BIJECTIONS AND METRIC SPACES INDUCED BY SOME COLLECTIVE PROPERTIES OF CONCAVE YOUNG-FUNCTIONS

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Abstract. For each \( b \in (0, \infty) \) we intend to generate a decreasing sequence of subsets \((Y_b^{(n)}) \subseteq Y_{\text{conc}}\) depending on \( b \) such that whenever \( n \in \mathbb{N} \), then \( A \cap Y_b^{(n)} \) is dense in \( Y_b^{(n)} \) and the following four sets \( Y_b^{(n)} \), \( Y_b^{(n)} \setminus (A \cap Y_b^{(n)}) \), \( A \cap Y_b^{(n)} \) and \( Y_{\text{conc}} \) are pairwise equinumerous. Among others we also show that if \( f \) is any measurable function on a measure space \((\Omega, \mathcal{F}, \lambda)\) and \( p \in [1, \infty) \) is an arbitrary number then the quantities \( \|f\|_{L^p} \) and \( \sup_{\Phi \in Y_{\text{conc}}} (\Phi(1))^{-1} \|\Phi \circ |f|\|_{L^p} \) are equivalent, in the sense that they are both either finite or infinite at the same time.

1. Introduction

We know that concave functions play major roles in many branches of mathematics for instance probability theory ([4], [6], [10], say), interpolation theory (cf. [13], say), weighted norm inequalities (cf. [5], say), and functions spaces (cf. [12], say), as well as in many other branches of sciences. In the line of [4], [6] and [10], the present author also obtained in martingale theory some results in connection with certain collective properties or behaviors of concave Young-functions (cf. [1], [2]). The study presented in [3] was mainly motivated by the question why strictly concave functions possess so many properties, worth to be characterized using appropriate tools that await to be discovered.

We say that a function \( \Phi : [0, \infty) \rightarrow [0, \infty) \) belongs to the set \( Y_{\text{conc}} \) (and is referred to as a concave Young-function) if and only if it admits the integral representation

\[
\Phi(x) = \int_0^x \varphi(t) \, dt,
\]

(1.1)

(where \( \varphi : (0, \infty) \rightarrow (0, \infty) \) is a right-continuous and decreasing function such that it is integrable on every finite interval \((0, x)\) and \( \Phi(\infty) = \infty \). It is worth to note that every function in \( Y_{\text{conc}} \) is strictly concave.

We will remind some results obtained so far in [3].

We shall say that a concave Young-function \( \Phi \) satisfies the \textit{density-level property} if \( A_{\Phi}(\infty) < \infty \), where \( A_{\Phi}(\infty) := \int_1^\infty \frac{\varphi(t)}{t} \, dt \). All the concave Young-functions possessing the density-level property will be grouped in a set \( A \).

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In Theorems 1 and 2 (cf. [3]), we showed that the composition of any two concave Young-functions satisfies the density-level property if and only if at least one of them satisfies it. These two theorems show that concave Young-functions with the density-level property behave like left and right ideal with respect to the composition operation.

We also proved ([3], Lemma 5, page 12) that if \( \| f \| > 0 \) and \( B = 0 \), then use this subset to express the value of \( F \). Theorem 1. The sets \( \mathcal{Y}_{\text{conc}} \) and \( \mathcal{Y}_{\text{conc}} \setminus \mathcal{A} \) really are.

We first anticipate that there are as many elements in each of the sets \( \mathcal{A} \) and \( \mathcal{Y}_{\text{conc}} \setminus \mathcal{A} \) as there exist in \( \mathcal{Y}_{\text{conc}} \), showing how broad the set of concave Young-functions possessing the density-level property and its complement really are.

**Theorem 1.** The sets \( \mathcal{A} \), \( \mathcal{Y}_{\text{conc}} \) and \( \mathcal{Y}_{\text{conc}} \setminus \mathcal{A} \) are pairwise equinumerous.
Proof. We first show that there is a bijection between \( A \) and \( \mathcal{Y}_{\text{conc}} \). In fact, since \( A \) is a proper subset of \( \mathcal{Y}_{\text{conc}} \) there is an injection from \( A \) to \( \mathcal{Y}_{\text{conc}} \), as a matter of fact, the identity mapping from \( A \) into \( \mathcal{Y}_{\text{conc}} \) will do. Fix any number \( \alpha \in (0, 1) \) and define the mapping \( S_\alpha : \mathcal{Y}_{\text{conc}} \to A \) by \( S_\alpha (\Phi) = \Phi^\alpha \). We point out that this mapping exists in virtue of Theorem 2 in [3]. It is not hard to see that \( S_\alpha \) is an injection. Then the Schröder-Bernstein theorem entails that there exists a bijection between \( A \) and \( \mathcal{Y}_{\text{conc}} \). To complete the proof it is enough to show that there is a bijection between \( A \) and \( \mathcal{Y}_{\text{conc}} \). In fact, fix some \( \Phi \in \mathcal{Y}_{\text{conc}} \setminus A \) and define the function \( h_\Phi : A \to \mathcal{Y}_{\text{conc}} \setminus A \) by \( h_\Phi (\Delta) = \Delta + \Phi \). Obviously, \( h_\Phi \) is an injection. Now, fix any \( \Delta \in A \) and define the function \( f_\Delta : \mathcal{Y}_{\text{conc}} \setminus A \to A \) by \( f_\Delta (\Phi) = \Delta \circ \Phi \). We point out that this function always exists due to Theorem 2 in [3]. It is not difficult to show that \( f_\Delta \) is an injection if we take into account that \( \Delta \) is an invertible function. Consequently, the Schröder-Bernstein theorem guarantees the existence of a bijection between \( A \) and \( \mathcal{Y}_{\text{conc}} \setminus A \). Therefore, we can conclude on the validity of the argument. \( \square \)

Write \( A_b := \{ \Phi \in A : \Phi (b) = b \} \) and \( \mathcal{Y}_b := \{ \Phi \in \mathcal{Y}_{\text{conc}} : \Phi (b) = b \} \) for every number \( b \in (0, \infty) \).

Let us denote by \( Z := \{ A_b : b \in (0, \infty) \} \) and \( Z^* := \{ \mathcal{Y}_b : b \in (0, \infty) \} \).

It is obvious that \( A_b \subset \mathcal{Y}_b \) for every number \( b \in (0, \infty) \) and \( Z \cap Z^* = \emptyset \).

**Lemma 1.** For every number \( b \in (0, \infty) \) the identities \( A_b = \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in A \right\} \) and \( \mathcal{Y}_b = \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in \mathcal{Y}_{\text{conc}} \right\} \) hold true.

**Proof.** Pick any function \( \Psi \in A_b \). Then \( \Psi \in A \) and \( \Psi (b) = b \), so that \( \Psi = \frac{\Psi}{\Psi(b)} \in \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in A \right\} \), i.e. \( A_b \subset \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in A \right\} \). To show the reverse inclusion consider any function \( \Psi \in \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in A \right\} \). Then necessarily there must exist some \( \Phi \in A \) such that \( \Psi = \frac{\Phi}{\Phi(b)} \). It is obvious that \( \Psi \in A \) and \( \Psi (b) = b \), i.e. \( \Psi \in A_b \). Hence, \( \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in A \right\} \subset A_b \). These two inclusions yield that \( A_b = \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in A \right\} \). The proof of identity \( \mathcal{Y}_b = \left\{ \frac{\Phi}{\Phi(b)} : \Phi \in \mathcal{Y}_{\text{conc}} \right\} \) can be similarly carried out. \( \square \)

**Definition 1.** A proper subset \( \mathcal{G} \) of \( A \) is said to be maximally bounded if each of the sets \( \mathcal{G} \) and \( A \setminus \mathcal{G} \) is equinumerous with \( A \), i.e. there is a bijection between \( A \) and \( \mathcal{G} \), and \( \text{diam}(\mathcal{G}) < \infty \), where \( \text{diam}(\mathcal{G}) := \sup \{ d (\Phi_1, \Phi_2) : \Phi_1, \Phi_2 \in \mathcal{G} \} \) is the diameter of \( \mathcal{G} \).

We note that Definition 1 makes sense for the two reasons here below.

On the one hand we assert that \( \text{diam}(A) = \sup \{ d (\Phi_1, \Phi_2) : \Phi_1, \Phi_2 \in A \} = \infty \).

In fact, fix some \( \Phi \in A \) and define a sequence \( (\Phi_n) \subset \mathcal{Y}_{\text{conc}} \) by \( \Phi_{2n} = 4n\Phi \) and \( \Phi_{2n-1} = (2n-1)\Phi \), \( n \in \mathbb{N} \). It is clear that \( (\Phi_n) \subset A \) and \( d (\Phi_{2n}, \Phi_{2n-1}) = (2n+1)\|\Phi\| \), \( n \in \mathbb{N} \). Hence, \( \text{diam}(A) = \infty \).

On the other hand the set \( \{ (\Phi (1))^{-1} \Phi : \Phi \in \mathcal{Y}_{\text{conc}} \} \) is of finite diameter. In fact for any \( \Phi, \Psi \in \mathcal{Y}_{\text{conc}} \) we have, via Lemma 3 in [3], that

\[
 d \left( (\Phi (1))^{-1} \Phi, (\Psi (1))^{-1} \Psi \right) \leq \left\| (\Phi (1))^{-1} \Phi \right\| + \left\| (\Psi (1))^{-1} \Psi \right\| \leq 2 \|S\| < \infty.
\]

Let us define two relations \( \perp \subset A \times A \) and \( \perp^* \subset \mathcal{Y}_{\text{conc}} \times \mathcal{Y}_{\text{conc}} \) as follows:
(1) We say that $(\Phi, \Psi) \in \perp$, where $(\Phi, \Psi) \in \mathcal{A} \times \mathcal{A}$, (and write $\Phi \perp \Psi$) if and only if there is some constant $c \in (0, \infty)$ such that $\Psi(x) = c\Phi(x)$ for all $x \in (0, \infty)$.

(2) We say that $(\Phi, \Psi) \in \perp^*$, where $(\Phi, \Psi) \in \mathcal{Y}_{\text{conc}} \times \mathcal{Y}_{\text{conc}}$, (and write $\Phi \perp^* \Psi$) if and only if there is some constant $c \in (0, \infty)$ such that $\Psi(x) = c\Phi(x)$ for all $x \in (0, \infty)$.

It is not hard to see that $\perp$ and $\perp^*$ are equivalence relations on $\mathcal{A}$ and $\mathcal{Y}_{\text{conc}}$ respectively, i.e. they are reflexive, symmetric and transitive. Their corresponding equivalence classes are respectively

\[ p_{\perp}(\Psi) := \{ \Phi : \Phi \in \mathcal{A} \text{ and } \Phi \perp \Psi \}, \quad \Psi \in \mathcal{A} \]

\[ p_{\perp^*}(\Delta) := \{ \Phi : \Phi \in \mathcal{Y}_{\text{conc}} \text{ and } \Phi \perp^* \Delta \}, \quad \Delta \in \mathcal{Y}_{\text{conc}} \]

and their respective induced factor (or quotient) sets can be given by

\[ \mathcal{A}/\perp := \{ C : C \subset \mathcal{A} \text{ and } C = p_{\perp}(\Psi) \text{ for some } \Psi \in \mathcal{A} \}, \]

\[ \mathcal{Y}_{\text{conc}}/\perp^* := \{ C : C \subset \mathcal{Y}_{\text{conc}} \text{ and } C = p_{\perp^*}(\Delta) \text{ for some } \Delta \in \mathcal{Y}_{\text{conc}} \} \]

One can easily verify that for all $\Psi \in \mathcal{A}$ and $\Delta \in \mathcal{Y}_{\text{conc}}$ the equivalence classes $p_{\perp}(\Psi)$ and $p_{\perp^*}(\Delta)$ are of continuum size or magnitude.

**Theorem 2.** Let $b \in (0, \infty)$ be any fixed number.

**Part I.** Define the mapping $f : \mathcal{A} \to \mathcal{A}_b$ by $f(\Phi) = \frac{b}{\Psi(0)}\Phi$. Then there is a unique mapping $g : \mathcal{A}/\perp \to \mathcal{A}_b$ for which the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{p_{\perp}} & \mathcal{A}/\perp \\
\downarrow f & & \downarrow g \\
b & &
\end{array}
\]

commutes (i.e. $f = g \circ p_{\perp}$) and moreover, the mapping $g$ is a bijection.

**Part II.** Define the mapping $f^* : \mathcal{Y}_{\text{conc}} \to \mathcal{Y}_b$ by $f^*(\Delta) = \frac{b}{\Psi(0)}\Delta$. Then there is a unique mapping $g^* : \mathcal{Y}_{\text{conc}}/\perp^* \to \mathcal{Y}_b$ for which the diagram

\[
\begin{array}{ccc}
\mathcal{Y}_{\text{conc}} & \xrightarrow{p_{\perp^*}} & \mathcal{Y}_{\text{conc}}/\perp^* \\
\downarrow f^* & & \downarrow g^* \\
b^* & &
\end{array}
\]

commutes (i.e. $f^* = g^* \circ p_{\perp^*}$) and moreover, the mapping $g^*$ is a bijection.

We point out that the proof of Theorem 2 is obvious.

**Proposition 1.** Let $b \in (0, \infty)$ be an arbitrarily fixed number.

**Part I.** There is a bijection between $\mathcal{Y}_b$ and $\mathcal{Y}_{\text{conc}}$.

**Part II.** There is a bijection between $\mathcal{A}_b$ and $\mathcal{A}$.

**Proof.** We shall only show the first part because the other case can be similarly proved. To this end, write $\mathcal{Y}_{bb} := \{ b\Phi : \Phi \in \mathcal{Y}_{\text{conc}} \}$. We note that $\mathcal{Y}_{bb}$ and $\mathcal{Y}_{\text{conc}}$ are equinumerous for the reasons that $\mathcal{Y}_{bb} \subset \mathcal{Y}_{\text{conc}}$ and the function $F : \mathcal{Y}_{\text{conc}} \to \mathcal{Y}_{bb}$, defined by $F(\Phi) = b\Phi$, can be easily shown to be an injection. Thus it will be enough to prove that $\mathcal{Y}_{bb}$ and $\mathcal{Y}_b$ are equinumerous. In fact, consider the function...
Let $b \in (0, \infty)$ be arbitrary. Then the following six sets $\mathcal{A}$, $\mathcal{A}_b$, $\mathcal{Y}_b$, $\mathcal{A}/\bot$, $\mathcal{Y}_{\text{conc}}/\bot^*$ and $\mathcal{Y}_{\text{conc}}$ are equinumerous.

Proof. We note that $\mathcal{A}$ and $\mathcal{Y}_{\text{conc}}$ are equinumerous (by Theorem 11 and, by Theorem 2, $\mathcal{A}/\bot$ and $\mathcal{A}_b$ are equinumerous. On the other hand $\mathcal{A}$ and $\mathcal{A}_b$ are equinumerous as well as $\mathcal{Y}_b$ and $\mathcal{Y}_{\text{conc}}$ are (by Proposition 11). Thus $\mathcal{A}_b$ and $\mathcal{Y}_{\text{conc}}$ are equinumerous. Therefore, as $\mathcal{Y}_b$ and $\mathcal{Y}_{\text{conc}}/\bot^*$ are equinumerous (by Theorem 2), we can conclude on the validity of the argument.

Remark 1. Let $b_1$ and $b_2 \in (0, \infty)$ be two arbitrary distinct numbers. Then $\mathcal{A}_{b_1} \cap \mathcal{A}_{b_2}$ and $\mathcal{Y}_{b_1} \cap \mathcal{Y}_{b_2}$ are empty sets.

Remark 2. Let $b_1$ and $b_2 \in (0, \infty)$ be two arbitrary distinct numbers. Then $\mathcal{A}_{b_1} \cup \mathcal{A}_{b_2} \notin \mathcal{Z}$ and $\mathcal{Y}_{b_1} \cup \mathcal{Y}_{b_2} \notin \mathcal{Z}^*$.

Remark 3. Fix arbitrarily a number $b \in (0, \infty)$. Then it is easily seen that the function $h_b : [0, \infty) \to [0, \infty)$, defined by $h_b(x) = x + b$, is square integrable with respect to measure $\mu$ and, moreover, $C_b := \int_0^\infty \frac{(h_b(x))^2}{(x+1)^2} dx = \frac{1}{3} (b^2 + b + 1) < \infty$.

Remark 4. If $\Phi \in \mathcal{Y}_b$, then $\Phi(x) \leq h_b(x)$ for all $x \in [0, \infty)$.

Proof. Fix any $\Phi \in \mathcal{Y}_b$. As $\Phi$ is a concave function its graph must lie below the tangent of equation $y = \varphi(b)(x - b) + b$ at point $(b, b)$ since $\Phi(b) = b$. Consequently, for all $x \in [0, \infty)$ we have:

$$\Phi(x) \leq \varphi(b)(x - b) + b \leq \varphi(b)x + b = b\varphi(b) \frac{x}{b} + b \leq \Phi(b) \frac{x}{b} + b = h_b(x).$$

Proposition 2. Let $b \in (0, \infty)$ be any number. Then $\mathcal{Y}_b$ is of finite diameter.

Proof. Let $b \in (0, \infty)$ be the source of $\mathcal{Y}_b \in \mathcal{Z}^*$. We need to prove that $\mathcal{Y}_b$ has a finite diameter. In fact, consider two arbitrary functions $\Phi_1, \Phi_2 \in \mathcal{Y}_b$. Then

$$d(\Phi_1, \Phi_2) = ||\Phi_1 - \Phi_2|| \leq ||\Phi_1|| + ||\Phi_2|| \leq \sqrt{2C_b},$$

via Remarks 11 and 3. Therefore,

$$\text{diam}(\mathcal{Y}_b) := \sup\{d(\Phi_1, \Phi_2) : \Phi_1, \Phi_2 \in \mathcal{Y}_b\} \leq \sqrt{2C_b} < \infty.$$

Theorem 3. Let $b \in (0, \infty)$ be any number. Then $\mathcal{Y}_b$ is maximally bounded.

Proof. We just point out that the proof follows from the conjunction of both Propositions 2 and 11.

In the sequel $H_{[0, 1]}$ will stand for the collection of all finite sequences $(t_1, \ldots, t_k) \subset [0, 1]$ such that $t_1 + \ldots + t_k = 1$.

For any fixed $b \in (0, \infty)$ and every counting number $n \in \mathbb{N}$ write $\bigotimes_{i=1}^n \mathcal{A}_b$ (resp. $\bigotimes_{i=1}^n \mathcal{Y}_b$) for the $n$-fold Descartes product of $\mathcal{A}_b$ (resp. $\mathcal{Y}_b$).
For any pair of numbers \( n \in \mathbb{N} \) and whenever \( n \geq 2 \), write
\[
Y_b^{CO(n)} = \left\{ \Delta_1 \circ \Delta_2 \circ \ldots \circ \Delta_n : (\Delta_1, \Delta_2, \ldots, \Delta_n) \in X_b^n \right\},
\]
\[
A_b^{CO(n)} = \left\{ \Phi_1 \circ \ldots \circ \Phi_n : (\Phi_1, \ldots, \Phi_n) \in \prod_{i=1}^{n} Y_b \text{ and } \Phi_j \in A_b \text{ for some index } j \right\},
\]
\[
Y_b^{(n)} = \left\{ \sum_{i=1}^{k} t_i \Delta_i : \Delta_1, \Delta_2, \ldots, \Delta_k \in Y_b^{CO(n)}, (t_1, \ldots, t_k) \in H[0, 1] \right\},
\]
\[
A_b^{(n)} = \left\{ \sum_{i=1}^{k} t_i \Phi_i : \Phi_1, \Phi_2, \ldots, \Phi_k \in A_b^{CO(n)}, (t_1, \ldots, t_k) \in H[0, 1] \right\}.
\]

Further, for \( n = 1 \) write \( Z^{(1)} = Z \), \( Z^{* (1)} = Z^* \) and, for \( n \in \mathbb{N} \setminus \{1\} \) write \( Z^{(n)} := \left\{ A_b^{(n)} : b \in (0, \infty) \right\} \) and \( Z^{* (n)} := \left\{ Y_b^{(n)} : b \in (0, \infty) \right\} \).

**Remark 5.** For any pair of numbers \( n \in \mathbb{N} \) and \( b \in (0, \infty) \) the set \( A_b^{(n)} \) is a proper subset of \( Y_b^{(n)} \).

**Remark 6.** For any pair of numbers \( n \in \mathbb{N} \) and \( b \in (0, \infty) \) we have \( A_b^{(n)} \subset A_b^{(1)} = A_b \).

We point out that Remark 6 is a direct consequent of Theorem 2 in [3], page 6.

**Remark 7.** Let \( b \in (0, \infty) \), \( n \in \mathbb{N} \) and \( k \geq n \) be arbitrary numbers. Then
1. \( \Phi_1 \circ \Phi_2 \circ \ldots \circ \Phi_k \in A_b^{CO(n)} \) whenever \( \Phi_1, \Phi_2, \ldots, \Phi_k \in Y_b^{(1)} \) and \( \Phi_j \in A_b^{(1)} \)
   for some index \( j \in \{1, \ldots, k\} \)
2. \( \Delta_1 \circ \Delta_2 \circ \ldots \circ \Delta_k \in Y_b^{CO(n)} \) whenever \( \Delta_1, \Delta_2, \ldots, \Delta_k \in Y_b^{(1)} \).

**Proof.** Note that \( \Phi_1 \circ \Phi_2 \circ \ldots \circ \Phi_k = \Phi_1 \circ \Phi_2 \circ \ldots \circ \Phi_{n-1} \circ \Psi_1 \) and \( \Delta_1 \circ \Delta_2 \circ \ldots \circ \Delta_{n-1} \circ \Psi_2 \), where \( \Psi_1 = \Phi_n \circ \Phi_{n+1} \circ \ldots \circ \Phi_k \) and \( \Psi_2 = \Delta_n \circ \Delta_{n+1} \circ \ldots \circ \Delta_k \). From this simple observation the result easily follows. \( \square \)

From Remark 7 the following result can be easily derived, since it implies that \( A_b^{CO(n+1)} \) is a proper subset of \( A_b^{CO(n)} \) and, \( Y_b^{CO(n+1)} \) is also a proper subset of \( Y_b^{CO(n)} \).

**Lemma 2.** Let \( b \in (0, \infty) \) and \( n \in \mathbb{N} \) be arbitrary numbers. Then the following two assertions are valid.
1. The set \( A_b^{(n+1)} \) is a proper subset of \( A_b^{(n)} \).
2. The set \( Y_b^{(n+1)} \) is a proper subset of \( Y_b^{(n)} \).

**Theorem 4.** For any fixed pair of numbers \( n \in \mathbb{N} \) and \( b \in (0, \infty) \), the two sets \( A_b^{(n)} \) and \( A_b \) are equinumerous.

**Proof.** Throughout the proof we shall fix any counting number \( n \in \mathbb{N} \). We first note that the identity function \( I_{id} : A_b^{(n)} \rightarrow A_b \) is an injection, since \( A_b^{(n)} \subset A_b \). Next, pick any \( \Delta \in A_b \) and define the function \( f_\Delta : A_b \rightarrow A_b^{(n)} \) by \( f_\Delta (\Phi) = \underline{\Delta \circ \ldots \circ \Delta} \circ \Phi \).

We show that \( f_\Delta \) is an injection. In fact, let \( \Phi_1, \Phi_2 \in A_b \) be arbitrary and assume that \( f_\Delta (\Phi_1) = f_\Delta (\Phi_2) \). Then taking into account that \( \Delta \) is an invertible function we can easily deduce that \( \Phi_1 = \Phi_2 \), i.e. \( f_\Delta \) is an injection. Therefore, the Schröder-Bernstein theorem entails that there is a bijection between \( A_b \) and \( A_b^{(n)} \). This was to be proved. \( \square \)
Proposition 3. For any pair of numbers $n \in \mathbb{N}$ and $b \in (0, \infty)$ the sets $\mathcal{A}_b^{(n)}$ and $\mathcal{Y}_b^{(n)} \setminus \mathcal{A}_b^{(n)}$ are equinumerous.

Proof. Let $\Phi \in \mathcal{Y}_b^{(n)} \setminus \mathcal{A}_b^{(n)}$ and $(\alpha, \beta) \in H[0, 1]$ be arbitrarily fixed. Define the function $h_{\Phi}^{(\alpha, \beta)} : \mathcal{A}_b^{(n)} \to \mathcal{Y}_b^{(n)} \setminus \mathcal{A}_b^{(n)}$ by $h_{\Phi}^{(\alpha, \beta)}(\Delta) = \alpha \Delta + \beta \Phi$. It is clear that $h_{\Phi}^{(\alpha, \beta)}$ is actually an injection. Now, fix any $\Delta \in \mathcal{A}_b^{(n)}$ and define the function $f_\Delta : \mathcal{Y}_b^{(n)} \setminus \mathcal{A}_b^{(n)} \to \mathcal{A}_b^{(n)}$ by $f_\Delta(\Phi) = \Delta \circ \Phi$. We note that this function always exists because of the inclusion $\mathcal{A}_b^{(n)} \subset \mathcal{A}$ and Theorem 2 in [3]. Here too we can easily check that $f_\Delta$ is an injection. Therefore, the Schröder-Bernstein theorem yields the result to be proven. \[\square\]

Corollary 2. For any pair of numbers $n \in \mathbb{N}$ and $b \in (0, \infty)$ the following five sets $\mathcal{Y}_b^{(n)}$, $\mathcal{A}_b^{(n)}$, $\mathcal{Y}_b^{(n)} \setminus \mathcal{A}_b^{(n)}$, $\mathcal{A}$ and $\mathcal{Y}_{\text{conc}}$ are pairwise equinumerous.

3. THE METERIZATION OF SETS $Z^{(n)}$ AND $Z^{*^{(n)}}$

We shall only deal with the meterization of sets $Z$ and $Z^*$ since all the results in this section can be easily extended to the sets $Z^{(n)}$ and $Z^{*^{(n)}}$.

Whenever $\Phi \in \mathcal{Y}_{\text{conc}}$ write $G_{\Phi} := \{(x, \Phi(x)) : x \in (0, \infty)\}$ for the graph of $\Phi$ on $(0, \infty)$ and $G_{\Phi}^{a,b} := \{(x, \Phi(x)) : x \in [a, b]\}$ for the graph of $\Phi$ on the interval $[a, b]$ where $a < b$ are any non-negative numbers.

Remark 8. Let $b_1$ and $b_2 \in (0, \infty)$ be two arbitrary distinct numbers. If $b_1 < b_2$, then the following two assertions hold true:
1. For all $\Phi_1 \in \mathcal{A}_{b_1}$ and $\Phi_2 \in \mathcal{A}_{b_2}$ the inequality $\Phi_1(b_2) < \Phi_2(b_1)$ holds.
2. For all $\Phi_1 \in \mathcal{Y}_{b_1}$ and $\Phi_2 \in \mathcal{Y}_{b_2}$ the inequality $\Phi_1(b_2) < \Phi_2(b_1)$ holds.

Proof. Suppose that $b_1 < b_2$ and fix arbitrarily two functions $\Phi_1 \in \mathcal{Y}_{b_1}$ and $\Phi_2 \in \mathcal{Y}_{b_2}$. Obviously, $\Phi_1$ must hit $\Phi_2$ prior to $\Phi_2$. Hence, $G_{\Phi_1}^{b_1} \subset G_{\Phi_2}^{b_2}$. But since $G_{\Phi_1}^{b_1} \subset G_{\Phi_2}^{b_2}$ lies above the graph of the line of equation $y = b_1$ in the interval $(b_1, \infty)$, we have as an aftermath that $\Phi_1(b_1) < \Phi_2(b_2)$ and by the proof we note that assertion (2) can be similarly shown. \[\square\]

The binary relations $<$ and $\leq$, defined on $Z$ respectively by $\mathcal{A}_{b_1} \prec \mathcal{A}_{b_2}$ if and only if $\Phi_1(b_2) < \Phi_2(b_1)$ for all pairs $(\Phi_1, \Phi_2) \in \mathcal{A}_{b_1} \times \mathcal{A}_{b_2}$, and by $\mathcal{A}_{b_1} \preceq \mathcal{A}_{b_2}$ if and only if $\mathcal{A}_{b_1} \prec \mathcal{A}_{b_2}$ or $\mathcal{A}_{b_1} = \mathcal{A}_{b_2}$. We point out that The binary relations $<$ and $\leq$ can be similarly defined on $Z^*$.

We point out that the law of trichotomy is valid on $(Z, \preceq)$ and $(Z^*, \preceq)$, i.e. whenever $(\mathcal{A}_{b_1}, \mathcal{A}_{b_2}) \in Z \times Z$ or $(\mathcal{A}_{b_1}, \mathcal{A}_{b_2}) \in Z^* \times Z^*$, then precisely one of the following holds: $\mathcal{A}_{b_1} = \mathcal{A}_{b_2}$, $\mathcal{A}_{b_1} \prec \mathcal{A}_{b_2}$, $\mathcal{A}_{b_2} \prec \mathcal{A}_{b_1}$. Hence, we can easily check that $(Z, \preceq)$ and $(Z^*, \preceq)$ are chains, i.e. they are totally ordered sets.

Theorem 5. The functions $f_1 : (0, \infty) \to Z$ and $f_2 : (0, \infty) \to Z^*$, defined respectively by $f_1(p) = \mathcal{A}_p$ and $f_2(p) = \mathcal{Y}_p$, are order preserving bijections.

Proof. We show that the function $f_1 : (0, \infty) \to Z$, $f_1(p) = \mathcal{A}_p$, is an order preserving bijection. In fact, it is not hard to see via Remark 8 that $f_1$ is an injection. Now pick any element $C \in Z$. Obviously, there must exist some number $p \in (0, \infty)$ such that $C = \mathcal{A}_p = f_1(p)$, i.e. $f_1$ is a surjection. Consequently, $f_1$ is a bijection. To end the proof of this part we simply point out that the bijection $f_1$
is order preserving in virtue of Remark ??.

Finally, we note that we can similarly prove that \( f_2 \) is also an order preserving bijection. \( \square \)

Since the sets \( (\mathcal{Z}, \leq) \) and \( (\mathcal{Z}^*, \leq) \) are chains it is natural to look for a metric on them. We shall do this in the following two results. But before that let us recall the definitions of some distances known in the literature (cf. [4], say). If \( \Phi \in \mathcal{Y}_{\text{conc}} \) is any function and \( \mathcal{F}, \mathcal{G} \subset \mathcal{Y}_{\text{conc}} \) are arbitrary non-empty subsets, then we define the distance from the point \( \Phi \) to the set \( \mathcal{G} \) by
\[
\rho (\Phi, \mathcal{G}) := \inf \{ d (\Phi, \Psi) : \Psi \in \mathcal{G} \} = \inf \{ d (\Psi, \Phi) : \Psi \in \mathcal{G} \} = \rho (\mathcal{G}, \Phi)
\]
and the distance between the two sets \( \mathcal{F} \) and \( \mathcal{G} \) by
\[
\text{dist} (\mathcal{F}, \mathcal{G}) := \sup \{ \inf \{ d (\Phi, \Psi) : \Psi \in \mathcal{G} \} : \Phi \in \mathcal{F} \}
\]
\[
= \sup \{ \inf \{ d (\Phi, \Psi) : \Phi \in \mathcal{F} \} : \Psi \in \mathcal{G} \}.
\]

First we find sufficient conditions for which the distance from a point to a subset (both in \( \mathcal{Y}_{\text{conc}} \)) should be positive, in order to guarantee that the distance between two sets in \( \mathcal{Y}_{\text{conc}} \) have sense.

**Lemma 3.** Let \( b_1 \) and \( b_2 \in (0, \infty) \) be two arbitrary distinct numbers. Then \( \rho (\mathcal{Y}_{b_1}, \Phi_2) > 0 \) and \( \rho (\mathcal{A}_{b_1}, \Phi_2) > 0 \) whenever \( \Phi_2 \in \mathcal{Y}_{b_2} \).

**Proof.** It is enough to show that \( \rho (\mathcal{A}_{b_1}, \Phi_2) > 0 \) whenever \( \Phi_2 \in \mathcal{Y}_{b_2} \). In fact, suppose in the contrary that \( \rho (\mathcal{A}_{b_1}, \Phi_2) = 0 \) for some \( \Phi_2 \in \mathcal{Y}_{b_2} \). Then there can be extracted some sequence \( (\Delta_n) \subset \mathcal{A}_{b_1} \) such that \( d (\Delta_n, \Phi_2) = 0 \) for all \( n \in \mathbb{N} \), and \( \lim_{n \to \infty} d (\Delta_n, \Phi_2) = 0 \). We point out that this can be done because of the definition of the infimum. For each \( n \in \mathbb{N} \) let us set \( \Gamma_n := \inf_{k \geq n} (\Delta_k - \Phi_2)^2 \).

Clearly, \( (\Gamma_n) \) is a non-decreasing sequence of integrals with its corresponding sequence of integrals \( \int_{0}^{\infty} \Gamma_n d\mu \) been bounded above by \( C_{b_1} + C_{b_2} < \infty \), see Remark [4]. Then by the Beppo Levi’s Theorem we can derive that sequence \( (\Gamma_n) \) converges almost everywhere to some integrable measurable function \( \Gamma \) and \( \int_{0}^{\infty} \Gamma d\mu = \lim_{n \to \infty} \int_{0}^{\infty} \Gamma_n d\mu \leq \lim_{n \to \infty} d (\Delta_n, \Phi_2) = 0 \), meaning that \( \lim_{n \to \infty} \inf_{k \geq n} \Delta_k = \Phi_2 \) almost everywhere. There are two cases to be clarified. First assume that \( b_1 < b_2 \). Obviously, \( \mu ((b_1, b_2)) > 0 \), so that there must be at least one point \( x_0 \in (b_1, b_2) \) such that \( \lim_{n \to \infty} \inf_{k \geq n} \Delta_k (x_0) = \Phi_2 (x_0) \). But since \( b_1 < b_2 \) the concave property implies that the graph of \( \Phi_2 \) (resp. the graph of each function \( \inf_{k \geq n} \Delta_k \)) lies above (resp. below) the graph of the line of equation \( y = x \) in the interval \( (b_1, b_2) \). Consequently, \( \lim_{n \to \infty} \inf_{k \geq n} \Delta_k (x_0) \leq x_0 < \Phi_2 (x_0) \). This, however, is absurd since \( \lim_{n \to \infty} \inf_{k \geq n} \Delta_k (x_0) = \Phi_2 (x_0) \). Considering the second case when \( b_1 > b_2 \) we can similarly get into a contradiction, therefore, the statement is valid. \( \square \)

**Lemma 4.** Let \( b \) and \( c \in (0, \infty) \) be two arbitrary numbers. Then the following assertions are equivalent:

1. The equality \( b = c \) holds.
2. The sets \( \mathcal{Y}_b \) and \( \mathcal{Y}_c \) are equal.
3. The equality \( \text{dist} (\mathcal{Y}_b, \mathcal{Y}_c) = 0 \) holds.

**Proof.** We first note that the chain of implications \( (1) \to (2) \to (3) \) is obviously true. Thus we need only show the conditional \( (3) \to (1) \). In fact, assume that
\[
\text{dist} (\mathcal{Y}_b, \mathcal{Y}_c) = 0 \quad \text{but} \quad b \neq c.
\]
Then \( \rho (\mathcal{Y}_b, \Delta) = 0 \) for all \( \Delta \in \mathcal{Y}_c \). Nevertheless, this contradicts Lemma [4] since \( b \neq c \). Therefore, the argument is valid. \( \square \)

We can similarly prove that:
Lemma 5. Let $b$ and $c$ ∈ $(0, \infty)$ be two arbitrary numbers. Then the following assertions are equivalent:

1. The equality $b = c$ holds.
2. The sets $A_b$ and $A_c$ are equal.
3. The equality $\text{dist}(A_b, A_c) = 0$ holds.

Theorem 6. Let $b$ and $c$ ∈ $(0, \infty)$ be two arbitrary numbers. Then the quantities $\text{dist}(A_b, A_c)$ and $\text{dist}(Y_b, Y_c)$ define metrics on $\mathcal{Z}$ and $\mathcal{Z}^*$ respectively. Hence, the couples $(\mathcal{Z}, \text{dist})$ and $(\mathcal{Z}^*, \text{dist})$ are metric spaces.

Proof. We need only show that $\text{dist}(Y_b, Y_c)$ is a metric on the set $\mathcal{Z}^*$, because the other case can be similarly proved. In fact, we first point out that the condition $\text{dist}(Y_b, Y_c) \geq 0$ is obvious and, by Lemma 4 the equality holds if and only if $Y_b = Y_c$. We also note that the symmetry property trivially holds true. We are now left with the proof of the triangle inequality. In fact, let $Y_b \in \mathcal{Z}^*$ and $\Phi_j \in Y_b$ ($j \in \{1, 2, 3\}$) be arbitrary. Then by Proposition 5 (cf. [3], page 6) we have that $d(\Phi_1, \Phi_4) \leq d(\Phi_1, \Phi_2) + d(\Phi_2, \Phi_3)$. Next, by taking the infimum over $\Phi_3 \in Y_b$ it follows that

$$\rho(\Phi_1, Y_b) \leq d(\Phi_1, \Phi_2) + \rho(\Phi_2, Y_b) \leq d(\Phi_1, \Phi_2) + \text{dist}(Y_b, Y_b) ,$$

i.e. $\rho(\Phi_1, Y_b) \leq d(\Phi_1, \Phi_2) + \text{dist}(Y_b, Y_b)$. Finally, taking the infimum over $\Phi_2 \in Y_b$ yields $\rho(\Phi_1, Y_b) \leq \rho(\Phi_1, \Phi_2) + \text{dist}(Y_b, Y_b)$, so that

$$\text{dist}(Y_b, Y_b) \leq \text{dist}(Y_b, Y_b) + \text{dist}(Y_b, Y_b) .$$

This was to be proven.

By the law of trichotomy it is not hard to see that $(\mathcal{Z}, \leq)$ and $(\mathcal{Z}^*, \leq)$ are lattices. Here too, the supremum and infimum binary operations on the lattices $(\mathcal{Z}, \leq)$ and $(\mathcal{Z}^*, \leq)$ will be denoted by the usual symbols $\vee$ and $\wedge$ respectively. We also point out that $(\mathcal{Z}, \leq)$ and $(\mathcal{Z}^*, \leq)$ are infinite graphs. Between two vertices $A_{b_1}, A_{b_2} \in \mathcal{Z}$ we can define the edge in two different ways: one by $e = \text{dist}(A_{b_1}, A_{b_2}) \in (0, \infty)$ and the other one by $A_c \in \mathcal{Z}$ where $e = \text{dist}(A_{b_1}, A_{b_2})$. These two edges can apply for the vertices of $\mathcal{Z}^*$ as well.

4. Dense subsets in $\mathcal{Y}_b^{(n)}$

Theorem 7. Let $b \in (0, \infty)$ be an arbitrary number. Then $A_b$ is a dense set in $\mathcal{Y}_b$.

Proof. Fix arbitrarily any function $\Psi \in \mathcal{Y}_b$. Then there is some $\Phi \in \mathcal{Y}_{\text{conc}}$ such that $\Psi = \frac{\Phi(b)}{\Phi(x)}$ (by Lemma 4). Define $\Psi_n(x) = \frac{b(\Phi(x))^{1-1/(n+1)}}{(\Phi(b))^{1-1/(n+1)}}$, for all $x \in [0, \infty)$ and $n \in \mathbb{N}$. As we know from Theorem 2 (cf. [3], page 6) function $\Phi^{1-1/(n+1)} \in A$ for all $\Phi \in \mathcal{Y}_{\text{conc}}, n \in \mathbb{N}$. Then $(\Psi_n) \subset A$ (via Lemma 4). Hence, $(\Psi_n) \subset A_b$, since $\Psi_n(b) = b$ for all $n \in \mathbb{N}$. We can easily show that $(\Psi_n)$ converges pointwise to $\Psi$. By Remark 4 it ensues that $\Psi(x) \leq h_b(x)$ and $\Psi_n(x) \leq h_b(x)$ for all $x \in [0, \infty)$ and $n \in \mathbb{N}$, where $h_b(x) = x + b, x \in [0, \infty)$. We know via Remark 5 that function $h_b$ is square integrable. Then by applying twice the Dominated Convergence Theorem one can verify that

$$\lim_{n \to \infty} \int_{[0, \infty)} \Psi_n^2 d\mu = \int_{[0, \infty)} \Psi^2 d\mu,$$

and

$$\lim_{n \to \infty} \int_{[0, \infty)} \Psi_n d\mu = \int_{[0, \infty)} \Psi d\mu.$$
so that \( \lim_{n \to \infty} d(\Psi, \Psi_n) = 0 \), because \( \Psi(x) \Psi_n(x) \leq (h_b(x))^2 \) for all \( x \in [0, \infty) \) and \( n \in \mathbb{N} \) (by Remark 3). This was to be proven.

\[ \square \]

**Theorem 8.** Fix any pair of numbers \( n \in \mathbb{N} \setminus \{1\} \) and \( b \in (0, \infty) \). Then \( A_b^{(n)} \) is dense in \( Y_b^{(n)} \).

**Proof.** Pick arbitrarily some \( \Delta \in Y_b^{(n)} \). Since obviously \( Y_b^{CO(n)} \) is a proper subset of \( Y_b^{(n)} \), we will have two cases to take into consideration. First assume that \( \Delta \in Y_b^{CO(n)} \). This means that there can be found a counting number \( k \geq n \) and a finite sequence \( \Phi_1, \ldots, \Phi_k \in Y_b^{(1)} = Y_b \) such that \( \Delta = \Phi_1 \circ \ldots \circ \Phi_k \). Fix any integer \( j \in \mathbb{N} \) and write \( \Delta_j = \Psi_j \circ \Delta \), where \( \Psi_j(x) = \left(b^{1/j}x\right)^j, x \in [0, \infty) \). Clearly, \( \Psi_j \in A_b^{(1)} \) for all \( j \in \mathbb{N} \). Then applying Theorem 2 in [3] and via the structure of set \( A_b^{CO(n)} \), we can deduce that \( \Delta_j \in A_b^{CO(n)} \) for all \( j \in \mathbb{N} \). It is not difficult to see that sequence \( \{\Delta_j\} \) converge pointwise to \( \Delta \). By Remark 4 we observe that \( \Delta \leq h_b, \Delta_j \leq h_b \) and hence, \( \Delta \Delta_j \leq (h_b)^2 \) on \([0, \infty)\). Then recalling twice the Dominated Convergence Theorem we can easily verify that

\[
\lim_{j \to \infty} \int_{[0, \infty)} (\Delta_j)^2 \, d\mu = \int_{[0, \infty)} \Delta^2 \, d\mu = \lim_{j \to \infty} \int_{[0, \infty)} \Delta \Delta_j \, d\mu.
\]

Consequently, \( \lim_{j \to \infty} d(\Delta, \Delta_j) = 0 \). In the second case we can suppose that \( \Delta \in Y_b^{(n)} \setminus Y_b^{CO(n)} \). Then without loss of generality we may choose \( \Phi_1, \ldots, \Phi_k \in Y_b^{CO(n)} \), whose graphs are pairwise distinct, and some finite sequence \( (t_1, \ldots, t_k) \in H_{[0,1]} \) with \( (t_1, \ldots, t_k) \subset (0, 1) \) such that \( \Delta = \sum_{i=1}^{k} t_i \Phi_i \). Consider \( \Delta_j = \sum_{i=1}^{k} t_i (\Psi_j \circ \Phi_i) \), where \( \Psi_j(x) = \left(b^{1/j}x\right)^j, x \in [0, \infty), j \in \mathbb{N} \). Clearly, on the one hand we have that \( \Delta_j \subset A_b^{(n)} \) because \( (\Psi_j \circ \Phi_i) \subset A_b^{CO(n)} \) for every fixed index \( i \in \{1, \ldots, k\} \) and on the other hand \( \lim_{j \to \infty} d(\Phi_i, \Psi_j \circ \Phi_i) = 0 \), \( i \in \{1, \ldots, k\} \), because of the first part of this proof. Consequently, by the Minkowski inequality we can observe that \( \lim_{j \to \infty} d(\Delta, \Delta_j) \leq \sum_{i=1}^{k} t_i \lim_{j \to \infty} d(\Phi_i, \Psi_j \circ \Phi_i) = 0 \). This completes the proof.

\[ \square \]

5. SOME CRITERION ON THE \( L^p \)-NORM

The result here below is worth being mentioned, which is an answer to the second open problem in [3].

**Theorem 9.** Let \( \Phi \in Y_{conc} \) be arbitrary. Then the following assertions are equivalent.

1. \( \lim_{t \to \infty} \frac{\Phi(t)}{\varphi(t)} = \lim_{t \to \infty} \varphi(t) \in (0, \infty) \).
2. There is some constant \( c \in [1, \infty) \) such that \( c\Phi > \Phi_{id} \) on \((0, \infty)\).
3. There is some constant \( c \in [1, \infty) \) and some strictly concave function \( \Delta : [0, \infty) \to [0, \infty) \), differentiable on \((0, \infty)\) and vanishing at the origin such that \( c\Phi = \Phi_{id} + \Delta \) on \([0, \infty)\).

**Proof.** We first prove the conditional \( 1 \rightarrow 2 \). In fact, assume that \( \lim_{t \to \infty} \frac{\Phi(t)}{\varphi(t)} \in (0, \infty) \) but in the contrary for every counting number \( k \in \mathbb{N} \) there is some \( x_k \in (0, \infty) \) for which \( k\Phi(x_k) \leq x_k \). Obviously, \( \limsup_{k \to \infty} \frac{\Phi(x_k)}{x_k} \leq \lim_{k \to \infty} k^{-1} = 0 \) which
is absurd since $\limsup_{k \to \infty} \frac{\Phi(x_k)}{x_k} \in (0, \infty)$ by the assumption. Next we show the implication (2)$\Rightarrow$(3). In fact, assume that there is some constant $c \in [1, \infty)$ such that $c\Phi > \Phi_{id}$ on $(0, \infty)$ and write $\Delta := c\Phi - \Phi_{id}$. Clearly, $\Delta : [0, \infty) \to [0, \infty)$ is a function such that $\Delta (0) = 0$ and $\Delta$ is positive on $(0, \infty)$. We also note that $\Delta$ is differentiable on $(0, \infty)$. Writing $\delta$ for the derivative of $\Delta$, we can observe that $\delta = c\varphi - 1$ on $(0, \infty)$. To show that $\Delta$ is strictly concave it is enough if we prove that

$$(y - x)\delta (y - 0) < \Delta (y) - \Delta (x) < (y - x)\delta (x + 0) = (y - x)\delta (x)$$

for all $x, y \in (0, \infty)$ with $x < y$ (where, $\delta (t - 0)$ respectively is the left derivative and $\delta (t + 0)$ the right derivative of $\Delta$ at point $t$). In fact, fix arbitrarily two numbers $x, y \in (0, \infty)$ such that $x < y$. But since $\Phi$ is strictly concave we have that

$$(y - x) \varphi (y - 0) < \Phi (y) - \Phi (x) < (y - x) \varphi (x + 0) = (y - x) \varphi (x)$$

which easily leads to

$$c\varphi (y - 0) < \frac{c\Phi (y) - c\Phi (x)}{y - x} < c\varphi (x + 0) = c\varphi (x).$$

Hence,

$$c\varphi (y - 0) - 1 < \frac{c\Phi (y) - c\Phi (x)}{y - x} - 1 < c\varphi (x) - 1,$$

i.e.

$$(y - x)\delta (y - 0) < \Delta (y) - \Delta (x) < (y - x)\delta (x).$$

This ends the proof of the implication (2)$\Rightarrow$(3). In the last step, we just point out that the conditional (3)$\Rightarrow$(1) is obvious. Therefore, we can conclude on the validity of the argument.

Denote $\widetilde{\mathcal{Y}}_{conc} := \{\Phi \in \mathcal{Y}_{conc} : \lim_{t \to \infty} \frac{\Phi(t)}{t} > 0\}$. It is not difficult to check that $\widetilde{\mathcal{Y}}_{conc} = \{\Delta \in \mathcal{Y}_{conc} : c\Delta > \Phi_{id} \text{ on } (0, \infty) \text{ for some } c \in [1, \infty)\}$. Write $T_\Delta = \{c \in [1, \infty) : c\Delta > \Phi_{id} \text{ on } (0, \infty)\}$, $\Delta \in \widetilde{\mathcal{Y}}_{conc}$.

Some few words about set $\widetilde{\mathcal{Y}}_{conc}$.

**Remark 9.** Let $\alpha \in (0, \infty)$ be arbitrary. Then $\alpha \Delta \in \widetilde{\mathcal{Y}}_{conc}$ provided that $\Delta \in \widetilde{\mathcal{Y}}_{conc}$.

**Proof.** Whenever $\Delta \in \widetilde{\mathcal{Y}}_{conc}$ we can choose a corresponding $c \in T_\Delta$ such that $c\Delta > \Phi_{id} \text{ on } (0, \infty)$. Now choose a constant $t_0 \in (1, \infty)$ such that $\alpha t_0 \geq c$. Hence, $t_0 (\alpha \Delta) \geq c \Delta > \Phi_{id} \text{ on } (0, \infty)$, i.e. $\alpha \Delta \in \widetilde{\mathcal{Y}}_{conc}$.

**Remark 10.** Every function $\Delta \in \widetilde{\mathcal{Y}}_{conc}$ can be written as the sum of a finite number of elements of $\widetilde{\mathcal{Y}}_{conc}$. Conversely, the sum of a finite number of elements of $\widetilde{\mathcal{Y}}_{conc}$ also belongs to $\widetilde{\mathcal{Y}}_{conc}$.

Next, we show that the quantities $\|f\|_{L^p}$ and $\sup_{\Phi \in \widetilde{\mathcal{Y}}_{conc}} (\Phi (1))^{-1} \|\Phi \circ f\|_{L^p}$ are equivalent, in the sense that they are both either finite or infinite at the same time. This provides a kind of criterion for a measurable function to belong to $L^p$.

**Theorem 10.** Let $f$ be any measurable function on an arbitrarily fixed measure space $(\Omega, \mathcal{F}, \lambda)$ and $p \in [1, \infty)$ be any number. Then

$$\|f\|_{L^p} \leq \sup_{\Phi \in \widetilde{\mathcal{Y}}_{conc}} (\Phi (1))^{-1} \|\Phi \circ f\|_{L^p} \leq \|f\|_{L^p} + \lambda (\Omega).$$
Proof. Pick any function $\Phi \in \mathcal{Y}_{\text{conc}}$. Then
\[
\int_{\Omega} \left( (\Phi(1))^{-1} \Phi \circ |f| \right)^p d\lambda \leq \int_{\Omega} (|f| + 1)^p d\lambda
\]
because $\Delta \leq (\Phi_{\text{id}} + 1) \Delta (1)$ for all $\Delta \in \mathcal{Y}_{\text{conc}}$. Consequently, via the Minkowski inequality, it follows that $(\Phi(1))^{-1} \| \Phi \circ |f| \|_{L^p} \leq \|f\|_{L^p} + \lambda(\Omega)$, which proves the inequality on the right hand-side of the above chain. To show the left side inequality, it follows that $(\Phi(1))^{-1} \Phi \circ |f| \in Y$.

Passing to the limit yields sup $\lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \|_{L^p} \geq (1 + n^{-1})^{-1} \|f\|_{L^p} \}$

\[= (1 + n^{-1})^{-1} \|f\| + n^{-1} \|f\|_{L^p} \geq (1 + n^{-1})^{-1} \|f\|_{L^p}.\]

Passing to the limit yields sup $\Phi \in \mathcal{Y}_{\text{conc}}$ $(\Phi(1))^{-1} \| \Phi \circ |f| \|_{L^p} \geq \|f\|_{L^p}$. Therefore, we have obtained a valid argument.

\[\square\]

Theorem 11. Let $(\Omega, \mathcal{F}, \lambda)$ be any measure space and on it let $f$ be any measurable function. Then
\[
\lambda(\{|f| \geq \varepsilon\}) = \inf \left\{ \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \}) : c \in T_{\Delta} \right\} = \frac{\lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \})}{\lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq (\lambda \Delta) \})}
\]
for every number $\varepsilon \in (0, \infty)$.

Proof. Throughout the proof $\varepsilon \in [0, \infty)$ will be any fixed number. We first note that the assertion is trivial when $\{|f| = \infty\} \neq \emptyset$. We shall then prove it when $\{|f| < \infty\} \neq \emptyset$. Pick some $\Delta \in \mathcal{Y}_{\text{conc}}$ and $c \in T_{\Delta}$ such that $c \Delta > \Phi_{\text{id}}$ on $[0, \infty)$. It is not hard to see that $\{ (f \geq \varepsilon) = (\Delta \circ |f| \geq \Delta (\varepsilon)) \} \subset (\Delta \circ |f| \geq \varepsilon c^{-1})$ and thus
\[
\lambda(\{|f| \geq \varepsilon\}) = \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \lambda \Delta \}) \leq \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \Delta (\varepsilon) \}) \leq \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon^{-1} \}).
\]

Consequently,
\[
\lambda(\{|f| \geq \varepsilon\}) \leq \inf \left\{ \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \}) : c \in T_{\Delta} \right\} = \frac{\lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \})}{\lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon \}).
\]

To prove the converse statement, we need show that
\[
\lambda(\{|f| \geq \varepsilon\}) \geq \inf \left\{ \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \}) : c \in T_{\Delta} \right\} = \frac{\lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \})}{\lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon \}).
\]

In fact, for any $n \in \mathbb{N}$ set $\Delta_n = \Phi_{\text{id}} + n^{-1} (1 - e^{-\Phi_{\text{id}}})$. It is not difficult to see that $\Delta_n \in \mathcal{Y}_{\text{conc}}$ and $\Delta_n > \Phi_{\text{id}}$ on $[0, \infty)$, $n \in \mathbb{N}$. This means that $(\Delta_n) \subset \mathcal{Y}_{\text{conc}}$ and moreover, $1 \in T_{\Delta_n}$, $n \in \mathbb{N}$. Consequently,
\[
\inf \left\{ \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \}) : c \in T_{\Delta} \right\} = \frac{\lambda(\Delta_n \circ |f| \geq \varepsilon \Delta_n)}{\lambda(\{|f| \geq \varepsilon\})}
\]
\[
= \lambda(\{|f| + n^{-1} (1 - e^{-|f|}) \geq \varepsilon\}).
\]

However, as $(\Delta_n)$ is a decreasing sequence it is obvious that $(\Delta_{n+1} \circ |f| \geq \varepsilon) \subset (\Delta_n \circ |f| \geq \varepsilon), n \in \mathbb{N}$. Thus having passed to the limit we can observe that
\[
\inf \left\{ \lambda(\{ (\Phi(1))^{-1} \Phi \circ |f| \geq \varepsilon c^{-1} \}) : c \in T_{\Delta} \right\} \leq \lambda(\{|f| \geq \varepsilon\}).
\]

Therefore, the proof is a valid argument. \[\square\]
Theorem 12. Let $f \in L^p(\Omega, \mathcal{F}, \lambda)$, $p \geq 1$, where $(\Omega, \mathcal{F}, \lambda)$ is any given measure space. Then

$$
\|f\|_{L^p} = \inf \left\{ \inf \{c \|\Delta \circ |f|\|_{L^p} : c \in T_\Delta \} : \Delta \in \mathcal{Y}_{\text{conc}} \right\}.
$$

Proof. Pick arbitrarily some $\Delta \in \mathcal{Y}_{\text{conc}}$ and $c \in T_\Delta$ such that $c\Delta > \Phi_{\text{id}}$ on $(0, \infty)$. Clearly, $c\|\Delta \circ |f|\|_{L^p} \geq \|f\|_{L^p}$. We can then easily observe that

$$
\inf \left\{ \inf \{c \|\Delta \circ |f|\|_{L^p} : c \in T_\Delta \} : \Delta \in \mathcal{Y}_{\text{conc}} \right\} \geq \|f\|_{L^p}.
$$

To prove the converse of this inequality consider the sequence $(\Delta_n) \subset \mathcal{Y}_{\text{conc}}$, where $\Delta_n = \Phi_{\text{id}} + n^{-1}(1 - e^{-\Phi_{\text{id}}}) > \Phi_{\text{id}}$ on $(0, \infty)$, $n \in \mathbb{N}$. Then as $1 \in T_{\Delta_n}$, $n \in \mathbb{N}$, we have

$$
\inf \left\{ \inf \{c \|\Delta \circ |f|\|_{L^p} : c \in T_\Delta \} : \Delta \in \mathcal{Y}_{\text{conc}} \right\} \leq \|\Delta_n \circ |f|\|_{L^p}.
$$

Since $(\Delta_n)$ is a decreasing sequence it ensues that $(\Delta_n \circ |f|)$ is also a decreasing sequence which tends to $|f|$. As every member of sequence $(\Delta_n \circ |f|)$ is dominated by $\Delta_1 \circ |f| \in L^p$, then by applying the Dominated Convergence Theorem it will entail that

$$
\inf \left\{ \inf \{c \|\Delta \circ |f|\|_{L^p} : c \in T_\Delta \} : \Delta \in \mathcal{Y}_{\text{conc}} \right\} \leq \|f\|_{L^p}.
$$

This completes the proof. \hfill \Box

Corollary 3. Suppose that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then

$$
|h| = \inf \left\{ \inf \{c \|h\| : c \in T_\Delta \} : \Delta \in \mathcal{Y}_{\text{conc}} \right\}.
$$

Proof. Fix any number $x \in \mathbb{R}$ and let $f \in L^p(\Omega, \mathcal{F}, \lambda)$ be the constant function defined by $f \equiv h(x)$ on $\Omega$. Then by applying Theorem 12 we can easily deduce the result. \hfill \Box

Open problem 1. Given any number $k \in \mathbb{N}$ characterize all pairs of functions $\Phi$ and $\Delta \in \mathcal{Y}_{\text{conc}}$ such that $|\{x \in (0, \infty) : \Phi(x) = \Delta(x)\}| = k$.

Open problem 2. Characterize all pairs of functions $\Phi$ and $\Delta \in \mathcal{Y}_{\text{conc}}$ such that the sets $(0, \infty)$ and $\{x \in (0, \infty) : \Phi(x) = \Delta(x)\}$ should be equinumerous.

References

[1] Agbeko, N. K.: Concave function inequalities for sub-(super)martingales, Annales Univ. Sci. Budapest, Sectio Mathematica, 29 (1986), 9-17.
[2] Agbeko, N. K.: Necessary and sufficient condition for the maximal inequality of concave Young-functions, Annales Univ. Sci. Budapest, Sectio Mathematica, 32 (1989), 267-270.
[3] Agbeko, N. K.: Studies on concave Young-functions, Miskolc Math. Notes (6)2005, No. 1, 3 - 18. (Available online at: http://mat76.mat.uni-miskolc.hu/~mnotes/files/6-1/).
[4] Burkholder, D. L.: Distribution function inequalities for martingales, Annals of Probability, 1(1973), 19 - 42.
[5] Garcia-Cuerva, J. and Rubio De Francia, J. L.: Weighted Norm Inequalities and Related Topics. North-Holland, Amsterdam, 1985.
[6] Garsia, A. M.: Martingale inequalities, Seminar Notes on recent progress, Benjamin, Reading, Massachusetts, 1973.
[7] Hamilton, A. G.: Numbers, sets and axioms: The apparatus of mathematics, Cambridge University Press, 1982.
[8] Kuratowski, K.: Topology, vol. 1, Academic Press, New York, etc., 1966.
[9] MacLane, S. and Birkhoff G.: Algèbre, Tome 1, Structures fondamentales, Gauthier-Villars, 1971.
[10] MOGYORÓDI, J.: On a concave function inequality for martingales, Annales Univ. Sci. Bud. Sect. Math. 24(1981), 255 - 271.
[11] REED, M. C.: Fundamental ideas of analysis, John Wiley & Sons, New York ..., 1998.
[12] Sinnamon, G.: Embeddings of concave functions and duals of Lorentz spaces, Publ. Math. 46(2002), 489-525.
[13] TRIEBEL, H.: Interpolation theory, function spaces, differential operators, North-Holland, 1978.

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