On the solutions of a singular elliptic equation concentrating on a circle

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Abstract

Let \( A = \{ x \in \mathbb{R}^{2N+2} : 0 < a < |x| < b \} \) be an annulus. Consider the following singularly perturbed elliptic problem on \( A \)

\[
-\varepsilon^2 \Delta u + |x|^{\alpha} u = |x|^{\alpha} u^p, \quad \text{in } A \\
u > 0, \quad \text{in } A \\
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial A
\]

\( 1 < p < 2^{*} - 1. \) We shall show that there exists a positive solution \( u_\varepsilon \) concentrating on an \( S^1 \) orbit as \( \varepsilon \to 0. \) We prove this by reducing the problem to a lower dimensional one and analyzing a single point concentrating solution in the lower dimensional space. We make precise how the single peak concentration depends on the parameter \( \alpha. \)

1 Introduction

Consider the following singularly perturbed elliptic equation with super linear nonlinearity on an annulus in \( \mathbb{R}^{2N} \)

\[
-\varepsilon^2 \Delta u + |x|^{\alpha} u = |x|^{\alpha} u^p, \quad \text{in } A \\
u > 0, \quad \text{in } A \\
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial A
\]

\( \star \)

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$1 < p < 2^* - 1$, $\varepsilon$ is a singular perturbation parameter. $A = \{x \in \mathbb{R}^{2N+2} : 0 < a < |x| < b\}$. $\alpha$ any real number.

The result of point concentration on bounded domains has been well established by several authors [6, 7, 8]. In these works, the behavior of the least energy solutions and their concentration phenomena has been studied. For the Dirichlet problem W.-M. Ni and J. Wei [8], have shown that the least energy solution can have at most one local maximum and the point of maximum converges to a point which stays at maximum distance from the boundary. W.-M. Ni and Takagi in [6, 7] have analyzed the Neumann problem, where they have also shown that a least energy solution can have at most one local maximum but it will lie on the boundary for sufficiently small $\varepsilon$ and it will converge to a point of maximum mean curvature of the boundary. Later J. Byeon and J. Park in [2] have generalized the same results for both boundary conditions on a Riemannian manifold.

Also the $N - 1$ dimensional (sphere) concentration of the problem in the presence of a potential has been studied by A. Ambrosetti, A. Malchiodi and W.-M. Ni in [11] where they have looked at the radial solutions and established the concentration phenomena which depends upon the behavior of the potential.

The inspiration for the work comes from the result by Bernhard Ruf and P. N. Srikanth [10] where the authors have found a solution concentrating on a circle in the case of Dirichlet data. The problem was considered in dimension 4 and using the $S^1$ action on $S^3$ the problem is reduced to another Singularly perturbed elliptic problem on an annulus in dimension 3. In a recent work, [11] Pacella and Srikanth have generalized this result to find solutions concentrating on $S^{N-1}$ orbit where the domain is an annulus in $\mathbb{R}^{2N}$.

We adapt ideas from [2] and [10] in the present case where the reduced problem is studied on a warped-product manifold. The main theorem we prove here is the following:

**Theorem 1.1.** There is a solution of (1.1) concentrating on an $S^1$ orbit, which lies on the inner boundary for $\alpha < \frac{2}{2N-1}$, on the outer boundary for $\alpha \geq \frac{2}{2N-1}$.

We can re-write the equation (1.1) as

$$
\begin{align*}
-\varepsilon^2 \Delta u + |x|^\alpha u &= |x|^\alpha f(u), & \text{in } A \\
u > 0 & & \text{in } A \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial A \\
\end{align*}
$$

(1.2)
where \( f(t) = t^p \), for \( t > 0 \) and \( = 0 \), for \( t \leq 0 \). Then any solution of (1.2) is positive and hence a solution of (1.1) also.

The basic idea is to reduce the problem to lower dimension using an \( S^1 \) action which leads to the Hopf Fibration. Recall that the annulus has an warped product structure as

\[
I \times S^{2N+1}
\]

with the product metric

\[
g_A = dr^2 + r^2 dS_{2N+1}
\]

We write the co-ordinates of \( S^{2N+1} \) as \((z_1, z_2, \ldots, z_{n+1})\). The Hopf map \( S^1 \hookrightarrow S^{2n+1} \to \mathbb{CP}^n \) can be described as \((z_1, z_2, \ldots, z_{n+1}) \to (\frac{z_1}{z_2}, \frac{z_3}{z_4}, \ldots, 1, \ldots, \frac{z_{2n+1}}{z_1}) \) provided \( z_i \neq 0 \). Also under this transformation \( \Delta_{S^{2N+1}} \) goes to \( \Delta_{\mathbb{CP}^N} \) (Details can be found in [11]). Also choosing a proper scaling of the radius we reduce the problem in a lower dimensional singularly perturbed problem on the warped product manifold \( M = I' \times_f \mathbb{CP}^N \) with the product metric \( g_M = ds^2 + \frac{2N-1}{2N} r^2 g_{\mathbb{CP}^N} \). Where \( I' = (\frac{2N}{2N-1})^{\frac{1}{2N-1}} (a^{\frac{2N}{2N-1}}, b^{\frac{2N}{2N-1}}) \), and \( g_{\mathbb{CP}^N} \) is the Fubini Study metric on \( g_{\mathbb{CP}^N} \) (for details please look at the appendix).

We shall seek for a solution of the reduced equation and try to get the behavior of the sequence of solutions. We shall prove that as \( \varepsilon \to 0 \) ‘up-to a subsequence’ the solutions concentrates at a single point on the boundary. We lift the solution in the annulus to get solutions concentrating on \( S^1 \).

### 2 The group action and reduction

Let \( A = \{ x \in \mathbb{R}^{2N+2} : 0 < a < |x| < b \} \) is an annular domain in \( \mathbb{R}^{2N+2} \). We can express \( A \) as a product manifold \( A = I \times_r S^{2N+1} \), where \( I = (a, b) \), with the product metric

\[
g = dr^2 + r^2 dS_{2N+1}
\]

Let \( H^1_0, rad \subset H^1_0(A) \) where \( H^1_{rad} \) denote the space of radial functions in \( H^1(A) \) consists of radial functions.
Consider a suitable co-ordinate representation of the annulus $A$ such that any point $z \in A$ can be written as

$$z \equiv z(r, t_1, ..., t_n, \theta_1, ..., \theta_{N+1})$$

where $a < r < b$ and $0 \leq t_i < \pi/2, (i = 1, 2, ..., n)$ and $0 \leq \theta_j < 2\pi, (j = 1, 2, ..., n + 1)$.

Note that the angles $0 \leq \theta_i < 2\pi (i = 1, 2)$ represents the angle between $(x_{2i-1}, x_{2i})$ in the $x_{2i-1}, x_{2i}$, and $0 \leq t_j < \pi/2$ is the angle between the respective planes.

Now consider the following one parameter group action $T_\tau$ on $A$: Define

$$z(r, t_1, ..., t_n, \theta_1, ..., \theta_{N+1}) = (x_1, x_2, ..., x_{2N+2})$$

Then let $T_\tau(z) = z(r, t_1, ..., t_n, \theta_1 + \tau, ..., \theta_{N+1} + \tau)$ for $\tau \in [0, 2\pi)$. Define $H_1^1(A) \subset H^1(A)$ by

$$H_1^1(A) = \{ u \in H^1(A) : u(T_\tau(z)) = u(z), \forall \tau \in [0, 2\pi) \}. \tag{2.2}$$

**Lemma 2.1.** $T_\tau : A \to A$ is a fixed point free group action.

**Remark.** The above lemma is important. As any solution concentrating on a fixed point shall not give any concentrating orbit.

For $u \in H_1^1(A)$ we see that $u(T_\tau(z)) = u(z), \forall \tau \in [0, 2\pi)$, so $u(T_{-\theta_{N+1}}(z)) = u(z)$ Let us define new variables $\psi_i = \theta_i - \theta_{N+1}$ and define $v(r, t_1, ..., t_n, \psi_1, ..., \psi_N) = u(T_{-\theta_{N+1}}(z))$

**Lemma 2.2.** $v$ is well defined.

The proofs of lemma 2.1 and lemma 2.2 are given in the appendix.

Note that any solution of (1.2) is a critical point of the functional

$$J_\varepsilon(u) = \int_A \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + |x|^\alpha \frac{u^2}{2} - |x|^\alpha F(u) \right) dx$$

$$= \int_{I \times S^{2N+1}} \left[ \frac{\varepsilon^2}{2} (u_r^2 + \frac{1}{r^2} |\nabla S^{2N+1} u|^2) + r^\alpha \frac{u^2}{2} - r^\alpha F(u) \right] r^{2N+1} dr d\sigma_{S^{2N+1}}$$
Let $\Psi : A \to M$ be the Riemann submersion, under the metric (5.4), given by

$$\Psi(r, \theta) = (s, \psi)$$

(2.3)

where $\theta$ is a point on $S^{2N+1}$ and $\psi$ is the image of $\theta$ of the map (5.3), (see appendix A). Denote the projections by $s, \sigma$ from $M$ onto $I'$ and $\mathbb{CP}^N$ respectively. Then under this change of co-ordinates the energy functional $J_\varepsilon(u)$ takes the form (see appendix B)

$$2\pi \int_{I' \times \mathbb{CP}^N} \left[ \frac{\varepsilon^2}{2} \left( v_s^2 + \left( \frac{2N}{2N-1} s \right)^{-2} |\nabla_{\mathbb{CP}^N} v|^2 \right) ight. + \left. \left( \frac{2N-1}{2N} \right)^{\frac{4N+2+\alpha}{2N}} \left( \frac{v^2}{2s^{-\frac{2+2N\alpha}{2N}}} - \frac{F(v)}{s^{-\frac{\alpha+2-2N\alpha}{2N}}} \right) \right] s^{2N} ds dV_{\mathbb{CP}^N}$$

(2.4)

So the critical points of $J_\varepsilon$ in $H^1_\varepsilon(A)$ corresponding to solutions of the following equation

$$-\varepsilon^2 \Delta_g u + \left( \frac{2N-1}{2N} \right)^{\frac{4N+2+\alpha}{2N}} \left( \frac{v}{|s(p)|^\eta} - \frac{f(v)}{|s(p)|^\eta} \right) = 0 \quad \text{in } M$$

$$u > 0 \quad \text{in } M$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial M$$

where $\eta = \frac{\alpha+2-2N\alpha}{2N}$ and $s$ is the projections from $M$ onto $I'$.

Or equivalently replacing $\varepsilon$ by $\left( \frac{2N-1}{2N} \right)^{\frac{4N+2+\alpha}{2N}} \varepsilon$ we have

$$-\varepsilon^2 \Delta_g u + \frac{v}{|s(p)|^\eta} - \frac{f(v)}{|s(p)|^\eta} = 0 \quad \text{in } M$$

$$u > 0 \quad \text{in } M$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial M$$

(2.5)

where $f(t) = 0$ for $t < 0$, $t^\alpha$ for $t \geq 0$.

We look for a single peak solution for the equation (2.5). In order to do this we will find a Mountain Pass solution to equation (2.5) and analyze its behavior to prove our theorem. We shall first analyze the limit equation of (2.5). It turns out that (as we shall see in the next section), for a mountain pass solution $u_\varepsilon$ of (2.5) the transformed function $U_\varepsilon(x) = u_\varepsilon(\varepsilon x_{\mu_0}^{-1}(x))$ converges uniformly to a solution of the equation (in $C^2_{\text{loc}}$ sense).

$$\Delta U - \frac{U}{|s(\mu_0)|^\eta} - \frac{f(U)}{|s(\mu_0)|^\eta} = 0, U > 0 \quad \text{in } \{ x \in \mathbb{R}^{2N+1} : \mu \cdot x > 0 \}$$

$$\frac{\partial U}{\partial \mu} = 0 \quad \text{on } \{ x \in \mathbb{R}^{2N+1} : \mu \cdot x = 0 \}$$

$$\lim_{|x| \to \infty} U(x) = 0$$

(2.6)
for some unit vector $\mu \in \mathbb{R}^{2N+1}$, $P_0 \in \partial M$ and for some local co-ordinate $x_{P_0}$ around $P_0$. Let $|s(P_0)|^2 = \kappa$.

Let $V(x) = U(\frac{x}{\sqrt{\kappa}})$, the for $U$ solving (2.6), $V$ satisfies
\[
\begin{align*}
\Delta V - V + f(V) &= 0, \quad V > 0 \quad \text{in } \{x \in \mathbb{R}^{2N+1} : \mu \cdot x > 0\} \\
\frac{\partial V}{\partial \mu} &= 0 \quad \text{on } \{x \in \mathbb{R}^{2N+1} : \mu \cdot x = 0\} \\
\lim_{|x| \to \infty} V(x) &= 0
\end{align*}
(2.7)
\]

With choosing proper co-ordinate chart around $P_0$ we can have $\mu = (0, \ldots, 0, 1)$. For $U \in H^1(\mathbb{R}_+^{2N+1})$ define
\[
\Gamma(U) = \int_{\mathbb{R}^{2N+1}_+} \left[ \frac{1}{2} |\nabla U|^2 + \frac{U^2}{2\kappa} - \frac{F(U)}{\kappa} \right] dx
(2.8)
\]

Let $B$ be the set of all solutions $U$ of equation (2.6) with $\mu = (0, 0, \ldots, 0, 1)$ satisfying
\[
U(0) = \max_{x \in \mathbb{R}^{2N+1}_+} U(x)
\]

Then the following results are well known about $U$

**Proposition 2.3.** Any $V \in B$ is radially symmetric and $V'(r) < 0$ for $r > 0$. Moreover there exist a $C, c > 0$ such that $V(x) + |\nabla V(x)| \leq C \exp(-c|x|)$. Also the set $B$ is compact in $H^1(\mathbb{R}_+^{2N+1})$.

**Proposition 2.4.** For $U \in B$ we have for $1 \leq j \leq 2N, \ m \geq 0$ the following identities
\[
\begin{align*}
(i) & \quad \int_{\mathbb{R}_+^{2N+1}} \left[ \frac{1}{2} |\nabla U|^2 + \frac{U^2}{2\kappa} - \frac{F(U)}{\kappa} \right] x_{2N+1}^m dx = \frac{m + 1}{2N + m + 1} \int_{\mathbb{R}_+^{2N+1}} |\nabla U|^2 x_{2N+1}^m dx \quad (2.9) \\
(ii) & \quad \int_{\mathbb{R}_+^{2N+1}} \left( \frac{\partial U}{\partial x_j} \right)^2 x_{2N+1}^m dx = \frac{1}{2N + m + 1} \int_{\mathbb{R}_+^{2N+1}} |\nabla U|^2 x_{2N+1}^m dx \quad (2.10) \\
(iii) & \quad \int_{\mathbb{R}_+^{2N+1}} \left( \frac{\partial U}{\partial x_{2N+1}} \right)^2 x_{2N+1}^m dx = \frac{m + 1}{2N + m + 1} \int_{\mathbb{R}_+^{2N+1}} |\nabla U|^2 x_{2N+1}^m dx \quad (2.11) \\
\end{align*}
\]
and the Pohozaev identity
\[
(iv) \quad \int_{\mathbb{R}_+^{2N+1}} \left[ \frac{2N - 1}{2} |\nabla U|^2 - (2N + 1) \frac{U^2}{2\kappa} - (2N + 1) \frac{F(U)}{\kappa} \right] dx = 0 \quad (2.12)
\]
For \( u \in C^\infty \mathcal{M} \) define
\[
\|u\|_\varepsilon = \int_{\mathcal{M}} \left[ \varepsilon^2 |\nabla g u|^2 + \frac{u^2}{|s(p)|^\eta} \right] dv_g
\] (2.13)

We can easily verify that \(|\cdot|_\varepsilon\) defines an equivalent norm on \( C^\infty(\mathcal{M}) \) as the usual \( H^1 \) norm on \( \mathcal{M} \). Let \( H_{\varepsilon}(\mathcal{M}) \) be the completion of \( C^\infty(\mathcal{M}) \) in the norm \(|\cdot|_\varepsilon\). The for \( u \in H_{\varepsilon}(\mathcal{M}) \) we have
\[
\Gamma_{\varepsilon}(u) = \frac{1}{2} \|u\|_\varepsilon^2 - \int_{\mathcal{M}} \frac{F(u)}{|s(p)|^\eta} dv_g
\] (2.14)

3 Some geometric preliminaries

For \( P_0 \in \partial \mathcal{M}, \) let \( (x_1, x_2, ..., x_{2N}) \) be a Riemann normal coordinates on \( \partial \mathcal{M} \) at \( P_0 \). For a point \( q \) close enough to \( P_0 \), let \( x_{2N+1} \) be the distance of \( q \) from \( \partial \mathcal{M} \). The chart \( x^\partial_{P_0} = (x_1, ..., x_{2N}, x_{2N+1}) \) is known as Fermi co-ordinate at \( P_0 \). In these co-ordinates the arc length \( dl^2 \) can be written as:
\[
dl^2 = dx_{2N+1}^2 + \sum_{i,j=1}^{2N} g_{ij}(x', x_{2N+1}) dx_i dx_j \] (3.1)

where \( g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})|_{(x', x_{2N+1})} \).

**Lemma 3.1.** For \( p \) close enough to \( P_0 \) we have
\[
x_{2N+1}(p) = |s(p) - s(P_0)|
\] (3.2)

**Proof.** We have \( \mathcal{M} = B \times_f F \), where \( B = I' \) and \( F = \mathbb{CP}^N \) and the metric \( g = ds^2 + (\frac{2N - 1}{2N})^2 g_{\mathbb{CP}^N} \). Now \( x_{2N+1}(p) = dist_g(p, \partial \mathcal{M}) \). Let \( \sigma(p) = p' \) and take the point \( (s(P_0), p') \in \partial \mathcal{M} \). Consider the path \( \gamma(t) = (s(p) + t(s(P_0) - s(P), p'), t \in [0, 1] \) joining \( p \) and \( (s(P_0), p') \in \partial \mathcal{M} \). Then
\[
x_{2N+1}(p) \leq dist_g(p, (s(P_0), p')) \leq \int_0^1 (g(\gamma'(t), \gamma'(t)))^{1/2} dt
\]
Now \( \gamma'(t) = (s(P_0) - s(p)) \frac{\partial}{\partial r} \) implies \( g(\gamma'(t), \gamma'(t)) = |s(P_0) - s(p)|^2 \) So we have
\[
x_{2N+1}(p) \leq \int_0^1 |s(P_0) - s(p)| dt = |s(P_0) - s(p)|
\]

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Now from the compactness of $\mathbb{CP}^N$ we have a $\tilde{p} \in \partial M$ such that

$$x_{2N+1}(p) = \text{dist}_g(p, \partial M) = \text{dist}_g(p, \tilde{p})$$

Let $\eta(t)$ be a geodesic joining $p$ and $\tilde{p}$ which is length minimizing. Then

$$x_{2N+1}(p) = \int_0^1 (g(\eta'(t), \eta'(t)))^{1/2} dt$$

$$= \int_0^1 ((dr)^2 + \text{some positive terms})^{1/2} dt$$

$$\geq \int_0^1 dr = |s(1) - s(0)| = |s(p) - s(P_0)|$$

$$\square$$

Now for $P_0 \in \partial M$, let $P(P_0)$ be the projection of $T_{P_0}(M)$ onto $T_{P_0}(\partial M)$. The second fundamental form $\Pi(X, Y)$ is defined as $\Pi(X, Y) = \nabla_X Y - (\nabla_X Y)$ for $X, Y \in T(\partial M)$. The mean curvature of $\partial M$ at $P_0 \in \partial M$ is defined as the trace of $\Pi$ at $P_0 \in \partial M$. Let $(X_1, \ldots, X_{2N})$ be an orthogonal vector field in a neighborhood of $P_0$ in $\partial M$. It is well known that the second fundamental form

$$\Pi(X_i, X_j)(P_0) = -\frac{\nabla f}{f} |_{P_0} = -\frac{1}{s(P_0)} \quad (3.3)$$

Let $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, corresponding to the Fermi co-ordinate $x_{P_0}^\partial$ at $P_0$. Let $g^{kl} = ((g_{ij}))^{-1}_{kl}$ and $|g| = \text{det}(g_{ij})$. Then it is well known that

$$g^{ij}(x) = \delta_{ij} + 2h_{ij}x_{2N+1} + O(|x|^2) \quad (3.4)$$

$$\sqrt{|g|} = 1 - 2NHx_{2N+1} + O(|x|^2) \quad (3.5)$$

for $x$ small enough, $(h_{ij})_{i,j \leq 2N}$ is the second fundamental form and $H(p) = -\frac{1}{s(p)}$ is the mean curvature at $P \in \partial M$.

Define the functional $L : \partial M \times \mathcal{B} \to \mathbb{R}$ by

$$H(p, U) = -\sum_{i,j = 1}^{2N} h_{ij}(p) \int_{\mathbb{R}^{2N+1}} \partial_i U \partial_j U x_{2N+1} dx$$

$$+ 2NH(p) \int_{\mathbb{R}^{2N+1}} \left( \frac{1}{2} |\nabla U|^2 + \frac{U^2}{2|s(p)|^{\eta}} - \frac{F(U)}{|s(p)|^{\eta}} \right) x_{2N+1} dx \quad (3.6)$$

Then we have the following
Proposition 3.2. Let \( V \in \mathcal{B} \) be a radially symmetric solution of (2.6) with \( \mu = (0, \ldots, 0, 1) \). Then for any \( p \in \partial M \) we have

\[
H(p, V) = \frac{N}{N + 1} H(p) \int_{\mathbb{R}^{2N+1}} |\nabla V|^2 x_{2N+1} dx
\]  

(3.7)

In lemma (3.1) we have shown that for co-ordinate \( x_{P_0}, x_{2N+1}(p) = |s(p) - s(P_0)| \). Now the boundary of \( M \) is the two disjoint copies of \( \mathbb{CP}^N \). Let us denote the component of the boundary corresponding to \( s = (\frac{2N}{2N-1})^{\frac{1}{2N-1}} a^{\frac{2N}{2N-1}} \) as \( \partial M_a \) and the component of the boundary corresponding to \( s = (\frac{2N}{2N-1})^{\frac{1}{2N-1}} b^{\frac{2N}{2N-1}} \) as \( \partial M_b \). Then for \( q \) near \( \partial M_a \) from lemma (3.1) we have

\[
|s(q)|^{-\eta} = |s(P_0)|^{-\eta} - \eta |s(P_0)|^{-\eta-1} x_{2N+1} + O(|x_{2N+1}|^2)
\]

(3.8)

and for \( q \) near \( \partial M_b \) we have

\[
|s(q)|^{-\eta} = |s(P_0)|^{-\eta} + \eta |s(P_0)|^{-\eta-1} x_{2N+1} + O(|x_{2N+1}|^2)
\]

(3.9)

4 The MP solution and proof of Theorem(1.1)

Here we shall work with the Fermi coordinate as we have discussed earlier around a point \( P_0 \) on the boundary of the manifold. We denote it by \( x_{\partial P_0} \). Let \( \delta \) be small enough such that \( x_{\partial P_0} \) is a diffeomorphism from \( \{ x \in M \setminus \partial M : \text{dist}_g(P_0, x) < \delta \} \) to an open neighborhood of 0 in \( \mathbb{R}^{2N+1} \). Note that \( x_{\partial P_0} \) maps \( \partial M \) into \( \mathbb{R}^{2N} = \partial \mathbb{R}^{2N+1} \) locally around \( P_0 \). Define \( \phi_\gamma \in C_0^\infty ([\mathbb{R}^{2N+1}, [0, 1]) \) as

\[
\phi_\gamma(x) = \begin{cases} 
1 & \text{if } |x| \leq \gamma \\
0 & \text{if } |x| \geq 2\gamma
\end{cases}
\]

For \( V \in \mathcal{B} \) define \( Z_{\varepsilon,t}(p) = \phi_{2t}(x_{\partial P_0}(p)) V(x_{\partial P_0}(p)) \). Let us consider that \( B_+(0, 2t\gamma) \subset x_{\partial P_0}(B_g(P_0, \delta)) \). Then

\[
\Gamma_\varepsilon(Z_{\varepsilon,t}) = \int_M \left( \frac{\varepsilon^2}{2} \left| \nabla g Z_{\varepsilon,t} \right|^2 + \frac{|Z_{\varepsilon,t}|^2}{2|s(p)|^\eta} - \frac{F(Z_{\varepsilon,t})}{|s(p)|^\eta} \right) dv_g
\]

(4.1)
to get the Mountain pass solution we need to first simplify the terms of the above expressions

\[
\begin{align*}
\int_{\mathcal{M}} \frac{\varepsilon^2}{2} |\nabla_g Z_{\varepsilon,t}^\gamma|^2 dv_g \\
&= \int_{B_g(P_0, \delta)} \frac{\varepsilon^2}{2} |\nabla_g Z_{\varepsilon,t}^\gamma|^2 dv_g \\
&= \frac{\varepsilon^2}{2} \int_{B_+(0, 2\varepsilon)} 2N+1 \sum_{i,j=1}^{2N+1} g^{ij}(x) \partial_i(\phi_\gamma(\frac{x}{\varepsilon t})V(\frac{x}{\varepsilon t})) \partial_j(\phi_\gamma(\frac{x}{\varepsilon t})V(\frac{x}{\varepsilon t})) \sqrt{|g|}(\varepsilon y) dx \\
&= \frac{1}{2} \varepsilon^{2N+1} t^{2N-1} \int_{B_+(0, \frac{2\varepsilon}{\varepsilon t})} 2N+1 \sum_{i,j=1}^{2N+1} g^{ij}(\varepsilon ty) \partial_i(\phi_\gamma(\varepsilon y)V(y)) \partial_j(\phi_\gamma(\varepsilon y)V(y)) \sqrt{|g|}(\varepsilon ty) dy
\end{align*}
\]

Where \( x = \varepsilon ty \). From the expressions of \( |g| \) and \( g^{ij} \) in (3.4) and (3.5) we get

\[
\sum_{i,j=1}^{2N+1} g^{ij}(\varepsilon ty) \partial_i(\phi_\gamma(\varepsilon y)V(y)) \partial_j(\phi_\gamma(\varepsilon y)V(y)) \sqrt{|g|}(\varepsilon ty)
\]

\[
= |\nabla(\phi_\gamma(\varepsilon y)V(y))|^2 - 2NH(P_0)\varepsilon t |\nabla(\phi_\gamma(\varepsilon y)V(y))|^2 y_{2N+1} \\
+ 2\varepsilon t \sum_{i,j=1}^{2N} h_{ij}(P_0) \partial_i(\phi_\gamma(\varepsilon y)V(y)) \partial_j(\phi_\gamma(\varepsilon y)V(y)) y_{2N+1} \\
+ \sum_{i,j=1}^{2N} O(\varepsilon t y^2) \partial_i(\phi_\gamma(\varepsilon y)V(y)) \partial_j(\phi_\gamma(\varepsilon y)V(y))
\]

So we get

\[
\begin{align*}
\int_{\mathcal{M}} \frac{\varepsilon^2}{2} |\nabla_g Z_{\varepsilon,t}^\gamma|^2 dv_g \\
&= \varepsilon^{2N+1} \left[ \int_{B_+(0, \frac{2\varepsilon}{\varepsilon t})} \frac{t^{2N-1}}{2} |\nabla(\phi_\gamma(\varepsilon y)V(y))|^2 dy \\
&\quad - 2NH(P_0)\varepsilon^t^{2N} \int_{B_+(0, \frac{2\varepsilon}{\varepsilon t})} |\nabla(\phi_\gamma(\varepsilon y)V(y))|^2 y_{2N+1} dy \\
&\quad + \varepsilon t^{2N} \sum_{i,j=1}^{2N} h_{ij}(P_0) \int_{B_+(0, \frac{2\varepsilon}{\varepsilon t})} \partial_i(\phi_\gamma(\varepsilon y)V(y)) \partial_j(\phi_\gamma(\varepsilon y)V(y)) y_{2N+1} dy \right]
\end{align*}
\]
We got the above inequality using the decay estimate of \( V \) and \( DV \) near infinity. Again using the same decay estimate we can easily show the following estimates

\[
\int_{B_{+}(0, \frac{2\varepsilon t}{\gamma})} |\nabla V|^2 dy + \int_{B_{+}(0, \frac{2\varepsilon t}{\gamma}) \setminus B_{+}(0, \frac{2\varepsilon t}{\gamma})} |\nabla (\phi_{\gamma}(\varepsilon y)V(y))|^2 dy \]

\[
= \int_{B_{+}(0, \frac{2\varepsilon t}{\gamma})} |\nabla V|^2 dy + O(\varepsilon^2)
\]

\[
= \int_{B_{+}(0, \frac{2\varepsilon t}{\gamma})} |\nabla V|^2 dy + O(\varepsilon^2)
\]

Estimating all the terms in the same way we get the expression for the first integral as

\[
\frac{\varepsilon^2}{2} \int_{M} |\nabla g Z_{\varepsilon,t}^\gamma|^2 dv_g
\]

\[= \varepsilon^{2N+1} \left[ \frac{t^{2N-1}}{2} \int_{\mathbb{R}_{+}^{2N+1}} |\nabla V|^2 dy - 2NH(P_0) \frac{\varepsilon t^{2N}}{2} \int_{\mathbb{R}_{+}^{2N+1}} |\nabla V|^2 y_{2N+1} dy \right. \]

\[+ \varepsilon t^{2N} \sum_{i,j=1}^{2N} h_{ij}(P_0) \int_{\mathbb{R}_{+}^{2N+1}} \partial_i V \partial_j V y_{2N+1} dy + t^{2N+1} O(\varepsilon^2) \left. \right] \tag{4.3}
\]

The second term is

\[
\int_{M} \frac{|Z_{\varepsilon,t}^\gamma|^2}{2|s(p)|^\eta} dv_g
\]

\[= \frac{1}{2} \int_{B_{g}(P_0, \delta)} \frac{|Z_{\varepsilon,t}^\gamma|^2}{|s(p)|^\eta} dv_g \]

\[= \frac{1}{2} \int_{B_{+}(0, 2\varepsilon t)} \frac{1}{|s(p)|^\eta} (\phi_{\gamma}(\frac{x}{\varepsilon t}) V(\frac{x}{\varepsilon t}) \frac{1}{2|s(x_{p_0})^{-1}(\varepsilon ty)|^\eta} (\phi_{\gamma}(\varepsilon y)V(y))^2 \sqrt{|g(\varepsilon ty)|} dy \]

Here we have made the change of variable \( x = \varepsilon ty \) as before. To simplify the above expression let us expand \( s(p) \) around \( s(P_0) \). W.L.O.G we can take \( P_0 \) on the inner boundary. The same approach shall work for \( P_0 \) on the outer boundary.
Similarly we have

\[
\int_{B_1(0, \frac{2}{\varepsilon})} \varepsilon^{2N+1} \frac{1}{2} |s(P)|^\eta (\phi_\gamma(\varepsilon y)V(y))^2 \sqrt{|g(\varepsilon y)|} dy
\]

\[
= \varepsilon^{2N+1} \left[ \int_{B_1(0, \frac{2}{\varepsilon})} \left( \frac{t^{2N+1} (\phi_\gamma(\varepsilon y)V(y))^2}{2 |s(P)|^\eta} - 2NH(P_0)\varepsilon t \left( \frac{\phi_\gamma(\varepsilon y)V(y)^2}{|s(P)|^\eta} y_{2N+1} \right) \right) dy + O(\varepsilon^2) \right]
\]

\[
= \varepsilon^{2N+1} \left[ t^{2N+1} \int_{\mathbb{R}^{2N+1}_+} \frac{V^2(y)}{2|s(P)|^\eta} dy - 2NH(P_0)\varepsilon t^{2N+2} \int_{\mathbb{R}^{2N+1}_+} \frac{V^2(y)}{2|s(P)|^\eta} y_{2N+1} dy - \varepsilon t^{2N+2} \int_{\mathbb{R}^{2N+1}_+} \frac{V^2(y)}{2|s(P)|^\eta} y_{2N+1} dy + O(\varepsilon^2) \right]
\]

Similarly we have

\[
\int_{\mathcal{M}} \frac{F(Z_{\varepsilon,t}^\gamma)}{|s(P)|^\eta} dv_g
\]

\[
= \varepsilon^{2N+1} \left[ t^{2N+1} \int_{\mathbb{R}^{2N+1}_+} \frac{F(V)}{2|s(P)|^\eta} dy - 2NH(P_0)\varepsilon t^{2N+2} \int_{\mathbb{R}^{2N+1}_+} \frac{F(V)}{2|s(P)|^\eta} y_{2N+1} dy - \varepsilon t^{2N+2} \int_{\mathbb{R}^{2N+1}_+} \frac{F(V)}{2|s(P)|^\eta} y_{2N+1} dy + O(\varepsilon^2) \right]
\]

Finally combining (4.3), (4.5), (4.6) we get

\[
\varepsilon^{-(2N+1)} \Gamma_\varepsilon(Z_{\varepsilon,t}^\gamma) = I_1(t) - \varepsilon I_2(t) - \varepsilon I_3(t) + O(\varepsilon^2)
\] (4.7)

where

\[
I_1(t) = \int_{\mathbb{R}^{2N+1}_+} \left( \frac{t^{2N-1}}{2} \nabla V^2 dy + \frac{t^{2N+1}}{2} \frac{V^2(y)}{|s(P)|^\eta} - t^{2N+1} \frac{F(V)}{|s(P)|^\eta} \right) dy
\]

\[
I_2(t) = t^{2N+2} \int_{\mathbb{R}^{2N+1}_+} \left( \frac{V^2(y)}{2|s(P)|^\eta} - \frac{F(V)}{|s(P)|^\eta} \right) y_{2N+1} dy
\]

\[
I_3(t) = 2NH(P_0) \int_{\mathbb{R}^{2N+1}_+} \left( \frac{t^{2N+1}}{2} |\nabla V|^2 + \frac{t^{2N+2}}{2} \frac{V^2(y)}{|s(P)|^\eta} - t^{2N+2} \frac{F(V)}{|s(P)|^\eta} \right) y_{2N+1} dy
\]
\[ + t^{2N} \sum_{i,j=1}^{2N} h_{ij}(P_0) \int_{\mathbb{R}^{2N+1}_+} \partial_i V \partial_j V y_{2N+1} dy \]

Now note that \( \Gamma_{\varepsilon}(Z_{\varepsilon,t}^\gamma) = \varepsilon^{2N+1}[I_1(t) + O(\varepsilon)] \) uniformly for \( P_0 \in \partial \mathcal{M} \) and \( t \in \mathbb{R} \)
(here we can take any of the boundaries and can get the same expression up-to order of \( \varepsilon \)). From the Pohozaev’s identity we see that \( \beta \) from the Pohozaev identity we get that \( t \). Moreover we see from (4.7) that \( \beta \rightarrow t \in [0,1] \). Then it follows that \( \beta(0) = 0, \beta(1) = Z_{\varepsilon,t_0}^\gamma \).

**Lemma 4.1.** \( c_\varepsilon \) does not depend upon \( p \in \partial \mathcal{M} \) and \( V \in \mathcal{B} \)

**Proof.** Same proof as given in lemma 3.1 in [6].

Let \( \beta(t) = Z_{\varepsilon,t_0}^\gamma, t \in [0,1] \), then it follows that \( \beta(0) = \lim_{t \to 0} \beta(t) = 0 \) and \( \beta(1) = Z_{\varepsilon,t_0}^\gamma \). Moreover we see from (4.7) that

\[
\Gamma_{\varepsilon}(\beta(t)) = \varepsilon^{2N+1} \left[ (tt_0)^{2N-1} \int_{\mathbb{R}^{2N+1}_+} \frac{1}{2} |\nabla V|^2 dy + (tt_0)^{2N+1} \int_{\mathbb{R}^{2N+1}_+} \frac{V^2(y)}{\kappa} dy \right]
\]

uniformly for \( t \in [0,1] \). So

\[
\Gamma_{\varepsilon}(\beta(t)) \leq \max_{t \in [0,t_0]} \left[ \frac{t^{2N-1}}{2} \int_{\mathbb{R}^{2N+1}_+} |\nabla V|^2 dy + t^{2N+1} \int_{\mathbb{R}^{2N+1}_+} \left( \frac{V^2(y)}{\kappa} - \frac{F(V)}{\kappa} \right) dy + O(\varepsilon) \right]
\]

i.e. \( \lim_{\varepsilon \to 0} c_\varepsilon \leq \max_{t \in [0,t_0]} \left[ I_1(t) + O(\varepsilon) \right] \) (4.9)

From the Pohozaev identity we get that \( t = 1 \) is the unique maximum point of the \( \mathcal{R} \) of (4.9), and hence we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{-(2N+1)} c_\varepsilon \leq \Gamma(V)
\]

(4.10)
Lemma 4.2. For a mountain pass solution $u_{\varepsilon}$ any local maxima $P_{\varepsilon}$ of $u_{\varepsilon}$ in $\mathcal{M}$ converges to the boundary of $\mathcal{M}$ as $\varepsilon \to 0$.

Proof. First note that (2.5) has constant solutions 0 and 1. with $\Gamma_{\varepsilon}(0) = 0$ and $\Gamma_{\varepsilon}(1) = (\frac{1}{2} - \frac{1}{p+1})\int_{\mathcal{M}} \frac{d\sigma_{\gamma}}{\|\sigma_{\gamma}\|^n} > 0$. But $c_{\varepsilon} = O(\varepsilon^{2N+1})$. So $u_{\varepsilon}$ is not constant. Clearly $u_{\varepsilon}(P_{\varepsilon}) \geq 1$. We claim that there exists a constant $C$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \text{dist}_{g}(P_{\varepsilon}, \partial \mathcal{M}) < C$$

(4.11)

If not, let $\exists \varepsilon_{m} \to 0$ such that $\frac{1}{\varepsilon_{m}} \text{dist}_{g}(P_{\varepsilon_{m}}, \partial \mathcal{M}) \geq 2m$. Consider a normal coordinate $x_{P_{\varepsilon_{m}}} : B_{g}(P_{\varepsilon_{m}}, m \varepsilon_{m}) \to \mathbb{R}^{2N+1}$ and define $w_{\varepsilon_{m}}(x) = u_{\varepsilon_{m}}(x_{P_{\varepsilon_{m}}}(\varepsilon_{m}x))$ in $B(0, m)$. Then we have

$$\frac{1}{\sqrt{|g(\varepsilon_{m}x)|}} \sum \partial_{k} \left( g^{ik}(\varepsilon_{m}x) \sqrt{|g(\varepsilon_{m}x)|} \partial_{k} w_{\varepsilon_{m}} \right) + \frac{w_{\varepsilon_{m}}}{|\kappa_{m}(x)|^{\gamma}} - \frac{f(w_{\varepsilon_{m}})}{|\kappa_{m}(x)|^{\eta}} = 0$$

in $B(0, m)$

Where $\kappa_{m}(x) = s(x_{P_{\varepsilon_{m}}}(\varepsilon_{m}x))$ in $B(0, m)$. Let $P_{\varepsilon_{m}} \to \tilde{P}$ up-to a subsequence (using compactness argument), and take $\tilde{\kappa} = |s(\tilde{P})|^{\eta}$. Then by standard elliptic estimate and Sobolev embedding we have $\{w_{\varepsilon_{m}}\}$ bounded in $C^{2,\theta}(B(0, m))$ for some $0 < \theta < 1$ and up-to a subsequence $w_{\varepsilon_{m}} \to w$ in $C^{2}_{loc}(\mathbb{R}^{2N+1})$ where $w$ satisfies

$$-\Delta w + \frac{w}{\tilde{\kappa}} - \frac{f(w)}{\tilde{\kappa}} = 0, \quad w > 0$$

(4.12)

let

$$\tilde{J}(w) = \int_{\mathbb{R}^{2N+1}_{+}} \left( \frac{1}{2} |\nabla w|^{2} + \frac{w^{2}}{2\tilde{\kappa}} - \frac{F(w)}{\tilde{\kappa}} \right) dx$$

(4.13)

Define $w_{1}(y) = w(\sqrt{\frac{2}{\tilde{\kappa}}} y)$. Then $w_{1}$ satisfies

$$-\Delta w_{1} + \frac{w_{1}}{\kappa} - \frac{f(w_{1})}{\kappa} = 0, \quad w_{1} > 0$$

(4.14)

Now using the change of variable $x = \sqrt{\frac{2}{\tilde{\kappa}}} y$ we have

$$\tilde{J}(w) = \int_{\mathbb{R}^{2N+1}_{+}} \left( \frac{1}{2} |\nabla w|^{2} + \frac{w^{2}}{2\tilde{\kappa}} - \frac{F(w)}{\tilde{\kappa}} \right) dx$$

$$= \int_{\mathbb{R}^{2N+1}_{+}} \left( \frac{1}{2} \kappa |\nabla w_{1}|^{2} + \frac{w_{1}^{2}}{2\tilde{\kappa}} - \frac{F(w_{1})}{\tilde{\kappa}} \right) \left( \frac{\kappa}{\tilde{\kappa}} \right)^{2N+1} dx$$

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\[
\tilde{J}(w) = 2 \left( \frac{\tilde{\kappa}}{\kappa} \right)^{\frac{2N-1}{2}} \int_{\mathbb{R}^{2N+1}} \left( \frac{1}{2} |\nabla \omega_1|^2 + \frac{w_1^2}{2\kappa} - \frac{F(w_1)}{\kappa} \right) dx
\]

Case I: Let \( \alpha \leq \frac{2}{2N-1} \), then we have \( \eta > 0 \) and take \( P_0 \) on the inner boundary. The \( \tilde{\kappa} \geq 1 \). Take \( V = w_1|_{\mathbb{R}^{2N+1}} \). Clearly \( V \) satisfies (2.6) with \( \mu = (0, \ldots, 0, 1) \) and \( \kappa = |s(P_0)|^\eta \). Then by symmetry we get

\[
\tilde{J}(w) = 2 \left( \frac{\tilde{\kappa}}{\kappa} \right)^{\frac{2N-1}{2}} \Gamma(V)
\]

Now \( \varepsilon^{-(2N+1)}c_\varepsilon = \varepsilon^{-(2N+1)}\Gamma_\varepsilon(u_\varepsilon) \to \tilde{J}(w) \) as \( \varepsilon \to 0 \). Then we have the following contradictory argument

\[
l_{\lim_{\varepsilon_m \to 0}}^{\varepsilon_m^{-(2N+1)}}c_{\varepsilon_m} = \tilde{J}(w) \geq 2\Gamma(V) \geq 2\lim_{\varepsilon \to 0} \varepsilon^{-(2N+1)}c_\varepsilon
\]

Case II: Let \( \alpha > \frac{2}{2N-1} \), then we have \( \eta > 0 \) and take \( P_0 \) on the outer boundary. Then similarly as above we shall arise at a contradiction.

To prove the next lemma we need the following result from Ni and Takagi

**Lemma 4.3.** Let \( \phi \in C^2(B_a) \) be radial function satisfying \( \phi'(0) = 0 \) and \( \phi''(0) < 0 \) in \([0, a]\) the exists a \( \delta > 0 \) such that, if \( \psi \in C^2(\overline{B(0, a)}) \) satisfies (i) \( \nabla \psi(0) = 0 \) and (ii) \( ||\psi - \phi||_{C^2(\overline{B(0, a)})} < \delta \) then \( \nabla \psi \neq 0 \) for \( x \neq 0 \)

**Lemma 4.4.** \( P_\varepsilon \in \partial \mathcal{M} \) for \( \varepsilon \) small enough

**Proof.** Let \( \varepsilon_k \downarrow 0 \) be a decreasing sequence such that \( P_k := P_{\varepsilon_k} \in \mathcal{M} \). From lemma (4.2) we have \( P_k \to \tilde{P}(\text{say}) \in \partial \mathcal{M} \) (up-to a subsequence). Take the Fermi co-ordinate \( x_{\tilde{P}}^\partial \) on a neighborhood of \( \tilde{P} \) and let \( (x_{\tilde{P}}^\partial)^{-1} \) is defined on a set containing the closed half-ball \( B_+(0, 2\eta) \), \( \eta > 0 \) and \( q_k := (x_{\tilde{P}}^\partial)^{-1}(P_k) \in B_+(0, \eta) \) for all \( k \). Let \( v_k(y) := u_{\varepsilon_k}((x_{\tilde{P}}^\partial)^{-1}(y)) \) for \( y \in B_+(0, 2\eta) \). Extend \( v_k \) to all of \( B(0, 2\eta) \) by

\[
\tilde{v}_k(y) = \begin{cases} v_k(y), & \text{if } y \in B_+(0, 2\eta) \\ v_k(y' - y_{2N+1}), & \text{if } y \in B_-(0, 2\eta) \end{cases}
\]
Define \( w_k(z) = \tilde{v}_k(q_k + \varepsilon_k z) \) for \( z \in \overline{B(0, \frac{1}{k})} \). Let \( q_k = (q_k', \theta_k \varepsilon_k) \), \( q_k' \in \mathbb{R}^{2N} \) and \( \theta_k > 0 \). Then from (4.11) \( \theta_k \) is bounded. Then it can be easily shown that \( w_k \rightarrow W \) in \( C^2_{loc} \) where \( W \) satisfies
\[
- \Delta W + \frac{W}{|s(P)|^\eta} - \frac{f(W)}{|s(P)|^\eta} = 0 \quad \text{in} \ \mathbb{R}^{2N+1}
\] (4.15)
Let \( R > 0 \) be sufficiently large and define \( \varepsilon_R = C_0 \exp(-R/2) \). Then \( \exists k_R \) such that
\[
||w_k - W||_{C^2(B_R)} \leq \varepsilon_R \quad \text{for} \ k \geq k_R
\] (4.16)
We choose \( R > \eta_k, \forall k \).

Now choose \( c, d(0 < c < d) \) such that \( W''(r) < 0 \) for \( r \in [0, c] \) (as \( W''(0) < 0 \)) and \( W(d) < 1 \). Since \( W' < 0 \) for \( r > 0 \) one sees \( C_* := \min\{|W'(r)| r \in [c, d]\} > 0 \). If \( c \leq |z| \leq d \), the from (4.16) we have
\[
|\nabla w_k(z)| \geq |\nabla W(z)| - |\nabla w_k(z) - \nabla W(z)| \geq C_* - \varepsilon_R > 0
\]
provided \( C_* > \varepsilon_R \). Apply lemma (4.3) in the ball \( \overline{B_b} \) we conclude that \( z = 0 \) is only maximum point of \( w_k \) in \( B_0 \). If \( z_k \) is a local maximum point of \( w_k \) in \( B_R \) then \( w_k(z_k) \geq \pi \equiv 1 \). choosing \( R \) so large that \( \varepsilon_R < 1 - w(b) \) one has if \( |z| > b \) then \( w_k(z) = W(z) + \varepsilon_R < \pi \equiv 1 \). Hence \( z_k \in B_b \). Consequently \( z_k = 0 \). Now if \( \theta_k > 0 \), \( \forall k \) then by definition of \( \tilde{v}_k q_k = (q_k', -\theta_k \varepsilon_k) \) is also a local maximum of \( \tilde{v}_k \) and hence \( (0, -\theta_k) \) is another local maximum point of \( w_k \) in \( B_R \), which is contradictory. So \( \theta_k = 0 \) for \( k \) large enough. \( \square \)

Now let \( P_k \in \partial \mathcal{M} \) be a local max of \( u_* \). Take the Fermi co-ordinate \( x_{P_k}^0 \) around \( P_k \) such that \( x_{P_k}^0 \) maps \( B_0(P_k, 2\delta) \) onto a half ball \( B_+(0, 2\delta) \cup (\partial \mathbb{R}^{2N+1} \cap B(0, 2\delta)) = B \) (say) diffeomorphically. Define \( \phi \in C^\infty(\mathbb{R}) \) such that \( \phi(r) = 0 \) if \( r > 2\delta \), \( 1 \) if \( r < \delta \) and define
\[
v_{\varepsilon_k}(x) = \phi(|x|)u_{\varepsilon_k}(x_{P_k}^0)^{-1}(\varepsilon_k x) \quad \text{in} \ \frac{1}{\varepsilon_k} B
\] (4.17)
It can be easily shown that \( v_{\varepsilon_k} \rightarrow V \) in \( C^2_{loc}(\mathbb{R}^{2N+1}_+) \) with \( V \in (\mathbb{R}^{2N+1}_+) \). Clearly \( V \geq 0 \) and \( V \) satisfies (2.7) with \( P_0 \) lies on the same boundary where \( P_k \) lies and \( \mu = (0, ..., 0, 1) \). Also we have for \( \varepsilon_R := C_0 e^{-R} \) with some constant \( C_0 \), \( \exists \) integer \( k_R \) such that for \( k > k_R \) we have
\[
||v_{\varepsilon_k} - V||_{C^2(B_+(0, 2\delta))} \leq \varepsilon_R \quad \text{for} \ k \geq k_R
\] (4.18)
Lemma 4.5. If \( u_\varepsilon \) attains a local maximum at \( x_0 \in \mathcal{M} \) the there exists a positive constant \( \eta_0 \) independent of \( x_0 \) and \( \varepsilon \) such that \( u_\varepsilon(x) \geq \eta_0 \) for \( x \in B_g(x_0, \varepsilon) \cup \mathcal{M} \) provided \( \varepsilon \) is sufficiently small.

Proof. Easily obtained by Harnack Inequality. (See Ni, Takagi (7)).

Lemma 4.6. \( u_\varepsilon \) can have only one local maximum.

Proof. If possible let us consider that there is a decreasing sequence \( \varepsilon_m \downarrow 0 \) such that \( u_{\varepsilon_m} \) has two local maxima say \( P_1 \) and \( P_2 \). From the previous lemmas we have both \( P_1 \) and \( P_2 \) are on \( \partial \mathcal{M} \) for large \( m \). Also as the scaled function \( \tilde{v} \) constructed in lemma (4.4), can’t have two local maximas in \( B_R \) for any \( R > 0 \), we see that \( \frac{1}{\varepsilon} \text{dist}_g(P_1, P_2) \to \infty \) as \( m \to \infty \). Take the co-ordinate \( x_{P_{\varepsilon_m}}^\partial \) around \( P_{\varepsilon_m} \) and define \( v_{P_{\varepsilon_m}} \) as in (4.17).

In the next step we shall give a lower estimate of the energy functional in order to prove the lemma. Note that

\[
c_{\varepsilon_m} = \int_{\mathcal{M}} \left( \frac{u_{\varepsilon_m} f(u_{\varepsilon_m})}{2|s(q)|^q} - \frac{F(u_{\varepsilon_m})}{|s(q)|^q} \right) dv_g
\]

\[
= \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathcal{M}} \frac{u_{\varepsilon_m}^{p+1}}{|s(q)|^q} dv_g
\]

\[
= \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \int_{B_1} \frac{u_{\varepsilon_m}^{p+1}}{|s(q)|^q} dv_g + \int_{\mathcal{M}\setminus B_1} \frac{u_{\varepsilon_m}^{p+1}}{|s(q)|^q} dv_g \right)
\]

where \( B_1 = B_g(P_m, R_{\varepsilon_m}) \) and \( m \) is so large that \( R_{\varepsilon_m} < \delta \). Now

\[
\int_{B_1} \frac{u_{\varepsilon_m}^{p+1}}{|s(q)|^q} dv_g
\]

\[
= \int_{B_+ (0, R_{\varepsilon_m})} \frac{v_{\varepsilon_m}^{p+1}}{|s(x_{P_{\varepsilon_m}}^\partial)^{-1}(x)|^q} \sqrt{|g|(x)} dx
\]

\[
= \varepsilon_m^{2N+1} \int_{B_+ (0, R_{\varepsilon_m})} \frac{v_{\varepsilon_m}^{p+1} (\varepsilon_m y)}{|s(x_{P_{\varepsilon_m}}^\partial)^{-1}(\varepsilon_m y)|^q} \sqrt{|g|((\varepsilon_m y)} dx
\]

\[
= \varepsilon_m^{2N+1} \int_{B_+ (0, R_{\varepsilon_m})} v_{\varepsilon_m}^{p+1} (\varepsilon_m y) \left( \frac{1}{|s(P_{\varepsilon_m})|^q} + O(|\varepsilon_m y|) \right) dy
\]

using the change of variable \( x = \varepsilon_m y \) and using (3.5) and (3.8). Note that \( v_{\varepsilon_m}^{p+1} \geq V_{p+1} - |v_{\varepsilon_m}^{p+1} - V_{p+1}| \) and we have

\[
\int_{B_1} \frac{u_{\varepsilon_m}^{p+1}}{|s(q)|^q} dv_g
\]
\[\geq \varepsilon_m^{2N+1} \int_{B_+(0,R)} \left( \frac{V_{p+1}^{p+1} - |v_{\varepsilon_m}^{p+1} - V_{p+1}^{p+1}|}{|s(P_{\varepsilon_m})|^\eta} \right) dx + O(|v_{\varepsilon_m}^{p+1} - V_{p+1}^{p+1}|) R^{2N+1}\]

\[\geq \varepsilon_m^{2N+1} \int_{B_+(0,R)} \frac{V^{p+1}}{|s(P_{\varepsilon_m})|^\eta} dy - C_1 \varepsilon R^{2N+1}\]

\[\geq \varepsilon_m^{2N+1} \int_{R^{2N+1}} \frac{V^{p+1}}{|s(P_{\varepsilon_m})|^\eta} dy - C_1 \varepsilon R^{2N+1} - C_2 \varepsilon\]

Using decay estimate of \(V\). On the other hand from lemma (4.5) we have

\[\int_{M \setminus B_1} \frac{u_{\varepsilon_m}^{p+1}}{|s(q)|^\eta} dv_g \geq \int_{B_0(P_{\varepsilon_m}, \varepsilon_m)} \frac{u_{\varepsilon_m}^{p+1}}{|s(q)|^\eta} dv_g \geq \eta_0 \int_{B_0(P_{\varepsilon_m}, \varepsilon_m)} dv_g = C_0 \varepsilon^{2N+1}\]

So finally we have the lower estimate as

\[c_{\varepsilon_m} \geq \varepsilon_m^{2N+1} \left( \Gamma(V) + C_0 - C_1 r^{-2N} e^{-R} + C_2 \varepsilon_m \right) \] (4.19)

Now define \(Z_{\varepsilon, t}\) as in section 4 by taking \(P_0 = P_{\varepsilon_m}\) and \(W\) to be a least energy solution of (2.6) with \(\mu = (0, ..., 0, 1)\). then from (4.10) we get

\[\lim_{\varepsilon \to 0} \varepsilon^{-2N+1} c_{\varepsilon} \leq \Gamma(W) \] (4.20)

Now \(\Gamma(V) \geq \Gamma(W)\) implies (4.19) and (4.19) are contradictory. Hence \(u_{\varepsilon}\) can have only one maximum point which lies on the boundary point of \(M\) for \(\varepsilon\) small enough.

Proof. We know that for \(\varepsilon\) small enough \(P_{\varepsilon}\) is on \(\partial M\).

\[u_{\varepsilon}(x) + |\nabla u_{\varepsilon}(x)| \leq C \exp\left(-\frac{c}{\varepsilon} \text{dist}_g(c, P_{\varepsilon})\right) \] (4.21)

for some constants \(C, c > 0\).

**Proposition 4.7.** For \(\varepsilon\) small enough the following holds

(i) for \(\eta > 0\), i.e. \(\alpha < \frac{2}{2N-1}\), we have \(s(P_{\varepsilon}) = (\frac{2N}{2N-1})^{\frac{1}{2N-1}} a^{\frac{2N}{2N-1}}\)

(ii) for \(\eta \leq 0\), i.e. \(\alpha \geq \frac{2}{2N-1}\), we have \(s(P_{\varepsilon}) = (\frac{2N}{2N-1})^{\frac{1}{2N-1}} b^{\frac{2N}{2N-1}}\)

**Proof.** We know that for \(\varepsilon\) small enough \(P_{\varepsilon}\) is on \(\partial M\).
Case 0: For $\eta = 0$ we have the result of Byeon and Park [2]. So we have that a maximum point converges to a point of $\partial M$ which have maximum mean curvature. From (3.3) we have $s(P_\varepsilon) = (\frac{2N}{2N-1})^{\frac{2N-1}{2N}}b^{\frac{2N}{2N-1}}$.

Case I: Let $\eta > 0$. Let $V$ be a least energy solution of (2.6) with $\mu = (0,...,0,1)$. Define $Z_{\varepsilon,t}$ as in section 4 with $P_0$ on the inner boundary and taking the co-ordinate around $P_0$. Then using (4.7) and the Pohozaev identity we get

$$\varepsilon^{-(2N+1)}\Gamma_\varepsilon(Z_{\varepsilon,t}) \leq \int_{\mathbb{R}^{2N+1}_+} \frac{1}{2} |\nabla V|^2 dy + \int_{\mathbb{R}^{2N+1}_+} \left( \frac{V^2(y)}{2\kappa} - \frac{F(V)}{\kappa} \right) dy + C\varepsilon$$

$$= (\kappa)^{\frac{2N-1}{2}} I(U) + C\varepsilon \tag{4.22}$$

where $\kappa = (\frac{2N}{2N-1})^{\frac{2N-1}{2N}}b^{\frac{2N}{2N-1}}$, $I(U) = \int_{\mathbb{R}^{2N+1}_+} \frac{1}{2} |\nabla U|^2 dy + \frac{U^2(y)}{2} - F(U) dy$ and $U$ is a least energy solution of (2.7). Now if possible let $s(P_\varepsilon) = (\frac{2N}{2N-1})^{\frac{2N-1}{2N}}b^{\frac{2N}{2N-1}}$. Then from (4.19) we have

$$\varepsilon^{-(2N+1)}c_{\varepsilon m} \geq \Gamma(W) - C_1 r^{2N}e^{-R} + C_2 \varepsilon_m$$

where $W$ solves (2.6) with $s(P_0)$ replaced by $\tilde{\kappa} := s(P_\varepsilon)$. Then as before we can easily show

$$\varepsilon^{-(2N+1)}c_{\varepsilon m} \geq (\tilde{\kappa})^{\frac{2N-1}{2}} I(\tilde{U}) - C_1 r^{2N}e^{-R} + C_2 \varepsilon_m \tag{4.23}$$

Where $\tilde{U}$ is a solution of (2.7). Note that for $\eta > 0$ $\tilde{\kappa} > \kappa$. Also noting that $U$ is a least energy solution of (2.7) we reach two contradictory inequalities (4.22) and (4.23) when $R$ is large enough.

Case II: Follows similarly as above by taking $P_0$ on the outer boundary for the test function.

Proof of theorem (1.1):

Proof. It is clear that one point concentrating solutions of equation (2.5) can be lifted to $S^1$ concentrating solutions of (1.1) with the required properties.
5 Appendices

5.1 Appendix A : Hopf fibration, Fubini Study metric and the Warped product.

Here we shall discuss some well known facts about hops fibration on $S^{2N+1}$. All the details can be found in [9]. The $2N+1$ sphere $S^{2N+1}$ can be represented as $S^{2N+1} = I \times (S^{2N-1} \times S^1)$. Also the metric $g$ in (1.3) can be represented by another representation (doubly warped product metric) as

$$g = dt^2 + \sin^2 t_1 dS^2_{2N-1} + \cos^2 t_1 dS^2_1 \quad (5.1)$$

The unit circle acts on both the spheres by complex scalar multiplication as, for $\lambda \in S^1$ and $(z, w) \in S^{2N-1} \times S^1$ we have $\lambda \cdot (z, w) = (\lambda z, \lambda w)$, which induces a fixed point free isometric action on the space. The quotient map

$$I \times (S^{2N-1} \times S^1) \longrightarrow I \times ((S^{2N-1} \times S^1)/S^1) \quad (5.2)$$

can be made into Riemann submersion by choosing an appropriate metric on the quotient space. To find this metric we split the canonical metric

$$dS_{2N-1} = h + g$$

where $h$ corresponds to the metric along the Hopf fiber and $g$ is the orthogonal complement. Then we got the generalized Hopf fibration $S^{2N+1} \rightarrow \mathbb{C}P^N$, defined by

$$(0, \frac{\pi}{2}) \times (S^{2N-1} \times S^1) \longrightarrow (0, \frac{\pi}{2}) \times ((S^{2N-1} \times S^1)/S^1) \quad (5.3)$$

as a Riemann submersion and the corresponding metric is given by

$$g_{\mathbb{C}P^N} = dt^2 + \sin^2 t_1 (g + \cos^2 t_1 h) \quad (5.4)$$

Now let us take the manifolds $I'$ and $\mathbb{C}P^N$ with the metrics $ds^2$ and $g^N_{\mathbb{C}P}$ respectively where $I'$ is the interval $(\frac{2N}{2N-1})^{\frac{1}{2N-1}}(a^{\frac{2N}{2N-1}}, b^{\frac{2N}{2N-1}})$. Consider the product manifold $\mathcal{M} = I' \times f \mathbb{C}P^N$. with the warping function $f = \frac{2N}{2N-1}s$. The warped product metric $g$ is of the form

$$g = s^*(ds^2) + (f \circ r)^2 \sigma^*(g_{\mathbb{C}P^N})$$
\[ ds^2 + \left( \frac{2N}{2N - 1} \right)^2 s^2 g_{CP^N} \]  
(5.5)

Where \( s \) and \( \sigma \) are projections from \( \mathcal{M} \) onto \( I' \) and \( CP^N \) respectively. For \( v \in H^1(\mathcal{M}) \) we have

\[ |\nabla g v|^2 = |v_s|^2 + \left( \frac{2N}{2N - 1} s \right)^2 |\nabla_{CP^N} v|^2 \]  
(5.6)

5.2 Proof of lemma 2.1

Proof. If possible let \( z^0 = z(r^0, t^0_1, ..., t^0_n, \theta^0_1, ..., \theta^0_{N+1}) \) be a fixed point of \( T_\tau \). So we have

\[ z(r^0, t^0_1, ..., t^0_n, \theta^0_1, ..., \theta^0_{N+1}) = z(r^0, t^0_1, ..., t^0_n, \theta^0_1 + \tau, ..., \theta^0_{N+1} + \tau) \]

for all \( \tau \in [0, 2\pi) \). Now equating the last two coordinates we get \( \cos(t_1) = 0 \). So \( \sin(t_1) \neq 0 \). Now compare \( x_{2N} \) and use the fact that \( \sin(t_1) \neq 0 \) and we get \( \cos(t_2) = 0 \). In the same way we shall get \( \cos(t_j) = 0 \) for all \( j = 1, 2, ..., n \). Now comparing the first coordinates we arrived at contradiction. So \( T_\tau \) is a fixed point free group action.

5.3 Proof of lemma 2.2

Proof. Let \( z = z(r, t_1, ..., t_n, \theta_1, ..., \theta_{N+1}) \) and \( z = z(r, t_1, ..., t_n, \theta'_1, ..., \theta'_{N+1}) \) such that \( \theta_i - \theta_{N+1} = \theta'_i - \theta'_{N+1} = \sigma_i \). We need to show that \( u(z) = u(z') \). Let \( \theta = \theta'_1 - \theta_1 \). Then we have \( \theta'_1 = \theta_1 + \theta \) for some \( \theta \in [0, 2\pi) \). Hence \( \theta'_{N+1} = \theta'_1 - \theta_1 + \theta_{N+1} = \theta + \theta_{N+1} \) and we get \( \theta'_{N+1} - \theta_{N+1} = \theta \). Similarly we can show \( \theta'_i - \theta_i = \theta'_{N+1} - \theta_{N+1} = \theta \). So \( v \) is well defined for \( u \in H^1_\alpha(A) \).

5.4 Appendix B : The reduction.

In this polar co-ordinate the energy functional \( J_\varepsilon(u) \) of (1.1) takes the form

\[ J_\varepsilon(u) = \int_A \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + |x|^\alpha \frac{u^2}{2} - |x|^\alpha F(u) \right) dx \]
\[
= \int_{I \times S^{2N+1}} \frac{\varepsilon^2}{2} (u_r^2 + \frac{1}{r^2} |\nabla_{S^{2N+1}} u|^2) + r^\alpha \frac{u^2}{2} - r^\alpha F(u) [r^{2N+1} dr d\sigma_{S^{2N+1}}]
\]

where \( I = [a, b] \). First we shall do the reduction on the first part of the above integral as

\[
\int_{I \times S^{2N+1}} (u_r^2 + \frac{1}{r^2} |\nabla_{S^{2N+1}} u|^2) r^{2N+1} dr d\sigma_{S^{2N+1}}
= \int_I \left\{ \int_{S^{2N+1}} (u_r^2 + \frac{1}{r^2} |\nabla_{S^{2N+1}} u|^2) \sigma_{S^{2N+1}} \right\} r^{2N+1} dr
\]

Now use the change of variable \( \phi(r, t_1, \ldots, t_n, \theta_1, \ldots, \theta_{N+1}) = (r, t_1, \ldots, t_n, \psi_1, \ldots, \psi_N) \), as defined above and we get

\[
\int_{S^{2N+1}} (u_r^2 + \frac{1}{r^2} |\nabla_{S^{2N+1}} u|^2) d\sigma_{S^{2N+1}} = 2\pi \int_{C_P N} (v_r^2 + \frac{1}{r^2} |\nabla_{C_P N} v|^2) dV_{C_P N}
\]

So we have

\[
\int_A |\nabla u|^2 dx = 2\pi \int_I \left\{ \int_{C_P N} (v_r^2 + \frac{1}{r^2} |\nabla_{C_P N} v|^2) dV_{C_P N} \right\} r^{2N+1} dr \quad (5.7)
\]

let \( r = \left( \frac{2N-1}{2N} \right)^{\frac{1}{2N}} s^{\frac{2N+1}{2N}} \). Then

\[
dr = \left( \frac{2N - 1}{2N} \right)^{\frac{2N+1}{2N}} s^\frac{1}{2N} \frac{ds}{s^{2N-\frac{1}{N}}}
\]

\[
r^{2N+1} dr = \left( \frac{2N - 1}{2N} \right)^{\frac{2N+1}{N}} s^{2N-\frac{1}{N}} ds
\]

Also \( |v_r|^2 = \left( \frac{2N}{2N-1} \right)^{\frac{2N+1}{2N}} s^{\frac{1}{2N}} |v_s|^2 \) and \( \frac{1}{r^2} = \left( \frac{2N}{2N-1} \right)^{\frac{1}{2N}} s^{-2} \). Then finally we get

\[
\int_A |\nabla u|^2 dx = 2\pi \int_{C_P N} \left\{ \int_{C_P N} (v_s^2 + \left( \frac{2N}{2N-1} \right)^{-2} |\nabla_{C_P N} v|^2) dV_{C_P N} \right\} s^{2N} ds \quad (5.8)
\]
where \( I' = \left( \frac{2^{N}}{2^N} \right)^{\frac{1}{2N+1}} \left( a^{\frac{2N}{2N+1}}, b^{\frac{2N}{2N+1}} \right) \). Also using the same change of variables we get

\[
\begin{align*}
\int_{A} (|x|^\alpha \frac{u^2}{2} - |x|^\alpha F(u)) \, dx &= \int_{I} \{ \int_{S^{2N+1}} (\frac{u^2}{2} - F(u)) \, d\sigma_{S^{2N+1}} \} r^{2N+\alpha+1} \, dr \\
&= 2\pi \int_{I} \{ \int_{\mathbb{CP}^N} (\frac{v^2}{2} - F(v)) \, dV_{\mathbb{CP}^N} \} r^{2N+\alpha+1} \, dr \\
&= 2\pi \int_{I^{\prime} \times \mathbb{CP}^N} \left( \frac{2^N - 1}{2} \right)^{\frac{\alpha}{2N}} s^{\frac{2N+2-\alpha}{2N}} \left( \frac{v^2}{2} - F(v) \right) s^{2N} \, ds \, dv_{\mathbb{CP}^N} \\
&\quad \times \left( v^{\alpha+2-2\alpha} \right) s^{\frac{\alpha}{2N}} \left( s^{\frac{2N}{2N+1}} - \frac{F(v)}{s^{\frac{2N}{2N+1}}} \right) s^{2N} \, ds \, dv_{\mathbb{CP}^N}
\end{align*}
\]

So finally

\[
J_s(u) = 2\pi \int_{I^{\prime} \times \mathbb{CP}^N} \left[ \frac{\varepsilon^2}{2} (v_s^2 + (\frac{2N}{2N+1} s)^{-2} |\nabla_{\mathbb{CP}^N} v|^2) + \left( \frac{2^N - 1}{2} \right)^{\frac{\alpha}{2N}} s^{\frac{2N+2-\alpha}{2N}} \left( \frac{v^2}{2} - F(v) \right) s^{2N} \, ds \, dv_{\mathbb{CP}^N} \right] (5.10)
\]

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