Topological Landau-Ginzburg Matter from $\text{Sp}(N)_K$ Fusion Rings

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Abstract

We find and analyze the Landau-Ginzburg potentials whose critical points determine chiral rings which are exactly the fusion rings of $\text{Sp}(N)_K$ WZW models. The quasi-homogeneous part of the potential associated with $\text{Sp}(N)_K$ is the same as the quasi-homogeneous part of that associated with $\text{SU}(N+1)_K$, showing that these potentials are different perturbations of the same Grassmannian potential.

Twisted $N = 2$ topological Landau-Ginzburg theories are derived from these superpotentials. The correlation functions, which are just the $\text{Sp}(N)_K$ Verlinde dimensions, are expressed as fusion residues.

We note that the $\text{Sp}(N)_K$ and $\text{Sp}(K)_N$ topological Landau-Ginzburg theories are identical, and that while the $\text{SU}(N)_K$ and $\text{SU}(K)_N$ topological Landau-Ginzburg models are not, they are simply related.

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1. Introduction

From the beginning, the Landau-Ginzburg (LG) approach has provided a useful picture of rational conformal field theories. More recently the Landau-Ginzburg formulation of $\mathcal{N} = 2$ superconformal theories has yielded detailed results since the equation of motion for the superpotential $dW = 0$ is not subject to renormalizations. A subset of the fields in such LG theories form a closed, non-singular operator algebra, the chiral ring. Such fields have the property that their correlation functions are topological. The non-topological fields can be removed by twisting these theories to obtain LG models which are entirely topological.

It is remarkable that there are deformations of the $\mathcal{N} = 2$ superconformal theories to non-conformal (but still $\mathcal{N} = 2$ supersymmetric) theories which retain a closed topological ring. They can also be twisted to obtain topological LG models. These twisted $\mathcal{N} = 2$ models are important since they serve as topological matter that can be coupled to topological gravity.

A particularly interesting example, a set of LG potentials whose chiral rings are isomorphic to the SU($N$)$_K$ WZW fusion rings, has been presented in ref. [10]. These potentials, considered as the superpotentials of $\mathcal{N} = 2$ supersymmetric theories, are obtained by perturbing the Grassmannian superconformal field theories. It is possible to twist these perturbed theories to obtain topological LG theories.

One might expect, due to the prominence of the simply-laced Dynkin diagrams in singularity theory, for example, that only the simply-laced groups such as SU($N$)$_K$ would be amenable to this approach in a natural way. In this paper we show that all the constructions mentioned above apply just as naturally to the non-simply laced case of Sp($N$)$_K$. (Two recent papers [12, 13] appeared during the preparation of this letter which suggest that LG potentials exist for arbitrary WZW fusion rings.)

Here we give the first detailed description of a fusion ring of a non-simply laced group, Sp($N$)$_K$, as a LG ring. As an immediate application of this result, we construct the twisted $\mathcal{N} = 2$ topological LG model by using these Sp($N$)$_K$ potentials as superpotentials. We find that the theory characterized by an Sp($N$)$_K$ potential is a deformation of the same Grassmannian that can be perturbed (in a different way) to obtain the SU($N + 1$)$_K$ based LG models. A further result is that the twisted versions of the Sp($N$)$_K$ and Sp($K$)$_N$ based topological LG theories are identical (on surfaces of any genus). We also find that although the SU($N$)$_K$ and SU($K$)$_N$ theories are only identical on genus zero, their correlation functions on arbitrary genus surfaces are closely related.

2. Landau-Ginzburg Potentials for the Sp($N$)$_K$ Fusion Ring

First, we exhibit the Sp($N$)$_K$ fusion ring as the quotient of an unrestricted polynomial ring by an ideal. We then present a Landau-Ginzburg potential and show that its chiral ring is the fusion ring.

The primary fields of Sp($N$)$_K$ ($N = \text{rank}(\text{Sp}(N))$) are naturally described by the Young tableaux with at most $N$ rows and $K$ columns. Let $l_i$ denote the $i^{\text{th}}$ row length and $k_i$ the $i^{\text{th}}$ column length of a given tableau. The row lengths are related to Dynkin indices by $l_i = \sum_{j=1}^{N} a_i$. The representation of Sp($N$) with fundamental
highest weight $\Lambda_j$ is described by a tableau with a single column of length $j$. Let $\chi_1, \ldots, \chi_N$ denote the characters of these fundamental representations.

In order to obtain the fusion ring as a quotient of a free ring in these fundamental variables by an ideal, first consider the free ring generated by the infinite set \{\chi_i \mid i = 0, \ldots, \infty\}. In this context define the character of an arbitrary tableau $\lambda$ with first row length $l_1 = s$ by the Giambelli type formula,

\[
\text{char}(\lambda) = \det \begin{vmatrix}
\chi_{k_1} & (\chi_{k_1+1} + \chi_{k_1-1}) & \cdots & (\chi_{k_1-s} + \chi_{k_1-s+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{k_i - i + 1} & (\chi_{k_i-i+2} + \chi_{k_i-i}) & \cdots & (\chi_{k_i-i-s} + \chi_{k_i-i-s+2}) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{k_s - s + 1} & (\chi_{k_s-s+2} + \chi_{k_s-s}) & \cdots & (\chi_{k_s} + \chi_{k_s-2s+2})
\end{vmatrix}
\]

(2.1)

with $\chi_0 = 1$ and $\chi_j = 0$ for $j < 0$. An alternate expression for $\text{char}(\lambda)$ can be given in terms of the tableaux with just one row. Let $J_i$ denote the character of a tableau consisting of a single row of length $i$. Then for an arbitrary tableau with first column length $k_1 = p$

\[
\text{char}(\lambda) = \det \begin{vmatrix}
\vdots & \vdots & \ddots & \vdots \\
J_{i_1-i+1} & (J_{i_1-i+2} + J_{i_1-i}) & \cdots & (J_{i_1-i+p} + J_{i_1-i-p+2}) \\
\vdots & \vdots & \ddots & \vdots \\
\end{vmatrix}
\]

(2.2)

Even though a tableau with $p > N$ does not correspond to an Sp($N$) representation, these formulas define a character for all such tableaux in terms of the $\chi_j$ or $J_j$.

The Sp($N$)$_K$ fusion algebra operation is defined by

\[
\phi_a \phi_b = \sum_c N_{ab}^c \phi_c
\]

(2.3)

where the non-negative integers $N_{ab}^c$ can be calculated as follows. The product of the characters of arbitrary tableaux can be computed in any given case from the Pieri type formula,

\[
\text{char}(\lambda)\text{char}(\mu) = \sum_\nu \text{char}( (\lambda/\nu) \cdot (\mu/\nu) )
\]

(2.4)

where $(\alpha/\beta)$ denotes the sum of all tableaux $\gamma$ such that $\alpha \in \beta \otimes \gamma$, and the dot indicates the Littlewood-Richardson product. This result follows ultimately from the definition of the characters in terms of the $\chi_j$ given in eq. (2.1). Tableaux not corresponding to Sp($N$)$_K$ primaries (i.e., those with more than $N$ rows or $K$ columns) can appear in eqs. (2.1) and (2.4)

In order for eq. (2.4) to reproduce the Sp($N$) tensor ring, the characters of arbitrary tableaux defined by the determinants above must satisfy, in addition, a set of identities (the rank modification rules) which allow elimination of all tableaux with more than $N$ rows. This elimination means that the characters of tableaux
corresponding to actual Sp($N$) representations are functions of a finite number of fundamental variables, specifically, $\chi_0, \ldots, \chi_N$.

The characters $\chi_0, \chi_1, \ldots$ are then connected with certain elementary symmetric functions $E_j$ by the following equation, which holds for all $j$:

$$\chi_j = E_j - E_{j-2}, \quad (2.5)$$

with $E_j = 0$ for $j < 0$. The $E_j$ with $j \geq 0$ are defined in terms of the auxiliary variables $q_i (i = 1, \ldots, N)$ via the generating function:

$$\sum_{j=0}^{\infty} E_j t^j = \prod_{i=1}^{N} (1 + q_i t)(1 + q_i^{-1} t) . \quad (2.6)$$

It follows that $E_0 = 1$, $E_{2N-j} = E_j$ for $j = 0, \ldots, 2N$, and $E_j = 0$ if $j > 2N$.

From this and eq. 2.5 the generating characters $\chi_j$ clearly satisfy the relations

$$\begin{align*}
\chi_{N+1} &= 0 \\
\chi_{N+2} + \chi_N &= 0 \\
& \quad \vdots \\
\chi_j + \chi_{2N+2-j} &= 0 \\
& \quad \vdots
\end{align*} \quad (2.7)$$

These relations generate the classical (rank) modification rules mentioned above. As a result, the free ring of polynomials in the fundamental characters $\chi_0, \chi_1, \ldots, \chi_N$ generates the classical tensor ring.

Equations 2.4 and 2.5 also imply that the character of the single row tableau of length $i$, $J_i$, can be written as the $i^{th}$ complete symmetric function of the variables $q_i$ via the generating function:

$$\sum_{j=0}^{\infty} J_j t^j = \prod_{i=1}^{N} \frac{1}{(1 - q_i t)(1 - q_i^{-1} t)} . \quad (2.8)$$

One can impose a further set of identities (the fusion modification rules) that allow the elimination of all characters of tableaux with more than $K$ columns in favor of those corresponding to primary fields of $\text{Sp}(N)_K$. The ring structure of eq. 2.3 is thus determined by considering the ring defined in eq. 2.4 modulo the ideal whose irreducible elements are exactly given by the modification rules. Equivalently, this is the ring of polynomials in the basic variables $\chi_i, \ldots, \chi_N$ modulo the (fusion) modification rules, also written in terms of these basic variables:

$$\mathcal{R} = \mathcal{O}[\chi_1, \ldots, \chi_N]/\mathcal{I}_f . \quad (2.9)$$
The following relations generate this quantum or fusion ideal \( \mathcal{I}_f \) since they imply the fusion modification rules (the proof of which will appear elsewhere)

\[
\begin{align*}
J_{K+1} &= 0 \\
J_{K+2} + J_K &= 0 \\
&\vdots \\
J_{K+N} + J_{K-N+2} &= 0
\end{align*}
\] (2.10)

The remarkable fact which we describe in this paper is that the ring \( \mathcal{R} \) can be obtained from a Landau-Ginzburg potential \( V \), i.e.,

\[
\mathcal{R} = \mathcal{A}[\chi_1, \ldots, \chi_N]/dV .
\] (2.11)

To express the potential in a compact form we use the connection between the fundamental characters \( \chi_j \) and \( J_j \) and the symmetric functions described above.

It is then natural to consider (in the light of ref. [10])

\[
V_{N+K+1} = \frac{1}{N + K + 1} \sum_{i=1}^{N} (q_i^{N+K+1} + q_i^{-(N+K+1)})
\] (2.12)

as candidates for potentials for the \( \text{Sp}(N)_K \) fusion ring. Although eq. 2.12 involves inverse powers of the auxiliary variables, it is a polynomial in the fundamental variables \( \chi_j \). (This is clear from eq. 2.16 below.)

First we define a generating functional for these potentials

\[
V(t) = \sum_{i=1}^{N} \log(1 + q_i t)(1 + q_i^{-1} t) = \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m
\] (2.13)

so that

\[
V_m = \frac{1}{m} \sum_{i=1}^{N} (q_i^m + q_i^{-m})
\] (2.14)

Then a calculation similar to that of ref. [10] shows that

\[
\frac{\partial V_m}{\partial E_i} = \sum_{0 \leq j \leq 2N} (-1)^{j+1} J_{m-j} \frac{\partial E_j}{\partial E_i}
\] . (2.15)

Since \( E_j = E_{2N-j} \) (from eq. 2.8), we find

\[
\frac{\partial V_m}{\partial E_i} = \begin{cases} 
(-1)^{i+1}(J_{m-i} + J_{m+i-2N}) & \text{for } 1 \leq i \leq N - 1 \\
(-1)^{i+1} J_{m-i} & \text{for } i = N
\end{cases}
\] (2.16)

Since \( \det(\partial \chi_j / \partial E_i) = 1 \), we observe, upon setting \( m = N + K + 1 \), that the \( N \) critical point conditions of \( V_{N+K+1} \) obtained from eq. 2.16 \( (\partial V_m / \partial E_i = 0) \) exactly coincide with the generators of the ideal in eq. 2.10. This shows that the fusion ring
of \( \text{Sp}(N)_K \) is, in a natural way, a Landau-Ginzburg ring of the potential \( V_{K+N+1} \) when written in terms of the fundamental variables \( \chi_1, \ldots, \chi_N \).

As an example, consider the potentials \( V_{K+3} \) (i.e., \( N = 2 \)) for \( \text{Sp}(2)_K \). Because the polynomial in \( q \),

\[
\prod_{i=1}^{N} (q - q_i)(q - q_i^{-1}) = 0,
\]

is satisfied by any \( q_i^{\pm 1} \), we can find a recursion relation for the potentials in terms of the fundamental variables \( x = \chi_1 \) and \( y = \chi_2 \),

\[
(k + 4)V_{k+4} - x(k + 3)V_{k+3} + (y + 1)(k + 2)V_{k+2} - x(k + 1)V_{k+1} + \left\{ \begin{array}{ll}
4 & k = 0 \\
kV_k & k > 0
\end{array} \right\} = 0
\]

(2.18)

where \( V_1 = x \), \( V_2 = \frac{1}{4}x^2 - y - 1 \), and \( V_3 = \frac{1}{4}x^3 - xy \) are the proper initial conditions. If this equation is scaled by \( x \to \lambda x \) and \( y \to \lambda^2 y \), we obtain a recursion relation for the quasi-homogeneous part which is identical to the recursion relation for the quasi-homogeneous part of the \( \text{SU}(3)_K \) potential. (In general, the same is true for \( \text{Sp}(N)_K \) and \( \text{SU}(N+1)_K \).)

It will also be useful to describe the critical point conditions of \( V_{N+K+1} \) in eq. 2.16 in terms of the auxiliary variables \( q_i \). From eq. 2.12 we find that these conditions are given by

\[
q_j^{2(N+K+1)} = 1 \quad \text{for} \quad j = 1, \ldots, N.
\]

(2.19)

If \( q_i \neq q_j^{\pm 1} \) for \( i \neq j \) and \( q_i^2 \neq 1 \), then the Jacobian of the variable change \( \chi_j \to q_j \) is non-singular.

The \( \text{Sp}(N)_K \) fusion ring has a discrete symmetry generated by the operation \( \rho \), which is just the map of representations induced by the \( \mathbb{Z}_2 \) automorphism of the extended Dynkin diagram of \( \text{Sp}(N)_K \). In terms of tableaux, \( \rho(i) \) is the representation whose tableau is the complement in the \( N \) by \( K \) rectangle of the tableau of \( i \). Then for integrable \( \text{Sp}(N)_K \) representations \( a, b, \) and \( c \) (from eq. 3.10 of ref. [17])

\[
N_{ab}^c = N_{\rho(a)b}^{\rho(c)}
\]

(2.20)

Let us denote by \( \hat{c} \) the representation \( \rho(0) \), whose tableau is the complement of the identity, i.e. the rectangular tableau with \( N \) rows and \( K \) columns. Eq. 2.20 then implies that the fusion rule for the field \( \phi_{\hat{c}} \) with any primary field \( i \) satisfies

\[
\phi_{\hat{c}} \cdot \phi_i = \phi_{\rho(i)}.
\]

(2.21)

The presence of this symmetry permits the construction of a nontrivial topological LG model whose chiral ring is exactly the \( \text{Sp}(N)_K \) fusion ring.

3. Twisted \( N = 2 \) Topological Field Theories Based on \( \text{Sp}(N)_K \) The potential in eq. 2.12 is not quasi-homogeneous, which is expected, since \( \text{Sp}(N)_K \) is
a rational conformal field theory. Under the scaling \( \chi_j \to \lambda^j \chi_j \), one finds that the quasi-homogeneous part of eq. (2.12) is generated by

\[
(\mathcal{V}_{N+K+1})_{\text{quasi-hom}} = \frac{1}{N+K+1} \sum_{i=1}^{N} q_i^{N+K+1}
\]

which is the potential of the Grassmannian

\[
\frac{U(N+K)}{U(N) \times U(K)}
\]

(as well as being related to the Kazama-Suzuki coset \( \text{SU}(N+1)_K/\text{SU}(N) \otimes \text{U}(1) \)) so that the Landau-Ginzburg potentials of the \( \text{Sp}(N)_K \) and \( \text{SU}(N+1)_K \) fusion rings are different perturbations of the same \( N = 2 \) superconformal field theory.

These results will now be used to construct a topological Landau-Ginzburg model by twisting the \( N = 2 \) Landau-Ginzburg model associated with \( \text{Sp}(N)_K \). The starting point is the theory characterized by the superpotential \( W(\Phi_i) \) of an \( N = 2 \) supersymmetric LG theory, where the \( \Phi_i \) denote chiral superfields. We take the superpotentials to be the functions in eq. (2.12) expressed in terms of the fundamental fields (labeled by single column Young tableaux) which are then replaced by the chiral superfields, \( \text{i.e.} \chi_i \to \Phi_i \).

A finite number of states are topological and form a closed ring

\[
\mathcal{R} = \mathcal{D}[\Phi_1, \ldots, \Phi_N]/dW
\]

which is isomorphic to the \( \text{Sp}(N)_K \) fusion ring in eq. (2.9). This result shows that the remarkable correspondence between deformations of \( N = 2 \) superconformal models defined by Landau-Ginzburg potentials and rational conformal field theories also extends to the case of \textit{non-simply laced} groups in a natural way.

The twisted version of these \( N = 2 \) theories contains \textit{only} these chiral primary fields, which are the topological fields. The genus \( g \) correlation functions with an insertion of an arbitrary function \( F \) of chiral superfields \( \Phi_i \), with superpotential \( W \), is

\[
\langle F(\Phi_i) \rangle_g = \sum_{dW=0} h^{g-1} F(\Phi_i)
\]

where the handle operator \( h \) is (using the normalization of ref. [11])

\[
h = (-1)^{N(N-1)/2} \det(\partial_i \partial_j W).
\]

The sum is over the critical points at which \( dW = 0 \). Here, there is one critical point for each primary field of \( \text{Sp}(N)_K \). At the \( a \)th critical point of \( dW = 0 \) one has \( \Phi_i(a) = S_{ia}/S_{aa} \), which forms a diagonal representation of the fusion rules. We now compute the Hessian \( H \) of \( W \) at these critical points:

\[
H = \det \left( \frac{\partial^2 W}{\partial \chi_j \partial \chi_k} \right) = \det \left( \frac{\partial^2 W}{\partial E_j \partial E_k} \right)_{dW=0} = \frac{1}{\Delta^2} \det \left( \frac{\partial^2 W}{\partial q_j \partial q_k} \right)_{dW=0}
\]
where (using eq. 2.6)

\[ \Delta = \prod_{i=1}^{N} q_i^{-1} \prod_{i=1}^{N} (q_i - q_i^{-1}) \prod_{i<j} (q_i + q_i^{-1} - (q_j + q_j^{-1})) \] (3.7)

A direct calculation using eq. 2.19 gives

\[ \det \left( \frac{\partial^2 W}{\partial q_j \partial q_k} \right)_{dW=0} = 2^N (N + K + 1)^N \prod_{j=1}^{N} q_j^{N+K-1}. \] (3.8)

The Hessian \( H \) can be related to the modular transformation matrices \( S_{oa} \) as follows. From ref. 19 we know that

\[ S_{ab} = (-)^{N(N-1)/2} \left( \frac{2}{K + N + 1} \right)^{N/2} \det \text{Mat}(a, b) \] (3.9)

where

\[ \text{Mat}_{ij}(a, b) = \sin \left( \frac{\pi \theta_i(a) \theta_j(b)}{K + N + 1} \right), \] (3.10)

for \( i, j = 1, \ldots, N \), and

\[ \theta_i(a) = l_i(a) - i + N + 1, \quad i = 1, \ldots, N \] (3.11)

where \( l_i(a) \) are the row lengths of the tableau \( a \).

Then with

\[ q_j(a) = e^{i\pi \theta_j(a)/(N+K+1)} \] (3.12)

we find that (the \( a \) dependence of the \( q_j \) will be suppressed in what follows)

\[ S_{0a} = \frac{(-i)^N}{[2(N + K + 1)]^{N/2}} \prod_{i=1}^{N} (q_i - q_i^{-1}) \prod_{i<j} (q_i + q_i^{-1} - (q_j + q_j^{-1})) \] (3.13)

From eq. 3.12 we also find

\[ \prod_{j=1}^{N} q_j^{N+K-1} = \prod_{j=1}^{N} (-1)^{\theta_j(a)} q_j^{-2} = (-1)^{N(N+1)/2} (-1)^{r(a)} \prod_{j=1}^{N} q_j^{-2} \] (3.14)

where \( r(a) \) is the number of boxes of the tableau \( a \). Combining eqs. 3.6, 3.8, and eqs. 3.13, 3.14 we find

\[ \det \left( \frac{\partial^2 W}{\partial \chi_j \partial \chi_k} \right)_{a-\text{th crit}} = \frac{(-1)^{N(N-1)/2} (-1)^{r(a)}}{(S_{0a})^2} \] (3.15)

Therefore using eq. 3.5 we find that

\[ (h)_{a-\text{th}} = \frac{(-1)^{r(a)}}{(S_{0a})^2}. \] (3.16)
The value of the field \( \Phi_b \) at the \( a^{th} \) critical point is given by the character at that point
\[
\Phi_b(a) = \text{char}_a(b) = \frac{S_{ba}}{S_{0a}}. \tag{3.17}
\]
Since (from eq. 2.10 of ref. [17])
\[
S_{a\rho(b)} = (-1)^{r(a)} S_{ab} \tag{3.18}
\]
and since \( \hat{\epsilon} = \rho(0) \)
\[
\Phi_{\hat{\epsilon}}(a) = \text{char}_a(\hat{\epsilon}) = \frac{S_{\rho(0)a}}{S_{0a}} = (-1)^{r(a)}. \tag{3.19}
\]
Therefore the handle operator in eqs. 3.3 and 3.16 satisfies
\[
(h)_{a-\text{th}} = (S_{0a})^{-2} \Phi_{\hat{\epsilon}}(a) \tag{3.20}
\]
for all critical points \( a \). Due to the non-singularity of the matrix \( S_{ab} \) this uniquely specifies the operators and we have \( h = h_0 \Phi_{\hat{\epsilon}} \), where the untwisted handle operator satisfies \( h_0(a) = S_{0a}^{-2} \).

Since \( \Phi_{\hat{\epsilon}} \) is the unique chiral primary field with maximal charge,
\[
\langle \Phi_{\hat{\epsilon}} \rangle_{g=0} = 1 \tag{3.21}
\]
The topological metric for the theory based on the Sp\((N)K\) superpotential is
\[
\eta_{ij} = \langle \Phi_i \Phi_j \rangle_{g=0} = N_{ij} \hat{\epsilon} = \delta_{j,\rho(i)} \tag{3.22}
\]
where \( \rho(i) \) is the complement tableau of \( i \) in the \( N \times K \) rectangle, while the topological metric in the unperturbed theory (the Grassmannian) is
\[
\eta_{ij}^{(0)} = \delta_{j,\text{comp}(i)} \tag{3.23}
\]
where \( \text{comp}(i) \) is again the complement tableaux of \( i \) in an \( N \times K \) rectangle. Thus the metric is unchanged upon deformation
\[
\eta_{ij} = \eta_{ij}^{(0)} \tag{3.24}
\]
as is required for a perturbed topological theory.

Note that in the twisted SU\((N+1)K\) theory, where the \( \mathbb{Z}_{N+1} \) discrete symmetry allows construction of a nontrivial topological LG theory, the topological metric involves exactly the same operation of tableau complement. The operation \( \sigma \) that generates this symmetry adds a row of length \( K \) to the top of the tableau \( a \), so that it connects representations that are cominimally equivalent (in the terminology of ref. [17]). Then \( \sigma^N(7) \) implements exactly the tableau complement in an \( N \times K \) rectangle.
4. Rank-Level Duality of Topological Field Theories

The pair of topological field theories built by twisting the $N = 2$ models associated with $\text{Sp}(N)_K$ and $\text{Sp}(K)_N$ Landau-Ginzburg potentials are in fact equivalent. For the (untwisted) $N = 2$ supersymmetric (non-conformal) theories characterized by these superpotentials associated with rational conformal field theories, the correlation functions of the chiral primary fields $\Phi_{i_1}, \ldots, \Phi_{i_n}$ can be expressed in terms of the modular transformation matrices

$$
\langle \Phi_{i_1}, \ldots, \Phi_{i_n} \rangle^{(u)}_g = \sum_{p} S_{op}^{-2(g-1)} S_{i_1p} \cdots S_{i_np} \overline{S_{0p}} \tag{4.1}
$$

where the superscript $(u)$ indicates the untwisted theory. The correlation functions in the twisted theory can be calculated for any genus in terms of the untwisted theory by inserting $\Phi_c$

$$
\langle \Phi_{i_1} \cdots \Phi_{i_n} \rangle^{(t)} = \langle \Phi_{i_1} \cdots \Phi_{i_n} (\Phi_c)^{g-1} \rangle^{(u)} \tag{4.2}
$$

From the identity

$$
(S_{ab})_{\text{Sp}(N)_K} = (S_{\tilde{a} \tilde{b}})_{\text{Sp}(K)_N}, \tag{4.3}
$$

where $\tilde{a}$ denotes the transpose tableau of $a$, and the one-to-one correspondence of primary fields of $\text{Sp}(N)_K$ and $\text{Sp}(K)_N$ given by transposition, it follows directly from eqs. 4.1 and 4.2 that

$$
\langle \Phi_{a_1} \Phi_{a_2} \cdots \Phi_{a_n} \rangle_{\text{Sp}(N)_K} = \langle \Phi_{a_1} \Phi_{a_2} \cdots \Phi_{a_n} \rangle_{\text{Sp}(K)_N} \tag{4.4}
$$

for correlation functions on any genus surface of the twisted $\text{Sp}(N)_K$ theory.

To obtain the analogous results for $\text{SU}(N)_K$ we can use \cite{22} (where $r(a)$ denotes the number of boxes in the tableau $a$)

$$
(S_{ab})_{\text{SU}(N)_K} = \sqrt{\frac{K}{N}} e^{-2\pi i r(a)/NK} (S_{\tilde{a} \tilde{b}})_{\text{SU}(K)_N} \tag{4.5}
$$

in eq. 4.1. Lemma 1 of ref. \cite{22} allows one to relate a sum over primaries in one theory to a sum over primaries in the other at the cost of one factor of $N/K$ (which results from the different sizes of the cominimal orbits in $\text{SU}(N)_K$ and $\text{SU}(K)_N$) on the right hand side of eq. 4.6 below, so that for untwisted $\text{SU}(N)_K$ and $\text{SU}(K)_N$ the duality of the non-zero correlation functions of chiral primaries on a genus $g$ surface is

$$
\langle \Phi_{i_1} \cdots \Phi_{i_n} \rangle_{\text{SU}(N)_K} = \left( \frac{N}{K} \right)^g \langle \Phi_{\sigma^{p_1} \tilde{i}_1} \cdots \Phi_{\sigma^{p_n} \tilde{i}_n} \rangle_{\text{SU}(K)_N} \tag{4.6}
$$

for any set of integers $p_1, \ldots, p_n$ such that $\sum_{i=1}^n p_i = \Delta \mod N$ with $\Delta = \sum_{j=1}^{p_i} r(i_j)/N$. Here $\sigma^p(i)$ denotes the $p$-th representation in the cominimal orbit (cf. ref. \cite{17}). (That
Δ is an integer follows from properties of the SU(N)K fusion ring.) The fact that any set of integers that sum to \(- Δ \mod N\) can be chosen on the right hand side of eq. 4.6 is an example of the general symmetry of the Verlinde dimensions on arbitrary genus surfaces under the discrete \(\mathbb{Z}_N\) symmetry of SU(N)K. For \(g = 0\) and \(n = 3\) we recover the special cases of the duality of non-zero fusion coefficients \(N_{abc} = N_{\tilde{a}\tilde{b}\tilde{c} N - \Delta}\) and their cominimal symmetry \(N_{\sigma^2(a)\sigma^2(b)\sigma^2(c)} = N_{\sigma^2(a)\sigma^2(b)\sigma^2(c)}\) for any \(p_i\) and \(q_i\) such that \(\sum p_i = \sum q_i \mod N\), which are described in refs. 22, 21 and 17. If \(g \neq 0\) then we do not have an exact equality due to the different size of the cominimal equivalence classes (alternately, of the orbits of the simple currents). The correlation functions in the twisted SU(N)K theory are related to those of the twisted SU(K)N theory by exactly the same formula 4.6 except that the insertion of \(\Phi\) in eq. 4.2 shifts \(\Delta\) to \(\Delta + (g - 1)K\) in the condition on the \(p_i\) below eq. 4.6.

Therefore the twisted \(N = 2\) Landau-Ginzburg models based on Sp(N)K and Sp(K)N are exactly dual, and the twisted \(N = 2\) theories based on SU(N)K and SU(K)N are simply related, although only exactly dual on genus zero.

5. Concluding Remarks In this paper we have explained how certain LG potentials have chiral rings that exactly reproduce the Sp(N)K fusion ring. As applications of this result, we also examined the twisted \(N = 2\) topological LG theories given by interpreting these potentials as superpotentials, and then demonstrated the presence of rank-level duality in the \(N = 2\) topological LG theories based on Sp(N)K and SU(N)K.

The extent to which the remarkable pattern of connections between many disparate problems in low dimensional physics in the SU(N)K case is reproduced for Sp(N)K is presently under investigation.

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