THE MEAN: AXIOMATICs, GENERALIZATIONs, APPLICATIONs

John E. Gray and Andrew Vogt

Abstract. We present an axiomatic approach to the mean and discuss generalizations of the mean, including one due to Kolmogorov based on the Weak Law of Large Numbers. We offer examples and counterexamples, describe conventional and unconventional uses of the mean in statistical mechanics, and resolve an anomaly in quantum theory concerning apparent simultaneous coexistence of means and variances of observables. These issues all arise from the familiar definition of the mean.

1. Introduction

The most important number summarizing a data set is generally thought to be the mean. Some have questioned its utility, comparing it unfavorably with the median, the mode, the midrange. Capitalists and communists used to argue over whether mean income or median income was the truer measure of citizen well-being. For another example, see Kosko [9]. The mean is not robust against outliers: it can be strongly influenced by a single observation. This is both a strength and a weakness. Kosko objected that not only does a Cauchy random variable not have a well-defined mean but the average of independent identically distributed Cauchy random variables is itself a Cauchy variable with the same distribution and thus averaging does not reduce variability at all. Investigators often pursue the quest for a single number or a small set of numbers that capture the essence of a data set, make multiple data sets comparable, and provide order to the world of data sets. As data sets get larger and larger, thanks to the digital explosion, scrutiny of measures that compress data becomes more important. Candidates, in addition to those mentioned above, include entropy and various generalized means, but no one has arrived at measures clearly superior to the mean and its associated measure, the root mean squared deviation or standard deviation.

In work with sample data the mean is easy to understand, in contrast with other notions from probability theory - such as independence, conditional probability, and even probability itself. Some have argued (e.g., de Finetti [2], Pollard [12], and Whittle [14]) that the mean is the fundamental notion in probability theory and should occupy the central place in all treatments of probability.

In this note we review some properties of the mean, consider some generalizations for cases when the ordinary mean does not exist, and investigate the significance of the mean in state space theory and quantum mechanics.

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We begin by axiomatizing the notion of sample mean. Along with familiar axioms for symmetry, homogenity, and translation invariance, we introduce a condensation axiom that describes the result of replacing arbitrary values by their sample mean. We then use the Strong Law of Large Numbers to arrive at the familiar mathematical notion of mean, \( E(X) \). Thereafter we consider generalizations of the mean. These are not needed for bounded or semi-bounded random variables, but really only for variables that have heavy-tailed distributions on both right and left, with tails of similar size. We consider what happens when a random variable is restricted to an interval \([c - M, c + M]\) and \( M \) is allowed to tend to infinity. We state a theorem (Theorem 3.1) describing the different kinds of behavior possible and provide examples of each. One generalization, which is due to Kolmogorov, is what we have chosen to call the weak mean, \( E_w(X) \), and corresponds precisely to validity of the Weak Law of Large Numbers. Yet another generalization, the doubly weak mean, \( E_{ww}(X) \), applies to the Cauchy distribution. We also discuss multipliers that can be applied to a variable \( X \) to finitize the mean in the spirit of Feynman and note the dangers of such finitizations. Nonetheless, we recognize that attempts to scrutinize the notion of mean in connection with the Cauchy distribution and other long-tailed distributions are timely.

Turning to applications, we point out that the mean is a natural tool in state space theory for the transition from deterministic models to statistical models. We discuss entropy and observe that although it is regarded as a mean it is very different from means arising from ordinary observables. We recall Jaynes’ Maximum Entropy Principle, which seeks to maximize entropy subject to given values of conventional means.

Lastly, we discuss the role the mean plays in quantum theory, and provide a precise answer to the question of when the mean and variance exist for a particular quantum state and a particular quantum observable.

The conclusion, implicit in this discussion, is that the mean is the paramount measure, of great and wide utility, instructive even when it falls short. There is little prospect of it losing its longtime preeminence.

2. Axiomatics for the Sample Mean and the Strong Law of Large Numbers

Prior to introducing probability measures, let us consider potential axioms for the mean of a finite set. In this setting, with \( \mathcal{R} = (-\infty, \infty) \), the mean can be thought of as a family of functions \( \{f_n\} \) for \( n \geq 1 \) with \( f_n : \mathcal{R}^n \to \mathcal{R} \). Its properties include the following:

M-1) (Homogeneity) \( f_n(\lambda x_1, \ldots, \lambda x_n) = \lambda f_n(x_1, \ldots, x_n) \) for all \( (x_1, \ldots, x_n) \in \mathcal{R}^n \) and all \( \lambda \in \mathcal{R} \);

M-2) (Symmetry) \( f_n(x_1, \ldots, x_n) = f_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \) for all permutations \( \sigma \) of the set \( \{1, 2, \ldots, n\} \);

M-3) (Translation Invariance) \( f_n(x_1 + c, \ldots, x_n + c) = f_n(x_1, \ldots, x_n) + c \) for all \( (x_1, \ldots, x_n) \in \mathcal{R}^n \) and all \( c \in \mathcal{R} \).
Other properties are the following:

(Positive Homogeneity) \( f_n(\lambda x_1, \ldots, \lambda x_n) = \lambda f_n(x_1, \ldots, x_n) \) for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) and all \( \lambda > 0 \);

(Nonnegativity) If for some \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \in \mathbb{R}^n \) \( x_1 \leq y_1, \ldots, x_n \leq y_n \), then \( f_n(x_1, \ldots, x_n) \leq f_n(y_1, \ldots, y_n) \);

(Positivity) If for some \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \in \mathbb{R}^n \) \( x_1 \leq y_1, \ldots, x_n \leq y_n \), and \( x_i < y_i \) for some \( i \), then \( f_n(x_1, \ldots, x_n) < f_n(y_1, \ldots, y_n) \).

(Strict Positivity) If for some \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \in \mathbb{R}^n \) \( x_i < y_i \) for all \( i = 1, \ldots, n \), then \( f_n(x_1, \ldots, x_n) < f_n(y_1, \ldots, y_n) \).

(Additivity) \( f_n(x_1 + y_1, \ldots, x_n + y_n) = f_n(x_1, \ldots, x_n) + f_n(y_1, \ldots, y_n) \) for all \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \in \mathbb{R}^n \);

The above axioms seem reasonable except for additivity. The measure should be independent of units, thus homogeneous, and independent of the choice of zero point, and a function of the set rather than the ordered set. In addition, to capture characteristics of the data, ordering properties - nonnegativity and perhaps positivity - are not unreasonable. However, additivity asserts a relationship between the ordering of two data sets that survives reordering of one set, and this seems much too restrictive.

Consider a rival measure to the mean, namely, the median. The median of a finite data set \( \{x_1, \ldots, x_n\} \) is defined as the midmost of the numbers when they are arranged in increasing order if \( n \) is odd, and half the sum of the two midmost numbers in such an arrangement if \( n \) is even.

The median satisfies homogeneity, symmetry, translation invariance, and nonnegativity. Furthermore, any fixed convex combination of the mean and the median other than the median itself satisfies homogeneity, symmetry, translation invariance, nonnegativity, positivity, and strict positivity. Indeed, not only the median, but the maximum and the minimum of \( \{x_1, \ldots, x_n\} \) (and other rank functions and convex combinations) satisfy positive homogeneity, symmetry, translation invariance, and nonnegativity.

**Proposition 2.1.** Let \( f_n : \mathbb{R}^n \to \mathbb{R} \) be a function satisfying homogeneity, symmetry, and translation invariance.

1) If \( n = 1 \), then \( f_1(x) = x \) for all \( x \in \mathbb{R} \).

2) If \( n = 2 \), then \( f_2(x_1, x_2) = \frac{x_1 + x_2}{2} \) for all \( (x_1, x_2) \in \mathbb{R}^2 \).

**Proof.** Homogeneity implies that \( f_1(0) = f_2(0, 0) = 0 \). Translation invariance then indicates that \( f_1(x) = f_1(0 + x) = f_1(0) + x = x \). When \( n = 2 \),
\[ f_2(a, b) = f_2\left(-\frac{b-a}{2} + \frac{a+b}{2}, \frac{b-a}{2} + \frac{a+b}{2}\right) \]
\[ = f_2\left(-\frac{b-a}{2}, \frac{b-a}{2}\right) + \frac{a+b}{2} \]
\[ = \left(\frac{b-a}{2}\right)f_2(-1, 1) + \frac{a+b}{2}. \]

However, by homogeneity and symmetry \( f_2(-1, 1) = -f_2(1, -1) \)
\[ = -f_2(-1, 1), \text{ and } f_2(-1, 1) = 0. \]

When \( n = 1 \) or \( 2 \), the median and the mean coincide. However, it is obvious that they do not coincide in general when \( n \) is 3 or larger. Without the requirement of additivity it is natural to inquire whether there is another suitable property that will distinguish between the median and the mean. One property that we consider and reject is that \( f_n \) shall have continuous partial derivatives.

**Proposition 2.2.** Let \( f_n : \mathbb{R}^n \to \mathbb{R} \) be a function satisfying homogeneity, symmetry, and translation invariance that has partial derivatives at each point with the partial derivatives continuous at \((0, \ldots, 0) \in \mathbb{R}^n\). Then

\[ f_n(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n} \]

for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\).

**Proof.** If we differentiate the equation \( f_n(\lambda x_1, \ldots, \lambda x_n) = \lambda f_n(x_1, \ldots, x_n) \) with respect to \( x_i \), we obtain:

\[ \lambda \frac{\partial f_n}{\partial x_i}(\lambda x_1, \ldots, \lambda x_n) = \lambda \frac{\partial f_n}{\partial x_i}(x_1, \ldots, x_n). \]

Cancelling \( \lambda \) from each side and taking a limit as \( \lambda \) approaches 0, we obtain:

\[ \frac{\partial f_n}{\partial x_i}(0, \ldots, 0) = \frac{\partial f_n}{\partial x_i}(x_1, \ldots, x_n). \]

Thus all partial derivatives are constant. Since \( f_n(0, \ldots, 0) = 0 \) by homogeneity, \( f_n \) has the form:

\[ f_n(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n. \]

Symmetry now dictates that \( a_1 = \ldots = a_n \) and the fact that \( f_n(1, \ldots, 1) = f_n(0, \ldots, 0) + 1 = 0 + 1 = 1 \) accordingly implies that each \( a_i = \frac{1}{n}. \)

The continuous differentiability assumption seems to be aimed primarily at elimination of the median. So we reject it. Instead we offer as an axiom a different property characteristic of the mean.

**M-4** (Condensation) For \( n > m \),

\[ f_n(x_1, \ldots, x_n) = f_m(x_1, \ldots, x_m), \ldots, f_m(x_1, \ldots, x_m), x_{m+1}, \ldots, x_n \]

for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\).
This property asserts that if a subset of data is replaced by its “mean”, the grand “mean” is not changed. This is the first property that proposes a definite relationship between means of sets of different sizes. In view of the symmetry axiom (M-3), the statement does not really restrict the order of the subset, and as we shall see shortly the statement is only really needed in special cases.

**Proposition 2.3.** Let \( f_n : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function satisfying homogeneity, symmetry, translation invariance, and condensation, that is, M-1, M-2, M-3, and M-4. Then

\[
f_n(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n}
\]

for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\).

**Proof.** In view of Proposition 2.1 we need only perform an inductive step showing that the mean formula holds for \( n \geq 3 \) when it holds for \( n - 1 \). Consider

\[
f_n(x_1, \ldots, x_n) = f_n(\frac{1}{n-1}(x_1 + \ldots + x_{n-1}), \ldots, \frac{1}{n-1}(x_1 + \ldots + x_{n-1}), x_n)
\]

\[
= f_n(0, \ldots, 0, x_n - \frac{1}{n-1}(x_1 + \ldots + x_{n-1})), 0, \ldots, 0) + \frac{1}{n-1}(x_1 + \ldots + x_{n-1})
\]

\[
= (x_n - \frac{1}{n-1}(x_1 + \ldots + x_{n-1}))f_n(0, \ldots, 0, 1) + \frac{1}{n-1}(x_1 + \ldots + x_{n-1})
\]

\[
= a_1x_1 + \ldots + a_nx_n.
\]

This shows that \( f_n \) is a linear function of \( x_1, \ldots, x_n \). It now follows from Proposition 2.2 that it is the mean. □

The proof of Proposition 2.3 requires that M-4 holds in the case when \( m = n - 1 \). In fact we can get by with the assumption that M-4 holds when \( m = 2 \). It is easy to see that in this case M-4 also holds for \( m = 2^k \). Now set \( n = 2^k + j \), where \( 0 \leq j < 2^k \). If \( j = 0 \), \( f_n(x_1, \ldots, x_n) = f_n(c, \ldots, c) = cf_n(1, \ldots, 1) \) where \( c = (x_1 + \ldots + x_n)/n \). If \( j > 0 \), then \( f_n(x_1, \ldots, x_n) = f_n(c, \ldots, c, x_m+1, \ldots, x_n) \) where \( m = 2^k \) and \( c = (x_1 + \ldots + x_n)/m \). We now replace \( x_1, \ldots, x_m \) by \( x'_1, \ldots, x'_{m-j}, 0, \ldots, 0 \) so that \( c = (x'_1 + \ldots + x'_{m-j} + 0 + \ldots + 0)/m \). Using the symmetry and homogeneity axioms, we obtain: \( f_n(x_1, \ldots, x_n) = f_n((mc + x_{m+1} + \ldots + x_n)/m, \ldots, (mc + x_{m+1} + \ldots + x_n)/m, 0, \ldots, 0) = (x_1 + \ldots + x_n)/m) f_n(1, \ldots, 1, 0, \ldots, 0) \). Thus we have established that \( f_n \) is linear in \( x_1, \ldots, x_n \).

By Proposition 2.3 \( f_n \) is the ordinary mean.

A further note on axiomatics is that the translation invariance axiom can be replaced by \( f_n(1, \ldots, 1) = 1 \) if we also assume that \( f_2(x_1, x_2) = (x_1 + x_2)/2 \).

To pass from the sample mean of a finite set to the usual general notion of mean, we introduce a real-valued random variable \( X \). We suppose that associated with \( X \) is a Borel probability measure \( P_X \) taking each Borel subset \( A \) of the real numbers to:

\[
P_X(A) = \text{the probability that } X \text{ belongs to the set } A.
\]
The mean of $X$, denoted by $E(X)$ or $\mu_X$, is defined when $x$ is integrable with respect to $P_X$ to be:

$$E(X) = \int \mathbb{R} xP_X(dx).$$

One direction of the remarkable Strong Law of Large Numbers (see Pollard [12, p. 78 and pp. 37-8]) states that if $\{X_n\}$ is a sequence of independent random variables with common distribution $P_X$ and there exists a constant $m$ such that

$$\frac{X_1 + \ldots + X_n}{n}$$

converges almost surely to $m$ as $n \to \infty$, then each $X_n$ has mean $m$. Here “almost surely” means outside a set of measure zero in the countably infinite product space induced by the measure $P_X$ (see [12, pp. 99-102]). More briefly, if sample means of independent copies of $X$ settle down to something, then that something is $E(X)$. This can be regarded as the motivation for the transition from the sample mean to the mathematical mean $E(X)$. The general notion of mean is derived from the finitary notion considered earlier.

The other direction of the Strong Law of Large Numbers asserts that if $E(X)$ exists, then the sample mean of $n$ identical independent copies of $X$ converges almost surely to $E(X)$ as $n$ tend to infinity. For a proof of both directions of the Strong Law, see [12] pp. 95-102, p. 105. For an alternate proof due to N. Etemadi, see [12] pp. 106-7.

The transition here from finite samples to infinite populations distinguishes the deductive method from the inductive method. While true science deals comfortably with induction based on finite samples, the deductive method of the Greeks (and Isaac Newton) relies on axioms whose relationship to reality is only approximate and always contingent.

Indeed, in using the Strong Law of Large Number we are admittedly introducing the full panoply of probability theory. It is possible, as noted in the Introduction, to represent all of probability theory using the mean as the primitive notion. Thus $P_X(A)$ can be defined as $E(\chi_A(X))$, the mean of $\chi_A(X)$, where $\chi_A(X)$ is the random variable that equals 1 when $X$ is in $A$ and 0 when $X$ is not in $A$. However, since in what follows we plan to use probability theory in its conventional form (i.e., according to the axioms of Kolmogorov [6]), we see no reason to restate measure-theoretic facts in terms of the mean as primitive. Indeed, a reason not to do so is that the concept of independence, which is also fundamental in probability, is awkward when expressed exclusively in terms of means.

### 3. Extending the Mean

When $x$ is not integrable with respect to $P_X$, the notion $E(X)$ above is inapplicable and we must rely on other notions of mean. Richard Feynman was famous for his integration tricks, and some of these are recorded in the book of Mathews and Walker [11], based on lectures Feynman gave at Cornell. Feynman’s tricks partly motivated our investigation.

Perhaps the most obvious generalization is the following:
Let \( L(c) = \lim_{M \to \infty} \int_{[c-M, c+M]} x P_X(dx) \)
for a real number \( c \).

By the Lebesgue Dominated Convergence Theorem this notion coincides with
the ordinary mean when \( x \) is integrable with respect to \( P_X \). Kolmogorov [6, p.
40], in his great foundational work, noted this option in the case when \( c = 0 \) and
observed that it does not require integrability of \( |x| \). Indeed if \( X \) is a random
variable obeying the Cauchy distribution \( f(x) = 1/\pi(1 + x^2) \), then \( X \) satisfies
\( L(c) \equiv 0 \) for any choice of \( c \).

We mention two related notions of mean:
L-1) \( \lim_{M \to \infty} \int_{[a-M, b+M]} x P_X(dx) \), and
L-2) \( \lim_{\min\{M,K\} \to \infty} \int_{[a-M, b+K]} x P_X(dx) \),
where \( a \leq b \).

It is easily seen that L-1 coincides with \( L((a + b)/2) \) since
\[
[a - M, b + M] = \left[ \frac{a + b}{2} - (M + \frac{b - a}{2}), \frac{a + b}{2} + (M + \frac{b - a}{2}) \right].
\]
As for L-2, we have the following result.

**Proposition 3.1.** Let \( X \) be a random variable with probability measure \( P_X \). Then
for some \( a \) and \( b \) with \( a \leq b \)
\[
\lim_{\min\{K,M\} \to \infty} \int_{[a-M, b+K]} x P_X(dx)
\]
exists if and only if \( x \) is integrable with respect to \( P_X \).

**Proof.** If \( x \) is integrable on \( \mathcal{R} \), the limit exists and equals the mean of \( x \) by
Lebesgue’s Dominated Convergence Theorem. Conversely, if the limit exists, then
\[
0 \leq \int_{[b+K, b+K']} x P_X(dx) < \epsilon
\]
for \( K < K' \), both sufficiently large, and any given \( \epsilon \). Likewise
\[
-\epsilon < \int_{[a-M', a-M]} x P_X(dx) \leq 0
\]
for \( M < M' \), both sufficiently large. Fatou’s Lemma or Levi’s Theorem [4, p. 172]
thus implies that
\[
0 \leq \int_{(b+K, \infty)} x P_X(dx) \leq \epsilon, \quad -\epsilon \leq \int_{(-\infty, a-M]} x P_X(dx) \leq 0
\]
and thus \( x \) is integrable on \( [0, \infty) \) as well as \( (\infty, 0] \) and so is integrable on \( \mathcal{R} = (-\infty, \infty) \).

So, when L-2 exists, it coincides with \( E(X) \). \( \Box \)

We now return to the study of \( L(c) \). We shall allow \(-\infty \leq L(c) \leq \infty \). This gives
us a bit more flexibility in characterizing what can happen.
Lemma 3.2. Let $X$ be a random variable with probability measure $P_X$, and let $c_1$ and $c_2$ be real numbers with $c_1 < c_2$. Then there are three possibilities:

i) If $L(c_1)$ exists in $[-\infty, \infty]$, then

$$L(c_1) \leq \liminf_{M \to \infty} \int_{[c_2-M,c_2+M]} x P_X(dx);$$

ii) If $L(c_2)$ exists in $[-\infty, \infty]$, then

$$L(c_2) \geq \limsup_{M \to \infty} \int_{[c_1-M,c_1+M]} x P_X(dx);$$

iii) If $L(c_1)$ and $L(c_2)$ both exist in $[-\infty, \infty]$, then $L(c_1) = L(c_2)$.

Proof. Suppose $c_1 < c_2$. Then

$$\int_{[c_2-M,c_2+M]} x P_X(dx) = \int_{[c_1-M,c_1+M]} x P_X(dx) + \int_{(c_1+M,c_2+M]} x P_X(dx) - \int_{[c_1-M,c_2-M)} x P_X(dx).$$

The second and third terms on the right are both non-negative and accordingly i) and ii) follow. In the case of iii), note that i) and ii) imply that if both $L(c_1)$ and $L(c_2)$ exist, then $L(c_1) \leq L(c_2)$.

If $L(c_2) - L(c_1) > 0$, then there is a positive constant $K$ (for example, any positive number < $L(c_2) - L(c_1)$) such that for $M$ sufficiently large:

$$K < \int_{(c_1+M,c_2+M]} x P_X(dx) - \int_{[c_1-M,c_2-M)} x P_X(dx)$$

$$\leq (c_2 + M) P_X((c_1 + M, c_2 + M]) + (M - c_1) P_X([c_1 - M, c_2 - M))$$

$$\leq (M + d)(P_X((c_1 + M, c_2 + M] \cup [c_1 - M, c_2 - M))$$

where $d = \max \{|c_2|, |c_1|\}$. Thus

$$\frac{K}{M + d} < P_X((c_1 + M, c_2 + M] \cup [c_1 - M, c_2 - M)).$$

Now replace $M$ by $M_j = M + j(c_2 - c_1)$ for each integer $j \geq 0$ to get:

$$\frac{K}{M_j + d} < P_X((c_1 + M_j, c_2 + M_j] \cup [c_1 - M_j, c_2 - M_j)).$$

Summing over these inequalities and noting that $c_2 + M_j = c_1 + M_{j+1}$ and $c_1 - M_j = c_2 - M_{j+1}$, we obtain:

$$\infty = \sum_{j=0}^{\infty} \frac{K}{M + d + j(c_2 - c_1)} \leq P_X((c_1 + M, \infty) \cup (-\infty, c_2 - M)) \leq 1.$$
for a contradiction. Thus, this case is eliminated. So $L(c_1) = L(c_2)$. □

**Theorem 3.3.** Let $X$ be a random variable with probability measure $P_X$. Then exactly one of the following possibilities holds:

i) $L(c)$ does not exist in $[-\infty, \infty]$ for any real number $c$;

ii) $L(c)$ exists in $(-\infty, \infty)$ for exactly one real number $c$;

iii) $L(c)$ exists in $[-\infty, \infty]$ for all real numbers $c$ and is independent of $c$;

iv) there is a number $c_0$ such that $L(c) = \infty$ for $c > c_0$ and $L(c)$ does not exist for $c < c_0$; or

v) there is a number $c_0$ such that $L(c) = -\infty$ for $c < c_0$ and $L(c)$ does not exist for $c > c_0$.

**Proof.** By Lemma 3.1 it suffices to show what happens when $L(c_2) = L(c_1)$ is finite. In this case the last two terms in the equation at the beginning of the proof of Lemma 3.1 each tend to 0 as $M$ tends to infinity. By a change of variable, we obtain for the positive number $c = c_2 - c_1$.

$$\lim_{M \to \infty} \int_{(M,c+M]} xP_X(dx) = \lim_{M \to \infty} \int_{[-M-c,-M]} xP_X(dx) = 0;$$

Assume $0 < d < c$ and $M \geq 0$. Then

$$0 \leq \int_{(M,d+M]} xP_X(dx) \leq \int_{(M,c+M]} xP_X(dx)$$

and

$$\int_{[-M-c,-M]} xP_X(dx) \leq \int_{[-M-d,-M]} xP_X(dx) \leq 0$$

Thus if ii) holds for $c$, it holds for $d$. On the other hand, if ii) holds for $c$ it also holds for $nc$ where $n$ is any fixed positive integer since

$$\int_{(M,nc+M]} xP_X(dx) = \sum_{j=1}^{n} \int_{((j-1)c+M,jc+M]} xP_X(dx)$$

and

$$\int_{[-M-nc,-M]} xP_X(dx) = \sum_{j=1}^{n} \int_{[-(n+1-j)c,-M-(n-j)c]} xP_X(dx),$$

and if the $M$’s are chosen far enough out so that the individual integrals are closer to zero than $\epsilon/n$, the sum integral is within $\epsilon$ of 0. Finally since any positive real number $d$ is smaller than $nc$ for some positive integer $n$, all cases are covered. Accordingly, ii) implies iii).

The argument for i) implies ii) can now be used to show that for any two real numbers $c_1$ and $c_2$, if either $L(c_1)$ or $L(c_2)$ exists in $[-\infty, \infty]$, then the other exists and equals it since the approximating integrals differ by two integrals on intervals of length $|c_1 - c_2|$ that tend to zero as $M$ tends to infinity. Thus iii) implies iv). □

We give some examples to illustrate that each of the possibilities enumerated in Theorem 3.1 can occur.

Consider a random variable $X$ whose probability measure is of the form
\[ P_X(A) = \sum_{n=1}^{\infty} \left( \frac{1}{2^{2n}} \delta_{2^n}(A) + \frac{1}{2^{2n-1}} \delta_{-2^{n-1}}(A) \right) \]

where \( \delta_z \) is the (Dirac) probability measure whose value is 1 on any Borel subset \( A \) of \( \mathbb{R} \) that contains the real number \( z \) and whose value is zero otherwise. The sum of the nonzero values is one, so this obviously defines a probability measure. However the integral of \( x \) over the interval \([c-M, c+M]\) is the difference between the size of the first set and the size of the second set below:

\[
\text{The size of the set } \{n : 1 \leq n, 2^n \leq (c+M)\} = \left\lfloor \frac{\log(c+M)}{2 \log 2} \right\rfloor
\]

\[
\text{The size of the set } \{n : 1 \leq n, 2^{n-1} \leq (M-c)\} = \left\lfloor \frac{\log(M-c)}{2 \log 2} + \frac{1}{2} \right\rfloor
\]

where \( \lfloor \cdot \rfloor \) is the floor function. For fixed \( c \) and sufficiently large \( M \) the difference of the above quantities can assume the values 0 and \(-1\) and the integral does not settle down to either one. This is an instance of Theorem 3.1, part i).

A random variable \( X \) can also be defined with probability measure of the form

\[ P_X(A) = \sum_{n=1}^{\infty} \left( \frac{1}{2^{n+1}} (\delta_{2^n}(A) + \delta_{-2^n}(A)) \right). \]

So \( P_X \) is concentrated at the points \( \pm 2^n \) and assigns probability \( 1/(2^{n+1}) \) to such points. For this measure, \( L(0) \) equals 0 by symmetry. However, \( L(c) \) does not exist for other choices of \( c \). If \( c \) is positive, the integral of \( x \) over the closed interval \([c-M, c+M]\) reduces to its integral over the open interval \((M-c, M+c]\) and this integral oscillates between 0 and \( \frac{1}{2} \) for large \( M \) depending on whether \( 2^n \) is in the interval \((M-c, M+c]\) or not. Similar behavior occurs when \( c < 0 \). This example is an instance of Theorem 3.1, part ii).

Now consider a random variable \( X \) having a probability density (with respect to Lebesgue measure on the real line) of the form

\[ f(x) = \begin{cases} \frac{1}{1+C|x|^a} & \text{if } x \geq 0 \\ \frac{1}{1+D|x|^b} & \text{if } x < 0 \end{cases} \]

where \( a \) and \( b \) are numbers in \((1, 2)\) and \( C \) and \( D \) are suitable positive constants that guarantee that the density integrates to 1. Notice that this random variable satisfies iii) of Proposition 3.1. It is easy to see that \( L(c) \equiv \infty \) for all \( c \) or \(-\infty \) for all \( c \) according as \( b > a \) or \( a > b \). The example illustrates part iii) of Theorem 3.1 (as does the Cauchy distribution with \( L(c) \equiv 0 \)).

The probability measure

\[ P_X(A) = \sum_{n=1}^{\infty} \left( \frac{2^{n-1}}{3^{n+1}} \delta_{3^n}(A) + \frac{2^{n-2}}{3^{n+1}} \delta_{-3^n}(A) \right) \]

illustrates part iv) of Theorem 3.1. If \( c \geq 0 \), the integral of \( x \) over \([c-M, c+M]\) is given by:

\[
\sum_{\{n : 1 \leq n, 3^n \leq c+M\}} \frac{2^{n-1}}{3} - \sum_{\{n : 1 \leq n, 3^n \leq M-c\}} \frac{2^{n-2}}{3},
\]
and this expression has the value \((2^{n_0 - 1} - 2^{-1})/3\) or \((2^{n_0 - 1} + 2^{n_0 - 2} - 2^{-1})/3\) where \(n_0 \approx (\log c + M)/\log 3\) for large \(M\). Since \(M\) and \(n_0\) tend to infinity together, it follows that \(L(c) \equiv \infty\) for \(c \geq 0\).

On the other hand, if \(c = -d\) where \(d > 0\), the integral of \(x\) over \([c - M, c + M] = [-M - d, M - d]\) is given by:

\[
\sum_{\{n : \ 1 \leq n, 3^n \leq M - d\}} \frac{2^{n-1}}{3} - \sum_{\{n : \ 1 \leq n, 3^n \leq M + d\}} \frac{2^{n-2}}{3},
\]

and this reduces to \((2^{n_0 - 1} - 2^{-1})/3\) or to \((-1)/6\) for large \(M\) depending on whether a positive integer \(n_0\) lies in the interval \((\log (M - d))/\log 3, \log (M + d)/\log 3\) or not. Thus, \(L(c)\) does not exist for \(c < 0\).

A final example (also for part iv) of Theorem 3.1 is the case where the probability measure is given by:

\[
P_X(A) = K \sum_{n=1}^{\infty} \left( \frac{2^n}{3^n + (1/n)} \delta_{3^n+(1/n)}(A) + \frac{2^{n-1}}{3^n} \delta_{3^n}(A) \right).
\]

Here \(K\) is a suitably chosen positive normalizer, which is easily seen to be smaller than 1/3. For \(c > 0\), the integral of \(x\) over \([c - M, c + M]\) is

\[
K \sum_{\{n : \ 1 \leq n, 3^n+1/n \leq M+c\}} 2^n - K \sum_{\{n : \ 1 \leq n, 3^n \leq M-c\}} 2^{n-1},
\]

and this reduces to \(K(2^{n_0} - 1)\) for \(M\) sufficiently large where \(n_0\) is the largest integer such that \(3^{n_0} \leq M - c\). Since \(n_0\) and \(M\) tend to infinity together, \(L(c) \equiv \infty\) for all \(c > 0\).

When \(c = 0\), the integral of \(x\) over \([-M, M]\) reduces to \(K(2^{n_0} - 1)\) or to \(-K\) where \(n_0\) is the largest integer such that \(3^{n_0} + (1/n_0) \leq M\) and the first or second reduction occurs according as \(M < 3^{n_0+1}\) or not. Thus \(L(0)\) does not exist. By Theorem 3.1 \(L(c)\) does not exist for \(c < 0\).

Other cases arising in Theorem 3.1, such as part v), are obtained by modifying the examples above, e.g., replacing \(X\) by \(-X\) or by \(X + a\).

4. Weak Means and Multipliers

One of the implications of Theorem 3.1 is that if \(L(c)\) exists for more than one choice of \(c\) and is finite in some case then it is finite for all \(c\) and is independent of \(c\). The case of the Cauchy distribution shows that this can happen without the ordinary mean existing. Accordingly for a random variable \(X\), we define the doubly weak mean of \(X\), denoted by \(E_{ww}(X)\), to be the common value of \(L(c)\) for all \(c\) when this common value exists and is in \((-\infty, \infty)\).

We also introduce an intermediate notion due to Kolmogorov between the ordinary mean and the doubly weak that motivates our terminology. The weak mean of \(X\), denoted by \(E_w(X)\), is defined as follows: \(E_w(X)\) is the quantity \(L(0)\) provided the latter exists in \((-\infty, \infty)\) and provided \(\lim_{n \to \infty} nP_X(|X| > n) = 0\).

The following proposition is due to Kolmogorov. It indicates that existence of the weak mean coincides precisely with the existence of a number for which the Weak Law of Large Numbers holds.
Proposition 4.1 (Kolmogorov, 1928). Let $X$ be a random variable. Suppose that \{\(X_1, \ldots, X_n, \ldots\)\} are independent identically distributed copies of $X$ with $P_n$ the $n$-fold product distribution. Then there is a real number $m$ such that for each $\epsilon > 0$

$$\lim_{n \to \infty} P_n(\left| \frac{X_1 + \ldots + X_n}{n} - m \right| > \epsilon) = 0$$

if and only if $X$ has weak mean $E_w(X) = m$.

Proof. See [6, p. 65], [7], and [8, Theorems XII and XIII]. \hfill \Box

Corollary 4.2. Let $X$ be a random variable.

i) If $X$ has a mean, then $X$ has a weak mean and $E(X) = E_w(X)$;

and

ii) if $X$ has a weak mean, then $X$ has a doubly weak mean and $E_w(X) = E_{ww}(X)$.

Proof. In case $X$ has a mean, then the identity function $x \mapsto x$ is integrable with respect to the probability measure $P_X$ on the real line. In particular the tail integrals

$$\int_{[n, \infty)} xP_X(dx) \text{ and } \int_{(-\infty, -n]} xP_X(dx)$$

tend to zero as $n$ tends to infinity. Since the absolute values of these integrals are larger respectively than $nP_X(|X| > n)$ and $nP_X(|X| \leq -n)$, it follows that $\lim_{n \to \infty} nP_X(|X| > n) = 0$. Likewise by Lebesgue’s Dominated Convergence Theorem, $L(0) = E(X)$, Suppose $X$ has a weak mean. Then if $c_1 < c_2$ and $\epsilon > 0$ and a sufficiently large $M$ are given,

$$0 \leq \int_{(c_1 + M, c_2 + M]} xP_X(dx) \leq (c_2 + M)P_X((c_1 + M, c_2 + M]) \leq (c_2 - c_1 + M)P_X((|X| > c_1 + M) \leq (c_2 - c_1 + M)P_X(|X| \geq n) \leq \left( c_2 - c_1 \right) \frac{M}{n} \epsilon$$

where $n = |c_1 + M|$. For $M$ sufficiently large, the right side is as close to $\epsilon$ as we like. Thus

$$\lim_{M \to \infty} \int_{(c_1 + M, c_2 + M]} xP_X(dx) = 0.$$ 

Similarly,

$$\lim_{M \to \infty} \int_{(c_1 - M, c_2 - M]} xP_X(dx) = 0.$$ 

Accordingly from the first equation in the proof of Lemma 3.1, it follows that when one of $L(c_2)$ or $L(c_1)$ exists and is finite, the other exists and is equal to it. Since $L(0) = m$, it follows that $L(c)$ exists for all $c$, $L(c) \equiv m$ and $m$ is the doubly weak mean of $X$. \hfill \Box
Kolmogorov in [6, p. 66] gives an example where the Weak Law holds but the Strong Law does not. Cauchy random variables have $L(c)$ existing for all $c$, independent of $c$, but violate the Weak Law by not decaying rapidly enough at infinity. Thus the mean, weak mean, and doubly weak mean are strictly distinct notions.

We make one more observation on generalizations of the mean, based on using multipliers to attempt to finitize the mean. These multipliers are a type of “mollifier.” Usually mollifiers are used to aid approximation of the delta function and to smooth functions, but another use is to regularize behavior at $\pm \infty$. The idea is to introduce a function $\phi_\lambda(x)$ that depends on a parameter $\lambda$ so that $x \mapsto \phi_\lambda(x)x$ is integrable with respect to $P_X$ for $\lambda \neq \lambda_0$ and $\phi_\lambda(x) \to 1$ for a.e. $x$ as $\lambda \to \lambda_0$. In the case of $L(c)$, the multiplier can be taken to be

$$\phi_\lambda(x) = \chi_{(c-1/\lambda,c+1/\lambda]}(x)$$

where $\chi_A$ is the characteristic function of the set $A$ and $\lambda = 1/M$.

Multipliers, and indeed other straight-forward generalizations of the mean including the weak and doubly weak mean, are useful only when the following equations hold:

$$\int_{[0,\infty)} xP_X(dx) = \infty$$
$$\int_{(-\infty,0]} xP_X(dx) = -\infty.$$  

If neither of these equations holds, $x$ is integrable and the mean is well-defined. If only the first equation holds, the mean is $+\infty$, and if only the second equation holds, the mean is $-\infty$. If both equations hold, then there is some room for maneuver. $L(c)$ cannot exist finitely unless the infinities on each end are of the same order. If for example $P_X$ is given by a density function $f$ with respect to Lebesgue measure such that $f(x)$ decays as $1/x^2$ as $x \to \infty$ and decays as $1/|x|^{3/2}$ as $x \to -\infty$, then $L(c) \equiv -\infty$ for all $c$.

Multipliers offer possibilities for extending the notion of the mean. They can be of use in such activities as renormalization where the aim is to reinterpret integrals to make them finite. In our case we set:

$$E_{\text{mult}}(X) = \lim_{\lambda \to \lambda_0} E(\phi_\lambda(X)X)$$

provided this limit exists. This method is used to “evaluate” the integrals of $\sin bx$ and $\sin x/x$ on $[0,\infty)$ in [11, p. 60 and p. 91].

However, there are dangers that the following example illustrates.

Define a function $\phi_{\lambda,c}$ for $\lambda > 0$ and $c$ in $\mathbb{R}$ by:

$$\phi_{\lambda,c}(x) = \begin{cases} e^{-\lambda x} & \text{if } x > 0 \\ e^{\lambda x}(1 + \pi c\lambda x) & \text{if } x < 0. \end{cases}$$

Here $c$ is an arbitrary constant. Evidently $\phi_{\lambda,c}$ is a well-behaved function, integrable and dying off at $\pm \infty$. Also $\{\phi_{\lambda,c}\}$ converges pointwise to the constant function one as $\lambda$ tend to $0^+$ with fixed $c$.

Suppose we use this family of functions as a multiplier to determine a mean for a variable obeying the Cauchy distribution. Let $m(\lambda,c)$ be defined by:
\[ m(\lambda, c) = \int_{-\infty}^{\infty} \phi_{\lambda,c}(x) \frac{x}{\pi(1 + x^2)} \, dx = \int_{-\infty}^{0} \frac{c \lambda e^{\lambda x} x^2}{1 + x^2} \, dx = cm(\lambda, 1). \]

Now

\[ 1 = e^{\lambda x}|_{0}^{\infty} = \int_{-\infty}^{0} \lambda e^{\lambda x} \, dx \]
\[ \geq \int_{-\infty}^{0} \frac{\lambda e^{\lambda x} x^2}{1 + x^2} \, dx = m(\lambda, 1) = \int_{0}^{\infty} \frac{\lambda e^{-\lambda x} x^2}{1 + x^2} \, dx \]
\[ \geq \int_{K}^{\infty} \frac{\lambda e^{-\lambda x} x^2}{1 + x^2} \, dx \geq \frac{K^2 e^{-\lambda K}}{1 + K^2} \]

for any positive real number \( K \). Thus

\[ 1 \geq \lim_{\lambda \to 0^+} \sup m(\lambda, 1) \geq \lim_{\lambda \to 0^+} \inf m(\lambda, 1) \geq \frac{K^2}{1 + K^2}. \]

Letting \( K \) tend to infinity, we find that \( \lim_{\lambda \to 0^+} m(\lambda, 1) = 1. \)

Hence the multiplier-induced mean of the standard Cauchy distribution is:

\[ E_{\text{mult}}(X) = \lim_{\lambda \to 0^+} \int_{-\infty}^{\infty} \phi_{\lambda,c}(x) \frac{x}{\pi(1 + x^2)} \, dx = \lim_{\lambda \to 0^+} m(\lambda, c) \]
\[ = \lim_{\lambda \to 0^+} cm(\lambda, 1) = c \lim_{\lambda \to 0^+} m(\lambda, 1) = c. \]

However, \( c \) was arbitrary depending on the choice of the multiplier!

Although some may consider the Cauchy distribution anomalous, we remind the reader that its legitimacy and importance stem in part from the fact that it is the quotient of two independent standard normal random variables. It has application in physics under the name of the Lorentz distribution. Indeed long-tailed and counter-intuitive distributions are increasingly important in recent times (see Gumble \[3\] or Taleb \[13\]) in financial mathematics, the study of natural and man-made disasters, and computer network analysis. Extending the notion of mean to such distributions, and investigating the limits of the notion of mean in such settings, are among the ways of moving beyond the normal regime.

5. State Space Theory

State Space Theory or System Theory is widely used to provide a mathematical description of physical systems including those of classical mechanics as well as other systems such as biological and social systems. The state of the system at any time is taken to be an element of a set \( S \) called state space. The evolution of the state is given by a function \( T_t : S \to S \) taking the state \( s \) at time 0 to the state \( T_t(s) \) at time \( t \). A (real-valued) observable is any function \( f : S \to R \) which assigns to each state \( s \) a number \( f(s) \) (see Mackey \[10\]). All observables may be determined from the state, and indeed the state can be viewed as a maximal independent set of observables that characterize the system at a given time. The dynamic evolution of the state is deterministic and time may be taken to be either discrete or continuous. Evolution of an observable \( f \) can be expressed by \( t \mapsto f \circ T_t(s) \), i.e., the value of the observable at time \( t \) is obtained by applying the observable function to the state at time \( t \).
A familiar example of the state space approach is Hamiltonian mechanics. The state space in this case is phase space, and a state is a 2n-tuple \((q,p) = (q_1, ..., q_n, p_1, ..., p_n)\) consisting of position coordinates \(q_i\) and momentum coordinates \(p_i\). The evolution is \(T_t(q,p) = (q(t), p(t))\), where the latter is the solution to Hamilton’s equations with initial data \((q(0), p(0)) = (q, p)\):

\[
\begin{align*}
\frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\
\frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}
\end{align*}
\]

for \(i = 1, 2, ..., n\). Here \(H(q,p)\) is the Hamiltonian function of the system, which is assumed to be a continuously differentiable function on state space representing the total energy of the system. The function \(H\) is an example of an observable, as are the position and momentum coordinates, angular momenta \(q_i p_j - q_j p_i\), etcetera. A differentiable observable \(f\) evolves according to the equation:

\[
\frac{df}{dt} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right),
\]

the right-hand side being the definition of the Poisson bracket \([f, H]\), under which operation \(C^\infty\) observables form a Lie algebra.

Given a deterministic state space it is natural to pass to a statistical setting as follows. We replace the old states \(s\) by new states that are (Borel) probability measures \(P\) on the state space \(S\). The old observables \(f\) on the original state space are replaced by new observables that are the means of the old observables with respect to the probability measure \(P\). Thus for any original state space observable \(f\), the map

\[
P \mapsto E_P(f)
\]

defines an observable on the set of probability measures. If \(f\) is bounded, this observable is defined for all probability measures. If not, it is defined for those measures with respect to which \(f\) is integrable.

If the only observables allowed were obtained in this manner, this would appear to be a severe limitation. However, the variance of \(f\) and all moments of \(f\) can themselves be regarded as means of original observables. Indeed, even the probability distribution for \(f\) can be regarded as a mean. This is because on a Borel subset \(A\) of the reals, the probability that \(f\) takes a value in \(A\) is given by \(E_P(\chi_A \circ f)\), where \(\chi_A\) is the characteristic function of the set \(A\).

The evolution of the probabilistic state can be induced by an underlying deterministic evolution. The probability measure at time \(t\), \(P_t\), is given by \(P_t(A) = P(T_{-t}(A))\) where \(A\) is any (Borel) subset of \(S\). This permits us to talk about the evolution of observables since the mapping \(t \mapsto E_{P_t}(f)\) describes such an evolution. In the Hamiltonian formalism phase space has a natural 2n-dimensional Lebesgue measure \(\lambda\) called Liouville measure with infinitesimal volume element \(dq_1 ... dq_n dp_1 ... dp_n\), and \(\lambda(T_t(A)) = \lambda(A)\) for all Borel subsets \(A\) of \(S\) and all times \(t\). Dynamics in phase space can be thought of as a fluid flow that permits change of shape but no change in volume. The probability state \(P\) can often be taken to be the integral of a probability density function \(\rho(q,p)\) with respect to \(\lambda\). At the other
extreme $P$ can be taken to be a delta function $\delta(q - q_0)\delta(p - p_0)$, which reduces to the deterministic theory with state $s = (q_0, p_0)$. The probabilistic setting also permits us to abandon the deterministic evolution $\{T_t\}$ and work with a stochastic evolution exclusively, e.g., one of Markov type.

A use of means in state space theory that we have not touched on here relates to ergodic theory, in which time averages of observables over trajectories are compared with averages over state space regions using a suitable normalized volume measure.

The essential point is that means provide the transition from classical observables to statistical observables for stochastic systems.

6. Entropy

A subtlety occurs in statistical mechanics that is not present in ordinary probability theory. An observable is commonly defined as a real-valued function of the state, and in statistical mechanics the state is a probability measure $P$ on state space. Thus any real-valued function of $P$ can be taken to be an observable, e.g., $P \mapsto P(B)$ is an observable where $B$ is any fixed Borel set in the state space $S$. This observable is an expected value since $P(B) = E_P(\chi_B)$. However, not all observables arise as expected values of original observables. The most familiar example of such an observable is the entropy function, which can be interpreted as an expected value (mean) but is not a conventional mean.

To avoid certain difficulties associated with the continuous case we will confine our attention to the case where the underlying state space is a finite set. Let $S$ be a finite state consisting of $n$ states. A classical observable is a function $f : S \to (\infty, \infty)$. A discrete classical evolution might be a function $T : S \to S$ such that if $i$ is the state at a given time then $T(i)$ is the state one time unit later. (A continuum of times presents a problem for deterministic evolution in a finite state space, although that problem does not arise in the probabilistic setting.)

When we pass to a statistical notion of state, we arrive at a probability vector $p = (p_1, \ldots, p_n)$ where $p_i$ is the probability that the system is in the deterministic state $i$. We can now form expected values of classical observables $f$, i.e.,

$$E(f) = \sum_i p_i f(i)$$

as noted before. We can also form such expressions as the entropy:

$$H(p) = \sum_i p_i \log(\frac{1}{p_i}).$$

Superficially the entropy appears to be another mean value, the mean value of the “uncertainty” $\log(1/p_i)$, also called the “surprise value” (The log here is usually taken with base $2$.) Thus the entropy of a probability state is the mean uncertainty of the state. This is not the mean of a classical observable since the function $i \mapsto \log(1/p_i)$ is not a classical observable. Classical observables should exist and be measurable prior to assignment of probabilities, but it makes no sense to consider the uncertainty function until probabilities have been introduced. The dependency of the uncertainty function on $i$ is not intrinsic and is only determined through the postulated probability state $p_i$.

It happens that entropy has another relationship to means of considerable importance, namely through the Maximum Entropy Principle (MEP), also known as
Jaynes’ Principle. In the absence of an evolutionary law $T$ and an initial assignment, we are faced with the problem of determining the probability state $p_i$, i.e., an assignment of probabilities to the deterministic states $i$. The MEP [5, p. 370] asserts that:

The probability state $p$ maximizing entropy subject to the given values $\alpha_1, \ldots, \alpha_k$ for the means of known classical observables $g_1, \ldots, g_k$ provides predictions “most strongly indicated by our present information.”

Using the calculus of variations, we can in general determine a unique distribution among those that satisfy the constraints

$$\sum_i p_i g_j(i) = \alpha_j$$

for $j = 1, \ldots, k$ and maximizing $H(p)$, namely, the one with the probability assignment

$$p_i = C \exp\{-\sum_j \beta_j g_j(i)\}$$

for $i = 1, \ldots, n$ where $\beta_1, \ldots, \beta_k$ are constants determined from the $\alpha_i$’s, and $C$ is a positive normalizing constant chosen so that the sum of the $p_i$’s is 1.

The interpretation of this result takes two forms (at least). Suppose the states are those of an individual particle in a gas of $N$ particles. Then the quantities $\alpha_1, \ldots, \alpha_k$ represent measured values of the total value of $g_1, \ldots, g_k$ over the entire gas divided by $N$. The probabilities $p_i$, derived from the MEP, are the probabilities that a particle picked at random from among the $N$ particles is in the $i$-th state. They may also be regarded as the fraction of particles that are in the $i$-th state. We may not care about individual particles but we do care about these fractions, which can be taken to define the macroscopic state of the gas (volume, pressure, temperature, and the like). This is the ensemble viewpoint of Gibbs. Yet another perspective is to regard what we usually observe as a small perturbation about values induced by means.

7. Quantum Issues

The mean plays a pivotal role in quantum theory, even if this role has not been examined closely in most treatments of quantum theory. In quantum mechanics the state of a physical system is described by a wave function $\psi$ that is an element of a Hilbert space $\mathcal{H}$. (Strictly speaking $\psi$ is not a function but an equivalence class of functions, and in addition each state is associated with a ray in Hilbert space.) Each physical observable that takes on real-number values (e.g., a position coordinate, a momentum coordinate, the energy, a spin component) is associated with a self-adjoint operator $A$ in $\mathcal{H}$. For simplicity each observable is denoted by the same symbol “$A$” as the associated operator. Any self-adjoint operator $A$ has in turn an associated projection-valued measure $P_A$ (see, for example, [10]) that assigns to each Borel set $S$ in $\mathcal{R}$ an orthogonal projection $P_A(S)$ in the Hilbert space:

$$S \mapsto P_A(S)$$
in such a way that $A$ is an integral combination of these orthogonal projections, represented symbolically by:

$$A = \int_R x P_A(dx),$$

or by:

$$A(\psi) = \int_R x P_A(dx)(\psi),$$

where $\psi$ is in the domain of $A$. If a measurement is made, the probability that the value of $A$ is in the set $S$ when the system state is $\psi$ defined to be:

$$\langle P_A(S)(\psi), \psi \rangle = ||P_A(S)(\psi)||^2$$

where $\langle, \rangle$ is the inner product on $\mathcal{H}$, linear in the first variable and conjugate-linear in the second variable, and $|| \|$ is the norm on $\mathcal{H}$.

Quantum Mechanics is thus a statistical theory based on a family of probability measures defined by:

$$S \mapsto ||P_A(S)(\psi)||^2.$$

These are the Borel probability measures associated with observables $A$ when the system state is $\psi$. One consequence of this is that the set of possible values of $A$ is the spectrum of the operator $A$, and another is that the mean of $A$, when the state is $\psi$, is given by:

$$\langle A(\psi), \psi \rangle = \int_R x \langle P_A(dx)(\psi), \psi \rangle = \int_R x ||P_A(dx)(\psi)||^2.$$

In particular the quantity $\langle P_A(S)(\psi), \psi \rangle = ||P_A(S)(\psi)||^2$ can be interpreted as the mean of the observable $P_A(S)$ when the state is $\psi$. The observable $P_A(S)$ is an orthogonal projection, taking the value 1 when the value of $A$ is in is $S$ and the value 0 when the value of $A$ is not in $S$. Thus $||P_A(S)(\psi)||^2$ also represents the probability that $A$ is in $S$ when the state is $\psi$. This is a reminder that all probabilities are means.

The mean, $\langle A(\psi), \psi \rangle$, is the integral over the real line of the real variable $x$ with respect to the Borel probability measure $||P_A(\cdot)(\psi)||^2$. Thus the mean exists, it appears, if and only if $x$ is integrable with respect to this measure, thus if and only if $\psi$ is in the domain of $A$. Self-adjoint operators have domains that are dense in $\mathcal{H}$ but many of the most prominent ones (e.g., those associated with position and momentum and often energy) do not have domain equal to $\mathcal{H}$. Hence there will be states for which the means of some observable may not be well-defined. Whether these states are realizable in practice is uncertain, but there is no good theoretical reason why they should be ignored. (Our discussion focuses on mathematical definition and characterization. The spectrum of a self-adjoint operator is identified with the possible values of a measured quantity. If the spectrum is discrete, a measurement may be able to distinguish one value from another; if the spectrum is continuous, measurement will only be able to determine an interval that contains the value, not the exact value. Repeated measurements when the system is in the same state thus only arrive at a rough approximation of the distribution and a rough estimate of the mean for a state.)
A curiosity in quantum mechanics, not ordinarily seen in other applications of probability, is the following. Suppose $\mu = \langle A(\psi), \psi \rangle$ is the mean of some observable $A$ when the state is $\psi$. Then the variance of the observable in this state is naturally given by:

$$\int_R (x - \mu)^2 \langle P_A(dx)\psi, \psi \rangle = \langle (A - \mu I)^2(\psi), \psi \rangle = ||(A - \mu I)(\psi)||^2.$$ 

So the variance exists if and only if $\psi$ is in the domain of $A$. The condition for the mean to exist is the same as the condition for the variance to exist. In quantum mechanics we are led to think that the only distributions for which the mean is finite are ones in which the variance is also finite. However, a closer look at this situation reveals some discrepancies.

The chief discrepancy is the following. Suppose that the original observable $A$ can be written in the form $A = C - D$ where $C$ and $D$ are non-negative self-adjoint operators. Non-negative self-adjoint operators can be written as squares of self-adjoint operators, so that $C = E^2$ and $D = F^2$ with $E$ and $F$ self-adjoint. Then

$$\langle A(\psi), \psi \rangle = \langle (C - D)(\psi), \psi \rangle = \langle (E^2 - F^2)(\psi), \psi \rangle$$

$$= \langle E^2(\psi), \psi \rangle - \langle F^2(\psi), \psi \rangle = ||E(\psi)||^2 - ||F(\psi)||^2.$$ 

Thus, the mean of $A$ exists if and only if $\psi$ is in the intersection of the domain of $E$ and the domain of $F$. It is easy, incidentally, to construct examples of elements of $\mathcal{H}$ that are in the domain of a self-adjoint operator $E$ but are not in the domain of its square $E^2$. In addition, as it happens, it is possible to offer explicit candidates for the operators $E$ and $F$ given $A$. Set

$$E = \int_{0,\infty} \sqrt{x} P_A(dx) \quad \text{and} \quad F = \int_{-\infty,0} \sqrt{-x} P_A(dx).$$ 

We conclude that the mean of $A$ exists when $\psi$ is in $\text{dom } E \cap \text{dom } F$ and the variance of $A$ exists when $\psi$ is in $\text{dom } A$. If $\psi$ is not in $\text{dom } E$ but is in $\text{dom } F$, then it is reasonable to say that the mean of $A$ is $\infty$. Likewise if $\psi$ is in $\text{dom } E$ but not in $\text{dom } F$, the mean of $A$ is $-\infty$. If $\psi$ is in neither $\text{dom } E$ nor $\text{dom } F$, then the ordinary mean does not exist. In the spirit of our discussion of $L(c)$ earlier, it is possible to truncate the integrals for $E$ and $F$ in the last display, replacing $\infty$ by $M$ and $-\infty$ by $-M$ and investigate the existence of an appropriate combined limit as $M$ tend to infinity.

Similar considerations can be applied when the “pure” state $\psi$ is replaced by a density matrix representing a statistical ensemble of pure states, or in rigged Hilbert spaces where the existence of states varies according to the properties of the observable, or to cases arising by the use of positive-operator-valued measures generalizing the projection-valued measures treated above.

8. Conclusion

The mean, as we have seen, is ubiquitous in scientific explanation. Not only does it provide a summary of sample data and, when it exists, of data from the
entire population, but it establishes a connection between samples and the whole population. Furthermore, it facilitates generalization of deterministic observables that are functions of the deterministic state to probabilistic observables that are functions of the probabilistic state. The Maximum Entropy Principle then makes use of constrained means to identify macroscopic distribution of physical importance parametrized by these means. While quantum mechanics abandons determinism, it retains the notion of mean to summarize the possible results of experiments and the measurement of quantum observables. Although not all observables have finite means, weak and doubly weak means and the alternatives identified in Theorem 3.1 provide an enumeration of possible behaviors of variables and associated probability distributions, and give further insight into potentialities associated with large data sets.

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