Langevin Dynamics for Inverse Reinforcement Learning of Stochastic Gradient Algorithms

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Abstract

Inverse reinforcement learning (IRL) aims to estimate the reward function of optimizing agents by observing their response (estimates or actions). This paper considers IRL when noisy estimates of the gradient of a reward function generated by multiple stochastic gradient agents are observed. We present a generalized Langevin dynamics algorithm to estimate the reward function \( R(\theta) \); specifically, the resulting Langevin algorithm asymptotically generates samples from the distribution \( \exp(R(\theta)) \). The proposed IRL algorithms use kernel-based passive learning schemes. We also construct multi-kernel passive Langevin algorithms for IRL which are suitable for high dimensional data. The performance of the proposed IRL algorithms are illustrated on examples in adaptive Bayesian learning, logistic regression (high dimensional problem) and constrained Markov decision processes. We prove weak convergence of the proposed IRL algorithms using martingale averaging methods. We also analyze the tracking performance of the IRL algorithms in non-stationary environments where the utility function \( R(\theta) \) jump changes over time as a slow Markov chain.

Keywords. stochastic gradient algorithm, inverse reinforcement learning, weak convergence, martingale averaging theory, Langevin dynamics, stochastic sampling, inverse Bayesian learning, Constrained Markov Decision process, logistic regression, variance reduction

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I. INTRODUCTION

Inverse reinforcement learning (IRL) aims to estimate the reward function of optimizing agents by observing their actions (estimates). Classical IRL is off-line: given a data set of actions chosen according to the optimal policy of a Markov decision process, Ng and Russell [2000] formulated a set of inequalities that the reward function must satisfy. In comparison, this paper constructs and analyzes real time IRL algorithms by observing optimizing agents that are performing real time reinforcement learning (RL). The problem we consider is this: Suppose we observe estimates of multiple (randomly initialized) stochastic gradient algorithms (reinforcement learners) that aim to maximize a (possibly non-concave) expected reward. How to design another stochastic gradient algorithm (inverse learner) to estimate the expected reward function?

A. RL and IRL Algorithms

To discuss the main ideas, we first describe the point of view of multiple agents performing reinforcement learning (RL). These agents act sequentially to perform RL by using stochastic gradient algorithms to maximize a reward function. Let \( n = 1, 2, \ldots \) index agents that perform RL sequentially. The sequential protocol is as follows. The agents aim to maximize a possibly non-concave reward \( R(\theta) = \mathbb{E}(r_k(\theta)) \) where \( \theta \in \mathbb{R}^N \). We assume without loss of generality that reward \( R(\theta) \geq 0 \).

Each agent \( n \) runs a stochastic gradient algorithm over the time horizon \( k \in \{\tau_n, \tau_n + 1, \ldots, \tau_{n+1} - 1\} \):

\[
\theta_{k+1} = \theta_k + \varepsilon \nabla r_k(\theta_k), \quad k = \tau_n, \tau_n + 1, \ldots, \tau_{n+1} - 1
\]

initialized independently by \( \theta_{\tau_n} \sim \pi(\cdot) \) \hspace{1cm} (1)

Here \( \nabla r_k(\theta_k) \) denotes the sample path gradient evaluated at \( \theta_k \), and \( \tau_n, n = 1, 2, \ldots \) denote stopping times measurable wrt the \( \sigma \)-algebra generated by \( \{\theta_{\tau_n}, \nabla r_k(\theta_k), k = \tau_n, \tau_n + 1, \ldots\} \). The initial estimate \( \theta_{\tau_n} \) for agent \( n \) is sampled independently from probability density function \( \pi(\cdot) \) defined on \( \mathbb{R}^N \). Finally, \( \varepsilon \) is a small positive constant step size.

Next we consider the point of view of an observer that performs inverse reinforcement learning (IRL) to estimate the reward function \( R(\theta) \). The observer (inverse learner) knows initialization density \( \pi(\cdot) \) and only has access to the estimates \( \{\theta_k\} \) generated by RL algorithm (1). The observer reconstructs the gradient \( \nabla r_k(\theta_k) \) as \( \nabla r_k(\theta_k) = (\theta_k - \theta_{k+1})/\mu \) for some positive step size \( \mu \). The main idea of this paper is to propose and analyze the following IRL algorithm (which is a passive Langevin dynamics algorithm) deployed by the observer:

\[
\alpha_{k+1} = \alpha_k + \mu \left[ \frac{1}{\Delta N} K(\theta_k - \theta_k) \frac{\beta}{\Delta} \nabla r_k(\theta_k) + \nabla \alpha \pi(\alpha_k) \right] \pi(\alpha_k) + \sqrt{\mu} \pi(\alpha_k) w_k, \quad k = 1, 2, \ldots \] \hspace{1cm} (2)

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initialized by $\alpha_0 \in \mathbb{R}^N$. Here $\mu$ and $\Delta$ are small positive constant step sizes, $\{w_k, k \geq 0\}$ is an i.i.d. sequence of standard $N$-variate Gaussian random variables, and $\beta = \varepsilon / \mu$ is a fixed constant. Note that we have expressed (2) in terms of $\nabla_\theta r_k(\theta_k)$ (rather than $\nabla_\theta r_\alpha(\theta_k)$) since we have absorbed the ratio of step sizes into the scale factor $\beta$.

The key construct in (2) is the kernel function $K(\cdot)$. This kernel function is chosen by the observer such that $K(\cdot)$ decreases monotonically to zero as any component of the argument increases to infinity,

$$K(\theta) \geq 0, \quad K(\theta) = K(-\theta), \quad \int_{\mathbb{R}^N} K(\theta) d\theta = 1. \quad (3)$$

An example is to choose the kernel as a multivariate normal $\mathcal{N}(0, \sigma^2 I_N)$ density with $\sigma = \Delta$, i.e.,

$$\frac{1}{\Delta^N} K(\frac{\theta}{\Delta}) = (2\pi)^{-N/2} \Delta^{-N} \exp\left(-\frac{||\theta||^2}{2\Delta^2}\right),$$

which is essentially like a Dirac delta centered at 0 as $\Delta \to 0$. Our main result stated informally is as follows; see Theorem 1 in Sec.IV for the formal statement.

**Informal Statement of Result.** Based on the estimates $\{\theta_k\}$ generated by RL algorithm (1), the IRL algorithm (2) asymptotically generates samples $\{\alpha_k\}$ from the Gibbs measure

$$p(\alpha) \propto \exp(\beta R(\alpha)), \quad \alpha \in \mathbb{R}^N, \quad \text{where } \beta = \varepsilon / \mu. \quad (4)$$

To explain the above result, let $\hat{p}$ denote the empirical density function constructed from samples $\{\alpha_k\}$ generated by IRL algorithm (2). Then clearly

$$\log \hat{p}(\alpha) \propto R(\alpha) \text{.}$$

Thus IRL algorithm (2) serves as a non-parametric method for reconstructing reward $R$, given the estimates $\{\theta_k\}$ of RL algorithm (1). Alternatively, since IRL algorithm (2) generates samples $\alpha_k \sim \hat{p}(\cdot)$, a MCMC sampler can simulate samples from the density function proportional to $R(\alpha)$. (Since reward $R(\alpha) \geq 0$ by assumption, the normalized reward can be viewed as a probability density on $\mathbb{R}^N$.) Hence based on the estimates $\{\theta_k\}$ generated by RL algorithm (1), IRL algorithm (2) serves as a randomized sampling method for exploring the reward $R(\alpha)$ by simulating random samples from it. Finally, in adaptive Bayesian learning discussed in Sec.III, the RL agents maximize $\log R(\alpha)$ using gradient algorithm (1); then IRL algorithm (2) directly yields samples from $\beta R(\alpha)$.

**B. Context and Discussion**

The RL algorithm (1) together with IRL algorithm (2) constitute our main setup. More abstractly, the IRL problem we address is this: given a sequence of noisy sample path gradients $\{\nabla_\theta r_k(\theta_k)\}$, how to estimate the expected reward $R(\theta)$? To give additional insight we now discuss their context, useful generalizations of IRL algorithm (2), and related works in the literature.

(i) **Multiple agents evaluating gradients.** The multiple agent RL algorithm (1) is natural in non-convex stochastic optimization problems. Starting from various randomly chosen initial conditions $\theta_{r_\alpha} \sim \pi(\cdot)$, the agents evaluate the gradients $\nabla_\theta r_k(\theta_k)$ at various points $\theta_k$ to estimate the global maximizer. Since $\theta_{r_\alpha} \sim \pi(\cdot)$ are statistically independent, the RL agents can act in parallel (instead of sequentially). Several types of policy gradient algorithms for RL in the Markov decision process literature fit this framework. Given this sequence of gradients $\{\nabla_\theta r_k(\theta_k)\}$, the aim of this paper is to construct IRL algorithms to estimate $R(\theta)$.

(ii) **Passive IRL.** The IRL algorithm (2) is a Langevin dynamics based gradient algorithm with injected noise $\{w_k\}$. It is a passive learning algorithm since the gradients are not evaluated at $\alpha_k$ by the inverse learner; instead the gradients are evaluated at the random points $\theta_k$ chosen by the RL algorithm. This passive framework is natural in an IRL. The inverse learner passively observes the RL algorithm and aims to estimate its utility. The kernel $K(\cdot)$ in (2) effectively weights the usefulness of the gradient $\nabla_\theta r_k(\theta_k)$ compared to the required gradient $\nabla_\theta r_k(\alpha_k)$. If $\theta_k$ and $\alpha_k$ are far apart, then kernel $K((\theta_k - \alpha_k)/\Delta)$ will be small. Then only a small proportion of the gradient estimate $\nabla_\theta r_k(\theta_k)$ is added to the IRL iteration. On the other hand, if $\alpha_k = \theta_k$, then $\frac{1}{\Delta^2} K(\cdot) = 1$ and (2) becomes a standard Langevin dynamics type algorithm. We refer to Révész [1977], Hardle and Nixdorf [1987], Nazin et al. [1989], Yin and Yin [1996] for the analysis of passive stochastic gradient algorithms. The key difference compared to these works is that we are dealing with a passive Langevin dynamics algorithm, i.e., there is an extra injected noise term involving $w_k$.

(iii) **Multi-kernel IRL.** IRL algorithm (2) requires the gradient $\nabla_\theta r(\theta_k)$ and knowing the density $\pi(\cdot)$. In Sec.II-B2 we will discuss a two-time scale multi-kernel IRL algorithm, namely (15), that does not require knowledge of the density $\pi(\cdot)$. All that is required is a sequence of samples $\{\nabla_\theta r_k(\theta_i)\}$ when the IRL estimate is $\alpha_k$. The multi-kernel IRL algorithm (15) incorporates variance reduction and is suitable for high dimensional inference. In Sec.II-B, we also discuss several other

1Since the IRL algorithm does not know the step size $\varepsilon$ of the RL, it can only estimate $R(\cdot)$ up to a proportionality constant $\beta$. In classical Langevin dynamics $\beta$ denotes an inverse temperature parameter.
variations of IRL algorithm (2) including a mis-specified active IRL algorithm where the gradient is evaluated at a point \( \theta_k \) that is a corrupted value of \( \alpha_k \).

(iv) **Global Optimization vs IRL.** Langevin dynamics based gradient algorithms have been studied as a means for achieving global minimization for non-convex stochastic optimization problems, see for example Gelfand and Mitter [1991]. The papers Teh et al. [2016], Raginsky et al. [2017] give a comprehensive study of convergence of the Langevin dynamics stochastic gradient algorithm in a non-asymptotic setting. Also Welling and Teh [2011] studies Bayesian learning, namely, sampling from the posterior using stochastic gradient Langevin dynamics.

Langevin dynamics for global optimization considers the limit as \( \beta \to \infty \). In comparison, the IRL algorithms in this paper consider the case of fixed \( \beta = \epsilon/\mu \), since we are interested in sampling from the reward \( R(\cdot) \). Also, we consider passive Langevin dynamics algorithms in the context of IRL. Thus the IRL algorithm (2) is non-standard in two ways. First, as mentioned above, it has a kernel to facilitate passive learning. Second, the IRL algorithm (2) incorporates the initialization probability \( \pi(\cdot) \) which appears in the RL algorithm (1). Thus (2) is a non-standard generalized Langevin dynamics algorithm (which still has reversible diffusion dynamics).

(v) **Constant step size IRL.** An important feature of the IRL algorithm (2) and other IRL algorithms proposed in this paper is the constant step size \( \mu \) (as opposed to a decreasing step size). This facilities estimating (tracking) rewards that evolve over time. Sec.V gives a formal weak convergence analysis of the asymptotic tracking capability of the IRL algorithm (2) when the reward \( R(\cdot) \) jump changes over time according to a slow (but unknown) Markov chain. The most interesting case considered in Sec.V is when the reward changes at the same rate as the IRL algorithm. Then stochastic averaging theory yields a Markov switched diffusion limit as the asymptotic behavior of the IRL algorithm. Due to the constant step size, the appropriate notion of convergence is weak convergence [Kushner and Yin, 2003, Ethier and Kurtz, 1986, Billingsley, 1999]. The Markovian hyper-parameter tracking analysis generalizes our earlier work Yin et al. [2004, 2009] in stochastic gradient algorithms to the current case of passive Langevin dynamics with a kernel.

(vi) **Estimating utility functions.** Estimating a utility function given the response of agents is studied under the area of revealed preferences in microeconomics. Afriat’s theorem [Afriat, 1967, Diewert, 2012, Varian, 2012] in revealed preferences uses the response of a linearly constrained optimizing agent to construct a set of linear inequalities that are necessary and sufficient for an agent to be an utility maximizer; and gives a set valued estimate of the class of utility functions that rationalize the agents behavior. Different to revealed preferences, the current paper uses noisy gradients to recover the utility function and that too in real time via a constant step size Langevin diffusion algorithm.

(vii) **Interpretation as a numerical integration algorithm.** Finally, it is helpful to view IRL algorithm (2) as a numerical integration algorithm when the integrand (gradients to be integrated) are presented at random points and the integrand terms are corrupted by noise (noisy gradients). One possible offline approach is to discretize \( \mathbb{R}^N \) and numerically build up an estimate of the integral at the discretized points by rounding off the evaluated integrands terms to the nearest discretized point. However, such an approach suffers from the curse of dimensionality: one needs \( O(2^N) \) points to construct the integral with a specified level of tolerance. In comparison, the passive IRL algorithm (2) provides a principled real time approach for generating samples from the integral, as depicted by main result (4).

C. **Organization**

The rest of the paper is organized as follows:

1) Sec.II discusses the IRL algorithm (2), related works in the literature and gives an informal proof of convergence based on averaging theory arguments. Also the following IRL algorithms are discussed:
   a) A two time scale multi-kernel IRL algorithm with variance reduction. This IRL algorithm is illustrated in a high dimensional example.
   b) An active IRL algorithm with mis-specified gradient. That is, given the current estimate \( \alpha_k \), the IRL is given a gradient estimate at \( \nabla_\theta r_k(\alpha_k + v_k) \) where \( v_k \) is a noise process, and the mis-specified point \( \alpha_k + v_k \) is known to the IRL algorithm.
   c) A non-reversible diffusion IRL where a skew symmetric matrix yields a larger spectral gap and therefore faster convergence to the stationary distribution (at the expense of increased computational cost).
2) Sec.III gives three classes of numerical examples that illustrate our proposed IRL algorithms:
   a) Inverse Adaptive Bayesian learning
   b) IRL on a logistic regression classifier involving the adult dataset; this is a large dimensional example with \( N = 124 \) and requires careful use of the proposed multi-kernel IRL algorithm.
   c) IRL for reconstructing the cumulative reward of an finite horizon constrained Markov decision process (CMDP). Such CMDPs are non-convex in the action probabilities and have optimal polices that are randomized. We demonstrate how the Langevin-based IRL can learn a from a policy gradient RL algorithm.
3) Sec.IV gives a complete weak convergence proof of IRL algorithm (2) using martingale averaging methods. Sec.VI gives a formal proof of convergence of the multi-kernel algorithm (15).
4) Sec.V gives a formal weak convergence analysis of the asymptotic tracking capability of the IRL algorithm (2) when the utility function jump changes according to a slow (but unknown) Markov chain.
5) Finally, the appendix gives Matlab source codes for the three numerical examples presented in the paper. So the numerical results of this paper are fully reproducible.

II. INFORMAL PROOF AND ALTERNATIVE IRL ALGORITHMS

The RL algorithm (1) together with IRL algorithm (2) constitute our main setup. In this section we first start with an informal proof of convergence of (2) based on stochastic averaging theory; the formal proof is in Sec.IV. The informal proof is useful since gives additional insight into the design of related IRL algorithms. We will discuss several related IRL algorithms including a multi-kernel version with variance reduction.

A. Informal Proof of Main Result (4)

Since the IRL algorithm (2) uses a constant step size, the appropriate notion of convergence is weak convergence. Weak convergence [Ethier and Kurtz, 1986] is a function space generalization of convergence in distribution; function space because we prove convergence of the entire trajectory (stochastic process) rather than simply the estimate at a fixed time (random variable). As is typically done in weak convergence analysis, we first represent the sequence of estimates \( \{ \alpha_k \} \) generated by the IRL algorithm as a continuous-time random process. This is done by constructing the continuous-time trajectory via piecewise constant interpolation as follows: For \( t \in [0, T] \), define the continuous-time piecewise constant interpolated processes parametrized by the step size \( \mu \) as

\[
\theta^\mu(t) = \theta_k, \quad \alpha^\mu(t) = \alpha_k, \quad \text{for} \quad t \in [\mu k, \mu k + \mu). \tag{5}
\]

Sec.IV gives the detailed weak convergence proof using the martingale problem formulation of Strook and Varadhan [Ethier and Kurtz, 1986].

Our informal proof of the main result (4) proceeds in two steps:

Step I. We first fix the kernel step size \( \Delta \) and apply stochastic averaging theory arguments: this says that at the slow time scale, we can replace the fast variables by their expected value. For small step sizes \( \varepsilon \) and \( \mu = \varepsilon / \beta \), there are three time scales in IRL algorithm (2):

1) \( \{ \theta_k \} \) evolves slowly on intervals \( k \in \{ \tau_n, \tau_{n+1} - 1 \} \), and \( \{ \alpha_k \} \) evolves slowly versus \( k \).
2) We assume that the run-time of the RL algorithm (1) for each agent \( n \) is bounded by some finite constant, i.e., \( \tau_{n+1} - \tau_n < M \) for some constant \( M \). So \( \{ \tau_n \} \sim \pi \) is a fast variable compared to \( \{ \alpha_k \} \).
3) Finally the noisy gradient process \( \nabla \theta r_k(\cdot) \) evolves at each time \( k \) and is a faster variable than \( \{ \tau_n \} \) which is updated at stopping times \( \tau_n \).

With the above time scale separation, there are two levels of averaging involved. First averaging the noisy gradient \( \nabla \theta r_k(\cdot) \) yields \( \nabla \theta R(\cdot) \). Next averaging \( \{ \tau_n \} \) yields \( \theta \sim \pi \). Thus applying averaging theory to IRL algorithm (2) yields the following averaged system:

\[
\bar{\alpha}_{k+1} = \bar{\alpha}_k + \mu \mathbb{E}_\theta \sim \pi \left[ \frac{1}{\Delta N} K \left( \frac{\theta - \bar{\alpha}_k}{\Delta} \right) \beta \nabla \theta R(\theta) + \nabla \alpha \pi(\bar{\alpha}_k) \right] \pi(\bar{\alpha}_k) + \sqrt{\mu} \pi(\bar{\alpha}_k) w_k
\]

\[
= \bar{\alpha}_k + \mu \int_{\mathbb{R}^N} \frac{1}{\Delta N} K \left( \frac{\theta - \bar{\alpha}_k}{\Delta} \right) \beta \pi(\bar{\alpha}_k) \nabla \theta R(\theta) \pi(\theta) d\theta + \pi(\bar{\alpha}_k) \nabla \alpha \pi(\bar{\alpha}_k) + \sqrt{\mu} \pi(\bar{\alpha}_k) w_k \tag{6}
\]

Given the sequence \( \{ \bar{\alpha}_k \} \), define the interpolated continuous time process \( \bar{\alpha}^\mu \) as in (5). Then as \( \mu \) goes to zero, \( \bar{\alpha}^\mu \) converges weakly to the solution of the stochastic differential equation

\[
d\alpha(t) = \int_{\mathbb{R}^N} \frac{1}{\Delta N} K \left( \frac{\theta - \alpha}{\Delta} \right) \beta \pi(\alpha) \nabla \theta R(\theta) \pi(\theta) d\theta + \nabla \alpha \pi(\alpha) d\alpha + \pi(\alpha) dW(\alpha), \quad \alpha(0) = \alpha_0 \tag{7}
\]

where \( W(\alpha) \) is standard Brownian motion. Put differently, the Euler-Maruyama time discretization of (7) yields (6). To summarize (7) is the continuous-time dynamics of IRL algorithm (2).

Step II. Next, we set the kernel step size \( \Delta \rightarrow 0 \). Then \( K(\cdot) \) mimics a Dirac delta function and so the asymptotic dynamics of (7) become the diffusion

\[
d\alpha(t) = \left[ \frac{1}{2} \beta \pi(\alpha) \nabla \alpha R(\alpha) d\alpha + \nabla \alpha \pi(\alpha) d\alpha + dW(\alpha) \right] \pi(\alpha), \quad \alpha(0) = \alpha_0 \tag{8}
\]

Finally, (8) is a reversible diffusion and its stationary measure is the Gibbs measure \( \mu(\alpha) \) defined in (4). Showing this is straightforward: Recall [Karatzas and Shreve, 1991] that for a generic diffusion process denoted as \( d\alpha(t) = f(\alpha) dt + \sigma(\alpha) dW(t) \), the stationary distribution \( \mu \) satisfies

\[
\mathbb{L}^* \mu = \frac{1}{2} \text{Tr}[\nabla^2(\Sigma \mu)] - \text{div}(f \mu) = 0, \quad \text{where} \quad \Sigma = \sigma \sigma' \tag{9}
\]

\[\text{Note Stramer and Tweedie [1999, Eq.34] has a typographic error in specifying the determinant.}\]
and $\mathcal{L}^*$ is the forward operator. From (8), $f(\alpha) = [\beta R(\alpha) + \nabla_\alpha \pi(\alpha)]\pi(\alpha)$, $\sigma = \pi(\alpha)I$. Then it is verified by elementary calculus that $p(\alpha) \propto \exp(\beta R(\alpha))$ satisfies (9).

To summarize, we have shown informally that IRL algorithm (2) generates samples from (4). Sec.IV gives the formal weak convergence proof.

(v) Why not use classical Langevin dynamics? The passive version of the classical Langevin dynamics algorithm reads:

$$
\alpha_{k+1} = \alpha_k + \mu \frac{1}{\Delta N} K(\theta_k - \alpha_k) \nabla R(\theta_k) + \sqrt{\mu} \sqrt{\frac{2}{\beta}} w_k, \quad k = 1, 2, \ldots
$$

(10)

where $\theta_k$ are computed by RL (1). Then averaging theory (as $\mu \to 0$ and then $\Delta \to 0$) yields the following asymptotic dynamics (where $W(t)$ denotes standard Brownian motion)

$$
d\alpha(t) = \nabla R(\alpha) \pi(\alpha) dt + \sqrt{\frac{2}{\beta}} dW(t), \quad \alpha(0) = \alpha_0
$$

(11)

Then the stationary distribution of (11) is proportional to $\exp(\beta \int \nabla R(\alpha) \pi(\alpha) d\alpha)$. Unfortunately, this is difficult to relate to $R(\alpha)$ and therefore less useful. In comparison the generalized Langevin algorithm (2) yields samples from stationary distribution proportional to $\exp(\beta R(\alpha))$ from which $R(\alpha)$ is easily estimated (as discussed below (4)). This is the reason why we will use the passive generalized Langevin dynamics (2) for IRL instead of the passive classical Langevin dynamics (10).

B. Alternative IRL Algorithms

IRL algorithm (2) is the vanilla IRL algorithm considered in this paper and its formal proof of convergence is given in Sec.IV. In this section we discuss several variations of IRL algorithm (2). The algorithms discussed below include a passive version of the classical Langevin dynamics, a two-time scale multi-kernel MCMC based IRL algorithm (for variance reduction) and finally, a non-reversible diffusion algorithm. The construction of these algorithms are based on the informal proof discussed above.

1) Passive Langevin Dynamics Algorithms for IRL: IRL algorithm (2) can be viewed as a passive modification of the generalized Langevin dynamics proposed in Stramer and Tweedie [1999]. Since generalized Langevin dynamics includes classical Langevin dynamics as a special case, it stands to reason that we can construct a passive version of the classical Langevin dynamics algorithm. Indeed, instead of (2), the following passive Langevin dynamics can be used for IRL (initialized by $\alpha_0 \in \mathbb{R}^N$):

$$
\alpha_{k+1} = \alpha_k + \frac{1}{\Delta N} K(\theta_k - \alpha_k) \frac{\beta}{2 \pi(\alpha)} \nabla R(\theta_k) + \sqrt{\mu} w_k, \quad k = 1, 2, \ldots
$$

(12)

Note that this algorithm is different to (10) due to the term $\pi(\alpha_k)$ in the denominator; this makes a crucial difference. Indeed, unlike (10), algorithm (12) generates samples from (4), as we now explain: By stochastic averaging theory arguments as $\mu$ goes to zero, the interpolated processes $\alpha^\mu$ converges weakly to (where $\mathcal{W}(t)$ below is standard Brownian motion)

$$
d\alpha(t) = \int_{\mathbb{R}^N} \frac{1}{\Delta N} K(\theta - \alpha) \left[ \frac{\beta}{2 \pi(\alpha)} \nabla R(\theta) dt \right] \pi(\theta) d\theta + d\mathcal{W}(t), \quad \alpha(0) = \alpha_0
$$

(13)

Again as $\Delta \to 0$, $K(\cdot)$ mimics a Dirac delta function and so the $\pi(\cdot)$ in the numerator and denominator cancel out. Therefore the asymptotic dynamics become the reversible diffusion

$$
d\alpha(t) = \frac{\beta}{2} \nabla R(\alpha) dt + d\mathcal{W}(t), \quad \alpha(0) = \alpha_0
$$

(14)

Note that (14) is the classical Langevin diffusion and has stationary distribution $p$ specified by (4). So algorithm (12) asymptotically generates samples from (4).

Finally, we note that Algorithm (12) can be viewed as a special case of IRL algorithm (2) since its limit dynamics (14) is a special case of the limit dynamics (8) with $\pi(\cdot) = 1$.

2) Variance Reduction for High Dimensional IRL: For large dimensional problems (e.g., $N = 124$ in the numerical example of Sec.III), the passive IRL algorithm (2) can take a very large number of iterations to converge to its stationary distribution. This is because with high probability, the kernel $K(\theta_k, \alpha_k)$ will be close to zero and so updates of $\alpha_k$ will occur very rarely.

There is strong motivation to introduce variance reduction in the algorithm. Below we propose a two time step, multi-kernel variance reduction IRL algorithm motivated by importance sampling. Apart from the ability to deal with high dimensional problems, the algorithm also does not require knowledge of the initialization probability density $\pi(\cdot)$.

Suppose the IRL operates at a slower time scale than the RL algorithm. At each time $k$ (on the slow time scale), by observing the RL algorithm, the IRL obtains a pool of samples of the gradients $\nabla \theta r_k(\theta_{k,i})$ evaluated at a large number of points $\theta_{k,i}$,
\[ \alpha_{k+1} = \alpha_k + \mu \frac{\beta}{2} \sum_{i=1}^{L} p(\theta_{k,i}|\alpha_k) \nabla_{\theta} r_k(\theta_{k,i}) + \sqrt{\mu} w_k, \quad \theta_{k,i} \sim \pi(\cdot) \]  
(15)

In (15), we choose the conditional probability density function \( p(\theta|\alpha) \) as follows:

\[ p(\theta|\alpha) = p_{\nu}(\theta - \alpha) \quad \text{where} \quad p_{\nu}(\cdot) = \mathcal{N}(0, \sigma^2 I_N). \]  
(16)

For notational convenience, for each \( \alpha \), denote the normalized weights in (15) as

\[ \gamma_{k,i}(\alpha) = \frac{p(\theta_{k,i}|\alpha_k)}{\sum_{i=1}^{L} p(\theta_{k,i}|\alpha_k)}, \quad i = 1, \ldots, L \]  
(17)

Then these \( L \) normalized weights qualify as symmetric kernels in the sense of (3). Thus IRL algorithm (15) can be viewed as a multi-kernel passive stochastic approximation algorithm. Note that the algorithm does not require knowledge of \( \pi(\cdot) \).

Since for each \( k \), the samples \( \{\theta_{k,i}, i = 1, \ldots, L\} \) are generated i.i.d., it is well known from self-normalized importance sampling Cappe et al. [2005] that as \( L \to \infty \), then for fixed \( \alpha \),

\[ \sum_{i=1}^{L} \gamma_{k,i}(\alpha) \nabla_{\theta} r_k(\theta_{k,i}) \to \mathbb{E}\{\nabla_{\theta} r_k(\theta)|\alpha\} \quad \text{w.p.1}, \]  
(18)

provided \( \mathbb{E}|\nabla r(\theta)| < \infty \). Similar results can also be established more generally if \( \{\theta_{k,i}, i = 1, \ldots, L\} \) is a geometrically ergodic Markov process with stationary distribution \( \pi(\cdot) \).

Remark: Clearly the conditional expectation \( \mathbb{E}\{\nabla_{\theta} r_k(\theta)|\alpha\} \) always has smaller variance than \( \nabla_{\theta} r_k(\theta) \); therefore variance reduction is achieved in IRL algorithm (15).

In sequential Markov chain Monte Carlo (particle filters), to avoid degeneracy, one resamples from the pool of “particles” \( \{\theta_i, i = 1, \ldots, L\} \) according to the probabilities (normalized weights) \( \gamma_i \). For large \( L \), the resulting resampled particles have a density \( p(\theta|\alpha_k) \). However, we are only interested in computing an estimate of the gradient (not in propagating particles over time). So we use the estimate \( \sum_{i=1}^{L} \gamma_{k,i} \nabla_{\theta} r_k(\theta_{k,i}) \) in (15); this always has a smaller variance than resampling and then estimating the gradient; see Ross [2013, Sec.12.6] for an elementary proof.

Why not use the popular MCMC tool of sequential importance sampling with resampling? Such a process resamples from the pool of particles and pastes together components of \( \theta_i \) from other more viable candidates \( \theta_j \). As a result \( L \) composite vectors are obtained which are more viable. However, since our IRL framework is passive, this is of no use since we cannot obtain the gradient for these \( L \) composite vectors. Recall that in our passive framework, the IRL has no control over where the gradients \( \nabla_{\theta} r_k(\theta) \) are evaluated.

Informal Analysis of IRL algorithm (15). By stochastic averaging theory arguments as \( \mu \) goes to zero, the interpolated process \( \alpha^\mu \) from IRL algorithm (15) converges weakly to

\[ d\alpha(t) = \int_{\mathbb{R}^N} \beta \nabla_\alpha R(\theta) p(\theta|\alpha(t)) \, d\theta \, dt + dW(t), \quad \alpha(0) = \alpha_0 \]  
(19)

where \( W(t) \) is standard Brownian motion. Notice that even though \( \theta_i \) are sampled from the density \( \pi(\cdot) \), the above averaging is w.r.t. the conditional density \( p(\theta|\alpha) \) because of (18). For small variance \( \sigma^2 \), the conditional density (symmetric kernel) \( p(\theta|\alpha) \) in (19) acts as a Dirac delta yielding the classical Langevin diffusion

\[ d\alpha(t) = \frac{\beta}{2} \nabla_\alpha R(\alpha(t)) \, dt + dW(t) \]  
(20)

Therefore algorithm (15) generates samples from distribution (4). The formal proof is in Sec.VI.

3) Active IRL with Mis-specified Gradient: Thus far we have considered the case where the RL algorithm provides estimates \( \nabla_{\theta} r_k(\theta_k) \) at randomly chosen points independent of the IRL estimate \( \alpha_k \). In other words, the IRL is passive and has no role in determining where the RL algorithm evaluates gradients.

We now consider a modification where the RL algorithm gives a noisy version of the gradient evaluated at a stochastically perturbed value of \( \alpha_k \). That is, when the IRL estimate is \( \alpha_k \), it requests the RL algorithm to provide a gradient estimate \( \nabla_{\theta} r_k(\alpha_k) \). But the RL algorithm evaluates the gradient at a mis specified point \( \theta_k = \alpha_k + v_k \), namely, \( \nabla_{\theta} r_k(\alpha_k + v_k) \). Here \( v_k \sim \mathcal{N}(0, \sigma^2 I_N) \) is an i.i.d. sequence. The RL algorithm then provides the IRL algorithm with \( \theta_k \) and \( \nabla_{\theta} r_k(\theta_k) \). So, instead of \( \theta_k \) being independent of \( \alpha_k \), now \( \theta_k \) is conditionally dependent on \( \alpha_k \) as

\[ p(\theta_k|\alpha_k) = \frac{1}{(2\pi)^N \sigma^N} \exp\left(-\frac{1}{2\sigma^2} \|\theta_k - \alpha_k\|^2\right), \quad \theta_k, \alpha_k \in \mathbb{R}^N \]  
(21)

In other words, the IRL now actively specifies where to evaluate the gradient; however, the RL algorithm evaluates a noisy gradient and that too at a stochastically perturbed (mis-specified) point \( \theta_k \).
The active IRL algorithm we propose is as follows:
\[ \alpha_{k+1} = \alpha_k + \mu \frac{1}{\Delta N} K\left(\frac{\theta_k - \alpha_k}{\Delta}\right) \frac{\beta}{2 p(\theta_k | \alpha_k)} \nabla_{\theta} r_k(\theta_k) + \sqrt{\mu} w_k, \quad \text{where} \quad \theta_k = \alpha_k + v_k \] (22)

The proof of convergence again follows using averaging theory arguments. Since \( \{\theta_k\} \sim p(\theta | \alpha_k) \) is the fast signal and \( \{\alpha_k\} \) is the slow signal, the averaged system is
\[ d\alpha(t) = \int_{\mathbb{R}^N} \frac{1}{\Delta N} K\left(\frac{\theta - \alpha}{\Delta}\right) \frac{\beta}{2 p(\theta | \alpha(t))} \nabla_{\theta} R(\theta) p(\theta | \alpha(t)) \, d\theta \, dt + dW(t) \]
So the \( p(\theta | \alpha(t)) \) cancel out in the numerator and denominator. As \( \Delta \to 0 \), the kernel acts as a Dirac delta thereby yielding the classical Langevin diffusion (20).

**Remark:** The active IRL algorithm (22) can be viewed as an idealization of the multi-kernel IRL algorithm (15). The multi-kernel algorithm constructs weights to approximate sample from the conditional distribution \( p(\theta | \alpha) \). In comparison, the active IRL has direct measurements from this conditional density. So the active IRL algorithm can be viewed as an upper bound to the classical Langevin diffusion (20).

4) **Non-reversible Diffusion for IRL:** So far we have defined four different passive Langevin dynamics algorithms for IRL, namely (2), (12), (15), and (22). These algorithms yield reversible diffusion processes that asymptotically sample from the stationary distribution (4). It is well known [Hwang et al., 1993, 2005, Pavliotis, 2014] that adding a skew symmetric matrix to the gradient always improves the convergence rate of Langevin dynamics to its stationary distribution. That is for any \( N \times N \) dimensional skew symmetric matrix \( S = -S^T \), the non-reversible diffusion process
\[ d\alpha(t) = \frac{\beta}{2} (I_N + S) \nabla_{\alpha} R(\alpha) \, dt + dW(t), \quad \alpha(0) = \alpha_0 \] (23)
has a larger spectral gap and therefore converges to stationary distribution \( \pi(\alpha) \) faster than (8). The resulting IRL algorithm obtained by a Euler-Maruyama time discretization of (23) and then introducing a kernel \( K(\cdot) \) is
\[ \alpha_{k+1} = \alpha_k + \mu \frac{1}{\Delta N} K\left(\frac{\theta_k - \alpha_k}{\Delta}\right) \frac{\beta}{2 \pi(\alpha_k)} \nabla_{\theta} r_k(\theta_k) + \sqrt{\mu} w_k, \quad k = 1, 2, \ldots \] (24)
initialized by \( \alpha_0 \in \mathbb{R}^N \). Again a stochastic averaging theory argument shows that IRL algorithm (24) converges weakly to the non-reversible diffusion (23). In numerical examples, we found empirically that the convergence of (24) is faster than (2) or (12). However, the faster convergence comes at the expense of an order of magnitude increased computational cost. The computational cost of IRL algorithm (24) is \( O(N^2) \) at each iteration due to multiplication with skew symmetric matrix \( S \). In comparison the computational costs of IRL algorithms (2) and (12) are each \( O(N) \).

**III. NUMERICAL EXAMPLES**

This section presents three examples to illustrate the performance of the proposed IRL algorithms.

**A. Example 1. Inverse Adaptive Bayesian Learning**

The framework described here comprises of the RL algorithm (1) that uses multiple randomly initialized stochastic gradient algorithms to estimate the maxima of a possibly-multi-modal expected posterior distribution; see Welling and Teh [2011]. By viewing these RL estimates, the inverse learner deploys an IRL algorithm (2) to estimate the maxima of the multi-modal posterior density and also explore high probability regions of the posterior.

**Setup:** A RL system aims to estimate the optimal randomized policy
\[ u^*(y_1, \ldots, y_T) = \max_\theta p(\theta | y_1, \ldots, y_T) \]
by solving the following non-concave stochastic optimization problem: Find
\[ \theta^* = \arg \max_\theta J(\theta, \theta^o) = E_{\theta^o} \{ \log p(\theta | y_1, \ldots, y_T) \} \]
Here \( \theta = [\theta(1), \theta(2)]' \in \mathbb{R}^2 \) and \( \theta^o \) is the true parameter value which is unknown to the learner. The prior is \( p(\theta) = \mathcal{N}(0, \Sigma) \) where \( \Sigma = \text{diag}[10, 2] \). The observations \( y_k \) are independent and distributed according to the multi-modal mixture likelihood
\[ y_k \sim p(y | \theta) = \frac{1}{2} \mathcal{N}(\theta(1), 2) + \frac{1}{2} \mathcal{N}(\theta(1) + \theta(2), 2) \]
For true parameter value \( \theta^o = [-1, 1]' \), it can be verified that the objective \( J(\theta, \theta^o) \) is non-concave in \( \theta \) and has two maxima at \( \theta = [0, 1]' \) and \( \theta = [1, -1]' \). Figure 1 plots the empirical density and contours of \( p(\theta | y_1, \ldots, y_T) \) for \( T = 100 \) using the Metropolis Hastings algorithm.
**RL Algorithm:** Since $y_1, \ldots, y_T$ are independent and identically distributed, the objective $J(\theta, \theta^o)$ can be re-expressed as

$$J(\theta, \theta^o) = E_{\theta^o}\{\log p(\theta) + T \log p(y|\theta)\} \quad (25)$$

The RL agents aim to estimate the optimal parameter $\theta^*$. To do so, the RL agents use the stochastic gradient algorithm (1):

$$\theta_{k+1} = \theta_k + \varepsilon \left[ \nabla_{\theta} \log p(\theta_k) + T \nabla_{\theta} \log p(y_k|\theta_k) \right] \quad (26)$$

with multiple random initializations, depicted by agents $n = 1, 2, \ldots$. For each agent $n$, the initial estimate $\theta_{\tau_n} \sim \pi(\cdot) = N(0, I_{2 \times 2})$. Each agent runs the gradient algorithm for 100 iterations with step size $\varepsilon = 10^{-3}$ and the number of agents is $10^5$. Thus the sequence $\{\theta_k; k = 1, \ldots, 10^7\}$ is generated.

**IRL algorithm and performance:** Given the sequence $\{\theta_k\}$ generated above, and initialization density $\pi$, the inverse learner aims to generate samples from suitably normalized $J(\theta, \theta^o)$ in (25). Since the inverse learner has no control of where the reinforcement learner evaluates its gradients, we are in passive IRL setting.

To benchmark the IRL algorithms we first ran the classical Langevin dynamics algorithm:

$$\alpha_{k+1} = \alpha_k + \mu \beta \nabla r(\alpha_k) + \sqrt{\mu} w_k, \quad k = 1, 2, \ldots$$

which corresponds to the ground truth (since the gradients are evaluated at $\alpha_k$). Figure 2 displays the both the empirical histogram and a contour plot of the estimate $J(\theta, \theta^o)$ generated by classical Langevin dynamics; this can be viewed as the ground truth.

Next, we illustrate the performance of the IRL algorithm (2) with kernel $K(\theta, \alpha) \propto \exp(-\frac{\|\alpha - \theta\|^2}{0.02^2})$, step size $\mu = 5 \times 10^{-4}$, $\beta = 1$. Figure 3 displays both the empirical histogram and a contour plot. Notice that the performance of the IRL is very similar to classical Langevin dynamics (ground truth).

Finally, we illustrate the performance of the two time-scale multikernel algorithm (15). Recall this algorithm does not require knowledge of the initialization probabilities $\pi(\cdot)$. Figure 4 displays the both the empirical histogram and a contour plot. Again the performance of the IRL is very similar to classical Langevin dynamics.

**Multiple Inverse Learners:** We also considered the case where multiple inverse learners act in parallel. Suppose each inverse learner $l \in \{1, 2, \ldots, L\}$ deploys IRL algorithm (2) with its own noise sample path denoted $\{w_k^{(l)}\}$, which is independent of that of other inverse learners. Obviously, if the estimate $\alpha_k^{(l)}$ of one of the inverse learners (say $l$) is close to $\theta_k$, then $\nabla r_k(\alpha_k^{(l)})$ is a more accurate gradient estimate for $\nabla r_k(\theta_k)$. However, for high dimensional problems, we found very little benefit unless the number of inverse learners is chosen as $L = O(2^N)$ which is intractable.

B. Example 2. IRL with Logistic Regression Classifier

Here we consider a high dimensional IRL problem ($N = 124$) on the benchmark adult a9a dataset. Performing IRL, i.e., generating samples from a 124 dimensional probability density that represents the utility, is challenging and requires use of the multi-kernel variance reduced IRL algorithm (15).
Fig. 2: Classical Langevin dynamics (ground truth)

Fig. 3: IRL Algorithm (2)

Fig. 4: Two time scale multi-kernel IRL Algorithm (15)
Setup: In a logistic regression model parameterized by $\theta \in \mathbb{R}^N$, the observations (labels) $y_k \in \{0, 1\}$ are assumed to be generated probabilistically from

$$P(y_k = 1|\theta) = \sigma(\psi_k^T \theta) = \frac{1}{1 + \exp(-\psi_k^T \theta)}, \quad \theta \in \mathbb{R}^N$$

Here $\psi_k \in \mathbb{R}^N$ is known input vector at time $k$ and is called the feature.

We consider a Bayesian setting where the prior of $\theta$ is assumed to be an $N$-variate Laplacian density with independent components. So the prior is

$$p(\theta) \propto \exp\left(-\sum_{i=1}^{N} |\theta(i)|\right)$$

As in Sec.III-A, we assume that the RL aims to maximize

$$J(\theta, \theta^o) = \mathbb{E}_{\theta^o}\{\log p(\theta) + T \log p(y|\theta)\}$$

The RL agents aim to estimate the optimal parameter $\theta^o$. To do so, the RL agents use the stochastic gradient algorithm

$$\theta_{k+1} = \theta_k + \epsilon \left[\nabla_\theta \log p(\theta_k) + T \nabla_\theta \log p(y_k|\theta_k)\right]$$

Note that for the logistic model, $\nabla_\theta \log p(\theta_k) = -\sigma(\psi_k^T \theta) \text{ elementwise}$ and $\nabla_\theta \log p(y_k|\theta_k) = \psi_k(y_k - \sigma(\psi_k^T \theta_k))$.

Dataset: We consider the benchmark adult a9a dataset which can be downloaded from https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html.

The dataset consists of a time series of binary valued (categorical) observations $y_k \in \{0, 1\}$ and a time series of regression vectors $\psi_k \in \mathbb{R}^{123}$ for $k = 1, \ldots, 32651$. To model the bias, we add one additional component; so the unknown parameter vector is $\theta \in \mathbb{R}^{124}$ and the augmented regression vectors are $\psi_k = \begin{bmatrix} 1 \\ \psi_k \end{bmatrix} \in \mathbb{R}^{124}$, for $k = 1, \ldots, 32651$.

Performance of IRL Algorithm (15): Since $N = 124$, the IRL algorithm needs to explore and sample from a 124-variate distribution which is a formidable task. The vanilla IRL algorithm (2) is not tractable since it would take a prohibitive number of iterations to converge. We illustrate the performance of the multi-kernel variance reduction IRL algorithm (15).

We ran multi-kernel IRL algorithm (15) and active IRL algorithm (22) on the a9a dataset. As mentioned in Sec.II-B3, the active IRL (22) is an idealization of the multikernel IRL algorithm (15) and so forms a benchmark for it. The parameters were chosen as $\mu = 2.5 \times 10^{-4}$, $\pi(\theta) = \mathcal{N}(0, I)$, $\sigma = 0.1$, $L = 100$ in (15) and $T = 10$ in (27). As in Welling and Teh [2011], we ran 10 “sweeps” through the dataset. That is, we appended 9 repetitions of the data set resulting in a single dataset of $10 \times 32651$ time points; and then ran the IRL algorithms on this appended dataset.

To benchmark these algorithms, we also ran the classical Langevin dynamics algorithm:

$$\alpha_{k+1} = \alpha_k + \mu \beta \nabla r_k(\alpha_k) + \sqrt{\mu} w_k, \quad k = 1, 2, \ldots$$

which corresponds to the ground truth (since the gradients are evaluated at $\alpha_k$).

IRL Algorithms (15) and (22) generate samples $\{\alpha_k\}$ from a 124-dimensional distribution. To visualize the performance, we used the output sequence $\{\alpha_k\}$ from these algorithms to compute the empirical cumulative distribution functions for each of the 124 marginal distributions, denoted $F_i(\alpha(i))$, $i = 1, \ldots, 124$. For each such marginal empirical distribution, we then computed the corresponding marginal from the classical Langevin dynamics (28), denoted as $F_i(\alpha(i))$; this can be viewed as the ground truth. Finally, we computed the $L_1$ distance (Wasserstein 1-metric)

$$d(i) = \int |\hat{F}_i(\alpha(i)) - F_i(\alpha(i))| \, d\alpha(i), \quad i = 1, \ldots, 124$$

This $L_1$ distance is more appropriate for our purposes than the Kolmogorov-Smirnov distance since typically the constant or proportionality $\beta$ is not known and so the regions of support of the empirical cdfs can vary substantially.

Figure 5(a) and (b) displays the $L_1$ distance $d(i)$ vs $i = 1, 2, \ldots, 124$ for the IRL Algorithms (15) and (22). In a sense, Algorithm (22) can be viewed as an upper bound for the performance of Algorithm (15) since the conditional density $p(\theta|\alpha_k)$ used to generate $\theta$ in Algorithm (22) is exactly the same kernel used in Algorithm (15). As can be seen from Figure 5(a) and (b), the two algorithms perform similarly, despite the fact that Algorithm (15) has no control over where the derivative is evaluated. This shows that the IRL algorithm is a viable method for sampling from the high-dimensional Bayesian posterior; or equivalently estimating $J(\theta, \theta^o)$ in (27). Finally, Figure 5(c) shows the marginal distribution for the 117-th component of $\theta$ for the classical Langevin (ground truth), active IRL (mis-specified), multi-kernel IRL and a naive Langevin. By naive Langevin we mean the Langevin algorithm that uses the gradient $\nabla_\theta r_k(\theta_k)$ instead of $\nabla_\theta r_k(\alpha_k)$ at the estimate $\alpha_k$, without any kernel. We see that the multi-kernel and active IRL are close to the ground truth (Langevin) while the naive IRL performs very poorly (since it completely disregards the fact that the gradients evaluated at $\alpha_k$ and $\theta_k$ are different).
Fig. 5: Comparison of multi-kernel IRL Algorithm (15) and active IRL algorithm (22) with classical Langevin (28) (ground truth)

C. Example 3. IRL for Constrained Markov Decision Process (CMDP)

In this section we illustrate the performance of the IRL algorithms for reconstructing the cumulative reward of a constrained Markov decision process (CMDP) given gradient information from a RL algorithm. This is in contrast to classical IRL [Ng and Russell, 2000] where the transition matrices of the MDP are assumed known to the inverse learner.

Consider a unichain\(^3\) average reward CMDP \(\{x_n\}\) with finite state space \(\mathcal{X} = \{1, \ldots, X\}\) and action space \(\mathcal{U} = \{1, 2, \ldots, U\}\). The CMDP evolves with transition probability matrix \(P(u)\) where

\[
P_{ij}(u) \triangleq \mathbb{P}[x_{n+1} = j|x_n = i, u_n = u], \quad u \in \mathcal{U}. \tag{30}
\]

When the system is in state \(x_n \in \mathcal{X}\), an action \(u_n = u(x_n) \in \mathcal{U}\) is chosen, where \(u\) denotes (a possible randomized) stationary policy. The reward incurred at stage \(n\) is \(r(x_n, u_n) \geq 0\).

Let \(D\) denote the class of stationary randomized Markovian policies. For any stationary policy \(u \in D\), let \(\mathbb{E}_u\) denote the corresponding expectation and define the infinite horizon average reward

\[
J(u) = \lim_{T \to \infty} \inf \frac{1}{T} \mathbb{E}_u \left[ \sum_{n=1}^{T} r(x_n, u_n) \mid x_0 = x \right]. \tag{31}
\]

\(^3\)By *unichain* [Puterman, 1994, pp. 348] we mean that every policy where \(u_n\) is a deterministic function of \(x_n\) consists of a single recurrent class plus possibly an empty set of transient states.
Motivated by modeling fairness constraints in network optimization [Ngo and Krishnamurthy, 2010], we consider the reward (31), subject to the average constraint:

\[ B(u) = \lim_{T \to \infty} \inf \frac{1}{T} \mathbb{E}_u \left[ \sum_{n=1}^{T} \beta(x_n, u_n) \right] \leq \gamma, \]  

(32)

(31), (32) constitute a CMDP. Solving a CMDP involves computing the optimal policy \( u^* \in \mathcal{D} \) that satisfies

\[ J(u^*) = \sup_{u \in \mathcal{D}} J(u) \quad \forall x_0 \in \mathcal{X}, \text{ subject to } (32) \]  

(33)

To solve a CMDP, it is sufficient to consider randomized stationary policies:

\[ u(x) = u \text{ with probability } \phi(u|x) = \frac{\tilde{\phi}(x, u)}{\sum_{\tilde{u} \in U} \phi(x, \tilde{u})}. \]  

(34)

Here the conditional probabilities \( \phi \) and joint probabilities \( \tilde{\phi} \) are defined as

\[ \phi(u|x) = \mathbb{P}(u_n = u|x_n = x), \quad \tilde{\phi}(x, u) = \mathbb{P}(u, x). \]  

(35)

Then the optimal policy \( u^* \) is obtained as the solution of a linear programming problem in terms of the \( X \times U \) elements of \( \phi \); see Puterman [1994] for the precise equations. Also [Altman, 1999], the optimal policy \( u^* \) of the CMDP is randomized for at most one of the states. That is,

\[ u^*(x) = p \ u^*_1(x) + (1 - p) \ u^*_2(x) \]  

(36)

where \( p \in [0,1] \) denotes the randomization probability and \( u^*_1, u^*_2 \) are pure (non-randomized) policies. Of course, when there is no constraint (32), the CMDP reduces to classical MDP and the optimal stationary policy \( u^*(x) \) is a pure policy. That is, for each state \( x \in \mathcal{X} \), there exists an action \( u \) such that \( \phi(u|x) = 1 \).

Remark. (32) is a global constraint that applies to the entire sample path [Altman, 1999]. Since the optimal policy is randomized, classical value iteration based approaches and Q-learning cannot be used to solve CMDPs as they yield deterministic policies. One can construct a Lagrangian dynamic programming formulation [Altman, 1999] and Lagrangian Q-learning algorithms [Djouadj and Krishnamurthy, 2007]. Below for brevity, we consider a policy gradient RL algorithm.

1) Policy Gradient for RL of CMDP: Having specified the CMDP model, we next turn to the RL algorithm. RL algorithms\(^4\) are used to estimate the optimal policy of an MDP when the transition matrices are not known. Then the LP formulation in terms of joint probabilities \( \phi \) is not useful since the constraints depend on the transition matrix. In comparison, policy gradient RL algorithms are stochastic gradient algorithms of the form (1) that operate on the conditional action probabilities \( \phi(u|x) \) defined in (35) instead of the joint probabilities \( \tilde{\phi}(x, u) \).

Note that (33) written as a minimization (in terms of \( -J \)), together with constraint (32) is in general, no longer a convex optimization problem in the variables \( \phi \); see Figure 6 for an illustration. So it is not possible to guarantee that simple gradient descent schemes\(^5\) can achieve the global optimal policy. This motivates the setting of (1) where multiple agents that are initialized randomly aim to estimate the optimal policy.

Since the problem is non-convex, and the inequality constraint is active (i.e. achieves equality) at the global maximum, we assume that the RL agents use a quadratic penalty method: For \( \lambda \geq 0 \), denote the quadratic penalized objective to be maximized as

\[ R(\phi) = J(\phi) - \lambda (B^2(\phi) - \gamma) \]  

(37)

Such quadratic penalty functions are used widely for equality constrained non-convex problems.

The RL agents aim to minimize the \( T \)-horizon sample path penalized objective which at batch \( k \) is

\[ r_k(\phi) = J_{k,T}(\phi) + \lambda \left( B^2_{k,T}(\phi) - \gamma \right), \quad \lambda \in \mathbb{R}_+ \]

\[ J_{k,T} = \frac{1}{T} \sum_{n=1}^{T} r(x_n, u_{\phi}(u_n)), \quad B_{k,T} = \frac{1}{T} \sum_{n=1}^{T} \beta(x_n, u_{\phi}(u_n)) \]  

(38)

There are several methods for estimating the policy gradient \( \nabla_{\phi} r_k(\phi_k) \) [Pflug, 1996] including the score function method, weak derivatives [Abad and Krishnamurthy, 2003] and finite difference methods. A useful finite difference gradient estimate is given by the SPSA algorithm [Spall, 2003]; useful because SPSA evaluates the gradient along a single random direction.

\(^4\)In adaptive control, RL algorithms such as policy gradient are viewed as simulation based implicit adaptive control methods that bypass estimating the MDP parameters (transition probabilities) and directly estimate the optimal policy.

\(^5\)Consider minimizing the negative of the objective function, namely \( -J \) without constraint (32). Even though \( -J \) is nonconvex in \( \phi \), one can show (using Lypunov function arguments) that for this unconstrained MDP case, the gradient algorithm will converge to a global optimum. However for the constrained MDP case this is not true; the nonconvex objective and constraints results in a duality gap.
2) IRL for CMDP: Consider the CMDP (30), (31), (34). Assume we are given a sequence of gradient estimates \( \{\nabla_{\phi} r_k(\phi_k)\} \) of the sample path with respect to the parametrized policy \( \phi \) from (38). The aim of the inverse learner is to reconstruct the reward \( R(\phi) \) in (37). Since by construction the constraint is active at the optimal policy, the aim of the inverse learner is to explore regions of \( \phi \) in the vicinity where the constraint \( \{\phi : B(\phi) \approx \gamma\} \) is active in order to estimate \( R(\phi) \).

A naive application of Langevin IRL algorithm (2) to update the conditional probabilities \( \{\phi_k\} \) will not work. This is because there is no guarantee that the estimate sequence \( \{\phi_k\} \) generated by the algorithm are valid probability vectors, namely

\[
\phi_k(u|x) = \left\{ \begin{array}{ll}
\text{if } u = 1 \\
y \in [0,1], \sum_{u \in \mathcal{U}} \phi_k(u|x) = 1, \ x \in \mathcal{X}.
\end{array} \right.
\]

(39)

We will use spherical coordinates\(^\text{a}\) to ensure that the conditional probability estimates \( \phi_k \) generated by the IRL algorithm satisfy (39) at each iteration \( k \). The idea is to parametrize \( \phi_k(u|x) \) to lie on the unit hyper-sphere in \( \mathbb{R}^U \). Then all we need are the \( U - 1 \) angles for each \( x \), denoted as \( \theta(i,1), \ldots \theta(i, U - 1) \). Define the spherical coordinates in terms of the mapping:

\[
\phi = \mathcal{E}(\theta), \quad \text{where } \phi(u|x) = \left\{ \begin{array}{ll}
\cos^2 \theta(i,1) & \text{if } u = 1 \\
\cos^2 \theta(i, u) \prod_{p=1}^{u-1} \sin^2 \theta(i, p) & \text{if } u \in \{2, \ldots, U - 1\} \\
\sin^2 \theta(i, U - 1) \prod_{p=1}^{U-2} \sin^2 \theta(i, p) & \text{if } u = U
\end{array} \right.
\]

(40)

Then clearly \( \phi(u|x) \) in (40) always satisfies feasibility (39) for any real-valued (un-constrained) \( \theta(x,u) \). To summarize, there are \( (U - 1)X \) unconstrained parameters in \( \theta \). Also for \( \theta(i, u) \in [0, \pi/2] \), the mapping \( \mathcal{E} : \mathbb{R}^U \times X \rightarrow \mathbb{R}^U \times X \) in (40) is one-to-one and therefore invertible. We denote the inverse as \( \mathcal{E}^{-1} \).

Remark: As an example, consider \( U = 2 \). Then in spherical coordinates \( \phi(1|i) = \sin^2 \theta(i,1), \phi(2|i) = \cos^2 \theta(i,1), \) where \( \theta(i,1) \) is un-constrained. Clearly \( \phi(1|i) + \phi(2|i) = 1, \phi(u|x) \geq 0 \).

With the above re-parametrization, we can run any of the passive Langevin dynamics IRL algorithms proposed in this paper. In the numerical example below, we ran the two-time scale multi-kernel IRL algorithm (15). Recall this does not require knowledge of \( \pi(\cdot) \) and also provides variances reduction: Given the current IRL estimate \( \alpha_k \), the RL gives us a sequence \( \{\phi_i, \nabla_{\phi} r_k(\phi_i), i = 1, \ldots, L\} \) The IRL algorithm (15) operating on the \( (U - 1)X \) unconstrained parameters of \( \theta \) is:

\[
\alpha_{k+1} = \alpha_k + \mu \frac{\beta}{2} \sum_{i=1}^{L} p(\theta|i, \alpha_k) \nabla_{\phi} r_k(\theta_i) + \sqrt{\mu} \nu_k, \quad \phi_i \sim \pi(\cdot)
\]

(41)

In the second line of (41), we transformed \( \nabla_{\phi} r_k(\phi_k) \) to \( \nabla_{\phi} r_k(\theta_i) \) to use in the IRL algorithm.

To summarize, the IRL algorithm (41) generates samples \( \phi_k \sim \exp(\mathcal{E}(\theta_i)) \). Equivalently, \( \phi_k = \mathcal{E}(\alpha_k) \sim \exp(R(\phi)) \) where \( R(\phi) \) is defined in (37). Thus given only gradient information from a RL algorithm, we can reconstruct (sample from) the penalized reward \( R(\cdot) \) of the CMDP without any knowledge of the CMDP parameters.

3) Numerical Example: We generated a CMDP with \( X = 2 \) (2 states), \( U = 2 \) (2 actions) and 1 constraint with

\[
P(1) = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \ P(2) = \begin{bmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}, \ r = \begin{bmatrix} 1 & 100 \\ 30 & 2 \end{bmatrix}, \ \beta = \begin{bmatrix} 0.2 & 0.3 \\ 2 & 1 \end{bmatrix}, \ \bar{\gamma} = 1, \ \lambda = 10^5
\]

(42)

Recall the transition matrices \( P(u) \) are defined in (30), the reward matrix \( (r(x,u)) \) in (31), constraint matrix \( (\beta(x,u)) \) and \( \gamma \) in (32), and penalty multiplier \( \lambda \) in (38).

The randomized policy \( \phi(u|x), u \in \{1,2\}, x \in \{1,2\} \) is a \( 2 \times 2 \) matrix. It is completely determined by \( \phi(1|1), \phi(1|2) \in [0,1] \times [0,1] \); so it suffices to estimate \( R(\phi) \) over \([0,1] \times [0,1] \).

Figure 6(a) displays the cumulative reward \( J(\phi) \); this constitutes the ground truth. To obtain this figure, we computed the average reward MDP value function \( J(\phi) \) and constraint \( B(\phi) \) for each policy \( \phi \) where \( \phi \) sweeps over \([0,1] \times [0,1] \). Given a policy \( \phi \), \( J(\phi) \) and \( B(\phi) \) are computed by first evaluating the joint probability \( \tilde{\phi} \) as [Ross, 1983, pp.101]

\[
\tilde{\phi}(j,a) = \sum_i \sum_{\tilde{a}} \tilde{\phi}(i,\tilde{a}) P_{ij}(\tilde{a}) \phi(a|j), \quad \sum_j \sum_{a} \tilde{\phi}(j,a) = 1
\]

and then \( J(\phi) = \sum_x \sum\limits_u \tilde{\phi}(x,u) r(x,u), B(\phi) = \sum_x \sum\limits_u \phi(x,u) \beta(x,u) \).

For values of \( \phi \) that do not satisfy the constraint \( B(\phi) < \gamma \), we plot \( J(\phi) = 0 \). Figure 6(a) illustrates the non-convex nature of the constraint set.

Figure 6(b) displays the penalized cumulative reward \( R(\phi) = J(\phi) - \lambda (B(\phi) - \gamma)^2 \) where the quadratic penalty function is \( \lambda (B(\phi) - \gamma)^2 \). As mentioned earlier, since we know that the constraint is active at the optimal policy, we want the IRL to explore the vicinity of the region where the constraint is active.

\(^{a}\)Another parametrization widely used in machine learning is exponential coordinates: \( \phi(u|x) = \frac{\exp(\theta(x,u))}{\sum_u \exp(\theta(x,u))} \), where \( \theta(x,u) \in \mathbb{R} \) is unconstrained. However, as shown in Krishnamurthy [2016], Krishnamurthy and Vazquez Abad [2018], spherical coordinates typically yield faster convergence. We also found this in numerical studies on IRL (not presented here).
We then ran the IRL algorithm (41) using spherical coordinates with parameters \( \mu = 5 \times 10^{-6}, \sigma = 0.1, L = 50 \) for \( T = 10^5 \) iterations. Figure 6(c) displays 3-dimensional stem plots of the log of the empirical distribution of \( \phi_k = E(\alpha_k) \).

IV. WEAK CONVERGENCE ANALYSIS OF IRL ALGORITHM

This section discusses the main assumptions, weak convergence theorem and proof regarding IRL algorithm (2). (Recall the informal proof in Sec.II-A for the motivation of weak convergence.)

Notation:
- Since \( \nabla \theta r_k(\theta_k) \) is a noise corrupted estimate of the gradient \( \nabla \theta R(\theta) \), we write it in more explicit notation as \( \tilde{r}(\theta_k, \xi_k) \), where \( \{\xi_k\} \) is a sequence of random variables satisfying appropriate conditions specified below.
- We use \( \pi_n(\cdot) \) to denote \( \nabla_\alpha \pi(\cdot) \).
- Finally, \( E_m \) denotes the conditional expectation (conditioning up to time \( m \)), i.e., conditioning wrt the \( \sigma \)-algebra \( F_m = \sigma\{\alpha_0, \alpha_1, \ldots, \alpha_j; j < m\} \).

Algorithm: There are two possible implementations of IRL algorithm (2). The first implementation is (2), namely,

\[
\alpha_{k+1} = \alpha_k + \frac{\mu}{\Delta N} K(\frac{\theta_k - \alpha_k}{\Delta}) \frac{\beta}{2} \tilde{r}(\theta_k, \xi_k) \pi(\alpha_k) + \mu \pi(\alpha_k) \pi(\alpha_k) + \sqrt{\mu} \pi(\alpha_k) w_k, \tag{43}
\]

and the second implementation is

\[
\alpha_{k+1} = \alpha_k + \frac{\mu}{\Delta N} K(\frac{\theta_k - \alpha_k}{\Delta}) \left[ \frac{\beta}{2} \tilde{r}(\theta_k, \xi_k) \pi(\theta_k) + \pi(\alpha_k) \right] + \sqrt{\frac{\mu}{\Delta N}} K(\frac{\theta_k - \alpha_k}{\Delta}) \pi(\theta_k) w_k, \tag{44}
\]

where \( \mu \) is the stepsize and \( \Delta = \Delta(\mu) \) is chosen so \( \mu/\Delta N \to 0 \) as \( \mu \to 0 \).

Both the above algorithms converge to the same limit. The proof below is devoted to (43), but (44) can be handled similarly. Also the proofs of the other two proposed IRL algorithms, namely (12) and (24) are similar.

Taking a continuous-time interpolation

\[
\alpha^\mu(t) = \alpha_k \quad \text{for} \quad t \in [\mu k, \mu k + \mu), \tag{45}
\]

we aim to show that the sequence \( \alpha^\mu(\cdot) \) converges weakly to \( \alpha(\cdot) \), which give the desired limit.

A. Assumptions

We begin by stating the conditions needed.

(H1) For each \( \xi, \tilde{r}(\cdot, \xi) \) has continuous partial derivatives up to the second order such that the second partial \( \tilde{r}_{\alpha\alpha}(\cdot, \xi) \) is bounded. For each \( b < \infty \) and \( T < \infty \), \( \{\tilde{r}(\alpha, \xi); |\alpha| \leq b, j \mu \leq T\} \) is uniformly integrable.

(H2) The sequences \( \{\theta_k\} \) is stationary and independent of \( \{\xi_k\} \). For each \( k \geq n \), there exists a conditional density of \( \theta_k \) given \( F_n \), denoted by \( \pi_k(\theta|F_n) \) such that \( \pi_k(\theta|F_n) > 0 \) for each \( \theta \) and that \( \pi_k(\cdot|F_n) \) is continuous. The sequence \( \{\pi_k(\cdot|F_n)\}_{k \geq n} \) is bounded uniformly. The probability density \( \pi(\cdot) \) is continuous and bounded with \( \pi(\theta) > 0 \) for each \( \theta \) such that

\[
\lim_{k-n \to \infty} \mathbb{E}[\pi_k(\theta|F_n) - \pi(\theta)] = 0. \tag{46}
\]

(H3) The measurement noise \( \{\xi_n\} \) is exogenous, and bounded stationary mixing process with mixing measure \( \varphi_k \) such that \( \mathbb{E}\tilde{r}(\alpha, \xi_k) = R_\alpha(\alpha) \) for each \( \alpha \) and \( \sum_k \varphi_k < \infty \). The \( \{w_k\} \) is a sequence of \( \mathbb{R}^N \)-valued i.i.d. random variables with mean 0 and covariance matrix \( I \) (the identity matrix); \( \{w_k\} \) and \( \{\xi_k\} \) are independent.

(H4) The kernel \( K(\cdot) \) has a bounded support, i.e., \( K(x) = 0 \) for \( |x| > \kappa_0 \) for some \( \kappa_0 > 0 \) and

\[
K(x) \geq 0, \quad K(x) = K(-x), \quad \text{and} \quad \int K(x)dx = 1. \tag{47}
\]

Remarks: We briefly comment on the assumptions (H1)-(H4).

- Assumption (H1) requires the smoothness of \( \tilde{r}(\cdot, \xi) \), which is natural because we are using \( \tilde{r}(\cdot, \xi_k) \) to approximate the smooth function \( \nabla R \). We consider a general noise so the uniform integrability is used. If the noise is additive in that \( \tilde{r}(\theta, \xi) = \nabla R(\theta) + \xi \), then we only need the finite \( \tilde{p} \)-moments of \( \xi_k \) for \( \tilde{p} > 1 \).

- Assumption (H3) requires the stochastic process \( \{\xi_n\} \) to be exogenous, and bounded stationary mixing. Thus for each \( \alpha, \{\tilde{r}(\alpha, \xi_k)\} \) is also a mixing sequence. A mixing process is one in which remote past and distant future are asymptotically independent. It covers a wide range of random processes such as i.i.d. sequences, martingale difference sequences, moving average sequences driving by a martingale difference sequence, and functions of stationary Markov processes with a finite state space [Billingsley, 1999], etc. The case of \( \{w_k\} \) and \( \{\xi_k\} \) being dependent can be handled, but for us \( \{w_k\} \) is the added perturbation to get the desired Brownian motion so independence is sufficient.
(a) Cumulative Reward $J(\phi)$ with active constraint $B(\phi) \leq 1$. The non-convexity of the constraint set is clearly seen.

(b) Penalized Cumulative Reward with Quadratic Penalty $R(\phi) = J(\phi) - \lambda (B(\phi) - 1)^2$. The lighter green shade on top shows the active constraint. This plot constitutes the ground truth.

(c) IRL algorithm estimate. Snapshot 1 shows that the IRL estimates $R(\phi)$ in the vicinity of the active constraint. Snapshot 2 shows that the IRL explores regions in the vicinity of the active constraint. Specifically the curve is close to the lighter shade green in Fig (b).

Fig. 6: IRL for Constrained MDP
• By exogenous in (H3), we mean that
  \[ P(\xi_{n+1} \in A_1, \ldots, \xi_{n+k} \in A_k | \alpha_0, \xi_j; j \leq n) = P(\xi_{n+1} \in A_1, \ldots, \xi_{n+k} \in A_k | \alpha_0, x_j, \xi_j, \alpha_{j+1}; j \leq n), \]
  for all Borel sets \( A_i \), \( i \leq k \), and for all \( k \) and \( n \).
• In view of the mixing condition (H3) on \( \{\xi_k\} \), for each \( b < \infty \) and \( T < \infty \), \( \{\tilde{r}(\alpha, \xi_j); |\alpha| \leq b, j \mu \leq T\} \) and \( \{\tilde{r}_\alpha(\alpha, \xi_j); |\alpha| \leq b, j \mu \leq T\} \) are uniformly integrable.
• Again, using the mixing condition, for each \( \alpha \), as \( n \to \infty \),
  \[ \frac{1}{n} \sum_{j=m}^{m+n-1} E_m \tilde{r}(\alpha, \xi_j) \to R_\alpha(\alpha) \text{ in probability.} \] (48)
• For a Borel set \( A \), we have \( P(\theta_k \in A | F_n) = \int_{\theta \in A} \pi_k(\theta | F_n) d\theta \). If \( \{\theta_n\} \) is itself a stationary \( \phi \)-mixing sequence with a continuous density, and if \( \mathbb{E} |\theta_n|^2 < \infty \), then by virtue of a well-known mixing inequality, some \( c_0 > 0 \), [Ethier and Kurtz, 1986, Corollary 2.4 in Chapter 7],
  \[ \mathbb{E} \{ \| \int \theta \pi_k(\theta | F_n) d\theta - \int \theta \pi(\theta) d\theta \| \} \leq c_0 \varphi_\theta^{1/2}(k-n) \mathbb{E}^{1/2} |\theta_k|^2 \to 0 \text{ as } k-n \to \infty, \]
  where \( \varphi_\theta(\cdot) \) denotes the mixing measure.
• (H4) is simply a restatement of (3) regarding the properties of the kernel used in passive stochastic approximation.

B. Main Result and Proof

As is well known [Kushner and Yin, 2003], a classical fixed step size stochastic gradient algorithm converges weakly to a deterministic ordinary differential equation (ODE) limit; this is the basis of the so called ODE approach for analyzing stochastic gradient algorithms. In comparison, the discrete time IRL algorithm (2) converges weakly to a stochastic process limit \( \alpha(\cdot) \). In this section we prove weak convergence of the interpolated process \( \{\alpha^\mu(\cdot)\} \) to the stochastic process limit \( \alpha(\cdot) \) as \( \mu \to 0 \).

Proving weak convergence requires first that the tightness of the sequence be verified and then the limit be characterized via the so called martingale problem formulation. For a comprehensive treatment of the martingale problem of Stroock and Varadhan, see Ethier and Kurtz [1986].

**Theorem 1:** Assume conditions (H1)-(H4). Then the interpolated process \( \alpha^\mu(\cdot) \) (defined in (45)) for IRL algorithm (2) has the following properties:
1) \( \{\alpha^\mu(\cdot)\} \) is tight in \( D^d[0, \infty) \).
2) Any weakly convergent subsequence of \( \{\alpha^\mu(\cdot)\} \) has a limit \( \alpha(\cdot) \) that satisfies
  \[ d\alpha(t) = \left[ \frac{\beta}{2} \pi^2(\alpha(t)) R_\alpha(\alpha(t)) + \pi_\alpha(\alpha(t)) \pi(\alpha(t)) \right] dt + \pi(\alpha(t)) dW(t), \]
  \[ \alpha(0) = \alpha_0, \] (49)
  where \( W(\cdot) \) is a standard Brownian motion with mean 0 and covariance being the identity matrix \( I \in \mathbb{R}^{N \times N} \), provided (49) has a unique weak solution (in a distributional sense) for each initial condition.

For sufficient conditions leading to unique weak solutions of stochastic differential equation and uniqueness of martingale problem, see Ethier and Kurtz [1986, p. 182] or Karatzas and Shreve [1991].

**Proof.** The proof is divided into 4 steps.

**Step 1. Use a truncation device.** Because the sequence \( \{\alpha_k\} \) is not a priori bounded, the main idea is to use a truncation device [Kushner and Yin, 2003, p.284]. (Step 4 below deals with the un-truncated process.) Let \( M > 0 \) be fixed arbitrary constant. Denote by \( S_M = \{\alpha \in \mathbb{R}^N : |\alpha| \leq M\} \) the \( N \)-dimensional ball centered at the origin with radius \( M \). Consider the truncated algorithm
  \[ \alpha^M_{k+1} = \alpha^M_k + \mu \left[ \frac{1}{\Delta N} K(\frac{\theta_k - \alpha^M_k}{\Delta}) - \frac{\beta}{2} \pi(\theta_k, \xi_k) + \pi_\alpha(\alpha^M_k) \right] \pi(\alpha^M_k) q_M(\alpha^M_k) + \sqrt{\mu} \pi(\alpha^M_k) q_M(\alpha^M_k) w_k, \] (50)
where
  \[ q_M(\alpha) = \begin{cases} 1, & \alpha \in S_M; \\ 0, & \alpha \in \mathbb{R}^N - S_{M+1}; \\ \text{smooth otherwise.} \end{cases} \]
By (H4), \( K((\theta_k - \alpha^M_k)/\Delta) \neq 0 \) only when \( |\theta_k - \alpha^M_k| \leq \kappa_0 \Delta \) or \( |\theta_k| \leq |\alpha^M_k| + \kappa_0 \Delta \leq M + \kappa_0 \Delta \). Therefore, the kernel function forces the iterates to be added corresponding to bounded \( \theta_k \) only.

**Remark.** Define \( \alpha^{\mu,M}(t) = \alpha^M_0 \) on \( [\mu k, \mu k + \mu) \). Then \( \alpha^{\mu,M}(\cdot) \in D^N[0, \infty) \) and is an \( M \)- truncation for \( \alpha^\mu(\cdot) \) [Kushner and Yin, 2003, p.284]. We proceed to prove the tightness and weak convergence of the truncated sequence \( \{\alpha^{\mu,M}(\cdot)\} \) first and then complete the proof by letting \( M \to \infty \) in Step 4.
Step 2. Prove the tightness of \( \{\alpha^{\mu,M}(\cdot)\} \). Note that in view of Nazin et al. [1989, Lemma 1], by virtue of (A4), for a function \( h(\cdot) \) that is twice continuously differentiable with bounded second derivative, it follows that
\[
\left| \frac{1}{\Delta^N} \int K\left( \frac{\theta - \alpha}{\Delta} \right) h(\theta) d\theta - h(\alpha) \right| = O(\Delta^2). \tag{51}
\]
Using (51), (H1), and noting that \( \{w_k\} \) is an i.i.d. sequence with mean 0 and covariance matrix \( I \), we can show that \( \left\{ \Delta^{-N} K\left( \frac{\theta_k - \alpha^M_k}{\Delta} \right) q(\theta_k, \xi_k) + \pi_n(\alpha^M_k) \pi(\alpha^M_k) q_M(\alpha^M_k) \right\} \) is uniformly integrable and also \( \{\pi(\alpha^M_k) q_M(\alpha^M_k) w_k\} \) is uniformly integrable. Then using Kushner [1984, p.51, Lemma 7] (or use a perturbed test function methods as in Kushner and Yin [2003, Chapter 7]), it can be shown that \( \{\alpha^{\mu,M}(\cdot)\} \) is tight in \( D([0, \infty), \mathbb{R}^N) \), the space of \( \mathbb{R}^N \)-valued functions that are right continuous, have left limits, endowed with the Skorohod topology.

Step 3. Characterize the limit process. Because \( \{\alpha^{\mu,M}(\cdot)\} \) is tight, by virtue of Prohorov’s theorem [Billingsley, 1999], we can extract a weakly convergent subsequence. To simplify notation, denote the sequence by \( \{\alpha^{\mu,M}(\cdot)\} \) with limit denoted by \( \alpha^M(\cdot) \). By Skorohod representation [Kushner and Yin, 2003, p. 230] with a slight abuse of notation, we may assume that \( \alpha^{\mu,M}(\cdot) \) converges to \( \alpha^M(\cdot) \) w.p.1. To complete the proof, we need only characterize the limit process by showing that the limit \( \alpha^M(\cdot) \) is a solution of the martingale problem with backward operator
\[
L^M f(\cdot) = f'_\alpha(\cdot) \left[ \frac{\beta}{2} \pi^2(\alpha) R_n(\alpha) + \pi_n(\alpha) \pi(\alpha) \right] q_M(\alpha) + \frac{1}{2} \pi^2(\alpha) \text{Tr}[f_{\alpha n}(\alpha)] q_M(\alpha) \tag{52}
\]
for any real-valued function \( f(\cdot) \in C_b^2 \) (Etherd and Kurtz [1986, Lemma 8.1, p.225]), where \( f' \) denotes the transpose of \( f \).

To verify the martingale property, we show that for any bounded and continuous test function \( g(\cdot) \), any \( t, s > 0 \), any positive integer \( k \), and any \( t_i \leq t \),
\[
\mathbb{E} \left\{ g(\alpha^M(t_s) \leq k \right\} \left[ f(\alpha^M(t_s) - f(\alpha^M(t)) - \int_t^{t+s} L^M f(\alpha^M(u)) du \right] = 0. \tag{53}
\]
Note that (53), namely, the solution of the martingale problem, is a statement about the finite dimensional distributions of \( \alpha^{\mu,M}(\cdot) \) at times \( t_1, \ldots, t_k \).

To verify (53), we work with the sequence indexed by \( \mu \). By the continuity of \( f(\cdot) \), the weak convergence, and the Skorohod representation, we have that as \( \mu \to 0 \),
\[
\mathbb{E} g(\alpha^{\mu,M}(t_s) \leq k \right\} \left[ f(\alpha^{\mu,M}(t_s) - f(\alpha^{\mu,M}(t)) \right]
\to \mathbb{E} g(\alpha^{\mu,M}(t_s) \leq k \right\} \left[ f(\alpha^M(t_s) - f(\alpha^M(t)) \right]. \tag{54}
\]

To simplify notation, we denote \( q^M_k = q_M(\alpha^M_k) \) in what follows whenever there is no confusion and retain the notation \( q^M_k \) whenever it is needed. Dividing the segment
\[
\left[ t/\mu \right] \leq k \leq \left[ (t + s)/\mu \right]
\]
into sub-blocks of size \( m_\mu \), each so that \( m_\mu \to \infty \) as \( \mu \to 0 \) and \( \delta_\mu = \mu m_\mu \to 0 \). Then we obtain
\[
\mathbb{E} g(\alpha^{\mu,M}(t_s) \leq k \right\} \left[ f(\alpha^{\mu,M}(t_s) - f(\alpha^{\mu,M}(t)) \right]
= \mathbb{E} g(\alpha^{\mu,M}(t_s) \leq k \right\} \left( \mathbb{E}_{l_\mu} \sum_{l=t/\delta_\mu}^{(t+s)/\delta_\mu} f'_\alpha(\alpha^{l_\mu}) \right. \\
\left. \times \sum_{l=t/\delta_\mu}^{(t+s)/\delta_\mu} \left[ \frac{\mu}{\Delta^N} K\left( \frac{\theta_k - \alpha^M_k}{\Delta} \right) \right] \right. \\
\left. \sum_{k=lm_\mu}^{lm_\mu + m_\mu - 1} \pi(\alpha^M_k) q^M_k \right) \tag{55}
\]
where \( \mathbb{E}_{l_\mu} \) denotes the conditional expectation with respect to the past information up to the time \( l_\mu \) (i.e., the \( \sigma \)-algebra generated by \( \{ \alpha_k, \theta_k, \xi_k : k < l_\mu \} \)), and \( e_i^\mu \) is an error term. It can be shown that
\[
\mathbb{E} g(\alpha^{\mu,M}(t_s) \leq k \right\} \left( \mathbb{E}_{l_\mu} \sum_{l=t/\delta_\mu}^{(t+s)/\delta_\mu} e_i^\mu \right)^2 \to 0 \quad \text{as} \quad \mu \to 0. \tag{56}
\]
Noting that \( \{ w_k \} \) is an i.i.d. sequence with mean 0 and covariance \( I \) (the identity matrix), using the continuity of \( \pi(\cdot) \), the limit of
\[
\mathbb{E} g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \frac{1}{2} \mathbb{E}_{l_m, \mu} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \sum_{k=l_m}^{l_m + \mu - 1} \pi^2(\alpha_k^M) \text{Tr}[f_{\alpha}(\alpha_{l_m})w_k w_k^T] \right\},
\]
is the same as
\[
\mathbb{E} g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \frac{1}{2} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \sum_{k=l_m}^{l_m + \mu - 1} \pi^2(\alpha_k^M) \text{Tr}[f_{\alpha}(\alpha_{l_m})]\right\}
= \mathbb{E} g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \frac{1}{2} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \pi^2(\alpha_k^M) \text{Tr}[f_{\alpha}(\alpha_{l_m})]\right\}.
\]

It then follows from weak convergence of \( \alpha_{\mu,M}(\cdot) \) to \( \alpha^M(\cdot) \) and the Skorohod representation, that
\[
\mathbb{E} g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \frac{1}{2} \mathbb{E}_{l_m, \mu} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \sum_{k=l_m}^{l_m + \mu - 1} \pi^2(\alpha_k^M) \text{Tr}[f_{\alpha}(\alpha_{l_m})w_kw_k^T] \right\}
\rightarrow \mathbb{E} g(\alpha^M(t_s) : t \leq \kappa_1) \left\{ \frac{1}{2} \int_t^{t+s} \pi^2(\alpha^M(u)) \text{Tr}[f_{\alpha}(\alpha^M(u))]q^M(\alpha^M(u)du) \right\} \text{ as } \mu \rightarrow 0.
\]

Using the condition on the i.i.d. noise \( \{ w_k \} \), it is readily seen that
\[
\mathbb{E} g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \mathbb{E}_{l_m, \mu} \sqrt{\mu} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \mathbb{E}_{l_m, \mu} \sum_{k=l_m}^{l_m + \mu - 1} \pi(\alpha_k^M)w_k q_k^M \right\}
= \mathbb{E} g(\alpha^M(t_s) : t \leq \kappa_1) \left\{ \sqrt{\mu} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \mathbb{E}_{l_m, \mu} \sum_{k=l_m}^{l_m + \mu - 1} \pi(\alpha_k^M)q_k^M \right\}
\rightarrow 0 \text{ as } \mu \rightarrow 0.
\]

Next, using the continuity of \( \pi(\cdot) \), \( \pi(\cdot) \), \( f_{\alpha}(\cdot) \), together with the weak convergence of \( \alpha_{\mu,M}(\cdot) \) to \( \alpha^M(\cdot) \), the Skorohod representation, the notation \( q_k^M \) defined before, and the notation convention \( q^M_{l_m, \mu} = q^M(\alpha_{l_m}) \) and \( q^M = q^M(\alpha^M(l\delta_{\mu})) \), we have
\[
\lim_{\mu \rightarrow 0} \mathbb{E} \left[ g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \mathbb{E}_{l_m, \mu} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \mathbb{E}_{l_m, \mu} \sum_{k=l_m}^{l_m + \mu - 1} \pi(\alpha_k^M)q_k^M \right\} \right]
= \lim_{\mu \rightarrow 0} \mathbb{E} \left[ g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \mathbb{E}_{l_m, \mu} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \mathbb{E}_{l_m, \mu} \sum_{k=l_m}^{l_m + \mu - 1} \pi(\alpha_k^M)q_k^M \right\} \right]
= \mathbb{E} \left[ g(\alpha^M(t_s) : t \leq \kappa_1) \left\{ \int_t^{t+s} \frac{f_{\alpha}(\alpha^M(u))\pi(\alpha^M(u))q^M(\alpha^M(u)du)}{\mu} \right\} \right].
\]

Note that
\[
\mathbb{E} \left[ g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \mathbb{E}_{l_m, \mu} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \mathbb{E}_{l_m, \mu} \sum_{k=l_m}^{l_m + \mu - 1} \frac{\mu}{\Delta^N} K(\theta_k - \alpha_k^M) \delta_{\mu}\right\} \right]
= \mathbb{E} \left[ g(\alpha_{\mu,M}(t_s) : t \leq \kappa_1) \left\{ \frac{\beta}{2} \sum_{t=t/\delta_{\mu}}^{(t+s)/\delta_{\mu}} \delta_{\mu}\right\} \right]
\times \frac{1}{\Delta^N} m_{\mu} \sum_{k=l_m}^{l_m + \mu - 1} \mathbb{E}_{l_m, \mu} K\left(\theta_k - \alpha_k^M\right) \delta_{\mu} + o(1),
\]
where \( o(1) \to 0 \) as \( \mu \to 0 \) uniformly in \( t \). By the continuity of \( \pi(\cdot) \) and \( \bar{\pi}(\cdot, \xi) \) for each \( \xi \),

\[
\psi_\mu = \frac{1}{\Delta^N} \frac{1}{m_\mu} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} K\left( \frac{\theta_k - \alpha_k^M}{\Delta} \right) \bar{\pi}(\theta_k, \xi_k) q_k^M
\]

\[
= \frac{1}{\Delta^N} \frac{1}{m_\mu} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} \left[ \int K\left( \frac{\theta - \alpha_k^M}{\Delta} \right) \bar{\pi}(\theta, \xi_k) \pi(\theta) d\theta \right]_{\theta=\theta_k} q_k^M
\]

\[
+ \frac{1}{\Delta^N} \frac{1}{m_\mu} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} \left[ \int K\left( \frac{\theta - \alpha_k^M}{\Delta} \right) \bar{\pi}(\theta, \xi_k) \pi(\theta | F_{l_{m_\mu}} - \pi(\theta)) d\theta \right]_{\theta=\theta_k} q_k^M
\]

In view of (H2), the last term above contributes nothing to the limit. By virtue of (51),

\[
\frac{1}{\Delta^N} \frac{1}{m_\mu} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} \left[ \int K\left( \frac{\theta - \alpha_k^M}{\Delta} \right) \bar{\pi}(\theta, \xi_k) \pi(\theta) d\theta \right]_{\theta=\theta_k} q_k^M
\]

\[
= \frac{1}{m_\mu} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} \bar{\pi}(\alpha_k^M, \xi_k) \pi(\alpha_k^M) q_k^M + o(1),
\]

where \( o(1) \to 0 \) in probability. Thus we have

\[
\psi_\mu = \frac{1}{m_\mu} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} \bar{\pi}(\alpha_k^M, \xi_k) \pi(\alpha_k^M) q_k^M + o(1)
\]

\[
= \frac{1}{m_\mu} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} \bar{\pi}(\alpha_k^M, \xi_k) \pi(\alpha_k^M) q_k^M + o(1)
\]

(61)

where \( o(1) \to 0 \) in probability as \( \mu \to 0 \), because of the continuity of \( \pi(\cdot) \) and \( \bar{\pi}(\cdot, \xi) \) for each \( \xi \). Letting \( l_{\delta_k} \to u \) as \( \mu \to 0 \), then for any \( l_{m_\mu} \leq k \leq m_\mu + m_\mu, l_{\delta_k} \to u \). Using the weak convergence of \( \alpha^{\mu,M}(\cdot) \) to \( \alpha^M(\cdot) \) and the Skorohod representation, we can approximate \( \bar{\pi}(\alpha^{\mu,M}(l_{\delta_k}), \xi_k) \pi(\alpha^{\mu,M}(l_{\delta_k})) q_k^M(\alpha^{\mu,M}(l_{\delta_k})) \) by \( \bar{\pi}(\alpha^M(u), \xi_k) \pi(\alpha^M(u)) q_k^M(\alpha^M(u)) \) with an error going to 0. Because \( \alpha^M(\cdot) \) is bounded, for each \( \gamma \geq 0 \), we can choose \( \{O^\gamma_i : i \leq i_\gamma\} \) as a finite collection of disjoint sets of diameter no larger than \( \gamma \) whose union covers the range of \( \alpha^M(\cdot) \). Thus, \( \alpha^M(\cdot) \) can be approximated by \( \sum_{i=1}^{i_\gamma} \alpha^1_i(\alpha^M(u) \in O^\gamma_i) \). Consequently,

\[
\psi_\mu = \frac{1}{m_\mu} \sum_{i=1}^{i_\gamma} \sum_{k=l_{m_\mu}}^{l_{m_\mu}+m_\mu-1} \mathbb{E}_{l_{m_\mu}} \bar{\pi}(\alpha^1_i, \xi_k) \pi(\alpha^1_i) 1_{\alpha^M(u) \in O^\gamma_i} q_k^M(\alpha^M(u)) + o(1),
\]

(62)

where \( o(1) \to 0 \) in probability. Now it is clear that condition (H3) and hence (48) can be used. Using (62) and (48) together with (60) and detailed calculation yields that

\[
\mathbb{E}\left[ g(\alpha^M(t_i) : i \leq \kappa_1) \left\{ \mathbb{E}_{l_{\delta_k}} \left[ \int \frac{(t_i + \delta_k) \beta}{\Delta^N} \right]_{\delta_k}^{l_{\delta_k}} K\left( \frac{\theta_k - \alpha_k^M}{\Delta} \right) \bar{\pi}(\theta_k, \xi_k) \pi(\alpha_k^M, q_k^M) \right\} \right]
\]

\[
\to \mathbb{E}\left[ g(\alpha^M(t_i) : i \leq \kappa_1) \left\{ \frac{\beta}{\Delta^N} \int_{l_{\delta_k}}^{l_{\delta_k}} R_{\alpha}(\alpha^M(u)) \pi(\alpha^M(u)) q_k^M(u) du \right\} \right].
\]

Using (54) and (55), and combining the estimates and calculation in (56)-(63) lead to (53). Therefore, we arrive at that \( \alpha^M(\cdot) \) is the solution of the martingale problem with operator \( \mathcal{L}^M \) given in (52).

**Step 4.** Let the truncation level \( M \to \infty \). In the last step, we let \( M \to \infty \) to obtain the convergence of the un-truncated process \( \alpha^M(\cdot) \). The details are as in Kushner [1984, pp. 44-46]. The verbatim argument is thus omitted.

Now, our arguments in Steps 1-4 yield the desired result Theorem 1. The proof of the theorem is concluded.

**C. Comments**

We make several comments below.

- We proved Theorem 1 above for algorithm (43); equivalently (2). The proof of convergence of (44) can be carried out similarly. The main difference is that we are utilizing the kernel \( K(\cdot) \) to incorporate \( \theta_k \) used in the algorithm. There is no additional technical difficulty.
Note that in a way, (43) can be considered to be more efficient than (44). First, because $\pi(\alpha)$ is available, (43) is more direct. Second, using $\pi(\alpha)$ and $\pi(\alpha)$ in lieu of using $\pi(\theta)$ and $\pi(\alpha)$ together with the kernel $K(\cdot)$ avoids an additional averaging and the involvement of a Dirac $\delta$-like function.

V. TRACKING ANALYSIS OF IRL IN NON-STATIONARY ENVIRONMENT

An important feature of the IRL algorithm (2) is its constant step size $\mu$ (as opposed to a decreasing step size). This facilitates estimating (tracking) time evolving reward functions. This section analyzes the ability of IRL algorithm to track a time-varying reward function.

Since we are estimating a time evolving reward, we first give a model for the evolution of the reward $R(\theta)$ over time. Below, the Markov chain $\{x_k\}$ will be used as a hyper-parameter to model the evolution of the time varying reward, which we will denote as $R(\theta, x_k)$. By hyper-parameter we mean that the Markov chain model is not known or used by the IRL algorithm (2). The Markov chain assumption is used only in our convergence analysis to determine how well does the IRL algorithm estimates (tracks) the reward $R(\theta, x_k)$ that jump changes (evolves) according to an unknown Markov chain $x_k$

We assume that the RL agents perform gradient algorithm (1) by evaluating the sequence of gradients $\{\nabla_\alpha r_k(\theta_k, x_k)\}$. Note that both the RL and IRL do not know the sample path $\{x_k\}$. We will use similar notation to Sec.IV:

- Denote $\nabla_\alpha r_k(\theta_k, x_k)$ as $\bar{r}(\theta_k, \xi_k, x_k)$.
- We use $\pi(\alpha)$ to denote $\nabla_\alpha \pi(\cdot)$.

A. Assumptions

We focus on the following algorithm

$$
\alpha_{k+1} = \alpha_k + \frac{\mu}{\Delta_N} K \left( \frac{\theta_k - \alpha_k}{\Delta} \right) \bar{r}(\theta_k, \xi_k, x_k) \pi(\alpha_k) + \mu \pi(\alpha) \pi(\alpha_k) + \sqrt{\mu \pi(\alpha_k)} w_k,
$$

(64)

The main assumptions are as follows.

(M1) (Markovian hyper-parameter) Let $\{x_k, k \geq 0\}$ be a Markov chain with finite state space $\mathcal{X} = \{1, \ldots, X\}$ and transition probability matrix $I + \eta QQ$, where $\eta > 0$ is a small parameter and $Q = (q_{ij})$ is an $X \times X$ irreducible generator (matrix) [Yin and Zhang, 2013, p.23] with

$$
q_{ij} \geq 0, \quad i \neq j, \quad \sum_j q_{ij} = 0, \quad i \in \mathcal{X},
$$

also $\{x_k\}$ is independent of $\{\theta_k\}$ and $\{w_k\}$.

(M2) Assumption (H1) holds on $\bar{r}(\cdot, \xi, i)$ for each fixed state $i \in \mathcal{X}$. Also (H2), (H3), (H4) hold.

B. Main Result

Recall that $\mu$ is the step size of the IRL algorithm while $\eta$ reflects the rate at which the hyper-parameter Markov chain $x_k$ evolves. In the following tracking analysis of IRL algorithm (2), we will consider three cases, $\mu = O(\eta)$, $\mu \ll \eta$, and $\mu \gg \eta$. The three cases represent three different types of asymptotic behavior. If $\mu \gg \eta$, the frequency of changes of the Markov chain is very slow. Thus essentially, we are treating a case similar to a constant parameter, or the objective function is essentially a first order averaging and the involvement of a Dirac $\delta$-like function.

Theorem 2: Consider the IRL algorithm (64). Under Assumptions (M1) and (M2), assuming that (65), or (66), or (67) has a unique solution in the sense in distribution. Then the following results hold.

1. Assume $\mu = \eta$. Then as $\mu \downarrow 0$, the interpolated process $(\alpha^\mu(\cdot), \pi^\mu(\cdot))$ converges weakly to the switching diffusion $(\alpha(\cdot), \pi(\cdot))$ satisfying

$$
d\alpha(t) = \left[ \frac{\beta}{2} \pi^2(\alpha(t)) R_\alpha(\alpha(t), x(t)) + \pi(\alpha(t)) \right] dt + \pi(\alpha(t)) dW(t),
$$

(65)

where $W(\cdot)$ is a standard Brownian motion with mean 0 and covariance being the identity matrix $I \in \mathbb{R}^{N \times N}$, and $x(\cdot)$ is a continuous-time Markov chain with generator $Q$.

2. Suppose $\eta = \mu^{1+\Delta}$ with $\Delta > 0$ and denote the initial distribution of $x^\eta(0)$ by $p_t$ (independent of $\eta$) for each $c \in \mathcal{X}$. Then as $\mu \downarrow 0$, the interpolated process $(\alpha^\mu(\cdot))$ converges weakly to the following diffusion process

$$
d\alpha(t) = \left[ \frac{\beta}{2} \pi^2(\alpha(t)) \sum_{i \in \mathcal{X}} R_\alpha(\alpha(t), i) p_t + \pi(\alpha(t)) \right] dt + \pi(\alpha(t)) dW(t),
$$

(66)
3. Suppose that \( \eta = \mu^\Delta \) with \( 0 < \Delta < 1 \) and denote the stationary distribution associated with the continuous-time Markov chain with generator \( Q \) by \( \nu = (\nu_1, \ldots, \nu_X) \). Then as \( \mu \downarrow 0 \), the interpolated process \((\alpha^\nu(t))\) converges weakly to the following diffusion process

\[
d\alpha(t) = \left[ \frac{\beta}{2} \pi^2(\alpha(t)) \sum_{i \in X} R_\alpha(\alpha(t), i) \nu_i + \pi_\alpha(\alpha(t))\pi(\alpha(t)) \right] dt + \pi(\alpha(t))dW(t).
\]

**Remark.** Theorem 2 presented the asymptotic behavior of the IRL algorithm (64) with Markovian switching. In accordance with the rates of variations of the adaptation rates (represented by the stepsize \( \mu \)) and the switching rate (represented by the stepsize \( \eta \)), three cases are considered. Case 1 indicates that when \( \mu \) is in line with \( \eta \), the limit differential equation is a switching diffusion. Case 2 concentrates on the case that the switching is much slower than the stochastic approximation generated by the recursion. Thus, the limit Lagrange equation is one in which the drift and diffusion coefficients are averaged out with respect to the initial distribution of the limit Markov chain. Roughly, it reveals that the “jump change” parameter \( x(t) \) is more or less as a constant in the sense the coefficients are averaged w.r.t. the initial distribution. Case 3 is the one that the Markov chain is changing much faster than the stochastic approximation rate. As a result, the “jump change” behavior is replaced by an average with respect to the stationary distribution of the Markov chain. Then we derive the associated limit Lagrange equation. Again, the limit has no switching in it.

**C. Proof of Theorem 2**

We will prove Statement 1 for the case \( \mu = \eta \). Consider \((\alpha^\mu(\cdot), x^\mu(\cdot))\), the pair of interpolated processes. We show that this pair of processes converges weakly to \((\alpha(\cdot), x(\cdot))\) such that the limit is a solution of (65) or equivalently, \((\alpha(\cdot), x(\cdot))\) is a solution of the martingale problem with an operator redefined by

\[
\mathcal{L}f(\alpha, i) = f'_\alpha(\alpha, i) \left[ \frac{\beta}{2} \pi^2(\alpha, i) R_\alpha(\alpha) + \pi_\alpha(\alpha)\pi(\alpha) \right] + \frac{1}{2} \pi^2(\alpha) \text{Tr} [f_{\alpha\alpha}(\alpha, i)] + Qf(\alpha, \cdot)(i),
\]

where

\[
Qf(\alpha, \cdot)(i) = \sum_{j \in X} q_{ij} f(\alpha, j), \quad \text{for each } i \in X.
\]

We still need to use an \( M \) truncation device (truncation on \( \alpha \)). However, for notation simplicity, we suppress the \( M \) truncation. From (64), it is easily seen that

\[
\alpha_{k+1} = \alpha_k + \mu \Delta N K \left( \frac{\theta_k - \alpha_k}{\Delta} \right) \frac{\beta}{2} \sum_{i \in X} (\theta_k, \xi_k, i) \pi(\alpha_k)1_{\{x_k = i\}} + \mu \pi_\alpha(\alpha_k)\pi(\alpha_k) + \sqrt{\mu} \pi(\alpha_k)w_k.
\]

To prove the tightness of \((\alpha^\mu(\cdot), x^\mu(\cdot))\), we prove the tightness \(\{x^\mu(\cdot)\}\) first. This can be done by considering \(x^\mu = (1_{\{x_k = 1\}}, \ldots, 1_{\{x_k = X\}}) \in \mathbb{R}^{1 \times X}\), and defining \(\chi^\mu(t) = \chi_k \) for \( t \in [\mu k, \mu k + \mu) \). Denote by \( \mathcal{F}_t^\mu \), the \( \sigma \)-algebra generated by \(\{\xi_k, \theta_k, x_k, \alpha_0 : k \leq t / \mu\} \), and denote the corresponding conditional expectation by \( \mathbb{E}_t^\mu \). Because \( \alpha \) is a Markov chain and because of the independence of \( x_k \) with \( \xi_k \) and \( \theta_k \), we can show for any \( \delta > 0 \), \( t > 0 \), \( s > 0 \) with \( s \leq \delta \), for some \( \tilde{\gamma}_t^\mu > 0 \) that is \( \mathcal{F}_t^\mu \) measurable,

\[
\sup_{0 \leq s \leq \delta} \mathbb{E}_t^\mu [||\chi^\mu(t + s) - \chi^\mu(t)||^2 | \mathcal{F}_s^\mu] \leq \tilde{\gamma}_t^\mu = O(\delta).
\]

It then follows that

\[
\lim_{\delta \to 0} \sup_{\mu \to 0} \mathbb{E}_t^\mu \leq 0,
\]

which implies the tightness of \(\{\chi^\mu(\cdot)\}\) (see [Kushner, 1984, p. 47, Theorem 3]) and hence the tightness of \(\{x^\mu(\cdot)\}\). We can also prove the tightness of \(\{\alpha^\mu(\cdot)\}\). Then the tightness of \(\alpha^\mu(\cdot), x^\mu(\cdot)\) can be proved. The rest of the averaging procedure is similar to that of the proof of Theorem 1.

For the proofs of Statement 2, the case \( \eta = \mu^{1+\Delta} \), and Statement 3, the case \( \eta = \mu^\Delta \), the arguments are similar to Yin et al. [2013] Section 4.1 and Section 4.2, respectively. We thus omit the details.

**VI. Proof of Convergence of Algorithm (15)**

We recall that \( \nabla r_k(\theta) \) can be written as \( \bar{r}(\theta, \xi_k) \) as in the proof of Theorem 1. The algorithm can be written as

\[
\alpha_{k+1} = \alpha_k + \frac{\beta}{2} \sum_{i=1}^n \frac{p(\theta_i|\alpha_k)\bar{r}(\theta_i, \xi_k)}{\sum_{i=1}^n p(\theta_i|\alpha_k)} + \sqrt{\mu} w_k,
\]

where \( L_\mu \) is so chosen that \( L_\mu \to \infty \) as \( \mu \to 0 \). We illustrate how the convergence can be carried out. Let us begin by assuming the following assumptions holds.

(a) For each \( \xi, \bar{r}(\cdot, \xi) \) has continuous partial derivatives up to the second order.
b) The \{\theta_i\} is a stationary sequence \(\theta_i \sim \pi(\cdot)\); \{\theta_i\} is independent of \{\xi_k\} and \{w_k\}, where \{\xi_k\} and \{w_k\} are as in (H3).

c) For each fixed \(\alpha\), and each \(i = 1, \ldots, L_\mu\), define
\[
\gamma_i(\alpha) = \frac{p(\theta_i|\alpha)}{\sum_{l=1}^{L_\mu} p(\theta_l|\alpha)}.
\]

For each \(\xi\) and each \(\alpha\), as \(\mu \to 0\),
\[
\sum_{i=1}^{L_\mu} \gamma_i(\alpha) \overline{r}(\theta_i, \xi) \to \mathbb{E}\overline{r}(\theta, \xi|\alpha) \quad \text{w.p.1.} \tag{71}
\]

d) \(\mathbb{E}(|\overline{r}(\theta_i, \xi_k)|^2) < \infty\) for each \(i\) and each \(k\).

Remark. We briefly comment on the assumptions. Part (a) is on the smoothness of \(\overline{r}(\cdot, \xi)\). Part (b) requires that the distribution of \(\theta_i\) to be stationary with distribution \(\pi(\cdot)\). Part (c) is an averaging condition; i.i.d. sample \(\{\theta_i\}\) is a special case. In fact, we only need the convergence to be in the sense of convergence in probability. Part (d) because \(\{\theta_i\}\) and \(\{\xi_k\}\) are stationary, \(\{\overline{r}(\theta_i, \xi_k)\}\) is also stationary. The requirement here is on the moment of this sequence.

In what follows, we present the main ideas leading to the weak convergence, define \(\alpha^\mu(t) = \alpha_k, \text{ for } t \in [k\mu, k\mu + \mu)\). We should still use the martingale problem formulation. However, for simplicity, we will illustrate the idea without using this function \(f(\cdot)\).

Denote by \(\mathcal{F}_t^\mu\), the \(\sigma\)-algebra generated by \(\{\theta_k, \xi_k, \alpha_0 : k \leq \lceil t/\mu \rceil\}\), where \(\lceil s \rceil\) denotes the integer part of \(s\). In what follows, we shall suppress the floor function notation. Denote by \(\mathbb{E}_t^\mu\), the conditional expectation with respect to \(\mathcal{F}_t^\mu\). We also use \(\mathbb{E}_{\xi_k}\) to denote the conditioning on \(\{\xi_j : j \leq k\}\). For any \(\delta > 0, t > 0, s > 0\) and \(s \leq \delta\), we have
\[
\mathbb{E}_t^\mu|\alpha^\mu(t + s) - \alpha^\mu(t)|^2 \\
\leq K \left[ \sum_{k=t/\mu}^{(t+s)/\mu - 1} \sum_{i=1}^{L_\mu} \gamma_i(\alpha_k) \overline{r}(\theta_i, \xi_k) \right]^2 + \mathbb{E}_t^\mu \left[ \sqrt{\mu} \sum_{k=t/\mu}^{(t+s)/\mu - 1} w_k \right]^2 \\
\leq Ks \mu \sum_{k=t/\mu}^{(t+s)/\mu - 1} \mathbb{E}_k^\mu \left[ \sum_{i=1}^{L_\mu} \gamma_i(\alpha_k) \overline{r}(\theta_i, \xi_k) \right]^2 + K \mu \mathbb{E}_t^\mu \sum_{k=t/\mu}^{(t+s)/\mu - 1} w_k^2 w_k \leq \hat{\gamma}_t^\mu
\]

such that \(\hat{\gamma}_t^\mu\) is \(\mathcal{F}_t^\mu\) measurable and
\[
\lim_{\delta \to 0} \lim_{\mu \to 0} \mathbb{E}\hat{\gamma}_t^\mu = 0.
\]

Thus the tightness of \(\{\alpha^\mu(\cdot)\}\) is obtained; see [Kushner, 1984, p. 47].

By Prohorov’s theorem, we can extract a weakly convergent subsequence. Select such a sequence and still use \(\mu\) as the index with limit \(\alpha(\cdot)\). By Skorohod representation (without changing notation), \(\alpha^\mu(\cdot)\) converges w.p.1 to \(\alpha(\cdot)\). Now for any \(t > 0\) and \(s > 0\),
\[
\alpha^\mu(t + s) - \alpha^\mu(t) = \frac{\beta}{2} \mu \sum_{k=t/\mu}^{(t+s)/\mu - 1} \sum_{i=1}^{L_\mu} \gamma_i(\alpha_k) \overline{r}(\theta_i, \xi_k) + \sqrt{\mu} \sum_{k=t/\mu}^{(t+s)/\mu - 1} w_k. \tag{72}
\]

Define
\[
W^\mu(t) = \sqrt{\mu} \sum_{k=0}^{t/\mu - 1} w_k.
\]

By using the classical functional invariance theorem, it is easily seen that \(W^\mu(\cdot)\) converges weakly to \(W(\cdot)\) a standard Brownian motion. As a consequence,
\[
W^\mu(t + s) - W^\mu(t) = \sqrt{\mu} \sum_{k=t/\mu}^{(t+s)/\mu - 1} w_k \\
\to W(t + s) - W(t)
\]
by the weak convergence and the Skorohod representation. To figure out the limit of the drift term, we do it as in the proof of Theorem 1. In view of (16), \( \gamma_i(\alpha) \) is continuous (and in fact smooth) w.r.t. \( \alpha \). Thus using the notation as in the proof of Theorem 1, and choosing any positive integer \( \kappa_1 \) and \( t_s \leq t \) with \( t \leq \kappa_1 \),

\[
\lim_{\mu \to 0} \mathbb{E}g(\alpha^\mu(t_s) : t \leq \kappa_1)
\begin{bmatrix}
\frac{(t+s)/\mu - 1}{L_{\mu}} L_{\mu} \sum_{k=l_{\mu}} \sum_{i=1} \gamma_i(\alpha_k)^2(\theta_1, \xi_k)
\end{bmatrix}
\]

\[
= \lim_{\mu \to 0} \mathbb{E}g(\alpha^\mu(t_s) : t \leq \kappa_1)
\begin{bmatrix}
\sum_{k=l_{\mu}} \sum_{i=1} \gamma_i(\alpha_k)^2(\theta_1, \xi_k)
\end{bmatrix}
\]

\[
= \lim_{\mu \to 0} \mathbb{E}g(\alpha^\mu(t_s) : t \leq \kappa_1)
\begin{bmatrix}
\sum_{k=l_{\mu}} \sum_{i=1} \gamma_i(\alpha_k)^2(\theta_1, \xi_k)
\end{bmatrix}
\]

\[
(73)
\]

\[
= \lim_{\mu \to 0} \mathbb{E}g(\alpha^\mu(t_s) : t \leq \kappa_1)
\begin{bmatrix}
\sum_{k=l_{\mu}} \sum_{i=1} \gamma_i(\alpha_k)^2(\theta_1, \xi_k)
\end{bmatrix}
\]

\[
= \lim_{\mu \to 0} \mathbb{E}(\alpha(t_s) : t \leq \kappa_1)
\begin{bmatrix}
\int_t^{t+s} \int_{\mathbb{R}^N} \nabla R_\theta p(\theta) d\theta d\mu
\end{bmatrix}
\]

where \( o(1) \to 0 \) uniformly, and \( \mathbb{E}_{\xi_k} \) denotes the conditioning on \( \{\xi_j : j \leq k\} \). In the above, we used (71), and noted that letting \( \mu l_{\mu} \to u \) yields \( \mu k \to u \) for \( l_{\mu} k \leq k \leq l_{\mu} + m_{\mu} \). We also used the finite value approximation argument as just above (62). That is, for each \( \eta > 0 \), we can choose \( \{\xi_j^\eta : j \leq j_\eta\} \) as a finite collection of disjoint sets of diameter no larger than \( \eta \) whose union covers the range of \( \alpha^\mu(u) \) so \( \alpha^\mu(u) \) can be approximated by \( \sum_{j=1}^{j_\eta} \alpha_j^\eta 1_{\{\alpha^\mu(u) \in O_j^\eta\}} \). Putting the estimates together, we obtain the limit (19).

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G. G. Yin and Q. Zhang. Continuous-time Markov chains and applications: a two-time-scale approach, volume 37. Springer Science & Business Media, 2013.
% IRL algorithm for mutimodal mixture
mixture_weight = 0.5; nsamples=100; sigp1=sqrt(10); sigp2 = 1; sigl1 = sqrt(2); sigl2 = sqrt(2);
mixture_mean1 = 0; mixture_mean2 = 1;
T = 800000000; th=0.01; alfa = randn(2,1);
step =1e-3; lang_step1 = 1e-5; lang_step2 = sqrt(lang_step1); kernelstep=0.2;
Cker = 1/(2*pi*kernelstep^2); kernelstepq = 2*kernelstep^2;
alphIRL = zeros(2,T); est=zeros(2,T);
for iter = 1: T
    th = randn(2,1);
    % simulate data
    if rand < mixture_weight
        y = sigl1 * randn + mixture_mean1;
    else
        y = sigl2 * randn + mixture_mean1 + mixture_mean2;
    end;
    t1 = th(1); t2 = th(2);
    % evaluate gradients
    grad1 = nsamples*(mix_weight*exp(-(t1 - y)^2/(2*(sigl1^2)))*(2*t1 - 2*y))/(2*sigl1^3) - (exp(-(t1 + t2 - y)^2/(2*(sigl2^2)))*(mix_weight - 1)*((2*t1 + 2*t2 - 2*y)))/(2*sigl2^3)*((exp(-(t1 + t2 - y)^2/(2*(sigl2^2)))*(mix_weight - 1))/sigl2 - (mix_weight*exp(-(t1 - y)^2/(2*(sigl1^2))))/sigl1) - t1/(sigl1^2);
    grad2 = - t2/(sigl2^2) - nsamples*(exp(-(t1 + t2 - y)^2/(2*(sigl2^2)))*(mix_weight - 1)*(2*t1 + 2*t2 - 2*y)))/(2*sigl2^3)*((exp(-((t1 + t2 - y)^2/(2*(sigl2^2)))*(mix_weight - 1))/sigl2 - (mix_weight*exp(-(t1 - y)^2/(2*(sigl1^2)))))/sigl1)) ;

% passive Langevin dynamics
    gaus = exp(-alfa(1)/2) .* exp(-alfa(2)/2) / ( 2 * pi);
    alfa = alfa + Cker*exp(-(norm(th-alfa))^2/kernelstepq)*([lang_step1/2 *[grad1; grad2]]/gaus) + lang_step2 * randn(2,1);
alphIRL(:,iter) = alfa;
% classical stochastic gradient
    th = th + step *[grad1; grad2];
est(:,iter) = th;
end;

% Plotting
    figure (4); histogram2 (alphIRL(1,5000:end),alphIRL(2,5000:end), 'Normalization', 'probability');
    axis([-3.3,-3.3]); xlabel('theta(1)', 'Interpreter', 'latex', 'FontSize',18); ylabel('theta(2)', 'Interpreter', 'latex', 'FontSize',18);
    figure (5); [M1,N1] = hist3 (alphIRL(:,5000:end),[20,20]);
    contour(M1,[N1(1), N1(2), M1]); axis([-3.3,-3.3]);
    xlabel('theta(1)', 'Interpreter', 'latex', 'FontSize',18);
    ylabel('theta(2)', 'Interpreter', 'latex', 'FontSize',18); grid on; colormap(jet);

% Multi-kernel IRL for logistic regression a9a dataset
%Algorithm parameters
    lang_step1 = 0.25e-3; lang_step2 = sqrt(lang_step1); sigma_theta=1;
    sigma_kernel = 0.1; L =100; incrementsig = 0.1;
% Read a9a Dataset
    load datasetFeatures.mat; load datasetLabels.mat;
    thdim=size (features,1) + 1; T = 32400; Nsweep =10; nsamples=10;
lables (labels < 0) = 0; % set all -1 to 0
    est = zeros (thdim,T*Nsweep); th = zeros (thdim,1); alfa = th;
for sweep = 1: Nsweep
    for iter=1:T
        th = sigma_theta * randn(thdim,L); % RL chooses th randomly
        d = vecnorm(th-alfa);
        weight =10^(-2*thdim) * exp(- d.^2/(2*sigma_kernel));
        if (sum(weight) < 1e-60)
            alfa = 0.1*randn(thdim,1); %reset alfa if stuck
end;

nweight = weight / sum(weight);

psi = [1:features(:,iter)]; y = labels(iter);
sigmoidy = 1./(1 + exp(-psi'*th));
wgrad = (nsamples*psi.* (y - sigmoidy) - sign(th))*nweight';

k = (sweep-1) * T + iter; k,

% logistic regression step for IRL algorithm
alfa = alfa +0.5*lang_step1 * wgrad + lang_step2 * randn(thdim,1);
est(:,k) = alfa ;
end;

figure(4); histogram(est(1,:), 'Normalization', 'probability');
title('Multi-kernel algorithm')

Remarks: Out of 10 sweeps, where each sweep has 32000 iterations, only 14 resets (line 25) were required for $L = 100$ in IRL algorithm (15).
```matlab
% Multi-kernel IRL for MDP
T = 150000; gridpoints = 100;
statedim = 2; actiondim = 2;
lang_step1 = 5e-6; lang_step2 = sqrt(lang_step1);
l*step = 1/2*lang_step1;
cost = [1 100; 30 2]; constraint = [0.2 0.3; 2 1]; lambda = 1e5;
inversestep = gridpoints/2;

% Solve avg cost MDP exactly, over a grid of 100x100 possible randomized policies
for i=1:gridpoints-1,
    for j=1:gridpoints-1,
        k = gridpoints*(i-1)+j;
        policy = [i/gridpoints, 1-i/gridpoints; j/gridpoints, 1-j/gridpoints];
        pol(:,k) = [policy(1,1); policy(2,1)];
        pol_eval = mdp_barrier(cost, constraint, policy, tp);
        Penalty_Reward(k) = polval(pol, pol_con(k) = polcon);
        Penalty_Reward(k) = polval - lambda * ((1 - polcon)^2);
end
end

% Evaluate and store finite difference gradients of MDP over a 100x100 grid
for i=2:gridpoints-1,
    for j=2:gridpoints-1,
        grad(i,j,1) = (pol_eval(i+1,j) - pol_eval(i-1,j)) * inversestep;
        grad(i,j,2) = (pol_eval(i,j+1) - pol_eval(i,j-1)) * inversestep;
end
end

% IRL algorithm
kernelstep = 0.1;
alphabet = [1;1]; Cker = 1/(2*pi*kernelstep^2); L=50;

for k=1:T,
    thbar = pi/2+rand(2,L);
    d = vecnorm(alphabet-thbar);
    weight = 10*exp(-d.^2/(2*kernelstep^2));
    if (sum(weight) < 1e-6)
        alphabet = 0.1*randn(2,1); %reset alphabet if stuck
    end;
    newweight = weight/sum(weight);
    p = (sin(thbar))' ;
    pindex = min(max(round(gridpoints*p), [1;1]),[gridpoints-1;gridpoints-1]);
    ghatp = zeros(2,1);
    for i=1:L
        ghatp = ghatp + grad(pindex(i,1),pindex(2,i),1); grad(pindex(i,1),pindex(2,i),2)) . * sin(thbar(i)) * cos(thbar(i)) * newweight(i) + ghatp;
    end;
    ghatm = 2*ghatp; % weighted gradient

% Passive Langevin dynamics
alphabet = alphabet + (l*step * ghatm) + lang_step2 * randn(2,1); alphabet = abs(alphabet);
alpha(:,k) = alphabet;
cond_prob(:,k) = (sin(alphabet))^2; %policy conditional probabilities
end;

% plot log of empirical density
[M,N] = hist3(cond_prob(:,T/2:end)',[100,100]);
figure(3); stem3(flip(N{1}),flip(N{2}),log(M),'MarkerFaceColor','g');
xlabel('pol'); ylabel('pol2');
end

% External Function used in above program
function [avg_cost, avg_constraint] = mdp_barrier(cost, constraint, policy, tp)
```

```
% evaluate MDP policy for average cost MDP

\[ \text{statedim} = \text{size}(\text{cost},1); \quad \text{actiondim} = \text{size}(\text{cost},2); \]

\[ \text{tp}(\cdot,\cdot,1) = [0.8 \ 0.2; \ 0.3 \ 0.7]; \quad \text{tp}(\cdot,\cdot,2) = [0.6 \ 0.4; \ 0.1 \ 0.9]; \]

\[ \text{pol} = [0.8, \ 0.2; \ 0.3 \ 0.7]; \]

\[ \text{cost} = [1 \ 10; \ 3 \ 2]; \]

\begin{verbatim}
for i=1:statedim
    for a=1:actiondim
        l = a + (i-1)*actiondim;
        for j=1:statedim
            for abar=1:actiondim
                m = abar + (j-1)*actiondim;
                \text{tpcomposite}(l,m) = \text{tp}(i,j,a) * \text{pol}(j,abar);
            end;
            \text{costvector}(l) = \text{cost}(i,a);
            \text{constraintvector}(l) = \text{constraint}(i,a);
        end;
    end;
end;

[aa,bb] = \text{eig}(\text{tpcomposite}');
\text{jointprob} = aa(:,1)/\text{sum}(aa(:,1));

\text{avgconstraint} = \text{constraintvector} * \text{jointprob};
\text{avgcost} = \text{costvector} * \text{jointprob};
\end{verbatim}