The stationary Weyl equation and Cosserat elasticity

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Received 15 April 2010, in final form 28 June 2010
Published 19 July 2010
Online at stacks.iop.org/JPhysA/43/335203

Abstract
The paper deals with the Weyl equation which is the massless Dirac equation. We study the Weyl equation in the stationary setting, i.e. when the spinor field oscillates harmonically in time. We suggest a new geometric interpretation of the stationary Weyl equation. We think of our three-dimensional space as an elastic continuum and assume that material points of this continuum can experience no displacements, only rotations. This framework is a special case of the Cosserat theory of elasticity. The rotations of material points of the space continuum are described mathematically by attaching to each geometric point an orthonormal basis which gives a field of orthonormal bases called the coframe. As the dynamical variables (unknowns) of our theory, we choose the coframe and a density. We choose a particular potential energy which is conformally invariant and then incorporate time into our action in the standard Newtonian way, by subtracting kinetic energy. The main result of our paper is the theorem stating that in the stationary setting our model is equivalent to a pair of Weyl equations.

PACS numbers: 11.10.Lm, 46.05.+b, 14.60.Lm

1. Introduction

Throughout this paper we work on a 3-manifold $M$ equipped with the local coordinates $x^\alpha$, $\alpha = 1, 2, 3$, and prescribed positive metric $g_{\alpha\beta}$ which does not depend on time. We extend the Riemannian 3-manifold $M$ to a Lorentzian 4-manifold $\mathbb{R} \times M$ by adding the time coordinate $x^0 \in \mathbb{R}$. The metric on $\mathbb{R} \times M$ is defined as

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix}. \quad (1)$$
Here and further on we use bold type for extended quantities. Say, the use of bold type in the tensor indices \( \alpha, \beta \) appearing on the lhs of formula (1) indicates that these run through the values 0, 1, 2, 3, whereas the use of normal type in the tensor indices \( \alpha, \beta \) appearing on the rhs of formula (1) indicates that these run through the values 1, 2, 3.

All constructions presented in the paper are local, so we do not make a priori assumptions on the geometric structure of \( \{ M, g \} \). Of course, as we are working on a 3-manifold, moving to a global setting is not that difficult: an orientable 3-manifold is parallelizable. However, in the current paper we will be switching between frames and spinors, the latter effectively being the ‘square root’ of the frame, see formulae (B.1), (B.3) and (B.4), and this leads to a separate problem of whether one can define a single-valued ‘square root’ of a given frame. This also gets us involved in the old debate on whether the sign of a spinor has a physical meaning; see, for example, section 19 in [1], section 3.5 in [2] or [3]. We do not want to deal with these issues in the current paper and, therefore, stick with the local analysis. We note only that a similar issue (extraction of a single-valued ‘square root’ of a tensor) arises in the mathematical theory of liquid crystals [4].

The Weyl equation is the massless Dirac equation. It is the accepted mathematical model for a massless neutrino field. The dynamical variable (unknown quantity) in the Weyl equation is a two-component complex-valued spinor field \( \xi \) which is a function of time \( x^0 \in \mathbb{R} \) and local coordinates \( x^a \) on \( M \). The explicit form of the Weyl equation is

\[
\mathbf{i}(\pm \sigma^0 \partial_0 + \sigma^a \nabla_a)\mathbf{\xi}^b = 0. \tag{2}
\]

Here \( \sigma \) are the Pauli matrices, \( \partial_0 \) is the time derivative and \( \nabla_a \) is the covariant spatial derivative; see appendix A for details. Summation in (2) is carried out over the tensor index \( a = 1, 2, 3 \) as well as over the spinor index \( b = 1, 2 \). The use of the partial derivative \( \partial_0 = \partial / \partial x^0 \) in equation (2) is justified because our time coordinate \( x^0 \) is fixed and we allow only changes of coordinates \((x^1, x^2, x^3)\) which do not depend on \( x^0 \).

We see that the Weyl equation (2) is a system of two \((\dot{a} = 1, 2)\) complex linear partial differential equations on the 4-manifold \( \mathbb{R} \times M \) for two complex unknowns \( \xi^b, b = 1, 2 \). The two choices of sign in (2) give two versions of the Weyl equation which differ by time reversal. Thus, we have a pair of Weyl equations.

We will be interested in the spinor fields of the form

\[
\xi(x^0, x^1, x^2, x^3) = \mathbf{e}^{-\mathbf{i}p_0 x^0} \eta(x^1, x^2, x^3), \tag{3}
\]

where

\[
p_0 \neq 0 \tag{4}
\]

is a real number. Substituting (3) into (2) we get the equation

\[
\pm p_0 \sigma^0 \partial_0 \eta^b + \sigma^a \nabla_a \eta^b = 0 \tag{5}
\]

which we call the stationary Weyl equation. The difference between equations (2) and (5) is that in equation (2) the spinor field \( \xi \) ‘lives’ on the Lorentzian 4-manifold \( \mathbb{R} \times M \), whereas in equation (5) the spinor field \( \eta \) ‘lives’ on the Riemannian 3-manifold \( M \).

The stationary Weyl equation (5) is the object of study of this paper. Note that the separation of the time variable \( x^0 \) has a clear physical meaning: the real number \( p_0 \) appearing in (3) and (5) is the quantum mechanical energy.

The aim of our paper is to show that the stationary Weyl equation (5) can be reformulated in an alternative (but mathematically equivalent) way using a different set of dynamical variables instead of a spinor field. Namely, we view our 3-manifold \( M \) as an elastic continuum whose material points can experience no displacements, only rotations, with rotations of different material points being totally independent. The idea of rotating material points may seem

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exotic; however, it has long been accepted in continuum mechanics within the Cosserat theory of elasticity [5]. An elastic continuum with no displacements, only rotations, is a limit case of Cosserat elasticity, the other limit case being classical elasticity with displacements only and no (micro)rotations.

The rotations of material points of the three-dimensional elastic continuum are described mathematically by attaching to each geometric point of the manifold $M$ an orthonormal basis, which gives a field of orthonormal bases called the frame or coframe, depending on whether one prefers dealing with vectors or covectors. Our model will be built on the basis of exterior calculus, so for us it will be more natural to use the coframe.

Our model is described in section 2. Subsequent sections contain mathematical analysis of this model culminating in theorem 5.1 which states that in the stationary setting our model is equivalent to a pair of Weyl equations (5).

Our analysis exhibits certain similarities with [6, 7] in that a three-dimensional (co)frame is used as a dynamical variable and that a second-order partial differential equation is reduced to a first-order equation.

2. Our model

The coframe $\vartheta$ is a triple of orthonormal covector fields $\vartheta^j$, $j = 1, 2, 3$, on the 3-manifold $M$. Each covector field $\vartheta^j$ can be written more explicitly as $\vartheta^j_\alpha$ where the tensor index $\alpha = 1, 2, 3$ enumerates the components. The orthonormality condition for the coframe can be represented as a single tensor identity:

$$ g = \delta_{jk} \vartheta^j \otimes \vartheta^k. \quad (6) $$

We view the identity (6) as a kinematic constraint: the metric $g$ is given (prescribed) and the coframe elements $\vartheta^j$ are chosen so that they satisfy (6), which leaves us with three real degrees of freedom at every point of $M$.

As dynamical variables we choose the coframe $\vartheta$ and a positive density $\rho$. Our coframe and density are functions of local coordinates $x^\alpha$ on $M$ as well as of time $x^0$.

We need to write down the potential energy of a deformed Cosserat continuum. The natural measure of deformations caused by the rotations of material points is the torsion tensor defined by the explicit formula

$$ T := \delta_{jk} \vartheta^j \wedge d\vartheta^k, \quad (7) $$

where $d$ denotes the exterior derivative. Here ‘torsion’ means ‘torsion of the teleparallel connection’ with ‘teleparallel connection’ defined by the condition that the covariant derivative of each coframe element $\vartheta^j$ is zero; see appendix A of [8] for a concise exposition.

Our construction of potential energy follows the logic of classical linear elasticity [9], the only difference being that instead of a rank 2 tensor (strain) we deal with a rank 3 tensor (torsion). The logic of classical linear elasticity dictates that we must first decompose our measure of deformation (torsion) into irreducible pieces, with irreducibility understood in terms of invariance under changes of local coordinates preserving the metric $g_{\alpha\beta}$ at a given point $P \in M$. It is known [10] that torsion has three irreducible pieces labelled by the adjectives axial, vector and tensor. (Vector torsion is sometimes called trace torsion.) The general formula for the potential energy of a linear elastic material contains squares of all irreducible pieces with some constant coefficients in front. We, however, choose to construct our potential energy using only one piece of torsion, namely the axial piece given by the explicit formula

$$ T^{\text{ax}} := \frac{1}{2} \delta_{jk} \vartheta^j \wedge d\vartheta^k. \quad (8) $$
Comparing (8) with (7) we see that axial torsion is the totally antisymmetric part of the torsion tensor (\( T \) is antisymmetric only in the last pair of indices whereas, \( T^{ax} \) is antisymmetric in all three). In other words, \( T^{ax} \) is a 3-form.

Note that axial torsion possesses the property of conformal covariance, i.e. scales nicely under conformal rescalings of the metric. Indeed, if we rescale our coframe as
\[ \vartheta^j \mapsto e^h \vartheta^j, \]
where \( h : M \to \mathbb{R} \) is an arbitrary scalar function, then our metric is scaled as
\[ g_{\alpha \beta} \mapsto e^{2h} g_{\alpha \beta}, \]
and axial torsion is scaled as
\[ T^{ax} \mapsto e^{2h} T^{ax} \]
without the derivatives of \( h \). The fact that axial torsion is conformally covariant was previously observed by Obukhov [11] and Nester [12].

We take the potential energy of our continuum to be
\[ P(x^0) := \int_M \| T^{ax} \|^2 \rho \, dx^1 \, dx^2 \, dx^3. \]
(12)

It is easy to see that the potential energy (12) is conformally invariant: it does not change if we rescale our coframe as (9) and our density as \( \rho \mapsto e^{2h} \rho \). This follows from formulae (11), (10) and \( \| T^{ax} \|^2 = \frac{1}{3!} T^{ax}_{\alpha \beta \gamma} T^{ax}_{\kappa \lambda \mu} g^{\alpha \kappa} g^{\beta \lambda} g^{\gamma \mu} \).

We take the kinetic energy of our continuum to be
\[ K(x^0) := \int_M \| D_0 \vartheta \|^2 \rho \, dx^1 \, dx^2 \, dx^3, \]
(13)

where \( D_0 \vartheta \) is the 2-form
\[ D_0 \vartheta := \frac{1}{2} \delta_{jk} \vartheta^j \wedge \partial_0 \vartheta^k, \]
(14)
(compare with (8)). The 2-form (14) can, of course, be written as
\[ D_0 \vartheta = \frac{1}{3} \ast \omega, \]
(15)
where
\[ \omega := \frac{1}{2} \ast (\delta_{jk} \vartheta^j \wedge \partial_0 \vartheta^k) \]
(16)
is the (pseudo)vector of angular velocity. Hence, (13) is the standard expression for the kinetic energy of a Cosserat continuum in the special case with no displacements. In writing formula (13) we assumed homogeneity (properties of the material are the same at all points of the manifold \( M \)) and isotropy (properties of the material are invariant under the rotations of the local coordinate system). We think of each material point as a uniform ball possessing a moment of inertia and without a preferred axis of rotation.

We now combine the potential energy (12) and kinetic energy (13) to form the action (variational functional) of our dynamic problem:
\[ S(\vartheta, \rho) := \int_\mathbb{R} (P(x^0) - K(x^0)) \, dx^0 = \int_{\mathbb{R} \times M} L(\vartheta, \rho) \, dx^0 \, dx^1 \, dx^2 \, dx^3, \]
(17)
where
\[ L(\vartheta, \rho) := (\| T^{ax} \|^2 - \| D_0 \vartheta \|^2) \rho \]
(18)
is our Lagrangian density. Note that our construction of the action (17) out of potential and kinetic energies is Newtonian (analogous to that in classical elasticity).
Our field equations (Euler–Lagrange equations) are obtained by varying the action (17) with respect to the coframe $\vartheta$ and density $\rho$. Varying with respect to the density $\rho$ is easy: this gives the field equation $\|T^a\|^2 - \|D_0 \vartheta\|^2 = 0$ which is equivalent to $L(\vartheta, \rho) = 0$. Varying with respect to the coframe $\vartheta$ is more difficult because we have to maintain the kinematic constraint (6); recall that the metric is assumed to be prescribed (fixed). A technique for varying the coframe with kinematic constraint (6) is described in appendix B of [8] but we do not use it in the current paper.

**Remark 2.1.** The 3-form $T^a$ and 2-form $D_0 \vartheta$ are invariant under the rigid rotations of the coframe, i.e. under special (det $O^{jk} = +1$) orthogonal transformations $\vartheta^j \mapsto \tilde{\vartheta}^j = O^{jk} \vartheta^k$ with constant $O^{jk}$. Hence, our Lagrangian density (18) is invariant under the rigid rotations of the coframe and, accordingly, solutions of our field equations whose coframes differ by rigid rotations can be collected into equivalence classes.

### 3. Switching to the language of spinors

As pointed out in the previous section, varying the coframe subject to the kinematic constraint (6) is not an easy task. This technical difficulty can be overcome by switching to a different dynamical variable. Namely, it is known, see appendix B, that in dimension 3 a coframe $\vartheta$ and a (positive) density $\rho$ are equivalent to a nonvanishing spinor field $\xi$ modulo the sign of $\xi$. The advantage of switching to a spinor field $\xi$ is that there are no kinematic constraints on its components, so the derivation of field equations becomes straightforward.

We now need to substitute formulae (B.1), (B.3) and (B.4) into (8) and (14) to get explicit expressions for $T^a$ and $D_0 \vartheta$ in terms of the spinor field $\xi$. The results are presented in appendix B. Formula (B.5) gives the spinor representation of the 3-form $T^a$, whereas formulae (B.6) and (15) give the spinor representation of the 2-form $D_0 \vartheta$. We also know the spinor representation for our density $\rho$; see formulae (B.1) and (B.2). Substituting all these into formula (18) we arrive at the following self-contained explicit spinor representation of our Lagrangian density:

$$L(\xi) = \frac{4}{9 \varepsilon^a_\sigma \varepsilon^b_\sigma} ([((\tilde{\vartheta}^a_\sigma \sigma^a_\sigma \vartheta^b_\sigma + \varepsilon^b_\sigma \sigma^a_\sigma \vartheta^a_\sigma)]^2 - \|i(\tilde{\vartheta}^a_\sigma \sigma_\sigma \vartheta^b_\sigma + \varepsilon^b_\sigma \sigma_\sigma \vartheta^a_\sigma\|)^2) \sqrt{\det g}. \quad (19)$$

Here and further on we write our Lagrangian density and our action as $L(\xi)$ and $S(\xi)$ rather than $L(\vartheta, \rho)$ and $S(\vartheta, \rho)$, thus indicating that we have switched to spinors. The nonvanishing spinor field $\xi$ is the new dynamical variable.

The field equation for our Lagrangian density (19) is

$$-\frac{4i}{3} ((\ast T^a) \sigma^a_\sigma \vartheta^b_\sigma \ast \vartheta^b_\sigma + \sigma^a_\sigma \vartheta^b_\sigma ((\ast T^a) \xi^b_\sigma)) - \frac{8i}{9} (\omega_\sigma \sigma^a_\sigma \vartheta^b_\sigma + \sigma^a_\sigma \vartheta^b_\sigma \vartheta^b_\sigma) - \rho^{-1} L \sigma_\sigma \vartheta^b_\sigma = 0, \quad (20)$$

where the quantities $\ast T^a$, $\omega$, $\rho$ and $L$ are expressed via the spinor field $\xi$ in accordance with formulae (B.5), (B.6), (B.1), (B.2) and (19). We shall refer to (20) as the **dynamic** field equation, with ‘dynamic’ indicating that it contains the time derivative $\partial_0$. 

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4. Separating out time

Our dynamic field equation (20) is highly nonlinear, and one does not expect it to admit separation of variables. Nevertheless, we seek solutions of the form (3). Substituting formula (3) into (B.5), (B.6), (B.1), (B.2) and (19) and using the identity (A.5) we get

\[ s^a T^a_{\alpha} = -\frac{2i}{3\bar{\eta}^\sigma \sigma_{a\beta} \eta^\beta} (\bar{\eta}^\sigma \sigma_{\alpha\beta} \eta^\beta - \eta^b \sigma_{a\beta} \nabla_\alpha \eta^b), \]

(21)

\[ \omega_\alpha = \frac{2p_0 \bar{\eta}^\sigma \sigma_{a\beta} \eta^b}{\bar{\eta}^\sigma \sigma_{a\beta} \eta^b}, \]

(22)

\[ \rho = \bar{\eta}^\sigma \sigma_{a\beta} \eta^b \sqrt{\det g}, \]

(23)

\[ L(\eta) = \frac{16}{9\bar{\eta}^\sigma \sigma_{a\beta} \eta^b} \left( \left[ \frac{i}{2} \bar{\eta}^\sigma \sigma_{a\beta} \nabla_\alpha \eta^b - \eta^b \sigma_{a\beta} \nabla_\alpha \bar{\eta}^\sigma \right]^2 - (p_0 \bar{\eta}^\sigma \sigma_{a\beta} \eta^b)^2 \right) \sqrt{\det g}. \]

(24)

Note that the quantities (21)–(24) do not depend on time \( x^0 \), which simplifies the next step: substituting (3) into our dynamic field equation (20), using the identity (A.5) and dividing through by the common factor \( e^{-ip_0 x^0} \), we get

\[ -4i \left( (s^a T^a_{\alpha} \sigma^a_{a\beta} \nabla_\alpha \eta^b + \sigma^a_{a\beta} \nabla_\alpha ((s^a T^a_{\alpha}) \eta^b)) - \frac{32p_0^2}{9} \sigma_{a\beta} \eta^b - \rho^{-1} L \sigma_{a\beta} \eta^b = 0. \]

(25)

Note that formulae (21)–(25) do not contain time \( x^0 \). Thus, we have shown that our dynamic field equation (20) admits separation of variables, i.e. one can seek solutions in the form (3). We shall refer to (25) as the stationary field equation, with ‘stationary’ indicating that time \( x^0 \) has been separated out.

Consider now the action

\[ S(\eta) := \int_M L(\eta) \, dx^1 \, dx^2 \, dx^3, \]

(26)

where \( L(\eta) \) is our stationary Lagrangian density (24). It is easy to see that our stationary field equation (25) is the Euler–Lagrange equation for our stationary action (26).

In the remainder of the paper we do not use the explicit form of the stationary field equation (25), dealing only with the stationary Lagrangian density (24) and the stationary action (26). We needed the explicit form of field equations, dynamic and stationary, only to justify separation of variables.

It appears that the underlying group-theoretic reason for our nonlinear dynamic field equation (20) admitting separation of variables is the fact that our model is \( U(1) \)-invariant, i.e. it is invariant under the multiplication of a spinor field by a complex constant of modulus 1. Hence, it is feasible that one could have performed the separation of variables argument without even writing down the explicit form of field equations.

We give for reference a more compact representation of our stationary Lagrangian density (24) in terms of the axial torsion \( T^a_{\alpha} \) (see formula (21)) and density \( \rho \) (see formula (23)):

\[ L(\eta) = \left( \| T^a_{\alpha} \|^2 - \frac{16}{9} p_0^2 \right) \rho. \]

(27)

Of course, formula (27) is our original formula (18) with time separated out. The choice of dynamical variables in the stationary Lagrangian density (27) is up to the user: one can use either the time-independent spinor field \( \eta \) or, equivalently, the corresponding time-independent coframe and time-independent density (the latter are related to \( \eta \) by formulae (B.1)–(B.4) with \( \xi \) replaced by \( \eta \)).

The fact that we use the same notation \( L \) for both the dynamic and stationary Lagrangian densities should not cause problems as in all subsequent sections, apart form section 7, we deal with the stationary case only.
5. Main result

The main result of our paper is as follows.

**Theorem 5.1.** A nonvanishing time-independent spinor field \( \eta \) is a solution of the field equation for our stationary Lagrangian density (27) if and only if it is a solution of one of the two stationary Weyl equations (5).

Theorem 5.1 provides an elementary, in terms of Newtonian mechanics, interpretation of the stationary Weyl equation. The only technical assumption contained in the statement of theorem 5.1 is that the density \( \rho \) does not vanish which is equivalent to the nonvanishing of the spinor field \( \eta \). We do not know how to drop this assumption. We can only remark that generically one would not expect a spinor field \( \eta \) ‘living’ on a 3-manifold to vanish as this would mean satisfying four real equations \( \text{Re} \, \eta^1 = \text{Im} \, \eta^1 = \text{Re} \, \eta^2 = \text{Im} \, \eta^2 = 0 \) having at our disposal only three real variables \( x^\alpha, \alpha = 1, 2, 3 \).

**Proof of theorem 5.1.** Put

\[
L_\pm(\eta) := \left[ \frac{i}{2} (\widehat{\eta}^a \sigma_{ab} \nabla_a \eta^b - \eta^a \sigma_{ab} \nabla_a \eta^b) \pm p_0 \eta^0 \sigma_{ab} \eta^b \right] \sqrt{\det g}. \tag{28}
\]

This is the Lagrangian density for the stationary Weyl equation (5). Formula (28) can be written in a more compact form as

\[
L_\pm(\eta) = \left( -\frac{3}{4} \ast T^{ax} \mp p_0 \right) \rho, \tag{29}
\]

where \( \ast T^{ax} \) is the Hodge dual of axial torsion, see formula (21), and \( \rho \) is the density, see formula (23). Comparing formulae (27) and (29) we get

\[
L(\eta) = -\frac{32}{9} p_0 \frac{L_+ (\eta) L_- (\eta)}{L_+ (\eta) - L_- (\eta)}. \tag{30}
\]

Recall that we assume that the density \( \rho \) does not vanish. In view of formulae (29) and (4) the assumption \( \rho \neq 0 \) can be equivalently rewritten as

\[
L_+ (\eta) \neq L_- (\eta), \tag{31}
\]

so the denominator in (30) is nonzero.

We will now show that the nonlinear second-order field equation corresponding to the Lagrangian density \( L(\eta) \) reduces to a pair of linear first-order field equations corresponding to the Lagrangian densities \( L_\pm (\eta) \). The argument presented below is of an abstract nature and does not depend on the physical nature of the dynamical variable \( \eta \), the only requirement being that it is an element of a vector space so that scaling makes sense. To our knowledge, our construction is a new mathematical result which does not fit into the standard schemes of the theory of integrable systems.

Observe that the Lagrangian densities \( L_\pm \) defined by formula (28) possess the property of scaling covariance:

\[
L_\pm (e^h \eta) = e^{2h} L_\pm (\eta), \tag{32}
\]

where \( h : M \rightarrow \mathbb{R} \) is an arbitrary scalar function. In fact, the Lagrangian density of any formally self-adjoint (symmetric) linear first-order partial differential operator has the scaling covariance property (32). We claim that the statement of theorem 5.1 follows from formulae (30) and (32).

Note that formulae (30) and (32) imply that the Lagrangian density \( L \) possesses the property of scaling covariance, so all three of our Lagrangian densities, \( L, L_+ \) and \( L_- \), have
this property. Note also that if \( \eta \) is a solution of the field equation for some Lagrangian density \( \mathcal{L} \) possessing the property of scaling covariance, then \( \mathcal{L}(\eta) = 0 \). Indeed, let us perform a scaling variation of our dynamical variable

\[
\eta \rightarrow \eta + h\eta,
\]

(33)

where \( h : M \rightarrow \mathbb{R} \) is an arbitrary ‘small’ scalar function with compact support. Then \( 0 = \delta \mathcal{L}(\eta) = 2 \int h \mathcal{L}(\eta) \) which holds for arbitrary \( h \) only if \( \mathcal{L}(\eta) = 0 \).

In the remainder of the proof the variations of \( \eta \) are arbitrary and not necessarily of the scaling type (33).

Suppose that \( \eta \) is a solution of the field equation for the Lagrangian density \( \mathcal{L}^+ \). (The case when \( \eta \) is a solution of the field equation for the Lagrangian density \( \mathcal{L}^- \) is handled similarly.)

Then \( \mathcal{L}^+(\eta) = 0 \) and, in view of formula (31), \( \mathcal{L}^-(\eta) \neq 0 \). Varying \( \eta \) we get

\[
\delta \int \mathcal{L}(\eta) = -\frac{32 p_0}{9} \left( \int \mathcal{L}^-(\eta) \delta \mathcal{L}^+ + \int \mathcal{L}^+(\eta) \delta \mathcal{L}^- \right) = \frac{32 p_0}{9} \int \delta \mathcal{L}^+(\eta) = \frac{32 p_0}{9} \delta \int \mathcal{L}^+(\eta)
\]

(34)

We assumed that \( \eta \) is a solution of the field equation for the Lagrangian density \( \mathcal{L}^+ \), so \( \delta \int \mathcal{L}^+(\eta) = 0 \) and formula (34) implies that \( \delta \int \mathcal{L}(\eta) = 0 \). As the latter is true for an arbitrary variation of \( \eta \), this means that \( \eta \) is a solution of the field equation for the Lagrangian density \( \mathcal{L} \).

Suppose that \( \eta \) is a solution of the field equation for the Lagrangian density \( \mathcal{L} \). Then \( \mathcal{L}^+(\eta) = 0 \) and formula (30) implies that either \( \mathcal{L}^+(\eta) = 0 \) or \( \mathcal{L}^-(\eta) = 0 \); note that in view of (31) we cannot have simultaneously \( \mathcal{L}^+(\eta) = 0 \) and \( \mathcal{L}^-(\eta) = 0 \). Assume for definiteness that \( \mathcal{L}^+(\eta) = 0 \). (The case when \( \mathcal{L}^-(\eta) = 0 \) is handled similarly.) Varying \( \eta \) and repeating the argument from the previous paragraph we arrive at (34). We assumed that \( \eta \) is a solution of the field equation for the Lagrangian density \( \mathcal{L} \), so \( \delta \int \mathcal{L}(\eta) = 0 \) and formula (34) implies that \( \delta \int \mathcal{L}(\eta) = 0 \). As the latter is true for an arbitrary variation of \( \eta \), this means that \( \eta \) is a solution of the field equation for the Lagrangian density \( \mathcal{L}^+ \).  

\[\square\]

6. Plane wave solutions

Suppose that \( M = \mathbb{R}^3 \) is Euclidean 3-space equipped with Cartesian coordinates \( x = (x^1, x^2, x^3) \) and standard Euclidean metric

\[
g_{\alpha\beta} = \text{diag}(1, 1, 1)
\]

(35)

In this section we construct a special class of explicit solutions of the field equations for our Lagrangian density (18). This construction is presented, initially, in the language of spinors.

Let us choose the traditional Pauli matrices

\[
\sigma_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2ab} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{3ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(36)

and seek the solutions of the form

\[
\xi(x^0, x^1, x^2, x^3) = e^{-i(p_0 x^0 + p \cdot x)} \zeta,
\]

(37)

where \( p_0 \) is a real number as in formulae (3) and (4), \( p = (p_1, p_2, p_3) \) is a real constant covector and \( \zeta \neq 0 \) is a constant spinor. We shall call solutions of the type (37) plane wave.
In seeking plane wave solutions what we are doing is separating out all the variables, namely the time variable $t$ and the spatial variables $x = (x^1, x^2, x^3)$.

Our dynamic field equation (20) is highly nonlinear, so it is not a priori clear that one can seek solutions in the form of plane waves. However, plane wave solutions are a special case of stationary solutions (3), and these have already been analysed in the preceding sections. We know that according to theorem 5.1 in the stationary case our model reduces to a pair of Weyl equations (2). Substituting formulae (A.2), (36) and (37) into equation (2) we get

$$
\left( \mp p_0 + p_3, p_1 - ip_2, p_1 + ip_2, \mp p_0 - p_3 \right) \left( \frac{\xi^1}{\xi^2} \right) = 0.
$$

The determinant of the matrix on the lhs of equation (38) is $p_0^2 - p_1^2 - p_2^2 - p_3^2$, so this system has a nontrivial solution $\xi$ if and only if $p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0$. Our model is invariant under the rotations of the Cartesian coordinate system (orthogonal transformations of the coordinate system preserving orientation), so without loss of generality we can assume that

$$
p_1 = p_2 = 0, \quad p_3 = \pm p_0,
$$

where the ± sign is chosen to agree with that in equation (38), i.e. upper sign in (39) corresponds to upper sign in (38) and same for lower signs. Substituting formulae (39) into equation (38) and recalling our assumption (4) we conclude that, up to scaling by a nonzero complex factor, we have

$$
\xi^d = \left( \frac{1}{0} \right).
$$

Combining formulae (37), (39) and (40) we conclude that for each real $p_0 \neq 0$ our model admits, up to a rotation of the coordinate system and complex scaling, two plane wave solutions and that these plane wave solutions are given by the explicit formula

$$
\xi^d = \left( \frac{1}{0} \right) e^{-ip_0(x^0 \pm x^3)}.
$$

Let us now rewrite the plane wave solutions (41) in terms of our original dynamical variables, coframe $\vartheta$ and density $\rho$. Substituting formulae (A.2), (36) and (41) into formulae (B.1)–(B.4) we get $\rho = 1$ and

$$
\vartheta^1_a = \begin{pmatrix} \cos 2p_0(x^0 \pm x^3) \\ -\sin 2p_0(x^0 \pm x^3) \\ 0 \end{pmatrix}, \quad \vartheta^2_a = \begin{pmatrix} -\sin 2p_0(x^0 \pm x^3) \\ \cos 2p_0(x^0 \pm x^3) \\ 0 \end{pmatrix}, \quad \vartheta^3_a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
$$

Note that scaling the spinor $\xi$ by a nonzero complex factor is equivalent to scaling the density $\rho$ by a positive real factor and time shift $x^0 \mapsto x^0 + \text{const}$.

We will now establish how many different (ones that cannot be continuously transformed into one another) plane wave solutions we have. To this end, we rewrite formula (42) in the form

$$
\vartheta^1_a = \begin{pmatrix} \cos 2|p_0|(x^0 + bx^3) \\ a \sin 2|p_0|(x^0 + bx^3) \\ 0 \end{pmatrix}, \quad \vartheta^2_a = \begin{pmatrix} -a \sin 2|p_0|(x^0 + bx^3) \\ \cos 2|p_0|(x^0 + bx^3) \\ 0 \end{pmatrix}, \quad \vartheta^3_a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
$$

where $a$ and $b$ can, independently, take values $\pm 1$. It may seem that we have a total of four different plane wave solutions. Recall, however (see remark 2.1), that we can perform rigid rotations of the coframe and that we have agreed to view coframes that differ by a rigid rotation as equivalent. Let us perform a rotation of the coordinate
system \((x^1 x^2 x^3) \mapsto (x^2 x^1 - x^3)\) simultaneously with a rotation of the coframe \((\theta^1 \theta^2 \theta^3) \mapsto (\theta^2 \theta^1 - \theta^3)\). It is easy to see that the above transformations turn a solution of the form (43) into a solution of this form again only with \(a \mapsto -a, b \mapsto -b\). Thus, the numbers \(a\) and \(b\) on their own do not characterize different plane wave solutions. Different plane wave solutions are characterized by the number \(c := ab\) which can take two values, \(+1\) and \(-1\).

We have established that for a given positive frequency \(|p_0|\), we have two essentially different types of plane wave solutions. These can be written, for example, as

\[
\vartheta^1_{\alpha} = \begin{pmatrix} \cos 2|p_0|(x^0 + x^3) \\ \pm \sin 2|p_0|(x^0 + x^3) \\ 0 \end{pmatrix}, \quad \vartheta^2_{\alpha} = \begin{pmatrix} \mp \sin 2|p_0|(x^0 + x^3) \\ \cos 2|p_0|(x^0 + x^3) \\ 0 \end{pmatrix}, \quad \vartheta^3_{\alpha} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{44}
\]

The plane wave solutions (44) describe travelling waves of rotations. Both waves travel with the same velocity (speed of light) in the negative \(x^3\)-direction. The difference between the two solutions is in the direction of rotation of the coframe: if we fix the spatial coordinate \(x^3\) and look at the evolution of (44) as a function of time \(x^0\) or if we fix time \(x^0\) and look at the evolution of (44) as a function of the spatial coordinate \(x^3\), then one solution describes a clockwise rotation, whereas the other solution describes an anticlockwise rotation. We identify one of the solutions (44) with a left-handed massless neutrino and the other with a right-handed massless antineutrino.

7. Relativistic representation of our Lagrangian

In this section we work on the 4-manifold \(\mathbb{R} \times M\) equipped with Lorentzian metric (1). This manifold is an extension of the original 3-manifold \(M\). We use bold type for extended quantities.

We extend our coframe as

\[
\vartheta^0 = \begin{pmatrix} 1 \\ 0_a \end{pmatrix}, \quad \vartheta^j = \begin{pmatrix} 0 \\ \vartheta^j_a \end{pmatrix}, \quad j = 1, 2, 3, \tag{45}
\]

where the bold tensor index \(\alpha\) runs through the values 0, 1, 2, 3, whereas its non-bold counterpart \(\alpha\) runs through the values 1, 2, 3. In particular, \(0_a\) in formula (45) stands for a column of three zeros.

Throughout this section our original three-dimensional coframe \(\vartheta\) is allowed to depend on time \(x^0\) in an arbitrary (not necessarily harmonic) manner, as long as the kinematic constraint (6) is maintained. Thus, our only restriction on the choice of the extended four-dimensional coframe \(\vartheta\) is formula (45) which states that the zeroth element of the coframe is prescribed as the conormal to the original Riemannian 3-manifold \(M\).

The extended metric (1) is expressed via the extended coframe (45) as

\[
g = o_{jk} \vartheta^j \otimes \vartheta^k, \tag{46}
\]

where \(o_{jk} = o^{jk} := \text{diag}(-1, +1, +1, +1)\) (compare with formula (6)). The extended axial torsion is

\[
T^{ax} := \frac{1}{3} o_{jk} \vartheta^j \wedge d \vartheta^k = \frac{1}{3} (-\vartheta^0 \wedge d \vartheta^0 + \vartheta^1 \wedge d \vartheta^1 + \vartheta^2 \wedge d \vartheta^2 + \vartheta^3 \wedge d \vartheta^3), \tag{47}
\]

where \(d\) denotes the exterior derivative on \(\mathbb{R} \times M\) (compare with formula (8)). Formula (47) can be rewritten as

\[
T^{ax} = T^{ax} - \vartheta^0 \wedge D_0 \vartheta \tag{48}
\]
with $T^{\alpha \beta}$ and $D_t \theta$ defined by formulae (8) and (14) respectively. Squaring (48) we get 
\[ \|T^{\alpha \beta}\|^2 = \|T^{\alpha \beta}\|^2 - \|D_t \theta\|^2 \] which means that our Lagrangian density (18) can be rewritten as 
\[ L(\theta, \rho) = \|T^{\alpha \beta}\|^2 \rho. \] (49)

The point of the arguments presented in this section was to show that if one adopts the relativistic point of view, then our Lagrangian density (18) takes the especially simple form (49). A consistent pursuit of the relativistic approach would require the variation of all four elements of the extended coframe, giving three extra dynamical degrees of freedom (Lorentz boosts in three directions). We do not do this in the current paper. In the long run we hope to analyse the relativistic version of our model and also perform a comprehensive comparison with the established model of relativistic elasticity (sometimes called ‘relasticity’) as described in [13–15].

Acknowledgments

The authors are grateful to C Böhmer, J Burnett, E V Ferapontov, R Halburd, F W Hehl, F E A Johnson and Yu N Obukhov for stimulating discussions and to the referees of this paper for constructive suggestions.

Appendix A. Notation

Our notation mostly follows [8, 16, 17], the only major difference being that we changed the signature of Lorentzian metric $g_{\alpha \beta}$ from $+---$ to $-++$. The latter is more natural when promoting the Newtonian continuum mechanics approach.

We use Greek letters for tensor (holonomic) indices and Latin letters for frame (anholonomic) indices.

We identify differential forms with covariant antisymmetric tensors. Given a pair of real covariant antisymmetric tensors $P$ and $Q$ of rank $r$, we define their dot product as 
\[ P \cdot Q := \frac{1}{r!} P_{\alpha_1 \cdots \alpha_r} Q_{\beta_1 \cdots \beta_r} g^{\alpha_1 \beta_1} \cdots g^{\alpha_r \beta_r}. \] We also define 
\[ \|P\|^2 := P \cdot P. \]

All our constructions are local and occur in a neighbourhood of a given point $P$ of the 3-manifold $M$. We allow only changes of the local coordinates $x^\alpha$, $\alpha = 1, 2, 3$, which preserve orientation.

Working in local coordinates with specified orientation allows us to define the Hodge star: we define the action of $\ast$ on a rank $r$ antisymmetric tensor $R$ as 
\[ (\ast R)_{\alpha_1 \cdots \alpha_r} := (r!)^{-1} \det g R^{\alpha_1 \cdots \alpha_r} \varepsilon_{\alpha_1 \cdots \alpha_r}, \] where $\varepsilon$ is the totally antisymmetric quantity, $\varepsilon_{123} := +1$.

The coframes $\vartheta$ fall into two separate categories, depending on the sign of $\det \vartheta^j_a$. We choose to work with the coframes satisfying the condition
\[ \det \vartheta^j_a > 0. \] (A.1)

This condition means that orientation encoded in our coframe agrees with that encoded in our coordinate system.

We use two-component complex-valued spinors (Weyl spinors) whose indices run through the values 1, 2 or 1, 2. Complex conjugation makes the undotted indices dotted and vice versa.

We define the ‘metric spinor’ $\epsilon_{ab} = \epsilon_{\dot{a} \dot{b}} = \epsilon^{\dot{a} \dot{b}} = \epsilon_{\dot{a} \dot{b}} = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)$ with the first index enumerating rows and the second enumerating columns; we will be using this spinor for lowering and raising spinor indices. We define 
\[ \sigma_{a\dot{b}} = \sigma_{\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{0}_{a\dot{b}} = \sigma^{\dot{a}b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (A.2)
The spinor (A.2) can also be used for raising and lowering spinor indices. This is a feature of the nonrelativistic setting, when we have a specified time coordinate $x^0$ and the transformations of spatial local coordinates $x^a, a = 1, 2, 3$, do not involve time.

Let $\nu$ be the real vector space of trace-free Hermitian $2 \times 2$ matrices $\sigma_{ab}$. Pauli matrices $\sigma_{a\alpha b}, \alpha = 1, 2, 3$, are a basis in $\nu$ satisfying

$$\sigma_{a\alpha b} \sigma_{b\beta c} + \sigma_{b\beta a} \sigma_{a\alpha c} = -2 g_{a\beta} \delta_\alpha^c,$$

(A.3)

where $\sigma_{\beta c} := \epsilon^{abc} \sigma_{\beta e} \epsilon^{ed}$.

Note that formula (A.3) automatically implies an analogous formula for the extended metric (1):

$$\sigma_{a\alpha b} \sigma_{b\beta c} + \sigma_{b\beta a} \sigma_{a\alpha c} = -2 g_{a\beta} \delta_\alpha^c,$$

(A.4)

where the bold tensor indices $\alpha, \beta$ run through the values 0, 1, 2, 3.

Of course, our Pauli matrices $\sigma_{\alpha}, \alpha = 1, 2, 3$, are not uniquely defined: if $\sigma_{\alpha} = \sigma_{a\alpha b}$ are the Pauli matrices, then so are the matrices $U^* \sigma_{\alpha} U$ where $U$ is an arbitrary special ($\det U = 1$) unitary matrix-function. Note also that under coordinate transformations our Pauli matrices $\sigma_{a\alpha b}$ transform as components of a covector; this is indicated by the Greek subscript $\alpha$.

Let us mention a useful identity for Pauli matrices, very similar to (A.9) in [17]. This is because we changed the formula for the extended metric (1):

$$\sigma_{a\alpha b} \sigma_{b\beta c} + \sigma_{b\beta a} \sigma_{a\alpha c} = -2 g_{a\beta} \delta_\alpha^c,$$

(A.4)

where the bold tensor indices $\alpha, \beta$ run through the values 0, 1, 2, 3.

For $\vec{\omega}$ the real vector space of trace-free Hermitian $2 \times 2$ matrices $\sigma_{ab}$ the nonrelativistic setting, when we have a specified time coordinate $x^0$, the above formulae are a special case of those from [18].

In dimension 3 a coframe $\vartheta$ and a (positive) density $\rho$ are equivalent to a nonvanishing spinor field $\xi$ in accordance with the formulæ

$$s = \tilde{\xi}^a \sigma_{a\alpha b} \xi^b, \quad \rho = s \sqrt{\det g}, \quad (\vartheta^1 + i \vartheta^2)_a = s^{-1} \epsilon^{cb} \sigma_{a\alpha b} \xi^a \sigma_{c\alpha d} \xi^d, \quad \bar{\vartheta}^3_\alpha = s^{-1} \tilde{\xi}^a \sigma_{a\alpha b} \xi^b.$$
of det $\theta^{\alpha}_{\beta}$ can always be changed by switching from original Pauli matrices to their complex conjugates.

We will now show that the Hodge dual of axial torsion (8) is expressed via the spinor field $\xi$ as

$$
* T^{ax} = -\frac{2i(\xi^a \sigma^a_{ab} \nabla_a \xi^b - \xi^b \sigma^a_{ab} \nabla_a \xi^a)}{3\xi^i \sigma_{i0d} \xi^d}
$$

(B.5)

and that angular velocity (16) is expressed via the spinor field $\xi$ as

$$
\omega_\alpha = \frac{i(\xi^a \sigma_{aub} \partial_b \xi^b - \xi^b \sigma_{aub} \partial_b \xi^a)}{\xi^i \sigma_{i0d} \xi^d}.
$$

(B.6)

Note that formulae (B.5) and (B.6) are invariant under the rescaling of our spinor field by an arbitrary nonvanishing real scalar function.

Formulae (B.5) and (B.6) are proved by direct substitution of formulae (B.1), (B.3) and (B.4) into (8) and (16) respectively. In order to simplify calculations we observe that the expressions on the left- and right-hand sides of formulae (B.5) and (B.6) have an invariant nature; hence, it is sufficient to prove these formulae for the standard Euclidean metric (35), traditional Pauli matrices (36) and at a point at which the spinor field takes the value $\xi^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then at this point we have

$$
\theta^{\alpha}_{\beta} = \delta^{\alpha}_{\beta},
$$

(B.7)

$$
[\partial_\alpha(\theta^1 + i \theta^2)]_\beta = \begin{pmatrix}
\partial_\alpha \xi^1 - \partial_\alpha \xi^1 \\
i \partial_\alpha \xi^1 - i \partial_\alpha \xi^1 \\
-2i \partial_\alpha \xi^2
\end{pmatrix},
$$

$$
[\partial_\alpha(\theta^3)]_\beta = \begin{pmatrix}
\partial_\alpha \xi^2 + \partial_\alpha \xi^2 \\
-2i \partial_\alpha \xi^2 + i \partial_\alpha \xi^2 \\
0
\end{pmatrix},
$$

(B.8)

where $\alpha = 0, 1, 2, 3$. Note that formulae (B.8) with $\alpha = 1, 2, 3$ imply

$$
[curl(\theta^1 + i \theta^2)]_\beta = \begin{pmatrix}
-2i \partial_2 \xi^2 - \partial_1 (i \xi^1 - i \xi^1) \\
2i \partial_2 \xi^2 + \partial_1 (i \xi^1 - i \xi^1) \\
\partial_1 (i \xi^1 - i \xi^1) - \partial_2 (i \xi^1 - i \xi^1)
\end{pmatrix},
$$

(B.9)

$$
[curl(\theta^3)]_\beta = \begin{pmatrix}
-\partial_3 (-i \xi^2 + i \xi^2) \\
\partial_3 (i \xi^2 + i \xi^2) \\
\partial_1 (-i \xi^2 + i \xi^2) - \partial_2 (i \xi^2 + i \xi^2)
\end{pmatrix},
$$

(B.10)

where $\text{curl} u := \ast du$.

We rewrite the formulae for $\ast T^{ax}$ and $\omega$ in the form

$$
\ast T^{ax} = \frac{1}{6}(\theta^1 - i \theta^2) \cdot \text{curl}(\theta^1 + i \theta^2) + \frac{1}{6}(\theta^1 + i \theta^2) \cdot \text{curl}(\theta^1 - i \theta^2) + \frac{1}{3} \theta^3 \cdot \text{curl} \theta^3,
$$

(B.11)

$$
\omega = \frac{1}{2}(\theta^1 - i \theta^2) \times \partial_0 (\theta^1 + i \theta^2) + \frac{1}{2}(\theta^1 + i \theta^2) \times \partial_0 (\theta^1 - i \theta^2) + \frac{1}{2} \theta^3 \times \partial_0 \theta^3,
$$

(B.12)

where $u \cdot v := \eta_{\mu\nu} u^\mu v^\nu$ (note the absence of complex conjugation) and $u \times v := \ast (u \wedge v)$.

Substituting formulae (B.7), (B.9) and (B.10) into formula (B.11) we get

$$
\ast T^{ax} = -\frac{2i}{3} [\partial_2 \xi^1 + (\partial_1 - i \partial_2) \xi^2 - \partial_2 \xi^1 - (\partial_1 + i \partial_2) \xi^2]
$$
which coincides with the rhs of formula (B.5). Substituting formulae (B.7) and (B.8) with \( \alpha = 0 \) into formula (B.12) we get

\[
\omega_{\alpha} = i \begin{pmatrix}
\partial_0 \xi^2 - \partial_0 \bar{\xi}^2 \\
-\partial_1 \xi^2 - \partial_1 \bar{\xi}^2 \\
\partial_1 \xi^1 - \partial_1 \bar{\xi}^1
\end{pmatrix}
\]

which coincides with the rhs of formula (B.6).

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