REALIZATIONS OF $\mu-\tau$ INTERCHANGE SYMMETRY

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Some models for the lepton sector, based on seesaw extensions of the Standard Model, are discussed in which the $\mu-\tau$ interchange symmetry is realized in various ways. The symmetries defining such models and their characteristic predictions for lepton mixing are presented.

Keywords: Neutrino mass matrices; Lepton mixing; $\mu-\tau$ (anti)symmetry.

1. Mass matrices with $\mu-\tau$ interchange symmetry

At present, all data from neutrino oscillation measurements are compatible with the lepton mixing angles $^{1}$

$$\theta_{23} = 45^\circ \text{ and } \theta_{13} = 0^\circ. \quad (1)$$

A neutrino mass matrix with these properties, in the basis in which the charged-lepton mass matrix $M_e$ is diagonal, exhibits a $\mu-\tau$ (interchange) symmetry—for early papers on this issue see Refs. 2, 3. Assuming that the lepton flavours $\alpha$ are ordered in the usual way with $\alpha = e, \mu, \tau$, the $\mu-\tau$ symmetry for a matrix $M^{(S)}$ is formulated as $^4$

$$TM^{(S)}T = M^{(S)} \Rightarrow M^{(S)} = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \quad (2)$$

where $T$ is a permutation matrix, performing the flavour exchange $\mu \leftrightarrow \tau$. If $M^{(S)}$ is conceived as a Majorana neutrino mass matrix $M_{\nu}$, it is easy to see that it predicts the mixing angles of Eq. (1). There are no further predictions of $M^{(S)}$. $^3$

With $\mu-\tau$ antisymmetry $^5,^6$, one obtains

$$TM^{(AS)}T = -M^{(AS)} \Rightarrow$$

$$M^{(AS)} = \begin{pmatrix} 0 & p & -p \\ p & q & 0 \\ -p & 0 & -q \end{pmatrix}. \quad (3)$$

Assuming $M_{\nu} = M^{(AS)}$, this matrix gives

$$\theta_{12} = \theta_{23} = 45^\circ, \quad m_1 = m_2, \quad m_3 = 0. \quad (4)$$

We gather from this result that $M^{(AS)}$ is not suitable as a neutrino mass matrix, however, its predictions are not excessively far from reality, if we assume $\theta_{13}$ to be sufficiently small.

2. $M_{\nu}$ versus $M_{\nu}^{-1}$

(1) Denoting the diagonalizing matrix of $M_{\nu}$ by $U$ and assuming det $M_{\nu} \neq 0$, we observe the relationship

$$U^T M_{\nu} U = \hat{m} \equiv \text{diag} (m_1, m_2, m_3) \Leftrightarrow U^1 (M_{\nu})^{-1} U^* = (\hat{m})^{-1} \cdot (5)$$

(2) Next, we assume the validity of the seesaw mechanism: $M_{\nu} = -M^R_D M^{-1}_R M_D$ with the neutrino Dirac-mass matrix $M_D$ and the mass matrix $M_R$ of the right-handed neutrino singlet fields $\nu_R$ whose mass Lagrangian is given by

$$\mathcal{L}_M(\nu_R) = \frac{1}{2} \nu_R^T C^{-1} M^*_R \nu_R + \text{H.c.} \quad (6)$$

If $M_\ell$ is diagonal and $M_D$ has the form $M_D = \text{diag} (a, b, b)$, then $\mu-\tau$ symmetry (antisymmetry) of $(M_{\nu})^{-1}$ is equivalent to $\mu-\tau$ symmetry (antisymmetry) of $M_R$. To impose $\mu-\tau$ symmetry on $M_R$ we simply have to require invariance of $\mathcal{L}_M(\nu_R)$ under $\nu_R \rightarrow T \nu_R$; for $\mu-\tau$ antisymmetry the transformation is $\nu_R \rightarrow i T \nu_R$. 

1.
With the assumptions of Item (2) it makes sense to impose conditions on the inverse mass matrix instead of on the mass matrix itself. Obviously, $(\mathcal{M}_\nu)^{-1}$ can be decomposed as
\begin{equation}
(\mathcal{M}_\nu)^{-1} = M^{(8)} + M^{(AS)}.
\end{equation}
This rather trivial observation is the point of departure for the following discussion where we will present several models which realize the $\mu-\tau$ interchange symmetry within seesaw extensions of the Standard Model (SM).

3. The framework

We consider the lepton sector of the SM, enlarge the scalar sector to three Higgs doublets $\phi_j$ ($j = 1, 2, 3$) and add three right-handed neutrino singlets $\nu_{\alpha R}$ for the purpose of the seesaw mechanism. The left-handed lepton doublets are denoted by $D_{\alpha L}$ and the right-handed charged-lepton singlets by $\ell_{\alpha R} \equiv \alpha_R$. We impose the following symmetries:

1. The groups $U(1)_{L_{\alpha}}$ ($\alpha = e, \mu, \tau$) associated with the family lepton numbers $L_{\alpha}$, or, alternatively, $D_L \to \text{diag}(1, \omega, \omega^2)D_L$ and, analogously, for $\ell_R$ and $\nu_R$, with $\omega = e^{2\pi i/3}$, corresponding to the symmetry group $Z_3$.

2. The symmetry transformation $D_L \to i^kTD_L$, $\ell_R \to i^kT\ell_R$, $\nu_R \to i^kT\nu_R$, $\phi_3 \to -\phi_3$, which either corresponds to the $\mu-\tau$ symmetry for $k = 0$ or to the $\mu-\tau$ antisymmetry for $k = 1$.

3. An auxiliary symmetry $Z_2^{(aux)}$ defined by the sign change of the fields $\nu_{\alpha R}$ ($\alpha = e, \mu, \tau$), $\phi_1$, $\phi_2$, $e_R$.

It is easy to check that the most general Yukawa Lagrangian compatible with these symmetries is given by
\begin{equation}
\mathcal{L}_Y(\phi) = -y_1 \bar{D}_{eL} \nu_{eR} \tilde{\phi}_1 - y_2 (\bar{D}_{\mu L} \nu_{\mu R} + \bar{D}_{\tau L} \nu_{\tau R}) \tilde{\phi}_1 - y_3 \bar{D}_{eL} e_R \phi_1 - y_4 (\bar{D}_{\mu L} \mu_R + \bar{D}_{\tau L} \tau_R) \phi_2 - y_5 (\bar{D}_{\mu L} \mu_R - \bar{D}_{\tau L} \tau_R) \phi_3 + \text{H.c.}
\end{equation}

Note that the symmetries of Item (1) enforce diagonal Yukawa couplings, Item (2) provides the $\mu-\tau$-symmetric structure of the couplings, and Item (3) makes sure that $\phi_3$ does not couple to $\nu_R$; the latter point is important for supplying the $\mu-\tau$-symmetric form $M_D = \text{diag}(a, b, b)$ of the neutrino Dirac mass matrix.

The $\mu-\tau$ (anti)symmetry is spontaneously broken by the VEV of $\phi_3$, which allows for $m_\mu \neq m_\tau$.

4. A model based on $S_3 \times Z_2^{(aux)}$

The model presented in this section is based on the group $S_3$. We have the following representations: $D_L, \ell_R, \nu_R \in 1 \oplus 2, \phi_{1, 2} \in 1, \phi_3 \in 1'$, We add a complex scalar $\chi$ such that $(\chi, \chi^*) \in 2$. The connex of $S_3$ with the symmetries of the previous section is obtained via
\begin{equation}
2 : \{(12) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (123) \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \end{pmatrix}
\end{equation}

where $(12), (123) \in S_3$. The cyclic permutation represents the $Z_3$ symmetry of Item (1), whereas the transposition $(12)$ is mapped into the $\mu-\tau$ symmetry of Item (2). The trivial one-dimensional representation is denoted by $1$ and $1'$ is given by $(12) \to -1, (123) \to 1$.

Apart from the Lagrangian (8), the symmetries allow Yukawa couplings of the singlet scalar, described by the Lagrangian
\begin{equation}
\mathcal{L}_Y(\chi) = y_1^* \nu_{eR} C^{-1} (\nu_{\mu R} \chi^* + \nu_{\tau R} \chi) + \frac{1}{2} z_1^* (\nu_{\mu R} C^{-1} \nu_{\mu R} + \nu_{\tau R} C^{-1} \nu_{\tau R} + \nu_{\tau R} C^{-1} \nu_{\tau R}) + \text{H.c.},
\end{equation}

and a $\nu_R$ mass term
\begin{equation}
\mathcal{L}_{\text{mass}} = \frac{1}{2} m^* \nu_{eR} C^{-1} \nu_{eR} + m'^* \nu_{\mu R} C^{-1} \nu_{R} + \text{H.c.}
\end{equation}

We assume the VEV of $\chi$ and $m, m'$ to be of the order of the seesaw scale.
depends on the type of symmetry breaking of \( \mu \) and \( \pi \). Still, the matrix of absolute values in \((\mathcal{M}_\nu)\) is still \( \mu - \tau \)-symmetric. Which case is realized, depends on the type of symmetry breaking of \( S_3 \): If it is broken spontaneously, then \( \psi = 0 \) or \( \pi \); if, in addition, its \( \mathbb{Z}_3 \) subgroup is broken softly via terms of dimension one and two in the scalar potential, sin \( \psi \) is non-zero. In the latter case, there are correlated deviations from Eq. (1), approximately given by\(^3\)

\[
\cos 2\theta_{23} \simeq -2c_{12}s_{12}
\]

\[
\Delta m^2_{\odot} \left( \sum_{\alpha,\beta} z_{\alpha\beta}^* v^T_{\alpha R} C^{-1} v_{\beta R} \chi_{\alpha\beta} + \text{H.c.} \right)
\]

where \( \Delta m^2_{\odot} = m_2^2 - m_1^2 \) is the solar mass-squared difference and \( \delta \) is the CKM-type phase in \( U \). For an inverted neutrino mass spectrum, \( \theta_{23} \) is still maximal for all practical purposes. For a normal spectrum, possible deviations of \( \theta_{23} \) from 45° are most pronounced in the hierarchical case, namely \( \cos 2\theta_{23} \simeq -3 s_{13} \cos \delta \).

### 5. A class of models based on \( \mu - \tau \) antisymmetry

Here we discuss a class of models based on conserved lepton numbers and \( \mu - \tau \) antisymmetry—see Sec. 3, Items (1) and (2). Since a \( \mu - \tau \)-antisymmetric \( M_R \) is singular, we add complex scalar gauge singlets which carry lepton numbers. Such scalars have the general Yukawa couplings

\[
\mathcal{L}_Y(\chi) = \frac{1}{2} \sum_{\alpha,\beta} z_{\alpha\beta}^* v^T_{\alpha R} C^{-1} v_{\beta R} \chi_{\alpha\beta} + \text{H.c.}
\]

In Table 1 we have listed the four basic cases of scalar singlets compatible with the family symmetries. Their VEVs make \( M_R \) non-singular and induce a \( \mu - \tau \)-symmetric contribution in \((\mathcal{M}_\nu)\)\(^{-1}\)—cf. Eq. (7)—as shown in the last column of Table 1.

Combining \( M(\mathcal{A}) \) with one or two of the cases in Table 1 for the construction of \( M_R \) leads to ten models—see Table 2. Of these models, only five are compatible with the data, as indicated in this table. Each of the five viable models has six physical parameters in \( \mathcal{M}_\nu \). Models (1)–(4) (four parameters in \( \mathcal{M}_\nu \)) and model (10) (five parameters in \( \mathcal{M}_\nu \)) are ruled out; properties of these models which lead to contradiction with the data are found in the last column of Table 2. For the viable models, the preferred or predicted neutrino mass spectrum is indicated in that column.

Let us make some comments on the mod-


Table 2. The models which can be constructed by using one or two of the scalar multiplets of Table 1. A cross (tick) indicates that the model is ruled out (allowed) by the data. The numbers (1)–(4) refer to the cases in Table 1.

| Case   | Validity | Comment                          |
|--------|----------|----------------------------------|
| (1)    | –        | $s_{13}^2 > 0.1$                |
| (2)    | –        | $\Delta m^2_{23} > 1$            |
| (3)    | –        | $\Delta m^2_{23} = 0$            |
| (4)    | –        | $\Delta m^2_{23} > 1$            |
| (5)    | (1)+(2)  | any spectrum                     |
| (6)    | (1)+(3)  | normal preferred                 |
| (7)    | (1)+(4)  | any spectrum                     |
| (8)    | (2)+(3)  | inverted spec.                   |
| (9)    | (2)+(4)  | inverted spec.                   |
| (10)   | (3)+(4)  | $\Delta m^2_{23} > 1$            |

6. Conclusions

In this report we have combined a $\mu$-$\tau$ interchange symmetry with family symmetries which give diagonal Yukawa couplings in order to obtain a predictive neutrino mass matrix. We have considered extensions of the SM which have three Higgs doublets and right-handed neutrino singlets for the seesaw mechanism. Other important ingredients are scalar gauge singlets which induce, upon acquiring VEVs, contributions to $M_R$. With diagonal Yukawa couplings, lepton mixing stems solely from a non-diagonal $M_R$ and conditions on $M_R$ are translated into conditions on $(M_\nu)^{-1}$. While exact $\mu$-$\tau$ symmetry in $M_\nu$ or $(M_\nu)^{-1}$ leads to Eq. (1), deviations from exact $\mu$-$\tau$ symmetry can lead to interesting correlations between atmospheric mixing and $\theta_{13}$. Though exact $\mu$-$\tau$ antisymmetry in $M_\nu$ or $(M_\nu)^{-1}$ is not viable, it is nevertheless a useful concept, in combination with the above-mentioned scalar singlets, for producing predictive models.

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