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FEW REMARKS ON THE USE OF LOVE WAVES IN NON DESTRUCTIVE TESTING

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ABSTRACT. This paper concerns a theoretical study on the possibility of using Love waves for non destructive testing. A mathematical model is presented and analyzed. Several numerical tests are given in order to show the mechanical behaviour of this model.

1. Introduction. The detection of cracks at the interface between two materials, for instance in pipes with a coating, is often treated using ultrasonic waves or Foucault currents propagating transversally to the interface. The detection is based on the analysis of the return signal which is different depending if the wave meets a crack or not. The possibility of using Love waves is a new idea suggested by many authors (for instance one can consult [9]-[11]-[12]-[14]-[15]-[18]). Such waves could enable one to explore larger area from a single shot. But the signal processing which should be implied is much more complicated. Therefore, a mathematical model can be useful in order to determine if there is a crack and where it is localized. One solution is to have a wave simulator model set on a finite part of the structure, but the boundary conditions should avoid artificial reflections. The goal of this paper is to make explicit these remarks and to suggest appropriate numerical transparent boundary conditions for this challenge. Nevertheless, this study is only a discussion on a mathematical model which could be used in order to improve the understanding of Love waves for NDT. Concerning the details of the mathematical proofs and the application to an operational use, we refer the reader to the references. Let us consider a rectangular open set as the one drawn on figure 1. There are two subsets $\Omega_-$ and $\Omega_+$ corresponding to two different materials. The wave velocities are respectively $c_-$ and $c_+$. For sake of clarity, it is assumed that $c_- < c_+$. The upper and lower boundaries -say $\Gamma_+$ and $\Gamma_-$- are free edges and the lateral boundaries $\Gamma_e$ and $\Gamma_s$ are artificial cuts on which transparent boundary conditions are applied in order to simulate the propagation on an infinite strip. The separation line between the two media is denoted by $\Gamma_i$. A crack may exist somewhere on this line. Its two extremities are positioned at points $A$ and $B$ with abscissa $x_1 = x_A$ and $x_1 = x_B$. A mechanical excitation is produced on a subdomain of $\Omega_+$ or $\Omega_-$ which is assumed to be remote from the crack tips. It can be an initial condition or a time dependent force denoted

\[ \frac{\partial u}{\partial t} - \nabla \cdot (c \nabla u) + f \quad \text{in} \quad \Omega \times (0, T) \]

\[ u = 0 \quad \text{on} \quad \partial \Omega \times (0, T) \]

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The mathematical model that we discuss consists in finding $u(x, t)$ solution for any $t$ of the following system:

\begin{align}
\frac{\partial^2 u}{\partial t^2} - \text{div} \left( c^2 \nabla u \right) &= f \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \Gamma^- \cup \Gamma^+, \text{ and } \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial \nu} &= 0 \text{ on } \Gamma_e \cup \Gamma_s, \\
u(x, 0) &= u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x) \text{ in } \Omega.
\end{align}

(1)

The existence and uniqueness of a solution is not exactly the result contained in most books on partial differential equations, because of the dissipation term which appears on the boundaries $\Gamma_e$ and $\Gamma_s$. The proof method consists in constructing a sequence of approximations of the model (1) using a Galerkin approximation with the variational formulation and the family of eigenfunctions $w_n$, $n \geq 1$ solution of:

\begin{align}
-\text{div} \left( c^2 \nabla w_n \right) &= \lambda_n w_n \text{ in } \Omega, \quad 0 \leq \lambda_1 \leq \ldots \lambda_n \leq \lambda_{n+1} \ldots, \\
\frac{\partial w_n}{\partial \nu} &= 0 \text{ on } \Gamma_+ \cup \Gamma_- \cup \Gamma_e \cup \Gamma_s, \\
\int_{\Omega} w_n^2(x)dx &= 1.
\end{align}

(2)

The first term $w_1$ is the constant function corresponding to $\lambda_1 = 0$. One can define a sequence of approximate solutions $u^n$ to the equation (1) using the space $V^N$ span by the $N$-first eigenvectors solution of (2). This is obtained from the variational formulation using...
test functions in $V^N$. This solution can be written:

$$\sum_{i=1}^{N} \alpha_n(t) w_n(x),$$

(3)

and one can prove that it is bounded in the space $L^\infty([0,T]; H^1(\Omega))$. By extracting a subsequence, one proves the existence and then the uniqueness of a solution to (1). The time regularity is obtained by taking the time derivative of the model. The functional space used for the space dependence of the solution is $H^1(\Omega)$ and $L^2(\Omega)$ for its time derivative.

For instance, one has the following result:

**Theorem 1.1.** Let us assume that $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and:

$$f \in L^2([0,T] \times \Omega).$$

Then there is a unique solution $u$ to (1) such that:

$$u \in C^0([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

**Remark 1.** It is worth noting that the convergence of the series (3) can’t be proved in the space $H^2(\Omega)$. This can be justified by the fact that $u^n$ satisfies on the boundary of $\Omega$ the condition:

$$\frac{\partial u^n}{\partial \nu} = 0,$$

which is incompatible with the boundary condition which should be satisfied by $u$ at any time on $\Gamma_e \cup \Gamma_s$.

2. A more refined regularity analysis for $u$. Let us discuss locally what happens in the vicinity of points $E$ (or/and $F$) (see figure 2). On a close neighbourhood $O$ of $E$ (same is true for $F$), the solution $u$ can be locally split into its symmetrical and unsymmetrical parts with respect to the coordinate $x_2$ as shown on figure 2. They are respectively denoted by $u_s$ and $u_a$. From a classical result, the function $u_s$ and its derivative with respect to $x_2$ are continuous across $\Gamma_1$. Therefore one can prove from the local equation satisfied by $u_s$, that

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The neighbourhood $O$ near the point $E$}
\end{figure}
it is locally in the space $H^2(\Omega)$. The discontinuity of $\frac{\partial u}{\partial x_1}$ is therefore taken into account by the unsymmetrical term $\frac{\partial u}{\partial x_1}$ as it is explained hereafter. Because:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_e,$$

the function $\frac{\partial u}{\partial x_1}$ is piecewise smooth as far as:

$$\frac{\partial u}{\partial t} \in C^0([0, T]; H^{1/2}(\Gamma_e))$$

which can’t be discontinuous. Since the velocity $c$ is discontinuous, $\frac{\partial u}{\partial x_1}$ is also discontinuous. In fact, one can introduce the following function defined on $\Omega$ and which takes into account this discontinuity ($c_+ = c_-, c_\pm = c_+$ for $x_2 > 0$ and $c_+ = c_+, c_\pm = c_-$ for $x_2 < 0$):

$$q(x_1, x_2) = \left(1 + 2 \cos(x_2 \pi) e^{-\pi x_1} + e^{-2x_1 \pi}\right) \left(1 - 2 \cos(x_2 \pi) e^{-\pi x_1} + e^{-2x_1 \pi}\right)$$

A primitive of $q$ with respect to $x_1$ is a obtained using the so-called Dilog function \cite{22}. More precisely \cite{5}, let us set:

$$D(z) = \sum_{n \geq 1} \frac{z^n}{n^2} \text{ with } z = e^{-\pi(x_1-ix_2)}.$$  

The singular function $S$ involved in our problem is:

$$S(x_1, x_2) = \text{Im}(D(z) - D(-z))/2, \quad \text{and} \quad q(x_1, x_2) = \left(\frac{c_+ - c_-}{\pi(c_+^2 + c_-^2)}\right)\left(\frac{c_\pm}{c_\pm^2}\right) \frac{\partial S}{\partial x_2}.$$  

From a simple computation one can check that this function is locally equivalent to $\text{Log}(r)$ ($r$ is the distance from $E$). Furthermore it can be truncated symmetrically around $\Gamma_i$ (the interface) such that its integral over $\Omega$ is zero. $S$ and its derivative with respect to the coordinate $x_2$ have been plotted on figure 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure3}
\caption{The singular function $S$ on left and $\partial S/\partial x_2$ on right}
\end{figure}

\textbf{Theorem 2.1.} Let $Q$ be a sub-open set strictly included in $\Omega_+ \cup \Omega_-$. In other words, $\overline{Q}$ doesn’t cross the interface $\Gamma_i$. Assuming that the initial data are such that $u_0 \in H^2(\Omega)$
and $H^1(\Omega)$, and that the right hand side satisfies $f \in H^1([0, T]; L^2(\Omega))$, with support in $Q$, one has:

$$u = \alpha(t)S(x) + u_R(x, t), \text{ with } u_R(x, t) \in W^{1,\infty}([0, T]; H^1(\Omega)) \cap W^{2,\infty}([0, T]; L^2(\Omega)).$$

More generally, the following array using the same notations as above and the assumptions of theorem 2.1, describes the regularity of $u$ in the different parts of $\Omega$:

| Function | $\frac{\partial u}{\partial t}$ | $\frac{\partial u}{\partial x_1}$ | $\frac{\partial u}{\partial x_2}$ |
|---|---|---|---|
| $U \subset \Omega_\pm$ | $L^\infty([0, T]; H^1(\Omega))$ | $L^\infty([0, T]; H^1(U))$ | $L^\infty([0, T]; H^1(U))$ |
| $Q \subset \Gamma_{\epsilon \pm}$ | $L^\infty([0, T]; H^{3/2}(\Gamma_{\epsilon}))$ | $L^\infty([0, T]; H^{3/2}(Q))$ | $L^\infty([0, T]; H^{3/2}(Q))$ |
| $P \subset \Gamma_i$ | $L^\infty([0, T]; H^{3/2}(\Gamma_i))$ | $L^\infty([0, T]; H^{3/2}(P))$ | $L^\infty([0, T]; H^{-1/2}(\Gamma_i))$ |

With another respect, the time regularity can be derived up to any order assuming regularity on the initial data $u_0$ et $u_1$ and the right hand side $f$ of the wave equation.

**Remark 2.** Let us consider a vertical line -say $\Gamma_v$ (corresponding to $0 < x_1 < L$). Along $\Gamma_v^+(x_2 > 0)$ or $\Gamma_v^-(x_2 < 0)$ the function $\frac{\partial u}{\partial x_2}$ belongs to the space:

$$L^\infty([0, T]; H^{3/2}(\Gamma_v^+ \cup \Gamma_v^-)) \quad \text{(it is discontinuous across } \Gamma_i).$$

Hence there is a finite limit for $x_2 = 0^+$ and $x_2 = 0^-$. This property is no more true along the boundary $\Gamma_e \cup \Gamma_s$ because of the singularity $S$ given at (6). This remark is meaningful in our discussion.

For instance, one has:

$$\frac{\partial u}{\partial t} \in L^\infty([0, T]; H^{1/2}(\Gamma_e \cup \Gamma_s))$$

and therefore:

$$c \frac{\partial u}{\partial \nu}_{|\Gamma_e \cup \Gamma_s} \in L^\infty([0, T]; H^{1/2}(\Gamma_e \cup \Gamma_s)).$$

But $c$ is discontinuous across $\Gamma_i$ and therefore one has:

$$\frac{\partial u}{\partial \nu}_{|\Gamma_e \cup \Gamma_s} \notin L^\infty([0, T]; H^{1/2}(\Gamma_e \cup \Gamma_s)),$$

but (assuming an additional regularity on the initial data and on the right hand side of (1)), one has (and even more):

$$\frac{\partial u}{\partial \nu}_{|\Gamma_e \cup \Gamma_s} \in L^\infty([0, T]; L^2(\Gamma_e \cup \Gamma_s)).$$
2.1. Effect of the transparent boundary condition on $u$. For sake of simplicity in this subsection, let us assume that $f = 0$. But this is just a facility, not a restriction. The mechanical energy of the structure is defined by:

$$E(t) = \frac{1}{2} \int_{\Omega} |\frac{\partial u}{\partial t}|^2(x,t)dx + \frac{1}{2} \int_{\Omega} c^2 |\nabla u|^2(x,t)dx.$$ 

The mechanical energy is decreasing with respect to the time variable because there are only initial conditions and due to the transparent boundary condition which acts as a damping. Let us check it. One has, by multiplying the wave equation by $\frac{\partial u}{\partial t}$ and integrating over $\Omega \times ]0, t[$:

$$E(t) + \int_0^t \int_{\Gamma_e \cup \Gamma_s} c |\frac{\partial u}{\partial t}|^2(x_2, t)dx_2dt \leq E(0) \ \forall t > 0. \quad (7)$$

This proves two properties:

1. the function $t \rightarrow E(t)$ which is continuous, is decreasing when $t \rightarrow \infty$. Hence there is a lower limit -say $E_\infty$- such that $\lim_{t \rightarrow \infty} E(t) = E_\infty$;

2. There exists a constant $c$ independent on time such that:

$$\forall t > 0, \ ||\frac{\partial u}{\partial t}\||_{0,2,\Omega} \leq c, \ \ ||\nabla u\||_{1,\Omega} \leq c.$$ 

Let us point out that the estimate (7) proves a somewhat hidden regularity concerning the term $\frac{\partial u}{\partial t}$ on the boundary $\Gamma_e \cup \Gamma_s$. Let us introduce the sequence of functions $u^{t_0}(x, t) = u(x, t + t_0)$. The previous estimate enables one to extract a subsequence which converges to a a term denoted by $u^*$ (with respect to the time $t_0$ and the uniqueness of the limit will prove that all the sequence converges) such that:

$$\begin{cases} 
    u^{t_0}(x, t) \rightarrow_{t_0 \rightarrow \infty} u^* & \text{in } L^\infty(]0, T[; H^1(\Omega)) \text{ weakly}, \\
    \frac{\partial u^{t_0}}{\partial t}(x, t) \rightarrow_{t_0 \rightarrow \infty} \frac{\partial u^*}{\partial t} & \text{in } L^\infty(]0, T[; L^2(\Omega)) \text{ weakly}, \\
    \frac{\partial u^{t_0}}{\partial t}(x, t) \rightarrow_{t_0 \rightarrow \infty} \frac{\partial u^*}{\partial t} & \text{in } L^2(]0, T[ \times \{\Gamma_e \cup \Gamma_s\}) \text{ weakly}. 
\end{cases} \quad (8)$$

From the convergence of $E(t)$ to $E_\infty$ one can state that:

$$\forall \varepsilon > 0, \ \exists \varepsilon > 0, \ \forall t \geq t_\varepsilon : \ E(t) \leq E_\infty + \varepsilon. \quad (9)$$

Therefore:

$$\forall t \geq t_\varepsilon, \ E_\infty \leq \liminf_{t \rightarrow \infty} E(t) + \int_{t_\varepsilon}^\infty \int_{\Gamma_e \cup \Gamma_s} c |\frac{\partial u}{\partial t}|(x_2, t)^2dx_2dt \leq E(t_0) \leq E_\infty + \varepsilon.$$ 

This implies that:

$$\forall \varepsilon > 0, \ \exists \varepsilon > 0, \ \forall t \geq t_\varepsilon, \ \int_{t_\varepsilon}^{t_1} \int_{\Gamma_e \cup \Gamma_s} c |\frac{\partial u}{\partial t}|(x_2, t)^2dx_2dt \leq \varepsilon,$$

and finally, because of the lower semi-continuity of a continuous convex function:

$$\frac{\partial u^*}{\partial t} = 0 \text{ on }]0, T[ \times \{\Gamma_e \cup \Gamma_s\}. \quad (10)$$
From the equations satisfied by $u$, one obtains that $u^*$ is solution of the limit model:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial^2 u^*}{\partial t^2} - \text{div}(c^2 \nabla u^*) = 0 & \text{in } [0, T] \times \Omega, \\
\frac{\partial u^*}{\partial \nu} = 0 & \text{on } [0, T] \times \Gamma_0, \\
\frac{\partial u^*}{\partial t} = 0 & \text{on } [0, T] \times \Gamma_e \cup \Gamma_s.
\end{array} \right. 
\tag{11}
\end{align*}
\]

The Holmgren theorem [13] enables one to conclude that the function $u^*$ (in a first step one proves that the time derivative of $u^*$ is zero) is a constant with respect to space and time variables on $[0, T] \times \Omega$. Such a constant doesn’t generate wave. Hence when $t \to \infty$ the waves disappears. One can get rid of this constant as follows. By integrating the wave equation (1) on $[0, t] \times \Omega$, one obtains:

\[
\forall t > 0, \quad \frac{\partial}{\partial t} \left( \int_\Omega u(x, t) dx_1 dx_2 + \int_{\Gamma_e \cup \Gamma_s} cu(x, t) dx_2 \right) = \int_\Omega u_1(x) dx_1 dx_2 + \int_{\Gamma_e \cup \Gamma_s} cu_0(x) dx_2.
\]

If the initial conditions satisfy:

\[
\int_\Omega u_1 + \int_{\Gamma_e \cup \Gamma_s} cu_0 = 0,
\]

one deduces that when $t \to \infty$ and because $u^* = \text{constant}$:

\[
\int_{\Gamma_e \cup \Gamma_s} cu^* = 0 \implies u^* = 0.
\]

A similar analysis can be derived if $f \neq 0$ but with some conditions on this term.

3. The Fourier transform for the operational use. For sake of simplicity in the notations, it is assumed in this section that the initial conditions on $u$ are homogeneous and that $u = 0$ for $t < 0$. The only external force applied is $f$ at the right hand side of the wave equation. Furthermore, it is assumed to decrease sufficiently fast when $t \to \infty$ in order to make sense to the following. Let us set:

\[
\hat{u}(\omega, x) = \int_0^\infty e^{-i\omega t} u(x, t) dt = u_r + iu_i, \quad (\text{real and imaginary parts of } \hat{u}). \tag{12}
\]

One has:

\[
\left\{ \begin{array}{l}
-\omega^2 \hat{u} - \text{div}(c^2 \nabla \hat{u}) = f & \text{in } \Omega, \\
\frac{\partial \hat{u}}{\partial \nu} = 0 & \text{on } \Gamma_0, \\
i\omega \hat{u} + c \frac{\partial \hat{u}}{\partial \nu} = 0 & \text{on } \Gamma_e \cup \Gamma_s.
\end{array} \right. 
\tag{13}
\]

By splitting these equations into the real and imaginary parts, one deduces that:

\[
\left\{ \begin{array}{l}
-\omega^2 \hat{u}_r - \text{div}(c^2 \nabla \hat{u}_r) = f_r & \text{in } \Omega, \\
\frac{\partial \hat{u}_r}{\partial \nu} = 0 & \text{on } \Gamma_0, \\
\omega u_i = c \frac{\partial \hat{u}_r}{\partial \nu} & \text{on } \Gamma_e \cup \Gamma_s. 
\end{array} \right. \tag{14}
\]

\[
\left\{ \begin{array}{l}
-\omega^2 \hat{u}_i - \text{div}(c^2 \nabla \hat{u}_i) = f_i & \text{in } \Omega, \\
\frac{\partial \hat{u}_i}{\partial \nu} = 0 & \text{on } \Gamma_0, \\
\omega u_r + c \frac{\partial \hat{u}_i}{\partial \nu} = 0 & \text{on } \Gamma_e \cup \Gamma_s. 
\end{array} \right. \tag{15}
\]
3.1. **Property of the Fourier model.** Let us consider \( \hat{u} \) the Fourier transform of \( u \) defined at (12); \( u \) is decreasing to zero when \( t \to \infty \) because of the transparent boundary condition which acts as a damping as shown in the previous section as soon as a compatibility condition is satisfied by the data: One can prove the following result.

**Theorem 3.1.** Let us denote by \((\lambda_n, w_n)\) the eigenvalues and eigenfunctions solution of the model (2). Let us set \( \Lambda = \left\{ \sqrt{\lambda_n}, n \in \mathbb{N}^* \right\} \). For any \( \omega \in \mathbb{R} \setminus \Lambda \), the coupled system (15) and (14) has a unique solution.

**Proof** Let us introduce the bilinear form \( a(.,.) \) defined on the space \( H^1(\Omega) \times H^1(\Omega) \) and the linear form \( l(.) \) on \( H^1(\Omega) \) by:

\[
\begin{align*}
\forall \tilde{u} = (u_r, u_i), \tilde{v} = (v_r, v_i) & \in H^1(\Omega) \times H^1(\Omega), \\
 a(\tilde{u}, \tilde{v}) & = -\omega^2 \int_\Omega u_r v_r + u_i v_i \\
 & + \int_\Omega c^2 (\nabla u_r \cdot \nabla v_r + \nabla u_i \cdot \nabla v_i) + \omega \int_{\Gamma_e \cup \Gamma_s} c (u_r v_i - u_i v_r), \quad (16) \\
 l(\tilde{v}) & = \int_\Omega (f_r v_r + f_i v_i)(x, \omega)dx.
\end{align*}
\]

One has:

\[
a(\tilde{v}, \tilde{v}) = -\omega^2 \int_\Omega (v_r^2 + v_i^2) + \int_\Omega c^2 (|\nabla v_r|^2 + |\nabla v_i|^2).
\]

Therefore the bilinear form \( a(.,.) \) is Garding-coercive [16]. This enables one to apply the Fredholm alternative [17] ensuring that the model (14)-(15) which is equivalent to find \( \tilde{u} \in H^1(\Omega) \times H^1(\Omega) \) such that:

\[
\forall \tilde{v}, \ a(\tilde{u}, \tilde{v}) = l(\tilde{v}), \quad (17)
\]

has a unique solution as soon as \( \omega \neq \sqrt{\lambda_n}, \forall n \in \mathbb{N}. \)

From (17), it can be deduced that if one uses Shannon wavelet [21], the solution \( \hat{u} \) is zero excepted for the frequencies in the bandwidth concerned.

4. **Local behaviour of the solution \((u_r, u_i)\) near the crack tip.** Let us assume that there is a crack on the interface \( \Gamma_i \) as shown on figure 1. The two extremities are denoted by \( A \) and \( B \) and their abscissa on \( \Gamma_i \) are respectively \( x_1 = a \) and \( x_1 = b \). From classical results (see [4], [3]) one can write in the vicinity of each crack tip \((R_A, \theta_A), \) -respectively \((R_B, \theta_b), \) -are the polar coordinates from \( A \) -respectively \( B \) - the reference axis for the polar angle being the line \( \Gamma_i \) as shown on figure 4):
On a neighbourhood $V_A$ of $A$, one can write:

$$u(x,t) = K_A(t) S_A(x) + u_{RA}, \quad u_{RA} \in L^\infty([0,T]; H^2(V_A)),$$

with $S_A(x) = \sqrt{r_A} \sin(\theta_A / 2)$, $c = c_+ \text{ if } \theta_A > 0$, $c = c_- \text{ if } \theta_A < 0,$

and on a neighbourhood $V_B$ of $B$:

$$u(x,t) = K_B(t) S_B(x) + u_{RB}, \quad u_{RB} \in L^\infty([0,T]; H^2(V_B)),$$

with $S_B(x) = \sqrt{r_B} \sin(\theta_A / 2)$, $c = c_+ \text{ if } \theta_B > 0$, $c = c_- \text{ if } \theta_B < 0.$

The coefficients $K_A$ and $K_B$ are called the stress intensity factors and they depend on the time variable. Concerning the Fourier transform of $u$, the previous result becomes for instance around the extremity $A$:

$$\hat{u}(x,\omega) = \hat{K}_A(\omega) S_A(x) + \hat{u}_{RA},$$

with:

$$\hat{K}_A(\omega) = \int_0^\infty e^{-i\omega t} K_A(t)dt \quad \text{and} \quad \hat{u}_{RA}(x,\omega) = \int_0^\infty e^{-i\omega t} u(x,t)dt.$$

The same result is also true for the extremity $B$ of the crack tip.

5. **An energy release rate for crack detection.** Let us now introduce the so-called multiplier method for partial differential equations which in fact is nothing else but a domain derivative of the elastic energy with respect to a displacement of the domain $\Omega$ (see [8]).

Several integrations by parts are used. Multiplying (15)-(14) by $\frac{\partial u_r}{\partial x_1}$ or $\frac{\partial u_i}{\partial x_1}$ and by integrating over $\Omega$, one obtains the following relations ($\nu_1$ is +1 on $\Gamma_s$ and −1 on $\Gamma$:)

$$- \int_{\Gamma_+ \cup \Gamma_-} (\omega^2 (u_r^2 + u_i^2) ) \nu_1 + \frac{c^2}{2} (|\frac{\partial u_r}{\partial x_2}|^2 + |\frac{\partial u_i}{\partial x_2}|^2) \nu_1 - \int_{\Omega} (f_r \frac{\partial u_r}{\partial x_1} + f_i \frac{\partial u_i}{\partial x_1}) = G(\omega) = \frac{\pi}{4} (|\hat{K}_B|^2 - |\hat{K}_A|^2)$$

(20)
Let us discuss briefly this relation. This is an important point in the detection criterion of the crack located on the interface line $\Gamma_\alpha$.

1. There is a formula which gives the expression of the stress intensity factors with respect to the solution $\hat{u}$ in a neighbourhood of the crack tip. Let us recall it (\cite{8}).

First of all we denote by $\partial V_A$ the boundary of the neighbourhood of $A$ as shown on figure 4. Let us denote by $\hat{K}_A$, the stress intensity factor at point $A$ for $u_r$. If $\nu' = (\nu'_1, \nu'_2)$ is the unit normal to $\partial V_A$ inside $V_A$, one has (a similar result is also true in the vicinity of point $B$):

$$\frac{\pi}{4} |\hat{K}_A|^2 = \int_{\partial V_A} \left[ -\frac{\omega^2}{2} u_r^2 \nu'_1 - c^2 \frac{\partial u_r}{\partial \nu'} \frac{\partial u_r}{\partial x_1} + \frac{c^2}{2} |\nabla u_r|^2 \nu'_1 \right].$$  \hspace{1cm} (21)

This formula enables one to compute accurately the stress intensity factor $K_A$.

2. If there is no crack ($\hat{K}_A = \hat{K}_B = 0$) one has always $G = 0$;

3. Let us notice that if:

$$G(\omega) = 0, \ \forall \omega \in [\omega_1, \omega_2]$$

then (it is assumed that $\omega_1 \neq \omega_2$):

$$|\hat{K}_A(\omega)| = |\hat{K}_B(\omega)|, \ \forall \omega \in [\omega_1, \omega_2].$$

4. If $G(\omega) = 0$ on $[\omega_1, \omega_2]$, and if the function $f$ is correctly chosen, one can prove that there is no crack. It looks like a controllability result. Let us give few ideas on the justification of this property. For sake of simplicity we only consider $u_r$ assuming for instance that $u_i = 0$ and we restrict the discussion to the case $c_+ = c_-$ in order to avoid the two singularities appearing at $E$ and $F$ (see \cite{7} for the complete case).

In fact one can use the characterization of the stress intensity factor from the dual singular functions (one for each crack tip) \cite{10} -say $S^*_A$ and $S^*_B$- which leads to the expressions:

$$K_A = \int_{\Omega} \left[ f_r S_A^* + \omega^2 u_r S_B^* \right] dx, \quad K_B = \int_{\Omega} \left[ f_r S_B^* + \omega^2 u_r S_A^* \right] dx. \hspace{1cm} (22)$$

In fact, one can use the eigenvector basis $\{w_n\}$ defined at (2) in order to represent the solution $u_r$. Hence one has for any $\omega \notin \Lambda = \{\lambda_n\}$ (see (2)):

$$u_r(x, \omega) = \sum_{n \geq 1} \xi_n(\omega) w_n(x) \quad \text{with} \quad \xi_n(\omega) = \frac{\int f_r w_n}{\lambda_n - \omega^2}.$$  

Thus, setting:

$$\delta_A = \frac{1}{\int_{\Omega} \text{div}(c^2 \nabla S_A^*) S_A^*}, \quad \delta_B = \frac{1}{\int_{\Omega} \text{div}(c^2 \nabla S_B^*) S_B^*},$$

one has:

$$K_A(\omega) = \delta_A \sum_{n \geq 1} \frac{\lambda_n}{\lambda_n - \omega^2} \frac{\int f_r w_n \int S_A^* w_n}{\lambda_n} \quad \text{and} \quad K_B(\omega) = \delta_B \sum_{n \geq 1} \frac{\lambda_n}{\lambda_n - \omega^2} \frac{\int f_r w_n \int S_B^* w_n}{\lambda_n}.$$
Let us consider that \( f_r = \chi_{[\omega_1, \omega_2]}(x) \). Let us introduce a function of the space \( L^2(\Omega) \)-say \( h(x, \omega) \)- such that:

\[
h(x, \omega) = \sum_{n \geq 1} \frac{\lambda_n}{\lambda_n - \omega^2} \int_{\Omega} d(x) w_n(x) dx \]

For instance

\[
h(x, 0) = d(x).
\]

If:

\[
\forall \omega \in [\omega_1, \omega_2], \quad |K_{Ar}| = |K_{Br}|,
\]

one has, assuming that \( f_r = \chi_{[\omega_1, \omega_2]}(x) \), one can deduce:

\[
\int_{\Omega} h(\omega) S^*_A = \pm \int_{\Omega} h(\omega) S^*_B = 0.
\]

The expressions of \( S^*_A \) and \( S^*_B \) are known (for instance using the polar coordinate around the point \( A \) one has [10]):

\[
S^*_A(r_A, \theta_A) = \frac{1}{c\sqrt{r_A}} \sin\left(\frac{\theta_A}{2}\right).
\]

Therefore, if \( d \) is chosen such that one can conclude that this is impossible for any position of \( A \) and \( B \), one can claim that there is no crack.

5. Let us denote by \( G_{\text{obs}} \) the quantity \( G \) but computed from an experimental measurement (or an estimation of it). With another respect, for a given position of the crack (i.e. \( x_A \) and \( x_B \) given) one can compute the numerical expression of \( G(\omega, x_A, x_B) \) and compare it to \( G_{\text{obs}} \). Therefore, one can define an observability criterion of a crack by:

\[
\text{CND}(x_A, x_B) = \int_{\omega_1}^{\omega_2} |G(\omega, x_A, x_B) - G_{\text{obs}}|^2 d\omega.
\]

And the localisation of the crack can be handle through the following optimisation problem for \( x_C \) given between \( [0, L] \) and \((x_A, x_B) \in \mathbb{R}^2\) being the control variables:

\[
\min_{0 \leq x_A \leq x_C \leq x_B \leq L} \text{CND}(x_A, x_B) + \epsilon[(x_A - x_C)^2 + (x_B - x_C)^2].
\]

The last term is introduced in order to select the solution for which the middle of the crack is the nearest of the point with abscissa \( x_1 = x_C \) and \( \epsilon \) is a small parameter (inducing a Tikhonov regularisation because of the \textit{a priori} non uniqueness of the solution [20]). The continuity of \( \text{CND} \) with respect to \((x_A, x_B)\) and its positiveness ensure the existence of a solution (but not the uniqueness). In fact, the main point is to detect the presence of a crack. Its exact position is a second step much more complicated from the computational point of view. It can performed numerically by using an adjoint state which leads to the expression of the gradient of the criterion to be minimized. This will be discussed in future works.
The numerical simulation in time. The Fourier transform can be computed directly or derived from a time simulation which is the choice we did. In fact we choose to solve numerically in time in order to have a full description of the mechanical phenomenon. Furthermore, in order to restrict the frequency range, we use a pseudo-Shannon wavelet setting:

\[ f(x, t) = d(x) \left[ \frac{\sin(at)}{\pi t} - \frac{\sin(bt)}{\pi t} \right], \quad a > b, \]  

(25)

where \( d(x) \) is a function the support of which is included in the soft media and which interacts with Love waves as shown on the left of figure 5. On the right of figure 5, we have plotted the solution representing the wave propagation for two materials with the velocities \( c_+ = 2c_- \) and the soft media corresponds to a a small strip (thickness = .1L and localized downwards on the figure)

\[ \text{Figure 5. Right hand side (left); solution with } c_+ = 2c_- \text{ (right)} \]

In order to compare the influence of the soft media we have drawn on figure 6 two cases of ratio between \( c_+ \) and \( c_- \). One can see the trapping effect in the softest media (Love waves) which increases with the ratio \( c_+/c_- \) compared to the figure 5 on the right.

\[ \text{Figure 6. Left : } c_+ = c_- \quad \text{Right: } c_+ = 4c_- \]
The Fourier transform of $f$ is the product of the characteristic function $\chi[−b,−a]∪[a,b](\omega)$ of the segment $±[a, b]$ by $q(x)$. In particular one has $f_i = 0$. The Fourier transform of $u$ is therefore computed by a symmetrisation with respect to the time (the initial conditions are zero). The values of $a$ and $b$ are chosen such that the segment $[a, b]$ contains Love wave frequencies. The figures 7 represents the wave propagation for two examples of frequencies ranges. The terms implying $\frac{\partial u}{\partial x_2}$ on the boundaries $\Gamma_e$ and $\Gamma_s$ which are plotted on figures 8 (without crack) and 9 (with a crack), show the effects of the artificial singularities appearing at points $E$ and $F$ compared to what happens inside the open set $\Omega$. These terms are the most important in the computation of the detection criterion $G$ explicited at (20).

**Figure 7.** Without crack Solution of (1): low frequencies (left), $\frac{\partial u}{\partial x_2}$ (right); $f$ is given at (25): $a = 50$ $b = 10$. Comparing with figure 8 one can see that the energy is less localized in the softest media than with high frequencies.

**Figure 8.** Without crack $\frac{\partial u}{\partial x_2}$ on the entry $\Gamma_e$ (left) on the exit $\Gamma_s$ (right), $c_+ = 2c_-$ the initial data are zero, $f$ is given at (25): $a = 260$ $b = 360$. 
Figure 9. With a crack on the interface between the two materials
The left figure represents the wave equation solution. The right one is
\( \frac{\partial u}{\partial x} \) on \( \Gamma_e \) and on \( \Gamma_s \). Same conditions as on figure 8 but with a
crack between \( x_1 = 0.4L \) and \( x_1 = 0.6L \). One can see the influence of the
crack on this quantity and therefore on the criterion CND defined at (23)

7. The Fix method for the singularities at the points \( E \) and \( F \). In order to improve a
finite element scheme for solving a PDEs, Fix [19] has suggested in fracture mechanics
modelling, to add the singular function \( S \) to the classical f.e.m. space \( V_h \) spanned by
first order polynomials (using here quadrangular elements). This leads to an approximate
solution denoted by \( u^h \) and such that:

\[
  u^h(x, t) = \alpha^h(t)S(x) + u^h_r(x, t), \quad u^h_r(x, t) \in V_h.
\]  

(26)

Our goal is to discuss this method in our framework. In particular, the error estimate be-
tween \( \alpha(t) \) and \( \alpha^h(t) \) is analyzed in terms of error with respec to the mesh size \( h \).
Finally, this suggests to subtract the term \( \alpha^h(t)S(x) \) to the approximate solution \( u^h \) in
order to try to eliminate the effect of this non-physical singularity in the computaion of the
criterion \( CND(x_A, x_B) \).

For sake of simplicity, we discuss the case of the following model (the functions \( f \) and
\( g \) are given):

\[
\begin{aligned}
  -\text{div}(c^2 \nabla u) &= f \text{ in } \Omega \quad \text{assuming that: } \int_{\Omega} f + \int_{\Gamma_e \cup \Gamma_s} gv = 0, \\
  \frac{\partial u}{\partial \nu} &= 0 \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial \nu} = g \text{ on } \Gamma_e \cup \Gamma_s, \quad \int_{\Omega} u = 0.
\end{aligned}
\]  

(27)

The variational formulation consists in finding \( u \in V_0 = \{ v \in H^1(\Omega) \text{ and } \int_{\Omega} v = 0 \} \)
such that:

\[
\forall v \in V_0, \quad \int_{\Omega} c^2 \nabla u \cdot \nabla v = \int_{\Omega} f v + \int_{\Gamma_e \cup \Gamma_s} gv. \quad (28)
\]

Let us denote by \( V^h \) the approximation space spanned by the first degree Lagrange finite
element with the classical assumptions. \( V^h_0 \) is the subspace of functions \( v \in V^h \) satisfying
the additional condition $\int_{\Omega} v = 0$. Setting:

$$V^h_s = \{ \mathbb{R} \times s \} \oplus V^h, \text{ and } V^h_{0s} = \{ v \in V^h_s, \int_{\Omega} v = 0 \}. \quad (29)$$

The upgrade approximate model consists in finding $u^h \in V^h_{0s}$ such that:

$$\forall v \in V^h_{0s}, \int_{\Omega} c^2 \nabla u^h \cdot \nabla v = \int_{\Omega} f v + \int_{\Gamma_s \cup \Gamma_s} g v. \quad (30)$$

**Theorem 7.1.** Let $u \in V_0$ and $u^h \in V^h_{0s}$ be the solutions of (28) and (30). Let us set:

$$u = \alpha S + u^r, \quad u^h = \alpha^h S + u^{rh} \text{ where } u^r \in H^2(\Omega) \cap V_0, \quad u^{rh} \in V^h_0.$$

The family of meshes is assumed to satisfy the classical regularity assumptions (see Ciarlet [4] or Raviart-Thomas [17]). Then there exists a constant $C$ independent on the mesh size $h$ such that:

$$|\alpha - \alpha^h| \leq \frac{c}{\log(h)} |u^r|_{2,2,\Omega}.$$

**Indications on the proof following the method used in Amara-Destuynder-Djaoua [1].** From the variational formulation characterizing $u$ at (28), one gets: Let us set (see figure 8): $K^h = [0, h[ \times h], h[\times ]$, which are the two elements near the singular point $E$ for instance. Furthermore, we introduce the projection -say $P^\gamma$ from $V_0$ onto $V^h_0$ with respect to the scalar product induced by the bilinear form $a(\cdot, \cdot)$. One has for instance (constant functions disappear in the expression of the bilinear form $a(\cdot, \cdot)$):

$$\forall v \in V^h_0, \quad a(S - PS, v) = 0.$$

Let us define by $\pi$ the Lagrange interpolation operator from $V \cap C^0(\overline{\Omega})$ into $V^h$. From the relations satisfied by $u$ and $u^h$ one has, using classical error estimates (see Strang-Fix [19] and P.A. Raviart-J.M. Thomas [17]):

$$\forall v \in V^h_0, \quad a(u^h - u, v) = 0 \implies \sqrt{a(u^h - u, u^h - u)} \leq \sqrt{a(u - \pi u, u - \pi u)} \leq c |u^r|_{2,2,\Omega}.$$

And by separating the term in $V^h_0$ from the singular function $S$, one obtains:

$$\forall v \in V^h_0, \quad (\alpha^h - \alpha)a(S, v) + a(u^h - u^r, v) = 0, \quad \forall v \in V^h_0, \quad a(S - PS, v) = 0.$$

Setting:

$$v = S - PS,$$

one deduces that (with a new constant $C$):

$$|\alpha^h - \alpha| \leq \frac{ch}{\sqrt{a(S - PS, S - PS)}}\inf_{z = \alpha x_1 + bx_2 + cx_1 x_2} \frac{1}{c^2} \left| \frac{\partial(S - z)}{\partial x_2} \right|^2.$$
The computation of the previous infimum using a symbolic software (Maple) enables one to complete the proof, because $S$ is known analytically.

**Remark 3.** The extension to the stationary case ($\omega \neq 0$) is based on the same method as the one described quickly here. Once, $\alpha^h$ known, the idea is to substract $\alpha^h S$ from $u^h$ in order to avoid the effect of the singularity $S$ in the computation of the detection criterion $G$ defined at (20). The effect is to smooth the behaviour of the solution of the wave equation near the singular points.

**Remark 4.** This theoretical result suggests to use $u^h$ in the CND criterion for detecting the crack. It appears has a way for eliminating the artifical singularity $S$ even if the error estimate on $\alpha - \alpha^h$ is very coarse.

Another possibility is to project the finite element solution (without adding the singularity) orthogonally to the singularity $S$. This is done by setting for instance:

$$ R_0 u^h = u^h - \int_\Omega S u^h S \quad \text{or} \quad R_1 u^h = u^h - \int_\Omega c^2 \nabla u^h \nabla S S. $$

We have plotted $u^h$ and $R_0 u^h$ on figure 10 the results obtained by this simple method in the computation of the term $\frac{\partial u}{\partial x_2}$. But up to now this is just an idea without mathematical foundations.

**Figure 10.** Comparison between $u^h$ (left) and $R u^h$ (right) inside $\Omega$ at the same time

8. **Conclusion and future work.** The non destructive testing is a challenge which requires efficient and reliable methods. The large amount of structures which should be tested requires new methods which enable to perform this duty faster than classical ones. The researches in this field are very active and a lot of problems are still to be solved from both the experimental aspect and the signal processing algorithms. In this paper, we have only suggested few remarks based on mathematical modelling but which are to be continued in order to confirm the operational possibilities of the tool considered here.
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