Edgeworth Expansion of the Largest Eigenvalue Distribution Function of GOE

Leonard N. Choup
Department of Mathematical Sciences
University Alabama in Huntsville
Huntsville, AL 35899, USA
e-mail: Leonard.Choup@uah.edu

February 2, 2008

Abstract

In this paper we focus on the large $n$ probability distribution function of the largest eigenvalue in the Gaussian Orthogonal Ensemble of $n \times n$ matrices (GOE$_n$). We prove an Edgeworth type Theorem for the largest eigenvalue probability distribution function of GOE$_n$. The correction terms to the limiting probability distribution are expressed in terms of the same Painlevé II functions appearing in the Tracy-Widom distribution. We conclude with a brief discussion of the GSE$_n$ case.

1 Introduction

Limiting probability distributions laws from Random Matrix Theory have found many applications outside their initial domain of discovery; the length of the longest increasing subsequence (P. Deift et al. [1]) properly scaled converges in distribution to the Unitary Tracy-Widom law, the properly scaled largest principal component of a white Wishart converges in distribution to the Orthogonal Tracy-Widom law (I. M. Johnstone [14]). For recent reviews we refer the reader to [4, 5, 6, 13, 27]. In these applications of Tracy-Widom distributions, one would like to control quantitatively the range of validity of the various limit laws. One therefore needs finite $n$ correction to these limiting distributions. In a previous work [2] we initiated the study of this problem for the Gaussian Unitary Ensemble of $n$ by $n$ matrices (GUE$_n$). Following this work, we will derive the analogous result for the Gaussian Orthogonal Ensemble (GOE$_n$) in this paper. The derivation of the probability distribution function of the largest eigenvalue for Gaussian Symplectic Ensemble (GSE$_n$) is similar to the GOE$_n$ case up to the parity of the size $n$ of matrices in consideration. We will therefore mention at each step of the derivation the corresponding result without much explanations except when it is necessary. We seek a large $n$ expansion of the probability
distribution function of the largest eigenvalue in GOE, and GSE, similar to the following Edgeworth Expansion arising in probability in applications of the Central Limit Theorem.

If $S_n$ is a sum of i.i.d. random variables $X_j$, each with mean $\mu$ and variance $\sigma^2$, then the distribution $F_n$ of the normalized random variable $(S_n - n\mu)/(\sigma \sqrt{n})$ satisfies the Edgeworth expansion

$$F_n(x) - \Phi(x) = \phi(x) \sum_{j=3}^{r} n^{-\frac{j}{2}+1} R_j(x) + o(n^{-\frac{r}{2}+1})$$

uniformly in $x$. Here $\Phi$ is the standard normal distribution with density $\phi$, and $R_j$ are polynomials depending only on $E(X^k_j)$ but not on $n$ and $r$ (or the underlying distribution of the $X_j$).

If we view the random matrix ensembles of $n$ by $n$ matrices in terms of the associated eigenvalues, then the Gaussian $\beta$-ensembles are probability spaces on $n$-tuples of random variables $\{\lambda_1, \ldots, \lambda_n\}$ (think of them as eigenvalues of a randomly chosen matrix from the ensemble.) The probability density that the eigenvalues lie in an infinitesimal intervals about the points $x_1, \ldots, x_n$ is

$$P_{n\beta}(x_1, \ldots, x_n) = C_{n\beta} \exp \left( -\frac{\beta}{2} \sum_{j=1}^{n} x_j^2 \right) \prod_{i<j} |x_i - x_j|^\beta,$$

with

$$-\infty < x_i < \infty, \text{ for } i = 1, \ldots, n.$$

Here $C_{n\beta}$ is the normalizing constant such that the total integral over the $x_i$'s is one. The cases $\beta = 1, 2, 4$ correspond to the GOE, GUE and GSE respectively. We denote the largest eigenvalue by $\lambda_{\text{Max}}$, and

$$F_{n,\beta}(t) = P(\lambda_{\text{Max}} \leq t)$$

the probability distribution function.

When $\beta = 2$, the harmonic oscillator wave functions

$$\varphi_k(x) = \frac{1}{(2^kk!\sqrt{\pi})^{1/2}} H_k(x) e^{-x^2/2} \quad k = 0, 1, 2, \ldots$$

obtained by orthonormalizing the sequence $x^k e^{-x^2}$ (with $H_k(x)$ the Hermite polynomials of degree $k$) play an important role. We also have the Hermite kernel

$$K_{n,2}(x, y) = \sum_{k=0}^{n-1} \varphi_k(x) \varphi_k(y) = \sqrt{\frac{n}{2}} \frac{\varphi_n(x)\varphi_{n-1}(y) - \varphi_n(y)\varphi_{n-1}(x)}{x-y},$$

which is the kernel of an integral operator $K_2$ acting on $L^2(t, \infty)$, with resolvent

$$R_n(x, y; t) = (I - K_n)^{-1} K_n(x, y).$$

---

1 We assume, of course, the moments $E(X^k_j), k = 3, \ldots, r$, exist; and as well, the condition $\lim_{|\zeta| \to \infty} \sup |\wp(\zeta)| < \infty$ where $\wp$ is the characteristic function of $X_j$, see [7].
The product on the right is operator multiplication. We have the following representation of (1.2), (see for example, [16])

\[ P_{n_2}(x_1, \cdots, x_n) = \det(K_n(x_i, x_j))_{1 \leq i, j \leq n}. \]

Following Tracy and Widom [24], we define

\[ \varphi(x) = \left( \frac{n}{2} \right)^{\frac{1}{4}} \varphi_n(x), \quad \psi(x) = \left( \frac{n}{2} \right)^{\frac{1}{4}} \varphi_{n-1}(x), \] (1.7)

by \( \varepsilon \) the integral operator with kernel

\[ \varepsilon_t(x) = \frac{1}{2} \text{sgn}(x - t), \] (1.8)

\( D \) the differentiation with respect to the independent variable,

\[ Q_{n,i}(x; t) = \left( (I - K_n)^{-1}, x^i \varphi_n \right) \] (1.9)

and

\[ P_{n,i}(x; t) = \left( (I - K_n)^{-1}, x^i \varphi_{n-1} \right). \] (1.10)

We introduce the following quantities

\[ q_{n,i}(t) = Q_{n,i}(t; t), \quad p_{n,i}(t) = P_{n,i}(t; t) \] (1.11)

\[ u_{n,i}(t) = (Q_{n,i}, \varphi_n), \quad v_{n,i}(t) = (P_{n,i}, \varphi_n), \] (1.12)

\[ \tilde{v}_{n,i}(t) = (Q_{n,i}, \varphi_{n-1}), \quad \text{and} \quad w_{n,i}(t) = (P_{n,i}, \varphi_{n-1}). \] (1.13)

Here \((\cdot, \cdot)\) denotes the inner product on \( L^2(t, \infty) \). In our notation, the subscript without the \( n \) represents the scaled limit of that quantity when \( n \) goes to infinity, and we dropped the second subscript \( i \) when it is zero.

If \( Ai \) is Airy function, the kernel \( K_{n,2}(x, y) \) then scales\(^2\) to the Airy kernel

\[ K_{Ai}(X, Y) = \frac{Ai(X) Ai'(Y) - Ai(Y) Ai'(X)}{X - Y}. \] (1.14)

Our conventions are as follows:

\[ Q_i(x; s) = \left( (I - K_{Ai})^{-1}, x^i Ai \right), \quad Q_0(x; s) = Q(x; s), \] (1.15)

\[ P_i(x; s) = \left( (I - K_{Ai})^{-1}, x^i Ai' \right), \quad P_0(x; s) = P(x; s), \] (1.16)

\[ q_i(s) = Q_i(s; s), \quad q_0(s) = q(s), \quad p_i(s) = P_i(s; s), \quad p_0(s) = p(s), \] (1.17)

\[ u_i(s) = (Q_i, Ai), \quad u_0(s) = u(s), \quad v_i(s) = (P_i, Ai), \quad v_0(s) = v(s), \] (1.18)

\[ \tilde{v}_i(s) = (Q_i, Ai'), \quad \tilde{v}_0(s) = \tilde{v}(s), \quad w_i(t) = (P_i, Ai'), \quad \text{and} \quad w_0(s) = w(s). \] (1.19)

\(^2\)For the precise definition of this scaling, see the next section.
Here $(\cdot, \cdot)$ denotes the inner product on $L^2(s, \infty)$ and $i = 0, 1, 2$.

We also note that $q(s)$ is the solution to the Pailevé II equation $q''(s) = sq(s) + 2q^3(s)$ with the boundary condition $q(s) \sim Ai(s)$ as $s \to \infty$.

We use the subscript $n$ for unscaled quantities only.

\[ R_{n,1} := \int_{-\infty}^{t} R_n(x, t; t) dx, \quad P_{n,1} := \int_{-\infty}^{t} P_n(x; t) dx, \quad Q_{n,1} := \int_{-\infty}^{t} Q_n(x; t) dx, \quad (1.20) \]

and

\[ R_{n,4}(t) := \int_{-\infty}^{\infty} \varepsilon_t(x) R_n(x, t; t) dx, \quad P_{n,4}(t) := \int_{-\infty}^{\infty} \varepsilon_t(x) P_n(x; t) dx, \quad Q_{n,4}(t) := \int_{-\infty}^{\infty} \varepsilon_t(x) Q_n(x; t) dx. \quad (1.21) \]

The epsilon quantities are

\[ Q_{n,\varepsilon}(x; t) = ((I - K_n)^{-1}(x, y), \varepsilon\varphi(y)), \quad q_{n,\varepsilon}(t) = Q_{n,\varepsilon}(t; t) \quad (1.22) \]

\[ u_{n,\varepsilon}(t) = (Q_{n,\varepsilon}(x; t), \varphi(x)), \quad \tilde{v}_{n,\varepsilon}(t) = (Q_{n,\varepsilon}(x; t), \psi(x)), \quad (1.23) \]

where $(\cdot, \cdot)$ denotes the inner product on $L^2(t, \infty)$.

The GOE and GSE$_n$ analogue of (1.28) in Theorem 1.1 below will follow from representations (equations (40) and (41) of [21].)

\[ F_{n,1}(t)^2 = F_{n,2}(t) \cdot \left( (1 - \tilde{v}_{n,\varepsilon}(t))(1 - \frac{1}{2}R_{n,1}(t)) - \frac{1}{2}(q_{n,\varepsilon}(t) - c\varphi)P_{n,1}(t) \right) \quad (1.24) \]

and

\[ F_{n,4}(t/\sqrt{2})^2 = F_{n,2}(t) \cdot \left( (1 - \tilde{v}_{n,\varepsilon}(t))(1 + \frac{1}{2}R_{n,4}(t)) + \frac{1}{2}q_{n,\varepsilon}(t)P_{n,4}(t) \right). \quad (1.25) \]

Here we first derive a large $n$-expansion of $R_{n,1}$, $R_{n,4}$, $P_{n,4}$, $P_{n,4}$, $\tilde{v}_{n,\varepsilon}$, and $q_{n,\varepsilon}$ in terms of $p_n$ and $q_n$, by solving the associated systems of differential equations. We then substitute the resulting expressions in (1.24) and (1.25). We will need the following result which gives the large $n$ expansion of $F_{n,2}$.

**Theorem 1.1.** [2] If we set

\[ t = (2(n + c))^\frac{1}{2} + 2^{-\frac{1}{2}} n^{-\frac{1}{2}} s \quad \text{and} \quad (1.26) \]

\[ E_{c,2} := E_{c,2}(s) = 2w_1 - 3u_2 + (-20c^2 + 3)v_0 + u_1v_0 - u_0v_1 + u_0v_1^2 - u_0^2w_0. \quad (1.27) \]

Then as $n \to \infty$

\[ F_{n,2}(t) = F_2(s) \left\{ 1 + c u_0(s) n^{-\frac{1}{2}} - \frac{1}{20} E_{c,2}(s) n^{-\frac{3}{2}} \right\} + O(n^{-1}) \quad (1.28) \]

uniformly in $s$, and

\[ F_2(s) = \lim_{n \to \infty} F_{n,2}(t) = \exp \left( - \int_s^\infty (x-s)q(x)^2 \, dx \right) \quad (1.29) \]

is the Tracy-Widom distribution.
1.1 Statement of our results

To state our main result we need the following definitions

\[ \alpha := \alpha(s) = \int_s^\infty q(x) u(x) \, dx, \]

\[ \mu := \mu(s) = \int_s^\infty q(x) \, dx, \]

\[ \nu := \nu(s) = \int_s^\infty p(x) \, dx = \alpha(s) - q(s), \]

\[ \eta := \eta(s) = \frac{1}{20\sqrt{2}} \int_s^\infty (6qv + 3pu + 2p_2 + 2p_1v + 2pv - 2q_2u - 2q_1u_1 - 2qu_2) \, dx - \frac{20c^2q'(s) + 3p(s)}{20\sqrt{2}}. \]

\[ E_{c,1}(s) = -\frac{1}{20} E_{c,2}(s) e^{-\mu} - \frac{c\alpha}{2\mu^2} + \frac{cp}{2\mu} + \frac{(2c - 1)\nu^2}{4\mu} + cu \left( c q e^{-\mu} - \frac{\nu}{2\mu} (1 - e^{-\mu}) \right) + e^{-2\mu} \left( \frac{\nu (\nu + 8c q)}{32\mu} - \frac{\eta}{4\sqrt{2}} \right) + e^{-\mu} \left( \frac{2\sqrt{2}c^2q^2 - 3\eta}{4\sqrt{2}} + \frac{\nu^2 - 8(2cp + c^2q^2) - 4c^2\alpha^2}{32\mu} \right) - \frac{c^2q^2}{8\mu^2} + \frac{2 - \mu}{2\mu^2} \left( c q \alpha + \frac{1}{4} \nu^2 + (c^2 - c)q^2 \right) - (4c^2\alpha^2 + 3c^2q^2 - \nu^2) \frac{\cosh(\mu)}{8\mu^2}. \]

Unfortunately, we did not find a simple representation of \( \eta \) and \( E_{c,1} \). Nevertheless the quantities \( \alpha(s), \mu(s), \nu(s), \eta(s) \) and \( E_{c,1} \) are easy to compute numerically. For \( E_{c,1} \) and \( \eta \) we only need the recurrences relations defining \( p_i(s) \) and \( q_i(s) \) in term of \( q(s) \) and \( q'(s) \). We find the following representation of the large \( n \) probability distribution function for \( \lambda_{\text{max}} \) in GOE\(_n\). The derivation of the analogous result for the GSE\(_n\) follows exactly the same steps as the one given in this paper. Here is our main result.

**Theorem 1.2.** We set

\[ t = (2(n + c))^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{2}} s. \]

Then as \( n \to \infty \)

\[ F_{n,1}(t)^2 = F_2(s) \cdot \left\{ e^{-\mu(s)} + \left[ c(q(s) + u(s))e^{-\mu(s)} - \frac{\nu(s)}{2\mu(s)} (1 - e^{-\mu(s)}) \right] n^{-\frac{1}{2}} + E_{c,1} n^{-\frac{3}{4}} \right\} + O(n^{-1}) \]

uniformly for bounded \( s \).
Note that unlike the GUE case where for \( c = 0 \) the \( n^{-\frac{1}{3}} \) correction term vanishes as shown in equation (1.28), the \( n^{-\frac{1}{3}} \) correction term does not vanish in the GOE no matter what the fine tuning constant \( c \) is.

In §2 we reproduce the derivation of (1.24) following Tracy and Widom in [24]. In §3 we solve the system of equations satisfied by the various functions on the right of (1.24) for our derivation of the Edgeworth expansion of the probability distribution of the largest eigenvalue in the GOE.

2 Derivation of \( F_{n,\beta} \)

We treat here the case \( n \) even. If we set

\[
K_{n,1} = \begin{pmatrix}
K_{n,2} + \psi \otimes \varepsilon \varphi & K_{n,2} D - \psi \otimes \varphi \\
\varepsilon K_{n,2} - \varepsilon + \varepsilon \psi \otimes \varepsilon \varphi & K_{n,2} + \varepsilon \psi \otimes \varphi
\end{pmatrix}
\]  

(2.1)

then \( F_{n,1}^2(t) \) is the Fredholm determinant of \( K_{n,1} \) on the set \( J = (t, \infty) \). If we denote by \( \chi \) the multiplication by the function \( \chi_J(x) \), then \( F_{n,1}^2(t) \) is the Fredholm determinant of the integral operator with kernel

\[
K_{n,1} = \chi_J(x) \begin{pmatrix}
K_{n,2} + \psi \otimes \varepsilon \varphi & K_{n,2} D - \psi \otimes \varphi \\
\varepsilon K_{n,2} - \varepsilon + \varepsilon \psi \otimes \varepsilon \varphi & K_{n,2} + \varepsilon \psi \otimes \varphi
\end{pmatrix} \chi_J(y)
\]  

(2.2)

on \( \mathbb{R} \), see for example [24] equation (31). Using the following commutators,

\[
[K_{n,2},D] = \varphi \otimes \psi + \psi \otimes \varphi, \quad [\varepsilon, K_{n,2}] = -\varepsilon \varphi \otimes \varepsilon \psi - \varepsilon \psi \otimes \varepsilon \varphi \]  

(2.3)

(\( \psi \) and \( \varphi \) appear as a consequence of the Christoffel Darboux formula applied to \( K_{n,2} \)) we have

\[
K_{n,2} + \psi \otimes \varphi = D \varepsilon K_{n,2} + D \varepsilon \psi \otimes \varepsilon \varphi = D(\varepsilon K_{n,2} + \varepsilon \psi \otimes \varepsilon \varphi) = D(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi)
\]

\[
K_{n,2} D - \psi \otimes \varepsilon = D K_{n,2} + \varphi \otimes \psi = D K_{n,2} + D \varepsilon \psi \otimes \varepsilon = D(K_{n,2} + \varepsilon \varphi \otimes \psi)
\]

and

\[
\varepsilon K_{n,2} - \varepsilon + \varepsilon \psi \otimes \varepsilon \varphi = K_{n,2} \varepsilon - \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi
\]

as \( D \varepsilon = I \). Our kernel is now

\[
K_{n,1} = \chi \begin{pmatrix}
D(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) & D(K_{n,2} + \varepsilon \varphi \otimes \psi) \\
K_{n,2} \varepsilon - \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi & K_{n,2} + \varepsilon \psi \otimes \varphi
\end{pmatrix} \chi
\]  

(2.4)

\[
= \begin{pmatrix}
\chi D & 0 \\
0 & \chi
\end{pmatrix} \begin{pmatrix}
(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi & (K_{n,2} + \varepsilon \varphi \otimes \psi) \chi \\
(K_{n,2} \varepsilon - \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi & (K_{n,2} + \varepsilon \psi \otimes \varphi) \chi
\end{pmatrix}
\]  

(2.5)

3To simplify notation we kept the same notation for the integral operator as well as the kernel.
Since $K_{n,1}$ is of the form $AB$, we can use the fact that $\det(I-AB) = \det(I-BA)$ and deduce that the Fredholm determinant of $K_{n,1}$ is unchanged if instead we take $K_{n,1}$ to be

\[
\begin{pmatrix}
(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi) \chi & (K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi \\
(K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon ) \chi & (K_{n,2} + \varepsilon \varphi \otimes \varepsilon \varphi ) \chi
\end{pmatrix}
\begin{pmatrix}
\chi D & 0 \\
0 & \chi
\end{pmatrix}
\]

(2.6)

\[
\det(I - K_{n,1}) = \det \left( I - (K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi ) \chi D - (K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi - (K_{n,2} + \varepsilon \varphi \otimes \varepsilon \varphi ) \chi \\
\varepsilon \chi D
\right). \tag{2.7}
\]

Performing row and column operations on the matrix\footnote{This does not change the determinant, for more details see [24]} does not change the Fredholm determinant. We first subtract row 1 from row 2, next we add column 2 to column 1 to have the following matrix

\[
\begin{pmatrix}
I - (K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi ) \chi D - (K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi & - (K_{n,2} + \varepsilon \varphi \otimes \varepsilon \varphi ) \chi \\
\varepsilon \chi D
\end{pmatrix}
\]

(2.9)

Right-multiply column 2 by $-\varepsilon \chi D$ and add it to column 1, and multiply row 2 by $(K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi$ and add it to row 1 to have

\[
\begin{pmatrix}
I - (K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi ) \chi D - (K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi & (K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi \varepsilon \chi D - (K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi \\
0 & \chi
\end{pmatrix} \tag{2.10}
\]

We therefore have,

\[
\det(I - K_{n,1}) = \det \left( I - (K_{n,2} \varepsilon - \varepsilon \varphi \otimes \varepsilon \psi ) \chi D + (K_{n,2} + \varepsilon \varphi \otimes \varepsilon ) \chi (\varepsilon \chi D - I) \right)
\]

(2.11)

\[
= \det \left( I - K_{n,2} \chi - K_{n,2} (I - \chi) \varepsilon \chi D - (\varepsilon \varphi \otimes \varepsilon ) (\chi - \chi \varepsilon \chi D) - \varepsilon \varphi \otimes \psi \varepsilon \chi D \right). \tag{2.12}
\]

We used the fact that $\varepsilon$ is antisymmetric to have

\[
\varepsilon \varphi \otimes \varepsilon \psi \chi D = \varepsilon \varphi \otimes \varepsilon \psi \chi D = -\varepsilon \varphi \otimes \psi \varepsilon \chi D,
\]

and if we note that $\chi$ is multiplication, then the determinant is

\[
= \det \left( I - K_{n,2} \chi - K_{n,2} (I - \chi) \varepsilon \chi D - \varepsilon \varphi \otimes \psi (1 - \chi) \varepsilon \chi D - \varepsilon \varphi \otimes \chi \psi \right).
\]

Next we factor out $I - K_{n,2}$ and note that $(I - K_{n,2})^{-1} = I + R_{n,2}$, where $R_{n,2}$ was defined as the resolvent of $K_{n,2}$, and $(I - K_{n,2})^{-1} \varepsilon \varphi = Q_{n,\varepsilon}$. We are interested on the determinant of the following operator

\[
(I - K_{n,2} \chi) (I - (K_{n,2} + R_{n,2} K_{n,2}) (I - \chi) \varepsilon \chi D - Q_{n,\varepsilon} \otimes \psi (1 - \chi) \varepsilon \chi D - Q_{n,\varepsilon} \otimes \chi \psi). \tag{2.13}
\]

We have a large $n$-expansion of $\det(I - K_{n,2} \chi) = F_{n,2}$ from the author work in GUE$_n$ see [2] or [3]. We will therefore focus our attention on the second factor of
We will represent this factor in the form \((I - \sum_{j=1}^{k} \alpha_j \otimes \beta_j)\) and use the well known formula \(\det(I - \sum_{j=1}^{k} \alpha_j \otimes \beta_j) = \det(\delta_{ij} - (\alpha_i, \beta_j))_{i,j=1,\ldots,k}\) to expand the Fredholm determinant. First we need to find a representation of \(\varepsilon \chi D\) as a finite rank operator. To this end we introduce in this section the following notation,

\[\varepsilon_k(x) = \varepsilon(x-a_k), \quad R_k(x) = R_{n,2}(x,a_k), \quad \delta_k(x) = \delta(x-a_k), \quad a_1 = t, \quad \text{and} \quad a_2 = \infty.\]

With the new notation \(J = (t, \infty) = (a_1, a_2)\), and the commutator

\[ [\chi D] = -\delta_1 \otimes \delta_1 + \delta_2 \otimes \delta_2, \]

we have

\[\varepsilon[\chi D] = -\varepsilon_1 \otimes \delta_1 + \varepsilon_2 \otimes \delta_2.\]

Next we use the identity \(\varepsilon D = I\) to have

\[ (I - \chi)\varepsilon \chi D = (I - \chi)\varepsilon[\chi D] = -(I - \chi)\varepsilon_1 \otimes \delta_1 + (I - \chi)\varepsilon_2 \otimes \delta_2, \tag{2.14} \]

and the representation

\[ (K_{n,2} + R_{n,2} K_{n,2})(I - \chi)\varepsilon \chi D = \sum_{k=1,2} (-1)^k (K_{n,2} + R_{n,2} K_{n,2}) (I - \chi)\varepsilon_k \otimes \delta_k. \tag{2.15} \]

We substitute (2.14) and (2.15) in the second factor of (2.13) and have,

\[ I - \sum_{k=1,2} (-1)^k (K_{n,2} + R_{n,2} K_{n,2})(I - \chi)\varepsilon_k \otimes \delta_k - \sum_{k=1,2} (-1)^k Q_{n,\varepsilon} \otimes \psi \cdot (1 - \chi)\varepsilon_k \otimes \delta_k - Q_{n,\varepsilon} \otimes \chi \psi. \tag{2.16} \]

The dot in this formula represent operator multiplication. In this case we just multiply the kernels using the formula \((\alpha \otimes \beta)(\gamma \otimes \delta) = (\beta, \gamma)\alpha \otimes \delta\), to have the following form of (2.16)

\[ I - \sum_{k=1,2} (-1)^k (K_{n,2} + R_{n,2} K_{n,2})(I - \chi)\varepsilon_k \otimes \delta_k - \sum_{k=1,2} (-1)^k (\psi, (I - \chi)\varepsilon_k) Q_{n,\varepsilon} \otimes \delta_k - Q_{n,\varepsilon} \otimes \chi \psi. \tag{2.17} \]

We have

\[ \varepsilon_2 = -\frac{1}{2}, \quad (I - \chi)\varepsilon_1 = (I - \chi)\varepsilon_2 = -\frac{1}{2}, \quad \text{and} \quad R_2 = R_{n,2}(x, \infty) = 0.\]

If we substitute these value in (2.17), it then becomes,

\[ I - Q_{n,\varepsilon} \otimes \chi \psi - \frac{1}{2} [ (k_{n,2} + R_{n,2} K_{n,2})(I - \chi) + (\psi, (I - \chi) Q_{n,\varepsilon}) ] \otimes (\delta_1 - \delta_2). \tag{2.18} \]

This operator is of the desired form

\[ I - \sum_{k=1,2} \alpha_k \otimes \beta_k \]
with
\[ \alpha_1 = Q_{n,\varepsilon}, \quad \alpha_2 = \frac{1}{2} [(k_{n,2} + R_{n,2} K_{n,2})(I - \chi) + a_1 Q_{n,\varepsilon}], \quad \beta_1 = \chi \psi, \quad \beta_2 = \delta_1 - \delta_2, \quad (2.19) \]
and
\[ a_1 = \langle \psi, (I - \chi) \rangle. \]
The corresponding inner product are;
\[ (\alpha_1, \beta_1) = \tilde{v}_{n,\varepsilon}, \quad (\alpha_1, \beta_2) = q_{n,\varepsilon} + c_\varphi \quad (2.20) \]
with
\[ c_\varphi = \varepsilon \varphi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi(x) \, dx, \quad c_\psi = \varepsilon \psi(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(x) \, dx, \quad (2.21) \]
and for \( n \) even
\[ c_\varphi = (\pi n)^{1/4} 2^{-3/4 - n/2} (n!)^{1/2} \left( \frac{n}{2} \right)! , \quad (2.22) \]
and
\[ (\alpha_2, \beta_1) = \frac{1}{2} (P_{n,1} - a_1 + a_1 \tilde{v}_{n,\varepsilon}), \quad (\alpha_2, \beta_2) = \frac{1}{2} (R_{n,1} + a_1 q_{n,\varepsilon} - a_1 c_\varphi). \quad (2.23) \]
The determinant of (2.17) is therefore
\[ (1 - \tilde{v}_{n,\varepsilon})(1 - \frac{1}{2} R_{n,1}) - \frac{1}{2} (q_{n,\varepsilon} - c_\varphi) P_{n,1}. \quad (2.24) \]
In a similar way we obtain the second factor on the right side of (1.25) for the GSE\(_n\) case
\[ (1 - \tilde{v}_{n,\varepsilon}) (1 + \frac{1}{2} R_{n,4}) + \frac{1}{2} q_{n,\varepsilon}(t) P_{n,4}(t). \quad (2.25) \]
We will derive differential equations involving the various terms in equation (2.24) and solutions in terms of \( q_n \) and \( p_n \). Then used known asymptotic of \( q_n \) and \( p_n \) for the derivation of a large \( n \) expansion of \( F_{n,1} \).

3 Differential Equations

3.1 System of Differential Equations
This section will be devoted to finding expressions of
\[ R_{n,1}(t), \quad P_{n,1}(t), \quad q_{n,\varepsilon}(t), \quad \text{and} \quad \tilde{v}_{n,\varepsilon}(t), \]
and show how to obtain the corresponding quantities
\[ R_{n,4}(t), \quad \text{and} \quad P_{n,4}(t) \]
To solve the associated system of differential equations it is convenient to introduce the following quantities

\[ Q_n(t), \quad u_{n,\varepsilon}(t), \]

and

\[ \rho_{n,2}(x, y) \quad \text{the kernel of} \quad (I - K_{n,2} \chi)^{-1}. \]  

We have

\[ \rho_{n,2}(x, y) = \delta(x - y) + R_{n,2}(x, y), \]  

and

\[ \frac{d}{dt} R_{n,1}(t) = \frac{d}{dt} \int_{-\infty}^{t} R_{n,2}(x, t) \, dx = R_{n,2}(t, t) + \int_{-\infty}^{t} \frac{d}{dt} R_{n,2}(x, t) \, dx. \]  

Formula (45) of [24] gives

\[ \frac{d}{dt} R_{n,2}(x, t) = -\frac{d}{dx} R_{n,2}(x, t) - p_{n}(t) Q_{n}(x; t) - q_{n} P_{n}(x; t), \]  

and we find that

\[ R'_{n,1}(t) = \frac{d}{dt} R_{n,1}(t) = -p_{n}(t) Q_{n,1}(t) - q_{n}(t) P_{n,1}(t). \]  

We also have

\[ Q'_{n,1}(t) = \frac{d}{dt} Q_{n,1}(t) = \frac{d}{dt} \int_{-\infty}^{t} Q_{n}(x; t) \, dx = q_{n}(t)(1 - R_{n,1}(t)), \]  

and

\[ P'_{n,1}(t) = \frac{d}{dt} P_{n,1}(t) = \frac{d}{dt} \int_{-\infty}^{t} P_{n}(x; t) \, dx = p_{n}(t)(1 - R_{n,1}(t)), \]  

where we used

\[ \frac{d}{dt} Q_{n}(x; t) = -q_{n}(t) R_{n,2}(x, t), \quad \text{and} \quad \frac{d}{dt} P_{n}(x; t) = -p_{n}(t) R_{n,2}(x, t). \]  

The other derivatives are

\[ \frac{d}{dt} u_{n,\varepsilon}(t) = \frac{d}{dt} \int_{t}^{\infty} Q_{n}(x, t) \varepsilon \varphi(x) \, dx = -q_{n}(t) \varepsilon \varphi(t) + \int_{t}^{\infty} \frac{d}{dt} Q_{n}(x, t) \varepsilon \varphi(x) \, dx \]  

\[ = -q_{n}(t) \varepsilon \varphi(t) + \int_{t}^{\infty} \frac{d}{dt} R_{n,2}(x, t) \varepsilon \varphi(x) \, dx = -q_{n}(t) \int_{t}^{\infty} \rho_{n,2}(x, t) \varepsilon \varphi(x) \, dx, \]  

and therefore

\[ u'_{n,\varepsilon}(t) = -q_{n}(t) q_{n,\varepsilon}(t). \]  

Similarly

\[ \frac{d}{dt} v_{n,\varepsilon}(t) = \frac{d}{dt} \int_{t}^{\infty} P_{n}(x, t) \varepsilon \varphi(x) \, dx = -p_{n}(t) \varepsilon \varphi(t) + \int_{t}^{\infty} \frac{d}{dt} P_{n}(x, t) \varepsilon \varphi(x) \, dx, \]
or
\[ \tilde{v}'_{n,\varepsilon}(t) = -p_n(t) q_{n,\varepsilon}(t). \] (3.13)

The last of these is
\[ \frac{d}{dt} q_{n,\varepsilon}(t) = \frac{d}{dt} \int \rho_{n,2}(t,y) \varepsilon \varphi(y) dy \] (3.14)
\[ = - \int \frac{\partial}{\partial y} \rho_{n,2}(t,y) \varepsilon \varphi(y) dy - q_n(t) \left( \chi P_n(y; t) \varepsilon \varphi(y) \right) - p_n(t) \left( \chi Q_n(y; t) \varepsilon \varphi(y) \right). \] (3.15)

Integration by parts together with the boundary conditions and \( D\varepsilon = I \) gives
\[ - \int \frac{\partial}{\partial y} \rho_{n,2}(t,y) \varepsilon \varphi(y) dy = \int \rho_{n,2}(t,y) \chi \varepsilon \varphi(y) dy = q_n(t), \] (3.16)
which in turn gives
\[ q_{n,\varepsilon}'(t) = q_n(t) - \tilde{v}_{n,\varepsilon}(t) q_n(t) - u_{n,\varepsilon}(t) q_n(t). \] (3.17)

The boundary conditions at \( t = \infty \) for these function are,
\[ \mathcal{R}_{n,1}(\infty) = 0, \quad \mathcal{Q}_{n,1}(\infty) = 2c_\varphi \quad \text{and} \quad \mathcal{P}_{n,1}(\infty) = 2c_\psi = 0 \quad \text{as} \ n \ \text{is even}. \] (3.18)

and
\[ \tilde{v}_{n,\varepsilon}(\infty) = 0, \quad u_{n,\varepsilon}(\infty) = 0, \quad \text{and} \quad q_{n,\varepsilon}(\infty) = c_\varphi. \] (3.19)

The associated systems of equations are;
\[ \begin{cases} q_{n,\varepsilon}'(t) = q_n(t) \left( 1 - \tilde{v}_{n,\varepsilon}(t) \right) - p_n(t) u_{n,\varepsilon}(t); \\ (1 - \tilde{v}_{n,\varepsilon})' = p_n(t) q_{n,\varepsilon}(t); \\ u_{n,\varepsilon}'(t) = -q_n(t) q_{n,\varepsilon}(t). \end{cases} \] (3.20)

and
\[ \begin{cases} (1 - \mathcal{R}_{n,1})' = p_n(t) \mathcal{Q}_{n,1}(t) + q_n(t) \mathcal{P}_{n,1}(t); \\ \mathcal{Q}_{n,1}' = q_n(t) \left( 1 - \mathcal{R}_{n,1}(t) \right); \\ \mathcal{P}_{n,1}(t) = p_n(t) \left( 1 - \mathcal{R}_{n,1}(t) \right). \end{cases} \] (3.21)

### 3.2 Asymptotic solutions

We will define in this subsection only
\[ V_{n,\varepsilon}(t) = 1 - \tilde{v}_{n,\varepsilon}(t), \quad \text{and} \quad \bar{\mathcal{R}}_{n,1}(t) = 1 - \mathcal{R}_{n,1}(t). \] (3.22)

With this notation, system (3.20) is
\[ \frac{d}{dt} \begin{pmatrix} u_{n,\varepsilon}(t) \\ V_{n,\varepsilon}(t) \\ q_{n,\varepsilon}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -q_n(t) \\ 0 & 0 & p_n(t) \\ -p_n(t) & q_n(t) & 0 \end{pmatrix} \begin{pmatrix} u_{n,\varepsilon}(t) \\ V_{n,\varepsilon}(t) \\ q_{n,\varepsilon}(t) \end{pmatrix}. \] (3.23)
and \(3.21\) is
\[
\frac{d}{dt} \begin{pmatrix}
Q_{n,1}(t) \\
\mathcal{P}_{n,1}(t) \\
\tilde{\mathcal{R}}_{n,1}(t)
\end{pmatrix} = \begin{pmatrix}
0 & 0 & q_n(t) \\
0 & 0 & p_n(t) \\
p_n(t) & q_n(t) & 0
\end{pmatrix} \cdot \begin{pmatrix}
Q_{n,1}(t) \\
\mathcal{P}_{n,1}(t) \\
\tilde{\mathcal{R}}_{n,1}(t)
\end{pmatrix}.
\]
(3.24)

If we let
\[
X' = (u_{n,e}(t), v_{n,e}(t), q_{n,e}(t)) \quad \text{and} \quad Y' = (Q_{n,1}(t), \mathcal{P}_{n,1}(t), \tilde{\mathcal{R}}_{n,1}(t)),
\]
then \(3.20\) and \(3.21\) have the following representations
\[
X'(t) = A(t) X(t) \quad \text{and} \quad Y'(t) = B(t) Y(t)
\]
(3.25)
with
\[
X'(\infty) = (0, 1, c_\varphi), \quad \text{and} \quad Y'(\infty) = (2c_\varphi, 0, 1).
\]
(3.26)

We note that \(A(t)\) is continuous for \(t\) bounded away from \(-\infty\). We need to show that our matrix \(A(t)\) is bounded as an operator on \(L^1(t, \infty) \otimes L^1(t, \infty) \otimes L^1(t, \infty)\) for this end we will use the Max norm. The entries of \(A(t)\) are \(\pm q_n(x)\) and \(\pm p_n(x)\).

\[
\int_t^\infty |q_n(x)| \, dx = \frac{1}{\sqrt{2}} \int_s^\infty \left| q(x) + f(x)n^{-\frac{3}{2}} \right| \, dx = M_1 \quad \text{with} \quad M_1 < \infty
\]
(3.27)

We made use of the following change of variables together with formula \((2.29)\) of [3]
\[
x = \sqrt{2n} + \frac{X}{\sqrt{2n^2}}, \quad \text{and} \quad t = \sqrt{2n} + \frac{s}{\sqrt{2n^2}},
\]
(3.28)
the fact that the asymptotics for \(p\) at infinity can be obtained from the following representation \(p = q + uq\) where \(u(\infty) = 0\) or that \(u(x)\) is bounded for \(x\) away from minus infinity. We also assumed without lost of generalities for this section that \(p(x) \sim \text{Const} \cdot x^{1/4} e^{-\frac{3}{8}x^{1/2}}\) as \(x \to \infty\) which is a consequence of the asymptotics \(q(x) \sim \frac{1}{2\sqrt{\pi x}} e^{-\frac{3}{8}x^{1/2}}\) as \(x \to \infty\). We also remarked that the scaled value of \(q_n\) in \((2.29)\) of [3] is represented in terms of finite combinations of bounded functions (the \(u_i, v_i, w_i, 1, 2, \ldots\) ) \(x\), \(i = 0, 1, 2\), with \(p\) and \(q\). So again we assumed that the scaled value of \(q_n(x)\) was of order \(n^{\frac{1}{2}, x^2} e^{-\frac{3}{8}x^{1/2}}\) as \(x \to \infty\). A similar argument hold for \(p_n\),

here we use formula \((2.30)\) of [3] instead.

\[
\int_t^\infty |p_n(x)| \, dx = \frac{1}{\sqrt{2}} \int_s^\infty \left| q(x) + g(x)n^{-\frac{3}{2}} \right| \, dx = M_2 \quad \text{with} \quad M_2 < \infty.
\]
(3.29)

Note that \(||A||_{\text{Max}}\) and \(||B||_{\text{Max}}\) are at most \(2M_1 + 2M_2\). The fundamental local existence of solution for linear Ordinary Differential Equation says that equations \((3.25)\) have solutions in \((a, \infty)\) with \(a\) bounded away from infinity given by

\[
X(t) = \exp \left( - \int_t^\infty A(x) \, dx \right) \cdot X(\infty), \quad \text{and} \quad Y(t) = \exp \left( - \int_t^\infty B(x) \, dx \right) \cdot Y(\infty).
\]
(3.30)
This solution is convenient for the large \( n \) expansion of the probability distribution since it allows us to give a series expansion of the solutions of our solution in terms of \( q_n \) and \( p_n \). The other advantage is the built in symmetries in matrices \( A \) and \( B \). These symmetries make the computation of the matrix exponential very easy. We will start with the first system \( X(t) = \exp \left( - \int_t^\infty A(x)dx \right) \cdot X(\infty) \). We set

\[
\exp \left( - \int_t^\infty A(x)dx \right) = \exp \left\{ \begin{pmatrix} 0 & 0 & \int_t^\infty q_n(x) \, dx \\ 0 & 0 & -\int_t^\infty p_n(x) \, dx \\ \int_t^\infty p_n(x) \, dx & -\int_t^\infty q_n(x) \, dx & 0 \end{pmatrix} \right\}
= \exp(M).
\]

\( M \) is of the form

\[
M = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & -b \\ b & -a & 0 \end{pmatrix}
\]

with

\[
\exp(M) = \begin{pmatrix} 1 + \sum_{k \geq 1} \frac{2^{k-1}a^{k}b^{k}}{(2k)!} & -\sum_{k \geq 1} \frac{2^{k-1}a^{k+1}b^{k-1}}{(2k)!} & \sum_{k \geq 0} \frac{2^{k}a^{k+1}b^{k}}{(2k+1)!} \\ -\sum_{k \geq 1} \frac{2^{k-1}a^{k-1}b^{k+1}}{(2k)!} & 1 + \sum_{k \geq 1} \frac{2^{k-1}a^{k}b^{k}}{(2k)!} & -\sum_{k \geq 0} \frac{2^{k}a^{k}b^{k+1}}{(2k+1)!} \\ \sum_{k \geq 0} \frac{2^{k}a^{k}b^{k+1}}{(2k+1)!} & -\sum_{k \geq 0} \frac{2^{k}a^{k}b^{k}}{(2k+1)!} & 1 + \sum_{k \geq 1} \frac{2^{k}a^{k}b^{k}}{(2k+1)!} \end{pmatrix}
\]

\[
= \begin{pmatrix} \exp(M)_{11} & \exp(M)_{12} & \exp(M)_{13} \\ \exp(M)_{21} & \exp(M)_{22} & \exp(M)_{23} \\ \exp(M)_{31} & \exp(M)_{32} & \exp(M)_{33} \end{pmatrix}
\]

We have

\[
u_{n,\varepsilon}(t) = \exp(M)_{12} + c_\varphi \exp(M)_{13},
\]

\[
V_{n,\varepsilon}(t) = \exp(M)_{22} + c_\varphi \exp(M)_{23},
\]

and

\[
q_{n,\varepsilon}(t) = \exp(M)_{32} + c_\varphi \exp(M)_{33}.
\]

### 3.2.1 Scaling

At this point we scale the functions involved in (3.33) in terms of \( n \) at the point corresponding to the expected value of the largest eigenvalue. If we set

\[
t = \tau(s) = \sqrt{2(n + c)} + \frac{s}{2\pi n^\frac{3}{2}},
\]

then equations (2.29) and (2.30) of [3] are

\[
q_n(\tau(s)) = Q_n(\tau(s); \tau(s)) = n^\frac{1}{3} \left( q(s) + \left[ \frac{2c - 1}{2} p(s) - c q(s) u(s) \right] n^\frac{1}{3} \right).
\]
\[
\begin{align*}
+ & \left[ (10c^2 - 10c + \frac{3}{2})q_1(s) + p_2(s) + (-30c^2 + 10c + \frac{3}{2})q(s)u(s) \\
+ & p_1(s)v(s) + p(s)v_1(s) - q_2(s)u(s) - q_1(s)u_1(s) - q(s)u_2(s) \\
+ & (-10c^2 + \frac{3}{2})p(s)u(s) + 20c^2q(s)u^2(s) \right] \frac{n^{-\frac{5}{3}}}{20} + O(n^{-1})e_q(s) , \quad (3.38)
\end{align*}
\]

and
\[
\begin{align*}
p_n(\tau(s)) = P_n(\tau(s); \tau(s)) = n^\frac{1}{3} \left( q(s) + \left[ \frac{2c + 1}{2}p(s) - cq(s)u(s) \right] n^\frac{1}{3} \\
+ & \left[ (10c^2 + 10c + \frac{3}{2})q_1(s) + p_2(s) + (-30c^2 - 10c + \frac{3}{2})q(s)v(s) \\
+ & p_1(s)v(s) + p(s)v_1(s) - q_2(s)u(s) - q_1(s)u_1(s) - q(s)u_2(s) \\
+ & (-10c^2 + \frac{3}{2})p(s)u(s) + 20c^2q(s)u^2(s) \right] \frac{n^{-\frac{5}{3}}}{20} + O(n^{-1})e_p(s) \right) , \quad (3.39)
\end{align*}
\]

If we change the variable in \(a\) and \(b\) by setting \(x := \tau(x)\), we obtain
\[
\begin{align*}
a = & \int_t^\infty q_n(x) \, dx = \frac{1}{\sqrt{2}} \int_s^\infty \left( q(x) + \left[ \frac{2c - 1}{2}p(x) - cq(x)u(x) \right] n^{\frac{1}{3}} \\
+ & \left[ (10c^2 - 10c + \frac{3}{2})q_1(x) + p_2(x) + (-30c^2 + 10c + \frac{3}{2})q(x)v(x) \\
+ & p_1(x)v(x) + p(x)v_1(x) - q_2(x)u(x) - q_1(x)u_1(x) - q(x)u_2(x) \\
+ & (-10c^2 + \frac{3}{2})p(x)u(x) + 20c^2q(x)u^2(x) \right] \frac{n^{-\frac{5}{3}}}{20} + O(n^{-1})e_q(x) \right) \, dx , \quad (3.40)
\end{align*}
\]

and
\[
\begin{align*}
b = & \int_t^\infty p_n(x) \, dx = \frac{1}{\sqrt{2}} \int_s^\infty \left( q(x) + \left[ \frac{2c + 1}{2}p(x) - cq(x)u(x) \right] n^{\frac{1}{3}} \\
+ & \left[ (10c^2 + 10c + \frac{3}{2})q_1(x) + p_2(x) + (-30c^2 - 10c + \frac{3}{2})q(x)v(x) \\
+ & p_1(x)v(x) + p(x)v_1(x) - q_2(x)u(x) - q_1(x)u_1(x) - q(x)u_2(x) \\
+ & (-10c^2 + \frac{3}{2})p(x)u(x) + 20c^2q(x)u^2(x) \right] \frac{n^{-\frac{5}{3}}}{20} + O(n^{-1})e_p(x) \right) \, dx \quad (3.41)
\end{align*}
\]

\[a_0(s) + a_1(s)n^{-1/3} + a_2(s)n^{-2/3} + a_3(s)n^{-1}\]

\[b_0(s) + b_1(s)n^{-1/3} + b_2(s)n^{-2/3} + b_3(s)n^{-1}\]

\[\text{We use the same letter in both sides in the change here to simplify notation.}\]
We next focus on the following expression

\[ a^k b^k = (ab)^k = (a_0^2 + a_0(a_1 + b_1)n^{-\frac{1}{3}} + (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1})^k. \]

An expansion of this expression is

\[ a^k b^k = (ab)^k = a_0^{2k} + k a_0^{2k-2} (a_0(a_1 + b_1)n^{-\frac{1}{3}} + (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1}) + \frac{k(k-1)}{2} a_0^{2k-4} \left(a_0(a_1 + b_1)n^{-\frac{1}{3}} + (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1}\right)^2 + \sum_{i=3}^{k} \binom{k}{i} a_0^{2k-2i} (a_0(a_1 + b_1)n^{-\frac{1}{3}} + (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1})^i. \]

If we note that for \( i \geq 3 \)

\[ \left(a_0(a_1 + b_1)n^{-\frac{1}{3}} + (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1}\right)^i = O(n^{-1}), \]

then the sum in this last term can be represented as

\[ \sum_{i=3}^{k} \binom{k}{i} a_0^{2k-2i} O(n^{-1}) = \left( \sum_{i=3}^{k} \binom{k}{i} a_0^{2k-2i} \right) + a_0^{2k} + k a_0^{2k-2} + \frac{k(k-1)}{2} a_0^{2k-4} - a_0^{2k} - k a_0^{2k-2} \]

\[ - \frac{k(k-1)}{2} a_0^{2k-4} O(n^{-1}) = \left( (a_0^2 + 1)^k - a_0^{2k} - k a_0^{2k-2} - \frac{k(k-1)}{2} a_0^{2k-4} \right) O(n^{-1}). \]

We have

\[ \frac{ka_0^{2k-2}}{(2k)!} \left(a_0(a_1 + b_1)n^{-\frac{1}{3}} + (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1}\right) = \sum_{i=3}^{k} \binom{k}{i} a_0^{2k-2i} (a_0(a_1 + b_1)n^{-\frac{1}{3}} + (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1})^i. \]

We have at this stage,

\[ 1 + \sum_{k \geq 1} \frac{2^{k-1} a_0^{2k}}{(2k)!} = 1 + \sum_{k \geq 1} \frac{a_0^{2k}}{(2k)!} + \sum_{k \geq 1} \frac{a_0^{2k-1}}{2(2k-1)!} (a_0(a_1 + b_1)n^{-\frac{1}{3}} + \sum_{k \geq 1} \frac{2^{k-1} a_0^{2k-1}}{2(2k-1)!} (a_0(a_1 + b_1)n^{-\frac{1}{3}} + \sum_{k \geq 1} \frac{a_0^{2k}}{2(2k)!} (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1})^i. \]

\[ + \sum_{k \geq 1} \frac{a_0^{2k}}{2(2k-1)!} a_1 b_1 + \left( \sum_{k \geq 2} \frac{2^{k-1} a_0^{2k-2}}{8(2k-2)!} - \sum_{k \geq 2} \frac{a_0^{2k-2}}{8(2k-1)!} \right) (a_0(a_1 + b_1)n^{-\frac{1}{3}} + \sum_{k \geq 1} \frac{2^{k-1} a_0^{2k-2}}{2(2k-1)!} (a_0(b_2 + a_2) + a_1 b_1)n^{-\frac{2}{3}} + Dn^{-1})^i. \]
\[
\sum_{k \geq 1} \frac{2^{k-1}(a_2^2 + 1)^k}{(2k)!} O(n^{-1})
\] (3.44)

\[
= 1 + \frac{1}{2} \sum_{k \geq 1} \frac{(\sqrt{2}a_0)^{2k}}{(2k)!} + \frac{(a_1 + b_1)}{2\sqrt{2}} \sum_{k \geq 1} \frac{(\sqrt{2}a_0)^{2k-1}}{(2k-1)!} n^{-\frac{1}{2}} + \left[ \frac{(a_2 + b_2)}{2\sqrt{2}} \sum_{k \geq 1} \frac{(\sqrt{2}a_0)^{2k-1}}{(2k-1)!} + \frac{a_1b_1}{2\sqrt{2}a_0} \sum_{k \geq 1} \frac{(\sqrt{2}a_0)^{2k-1}}{(2k-1)!} \right] n^{-\frac{1}{2}} + \sum_{k \geq 1} \frac{(2a_0^2 + 2b)^k}{(2k)!} O(n^{-1})
\] (3.45)

\[
= \frac{1}{2} \left( 1 + \cosh(\sqrt{2}a_0) \right) + \frac{(a_1 + b_1)}{2\sqrt{2}} \sinh(\sqrt{2}a_0) n^{-\frac{1}{2}} + \left[ \frac{(a_2 + b_2)}{2\sqrt{2}} \sinh(\sqrt{2}a_0) + \frac{a_1b_1}{2\sqrt{2}a_0} \sinh(\sqrt{2}a_0) + \frac{1}{8} \left( \cosh(\sqrt{2}a_0) - \frac{1}{\sqrt{2}a_0} \sinh(\sqrt{2}a_0) \right) (a_1 + b_1)^2 \right] n^{-\frac{1}{2}} + \cosh(2a_0^2 + 1) O(n^{-1}) = \exp(M)_{11} = \exp(M)_{22}
\] (3.46)

A similar argument gives

\[
- \sum_{k \geq 1} \frac{2^{k-1}a^{k+1}b^{k-1}}{(2k)!} = \frac{1}{2} \left( 1 - \cosh(\sqrt{2}a_0) \right) + \left[ \frac{a_1 - b_1}{2a_0} (1 - \cosh(\sqrt{2}a_0)) - \frac{a_1 + b_1}{2\sqrt{2}} \sinh(\sqrt{2}a_0) \right] n^{-\frac{1}{2}} + \left[ \frac{b_1(b_1 - a_1)}{2a_0^2} + \frac{a_2 - b_2}{2a_0} + \left( \frac{b_1(a_1 - b_1)}{2a_0^2} - \frac{(a_1 + b_1)^2}{8} \right) + \frac{b_2 - a_2}{2a_0} \right] \cosh(\sqrt{2}a_0) + \left( \frac{5\sqrt{2}a_1^2}{16a_0} - \frac{3\sqrt{2}a_1}{16a_0} - \sqrt{2}a_1b_1 - \sqrt{2}(a_2 + b_2) \right) \sinh(\sqrt{2}a_0) \right] n^{-\frac{1}{2}} + \cosh(\sqrt{2}a_0) O(n^{-1}) = \exp(M)_{12},
\] (3.47)

if we interchange \( a \) and \( b \), then

\[
- \sum_{k \geq 1} \frac{2^{k-1}a^{k+1}b^{k-1}}{(2k)!} = \frac{1}{2} \left( 1 - \cosh(\sqrt{2}a_0) \right) + \left[ \frac{b_1 - a_1}{2a_0} (1 - \cosh(\sqrt{2}a_0)) - \frac{a_1 + b_1}{2\sqrt{2}} \sinh(\sqrt{2}a_0) \right] n^{-\frac{1}{2}} + \left[ \frac{a_1(a_1 - b_1)}{2a_0^2} + \frac{b_2 - a_2}{2a_0} + \left( \frac{a_1(b_1 - a_1)}{2a_0^2} - \frac{(a_1 + b_1)^2}{8} \right) + \frac{a_2 - b_2}{2a_0} \right] \cosh(\sqrt{2}a_0) + \left( \frac{5\sqrt{2}a_1^2}{16a_0} - \frac{3\sqrt{2}a_1}{16a_0} - \sqrt{2}a_1b_1 - \sqrt{2}(a_2 + b_2) \right) \sinh(\sqrt{2}a_0) \right] n^{-\frac{1}{2}} + \cosh(\sqrt{2}a_0) O(n^{-1}) = \exp(M)_{21},
\] (3.48)

We also have

\[
\sum_{k \geq 0} \frac{a^{k+1}b^k}{(2k + 1)!} = \frac{1}{\sqrt{2}} \sinh(\sqrt{2}a_0) + \left[ \frac{a_1 + b_1}{2} \cosh(\sqrt{2}a_0) + \frac{a_1 - b_1}{2\sqrt{2}a_0} \sinh(\sqrt{2}a_0) \right] n^{-\frac{1}{2}} +
\]
The last term of the exponential matrix (3.33) is
\[
\left[ \left( \frac{(a_1 + b_1)^2}{8a_0} - \frac{b_1^2}{2a_0} + \frac{a_2 + b_2}{2} \right) \cosh\left( \sqrt{2}a_0 \right) + \left( \frac{(a_1 + b_1)^2}{4\sqrt{2}} - \frac{(a_1 + b_1)^2}{8\sqrt{2}a_0^3} + \frac{b_1^2}{2\sqrt{2}a_0^3} \right) \cosh\left( \sqrt{2}a_0 \right) \right] n^{-\frac{3}{2}} + \sinh(\sqrt{2}a_0)O(n^{-1}) = \exp(M)_{13} = -\exp(M)_{32},
\]
and if we interchange \(a\) and \(b\) in this last formula,
\[
\sum_{k \geq 0} \frac{2^k a^k b^{k+1}}{(2k+1)!} = \frac{1}{\sqrt{2}} \sinh\left( \sqrt{2}a_0 \right) + \left[ \frac{a_1 + b_1}{2} \cosh\left( \sqrt{2}a_0 \right) + \frac{b_1 - a_1}{2\sqrt{2}a_0} \sinh\left( \sqrt{2}a_0 \right) \right] n^{-\frac{1}{2}} + \left[ \left( \frac{(a_1 + b_1)^2}{8a_0} - \frac{a_1^2}{2a_0} + \frac{a_2 + b_2}{2} \right) \cosh\left( \sqrt{2}a_0 \right) + \left( \frac{(a_1 + b_1)^2}{4\sqrt{2}} - \frac{(a_1 + b_1)^2}{8\sqrt{2}a_0^3} + \frac{a_1^2}{2\sqrt{2}a_0^3} \right) \cosh\left( \sqrt{2}a_0 \right) \right] n^{-\frac{1}{2}} + \sinh\left( \sqrt{2}a_0 \right)O(n^{-1}) = \exp(M)_{31} = -\exp(M)_{23}.
\]

The last term of the exponential matrix (3.33) is
\[
1 + \sum_{k \geq 1} \frac{2^k a^k b^k}{(2k)!} = \cosh\left( \sqrt{2}a_0 \right) + \frac{a_1 + b_1}{\sqrt{2}} \sinh\left( \sqrt{2}a_0 \right) n^{-\frac{1}{2}} + \left[ \frac{(a_1 + b_1)^2}{4} \cosh\left( \sqrt{2}a_0 \right) + \left( \frac{a_1 b_1}{\sqrt{2}a_0} + \frac{a_2 + b_2}{\sqrt{2}a_0} - \frac{(a_1 + b_1)^2}{4\sqrt{2}a_0} \right) \sinh\left( \sqrt{2}a_0 \right) \right] n^{-\frac{3}{2}} + \sinh\left( \sqrt{2}a_0 \right)O(n^{-1}) = \exp(M)_{33}.
\]

We note that \(\sqrt{2}a_0\) is exactly the quantity \(\mu\) defined in [24] as the new variable when solving for the limiting system of equation as \(n\) goes to infinity.

We then use equations (3.35), (3.40), (3.50) together with the numerical value of \(c_\varphi\) for \(n\) even given by (2.22) to have the following expansion of \(V_{n,\varepsilon}\) and \(q_{n,\varepsilon}\).

\[
V_{n,\varepsilon}(\tau(s)) = \frac{1}{2} \left( 1 - \exp(-\sqrt{2}a_0) \right) + \left[ \frac{a_1 - b_1}{2} \sinh\left( \sqrt{2}a_0 \right) - \frac{a_1 + b_1}{\sqrt{2}a_0} \exp(-\sqrt{2}a_0) \right] n^{-\frac{3}{2}} + \left[ \left( \frac{(a_1 + b_1)^2}{8} - \frac{1}{\sqrt{2}a_0} \right) - \frac{\sqrt{2}(a_2 + b_2)}{4} + \frac{a_1^2}{2\sqrt{2}a_0} \right] \cosh\left( \sqrt{2}a_0 \right) - \left[ \frac{(a_1 + b_1)^2}{8} (1 + \frac{1}{\sqrt{2}a_0}) \right] \sinh\left( \sqrt{2}a_0 \right) n^{-\frac{3}{2}} + O\left( \frac{1}{n} \right)
\]

and

\[
q_{n,\varepsilon}(\tau(s)) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}a_0) + \left[ -\frac{a_1 + b_1}{2} \exp(-\sqrt{2}a_0) - \frac{a_1 - b_1}{2} \sinh\left( \sqrt{2}a_0 \right) \right] n^{-\frac{3}{2}} + \left[ \frac{\sqrt{2}(a_1 + b_1)^2}{8} (1 - \frac{1}{\sqrt{2}a_0}) + \frac{b_1^2}{2a_0} - \frac{a_2 + b_2}{2} \right] \cosh\left( \sqrt{2}a_0 \right) + \left[ \frac{b_2 - a_2}{2\sqrt{2}a_0} + \frac{a_2 + b_2}{2} \right] \sinh\left( \sqrt{2}a_0 \right) n^{-\frac{3}{2}} + O\left( \frac{1}{n} \right).
\]
3.2.2 Second system of equations involving the calligraphic variables for GOE\(_n\)

The system involving the calligraphic variables is

\[ Y(t) = \exp\left(-\int_t^\infty B(x)dx\right) \cdot Y(\infty). \quad (3.52) \]

We set

\[
\exp\left(-\int_t^\infty B(x)dx\right) = \exp\left\{ \begin{pmatrix} 0 & 0 & -\int_t^\infty q_n(x) \, dx \\ 0 & 0 & -\int_t^\infty p_n(x) \, dx \\ -\int_t^\infty p_n(x) \, dx & -\int_t^\infty q_n(x) \, dx & 0 \end{pmatrix} \right\} = \exp(M).
\]

\(M\) is of the form

\[
M = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & -b \\ -b & -a & 0 \end{pmatrix} \quad (3.53)
\]

with

\[
\exp(M) = \left( 1 + \sum_{k \geq 1} \frac{2^{k-1} a^k b^k}{(2k)!} \right) \sum_{k \geq 1} \frac{2^{k-1} a^{k+1} b^{k+1}}{(2k)!} - \sum_{k \geq 0} \frac{2^k a^{k+1} b^{k+1}}{(2k+1)!} \\
\sum_{k \geq 1} \frac{2^{k-1} a^k b^{k+1}}{(2k)!} \right) \right) + \sum_{k \geq 1} \frac{2^{k-1} a^{k+1} b^{k+1}}{(2k+1)!} - \sum_{k \geq 0} \frac{2^k a^{k+1} b^{k+1}}{(2k+1)!}
\]

\[
= \begin{pmatrix} \exp(M)_{11} & \exp(M)_{12} & \exp(M)_{13} \\ \exp(M)_{21} & \exp(M)_{22} & \exp(M)_{23} \\ \exp(M)_{31} & \exp(M)_{32} & \exp(M)_{33} \end{pmatrix} \quad (3.54)
\]

We have

\[
Q_{n,1}(t) = 2c_\varphi \exp(M)_{11} + \exp(M)_{13}, \quad (3.55)
\]

\[
P_{n,1}(t) = 2c_\varphi \exp(M)_{21} + \exp(M)_{23}, \quad (3.56)
\]

and

\[
\hat{R}_{n,1}(t) = 2c_\varphi \exp(M)_{31} + \exp(M)_{33}. \quad (3.57)
\]

We note that \(\exp(M)_{21} = -\exp(M)_{21}\), \(\exp(M)_{23} = \exp(M)_{23}\), \(\exp(M)_{31} = -\exp(M)_{31}\) and \(\exp(M)_{33} = \exp(M)_{33}\). The solutions (3.55) and (3.57) follow directly from the large \(n\) expansion obtained in the last subsection. We therefore have the following solutions for \(P_{n,1}\) and \(\hat{R}_{n,1}\)

\[
P_{n,1}(\tau(s)) = \frac{1}{\sqrt{2}}(\exp(-\sqrt{2}a_0) - 1) + \left[ \frac{a_1 - b_1}{\sqrt{2}a_0} + \frac{a_1 + b_1}{2} \exp(-\sqrt{2}a_0) + \frac{b_1 - a_1}{\sqrt{2}a_0} \cosh(\sqrt{2}a_0) \right]
\]

18
\[
\begin{align*}
&\frac{b_1 - a_1 \sinh(\sqrt{2}a_0)}{\sqrt{2}a_0} \cdot n^{-\frac{1}{2}} + \left( -\frac{\sqrt{2}a_1^2}{2a_0^2} + \frac{\sqrt{2}a_1 b_1}{2a_0^2} + \frac{a_2 - b_2}{\sqrt{2}a_0} + \frac{(a_1 + b_1)^2}{4\sqrt{2}} + \frac{\sqrt{2}(a_1^2 + a_1 b_1)}{2a_0^2} \right) \\
&\quad - \frac{a_1^2}{4a_0} - \frac{a_2 - b_2}{\sqrt{2}a_0} + \frac{b_1^2}{4a_0} \cosh(\sqrt{2}a_0) + \left[ \frac{(a_1 + b_1)^2}{4\sqrt{2}} + \frac{(a_1 + b_1)^2}{8\sqrt{2}a_0^2} - \frac{a_1^2}{2\sqrt{2}a_0^2} + \frac{a_2 - b_2}{2\sqrt{2}a_0} \right] \sinh(\sqrt{2}a_0) \right) n^{-\frac{2}{3}} + O\left( \frac{1}{n} \right)
\end{align*}
\]

and
\[
\bar{R}_{n,1}(\tau(s)) = \exp(-\sqrt{2}a_0) + \left[ \frac{a_1 - b_1 \sinh(\sqrt{2}a_0)}{\sqrt{2}} - \frac{a_1 + b_1}{\sqrt{2}} \exp(-\sqrt{2}a_0) \right] n^{-\frac{1}{2}}
\]
\[
\left[ \left( \frac{(a_1 + b_1)^2}{4} - \frac{a_2 + b_2}{\sqrt{2}} \right) \exp(-\sqrt{2}a_0) - \frac{(a_1 + b_1)^2}{4\sqrt{2}a_0} \cosh(\sqrt{2}a_0) - \frac{(a_1 - b_1)^2}{4\sqrt{2}a_0} \sinh(\sqrt{2}a_0) \right]
\]
\[
\left[ \frac{(a_1 + b_1)^2}{8a_0^2} \sinh(\sqrt{2}a_0) + \frac{a_2 - b_2}{2a_0} \sinh(\sqrt{2}a_0) + \frac{a_1^2}{\sqrt{2}a_0} \cosh(\sqrt{2}a_0) - \frac{a_1^2}{2a_0^2} \sinh(\sqrt{2}a_0) \right] n^{-\frac{2}{3}}
\]
\[
+ O\left( \frac{1}{n} \right)
\]

### 3.2.3 Calligraphic variables for GSE\(_n\)

We note that the GSE\(_n\) case is identical to the GOE\(_n\) up to a sign change and the parity of \(n\) for the calligraphic variables. The large \(n\) expansion for \(u_{n,\epsilon}(t)\) and \(\bar{v}_{n,\epsilon}(t)\) follows from the matrix exponential \((3.33)\). The boundary conditions need to be change to \(u_{n,\epsilon}(\infty) = 0\) and \(\bar{v}_{n,\epsilon}(\infty) = 0\) and \(q_{n,\epsilon}(\infty) = 0\) as \(n\) is odd. Therefore in this case

\[
u_{n,\epsilon}(t) = \exp(M)_{12}
\]
\[
V_{n,\epsilon}(t) = \exp(M)_{22}
\]

and

\[
q_{n,\epsilon}(t) = \exp(M)_{32}
\]

The large \(n\) expansions of these quantities is given by \((3.47), (3.46)\) and \((3.49)\) respectively.

The system of equations satisfied by the calligraphic variables is

\[
\frac{d}{dt} \begin{pmatrix} Q_{n,4}(t) \\ P_{n,4}(t) \\ \bar{R}_{n,4}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -q_n(t) \\ 0 & 0 & -p_n(t) \\ -p_n(t) & -q_n(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} Q_{n,4}(t) \\ P_{n,4}(t) \\ \bar{R}_{n,4}(t) \end{pmatrix}
\]

where \(\bar{R}_{n,4}(t) = 1 + \bar{R}_{n,4}(t)\). The boundary conditions in this case are

\[
\begin{pmatrix} Q_{n,4}(\infty) \\ P_{n,4}(\infty) \\ \bar{R}_{n,4}(\infty) \end{pmatrix} = \begin{pmatrix} -c_\varphi \\ -c_\psi \\ -c_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ -c_\psi \\ 1 \end{pmatrix}
\]
as \(n\) is odd

We can use the same technique as for the Orthogonal case to find a large \(n\) expansion of \(Q_{n,4}(t), P_{n,4}(t)\) and \(\bar{R}_{n,4}(t)\).
3.3 Large $n$ expansion of the probability distribution of the largest eigenvalue

We recall that the quantity of interest is \( \langle 2.24 \rangle \). Under our change of variables it reads

\[
(1 - \kappa_n)(1 - \frac{1}{2} \mathcal{R}_n) - \frac{1}{2}(q_n - c_\nu) \mathcal{P}_n = \frac{1}{2} \left[ \mathcal{V}_{n,\xi} \left( 1 + \mathcal{R}_n \right) - \mathcal{P}_n(q_n - c_\nu) \right]. \quad (3.65)
\]

Upon substitutions of the newly derived expressions in the right of \( \langle 3.65 \rangle \), the right side of \( \langle 3.65 \rangle \) takes the form

\[
\exp(-\sqrt{2a_0}) + \left[ a_1 - \frac{b_1}{2a_0} (1 - \exp(-\sqrt{2a_0})) - \frac{a_1 + b_1}{\sqrt{2}} \exp(-\sqrt{2a_0}) \right] n^{-\frac{1}{2}}
\]

\[
\left\{ \frac{a_1b_1 - a_1^2}{2a_0^2} + \frac{a_2 - b_2}{2a_0} + \left( \frac{3a_1^2 - b_1^2}{8\sqrt{2a_0}} - \frac{(a_1 + b_1)^2}{16\sqrt{2a_0}} - \frac{a_2 + b_2}{4\sqrt{2}} \right) \exp(-2\sqrt{2a_0}) \right. \\
\left. + \frac{(a_1 + b_1)^2}{4} - \frac{3(a_2 + b_2)}{4\sqrt{2}} + \frac{a_1^2}{2a_0} - \frac{(a_1 + b_1)^2}{16\sqrt{2a_0}} + \frac{a_1^2}{2\sqrt{2a_0}} - \frac{a_1b_1}{4\sqrt{2a_0}} - \frac{a_2 - b_2}{2a_0} \right) \exp(-\sqrt{2a_0}) \right\} n^{-\frac{3}{2}} + O\left( \frac{1}{n} \right) \quad (3.66)
\]

We follow Tracy and Widom \( \langle 24 \rangle \) and denote by $\mu(s)$ the quantity $\sqrt{2a_0}(s)$,

\[
\mu := \mu(s) = \sqrt{2a_0}(s) = \int_s^\infty q(x)dx,
\]

and we introduce the following notations,

\[
\nu := \nu(s) = \int_s^\infty p(x)dx, \quad \alpha := \alpha(s) = \int_s^\infty q(x)u(x)dx,
\]

\[
a_1(s) = \frac{1}{\sqrt{2}} \int_s^\infty \left( \frac{2c - 1}{2} p(x) - cq(x)u(x) \right) dx,
\]

\[
b_1(s) = \frac{1}{\sqrt{2}} \int_s^\infty \left( \frac{2c + 1}{2} p(x) - cq(x)u(x) \right) dx,
\]

\[
a_2(s) = \frac{1}{20\sqrt{2}} \int_s^\infty \left( (10c^2 - 10c + \frac{3}{2})q_1(x) + p_2(x) + (-30c^2 + 10c + \frac{3}{2})q(x)v(x) + p_1v(x) \\
+ p(x)v_1(x) - q_2(x)u(x) - q_1(x)u_1(x) - q(x)u_2(x) + \left( \frac{3}{2} - 10c^2 \right)p(x)u(x) + 20c^2q(x)u^2(x) \right) dx,
\]

\[
b_2(s) = \frac{1}{20\sqrt{2}} \int_s^\infty \left( (10c^2 + 10c + \frac{3}{2})q_1(x) + p_2(x) + (-30c^2 - 10c + \frac{3}{2})q(x)v(x) + p_1v(x) \right)
\]

20
\[ + p(x)v_1(x) - q_2(x)u(x) - q_1(x)u_1(x) - q(x)u_2(x) + \left( \frac{3}{2} - 10c^2 \right)p(x)u(x) + 20c^2q(x)u^2(x) \] \, dx.

We note that
\[
    a_1(s) - b_1(s) = -\frac{1}{\sqrt{2}} \int_s^\infty p(x) \, dx, \tag{3.69}
\]
\[
    a_1(s) + b_1(s) = \frac{1}{\sqrt{2}} \int_s^\infty (2cp(x) - 2cq(x)u(x)) \, dx = -\sqrt{2} c q(s), \tag{3.70}
\]
and
\[
    a_2(s) - b_2(s) = \frac{c}{\sqrt{2}} \int_s^\infty (q(x)v(x) - q_1(x)) \, dx = \frac{c}{\sqrt{2}} p(s). \tag{3.71}
\]

We set
\[
    \eta(s) = a_2(s) + b_2(s) = \frac{1}{20\sqrt{2}} \int_s^\infty \left( (20c^2+3)q_1(x) + 2p_2(x) + (-60c^2+3)q(x)v(x) + 2p_1v(x) \right.
\]
\[
    + 2p(x)v_1(x) - 2q_2(x)u(x) - 2q_1(x)u_1(x) - 2q(x)u_2(x) + (3 - 20c^2)p(x)u(x)
\]
\[
    + 40c^2q(x)u^2(x) \right) \, dx. \tag{3.72}
\]

This last equality comes from
\[
    (20c^2 + 3)q_1(x) + (3 - 60c^2)q(x)v(x) + (3 - 20c^2)p(x)u(x) + 40c^2q(x)u^2(x) = 20c^2( q_1(x) - 3q(x)v(x) - p(x)u(x) + 2q(x)u^2(x) ) + 3(q_1(x) + q(x)v(x) + p(x)u(x) ), \tag{3.73}
\]
the substitutions
\[
    q_1(x) = xq(x) - q(x)v(x) + p(x)u(x), \quad \text{and} \quad q_1(x) = p'(x) + q(x)v(x), \tag{3.74}
\]
in \[3.74\]
\[
    20c^2( xq(x) + 2q(x)u(x) ) + 3(p'(x) + 2q(x)v(x) + p(x)u(x))
\]
\[
    = 20c^2( xq(x) + 2q(x)(-2v(x) + u^2(x)) ) + 3(p'(x) + 2q(x)v(x) + p(x)u(x)), \tag{3.76}
\]
and the substitutions
\[
    q^2(x) = u^2(x) - 2v(x), \quad \text{and} \quad q''(x) = xq(x) + 2q^3(x) \tag{3.77}
\]
in \[3.77\]
\[
    20c^2( xq(x) + 2q^3(x) ) + 3(p'(x) + 2q(x)v(x) + p(x)u(x))
\]
\[
    = 20c^2q''(x) + 3p'(x) + 6q(x)v(x) + 3p(x)u(x). \tag{3.78}
\]

21
With these representations, (3.66) is
\[ e^{-\mu} + \left[ c q(s) e^{-\mu} - \frac{1}{2\mu} \nu (1 - e^{-\mu}) \right] n^{-\frac{3}{2}} + \left\{ \frac{2c-1}{4\mu^2} \nu^2 - \frac{c}{2\mu^2} \nu \int_s^\infty q(x) u(x) dx + \frac{c p(s)}{2\mu} \right. \]
\[ \left[ \frac{c^2 - 2c + 1/4}{8\mu} \nu^2 - \frac{c(c-1)}{4\mu} \nu \int_s^\infty q(x) u(x) dx + \frac{c^2}{8\mu} \left( \int_s^\infty q(x) u(x) dx \right)^2 - \frac{c^2 q^2(s)}{8\mu} \right] \]
\[ - \frac{n}{4\sqrt{2}} e^{-2\mu} + \left[ \frac{c^2 q^2(s)}{2} - \frac{3\eta}{4\sqrt{2}} + \frac{2 - \mu}{2\mu^2} \left( \frac{c^2 - c + 1/4}{2} \nu^2 - \frac{2c^2 - c}{2} \nu \int_s^\infty q(x) u(x) dx \right) \right. \]
\[ + \frac{c^2}{4\mu} \left( \int_s^\infty q(x) u(x) dx \right)^2 - \frac{c^2 q^2(s)}{2\mu} \]
\[ \left. - \frac{c^2 q^2(s)}{2\mu^2} \right] e^{-\mu} + \frac{c^2 q^2(s)}{2\mu^2} \sinh(\mu) - \left( \frac{c^2 - 1/4}{2} \nu^2 - c \nu \int_s^\infty q(x) u(x) dx \right) \]
\[ \frac{c^2 (\int_s^\infty q(x) u(x) dx)^2}{\mu^2} \cosh(\mu) \right\} n^{-\frac{3}{2}} + O(\frac{1}{n}) \quad (3.79) \]
as \( n \) goes to infinity uniformly for \( s \) bounded away from minus infinity.

Finally we combine (3.79) with Theorem 1.1 to have the following version of Theorem 1.2.

If we set \( t = \tau(s) \), then as \( n \to \infty \)
\[ F_{n,s}^2(t) = F_2(s) \cdot \left\{ e^{-\mu} + \left[ c (q(s) + u(s)) e^{-\mu} - \frac{\nu}{2\mu} (1 - e^{-\mu}) \right] n^{-\frac{3}{2}} + \right. \]
\[ \left[ - \frac{1}{20} E_{c,2}(s) e^{-\mu} - \frac{c \alpha(s)}{2\mu} + \frac{c p(s)}{2\mu} + \frac{(2c - 1) \nu^2}{4\mu^2} + c u(s) \left( c q(s) e^{-\mu} - \frac{\nu}{2\mu} (1 - e^{-\mu}) \right) \right. \]
\[ + e^{-2\mu} \left( - \frac{\eta}{4\sqrt{2}} + \frac{c^2 \alpha^2(s)}{8\mu} - \frac{c^2 q^2(s)}{8\mu} - \frac{(c^2 - c) \nu \alpha(s)}{4\mu} + \frac{\left( \frac{1}{4} - 2c + c^2 \right) \nu^2}{8\mu} \right) + \]
\[ e^{-\mu} \left( \frac{c^2 q^2(s)}{2} - \frac{3\eta}{4\sqrt{2}} - \frac{c^2 \alpha^2(s)}{4\mu} - \frac{c p(s)}{2\mu} - \frac{c^2 q^2(s)}{8\mu} + \frac{c^2 \nu \alpha(s)}{4\mu} - \frac{\left( \frac{1}{4} + c^2 \right) \nu^2}{8\mu} \right) \]
\[ + \frac{2 - \mu}{2\mu^2} \left( \frac{c^2 \alpha^2(s)}{2} - \frac{(2c^2 - c) \nu \alpha(s)}{2} + \frac{\left( \frac{1}{4} - c + c^2 \right) \nu^2}{2} \right) \right. \]
\[ \left. - \left( c^2 \alpha^2(s) - c^2 \nu \alpha(s) + \frac{\left( \frac{1}{4} + c^2 \right) \nu^2}{2} \right) \cosh(\mu) + \frac{c^2 q^2(s)}{8\mu^2} \sinh(\mu) \right] n^{-\frac{3}{2}} \} + O(n^{-1}) \quad (3.80) \]
uniformly in \( s \).

To simplify the \( n^{-\frac{3}{2}} \) term in (3.80), we use the representation \( p(s) = q'(s) + q(s) u(s) \) which says in this setting that \( \nu(s) = \alpha(s) - q(s) \). The result of this substitution is Theorem 1.2.

22
4 Conclusion

We note that unlike $F_{n,2}(t)$ for the GUE$_n$, the GOE$_n$ large $n$ expansion of the probability distribution of the largest eigenvalue $F_{n,1}(t)$ has a non vanishing $n^{-\frac{1}{2}}$ correction term. Thus the convergence to the limiting Tracy-Widom distribution $F_1(t)$ is slower. Numerical applications of $F_{n,1}(t)$ follows easily from $q(s)$ this is one consequence of our representation of $F_{n,1}(t)$ in Theorem 1.2. All the terms on the right side of (1.36) can be expressed in terms of $q(s)$ and $q'(s)$.

The GSE$_n$ largest eigenvalue distribution is derived in a similar way (the only major difference being that $n$ needs to be odd in this case.)
Acknowledgements: The author would like to thank Professor Craig Tracy for the discussions that initiated this work and for the invaluable guidance, and the Department of Mathematical Sciences at the University of Alabama in Huntsville.

References

[1] J. Baik, P. A. Deift and K. Johansson. On the distribution of the length of the longest increasing subsequence in a random permutation J. Amer. Math. Soc., 12 (1999), 1119-1178.

[2] L. N. Choup. Edgeworth Expansion of the Largest Eigenvalue Distribution Function of GUE and LUE IMRN Volume 2006, ID 61049, Pages 1-33.

[3] L. N. Choup. Edgeworth Expansion of the Largest Eigenvalue Distribution Function of GUE Revisited arXiv:0711.4206v1

[4] P. Deift. Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach. American Mathematical Society. Courant Lecture Notes 3, 2000.

[5] P. Deift, Universality for mathematical and physical systems. International Congress of Mathematicians, Vol.1, 125-152, Eur.Math.Soc., Zürich, 2007.

[6] M. Dieng and C. A. Tracy. Application of random matrix theory to multivariate statistics. preprint, Arxiv:math.PR/0603543

[7] W. Feller. An Introduction to Probability Theory and Its Applications , Vol.II. Second edition, John Wiley, 1971.

[8] T. M. Garoni, P. J. Forrester and N. E. Frankel. Asymptotic corrections to the eigenvalue density of the GUE and LUE arXiv:math-ph/0504053v1

[9] I. Gohberg, S. Goldberg, and M. A. Kaashoek. Classes of Linear Operators, Vol. I, volume 49 of Operator Theory: Advances and Applications. Birkhäuser, 1990.

[10] I. Gohberg, S. Goldberg, and M. A. Kaashoek. Classes of Linear Operators, Vol. II, volume 63 of Operator Theory: Advances and Applications. Birkhäuser, 1993.

[11] I. C. Gohberg, M. G. Kreĭn. Introduction to the Theory of Linear Nonselfadjoint Operators, volume 18 of Translations of Mathematical Monographs. American Mathematical Society, 1969.

[12] H. Hochstadt. The Functions of Mathematical Physics, volume 23 of Pure and Applied Mathematics: A series of texts and Monographs. Wiley-Interscience, 1971.

[13] K. Johansson. Toeplitz determinants, random growth and determinantal processes. Proceedings of the ICM, Beijing 2002, vol. 3, 53–62, math.PR/0304368
[14] I. M. Johnstone, On the distribution of the largest eigenvalue in principal component analysis, *Ann. Stats.*, 29(2):295–327, 2001.

[15] P. D. Lax. *Functional Analysis* Wiley-Interscience, 2002.

[16] M. L. Mehta. *Random Matrices, Revised and Enlarged Second Edition*. Academic Press, 1991.

[17] F. W. J. Olver. Asymptotics and Special Functions Academic Press, New York, 1974.

[18] M. Plancherel and W. Rotach. Sur les valeurs asymptotiques des polynomes d’Hermite Comm. Math. Helv. 1 (1929)227-254.

[19] A. Soshnikov. Universality at the Edge of the Spectrum in Wigner Random Matrices. *J. Stat. Phys.*, 108(5–6):1033–1056, 2002.

[20] G. Szegö. Orthogonal Polynomials. American Mathematical Society Colloquium Publications Volume 23

[21] M. E. Taylor Partial Differential Equations. Springer-Verlag, New York, 1996

[22] C. A. Tracy and H. Widom. Level–spacing distributions and the Airy kernel. *Commun. Math. Physics*, 159:151–174, 1994.

[23] C. A. Tracy and H. Widom. Fredholm determinants, differential equations and matrix models. *Commun. Math. Physics*, 163:33–72, 1994.

[24] C. A. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Commun. Math. Physics*, 177:727–754, 1996.

[25] C. A. Tracy and H. Widom. Correlation functions, cluster functions, and spacing distributions for random matrices. *J. Stat. Phys.*, 92(5–6):809–835, 1998.

[26] C. A. Tracy and H. Widom. Airy kernel and Painlevé II. In *Isomonodromic deformations and applications in physics*, volume 31 of CRM Proceedings & Lecture Notes, pages 85–98. Amer. Math. Soc., Providence, RI, 2002.

[27] C. A. Tracy and H. Widom. Distribution functions for largest eigenvalues and their applications. In *Proceedings of the International Congress of Mathematicians, Beijing 2002*, Vol. I, ed. LI Tatsien, Higher Education Press, Beijing, pgs. 587–596, 2002.

[28] C. A. Tracy and H. Widom. Matrix kernels for the Gaussian orthogonal and symplectic ensembles. *Ann. Inst. Fourier, Grenoble*, 55, 2197–2207, 2005.

[29] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis* Fourth Edition Cambridge University Press, 2004.