Lie symmetry analysis and exact solutions of the one-dimensional heat equation with power law diffusivity

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Abstract
A heat equation with non-constant diffusivity depending as a power law on the spatial variable is analysed using Lie’s method to identify classical point symmetries. It is shown that the group invariant solutions of a four-dimensional symmetry subgroup can be decomposed into three different classes. These admit explicit solutions which can either be expressed in terms of Bessel functions, confluent hypergeometric functions or Coulomb wave functions.

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1. Introduction
The purpose of this paper is to find Lie point symmetries and associated exact solutions of the one-dimensional heat equation with power law diffusivity

\[ u_t - x^{2-1/a} u_{xx} = 0, \quad a \in \mathbb{R}\setminus\{0\}. \] (1.1)

If \( a = 1/2 \) one obtains the heat equation with constant diffusivity whose Lie point symmetries are well-known \([1,2]\). For \( a \to \pm\infty \) the equation can be transformed to the constant coefficient heat equation \([3]\). The transformation \( y = x^{-1}, v = y u(t,x(y)) \) yields an equation of similar form for \( v(t,y) \) with the parameter \( a \) replaced by \( -a \) \([3]\). Hence in the following we assume that \( a \in \mathbb{R}\setminus\{1/2\} \).

A second order partial differential equation (PDE) of the above type is used in stationary two-dimensional diffusion boundary layer problems to model the evaporation of particles into a turbulent medium \([3,4]\). It also plays a major role in the theory of accretion discs, an astrophysical research field which deals with rotating fluid flows of gaseous discs under the influence of friction and gravity \([4]\).

The general solution of equation (1.1) in terms of Green’s function is well known. It is obtained with help of the transformation \( y = 2ax^{1/(2a)}, v(t,y) = x^{-a} u(t,x(y)) \) for \( a \in \mathbb{R}^+ \) and application of Hankel transforms of order \( a \) with respect to the new variable \( y \) \([3,4,5]\). Some exact solutions including similarity solutions are also known \([3,5,6]\).
2. Generator of the Lie group of point transformations

Let’s consider the one-parameter Lie group of infinitesimal point transformations
\[ i = t + \varepsilon \tau(t, x, u) + \mathcal{O}(\varepsilon^2), \quad \dot{x} = x + \varepsilon \xi(t, x, u) + \mathcal{O}(\varepsilon^2), \quad \dot{u} = u + \varepsilon \eta(t, x, u) + \mathcal{O}(\varepsilon^2) \tag{2.1} \]
and its generator
\[ X = \tau \partial_t + \xi \partial_x + \eta \partial_u. \tag{2.2} \]

As usual one computes the prolongation up to second order in the derivatives of \( u \) and applies this operator to the PDE (1.1) to derive the linearized symmetry condition \([1, 2, 7]\)
\[ \eta(t) - (2 - 1/a)x^{1-1/a}\xi_{uu} - x^{2-1/a}\eta_{xx} = 0 \tag{2.3} \]
with
\[ \eta(t) = [\eta_t - \xi_t u_x + (\eta_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u^2_x] \tag{2.4} \]
\[ \eta^{(xx)} = \eta_{xx} + (2\eta_{xx} - \xi_{xx})u_x - \tau_{xx} u_t + (\eta_{uu} - 2\xi_{xx} + 2\tau_{xx} u_x u_t - \xi_{uu} u_x^2 - \tau_{uu} u_x u_t - \xi_{uu} u^2_x - \tau_{uu} u^2_x) \tag{2.5} \]

Inserting these expressions in (2.3), collecting terms of like derivatives of \( u \) and equating the coefficient functions with zero yields the determining equations
\[ \tau_u = 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \eta_{uu} = 0, \quad \eta_t - x^{2-1/a}\eta_{xx} = 0, \tag{2.6} \]
\[ \xi_t - x^{2-1/a}\xi_{xx} = 2x^{2-1/a}\eta_{xu}, \quad 2x\xi_x - (2 - 1/a)\xi = x\tau_t. \tag{2.7} \]

This is already a reduced set of equations in which some obvious simplifications were applied to eliminate redundancies. From the first four equations in (2.4) one concludes immediately that \( \tau = \tau(t) \), \( \xi = \xi(t, x) \), and \( \eta = V(t, x)u + W(t, x) \). Inserting the last expression for \( \eta \) in the fifth equation shows that \( V(t, x) \) and \( W(t, x) \) must be particular solutions of the original PDE. If \( a \neq \pm 1/2 \) the system is solved by
\[ \tau(t) = k_1 + k_2 t + k_3 t^2, \quad \xi(t, x) = a(k_2 + 2k_3 t)x, \]
\[ \eta(t, x, u) = (k_4 - k_3 (1 - a)t + a^2 x^{1/a}) u + W(t, x) \tag{2.8} \]
where the \( k_j \) are arbitrary constants. The Lie algebra associated with this symmetry group of infinitesimal point transformations is spanned by
\[ \{X_W = W(t, x)\partial_u : W_t = x^{2-1/a}W_{xx}\}, \quad X_1 = \partial_t, \quad X_2 = t\partial_t + ax\partial_x, \]
\[ X_3 = t^2\partial_t + 2atx\partial_x - ((1 - a)t + a^2 x^{1/a})u\partial_u, \quad X_4 = u\partial_u. \tag{2.9} \]

In the subsequent sections we will only consider symmetries of the four-dimensional subalgebra \( \mathcal{L}_4 \) generated by \( \{X_1, X_2, X_3, X_4\} \) whose non-vanishing structure constants \( c_{ij}^k \) which are defined according to
\[ [X_i, X_j] = \sum_{k=1}^{4} c_{ij}^k X_k \quad \text{with} \quad i, j = 1, \ldots, 4 \tag{2.10} \]
are given by
\[[X_1, X_2] = X_1, \quad [X_1, X_3] = X_2 + (a - 1)X_4, \quad [X_2, X_3] = X_3.\] (2.11)

**Remark.** The subalgebra spanned by \(\{X_1, X_2, X_3\}\) is the special linear Lie algebra \(sl(2)\). If \(a = 1\) the algebra \(L_4\) decomposes into the \(sl(2)\) and the one-dimensional subalgebra generated by \(X_4\).

### 3. Optimal system of generators

Before proceeding with the construction of the group invariant solutions we first examine the structure of the algebra and deduce an optimal system of generators. This allows for classification of the solutions into classes of equivalent solutions. Within each class we try to find the most simple representative to which any other solution in the class is related by the adjoint group action given by the Lie series

\[\tilde{X} = \text{Ad}(\exp(\epsilon X_i))X = X - \epsilon[X_i, X] + \frac{\epsilon^2}{2}[X_i, [X_i, X]] - \cdots.\] (3.1)

This maps the generator
\[X = \sum_{j=1}^{4} k_j X_j\] (3.2)

having components \(k_j\) with respect to the basis \(\{X_1, X_2, X_3, X_4\}\) to some other generator \(\tilde{X}\) with components \(\tilde{k}_j\). To obtain the optimal representative one tries to make as many of the \(\tilde{k}_j\) as possible zero. Usually it is necessary to apply different adjoint maps successively. Further simplification can be achieved by rescaling \(X\).

The crucial point is that this mapping does not necessarily connect any two generators. Instead there are restrictions due to the existence of invariants of the adjoint action, i.e., vector-valued functions \(\Phi(X)\) satisfying the following system of linear first order PDEs:

\[\sum_{j=1}^{4} k_m c^j_m \partial_{k_j} \Phi = 0 \quad \text{for} \quad i = 1, \ldots, 4.\] (3.3)

The solutions of the system (3.3) with \(c^j_m\) from (2.11) can be obtained using the method of characteristics which gives two invariants in this case

\[\Phi_1 = k_2^2 - 4k_1k_3, \quad \Phi_2 = \begin{cases} k_4 & \text{if} \quad a = 1 \\ k_2 + \frac{1}{1-a}k_4 & \text{if} \quad a \in \mathbb{R}^+ \setminus \{\frac{1}{2}, 1\} \end{cases}\] (3.4)

The existence of these invariants is the actual reason for the group invariant solutions being subdivided into subsets of equivalent solutions.

Although these invariants are preserved under the adjoint group action they are affected by rescaling \(X\). If, e.g., \(\Phi_1 = 0\) this has no effect, but if \(\Phi_1 \neq 0\) it is rescaled by a positive constant. In case of \(\Phi_2\) one can also alter the sign. Thus we must distinguish the six cases \(\Phi_1 > 0, \Phi_1 < 0, \Phi_1 = 0\) combined with either \(\Phi_2 \neq 0\) or \(\Phi_2 = 0\) to obtain the optimal system of generators

\[X_1 + \mu X_4, \quad X_2 + \mu X_4, \quad X_3 + \mu X_4, \quad X_4\] (3.5)
with $\mu \in \mathbb{R}$. The derivation for $a = 1$ can be found in [7], examples 10.2 and 10.4. The considerations for $a \neq 1$ are similar but need a little more case-by-case analysis.

4. Differential invariants and group invariant solutions

A solution $u(t, x)$ of the PDE (1.1) defined implicitly by $F(t, x, u) = 0$ is invariant with respect to the four-dimensional Lie group of point transformations (2.8) with $W(t, x) = 0$ if it satisfies the invariant surface condition $XF = 0$ with $X$ given by (3.2). Accordingly we derive the group invariant solutions admitted by the generators (3.5) for each subgroup of the optimal system. Since the invariant surface condition yields a linear first order PDE its solution is obtained solving the characteristic equations which gives the differential invariants.

The group invariant solution with respect to the generator $X_4 = u \partial_u$ is only the trivial solution $u = 0$. Thus three different classes of non-trivial group invariant solutions remain.

Case 1. Generator $X = X_1 + \mu X_4$

This group comprises the stationary and the separable solutions [3]. The differential invariants are $x$ and $u e^{\mu t}$. If $\mu = 0$ the group invariant solutions are the stationary solutions $u = C_1 x + C_2$. Otherwise we define $\mu = \pm \kappa^2$ with $\kappa > 0$ and obtain the separable solutions

$$u(t,x) = e^{\pm \kappa^2 t} \xi^a y(\xi), \quad \text{with} \quad \xi(x) = 2\kappa ax^{1/2a}$$

where $y(\xi)$ is a solution of the (modified) Bessel differential equation [10]

$$\xi^2 y'' + \xi y' + (\mp \xi^2 - a^2) y = 0. \quad (4.2)$$

Case 2. Generator $X = X_2 + \mu X_4$

This yields the scale invariant similarity solutions recently found by [6]. Some particular cases are also listed in [3, 5]. The differential invariants are $x/t^a$ and $u/t^\mu$ and the group invariant solutions are given by

$$u(t,x) = t^\mu \xi^a e^{-\xi} y(\xi), \quad \text{with} \quad \xi = a^2 x^{1/a} t^{-1}$$

where $y(\xi)$ is a solution of the confluent hypergeometric differential equation [10]

$$\xi y'' + (1 + a - \xi) y' - (1 + \mu) y = 0. \quad (4.4)$$

Case 3. Generator $X = X_1 + X_3 + \mu X_4$

With the differential invariants $x/(1 + t^2)^{1/2}$ and

$$u \left(1 + t^2\right)^{-1/2} \exp \left(-\mu \arctan t + a^2 x^{1/a} t \right). \quad (4.5)$$

the group invariant solutions of this class become

$$u(t,x) = \left(a x^{1/2a}\right)^{a-1} e^{(\mu \arctan t - \xi)} y(\xi), \quad \text{with} \quad \xi = a^2 x^{1/a} (1 + t^2)^{-1} \quad (4.6)$$
where $y(\xi)$ is a solution of the Coulomb wave equation \[10\]

$$y'' + \left(1 - \frac{\mu}{\xi} - \frac{\ell(\ell + 1)}{\xi^2}\right)y = 0,$$

with $\ell = \frac{a^2}{2} - \frac{1}{2}$. (4.7)

More general three-parameter solutions which are invariant with respect to the full group action can be obtained from these solutions applying suitable group transformations \[2, 7\]. If $F(t, x, u) = 0$ is one of the basic solutions given above then $\hat{F}(\hat{t}, \hat{x}, \hat{u}) = 0$ with

$$\hat{t} = e^{\varepsilon X_1}t, \quad \hat{x} = e^{\varepsilon X_2}x, \quad \hat{u} = e^{\varepsilon X_3}u$$

is also a group invariant solution. One easily proves with help of the Lie series, that

$$e^{\varepsilon X_1}t = t + \varepsilon, \quad e^{\varepsilon X_1}x = x, \quad e^{\varepsilon X_1}u = u$$

(4.8)

$$e^{\varepsilon X_2}t = e^\varepsilon t, \quad e^{\varepsilon X_2}x = e^{\varepsilon^2}x, \quad e^{\varepsilon X_2}u = u$$

(4.9)

$$e^{\varepsilon X_3}t = t \sum_{j=0}^{\infty} (\varepsilon t)^j, \quad e^{\varepsilon X_3}x = e^{\varepsilon^2}t x, \quad e^{\varepsilon X_3}u = e^{-\varepsilon(1-a)t + a^2 x^{1/a}}u.$$ (4.10)

Since $X_4$ commutes with all generators, this transformation (which just rescales $u$) does not yield any new solutions. The generator $X_3$ induces a transformation of time $t$ which only maps to finite values if $|\varepsilon t| < 1$. Otherwise the geometric series in (4.10) diverges. Thus depending on the time-dependence, solutions obtained applying this transformation may become infinitely large or vanish everywhere if $t$ exceeds a certain value.

It can be shown that time shifts and scalings applied to separable solutions map to other separable solutions. Therefore only transformation (4.10) generates new solutions in this case. Scale invariant similarity solutions can be transformed to new solutions applying time shifts (4.8) and the transformations (4.10). The third class of basic solutions admits new solutions if scalings (4.9) or transformations generated by $\varepsilon X_1 + X_3$ with $\varepsilon \neq 1$ are applied.

5. Conclusions

The heat equation with non-constant power-law diffusivity has been analysed using Lie group methods. A classical four-dimensional symmetry group has been derived which admits the construction of explicit solutions. The analysis of the associated Lie algebra yields three different classes of equivalent solutions. Thereby the first and second class lead to the well-known separable and scale invariant solutions. To the best of our knowledge, a solution of the third type has not been reported before. Application of the group transformations to the basic solutions yields further two- and three-parameter solutions which have not been described either.

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