ON THE INTEGRAL SYSTEMS WITH NEGATIVE EXPOUNTS

YUTIAN LEI
Jiangsu Key Laboratory for NSLSCS
School of Mathematical Sciences
Nanjing Normal University, Nanjing, 210023, China

(Communicated by Congming Li)

Abstract. This paper is concerned with the integral system
\begin{align*}
\begin{cases}
  u(x) = \int_{R^n} \frac{|x-y|^\lambda dy}{v^q(y)}, & u > 0 \text{ in } R^n, \\
  v(x) = \int_{R^n} \frac{|x-y|^\lambda dy}{u^p(y)}, & v > 0 \text{ in } R^n,
\end{cases}
\end{align*}
where \( n \geq 1, \ p, q, \lambda \neq 0 \). Such an integral system appears in the study of the conformal geometry. We obtain several necessary conditions for the existence of the \( C^1 \) positive entire solutions, particularly including the critical condition
\[
\frac{1}{p-1} + \frac{1}{q-1} = \frac{\lambda}{n},
\]
which is the necessary and sufficient condition for the invariant of the system and some energy functionals under the scaling transformation. The necessary condition \( \frac{1}{p-1} + \frac{1}{q-1} = \frac{\lambda}{n} \) can be relaxed to another weaker one \( \min\{p, q\} > \frac{n+\lambda}{\lambda} \) for the system with double bounded coefficients. In addition, we classify the radial solutions in the case of \( p = q \) as the form
\[ u(x) = v(x) = a(b^2 + |x - x_0|^2)^{\frac{\lambda}{2}} \]
with \( a, b > 0 \) and \( x_0 \in R^n \). Finally, we also deduce some analogous necessary conditions of existence for the weighted system.

1. Introduction. In this paper, we study the existence and the asymptotic behavior for the integral system
\begin{align*}
\begin{cases}
  u(x) = \int_{R^n} \frac{|x-y|^\lambda dy}{v^q(y)}, & u > 0 \text{ in } R^n, \\
  v(x) = \int_{R^n} \frac{|x-y|^\lambda dy}{u^p(y)}, & v > 0 \text{ in } R^n,
\end{cases}
\end{align*}
where \( n \geq 1, \ p, q > 0 \) and \( \lambda \neq 0 \).

When \( p = q \) and \( u \equiv v \), (1.1) becomes
\[ u(x) = \int_{R^n} \frac{|x-y|^\lambda dy}{u^p(y)}, \quad u > 0 \quad \text{in} \quad R^n. \]

This equation is related to the study of the conformal geometry and the nonlinear elliptic PDEs with the negative exponent (cf. [2], [9], [10], [12], [21], [22] and [25]).

2010 Mathematics Subject Classification. 45E10, 45G15, 45M05, 45M20.
Key words and phrases. Singular integral equation, conformal invariant, asymptotic behavior, classification of radial solutions.
This work was supported partly by NSF of China (No.11171158, No.11471164) and the Natural Science Foundation of Jiangsu (No. BK2012846).
A problem posed by Li [18] is whether or not does (1.2) admit any positive (regular) solutions for all \( n \geq 1, \lambda > 0 \) and \( p > (2n + \lambda)/\lambda \). Xu [26] gave a positive answer and obtained the following results.

(R1) Let \( \lambda > 0 \). Eq. (1.2) has a \( C^1 \) positive solution if and only if \( 2n + \lambda = p\lambda \).

Now, \( u \) is given by
\[
u(x) = a(b^2 + |x - x_0|^2)^{\lambda/2}
\]  
with \( a, b > 0 \) and \( x_0 \in \mathbb{R}^n \).

(R2) If \( \lambda \in (-n, 0) \), then (1.2) has no \( C^1 \) positive solution.

An analogous integral system is
\[
\begin{cases}
  u(x) = \int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x - y|^{n-\alpha}}, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\
  v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x - y|^{n-\alpha}}, & v > 0 \quad \text{in} \quad \mathbb{R}^n.
\end{cases}
\]  

Here \( p, q > 0 \) and \( \alpha \in (0, n) \). When \( p = q \) and \( u \equiv v \), (1.4) is reduced to a single equation
\[
u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x - y|^{n-\alpha}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n.
\]  

Both (1.4) and (1.5) are associated with the study of the sharp constants in the Hardy-Littlewood-Sobolev inequality (cf. [7], [18] and [20]). In 1981, Gidas and Spruck [11] proved that (1.5) with \( \alpha = 2 \) has no positive solution in the subcritical condition \( p < \frac{n+2}{n-2} \). This Liouville type result for (1.5) with \( \alpha \in (0, n) \) was also obtained in [27]. If \( p \) is the critical exponent \( \frac{n+\alpha}{n-\alpha} \), according to the classification results in [3], [7] and [18], \( u \) is given by
\[
u(x) = a(b^2 + |x - x_0|^2)^{(\alpha-n)/2}
\]  
with \( a, b > 0 \) and \( x_0 \in \mathbb{R}^n \). Different from (R1), (1.5) still has positive solutions in the supercritical case.

For the system (1.4), there exists positive solution when \( p, q \) satisfy the critical condition (cf. [20])
\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\alpha}{n}.
\]  

Furthermore, (1.4) has positive solution in \( L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n) \) if and only if the critical condition (1.6) holds (cf. [15]). According to the results in [24], the positive solutions \( u, v \) are bounded and decay with the fast rates as long as \( (u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n) \). In the subcritical case
\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-\alpha}{n},
\]  
the Hardy-Littlewood-Sobolev conjecture states that (1.4) has no positive solution (cf. [1] and [5]). When \( \alpha = 2 \) and \( n \leq 4 \), this problem (Lane-Emden conjecture) was resolved (cf. [23]). The corresponding results on the discrete Hardy-Littlewood-Sobolev inequality can be seen in [8].

In this paper, we also consider the problems mentioned above for (1.1). The results listed as follows will be proved in section 2.

**Theorem 1.1.** Let \( u, v \) be the positive entire solutions of (1.1) in \( C^1(\mathbb{R}^n) \). If \( \lambda > 0 \) and \( \max\{p, q\} > (n + \lambda)/\lambda \), then

(R1) \( u^{-1}, v^{-1} \in L^s(\mathbb{R}^n) \) for all \( s > n/\lambda \).
(Rt2) $p, q$ must satisfy $\min\{p, q\} > (n + \lambda)/\lambda$ and
\[
\frac{1}{p - 1} + \frac{1}{q - 1} = \frac{\lambda}{n}.
\]
(1.7)

(Rt3) $\lim_{|x| \to \infty} \frac{u(x)}{|x|^{n}} = \|v^{-1}\|^q$; $\lim_{|x| \to \infty} \frac{v(x)}{|x|^{n}} = \|u^{-1}\|^p$.

Two interesting problems are posed naturally. One is whether (1.1) with $\lambda > 0$ has no positive solution in the case of $\max\{p, q\} \leq (n + \lambda)/\lambda$. The other is whether the necessary condition (1.7) is still the sufficient condition for the existence of positive solutions.

Eq. (1.7) can be viewed as the necessary and sufficient condition ensuring the conformal invariant. Namely, the system (1.1) and the energy functionals $\|u^{-1}\|_{p-1}$ and $\|v^{-1}\|_{q-1}$ are invariant under the scaling transformation (see section 3). Here we deduce (1.7) by using the integral estimates instead of the ideas in [26]. Certainly those ideas in [26] also work and the readers can refer to [13].

Clearly, if $p = q$ in (1.7), then $p$ is the critical exponent $(2n + \lambda)/\lambda$ in (R1).

When $p = q$, we will prove $u \equiv v$ in section 4 by the same ideas in [4]. Thus, applying the result (R1), we can also classify the positive $C^1$ entire solutions as the form (1.3).

**Theorem 1.2.** Let $\lambda > 0$. Assume the positive entire solutions $u, v \in C^1(R^n)$ of (1.1) are radially symmetric about some point $x_0 \in R^n$. If $p = q$, then
\[
u(x) = \frac{a}{b^2 + |x - x_0|^2}^{1/2}
\]
with $a, b > 0$ and $x_0 \in R^n$.

We expect to remove the assumption of radial property. However, it seems difficult to verify the radial property for the positive solutions of (1.1). For the system (1.4) with positive exponents, the radial symmetry of the positive solutions was proved by the method of moving planes as long as $(u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ (cf. [6]). According to Theorem 1.1, the pair of positive solutions $(u, v)$ of (1.1) belongs to $L^{1-p}(R^n) \times L^{1-q}(R^n)$ naturally. Although the Hardy-Littlewood-Sobolev inequality does not work for $\lambda, p, q > 0$ as in [6], we conjecture that $u, v$ are still radially symmetric by the ideas in [19].

Theorem 1.1 shows that (1.7) is the necessary condition of the existence of positive $C^1$ entire solutions. For the following system
\[
\begin{align*}
u(x) &= c_1(x) \int_{R^n} \frac{|x - y|^{\lambda}dy}{v^q(y)}, \quad u > 0 \text{ in } R^n, \\
v(x) &= c_2(x) \int_{R^n} \frac{|x - y|^{\lambda}dy}{u^p(y)}, \quad v > 0 \text{ in } R^n,
\end{align*}
\]
the necessary condition (1.7) can be relaxed to another weaker one $\min\{p, q\} > (n + \lambda)/\lambda$. Here $c_1(x)$ and $c_2(x)$ are double bounded functions. Namely, there exists $C > 0$ such that
\[
\frac{1}{C} \leq c_1(x), c_2(x) \leq C, \quad \forall x \in R^n.
\]

In section 5, we will prove the following results.

**Theorem 1.3.** (Rt4) When $\lambda > 0$ and $\min\{p, q\} > (n + \lambda)/\lambda$, there exists a radial solution of (1.8) for some double bounded functions $c_1(x)$ and $c_2(x)$.
Critical Case

(u,v)

Integrability intervals of positive solutions of the weighted Hardy-Littlewood-Sobolev inequality (cf. [20]). The optimal

where

When

\( n \in \mathbb{N} \)

were established in [14]. Based on this result, the authors of [16] obtained the fast asymptotic rates of \( u,v \) of (1.1), then \( \bar{u}, \bar{v} \) solve the following weighted system

\[
\begin{align*}
  u(x) &= \int_{\mathbb{R}^n} \frac{|x-y|^\lambda}{|x|^\alpha |y|^{\beta}} \, dy, \quad u > 0 \text{ in } \mathbb{R}^n, \\
  v(x) &= \int_{\mathbb{R}^n} \frac{|x-y|^\lambda}{|x|^\beta |y|^{\alpha}} \, dy, \quad v > 0 \text{ in } \mathbb{R}^n,
\end{align*}
\]

(1.9)

where \( n \geq 1, \lambda, p, q > 0 \) and \( \max\{\alpha, \beta\} < n \).

Another interesting problem is whether (1.8) with \( \lambda > 0 \) has no positive solution for any double bounded \( c_1(x), c_2(x) \) in the case of \( \min\{p, q\} \leq (n + \lambda)/\lambda \).

After completing this work, the author learned that Jiankai Xu recently completed a preprint where he proves Theorem 1.2 without the assumption of radial constraint by the ideas in [19] and [26].

Remark. After completing this work, the author learned that Jiankai Xu recently completed a preprint where he proves Theorem 1.2 without the assumption of radial constraint by the ideas in [19] and [26].

2. Necessary conditions of existence. Assume \( u, v \) are the positive \( C^1 \) entire solutions of (1.1). We will give the proof of Theorem 1.1.

Before verifying (Rt1), we need the following result.
Proposition 2.1. If \( \lambda > 0 \) and \( \max\{p, q\} > (n + \lambda)/\lambda \), then \( u, v \) are increasing with the fast rate \( \lambda \):
\[
\begin{align*}
u(x), v(x) & \simeq |x|^{\lambda} \\
\text{when } |x| & \to \infty.
\end{align*}
\]

Proof. Without loss of generality, we suppose \( p \geq q \). By the condition of Proposition 2.1, we have
\[
p > (n + \lambda)/\lambda. \tag{2.1}
\]

When \( |x| > 1 \), there holds
\[
u(x) \geq \int_{B_1(0)} \frac{|x - y|^{\lambda} dy}{u^q(y)} \geq c \left( \min_{B_1(0)} u^{-q} \right) \int_{B_1(0)} |x - y|^{\lambda} dy \geq c|x|^{\lambda}. \tag{2.2}
\]

Similarly, we also obtain
\[
v(x) \geq \int_{B_1(0)} \frac{|x - y|^{\lambda} dy}{u^p(y)} \geq c \left( \min_{B_1(0)} u^{-p} \right)|x|^{\lambda} \geq c|x|^{\lambda}. \tag{2.3}
\]

and
\[
v_1(x) := \int_{B_1(0)} \frac{|x - y|^{\lambda} dy}{u^p(y)} \leq C \left( \max_{B_1(0)} u^{-p} \right)|x|^{\lambda} \leq C|x|^{\lambda}. \tag{2.4}
\]

When \( y \in B_{2|x|}(0), |x - y| \leq |x| + |y| \leq 3|x| \). Therefore, by (2.1) and (2.2) we get
\[
v_2(x) := \int_{B_{2|x|}(0) \setminus B_1(0)} \frac{|x - y|^{\lambda} dy}{u^p(y)} \leq C|x|^{\lambda} \int_{B_{2|x|}(0) \setminus B_1(0)} |y|^{-p\lambda} dy \leq C|x|^{\lambda}. \tag{2.5}
\]

When \( y \in R^n \setminus B_{2|x|}(0), |x - y| \leq |x| + |y| \leq 3|y|/2 \). Therefore, by (2.1) and (2.2) we get
\[
v_3(x) := \int_{R^n \setminus B_{2|x|}(0)} \frac{|x - y|^{\lambda} dy}{u^p(y)} \leq C \int_{R^n \setminus B_{2|x|}(0)} |y|^{-p\lambda} dy \leq C|x|^{n+\lambda-p\lambda}. \tag{2.6}
\]

Noting \( n + \lambda - p\lambda < 0 \) which is implied by (2.1), from the estimates of (2.4)-(2.6), it follows
\[
v(x) \leq C|x|^{\lambda}. \tag{2.7}
\]

Combining this result with (2.3) we have
\[
v(x) \simeq |x|^{\lambda}
\]
when \( |x| \to \infty \).

When \( y \in R^n \setminus B_{2|x|}(0), |x - y| \geq |y| - |x| \geq |y|/2 \). By (2.7) we obtain
\[
u(x) \geq \int_{R^n \setminus B_{2|x|}(0)} \frac{|x - y|^{\lambda} dy}{u^q(y)} \geq c \int_{2|x|}^\infty r^{n+\lambda-q\lambda} dr / r. \tag{2.8}
\]

Since \( u \) is an entire solution, we also see from the result above
\[
n + \lambda - q\lambda < 0.
\]

Therefore, we can estimate the upper bound of \( u \) by the same process for \( v \), and still obtain
\[
u(x) \simeq |x|^{\lambda}
\]
when \(|x| \to \infty\). Thus, Proposition 2.1 is proved. \(\square\)

**Proof of (Rt1).** By Proposition 2.1, there exists \(R > 0\) sufficiently large, such that
\[ u(x) \simeq |x|^\lambda, \quad \text{for } |x| > R. \]

Since \(u\) is a positive \(C^1\) solution, we have
\[
\int_{\mathbb{R}^n} u^{-s}(x) dx = \int_{B_N(0)} u^{-s}(x) dx + \int_{\mathbb{R}^n \setminus B_N(0)} u^{-s}(x) dx \\
\leq C + C \int_{R}^{\infty} r^{n-\lambda} dr.
\]

Therefore, \(u^{-1} \in L^s(\mathbb{R}^n)\) as long as \(s > n/\lambda\).

Similarly, \(v\) has the same consequence. (Rt1) is proved. \(\square\)

**Proof of (Rt2).** From (2.1) and (2.8), it follows that
\[
\min\{p-1, q-1\} > n/\lambda.
\]

Thus, (Rt1) implies
\[
(u^{-1}, v^{-1}) \in L^{p-1}(\mathbb{R}^n) \times L^{q-1}(\mathbb{R}^n).
\]

By the definition of the improper integral, we have
\[ 0 = \lim_{\rho \to \infty} \int_{B_{2\rho} \setminus B_{\rho}} u^{1-p} dx = \lim_{\rho \to \infty} \int_{\partial B_r} u^{1-p} dx \frac{dr}{r}, \]
and hence
\[ \inf_{[\rho, 2\rho]} r \int_{\partial B_r} u^{1-p} dx \to 0 \]
when \(\rho \to \infty\). This implies that there exists \(R = R_j \to \infty\) such that
\[ R \int_{\partial B_R(0)} u^{1-p} dx \to 0. \]

Therefore, integrating by parts we get
\[
\int_{\mathbb{R}^n} x \cdot \nabla u^{1-p} dx = -n \int_{\mathbb{R}^n} u^{1-p} dx, \tag{2.10}
\]
and hence \((x \cdot \nabla u^{1-p}) \in L^1(\mathbb{R}^n)\). Using (2.10) we have
\[
\int_{\mathbb{R}^n} x \cdot \nabla u(x) dx = \frac{1}{1-p} \int_{\mathbb{R}^n} x \cdot \nabla u^{1-p} dx = \frac{n}{p-1} \int_{\mathbb{R}^n} u^{1-p} dx. \tag{2.11}
\]

Similarly, we also have \((x \cdot \nabla v^{1-q}) \in L^1(\mathbb{R}^n)\). By (1.1) and the Fubini theorem,
\[
\frac{q}{q-1} \int_{\mathbb{R}^n} z \cdot \nabla v^{1-q}(z) dz = \int_{\mathbb{R}^n} (z \cdot \nabla v^{-q}(z)) v(z) dz \\
= \int_{\mathbb{R}^n} (z \cdot \nabla v^{-q}(z)) \int_{\mathbb{R}^n} \frac{|x-z|^\lambda dx}{wp(x)} dz \\
= \int_{\mathbb{R}^n} u^{-p}(x) \int_{\mathbb{R}^n} |x-z|^\lambda (z \cdot \nabla v^{-q}(z)) dz dz. \tag{2.12}
\]

On the other hand, for \(\mu > 0\),
\[
u(\mu x) = \int_{\mathbb{R}^n} \frac{|\mu x - y|^\lambda dy}{v^q(y)} = \mu^{n+\lambda} \int_{\mathbb{R}^n} |x-z|^\lambda dz \frac{dz}{v^q(\mu z)}
\]
Differentiate it with respect to $\mu$, and then let $\mu = 1$. Thus,

$$x \cdot \nabla u(x) = (n + \lambda)u(x) + \int_{R^n} |x - z|^\lambda (z \cdot \nabla v^{-q}(z))dz.$$  

Multiplying $u^{-p}$ and integrating on $R^n$, and using (2.12), we obtain

$$\int_{R^n} x \cdot \nabla u(x)u^{-p}(x)dx = (n + \lambda)\int_{R^n} u^{-p}(x)dx + \frac{q}{q - 1} \int_{R^n} z \cdot \nabla v^{-q}(z)dz. \quad (2.13)$$

Similar to the derivation of (2.10), we also have

$$\int_{R^n} x \cdot \nabla v^{-q}(x)dx = -n \int_{R^n} v^{-q}(x)dx.$$  

Inserting this result and (2.11) into (2.13) yields

$$\frac{n}{p - 1} \int_{R^n} u^{-p}dx = (n + \lambda) \int_{R^n} u^{-p}dx - \frac{qn}{q - 1} \int_{R^n} v^{-q}dx. \quad (2.14)$$

Applying (1.1) and the Fubini theorem, we also have

$$\int_{R^n} u^{-p} dx = \int_{R^n} u^{-p} \int_{R^n} \frac{|x - y|^\lambda dy}{v^q(y)} dx = \int_{R^n} v^{-q} \int_{R^n} \frac{|x - y|^\lambda dx}{u^p(y)} dy = \int_{R^n} v^{-q} dy.$$  

Substituting this result into (2.14), we get

$$\frac{n}{p - 1} + \frac{qn}{q - 1} = n + \lambda,$$

which leads to the critical condition (1.7). (Rt2) is proved.

Finally, we prove (Rt3).

**Proposition 2.2.** Assume $u, v$ are positive $C^1$ entire solutions of (1.1). If $\lambda > 0$ and $\max\{p, q\} > (n + \lambda)/\lambda$, then

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^\lambda} = \int_{R^n} \frac{dy}{v^q(y)}; \quad \lim_{|x| \to \infty} \frac{v(x)}{|x|^\lambda} = \int_{R^n} \frac{dy}{u^p(y)}.$$

**Proof.** By virtue of (2.1) and (2.8), we have $p, q > n/\lambda$. Thus, (Rt1) implies

$$v^{-1} \in L^p(R^n), \quad v^{-1} \in L^q(R^n).$$

When $y \in B_R$ for $R > 0$, $\frac{|x - y|^\lambda}{|x|^\lambda} - 1 \leq 2$ for large $|x|$. Noting $v^{-1} \in L^q(R^n)$, we have

$$\lim_{R \to \infty} \lim_{|x| \to \infty} \int_{B_R} \frac{|x - y|^\lambda}{|x|^\lambda} - 1 \frac{dy}{v^q(y)} = 0.$$

When $y \in B_{2|x|}(0)$, $|x - y| \leq 3|x|$. Thus from $v^{-1} \in L^q(R^n)$, it follows

$$\lim_{R \to \infty} \lim_{|x| \to \infty} \int_{B_{2|x|}(0) \setminus B_{R}} \frac{|x - y|^\lambda}{|x|^\lambda} \frac{dy}{v^q(y)} = 0.$$
When \( y \in \mathbb{R}^n \setminus B_{2|x|}(0), |x - y| \leq 3|y|/2 \). Thus, by (2.3),

\[
\lim_{|x| \to \infty} \int_{\mathbb{R}^n \setminus B_{2|x|}(0)} \frac{|x - y|}{|x|^\lambda} \frac{dy}{v^q(y)} \\
\leq c \lim_{|x| \to \infty} |x|^{-\lambda} \int_{|x|}^\infty r^{n+\lambda-q\lambda} \frac{dr}{r} \\
= c \lim_{|x| \to \infty} |x|^{n-q\lambda} = 0.
\]

Combining these three estimates, we see that

\[
\lim_{|x| \to \infty} \frac{u(x)}{|x|^{\lambda}} = \int_{\mathbb{R}^n} \frac{dy}{v^q(y)}
\]

Similarly, we also have

\[
\lim_{|x| \to \infty} \frac{v(x)}{|x|^{\lambda}} = \int_{\mathbb{R}^n} \frac{dy}{u^q(y)}.
\]

Proposition 2.2 is proved. \( \square \)

3. Critical condition and conformal invariant. Clearly, the system (1.1) is invariant under the translation transformation. Namely, if we write \( \tilde{u}(x) = u(x+x_0) \) and \( \tilde{v}(x) = v(x+x_0) \) for any \( x_0 \in \mathbb{R}^n \), then \( \tilde{u}, \tilde{v} \) still solve (1.1).

In fact,

\[
\tilde{u}(x) = \int_{\mathbb{R}^n} \frac{|x + x_0 - y|^\lambda}{v^q(y)} dy = \int_{\mathbb{R}^n} \frac{|x - z|^\lambda}{v^q(z + x_0)} dz = \int_{\mathbb{R}^n} \frac{|x - z|^\lambda}{v^q(z)} dz.
\]

Similarly, \( \tilde{v}(x) = \int_{\mathbb{R}^n} \frac{|x - z|^\lambda}{w^q(z)} dz. \)

Next, the system (1.1) is invariant under the rotation transformation. Namely, if we write

\[
u_\theta(x) = u(\tilde{x}), \quad v_\theta(x) = v(\tilde{x}),
\]

where \( \tilde{x} = (x_1, \cdots, \tilde{x}_j, \cdots, \tilde{x}_k, \cdots, x_n) \) with

\[
\tilde{x}_j = x_j \cos \theta + x_k \sin \theta, \quad \tilde{x}_k = -x_j \sin \theta + x_k \cos \theta,
\]

then \( \tilde{u}, \tilde{v} \) still solve (1.1).

In fact, if we set \( y = \tilde{z} \), then the modulus of the Jacobi determinant is +1. Therefore,

\[
u_\theta(x) = \int_{\mathbb{R}^n} \frac{|\tilde{x} - y|^\lambda}{v^q(y)} dy = \int_{\mathbb{R}^n} \frac{|x - z|^\lambda}{v^q(z)} dz.
\]

Similarly, \( v_\theta(x) = \int_{\mathbb{R}^n} \frac{|x - z|^\lambda}{w^q(z)} dz. \)

By the same argument, we can also see the system is invariant under the reflection transformation.

In the following, we show that (1.7) is the necessary and sufficient condition of the fact that the system (1.1) and the energy are invariant under the scaling transformation.

For \( r > 0 \), set the scaling of \( u, v \) as follows:

\[
u_r(x) = r^{\sigma_1} u(rx), \quad v_r(x) = r^{\sigma_2} v(rx),
\]

where \( \sigma_1, \sigma_2 \neq 0 \) will be determined later to ensure that \( u_r, v_r \) still solve (1.1).

Clearly,

\[
u_r(x) = r^{\sigma_1} \int_{\mathbb{R}^n} \frac{|rx - y|^\lambda}{v^q(y)} dy = r^{n+\lambda+\sigma_1+q\sigma_2} \int_{\mathbb{R}^n} \frac{|x - z|^\lambda}{v^q(z)} dz.
\]
and similarly,

\[ v_r(x) = r^{n+\lambda+\sigma_2+p\sigma_1} \int_{R^n} \frac{|x-z|^\lambda}{w_r(z)} \, dz. \]

These imply \( n + \lambda + \sigma_1 + q\sigma_2 = 0 \) and \( n + \lambda + \sigma_2 + p\sigma_1 = 0 \). Thus,

\[ \sigma_1 = \frac{(q-1)(n+\lambda)}{1-pq}, \quad \sigma_2 = \frac{(p-1)(n+\lambda)}{1-pq}. \]

If (1.7) holds, by noting

\[ pq - 1 = (p - 1)(q - 1) + (p - 1) + (q - 1), \]

we get

\[ \sigma_1 = \frac{n}{1-p}, \quad \sigma_2 = \frac{n}{1-q}. \quad (3.1) \]

Thus, we claim that the norms of \( L^{p-1}(R^n) \) and \( L^{q-1}(R^n) \) of \( u_r^{-1} \) and \( v_r^{-1} \) are invariant. In fact,

\[ \int_{R^n} u_r^{1-p}(x) \, dx = r^{\sigma_1(1-p)-n} \int_{R^n} u^{1-p}(z) \, dz, \]

\[ \int_{R^n} v_r^{1-q}(x) \, dx = r^{\sigma_2(1-q)-n} \int_{R^n} v^{1-q}(z) \, dz. \]

From (3.1) we see our claim. Thus, the sufficiency is proved.

The necessity can also be proved by the same calculation.

Finally, we should point out that (1.1) is not invariant under the inversion transformation

\[ \bar{u}(x) = |x|^{\lambda_1} u\left(\frac{x}{|x|^2}\right), \quad \bar{v}(x) = |x|^{\lambda_2} v\left(\frac{x}{|x|^2}\right) \]

except \( \lambda_1 = \lambda_2 = \lambda \) and \( p = q = (2n+\lambda)/\lambda \). If the system is replaced by the weighted one (1.9), then it is invariant under the inversion transformation

\[ \tilde{u}(x) = |x|^{\frac{2n}{p-1}} u\left(\frac{x}{|x|^2}\right), \quad \tilde{v}(x) = |x|^{\frac{2n}{q-1}} v\left(\frac{x}{|x|^2}\right) \quad (3.2) \]

with the new \( \tilde{\alpha}, \tilde{\beta} \) which are dual to the origin ones. In fact,

\[ \tilde{u}(x) = |x|^{\frac{2n}{p-1}} \int_{R^n} \frac{|x|^2 - y|^\lambda}{|x|^\alpha v^q(y)|y|^\beta} \, dy \]

\[ = |x|^{\frac{2n}{p-1}} \int_{R^n} \frac{|x-z|^\lambda}{|x|^\tilde{\alpha} v^q(\frac{z}{|z|^2}) |z|^\tilde{\beta} + 2n} \, dz \]

\[ = \int_{R^n} \frac{|x-z|^\lambda}{|x|^\tilde{\alpha} \tilde{v}(z) |z|^\tilde{\beta}}, \]

where \( \tilde{\alpha} = \lambda - \alpha - \frac{2n}{p-1}, \tilde{\beta} = \lambda - \beta - \frac{2n}{q-1} \). Similarly,

\[ \tilde{v}(x) = \int_{R^n} \frac{|x-z|^\lambda}{|x|^\tilde{\alpha} \tilde{u}(z) |z|^\tilde{\beta}}, \]

However, the invariant of the weighted system is clearly absent under the translation transformation.
4. Classification of radial solutions when \( p=q \). In this section, we will prove Theorem 1.2. Assume \( u, v \) solve (1.1). According to Theorem 1.1, (1.7) holds. In view of \( p = q \), we have

\[
p = q = \frac{2n + \lambda}{\lambda}.
\]

We can see later that this exponent ensures us to use the properties of the conformal invariant.

Define

\[
\begin{align*}
  u(\infty) &= \lim_{|x| \to \infty} \frac{u(x)}{|x|^\lambda}; & v(\infty) &= \lim_{|x| \to \infty} \frac{v(x)}{|x|^\lambda}.
\end{align*}
\]

Clearly, for solutions \( u, v \) in Proposition 2.2, \( u(\infty) = \|v^{-1}\|^q_q \) and \( v(\infty) = \|u^{-1}\|^p_p \).

For any \( a \in \mathbb{R}^n \), let \( r > 0 \) satisfy

\[
r^{-\lambda} = \frac{u(\infty)}{u(a)}. \tag{4.2}
\]

Our main aim is to prove

\[
u(a + rx) = |x|^\lambda u(a + \frac{rx}{|x|^2}). \tag{4.3}
\]

Proof of (4.3). At first, for \( a = 0 \), consider

\[
u_t(x) = t^{\frac{\lambda}{2}} u(tx), \quad \nu_l(x) = t^{\frac{\lambda}{2}} v(tx).
\]

Thus,

\[
\begin{align*}
u_r(\infty) &= \lim_{|x| \to \infty} |x|^{-\lambda} u_r(x) \\
&= \lim_{|x| \to \infty} |x|^{-\lambda} r^{-\lambda} u(rx) \\
&= r^{\frac{\lambda}{2}} \lim_{|y| \to \infty} |y|^{-\lambda} u(y) \\
&= r^{\frac{\lambda}{2}} u(\infty) \\
&= r^{\frac{\lambda}{2}} u(0), \quad (\text{by (4.2)}) \\
&= u_r(0).
\end{align*}
\]

Let \( e \) be an arbitrary unit vector. Take the Kelvin type transforms

\[
\begin{align*}
\tilde{u}(x) &= |x|^\lambda u_r\left(\frac{x}{|x|^2} - e\right), \\
\tilde{v}(x) &= |x|^\lambda u_r\left(\frac{x}{|x|^2} - e\right).
\end{align*} \tag{4.5}
\]

Thus,

\[
\begin{align*}
\tilde{u}(0) &= \lim_{|x| \to 0} |x|^\lambda u_r\left(\frac{x}{|x|^2} - e\right) \\
&= r^{\frac{\lambda}{2}} \lim_{|x| \to 0} |x|^\lambda u\left(\frac{rx}{|r|^2} - re\right) \\
&= r^{\frac{\lambda}{2}} \lim_{|y| \to \infty} |y|^{-\lambda} u(ry - re) \\
&= \lim_{|y| \to \infty} |y|^{-\lambda} u_r(y - e) \\
&= u_r(\infty).
\end{align*}
\]

This result, together with (4.4) and (4.5), implies

\[
\tilde{u}(0) = u_r(\infty) = u_r(0) = \tilde{u}(e). \tag{4.6}
\]
Claim 1. Both \( \bar{u}^{-1} \) and \( \bar{v}^{-1} \) belong to \( L^{p-1}(R^n) \).

From (4.5), we see that
\[
\bar{u}(x) = |x|^{\lambda r} \frac{r_x}{|x|^2} u(r_x - re).
\]

Noting \( p - 1 = \frac{2n}{\lambda} \) which is implied by (4.1), we deduce that
\[
\int_{R^n} |x|^{2n} - u^{-p} \frac{r_x}{|x|^2} - re)dx \]
\[
= \int_{R^n} u^{-p} dy, \quad (\text{where} \quad y = \frac{r_x}{|x|^2} - re).
\]

This means \( \|\bar{u}^{-1}\|_{p-1} = \|u^{-1}\|_{p-1} < \infty \). Similarly, \( \bar{v} \) has the same property.

Claim 2. The pair \((\bar{u}, \bar{v})\) solves (1.1).

In view of (4.5) and (1.1), we obtain by using \( q = (2n + \lambda)/\lambda \) that
\[
\bar{u}(x) = |x|^{\lambda r} \frac{r_x}{|x|^2} \int_{R^n} \frac{r_x}{|x|^2} - y|^{\lambda - q} dy = \int_{R^n} \frac{|x - z|^{\lambda} dz}{\bar{v}^{p}(z)}.
\]

Similarly, we also have
\[
\bar{v}(x) = \int_{R^n} \frac{|x - z|^{\lambda} dz}{\bar{u}^{p}(z)}.
\]

These results show that \( \bar{u} \) and \( \bar{v} \) are solutions of (1.1).

Claims 1 and 2 imply that \( \bar{u} \) and \( \bar{v} \) are also radially symmetric and decreasing about some point \( x_* \in R^n \). By virtue of (4.6), we get \( x_* = \frac{e}{2} \). Therefore,
\[
\bar{u}(\frac{1}{2} - h)e) = \bar{u}(\frac{1}{2} + h)e)
\]
for any \( h \in R \). The definition (4.5) of \( \bar{u} \) leads to
\[
\left(\frac{1}{2} - h\right) \frac{r_x}{\sqrt{r}} u(r \frac{1}{2} - h) = \left(\frac{1}{2} + h\right) \frac{r_x}{\sqrt{r}} u(r \frac{1}{2} + h).
\]

Write \( \Lambda = \frac{\frac{1}{2} - h}{\frac{1}{2} + h} \), then
\[
u(r \Lambda e) = \Lambda^\lambda u(r \frac{\Lambda}{\Lambda} e).
\]

Denote \( \Lambda e \) by \( x \). Since \( h \) and \( e \) are arbitrary, there holds
\[
u(r x) = |x|^{\lambda r} u(r \frac{r_x}{|x|^2}).
\]

By a translation, we see that (4.3) holds for any \( a \in R^n \).

Similarly, we also get
\[
u(a + r x) = |x|^{\lambda r} v(a + \frac{r_x}{|x|^2}).
\]

By the same argument above, we also deduce that
\[
u(a + s x) = |x|^{\lambda r} u(a + \frac{s_x}{|x|^2}), \quad (4.7)
\]
as long as $s > 0$ satisfies
\[
s^{-\lambda} = \frac{v(\infty)}{v(a)}. \tag{4.8}
\]

**Claim 3.** $r = s$.

Set $x = he$, where $e$ is a unit vector. Combining (4.3) with (4.7), we have
\[
h^{-\lambda}u(a + rhe) = u(a + \frac{r}{h}e),
\]
\[
h^{-\lambda}u(a + she) = u(a + \frac{s}{h}e).
\]
Therefore,
\[
h^{-\lambda}u(a + she) = u(a + r\frac{s}{h}e)
\]
\[
= \left(\frac{s}{h}\right)^{\lambda}u(a + \frac{r}{s/(rh)}e) = \left(\frac{rh}{s}\right)^{-\lambda}u(a + r^2h e).
\]
This means
\[
u(a + she) = \left(\frac{r}{s}\right)^{-\lambda}u(a + \left(\frac{r}{s}\right)^2she). \tag{4.9}
\]

Suppose $r \neq s$. Without loss of generality, assume $t := \frac{r}{s} > 1$. Denote $sh = \tau$.

Then, (4.9) becomes
\[
u(a + \tau e) = t^{-\lambda}u(a + t^2 \tau e). \tag{4.10}
\]

Take $\tau = 1$ in (4.10), we have
\[
u(a + e) = t^{-\lambda}u(a + t^2 e). \tag{4.11}
\]

Take $\tau = t^{2k}$ in (4.10), $k = 1, 2, \cdots$, then
\[
u(a + t^{2k} e) = t^{-\lambda}u(a + t^{2(k+1)} e).
\]

Combining this with (4.11), we obtain
\[
u(a + e) = t^{-k\lambda}u(a + t^{2k} e). \tag{4.12}
\]

In view of the definition of $u(\infty)$ and $t > 1$, there holds
\[
u(\infty) = \lim_{k \to \infty} t^{-2k\lambda}u(a + t^{2k} e).
\]
Therefore, letting $k \to \infty$ in (4.12), we get
\[
u(a + e) = \nu(\infty) \lim_{k \to \infty} t^{k \lambda} = \infty.
\]

It is impossible. This contradiction leads to $r = s$.

Substituting $r = s$ into (4.2) and (4.8) yields
\[
u(x) = \frac{v(\infty)}{u(\infty)} v(x) := c_0 v(x), \quad \forall x \in \mathbb{R}^n. \tag{4.13}
\]

Clearly, $c_0 > 0$. Using (1.1), for any $x \in \mathbb{R}^n$ we have
\[
(c_0 - 1) v(x) = u(x) - v(x) = \int_{\mathbb{R}^n} |x - y|^p [v^{-p}(y) - u^{-p}(y)] dy
\]
\[
= \frac{c_0^p - 1}{c_0^p} \int_{\mathbb{R}^n} |x - y|^p dy = \frac{c_0^p - 1}{c_0^p} u(x) = \frac{c_0^p - 1}{c_0^p} v(x).
\]

This implies $c_0 - 1 = \frac{c_0^p - 1}{c_0^p}$, which leads to $c_0 = 1$. Thus, from (4.13) it follows $u \equiv v$ in $\mathbb{R}^n$.

Now, (1.1) is reduced to (1.2) with (4.1). According to (R1), all the solutions are classified as the form (1.3). Theorem 1.2 is proved. \qed
5. System with double bounded coefficients. In this section, we prove Theorem 1.3.

Proof of (Rt4). For \( \theta_1, \theta_2 \neq 0 \), set
\[
u(x) = (1 + |x|^2)^{\theta_1/2}, \quad v(x) = (1 + |x|^2)^{\theta_2/2}.
\] (5.1)

When \(|x| \leq 2R\) for some large \( R > 0 \), \( v(x) \) is proportional to \( \int_{R^n} \frac{|x-y|^{\lambda}}{u^p(y)} dy \). Thus, we only consider the case of \(|x| > 2R\).

By the condition of Theorem 1.3, we see
\[p, q > n + \frac{\lambda}{\lambda}.
\] (5.2)

Clearly,
\[
\int_{R^n} \frac{|x-y|^{\lambda}}{u^p(y)} dy = \int_{B_{i}(0)} \frac{|x-y|^{\lambda}}{u^p(y)} dy + \int_{B_{2i}(0) \setminus B_{i}(0)} \frac{|x-y|^{\lambda}}{u^p(y)} dy + \int_{R^n \setminus B_{2i}(0)} \frac{|x-y|^{\lambda}}{u^p(y)} dy := I_1 + I_2 + I_3.
\]

First, there exists \( C > 0 \) such that
\[
\frac{|x|^\lambda}{C} \leq I_1 \leq C|x|^\lambda.
\] (5.3)

By (5.1), from \(|x-y| \leq 3|x|\) as \(|y| \leq 2|x|\), we get
\[
I_2 \leq C|x|^\lambda \int_1^{2|x|} r^{n-p\theta_1} \frac{dr}{r}.
\]

In addition, from \(|y|/2 \leq |x-y| \leq 3|y|/2\) as \(|y| \geq 2|x|\), we also get
\[
\frac{1}{C} \int_{2|x|}^{\infty} r^{n+\lambda-p\theta_1} \frac{dr}{r} \leq I_3 \leq C \int_{2|x|}^{\infty} r^{n+\lambda-p\theta_1} \frac{dr}{r}.
\]

In order to ensure \( I_3 < \infty\), we require
\[
\theta_1 > \frac{n+\lambda}{p},
\] (5.4)

and hence both the integral orders in \( I_2 \) and \( I_3 \) are negative:
\[n - p\theta_1 < n + \lambda - p\theta_1 < 0.
\]

Thus, combining the estimates of \( I_1, I_2, I_3 \), we see that
\[
\frac{|x|^\lambda}{C} \leq \int_{R^n} \frac{|x-y|^{\lambda}}{u^p(y)} dy \leq C|x|^\lambda.
\] (5.5)

Let \( \theta_1 = \theta_2 = \lambda \). Then (5.2) ensures that (5.4) holds. By (5.1) and (5.5), we obtain
\[
\frac{1}{C} \int_{R^n} \frac{|x-y|^{\lambda}}{u^p(y)} dy \leq v(x) \leq C \int_{R^n} \frac{|x-y|^{\lambda}}{u^p(y)} dy.
\]

Write \( K_2(x) = v(x)[\int_{R^n} \frac{|x-y|^{\lambda}}{u^p(y)}]^{-1} \). Then, \( K_2(x) \) is double bounded and
\[
v(x) = K_2(x) \int_{R^n} \frac{|x-y|^{\lambda}}{u^p(y)}.
\]
Similarly, we can also deduce that
\[ u(x) = K_1(x) \int_{R^n} \frac{|x - y|^\lambda dy}{v^q(y)} \]
where \( K_1(x) = u(x)[\int_{R^n} \frac{|x - y|^\lambda dy}{v^q(y)}]^{-1} \) is double bounded. (Rt4) is proved.

**Proof of (Rt5).** Let \( p, q > 0 \) and \( \lambda \in (-n, 0) \). We prove (Rt5) by contradiction.

Suppose that there exist \( \theta_1, \theta_2 \in R \) such that the solutions \( u, v \) of (1.8) satisfy
\[ u(x) \simeq |x|^{\theta_1}, \quad v(x) \simeq |x|^{\theta_2}. \quad (5.6) \]
Therefore, from (1.8) it follows
\[ u(x) \geq c \int_{R^n \setminus B_{1/2}(0)} \frac{|x - y|^\lambda dy}{v^q(y)} \geq c \int_{2|x|}^{\infty} r^{n+\lambda-q\theta_2} \frac{dr}{r}, \]
which implies
\[ n + \lambda - q\theta_2 < 0. \]
Similarly, we can also obtain
\[ n + \lambda - p\theta_1 < 0. \]
Noting \( \lambda \in (-n, 0) \), we see that \( \theta_1, \theta_2 > 0. \) \quad (5.7)

By the same argument of the estimates of \( I_1, I_2, I_3 \), from (5.6) we also deduce
\[ v(x) \leq C|x|^\lambda + C|x|^{n+\lambda-p\theta_1}. \]
Comparing this with (5.6), we get
\[ \theta_2 \leq \max\{\lambda, n + \lambda - p\theta_1\} < 0. \]
This contradicts with (5.7). (Rt5) is proved.

6. **Weighted system.** In this section, we prove Theorem 1.4.

**Proof of (Rt6).** **Step 1.** Without loss of generality, we suppose \( p(\lambda - \alpha) + \alpha \geq q(\lambda - \beta) + \beta \). By the condition of Theorem 1.4, we have
\[ p(\lambda - \alpha) > n + \lambda - \alpha. \] \quad (6.1)

When \( |x| \gg 1 \), there holds
\[ u(x) \geq c \frac{|x|^{\alpha}}{|x|^\beta} \int_{B_1(0) \setminus B_{1/2}(0)} \frac{|x - y|^\lambda dy}{v^q(y)|y|^\beta} \geq c|x|^{\lambda - \alpha} \] \quad (6.2)
by virtue of \( \beta < n \). Similarly, we also obtain
\[ v(x) \geq c|x|^{\lambda - \beta}. \] \quad (6.3)
and
\[ v_1(x) := \frac{1}{|x|^\beta} \int_{B_1(0)} \frac{|x - y|^\lambda dy}{v^q(y)|y|^\alpha} \leq C|x|^{\lambda - \beta}. \] \quad (6.4)

By (6.2), we get
\[ v_2(x) := \frac{1}{|x|^\beta} \int_{B_{2|x|}(0) \setminus B_1(0)} \frac{|x - y|^\lambda dy}{v^p(y)|y|^\alpha} \leq C|x|^{\lambda - \beta} \int_{B_{2|x|}(0) \setminus B_1(0)} |y|^{-\alpha - p(\lambda - \alpha)} dy. \]
Thus,

\[
\begin{align*}
    v_2(x) &\leq C|x|^{\lambda - \beta}, &\text{as } p(\lambda - \alpha) > n - \alpha; \\
    v_2(x) &\leq C|x|^{\lambda - \beta} \log |x|, &\text{as } p(\lambda - \alpha) = n - \alpha; \\
    v_2(x) &\leq C|x|^{n + \lambda - \alpha - \beta - p(\lambda - \alpha)}, &\text{as } p(\lambda - \alpha) < n - \alpha. \\
\end{align*}
\]  
(6.5)

By (6.1) and (6.2), we get

\[
v_3(x) := \frac{1}{|x|^{\beta}} \int_{R^n \setminus B_{2|x|}(0)} \frac{|x - y|^{\lambda} \, dy}{u^p(y)|y|^{\alpha}} \leq C \int_{R^n \setminus B_{2|x|}(0)} |y|^{\lambda - \alpha - p(\lambda - \alpha)} \, dy \leq C|x|^{n + \lambda - \alpha - \beta - p(\lambda - \alpha)}. \tag{6.6}
\]

Step 2. In view of \( n > \alpha \), (6.1) leads to \( \lambda - \alpha > 0 \). We claim that

\[
\lambda - \beta > 0 \tag{6.7}
\]

and \( v(x) \simeq |x|^{\lambda - \beta} \) when \( |x| \to \infty \).

If \( \beta \leq 0 \), we see (6.7) immediately. If \( \beta > 0 \), then

\[
n + \lambda - \alpha - \beta - p(\lambda - \alpha) := \theta_0 < 0. \tag{6.8}
\]

By the estimates of \( v_1, v_2, v_3 \), we obtain three increasing rates for \( v \). We compare those rates in three cases.

Case (i). If \( \lambda - \beta < \theta_0 \) which is equivalent to \( p(\lambda - \alpha) < n - \alpha \), then (6.5) gives the rate of \( v_2 \):

\[
v_2(x) \leq C|x|^\theta_0. \]

Combining the estimates of \( v_1, v_2 \) and \( v_3 \) together, we see

\[
v(x) \leq C|x|^{\theta_0}. \]

Therefore,

\[
u(x) \geq \frac{c}{|x|^\alpha} \int_{R^n \setminus B_{2|x|}(0)} |x - y|^{\lambda} \, dy \geq C \int_{2|x|}^\infty r^{n + \lambda - \beta - q\theta_0} \, dr. \tag{6.9}
\]

Noting (6.8), we have \( n + \lambda - \beta - q\theta_0 > 0 \) and hence \( u \) blows up not only at the origin. It is impossible since \( u \) is an entire solution in \( C^1(R^n \setminus \{0\}) \). Case (i) does not happen.

Case (ii). If \( \lambda - \beta = \theta_0 \) which is equivalent to \( p(\lambda - \alpha) = n - \alpha \), then for large \( |x| \),

\[
v(x) \leq C|x|^\theta_0 \log |x| \leq C|x|^\theta_0 + \varepsilon.
\]

Here \( \varepsilon > 0 \) is sufficiently small. By an analogous calculation of (6.9), we know \( u \) also blows up. It is impossible and Case (ii) does not happen.

Case (iii). If \( \lambda - \beta > \theta_0 \) which is equivalent to \( p(\lambda - \alpha) > n - \alpha \), then

\[
v(x) \leq C|x|^{\lambda - \beta}. \]

Combining with (6.3), we have

\[
v(x) \simeq |x|^{\lambda - \beta} \quad \text{as } |x| >> 1. \tag{6.10}
\]

If \( \lambda - \beta \leq 0 \), by an analogous calculation of (6.9), we know \( u \) blows up. Thus, \( \lambda - \beta > 0 \).

Step 3. We claim that \( u(x) \simeq |x|^{\lambda - \alpha} \) as \( |x| \to \infty \).
By (6.13), we can find
\[
\text{and} \quad \text{Similarly, for} \quad |y| = |x|/2. \text{ By Step 3, we have}
\]
\[
u(x) \geq \frac{c}{|x|^\alpha} \int_{R^n \setminus B_{2|x|}(0)} \frac{|x-y|^\lambda \, dy}{\nu^q(y)|y|^\beta} \geq \frac{c}{|x|^\alpha} \int_{2|x|}^\infty r^{n+\lambda-\beta-q(\lambda-\beta)} \frac{dr}{r}.
\]
Since \( u \) is an entire solution, we also see from the result above
\[
n + \lambda - \beta - q(\lambda - \beta) < 0. \tag{6.11}
\]
Therefore, we can estimate the upper bound of \( u \) by the same process for \( v \), and still obtain
\[
u(x) \simeq |x|^{\lambda-\alpha}, \tag{6.12}
\]
when \(|x| \to \infty\). In addition, the conclusions of Steps 2 and 3 and (6.11) show that (Rt6) is true.

Next, We consider the asymptotic rates of \( u, v \) when \(|x| << 1\).

From (6.10) we can find
\[
\text{when } R > 0 \text{ sufficiently large such that}
\]
\[
v(y) \leq C|y|^{\lambda-\beta}, \quad \text{for } |y| > R.
\]

Thus, when \(|x| << 1\), using (6.11) we get
\[
u(x) = \frac{c}{|x|^\alpha} \int_{R^n \setminus B_R(0)} \frac{|x-y|^\lambda \, dy}{\nu^q(y)|y|^\beta} \geq \frac{c}{|x|^\alpha} \int_{R}^{\infty} r^{n+\lambda-\beta-q(\lambda-\beta)} \frac{dr}{r} \geq c|x|^{-\alpha}. \tag{6.13}
\]

Similarly, for \(|x| << 1\), from (6.12) and (6.1) we can also deduce
\[
u(x) \geq c|x|^{-\beta}, \tag{6.14}
\]
and
\[
v_1(x) := \frac{c}{|x|^\beta} \int_{R^n \setminus B_R(0)} \frac{|x-y|^\lambda \, dy}{\nu^q(y)|y|^\alpha} \leq C, \tag{6.15}
\]
By (6.13), we can find \( \delta > 0 \) suitably small such that
\[
u(y) \geq C|y|^{-\alpha}, \quad \text{for } |y| < \delta.
\]
Since \( u \in C^1(R^n \setminus B_\delta(0)) \), we see
\[
u(y) \geq c, \quad \text{for } \delta \leq |y| \leq R.
\]
Thus, for \(|x| << 1\) we have
\[
v_2(x) := \frac{1}{|x|^\beta} \int_{B_R(0) \setminus B_{2|x|}(0)} \frac{|x-y|^\lambda \, dy}{\nu^p(y)|y|^\alpha} \leq C, \tag{6.16}
\]
and
\[
v_3(x) := \frac{1}{|x|^\beta} \int_{2|x|}^{\infty} \frac{|x-y|^\lambda \, dy}{\nu^p(y)|y|^\alpha} \leq \frac{C}{|x|^\beta} \int_{2|x|}^{\infty} r^{n+\lambda-\alpha-\beta+\alpha} \frac{dr}{r} \leq C|x|^{-\beta}. \tag{6.17}
\]
Combining the estimates of \( v_1, v_2, v_3 \) with (6.14), we can find \( C > 0 \) such that as \(|x| << 1\),
\[
\frac{1}{C} \leq v(x)|x|^\beta \leq C. \tag{6.18}
\]
Using (6.15) we also obtain
\[ u(x) \leq C|x|^{-\alpha} \]
when \(|x| \ll 1\). Combining with (6.13) yields
\[ \frac{1}{C} \leq u(x)|x|^\alpha \leq C. \]  
(6.16)
The proof of (Rt6) is complete. \( \square \)

**Remark 6.1.** From the kelvin type transformation (3.2), we also see the rates of \( u \) and \( v \) are \(-\alpha\) and \(-\beta\) respectively. In fact, when \(|x| \to 0\), we have \(|y| \to \infty\) if setting \( y = \frac{x}{|x|^\frac{1}{2\alpha}}\). Thus, from (3.2) and (6.12), we deduce
\[ u(x) = |x|^\frac{2\alpha}{\alpha-1} \tilde{u}(\frac{x}{|x|^\alpha}) = |y|^{-\frac{2\alpha}{\alpha-1}} \tilde{u}(y) \simeq |y|^{-\frac{2\alpha}{\alpha-1}} |y|^\lambda = |y|^\alpha = |x|^{-\alpha}. \]
Similarly, \( v \) has an analogous conclusion.

**Proof of (Rt7).** For small \( \delta > 0 \) and large \( R > 0 \), we can deduce from (6.12) and (6.16) that
\[ \int_{R^n} u^{-r}(x)dx \leq C \int_0^\delta \rho^{n+r\alpha} \frac{d\rho}{\rho} + C + C \int_R^\infty \rho^{n-r(\lambda-\alpha)} \frac{d\rho}{\rho} < \infty \]
as long as \( r > \frac{n\alpha}{\lambda-\alpha} \). This shows \( u^{-1} \in L^r(R^n) \) for all \( r > \frac{n\alpha}{\lambda-\alpha} \).
Similarly, we also have \( v^{-1} \in L^s(R^n) \) for all \( s > \frac{n\alpha}{\lambda-\beta} \). \( \square \)

**Proof of (Rt8).** (I) By (6.10) and (6.15), we have
\[ \int_{R^n} \frac{|y|^\alpha dy}{v^\beta(y)|y|^\beta} \leq C \int_0^\delta r^{n+\lambda+\beta-\beta} \frac{dr}{r} + C + C \int_R^\infty r^{n+\lambda-\beta-\beta} \frac{dr}{r} < \infty. \]
Thus, when \(|x| \to 0\),
\[ |x|^\alpha u_1(x) := \int_{R^n \setminus B_\delta(0)} \frac{|x-y|^\lambda dy}{v^\beta(y)|y|^\beta} \to \int_{R^n \setminus B_\delta(0)} \frac{|y|^\lambda dy}{v^\beta(y)|y|^\beta}. \]
In addition, by (6.15), we have
\[ |x|^\alpha u_2(x) := \int_{B_\delta(0) \setminus B_\delta(0)} \frac{|x-y|^\lambda dy}{v^\beta(y)|y|^\beta} \leq C \int_{2|x|}^\delta r^{n+\lambda+\beta-\beta} \frac{dr}{r} \to 0 \]
as \(|x| \to 0\) and \( \delta \to 0 \). Similarly,
\[ |x|^\alpha u_3(x) := \int_{B_\delta(0) \setminus B_\delta(0)} \frac{|x-y|^\lambda dy}{v^\beta(y)|y|^\beta} \leq C|x|^\lambda \int_0^{2|x|} r^{n+\beta-\beta} \frac{dr}{r} \to 0 \]
as \(|x| \to 0\). Therefore,
\[ \lim_{\delta \to 0} \lim_{|x| \to 0} |x|^\alpha u(x) = \lim_{\delta \to 0} \lim_{|x| \to 0} |x|^\alpha (u_1(x) + u_2(x) + u_3(x)) = \int_{R^n} \frac{|y|^\lambda dy}{v^\beta(y)|y|^\beta}. \]
Similarly, we can also prove
\[ \lim_{|x| \to 0} |x|^\beta v(x) = \int_{R^n} \frac{|y|^\lambda dy}{u^\beta(y)|y|^\alpha} \]  
(6.17)

(II) By (6.10) and (6.15), we have
\[ \int_{R^n} \frac{dy}{v^\beta(y)|y|^\beta} \leq C \int_0^\delta r^{n+\beta-\beta} \frac{dr}{r} + C + C \int_R^\infty r^{n-\beta-\beta} \frac{dr}{r} < \infty. \]
Thus, when $|x| \to \infty$,
\[
|x|^{\alpha - \lambda} u_4(x) := \int_{B_R(0)} \frac{|x - y|^\lambda dy}{|x|^{\lambda + \beta}(y)|y|^\beta} \to \int_{R^n \setminus B_{R_0}(0)} \frac{dy}{v^\beta(y)|y|^\beta},
\]
In addition,
\[
|x|^{\alpha - \lambda} u_5(x) := \int_{B_{2|x|}(0) \setminus B_R(0)} \frac{|x - y|^\lambda dy}{|x|^{\lambda + \beta}(y)|y|^\beta} \leq C \int_{R^n \setminus B_{R_0}(0)} \frac{dy}{v^\beta(y)|y|^\beta} \to 0
\]
as $|x| \to \infty$ and $R \to \infty$. Similarly, by (6.15), we have
\[
|x|^{\alpha - \lambda} u_6(x) := \int_{R^n \setminus B_{2|x|}(0)} \frac{|x - y|^\lambda dy}{|x|^{\lambda + \beta}(y)|y|^\beta} \leq C|x|^{-\lambda} \int_{2|x|}^\infty r^{n + \lambda - q(\lambda - \beta) - \beta} \frac{dr}{r} \leq C|x|^{n - q(\lambda - \beta) - \beta} \to 0
\]
as $|x| \to \infty$. Therefore,
\[
\lim_{|x| \to \infty} |x|^{\alpha - \lambda} u(x) = \lim_{R \to \infty} \lim_{|x| \to \infty} |x|^{\alpha - \lambda}(u_4(x) + u_5(x) + u_6(x)) = \int_{R^n} \frac{dy}{v^\beta(y)|y|^\beta}.
\]
Similarly, we can also prove
\[
\lim_{|x| \to \infty} |x|^{\beta - \lambda} v(x) = \int_{R^n} \frac{dy}{u^\rho(y)|y|^\alpha}.
\] (6.18)
The proof of (Rt8) is complete. \[\square\]

Remark 6.2. By the Kelvin transformation, we can also calculate the limits of $u, v$ when $|x| \to \infty$ by the limits of $|x| \to 0$. In fact, by (6.17),
\[
\lim_{|x| \to \infty} \frac{v(x)}{|x|^{\lambda - \beta}} = \lim_{|x| \to \infty} \frac{v(\frac{x}{|x|^\alpha})}{|x|^{\lambda - \beta - \frac{2n}{\alpha}} = \lim_{|y| \to 0} \frac{|y|^{\beta} v(y)}{|y|^{\beta + \rho - \lambda + \frac{2n}{\alpha}}} = \int_{R^n} \frac{dy}{u^\rho(y)|y|^\alpha} = \int_{R^n} \frac{dz}{w^\rho(z)|z|^\alpha}.
\]
This is (6.18). By the same calculation, we also deduce (6.17) from (6.18). Similarly, $u$ has also the analogous results.

Acknowledgments. This research was supported by the Visiting Scholar Program of CIM in Nankai University and PAPD of Jiangsu Higher Education Institutions.

REFERENCES

[1] G. Caristi, L. D’Ambrosio and E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, Milan J. Math., 76 (2008), 27–67.
[2] A. Chang and M. del Mar Gonzalez, Fractional Laplacian in conformal geometry, Adv. Math., 226 (2011), 1410–1432.
[3] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J., 63 (1991), 615–622.
[4] W. Chen and C. Li, Regularity of solutions for a system of integral equations, Commun. Pure Appl. Anal., 4 (2005), 1–8.
[5] W. Chen and C. Li, An integral system and the Lane-Emden conjecture, Discrete Contin. Dyn. Syst., 24 (2009), 1167–1184.
[6] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, Comm. Partial Differential Equations, 30 (2005), 59–65.
[7] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math., 59 (2006), 330–343.
[8] Z. Cheng and C. Li, An extended discrete Hardy-Littlewood-Sobolev inequality, Discrete Contin. Dyn. Syst., 34 (2014), 1951–1959.
[9] Y. Choi and X. Xu, Nonlinear biharmonic equations with negative exponents, J. Differential Equations, 246 (2009), 216–234.
[10] J. Davila, I. Flores and I. Guerra, Multiplicity of solutions for a fourth order problem with power-type nonlinearity, Math. Ann., 348 (2010), 143–193.
[11] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math., 34 (1981), 525–598.
[12] Z. Guo and J. Wei, Liouville type results and regularity of the extremal solutions of biharmonic equation with negative exponents, Discrete Contin. Dyn. Syst., 34 (2014), 2561–2580.
[13] Y. Hua and X. Yu, Necessary conditions for existence results of some integral system, Abstr. Appl. Anal., (2013), Art. ID 504282, 5 pp.
[14] C. Jin and C. Li, Qualitative analysis of some systems of integral equations, Calc. Var. Partial Differential Equations, 26 (2006), 447–457.
[15] Y. Lei and C. Li, Sharp Criteria of Liouville Type for some Nonlinear Systems, arXiv:1301.6235, 2013.
[16] Y. Lei, C. Li and C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system, Calc. Var. Partial Differential Equations, 45 (2012), 43–61.
[17] Y. Lei and Z. Lü, Axisymmetry of locally bounded solutions to an Euler-Lagrange system of the weighted Hardy-Littlewood-Sobolev inequality, Discrete Contin. Dyn. Syst., 33 (2013), 1987–2005.
[18] Y. Li, Remark on some conformally invariant integral equations: The method of moving spheres, J. Eur. Math. Soc., 6 (2004), 153–180.
[19] Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J., 80 (1995), 383–417.
[20] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math., 118 (1983), 349–374.
[21] L. Ma and J. Wei, Properties of positive solutions to an elliptic equation with negative exponent, J. Funct. Anal., 254 (2008), 1058–1087.
[22] P. J. McKenna and W. Reichel, Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry, Electron. J. Differential Equations, (2003), 1–13.
[23] Ph. Souplet, The proof of the Lane-Emden conjecture in 4 space dimensions, Adv. Math., 221 (2009), 1409–1427.
[24] S. Sun and Y. Lei, Fast decay estimates for integrable solutions of the Lane-Emden type integral systems involving the Wolff potentials, J. Funct. Anal., 263 (2012), 3857–3882.
[25] X. Xu, Exact solution of nonlinear conformally invariant integral equations in $\mathbb{R}^3$, Adv. Math., 194 (2005), 485–503.
[26] X. Xu, Uniqueness theorem for integral equations and its application, J. Funct. Anal., 247 (2007), 95–109.
[27] X. Yu, Liouville type theorems for integral equations and integral systems, Calc. Var. Partial Differential Equations, 46 (2013), 75–95.

Received April 2014; revised August 2014.
E-mail address: leiyutian@njnu.edu.cn