Covariant Field Equations of the M Theory Five-Brane

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Abstract

The component form of the equations of motion for the 5-brane in eleven-dimensions is derived from the superspace equations. These equations are fully covariant in six-dimensions. It is shown that double-dimensional reduction of the bosonic equations gives the equations of motion for a 4-brane in ten dimensions governed by the Born-Infeld action.

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1 Introduction

It is now widely believed that there is a single underlying theory which incorporates all superstring theories and which also has, as a component, a new theory in eleven-dimensions which has been christened “M-theory”. Opinion is divided as to whether M-theory is itself the fundamental theory or whether it is one corner of a large moduli space which has the five consistent ten-dimensional superstring theories as other corners. Whichever viewpoint turns out to be correct it seems certain that M-theory will play a crucial rôle in future developments. Not much is known about this theory at present, apart from the fact that it has eleven-dimensional supergravity as a low energy limit and that it has two basic BPS p-branes, the 2-brane and the 5-brane, which preserve half-supersymmetry. The former can be viewed as a fundamental (singular) solution to the supergravity equations whereas the latter is solitonic. It is therefore important to develop a better understanding of these branes and in particular the 5-brane, since the Green-Schwarz action for the 2-brane has been known for some time.

In a recent paper [1] it was shown that all branes preserving half-supersymmetry can be understood as embeddings of one superspace, the worldsurface, into another, the target superspace, which has spacetime as its body, and that the basic embedding condition which needs to be imposed is universal and geometrically natural. The results of [1] were given mainly at the linearised level; in a sequel [2] the eleven-dimensional 5-brane was studied in more detail and the full non-linear equations of motion were derived. However, these were expressed in superspace notation. It is the purpose of this paper to interpret these equations in a more familiar form, in other words to derive their component equivalents. In the context of superembeddings the component formalism means the Green-Schwarz formalism since the leading term in the worldsurface $\theta$-expansion of the embedding describes a map from a bosonic worldsurface to a target superspace.

Partial results for the bosonic sector of the eleven-dimensional fivebrane have been obtained in [3, 4, 5, 6]. More recently, a non-covariant bosonic action has been proposed [7, 8]. In this approach, only the five-dimensional covariance is manifest. In [9], a complete bosonic action has been constructed. The action contains an auxiliary scalar field, which can be eliminated at the expense of sacrificing the six dimensional covariance, after which it reduces to the action of [7, 8].

In this paper we will show that the covariant superfield equations of motion of the eleven-dimensional superfivebrane presented in [1, 2] can be written in $\kappa$-invariant form, and that they do have the anticipated Born-Infeld structure. The $\kappa$-symmetry emerges from the worldsurface diffeomorphism invariance of the superspace equations, the parameter of this symmetry being essentially the leading component in the worldsurface $\theta$-expansion of an odd diffeomorphism. We find neither the need to introduce a scalar auxiliary field, nor the necessity to have only five-dimensional covariance. As long as one does not insist on having an action, it is possible to write down six dimensional covariant equations, as one normally expects in the case of chiral $p$-forms.

In order to show that our equations have the expected Born-Infeld form we perform a double-dimensional reduction and compare them with the equations of motion for a 4-brane in ten dimensions. In section 4, we do this comparison in the bosonic sector, and flat target space, and show that the Born-Infeld form of the 4-brane equations of motion does indeed emerge. The work of refs [1, 2] is briefly reviewed in the next section, and in section 3 the equations of motion are described in Green-Schwarz language.
2 Equations of Motion in Superspace

The 5-brane is described by an embedding of the worldsurface $M$, which has (even|odd) dimension ($6$|$16$) into the target space, $\underline{M}$, which has dimension ($11$|$32$). In local coordinates $z^M = (x^m, \theta^\mu)$ for $M$ and $z^\underline{M}$ for $\underline{M}$ the embedded submanifold is given as $z^\underline{M}(z)^\underline{A}$. We define the embedding matrix $E_A^\underline{A}$ to be the derivative of the embedding referred to preferred bases on both manifolds:

$$E_A^\underline{A} = E_A^M \partial_M z^M E_M^\underline{A},$$

where $E_M^A$ ($E_A^M$) is the supervielbein (inverse supervielbein) which relates the preferred frame basis to the coordinate basis, and the target space supervielbein has underlined indices. The notation is as follows: indices from the beginning (middle) of the alphabet refer to frame (coordinate) indices, latin (greek) indices refer to even (odd) components and capital indices to both, non-underlined (underlined) indices refer to $M$ ($\underline{M}$) and primed indices refer to normal directions. We shall also employ a two-step notation for spinor indices; that is, for general formulae a spinor index $\alpha$ (or $\alpha'$) will run from 1 to 16, but to interpret these formulae, we shall replace a subscript $\alpha$ by a subscript pair $\alpha i$ and a subscript $\alpha'$ by a pair $\alpha' i$, where $\alpha = 1, \ldots, 4$ and $i = 1, \ldots, 4$ reflecting the $\text{Spin}(1,5) \times USp(4)$ group structure of the $N = 2, d = 6$ worldsurface superspace. (A lower (upper) $\alpha$ index denotes a left-handed (right-handed) $d = 6$ Weyl spinor and the $d = 6$ spinors that occur in the theory are all symplectic Majorana-Weyl.)

The torsion 2-form $T^A$ on $\underline{M}$ is given as usual by

$$T^A = dE_A^\underline{A} + E_B^\underline{A} \Omega^B^A,$$

where $\Omega$ is the connection 1-form. The pull-back of this equation onto the worldsurface reads, in index notation,

$$\nabla_A E_B^C - (-1)^{AB} \nabla_B E_A^C + T_{AB}^C E_C^D E_D^C = (-1)^{A(B+C)} E_B^B E_A^D T_{AB}^C,$$

where the derivative $\nabla_A$ is covariant with respect to both spaces, i.e. with respect to both underlined and non-underlined indices, the connection on $M$ being, at this stage at least, independent of the target space connection.

The basic embedding condition is

$$E_\alpha^a = 0,$$

from which it follows that (using (3))

$$E_\alpha^a E_\beta^b T_{ab}^{\beta} = T_{\alpha \beta}^\gamma E_\gamma^\beta .$$

If the target space geometry is assumed to be that of (on-shell) eleven-dimensional supergravity equation (4) actually determines completely the induced geometry of the worldsurface and the dynamics of the 5-brane. In fact, as will be discussed elsewhere, it is not necessary to be so specific about the target space geometry, but it will be convenient to adopt the on-shell geometry in the present paper. The structure group of the target superspace is $\text{Spin}(1,10)$ and the non-vanishing parts of the target space torsion are $\Gamma_\alpha^\beta_\gamma$.

$$T_{\alpha \beta}^{\gamma} = -i (\Gamma_{\alpha \beta})_\gamma,$$

$$T_{\alpha \beta}^{\gamma} = -\frac{1}{36} (\Gamma_{\alpha \beta \gamma \delta})^{\alpha \beta \gamma \delta} H_{\alpha \beta \gamma \delta} - \frac{1}{288} (\Gamma_{\alpha \beta \gamma \delta \epsilon})^{\alpha \beta \gamma \delta \epsilon} H_{\alpha \beta \gamma \delta \epsilon} ,$$

\(^1\)We shall also denote the coordinates of $M(\underline{M})$ by $z = (x, \theta)(\underline{z} = (x, \underline{\theta}))$ if it is not necessary to use indices
where \( H_{abcd} \) is totally antisymmetric, and the dimension 3/2 component \( T_{ab} \). \( H_{abcd} \) is the dimension one component of the closed superspace 4-form \( H_4 \) whose only other non-vanishing component is
\[
H_{ab,c} = -i(\Gamma_{ab})_{cd}.
\]
With this target space geometry equation (5) becomes
\[
E_{\alpha \alpha} E_{\beta \beta} (\Gamma^c)_{\alpha \beta} = iT_{\alpha \beta} E_{c \epsilon}.
\]
The solution to this equation is given by
\[
E_{\alpha \alpha} = u_{\alpha \alpha} + h_{\alpha \beta} u_{\beta \alpha} ,
\]
and
\[
E_{a \alpha} = m_{a \beta} u_{\beta \alpha} ,
\]
where \( m_{a \beta} = \delta_{a \beta} - 2h_{abcd} h^{bcd} \).

This solution is determined up to local gauge transformations belonging to the group \( Spin(1,5) \times USp(4) \), the structure group of the worldsurface. One also has the freedom to make worldsurface super-Weyl transformations but one can consistently set the conformal factor to be one and we shall do this throughout the paper.

It is useful to introduce a normal basis \( E_A \equiv E_A \Delta E_A \) of vectors at each point on the worldsurface. The inverse of the pair \( (E_A, E_A') \) is denoted by \( (E_A, E_A') \). The odd-odd and even-even components of the normal matrix \( E_A \) can be chosen to be
\[
E_{\alpha \alpha} = u_{\alpha \alpha} ,
\]
and
\[
E_{a \alpha} = u_{a \alpha} .
\]
Together with (11) and (16), it follows that the inverses in the odd-odd and even-even sectors are
\[
E_{\alpha}^{a} = u_{\alpha}^{a} , \quad E_{a}^{\alpha'} = u_{a}^{\alpha'} - u_{\beta}^{\alpha} h_{\beta}^{\alpha'} ,
\]
and
\[
E_{a}^{a} = u_{a}^{b}(m^{-1})_{b}^{a} , \quad E_{a}^{a'} = u_{a}^{a'} .
\]

2We have rescaled the \( H_{abc} \) and \( h_{abc} \) of refs [1, 2] by a factor of 6.
Later, we will also need the relations \[ 2 \]

\[
\begin{align*}
\alpha \alpha & (\Gamma \alpha \beta) = (\Gamma \alpha \beta), \\
\alpha \beta & (\Gamma \alpha \beta) = (\Gamma \alpha \beta), \\
\alpha \beta & (\Gamma \alpha \beta) = (\Gamma \alpha \beta), 
\end{align*}
\]

which follow from the fact that the \( u \)'s form a 32 \( \times \) 32 matrix that is an element of \( \text{Spin}(1,10) \).

The field \( h_{abc} \) is a self-dual antisymmetric tensor, but it is not immediately obvious how it is related to a 2-form potential. In fact, it was shown in \[ 2 \] that there is a superspace 3-form \( H_3 \) which satisfies

\[
dH_3 = -\frac{1}{4} H_4, 
\]

where \( H_4 \) is the pull-back of the target space 4-form, and whose only non-vanishing component is \( H_{abc} \) where

\[
H_{abc} = m_a^d m_b^e h_{cde}. 
\]

The equations of motion of the 5-brane can be obtained by systematic analysis of the torsion equation \( 3 \), subject to the condition \( 4 \) \[ 2 \]. The bosonic equations are the scalar equation

\[
\eta^{ab} K_{ab} c' = \frac{1}{8} (\gamma_{c'})^{jk} (\gamma^a)^{\beta j} Z_{a,\beta j,\gamma k}, 
\]

and the antisymmetric tensor equation

\[
\nabla^c h_{abc} = -\frac{\eta^{jk}}{16} (\gamma_{[a})^{\beta j} Z_{b],\beta j,\gamma k} + \frac{1}{2} (\gamma^a)^{\beta j} Z_{c,\beta j,\gamma k}), 
\]

where

\[
Z_{ab} = E_a^\alpha E_b^\beta T_{\alpha \beta}^c E_a^\gamma - E_b^\alpha E_a^\beta T_{\alpha \beta}^c E_a^\gamma, 
\]

and

\[
\nabla_a h_{bcd} = \nabla_a h_{bcd} - 3X_{a,bc} h_{cde}, 
\]

with

\[
X_{a,bc} = (\nabla_a u_b) u_c. 
\]

In the scalar equation we have introduced a part of the second fundamental form of the surface which is defined to be

\[
K_{AB} C' = (\nabla A E_B C') E_C C'. 
\]

Finally, the spin one-half equation is simply

\[
(\gamma^a)^{\alpha \beta} \chi_{a \beta}^j = 0, 
\]

where

\[
\chi_{a \alpha}^j = E_a^{\alpha} E_a^{\alpha}. 
\]

We end this section by rewriting the equations of motion \( 24 \), \( 25 \) and \( 30 \) in an alternative form that will be useful for the purposes of the next section:

\[
E_a^\alpha E_a^\beta (\Gamma^a)^{\beta \gamma} = 0, 
\]

\[
\eta^{ab} \nabla_a E_b^\alpha E_a^\beta = -\frac{1}{8} (\Gamma^a)^{\gamma \beta} Z_{a \beta}^\gamma, 
\]

\[
\nabla^c h_{abc} = -\frac{1}{32} (\Gamma^c \nabla a)^{\gamma \beta} Z_{c \beta}^\gamma. 
\]
It will also prove to be useful to rewrite \((26)\) as

\[
Z_{a\beta}^{\gamma'} = E_{\beta}^{\gamma} \left( T_{a\beta}^{\gamma} - K_{a\beta}^{\gamma} \right) E_{\gamma}^{\gamma'},
\]

with the matrices \(T_a\) and \(K_a\) defined as

\[
T_{a\beta}^{\gamma} = E_{\alpha a}^{\gamma} T_{a\beta}^{\gamma},
\]

\[
K_{a\beta}^{\gamma} = E_{\alpha \delta}^{\gamma} \left( \nabla_{\gamma} E_{\delta}^{\delta'} \right) E_{\delta}^{\gamma}.
\]

3  Equations of Motion in Green-Schwarz Form

3.1 Preliminaries

In this section we derive the component equations of motion following from the superspace equations given in the last section. The idea is to expand the superspace equations as power series in \(\theta^\mu\) and to evaluate them at \(\theta = 0\). We may choose a gauge in which the worldsurface supervielbein takes the form

\[
E_{m\alpha}(x, \theta) = E_{m\alpha}(x) + O(\theta)
\]

\[
E_{\mu\alpha}(x, \theta) = 0 + O(\theta)
\]

and the inverse takes the form

\[
E_{a\alpha}(x, \theta) = E_{a\alpha}(x) + O(\theta)
\]

\[
E_{\alpha\mu}(x, \theta) = \delta_{\alpha\mu} + O(\theta),
\]

where \(E_{a\alpha}(x)\) is the inverse of \(E_{m\alpha}(x)\). The component field \(E_{m\alpha}(x)\) is the worldsurface gravitino, which is determined by the embedding, but which only contributes terms to the equations of motion which we shall not need for the purpose of this section. The field \(E_{\alpha\mu}(x)\) is linearly related to the gravitino. From the embedding condition \(\Pi\) we learn that

\[
\partial_{\mu} E_{M}^{\alpha} = 0 \quad \text{at} \quad \theta = 0,
\]

so that

\[
E_{a}^{\alpha} = E_{m}^{\alpha} \mathcal{E}_{m}^{\alpha} \quad \text{at} \quad \theta = 0,
\]

\[
E_{a}^{\alpha} = E_{m}^{\alpha} \mathcal{E}_{m}^{\alpha} \quad \text{at} \quad \theta = 0,
\]

where we have used the definitions

\[
\mathcal{E}_{m}^{\alpha}(x) = \partial_{m} E_{M}^{\alpha} \quad \text{at} \quad \theta = 0,
\]

\[
\mathcal{E}_{m}^{\alpha}(x) = \partial_{m} E_{M}^{\alpha} \quad \text{at} \quad \theta = 0.
\]

These are the embedding matrices in the Green-Schwarz formalism, often denoted by \(\Pi\). From \(\Pi\) we have

\[
E_{a}^{\alpha} E_{b}^{\beta} \eta_{ab} = m_{a}^{c} m_{b}^{d} \eta_{cd},
\]

this equation being true for all \(\theta\) and in particular for \(\theta = 0\). Therefore, if we put

\[
e_{a}^{m} = ((m^{-1})_{a}^{b} E_{b}^{m})(x),
\]
we find that $e_m^a$ is the sechsbein associated with the standard GS induced metric

$$g_{mn}(x) = E_m^a E_n^b \eta_{ab}. \quad (47)$$

There is another metric, which will make its appearance later, which we define as

$$G^{mn} = E_m^a(x) E_n^b(x) \eta^{ab} \quad (48)$$

$$= (m^2)^{ab} e_a^m e_b^n(x). \quad (49)$$

We also note the relation

$$u_a^a = e_a^m E_m^a, \quad (50)$$

which follows from (11), (41) and (46).

For the worldsurface 3-form $H^3$ we have

$$H_{MNP} = E_P^C E_B^N E_A^M H_{ABC}(-1)^{(M+N)(P+C)}(M+N)) \quad (51)$$

Evaluating this at $\theta = 0$ one finds

$$H_{mnp}(x) = (E_m^a E_n^b E_p^c H_{abc})(x) \quad (52)$$

so that, using (23) and (46), one finds

$$h_{abc}(x) = m_a^d e_d^m e_b^n e_c^p H_{mnp}(x). \quad (53)$$

We are now in a position to write down the equations of motion in terms of $E_m^a$, $E_m^\dot{a}$ and $H_{mnp}(x)$. The basic worldsurface fields are $x_m^a, \theta^a$ and $B_{mn}(x)$, where $B_{mn}$ is the 2-form potential associated with $H_{mnp}$ as $H^3 = dB_2 - \frac{1}{4} C_3$ and $C_3$ is the pull-back of the target space 3-form. We begin with the Dirac equation (30).

### 3.2 The Dirac Equation

In order to extract the Dirac equation in $\kappa$-invariant component form, it is convenient to define the projection operators

$$E_\alpha^a E_\alpha^a = \frac{1}{2} (1 + \Gamma)^\alpha_\alpha, \quad (54)$$

$$E_\alpha^a E_\alpha^\dot{a} = \frac{1}{2} (1 - \Gamma)^\alpha_\alpha. \quad (55)$$

The $\Gamma$-matrix, which clearly satisfies $\Gamma^2 = 1$, can be calculated from these definitions as follows. We expand

$$E_\alpha^a E_\alpha^\dot{a} = \sum_{n=0}^{5} C_{\alpha_1 \cdots \alpha_n}^a \Gamma_{\alpha_1 \cdots \alpha_n}^a \quad (56)$$

where $C$’s are the expansion coefficients that are to be determined. Tracing this equation with suitable $\Gamma$-matrices, and using the relations (19-21), we find that the only non-vanishing coefficients are

$$C = \frac{1}{2}, \quad (57)$$

$$C_{abc} = \frac{1}{6} h_{abc} u_a^a u_b^b u_c^c, \quad (58)$$

$$C_{a_1 \cdots a_6} = -\frac{1}{6!2} e_{a_1 \cdots a_6} u_a^a u_{a_1}^{a_1} \cdots u_{a_6}^{a_6}. \quad (59)$$
Substituting these back, and comparing with (54), we find
\[ \Gamma = \frac{1}{6!\sqrt{-g}} e^{m_1 \cdots m_6} ( - \Gamma_{m_1 \cdots m_6} + 40 \Gamma_{m_1 \cdots m_3} h_{m_4 \cdots m_6} ) , \] (60)
where we have used (50) and the definitions
\[ \Gamma_m = \epsilon_m{}^a \Gamma_a , \] (61)
\[ h_{mnp} = e_m^a e_n^b e_p^c h_{abc} . \] (62)

The matrix \( \Gamma \) can also be written as
\[ \Gamma = ( -1 + \frac{1}{3} \Gamma_{mnp} h_{mnp} ) \Gamma_{(0)} , \] (63)
where
\[ \Gamma_{(0)} = \frac{1}{6!\sqrt{-g}} e^{m_1 \cdots m_6} \Gamma_{m_1 \cdots m_6} . \] (64)

It is now a straightforward matter to derive the component for of the Dirac equation (32). We use (19) to replace the worldsurface \( \Gamma \)-matrix by the target space \( \Gamma \)-matrix multiplied by factors of \( u \), and recall (15), (17), (50) and (55) to find
\[ \eta^{bc}(1 - \Gamma)\gamma^\beta (\Gamma^a \gamma^\alpha E_{c \gamma}^a \gamma^b E_{b \alpha}^a) = 0 . \] (65)

We recall that \( \mathcal{E}_{b \alpha}^a = e_b^m \epsilon_{m \alpha}^a \) and that \( E_{c \gamma}^a = m_c^d e_d^a \epsilon_{n \gamma}^c \). Using these relations, the Dirac equation can be written as
\[ \mathcal{E}_a (1 - \Gamma) \Gamma^b m_b^a = 0 , \] (66)
where \( \Gamma^b = \Gamma^m e_m^a \) and the target space spinor indices are suppressed.

The Dirac equation obtained above has a very similar form to those of D-branes in ten dimensions [12, 13, 14, 15, 16], and indeed we expect that a double dimensional reduction would yield the 4-brane Dirac equation.

The emergence of the projection operator \( (1 - \Gamma) \) in the Dirac equation in the case of D-branes, and the other known super p-branes is due to the contribution of Wess-Zumino terms in the action (see, for example, ref. [17] for the eleven dimensional supermembrane equations of motion). These terms are also needed for the \( \kappa \)-symmetry of the action. It is gratifying to see that the effect of Wess-Zumino terms is automatically included in our formalism through a geometrical route that is based on considerations of the embedding of a world superspace into target superspace.

### 3.3 The Scalar Equation

By scalar equation we mean the equation of motion for \( x^m(x) \), i.e. the coordinates of the target space, which are scalar fields from the worldsurface point of view. In a physical gauge, these describe the five scalar degrees of freedom that occur in the worldsurface tensor supermultiplet.

The scalar equation is the leading component of the superspace equation (33) which we repeat here for the convenience of the reader:
\[ \eta^{ab} \nabla_a E_b^a = - \frac{1}{8} (\Gamma^{b' \alpha})_{\gamma \beta} Z_{a \beta} \gamma' . \] (67)
The superspace equation for the covariant derivative
\[ \nabla_a = E_a^m \nabla_m + E_a^\mu \nabla_\mu, \] (68)
when evaluated at \( \theta = 0 \) involves the worldsurface gravitino \( E_a^\mu(x) \) which is expressible in terms of the basic fields of the worldsurface tensor multiplet. Since it is fermionic it follows that the second term in the covariant derivative will be bilinear in fermions (at least), and we shall henceforth drop all such terms from the equations in order to simplify life a little. We shall temporarily make a further simplification by assuming that the target space is flat. The tensor \( Z \), as we saw earlier, has two types of contribution, one \( (T_a) \) involving \( H_{abcd} \), and the other \( (K_a) \) involving only terms which are bilinear or higher order in fermions. In accordance with our philosophy we shall henceforth ignore these terms.

To this order the right-hand side of the scalar equation vanishes as does the right-hand side of the tensor equation \((25)\). Multiplying the scalar equation \((67)\) with \( E_b^c \), we see that it can be written in the form
\[ \eta^{ab}(\nabla_a E_b^c - K_{ab}^c E_c^c) = 0, \] (69)
where \( K_{ab}^c \) is defined below. Using the relation \( E_b^c = m_b^d u_d^c \) and the definition of \( X_{ab}^c \) in \((28)\) we find that
\[ K_{ab}^c := \nabla_a E_b^d E_d^c = (\nabla_a m_b^d)(m^{-1})_d^c + X_{ab}^c. \] (70)
Using the relation
\[ \eta^{ab} \hat{\nabla}_a E_b^c = 0, \] (71)
which we will prove later, we conclude that \( \eta^{ab} K_{ab}^c = \eta^{ab} X_{ab}^c \). As a result, we can express the scalar equation of motion in the form
\[ \eta^{ab} \hat{\nabla}_a E_b^c = 0. \] (72)
where \( \hat{\nabla}_a E_b^c = \nabla_a E_b^c - X_{ab}^d E_d^c \). The relation \((71)\) allows us to rewrite the scalar equation of motion in the form
\[ m^{ab} \hat{\nabla}_a E_b^c = 0. \] (73)

The next step is to find an explicit expression for the spin connection \( \hat{\omega}_{ab}^c \) associated with the hatted derivative. Using the definition of \( X_{ab}^d \) given in \((28)\), we find that this spin connection is given by
\[ \hat{\omega}_{ab}^c = \Omega_{ab}^c + X_{ab}^c = E_a^m (\partial_m u_b^c) u_c^c. \] (74)
Recalling \((50)\) and \((46)\), we find that the hatted spin connection takes the form
\[ \hat{\omega}_{ab}^c = m_a^j e_j^a (\partial_n e_b^m g_{mp} e^{cp} + e_b^m \partial_n E_m^d \epsilon_{pd} e^{cp}). \] (75)
From this expression it is straightforward to derive the following result; given any vector \( V_m \) one has
\[ \hat{\nabla}_a V_b = m_a^d e_d^a e_b^m \nabla_n V_m \] (76)
where
\[ \nabla_n V_m = \partial_n V_m - \Gamma_{nm}^p V_p \] (77)
and
\[ \Gamma_{nm}^p = \partial_n E_m^c \epsilon_{cp} g^{sp}. \] (78)
It is straightforward to verify that to the order to which we are working this connection is indeed the Levi-Civita connection for the induced metric $g_{mn}$.

We are now in a position to express the scalar equation in its simplest form which is in a coordinate basis using the hatted connection. Using the above result we find that (73) can be written as

$$G^{mn} \nabla_m E_n = 0. \quad (79)$$

It remains to prove (74). Using the expression for $m_a^b$ given in (14) we find that

$$\eta^{ab} \hat{\nabla}_a m_b^c = -2 \hat{\nabla}^b (h_{bcd} h^{cde})$$

$$= -2 h_{bde} \hat{\nabla}^b h^{cde} = -\frac{2}{3} h_{bde} \hat{\nabla}^c h_{bde} = -\frac{1}{3} \hat{\nabla}^c (h_{bde} h^{bde}) = 0. \quad (80)$$

In carrying out the above steps we have used the $\eta_{abc}$ equation of motion and the self-duality of this field.

In the case of a non-flat target space the derivation is quite a bit longer and the steps will be discussed elsewhere. One finds that the right hand side of the scalar equation in the form of (67) is given by

$$\eta^{ab} \nabla_a E_b^a E_c^c = -\frac{1}{144} \left(1 - \frac{2}{3} \text{tr} k^2\right) \epsilon^{c'd'e'd'}\epsilon_{c'd'e'd'} H_{c'd'e'd'} + \frac{2}{3} m_a^b H_{bcd} h^{acd}, \quad (81)$$

where

$$k_{ab} := h_{acd} h^{bcd}. \quad (82)$$

Using the steps given above this result can be expressed in the form

$$G^{mn} \nabla_m E_n = \frac{1}{\sqrt{-g}} \left(1 - \frac{2}{3} \text{tr} k^2\right) \epsilon^{m_1 \cdots m_6} \left( \frac{1}{6^2 4.5} H_{a_{m_1 \cdots m_6}}^a + \frac{2}{3} m_a^b H_{bcd} h^{acd} \right), \quad (83)$$

where the target space indices on $H_4$ and $H_7$ have been converted to worldvolume indices with factors of $E_7$ and

$$H_{a_{\xi_1 \cdots \xi_7}} = \frac{1}{7!} \epsilon_{a_{\xi_1 \cdots \xi_7}} H_{\xi_1 \cdots \xi_7}. \quad (84)$$

where $H_7$ is the seven form that occurs in the dual formulation of eleven dimensional supergravity. One can verify that the ratio between the two terms on the right hand side is precisely what one expects were this term to have been derived from the expected gauge invariant Wess-Zumino term of the form $C_6 + 4 C_3 \wedge H_3$. We also note that the last factor in (83) implies that the RHS of the equation vanishes identically when multiplied with $E_7^q$, as it should, indicating that only five of the eleven equations, which correspond to the Goldstone scalars, are independent.

### 3.4 The Tensor Equation

The tensor equation can be manipulated in a similar fashion. If we consider the simplest case of ignoring the fermion bilinears and assuming the target space to be flat we have, from (34)

$$\eta^{ab} \hat{\nabla}_a h_{bcd} = 0. \quad (85)$$

We can relate $h$ to $H$ using (53) and take the factor of $m$ past the covariant derivative using (74) to get

$$m^{ab} \hat{\nabla}_a (e_b^m e_c^n e_c^p H_{mnp}) = 0. \quad (86)$$
Using similar steps to those given in the proved in the previous subsection and converting to a coordinate basis we find the desired form of the tensor equation in this approximation, namely

\[ G^{mn} \nabla_m H_{npq} = 0. \]  

(87)

In the case of a non-trivial target space a lengthy calculation is required to find the analogous result. One first finds that

\[ \hat{\nabla}^c h_{abc} = \frac{1}{288} m_c^f m_b^g \epsilon_{fg e_1 e_2 e_3 e_4} H^{e_1 e_2 e_3 e_4} - \frac{1}{12} \epsilon_{abcdef e_1 e_2 e_3} m_f^d H^{e_1 e_2 e_3} + 6 h^{e_1 e_2} [a h_{bc}] e^3 m_{c c}^1 H_{e_1 e_2 e_3} + \frac{4}{3} h_{abc} h^{e_1 e_2 e_3} m_{c c}^1 H_{e_1 e_2 e_3} \equiv Y_{ab}. \]  

(88)

It is possible to rewrite \( Y_{ab} \) in the form

\[ Y_{ab} = (\tilde{K} + m \tilde{K} + \frac{1}{4} m m \tilde{K})_{ab} \]  

(89)

where \( \tilde{K}_{ab} = -\frac{1}{360} \epsilon_{abcdef} H^{cde f}, \) \((m \tilde{K})_{ab} = m^c_{[a} \tilde{K}^b_{c]}, \) \((m m \tilde{K})_{ab} = m^c_{[a} m^d_{b]} \tilde{K}^c_{d]. \) The scalar equation of motion can also be expressed in the form

\[ G^{mn} \nabla_m H_{npq} = \frac{1}{\left(1 - \frac{4}{3} \text{tr} k^2\right)} v^a_p E^b_q (4Y + 4mY + mmY)_{ab}, \]  

(90)

where \( mY \) and \( mmY \) are defined in a similar way to the \( m \tilde{K} \) and \( mm \tilde{K} \) terms above.

### 3.5 The \( \kappa \)-Symmetry Transformations

The \( \kappa \)-symmetry transformations are related to odd worldsurface diffeomorphisms. Under an infinitesimal worldsurface diffeomorphism \( \delta z^M = -v^M \) the variation of the embedding expressed in a preferred frame basis is

\[ \delta z^A \equiv \delta z^M E^A_M = v^A E^A_A. \]  

(91)

For an odd transformation \( (v^a = 0) \) one has

\[ \delta z^a = 0, \]  

\[ \delta z^\alpha = v^\alpha E^\alpha_\alpha. \]  

(92)

The vanishing of the even variation \( \delta z^a \) is typical of \( \kappa \)-symmetry and follows from the basic embedding condition (4).

The relation between the parameter \( v^a \) and the familiar \( \kappa \) transformation parameter \( \kappa^\alpha \) can be expressed as

\[ v^a = \kappa^\alpha E^\alpha_\alpha. \]  

(93)

Therefore, recalling (54), the \( \kappa \) transformation rule (92) takes the form

\[ \delta z^\alpha = \kappa^\alpha (1 + \Gamma)^\alpha_\alpha, \]  

(94)

where we have absorbed a factor of two into the definition of \( \kappa. \) It is understood that these formulae are to be evaluated at \( \theta = 0, \) so that they are component results.

There remains the determination of the \( \kappa \)-symmetry transformation of the antisymmetric tensor field \( B_{mn}. \) It is more convenient to compute the \( \kappa \) transformations rule for the field \( h_{abc}(x). \) (The relation between the two fields is described earlier.) Thus we need to consider

\[ \delta h_{abc} = \kappa^\alpha E^\alpha_\alpha \nabla_\alpha h_{abc} \text{ at } \theta = 0. \]  

(95)
By including a Lorentz transformation we may write this transformation as

\[ \delta h_{abc} = \kappa \tilde{E}_\alpha \hat{\nabla}_\alpha h_{abc}. \]  

(96)

We have calculated \( \hat{\nabla}_\alpha h_{abc} \), and the derivation of the result will be given elsewhere \[18\]. Using this result, we find

\[ \delta h_{abc} = -\frac{i}{16} m[a] d \mathcal{E}_d (1 - \Gamma) \Gamma_{[bc]} \kappa, \]

(97)

where \( \Gamma_a = \Gamma^m e_{ma} \) and the target space spinor indices are suppressed. One can check that the RHS is self-dual, modulo the Dirac equation (66).

### 4 Double Dimensional Reduction

The procedure we shall adopt now is to use double-dimensional reduction \[13\] to obtain a set of equations for a 4-brane in ten dimensions and then to compare this set of equations with the equations that one derives by varying the Born-Infeld action. We shall take the target space to be flat and we shall ignore the terms bilinear in fermions on the right-hand-side of (24) and (25), that is we drop the terms in these equations that involve the quantity \( Z \) defined by (26) and we also ignore terms involving the worldsurface gravitino. From the previous section, we read off the resulting equations of motion:

\[ G^{mn} \nabla_m \mathcal{E}_n = 0, \]

(98)

\[ G^{mn} \nabla_m H_{npq} = 0. \]

(99)

We can further simplify matters by considering the corresponding bosonic problem, i.e. by neglecting \( \theta \) as well. In this limit, and recalling that we have assumed that the target space is flat, one has

\[ \mathcal{E}_m = \partial_m x. \]

(100)

In order to carry out the dimensional reduction we shall, in this section, distinguish 6 and 11 dimensional indices from 5 and 10 dimensional indices by putting hats on the former. We have

\[ x^\hat{m} = (x^m, y) \]

(101)

and

\[ x^{\hat{m}} = (x^m, y) \]

(102)

so that the sixth dimension of the worldsheet is identified with the eleventh dimension of the target space; moreover, this common dimension is taken to be a circle, and the reduction is effected by evaluating the equations of motion at \( y = 0 \). The metric is diagonal:

\[ g_{\hat{m} \hat{n}} = (\delta_{mn}, 1), \]

(103)

and the sechsbein can be chosen diagonal as well:

\[ e_{\hat{m}}^a = (e_m^a, 1), \]

(104)

where both the five-dimensional metric and its associated fünfbein are independent of \( y \). Since the fields do not depend on \( y \), and since the connection has non-vanishing components only if all of its indices are five-dimensional, the equations of motion reduce to

\[ G^{mn} \nabla_m \partial_n x = 0 \]

(105)

\[ G^{mn} \nabla_m F_{np} = 0, \]

(106)
where

\[ F_{mn} = H_{mny}. \tag{107} \]

Since \( h \) in six-dimensions is self-dual, and since \( H \) is related to \( h \) it follows that we only need to consider the \( py \) component of the tensor equation. It will be convenient to rewrite these equations in an orthonormal basis with respect to the five-dimensional metric; this basis is related to the coordinate basis by the \( \text{f"unftbein} \). Using \( a, b, \) etc. to denote orthonormal indices, the equations of motion become

\[
\begin{align*}
G^{ab} \nabla_a \partial_b x^a &= 0 \tag{108} \\
G^{ab} \nabla_a F_{bc} &= 0, \tag{109}
\end{align*}
\]

where

\[ G^{ab} = (\hat{m}^2)^{ab}, \tag{110} \]

and where we have introduced a hat for the six-dimensional \( m \)-matrix for later convenience.

The claim is that these equations are equivalent to the equations of motion arising from the five-dimensional Born-Infeld Lagrangian,

\[ L = \sqrt{-\det K} \tag{111} \]

where

\[ K_{mn} = g_{mn} + F_{mn}, \tag{112} \]

\( g_{mn} \) being the induced metric. To prove this we first show that the Born-Infeld equations can be written in the form

\[
\begin{align*}
L^{mn} \nabla_m \partial_n x^a &= 0 \tag{113} \\
L^{mn} \nabla_m F_{np} &= 0, \tag{114}
\end{align*}
\]

where

\[ L = (1 - F^2)^{-1}. \tag{115} \]

When matrix notation is used, as in the last equation, it is understood that the first index is down and the second up, and \( F^2 \) indicates that the indices are in the right order for matrix multiplication. \( L^{mn} \) is then obtained by raising the first index with the inverse metric as usual. To complete the proof we shall then show that \( G \) is proportional to \( L \) up to a scale factor.

The matrix \( K \) is \( 1 + F \) so that its inverse is

\[ K^{-1} = (1 + F)^{-1} = (1 - F)L, \tag{116} \]

from which we find

\[
(K^{-1})^{mn} = L^{mn} \quad (K^{-1})^{[mn]} = -(FL)^{mn}, \tag{117}
\]

the right-hand side of the second equation being automatically antisymmetric. Varying the Born-Infeld Lagrangian with respect to the gauge field \( A_m, \ (F = dA) \), gives

\[
\partial_n (\sqrt{-\det K} (K^{-1})^{[mn]}) = 0. \tag{118}
\]

Carrying out the differentiation of the determinant, switching to covariant derivatives, and using the Bianchi identity for \( F \), one finds

\[
\nabla_n (K^{-1})^{[mn]} + (K^{-1})^{[pq]} \nabla_p F_{qn} (K^{-1})^{[mn]} = 0. \tag{119}
\]
Using the identity
\[(K^{-1})^{[mn]}F_{np} = \delta^m_p L^m_p\] (120)
and the expression for \((K^{-1})^{[mn]}\) in terms of \(L\) and \(F\) one derives from (119)
\[L_n^q \nabla_q (F_{mp} L^m_p) + F^{pm} L_n^q \nabla_q L^m_p = 0.\] (121)
On differentiating the product in this equation one finds that the two terms with derivatives of \(L\) vanish by symmetry. Multiplying the remaining term by \((L^{-1})^m_r\) then yields the claimed result, namely (114). A similar calculation is used to derive (113).

To complete the proof we need to show that \(G_{mn}\) is proportional to \(L_{mn}\). We begin by setting
\[f_{ab} = h_{ab5},\] (122)
\[F_{ab} = e^m_a e^n_b F_{mn}.\] (123)
We then find
\[h_{abc} = \frac{1}{2} \epsilon_{abcde} f^{de},\] (124)
\[F_{ab} = (m^{-1})^c_a f_{cb},\] (125)
where \(m^b_a = \hat{m}^b_a b\). The first equation follows from the self-duality of \(h_{abc}\), while the second equation follows from (53), (104), (107) and (122).

We set
\[\hat{m}^a_b = (\hat{m}^a_b, \hat{m}^5_a, \hat{m}^5_b, \hat{m}^5_5)\] (126)
\[= (m^b_a, M_a, M^b, N).\] (127)
Recalling that
\[\hat{m}^a_b = (1 - 2h^2)_{\hat{a}b}\] (128)
one finds
\[m^a_b = \delta^a_b (1 - 2t_1) + 8(f^2)_{ab}^b\] (129)
\[M_a = -\epsilon_{abcde} f^{bc} f^{de}\] (130)
\[N = (1 + 2t_1),\] (131)
where \(t_1 = \text{tr}(f^2)\). Noting that \(f^a_b M_b = 0\), as can be seen by symmetry arguments, it follows from (123) that
\[F_{ab} M_b = 0.\] (132)
Now, by a direct calculation, starting from (110) one finds that
\[G_{ab} = A \eta_{ab} + 16(f^2)_{ab},\] (133)
where
\[A = 1 - 4t_1 - 4(t_1)^2 + 16t_2,\] (134)
and we have defined \(t_2 = \text{tr}(f^4)\). Now, multiplying \(G_{ab} = (m^2)_{ab} + M_a M_b\) with \((F^2)^{bc}\), and recalling (125), one finds
\[GF^2 = f^2.\] (135)
Using this relation in (133) we find

\[ G = A(1 - 16F^2)^{-1}. \]  

Therefore we have shown that (after a suitable rescaling of \( F \)), \( G \) is proportional to \( L \) and hence the equations of motion arising from the superspace formulation of the 5-brane, when reduced to a 4-brane in ten dimensions, coincide with those that one derives from the Born-Infeld Lagrangian.

5 Conclusions

The component form of the equations of motion for the 5-brane in eleven-dimensions are derived from the superspace equations. They are formulated in terms of the worldsurface fields \( x^\mu, \theta^\mu, B_{mn} \). These equations are fully covariant in six-dimensions; they possess six dimensional Lorentz invariance, reparametrization invariance, spacetime supersymmetry and \( \kappa \) symmetry. We have also derived the \( \kappa \) transformations of the component fields. The fivebrane equations are derived from the superspace embedding condition for p-branes which possess half the supersymmetry found previously \([1]\) and used to find superspace equations for the 5-brane in eleven-dimensions in \([2]\). In the superembedding approach advocated here, the \( \kappa \)-symmetry is nothing but the odd diffeomorphisms of the worldsurface and as such invariance of the equations of motion under \( \kappa \)-symmetry is guaranteed.

We have also carried out a double dimensional reduction to obtain the 4-brane in ten dimensions. We find agreement with the known Born-Infeld formulation for this latter theory. The result in ten dimensions which emerges from eleven dimensions appears in an unexpected form and that generalises the Born-Infeld structure to incorporate the worldsurface chiral 2-form gauge field.

In a recent paper \([7]\) it was suggested that it was impossible to find a covariant set of equations of motion for a self dual second rank tensor in six dimensions. However, in this paper we have presented just such a system whose internal consistency is ensured by the manner of its derivation. We would note that although the field \( h_{abc} \) which emerges from the superspace formalism obeys a simple duality condition, the field strength \( H_{mnp} \) of the gauge field inherits a version of this duality condition which is rather complicated. Using the solution of the chirality constraint on the 2-form, we expect that our bosonic equations of motion will reduce to those of \([7, 8]\).

In reference \([9]\), an auxiliary field has been introduced to write down a 6D covariant action. It would be interesting to find if this field is contained in the formalism considered in this paper. We note, however, that in the approach of reference \([9]\) one replaces the nonmanifest Lorentz symmetry with another bosonic symmetry that is equally nonmanifest, but necessary to eliminate the unwanted auxiliary field and that the proof the new symmetry involves steps similar to those needed to prove the nonmanifest Lorentz symmetry \([9]\). Further, it is not clear if a 6D covariant gauge fixing procedure is possible to gauge fix this extra symmetry.

In a forthcoming publication, we shall give in more detail the component field equation and the double dimensional reduction \([18]\). We also hope to perform a generalized dimensional reduction procedure to the worldsurface, but staying in eleven dimensions. In the approach of this paper, there is little conceptual difference in whether the worldsurface multiplet is a scalar multiplet (Type I branes), or vector multiplets (D branes), or indeed tensor multiplets (M branes) and we hope to report on the construction of all p-brane solutions from this viewpoint.
We conclude by mentioning some open problems that are natural to consider, given the fact that we now know the 6D covariant field equations of the M theory five-brane. It would be interesting to consider solitonic p-brane solutions of these equations, perform a semiclassical quantization, explore the spectral and duality properties of our system and study the anomalies of the chiral system. Finally, given the luxury of having manifest worldsurface and target space supersymmetries at the same time, it would be instructive to consider a variety of gauge choices, such as a static gauge, as was done recently for super D-branes [14], which would teach us novel and interesting ways to realize supersymmetry nonlinearly. This may provide useful tools in the search for the “different corners of M theory”.

Note Added

While this paper was in the final stages of being written up, we saw two related papers appear on the net [20, 21]. We hope to comment on the relationship between these papers and the work presented in a subsequent publication.
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