ALGEBRAIC INTEGRABILITY OF LOTKA-VOLterra
EQUATIONS IN THREE DIMENSIONS

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Abstract. We examine the algebraic complete integrability of Lotka-Volterra equations in three dimensions. We restrict our attention to Lotka-Volterra systems defined by a skew symmetric matrix. We obtain a complete classification of such systems. The classification is obtained using Painlevé analysis and more specifically by the use of Kowalevski exponents. The imposition of certain integrality conditions on the Kowalevski exponents gives necessary conditions for the algebraic integrability of the corresponding systems. We also show that the conditions are sufficient.

1. Introduction

The Lotka-Volterra model is a basic model of predator-prey interactions. The model was developed independently by Alfred Lotka (1925), and Vito Volterra (1926). It forms the basis for many models used today in the analysis of population dynamics. In three dimensions it describes the dynamics of a biological system where three species interact.

The most general form of Lotka-Volterra equations is

\[ \dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad i = 1, 2, \ldots, n. \]

We consider Lotka-Volterra equations without linear terms (\( \varepsilon_i = 0 \)), and where the matrix of interaction coefficients \( A = (a_{ij}) \) is skew-symmetric. This is a natural assumption related to the principle that crowding inhibits growth. The special case of Kac-van Moerbeke system (KM-system) was used to describe population evolution in a hierarchical system of competing individuals. The KM-system has close connection with the Toda lattice. The Lotka-Volterra equations were studied by many authors in its various aspects, e.g. complete integrability [7] Poisson and bi-Hamiltonian formulation ([10], [11], [16]), stability of solutions and Darboux polynomials ([9], [18]).

In this paper we examine the algebraic complete integrability of such Lotka-Volterra equations in three dimensions. The basic tools for the required classification are, the use of Painlevé analysis, the examination of the eigenvalues of the Kowalevski matrix and other standard Lax pair and Poisson techniques. The Kowalevski exponents are useful in establishing integrability or non-integrability of Hamiltoninan systems; see [1], [2], [6], [13], [14], [17], [21]. The first step is to impose certain conditions on the exponents, i.e., we require that all the Kowalevski exponents be integers. This gives a finite list of values of the parameters satisfying such conditions. This step requires some elementary number theoretic techniques as is usual with such type of classification. In the three-dimensional case the general expressions for the Kowalevski exponents are rational and therefore the number theoretic analysis is manageable.

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The second step is to check that the leading behavior of the Laurent series solutions agrees with the weights of the corresponding homogeneous vector field defining the dynamical system. In our case the weights are all equal to one and therefore we must exclude the possibility that some of the Laurent series have leading terms with poles of order greater than one. This is a step usually omitted by some authors due to its complexity, but in this paper we analyse this in detail. To accomplish this step we use old-fashioned Painlevé Analysis, i.e., Laurent series. The application of Painlevé analysis and especially of the ARS algorithm (see [3] [4], [5], [7], [8], [14]) is useful in calculating the Laurent solution of a system and check if there are \((n-1)\) free parameters.

In performing Painlevé analysis we use the fact that the sum of the variables is always a first integral. Surprisingly the Painlevé analysis does not reveal any additional cases besides the ones already found by using the Kowalevski exponents. In this classification of the algebraic completely integrable Lotka-Volterra systems we discover, as expected, some well known integrable systems like the open and periodic Kac-van Moerbeke systems.

To make sure that our conditions are not only necessary but also sufficient we verify that the systems obtained are indeed algebraically completely integrable by checking the number of free parameters. We also have to point out that our classification is up to isomorphism. In other words, if one system is obtained from another by an invertible linear change of variables, we do not consider them as different. Modulo this identification we obtain only six classes of solutions.

The Lotka-Volterra system can be expressed in hamiltonian form as follows: Define a quadratic Poisson bracket by the formula

\[
\{x_i, x_j\} = a_{ij} x_i x_j, \quad i, j = 1, 2, \ldots, n.
\]

Then the system can be written in the form \(\dot{x}_i = \{x_i, H\}\), where \(H = \sum_{i=1}^{n} x_i\). The Liouville integrability in the three-dimensional case can be easily established. In addition to the Hamiltonian function \(H\), there exists a second integral, in fact a Casimir \(F\). The formula for this Casimir is given afterwards. We have to point out that in general algebraic integrability does not imply Liouville integrability and vice versa.

In this paper we restrict our attention to the three dimensional case. For \(n = 3\) the system is defined by the matrix

\[
A = \begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{pmatrix},
\]

where \(a, b, c\) are constants. We use the notation \((a, b, c)\) to denote this system. It turns out that the algebraically integrable Lotka-Volterra systems fall either into two infinite families or four exceptional cases:

**Theorem 1.** The Lotka-Volterra equations in three dimensions are algebraically complete integrable if and only if \((a, b, c)\) is in the class of

\[
(l_2) \quad (1, 0, 1) \\
(l_3) \quad (1, -1, 1) \\
(l_4) \quad (1, -1, 2) \\
(l_6) \quad (1, -2, 3) \\
(l_\lambda) \quad (1, 1, \lambda) \quad \lambda \in \mathbb{Z} \setminus 0. \\
(l_0) \quad (1, 1 + \mu, \mu) \quad \mu \in \mathbb{R} \setminus 0.
\]

We use the notation \(l_j\) to indicate that the system has an invariant of degree \(j\) or equivalently that the largest Kovalevski exponent is \(j\).
In Section 2 we give the basic definitions of weight-homogeneous vector fields and the Kowalevski matrix. The definition of Kowalevski exponents and relevant results follow the recent book [2]. See also the review article of Goriely [13] where one can find more properties of these exponents. In Section 3 we give some related properties of the Kowalevski exponents and a criterion of algebraic complete integrability. In Section 4 we define the three dimensional Lotka-Volterra systems and find necessary conditions for their algebraic integrability by analyzing the corresponding Kowalevski exponents. In Section 5 we show that our classification is complete. Finally, in Section 6 we exclude any solutions that may exist due to higher order poles.

2. Basic definitions

We begin by defining what is a weight homogeneous polynomial. We follow the notation from [2].

**Definition 1.** A polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is called a weight-homogeneous polynomial of weight $k$ with respect to a vector $v = (v_1, v_2, \ldots, v_n)$ if
\[
f(t^{v_1}x_1, \ldots, t^{v_n}x_n) = t^k f(x_1, x_2, \ldots, x_n).
\]
The vector $v$ is called the weight vector. The $v_i$ are all positive integers without a common divisor. The weight $k$ is denoted by $\omega(f)$.

**Definition 2.** A polynomial vector field on $\mathbb{C}^n$,
\[
\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n) \\
\vdots \\
\dot{x}_n = f_n(x_1, x_2, \ldots, x_n)
\]
is called a weight-homogeneous vector field of weight $k$ (with respect to a weight vector $v$), if $\omega(f_i) = v_i + k = \omega(x_i) + k$ for $i = 1, 2, \ldots, n$. A weight-homogeneous vector field of weight 1 is called a homogeneous vector field. Furthermore, when all the weights are equal to 1, this is simply called a homogeneous vector field.

**Example 1.** We consider the periodic 5-particle Kac-van Moerbeke lattice that is given by the quadratic vector field
\[
\dot{x}_i = x_i(x_i-1 - x_{i+1}), \quad i = 1, \ldots, 5,
\]
with $x_i = x_{i+5}$. This system has three independent constants of motion,
\[
F_1 = x_1 + x_2 + x_3 + x_4 + x_5, \\
F_2 = x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2, \\
F_3 = x_1x_2x_3x_4x_5.
\]
Taking $v = (1, 1, 1, 1, 1)$, (3) becomes a homogeneous vector field and the weights of the integrals of motion are $\omega(F_1) = 1$, $\omega(F_2) = 2$ and $\omega(F_3) = 5$.

We now give the definition of an algebraically completely integrable system following [13] [14]. Note that this definition differs from the one given in [2]. However, the definition in [2] implies the definition of this paper.

**Definition 3.** A vector field,
\[
\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n) \\
\vdots \\
\dot{x}_n = f_n(x_1, x_2, \ldots, x_n),
\]
is called an algebraically completely integrable system (a.c.i.) if its solution can be expressed as Laurent series
\[ x_i(t) = \frac{1}{t^{v_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \quad i = 1, 2, \ldots, n, \]
where \( n - 1 \) of the coefficients \( x_i^{(k)} \) are free parameters.

2.1. Kowalevski Exponents. The following Proposition is important for two reasons. First, it gives an induction formula for finding the Laurent solution of a weight-homogeneous vector field and second it defines the Kowalevski exponents which is an important tool for our classification.

**Proposition 1.** Suppose that we have a weight-homogeneous vector field on \( \mathbb{C}^n \) given by
\[ \dot{x}_i = f_i(x_1, \ldots, x_n), \quad i = 1, 2, \ldots, n, \]
and suppose that
\[ x_i(t) = \frac{1}{t^{v_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \quad i = 1, 2, \ldots, n \]
is a weight-homogeneous Laurent solution for this vector field. Then the leading coefficients, \( x_i^{(0)} \), satisfy the non-linear algebraic equations
\[ v_1 x_1^{(0)} + f_1(x_1^{(0)}, \ldots, x_n^{(0)}) = 0, \]
\[ \vdots \]
\[ v_n x_n^{(0)} + f_n(x_1^{(0)}, \ldots, x_n^{(0)}) = 0, \]
while the subsequent terms \( x_i^{(k)} \) satisfy
\[ (kI_{n} - K(x^{(0)})) x^{(k)} = R^{(k)}, \]
where \( x^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix} \) and \( R^{(k)} = \begin{pmatrix} R_1^{(k)} \\ \vdots \\ R_n^{(k)} \end{pmatrix} \). \( R^{(k)} \) is a polynomial, which depends on the variables \( x_1^{(l)}, \ldots, x_n^{(l)} \) with \( 0 \leq l < k \) only. The elements of the \( n \times n \) matrix \( K \) are given by
\[ K_{i,j} := \frac{\partial f_i}{\partial x_j} + v_i \delta_{ij}, \]
where \( \delta \) is the Kronecker delta.

**Remark 1.** The pole order \( v_i \) of \( x_i \) in (2) coincides with the \( i \)th component of the weight vector. The number, \( v_i \), is not necessarily the pole order of \( x_i \) because some of the \( x_i^{(0)} \) that can be calculated solving (5) may be equal to zero.

**Definition 4.** The system (5) is called the indicial equation and its solution set is called the indicial locus and it is denoted by \( \mathcal{I} \). The \( n \times n \) matrix \( K \), defined by (8), is called the Kowalevski matrix and its eigenvalues are called Kowalevski exponents (a terminology due to Yoshida).

A necessary condition for algebraic integrability is that \( n - 1 \) eigenvalues of \( K \) should be integers. It turns out that the last eigenvalue is always \(-1\). The eigenvector that corresponds to \(-1\) is also known. We have the following Proposition which can be found in [2].
Proposition 2.
For any $m$ which belongs to the indicial locus $I$, except for the trivial element, the Kowalevski matrix $K(m)$ of a weight homogeneous vector field always has $-1$ as an eigenvalue. The corresponding eigenspace contains $(v_1m_1, \ldots, v_nm_n)^T$ as an eigenvector.

3. Properties of Kowalevski exponents

In this section we state some properties of Kowalevski exponents clarifying the connection with the degrees of the first integrals. We also give a necessary condition for a system to be algebraically completely integrable. The following results can be found in [12, 13, 17, 19, 21].

Theorem 2. If the weight-homogeneous system $\dot{x} = f(x)$ has $k$ independent algebraic first integrals $I_1, \ldots, I_k$ of weighted degrees $d_1, \ldots, d_k$ and Kowalevski exponents $\rho_2, \ldots, \rho_n$, then there exists a $k \times (n - 1)$ matrix $N$ with integer entries, such that

$$\sum_{j=2}^{n} N_{ij} \rho_j = d_i, \quad i = 1, \ldots, k.$$ 

From this theorem we have the two following corollaries:

Corollary 1. If the Kowalevski exponents are $\mathbb{Z}$-independent, then there is no rational first integral.

Corollary 2. If the Kowalevski exponents are $\mathbb{N}$-independent, then there is no polynomial first integral.

We also have the following theorem, see [12, 13, 20].

Theorem 3. Suppose that the system (1) possesses a homogeneous first integral $F_m$ of degree $m$. Then there exists a set of non-negative integers $k_2, \ldots, k_n$ such that

$$\sum_{j=2}^{n} k_j \rho_j = m \quad k_2 + k_3 + \cdots + k_n \leq m.$$

The next theorem which can be found in [2] gives us a necessary condition for a system to be algebraically integrable. This criterion can be checked easily simply by computing the Kowalevski exponents.

Theorem 4. Let $\rho_1 = -1$. A necessary condition for a system of the form (2) to be algebraically completely integrable is that all the Kowalevski exponents $\rho_2, \ldots, \rho_n$ should be integers.

4. Lotka-Volterra systems

4.1. Hamiltonian formulation. Consider a Lotka-Volterra system of the form

$$\dot{x}_j = \sum_{k=1}^{n} a_{jk}x_jx_k, \quad \text{for } j = 1, 2, \ldots, n,$$

where the matrix $A = (a_{ij})$ is constant and skew symmetric.
There is a sympletic realization of the system which goes back to Volterra. In other words a projection from $\mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ from a symplectic space to a Poisson space. Volterra defined the variables

$$q_i(t) = \int_0^t u_i(s) ds$$

and

$$p_i(t) = \ln(\dot{q}_i) - \frac{1}{2} \sum_{k=1}^n a_{ik} q_k,$$

for $i = 1, 2, \ldots, n$. Now the number of variables is doubled and Volterra’s transformation is given explicitly by

$$x_i = e^{p_i + \frac{1}{2} \sum_{k=1}^n a_{ik} q_k} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

The Hamiltonian in these coordinates becomes

$$H = \sum_{i=1}^n x_i = \sum_{i=1}^n q_i = \sum_{i=1}^n e^{p_i + \frac{1}{2} \sum_{k=1}^n a_{ik} q_k}.$$

The equations (9) can be written in Hamiltonian form

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\},$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\},$$

$i = 1, 2, \ldots, n$, and the bracket $\{\cdot, \cdot\}$ is the standard symplectic bracket on $\mathbb{R}^{2n}$:

$$\{q_i, p_j\} = \delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \quad i, j = 1, 2, \ldots, n; \end{cases}$$

all other brackets are zero. The corresponding Poisson bracket in $x$ coordinates is quadratic

$$\{x_i, x_j\} = a_{ij} x_i x_j, \quad i, j = 1, 2, \ldots, n.$$

Equations (9) in $x$ coordinates are obtained by using this Poisson bracket and the Hamiltonian, $H = x_1 + x_2 + \cdots + x_n$.

4.2. The three-dimensional case.

In this paper we restrict our attention to the three dimensional case. For $n = 3$ the system is defined by the matrix

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

where $a, b, c$ are real constants.

Using equations (8) we obtain the Kowaleski matrix

$$\begin{pmatrix} ax_2^{(0)} + bx_3^{(0)} + 1 \\ -ax_1^{(0)} + cx_3^{(0)} + 1 \\ -bx_1^{(0)} + cx_2^{(0)} + 1 \end{pmatrix},$$

where $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ is an element of the indicial locus, i.e., a solution of the simultaneous equation (6), which in this case is written as

$$\begin{align*}
  x_1^{(0)} + ax_1^{(0)} x_2^{(0)} + bx_1^{(0)} x_3^{(0)} &= 0, \\
  x_2^{(0)} - ax_2^{(0)} x_1^{(0)} + cx_2^{(0)} x_3^{(0)} &= 0, \\
  x_3^{(0)} - bx_3^{(0)} x_1^{(0)} - cx_3^{(0)} x_2^{(0)} &= 0.
\end{align*}$$
In Table 1 we list the corresponding Kowalevski exponents for each element of the indicial locus.

| Vector $x^{(0)}$ | Kowalevski exponents | Vector $x^{(0)}$ | Kowalevski exponents |
|------------------|----------------------|------------------|----------------------|
| $(0,0,0)$         | 1,1,1                | $(0,\frac{1}{c},\frac{1}{c})$ | $-1,1,\frac{a-b+c}{c}$ |
| $(\frac{1}{b},0,\frac{1}{b})$ | $-1,1,-\frac{a-b+c}{b}$ | $(\frac{1}{a},\frac{1}{a},0)$ | $-1,1,\frac{a-b+c}{a}$ |

**Table 1.** Kowalevski exponents of 3x3 Lotka-Volterra equations

A necessary condition of algebraic integrability is that all the Kowalevski exponents must be integers. So we have to solve the simultaneous Diophantine equations

\begin{align}
\frac{a-b+c}{a} &= k_1, & \frac{a-b+c}{c} &= k_2, & -\frac{a-b+c}{b} &= k_3,
\end{align}

where $k_1, k_2, k_3 \in \mathbb{Z}$. The case $b = a + c$ for which $k_1 = k_2 = k_3 = 0$ is investigated below. Solving (13) we find that

\begin{align}
k_3 &= \frac{k_1k_2}{k_1k_2 - k_1 - k_2}, \quad \begin{cases}
c = \frac{k_1}{k_2}a, & b = \frac{k_1+k_2-k_1k_2}{k_2}, \\
a = k_2, & b = \frac{k_1+k_2-k_1k_2}{k_1}c
\end{cases} \\
k_2 &= \frac{k_1k_3}{k_1k_3 - k_1 - k_3}, \quad \begin{cases}
b = -\frac{k_2}{k_3}a, & c = \frac{k_1+k_2-k_1k_2}{k_3}, \\
a = \frac{k_3}{k_1}b, & c = \frac{k_1+k_2-k_1k_2}{k_1}b
\end{cases} \\
k_1 &= \frac{k_2k_3}{k_2k_3 - k_2 - k_3}, \quad \begin{cases}
b = -\frac{k_2}{k_3}c, & a = \frac{k_1+k_2-k_1k_2}{k_3}, \\
c = \frac{k_2}{k_2}b, & a = \frac{k_1+k_2-k_1k_2}{k_2}b
\end{cases}
\end{align}

We assume first, that the Kowalevski exponents are not zero. We examine the solution

\begin{align}
k_3 &= \frac{k_1k_2}{k_1k_2 - k_1 - k_2}, & b &= \frac{k_1+k_2-k_1k_2}{k_2}, & c &= \frac{k_1}{k_2}a.
\end{align}

We determine the values of $k_1$ and $k_2$ so that the fraction,

\begin{align}
k_3 &= \frac{k_1k_2}{k_1k_2 - k_1 - k_2},
\end{align}

is an integer. We first consider the case $k_1k_2 - k_1 - k_2 \neq 0$.

**Case I** Assume positive values for both $k_1$ and $k_2$.

Since

\begin{align}
\frac{k_1k_2}{k_1k_2 - k_1 - k_2} = 1 + \frac{k_1+k_2}{k_1k_2 - k_1 - k_2}
\end{align}

it is enough to to solve the Diophantine equation

\begin{align}
\frac{x+y}{xy - x - y} = z
\end{align}

for $x, y$ positive integers and $z \in \mathbb{Z}$. 
Lemma 1. Let $x, y \in \mathbb{Z}^+$ with $x \leq y$. Then

$$\frac{x + y}{xy - x - y} \in \mathbb{Z}$$

if and only if $(x, y)$ is one of the following: $(1, \lambda)$, $\lambda \in \mathbb{Z}^+$, $(2, 3)$, $(2, 4)$, $(2, 6)$, $(3, 3)$, $(3, 6)$, $(4, 4)$.

Proof. Since

$$xy - x - y \leq x + y$$

we have

$$xy \leq 2(x + y) \leq 4y .$$

Since $y \neq 0$ we get $x \leq 4$. Therefore $x = 1, 2, 3, 4$. We examine each case separately.

- If $x = 1$

  $$\frac{x + y}{xy - x - y} = \frac{1 + y}{-1} = -1 - y \in \mathbb{Z} .$$

  Therefore $(1, \lambda)$, $\lambda \in \mathbb{Z}^+$ is always a solution.

- Suppose $x = 2$. Then

  $$\frac{x + y}{xy - x - y} = \frac{2 + y}{y - 2} = 1 + \frac{4}{y - 2}$$

  should be an integer. Therefore $y - 2 = \pm 1, \pm 2, \pm 4$. We obtain the solutions $(2, 3)$, $(2, 4)$ and $(2, 6)$.

- Suppose $x = 3$. Then

  $$\frac{x + y}{xy - x - y} = \frac{y + 3}{2y - 3}$$

  should be an integer. Therefore

  $$2y - 3 \leq y + 3$$

  and we obtain $y \leq 6$. We obtain the solutions $(3, 3)$ and $(3, 6)$.

- Suppose $x = 4$. Then

  $$\frac{x + y}{xy - x - y} = \frac{y + 4}{4y - 4}$$

  should be an integer. Therefore

  $$3y - 4 \leq y + 4$$

  and we obtain $y \leq 4$. We obtain the solution $(4, 4)$.

  $\square$

Of course, since the fraction

$$\frac{x + y}{xy - x - y}$$

is symmetric with respect to $x$ and $y$, we easily obtain all solutions in positive integers.

We summarize:
(18) For $1 \leq k_1 \leq k_2$ \[
\begin{align*}
  k_1 &= 1 \implies k_2 = \lambda \in \mathbb{Z}^+ \\
  k_1 &= 2 \implies k_2 \in \{3, 4, 6\} \\
  k_1 &= 3 \implies k_2 \in \{3, 6\} \\
  k_1 &= 4 \implies k_2 = 4.
\end{align*}
\]

Note that the case $k_1 = 3, k_2 = 3$ implies $k_3 = 3$ and we obtain the periodic KM-system $(1, -1, 1)$.

**Case II**

Suppose one of them, say $k_1$, is positive while the other, $k_2$, is negative. Let $k_2 = -x, x > 0$. Then

\[
  k_3 = \frac{-k_1 x}{-k_1 x - k_1 + x} = \frac{k_1 x}{k_1 x + k_1 - x} = 1 + \frac{x - k_1}{k_1 x + k_1 - x}
\]

It is enough to to solve the Diophantine equation

\[
  \frac{x - y}{xy + y - x} = z
\]

for $x, y$ positive integers and $z \in \mathbb{Z}$.

**Lemma 2.** Let $x, y \in \mathbb{Z}^+$. Then

\[
  \frac{x - y}{xy + y - x} \in \mathbb{Z}
\]

if and only if $(x, y)$ is of the form $(\lambda, 1)$ or $(\lambda, \lambda)$ with $\lambda \in \mathbb{Z}^+$.

**Proof.** If $y = 1$ then

\[
  \frac{x - y}{xy + y - x} = x \in \mathbb{Z}.
\]

Therefore a pair of the form $(\lambda, 1)$ is always a solution. Assume $y > 1$. We note that

\[
  \frac{xy}{xy + y - x} = 1 + \frac{x - y}{xy + y - x}
\]

and therefore $xy + y - x \leq xy$ implies $y \leq x$. If $x = y$ then our fraction is clearly an integer. On the other hand, if $y < x$, then the fraction

\[
  \frac{x - y}{xy + y - x} \notin \mathbb{Z}
\]

since

\[
  (y - 1)x + y \geq x + y > x - y.
\]

\[\square\]

If $k_1 = 1$ then $k_2 = -\lambda$ and $k_3 = \lambda$. Similarly, if $k_1 = \lambda$ then $k_2 = -\lambda$ and $k_3 = 1$. The two cases are isomorphic and correspond to Case 5 in Table 3.

**Case III**

If we take negative values for both $k_1$ and $k_2$, then

\[
  k_3 = \frac{xy}{xy + x + y} = 1 - \frac{x + y}{xy + x + y}.
\]
where $k_1 = -x$ and $k_2 = -y$ with $x, y > 0$. We have that $xy > 0$ and $xy + x + y > 0$ so that the Kowalevski exponent is an integer if
\[ xy + x + y \leq x + y \]
which implies $xy \leq 0$, a contradiction. Therefore, in this case $k_3$ cannot be an integer.

This completes the analysis of the case $k_1 k_2 - k_1 - k_2 \neq 0$.

Now suppose $k_1 k_2 - k_1 - k_2 = 0$.

In this case we have $k_1 + k_2 = k_1 k_2$ and obviously (since we assume non-zero Kowalevski exponents) we must have $k_1 = k_2 = 2$. We easily obtain $a = c$ and $b = 0$. This system is equivalent to the open KM-system (also known as the Volterra lattice). This is Case 1 in Table 3.

This concludes our analysis. We have obtained necessary conditions for the algebraic integrability of Lotka-Volterra systems in three dimensions and the results are summarized in Table 2. In Table 3 we also include the case of a zero exponent i.e. $b = a + c$. Note that the case $b = a + c$ which is equivalent to $(1, 1 + \mu, \mu)$ for $\mu \in \mathbb{R} \setminus 0$ was also considered in [7] from a different point of view.

### 4.3. Equivalence.

In order to have a more compact classification, we define an equivalence between two Lotka-Volterra systems. To begin with, common factors can be removed. In other words, suppose that matrix $A = (a_{ij})$ in (9) has a common factor $a$. Precisely, if
\[ a_{ij} = C_{ij}a, \quad \text{where } C_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \ldots, n, \]
then the Lotka-Volterra system (9) can be simplified to
\[
\dot{u}_i = \sum_{j=1}^{n} C_{ij} u_i u_j, \quad i = 1, 2 \ldots, n,
\]
using the transformation
\[ u_i = ax_i, \quad i = 1, 2 \ldots, n. \]

More generally, we consider two systems to be isomorphic if there exists an invertible linear transformation mapping one to the other. Special cases of isomorphic systems are those that are obtained from a given system by applying a permutation of the coordinates. Let $\sigma \in S_n$, and define a transformation
\[ X_i \mapsto x_{\sigma(i)}, \quad i = 1, 2, \ldots, n. \]

The transformed system is then considered equivalent to the original system. We illustrate with an example for $n = 3$.

**Example 2.** We prove that the system
\[
\begin{align*}
\dot{x}_1 &= ax_1 x_2 - \frac{a}{7} x_1 x_3 \\
\dot{x}_2 &= -ax_1 x_2 + \frac{2a}{3} x_2 x_3 \\
\dot{x}_3 &= \frac{a}{3} x_1 x_3 - \frac{2a}{3} x_2 x_3
\end{align*}
\]
is isomorphic to the system
\[ \dot{x}_1 = ax_1 x_2 - 2ax_1 x_3 \]
\[ \dot{x}_2 = -ax_1 x_3 + 3ax_2 x_3 \]
\[ \dot{x}_3 = 2ax_1 x_3 - 3ax_2 x_3 \]

Applying \( \sigma = (1\ 3\ 2) \) we have that

\[ \dot{X}_1 = \dot{x}_{\sigma(1)} = \dot{x}_3 = x_1 x_3 - 2x_2 x_3 = X_2 X_1 - 2X_3 X_1 \]
\[ \dot{X}_2 = \dot{x}_{\sigma(2)} = \dot{x}_1 = 3x_1 x_2 - x_1 x_3 = 3X_2 X_3 - X_2 X_1 \]
\[ \dot{X}_3 = \dot{x}_{\sigma(3)} = \dot{x}_2 = -3x_1 x_2 + 2x_2 x_3 = -3X_2 X_3 + 2X_3 X_1 \]

which is the second vector field.

**Example 3.** Note that the system

\[
\begin{align*}
\dot{x}_1 &= -x_2 x_3 \\
\dot{x}_2 &= x_2 x_3 \\
\dot{x}_3 &= x_1 x_3 - x_2 x_3 - x_3^2
\end{align*}
\]

is equivalent to the open KM-system \((1,0,1)\) under the transformation

\[
(x_1, x_2, x_3) \rightarrow (x_2 + x_3, x_1, x_2)
\]

but it is not a Lotka-Volterra system.

In Table 2 we display the different values of \((a, b, c)\) of the solutions \([14], [15]\) and \([16]\) of the simultaneous equations \([13]\) which ensure integer Kowalevski exponents for the Lotka-Volterra system in three dimensions. We also list the elements of the symmetric group \(S_3\) which realize the isomorphism. Note that \(\lambda \in \mathbb{Z}\setminus\{0\}\). The final six non-isomorphic systems are displayed in Table 3.

| Vector \((a, b, c)\) | Kowalevski exponents | \(\sigma\) |
|----------------------|----------------------|----------|
| \((a, \frac{a}{3}, \frac{a}{3})\) | \(-1, 1, 1\) | \(\sigma = (1\ 3)\) |
| \((a, a, \lambda a)\) | \(-1, 1, \lambda\) | \(\sigma = (2\ 3)\) |
| \((a, \lambda a, -a)\) | \(-1, 1, -\lambda\) | |
| \((a, -\frac{a}{2}, \frac{a}{2})\) | \(-1, 1, 2\) | \(\sigma = (1\ 3)\) |
| \((a, -a, 2a)\) | \(-1, 1, 4\) | \(\sigma = (1\ 3\ 2)\) |
| \((a, -2a, a)\) | \(-1, 1, 3\) | |
| \((a, -\frac{a}{3}, \frac{a}{3})\) | \(-1, 1, 2\) | \(\sigma = (1\ 3\ 2)\) |
| \((a, -\frac{2a}{3}, \frac{a}{3})\) | \(-1, 1, 3\) | \(\sigma = (1\ 2\ 3)\) |
| \((a, -\frac{a}{3}, \frac{2a}{3})\) | \(-1, 1, 6\) | \(\sigma = (2\ 3)\) |
| \((a, -2a, 3a)\) | \(\sigma = (1\ 2)\) | |
| \((a, -3a, 2a)\) | \(\sigma = (1\ 3\ 2)\) | |
| \((a, 0, a)\) | \(\sigma = (1\ 3\ 2)\) | |
| \((a, -a, 0)\) | \(-1, 1, 2\) | \(\sigma = (1\ 2\ 3)\) |
| \((0, b, -b)\) | \(\sigma = (1\ 2\ 3)\) | |

**Table 2.** Systems with integer Kowalevski exponents
Example 4. The periodic KM system (15) in three dimensions is the system

\[ \dot{x}_i = \sum_{j=1}^{3} a_{ij} x_i x_j, \quad i = 1, 2, 3, \]

where \( A \) is the \( 3 \times 3 \) skew-symmetric matrix

\[ A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \]

This system is a special case of the system (9) where \((a, b, c) = (-1, 1, -1)\). This is Case 2 in Table 3. The Kowalevski exponents of this system are \(-1, 1, 3\). The system can be written in the Lax-pair form \( \dot{L} = [L, B] \), where

\[ L = \begin{pmatrix} 0 & x_1 & 1 \\ 1 & 0 & x_2 \\ x_3 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & x_1 x_2 \\ x_2 x_3 & 0 & 0 \\ 0 & x_1 x_3 & 0 \end{pmatrix}. \]

We have the constants of motion

\[ H_k = \text{trace} \left( L^k \right), \quad k = 1, 2, \ldots \]

The functions

\[ H_2 = x_1 + x_2 + x_3 \]
\[ H_3 = 1 + x_1 x_2 x_3 \]

are independent constants of motion in involution with respect to the Poisson bracket

\[ \pi = \begin{pmatrix} 0 & -x_1 x_2 & x_1 x_3 \\ x_1 x_2 & 0 & -x_2 x_3 \\ -x_1 x_3 & x_2 x_3 & 0 \end{pmatrix}. \]

We note that the positive Kowalevski exponents, 1 and 3, correspond to the degrees of the constants of motion.

We have to point out that all Lotka-Volterra systems in three dimensions are integrable in the sense of Liouville since there exist two constants of motion which are independent and in involution. The function

\[ H = x_1 + x_2 + x_3 \]

is the Hamiltonian for these systems using the quadratic Poisson bracket

\[ \pi = \begin{pmatrix} 0 & ax_1 x_2 & bx_1 x_3 \\ -ax_1 x_2 & 0 & cx_2 x_3 \\ -bx_1 x_3 & -cx_2 x_3 & 0 \end{pmatrix}. \]

The equations of motion can be written in Hamiltonian form

\[ \dot{x}_i = \{x_i, H\}, \quad i = 1, 2, 3. \]

The second constant of motion, independent of \( H \) always exists. It is straightforward to check that the function

\[ F = x_1^a x_2^b x_3^c \]

is always a Casimir. Therefore the system is Liouville integrable for any value of \( a, b, c \). This is not the case if \( n \geq 4 \).
5. Free Parameters

We would like to classify the algebraically completely integrable Lotka-Volterra equations in three dimensions. In order to use Proposition 1 we have to assume Laurent solutions of the form

$$x_i(t) = \frac{1}{t^{v_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \quad i = 1, 2, \ldots, n,$$

where $v_i$ are the components of the weight vector $v$ that makes the vector field

$$\dot{x}_i = f_i(x_1, \ldots, x_n), \quad i = 1, 2, \ldots, n,$$

to be weight homogeneous. In our case the weight vector is $v = (1, 1, 1)$.

To make sure that our classification is complete we must check that each system obtained by imposing these necessary conditions is indeed a.c.i. This means that the Laurent series of the solutions $x_1, x_2$ and $x_3$ must have $n - 1 = 2$ free parameters. Using the results in [2], the free parameters appear in a finite number of steps of calculation. The first thing to do is to substitute (19) into equations (9). After that we equate the coefficients of $t^k$. We have already equated the coefficients of $t^{-v_i-1}$ by solving the indicial equation to find $x_i^{(0)}$. Then we call Step $m$ ($m \in \mathbb{N}$) when we equate the coefficients of $t^{-v_i-1+m}$ to find $x_i^{(m)}$. According to [2] all the free parameters appear in the first $k_p$ Steps, where $k_p$ is the largest (positive) Kowalevski exponent of the system. The calculations are straightforward and we omit the details.

| Vector      | Kowalevski exponents | Free parameters | Degree of invariant |
|-------------|-----------------------|-----------------|---------------------|
| $(a, 0, a)$ | $-1, 1, 2$            | $x_3^{(1)}$, $x_3^{(2)}$ | 2                   |
| $(a, -a, a)$| $-1, 1, 3$            | $x_3^{(1)}$, $x_3^{(3)}$ | 3                   |
| $(a, -\frac{a}{2}, \frac{a}{2})$ | $-1, 1, 2$            | $x_3^{(1)}$, $x_3^{(2)}$ | 4                   |
|              | $-1, 1, 4$            | $x_3^{(1)}$, $x_3^{(2)}$ |                     |
| $(a, -2a, 3a)$ | $-1, 1, 2$            | $x_3^{(1)}$, $x_3^{(2)}$ | 6                   |
|              | $-1, 1, 3$            | $x_3^{(1)}$, $x_3^{(2)}$ |                     |
|              | $-1, 1, 6$            | $x_3^{(1)}$, $x_3^{(2)}$ |                     |
| $(a, \frac{a}{\lambda}, \frac{a}{\lambda})$ | $-1, 1, 1$            | $x_1^{(1)}$, $x_2^{(1)}$ | $\lambda$          |
|              | $-1, 1, \lambda$      | $x_1^{(1)}$, $x_2^{(1)}$ |                     |
|              | $-1, 1, -\lambda$     | $x_1^{(1)}$, $x_2^{(1)}$ |                     |
| $(a, a + c, c)$ | $-1, 1, 0$            | $x_1^{(1)}$, $x_3^{(0)}$ | 0                   |

Table 3. Free parameters for the algebraic complete integrability of each system

All the systems that we have obtained turn out to be a.c.i. We summarize the results in Table 3 where we display the 2 free parameters in each case. We note that the six Cases of Theorem 1 are non-isomorphic by examining the degree of the Casimir.
6. Higher Order Poles

In our classification, using the Kowalevski exponents, we assume that the order of the poles agrees with the components of the weight vector, in our case all equal to 1. We have to exclude the possibility of missing some cases due to solutions with higher order poles. We show that no such new cases appear.

Suppose that the Laurent solution of the system is

\[ x_1(t) = \frac{1}{\nu_1} \sum_{k=0}^{\infty} x_1^{(k)} t^k, \quad \text{with} \quad x_1^{(0)} \neq 0, \]
\[ x_2(t) = \frac{1}{\nu_2} \sum_{k=0}^{\infty} x_2^{(k)} t^k, \quad \text{with} \quad x_2^{(0)} \neq 0, \]
\[ x_3(t) = \frac{1}{\nu_3} \sum_{k=0}^{\infty} x_3^{(k)} t^k, \quad \text{with} \quad x_3^{(0)} \neq 0. \]  \hspace{1cm} (20)

If \( \nu_1, \nu_2, \nu_3 \leq 1 \), then these systems have been already investigated using Proposition 1. On the other hand, keeping in mind that \( H = x_1 + x_2 + x_3 \) is always a constant of motion, we end-up with the following four cases to consider:

(i) \( \nu_1 = \nu_2 = \nu > 1 \) and \( \nu_3 < \nu \), or
(ii) \( \nu_1 = \nu_3 = \nu > 1 \) and \( \nu_2 < \nu \), or
(iii) \( \nu_2 = \nu_3 = \nu > 1 \) and \( \nu_1 < \nu \), or
(iv) \( \nu_1 = \nu_2 = \nu_3 = \nu > 1 \).

Recall that equations (9) in three dimensions are:

\[ \dot{x}_1 = ax_1 x_2 + bx_1 x_3, \] \hspace{1cm} (21)
\[ \dot{x}_2 = -ax_1 x_2 + cx_2 x_3, \] \hspace{1cm} (22)
\[ \dot{x}_3 = -bx_1 x_3 - cx_2 x_3. \] \hspace{1cm} (23)

We examine each of the four cases:

(i) \( \nu_1 = \nu_2 = \nu > 1 \) and \( \nu_3 < \nu \)

Since \( \nu_1 = \nu_2 = \nu \) and using the fact that \( H = x_1 + x_2 + x_3 \) is a constant of motion we have that

\[ x_1^{(0)} = -x_2^{(0)} = \alpha \neq 0. \]

We also note that \( \nu + \nu_3 < 2 \nu \) and \( x_1^{(0)} x_2^{(0)} \neq 0 \). Equating the coefficients of \( t^{\nu} \) of the LHS and RHS of (21) or (22), we are led to \( a x_1^{(0)} x_2^{(0)} = 0 \). Therefore \( a = 0 \).

As we know that \( \nu_3 + 1 < \nu + \nu_2 \), the coefficient of \( t^{\nu + \nu_3} \) of the RHS of (23) must be equal to zero. So

\[ x_3^{(0)} (-bx_1^{(0)} - cx_2^{(0)}) = 0, \]

but \( x_3^{(0)} \neq 0 \) and \( x_2^{(0)} = -x_1^{(0)} \neq 0 \); therefore \( b = c \).

If \( b = 0 \), then from (21) and (22) we have that

\[ \dot{x}_1 = \dot{x}_2 = 0 \implies x_1, \ x_2 \] are constant functions,

that is a contradiction because \( \nu_1 = \nu_2 = \nu > 1 \).
If $b$ and $c$ are non-zero, then the equations (21) and (22) become

(24) $\dot{x}_1 = bx_1x_3,$

(25) $\dot{x}_2 = bx_2x_3.$

Using equation (21) we obtain

$$\nu + 1 = \nu + \nu_3$$

therefore $\nu_3 = 1$, since $x_1^{(0)} x_3^{(0)} \neq 0$ and $x_2^{(0)} x_3^{(0)} \neq 0.$

It follows from (24) and (25) that

$$\dot{x}_1 = \dot{x}_2 \Rightarrow x_1 = \kappa x_2, \quad \kappa \text{ is a constant}.$$ 

However, we know that

$$x_1^{(0)} = -x_2^{(0)} \Rightarrow \kappa = -1 \Rightarrow x_1 = -x_2.$$ 

Equation (23) becomes

$$\dot{x}_3 = -b(-x_2)x_3 - bx_2x_3 = 0 \Rightarrow x_3 = c, \quad c \text{ is a constant}.$$ 

This is a contradiction since $\nu_3 = 1$ and $x_3^{(0)} \neq 0.$

(ii) $\nu_1 = \nu_3 = \nu > 1 \text{ and } \nu_2 < \nu$

It leads to a contradiction, as in case (i).

(iii) $\nu_2 = \nu_3 = \nu > 1 \text{ and } \nu_1 < \nu$

It leads to a contradiction, as in case (i).

(iv) $\nu_1 = \nu_2 = \nu_3 = \nu > 1$

In this case, for $i = 1, 2, 3,$

$$x_i(t) = \frac{1}{\nu} \sum_{k=0}^{\infty} x_i^{(k)} t^k,$$

we have that the degrees of the leading term of the LHS of the equations (21), (22) and (23) are equal to $\nu + 1$, but the degrees of the leading term RHS of these equations are equal to $2\nu$ and so the coefficients of $\frac{1}{\nu+k}$ of the RHS of these equations must be zero for $k = 2, 3, \ldots, \nu.$

The coefficients of $\frac{1}{\nu+k}, \ k = 1, 2, \ldots, \nu,$ are given by the sums

(26) $S_{i,k} = \sum_{\lambda=0}^{\nu-k} x_i^{(\lambda)} u_{i,k}^{(\lambda)}, \text{ for } i = 1, 2, 3,$

where

$$u_{1,k}^{(\lambda)} = ax_2^{(\nu-k-\lambda)} + bx_3^{(\nu-k-\lambda)},$$

$$u_{2,k}^{(\lambda)} = -ax_1^{(\nu-k-\lambda)} + cx_3^{(\nu-k-\lambda)},$$

$$u_{3,k}^{(\lambda)} = -bx_2^{(\nu-k-\lambda)} - cx_3^{(\nu-k-\lambda)}.$$
Note that
\[(27)\]
\[u_{i,k}^{(\lambda)} = u_{i,j}^{(m)}, \text{ if } k + \lambda = j + m.\]

In addition
\[S_{i,k} = 0, \text{ for } i = 1, 2, 3 \text{ and } k = 2, 3, \ldots, \nu.\]

For \(k = n\) sum \((26)\) becomes
\[S_{i,\nu} = x_i^{(0)} u_{i,\nu}^{(0)} = 0 \implies u_{i,\nu}^{(0)} = 0\]
since \(x_i^{(0)} \neq 0\).

For \(k = \nu - 1\) we have that
\[S_{i,\nu - 1} = x_i^{(0)} u_{i,\nu - 1}^{(0)} + x_i^{(1)} u_{i,\nu - 1}^{(1)} = 0\]
\[\implies x_i^{(0)} u_{i,\nu - 1}^{(0)} + x_i^{(1)} u_{i,\nu}^{(0)} = x_i^{(0)} u_{i,\nu - 1}^{(0)} = 0\]
\[\implies x_i^{(0)} u_{i,\nu - 1}^{(0)} = 0 \text{ because } x_i^{(0)} \neq 0.\]

Let \(m \in \{1, 2, \ldots, \nu - 1\}\) and assume that \(u_{i,k}^{(0)} = 0\) for \(k > m\).

For \(k = m\) we have that
\[S_{i,m} = x_i^{(0)} u_{i,m}^{(0)} + \sum_{\lambda=1}^{\nu - m} x_i^{(\lambda)} u_{i,m+\lambda}^{(\lambda)} = x_i^{(0)} u_{i,m}^{(0)} + \sum_{\lambda=1}^{\nu - m} x_i^{(\lambda)} u_{i,m+\lambda}^{(\lambda)}.\]

Since \(S_{i,m} = 0\) for \(m > 1\) and, since \(x_i^{(0)} \neq 0\), then \(u_{i,m}^{(0)} = 0\).

Now we equate the coefficients of \(t^{-\nu}\) on both sides of the equations \((21)-(23)\) to obtain
\[S_{i,1} = x_i^{(0)} u_{i,1}^{(0)} = -\nu x_i^{(0)}.\]

Therefore, \(\nu + u_{i,1}^{(0)} = 0\).

Therefore we have that
\[(28)\]
\[ax_2^{(\nu-1)} + bx_3^{(\nu-1)} = -\nu,\]
\[-ax_1^{(\nu-1)} + cx_3^{(\nu-1)} = -\nu,\]
\[-bx_2^{(\nu-1)} - cx_3^{(\nu-1)} = -\nu.\]

These simultaneous equations have solutions only if
\[b = a + c.\]

If \(a = 0\), then \(b = c\) (obviously \(b = c \neq 0\)). Then the system is isomorphic to the following \((0, 1, 1)\) system:
\[\dot{x}_1 = x_1 x_3,\]
\[\dot{x}_2 = x_2 x_3,\]
\[\dot{x}_3 = -x_1 x_3 - x_2 x_3.\]

Equating the coefficients of \(t^{-2\nu}\) \((\nu > 1)\) in the first and second equations we have that
\[x_1^{(0)} x_3^{(0)} = x_2^{(0)} x_3^{(0)} = 0.\]
This is impossible because \( x_i^{(0)} \neq 0 \), for \( i = 1, 2, 3 \).

The same happens if \( bc = 0 \). So in the following calculations we assume that \( abc \neq 0 \).

We will show that there exists no such solution with \( \nu \geq 2 \). Since \( b = a + c \) and the function \( H = x_1 + x_2 + x_3 \) is a constant of motion, the Lotka-Volterra equations in three dimensions can be written in the form

\[
\begin{align*}
\dot{x}_1 &= akx_1 - ax_1^2 + cx_1x_3, \\
\dot{x}_2 &= -\dot{x}_1 - \dot{x}_3, \\
\dot{x}_3 &= -ckx_3 + cx_2^2 - ax_1x_3,
\end{align*}
\]

where \( k \) is the constant value of the function \( H \). It is straightforward to see that if \( k \neq 0 \), then the solution is

\[
\begin{align*}
x_1 &= \frac{kC_1e^{akt}}{C_1e^{akt} + ae^{-ckt} - C_2}, \\
x_3 &= \frac{kae^{-ckt}}{C_1e^{akt} + ae^{-ckt} - C_2}, \\
x_2 &= k - x_1 - x_3 = \frac{-kC_2}{C_1e^{akt} + ae^{-ckt} - C_2}.
\end{align*}
\]

Obviously \( C_2 \neq 0 \). The pole \( t_s \) satisfies

\[
C_1e^{akt_s} + ae^{-ckt_s} - C_2 = 0 \Rightarrow C_2 = C_1e^{akt_s} + ae^{-ckt_s} \neq 0
\]

Hence using De l’Hôpital Rule we are led to the fact that

\[
\lim_{t \to t_s} (t - t_s)x_2(t) = \frac{C_2}{aC_1e^{akt_s} - ace^{-ckt_s}}.
\]

Since the pole order is greater than 1, we have that

\[
\lim_{t \to t_s} (t - t_s)x_2(t) = \infty.
\]

Therefore

\[
C_1 = ce^{-(a+c)kt} = ce^{-bkt}
\]

The solution (30) possesses only one arbitrary constant \( k \), but we need 2.

Now if \( k = 0 \) the solutions of (29) are

\[
\begin{align*}
x_3(t) &= 0, \quad x_1(t) = \frac{1}{at + C_1}, \\
\end{align*}
\]

or

\[
\begin{align*}
x_1(t) &= \frac{C_1 - c}{a(C_1t + C_2)}, \quad x_3(t) = -\frac{1}{C_1t + C_2}.
\end{align*}
\]

Both solutions lead to a contradiction since the pole order of \( x_1 \) and \( x_3 \) is assumed to be greater than 1.

Therefore the case \( \nu_1 = \nu_2 = \nu_3 = \nu > 1 \) does not give us any new algebraically integrable systems. The conclusion is that the case \( b = a + c \) is algebraically integrable only when \( \nu_1 = \nu_2 = \nu_3 = 1 \).

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