LARGE-TIME BEHAVIOR OF MATURED POPULATION IN AN AGE-STRUCTURED MODEL

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ABSTRACT. In this paper, we model a mosquito plasticity problem and investigate the large time behavior of matured population under different control strategies. We prove that when the control is small, then the matured population will become large for large time and when the control is large, then the matured population will become small for large time. In the intermediate case, we derive a time-delayed model for the matured population which can be governed by a sub-equation and a super-equation. We prove the existence of traveling fronts for the sub-equation and use it to prove that the matured population will finally be between the positive states of the sub-equation and super-equation. At last, we present numerical simulations.

1. Introduction. The transmission mechanism of Malaria, one of the most intractable infectious diseases in the world, has been widely studied by biologists. This disease is caused by Plasmodium which is transmitted by female Anopheles mosquitoes when they bite. The control of transmission of Malaria mainly relies on the control of the vector species, which is mainly by using indoor residual spraying (IRs) and insecticide-treated nets (ITNs). These strategies turned out to be very efficient to reduce malaria transmission in Africa. However, mosquitoes are adapting due to insecticide pressure relates to ITNs and RIs usage. The most relevant adaptations are mutations which may grant insecticide resistance to these vectors [36], alterations in vector population [18] and also behavioral adaptation [26], such as changing main preferential biting times in order to evade ITNs presence [13, 24].

In order to study the mosquitoes population adaptation by the usage of ITNs and RIs, we develop a mathematical model describing the dynamics of a single species population with age dependence and spatial structure. Let $p(a, t, x)$ be the distribution of individuals of age $a > 0$ at time $t \geq 0$ and biting at time $x \in \mathbb{R}$. Let $a_0$ be the matured age and let $a_\dagger$ be the life expectancy of an individual. Let
$\beta(a) \geq 0$ be the natural fertility-rate and $\mu(a, w) \geq 0$ be the natural death-rate of individuals of age $a$ and matured population $w(t, x) = \int_{a_0}^{a_1} p(a, t, x) da$. The new generation is able to adapt the biting time in order to maximize its fitness, which is modeled by the kernel $K$ in the renewal equation below. Let the parameter $\eta$ be the maximum biting time difference the new generation can reach. The kernel $K(x, s)$ means the probability of that the new generation of mosquitoes adapts its biting time from $s$ to $x$ and the parameter $\eta$ means that this adaptation can only happen in the maximal interval $s \in [x - \eta, x + \eta]$. Let $u(a, w)$ be an additional mortality because of the use of ITNs and IRs. We set that the ITNs and RIs are only useful to matured population, that is, $u(a, w) = 0$ for $a \in (0, a_0)$ and $u(a, w) = u(w)$ for $a \in [a_0, a_1)$. Moreover, mosquitoes can adapt their biting time and we set their adapting model to be a $\Delta$ diffusion with a diffusive coefficient $\delta$. The diffusion term $\Delta p$ means that the mosquitoes can locally change their biting time to resist the usage of ITNs and RIs. If we assume that the mosquitoes are smart enough and their response to ITNs and RIs is intensive, the diffusion could be modeled by a nonlocal term which will be our further investigation. Therefore, we model the mosquito plasticity problem as the following system

$$\begin{cases}
Dp - \delta \Delta p + \mu(a, w(t, x))p = u(a, w(t, x))p, \\
p(0, t, x) = \int_{0}^{a_1} \beta(a) \int_{x-\eta}^{x+\eta} K(x, s)p(a, t, s)dsda, \\
p(a, 0, x) = p_0(a, x),
\end{cases}$$

where $Dp(a, t, x)$ represents the directional derivative of $p$ with respect to the direction $(1, 1, 0)$, which is defined as below

$$Dp(a, t, x) = \lim_{\varepsilon \to 0} \frac{p(a + \varepsilon, t + \varepsilon, x) - p(a, t, x)}{\varepsilon},$$

and the kernel $K(x, s)$ is defined by

$$K(x, s) = (x - s)^2 e^{- (x-s)^2}, \ (x, s) \in \mathbb{R}^2.$$  

In our paper, we consider $p(a, t, x)$ smooth enough. It means that

$$Dp(a, t, x) = \frac{\partial p(a, t, x)}{\partial t} + \frac{\partial p(a, t, x)}{\partial a}.$$  

Let us recall some history about the single species with age structure researches. Before 1990, many researchers considered diffusion into a time delay model by simply adding a diffusion term to the corresponding delay ordinary differential equation model, see Memory [22] and Yoshida [37]. But, in the nature biology, individuals have not been at the same point in time at previous times. Thereupon, in 1990, Britton [7] first proposed to address the problem for a delayed Fisher equation on an infinite domain. More details in Section 3 will be given to see how we derive our problem into a time-delayed problem. Since it is so important for an age structure model to derive a reaction diffusion equation with time delay, more and more researchers have widely concerned and extensively studied about this problem in the past few years, see [16, 28, 30, 31].

Meanwhile, as for the reaction diffusion equations with time delay, there are rich results about local delay and nonlocal delay. For the reaction diffusion equations with local time delay, the KPP and bistable nonlinear diffusion equations with a discrete delay were considered by Schaal [27]. In [35], more general reaction-diffusion systems with finite delay were studied by using the classical monotone iteration
technique and the sub- and supersolutions method. For more rich results about the reaction-diffusion equations with discrete delay, one can refer to [8, 9, 11, 14, 29] and references cited therein. It is worth mentioning that the research of Ma and Zou [21] provided a more generalized method than Chen [8, 9] for a class of discrete reaction-diffusion monostable equation with delay.

By the practical significance of biology, it is advantageous to consider the reaction diffusion equations with nonlocal delays. We would like to mention the work of Britton [6, 7], since they first attempted to study the periodic traveling wave solutions in reaction-diffusion equations with nonlocal delays. Since then, there are many researchers devoted to proving the existence of traveling wave solutions of these type equations mainly by three methods: the perturbation theory [3, 16], the geometric singular perturbation theory [1, 5, 16, 25], the monotone iteration method [12, 23, 31, 33, 34]. In fact, the posterior results concerning about the existence of traveling wave solutions in this paper are due to the monotone iteration method [31, 34].

In this paper, we consider that the biting behavior of mosquitoes is periodic with 24 hours a day. It means that we consider the initial value $p_0(a, x)$ satisfying
$$p_0(a, x + 24) = p_0(a, x), x \in \mathbb{R},$$
and the solution $p(a, t, x)$ satisfying
$$p(a, t + 24) = p(a, t, x), a \in (0, a_1), t \in \mathbb{R}^+, x \in \mathbb{R}.$$
We are interested in how the insecticidal control $u(a, w)$ (such as ITNs and RIs) effects the matured population of mosquitoes, that is, the large time behavior of $w(t, x)$. From the biological point of view, we make the following hypotheses throughout this paper:

**J1:** The death rate $\mu(a, w) \geq 0$ satisfies that
$$\mu(a, w) = \begin{cases} 
\mu_1(a), & a \in [0, a_0), \\
\mu_2(a, w), & a \in [a_0, a_1), 
\end{cases}$$
where $\mu_1(a) \in L^\infty(0, a_0)$, $\mu_2(a, w)$ is continuous with respect to $a$ and $w$, $\mu_2(a, w) \in L^\infty_{loc}(a_0, a_1)$ for every $w \geq 0$ and $\int_0^{a_1} \mu(a, w) da = +\infty$ for every $w \geq 0$. As a matter of fact, the natural death population can not exceed the amount of matured population, that is, $0 \leq \int_{a_0}^{a_1} \mu(a, w)p(a, t, x) da \leq w(t, x)$. Thus, we assume $0 \leq \int_{a_0}^{a_1} \mu(a, w)p(a, t, x) da \leq g(w)$ for some smooth continuous function $g(w)$.

**J2:** The birth rate $\beta(a)$ satisfies
$$\beta(a) = \begin{cases} 
0, & a \in [0, a_0), \\
\beta, & a \in [a_0, a_1), 
\end{cases}$$
where $\beta$ is a positive constant.

**J3:** The insecticidal control $u(a, w) \leq 0$ satisfies that
$$u(a, w) = \begin{cases} 
0, & a \in [0, a_0), \\
u(w), & a \in [a_0, a_1), 
\end{cases}$$
where $u(w)$ is a $C^2$ function in $w$.

**J4:** $p_0(a, x) \in L^\infty([0, a_1] \times \mathbb{R})$, $p_0(a, \cdot) \geq 0$ for every $a \in [0, a_1]$ and $p_0(a, x + 24) = p_0(a, x)$ for $x \in \mathbb{R}$.
(J5): We assume that \( \sup_{w \geq 0} \mu(a, w) \leq \tilde{\mu}(a) \) where \( \tilde{\mu}(a) \in L_{loc}^\infty([0, a_1]) \) and \( \beta(a) \) is sufficiently large such that \( \int_0^{a_1} \beta(a)e^{-\int_0^a \tilde{\mu}(\rho)d\rho} da \) is sufficiently large for every \( w \geq 0 \) which can ensure that there are mosquitoes surviving forever. We give some comments about these hypotheses. We assume that the natural death rate \( \mu(a, w) \) is only depending on the age for immature mosquitoes since immature mosquitoes do not have to compete with each other, and \( \mu(a, w) \) is depending on both the age \( a \) and the mature population \( w \) for mature mosquitoes. Notice that the conditions in (J1), imply that \( \mu \) both the age \( \mu \) and the total death-rate growing. On the other hand, if the insecticidal control is very large, it means that the death-rate is large and the natural death-rate is small and then the population will keep growing. In biological meaning, it means that the death rate is really high when the age approaches \( a_1 \). The condition \( \int_0^{a_1} \mu(a, w) da = +\infty \) for every \( w \geq 0 \) ensures \( a_1 \) being the maximal age of an individual, that is, \( p(a_1, t, x) = 0 \) for any \( t \in \mathbb{R}^+ \) and \( x \in \mathbb{R} \), see [4] for rigorous proof. From Theorem 1.1, the last assumption (J5) implies that if there is only less insecticidal control, the matured population of mosquitoes will go to infinity. In biological meaning, (J5) means that the fertility-rate is large and the natural death-rate is small and then the population will keep growing. On the other hand, if the insecticidal control is very large, it means that the total death-rate \( \mu(a, w) + u(a, w) \) is large and \( \int_0^{a_1} \beta(a)e^{-\int_0^a (\mu(\rho, w) + u(\rho, w))d\rho} da \) is small in some sense. Then, the population will be decaying to 0 for large time. Such threshold can be more clear for some other population models, one can refer to [2].

**Theorem 1.1.**  
(i) If \( \sup_{w \geq 0} |u(w)| \) is small enough, then one has that \( w(t, x) \to +\infty \), as \( t \to +\infty \).

(ii) If \( \inf_{w \geq 0} |u(w)| \) is large enough, then one has that \( w(t, x) \to 0 \), as \( t \to +\infty \).

In fact, a more realistic situation is that the population can not be infinity large or very small because of the limitation of insecticidal control strategy. It means that the matured population \( w(t, x) \) may reach some balanced states. In Section 3, we will derive a time-delayed model for \( w(t, x) \), that is, \( w(t, x) \) satisfies

\[
\begin{align*}
\frac{w_t - \delta \Delta w}{\int_{a_0}^{a_1} \left( -\mu(a, w) + u(a, w) \right) p(a, t, x) da} & + M\beta \int_{-\infty}^{+\infty} \left[ \int_{-\frac{\eta}{\eta}}^{\eta} z^2 e^{-z^2} w(t - a_0, x - z - s) dz \right] f_{\delta a_0}(s) ds, \\
\end{align*}
\]

for \( t > 0, x \in \mathbb{R} \) and

\[
\begin{align*}
w(s, x) = \int_{a_0}^{a_1} p_0(a + s, x) da, \quad \text{for } s \in [-a_0, 0] \text{ and } x \in \mathbb{R}. \\
\end{align*}
\]

Since \( 0 \leq \int_{a_0}^{a_1} \mu(a, w)p(a, t, x) da \leq g(w) \) by (J1), one has that \( w(t, x) \) is governed by the following sub-equation and super-equation

\[
\begin{align*}
\frac{w_t - \delta \Delta w - g(w) + u(w)w}{\int_{a_0}^{a_1} \left( -\mu(a, w) + u(a, w) \right) p(a, t, x) da} & + M\beta \int_{-\infty}^{+\infty} \left[ \int_{-\frac{\eta}{\eta}}^{\eta} z^2 e^{-z^2} w(t - a_0, x - z - s) dz \right] f_{\delta a_0}(s) ds, \\
\end{align*}
\]

and

\[
\begin{align*}
\frac{w_t - \delta \Delta w}{\int_{a_0}^{a_1} \left( -\mu(a, w) + u(a, w) \right) p(a, t, x) da} & + M\beta \int_{-\infty}^{+\infty} \left[ \int_{-\frac{\eta}{\eta}}^{\eta} z^2 e^{-z^2} w(t - a_0, x - z - s) dz \right] f_{\delta a_0}(s) ds. \\
\end{align*}
\]
As we announced, the insecticidal control will be enhanced as the population of mosquitoes increasing and finally the population of mosquitoes will reach a balanced state, that is, the death-rate will offset the birth-rate in some sense. Therefore, we assume that both sub-equation and super-equation can reach a balanced state, that is,

(H1) let $G(w) = -g(w) + u(w)w + M\beta M_1 w$. There is $w_2 > 0$ such that $G(w_2) = 0$, $G(w) > 0$ for $0 < w < w_2$, $G'(w_2) < 0$ and $g(0) = g'(0) = u(0) = 0$,

(H2) there is $w_3 > 0$ such that $u(w_3)w_3 + M\beta M_1 w_3 = 0$,

where $M_1 = \int_{-\infty}^{+\infty} \int_{-\eta}^{\eta} z^2 e^{-z^2} dz f_{\delta a_0}(s) ds$. Since $g(w) \geq 0$ for $w \geq 0$, it is obvious that $w_2 \leq w_3$.

We then derive the existence of traveling fronts of the sub-equation. In fact, the traveling front can describe the invasion of one steady state to another. It implies that the area in which the matured population of mosquitoes is close to $w_2$ will invade the area with less mosquitoes.

**Theorem 1.2.** Assume that (H1) hold. There exists a $c^* > 0$ such that for every $c > c^*$, the equation

$$w_t - \delta \Delta w = -g(w) + u(w)w + M\beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0, x-z-s) dz \right] f_{\delta a_0}(s) ds,$$

for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, admits a traveling front $\phi(x + ct)$ connecting 0 and $w_2$.

**Remark 1.** The traveling front implies that the matured mosquitoes diffuse with a speed no less than $c^*$ on the biting time line. It implies that the matured mosquitoes do not only exist at one biting time, but on all biting time line as $t \to +\infty$. Here, we should notice that the control $u$ is homogeneous in $x$, so that the saturated state $w_2$ is homogeneous in $x$.

Now, let us come back to the problem (2) with (3). Notice that $w(s, x + 24) = w(s, x)$ for $s \in [-a_0, 0]$ and $x \in \mathbb{R}$ since $p_0(a, x + 24) = p_0(a, x)$ for $a \in [0, a_1]$ and $x \in \mathbb{R}$. We assume that the matured population is not large at the initial time, that is, $w(s, x) \leq w_3$ for $s \in [-a_0, 0]$ and $x \in \mathbb{R}$. Assume further that $-g(w) + u(w)w$ satisfies

(H3) for every $\gamma \in (0, 1)$, there exist $a = a(\gamma) > 0$ and $\alpha = \alpha(\gamma) > 0$ such that for any $\theta \in (0, \gamma]$ and $w \in [0, w_2]$,

$$(1 - \theta)(-g(w) + u(w)w) - (-g((1 - \theta)w) + u((1 - \theta)w)(1 - \theta)w) \leq -a\theta w^\alpha.$$

**Theorem 1.3.** Assume that (H1), (H2) and (H3) hold. For problem (2) with the initial value (3), it holds that for each $\gamma \in (0, 1)$, there exist $T_0 > 0$, $\rho > 0$ and $\sigma > 0$ such that for each $\epsilon \in [0, \gamma]$, the following functions

$$w(t, x) \geq (1 - \epsilon e^{-\rho t}) \phi(x + ct + \sigma \epsilon e^{-\rho t}), \text{ for } t \geq T,$$

and

$$w_2 \leq \liminf_{t \to +\infty} \inf_{x \in \mathbb{R}} w(t, x) \leq \limsup_{t \to +\infty} \sup_{x \in \mathbb{R}} w(t, x) \leq w_3.$$

In fact, since the equation (2) is monotone, the inequality (7) can also be established by the periodicity of the solution $w(t, x)$ on $x \in \mathbb{R}$ and the theory of the spreading speed for the monotone semiflow, see [20]. Theorem 1.3 means that the
population of matured mosquitoes will finally reach an equilibrium or an oscillatory
solution of (2) between $w_2$ and $w_3$.

This paper is organized as follows. In section 2, we analyze the large time be-
behavior of the matured population when the control is small and when it is large,
that is, we prove Theorem 1.1. Section 3 is devoted to the proof of the existence of
traveling fronts for the sub-equation and the proof of Theorem 1.3. In Section 4,
we present numerical simulations of $w(t,x)$.

2. Proof of Theorem 1.1. In this section, our main job is to study the behavior
of the matured mosquitoes population when the control $|u(w)|$ is small enough and
large enough.

Let us start with the following assumptions:

(A1): $\mu^*(a) \in L^1_{loc}([0,a_1])$, $\int_0^a \mu^*(\rho) d\rho < \infty$, where $a < a_1$ and $\int_0^{a_1} \mu^*(\rho) d\rho = +\infty$;

(A2): $\beta^*(a) \in L^\infty((0,a_1))$, $\{\beta(a) > 0\} > 0$;

(A3): $p_0^*(a,x) \in L^\infty((0,a_1) \times (0,24))$, $p_0^*(a,x) \geq 0$.

Now, we first consider the following system

$$
\begin{aligned}
    Dq - \delta \Delta q + \mu^*(a)q &= 0, \\
    q(a,t,0) &= q(a,t,24), \\
    \partial_x q(a,t,0) &= \partial_x q(a,t,24), \\
    q(0,t,x) &= C \int_0^{a_1} \beta^*(a)q(a,t,x) da, \\
    q(a,0,x) &= p_0^*(a,x),
\end{aligned}
$$

where $Q_{a_1} = (0,a_1) \times \mathbb{R}^+ \times (0,24)$ and $C$ is a positive constant. Define the operator $F : X \rightarrow X$ as:

$$
F \phi(a,x) = -\frac{\partial \phi(a,x)}{\partial a} + \delta \Delta \phi(a,x) - \mu^*(a)\phi(a,x), \ \forall \phi(a,x) \in D(F),
$$

where $X = L^2((0,a_1) \times (0,24))$ and

$$
D(F) = \{\phi(a,x) | \phi, \partial_a \phi \in X, \phi(a,0) = \phi(a,24), \partial_x \phi(a,0) = \partial_x \phi(a,24),
$$

$$
\phi(0,x) = C \int_0^{a_1} \beta^*(a)\phi(a,x) da \}. \ 
$$

Then, we can write (8) as

$$
\begin{aligned}
    dq(a,t,x) &= Fq(a,t,x), \\
    q(a,0,x) &= p_0^*(a,x).
\end{aligned}
$$

Define an operator

$$
\mathcal{F}_\lambda = \int_0^{a_1} C \beta^*(a) e^{-\lambda a} e^{-\int_0^a \mu^*(\rho) d\rho} e^{\beta a} da,
$$

where the operator $B : L^2((0,24)) \rightarrow L^2((0,24))$ is defined as

$$
B(u) = \delta \Delta u(x), \ \forall u \in D(B),
$$

where

$$
D(B) = \{u(x)|u, Bu \in L^2(0,24), u(0) = u(24), u'(0) = u'(24)\}. \ 
$$

From [19, Lemma 2.1], one has the following lemma.
Lemma 2.1. The operator $F$ defined by (9).

(1): The operator $F$ has a real dominant eigenvalue $\tilde{\lambda}_0$, that is, $\tilde{\lambda}_0$ is greater than any real part of the eigenvalues of $F$.

(2): For the operator $F_{\lambda_0}$, 1 is an eigenvalue with the eigenfunction $\phi_0(x)$. Furthermore, $\gamma(F_{\lambda_0}) = 1$, where $\gamma(F_{\lambda_0})$ is the spectral radius of $F_{\lambda_0}$, that is, $\gamma(F_{\lambda_0}) = \sup\{|r| : r$ is an eigenvalue of $F_{\lambda_0}\}$.

Furthermore, one can get the following lemma.

Lemma 2.2. The eigenvalue $\tilde{\lambda}_0$ obtained in Lemma 2.1 is such that

(1): if $C \int_0^{a_1} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho} da > 1$, then $\tilde{\lambda}_0 > 0$.

(2): if $C \int_0^{a_1} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho} da < 1$, then $\tilde{\lambda}_0 < 0$.

Proof. We denote by $(\overline{\lambda}_i, \phi_i)_{i \geq 0}$ the eigenvalues and eigenfunctions of the following problem

\[
\begin{cases}
-\delta \Delta \phi_i(x) = \overline{\lambda}_i \phi_i(x), & x \in (0, 24), \\
\phi_i(0) = \phi_i(24), \\
\partial_x \phi_i(0) = \partial_x \phi_i(24),
\end{cases}
\]

where $\int_0^{24} \phi_i^2(x)dx = 1$, $i \geq 0$, and $\phi_0(x) > 0$ in $(0, 24)$. It is obvious that $\overline{\lambda}_0 = 0$ and $\phi_0(x)$ is a fixed positive constant. We also assume that $0 = \overline{\lambda}_0 < \overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \cdots$.

Let $H$ be the operator in $L^2(0, a_1)$ defined as

$$H \phi(a) = -\frac{d\phi(a)}{da} - \mu^*(a) \phi(a), \quad \forall \phi \in D(H),$$

where

$$D(H) = \{ \phi(a) | \phi, H \phi \in L^2(0, 24), \phi(0) = C \int_0^{a_1} \beta^*(a) \phi(a) da \}.$$ 

Let $\{\lambda_j\}_{j \geq 0}$ be the eigenvalues of $H$, that is, the solutions of the following equation

$$1 - C \int_0^{a_1} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho} da = 0.$$ 

We assume that $\hat{\lambda}_0 > \text{Re}\hat{\lambda}_1 \geq \text{Re}\hat{\lambda}_2 \geq \cdots$, even if it means re-arrange $\hat{\lambda}_j$. From [19, Lemma 2.1], one knows that $\lambda_0 = \hat{\lambda}_0 - \overline{\lambda}_0 = \hat{\lambda}_0$. It is obvious that if $C \int_0^{a_1} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho} da > 1$, then $\lambda_0 > 0$ and if $C \int_0^{a_1} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho} da < 1$, then $\lambda_0 < 0$. \hfill \Box

Lemma 2.3. Let $q(a, t, x)$ be the solution of

\[
\begin{cases}
Dq - \delta \Delta q + \mu^*(a)q = 0, & (a, t, x) \in Q_{a_1}, \\
q(a, t, 0) = q(a, t, 24), & (a, t) \in (0, a_1) \times \mathbb{R}^+, \\
\partial_x q(a, t, 0) = \partial_x q(a, t, 24), & (a, t) \in (0, a_1) \times \mathbb{R}^+, \\
q(0, t, x) = \int_0^{a_1} \beta^*(a) \int_{x-a}^{x+\eta} K_1(x, s) q(a, t, s) ds da, & (t, x) \in \mathbb{R}^+ \times (0, 24), \\
q(a, 0, x) = p_0(a), & (a, x) \in (0, a_1) \times (0, 24).
\end{cases}
\]

(1): If $C \int_0^{a_1} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho} da$ is sufficiently large and

$$K_1(x, s) = \begin{cases} (x-s)^2 e^{-(x-s)^2}, & (x, s) \in (0, 24) \times (0, 24), \\
0, & \text{else}, \end{cases}$$

we have $\hat{\lambda}_0 > 0$.\hfill \Box
then
\[ q(a,t,x) \to +\infty, \text{ as } t \to +\infty \text{ for every } a \in [0,a_1] \text{ and } x \in [0,24], \]
where \( a_1 \in (a_0,a_1) \).

(2): If \( \int_0^{a_1} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da \) is sufficiently small and \( K_1(x,s) \equiv K_1 \) where \( K_1 \)

is a positive constant, then
\[ q(a,t,x) \to 0, \text{ as } t \to +\infty \text{ for every } a \in [0,a_1] \text{ and } x \in [0,24]. \]

**Proof.** By referring to [19, Theorem 1.1], one knows that \( q(a,t,x) \) has an asymptotic expression
\[ q(a,t,x) = e^{\lambda_0 t} e^{-\lambda_0 a} T(0,a) C_{\lambda_0} \int_0^{a_1} \beta^*(a) \int_{x-\eta}^{x+\eta} K_1(x,s) \int_0^a e^{-\lambda_0 (a-\sigma)} F(\sigma,a) \]
\[ p_0(\sigma,s) ds d\sigma + o(e^{(\lambda_0-\epsilon)t}). \]

Here, \( \lambda_0 \) is the algebraically simple real eigenvalue of the operator \( \mathcal{A} : L^2((0,a_1) \times (0,24)) \to L^2((0,a_1) \times (0,24)) \) defined as
\[ \mathcal{A}\phi(a,x) = -\frac{\partial \phi(a,x)}{\partial a} + \delta \Delta \phi(a,x) - \mu^*(a) \phi(a,x), \forall \phi(a,x) \in D(\mathcal{A}), \]
where
\[ D(\mathcal{A}) = \{ \phi(a,x) | \phi, \mathcal{A}\phi \in X, \phi(a,0) = \phi(a,24), \partial_x \phi(a,0) = \partial_x \phi(a,24) \}, \]
\[ \phi(0,x) = \int_0^{\alpha_1} \beta^*(a) \int_{x-\eta}^{x+\eta} K_1(x,s) \phi(a,s) ds da \}. \]

And \( \lambda_0 \) is larger than the real part of any other eigenvalue of the operator \( \mathcal{A} \). Let \( \mathcal{\tau}(\tau,s) = e^{-\int_0^\tau \mu^*(\rho)d\rho} e^{\beta(s-\tau)} \). \( C_{\lambda_0} = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)(I - \mathcal{B}_\lambda)^{-1} \), where the operator \( \mathcal{B}_\lambda : L^2((0,24)) \to L^2((0,24)) \) is defined as
\[ \mathcal{B}_\lambda(\phi(x)) = \int_0^{a_1} \beta^*(a) \int_{x-\eta}^{x+\eta} K_1(x,s) e^{-\lambda \tau} \mathcal{F}(0,a) \phi(s) ds da. \]

(1) From the proof of [19, Theorem 3.1], one can take \( C \) in (8) sufficiently small such that \( \lambda_0 \geq \tilde{\lambda}_0 \). Note that the choice of \( C \) depends on \( K_1(x,s) \). Then, if \( \int_0^{a_1} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da \) is sufficiently large such that \( C \int_0^{a_1} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da > 1 \), by the result of Lemma 2.2 (1), one has that \( \lambda_0 \geq \tilde{\lambda}_0 > 0 \). From the asymptotic expression, one gets that
\[ q(a,t,x) \to +\infty, \text{ as } t \to +\infty \text{ for every } a \in [0,a_1] \text{ and } x \in [0,24]. \]

(2) With similar arguments to the proof of [19, Theorem 3.1], one can take \( C \) in (8) sufficiently large and prove that \( \lambda_0 \leq \tilde{\lambda}_0 \). Then, if \( \int_0^{a_1} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da \) is sufficiently small such that \( C \int_0^{a_1} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da < 1 \), by the result of Lemma 2.2 (2), one has that \( \lambda_0 \leq \tilde{\lambda}_0 < 0 \). From the asymptotic expression, one has that
\[ q(a,t,x) \to 0, \text{ as } t \to +\infty \text{ for every } a \in [0,a_1] \text{ and } x \in [0,24]. \]

This completes the proof. \( \square \)

To study the behavior of the matured mosquitoes population, we need the following lemma.
Lemma 2.4. If \( p_i(i \in 1, 2) \) are the solutions of the following systems

\[
\begin{align*}
Dp_i - \delta p_i + \mu_i(a, w)p_i &= 0, & (a, t, x) \in Q_{a_t}, \\
p_i(0, t, x) &= \int_0^{a_i} \beta_i(a) \int_{x-n}^{x+n} K_i(x, s)p_i(a, t, s)dsda, & (t, x) \in \mathbb{R}^+ \times (0, 24), \\
p_i(a, 0, x) &= p_{0i}(a, x), & (a, x) \in (0, a_t) \times (0, 24),
\end{align*}
\]

where \( w(t, x) = \int_0^{a_t} p(a, t, x)da, \mu_1(a, w), \mu_2(a, w) \in L^\infty([0, a_t]) \) for every \( w \geq 0 \) and are locally Lipschitz functions with respect to \( w, \beta_1, \beta_2 \) satisfy (A2), \( K_1, K_2 \in L^2([0, 24]^2), p_{01}, p_{02} \) satisfy (A3) and \( \mu_1 \geq \mu_2, \beta_1 \leq \beta_2, K_1 \leq K_2, p_{01} \leq p_{02}, \) then

\[ 0 \leq p_1(a, t, x) \leq p_2(a, t, x) \text{ a.e. in } Q_{a_t}. \]

Proof. Following [4, Theorem 4.2.2], one can easily get the previous comparison principle. Thus, we omit the details.

We now extend \( q(a, t, x) \) to \( x \in \mathbb{R} \) periodically such that

\[
\begin{align*}
\hat{q}(a, t, x) &= q(a, t, x), & x \in [0, 24], \\
\hat{q}(a, t, x + 24) &= \hat{q}(a, t, x), & x \in \mathbb{R}.
\end{align*}
\]

Then, \( \hat{q}(a, t, x) \) satisfies

\[
\begin{align*}
D\hat{q} - \delta \hat{q} + \mu^*(a)\hat{q} &= 0, & (a, t, x) \in (0, a_t) \times \mathbb{R}^+ \times \mathbb{R}, \\
\hat{q}(0, t, x) &= \int_0^{a_t} \beta^*(a) \int_{x-n}^{x+n} \hat{K}_1(x, s)\hat{q}(a, t, s)dsda, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\hat{q}(a, 0, x) &= \hat{q}_0(a, x), & (a, x) \in (0, a_t) \times \mathbb{R},
\end{align*}
\]

where

\[
\begin{align*}
\hat{q}_0(a, x) &= p_{0}(a, x), & x \in [0, 24], \\
\hat{q}(a, x + 24) &= \hat{q}(a, x), & x \in \mathbb{R}, \\
\hat{K}_1(x, s) &= K_1(x, s), & x \in [0, 24], s \in [0, 24], \\
\hat{K}_1(x, s) &= 0, & x \in [0, 24], s < 0 \text{ and } s > 24, \\
\hat{K}_1(x + 24, s + 24) &= \hat{K}_1(x, s), & x \in \mathbb{R}, s \in \mathbb{R}.
\end{align*}
\]

By Lemma 2.3, one has the following lemma.

Lemma 2.5. Let \( \hat{q}(a, t, x) \) be the solution of system (10).

(1) If \( \int_0^{a_t} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho}da \) is sufficiently large and

\[
K_1(x, s) = \begin{cases}
(x-s)^2e^{-(x-s)^2}, & (x, s) \in (0, 24) \times (0, 24), \\
0, & \text{else},
\end{cases}
\]

then

\[ \hat{q}(a, t, x) \to +\infty, \text{ as } t \to +\infty \text{ for every } a \in [0, a_t] \text{ and } x \in \mathbb{R}. \]

(2) If \( \int_0^{a_t} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho}da \) is sufficiently small and \( K_1(x, s) \equiv K_1 \) where \( K_1 \) is a positive constant, then

\[ \hat{q}(a, t, x) \to 0, \text{ as } t \to +\infty \text{ for every } a \in [0, a_t] \text{ and } x \in \mathbb{R}. \]
Proof of Theorem 1.1. (i) Let \( p(a,t,x) \) be the solution of (1). From the assumption (J5), one can take \( |u(w)| \) small enough such that
\[
\mu(a,w) - u(w) \leq \sup_{w \geq 0} (\mu(a,w) - u(a,w)) := \mu^* (a),
\]
and \( \int_0^{a_1} \beta(a)e^{-\int_0^a \mu^*(\rho)d\rho} \) is sufficiently large. Let \( \tilde{q}(a,t,x) \) be the solution of system (10) with \( \mu^*(a), \beta(a), \tilde{K}_1(x,s) \) and \( \tilde{q}_0(a,x) \) where \( \tilde{K}_1(x,s) \) is the periodic extension of \( K_1(x,s) \) defined by (11) and \( \tilde{q}_0(a,x) = p_0(a,x) \). Obviously, one has \( \tilde{K}_1(x,s) \leq K(x,s) \). By Lemma 2.4, one has that
\[
p(a,t,x) \geq \tilde{q}(a,t,x).
\]
Then, from Lemma 2.5 (i), one has that
\[
w(t,x) \geq \int_0^{a_1} \tilde{q}(a,t,x) da \geq \int_0^{a_1} \tilde{q}(a,t,x) da \to +\infty, \text{ as } t \to +\infty.
\]
(ii) By the assumptions (J1), (J3), we know that
\[
\mu(a,w) - u(a,w) \geq \inf_{w \geq 0} \mu(a,w) + \inf_{w \geq 0} (-u(a,w)).
\]
Then let \( \mu^*(a) = \inf_{w \geq 0} \mu(a,w) + \inf_{w \geq 0} (-u(a,w)) \) and \( |u(a,w)| \) be large enough such that \( \int_0^{a_1} \beta(a)e^{-\int_0^a \mu^*(\rho)d\rho} \) is small enough. Let \( \tilde{q}(a,t,x) \) be the solution of system (10) with \( \mu^*(a), \beta(a), \tilde{K}_1(x,s) \) and \( \tilde{q}_0(a,x) \) where \( \tilde{q}_0(a,x) = p_0(a,x) \) and \( \tilde{K}_1(x,s) \equiv K_1 := \sup_{(x,s) \in \mathbb{R}^2} K(x,s) \). By Lemma 2.4, one has that
\[
p(a,t,x) \leq \tilde{q}(a,t,x).
\]
Then, from Lemma 2.5 (ii), one has that
\[
w(t,x) \leq \int_0^{a_1} \tilde{q}(a,t,x) da \to 0, \text{ as } t \to +\infty.
\]
This completes the proof. \( \square \)

2.1. Existence of traveling fronts. In this subsection, we study the sub-equation (6) and prove Theorem 1.2. The purpose of this section is to establish the existence of traveling fronts of (6).

A traveling front of (6) is a solution \( w(t,x) = \phi(x + ct) \), where \( c > 0 \) is the wave speed and \( \phi \in C(\mathbb{R}; \mathbb{R}) \) is a non-decreasing function satisfying the following equation
\[
c\phi' (\xi) = \delta \phi'' (\xi) - g(\phi) + u(\phi)\phi(\xi) + M\beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} \phi(\xi - s - z - c\eta) dz \right] f_{\delta\eta}(s) ds,
\]
with the boundary conditions
\[
\lim_{\xi \to -\infty} \phi(\xi) = 0, \quad \lim_{\xi \to +\infty} \phi(\xi) = w_2,
\]
where \( \xi = x + ct \). Then, for \( \xi \in \mathbb{R} \), we define the following profile set
\[
\Gamma = \left\{ \phi \in C(\mathbb{R}; \mathbb{R}) \mid (1) \phi(\xi) \text{ is non-decreasing; (2)} \lim_{\xi \to -\infty} \phi(\xi) = 0; \lim_{\xi \to +\infty} \phi(\xi) = w_2 \right\}.
\]
Notice that since \( g(w), u(w) \) and their derivatives are continuous, there exists a constant \( b \geq 0 \) such that
\[
(-g(\phi) + u(\phi)\phi) - (-g(\psi) + u(\psi)\psi) + b(\phi - \psi) \geq 0,
\]
where \( \phi, \psi \in \Gamma \) and \( 0 \leq \psi(\xi) \leq \phi(\xi) \leq w_2 \). Furthermore, define \( H : C(\mathbb{R}; \mathbb{R}) \to C(\mathbb{R}; \mathbb{R}) \) by
\[
H(\phi)(\xi) = b\phi(\xi) - g(\phi) + u(\phi)\phi(\xi) + M \int_{-\infty}^{+\infty} \beta \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} \phi(\xi - s - z - ca_0)dz \right] f_{\delta a_0}(s)ds.
\]
It follows that the equation of (12) involves the following nonhomogeneous system of ordinary differential equation

\[
c\phi'(\xi) = \delta\phi''(\xi) - b\phi(\xi) + H(\phi)(\xi), \quad \xi \in \mathbb{R}.
\]

By exploring \( H(\phi) \), we get the following lemma directly.

**Lemma 2.6.** For any \( \phi, \psi \in \Gamma \), we have

1. \( H(\phi)(\xi) \geq 0 \), for all \( \xi \in \mathbb{R} \).
2. \( H(\phi)(\xi) \) is non-decreasing in \( \xi \in \mathbb{R} \).
3. \( H(\psi)(\xi) \leq H(\phi)(\xi) \), for all \( \xi \in \mathbb{R} \), provided \( \psi \in C(\mathbb{R}; \mathbb{R}) \) is such that \( 0 \leq \psi(\xi) \leq \phi(\xi) \leq w_2 \), for all \( \xi \in \mathbb{R} \).

**Proof.** By some simple calculations, it is easy to get these results. Thus, we omit the proof.

Now, we define subsolutions and supersolutions for (14) as follows.

**Definition 2.7.** A function \( \phi \in C(\mathbb{R}; \mathbb{R}) \) is called a supersolution of (14) if \( \phi' \) and \( \phi'' \) exist almost everywhere and are essentially bounded on \( \mathbb{R} \), and \( \phi \) satisfies

\[
c\phi'(\xi) \geq \delta\phi''(\xi) - b\phi(\xi) + H(\phi)(\xi), \quad a.e \ in \ \mathbb{R},
\]

and for every discontinuous point \( \xi \) of \( \phi'(\xi) \), \( \phi'(\xi+) \leq \phi'(\xi-) \). A subsolution of (14) is defined in a similar way by reversing the inequality in (2.7).

In the following, we first assume that there exists a pair \( (\overline{\phi}, \phi) \), where \( \overline{\phi} \in \Gamma \) is a supersolution and \( \phi \) is a subsolution of (14) (which is not necessarily in \( \Gamma \)), such that

\[
\text{(G1):} \quad 0 \leq \phi(\xi) \leq \overline{\phi}(\xi) \leq w_2, \quad \text{for all } \xi \in \mathbb{R};
\]
\[
\text{(G2):} \quad \overline{\phi}(\xi) \not\equiv 0.
\]

Our goal is to prove that the equation of (12) has a solution \( \phi(\xi) \) satisfying the boundary conditions (13) by the iterative method. It is equivalent to verify that the equation of (14) has a solution \( \phi(\xi) \) satisfying (13). Naturally, we start our iteration with a supersolution of (12) as the following iteration scheme:

\[
c\phi'_n(\xi) = \delta\phi''_n(\xi) - b\phi_n(\xi) + H(\phi_{n-1})(\xi), \quad \xi \in \mathbb{R}, \quad n = 1, 2, \ldots,
\]

with the boundary conditions
\[
\lim_{\xi \to -\infty} \phi_n(\xi) = 0, \quad \lim_{\xi \to \infty} \phi_n(\xi) = w_2,
\]

where \( \phi_0(\xi) = \overline{\phi}(\xi) \). Among all solutions of (15), we choose a special one and explore its properties as below

\[
\begin{cases}
\phi_n(\xi) = \frac{1}{\delta(\beta_1 - \beta_2)} \left[ \int_{-\infty}^{\xi} e^{\beta_1(\xi-s)}H(\phi_{n-1})(s)ds + \int_{\xi}^{\infty} e^{\beta_2(\xi-s)}H(\phi_{n-1})(s)ds \right], \\
\phi_0(\xi) = \overline{\phi}(\xi),
\end{cases}
\]
where \( \xi \in \mathbb{R}, n = 1, 2, \ldots, \) and
\[
\beta_1 = \frac{c - \sqrt{c^2 + 4\delta b}}{2\delta}, \quad \beta_2 = \frac{c + \sqrt{c^2 + 4\delta b}}{2\delta}.
\]

Following the proof of lemma 3.3, lemma 3.4 and proposition 3.5 in [35] step by step, we can get the following Lemma.

**Lemma 2.8.** The sequence of functions \( \{\phi_n(\xi)\}_{n=0}^\infty \) satisfies

1. \( \phi_n \in \Gamma, \) for all \( n = 1, 2, \ldots; \)
2. \( 0 \leq \phi(\xi) \leq \phi_n(\xi) \leq \phi_{n-1}(\xi) \leq \phi(\xi) \leq w_2, \) for all \( \xi \in \mathbb{R}, n = 1, 2, \ldots; \)
3. Each \( \phi_n(\xi) \) is a supersolution of (12);
4. \( \phi(\xi) = \lim_{n \to \infty} \phi_n(\xi) \) is a solution of (12) satisfying (13).

Now, we summarize the above lemmas and obtain the following Theorem.

**Theorem 2.9.** Suppose that (12) has a supersolution \( \phi \in \Gamma \) and a subsolution \( \phi \) (which is not necessarily in \( \Gamma \)) satisfying (G1), (G2). Then (12) has a solution satisfying the boundary conditions (13). That is, (13) has a traveling wavefront solution \( \phi \), which connects 0 and the positive equilibrium \( w_2 \).

We see that it is significant for us to prove the existence of a pair of supersolution and subsolution of (12) satisfying (G1), (G2). In the rest of this subsection, we will construct such a pair of supersolution and subsolution. Following the theory of Wang et al. [34], we define the function
\[
\Delta_c(\lambda) = \delta \lambda^2 - c\lambda + \tilde{M}_c(\lambda),
\]
where \( \tilde{M}_c(\lambda) = M\beta e^{\delta \alpha_0 \lambda^2 - \lambda \alpha_0} \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z} dz. \) Then, we can get the following lemma.

**Lemma 2.10.** There exist \( c^* \) and \( \lambda^* \) such that
1. \( \Delta^c_\lambda(\lambda^*) = 0 \) and \( \frac{\partial}{\partial \lambda} \Delta_c^\lambda(\lambda)|_{\lambda = \lambda^*} = 0; \)
2. If \( 0 < c < c^* \), then \( \Delta_c(\lambda) > 0 \) for any \( \lambda > 0; \)
3. If \( c > c^* \), then the equation \( \Delta_c(\lambda) = 0 \) has two positive real roots \( \lambda_1, \lambda_2 \) such that \( 0 < \lambda_1 < \lambda_2 \) and
\[
\Delta_c(\lambda) = \begin{cases} 
> 0, & \lambda < \lambda_1, \\
< 0, & \lambda_1 < \lambda < \lambda_2, \\
< 0, & \lambda > \lambda_2.
\end{cases}
\]

**Proof.** By some calculations, we obtain
\[
\frac{\partial}{\partial \lambda} \Delta_c(\lambda) = 2\delta \lambda - c + M\beta e^{\delta \alpha_0 \lambda^2 - \lambda \alpha_0} \int_{-\eta}^{\eta} z^2 e^{-z^2} (-z)e^{-\lambda z} dz \\
+ M\beta e^{\delta \alpha_0 \lambda^2 - \lambda \alpha_0} (\delta \alpha_0 \lambda - \alpha_0) \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z} dz.
\]
\[
\frac{\partial^2}{\partial \lambda^2} \Delta_c(\lambda) = 2\delta + M\beta e^{\delta a_0\lambda^2 - \lambda ca_0} \int_{-\eta}^{\eta} z^2 e^{-z^2} \left( z - (2\delta a_0\lambda - ca_0) \right)^2 e^{-\lambda z} dz \\
+ M\beta e^{\delta a_0\lambda^2 - \lambda ca_0} 2\delta a_0 \lambda \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z} dz \\
> 0,
\]

\[
\frac{\partial}{\partial c} \Delta_c(\lambda) = -\lambda + M\beta (\lambda a_0) e^{\delta a_0\lambda^2 - \lambda ca_0} \int_{-\eta}^{\eta} z^2 e^{-z^2} (-z) e^{-\lambda z} dz < 0,
\]

\[
\Delta_c(0) = \tilde{M}_c(0) = M \beta \int_{-\eta}^{\eta} z^2 e^{-z^2} dz > 0,
\]

\[
\frac{\partial}{\partial \lambda} \Delta_c(0) = -c + M\beta \int_{-\eta}^{\eta} z^2 e^{-z^2} (-z) dz + M\beta (-ca_0) \int_{-\eta}^{\eta} z^2 e^{-z^2} dz \leq 0,
\]

\[
\Delta_0(\lambda) = \delta \lambda^2 + \tilde{M}_0(\lambda) = \delta \lambda^2 + M\beta \delta a_0 \lambda^2 \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z} dz > 0,
\]

\[
\lim_{\lambda \to +\infty} \Delta_c(\lambda) = +\infty.
\]

Then it is easy to see that the lemma holds. \(\square\)

**Lemma 2.11.** Let \(c^*, \lambda_1 \) and \(\lambda_2 \) be defined as in Lemma 2.10, and choose \(\rho > 0\) sufficiently small so that \(\rho < \lambda_1 < \lambda_1 + \rho < \lambda_2 \). Then for fix \(c > c^*\), there exists a constant \(L > 1\) such that the functions \(\overline{\phi}\) and \(\underline{\phi}\) defined by

\[
\overline{\phi}(\xi) = \min \{ w_2, w_2 e^{\lambda_1 \xi} \}, \xi \in \mathbb{R}
\]

\[
\underline{\phi}(\xi) = \max \{ 0, w_2(1 - L e^{\rho s_1}) e^{\lambda_1 \xi} \}, \xi \in \mathbb{R}
\]

are a supersolution and a subsolution of (12), respectively.

**Proof.** First of all, we see that it is easy to verify that \(\overline{\phi}, \underline{\phi}\) satisfy (G1), (G2). Now, We begin by proving that \(\overline{\phi}\) and \(\tilde{\phi}\) are a pair of supersolution and subsolution of (12). Our strategy here is to prove this part into two steps: (i) \(\overline{\phi}\) is a supersolution of (12) satisfying \(\overline{\phi} \in \Gamma\); (ii) there exists a sufficiently large \(L\) such that \(\tilde{\phi}\) is a subsolution of (12).

**Step(i):** Note that \(\overline{\phi} \in \Gamma\) is obvious. If \(\xi \in (0, +\infty)\). Then \(\overline{\phi}(\xi) = w_2, \overline{\phi}(\xi) = \overline{\phi}''(\xi) = 0\). Since the definition of \(\overline{\phi}(\xi)\), \(0 \leq \overline{\phi}(\xi - s - z - ca_0) \leq w_2\). Recalling (A1), we have

\[
c\overline{\phi}(\xi) - \delta \overline{\phi}''(\xi) + b\overline{\phi}(\xi) - H(\overline{\phi})(\xi)
\]

\[
=g(w_2) - u(w_2)w_2 - M\beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} \overline{\phi}(\xi - s - z - ca_0) dz \right] f(s)ds
\]

\[
\geq g(w_2) - u(w_2)w_2 - M\beta M_1 w_2
\]

\[
=0.
\]

If \(\xi \in (-\infty, 0)\). Then \(\overline{\phi}(\xi) = w_2 e^{\lambda_1 \xi}, \overline{\phi}(\xi) = w_2 \lambda_1 e^{\lambda_1 \xi}, \overline{\phi}''(\xi) = w_2 \lambda_1^2 e^{\lambda_1 \xi}\). Recalling (12), (16) and the assumptions (J1), (J3), one obtains

\[
c\overline{\phi}(\xi) - \delta \overline{\phi}''(\xi) + b\overline{\phi}(\xi) - H(\overline{\phi})(\xi)
\]
\[ \begin{align*}
&= cw_2 \lambda_1 e^{\lambda_1 \xi} - \delta w_2 \lambda_2^2 e^{\lambda_1 \xi} + g(\phi) - u(\phi)w_2 e^{\lambda_1 \xi} - M \beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} w_2 e^{\lambda_1 (\xi - s - z - ca_0)} d\xi \right] f_{\theta a_0} \phi(s) ds \\
&\geq (\Delta_c(\lambda_1) + \widetilde{M}_c(\lambda_1)) w_2 e^{\lambda_1 \xi} - M \beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} w_2 e^{\lambda_1 (\xi - s - z - ca_0)} d\xi \right] f_{\theta a_0} \phi(s) ds \\
&= \widetilde{M}_c(\lambda_1) w_2 e^{\lambda_1 \xi} - \widetilde{M}_c(\lambda_1) w_2 e^{\lambda_1 \xi} = 0.
\end{align*} \]

Therefore, \( \widetilde{\phi} \in \Gamma \) is a supersolution of (12).

**Step (ii):** If \( \xi \in \left( \frac{1}{2} \ln \frac{1}{\delta}, +\infty \right) \). Then \( \phi'(\xi) = 0, \phi''(\xi) = 0 \). Since the definition of \( \phi(\xi), \phi(\xi - s - z - ca_0) \geq 0 \), thus,

\[ \begin{align*}
&\frac{c\phi'(\xi)}{\phi'(\xi)} - \delta \phi''(\xi) + b\phi(\xi) - H(\phi)(\xi) \\
&= - M \beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} \phi(\xi - s - z - ca_0) d\xi \right] f_{\theta a_0} \phi(s) ds \\
&\leq 0.
\end{align*} \]

If \( \xi \in \left( -\infty, \frac{1}{2} \ln \frac{1}{\delta} \right) \). Then \( \phi(\xi) = w_2(1 - Le^{\xi})e^{\lambda_1 \xi}, \phi'(\xi) = w_2(\lambda_1 - L(\rho + \lambda_1)e^{\xi})e^{\lambda_1 \xi} \) and \( \phi''(\xi) = w_2(\lambda_2^2 - L(\rho + \lambda_1)^2 e^{\xi})e^{\lambda_1 \xi} \). By Lemma 2.10, one obtains \( \Delta_c(\rho + \lambda_1) = \delta(\rho + \lambda_1)^2 - c(\rho + \lambda_1) + M_c(\rho + \lambda_1) < 0 \). Notice from the Taylor expansion that \( g(\phi) - u(\phi)\phi = g(0) + g'(0)\phi + \frac{1}{2} g''(\theta_1)\phi^2 - (u(0) + u'(\theta_2)\phi)\phi \), where \( 0 \leq \theta_1, \theta_2 \leq \phi \). Then by the assumption (H1), we have \( g(\phi) - u(\phi)\phi \leq \bar{L} \phi^2 \leq \bar{L} w_2^2 e^{2\lambda_1 \xi} \) where \( \bar{L} = \max_{w \in [0, w_2]} \{|g''(w)| + |u'(w)|\} \). Since that \( \rho \) is small such that \( \rho < 1, g(\phi) - u(\phi)\phi \leq \bar{L} w_2^2 e^{(\rho + \lambda_1) \xi} \). Thus,

\[ \begin{align*}
&\frac{c\phi'(\xi)}{\phi'(\xi)} - \delta \phi''(\xi) + b\phi(\xi) - H(\phi)(\xi) \\
&= cw_2(\lambda_1 - L(\rho + \lambda_1)e^{\xi})e^{\lambda_1 \xi} - \delta w_2(\lambda_2^2 - L(\rho + \lambda_1)^2 e^{\xi})e^{\lambda_1 \xi} + g(\phi) \\
&- u(\phi)\phi - M \beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} \phi(\xi - s - z - ca_0) d\xi \right] f_{\theta a_0} \phi(s) ds \\
&\leq cw_2(\lambda_1 - L(\rho + \lambda_1)e^{\xi})e^{\lambda_1 \xi} - \delta w_2(\lambda_2^2 - L(\rho + \lambda_1)^2 e^{\xi})e^{\lambda_1 \xi} + \bar{L} w_2^2 e^{(\rho + \lambda_1) \xi} \\
&- M \beta \int_{-\infty}^{+\infty} \left[ \int_{-\eta}^{\eta} z^2 e^{-z^2} w_2(1 - Le^{\xi})e^{\lambda_1 (\xi - s - z - ca_0)} d\xi \right] f_{\theta a_0} \phi(s) ds \\
&= c\lambda_1 w_2 e^{\lambda_1 \xi} - cw_2 L(\rho + \lambda_1)e^{(\rho + \lambda_1) \xi} - \delta w_2 \lambda_2^2 e^{\lambda_1 \xi} + \delta w_2 L(\rho + \lambda_1)^2 e^{(\rho + \lambda_1) \xi} \\
&+ \bar{L} w_2^2 e^{(\rho + \lambda_1) \xi} - M \beta w_2 e^{\lambda_1 \xi} w_2^2 e^{\lambda_1 \xi} \delta_{a_0} - \lambda c a_0 \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-z\lambda_1} d\xi \\
&+ M \beta w_2 L(\rho + \lambda_1)e^{(\rho + \lambda_1) \xi} \delta_{a_0} - (\rho + \lambda_1)ca_0 \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{(\rho + \lambda_1) \xi} d\xi \\
&= - w_2 e^{\lambda_1 \xi} \Delta_c(\lambda_1) + w_2 L(\rho + \lambda_1) \Delta_c(\lambda_1) + \bar{L} w_2^2 e^{(\rho + \lambda_1) \xi} \\
&= w_2 e^{(\rho + \lambda_1) \xi} \Delta_c(\rho + \lambda_1) \left( L + \frac{w_2 \bar{L}}{\Delta_c(\rho + \lambda_1)} \right) < 0.
\end{align*} \]
Here, $L$ is a sufficiently large positive constant. Therefore, $\underline{\phi}$ is a subsolution of (12). The proof is complete. \hfill $\square$

2.2. Proof of Theorem 1.3. In this subsection, we also assume that (H1), (H2), (H3) hold. In order to study the population of matured mosquitoes, we need the following two results which are established by Wang et al. [34].

**Lemma 2.12.** [34, Theorem 3.3] *Equation (6) has a unique mild solution $w(t,x)$ on $[0, +\infty]$ and $w(t,x)$ is a classical solution to (6) for $(t,x) \in (a_0, +\infty) \times \mathbb{R}$. Furthermore, for any pair of supersolution $\overline{w}(t,x)$ and subsolution $\underline{w}(t,x)$ of (6) on $[0, +\infty)$ with $0 \leq w(t,x)$, $\overline{w}(t,x) \leq w_2$ for $t \in [-a_0, +\infty)$, $x \in \mathbb{R}$, and $\overline{w}(s,x) \geq \overline{w}(s,x)$ for $x \in \mathbb{R}$, $s \in [-a_0, 0]$, there holds $\overline{w}(t,x) \geq \underline{w}(t,x)$ for $x \in \mathbb{R}$, $t \geq 0$, and

$$
\overline{w}(t,x) - \underline{w}(t,x) \geq \Theta(J, t - t_0) \int_x^{x+1} (\overline{w}(t_0, y) - \underline{w}(t_0, y))dy
$$

for any $J \geq 0$, $x$ and $z \in \mathbb{R}$ with $|x - z| \leq J$, and $t > t_0 \geq 0$, where

$$
\Theta(J,t) = \frac{1}{\sqrt{4\pi dt}} \exp \left( -L_1 t - \frac{(J+1)^2}{4dt} \right), \quad J \geq 0, \quad t > 0
$$

and $L_1 = \max_{0 \leq w \leq w_2} |g'(w) + u'(w)w + u(w)|$. In particular, if there exists $x_0 \in \mathbb{R}$ such that $\overline{w}(0, x_0) > \underline{w}(0, x_0)$, then $\overline{w}(t,x) > \underline{w}(t,x)$ for any $x \in \mathbb{R}$ and $t > 0$.

**Lemma 2.13.** [34, Lemma 3.7] For each $\gamma \in (0, 1)$, there exist $\rho > 0$ and $\sigma > 0$ such that for each $\epsilon \in [0, \gamma]$, the following function

$$
\overline{w}(t,x) = (1 - \epsilon e^{-\rho t})\phi(x + ct + \sigma \epsilon e^{-\rho t})
$$

is a subsolution of (6), where $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $\phi$ is a traveling front of (6).

*Proof of Theorem 1.3.* Recalling the equation (3) which is the definition of $w(s,x)$, we have $w(s,x) = \int_{a_0}^s p_0(a + s,x)da \not\equiv 0$ for every $s \in [-a_0, 0]$ and $w(s,x) \leq w_3$. Then, by Lemma 2.12, one has that

$$
w(t,x) \leq w_3, \text{ for } t > 0 \text{ and } x \in \mathbb{R}.
$$

$$
w(t,x) > 0, \text{ for } t > 0 \text{ and } x \in \mathbb{R}.
$$

It follows that $\inf_{x \in \mathbb{R}} w(T + s,x) > 0$ for some fixed $T > 0$ and $s \in [-a_0, 0]$. Then, by Lemma 2.13, one can pick $\epsilon > 0$ close 1 enough such that

$$
\overline{w}(s,x) = (1 - \epsilon e^{-\rho s})\phi(x + cs + \sigma \epsilon e^{-\rho s})
\leq (1 - \epsilon)\phi(x + cs + \sigma e^{-\rho s})
\leq \inf_{x \in \mathbb{R}} w(T + s,x)
\leq w(T + s,x), \text{ for } s \in [-a_0, 0] \text{ and } x \in \mathbb{R}.
$$

Therefore, by Lemma 2.12, it follows that

$$
w(T + t,x) \geq \overline{w}(t,x), \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}.
$$

Then, using (19), one can obtain

$$
w(T + t,x) \geq w_2, \text{ as } t \to +\infty.
$$

This completes the proof. \hfill $\square$
3. Numerical simulations. In the following, we provide some numerical simulations to illustrate the interaction between the matured population and the control. We rescale the biting time variable $x \in [0, 24]$ into $x \in [0, 1]$ and we assume that $a^+ = 1$, that is, $a \in [0, 1)$. We take the matured age $a_0 = 0.1$. We consider system (1) with the parameters taking the values as follows

$$\delta = 0.001, \ \eta = 0.1 \ \text{and} \ \beta(a) = \begin{cases} 0, & a \in [0, a_0), \\ 200, & a \in [a_0, 1). \end{cases}$$

Firstly, we take that

$$\mu(a, w) = \begin{cases} 0.1a, & a \in [0, a_0), \\ 0.1a_0 + \frac{0.1}{1-a}, & a \in [a_0, 1), \end{cases}$$

and consider (1) under no control, that is, $u(a, w) = 0$. Then, in figures 1 and 2, we plot the matured population $w(t, x)$. We can see that $w(t, x)$ becomes very large as time goes. It implies that if there is no control, the matured population will be very large.

**Figure 1.** the matured population $w(t, x)$ for $t = 0$ and $t = 0.25$ with no control.

**Figure 2.** the matured population $w(t, x)$ for $t = 0.5$ and $t = 1$ with no control.
Now, we still take $\mu(a, w)$ be (20) and take the control large as

$$u(a, w) = \begin{cases} 
0, & a \in [0, a_0), \\
-w^2 - 90, & a \in [a_0, 1),
\end{cases}$$

Then, in following figures 3 and 4, we plot the matured population $w(t, x)$. We can see that $w(t, x)$ becomes very small as time goes. It means that under large control, the matured population will extinct.

![Figure 3](image3.png)

**Figure 3.** the matured population $w(t, x)$ for $t = 0$ and $t = 0.25$ with control $u(a, w)$.

![Figure 4](image4.png)

**Figure 4.** the matured population $w(t, x)$ for $t = 0.5$ and $t = 1$ with control $u(a, w)$.

Finally, we take that

$$\mu(a, w) = \begin{cases} 
0.1a, & a \in [0, a_0), \\
0.1a_0 + \frac{0.1}{1-a}w, & a \in [a_0, 1),
\end{cases}$$

$$u(a, w) = \begin{cases} 
0, & a \in [0, a_0), \\
-w^2 - 50, & a \in [a_0, 1).
\end{cases}$$

In following figures 5, 6 and 7, we plot the matured population $w(t, x)$. We can see that $w(t, x)$ is in $[0.4, 1.2]$ as time goes. It implies that under suitable control, the matured population will be controlled to be bounded and will not extinct.
4. **Discussion.** In this paper, we established an age-structured model to study the mosquito plasticity. In our model, we modeled that the mosquitoes can adapt their biting time to resist the usage of insecticide methods such as IRs and ITNs by the diffusion term $\delta \Delta p$. Indeed, this adaption could also be modeled by a nonlocal diffusion which will be our further study. We also assumed that the new generation
of mosquitoes can also adapt its biting time up to a maximal biting time difference \( \eta \). This was modeled by an integral term with a kernel \( K \).

We analyzed our model to study the effect of different control strategies for the matured population of mosquitoes. We proved that if the control term is small, then the matured population will become large for large time and if the control term is large, then the matured population will become small for large time. In the intermediate case, we derive a time-delayed model for the matured population which can be governed by a sub-equation and a super-equation. We proved the existence of traveling fronts for the sub-equation and use it to prove that the matured population will finally be between the positive states of the sub-equation and super-equation. The existence of traveling fronts implies that the matured mosquitoes can actually spread from one biting time to all the biting time line. Notice that this is also because the control term \( u \) is homogeneous in the biting time \( x \). If \( u \) is heterogeneous in \( x \), then the matured mosquitoes may not be existing on the whole biting time and the proof of the existence of traveling fronts will be more difficult. This will also be our further study.

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