Borcherds and Kac-Moody extensions of simple finite-dimensional Lie algebras

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Abstract

We study the Borcherds superalgebra obtained by adding an odd (fermionic) null root to the set of simple roots of a simple finite-dimensional Lie algebra. We compare it to the Kac-Moody algebra obtained by replacing the odd null root by an ordinary simple root, and then adding more simple roots, such that each node that we add to the Dynkin diagram is connected to the previous one with a single line. This generalizes the situation in maximal supergravity, where the $E_n$ symmetry algebra can be extended to either a Borcherds superalgebra or to the Kac-Moody algebra $E_{11}$, and both extensions can be used to derive the spectrum of $p$-form potentials in the theory. We show that also in the general case, the Borcherds and Kac-Moody extensions lead to the same ‘$p$-form spectrum’ of representations of the simple finite-dimensional Lie algebra.

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1 Introduction

Maximal supergravity in $D$ dimensions contains $p$-form potentials that transform in representations of a global symmetry group. Including also the non-dynamical $(D-1)$- and $D$-forms that are possible to add to the theory, all these representations can be derived from the infinite-dimensional Kac-Moody algebra $E_{11}$ \cite{[1-6]}. Considering $E_{11}$ as the ‘very extension’ of (the split real form of) the exceptional Lie algebra $E_8$, the corresponding derivation also works for half-maximal supergravity theories, where the role of $E_8$ is played by $B_7$, $B_8$ or $D_8$ \cite{[7]}. The spectrum of $p$-form representations for maximal supergravity in $D$ dimensions can alternatively be derived from a Borcherds algebra which depends on $D$ \cite{[8-12]}. The fact that these Borcherds algebras lead to the same $p$-form spectrum as $E_{11}$ was explained in \cite{[10]}, and an alternative explanation was given in \cite{[13]}. Since Borcherds algebras arise from Bianchi identities in half-maximal supergravity \cite{[14]} as well as in maximal theories \cite{[11, 12]} it raises the question whether also these Borcherds algebras lead to the same $p$-form spectra as the corresponding very extended Kac-Moody algebras. The present paper gives an affirmative answer to that question, by generalizing the result in \cite{[13]}.

The Borcherds algebra associated to maximal supergravity in $3 \leq D \leq 7$ dimensions, with a Lie algebra $\mathfrak{g}$ corresponding to the global symmetry group, can be constructed from $\mathfrak{g}$ by adding an extra simple root in a certain way (or equivalently, an extra node to the Dynkin diagram of $\mathfrak{g}$). It is in fact not a Lie algebra but a Lie superalgebra, where the eigenvectors corresponding to the extra simple root are odd elements, and furthermore the eigenvalues are zero. If we instead add $N \geq 1$ ordinary simple roots (such that each node that we add to the Dynkin diagram is connected to the previous one with a single line), then we obtain a Kac-Moody algebra, which for $N = D$ is the very extended Kac-Moody algebra $E_{11}$. Up to level $p = N$, the level decomposition of this Kac-Moody algebra under $\mathfrak{g} \oplus \mathfrak{sl}_N$, restricted to antisymmetric $\mathfrak{sl}_N$ tensors, gives the same ‘$p$-form spectrum’ of $\mathfrak{g}$ representations as the level decomposition of the Borcherds algebra under $\mathfrak{g}$. We will show that the corresponding result holds for any such Borcherds and Kac-Moody extensions of a simple finite-dimensional Lie algebra $\mathfrak{g}$. More precisely, for any such $\mathfrak{g}$, any way of adding the first extra node (the only extra node for the Borcherds algebra), and any total number $N$ of extra nodes for the Kac-Moody algebra, the two algebras lead to the same $p$-form spectrum (up to level $p = N$). The fact that $N$ can be larger than the spacetime dimension $D$ is important for applications to the superspace approach that has been employed recently in \cite{[11,12,14]}, since $p$-form superfields with $p > D$ need not be zero.

In this paper we denote the Borcherds and Kac-Moody extensions by $U$ and $W$, and we let $V$ be an intermediate Kac-Moody algebra. The algebras $U$, $V$ and $W$ are described in section 2, 3 and 4, respectively, and in the end we show that $W$ gives the same $p$-form spectrum as $U$. The reader who finds the paper difficult to follow is invited to read \cite{[13]} first, where $U$, $V$ and $W$ corresponds $U_{n+1}$, $E_{n+1}$ and $E_{11}$.
2 The Borcherds algebra $U$

As indicated in the introduction, our definition of Borcherds algebras include also the generalization \cite{15} of the original Borcherds algebras \cite{16} to superalgebras. However, we will here only define a very special case of such superalgebras, and refer to \cite{17} for the full definition. As noted in \cite{10}, footnote 8, there is an error in the definition in \cite{17}, but this has no importance for the special cases we consider here.

A Borcherds algebra is given by a (generalized) Cartan matrix $a_{IJ}$, which is a non-degenerate symmetric real matrix, where the rows and columns are labelled by some index set. This set can in general be infinite, but here we restrict it to be finite and write $I, J, \ldots = 0, 1, \ldots, r$ for some $r$. For each value $I$ of the indices we associate two Chevalley generators $e_I$ and $f_I$ which are both either odd (fermionic) or even (bosonic) elements of the Borcherds algebra. We assume $e_0$ and $f_0$ to be odd, and use the indices $i, j, \ldots = 1, 2, \ldots, r$ for the even generators. Furthermore, we assume that $a_{00} = 0$ and $a_{ii} > 0$. With these restrictions, the conditions that define $a_{IJ}$ to be the Cartan matrix of a Borcherds algebra are

$$I \neq J \Rightarrow a_{IJ} \leq 0, \quad 2 \frac{a_{iJ}}{a_{ii}} \in \mathbb{Z}. \quad (2.1)$$

Note that this matrix is symmetric, unlike general Cartan matrices of Kac-Moody algebras with the standard definition (see for example \cite{15,19}). However, we can ‘de-symmetrize’ $a_{IJ}$ and define an in general non-symmetric matrix $A_{IJ}$ by

$$A_{iJ} = 2 \frac{a_{iJ}}{a_{ii}}, \quad A_{0i} = a_{0i}, \quad A_{00} = a_{00} = 0. \quad (2.2)$$

Any multiple of $a_{IJ}$ gives the same Borcherds algebra as $a_{IJ}$. Together with the second condition in (2.1), this implies that we can assume all the diagonal entries $a_{ii}$ to be even integers. It then follows from the same condition that all the entries in $a_{IJ}$ are integers, in particular $a_{00}$. We conclude that $A_{IJ}$ is an integer-valued matrix, with $A_{00} = 0$ and $A_{ii} = 2$. The off-diagonal entries are non-positive integers, in general with $A_{IJ} \neq A_{JI}$, but if $A_{IJ} = 0$, then $A_{JI} = 0$ as well. Thus $A_{ij}$ satisfies the definition of a Cartan matrix of a Kac-Moody algebra, and, as a last restriction, we require this Kac-Moody algebra to be finite, that is, a simple finite-dimensional Lie algebra $g$.

The Borcherds algebra $U$ associated to $a_{IJ}$ (or $A_{IJ}$) is now defined as the Lie superalgebra generated by the Chevalley generators $e_I, f_I$ and $h_I = \left[ e_I, f_I \right]$ modulo the relations

$$\left[ h_I, e_J \right] = A_{IJ}e_J, \quad \left[ h_I, f_J \right] = -A_{IJ}f_J, \quad \left[ e_I, f_J \right] = \delta_{IJ}h_J, \quad \left[ e_0, e_0 \right] = \left[ f_0, f_0 \right] = (\text{ad } e_i)^{1-A_{ij}}(e_j) = (\text{ad } f_i)^{1-A_{ij}}(f_j) = 0, \quad (2.3)$$
where \( i \neq J \), and \([x, y]\) denotes the supercommutator of two elements \( x \) and \( y \). This is a symmetric anticommutator \([x, y] \equiv \{ x, y \} = \{ y, x \}\) if both \( x \) and \( y \) are odd elements, and an ordinary antisymmetric commutator \([x, y] \equiv [x, y] = -[y, x]\) if at least one of the elements is even.

The Borcherds algebra \( U \) has a bilinear form, which we write as \( \langle x|y \rangle \) for two elements \( x \) and \( y \), and define by

\[
\langle h_I|h_J \rangle = A_{IJ}, \quad \langle e_I|f_J \rangle = \delta_{IJ}, \quad \langle e_I|e_J \rangle = \langle f_I|f_J \rangle = \langle h_I|e_J \rangle = \langle h_I|f_J \rangle = 0. \quad (2.4)
\]

The definition can then be extended to the full algebra \( U \) in such a way that the bilinear form is invariant

\[
\langle [x, y]|z \rangle = \langle x|[y, z] \rangle, \quad (2.5)
\]

and supersymmetric, which means that \( \langle x|y \rangle = -\langle y|x \rangle \) if both elements are odd, and \( \langle x|y \rangle = \langle y|x \rangle \) if at least one of them is even.

The odd generators \( e_0 \) and \( f_0 \) give rise to a \( \mathbb{Z}_2 \)-grading of \( U \) which is consistent with the \( \mathbb{Z}_2 \)-grading that \( U \) naturally is equipped with as a superalgebra. This means that it can be written as a direct sum of subspaces \( U_p \) for all integers \( p \), such that

\[
[U_p, U_q] \subseteq U_{p+q}, \quad (2.6)
\]

where \( U_p \) consists of odd elements if \( p \) is odd, and of even elements if \( p \) is even. (These subspaces should not be confused with \( U_{n+1} \) in \([13]\), which is simply \( U \) here.) Among the Chevalley generators \( e_0 \) belongs to \( U_{-1} \), whereas \( f_0 \) belongs to \( U_1 \), and all the others belong to \( U_0 \).

It follows from the grading \((2.6)\) that each subspace \( U_p \) constitute a representation \( r_p \) of \( g \) (called \( s_p \) in \([13]\)). One can easily see that there is an isomorphism between the subspaces \( U_1 \) and \( U_{-1} \), such that elements mapped to each other have eigenvalues with opposite signs under the adjoint action of \( h_i \), and therefore the representations \( r_1 \) and \( r_{-1} \) are conjugate to each other. Accordingly, we introduce indices

\[
\mathcal{M}, \mathcal{N}, \ldots = 1, 2, \ldots, \dim r_1, \quad (2.7)
\]

and write the basis elements of \( U_{-1} \) and \( U_1 \) as \( E_{\mathcal{M}} \) and \( F_{\mathcal{N}} \), respectively, chosen such that \( \langle E_{\mathcal{M}}|F_{\mathcal{N}} \rangle = \delta_{\mathcal{M} \mathcal{N}} \). For \( p \geq 2 \) the subspace \( U_{-p} \) is then spanned by the elements

\[
E_{\mathcal{M}_1 \ldots \mathcal{M}_p} \equiv [E_{\mathcal{M}_1}, [E_{\mathcal{M}_2}, \ldots, [E_{\mathcal{M}_{p-1}}, E_{\mathcal{M}_p}] \ldots]] \quad (2.8)
\]

and \( U_p \) by the elements

\[
F_{\mathcal{N}_1 \ldots \mathcal{N}_p} \equiv [F_{\mathcal{N}_1}, [F_{\mathcal{N}_2}, \ldots, [F_{\mathcal{N}_{p-1}}, F_{\mathcal{N}_p}] \ldots]]. \quad (2.9)
\]
As explained in [13], each representation \( r \) is determined by the lower (or upper) indices in the tensor

\[
f_{M_1 \cdots M_p}^{N_1 \cdots N_p} = \langle E_{M_1 \cdots M_p} | F^{N_1 \cdots N_p} \rangle,
\]

and all such tensors can be computed recursively, starting from the constants

\[
f_M^{N} P Q = \langle [\{E_M, F^N\}, E_P] | F^Q \rangle,
\]

which are the structure constants of \( U_{-1} \) considered as a (generalized Jordan) triple system with the triple product \([\{E_M, F^N\}, E_P]\). To find the recursion formula, we first use the Jacobi identity to compute

\[
[F^N, E_{M_1 \cdots M_p}] = \langle [\{F^N, E_{M_1}\}, E_{M_2 \cdots M_p}] - \langle [E_{M_1}, [F^N, E_{M_2 \cdots M_p}]] - \langle [\{F^N, E_{M_1}\}, E_{M_2 \cdots M_p}] - \langle [E_{M_1}, [F^N, E_{M_2 \cdots M_p}]] + \langle [E_{M_1}, [E_{M_2}, [F^N, E_{M_3 \cdots M_p}]]] + \langle [E_{M_1}, [E_{M_2}, [F^N, E_{M_3 \cdots M_p}]]] + \cdots + (-1)^{p+1} \langle [E_{M_1}, [E_{M_2}, \ldots, [E_{M_{p-1}}, [F^N, E_{M_p}]]] \cdots] \rangle
\]

\[
= \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} (-1)^{i+1} f_{M_i}^{N_i} M_j P E_{M_1 \cdots M_{i-1} M_{i+1} \cdots M_{j-1} P M_{j+1} \cdots M_p}
\]

\[+ (-1)^p f_{M_p}^{N_p} M_{p-1} P E_{M_1 \cdots M_{p-2} P},\]

and then, using the invariance of the bilinear form, we obtain

\[
f_{M_1 \cdots M_p}^{N_1 \cdots N_p} = \langle E_{M_1 \cdots M_p} | F^{N_1 \cdots N_p} \rangle = (-1)^{p+1} \langle [F^{N_i}, E_{M_1 \cdots M_p}] | F^{N_2 \cdots N_p} \rangle = \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} (-1)^{i+p} f_{M_i}^{N_i} M_j P f_{M_1 \cdots M_{i-1} M_{i+1} \cdots M_{j-1} P M_{j+1} \cdots M_p}^{N_2 \cdots N_p} - f_{M_p}^{N_p} M_{p-1} P f_{M_1 \cdots M_{p-2} P}^{N_2 \cdots N_p}.
\]

The subspace \( U_0 \) is spanned by \( g \) and \( h_0 \). Since \( U_0 \) is a finite-dimensional representation of \( g \) it must be fully reducible, and since its dimension is \((\dim g + 1)\) it must (as a Lie algebra) be the direct sum of \( g \) and a one-dimensional abelian subalgebra.
spanned by an element $c$. It then follows from the invariance of the bilinear form that the commutation relations between the elements in $U_0$ and $U_{\pm 1}$ are

\[
\{E_M, F^N\} = (t^\alpha)_{M^N} t^\alpha + \delta_{M^N} c, \quad [t^\alpha, c] = 0,
\]

\[
[t^\alpha, E_M] = (t^\alpha)_{M^N} E_N, \quad [c, E_M] = \langle c|c\rangle E_M,
\]

\[
[t^\alpha, F^N] = -(t^\alpha)_{M^N} F^M, \quad [c, F^N] = -\langle c|c\rangle F^N,
\]

(2.14)

where $t^\alpha$ are the basis elements of $\mathfrak{g}$, and $(t^\alpha)_{M^N}$ are the components of $t^\alpha$ in the representation $r_1$. The adjoint index $\alpha$ has been lowered with the restriction of the invariant bilinear form to $\mathfrak{g}$ (the Killing form), so that $\langle t^\alpha|t^\beta\rangle = \delta^\alpha_\beta$, and the normalization of $c$ has been fixed by the first equation in (2.14) as we will see in the next section. Thus we end up with the expression

\[
f_{M^N P^Q} = \langle [\{E_M, F^N\}, E_P]|F^Q\rangle = (t^\alpha)_{M^N} (t^\alpha)_{P^Q} + \langle c|c\rangle \delta_{M^N} \delta_{P^Q}
\]

(2.15)

for the structure constants $f_{M^N P^Q}$, which can then be inserted in (2.13).

### 3 The Kac-Moody algebra $V$

Let $B_{IJ}$ be the matrix obtained from $A_{IJ}$ by replacing the entry $A_{00} = 0$ by $B_{00} = 2$. Thus we have

\[
B_{00} = 2, \quad B_{II} = A_{II}, \quad B_{IJ} = A_{IJ}.
\]

(3.1)

We then define the Kac-Moody algebra $V$ associated to the Cartan matrix $B_{IJ}$ as the Lie algebra generated by $e_I, f_I$ and $h_I = [e_I, f_I]$ modulo the relations (2.3), but now with $A_{IJ}$ replaced by $B_{IJ}$, and all Chevalley generators being even elements, so that the supercommutators are ordinary antisymmetric commutators. The relations corresponding to (2.4) define a bilinear form on $V$ which is invariant and, unlike the one on $U$, fully symmetric. We write it as $(x|y)$ for two elements $x$ and $y$ to distinguish it from the invariant bilinear form on $U$. Note that we have $(h_0|h_0) = 2$, whereas $(h_0|h_0) = 0$.

In the same way as for $U$, the generators $e_0$ and $f_0$ give rise to a $\mathbb{Z}$-grading of $V$, where each subspace $V_p$ constitutes a representation $s_p$ of $\mathfrak{g}$. The difference between $A_{IJ}$ and $B_{IJ}$ does not affect the commutation relations between $\mathfrak{g}$ and $e_I$ or $f_I$, and therefore we have $s_{\pm 1} = r_{\pm 1}$. Furthermore, the Lie algebra $V_0$ is, in the same way as $U_0$, the direct sum of $\mathfrak{g}$ and a one-dimensional abelian subalgebra spanned by an element $d$. Using the same notation for the basis elements of $V_{\pm 1}$ as for $U_{\pm 1}$, the commutation relations between the elements in $V_0$ and $V_{\pm 1}$ are then

\[
[E_M, F^N] = (t^\alpha)_{M^N} t^\alpha + \delta_{M^N} d, \quad [t^\alpha, d] = 0,
\]
\[
\begin{align*}
[t^\alpha, E_M] &= (t^\alpha)_M^N E_N, & [d, E_M] &= (d|d) E_M, \\
[t^\alpha, F^N] &= -(t^\alpha)_M^N F^M, & [d, F^N] &= -(d|d) F^N.
\end{align*}
\] (3.2)

Let us compare \(d\) in \(V_0\) with the corresponding element \(c\) in \(U_0\). From the invariance of the bilinear form it follows that \(c\) and \(d\) are determined up to normalization by the conditions \(\langle c|g \rangle = 0\) and \((d|g) = 0\), respectively. This implies in turn that both \(c\) and \(d\) are linear combinations
\[
c = c_0 h_0 + c_1 h_1 + \cdots + c_r h_r,
\]
\[
d = d_0 h_0 + d_1 h_1 + \cdots + d_r h_r
\] (3.3)

(identifying the Chevalley generators of \(U\) and \(V\) with each other). The first equations in (2.14) and (3.2) fix the coefficients \(c_0\) and \(d_0\) to \(c_0 = d_0 = 1\). Furthermore, the conditions \(\langle c|g \rangle = 0\) and \((d|g) = 0\) do not involve \(A_{00}\) or \(B_{00}\), which are the only entries that differ between \(A_{IJ}\) or \(B_{IJ}\), so they are in fact equivalent, and we conclude that \(c = d\). Now we have
\[
[c, e_0] = c_0 [h_0, e_0] + c_1 [h_1, e_0] + \cdots + c_r [h_r, e_0]
= (c_0 A_{00} + c_1 A_{10} + \cdots + c_r A_{r0}) e_0 \\
= (c_1 A_{10} + \cdots + c_r A_{r0}) e_0
\] (3.4)
in \(U\), and
\[
[d, e_0] = d_0 [h_0, e_0] + d_1 [h_1, e_0] + \cdots + d_r [h_r, e_0]
= (d_0 B_{00} + d_1 B_{10} + \cdots + d_r B_{r0}) e_0 \\
= (2 + c_1 A_{10} + \cdots + c_r A_{r0}) e_0
\] (3.5)
in \(V\). On the other hand, from (2.14) and (3.2) we have \([c, e_0] = \langle c|c \rangle e_0\) in \(U\), and \([d, e_0] = (d|d) e_0\) in \(V\), so we conclude that \((d|d) = \langle c|c \rangle + 2\). It follows that the structure constants of \(V_{-1}\) considered as a triple system are
\[
g_M^N p^Q = ([E_M, F^N], E_P)|F^Q) = (t^\alpha)_M^N (t^\alpha)_P^Q + (\langle c|c \rangle + 2) \delta_M^N \delta_P^Q
\]
\[
= f_M^N p^Q + 2 \delta_M^N \delta_P^Q.
\] (3.6)

4 The extended Kac-Moody algebra \(W\)

Let \(C\) be the matrix obtained from \(B\) by adding \(N - 1\) more rows and columns, labelled by \(m, n, \ldots = -N + 1, -N + 2, \ldots, -1\), so that
\[
\begin{align*}
C_{IJ} &= B_{IJ}, & C_{mI} &= C_{Im} = 0,
\end{align*}
\] (4.1)
and \( C_{mn} \) is the well known Cartan matrix of \( A_{N-1} = \mathfrak{s}l_N \). Let \( W \) be the Kac-Moody algebra given by the Cartan matrix \( C \). This corresponds to adding \( N - 1 \) more nodes to the Dynkin diagram of \( V \), each connected to the previous one by a single line.

In the same way as for \( U \) and \( V \), the generators \( e_0 \) and \( f_0 \) give rise to a \( \mathbb{Z} \)-grading of \( W \), where each subspace \( W_p \) constitutes a representation \( t_p \) of \( \mathfrak{g} \), but also a representation of \( \mathfrak{s}l_N \). Considering \( V \) as a subalgebra of \( W \) we can write the basis elements of \( W_1 \) and \( W_{-1} \) as \( E_{Ma} \) and \( F_{Mb} \), respectively, where \( a, b, \ldots = 0, 1, \ldots, N - 1 \), and

\[
E_{M0} = E_M, \quad E_{M(-m)} = \left[ \cdots \left[ e_m, e_{m+2}, \ldots, e_{-1} \right], E_M \right], \\
F_{M0} = F_M, \quad F_{M(-m)} = (\pm 1)^m \left[ \cdots \left[ f_m, f_{m+2}, \ldots, f_{-1} \right], F_M \right]. \quad (4.2)
\]

For \( p \geq 2 \), the subspace \( W_p \) is then spanned by the elements

\[
E_{M_1 \cdots M_p} a_1 \cdots a_p = \left[ E_{M_1} a_1, \left[ E_{M_2} a_2, \cdots, \left[ E_{M_{p-1}} a_{p-1}, E_{M_p} a_p \right] \cdots \right] \right], \quad (4.3)
\]
and \( W_{-p} \) by the elements

\[
F_{M_1 \cdots M_p} a_1 \cdots a_p = \left[ F_{M_1} a_1, \left[ F_{M_2} a_2, \cdots, \left[ F_{M_{p-1}} a_{p-1}, F_{M_p} a_p \right] \cdots \right] \right]. \quad (4.4)
\]

Following the steps in [20] it is straightforward to show that the structure constants of the triple system \( W_{-1} \) are related to those of \( V_{-1} \) as

\[
h_{M}^{N_p} p Q^{b-d} = (\left[ E_{Ma}, F_{N_p} \right], E_{Pc}) | Q^d \rangle \\
g_m^{N_p} Q^d \delta_{a}^{b-d} - \delta_m^{N_p} \delta_{P}^{Q} \delta_{a}^{b} \delta_{c}^{d} + \delta_m^{N_p} \delta_{P}^{Q} \delta_{c}^{b} \delta_{a}^{d}, \quad (4.5)
\]
and if we antisymmetrize in \( a \) and \( c \) we obtain

\[
h_{M}^{N_p} p Q[a|c] b-d = (g_m^{N_p} Q - 2 \delta_m^{N_p} \delta_{P}^{Q}) \delta_{[a}^{b} \delta_{c]}^{d} = f_m^{N_p} Q \delta_{[a}^{b} \delta_{c]}^{d}. \quad (4.6)
\]

Thus we get back the structure constants (2.15) for the triple system \( U_{-1} \), times \( \delta_{[a}^{b} \delta_{c]}^{d} \).

As we will see next, this relation between the two triple systems can be viewed as the reason why \( U \) and \( W \) lead to the same \( p \)-form spectrum, or to be precise, why \( t_p = r_p \) for \( 1 \leq p \leq N \), which is the main result of this paper.

As for \( U \), each representation \( t_p \) is determined by the lower indices in the tensor

\[
h_{M_1 \cdots M_p} N_1 \cdots N_p[a_1 \cdots a_p] b_1 \cdots b_p = (E_{M_1 \cdots M_p} a_1 \cdots a_p | F_{N_1 \cdots N_p} b_1 \cdots b_p). \quad (4.7)
\]

In the same way as we obtained (2.13) for \( U \), we now obtain

\[
h_{M_1 \cdots M_p} N_1 \cdots N_p[a_1 \cdots a_p] b_1 \cdots b_p = (E_{M_1 \cdots M_p} a_1 \cdots a_p | F_{N_1 \cdots N_p} b_1 \cdots b_p) \\
= \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} h_{M_i a_i \cdots a_j c}^{N_1 \cdots N_p} b_1 \cdots b_p \\
\times h_{M_1 \cdots M_{p-i}} N_1 \cdots N_{p-i-1} a_{i+1} \cdots a_{j-1} | a_{j+1} \cdots a_p | b_{i} \cdots b_{p-1} c \times h_{M_{p-i} a_i \cdots a_{j-1} c}^{N_1 \cdots N_p} b_{i} \cdots b_{p-1} c \\
- h_{M_p a_i \cdots a_{p-i-1} c}^{N_1 \cdots N_p} b_{i} \cdots b_{p-i} c \times h_{M_{p-i} a_i \cdots a_{j-1} c}^{N_1 \cdots N_p} b_{i} \cdots b_{p-i} c. \quad (4.8)
\]
for \( W \), where we have simplified the notation by writing
\[
M_1 \cdots P \cdots M_p = M_1 \cdots M_{i-1} M_{i+1} \cdots M_{j-1} P M_{j+1} \cdots M_p.
\]
(4.9)
The difference compared to (2.13) is that \( f \) is replaced by \( h \), that each \( r_1 \) index is accompanied by an \( \mathfrak{sl}_N \) index and, most important, that the prefactor \((-1)^{i+p}\) is replaced by 1. We will now show, by induction over \( p \), that
\[
h_{M_1 \cdots M_p} N_1 \cdots N_p \prod_{[a_1, \ldots, a_p]} b_1 \cdots b_p = (-1)^{\sigma(p)} \delta_{[a_1, \ldots, a_p]} b_1 \cdots b_p \prod_{M_1 \cdots M_p} N_1 \cdots N_p,
\]
(4.10)
for all integers \( p \geq 1 \), where \( \sigma(p) = p(p - 1)/2 \). For \( p = 1 \) we have
\[
h_{M} N_1 b = (E_{M_1} | F N) = \delta_a b \delta_{M} N_1 = \delta_a b (E_{M_1} | F N) = \delta_a b g_{M} N_1.
\]
(4.11)
Assume now that (4.10) holds for \( p = q - 1 \), where \( q \) is some integer \( q \geq 2 \). Then
\[
h_{M_1 \cdots M_q} N_1 \cdots N_q \prod_{[a_1, \ldots, a_q]} b_1 \cdots b_q = (E_{M_1 \cdots M_q} [a_1, \ldots, a_q] | F N_1 \cdots N_q b_1 \cdots b_q)
\]
\[
= \sum_{i=1}^{q-1} \sum_{j=i+1}^q h_{M_i} N_i M_j \prod_{[a_1, \ldots, a_j]} b_1 \cdots b_j \times \sum_{i=1}^{q-1} \sum_{j=i+1}^q h_{M_i} N_i M_{j-1} \prod_{[a_1, \ldots, a_{j-1}]} b_1 \cdots b_j
\]
\[
= \sum_{i=1}^{q-1} \sum_{j=i+1}^q f_{M_i} N_i M_j \prod_{[a_1, a_j]} b_1 \cdots b_j \times (-1)^{\sigma(q-1)} f_{M_1 \cdots M_q} N_1 \cdots N_q \prod_{[a_1, \ldots, a_q]} b_1 \cdots b_q
\]
\[
= \sum_{i=1}^{q-1} \sum_{j=i+1}^q (-1)^{\sigma(q-1)} f_{M_i} N_i M_j \prod_{[a_1, a_j]} b_1 \cdots b_j
\]
\[
= \sum_{i=1}^{q-1} \sum_{j=i+1}^q (-1)^{\sigma(q-1)+i+1} f_{M_i} N_i M_j \prod_{[a_1, a_j]} b_1 \cdots b_j
\]
\[
= \sum_{i=1}^{q-1} \sum_{j=i+1}^q (-1)^{\sigma(q-1)+q-1} \delta_{[a_1, \ldots, a_q]} b_1 \cdots b_q \prod_{M_1 \cdots M_q} N_1 \cdots N_q
\]
\[
= (-1)^{\sigma(q)} \delta_{[a_1, \ldots, a_q]} b_1 \cdots b_q \prod_{M_1 \cdots M_q} N_1 \cdots N_q,
\]
(4.12)
where we first have inserted the assumption of the induction, and then used (2.13). By the principle of induction, it follows that (4.10) holds for all integers \( p \geq 1 \). Since the lower \( r_1 \) indices on the left hand side of (4.10) determine \( r_p \), and those on the right hand side determine \( t_p \), we conclude that \( r_p = t_p \) as long as the delta factor does not vanish, that is, for \( 1 \leq p \leq N \).
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