Entangling power of quantum chaotic evolutions via operator entanglement

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We study operator entanglement of the quantum chaotic evolutions. This study shows that properties of the operator entanglement production are qualitatively similar to the properties reported in literature about the pure state entanglement production. This similarity establishes that the operator entanglement quantifies intrinsic entangling power of an operator. The term ‘intrinsic’ suggests that this measure is independent of any specific choice of initial states.

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Entanglement is a unique quantum phenomenon which continues to baffle and surprise. Recently, this remarkable property of quantum mechanical systems has been identified as a quantum resource whose production is an elementary prerequisite for any quantum informational and computational tasks. This basic task is accomplished by unitary transformations, i.e., a given unitary operator acts on a product state and transforms that state into an entangled state. Then this given operator is referred to as an entangling operator. However, entangling power of different unitary operators are naturally not the same. Two different unitary operators operating on two identical initial states can produce different entanglement.

Investigations of different quantum signatures, like spectral statistics of the quantum chaotic Hamiltonian, phase space scarring, fidelity decay, etc., of classically chaotic systems is the subject of “quantum chaos.” Recent studies have shown that entanglement in chaotic systems can also be a good indicator of the regular to chaotic transition in its classical counterpart. Some of these studies have reported that the presence of chaos enhances the entanglement production rate. However, in our earlier works, we have observed saturation of entanglement production for strongly coupled strongly chaotic systems. This saturation of the entanglement production is a statistical property which we have modeled by random matrix theory (RMT). Coupling strength between two chaotic subsystems is another important parameter for the entanglement production. For example, in case of weak coupling, the entanglement production is higher for sufficiently long time corresponding to non-chaotic cases.

Recently, rather than focusing on the entanglement evolution of single initial product states, the authors studied global entangling properties of coupled chaotic systems using the coupled kicked tops model. Following Ref. \textsuperscript{12}, they have considered different ensembles of initial product states and have studied ensemble average of the entanglement production as a measure of entangling power of coupled kicked tops time evolution operator $U_T$. Their results have shown that the entangling power of $U_T$ is quantitatively, as well as qualitatively different for two different ensembles of initial states, one that is averaged over all product coherent states, the other including all appropriate product states. Thus this averaging gives a more global measure of entanglement.

In the present paper, we study operator entanglement of quantum chaotic evolution operator. We find that the operator entanglement production behaves qualitatively similar to pure state entanglement production. This similarity justifies that the operator entanglement is an intrinsic measure of the entangling power of a given operator. The term ‘intrinsic’ is referred to the fact that this measure is independent of any specific choice of single initial state or some ensemble of initial states. Indeed the operator entanglement is only a property of the operator and makes no reference to states it acts on. Furthermore, it has been shown in Ref. \textsuperscript{17} that the operator entanglement is related to the measure proposed in \textsuperscript{16}. However, the operator entanglement is easier to calculate than the measure proposed in \textsuperscript{14}.

For a formal definition of this measure, let us start with a simple algebraic transformation on matrices called matrix reshaping. Consider a rectangular matrix $A$ with elements $A_{ij}$, $i = 1, \ldots, K$ and $j = 1, \ldots, L$. This matrix can be reshaped into an one-dimensional vector $|A\rangle$ by putting its elements row after row into lexicographical order of size $KL$, i.e.

$$a_m = \langle m | A \rangle = A_{ij} \quad \text{where} \quad m = (i - 1)L + j,$$

$$i = 1, \ldots, K; \quad j = 1, \ldots, L.$$  \hspace{1cm} (1)

Following is a very simple example of the matrix reshaping:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \leftrightarrow |A\rangle = \{A_{11}, A_{12}, A_{21}, A_{22}\}^T.$$  \hspace{1cm} (2)

Vector $|A\rangle$ can be considered as an element of operator Hilbert space or Hilbert-Schmidt (HS) space. The scalar product between any two elements $(A, B)$ of a HS space $\mathcal{H}_{HS}$ is defined as $\langle A | B \rangle = \text{Tr} A^\dagger B$. Consequently, the
HS norm of a matrix is equal to just the norm of the associated vector, i.e. $|A|_{HS}^2 = \langle A | A \rangle = \sum_{m} |a_m|^2$.

Let us now consider an arbitrary unitary operator $U$ operating on a bipartite state space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\dim \mathcal{H}_1 = N \leq \dim \mathcal{H}_2 = M$, and $U \in U(\mathcal{H})$. We are interested to measure entangling power of the unitary operator $U$. This unitary operator can be expanded in terms of complete orthonormal operator basis states $\{|A_m\rangle \otimes |B_n\rangle\}$ as

$$|U\rangle = \sum_{m=1}^{N^2} \sum_{a=1}^{M^2} X_{ma} |A_m\rangle \otimes |B_a\rangle,$$

where $|U\rangle$, $|A_m\rangle$ and $|B_n\rangle$ are the associated reshaped vectors of the matrices $U$, $A_m$ and $B_n$. The operator basis states $\{|A_m\rangle\}$ and $\{|B_n\rangle\}$ are orthonormal in the sense that they satisfy $\langle A_m | A_n \rangle = \text{Tr}(A_a^\dagger A_a) = \delta_{mn}$ and $\langle B_a | B_b \rangle = \text{Tr}(B_b^\dagger B_b) = \delta_{ab}$. A very simple example of a complete orthonormal operator basis is $\{|I_2, \sigma_i\rangle\}/\sqrt{2}$, where $I_2$ is the $2 \times 2$ unit matrix and $\sigma_i$'s are the Pauli spin matrices. Now $|U\rangle$ can be considered as a vector in the composite HS space, $\mathcal{H}_{HS} \otimes \mathcal{H}_{HS}$. We can apply Schmidt decomposition to $|U\rangle$ and get

$$|U\rangle = \sum_{m=1}^{N^2} \sqrt{\lambda_m} |\tilde{A}_m\rangle \otimes |\tilde{B}_m\rangle,$$

where $\{|\lambda_m\rangle\}$ are the singular values of the rectangular matrix $X$, and $\{|\tilde{A}_m\rangle\}$ and $\{|\tilde{B}_m\rangle\}$ are the new orthonormal basis states. $\langle \lambda_m \rangle$ can also be identified as the nonzero eigenvalues of operator reduced density matrices. The operator Schmidt decomposition has also been proposed by Nielsen et al. However, application of the matrix reshaping operation has put operator Schmidt decomposition and state Schmidt decomposition on a same footing. Here we notice $\langle U | U \rangle = \sum_{m=1}^{N^2} \lambda_m = \text{Tr}(U^\dagger U) = NM$, i.e. the vector $|U\rangle$ is not normalized. To normalize $|U\rangle$, we define $\tilde{\lambda}_m \equiv \lambda_m/NM$. We now define von Neumann entropy $S_V(U)$ and linear entropy $S_L(U)$ of the operator entanglement respectively as

$$S_V(U) \equiv - \sum_{m=1}^{N^2} \tilde{\lambda}_m \ln \tilde{\lambda}_m = - \sum_{m=1}^{N^2} \frac{\lambda_m}{NM} \ln \frac{\lambda_m}{NM},$$

$$S_L(U) \equiv 1 - \sum_{m=1}^{N^2} \tilde{\lambda}_m^2 = 1 - \sum_{m=1}^{N^2} \frac{\lambda_m^2}{NM^2}.$$

These measures have already been utilized to study entanglement capability of qudit gates. Both the measures give qualitatively similar results, but the von Neumann entropy is a more acceptable measure. Therefore, in the present paper we prefer $S_V(U)$ to investigate the entangling power of quantum chaotic unitary operators.

We use coupled kicked tops as our model of coupled chaotic system. The time evolution operator, defined in between two consecutive kicks, corresponding to the coupled kicked tops is given by

$$U_T = U_{12}^\dagger (U_1 \otimes U_2) = U_{12}^\dagger[(U_k^1 U_1^f) \otimes (U_k^2 U_2^f)]$$

where the different terms are given by,

$$U_i^f \equiv \exp \left(-i \frac{\pi}{2} J_{y_i}\right), \quad U_i^k \equiv \exp \left(-i \frac{k}{2j_i z_i^2}\right),$$

and $i = 1, 2$ represents two different tops. The term $U_i^f$ describes free precession of the top around $y$ axis with angular frequency $\pi/2$, $U_i^k$ represents a torsion about $z$ axis by an angle proportional to $J_z$, with the proportionality factor $k/2j_i$, and $U_{12}^\dagger$ is the coupling between the tops using spin-spin interaction term with a coupling strength of $\epsilon/\sqrt{j_1 j_2}$.

Here we study the evolution of the operator entanglement of $U_T$. More precisely, we study the operator entanglement of $U_T^n$ as a function of the time step $n$. In our calculation, we choose complete orthonormal operator bases corresponding to each subsystem as,

$$A_\alpha = |m_1\rangle \langle n_1| \quad \text{and} \quad B_\beta = |m_2\rangle \langle n_2|$$

where $\alpha \equiv N(m_1 + j_1) + (n_1 + j_1 + 1); \quad (m_1, n_1) = -j_1, \ldots, j_1 \quad \beta \equiv M(m_2 + j_2) + (n_2 + j_2 + 1); \quad (m_2, n_2) = -j_2, \ldots, j_2 \quad \text{and} \quad N = 2j_1 + 1, \quad M = 2j_2 + 1$. Applying matrix reshaping operation to $U_T^n$ and to the orthonormal basis states, we can expand $|U_T^n\rangle$ as,

$$|U_T^n\rangle = \sum_{\alpha=1}^{N^2} \sum_{\beta=1}^{M^2} u_{\alpha\beta}(n) |A_\alpha\rangle \otimes |B_\beta\rangle.$$

Following the procedure discussed above, we investigate the operator von Neumann entropy $S_V(U_T^n)$ as a function of $n$. This study will show how entangling power of chaotic evolution changes with time steps.

In Fig. 1, we have presented our results for the operator entanglement production in coupled kicked tops for the spin $j_1 = j_2 = j$ (say) = 10. As we go from top to bottom windows, coupling strength is increasing by a factor of ten ($\epsilon = 10^{-4}$ to $\epsilon = 1.0$). For each coupling strength, we have studied operator entanglement production for four different single top kicked strengths $k$, whose corresponding classical phase space picture has been presented in Ref. It. Here we are presenting a brief qualitative description of those phase space pictures to correlate the effect of underlying classical dynamics on the operator entanglement production. For $k = 1.0$, the phase space was mostly covered by regular orbits, without any visible stochastic region. For $k = 2.0$, most of the phase space was covered by the regular region, but with
FIG. 1: Time evolution of the operator von Neumann entropy corresponding to the coupled kicked tops is presented for different coupling strengths and for different underlying classical dynamics. Solid line represents \( k = 1.0 \), dotted line corresponds to \( k = 2.0 \), dashed line is for \( k = 3.0 \) and dash-dot line represents \( k = 6.0 \).

A thin stochastic layer at the separatrix. For the change in the parameter value from \( k = 2.0 \) to \( k = 3.0 \), there was significant change in the phase space. At \( k = 3.0 \), the phase space was of truly mixed type. However, the size of the chaotic region was much larger than the regular region. Finally, when \( k = 6.0 \), the phase space was mostly covered by the chaotic region, with few very tiny regular islands.

Let us first discuss the case of weaker coupling \( \epsilon = 10^{-3} \), whose results are presented in the topmost window of Fig. 1. Here we observe larger operator entanglement production for the nonchaotic cases than the chaotic one. Therefore, for very weak coupling, the presence of chaos actually suppresses the entangling power of \( U_n \). This property has already been observed in \[6, 15\]. Present observation shows that the suppression of entanglement by chaos is an intrinsic property of coupled chaotic systems and is independent of any specific choice of initial state.

The results corresponding to \( \epsilon = 10^{-2} \) are presented in the second window from the top of Fig. 1. In this case, at least for initial time steps \( n \leq 100 \), we have observed suppression of operator entanglement by chaos. However, for larger \( n \), the operator entanglement corresponding to both the nonchaotic cases \( (k = 1.0, 2.0) \) eventually saturates at a value lower than the values corresponding to the mixed \( (k = 3.0) \) and the chaotic \( (k = 6.0) \) cases. Here we also notice that the operator entanglement production corresponding to the chaotic case \( (k = 6.0) \) is always larger than the mixed case \( (k = 3.0) \).

We now come to a reasonably stronger coupling strength \( \epsilon = 10^{-1} \), for which results are presented in the third window from the top of Fig. 1. Here again we observe saturation of the operator entanglement production for all the cases. However, the operator entanglement production rate is now much higher than the previous cases. These saturation values are clearly different for nonchaotic, mixed and chaotic cases. For the chaotic case \( k = 6.0 \), our numerical estimation shows that the saturation value is approximately \( \ln(0.6N^2) = \ln(0.6 \times 441) \approx 5.57 \). On the other hand, the saturation value corresponding to the mixed case \( k = 3.0 \) is slightly lower than this value. But the saturation values corresponding to the nonchaotic cases are distinctly lower than the other two cases.

Finally, we discuss the case of very strong coupling \( (\epsilon = 1.0) \) for which results are presented in the bottom window of Fig. 1. Here, due to the strong coupling, the over all coupled system is chaotic irrespective of underlying classical dynamics of the individual subsystems. Consequently the saturation values of the operator entanglement production are almost equal to \( \ln(0.6N^2) \approx 5.57 \) for all different values of \( k \).

An important outcome of this study is the observation of operator entanglement saturation at around \( \ln(0.6N^2) \) in case of chaotic subsystems which are coupled very strongly. Consequently, from our previous knowledge of the saturation of the pure state entanglement \[7, 8, 15\], we expect that the distribution of the eigenvalues of the operator reduced density matrices (RDMs) will follow RMT prediction. In Fig. 2, we have presented the distribution of the eigenvalues of the operator RDM corresponding to coupled kicked tops for different Hilbert space dimension \( M \) of the second top. Here we have fixed the dimension of the Hilbert space of the first top at \( N = 2j_1 + 1 = 21 \). The solid curve is representing the RMT predicted Laguerre distribution, i.e.,

\[
f(\lambda) = \frac{N^2Q}{2\pi} \frac{\sqrt{(\lambda_{\text{max}} - \lambda)(\lambda - \lambda_{\text{min}})}}{\lambda} \]

\[
\lambda_{\text{max}} = \frac{1}{2N^2} \left(1 + \frac{1}{Q} \pm \frac{2}{\sqrt{Q}}\right)
\]

where \( \lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}] \) and \( Q = M^2/N^2 \). The histograms are the numerical results obtained from coupled kicked tops model. This figure shows clear agreement between RMT prediction and numerical results. Using above distribution and following the procedure presented in \[6\], we can analytically estimate the operator entanglement saturation for different dimension ratios \( Q \). This calculation is straightforward, therefore we do not pursue this further here.

Above studies on the operator entanglement production show some qualitative properties which are common to the observed properties of the pure state entanglement production (see \[8, 15\]). For instance, (1) the operator entanglement, in general, supports ‘more chaos more entanglement’ hypothesis; (2) the operator entanglement production has also one statistical upper bound
which can be explained by RMT, and (3) for the weakly coupled cases, chaos suppresses the operator entanglement production. These similarities confirm that these are generic intrinsic properties of the entanglement production under chaotic evolution.

Eigenstates of any entangling operator also show some intrinsic entanglement properties of that operator. However, in many cases, eigenstates fail to give any clue of the entangling power of the operator. Consider an entangling operator $U = \exp(-i\alpha J_{z_1} \otimes J_{z_2})$ where $J_{z_1}$'s are usual angular momentum operators. Its entangling power is determined by the parameter $\alpha$. For a definite $\alpha$, this operator can create a maximally entangled state in case of a very special initial state. The eigenstates of $J_{z_1}$'s, i.e. $\{|m_1, m_2\}$, are also the eigenstates of $U$ with eigenvalues $\{\exp(-i\alpha m_1 m_2)\}$. These eigenstates are all unentangled, but many of them are degenerate. Any linear superposition of degenerate eigenstates can form entangled eigenstates, which are independent of the parameter $\alpha$. For example, $|\phi\rangle = (|m_1, m_2\rangle + |-m_1, -m_2\rangle)/\sqrt{2}$ are the entangled eigenstates of $U$ for $m_1 = 0$ or $m_2 = 0$, but $m_1, m_2$ are not simultaneously equal to zero. These eigenstates belong to the largest degenerate subspace of dimension $4j + 1$ with eigenvalues of unity. However, the maximum possible von Neumann entropy of these entangled eigenstates can be $\ln 2$. This is a maximally entangled state only for the spin $j = 1/2$. Besides, there are some degenerate subspaces with dimensions less than $4j + 1$, in which we can construct eigenstates with von Neumann entropy larger than $\ln 2$ but much less than the maximum possible value $\ln(2j + 1)$. Above all entanglement of these eigenstates are independent of $\alpha$. On the other hand, operator entanglement of $U$ is a function of $\alpha$ and it shows qualitatively similar behav-

ior to the pure state entanglement production by $U$. In case of the operator $U^n_p$, we have seen that its entangling power increases with $n$. But the eigenstates of $U^n_p$ are independent of $n$ and consequently it is impossible to distinguish the entangling power of $U^n_p$ for different $n$. Let us consider an interesting unitary operator $U_p = \exp(-i\alpha J_{z_1}) \otimes \exp(-i\alpha J_{z_2}) = \exp(-i\alpha (J_{z_1} + J_{z_2}))$ whose eigenstates are again $\{|m_1, m_2\}\}. This operator is clearly not an entangling operator. However, due to degeneracy, there are some eigenstates like $|\phi_p\rangle = N^{-\frac{1}{2}} \sum_m |m, -m\rangle$ are maximally entangled. Moreover, from different degenerate subspaces, we can construct eigenstates with von Neumann entropies $\ln X$ where $X$ can be any real number between $2j + 1$ to 1. Therefore, in this case, entanglement of the eigenstates gives an incorrect estimation of the entangling power of $U_p$. However, operator entanglement of $U_p$ is equal to zero.

In summary, we have studied operator entanglement production of the chaotic evolutions. This study shows that the behaviors of the operator entanglement production are qualitatively similar to the reported behaviors of the pure state entanglement production by the chaotic evolution operators. Consequently, this study establishes that the operator entanglement is an intrinsic measure of the entangling power of an operator. Moreover, we have pointed out that, when the eigenstates of a given operator fail to give any clue of its entangling power, the operator entanglement can give some estimation. As far as RMT is concerned, we have demonstrated another realization of the Laguerre distribution.

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