Variation formulas for transversally harmonic and bi-harmonic maps

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Abstract. In this paper, we study variation formulas for transversally harmonic maps and bi-harmonic maps, respectively. We also study the transversal Jacobi field along a map and give several relations with infinitesimal automorphisms.

1 Introduction

Let \((M, \mathcal{F})\) and \((M', \mathcal{F}')\) be two foliated Riemannian manifolds and let \(\phi : M \to M'\) be a smooth foliated map, i.e., \(\phi\) is a leaf-preserving map. Then \(\phi\) is transversally harmonic if \(\phi\) is a critical point of the transversal energy functional on any compact domain of \(M\), which is defined in Section 3 (cf. \cite{5,9,10}). Equivalently, it is a solution of \(\tau_b(\phi) = 0\), where \(\tau_b(\phi)\) is a transversal tension field, which is given by \(\tau_b(\phi) = \text{tr}_Q \tilde{\nabla} d_T \phi\) (see \cite{2} for more details). That is, transversally harmonic maps are considered as harmonic maps between the leaf spaces \cite{9,10}. For harmonic maps, see \cite{3,15}. Also, we study the transversally bi-harmonic map as the critical point of the transversal bi-energy functional on any compact domain of \(M\) (Section 6). In this paper, we study the second variation formulas for the transversal energy and transversal bi-energy of \(\phi\). And we give some applications. This paper is organized as follows. In Section 2, we recall the basic facts on foliated manifolds. In Section 3, we review transversally harmonic maps and the first variation formula. In Section 4, we give the second variation formula for

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the transversal energy. In Section 5, we define the transversal Jacobi operator along the foliated map and study its relation with infinitesimal automorphisms. In Section 6, we study transversally bi-harmonic maps and their applications. In Section 7, we give the second variation formula for the transversal bi-energy. Note that some results in Section 6 and Section 7 of the present paper can be found in [2], but the approach is different in a technical sense. Throughout this paper, \((M, \mathcal{F})\) is considered as a foliated Riemannian manifold, i.e., a Riemannian manifold with a Riemannian foliation, and all leaves of \(\mathcal{F}\) are compact.

2 Preliminaries

Let \((M, g, \mathcal{F})\) be a \((p+q)\)-dimensional foliated Riemannian manifold with foliation \(\mathcal{F}\) of codimension \(q\) and a bundle-like metric \(g\) with respect to \(\mathcal{F}\). Let \(TM\) be the tangent bundle of \(M\), \(L\) the tangent bundle of \(\mathcal{F}\), and \(Q = TM/L\) the corresponding normal bundle of \(\mathcal{F}\). Let \(g_Q\) be the holonomy invariant metric on \(Q\) induced by \(g\). We denote by \(\nabla^Q\) the transverse Levi-Civita connection on the normal bundle \(Q\) [13,14]. Let \(R^Q, K^Q, \text{Ric}^Q, \sigma^Q\) be the transversal curvature tensor, transversal sectional curvature, transversal Ricci operator and transversal scalar curvature with respect to \(\nabla^Q\) respectively. Let \(\Omega^r_B(\mathcal{F})\) be the space of all basic \(r\)-forms, i.e., \(\omega \in \Omega^r_B(\mathcal{F})\) if and only if \(i(X)\omega = 0 = i(X)d\omega\) for any \(X \in \Gamma L\), where \(i(X)\) is the interior product. Then \(\Omega^r_\ast(M) = \Omega^r_B(\mathcal{F}) \oplus \Omega^r_B(\mathcal{F})^\perp\) [1]. Let \(\kappa_B\) be the basic part of \(\kappa\), the mean curvature form of \(\mathcal{F}\). Then \(\kappa_B\) is closed, i.e., \(d\kappa_B = 0\) [1]. The basic Laplacian \(\Delta_B\) acting on \(\Omega^r_B(\mathcal{F})\) is defined by

\[
\Delta_B = d_B \delta_B + \delta_B d_B, \tag{2.1}
\]

where \(\delta_B\) is the formal adjoint of \(d_B = d|_{\Omega^r_B(\mathcal{F})}\) [12,14]. Let \(V(\mathcal{F})\) be the space of all transversal infinitesimal automorphisms \(Y\) of \(\mathcal{F}\), i.e., \([Y, Z] \in \Gamma L\) for all \(Z \in \Gamma L\). Let \(\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) | Y \in V(\mathcal{F})\}\), where \(\pi : TM \to Q\) is a projection. Trivially, \(\bar{V}(\mathcal{F}) \cong \Omega^1_B(\mathcal{F})\) [11]. For later use, we recall the transversal
divergence theorem \[16\] on a foliated Riemannian manifold.

**Theorem 2.1** Let \((M, g_M, \mathcal{F})\) be a closed, oriented Riemannian manifold with a transversally oriented foliation \(\mathcal{F}\) and a bundle-like metric \(g_M\) with respect to \(\mathcal{F}\). Then

\[
\int_M \text{div}_\nabla \bar{X} = \int_M g_Q(\bar{X}, \kappa_B^\sharp)
\]

(2.2)

for all \(X \in V(\mathcal{F})\), where \(\text{div}_\nabla \bar{X}\) denotes the transversal divergence of \(\bar{X}\) with respect to the connection \(\nabla_Q\).

Now, we define the bundle map \(A_Y : \Lambda^r Q^* \to \Lambda^r Q^*\) for any \(Y \in V(\mathcal{F})\) \[7\] by

\[
A_Y \omega = \theta(Y) \omega - \nabla_Y \omega \quad \forall \omega \in \Lambda^r Q^*,
\]

(2.3)

where \(\theta(Y)\) is the transverse Lie derivative. It is well-known \[7\] that, on \(\Gamma Q\)

\[
A_Y s = -\nabla_{Y_s} \bar{Y} \quad \forall s \in \Gamma Q,
\]

(2.4)

where \(Y_s\) is the vector field such that \(\pi(Y_s) = s\). So \(A_Y\) depends only on \(\bar{Y} = \pi(Y)\).

Since \(\theta(X) \omega = \nabla_X \omega\) for any \(X \in \Gamma L\), \(A_Y\) preserves the basic forms and depends only on \(\bar{Y}\). Let \(E \to M\) be a vector bundle over \(M\) and \(\Omega^r_B(E) \equiv \Omega^r_B(\mathcal{F}) \otimes E\) be the space of all \(E\)-valued basic \(r\)-forms. Let \(\nabla\) be also the connection on \(E\). Then the operator \(A_X\) is extended to \(\Omega^r_B(E)\) \[5\]. Now we define \(d_{\nabla} : \Omega^r_B(E) \to \Omega^{r+1}_B(E)\) by

\[
d_{\nabla}(\omega \otimes s) = d_{\nabla} \omega \otimes s + (-1)^r \omega \wedge \nabla s
\]

(2.5)

for any \(\omega \in \Omega^r_B(\mathcal{F})\) and \(s \in E\). Let \(\delta_{\nabla}\) be the formal adjoint of \(d_{\nabla}\). Then we define the Laplacian \(\Delta\) on \(\Omega^r_B(E)\) by

\[
\Delta = d_{\nabla} \delta_{\nabla} + \delta_{\nabla} d_{\nabla}.
\]

(2.6)

From now on, let \(\{E_a\}(a = 1, \cdots, q)\) be a local orthonormal frame on \(Q\) and \(\theta^a\) be the \(g_Q\)-dual 1-form to \(E_a\). Then the generalized Weitzenböck type formula on \(\Omega^r_B(E)\) is given by \[5\]

\[
\Delta \Phi = \nabla^*_{tr} \nabla_{tr} \Phi + F(\Phi) + A_{\kappa_B^\sharp} \Phi, \quad \forall \Phi \in \Omega^r_B(E),
\]

(2.7)
\[ \nabla_{\text{tr}} \nabla_{\text{tr}} = -\sum a \nabla_{E_a, E_a} + \nabla_{\kappa_B^*} \] and \( F = \sum_{a, b=1}^q \theta^a \wedge i(E_b)R^\nabla(E_b, E_a) \). From (2.7), we also have
\[ \frac{1}{2} \Delta_B |\Phi|^2 = \langle \Delta \Phi, \Phi \rangle - |\nabla_{\text{tr}} \Phi|^2 - \langle A_{\kappa_B^*}, \Phi, \Phi \rangle - \langle F(\Phi), \Phi \rangle, \tag{2.8} \]
where \( \langle \cdot, \cdot \rangle \) is an inner product on \( \Omega^r_B(E) \).

Now, we recall the following generalized maximum principles.

**Lemma 2.2** \[6\] Let \( F \) be a Riemannian foliation on a closed, oriented Riemannian manifold \((M, g_M)\). If \((\Delta_B - \kappa_B^*) f \geq 0 \) (or \( \leq 0 \)) for any basic function \( f \), then \( f \) is constant.

### 3 Transversally harmonic maps

Let \((M, g, \mathcal{F})\) and \((M', g', \mathcal{F}')\) be two foliated Riemannian manifolds and all leaves of \( \mathcal{F} \) are compact. Let \( \nabla^M \) and \( \nabla^{M'} \) be the Levi-Civita connections on \( M \) and \( M' \), respectively. And \( \nabla \) and \( \nabla' \) be the transverse Levi-Civita connections on \( Q \) and \( Q' \), respectively. Let \( \phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}') \) be a smooth foliated map, i.e., \( d\phi(L) \subset L' \). We define \( d_T \phi : Q \to Q' \) by
\[ d_T \phi := \pi' \circ d\phi \circ \sigma, \tag{3.1} \]
where \( \sigma : Q \to L^\perp \) is a bundle map satisfying \( \pi \circ \sigma = \text{id} \). Then \( d_T \phi \) is a section in \( Q^* \otimes \phi^{-1}Q' \). Let \( \nabla^\phi \) and \( \nabla' \) be the connections on \( \phi^{-1}Q' \) and \( Q^* \otimes \phi^{-1}Q' \), respectively. Then \( \phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}') \) is called transversally totally geodesic if it satisfies
\[ \nabla_{\text{tr}} d_T \phi = 0, \tag{3.2} \]
where \( (\nabla_{\text{tr}} d_T \phi)(X, Y) = (\nabla_X d_T \phi)(Y) \) for any \( X, Y \in \Gamma Q \). And the transversal tension field of \( \phi \) is defined by
\[ \tau_b(\phi) = \text{tr}_Q \nabla d_T \phi = \sum_{a=1}^q (\nabla_{E_a} d_T \phi)(E_a), \tag{3.3} \]
where \( \{E_a\} (a = 1, \ldots, q) \) is a local orthonormal frame on \( Q \). Trivially, the transversal tension field \( \tau_b(\phi) \) is a section of \( \phi^{-1}Q' \).

**Definition 3.1** Let \( \phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}') \) be a smooth foliated map. Then \( \phi \) is said to be \textit{transversally harmonic} if the transversal tension field vanishes, i.e., \( \tau_b(\phi) = 0 \).

Let \( \text{vol}_L : M \to [0, \infty] \) be the volume map for which \( \text{vol}_L(x) \) is the volume of the leaf passing through \( x \in M \). It is trivial that \( \text{vol}_L \) is a basic function. And it holds \( \text{(3.4)} \) that

\[
d_B \text{vol}_L + (\text{vol}_L)\kappa_B = 0.
\]

The transversal energy of \( \phi \) on a compact domain \( \Omega \subset M \) is defined by

\[
E_B(\phi; \Omega) = \frac{1}{2} \int_{\Omega} |d_T\phi|^2 \frac{1}{\text{vol}_L} \mu_M,
\]

where \( \mu_M \) is the volume element of \( M \). Let \( V \in \phi^{-1}Q' \). Obviously, \( V \) may be considered as a vector field on \( Q' \) along \( \phi \). Then there is a 1-parameter family of foliated maps \( \phi_t \) with \( \phi_0 = \phi \) and \( \frac{d\phi_t}{dt}|_{t=0} = V \). The family \( \{\phi_t\} \) is said to be a \textit{foliated variation} of \( \phi \) with the normal variation vector field \( V \). Then we have the first variation formula.

**Theorem 3.2** \( \text{(The first variation formula)} \) Let \( \phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}') \) be a smooth foliated map, and all leaves of \( \mathcal{F} \) be compact. Let \( \{\phi_t\} \) be a smooth foliated variation of \( \phi \) supported in a compact domain \( \Omega \). Then

\[
\frac{d}{dt} E_B(\phi_t, \Omega)|_{t=0} = -\int_{\Omega} \langle V, \tau_b(\phi) \rangle \frac{1}{\text{vol}_L} \mu_M,
\]

where \( V(x) = \frac{d\phi_t}{dt}(x)|_{t=0} \) is the normal variation vector field of \( \{\phi_t\} \).

**Definition 3.3** Let \( \phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}') \) be a smooth foliated map. Then the transversal stress-energy tensor \( S_T(\phi) \) of \( \phi \) is defined by

\[
S_T(\phi) = \frac{1}{2} |d_T\phi|^2 g_Q - \phi^* g_{Q'},
\]

where \( \phi^* \) is the pull-back of \( \phi \).
Trivially, $S_T(\phi) \in \otimes^2 Q^*$ is the symmetric 2-covariant normal tensor field on $M$.

**Proposition 3.4** \[2\] Let $\phi : (M, g, F) \to (M', g', F')$ be a smooth foliated map. Then, for any vector field $X \in \Gamma Q$,

$$(\text{div}_{\nabla} S_T(\phi))(X) = -\langle \tau_b(\phi), d_T \phi(X) \rangle,$$  \hspace{1cm} (3.8)

where $(\text{div}_{\nabla} S_T(\phi))(\cdot) = \sum_{a=1}^{q} (\nabla_{E_a} S_T(\phi))(E_a, \cdot)$.

**Proof.** Note that $[d_T \phi(X), d_T \phi(Y)] = d_T \phi([X, Y])$ for any $X, Y \in \Gamma Q$. So, by direct calculation, the proof follows. □

If $\text{div}_{\nabla} S_T(\phi) = 0$, then we say that $\phi$ satisfies the **transverse conservation law** \[2\]. The foliated map satisfying the transverse conservation law is said to be **transversally relatively harmonic**. Then we have the following.

**Corollary 3.5** Any transversally harmonic map is transversally relatively harmonic.

The converse of Corollary 3.5 does not hold. For the converse, see Theorem 7.4 below.

**Remark.** \[5\] Let $\phi : (M, g, F) \to (M', g', F')$ be a smooth foliated map. Then

$$d_{\nabla} d_T \phi = 0, \quad \tilde{\delta} d_T \phi = -\tau_b(\phi),$$  \hspace{1cm} (3.9)

where $\tilde{\delta} = \delta - i(\kappa^*_B)$.

### 4 The second variation formula for the transversal energy

Let $(M, g, F)$ and $(M', g', F')$ be two foliated Riemannian manifolds, and all leaves of $F$ be compact. Let $\phi : (M, g, F) \to (M', g', F')$ be a transversally
harmonic map. For any $V, W \in \phi^{-1}Q'$, there exists a family of foliated maps $\phi_{t,s}(-\epsilon < s, t < \epsilon)$ satisfying

$$
\begin{align*}
V &= \frac{\partial \phi_{t,s}}{\partial t} \bigg|_{(t,s)=(0,0)}, \\
W &= \frac{\partial \phi_{t,s}}{\partial s} \bigg|_{(t,s)=(0,0)}, \\
\phi_{0,0} &= \phi.
\end{align*}
$$

Then $\{\phi_{t,s}\}$ is said to be the foliated variation of $\phi$ with the normal variation vector fields $V$ and $W$. Then we have the second variation formula for the transversal energy.

**Theorem 4.1** (The second variation formula) Let $\phi : (M, g, F) \to (M', g', F')$ be a transversally harmonic map with $M$ compact without boundary, and all leaves of $F$ be compact. Let $\{\phi_{t,s}\}$ be the foliated variation of $\phi$ with the normal variation vector fields $V$ and $W$. Then

$$
\frac{\partial^2}{\partial t \partial s} E_B(\phi_{t,s}) \bigg|_{(t,s)=(0,0)} = \int_M \langle (\nabla^\phi_{tt})^\ast (\nabla^\phi_{ts}) V - \nabla^\phi_{t} V - \text{tr}_Q R^Q(\nabla d_T \phi) d_T \phi, W \rangle \frac{1}{\text{vol}_L} \mu_M,
$$

where $\text{tr}_Q R^Q(\nabla d_T \phi) d_T \phi = \sum_{a=1}^q R^Q(\nabla d_T (E_a)) d_T (E_a)$.

**Proof.** Let $\Phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M'$ be a smooth map, which is defined by $\Phi(x, t, s) = \phi_{t,s}(x)$. Let $\nabla^\Phi$ be the pull-back connection on $\Phi^{-1}Q'$. It is trivial that $[X, \frac{\partial}{\partial t}] = [X, \frac{\partial}{\partial s}] = 0$ for any vector field $X \in TM$. From the first normal variation formula (Theorem 3.2), we have

$$
\frac{\partial}{\partial s} E_B(\phi_{t,s}) = - \int_M \langle \frac{\partial \Phi}{\partial s}, \tau_b(\phi_{t,s}) \rangle \frac{1}{\text{vol}_L} \mu_M.
$$

By differentiating (4.3) with respect to $t$, we have

$$
\frac{\partial^2}{\partial t \partial s} E_B(\phi_{t,s}) = - \int_M \left\{ \langle \frac{\partial^2 \Phi}{\partial t \partial s}, \tau_b(\phi_{t,s}) \rangle + \langle \frac{\partial \Phi}{\partial s}, \nabla^\Phi_{tt} \tau_b(\phi_{t,s}) \rangle \right\} \frac{1}{\text{vol}_L} \mu_M.
$$
At \((t, s) = (0, 0)\), the first term vanishes since \(\tau_b(\phi) = 0\). Hence we have

\[
\frac{\partial^2}{\partial t \partial s} E_B(\phi_{t,s}) \bigg|_{(t,s) = (0,0)} = -\int_M \langle W, \nabla^\phi \tau_b(\phi_{t,s}) \rangle \bigg|_{(t,s) = (0,0)} \frac{1}{\text{vol}_L} \mu_M.
\]

We choose a local orthonormal basic frame field \(\{E_a\}\) with \((\nabla E_a)(x) = 0\). Then, at \(x \in M\),

\[
\nabla^\phi_T \tau_b(\phi_{t,s}) = \sum_a \{\nabla^\phi_{E_a} \nabla^\phi_{E_a} d_T \Phi(E_a) - \nabla^\phi_{\partial_t} d_T \Phi(\nabla E_a E_a)\}
\]

\[
= \sum_a \{\nabla^\phi_{E_a} \nabla^\phi_{E_a} d_T \Phi(E_a) + R^\phi(\partial_t, E_a) d_T \Phi(E_a) - \nabla^\phi_{\partial_t} d_T \Phi(\nabla E_a E_a)\}
\]

\[
= \sum_a \{\nabla^\phi_{E_a} \nabla^\phi_{E_a} \frac{\partial}{\partial t} - \nabla^\phi_{\nabla E_a E_a} \frac{\partial}{\partial t} + R^\phi_{\Phi}(\partial_t, d_T \Phi(E_a)) d_T \Phi(E_a)\}.
\]

Hence, at \((t, s) = (0, 0)\), we have

\[
\nabla^\phi_T \tau_b(\phi_{t,s}) \bigg|_{(t,s) = (0,0)} = \sum_a \{\nabla^\phi_{E_a} \nabla^\phi_{E_a} V - \nabla^\phi_{\nabla E_a E_a} V + R^\phi_{\Phi}(V, d_T \phi(E_a)) d_T \phi(E_a)\}. \tag{4.5}
\]

Hence the proof is complete. \(\square\)

Let \(\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')\) be a foliated map with \(M\) compact. Then we define the transversal Hessian \(T\text{Hess}_\phi\) of \(\phi\) by

\[
T\text{Hess}_\phi(V, W) = \frac{\partial^2}{\partial t \partial s} E_B(\phi_{t,s}) \bigg|_{(t,s) = (0,0)},
\]

where \(\{\phi_{t,s}\}\) is a foliated variation of \(\phi\) with the normal variation vector fields \(V\) and \(W\). Then we have the following corollary.

**Corollary 4.2** Let \(\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')\) be a transversally harmonic map with \(M\) compact without boundary, and all leaves of \(\mathcal{F}\) be compact. Then, for any \(V, W \in \phi^{-1}Q'\),

\[
T\text{Hess}_\phi(V, W) = \int_M \langle \nabla^\phi_{\partial_t} V, \nabla^\phi_{\partial_t} W \rangle \frac{1}{\text{vol}_L} \mu_M - \int_M \langle \text{tr}_Q R^\phi(V, d_T \phi)d_T \phi, W \rangle \frac{1}{\text{vol}_L} \mu_M \tag{4.7}
\]
and $T\text{Hess}_\phi$ is symmetric, i.e., $T\text{Hess}_\phi(V,W) = T\text{Hess}_\phi(W,V)$ for any normal vector fields $V$ and $W$ along $\phi$.

**Proof.** From (4.2) and (4.6), we have

$$T\text{Hess}_\phi(V,W) = \sum_a \int_M \langle \nabla_{E_a}^\phi V, \nabla_{E_a}^\phi (\frac{1}{\text{Vol}_L} W) \rangle \mu_M - \int_M \langle \nabla_{(\text{Vol}_L)^{-1}\kappa_B} \phi V, W \rangle \mu_M$$

$$- \int_M \langle \text{tr}_Q R^Q(V, dT\phi) dT\phi, W \rangle \frac{1}{\text{Vol}_L} \mu_M$$

$$= \int_M \sum_a \langle \nabla_{E_a}^\phi V, \nabla_{E_a}^\phi W \rangle \frac{1}{\text{Vol}_L} \mu_M - \int_M \langle \text{tr}_Q R^Q(V, dT\phi) dT\phi, W \rangle \frac{1}{\text{Vol}_L} \mu_M$$

$$- \int_M \langle \nabla_{d_B \text{Vol}_L^2 + (\text{Vol}_L)^{-1}\kappa_B} V, W \rangle \frac{1}{\text{Vol}_L^2} \mu_M.$$

By (3.4), the last term in the last equality above vanishes. So the proof is completed. $\square$

If the transversal Hessian of $\phi : (M, g, F) \to (M', g', F')$ is positive semi-definite, i.e., $T\text{Hess}_\phi(V,V) \geq 0$ for any normal vector field $V$ along $\phi$, then $\phi$ is said to be **transversally stable**. From (4.7), we have the following corollary.

**Corollary 4.3 (Stability)** Any transversally harmonic map from a compact (without boundary) foliated Riemannian manifold to a foliated Riemannian manifold of non-positive transversal sectional curvature is transversally stable.

## 5 Transversal Jacobi operator along a map

Let $(M, g, F)$ be a compact foliated Riemannian manifold, and all leaves of $F$ are compact.

**Definition 5.1** Let $\phi : (M, g, F) \to (M', g', F')$ be a foliated map. Then the **transversal Jacobi operator** $J^T_\phi : \Gamma \phi^{-1}Q' \to \Gamma \phi^{-1}Q'$ along $\phi$ is defined by

$$J^T_\phi(V) = (\nabla_{\text{tr}}^\phi)^*(\nabla_{\text{tr}}^\phi) V - \nabla_{\kappa_B}^\phi V - \text{tr}_Q R^Q(V, dT\phi) dT\phi. \quad (5.1)$$
Any \( V \in \text{Ker}J^T_\phi \) is called a \textit{transversal Jacobi field} along \( \phi \) for the transversal energy.

From (2.4), the transversal Jacobi operator \( J^T_\chi \equiv J^T_\id \) along the identity map is given by

\[
J^T_\chi(Y) = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \bar{Y} - \rho \nabla (\bar{Y}) + A_Y \kappa^\sharp_B
\]  

(5.2)

for any \( Y \in V(F) \), which is called to \textit{generalized Jacobi operator} of \( F \) on \( M \).

From (4.2) and (4.6), if \( M \) is compact without boundary, then we have

\[
\text{THess}_\phi(V, W) = \int_M \langle J^T_\phi(V), W \rangle \frac{1}{vol_L} \mu_M.
\]  

(5.3)

Let \( \{\phi_t\} \) be a smooth foliated variation of \( \phi \) with the normal variation vector field \( V \). From (4.4) and (5.3), we have

\[
J^T_\phi(V) = -\nabla^*_{\text{tr}} \tau_b(\phi_t) \bigg|_{t=0}.
\]  

(5.4)

Hence we have the following proposition.

\textbf{Proposition 5.2} \ Let \( \phi : (M, g, F) \to (M', g', F') \) be a transversally harmonic map and \( \{\phi_t\} \) be a smooth foliated variation of \( \phi \) with the normal variation vector field \( V \). Then \( J^T_\phi(V) = 0 \), i.e., \( V \) is a transversal Jacobi field along \( \phi \) for the energy.

Let \( Y \in V(F) \) be an infinitesimal automorphism on \( (M, F) \). It is well-known [4] that, if \( \bar{Y} \) is a transversal affine field, i.e., \( \theta(Y)\nabla = 0 \), then

\[
\nabla^*_{\text{tr}} \nabla_{\text{tr}} \bar{Y} - \rho \nabla (\bar{Y}) + A_Y \kappa^\sharp_B = 0.
\]  

(5.5)

Hence we have the following proposition.

\textbf{Proposition 5.3} \ On \( (M, F) \), any transversal affine field is a generalized Jacobi field for \( F \).
Remark. Since any transversal Killing field is transversal affine \cite{7}, any transversal Killing field is also generalized Jacobi field of \( \mathcal{F} \). And the converse in Proposition 5.3 does not hold unless \( \mathcal{F} \) is harmonic.

**Corollary 5.4** (cf.\cite{4}) On \((M, \mathcal{F})\) with \( M \) compact without boundary, the followings are equivalent: for any vector field \( Y \in V(\mathcal{F}) \),

1. \( Y \) is a transversal Killing field, that is, \( \theta(Y) g_Q = 0 \);
2. \( Y \) is a generalized Jacobi field for \( \mathcal{F} \) satisfying (i) \( \text{div}_\nabla(Y) = 0 \) and (ii) \( \int_M g_Q(B_Y \tilde{Y}, \kappa_B^Y) \geq 0 \), where \( B_Y = A_Y + A_Y^t \).

Now, we recall the *transversal Jacobi operator* \( J_\nabla : \Gamma Q \to \Gamma Q \) of \( \mathcal{F} \) \cite{7} by

\[
J_\nabla = \nabla^*_\nabla \nabla - \rho^\nabla. \tag{5.6}
\]

Then \( \tilde{Y} \in \text{Ker} J_\nabla \) is called a *transversal Jacobi field for \( \mathcal{F} \)*. Note that two operators \( J_\nabla \) and \( J_T^\nabla \) are related by

\[
J_T^\nabla(\tilde{Y}) = J_\nabla(\tilde{Y}) + A_Y \kappa_B^Y. \tag{5.7}
\]

On a harmonic foliation \( \mathcal{F} \), \( J_T^\nabla = J_\nabla \). Hence, from Proposition 5.3 and Corollary 5.4, we have the following corollary.

**Corollary 5.5** \[8\] Let \( \mathcal{F} \) be a harmonic foliation on a compact Riemannian manifold \((M, g)\). Then the following are equivalent:

1. \( \tilde{Y} \) is a transversal Killing field, i.e., \( \theta(Y) g_Q = 0 \);
2. \( \tilde{Y} \) is a transversal Jacobi field of \( \mathcal{F} \) and \( \text{div}_\nabla(\tilde{Y}) = 0 \);
3. \( \tilde{Y} \) is transversal affine field, i.e., \( \theta(Y) \nabla = 0 \).

Now, we have the vanishing theorem about the transversal Jacobi field along the map.

**Theorem 5.6** Let \((M, g, \mathcal{F})\) be a closed, connected Riemannian manifold with a foliation \( \mathcal{F} \) and a bundle-like metric \( g \). Assume the transversal Ricci operator is non-positive and negative at some point. Then any generalized Jacobi field \( \tilde{Y} \) for \( \mathcal{F} \) is trivial, i.e, \( Y \) is tangential to \( \mathcal{F} \).
Proof. It is well-known [4] that
\[ \frac{1}{2} (\Delta_B - \kappa_B^\sharp) |\bar{Y}|^2 = g_Q(J_T^\nabla (\bar{Y}), \bar{Y}) - |\nabla_u \bar{Y}|^2. \] (5.8)
Let \( \bar{Y} \) be a generalized Jacobi field for \( \mathcal{F} \). Then we have
\[ \frac{1}{2} (\Delta_B - \kappa_B^\sharp) |\bar{Y}|^2 = g_Q(\rho^\nabla (\bar{Y}), \bar{Y}) - |\nabla_u \bar{Y}|^2. \] (5.9)
Since the transversal Ricci curvature is non-positive, we have
\[ (\Delta_B - \kappa_B^\sharp) |\bar{Y}|^2 \leq 0. \] (5.10)
Hence, by the generalized maximum principle (Lemma 2.2), \( |\bar{Y}| \) is constant. Again, from (5.9), \( \bar{Y} \) is parallel. Moreover, since \( \rho^\nabla \) is negative at some point, \( \bar{Y} \) is trivial. Equivalently, \( Y \) is tangential to \( \mathcal{F} \).

If \( \bar{Y} \) is a transversal Jacobi field of \( \mathcal{F} \), i.e., \( J^\nabla (\bar{Y}) = 0 \), then \( J_T^\nabla (\bar{Y}) = -\nabla_{\kappa_B^\sharp} \bar{Y} \).

Therefore, from (5.8) we have
\[ \frac{1}{2} \Delta_B |\bar{Y}|^2 = g_Q(\rho^\nabla (\bar{Y}), \bar{Y}) - |\nabla_u \bar{Y}|^2. \] (5.11)
From (5.11), we have the following corollary [7, p535].

**Corollary 5.7** [7] Let \((M, g, \mathcal{F})\) be as in Theorem 5.6. Assume the transversal Ricci operator is non-positive and negative at some point. Then any transversal Jacobi field of \( \mathcal{F} \) is trivial, i.e., \( Y \) is tangential to \( \mathcal{F} \).

**Proof.** Let \( \bar{Y} \) be a transversal Jacobi field of \( \mathcal{F} \). Since \( \rho^\nabla \) is non-positive, from (5.11), \( \Delta_B |\bar{Y}|^2 \leq 0 \). Since \( \Delta = \Delta_B \) on a basic function, by the maximum principle, \( |\bar{Y}| \) is constant. Since \( \rho^\nabla \) is negative at some point, from (5.11), \( \bar{Y} \) is trivial. \( \square \)

6 Transversally bi-harmonic maps

Let \((M, g, \mathcal{F})\) and \((M', g', \mathcal{F}')\) be two foliated Riemannian manifolds. Let \( \phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}') \) be a smooth foliated map. Now we define the transversal
bi-tension field \((\tau_2)_b(\phi)\) of \(\phi\) by

\[
(\tau_2)_b(\phi) = J^T_\phi(\tau_b(\phi)).
\] (6.1)

**Definition 6.1** Let \(\phi : (M, g, F) \to (M', g', F')\) be a smooth foliated map. Then \(\phi\) is said to be *transversally bi-harmonic* if the transversal bi-tension field vanishes, i.e., \((\tau_2)_b(\phi) = 0\).

Trivially, \(\phi\) is a transversally bi-harmonic map if and only if the transversal tension field \(\tau_b(\phi)\) is a transversal Jacobi field along \(\phi\). Moreover, any transversal harmonic map is a transversal bi-harmonic map. Now, we define the *transversal bi-energy* of \(\phi\) supported in a compact domain \(\Omega\) by

\[
(E_2)_B(\phi, \Omega) = \frac{1}{2} \int_\Omega |\delta d_T \phi|^2 \frac{1}{\text{vol}_L} \mu_M. \quad (6.2)
\]

Then we have the following theorem.

**Theorem 6.2** (The first variation formula for the transversal bi-energy) Let \(\phi : (M, g, F) \to (M', g', F')\) be a smooth foliated map, and all leaves of \(F\) be compact. Let \(\{\phi_t\}\) be a foliated variation of \(\phi\) with the variation vector field \(V\) in a compact domain \(\Omega\). Then we have

\[
\frac{d}{dt}(E_2)_B(\phi_t, \Omega)\bigg|_{t=0} = -\int_\Omega \langle (\tau_2)_b(\phi), V \rangle \frac{1}{\text{vol}_L} \mu_M. \quad (6.3)
\]

**Proof.** Let \(\{\phi_t\}\) be a foliated variation of \(\phi\) such that \(\frac{d\phi_t}{dt}\bigg|_{t=0} = V\) and \(\phi_0 = \phi\). Choose a local orthonormal basic frame \(\{E_a\}\) with \((\nabla E_a)(x) = 0\). Define \(\Phi : M \times (-\epsilon, \epsilon) \to M'\) by \(\Phi(x, t) = \phi_t(x)\). Let \(\nabla^\Phi\) be the pull-back connection on \(\Phi^{-1}Q'\). Obviously, \(d_T \Phi(E_a) = d_T \phi(E_a)\) and \(d\Phi(\frac{d}{dt}) = \frac{d\phi}{dt}\). Moreover, it is trivial that \(\nabla^{\frac{d}{dt}} E_a = \nabla E_a = \nabla E_a^{\frac{d}{dt}} = 0\). Hence, from (3.9), we have

\[
\frac{d}{dt}(E_2)_B(\phi_t, \Omega) = \int_\Omega \langle \nabla^{\frac{d}{dt}} \tau_b(\phi_t), \tau_b(\phi_t) \rangle \frac{1}{\text{vol}_L} \mu_M. \quad (6.4)
\]
From (5.4), it follows that
\[
\frac{d}{dt}(E_2)_B(\phi_t, \Omega)|_{t=0} = -\int_{\Omega} \langle J^T_\phi(V), \tau_b(\phi) \rangle \frac{1}{\text{vol}_L} \mu_M \\
= -\int_{\Omega} \langle J^T_\phi(\tau_b(\phi)), V \rangle \frac{1}{\text{vol}_L} \mu_M.
\]

The last equality above follows from (5.3) and the symmetry of the transversal Hessian \(THess\) of \(\phi\). From (6.1), the proof is complete. \(\square\)

**Corollary 6.3** Let \(\phi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')\) be a smooth foliated map, and all leaves of \(\mathcal{F}\) be compact. Then \(\phi\) is transversally bi-harmonic if and only if it is a critical point of the transversal bi-energy \((E_2)_B(\phi)\) of \(\phi\) on any compact domain.

Then we have the following (cf. [2]).

**Theorem 6.4** Let \(\phi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')\) be a transversally bi-harmonic map with \(M\) compact without boundary, and all leaves of \(\mathcal{F}\) be compact. Assume that the transversal sectional curvature \(K^{Q'}\) of \(\mathcal{F}'\) is non-positive. Then \(\phi\) is transversally harmonic.

**Proof.** Let \(\{E_a\}\) be a local orthonormal basic frame of \(Q\). Then \((\tau_2)_b(\phi) = 0\) implies that
\[
(\nabla^\phi_{\text{tr}}) \nabla^\phi_{\text{tr}} \tau_b(\phi) - \nabla^\phi_{\kappa^2_B} \tau_b(\phi) - \sum_a R^{Q'}(\tau_b(\phi), d_T \phi(E_a)) d_T \phi(E_a) = 0. \tag{6.5}
\]

From (2.7) and (2.8), we have \(\Delta_B |\tau_b(\phi)|^2 = 2(\langle \nabla^\phi_{\text{tr}} \nabla^\phi_{\text{tr}} \tau_b(\phi), \tau_b(\phi) \rangle) - 2 |\nabla_{\text{tr}} \tau_b(\phi)|^2\). Hence from (6.5), we have
\[
\frac{1}{2} (\Delta_B - \kappa^2_B) |\tau_b(\phi)|^2 = -|\nabla_{\text{tr}} \tau_b(\phi)|^2 \\
+ \sum_a \langle R^{Q'}(\tau_b(\phi), d_T \phi(E_a)) d_T \phi(E_a), \tau_b(\phi) \rangle. \tag{6.6}
\]

Since the transversal sectional curvature \(K^{Q'}\) of \(\mathcal{F}'\) is non-positive, we have
\[
(\Delta_B - \kappa^2_B) |\tau_b(\phi)|^2 \leq 0.
\]

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Hence, by the generalized maximum principle (Lemma 2.3), \( |\tau_b(\phi)| \) is constant. Again, from (6.6), we have that for all \( a \),

\[
\nabla_{E_a} \tau_b(\phi) = 0.
\] (6.7)

Now, we define the normal vector bundle \( X \) by

\[
X = \frac{1}{\text{vol}_L} \sum_a \langle d_T \phi(E_a), \tau_b(\phi) \rangle E_a.
\]

Then we have

\[
\text{div}_\tau(X) = -\frac{1}{\text{vol}_L^2} \langle d_T \phi(d_B \text{vol}_L^2), \tau_b(\phi) \rangle + \frac{1}{\text{vol}_L} |\tau_b(\phi)|^2
\]

\[
= \frac{1}{\text{vol}_L} \langle d_T \phi(\kappa_B^2), \tau_b(\phi) \rangle + \frac{1}{\text{vol}_L} |\tau_b(\phi)|^2.
\]

The last equality above follows from Lemma 2.2. By integrating and by using the transversal divergence theorem (Theorem 2.1), we have

\[
\int_M |\tau_b(\phi)|^2 \frac{1}{\text{vol}_L} \mu_M = 0,
\] (6.8)

which implies that \( \tau_b(\phi) = 0 \). So \( \phi \) is transversally harmonic. \( \square \)

7 The second variation formula for the transversal bi-energy

Let \((M, g, \mathcal{F})\) and \((M', g', \mathcal{F}')\) be two foliated Riemannian manifolds. Let \( \phi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}') \) be a transversally bi-harmonic map. Then we have the second variational formula of the transversal bi-energy as follows.

**Theorem 7.1** (The second variation formula for the transversal bi-energy) Let \( \phi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}') \) be a smooth foliated map with \( M \) compact without
boundary and all leaves be compact. Let \( \{ \phi_t \} \) be a foliated variation of \( \phi \) with the normal variation vector field \( V \). Then

\[
\frac{d^2}{dt^2} (E_2)_B(\phi_t) \Big|_{t=0} = \int_M \{ |(\tau_2)_b(\phi)|^2 - (\tau_2)_b(\phi), \nabla_V V \} - \langle R_{\phi}^{\phi}(V, \tau_b(\phi)) \tau_b(\phi), V \rangle \frac{1}{\text{vol}_L \mu_M} \\
- 2 \sum_a \int_M \langle (\nabla_{E_a} R_{\phi})^a(V, d_T \phi(E_a)) \tau_b(\phi), V \rangle \frac{1}{\text{vol}_L \mu_M} \\
+ \sum_a \int_M \langle (\nabla_{\tau_b(\phi)} R_{\phi})^a(V, d_T \phi(E_a))d_T \phi(E_a), V \rangle \frac{1}{\text{vol}_L \mu_M} \\
- 4 \sum_a \int_M \langle R_{\phi}^{\phi}(\nabla_{E_a}^V, \tau_b(\phi))d_T \phi(E_a), V \rangle \frac{1}{\text{vol}_L \mu_M}.
\]

**Proof.** Let \( V \in \phi^{-1}Q' \) and let \( \{ \phi_t \} \) be a foliated variation of \( \phi \) such that \( \frac{d\phi_t}{dt} \big|_{t=0} = V \) and \( \phi_0 = \phi \). We choose a local orthonormal basic frame \( \{ E_a \} \) such that \( (\nabla E_a)(x) = 0 \) at point \( x \). Define \( \Phi : M \times (-\epsilon, \epsilon) \to M' \) by \( \Phi(x, t) = \phi_t(x) \). Let \( \nabla^\phi \) be the pull-back connection on \( \phi^{-1}Q' \). Obviously, \( d_T \Phi(E_a) = d_T \phi(E_a) \) and \( \frac{d\phi_t}{dt} = \frac{d\phi}{dt} \). Trivially, \( \nabla_{\frac{dA}{dt}} = \nabla_{\frac{dt}{dt}} E_a = \nabla E_a \frac{dt}{dt} = 0 \). Hence, from (6.3) we have

\[
\frac{d^2}{dt^2} (E_2)_B(\phi_t) = \int_M \left( \langle \nabla_{\frac{dt}{dt}}^\phi \nabla_{\frac{dt}{dt}}^\phi \tau_b(\phi_t), \tau_b(\phi_t) \rangle + |\nabla_{\frac{dt}{dt}}^\phi \tau_b(\phi_t)|^2 \right) \frac{1}{\text{vol}_L \mu_M}.
\]

From (4.5), we have

\[
\nabla_{\frac{dt}{dt}}^\phi \tau_b(\phi_t) = \sum_a (\nabla_{\frac{dt}{dt}}^\phi)_{E_a}^a \frac{\partial \phi}{\partial t} + \sum_a R_{\phi}^{\phi}(\frac{\partial \phi}{\partial t}, d_T \phi_t(E_a))d_T \phi_t(E_a),
\]

and then

\[
\nabla_{\frac{dt}{dt}}^\phi \nabla_{\frac{dt}{dt}}^\phi \tau_b(\phi_t) = \sum_a (\nabla_{\frac{dt}{dt}}^\phi)_{E_a}^a \frac{\partial \phi}{\partial t} + R_{\phi}^{\phi}(\frac{\partial \phi}{\partial t}, d_T \phi_t(E_a))d_T \phi_t(E_a).
\]

By a long calculation, we get

\[
\nabla_{\frac{dt}{dt}}^\phi (\nabla_{\frac{dt}{dt}}^\phi)_{E_a}^a \frac{\partial \phi}{\partial t} = (\nabla_{\frac{dt}{dt}}^\phi)_{E_a}^a \nabla_{\frac{dt}{dt}}^\phi \frac{\partial \phi}{\partial t} + \nabla_{E_a} R_{\phi}^{\phi}(\frac{d}{dt}, E_a) \frac{\partial \phi}{\partial t} \\
+ R_{\phi}^{\phi}(\frac{d}{dt}, E_a) \nabla_{E_a} \frac{\partial \phi}{\partial t} + R_{\phi}^{\phi}(\nabla_{E_a} E_a, \frac{d}{dt}) \frac{\partial \phi}{\partial t}
\]
and
\[
\nabla_{\phi} R' (\frac{\partial \Phi}{\partial t}, d_T \phi_t(E_a))d_T \phi_t(E_a)
\]
\[
= (\nabla_{\phi} R' (\frac{\partial \Phi}{\partial t}, d_T \phi_t(E_a))d_T \phi_t(E_a) + R' (\nabla_{\phi} \frac{\partial \Phi}{\partial t}, d_T \phi_t(E_a))d_T \phi_t(E_a)
\]
\[
+ R' (\frac{\partial \Phi}{\partial t}, \nabla_{\phi} d_T \phi_t(E_a))d_T \phi_t(E_a) + R' (\frac{\partial \Phi}{\partial t}, d_T \phi_t(E_a))\nabla_{\phi} d_T \phi_t(E_a).
\]

Since \([\frac{d}{dt}, E_a] = 0\), we have
\[
\nabla_V d_T \phi(E_a) = \nabla_{d_T \phi(E_a)} V = \nabla_{E_a} \phi V.
\]
Hence from the equations above, we have
\[
\nabla_{\phi} \frac{\partial \Phi}{\partial t} \nabla_{\phi} \frac{\partial \Phi}{\partial t} \tau_b(\phi)\big|_{t=0}
\]
\[
= -J^T (\nabla_V V) + \sum_a \nabla_{E_a} R' (V, d_T \phi(E_a)) V
\]
\[
+ 2 \sum_a R' (V, d_T \phi(E_a)) \nabla_{E_a} V + \sum_a R' (d_T \phi(E_a), V) V
\]
\[
+ \sum_a (\nabla_V R') (V, d_T \phi(E_a))d_T \phi(E_a) + \sum_a R' (V, \nabla_{E_a} V)d_T \phi(E_a).
\]

So, by the first and second Bianchi identities, we have
\[
\langle \nabla_{\phi} \frac{\partial \Phi}{\partial t} \nabla_{\phi} \frac{\partial \Phi}{\partial t} \tau_b(\phi)\rangle|_{t=0} = -\langle J^T (\nabla_V V), \tau_b(\phi)\rangle + \langle R' (V, \tau_b(\phi)) V, \tau_b(\phi)\rangle
\]
\[
+ \sum_a \langle (\nabla_V R') (V, d_T \phi(E_a))d_T \phi(E_a), \tau_b(\phi)\rangle
\]
\[
+ \sum_a \langle (\nabla_{E_a} R') (V, d_T \phi(E_a)) V, \tau_b(\phi)\rangle
\]
\[
+ 4 \sum_a \langle R' (V, d_T \phi(E_a)) \nabla_{E_a} V, \tau_b(\phi)\rangle.
\]
By integrating together with (5.4), we have

\[
\int_M \langle \nabla_{\phi} \nabla_{\phi} \tau_b(\phi) \rangle_{t=0, \tau_b(\phi)} \frac{1}{\text{vol}_L} \mu_M \\
= - \int_M \langle J_\phi^T(\tau_b(\phi)), \nabla_V V \rangle \frac{1}{\text{vol}_L} \mu_M - \int_M \langle R^Q(\nabla_{\tau_b(\phi)} \tau_b(\phi), V) \rangle \frac{1}{\text{vol}_L} \mu_M \\
- 2 \sum_a \int_M \langle (\nabla_{E_a} R^Q)(V, d_T \phi(E_a)) \tau_b(\phi), V \rangle \frac{1}{\text{vol}_L} \mu_M \\
+ \sum_a \int_M \langle (\nabla_{\tau_b(\phi)} R^Q)(V, d_T \phi(E_a)) d_T \phi(E_a), V \rangle \frac{1}{\text{vol}_L} \mu_M \\
- 4 \sum_a \int_M \langle R^Q(\nabla_{E_a}^\phi V, \tau_b(\phi)) d_T \phi(E_a), V \rangle \frac{1}{\text{vol}_L} \mu_M.
\]

From (6.1) and (7.1), the proof follows. □

Then we have the following corollary (cf. [2]).

**Corollary 7.2** Let \( \phi : (M, g, F) \to (M', g', F') \) be a transversally bi-harmonic map with \( M \) compact without boundary. Let \( \{\phi_t\} \) be a foliated variation of \( \phi \) with the normal variation vector field \( V \). Then

\[
\frac{d^2}{dt^2}(E_2)_B(\phi_t) \bigg|_{t=0} = - \int_M \langle R^Q(\nabla_{\tau_b(\phi)} \tau_b(\phi), V) \rangle \frac{1}{\text{vol}_L} \mu_M \\
- 2 \sum_a \int_M \langle (\nabla_{E_a} R^Q)(V, d_T \phi(E_a)) \tau_b(\phi), V \rangle \frac{1}{\text{vol}_L} \mu_M \\
+ \sum_a \int_M \langle (\nabla_{\tau_b(\phi)} R^Q)(V, d_T \phi(E_a)) d_T \phi(E_a), V \rangle \frac{1}{\text{vol}_L} \mu_M \\
- 4 \sum_a \int_M \langle R^Q(\nabla_{E_a}^\phi V, \tau_b(\phi)) d_T \phi(E_a), V \rangle \frac{1}{\text{vol}_L} \mu_M.
\]

**Definition 7.3** If the transversally bi-harmonic map \( \phi : (M, g, F) \to (M', g', F') \) satisfies \( \frac{d^2}{dt^2}(E_2)_B(\phi_t) \bigg|_{t=0} \geq 0 \), then \( \phi \) is said to be stable.

Note that any transversally harmonic map can be considered as a stable transversally bi-harmonic map. We also have the following theorem ([2]).
Theorem 7.4 Let \((M, \mathcal{F})\) be a closed Riemannian manifold with a foliation \(\mathcal{F}\) and let \((M', \mathcal{F}')\) be a Riemannian manifold with a constant transversal sectional curvature \(C > 0\). If a foliated map \(\phi: (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')\) is stable transversally bi-harmonic and relatively harmonic, then \(\phi\) is transversally harmonic.

Proof. Since the transversal sectional curvature \(K^{\mathcal{F}'}\) of \(\mathcal{F}'\) is constant \(C > 0\), from Corollary 7.2, we have
\[
\frac{d^2}{dt^2} (E_2)_B(\phi_t) \bigg|_{t=0} = - \int_M \langle R^{\mathcal{F}'}(V, \tau_b(\phi))\tau_b(\phi), V \rangle \frac{1}{\text{vol}_L} \mu_M \\
- 4 \sum_a \int_M \langle R^{\mathcal{F}'}(\nabla_{E_a} V, \tau_b(\phi)) d_T \phi(E_a), V \rangle \frac{1}{\text{vol}_L} \mu_M.
\]

Let \(V = \tau_b(\phi)\). Since \(\phi\) is relatively harmonic, i.e., \(\langle \tau_b(\phi), d_T \phi(X) \rangle\) for any vector field \(X \in \Gamma \mathcal{F}'\), we have
\[
\frac{d^2}{dt^2} (E_2)_B(\phi_t) \bigg|_{t=0} = - 4 \sum_a \int_M \langle R^{\mathcal{F}'}(\nabla_{E_a} \tau_b(\phi), \tau_b(\phi)) d_T \phi(E_a), \tau_b(\phi) \rangle \frac{1}{\text{vol}_L} \mu_M \\
= 4C \sum_a \int_M \langle \nabla_{E_a} \tau_b(\phi), d_B \phi(E_a) \rangle \tau_b(\phi) \frac{1}{\text{vol}_L} \mu_M \\
= -4C \int_M |\tau_b(\phi)|^4 \frac{1}{\text{vol}_L} \mu_M.
\]

This stability implies that \(\tau_b(\phi) = 0\), i.e., \(\phi\) is transversally harmonic. \(\square\)

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