Entanglement entropy in (2 + 1)-dimensional interacting theory: A dimensional reduction approach

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A formidable perspective in understanding collective quantum phenomena of a given many-body system is through its entanglement contents. Yet apart from well-established knowledge for free theories, so far much less is known about entanglement structure of interacting particles, especially for the cases beyond (1 + 1) dimension. Here, we develop an efficient scheme to study the entanglement entropy for (2+1)-dimensional quantum field theories, which is able to go beyond the non-interacting or conformal settings. Within this framework, we exactly derive the area-law entanglement entropy for (2 + 1)-dimensional free scalar field and Dirac field, which are consistent with the expectations from existing studies. As a concrete example of interacting theory, we investigate the entanglement entropy of (2 + 1)-dimensional Dirac fermion under a random magnetic field, which cannot be straightforwardly solved via previous approaches. We analytically prove the area-law entanglement entropy remains, with a minor modification of the area-law coefficient by disorder. Our analytical solution is further validated by the corresponding lattice simulation. Moreover, our scheme opens a window to derive the universal sub-leading terms in the entanglement entropy, which is connected to the dynamics of the renormalization group flows of the underlying theories. Therefore, this methodology advance not only offers a tool to exploring the correlations and quantum criticality, but also achieves a deepened understanding of the entanglement structure of quantum many-body systems.

I. INTRODUCTION

Entanglement expresses non-local connotations inherent to quantum mechanics, which has prompted remarkable insights into various fields of modern physics, bridging microscopic laws in quantum matters and macroscopic structure of spacetime [1–10]. Compared to the traditional methods by inspecting various order parameters and their responses to external perturbations, the study of many-body wave function via entanglement-based analysis developed in quantum information science is able to unveil novel properties in a large variety of collective quantum phenomena, ranging from the presence of topological order to the onset of quantum criticality [11–19]. Indeed, the overwhelming majority of works done so far are in support of entanglement-based analysis as a profitable tool to diagnose strong correlations for both in and out-of-equilibrium systems [20–26].

A simple way to analyze the entanglement structure is to separate a target system into subsystem \( A \) and its complement \( \overline{A} \), then a measure of the entanglement between \( A \) and \( \overline{A} \) is given by the von Neumann entropy associated with reduced density matrix \( \rho_{A} \): \( S = - \text{Tr}[\rho_{A} \ln \rho_{A}] \), which is also referred as the entanglement entropy (EE). Intriguingly, the EE is typically not an extensive quantity for many-body ground states, instead it usually satisfies an area-law [1, 2]. That is, the EE is proportional to the area of surface separating two subsystems, in sharp contrast with the thermal entropy that should obey the volume-law. The emergent area-law EE partially reflects a decay of correlation associated with quantum many-body states [27, 28]. Especially, in (1 + 1) dimension, the area-law is a character of massive theories with exponentially decaying correlations [29–32], and the logarithmic correction on the EE is expected for critical systems [11–13]. In higher dimensions, the area-law is believed to be generally hold in quantum field theories (QFTs), as a consequence of the locality of physical interactions. Such strong restriction of the entanglement alludes a deep connection with black hole physics [5, 33–35], and also offers crucial implications on the numerical computations on lattice models [22, 36–39], thus it is of vital importance.

In addition to the area-law contribution, the EE may host a sub-leading correction that encodes universal constraints of underlying QFTs free of ultraviolet (UV) cut-offs. It gives a unique global constant (see Eq. (1)) that measures the effective degrees of freedom of the theory, which should monotonically decrease along the renormalization group (RG) flows. This fact motivates an idea to inspect irreversible RG flows in general dimensions from the point view of quantum entanglement [55–60]. To be specific, for (2 + 1) dimension, this is related to a proposal of the irreversibility theorem under RG transformations, dubbed by the F-theorem [56, 57]. In this regard, an exact calculation the sub-leading correction of EE is quite informative for understanding the dynamics of RG flows of the underlying theory.

Nevertheless, to rigidly compute the EE of QFTs is challenging, despite some numerical answers have been found. In discrete (quasi)free bosonic or fermionic models, information-theoretical tools can largely capture an area-law EE, with mostly a logarithmic viola-

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In this paper, we aim to address the aforementioned questions, and explore an efficient scheme to study the EE for generic $(2 + 1)$-dimensional $(2 + 1)$D QFTs in flat spacetime. In particular, we develop a dimensional reduction method that transforms the calculation of higher-dimensional replicated Green’s function to infinite series of diagrams on lower-dimensional replica manifold. As a benchmark, we apply this method to free scalar field and Dirac field, and successfully reproduce the area-law EE that in agreement with previously known results. Moreover, as a concrete example beyond free theories, we investigate a $(2 + 1)$D Dirac field subjected to a random static gauge field (magnetic field). We analytically derive an area-law scaling of the EE, which signals the critical behavior of the ground state. In addition, we perform a numerical simulation on the corresponding lattice model after discretization, and obtain consistent entanglement scaling with the analytical solution. Last but not least, our calculations validate the F-theorem from a perturbative viewpoint, which sheds lights on the dynamics of RG flow of the $(2+1)$D interacting theories. In this context, our work not only offers a tool to faithfully calculate the EE for interacting QFTs in $(2 + 1)$ dimension, but also gives affirmative answers to many long-standing problems.

This paper is structured as follows. Sec. II A summarizes the existing methods of calculating EE in QFTs. We then discuss the general strategy of the dimensional reduction scheme in Sec. II B, and show how our idea is developed. As a benchmark, we apply our method to $(2 + 1)$D free scalar field in Sec. III and free Dirac field in Sec. IV, which faithfully recover the previously known area-law behavior of the EE. The calculation is further extended into an interacting theory of $(2 + 1)$D Dirac fermion under a random magnetic field in Sec. V, with an introduction to the background of investigating this model presented in Sec. V A. The derivation of an analytical solution of the EE in this interacting theory is addressed in Sec. V B, which is validated by the corresponding lattice simulation in Sec. V C. Furthermore, we discuss a quasiparticle picture to understand the observed area-law in the point of view of correlations in Sec. V D. At last, by connecting with the irreversibility

| Method | Advantages | Limitations |
|--------|------------|-------------|
| Real-Time Approach | Numerical determination of $\rho_A$ | Applicable to any theory with discretization on a lattice | Exponential growth of computational complexity in non-integrable systems |
| | Correlation matrix technique [40–42] | Polynomial computational complexity in system size | Restricted to the Gaussian states |
| | Resolvent technique [43] | Applicable to analytical solution with multi-regions | Restricted to $(1 + 1)$D free massless fermions and chiral bosons |
| Euclidean-Time Approach (Replica Trick) [3, 13, 44] | Heat-kernel technique [4, 45–47] | Applicable to analytical solution | Restricted to the quadratic order in quantum fluctuations |
| | Green’s function technique [13, 44] | Applicable to analytical solution; Applicable to higher dimensions | Capability and feasibility in interacting theories are yet to be explored |
| CFT Approach [11, 13, 45, 48, 49] | Applicable to universal prediction of EE in (1+1)D critical systems | Hard to be extended into massive theories and higher dimensions |
| Holographic Approach [5, 10, 50] | Reduced to a geometric problem | Limited by poor knowledge on the gravitational dual of given QFTs |
| Extensive mutual information model [51, 52] (a quasiparticle picture of entangled pairs) [27, 28] | Reduced to a geometric problem (much simpler than holography) | Does not correspond to an actual CFT beyond $(1 + 1)$ dimensions |
| Dimensional Reduction | Summation of $(1 + 1)$D EE with (effective) mass [53] | Quick evaluation of EE in free theories | Assuming EE to be extensive |
| | Summation of $(1 + 1)$D entropic-c function [54] | Quick evaluation of EE in free theories | Assuming EE to be extensive |
of RG flows, we point out the physical meaning of computing the universal sub-leading term of EE in Sec. VI. These results are concluded in Sec. VII, with outlooks for some open questions. Appendixes contain technical details about the current calculation and known results of the investigated model.

II. TECHNICAL OVERVIEW

In this work, we focus on the EE of a pure ground state. It is expected that the EE of a \((d + 1)D\) QFT \((d > 1)\) satisfies an area-law scaling \([13, 44]\)

\[
S \sim c_{\text{cut-off}} A / \epsilon^{d-1} + \gamma_d(m),
\]

where \(A\) is the area of the codimension-one entangling surface, \(\epsilon\) is a microscopic cut-off. Here, the leading term of EE depends on the UV cut-off, and its coefficient \(c_{\text{cut-off}}\) is sensitive to the choice of regularization scheme. It reflects the intrinsic nature of the system only when it becomes a function of the coupling constants. The second sub-leading term is expected to be finite as

\[
\gamma_d(m) \propto r_d A m^{d-1},
\]

where the coefficient \(r_d\) is expected to be universal. It appears when perturbing away from a quantum critical point by a finite correlation length, and the scaling might provide useful information for characterizing the universality. \([47, 71, 73]\]

As mentioned in introductory part, to determine the form of Eq. (1) for an interacting theory in dimensions higher than \((1 + 1)D\), is generally hard. Due to the difficulty of dealing with the intrinsic non-Gaussian features and possible divergence in loop corrections, the currently existing methods fail to give an exact answer of EE for general situations, which motivates us to develop a novel scheme for calculating it. Before presenting the detailed implementation, we start from a brief review of existing methods on calculating EE in QFTs (see Table. I). Based on this, we will show how the previous investigations inspire us to propose an exact dimensional reduction method. The connection to and distinction from the existing studies will be also addressed in detail.

A. Existing methods of calculating EE

1. Real-time approach

By definition, the calculation of EE requires the spectrum information of the reduced density matrix \(\rho_A\). The most straightforward way is to diagonalize it directly in Minkowski spacetime, which is so-called real-time approach. In principle, numerical methods (e.g. exact diagonalization technique) can determine the spectrum of \(\rho_A\) for any discretized system (lattice model). However, due to the exponentially growing Hilbert space, the computationally accessible size (typically about 10 – 20 qubits) is extremely small comparing with the realistic systems.

For free theories, the full information of their ground state is encoded in two-point correlators \(C_{ij} = \langle \Psi | c_i^{\dagger} c_j | \Psi \rangle\). This fact leads to the implementation of correlation matrix method \([40–42]\) for calculating EE

\[
S = - \text{Tr} [C_A \ln C_A + (1 - C_A) \ln (1 - C_A)],
\]

where \(C_A\) is the correlation matrix for subsystem \(A\). It only requires diagonalization of a \(N \times N\) matrix and \(N\) is number of lattice sites. This method has been widely used in numerical simulations.

Notably, for certain cases, the correlation matrix method can give an analytical solution of EE and entire spectrum of the reduced density matrix \([43]\). By taking Eq. (3) as an integral operator with kernel \(C_A\) inside certain intervals, the EE can be written in terms of a contour integral of its resolvent. This technique is valuable to determine multi-interval EE of \((1 + 1)D\) free massless fermions and chiral bosons, however it is restricted to these cases due to the mathematical difficulty on calculating the exact resolvent.

2. Euclidean approach: Replica trick

Direct calculation of the EE in Minkowski spacetime is mainly limited to finite-size numerical simulation for discrete lattices instead of continuous spacetime. This leads to the difficulty on determining the scaling behavior of EE. By contrast, the Euclidean approach via replica trick \([3, 13, 44]\), is powerful for solving EE analytically. The replica trick is introduced to avoid the difficulty of taking logarithm to the reduced density operator \(\rho_A\). With introducing a replica index of \(n\), the EE can be rewritten as

\[
S = - \frac{\partial}{\partial n} \ln \text{Tr} (\rho_A^n) \bigg|_{n \to 1}.
\]

The physical meaning of the index \(n\) is to make \(n\) decoupled identical copies of the theory. Analytic continuation of \(n\) is then assumed before taking the replica limit \(n \to 1\).

Since we are interested in the case of ground state, the trace of \(\rho_A^n\) has a natural Euclidean path integral representation \([9, 13, 21]\)

\[
\text{Tr} (\rho_A^n) = \frac{Z^{(n)}}{Z^{(1)}},
\]

where \(Z^{(n)}\) represents the partition function defined on the \(n\)-fold replica spacetime manifold with the entanglement cut along \(A\). The calculation of EE is then reduced to the problem of solving the partition function \(Z^{(n)}\) on a certain \(n\)-fold non-smooth manifold as

\[
S = - \frac{\partial}{\partial n} \left[ \ln Z^{(n)} - n \ln Z^{(1)} \right] \bigg|_{n \to 1}.
\]
Geometrically, the manifold is equivalent to an Euclidean spacetime with conical singularities at coincident points that is described by the metric [4, 9, 13]

$$ds^2 = dp^2 + n^2 \rho^2 d\theta^2 + \sum_{i=3}^{d_M} dx_i^2,$$  \hspace{1cm} (7)

where $n$ is the replica index that characterizes this metric, $d_M$ is the spacetime dimension, and the $(x_1, x_2)$ plane is written in terms of the polar coordinates $(\rho, \theta)$. In this paper, we focus on the case of an infinite cone, i.e. $\rho \in [0, \infty)$, $\theta \in [0, 2\pi)$ and $\{x_j\}$ in the whole space. For solving the functional integral and differential equations, this metric can be described by changing boundary condition from the ordinary period of $\theta \sim \theta + 2\pi$ to $\theta \sim \theta + 2\pi n$.

For free theories, the partition function is one-loop divergent, so that the heat-kernel technique is quite standard for calculating it [79]. Several models, including free scalar, Dirac and Maxwell fields with/without curvature coupling, were investigated in previous works [3, 4, 46, 47, 69, 80]. However, the heat-kernel technique meets difficulty when dealing with generic interacting theories on a manifold with conical singularities, since it captures only the quadratic order of quantum fluctuations (effective action at one-loop level) and there is no closed analytical expression for higher-order heat-kernel coefficients on replica manifold with conical singularities.

Another possible way to estimate the partition function $Z(\theta)$ is through the Green’s function $G(\theta)$ on replica manifold [13, 44, 71, 74]. They are related by taking a derivative with respect to the mass

$$\frac{\partial}{\partial m^2} \ln Z(\theta) = \frac{-1}{2} \text{Tr} G(\theta).$$  \hspace{1cm} (8)

Here the concept of the Green’s function is not limited to its original meaning in solving differential equations, but is extended to the two-point correlation function of QFTs. This is important for calculating EE in the theories with no direct field-equation representation, e.g. disordered systems. However, unlike the universal expansion procedure in heat-kernel technique, there is no general way for calculating replicated Green’s function in higher-dimensional interacting theories. Fortunately, the calculation of Green’s function of QFTs in curved spacetime with conical singularities has been attracted considerable attentions in various contexts, such as scattering of electromagnetic waves [81–84] and orbifold conformal field theory [85–89]. These studies provide valuable knowledge for calculating EE in QFTs.

3. Conformal field theory approach

The above discussion are made for generic QFTs. For critical systems described by CFT, there are some universal behaviors of the EE that are analytically accessible. In $(1 + 1)$ dimension, CFT techniques (combine with the replica trick) have received great achievement of calculating the EE in critical systems, demonstrating a logarithmic divergent EE with a prefactor of central charge $c$ that characterizes universality of the quantum criticality [11, 13, 48]

$$S_{2d \text{ CFT}} = \frac{c}{3} \ln \frac{l}{\epsilon} + c_{\text{finite}},$$  \hspace{1cm} (9)

where $l$ is the size of a single-interval subsystem in an infinite chain, $\epsilon$ is a UV cut-off of lattice constant, and $c_{\text{finite}}$ is a non-universal finite term.

For higher dimensions, the conformal symmetry is generally not so strong as 2D to fully determine the scaling behavior of the EE. For spherical entangling surface in $\mathbb{R}^{1,d}$ flat Minkowski spacetime, the problem of the EE of a CFT can be conformally mapped to the solution of thermal entropy in a $\mathbb{H}^{d+1}$ hyperbolic space [45, 49, 90–92], where an infrared (IR) cutoff leads to the area-law EE of the quantum fields in $\mathbb{R}^{1,d}$ at UV. However, this approach cannot be extended to generic geometries, where the local form of modular Hamiltonian is unknown.

4. Holographic approach

The difficulty of calculating the EE in higher dimensional QFTs motivates a holographic interpretation of the EE based on the conjecture of AdS/CFT correspondence, which bridges the $(d+2)$D AdS space and a $(d+1)$D CFT [93]. It was proposed that the calculation of EE can be reduced to the problem of finding extreme surface inside the AdS space, for which a Bekenstein-Hawking-like formula (the RT formula) naturally gives an area-law [5, 50]. Nevertheless, the RT formula is still far away from the answer to entanglement in QFTs. The use of RT formula requires the dictionary between field theorems and its gravitational dual, however, only few cases are known. Meanwhile, in the AdS calculation, although solving the extreme surface is a classical task, in most cases we can only perform a numerical estimation on it. More importantly, there is no rigorous proof of the holographic principle, and the sufficient condition for the establishment of RT formula remains an open question.

5. Quasi-particle picture

The area-law EE can be understood within a quasiparticle picture, which assumes that the entanglement is made of the correlations between entangled quasiparticles in the system [20, 27, 28]. The simplification of...
this picture is made by assuming the mutual information to be extensive, which reduces the calculation of EE to be a geometric problem that is much simpler than the holographic approach. In parallel to the aforementioned quasiparticle picture that comes from a dynamical diffusion-annihilation process of free fermions, the observation of extensive mutual information in the ground state of (1 + 1)D massless Dirac fermions motivates investigations on an “extensive mutual information” model, which has been used for understanding the entanglement structure with various applications. Recently, it is proven that the extensive mutual information model does not correspond to an actual CFT beyond (1 + 1) dimensions, so that fail to be an exact solution of EE in higher dimensions. However, it does capture the leading scaling behavior of entanglement and provide significant understanding in fermionic scale-invariant systems.

B. General strategy of dimensional reduction

The general idea of dimensional reduction is to use low-dimensional results (which is known) to calculate higher-dimensional results (which is hard to know). For non-interacting cases, one can consider that the higher-dimensional theories are constructed by infinite many (1 + 1)D modes with an effective mass that is associated with its momentum. This fact motivates a direct reduction of higher-dimensional entropy to a sum of (1 + 1)D entropic c-function (defined as \( c(L) = L \frac{\Delta S(L)}{\Delta L} \)) for the sub-system with spatial size \( L \). These calculations are quite simple and provide an intuitive picture on entanglement structure of (2+1)D many-body states. However, this procedure has two drawbacks. First, this calculation of the EE requires the additivity of the entropic function, which is mathematically less evident. Second, this method is hard to be extended into generic models for exact results, so the dimensional reduction scheme in previous works only has phenomenological meaning. Therefore, seeking for possible exact dimensional reduction approach on a firm ground is highly desired.

In order to solve the aforementioned problems, let us consider one question first: Which physical quantity is highly desired. Apparently, the most suitable one for QFTs is the Green’s function on replica manifold \( G^{(n)} \), which has great advantages on computation with the help of tools from conventional perturbation theory such as diagram technique and renormalization group analysis.

Here we explain how this works for the simplest case of constructing (d+1)D Green’s function of free scalar field in usual flat Minkowski spacetime from its (d+0)D reduction. Start from its action

\[
I^{[d+1]} = \int dt \int d^d x \left( -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 \right)
\]

\[
= \int dt \int d^d x \int \frac{d\omega}{2\pi} e^{-i\omega t} \left[ \mathcal{L}_0^d + \mathcal{L}_{\text{int}}^d(\omega) \right]
\]

where \( \mathcal{L}_0^d = -\frac{1}{2} \partial^\mu \phi(\omega) \partial_\mu \phi(\omega) - \frac{m^2}{2} \phi^2(\omega) \) is the free Lagrangian density and \( \mathcal{L}_{\text{int}}^d(\omega) = \frac{\alpha}{\pi} \phi^2(\omega) \) is the interacting term in (d+0)D in a quadratic form, with the index \( \mu \) runs over the spacetime dimensions and \( i \) only for spatial. This action has the exact solution of the Green’s function as \( G_0^{[d+1]}(k, \omega) = (-\omega^2 + k^2 + m^2)^{-1} \), and can be represented as a sum of all tree-level diagrams with respect to \( \omega \)

\[
G_0^{[d+1]}(k, \omega) = g_0^{[d+0]}(k) \sum_{l=0}^{\infty} \left[ \omega^2 g_0^{[d+0]}(k) \right]^l.
\]

Here the (d+0)D Green’s function \( g_0^{[d+0]}(k, 0) \) is regarded as the “free” solution of \( \mathcal{L}_0^d \), and Eq. (11) actually defines an alternative approach of dimensional reduction, with the quadratic construction as an inherent regulator. From here on, we use the upper index in \([..]\) to represent the spacetime dimension, while that in (…) to denote the replica index. For simplicity, \( g^{[d+0]} \equiv g \) stands for the Green’s function in (d + 0) dimension, and symbol \( G^{[d+1]} \equiv G \) is used for the full Green’s function in (d+1) dimension.

The advance of the above dimensional reduction of Green’s function can also deal with possible interactions, at least in the perturbative region. In particular, for the case of adding a static potential without dynamics, i.e. the interaction with an field that does not depend on time, the extension can be made by considering the effect of interactions as quantum corrections to the (d+0)D Green’s function \( g^{[d+0]}(k) \) as

\[
G_{\text{int}}^{[d+1]}(k, \omega) = g_{\text{int}}^{[d+0]}(k) \sum_{l=0}^{\infty} \left[ \omega^2 g_{\text{int}}^{[d+0]}(k) \right]^l.
\]

Here we note that this equation is just a formal representation, and we are not limited to this concrete construction. Importantly, the above discussions are not restricted to the flat spacetime, but can be directly extended to the case of curved spacetime with certain singularities. Therefore, our task of calculating higher-dimensional EE turns into the calculation of lower-dimensional interacting Green’s functions on the replica spacetime manifold. This is of course a difficult problem, however, we will show that the calculation at replica limit \( n \to 1 \) is very much similar to the usual perturbation theory in flat spacetime, but with a term of the conical singularity.

At the end of this section, let us emphasize the motivation to apply the dimensional reduction scheme to interacting theories, instead of calculating perturbative
expansions directly in high dimensions [72]. Generally speaking, one important consequence of interactions is the breakdown of Gaussianity of the many-body ground state, which usually requires a higher-order (multi-loop) calculation to capture the non-Gaussian features. Unfortunately, a proper renormalization scheme for these higher-order corrections in (2+1)D is less known in the context of calculating the EE, since the fields are living on the replica manifold with conical singularities instead of the usual flat spacetime [76–78]. Moreover, it is worth noting that there are celebrated approaches to addressing non-perturbative results has shown a powerful perspective in understanding low-energy collective excitations in condensed matter [99]. The key of our dimensional reduction method is to access the entanglement structure of interacting theories via a similar manner. Later we will see that this construction does reproduce the explicit form of (2+1)D results through conventional field theory techniques.

III. (2+1)D FREE SCALAR FIELD

We start the calculation of EE in (2+1)D QFT by the free scalar field using the above dimensional reduction method, as a benchmark. This process is instructive and provides insights on the further calculation for (2+1)D Dirac field.

A. Direct solution

Let us start from a brief review of the area-law EE of a free scalar field living on a “waveguide” geometry $\mathbb{R}^2 \times \mathbb{I}$, where the wavefunction propagates as a plane wave on the finite interval $\mathbb{I}$ [13, 47, 69]. The non-interacting Green’s function of scalar fields $G^{(n)}$ satisfies the Helmholtz equation that is defined on the corresponding 3D replica spacetime manifold $M^{(n)}$

$$\nabla^2 - m^2 G^{(n)}(\mathbf{r}, \mathbf{r}') = -\delta^{(3)}(\mathbf{r}, \mathbf{r}'),$$

(13)

where $\mathbf{r}, \mathbf{r}'$ are 3D vectors and $\delta^{(3)}$ is the Dirac-delta function in 3D. In our case, the replica manifold follows the waveguide construction $M^{(n)} = C^2 \times \mathbb{I}$ as the product of a 2D cone $C^2$ and the interval $\mathbb{I}$, with the metric in Eq. (7). The solution of $G^{(n)}$ in cylindrical coordinates $\mathbf{r} = (\rho, \theta, r_\perp)$ is then given by

$$G^{(n)}(\mathbf{r}, \mathbf{r}') = \int \frac{dk_\perp}{2\pi} e^{ik_\perp(r_\perp - r'_\perp)} \frac{1}{2\pi n} \sum_{q=0}^{\infty} d_q \cos \left[ q n (\theta - \theta') \right] \int_0^\infty J_q(n\lambda\rho) J_{q/n}(\lambda\rho') \frac{\lambda d\lambda}{\lambda^2 + m^2 + k_\perp^2}$$

(14)

where $k_\perp$ is the momentum of the translation-invariant $r_\perp$-direction that perpendicular to the plane of polar coordinates $\mathbf{r} = (\rho, \theta)$, $q$ is the angular momentum in the $(\rho, \theta)$ plane that takes integer values, $d_q = 1$, $d_{q>0} = 2$, and $J_q(\lambda\rho)$ is the Bessel function of first kind at $q$-th order with the eigenvalue $\lambda$ in the radial equation.

Taking trace of $G^{(n)}$ requires the information at coincident points $(\rho, \theta) \rightarrow (\rho', \theta')$, where the Green’s function is generally UV divergent. In our case, the divergence comes from the sum over angular momentum $q$, and can be regularized in the calculation of the normalized partition function $Z^{(n)}/|Z^{(1)}|^n$. Mathematically, this is achieved by using the Euler-Maclaurin formula that translates the summation to an improper integral with remaining terms (see Appendix A). It gives

$$G^{(n)} - G^{(1)} = \int \frac{dk_\perp}{2\pi} \frac{1 - n^2}{12\pi^2 n^2} \left[ K_0(\sqrt{k_\perp^2 + m^2 \rho}) \right]^2,$$

(15)

where $K_0(x)$ is the zero-th order modified Bessel function of the second kind, and all the higher-order remaining terms in Euler-Maclaurin expansion vanish at the coincident points. Then it leads to

$$\frac{\partial}{\partial m^2} \ln \frac{Z^{(n)}}{Z^{(1)}} = -\frac{1}{2} \text{Tr}^{(n)} \left[ G^{(n)} - G^{(1)} \right]$$

$$= - \int d\mathbf{r}_\perp \int \frac{dk_\perp}{2\pi} \frac{1 - n^2}{24n(k_\perp^2 + m^2)},$$

(16)

where Tr$^{(n)}$ represents that the integral over full spacetime is taking on the $n$-fold manifold. The integral $\int d\mathbf{r}_\perp = A$ gives the area of the entangling surface in the finite interval $\mathbb{I}$. Here the integral over $k$ will lead to logarithmic divergence that requires a cut-off of $\epsilon^{-1}$ ($\epsilon \ll 1$ plays the role of lattice constant). These give the regularized area-law EE

$$S = -\frac{A}{12} \int_{-\infty}^{\infty} \frac{dk_\perp}{2\pi} \ln \frac{k_\perp^2 + m^2}{k_\perp^2 + \epsilon^{-2}} = \frac{A}{12} \left(\epsilon^{-1} - m \right).$$

(17)

For the massless case $m = 0$, we simply have $S = \frac{A\epsilon}{12\pi}$ as the leading UV-divergent area-law scaling.

B. Dimensional reduction calculation

In this section, we show that the above result of EE can be reproduced through the dimensional reduction method. As we have introduced in Sec. II B, for dimensional reduction we need to calculate the products of the Green’s function on 2D replica manifold. In real-space representation, they are

$$P^{(n)}_i(\mathbf{r}_{||}, \mathbf{r}'_{||}) = \int \int \ldots \int \frac{d^2 \mathbf{r}_{||,1}}{2\pi} \cdots \frac{d^2 \mathbf{r}_{||,l}}{2\pi} g^{(n)}(\mathbf{r}_{||,1}) \cdots g^{(n)}(\mathbf{r}_{||,l}, \mathbf{r}'_{||})$$

(18)
\[ G^{(n)}_i(r_k, r'_k) - G^{(1)}_i(r_k, r'_k) \]
\[ = \sum_{l=0}^{\infty} (-k^2)^l \left[ P^{(n)}_l(r_k, r'_k) - P^{(1)}_l(r_k, r'_k) \right] \]
\[ = \int d^2 r_{\perp} r_{\perp}^{\prime} ∙ r_{\perp}^{\prime} \int d^2 r_{\perp} r_{\perp}^{\prime} + \int d^2 r_{\perp} r_{\perp}^{\prime} r_{\perp}^{\prime} + \cdots \]

FIG. 1. The diagram representation of the replicated Green’s function of (2+1)D free scalar field via the dimensional reduction method, with ignoring higher-order terms of \( O(1 - n)^2 \). Here the lines are the usual flat Green’s function in its real-space representation, the dot with label \( r_{\perp}^{\prime} \) represents a vertex of \(-\omega^2 \int (d^2 r_{\perp}, l)\), and the x at original point denotes a factor of \( 2\pi \frac{1}{\omega^2} \).

which gives the \( l \)-order perturbation \((-k^2)^l P^{(n)}_l\). Here \( f^{(n)} \) represents that the integral is performed on the \( n \)-fold manifold. The exact calculation of these products are hard, since for \( g^{(n)} \) at general two points we do not have a simple relation as Eq. (15). However, if one consider the approximate expansion near the coincident points that contributes to the entanglement entropy [72], the calculation can be simplified again by using the Euler–Maclaurin formula, which gives

\[ g^{(n)}(r_{||}, r'_{||}) \sim g^{(1)}(r_{||}, r'_{||}) + \frac{1 - n^2}{12\pi^2} K_0(m\rho)K_0(m\rho'). \]

This relation reduces the product of \( g^{(n)} \) into

\[ P^{(n)}_l(r_{||}, r'_{||}) = (l + 1) \int d^2 r_{\perp} g^{(n)}(r_{||}, r_{||}) \]

\[ = (l + 1) \frac{1 - n^2}{12\pi^2} K_0(m\rho) \int d^2 r_{\perp} d^2 r'_{\perp} g^{(1)}(r_{||}, r_{||}) + O((1 - n)^2), \]

where the factor of \((l + 1)\) is the symmetry factor, and the higher-order terms of \( O((1 - n)^2) \) vanish in the EE as taking the derivative and the replica limit \( n \to 1 \). The Eq. (20) actually defines an expansion of the products \( P^{(n)}_l \) around \((1 - n)\), which leads to the simplification of calculating \( P^{(n)}_l \) via conventional diagram techniques as we will demonstrate below. Here \( g^{(n)} \) should be written in terms of \( g^{(1)} \) and the remaining term, since we have changed the integral measure onto a single copy instead of the entire \( n \)-fold manifold. It gives

\[ P^{(n)}_l(r_{||}, r'_{||}) = (l + 1) \frac{1 - n^2}{12\pi^2} K_0(m\rho) \int d^2 r_{\perp} d^2 r'_{\perp} g^{(1)}(r_{||}, r_{||}) \]

\[ = (l + 1) \frac{1 - n^2}{12\pi^2} K_0(m\rho_1)g^{(1)}(r_{||, 1}, r'_{||, 1}) \cdots g^{(1)}(r_{||, l}, r'_{||, l}). \]

It is important to notice that elements in the above expansion contains only the usual flat Green’s function of free scalar field in 2D spacetime,

\[ g^{(1)}(r_{||}, r'_{||}) = \int d^2 k \frac{e^{i k_0 (r_{||} - r'_{||})}}{(2\pi)^2 k^2 + m^2} = \frac{1}{2\pi} K_0(|m\rho - r_{||}'|). \]

This means that we are dealing with nothing unusual but the ordinary diagrams with a non-trivial additional vertex that comes from the conical singularity, see Fig. 1. Now the calculation of EE is fully reduced to conventional perturbation theory that we are familiar with, Eq. (21) is then simplified as

\[ P^{(n)}_l(r_{||}, r'_{||}) = \frac{l + 1}{(l + 1) \frac{1 - n^2}{12\pi^2} K_0(m\rho) \int d^2 k \frac{e^{i k_0 (r_{||} - r'_{||})}}{(2\pi)^2 k^2 + m^2}}. \]

\[ = \frac{l + 1}{12\pi^2} K_0(m\rho) \int d^2 k \frac{1}{k^2 + m^2}. \]

In practice, we find that it is more convenient to trace over the reduced two-dimensions before taking the summation over the perturbation levels \( l \), which gives

\[ \frac{l + 1}{12\pi^2} K_0(m\rho) \int d^2 k \frac{1}{k^2 + m^2}. \]

Its sum over \( l \) is just a geometric sequence, and gives to the higher-dimensional Green’s function as shown diagrammatically in Fig. 1. This leads to

\[ \frac{l + 1}{12\pi^2} K_0(m\rho) \int d^2 k \frac{1}{k^2 + m^2}. \]

which is identical to the calculation in Eq. (16), and leads the same result of EE in Eq. (17).

Here, we stress that the key step in the above calculation is an integral of the replicated Green’s function \( G^{(n)} \) over the cone \( C^2 \), as shown in Eq. (21) and (23). It is usually UV divergent and requires introducing a microscopic cutoff for accessing its finite contribution. Fortunately, to distinguish these singularities on the replica \( n \)-fold manifold is quite straightforward in dimensional reduction scheme, with the aid of experience in \((2 + 0)D \) [13]. In a word, this example shows the proposed dimensional reduction scheme correctly captures the singularity contributed to the EE.

IV. \((2+1)D\) FREE DIRAC FIELD

In previous section we have shown that our proposed dimensional reduction method faithfully recovers the area-law EE for free scalar field. Now we will present an exact derivation of the area-law EE in \((2+1)D\) free Dirac field in a similar manner.

\[ \text{We have tested that exchanging the order of } T^{(n)}_{3D} \text{ and sum over } l \text{ does not influence on the result, see details in Appendix B.} \]
The action of free Dirac field in 2D Euclidean space is
\[ I_D^{[2]} = \int d^2 \vec{r} \bar{\Psi} (\gamma^\mu \partial_\mu + m) \Psi, \] (26)
and the corresponding spinor Green’s function satisfies
\[ (\gamma^\mu \partial_\mu + m) G^{(n)}_D (\vec{r}, \vec{r}') = -\delta^{[2]}(\vec{r}, \vec{r}'), \] (27)
where the index of spacetime dimensions \( \mu = 1, 2 \) with \( \gamma^1 = \sigma_1 \) and \( \gamma^2 = \sigma_2 \). The solution on polar coordinates \( \vec{r} = (\rho, \theta) \) is (see details in Appendix C)
\[ G^{(n)}_D (\vec{r}, \vec{r}') = \frac{1}{4\pi n} \sum_{q=-\infty}^{\infty} e^{i q (\theta-\theta')} \int_0^\infty \frac{\lambda d\lambda}{\lambda^2 + m^2} \]
\[ \left( m J_{q} (\lambda \rho) J_{q} (\lambda \rho') + i \lambda e^{i \theta'} J_{q+1} (\lambda \rho) J_{q+1} (\lambda \rho') \right) \]
\[ \left( i \lambda e^{\theta} J_{q+1} (\lambda \rho) J_{q+1} (\lambda \rho') \right). \] (28)
As taking \( n = 1 \), it reduces to the usual spinor Green’s function with the difference on a global factor of \( \frac{1}{2} \) that comes from the choice of normalizing the entire spinor. This ensures that the EE of each spinor component of the free fermion is the half of the scalar case in 2D.

For constructing the 3D spinor Green’s function, we need to introduce an additional Dirac-\( \gamma \) matrix in higher dimension \( \gamma^0 = \sigma_3 \). Similar to the case of scalar field, the 3D spinor Green’s function is represented as
\[ G^{(n)}_D (\vec{r}, \vec{r}'; k_\perp) = \sum_{l=0}^{\infty} (i k_\perp)^l P^{(n)}_{D,l} (\vec{r}, \vec{r}'), \] (29)
with the l-product of 2D functions \( g^{(n)}_{D,l} \)
\[ P^{(n)}_{D,l} (\vec{r}, \vec{r}') = \int d^2 r_{\perp,1} \cdots \int d^2 r_{\perp,l} g^{(n)}_D (\vec{r}, \vec{r}_{\perp,1}) \cdots \left[ \gamma^0 g^{(n)}_D (\vec{r}_{\perp,l-1}, \vec{r}') \right] \]
\[ \cdots \left[ \gamma^0 g^{(n)}_D (\vec{r}_{\perp,2}, \vec{r}') \right] \]. (30)

Analog to the free scalar field, here we would like to transform the real-space Green’s function into momentum representation. The off-diagonal components in replicated spinor Green’s function \( G^{(n)}_D \) is generally hard to be dealt with, due to the non-trivial spin structure. However, it is important to notice that the double product of the spinor function is diagonal and identical to the scalar case [47]. Meanwhile, the odd-order terms all vanish in the later trace of the higher dimension (the integral over \( k_\perp \)), since they are odd functions of \( k_\perp \). These facts lead to the simplification of
\[ G^{(n)}_D - G^{(1)}_D = \sum_{l=0}^{\infty} (-k_\perp^2)^l \left[ P^{(n)}_{D,2l} - P^{(1)}_{D,2l} \right] \] (31)
with
\[ P^{(n)}_{D,2l} (\vec{r}, \vec{r}') - P^{(1)}_{D,2l} (\vec{r}, \vec{r}') \]
\[ = (l + 1) \frac{1 - n^2}{6n^2} \frac{m}{2\pi} K_0 (mp) \int d^2 k_{\|} \frac{e^{i k_{\|} r'}}{2\pi} \left| k_{\|} + m^2 \right|^{l+1}, \] (32)
where \( \text{Id} \) is a two-by-two identity matrix, and the higher-order terms of \( O((1-n^2)^2) \) are ignored. Analog to the free scalar field, it leads to the trace on 3D replica manifold
\[ \text{Tr}^{(n)}_{3D} \left[ G^{(n)}_D - G^{(1)}_D \right] = \frac{1 - n^2}{12n} \int dr_{\perp} \int dk_{\perp} \frac{2m}{2\pi} \frac{k_{\perp}^2 + m^2}{k_{\perp}^2 + m^2} \] (33)
and the corresponding normalized partition function
\[ \ln \frac{Z^{(n)}}{Z^{(1)}} = - \int dm \text{Tr}^{(n)}_{3D} \left[ G^{(n)}_D - G^{(1)}_D \right] \]
\[ = - \frac{1 - n^2}{12n} \frac{A}{2\pi} \int dm \int k_{\perp}^2 \frac{1}{k_{\perp}^2 + m^2}. \] (34)
Finally, we have the EE in \((2 + 1)D\) free Dirac field
\[ S = \frac{1}{6} A (\epsilon^{-1} - m). \] (35)
Comparing with the free scalar case, we obtain
\[ r_{\text{dirac}} = 2r_{\text{scalar}}, \] (36)
where \( r \) is the coefficient of the mass scaling in Eq. (2). Here we see, through the dimensional reduction calculation, the EE of \((2 + 1)D\) free Dirac field is observed to exhibit an area-law behavior, consistent with the previous results of calculating the entropic \( c \)-function [50, 54, 67, 68] and the heat-kernel on replica manifold [47].

At last, we would like to comment on the difficulty of performing a direct calculation of solving the eigenvalue problem on replica manifold. Opposite to the scalar case, the spinor wavefunction on replicated waveguide geometry \( C^2 \times \mathbb{I} \) cannot be separated into the product of two individual eigenfunctions on \( C^2 \) and \( \mathbb{I} \). However, analog to the previous investigation on the heat kernel [47], we find that the dimensional reduction of the spinor Green’s function does not require the separation of eigenfunctions (see Appendix C).

V. \((2 + 1)D\) DIRAC FERMIONS UNDER A RANDOM MAGNETIC FIELD

In addition to the area-law EE in the above two examples, it is more interesting to investigate the cases beyond free theories. As perturbing away from the free fixed point, the area-law coefficient becomes a function of the coupling constant. This physical consequence of interactions is hardly studied before. In this section, as a concrete example, we focus on \((2 + 1)D\) Dirac fermions under a random magnetic field (static gauge field).

We would like to highlight that this example is quite meaningful. First, in the presence of randomness, one cannot exactly solve the eigenvalue problem of Dirac spinor due to the lack of a straightforward field-equation description. Second, different from the ordinary static gauge field, the random field is equivalent to a certain type of effective interaction of the Dirac field.
$g_A (\overline{\Psi} \gamma^\mu \Psi)^2$ [100, 101]. So this model can be regarded as an example of interacting theory. Thus, many established tools such as heat-kernel technique are fail to give the EE for this example. Third, one may wonder why we choose such a model with "effective" interactions rather than a model with explicit interactions. The reason is, the EE of this model can be numerically calculated up to $\sim 10^4$ lattice sizes (see Sec. V C below), which provides an unbiased way to validate our analytical results. As a comparison, for a model with explicit interactions, the numerical calculation of the EE may suffer from strong finite-size effect. Fourth, it is conjectured that the random magnetic field leads to a multifractal critical ground state [102–105], based on the traditional numerical/theoretical methods. We anticipate to uncover this criticality from its internal entanglement structure. In a word, this is a good example to demonstrate the power of our dimensional reduction method.

A. Preliminary results

The study on the localization-delocalization transition induced by disorder is a central subject in condensed matter physics [106–110]. It is well known that localization property depends on the dimensionality and underlying symmetry [107, 108, 111, 112]. In history, (2 + 1)D Dirac fermion exposed to a random magnetic field or transverse gauge-field randomness received much attention, which is expected to describe the universality class of the metal-insulator transition in the integer quantum Hall effect [102, 103, 113–115], the quantum fluctuations in quantum spin liquids [116], and disordered graphene [117]. Interestingly, it has been proposed that this problem has an exactly solvable zero-energy wavefunction with multifractal critical scaling behaviors [103–105], which could be immune to randomness and thus escape from localization.

The theory has a natural Euclidean description in 2D

$$\mathcal{L} = \overline{\Psi} \gamma^\mu (\partial_\mu + i \sqrt{g_A} A_\mu) \Psi + \overline{\Psi} (i \omega + 0) \Psi,$$

where $A_\mu$ describes the random gauge field (vector potential). For simplicity, here $A_\mu$ is chosen to be Gaussian-distributed

$$\mathcal{P}(A_\mu) \propto e^{-\frac{1}{2} \int \! d^2 \tau_1 A_\mu^2 (\tau_1)},$$

with vanishing gauge flux on average. By absorbing the coupling constant $\sqrt{g_A}$ into the gauge field $A_\mu$, it is clear that $g_A$ plays the role of the variance of the disorders.

Especially, when $\omega = 0$ the random gauge field preserves the chiral symmetry, so that the zero-energy wavefunction of this model remains critical under the perturbation. It can be exactly solved within a non-unitary CFT, and the multifractal scaling exponents of zero-energy state is determined to be $\Delta = 1 - \frac{g_A}{2\pi}$ [103]. The exponent is continuously tunable as changing the randomness strength $g_A$, and it becomes negative at $g_c = 2\pi$, indicating a spontaneous symmetry breaking. Moreover, the dynamical exponent $z$ of this model smoothly varies as a function of $z(g_A) = 1 + \frac{4\pi}{g_A}$.

For solving the zero-mode, it is beneficial to apply the Hodge decomposition to the 2D gauge field

$$A_\mu = \epsilon_{\mu \nu} \partial_\nu \Phi_1 (x) + \partial_\nu \Phi_2 (x)$$

and introducing the axial gauge transformation

$$\Psi = e^{i \frac{\omega}{\sqrt{g_A}} \Phi_1 + i \sqrt{g_A} \Phi_2},$$

The original Lagrangian density becomes

$$\mathcal{L} = \overline{\Psi} (\gamma^\mu \partial_\mu + M) \Psi + i \omega \overline{\Psi} e^{2\gamma^5 \sqrt{g_A} \Phi_1} \gamma^0 \Psi.$$

Here we impose a “mass” term $M\overline{\Psi} \Psi$ into the theory, which measures the gap between ground state and the first excited state in the chiral representation. Rather than dealing with the real mass of the original Dirac field, this treatment does not break the chiral symmetry of the fixed points. This leads to a simple calculation of the partition function and a reasonable estimation on the scaling behavior with respect to the finite correlation length, which is important for further analysis on the RG flows (see the next Sec. VI). The first term is just a free theory of the axial spinor filed $\{\Psi, \overline{\Psi}\}$, and the second term can be calculated perturbatively. Since there is no dynamical term of the gauge field, the components after Hodge decomposition can be treated as real scalars. Our choice of the Gaussian-distributed probability $\mathcal{P}(A_\mu)$ leads to the equivalence with a massless free scalar theory for both of $\Phi_1$ and $\Phi_2$. This ensures the exact representation of the axial transformation and leads to non-perturbative solution of the zero-mode.

B. Explicit derivation of the EE for Dirac fermions in presence of the random magnetic field

Here we calculate the EE in this model by using the dimensional reduction method. To achieve this, we need to solve the replicated Green’s function $g_{D, \text{int}} (r_1, r_1')$ for the Lagrangian in Eq. (41). The situation is more complicated than the previous free cases, since now we have to deal with randomness (or interaction) on $g_{D, \text{int}}^{(n)} (r_1, r_1')$. The effect of gauge field is considered as the perturbative correction to $g_{D, \text{int}}^{(n)} (r_1, r_1')$, and then $g_{D, \text{int}}^{(n)} (r_1, r_1')$ will be the base for the construction of higher-dimensional theory as a geometric sequence in the additional dimension.

The replicated Green’s function of massive free Dirac theory has been shown in Eq. (28), we now consider the perturbation with respect to $g_A$ that is contributed from the second term in Eq. (41). The effect of random magnetic field can be considered as the correction to internal lines in the construction of higher-dimensional theory, which appears in the form of an additional vertex correlator of the longitudinal axial field $\Phi_1$. 
Let us start from the lowest-order perturbation of $g_A$. Note that all odd-order contributions are ruled out due to the vanishing vertex correlation with non-zero charge. For instance, the lowest-order perturbation $-\omega^2 P_{D,l=2,\text{int}}^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel')$ is at $l = 2$, with the following explicit form of the product

$$P_{D,l=2,\text{int}}^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel') = \int d^2 \mathbf{r}_\parallel, 1 \int d^2 \mathbf{r}_\parallel', 2 g_{D,0}^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel, 1) g_{D,0}^{(n)}(\mathbf{r}_\parallel', 2, \mathbf{r}_\parallel') \gamma_0 (\mathbf{r}_\parallel, \mathbf{r}_\parallel, 1) \gamma_0 (\mathbf{r}_\parallel', 2, \mathbf{r}_\parallel')$$

where $\gamma_0$ is the two-point correlation function of vertex operator of the field $\Phi_1$ on 2D replica manifold

$$\gamma_0^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel') = e^{-\sqrt{2\pi} \Phi_1(\mathbf{r}_\parallel)} e^{-\sqrt{2\pi} \Phi_1(\mathbf{r}_\parallel')} \mathcal{R}^{(n)}$$

It is important to note that the prefactor on the exponential of this vertex operator is real, instead of the imaginary one for usual vertex operator in CFT. As a consequence, different from the usual power-law divergence, this nontrivial vertex correlation vanishes at the coincident points without singular behavior. Meanwhile, the bulk behavior of the replicated functions (at distant two points) is expected to be indistinguishable from their flat form. Based on these, in further treatment we will consider the following approximation

$$\gamma_0^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel') \sim \gamma_0^{(1)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel') = |\mathbf{r}_\parallel - \mathbf{r}_\parallel'|^{2\Delta}$$

without counting its conical contribution.

We next move to the construction of $(2 + 1)$D theory. To achieve this, we introduce infinite series that are generated by the above discussed lowest-order correction

$$C_{D}^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel') = \sum_{l=0}^\infty (-\omega^2)^l P_{D,2l,\text{int}}^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel')$$

with the $2l$-th order product

$$P_{D,2l,\text{int}}^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel') = \int d^2 \mathbf{r}_\parallel, 1 \cdots \int d^2 \mathbf{r}_\parallel', 2 g_{D,0}^{(n)}(\mathbf{r}_\parallel, \mathbf{r}_\parallel, 1) g_{D,0}^{(n)}(\mathbf{r}_\parallel', 2, \mathbf{r}_\parallel') \gamma_0 (\mathbf{r}_\parallel, \mathbf{r}_\parallel, 1) \gamma_0 (\mathbf{r}_\parallel', 2, \mathbf{r}_\parallel')$$

where $a = \sqrt{2\Delta}$, $R_{l,1,2} = |\mathbf{r}_\parallel - \mathbf{r}_\parallel'|$, and the higher-order terms of $O(1 - n)^2$ is ignored. This construction is known as the Born approximation, and widely adopted in the study of disordered electron systems. We note that there are other irreducible diagrams that are ignored in the Born approximation, e.g. the crossing diagrams. However, as we will show by comparing with the numerical simulation, the main feature of adding a random magnetic field is well captured in the current construction.

Here the expansion of Eq. (46) involves a scalar function that comes from the Gaussian-distributed random magnetic field (static gauge field), which does not influence the spin structure of the two-point function. The presence of this vertex correlation only leads to the correction of the scalar kernel of the double product of spinor Green’s function. Meanwhile, the scalar nature of the double product implies that the vertex correlation gives an identical correction factor for both the zero-th order usual flat function and the term in order of $(1 - n)$. These leads to the following expansion

$$P_{D,2l,\text{int}}^{(n)} - P_{D,2l,\text{int}}^{(1)} = (l + 1) \frac{1 - n^2}{6} M K_0(M \rho) \int \frac{d^2 \mathbf{k}'}{2\pi} e^{i \mathbf{k} \cdot \mathbf{r}'}$$

$$\approx \frac{1}{6} A (\epsilon^{-1} - M) \left[ 1 - g_A \mu_{\text{gauge}} \frac{\pi}{2a} \right],$$

where $\mu_{\text{gauge}} \approx 0.3$ is a positive constant, see Appendix D for the detailed derivations.

Eq. (48) is one of main results of the current work. It shows that the EE of Dirac fermions under random magnetic field remains the area-law scaling. The randomness slightly modifies the area-law scaling coefficient. To be specific, for a typical value $g_A = 0.1$, the correction to the area-law coefficient is less than 0.5%, which is sufficiently small compared to its bare (free field without randomness) value. Crucially, this area-law EE is in line with the prediction of quantum criticality under random magnetic field [103–105], which escapes from localization due to the chiral symmetry. In this regard, we believe the proof of area-law EE is a strong evidence of critical ground state under the random gauge field.

C. Area-law scaling in the lattice realization

To validate the analytical result Eq. (48), here we consider the $\pi$-flux model on square lattice as a typical lattice realization of the Dirac fermion, and an additional random hopping is introduced to mimic the random magnetic field [102, 118]:

$$H = - \sum_{\langle i, j \rangle} (-1)^{2x_i + 4y_i} c_{i_x+i_y}^\dagger c_{j_x+j_y} + \sum_{\langle i, j \rangle} W_{ij} c_i^\dagger c_j,$$
be Gaussian-distributed with the randomness strength (variance) $g_A$. At low-energy limit, this model leads to the massless theory of Eq. (37).

We numerically calculate the EE of the ground state of this lattice model. Its tight-binding nature allows a large-scale simulation by using the correlation matrix technique [40–42]. In the left panel of Fig. 2, the half-cut EE approximately linearly scales with the boundary size $L$. We confirm that, for a moderate value of randomness strength $g_A \in [0, 0.1]$, the linear scaling behavior is robust. Moreover, we also investigate the dependence of area-law coefficient on the variance $g_A$ in the region $g_A \in [0, 0.1]$. As shown in the upper inset of Fig. 2, the area-law coefficient exhibits a slowly linear decay, which is consistent with analytical prediction Eq. (48). Therefore, we conclude that our dimensional reduction scheme fairly captures the main features of the EE for $(2+1)$D Dirac fermion subjected to a random magnetic field.

### D. Correlation, entanglement and criticality

We now turn to discuss the entanglement structure in the $(2+1)$D Dirac field under random magnetic field, and its relation to quantum correlation of the field operator. In the right panel of Fig. 2, we show the squared two-point correlator after average, which exhibits a power-law scaling at long distance $|C(1, r)|^2 \propto r^{-k}$. Moreover, we numerically find that the power-law correlation has little change when adjusting the strength $g_A$ of random magnetic field. This motivates us to think about some universal connections between the EE and intrinsic correlations.

Here, we adopt a quasiparticle picture to describe the EE in scale-invariant fermionic systems [27, 28], where the entanglement is considered to be produced by quasiparticle entangled-pairs in the system. The only control parameter in this picture is the distribution function of those pairs $P(r)$, which gives the EE

$$S_A \sim \int_A dV_A \int_{\bar{A}} dV_{\bar{A}} P(r_A, \bar{r})$$

where $A$ and $\bar{A}$ are complementary to the total system, and $r_A, \bar{r}$ is the distance between the (lattice) points in the two subsystems $A$ and $\bar{A}$. Although the current case is a ground state that different from the dynamical steady state with excitations, it is still naturally to understand $P(r_A, \bar{r})$ as the squared two-point correlation function of the fermion operator, which gives a power-law decay of $P(r_A, \bar{r}) \propto r_A^{-k}$ for scale-invariant systems. An estimation of the integral in Eq. (50) indicates that an area-law EE occurs when $k > 3$ for the spatial dimension $d = 2$ (see details in Appendix F). It turns out that the exponent of $k$ determines the scaling behavior of EE, so that a numerical estimation of the power-law scaling becomes much more meaningful than the ordinary detection of the scale invariance.

For $(2+1)$D free Dirac field, the asymptotic behavior of two-point correlator at the long-distance limit, as $|C(1, r)|^2 \propto r^{-4}$. In our finite-size numerics on a lattice model with size $160 \times 160$, we find a close power-law scaling of $|C(1, r)|^2 \propto r^{-4.3}$. When varying the randomness strength (the variance of the random gauge field) $g_A$, the exponent is found to be almost unchanged. Plugging in these observations into the quasiparticle picture, it indicates a robust area-law scaling of the EE. This is exactly what we have observed in both the field theory calculation and numerical lattice simulation. In this context, the current model is one more example that can be understood in the quasiparticle picture phenomenologically.

### VI. ENTANGLEMENT ENTROPY AND RENORMALIZATION GROUP IN $(2+1)$D QFT

Besides the area-law scaling of the EE, our scheme is also capable of deriving the sub-leading term of the EE that is relevant with the dynamics of RG flow. The RG flow serves as a coarse-graining of the microscopic degrees of freedom of a physical system, so that it is expected to be an irreversible process between the fixed points. For $(1+1)$D QFTs, the irreversibility theorem of RG flows is known as the famous Zamolodchikov’s c-theorem [119], which proves the existence of a $c$-function (the central charge of CFT that describes the fixed points) that monotonically decreases during RG flows. Seeking for possible extensions of the c-theorem to generic dimensions is a long-standing challenge, especially for odd spacetime dimensions without the concept of central charges [56–58, 60, 120–123].
FIG. 3. The absolute value of the finite sub-leading term in the EE, plotted as a function of the scaling parameter of the correlation length $\xi$. Curve with higher transparency represents the result for higher value of $g_A$.

As mentioned in the introductory part, an important attempting is to understand the irreversibility of RG flows from EE. In $(1 + 1)$D CFT, the EE is fully determined by the central charge, therefore it is natural to construct an entropic $c$-function that points to the irreversibility of RG flows [55]. Furthermore, this idea is extended into higher dimensions, for which the universal finite term in EE (the $\gamma$ term in Eq. (1)) is expected to be an analog of the $c$-function [56–60]. Specifically, for $(2 + 1)$D QFTs, the $\gamma$ term is expected to be negative and satisfying the irreversibility relation [58, 59, 124, 125]

$$|\gamma_{\text{UV}}| \geq |\gamma_{\text{IR}}|,$$

(51)

which serves as a concrete construction of the $F$-function that is expected to exist in the $F$-theorem.

Now, let us recall the result of Dirac fermions in Eq. (35) and (48). In presence of an external random magnetic field, the finite sub-leading term of EE that responses to the imposed “mass” term behaves as

$$\gamma_{\text{gauge}} \approx -\frac{1}{6} \left[ 1 - g_A \frac{\mu_{\text{gauge}}}{2\pi} \right] A M \sim r_{\text{gauge}}(g_A) \frac{A}{\xi},$$

(52)

which gives the response of EE to a finite correlation length $\xi$. Since the entangling surface $A$ is spatially one-dimensional, the term $\frac{A}{\xi}$ naturally plays a role of the dimensionless parameter that is rescaled under RG transformations.

We graphically present the obtained result in Fig. 3. As it shows, the absolute value of $|\gamma_{\text{gauge}}|$ indeed reduces monotonically by approaching the IR limit (increasing $\xi$), which is in line with the proposal of $F$-theorem. Furthermore, if we focus on the dependence on the randomness strength $g_A$, we will observe a monotonically decreasing of $|\gamma_{\text{gauge}}|$ as increasing $g_A$ (at least in the perturbative region). Combining with the expectation of an irreversibility relation of Eq. (51), this result implies the formation of an unidirectional critical line (a line of fixed points) of the RG transformation. Importantly, this critical line is also observed in previous investigations from the ordinary RG analysis [102]. The consistency of various independent approaches demonstrates the existence of a critical line of RG transformation for $(2 + 1)$D Dirac fermions in presence of a random magnetic field. In particular, our result strongly supports the generality of the extended irreversibility relation of Eq. (51) in $(2 + 1)$ dimensions. This connection to (perturbative) RG gives a clear physical meaning to the negative correction of the EE from the disorder: Arbitrarily weak disorder shifts the fixed point and thus the conformal critical point is generically unstable.

VII. CONCLUSIONS AND OUTLOOKS

For a $(2+1)$D QFT in the presence of interactions, analytical calculation of the EE is generally difficult. In the present work, we have developed a dimensional reduction method to calculate the EE, which is able to deal with $(2 + 1)$D QFTs with interactions. In particular, we transform the $(2 + 1)$D replicated Green’s function to infinite series of the $(2 + 0)$D (interacting) replicated Green’s function, which can be calculated via conventional field theory techniques. The derivation can be greatly simplified in the replica limit, albeit difficult in evaluation of the interactions on the $n$-fold replica manifold.

We first apply this method to the EE in non-interacting $(2 + 1)$D theories, including free scalar field and Dirac field. It faithfully reproduces the area-law EE that are obtained by directly solving the $(2 + 1)$D field equation. We then extend the calculation to interacting theories with the help of perturbation theory. As a concrete example, we consider a non-trivial case of Dirac fermions subjected to a random magnetic field, where the traditional methods (in Tab. I) fail to give a straightforward derivation of the EE. Within the framework of the dimensional reduction approach, we explicitly derive the area-law EE, with observing a slight modification to its coefficient from the disorder (effective interaction). We further utilize numerical simulation on the lattice model to validate our analytical solution. Additionally, we attempt to understand the emergent area-law EE from the microscopic details of quantum correlation, pointing to the critical scaling behavior of the ground state. Last but not least, we give an affirmative evidence that the sub-leading term of the EE monotonically reduces in the IR limit. This provides a specific construction to validate the $F$-theorem in $(2 + 1)$ dimension. Moreover, by relating to the irreversibility theorem of RG flows, we clarify the physical meaning of the disorder-induced correction to the mass-scaling coefficient of the EE: It stipulates a line of fixed points for any finite weak disorder.

Here we would like to stress that, the current dimensional reduction scheme is distinct from the existing literature. In existing works [44, 53, 54, 97, 98], the starting point relies on the known EE function in $(1 + 1)$D, summation of which gives the EE in $(2 + 1)$D. This process is conceptually intuitive, however, it is in against to the fact that the EE (of a many-body ground state) is not an extensive quantity, so that it is difficult to be
extended into generic cases. In this work, to overcome this issue, we explore a distinguished path, based on constructing the \((2+1)\)D Green’s function using the dimensional reduction method. Compared to the aforementioned methods [53, 54] (see Tab. I), this Green’s function based scheme is quite feasible, without any prior knowledge about the EE.

Additionally, we compare the dimensional reduction method with other Green’s function based calculations [71–74]. A series of works [71, 73, 74] consider the \(O(N)\) model near CFT fixed points, and roughly estimation of the Green’s function on the replica manifold. These results are hard to be extended to generic interacting models. Besides, in Ref. [72], it was proposed that the effect of interaction on the EE can be understood as the usual flat spacetime renormalization to the mass. An important consequence of this statement is that the leading UV divergent term of EE will be independent on the coupling constant. However, as discussed in the previous RG analysis [71], this term generally becomes an analytical function of the coupling constant as perturbing away from the fixed point. This is also what we have observed in current calculation of Dirac fermions under a random magnetic field. The treatment of quantum corrections distinguishes our method from these calculations. The dimensional reduction allows the construction of an effective theory that starts with non-perturbative result at a fixed point. This is also what we have observed in current calculation of Dirac fermions under a random magnetic field. The treatment of quantum corrections distinguishes our method from these calculations. The dimensional reduction allows the construction of an effective theory that starts with non-perturbative result at vanishing energy, which leads to the great advantage of approaching non-Gaussian features that are intrinsic for interacting theories.

Finally, we believe the proposed advance here will lead to new perspective on studies of entanglement structure in condensed matters, especially for the \((2+1)\)D systems with possible criticalities. These systems typically have a conformal-invariant zero-energy wavefunction, for which CFT techniques could faithfully produce exact results. Through a construction of higher-dimensional (effective) theory based on these non-perturbative results, one can estimate the scaling behavior of the EE, which could provide valuable information for understanding low-energy collective behaviors of the system. For example, nearby a fixed point, the analytical solution of the EE and its dependence on the interaction coupling parameter can be derived. Further comparison with the numerical simulation on the discretized lattice models would bring more insights to the emergent quantum criticality. More importantly, as discussed in Sec. VI, the determination of the mass-scaling coefficient is instructive for investigating the dynamics of RG flows of \((2+1)\)D QFTs.

In particular, it is quite uplifting to imagine how our method could be applied to models of interest in high-energy physics. As we are constructing effective field theories, the resummation of higher-dimensional theory actually implies the choice of an energy scale as a consequence of summing up the physically dominate scattering channels. It is interesting to consider whether there is a well-defined RG flow that is reflected by the behaviors of EE at different energy scales.

Furthermore, there are some more extensions of the proposed dimensional reduction method. For example, our calculation could be extended to calculate the mutual information, which is another important entanglement measure that provides a upper bound of correlations in the systems. Exploring more detailed entanglement structure, such as the (logarithmic) universal sub-leading finite terms in the EE, is also deserved to study in the future.

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**Appendix A: Regularization of the UV divergent replicated Green’s function at coincident points**

In this appendix, we show how the Euler-Maclaurin formula gives a regularization of the UV divergent replicated Green’s function at coincident points. Here the derivation follows the previous work [13] by Calabrese and Cardy.

The 2D Green’s function of the free scalar field on replica manifold is

\[
g^{(n)}(\rho, \theta; \rho', \theta') = \frac{1}{2\pi n} \sum_{q=0}^{\infty} d_q \cos \left[ \frac{q}{n} (\theta - \theta') \right] \int_0^\infty \frac{J_{q/n}(\lambda \rho) J_{q/n}(\lambda \rho')}{\lambda^2 + m^2 + k^2} \lambda d\lambda, \tag{A1}
\]

where \(d_0 = 2\) and \(d_q = 1\) for \(q \geq 1\). At coincident points, it becomes

\[
g^{(n)}(\rho, \theta; \rho, \theta) = \frac{1}{2\pi n} \sum_{q=0}^{\infty} d_q \int_0^\infty \frac{J_{q/n}(\lambda \rho) J_{q/n}(\lambda \rho)}{\lambda^2 + m^2 + k^2} \lambda d\lambda = \frac{1}{2\pi n} \sum_{q=0}^{\infty} d_q J_{q/n}(m \rho) K_{q/n}(m \rho), \tag{A2}
\]

which shows a UV divergence due to summation over infinite many modes labeled by the angular momentum \(q\). To
regularize it we use the Euler-Maclaurin formula
\[
\int_0^\infty f(q) dq = h \left\{ \frac{f(0)}{2} + f(h) + f(2h) + \ldots \right\} + \sum_{k=1}^{\infty} \frac{h^{2k} B_{2k}}{(2k)!} \left[ - (\partial_q)^{2k-1} f(0) \right],
\]
where \( B_k \) is Bernoulli number. We are interested in the case of \( h = 1 \) and \( f(q) = I_{q/n}(mp)K_{q/n}(mp) \), which is divergent under an integral over \( q \). For regularization, we insert a function \( F(\frac{q}{n\Lambda}) \) into \( f(q) \), i.e. let \( f(q) = I_{q/n}(mp)K_{q/n}(mp)F(\frac{q}{n\Lambda}) \). The function \( F(\frac{q}{n\Lambda}) \) is chosen that \( F(0) = 1 \) and \( (\partial_q)^i F(0) = 0, i \geq 1 \). Now, the integral of \( \int_0^\infty f(q) dq \) is controlled by the parameter \( \Lambda \), and goes back to the original form at the limit of \( \Lambda \rightarrow \infty \). Then
\[
g^{(n)}(\rho, \theta; \rho, \theta) = \frac{1}{2\pi n} \sum_{q=0}^\infty dq I_{q/n}(mp)K_{q/n}(mp)F(\frac{q}{n\Lambda})
\]
\[
= \frac{1}{2\pi n} \left[ 2 \int_0^\infty I_{q/n}(mp)K_{q/n}(mp)F(\frac{q}{n\Lambda}) dq - \frac{B_2}{2} \frac{\partial}{\partial q} \left| I_{q/n}(mp)K_{q/n}(mp) \right|_{q=0} \right]
\]
\[
- \frac{1}{2\pi n} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \left( \partial_q \right)^{2k-1} \left| I_{q/n}(mp)K_{q/n}(mp) \right|_{q=0},
\]
where we have used \( B_2 = \frac{1}{6}, \left. \frac{\partial I_q(z)}{\partial \nu} \right|_{v=0} = -K_0(z), \left. \frac{\partial K_0(z)}{\partial \nu} \right|_{v=0} = 0 \). It should be noticed that although the higher-order derivatives of \( I_{q/n}(mp)K_{q/n}(mp) \) at \( q = 0 \) do not vanish, the their integral over \( \rho \) all vanishes in the later trace over the plane, as
\[
\int_0^\infty \rho d\rho \frac{d^k}{dq^k} \left| I_{q/n}(mp)K_{q/n}(mp) \right| = \frac{d^k}{dq^k} \int_0^\infty \rho d\rho \left| I_{q/n}(mp)K_{q/n}(mp) \right| = \frac{d^k}{dq^k} \left[ \frac{1}{2m^2 n} q \right] = 0 \quad \text{for } k > 1.
\]
The above regularization of the replicated Green’s function gives
\[
g^{(n)}(\rho, \theta; \rho, \theta) = g^{(1)}(\rho, \theta; \rho, \theta) + \frac{1}{12\pi n^2} [K_0(mp)]^2.
\]
The first term is just the flat divergence of the Green’s function at coincident points, and the second term is the contribution from the conical singularity. Moreover, in the calculation of the products of these Green’s functions, one can first integrate out the integral measure on the vertex as taking the trace over whole plane, then the higher-order derivatives of \( q \) all vanish as the same. This makes an approximation of \( g^{(n)} \) for general two points in the \( (\rho, \theta) \) plane reasonable in the calculation of dimensional reduction method that is discussed in the main text.

Appendix B: Calculation of constructing of 3D replicated partition function from 2D Green’s function

In this appendix, we show that exchanging the order of summation over perturbation levels \( l \) and the outside trace does not influence on the result of constructing higher-dimensional (3D) replicated partition function. Start from Eq. (23), before integral out \( k \), here we perform the summation over \( l \). It gives
\[
G^{(n)} - G^{(1)} = \sum_{l=0}^{\infty} (-k_\perp^2)^l \left[ P_l^{(n)} - P_l^{(1)} \right] = \frac{1}{12\pi n^2} K_0(mp) \int d^2k_{\parallel} e^{ik_\parallel r_\parallel} \sum_{l=0}^{\infty} (l + 1) \left( \frac{-k_\perp^2}{k_\perp^2 + m^2} \right)^l
\]
\[
= \frac{1}{12\pi n^2} K_0(mp) \left[ K_0(\sqrt{m^2 + k_\perp^2 \rho^2}) - \frac{\rho' k_\perp^2}{2\sqrt{m^2 + k_\perp^2}} K_1(\sqrt{m^2 + k_\perp^2 \rho^2}) \right].
\]
The trace in 3D replica spacetime is then given by
\[
\text{Tr}^{(n)} G^{(n)} - n \text{Tr}^{(1)} G^{(1)} = \text{Tr}^{(n)} \left[ G^{(n)} - G^{(1)} \right] = \int dr_\perp \int \frac{dk_{\perp}}{2\pi} \int d^2r_{\parallel} \left[ G^{(n)} - G^{(1)} \right]
\]
\[
= \frac{1}{12n} \int dr_\perp \int \frac{dk_{\perp}}{2\pi} \frac{1}{k_\perp^2 + m^2}.
\]
It is clear to see that the result of \( \text{Tr} G^{(n)} \) is identical to the calculation that is presented in the main text, and of course leads the same result of EE.
Appendix C: The solution of the replicated Green’s function for (1 + 1)D massive free Dirac field

1. A direct derivation of the spinor Green’s function on 2D replica manifold

In this section, we present a detailed derivation of the replicated Green’s function for the (1 + 1)D free Dirac field. The Lagrangian density of Dirac field in 2D (Euclidean) space is

\[ \mathcal{L} = \overline{\Psi} (\gamma^\mu \partial_\mu + m) \Psi. \]  

We choose the representation of gamma matrices to be

\[ \gamma^0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = i \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \]  

By applying variation to the Lagrangian, we have the spinor Green’s function satisfies

\[ (\gamma^\mu \partial_\mu + m) G^{(n)}_D(r, r') = \delta^{(2)}(r - r'), \]  

its explicit matrix form is

\[ \begin{pmatrix} m & \partial_x - i \partial_y \\ \partial_x + i \partial_y & m \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} \delta^2(r - r') & 0 \\ 0 & \delta^2(r - r') \end{pmatrix}. \]  

To calculate the Green’s function we solve the eigenvalue problem

\[ \gamma^\mu \partial_\mu \Psi = -\lambda \Psi, \]  

It is important to notice that the spinor differential operator \( \gamma^\mu \partial_\mu \) is anti-hermitian, so that its eigenvalue is purely imaginary. For convenience, we rewrite the above equation to be

\[ \gamma^\mu \partial_\mu \Psi = -i \lambda \Psi, \]  

with \( \lambda \) real.

Write it explicitly in the matrix form we have

\[ \begin{pmatrix} 0 & \partial_x - i \partial_y \\ \partial_x + i \partial_y & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = -i \lambda \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \]  

In the polar coordinates, we have

\[ \partial_x - i \partial_y = e^{-i\theta} \left[ \partial_\rho - \frac{i}{\rho} \partial_\theta \right], \quad \partial_x + i \partial_y = e^{i\theta} \left[ \partial_\rho + \frac{i}{\rho} \partial_\theta \right], \]  

use this to translate the eigenvalue problem, it becomes

\[ e^{i\theta} \left[ \partial_\rho - \frac{i}{\rho} \partial_\theta \right] \Psi_2(\rho, \theta) = -i \lambda \Psi_1(\rho, \theta), \]
\[ e^{-i\theta} \left[ \partial_\rho + \frac{i}{\rho} \partial_\theta \right] \Psi_1(\rho, \theta) = -i \lambda \Psi_2(\rho, \theta). \]  

Assuming the solution of \( \Psi_1 \) has the form

\[ \Psi_1(\rho, \theta) = A e^{i\nu \theta} R_1(\rho), \]  

where \( \nu = q/n \) with integer \( q \) (take both negative and non-negative values). This form satisfies the periodic boundary condition in the angular direction \( \Psi_1(\rho, \theta + 2\pi n) = \Psi_1(\rho, \theta) \), and it gives

\[ e^{i\theta} \left[ \partial_\rho + \frac{i}{\rho} \partial_\theta \right] \Psi_1(\rho, \theta) = e^{i\theta} \left[ \partial_\rho + \frac{i}{\rho} \partial_\theta \right] \left[ A e^{i\nu \theta} R_1(\rho) \right] = A e^{i(\nu + 1)\theta} \left[ \partial_\rho R_1(\rho) - \frac{\nu}{\rho} R_1(\rho) \right]. \]  

According to this, we assume

\[ \Psi_2(\rho, \theta) = A e^{i(\nu + 1)\theta} R_2(\rho). \]
We then have
\[-i\lambda R_1(\rho) = \left[ \frac{d}{d\rho} + \frac{\nu + 1}{\rho} \right] R_2(\rho), \quad -i\lambda R_2(\rho) = \left[ \frac{d}{d\rho} - \frac{\nu}{\rho} \right] R_1(\rho).\] (C13)

This gives
\[\rho^2 \frac{d^2 R_1(\rho)}{d\rho^2} + \rho \frac{dR_1(\rho)}{d\rho} + (\lambda^2 \rho^2 - \nu^2) R_1(\rho) = 0,\]
\[\rho^2 \frac{d^2 R_2(\rho)}{d\rho^2} + \rho \frac{dR_2(\rho)}{d\rho} + (\lambda^2 \rho^2 - (\nu + 1)^2) R_2(\rho) = 0,\] (C14)

which is the \((\nu + 1)\)-th order Bessel equation, its solution is the Bessel function
\[R_1(\rho) = J_\nu(\lambda \rho), \quad R_2(\rho) = J_{\nu+1}(\lambda \rho).\] (C15)

Finally, we have the solution of the eigenvalue problem as
\[\Psi = A \left( e^{i\nu \theta} J_\nu(\lambda \rho) \right),\] (C16)
where \(A\) is the normalization factor. Note, there is still no constrain on the value that the eigenvalue \(\lambda\) can take, and \(\nu = q/n\) that \(q\) runs over all integers (including negative and zero).

We then require the boundary condition that \(R_1(\rho)\) vanishes at the boundary \(\rho = L\), which means that the eigenvalues satisfy
\[\lambda_{\nu,i} = \frac{\alpha_{\nu,i}}{L},\] (C17)

where \(\alpha_{\nu,i}\) is the zeros of \(\nu\)-th order Bessel function of the first kind. It is important to notice that the solution of eigenfunction has a “particle-hole” symmetry with respect to the sign of the eigenvalue \(\lambda_{\nu,i}\) (it is also the sign of the “angular momentum” \(\nu\)). When we switch the sign of the eigenvalue \(\lambda \rightarrow -\lambda\), the choice of \(R_1(\rho) \rightarrow -R_1(\rho)\) makes the differential equation of Eq. (C14) unchanged. Require the form of Eq. (C16) is only valid for the positive eigenvalues, then for negative eigenvalues we have
\[\Psi^- = A \left( e^{i(\nu+1)\theta} J_{\nu+1}(\lambda \rho) \right).\] (C18)

The normalization factor \(A_{\nu,i}\) can be calculated as
\[1 = \int_0^L \rho d\rho \int_0^{2\pi} d\theta \Psi(\rho, \theta) \Psi(\rho, \theta) = |A_{\nu,i}|^2 \int_0^L \rho d\rho \int_0^{2\pi} \left[ \left| J_\nu(\lambda_{\nu,i} \rho) \right|^2 + \left| J_{\nu+1}(\lambda_{\nu,i} \rho) \right|^2 \right] \]
\[= |A_{\nu,i}|^2 2\pi n \left\{ \frac{L^2}{2} \left| J_{\nu+1}(\lambda_{\nu,i} L) \right|^2 + \frac{L^2}{2} \left| J_\nu(\lambda_{\nu,i} L) \right|^2 \right\} \]
\[\implies \quad |A_{\nu,i}|^2 = \frac{1}{2\pi n L^2 \left| J_{\nu+1}(\lambda_{\nu,i} L) \right|^2} \] (C19)

Note that the above equation has no typo, the two integrals of Bessel functions with different order indeed give the same result due to the nice properties that \(\lambda_{\nu,i} L\) is the zero of \(J_\nu\). This is ensured by the following fact. First, recall that we have the recurrence relationship
\[\frac{d}{dx} \left[ J_\nu(x) x^{-\nu} \right] = -J_{\nu+1}(x) x^{-\nu}, \quad \frac{d}{dx} \left[ x^\nu J_\nu(x) \right] = x^\nu J_{\nu-1}(x).\] (C20)

If we take the zeros \(x = \alpha_{\nu,i}\), it becomes
\[J'_\nu(\alpha_{\nu,i}) = -J_{\nu+1}(\alpha_{\nu,i}), \quad J'_\nu(\alpha_{\nu,i}) = J_{\nu-1}(\alpha_{\nu,i}).\] (C21)

Now we see that the \((\nu + 1)\)-th and \((\nu - 1)\)-th Bessel functions are just different in a sign. Second, we have the integral of the double product of Bessel functions as
\[\int t dt [J_\nu(at)]^2 = \frac{t^2}{2} \left\{ [J_\nu(at)]^2 - J_{\nu+1}(at) J_{\nu-1}(at) \right\}.\] (C22)
By using
\[-J_{\nu+1}(\alpha_{\nu,i}) = J_{\nu-1}(\alpha_{\nu,i}),\quad (C23)\]
we have
\[
\int_0^L t dt [J_{\nu}(\frac{\alpha_{\nu,i}}{L})]^2 = -\frac{L^2}{2} J_{\nu+1}(\alpha_{\nu,i}) J_{\nu-1}(\alpha_{\nu,i}) = \frac{L^2}{2} [J_{\nu+1}(\alpha_{\nu,i})]^2.
\quad (C24)
\]
On the other hand
\[
\int_0^L t dt [J_{\nu+1}(\frac{\alpha_{\nu,i}}{L})]^2 = \frac{L^2}{2} [J_{\nu+1}(\alpha_{\nu,i})]^2.
\quad (C25)
\]
We move to the calculation of Green’s function, let
\[
G_D^{(n)}(\rho, \theta; \rho', \theta') = \sum_{\nu,i} C_{\nu,i,n} \psi_{\nu,i,n}(\rho, \theta).
\quad (C26)
\]
Substitute it back to the Dirac equation, we have
\[
\sum_{\nu,i} (i\lambda_{\nu,i} + m) C_{\nu,i,n} \psi_{\nu,i,n} = \delta(\rho - \rho') \delta(\theta - \theta'),
\quad (C27)
\]
which leads to
\[
G_D^{(n)}(\rho, \theta; \rho', \theta') = \sum_{\nu,i} \frac{1}{i\lambda_{\nu,i} + m} \psi_{\nu,i,n}(\rho, \theta) \psi_{\nu,i,n}^\dagger(\rho', \theta').
\quad (C28)
\]
Note that here the eigenvalues \(\lambda_{\nu,i} = \alpha_{\nu,i}/L\) can take both negative and non-negative values.
Separation of the negative and non-negative will simplify the calculation of the Green’s function, as
\[
G_D^{(n)}(\rho, \theta; \rho', \theta') = \sum_{\nu,i} \frac{\psi_{\nu,i,n,+}(\rho, \theta) \psi_{\nu,i,n,+}^\dagger(\rho', \theta')}{i\lambda_{\nu,i} + m} + \sum_{\nu,i} \frac{\psi_{\nu,i,n,-}(\rho, \theta) \psi_{\nu,i,n,-}^\dagger(\rho', \theta')}{-i\lambda_{\nu,i} + m}
\]
\[
= \sum_{\nu,i} |A_{\nu,i}|^2 \left(\frac{e^{i\nu(\theta' - \theta)} J_{\nu}(\lambda\rho) J_{\nu}(\lambda\rho')}{e^{i\nu(\theta - \theta')} e^{i\theta} J_{\nu+1}(\lambda\rho) J_{\nu}(\lambda\rho')} - \frac{e^{i\nu(\theta - \theta')} e^{-i\theta} J_{\nu}(\lambda\rho) J_{\nu+1}(\lambda\rho')}{e^{i(\nu+1)(\theta - \theta')} J_{\nu+1}(\lambda\rho) J_{\nu+1}(\lambda\rho')} \right)
\quad (C29)
\]
\[
+ \sum_{\nu,i} |A_{\nu,i}|^2 \left(\frac{-e^{i\nu(\theta - \theta')} e^{i\theta} J_{\nu}(\lambda\rho) J_{\nu+1}(\lambda\rho')}{-e^{i\nu(\theta' - \theta)} e^{-i\theta} J_{\nu+1}(\lambda\rho) J_{\nu}(\lambda\rho')} + \frac{2i e^{i(\nu+1)(\theta - \theta')} e^{i\theta} J_{\nu+1}(\lambda\rho) J_{\nu+1}(\lambda\rho')} {2i e^{i(\nu+1)(\theta' - \theta)} e^{i\theta} J_{\nu+1}(\lambda\rho) J_{\nu+1}(\lambda\rho')} \right)
\]
where we have set \(\lambda_{\nu,i} \geq 0\). Note that here the index \(\nu = q/n\), and \(q\) takes both negative and non-negative integers.
Here we do not need to worry about the problem of double counting on the zero eigenvalue. The double counted terms of \(\lambda = 0\) is the zero of all non-zeroth order Bessel functions, so that all double counted terms vanish.
Now we extend the solution into thermodynamic limit \(L \to \infty\). The normalization factor becomes
\[
\lim_{L \to \infty} |A_{\nu,i}|^2 = \lim_{L \to \infty} \frac{1}{2\pi n L^2 [J_{\nu+1}(\lambda L)]^2} = \frac{\lambda}{4\pi n L},
\quad (C30)
\]
and the value of \(\lambda\) becomes continuous for each given order \(\nu\). This means that the summation of index \(i\) changes into the integral over \(\lambda\) via \(\sum_i \to \frac{1}{2\pi} \int_0^\infty d\lambda\), then we have the spinor Green’s function
\[
G_D^{(n)}(\rho, \theta; \rho', \theta') = \frac{1}{4\pi n} \sum_\nu e^{i\nu(\theta - \theta')} \int_0^\infty d\lambda \frac{\lambda}{\lambda^2 + m^2} \left( \begin{array}{cc}
J_{\nu}(\lambda\rho) J_{\nu}(\lambda\rho') & e^{i\nu(\theta - \theta')} J_{\nu}(\lambda\rho) J_{\nu+1}(\lambda\rho') \\
e^{-i\theta} J_{\nu+1}(\lambda\rho) J_{\nu}(\lambda\rho') & e^{i(\nu+1)(\theta - \theta')} J_{\nu+1}(\lambda\rho) J_{\nu+1}(\lambda\rho')
\end{array} \right),
\quad (C31)
\]
where \(\nu = q/n\) with integer \(q\).
2. Entanglement entropy of (1 + 1)D free Dirac field

As long as the 2D replicated Green's function is obtained, we can calculate the entanglement entropy of (1 + 1)D free Dirac field. The relation between the partition function and the Green’s function is

\[- \frac{\partial \ln Z_D}{\partial m} = \text{Tr} G_D.\]

Here the trace contains the sum over diagonal components of the spinor Green's function, so that requires the explicit form of them. The integral of diagonal components has been already calculated in the free scalar field. The sum over index \(\nu\) has the following form for both \(G_{11}\) and \(G_{22}\)

\[
\frac{m}{4\pi n} \sum_{q=-\infty}^{\infty} e^{i\frac{\pi}{2}(\theta - \theta')} \int_0^\infty d\lambda \frac{\lambda}{\lambda^2 + m^2} J_q/n(\lambda \rho) J_q/n(\lambda \rho')
\]

\[
= \frac{m}{4\pi n} \sum_{q=-\infty}^{\infty} e^{i\frac{\pi}{2}(\theta - \theta')} \frac{\pi i}{2} \left[ \theta(\rho' - \rho) J_q/n(im \rho) H_n^{(1)}(im \rho') + \theta(\rho - \rho') J_q/n(im \rho) H_n^{(1)}(im \rho) \right]
\]

(C33)

Then the trace of the Green’s function becomes

\[
\text{Tr} G_D^{(n)} = 2 \left( \frac{m}{4\pi n} \right) \int_0^{2\pi/n} \rho d\theta \int_0^\infty \rho d\rho \sum_{q=0}^{\infty} d_q I_q/n(m \rho) K_q/n(m \rho) + \int_0^\infty \rho d\rho \sum_{q=0}^{\infty} d_q I_q/n(m \rho) K_q/n(m \rho),
\]

(C34)

The summation over \(q\) is UV divergent, so that we introduce a renormalization function \(F(q/m)\), which is chosen that \(F(0) = 1\) and \(F(k)(0) = 0\). Similar to the 2D free scalar field, from the Euler-Maclaurin formula, we have

\[
\sum_{q=0}^{\infty} d_q I_q/n(m \rho) K_q/n(m \rho) F(\frac{q}{n \Lambda}) = 2 \int_0^\infty I_q/n(m \rho) K_q/n(m \rho) F(\frac{q}{n \Lambda}) dq + \frac{1}{6n} [K_0(m \rho)]^2.
\]

(C35)

Then

\[
\text{Tr} G_D^{(n)} = m \int_0^\infty \rho d\rho \left[ 2 \int_0^\infty I_q/n(m \rho) K_q/n(m \rho) F(\frac{q}{n \Lambda}) dq + \frac{1}{6n} [K_0(m \rho)]^2 \right] + \frac{m}{6n} \int_0^\infty \rho d\rho [K_0(m \rho)]^2
\]

(C36)

\[
= 2mnC(m) + \frac{m}{6n} \left[ \frac{1}{2m^2} \right] = 2mnC(m) + \frac{1}{12mn}
\]

From this we have

\[
- \frac{\partial \ln Z_D^{(n)}}{\partial m} = \text{Tr}^{(n)}(n) G_D^{(n)} - n \text{Tr}^{(1)} G_D^{(1)} = \frac{1 - n^2}{12mn}.
\]

(C37)

We intermediately notice that this will lead to the same partition function as the free scalar field

\[
\ln \left[ \frac{Z_D^{(n)}}{Z_D^{(1)}} \right] = \frac{n^2 - 1}{12n} \ln m + C.
\]

(C38)

By letting \(C = \frac{n^2 - 1}{12n} \ln \epsilon\), it gives the entanglement entropy

\[
S_D = - \frac{\partial \ln Z_D^{(n)}}{\partial m} \bigg|_{n=1} = \frac{1}{6} \ln(m \epsilon),
\]

(C39)

where \(\epsilon\) plays the role of UV cutoff of the theory (lattice constant).
3. Reduction to the usual flat spinor Green’s function at the replica limit $n \to 1$

In this section, we show that by taking the replica limit $n \to 1$, the above solved replicated Green’s function is reduced to the usual flat one. The key is using the Addition Theorem of the Bessel function.

Starting from the diagonal component, they are

$$G_{11}^{(1)} = G_{22}^{(1)} = \frac{m}{4\pi} \sum_{q} e^{iq(\theta-\theta')} \int_{0}^{\infty} \frac{\lambda J_{q}(\lambda\rho) J_{q}(\lambda\rho')}{\lambda^{2} + m^{2}} \, d\lambda \quad (C40)$$

One can exchange the summation and the integral, it gives

$$G_{11}^{(1)} = G_{22}^{(1)} = \frac{m}{4\pi} \int_{0}^{\infty} d\lambda \frac{\lambda}{\lambda^{2} + m^{2}} \sum_{q} e^{iq(\theta-\theta')} J_{q}(\lambda\rho) J_{q}(\lambda\rho') = \frac{m}{4\pi} \int_{0}^{\infty} \frac{\lambda J_{0}(\lambda R)}{\lambda^{2} + m^{2}} \, d\lambda = \frac{m}{4\pi} K_{0}(mR), \quad (C41)$$

where we have applied the (Neumann) Addition Theorem of the Bessel function

$$J_{0}(R) = \sum_{0}^{\infty} d_{q} \cos[q(\theta - \theta')] J_{q}(\rho) J_{q}(\rho'),$$

with $R = \sqrt{\rho^{2} + \rho'^{2} - 2\rho\rho' \cos(\theta - \theta')}$ is the distance between $(\rho, \theta)$ and $(\rho', \theta')$.

For the off-diagonal component, we have

$$G_{12}^{(1)} = \frac{i}{4\pi} \sum_{q} e^{iq(\theta-\theta')} \int_{0}^{\infty} d\lambda \frac{\lambda^{2}}{\lambda^{2} + m^{2}} e^{-i\theta'} J_{q}(\lambda\rho) J_{q+1}(\lambda\rho')$$

$$= \frac{i}{4\pi} \int_{0}^{\infty} d\lambda \frac{\lambda^{2}}{\lambda^{2} + m^{2}} e^{-i\theta'} \sum_{q} e^{iq(\theta-\theta')} J_{q}(\lambda\rho) J_{q+1}(\lambda\rho') = \frac{i}{4\pi} \int_{0}^{\infty} \frac{\lambda^{2} J_{1}(\lambda R)}{\lambda^{2} + m^{2}} e^{-i\theta'} \left[ \frac{\rho' - \rho e^{-i(\theta-\theta')}}{\rho' - \rho e^{i(\theta-\theta')}} \right]^{\frac{1}{2}} \quad (C43)$$

Similarly

$$G_{21}^{(1)} = \frac{i}{4\pi} \sum_{q} e^{iq(\theta-\theta')} \int_{0}^{\infty} d\lambda \frac{\lambda^{2}}{\lambda^{2} + m^{2}} e^{i\theta} J_{q+1}(\lambda\rho) J_{q}(\lambda\rho')$$

$$= \frac{i}{4\pi} \int_{0}^{\infty} d\lambda \frac{\lambda^{2}}{\lambda^{2} + m^{2}} e^{i\theta} \sum_{q} e^{iq(\theta-\theta')} J_{q}(\lambda\rho) J_{q-1}(\lambda\rho') = \frac{i}{4\pi} \int_{0}^{\infty} \frac{\lambda^{2} J_{-1}(\lambda R)}{\lambda^{2} + m^{2}} e^{i\theta} \left[ \frac{\rho' - \rho e^{-i(\theta-\theta')}}{\rho' - \rho e^{i(\theta-\theta')}} \right]^{\frac{1}{2}} \quad (C44)$$

This is identical to the solution of usual Green’s function in flat Euclidean spacetime, which has the following form

$$G_{ab}(x, x') = \int \frac{d^{2}k}{(2\pi)^{2}} e^{ik(x-x')} (i\gamma^{k} k_{i} + m)_{ab} = \frac{m}{2\pi} \left( K_{0}(mR) e^{-i\arctan \frac{\rho}{\rho'}} K_{1}(mR) \right). \quad (C45)$$

The difference of a factor $\frac{1}{2}$ comes from the choice of normalization condition. In our approach, we choose to normalize the entire spinor. In the usual convention, the normalization is taken for each component of the spinor.

Appendix D: The Green’s function of the Dirac field under a random static gauge field

In this section, we present a quick derivation of some known results of the Dirac field under a random static gauge field (magnetic field), mainly the usual flat Green’s function in Minkowski spacetime. The Lagrangian density in 2D Euclidean space is

$$\mathcal{L}[\psi] = \bar{\psi} \gamma^{\mu} (\partial_{\mu} + i \sqrt{g_{A}} A_{\mu}) \psi + \bar{\psi} (i \omega \gamma^{0}) \psi, \quad (D1)$$
where $\omega$ is the frequency (energy scale) and $A_\mu$ is the 2D Gaussian-distributed random static gauge field

$$\mathcal{P}(A_\mu) \propto e^{-\frac{1}{2} \int d^2 r_1 A_\mu^2(r_1)}, \quad (D2)$$

with vanishing mean value and variance $g_A$.

Start from the case of $\omega = 0$. First, we apply the Hodge decomposition for the 2D static gauge field

$$A_\mu = \epsilon \partial_\mu \Phi_1(x) + \partial_\mu \Phi_2(x), \quad (D3)$$

where $\epsilon_{\mu\nu}$ is the Levi-Cevita tensor, $\Phi_1$ and $\Phi_2$ are longitudinal and transverse components. This gives

$$\mathcal{L}[\omega = 0] = \overline{\Psi} \{ i \gamma^\mu \left[ \partial_\mu - i \sqrt{g_A} (\epsilon_{\mu\nu} \partial_\nu \Phi_1 + \partial_\nu \Phi_2) \right] \} \Psi \quad (D4)$$

Second, we introduce the following axial gauge transformation

$$\Psi = \overline{\Psi}' e^{i \gamma^5 \sqrt{g_A} \Phi_1 + i \sqrt{g_A} \Phi_2}, \quad \overline{\Psi}' = e^{i \gamma^5 \sqrt{g_A} \Phi_1 + i \sqrt{g_A} \Phi_2} \Psi'. \quad (D5)$$

Here we have chosen $\gamma^5 = \gamma^0 = i \gamma^1 \gamma^2 = \sigma_3$ with $\gamma^1 = i \sigma_1$ and $\gamma^2 = i \sigma_2$. After some straightforward algebra, we have

$$\mathcal{L}_0 = \overline{\Psi}' (i \gamma^\mu \partial_\mu) \Psi'. \quad (D6)$$

To make it easy to be calculated, we further transform the theory into the chiral basis

$$\Psi'_\pm = \Psi'_\pm e^{\mp \sqrt{g_A} \Phi_1 - i \sqrt{g_A} \Phi_2}, \quad \Psi'_\pm = e^{\pm \sqrt{g_A} \Phi_1 + i \sqrt{g_A} \Phi_2} \Psi'_\pm, \quad (D8)$$

and their two-point correlation function

$$\left\langle \Psi'_\pm(z, \sigma) \overline{\Psi}'_\pm(w, \bar{\sigma}) \right\rangle = \left\langle e^{\pm \sqrt{g_A} \Phi_1(z, \sigma) - i \sqrt{g_A} \Phi_2(z, \sigma)} \Psi'_\pm(z, \sigma) \overline{\Psi}'_\pm(w, \bar{\sigma}) e^{\mp \sqrt{g_A} \Phi_1(w, \bar{\sigma}) + i \sqrt{g_A} \Phi_2(w, \bar{\sigma})} \right\rangle \quad (D9)$$

where the correlation function of the chiral Dirac field is

$$\left\langle \Psi'_\pm(z, \sigma) \overline{\Psi}'_\pm(w, \bar{\sigma}) \right\rangle \sim \frac{1}{2\pi} \frac{1}{z - w}, \quad \left\langle \Psi'_\pm(z, \sigma) \overline{\Psi}'_\pm(w, \bar{\sigma}) \right\rangle \sim \frac{1}{2\pi} \frac{1}{\overline{z} - \overline{w}}, \quad (D10)$$

and the correlation function of the axial field is simply the correlation function of the vertex operator of the free scalar field

$$\left\langle e^{+i f \Phi(z, \sigma)} e^{-i f \Phi(w, \bar{\sigma})} \right\rangle \sim |z - w|^{-\frac{2\sqrt{2}}{f}}. \quad (D11)$$

Interestingly, here we will see that, in $\left\langle \Psi'_\pm(z, \sigma) \overline{\Psi}'_\pm(w, \bar{\sigma}) \right\rangle$ the contribution from the longitudinal field $\Phi_1$ cancels with the contribution from the transversal field $\Phi_2$, i.e.

$$\left\langle \Psi'_\pm(z, \sigma) \overline{\Psi}'_\pm(w, \bar{\sigma}) \right\rangle = \left\langle \Psi'_\pm(z, \sigma) \Psi'_\pm(w, \bar{\sigma}) \right\rangle. \quad (D12)$$

We now turn to consider the case of $\omega \neq 0$, where the frequency term can be understood as an interaction

$$\omega \overline{\Psi} \gamma^0 \Psi = \omega \overline{\Psi} e^{2\gamma^5 \sqrt{g_A} \Phi_1} \gamma^0 \Psi' \quad (D13)$$

The $l$-th order tree level diagram is

$$\omega^{l} P'_l(x, x') = (\omega)^l \int d^2 y_1 \cdots d^2 y_l \{ e^{2\gamma^5 \sqrt{g_A} \Phi_1(y_1)} \gamma^0 g'(y_1, y_2) \} \cdots \{ e^{2\gamma^5 \sqrt{g_A} \Phi_1(y_{l-1})} \gamma^0 g'(y_{l-1}, y_l) \} \{ e^{2\gamma^5 \sqrt{g_A} \Phi_1(y_l)} \gamma^0 g'(y_l, x') \} \quad (D14)$$
It is important to notice that the odd order perturbations vanish since the expectation value of the charged vertex operators vanishes. Therefore only even order perturbations contribute to the final result, which are

\[
\left[ e^{2\gamma \phi y} \Phi_1(y_{i-1}) \Phi_1(y_{i}) g'(y_{i+1}, y_i) \right] \left[ e^{2\gamma \phi y} \Phi_1(y_{i-1}) \Phi_1(y_{i}) g'(y_{i+1}, y_i) \right] = \left[ e^{2\gamma \phi y} \Phi_1(y_{i-1}) \Phi_1(y_{i}) g'(y_{i+1}, y_i) \right] \left[ e^{2\gamma \phi y} \Phi_1(y_{i-1}) \Phi_1(y_{i}) g'(y_{i+1}, y_i) \right] \]

\[
\left( f(y_{i+1}, y_i, y_{i+2}) \cdots f(y_{i+1}, y_i, y_{i+3}) \right) = \left( \bar{f}(y_{i+1}, y_i, y_{i+2}) \cdots \bar{f}(y_{i+1}, y_i, y_{i+3}) \right),
\]

where

\[
\begin{align*}
    f(y_{i+1}, y_i, y_{i+2}) &= e^{\gamma \phi y} \Phi_1(y_{i+1}) \Phi_1(y_{i}) e^{-\gamma \phi y} g(y_{i+1}, y_i) g'(y_{i+1}, y_i) \\
    &= \left( e^{\gamma \phi y} \Phi_1(y_{i+1}) e^{-\gamma \phi y} \right) g(y_{i+1}, y_i) g'(y_{i+1}, y_i), \\
    \bar{f}(y_{i+1}, y_i, y_{i+2}) &= e^{-\gamma \phi y} \Phi_1(y_{i+1}) \Phi_1(y_{i}) e^{\gamma \phi y} g(y_{i+1}, y_i) g'(y_{i+1}, y_i).
\end{align*}
\]

Note that

\[
\left( e^{\gamma \phi y} \Phi_1(y_{i+1}) e^{-\gamma \phi y} \right) = \left( e^{-\gamma \phi y} \Phi_1(y_{i+1}) e^{\gamma \phi y} \right) = (r_{y_{i+1}, y_i})^\frac{g_A}{\phi}.
\]

This means that for even \( l \), we have

\[
(\omega)^l P^l(x, x') = (\omega)^l \int d^2y_1 \cdots d^2y_3 g'(x, y_i) F(y_{i+1}, y_i, y_{i+2}) F(y_{i+1}, y_i, y_{i+2}) \cdots F(y_{i+1}, y_i, x')
\]

where

\[
F(y_{i+1}, y_i, y_{i+2}) = \left( f(y_{i+1}, y_i, y_{i+2}) \right) = \left( e^{\gamma \phi y} \Phi_1(y_{i+1}) e^{-\gamma \phi y} \right) g'(y_{i+1}, y_i) g'(y_{i+1}, y_i).
\]

The most important part is

\[
g_{\text{gauge}}(y_{i+1}, y_i) = \left( e^{\gamma \phi y} \Phi_1(y_{i+1}) e^{-\gamma \phi y} \right) g'(y_{i+1}, y_i) = r_{y_{i+1}, y_i}^{1+\frac{g_A}{\phi}} \left( 0 e^{i\theta} \right).
\]

Its Fourier transformation is

\[
g_{\text{gauge}}(k) = \int_0^\infty r dr \int_0^{2\pi} d\theta e^{-ikr \sin(\theta + \arctan(\frac{k}{2\pi}))} r_{y_{i+1}, y_i}^{1+\frac{g_A}{\phi}} \left( 0 e^{i\theta} \right).
\]

The integral is divergent when \( \frac{g_A}{\phi} \geq \frac{1}{2} \). For \( 0 \leq \frac{g_A}{\phi} < \frac{1}{2} \), we have

\[
g_{\text{gauge}}(k) = 2 \frac{g_A}{\phi} \frac{\Gamma(1 + \frac{g_A}{\phi})}{\Gamma(1 - \frac{g_A}{\phi})} k^{-(1 + \frac{g_A}{\phi})} \left( 0 e^{i\arctan(\frac{k}{2\pi})} \right) = 2 \frac{g_A}{\phi} \frac{\Gamma(1 + \frac{g_A}{\phi})}{\Gamma(1 - \frac{g_A}{\phi})} k^{\frac{g_A}{\phi}} g(k).
\]

Then, for \( 0 < \frac{g_A}{\phi} < \frac{1}{2} \), the summation over tree level diagrams then becomes

\[
G(x, x'; \omega) = \sum_{\text{even } l} (\omega)^l \int \frac{d^2k}{(2\pi)^2} e^{ik(x-x')} \left| g'(k) \right|^{l+1} \left[ 2 \frac{g_A}{\phi} \frac{\Gamma(1 + \frac{g_A}{\phi})}{\Gamma(1 - \frac{g_A}{\phi})} k^{\frac{g_A}{\phi}} g'(k) \omega \right]^l
\]

\[
= \int \frac{d^2k}{(2\pi)^2} e^{ik(x-x')} g'(k) \sum_{l=0}^\infty \sqrt{C(g(k))} k^{\frac{g_A}{\phi}} g'(k) \omega
\]

\[
= \int \frac{d^2k}{(2\pi)^2} e^{ik(x-x')} G^l(k) \left( 1 + \sqrt{C(g(k))} k^{\frac{g_A}{\phi}} g'(k) \omega \right)
\]

\[
= \int \frac{d^2k}{(2\pi)^2} e^{ik(x-x')} \frac{-\gamma k_i}{k^2 - C(g(k)) k^{\frac{g_A}{\phi}} \omega^2}.
\]
where
\[ C(g_A) = 2 \pi i \frac{\Gamma(1 + \frac{g_A}{2})}{\Gamma(1 - \frac{g_A}{2})}. \] (D24)

The Green’s function provides a lot of information of the theory. First, its pole is located at \( C(g_A) k^{2 + \frac{g_A}{2}} = \omega^2 \), this indicates the dispersion relation as
\[ E(k) \propto |k|^{1 + \frac{g_A}{2}}. \] (D25)

Second, we can calculate the density of states from it
\[ \rho(\omega) = \frac{1}{2\pi i} \lim_{x \to x'} [G(x, x'; \omega)_{\text{adv}} - G(x, x'; \omega)_{\text{rest}}] = \frac{1}{\pi i} \Im \left[ \lim_{x \to x'} G(x, x'; \omega)_{\text{adv}} \right] \]
\[ = \frac{1}{\pi i} \frac{1}{2\pi} \Im \left[ \int_0^\infty dk \frac{(-i\gamma^i k_i) \left[ C(g_A) k^{\frac{g_A}{2}} \right]}{C(g_A) k^{2 + \frac{g_A}{2}} - \omega^2 - i\delta} \right]. \] (D26)

The scaling behavior of \( \rho(\omega) \) does not depend on the phase factor and the constant \( C(g_A) \), so that the task is reduced to evaluate the following integral
\[ \int_0^\infty dk \frac{k^{2 + \frac{g_A}{2}}}{k^{1 + \frac{g_A}{2}} - \omega^2 - i\delta} = \frac{1}{2\pi} \int_0^\infty dk \left[ \frac{k^{2 + \frac{g_A}{2}}}{k^{1 + \frac{g_A}{2}} - \omega - i\delta} - \frac{k^{2 + \frac{g_A}{2}}}{k^{1 + \frac{g_A}{2}} + \omega + i\delta} \right] \]
\[ = \frac{1}{2\pi} \int_0^\infty dt \left[ \frac{\delta^{1 + (1 + \frac{g_A}{2})^{-1}}}{t - \omega - i\delta} - \frac{\delta^{1 + (1 + \frac{g_A}{2})^{-1}}}{t + \omega + i\delta} \right] = \frac{1}{2\pi} \left[ I_1 + \pi i \omega^{1 + (1 + \frac{g_A}{2})^{-1}} - I_2 + \pi i \omega^{1 + (1 + \frac{g_A}{2})^{-1}} \right] \] (D27)

The imaginary part of the above integral gives the density of states
\[ \rho(\omega) \propto \omega^{\frac{1}{1 + \frac{g_A}{2}}}. \] (D28)

**Appendix E: The replicated Green’s function of the Dirac field under a random static gauge field**

**1. The Fourier transformation of the corrected internal spinor propagator**

The presence of a random static gauge field leads to the corrected internal 2D propagator of the Dirac field
\[ \tilde{g}_D^{(1)}(r_{||,1}, r_{||,2}) = g_D^{(1)}(r_{||,1}, r_{||,2}) R_1^a. \] (E1)

Evaluating its Fourier transformation is the key to access the higher-dimensional construction
\[ \tilde{g}_D^{(1)}(k) = \int d^2 R e^{i k \mathbf{r}_{||,1} - \mathbf{r}_{||,2}} \tilde{G}_D^{(1)}(r_{||,1}, r_{||,2}). \] (E2)

The diagonal components are
\[ \left[ \tilde{g}_D^{(1)}(k) \right]_{11} = \left[ \tilde{g}_D^{(1)}(k) \right]_{22} = \int d^2 R e^{-i k \mathbf{R}} R^a \frac{M}{2\pi} K_0(MR) \]
\[ = \int_0^\infty dR R^a J_0(kR) K_0(MR) dR \]
\[ = M \left( 2^a M^{-2-a} \right) \left[ \Gamma \left( 1 + \frac{a}{2} \right) \right]^2 \frac{\Gamma \left( \frac{a}{2} + 1, \frac{a}{2} + 1; - \frac{k^2}{M^2} \right)}{2 F_1 \left( \frac{a}{2} + 1, \frac{a}{2} + 1; - \frac{k^2}{M^2} \right)}. \] (E3)
and the off-diagonal components are
\[
\left[\hat{g}_D^{(1)}(k)\right]_{12} = -\left[\bar{g}_D^{(1)}(k)\right]^{*}_{21} = \int d^2R e^{-ikR} R^a(i) \frac{M}{2\pi} e^{i\arctan \frac{\delta}{\pi} / 2} K_1(MR) \\
= \frac{iM}{2\pi} \int_0^{2\pi} R(dR) \int_0^{\pi} d\theta e^{-ikR\sin(\theta + \arctan \frac{\delta}{\pi})} e^{i\theta} K_1(MR) R^a \\
= \frac{iM}{2\pi} \int_0^{\infty} dR R^{3+a} K_1(MR) \int_0^{2\pi} d\theta e^{i\theta} \sum_{n=-\infty}^{\infty} (i)^n J_n(-kR) e^{i(n-\frac{a}{2} + \arctan \frac{\delta}{\pi})} \\
= \frac{iM}{2\pi} \int_0^{\infty} dR R^{3+a} K_1(MR)(2\pi)(i)^{-1} J_{-1}(-kR) e^{-i(n-\frac{a}{2} + \arctan \frac{\delta}{\pi})} \\
(E4)
= iMe^{-i\arctan \frac{\delta}{\pi}} \int_0^{\infty} R^{3+a} K_1(MR) J_1(kR) dR \\
= iMe^{-i\arctan \frac{\delta}{\pi}} 2\pi k M^{-a-a} \Gamma \left(1 + \frac{a}{2}\right) \Gamma \left(2 + \frac{a}{2}\right) _2\mathrm{F}_1 \left(\frac{a}{2} + 1, \frac{a}{2} + 2; -\frac{k^2}{M^2}\right) \\
= (ik_1 + k_2) 2\pi M^{-2-a} \Gamma \left(1 + \frac{a}{2}\right) \Gamma \left(2 + \frac{a}{2}\right) _2\mathrm{F}_1 \left(\frac{a}{2} + 1, \frac{a}{2} + 2; -\frac{k^2}{M^2}\right).
\]

The above is the non-perturbative results of the internal propagator, we then expand them to the lowest order with respect to \(a\). For that, on the one hand, the series of momentum-independent coefficients are
\[
2^a M^{-2-a} \left[\Gamma \left(1 + \frac{a}{2}\right)\right]^2 = \frac{1}{M^2} + \frac{(-\gamma + \ln 2 - \ln M)a}{M^2} + \mathcal{O}(a^2) \quad (E5)
\]

and
\[
2^a M^{-2-a} \Gamma \left(1 + \frac{a}{2}\right) \Gamma \left(2 + \frac{a}{2}\right) = \frac{1}{M^2} + \frac{(-\gamma + \ln 2 - \ln M)a}{M^2} + \frac{a}{2M^2} + \mathcal{O}(a^2), \quad (E6)
\]
where \(\gamma \simeq 0.577216\) is the Euler’s constant. On the other hand, for expanding the hypergeometric function, we need to calculate its derivative with respect to the parameters
\[
\frac{\partial}{\partial \alpha} _2\mathrm{F}_1 \left(\alpha, 1; 1; -\frac{k^2}{M^2}\right) \bigg|_{\alpha=1} = \frac{\partial}{\partial \beta} _2\mathrm{F}_1 \left(1, \beta; 1; -\frac{k^2}{M^2}\right) \bigg|_{\beta=1} = \frac{\partial}{\partial \alpha} _2\mathrm{F}_1 \left(\alpha, 2; 2; -\frac{k^2}{M^2}\right) \bigg|_{\alpha=1} \quad (E7)
\]
\[
= -\frac{k^2}{M^2} \sum_{n=0}^{\infty} \frac{(1)_n (1)_n (2)_n}{(2)_n} \frac{(-\frac{k^2}{M^2})^n}{n!} _3\mathrm{F}_2 \left(1, n + 2, n + 2; n + 2, n + 2; -\frac{k^2}{M^2}\right) \\
= -\frac{k^2}{M^2} \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(2)_n} \frac{(-\frac{k^2}{M^2})^n}{n!} \frac{1}{1 + \frac{k^2}{M^2}} = -\frac{k^2}{k^2 + M^2} _2\mathrm{F}_1 \left(1, 1; 2; -\frac{k^2}{M^2}\right) \\
= -\frac{k^2}{k^2 + M^2} \left(\frac{k^2}{M^2}\right)^{-1} \ln \left(1 + \frac{k^2}{M^2}\right) = -\left(1 + \frac{k^2}{M^2}\right)^{-1} \ln \left(1 + \frac{k^2}{M^2}\right),
\]
and
\[
\frac{\partial}{\partial \beta} _2\mathrm{F}_1 \left(1, \beta; 2; -\frac{k^2}{M^2}\right) \bigg|_{\beta=2} = -\frac{k^2}{M^2} \sum_{n=0}^{\infty} \frac{(1)_n (2)_n (2)_n}{(3)_n} \frac{(-\frac{k^2}{M^2})^n}{n!} _3\mathrm{F}_2 \left(1, n + 2, n + 3; n + 2, n + 3; -\frac{k^2}{M^2}\right) \quad (E8)
\]
\[
= -\frac{1}{2} \frac{k^2}{k^2 + M^2} _2\mathrm{F}_1 \left(1, 2; 3; -\frac{k^2}{M^2}\right) = -\frac{1}{2} \frac{k^2}{k^2 + M^2} 2 \frac{k^2 M^2 - M^4 \ln \left(1 + \frac{k^2}{M^2}\right)}{k^4} \\
= -\left(1 + \frac{k^2}{M^2}\right)^{-1} + \left(\frac{k^2}{M^2}\right)^{-1} \ln \left(1 + \frac{k^2}{M^2}\right),
\]
with the zero-th order contribution

\[ zF_1 \left( 1, 1; 1; -\frac{k^2}{M^2} \right) = zF_1 \left( 1, 2; 2; -\frac{k^2}{M^2} \right) = \left( 1 + \frac{k^2}{M^2} \right)^{-1}. \]  

(E9)

These lead to the expansion of \( \tilde{g}_D^{(1)}(k) \) at the lowest-order with respect to \( a \)

\[
\left[ \tilde{g}_D^{(1)}(k) \right]_{11} = \left[ \tilde{g}_D^{(1)}(k) \right]_{22} = M \left( \frac{1}{M^2} + \frac{-\gamma + \ln 2 - \ln M}{M^2} \right) \left( 1 + \frac{k^2}{M^2} \right)^{-1} \left[ 1 - a \ln \left( 1 + \frac{k^2}{M^2} \right) + \mathcal{O}(a^2) \right]
\]

(E10)

and

\[
\left[ \tilde{g}_D^{(1)}(k) \right]_{12} = -\left[ \tilde{g}_D^{(1)}(k) \right]_{21} = i k_1 + k_2 \left\{ \frac{k^2}{M^2} + \frac{a k_1 + k_2}{k^2} \ln \left( 1 + \frac{k^2}{M^2} \right) + \mathcal{O}(a^2) \right\}
\]

(E11)

2. The scalar nature of the product of corrected and free spinor propagators

The ordinary double product of spinor propagators for the free Dirac field gives a diagonal scalar propagator

\[
\gamma^0 g_D^{(1)}(k) \gamma^0 g_D^{(1)}(k) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{M}{k^2 + M^2} & \frac{i k_1 + k_2}{k^2 + M^2} \\ \frac{i k_1 - k_2}{k^2 - M^2} & \frac{-2}{k^2 + M^2} \end{pmatrix} = \frac{\text{Id}}{k^2 + M^2}.
\]

(E12)

Consider the correction from the static gauge field, its contribution to diagonal components of \( g_D^{(1)}(k) \) is

\[
a \left[ -\gamma + \ln 2 + \ln M - \ln \left( k^2 + M^2 \right) \right] \gamma^0 g_D^{(1)}(k) \gamma^0 g_D^{(1)}(k) = \frac{a}{k^2 + M^2} \left[ -\gamma + \ln 2 + \ln M - \ln \left( k^2 + M^2 \right) \right] \text{Id}. \]

(E13)

Besides, there is an anisotropic term between the diagonal and off-diagonal components, which contributes

\[
\frac{a}{2} \ln \left( 1 + \frac{k^2}{M^2} \right) \gamma^0 \begin{pmatrix} 0 & i k_1 + k_2 \\ -i k_1 - k_2 & 0 \end{pmatrix} \gamma^0 g_D^{(1)}(k) = \frac{a}{2 \left( k^2 + M^2 \right)} \ln \left( 1 + \frac{k^2}{M^2} \right) \text{Id}
\]

In summary, we have the correction to the double product of \( G_D^{(1)} \)

\[
\gamma^0 \tilde{g}_D^{(1)}(k) \gamma^0 g_D^{(1)}(k) = \frac{1}{k^2 + M^2} \left\{ 1 + a \left[ -\gamma + \ln 2 - \frac{1}{2} \ln \left( k^2 + M^2 \right) \right] \right\} \text{Id} + \mathcal{O}(a^2).
\]

(E15)

3. The construction from the 2D corrected propagator to 3D theory

The above calculation leads to the elements of the infinite series that are used to construct the higher-dimensional theory, namely the interacting two-point correlator on 2D replica manifold

\[
(\omega^2)^l P_{D,2l,\text{int}}^{(n)}(r_{\parallel}, r'_{\parallel}) = (l + 1) (\omega^2)^l \int d^2r_{\parallel,1} \cdots \int d^2r_{\parallel,2l} g_D^{(1)}(r_{\parallel,1}, r_{\parallel,1}) \gamma^0 g_D^{(1)}(r_{\parallel,2}, r_{\parallel,2}) R_{1,2}^{n} \gamma^0 \tilde{g}_D^{(1)}(r_{\parallel,2l-1}, r_{\parallel,2l}) R_{2l-1,2l}^{n} \gamma^0 g_D^{(1)}(r_{\parallel,2l}, r'_{\parallel})
\]

\[
= (l + 1) (\omega^2)^l \int \frac{d^2k'}{(2\pi)^2} e^{ik'_1 r_{\parallel,1}} \tilde{g}_D^{(n)}(k') \left[ \gamma^0 g_D^{(1)}(k) \gamma^0 g_D^{(1)}(k) \right]^l
\]

\[
= (l + 1) (\omega^2)^l \int \frac{d^2k'}{(2\pi)^2} e^{ik'_1 r_{\parallel,1}} g_D^{(n)}(k) \frac{1 + la \left[ -\gamma + \ln 2 - \frac{1}{2} \ln \left( k^2 + M^2 \right) \right]}{(k^2 + M^2)^l} + \mathcal{O}(a^2).
\]

(E16)
Its regularization at lowest-order perturbation with respect to \( a \) is

\[
(-\omega^2)^l \left[ P^{(n)}_{D,2l,\text{int}}(\mathbf{r}_\parallel, \mathbf{r}'_\parallel) - P^{(1)}_{D,2l,\text{int}}(\mathbf{r}_\parallel, \mathbf{r}'_\parallel) \right]
\]

\[
= (l + 1)(-\omega^2)^l \frac{1 - n^2 M}{6n^2} \frac{M K_0(M \rho)}{2\pi} \int_0^\infty \frac{k J_0(k \rho')}{(k^2 + M^2)^{l+1}} dk \left( 1 + l a \left[ -\gamma + \ln 2 - \frac{1}{2} \ln \left( k^2 + M^2 \right) \right] \right)^{(l+1)}. \quad \text{(E17)}
\]

4. The partition function of constructed 3D theory

To calculate the partition function, we start from evaluating the trace of the corrected replicated Green’s function \((-\omega^2)^l P^{(n)}_{D,2l,\text{int}}\). In the previous section, we have presented the explicit evaluation of its lowest-order perturbation with respect to the randomness strength \( \alpha = \frac{\int \gamma}{2\pi} \). We now separate it into three parts, the first one is

\[
(-\omega^2)^l \operatorname{Tr}_{2D} \left[ P^{(n)}_{D,2l,\text{int}}(\mathbf{r}_\parallel, \mathbf{r}'_\parallel) - P^{(1)}_{D,2l,\text{int}}(\mathbf{r}_\parallel, \mathbf{r}'_\parallel) \right]_{\text{first}}
\]

\[
= \operatorname{Tr}_{2D} \left[ (l + 1)(-\omega^2)^l \frac{1 - n^2 M}{6n^2} \frac{M K_0(M \rho)}{2\pi} \int_0^\infty \frac{k J_0(k \rho')}{(k^2 + M^2)^{l+1}} dk \right]_{\text{first}} \operatorname{Tr}_{2D} \left[ P^{(n)}_{D,2l,\text{int}} - P^{(1)}_{D,2l,\text{int}} \right] = \frac{1 - n^2}{6n} \frac{M}{\omega^2 + M^2}, \quad \text{(E18)}
\]

The summation over perturbation levels \( l \) gives

\[
\operatorname{Tr}_{2D} \left[ G^{(n)}_{D,\text{int}} - G^{(1)}_{D,\text{int}} \right]_{\text{first}} = \sum_{l=0}^\infty \operatorname{Tr}_{2D} \left[ (-\omega^2)^l \operatorname{Tr}_{2D} \left[ P^{(n)}_{D,2l,\text{int}} - P^{(1)}_{D,2l,\text{int}} \right] \right]_{\text{first}} = \frac{1 - n^2}{6n} \frac{M}{\omega^2 + M^2}. \quad \text{(E19)}
\]

which leads to the free contribution of the EE

\[
S_{\text{first}} = \frac{1}{6} A(\epsilon^{-1} - M). \quad \text{(E20)}
\]

The second part is

\[
(-\omega^2)^l \operatorname{Tr}_{2D} \left[ P^{(n)}_{D,2l,\text{int}}(\mathbf{r}_\parallel, \mathbf{r}'_\parallel) - P^{(1)}_{D,2l,\text{int}}(\mathbf{r}_\parallel, \mathbf{r}'_\parallel) \right]_{\text{second}}
\]

\[
= \operatorname{Tr}_{2D} \left[ (l + 1) I a \left( -\gamma + \ln 2 \right) (-\omega^2)^l \frac{1 - n^2 M}{6n^2} \frac{M K_0(M \rho)}{2\pi} \int_0^\infty \frac{k J_0(k \rho')}{(k^2 + M^2)^{l+1}} dk \right]
\]

\[
= a \left( -\gamma + \ln 2 \right) \frac{1 - n^2}{6n} \frac{M}{M} \left( -\omega^2 \right)^l. \quad \text{(E21)}
\]

The summation over perturbation levels \( l \) gives

\[
\operatorname{Tr}_{2D} \left[ G^{(n)}_{D,\text{int}} - G^{(1)}_{D,\text{int}} \right]_{\text{second}} = \sum_{l=0}^\infty \operatorname{Tr}_{2D} \left[ (-\omega^2)^l \operatorname{Tr}_{2D} \left[ P^{(n)}_{D,2l,\text{int}} - P^{(1)}_{D,2l,\text{int}} \right] \right]_{\text{second}} = \frac{1 - n^2}{6n} a \left( -\gamma + \ln 2 \right) \frac{M}{\left( \omega^2 + M^2 \right)^2}. \quad \text{(E22)}
\]

which leads to the correction to EE as

\[
S_{\text{second}} = \left( -\frac{a}{2} \right) \left( -\gamma + \ln 2 \right) \frac{1}{6} A(\epsilon^{-1} - M). \quad \text{(E23)}
\]
The third part is much more complicated

\[ (-\omega^2)^l \text{Tr}_{2D}^{(n)} \left[ P_{D,2l,\text{int}}^{(n)}(r_||, r'_||) - P_{D,2l,\text{int}}^{(1)}(r_||, r'_||) \right]_{\text{third}} \]

\[ = -\frac{1}{2} \text{Tr}_{2D}^{(n)} \left[ (l+1)la(-\omega^2)^l \frac{1-n^2 M}{6n^2} K_0(M) \int_0^\infty k \ln \left( \frac{k^2 + M^2}{(k^2 + M^2)^{l+1}} \right) dk \right] \]

\[ = 2 \left( -\frac{a}{2} \right) \frac{1-n^2 M}{6n^2} \int_0^\infty \rho d\rho \int_0^{2\pi} d\theta K_0(M) \frac{1}{\Gamma(l+1)} 2^{-l} M^{-l} \rho^l \]

\[ \times \left\{ K_{-l}(M) \ln 2 + \ln M - \ln \rho + \psi(l+1) + \frac{\partial}{\partial \rho} K_{-l}(M) |_{\rho=1} \right\} \]

\[ = -\frac{a}{6n^2} \frac{1-n^2 M}{(l+1) \Gamma(l+1) 2^{-l} M^{-l-1}} (-\omega^2)^l \int_0^\infty \rho d\rho \int_0^{2\pi} d\theta K_0(M) \rho^l \]

\[ \times \left\{ K_{-l}(M) \ln 2 + \ln M - \ln \rho + \psi(l+1) - \frac{l!}{2(\frac{1}{2} M)^l} \sum k=0^{l-1} \frac{\frac{1}{2} M \rho^k K_{-l}(M)}{k!(l-k)!} \right\} \].

(E24)

We further separate it into three parts. The first one is

\[ (-\omega^2)^l \text{Tr}_{2D}^{(n)} \left[ P_{D,2l,\text{int}}^{(n)}(r_||, r'_||) - P_{D,2l,\text{int}}^{(1)}(r_||, r'_||) \right]_{\text{third},1} \]

\[ = -\frac{1}{6n^2} \frac{1-n^2 M}{2M} \sum l \left[ \ln 2 + \ln M + \psi(l+1) \right] \left( \frac{-\omega^2}{M^2} \right)^l \]

\[ = -\frac{1}{6n^2} \frac{1-n^2 M}{2M} \left[ \ln 2 + \ln M \right] \sum l \left( \frac{-\omega^2}{M^2} \right)^l \]

\[ = -\frac{1}{6n^2} \frac{1-n^2 M}{2M} \left[ \ln 2 + \ln M \right] \frac{-M^2 \omega^2}{(M^2 + \omega^2)^2} + \frac{M^2 \omega^2}{(M^2 + \omega^2)^2} \]

(E26)

which leads to the correction to EE as

\[ S_{\text{third},1} = -\frac{a}{4} (1 + \gamma + \ln 2) \frac{1}{6} A \left( \epsilon^{-1} - M \right) - \frac{a}{4} \frac{1}{6} A \left( -\epsilon^{-1} \ln \epsilon^{-1} + M \ln M \right) . \]

(E27)

The second one is

\[ (-\omega^2)^l \text{Tr}_{2D}^{(n)} \left[ P_{D,2l,\text{int}}^{(n)}(r_||, r'_||) - P_{D,2l,\text{int}}^{(1)}(r_||, r'_||) \right]_{\text{third},2} \]

\[ = -\frac{1}{6n^2} \frac{1-n^2 M}{2M} \left[ \ln 2 + \ln M \right] \int_0^\infty \rho d\rho K_0(M) K_{-l}(M) \rho^l \]

\[ = -\frac{1}{6n^2} \frac{1-n^2 M}{2M} \left[ \ln 2 + \ln M \right] \frac{-M^2 \omega^2}{(M^2 + \omega^2)^2} + \frac{M^2 \omega^2}{(M^2 + \omega^2)^2} \]

(E28)
Its summation over perturbation levels \( l \) gives

\[
(-\omega^2)^l T_{2D}^{(n)} \left[ P_{D,2l,\text{int}}^{(n)}(r_\parallel, r_\parallel') - P_{D,2l,\text{int}}^{(1)}(r_\parallel, r_\parallel') \right]_{\text{third,2}} = a \frac{1 - n^2}{6n} \frac{1}{2M} \left[ -\sum_{l=0}^{\infty} \frac{l}{l+1} \left( -\frac{\omega^2}{M^2} \right)^l + (-\gamma + \ln 2 - M) \sum_{l=0}^{\infty} l \left( -\frac{\omega^2}{M^2} \right)^l \right] \tag{E29}
\]

\[
= a \frac{1 - n^2}{6n} \frac{1}{2M} \left[ M^2 \left[ -\omega^2 + (M^2 + \omega^2) \ln \left( 1 + \frac{\omega^2}{M^2} \right) \right] + (-\gamma + \ln 2 - M) \frac{-M^2 \omega^2}{(M^2 + \omega^2)^2} \right]
\]

which leads to the correction to EE as

\[
S_{\text{third,2}} = \frac{a}{4} (1 + \gamma - \ln 2) \frac{1}{6} A (\epsilon^{-1} - M) + \frac{a}{4} \frac{1}{6} A (-\epsilon^{-1} \ln \epsilon^{-1} + M \ln M). \tag{E30}
\]

The third one is

\[
(-\omega^2)^l T_{2D}^{(n)} \left[ P_{D,2l,\text{int}}^{(n)}(r_\parallel, r_\parallel') - P_{D,2l,\text{int}}^{(1)}(r_\parallel, r_\parallel') \right]_{\text{third,3}} = a \frac{1 - n^2}{6n} \frac{M}{2} l(l+1) \left( -\frac{\omega^2}{M^2} \right)^l \int_0^\infty \rho d\rho \sum_{k=0}^{l-1} \frac{1}{k!(l-k)!} 2^{-2-k} M^k \rho^k K_k(M\rho). \tag{E31}
\]

By exchanging the summation of \( k \) and the integral over \( \rho \), we have

\[
(-\omega^2)^l T_{2D}^{(n)} \left[ P_{D,2l,\text{int}}^{(n)}(r_\parallel, r_\parallel') - P_{D,2l,\text{int}}^{(1)}(r_\parallel, r_\parallel') \right]_{\text{third,3}} = a \frac{1 - n^2}{6n} \frac{M}{4} l(l+1) \left( -\frac{\omega^2}{M^2} \right)^l \sum_{k=0}^{l-1} \frac{1}{(k+1)(l-k)!} 2^{k-1} M^{-k-2} \tag{E32}
\]

Its summation over perturbation levels \( l \) gives

\[
T_{2D}^{(n)} \left[ G_{D,\text{int}}^{(n)} - G_{D,\text{int}}^{(1)} \right]_{\text{third,3}} = \sum_{l=0}^{\infty} (-\omega^2)^l T_{2D}^{(n)} \left[ P_{D,2l,\text{int}}^{(n)} - P_{D,2l,\text{int}}^{(1)} \right]_{\text{third,3}} = a \frac{1 - n^2}{6n} \frac{1}{4M} \sum_{l=0}^{\infty} l(l+1) \left( -\frac{\omega^2}{M^2} \right)^l \sum_{k=0}^{l-1} \frac{1}{(k+1)(l-k)!}, \tag{E33}
\]

Consider an auxiliary function

\[
f(x) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} \frac{x^l}{(k+1)(l-k)!} = \left[ \sum_{l=0}^{\infty} \frac{1}{l+1} x^l \right] \left[ \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right] = \ln(1-x) \frac{e^x}{x}. \tag{E34}
\]

Its first-order derivative is

\[
f'(x) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} \frac{l x^{l-1}}{(k+1)(l-k)!} = \frac{1}{x} \sum_{l=0}^{\infty} \sum_{k=0}^{l} \frac{l x^l}{(k+1)(l-k)!} = -\frac{e^x}{x} - \frac{e^x \ln(1-x)}{x^2} + \frac{e^x \ln(1-x)}{x}, \tag{E35}
\]

and the second-order derivative is

\[
f''(x) = \frac{2 e^x}{(1-x)x^2} - \frac{e^x}{(1-x)^2 x} - \frac{2 e^x \ln(1-x)}{x^3} + \frac{2 e^x \ln(1-x)}{x^2} + \frac{e^x \ln(1-x)}{x}. \tag{E36}
\]
These gives
\[
\text{Tr}^{(n)}_{2D} \left[ G_{D, \text{int}}^{(n)} - G_{D, \text{int}}^{(1)} \right]_{\text{third,3}} = a \frac{1 - n^2}{6n} \left( \frac{1}{4M} \right) \left\{ \left[ \left( -\frac{\omega^2}{M^2} \right)^2 f'' \left( -\frac{\omega^2}{M^2} \right) + 2 \left( -\frac{\omega^2}{M^2} \right) f' \left( -\frac{\omega^2}{M^2} \right) \right] - \frac{M^2 w^2}{(M^2 + w^2)^2} \right\},
\]
(E37)
which leads to the correction to EE as
\[
S_{\text{third,3}} = \frac{a}{8} \left[ \frac{\gamma}{\sqrt{\pi}} + e \text{Erfc}(1) - \sqrt{\pi} \text{Erfi}(1) + \frac{2}{\sqrt{\pi}} \text{F}_2 \left( 1, 1; \frac{3}{2}, 2; 1 \right) + \frac{2}{\sqrt{\pi}} \ln 2 - 1 \right] \frac{1}{6} A (\epsilon^{-1} + M) \] (E38)
where Erfc and Erfi are the complementary and imaginary error functions.
In summary, we have
\[
S = S_{\text{first}} + S_{\text{second}} + S_{\text{third,1}} + S_{\text{third,2}} + S_{\text{third,3}} = \frac{1}{6} A (\epsilon^{-1} - M) \left[ 1 - a \left( -\frac{\gamma}{2} + \ln 2 + \mu_{\text{third,3}} \right) \right]
S = \frac{1}{6} A (\epsilon^{-1} - M) \left[ 1 - a \left( 0.309646 \ldots \right) \right].
\]
(E39)

**Appendix F: Estimation of the entanglement entropy in the quasiparticle picture**

In this section we present the detailed estimation of the EE in the quasiparticle picture, where the entanglement is considered to be produced by quasiparticle entangled pairs in the system. This picture was proposed by Skinner and Nahum [27] to describe a Majorana dynamics with diffusion-annihilation process. Later it is adopted to capture the entanglement structure in a mixed nonunitary dynamics of free fermions [28].
Let A be a subsystem and B be its complementary, and define the probability of two specific particles with distance r paired (entangled) to be \( P(r) \), the entanglement entropy between A and B is
\[
S_A = S_B = \sum_{\{\text{point in } A\}} \sum_{\{\text{point in } B\}} P(\text{distance between the point in } A \text{ and the point in } B).
\]
(F1)
For scale-invariant systems, we have a power-law decaying \( P(r) \propto r^{-k} \), and the exponent \( k \) is the only control parameter of the scaling behavior of EE.
To make it analytically trackable, we do continuum extension for the summation as
\[
S_A \sim \int_A dV_A \int_B dV_B P(r_{A,B}) = \int_A dV_A \int_B dV_B \frac{1}{r_{A,B}^{k}}.
\]
(F2)
It is convenient to work in polar coordinate with disc geometry, then the above integral becomes
\[
S_A \sim \int_{L_A}^{L_A} r' dr' \int_{L_{A,+}}^{2\pi} d\varphi' \int_0^\infty r dr \int_0^{2\pi} d\varphi \frac{1}{|\vec{r}' - \vec{r}|^k}.
\]
(F3)
There are two set of angular variable, and one of them can be always removed. For instance, we have
\[
S_A \sim 2\pi \int_0^{L_A} r' dr' \int_{L_{A,+}}^{\infty} r dr \int_0^{2\pi} d\varphi \frac{1}{|\vec{r}' - \vec{r}|^k} \sim 2\pi L_A^k \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \left\{ L_A^2 \left[ (u_x - v_x)^2 + v_y^2 \right] \right\}^{-\frac{k}{2}},
\]
(F4)
where we have used \( \vec{r}' = L_A \vec{u} \) and \( \vec{r} = L_A \vec{v} \). For \( k > 3 \), this integral gives a robust area-law EE
\[
S_A \sim 2\pi (L_A)^{4-k} \left( \frac{\epsilon}{L_A} \right)^{3-k} \propto L_A.
\]
(F5)
For \( k = 3 \), this integral gives a logarithmic violation of the area-law
\[
S_A \sim -4\pi L_A \ln \frac{\epsilon}{L_A} \sim 4\pi L_A \ln L_A \propto L_A \ln L_A.
\]
(F6)
[1] Luca Bombelli, Rabinder K. Koul, Joohan Lee, and Rafael D. Sorkin, “Quantum source of entropy for black holes,” Phys. Rev. D 34, 373–383 (1986).
[2] Mark Srednicki, “Entropy and area,” Phys. Rev. Lett. 71, 666–669 (1993).
[3] Curtis Callan and Frank Wilczek, “On geometric entropy,” Physics Letters B 333, 55–61 (1994).
[4] Daniel Kabat, “Black hole entropy and entropy of entanglement,” Nuclear Physics B 453, 281–299 (1995).
[5] Shinsei Ryu and Tadashi Takayanagi, “Holographic derivation of entanglement entropy from the Anti-de Sitter space/conformal field theory correspondence,” Phys. Rev. Lett. 96, 181602 (2006).
[6] Luigi Amico, Rosario Fazio, Andreas Osterloh, and Vlatko Vedral, “Entanglement in many-body systems,” Rev. Mod. Phys. 80, 517–576 (2008).
[7] J. Eisert, M. Cramer, and M. B. Plenio, “Colloquium: Area laws for the entanglement entropy,” Rev. Mod. Phys. 82, 277–306 (2010).
[8] Brian Swingle, “Entanglement renormalization and holography,” Phys. Rev. D 86, 065007 (2012).
[9] Tatsuma Nishioka, “Entanglement entropy: Holographic derivation,” Phys. Rev. Lett. 90, 035007 (2008).
[10] Mugund Rangamani and Tadashi Takayanagi, Holographic Entanglement Entropy (Springer, 2017).
[11] Christoph Holzhey, Finn Larsen, and Frank Wilczek, “Geometric and renormalized entropy in conformal field theory,” Nuclear Physics B 424, 443 – 467 (1994).
[12] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, “Entanglement in quantum critical phenomena,” Phys. Rev. Lett. 90, 227902 (2003).
[13] Pasquale Calabrese and John Cardy, “Entanglement entropy and quantum field theory,” Journal of Statistical Mechanics: Theory and Experiment 2004, P06002 (2004).
[14] Alexei Kitaev and John Preskill, “Topological entanglement entropy,” Phys. Rev. Lett. 96, 110404 (2006).
[15] Michael Levin and Xiao-Gang Wen, “Detecting topological order in a ground state wave function,” Phys. Rev. Lett. 96, 110405 (2006).
[16] Eduardo Fradkin and Joel E. Moore, “Entanglement entropy of 2d conformal quantum critical points: Hearing the shape of a quantum drum,” Phys. Rev. Lett. 97, 050404 (2006).
[17] Hui Li and F. D. M. Haldane, “Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-abelian fractional quantum hall effect states,” Phys. Rev. Lett. 101, 010504 (2008).
[18] Xiao-Liang Qi, Hosho Katsura, and Andreas W. W. Ludwig, “General relationship between the entanglement spectrum and the edge state spectrum of topological quantum states,” Phys. Rev. Lett. 108, 196402 (2012).
[19] Pablo Bueno, Robert C. Myers, and William Wits zak-Krempa, “Universality of corner entanglement in conformal field theories,” Phys. Rev. Lett. 115, 021602 (2015).
[20] Pasquale Calabrese and John Cardy, “Evolution of entanglement entropy in one-dimensional systems,” Journal of Statistical Mechanics: Theory and Experiment 2005, P04010 (2005).
[21] Pasquale Calabrese and John Cardy, “Entanglement entropy and conformal field theory,” Journal of Physics A: Mathematical and Theoretical 42, 504005 (2009).
[22] Nicolas Laflorencie, “Quantum entanglement in condensed matter systems,” Phys. Rep. 646, 1–59 (2016).
[23] Xiao-Gang Wen, “Colloquium: Zoo of quantum-topological phases of matter,” Rev. Mod. Phys. 89, 041004 (2017).
[24] C. W. von Keyserlingk, Tibor Rakovszky, Frank Pollmann, and S. L. Sondhi, “Operator hydrodynamics, otocs, and entanglement growth in systems without conservation laws,” Phys. Rev. X 8, 021013 (2018).
[25] Dmitry A. Abanin, Elad Altman, Immanuel Bloch, and Maksym Serbyn, “Colloquium: Many-body localization, thermalization, and entanglement,” Rev. Mod. Phys. 91, 021001 (2019).
[26] R. J. Lewis-Swan, A. Safavi-Naini, A. M. Kaufman, and A. M. Rey, “Dynamics of quantum information,” Nature Reviews Physics 1, 627–634 (2019).
[27] Adam Nahum and Brian Skinner, “Entanglement and dynamics of diffusion-annihilation processes with majorana defects,” Phys. Rev. Research 2, 023288 (2020).
[28] Qicheng Tang, Xiao Chen, and W. Zhu, “Quantum criticality in the nonunitary dynamics of (2+1)-dimensional free fermions,” Phys. Rev. B 103, 174303 (2021).
[29] M. B. Hastings, “Entropy and entanglement in quantum ground states,” Phys. Rev. B 76, 035114 (2007).
[30] Michael M. Wolf, Frank Verstraete, Matthew B. Hastings, and J. Ignacio Cirac, “Area laws in quantum systems: Mutual information and correlations,” Phys. Rev. Lett. 100, 070502 (2008).
[31] Fernando G. S. L. Brandão and Michal Horodecki, “An area law for entanglement from exponential decay of correlations,” Nature Physics 9, 721–726 (2013).
[32] Jaeyoon Cho, “Realistic area-law bound on entanglement from exponentially decaying correlations,” Phys. Rev. X 8, 031009 (2018).
[33] Mark Van Raamsdonk, “Building up spacetime with quantum entanglement,” General Relativity and Gravitation 42, 2323–2329 (2010).
[34] Thomas Faulkner, Monica Guica, Thomas Hartman, Robert C. Myers, and Mark Van Raamsdonk, “Gravitation from entanglement in holographic cfts,” Journal of High Energy Physics 2014, 51 (2014).
[35] Brian Swingle and Mark Van Raamsdonk, “Universality of gravity from entanglement,” (2014), arXiv:1405.2933 [hep-th].
[36] Steven R. White, “Density matrix formulation for quantum renormalization groups,” Phys. Rev. Lett. 69, 2863–2866 (1992).
[37] F. Verstraete and J. I. Cirac, “Renormalization algorithms for quantum-many body systems in two and higher dimensions,” (2004), arXiv:cond-mat/0407066 [cond-mat].
[38] G. Vidal, “Class of quantum many-body states that can be efficiently simulated,” Phys. Rev. Lett. 101, 110501 (2008).
[39] Ulrich Schollwöck, “The density-matrix renormalization group in the age of matrix product states,” Annals of Physics 326, 96–192 (2011), January 2011 Special Issue.
Aitor Lewkowycz, Robert C. Myers, and Michael Sergey N. Solodukhin, “Entanglement entropy of black holes,” Living Reviews in Relativity 14 (2011).

Hong Liu and Mark Mezei, “A refinement of entanglement entropy for disconnected regions,” Journal of High Energy Physics 2015, 085004 (2015).

Satoshi Iso, Takato Mori, and Katsuta Sakai, “Non-universal terms for the entanglement entropy in 2+1 dimensions,” Nuclear Physics B 764, 183–201 (2007).

H. Casini, M. Huerta, and L. Leitao, “Entanglement entropy for a dirac fermion in three dimensions: Vertex contribution,” Nuclear Physics B 814, 594–609 (2009).

Mark P. Hertzberg and Frank Wilczek, “Some calculable contributions to entanglement entropy,” Phys. Rev. D 90, 104035 (2015).

Max A. Metlitski, Carlos A. Fuertes, and Subir Sachdev, “Entanglement entropy in the O(N) model,” Phys. Rev. B 80, 115122 (2009).

Mark P Hertzberg, “Entanglement entropy in scalar field theory,” Journal of Physics A: Mathematical and Theoretical 46, 015402 (2012).

Seth Whitsitt, William Witeczak-Krempa, and Subir Sachdev, “Entanglement entropy of large-N Wilson-Fisher conformal field theory,” Phys. Rev. B 95, 045148 (2017).

Ling-Yan Hung, Yikun Jiang, and Yixu Wang, “Area term of the entanglement entropy of a supersymmetric O(N) vector model in three dimensions,” Phys. Rev. D 95, 085004 (2017).

Arpan Bhattacharyya, Ling-Yan Hung, and Charles M. Melby-Thompson, “Instantons and entanglement entropy,” Journal of High Energy Physics 2017, 011 (2017).

Yangang Chen, Lucas Hackl, Ravi Kunjwal, Heidar Moradi, Yasaman K. Yazdi, and Miguel Zilhão, “Towards spacetime entanglement entropy for interacting theories,” Journal of High Energy Physics 2020, 114 (2020).

Satoshi Iso, Takato Mori, and Katsuta Sakai, “Entanglement entropy in scalar field theory and $Z_N$ gauge theory on feynman diagrams,” Phys. Rev. D 103, 105010 (2021).

Satoshi Iso, Takato Mori, and Katsuta Sakai, “Non-gaussianity of entanglement entropy and correlations of composite operators,” Phys. Rev. D 103, 125019 (2021).

D.V. Vassilevich, “Heat kernel expansion: user’s manual,” Physics Reports 388, 279–360 (2003).

Dmitry Nesterov and Sergey N. Solodukhin, “Short-distance regularity of green’s function and uv divergences in entanglement entropy,” Journal of High En-
[119] Alexander B. Zamolodchikov, “Irreversibility of the flux of the renormalization group in a 2d field theory,” Jetp Letters 43, 565–567 (1986).
[120] John L. Cardy, “Is there a c-theorem in four dimensions?” Physics Letters B 215, 749–752 (1988).
[121] Daniel L. Jafferis, Igor R. Klebanov, Silviu S. Pufu, and Benjamin R. Safdi, “Towards the F-theorem: $\mathcal{N} = 2$ field theories on the three-sphere,” Journal of High Energy Physics 2011, 102 (2011).
[122] Igor R. Klebanov, Silviu S. Pufu, Subir Sachdev, and Benjamin R. Safdi, “Entanglement entropy of 3-d conformal gauge theories with many flavors,” Journal of High Energy Physics 2012, 36 (2012).
[123] Simone Giombi, Igor R Klebanov, and Grigory Tarnopolsky, “Conformal QED$_d$, F-theorem and the $\varepsilon$ expansion,” Journal of Physics A: Mathematical and Theoretical 49, 135403 (2016).
[124] Tarun Grover, Ari M. Turner, and Ashvin Vishwanath, “Entanglement entropy of gapped phases and topological order in three dimensions,” Phys. Rev. B 84, 195120 (2011).
[125] Tarun Grover, “Entanglement monotonicity and the stability of gauge theories in three spacetime dimensions,” Phys. Rev. Lett. 112, 151601 (2014).