D-instanton derivation of multi-fermion $F$-terms in supersymmetric QCD

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Abstract

We investigate effects of field theory instantons by considering D-instantons in a suitable D3-brane background. In supersymmetric QCD with $SU(N_c)$ gauge group with $N_f = N_c$ flavors, the moduli space of vacua is deformed by instantons. This effect can be described by the chiral interactions which are called multi-fermion $F$-terms. We derive these chiral interaction terms as D-instanton effects in the presence of D3-branes. For $SU(2)$, the obtained result agrees with the previous result worked out by Beasley and Witten \cite{Beasley:1989}. We also explicitly work out those for the case of the symplectic gauge group, and show that they describe the deformation of the moduli space.

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1 Introduction

Instanton physics in string theory is an interesting area to study. There are various nonperturbative objects such as D-branes, membranes and 5-branes and one can consider the instanton effects arising from such objects wrapping on some suitable cycles \([2, 3, 4]\). Spacetime approach or the physical gauge approach was initiated by \([5]\) and various interesting works were done. Since some field theories can be embedded in a string theory, field theory instantons can be understood in terms of instantons in string theory and this approach shed much light on the understanding of the structures of gauge theory instantons, such problem as the measure of the instanton moduli space. One of the important discovery is the D5-brane as a small instanton in Type I theory \([2]\) and this led to the much progress in our understanding of the field theory instantons. Also in the string theory set up, one can have truly stringy instantons in the embedded field theory \([6, 7, 8, 9, 10, 11]\), whose effect cannot be reproduced within field theory and this leads to many interesting physics such as dynamical supersymmetry breaking combined with the other effects.

In this work we will use the string theory setup to understand one aspect of gauge theory instanton \([12]\). In particular we are interested in \(\mathcal{N} = 1\) supersymmetric QCD and derive one interesting effect coming from the gauge theory instanton. We realize the gauge theory as a suitable D3-brane configuration \([13]\) where D-instanton plays the role of the usual gauge theory instanton. In \(\mathcal{N} = 1\) \(SU(N_c)\) gauge theory with \(N_f\) fundamental flavors, the well known instanton effect is the generation of the ADS superpotential \([14]\) for \(N_c = N_f - 1\), which lifts the moduli of vacua (for a review, see \([15]\)). This superpotential can be reproduced by the D-brane effective theory \([16]\). For \(N_f = N_c\), no superpotential is generated but a quantum effect deforms the complex structure of the moduli space \([17]\). In \([1]\), it was pointed out that the deformation of the moduli space is still due to the usual gauge theory instanton, which gives rise to a chiral interaction. This interaction is a four-fermion interaction term and called multi-fermion \(F\)-terms. For \(N_f > N_c\), instantons generate interaction terms with more fermions.

In this paper, we reproduce these multi-fermion \(F\)-terms from the D-brane effective theory, which is realized as D3-branes at a particular orbifold singularity. Though only the scalars which parametrize the moduli of \(\mathcal{N} = 1\) SQCD are involved in the calculation of the ADS superpotential, we have to include interactions of fermions in the quark superfields. These \(F\)-terms are given in the form of the integral with respect to the moduli of the instanton. It is difficult to perform the integration for general case. Hence, we calculate the simplest case \(N_c = N_f = 2\). In this case, we can easily perform the integration and obtain the multi-fermion \(F\)-terms which are equivalent to the result of \([1]\). We study the case of the symplectic group as another example, and make a connection with the deformation of the moduli space. In this case, we can also explicitly carry out the integration, since the ADHM constraint is absent and
the structure of the instanton moduli is simple.

This paper is organized as follows. In section 2 we briefly recall the basic setup of the D-brane effective theory which describes instantons in the $\mathcal{N} = 4$ super Yang-Mills [12]. In section 3 we introduce D3-branes on one particular orbifold and realize the $\mathcal{N} = 1$ SQCD with $SU(N_c)$ gauge group [13]. After that, we discuss basic facts about the derivation of superpotentials and show the general expression of multi-fermion $F$-terms for $N_f = N_c$. In section 4 we briefly review the multi-fermion $F$-terms and their relation to the deformation of the moduli space [1]. In section 5 we describe details of calculations in the simplest case of $N_f = N_c = 2$. We show that our result coincides with that in [1]. In section 6 we consider the more general case of $USp(N_c)$. We derive an expression of multi-fermion $F$-terms and show that they describe the deformation of the moduli space for $N_f = N_c + 1$. We also consider multi-fermion $F$-terms with more fermions, which appear for $N_f > N_c + 1$. Section 7 is devoted for conclusions and discussions.

2 Preliminaries

In this section, we explain the basic setup of the system. It is well known that the D-instanton is the gauge theory instanton if the gauge theory in consideration is realized by D3-brane configurations [12]. Here we describe the massless spectrum of various sectors coming from D3 branes and D-instantons in the flat space and specify the interaction terms between various sectors. We divide open string fields into the following three sectors.

D3-D3 sector This sector consists of the open strings whose both ends are on the D3-branes. At low energy, only massless modes of these open strings contribute to the theory. These massless fields form $\mathcal{N} = 4$ super Yang-Mills multiplets. Let $x^\mu$ denote the world-volume coordinates of D3-branes and $x^a$ denote the transverse coordinates of the remaining six dimensions. The bosonic components are denoted by $A^\mu$ and $X^a$, and their fermionic partners by $\Psi^A_\alpha$ and $\bar{\Psi}^\dot{\alpha}_A$. Indices of $A$ are those of $SU(4)$ denoting the chirality of the transverse six dimensions while $\alpha$ and $\dot{\alpha}$ denote the usual four dimensional chirality. We also write six scalars $X^a$ in the antisymmetric representation

$$\Phi_{AB} \equiv \bar{\Sigma}_{AB} X^a.$$

(2.1)

where $\bar{\Sigma}_a$ (and $\Sigma^a$) realize six-dimensional Clifford algebras, and appear in the expression of six-dimensional $\gamma$-matrices,

$$\Gamma^a = \begin{pmatrix} 0 & \Sigma^a \\ \Sigma^a & 0 \end{pmatrix}.$$  

(2.2)

When we put $N$ D3-branes, all of theses fields are in the adjoint representation of $SU(N)$, and can be written as $N \times N$ hermitian matrices.
D(-1)-D(-1) sector  This sector consists of the open strings with both ends on the D-instantons. These are obtained by the dimensional reduction of ten-dimensional super Yang-Mills theory. The bosonic fields are written as $a^\mu, \chi^a$ and the fermionic modes are denoted as $M^{\alpha A}, \lambda_{\dot{\alpha} A}$. Here we have adopted the ADHM inspired notation. We also introduce the triplet of the auxiliary fields $D_c$. These are expressed in terms of $k \times k$ matrices for a background with $k$ D-instantons. It is argued in [12] that there are subtleties to obtain the D-instanton action in the presence of D3-branes by taking $\alpha' \to 0$ limit. This is because the coupling constant dependence on $\alpha'$ is different for D3-branes and D-instantons. If we take the coupling constant of D3-branes to be constant, the coupling constant of D-instanton diverges. In order to obtain the usual interacting theory for D(-1) massless modes, we need a suitable rescaling of the fields on D-instantons. Here we assume that such rescaling is already taken. We will refer to the rescaled fields by using the above notations.

D3-D(-1) sector  This sector includes the massless modes of the open strings stretching between the D3-branes and D-instantons. From the Neveu-Schwarz (NS) sector we have a bosonic spinor in the first four-directions where the GSO projection picks up the negative chirality. In the conjugate sector we obtain an independent bosonic spinor with the same chirality. We will refer to them as $\omega_{\dot{\alpha}}$ and $\bar{\omega}_{\dot{\alpha}}$, respectively. From the Ramond sector and its conjugate sector, we obtain a pair of fermions $\mu^A$ and $\bar{\mu}^A$. These fields are $N \times k$ and $k \times N$ matrices, respectively.

We are specifying the action of various sectors and the SUSY transformations are worked out in appendix [A]. Massless modes of the D(-1)-D(-1) and D3-D(-1) sectors correspond to the moduli of the gauge instanton which appear in the ADHM construction [18, 19]. When carrying out the path integral, one obtains the measure of the instanton moduli space naturally. The action is given by

$$S_1 = \text{tr} \left( -[\alpha^a, \chi^a]^2 + \chi^a \bar{\omega}_{\dot{\alpha}} \omega^A a + \frac{1}{2} (\Sigma^a)_{AB} \bar{\mu}^A \mu^B \chi^a - \frac{i}{4} (\Sigma^a)_{AB} M^{\alpha A} \chi^a, M^B \right)$$

$$+ i (\bar{\mu}^A \omega_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}} \mu^A + \sigma_{\dot{\beta} \dot{\alpha}} [M^{B A}, a^\mu]) \lambda^\alpha_A - i D^c (\bar{\omega}_{\dot{\alpha}} (\tau^c)^{\dot{\beta}} \omega_{\dot{\beta}} + \bar{\eta}_{\mu} \epsilon_{\mu} [a^u, a^v]) \right) \quad (2.3)$$

where $\eta, \bar{\eta}$ are the usual ’t Hooft symbols and $\tau^c$ denote the usual Pauli matrices.

Including interaction terms with the scalars in $\mathcal{N} = 4$ super Yang-Mills theory, we can also obtain the instanton effective action with non-zero VEVs of the scalars, which is given by,

$$S_2 = \text{tr} \left( \frac{1}{8} \epsilon^{ABCD} \bar{\omega}_{\dot{\alpha}} \Phi_{AB} \Phi_{CD} \omega^\dot{\alpha} + \frac{i}{2} \bar{\mu}^A \Phi_{AB} \mu^B + \frac{1}{4} \epsilon^{ABCD} \bar{\omega}_{\dot{\alpha}} \chi_{AB} \Phi_{CD} \omega^\dot{\alpha} \right). \quad (2.4)$$

In this paper, we also include contributions from the fermions in the D3-brane field theory.
The interaction terms with the fermions (and gauge fields) are as follows:

\[
S_3 = \text{tr} \left( i \tilde{\omega}_\alpha \bar{\Psi}_A \gamma^A \left( -i \bar{\mu}^A \bar{\Psi}_A + \frac{1}{2} \bar{\omega}_\alpha \sigma^\mu_{\dot{\alpha}} \bar{\omega}^\dot{\beta} F_{\mu \nu} \right) \right). \tag{2.5}
\]

We will mainly consider nonzero fermionic background with no gauge field background.

## 3 Multi-fermion F-terms from D-instantons

In this paper, we study nonperturbative effects of instantons in an \( \mathcal{N} = 1 \) gauge theory. One way to obtain \( \mathcal{N} = 1 \) theory is the orbifolding with putting D3-branes at the orbifold singularity \[13\]. Here we take a \( \mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold. We also introduce fractional D3-branes which are D5-branes wrapped on the singularity. Then, the gauge group is broken into \( U(N_1) \times U(N_2) \times U(N_3) \times U(N_4) \). We assume that some of \( U(1) \) factors can get massive, e.g., via Green-Schwarz mechanism and we can get \( SU(N) \) factors. When we discuss \( SU(N) \) theory, we assume that \( U(1) \) is decoupled. We will refer to D3-brane configurations as \((N_1, N_2, N_3, N_4)\). The configuration of \((N_c, N_f, 0, 0)\) yields the \( \mathcal{N} = 1 \) SQCD with gauged flavor symmetry. We neglect gauging of flavor symmetry and treat the model as the usual \( \mathcal{N} = 1 \) SQCD. Configurations of D-instantons (and fractional D-instantons) are characterized by a similar fashion, \((k_1, k_2, k_3, k_4)\). It is taken to be \((1, 0, 0, 0)\) in order to describe the one-instanton background.

In order to obtain the \( \mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold, we introduce the following complex coordinates,

\[
z^1 = x^4 + ix^5, \quad z^2 = x^6 + ix^7, \quad z^3 = x^8 + ix^9. \tag{3.1}
\]

The orbifold is constructed from the following two projections

\[
g_1 : \quad z^2 = -z^2 \quad \text{and} \quad z^3 = -z^3, \tag{3.2a}
\]

\[
g_2 : \quad z^1 = -z^1 \quad \text{and} \quad z^3 = -z^3. \tag{3.2b}
\]

The low energy effective theory of the D3-brane becomes the \( \mathcal{N} = 1 \) quiver gauge theory with bi-fundamental matters. This theory can be obtained by imposing an orbifold projection on the \( \mathcal{N} = 4 \) SYM. Since the orbifolding is abelian, the orbifold projection act on the Chan-Paton factor as follows:

\[
A_\mu = \gamma(g_i) A_\mu \gamma(g_i)^{-1}, \quad \Phi_{AB} = \pm \gamma(g_i) \Phi_{AB} \gamma(g_i)^{-1}, \quad i = 1, 2 \tag{3.3}
\]

where the \( \pm \) sign must be chosen according to the projections \[3.2\]. Since the orbifold is abelian, the representation matrix \( \gamma(g_i) \) can be diagonalized and written in terms of the block diagonal matrices:

\[
\gamma(g_1) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \gamma(g_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \tag{3.4}
\]
where size of each blocks is determined by the D3-brane and D-instanton configurations. They have the size of \((N_1, N_2, N_3, N_4)\) for the Chan-Paton factors on D3-branes, and \((k_1, k_2, k_3, k_4)\) for those on D-instantons.

Now, we take the configuration of \(\mathcal{N} = 1\) SQCD, namely, \((N_c, N_f, 0, 0)\) for D3-branes and \((1, 0, 0, 0)\) for D-instantons. The fields which survive in the orbifolding are as follows. First, we consider fields on the D3-branes. We have the gauge group \(SU(N_c) \times SU(N_f)\) and regard the former as a color gauge group and the latter as a flavor group. Among the scalar fields, only two real (one complex) scalars have non-zero components. In terms of \(SU(4)\) R-symmetry, \(\Phi_{23}\) and \(\Phi_{14}\) survive, and they are hermitian conjugate to each other. Non-zero components are

\[
\Phi_{23} = \begin{pmatrix}
0 & Q & 0 & 0 \\
\tilde{Q} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \Phi_{14} = \begin{pmatrix}
0 & \tilde{Q}^\dagger & 0 & 0 \\
Q^\dagger & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

These two scalars are bi-fundamental in \((N_c, N_f)\). Concerning fermions, two fermions \(\Psi^1\) and \(\Psi^4\), (and their hermitian conjugate) survive. The fermion \(\Psi^1\) has a non-zero component in a similar fashion to the scalar fields:

\[
\Psi^1 = \begin{pmatrix}
0 & \psi & 0 & 0 \\
\tilde{\psi} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \bar{\Psi}_1 = \begin{pmatrix}
0 & \tilde{\psi} & 0 & 0 \\
\psi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Fermions \(\psi\) and \(\tilde{\psi}\) are the superpartner of the scalars \(Q\) and \(\tilde{Q}\) respectively. The other surviving fermion \(\Psi^4\) becomes the gaugino \(\lambda^{(g)}\) and has two non-zero block-diagonal components.

For the fields in the D(-1)-D(-1) sector, the situation is similar, but much more components vanish. Since we set all block to be zero except for the first, only the first diagonal block survives. Furthermore, \(a_\mu\) and \(M^{\alpha \beta}\) do not appear in the action. Then, relevant fields in this sector are \(\lambda^{\dot{\alpha}} \equiv \lambda^{\dot{\alpha}}_1\) and \(D^c\). In the D3-D(-1) sector, the first diagonal components of \(\omega_{\dot{\alpha}}, \tilde{\omega}_{\dot{\alpha}}, \mu \equiv \mu^4, \tilde{\mu} \equiv \tilde{\mu}^4\) survive, and are in the fundamental and anti-fundamental representations of \(SU(N_c)\), respectively. Off-diagonal components of \(\mu' \equiv \mu^1\) and \(\tilde{\mu}' \equiv \tilde{\mu}^1\) also survive and they are in the fundamental and anti-fundamental representations of \(SU(N_f)\), respectively.

After the \(\mathbb{Z}_2 \times \mathbb{Z}_2\) orbifold projection, the action is obtained from the action of D3 and D(-1) branes after keeping terms compatible with the orbifold projections. The part containing only the fields in D(-1)-D(-1) and D3-D(-1) sectors is greatly simplified and turns out to be

\[
S_1 = i(\bar{\mu}_u \omega_{\bar{u} u} + \tilde{\omega}_{\bar{u} u} \mu^4) \lambda^{\dot{\alpha}} - iD_c \bar{\omega}^{\dot{\alpha}}_u (\tau^c)^{\dot{b}}_\beta \omega^{u \beta}.
\]

The interaction terms with scalars are

\[
S_2 = \frac{1}{2} \bar{\omega}_{\bar{u} u} (Q_f^u \bar{Q}_v^f + \bar{Q}_f^u \bar{Q}_v^f) \omega^{\dot{\alpha} \nu} - \frac{i}{2} \bar{\mu}_u \bar{Q}_f^f \mu' + \frac{i}{2} \tilde{\mu}_f \bar{Q}_u^f \mu'.
\]
The above action is equivalent to those appearing in the ADHM construction. Here, we also include the interaction terms with fermions. These terms and the interactions with gauge fields and gauginos are

\[ S_3 = -i\tilde{\mu}^i \psi^{\dagger}_{\alpha u} \omega^{\dagger u} + i\tilde{\omega}_{\dot{\alpha} u} \psi_f^{\mu f} - i\tilde{\mu}^{(g)}_{u} \omega^{\dagger u} \]

\[ + i\omega_{\dot{\alpha} u} \tilde{\lambda}^{(g)}_{u} \dot{\alpha} u + \tilde{\omega}_{\dot{\alpha} u} \sigma^{\mu\nu\dot{\alpha}} \beta (F_{\mu\nu}) u \omega^{\dagger u} \]

(3.9)

In this paper, we mainly investigate the contribution from the fermionic components of quarks, \( \psi \) and \( \tilde{\psi} \). We do not consider the terms with the gauge fields and gauginos, and \( S_3 \) indicates only terms with \( \tilde{\psi} \) or \( \tilde{\psi} \), hereafter.

The space-time approach to obtain the superpotential or \( F \)-term contribution from D-instanton was explained in [20]. Following that, we should evaluate the following path integral for one-instanton

\[ Z = C \int d\{a, \chi, M, \lambda, D, \omega, \bar{\omega}, \mu, \bar{\mu}\} e^{-S_1 - S_2 - S_3}, \]

(3.10)

where, \( C \) represents some overall constant. In the path integral evaluation, \( a^\mu \) gives rise to four bosonic translation modes \( x^\mu \) which represent position of the instanton. The upper left component of \( M^{\alpha 4} \) gives rise to super-translation modes \( \theta^\alpha \), since they are super-transform of \( a^\mu \). Then, we can interpret the path integral over these fields as the integration with respect to (super-)coordinate. Then, the \( F \)-term contributions from D-instantons can be expressed as

\[ Z = \int d^4 x d^2 \theta \ W, \]

(3.11)

where \( W \) can be interpreted as a correction to the superpotential, and is given by

\[ W = C \int d\{\lambda, D, \omega, \bar{\omega}, \mu, \bar{\mu}\} e^{-S_1 - S_2 - S_3}. \]

(3.12)

The integrals over \( D \) and \( \lambda \) enforces the bosonic and fermionic ADHM constraints. If we carry out these integration we have

\[ W = C \int d\{\omega, \bar{\omega}, \mu, \bar{\mu}\} \delta(\bar{\mu}^{\mu u} + \omega_{\dot{\alpha} u} \mu^v) \delta(\omega^{\dagger u}(\tau^c)^{\dot{\alpha}} \bar{\omega}_{\dagger \beta} \beta) e^{-S_2 - S_3}. \]

(3.13)

In the computation of the ADS superpotential, one can consider only the contribution from \( S_2 \). If we consider the \( SU(N_c) \) gauge theory, one can easily see that \( N_f = N_c - 1 \) is the condition for nonvanishing fermionic integration from the structure of \( S_2 \). From \( S_2 \) one pair of \( \mu^u, \mu^{\mu f} \) comes down to the integrand simultaneously and \( \delta(\bar{\mu}^{\mu u} + \omega_{\dot{\alpha} u} \mu^v) \) will give additional \( \mu^u \) to the integration. Thus we need \( N_f = N_c - 1 \) for the nonzero fermionic integration. Now we consider the \( N_f = N_c \) case. For nonzero fermionic integration, we need \( S_3 \) to pull down two more fermion terms after pulling down fermion terms up to \( N_c - 1 \) from \( S_2 \). After carrying out
these integration, we obtain the following expression:
\[
W = C \int d\omega^2 d\tilde{\omega}^2 \delta^{(3)}(\tilde{\omega}_\alpha v(\tau^c)_{\beta}^\alpha \omega_{\beta}^\nu) \ e^{-\frac{i}{2} \tilde{\omega}_\alpha \nu (\tilde{Q} \tilde{Q})_{\alpha}^\nu \bar{\omega} \bar{\nu}}
\]
\[
\times \left\{ \frac{1}{2N!(N-2)!} \left[ \varepsilon_{\mu_1 \cdots \mu_{N-2}} \tilde{\varepsilon} \tilde{\varepsilon}_{h_1 \cdots h_{N-2}} \varepsilon_{s_1 \cdots s_N} \varepsilon_{k_1 \cdots k_N} 
\times \omega_\beta^\nu \omega_\gamma^\nu \tilde{\omega}_\delta^\nu \tilde{\omega}_\epsilon^\nu \tilde{\varepsilon}_{\delta \epsilon} \tilde{\varepsilon}_{\gamma \beta} \tilde{\varepsilon}_{\alpha \alpha} \tilde{\varepsilon}_{\nu \nu} \tilde{Q}_{h_1} \cdots \tilde{Q}_{h_{N-2}} Q_{s_1} \cdots Q_{s_N} \right] 
- \frac{2}{(N-1)!(N-1)!} \left[ \varepsilon_{\mu_1 \cdots \mu_{N-1}} \tilde{\varepsilon} \tilde{\varepsilon}_{h_1 \cdots h_{N-2}} \varepsilon_{s_1 \cdots s_N} \varepsilon_{g_1 \cdots k_{N-2}} 
\times \omega_\beta^\nu \omega_\gamma^\nu \tilde{\omega}_\delta^\nu \tilde{\omega}_\epsilon^\nu \tilde{\varepsilon}_{\delta \epsilon} \tilde{\varepsilon}_{\gamma \beta} \tilde{\varepsilon}_{\nu \nu} \tilde{Q}_{h_1} \cdots \tilde{Q}_{h_{N-2}} Q_{s_1} \cdots Q_{s_N} \right] 
\right\}
\]
\]
(3.14)

where \( N = N_f = N_c \). As in the case of the ADS superpotential, this expression represents lowest components of the effective superpotential. The fermions \( \tilde{\psi}_\alpha \) and \( \tilde{\bar{\psi}}_\alpha \) can be regarded as the lowest component of the superfield \( \tilde{D}_\alpha Q \) and \( \tilde{D}_\alpha \tilde{Q} \), where \( Q_i \)'s are promoted to the superfields. Schematically, this is in the form of
\[
\tilde{\psi}_\alpha \tilde{\bar{\psi}}_\alpha F(Q, Q^\dagger) \rightarrow \tilde{D}_\alpha Q \tilde{D}_\alpha \tilde{Q} F(Q, \tilde{Q}).
\]
(3.15)

This term is not the superpotential in the usual sense, and is not manifestly supersymmetric. However, this “superpotential” is chiral in the on-shell supersymmetry algebra because it is related to a representative of a Dolbeault cohomology whose elements parametrize infinitesimal deformations of moduli space \[1\]. Since this interaction term contains a four-fermion interaction, it is called multi-fermion F-term\[1\].

Similarly one can clearly see that the nonvanishing contribution can be obtained even for \( N_f > N_c \). We just have to pull down the necessary terms from \( S_3 \). It is a formidable task to perform integration in general cases, and we consider the simplest cases. We will consider the case of SU(2) with \( N_f = N_c \). In this case, the superpotential can be connected to the moduli deformation. In the next section, we review the relation between multi-fermion F-terms and the moduli space deformation.

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1 This type of the effective potential was also considered in [21].
4 Review on multi-fermion $F$-terms

In this section, we briefly review the multi-fermion $F$-terms. The simplest multi-fermion $F$-term is obtained from the superpotential which has two fermions in the lowest component. In this case, the superpotential can be regarded as a correction to the Kähler potential and connected to the deformation of the moduli space of vacua.

Let us consider a model whose moduli space is classically described by the complex fields $\phi^I$ satisfying a holomorphic equation,

$$C(\phi) = 0.$$  \hspace{1cm} (4.1)

The effective action is written in terms of $\phi$ as

$$S = \int d^4x d^4\theta K(\Phi, \bar{\Phi})$$  \hspace{1cm} (4.2)

where $\Phi^I$ and $\bar{\Phi}^I$ are chiral and anti-chiral superfields whose lowest components are $\phi^I$ and $\bar{\phi}^I$, respectively. Then, the bosonic part of this action is

$$S = \int d^4x g_{IJ} \partial \phi^I \partial \bar{\phi}^J,$$  \hspace{1cm} (4.3)

where $g_{IJ}$ is metric on the moduli space, and obtained from the Kähler potential as

$$g_{IJ} = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^I \partial \bar{\phi}^J}.$$  \hspace{1cm} (4.4)

Suppose that the governing equation on the moduli space is deformed to

$$C(\phi) = \epsilon.$$  \hspace{1cm} (4.5)

This deformation of the complex structure can be represented as the change of the basis of holomorphic one forms

$$d\phi^I \rightarrow d\phi^I - \omega^I_J d\bar{\phi}^J.$$  \hspace{1cm} (4.6)

Then, the metric on the moduli space changes as

$$g_{IJ} d\phi^I d\bar{\phi}^J \rightarrow g_{IJ} \left( d\phi^I - \omega^I_J d\bar{\phi}^J \right) d\bar{\phi}^J,$$  \hspace{1cm} (4.7)

and the sigma model action receives the correction term

$$\delta S = \int d^4x d^2\theta \omega_{IJ} D_\alpha \Phi^I \bar{D}^\alpha \bar{\Phi}^J = \int d^4x \omega_{IJ} \partial \phi^I \partial \bar{\phi}^J + \cdots$$  \hspace{1cm} (4.8)

$$\omega_{IJ} = g_{IK} \omega^K_J.$$  \hspace{1cm} (4.9)
Thus, the deformation of the moduli space is related to the multi-fermion $F$-terms. Note that away from the singularities on the moduli space, this deformation can be converted to the non-holomorphic change of the valuable:

$$ \phi^I \rightarrow \tilde{\phi}^I = \phi^I + \delta \phi^I (\phi, \tilde{\phi}), \quad (4.10) $$

where $\tilde{\phi}^I$ satisfies the classical constraint

$$ C(\tilde{\phi}) = 0. \quad (4.11) $$

In the next section, we work out the $SU(2)$ case and see that the result agrees with the computation carried out in [1].

5 Computation of multi-fermion $F$-terms

In the previous section, we have seen that multi-fermion $F$-terms in the form of the integral with respect to instanton moduli. However, it is difficult to evaluate the integral for general $SU(N)$. In this section, we investigate the simplest case of multi-fermion $F$-terms, i.e. $SU(N)$ SQCD for $N_f = N_c = 2$. In this case, multi-fermion $F$-terms are calculated in [1]. We show that our D-instanton derivation correctly reproduces their result, which is related to the moduli space deformation.

We start with the expression of (3.12),

$$ W = C \int d\{ \lambda, D, \omega, \bar{\omega}, \mu, \bar{\mu} \} e^{-S_1 - S_2 - S_3}. \quad (5.1) $$

We first perform integrations with respect to fermions. Integrals over $\lambda_\alpha$ enforce the fermionic ADHM constraints. Here, we do not consider the terms with gauginos (and gauge fields). By neglecting these terms, integrals over $\mu$’s determine terms which contribute to the superpotential. After these integrations, we obtain the following expression of the superpotential, up to an overall integration constant:

$$ W = \int d^2 \omega \, d^2 \bar{\omega} \, d^3 D \, \tilde{W} \, e^{-\frac{1}{2} \omega (Q Q^\dagger + \bar{Q}^\dagger \bar{Q}) \omega + i D \bar{\omega} \tau_k \omega}, \quad (5.2) $$

$$ \tilde{W} = \frac{1}{4} \varepsilon_{u_1 u_2} \varepsilon_{f_1 f_2} \varepsilon_{g_1 g_2} Q_{v_1}^1 Q_{v_2}^1, \quad (5.3) $$

where $\varepsilon_{u_1 u_2}$ denotes the Levi-Civita symbol, and $Q_{v_1}^1 Q_{v_2}^1$ is a determinant of the matrices $Q_{v_1}^1$ and $Q_{v_2}^1$. This expression includes the contributions from instanton quiver configurations, and it is related to the moduli space deformation.
Using the Fierz identity\(^\text{2!}\) for the first and the third terms, we obtain an expression with one pair of \(\omega, \bar{\omega}\) with their spinor indices contracted, i.e.,

\[
(\omega^\mu \omega^\nu)(\bar{\omega}_x \psi_f) = -(\omega^\nu \bar{\psi}_f)(\omega^\mu \bar{\omega}_x) + (\omega^\nu \bar{\psi}_f)(\omega^\mu \bar{\omega}_x),
\]

(5.5)

where spinor indices are contracted in each parenthesis. Then, \(\bar{W}\) can be expressed as

\[
\bar{W} = \bar{\omega}_{\dot{\alpha}u} \omega^{\dot{\alpha}v} \bar{\bar{\omega}}_{\dot{\beta}u} \omega^{\dot{\beta}v} \left[ \frac{1}{2} \varepsilon_{v_1 v_2} \varepsilon f_1 f_2 \varepsilon \bar{\psi}_f \psi_f \varepsilon^{u_1 u_2} \varepsilon g_1 g_2 Q^{1}_{u_1} Q^{2}_{u_2} \right. \\
- 2 \varepsilon_{v_1 v_2} \varepsilon f_1 f_2 \bar{Q}^{1}_{v_1} \bar{Q}^{2}_{v_2} \varepsilon g_1 g_2 \bar{\psi}_f \psi_f \varepsilon^{u_1 u_2} \\
+ \left. \frac{1}{2} \varepsilon_{v_1 v_2} \varepsilon g_1 g_2 \bar{Q}^{1}_{v_1} \bar{Q}^{2}_{v_2} \varepsilon f_1 f_2 \bar{\psi}_f \psi_f \right].
\]

(5.6)

Next, we perform the integration with respect to \(\omega\) and \(\bar{\omega}\). After some algebras, we obtain the following relation (see Appendix C):

\[
F(A) = \int d^2 \omega d^2 \bar{\omega} d^3 D e^{-\bar{\omega}\omega A + i D^c \bar{\omega} \tau^c \omega} \\
= \int d^3 D \frac{1}{\det(A^2 + D^2)} \\
= (\text{tr} A)^{-1}.
\]

(5.7)

where the matrix \(A\) is a \(2 \times 2\) hermitian matrix, and given by \(A = \frac{1}{2}(QQ^\dagger + \bar{Q}\bar{Q}^\dagger)\) in the present case. Picking up relevant parts in (5.2) and (5.6), we obtain

\[
\int d\omega d\bar{\omega} d^3 D \bar{\omega}_{\dot{\alpha}u} \omega^{\dot{\alpha}v} \bar{\bar{\omega}}_{\dot{\beta}u} \omega^{\dot{\beta}v} e^{-\bar{\omega}\omega A + i D^c \bar{\omega} \tau^c \omega} = \frac{1}{2} \varepsilon_{\dot{\beta} \dot{\gamma}} \int d\omega d\bar{\omega} d^3 D \bar{\omega}_{\dot{\alpha}u} \omega^{\dot{\alpha}v} \bar{\bar{\omega}}_{\dot{\alpha}u} \omega^{\dot{\gamma}v} e^{-\bar{\omega}\omega A + i D^c \bar{\omega} \tau^c \omega} \\
= \frac{1}{2} \varepsilon_{\dot{\beta} \dot{\gamma}} \frac{\partial}{\partial A_{u}^{\dot{\nu}}} \frac{\partial}{\partial A_{\dot{\nu}}^{u}} F(A) = \varepsilon_{\dot{\beta} \dot{\gamma}} \left( \frac{\partial}{\partial A_{u}^{\dot{\nu}}} \frac{\partial}{\partial A_{\dot{\nu}}^{u}} \right) F(A).
\]

(5.8)

Using this, the superpotential is given by

\[
W = (\text{tr} A)^{-3} \left( -\frac{1}{2} \varepsilon^{u v} \varepsilon f_g \bar{\psi}_f \bar{\psi}_f \varepsilon^{u v} f_g Q_{u}^{f} Q_{v}^{g} \\
+ 2 \varepsilon_{u v} \varepsilon f_k \varepsilon g_k \bar{\psi}_f \bar{\psi}_f g_k Q_{u}^{f} Q_{v}^{g} \\
- \frac{1}{2} \varepsilon^{u v} \varepsilon f_g \bar{Q}_{f}^{k} Q_{f}^{k} \varepsilon^{u v} f_g \bar{\psi}_f \bar{\psi}_f g_k Q_{u}^{f} Q_{v}^{g} \right). 
\]

(5.9)

In the case of \(SU(2)\), the fundamental and anti-fundamental representations coincide, and the \(N_f\) flavors can be treated as \(2N_f\) flavors. Here, we treat the flavor symmetry as global symmetry

\(^2\) For bosonic spinors \(A, B, C\) and \(D\), we have the following relation:

\[
A_{\dot{\alpha}} B_{\dot{\beta}} C_{\dot{\gamma}} D_{\dot{\delta}} = (AB)(CD) = -(AD)(BC) + (AC)(BD).
\]

(5.4)

This can be used for color and flavor \(SU(2)\) indices as well.
by neglecting gauge fields of flavor symmetry. Then, the flavor symmetry becomes $SU(4)$. First, we rewrite the quarks $\tilde{Q}$ in the fundamental representation of $SU(2)$ color symmetry as

$$Q^{uf} = \varepsilon^{uv} \tilde{Q}_v,$$
$$\psi'^{uf}_\alpha = \varepsilon^{uv} \tilde{\psi}^{f}_v.$$

By applying the Fierz identity to color $SU(2)$ indices, each term in the parenthesis becomes

$$-\varepsilon^{f g} \varepsilon^{f' g'} (Q'^{f f'} \tilde{\psi}^g (Q'^{g g'} \tilde{\psi})),
2\varepsilon^{f g} \varepsilon^{f' g'} ((\tilde{\psi}^g Q'^{f}) (\tilde{\psi}^{g'} Q'^{f'})) + (\tilde{\psi}^g Q'^{f'}) (\tilde{\psi}^{g'} Q'^{f})
-\varepsilon^{f g} \varepsilon^{f' g'} (Q'^{f f'} \tilde{\psi}^g) (Q'^{g g'} \tilde{\psi}).
$$

respectively, where $SU(2)$ color indices are contracted inside each parenthesis and are not written explicitly. Thus, the superpotential is given by

$$W = (\text{tr } A)^{-3} \varepsilon^{f g} \varepsilon^{f' g'} \left[ - (Q'^{f f'} \tilde{\psi}^g (Q'^{g g'} \tilde{\psi})
+ 2 \left( (\tilde{\psi}^g Q'^{f}) (\tilde{\psi}^{g'} Q'^{f'}) + (\tilde{\psi}^g Q'^{f'}) (\tilde{\psi}^{g'} Q'^{f}) \right)
- (Q'^{f f'} \tilde{\psi}^g) (Q'^{g g'} \tilde{\psi}). \right].
$$

By promoting the quarks $Q$ and their superpartners $\psi$ to the superfields, fermion $\tilde{\psi}_{\dot{\alpha}}$ can be interpreted as the lowest component of $\tilde{D}_{\dot{\alpha}} \tilde{Q}$. The quarks can be combined into $Q$ as

$$Q^i = \begin{cases} Q^i_u, & (i = 1, 2) \\ Q'^{i f}, & (i = 3, 4) \end{cases}
$$

Then, the superpotential can be expressed in the following compact form

$$W = (\text{tr } A)^{-3} \varepsilon^{ijkl} \tilde{D}_{\dot{\alpha}} (\varepsilon^{uv} \tilde{Q}^i_u \tilde{Q}^j_v) \tilde{D}^{\dot{\alpha}} (\varepsilon^{uv'} \tilde{Q}^k_u \tilde{Q}^{l'}_v),
$$

where the matrix $A$ can be written as $A = Q \tilde{Q}$. In the moduli space of SQCD, we have the $D$-flatness condition, $D^a = \text{tr } Q^a = 0$, and we can rewrite $\text{tr } A$ as

$$\text{tr } A = \sqrt{\det Q \tilde{Q}}.
$$

Introducing the “meson” field $M_{ij} = \varepsilon_{uv} Q^i_u Q^j_v$, we obtain the following expression for the superpotential

$$W = (\text{tr } M)^{-3} \varepsilon^{ijkl} \tilde{D}_{\dot{\alpha}} \tilde{M}_{ij} \tilde{D}^{\dot{\alpha}} \tilde{M}_{kl}.
$$

In [1], the superpotential is derived by the usual field theory instanton calculation and its relation to the moduli space deformation is explained. In the case of $SU(2)$, the classical moduli space is described by the meson $M$ satisfying the classical constraint

$$\text{Pf } M = 0.
$$
This constraint is modified by the quantum effect to
\[ \text{Pf } M = \epsilon. \]  

(5.20)

This quantum moduli space can be converted to the classical moduli space by a non-holomorphic change of variables. Introducing the new coordinate \( \tilde{M} = M - \delta M \), the deformation of the complex structure gives rise to the additional superpotential of the form
\[ \omega_{ijkl}G_{mn} \tilde{D} \tilde{M}^{kl} \tilde{D} \tilde{M}^{mn}, \]

(5.21)

where \( G_{kl} \) is the metric of the moduli space, and \( \omega_{ijkl} \) is given by
\[ \left( \frac{\partial}{\partial M^{kl}} + \omega_{ijkl} \frac{\partial}{\partial M_{ij}} \right) \tilde{M}_{mn} = 0. \]

(5.22)

In this case, the new coordinate is given by
\[ M_{ij} \rightarrow M_{ij} - \frac{\epsilon}{2} \frac{\varepsilon_{ijkl}\tilde{M}^{kl}}{(\text{tr } \tilde{M}M)}, \]

(5.23)

Then, \( \omega_{ijkl} \) becomes
\[ \omega_{ijkl} = \frac{\epsilon}{2} \left( \frac{\varepsilon_{ijkl}}{\text{tr } \tilde{M}M} - \frac{\varepsilon_{ijmn}\tilde{M}^{mn}M_{kl}}{(\text{tr } \tilde{M}M)^2} \right). \]

(5.24)

The metric can be determined by the asymptotic form of the Kähler potential \( K = \sqrt{\text{tr } \tilde{M}M} \). The additional factor of \( (\text{tr } \tilde{M}M)^{-1/2} \) comes from this metric. Then, the superpotential becomes,
\[ W = (\text{tr } \tilde{M}M)^{-\frac{3}{2}} \left( \varepsilon_{ijkl} - \frac{\varepsilon_{ijmn}\tilde{M}^{mn}M_{kl}}{\text{tr } \tilde{M}M} - \frac{\varepsilon_{mkln}\tilde{M}^{mn}M_{ij}}{\text{tr } \tilde{M}M} \right) \tilde{D} \tilde{M}^{ij} \tilde{D} \tilde{M}^{kl}, \]

(5.25)

where the third term has been added so that the superpotential is manifestly symmetric to the exchange of two \( \tilde{D}\tilde{M} \)'s. It can be done because the second and third term vanish on the moduli space since
\[ \varepsilon_{ijkl} \tilde{M}_{kl} d\tilde{M}_{ij} = d(\varepsilon_{ijkl} \tilde{M}_{ij} \tilde{M}_{kl}). \]

(5.26)

Using this relation (5.26) again, the superpotential (5.18) is equivalent to (5.25) which is based on the moduli space deformation.

6 Case of symplectic gauge group

In this section, we consider the case of the symplectic gauge group, which can be obtained by introducing the orientifold \cite{22}. Here, we take the orbifolding with \( (2N_c, 2N_f, 0, 0) \) for D3-branes
and \((1, 0, 0, 0)\) for D-instantons, and put the O3-plane on the orbifold singularity. Then, the color and flavor gauge groups becomes \(USp(2N_c)\) and \(USp(2N_f)\), respectively. If one neglects the flavor gauge symmetry and treats this as a global symmetry, this remains \(U(2N_f)\).

The orientifold acts on the Chan-Paton factor as a matrix \(\gamma(\Omega)\). This matrix can be different for different D-branes, and have opposite symmetries (symmetric or anti-symmetric) for the D3-brane and D-instanton \([23]\). We take the anti-symmetric one for the D3-brane to obtain the symplectic gauge group. The matrix \(\gamma(\Omega)\) must satisfy the following consistency condition:

\[
\gamma(g)\gamma(\Omega)\gamma(g) = \gamma(\Omega).
\]

Then, the action of the orientifold is

\[
\gamma_- = \begin{pmatrix}
J & 0 & 0 & 0 \\
0 & \tilde{J} & 0 & 0 \\
0 & 0 & J^{(3)} & 0 \\
0 & 0 & 0 & J^{(4)}
\end{pmatrix},
\]

where, \(J\)'s are anti-symmetric matrices which satisfies \(J^2 = -1\). We use the notation of \(J_{uv} = -J_{uv} = (J^{-1})_{uv}\), and similarly for \(\tilde{J}\).

The orientifold imposes the following additional conditions on massless fields in the D3-D3 sector:

\[
\begin{align*}
A_\mu &= -\gamma_- A_\mu^T \gamma_-^{-1}, \\
\Phi &= -\gamma_- \Phi^T \gamma_-^{-1}, & \Phi^t &= -\gamma_- \Phi^T \gamma_-^{-1}, \\
\Psi^A &= -R^A_B \gamma_- (\Psi^B)^T \gamma_-^{-1}, & \bar{\Psi}_A &= \gamma_- (\bar{\Psi}_B)^T \gamma_-^{-1} R_B^A,
\end{align*}
\]

where the minus sign for the scalars is due to the spacetime reflection, and those for the gauge fields and the fermions come from the worldsheet reflection. The matrix \(R^A_B\) is the action of the reflection on six dimensional spinors:

\[
R = -i\Gamma^{456789}.
\]

The projection on the gauge fields (and gauginos) restrict the color and flavor gauge groups to the symplectic gauge group. For the chiral scalars and fermions, the conditions \([6,3]\) make connections between \(Q\) and \(\bar{Q}\), \(\psi\) and \(\bar{\psi}\) as

\[
\begin{align*}
\bar{Q}_u^f &= -\tilde{f}g Q^v_g J^v_j, & \bar{Q}_j^u &= -J^{uv} Q^v_g J^{j}_j, & \bar{\psi}_u^f &= -\tilde{f}g \psi^v_g J^v_j, & \bar{\psi}_j^u &= -J^{uv} \psi^v_g J^j_j. \tag{6.5}
\end{align*}
\]

Since these conditions do not give any restriction on the flavor symmetry, the global symmetry itself remains to be \(U(2N_f)\).
For the Chan-Paton factor on the D-instanton, the action of the orientifold is a symmetric matrix, which can be taken to be the unit matrix

$$\gamma_+ = 1.$$ \(\text{(6.6)}\)

For the fields in the D(-1)-D(-1) sector, the orientifold imposes the following conditions:

$$a_\mu = a_{\mu}^T, \quad \chi = -\chi^T, \quad M_\alpha^A = R_B^A(M_\alpha^B)^T, \quad \lambda_{\dot\alpha A} = (\lambda_{\dot\alpha B})^T R_A^B.$$ \(\text{(6.7)}\)

After the orbifolding for the one-instanton condition, \((1,0,0,0)\), these relations yield

$$\lambda_{\dot\alpha} = -\lambda_{\dot\alpha}, \quad D^c = -D^c.$$ \(\text{(6.8)}\)

Thus, \(\lambda\) and \(D^c\) must vanish. It implies that there are no bosonic and fermionic ADHM constraints in \(USp(2N_c)\) gauge theory as is well-known.

For the fields in the D3-D(-1) sector, the orientifold projection gives relations between \(\omega\) and \(\bar{\omega}\), and, \(\mu\) and \(\bar{\mu}\):

$$\bar{\omega} = \omega^T \gamma^{-1}, \quad \bar{\mu}^A = R_B^A (\mu^B)^T \gamma^{-1}.$$ \(\text{(6.9)}\)

These relations become the following relations after the orbifolding:

$$\bar{\omega}_u = \omega^v J_{vu}, \quad \bar{\mu}_u = \mu^v J_{vu}, \quad \bar{\mu}'_f = \mu^g \bar{J}_{gf}.$$ \(\text{(6.10)}\)

The superpotential is greatly simplified since the ADHM constraints are absent in this case. The action of our interests becomes

$$S = \omega_\alpha^u J_{uv} Q_{j\dot{\alpha}}^v Q_{j\dot{\alpha}}^u \omega_{\dot{\alpha}u} + i \mu_{\alpha}^j J_{fg} Q_{\dot{\alpha}}^g \mu_{\dot{\alpha}}^j + 2i \mu_{\alpha}^j \bar{J}_{fg} \bar{\psi}_{\dot{\alpha}}^g \omega_{\dot{\alpha}u}.$$ \(\text{(6.11)}\)

6.1 Multi-fermion F-terms and the moduli space deformation

In this section, we consider the simplest case of the multi-fermion F-term. As we have seen in the case of \(SU(2)\), the superpotential is related to the deformation of the moduli space. We generalize the analysis to the symplectic case, and show that our superpotential reproduces the moduli deformation.

The superpotential \(W\) is given by the path integral of the action \((6.11)\):

$$W = \int d\{\omega, \mu\} e^{-S}.$$ \(\text{(6.12)}\)

We neglect contributions from gauginos. Then, all of \(\mu\) in the integrand must come from the second term of \((6.11)\), because the ADHM constraints are absent in this case. If \(N_c = N_f\), there are no additional contribution of \(\mu'\), and it yields an ADS-type superpotential, as is well known \cite{24}. In the case of \(N_f > N_c\), \(N_c\) of \(\mu'\) are supplied by the second term of \((6.11)\), and the rest of
\( \mu' \) are by the third term. Then, the superpotential has the \( 2(N_f - N_c) \) of the fermions \( \bar{\psi} \). We obtain the simplest multi-fermion \( F \)-terms in the case of \( N_f = N_c + 1 \). In this case, we have the following expression up to an overall constant:

\[
W = \int d^2 \omega \ e^{-\omega_a (JQ^\dagger)_{\omega^a}} \times \varepsilon^{u_1 \ldots u_N} \varepsilon^{f_1 \ldots f_{N+2}} (JQ^\dagger)_{u_1 f_1} \ldots (JQ^\dagger)_{u_N f_N} (\bar{\psi} \omega)_{f_{N+1}} (\bar{\psi} \omega)_{f_{N+2}} \]

\[
\times (\bar{\psi} \omega)^{g_{N+1}} (\bar{\psi} \omega)^{f_{N+2}} ((Q Q^\dagger)_{uv})^{vw} \]

(6.13)

where, \( N = 2N_c = 2(N_f - 1) \). Using the Bose statistics of the quarks \( Q \), we can rewrite this superpotential in terms of the “mesons” \( M_{fg} = J_{uv} Q^u f Q^v g \). Then, we obtain the following compact form of the superpotential:

\[
W = \frac{C'(\bar{M})_{ijkl}}{C(M)_{fg} C(M)_{fg}} \bar{G}^{ijkl} \bar{D} M^{kl} \bar{D} M^{mn}, \quad (6.14)
\]

\[
\bar{G}^{ijkl} = Q^{ij}_u \left( (Q Q^\dagger)^{-2} \right)^{uv} \delta^{ijkl}_v, \quad (6.15)
\]

where we define \( C, \bar{C} \) and \( \bar{C}' \) as

\[
C(M)^{ij} = \varepsilon^{ij f_1 \ldots f_N} M_{f_1 f_2} \ldots M_{f_{N-1} f_N}, \quad (6.16)
\]

\[
\bar{C}(\bar{M})_{ij} = \varepsilon_{ij f_1 \ldots f_N} \bar{M}_{f_1 f_2} \ldots \bar{M}_{f_{N-1} f_N}, \quad (6.17)
\]

and

\[
\bar{C}'(\bar{M})_{ijkl} = \varepsilon_{ijkl f_1 \ldots f_{N-2}} \bar{M}_{f_1 f_2} \ldots \bar{M}_{f_{N-3} f_{N-2}}. \quad (6.18)
\]

Let us consider the relation between our superpotential and the moduli space deformation [24]. We treat only the color symmetry as the gauge symmetry, but the flavor symmetry as a global symmetry by neglecting the flavor gauge field. Then, our model has \( USp(2N_c) \) color gauge symmetry and \( U(2N_f) \) flavor global symmetry. The classical moduli space is described by the “mesons” \( M_{fg} \). For \( N_f > N_c \), these mesons satisfy the classical constraint

\[
Pf M = 0, \quad (6.19)
\]

which is a trivial consequence of the Bose statistics of the quark fields \( Q \). In the case of \( N_f = N_c + 1 \), this constraint is modified to

\[
Pf M = \Lambda \quad (6.20)
\]

by the quantum effect of instantons.
The quantum moduli space can be converted to the classical moduli space by a non-holomorphic change of variables. Introducing the new coordinates \( \tilde{M} = M - \delta M \), the deformation of the complex structure gives rise to the additional superpotential of the form

\[
\omega_{ijkl} G_{mn}^{ij} \tilde{D}^{kl} \tilde{D}^{mn},
\]  

(6.21)

where \( G_{kl} \) is the metric of the moduli space, and \( \omega_{ijkl} \) is given by

\[
\left( \frac{\partial}{\partial M^{kl}} + \omega_{ijkl} \frac{\partial}{\partial M^{ij}} \right) \tilde{M}_{mn} = 0.
\]  

(6.22)

In this case, the moduli deformation is described by

\[
\delta M_{ij} = \bar{C}(\bar{M})_{ij} \bar{C}(\bar{M}) \bar{C}(\bar{M})^{fg} C(M)^{fg},
\]  

(6.23)

where \( \bar{C} \) is defined in (6.18). We do not know the explicit form of the metric \( G \) but we can estimate it from the asymptotic form of the Kähler potential \( K \). The Kähler potential for mesons \( M \) equals to that of quarks \( Q \) at the asymptotic region of the moduli space. Since the \( D \)-flatness condition for the symplectic gauge group is

\[
J_{uw} Q^w_f \bar{Q}_{u}^f = Q^w_u \bar{Q}_{w}^u J_{uw},
\]  

(6.25)

the Kähler potential can be written as

\[
K = \text{tr} Q^\dagger Q = \text{tr} \sqrt{JQ^\dagger Q J^{-1} Q^\dagger}.
\]  

(6.26)

We can read off an expression of the metric from the last expression by using the following relation:

\[
\frac{\partial^2 K}{\partial Q^u_{v}^{ij} \partial Q^u_j} = \frac{\partial^2 K}{\partial M^{kl} \partial M_{ij}^{kl} \partial \tilde{D}^{kl}} = G^{ij}_{kl} \frac{\partial M_{ij}^{kl}}{\partial \tilde{D}^{kl}},
\]  

(6.27)

Thus, we obtain

\[
G^{ij}_{kl} = \delta^i_k \bar{B}_i^j + \bar{B}_k^j \delta^i_k - \frac{1}{2} (D_i^k \bar{B}_l^i + \bar{B}_k^j D^i_l) + \frac{1}{2} \sum_{r=1}^{\infty} (-1)^r \left[ (B^r)^j_k (\bar{B}^{r+1})^i_l + (\bar{B}^{r+1})^j_k (B^r)^i_l \right]
\]  

(6.28)
where

\[ B^i_j = Q^i_u Q^u_j, \quad \tilde{B}^i_j = Q^{\dagger i}_u \left( (QQ^\dagger)^{-2} \right)_v^u Q^v_j, \quad D^i_j = Q^{\dagger i}_u \left( (QQ^\dagger)^{-1} \right)_v^u Q^v_j. \tag{6.29} \]

Substituting (6.24) and (6.28) into (6.21), we obtain the superpotential generated by the quantum effect of instantons. Using the relation \( M_{fg} = J_{uv} Q^u_f Q^v_g \) and the Bose statistics of \( Q \), this superpotential is simplified to the following form:

\[ W = \frac{\tilde{G}^{ij}(M)_{ijkl}}{C(M)_{fg} C(M)_{fg}} \tilde{G}^{ij}_{mn} \bar{D} \tilde{M}^{kl} \tilde{D} \tilde{M}^{mn}, \tag{6.30} \]

\[ \tilde{G}^{ij}_{kl} = B^{i}_{k} \delta^{j}_{l}, \tag{6.31} \]

which is equivalent to our superpotential (6.14).

### 6.2 Higher multi-fermion F-terms

In the case of \( N_f > N_c + 1 \), the moduli space is not deformed by effects of instantons, and we cannot relate multi-fermion F-terms to the moduli space deformation. However, multi-fermion F-terms themselves exist even in these cases. These multi-fermion F-terms are also found from the purely field theoretical analysis. Here, we restrict ourselves to the simplest gauge group of \( USp(2) \simeq SU(2) \), and show that our analysis reproduces the result from field theory.

In general cases of \( N_f > N_c \), we have the following expression of the superpotential:

\[
W = \int d^2 \omega \ e^{-\omega_a (JQQ^\dagger)} \omega^a \\
\times \varepsilon^{u_1 \cdots u_{2N_c}} \varepsilon^{f_1 \cdots f_{2N_f}} (JQ^\dagger)_{u_1 f_1} \cdots (JQ^\dagger)_{u_{2N_c} f_{2N_c}} (\bar{J} \psi \omega)_{f_{2N_c+1}} \cdots (\bar{J} \psi \omega)_{f_{2N_f}} \\
= \sum_{\sigma} \frac{1}{\text{det}(JQQ^\dagger)} \varepsilon^{u_1 \cdots u_{2N_c} \varepsilon^{f_1 \cdots f_{2N_f}}} (JQ^\dagger)_{u_1 f_1} \cdots (JQ^\dagger)_{u_{2N_c} f_{2N_c}} \\
\times (\bar{J} \psi)_{\tilde{\alpha}_{f_1} u_{2N_c+1}} \cdots (\bar{J} \psi)_{\tilde{\alpha}_{f_{2N_f}} u_{2N_c}} (\bar{J} \psi)_{\tilde{\alpha}_{f_{2N_c+1}} v_1 \cdots (\bar{J} \psi)_{\tilde{\alpha}_{f_{2N_f}} v_{2N_c+1}} \\
\times (QQ^\dagger)_{v_1 \sigma(v_1)} \cdots (QQ^\dagger)_{v_{2N_c} \sigma(v_{2N_c})}. \tag{6.32} \]

where, \( \sigma \) indicates the permutation of indices, which can be removed by using the Fierz identity. Using the relation \( M_{fg} = Q^u_f Q^v_g \) and the Bose statistics of quarks \( Q \), we can simplify it into the following compact form:

\[
W = \frac{1}{\text{det}(JQQ^\dagger)} \varepsilon^{f_1 \cdots f_{2N_f}} \bar{M}^{f_1 f_2} \cdots \bar{M}^{f_{2N_c} - 1 f_{2N_c}} \mathcal{O} f_{2N_c + 1 f_{2N_c + 2}} \cdots \mathcal{O} f_{2N_f - 1 f_{2N_f}}, \tag{6.33} \]

where

\[
\mathcal{O}_{fg} = \bar{\psi}^f_{\tilde{\alpha}_u} \left[ (QQ^\dagger)^{-1} \right]_w^u J^w v \bar{\psi}^g_v. \tag{6.34} \]
Using the Bose statistics of $Q$ again, and extending $\bar{\psi}$ to the superfield $\bar{D}Q$, we can replace $O$ by $\bar{O}$ which is defined as

$$\bar{O}^{fg} = \bar{D}M^{ff'} \left[ Q^T J(QQ^t)^{-1} \right] \bar{f}_{g'} \bar{D}M^{g'g}. \quad (6.35)$$

In the case of $USp(2) = SU(2)$, $D$-flatness condition gives the following relation:

$$(QQ^t)^u_v^u = \delta^u_v \text{tr}(QQ^t) = \delta^u_v \sqrt{\text{tr} \bar{M}M}. \quad (6.36)$$

Then, we obtain

$$\bar{O}^{fg} = (\text{tr} \bar{M}M)^{-3/2} M^{f'g'} \bar{D}M^{ff'} \bar{D}M^{g'g'}. \quad (6.37)$$

Using this expression, we can rewrite the superpotential as

$$W = (\text{tr} \bar{M}M)^{-3/2} \varepsilon_{f_1 \cdots f_{2N_f}} M^{f_1 f_2} \bar{O}^{f_3 f_4} \ldots \bar{O}^{f_{2N_f} - 1 f_{2N_f}}, \quad (6.38)$$

$$\bar{O}^{ij} = M_{kl} \bar{D}_\alpha \bar{M}^{ik} \bar{D}^\alpha \bar{M}^{lj}. \quad (6.39)$$

Therefore, this multi-fermion $F$-term is equivalent to that in \[1\].

### 7 Conclusions and discussions

In this paper, we have investigated the instanton in $\mathcal{N} = 1$ SQCD by using the D-brane effective theory. In SQCD with gauge group $SU(N_c)$ and $N_f = N_c$ flavors, instantons modify the moduli space of vacua. This effect can be described by the multi-fermion $F$-terms, and we have derived these terms from the D-brane effective action. SQCD can be obtained by introducing the orbifolding into the D3-brane effective theory, and the gauge instanton corresponds to the D-instanton on the D3-brane. The effective potential generated by instantons can be obtained from the D-instanton effective action. The multi-fermion $F$-terms are given in terms of integral with respect to the instanton moduli. This integration generally gives complicated expression. We have considered the simplest case of $SU(2)$ gauge group, for which the potential was calculated in \[1\] by using purely field theoretical techniques. Our result correctly reproduced that in \[1\].

We have also considered the case of symplectic gauge group. In this case, we have obtained much simpler results than those in the case of the unitary group. This is due to the fact that the ADHM constraints are absent for the symplectic gauge group. We have shown that the deformation of the moduli space is described by the multi-fermion $F$-terms derived from D-instantons, for $N_f = N_c + 1$. We have also calculated multi-fermion $F$-terms with more fermions, which appear for the theory with more flavors. For $USp(2) \sim SU(2)$, our result agrees with that in \[1\], again.

We would like to comment on the case of the orthogonal group. SQCD with this gauge group can be obtained by introducing the orientifold which is opposite to that for the symplectic group; symmetric projection is imposed on the Chan-Paton factor of D3-branes and the anti-symmetric
projection on that of D-instantons. We can obtain multi-fermion F-terms similar to those in the case of unitary gauge group for $N_f = N_c$ or symplectic gauge group for $N_f = N_c + 1$, i.e. those in the form of (3.15). Due to the anti-symmetric projection, the size of the matrices for D-instantons must be even, and we should take $k = 2$ for one-instanton. Then, the Chan-Paton factor for D-instantons becomes $USp(2) \sim SU(2)$, and consequently, there are three sets of the ADHM constraints corresponding to three $SU(2)$ generators. The fermionic ADHM constraints supply six fermions $\mu$. Since there are $2N_c$ fermions $\mu$ and $2N_f$ fermions $\mu'$, we obtain the multi-fermion $F$-terms in the form of (3.15), if $N_f = N_c - 2$. However, there are no constraint on the moduli space for $N_f = N_c - 2$, and therefore, we cannot see any relation between the multi-fermion $F$-terms and the deformation of the moduli space. Furthermore, a large number of the ADHM constraint makes the integration in the expression of multi-fermion $F$-terms much more complicated.

It would be interesting to study the relation between multi-fermion $F$-terms and the deformation of the moduli space in the case of $SU(N_c)$ for $N_c > 2$. In order to study this relation, we have to obtain an explicit form of the metric on the moduli space. Even in the case of $SU(2)$ and symplectic group, the metric is determined in the asymptotic region by using symmetries. It is interesting to describe deformation of the moduli space in full detail. These issues are left for future studies.

Another interesting problem is a generalization to other models. In this paper, we have demonstrated the D-instanton derivation of multi-fermion $F$-terms which are related to the deformation of the moduli space. This method can be applied to other models and will show how the moduli space is deformed by instantons. It would also be interesting to study stringy effects of multi-fermion $F$-terms.

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A Supersymmetry transformations

In this appendix, we describe the supersymmetry transformation of the effective theory. The D3-brane effective theory is the $\mathcal{N} = 4$ super Yang-Mills theory. The followings are their
transformations:

\[ \delta A_\mu = i \xi_{\dot{\alpha} A} \tilde{\sigma}^{\mu}{}_{\dot{\alpha}} \Psi^A + \xi^\alpha A \bar{\sigma}_{\alpha} \tilde{\Psi}^A, \]  
(A.1)

\[ \delta \Psi^A = i \frac{\sigma^\mu}{2} \xi^\beta A F_{\mu \nu} + i \tilde{\xi}^{ABC} D_{\mu} \Phi_{AB} + \left( \frac{1}{2} \varepsilon^{\alpha \beta \gamma} \xi_{\alpha} A \bar{\Phi} - \frac{1}{2} \varepsilon^{\alpha \beta \gamma} \xi_{A} \Phi_{BC} \right), \]  
(A.2)

\[ \delta \bar{\Psi}^\dot{\alpha} = \frac{i}{2} \xi_{\dot{\alpha}} A \sigma A \bar{\mu} \bar{\bar{\mu}} F_{\mu \nu} + i \bar{\xi}^{AB} D_{\mu} \Phi_{AB} + \varepsilon^{ABC} \left( \frac{1}{2} \xi^C A \bar{\Phi}_{BC} \phi_{DE} - \frac{1}{2} \xi^C A \phi_{AB} \phi_{CD} \right), \]  
(A.3)

\[ \delta \Phi_{AB} = 2 i \varepsilon^{ABCD} \xi^C A \phi_{AB} - 2 i \left( \xi_{\dot{\alpha}} A \bar{\phi}_{B} - \xi_{\dot{\alpha}} A \bar{\phi}^A \right). \]  
(A.4)

where, we rewrote the \( R \)-symmetry in terms of \( SU(4) \) using the definition, \( \Phi_{AB} = \Sigma_{a} A_{B} \Phi_{a} \).

Introducing the D-instanton, a half of the supersymmetry is broken and the unbroken symmetry is generated by supercharges \( \bar{Q}_{\dot{A}} \). We list these supersymmetry transformations of the fields on the D-instanton. The fields in the D(-1)-D(-1) sector are given by dimensional reduction of the \( N = 4 \) super Yang-Mills theory. Their transformations are

\[ \delta a_\mu = \frac{i}{2} \xi_{\dot{\alpha}} A \tilde{\sigma}^{\mu}{}_{\dot{\alpha}} M^A, \]  
(A.5)

\[ \delta \chi^A = -i \xi_{\dot{\alpha}} A \sum_{AB} \phi_{\dot{A} B}, \]  
(A.6)

\[ \delta M^\alpha A = -\frac{i}{2} \xi_{\dot{\alpha}} A \sum_{AB} \phi^{\alpha A} [\chi^A, a_\mu], \]  
(A.7)

\[ \delta \lambda_{\dot{A}} A = \frac{i}{2} \xi_{\dot{\alpha}} A \sum_{AB} [\chi^A, a_\mu] + \xi_{\dot{\alpha}} A \tilde{\sigma}^{\mu}{}_{\dot{\alpha}} [a_\mu, a_\nu]. \]  
(A.8)

By introducing the auxiliary fields \( D^c \), the transformation of \( \lambda_{\dot{A}} A \) becomes

\[ \delta \lambda_{\dot{A}} A = \frac{i}{2} \xi_{\dot{\alpha}} A \tilde{\sigma}^{\beta A} D^c. \]  
(A.9)

And the transformations in the D3-D(-1) sector are the followings:

\[ \delta \omega_{\dot{A}} = -i \xi_{\dot{\alpha}} A \mu^A, \]  
(A.10)

\[ \delta \bar{\omega}_{\dot{A}} = i \xi_{\dot{\alpha}} A \bar{\mu}^A, \]  
(A.11)

\[ \delta \mu^A = \varepsilon^{ABC} \xi_{\dot{A}} B \left( \omega^A \chi_{CD} + \Phi_{CD} \omega_{\dot{A}} \right), \]  
(A.12)

\[ \delta \bar{\mu}^A = \varepsilon^{ABC} \xi_{\dot{A}} B \left( \chi_{CD} \bar{\omega}^A + \bar{\omega}_{\dot{A}} \Phi_{CD} \right), \]  
(A.13)

where, \( \chi_{AB} = \Sigma_{a} A_{B} \chi^a \).

## B Multi-fermion F-terms for general \( SU(N) \)

In this appendix, we show an explicit expression of the superpotential for general \( N_f \geq N_c \) in the case of unitary gauge group. The expression is quite complicated if all fields in the D3-D(-1) sector are integrated out. Therefore, we do not perform all of the integration. The superpotential
can be expressed in terms of the integral with respect to $\omega$:

$$W = \int d\omega d\bar{\omega} \delta^{(3)}(\bar{\omega}_\alpha (\tau^c \alpha)^{\hat{\theta}} \bar{\phi} \beta^\mu) e^{-\frac{i}{2} \bar{\omega}_\alpha (Q_3^I \phi^f + \bar{Q}_3^I \bar{\phi}_c) \omega^{\alpha \nu} \epsilon_{\nu 1 \cdots u_{Nc} \epsilon_{v_1 \cdots v_{Nc}} \epsilon_{f_1 \cdots f_{Nf}}} e^{\bar{\epsilon} g_{1 \cdots g_{Nf}}}$$

$$\times \left[ \frac{1}{N_c! (N_c - 2)! (N_f - N_c)! (N_f - N_c + 2)!} \times Q_{u_{Nc-2}}^{f_{Nc-1}} Q_{g_{Nc-1}}^{f_{Nc}} Q_{g_{Nc}}^{f_{Nc+2}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}$$

$$+ \frac{1}{(N_c - 1)! (N_f - N_c)! (N_f - N_c + 1)!} \times Q_{u_{Nc-2}}^{f_{Nc-1}} Q_{g_{Nc-1}}^{f_{Nc}} Q_{g_{Nc}}^{f_{Nc+1}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}$$

$$+ \frac{1}{N_c! (N_c - 2)! (N_f - N_c)! (N_f - N_c + 2)!} \times Q_{u_{Nc-2}}^{f_{Nc-1}} Q_{g_{Nc-1}}^{f_{Nc}} Q_{g_{Nc}}^{f_{Nc+2}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}$$

$$\times \bar{\omega}_{\beta y_{Nc-1}} \bar{\omega}_{\gamma y_{Nc}} \tilde{f}_{\alpha_1 \nu_{1f}} \bar{\psi}_{\alpha_{1f} \nu_{1f}} \omega_{\alpha_{1f} \nu_{1f}} \tilde{f}_{\alpha_{1f} \nu_{1f}}} \right].$$

(B.1)

The fields on the D-instanton correspond to the moduli of the instanton, and $\omega$’s are the gauge direction and the size of the instanton. For example, the instanton solution of the gauge field $A_\mu$ is written in terms of $\omega$ as

$$A_\mu(x) = \omega_\alpha^\mu (\bar{\sigma}_{\mu \nu})^{\hat{\theta}} \bar{\phi}^\beta \frac{(x - x_0)^\nu}{(x - x_0)^2 \left[ (x - x_0)^2 + \frac{1}{2} \bar{\omega}_{\gamma \mu} \omega^{\gamma \nu} \right]}$$

(B.2)

where, we have taken the singular gauge, and $x_0^\mu = a^\mu$ is the position of the instanton.

C Constrained Gaussian integral

In this appendix, we describe the calculation of the constraint Gaussian integral. Let us define $F$ by

$$F = \int d\omega \ d\bar{\omega} \delta^{(3)}(\bar{\omega} \tau^c \omega) e^{-\bar{\omega}_\alpha A^\alpha}$$

(C.1)

where $A$ is an $N \times N$ matrix. Writing the $\delta$-function in terms of the contour integral, we obtain

$$F = \int d\omega \ d\bar{\omega} \ d^3k \ e^{-\bar{\omega}_\alpha A^\alpha + i k_\alpha \bar{\omega}_\alpha \tau^\alpha \omega^\beta} = \int d\Omega \ d\bar{\Omega} \ d^3k \ e^{-\bar{\Omega} \bar{\Omega}}$$

(C.2)
where $\tilde{A}$ is the $2N \times 2N$ matrix which is defined as

$$
\tilde{A} = \begin{pmatrix}
A - ik_3 & -k_1 - k_2 \\
-ik_1 + k_2 & A + ik_3
\end{pmatrix},
$$

(C.3)

and $\Omega$ and $\bar{\Omega}$ are defined as

$$
\Omega = \begin{pmatrix}
\omega^1 \\
\omega^2
\end{pmatrix},
\bar{\Omega} = \begin{pmatrix}
\bar{\omega}_1 \\
\bar{\omega}_2
\end{pmatrix}.
$$

(C.4)

Then integrating over $\Omega$ and $\bar{\Omega}$, we have

$$
F = \int d^3k \frac{1}{\det \tilde{A}} = \int dk \frac{1}{\det(A^2 + k^2)}
$$

(C.5)

where we used the following formula:

$$
\det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \det(AD - BC)
$$

(C.6)

if $A, B, C, D$ mutually commute. Diagonalizing $A$ into $\text{diag}(a_1, a_2 \ldots a_N)$ and picking up all residues in the upper half plane of $k$, we obtain

$$
F = \int dk k^2 \prod_{m=1}^{N} (a_m^2 + k^2) = \sum_{n=1}^{N} \frac{a_n}{\prod_{m \neq n}(a_m^2 - a_n^2)}
$$

(C.7)

This can be written in terms of $A$ as

$$
F = \text{tr} \frac{A}{\sum_{r=0}^{N-1} A^{2r} C^{(r)}}
$$

(C.8)

where

$$
C^{(r)}_{ij} = \frac{1}{(N - r - 1)! r!} \varepsilon_{jn_1 \ldots n_r l_1 \ldots l_{N-(r+1)}} \varepsilon_{j_1 n_1 \ldots n_r k_1 \ldots k_{N-(r+1)}} (A^2)^{k_1}_{l_1} \cdots (A^2)^{k_{N-(r+1)}}_{l_{N-(r+1)}}
$$

(C.9)

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