Higher-spin strings and $W$ minimal models

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ABSTRACT

We study the spectrum of physical states for higher-spin generalisations of string theory, based on two-dimensional theories with local spin-2 and spin-$s$ symmetries. We explore the relation of the resulting effective Virasoro string theories to certain $W$ minimal models. In particular, we show how the highest-weight states of the $W$ minimal models decompose into Virasoro primaries.

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1. Introduction

In a recent paper it was shown that the BRST operator $Q_B$ for the $W_3$ algebra, in the case where the matter currents are realised in terms of scalar fields, admits a considerable simplification if one performs a redefinition that mixes the ghost fields and one of the matter fields, called $\varphi$, that plays a special role in the realisation [1]. In particular, the BRST operator written in terms of the redefined fields has a double grading; i.e. $Q_B = Q_0 + Q_1$, where $Q_0$ has grade $(1,0)$ and $Q_1$ has grade $(0,1)$, with $(p,q)$ denoting the grading of an operator with ghost number $p$ for the spin-2 ghost system and ghost number $q$ for the spin-3 ghost system.

The form of the BRST operator for the $W_3$ algebra immediately suggests a generalisation to the case where the two matter currents have spins 2 and $s$ rather than 2 and 3 [2]. In [2], such BRST operators were explicitly constructed, for the cases $s = 4, 5$ and 6, and some aspects of the physical spectra of the associated generalisations of string theory were explored. Presumably the construction would work for higher values of $s$ too. It should be emphasised that such BRST operators can apparently be constructed for general $s$, even though closed $W$ algebras of the form $W_{(2,s)}$ do not exist for general values of $s$. Evidently the conditions for the existence of a nilpotent BRST operator are weaker than the conditions for the existence of a closed operator algebra. (A previous example of this kind of phenomenon, where a BRST operator exists even when the matter system does not generate a closed algebra at the quantum level, was found in the context of the “non-critical” $W_3$ string discussed in [3,4,5].)

It was shown in [2] that for $s = 3$, 4, 5 and 6, there is a BRST operator for a spin-2 plus spin-$s$ system that gives rise to a higher-spin string theory that appears to be unitary. It was conjectured that this theory admits an interpretation as an effective Virasoro string theory, with central charge $c^{\text{eff}} = 26 - \frac{2(s-2)}{(s+1)}$, coupled to a certain minimal model with central charge $\frac{2(s-2)}{(s+1)}$ realised by the special matter field $\varphi$ and the spin-s ghost system. Some arguments were presented in [2] that suggested that the minimal model in question is the lowest unitary $W_{s-1}$ minimal model (i.e. the $W_{s-1}$ generalisation of the Ising model). It is expected that the results should generalise to all $s$.

In this paper, we develop the idea further, and study the physical states in detail for $s = 4, 5$ and 6. In particular, for the spin-2 plus spin-4 string, we find further evidence that it is associated with the lowest $W_3$ minimal model, which has central charge $c = \frac{4}{5}$. We do this by finding a primary spin-3 operator in the physical spectrum which, together with the energy-momentum tensor, generates the $W_3$ algebra at $c = \frac{4}{5}$. We then check that all the low-lying physical states (up to level 9) are either highest-weight states of this $W_3$ minimal model with $c = \frac{4}{5}$, or else they are $W$ descendants. This example is somewhat non-generic in having $c < 1$, so the $W$ fields decompose into a finite number of Virasoro primaries, namely into a subset of those of the 3-state Potts model. Nonetheless, it provides a nice illustration
of the manner in which the highest-weight states of the $W$ model decompose into a larger set of Virasoro primaries, the extra Virasoro primaries corresponding to operators that can be written as descendants under the $W$ currents. Similarly, for the spin-2 plus spin-5 string, and the spin-2 plus spin-6 string, we show that their physical spectra include primary operators that generate the $W_4$ algebra at $c = 1$, and the $W_5$ algebra at $c = \frac{8}{7}$ respectively. Again, we check that the low-lying physical states are either highest weight under the currents of the $W$ algebra, or else $W$ descendants.

We have made extensive use of the Mathematica package OPEdefs [6] for the conformal field theoretic calculations in this paper.

2. Spin-2 plus spin-$s$ Strings

2.1 Generalities

We begin by introducing the usual $(b,c)$ ghost system for the spin-2 current, and the $(\beta, \gamma)$ ghost system for the spin-$s$ current. Note that $\beta$ therefore has spin $s$, and $\gamma$ has spin $(1 - s)$. The BRST operator for the spin-2 plus spin-$s$ string then takes the form [2]:

$$Q_B = Q_0 + Q_1,$$

$$Q_0 = \oint dz c \left( T^{\text{eff}} + T_\varphi + T_{\gamma,\beta} + \frac{1}{2} T_{c,\beta} \right),$$

$$Q_1 = \oint dz \gamma F(\varphi, \beta, \gamma),$$

where the energy-momentum tensors are given by

$$T_\varphi \equiv -\frac{1}{2} (\partial \varphi)^2 - \alpha \partial^2 \varphi,$$

$$T_{\gamma,\beta} \equiv -s \beta \partial \gamma - (s - 1) \partial \beta \gamma,$$

$$T_{c,\beta} \equiv -2b \partial c - \partial b c,$$

$$T^{\text{eff}} \equiv -\frac{1}{2} \partial X^\mu \partial X^{\nu} \eta_{\mu\nu} - ia_\mu \partial^2 X^\mu.$$

The operator $F(\varphi, \beta, \gamma)$ has spin $s$ and ghost number zero. Because of the grading discussed in the introduction, it follows that one will have $Q_B^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. The first of these conditions is satisfied provided that the total central charge vanishes, i.e.

$$0 = -26 - 2(6s^2 - 6s + 1) + 1 + 12\alpha^2 + c^{\text{eff}}.$$

The remaining two nilpotency conditions determine the precise form of the operator $F(\varphi, \beta, \gamma)$ appearing in (2.3). Solutions for $s = 4, 5$ and 6 were found in [2]. (In fact in [2] it was found that there are two different nilpotent BRST operators when $s = 4$, and four different ones when $s = 6$. Presumably this non-uniqueness is a rather general feature for arbitrary $s$. However, only one BRST operator for each $s$, for which the $(\varphi, \beta, \gamma)$ system has the central
charge $\frac{2(s-2)}{(s+1)}$ discussed above, seems to be associated with a unitary theory. It is this choice that we shall be concentrating on in the present paper.)

2.2 The spin-2 plus spin-4 string

Let us consider first the spin-2 plus spin-4 string. The BRST operator is then given by (2.1)–(2.7), with $\alpha^2 = \frac{243}{20}$ and the operator

$$F(\varphi, \beta, \gamma) = (\partial \varphi)^4 + 4\alpha \partial^2 \varphi (\partial \varphi)^2 + \frac{41}{9} \alpha \partial^2 \varphi (\partial \varphi)^2 + \frac{32}{9} \partial^2 \varphi (\partial \varphi)^2 + \frac{46}{135} \partial^4 \varphi + 8(\partial \varphi)^2 \beta \partial \gamma - \frac{16}{9} \alpha \partial^2 \varphi \beta \partial \gamma - \frac{32}{9} \partial \varphi \beta \partial^2 \gamma - \frac{8}{9} \beta \partial^3 \gamma + \frac{16}{3} \partial^2 \beta \partial \gamma.$$  

(2.9)

As usual, physical states $|\chi\rangle$ are determined by the requirement that they be annihilated by the BRST operator, and that they be BRST non-trivial. In other words, $Q_B |\chi\rangle = 0$ and $|\chi\rangle \neq Q_B |\psi\rangle$ for any $|\psi\rangle$. It was conjectured in [1] and [2] that all continuous-momentum physical states for multi-scalar string theories of the kind we are discussing can be described by physical operators of the form

$$V_\Delta = c U(\varphi, \beta, \gamma) V^{\text{eff}}(X),$$  

(2.10)

acting on the $SL(2, C)$ vacuum, where the operator $V^{\text{eff}}(X)$ creates an effective spacetime physical state $|\text{phys}\rangle_{\text{eff}} \equiv V^{\text{eff}}(X(0))|0\rangle$ satisfying the highest-weight conditions

$$L_n^{\text{eff}} |\text{phys}\rangle_{\text{eff}} = 0, \quad n > 0,$$

$$\left(L_0^{\text{eff}} - \Delta\right) |\text{phys}\rangle_{\text{eff}} = 0.$$  

(2.11)

For simplicity, one can always take the effective-spacetime operator $V^{\text{eff}}(X)$ to be tachyonic, since the discussion of physical states with excitations in the effective spacetime proceeds identically to that of ordinary string theory. The interesting new features of the $W$ string theories are associated with excitations in the $(\varphi, \beta, \gamma)$ fields. Thus we are primarily concerned with solving for the operators $U(\varphi, \beta, \gamma)$ that are highest weight under $T \equiv T_\varphi + T_{\gamma, \beta}$ with conformal weights $h = 1 - \Delta$, and that in addition satisfy $[Q_1, U] = 0$. Solving these conditions for $U(\varphi, \beta, \gamma)$, with $V^{\text{eff}}$ being highest weight under $T^{\text{eff}}$ with conformal weight $\Delta = 1 - h$, is equivalent to solving the physical-state conditions for $V_\Delta$ in (2.10).

In [2], all physical states up to and including level $\ell = 9$ in $(\varphi, \beta, \gamma)$ excitations were studied for the spin-2 plus spin-4 string. It was found that all the physical states fall into a set of different sectors, characterised by the value $\Delta$ of the effective spacetime intercept. (There is also a sector of discrete physical states, with zero momentum in the effective spacetime, which will not be of relevance to us here [2].) Specifically, for the spin-2 plus spin-4 string, $\Delta$ can take values in the set $\Delta = \{1, \frac{14}{15}, \frac{3}{5}, \frac{1}{3}, -\frac{2}{5}, -2\}$. As one goes to higher and higher levels $\ell$, one just encounters repetitions of these same intercept values, with more and more complicated operators $U(\varphi, \beta, \gamma)$. These operators correspondingly have conformal weights
that are conjugate to $\Delta$, i.e. $h = 1 - \Delta = \{0, \frac{1}{15}, \frac{2}{3}, \frac{7}{5}, 3\}$. For convenience, we reproduce here the table of results up to level 9, giving the $(\beta, \gamma)$ ghost number $g$ of the operators $U(\varphi, \beta, \gamma)$, their conformal weights $h$, and their $\varphi$ momenta $\mu$:

| $\ell$  | $g$ | $h$ | $\mu$ (In units of $\alpha/27$) |
|--------|-----|-----|---------------------------------|
| $\ell = 0$ | 3   | $\frac{1}{15}$ | 0 | $(-28, -26)$ | $(-30, -24)$ |
| $\ell = 1$ | 2   | $\frac{2}{3}$ | $\frac{1}{15}$ | $-20$ | $-18$ | $-16$ |
| $\ell = 2$ | 2   | $\frac{2}{3}$ | $\frac{1}{15}$ | $-18$ | $-14$ |
| $\ell = 3$ | 1   | $\frac{3}{5}$ | $\frac{1}{15}$ | $-10$ | $-8$ |
| $\ell = 4$ | 1   | $\frac{3}{5}$ | $\frac{1}{15}$ | $-6$ | $-4$ |
| $\ell = 5$ | 0   | 0 | 0 | |
| $\ell = 6$ | 0   | 0 | 0 | |
| $\ell = 7$ | 0   | $\frac{1}{15}$ | 2 | |
| $\ell = 8$ | 0   | $\frac{1}{15}$ | 4 | |
| $\ell = 9$ | 0   | 3 | 0 | 6 |

Table 1. $U(\varphi, \beta, \gamma)$ operators for the spin-2 plus spin-4 string

The explicit expressions for the operators $U(\varphi, \beta, \gamma)$ can be quite complicated, and we shall not give them all here. Some simple examples are as follows. We find $U = c \partial^2 \gamma \partial \gamma e^{\mu \varphi}$ at level $\ell = 0$; $U = c \partial \gamma \gamma e^{\mu \varphi}$ at $\ell = 1$; $U = \left(10 \partial \varphi \partial \gamma \gamma - (\mu + 2\alpha)\partial^2 \gamma \gamma \right) e^{\mu \varphi}$ at $\ell = 2$; and $U = 1$ at $\ell = 6$. The values of the momentum $\mu$ are given in the table.

We wish to show that the complete set of operators $U(\varphi, \beta, \gamma)$ can be associated with the highest-weight states of the lowest $W_3$ minimal model. The key to doing this is to identify the spin-2 and spin-3 currents $T$ and $W$ of the associated $W_3$ algebra, realised on the $(\varphi, \beta, \gamma)$ system. For $T$, this is straightforward; it is simply given by

$$T = T_\varphi + T_{\gamma, \beta},$$

(2.12)

where $T_\varphi$ and $T_{\gamma, \beta}$ are given in (2.4) and (2.5). For $W$, we observe from the results in [2] that at level $\ell = 9$ there is an operator $U(\varphi, \beta, \gamma)$ with conformal weight 3, ghost number $g = 0$, and momentum $\mu = 0$. Clearly this is the required primary spin-3 current. Its detailed form is

$$W = \sqrt{\frac{2}{13}} \left\{ \frac{5}{3} (\partial \varphi)^3 + 5\alpha \partial^2 \varphi \partial \varphi + \frac{25}{4} \partial^3 \varphi + 20 \partial \varphi \beta \partial \gamma \\
+ 12 \partial \varphi \partial \beta \gamma + 12 \partial^2 \varphi \beta \gamma + 5\alpha \partial \beta \partial \gamma + 3\alpha \partial^2 \beta \gamma \right\},$$

(2.13)

where we have given it the canonical normalisation in which $W(z)W(w) \sim \frac{c/3}{(z-w)^6} + \text{more}$, with the central charge $c = \frac{4}{5}$. It is now a straightforward matter to compute the OPEs of the $T$ and $W$ currents and verify that they do indeed generate the $W_3$ algebra at $c = \frac{4}{5}$. The only noteworthy point in the verification is that at the second-order pole in the OPE of $W$ with $W$ there is an additional spin-4 primary current $H$ over and above the expected
terms. However it turns out that $H$ is null, and so it does not upset the fact that $T$ and $W$ generate the $W_3$ algebra. In fact, $H$ is the BRST-trivial current $\{Q_1, \beta\}$.

Having found the currents that generate the $W_3$ algebra, we are now in a position to see how they act on the operators $U(\varphi, \beta, \gamma)$ of the physical states of the spin-2 plus spin-4 string. Of course we already know that the operators $U(\varphi, \beta, \gamma)$ are primary fields under $T$. Acting with $W$, we find that when $h$ takes values in the set $\{0, \frac{1}{15}, \frac{2}{5}, \frac{2}{3}\}$, the corresponding operators are highest-weight under $W$, i.e. $[W_n, U(0)] = 0$ for $n > 0$, and $[W_0, U(0)] = \omega U(0)$. We find that the weights of the highest-weight operators are as follows*:

\[
L_0 : \quad \{0, \frac{1}{15}, \frac{2}{5}, \frac{2}{3}\},
\]

\[
\frac{243}{\alpha} \sqrt{\frac{13}{8}} W_0 : \quad \{0, \pm 1, 0, \pm 26\}.
\]

Comparing with the results in [7], we see that these $L_0$ and $W_0$ weights are precisely those for the lowest $W_3$ minimal model, with $c = \frac{4}{5}$. The remaining operators $U(\varphi, \beta, \gamma)$ in the physical states of the spin-2 plus spin-4 string have $L_0$ weights $h = \frac{7}{5}$ and 3. We find that these are not highest weight under the $W$ current. In fact, they are $W$ descendant fields; those with $h = \frac{7}{5}$ can be written as $W_{-1} + \cdots$ acting on operators $U(\varphi, \beta, \gamma)$ with $h = \frac{2}{5}$, and those with $h = 3$ can be written as $W_{-3} + \cdots$ acting on operators $U(\varphi, \beta, \gamma)$ with $h = 0$.

The conclusion of the above discussion is that the $U(\varphi, \beta, \gamma)$ operators appearing in the physical states of the spin-2 plus spin-4 string are precisely those associated with the $c = \frac{4}{5}$ lowest $W_3$ minimal model. Those with $h = \{0, \frac{1}{15}, \frac{2}{5}, \frac{2}{3}\}$ are $W_3$ highest-weight fields, whilst those with $h = \frac{7}{5}$ and 3 are $W_3$ descendants. Viewed as purely Virasoro fields, they are all primaries. In fact, what we are seeing is an explicit example of the phenomenon under which the set of highest-weight fields of a $W$ minimal model decomposes into a larger set of highest-weight fields with respect to the Virasoro subalgebra. In this example, since $c$ is less than 1, the $W$ fields decompose into a finite number of Virasoro primaries (namely a subset of the primaries of the $c = \frac{4}{5}$ 3-state Potts model). In a more generic example, where the $W$ minimal model has $c \geq 1$, the finite number of $W$ highest-weight states will decompose into an infinite number of Virasoro primaries, with infinitely many of them arising as $W$ descendants. We shall encounter explicit examples of this when we study the spin-2 plus spin-$s$ strings with $s = 5$ and $s = 6$.

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* To be precise, we find that for $h = \frac{1}{15}$ the $W_0$ weight is positive for those operators $U(\varphi, \beta, \gamma)$ that have $(\frac{2}{27})^{-1} \mu = 4 \mod 6$, and negative when $(\frac{2}{27})^{-1} \mu = 2 \mod 6$. Similarly, for operators with $h = \frac{2}{5}$, we find the $W_0$ weight is positive when $(\frac{2}{27})^{-1} \mu = 2 \mod 6$, and negative when $(\frac{2}{27})^{-1} \mu = 4 \mod 6$. These results accord with the observation in [2] that there are two independent towers of $h = \frac{1}{15}$ operators, and two independent towers of $h = \frac{2}{5}$ operators, with the screening operator $\beta e^{\frac{2}{9} \omega \varphi}$ generating each tower from its lowest-level member.
2.3 The spin-2 plus spin-5 string

Let us now turn to the example of the spin-2 plus spin-5 string. The operator $F(\varphi, \beta, \gamma)$ appearing in (2.3) is now given by [2]

$$
F(\beta, \gamma, \varphi) = (\partial \varphi)^5 + 5\alpha \partial^2 \varphi (\partial \varphi)^3 + \frac{305}{8} (\partial^2 \varphi)^2 \partial \varphi + \frac{115}{6} \partial^3 \varphi (\partial \varphi)^2 + \frac{10}{3} \alpha \partial^3 \varphi \partial^2 \varphi
+ \frac{55}{48} \alpha \partial^4 \varphi \partial \varphi + \frac{251}{576} \partial^5 \varphi + \frac{25}{2} (\partial \varphi)^3 \beta \partial \gamma + \frac{25}{4} \alpha \partial^2 \varphi \partial \varphi \beta \partial \gamma + \frac{25}{4} \alpha (\partial \varphi)^2 \partial \beta \partial \gamma
+ \frac{125}{16} \partial^3 \varphi \beta \partial \gamma + \frac{325}{12} \partial^2 \varphi \partial \beta \partial \gamma + \frac{375}{16} \partial \varphi \partial^2 \beta \partial \gamma - \frac{175}{48} \partial \varphi \beta \partial^3 \gamma
+ \frac{5}{3} \alpha \partial^3 \beta \partial \gamma - \frac{35}{48} \alpha \partial \beta \partial^3 \gamma,
$$

with $\alpha^2 = \frac{121}{6}$. Here, we have $c_{\text{eff}} = 25$, and the associated minimal model, with $c = 1$, is expected to be the lowest $W_4$ minimal model [2]. Following the same strategy as before, we should therefore begin by looking amongst the operators $U(\varphi, \beta, \gamma)$ associated with the physical states for examples with zero $\varphi$ momentum, and ghost number $g = 0$, at spins 3 and 4. These will be the candidate spin 3 and spin 4 primary fields of the $W_4$ algebra. In [2], all physical states of the spin-2 plus spin-5 string up to level $\ell = 13$ were obtained. In fact one can easily see that the required physical states associated with the spin-3 and spin-4 currents will occur at levels 13 and 14 respectively. Thus we seek such physical states of the form (2.10), with $U(\varphi, \beta, \gamma)$ having zero ghost number and zero $\varphi$ momentum. In other words, $U(\varphi, \beta, \gamma)$ should be primary under $T = T_\varphi + T_{\gamma, \beta}$, with weight 3 or 4 respectively, and satisfy $\{Q_1, U\} = 0$. We find the following expressions for the primary spin-3 current $W$ and spin-4 current $V$ of the $W_4$ algebra at $c = 1$:

$$
W = \frac{1}{2} (\partial \varphi)^3 + \frac{3}{2} \alpha \partial^2 \varphi \partial \varphi + \frac{31}{18} \partial^3 \varphi + \frac{15}{2} \partial \varphi \beta \partial \gamma
+ 5 \partial \varphi \beta \partial \gamma + 5 \partial^2 \varphi \beta \gamma + \frac{3}{2} \alpha \partial \beta \partial \gamma + \alpha \partial^2 \beta \gamma,
$$

(2.16)

$$
V = -\frac{25}{\sqrt{654}} \left\{ (\partial \varphi)^4 + 4\alpha \partial^2 \varphi (\partial \varphi)^2 + \frac{237}{150} (\partial^2 \varphi)^2 + \frac{277}{25} \partial^3 \varphi \partial \varphi + \frac{617}{1650} \alpha \partial^4 \varphi
+ \frac{108}{5} \partial^2 \varphi \partial \varphi \beta \gamma + 20 (\partial \varphi)^2 \beta \partial \gamma + \frac{292}{25} (\partial \varphi)^2 \partial \beta \gamma + \frac{62}{11} \alpha \partial^2 \varphi \beta \partial \gamma
+ \frac{2104}{275} \alpha \partial^2 \varphi \partial \beta \gamma + \frac{2116}{55} \alpha \partial \varphi \partial^2 \beta \gamma + \frac{375}{55} \alpha \partial \varphi \partial \beta \partial \gamma + \frac{108}{55} \alpha \partial^3 \varphi \beta \gamma
- \frac{44}{15} \beta \partial^3 \gamma - \frac{132}{25} \partial \beta \partial^2 \gamma + \frac{321}{25} \partial^2 \beta \partial \gamma + \frac{544}{75} \partial^3 \beta \gamma + \frac{44}{5} \partial \beta \partial \beta \partial \gamma \right\}.
$$

(2.17)

We have normalised these currents canonically, so that the coefficient of the highest-order pole in the OPE of a spin-$s$ current with itself is $c/s$, where the central charge is $c = 1$ in the present case.

It is now a straightforward matter to check that $T$, $W$ and $V$ indeed generate the $W_4$ algebra at $c = 1$. We find complete agreement with the algebra given in [8], again modulo the appearance of certain additional primary fields that are BRST exact, and hence null. Specifically, we find a spin-5 null primary field $\{Q_1, \beta\}$ and its Virasoro descendants in
the OPE of $W$ with $V$, and a spin-6 null primary field $\{Q_1, (30 \partial \varphi \beta + 11\sqrt{6} \partial \beta)\}$ and its descendants in the OPE of $V$ with $V$.

Having obtained the currents that generate the $W_4$ algebra, we may now examine the $U(\varphi, \beta, \gamma)$ operators in the physical states of the spin-2 plus spin-5 string, in order to compare their weights with those of the lowest $W_4$ minimal model. Specifically, this model has primary fields with conformal weights $h = \{0, \frac{1}{16}, \frac{1}{12}, \frac{1}{3}, \frac{9}{16}, \frac{3}{4}, 1\}$. The results presented in [2], extended to level $\ell = 14$, are given in Table 2 below.

| $\ell$ | $g$ | $h$ | $\mu$ (In units of $\alpha/22$) |
|-------|-----|-----|------------------|
| 0     | 4   | $\frac{1}{17}$ | $\frac{1}{17}$ | 0 | $-22$ | $-23$, $-21$ | $-24$, $-20$ |
| 1     | 3   | $\frac{2}{5}$ | $\frac{2}{5}$ | $\frac{2}{5}$ | $-18$ | $-17$ | $-16$ | $-15$ |
| 2     | 3   | $\frac{23}{10}$ | $\frac{23}{10}$ | $\frac{23}{10}$ | $-17$ | $-16$ | $-14$ |
| 3     | 2   | 1 | $\frac{9}{17}$ | $\frac{9}{17}$ | $-12$ | $-11$ | $-10$ |
| 4     | 2   | $\frac{27}{10}$ | $\frac{27}{10}$ | $\frac{27}{10}$ | $-11$ | $-9$ |
| 5     | 2   | 3 | $\frac{25}{10}$ | $\frac{25}{10}$ | $1$ | $-12$ | $-10$ | $-9$ | $-8$ |
| 6     | 1   | $\frac{1}{17}$ | $\frac{1}{17}$ | $\frac{1}{17}$ | $-6$ | $-5$ |
| 7     | 1   | $\frac{1}{17}$ | $\frac{1}{17}$ | $\frac{1}{17}$ | $-4$ |
| 8     | 1   | $\frac{1}{17}$ | $\frac{1}{17}$ | $\frac{1}{17}$ | $-4$ | $-3$ |
| 9     | 1   | $\frac{49}{10}$ | $\frac{49}{10}$ | $\frac{49}{10}$ | $-5$ | $-3$ | $-2$ |
| 10    | 0   | 0 | 0 | 0 |
| 11    | 0   | 0 | $\frac{1}{17}$ | $\frac{1}{17}$ | 1 |
| 12    | 0   | 0 | $\frac{1}{17}$ | $\frac{1}{17}$ | 2 |
| 13    | 0   | 3 | $\frac{1}{17}$ | $\frac{1}{17}$ | 0 | 3 |
| 14    | 0   | 4 | $\frac{49}{10}$ | $\frac{49}{10}$ | 0 | 1 | 2 | 4 |

Table 2. $U(\varphi, \beta, \gamma)$ operators for the spin-2 plus spin-5 string

One can see from the results in Table 2 that indeed all the conformal weights of the primary fields of the lowest $W_4$ minimal model occur in the spin-2 plus spin-5 string. We find that the corresponding weights under the $W_4$ currents (2.16) and (2.17) are:

$$L_0 : \{0, \frac{1}{16}, \frac{1}{12}, \frac{1}{3}, \frac{9}{16}, \frac{3}{4}, 1\},$$

$$\frac{352}{\alpha}W_0 : \{0, \pm 1, 0, 0, \pm 11, \pm 32, 0\}.$$ (2.18)

$$6912\sqrt{6}V_0 : \{0, 27, -64, 128, -405, 1728, -6912\}.$$

We have checked that these weights agree with those that one finds using the highest-weight vertex-operators of the $W_N$ minimal models in the “Miura” realisations discussed in [9], after converting from the non-primary basis of Miura currents to the primary basis that we are
using here.* The remaining $U(\varphi, \beta, \gamma)$ operators obtained here and in [2], with conformal weights $h$ that lie outside the set of weights for the $W_4$ minimal model, correspond to $W$ and $V$ descendant states. In other words, they are secondaries of the $W_4$ minimal model, but they are primaries with respect to a purely Virasoro $c=1$ model. In this more generic case, with $c \geq 1$, the number of primaries in the purely Virasoro model will be infinite. Thus if we would go on solving the physical-state conditions at higher and higher levels $\ell$, we would find a set of operators $U(\varphi, \beta, \gamma)$ with conformal weights $h$ that increased indefinitely. All those lying outside the set $h = \{0, \frac{1}{16}, \frac{1}{12}, \frac{1}{3}, \frac{9}{16}, \frac{3}{4}, 1\}$ would be given by certain integers added to values lying in the set, corresponding to $W$ and $V$ descendant fields.

2.4 The spin-2 plus spin-6 string

In [2], it was found that there are four different nilpotent BRST operators of the form (2.1)–(2.7), corresponding to different values of $\alpha$, and hence $c_{\text{eff}}$. As usual, we shall be concerned with the case which seems to be associated with a unitary string theory. This is given by $\alpha^2 = \frac{845}{20}$, implying $c_{\text{eff}} = \frac{174}{4}$ and hence the $(\varphi, \beta, \gamma)$ system describes a model with $c = \frac{8}{7}$. We expect this to be the lowest $W_5$ minimal model. The operator $F(\varphi, \beta, \gamma)$ in this case takes the form [2]:

$$F(\beta, \gamma, \varphi) = (\partial \varphi)^6 + 6\alpha \partial^2 \varphi (\partial \varphi)^4 + \frac{765}{7} (\partial^2 \varphi)^2 (\partial \varphi)^2 + \frac{26}{35} \partial^3 \varphi (\partial \varphi)^3 + \frac{174}{35} \alpha (\partial^2 \varphi)^3$$
$$+ \frac{528}{35} \alpha \partial^3 \varphi \partial^2 \varphi \partial \varphi + \frac{18}{7} \alpha \partial^4 \varphi (\partial \varphi)^2 + \frac{1514}{245} (\partial^3 \varphi)^2 + \frac{2061}{245} \partial^4 \varphi \partial^2 \varphi + \frac{2736}{1225} \partial^5 \varphi \partial \varphi$$
$$+ \frac{142}{625} \alpha \partial^6 \varphi + 18 (\partial^2 \varphi)^4 \partial \beta \gamma + \frac{72}{5} \alpha \partial^2 \varphi (\partial \varphi)^2 \partial \beta \gamma + \frac{48}{5} \alpha (\partial \varphi)^3 \partial \beta \gamma$$
$$+ \frac{216}{5} \alpha \partial^3 \varphi \partial \beta \gamma + \frac{1941}{35} (\partial^2 \varphi)^2 \partial \beta \gamma + \frac{5256}{35} \partial^2 \varphi \partial \varphi \partial \beta \gamma + \frac{1941}{245} (\partial \varphi)^2 \partial^2 \beta \partial \gamma$$
$$- \frac{72}{5} (\partial \varphi)^2 \beta \partial^3 \gamma + \frac{204}{175} \alpha \partial^4 \varphi \beta \partial \gamma + \frac{102}{175} \alpha \partial^3 \varphi \partial \beta \partial \gamma + \frac{2176}{175} \alpha \partial^2 \varphi \partial^2 \beta \partial \gamma$$
$$- \frac{144}{175} \alpha \partial^2 \varphi \beta \partial^3 \gamma + \frac{1926}{175} \alpha \partial \varphi \partial \beta \partial^3 \gamma + \frac{576}{175} \alpha \partial \varphi \partial \beta \partial^3 \gamma + \frac{1614}{175} \alpha \partial^4 \varphi \beta \partial \gamma$$
$$- \frac{216}{35} \alpha \partial^2 \beta \partial^3 \gamma + \frac{144}{175} \beta \partial^5 \gamma + \frac{144}{35} \partial \beta \partial^2 \gamma \partial \gamma.$$

(2.19)

In [2], physical states in this theory up to and including level $\ell = 6$ were studied. Here, we are primarily concerned with finding the physical states associated with the expected spin-3, spin-4 and spin-5 primary fields of the $W_5$ minimal model. These should occur at levels $\ell = 18, 19$ and 20 respectively. It is a straightforward matter to solve for such physical states corresponding to operators $U(\varphi, \beta, \gamma)$ with zero ghost number, and zero $\varphi$ momentum. We find the following results for the spin-3, spin-4 and spin-5 operators $W$, $V$ and $Y$:

$$W = \sqrt{\frac{2}{37}} \left\{ \frac{7}{3}(\partial \varphi)^3 + 7\alpha \partial^2 \varphi \partial \varphi + \frac{185}{12} \partial^3 \varphi + 42 \partial \varphi \beta \partial \gamma$$
$$+ 30 \partial \varphi \partial \beta \gamma + 30 \partial^2 \varphi \beta \gamma + 7\alpha \partial \beta \partial \gamma + 5\alpha \partial^2 \beta \gamma \right\}.$$  

(2.20)

* If $h = \frac{1}{16}$, the weight under the spin-3 current $W$ is positive when $(\frac{\alpha}{27})^{-1} \mu = 3 \text{ mod } 4$, and negative when $(\frac{\alpha}{27})^{-1} \mu = 1 \text{ mod } 4$. If $h = \frac{9}{16}$, the weight under the spin-3 current $W$ is positive when $(\frac{\alpha}{27})^{-1} \mu = 1 \text{ mod } 4$, and negative when $(\frac{\alpha}{27})^{-1} \mu = 3 \text{ mod } 4$. If $h = \frac{3}{4}$, for which $(\frac{\alpha}{27})^{-1} \mu = 12n - 18$ or $12n - 14$, with $n$ a non-negative integer (see [2]), the spin-3 weight is positive when $n$ is odd, and negative when $n$ is even.
\begin{align}
V &= -\sqrt{\frac{7}{60819}} \left\{ \frac{427}{8} (\partial \varphi)^4 + \frac{427}{2} \alpha \partial^2 \varphi (\partial \varphi)^2 + \frac{10619}{8} (\partial^2 \varphi)^2 + 743 \partial^3 \varphi \partial \varphi + \frac{3313}{156} \alpha \partial^4 \varphi \\
&\quad + 1455 \partial^2 \varphi \partial \varphi \beta \gamma + 1281 (\partial \varphi)^2 \beta \gamma + 825 (\partial \varphi)^2 \partial \beta \gamma + \frac{5370}{13} \alpha \partial^2 \varphi \beta \partial \gamma \\
&\quad + \frac{6900}{13} \alpha \partial^2 \varphi \partial \beta \gamma + \frac{2910}{13} \alpha \partial \varphi \partial^2 \beta \gamma + \frac{4656}{13} \alpha \partial \varphi \partial \beta \partial \gamma + \frac{1455}{13} \alpha \partial^3 \varphi \beta \gamma \\
&\quad - 247 \beta \partial^3 \gamma - 494 \beta \partial^2 \gamma + \frac{6901}{7} \partial^2 \beta \partial \gamma + \frac{7785}{14} \partial^3 \beta \gamma + 1170 \partial \beta \beta \partial \gamma \right\}. \tag{2.21}
\end{align}

\begin{align}
Y &= \sqrt{\frac{7}{122}} \left\{ \frac{749}{165} (\partial \varphi)^5 + \frac{749}{33} \alpha (\partial \varphi)^3 \partial^2 \varphi + \frac{6901}{22} (\partial^2 \varphi)^2 \partial \varphi + \frac{1361}{66} \alpha \partial^3 \varphi \partial^2 \varphi \\
&\quad + \frac{13551}{132} \partial^3 \varphi (\partial \varphi)^2 + \frac{2330}{143} \alpha \partial^4 \varphi \partial \varphi + \frac{4825}{1848} \partial^5 \varphi + \frac{1498}{11} (\partial \varphi)^3 \beta \partial \gamma \\
&\quad + \frac{2570}{33} (\partial \varphi)^3 \beta \partial \gamma + \frac{11382}{143} \alpha \partial^2 \varphi \partial \varphi \partial \beta \partial \gamma + \frac{51670}{143} \alpha \partial^2 \varphi \partial \varphi \partial \beta \partial \gamma + \frac{430}{11} \alpha (\partial^2 \varphi)^2 \beta \gamma \\
&\quad + \frac{1420}{143} \alpha \partial^3 \varphi \partial \varphi \eta \gamma + \frac{2350}{11} \partial^2 \varphi (\partial \varphi)^2 \beta \gamma + \frac{7840}{143} \alpha (\partial \varphi)^2 \partial \beta \partial \gamma + \frac{4380}{143} \alpha (\partial \varphi)^2 \partial \beta \gamma \\
&\quad - 52 \partial \varphi \beta \partial^3 \gamma - \frac{624}{7} \partial \varphi \partial \beta \partial^2 \gamma + \frac{17331}{77} \alpha \partial \varphi \partial \beta \partial \gamma + \frac{775}{77} \partial \varphi \partial^3 \beta \gamma \\
&\quad - \frac{624}{7} \partial^2 \varphi \beta \partial^2 \gamma + \frac{18541}{77} \partial^2 \varphi \partial \beta \partial \gamma + \frac{3390}{11} \partial^2 \varphi \partial^2 \beta \gamma + \frac{7957}{144} \partial^3 \varphi \beta \partial \gamma \\
&\quad + \frac{77705}{462} \partial^3 \varphi \beta \partial \gamma + \frac{11575}{462} \partial^4 \varphi \beta \gamma - \frac{26}{3} \alpha \partial \beta \partial^3 \gamma - \frac{104}{13} \alpha \partial^2 \beta \partial^2 \gamma \\
&\quad + \frac{6725}{1001} \alpha \partial^3 \beta \partial \gamma + \frac{5365}{1001} \alpha \partial^3 \beta \gamma + 120 \partial \varphi \beta \partial \beta \partial \gamma \right\}. \tag{2.22}
\end{align}

We have as usual given these currents their canonical normalisations. We have checked that they indeed, together with $T = T_\varphi + T_{\gamma, \beta}$, generate the $W_5$ algebra, given in [10], with central charge $c = \frac{8}{9}$. Again, one finds additional BRST exact fields appearing on the right-hand sides of the OPEs of the primary currents. These fields are primaries (and their descendants) except in the case of the OPE $Y(z)Y(w)$. The new field occuring at the second order pole of this OPE is only primary up to BRST exact terms.

It was found in [2] that the physical states of the spin-2 plus spin-6 string were associated with operators $U(\varphi, \beta, \gamma)$ whose conformal weights included those of the highest-weight fields of the lowest $W_5$ minimal model, which has $c = \frac{8}{9}$. Indeed, here we find that the highest-weight fields have the weights

\begin{align}
L_0 &:= \{0, \frac{2}{35}, \frac{3}{35}, \frac{2}{7}, \frac{17}{35}, \frac{23}{35}, \frac{4}{5}, \frac{6}{7}, \frac{6}{5}\}, \\
\frac{325}{\alpha} \sqrt{\frac{57}{8}} W_0 &:= \{0, \pm2, \pm1, 0, \pm13, \pm39, \pm76, 0, \pm38\}, \tag{2.23} \\
25 \sqrt{\frac{141911}{3}} V_0 &:= \{0, 11, -14, 50, -74, 46, 836, -1100, -2299\}, \\
\frac{89375}{\alpha} \sqrt{\frac{427}{32}} Y_0 &:= \{0, \pm11, \mp48, 0, \pm314, \mp902, \pm1452, 0, \mp16621\}.
\end{align}
Note that the ± signs for the weights under the spin-3 current $W$ are correlated with those for the weights under the spin-5 current $Y$. Again we have checked that these weights agree with those calculated from the realisations of the $W_N$ minimal models given in [9]. All the physical states of the spin-2 plus spin-6 string are presumably associated with operators $U(\varphi, \beta, \gamma)$ that are either highest-weight under the $W_5$ algebra, as given in (2.23), or they are $W, V$ or $Y$ descendants of such operators. Some examples of descendant operators were found in [2]. Again one expects, since the $W_5$ minimal model has $c = \frac{\Delta}{7} \geq 1$, that there will be an infinite number of descendant operators.

3. Discussion

In this paper, we have investigated the physical states of a string theory based on constraints imposed by holomorphic currents of spins 2 and $s$. In [2], where these models were first proposed, it was conjectured that the physical states of the spin-2 plus spin-$s$ string are described by effective Virasoro strings whose spacetime energy-momentum tensor has central charge $26 - \frac{2(s-2)}{(s+1)}$. The physical states factorise as in (2.10), with the intercepts $\Delta$ for the effective-spacetime states being conjugate to the conformal weights $h$ of the operators $U(\varphi, \beta, \gamma)$, in the sense that $\Delta = 1 - h$. It was conjectured in [2] that the $(\varphi, \beta, \gamma)$ system, which has central charge $c = \frac{2(s-2)}{(s+1)}$, should describe the lowest unitary minimal model of the $W_{s-1}$ algebra.

We have tested the above conjecture in detail for the cases $s = 4, 5$ and 6 of the spin-2 plus spin-$s$ string. We have shown, for the lowest few levels, that indeed the operators $U(\varphi, \beta, \gamma)$ that arise in the physical states (2.10) are associated with the highest-weight fields of the lowest unitary $W_{s-1}$ minimal models. Specifically, we find in all physical states (2.10) that $U(\varphi, \beta, \gamma)$ is either a highest-weight field of the corresponding $W_{s-1}$ minimal model, or else it is a descendant field in the sense that it is obtained from a highest-weight field by acting with the negative modes of the primary currents of the $W_{s-1}$ algebra. Since the central charge $c = \frac{2(s-2)}{(s+1)}$ of the $(\varphi, \beta, \gamma)$ system satisfies $c \geq 1$ for $s \geq 5$, it follows that in these cases there are infinite numbers of such descendant fields in the models. Thus the $W_{s-1}$ currents provide a strikingly powerful organising symmetry in these cases.

The original realisations of the $W_N$ algebras were the $(N-1)$-scalar realisations from the Miura transformation, introduced in [7,9]. By contrast, the realisations of the lowest unitary $W_{s-1}$ minimal models that we find here are all given in terms of just one scalar field $\varphi$, and the $(\beta, \gamma)$ ghost system for spin-$s$. This ghost system can be bosonised, yielding two-scalar realisations. However even when $s = 4$, our two-scalar realisation is quite different from the usual Miura realisation of $W_3$. In particular, our realisations close on the $W_{s-1}$ algebras modulo the appearance of certain null primary fields in the OPEs of the currents, whereas no such null fields arise in the Miura realisations. Presumably the realisations that we find here are very specific to the particular unitary minimal models that arise in these higher-spin
string theories. As an example, we present the spin-2 and spin-3 currents (2.12) and (2.13) for the $W_3$ algebra at $c = \frac{4}{5}$ in the bosonised language, where $\gamma = e^{i\rho}$ and $\beta = e^{-i\rho}$.

\[ T = -\frac{1}{2} (\partial \varphi)^2 - \frac{1}{2} (\partial \rho)^2 - \alpha \partial^2 \varphi + \frac{7}{2} i \partial^2 \rho, \]

\[ W = \sqrt{\frac{2}{13}} \left\{ \frac{5}{3} (\partial \varphi)^3 + 5\alpha \partial^2 \varphi \partial \varphi + \frac{25}{4} \partial^3 \varphi + 4 \partial \varphi (\partial \rho)^2 - 16i \partial \varphi \partial^2 \rho \right. \]

\[ - 12i \partial^2 \varphi \partial \rho - \frac{2}{3} i \alpha (\partial \rho)^3 - 3\alpha \partial^2 \rho \partial \rho - \frac{11}{6} i \alpha \partial^3 \rho \right\}. \tag{3.1} \]

where $\alpha^2 = \frac{243}{20}$. It is interesting to note that this realisation of the $W_3$ algebra at $c = \frac{4}{5}$ is precisely the one obtained in [11] (case I, after an $SO(1,1)$ rotation of the two scalars), where more general scalar realisations of $W_3$ modulo a null spin-4 operator were considered.

Finally, we remark that, as observed in [2], the spectrum of physical states for the spin-2 plus spin-$s$ string becomes more complicated if there is just one $X^\mu$ coordinate in the effective energy-momentum tensor $T^{\text{eff}}$ (2.7). In particular, there are additional physical states over and above those of the form (2.10), which do not factorise into the product of effective-spacetime physical states times operators $U(\varphi, \beta, \gamma)$. Examples of these were found for the $W_3$ string in [12], and for spin-2 plus spin-$s$ strings in [2]. A general discussion of the BRST cohomology for the two-scalar $W_3$ string is given in [13]. It may well be that the spin-2 plus spin-$s$ strings with just one additional coordinate $X^\mu$ capture the more subtle aspects of the underlying higher-spin geometry.

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