An inexact proximal point method for variational inequality on Hadamard manifolds

G. C. Bento * O. P. Ferreira * E.A. Papa Quiroz †

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Abstract

In this paper we present an inexact proximal point method for variational inequality problem on Hadamard manifolds and study its convergence properties. The proposed algorithm is inexact in two sense. First, each proximal subproblem is approximated by using the enlargement of the vector field in consideration and then the next iterated is obtained by solving this subproblem allowing a suitable error tolerance. As an application, we obtain an inexact proximal point method for constrained optimization problems, equilibrium problems and nonlinear optimization problems on Hadamard manifolds.

Keywords: Inexact proximal method, equilibrium problem, optimization problem, Hadamard manifold

1 Introduction

Extensions of concepts and techniques of optimization from the Euclidean space to the Riemannian context have been a subject of intense research in recent years. An special attention has been given to methods of Riemannian mathematical programming; papers published on this topic involving proximal point methods include, but are not limited to, [2, 3, 6, 7, 29, 35, 42, 46, 47]. It is well known that one of the reasons for this extension is the possibility of transforming non-convex or non-monotone problems in the Euclidean context into Riemannian convex or monotone problems, by introducing a suitable metric, which enables modified numerical methods to find solutions for these problems; see [7, 8, 13, 15, 20, 37]. Moreover, constrained optimization problems can be viewed as unconstrained ones from a Riemannian geometry point of view. In particular, many Euclidean optimization problems are naturally posed on the Riemannian context; see, e.g., [1, 13, 18, 25, 26, 31, 34, 40, 41, 45, 49].

In this paper, we consider the problem of finding a solution of a variational inequality problem defined on Riemannian manifolds. Variational inequality on Riemannian manifolds were first introduced and studied by Németh in [33], for univalued vector fields on Hadamard manifolds, and for multivalued vector fields on general Riemannian manifolds by Li and Yao in [29]; for recent

*Instituto de Matemática e Estatística, Universidade Federal de Goiás, CEP 74001-970 - Goiânia, GO, Brazil, E-mails: glaydston@ufg.br, orizon@ufg.br. The authors was supported in part by CNPq grants 423737/2016-3, 310864/2017-8, 305158/2014-7, 408151/2016-1 and 302473/2017-3, FAPEG/PRONEM- 201710267000532.

†Universidad Nacional Mayor de San Marcos and Universidad Privada del Norte, Lima, Peru, erikpapa@gmail.com
works addressing this subject see [19,30,43,44]. It is worth noting that constrained optimization problems and the problem of finding the zero of a multivalued vector field on Riemannian manifolds, which were studied in [2,7,16,21,28,46], are particular instances of the variational inequality problem.

The aim of this paper is to introduce an inexact proximal point method for variational inequality problem in Hadamard manifolds and to study its convergence properties. The proposed algorithm combine ideas from the papers [4] and [46] to obtain in inexact algorithm in two sense. First, each proximal subproblem is approximated by using the enlargement of the vector field in consideration and then the next iterated is obtained by solving this subproblem allowing a suitable error tolerance. This algorithm has as particular instances some algorithms previously studied. For instance, it generalize the algorithm studied in [38] to Riemannian setting, by considering two of the four errors considered there. Considering that the Riemannian algorithm studied in [46,47] does not use the enlargement of the vector field in consideration, then in this sense our algorithm has it as particular instance. Moreover, our algorithm also merges into algorithms studied in [5,42]. It is worth highlighting that the use of enlargement in the proximal subproblem to define the next iteration of the algorithm has the advantage of providing more latitude and more robustness to the algorithm, as explained in [11]. The concept of enlargement of monotone operators in linear spaces has been successfully employed for a wide range of purposes; see [10] and its reference therein. The extension of this concept to Riemannian context has been presented in [5]. As an application, from the our iterative scheme we obtain an inexact proximal point method for constrained optimization problems, equilibrium problems and nonlinear optimization problems on Hadamard manifolds. To the best of our knowledge, our approach brings a first proposal of an inexact proximal method for equilibrium problems on Hadamard manifolds.

It is important to note that an exact version was first introduced in [12] and, by using the theory of variational inequality, has been reaffirmed for genuine Hadamard manifolds in [48].

The organization of the paper is as follows. In Section 2, notations basic results used thought the paper are presented. In Section 3, the inexact proximal point method for variational inequalities is presented and its convergence properties are studied. As an application, in Section 4 an inexact proximal point method for constrained optimization problems, equilibrium problems and nonlinear optimization problems are obtained. In Section 5 concluding remarks are presented.

2 Preliminaries

The aim of the section is to recall some fundamental properties and notations of Riemannian geometry, as well as the notions of monotonicity and maximal monotonicity and enlargement of multivalued vector fields on Hadamard manifolds; for more details see [4].

2.1 Notation and terminology

In this paper, all manifolds $M$ are assumed to be Hadamard finite dimensional. Next we recall a fundamental inequality of Hadamard manifolds that we will need

$$d^2(p_1,p_3) + d^2(p_3,p_2) - 2 \left( \exp_{p_3}^{-1} p_1, \exp_{p_3}^{-1} p_2 \right) \leq d^2(p_1,p_2), \quad p_1,p_2,p_3 \in M, \quad (1)$$

where $\exp_p(\cdot)$ denotes the exponential map, $\exp_p^{-1}(\cdot)$ its inverse and $d(\cdot,\cdot)$ is the Riemannian distance. The function $f : M \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be proper if $\text{dom} f := \{ p \in M : f(p) < +\infty \} \neq \emptyset$ and it is convex on a convex set $\Omega \subset \text{dom} f$ if for any geodesic segment $\gamma$ in
Denotes by \( \mathbf{V} \) valued vector fields \( X, Y \) and \( \sum_k \) for all Lemma 5. Let \( m \) of The indicator function \( \in \{ p, q \} \Omega \), the composition \( f \) λ > \( \langle X \rangle \) satisfying \( X \). Moreover, a monotone vector field \( \lambda > \) (Lemma 3. Let \( \delta \) Proposition 4. Lemma 3 is a monotone multivalued vector field. Furthermore, if \( \lambda > \) dom \( X \) dom \( \lambda > \) \( \delta \). If \( \lambda > \) \( \delta \) \( \lambda > \) \( \delta \). The proof of the next result follows from \( \lambda > \) \( \delta \) \( \lambda > \) \( \delta \). The proof of the next result can be found in \( \lambda > \) Theorem 5.1.[1] Theorem 1. Let \( f \) be a proper, lower semicontinuous and convex function on \( M \). The subdifferential \( \partial f \) is a monotone multivalued vector field. Furthermore, if \( \lambda > \) \( \delta \) \( \lambda > \) \( \delta \). Let \( \lambda > \) \( \delta \). The function \( \lambda > \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \Omega \) \( \delta \O
2.2 Enlargement of Monotone Vector Fields

In this section we recall some concepts and results related to enlargement of vector fields in the Hadamard manifolds setting, for details see [5]. Throughout this section $X$ and $Y$ denote multivalued monotone vector fields on $M$ and $\epsilon \geq 0$.

**Definition 1.** The enlargement of vector field $X^\epsilon : M \rightrightarrows TM$ associated to $X$ is defined by

$$X^\epsilon (p) := \left\{ u \in T_p M : \left\langle P_{qq}^{-1} u - v, \exp_q^{-1} p \right\rangle \geq -\epsilon, \quad q \in \text{dom} X, \quad v \in X(q) \right\}, \quad p \in \text{dom} X.$$

Next proposition shows that $X^\epsilon$ effectively constitutes an enlargement to $X$.

**Proposition 6.** $X \subset X^\epsilon$ and $\text{dom} X \subset \text{dom} X^\epsilon$. In particular, if $\text{dom} X = M$ then $\text{dom} X^\epsilon = \text{dom} X$. Moreover, if $X$ is maximal then $X^0 = X$.

In the next three propositions we state the main properties used throughout our presentation, which are extensions to the Riemannian context of the corresponding one of linear setting; see [11].

**Proposition 7.** $X^{\epsilon_2} \subset X^{\epsilon_1}$, for all $\epsilon_1 \geq \epsilon_2 \geq 0$, and $X^{\epsilon_1} + Y^{\epsilon_2} \subset (X + Y)^{\epsilon_1 + \epsilon_2}$.

**Proposition 8.** Let $\{\epsilon^k\}$ be a sequence of positive numbers, and $\{(p^k, u^k)\}$ be a sequence in $TM$. If $\tau = \lim_{k \to \infty} \epsilon^k$, $\overline{\tau} = \lim_{k \to \infty} \overline{p^k}$, $\overline{\tau} = \lim_{k \to \infty} u^k$, and $u^k \in X^{\epsilon^k}(p^k)$ for all $k$, then $\overline{\tau} \in X^{\overline{\tau}}(\overline{\tau})$.

**Proposition 9.** If $X$ is maximal monotone and $\text{dom} X = M$, then $X^\epsilon$ is bounded on bounded sets, for all $\epsilon \geq 0$.

3 Inexact Proximal Point Method for Variational Inequalities

In this section, we introduce an inexact version of the proximal point method for variational inequalities in Hadamard manifolds. It is worth noting that, the variational inequality problem was first introduced in [33], for single-valued vector fields on Hadamard manifolds, and in [29] for multivalued vector fields in Riemannian manifolds.

Let $X : M \rightrightarrows TM$ be a multivalued vector field and $\Omega \subset M$ be a nonempty set. The *variational inequality problem* for $X$ and $\Omega$, denoted by VIP$(X,\Omega)$, consists of finding $p^* \in \Omega$ such that there exists $u \in X(p^*)$ satisfying

$$\langle u, \exp_{p^*}^{-1} q \rangle \geq 0, \quad q \in \Omega.$$ (3)

Using (2), i.e., the definition of normal cone to $\Omega$, VIP$(X,\Omega)$ becomes the problem of finding an $p^* \in \Omega$ that satisfies the inclusion

$$0 \in X(p) + N_\Omega(p).$$ (4)

**Remark.** In particular, if $\Omega = M$, then $N_\Omega(p) = \{0\}$ and VIP$(X,\Omega)$ are problems with regard to finding $p^* \in \Omega$ such that $0 \in X(p^*)$.

Hereafter, $S(X, \Omega)$ denotes the solution set of the inclusion (4). We require the following three assumptions:

**A1.** $\text{dom} X = M$ and $\Omega$ closed and convex;

**A2.** $X$ is maximal monotone;
A3. $S(X, \Omega) \neq \emptyset$.

In the following we state two algorithms to solve (3) or equivalently (4). To state the algorithms take two real numbers $\tilde{\lambda}$ and $\hat{\lambda}$ satisfying $0 < \tilde{\lambda} \leq \hat{\lambda}$ and four exogenous sequences of positive real numbers, $\{\lambda_k\}, \{\sigma_k\}, \{\theta_k\}$ and $\{\epsilon_k\}$ satisfying

$$
\hat{\lambda} \leq \lambda_k \leq \tilde{\lambda}, \quad \sum_{k=1}^{+\infty} \epsilon_k < +\infty, \quad \sum_{k=1}^{+\infty} \sigma_k < +\infty, \quad \sum_{k=1}^{+\infty} \theta_k < +\infty. \tag{5}
$$

The first version of inexact proximal point method for solving (4) is defined as follows:

**Algorithm 1. Inexact proximal point method with absolute error tolerance**

0. Take $\{\lambda_k\}, \{\epsilon_k\}$ and $\{\theta_k\}$ satisfying (5), and $p^0 \in \Omega$. Set $k = 0$.

1. Given $p^k \in \Omega$, compute $p^{k+1} \in \Omega$ and $e^{k+1} \in T_{p^{k+1}} M$ such that

$$
e^{k+1} \in X\epsilon_k(p^{k+1}) + N_{\Omega}(p^{k+1}) - \lambda_k \exp_{p^{k+1}}^{-1} p^k, \tag{6}
$$

$$
\|e^{k+1}\| \leq \theta_k. \tag{7}
$$

2. If $p^k = p^{k+1}$, then stop; otherwise, set $k \leftarrow k + 1$, and go to step 1.

It is worth noting that Algorithm 1 is inexact in two sense, namely, $X\epsilon_k$ is an enlargement of the vector field $X$ and each iteration $p^{k+1}$ is an approximated solution of the vectorial inclusion $0 \in X\epsilon_k(p) + N_{\Omega}(p) - \lambda_k \exp_{p^{k+1}}^{-1} p^k$ satisfying the error criterion (7). Note that for $\theta_k = 0$ in (7), Algorithm 1 merges into algorithm introduced in [5]. The error criterion (7) was introduced in the celebrated paper [38] to analyze an inexact version of the proximal point method to find zeroes of maximal monotone operators in linear context; see [46, 47] for a generalization to Riemannian setting.

The second version of inexact proximal point method for solving (4) is defined as follows:

**Algorithm 2. Inexact proximal point method with relative error tolerance**

0. Take $\{\lambda_k\}, \{\epsilon_k\}$ and $\{\sigma_k\}$ satisfying (5), and $p^0 \in \Omega$. Set $k = 0$.

1. Given $p^k \in \Omega$, compute $p^{k+1} \in \Omega$ and $e^{k+1} \in T_{p^{k+1}} M$ such that

$$
e^{k+1} \in X\epsilon_k(p^{k+1}) + N_{\Omega}(p^{k+1}) - \lambda_k \exp_{p^{k+1}}^{-1} p^k, \tag{8}
$$

$$
\|e^{k+1}\| \leq \sigma_k d(p^k, p^{k+1}). \tag{9}
$$

2. If $p^k = p^{k+1}$, then stop; otherwise, set $k \leftarrow k + 1$, and go to step 1.
First we remark that Algorithm 2 differs from Algorithm 1 only in the errors criterion adopted, more precisely, between (7) and (9). The error criterion (9) was also introduced in [38] in linear setting. When $\sigma_{k+1} = 0$ in (9), Algorithm 2 merges into algorithm introduced in [5]. A variant of error criterium (9) to analyze (8), for the particular case $\epsilon_k \equiv 0$, has appeared in [42].

In the following we present the well-definedness and convergence properties of the sequence $\{p^k\}$ generated by Algorithms 1 and 2. We begin with the well-definition.

**Theorem 10.** Each sequence $\{p^k\}$ generated by Algorithms 1 or 2 is well defined.

**Proof.** First of all note that for each $p^k \in \Omega$ and $\lambda_k > 0$, Lemma 3 implies that

$$0 \in X(p) + N_\Omega(p) - \lambda_k \exp_{p^k}^{-1} p^k,$$

has an unique solution in $\Omega$. Since $\text{dom} X = M$, Proposition 6 and item (i) of Proposition 9 imply that $X(p) \subseteq X^\epsilon(p)$ for all $p \in M$ and $\epsilon \geq 0$. Therefore, letting $\epsilon^{k+1} = 0$ and $p^{k+1}$ as the solution of (10), we conclude they also satisfy (6)-(9), which prove the well definition. \hfill $\square$

**Remark.** Using Proposition 7 we conclude that $N_\Omega \subset N_\Omega^0$. Thus, from second. part of Proposition 7 we have $X^\epsilon_k + N_\Omega \subset (X + N_\Omega)^\epsilon_k$, for all $k = 0, 1, \ldots$. Therefore, using (8), the following inequality holds

$$e^{k+1} \in (X + N_\Omega)^\epsilon_k (p^{k+1}) - \lambda_k \exp_{p^{k+1}}^{-1} p^k, \quad k = 0, 1, \ldots.$$  

Note that the condition (11) is less restrictive than (6) and (8).

From now on, unless explicitly stated, $\{p^k\}$ denotes the sequence generated by Algorithm 1 or 2. It is worth to noting that $p^k = p^{k+1}$ implies $p^{k+1} \in S(X, \Omega)$. Thus, without loss of generality, we assume that $\{p^k\}$ is infinite.

### 3.1 Convergence analysis

In this section our aim is to prove the convergence of the sequence $\{p^k\}$ to a point in $S(X, \Omega)$. For that we first need some auxiliary results. We begin establishing a useful inequality.

**Lemma 11.** For all $\eta > 0$, the following inequality holds

$$\left[1 - \frac{\eta}{\lambda_k}\right] \| d^2(q, p^{k+1}) - d^2(p^k, p^{k+1}) + \frac{1}{\eta \lambda_k} \| \epsilon^{k+1} \|^2 + \frac{2}{\lambda_k} \epsilon_k, \quad k = 0, 1, \ldots.$$  

**Proof.** First we note that (8) or (6) is equivalent to

$$e^{k+1} + \lambda_k \exp_{p^{k+1}}^{-1} p^k \in (X + N_\Omega)^\epsilon_k (p^{k+1}), \quad k = 0, 1, \ldots.$$  

Considering that $\exp_{p^{k+1}}^{-1} \exp_q^{-1} p^{k+1} = -\exp_p^{-1} q$ and the parallel transport being isometric, the last inclusion together with Definition 4 yields

$$-\left\langle e^{k+1} + \lambda_k \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}} q \right\rangle + \left\langle v, -\exp_q^{-1} p^{k+1} \right\rangle \geq -\epsilon_k,$$

for all $q \in \Omega$, $v \in (X + N_\Omega)(q)$ and all $k = 0, 1, \ldots$. In particular, if $q \in S(X, \Omega)$, then $0 \in (X + N_\Omega)(q)$ and the last inequality becomes

$$-\left\langle e^{k+1} + \lambda_k \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} q \right\rangle \geq -\epsilon_k.$$
for all $q \in S(X, \Omega)$ and all $k = 0, 1, \ldots$. Using the last inequality and (1) with $p_1 = p^k$, $p_2 = q$, and $p_3 = p^{k+1}$, along with some algebraic calculations, we obtain

$$
\frac{2}{\lambda_k} \left( e^{k+1} \exp^{-1}_{p^{k+1}} q - \epsilon_k \right) \leq d^2(q, p^k) - d^2(p^k, p^{k+1}) - d^2(q, p^{k+1}),
$$
for all $q \in S(X, \Omega)$ and all $k = 0, 1, \ldots$. The last inequality gives

$$
d^2(q, p^{k+1}) \leq d^2(q, p^k) - d^2(p^k, p^{k+1}) + \frac{2}{\lambda_k} \left( e^{k+1} \exp^{-1}_{p^{k+1}} q \right) + \frac{2\epsilon_k}{\lambda_k}
$$
for all $q \in S(X, \Omega)$ and all $k = 0, 1, \ldots$. On the other hand, some algebraic manipulations yields

$$
- \left( e^{k+1} \exp^{-1}_{p^{k+1}} q \right) \leq \frac{1}{2\eta} \|e^{k+1}\|^2 + \frac{1}{2}\eta d^2(p^{k+1}, q).
$$

Therefore, combining two previous last inequalities yields the inequality of the lemma. \hfill \Box

**Corollary 12.** Let $\{p^k\}$ be generated by Algorithms 7. Then, there exists a $\bar{k} \in \mathbb{N}$ such that, for all $k \geq \bar{k}$, there holds

$$
d^2(q, p^{k+1}) \leq \left( 1 + \frac{2\theta_k}{\lambda} \right) d^2(q, p^k) - d^2(p^k, p^{k+1}) + \frac{2\theta_k}{\lambda}(\theta_k + 2\epsilon_k).
$$

**Proof.** First, applying Lemma 11 with $\eta = \theta_k$ and then using (7) yields

$$
\left( 1 - \frac{\theta_k}{\lambda_k} \right) d^2(q, p^{k+1}) \leq d^2(q, p^k) - d^2(p^k, p^{k+1}) + \frac{\theta_k}{\lambda_k} + \frac{2\epsilon_k}{\lambda_k}, \quad k = 0, 1, \ldots,
$$

It follows from (5) that there exists a $\bar{k} \in \mathbb{N}$ such that $0 \leq \theta_k < \lambda_k/2$, for all $k \geq \bar{k}$. Thus, we conclude from the last inequality that

$$
d^2(q, p^{k+1}) \leq \left( 1 + \frac{\theta_k}{\lambda_k} \right) d^2(q, p^k) - \frac{\lambda_k}{\lambda_k - \theta_k} d^2(p^{k+1}, p^k) + \frac{1}{\lambda_k - \theta_k}(\theta_k + 2\epsilon_k), \quad k \geq \bar{k},
$$
and the deride inequality follows by using again $0 \leq \theta_k < \lambda_k/2$ and first inequality in (5). \hfill \Box

**Corollary 13.** Let $\{p^k\}$ be generated by Algorithms 2. Then, there exists a $\bar{k} \in \mathbb{N}$ such that, for all $k \geq \bar{k}$, there holds

$$
d^2(q, p^{k+1}) \leq \left( 1 + \frac{2\sigma_k}{\lambda} \right) d^2(q, p^k) - d^2(p^{k+1}, p^k) + \frac{4}{\lambda} \epsilon_k.
$$

**Proof.** Applying Lemma 11 with $\eta = \sigma_k$ and using (11), we conclude that

$$
\left( 1 - \frac{\sigma_k}{\lambda_k} \right) d^2(q, p^{k+1}) \leq d^2(q, p^k) - \left( 1 - \frac{\sigma_k}{\lambda_k} \right) d^2(p^{k+1}, p^k) + \frac{2}{\lambda_k}\epsilon_k, \quad k = 0, 1, \ldots
$$
On the other hand, the forth inequality in (5) implies that there exists a $\bar{k} \in \mathbb{N}$ such that $0 < \sigma_k < \lambda_k/2$, for all $k \geq \bar{k}$. Hence, using (14) together with the first inequality in (5), we obtain the desired inequality. \hfill \Box
Proposition 14. Let \( \{p^k\} \) be a sequence generated by Algorithms 1 or 2. Then, the following statement hold:

(a) The sequence \( \{d(p^k, q)\} \) converges, for all \( q \in S(X, \Omega) \);

(b) The sequence \( \{p^k\} \) is bounded;

(c) \( \lim_{k \to \infty} d(p^{k+1}, p^k) = 0 \).

Proof. If \( \{p^k\} \) is generated by Algorithm 1 then Corollary 12 implies that

\[
d^2(q, p^{k+1}) \leq \left( 1 + \frac{2\theta_k}{\lambda} \right) d^2(q, p^k) + \frac{2}{\lambda} \left[ \theta_k + 2\epsilon_k \right].
\]

Hence, item (a) follows by applying Lemma 5 with \( \gamma_k = 2\theta_k/\lambda, \beta_k = 2[\theta_k + 2\epsilon_k]/\lambda, \zeta_k = d^2(q, p^k) \) and \( \zeta_{k+1} = d^2(q, p^{k+1}) \). On the other hand, if \( \{p^k\} \) is a sequence generated by Algorithm 2 then Corollary 13 gives

\[
d^2(q, p^{k+1}) \leq \left( 1 + \frac{2\sigma_k}{\lambda} \right) d^2(q, p^k) + \frac{4}{\lambda} \epsilon_k.
\]

Thus, item (a) follows by applying Lemma 5 with \( \gamma_k = 2\sigma_k/\lambda, \beta_k = 4\epsilon_k/\lambda, \zeta_k = d^2(q, p^k) \) and \( \zeta_{k+1} = d^2(q, p^{k+1}) \). The item (b) is an immediate consequence of item (a). The next task is to prove item (c). If \( \{p^k\} \) is generated by Algorithm 1 then using again Corollary 12 we have

\[
d^2(p^{k+1}, p^k) \leq d^2(q, p^k) - d^2(q, p^{k+1}) + \frac{2\theta_k}{\lambda} d^2(q, p^k) + \frac{2}{\lambda} \left[ \theta_k + 2\epsilon_k \right].
\]

Now, note that (5) implies \( \lim_{k \to \infty} \sigma_k = 0 \) and \( \lim_{k \to \infty} \epsilon_k = 0 \). Therefore, item (a) together with the last inequality imply item (c). If \( \{p^k\} \) is generated by Algorithm 2 then it follows from Corollary 13 that

\[
d^2(p^{k+1}, p^k) \leq d^2(q, p^k) - d^2(q, p^{k+1}) + \frac{2\sigma_k}{\lambda} d^2(q, p^k) + \frac{4}{\lambda} \epsilon_k.
\]

On the other hand, using (6) we have \( \lim_{k \to \infty} \sigma_k = 0 \) and \( \lim_{k \to \infty} \epsilon_k = 0 \). Therefore, item (a) together with the last inequality imply item (c).

Theorem 15. Let \( \{p^k\} \) be generated by Algorithms 1 or 2. Then, \( \{p^k\} \) converges to a point \( p^* \in S(X, \Omega) \).

Proof. Since \( \{p^k\} \subset \Omega \) and \( \Omega \) is closed, item (b) of Proposition 14 implies that there exists \( \bar{p} \in \Omega \) a cluster point of \( \{p^k\} \). Let \( \{p^{k_j}\} \) be a subsequence of \( \{p^k\} \) such that \( \lim_{j \to \infty} p^{k_j} = \bar{p} \). Our first aim is to prove that \( \bar{p} \in S(X, \Omega) \). For that, using inclusion (5) or (6), there exist \( u^{k_j+1} \in X^{\epsilon_k}(\bar{p}^{k_j+1}) \) such that

\[
e^{k_j+1} + \lambda_{k_j} \exp^{-1} p^{k_j+1} u^{k_j+1} \in N_{\Omega}(\bar{p}^{k_j+1}), \quad j = 0, 1, \ldots \quad (15)
\]

On the other hand, item (c) of Proposition 14 implies that \( \lim_{j \to \infty} u^{k_j+1} = \bar{u} \). Moreover, considering that \( \{\theta_k\} \) and \( \{\sigma_k\} \) are bounded, it follows from (7), respectively (9), and item (c) of Proposition 14 that \( \lim_{k \to \infty} \epsilon_k = 0 \). Letting \( \tilde{\epsilon} = \sup_{k \geq k_0} \epsilon_k \), the first part of Proposition 7 implies that \( u^{k_j+1} \in X^{\epsilon_{k_j}}(\bar{u}^{k_j+1}) \subset X^{\epsilon_{k_j}}(p^{k_j+1}) \), for all \( j = 0, 1, \ldots \). Thus, considering that \( \{p^k\} \) is bounded, we conclude from Proposition 9 that \( \{u^{k_j+1}\} \) is also bounded. Without loss of generality we assume that \( \lim_{j \to \infty} u^{k_j+1} = \bar{u} \). Hence, taking into account that \( \lim_{k \to \infty} \epsilon_k = 0 \), \( \lim_{j \to \infty} \epsilon_{k_j} = 0 \).
\[ u^{k+1} \in X^{\varepsilon_{kj}}(p^{k+1}), \text{ for all } j = 0, 1, \ldots, \] it follows from Proposition 6 and Proposition 8 that \( \bar{u} \in X^{0}(\bar{p}) = X(\bar{p}) \). Therefore, taking limit in (15) and considering Proposition 4 we conclude that \( -\bar{u} \in N_{\Omega}(\bar{p}) \). Due to \( \bar{u} \in X(\bar{p}) \) we have \( 0 \in X(\bar{p}) + N_{\Omega}(\bar{p}) \), which implies that \( \bar{p} \in S(X, \Omega) \). Moreover, using item (a) of Proposition 14 we obtain that the sequence \( \{d(p^{k}, \bar{p})\} \) converges. Considering that \( \lim_{j \to \infty} p^{kj} = \bar{p} \), we have \( \lim_{k \to \infty} d(p^{k}, \bar{p}) = 0 \). Therefore, we conclude that \( \lim_{k \to \infty} b^{k} = \bar{p} \), which concludes the proof.

\( \square \)

**Remark.** In [46,47] is presented an inexact version of the proximal point method for to find singularity of a vector field on Hadamard manifolds. These papers differs from the present paper in two ways, namely, [46,47] use only absolute summable error criteria and the enlargement \( X^\varepsilon \) of \( X \) was not considered. It is worth noting that, the enlargement \( X^\varepsilon \) is an (outer) approximation to \( X \). Consequently, even in the linear setting, the proximal subproblem using the enlargement has the advantage of providing more latitude and more robustness to the methods used for solving it; see [10,11].

### 4 Applications

The general Problem (11) has as particular instances the optimization problem, equilibrium problem and nonlinear optimization problem. The aim of this section is to apply the results obtained in the previous section to these particular instances. For each problem studied, a version of the Algorithm 2 is stated to solve it. Since a version of the Algorithm 2 can be stated following the same idea, it will be omitted.

#### 4.1 Inexact proximal point method for optimization

In this section, we apply the results of the previous section to obtain an inexact proximal point method for the constrained optimization problems in Hadamard manifolds. Given a closed and convex set \( \Omega \subset M \) and a convex function \( f : M \to \mathbb{R} \), the constrained optimization problem consists of

\[
\min f(p), \; p \in \Omega. \tag{16}
\]

The problem in (16) is equivalently stated as follows

\[
\min (f + \delta_{\Omega})(p), \; p \in M. \tag{17}
\]

where \( \delta_{\Omega} \) is the indicate function of \( \Omega \). Hereafter, \( S(f, \Omega) \) denotes the solution set of the problem in (16). It well know that (17) can be stated as the variational inequality problem (11). In fact, first note that due to convexity of the set \( \Omega \) and of the function \( f \) we conclude that \( f + \delta_{\Omega} \) is also convex. Thus, by using Proposition 2 we have

\[
\partial(f + \delta_{\Omega})(p) = \partial f(p) + N_{\Omega}(p), \quad p \in \Omega.
\]

Therefore, \( p^* \in S(f, \Omega) \) if, and only if, \( 0 \in \partial f(p^*) + N_{\Omega}(p^*) \). Therefore, (17) is equivalent to find an \( p^* \in \Omega \) satisfying the inclusion

\[
0 \in \partial f(p) + N_{\Omega}(p). \tag{18}
\]

In order to present a version of Algorithm 2 to solve (16) or equivalently (18), we need to consider the enlargement of the subdifferential of \( f \), denoted by \( \partial^\varepsilon f : M \rightrightarrows TM \), which is defined by

\[
\partial^\varepsilon f(p) := \left\{ u \in T_p M : \left\langle P_q^{-1}u - v, \exp_q^{-1}p \right\rangle \geq -\varepsilon, \; q \in M, \; v \in \partial f(q) \right\}, \quad \varepsilon \geq 0.
\]
To state the version of Algorithm 2 to solve (16) or equivalently (18), take three exogenous sequences of nonnegative real numbers, \( \{\lambda_k\}, \{\epsilon_k\} \) and \( \{\sigma_k\} \) satisfying (5). Then, the inexact proximal point method for the optimization problem (18) is introduced as follows:

Algorithm 3. Inexact proximal point method for optimization problems

0. Take \( \{\lambda_k\}, \{\epsilon_k\} \) and \( \{\sigma_k\} \) satisfying (5) and \( p^0 \in \Omega \). Set \( k = 0 \).

1. Given \( p^k \in \Omega \), compute \( p^{k+1} \in \Omega \) and \( e^{k+1} \in T_{p^{k+1}} M \) such that
\[
e^{k+1} \in \partial^\epsilon f(p^{k+1}) + N_\Omega(p^{k+1}) - 2\lambda_k \exp^{-1}_{p^{k+1}} x^k,
\]
\[
\|e^{k+1}\| \leq \sigma_k d(p^k, p^{k+1}),
\]
(19)
(20)

2. If \( p^k = p^{k+1} \), then stop; otherwise, set \( k \leftarrow k + 1 \), and go to step 1.

Remark. In case, \( \epsilon_k = 0 \), \( e^{k+1} \equiv 0 \) and \( \Omega = M \), the Algorithm 3 generalize the algorithm proposed by Ferreira and Oliveira [21], and the method (5.15) of Chong Li et. al. [28]. For \( \epsilon_k = 0 \), inexact variations of (19) with absolute errors can be found in [46] and [42] for relative erro. Finally, letting \( e^{k+1} \equiv 0 \), the Algorithm 3 retrieves the one presented in [5].

In the following we state a convergence result for the sequence generated by (19) and (20). First note that, for considering that \( \text{dom} f = M \), Theorem 1 implies that \( \partial f \) is maximal monotone. Hence, from Proposition 2 we have \( N_\Omega = \partial \delta_\Omega \). Therefore, by applying Theorems 10 and 15 with \( X = \partial f \) we obtain the following theorem.

**Theorem 16.** Assume that \( S(f, \Omega) \neq \emptyset \). Then, the sequence \( \{p^k\} \) generated by (19) and (20) is well defined and converges to a point \( p^* \in S(f, \Omega) \).

In the next remark, we highlight the advantage of using the enlargement of the subdifferential instead of the \( \epsilon \)-subdifferential of \( f \).

**Remark.** The \( \epsilon \)-subdifferential of \( f \), denoted by \( \partial_\epsilon f : M \rightrightarrows TM \), is given by
\[
\partial_\epsilon f(p) := \left\{ u \in T_p M : f(q) \geq f(p) + \left\langle u, \exp^{-1}_p q \right\rangle - \epsilon, \ q \in M \right\}, \quad \epsilon \geq 0.
\]

As can be seen in [5], the enlargement of the subdifferential of \( f \) is bigger than its \( \epsilon \)-subdifferential, i.e., for each \( p \in M \), there holds \( \partial_\epsilon f(p) \subseteq \partial f(p) \). Taking into account that this inclusion may be strict, we can to state that the iteration in (19) using the enlargement of \( \partial f(\cdot) \) has the advantages of providing more latitude and more robustness than a method using the \( \epsilon \)-subdifferential of \( f \).
4.2 Inexact proximal point method for equilibrium problems

In this section, by using the results of Section 3, we present a version of the inexact proximal point method for equilibrium problems in Hadamard manifolds. For that we need some preliminaries. Let \( C \subset M \) be a nonempty, closed and convex set and \( F : M \times M \to \mathbb{R} \) be a bifunction satisfying the following standard assumptions:

**H1.** \( F(\cdot, y) : M \to \mathbb{R} \) is upper semicontinuous for all \( y \in M \);

**H2.** \( F(x, \cdot) : M \to \mathbb{R} \) is convex, for all \( x \in M \);

**H3.** \( F \) is monotone on \( C \), i.e., \( F(x, y) + F(y, x) \leq 0 \), for all \( x, y \in C \);

**H4.** \( F(x, x) = 0 \), for all \( x \in C \).

The equilibrium problem associated to the set \( C \) and the bifunction \( F \), denoted by EP\((F, C)\), is stated as follows: Find \( x^* \in C \) such that

\[
F(x^*, y) \geq 0, \quad y \in C. \tag{21}
\]

Denote by \( S(F, C) \) the solution set of the EP\((F, C)\). By using [48, Proposition 3.3] we obtain that (21) is equivalent to find an \( p^* \in \Omega \) satisfying the inclusion

\[
0 \in \partial_2 F(p, \cdot)(p) + N_{\Omega}(p), \tag{22}
\]

where \( \partial_2 F(p, \cdot) \) denotes the subdifferential of \( F \) with respect to the second argument. We also assume that

**H5.** The set \( S(F, C) \) is nonempty.

**Remark.** Assumptions H1-H4 are standard for the study of equilibrium problems in linear spaces, see [22-24]. It is worth noting that assumption H5 can be reached under suitable condition on the set \( C \) or the bifunction \( F \); papers addressing this issue include, but are not limited to, [4,13,27,48,50]. To the best of our knowledge, our approach brings a first proposal of an inexact proximal method for equilibrium problems. It is worth noting that an exact version has been first introduced in [12] and, by using variational inequality theory, reaffirmed for genuine Hadamard manifolds in [48].

Before presenting a version of Algorithm 2 to solve (21), or equivalently (22), we need to consider the enlargement of the subdifferential of \( F \) with respect to the second argument, denoted by \( \partial_2^e F(z, \cdot) : M \rightrightarrows TM \), for each fixed \( z \in C \), which is introduced as follows

\[
\partial_2^e F(z, x) := \left\{ w \in T_p M : F(z, y) \geq F(z, x) + \langle w, \exp_x^{-1} y \rangle - \epsilon, \; y \in M \right\}. \tag{23}
\]

To state the version of Algorithm 2 to solve the equilibrium problem (21) or equivalently (22), take three exogenous sequences of nonnegative real numbers, \( \{\lambda_k\} \), \( \{\epsilon_k\} \) and \( \{\sigma_k\} \) satisfying (5). In case, the inexact proximal point method for solving (21) is introduced as follows:

**Algorithm 4.** Inexact proximal point method for equilibrium problems
0. Take \( \{\lambda_k\}, \{\epsilon_k\} \) and \( \{\sigma_k\} \) satisfying (5), \( x^0 \in C \) and \( \sigma > 1 \). Set \( k = 1 \).

1. Given \( x^k \in C \), compute \( x^{k+1} \in C \) and \( e^k \in T_xM \) such that

\[
e^{k+1} \in \partial^2_2 F(x^{k+1}, x^{k+1}) + N_{\Omega}(x^{k+1}) - \lambda_k \exp^{-1}_{x^{k+1}} x^k, \quad (24)
\]

\[
\|e^{k+1}\| \leq \sigma_k d(p^k, p^{k+1}). \quad (25)
\]

2. If \( x^{k-1} = x^k \) or \( x^k \not\in \), then stop; otherwise, set \( k \leftarrow k + 1 \), and go to step 1.

Remark. The Algorithm 4 can be seen as an inexact version of the following iterative scheme considered in [12]: For \( x^k \in C \), compute \( x^{k+1} \in C \) such that

\[
F(x^{k+1}, x) - \lambda_k (\exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} x) \geq 0, \quad x \in C.
\]

Indeed, given \( x^k \) and \( x^{k+1} \in C \) satisfying (26) we have

\[
F(x^{k+1}, x) + \delta_C(x) - \left(F(x^{k+1}, x^{k+1}) + \delta_C(x^{k+1}) + \left(\lambda_k \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} x\right)\right) \geq 0, \quad (27)
\]

for all \( x \in M \). Since the function \( p \mapsto (F(p^{k+1}, \cdot) + \delta_C(\cdot))(p) \) is convex, it follows from the definition of the subdifferential that \( \lambda_k \exp^{-1}_{x^{k+1}} x^k \in \partial^2_2 F(x^{k+1}, \cdot) + \delta_C(\cdot)(x^{k+1}) \). Hence, by using Proposition 2 we obtain

\[
0 \in \partial^2_2 F(x^{k+1}, x^{k+1}) - \lambda_k \exp^{-1}_{x^{k+1}} x^k + N_{\Omega}(x^{k+1}),
\]

which implies that \( x^k \) and \( x^{k+1} \) also satisfy (24) and (25) with \( e^{k+1} = 0 \) and \( \epsilon_k = 0 \).

In the following we state a convergence result for the sequence generated by (24) and (25). First note that, for considering that \( \text{dom} F(p, \cdot) = M \), Theorem 1 implies that \( \partial^2_2 F(p, \cdot) \) is maximal monotone, for all \( p \in M \). Moreover, Proposition 2 implies that \( N_{\Omega} = \partial \delta_{\Omega} \). Therefore, by applying Theorems 10 and 15 with \( X = \partial^2_2 F(p, \cdot) \) we obtain the following theorem.

**Theorem 17.** The sequence \( \{p^k\} \) generated by (24) and (25) is well defined and converges to a point \( p^* \in S(f, \Omega) \).

**Proof.** First note that, for considering that \( \text{dom} F(p, \cdot) = M \), Theorem 1 implies that \( \partial^2_2 F(p, \cdot) \) is maximal monotone, for all \( p \in M \). Moreover, Proposition 2 implies that \( N_{\Omega} = \partial \delta_{\Omega} \). Therefore, by applying Theorems 10 and 15 with \( X = \partial^2_2 F(p, \cdot) \) we obtain the following theorem.

4.3 Inexact proximal point method for nonlinear optimization problem

In this section, we apply the results of the previous section to obtain an inexact proximal point method for the nonlinear optimization problem in the form

\[
\min f(p), \quad p \in \{p \in M : g(p) \leq 0, \quad h(p) = 0\},
\]

where \( M \) is a Hadamard manifold, the objective function \( f : M \rightarrow \mathbb{R} \) and the constraint functions \( g = (g_1, \ldots, g_m) : M \rightarrow \mathbb{R}^m \) and \( h = (h_1, \ldots, h_{\ell}) : M \rightarrow \mathbb{R}^\ell \) are assumed to be continuously
differentiable and convex. In order to state the problem (28) as the variational inequality problem in (4), we first recall the first-order necessary optimality conditions in Karush-Kuhn-Tucker (KKT) form. It is worth noting that recently the KKT conditions were addressed in [9]. Let \( \mathcal{L} : M \times \mathbb{R}^m_+ \times \mathbb{R}^\ell \rightarrow \mathbb{R} \) be the Lagrangian associated with (28) defined by

\[
\mathcal{L}(p, \mu, \lambda) := f(p) + \sum_{i=1}^m \mu_ig_i(p) + \sum_{j=1}^\ell \lambda_jh_j(p).
\]

(29)

Since the functions \( f, g \) and \( h \) are continuously differentiable, it follows from (29) that KKT conditions are given by

\[
\text{grad}_p \mathcal{L}(p, \mu, \lambda) := \text{grad}_f(p) + \sum_{i=1}^m \mu_ig_i(p) + \sum_{j=1}^\ell \lambda_jh_j(p) = 0 \quad (30)
\]

\[
g_i(p) \leq 0, \quad i = 1, \ldots, m \quad (31)
\]

\[
h_j(p) = 0, \quad j = 1, \ldots, \ell \quad (32)
\]

\[
\mu_ig_i(p) = 0, \quad i = 1, \ldots \quad (33)
\]

\[
\mu_i \geq 0, \quad i = 1, \ldots, m. \quad (34)
\]

Let \( \tilde{M} := M \times \mathbb{R}^m_+ \times \mathbb{R}^\ell \) be the product manifold with the induced product metric, for more details see [39]. Then, the tangent plane at \( \tilde{p} := (p, \mu, \lambda) \in \tilde{M} \) is \( T_{\tilde{p}}\tilde{M} \supset \mathbb{R}^m_+ \times \mathbb{R}^\ell \) and the exponential map \( \tilde{\exp}_{\tilde{p}} : T_{\tilde{p}}\tilde{M} \rightarrow \tilde{M} \) is given by

\[
\tilde{\exp}_{\tilde{p}}\tilde{w} := (\exp_p w, \mu + u, \lambda + v), \quad \tilde{w} := (w, u, v) \in T_{\tilde{p}}\tilde{M}.
\]

where \( \exp_p \) is the exponential map of \( M \) a \( p \in M \). Consequently, the inverse of \( \tilde{\exp}_{\tilde{p}} \) is given by

\[
\tilde{\exp}_{\tilde{p}}^{-1}\tilde{q} := (\exp_p^{-1}q, \nu - \mu, \zeta - \lambda), \quad \tilde{q} := (q, \nu, \zeta) \in \tilde{M}.
\]

Let \( \tilde{\Omega} := M \times \mathbb{R}^m_+ \times \mathbb{R}^\ell \subset \tilde{M} \), which is convex set in \( \tilde{M} \). In this case, the normal cone of the set \( \tilde{\Omega} \) at a point \( \tilde{p} \in \tilde{\Omega} \) is given by

\[
N_{\tilde{\Omega}}(\tilde{p}) := \left\{ \tilde{w} \in T_{\tilde{p}}\tilde{M} : \langle \tilde{w}, \tilde{\exp}_{\tilde{p}}^{-1}\tilde{q} \rangle \leq 0, \quad \tilde{q} \in \tilde{\Omega} \right\}.
\]

Since the functions \( f, g \) and \( h \) are continuously differentiable and convex, the definition [29] implies that the vector field \( X : M \times \mathbb{R}^m_+ \times \mathbb{R}^\ell \rightarrow TM \times \mathbb{R}^m_+ \times \mathbb{R}^\ell \) defined by

\[
X(\tilde{p}) := \begin{bmatrix}
\text{grad}_p \mathcal{L}(\tilde{p}) \\
-g(p) \\
h(p)
\end{bmatrix} \in T_{\tilde{p}}\tilde{M}, \quad \tilde{p} := (p, \mu, \lambda) \in \tilde{M}.
\]

(35)

is maximal monotone. Moreover, using the definition of normal cone of the set \( \tilde{\Omega} \) we conclude that (30)-(34) is equivalent to

\[
0 \in X(\tilde{p}) + N_{\tilde{\Omega}}(\tilde{p}), \quad \tilde{p} := (p, \mu, \lambda) \in \tilde{M}.
\]

(36)
To the definition of enlargement of $X$, consider the parallel transport on $\tilde{M}$ from $\tilde{p}$ to $\tilde{q}$ as being
\[
P_{\tilde{pq}}\tilde{w} := (P_{pq}w, u, v), \quad \tilde{w} := (w, u, v) \in T_{\tilde{p}}\tilde{M}.
\]
where $P_{pq}$ is the parallel transport on $M$ from $p$ to $q$. Then, the enlargement of vector field $X^\varepsilon : M \to TM$ associated to $X$ is defined by
\[
X^\varepsilon(\tilde{p}) := \left\{ \tilde{w} := (w, u, v) \in T_{\tilde{p}}\tilde{M} : \left\langle P_{\tilde{pq}}^{-1}\tilde{w} - \tilde{z}, \exp^{-1}_q\tilde{p} \right\rangle \geq -\varepsilon, \quad \tilde{q} \in M, \quad \tilde{z} \in X(\tilde{q}) \right\},
\]
for all $\tilde{p} := (p, \mu, \lambda) \in \tilde{M}$. Finally, taking three exogenous sequences of nonnegative real numbers, $\{\lambda_k\}$, $\{\varepsilon_k\}$ and $\{\sigma_k\}$ satisfying (5), the inexact proximal point method for solving (28) is introduced as follows:

Algorithm 5. Inexact proximal point method for nonlinear optimization problem

0. Take $\{\lambda_k\}$, $\{\varepsilon_k\}$ and $\{\sigma_k\}$ satisfying (5), $\tilde{p}^0 \in \tilde{\Omega}$ and $\sigma > 1$. Set $k = 1$.

1. Given $\tilde{p}^k \in \tilde{\Omega}$, compute $\tilde{p}^{k+1} \in C$ and $\tilde{e}^k \in T_{\tilde{p}^k}M$ such that
\[
\tilde{e}^{k+1} \in X^\varepsilon(\tilde{p}^{k+1}) + N_{\tilde{\Omega}}(\tilde{p}^{k+1}) - \lambda_k \exp^{-1}_{\tilde{p}^{k+1}}\tilde{p}^k,
\]
\[
\|\tilde{e}^{k+1}\| \leq \sigma_k d(\tilde{p}^k, \tilde{p}^{k+1}).
\]

2. If $\tilde{p}^{k-1} = \tilde{p}^k$ or $\tilde{p}^k \in \tilde{\Omega}$, then stop; otherwise, set $k \leftarrow k + 1$, and go to step 1.

First note that $X$ defined in (35) satisfies A1 and A2. Moreover, under suitable constraint qualifications $X$ also satisfies A2, see [9, Theorem 11]. Therefore, we can apply Theorem 15 to obtain the following result.

Theorem 18. The sequence $\{p^k\}$ generated by (38) and (39) is well defined and converges to a point $p^* \in S(f, \Omega)$.

5 Conclusions

In this paper we combine the ideas in [38] and [11] to introduce an inexact proximal point method for solving variational inequality problems on Hadamard manifolds. As a proposal of future work it would be interesting to study local version of our results on arbitrary Riemannian manifolds. Note that for this purpose, a local version of the formula (1) will be required.

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