Numerical study of a mathematical model of internal erosion of soil

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Abstract. The process of internal erosion in a three-phase saturated soil is studied. A mathematical model describing the process consists of the equations of mass conservation, Darcy’s law and equation for capillary pressure. The original system of equations is reduced to a system of three equations for porosity, pressure and water saturation. Obtained equation for the water saturation is degenerate. The degenerate problem in an one-dimensional domain is solved numerically using the finite-difference method.

1. Introduction

The problems associated with soil erosion have been extensively studied throughout the last century. This process is of great importance in applied problems in agriculture: irrigation and drainage of agricultural fields [1] and canal irrigation of soil [2]. The erosion process must be taken into account in studies related to the forecast of the spread of contamination, filtration near reservoirs and other hydraulic structures [3]. Similar problems related to the process of soil erosion occur in other areas, including oil and gas production [4].

Several different approaches are existed for modeling the process of internal erosion and motion of fluidized soil particles. In the works of R. J. Garde [5], the process of motion of a mixture of gas and mobile solid particles is described without taking into account the process of internal suffusion. Is is noted that for the considering the erosion process the M.A. Biot model for water-saturated soils [6] can be used.

Fundamentally different approaches in erosion modeling are used in the series of works by S. Bonelli [7]. This approach involves the determination of an unknown interface between a solid skeleton and an one-velocity mixture of the water and fluidized solid particles with standard rheology. To determine the unknown boundary the common method is to introduce analogies of the kinematic and dynamic conditions in the problems with free boundaries.

The most complete models of the motion of water and moving solid particles in a porous medium are proposed in a series of works by I. Vardoulakis and his followers [8]. Soil in these works are considered as a multiphase medium on the one hand, on the other it is assumed that velocities of water and moving soil particles are proportional. This assumption is a consequence of the hypotheses that the pressures of moving phases are equal.

Last but not the least important thing of the modeling is that there is no single approach describing phase transitions. In this paper to guarantee the satisfaction of the physical maximum
principles for porosity and phase concentration correct description of the phase transitions is considered.

2. Formulation of the problem

A mathematical model of internal erosion of soil in three-dimensional domain is formulated under the theory of multiphase filtration in a porous medium. Saturated soil consists of water \((i=1, \text{ first phase})\), fluidized solid particles \((i=2, \text{ second phase})\) and solid skeleton \((i=3, \text{ third phase})\) [8]. Constitutive equations of the model in the general case are the equations of mass conservation for each phase with phase transitions, generalized Darcy’s law and equation of capillary pressure for water and fluidized solid particles [10]:

\[
\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \vec{u}_i) = \sum_{j=1}^{3} I_{ij}, \quad i = 1, 2, 3, \quad I_{ji} = -I_{ij}, \quad \sum_{i,j=1}^{3} I_{ij} = 0; \tag{1}
\]

\[
\vec{v}_i = s_i \phi \vec{u}_i = -K_0(\phi) \frac{\bar{k}_{0i}}{\mu_i} (\nabla p_i + \rho_i \vec{g}), \quad i = 1, 2; \quad p_2 - p_1 = p_c(s_1) \geq 0, \quad s_1 + s_2 = 1. \tag{2}
\]

Here \(\vec{u}_i\) is a velocity of the \(i\)th phase \((\vec{u}_3 = 0, \text{ the third phase is considered motionless medium})\), \(\rho_i\) is a reduced density related to a true density \(\rho_i^0\) and a volumetric concentration \(\alpha_i\) by \(\rho_i = \alpha_i \rho_i^0\) (identity \(\sum_{i=1}^{3} \alpha_i = 1\) is a consequence of the definition of \(\rho_i\)), \(I_{ij}\) is a rate of mass transfer from the \(j\)th to the \(i\)th phase per unit volume in unit of time, \(\vec{v}_i = \phi s_i \vec{u}_i\) is a filtration velocity of the \(i\)th phase \((i=1,2)\), \(\phi\) is porosity, \(s_1\) and \(s_2\) are water and fluidized solid particles saturations \((\alpha_1 = \phi s_1, \alpha_2 = \phi s_2, \alpha_3 = 1 - \phi)\), \(K_0\) is the filtration tensor; \(\bar{k}_{0i}\) is a relative phase permeability \(\bar{k}_{0i} = \bar{k}_{0i}(s_i) \geq 0, \bar{k}_{0i}|_{s_i=0} = 0\), \(\mu_i\) is a dynamic viscosity, \(p_i\) is a phase pressure \((i=1,2)\): \(p_c\) is the capillary pressure, \(\vec{g}\) is the acceleration vector due to gravity.

The system \((1)-(2)\) is considered under the following assumptions for the densities: \(\rho_i^0 = \text{const}, \rho_2^0 = \rho_3^0, I_{ij} = -I_{ji}\). For the phase transitions it is postulated that \(I_{12} = I_{13} = 0, I_{23} = \rho_3^0 I\), and \(I\) is a given function of \(s_1, \phi, v_1, v_2\).

The most studied case of the described system is considering porosity as a given function, then it can be shown that the system \((1)-(2)\) reduces to the well-known Masket–Leverett equations of two-phase filtration for immiscible fluids. This study is concerned on the investigation of the problem with unknown and to be determined function of porosity.

In the one-dimensional domain and under the conditions set forth above the system \((1)-(2)\) reads [11]

\[
\phi \frac{\partial s}{\partial t} = \frac{\partial}{\partial x} \left( a(s, \phi) \frac{\partial s}{\partial x} + b(s) v + F(s, \phi) \right) - s \frac{\partial \phi}{\partial t}, \tag{3}
\]

\[
\frac{\partial v}{\partial x} = 0, \quad v = K(s, \phi) \frac{\partial p}{\partial x} + f(s, \phi), \tag{4}
\]

\[
\frac{\partial \phi}{\partial t} = I(s, \phi, p). \tag{5}
\]

Here \(s \equiv s_1, v = v_1 + v_2\) is a mixture filtration velocity (prescribed function of time) [5]. The rate of mass transfer is given as \(I = \lambda \delta(s) R(\phi) \max\{|v(t)| - v_k, 0\}\) [2, 10], where \(\delta(s) = 0\) if \(s \geq 1\), \(\delta(s) = 1 - s\) if \(0 < s < 1\), \(\delta(s) = 1\) if \(s \leq 0\); \(R(\phi) = 0\) if \(\phi \geq 1\), \(R(\phi) = \phi(1 - \phi)\) if \(0 < \phi < 1\), \(R(\phi) = 0\) if \(\phi \leq 0\), \(\lambda > 0\) is a dimension constant \([1/m]\), \(v_k\) is the critical velocity. The critical velocity indicates whether the erosion process has started or hasn’t. If the velocity
v reaches value of the critical speed then the theory suggests the erosion process will begin (the critical speed can be evaluated using experimental methods, see, e.g. \cite{11}).

The coefficients of the system (3) – (4) have the form

\[ a(s, \phi) = -K_0(\phi) \frac{k_{01}k_{02}}{k} \frac{\partial p_c}{\partial s} > 0, \quad k_{0i}(s) = \frac{\tilde{K}_{0i}}{\mu_i} \geq 0, \quad k(s) = k_{01} + k_{02} > 0, \]

\[ F(s, \phi) = K_0(\phi) \frac{k_{01}k_{02}}{k} (p_{01}^0 - p_{02}^0)g, \quad b(s) = \frac{k_{01}}{k} \geq 0. \]

The functions \( s(x, t) \) and \( \phi(x, t) \) satisfy the initial and boundary conditions

\[ s(x, 0) = s^0(x), \quad \phi(x, 0) = \phi^0(x), \]

\[ s(0, t) = s_0(t), \quad s(l, t) = s_l(t), \quad \frac{\partial p}{\partial x}(0, t) = p_0(t), \quad p(l, t) = p_l. \tag{6} \]

3. Method of the numerical solution

The system of the equations (3) – (5) is solved numerically for the initial and boundary conditions in the form (6) in a finite one-dimensional domain. The numerical study aims to investigate the behavior of the functions of porosity and water saturation as functions of space variable and time depending on given parameters of the problem.

The numerical solution of the problem (3) – (5) is obtained with the help of the finite-difference method. It is convenient to introduce a finite-difference grid with constant steps \( x_i = ih, \ i = 0...N \) and \( t_n = n\tau, \ n = 0...T \), where \( h \) is a mesh size, \( N \) is number of steps in space, \( \tau \) is time step, \( T \) is number of steps in time. The convective term in the equation (3) is approximated by the directed difference. The equation (4) is approximated by an implicit scheme of the second order of accuracy. The result is the system of finite-difference equations

\[ \frac{\phi^{n+1}_i - \phi^n_i}{\tau} = a^n_{i+1/2} (s^{n+1}_{i+1} - s^n_{i+1})/h^2 - a^n_{i-1/2} (s^{n+1}_{i-1} - s^n_{i-1})/h^2 + \]

\[ + \frac{(|G_i^n| + G_i^n) s^{n+1}_{i+1} - 2|G_i^n| s^{n+1}_{i} + (|G_i^n| - G_i^n) s^{n+1}_{i-1}}{2h} + F_{\phi_i} \phi^{n+1}_{i+1} - \phi^n_{i+1} - \phi^{n+1}_{i-1} - \phi^n_{i-1} \]

\[ = 0. \tag{7} \]

\[ K^n_{i+1/2} \frac{p^{n+1}_{i+1} - p^n_{i+1}}{h^2} - K^n_{i-1/2} \frac{p^n_{i} - p^{n-1}_{i}}{h^2} + f^n_{s_{i+1}} s^n_{i+1} - s^n_{i+1} + f^n_{s_{i-1}} \phi^{n+1}_{i+1} - \phi^{n+1}_{i-1} \]

\[ = 0. \tag{8} \]

The equation (5) is approximated by the Runge-Kutta scheme of the second order of accuracy. The value found in the first step in the form

\[ \tilde{\phi}_i^{n+1} = \phi^n_i + \tau I^n_i \tag{9} \]

is refined as

\[ \phi_i^{n+1} = \phi_i^n + \tau \frac{I(\phi^n_i, s^n_i) + I(\tilde{\phi}_i^n, s^n_i)}{2}. \tag{10} \]

Here \( i = 1, ..., N - 1, \ \tau = 1, ..., T - 1, \)

\[ a^n_{i+1/2} = \frac{2K_0(\phi_{i+1}^n) a(s^n_{i+1})K_0(\phi_i^n) a(s^n_i)}{K_0(\phi_{i-1}^n) a(s^n_{i-1}) + K_0(\phi_i^n) a(s^n_i)}, \quad a^n_{i+1/2} = \frac{2K_0(\phi_{i+1}^n) a(s^n_{i+1})K_0(\phi_i^n) a(s^n_i)}{K_0(\phi_{i+1}^n) a(s^n_{i+1}) + K_0(\phi_i^n) a(s^n_i)}. \]
The numerical algorithm consists of several steps. First, the initial conditions for the porosity \( \phi^0 \) and water saturation \( s_i^0 \) are substituted to the coefficients of the finite-difference scheme (8). Then, using sweep method in the scheme (8), values of the pressure \( p_i^0 \) are found. After that, using the values of the pressure, values of the water filtration rate \( v_i^0 \) and moving soil particles velocity \( \psi^0 \), the velocities of mobile soil particles [8]. In his work it is postulated that the velocity of soil

\[ F_{\phi i}^n = \frac{\partial F}{\partial \phi}(s_i^n, \phi_i^n), \quad f_{si}^n = \frac{\partial f}{\partial s}(s_i^n, \phi_i^n), \quad f_{\phi i}^n = \frac{\partial f}{\partial \phi}(s_i^n, \phi_i^n), \]

\[ G_i^n = \frac{\partial F}{\partial s}(s_i^n, \phi_i^n) + v(t_n) \frac{\partial b}{\partial s}(s_i^n), \]

\[ K_{n+1/2}^i = \frac{2K(\phi_{i-1}^n, s_{i-1}^n)K(\phi_i^n, s_i^n)}{K(\phi_{i-1}^n, s_{i-1}^n) + K(\phi_i^n, s_i^n)}, \quad K_{n+1}^i = \frac{2K(\phi_{i+1}^n, s_{i+1}^n)K(\phi_i^n, s_i^n)}{K(\phi_{i+1}^n, s_{i+1}^n) + K(\phi_i^n, s_i^n)}, \]

\[ \lambda_i = \begin{cases} \lambda(1 - \phi_i^n)(1 - s_i^n)\phi_i^n(|v_i^n| - |v_k|), & |v_i^n| \geq |v_k|; \\
0, & |v_i^n| < |v_k|. \end{cases} \]

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4. Numerical results

The porosity and the water saturation are calculated for the following set of the parameters of the problem: \( g = 0 \) m/s\(^2\), \( v_k = 0 \), \( K_0(\phi) = \phi^0/(1 - \phi^0) \), \( k_{0i} = s_i^2 \) if \( 0 \leq s_i^2 \leq 1 \), \( k_{0i} = 0 \) if \( s_i^2 < 0 \), \( k_{0i} = 1 \) if \( s_i \geq 1 \). The initial and boundary conditions are taken in the form \( \frac{\partial \phi}{\partial x}(0, t) = H \), \( \frac{\partial s_i^0}{\partial x}(0, t) = H + \frac{k_{0i}(s_i)}{k(s_i)} \frac{\partial \phi}{\partial s}(s_i^0) \), where \( H \) is water head and equals 2.4 in numerical calculations. At the points \( x = 0 \) there are no fluidized solid particles the boundary condition is \( s(0, t) = s_0 = 1 \). At the points \( x = l \) the boundary conditions are \( s(l, t) = s_h(t) \), where \( s_h(t) \) is function obtained by interpolating the experimental concentration value [9]. The initial condition for porosity is \( \phi(x, 0) = 3.5 \), the one for \( s \) is assumed to be a linear function of \( x \), \( s(x, 0) = (s_h(0) - s_0)/s + s_0 \). In presented results the numerical parameters are \( h = 0.00123 \), \( \tau = 0.0001 \), \( N = 1000 \), \( T = 1000 \).

Following values of the physical parameters are used in numerical calculations [9]:

\[ p_i^0 = 1000 \text{ kg/m}^3, \quad \rho_0^0 = 2600 \text{ kg/m}^3, \quad l = 123 \text{ mm}, \]
\[ \mu_1 = 0.001787 \text{ kg/ms}^2, \quad \mu_2 = 0.003574 \text{ kg/ms}^2. \]

In numerical calculations mathematical model taken into account the capillary forces is studied. The Leverett function has the form [12] \( p_c(s) = \gamma(1/s - 1) \), where \( \gamma \) – dimension parameter (in the calculations \( \gamma = 0.001 \)). Obtained results of the calculations were compared with the experimental data presented in the paper [9]. A. Chetti and co-authors investigated a mathematical model of internal suffusion based on the advection-diffusion equation. The main assumption of the model studied by A. Chetti and co-authors was equality of the water and moving soil particles velocities. Comparison of results obtained in this paper with the experimental data showed that their model gives highly overestimated values of the eroded soil mass from the filtration region for soils with low susceptibility to suffusion (\( \lambda = 4 \) and \( \lambda = 1 \)). I. Vardoulakis in his work proposed the relation \( v_1 = \beta v_2 \) (\( 0 < \beta < 1 \)) for determining the velocities of mobile soil particles [8]. In his work it is postulated that the velocity of soil
particles is less than the rate of water filtration, but method how the value of the coefficient $\beta$ could be chosen is not explained. In this paper the velocity of moving soil particles is determined during the solution of the problem using Darcy’s law (2) \cite{5}. The velocity modulus of the water filtration rate $v_1$ and the velocity modulus of the moving soil particles are shown in Figure 2. It is shown that the velocity of the moving soil particles is much less than the velocity of water. The comparison of the calculated values of the outward mass from the filtration region with the experimental data presented in \cite{9} is shown in Figure 1. The comparison of the calculated values of the filtration rates of water and moving soil particles shows that the water velocity far exceeds the velocity of the mobile soil particles. Effect of the unsteady velocity of the moving soil particles is significant for soils that are slightly susceptible to suffusion ($\lambda = 4$ and $\lambda = 1$).

5. Conclusion
The degenerate one dimensional problem are solved numerically by the finite-difference method. Calculations results are compared with the experimental data. Using the Leverett J-function the moving phases velocities are found. The water and moving soil particles filtration rates are compared.

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