On the Hurewicz homomorphism on the extensions of ideals in $\pi_*^S$ and spherical classes in $H_*Q_0S^0$

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Abstract

This note records some partial results on the Curtis conjecture on the image of the Hurewicz map $h : 2\pi_*Q_0S^0 \to H_*Q_0S^0; \mathbb{Z}/2$; the result might be well known, but we don't know of any published account. First, we show that if $f \in 2\pi_*^S$ is not of Adams filtration 1 or 2 with $h(f) \neq 0$ then $f$ is not a decomposable element in $2\pi_*^S$. Moreover, it is shown if $k$ is the least positive integer that $f$ is represented by a cycle in $\text{Ext}_A^{k,k+n}(\mathbb{Z}/2,\mathbb{Z}/2)$, then (i) if $\sigma_*h(f) \neq 0$ then $n \geq 2^k-1$; (ii) if $\sigma_*h(f) = 0$ then $n \geq 2^k-1$. Second, for $S \subseteq 2\pi_*^S$ we show that: (i) if the conjecture holds on $S$, then it holds on $(S)$; (ii) if $h(S) = 0$ then $h$ acts trivially on any extension of $S$ obtained by applying homotopy operations arising from $2\pi_*D_rS^n$ with $n > 0$. We also provide partial results on the extensions of $(S)$ by taking (possible) Toda brackets of its elements.

1 Introduction and statement of results

This note is circulated around Curtis conjecture. Let $Q_0S^0$ be the base point component of $QS^0 = \text{colim} \Omega^iS^i$. The conjecture then reads as follows.

Conjecture 1. In positive degrees only the Hopf invariant one and Kervaire invariant one elements survive under the Hurewicz homomorphism $h : 2\pi_*Q_0S^0 \to H_*Q_0S^0; \mathbb{Z}/2$.

Note that given $f, g \in 2\pi_*Q_0S^0$ with $h(f) \neq 0$ and $h(g) = 0$ then $h(f + g) = h(f) \neq 0$. Also, note that if $f : S^0 \to S^0$ is any map of odd degree, then $h(f) = h(f)$. Finally, note that the Hurewicz homomorphisms $\pi_*Q_0S^0 \to H_*Q_0S^0; \mathbb{Z}$ and $2\pi_*Q_0S^0 \to H_*Q_0S^0; \mathbb{Z}/2$ are $\mathbb{Z}$-module homorphims, and not ring homomorphisms (see Theorem 2 below). These hopefully will justify the way that we have stated our results.

Here and throughout the paper, we write $\langle S \rangle$ for the ideal generated by a set $S$ in a ring $R$, $\pi_*$ and $\pi_*^S$ for homotopy and stable homotopy respectively, and $p\pi_*^{sp}p\pi_*^S$ for their $p$-primary components, respectively. We write $\pi_*^{>0}$ for $\bigoplus_{i>0}\pi_*^i$. We shall use $f_+$ to denote the mapping induced in homotopy, and $f_*$ for the mapping induced in homology, where $f$ is a mapping of spaces or stable complexes, and $C_f$ will refer to (stable) mapping cone of $f$. To avoid confusion, we write $f$ for summation, $\Sigma$ for the suspension functor on the category of pointed spaces as well as spectra, and $\Sigma_* : \tilde{H}_nX \to H_{n+1}\Sigma X$ for the suspension isomorphism. Finally, $\eta, \nu, \sigma$ will denote the well known Hopf invariant one elements. Our fist observation is the following.
Theorem 2. (i) For $i, j > 0$, consider the composition

$$\pi_i Q_0 S^0 \otimes \pi_j Q_0 S^0 \rightarrow \pi_{i+j} Q_0 S^0 \xrightarrow{h} H_{i+j}(Q_0 S^0; \mathbb{Z})$$

where the first arrow is the product in $\pi_*$. Then $h(f g) \neq 0$ only if $f$ and $g$ live in the same grading, and both are detected by the unstable Hopf invariant.

(ii) For $i, j > 0$, consider the composition

$$2\pi_i Q_0 S^0 \otimes 2\pi_j Q_0 S^0 \rightarrow 2\pi_{i+j} Q_0 S^0 \xrightarrow{h} H_{i+j}(Q_0 S^0; \mathbb{Z}/2).$$

Then $h(f g) \neq 0$ only if $f = g$ with $f = \eta, \nu, \sigma$ or odd multiples of these elements, i.e. the image of this composition only consists of Kervaire invariant one elements $h(\eta^2), h(\nu^2), h(\sigma^2)$. Here, the first arrow on the left is the multiplication on the stable homotopy ring. Also, the image of the composite

$$\langle f : f = \eta, \nu, \sigma \rangle \rightarrow 2\pi_* Q_0 S^0 \xrightarrow{h} H_*(Q_0 S^0; \mathbb{Z}/2)$$

only consists of the Hurewicz image of the Hopf invariant one elements $\eta, \nu, \sigma$, and the Kervaire invariant one elements $\eta^2, \nu^2, \sigma^2$.

Our first application of this theorem is to provide some necessary conditions for an element $f \in 2\pi_*^s$ to map nontrivially under $h$ which we state as follows.

Theorem 3. Suppose $f \in 2\pi_*^s$ is not of Adams filtration $\leq 2$, and $h(f) \neq 0$. Then, $f$ is not a decomposable element in $2\pi_*^s$ and is represented by a permanent cycle in the $E_2$-term of ASS which is not a decomposable.

Note that the two conditions of being a decomposable permanent cycle in ASS and representing a decomposable element in $2\pi_*^s$ are not the same; for instance $h_3^2$ in ASS with $3 < j < 7$ represents a Kervaire invariant one element $\theta_j$ where by Theorem 2 the latter is not a decomposable in $2\pi_*^s$. The proof of the above theorem is very short, so we include it here.

Proof. Suppose $f \in 2\pi_*^s$ does not live in $F^2$, so that if written as a sum of decomposable terms in $2\pi_*^s$, it cannot involve terms such as $\eta^2, \nu^2$ or $\sigma^2$. Hence, by Theorem 2, $h(f) = 0$. On the other hand, recall the Lannes-Zarati homomorphism $\varphi_k : \text{Ext}_A^{k+i}(\mathbb{Z}/2, \mathbb{Z}/2) \rightarrow (\mathbb{Z}/2 \otimes D_k)_i^*$, which is meant to filter the Hurewicz homomorphism $\pi_* Q_0 S^0 \rightarrow H_*(Q_0 S^0; \mathbb{Z}/2)$ on the $E_{\infty}$-page level of the ASS; in particular $h(f) \neq 0$ implies that $\varphi(c_f) \neq 0$ where $\varphi = \oplus \varphi_k$ and $c_f$ is any permanent cycle representing $f$ in the ASS. According to Hung and Peterson [?, Proposition 5.4] $\varphi_k$ does vanish on decomposable classes in the spectral sequence when $k > 2$. Hence, if $f$ is represented by a permanent cycle which is a sum of decomposable elements, then $\varphi_k(c_f) = 0$, and we cannot have $h(f) \neq 0$. This completes the proof.

Next, together with some detailed calculations on $h(\kappa)$, the following becomes an immediate corollary of Theorem 2.

Theorem 4. Suppose that Curtis conjecture holds on $S \subset 2\pi_*^s$. Then the conjecture also holds on $\langle S \rangle$.

An ad hoc way to verify the conjecture, then is to try to choose the set $S$ as large as possible by feeding in more elements into $S$. For instance, writing $J$ for the fibre of the Adams operation $\psi^3 - 1 : \text{BSO} \rightarrow \text{BSO}$, it is known that the conjecture holds on $2\pi_* J$ [21 Theorem 1]. Write $F^t$ for the elements of $2\pi_*^s$ which are of Adams filtration $\leq t$, write $\eta_i \in 2\pi_*^s$, for the elements of Mahowald family which live in $F^2$, and $\tau_i \in 2\pi_*^{2i+1,1}$ for Bruner’s family which live in the 3-line of the ASS. The following provides an example for applying Theorem 4.
\textbf{Lemma 5.} Consider the composition

\[ (f : f \in F^2 + 2\pi_s J, \text{ or } f = \tau_i) \mapsto 2\pi^*_s \simeq 2\pi_s Q_0 S^0 \mapsto H_*(Q_0 S^0; \mathbb{Z}/2). \]

Then, only elements of Hopf invariant one or Kervaire invariant one map nontrivially under the above composition.

Let’s note that recently Gaudens \cite{6} Theorem 6.5 has used a BV-algebra structure on the homology of double loop spaces to verify Theorem 4 in the case \( S \) is the image of \( J \)-homomorphism \( \pi_* \text{SO} \mapsto 2\pi_* Q_0 S^0 \). Since, at the prime 2, \( \text{im} ((J_\mathbb{F})_\#) \subset 2\pi_* J \), the above Lemma provides a generalisation of Gaudens’ results, where we have made no use of string structures.

Next, note that for a given set \( S \) in a ring, the ideal \( \langle S \rangle \) is the largest that we can get from the ring multiplication. Hence, we have to look for other methods for extending \( S \) or \( \langle S \rangle \) into a larger set. Working in \( 2\pi^*_s \), we consider Toda brackets and homotopy operations. First we consider Toda brackets. An ideal result would be to show that “if the conjecture hold on \( S \), then it also holds on the extension of \( S \), or of \( \langle S \rangle \), by all (higher) Toda brackets” of course modulo our definition of a higher bracket! By a result of J. Cohen \cite{6} Theorem 4.5], any element of \( 2\pi^*_s \) might be written as a (higher) Toda bracket, as he defined, of \( 2, \eta, \nu, \sigma \). Hence, such a statement would imply the Conjecture \cite{6} as by Cohen’s result it is enough to choose \( S = \{ 2, \eta, \nu, \sigma \} \). We don’t have such a statement. In this direction, (1) we have obtained some necessary conditions for nonvanishing of the Hurewicz image of elements represented by Toda bracket; (2) we have observed that under expected stability conditions the Hurewicz image does indeed vanish.

\textbf{Proposition 6.} (i) Let \( \alpha \in 2\pi^*_s X_1 \), \( \beta : X_1 \to X_2 \) and \( \gamma : X_2 \to S^0 \) be stable maps with \( X_1 \) and \( X_2 \) being stable complexes of finite type. Suppose that the Toda bracket \( \{ \alpha, \beta, \gamma \} \) is defined and represents \( f \in 2\pi^*_s \) such that \( h(f) \neq 0 \). Then, at least one of \( \gamma \) or \( \alpha \) is nontrivial in homology, i.e. \( \alpha_* \neq 0 \) or \( \gamma_* \neq 0 \).

(ii) Suppose \( \alpha \in \pi^*_1 X_1 \). Let \( \beta : X_1 \to X_2 \), \( \gamma : X_2 \to S^0 \) be stable maps, with \( X_1, X_2 \) being (stable) complexes, \( i < 2 \text{conn}(X_1) \), \( \dim X_1 < 2 \text{conn}(X_2) \), \( i > \dim X_1 > \dim X_2 \), so that the Toda bracket \( \{ \alpha, \beta, \gamma \} \) is defined. Here, \( \text{conn}(X) = n \) if \( \pi^*_t X \simeq 0 \) for all \( t < n \). Then, the element of \( \pi_{i+1} Q_0 S^0 \) represented by Toda bracket \( \{ \alpha, \beta, \gamma \} \) maps trivially under the Hurewicz homomorphism \( h : \pi_{i+1} Q_0 S^0 \to H_{i+1}(Q_0 S^0; \mathbb{Z}) \). In particular, this is true on the 2-component of \( \pi^*_s \).

The idea here is that higher Toda brackets of maps among (wedge of) spheres, can be written as a triple Toda bracket among complexes so that the complexes somehow encode some of the maps in the higher bracket.

Next, we consider the possible extensions by available homotopy operations coming from \( \pi^*_n D_2 S^n \) for \( n > 0 \) where \( D_r = E\Sigma_r \ltimes \Sigma_r (\cdot)^{r^\gamma} \), with \( \Sigma_r \) being the permutation group on \( r \) elements, is the \( r \)-adic construction. We have the following which is more or less expected.

\textbf{Proposition 7.} Suppose \( \alpha \in 2\pi^*_m D_r S^n \) and \( f \in 2\pi_n Q_0 S^0 \) such that \( h(f) = 0 \). Then, for \( \alpha^*(f) \) defined by the composition

\[ S^n \xrightarrow{\alpha} D_r S^n \xrightarrow{D_r f} D_2 S^0 \xrightarrow{\mu} S^0 \]

with \( \mu : D_2 S^0 = P_r \to S^0 \) being induced by the \( H_\infty \) ring structure of \( S^0 \), we have \( h(\alpha^*(f)) = 0 \) where \( h : 2\pi_s Q_0 S^0 \to H_*(Q_0 S^0; \mathbb{Z}/2) \) is the Hurewicz homomorphism.
Remark 8. Note that, ideally, we would like to show that if the conjecture holds on $S \subseteq 2\pi_*>0$, then conjecture holds on the extension of $S$ by (iterated) application of homotopy operations available throughout $2\pi_3D_2S^n$. By the above theorem if $h(\alpha^*(f)) \neq 0$ then $h(f) \neq 0$. If $f \in S$ on which the conjecture holds, then $f$ is either a Hopf invariant one or a Kervaire invariant one element. The proof will be complete if we show that for any $\alpha$, $h(\alpha^*(\theta_j)) = 0$, and $h(\alpha^*(h_i)) = 0$ unless $\alpha^*(h_i)$ is a Hopf or Kervaire invariant one element. Here, we have used $h_i$ for one of the Hopf invariant one elements, and $\theta_j$ for Kervaire invariant one elements. We do not have a proof of this latter claim yet.

Finally, we consider the relation between dimension of a spherical class and the Adams filtration of homotopy classes that map to it (compare to [12, Corollary 1.5]). We have the following.

Theorem 9. Suppose $f \in 2\pi_nQ_0S^0$ of Adams filtration $k$, i.e. $k$ is the least positive integer where $f$ is represented by a cycle in $\text{Ext}^{k,k+n}(\mathbb{Z}/2,\mathbb{Z}/2)$, and $h(f) \neq 0$ where $h: 2\pi_nQ_0S^0 \to H_n(Q_0S^0;\mathbb{Z}/2)$ is the Hurewicz homomorphism. The following statement then hold.

(i) If $\sigma_*h(f) \neq 0$ then $n \geq 2^k-1$.
(ii) If $\sigma_*h(f) = 0$ then $n \geq 2^{k-1}$.

2 Preliminaries

2.1 Iterated loop spaces

We wish to recall some standard facts on iterated loop spaces, and refer the reader to [16] and [1] for more details. We refer to a space $X$ as a $n$-fold loop space, or $\Omega^n$-space for short, with $n \leq +\infty$, if there exists a collection of spaces $X_i$, $i = 0,1,2,\ldots,n$, together with homotopy equivalences $X_i \to \Omega X_{i+1}$ such that $X = X_0$. Obviously, an $\Omega^n$ space is also an $\Omega^i$-space for $i < n$. An $\Omega^n$-space is an $E_n$-algebra in the operadic language of [16] and admits a ‘structure map’ or ‘evaluation map’ $\theta_n(X): \Omega^n\Sigma^nX \to X$, briefly denoted by $\theta_n$ if there is no confusion, that itself is a $n$-fold loop map; spaces of the form $\Omega^n\Sigma^nX$ are ought to play the role of free objects in the category of $E_n$-algebras. Any map $f: Y \to X$ with $X$ being an $\Omega^n$-space, admits a unique extension to a $n$-fold loop map $\Omega^n\Sigma^nY \to X$ defined by the composite

$$\Omega^n\Sigma^nY \xrightarrow{\Omega^n\Sigma^nf} \Omega^n\Sigma^nX \xrightarrow{\theta_n} X.$$

In the case of $\Omega^\infty$-spaces, our main focus will be spaces $QX = \text{colim} \Omega^n\Sigma^nX$ which by definition satisfy $\Omega Q\Sigma X = QX$. By the adjointness of $\Sigma$ and $\Omega$, $\pi_*X \simeq \pi_*QX$ which we refer to as the stable adjointness isomorphism; this is induced by sending a stable map $S^n \to X$ to its stable adjoint $S^n \to QX$.

2.2 Homology of iterated loop spaces

We recall a description of $\mathbb{Z}/2$-homology of $\Omega^{n+1}$-loop spaces of the form $\Omega^{n+1}\Sigma^{n+1}X$. Let $Y$ be an $\Omega^{n+1}$-space, $0 \leq n \leq +\infty$ with the convention $+\infty + 1 = +\infty$. Since $Y$ is a loop space, the homology $H_*(Y;\mathbb{Z}/2)$ is a ring under Pontrjagin product. Moreover, there are group homomorphisms [5] Part I, Theorem I] and [5] Part III, Theorem 1.1] (see also [?])

$$Q_i : H_d(Y;\mathbb{Z}/2) \to H_{i+2d}(Y;\mathbb{Z}/2)$$

for $i < n+1$ with $Q_0$ acting as the squaring operation with respect to the Pontrjagin product, i.e. $Q_0\xi = \xi^2$. These operations fit into an algebra known as the Dyer-Lashof algebra, often
denoted by $R$. For a sequence $E = (e_1, \ldots, e_s)$ we may abbreviate $Qe_1 \cdots Qe_s$ to $Q_E$. For a path-connected space $X$, the homology ring $H_*(\Omega^{n+1}\Sigma^{n+1}X; \mathbb{Z}/2)$ is a ring, and as a module of $R$ in the case of $n = +\infty$, is described by (see [5, Part III, Lemma 3.8], [5, Part I, Lemma 4.10])

$$H_*(\Omega^{n+1}\Sigma^{n+1}X; \mathbb{Z}/2) \simeq \mathbb{Z}/2[Q_Ex_\mu : E \text{ nondecreasing}, e_1 > 0, e_s < n + 1]$$

where $\{x_\mu\}$ is an additive basis for the reduced homology $\tilde{H}_*(X; \mathbb{Z}/2)$, i.e. symbols as $Q_Ex_\mu$ are generators of this polynomial algebra. We allow the empty sequence $\phi$ to be nondecreasing with $Q_0\xi = \xi$. Sometimes, it is more convenient to work with upper indexed operations, $Q^i : H_d(\Omega^{n+1}Y; \mathbb{Z}/2) \to H_{d+i}(\Omega^{n+1}Y; \mathbb{Z}/2)$, known as the Kudo-Araki operations, defined by $Q^i\xi = Q_{i-1}\xi$. For $I = (i_1, \ldots, i_s)$, we say $I$ is admissible if $i_j \leq 2i_{j+1}$, and the excess is defined by excess($Q^Ix_\mu$) = $i_1 - (i_2 + \cdots + i_s + \dim x_\mu)$. We allow $\phi$ to be admissible with $Q^0\xi = \xi$ and excess($Q^0x_\mu$) = $+\infty$. Note that if $I$ is admissible for $Q^Ix_\mu = Q_Ex_\mu$ then $E$ is nondecreasing and vice versa; also excess($Q^Ix_\mu$) = $e_1$. In the case of the infinite loop space $QX$, for a path connected space $X$, using the upper indexed operations we have

$$H_*(QX; \mathbb{Z}/2) \simeq \mathbb{Z}/2[Q^Ix_\mu : I \text{ is admissible}, \text{ excess}(Q^Ix_\mu) > 0].$$

The operations $Q^i$ are additive homomorphisms with $Q^i a = a^2$ if $i = \dim a$ and $Q^i a = 0$ if $i < \dim a$; hence excess($Q^Ix_\mu$) = 0 means that $Q^Ix$ is a square in the polynomial ring $H_*(QX; \mathbb{Z}/2)$.

Write $X_+$ for $X$ with a disjoint base point. If $X$ is path connected, we may describe homology of $Q_0(X_+)$, the base point component of $Q(X_+)$, as follows. Write $[n]$ for the image of $n \in \pi_0Q(X_+) \simeq \pi_0^e(X_+) \simeq \mathbb{Z}$ in $H_0(Q(X_+); \mathbb{Z})$ under the Hurewicz homomorphism. Then, we have

$$H_*(Q_0(X_+); \mathbb{Z}/2) \simeq \mathbb{Z}/2[Q^Ix_\mu * [-2^I(I)] : I \text{ is admissible}, \text{ excess}(Q^Ix_\mu) > 0]$$

where $*$ is the Pontrjagin product in $H_*(Q(X_+); \mathbb{Z}/2)$ induced by the loop sum. Note that $Q^Ix_\mu * [-2^I(I)]$ is not a decomposable in $Q_0(X_+)$ whereas it is in $Q(X_+)$. If $f : X \to Y$ is given, we then obtain a map of $\Omega^{n+1}$-spaces, $n \leq +\infty$, as $\Omega^{n+1}\Sigma^{n+1}f : \Omega^{n+1}\Sigma^{n+1}X \to \Omega^{n+1}\Sigma^{n+1}Y$. The homology of this map on generators $Q^Ix$ is determined by

$$(\Omega^{n+1}\Sigma^{n+1}f)_*Q^Ix = Q^If_*x.$$ 

Similarly, for the homology of the induced map $Q(X_+) \to Q(Y_+)$ we have

$$(Q(f_+))_*(Q^Ix * [-2^I(I)]) = Q^If_*x * [-2^I(I)]$$

. The action of the Steenrod algebra on the generators $Q^Ix_\mu \in H_*(\Omega^{n+1}\Sigma^{n+1}X; \mathbb{Z}/2)$ is determined by (iterated application of) Nishida relations

$$Sq_*^aQ^b = \int_{t \geq 0} \left( b - a \atop a - 2t \right) Q^{b-a+r}Sq_*^t$$

(1)

where $Sq_*^i : H_*(-; \mathbb{Z}/2) \to H_{*-i}(-; \mathbb{Z}/2)$ is the operation dual to $Sq^i : H^*(-; \mathbb{Z}/2) \to H^{*+i}(-; \mathbb{Z}/2)$. Moreover, for $\xi, \zeta \in H_*(\Omega^{n+1}\Sigma^{n+1}X; \mathbb{Z}/2)$, there is a Catran formula [?; Remark 1.9] that

$$Sq_*^t(\xi \zeta) = \int_{i=0}^t (Sq_*^{t-i}(\xi)(Sq_*^i\zeta)).$$

These relations completely determine the action of Steenrod algebra on homology of $\Omega^{n+1}\Sigma^{n+1}X$ and $Q_0(X_+)$. 

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2.3 Homology suspension

Let $\varepsilon : \Sigma \Omega X \to X$ be the adjoint of $1_{\Omega X} : \Omega X \to \Omega X$. The homology suspension $\sigma_*$ is defined as

$$H_*\Omega X \xrightarrow{\sim} H_{*-1}\Sigma \Omega X \xrightarrow{\sigma_*} H_{*-1}X.$$  

Since $OX = \Omega Q\Sigma X$, we then may consider the homology suspension $\sigma_* : H_*QX \to H_{*-1}Q\Sigma X$; in this case $\sigma_*$ is characterised by the following properties: (1) $\sigma_* Q^I x = Q^I \Sigma_* x$ for $x \in H_* X$; (2) $\sigma_* \xi = 0$ if $\xi$ is a decomposable class in the polynomial algebra $H_*QX$. Moreover, by [5, Page 47] the homology suspension $H_* (Q_0 (X_+); \mathbb{Z}/2) \to H_{*-1} (Q\Sigma (X_+); \mathbb{Z}/2)$ is characterised by the following properties: (1) $\sigma_*$ acts trivially on decomposable terms; (2) on the generators it is given by

$$\sigma_* (Q^I x_\mu * [-2^{(I)}]) = Q^I \Sigma_* x_\mu.$$  

Similar statements hold for the homology suspension $\sigma_* : H_* \Omega^{n+1}\Sigma^{n+1}X \to H_{*-1}\Omega^n\Sigma^n X$. For a Hopf algebra $H$, we write $DH$ for the submodule of decomposable elements and $\text{Ind}(H) = H/DH$ for the quotient module of indecomposable elements. The following provides more information on the action of homology suspension.

**Lemma 2.1.** Let $X$ be a path connected space.

(i) The homology suspension $\sigma_* : H_* QX \to H_{*-1} Q\Sigma X$ does induce a monomorphism $\sigma_* : \text{Ind}(H_* QX) \to H_{*-1} Q\Sigma X$. In particular, if $\int Q^I x_\mu \in \text{ker} \sigma_*$ where sum runs over some $I, x_\mu$ then $Q^I x_\mu \in \text{ker} \sigma_*$ for all $I, x_\mu$ in the sum, i.e. $\text{excess}(Q^I x_\mu) = 0$.

(ii) The homology suspension $\sigma_* : H_* Q_0 (X_+) \to H_{*-1} Q\Sigma (X_+)$ does induce a monomorphism $\sigma_* : \text{Ind}(H_* Q_0 (X_+)) \to H_{*-1} Q\Sigma (X_+)$. In particular, if $\int Q^I x_\mu * [-2^{(I)}] \in \text{ker} \sigma_*$ where sum runs over some $I, x_\mu$ then $Q^I x_\mu * [-2^{(I)}] \in \text{ker} \sigma_*$ for all $I, x_\mu$ in the sum, i.e. $\text{excess}(Q^I x_\mu) = 0$.

**Proof.** (i) Let $\xi \in \text{Ind}(H_* QX)$, a nonzero element, be represented by a nonempty sum $\int Q^I x_\mu$ of monomial generators of $H_* QX$ with $\dim(Q^I x_\mu) = n$ and $\text{excess}(Q^I x_\mu) > 0$, modulo decomposable terms. Hence, as decomposables are killed by $\sigma_*$, we have

$$\sigma_* \xi = \sigma_* \int Q^I x_\mu = \int Q^I \Sigma_* x_\mu$$  

where every term on the right satisfies $\text{excess}(Q^I \Sigma_* x_\mu) \geq 0$, being distinct monomials in the polynomial algebra $H_* Q\Sigma X$ since $Q^I x_\mu$’s were distinct in $H_* QX$. Hence, $\sigma_* \xi \neq 0$ in $H_* Q\Sigma X$ which verifies that $\sigma_* : \text{Ind}(H_* QX) \to H_{*-1} Q\Sigma X$ is a monomorphism. In particular, the equation $\sigma_* \xi = \int Q^I \Sigma_* x_\mu$ shows that a sum $\int Q^I x_\mu \in \text{ker} \sigma_*$ only if all $Q^I x_\mu \in \text{ker} \sigma_*$. This completes the proof. The proof of (ii) is similar.

Finally, we note that the homology of a map $f : X \to \Omega Y$ and its adjoint $\tilde{f} : \Sigma X \to Y$ are related through $\sigma_*$ by the commutative diagram

$$\begin{array}{ccc}
H_* X & \xrightarrow{j_*} & H_* \Omega Y \\
\downarrow \Sigma_*(=) & & \downarrow \sigma_* \\
H_{*-1} \Sigma X & \xrightarrow{f_*} & H_{*-1} Y.
\end{array}$$
2.4 Spherical classes

Let \( k = \mathbb{Z}, \mathbb{Z}/p \) with \( p \) some prime number. A homology class \( \xi \in H_n(X; k) \) is call spherical if it is in the image of the Hurewicz homomorphism \( \pi_n X \to H_n(X; k) \). A spherical class \( \xi \) has some nice properties which follow from its definition and the basic properties of \( H_*(S^n; k) \):

1. \( \xi \) is a primitive in the coalgebra \( H_*(X; k) \) where the coproduct is induced by the diagonal \( X \to X \times X \);
2. \( \xi \) pulls back to a spherical class \( \xi_{-1} \in H_{n-1}(\Omega X; k) \) which is not necessarily unique;
3. when \( k = \mathbb{Z}/p \) then \( \xi \) is annihilated by all Steenrod operations \( P_i^* \).

We are mainly interested in the case \( p = 2 \). The property (3) read as \( Sq_i^* \xi = 0 \) for all \( i > 0 \).

2.5 Stable splitting of \( QX \)

According to May [16] (see also [2]), for a space \( X \), there exists a filtered space \( CX \), with filtration \( \{F_j CX\} \), such that the successive quotients \( F_r CX/F_{r-1} CX \) are the same as the \( r \)-adic construction on \( X \), \( D_r X \). Moreover, there is a inclusion mapping \( i : CX \to QX \) which is a weak equivalence when \( X \) is path connected, and a group completion in general [2, Proposition 5.1, Corollary 5.4]. Furthermore, for a map \( f : X \to Y \) there exist a commutative diagram as

\[
\begin{array}{ccc}
QX & \xrightarrow{Qf} & QY \\
\uparrow & & \uparrow \\
CX & \xrightarrow{Cf} & CY.
\end{array}
\]

The projection \( F_r CX \to D_r X \) admits an extension to a map \( j_r : CX \to CD_r X \) [3, Proposition 1.1], known as the \( r \)-th stable James-Hopf invariant. These maps assemble together to provide a homotopy equivalence \( QCX \to Q(\bigvee D_r X) \cong \prod_{r=0}^{\infty} QD_r X \) [3, Theorem A, Theorem B]. This yields the well known Snaith splitting [18, Theorem 1.1] (see also [3, Theorem C])

\[
\Sigma^\infty CX \to \bigvee_{r=1}^{\infty} \Sigma^\infty D_r X.
\]

For \( X \) path connected, throughout the homotopy equivalence \( CX \to QX \), the maps \( j_r \) give rise to \( QX \to QD_r \). When \( X \) is not path connected, since the group completion is functorial, by passing to group completion we obtain maps \( QX \to QD_r X \) which we continue to denote with \( j_r \); these maps, however, in general, do not result in a splitting such as Snaith’s for \( QX \).

We also note that the maps \( j_r \) are natural with respect to maps between space, that is, if \( f : X \to Y \) is given then we have a commutative diagram of spaces as

\[
\begin{array}{ccc}
QX & \xrightarrow{Qf} & QY \\
\downarrow{j_r} & & \downarrow{j_r} \\
QD_r X & \xrightarrow{QD_r f} & QD_r Y.
\end{array}
\]

Finally, note that when \( X \) is path connected, the evaluation map \( e : \Sigma QX \to Q\Sigma X \) yields a map \( \Sigma CX \to C\Sigma X \) through the homotopy equivalence which respects the filtration of \( CX \), hence providing a map \( \Sigma F_r CX \to F_r C\Sigma X \). This then induces \( \Sigma D_r X \to D_r \Sigma X \) which latter extends to a mapping of infinite loop spaces \( e' : Q\Sigma D_r X \to QD_r \Sigma X \) which is compatible with \( e \), i.e. \( e' \) fits into a commutative diagram as following

\[
\begin{array}{ccc}
\Sigma QX & \xrightarrow{\tilde{j_r}} & Q\Sigma D_r X \\
\downarrow{e} & & \downarrow{e'} \\
Q\Sigma X & \xrightarrow{j_r} & QD_r \Sigma X
\end{array}
\]
where $\tilde{j}_r : \Sigma QX \to Q\Sigma D_r X$ in the top row is the adjoint of the $r$-th James-Hopf invariant $j_r : QX \to QD_r X$. Let’s note that to compute $c'_i$ we only need to calculate the homology of $\Sigma D_r X \to D_r \Sigma X$ which is induced by the evaluation mapping.

We conclude this section by recalling the delooped Kahn-Priddy Theorem [7, Corollary 2.14] (see also [8, Corollary 6.8] for a more general case). According to this Theorem, there is a choice for a map $\lambda : QP \to Q_0 S^0$ which induces an epimorphism in $2\pi_*$-homotopy, deloops once, and its right inverse is given by $t = \Omega j_2$ with $j_2 : QS^1 \to QD_2 S^1 = Q\Sigma P$, i.e. $t\lambda$ is a $2\pi_*$-equivalence. Consequently, $t$ is a monomorphism in $\mathbb{Z}/2$-homology. Note that $D_r S^0 = B\Sigma_{r+}$.

3 Hurewicz homomorphism and products: Proof of Theorem [2]

We begin with the following.

**Lemma 3.1.** Let $f \in \pi_i Q_0 S^0, g \in \pi_j Q_0 S^0$ with $i, j > 0$ and $i \neq j$. Then $h(gf) = 0$.

**Proof.** The product $gf$ in $\pi_* S_0$ is determined by the composition of stable maps $S^{i+j} \overset{f}{\to} S^i \overset{g}{\to} S^0$ which we wish to compute homology of its stable adjoint $S^{i+j} \to QS^i \to QS^0$. For $i < j$ the map $S^{i+j} \to S^j$ is in the stable range, so it can be taken as a genuine map, i.e. $S^{i+j} \to QS^j$ does factorise as $S^{i+j} \to S^j \to QS^j$ where the part $S^{i+j} \to S^j$ is not necessarily unique. The stable adjoint of $gf$ then maybe viewed as $S^{i+j} \overset{f}{\to} S^i \overset{g}{\to} Q_0 S^0$ which is trivial in homology for dimensional reasons. The indeterminacy in choosing the pull back $S^{i+j} \to S^j$ is irrelevant here and will not effect the dimensional reason. Hence, $h(gf) = 0$. The case $i > j$ is similar, noting that $\pi_* = \pi_0 Q_0 S^0$ is commutative.

Note that the above result hold integrally, and consequently on $p\pi_* \pi_*$ for any prime $p$. According to the above lemma, $h(gf) \neq 0$ may occur if $f, g \in \pi_n Q_0 S^0$. Note that for $f \in \pi_* S_0$ there exists $\tilde{f} \in \pi_{2n+1} S^{n+1}$, not necessarily unique, which maps to $f$ under the stabilisation $\pi_{2n+1} S^{n+1} \overset{\sim}{\to} \pi_* S_0$. We shall say $f$ is detected by the unstable Hopf invariant, if $\tilde{f}$ is detected by cup-squaring operation in its mapping cone, i.e. $g^2_{n+1} \neq 0$ in $H^*(C_2; \mathbb{Z})$.

**Theorem 3.2.** Suppose $f, g \in \pi_n Q_0 S^0$ with $h(fg) \neq 0$ with $h : \pi_* Q_0 S^0 \to H_*(Q_0 S^0, \mathbb{Z})$ being the Hurewicz homomorphism. Then both $f$ and $g$ are detected by the unstable Hopf invariant.

**Proof.** We are interested in the stable adjoint of $S^{2n} \overset{f}{\to} S^n \overset{g}{\to} S^0$ given by

$$S^{2n} \overset{f_n}{\to} QS^n \overset{g}{\to} Q_0 S^0.$$ 

As noted above, for dimensional reasons, the mapping $f_n$ factors as $S^{2n+1} \overset{\tilde{f}_n}{\to} \Omega S^{n+1} \overset{\sim}{\to} QS^n$ where $\tilde{f}_n$, which is not necessarily unique, is the adjoint for an appropriate $f$; the map $i : \Omega S^{n+1} \to QS^n$ is the stablisation map. Hence, the stable adjoint of $fg$ can be seen as a composite

$$S^{2n} \to \Omega S^{n+1} \to QS^n \to Q_0 S^0.$$ 

Now, $h(fg) \neq 0$ implies that $h(\tilde{f}_n) \neq 0$. This shows that $h(\tilde{f}_n) = \lambda g^2_n$ for some nonzero $\lambda \in \mathbb{Z}$. This latter implies that $h(f_n) = \lambda g^2_n$. On the other hand, it is well know that $h(\tilde{f}_n) = \lambda g^2_n$ if and only if $g^2_{n+1} = \pm \lambda g_{2n+2}$ in $H^*(C_2; \mathbb{Z})$, i.e. $f$ is detected by the unstable Hopf invariant (see for example [10, Proposition 6.1.5]). Similarly, $g$ is also detected by the unstable Hopf invariant.

$\Box$
The result in integral case, implies the $p$-primary case. In particular, we have the following.

**Corollary 3.3.** Suppose $f, g \in 2\pi_nQ_0S^0$ with $h(fg) \neq 0$ then both $f$ and $g$ are Hopf invariant one elements, i.e. $f, g = \eta, \nu, \sigma$. Here, $h : 2\pi_nQ_0S^0 \rightarrow H_*Q_0S^0; \mathbb{Z}/2$ is the mod 2 Hurewicz homomorphism.

In order to complete the proof of Theorem 2 note that by above observations the nontrivial image of the Hurewicz homomorphism on the the ideal $\langle f : f = \eta, \nu, \sigma \rangle$ can only arise from elements $f, f^2$. Moreover, choosing $g : S^0 \rightarrow S^0$ to be a stable map of odd degree, $f = \eta, \nu, \sigma$, then $fg$ is an odd multiple of a Hopf invariant one element which we know map nontrivially under $h$. This completes the proof.

### 4 Proof of Theorem 4

Suppose $S \subseteq 2\pi^s_{>0}$ on which the Curtis conjecture holds, i.e. if $f \in S$ and $h(f) \neq 0$ then $f$ equals to a Hopf invariant one or Kervaire invariant one element, or an odd multiple of them, modulo other terms which all vanish under $h$. We wish to verify that the conjecture holds on $\langle S \rangle$.

Since $2\pi^s_*$ is a graded ring then a typical element in $\langle S \rangle$ might be written as $\int \alpha f$ with $f \in S$ and $\alpha \in 2\pi^s_*$ where we may arrange terms of this sum according to their grading. Since $h$ is a $\mathbb{Z}$-module homomorphism, it is then enough to verify the theorem at each grading. Consider a finite sum $\int_{i+j=n, i>0} \alpha_j f_i$ with $f_i \in 2\pi^s_i \cap S$ and $\alpha_j \in 2\pi^s_j$ such that $h(\int_{i+j=n, i>0} \alpha_j f_i) \neq 0$. Then, there is at least one term $\alpha_j f_i$ with $h(\alpha_j f_i) \neq 0$. We have two cases: $j = 0$ and $j > 0$.

**Case $j > 0$.** By Theorem 2, $f_i = \alpha_j$ is a Hopf invariant one element or an odd multiple of these elements. Therefore, $\alpha_j f_i$ is one of $\eta^2, \nu^2, \sigma^2$, living in gradings 2, 6, 14 respectively. Note that 2-component of 2-stem and 6-stem are known to be isomorphic to $\mathbb{Z}/2$. Therefore, in these case the sum $\int \alpha_j f_i$ has only one term which we determined above to be one of the Kervaire invariant one elements $\eta^2, \nu^2$. This proves the theorem for this case. If $\alpha_j f_i = \sigma^2$ then the sum $\int \alpha_j f_i$ lives in $2\pi^s_{14} \simeq (\mathbb{Z}/2)^{\oplus 2}$ generated respectively by $\sigma^2$ and $\kappa$. Consequently, the sum $\int \alpha_j f_i$ can be written as $\sigma^2 + \alpha \kappa$ with $\alpha \in 2\pi^s_0$ which could be a map of even or odd degree. In any case, as $h(\kappa) = 0$, the result follows.

**Case $j = 0$.** If $j = 0$ then $\alpha_j$ is a map of odd degree which means $h(f_i) = h(\alpha_j f_i) \neq 0$. By assumption $f$ is either a Hopf invariant one element or Kervaire invariant one element or an odd multiple of these elements, so $\alpha_j f_i$ is. If $af$ is a Hopf invariant one element, then the sum $\int \alpha_j f_i$ lives in one of 1-, 3-, or 7-stems which are generated by $\eta, \nu, \sigma$ respectively. Hence, the whole sum $\int \alpha_j f_i$ must be an odd multiple of these elements. This verifies the theorem in this case. The case of $f$ being a Kervaire invariant one is verified similar to the above case.

#### 4.1 Computing $h(\kappa)$ and $h(\overline{\nu})$

We begin with $\overline{\nu} \in 2\pi^s_8$ which is an element of order 2.

**Lemma 4.1.** Let $\overline{\nu} \in 2\pi^s_8$ be a generator represented by the Toda bracket $\{\nu, \eta, \nu\}$. Then it maps trivially under $h : 2\pi_8Q_0S^0 \rightarrow H_8Q_0S^0; \mathbb{Z}/2$. 


Proof. Consider the composition $S^7 \xrightarrow{\nu} S^4 \xrightarrow{\eta} S^3 \xrightarrow{\nu} Q_0 S^0$ where all maps are genuine maps of spaces and successive compositions are trivial. This yields extension and co-extension maps $\nu^3, \nu_2$ that we can compose as $S^3 \xrightarrow{\nu} C_\eta \xrightarrow{\nu} Q_0 S^0$ which represents $\overline{\nu} : S^3 \to Q_0 S^0$. Obviously, $(\nu^3)_* = 0$ as $C_\eta$ has its top cell in dimension 5. Hence, $h(\overline{\nu}) = 0$. Note that the vanishing happens for dimensional reasons and the indeterminacy in choosing the extension and co-extension maps is irrelevant here. \hfill $\Box$

Next consider the generator $\kappa \in 2\pi^s_{14}$ which is an element of order 2. According to Toda [19], it can be represented by a triple Toda bracket $\{\beta, \alpha, \nu\}$ (our order of maps in the bracket is opposite to Toda, and we have dropped the suspensions from our notation for simplicity). Here, $\alpha : \Sigma K \to S^0$ is an extension of $\eta$ implied by $2\eta = 0$ with $K$ being the stable complex $S^0 \cup e^1$, and $\beta : S^9 \to K$ is a coextension of $\eta$ implied by $2 \eta = 0$. Note that $\Sigma K$ is just $\mathbb{R}P^2$.

**Lemma 4.2.** By abuse of notation, write $\alpha : \Sigma^3(\Sigma K) \to S^3$ for the third suspension of $\alpha : \Sigma K \to S^0$. Then, for the mapping cone of $\Gamma^6 \alpha : \Gamma^6(\Sigma^4 K) \to \Gamma^6 S^3$ we have

$$H_*(C_{\Gamma^6 \alpha}; \mathbb{Z}/2) \simeq H_*(\Gamma^6 S^3; \mathbb{Z}/2) \oplus H_{*+1}(\Gamma^6(\Sigma^4 K); \mathbb{Z}/2)$$

where $\Gamma^n = \Omega^n \Sigma^n$. Write $g_3 \in H_3(\Gamma^6 S^3; \mathbb{Z}/2)$ for a generator coming from $S^3$, and $\Sigma^3 a_1, \Sigma^3 a_2 \in H_4(\Gamma^6(\Sigma^4 K); \mathbb{Z}/2)$ for the 4- and 5-dimensional generators coming from $\Sigma K = \mathbb{R}P^2$ with the relation $\Sigma^2 a_2 = a_1$. The action of the Steenrod operations $Sq_*^t$ on $H_*(C_{\Gamma^6 \alpha}; \mathbb{Z}/2)$ then is determined with its action on $\Gamma^6 S^3$, $\Gamma^6(\Sigma^4 K)$ and the additional relation

$$Sq^2_2 \Sigma_*(\Sigma^3 a_1) = g_3$$

together with Nishida relations where $\Sigma_*(\Sigma^3 a_1) \in H_5(C_{\Gamma^6 \alpha}; \mathbb{Z}/2)$ is a generator projecting onto a generator of $H_5(\Sigma \Gamma^6(\Sigma^4 K); \mathbb{Z}/2)$ under the pinch map $C_{\Gamma^6 \alpha} \to \Sigma \Gamma^6(\Sigma^4 K)$.

Proof. The mapping $\alpha : \Sigma^3(\Sigma K) \to S^3$ is in the stable range whose homology is trivial for dimensional reasons. The homology $\Gamma^6 \alpha$ is determined by $(\Gamma^6 \alpha)_* Q^I x = Q^I \alpha_* x$ where $x \in H_*(\Sigma^4 K)$ is a generator. Consequently, $(\Gamma^6 \alpha)_* = 0$. Therefore, the homology decomposes as claimed. For the action of the Steenrod operations, we only need to justify the additional relation and the others are standard. But, this comes form the fact that $\alpha$ is an extension of $\eta$ and there is nontrivial $Sq^2$ action in $\eta$, and that, at least stably, there is commutative diagram

$$\begin{array}{ccc}
\Gamma^6(\Sigma^4 K) & \xrightarrow{\Gamma^6 \alpha} & \Gamma^6 S^3 \\
\downarrow & & \downarrow \\
\Sigma^4 K & \xrightarrow{\alpha} & S^3
\end{array}$$

Since, Steenrod operations are stable this then completes the proof. \hfill $\Box$

**Lemma 4.3.** Let $\kappa \in 2\pi^s_{14}$ be a generator of order 2 given by the a triple Toda bracket $\{\beta, \alpha, \nu\}$. Then $h(\kappa) = 0$ for the Hurewicz homomorphism $h : \pi_{14} Q_0 S^0 \to H_{14}(Q_0 S^0; \mathbb{Z}/2)$.

Proof. The Toda bracket representing $\kappa$ is obtained from considering $S^{13} \to \Sigma^4 K \to S^3 \to S^0$ which yields extension and co-extension maps that fit together to represent $\kappa$ as $S^{14} \to C_\alpha \to S^0$. In order to work in the level of spaces, we need to realise the stable maps as maps of spaces. We proceed as follows. We think of $\nu$ as a genuine map $S^3 \to Q_0 S^0$. Also note that $\Sigma^4 K \to S^3$ is already in the stable range, so it can be taken as a map of spaces. For $\beta : S^9 \to K$, as $K$ has its bottom cell in dimension 0, we need to suspend 10 times which
yields $S^{10} \to \Sigma^{10}K$. We then think of $\beta$ in terms of its 6-th adjoint as $S^{13} \to \Omega^6 \Sigma^6(\Sigma^4K)$. Now, consider the composition

$$S^{13} \xrightarrow{\beta} \Gamma^6 \Sigma^4K \xrightarrow{\varepsilon} \Gamma^6 S^3 \xrightarrow{\nu} Q_0 S^0$$

where $\Gamma^6 = \Omega^6 \Sigma^6$ and the map $\Gamma^6 S^3 \to Q_0 S^0$ is the extension of $S^3 \to Q_0 S^0$ obtained from the fact that $Q_0 S^0$ is already a 6-fold loop spaces. The successive compositions are trivial. This provides extension and co-extension maps that we may compose as

$$S^{14} \xrightarrow{\beta} C\tau_6 \xrightarrow{\nu} Q_0 S^0$$

which represents $\kappa$ as an element in $2\pi_1 Q_0 S^0$. We claim $\beta^3 = 0$, that is there is no spherical class in $H_{14}C\tau_6\alpha$. If $h(\beta^3) \neq 0$ then $Sq^4 h(\beta^3) = 0$ for all $t > 0$. Applying the above lemma, we compute that $H_{14}(C\tau_6\alpha; \mathbb{Z}/2) \to H_{14}(\mathbb{C}^6 S^3)$. Since $\beta^3 \neq 0$, then it could be written as

$$\epsilon_1(Q^3 g_3)(Q^5 g_3) + \epsilon_2 g_3 Q^6 g_3 + \epsilon_3 \Sigma_s((\Sigma^3 a_1)^2(\Sigma^3 a_2)) + \epsilon_4 \Sigma_s((Q^5 \Sigma^3 a_1)(\Sigma^3 a_1)) + \epsilon_5 \Sigma_s Q^8 \Sigma^3 a_2$$

with $\epsilon_i \in \mathbb{Z}/2$ and at least one of then is nonzero. First, consider the component coming from $H_s \Gamma^6 S^3$. Since

$$Sq^2(Q^3 g_3)(Q^5 g_3) = (Q^3 g_3)^2 = g_3^4$$

we conclude that it is impossible to have $\epsilon_1 + \epsilon_2 = 1$ in $\mathbb{Z}/2$ as if one of $\epsilon_1$ or $\epsilon_2$ is nontrivial, then the component of $h(\beta^3)$ in $H_{14} \Gamma^6 S^3$ is not annihilated by $Sq^2$, so $h(\beta^3)$ is not annihilated by $Sq^2$ which is a contradiction. Moreover, if $\epsilon_1 = \epsilon_2 = 1$ then

$$Sq^2((Q^3 g_3)(Q^5 g_3) + g_3 Q^6 g_3) = g_3^4 + g_3 Q^6 g_3 \neq 0.$$ 

This shows that if $h(\beta^3) \neq 0$, then it cannot have any nontrivial component in $H_s \Gamma^6 S^3$. Therefore, if $h(\beta^3) \neq 0$ then it has to project nontrivially onto $H_s(\Gamma^6(\Sigma^4 K); \mathbb{Z}/2)$ under the pinch map, i.e.

$$h(p \circ \beta^3) = \epsilon_3 \Sigma_s(((\Sigma^3 a_1)^2(\Sigma^3 a_2)) + \epsilon_4 \Sigma_s((Q^5 \Sigma^3 a_1)(\Sigma^3 a_1)) + \epsilon_5 \Sigma_s Q^8 \Sigma^3 a_2 \neq 0$$

where $p : C\tau_6 \alpha \to \Gamma^6(\Sigma^4 K)$ is the pinch map. On the other hand, by the construction of Toda bracket, $p \circ \beta^3 = \beta^3$, hence

$$h(\Sigma \beta) = \epsilon_3 \Sigma_s(((\Sigma^3 a_1)^2(\Sigma^3 a_2)) + \epsilon_4 \Sigma_s((Q^5 \Sigma^3 a_1)(\Sigma^3 a_1)) + \epsilon_5 \Sigma_s Q^8 \Sigma^3 a_2$$

which implies that

$$h(\beta) = \epsilon_3 (\Sigma^3 a_1)^2(\Sigma^3 a_2) + \epsilon_4 (Q^5 \Sigma^3 a_1)(\Sigma^3 a_1) + \epsilon_5 Q^8 \Sigma^3 a_2 \in H_{13}(\Gamma^6(\Sigma^4 K); \mathbb{Z}/2)$$

has to be nontrivial. Note that the first two terms in the above sum, if nontrivial, are decomposable classes, so by Cartan formula for $Sq^2$ operations, map to decomposable class under $Sq^4$. Now, if $\epsilon_5 \neq 0$ then we compute that

$$Sq^4 h(\beta) = Q^7 \Sigma^3 a_2 + \text{decomposable terms} \neq 0$$

which is a contradiction, hence $\epsilon_5 = 0$. Therefore, $h(\beta)$ consists of decomposable classes in the Hopf algebra $H_s(\Gamma^6(\Sigma^4 K)); \mathbb{Z}/2$). Since $h(\beta)$ is also primitive, hence by [17] Proposition 4.21] it has to be a square. But, it is obvious non of the first two terms in $h(\beta)$ neither their sum is a square. This is a contradiction as it shows that $h(\beta) = 0$ which contradicts the requirement that $h(p \circ \beta^3) = h(\Sigma \beta) \neq 0$. Therefore, $h(\beta^3) = 0$. Moreover, our computations do not depend on a specific choice of $\beta^3$, hence the indeterminacy in choosing $\beta^3$ is irrelevant here, therefore, $h(\kappa) = 0$. \[\Box\]
Let us note that $\kappa$ can also be represented by a higher Toda bracket which consists only of the Hopf invariant one maps and 2, namely $\{\nu, 7, 2, \eta\} = \{\nu, \eta, \nu, \eta, 2, 2\}$. The above proof then shows that the Hurewicz image of the element represented by this Toda bracket is trivial, a direct proof is however more delicate and involves too much detailed computations.

5 Proof of Theorem 5

By [21 Theorem 1] Curtis conjecture holds on $2\pi_\ast J$. Assuming that the conjecture also holds on $F^2$, and that $h(\tau_i) = 0$, the theorem follows upon applying Theorem [4] Note that the Adams filtration of the elements in $2\pi_\ast J$, consequently the elements of $\langle 2\pi_\ast J \rangle$, is not bounded [14 Theorem 4.2]. Since the Theorem holds on $\langle 2\pi_\ast J \rangle$ then we see that the Adams filtration of the elements on which the conjecture hold is not bounded. Hence, in order to complete the proof we have to verify the conjecture on $F^2$; we postpone computation of $h(\tau_i)$ to another section. The elements of $F^2$ are known: the Hopf invariant one elements $\eta, \nu, \sigma$; the Kervaire invariant one elements $\theta_i$ with $i \leq 6$; elements of Mahowald’s family $\mu_i$; and the element $\nu^* \in 2\pi_{18}^\ast$. The conjecture then is verified on $F^2$ if we show that $h(\mu_i) = 0$ and $h(\nu^*) = 0$. First we deal with $h(\mu_i)$ which is straightforward.

Lemma 5.1. Let $\eta_i \in 2\pi_{2i}Q_0S^0$, $i \geq 3$, denote Mahowald’s family. Then $h(\eta_i) = 0$ where $h : 2\pi_{2i}Q_0S^0 \to H_{2i}Q_0S^0$ is the Hurewicz homomorphism.

Proof. Mahowald’s $\eta_i$ family, living in $2\pi_{2i}^\ast$ for $i \geq 3$, is constructed as a composite of maps $S^{2i} \xrightarrow{f_i} X_i \xrightarrow{g_i} S^0$ where $X_i := F(\mathbb{R}^2, 2^i - 3) \ltimes \Sigma_{j=1}^{2^i-3} (S^7)^{(2^{i-3})}$, with $F(\mathbb{R}^2, 2^i - 3)$ being the configuration space of $2^i - 3$ distinct points in $\mathbb{R}^2$, has its bottom cell is in dimension $2^i - 2^{i-3} = 2^{i-3}$ and top cell in a dimension less than $2^i$. Therefore, the complex $X_i$ is a genuine complex and the mapping $f_i$ can be taken as a genuine mapping. The stable adjoint of $\eta_i$ then is given by $S^{2i} \xrightarrow{\eta_i} X_i \xrightarrow{g_i} Q_0S^0$ where $f_i = 0$ for dimensional reasons. Hence, $h(\eta_i) = 0$.

5.1 Computing $h(\nu^*)$

We begin with a computational result.

Lemma 5.2. Consider $\sigma, 2\sigma \in 2\pi_2^\ast$ and let $\sigma : C_\sigma \to QS^3$ be an extension of $2\sigma$ implied by $(2\sigma) \circ \sigma = 0$. Then $\sigma_* = 0$.

Proof. We want to show that for $\sigma : S^{17} \to S^{10}$ and $2\sigma : S^{10} \to QS^3$, then $\sigma_* = 0$. Since $\sigma$ extends $2\sigma$, hence its homology on the bottom cell is determined by $2\sigma$ which is trivial in $\mathbb{Z}/2$-homology. For the top cell, $\sigma_* g_{18} \in H_{18}QS^3$. The only nonzero admissible monomials in $H_{18}QS^3$ are

$$Q^{15}g_3, Q^{10}Q^5g_3, Q^9Q^6g_3 = (Q^6g_3)^2.$$ 

Hence, if $\sigma_* g_{18} \neq 0$ then

$$\sigma_* g_{18} = \epsilon_1 Q^{15}g_3 + \epsilon_2 Q^{10}Q^5g_3 + \epsilon_3 Q^9Q^6g_3$$

with $\epsilon_i \in \mathbb{Z}/2$ such that at least one of them is nonzero. If $\epsilon_2 = 1$ then by Nishida relations $Sq^1 Q^{2t} = Q^{2t-1}$ and $Sq^1 Q^{2t+1} = 0$ we compute that

$$Sq^1 \sigma_* g_{18} = Q^9Q^5g_3$$
which contradicts the fact that \( Sq^3_2 \sigma_\ast g_{18} = \sigma_\ast Sq^3_\ast g_{18} = 0 \) as \( C_\sigma \) has cell only in dimensions 10 and 18. Hence, \( \sigma_\ast g_{18} = \epsilon_1 Q^{15} g_3 + \epsilon_3 Q^6 g_3 \). Now, applying \( Sq^3_\ast \), we obtain
\[
Sq^3_\ast \sigma_\ast g_{18} = \epsilon_3 Q^8 Q^5 g_3 = \epsilon_3 (Q^5 g_3)^2.
\]
Hence, if \( \epsilon_3 = 1 \) then we run into contradiction as by a similar reasoning, since \( C_\sigma \) has no cells in dimension 16 we must have \( Sq^3_\ast \sigma_\ast g_{18} \). Consequently, the only remaining possibility is
\[
\sigma_\ast g_{18} = Q^{15} g_3.
\]
In order to eliminate this last possibility, we proceed with some unstable computations. The element \( 2\sigma \in \pi^2_7 \cong \pi_{10} QS^3 \) pulls back to \( \pi_{15} S^8 \), hence \( 2\sigma : S^{10} \to QS^3 \) pulls back to a map \( S^{10} \to \Omega^5 S^8 \). The composition \( S^{17} \to S^{10} \to \Omega^5 S^8 \) is trivial which provides a map \( \sigma_1 : C_\sigma \to \Omega^5 S^8 \), providing a factorisation of \( \sigma \) as \( C_\sigma \to \Omega^5 S^8 \to QS^3 \). Hence, if \( \sigma_\ast g_{18} \neq 0 \) implies that \( (\sigma_1)_\ast g_{18} \neq 0 \). Now, we may list the elements of \( H_{18}(\Omega^5 S^8; \mathbb{Z}/2) \) which are
\[
Q^{10} Q^5 g_3, Q^9 Q^6 g_3 = (Q^6 g_3)^2
\]
and can contribute to terms in \( (\sigma_1)_\ast g_{18} \). But, these are eliminated as similar to above. Moreover, the class \( Q^{15} g_3 = Q_{12} g_3 \) cannot live in \( H_\ast(\Omega^5 S^8; \mathbb{Z}/2) \) as we need at least 13 loops to have \( Q_{12} \). Therefore, we cannot have \( \sigma_\ast g_{18} \neq 0 \). This completes the proof.

Now, we may complete computation of \( h(\nu^\ast) \).

**Theorem 5.3.** Let \( \nu^\ast \in 2\pi^8_{18} \) be the element determined by Toda bracket \( \{\sigma, 2\sigma, \nu\} \). Then it maps trivially under \( h : 2\pi_{18} Q_0 S^0 \to H_{18}(Q_0 S^0; \mathbb{Z}/2) \).

**Proof.** Consider the stable adjoint of \( 2\sigma \) and the infinite loop extension of the stable adjoint of \( \nu \) as
\[
2\sigma : S^{10} \to QS^3, \ \nu : QS^3 \to Q_0 S^0.
\]
For the composition \( S^{17} \xrightarrow{\sigma} S^{10} \xrightarrow{2\sigma} S^3 \xrightarrow{\nu} S^0 \), the map \( S^{17} \to S^{10} \) is in the stable range, hence the stable adjoint of this composition is the top row in the following diagram
\[
\begin{array}{cccccc}
S^{17} & \xrightarrow{\sigma} & S^{10} & \xrightarrow{2\sigma} & QS^3 & \xrightarrow{\nu} & Q_0 S^0 \\
\downarrow & & \downarrow \sigma & & \downarrow \nu & & \\
C_\sigma & & C_{2\sigma} & & C_\nu & & \\
\downarrow \circ \sigma & & \downarrow \circ \nu & & \downarrow \circ \nu & & \\
S^{18} & & & & & & \\
\end{array}
\]
where the successive compositions \( (2\sigma) \circ \sigma \) and \( \nu \circ (2\sigma) \) are trivial, and give rise to the extension and coextension maps \( \sigma^\circ, \nu_\circ \) whose composition is ought to realise \( \nu^\ast \), i.e.
\[
S^{18} \xrightarrow{\sigma^\circ} C_{2\sigma} \xrightarrow{\nu_\circ} Q_0 S^0
\]
represents the Toda bracket \( \{\sigma, 2\sigma, \nu\} \) for \( \nu^\ast \). Since the map \( \sigma : S^{17} \to S^{10} \) has trivial homology, therefore the generator \( g_{18} \in H_{18} S^{18} \) is in the image of the pinch map \( C_\sigma \to S^{18} \). Therefore, by the previous lemma we have
\[
(\nu_\circ \circ \sigma^\circ)_\ast g_{18} = (\nu_\circ \circ \sigma^\circ \circ \nu)_\ast g_{18} = \nu_\ast \sigma_\ast g_{18} = 0.
\]
We do not rely on specific choices of \( \sigma^\circ \) or \( \nu_\circ \). Therefore, \( h(\nu^\ast) = 0 \). This completes the proof. \( \square \)
6 Hurewicz homomorphism and Toda brackets

Examples such as computing $h(\kappa)$ and $h(\mathcal{I})$ show that computing Hurewics image of an element represented by a Toda bracket could be very tedious and involved. However, in some cases, it is possible to use dimensional arguments. We provide two partial results in direction. We first prove an integral version.

**Theorem 6.1.** (i) Let $\alpha \in \pi^i_s$, $\beta \in \pi^j_s$, and $\gamma \in \pi^k_s$ with $i < j + k$ and $j < k$ so that the Toda bracket $\{\alpha, \beta, \gamma\}$ is defined. Then the element of $\pi_{i+j+k+1}$ represented by Toda bracket $\{\alpha, \beta, \gamma\}$ maps trivially under the Hurewicz homomorphism $h : \pi_*Q_0S^0 \to H_*(Q_0S^0; \mathbb{Z})$.

(ii) Suppose $\alpha \in \pi^i_sX_1$. Let $\beta : X_1 \to X_2$, $\gamma : X_2 \to S^0$ be stable maps, with $X_1, X_2$ being (stable) complexes, $i < 2\text{conn}(X_1)$, $\dim X_1 < 2\text{conn}(X_2)$, $i > \dim X_1 > \dim X_2$, so that the Toda bracket $\{\alpha, \beta, \gamma\}$ is defined. Here, $\text{conn}(X) = n$ if $\pi^i_tX \simeq 0$ for all $t < n$. Then, the element of $\pi_{i+1}Q_0S^0$ represented by Toda bracket $\{\alpha, \beta, \gamma\}$ maps trivially under the Hurewicz homomorphism $h : \pi_{i+1}Q_0S^0 \to H_{i+1}(Q_0S^0; \mathbb{Z})$.

**Proof.** First, we prove (i). The conditions $i < j + k$ and $j < k$ ensure that the maps $\alpha : S^{i+j+k} \to S^{i+k}$ and $\beta : S^{i+j+k} \to S^k$ are in the stale range. Hence, in order to realise the Toda bracket $\{\alpha, \beta, \gamma\}$ as an element in $\pi_*Q_0S^0$ it is enough to consider the composition

$$S^{i+j+k} \to S^{i+k} \to S^k \to QS^0$$

where the successive compositions are trivial. This leads to extension and co-extension maps $\alpha^\flat : S^{i+j+k+1} \to C_\beta$ and $\gamma^\sharp : C_\beta \to QS^0$ with the composition $S^{i+j+k+1} \to C_\beta \to QS^0$ representing $\{\alpha, \beta, \gamma\}$ in $\pi_*QS^0$ up to indeterminacy in choosing the extension and coextension maps. Notice that the space $C_\beta$ is a genuine complex with its top cell in dimension $j + k + 1$.

Hence, regardless the choice of $\alpha^\flat$, for dimensional reasons $\alpha^\flat \gamma^\sharp = 0$, hence $\gamma \circ \alpha^\flat \gamma^\sharp = 0$. This implies that the Toda bracket $\{\alpha, \beta, \gamma\}$ represents an element which acts trivially in homology, i.e. it maps trivially under the Hurewicz homomorphism $h : \pi_*Q_0S^0 \to H_*(Q_0S^0; \mathbb{Z})$.

The proof of (ii) is similar. In order to realise $\{\alpha, \beta, \gamma\}$ as an element of $\pi_*Q_0S^0$, think of $\gamma$ as $\gamma : X_2 \to Q_0S^0$. The conditions on the connectivity and dimension of the complexes ensure that the maps $\alpha, \beta$ are genuine maps among complexes. We then may consider the composite

$$S^i \to X_1 \to X_2 \to Q_0S^0$$

where the successive compositions are trivial. This yields extension and coextension maps $\alpha^\flat : S^{i+1} \to C_\beta$ and $\gamma^\sharp : C_\beta \to Q_0S^0$ with the composition $S^{i+1} \to C_\beta \to Q_0S^0$ realising $\{\alpha, \beta, \gamma\}$ up to indeterminacy in choosing extension and coextension maps. However, the condition $\dim X_1 > \dim X_2$ implies that, regardless the indeterminacy, $C_\beta$ has its top cell in dimension $\dim X_1 + 1 < i + 1$. Therefore, $\alpha^\flat \gamma^\sharp = 0$ and consequently $(\gamma \circ \alpha^\flat \gamma^\sharp) = 0$. This completes the proof.

The integral case, implies the $p$-primary case. Next, we provide some necessary condition for nonvanishing of the Hurewicz image of an elements in $\pi_*Q_0S^0$ which is represented by a Toda bracket. We have the following.

**Proposition 6.2.** Let $\alpha \in 2\pi^i_sX_1$, $\beta : X_1 \to X_2$ and $\gamma : X_2 \to S^0$ be stable maps with $X_1$ and $X_2$ being stable complexes of finite type. Suppose that the Toda bracket $\{\alpha, \beta, \gamma\}$ is defined and represents $f \in 2\pi^i_s$. Then the following statements are equivalent.

(i) If $\alpha : S^i \to QX_1$ is trivial in homology and $f$ maps nontrivially under $h : 2\pi^i_s \simeq 2\pi_*Q_0S^0 \to H_*(Q_0S^0; \mathbb{Z}/2)$ then $\gamma : QX_2 \to QS^0$ is nontrivial in homology.

(ii) If $\gamma : S^i \to QX_1$ is trivial in homology and $f$ maps nontrivially under $h : 2\pi^i_s \simeq 2\pi_*Q_0S^0 \to H_*(Q_0S^0; \mathbb{Z}/2)$ then $\alpha : QX_2 \to QS^0$ is nontrivial in homology.
(iii) If $\alpha : S^i \to QX_1$ and $\gamma : QX_2 \to QS^0$ are trivial in homology, then $f$ maps trivially under the Hurewicz homomorphism $h : 2\pi_* \simeq 2\pi_* Q_0 S^0 \to H_*(Q_0 S^0; \mathbb{Z}/2)$.

(iv) If $h(f) \neq 0$ then either $\alpha_* \neq 0$ or $\gamma_* \neq 0$.

Note that the statements (i) to (iv) are logically equivalent, and we have only mentioned them for the sake of completeness.

Proof. We prove (i). Since the Toda bracket is defined, we may build a commutative diagram as

$$
\begin{array}{ccc}
S^i & \xrightarrow{\alpha} & QX_1 & \xrightarrow{\beta} & QX_2 & \xrightarrow{\gamma} & QS^0 \\
& \downarrow{\alpha'} & \downarrow{\beta} & \downarrow{\gamma} & \downarrow{\gamma'} & \downarrow{\gamma''} & \\
& C_\alpha & \to & C_\beta & \to & \\
& \downarrow{p} & \downarrow{\alpha'} & \downarrow{p} & \downarrow{\alpha'} & \\
S^{i+1} & \to & \\
\end{array}
$$

where the diagonal maps are the extensions implied by $\beta \alpha = 0$ and $\gamma \beta = 0$. The composition $\gamma_\alpha \circ \alpha^3$ represents $f$ up the indeterminacy in choosing the extension and co-extension maps. The assumption $h(f) \neq 0$, however, will imply that with any choice for $\alpha^3$ and $\gamma_\beta$ we must have $(\gamma_\alpha \circ \alpha^3)^* \neq 0$ which implies $(\gamma_\beta)^* \neq 0$. Since $\alpha_* = 0$ then the generator $g \in H_{i+j+k} S^{i+j+k}$ belongs to the image of $p_*$. The commutativity of the diagram then implies that $(\gamma_\alpha \circ \beta \circ \gamma_\beta)^* \neq 0$ which shows that $\gamma_* = (\gamma_\alpha \circ \gamma_\beta)^* \neq 0$.

Note 6.3. Proposition 6.2 leaves us with the cases $\alpha_* \neq 0$. The assumption $h(f) \neq 0$ implies that $\alpha_* \neq 0$. Since $(\Sigma \alpha)_* \neq 0$ then $q : C_\beta \to \Sigma QS^{j+k}$ is nontrivial in homology and in particular $q_* \alpha^3(g) \neq 0$. This latter means that $\alpha^3(g) \in H_* C_\beta$ on which $(\gamma_\beta)^*$ acts nontrivially does not come of the image of $(\gamma_\beta)^*$. Hence, in this case $\gamma$ could be trivial or nontrivial in homology.

We wish to conclude that somehow the above proposition has to be enough for the general case. By definition of higher Toda brackets (see [20] for a modified version of Cohen’s definition) it seems that a long Toda bracket of maps among spheres will reduce into a triple Toda bracket of three maps among complexes. Unfortunately, we do not have more detailed result in this direction, as in such cases the complexity of the complexes in the triple Toda bracket seems to increase which enforces to use more delicate and detailed computation such as those one appearing in the computation of $h(\kappa)$.

7 Hurewicz homomorphism and homotopy operations

The motivation for this section is again provided by Theorem 2 and Theorem 4. Given $S \subset \pi^*_s$, we consider extensions of $S$ by applying homotopy operations. The operations that we consider here, are of specific type; writing $D_r$ for the $r$-adic construction $(E \Sigma r)^{\wedge} \Sigma r$, then a given element $\alpha \in \pi_m D_r S^n$ determines an operation $\alpha^* : \pi_s \to \pi_s$ which sends $f \in \pi_s$ to the element given by the composite

$$
S^m \xrightarrow{D_r S^n} D_r S^0 \xrightarrow{D_r f} D_r S^0 = B \Sigma r_+ \to S^0
$$

where $D_r S^0 \to S^0$ is induced by the multiplication of $S^0$ as an $H_\infty$ spectrum. Of course, it is not guaranteed that such operations always will exist or give rise to new elements. But,
when they are defined then we can ask about their Hurewicz image. The following then is more or less expected.

**Proposition 7.1.** Let \( \alpha \in \pi_mD_rS^n \). Suppose \( f \in 2\pi^n \) such that it maps trivially under \( h : 2\pi_0Q_0S^0 \to H_*(Q_0S^0, Z/2) \). Then \( h(\alpha^*(f)) = 0 \).

**Proof.** In order to compute \( h(\alpha^*(f)) \in H_*(Q_0S^0; Z/2) \) we need to compute the homology of the stable adjoint of \( S^m \to D_rS^n \xrightarrow{D(f)} D_rS^0 = Q(B\Sigma_{r+}) \to S^0 \) given by

\[
S^m \to QD_rS^n \xrightarrow{D(f)} QD_rS^0 = Q(B\Sigma_{r+}) \to QS^0.
\]

The homology ring \( H_*(QD_rS^n) \) has monomial generators of the form \( Q^I\xi \) with \( \xi \) being an element of height \( r \) in \( H_*(Q^nS^n) \). Since, for \( S^n \), with \( n > 0 \), being path connected, the James-Hopf map \( j_r : QS^n \to QD_rS^n \) acts like projection onto the elements of height \( r \) in homology, then by naturality of James-Hopf maps we have

\[
(D_r f)_* Q^I\xi = (D_r f)_* Q^I(j_r)_*\xi = (j_r)_* (Q f)_* Q^I\xi = (j_r)_* Q^I f_* \xi = 0.
\]

This shows that \( (D_r f)_* = 0 \) and consequently \( h(\alpha^*(f)) = 0 \).

As an application, let \( \tau_1 \) denote Bruner’s family which is constructed using homotopy operation \( \cup_1 \) as \( \tau_1 := \cup_1(\eta_k) - \eta_{k+1} \) \([4]\). We have the following.

**Lemma 7.2.** The elements \( \tau_1 \) map trivially under the Hurewicz homomorphism \( 2\pi_{2^i+1}Q_0S^0 \to H_{2^i+1}Q_0S^0 \).

**Proof.** By the above proposition \( h(\cup_1(\eta_k)) = 0 \). Moreover, \( \eta_{k+1} \) is realised as \( S^{2^i+1} \to S^{2^i} \to Q_0S^0 \) which is trivial in homology for dimensional reasons. Hence, \( h(\tau_1) = 0 \).

## 8 Hurewicz homomorphism and Adams filtration

This section is devoted to obtain a numerical conditions on the elements of \( 2\pi^n \) which map nontrivially. Let \( D_k \) be the Dickson algebra on \( k \) variables and \( A \) be the mod 2 Steenrod algebra. Recall the Lannes-Zarati homomorphism

\[
\varphi_k : \text{Ext}^{k,k+1}_A(\mathbb{Z}/2, \mathbb{Z}/2) \to (\mathbb{Z}/2 \otimes_A D_k)^*
\]

where its domain is the \( E_2 \)-term of the Adams spectral sequence. By \([13]\) Corollary 3.3, Proposition 3.5, and decomposition of Steenrod squares into composition of \( Sq^2 \), the target of \( \varphi_k \) is the \( R \)-submodule of \( H_*(Q_0S^0; \mathbb{Z}/2) \) generated by elements of length \( k \), (compare \([?]\) for more detailed discussion). Here, we use

\[
H_*(Q_0S^0; \mathbb{Z}/2) \simeq \mathbb{Z}/2[Q^I[1] * [-2^{l(I)}] : I \text{ admissible}]
\]

so the image of a cycle in \( \text{Ext}^{k,k+1}_A(\mathbb{Z}/2, \mathbb{Z}/2) \) under \( \varphi_k \), modulo decomposable terms, will be a linear combination of classes \( Q^I[1] * [-2^{l(I)}] \) with \( l(I) = k \). Here, the action of the homology suspension on the generators is determined by

\[
\sigma_*(Q^I[1] * [-2^{l(I)}]) = Q^I g_1
\]

where \( g_1 \in H_1S^1 \) is a generator. We then have the following.
Lemma 8.1. Suppose \( f \in 2\pi_r Q_0 S^0 \) of Adams filtration \( k \), i.e. \( k \) is the least positive integer where \( f \) is represented by a cycle in \( \text{Ext}_A^{k,k+n}(\mathbb{Z}/2,\mathbb{Z}/2) \), and \( h(f) \neq 0 \) where \( h : 2\pi_n Q_0 S^0 \to H_n(Q_0 S^0,\mathbb{Z}/2) \) is the Hurewicz homomorphism. The following statement then hold.

(i) If \( \sigma_* h(f) \neq 0 \) then \( n \geq 2^k - 1 \).

(ii) If \( \sigma_* h(f) = 0 \) then \( n \geq 2^{k-1} \).

Proof. The Hurewicz homomorphism and Lannes-Zarati homomorphisms are compatible. This implies that \( h(f) \in \text{im}(\varphi_k) \). (i) Since \( h(f) \in \text{im}(\varphi_k) \) then

\[
h(f) = \int_{l(I) \geq k, \epsilon_I \in \mathbb{Z}/2} \epsilon_I Q^I [1] * [-2^{l(I)}] + \text{decomposable terms}
\]

such that there exists at least one \( I \) with \( \epsilon_I = 1 \). This yields

\[
\sigma_* h(f) = \int_{l(I) \geq k, \epsilon_I \in \mathbb{Z}/2} \epsilon_I Q^I g_1.
\]

Let \( l(f) := \max \{ l(I) : \epsilon_I = 1 \} \), and note that \( l(f) \geq k \). Write \( j_{2l(f)} \) for the \( 2^{l(f)} \)-th stable James-Hopf invariant \( QS^1 \to QD_{2l(f)} S^1 \); it acts like projection on the elements of height \( 2^{l(f)} \) such as \( Q^l g_1 \) with \( l(I) = l(f) \), and all term \( Q^g g_1 \) with \( l(J) < l(f) \) are killed under this map. Hence,

\[
(j_{2l(f)})_* h(f) = \int_{l(I) = l(f), \epsilon_I \in \mathbb{Z}/2} \epsilon_I Q^I g_1 \neq 0.
\]

The space \( D_{2l(f)} S^1 \) has its bottom cell in dimension \( 2^{l(f)} \), and the homology of \( H_* QD_{2l(f)} S^1 \) vanishes in dimensions below \( 2^{l(f)} \). Hence, a necessary condition for \( h(f) \neq 0 \) is that \( n + 1 \geq 2^{l(f)} \geq 2^k \). This proves the desired inequality.

(ii) Since \( \sigma_* h(f) = 0 \) then

\[
h(f) = \int_{l(I) \geq k, \epsilon_I \in \mathbb{Z}/2} \epsilon_I Q^I [1] * [-2^{l(I)}] + D
\]

such that there exists at least one \( I \) with \( \epsilon_I = 1 \), and for every \( I \) we have \( \text{excess}(I) = 0 \), and \( D \) is a sum of terms each of which is a finite product of elements of the form \( Q^I [1] * [-2^{l(I)}] \). Since, each term in \( h(f) \) is a decomposable, hence as a primitive element it has to be a sequence; noting that any term \( Q^I [1] * [-2^{l(I)}] \) with \( \text{excess}(I) = 0 \) is already a square we conclude that \( D \) is also a sum of square terms. This allows us to write

\[
h(f) = \left( \int_{l(I) \geq k, \epsilon_I \in \mathbb{Z}/2} \epsilon_I Q^I_1 [1] * [-2^{l(I)}] + D_1 \right)^2
\]

where \( I = (i_1, I_1) \). We first consider the looped 2nd stable James-Hopf invariant \( \Omega_{j_2} : QS^0 \to QP \), with \( j_2 : QS^1 \to QD_2 S^1 \), which we know is a monomorphism in \( 2\pi_* \) as well as \( \mathbb{Z}/2 \)-homology. Moreover, this map respects the filtration in the sense that its composition with the Kahn-Priddy map is the identity up to elements of higher height filtration, this follows from Kuhn’s delooped version of the Kahn-Priddy Theorem [?, Corollary 2.14](see also [8]). Noting that \( \Omega j_2 x_i = a_i \) with \( a_i \in H_* P \simeq \mathbb{Z}/2 \) denoting a generator in grading \( i \), and that \( (\Omega j_2)_* \) is multiplicative, we have

\[
(\Omega j_2)_* h(f) = \left( \int_{l(I) \geq k, \epsilon_I \in \mathbb{Z}/2} \epsilon_I Q^{I_1} a_i + D_1 \right)^2 + O^2
\]

where for \( I = (i_1, \ldots, i_s) \) we have \( I = (i_1, I_1, i_s) \), and \( O \) is a the sum of terms such as \( Q^{I_1} a_{j_1} \) with \( \dim a_{j_1} < \dim a_i = s \) and \( l(J_1) = l(I_1) = s - 2 \) given by Kuhn’s formula. The important
fact since $\Omega j_2$ is an injection, then the above sum is nontrivial. Now, consider the stable James-Hopf invariant $j_{2k-1} : QP \to QD_{2k-1}P$. Now, let $l_2(f) = \max(l(I^1) : \epsilon_I \neq 0)$ and consider the $2^l(f)^j$th James Hopf invariant $QP \to QD_{2^l(f)^j}P$. It then follows that

$$(j_{2^l(f)^j})_*(\Omega j_2)_*h(f) = \left(\int_{l(I)=l_2(f)+2, \epsilon_I \in \mathbb{Z}/2} \epsilon_I Q^{I^1} a_i + Q^{I^j} a_{ij} + O''\right)$$

The space $D_{2k-1}P$ has bottom cell in dimension $2^{k-1}$. It follows that $n \geq 2^{k-1}$ is a necessary condition so that $h(f)$ is nontrivial.

A immediate corollary is the following.

**Corollary 8.2.** Suppose $f \in 2\pi_nQ_0S^0$ is represented by a permanent cycle coming from the $k$-line of the Adams spectral sequence, i.e. it has Adams filtration $k$.

(i) If $n < 2^{k-1}$ then $h(f) = 0$.

(ii) If $n < 2^k - 1$ then either $h(f) = 0$ or $h(f) \neq 0$ with $2^{k-1} \leq n < 2^k - 1$.

The above corollary seems to be a $HZ/2$-version of a recent result of Kuhn [12, Corollary 1.5]. A similar result as above can be obtained regarding the Adams filtration of the spherical classes in $H_*QX$ for $X$ being path connected, which we leave to the reader.

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