HYPERBOLICITY OF HOLOMORPHIC FOLIATIONS WITH PARABOLIC LEAVES

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Abstract. In this paper, we study a notion of hyperbolicity for holomorphic foliations with 1-dimensional parabolic leaves, namely the non-existence of holomorphic cylinders along the foliation – holomorphic maps from $\mathbb{D}^{n-1} \times \mathbb{C}$ to the manifold sending each $\{\ast\} \times \mathbb{C}$ to a leaf. We construct a tensor, that is a leaf-wise holomorphic section of a bundle above the foliated manifold, which vanishes on an open saturated set if and only if there exists such a cylinder through each point of this open set. Thanks to this study, we prove that open sets saturated by compact leaves are union of holomorphic cylinders along the foliation. We also give some more specific results and examples in the case of a compact manifold, and of foliations with one compact leaf or without any compact leaves.

Introduction

This paper is a first step in the study of hyperbolicity of foliations. In the general case of complex manifolds, a way of tackling complex hyperbolicity is to study the existence of holomorphic curves from $\mathbb{C}$ to the manifold: if the only such curves are the constant-curves, then the manifold is said to be Brody-hyperbolic (see notably [8], [1] or [2]). However, in the special case of foliations, as soon as there exists a parabolic leaf (i.e. a leaf whose universal covering is the complex plane $\mathbb{C}$) this question becomes trivial. That is why in the case of foliations with parabolic leaves, we need to study some more appropriate and relevant notions of hyperbolicity. More precisely, this paper studies the existence of holomorphic maps from $\mathbb{D}^{n-1} \times \mathbb{C}$ to the manifold along the foliation, i.e. maps which send each $\{\ast\} \times \mathbb{C}$ to a leaf ($\mathbb{D}^{n-1}$ denoting the ball of radius 1 in $\mathbb{C}^{n-1}$). Then, if there does not exist any such map which is immersed (there always exist the trivial ones taking values into a single leaf), the foliated manifold is called hyperbolic.

This question is linked with the issue of measure-hyperbolicity (see [8] for the definitions of the different notions of hyperbolicity and their links). As the Kobayashi hyperbolicity corresponds to the non-degeneracy of the Kobayashi pseudo-distance (see [8]), the measure-hyperbolicity corresponds to the non-degeneracy of the Kobayashi pseudo-distance.

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measure. In a similar way, as the existence of non-constant holomorphic maps from $\mathbb{C}$ to a complex manifold implies its non-Kobayashi hyperbolicity, the existence of non-trivial holomorphic maps from $\mathbb{D}^{n-1} \times \mathbb{C}$ implies its non-measure-hyperbolicity. This is one of the motivations behind the study of the existence of such maps.

For this study, we will consider foliations by parabolic leaves (so that $\{\ast\} \times \mathbb{C}$ could be sent onto a leaf).

So let us consider a complex manifold $X$ of dimension $n$ (real dimension $2n$) provided with a 1-dimensional holomorphic foliation $\mathcal{F}$ whose leaves are all parabolic (the leaf passing through a point $x$ will be denoted by $L_x$). We would like to find conditions under which there exists a holomorphic cylinder along the foliation:

**Definition 0.1.** A holomorphic cylinder along the foliation is an immersed holomorphic map $F : \mathbb{D}^{n-1} \times \mathbb{C} \to X$ such that, for any $x \in \mathbb{D}$, $\{x\} \times \mathbb{C}$ is sent onto a leaf of the foliation.

We construct in this paper an invariant of the foliation: a leaf-wise holomorphic section of a holomorphic bundle of the foliation, which vanishes if and only if there exist such holomorphic cylinders along the foliation. More precisely:

**Theorem 0.2.** There exists a tensor $\Gamma$ which is an invariant associated to the foliation:

$$\Gamma|_x : T_x X \times T_x \mathcal{F} \to \mathbb{C}, \ \forall x \in X$$

and which vanishes on an open saturated set $U$ if and only if through any point of $U$ there exists a holomorphic cylinder $F : \mathbb{D}^{n-1} \times \mathbb{C} \to U$ along the foliation.

It is $\mathbb{C}$-antilinear with respect to the first variable and $\mathbb{C}$-linear with respect to the second one.

Moreover, $\Gamma$ is a section of the bundle $E = \Lambda^{(0,1)} T^* N \otimes \Lambda^{(1,0)} T^* \mathcal{F}$ over $(X, \mathcal{F})$, which is holomorphic along the leaves.

In this theorem, and more generally in this paper, $T \mathcal{N}$ denotes the transverse (or normal) vector bundle of the foliation: $T \mathcal{N} = T X / T \mathcal{F}$. The usual vocabulary and properties of holomorphic foliations are recalled in Appendix A.

This analysis allows us to get some new results in the theory of foliations. Notably, in the case of a foliation with compact leaves, we can deduce that through each point, there exists a holomorphic cylinder along the foliation (result which was already known in the case of Stein manifolds foliated by parabolic leaves). This implies that, in this case, the leaves depend holomorphically on the transversal disk (see section 3.1 for more details).

In order to simplify notations, the analysis of this paper will first be conducted in the case of complex dimension 2. But, as it will be underlined, section 2.5, the results can straightforwardly be generalized to the case of higher dimensions. Let us detail the structure of this paper.

First, in section 1.1, we introduce a particular set $\mathcal{C}$ of maps $F : \mathbb{D} \times \mathbb{C} \to X$ which naturally appear as candidates for this analysis. And we prove that if a map of the desired type exists, then it necessarily belongs to this set $\mathcal{C}$. Thus, the
problem reduces to the question of the holomorphy of the maps in $C$.

To tackle this problem, we associate to any map $F \in C$, a complex-valued function $\omega$ on $D \times \mathbb{C}$, which is holomorphic along the leaves $\{\ast\} \times \mathbb{C}$, and measures the lack of holomorphy of $F$: this function is zero if and only if $F$ is holomorphic. This function $\omega$ depends on the map $F$. However, we note that if one of the maps in $C$ is holomorphic, then so are all the other maps in $C$ (whose images are included in the one of $F$). This leads us to look for an invariant on $X$ measuring the lack of holomorphy of any maps in $C$.

To deal with this, we study how $\omega$, and the local differential forms on $X$ naturally associated to it, depend on $F$ (section 1.3, section 2.1 and 2.2 respectively).

This leads us to construct such an invariant $\Gamma$, a tensor on $X$, independent of any choice of $F$, which is zero on an open saturated set if and only if any map $F \in C$ taking values in this open set is holomorphic. This invariant is described in section 2.4. It is an invariant associated to the foliation which detects the existence of holomorphic cylinders along the foliation. More precisely, let us state our result:

**Theorem 0.3.** To any $F \in C$ is associated a tensor, defined on an open set $V_F$ around $x_0 = F(0,0)$,

$$\Gamma_{F,x} : T_x X \times T_x F \to \mathbb{C}, \ \forall x \in V_F.$$  

This tensor does not depend on $F$, i.e. if $F' \in C$ and $V_F \cap V_{F'} \neq \emptyset$, then $\Gamma_F = \Gamma_{F'}$ on $V_F \cap V_{F'}$. Thus, a tensor can be defined on $X$ by $\Gamma = \Gamma_F$ on each open set $V_F$. It is $\mathbb{C}$-antilinear with respect to the first variable and $\mathbb{C}$-linear with respect to the second one. This is an invariant of the foliation, that is zero if and only if the maps in $C$ are holomorphic. More precisely, for any $F \in C$, $\Gamma$ is zero on $F(D \times \mathbb{C})$ if and only if $F$ is holomorphic.

Moreover, $\Gamma$ is a section of the line bundle $E = \Lambda^{(0,1)}T^*N \otimes \Lambda^{(1,0)}T^*F$ over $(X, F)$ and it is holomorphic along the leaves.

Thus if $\Gamma = 0$ on the saturation $U$ of a transverse disk $D$, then there exists a holomorphic cylinder along the foliation $F : D \times \mathbb{C} \to X$ that sends $D \times \{0\}$ biholomorphically on $D$. In fact, any $F \in C$ whose image is included in $U$ is such a holomorphic cylinder.

Moreover, if $\Gamma = 0$ on $X$, all the maps in $C$ are holomorphic. This, in particular, is satisfied when the line bundle $E$ over $(X, F)$ does not admit non-zero leaf-wise holomorphic sections, since $\Gamma$ is a leaf-wise holomorphic section of $E$.

Finally, we study some particular cases and examples. First, we prove that if all the leaves of $F$ are compact, then $\Gamma$ identically vanishes, and therefore all the maps $F \in C$ are holomorphic (section 3.1). Similarly, if $F$ has one compact leaf $L_0$ with a finite holonomy group, then $\Gamma$ vanishes on an open saturated set $U$ around $L_0$, and all the maps $F \in C$ taking values in $U$ are holomorphic. In the case where $F$ has one compact leaf with an infinite holonomy group, we study the action of the holonomy group on the complex-valued function $\omega$. We also prove that in the case where the manifold $X$ is compact, even though the leaves are not, if the bundle
$E$ has a negative curvature along the leaves then the invariant $\Gamma$ vanishes. We also give an example of a manifold with non-compact leaves whose invariant $\Gamma$ is non-zero and therefore does not admit any holomorphic cylinder along the foliation (section 3.3).

1. Holomorphy of cylinders along the foliation

1.1. Cylinders along the foliation. Let us consider $X$ a 2-dimensional complex manifold provided with a holomorphic foliation of complex dimension 1 whose leaves are all parabolic. We aim to study the existence of holomorphic maps from $\mathbb{D} \times \mathbb{C}$ to $X$ along the foliation.

Let us first note that if such a holomorphic map $F : \mathbb{D} \times \mathbb{C} \rightarrow X$ along the foliation exists, then $D := F(\mathbb{D} \times \{0\})$ is a holomorphic disk transverse to the foliation. Moreover, for every $x \in \mathbb{D}$, $t_x := dF(x,0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in T_F$, and $x \rightarrow t_x$ is a holomorphic trivialization of $T_F$ along $D$.

Thus the cylinder-maps $F : \mathbb{D} \times \mathbb{C} \rightarrow X$ along the foliation (i.e. a map sending $\{\ast\} \times \mathbb{C}$ onto a leaf) such that $D = F(\mathbb{D} \times \{0\})$ is a holomorphic disk transverse to the foliation and $t_x = dF(x,0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a holomorphic trivialization of $T_F$ along $D$, appear as the natural and only candidates for our study. The question is: are these maps holomorphic?

We will denote by $C$ the sets of all these maps. Let us first describe the one-and-one correspondence between the data of such a map $F$, and the data of the transverse disk $D$ and of the trivialization $t$.

Let $D$ be a holomorphic disk transverse to the foliation, parametrized holomorphically by the unit disk $\mathbb{D}$ with which it will be identified. Along $D$, we fix a trivialization $t : \mathbb{D} \rightarrow T_{\mathcal{F}}|_D$ of the tangent bundle $T_{\mathcal{F}}$ of the foliation. For each $x \in \mathbb{D}$, the leaf $L_x$ passing through $x$ is parabolic. Thus, if $\widetilde{L}_x$ denotes its universal cover, $\widetilde{L}_x \simeq \mathbb{C}$. Then for any fixed covering map $\psi_x : \widetilde{L}_x \rightarrow L_x$, there exists an unique bi-holomorphism $F_x^0 : \mathbb{C} \rightarrow \widetilde{L}_x$ such that $F_x := \psi_x \circ F_x^0$ satisfies $F_x(0) = x$ and $dF_x(1) = t_x$. Thus we get a surjective immersion

$$F : \begin{cases} \mathbb{D} \times \mathbb{C} & \rightarrow X \\ (x,y) & \rightarrow F_x(y) \end{cases}$$

which is by construction holomorphic with respect to the second variable $y$ and holomorphic on $\mathbb{D} \times \{0\}$.

It is the only map from $\mathbb{D} \times \mathbb{C}$ along the foliation such that $F(\mathbb{D} \times \{0\}) = D$ (the parametrization of $D$ via $\mathbb{D}$ being fixed) and $dF|_{\mathbb{D} \times \{0\}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = t$.

If one of these maps in $C$ is holomorphic, then it is a map of the desired kind: a holomorphic cylinder along the leaves.
Reciprocally, if such a holomorphic map $F$ along the foliation exists, then, as noted in the beginning of this section, this map $F$ belongs to $\mathcal{C}$. That is why, for the analysis of our problem, we only have to study the holomorphy of these maps $F \in \mathcal{C}$.

1.2. Holomorphy of the maps $F \in \mathcal{C}$. Let $F$ be a map in $\mathcal{C}$, and let us consider $J' = F^* J$, with $J$ the complex structure on $X$. This structure $J'$ can be decomposed as $J' = J_0 + \Omega$ with $J_0$ the standard complex structure on $\mathbb{D} \times \mathbb{C}$ and $\Omega$ a tensor in $\text{End}(T(\mathbb{D} \times \mathbb{C}))$. This tensor $\Omega$ measures the lack of holomorphy of $F$, since it is zero if and only if $J' = J_0$, which means if and only if $F$ is holomorphic.

This tensor $\Omega$ takes values in $T\mathcal{F}_0$ because $J'$ and $J_0$ coincide on $\mathbb{C} \times \{0\} \subset T'(\mathbb{D} \times \mathbb{C})$ (since $F$ has been constructed along the holomorphic transverse disk $D$; for more details see the Appendix [4]). Moreover, it vanishes on vectors tangent to the foliation (because $F$ is by construction holomorphic with respect to $y$).

Furthermore, $(J')^2 = -\text{Id}$, which implies that $\Omega J_0 = J_0 \Omega$ and $\Omega$ is antilinear. Thus, there exists a function $\omega : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ with $\omega_{|\mathbb{D} \times \{0\}} = 0$ such that

$$\Omega_{(x,y)} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ \omega(x,y)v_x \end{pmatrix}. $$

This function $\omega$ satisfies:

**Lemma 1.1.** The function $\omega : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ defined above vanishes if and only if $F$ is holomorphic. It is holomorphic with respect to the second variable, that is to say for all $x \in \mathbb{D}$, the functions $\omega(x, \cdot)$ are holomorphic on $\mathbb{C}$. Moreover these functions vanish on $0$ and their derivatives also vanish on $0$.

**Proof.** First, by definition of $\Omega$, $F$ is holomorphic if and only if $\Omega = 0$, or equivalently if and only if $\omega = 0$.

As $F$ is holomorphic along $\mathbb{D} \times \{0\}$, for all $x \in \mathbb{D}$, $\omega(x,0) = 0$.

The leaf-wise holomorphy of $\omega$ comes from the integrability of the structure $J'$. Indeed this property translates as: the Nijenhuis vector field $N_{J'}(Z,Y)$ vanishes for all vector fields $Z$ and $Y$ (or, equivalently, for any constant vector fields $Z$ and $Y$). Since $J_0$ is integrable, and $J' = J_0 + \Omega$,

$$N_{J'}(Z,Y) = [\Omega Z, J_0 Y] + [J_0 Z, \Omega Y] + [\Omega Z, \Omega Y] - \Omega[J_0 Z, Y] - \Omega[\Omega Z, Y] - J_0[\Omega Z, Y] - J_0[\Omega Y, Z] - J_0[\Omega Z, \Omega Y].$$

Let us take $Z = (z_1, z_2)$ and $Y = (y_1, y_2)$ constant on $\mathbb{D} \times \mathbb{C}$. Since $\nabla J_0 = 0$, and since $\Omega$ is zero on $\mathbb{D} \times \{0\}$ and takes values in $\{0\} \times \mathbb{C}$, $N'_{J}(Z,Y) = 0$ is equivalent to:

$$d\omega(J_0 Y)z_1 - d\omega(J_0 Z)y_1 + d\omega(0, \omega y_1)z_1$$
$$-d\omega(0, \omega z_1)y_1 - J_0d\omega(Y)z_1 + J_0d\omega(Z)y_1 = 0.$$

Firstly fixing $y_1 = 0$ and $x_1 \neq 0$, we notice that if $J'$ is integrable then $\omega$ has to be holomorphic with respect to the second variable.

Reciprocally, if $\omega$ is holomorphic with respect to the second variable, one can straightforwardly check that $N_{J'}(Z,Y) = 0$ for all constant vector fields $Z$ and $Y$. The leaf-wise holomorphy of $\omega$ comes from the integrability of the structure $J'$. Indeed this property translates as: the Nijenhuis vector field $N_{J'}(Z,Y)$ vanishes for all vector fields $Z$ and $Y$ (or, equivalently, for any constant vector fields $Z$ and $Y$). Since $J_0$ is integrable, and $J' = J_0 + \Omega$,

$$N_{J'}(Z,Y) = [\Omega Z, J_0 Y] + [J_0 Z, \Omega Y] + [\Omega Z, \Omega Y] - \Omega[J_0 Z, Y] - \Omega[\Omega Z, Y] - J_0[\Omega Z, Y] - J_0[\Omega Y, Z] - J_0[\Omega Z, \Omega Y].$$

Let us take $Z = (z_1, z_2)$ and $Y = (y_1, y_2)$ constant on $\mathbb{D} \times \mathbb{C}$. Since $\nabla J_0 = 0$, and since $\Omega$ is zero on $\mathbb{D} \times \{0\}$ and takes values in $\{0\} \times \mathbb{C}$, $N'_{J}(Z,Y) = 0$ is equivalent to:

$$d\omega(J_0 Y)z_1 - d\omega(J_0 Z)y_1 + d\omega(0, \omega y_1)z_1$$
$$-d\omega(0, \omega z_1)y_1 - J_0d\omega(Y)z_1 + J_0d\omega(Z)y_1 = 0.$$

Firstly fixing $y_1 = 0$ and $x_1 \neq 0$, we notice that if $J'$ is integrable then $\omega$ has to be holomorphic with respect to the second variable. Reciprocally, if $\omega$ is holomorphic with respect to the second variable, one can straightforwardly check that $N_{J'}(Z,Y) = 0$ for all constant vector fields $Z$ and $Y$. The leaf-wise holomorphy of $\omega$ comes from the integrability of the structure $J'$.
Y, and therefore for every vector field. Thus, \( J' \) (or equivalently \( J \)) is integrable if and only if \( \omega \) is holomorphic with respect to the second variable.

Let us now translate the holomorphy of the map \( x \rightarrow t_x = dF_{(x,0)} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \). Let us consider a distinguished open set \( U \) containing \( F(D \times \{0\}) \) on which there exists a holomorphic foliated chart: \( \psi : U \rightarrow \mathbb{D} \times \mathbb{D} (J,J_0) \)-holomorphic (with \( J_0 \) the standard structure on \( \mathbb{D} \times \mathbb{D} \)), sending \( D \) on \( \mathbb{D} \times \{0\} \). (The definitions of the involved notions and terms are recalled Appendix A).

Then let us define \( \alpha := \psi \circ F \) on \( F^{-1}(U) \).

This map \( \alpha \) can be written as \( \alpha(x,y) = (\alpha_1(x), \alpha_2(x,y)) \), with \( \alpha_1 \) a bi-holomorphism of \( D \) and \( \alpha_2 \) a map holomorphic with respect to \( y \). Moreover, by writing that \( \alpha \) is holomorphic for the complex structures \( J' \) and \( J_0 \),

\[
\frac{d}{dx} \left( (\Omega + J_0) \left( \begin{array}{c} v_x \\ v_y \end{array} \right) \right) = J_0 \frac{d}{dy} \left( \begin{array}{c} v_x \\ v_y \end{array} \right)
\]

we get

\[
2i \frac{\partial \alpha_2}{\partial x} = \frac{\partial \alpha_2}{\partial y}.
\]

So, by differentiating:

\[
2i \frac{\partial^2 \alpha_2}{\partial y \partial x} = \frac{\partial \omega}{\partial y} \frac{\partial \alpha_2}{\partial y} + \frac{\partial^2 \alpha_2}{\partial y^2}.
\]

Besides, \( x \rightarrow t_x \) is holomorphic, that is to say, in the trivialization, \( x \rightarrow \frac{\partial \alpha_2}{\partial y}(x,0) \) is holomorphic, which reads as \( \frac{\partial \omega}{\partial y}(x,0) = 0 \). And since \( \omega(x,0) = 0 \), we deduce from this that \( \frac{\partial \omega}{\partial y} = 0 \) on \( \mathbb{D} \times \{0\} \).

Summing up, \( \omega(x,\cdot) \) are holomorphic functions on \( \mathbb{C} \) which vanish on \( 0 \) and whose derivatives also vanish on \( 0 \). Furthermore \( F \) is holomorphic if and only if these functions are identically zero, or equivalently if and only if \( \frac{\partial \omega}{\partial y} = 0 \) (because \( \omega = 0 \) on \( \mathbb{D} \times \{0\} \) and it is holomorphic with respect to \( y \)).

So to summarize, the function \( F \) is holomorphic if and only if the associated function \( \omega \) vanishes, or equivalently, if and only if \( \frac{\partial \omega}{\partial y} = 0 \), or also if and only if \( \frac{\partial^2 \omega}{\partial y^2} = 0 \). We are now going to study more this function \( \omega \).

1.3. How \( \omega \) depends on \( F \). The goal of this section is to see how the function \( \omega \) varies when one considers another function in \( \mathcal{C} \).

So let us consider another \( \tilde{F} \in \mathcal{C} \) constructed as in \ref{1.1} from the data of a holomorphic disk \( D \) transverse to the foliation, and of a holomorphic trivialization \( \tilde{t} \) of \( T\mathcal{F} \) along \( D \). Then as previously, we can associate to it the tensor \( \tilde{\Omega} \) and the function \( \tilde{\omega} \) measuring the lack of holomorphy of \( \tilde{F} \). We would like to compare the function \( \omega \) associated with \( F \), with this function \( \tilde{\omega} \). More precisely, if
\[ F(x, y) = \tilde{F}(\tilde{x}, \tilde{y}) \], we’d like to compare \( \omega(x, y) \) with \( \tilde{\omega}(\tilde{x}, \tilde{y}) \).

Since \( D \) and \( \tilde{D} \) are two transverse holomorphic disks, a classical construction explained in the Appendix (lemma \[ \text{B.1} \]) provides us with a bi-holomorphism \( \theta_1 : D_1 \to D_2 \), with \( D_1 \subset \mathbb{D} \) and \( D_2 \subset \mathbb{D} \), such that for any \( x \), \( L_{\tilde{F}(x, 0)} = L_{F(\theta_1(x), 0)} \). So, without any loss of generality (since we are only interested in the intersection of the images of \( F \) and \( \tilde{F} \)), possibly restricting the two transverse holomorphic disks \( D \) and \( \tilde{D} \), we can consider \( \theta_1 \) as a bi-holomorphism of \( \mathbb{D} \) such that for any \( x \in \mathbb{D} \), \( \theta_1(x) \in \mathbb{D} \) satisfies \( \tilde{L}(x, 0) = L_{F(\theta_1(x), 0)} \).

Then, for any \( x \in D \), we can consider the bi-holomorphism of \( C : \theta_x = F^{-1}_x \circ \tilde{F}_x \) (actually \( (F^{-1}_x \circ \tilde{F}_x) \)). There exist some complex numbers \( a(x) \), and \( b(x) \) such that it can be written: \( \theta_x(y) = a(x) y + b(x) \).

And we define \( \theta : \mathbb{D} \times \mathbb{C} \to \mathbb{D} \times \mathbb{C} \) (locally \( \theta = F^{-1} \circ \tilde{F} \)) by \( \theta(x, y) = (\theta_1(x), \theta_2(x, y)) \) with \( \theta_2(x, y) = \theta_x(y) \). Thus we prove:

**Lemma 1.2.** There exists a diffeomorphism \( \theta : \mathbb{D} \times \mathbb{C} \to \mathbb{D} \times \mathbb{C} \) satisfying:

\[
\begin{align*}
\bullet & \quad \tilde{F} = F \circ \theta \\
\mathbb{D} \times \mathbb{C} & \xrightarrow{\theta} \tilde{F} \\
\mathbb{D} \times \mathbb{C} & \xrightarrow{F} X
\end{align*}
\]

\[
\bullet \quad \theta(x, y) = (\theta_1(x), \theta_2(x, y)), \quad \text{with } \theta_1 \text{ a bi-holomorphism of the disk } \mathbb{D}, \text{ and for any } x \in \mathbb{D} \text{ } \theta_2(x, .) \text{ a bi-holomorphism of } \mathbb{C}.
\]

Moreover,

\[
\frac{\partial \theta_2}{\partial y} \omega(x, y) = \omega(\theta(x, y)) \frac{\partial \theta_1}{\partial x} + 2i \frac{\partial \theta_2}{\partial x}.
\]

Thus, the change in the function \( \omega \) is measured by the lack of holomorphy with respect to \( x \) of the function \( \theta_2 \).

**Proof.** Only \[ \text{B.3} \] remains to be proven. Let us compare \( \Omega \) and \( \tilde{\Omega} : J_0 + \Omega = (\tilde{F})^* J = (F \circ \theta)^* J = \theta^* (J_0 + \Omega) \). Therefore \( \tilde{\Omega} = \theta^* \Omega + \theta^* J_0 - J_0 \), so \( \theta^* \Omega = \Omega \circ \theta_* + J_0 \circ \theta_* - \theta_* \circ J_0 \), which exactly writes as \[ \text{(1.3)} \].

The last remark of the lemma is explained since, possibly by reparametrizing \( \tilde{D} \) by a bi-holomorphism of \( \mathbb{D} \), we can suppose \( \frac{\partial \theta_1}{\partial x} = 1 \). Besides the term \( \frac{\partial \theta_2}{\partial x} \) is a non-zero multiplicative term – corresponding to a reparametrization via a bi-holomorphism of \( \mathbb{C} \) of the maps \( \tilde{F}_x \) (but not necessarily holomorphic with respect to \( x \)). So the most relevant term in \[ \text{(1.3)} \] is \( \frac{\partial \theta_2}{\partial x} \).

Thus, if \( F \) is not holomorphic, then the pull-back in \( \mathbb{D} \times \mathbb{C} \), via \( F \), of the holomorphic disk \( \tilde{D} = \tilde{F}(\mathbb{D} \times \{0\}) \) and of the holomorphic trivialization \( \tilde{t} = d\tilde{F}(x, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) will not be holomorphic. Besides, since locally \( \theta = F^{-1} \circ \tilde{F} \), the pull-back by \( F \) of \( \tilde{D} \) is \( \theta(\mathbb{D} \times \{0\}) \), i.e. the graph of \( b(x) \); and the pull-back by \( F \) of \( \tilde{t} \) is \( d\theta(x, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), i.e the graph of \( a(x) \). So the functions \( a \) and \( b \), and so also the function \( \theta_2 = a(x) y + b(x) \) are not holomorphic with respect to the variable \( x \). This lemma tells us that, in
this case, in fact, the change in the function ω is roughly measured by the non-holomorphy of θ₂.

Moreover, by the same argument, it is interesting to note that if F is holomorphic, then F determines (locally) a holomorphic trivialization of the foliation. Thus, both the disk ˜D and the trivialization ˜ι read holomorphically in this trivialization. This implies that a(x) and b(x) are holomorphic, and so is θ₂. So, in regards of (1.3), ˜F is holomorphic. Thus

Lemma 1.3. If F ∈ C is holomorphic, then any ˜F ∈ C (whose image is included in the one of F) is holomorphic too.

In order to better understand the effect of changing the function F, let us study what the possible changes are, i.e., the set of functions θ which can be associated to a change of the function F in another function ˜F ∈ C. This is particularly relevant since, as we have seen, this function θ determines the change in ω (see (1.3)). So, let us consider some map

\[ \theta : \left\{ \begin{array}{c}
\mathbb{D} \times \mathbb{C} \\
(x, y)
\end{array} \right\} \to \left\{ \begin{array}{c}
\mathbb{D} \times \mathbb{C} \\
(\theta_1(x), \theta_2(x, y))
\end{array} \right\} \]

such that θ₁ is a bi-holomorphism of D and, that for every x ∈ D the map y → θ₂(x, y) is a bi-holomorphism of C (and so of the form a(x)y + b(x)). Then, what are the necessary and sufficient conditions such that ˜F := F ◦ θ belongs to C (i.e., that it might be obtained as in (1.3)?

These conditions are:

(1) \( F(\theta(\mathbb{D} \times \{0\})) \) is a holomorphic disk i.e. \( F_*(\theta_* \left( \begin{array}{c}
i \\
0
\end{array} \right) ) = JF_*(\theta_* \left( \begin{array}{c}
1 \\
0
\end{array} \right) ) \) on \( \mathbb{D} \times \{0\} \), which is equivalent to \(-2i \frac{\partial \theta_2}{\partial x}(x, 0) = \omega(\theta(x, 0)) \frac{\partial \theta_1}{\partial x}(x, 0), \forall x.\)

(2) \( x \to d\tilde{F}_{(x,0)} \left( \begin{array}{c}
1 \\
0
\end{array} \right) \) is holomorphic, i.e. \( x \to dF_{\theta(x,0)} \left( \begin{array}{c}
0 \\
1
\end{array} \right) \frac{\partial \theta_2}{\partial y}(x, 0) \) is holomorphic.

Let us assume (possibly by first considering some intermediate disks) that \( \tilde{D} := F(\theta(\mathbb{D} \times \{0\})) \) is sufficiently close to D, or more precisely that it is included in a distinguished open set U. Let us fix a foliated chart \( \psi : U \to \mathbb{D} \times \mathbb{D} \). How is condition (2) expressed in this trivialization?

Keeping the previous notation \( \alpha = \psi \circ F \) on \( F^{-1}(U) \), condition (2) reads in this chart: the map \( x \to \frac{\partial \alpha_2}{\partial y}(\theta(x, 0)) \frac{\partial \theta_1}{\partial x} + 2i \frac{\partial^2 \theta_2}{\partial y \partial x} \) is holomorphic. By writing that the \( \dot{\theta} \) of this function must be zero and by using the equality (1.2) and condition (1), we get that (if condition (1) is satisfied) condition (2) is equivalent to:

\[ \left( \frac{\partial \alpha_2}{\partial y} \circ \theta \right) \left( \frac{\partial \omega}{\partial y} \circ \theta \frac{\partial \theta_1}{\partial x} \frac{\partial \theta_2}{\partial y} + 2i \frac{\partial^2 \theta_2}{\partial y \partial x} \right) = 0 \text{ on } \mathbb{D} \times \{0\}. \]
Since $\frac{\partial \omega}{\partial y}$ never vanishes, and since $\frac{\partial \omega}{\partial y} \circ \theta$ is the only term depending on $y$ (neither does $\theta_1$ depend on $y$, nor $\frac{\partial \omega}{\partial y} = b(x)$), we get:

\[
(1.4) \quad \frac{\partial \omega}{\partial y} (\theta(x,0)) \left( \frac{\partial \theta_1}{\partial x} (x,y) + \frac{\partial \theta_2}{\partial y} (x,y) + 2i \frac{\partial^2 \theta_2}{\partial y \partial x} (x,y) \right) = 0 \text{ on } F^{-1}(U).
\]

So, to summarize, if we assume (possibly by composing with a bi-holomorphism of the disk) that $\theta_1 = id$, then conditions (1) and (2) (which read as condition (1) and (1.4)) provide with the relations:

1. $-2i \frac{\partial \omega}{\partial x} = \omega(x,a(x))$
2. $\frac{\partial \omega}{\partial y} (x,a(x)) b(x) + 2i \frac{\partial \omega}{\partial x} = 0.$

Thus, the function $a$ is solution of a $\bar{\partial}$-differential equation, and once a solution $a$ has been fixed, $b$ is also solution of a $\bar{\partial}$-differential equation. This determines the function $\theta$ and then the perturbation term in the expression of $\tilde{\omega}$ (compared with $\omega$) which is $\frac{\partial \omega}{\partial y} = \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial x} y$. Thus all the possible functions $\theta$ corresponding to changes of $F$, and so all the possible changes for $\omega$ are determined.

2. Invariants of the foliation

The tensor $\Omega$ and the function $\omega$ measure the lack of holomorphy of the function $F$. However, they depend on $F$. Since we have noticed that one $F$ is holomorphic if and only if all $F \in \mathcal{C}$ taking values in the image of $F$ are, this raises the issue of the existence of an invariant of the foliation measuring the lack of holomorphy of all the maps $F \in \mathcal{C}$. Looking at the relation (1.4) expressing how varies $\omega$ under change of $F$, we notice that by differentiating it two times with respect to $y$, the variation in $\frac{\partial^2 \omega}{\partial y^2}$ doesn’t contain any additive term anymore:

\[
(2.1) \quad \frac{\partial^2 \omega}{\partial y^2} = \frac{\partial^2 \omega}{\partial y^2} \circ \theta \frac{\partial \theta_2}{\partial y} \frac{\partial \theta_1}{\partial x}.
\]

Thus, by considering the local differential forms on $X$ got by pushing-forward the function $\omega$ (defined on $\mathbb{D} \times \mathbb{C}$), and studying how they depend on $F$, we can check that the tensor on $X$ got by differentiating them twice in an appropriate way does not depend on $F$. Thus we are able to construct an invariant of the foliation: a tensor, both global on $X$ and independent on $F$, vanishing if and only if the maps $F$ are holomorphic, as wanted.

2.1. Constructions of differential forms/tensors. The map $F$ being a local diffeomorphism, there exists, on a neighborhood $V_F$ of the point $z_0 := F(0,0)$, a local inverse for $F$, taking values in a neighborhood $U_0$ of the point $(0,0) \in \mathbb{D} \times \mathbb{C}$. It will be (abusively) denoted by $F^{-1} : V_F \rightarrow U_0$, and its coordinate-functions will be denoted by $F^{-1} = ((F^{-1})_1, (F^{-1})_2)$.

Thus the map $F$ provides us with a local trivialization of the bundle $T_F$ on $V_F$:

\[
\phi_F : \begin{cases} 
T_F|_{V_F} & \rightarrow V_F \times \mathbb{C} \\
\xi_z & \rightarrow (z, \pi_2(dF^{-1}_z(\xi_z)) = (z, d(F^{-1})_2(\xi_z))
\end{cases}
\]

(with $\pi_2$ the second coordinate-function). Possibly restraining the transverse disk $D$, let us assume the neighborhood $V_F$ contains $F(\mathbb{D} \times \{0\})$. Moreover, in the
following, we can assume that there exists a holomorphic foliated chart on \( V_F \) (possibly restraining \( V_F \) \( \psi : V_F \to \mathbb{D} \times \mathbb{D} \)).

This being done, the tensor \( \Omega \) can be read on \( V_F \) as a tensor taking values in \( T\mathcal{F}|_{V_F} \) (from now on the restriction to \( V_F \) will always be implicit):

\[
\Omega_0 = (F_\ast) \circ \partial \circ (F_\ast^{-1}) : \begin{cases} \mathbb{C} \to \mathbb{C} \\ TX \to \mathbb{C} \\ Z \to JZ - F_\ast(J_0 F_\ast^{-1} Z) \end{cases}
\]

Then composing by \( \phi_F \) we get a 1-form \( \lambda_F \) on \( V_F \):

\[
\lambda_F : \begin{cases} \mathbb{C} \to \mathbb{C} \\ TX \to \mathbb{C} \\ Z \to \pi_2\big(F_\ast^{-1}(JZ) - J_0(F_\ast^{-1} Z)\big) = \pi_2(\Omega(F_\ast^{-1} Z)) = (F^{-1})^\ast \Omega_2(Z)
\end{cases}
\]

with \( \Omega_2 = \pi_2 \circ \Omega \) a 1-form on \( \mathbb{D} \times \mathbb{C} \), which reads by definition of \( \omega \) as \( \Omega_2 \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) = \omega v_1 \).

This form \( \lambda_F \) is zero along the leaves and \( \lambda_F(JZ) = -J_0 \lambda_F(Z) \), that is to say \( \lambda_F \) is a \((0, 1)\)-form. By construction, \( \lambda_F \) is zero if and only if \( \Omega \) vanishes, thus if and only if \( F \) is holomorphic on \( V_F \). Furthermore, we can point out that \( \lambda_F = -2i \partial(\bar{\partial} F^{-1})_{21} \) with the aforementioned notation.

Finally, by definition of \( \Omega \), for every vector field \( Z \), \( \lambda_F(Z) = \pi_2(\Omega(F_\ast^{-1} Z)) = \omega(F^{-1}(z)) \frac{d(F^{-1})}{1}(\overline{Z}) = \omega(F^{-1}(z)) \epsilon(Z) \), with \( \epsilon \) the 1-form on \( V_F \), \( \epsilon(Z) = \frac{d(F^{-1})}{1}(\overline{Z}) \).

Let us notice that the form \( \epsilon \) is equal to \( (F^{-1})^\ast \delta \), with \( \delta \) the 1-form on \( F^{-1}(V_F) \subset \mathbb{D} \times \mathbb{C} \) defined by \( \delta(V) = \pi_1(V) \) (with \( \pi_1 \) the projection on the first coordinate of \( \mathbb{C} \times \mathbb{C} \)), i.e. \( \delta = dx \) (if \( (x, y) \) is the standard coordinates system on \( \mathbb{D} \times \mathbb{C} \)). Thus \( d\delta = 0 \) and \( d\epsilon = 0 \). So, the 1-form \( \epsilon \) on \( V_F \) is a basic 1-form of the foliation: for any \( Y \in T\mathcal{F} \), \( \epsilon(Y) = 0 \) and \( i_Y d\epsilon = 0 \).

Let us now consider the 2-form \( d\lambda_F \) on \( V_F \subset X \), \( d\lambda_F : TX \times TX \to \mathbb{C} \). Since \( d = \partial + \bar{\partial} \) and \( \partial^2 = 0 \), it satisfies: \( d\lambda_F = 2\partial\bar{\partial}(F^{-1})_{21} \) on \( V_F \) and \( d\lambda_F \in \Lambda^{1, 1}X \). In regards of the expression of \( \lambda_F \), \( d\lambda_F(Z, Y) = d\omega(d(F^{-1}(Z)) \epsilon(Y) - d\omega(d(F^{-1}(Y)) \epsilon(Z)) + \omega \circ F^{-1} d\epsilon(Z) \).

Thus considering \( Y \in T\mathcal{F} \), \( d\lambda_F(Z) = -\frac{1}{2\pi|y|} (F^{-1}(z)) d(F^{-1})_{21}(Y) \epsilon(Z) \).

Thus, restricted to \( TX \times T\mathcal{F} \), \( d\lambda_F \) defines a tensor:

\[
A_F : \begin{cases} \mathbb{C} \times T\mathcal{F} \to \mathbb{C} \\ TX \times T\mathcal{F} \to \mathbb{C} \\ (Z, Y) \to \phi_F^{-1}(d\lambda_F(Z, Y)) \end{cases}
\]

From the properties of \( \lambda_F \), we deduce that \( A_F \) is zero on \( T\mathcal{F} \times T\mathcal{F} \). Moreover \( A_F \) can be written in terms of \( \omega : A_F(Z, Y) = -\frac{1}{2\pi|y|} (F^{-1}(Z)) \epsilon(Z) Y = -\frac{\partial \omega}{\partial y}(F^{-1}(Z)) d(F^{-1})_{11}(Z) \).

It is \( \mathbb{C} \)-antilinear with respect to the first variable and \( \mathbb{C} \)-linear with respect to the second one. Furthermore, \( A_F \) is zero if and only if \( \frac{\partial \omega}{\partial y} \) is zero on \( V_F \), or equivalently if and only if \( \omega = 0 \) on \( \mathbb{D} \times \mathbb{C} \) i.e. if and only if \( F \) is holomorphic on \( \mathbb{D} \times \mathbb{C} \). In view of the above expression of \( d\lambda_F \), one can note that \( F \) is holomorphic if and only if \( \partial\bar{\partial}(F^{-1})_{21} = 0 \) on \( TX \times T\mathcal{F} \) (harmonic condition).
Finally, let us point out that $A_F$ can also be seen as a 1-form taking values in $\mathcal{E}nd(T\mathcal{F}) \simeq \mathbb{C}$ since $\text{dim}_{\mathbb{C}} T\mathcal{F} = 1$:

$$A_F : \begin{cases} TX \to \mathcal{E}nd(T\mathcal{F}) \simeq \mathbb{C} \\ X \to (Y \rightarrow A_F(Z,Y)) \end{cases}$$

This is a 1-form of type $(0,1)$ and can be expressed in function of $\omega$: $A_F(Z) = -\frac{\partial \omega}{\partial y}(F^{-1}(z)) d(F^{-1})_1(Z)$, i.e. according to the previous notation:

$$(2.2) \quad A_F = -\frac{\partial \omega}{\partial y}(F^{-1}(z)) \epsilon.$$

### 2.2. Modifications when one changes $F$ into $\tilde{F}$

As previously, we assume $F(\mathbb{D} \times \mathbb{C}) = \tilde{F}(\mathbb{D} \times \mathbb{C})$ and we introduce the function $\theta$, given by lemma 1.2, such that $F = \theta \circ \tilde{F}$. Then, we prove:

**Proposition 2.1.** If we consider $A_F$ and $A_{\tilde{F}}$ as 1-forms, then on $V_F \cap V_{\tilde{F}}$

$$\frac{\partial \theta_2}{\partial y}(\tilde{F}^{-1}(z)) A_{\tilde{F}}(Z) = \frac{\partial \theta_2}{\partial y}(\tilde{F}^{-1}(z)) A_F(Z) - 2i\frac{\partial^2 \theta_2}{\partial y \partial \bar{x}}(\tilde{F}^{-1}(z)) d(F^{-1})_1(Z),$$

**Proof.** By definition,

$$\begin{pmatrix} 0 \\ \lambda_{\tilde{F}}(Z) \end{pmatrix} = (\tilde{F}^{-1})_{\ast}(JZ) - J_0(\tilde{F}^{-1})_{\ast}(Z)$$

So

$$\theta_{\ast} \begin{pmatrix} 0 \\ \lambda_{\tilde{F}}(Z) \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_F(Z) \end{pmatrix} + J_0 \theta_{\ast}(\tilde{F}^{-1}_{\ast}Z) - \theta_{\ast}(J_0 \tilde{F}^{-1}Z)$$

That is, $\frac{\partial \theta_{\ast}}{\partial y}(\tilde{F}^{-1}(z)) \lambda_{\tilde{F}}(Z) = \lambda_F(Z) + 2i\frac{\partial \theta_{\ast}}{\partial y}(\tilde{F}^{-1}(z)) d(F^{-1})_1(Z)$.

Thus, differentiating and evaluating on $X \in TX$ and $Y \in T\mathcal{F}$, we get

$$\frac{\partial \theta_{\ast}}{\partial y}(\tilde{F}^{-1}(z)) d\lambda_{\tilde{F}}(Z,Y) = d\lambda_F(Z,Y) - 2i\frac{\partial^2 \theta_{\ast}}{\partial y \partial \bar{x}}(\tilde{F}^{-1}(z)) d(F^{-1})_1(Z) (\text{since } d(F_1)(Y) = 0).$$

Finally, as $\phi_F \circ \phi_{\tilde{F}}^{-1}(z,v) = (z, \frac{\partial \theta_{\ast}}{\partial y}(\tilde{F}^{-1}(z)) v)$, this implies:

$$\frac{\partial \theta_2}{\partial y}(\tilde{F}^{-1}(z)) A_{\tilde{F}}(Z,Y) = \frac{\partial \theta_2}{\partial y}(\tilde{F}^{-1}(z)) A_F(Z,Y) - 2i\frac{\partial^2 \theta_2}{\partial y \partial \bar{x}}(\tilde{F}^{-1}(z)) d(F^{-1})_1(Z) \cdot Y.$$

Therefore, if we see $A_F$ as a 1-form taking values in $\mathcal{E}nd(T\mathcal{F})$: $A_F : TX \to \mathcal{E}nd(T\mathcal{F}) \simeq \mathbb{C}$, this relation reads as:

$$(2.3) \quad \frac{\partial \theta_2}{\partial y}(\tilde{F}^{-1}(z)) A_{\tilde{F}}(Z) = \frac{\partial \theta_2}{\partial y}(\tilde{F}^{-1}(z)) A_F(Z) - 2i\frac{\partial^2 \theta_2}{\partial y \partial \bar{x}}(\tilde{F}^{-1}(z)) d(F^{-1})_1(Z).$$

\[ \square \]

Let us note that in regard with (1.4), this can be written:

$$A_{\tilde{F}}(Z) = A_F(Z) - \frac{\partial \omega_{\bar{x}}}{\partial y}(F^{-1}(\tilde{\pi}(z))) d(F^{-1})_1(Z) \frac{\partial \omega}{\partial x}(F^{-1}(z))$$

$$= A_F(Z) - \frac{\partial \omega_{\bar{x}}}{\partial y}(F^{-1}(\tilde{\pi}(z))) d(F^{-1})_1(Z),$$

where $\tilde{\pi}$ is the projection of $V_F$ on $\tilde{F}(\mathbb{D} \times \{0\})$ along the leaves (see Appendix):
Then, using the same argument as the one used above to prove that if \( A \) is holomorphic then \( \tilde{F} \) is too (on \( \tilde{F}^{-1}(F(\mathbb{D} \times \mathbb{C})) \)).

These differential forms associated to \( F \) have been constructed quite naturally. We would like to better understand their meaning. In the next section we shall explain how this invariant \( A_F \) is closely related with the \( \bar{\partial} \)-connection on \( TF \).

2.3. Connections on \( TF \). The canonical connection \( \bar{\partial}_0 \) of the trivial bundle with fiber \( \mathbb{C} \) on \( U_F \) is defined by \( \bar{\partial}_0(Z,\xi) = \bar{\partial}(Z) \). From this connection, the map \( F \) determines a \( \bar{\partial} \)-connection on the bundle \( TF|_V \): \( \bar{\partial}_F = (\phi_F)^{-1} \circ \bar{\partial}_0 \circ \phi_F \), which means for any section \( \xi \in TF \) and any vector field \( Z \in TX \), \( (\bar{\partial}_F)_Z(\xi) = (\phi_F)^{-1}(\bar{\partial}(\phi_F(\xi))(Z)) \).

Then \( \nabla_F = \frac{i}{2} A_F + \bar{\partial}_F \) – i.e. for any section \( \xi \in TF \) and any vector field \( Z \in TX \), \( (\nabla_F)_Z(\xi) = \frac{i}{2} A_F(Z)\xi + (\bar{\partial}_F)_Z(\xi) \) – defines a \( \bar{\partial} \)-connection on the bundle \( TF|_V \).

This is a \( \bar{\partial} \)-connection and:

**Proposition 2.2.** The connection \( \nabla_F = \frac{i}{2} A_F + \bar{\partial}_F \) is independent from the choice of \( F \). More precisely, this is the canonical \( \bar{\partial} \)-connection on the holomorphic bundle \( TF \) – that is to say the one defining the holomorphic structure of \( TF \).

**Proof.** Using that \( (\phi_F) \circ \phi_F^{-1}(z,v) = (z, \frac{\partial \theta_2}{\partial y} v) \), we check that

\[
(\bar{\partial}_F)_Z(\xi) = \phi_F^{-1}\left( \bar{\partial}\left( \frac{\partial \theta_2}{\partial y} \circ \bar{\partial}_F(\xi) \right) \right)
\]

and then since \( \phi_F^{-1}(v) = \phi_F^{-1}\left( \frac{1}{\psi_2'} v \right) \),

\[
(\bar{\partial}_F)_Z(\xi) = (\bar{\partial}_F)_Z(\xi) + \frac{1}{\psi_2'} \frac{\partial^2 \theta_2}{\partial x \partial y} \circ \bar{\partial}_F^{-1}(\bar{\partial}(\phi_F(\xi))(Z)) \xi
\]

This inequality and (2.2) prove that \( \nabla_F = \bar{\partial}_F \).

Moreover, if we introduce a foliated chart \( \psi : V_F \to \mathbb{D} \times \mathbb{D} \), the canonical \( \bar{\partial} \)-structure of the holomorphic bundle \( TF \) is \( (\bar{\partial}_F)_Z(\xi) = \psi_2^{-1} \left( \bar{\partial}(\psi_1(\xi))(Z) \right) \).

The same kind of calculations as in section 2.2 but using \( \beta = F^{-1} \circ \psi \) instead of \( \theta \) – allows to read all the forms defined in 2.1 in this chart and we get:

\[
\frac{i}{2} d\lambda_F(Z,Y) = -\frac{\partial^2}{\partial y \partial x} (\psi(z)) d\psi_1(Z) d\psi_2(Y).
\]

Then, using the same argument as the one used above to prove that \( \nabla_F = \bar{\partial}_F \) – but again using \( \beta \) instead of \( \theta \) – leads to \( \nabla_F = \bar{\partial}_F \), the canonical \( \bar{\partial} \)-connection on the holomorphic bundle \( TF \).
2.4. Invariant tensor field $\Gamma$. In light of our previous study, we can construct a tensor field, independent of $F$, an invariant of the foliation that vanishes if and only if the maps $F \in \mathcal{C}$ are holomorphic, i.e. if and only if there exist some holomorphic maps $F : \mathbb{D} \times \mathbb{C} \rightarrow X$ compatible with the foliation. This answers our initial question of the existence of holomorphic cylinders along the foliation.

**Definition-Theorem 2.1.** Let $\Gamma_F = (dA_F)_{TX \times T\mathcal{F}} :$

$$\Gamma_F |_{x} : \begin{cases} (T_x X \times T_x \mathcal{F}) & \rightarrow \mathbb{C} \\ (Z, Y) & \rightarrow dA_F(Z, Y) \end{cases}, \forall x \in V_F$$

It does not depend on $F$. Thus this defines a tensor $\Gamma : \Gamma = \Gamma_F$ on each $V_F$. This is a tensor on $X$ which is associated with the foliation, $\mathbb{C}$-antilinear with respect to the first variable and $\mathbb{C}$-linear with respect to the second one. It vanishes if and only if the maps $F$ are holomorphic. More precisely, for any $F \in \mathcal{C}$, $\Gamma$ vanishes on $F(\mathbb{D} \times \mathbb{C})$ if and only if $F$ is holomorphic.

Moreover, $\Gamma$ is a leaf-wise holomorphic section of the line bundle $E = \Lambda^{(0,1)}T^{*}N \otimes \Lambda^{(1,0)}T^{*}\mathcal{F}$, holomorphic above $(X, \mathcal{F})$.

**Proof.** As seen previously, $A_F$ is a $\mathbb{C}$-valued 1-form on $V_F$. So, $dA_F$ is a 2-form. Since $\frac{\partial \omega}{\partial y}$ is independent of $y$ and $\epsilon = d(F^{-1})$ is a basic 1-form, differentiating (2.3) proves that, if restricted to $TX \times T\mathcal{F}$, $dA_F$ does not depend on $F$. Thus, we can define $\Gamma$ on $X$.

More precisely, according to the expression (2.2) for $A_F$ (and since $\epsilon$ is a basic form), $\Gamma$ can be written locally as $\Gamma(Z, Y) = d\left(\frac{\partial \omega}{\partial y}\right)(d(F^{-1}(Y))) \epsilon(Z)$, and so:

$$\Gamma(Z, Y) = \frac{\partial^2 \omega}{\partial y^2}(F^{-1}(z))d(F^{-1})_2(Y)d(F^{-1})_1(Z).$$

Moreover, $\Gamma$ is antilinear with respect to the first variable, linear with respect to the second one. Thus $\Gamma$ belongs to $E = \Lambda^{(0,1)}T^{*}X \otimes \Lambda^{(1,0)}T^{*}\mathcal{F}$. Furthermore, from the results on $A_F$, we know that if the maps $F$ are holomorphic, then $\Gamma$ vanishes on $F(\mathbb{D} \times \mathbb{C})$.

Conversely, if $\Gamma$ vanishes on $F(\mathbb{D} \times \mathbb{C})$, it follows that $\frac{\partial^2 \omega}{\partial y^2} = 0$ (on $F^{-1}((V_F)$ and so on $\mathbb{D} \times \mathbb{C}$), and consequently, $\omega$ being holomorphic along the leaves, $\frac{\partial \omega}{\partial y}$ is constant on each leaf. However, as this function vanishes on $\mathbb{D} \times \{0\}$, it is identically zero. Thus, $\omega$ vanishes identically. Finally the maps $F$ are holomorphic if and only if $\Gamma = 0$.

Finally, since $\Gamma$ vanishes on $T\mathcal{F} \times T\mathcal{F}$, it can be considered as a form on $T\mathcal{N} \times T\mathcal{F}$, which is $J$-antilinear with respect to the first variable and $J$-linear with respect to the second one. So $\Gamma$ is a section of the leaf-wise holomorphic bundle $E = \Lambda^{(0,1)}T^{*}\mathcal{N} \otimes \Lambda^{(1,0)}T^{*}\mathcal{F}$ above $(X, \mathcal{F})$. Furthermore, in sight of (2.4), it reads in the holomorphic foliated chart $\psi : V_F \rightarrow \mathbb{D} \times \mathbb{D} : ((\psi^{-1})^*\Gamma)(v_x, v_y) = \frac{\partial^2 \omega}{\partial y^2}(\beta(z)) \frac{\partial \omega}{\partial y}(x, y) \frac{\partial \omega}{\partial y}(x) v_y \overline{v_x}$, with $\beta = (\beta_1, \beta_2) = F^{-1} \circ \psi^{-1} : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D} \times \mathbb{C}$. Thus, we notice that $\Gamma$ is holomorphic along the leaves.

The same reasoning also proves that the bundle $E$ is holomorphic along the leaves.

Therefore the issue of the existence of some holomorphic map $F : \mathbb{D} \times \mathbb{C} \rightarrow X$ that is compatible with the foliation can come down to the study of the existence
of leaf-wise holomorphic sections of the bundle $E$ above the foliation $\mathcal{F}$.

Let us make a small remark: we could have deduced that $\Gamma_{\mathcal{F}}$ is independent of $F$ from its expression \((2.4)\) in function of $F$, using the equality \((2.1)\).

Another remark can be made: at the end of the proof above, we have expressed $\Gamma$ in the foliated chart $\psi$ in terms of $\omega$ and $\beta$. In fact, it can be expressed only in terms of $\beta$. Indeed, writing the holomorphy of $\beta$ between $(\mathbb{D} \times \mathbb{D}, J_0)$ and $(\mathbb{D} \times \mathbb{C}, J_0 + \Omega)$ leads us to: $-2i \frac{\partial \beta_2}{\partial x} = \omega \circ \beta \frac{\partial \bar{\beta}_1}{\partial \bar{x}}$. So if we suppose, without any loss of generality that $\beta_1 = id$, then

$$\omega \circ \beta = -2i \frac{\partial \beta_2}{\partial x}.$$

(This could also have been deduced from the relation \((1.1)\)). Differentiating twice this equation, we get: $\frac{\partial^2 \beta_2}{\partial y \partial x} \circ \beta \left( \frac{\partial \beta_2}{\partial y} \right)^2 = - \left( \frac{\partial \omega}{\partial y} \circ \beta + 2i \frac{\partial \beta_2}{\partial x \partial y} \right)$. Thus if $\psi$ is a foliated chart sending $F(\mathbb{D} \times \{0\})$ to $\mathbb{D} \times \{0\}$ (identically, i.e. via $\alpha_1 = id$), then on this disk, $\Gamma$ reads in this chart:

$$((\psi^{-1})^* \Gamma)_{(x,0)}(v_x, v_y) = -2i \left( \frac{\partial \beta_2}{\partial y}(x,0) \right)^{-1} \frac{\partial^2 \beta_2}{\partial x \partial y^2}(x,0).$$

2.5. **Generalization to higher dimensions.** Our analysis, written in the case of a complex dimension 2 manifold $X$, can be straightforwardly generalized to the case of higher dimensions. Let $X$ be a manifold of complex dimension $n$ provided with a 1-dimensional foliation $\mathcal{F}$ with parabolic leaves. We can consider the cylinder-maps $F$ from $\mathbb{D}^{n-1} \times \mathbb{C}$ into $X$, constructed along a holomorphic transverse disk $D \simeq \mathbb{D}^{n-1}$ and in the direction of a trivialization $t$ of the line bundle $T\mathcal{F}$ along $D$. Then, as previously, we can consider the tensor $\Omega$ on $\mathbb{D}^{n-1} \times \mathbb{C}$ defined by $J_0 + \Omega = F^* J$. It can be written:

$$\Omega_{(x,y)}(v_x, v_y) = \begin{pmatrix} 0 \\ \omega_{(x,y)}(v_x) \end{pmatrix},$$

with for any $(x, y) \in \mathbb{D}^{n-1} \times \mathbb{C}$, $\omega_{(x,y)}$ a $\mathbb{C}$-antilinear 1-form on $\mathbb{C}^{n-1}$.

Considering two functions $F$ and $\tilde{F}$, we can, as before, construct a map $\theta : (x, y) \in \mathbb{D}^{n-1} \times \mathbb{C} \to (\theta_1(x), \theta_2(x, y)) \in \mathbb{D}^{n-1} \times \mathbb{C}$, with $\theta_1$ a bi-holomorphism of $\mathbb{D}^{n-1}$, and for any $x$, $\theta_2(x, \cdot)$ a bi-holomorphism of $\mathbb{C}$, which satisfies $\tilde{F} = F \circ \theta$. Then, as previously we check,

$$\frac{\partial \theta_2}{\partial y} \omega(x, y) = \left( \frac{\partial \theta_1}{\partial x} \right)^* \omega(\theta(x, y)) + 2i \frac{\partial \theta_2}{\partial x},$$

where $(\frac{\partial \theta}{\partial x})$ is a linear endomorphism of $\mathbb{C}^{n-1}$ and where $*$ denotes its conjugate (for the canonical hermitian product on $\mathbb{C}^{n-1}$).

Then, in the same way we can define, from each $F$, a local differential form $A_F : TX \to \text{End}(T\mathcal{F}) \simeq \mathbb{C}$, and an invariant $\Gamma$: 

Definition-Theorem 2.2. Let $\Gamma_F = (dA_F)|_{T_X \times T_F}$:

$$
\Gamma_F|_x : \begin{cases}
T_x X \times T_x F & \rightarrow \mathcal{E}(T_F) \simeq \mathbb{C} \\
(X,Y) & \rightarrow dA_F(X,Y)
\end{cases}, \forall x \in V_F
$$

It does not depend on $F$. Thus this defines a tensor $\Gamma = \Gamma_F$ on each $U_F$. This is a tensor on $X$ which is associated to the foliation, $\mathbb{C}$-antilinear with respect to the first variable and $\mathbb{C}$-linear with respect to the second one. It vanishes if and only if the maps $F$ are holomorphic. More precisely, for any $F \in \mathcal{C}$, $\Gamma$ vanishes on $F(\mathbb{D} \times \mathbb{C})$ if and only if $F$ is holomorphic. Moreover, $\Gamma$ can be seen as a leaf-wise holomorphic section of the vector bundle $E = \Lambda^{(0,1)} T^* N \otimes \Lambda^{(1,0)} T^* F$ above $(X,F)$.

After this general study, we would like to analyze a few particular cases in which some more specific results can be proven.

3. PARTICULAR CASES AND EXAMPLES

Let us recall that in the case of Stein manifolds, it has been proven that all the maps $F \in \mathcal{C}$ are holomorphic (that is $\Gamma = 0$), [10] (see also [11], [12]). Our study allows us to show that, in the case of a foliation with compact leaves too, all the maps $F$ are holomorphic (i.e. $\Gamma = 0$).

3.1. Foliations with compact leaves, and holonomy groups. In the particular case of foliations with compact leaves, several known results of stability are at our disposal. First, let us recall the equivalence (see [5]):

Proposition 3.1. Let $F$ be a foliation on a manifold $X$ whose leaves are all compact. Then the following properties are all equivalent:

1. each leaf has a finite holonomy group
2. each leaf has a fundamental system of saturated open neighborhoods
3. the space of leaves $M/F$ is Hausdorff
4. the saturation of a compact set is a compact set

Such a foliation is called stable.

It has been proven that these properties are satisfied in some special cases. Notably in the case of holomorphic foliation (see [6] and [7], compared also with [2]):

Proposition 3.2. Let $X$ be a complex manifold provided with a holomorphic foliation with compact leaves. If $X$ is Kähler, or if the foliation has complex codimension 1, then the foliation is stable.

Thanks to these results, we can deduce:

Proposition 3.3. Let $X$ be a complex manifold provided with a holomorphic foliation of dimension 1, such that $X$ is either Kähler, or has complex dimension 2. If $D$ is a transverse holomorphic disk whose saturation is foliated by compact leaves, then all the maps $F \in C$ constructed along this disk are holomorphic. In other words, the invariant $\Gamma$ associated to the foliation vanishes on the saturation of $D$. 
As a consequence, if $X$ is foliated by compact leaves then $\Gamma$ identically vanishes on $X$.

**Proof.** Let us denote the saturation of $D$ by $U_0$. And let us consider a map $F \in \mathcal{C}$, constructed along the disk $D$, with $t$ being the fixed trivialization of $T\mathcal{F}|_D$. Possibly changing the projection $\psi_x : \tilde{L}_x \simeq \mathbb{C} \to L_x$, we can assume that for every $x \in D$, $t_x = 1$ in $L_x$. Then, for all $(x,y) \in D \times \mathbb{C}$, (keeping the notation from section $1$) $F_x = \psi_x \circ F^0_y$ $F^0_x(y) = y \in \tilde{L}_x$. Moreover, $L_x$ is written as $\mathbb{C}/G_x$, with $G_x = \gamma_1(x)\mathbb{Z} \oplus \gamma_2(x)\mathbb{Z}$ being its fundamental group.

Let us fix a distinguished open set $U$ containing $D$ and a foliated chart $\psi : U \to D \times \mathbb{D}$ such that $\psi(D) = D \times \{0\}$. Then, we can consider the projection on the transverse $D$ along the leaves $(\mathcal{F}_U) \pi_0 : U \to D$ (see Appendix $A$).

Let us denote $x_0 = F(0,0)$. For every $\gamma$ in the fundamental group $G_0 = G_{x_0}$, $F(0,\gamma) = x_0$. Thus, for every $x \in D$ close enough to $x_0$, $F(x,\gamma) \in U$. So there exists a unique $\alpha_\gamma(x) \in \mathbb{D}$ such that $F(\alpha_\gamma(x),0) = \pi_0(F(x,\gamma))$. The map $\alpha_\gamma$, thus defined, is the holonomy map associated with $\gamma$. As explained in the Appendix, this map $\alpha_\gamma$ is holomorphic. First, let us show that

**Lemma 3.1.** There exists a disk $D_0 \subset D$ and $N \geq 1$ such that $\alpha_\gamma$ is well-defined on $D_0$ and takes values into $D_0$. Moreover, $\alpha_\gamma^N = \text{Id}$ on $D_0$.

**Proof.** According to the stability results in the case of foliations whose all leaves are compact, the foliation induced on $U_0$ is stable. So the leaf $L_{x_0}$ has a fundamental system of saturated open neighborhoods.

Since $L_{x_0}$ is compact, one can consider a sub-disk $D' \subset D$ such that $L_{x_0} \cap D' = \{x_0\}$. Then one can consider a saturated open set $V_0$ included in the saturation of $D'$ (and so included into $U_0$). Since $L_{x_0} \cap D' = \{x_0\}$, $V_0 \cap D'$ is connected. Let us denote this by $D_0$. By construction, for any $x \in D_0$, $\alpha_\gamma(x) \in D_0$. So the map $\alpha_\gamma : D_0 \to D_0$ can be defined. Moreover, this is a bi-holomorphism of the disk $D_0$ whose inverse is $\alpha_{-\gamma}$.

Then for any $x \in D_0$, $\{\alpha_\gamma^n(x), n \in \mathbb{N}\}$ is included in $L_x \cap D_0$. The leaf $L_x$ being compact, this is a compact, discrete set, and therefore finite. So, there must exist $N \geq 1$ such that $\alpha_\gamma^N(x) = x$. Therefore, $D_0 \subset \cup_{n \geq 1}\{x \in D_0 \text{ such that } \alpha_\gamma^n(x) = x\}$.

Since each of the sets of this union is closed, necessarily, one among them has a non-empty interior. So, there exists a $N$ such that $\alpha_\gamma^N(x) = x$ on an open set $V \subset D_0$. As $\alpha_\gamma^N$ is holomorphic, we can conclude that $\alpha_\gamma^N = \text{Id}$ on $D_0$. \hfill $\square$

This being proven, if the fundamental group $G(x_0) = G_0 = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$, we can fix a $N$ such that for $i = 1, 2$, $\alpha_\gamma^N = \text{Id}$ on $D_0$ (possibly restraining it to a smaller disk). Then, $\pi_0(F(x,N\gamma_i)) = F(x,0)$ on $D_0$. So, the functions $N\gamma_i(x_0)$ can be continuously extended on $D_0$ by a map $\beta_i(x) \in G_x$. Let $G'_x$ be the subgroup of $G_x$ generated by these maps $\beta_i(x)$. Then, the functions $F(x,.)$ are $G'_x$-periodical for $x \in D_0$. 


Thus for any fixed family $\gamma(x) \in G_x'$, continuous with respect to $x$, and for all $(x, y) \in \mathbb{D} \times \mathbb{C}$, $F(x, y + \gamma(x)) = F(x, y)$. Therefore, we check, either by differentiating the equality $F(x, y + \gamma(x)) = F(x, y)$ and applying $J$, or by applying the results of section 2.2 to the map $\tilde{F}(x, y) = F(x, y + \gamma(x))$ (that belongs to $\mathcal{C}$ and for which $\theta(x, y) = (x, y + \gamma(x)))$: 

$$\omega(x, y + \gamma(x)) = \omega(x, y) - 2i \frac{\partial \gamma}{\partial x}$$

This implies $\frac{\partial \omega}{\partial y}(x, \gamma(x)) = \frac{\partial \omega}{\partial y}(x, y)$. Thus for any fixed $x \in D_0$, the function $\frac{\partial \omega}{\partial y}(x, \cdot)$ is holomorphic on $\mathbb{C}$ and $G_x'$-periodical. Therefore it is constant. As it vanishes on 0, it vanishes identically on $\mathbb{C}$. Finally $\omega(x, \cdot)$ vanishes on $\mathbb{C}$ for any $x \in D_0$, neighborhood of $x_0$. This can be applied around any other point in $D$. Thus, $\omega$ vanishes identically on the saturation of $D$ and $F$ is holomorphic.

Let us note that $\omega$ measures the lack of holomorphy of the map $x \to G_y'$. Indeed, if we fix any continuous map on $\mathbb{D} \gamma(x) = j \beta_1(x) + k \beta_2(x) \in G_x'$, then $x \to F(x, \gamma(x)) = F(x, 0) \in L_x$ is $J$-holomorphic. Thus, $x \to (x, \gamma_x)$ is $J^t$-holomorphic, which reads as:

$$-2i \frac{\partial \gamma}{\partial x} = \omega(x, \gamma(x)).$$

So, as a side product, through this proof, we get that the groups $G_x'$ depend holomorphically on $x \in D$ i.e. on the transversal disk to the foliation. More precisely,

**Lemma 3.2.** Let $X$ be satisfying the same assumptions as in the previous proposition. Let us suppose that along a transverse holomorphic disk $D$, all the leaves are compact. Then if for any $x$, $G_x$ denotes the fundamental group of the leaf $L_x$, any continuous family $\gamma(x) \in G_x$, $x \in D$ depends holomorphically on $x$.

One of the main points of these results is the finiteness of the holonomy groups of the leaves. In the case of a foliation $\mathcal{F}$ possessing one compact leaf, two cases can be distinguished depending on the holonomy group of the leaf. If the holonomy group of the compact leaf is finite then we can use the result ([Ree] II.2.16.):

**Theorem 3.3** (Reeb). A compact leaf from a foliation $\mathcal{F}$ whose holonomy group is finite, possesses a fundamental system of neighborhoods which are saturated by compact leaves with finite holonomy groups.

In particular, if $L$ is a compact leaf with finite holonomy group then there exists a small transverse disk $D$ around $L$ such that all leaves crossing $D$ are compact. This result and the proposition [Ree] allow us to deduce:

**Corollary 3.1.** If $\mathcal{F}$ is a holomorphic foliation of $X$ containing a compact leaf $L_0$ with finite holonomy group, then there exists a holomorphic cylinder $F : \mathbb{D} \times \mathbb{C}$ along the foliation with $F(0, 0) \in L_0$.

If the holonomy group of the compact leaf $L_0$ is infinite then the same reasoning does not apply. However in this case, we can study how the holonomy group acts on $\omega$. Since $L_0$ is a compact leaf whose universal covering is $\mathbb{C}$, $L_0$ is a torus and $\pi_1(L_0) \simeq \mathbb{Z}^2$.

Let us consider $F \in \mathcal{C}, F : \mathbb{D} \times \mathbb{C} \to X$, constructed along a transverse holomorphic disk $D_0$ and a fixed trivialization $t$ of $T\mathcal{F}|_{D_0}$, and such that $x_0 = F(0, 0) \in L_0$. 


Let \( \omega \) be the function on \( \mathbb{D} \times \mathbb{C} \) associated with \( F \). Possibly by reparametrizing the universal coverings \( \tilde{L}_x \), we can assume that in \( \tilde{L}_x \) \( F(x, 0) = 0 \), \( \text{i.e.} \) \( F_x^0 = 0 \) - with the usual notations - , and \( t_x = 1 \), \( \text{i.e.} \) \( (F_x^0)'(0) = 1 \). Then, these parametrizations of \( \tilde{L}_x \) being fixed, for any \( x \in \mathbb{D} \), \( F_x^0(y) = y \in \tilde{L}_x \). And \( L_0 = L_{x_0} = \tilde{L}_{x_0}/G_0 \) with \( G_0 = \gamma_1 \mathbb{Z} \oplus \gamma_2 \mathbb{Z} = \pi_1(L_0) \) the fundamental group of \( L_0 \).

In the following, to simplify the notations, for \( x \in \mathbb{D} \), we will denote the point \( F(x, 0) \in D \) by \( x \) (\( \text{i.e.} \) we identify \( D \) with its holomorphic parametrization \( x \to F(x, 0) \)).

For any \( \gamma_0 \in G_0 \), \( F(0, \gamma_0) = F(0, 0) = x_0 \). One can consider \( \tilde{\gamma}_0 \), the element of the holonomy group associated with \( \gamma_0 \) (whose definition is explained Appendix \[1\]). Then (as in the proof of proposition \[3\]) for any \( x \) close to \( x_0 \) (identified with 0), there exists an element \( \gamma(x) \) close to \( \gamma_0 \) such that \( F(x, \gamma(x)) \in D \), thus defining a continuous map \( \gamma \) on a neighborhood \( D_1 \) of 0, such that \( \gamma(0) = \gamma_0 \). The map \( \tilde{\gamma} \) defined by \( \tilde{\gamma}(x) = F(x, \gamma(x)) \) is a local bi-holomorphism around 0 - from \( D_1 \) to another neighborhood \( D_2 \) around 0 - , whose derivative in 0 is the holonomy element \( \tilde{\gamma}_0 \).

Let us first note that since the map \( x \to F(x, \gamma(x)) = \tilde{\gamma}(x) (= F(\tilde{\gamma}(x), 0)) \) is holomorphic, the map \( x \to (x, \gamma(x)) \), defined on \( D_1 \subset \mathbb{D} \) and taking values in \( \mathbb{D} \times \mathbb{C} \), is \((J_0, J')\)-holomorphic (with \( J' = J_0 + \Omega \)). This reads as

\[
-2i \frac{\partial \gamma}{\partial x} = \omega(x, \gamma(x))
\]

(The map \( \omega \) measures the non-holomorphy of \( \gamma \)).

Let us consider the map \( \tilde{F} \) in \( \mathcal{C} \) constructed along \( D_2 \subset D \), parametrized by \( x \to F(x, \gamma(x)) = \tilde{\gamma}(x) (= F(\tilde{\gamma}(x), 0) \) with which it is identified), \( x \in D_1 \subset \mathbb{D} \), and with \( t_x = t_{\tilde{\gamma}(x)} \).

On one hand \( \tilde{F}(x, 0) = F(\tilde{\gamma}(x), 0) \) and \( \tilde{t}_x = t_{\tilde{\gamma}(x)} \). So for any \( x \in D_1 \), \( \tilde{F}(x, y) = F(\tilde{\gamma}(x), y) \).

On the other hand, \( F(x, \gamma(x)) = \tilde{F}(x, 0) \). So \( \tilde{F} = F \circ \theta \) with \( \theta_1 = \text{Id} \) and \( \theta_2(x, 0) = \gamma(x) \). Moreover, \( \theta(x, .) \) is a bi-holomorphism of \( \mathcal{C} \), and consequently a degree-1-polynomial. So finally, \( \theta(x, y) = (x, \gamma(x) + \phi(x) y) \). Moreover, since \( \tilde{F}(0, 0) = F(0, \gamma(0)) = F(0, 0) \) and \( \tilde{t}_0 = t_0 \), \( \tilde{F}(0, .) = F(0, .) \) and so \( \phi(0) = 1 \).

If \( \tilde{\omega} \) is the function associated with \( \tilde{F} \), according to the relation \( (1.3) \) in lemma \[1, 2\] this leads to:

\[
\tilde{\omega}(x, y) = \omega(\tilde{\gamma}(x), y) \overline{\frac{\partial \gamma}{\partial x}}(x).
\]

Moreover this also implies

\[
\phi(x) \tilde{\omega}(x, y) = \omega(x, \gamma(x) + \phi(x) y) + 2t \left( \frac{\partial \gamma}{\partial x} + \frac{\partial \phi}{\partial x} y \right),
\]
which according to (3.1) and (3.2) leads to

\[ \phi(x) \frac{\partial \gamma}{\partial x}(x) \omega(\gamma(x), y) = \omega(x, \gamma(x) + \phi(x)y) - \omega(x, \gamma(x)) + 2i \frac{\partial \phi}{\partial x} y. \]

Differentiating this equality two times with respect to \( y \), we get:

\[ \frac{\partial}{\partial x}(x) \frac{\partial^2 \omega}{\partial y^2}(\gamma(x), y) = \frac{\partial^2 \omega}{\partial y^2}(x, \gamma(x) + \phi(x)y) \phi(x). \]

Thus at \( x = 0 \),

\[ (3.3) \frac{\gamma_0}{\gamma_0} \frac{\partial^2 \omega}{\partial y^2}(0, y) = \frac{\partial^2 \omega}{\partial y^2}(0, \gamma_0 + y), \]

with \( \gamma_0 \) the holonomy element associated with \( \gamma_0 \). So to conclude, the holomorphic function \( h(y) = \frac{\partial^2 \omega}{\partial y^2}(0, y) \) satisfies: for any \( \gamma \in G_0 = \pi_1(L_0) = \gamma_1 \mathbb{Z} \oplus \gamma_2 \mathbb{Z} \),

\[ h(y + \gamma) = \gamma h(y) \]

with \( \gamma \) the holonomy element associated with \( \gamma \). As a consequence, there must exist some constants \( C \) and \( \rho \) such that \( h(y) = C \exp(\rho y) \) with \( \exp(\rho \gamma) = \gamma \) for any \( \gamma \in G_0 \).

Thus, in the case where there exists a compact leaf \( L_0 \) whose holonomy group \( \tilde{G}_0 \) is infinite, if there exists an exponential map between the fundamental group \( G_0 \subset \mathbb{C} \) of the leaf and the holonomy group \( \tilde{G}_0 \subset \mathbb{C}^* \), \( k : \left\{ \begin{array}{l} G_0 \to \tilde{G}_0 \\
\gamma \exp(\rho \gamma) \end{array} \right. \), then the function \( \omega \) grows exponentially along \( L_0 : \omega(y) = C \exp(\rho y) \).

Otherwise, \( \omega \) vanishes identically on \( L_0 \).

If the manifold \( M \) is compact – even though the leaves are not – there is still a case where one can deduce that the invariant \( \Gamma \) vanishes, and therefore there is no holomorphic cylinder along the foliation: the case where the bundle \( E = \Lambda^{(0,1)} T^* \mathcal{N} \otimes \Lambda^{(1,0)} T^* \mathcal{F} \) above \((X, \mathcal{F})\) has a negative curvature.

3.2. Case of bundle \( E \) with negative curvature.

**Proposition 3.4.** Let \( X \) be compact. If the bundle \( E = \Lambda^{(0,1)} T^* \mathcal{N} \otimes \Lambda^{(1,0)} T^* \mathcal{F} \) above \((X, \mathcal{F})\) has a strictly negative curvature along the leaves, then the invariant \( \Gamma \) vanishes identically and there is no holomorphic cylinder along the foliation.

**Proof.** Let us fix a Hermitian metric \( g \) of \( T \mathcal{F} \) on \( X \). This provides each leaf with a Riemannian metric. If the curvature of the bundle \( E \) is strictly negative, then there exists a constant \( \epsilon > 0 \) such that \( K < -\epsilon g \).

On one hand, the invariant \( \Gamma \) is a leaf-wise holomorphic section of the bundle \( E \). So, if \( \Gamma \) does not vanish identically along a leaf, the curvature \( K \) of \( E \) along this leaf is \( K = dd^c \log |\Gamma|^2 \) (except possibly on the enumerable set of points where \( \Gamma = 0 \)). Thus along such a leaf \( K = dd^c \log |\Gamma|^2 < -\epsilon g \).

On the other hand, the manifold \( X \) being compact, the curvature \( K_g \) of \((T \mathcal{F}, g)\) is bounded by a constant on \( X \). In particular, there exists a constant \( C \) such that
along the leaves \(-Cg < K_g < Cg\).

If \(\Gamma\) is not identically zero, let us consider a leaf \(L\) on which \(\Gamma\) does not vanish identically. This leaf (or its universal cover) is isomorphic to \(\mathbb{C}\). Thus \(\mathbb{C}\) is provided with the metric \(g = g_L\), whose curvature satisfies \(K_g > -Cg\). And we have a bundle \(E\) over \(\mathbb{C}\), with a holomorphic section \(\Gamma\), whose curvature \(K = \ddc \log |\Gamma|^2 < -\epsilon g\).

Moreover, \(K_g = \ddc \log |z|^2_g\) on \(\mathbb{C} \setminus \{0\}\), with \(z\) being a holomorphic section of \(T\mathbb{C}\). So,

\[
\ddc \log |\Gamma|^2 < \frac{c}{C} \ddc \log |z|^2_g,
\]

which reads as \(\ddc \left( \log |\Gamma|^2 - \frac{c}{C} \log |z|^2_g \right) < 0\). Thus \(\phi = \log |\Gamma|^2 - \frac{c}{C} \log |z|^2_g\) is a sub-harmonic function on \(\mathbb{C}\) (with maybe a numerable set of points where \(\phi\) is not defined and in which \(\phi\) goes to \(-\infty\)). In particular, by the maximum principle, for any \(r > 0\), \(\sup_{|z|=r} \phi \geq \phi(0)\). Since \(\Gamma\) does not vanish identically, we can suppose that \(|\Gamma(0)| \neq 0\). So, for any \(r > 0\), there exists \(z_r\) such that \(|z_r| = r\) and \(\phi(z_r) \geq \phi(0) = C_0\). Finally, this means

\[
\log |\Gamma(z_r)|^2 \geq \frac{c}{C} \log r^2 + C_0.
\]

This implies that \(|\Gamma(z_r)| \to \infty\) when \(r \to \infty\), which is not possible since \(X\) is compact and so \(|\Gamma|\) is bounded on \(X\). So \(\Gamma\) has to vanish identically. \(\Box\)

In the general case, the invariant \(\Gamma\) is not trivial: there exist some foliated manifolds whose invariant \(\Gamma\) does not vanish and so, does not admit any holomorphic functions \(F : \mathbb{D} \times \mathbb{C} \to X\) compatible with the foliation. Let us consider the following example 1.

3.3. An example with \(\omega \neq 0\). Let us consider a non-holomorphic function \(f : \mathbb{D} \to \mathbb{C} \cup \{\infty\} = \mathbb{P}^1\mathbb{C}\). Let \(\gamma\) be the graph of the function \(f\) and then let us consider the manifold \(X = (\mathbb{D} \times \mathbb{CP}^1) \setminus \gamma\). This manifold is provided with the natural complex structure restricting the one of \(\mathbb{D} \times \mathbb{CP}^1\) and it is holomorphically foliated by the leaves \(L_x = \mathbb{CP}^1 \setminus \{f(x)\} \simeq \mathbb{C}\).

Let \(F : \mathbb{D} \times \mathbb{C} \to X\) be the map in \(\mathcal{C}\) constructed from a transverse holomorphic disk \(D_0 \subset \mathbb{D} \times \{0\}\), and from the holomorphic trivialization \(t\) of the bundle \(T\mathcal{F}\) that is defined by \(t_x = 1 \in \mathbb{C} \simeq T_x\mathbb{CP}^1 \simeq T_xT\mathcal{F}\). Then, if we assume for example that \(f \neq 0\) and \(f \neq \infty\) on \(D_0\), \(F\) reads as :

\[
F : \begin{cases} 
D_0 \times \mathbb{C} \rightarrow \bigcup_{x \in D_0} L_x \\
(x, y) \rightarrow (x, \frac{f(x)y}{y+f(x)})
\end{cases}
\]

It follows :

\[
dF_{(x,y)}(v_x \ v_y) = \begin{pmatrix} v_x & v_y \end{pmatrix} = \begin{pmatrix} \frac{f(x)^2}{(y+f(x))^2} v_x + \frac{y^2}{(y+f(x))^2} \frac{\partial f}{\partial x}(x) v_x + \frac{\partial f}{\partial x}(x) v_x \end{pmatrix}
\]

1following suggestions of Alexei Glutsyuk and Etienne Ghys
Thus, it is straightforward to check that \( w(x, y) = 2i \frac{y^2}{f(x)} \frac{\partial f}{\partial \bar{x}}(x) \) and so does not vanish, since \( f \) is not holomorphic.

Furthermore, let us note, that this is, in a way, “the easiest” example. Indeed, \( \omega \) here is a polynomial of degree 2 with respect to \( y \) (\( \omega(x, y) = \frac{2i}{f(x)} \frac{\partial f}{\partial \bar{x}}(x) y^2 \)), which can be considered as the “easiest” example for a holomorphic function on \( \mathbb{C} \) that vanishes on 0 and whose derivative vanishes on 0.

4. Final remarks and openings

The Brody and Kobayashi-hyperbolicity, initially defined for complex manifolds (see [8]), can also be defined in the case of almost-complex structures. This issue has been tackled in the last few years and is still an active area of research, notably in the case of an almost-complex structure compatible with a symplectic structure (see [1], [2]). However, the notion of measure-hyperbolicity and the notion of hyperbolicity for foliations defined in this paper only make sense in the framework of integrable complex structures. Indeed, as soon as there exists a holomorphic map from \( \mathbb{D}^{n-1} \times \mathbb{C} \), or \( \mathbb{D}^n \), to an almost-complex manifold, then its structure has to be integrable.

That’s why, in the case of foliated manifolds provided with an almost-complex structure (for example compatible with a symplectic structure), such that the leaves are complex submanifolds, it would be interesting to tackle the issue of hyperbolicity following different approaches. The one developed in [1] and [2] suggests us to use a notion of foliated Floer homology.

Appendix A. Holomorphic foliations: definitions and notations

For more details, one can refer to [5] and [9].

A complex manifold \( X \) is provided with a holomorphic foliation \( F \) if there exists an atlas of holomorphic charts \( \psi_i : U_i \to \Omega_i \subset \mathbb{C}^p \times \mathbb{C}^k \) such that the transitions maps are bi-holomorphisms:

\[
\psi_i \circ \psi_j^{-1} : \begin{cases} 
\psi_j^{-1}(U_i \cap U_j) & \to \psi_i^{-1}(U_i \cap U_j) \\
(x, y) \in \mathbb{C}^p \times \mathbb{C}^k & \to (\phi_i^{1,j}(x), \phi_i^{2,j}(x, y)) \in \mathbb{C}^p \times \mathbb{C}^k.
\end{cases}
\]

Thus in each point, \( \phi_i^{1,j} \) is a local diffeomorphism of \( \mathbb{C}^p \). And geometrically, if for each \( x \in \mathbb{C}^p \) we call plaques the connected components of \( \psi_i^{-1}(\{x\} \times \mathbb{C}^k) \), then the transition map sends each plaque (into \( U_j \)) to another plaque (into \( U_i \)).

These charts are called foliated charts.

An open set \( U \) on which there exists a foliated chart \( \psi : U \to \mathbb{D}^p \times \mathbb{D}^k \) is called a distinguished open set.

In each point \( z \in X \), the tangent plane to the foliation \( (T F)_z \) is defined as \( d\psi^{-1}(\{0\} \times \mathbb{C}^k) \) for any foliated chart \( \psi \). An equivalence relation can then be defined: \( x \sim y \) if \( x \) and \( y \) can be linked by a path tangent to the plane field \( T F \).

The leaves of the foliation are the equivalence classes for this relation.

They are the smallest connected sets satisfying: if a plaque intersects this set, then this plaque is completely included into it.

Furthermore, by definition, the foliated charts send each leaf to a \( \{*\} \times \mathbb{C}^k \).
The above introduced plane field $T\mathcal{F}$ is a holomorphic vector bundle above the manifold. Then, another holomorphic vector field can naturally be defined: the \textit{transverse} or normal vector field $T\mathcal{N} = TX/T\mathcal{F}$.

Moreover, a subset of $X$ is said to be \textit{saturated} if it is the union of leaves. The \textit{saturation} of an arbitrary subset $A$ is the union of all leaves passing through $A$. Let us recall that the saturation of an open subset is an open set.

Furthermore, if $U$ is a distinguished open set, one can consider the foliation induced by $\mathcal{F}$ on $U$. This \textit{induced foliation} will be denoted by $\mathcal{F}^U$. The leaves of this foliation are the plaques of $U$.

Finally, let us introduce the projection along the leaves on a transverse disk. Let $D$ be a transverse disk to the foliation. If $U$ is a distinguished open set and $\psi: U \to \mathbb{D} \times \mathbb{D}$ is a foliated chart sending $D$ onto $\mathbb{D} \times \{0\}$, then we can consider on $U$ the \textit{projection} $\pi$ \textit{on the transverse $D$ along the leaves}. This projection $\pi$ sends each plaque $L^U_z$ (the leaf of the induced foliation on $U$ passing through $\psi^{-1}(x,0)$) on $\psi^{-1}(x,0)$. Moreover, via the chart $\psi$, the map $\pi$ reads as

$$\psi \circ \pi \circ \psi^{-1} : \begin{cases}
\mathbb{D} \times \mathbb{D}^p & \to \mathbb{D}_k \times \{0\} \\
(z = (x, y)) & \to z' = (x, 0)
\end{cases}$$

This is the projection on the first coordinate, so it is holomorphic. Therefore, the projection $\pi$ on the transverse $D$ is holomorphic on $U$.

\section*{Appendix B. Holomorphy of the projections along the leaves}

Let $D$ be a fixed transverse holomorphic disk parametrized by $\mathbb{D}$, and $F: \mathbb{D} \times \mathbb{C} \to X$ a map in $\mathbb{C}$ constructed along the disk $D$. Then

\textbf{Lemma B.1.} Let $D'$ be any transverse disk included in the image of $F$, which can be parametrized as $D' = \{F(x, \mathcal{Y}(x)), x \in \mathbb{D}\}$, with $\mathcal{Y}$ any continuous map from $\mathbb{D}$ to $\mathbb{C}$. Then, on a distinguished neighborhood $U$ of $D'$ can be defined a projection $\pi_0$ on $D$ satisfying:

$$\pi_0 : \begin{cases}
U & \to D \\
z = F(x, \mathcal{Y}(x)) & \to F(x, 0),
\end{cases}$$

It also sends the plaque of $U$ passing through $z = F(x, \mathcal{Y}(x))$ onto $F(x, 0)$. Moreover, it is holomorphic on $U$.

\textit{Proof.} On any distinguished open set $U_0$, one can consider the projection $\pi_0$ on $D$ along the leaves, defined the above section. This projection is holomorphic and sends each plaque of $U_0$ passing through $F(x, 0)$ on $F(x, 0)$. If $D'$ is included in $U_0$ (with a “small” parametrization $\mathcal{Y}$) this projection satisfies the wished properties. In order to deal with a general $D'$, the idea (used also to define the holonomy pseudo-groups) is to link $D'$ with $D$ (along for example $(F(x, t\mathcal{Y}(x)))_{t \in [0,1]}$) through a chain of distinguished open sets $(U_j)_{j=0, \ldots, n}, D \subset U_0, D' \subset U_n = V$, such that for any $j = 1, \ldots, n$, $U_{j-1} \cap U_j$ contains a transverse holomorphic disk $D_j$. 


Then on each $U_j$, we can, as above, define a holomorphic projection $\pi_j$ on the disk $D_j$. Then we can define on $V = U_0$ the projection $\pi_0 \circ \pi_1 \circ \ldots \circ \pi_n$. It is holomorphic and satisfies the desired properties. □

Let us notice that this projection map on $D$ is not necessarily unique since it might depend on the choice of the path linking $D'$ and $D$.

The idea of this lemma is similar to the one used to define the holonomy groups of the leaves: if $L_0$ is a leaf, and $\gamma$ is an element of $\pi_1(L_0)$ represented by a loop $\gamma_0 \subset L_0$ with base-point $x_0$, then one can consider a chain of distinguished open sets $(U_j)$ along $\gamma_0$, with some transverse holomorphic disks $D_j$ in the intersection $U_j \cap U_{j-1}$. Then defining $\pi_j$ the projection on $D_j$ along the leaves – defined on $U_j$ –, one can consider, as above, $\pi_\gamma$ the composition of these projections: $\pi_\gamma : D_0 \to D_0$ with $D_0$ a small transverse holomorphic disk around $x_0$. As seen above, this map is holomorphic (and even a local bi-holomorphism around $x_0$ of inverse $\pi_{-\gamma}$). This determines a germ of bi-holomorphism around $x_0$ which does not depend on the choices of the chain $(U_j)$ and of the representation $\gamma_0$. This defines a map from $\pi_1(L_0, x_0) \to Hol(D_0)$ with $Hol(D_0)$ the group of germs of local bi-holomorphism around $x_0$. The image of this map is the holonomy group of $L_0$ at $x_0$. For two distinct points in $L_0$, the holonomy groups of $L_0$ in these points are conjugated. So one can talk about the holonomy group of the leaf $L_0$.

Let us come back to the result of lemma B.1. From this lemma it immediately follows the existence and holomorphy of the function $\theta_1$ introduced in section 1.3.

Moreover, we can deduce:

**Lemma B.2.** The complex structures $J' = F^*J$ and $J_0$ of $\mathbb{D} \times \mathbb{C}$ coincide on the quotient $\mathbb{D}$.

**Proof.** On a (distinguished) neighborhood $V$ of a fixed point $z = (x_0, y_0)$, the projection $\pi_0$ introduced in the previous proposition) can be defined. It satisfies $\pi_0(F(x, y)) = F(x, 0)$ on $V$, and so differentiating

$$d(\pi_0)_{F(x,y)}(dF_{(x,y)}(v,0)) = dF_{(x,0)}(v,0).$$

(B.1)

Since $\pi_0$ is holomorphic, $d(\pi_0)_{F(x,y)}(JY) = Jd(\pi_0)_{F(x,y)}(Y)$. Applying this equality to $dF_{(x,y)}(v,0)$, we get $d(\pi_0)_{F(x,y)}(JdF_{(x,y)}(v,0)) = Jd(\pi_0)_{F(x,y)}(dF_{(x,y)}(v,0))$, which equals $JdF_{(x,0)}(v,0)$ according to (B.1). Considering that $F$ is holomorphic along $\mathbb{D} \times \{0\}$, this equals $dF_{(x,0)}(J_0v,0)$.

Finally, applying (B.1) one last time,

$$d(\pi_0)_{F(x,y)}(JdF_{(x,y)}(v,0)) = d(\pi_0)_{F(x,y)}(dF_{(x,y)}(J_0v,0)).$$

Since $\text{Ker}(d\pi_0) = T\mathcal{F}$, our result follows. □

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