of the fluid parcel, we need the radial component of velocity of the flow in the radial growth of the perturbation with the infall time scale. To calculate the infall time scale of the fluid parcel, we need the radial component of velocity of the flow in the Keplerian disk and it is given by (see e.g. [7])

$$v_r(R) = 2.7 \times 10^4 \times \alpha_s^{1/2} \left( \frac{\dot{M}}{10^{16}} \right)^{1/2} \left( \frac{M}{M_\odot} \right)^{-1/2} \left( \frac{R}{10^{10}} \right)^{-7/10} \left( R_* \right)^{1/2} \left( 1 - \left( \frac{R_*}{R} \right)^{1/2} \right)^{-7/10} \text{cm s}^{-1},$$

(18)

where $\alpha_s$ is the Shakura-Sunyaev viscosity parameter, $\dot{M}$ is the mass accretion rate, $M$ is the mass of the accretor, $R$ is the radius where the analysis is done, $R_*$ is the radius of the accretor and for a nonrotating black hole.
we have to multiply it with a factor regime for the first time at $t = 100\pi/\alpha_1$ in dimensionless unit. To make it dimensionful, we have to multiply the size of the box $(0.05R_s)$ with it. Those $\alpha_1$ which are greater than $2\pi$, therefore, describe the best dynamics of the fluid parcel inside the box. This is the reason behind showing the temporal evolution of $(Re(u))^2$ with corresponding $Im(\beta)_{\text{max}}$ and $Re(\beta)_{\text{max}}$ for fixed $Re$ and $\alpha_1 = 10 \ (> 2\pi)$ for $m/u_0 = 10$ and $10^2$.

VI. CONCLUSION

Instability and hence turbulence, become inevitable for the fluid parcel inside the shearing box at the small region of the accretion disk. This instability is also controlled by $Re$ and the strength of the extra force which is white noise with nonzero mean ($m$). The presence of noise is very natural. It may arise from small thermal fluctuation present in the systems (see e.g. [60]). The presence of the noise in the systems can be due to the disturbances of arbitrary origins [52]. However, in the astrophysical context, particularly in accretion disks, the examples of origin of such force could be: the interaction between the dust grains and fluid parcel in protoplanetary disks (e.g. [53]); back reactions of outflow/jet to accretion disks; external forcing of the disk, i.e. tidal forcing, shock wave debris, outburst, or internal forcing by nonlinear terms [54][55]. These forces are also expected to be stochastic in nature.

Once, the instability and therefore turbulence kick in inside the shearing box, we consider the shearing box repeatedly throughout the radial extension of the accretion disk and hence the transport of angular momentum can be interpreted in the Keplerian accretion disk. However, for plane Couette flow, there is no requirement of infall. Hence, in presence of noise, it is always expected to lead instability.

VII. ACKNOWLEDGEMENT

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The solutions of equations (6) and (7) in the Fourier space will be

$$\hat{\psi}_{k,\omega} = \left( \frac{1}{2\pi} \right)^4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) e^{i(\alpha x - \beta t)} e^{-i(k x - \omega t)} d^3 x dt$$

$$= \frac{1}{2\pi} \delta(\alpha_2 - k_y) \delta(\alpha_3 - k_z) \delta(\beta - \omega) \int_{-\infty}^{\infty} \psi(x) e^{i(\alpha_1 - k_x) x} dx,$$

where $\psi$ will be any of $u$ and $\zeta$.

We now integrate equations (6) and (7) with respect to $k$ and $\omega$. Each term of equation (6) after the integration, assuming WKB approximation, i.e. neglecting second and higher derivatives of $u$ and $\zeta$, is obtained given below.

1.

$$\int_{-\infty}^{\infty} k_y k^2 \frac{\partial \hat{u}_{k,\omega}}{\partial k} d^3 k d\omega = \int_{-\infty}^{\infty} k_y (k_x^2 + k_y^2 + k_z^2) \frac{\partial \hat{u}_{k,\omega}}{\partial k} d^3 k d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} k_y (k_x^2 + k_y^2 + k_z^2) \delta(\omega_2 - k_y) \delta(\omega_3 - k_z) \frac{\partial}{\partial k_x} \int_{-\infty}^{\infty} dx' u(x') e^{i(\alpha_1 - k_x) x'} dx' dk_x dk_y dk_z$$

$$= -2\alpha_1 \alpha_2 u(0) + 2i\alpha_2 u'(0),$$

2.

$$\int_{-\infty}^{\infty} i\omega k^2 \hat{u}_{k,\omega} d^3 k d\omega = i\beta (\alpha^2 u(0) - 2i\alpha_1 u'(0)],$$

3.

$$\int_{-\infty}^{\infty} 2k_x k_y \hat{u}_{k,\omega} d^3 k d\omega = -2i\alpha_2 (u'(0) + i\alpha_1 u(0)],$$

4.

$$\int_{-\infty}^{\infty} \frac{k^4}{Re} \hat{u}_{k,\omega} d^3 k d\omega = \frac{\alpha^4}{Re} u(0) - \frac{4}{Re} i\alpha_1 \alpha^2 u'(0),$$

5.

$$\int_{-\infty}^{\infty} \frac{2k_x}{q} \hat{\zeta}_{k,\omega} d^3 k d\omega = \frac{2i\alpha_3}{q} \zeta_0,$$

6.

$$\int_{-\infty}^{\infty} m_1 \delta(k) \delta(\omega) d^3 k d\omega = m_1.$$

We collect all these terms and obtain the first part of equation (9). Following the same method, we also obtain the second part of equation (9) from equation (7).