Gaussian black holes in Rastall Gravity

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In this short note we present the solution of Rastall gravity equations sourced by a Gaussian matter distribution. We find that the black hole metric shares all the common features of other regular, General Relativity BH solutions discussed in the literature: there is no curvature singularity and the Hawking radiation leaves a remnant at zero temperature in the form of a massive ordinary particle.

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1. Introduction

In a variety of papers\cite{1,2,4} regular solutions of Einstein field equations generated by non-point like distributions of matter have been presented. Apart from different choices of matter source they all share the same characteristic features:
• the curvature singularity at the origin is replaced by an inner core of regular
de Sitter vacuum.

• They posses inner and outer horizons and admit an extremal configuration.

• The end-point of the Hawking evaporation is a zero temperature massive
remnant.

Recently, a black hole solutions sourced by a Gaussian energy momentum tensor
have been considered in the framework of Rastall gravity. Despite the regularity
of the source, the authors argued that, in this particular case, the above features
appears not to be simply realized. More in detail, they find that:

(1) there is no de Sitter core;
(2) the curvature singularity is still present in \( r = 0 \);
(3) the tangential pressure diverges near \( r = 0 \);
(4) the BH will totally convert its mass into thermal radiation leaving no remnant.

In this note we shall critically analyze the above statements and show that Gaussian
BHs shares all the characteristics of regular BHs contrary to the conclusions in. The paper is organized as follows. In Sect.(2) we present the essential features of
Rastall gravity; in Sect.(3) we solve the field equations with a Gaussian source. We
discuss the regularity of the solution and its thermodynamical behavior. In Sect.(4)
we summarize and the main results and their robustness.

2. A short introduction to Rastall gravity

Rastall introduced an interesting modification of General Relativity, where the
covariant conservation condition \( T_{\mu \nu ; \nu} = 0 \) is relaxed to a more general relation
relation

\[
T_{\mu \nu ; \nu} = a^\mu
\]  

Consistency with Special Relativity requires \( a^\mu \to 0 \) in the limit of vanishing
space-time curvature. Thus, a convenient choice, compatible with this limit, for a
d four-vector \( a^\mu \) is

\[
a^\mu = \lambda R \cdot \mu
\]

where, \( R \) is the Ricci scalar and \( \lambda \) is a free parameter. Equations (1), (2) lead to a modified Einstein field equations which read

\[a\text{In order to avoid misunderstanding, please, note that } \lambda \text{ is not the cosmological constant, but a}\text{ new free parameter.} \]
\[ R_{\mu \nu} - \frac{1}{2} (1 - 2\kappa \lambda) g_{\mu \nu} R = \kappa T_{\mu \nu} , \]  
(3)

\[ R = \frac{\kappa}{(4 \kappa \lambda - 1)} T , \quad \kappa \Lambda \neq 1/4 \]  
(4)

\[ T_{\mu \nu} = \lambda R \, \xi^\mu \]  
(5)

It is clear that Eq. (4) is not defined for \( \kappa \lambda = 1/4 \). In this case, Eq. (3) reads:

\[ R_{\mu \nu} - \frac{1}{4} g_{\mu \nu} R = \kappa T_{\mu \nu} \quad \rightarrow \quad T_{\mu}^\mu = 0 , \quad \forall R \]  
(6)

Thus, the choice \( \kappa \lambda = 1/4 \) leads to a consistent set of field equations only for a traceless energy momentum tensor. Even if this can be an interesting problem by itself, it is not the one we are going to address in this note.

In order to reproduce Einstein equations in the limit \( \lambda \rightarrow 0 \), the gravitational coupling \( \kappa \) must be identified as \( \kappa = 8\pi G_N \), but to make the field equations (3), (4), (5) analytically solvable, the better choice is \( \kappa \lambda = 1/2, \kappa = 4\pi G_N \).

\[ R_{\mu \nu} = \kappa T_{\mu \nu} , \]  
(7)

\[ T_{\mu \nu} = \frac{1}{2} T_{\nu}^\mu \]  
(8)

where \( T \equiv T_{\mu}^\mu \).

A Lagrangian formulation for Rastall gravity with general \( \lambda \) has been proposed in 16 and we refer to this paper for more details. This generalized gravitational model has been recently used as an alternative, phenomenologically motivated, description in different cases.17–21

3. Rastall Gaussian BHs

In this Section we are going to look for the exact regular solutions of the field equations for Rastall gravity, in order to compare it to the known solutions in General Relativity. The source is given by an anisotropic fluid energy-momentum tensor \( T_{\mu}^\nu = Diag ( -\rho, p_r, p_\perp, p_\perp ) \), with matter density given by a Gaussian source

\[ \rho \equiv \frac{M}{(4 \pi l_0^2)^{3/2}} e^{-r^2/4l_0^2} \]  
(9)

\( M \) is the total mass/energy of the system, given by

\[ M \equiv 4\pi \int_{0}^{\infty} dr r^2 \rho (r) , \]  
(10)

and the width \( l_0 \) of the Gaussian distribution is kept as a free parameter. In the original paper, where Gaussian BHs were first introduced,23 length scale \( l_0 \) was related to the parameter \( \theta \) characterizing coordinate non-commutativity, but other
The fluid satisfies the equation of state

\[ p_r = -\rho , \]  

(11)

which at short distance reproduces the “vacuum” state equation \( \rho(0) = -p_r(0) < \infty \).

The generalized conservation (4) determines the remaining tangential pressure \( p_\perp \) as

\[ \frac{dp_r}{dr} + 2 \frac{2}{r} (p_r - p_\perp) = \frac{1}{2} \frac{d}{dr} (-\rho + p_r + 2p_\perp) \]  

(12)

This equation will be solved in the next section.

We are looking for a spherically symmetric metric of the general form

\[ ds^2 = -f(r) \, dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2, \]

\[ f(r) \to 1 \quad \text{as} \quad r \to \infty. \]

The function \( f \) is determined by the above matter source. One has to solve the single field equation

\[ R^r_r = \kappa T^r_r = -\kappa \rho \]

(13)

Oddly enough, since the \( R^r_r \) component of the Ricci tensor is

\[ R^r_r = -\frac{1}{2r^2} \frac{d}{dr} \left( r^2 f' \right), \]

(14)

it follows that the Rastall field equations collapse to the classical Poisson equation

\[ \Delta f(r) = 2\kappa \rho \]

(15)

which has the known solution\(^{25}\)

\[ f(r) = 1 - \frac{2MG_N}{r} \frac{\gamma \left( 1/2 ; r^2/4l_0^2 \right)}{\Gamma(1/2)} \]

(16)

where, \( \gamma \left( 1/2 ; r^2/4l_0^2 \right) \) is the lower incomplete gamma-function. It is important to recall that Gaussian BH solutions in 4D General Relativity have, instead, \( \gamma \left( 3/2 ; r^2/4l_0^2 \right). \)\(^{23, 26, 27}\) The presence of \( \gamma \left( 1/2 ; r^2/4l_0^2 \right) \) is a particular feature of the Rastall model and has nothing to do with any weak-field approximation, or classical limit of the field equations.

\(^b\) The most “traditional” choice is probably \( l_0 = l_{Pl} \), but other assignments are equally physically meaningful.\(^{23}\)

\(^c\) It is important to remark that this is not a classical vacuum. Rather, this state equation emulates the properties of some possible “quantum gravitational” vacuum at short distance. The negative pressure provides a stabilizing mechanism avoiding the complete collapse of the matter source and replaces the central singularity with a de Sitter core.
In Sect. we study the short-distance behavior of the solution and prove the existence of the de Sitter core. We solve equation and show that $p_\perp$ is regular everywhere. Finally, we study the Hawking process and find that the evaporation ends at zero temperature leaving a cold, massive, particle as a finite remnant of the original BH. Deviations from the standard “area law”, due to the extended nature of the source, are also computed in the “large” and “small” BH limits.

In Sect. we briefly conclude.

3.1. The de Sitter inner core

As already mentioned in the introduction, all known regular BH solutions exhibit a de Sitter geometry near the origin. Thus, it is important to see what happens in the case of Rastall gravity in the same region.

The short distance behavior of the $\gamma \left( \frac{1}{2}, \frac{r^2}{4l_0^2} \right)$ function is given by

$$\gamma \left( \frac{1}{2}, \frac{r^2}{4l_0^2} \right) = \frac{r}{l_0} + \frac{r^3}{6l_0^3} + \cdots \quad (17)$$

and one easily finds that the short distance form of the metric is

$$-g_{00} = g_{rr}^{-1} = 1 - \frac{2MG_N}{\sqrt{\pi}l_0} - \frac{MG_N}{6\sqrt{\pi}l_0^3} r^2 + O(r^4) \quad (18)$$
which is, again, a de-Sitter line element, characterized by an effective cosmological constant \( \Lambda = \frac{M G N}{2 \sqrt{\pi l_0^3}} \). The presence of the extra constant \(-2G_N M/\sqrt{\pi l_0}\) does not alter the geometry and, simply, amounts to a constant rescaling of the units of length and time. In other words, a general metric tensor of the form \( \eta_{\mu\nu} = \text{Diag} (-\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) with constant components, is the Minkowski metric and can be written with \( \lambda_i = 1 \) by a simple rescaling of the units along the four axis. The same reasoning applies to the metric (18). On the other hand, it is well known that de Sitter space-time has a constant, finite, Ricci scalar \( R = -4\Lambda \).

### 3.2. Regular Ricci scalar and the tangential pressure

Let us give a more in depth look at the energy momentum tensor components:

\[
T^\mu_\nu = \text{Diag} (-\rho, p_r, p_\perp, p_\perp) \tag{21}
\]

The state equation is

\[
\rho = -p_r \tag{22}
\]

and the tangential pressure \( p_\perp \) is determined in terms of the matter density \( \rho \) as

\[
\frac{dp_\perp}{dr} + \frac{2}{r} p_\perp = \frac{2}{r} \rho \tag{23}
\]

This equation must be solved with a boundary condition in \( r = 0 \) consistent with de Sitter metric, i.e.

\[
p_\perp(0) = p_r(0) = -\rho(0) \tag{24}
\]

Thus, we the correct tangential pressure turns out to be

\[
p_\perp(r) = \frac{M}{2 \pi^{3/2} l_0 r^2} \left[ e^{-r^2/4l_0^2} - 1 \right] \tag{25}
\]

Equation (25) is finite everywhere, vanishes as \( r \to \infty \), and satisfies short distance behavior.

\[\text{[In this regard, it is useful to remark a difference with respect to General Relativity. In Rastall gravity the vacuum field equations in the presence of a cosmological constants read]

\[
R_{\mu\nu} - \frac{1}{2} (R - 2\Lambda) g_{\mu\nu} + \kappa \lambda g_{\mu\nu} R = 0 \tag{19}
\]

If \( \kappa \lambda = 1/2 \) we obtain

\[
R_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{20}
\]

and \( R = -4\Lambda \). General Relativity one obtains \( R = +4\Lambda \).}
In fact, considering short distance behavior of the Ricci scalar one finds

\[ R = 2k (p_\perp - \rho) , \]
\[ R(0) = 2k (p_\perp(0) - \rho(0)) = -4k\rho(0) = -4\Lambda < \infty \]  

(27)

This result asserts that the regular solutions of Rastall gravity behaves in the same way as the corresponding ones in GR, as far as the properties of the curvature are concerned. No singular behavior appears whatsoever.

3.3. **Black hole evaporation**

Firstly, let us note that the mass spectrum of the Rastall Gaussian BHs has a lower bound. In fact, the mass-horizon relation is given by

\[ M = \frac{r_H}{2G_N} \frac{\Gamma(1/2)}{\gamma (1/2 : r_H^2)} \]  

(28)

This is a monotonically increasing function, for each \( r_H \geq 0 \) there is a unique value of \( M \geq M_0 \). Below \( M_0 \) there is no horizon and the line element (16) describes an *ordinary* particle-like object. By using equation (17) one finds the value of minimal BH mass to be
Fig. 3. Plot of the function (28). The intercept on the vertical axis represents the lower bound of the BH mass spectrum.

\[ M_0 = l_0 \sqrt{\frac{\pi}{2G_N}} = \sqrt{\frac{\pi}{4\pi G N}} l_0 \frac{l_0}{l_{Pl}} \]  

(29)

where, we use the standard definition of Planck length and mass, given by \( 2G_N = l_{Pl}^2 = M_{Pl}^2 = l_{Pl}/M_{Pl} \). The mass spectrum (28) is bounded from below by the minimum value \( M_0 \) needed to have an event horizon. Equation (29) shows the relation between \( M_0 \) and the width \( l_0 \). Two choices of \( l_0 \) can be useful to clarify the physical meaning of \( M_0 \). The most intuitive assignment is the identification \( l_0 = l_{Pl} \), leading to

\[ M_0 = \sqrt{\frac{\pi}{4}} M_{Pl} = 1.76 M_{Pl} \]  

(30)

giving the same minimal mass as in Figure(1).

However, one could argue that this choice is more appropriate in a full quantum gravity framework rather than in the semi-classical regime we considering here. A more precautionary value for the width of the Gaussian distribution is the Compton wavelength associated to the mass \( M \) itself. In this case

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\[ The use of the definitions 2G_N = l_{Pl}^2 = M_{Pl}^2 = l_{Pl}/M_{Pl} guarantees that, in natural units h = 1, c = 1, for M = M_{Pl} the Schwarzschild radius and the Compton length coincide.\]
\[ M_0 = \pi^{1/4} M_{Pl} = 1.33 M_{Pl}, \]  
\[ M_0 = \pi^{1/4} M_{Pl} = 1.33 M_{Pl}, \tag{31} \]

which is even closer to the Planck mass. Thus, for any value of \( l_0 \) larger than the Planck length itself, there is no way to produce a “sub-Planckian” black hole. May be that in some proper extension of Rastall gravity, including large extra dimensions, the production threshold can be lowered enough to be in the range of next generation of particle colliders\cite{28-32} but in the present minimal model the energy scale is still too high.

The Hawking temperature is given by
\[ T_H = \frac{1}{4\pi r_H} \left[ 1 - \frac{r_H}{l_0} \frac{e^{-r_H^2/4l_0^2}}{\gamma(1/2 ; r_H^2/4l_0^2)} \right]. \tag{32} \]

\[ T_H = \frac{1}{4\pi r_H} \left[ 1 - \frac{r_H}{l_0} \frac{e^{-r_H^2/4l_0^2}}{\gamma(1/2 ; r_H^2/4l_0^2)} \right]. \]

Fig. 4. Plot of the temperature (32). \( T_H = 0 \leftrightarrow r_H = 0 \), i.e. the horizon completely evaporates as \( M \to M_0 \).

It is interesting to describe the final stage of evaporation of Rastall BH. As \( r_H \to 0 \) temperature drops to zero
\[ T_H \sim \frac{5r_H}{48\pi l_0^2} \to 0 \]  
\[ T_H \sim \frac{5r_H}{48\pi l_0^2} \to 0 \tag{33} \]

But, this occurs when \( M \to M_0 \). Thus, when \( T_H \) reaches zero the horizon is completely evaporated away leaving a frozen, particle-like remnant of mass \( M = M_0 \) dismissing claims in\cite{13} It is also interesting to notice that the total amount of thermal energy radiated via the Hawking process is
\( E_{RAD} = M - M_0 \) (34)

The frozen remnant is a regular, stable, massive lump of matter. Another thermodynamical aspect of BH’s is its entropy. A quick integration of the “first law” gives the BH entropy as a function of the horizon radius

\[
S = \frac{\pi^{3/2}}{G_N} \frac{r_H^2}{\gamma(1/2 ; r_H^2/4l_0^2)} + \frac{\pi^{3/2}}{G_N} \int_{r_H}^{\infty} du \frac{u^2 e^{u^2/4l_0^2}}{\gamma(u^2/4l_0^2)}
\]

(35)

For large BHs the entropy is given by the canonical area law \( A_H/4 \) plus a small correction keeping the memory of the extended nature of the source

\[
S \simeq \frac{\pi r_H^2}{G_N} + \frac{2\pi^{3/2}l_0^2}{G_N}
\]

(36)

In the opposite limit of small, cold, BHs, we find a linearly vanishing entropy, as it would be expected from the “third law”,

\[
S \simeq \frac{2\pi^{3/2}r_Hl_0}{G_N}
\]

(37)

Again, regular BH solutions do not satisfy simple area law but receive corrections due to the extended character of the matter source. Rastall BH is no exception to this rule.

4. Conclusions

In this note we have shown that the Rastall gravity BHs behave in the same way as all known regular BH solutions, with the only difference that it exhibits a single horizon. The Gaussian “dirty”-type BH, originally introduced in [33] which has been recently used in order to obtain regular metric, is not necessary in the Rastall framework. We have shown that a singularity free BH can be obtained following standard procedure leading to other regular solutions in General Relativity, without the need to set on a more “exotic” path as in [33]. The choice \( 2\kappa\lambda = 1 \) was dictated by the fact that [8] can be solved exactly. However, for any \( \kappa\lambda \neq 1/4 \) the conclusions of this paper still hold. As it has been shown in [33], whenever the energy momentum tensor describes a regular matter/energy distribution the solution of the field equations cannot lead to a singular metric. This property is independent from the actual value of the numerical constant multiplying the Ricci scalar.

References

1. J. M. Bardeen, Non-singular general-relativistic gravitational collapse, in Proceedings of of International Conference GR5, Tbilisi, USSR, p. 174 (1968).
2. E. Ayon-Beato and A. Garcia, Phys. Lett. B493, 149 (2000).
3. I. Dymnikova, Gen. Rel. Grav. 24, 235 (1992).
4. I. Dymnikova, Class. Quant. Grav. 19, 725 (2002).
5. I. Dymnikova, Int. J. Mod. Phys. D12, 1015 (2003).
6. K. A. Bronnikov and J. C. Fabris, Phys. Rev. Lett. 96, 251101 (2006).
7. K. A. Bronnikov, V. N. Melnikov and H. Delmenn, Gen. Rel. Grav. 39, 973 (2007).
8. K. A. Bronnikov and I. G. Dymnikova, Class. Quant. Grav. 24, 5803 (2007).
9. S. A. Hayward, Phys. Rev. Lett. 96, 031103 (2006).
10. S. Ansoldi, Spherical black holes with regular center: A Review of existing models including a recent realization with Gaussian sources, in Conference on Black Holes and Naked Singularities Milan, Italy, May 10-12, 2007, (2008).
11. V. P. Frolov, M. A. Markov and V. F. Mukhanov, Phys. Lett. B216, 272 (1989).
12. V. P. Frolov, M. A. Markov and V. F. Mukhanov, Phys. Rev. D41, 383 (1990).
13. M. S. Ma and R. Zhao, “Noncommutative geometry inspired black holes in Rastall gravity,” arXiv:1706.08054 [hep-th].
14. R. Balbinot and E. Poisson, Phys. Rev. D41, 395 (1990).
15. P. Rastall, Phys. Rev. D 6, 3357 (1972).
16. R. V. d. Santos and J. A. C. Nogales, arXiv:1701.08203 [gr-qc].
17. E. R. Bezerra de Mello, J. C. Fabris and B. Hartmann, Class. Quant. Grav. 32, no. 8, 085009 (2015).
18. A. M. Oliveira, H. E. S. Velten, J. C. Fabris and L. Casarini, Phys. Rev. D 92, no. 4, 044020 (2015).
19. K. A. Bronnikov, J. C. Fabris, O. F. Piattella, D. C. Rodrigues and E. C. Santos, Eur. Phys. J. C 77, no. 6, 409 (2017).
20. H. Moradpour and N. Sadeghnezhad, “Traversable asymptotically flat wormholes in Rastall gravity,” arXiv:1606.00846 [gr-qc].
21. Y. Heydarzade, H. Moradpour and F. Darabi, “Black Hole Solutions in Rastall Theory,” arXiv:1610.03881 [gr-qc].
22. P. Nicolini, A. Smailagic and E. Spallucci, “The Fate of radiating black holes in noncommutative geometry,” ESA Spec. Publ. 637, 11.1 (2006).
23. P. Nicolini, A. Smailagic and E. Spallucci, Phys. Lett. B 632, 547 (2006).
24. E. Spallucci and A. Smailagic, Int. J. Mod. Phys. D 0, 1730013 (2017).
25. P. Nicolini, J. Phys. A 38, L631 (2005).
26. S. Ansoldi, P. Nicolini, A. Smailagic and E. Spallucci, Phys. Lett. B 645, 261 (2007).
27. P. Nicolini, Int. J. Mod. Phys. A 24, 1229 (2009).
28. P. Nicolini and E. Winstanley, JHEP 1111, 075 (2011).
29. J. Mureika, P. Nicolini and E. Spallucci, Phys. Rev. D 85, 106007 (2012).
30. P. Nicolini, J. Mureika, E. Spallucci, E. Winstanley and M. Bleicher, “Production and evaporation of Planck scale black holes at the LHC,” doi:10.1142/9789814623995_0478, arXiv:1302.2640 [hep-th].
31. M. F. Wondrak, P. Nicolini and M. Bleicher, “Planck scale black holes - Theory vs. observations,” arXiv:1612.08415 [hep-ph].
32. M. F. Wondrak, M. Bleicher and P. Nicolini, “Black Holes and High Energy Physics: From Astrophysics to Large Extra Dimensions,” arXiv:1708.06763 [gr-qc].
33. P. Nicolini and E. Spallucci, Class. Quant. Grav. 27, 015010 (2010).
34. P. Nicolini, A. Smailagic and E. Spallucci,
“Remarks on regular black holes.” [arXiv:1705.05359] [gr-qc].