Toward a theory of integrable hyperbolic equations of third order

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Abstract

Examples are considered of integrable third order hyperbolic equations with two independent variables. In particular, an equation is found which admits as evolutionary symmetries the Krichever–Novikov equation and the modified Landau–Lifshitz system. The problem of the choice of dynamical variables for the hyperbolic equations is discussed.

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1. Introduction

An important class of nonlinear integrable equations consists of the hyperbolic ones

$$u_{xy} = h(x, y, u, u_x, u_y).$$

Historically, this type of equation includes the very first integrable examples: the Liouville and the sine–Gordon equations. The modern concept of integrability based on the notion of a Lax pair first arose in the study of KdV type equations which were evolutionary, but its applicability to hyperbolic equations was established very soon. Indeed, both classes of equations are in close relation with one another and the existence of a hierarchy of evolutionary symmetries serves as the most convenient test (or definition) of integrability for hyperbolic equations. Some important classification results were obtained by using the symmetry approach [1–6], although the problem of a description of all integrable cases is not completely solved so far (see the review in [7]).

The development of the theory shows, on the other hand, that in some cases the class of equations under consideration should be extended at least to third order hyperbolic equations

$$u_{xxy} = f(x, y, u, u_x, u_y, u_{xy}, u_{xx}).$$

For instance, the hyperbolic symmetry for the KdV equation itself is of this form. This is the class of equations which we consider in this paper. It is not very well studied, and hence we do not aim to obtain any classification results or to derive the necessary integrability conditions.
We restrict ourself to the consideration of several interesting examples and discuss the problem of the choice of dynamical variables for the equation.

Let us explain briefly the content of the article. Section 2 contains the main definitions, in particular, the notion of a consistent pair of third order hyperbolic equations is introduced. These systems belong to an intermediate class between those of the second and third order equations. Its consideration is necessary, since the examples presented demonstrate that systems of this type arise from third order equations as a result of the degeneration of the parameters.

Sections 3 and 4 are devoted to examples related to the KdV and Kaup equations. These examples are not new, since the respective hyperbolic equations are equivalent to the Camassa–Holm and the Degasperis–Procesi [8–13] equations, up to transformations of hodograph type and potentiation. However, our treatment contains some new features since $y$-symmetries are considered as well. We hope that these sections have some methodological value in providing a uniform approach to these examples.

Section 5 is a continuation of section 2. Here, we analyze the consistency condition for a pair of third order hyperbolic equations and introduce the notion of Bäcklund variables. This provides an alternative and more convenient set of dynamical variables, not only for the consistent pair, but also for a single third order equation.

The main example is considered in section 6, completely in the Bäcklund variables. It is related to the Krichcever–Novikov equation and seems to be new.

## 2. Types of hyperbolic equations

As already mentioned in the introduction, second order hyperbolic equations, of the form

$$u_{xy} = h(x, y, u, u_x, u_y)$$

when written in light-cone variables, belong to the simplest and most well studied class of hyperbolic equations. The notion of higher order hyperbolic equations can be introduced in many ways which we do not discuss here, see e.g. [14]. The main object in this paper is a particular class of third order hyperbolic equations with multiple characteristics, namely of the form

$$u_{xxy} = f(x, y, u, u_x, u_y, u_{xy}, u_{xx}),$$

For short, we refer to equations of types (1) and (2) simply as second and third order hyperbolic equations. Another class of equation studied in this paper consists of systems of what we shall call the consistent type, defined as follows.

**Definition 1.** A pair of third order hyperbolic equations

$$u_{xxy} = f(x, y, u, u_x, u_y, u_{xy}, u_{xx}),$$

$$u_{xyy} = g(x, y, u, u_x, u_y, u_{xy}, u_{yy})$$

is called consistent if the identity holds

$$(D_y(f) - D_x(g))|_{u_{xxy}} = f = 0$$

$$u_{xyy} = g$$

where $D_x$ and $D_y$ are operators of total derivatives with respect to $x$ and $y$.

**Definition 2.** A consistent pair (3) is called reducible if its general solution solves some one-parametric family of hyperbolic equations

$$u_{xy} = h(\alpha; x, y, u, u_x, u_y),$$

otherwise the pair is called irreducible.
Consistent systems of type (3) are rather delicate generalizations of second order equations. This is clear from comparing the initial data for the Goursat problem. The role of the Goursat data for equation (1) can be played by a pair of functions $u(x, 0) = a(x)$, $u(0, y) = b(y)$ such that the consistency condition $a(0) = b(0)$ is fulfilled. In the case of system (3) just one additional value should be given, the mixed derivative at the origin: $u_{xy}(0, 0) = \text{const}$, while in the case of equation (2) an additional function $u_x(0, y) = c(y)$ is required. The black disks in figure 1 represent the dynamical variables for the different types of equations under consideration: that is the set of derivatives in a point $\partial^m x \partial^n y (u)$ which can be chosen independently. We will call such sets standard dynamical variables.

**Definition 3.** An equation of any of type: (1), (2), (3) is called integrable if it is compatible with an infinite hierarchy of evolutionary symmetries, that is equations of the form

$$u_t = f(x, y, [u]),$$

where the square brackets denote an arbitrarily large finite number of dynamical variables for the hyperbolic equation under consideration.

This definition of integrability is standard enough. More formal definitions of the evolutionary symmetry and applications of this notion to the classification of hyperbolic equations are discussed in detail in the references cited and in many other sources, so we will not cover this topic here. We only recall two facts. First, even the existence of just one symmetry of high order with respect to derivatives is a very strong condition which seems to be equivalent to the existence of a whole hierarchy (no example is known of a nontrivial equation possessing only one higher symmetry). For this reason, we consider only a few higher symmetries in the examples, omitting proof that there are infinitely many. Second, in all known examples, the symmetry algebra is decomposed into two subalgebras containing derivatives with respect to $x$ or $y$ only. Each evolutionary symmetry is itself an integrable equation. This property is similar for all classes of equations under consideration; however, one should bear in mind that $y$-symmetries for equations of type (2) correspond to coupled systems with two dependent variables, for instance, $u$ and $v = u_x$ (see the examples in later sections).

The notion of a consistent pair is the only thing which may seem unusual and in order to illustrate it we conclude this section with several examples.

First, let us discuss the question of irreducibility. It is clear that a consistent pair can be obtained by differentiating an equation of the form (1) and disguising the result by means of some invertible transformation, for instance, the following equations are consistent:

$$u_{xxy} = D_x(h) + A(u_{xy} - h), \quad u_{xyy} = D_y(h) + A(u_{xy} - h)$$

where $h = h(x, y, u, u_x, u_y)$ and $A(z)$ is an arbitrary function. However, these examples are reducible and therefore uninteresting. It is not clear at once how to construct irreducible pairs, but the following example shows that they do indeed exist.
Example 1. Consider the equations

\[
\begin{align*}
    u_{xxy} &= u_{x} u_{xx} + \frac{u_{xy}^2}{2u_y} + u_y, \\
    u_{xyy} &= \frac{u_{xy} u_{yy}}{u_y} + \frac{u_{yy}^2}{2u_x} + u_x.
\end{align*}
\]

(5)

It can be proved directly that the identity (4) holds (a simple program for such kind of computations is presented in the appendix).

In order to determine whether this pair is reducible, let us replace the derivatives by virtue of an equation \( u_{xy} = h(x, y, u, u_x, u_y) \) and see whether the obtained equations can be made to hold identically with respect to all the dynamical variables \( u, u_x, u_{xx}, u_y, u_{yy} \). Collecting coefficients of \( u_{xx} \) in the first equation and of \( u_{yy} \) in the second one requires the relations

\[
    u_x h_x = h, \quad u_y h_y = h
\]

which together imply that the function \( h \) must be of the form \( h = u_x u_y H(x, y, u) \). Then, the first equation of the system turns into

\[
    u_x^2 (H^2 + 2H_x) + 2u_x H_x = 2.
\]

Obviously, this equation cannot be satisfied by any function \( H(x, y, u) \) (let alone by a one-parameter family), therefore the pair (5) is irreducible.

This pair is integrable as well, being compatible with the Schwarzian KdV equation, for both characteristic directions:

\[
    u_t = u_{xxx} - \frac{3u_{x}^2}{2u_x}, \quad u_x = u_{yyy} = \frac{3u_{y}^2}{2u_y}.
\]

It should be remarked that the Schwarzian KdV equation serves as the evolutionary symmetry not only for the pair (5), but also for the second order equation

\[
    u_{xy} = \frac{2u_x u_y}{u_x^2 + 1},
\]

and also for several other hyperbolic equations, see e.g. [15]. In general, the correspondence between (integrable) hyperbolic and evolutionary equations is not one-to-one: a given hyperbolic equation possesses at most one evolutionary symmetry of a given order, but one and the same symmetry may correspond to different hyperbolic equations which are not pointwise equivalent.

Neither should one think that the compatibility condition (4) is related somehow with integrability in the sense of definition 3. We will see in section 5 that there are ‘approximately as many’ consistent pairs as for the usual hyperbolic equations and, apparently, integrable cases for the two classes are equally rare. In the example which follows, we consider a family of consistent pairs which contains an arbitrary function, but which is not in general integrable. This example also illustrates the simplest type of differential substitution: introducing a potential.

In general, the question of which substitutions are admissible for a given equation is difficult and its consideration is beyond the scope of this paper. In particular, we do not know an algorithm which allows us to check the irreducibility of a consistent pair not only in the sense of definition 1, but also modulo differential substitutions. Presumably, such an example is provided by the pair (36) belonging to the hierarchy of the Krichever–Novikov equation which is not related via differential substitutions to other KdV-type equations [16].

Example 2. The Klein–Gordon equation

\[
    q_{xy} = f'(q)
\]

admits the conservation law

\[
    D_x (f(q)) = D_x \left( \frac{1}{2} q_x^2 \right)
\]
which can be used to introduce a new variable (the potential) according to the equations

\[ u_x = \frac{1}{2} q_x^2, \quad u_y = f(q). \]

Solving the second equation with respect to \( q \) and substituting into the first one brings us to the equation

\[ u_{yy} = \frac{\sqrt{2}u_x}{a'(u_x)}, \quad a(f(q)) = q. \]

The potential can be defined also in another way, according to the relations

\[ u_x = qx, \quad u_y = f'(q) \]

which yields the equation

\[ u_{xy} = u/a'(u_x), \quad a(f'(q)) = q. \]

Finally, both substitutions can be mixed by adding the trivial conservation law to the above one:

\[ D_x(f(q) + kf'(q)) = D_y \left( \frac{1}{2} q_x^2 + kq_{xx} \right). \]

This gives rise to the substitution

\[ u_x = \frac{1}{2} q_x^2 + kq_{xx}, \quad u_y = f(q) + kf'(q) \]

and elimination of \( q \) (as before, the latter equation is assumed to be solvable with respect to \( q \)) brings us to the following third order equation:

\[ u_{xxy} = \frac{1}{ka'(u_x)} \left( u_x - (ka''(u_x) + \frac{1}{2} a'(u_x)^2 u_y^2) \right), \quad a(f(q) + kf'(q)) = q. \]

However, in this case the conservation law is not exactly equivalent to the original equation and substituting \( q \) intermediately into (7) provides one more third order equation

\[ u_{xxyy} = -\frac{a''(u_x)}{a'(u_x)} u_{xxy} + \frac{f'(a(u_x))}{a'(u_y)} u_{xyy}. \]

The consistency of the obtained hyperbolic pair follows from its construction and a check along the lines of the previous example shows that it is irreducible. Concerning the integrability property, one can prove that it occurs exactly in the cases when the original equation (7) is integrable, that is, if the function \( f \) is equal to \( e^q, e^q + e^{-q} \) or \( e^q + e^{-2q} \) (up to linear changes of \( q, x, y \)) corresponding to the Liouville, sine–Gordon or Tzitzeica equations [1].

The final example demonstrates a further extension of the classes of equations under consideration.

**Example 3.** The system

\[
\begin{align*}
4 \det \begin{pmatrix} u_{yy} & u_{xyy} & u_{xxyy} \\ u_x & u_y & u_x \\ u_x & u_x & u_{xx} \end{pmatrix} & = u^3, \\
3(u_y u_{xy} - u_x u_{xy}) & = u_x u_{xx} - uu_{xxxy}, \\
3(u_y u_{xy} - u_x u_{xy}) & = u_x u_{yy} - uu_{xxyy}.
\end{align*}
\]

defines a consistent triple of fourth order hyperbolic equations, that is, the cross-derivatives are equal identically,

\[ D_x(u_{xyy}) = D_y(u_{xxy}), \quad D_y(u_{xyy}) = D_x(u_{xxy}). \]

by virtue of the system itself. Comparing with the consistent pair (3), the set of dynamical variables for such a triple additionally contains the derivatives \( u_{xy} \) and \( u_{xxy} \). It can be proved
that the above system is irreducible, that is, it cannot be obtained from some consistent pair by differentiating. However, it is related via the substitution $v = -2(\log u)_{xy}$ to the Tzitzeica equation in algebraic form

$$v_{xy} - v_x v_y = v^3 - 1.$$ 

More precisely, this equation gives rise to the first, trilinear equation of the system (8) (see e.g. [17]), while the two bilinear ones are consequences of the conservation laws

$$\left(\frac{v_{xx}}{v}\right)_y = 3v_x, \quad \left(\frac{v_{xy}}{v}\right)_x = 3v_y.$$ 

Indeed, the latter relations can be integrated after the substitution:

$$v_{xx} = -6(\log u)_{xx} + a(x), \quad v_{yy} = -6(\log u)_{yy} + b(y),$$

moreover one can assume that $a = b = 0$ without loss of generality, since the function $u$ is defined up to multiplication by arbitrary factors depending on $x$ and on $y$. Now, replacing $v$ yields two last equations of the system (8).

3. The potential Korteweg–de Vries equation

It is known [15] that the potential KdV (pot-KdV) equation

$$u_t = u_{xxx} - 3u_x$$

(9)

does not admit compatible second order equations of the form (1). However, it is compatible with the following third order equation:

$$u_{xyy} = \frac{u_y^2 - c}{2u_x} + 2u_xu_y.$$  

(10)

One can prove by straightforward computation that this is the general form of equation (2) compatible with the pot-KdV, up to the transformation $u \rightarrow u + ax + by$. The parameter $c$ can be scaled either to 0 or to 1 by scaling $y$ and we will see that the properties of the equation in these two cases are quite different (in regard of the real solutions, one should also distinguish $c = 1$ and $c = -1$, but this is not important in what follows). Equation (10) is well known, although in different variables: it is the potential form of the associated Camassa–Holm equation [9, 10].

The full algebra of evolutionary symmetries for (10) is a direct sum of two hierarchies, as in the case of equations (1). One of them, the pot-KdV hierarchy, contains equations only with derivatives $u_t$, $u_{xx}$, $u_{xxx}$, . . . on the right-hand side, while equations belonging to the other hierarchy contain, besides $u_t$, $u_{xy}$, . . . also the mixed derivatives $u_{tx}$, $u_{xy}$, $u_{xxy}$, . . . . The first two members of this second hierarchy in the generic case $c \neq 0$ are the following:

$$u_{t_2} = u_{xy}u_{yy} - u_xu_{xxy} + u^3_x,$$  

(11)

$$u_{t_3} = u_{yyy} - \frac{3u_{xy}^2}{2u_x} + \frac{3}{2u_x}(u_{xy}u_{xy} - u_xu_{xxy} + u^3_x)^2.$$  

(12)

Differentiating these equations with respect to $x$ and replacing $u_{xxy}$, $u_{xxy}$ from (10) gives rise to coupled evolutionary systems with respect to $u$ and $u_t$. The commutativity of the corresponding flows holds without taking equation (10) into account. These systems look rather awkward. In particular, in the case (11) where the matrix at the leading derivatives $u_{xy}$, $u_{xxy}$ is not constant and is not diagonal. However, the differential substitution $(u, u_t) \rightarrow (u, v)$

$$v = \frac{k - u_{xy}}{ku_x} + \frac{u}{k}, \quad k^2 = c,$$
brings the equations to a more compact form:

\[ k^{-1} u_2 = u_{yy} + u_y^2 v_y, \quad k^{-1} v_2 = -v_{yy} + u_y^2, \]

\[ u_3 = u_{yyy} + 3u_yv_yu_{xy} + \frac{3}{2} u_y^3 v_y^2, \quad v_3 = v_{yyy} - 3u_yv_yv_{xy} + \frac{3}{2} u_y^2 v_y^3. \]

This is the potential form of the Kaup–Newell system, or the derivative nonlinear Schrödinger equation [18]. Up to our knowledge, its relation to equation (10) and, therefore, to the pot-KdV equation (9) was not remarked before.

Now, let us consider the case \( c = 0 \). First of all, notice that equation (10) acquires in this (and only this) case the first integral

\[ D_x \left( u_{yy}y - u_{xy}uxy + u^2 \right) = 0 \]

which we rewrite in the form

\[ u_{yyy} = \frac{u_{xy}u_{xy}}{u_y} + u_y^2 + \gamma. \]  

(13)

Without loss of generality, the integration constant \( \gamma \) will be assumed to be independent of \( y \); at \( c = 0 \), the original equation (10) becomes invariant with respect to the changes \( y \rightarrow \phi(y) \); it is possible to set \( \gamma = \text{const} \) by the appropriate use of such a transformation.

Equations (10|\( c = 0 \)), (13) constitute a consistent pair. It is irreducible in the sense of definition 1; this can easily be proved by a direct check along the lines of example 1. Nevertheless, this pair is very simply related to a hyperbolic equation, since the substitution \( u_y = e^q \) lowers the order of the equation (13) and brings it to the sinh-Gordon equation (or to the Liouville equation at the special value \( \gamma = 0 \) of the first integral)

\[ q_{xy} = e^q + \gamma e^{-q}. \]

This is a particular case of the substitution from example 2. It is well known that the pot-mKdV equation

\[ q_t = q_{yyy} - \frac{1}{2} q_y^3 \]  

(14)

is an evolutionary symmetry for this equation. Returning to the variable \( u \), we obtain from here the symmetry for equation (13), namely, the Schwarzian KdV equation

\[ u_t = u_{yyy} - \frac{3u_{xy}^2}{2u_y} \]  

(15)

which we have already met in example 1. It is directly proved that this symmetry is compatible, by virtue of (13), also with equations (10|\( c = 0 \)) and (9).

Thus, the existence of the compatible equation (13) in the case \( c = 0 \) brings about a conversion of the hierarchy of \( y \)-symmetries: it simplifies and becomes one-component. By the use of (13), the first symmetry (12) turns into the classical symmetry \( u_{xy} = -\gamma u_y \). Equation (12) (as well as all other equations of the hierarchy) contains the parameter \( c \) in the denominator, but this does not lead to the loss of this symmetry, since division by \( c \) can be compensated by the scaling \( \gamma \) in the numerator. As a result, the fractional term in equation (12) also becomes proportional to \( u_y \) and can be neglected. Of course, this is just a heuristic argument, since actually we cannot make use of the relation (13) until the parameter \( c \) turns into 0. However, as we have already said, a direct check shows that indeed equation (15) defines the \( y \)-symmetry.
4. The potential Kaup equation

The potential Kaup equation

\[ u_t = u_{xxxx} + 5u_x u_{xxx} + \frac{15}{2} u_{xx}^2 + \frac{5}{3} u_x^3 \]  

(16)
is compatible with the hyperbolic third order equation

\[ u_{xyy} = \frac{3u_x^2}{4u_y} - u_x u_y - c. \]  

(17)

This is the general form of such an equation, up to the change \( u \to u + \beta y \), and no compatible equation of second order exists. The parameter \( c \) can be scaled to 0 or 1 (the sign is not important, in contrast to what happens for the KdV equation). Equation (17) is related to the Degasperis–Procesi equation [12, 13].

In order to write down the \( y \)-symmetries denote

\[ S = \frac{u_{yyyy}}{u_y} - \frac{3u_y^2}{2u_y^2}, \quad P = u_y^{-3/2} \left( u_y u_{xyy} - u_{xxy} u_{y} + \frac{2}{3} u_x^3 \right). \]

then the first two higher symmetries take the form, at \( c \neq 0 \):

\[ u_{t_1} = u_y \left( S + \frac{3}{4c} P^2 \right), \]  

(18)

\[ u_{t_2} = u_y \left( S_{yy} + \frac{3}{2} S^2 + \frac{5}{4c} \left( 2SP^2 + P^2_y + 2PP_{yy} \right) + \frac{15}{16c^2} P^4 \right). \]  

(19)

These equations can be written as two-component evolutionary systems with respect to \( u \) and \( u_x \), but their form is rather cumbersome.

The value \( c = 0 \) is distinguished, as in this case there arises the first integral: \( D_{xy}(P) = 0. \) This yields the equation

\[ u_{xyy} = \frac{u_{xy} u_{xy}}{u_y} - \frac{2}{3} u_y^2 + \gamma u_y^{1/2} \]  

(20)

which, together with (17), constitutes a consistent pair. One can assume without loss of generality, by a suitable change of variable \( y \to \phi(y) \) if necessary, that \( \gamma \) does not depend on \( y \).

At \( c = 0 \), the hierarchy of \( y \)-symmetries becomes one-component, but, in contrast to the previous section, its structure depends on the value of the first integral. At the special value \( \gamma = 0 \), the flow (18) survives and turns into the Schwarzian KdV equation (15). If \( \gamma \neq 0 \) then the third order symmetry does not exist, while the flow (18) turns into the equation

\[ u_{t_3} = u_{yyyy} = \frac{5u_y u_{yyyy}}{4u_y} - \frac{15u_y^2}{4u_y} u_{yy} - \frac{65u_y^2 u_{xyy}}{4u_y} - \frac{135u_y^4}{16u_y^2} \]

which is compatible, modulo (20), with equations (17|\( c=0 \)) and (16). The dependence of the answer on \( \gamma \) becomes clear after the substitution \( u_y = e^q \), which turns equation (20) into the Tzitzeica equation

\[ q_{xy} = -\frac{2}{3} e^{q} + \gamma e^{-q/2}. \]

If \( \gamma = 0 \) then we obtain the Liouville equation again, with the symmetry (14).
5. Passage to the Bäcklund variables

Until now, we used standard dynamical variables: the set of derivatives with respect to $x, y$ which cannot be eliminated by virtue of the equation (see figure 1). At first glance, this set is the only reasonable one. However, it turns out to be unfit in more complicated examples like the Krichever–Novikov equation from the next section, bringing catastrophic computations and answers. The key to their simplification is given by the problem of describing the consistent pairs of equations (3). We restrict ourselves to the case of equations which are linear with respect to second order derivatives $u_{xx}, u_{yy}$. The analysis below demonstrates that such pairs can be conveniently represented by equations with two dependent variables $u, v$. In this analysis, the pair is not assumed to be integrable in the sense of the existence of evolutionary symmetries.

So, let us consider the pair of equations

$$
uxxy = au_{xx} + b, \quad u_{xyy} = cu_{yy} + d
$$

(21)

where $a, b, c, d$ are functions depending on $x, y, u, ux, uy, uxy$. The consistency condition is

$$
D_y(auxx + b) = D_x(cuyy + d)
$$

(22)

and after replacing $uxxy$ and $u_{xyy}$ by virtue of (21) this equation must hold as an identity in $x, y, u$ and derivatives of $u$.

**Theorem 1.** If a system (21) is consistent and irreducible then it is of the form

$$
uxxy = \frac{1}{h_{uxx}}(F(x, y, u, u_x, h) - h_x - h_u u_x - h_{ux} u_{xx} - h_{uxy} u_{xy}),
$$

$$
u_{xyy} = \frac{1}{h_{uyy}}(G(x, y, u, u_y, h) - h_y - h_u u_y - h_{uy} u_{yy} - h_{uyy} u_{xyy})
$$

(23)

where the function $h = h(x, y, u, u_x, u_y)$ is implicitly defined by the equation

$$
FhG - FGh + u_{xy}(Fu - Guy) + u_x Fu - u_y Gu + F_y - G_y = 0.
$$

(24)

**Proof.** Collecting terms multiplying $u_{xx}u_{yy}$ in equation (22) yields the relation

$$
a_{ux} - c_{uy} = a c_{uxx} - c a_{uyy}
$$

which may be represented as a compatibility condition for the system of equations

$$
h_x = -ah_{ux}, \quad h_y = -ch_{uy},
$$

with respect to some unknown function $h$. Therefore, if the pair is consistent then a function $h(x, y, u, u_x, u_y, u_{xy})$ exists such that the pair is represented as follows:

$$
uxxy = f_0 - \frac{h_x}{h_{uxx}} u_{xx}, \quad u_{xyy} = g_0 - \frac{h_y}{h_{uyy}} u_{yy}.
$$

It is convenient to redefine $f_0, g_0$ and to rewrite this in the form

$$
D_x(h) = \tilde{f}(x, y, u, u_x, u_y, u_{xy}), \quad D_y(h) = \tilde{g}(x, y, u, u_x, u_y, u_{xy}).
$$

Now, collecting the terms with $u_{xx}$ and $u_{xy}$ in equation (22) yields

$$
\tilde{f}_x h_{uxx} = \tilde{f}_{uxx} h_x, \quad \tilde{g}_y h_{uyy} = \tilde{g}_{uyy} h_y.
$$

Solving these equations requires the functions $\tilde{f}, \tilde{g}$ to be of the form

$$\tilde{f} = F(x, y, u, u_x, h), \quad \tilde{g} = G(x, y, u, u_y, h),$$
that is that the pair (21) is of the form (23). Moreover, the consistency condition now takes the form (24).

Now let us prove that if the pair is irreducible then equation (24) must be (locally) solvable with respect to \( h \). Assume that this is not the case, that is, the functions \( F \) and \( G \) are such that equation (24) holds identically on \( h \). Then the coefficient of \( u_{xy} \) must vanish: \( F_{ux} - G_{uy} = 0 \), because \( F \) and \( G \) do not themselves depend on \( u_{xy} \). This implies that \( F \) is linear with respect to \( u_x \), \( G \) is linear with respect to \( u_y \) and further analysis of equation (24) shows easily that these functions are of the form

\[
F = -su_{ux} + sx, \\
G = -su_{uy} + sy
\]

\( s = s(x, y, u, h) \).

However, in such a case the pair (23) is reducible because its equations are obtained by differentiation of the one-parametric family of second order equations \( s(x, y, u, h(x, y, u, u_x, u_y, u_{xy})) = \alpha \). \( \square \)

Equation (24) can be effectively solved with respect to \( h \) only for very special functions \( F, G \), for example, linear or quadratic with respect to \( h \). As a particular example, if we chose

\[
F = \frac{(u - h)^2}{u_x}, \\
G = \frac{(u - h)^2}{u_y}
\]

then equation (24) is linear with respect to \( h \) and we find

\[
h = u - 2\frac{u_xu_y}{u_{xy}}.
\]

Now, substitution into equations (23) gives the pair (5) from example 1.

If the functions \( F, G \) are of a more general form then representation (23) is practically useless. However, it suggests the idea that the ‘proper’ dynamical variable is not the mixed derivative \( u_{xy} \), but the function \( h \) itself. Indeed, if we introduce the new variable \( v = h(x, y, u, u_x, u_y, u_{xy}) \) then \( u_{xy} \) as the inverse function of \( v \) is found explicitly by solving the linear equation (24). It is easy to see that the result of this transformation is that equations (23) are written in the form

\[
u_x = F(x, y, u, u_x, v), \\
u_y = G(x, y, u, u_y, v),
\]

(25)

while equation (24) follows from here after eliminating the cross derivatives, so providing an equation for \( u_{xy} \):

\[
u_{xy} = H(x, y, u, u_x, u_y, v).
\]

(26)

These equations give the desired representation of the consistent pair. In order to return to the standard set of dynamical variables, one has to differentiate (26) with respect to \( x \) and \( y \) and to eliminate \( v \) from the resulting equations, using equations (25) and (26) again.

Notice that the roles of the variables \( u \) and \( v \) in system (25) are symmetric (for generic functions \( F, G \)) and the elimination of \( u \) instead of \( v \) yields a consistent pair of hyperbolic equations with respect to the variable \( v \). Exactly the same idea is used in the definition of the Bäcklund transformation (see e.g. [19]). This is why we call \( u \) and \( v \) the Bäcklund variables.

**Definition 4.** We say that the equations (25) define the representation of the pair (21) in the Bäcklund variables if the pair of equations (21) follows from (25) as a result of eliminating the variable \( v \).

Let us stress that the pair (21) which admits the representation (25) is automatically consistent.

In the above proof, we have seen that equation (24) may lose the dependence on \( h \) or \( u_{xy} \) for some special choices of \( F, G \). Such functions are unfit for definition of the pair (23). However,
one can waive these restrictions and consider systems (25) with arbitrary $F$ and $G$ as the basic object. From this point of view, different types of degeneration are admissible. In particular, equations (25) may define indeed a Bäcklund transformation between hyperbolic equations of second order. This corresponds to the situation when equation (26) does not contain $v$ and the analogous equation for $v_x$ does not contain $u$. Thus, the class of systems (25) is rather general and important. Functions $F$, $G$ depend on five arguments; that is the functional dimension of this class is the same as for the class of second order equations (1).

It should be remarked that representation in Bäcklund variables is not unique, in contrast to the representation in the standard dynamical variables. Indeed, variable $v$ can be replaced by any variable of the form

$$\tilde{v} = \varphi(x, y, u, v), \quad \varphi_v \neq 0$$

(27)

without changing the general form of equations (25). Certainly, eliminating $\tilde{v}$ results in the same equations for $u$ as before. This arbitrariness should be taken into account when bringing a given system (25) to a simpler form.

A remarkable feature is that Bäcklund variables may become convenient even in consideration of a single third order equation rather than a pair (21). In such a case, one equation of system (25) is replaced with an equation of the form (26). As an example, let us rewrite in the Bäcklund variables a system corresponding to the pot-KdV equation (9). The consistent pair $(10|c=0)$, (13) can be represented as the system

$$v_x + u_x = \frac{1}{2} (u - v)^2, \quad v_x y = -\gamma - \frac{1}{2} y$$

Notice that the first equation of this system defines the $x$-part of the Bäcklund transformation (with zero spectral parameter) for equation (9). At $c \neq 0$, the third order equation (10) is equivalent to the system

$$v_x + u_x = \frac{1}{2} (u - v)^2, \quad u_{xy} = u_y (u - v) + k, \quad \kappa^2 = c.$$  

This result is in close relation with the representation of the associated Camassa–Holm equation by compatible differential-difference equations, the dressing chain and a Volterra-type lattice [20, 21].

Analogously, the consistent pair $(17|c=0)$, (20) corresponding to the Kaup equation (16) can be represented as the system

$$v_x = \frac{1}{2} (u + v)^2, \quad v_x = -\frac{1}{2} u_{xx} - \gamma u_x^{-1/2},$$

while the single equation (17) is equivalent to the system

$$v_x = \frac{1}{2} (u + v)^2, \quad u_{xy} = -u_y (u + v) + k u_x^{1/2}, \quad k^2 = 4c.$$  

A more complicated example of an application of Bäcklund variables is presented in the next section.

6. Krichever–Novikov equation

It is known [15] that the Krichever–Novikov equation

$$u_t = u_{xxx} - \frac{3(u_{xx} - ru)}{2u}, \quad r^{(5)}(u) = 0$$

(28)

does not admit a consistent second order hyperbolic symmetry (1). In this example, the search for a compatible third order equation using the standard dynamical variables runs into inextricable computational difficulties. Computation in the Bäcklund variables however, brings us the following answer.
Theorem 2. Let $h = h(u, v)$ be a biquadratic polynomial and $r(u), s(v)$ be its discriminants with respect to $v, u$, respectively:

$$h_{uuu} = h_{vvv} = 0, \quad r(u) = h^2_u - 2hh_{uv}, \quad s(v) = h^2_v - 2hh_{uv}.$$ 

Then (1) equation (28) defines an evolutionary symmetry for the hyperbolic system (with arbitrary parameter $c$)

$$u_{xy} = \frac{u_x}{h}(h_u u_x + c(hh_{uv} - h_u h_v)),$$

$$u_x v_y = h(u, v);$$

(2) the following equation holds by virtue of equations (28), (29), (30):

$$v_t = v_{xxx} - \frac{3(u^2_{xx} - s(v))}{2v_x} + \frac{h}{h^2_u} u_x u_y - c \left( \frac{h}{h^2_u} u_x + c \left( \frac{h}{h^2_u} u_y \right) \right) v_y$$

(3) the variables $u$ and $v$ are on an equal footing: equation (29) can be replaced with

$$v_{xy} = \frac{v_x}{h}(h_v v_x - c(hh_{uv} - h_u h_v)).$$

Proof. In order to prove statement (1) one should verify that differentiating system (29), (30) with respect to $t$ and using equation (28) gives rise to an identity modulo the system. First, differentiating equation (29) yields the equality

$$(D_x D_y - \frac{h}{h} D_x - \frac{h}{h} u_x D_y - \left( \frac{h}{h} u_x u_y - c \left( \frac{h}{h} u_x + c \left( \frac{h}{h} u_y \right) \right) u_y \right))\times \left( u_{xxx} - \frac{3(u^2_{xx} - r(u))}{2u_x} \right) = u_x \left( \frac{h}{h} u_x u_y + c \left( \frac{h}{h} u_y \right) v_y \right) v_y$$

where we denote $\hat{h} = hh_{uv} - h_u h_v$, and an explicit expression for $v_t$ is obtained from here. It is simplified after replacing the derivatives of $u$ by using (29), (30), and equation (31) appears as a result of straightforward, although rather tedious, computations (a remarkable circumstance here is that the left-hand side of the equation is divisible by the expression in the brackets from the right-hand side).

Thus, statement (2) is proved and in order to complete the proof of statement (1), we have to check that differentiating equation (30), with respect to $t$, yields an identity, that is

$$(D_x D_y - \frac{h}{h} D_x - \frac{h}{h} u_x D_y) \left( u_{xxx} - \frac{3(u^2_{xx} - r(u))}{2u_x} \right) + (u_x D_x - h) \left( v_{xxx} - \frac{3(v^2_{xx} - s(v))}{2v_x} \right) = 0.$$

This is proved by a direct and relatively simple computation. Moreover, since the original equation (28) and the obtained equation (31) do not contain derivatives with respect to $y$, hence equation (29) is not actually needed in this computation and only relation (30) is used. This is exactly equivalent to the known result [22] that this relation defines the $x$-part of the Bäcklund transformation between equations (28) and (31).

Statement (3) is very simple:

$$v_{xy} = \left( \frac{h}{u_x} \right)_y = \frac{h_u u_y + h_v v_y}{u_x} - \frac{h u_{xy}}{u_x^2} = \frac{h u_y v_x - h v_x v_y}{h} - \frac{1}{u_x} (h_u u_y + c(hh_{uv} - h_u h_v))$$

$$= \frac{v_x}{h} (h_v v_x - c(hh_{uv} - h_u h_v)).$$

It is clear that, vice versa, (29) follows from (32).
Notice that the forms of equations \((28)\)–\((32)\) are invariant under Möbius transformations of \(u\) and \(v\). The orbits of the group of transformations depend, in particular, on the multiplicity of the zeroes of the polynomials \(r, s\) (the proper Krichever–Novikov equation corresponds to the generic case of simple zeros) and a detailed classification of the orbits can be found in [23].

Thus, the hyperbolic system \((29), (30)\) defines a certain extension of the Bäcklund transformation for the Krichever–Novikov equation (one might call it the \(J.\ Phys.\ A:\ Math.\ Theor.\) \(y\)-part). An important feature is that in the case of the usual Bäcklund transformation, the variables \(u\) and \(v\) cannot be explicitly expressed in terms of one another, while adding the new independent variable \(y\) makes the transformation explicit. Indeed, variable \(v\) is expressed through \(u, u_x, u_y\) and \(u_{xy}\) as a solution of equation \((29)\), and the inverse transformation is obtained from \((32)\). In particular, this explains why, in statement \((2)\), equation \((31)\) is uniquely derived from \((28)–(30)\) and should not be postulated in advance.

The polynomial \(h = h(u) - h_y = \text{const}\) in the right-hand side of equation \((29)\) is biquadratic as well (moreover, an algebraic identity holds \(h(u) = h_y = \text{const}\)). Therefore, in order to find \(v\), one has to solve a quadratic equation with rather cumbersome coefficients. In principle, after substituting the obtained expression into \((30)\), one can write down a hyperbolic third order equation in the standard dynamical variables (that is, of form \((2)\)), but it is so bulky that the inapplicability of these variables in this example becomes obvious. Moreover, the form of this third order equation depends on the particular choice of \(h\). In the most degenerate case \(r = 0\), \(h = \frac{1}{2}(u - v)^2\) corresponding to the Schwarzian KdV equation, an essential simplification occurs and one obtains the equation (compare with the pair \((5)\) and equation \((13)\))

\[
 u_{yy} - 2u_x + \frac{u_y^2 - c^2 u_x^2}{2u_y} + u_v.
\]

It is also natural to use the variables \(u, v\) for representing the \(y\)-symmetries for system \((29), (30)\) (recall that, in the standard dynamical variables, we used \(u_x\) instead of \(v\)). The two simplest higher flows presented in the following statement were found by an intermediate computation. These flows are defined just by one equation for \(u_v\), since the equation for \(v_v\) is derived automatically. However, for the sake of completeness, we write down both equations of the coupled system.

**Statement 1.** If \(c \neq 0\) then the system \((29), (30)\) admits the following symmetries:

\[
\begin{align*}
 u_v = u_{yy} & - \frac{1}{ch}(u_y^2 - c^2 r(u))(v_y + ch_u) - \frac{c^2}{2} r'(u), \\
 v_v = v_{yy} & + \frac{1}{ch}(u_y^2 - c^2 s(v))(u_y - ch_v) - \frac{c^2}{2} s'(v);
\end{align*}
\]

\[
\begin{align*}
 u_{vv} = u_{yyy} & - \frac{3u_x u_{vy}}{c h} (v_y + ch_u) + \frac{3u_v}{2c^2 h^2} (u_y^2 - c^2 r(u))(v_y + ch_u)^2 \\
 & + \frac{3c}{2h} u_x v_y r'(u) + \frac{3c^3}{2h^2} (h_y^2 r(u) - h^2) u_v, \\
 v_{vv} = v_{yyy} & + \frac{3u_x v_{xy}}{c h} (u_y - ch_v) + \frac{3v_v}{2c^2 h^2} (v_y^2 - c^2 s(v))(u_y - ch_v)^2 \\
 & - \frac{3c}{2h} u_x v_y s'(v) + \frac{3c^3}{2h^2} (h_y^2 s(v) - h^2) v_v.
\end{align*}
\]

These are equations from the modified Landau–Lifshitz hierarchy, written in terms of variables given by stereographic projection [20].
Like in the examples from sections 3, 4, the parameter \( c \) can be scaled to 0 or 1 and the properties of the equations are different in the two cases. It is easy to prove that if \( c = 0 \) then the system (29), (30) admits the first integral
\[
D_c \left( \frac{u_x v_y}{h} \right) = 0.
\]
Up to the changes \( y \to \varphi(y) \), the value of the integration constant can be chosen to be equal to 1, and this gives rise to the following hyperbolic pair written in the Bäcklund variables (see definition 4).

**Statement 2.** Let \( h(u, v) \) be a biquadratic polynomial in \( u, v \). Then the system
\[
\begin{align*}
  u_t v_x &= h(u, v), \\
  u_x v_y &= h(u, v)
\end{align*}
\]
(36)
admits the Krichever–Novikov equation as both \( x \)- and \( y \)-symmetries, that is, it is compatible with both the evolutions
\[
\begin{align*}
  u_t &= u_{xxx} - \frac{3(u_x^2 - r(u))}{2u_x}, \\
  v_t &= v_{xxx} - \frac{3(v_x^2 - s(v))}{2v_x}
\end{align*}
\]
and
\[
\begin{align*}
  u_t &= u_{yyy} - \frac{3(u_y^2 - r(u))}{2u_y}, \\
  v_t &= v_{yyy} - \frac{3(v_y^2 - s(u))}{2v_y}
\end{align*}
\]
where \( r(u) = h_x^2 - 2hh_{xx}, s(v) = h_y^2 - 2hh_{yy} \).

The consistency of (28) and (36) follows from the construction and the formula for \( y \)-symmetry is obvious since the independent variables \( x \) and \( y \) are now on an equal footing. Of course, the statement can also be proven directly. However, the passage to the limit from system (35) is more complicated in this example compared to the cases of the KdV and the Kaup equations. Indeed, the right-hand side of (35) does not even contain any term with \( u_x^2 \).

It is clear that this paradox is explained by the fact that the second equation (36) is not valid until \( c \) turns into 0 and it should be replaced with a certain formal power series with respect to \( c \), but we will not delve into this analysis. The flow (34) turns into the classical symmetry \( u_t = u_x^2 v_y / h = u_0 \) after multiplying the right-hand side by \( c \) and setting \( c = 0 \).

It may not be clear in the above exposition, where the hyperbolic system (29), (30) appears from, especially if one does not wish to employ the fact that (30) defines the \( x \)-part of the Bäcklund transformation for (28). Actually, no guess is needed. The search for a pair (25), compatible with a given evolutionary equation is just a matter of computation and if we start from the Krichever–Novikov equation then it quickly leads us to system (36). The only step which is not algorithmic here is the choice of a convenient gauge (27), but in this example it is quite obvious from symmetry arguments. When the Bäcklund variables are chosen, we may forget about the second equation (25) and search for a more general equation of the form (26) which is compatible with the \( t \)-dynamics. This leads to equation (29) with the additional parameter \( c \).

**7. Conclusion**

In this paper we have considered several examples of third order hyperbolic equations possessing higher evolutionary symmetries. These examples demonstrate that equations of such type may acquire a first integral under a parametric degeneration which is interpreted as a complementary hyperbolic equation consistent with the original one. Such consistent pairs of equations are interesting objects in themselves. Their study leads us to the notion of the Bäcklund variables, which provide convenient dynamical variables for the equation.
The example of the Krichever–Novikov equation suggests that introducing such variables is a natural and reasonable step if we wish to obtain a complete description of integrable hyperbolic equations. However, this classification problem seems rather difficult and it is unlikely that it will be solved in the near future. Therefore, realistic problems are the search for new examples, further study of the associated structures and the construction of explicit solutions.

The discrete analogues of hyperbolic equations (1) are lattice equations of the form
\[ h_{m,n}(u_{m,n}, u_{m+1,n}, u_{m,n+1}, u_{m+1,n+1}) = 0 \]
(see e.g. [23]). Recall that the theory behind them is also closely related with Bäcklund transformations: first, such equations define the nonlinear superposition principle for equations (1) and KdV type equations, second, they define the Bäcklund transformations for differential-difference equations of the Volterra lattice type. Like in the continuous case, some examples require the consideration of more general types of equations. In particular, it is interesting to extend our results to equations of the form
\[ f(u_{m,n}, u_{m+1,n}, u_{m+2,n}, u_{m,n+1}, u_{m+1,n+1}, u_{m+2,n+1}) = 0 \]
which play the role of discrete analogues of equations (2).

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Appendix

Here we present a sample Mathematica [24] program which allows us to check the compatibility of the Schwarzian KdV equation
\[ u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} \]
with hyperbolic equations of different types. The computation is performed in the standard set of dynamical variables (an implementation of the Bäcklund variables requires certain modifications). The partial derivative \( \partial^m_n \partial^o (u) \) is denoted \( u[m,n] \). First, the operators of the total derivatives with respect to \( x, y \) and \( t \) are defined, as well as the equation itself:
\[
\begin{align*}
\text{vars}[f_] &:= \text{Union}[\text{Cases}[f,\{u,\{0,\}\},\{\text{Infinity}\}]] \\
\text{diff}[f_] &:= \text{Apply}[[\text{Plus},\text{Map}[D[f,#] \cdot \text{diff}[#] &, \text{vars}[f]]]] \\
\text{dx}[f_] &:= D[f,x] + \text{diff}[f] / \cdot \text{diff}[u[m,n]] :> u[m+1,n] \\
\text{dy}[f_] &:= D[f,y] + \text{diff}[f] / \cdot \text{diff}[u[m,n]] :> u[m,n+1] \\
\text{dt}[f_] &:= \text{diff}[f] / \cdot \text{diff}[u[m,n]] :> \text{dx}[\text{dy}[ut,n],m] \\
\text{ut} &= u[3,0] - 3/2 \cdot u[2,0]^2 - 2/u[1,0];
\end{align*}
\]
In the next lines, the mixed derivatives are eliminated by using the second order hyperbolic equation (6) and its compatibility with the flow \( \delta \) is verified:
\[
\begin{align*}
\text{Clear}[u] \\
u[n_,1] &:= \text{dx}[uxy,n-1] /; n>0 \\
u[1,n_] &:= \text{dy}[uxy,n-1] /; n>0 \\
uxy &= 2 \cdot u[0,1] \cdot u[1,0] \cdot u[0,1] / (u[0,0]^2 + 1) \\
\text{Together}[\text{dx}[\text{dy}[ut]] - \text{dt}[uxy]]
\end{align*}
\]
Analogously, the following instructions handle the third order equation (33):

\[
\begin{align*}
\text{Clear}[u] \\
u[n_-,1] &:= \text{dx}[uxy,n-2]/;n>1 \\
u[2,n_-] &:= \text{dy}[uxy,n-1]/;n>0 \\
uxxy &= \frac{u[1,1]*u[2,0]/u[1,0]+(u[1,1]^2-c^2*u[1,0]^2)/(2*u[0,1])+u[0,1]}{(u[1,1]^2-2*c^2*u[1,0])^2/(2*u[0,1])^2+u[0,1]}; \\
\text{Together}[\text{dx}[dy[ut],2]-\text{dt}[uxxy]]
\end{align*}
\]

The last fragment is for the consistent pair (5). The last three lines verify the consistency of the pair itself, that is identity (4), and the compatibility of each equation of the pair with the flow \(\partial_t\):

\[
\begin{align*}
\text{Clear}[u] \\
u[n_-,1] &:= \text{dx}[uxy,n-2]/;n>1 \\
u[1,n_-] &:= \text{dy}[uxy,n-2]/;n>1 \\
uxxy &= \frac{u[1,1]*u[2,0]/u[1,0]+u[1,1]^2/(2*u[0,1])+u[0,1]}{u[0,1]}; \\
uxyy &= \frac{u[1,1]*u[0,2]/u[0,1]+u[1,1]^2/(2*u[1,0])+u[1,0]}{u[1,0]}; \\
\text{Together}[\text{dx}[uxy]-\text{dy}[uxy]] \\
\text{Together}[\text{dx}[dy[ut],2]-\text{dt}[uxy]] \\
\text{Together}[\text{dx}[dy[ut],2]-\text{dt}[uxy]]
\end{align*}
\]

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