BGG CATEGORY FOR THE QUANTUM SCHRÖDINGER ALGEBRA

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Abstract. In 1996, a \(q\)-deformation of the universal enveloping algebra of the Schrödinger Lie algebra was introduced in Dobrev et al. [J. Phys. A 29 (1996) 5909–5918]. This algebra is called the quantum Schrödinger algebra. In this paper, we study the Bernstein-Gelfand-Gelfand (BGG) category \(O\) for the quantum Schrödinger algebra \(U_q(\mathfrak{s})\), where \(q\) is a nonzero complex number which is not a root of unity. If the central charge \(\hat{z} \neq 0\), using the module \(B_{\hat{z}}\) over the quantum Weyl algebra \(H_q\), we show that there is an equivalence between the full subcategory \(O[\hat{z}]\) consisting of modules with the central charge \(\hat{z}\) and the BGG category \(O(\mathfrak{sl}_2)\) for the quantum group \(U_q(\mathfrak{sl}_2)\). In the case that \(\hat{z} = 0\), we study the subcategory \(A\) consisting of finite dimensional \(U_q(\mathfrak{s})\)-modules of type 1 with zero action of \(Z\). We directly construct an equivalence functor from \(A\) to the category of finite dimensional representations of an infinite quiver with some quadratic relations. As a corollary, we show that the category of finite dimensional \(U_q(\mathfrak{s})\)-modules is wild.

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1. Introduction. In this paper, we denote by \(\mathbb{Z}\), \(\mathbb{Z}_+\), \(\mathbb{N}\), \(\mathbb{C}\), and \(\mathbb{C}^*\) the sets of all integers, nonnegative integers, positive integers, complex numbers, and nonzero complex numbers, respectively. Let \(q\) be a nonzero complex number which is not a root of unity. For \(n \in \mathbb{Z}\), denote \([n] = \frac{q^n - q^{-n}}{q - q^{-1}}\).

The Bernstein-Gelfand-Gelfand (BGG) category \(O\) for complex semisimple Lie algebras was introduced by Joseph Bernstein, Israel Gelfand, and Sergei Gelfand in the early 1970s, see [3], and it includes all highest weight modules such as Verma modules and finite dimensional simple modules. This category is influential in many areas of representation theory. For more on category \(O\), see the recent monograph [15] for details. It should be mentioned that the category \(O\) for quantum groups was already introduced by Andersen and Mazorchuk in [1]. However, Andersen and Mazorchuk deal only with quantum groups corresponding to semisimple Lie algebras, whereas the Schrödinger Lie algebra is non-semisimple.

The Schrödinger Lie algebra \(\mathfrak{s}\) is the semidirect product of \(\mathfrak{sl}_2\) and the three-dimensional Heisenberg Lie algebra. This algebra can describe symmetries of the free particle Schrödinger equation, see [10, 21]. The representation theory of the Schrödinger algebra has been studied by many authors. A classification of the simple highest weight representations of the Schrödinger algebra was given in [10]. All simple weight modules with finite dimensional weight spaces were classified in [12], see also [17]. All simple weight modules of the Schrödinger algebra were classified in [5, 6]. The BGG category \(O\) of \(\mathfrak{s}\) was studied in [13].
In 1996, in order to research the $q$-deformed heat equations, a $q$-deformation of the universal enveloping algebra of the Schrödinger Lie algebra was introduced in [11]. This algebra is called the quantum Schrödinger algebra.

The quantum Schrödinger algebra $U_q(s)$ over $\mathbb{C}$ is generated by the elements $E, F, K, K^{-1}, X, Y, Z$ with defining relations:

$$
[E, F] = \frac{k - k^{-1}}{q - q^{-1}}, \quad KXX^{-1} = qX,
KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,
KYK^{-1} = q^{-1}Y, \quad qYX - XY = Z, \quad (1.1)
EX = qXE, \quad EY = X + q^{-1}YE,
FX = YK^{-1} + XF, \quad FY = YF,
$$

where $Z$ is central in $U_q(s)$. The element $K$ corresponds to the power $q^D$ in [11]. The subalgebra generated by $E, F, K, K^{-1}$ is the quantum group $U_q(sl_2)$. The subalgebra generated by $X, Y, Z$ is called the quantum Weyl algebra $H_q$, see [2]. All simple weight modules over $U_q(s)$ with zero action of $Z$ were classified in [4].

In [14], a similar algebra was also studied, which was called the quantized symplectic oscillator algebra of rank one. For the quantized symplectic oscillator algebra, the element $Z$ is in the center of $U_q(sl_2)$ rather than in the center of $U_q(s)$. The paper [14] gave the Poincaré-Birkhoff-Witt Theorem, and studied the structure of highest weight modules and finite dimensional modules. In the present paper, we will give specific characterizations for each block of $\mathcal{O}$ for $U_q(s)$ using some quivers.

The paper is organized as follows. In Section 2, we recall some basic facts about the category $\mathcal{O}$ for $U_q(s)$. For a $z \in \mathbb{C}$, we denote by $\mathcal{O}[z]$ the full subcategory of $\mathcal{O}$ consisting of all modules which are annihilated by some power of the maximal ideal $(Z - z)$ of $\mathbb{C}[Z]$. In Section 3, in case of $z \neq 0$, using the modules $B_z$ over the quantum Weyl algebra $H_q$, we show that the functor $- \otimes B_z$ gives an equivalence between $\mathcal{O}^{(sl_2)}$ and $\mathcal{O}[z]$, where $\mathcal{O}^{(sl_2)}$ is the BGG category of $U_q(sl_2)$, see Theorem 8. A weight $U_q(s)$-module $M$ is of type 1 if the Supp($M$) $\subset q \mathbb{Z}$. In Section 4, we study the category $\mathcal{A}$ of finite dimensional $U_q(s)$-modules of type 1 with zero action of $Z$. It is shown that there is an equivalence between $\mathcal{A}$ and the category of finite dimensional representations of an infinite quiver with some quadratic relations, see Theorem 17. Our method is different from that in [13]. In [13], the grading technique was used in the study of finite dimensional modules for the Schrödinger Lie algebra.

2. Basic properties of the category $\mathcal{O}$.

2.1. The definition of category $\mathcal{O}$. Let $U_q(n^+)$ be the subalgebra of $U_q(s)$ generated by the elements $E, X$ and let $U_q(n^-)$ be the subalgebra generated by $F, Y$. Moreover let $U_q(h)$ be the subalgebra generated by the elements $K, K^{-1}, Z$. We write $\otimes$ for $\otimes_C$.

Then, we have the following triangular decomposition:

$$U_q(s) = U_q(n^-) \otimes U_q(h) \otimes U_q(n^+). \quad (2.1)$$

A $U_q(s)$-module $V$ is called a weight module if $K$ acts diagonally on $V$, i.e.,

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_{\lambda},$$
where $V_{\lambda} = \{ v \in V \mid Kv = \lambda v \}$. For a weight module $V$, let 
\[
\text{supp}(V) = \{ \lambda \in \mathbb{C}^* \mid V_{\lambda} \neq 0 \}.
\]

Next, we introduce the category $O$ for $U_q(\mathfrak{s})$.

**Definition 1.** A left module $M$ over $U_q(\mathfrak{s})$ is said to belong to category $O$ if

1. $M$ is finitely generated over $U_q(\mathfrak{s})$;
2. $M$ is a weight module;
3. The action of $U_q(n^+)$ on $M$ is locally finite, i.e., $\dim U_q(n^+)v < \infty$ for any $v \in M$.

For a weight module $V$, a weight vector $v_\lambda \in V_{\lambda}$ is called the **highest weight vector** if $Ev_\lambda = Xv_\lambda = 0$. A module $M$ is called the highest weight module of highest weight $\lambda$ if there exists the highest weight vector $v_\lambda$ in $M$ which generates $M$. For $\lambda \in \mathbb{C}^*$ and $\hat{\lambda} \in \mathbb{C}$, let $\Delta(\lambda, \hat{\lambda})$ be the Verma module generated by $v_\lambda$, where $Kv_\lambda = \lambda v_\lambda$, $Zv_\lambda = \hat{\lambda} v_\lambda$. Then, $\{ Y^k F^l v_\lambda \mid k, l \in \mathbb{Z}_+ \}$ is a basis of $\Delta(\lambda, \hat{\lambda})$. Let $R(\lambda, \hat{\lambda})$ be the largest proper submodule of $\Delta(\lambda, \hat{\lambda})$. Hence, $L(\lambda, \hat{\lambda}) = \Delta(\lambda, \hat{\lambda})/R(\lambda, \hat{\lambda})$ is the unique simple quotient module of $\Delta(\lambda, \hat{\lambda})$.

**2.2. Basic properties of O.** By the similar arguments as those in [15], we can see that every module $M$ in $O$ has the following standard properties.

**Lemma 2.** The category $O$ is closed with respect to taking submodules, quotient modules, and finite direct sums. That is, the category $O$ is an abelian category.

**Lemma 3.** Let $M$ be any module in $O$.
1. The module $M$ has a finite filtration
   \[
   0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,
   \]
   such that each subquotient $M_j/M_{j-1}$ for $1 \leq j \leq n$ is the highest weight module.
2. Each weight space of $M$ is finite dimensional.
3. Any simple module in $O$ is isomorphic to some $L(\lambda, \hat{\lambda})$, for $\lambda, \hat{\lambda} \in \mathbb{C}$.

**3. Blocks of nonzero central charge.** In this section, we assume that $\hat{\lambda} \neq 0$. We denote by $O[\hat{\lambda}]$ the full subcategory of $O$ consisting of all modules which are annihilated by the maximal ideal $\langle Z - \hat{\lambda} \rangle$ of $\mathbb{C}[Z]$. Let $O^{(\hat{\lambda})}$ denote the BGG category for $U_q(\mathfrak{sl}_2)$, defined similarly to Definition 1. We will show that there is an equivalence between $O^{(\hat{\lambda})}$ and $O[\hat{\lambda}]$. Firstly, we find that the structure of Verma modules over $U_q(\mathfrak{s})$ is similar as that of Verma modules over $U_q(\mathfrak{sl}_2)$.

**3.1. The tensor product realizations of highest modules.** In this subsection, we will give tensor product realizations of Verma modules using Verma modules over $U_q(\mathfrak{sl}_2)$ and $H_q$. This construction is crucial to the study of the category $O[\hat{\lambda}]$.

For a nonzero $\hat{\lambda} \in \mathbb{C}$, let 
\[
B_{\hat{\lambda}} := H_q / (H_q (Z - \hat{\lambda}) + H_q X),
\]
which is a simple $H_q$-module. Denote the image of $Y^i$ in $B_{\hat{\lambda}}$ by $v_i$ for $i \in \mathbb{Z}_+$. We can see that 
\[
Xv_i = -\frac{\hat{\lambda} (q^i - 1)}{q - 1} v_{i-1}, \quad i \in \mathbb{Z}_+.
\] (3.1)
Define the action of $U_q(sl_2)$ on $B_\dot{z}$ by

$$Kv_i = q^{-\frac{1}{2} - i}v_i,$$
$$Fv_i = \frac{q^2}{2(q + 1)}v_{i+2},$$
$$Ev_i = -\dot{z}(q^i - 1)(q^{i-1} - 1)\frac{q^{i-2}(q^2 - 1)(q - 1)}{v_{i-2}}.$$  \hspace{1cm} (3.2)

Then, we can check that

$$(EF - FE)v_i = \frac{K - K^{-1}}{q - q^{-1}}v_i,$$
$$EYv_i = XV_i + q^{-1}YE v_i,$$
$$FXv_i = YK^{-1}v_i + X Fv_i.$$  

Thus, the action (3.2) indeed makes $B_\dot{z}$ to be a module over $U_q(s)$. We denote this $U_q(s)$-module by $\tilde{B}_\dot{z}$. In fact, $\tilde{B}_\dot{z} \cong L(q^{-\frac{1}{2}}, \dot{z})$.

We can make a $U_q(sl_2)$-module $N$ to be a $U_q(s)$-module by defining $H_qN = 0$. We denote the resulting $U_q(s)$-module by $\tilde{N}$.

By straightforward calculations, we can obtain the following lemma.

**Lemma 4.** The following map

$$\Delta : U_q(s) \to U_q(s) \otimes U_q(s)$$

defined by

$$\Delta(E) = 1 \otimes E + E \otimes K,$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(K) = K \otimes K,$$
$$\Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\Delta(X) = 1 \otimes X,$$
$$\Delta(Y) = 1 \otimes Y,$$
$$\Delta(Z) = 1 \otimes Z$$

defines an algebra homomorphism.

**Remark:** There is no algebra homomorphism $\epsilon : U_q(s) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccc}
U_q(s) & \xrightarrow{\Delta} & U_q(s) \otimes U_q(s) \\
\downarrow{id} & & \downarrow{1 \otimes \epsilon} \\
U_q(s) & \xrightarrow{\text{can}} & U_q(s) \otimes \mathbb{C}.
\end{array}$$

So we cannot define a bialgebra structure on $U_q(s)$ with the coproduct $\Delta$.

Via the map $\Delta$, the space $\tilde{N} \otimes \tilde{B}_\dot{z}$ can be defined as a $U_q(s)$-module for any $U_q(sl_2)$-module $N$. More precisely, for $u \in U_q(s)$, if $\Delta(u) = \sum u_i \otimes w_i$, then the action of $U_q(s)$ on $\tilde{N} \otimes \tilde{B}_\dot{z}$ is defined by

$$u(n \otimes b) = \sum u_i n \otimes w_i b,$$

$n \in \tilde{N}, b \in \tilde{B}_\dot{z}.$
Let $\Delta_{sl_2}(\lambda)$ be the Verma module over $U_q(sl_2)$ with the highest weight $\lambda$ whose unique simple quotient module is $L_{sl_2}(\lambda)$. It is well known that the module $\Delta_{sl_2}(\lambda)$ is reducible if and only if $\lambda^2 \in q^{2\mathbb{Z}+}$, see [16]. For each $d \in \mathbb{Z}_+$, we have non-split short exact sequences

$$0 \to \Delta_{sl_2}(q^{d-2}) \to \Delta_{sl_2}(q^d) \to L_{sl_2}(q^d) \to 0,$$

and

$$0 \to \Delta_{sl_2}(-q^{d-2}) \to \Delta_{sl_2}(-q^d) \to L_{sl_2}(-q^d) \to 0.$$

The structure of $\Delta(\lambda, z)$ was determined in [11]. The following proposition gives a constructive proof.

**Proposition 5.** [7] The following results hold.

1. If $z \neq 0$, then $\Delta(\lambda, z) \cong \tilde{\Delta}_{sl_2}(\lambda q^\frac{1}{2}) \otimes \tilde{B}_z$. Therefore, the Verma module $\Delta(\lambda, z)$ is reducible if and only if $\lambda^2 \in q^{2\mathbb{Z}+1}$. Moreover, $L(\lambda, z) \cong \tilde{L}_{sl_2}(\lambda q^{\frac{1}{2}}) \otimes \tilde{B}_z$.

2. For each $d \in \mathbb{Z}_+$, we have non-split short exact sequences

$$0 \to \Delta(q^{d-\frac{1}{2}}, z) \to \Delta(q^{d-\frac{1}{2}}, z) \to L(q^{d-\frac{1}{2}}, z) \to 0,$$

and

$$0 \to \Delta(-q^{d-\frac{1}{2}}, z) \to \Delta(-q^{d-\frac{1}{2}}, z) \to L(-q^{d-\frac{1}{2}}, z) \to 0.$$

**3.2. Equivalence between $O(sl_2)$ and $O[\hat{z}]$.**

**Lemma 6.** If $\hat{z} \neq 0$, then any module in $O[\hat{z}]$ has finite composition length.

**Proof.** According to (1) in Lemma 3, any nonzero module $M$ in $O[\hat{z}]$ has a finite filtration with sub-quotients given by highest weight modules. Hence, it suffices to treat the case that $M$ is the Verma module $\Delta(\lambda, \hat{z})$. By Proposition 5, $\Delta(\lambda, \hat{z})$ has finite composition length if $\hat{z} \neq 0$. □

**Proposition 7.** Suppose that $V$ is a module in $O[\hat{z}]$ with nonzero central charge $\hat{z}$. Then $V \cong \tilde{N} \otimes \tilde{B}_z$ for some $U_q(sl_2)$-module $N$.

**Proof.** By Lemma 6, $V$ has a finite composition length $l$. We will proceed by induction on $l$. Firstly, we consider the case $l = 1$. The fact that $V \in O$ forces that $V$ is a simple highest weight $U_q(sl_2)$-module. Then $V$ is a simple quotient module of some Verma module $\Delta(\lambda, \hat{z})$. Note that $\Delta(\lambda, \hat{z}) \cong \tilde{\Delta}_{sl_2}(\lambda q^\frac{1}{2}) \otimes \tilde{B}_z$. Thus, $V \cong \tilde{L}_{sl_2}(\lambda q^{\frac{1}{2}}) \otimes \tilde{B}_z$.

Next, we consider the general case. Let $M$ be a maximal submodule of $V$. By the induction hypothesis, we see that

$$M \cong \tilde{N}_1 \otimes \tilde{B}_z, \quad V/M \cong \tilde{N}_2 \otimes \tilde{B}_z,$$

where $N_1, N_2$ are $U_q(sl_2)$-modules.

As vector spaces, we can assume that $V = N \otimes B_z$, where $N$ is a vector space such that $N_1 \subseteq N$ and $N/N_1 \cong N_2$. Moreover, $u(w \otimes v) = w \otimes uv, K(w \otimes v) = (Kw) \otimes (Kv)$, for $u \in H_q, w \in N, v \in B_z$.

For $w \in N$, we can find $w_i, v_i \in N, 0 \leq i \leq k$ such that $E(w \otimes v_0) = \sum_{i=0}^{k} w_i \otimes v_i$, where $v_i$ is the image of $Y^i$ in $B_z$. From $X^kE = q^{-k}EX^k, Xv_0 = 0$, we have

$$X^kE(w \otimes v_0) = w_k \otimes X^k v_k = q^{-k}(w \otimes X^k v_0) = 0.$$
By (3.1), when $k > 0$, $X^k v_k \neq 0$. So we must have that $k = 0$. Denote $w_E = q^i w_0$. Then $E(w \otimes v_0) = w_E \otimes K v_0$.

From

\[ E v_i = \frac{-\hat{z}(q^i - 1)(q^{i-1} - 1)}{q^{i-2}(q^2 - 1)(q - 1)} v_{i-2} \]

and

\[ E Y^i = q^{-i} Y^i E + [i] Y^{i-1} X - \frac{(q^i - 1)(q^{i-1} - 1)}{q^{i-2}(q^2 - 1)(q - 1)} Z Y^{i-2}, \quad (3.4) \]

we obtain that

\[ E (w \otimes v_i) = E Y^i (w \otimes v_0) \]

\[ = \left( q^{-i} Y^i E + [i] Y^{i-1} X - \frac{(q^i - 1)(q^{i-1} - 1)}{q^{i-2}(q^2 - 1)(q - 1)} Z Y^{i-2} \right) (w \otimes v_0) \quad (3.5) \]

we have

\[ F(w \otimes v_0) = w_F \otimes v_0 + K^{-1} w \otimes F v_0 + \sum_{i=1}^{k} w'_i \otimes v_i, \]

where $\sum_{i=1}^{k} w'_i \otimes v_i \in N_1 \otimes \tilde{B}_2$.

Consequently, from $Y F = F Y$ and $Y^i v_0 = v_i$, we have

\[ F(w \otimes v_i) = w_F \otimes v_i + K^{-1} w \otimes F v_i, \quad i \in \mathbb{Z}_+. \quad (3.6) \]

We can define the action of $U_q(\mathfrak{sl}_2)$ on $N$ as follows:

\[ E \cdot w = w_E, \quad F \cdot w = w_F, \quad w \in N. \]

From (3.5) and (3.6), we can see that $V \cong \tilde{N} \otimes \tilde{B}_2$. The proof is complete. □

By Proposition 7, we have the following category equivalence.
THEOREM 8. If \( \hat{z} \neq 0 \), using the algebra homomorphism \( \Delta \) in \( (3.3) \), we can define a functor

\[
- \otimes \hat{B}_2 : O^{(s\ell_2)} \rightarrow O[\hat{z}].
\]

Moreover, this functor is an equivalence of categories.

Proof. By the definition of category \( O \), the functor \( F := - \otimes \hat{B}_2 \) maps modules in \( O^{(s\ell_2)} \) to modules in \( O[\hat{z}] \). By Proposition 7, the functor \( F \) is essentially surjective.

Next, we will show that for any \( M, N \in \text{Obj}(O^{(s\ell_2)}) \), the map

\[
F_{M,N} : \text{Hom}_{O^{(s\ell_2)}}(M, N) \rightarrow \text{Hom}_{O[\hat{z}]}(F(M), F(N))
\]

is a bijection.

For \( f \in \text{Hom}_{O^{(s\ell_2)}}(M, N) \) such that \( F_{M,N}(f) = f \otimes 1 = 0 \), we must have \( f = 0 \). So \( F_{M,N} \) is injective.

For any \( g \in \text{Hom}_{O[\hat{z}]}(F(M), F(N)) \), suppose that

\[
g(w \otimes v_0) = \sum_{i=0}^{k} w_i \otimes v_i, \quad w \in M, \quad w_k \in N,
\]

where \( v_i \) is the image of \( Y^i \) in \( \hat{B}_2 \). Since \( \hat{B}_2 \) is a simple \( H_q \)-module, by the density theorem, there exists \( u \in H_q \) such that

\[
u v_0 = v_0, \quad v_i = 0, \quad i = 1, \ldots, k.
\]

Form

\[
u g(w \otimes v_0) = g(u(w \otimes v_0)) = g(w \otimes uv_0),
\]

we have

\[
u \left( \sum_{i=0}^{k} w_k \otimes v_k \right) = \sum_{i=0}^{k} w_k \otimes u v_k = w_0 \otimes v_0 = g(w \otimes v_0).
\]

Define the map \( f : M \rightarrow N \) such that \( f(w) = w_0 \), i.e., \( g(w \otimes v_0) = f(w) \otimes v_0 \).

From \( Y^i g(w \otimes v_0) = g(w \otimes Y^iv_0) \), we have

\[
g(w \otimes v_i) = f(w) \otimes v_i, \quad \forall i \in \mathbb{Z}_+.
\]

By \( Eg(w \otimes v_0) = g(E(w \otimes v_0)) \) and \( Ev_0 = 0 \), we have that

\[
E f(w) \otimes Kv_0 = f(w) \otimes Ev_0 + f(Ew) \otimes Kv_0 = f(Ew) \otimes Kv_0,
\]

i.e., \( Ef(w) = f(Ew) \). Similarly, we can check that \( Ff(w) = f(Fw) \). Then \( f \in \text{Hom}_{O^{(s\ell_2)}}(M, N) \). So \( F_{M,N}(f) = g \), \( F_{M,N} \) is surjective.

Therefore \( F_{M,N} \) is bijective, and hence \( F := - \otimes \hat{B}_2 \) is an equivalence. \( \square \)

3.3. The description of \( O[\xi, \hat{c}, \hat{z}] \). Let \( C = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2} \) be the Casimir element of \( U_q(s\ell_2) \). Define the following element in \( U_q(s) \):

\[
\tilde{C} = \frac{1}{q^2-1} K^{-1} + XY(q^{-1}FE - qEF).
\]

In [7], the following lemma was proved.
LEMMA 9. The element $\tilde{C}$ belongs to the center of $U_q(s)$.

Remark: The element $\tilde{C}$ resembles (in the $q \to 1$ case) a similar central element studied in the paper [20].

For a module $M$ in $O$, from that the weight spaces of $M$ are finite dimensional, we can see that the action of $C[\tilde{C}, Z]$ on $M$ is locally finite.

For $\lambda \in \mathbb{C}$, $z \in \mathbb{C}$, let $v_\lambda$ be the highest vector of the $U_q(s)$-module $L(\lambda, z)$. We denote by $\tilde{c}_\lambda$ the scalar corresponding to the action of the central element $\tilde{C}$ on $v_\lambda$. Similarly, for the $U_q(sl_2)$-module $L_{sl_2}(\lambda)$, we denote scalar corresponding to the action of $C$ by $c_\lambda$.

LEMMA 10. We have the following:

1. $\tilde{c}_\lambda = \frac{z}{(q - q^{-1})^2} ((q + q^2)\lambda + (q^{-1} + q^2)\lambda^{-1})$.
2. $\tilde{c}_\lambda = \tilde{c}_{\lambda q^{-1}} \text{ iff } \lambda^2 = q^{k-3}$.

Let $\xi \in \mathbb{C}/q^\mathbb{Z}$. We denote by $O[\xi]$ the full subcategory of $O$ consisting of all $M$ such that supp$(M) \subset \xi$. For $\hat{c}, \hat{z} \in \mathbb{C}$, we denote by $O[\xi, \hat{c}, \hat{z}]$ the full subcategory of $O[\xi]$ consisting of all $M$ such that $M$ is annihilated by some power of the maximal ideal $(\hat{C} - \hat{c}, Z - \hat{z})$ of $C[\tilde{C}, Z]$. Since the action of $C[\tilde{C}, Z]$ on $M$ is locally finite, we have

$$O[\xi] \cong \bigoplus_{\hat{c}, \hat{z} \in \mathbb{C}} O[\xi, \hat{c}, \hat{z}]$$

Similarly, we can define the subcategory $O^{(s\mathfrak{l}_2)}[\xi, \hat{c}]$ of $O^{(s\mathfrak{l}_2)}$, where $\hat{c}$ is defined by the action of the Casimir element $C$ of $U_q(sl_2)$.

Using the equivalence between $O^{(s\mathfrak{l}_2)}$ and $O[\hat{z}]$ given in Theorem 8, we have the following equivalence.

LEMMA 11. The restriction functor

$$- \otimes \tilde{B}_z : O^{(s\mathfrak{l}_2)}[\xi, c_\lambda] \to O[\xi^{-\frac{1}{2}}, \tilde{c}, q\frac{\lambda}{q - q^{-1}}]$$

is an equivalence of categories, where $\lambda \in \xi$, $c_\lambda = \frac{q\lambda + q^{-1}\lambda^{-1}}{(q - q^{-1})^2}$.

Using the structure of $O^{(s\mathfrak{l}_2)}$ (see Section 5.3 in [19]), we give descriptions of each block $O[\xi, \hat{c}, \hat{z}]$ as follows.

PROPOSITION 12. Let $\xi = q^{\frac{1}{2} + \mathbb{Z}}$, $\hat{z} \in \mathbb{C}^*$. Then the following claims hold.

1. The module $\Delta(q^{-\frac{1}{2}}, \hat{z})$ is the unique simple object in $O[q^{\frac{1}{2} + \mathbb{Z}}, \tilde{c}, q^{-\frac{1}{2}}, \hat{z}]$. Moreover, the block $O[q^{\frac{1}{2} + \mathbb{Z}}, \tilde{c}, q^{-\frac{1}{2}}, \hat{z}]$ is semisimple.
2. For $n \in \mathbb{Z}_+$, $\Delta(q^{-n-\frac{1}{2}}, \hat{z})$ are all the simple objects in $O[q^{\frac{1}{2} + \mathbb{Z}}, \tilde{c}, q^{-\frac{1}{2}}, \hat{z}]$.
3. For $n \in \mathbb{Z}_+$, the subcategory $O[q^{\frac{1}{2} + \mathbb{Z}}, \tilde{c}, q^{-\frac{1}{2}}, \hat{z}]$ is equivalent to the category of finite dimensional representations over $\mathbb{C}$ of the following quiver with relations:

$$\begin{array}{cc}
\bullet & \bullet \\
\otimes & \\
\begin{array}{c}
\otimes \\
\begin{array}{c}
\Delta \\
\bullet
\end{array}
\end{array}
\end{array}$$

$$ab = 0.$$

PROPOSITION 13. Let $\xi = q^\mathbb{Z}$, $\hat{z} \in \mathbb{C}^*$. Then we have the following:

1. For any $n \in \mathbb{Z}$, the Verma module $\Delta(q^n, \hat{z})$ is simple.
2. The modules $\Delta(q^n, \hat{z})$, $\Delta(q^{-n-3}, \hat{z})$ are simple objects in $O[q^\mathbb{Z}, \tilde{c}, q^n, \hat{z}]$. 

BGG CATEGORY FOR THE QUANTUM SCHRÖDINGER ALGEBRA 273
The quantum plane $\mathbb{C}_q[X, Y]$ is a $U_q(s\mathfrak{l}_2)$-module via the following action

$$K \cdot X = qX, \quad E \cdot X = 0, \quad F \cdot X = Y,$$

$$K \cdot Y = q^{-1}Y, \quad E \cdot Y = X, \quad F \cdot Y = 0.$$ (4.1)

We will use the completely reducibility of finite dimensional $U_q(s\mathfrak{l}_2)$-modules and the Clebsch–Gordon rule to discuss the category $\mathcal{A}$. For convenience, let $L(i)$ denote the finite dimensional $U_q(s\mathfrak{l}_2)$-module with the highest weight $q^i$, $i \in \mathbb{Z}_+$. In fact, we can assume that $L(1) = \mathbb{C}X + \mathbb{C}Y$, whose $U_q(s\mathfrak{l}_2)$-module structure was defined by (4.1). It is well known that the tensor product $L(m) \otimes L(n)$ is a $U_q(s\mathfrak{l}_2)$-module under the action defined by the following co-multiplication:

$$\Delta'(E) = E \otimes 1 + K \otimes E, \quad \Delta'(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\Delta'(K) = K \otimes K, \quad \Delta'(K^{-1}) = K^{-1} \otimes K^{-1}.$$ (4.2)

Remark: The above co-multiplication $\Delta'$ is different from $\Delta$ in (3.3). Now $\Delta'$ can guarantee that $\tau(\theta)$ defined in Lemma 16 is a $U_q(s\mathfrak{l}_2)$-module homomorphism; however, $\Delta$ in (3.3) cannot. This is because we consider the left action of $L(1)$ on $L(i)$ in Lemma 16.

Next, we will introduce two lemmas which are used in the proof of Theorem 17.

Lemma 15. (1) $L(1) \otimes L(i) \cong L(i + 1) \oplus L(i - 1)$, for $i \geq 1$;

(2) Suppose that $v_i$ is the highest weight vector of $L(i)$. Then $X \otimes v_i$ is the highest weight vector of $L(1) \otimes L(i)$ whose highest weight is $q^{i+1}$, and $[i]Y \otimes v_i - q^{-1}X \otimes Fv_i$ is the highest weight vector of $L(1) \otimes L(i)$ whose highest weight is $q^i$;

(3) In $L(1) \otimes L(1) \otimes L(i)$, the elements

$$[i + 1]Y \otimes X \otimes v_i - q^{-1}X \otimes X \otimes Fv_i - q^{-i-1}X \otimes Y \otimes v_i,$$

and

$$[i]X \otimes Y \otimes v_i - q^{-1}X \otimes X \otimes Fv_i,$$

are highest weight vectors with the highest weight $q^i$.

Proof. (1) follows from the Clebsch–Gordon rule [16]:

$$L(m) \otimes L(n) \cong L(m + n) \oplus L(m + n - 2) \oplus \cdots \oplus L(m - n), \quad m \geq n.$$
(2) We can check that 
\[ E(X \otimes v_i) = (E \cdot X) \otimes v_i + (K \cdot X) \otimes Ev_i = 0, \]

\[
\begin{align*}
E([i]Y \otimes v_i - q^{-1}X \otimes Fv_i) \\
= [i](E \cdot Y) \otimes v_i + [i](K \cdot Y) \otimes Ev_i - q^{-1}(E \cdot X) \otimes Fv_i - q^{-1}(K \cdot X) \otimes EFv_i \\
= [i]X \otimes v_i - X \otimes EFv_i = 0.
\end{align*}
\]

Then (2) holds.

(3) follows from (1) and (2). □

Next, we can check the following Lemma.

**Lemma 16.** For any module \( V \in \mathcal{A} \), \( \theta_i \in \text{Hom}_{U_q(sl_2)}(L(i), V) \), the following map

\[
\tau(\theta_i) : L(1) \otimes L(i) \longrightarrow V \\
(aX + bY) \otimes v \mapsto (aX + bY)\theta_i(v)
\]

is a \( U_q(sl_2) \)-module homomorphism, where \( a, b \in \mathbb{C}, v \in L(i) \).

Consider the following quiver.

\[
Q_\infty : 0 \overset{a_0}{\underset{b_0}{\longrightarrow}} 1 \overset{a_1}{\underset{b_1}{\longrightarrow}} 2 \overset{a_2}{\underset{b_2}{\longrightarrow}} \ldots
\]

The following theorem is inspired by the ideas in [13, 18].

**Theorem 17.** The category \( \mathcal{A} \) is equivalent to the category \( \mathcal{B} \) of finite dimensional representations for the quiver \( Q_\infty \) satisfying the following quadratic relations:

\[
b_0a_0 = 0, \quad a_ib_i = b_{i+1}a_{i+1}, \quad i \in \mathbb{Z}_+.
\]

(4.2)

**Proof.** By Lemma 15, we can define \( U_q(sl_2) \)-module homomorphisms 

\[
t_{i+1} : L(i+1) \rightarrow L(1) \otimes L(i), \quad t'_{i-1} : L(i-1) \rightarrow L(1) \otimes L(i),
\]

such that

\[
t_{i+1}(v_{i+1}) = X \otimes v_i, \quad t'_{i-1}(v_{i-1}) = [i]Y \otimes v_i - q^{-1}X \otimes Fv_i,
\]

where each \( v_i \) is a fixed highest weight vector of \( L(i) \).

We will prove the theorem in three steps.

**Step 1.** We define a functor \( F \) from \( \mathcal{A} \) to \( \mathcal{B} \). Let \( V \) be a \( U_q(sl_2) \)-module which belongs to \( \mathcal{A} \).

(1) For every \( i \), we can associate it with a vector space \( V_i := \text{Hom}_{U_q(sl_2)}(L(i), V) \).

(2) For arrows \( a_i, b_i \), we can define linear maps \( V(b_i) : V_{i+1} \rightarrow V_i, V(a_i) : V_i \rightarrow V_{i+1} \) as follows:

\[
V(b_i)(\theta_{i+1}) = \tau(\theta_{i+1})t'_{i}, \quad V(a_i)(\theta_i) = \tau(\theta_i)t_{i+1}.
\]

Next, we check that:

\[
V(b_0)V(a_0) = 0, \quad V(a_{i-1})V(b_{i-1}) = V(b_i)V(a_i), \quad i \in \mathbb{N}.
\]
For \( \theta_0 \in V_0, \nu \in L(0) \), from \( F_\nu = 0 \), we have

\[
(V(b_0)V(a_0)(\theta_0)) (\nu) = \tau (\tau(\theta_0)) (Y \otimes X \otimes \nu - q^{-1}X \otimes X \otimes F_\nu - q^{-1}X \otimes Y \otimes \nu)
= \tau (\tau(\theta_0)) (Y \otimes X \otimes \nu - q^{-1}X \otimes Y \otimes \nu)
= YX\theta_0(\nu) - q^{-1}XY\theta_0(\nu) = 0.
\]

For \( \theta_i \in V_i \), from \( XY = qYX, [i + 1]q^{-1} - q^{-i-1} = [i] \), we have

\[
(V(a_{i-1})V(b_{i-1})(\theta_i)) (\nu_i)
= \tau (\tau(\theta_i)) ([i]X \otimes Y \otimes \nu_i - q^{-1}X \otimes X \otimes F_{\nu_i})
= [i]XY\theta_i(\nu_i) - q^{-1}XXF\theta_i(\nu_i)
= [i + 1]XY\theta_i(\nu_i) - q^{-1}XXF\theta_i(\nu_i) - q^{-i-1}XY\theta_i(\nu_i)
= \tau (\tau(\theta_i)) ([i + 1]X \otimes X \otimes \nu_i - q^{-1}X \otimes X \otimes F_{\nu_i} - q^{-i-1}X \otimes Y \otimes \nu_i)
= (V(b_i)V(a_i)(\theta_i)) (\nu_i).
\]

By the fact that the \( U_q(\mathfrak{sl}_2) \)-module \( L(i) \) is generated by \( \nu_i \) and \( V(b_i)V(a_i)(\theta_i), V(a_{i-1})V(b_{i-1})(\theta_i) \) are \( U_q(\mathfrak{sl}_2) \)-module homomorphisms, we see that \( V(b_i)V(a_i)(\theta_i) = V(a_{i-1})V(b_{i-1})(\theta_i) \).

Thus, \( (V_i, V(a_i), V(b_i), i \in \mathbb{Z}_+) \) is a representation of \( \mathcal{Q}_\infty \) satisfying the relation (4.2).

(3) We define a functor \( F \) from \( \mathcal{A} \) to \( \mathcal{B} \). For \( V, W \in \text{Obj}(\mathcal{A}), f \in \text{Hom}_{U_q(\mathfrak{sl}_2)}(V, W) \), define

\[
F(V) = (V_i, V(a_i), V(b_i), i \in \mathbb{Z}_+)
\]

\[
F(f) = (f^*_i : V_i \rightarrow W_i, i \in \mathbb{Z}_+)
\]

where \( f^*_i \) satisfies \( f^*_i(\theta_i) = f\theta_i, \theta_i \in V_i \).

We check the following diagrams

\[
\begin{array}{ccc}
V_{i-1} & \xrightarrow{V(b_{i-1})} & V_i \\
\downarrow{f^*_i-1} & & \downarrow{f^*} \\
W_{i-1} & \xrightarrow{W(b_{i-1})} & W_i,
\end{array}
\begin{array}{ccc}
V_i & \xrightarrow{V(a_i)} & V_{i+1} \\
\downarrow{f^*_i} & & \downarrow{f^*_{i+1}} \\
W_i & \xrightarrow{W(a_i)} & W_{i+1},
\end{array}
\]

are commutative.

Since \( \tau(\theta_i) \) is a \( U_q(\mathfrak{sl}_2) \)-module homomorphism,

\[
W(b_{i-1})f^*_i(\theta_i) = W(b_{i-1})(f\theta_i)
= \tau(f\theta_i)t^*_{i-1} = f \tau(\theta_i)t^*_{i-1} = f^*_i V(b_{i-1})(\theta_i),
\]

and

\[
W(a_i)f^*_i(\theta_i) = W(a_i)(f\theta_i)
= \tau(f\theta_i)t_{i+1} = f \tau(\theta_i)t_{i+1} = f^*_i V(a_{i+1})(\theta_i).
\]

So

\[
f^*_i V(b_i) = W(b_i)f^*_{i+1}, f^*_i V(a_i) = W(a_i)f^*_{i}.
\]
Therefore, $F(f) = (f^*: V_i \rightarrow W_i, \ i \in \mathbb{Z}_+)$ is a morphism from the representation $F(V)$ to $F(W)$.

**Step 2.** For a representation $(V_i, V(a_i), V(b_i), i \in \mathbb{Z}_+)$ of the quiver $Q_\infty$ satisfying the relation (4.2), there is a $U_q(\mathfrak{s})$-module $V$ such that $F(V) \cong (V_i, V(a_i), V(b_i), i \in \mathbb{Z}_+)$. Let $V = \bigoplus_{i \in \mathbb{Z}_+} V_i \otimes L(i)$. Next, we define the action of $U_q(\mathfrak{s})$ on $V$. Since $v_i$ is the highest weight vector of $L(i)$, $F^s v_i, \ s \in \mathbb{Z}_+$ is a basis of $L(i)$. For $u \in U_q(\mathfrak{sl}_2), \theta_i \in V_i, \ s \in \mathbb{Z}_+$, define

$$u(\theta_i \otimes F^s v_i) = \theta_i \otimes uF^s v_i,$$

$$X(\theta_i \otimes F^s v_i) = \frac{q^i[i + 1 - s]}{[i + 1]} V(a_i)(\theta_i) \otimes F^s v_{i+1} - \frac{q^{-i}[s]}{[i + 1]} V(b_{i-1})(\theta_i) \otimes F^{s-1} v_{i-1}, \ (4.3)$$

$$Y(\theta_i \otimes F^s v_i) = \frac{q}{[i + 1]} V(b_{i-1})(\theta_i) \otimes F^s v_{i-1} + \frac{1}{[i + 1]} V(a_i)(\theta_i) \otimes F^{s+1} v_{i+1}.$$ 

The reason for defining the action of $U_q(\mathfrak{s})$ on $V$ by (4.3) comes from the definitions of $V(a_i), V(b_i)$ in Step 1. We can check that the action (4.3) indeed defines a $U_q(\mathfrak{s})$-module through a little cumbersome calculation.

From the definition of $F$, we can see that $F(V) \cong (V_i, V(a_i), V(b_i), i \in \mathbb{Z}_+)$. **Step 3.** The map

$$F_{V,W}: \text{Hom}_A(V, W) \rightarrow \text{Hom}_B(F(V), F(W)),$$

is a bijection.

If $f \in \text{Hom}_{U_q(\mathfrak{s})}(V, W)$ satisfying that $F_{V,W}(f) = 0$, then for any $\theta_i \in \text{Hom}_{U_q(\mathfrak{sl}_2)}(L(i), V), i \in \mathbb{Z}_+$, we have $f\theta_i = 0$. Since $V$ is a sum of simple submodules $L(i)$. So $f = 0$, and $F_{V,W}$ is injective.

From the completely reducibility of finite dimensional $U_q(\mathfrak{sl}_2)$-modules,

$$V = \sum_{i \in \mathbb{Z}_+} \sum_{\theta_i V_i} \theta_i(L(i)).$$

For $g = (g_i: V_i \rightarrow W_i, \ i \in \mathbb{Z}_+) \in \text{Hom}_B(F(V), F(W))$, we define $f : V \rightarrow W$ as follows:

$$f(\theta_i(w_i)) = g_i(\theta_i(w_i)), \ \theta_i \in V_i, \ w_i \in L(i).$$

Since $\theta_i, g_i(\theta_i)$ is a $U_q(\mathfrak{sl}_2)$-module homomorphism, for $u \in U_q(\mathfrak{sl}_2)$, we have

$$f(u(\theta_i(w_i))) = f(\theta_i(uw_i)) = g_i(\theta_i)(uw_i) = u(g_i(\theta_i))(w_i) = uf(\theta_i(w_i)).$$

At the same time, using the following commutative diagram:

$$\begin{array}{ccc}
V_{i-1} & \xrightarrow{V(b_{i-1})} & V_i \\
\downarrow g_{i-1} & & \downarrow g_i \\
W_{i-1} & \xrightarrow{W(b_{i-1})} & W_i \\
\end{array} \quad \begin{array}{ccc}
V_i & \xrightarrow{V(a_i)} & V_{i+1} \\
\downarrow g_i & & \downarrow g_{i+1} \\
W_i & \xrightarrow{W(a_i)} & W_{i+1} \\
\end{array}$$

we obtain that

$$f(X\theta_i(v_i)) = f((V(a_i)(\theta_i))(v_{i+1})) = g_{i+1}(V(a_i)(\theta_i))(v_{i+1})$$

$$W(a_i)g_i(\theta_i)(v_{i+1}) = Xf(\theta_i(v_i)).$$
and
\[
f(Y\theta_i(v_i)) = f\left(\frac{q}{i+1} V(b_{i-1})(\theta_i)(v_{i-1}) + \frac{1}{i+1} V(a_i)(\theta_i)(Fv_{i+1})\right)
= \frac{q}{i+1} W(b_{i-1}) g_i(\theta_i)(v_{i-1}) + \frac{1}{i+1} W(a_i) g_i(\theta_i)(Fv_{i+1})
= Yg_i(\theta_i)(v_i) = Yf(\theta_i(v_i)).
\]

So \( f \) is a \( U_q(\mathfrak{s}) \)-module homomorphism such that \( F(f) = g \).

Therefore, \( F \) is an equivalence.

Let \( \mathbb{C}\langle x_1, x_2 \rangle \) be the free associative algebra over \( \mathbb{C} \) generated by two variables \( x_1, x_2 \). Recall that an abelian category \( \mathcal{C} \) is wild if there exists an exact functor from the category of representations of the algebra \( \mathbb{C}\langle x_1, x_2 \rangle \) to \( \mathcal{C} \) which preserves indecomposability and takes nonisomorphic modules to nonisomorphic ones, see Definition 2 in [18]. By [13], the category \( \mathcal{B} \) for the quiver \( Q_\infty \) is wild. Hence, we have the following corollary.

**Corollary 18.** The representation type of the category \( \mathcal{A} \) is wild.

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