Peeling Properties of Light–Like Signals in General Relativity

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Abstract

The peeling properties of a light–like signal propagating through a general Bondi–Sachs vacuum space–time and leaving behind another Bondi–Sachs vacuum space–time are studied. We demonstrate that in general the peeling behavior is the conventional one which is associated with a radiating isolated system and that it becomes unconventional if the asymptotically flat space–times on either side of the history of the light–like signal tend to flatness at future null infinity faster than the general Bondi–Sachs space–time. This latter situation occurs if, for example, the space–times in question are static Bondi–Sachs space–times.

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1 Introduction

The history of a light–like signal in General Relativity is a singular null hypersurface. The null hypersurface is called singular because in general the Ricci tensor and the Weyl tensor of the space–time contain Dirac $\delta$–function terms with the $\delta$–function singular on the null hypersurface. Such a singular null hypersurface can be used as a simplified model of a supernova [1] if the space–time before and after the emission of the light–like signal is a model of the vacuum field due to an isolated gravitating system and if the singular null hypersurface is asymptotically (as future null infinity is approached) a future–directed null–cone. For the model described in [1] the space–times before and after the emission of the light–like signal are two copies of the Weyl asymptotically flat, static space–times [3] but with different multipole moments. Thus the explosion is modelled by a sudden change in the multipole moments of the source. The coefficients of the $\delta$–function in the Weyl tensor display unconventional peeling properties. By peeling properties here we mean the dependence of the coefficients of the $\delta$–function in the Weyl tensor on an affine parameter $r$ along the generators of the singular null hypersurface, with $r \to +\infty$ as future null infinity is approached. In general the $\delta$–function in the Weyl tensor can be unambiguously split into a matter part (if it exists) of Petrov type II and a wave part (if it exists) of Petrov type N. This splitting, which was originally announced in [2], is fully developed in [1]. In the simplified supernova model described in [1] the matter part of the $\delta$–function in the Weyl tensor depends on $r$ in the form of $O(r^{-3})$ terms and smaller terms, while the wave part of the $\delta$–function in the Weyl tensor has a coefficient which is $O(r^{-4})$. We denote the components of the $\delta$–function in the Weyl tensor, in Newman–Penrose notation, by $\hat{\Psi}_A$ ($A = 0, 1, 2, 3, 4$) chosen in such a way that the first four of these describe the matter part of the Weyl tensor and $\hat{\Psi}_4$ describes the wave part. Defining $\hat{\Psi}_A$ in this way the conventional peeling behavior would be to have $\hat{\Psi}_A = O(r^{-5+A})$ ($A = 0, 1, 2, 3, 4$). This is the conventional peeling behavior when compared with models of vacuum gravitational fields due to isolated gravitating systems in general (cf. [1]–[8] and (2.20)–(2.24) below). The purpose of the present paper is to explain the unconventional peeling behavior described above by putting it in the context of light–like signals emitted by a general class of isolated gravitating systems.

We take the vacuum gravitational field outside an isolated system, before and after the emission of a light–like signal, to be modelled by a Bondi–Sachs [6] [7] asymptotically flat space–time. We also consider the most general matching of the two space–times on the singular null hypersurface separating them. How the two space–times are glued together influences the type of
signal whose history is the singular null hypersurface. We demonstrate that in general the peeling behavior is the conventional one which is associated with a radiating isolated system and that it becomes unconventional if the asymptotically flat space–times on either side of the history of the light–like signal tend to flatness at future null infinity faster than the general Bondi–Sachs space–time.

2 Light–Like Signal From Isolated Source

Throughout this paper the space–time model of the vacuum gravitational field outside an isolated source will be a Bondi–Sachs [3, 4] space–time. A convenient form of this space–time line–element is given by

$$ds^2 = (\theta^1)^2 + (\theta^2)^2 - 2 \theta^3 \theta^4,$$

with

$$\theta^1 = rp^{-1}(e^\alpha \cosh \beta dx + e^{-\alpha} \sinh \beta dy + a du),$$

$$\theta^2 = rp^{-1}(e^\alpha \sinh \beta dx + e^{-\alpha} \cosh \beta dy + b du),$$

$$\theta^3 = -dr - \frac{1}{2} c du,$$

$$\theta^4 = -du.$$  

The six function $\alpha, \beta, a, b, p, c$ depend on all coordinates $x, y, r, u$ and $u = \text{const.}$ are null hypersurfaces generated by the geodesic integral curves of the vector field $\frac{\partial}{\partial r}$ with $r$ an affine parameter along these geodesics. The following assumptions are made regarding the $r$–dependence of these functions:

$$\alpha = \frac{\alpha_1}{r} + \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r^3} + \ldots,$$

$$\beta = \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + \frac{\beta_3}{r^3} + \ldots,$$

$$a = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \ldots,$$

$$b = b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \ldots,$$

$$p = p_0 \left(1 + \frac{q_1}{r} + \frac{q_2}{r^2} + \frac{q_3}{r^3} + \ldots\right),$$

$$c = 1 - \frac{2m}{r} + \ldots.$$  

Here $p_0 = 1 + \frac{1}{4}(x^2 + y^2)$ and all the other coefficients of the inverse powers of $r$ displayed above are functions of $(x, y, u)$. All eighteen functions of $(x, y, u)$
appearing in (2.6)–(2.11) are required in order to have a knowledge of the metric tensor components up to and including $\frac{1}{r^2}$ terms. The vacuum field equations and an outgoing radiation condition [9] allow us to specialise the sixteen functions as follows:

$$\alpha_2 = \beta_2 = 0 \ , \quad (2.12)$$

$$a_0 = a_1 = 0 \ , \quad \text{and} \quad b_0 = b_1 = 0 \ , \quad (2.13)$$

$$a_2 = p_0^4 \left\{ \frac{\partial}{\partial x} (p_0^{-2} \alpha_1) + \frac{\partial}{\partial y} (p_0^{-2} \beta_1) \right\} \ , \quad (2.14)$$

$$b_2 = p_0^4 \left\{ \frac{\partial}{\partial x} (p_0^{-2} \beta_1) - \frac{\partial}{\partial y} (p_0^{-2} \alpha_1) \right\} \ , \quad (2.15)$$

$$q_1 = 0 \ , \quad q_2 = \frac{1}{2} (\alpha_1^2 + \beta_1^2) \ , \quad q_3 = 0 \ . \quad (2.16)$$

When these equations are satisfied there remain five further field equations. They are propagation equations for $m(u, x, y), \alpha_3(u, x, y), \beta_3(u, x, y), a_3(u, x, y)$ and $b_3(u, x, y)$ off $u = \text{const}$. The simplest reads

$$\dot{M} + |\dot{\gamma}|^2 = 0 \ , \quad (2.17)$$

where the dot denotes partial differentiation with respect to $u$, and

$$M = m - \dot{q}_2 - \frac{1}{2} \left\{ \frac{\partial}{\partial x} (p_0^{-2} a_2) + \frac{\partial}{\partial y} (p_0^{-2} b_2) \right\} \ , \quad (2.18)$$

with

$$\gamma = \alpha_1 + i \beta_1 \ . \quad (2.19)$$

When the equation (2.17) is averaged over the 2–sphere with line–element $dt^2 = p_0^{-2} (dx^2 + dy^2)$ the well–known Bondi–Sachs mass–loss formula results. The remaining equations involving $\dot{\alpha}_3, \dot{\beta}_3, \dot{a}_3$ and $\dot{b}_3$ are given in [9] and will not be used here. The curvature tensor components, in Newman–Penrose notation, for the space–time described above display the conventional peeling behavior:

$$\Psi_0 = -\frac{1}{r^5} \left\{ 6(\alpha_3 + i \beta_3) - \frac{3}{2} (\gamma + \bar{\gamma})^2 (\gamma - \bar{\gamma}) - 2\bar{\gamma}^3 \right\} + \ldots \ , \quad (2.20)$$

$$\Psi_1 = -\frac{1}{r^4 \sqrt{2}} \left\{ \frac{3}{2} p_0^{-1} (a_3 + i b_3) + 3 p_0^3 \gamma \frac{\partial}{\partial \bar{z}} (p_0^{-2} \bar{\gamma}) \right\} + \ldots \ , \quad (2.21)$$

$$\Psi_2 = -\frac{1}{r^3} \left\{ M + \gamma \frac{\partial \bar{\gamma}}{\partial u} + 2 p_0^2 \frac{\partial}{\partial \bar{z}} \left( \frac{\partial}{\partial \bar{z}} (p_0^{-2} \gamma) \right) \right\} + \ldots \ , \quad (2.22)$$
\[ \Psi_3 = -\frac{2}{r^2\sqrt{2}} p_0 \frac{\partial}{\partial u} \left( p_0 \frac{\partial}{\partial z} (p_0^{-2}\gamma) \right) + \ldots, \]  
(2.23)

\[ \Psi_4 = -\frac{1}{r} \frac{\partial^2 \gamma}{\partial u^2} + \ldots, \]  
(2.24)

where we have put \( z = x + iy \) and a bar denotes complex conjugation. Finally, the complex shear \( \sigma \) and the real expansion \( \vartheta \) of the null geodesic integral curves of the vector field \( \frac{\partial}{\partial r} \) are given by

\[ \sigma = -\frac{(\alpha_1 + i\beta_1)}{r^2} - \frac{3(\alpha_3 + i\beta_3) + 2\alpha_1\beta_1^2}{r^4} + O\left(r^{-5}\right), \]  
(2.25)

\[ \vartheta = \frac{1}{r} + \frac{2q_2}{r^3} + O\left(r^{-5}\right), \]  
(2.26)

respectively, demonstrating that asymptotically (as \( r \to +\infty \)) the null hypersurfaces \( u = \text{const.} \) are future-directed null–cones.

We now consider the space–time above to be subdivided into two halves \( M^-(u \leq 0) \) and \( M^+(u \geq 0) \) each with boundary the null hypersurface \( u = 0 \). Let \( x^\mu = (x_+, y_+, r_+, u) \) (with \( \mu = 1, 2, 3, 4 \)) be the local coordinate system in \( M^+ \) in which the line–element takes the form given by (2.1)–(2.5) with the coefficients of the powers of \( r_+^{-1} \) in (2.6)–(2.11) denoted with a superscript plus as \( \alpha_1^+, \alpha_2^+, \) etc.. Let \( x^\mu = (x, y, r, u) \) be the local coordinate system in \( M^- \) in terms of which the line–element of the space–time has the form given by (2.1)–(2.5). We take \( \xi^a = (x, y, r) \), with \( a = 1, 2, 3 \), as local intrinsic coordinates on \( u = 0 \). Now \( u = 0 \) is the history of a light–like signal emitted by the isolated source and propagating into the space–time \( M^- \) leaving the space–time \( M^+ \) behind. We apply the Barrabès–Israel \[2\] technique (see also \[10\] for some recent developments) to analyse the physical properties of the signal with history \( u = 0 \). We shall assume that the reader is familiar with \[2\]. While we are applying the technique to the current situation we will comment on what we are doing so as to guide the reader by example through \[4\]. The first requirement of \[2\] is that the metric tensors induced on the null hypersurface \( u = 0 \) by its embedding in \( M^+ \) and in \( M^- \) agree on \( u = 0 \). This is achieved if the space–times \( M^+ \) and \( M^- \) described above are attached on \( u = 0 \) with the following matching conditions:

\[ x_+ = f(x, y) + \frac{f_1(x, y)}{r} + O\left(r^{-2}\right), \]  
(2.27)

\[ y_+ = g(x, y) + \frac{g_1(x, y)}{r} + O\left(r^{-2}\right), \]  
(2.28)

\[ r_+ = r h(x, y) + h_0(x, y) + O\left(r^{-1}\right), \]  
(2.29)
with
\[ f_x = g_y, \quad f_y = -g_x, \quad h = \frac{P}{p_0} \left\{ f_x^2 + g_x^2 \right\}^{-\frac{1}{2}}, \quad (2.30) \]

where \( P = 1 + \frac{1}{2}(f^2 + g^2) \) and the subscripts on \( f, g \) denote partial differentiation. In addition we must have

\[ hP^{-2}(f_x^2 - f_y^2) \beta_1^+ + 2hP^{-2}f_x f_y \alpha_1^+ - p_0^{-2} \beta_1 = \]
\[ -\frac{1}{2}h^2 P^{-2} \left\{ f_x \left( \frac{\partial g_1}{\partial x} + \frac{\partial f_1}{\partial y} \right) + f_y \left( \frac{\partial f_1}{\partial x} - \frac{\partial g_1}{\partial y} \right) \right\}, \quad (2.31) \]

\[ hP^{-2}(f_x^2 - f_y^2) \alpha_1^+ - 2hP^{-2}f_x f_y \beta_1^+ - p_0^{-2} \alpha_1 = \]
\[ -\frac{1}{2}h^2 P^{-2} \left\{ f_x \left( \frac{\partial f_1}{\partial x} - \frac{\partial g_1}{\partial y} \right) - f_y \left( \frac{\partial g_1}{\partial x} + \frac{\partial f_1}{\partial y} \right) \right\}, \quad (2.32) \]

and

\[ -h_0 + \frac{1}{2}p_0^{-1}(f_x^2 + f_y^2)^{-\frac{1}{2}}(f f_1 + g g_1) = \]
\[ \frac{1}{2} P p_0^{-1}(f_x^2 + f_y^2)^{-\frac{1}{2}} \left\{ f_x \left( \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial y} \right) + f_y \left( \frac{\partial f_1}{\partial x} - \frac{\partial g_1}{\partial y} \right) \right\}, \quad (2.33) \]

Here \( \beta_1^+ = \beta_1^+(f, g, 0), \beta_1 = \beta_1(x, y, 0), \alpha_1^+ = \alpha_1^+(f, g, 0) \) and \( \alpha_1 = \alpha_1(x, y, 0) \). The complexity of these matching conditions suggests that we examine the light–like signal in two stages. The leading terms in (2.27)–(2.28) are constructed from the analytic function \( F(z) = f(x, y) + ig(x, y) \) with \( z = x + iy \). They describe a part of the gluing of \( M^+ \) to \( M^- \) on \( u = 0 \) which is a Penrose warp \[11\]. This particular gluing leads to an impulsive gravitational wave with history \( u = 0 \) having a line or directional singularity and we will consider it in section 3. Thus for the remainder of this section we shall take \( f(x, y) = x \) and \( g(x, y) = y \) in (2.27) and (2.28) and thus in (2.29) \( h = 1 \).

Now (2.31)–(2.33) simplify to read

\[ [\beta_1] = -\frac{1}{2} \left( \frac{\partial g_1}{\partial x} + \frac{\partial f_1}{\partial y} \right), \quad [\alpha_1] = -\frac{1}{2} \left( \frac{\partial f_1}{\partial x} - \frac{\partial g_1}{\partial y} \right), \quad (2.34) \]

and

\[ h_0 = -\frac{1}{2} p_0^2 \frac{\partial}{\partial x}(p_0^{-2} f_1) - \frac{1}{2} p_0^2 \frac{\partial}{\partial y}(p_0^{-2} g_1), \quad (2.35) \]

where the square brackets will henceforth denote the jump across \( u = 0 \) of the quantity within them, calculated in the coordinates \( x^- \). Thus, for example, \( [\beta_1] = \beta_1^+(x, y, u = 0) - \beta_1(x, y, u = 0) \).
The Barrabès-Israel technique is an extension to hypersurfaces of all types of the extrinsic curvature technique for non-null hypersurfaces (see, for example [12]). In the case of a null hypersurface the normal is tangent to the hypersurface so in order to obtain an analogous quantity to extrinsic curvature one first constructs a ‘transverse vector field’ which is any vector field defined on the hypersurface which is not tangent to the hypersurface and which is the same vector field when viewed in the coordinates \((x^\mu_+)\) and in the coordinates \((x^\mu_-)\). In our case the normal to \(u = 0\) is given via the 1–form

\[ n_\mu \, dx^\mu \big|_\pm = -du , \]  

(2.36)

where \(|_\pm\) means the quantity is calculated in the plus or minus coordinates. A natural choice of transversal on the minus side, in view of the form of the line–element given by (2.1)–(2.5), is

\[ -N_\mu \, dx^\mu_- = -dr - \frac{1}{2} \, c \, du , \]  

(2.37)

with \(c\) given by (2.11). To ensure that the transversal when viewed on the plus side, \(N_\mu^+\), is the same covariant vector field as \(-N_\mu\) we proceed as follows:

We pointed out above that we may use \(\xi^a = (x, y, r)\) as intrinsic coordinates on \(u = 0\). We then have three linearly independent tangent vectors to \(u = 0\) given by \(\partial / \partial \xi^a\). On the minus side of \(u = 0\) these have components

\[ e^\mu_\mu \big|_- = \frac{\partial x^\mu_-}{\partial \xi^a} = \delta^\mu_a , \]  

(2.38)

and on the plus side their components are

\[ e^\mu_\mu \big|_+ = \frac{\partial x^\mu_+}{\partial \xi^a} , \]  

(2.39)

with \(x^\mu_\pm\) given in terms of \(\xi^a\) by (2.27)–(2.29) [now with \(f = x, g = y, h = 1\)]. Now \(N_\mu^+\) is chosen so that

\[ [N_\mu \, e^\mu_\mu \big|_+ = 0 = [N_\mu, N^\mu] . \]  

(2.40)

With \(N_\mu^+\) thus calculated the ‘transverse extrinsic curvature’ on the plus or minus sides of \(u = 0\) is defined by

\[ K^\pm_{\alpha \beta} = -N_\mu \left( \frac{\partial e^\mu_\mu \big|_\pm}{\partial \xi^b} + \pm \Gamma^\mu_{\lambda \sigma} \, e^\lambda_\alpha \big|_\pm \, e^\sigma_\beta \big|_\pm \right) = K^\pm_{\beta \alpha} . \]  

(2.41)
where $\pm \Gamma_{\lambda\sigma}^{\kappa}$ are the components of the Riemannian connection calculated on the plus or minus sides of $u = 0$. We define

$$\gamma_{ab} = 2 \left[ K_{ab} \right], \tag{2.42}$$

and this is independent of the choice of transversal $[2]$. Now $\gamma_{ab}$ is extended to a 4–tensor $\gamma_{\mu\nu}$ on $u = 0$ by padding out with zeros (the only requirement on the extension being that its projection tangential to $u = 0$ be $\gamma_{ab}$). With

$$\gamma^{\mu} = \gamma^{\mu\nu} n_{\nu}, \quad \gamma^\dagger = \gamma^{\mu\nu} \bar{n}_{\nu}, \quad \gamma = g^{\mu\nu} \gamma_{\mu\nu}, \tag{2.43}$$

calculated in the plus or minus coordinates (we leave out the designation $|_\pm$ in such situations), the coefficient of the delta function $\delta(u)$ in the Einstein tensor of the space–time $M^+ \cup M^-$ gives the surface stress–energy tensor $[2]$

$$16 \pi \eta^{-1} S^{\mu\nu} = 2 \gamma^{(\mu} n^{\nu)} - \gamma n^{\mu} n^{\nu} - \gamma^\dagger g^{\mu\nu}, \tag{2.44}$$

where $\eta^{-1} = n^\mu N_\mu$. The coefficient of the delta function $\delta(u)$ in the Weyl tensor components is then $[2]$

$$\hat{C}^{\kappa\lambda}_{\mu\nu} = 2 \eta n^{[\kappa} \gamma^{\lambda]}_{[\mu} n_{\nu]} - 16 \pi \delta^{[\kappa}_{[\mu} S^{\lambda]}_{\nu]} + \frac{8 \pi}{3} S^\alpha_{[\nu} \delta^{\kappa\lambda}_{\mu]}.$$ \hspace{1cm} \tag{2.45}

If $m^\mu$ is a unit complex vector field defined on $u = 0$ which is tangential to $u = 0$ and also orthogonal to the transversal then the Newman–Penrose components of $\hat{C}^{\kappa\lambda}_{\mu\nu}$ are given by $[10]$

$$\hat{\Psi}_0 = 0, \quad \hat{\Psi}_1 = 0, \quad \hat{\Psi}_2 = -\frac{1}{6} \eta \gamma^\dagger,$$
$$\hat{\Psi}_3 = -\frac{1}{2} \eta \gamma_{\mu} \bar{m}_{\mu}, \quad \hat{\Psi}_4 = -\frac{1}{2} \eta \gamma_{\mu\nu} \bar{m}^{\mu} \bar{m}^{\nu}. \tag{2.46}$$

In general the signal is Petrov type II and contains a gravitational wave if $\hat{\Psi}_4 \neq 0$.

Carrying out this procedure (the calculations in this paper have been performed using GRTensorM version 1.2 for MATHEMATICA 3.x $[13]$) we find that $u = 0$ has a non–vanishing surface stress–energy tensor with components

$$S^{11} = O \left( \frac{1}{r^5} \right), \tag{2.47}$$
$$S^{22} = O \left( \frac{1}{r^5} \right), \tag{2.48}$$
$$S^{12} = O \left( \frac{1}{r^6} \right), \tag{2.49}$$

$$S^{13} = \frac{1}{16 \pi r^3} \left\{ 2 \left[ a_2 \right] + f_1 - 2 p_0^2 \frac{\partial h_0}{\partial x} - 2 f_1 \dot{\alpha}_1^+ + 2 g_1 \dot{\beta}_1^+ \right\} + \ldots, \tag{2.50}$$
$$S^{23} = \frac{1}{16 \pi r^3} \left\{ 2 \left[ b_2 \right] + g_1 - 2 p_0^2 \frac{\partial h_0}{\partial y} - 2 f_1 \dot{\beta}_1^+ + 2 g_1 \dot{\alpha}_1^+ \right\} + \ldots, \tag{2.51}$$
and
\[
S^{33} = -\frac{1}{4\pi r^2} \left\{ [m - \dot{q}_2] - \frac{1}{2} p_0^2 \frac{\partial}{\partial y} (p_0^2 [b_2]) - \frac{1}{2} p_0^2 \frac{\partial}{\partial x} (p_0^2 [a_2]) + \frac{1}{2} (h_0 + \Delta h_0) \right\} + \ldots ,
\]
(2.52)
where \( \Delta = p_0^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) and \( q_2 \) is given by (2.16). The jumps \([a_2], [b_2] \) can be written in terms of the jumps \([\alpha_1], [\beta_1] \) using the field equations (2.14) and (2.15) and thence in terms of the functions \( f_1, g_1 \) via (2.34) to arrive at
\[
[a_2] = -p_0 \frac{\partial p_0}{\partial x} \left( \frac{\partial g_1}{\partial y} - \frac{\partial f_1}{\partial x} \right) + p_0 \frac{\partial p_0}{\partial y} \left( \frac{\partial g_1}{\partial x} + \frac{\partial f_1}{\partial y} \right) - \frac{1}{2} \Delta f_1 ,
\]
(2.53)
\[
[b_2] = p_0 \frac{\partial p_0}{\partial x} \left( \frac{\partial g_1}{\partial x} + \frac{\partial f_1}{\partial y} \right) + p_0 \frac{\partial p_0}{\partial y} \left( \frac{\partial g_1}{\partial y} - \frac{\partial f_1}{\partial x} \right) - \frac{1}{2} \Delta g_1 .
\]
(2.54)
and finally at
\[
S^{33} = -\frac{1}{4\pi r^2} \left\{ [m - \dot{q}_2] + \frac{1}{4} p_0^2 \left( \frac{\partial}{\partial x} (p_0^2 f_1) + \frac{\partial}{\partial y} (p_0^2 g_1) \right) \right\} + \ldots .
\]
(2.55)
The surface energy–density of the shell measured by a radially moving observer (see [2]) is a positive multiple of \( S^{33} \). If we make the assumption that the functions \( p_0^{-2} f_1, p_0^{-2} g_1 \), defined on the 2–sphere with line–element \( dl^2 = p_0^{-2} (dx^2 + dy^2) \), are bounded then it follows from (2.55) that the leading term in the surface energy density averaged over the 2–sphere is proportional to the jump in the Bondi–Sachs mass across \( u = 0 \) (this latter, as mentioned following (2.19), is the average of \( M \) in (2.18) over the 2–sphere). The average surface energy–density is positive if there is a loss of Bondi–Sachs mass.

Finally we calculate \( \Psi_A \) for \( A = 2, 3, 4 \) in (2.46). The result can be put in the form
\[
\hat{\Psi}_2 = O \left( \frac{1}{r^3} \right) ,
\]
(2.56)
\[
\hat{\Psi}_3 = -4\pi \sqrt{2} r p_0^{-1} \left( S^{13} - i S^{23} \right) + O \left( \frac{1}{r^3} \right) = O \left( \frac{1}{r^2} \right) ,
\]
(2.57)
and
\[
\hat{\Psi}_4 = -\frac{[\dot{\alpha}_1 - i \dot{\beta}_1]}{r} + W r^2 + O \left( \frac{1}{r^3} \right) ,
\]
(2.58)
with
\[
W = 2 \frac{\partial}{\partial z} \left( p_0^2 \frac{\partial h_0}{\partial z} \right) - \frac{1}{2} \left[ \alpha_1 - i \beta_1 \right] + \frac{\partial}{\partial z} \left[ a_2 - i b_2 \right] + \left( \dot{\alpha}_1^+ - i \dot{\beta}_1^+ \right) \left( h_0 - i \left( \frac{\partial g_1}{\partial x} - \frac{\partial f_1}{\partial y} \right) \right) .
\]
(2.59)
From this we arrive at the main result of this section, namely, in general the coefficients of the delta function terms in the Weyl tensor display the conventional peeling behavior. The signal contains an impulsive gravitational wave part $\hat{\Psi}_4$ with the expected $\frac{1}{r}$-behavior provided $[\dot{\alpha}_1 - i\dot{\beta}_1] \neq 0$. This latter means that there is a jump in the Bondi ‘news’ across $u = 0$. This impulsive wave is accompanied by a light–like shell with surface stress–energy tensor given by (2.47)–(2.55). We see from (2.58) and (2.59) that if the matching (2.27)–(2.29) is the identity matching then the wave produced by the jump in the news across $u = 0$ is free from line singularities.

3 A General and a Special Example

We return now to the general matching conditions (2.27)–(2.33). A considerable computational effort is needed to establish the following orders of magnitude of the components of the stress–energy tensor on $u = 0$ in this case:

\[ S^{11} = O \left( \frac{1}{r^5} \right), \quad S^{22} = O \left( \frac{1}{r^5} \right), \quad S^{12} = O \left( \frac{1}{r^6} \right), \quad (3.1) \]

\[ S^{13} = O \left( \frac{1}{r^3} \right), \quad S^{23} = O \left( \frac{1}{r^3} \right), \quad S^{33} = O \left( \frac{1}{r^2} \right). \quad (3.2) \]

In addition $\hat{\Psi}_A$ for $A = 2, 3, 4$ in this case satisfy

\[ \hat{\Psi}_2 = O \left( \frac{1}{r^3} \right), \quad \hat{\Psi}_3 = O \left( \frac{1}{r^2} \right), \quad (3.3) \]

and

\[ \hat{\Psi}_4 = \frac{1}{r} \left\{ F_0^2 \left( H(z) - \frac{(F')^2}{1 + \frac{1}{4} |F(z)|^2} (\dot{\alpha}_1^+ - i\dot{\beta}_1^+) \right) + \dot{\alpha}_1 - i\dot{\beta}_1 \right\} + O \left( \frac{1}{r^2} \right). \quad (3.4) \]

Here, as in the paragraph following (2.33), $F(z) = f(x, y) + ig(x, y)$ with $z = x + iy$, $F' = dF/dz$ and

\[ H(z) = \frac{F'''}{F'} - \frac{3}{2} \left( \frac{F''}{F} \right)^2. \quad (3.5) \]

Clearly when $F(z) = z$ this reduces to (2.58). We see from (3.3) and (3.4) that in this general case the conventional peeling behavior is exhibited. However we also see from the first $\frac{1}{r}$ term in (3.4) that the Penrose wave has a directional or line singularity (as $z\bar{z} \to +\infty$).

We mentioned in the introduction that the principal motivation for the present study is to put into perspective the unconventional peeling behavior
of $\Phi_A$ for $A = 2, 3, 4$ encountered in a simple example of a supernova [1] in which the space–times $M^+$ and $M^-$ are two copies of the Weyl asymptotically flat, static space–times having different multipole moments. We shall now demonstrate how this example emerges as a special case of the general situation described in section 2.

An asymptotically flat Weyl static space–time has a line–element which can be put in the form (2.1)–(2.11) with

$$
\alpha_1 = \beta_1 = 0 \,, \quad a_2 = b_2 = 0 
$$

$$
\alpha_3 = \frac{1}{2} Q p_0^{-2} (x^2 - y^2) \,, \quad \beta_3 = Q p_0^{-2} x y 
$$

$$
a_3 = -2 D x \,, \quad b_3 = -2 D y 
$$

$$
c = 1 - \frac{2m}{r} - 2 \frac{D}{r^2} p_0^{-2} (1 - \frac{1}{4} (x^2 + y^2)) - \frac{Q}{r^3} p_0^{-2} \left\{ 2 - 2(x^2 + y^2) + \frac{1}{8} (x^2 + y^2)^2 \right\} + \ldots 
$$

The constant $m$ is interpreted as the mass of the source while the constants $D$ and $Q$ are taken to be the dipole and quadrupole moments of the source respectively. In $M^+$ the local coordinates are $x_+^\mu = (x_+, y_+, r_+, u)$ and the multipole moments are $m_+, D_+, Q_+$ etc.. In $M^-$ the local coordinates are $x_-^\mu = (x, y, r, u)$ and the multipole moments are $m, D, Q$ etc.. The metric tensors induced on $u = 0$ (the boundary between $M^+$ and $M^-$) by its embedding in $M^+$ and $M^-$ agree provided the following matching conditions are satisfied:

$$
x_+ = x + 2 \frac{[Q]}{r^3} x p_0^{-1} + \ldots 
$$

$$
y_+ = y + 2 \frac{[Q]}{r^3} y p_0^{-1} + \ldots 
$$

$$
r_+ = r + \frac{[Q]}{r^3} p_0^{-2} (x^2 + y^2 - 2) + \ldots 
$$

Applying the Barrabès–Israel technique yields the results given in [1] which in the coordinates $(x, y, r)$ read: the components of the stress–energy tensor (2.44) on $u = 0$ are given by

$$
16 \pi S^{11} = 16 \pi S^{22} = -\frac{12}{r^6} [Q] (x^2 + y^2 - 2) + \ldots 
$$

$$
16 \pi S^{12} = \frac{24}{r^9} [Q] p_0^{-2} x y (x^2 + y^2 - 2) + \ldots 
$$
\[16\pi S_{13} = -\frac{6[D]}{r^4} x + \frac{6[Q]}{r^5} x p_0^{-1}(x^2 + y^2 - 5) + \ldots, \quad (3.15)\]

\[16\pi S_{23} = -\frac{6[D]}{r^4} y + \frac{6[Q]}{r^5} y p_0^{-1}(x^2 + y^2 - 5) + \ldots, \quad (3.16)\]

and

\[16\pi S_{33} = -\frac{4[m]}{r^2} + \frac{3[D]}{r^3} p_0^{-1}(x^2 + y^2 - 4) \]

\[-\frac{3[Q]}{2r^4} p_0^{-2}(24 + (x^2 + y^2)(x^2 + y^2 - 20)) + \ldots. \quad (3.17)\]

The coefficients of the delta function in the Weyl tensor are

\[\hat{\Psi}_2 = -\frac{2[Q]}{r^4} p_0^{-2}(x^2 + y^2 - 2) + \ldots, \quad (3.18)\]

\[\hat{\Psi}_3 = \frac{3[D]}{\sqrt{2} r^3} p_0^{-1}(x - iy) - \frac{3[Q]}{\sqrt{2} r^4} p_0^{-2}(x - iy)(x^2 + y^2 - 5) + \ldots, \quad (3.19)\]

and

\[\hat{\Psi}_4 = \frac{9[Q]}{4 r^4} p_0^{-2}(x - iy)^2 + \ldots. \quad (3.20)\]

In (3.18)–(3.20) we have an unconventional peeling behavior. It is clear now by comparing the conditions (3.6)–(3.9) and the general formulas (2.56)–(2.59) that this case is unconventional because the space-times \(M^+\) and \(M^-\) tend to flatness faster, as future null infinity is approached, than the more general space-times \(M^+\) and \(M^-\) considered in section 2. We note that this signal is free of directional singularities. To see unambiguously why \(\hat{\Psi}_4\) describes the impulsive gravitational wave part of the signal and \(\hat{\Psi}_2\) and \(\hat{\Psi}_3\) describe the light–like shell of matter (neutrino burst, for example) the reader must consult [1].

### 4 Discussion

We noted that (2.56)–(2.59) and (3.3) and (3.4) exhibit the conventional peeling behavior. Normally one associates the radiative part of the field \((\Psi_4\) throughout this paper) with a dominant \(\frac{1}{r}\) behavior. We see in (3.20) that this is not the case in the multipole example. On the other hand the radiative part (3.20) of that signal is due primarily to the jump in the quadrupole moment of the source across the light–like signal and this is something that would be expected. The general formulas (2.56)–(2.59) allow plenty of scope to construct further examples with unconventional peeling behavior. For example we could take \(M^+\) and \(M^-\) to be both Schwarzschild space–times with
masses $m_+$ and $m$. These space–times can be attached on $u = 0$ asymptotically (for large $r$) with the matching (2.27)–(2.30) with $f = x, g = y, h = 1$ and with (2.34) and (2.35) holding but with $\alpha_1 = \beta_1 = 0$. Now the stress–energy tensor on $u = 0$ is given by (2.47)–(2.55) with $a_2 = b_2 = \alpha_1 = \beta_1 = 0$. Thus for example
\begin{equation}
S^{33} = -\frac{1}{4\pi r^2} \left( [m] - \frac{1}{2} h_0 \right) + \ldots ,
\end{equation}
in this case. With $\hat{\Psi}_4$ calculated from (2.56)–(2.59) we see that in particular
\begin{equation}
\hat{\Psi}_4 = \frac{2}{r^2} \frac{\partial}{\partial z} \left( p_0 \frac{\partial h_0}{\partial z} \right) + O \left( \frac{1}{r^3} \right) .
\end{equation}
In general this wave will exhibit directional singularities. For example if $f_1$ and $g_1$ are constants then
\begin{equation}
\hat{\Psi}_4 = -\frac{1}{8 r^2} (f_1 - i g_1) \bar{z} + O \left( \frac{1}{r^3} \right) ,
\end{equation}
with $z = x + i y$.

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