WLS optimal design for variable FIR filters with continuous parameter of fractional delay

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Abstract. A spectral parameterization method with discretization parameters will meet a tradeoff between the design accuracy and computational effort when designing a variable digital filter. Thus, a continuous spectral parameterization method is presented for obtaining the WLS optimal coefficients of variable fractional delay filters. Separated non-negative weighting function and some characters of matrix and matrix operations are applied to derive a closed-form solution, and Cholesky factorization is adopted to avoid the numerical problem of computing the inverses of a matrix. Its high accuracy and considerably reduced computational complexity of the proposed algorithm are experimentally verified. The given FIR filters can find their applications in high accuracy dynamic time compensation.

1. Introduction
Multi-channel systems are used extensively in the fields such as radar, communication and synchronous control system. However, some multi-channel systems are exceedingly sensitive to the time errors among channels, which are crucial to their performances. For instance, time error of the channels mismatch must be kept within 1 ps in a two-channel Time-Interleaved Analog-to-Digital Converters (TIADCs) sampling system [1] to obtain the Spurious-Free Dynamic Range (SFDR) up to 70 dB. So, Variable Fractional Delay (VFD) filters [2-8] find their application in multi-channel systems to provide exact time compensation. The Method of Spectral Parameterization (MSP) is more attractive in VFD filters designs [2-8]. MSP was first proposed by Zarour et al. [2] to design infinite impulse response filters with variable pole-zero plots. And then, a Farrow structure [3], which is named for the author and is widely applied at present, was proposed to implement these digital variable filters. Furthermore, discrete parameters were introduced to MSP by Tarczynski et al. [4] and then developed [5-8] via optimizing variable filters design under the Weighted Least Squares (WLS) criterion to exploit high accuracy of delay. Nonetheless, this scheme is of computationally expensive.

A new MSP with continuous parameter is presented in this paper and the WLS optimal solution of these filters’ coefficients is deduced in detail. The amount of calculation is reduced markedly without any lose of accuracy in the proposed method via directly employing the value of the filter delay as one continuous spectral parameter. Numerical experiments demonstrate it works effectively.

2. WLS Model of VFD Filter
Suppose that variable $\tau$ ($\tau \in \Delta$) denote the value of delay, then the problem in the study is to structure a model of the filter relating to $\tau$. Note that $h(n, \tau)$ is the linear combination of the base $\psi_m(\tau)$ and the sub-filters $c_m(n)$ if $\tau$ is employed as a continuous spectral parameter [5-8], where $h(n, \tau)$ is the
coefficients of the FIR filter to design. Therefore, \( h(n, \tau) \) has the following forms in the time domain and the Z domain, respectively.

\[
h(n, \tau) = \sum_{m=0}^{M} c_m(n) \psi_m(\tau) \quad n = 0, \ldots, N, \quad H(z, \tau) = \sum_{n=0}^{N} h(n, \tau) z^{-n} = \sum_{m=0}^{M} C_m(z) \psi_m(\tau)
\]  

(1)

where \( C_m(z) = \sum_{n=0}^{N} c_m(n) z^{-n} \). Denote the sequence number \( q = n + (N + 1) \times m \), and define \( \Psi_q(e^{j\omega \tau}) = \psi_m(\tau) \cdot e^{-j\omega n} \), \( a = [c_0(0) \cdots c_0(N) \cdots c_M(0) \cdots c_M(N)]^T \), \( \Psi = [\Psi_0, \Psi_1, \ldots, \Psi_{(N+1)(M+1)-1}]^T \), where \([\cdot]^T\) is the transpose of a matrix, thus the frequency response of \( h(n, \tau) \) can be formulated as \( H(e^{j\omega \tau}) = a^T \Psi \). It indicates that \( h(n, \tau) \) can be determined uniquely by \( a \).

Assume \( H_d(e^{j\omega \tau}) \) as the expected frequency response of \( h(n, \tau) \), where \( \omega \in \Omega \) is the normalized angle frequency, and define the error function \( e(\omega, \tau) = H(e^{j\omega \tau}) - H_d(e^{j\omega \tau}, \tau) \), so the cost function of the filter design can be expressed as Eq. 2 according to WLS criterion.

\[
J(a) = \int_{\Omega} \int_{\Delta} W(\omega, \tau) |e(\omega, \tau)|^2 d\omega d\tau = a^T R a - 2a^T b + J_0
\]  

(2)

where \( W(\omega, \tau) \) is nonnegative weighted function, \( R \) is a \((N + 1)(M + 1)\) square matrix with \( R = \int_{\Omega} \int_{\Delta} W(\omega, \tau) \Psi(\omega, \tau)^H d\omega d\tau \), \( b \) is a vector with \( b = \int_{\Omega} \int_{\Delta} W(\omega, \tau) Re[H_d(\omega, \tau)] \Psi(\omega, \tau)^H d\omega d\tau \), \( J_0 \) is a constant independent of \( a \) and with \( J_0 = \int_{\Omega} \int_{\Delta} W(\omega, \tau) |H_d(\omega, \tau)|^2 d\omega d\tau \). The notations aforementioned \([\cdot]^H\), \((\cdot)^*\) and \(Re[\cdot]\) are the conjugate transpose, the conjugate and the real part of a matrix, respectively.

Let \( \partial J(a) / \partial a = 0 \), the WLS optimal resolution of \( a \) is modeled as

\[
a_{WLS} = R^{-1} b
\]  

(3)

where \([\cdot]^{-1}\) is the inverse of a matrix. Select \( \psi_m(\tau) = \tau^m \) as the base, \( h(n, \tau) \) can be then implemented using a simple Farrow structure [3] shown in Fig. 1, in which \( \triangleright \) is a multiplier unit and \( C_m(z) \) is a sub-filter with constant coefficients. It is praiseworthy that the VFD filter in Fig. 1 can be adjusted continuously in the scope of a sample period controlled solely by the parameter \( \tau \).

![Figure 1. Farrow structure of the VFD FIR filter](image)

3. Optimal Solution

To deduce the optimal solution of \( a \), it is required to calculate \( R^{-1} \) and \( b \) in Eq. 3. It is well known that solving the inverse of a matrix is computationally expensive. So we first consider the approaches to reduce the calculation of \( R^{-1} \).
The function $\Psi$ can be rewritten as $\Psi(e^{i\omega \tau} e^{-i\omega \tau} \tau^m = e^{i\omega \tau} \tau^m$. It leads to $\Psi = E \otimes \omega$, where $E = [1 \cdots \tau^M]$, $\omega = [1 e^{-j\omega} \cdots e^{-j\omega \tau}]$, and $\otimes$ is the Kronecker product [9] with the property of $\Psi^H = \tau^T \otimes \omega^*$. Hence, we have

$$\Psi \Psi^H = (\tau \otimes \omega) (\tau \otimes \omega)^H = (\tau \tau^T) \otimes (\omega \omega^H).$$  \hfill (4)$$

Furthermore, the methods of separated variables and segmental variables can be used to simplify the weighted function. Then, $W(\omega, \tau)$ can be expressed as

$$W(\omega, \tau) = W_1(\omega) W_2(\tau)$$  \hfill (5)$$

with $W_1(\omega) = \rho_1$, $\omega \in \omega \subseteq \Omega$, $k = 1, 2, \cdots K$, $W_2(\tau) = \sigma_1$, $\tau \in \tau \subseteq \Delta$, $l = 1, 2, \cdots L$. Substitute Eq. 4 and Eq. 5 to $R$, we derive

$$R = \int_{\Delta} \int_{\omega} W_1(\omega) W_2(\tau) (\tau \tau^T) \otimes (\omega \omega^H) d\omega d\tau = \int_{\Delta} W_2(\tau) (\tau \tau^T) d\tau \otimes \int_{\omega} W_1(\omega) \omega \omega^H d\omega.$$  \hfill (6)$$

The two parts of Kronecker product in the right of Eq. 6 are analyzed subsequently. The first part takes the form

$$P = \int_{\Delta} W_2(\tau) (\tau \tau^T) d\tau = \sum_{j=1}^{L} \sigma_j \int_{\tau_j} \tau \tau^T d\tau = \sum_{j=1}^{L} \sigma_j [P_1(i, j)]$$  \hfill (7)$$

where $[P_1(i, j)]$ is a $M + 1$ Hankel matrix [9] with elements $P_1(i, j) = (\tau^{i+j-1})/(i+j-1)$, in which $f(x)|_{x_i}$ is Newton-Leibniz definite integral. $[P_1(i, j)]$ is known as of no more than $2M + 1$ elements different. Obviously, $P$ is also a Hankel matrix and has the property $P^T = P$. Thereby, the elements to compute in $P$ amount to $(2M + 1) \times L$.

Analyzing the second part in Eq. 6, we define

$$Q_{\omega} = \int_{\Omega} W_1(\omega) \omega \omega^H d\omega$$  \hfill (8)$$

where $\omega \omega^H$ is a Hermitian matrix [7], which leads to $Q_{\omega} = Q_{\omega}^H$. Rewrite Eq. 6 as $R = P \otimes Q_{\omega}$, and apply the properties of $P$ and $Q_{\omega}$, we can derive $(a^T Ra)^H = a^T Ra$. As a result, the first part in Eq. 2 is a real function, especially, the real part of $Q_{\omega}$ is only needed to calculate. So, we define

$$Q = \text{Re}[Q_{\omega}] = \int_{\Omega} W_1(\omega) \text{Re}[\omega \omega^H] d\omega = \sum_{k=1}^{K} \rho_k [Q_1(i, j)]$$  \hfill (9)$$

where $Q_1(0, j) = \delta(\omega_j)$, $j = 0$; $\sin(jw)|_{\omega_j} / j$, $j \neq 0$, in which $\delta(\omega_j)$ is the range of $\omega_j$. In fact, $[Q_1(i, j)]$ is a $N + 1$ symmetrical Toeplitz matrix [9], which can be constructed by its elements on the first row. Then the amount to compute in $Q$ is $(N + 1) \times K$.

According to the analysis above, $R$ can be formulated as $R = P \otimes Q$, where $P$ and $Q$ have the forms of Eq. 7 and Eq. 9, respectively.

Now, we explore the matrix decomposition of $R^{-1}$ to further reduce its complexity. Consider that $P$ and $Q$ are symmetrical positive definite, so Cholesky decomposition [9] can be used, namely $P = FF^T$ and $Q = GG^T$, where $F$ and $G$ are both lower triangular matrix. Eventually, $R^{-1}$ can be expressed as follow
where \( [\cdot]^T \) denotes the inverse of a transposed matrix.

The next problem we consider is the calculation of \( b \). Review the express aforementioned, one part of \( b \) can be re-expressed as

\[
Re[H'_d(e^{j\omega}, \tau)\Psi] = Re[(\tau \otimes \omega) \cdot e^{j\omega}] = \tau \otimes Re[e^{j\omega}] = \tau \otimes \Phi
\]

where \( \Phi = [\cos(\omega \tau) \cos(\omega(\tau - 1)) \cdots \cos(\omega(\tau - N))]^T \). Apply Eq. 5 and Eq. 11 to \( b \), we can get

\[
b = \int_\Omega \int_\Lambda W_1(\omega)W_2(\tau)(\tau \otimes \Phi)d\omega d\tau = \int_\Lambda W_2(\tau)(\tau \otimes \int_\Omega W_1(\omega)\Phi d\omega)d\tau.
\]

And there is

\[
\int_\Omega W_1(\omega)\Phi d\omega = \sum_{k=1}^K \rho_k \int_\Omega \Phi d\omega = \sum_{k=1}^K \rho_k [\eta_k(i)]
\]

where \( \eta_k(i) \) is the integral of \( i \)-th element of \( \Phi \) in the range of \( \omega_k \), namely \( \eta_k(i) = \delta(\omega_k), \tau - i + 1 = 0 \) and \( \eta_k(i) = \sin(\omega(\tau - i + 1)) \mid_{\omega_k} / (\tau - i + 1), \tau - i + 1 \neq 0 \). Then, \( b \) can be computed using Eq. 14 shown below.

\[
b = \int_\Lambda W_2(\tau)(\tau \otimes \sum_{k=1}^K \rho_k [\eta_k(i)])d\tau = \int_\Lambda W_2(\tau) \sum_{k=1}^K \rho_k (\tau \otimes [\eta_k(i)])d\tau
\]

\[
= \sum_{k=1}^K \rho_k \int_\Lambda W_2(\tau)(\tau \otimes [\eta_k(i)])d\tau = \sum_{k=1}^K \rho_k \sum_{l=1}^L \sigma_l \int_{\omega_k} [\tau \otimes [\eta_k(i)]d\tau = \sum_{k=1}^K \rho_k (b_{\omega_k}(j))
\]

where vector \( [b_{\omega_k}(j)] \) has elements as \( b_{\omega_k}(j) = \int_{\omega_k} \tau^{\omega_k} \eta_k(i) d\tau, j = m(N+1) + i \).

Finally, the WLS resolution of \( a \) comes out by substituting Eq. 10 and Eq. 14 to Eq. 3, that is

\[
a_{\text{WLS}} = R^{-1}b = (F^{-T} \otimes G^{-T})(F^{-1} \otimes G^{-1})b = (F^{-T} \otimes G^{-T})(F^{-1} \otimes G^{-1})K \sum_{k=1}^K \rho_k \sigma_l [b_{\omega_k}(j)]
\]

4. Numerical Experiments and Discussions

Two numerical experiments are carried out in this section to illuminate the performances of our method compared to that of in [4], [8]. First, a group of VDF FIR filters are designed, which are low-pass filters and have the parameter of \( N = 61, M = 7, \Delta = [-0.5 0.5] \) and \( \Omega = [0 \ 0.85\pi] \). To flatten the response curve in the whole range of frequency, the weight value are selected as shown in Tab. 1.

| \( \omega / \pi \) | \( W_1(\omega) \) | \( \tau \) | \( W_2(\tau) \) |
|-----------------|----------------|----------------|----------------|
| [0 0.4) | 0.63 | [-0.5 -0.4) | 50 |
| [0.4 0.75) | 4.7 | [-0.4 0.4) | 0.2 |
| [0.75 0.85) | 35 | [0.4 0.5) | 7.8 |

The errors of the designed filters shown in Fig. 2 are analyzed using MATLAB 7.0. We can see that the maximum absolute error of \( \tau \) is only \( 2.985 \times 10^{-4} \), the maximum relative error of \( \tau \in \Delta \) is 0.112%, and only 0.041% for \( \tau \in [-0.1 \ 0.1] \). And the filters also have the ideal response curve, with errors...
\[ 20 \log | e(\omega, \tau) | \text{ are all under } -100\text{dB}. \] Fig. 2 show that the designed filters hold the properties of high delay accuracy and ideal response curve.

![Graph](image1)

(a) Absolute error of delay value

![Graph](image2)

(b) Errors of the frequency response

**Figure 2.** Errors curve of the designed filters

Secondly, the complexity and the accuracy of our algorithm are tested in DSP and as shown in Tab. 2, in which there has \( e_{\max} = \max \left\{ 20 \log | e(\omega, \tau) | \right\} \). It shows that our method outperforms the method in [4], [8], i.e., the float multiplication of our algorithm is only 5.585\% and 42.366\% of that in [4], [8], introspectively, and with slightly enhanced accuracy.

| Performance | Method [4] | Method [8] | This work |
|-------------|------------|------------|-----------|
| Float multiplication | 77689195 | 10240472 | 4338484 |
| \( e_{\max} \) (dB) | -99.8203 | -100.1035 | -100.1862 |

**Table 2.** Performance running in DSP

5. **Summary**

In this paper, we focused on the design of the VDF FIR filters and proposed a new method of continuous spectral parameterization. We employ the delay value of filters as spectral parameter and deduce the WLS optimal coefficients of the filters. Theoretical analysis and numerical experiments demonstrate that the complexity of this method is reduced enormously without any lose of accuracy. The FIR filters designed using our method can be used in the areas of radar and communication as high accuracy dynamic time compensation.

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