Cognitive obstacles in interiorization of the Riemann’s Sum concept through APOS approach

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Abstract. The APOS approach in mathematics learning has been developed by Dubinsky and has been applied in many studies. Instructional to build the concept through the stages of Action - Process - Objects and Schema do not always run well as genetic decomposition that has been built. This study explores students' cognitive difficulties in learning advanced mathematics on the Riemanns Sum, particularly in the process of interiorization Action into Processes. The subject of research is first year student of UIN Walisongo Semarang and the study was conducted in the first half of 2018. The data were taken using observation, tests and interviews. The results of the qualitative analysis show that students got the Action well but failed to interiorize it into Process due to the inability to connect two-dimensional knowledge with the given partition. It was showed when students could not find rectangular elements on a partition built from the area under the curve with a particular function. Unexpectedly, even though students understand the problem and the student could not use the given function $y=f(x)$ to determine the partition height. Furthermore, students got an obstacle to imagine adding partitions when the number of partitions approach to infinity. The transition from school mathematics to advanced mathematics is perceived when students represent their knowledge partially, instead understanding the problem as a whole interrelated concept. This constraint also indicates the importance of mathematical connections and mastery of prerequisite materials to construct the further concepts.

1. Introduction

Learning outcomes for mathematics education has a transition between school mathematics into advance mathematics. In Indonesia, the highest level of learning outcomes for school mathematics are to have procedural knowledge, while on advanced mathematics is to master mathematical concept. On skill domain, school mathematics is expected to have the ability to think and act effectively and creatively in the abstract and concrete domains, while advanced mathematics are expected to perform mathematical problem solving, conduct abstraction, generalization and mathematical modelling, and perform computation and mathematical simulation [1]. Advance mathematics is focused to learn how to think mathematically in different way and specific, to think as mathematicians [2]. This change is a necessity because education is about learning new skills and increasing competence to do something. Therefore, mathematics in higher education is intended to develop thinking skills in order to solve new problems, which procedures are not yet known.

Many reviewed studies showed the limitations of students' cognition for the requirements of formal mathematical thinking, including abstraction [3]. Mathematical abstraction is always used in every
mathematical thinking [4]. Abstraction that experienced by students can be empirical, pseudo-empirical, or reflective abstraction. In mathematics education, reflective abstraction is a very important concept [5]. In accordance with the theory of reflective abstraction, Students' understanding of mathematical concepts develops when they make a reflection on the problem and its solution, which constructs certain mental structures. Dubinsky stated this mental structure in the form of Action, Process, Object and Schema (APOS), and happened in sequence [6].

Theoretically, the process of achieving this mental structure can be guided through genetic decomposition, but the data shows that this genetic decomposition does not always achieved. Difficulties in constructing Process will change the genetic decomposition as a whole because the sequence characteristics of APOS. This study will examine the difficulties or obstacles of learners through the interiorization Action so that the process cannot be constructed as expected.

2. Theoretical Background

2.1. Reflective Abstraction
Reflective abstraction is one of three types of abstraction mentioned by Piaget. The other are empirical abstractions and pseudo empirical. Piaget stated that reflective abstraction is the basis for all development of mathematical thinking and the highest form of human thinking [6]. Reflective abstraction is a mental mechanism whereby the structure of mathematical logic is developed in the thinking of an individual [7].

Piaget provides many examples that reflective abstraction occurs on young child mathematical thinking [6]. How the abstraction reflective emerge, there are several theories about it. Dreyfus said that it come from representation, generalization and synthesis [8]; Hershkowitz et.al which states that reflective abstraction occurs through recognition, building-with and construction [9]; and Dubinsky who said that it happens through interiorization, coordination, reserval, encapsulation, de-encapsulation, generalization, and thematization [6].

The study of reflective abstraction in advanced mathematics carried out by Nardi at 1996 who identified and explored tensions that occur when beginner mathematics meets mathematical abstractions [10]. While Sopamena et.al conducted a qualitative study to describe reflective abstractions in problem solving of number sequence [5]. Wiryanto identified the level of abstraction in problem solving of derivative application and finite integral [11]. Some studies have shown various points of view of reflective abstraction, such as thinking tension, description of the process, and levelling. This research explores the difficulties of learners in a reflective abstraction process guided by the APOS approach, especially in the of interiorization of Action, lead to Process mental.

2.2. APOS
Some research on the development and process of cognition, including abstraction, examines the instructional design of learning. The need for instructional design in research that identifies student development, and between the instructional design and research are interdependent. On the other hand, the design of a learning classroom environment is a research context. Research in instructional design has been developing, and APOS is a continuance study on construction of concept, conducted by Dubinsky, as a part of cognitive development study. Another study is Concept Image and Concept Definition (CID) by Vinner and Herskowitz at 1980, Register Semiotic Representation by Dufal at 1995, Three World of Mathematics (TWM) at 2004 by David Tall [12].

The need to consider the pedagogical aspects in abstraction research is confirmed by Ozmantar & Monaghan which stated that mathematical abstraction is situated [13]. The construction of abstractions depends on the context where the learning activities are running. Ozmantar stated that he prefers context terms than situated. They also stressed the need for scaffolding in supporting abstraction. Ozmantar noted that the research of Hershkowitz et al. that conducted in 2001 showed the existence of an interviewer who encouraged students to reflect on what had been done so that students' progress beyond
the point to be achieved without an interviewer. Encouragement from interviewers, cues, and transition focus shows the role of scaffolding in achieving abstraction.

Fakhrudin & Sukarwan have showed that geometry learning with origami as media would support the construction of empirical abstraction and and the process of theoritical abstraction occurred on assimilation and accomodation [14]. In other research, Borji et al. showed that application of APOS with Maple as learning media gave a better understanding of derivative [15]. Research of APOS in Calculus also conducted by Jojjo that stated that APOS gave a better understanding of the chain rule and hence more of calculus and its applications [16].

Reflective abstraction will come from a situation that requires a learner to understand a phenomenon or find a solution. Therefore, in learning, educators should not teach too much, but spend time working on assignments and discussing the task. The key to raising students’ reflective abstractions in the learning process, is about understanding how students understand mathematical concepts and using them to help students’ mental construction to a higher level. For this, Dubinsky has developed the APOS theory since 1980s and continues to develop until presents [6].

APOS theory is based on the principle that one's mathematical knowledge A person's mathematical knowledge is his tendency to respond to a mathematical problem situation by reflecting his problems and solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and arranging them in a scheme to be used in dealing with situations. In APOS theory, there are mental structures and mental mechanisms. Mental structures are Action, Process, Object, and Schema. These mental structures are built through mental mechanisms: Interiorization, coordination, encapsulation and thematization. Specifically, the Action is interiorized into Process mental. Two Processes can be coordinated to form a new Process, a process can be encapsulated to form a mental Object, and a Scheme can be thematized into Objects. Interiorization, coordination, reserval, encapsulation, de-encapsulation, generalization and thematization referred to Dubinsky are seven types of reflective abstractions, or mental mechanisms [6].

3. Methods
The focus of the study is the obstacles of interiorization to lead the Process of APOS on Riemann Sum. Qualitative descriptive is used as research method, and the subjects were 35 students of the Department of Mathematics Education. Nine people were taken from each of the Upper, middle and the lower group as sample. The study was conducted at Walisongo State Islamic University Semarang in March - June 2018.

Data is collected by two major methods: documentation to take on the student identity and student response on their worksheet, and depth interview to clarify the construction process.

4. Results and Discussion

4.1. Learning Process
Form four steps of APOS, this research focused on Interiorization of Action to lead to Process on Riemann Sum. Action is a repeatable physical or mental manipulation that transform Object, and Process is an action that takes place entirely in the mind [17]. Both of these mental structures can be classified as operational conception. Interiorization, in this research, refers to the ability to apply symbols, language, pictures and mental images to construct internal processes as a way of making sense out of perceived phenomena. An action is a transformation of an object according to an algorithm, and at least externally driven. When students repeat the action and reflect on it, the action is interiorized into the mental process [18].

On Riemann Sum, in this research, action was conducted to find Area under curve repeatedly. The object is Area, and the mental manipulation is finding area under the curve. The curves are shown on Figure 1.
Before the learning activity, students made apperceptions about the Area of Plane and it is ensured that all students in the class understood the formula of Area for triangle, trapezoid and rectangular. For the figures 1a and 1b, students asked to find the whole area with trapezoidal and triangle area approach. While in Figure 1c and 1d, students would find the area shaded by adding up the area of each partition. To determine the area, students must determine the elements of each plane refer to the formula. The classroom discussion was designed in groups, and in each group consist of 5-6 students, at least one of them is students from upper group, and the rest is heterogenic. This is intended to build initial strategy to solve the problems, and the next strategy is expected to develop from the results of the discussion.

The results of this study showed that:

(1) Action is designed in divergent problem to explore students’ understanding, thinking processes and mathematics representations. Unfortunately, it seems that students are not used to think divergently. Students need a more and more time to determine the solution of the problem.

(2) The group leaders are chosen from the upper group, and others are randomly distributed. Performance of each group shown that the more homogeneous group is more productive and creative. Groups that consist of middle-class students are more productive than groups consisting of upper, middle and lower students or upper and lower students only. Ideas flow more naturally and more mathematics creativity appears in lower variability groups. The interview results lead to the conclusion that in the Activity and Classroom Discussion there are communication barriers to convey mathematical ideas.

(3) Even though students understand the formula to find the Area, students cannot decide elements of the area on the context. It caused they can’t build some execute their plan to solve the problem.

For example, on the Figure 1a, to find the length of parallel line of trapezoid, students can find the first line but failed to find the second line, because they could not connect function $y = 1 + t$ and $t = x$. There is a group that failed to find the high of trapezoid. It should be $x$, but students write as $2 + x$. 

![Figure 1](image-url)
Other groups estimate the high as 2.75, without any reason. For the Figure 1b, all of the groups understand the problem and can make a plan to find out the solution. Unfortunately, the majority of groups cannot determine the base of the triangular but after got a deepth assisstance, they can determine the base of the triangle in the same way as problem Figure 1a.

For problem number 3, the majority of groups know that the base length of each rectangle is 0.5, but cannot determine the height of the rectangle. This is because (1) Like Figure 1a, they culdnot use the function to find the height of the triangle. They can not relate and connect the concept of function to the context. (2) Students known that the base of each rectangular is 0.5, but, again, they cannot connect this understanding to the context of the problem, the height of the rectangle cannot be found because the point on the x-axis is not recognized. (3) there is a group (group 4) that calculate the area by area of trapezoid minus total area of blank triangles. The problem in solving of Figure 1,d is same as Figure 1c.

5. Conclusion

From the description above, it can be concluded that students’ cognition obstacles in carrying out Activity and Classroom Discussions for the interiorization of Action to Process, are (a) connection of mathematical concepts that are already known to the problem context. Considering that the prerequisite concept is already understood, but students cannot use it, also indicates an obstacle in the process of recalling or remembering the schema that has been formed before, (b) Mathematics communication, oral and written, becomes important in the construction of reflective abstraction regarding on the Activity and Classroom Discussion processes ideas flow through this communication, and (c) to encourage the problem solving, students difficult to think divergently. They tend to view of mathematical problems as complicated one. This perception prevents the mind from simplifying mathematical problems, things that they should be done.

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