Calculable Leptonic CP Violation

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Abstract

We propose a possibility to estimate the (Dirac-type) CP phase in case that the neutrino mixing matrix is perturbatively modified tri-bimaximal or bimaximal mixing. The expressions for the CP phase are derived from the equivalence between the standard parametrization of the neutrino mixing matrix for the Majorana neutrino and modified tri-bimaximal or bimaximal mixing matrices with appropriate CP phases. Carrying out numerical analysis based on the current experimental results for neutrino mixing angles, we can predict the values of the CP phase for several possible cases.

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I. INTRODUCTION

The recent measurements of not-so-small value of the reactor neutrino mixing angle have opened up new windows to probe leptonic CP violation (LCPV) \[1\]. Establishing LCPV is one of the most challenging tasks in future neutrino experiments \[2\]. The PMNS neutrino mixing matrix \[3\] is presented by \(3 \times 3\) unitary matrix which contains, in addition to the three angles, a Dirac type CP violating phase in general as it exists in the quark sector, and two extra phases if neutrinos are Majorana particles. Although we do not yet have compelling evidence for LCPV, the current fit to neutrino data indicates nontrivial values of the Dirac-type CP phase \[4\]. Several experiments have been proposed or being scheduled to establish CP violation in neutrino oscillations \[5\]. In this situation, it must deserve to investigate possible size of LCPV detectable through neutrino oscillations. From the point of view of calculability, it is conceivable that a Dirac type LCPV phase may be calculable with regards to some observables \[6\]. In this letter, we propose possible forms of neutrino mixing matrix that lead us to estimate the possible size of LCPV phase, particularly, in terms of two neutrino mixing angles only, in the PDG-type standard parametrization \[7\].

The estimation of LCPV phase is carried out by the following procedures:

- Constructing the neutrino mixing matrix with appropriate CP phases so as to accommodate the current neutrino oscillation data in such a way to perturb conventional (tri-)bimaximal matrix.

- Deriving the master formulae linking the Dirac-type CP phase with neutrino mixing angles from the equivalence principle that any forms of neutrino mixing matrix should be equivalent to the standard parametrization of the PMNS mixing matrix.

As will be shown later, the neutrino mixing matrices we adopt at the first step contain a maximal mixing angle which plays a crucial role in deriving the relations among neutrino mixing angles and Dirac-type CP phase in the standard parametrization. Substituting values of neutrino mixing angles into those equations obtained at the second step, we perform numerical analysis on observables for the LCPV and present the results.
II. NEUTRINO MIXING MATRICES

In the leading order approximation, the conventional neutrino mixing matrices in the flavor basis can be given by

\[
U^{\text{PMNS}}_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]  

(1)

Taking \( \sin \theta \) to be either \( 1/\sqrt{3} \) or \( 1/\sqrt{2} \) leads to the so-called tri-bimaximal mixing \( U^{\text{TBM}}_0 \) or bimaximal mixing \( U^{\text{BM}}_0 \), respectively \[8, 9\]. Although the tri-bimaximal and/or bimaximal ones are theoretically well-motivated patterns of the neutrino mixing matrix, they are challenged by the current experimental results for three neutrino mixing angles. While the bimaximal mixing has already been ruled out by the non-maximal mixing for the solar angle, the current measurements of non-zero \( \theta_{13} \) definitely disfavor the exact tri-bimaximal mixing. Since the measured values of \( \theta_{13} \) are turned out to be of order of the required deviation of \( \theta_{12} \) from maximal, the tri-bimaximal mixing can be treated on the same footing with the bimaximal mixing as leading order approximation of the neutrino mixing matrix.

The simplest possible forms of the neutrino mixing matrix without CP phases deviated from the (tri-)bimaximal mixing patterns are given by

\[
\begin{cases}
U^{(T)BM}_0 \cdot U_{ij}(\theta), \\
U^\dagger_{ij}(\theta) \cdot U^{(T)BM}_0,
\end{cases}
\]

(2)

where \( U_{ij}(\theta) \) represents the unitary matrix corresponding to the rotation with the angle \( \theta \) in \((i, j)\) plain. Once the mixing angle \( \theta \) can perturbatively be treated \[10\], then Eq. (2) possibly gives rise to non-zero value of the reactor angle as well as deviation from the maximal for the solar angle. As will be shown later, eight forms among twelve possible ones in Eq. (2) are in consistent with present neutrino data within \( 3\sigma \) C.L. In this respect, we call those eight forms of the neutrino mixing matrix \textit{minimally-modified (tri-)bimaximal (M(T)BM) parameterizations}. It is worthwhile to notice that those forms of the neutrino mixing matrix keep a column or a row in (tri-)bimaximal mixing matrix unchanged, which may be regarded as a remnant of a possible horizontal symmetry leading to (tri-)bimaximal mixing. The column vectors orthogonal to the \( i \)-th and \( j \)-th ones in \( U^{(T)BM}_0 \) are unchanged for \( U^{(T)BM}_0 U_{ij}(\theta) \), whereas the row vectors orthogonal to the \( i \)-th and \( j \)-th ones are unchanged.
for $U_{ij}^\dagger(\theta)U_0^{(T)BM}$. The multiplication of $U(\theta)_{ij}$ represents unitary transformation of the symmetry operator which corresponds to the rotation of two column vectors in the mixing matrix. Thus, a symmetry argument can still be applied to the origin of the neutrino mixing matrices in the M(T)BM parameterizations [11].

Since a Dirac-type CP phase $\delta_D$ is accompanied by $\theta_{13}$ in the standard parametrization, it is natural to involve CP phases when construct neutrino mixing matrix so as to generate non-zero $\theta_{13}$. Interesting points in this work based on the simplest forms of neutrino mixing matrix aforementioned are that $\theta_{13}$ is related with either $\theta_{12}$ or $\theta_{23}$, and $\delta_D$ in the standard parametrization can be related with two neutrino mixing angles as long as we identify the M(T)BM parameterizations with the standard one. It is desirable to include a Dirac-type CP phase $\xi$ in $U(\theta)_{ij}$. Among the above twelve forms of the mixing matrix given in Eq. (2), the forms $U_0^{(T)BM}U_{12}(\theta, \xi)$ and $U_0^{(T)BM}U_{23}(\theta, \xi)$ still lead to vanishing reactor mixing angle, and thus predict no CP violation. We do not consider these cases any longer. Therefore, all the forms of the mixing matrix eligible for our aim are presented as follows;

$$V = \begin{cases} 
  U_0^{TB}U_{23}(\theta, \xi) & \text{(Case–A)}, \\
  U_0^{TB}U_{13}(\theta, \xi) & \text{(Case–B)}, \\
  U_{12}^\dagger(\theta, \xi)U_0^{TB} & \text{(Case–C)}, \\
  U_{13}^\dagger(\theta, \xi)U_0^{TB} & \text{(Case–D)}, \\
  U_{12}^\dagger(\theta, \xi)U_0^{BM} & \text{(Case–E)}, \\
  U_{13}^\dagger(\theta, \xi)U_0^{BM} & \text{(Case–F)}, \\
  U_0^{BM}U_{23}(\theta, \xi) & \text{(Case–G)}, \\
  U_0^{BM}U_{13}(\theta, \xi) & \text{(Case–H)}. 
\end{cases} \quad (3)$$

III. CALCULATION OF LEPTONIC CP VIOLATION

Now we demonstrate how to derive $\delta_D$ in terms of neutrino mixing angles in the standard parametrization. This can be done from the equivalence between one of the parameterizations in Eq. (3) and the standard parametrization, shown in Eq. (4).

Assuming that neutrinos are Majorana particles, we begin by explicitly presenting the
PMNS neutrino mixing matrix in the PDG-type standard parametrization as follows \[7\],

\[ U^{\text{ST}} = U^{\text{PMNS}} \cdot P_\phi \]

\[
\begin{pmatrix}
  c_{12}c_{13} & -s_{12}c_{13} & -s_{13}e^{i\delta D} \\
  s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta D} & c_{12}c_{23} + s_{12}s_{23}s_{13}e^{-i\delta D} & -s_{23}c_{13} \\
  s_{12}s_{23} + c_{12}c_{23}s_{13}e^{-i\delta D} & c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta D} & c_{23}c_{13}
\end{pmatrix}
\]

\[P_\phi, \tag{4}\]

where \( P_\phi \equiv \text{Diag.}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}) \) is a \( 3 \times 3 \) phase matrix. Note that one of three phases in \( P_\phi \) is redundant. Incorporating phase matrices \( P \) defined above, the neutrino mixing matrices in Eq. (3) are given by \( P_\alpha \cdot V \cdot P_\beta \). Without those two phase matrices \( P_\alpha \) and \( P_\beta \), in general, we cannot equate the M(T)BM parameterizations with the standard parametrization given in Eq. (4). Please note that such bi-unitary transformation is regarded as a general basis change of leptonic fields. The equivalence between both parameterizations dictates the following relation,

\[ V_{ij}e^{i(\alpha_i + \beta_j)} = U^{\text{ST}}_{ij} = U^{\text{PMNS}}_{ij}e^{i\phi_j}. \tag{5} \]

Applying \(|V_{13}| = |U^{\text{ST}}_{13}|\) and \(|V_{11}/V_{12}| = |U^{\text{ST}}_{11}/U^{\text{ST}}_{12}|\) to Cases A, B, G and H, we obtain the relations between the solar and reactor mixing angles,

\[ s_{12}^2 = \begin{cases} 
1 - \frac{2}{3(1-s_{13}^2)} & \text{(Case–A)}, \\
\frac{1}{3(1-s_{13}^2)} & \text{(Case–B)}, \\
1 - \frac{1}{2(1-s_{13}^2)} & \text{(Case–G)}, \\
\frac{1}{2(1-s_{13}^2)} & \text{(Case–H)}. 
\end{cases} \tag{6} \]

Those relations indicate that non-zero values of \( s_{13}^2 \) lead to \( s_{12}^2 < 1/3 \) for Case–A and \( s_{12}^2 > 1/3 \) for Case–B. While the results for Case–A are well consistent with the current experimental values of \( s_{12}^2 \) at 1\( \sigma \) C.L., those for Case–B are so at 2\( \sigma \) C.L. It turns out that the above relations for Cases G and H are not consistent with experimental results up to 3\( \sigma \) C.L., and thus ruled out.

Similarly, we get the relations between the atmospheric and reactor mixing angles from \(|V_{13}| = |U^{\text{ST}}_{13}|\) and \(|V_{23}/V_{33}| = |U^{\text{ST}}_{23}/U^{\text{ST}}_{33}|\),

\[ s_{23}^2 = \begin{cases} 
1 - \frac{1}{2(1-s_{13}^2)} & \text{(Cases–C and –E)}, \\
\frac{1}{2(1-s_{13}^2)} & \text{(Cases–D and –F)}. 
\end{cases} \tag{7} \]
We see that non-zero values of $s_{13}^2$ lead to the values of $s_{23}^2 < 1/2$ for Cases–C and –E and $s_{23}^2 > 1/2$ for Cases–D and– F. They are turned out to be consistent with experimental values of $s_{23}^2$ at 2$\sigma$ C.L.

Now, let us derive the relations among $\delta_D$ and neutrino mixing angles in the standard parametrization. Since the same method can be applied to all the cases, we only present how to derive the relation only for Case–A. From the components of the neutrino mixing matrix for Case–A, we see that

$$\frac{V_{23} + V_{33}}{V_{22} + V_{32}} = \frac{V_{13}}{V_{12}}. \quad (8)$$

Applying the relation (5) and $V_{21} = V_{31}$ to Eq. (8), we can get the relation

$$\frac{U_{13}^{ST}}{U_{12}^{ST}} = \frac{U_{23}^{ST}U_{31}^{ST} + U_{33}^{ST}U_{21}^{ST}}{U_{22}^{ST}U_{31}^{ST} + U_{32}^{ST}U_{21}^{ST}}. \quad (9)$$

Presenting $U_{ij}^{ST}$ in terms of the neutrino mixing angles as well as $\delta_D$, and taking the real part in Eq. (9), we get the equation for $\delta_D$ as

$$\cos \delta_D = \frac{1}{2\tan 2\theta_{23}} \cdot \frac{1 - 5s_{13}^2}{s_{13}\sqrt{2 - 6s_{13}^2}}. \quad (10)$$

Notice that the imaginary part in Eq. (9) is automatically cancelled. Using the above formulae, we can easily derive the leptonic Jarlskog invariant as follows;

$$J_{CP}^2 = (\text{Im}[U_{\mu 2}^{ST}U_{e 1}^{ST}U_{\mu 3}^{ST}U_{e 2}^{ST*}U_{\mu 3}^{ST*}])^2 = \frac{1}{12^{2}}[8s_{13}^2(1 - 3s_{13}^2) - \cos^2 2\theta_{23}c_{13}^4]. \quad (11)$$

**TABLE I.** Formulae for $\cos \delta_D$ and $J_{CP}^2$ for Cases B – F. The second column corresponds to the relation (8) for Case A. $\eta_{ij} = \frac{1}{2\tan 2\theta_{ij}}$, $\kappa_{ij} = \cos^2 2\theta_{ij} \cdot c_{13}^4$, $\zeta = \sin 2\theta_{12}$ and $\omega = (s_{13}^2(9s_{12}^2 - 4) - 3s_{12}^2 + 1)^2$
By taking the same procedure described above, we can obtain the formulae for $\delta_D$ and $J_{CP}^2$ for Cases B – F as presented in Table I. Note that the Cases G and H are experimentally ruled out as previously mentioned.

FIG. 1. Predictions of $\delta_D/\pi$ in terms of $s_{23}^2$ ((a): Cases A and B) and $s_{12}^2$ ((b): Cases C and D) ((c):Cases D and E) based on the experimental data at $3\sigma$ and $1\sigma$ (only for Case–A) C.L. Regions in blue (red) correspond to Cases A, C and E (B, D and F).
**Numerical Results**—For our numerical analysis, we take the current experimental data for three neutrino mixing angles as inputs, which are given at $1\sigma$ – $3\sigma$ C.L., as presented in Ref. [4]. Here, we perform numerical analysis and present results only for normal hierarchical neutrino mass spectrum. It is straight-forward to get numerical results for the inverted hierarchical case. Using experimental results for three neutrino mixing angles, we estimate

![Contour plots](image)

**FIG. 2.** Contour plots for each values of $J_{CP}$ in the plain ($s_{23}^2$, $s_{13}^2$) for (a) Cases A ($1\sigma$), (b) A ($3\sigma$) and (c) B ($3\sigma$).
the values of $\delta_D$ and $J_{CP}$ in terms of neutrino mixing angles through Eqs. (10) and (11), respectively, and the formulae presented in Table I.

Fig. 1 shows the predictions of $\delta_D/\pi$ in terms of $s_{23}^2$ ((a): Cases A and B) and $s_{12}^2$ ((b): Cases C and D) ((c): Cases E and F) based on the corresponding experimental data given at $3\sigma$ C.L. Regions in blue (red) correspond to Cases A, C and E (B, D and F). In particular, the region in dark-blue in Fig. 1(a) corresponds to the results obtained by using the experimental data at $1\sigma$ C.L. for Case–A, which apparently indicates CP violation. The width of each bands implies the variation of the other mixing angles, $s_{12}^2$ (Cases A and B) and $s_{23}^2$ (Cases C-F). We see that $\delta_D \sim \pi$ can be achieved by Case–C. Rather large values of CP phase are predicted as $s_{23}^2$ increases (decreases) for Case–A (B), and $s_{12}^2$ decreases (increases) for the Case–C (D).

In Figs. 2 and 3, we display contour plots for each value of $|J_{CP}|$ in the plains of $(s_{23}^2, s_{13}^2)$ (a-c) and $(s_{12}^2, s_{13}^2)$ (d,e). The panels (a) and (b) correspond to the results for Case–A obtained by using the experimental data at $1\sigma$ and $3\sigma$ C.L., respectively. The results for Cases B, C (D) and E (F) based on the experimental data at $3\sigma$ C.L. are displayed in the panels (c), (d) and (e), respectively. We note that the sizes of $|J_{CP}|$ in the lepton sector can be as large as $0.03 \sim 0.04$ which are much larger than the values of the quark sector as

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FIG. 3. Contour plots for each values of $J_{CP}$ in the plain $(s_{12}^2, s_{13}^2)$ for (d) Cases C, D and (e) E, F.
order of $10^{-5}$, and expected to be measurable in foreseeable future. We see from Fig. 3-(e) that the region of $s_{12}^2 < 0.32$ for Cases E and F is excluded because it leads to $|\cos \delta_D| > 1$ for the experimentally allowed region of $s_{13}^2$.

As a summary, we have proposed an Ansatz to estimate possible size of the Dirac-type CP phase $\delta_D$ with regards to two neutrino mixing angles in the standard parametrization of the neutrino mixing matrix. This has been achieved by equating one of M(T)BM parameterizations of the neutrino mixing matrix with the standard parametrization of the PMNS one. Through the procedure, we could obtained several relations expressed in terms of two or three neutrino mixing angles and Dirac-type CP phase. We have also discussed how the parameterizations of the neutrino mixing matrix can be related with symmetries associated with tri-bimaximal or bimaximal mixing. However, the scheme we proposed could not lead to the predictions on the Majorana phases yet in terms of the mixing angles.

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