Conformal field theory of dipolar SLE with the Dirichlet boundary condition

Nam-Gyu Kang · Hee-Joon Tak

Abstract We develop a version of dipolar conformal field theory based on the central charge modification of the Gaussian free field with the Dirichlet boundary condition and prove that correlators of certain family of fields in this theory are martingale-observables for dipolar SLE. We prove the restriction property of dipolar SLE(8/3) and Friedrich-Werner's formula in the dipolar case.

Keywords dipolar conformal field theory, dipolar SLE, martingale-observables

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1 Introduction

We implement a version of dipolar conformal field theory with the Dirichlet boundary condition in a simply connected domain $D$ with two marked boundary points $q_{-}, q_{+}$. Using this theory we study basic properties of a certain collection of dipolar SLE martingale-observables. In physics literature (e.g., [1]), it is well known that under the insertions of the boundary operator $\psi_{1;2}$ at $p \in \partial D$ and of the boundary operators $\psi_{0;1/2}$ at $q_{\pm}$, all correlations of the fields in a certain collection are martingale-observables for dipolar SLE from $p$ to the boundary arc $Q (p \notin Q)$ with endpoints $q_{\pm}$. We present its proof after we give a precise definition for $\Psi(p) = \Psi(p; q_{-}, q_{+}) (= \psi_{1;2}(p)\psi_{0;1/2}(q_{-})\psi_{0;1/2}(q_{+})$ up to boundary puncture operators) as a boundary vertex field rooted at $q_{\pm}$ of a single variable $p$.

A version of dipolar conformal field theory with the Neumann boundary condition can be implemented as the dual of theory with the Dirichlet boundary condition (see Subsection 4.1 below). For example, as a bi-variant field, a Gaussian free field $\Phi^{N}(z, z_{0})$ with the Neumann boundary condition can be defined as the dual boson $\tilde{\Phi}(z, z_{0})$ of the Gaussian free

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Nam-Gyu Kang E-mail: nkang@snu.ac.kr
Department of Mathematical Sciences, Seoul National University, Seoul, 151-747, Republic of Korea

Hee-Joon Tak E-mail: tdd502@snu.ac.kr
Department of Mathematical Sciences, Seoul National University, Seoul, 151-747, Republic of Korea
field $\Phi$ with the Dirichlet boundary condition. On the other hand, in [6] we study a dipolar conformal field theory of central charge one with mixed boundary condition (Dirichlet boundary condition on one boundary arc and Neumann boundary condition on the other arc) and its relation to dipolar $\text{SLE}_4$.

Definitions and theories developed in the study of the chordal case [8] and the radial case [7] are modified into the dipolar case. We also explain the similarities and differences of the chordal, the radial, and the dipolar conformal field theory. For example, similarly as in the radial case, the neutrality condition is required for the (rooted) multi-vertex fields to be well-defined Fock space fields. Under the neutrality condition, the (rooted) multi-vertex fields are Aut$(D,q_{\pm})$-invariant primary fields and their correlators with $\Psi(p)/E[\Psi(p)]$ are dipolar $\text{SLE}_K$ martingale-observables.

2 Main results

2.1 Dipolar SLE martingale-observables

Let us consider a simply connected domain $(D,p,Q)$ with a marked boundary point $p$ and a marked boundary arc $Q \subseteq \partial D$ such that $p \notin Q$. We denote by $q_-q_+$ two endpoints of $Q$ such that $q_-p,q_+$ are positively oriented. A dipolar Schramm-Loewner evolution ($\text{SLE}_K$) in $(D,p,Q)$ with a parameter $\kappa$ ($\kappa > 0$) is the conformally invariant law on random curves from the point $p$ to the arc $Q$ described by the solution $\psi_t(z)$ of the dipolar Loewner equation

$$\partial_t \psi_t(z) = \coth_2(\psi_t(z) - \xi_t), \quad (\xi_t = \sqrt{\kappa} B_t),$$

(2.1)

where $\coth_2(z) := \coth(\frac{z}{2})$ and $B_t$ is a one-dimensional standard Brownian motion with $B_0 = 0$. As an initial data, $\psi_0 : (D,p,Q) \to (\mathbb{S},0,\mathbb{R} + \pi i)$ is the conformal map from $D$ onto the strip $\mathbb{S} := \{ z \in \mathbb{C} | 0 < \Im z < \pi \}$. (The solution $\psi_t(z)$ of (2.1) exists up to a stopping time $\tau_c \in (0,\infty]$.) Then for all $t$,

$$w_t : (D_t, \gamma, Q) \to (\mathbb{S},0,\mathbb{R} + \pi i), \quad w_t(z) := \psi_t(z) - \xi_t,$$

is a conformal map from $D_t := \{ z \in D : \tau_c > t \}$ onto the strip $\mathbb{S}$. The dipolar SLE curve $\gamma$ is defined by

$$\gamma \equiv \gamma(t) := \lim_{z \to 0} w_t^{-1}(z).$$

It is well known ([13,15]) that the SLE trace exists in the dipolar case and that dipolar $\text{SLE}_K$ is equivalent to chordal $\text{SLE}_K(\frac{1}{2}(\kappa - 6), \frac{1}{2}(\kappa - 6))$ with two force points $q_{\pm}$. We plan to apply definitions and constructions developed in [8,7] to conformal field theory of $\text{SLE}_K(p)$. As a preliminary work, we study a version of dipolar conformal field theory in this paper.

To define dipole SLE martingale-observables, let us recall the definition of non-random conformal fields. See [8, Section 4.1] for more details. A non-random conformal field $M$ is an assignment of a (smooth) function $(M, \phi) : \phi U \to \mathbb{C}$ to each local chart $\phi : U \to \mathbb{C}$.
A non-random conformal Fock space field $M$ is a $[\lambda, \lambda_*]$-differential if for any two overlapping charts $\phi, \tilde{\phi}$, we have

$$(M \parallel \phi) = (\phi h') q \phi^{-1} (M \parallel \tilde{\phi}) \circ h,$$

where $h = \tilde{\phi} \circ \phi^{-1} : \phi(U \cap \tilde{U}) \to \tilde{\phi}(U \cap \tilde{U})$ is the transition map. A pair $[\lambda, \lambda_*]$ is called degrees or conformal dimensions of $M$. Non-random conformal Fock space fields $M$ are called pre-pre-Schwarzian forms, pre-Schwarzian forms, and Schwarzian forms of order $\mu \in \mathbb{C}$ if the following transformation laws hold:

$$(M \parallel \phi) = (M \parallel \tilde{\phi}) \circ h + \mu \log h', (M \parallel \phi) = h' (M \parallel \tilde{\phi}) \circ h + \mu h''/h',$$

and

$$(M \parallel \phi) = (h')^2 (M \parallel \tilde{\phi}) \circ h + \mu S_h,$$

respectively. Here, $S_h$ is the Schwarzian derivative of $h$, $S_h = (h''/h')' - \frac{1}{2} (h''/h')^2$.

A non-random (conformal) field $M$ of $n$ variables in $\mathbb{S}$ is said to be a martingale-observable for dipolar SLE if for any $z_1, \ldots, z_n \in D$, the process

$$M_t(z_1, \ldots, z_n) = (M_{D_t, \mathbb{S}} \parallel \text{id})(z_1, \ldots, z_n) = (M \parallel w_t^{-1})(z_1, \ldots, z_n)$$

is a local martingale on dipolar SLE probability space. (The process $M_t(z_1, \ldots, z_n)$ is stopped when any $z_j$ exits $D_t$.) For example, we can use the identity chart of $D$, then for $[h, h_*]$-differentials $M$ with boundary conformal dimensions $h_\pm$ at $q_\pm$, we have

$$M_t(z) = (w_t^I(z))^{h_+} (w_t^I(q_-))^{h_-} (w_t^I(q_+))^{h_+} M(w_t(z)).$$

If $M$ is a pre-Schwarzian form of order $\mu$, then

$$M_t(z) = w_t^I(z) M(w_t(z)) + \mu w_t''(z)/w_t'(z).$$

Similarly, for a Schwarzian form $M$ of order $\mu$, we have

$$M_t(z) = (w_t^I(z))^2 M(w_t(z)) + \mu S_{w_t}(z).$$
As a multivalued field, the bi-variant chiral bosonic field $\Phi \equiv \Phi_{(b)}$ of the Gaussian free field $\Phi_{(0)}$ by

$$
\Phi_{(b)} = \Phi_{(0)} - 2b \arg w',
$$

(cf. central charge modifications of the Gaussian free field in the chordal case, see e.g., [8, Section 10.1]). We denote by $\Phi_{(b)}$ the OPE family of $\Phi_{(0)}$, the algebra over $\mathbb{C}$ spanned by $1, \partial/\partial \bar{z} \Phi_{(b)}$, and derivatives of the vertex fields $\partial/\partial z q^\alpha_{(b)}$ under OPE multiplications $\ast$. We will recall the definition of operator product expansion and its basic properties in Subsection A.3. As in the chordal case ([8, Sections 3.3, 10.2]), vertex fields are defined as OPE-exponentials of the bosonic field $\Phi = \Phi_{(b)}$:

$$
\gamma^\alpha \equiv \gamma^\alpha_{(b)} = e^{\ast \alpha \Phi} = \sum_{a=0}^{\infty} \frac{C_a}{n!} \Phi^a z^n.
$$

As in the radial case ([7]), we extend the OPE family $\Phi_{(b)}$ to include the bi-variant chiral bosonic field $\Phi^\ast_{(b)}(z, z_0)$, its conjugate, and (derivatives of) chiral multi-vertex fields

$$\Theta^{(\sigma, \sigma_\ast)}(z) \equiv (\sigma = (\sigma_1, \cdots, \sigma_n), \sigma_\ast = (\sigma_1\ast, \cdots, \sigma_n\ast), z = (z_1, \cdots, z_n), z_j \in D)$$

(its precise definition is given in Subsection 4.2) with the neutrality condition

$$
\sum_{j=1}^n (\sigma_j + \sigma_{j\ast}) = 0.
$$

As a multivalued field, the bi-variant chiral bosonic field $\Phi^\ast_{(b)}(z, z_0)$ can be expressed in terms of 1-point formal field $\Phi^\ast_{(b)}$ as follows:

$$
\Phi^\ast_{(b)}(z, z_0) = \Phi^\ast_{(b)}(z) - \Phi^\ast_{(b)}(z_0), \quad \Phi^\ast_{(b)}(z) = \Phi^\ast_{(0)}(z) + ib \log \frac{w'(z)}{1 - w(z)},
$$

where $w$ is a conformal map from $(D, q_{-}, q_{+})$ onto $(\mathbb{H}, -1, 1)$ and the formal 1-point field $\Phi^\ast_{(0)}$ (which can be interpreted as a "holomorphic" part of $\Phi_{(0)}$ in the sense that $\Phi_{(0)}(z) = 2 \Re \Phi^\ast_{(0)}(z)$) has formal correlations

$$
E[\Phi^\ast_{(0)}(z) \Phi^\ast_{(0)}(z_0)] = \log \frac{1}{w(z) - w(z_0)}, \quad E[\Phi^\ast_{(0)}(z) \overline{\Phi^\ast_{(0)}(z_0)}] = \log(w(z) - \overline{w(z_0)}).
$$
The chiral multi-vertex field \( \Theta^{(\sigma, \sigma_\ast)}(z) \) can be interpreted as the OPE exponential of the formal field

\[
i \sum_j \sigma_j \Phi^{(+)}_{(j)}(z_j) = \Phi_{(0)}(z),
\]

where \( \Phi_{(j)} = \Phi^{(+)}_{(j)} \). It turns out that the formal fields \( \Theta^{(\sigma, \sigma_\ast)} \) are well-defined Fock space fields if and only if the neutrality condition holds. Under the neutrality condition, multi-vertex fields \( \Theta^{(\sigma, \sigma_\ast)} \) are \( Aut(D, q_\pm) \)-invariant primary fields. See Subsection A.4 for the definition of conformal invariance.

In the chordal (radial) case, under the insertion of Wick’s exponential

\[
e^{\omega \Phi^+_{(0)}(p, q)} (p, q \in \partial D, p \neq q), \quad \left( e^{\omega \Phi^+_{(0)}(p, q)} (p \in \partial D, q \in D) \right),
\]

with \( \omega = \sqrt{2/\kappa} \) and \( b = \omega(\kappa/4 - 1) \), all fields in the extended OPE family \( \mathcal{F}_{(b)} \) of \( \Phi_{(b)} \) satisfy the “field Markov property” with respect to chordal (radial) SLE filtration, respectively. (See [2,12] for the chordal case and [3,4] for the radial case from the physics perspective, cf. [8, Proposition 14.3] and [7, Theorem 1.1].) The dipolar version of this theorem can be stated as follows. (As we mentioned in Section 1, special cases of the following theorem are well known in physics literature, e.g., [1].)

**Theorem 2.1** For the tensor product \( X = X_1(z_1) \cdots X_n(z_n) \) of fields \( X_j \) in the OPE family \( \mathcal{F}_{(b)} \) of \( \Phi_{(b)} \), the non-random fields

\[
E[ e^{\alpha \Phi^+_{(0)}(p, q) + \Phi^+_{(0)}(p, q)} X] \quad (\alpha = \sqrt{2/\kappa}, b = \sqrt{\kappa/8 - \sqrt{2/\kappa}})
\]

are martingale-observables for dipolar SLE\(_\kappa\).

### 2.3 Boundary condition changing operators and rooted vertex fields

The boundary condition changing operator on Fock space functionals/fields is the insertion of

\[
e^{\sqrt{i/2} \omega (\Phi^+_{(0)}(p, q_\pm) + \Phi^+_{(0)}(p, q_\pm))}.
\]

While this field does not belong to the extended OPE family \( \mathcal{F}_{(b)} \), we further extend the OPE family to contain the multi-vertex fields \( \Theta^{(\sigma, \sigma_\ast, \sigma_\ast, \sigma_\ast)} \) rooted at two points \( q_-, q_+ \) with the neutrality condition \( (\sigma_- + \sigma_+ + \sum_{j=1}^n (\sigma_j + \sigma_j)) = 0 \) so that the field (2.3) is represented as

\[
\Psi(p) \sim \Psi(q_-) := \Theta^{(\sigma_-; q_-, q_+)} := \Theta^{(\sigma; q_-, q_+)}(\eta_-) \in \mathcal{F}_{(b)}.
\]

Similarly as in the radial case, the definition of rooted multi-vertex fields can be obtained by normalizing multi-vertex fields

\[
\Theta^{(\sigma, \sigma_\ast)} \Theta^{(\sigma_\ast)}(\eta_-) \Theta^{(\sigma_\ast)}(\eta_+)
\]

and taking a limit as \( (\eta_-, \eta_+) \) approaches \( (q_-, q_+) \). This rooting procedure also gives rise to the definition of the **normalized tensor product** of rooted vertex fields as

\[
\Theta^{(\sigma, \sigma_\ast, \sigma_\ast, \sigma_\ast)} \ast \Theta^{(t, \tau_\ast, \tau_\ast, \tau_\ast, \tau_\ast)} = \Theta^{(\sigma + t, \sigma_\ast, \sigma_\ast + t, \sigma_\ast, \sigma_\ast)}(\eta_-, \eta_+).
\]
For a given compact hull $K$, the rooted multi-vertex fields $\mathcal{E}^{(\sigma_1, \sigma_2, \ldots, \sigma_n)}(x)$ can be interpreted as the OPE exponential of the field

$$i\sigma_+ \Phi_{(b)}^0(q^-) + i\sigma_- \Phi_{(b)}^+ (q^+) + i \sum \sigma_j \Phi_{(b)}^+ (z_j) - \sigma_j, \Phi_{(b)}^- (z_j),$$

where $\Phi_{(b)}^0(q) = \Phi_{(0)}^0(q)$ and $\Phi_{(b)}^0(q)$ are not linearly independent. Indeed, they have the same distribution as the dipolar SLE

$$\mathcal{F}(b) \text{ extended OPE family of } \Phi_{(b)}.$$ For a rooted vertex field $\mathcal{E} \equiv \mathcal{E}^{(\sigma_1, \ldots, \sigma_n)}$ with the neutrality condition, the extended OPE family of $\Phi_{(b)}$ is not well-defined because both $\Psi(p) \equiv \Psi(p; q^-, q^+)$ and $\mathcal{E}$ have nodes at $q_{-}$. However, Theorem 2.1 can be modified for a rooted vertex field with the neutrality condition (cf. [7, Theorem 1.2] for its radial version).

**Theorem 2.2** For a rooted vertex field $\mathcal{E} \equiv \mathcal{E}^{(\sigma_1, \ldots, \sigma_n)}$ with the neutrality condition, the non-random field

$$E[\Psi(p) \cdot \mathcal{E}]$$

is a dipolar SLE martingale-observable.

Theorems 2.1 and 2.2 can be extended to the fields in the extended OPE family of $\Phi_{(b)}$.

2.4 Examples of dipolar SLE martingale-observables

We use conformal field theory to present the proof for the restriction property of dipolar SLE$\frac{8}{3}$: for a fixed compact hull $K$ (i.e., $K$ is a compact set such that $\mathbb{S} \setminus K$ is a simply connected subdomain of $\mathbb{S}$) with $K \cap (\mathbb{R} + \pi i) = \emptyset$, the dipolar SLE$\frac{8}{3}$ path in $(\mathbb{S}, 0, -\infty, \infty)$ conditioned to avoid $K$ has the same distribution as the dipolar SLE$\frac{8}{3}$ path in $(\mathbb{S} \setminus K, 0, -\infty, \infty)$.

It is equivalent to Theorem 2.3 below. To state it, let us recall the definition of the strip capacity (e.g., see [14]) of a compact hull $K$. For a compact hull $K$ with $K \cap (\mathbb{R} + \pi i) = \emptyset$, there is a unique conformal transformation from $(\mathbb{S} \setminus K, -\infty, \infty)$ onto $(\mathbb{S}, -\infty, \infty)$ such that

$$\lim_{z \to \pm \infty} \psi_K(z) - z = \pm s$$

for some $s \geq 0$. Here, $z \to \pm \infty$ means that $z \in \mathbb{S}$ and $\Re z \to \pm \infty$. This $s$ is called the strip capacity of $K$ and denoted by $\operatorname{scap}(K)$.

**Theorem 2.3** For a given compact hull $K$,

$$\mathbb{P}(\text{SLE}_{\frac{8}{3}} \text{ path avoids } K) = \psi_K(0)^5 e^{-2\mu \operatorname{scap}(K)}, \quad (\lambda = 5/8, \mu = 5/96).$$
In the half-plane uniformization, (2.5) reads as

$$P(\text{SLE}_{8/3} \text{ path avoids } K) = \psi_K'(0)^\lambda (\psi_K'(-1)\psi_K'(1))^\mu,$$

(2.6)

where $K$ is a fixed compact hull with $\partial K \cap \mathbb{R} \subseteq (-1, 1) \setminus \{0\}$ and $\psi_K$ is the conformal transformation from $(\mathbb{H} \setminus K, -1, 1)$ onto $(\mathbb{H}, -1, 1)$ such that $\psi_K'(-1) = \psi_K'(1)$. Compare (2.6) to the restriction property of chordal SLE$_{8/3}$, radial SLE$_{8/3}$, see [11, Theorem 6.1], [10, Theorem 6.26], respectively, and to the one-sided restriction property of chordal SLE$_{8/3}$($\frac{1}{2}(\kappa - 6)$), see [11, Theorem 8.4]. The restriction exponents $\lambda$ and $\mu$ can be explained in term of conformal dimensions of $\Psi$. Let us introduce the effective boundary condition changing operator $\Psi_{\text{eff}}$:

$$\Psi_{\text{eff}} := \Psi \mathcal{P}(q_-) \mathcal{P}(q_+),$$

(2.7)

where $\mathcal{P}(q_{\pm})$ is the “boundary puncture operator” defined as a $-\frac{1}{2}b^2$-boundary differential at $q_{\pm}$ and $\mathcal{P}(q_{\pm}) \equiv 1$ in the identity chart of $\mathbb{H}$. Then

$$\lambda = h(\Psi) := \frac{a^2}{2} - ab = \frac{6 - \kappa}{2\kappa}, \quad \mu = h_\pm(\Psi_{\text{eff}}) := \frac{a^2}{8} - \frac{b^2}{2} = \frac{(\kappa - 2)(6 - \kappa)}{16\kappa}.$$

As an application of the restriction property of dipolar SLE$_{8/3}$, we prove a dipolar version of Friedrich-Werner’s formula.

**Theorem 2.4** For distinct $x_j \in \mathbb{R} \setminus \{0\}$ ($j = 1, \ldots, n$) and for $b = -\sqrt{3}/6$,

$$\lim_{\varepsilon \to 0} e^{-2\pi \varepsilon P(\text{SLE}_{8/3} \text{ hits all slits } [x_j, x_j + i\varepsilon \sqrt{2}])} = \hat{E}[T(x_1) \cdots T(x_n) \parallel \text{id}_\mathbb{S}].$$

Compare Theorem 2.4 to its chordal and radial version (see [5, Proposition 1] and [7, Theorem 4.4], respectively).

In the last subsection we identify the probability that $z \in D$ is swallowed by the SLE$_\kappa(\kappa > 4)$ hulls and the probability that $z$ is to the left (right) of the SLE$_\kappa$ paths with correlations of primary observables (Cardy-Zhan’s observables) by the method of “screening.”

### 3 Dipolar CFT

After we discuss central charge modifications of the Gaussian free field in a simply connected domain $D$ with two marked boundary points $q_{\pm}$, we define a dipolar version of Ward’s functionals in terms of Lie derivatives. Based on this approach, we derive Ward’s equations in the dipolar case. For those who are not familiar with some of the definitions and concepts developed in [8, Lectures 1, 3 – 5, 7]), we review them in Appendix.
3.1 Central charge modification

For a given simply connected domain \( D \) with two marked points \( q_{\pm} \in \partial D \), we consider a conformal transformation

\[
w : (D, q_{-}, q_{+}) \rightarrow (\mathbb{H}, -1, 1)
\]

from \( D \) onto the half-plane \( \mathbb{H} = \{ z \in \mathbb{C} \mid \Im z > 0 \} \). For a parameter \( b = \sqrt{\kappa/8 - \sqrt{2}/\kappa} \), we define the central charge modifications \( \Phi \equiv \Phi_{(b)} \) of the Gaussian free field \( \Phi_{(0)} \) with the Dirichlet boundary condition by

\[
\Phi_{(b)} = \Phi_{(0)} + \varphi, \quad \varphi = -2b \arg\left( \frac{w'}{1 - w^2} \right).
\]

It is well-defined since \( \varphi \) does not depend on the choice of \( w \). We also define the current field \( J \equiv J_{(b)} \) by

\[
J_{(b)} = \partial \Phi_{(b)} = J_{(0)} + j, \quad j = ib\left( \frac{w''}{w'} + \frac{2ww'}{1 - w^2} \right).
\]

The current field is an \( \text{Aut}(D, q_{-}, q_{+}) \)-invariant pre-Schwarzian form of order \( ib \). Furthermore, the OPE family \( \mathcal{F}_{(b)} \) is \( \text{Aut}(D, q_{-}, q_{+}) \)-invariant. Compare the following proposition to [8, Proposition 10.1]. See Subsections A.5 – A.6 for the definitions of a stress tensor and the Virasoro field.

**Proposition 3.1** The bosonic field \( \Phi_{(b)} \) has a stress tensor, and the Virasoro field is given by

\[
T \equiv T_{(b)} = -\frac{1}{2} J * J + ib\partial J. \tag{3.1}
\]

**Proof** Let us define a holomorphic field \( A \) by

\[
A \equiv A_{(b)} = A_{(0)} + (ib\partial - j)J_{(0)}, \quad A_{(0)} = -\frac{1}{2} J_{(0)} \circ J_{(0)}. \tag{3.2}
\]

Then \( A \) is a quadratic differential. Indeed, as in the chordal and the radial cases, \( ib\partial J_{(0)} \) and \( jJ_{(0)} \) satisfy the following transformation laws:

\[
ib\partial J_{(0)} = ibh''f_{(0)} \circ h + ib(h')^2 \partial f_{(0)} \circ h,
\]

\[
jJ_{(0)} = ib\left( \frac{h''}{h'} \right) h'f_{(0)} \circ h + (h')^2 \left( jf_{(0)} \right) \circ h.
\]

Since \( \Phi_{(b)} \) is a real part of pre-pre-Schwarzian form of order \( ib \), it is enough to check Ward’s OPE in the \( (\mathbb{H}, -1, 1) \)-uniformization,

\[
A(\xi)\Phi(z) \sim \frac{ib}{(\xi - z)^2} + \frac{J_{(b)}}{\xi - z}.
\]

(We use the notation \( \sim \) for the singular part of the operator product expansion.) However, this is immediate from Ward’s OPE in the case \( b = 0 \) and the following operator product expansions:

\[
\partial J_{(0)}(\xi)\Phi_{(0)}(z) \sim \frac{1}{(\xi - z)^2}, \quad j(\xi)\Phi_{(0)}(\xi)(z) \sim \frac{j(z)}{\xi - z}.
\]
Finally, let us show that $T_{(b)}$ is the Virasoro field for the OPE family $\mathcal{F}_{(b)}$. Since $\mathcal{F}_{(b)}$ is closed under differentiation and OPE multiplication (see Subsection A.5 or [8, Proposition 5.8]), $T_{(b)}$ is in $\mathcal{F}_{(b)}$. Therefore, it has a stress tensor $(A_{(b)}, \overline{A}_{(b)})$. It follows from the expressions of $T$ and $J$ that

$$T = A + \frac{1}{12} S_w - \frac{j^2}{2} + ibj,$$

where $S_w = (w''/w)' - \frac{1}{2}(w''/w)^2$ is the Schwarzian derivative of $w$. The term $-\frac{1}{2}j^2 + ibj$ simplifies $-b^2 S_w - 2b^2 w^2/(1 - w^2)^2$. Thus we get

$$T = A + \frac{1 - 12b^2}{12} S_w - 2b^2 \left(\frac{w'}{1 - w^2}\right)^2$$

(3.3)

and therefore $T$ is a Schwarzian form of order $\frac{1}{2}c$. The central charge $c$ is given by

$$c \equiv 1 - 12b^2 = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

### 3.2 Ward’s functionals and Ward’s identities

In this subsection we modify the definition of Ward’s functionals in the chordal case (see [8, Sections 5.5–5.6]) into the dipolar case and derive Ward’s identities. For a given open set $U$ such that $\overline{U} \subset \overline{D} \setminus \{q_\pm\}$ and a smooth vector field $v$ on $\overline{U}$, Ward’s functional $W^\pm(v; U)$ is defined by

$$W^+(v; U) = \frac{1}{2\pi i} \int_{\partial U} \nu A - \frac{1}{\pi} \oint_{\partial U} (\partial_v A), \quad W^-(v; U) = \overline{W^+(v; U)},$$

where $A \equiv A_{(b)} = A_{(0)} + (ib\partial - j)J_{(0)}, A_{(0)} = -\frac{1}{2} J_{(0)} \odot J_{(0)}$, see (3.2), and $j = E[J_{(b)}]$. We also write $W(v; U) = 2\Re W^+(v; U)$. Then $E[W^+(v; U), \mathcal{X}]$ is a well-defined correlation function if $\mathcal{X}$ is a Fock space functional on $D \setminus \{q_\pm\}$ with nodes in $U$ and in the maximal open set $D_{\text{hol}}(v)$ where $v$ is holomorphic.

Recall that the following statements are equivalent (see [8, Propositions 5.3 and 5.10]):

- Ward’s OPE holds for a Fock space field $X$.
- The residue form of Ward’s identity for $X$

$$\mathcal{L}_v X(z) = \frac{1}{2\pi i} \oint_{\partial z} \nu A^+ X(z) - \frac{1}{2\pi i} \oint_{\partial z} \delta A^+ X(z)$$

holds on $D_{\text{hol}}(v) \cap U$ for all (local) smooth vector field $v$. (See Subsection A.4 for the definition of Lie derivatives $\mathcal{L}_v$ and their basic properties.)
- For all $z \in D_{\text{hol}}(v) \cap U$

$$E[\mathcal{Y} \mathcal{L}_v X(z)] = E[W(v; U) X(z) \mathcal{Y}]$$

holds for all correlation functional $\mathcal{Y}$ whose nodes are in $(D \setminus U)$.
The definition of Ward’s functionals can be extended to meromorphic vector fields. For
a meromorphic vector field \( v \) which is continuous up to the boundary and has a simple zero
at \( q_{\pm} \), we define Ward’s functional \( W^+(v; \bar{D} \setminus \{q_{\pm}\}) \) by
\[
W^+(v; \bar{D} \setminus \{q_{\pm}\}) = \lim_{\varepsilon \to 0} W^+(v; D_{\varepsilon}),
\]
where \( D_{\varepsilon} = D \setminus (B(q_{-}, \varepsilon) \cup B(q_{+}, \varepsilon) \cup \bigcup j B(p_{j}, \varepsilon)) \) and \( p_{j}'s \) are poles of \( v \). Thus we have
\[
W^+(v; \bar{D} \setminus \{q_{\pm}\}) = \frac{1}{2\pi i} \int_{\partial D} v A - \frac{1}{\pi} \int_D (\bar{\partial} v) A,
\]
where \( \bar{\partial} v \) is considered as a distribution. Note that \( v A \) has a removable singularity at \( q_{\pm} \).

Suppose \( X_{j}'s \) are in the OPE family \( \mathcal{F}(b_{(j)}) \).

Then the global Ward’s identity
\[
E[L_{v} X_{1}(z_{1}) \cdots X_{n}(z_{n})] = E[W^+(v; \bar{D} \setminus \{q_{\pm}\}) X_{1}(z_{1}) \cdots X_{n}(z_{n})]
\]
(3.4)
holds if all \( z_{j} \in D_{\text{hol}}(v) \). Compare (3.4) to [8, Proposition 5.9].

We now represent a quadratic differential
\[
A \equiv A(b_{(j)}) = A_{(0)} + (ib \partial - j)J_{(0)}
\]
in terms of Ward’s functionals associated with the dipolar Loewner vector field \( v_{\zeta} \):
\[
(v_{\zeta} | \text{id}_{\mathbb{H} \setminus \{\pm 1\}})(z) = \frac{1 - z^{2}}{2} \frac{1 - \zeta z}{\zeta - z}.
\]

**Proposition 3.2** In the identity chart \( \text{id}_{\mathbb{H}} \) of \( \mathbb{H} \), we have
\[
A(\zeta) = \frac{2}{(1 - \zeta^{2})^{2}} \left( W^+(v_{\zeta} \mid \mathbb{H} \setminus \{\pm 1\}) + W^-(v_{\zeta} \mid \mathbb{H} \setminus \{\pm 1\}) \right).
\]

**Proof** For \( \zeta \in \mathbb{H} \), by the definition of Ward’s functional,
\[
W^+(v_{\zeta} \mid \mathbb{H} \setminus \{\pm 1\}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} v_{\zeta} A - \frac{1}{\pi} \int_{\mathbb{H}} (\bar{\partial} v_{\zeta}) A.
\]

On the other hand, the reflected vector field
\[
v_{\zeta}^{R}(z) := \overline{v_{\zeta}(\bar{z})} = \overline{v_{\zeta}(z)}
\]
is holomorphic in \( \mathbb{H} \). Therefore, we have
\[
W^+(v_{\zeta} \mid \mathbb{H} \setminus \{\pm 1\}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} v_{\zeta} A.
\]

Since \( A \) is real on the boundary and \( \overline{v_{\zeta}} = v_{\zeta} \) on the boundary,
\[
W^-(v_{\zeta} \mid \mathbb{H} \setminus \{\pm 1\}) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} v_{\zeta} A.
\]

Proposition now follows from the fact that \( \bar{\partial} v_{\zeta} = -\frac{i}{2\pi}(1 - \zeta^{2})^{2} \delta_{\zeta} \).
3.3 Ward’s equations in the upper half-plane

We will use Ward’s equations below (Proposition 3.3) to prove Theorem 2.1. For \( Y \in \mathcal{F}_{(h)} \), let us express its (holomorphic part of) Lie derivative \( \mathcal{L}_\zeta^+ Y(z) \) in terms of the singular part of operator product expansion \( A(\zeta)Y(z) \sim \sum_{j \leq -1} C_j(z)(\zeta - z)^j \) as \( \zeta \to z \) and the OPE coefficients \( C_j(j = -1, -2, -3) \):

\[
\mathcal{L}_\zeta^+ Y(z) = \sum_{j = -3}^{-1} P_j(\zeta, z) C_j(z) + \frac{(1 - \zeta^2)^2}{2} \sum_{j \leq -1} C_j(z)(\zeta - z)^j \tag{3.5}
\]

where \( P_{-1}(\zeta, z) = \frac{1}{2}(2\zeta + z - \zeta^3 - \zeta^2 z - \zeta z^2) \), \( P_{-2}(\zeta, z) = \frac{1}{2}(1 - \zeta^2 - 2\zeta z) \), \( P_{-3}(\zeta, z) = -\frac{1}{2}\zeta \). Equation (3.5) can be shown by the identities

\[
\mathcal{L}_\zeta^+ Y(z) = \frac{1}{2\pi i} \oint (\zeta - z)^j v_\zeta(\eta) A(\eta) Y(z) \, d\eta = \sum_{j \leq -1} C_j(z) \frac{1}{2\pi i} \oint (\zeta - z)^j v_\zeta(\eta) \, d\eta,
\]

and

\[
\frac{1}{2\pi i} \oint (\zeta - z)^j v_\zeta(\eta) \, d\eta = \frac{(1 - \zeta^2)^2}{2} (\zeta - z)^j + P_j(\zeta, z) \quad (j \leq -1)
\]

if we set \( P_j(\zeta, z) \equiv 0 \) for \( j \leq -4 \).

We state Ward’s equation in terms of Virasoro generators \( L_n \). Let us recall the definition of \( L_n \):

\[
L_n(z) := \frac{1}{2\pi i} \oint (\zeta - z)^n T(\zeta) \, d\zeta. \tag{3.6}
\]

As operators acting on fields, the modes \( L_n \) can be viewed as OPE multiplications, \( L_n X = T \ast_{n-2} X \). See Subsection A.3 for the definition of \( \ast_n \).

**Proposition 3.3** For \( Y \in \mathcal{F}_{(h)} \) and \( X = X_1(z_1) \cdots X_n(z_n) \) (\( X_j \in \mathcal{F}_{(h)} \)),

\[
E[Y(z)\mathcal{L}_\zeta^+ X] + E[\mathcal{L}_\zeta^- Y(z)X] = \frac{(1 - z^2)^2}{2} E[(L_{-2} Y)(z)X] - \frac{3z(1 - z^2)}{2} E[(L_{-1} Y)(z)X] \]
\[
+ \frac{3z^2 - 1}{2} E[(L_0 Y)(z)X] + \frac{z}{2} E[(L_1 Y)(z)X] + b^2 E[Y(z)X],
\]

where all fields are evaluated in the identity chart of \( \mathbb{H} \).

**Proof** Subtracting the singular part of OPE and using (3.5), we have

\[
E[(A \ast Y)(z)X] = \lim_{\zeta \to \zeta} E[A(\zeta)Y(z)X] - \frac{2}{(1 - \zeta^2)^2} E[(L_{-1}^+ Y)(z)X] \]
\[
+ \frac{2}{(1 - z^2)^2} \frac{3z(1 - z^2)}{2} E[(A \ast_{-1} Y)(z)X] \]
\[
+ \frac{1 - 3z^2}{2} E[(A \ast_{-2} Y)(z)X] - \frac{z}{2} E[(A \ast_{-3} Y)(z)X)].
\]
We now use Proposition 3.2 and Leibniz’s rule for Lie derivatives to derive
\[
E[Y(z)\mathcal{L}_{\psi}^+X] + E[\mathcal{L}_{\psi}^-(Y(z)X)]
= \frac{(1 - z^2)^2}{2}E[(A + Y)(z)X] - \frac{3z(1 - z^2)}{2}E[(A *_{-1} Y)(z)X]
+ \frac{3z^2 - 1}{2}E[(A *_{-2} Y)(z)X] + \frac{z}{2}E[(A *_{-3} Y)(z)X].
\]

Proposition now follows since \( T = A - 2h^2 / (1 - z^2)^2 \) in the identity chart of \( \mathbb{H} \).

### 4 Vertex fields

In the next section the boundary condition changing operator \( \Psi \) will be introduced as a vertex field rooted at two marked boundary points \( q_{\pm} \). Also we expand our collection of OPE family of \( \Phi_{(0)}^\pm \) by considering the (rooted) multi-vertex fields with the neutrality condition. For this purpose, we introduce the formal bosonic fields first and then define the (formal) multi-vertex fields in terms of the formal bosonic fields. We also use them to describe the relation between the conformal field theory with the Dirichlet boundary condition and one with the Neumann boundary condition.

#### 4.1 Formal fields

The formal 1-point fields \( \Phi_{(0)}^+ \) and \( \Phi_{(0)}^- \) can be interpreted as the “holomorphic part” and the “anti-holomorphic part” of the Gaussian free field \( \Phi_{(0)} \) in the sense that \( \Phi_{(0)} = \Phi_{(0)}^+ + \Phi_{(0)}^- \) and \( \Phi_{(0)} = \overline{\Phi_{(0)}^-} \). By definition they have the following formal correlations
\[
E[\Phi_{(0)}^+(z)\Phi_{(0)}^+(z_0)] = \log \frac{1}{w(z) - w(z_0)}, \quad E[\Phi_{(0)}^+(z)\Phi_{(0)}^-(z_0)] = \log(w(z) - w(z_0)),
\]
where \( w \) is any conformal transformation from \( D \) onto \( \mathbb{H} \). Of course, neither \( \Phi_{(0)}^+ \) nor \( \Phi_{(0)}^- \) is a genuine Fock space field. However, the formal field
\[
\sum_{j=1}^n \sigma_j \Phi_{(0)}^+(z_j) - \sigma_j, \Phi_{(0)}^-(z_j)
\]
is a well-defined (multivalued) Fock space field if and only if the “neutrality condition”
\[
\sum_j (\sigma_j + \sigma_j^*) = 0
\]
holds. For example, as a bi-variant field, \( \Phi_{(0)}^+(z, z_0) = \Phi_{(0)}^+(z) - \Phi_{(0)}^+(z_0) \) is a multivalued Fock space field:
\[
\Phi_{(0)}^+(z, z_0) = \{ \Phi_{(0)}^+(\gamma) := \int_{\gamma} J_{(0)}(\xi) \, d\xi \mid \gamma \text{ is a curve from } z_0 \text{ to } z \}.
\]
We now explain why a version of dipolar conformal field theory with the Neumann boundary condition can be developed as the dual of theory with the Dirichlet boundary condition. First, let us recall the definition of the harmonic conjugate \( \Phi_{\partial}(z) \) of bosonic field \( \Phi_{(b)} \):

\[
\tilde{\Phi}_{(b)}(z, z_0) = 2 \Im \Phi_{(b)}^+(z, z_0).
\]

We write \( \tilde{\Phi}_{(0)}(z) = 2 \Im \Phi_{(0)}^+(z) \) so that \( \tilde{\Phi}_{(0)}(z, z_0) = \tilde{\Phi}_{(0)}(z) - \tilde{\Phi}_{(0)}(z_0) \). As a formal field of one variable, \( \tilde{\Phi}_{(0)} \) has the 2-point correlation function

\[
E[\tilde{\Phi}_{(0)}(\zeta)\tilde{\Phi}_{(0)}(z)] = -E[\Phi_{(0)}^+(\zeta)\Phi_{(0)}^+(z)\Phi_{(0)}^-(\zeta)\Phi_{(0)}^-(z)] = G_N(\zeta, z),
\]

where \( G_N(\zeta, z) = 2 \log|w(\zeta) - w(z)||w(\zeta) - \overline{w(z)}| \) is the (formal) Green’s function of \( D \) with the Neumann boundary condition. Therefore,

\[
E[\tilde{\Phi}_{(0)}(\zeta, z_0)\tilde{\Phi}_{(0)}(z, z_0)] = G_N(\zeta, z) - G_N(\zeta_0, z) - G_N(\zeta, z_0) + G_N(\zeta_0, z_0).
\]

It is well known that the difference of two Neumann Green’s function is well-defined, see [9]. Thus, as a bi-variant Fock space field, a Gaussian free field \( \Phi_{(0)}^N(z, z_0) \) with the Neumann boundary condition can be defined in terms of the dual boson \( \tilde{\Phi}_{(0)}(z, z_0) \) of the Gaussian free field \( \Phi_{(0)} \) with the Dirichlet boundary condition:

\[
\Phi_{(0)}^N(z, z_0) = \tilde{\Phi}_{(0)}(z, z_0).
\]

For a real parameter \( b \), we define the central charge modification \( \Phi_{(b)}^N(z, z_0) \) by

\[
\Phi_{(b)}^N(z, z_0) := \Phi_{(0)}^N(z, z_0) + 2b \log \left| \frac{w(z)}{1 - w(z)^2} \right| - 2b \log \left| \frac{w(z_0)}{1 - w(z_0)^2} \right|.
\]

Then \( \Phi_{(b)}^N(z, z_0) = \tilde{\Phi}_{(b)}(z, z_0) \). Now we define the current \( J_{(b)}^N \) and the Virasoro field \( T_{(b)}^N \) by

\[
J_{(b)}^N(z) = i\partial_z \Phi_{(b)}^N(z, z_0), \quad T_{(b)}^N = -\frac{1}{2} J_{(b)}^N * J_{(b)}^N + \Phi_{(b)}^N \partial_z \Phi_{(b)}^N,
\]

so that \( J_{(b)}^N = J_{(0)} \) and \( T_{(b)}^N = T_{(0)} \). Also we have \( \tilde{\Phi}_{(b)}^N(z, z_0) = -\Phi_{(b)}(z, z_0) \).

4.2 Multi-vertex fields

It is convenient to describe multi-vertex fields in terms of formal fields. We formally define

\[
\mathcal{O}^{(\sigma)} = M^{(\sigma)} e^{\partial \sigma \Phi^+_0}, \quad M^{(\sigma)} = E[\mathcal{O}^{(\sigma)}] = (w^\sigma)^h (1 - w^2)^\mu,
\]

where \( w \) is any conformal transformation from \( D, q, q_x \) into \( (\mathbb{H}, -1, 1) \), and \( h = \frac{1}{4} \sigma^2 - \sigma b, \mu = \sigma b \). As a multivalued Fock space field, a chiral bi-vertex field is defined by

\[
\mathcal{O}^{(\sigma)}(z, z_0) = \mathcal{O}^{(\sigma)}(z) \mathcal{O}^{(-\sigma)}(z_0) = M^{(\sigma)}(z) M^{(-\sigma)}(z_0) I^{(\sigma)}(z, z_0) e^{\partial \sigma \Phi^+_0(z, z_0)},
\]

(4.1)
where the interaction term \( I \) is given by
\[
I^\sigma(z, z_0) = e^{\sigma^2 \mathcal{E}[\Phi(z)\Phi(z_0)]} = (w - w_0)^{-\sigma^2}.
\]

Next we define a general 1-point vertex field \( \mathcal{O}^{(\sigma, \sigma_s)} \) by
\[
\mathcal{O}^{(\sigma, \sigma_s)}(z) = \mathcal{O}^{(\sigma)}(z) \mathcal{O}^{(\sigma_s)}(z) = M^{(\sigma)}(z) M^{(\sigma_s)}(z) I^{(\sigma, \sigma_s)}(z) e^{\sigma \Phi(z_0)} e^{-i \sigma \Phi(z)},
\]
where the interaction term \( I \) is given by
\[
I^{(\sigma, \sigma_s)}(z) = e^{\sigma^2 \mathcal{E}[\Phi(z_0)]} = (w - w_0)^{\sigma \sigma_s}.
\]
Thus \( M^{(\sigma, \sigma_s)} = \mathbb{E}[\mathcal{O}^{(\sigma, \sigma_s)}] = (w')^h (w')^{b_z} (1 - w')^{\mu} (w - w)^{\sigma \sigma_s} \) with the dimensions and exponents
\[
h = \frac{\sigma^2}{2} - \sigma b, \quad h_s = \frac{\sigma^2}{2} - \sigma_s b, \quad \mu = \sigma b, \quad \mu_s = \sigma_s b.
\]

Finally we define the multi-vertex field
\[
\mathcal{O}^{(\sigma, \sigma_s)}(z) = \prod_{j,k} M^{(\sigma_j, \sigma_{s_j})}(z_j) \prod_{j,k} I_{j,k}(z_j, z_k) e^{\sigma \Phi(z_0)} e^{-i \sigma \Phi(z)} e^{\sigma \Phi(z_0)} e^{-i \sigma \Phi(z)} = \prod_{j,k} M^{(\sigma_j, \sigma_{s_j})}(z_j) I_{j,k}(z_j, z_k),
\]
(4.2)

where \( \sigma = (\sigma_1, \ldots, \sigma_n), \sigma_s = (\sigma_{s_1}, \ldots, \sigma_{s_n}), \) and \( z = (z_1, \ldots, z_n). \) The interaction terms \( I_{j,k}(z_j, z_k) = e^{\mathcal{E}[\Phi(z_j)] - \sigma \Phi(z_0) - \sigma_s \Phi(z_0)} \) are given by:
\[
I_{j,k}(z_j, z_k) = (w_j - w_k)^{\sigma_1 \sigma_{s_1}} (w_j - w_k)^{\sigma_2 \sigma_{s_2}} (w_j - w_k)^{\sigma_3 \sigma_{s_3}} (w_j - w_k)^{\sigma_4 \sigma_{s_4}}.
\]
(4.3)

One can view a multi-vertex field as a product of general 1-point vertex fields:
\[
\mathcal{O}^{(\sigma, \sigma_s)}(z) = \mathcal{O}^{(\sigma_1, \sigma_{s_1})}(z_1) \mathcal{O}^{(\sigma_2, \sigma_{s_2})}(z_2) \cdots \mathcal{O}^{(\sigma_n, \sigma_{s_n})}(z_n).
\]

**Proposition 4.1** Under the neutrality condition, \( \mathcal{O} \equiv \mathcal{O}^{(\sigma, \sigma_s)} \) are well-defined \( \text{Aut}(D, q_\pm) \)-invariant Fock space fields. Moreover, they satisfy Ward’s OPE:
\[
T(\zeta) \mathcal{O}(z) \sim h_j \frac{\mathcal{O}(z)}{(\zeta - z_j)^2} + \frac{\partial_j \mathcal{O}(z)}{\zeta - z_j}, \quad \bar{T}(\zeta) \mathcal{O}(z) \sim \bar{h}_j \frac{\mathcal{O}(z)}{(\zeta - z_j)^2} + \frac{\partial_j \mathcal{O}(z)}{\zeta - z_j},
\]
as \( \zeta \to z_j. \)

**Proof** Under the neutrality condition, \( \sum_{j} \sigma_j \Phi(z_j) = \sum_{j} \sigma_j \Phi^\dagger(z_j) = \sum_{j} \sigma_j \Phi(z_j) = 0 \) is a well-defined Fock space field,
\[
\sigma_j \Phi(z_j) + \sigma_j \Phi^\dagger(z_j) = 0.
\]

Suppose \( \mathbb{E}[\mathcal{O}^{(\sigma, \sigma_s)}] \) is expressed as \( M^{(\sigma, \sigma_s)}, \tilde{M}^{(\sigma, \sigma_s)} \) in terms of two different conformal transformations \( \tilde{w}, \tilde{w} \) from \( (D, q_-, q_+) \) onto \( (\mathbb{H}, -1, 1) \). Since a nontrivial element \( h \) in \( \text{Aut}(\mathbb{H}, -1, 1) \) is of the form
\[
h(z) = \frac{az + 1}{z + a}, \quad (a \in \mathbb{R} \setminus [-1, 1]),
\]
the ratio of $\tilde{M}^{(\sigma, \sigma_i)}$ and $M^{(\sigma, \sigma_i)}$ is

$$\left(\frac{\alpha^2 - 1}{\beta_j + a}\right)^{\sum_j (\sigma_j + \sigma_{j'})} \prod (w_j + a)^{\sigma_j + \sigma_{j'} + \sum_k \sigma_j} (w_j + a)^{\sigma_j + \sigma_{j'}}$$

Thus the rooted multi-vertex field $O^{(\sigma, \sigma_i)}$ with the neutrality condition is $\text{Aut}(D, q_+)$-invariant.

Finally let us show that Ward’s OPE holds for $\tilde{\Theta}$. Since the multi-vertex field is a differential, it is enough to verify Ward’s OPE in the upper half-plane. In the identity chart of the upper half-plane,

$$J = J_0 + ib \frac{2z}{1 - z^2}, \quad T = T_0 - j J_0 + (ib \partial J + \frac{1}{2} j^2),$$

where $j = 2ibz/(1 - z^2)$ and $T_0 = \frac{1}{2} J_0 \circ J_0$. In the simplest case $b = 0$, let us show that the singular part of operator product expansion of $T_0(\zeta)$ and $\tilde{\Theta}(z)$ is

$$\frac{\sigma^2}{2} \tilde{\Theta}^{\frac{\sigma}{z - z_j}} + i \sigma \left( \frac{\sigma}{z - z_j} \right) \tilde{\Theta}^{\frac{\sigma}{z - z_j}} + \left( \frac{\sigma}{z - z_j} \right) \tilde{\Theta}^{\frac{\sigma}{z - z_j}}$$

as $\zeta \to z_j$. For this, let

$$F \equiv F(\zeta, z) := EJ(\zeta) \sum (i \sigma_j \Phi_0^+(z_j) - i \sigma_j \Phi_0^-(z_j)) = -i \sum \left( \frac{\sigma_j}{\sigma_j - z_j} + \frac{\sigma_j}{\sigma_j - z_j} \right).$$

It follows from Wick’s calculus that

$$T_0(\zeta) \tilde{\Theta}(z) = T_0(\zeta) \tilde{\Theta}(z) - F J_0(\zeta) \tilde{\Theta}(z) - \frac{1}{2} F^2 \tilde{\Theta}(z).$$

While the first term $T_0(\zeta) \tilde{\Theta}(z)$ has no contribution to Ward’s OPE for $\tilde{\Theta}$, the second term $-F(\zeta, z) J_0(\zeta) \tilde{\Theta}(z)$ has the singular part

$$i \sigma J_0(\zeta) \tilde{\Theta}(z) \frac{\sigma}{\sigma - z_j}, \quad (\zeta \to z_j).$$

The singular part of the last term $-\frac{1}{2} F(\zeta, z)^2 \tilde{\Theta}(z)$ is

$$\frac{\sigma^2}{2} \tilde{\Theta}^{\frac{\sigma}{z - z_j}} + \left( \frac{\sigma}{z - z_j} \right) \tilde{\Theta}^{\frac{\sigma}{z - z_j}} + \left( \frac{\sigma}{z - z_j} \right) \tilde{\Theta}^{\frac{\sigma}{z - z_j}}$$

This proves Ward’s OPE when $b = 0$. For $b \neq 0$, we just need to show that

$$j(\zeta) J_0(\zeta) \tilde{\Theta}(z) \sim 2 \sigma_j \frac{\zeta_j - \zeta_j}{1 - \zeta_j - \zeta_j}, \quad ib \partial J_0(\zeta) \tilde{\Theta}(z, \z_0) \sim -\sigma_j \frac{\tilde{\Theta}(z)}{(\zeta - z_j)^2}.$$

Both of the singular OPEs follow from

$$J_0(\zeta) \tilde{\Theta}(z, \z_0) \sim -i \sigma_j \frac{\tilde{\Theta}(z)}{(\zeta - z_j)}.$$

Ward’s OPE for $\tilde{\Theta}$ can be obtained in a similar way.
4.3 Rooted multi-vertex fields

In this subsection we introduce the (formal) multi-vertex fields \( \Theta^{(\sigma, \sigma, \ldots, \sigma)} \) rooted at two marked boundary points \( q_-, q_+ \). The definition of rooted vertex fields can be arrived to by normalizing the tensor product of formal vertex fields, \( \Theta^{(\sigma, \sigma)}(z) \), \( \Theta^{(\sigma, \ldots, \sigma)}(\zeta_-) \), and \( \Theta^{(\sigma, \ldots, \sigma)}(\zeta_+) \) (\( \zeta_- \in \partial D \)) so that the limit exists as \( (\zeta_-, \zeta_+) \) tends to \( (q_-, q_+) \). For example, we define the rooted vertex field \( \Theta^{(\sigma, \sigma, \ldots, \sigma)} \) by

\[
\Theta^{(\sigma, \sigma, \ldots, \sigma)}(z) = M^{(\sigma, \sigma, \ldots, \sigma)}(z) e^\frac{C}{2} \Theta^{(\sigma, \sigma)}(z) - \sigma, \Theta^{(\sigma)}(q_-) + \sigma, \Theta^{(\sigma)}(q_+) \),
\]

where \( M^{(\sigma, \sigma, \ldots, \sigma)} = E[\Theta^{(\sigma, \sigma, \ldots, \sigma)}] \) is given by

\[
M^{(\sigma, \sigma, \ldots, \sigma)}(z) = (1 - w)^{\sigma_+} (1 + w)^{\sigma_-} (w - \bar{w})^{\sigma_0} \times (w')^{h_+} (w')^{h_-}, \quad (w'_\pm = w'(q'_\pm)).
\]

The dimensions \([h, h_+; h_-; h_+; \ldots; h_+]\) and exponents are given by

\[
h = \frac{\sigma_0^2}{2} - \sigma b, \quad h_+ = \frac{\sigma_0^2}{2} - \sigma b, \quad h_- = \frac{\sigma_0^2}{2}.
\]

and \( v_\pm = \sigma(b + \sigma_\pm), \quad v_\pm = \sigma(b + \sigma_\pm). \)

Let us explain this definition. We express the correlation function \( E[\Theta^{(\sigma, \sigma)}(z) \Theta^{(\sigma, \ldots, \sigma)}(\zeta_-) \Theta^{(\sigma, \ldots, \sigma)}(\zeta_+)] \) as

\[ M^{(\sigma, \sigma)}(\zeta_-) M^{(\sigma, \sigma)}(\zeta_+) I_+(\zeta_-, \zeta_+) I_+(\zeta_-, \zeta_+) \]

where \( M^{(\sigma, \sigma)}(\zeta) = E[\Theta^{(\sigma, \sigma)}(\zeta)] = (w(\zeta))^k (1 - w(\zeta_\pm)^2)^{\mu_\pm}, (\lambda_\pm = \frac{1}{2} \sigma_\pm^2 - \sigma, \pm b, \mu_\pm = \sigma, \pm b) \) and the interaction terms \( I_+(\zeta_-, \zeta_+) \) are given by

\[ I_+(\zeta_-, \zeta_+) = (w - w(\zeta_\pm))^{\sigma_\pm}(w - w(\zeta_\pm))^{\sigma_\pm} \]

We now apply the following rooting rules to \( E[\Theta^{(\sigma, \sigma, \ldots, \sigma)}(z) \Theta^{(\sigma, \ldots, \sigma)}(\zeta_-) \Theta^{(\sigma, \ldots, \sigma)}(\zeta_+)] \):

1. The term

\[ (1 - w(\zeta_\pm)^2)^{\mu_\pm} \]

in \( M^{(\sigma, \sigma)}(\zeta) \) is replaced by

\[ (w(\zeta_\pm))^{\mu_\pm} \]

2. All other terms \( w(\zeta_\pm) \) are replaced by \( \pm 1 \).

Thus \( h_\pm = \lambda_\pm + \mu_\pm = \frac{1}{2} \sigma_\pm^2 \) and \( v_\pm = \mu + \sigma_\pm, v_\pm = \mu + \sigma_\pm, v_\pm = \mu + \sigma_\pm \). The above rooting rules are obtained (up to constant) by

\[
\Theta^{(\sigma, \sigma, \ldots, \sigma)}(z) := \lim_{\varepsilon \to 0} \Theta^{(\sigma, \sigma)}(z) \Theta^{(\sigma_-, \ldots, \sigma)}(\zeta_-) \Theta^{(\sigma_+, \ldots, \sigma)}(\zeta_+),
\]

where \( \zeta_\pm \) is at distance \( \varepsilon \) from \( q_\pm \) in a given chart. The definition of rooted 1-point vertex fields can be extended to the rooted multi-vertex fields as follows:

\[
\Theta^{(\sigma, \sigma, \ldots, \sigma)}(w'_- \ldots w'_+) \prod M_j I_{j,k} e^{\frac{C}{2}} \Theta^{(\sigma_+, \ldots, \sigma)}(q_-) + \sigma, \Theta^{(\sigma)}(q_+) \sum \sigma, \Theta^{(\sigma)}(\zeta_-) - \sigma, \Theta^{(\sigma)}(\zeta_+)).
\]
where the interaction term $I_{j,k}$ is the same as (4.3) and

$$M_j = (w_j^i)^{h_j} (w_j^r)^{h_j} (1 - w_j)^{v_j^i} (1 + w_j)^{v_j^r} (1 + \bar{w}_j)^{\nu_j^r} (w - \bar{w})^{\sigma_\sigma},$$

with the exponents $v_j^i = \sigma_j (b + \sigma_\pm), v_j^r = \sigma_j (b + \sigma_\pm)$. The dimensions $[\mathbf{h}, h_-, h_+]$ of rooted multi-vertex fields $\mathcal{O}^{(\sigma, \sigma_+, \sigma_-)}$ are given by

$$h_j = \frac{\sigma_+^2}{2} - \sigma_j h, \quad h_j^* = \frac{\sigma_-^2}{2} - \sigma_j h, \quad h_\pm = \frac{\sigma_+^2 \pm \sqrt{\sigma_+^2 - 4\sigma_j h}}{2}.$$

If the neutrality condition

$$\sigma_+ + \sigma_- + \sum (\sigma_j + \sigma_j^* ) = 0$$

holds, then a rooted vertex field $\mathcal{O}^{(\sigma, \sigma_+, \sigma_-)}$ is a well-defined $\text{Aut}(D, q_-, q_+)$-invariant Fock space field. While the rooted multi-vertex field $\mathcal{O}^{(\sigma, \sigma_+, \tau_-)}$ in the radial case can be viewed as the OPE exponential of the field

$$i \tau \Phi_{(b)}^+(q) - i \tau \Phi_{(b)}^-(q) + i \sum \sigma_j \Phi_{(b)}^+(z_j) - \sigma_j \Phi_{(b)}^-(z_j),$$

the field $\mathcal{O}^{(\sigma, \sigma_+, \sigma_+)}$ in the dipolar case can be interpreted as the OPE exponential of the field

$$i \sigma_- \Phi_{(b)}^-(q_-) + i \sigma_+ \Phi_{(b)}^+(q_+) + i \sum \sigma_j \Phi_{(b)}^+(z_j) - \sigma_j \Phi_{(b)}^-(z_j).$$

Applying the rooting procedure, we define the normalized tensor product of rooted multi-vertex fields by

$$\mathcal{O}^{(\sigma, \sigma_+, \sigma_+)} \ast \mathcal{O}^{(\tau, \tau, \tau)} = \mathcal{O}^{(\sigma + \tau, \sigma_+ + \tau_-, \sigma_+ + \tau_-, \sigma_+, \tau_+)}.$$

One can view $\sigma, \sigma_+, \tau, \tau_-$ as divisors, maps from $D \setminus \{q_-, q_+\}$ to $\mathbb{R}$ which take the value 0 at all but finitely many points.

### 4.4 Ward’s identity and equation for rooted multi-vertex fields

We now extend the OPE family $\mathcal{F}_{(b)}$ to include rooted multi-vertex fields. This extension is natural in the sense that Ward’s OPEs for multi-vertex fields survive under the rooting procedure.

**Proposition 4.2** For a rooted multi-vertex field $\mathcal{O} \equiv \mathcal{O}^{(\sigma, \sigma_+, \sigma_-)}$, Ward’s OPE

$$T(\zeta) \mathcal{O}(\zeta) \sim \bar{h}_j \frac{\mathcal{O}(\zeta)}{(\zeta - z_j)^2} + \frac{\partial_{z_j} \mathcal{O}(\zeta)}{\zeta - z_j}, \quad (\zeta \rightarrow z_j),$$

holds and similar equation holds (with $\bar{h}_j$) for $\bar{\mathcal{O}}$.

Applying the rooting procedure, we derive the following Ward’s identity for rooted multi-vertex fields.
Proposition 4.3 For a rooted vertex field $\vartheta(z) \equiv \vartheta^{(\sigma, \sigma_1, \ldots, \sigma_\lambda)}$ with the neutrality condition, we have Ward’s identity

$$E[W(v; \hat{D}\setminus\{q_{\pm}\}) \vartheta(z)] = E[\mathcal{L}_v \vartheta(z)] + (h_- + h_+) E[\vartheta(z)], \quad (h_\pm = \frac{1}{2} \sigma_\pm^2),$$  

where $v$ is a non-random local holomorphic vector field with $v(q_{\pm}) = 0, v'(q_{\pm}) = 1$, and $z_j \in \text{D}_{\text{hol}}(v)$. The Lie derivative operator $\mathcal{L}_v$ does not apply to the points $q_{\pm}$.

Proof Since vertex fields are differentials, it suffices to perform the computation in the $(\mathbb{H}, -1, 1)$-uniformization. Suppose that

$$\vartheta(z) = \lim_{\varepsilon \to 0} \frac{\vartheta^{(\sigma, \sigma_1)}(z) \vartheta^{(\sigma_\pm)}(\zeta_\pm^\varepsilon)}{(1 + \zeta_\pm^\varepsilon)^{-\mu}(1 - \zeta_\pm^\varepsilon)^{\mu_+}},$$

where $\zeta_\pm^\varepsilon$ is at distance $\varepsilon$ from $\pm 1$. We write $\vartheta(z, \zeta_-, \zeta_+)$ for the (unrooted) multi-vertex field $\vartheta^{(\sigma, \sigma_1)}(z) \vartheta^{(\sigma_\pm)}(\zeta_\pm)^{\sigma_\pm}(\zeta_\pm^\varepsilon)$.

Since Ward’s identity holds for $\vartheta(z, \zeta_-, \zeta_+)$, it is enough to show that

$$\lim_{\varepsilon \to 0} \frac{E[\mathcal{L}_v \vartheta(z, \zeta_-, \zeta_+)]}{(1 + \zeta_\pm^\varepsilon)^{-\mu}(1 - \zeta_\pm^\varepsilon)^{\mu_+}} = E[\mathcal{L}_v \vartheta(z)] + (h_- + h_+) E[\vartheta(z)],$$

Clearly,

$$\lim_{\varepsilon \to 0} \frac{E[\mathcal{L}_v \vartheta(z, \zeta_-, \zeta_+)]}{(1 + \zeta_\pm^\varepsilon)^{-\mu}(1 - \zeta_\pm^\varepsilon)^{\mu_+}} = E[\mathcal{L}_v \vartheta(z)],$$

where $\mathcal{L}_v(z) = \sum (v(z_j) \partial_j + v'(z_j) \overline{\partial_j} + h_j v'(z_j) + h_j \overline{v'(z_j)})$. Write $\mathcal{L}_v(\zeta_\pm)$ for the differential operator, $\sum (v(\zeta_\pm) \partial_{\zeta_\pm} + \lambda_\pm v'(\zeta_\pm))$. Then

$$\lim_{\varepsilon \to 0} \frac{E[\mathcal{L}_v(\zeta_\pm) \vartheta(z, \zeta_-, \zeta_+)]}{(1 + \zeta_\pm^\varepsilon)^{-\mu}(1 - \zeta_\pm^\varepsilon)^{\mu_+}} = \lim_{\varepsilon \to 0} \frac{\mu_1 \frac{v(\zeta_\pm^\varepsilon)}{\zeta_\pm^\varepsilon} + \lambda_\pm v'(\zeta_\pm^\varepsilon)}{(1 + \zeta_\pm^\varepsilon)^{-\mu}(1 - \zeta_\pm^\varepsilon)^{\mu_+}} E[\vartheta(z, \zeta_-, \zeta_+)],$$

which completes the proof.

Since the dipolar Loewner vector field

$$v_\varepsilon(z) = \frac{1 - \zeta^2}{\zeta - \zeta_\varepsilon}$$

in the upper half-plane satisfies $v_\varepsilon(\pm 1) = 0, v'_\varepsilon(\pm 1) = 1$, we can apply the previous proposition to the vector field $v_\varepsilon$ together with Proposition 3.3 and derive the following form of Ward’s equation for a rooted multi-vertex field.

Proposition 4.4 For a rooted multi-vertex field $\vartheta \equiv \vartheta^{(\sigma, \sigma_1, \ldots, \sigma_\lambda)}$ in the extended OPE family $\mathcal{F}_b$, we have

$$\frac{(1 - \zeta_\varepsilon^2)^2}{2} E[T_b(\zeta) \vartheta] = E[\mathcal{L}_{v_\varepsilon^+} + \mathcal{L}_{v_\varepsilon^-} \vartheta'] + (h_- + h_+ - b^2) E[\vartheta],$$

where $T_b$ and $\vartheta$ are evaluated in the identity chart of the upper half-plane.
We now generalize the previous proposition.

**Proposition 4.5** For a 1-point rooted vertex field \( V \), and a rooted multi-vertex field \( \mathcal{O} \) in \( \mathcal{F}_b \), in the identity chart of the upper half-plane, we have

\[
E'[V(z) \star \mathcal{L}^+_{\nu}(V(z) \star \mathcal{O})] + E'[\mathcal{L}^+_{\nu} V(z) \star \mathcal{O}]
\]

\[
= \frac{(1 - z^2)}{2} E[(L_{-2} V)(z) \star \mathcal{O}] - \frac{3z(1 - z^2)}{2} E[L_{-1} V(z) \star \mathcal{O}]
\]

\[
+ \left( \frac{3z^2 - 1}{2} h(V) + b^2 - (h_-(V \star \mathcal{O}) + h_+(V \star \mathcal{O})) \right) E[V(z) \star \mathcal{O}],
\]

where \( h(V) \) is the conformal dimension of \( V \) with respect to \( z \) and \( h_\pm(V \star \mathcal{O}) \) is the boundary dimension of \( V \star \mathcal{O} \) with respect to \( q_\pm \).

**Remark.** The Lie derivative operators \( \mathcal{L}^\pm_{\nu} \) in the above proposition do not apply to the points \( q_\pm \). For the definition of a field \( V \star \mathcal{L}^\pm_{\nu} \mathcal{O} \), the rooting rules can be applied to the tensor product of a bi-vertex field and the Lie derivative of a multi-vertex field.

**Proof** Since Leibniz’s rule applies to \( \star \)-products, i.e.,

\[
\mathcal{L}^+_{\nu}(V \star \mathcal{O}) = (\mathcal{L}^+_{\nu} V) \star \mathcal{O} + V \star (\mathcal{L}^+_{\nu} \mathcal{O}),
\]

it follows from Proposition 4.4 that

\[
E'[V(z) \star \mathcal{L}^+_{\nu}(V(z) \star \mathcal{O})] + E'[\mathcal{L}^+_{\nu} V(z) \star \mathcal{O}]
\]

\[
= \frac{(1 - \zeta^2)}{2} E[T(\zeta)V(z) \star \mathcal{O}] - E[(\mathcal{L}^+_{\nu} \mathcal{O})(z) \star V(z)] - (h_- + h_+ - b^2) E[V(z) \star \mathcal{O}],
\]

where \( h_\pm = h_\pm(V \star \mathcal{O}) \). By (3.5),

\[
\lim_{\zeta \to z} \frac{(1 - \zeta^2)}{2} T(\zeta)V(z) = \mathcal{L}^+_{\nu} V(z)
\]

\[
= \frac{(1 - z^2)}{2} T^* V(z) - \frac{3z(1 - z^2)}{2} T^*_{-1} V(z) + \frac{3z^2 - 1}{2} T^*_{-2} V(z).
\]

Since \( V \) is a primary field in \( \mathcal{F}_b \) with conformal dimension \([h, 0] , L_0 V = h V \), see Proposition A.1.

### 4.5 Level two degeneracy equations

In Subsection 2.2 we introduce insertion fields

\[
e^{\frac{1}{2}\text{Log}(\Phi^p(q))} \quad (p \in \partial D \setminus \tilde{Q}).
\]

These Wick’s exponentials can be normalized properly such that the normalized fields form a one-parameter family of \( \text{Aut}(D, q_{-1}, q_{+}) \)-invariant (Virasoro) primary fields

\[
\psi := \mathcal{O}(a, 0; -\frac{1}{a}, -\frac{1}{a})
\]
in the extended OPE family, $\mathcal{F}_{(b)}$. By definition of rooted vertex fields,

$$\Psi(z) = (w_+)^{-\frac{1}{2}}(w_-)^{\frac{1}{2}}(w_+)^{-\frac{1}{2}} w(z) h^b e^{\frac{i}{2} a h(\Phi(z) + \Phi^+(z))},$$

(4.7)

where $h = \frac{1}{2} a^2 - ab$, and $w$ is a conformal transformation from $(D, q_-, q_+)$ onto $(-1, 1)$.

(Recall that $w_+ = w'(q_+)$.) As rooted vertex fields, $\Psi$ are current primary fields (see Subsection A.6) with charges $q = a, q_+ = 0$, i.e.,

$$J_0\Psi = -ia\Psi, \quad J_0\Psi = 0, \quad J_n\Psi = J_n\bar{\Psi} = 0 (n \geq 1),$$

(4.8)

where the modes $J_n$ of the current field are defined as

$$J_n(z) := \frac{1}{2\pi i} \oint_{(z)} (\xi - z)^n J(\xi) d\xi.$$

(4.9)

To show (4.8), one needs to check the singular OPEs

$$J_{(0)}(\xi)\Psi(z) \sim -ia\frac{\Psi(z)}{\bar{z} - z}, \quad J_{(0)}(\xi)\bar{\Psi}(z) \sim 0$$

in the identity chart of the upper half-plane. The first singular OPE follows from Wick’s calculus

$$J_{(0)}(\xi)\Psi(z) = J_{(0)}(\xi) \circ \Psi(z) + \frac{ia}{2} \mathbb{E}[J_{(0)}(\xi)(\Phi_{(0)}^+ (z, -1) + \Phi_{(0)}^+(z, 1))]\Psi(z)$$

and $\mathbb{E}[J_{(0)}(\xi)(\Phi_{(0)}^+ (z, -1) + \Phi_{(0)}^+(z, 1))] = -2/\xi - z + 2\xi/(\xi^2 - 1)$. The second OPE follows from the similar Wick’s decomposition and the fact that the correlation

$$\mathbb{E}[J_{(0)}(\xi)(\Phi_{(0)}^+ (z, -1) + \Phi_{(0)}^+(z, 1))] = -2\frac{\xi}{\xi - z} + 2\frac{\xi}{\xi^2 - 1}$$

has no singular term.

**Proposition 4.6** If $2a(a + b) = 1$, then

$$T_{(b)} + \Psi = 1 \frac{1}{2a^2} \phi^2 \Psi.$$

This proposition follows immediately from the characterization of level two degenerate current primary fields, see Proposition A.2. We combine Proposition 4.6 with Ward’s equations to prove Theorems 2.1 and 2.2.

5 **Connection between dipolar SLE and CFT**

After we introduce the insertion fields $\Psi$ as boundary condition changing operators acting on Fock space functionals/fields, we prove that correlation functions of fields in the OPE family, $\mathcal{F}_{(b)}$, of $\Phi_{(b)}$ under the insertion of $\Psi(p)/\mathbb{E}[\Psi(p)]$ are dipolar SLE$_k(p \rightarrow Q)$ martingale-observables. The main ingredient for its proof is BPZ-Cardy equation which is derived from the level two degeneracy equation for $\Psi$ and Ward’s equation. As applications, we discuss the restriction property of dipolar SLE$_{8/3}$ and the dipolar version of Friedrich-Werner’s formula.
5.1 Boundary condition changing operator

We define a boundary condition changing operator \( \mathcal{X} \mapsto \tilde{\mathcal{X}} \) as a linear operator acting on Fock space functionals in the following way. By definition, \( \mathcal{X} \mapsto \tilde{\mathcal{X}} \) is given by the rules

\[
\partial \mathcal{X} \mapsto \partial \tilde{\mathcal{X}}, \quad \hat{\mathcal{X}} \mapsto \tilde{\mathcal{X}}, \quad \mathcal{X} \circ \mathcal{Y} \mapsto \tilde{\mathcal{X}} \circ \tilde{\mathcal{Y}},
\]

and the formula

\[
\sum \sigma_j \Phi^+_\sigma(\zeta_j) - \sigma_j \Phi^-_\sigma(\zeta_j) \mapsto \sum \frac{-i\sigma_\mu}{2} \log \frac{w_j^2}{1 - w_j^2} + \frac{i\sigma_\mu}{2} \log \frac{w_j^2}{1 - w_j^2} + \sum \sigma_j \Phi^+_\sigma(\zeta_j) - \sigma_j \Phi^-_\sigma(\zeta_j),
\]

where \( w \) is the conformal transformation from \((D,p,q_-)\) onto \((\mathbb{H},0,-1,1)\) and the neutrality condition \( \sum_j (\sigma_j + \sigma_{j*}) = 0 \) holds.

Let us denote by \( \tilde{\mathcal{R}}(b) \) the image of \( \mathcal{R}(b) \) under this boundary condition changing operator \( \mathcal{X} \mapsto \tilde{\mathcal{X}} \). Also we denote

\[
\tilde{E}[\mathcal{X}] := \frac{E[\Phi^+(p)\mathcal{X}]}{E[\Phi^+(p)]} = E[e^{\frac{1}{2}w(\Phi^+(p,q_-) + \Phi^+(p,q_+))\mathcal{X}}] = E[\mathcal{X}], \quad (5.1)
\]

where \( \mathcal{X} \) is the string in \( \mathcal{R}(b) \) with nodes in \( D \setminus \{q_{\pm}\} \).

**Examples.** Let \( w \) be the conformal transformation from \((D,p,q_{\pm})\) onto \((\mathbb{H},0,\pm 1)\).

(a) The bosonic field \( \tilde{\Phi} \) is a real part of pre-pre-Schwarzian form of order \( ib \),

\[
\tilde{\Phi} = \Phi + \sigma_{\mu} w^2 \frac{1}{1 - w^2} = \Phi_{(0)} + \sigma_{\mu} w^2 \frac{1}{1 - w^2} - 2b \arg \frac{w'}{1 - w^2};
\]

(b) The current field \( \tilde{J} \) is a pre-Schwarzian form of order \( ib \),

\[
\tilde{J} = J - ia \frac{w'}{w(1 - w^2)} = J_{(0)} - ia \frac{w'}{w(1 - w^2)} + ib \frac{w'}{w^2} + \frac{2ww'}{1 - w^2};
\]

(c) The Virasoro field \( \tilde{H} \) is a Schwarzian form of order \( \frac{1}{2}c \),

\[
\tilde{H} = A_{(0)} - jj_{(0)} + ib\tilde{j}_{(0)} + \frac{c}{12} S_{\nu} + h_1 \frac{w^2}{w^2(1 - w^2)} + 4h_{0,1/2}(\frac{w'}{1 - w^2})^2,
\]

where \( A_{(0)} = -\frac{1}{2} j_{(0)} \), \( \tilde{j} = E[\tilde{J}] \) and \( h = \frac{1}{2} a^2 - ab, h_{0,1/2} = \frac{1}{2} a^2 - \frac{1}{2} b^2 \);

(d) The multi-vertex field \( \tilde{\phi}_{(\sigma, \sigma_j)} \) is a \([h, h_b]\)-differential (\( h_j = \frac{1}{2} \sigma_j^2 - \sigma_j b, h_{js} = \frac{1}{2} \sigma_j^2 - \sigma_j b \),

\[
\tilde{\phi}_{(\sigma, \sigma_j)}(z) = \prod j_k^{\tilde{M}_{\sigma_j}(\sigma_j, \sigma_{j*})(z_j)} \prod l_k e^{\frac{i}{2} \sum \sigma_\mu \Phi^+_\sigma_j(\zeta_j) - \sigma_j \Phi^-_{\sigma_j}(\zeta_j)},
\]

where \( \tilde{M}_{\sigma_j}(\sigma_j, \sigma_{j*})(z_j) = (w_j')^{h_j} (1 - w_j^2) \tilde{\mu}_{j*} (\zeta_j) \Phi^+_\sigma_j, \tilde{\mu}_j w_j^2 \Phi^-_{\sigma_j} (w_j - \tilde{w}_j) \sigma_{\sigma} \) and interaction term \( l_k \) is the same as (4.3). The exponents are given by

\[
\tilde{\mu}_j = \mu_j - \frac{1}{2} \sigma_j a = \sigma_j (b - \frac{1}{2} a), \quad \tilde{\mu}_{js} = \mu_{js} - \frac{1}{2} \sigma_j a = \sigma_j (b - \frac{1}{2} a).
\]
5.2 BPZ-Cardy equations

We now derive BPZ-Cardy equations in the dipolar case. Suppose $X = X_1(z_1) \cdots X_n(z_n)$ is the tensor product of fields $X_j$ in the OPE family $\mathcal{F}_b$. For $\xi \in \mathbb{H}$, we denote

$$\hat{\mathbb{E}}_{\xi}[X] = \mathbb{E}[e^{\frac{i}{2}\int a(\Phi^+_{(0)}(\xi,-1)+\Phi^+_{(1)}(\xi,1))}].$$

Proposition 5.1 If $2a(a+b) = 1$, then we have

$$\hat{\mathbb{E}}_{\xi}[\mathscr{L}_{\xi}^\pm X] = \frac{1}{2a^2} \left( \frac{1-\xi^2}{2} \partial^2 - \xi(1-\xi^2) \partial \xi \right) \hat{\mathbb{E}}_{\xi}[X], \quad v_\xi(z) := \frac{1-z^2 - \bar{\xi}z}{2} \frac{1-\xi}{\bar{\xi} - z}, \quad (5.2)$$

where all fields are evaluated in the identity chart of $\mathbb{H}$ and $\partial \xi = \partial + \bar{\partial}$.

Proof In the $(\mathbb{H},0,-1,1)$-uniformization, the rooted vertex field $\Psi = \mathcal{O}(a,0;\frac{i}{2},a,-\frac{i}{2})$ is evaluated at $\xi$ as

$$\Psi(\xi) = (1-\xi^2)^{-h} e^{\frac{i}{4}a(\Phi_{(0)}(\xi,-1)+\Phi_{(1)}(\xi,1))},$$

where $h = \frac{1}{2}a^2 - ab$. For $\xi \in \mathbb{H}$, let

$$R_\xi \equiv R(\xi;z_1,z_2,\cdots,z_n) = \mathbb{E}[(1-\xi^2)^h \Psi(\xi)X].$$

By Ward’s equation (Proposition 3.3), $L^{-1} \Psi = \partial \Psi$, and level two degeneracy equation (Proposition 4.6) for the rooted vertex field $\Psi$, we have

$$\mathbb{E}[\Psi(\xi)(\mathscr{L}_{\xi}^+ X + \mathscr{L}_{\xi}^- X)] = \frac{1}{2a^2} \left( \frac{1-\xi^2}{2} \partial^2 \mathbb{E}[\partial^2 \Psi(\xi)X] - \frac{3\xi(1-\xi^2)}{2} \mathbb{E}[(\partial \Psi)(\xi)X] \right) + \left( \frac{3\xi^2}{2} - h + b^2 - \frac{a^2}{4} \right) \mathbb{E}[\Psi(\xi)X].$$

(We also use the fact that $\Psi$ is a holomorphic field, and therefore $\mathscr{L}_{\xi}^+ \Psi(\xi) = 0.$) Due to the numerology $2a(a+b) = 1$, it simplifies that

$$\mathbb{E}[(1-\xi^2)^h \Psi(\xi)(\mathscr{L}_{\xi}^+ X + \mathscr{L}_{\xi}^- X)] = \frac{1}{2a^2} \left( \frac{1-\xi^2}{2} \partial^2 R_\xi - \xi(1-\xi^2) \partial R_\xi \right),$$

where $\hat{\partial}$ is the operator of differentiation with respect to the complex variable $\xi$. We now take the limits of both sides as $\xi \to \xi$. Since $\xi$ is real, the left-hand side converges to

$$\mathbb{E}[(1-\xi^2)^h \Psi(\xi)X] = \hat{\mathbb{E}}_{\xi}[\mathscr{L}_{\xi}^+ X].$$

On the other hand, since $\partial \xi = \partial + \bar{\partial}$, and the rooted vertex field $\Psi$ is holomorphic, $\partial R_\xi$, and $\partial^2 R_\xi$ in the right-hand sides converge to $\partial_\xi R_\xi$ and $\partial_\xi^2 R_\xi$, respectively.
5.3 Dipolar SLE martingale-observables

It is convenient to describe dipolar SLEs in terms of the \((\mathbb{H}, -1, 1)\)-uniformization. Let \(\xi_t = (e^{\sqrt{\kappa} B_t} - 1)/(e^{\sqrt{\kappa} B_t} + 1)\) and let \(g_t\) be the dipolar SLE map from \((D, \gamma, Q)\) onto \((\mathbb{H}, \xi_t, \mathbb{R} \setminus [-1, 1])\). Then \(g_t\) satisfies

\[
\partial_t g_t(z) = \frac{1 - g_t^2(z)}{2} \frac{1 - \xi_t g_t(z)}{\xi_t - g_t(z)},
\]

(5.3)

where \(g_0 : (D, p, Q) \rightarrow (\mathbb{H}, 0, \mathbb{R} \setminus [-1, 1])\) is the conformal map from \(D\) onto the upper half-plane \(\mathbb{H}\). Let us restate Theorem 2.1 and present its proof.

**Theorem 5.2** If \(X_j\)'s are Fock space fields in the OPE family \(\mathcal{F}_{(\gamma)}\), then a non-random field

\[
M(z_1, \ldots, z_n) = \mathbb{E}[X_1(z_1) \cdots X_n(z_n)]
\]

is a martingale-observable for dipolar SLE\(\kappa\).

**Proof** Conformal invariance allows us to represent the process

\[
M_t(z_1, \ldots, z_n) \equiv M_{(D, \gamma, Q)}(z_1, \ldots, z_n)
\]

as

\[
M_t = m(\xi_t, t), \quad m(\xi, t) = (R_{\xi} \parallel g_t^{-1}),
\]

where \(g_t : (D, \gamma, q_-, q_+) \rightarrow (\mathbb{H}, \xi_t, -1, 1)\) is the dipolar SLE map driven by \(\xi_t\) and

\[
R_{\xi}(z_1, \ldots, z_n) = \mathbb{E}_{\xi}[X_1(z_1) \cdots X_n(z_n)].
\]

Itô’s formula can be applied to \(m(\xi, t)\) since the function \(m(\xi, t)\) is smooth in both \(\xi\) and \(t\). Since the driving process \(\xi_t = (e^{\sqrt{\kappa} B_t} - 1)/(e^{\sqrt{\kappa} B_t} + 1)\) satisfies

\[
d\xi_t = \sqrt{\kappa} (1 - \xi_t^2) dB_t - \frac{\kappa}{4} \xi_t (1 - \xi_t^2) dt,
\]

Itô’s formula shows that \(M_t\) is a semi-martingale with the drift term

\[
\frac{\kappa}{4} \left( \frac{1 - \xi_t^2}{2} \partial_\xi^2 - \xi (1 - \xi_t^2) \partial_\xi \right) m(\xi, t) dt + \left. \frac{d}{dx} \right|_{x=0} (R_{\xi} \parallel g_t^{-1}) dt,
\]

where \(L_t = \partial_t |_{x=0} (R_{\xi} \parallel g_t^{-1}) = \partial_t |_{x=0} (R_{\xi} \parallel g_t^{-1} \circ f_{s,t}^{-1})\), and \(f_{s,t} = g_{t+s} \circ g_t^{-1}\). It follows from (5.3) that the time-dependent flow \(f_{s,t}\) satisfies

\[
\frac{d}{dx} f_{s,t}(\zeta) = -v_{\zeta,s}(f_{s,t}(\zeta)), \quad v_{\zeta}(z) := \frac{1 - \zeta^2}{2} \frac{1 - \xi \zeta}{\xi - z},
\]

Thus \(f_{s,t}\) can be approximated by \(i d - sv_{\zeta} + o(s)\) as \(s \to 0\). Since \(X_j\)'s depend smoothly on local charts,

\[
L_t = -(\mathcal{L}_{v_{\zeta}} R_{\xi} \parallel g_t^{-1}).
\]

It follows from BPZ-Cardy equations that \(M_t\) is driftless.
5.4 The restriction property

In this subsection we present CFT theoretic proof for the restriction property of the dipolar SLE$_{8/3}$; the dipolar SLE$_{8/3}$ path in $(\mathbb{H}, -1, 1)$ conditioned to avoid a fixed compact hull $K$ with $\partial K \cap \mathbb{R} \subseteq (-1, 1) \setminus \{0\}$ has the same distribution as the dipolar SLE$_{8/3}$ path in $(\mathbb{H} \setminus K, -1, 1)$.

Let $\kappa \leq 4$. On the event $\gamma(0, \infty) \cap K = \emptyset$, we denote $\Omega_t = g_t(D_t \setminus K), \tilde{\gamma} = \psi_K \circ \gamma$ and define a conformal map $h_t : \Omega_t \to \mathbb{H}$ by

$$h_t := \tilde{g}_t \circ \psi_K \circ g_t^{-1},$$

where $\tilde{g}_t$ is a dipolar Loewner map from $(\mathbb{H} \setminus \tilde{\gamma}[0, t], -1, 1)$ onto $(\mathbb{H}, -1, 1)$ and $\psi_K$ is the conformal transformation from $(\mathbb{H} \setminus K, -1, 1)$ onto $(\mathbb{H}, -1, 1)$ such that $\psi_K^{-1}(1) = \psi_K(1)$.

Let

$$M_t = (1 - \frac{\varepsilon t^2}{2^2}) F(\Psi_{\text{eff}}(\tilde{g}_t), \|\text{id}_{\tilde{g}_t}\),$$

where $\Psi_{\text{eff}}$ is the effective boundary condition changing operator, see (2.7). Then

$$\frac{\lambda}{2} = h(t(\Psi_{\text{eff}}) = \frac{\mu}{2} - \frac{\kappa}{2}, \quad \mu = h_f(t(\Psi_{\text{eff}}) = \frac{\kappa^2}{8} - \frac{\kappa}{2}),$$

Let

$$M_t = \left(1 - \frac{\varepsilon t^2}{2^2}\right) S_{h_t}(\tilde{g}_t) M_t \, dt.$$  

**Lemma 5.3** The process $M_t$ is a semi-martingale with the drift term

$$\frac{\lambda}{2} = h(t(\Psi_{\text{eff}}) = \frac{\mu}{2} - \frac{\kappa}{2}, \quad \mu = h_f(t(\Psi_{\text{eff}}) = \frac{\kappa^2}{8} - \frac{\kappa}{2}),$$

**Proof** Let $F(z, t) := E(\Psi_{\text{eff}}(z) \| \text{id}_{\tilde{g}_t})$. Then $M_t = (1 - \frac{\varepsilon t^2}{2^2}) F(\tilde{g}_t, t)$. Application of Itô’s formula to the smooth function $F$ gives the drift term of $dM_t/M_t$:

$$\frac{\dot{F}(\tilde{g}_t, t)}{F(\tilde{g}_t, t)} = \frac{\kappa}{4} (1 + 2\lambda) (1 - \frac{\varepsilon t^2}{2^2}) F'(\tilde{g}_t, t) + \frac{\mu}{8} (1 - \frac{\varepsilon t^2}{2^2}) F''(\tilde{g}_t, t) + \frac{\lambda^2}{2} \frac{\varepsilon t^2}{2^2} - \frac{\kappa}{4}.$$

We represent

$$\frac{\dot{F}(z, t)}{F(z, t)} = \frac{d}{ds} |_{s=0} E(\Psi_{\text{eff}}(h_{t+1}) \| h_{t+1}) (z) = \frac{d}{ds} |_{s=0} E(\Psi_{\text{eff}}(h_{t+1} \circ f_{s,t}) \| h_{t+1}) (z), \quad (f_{s,t} = h_{t+1} \circ h_{t-1}).$$

in terms of Lie derivative

$$\frac{\dot{F}(z, t)}{F(z, t)} = E(L(v, \tilde{g}_t) \Psi_{\text{eff}} \| h_{t+1}) (z), \quad (v \| \text{id}_{\tilde{g}_t}) = \frac{d}{ds} |_{s=0} f_{s,t} = h_t \circ h_t^{-1},$$

where the Lie derivative operator $L(v, \tilde{g}_t)$ applies to the points $+1$. To compute the vector field, we apply the chain rule to $h_t = \tilde{g}_t \circ \psi_K \circ g_t^{-1}$ and compute the capacity changes. Indeed,

$$h_t(z) = \left(1 - \frac{\varepsilon t^2}{2^2}\right) v_{h_t}(\tilde{g}_t)(h_t(z)) + \frac{\kappa}{2 \varepsilon t^2}.$$

$$v_{\tilde{g}_t}(z) := \frac{1 - \varepsilon t^2}{2 \varepsilon t^2}.$$
Thus
\[
(v^ightharpoonup \text{id}_{\mathcal{H}})(\xi) = \left( \frac{1 - \xi^2}{1 - h_t(\xi)^2} \right)^2 v_{h(\xi)}(\xi) + h_t'(h_t^{-1}(\xi)) v_{h_t^{-1}(\xi)}.
\] (5.6)

By (5.5) and (5.6), we have
\[
\dot{F}(z, t) = -\left( \frac{1 - \xi^2}{1 - h_t(\xi)^2} \right)^2 h_t'(z)^3 \mathbf{E}(\mathcal{L}(v_{h(\xi)}, \mathcal{H}) \Psi_{\text{eff}}(h_t(z))) \| \text{id}_{\mathcal{H}}
\]
\[
+ \mathbf{E}(\mathcal{L}(v_{\xi}, \Omega \setminus \{ \pm 1 \}) \Psi_{\text{eff}} \| \text{id}_{\Omega})(z) + 2\mu \mathbf{E}(\Psi_{\text{eff}} \| \text{id}_{\Omega})(z),
\]
where the Lie derivative operator \(\mathcal{L}(v_{\xi}, \Omega \setminus \{ \pm 1 \})\) does not apply to the points \(\pm 1\). It follows from Ward’s equation that
\[
\dot{F}(z, t) = -\left( \frac{1 - \xi^2}{2} \right)^2 h_t'(z)^3 h_t'(z)^3 \mathbf{E}(T(h_t(\xi)) \Psi_{\text{eff}}(h_t(z))) \| \text{id}_{\Omega})(z)
\]
\[
+ \mathbf{E}(\mathcal{L}(v_{\xi}, \Omega \setminus \{ \pm 1 \}) \Psi_{\text{eff}} \| \text{id}_{\Omega})(z) + 2\mu \mathbf{E}(\Psi_{\text{eff}} \| \text{id}_{\Omega})(z).
\]
By conformal invariance,
\[
\dot{F}(z, t) = -\left( \frac{1 - \xi^2}{2} \right)^2 \mathbf{E}(T(h_t(\xi)) \Psi_{\text{eff}}(z) \| \text{id}_{\Omega}) + \frac{\sqrt{c}}{24}(1 - \xi^2)^2 S_{h}(\xi) \mathbf{E}(\Psi_{\text{eff}} \| \text{id}_{\Omega})(z)
\]
\[
+ \mathbf{E}(\mathcal{L}(v_{\xi}, \Omega \setminus \{ \pm 1 \}) \Psi_{\text{eff}} \| \text{id}_{\Omega})(z) + 2\mu \mathbf{E}(\Psi_{\text{eff}} \| \text{id}_{\Omega})(z).
\]
Using (3.5) and the fact that \(L^{-1} \Psi = \partial \Psi, L_0 \Psi = \lambda \Psi, L_1 \Psi = 0\), we have
\[
\frac{\dot{F}(\xi, t)}{F(\xi, t)} = -\left( \frac{1 - \xi^2}{2} \right)^2 \left( \frac{\mathbf{E}(T(h_t(\xi)) \Psi_{\text{eff}}(\xi) \| \text{id})}{\mathbf{E}(\Psi_{\text{eff}}(\xi) \| \text{id})} \right) + \frac{3\xi(1 - \xi^2)}{2} \frac{F'((\xi, t))}{F((\xi, t))}
\]
\[
- \left( \frac{3\xi^2 - 1}{2} \lambda - 2\mu \right) + \frac{\sqrt{c}}{24}(1 - \xi^2)^2 S_{h}(\xi).
\]
Plugging the above equation into the drift term of \(dM_t\), lemma now follows from the level two degeneracy equation for \(\Psi_{\text{eff}}\).

From now on, we use the \((\mathcal{S}, -\infty, \infty)\)-uniformization. By abuse of notation, let \(\xi = \sqrt{\kappa}B_t\) (cf. \(\xi\) in Subsection 5.3) and let \(g_t\) be the dipolar SLE map from \((D_t, \gamma, \mathcal{O})\) onto \((\mathcal{S}, \xi, \mathbb{R} + \pi i)\). As before, for a compact hull \(K\) with \(K \cap (\mathbb{R} + \pi i) = \emptyset\), we denote \(\Omega_t = g_t(D_t \setminus K), \gamma = \psi_K \circ \gamma\) and define a conformal map \(h_t : \Omega_t \rightarrow \mathcal{S}\) by
\[
h_t := g_t \circ \psi_K \circ g_t^{-1},
\]
where \(g_t\) is a dipolar Loewner map from \((\mathcal{S} \setminus \gamma[0, t], -\infty, \infty)\) onto \((\mathcal{S}, -\infty, \infty)\) and \(\psi_K\) is the conformal transformation from \((\mathcal{S} \setminus K, -\infty, \infty)\) onto \((\mathcal{S}, -\infty, \infty)\) such that
\[
\lim_{\xi \rightarrow \pm \infty} \psi_K(z) - z = \pm \text{scap}(K).
\]
Then the process (5.4) in the \((\mathbb{H}, -1, 1)\)-uniformization becomes
\[
M_t = \left( h_t'(\xi) \right)^4 e^{-2\mu \text{scap}(K)}, \quad K_t = \mathbb{H} \setminus \Omega_t
\]
in the \((\mathcal{S}, -\infty, \infty)\)-uniformization.
Proof (Proof of Theorem 2.3) By Lemma 5.3, the process \( M_t = (H_t(x))^k e^{-2\mu \text{scap}(K)} \) is a local martingale if \( \kappa = 8/3 \). We first claim that the process \( M_t \) is a bounded continuous martingale. Since \( \text{scap}(K) \geq 0 \) for a compact hull \( K \) with \( K \cap (\mathbb{R} + \pi i) = \emptyset \), it suffices to show that \( (0 <) \psi_t(x) \leq 1 \) for \( x \in \mathbb{R} \setminus K \). As in the chordal case, \( \exists \psi_t(z) \leq \exists z \) is a bounded harmonic function with non-positive boundary values. Thus \( \exists \psi_t(z) \leq \exists z \) for \( z \in \mathbb{S} \setminus K \) and \( \psi_t(x) \leq 1 \) for \( x \in \mathbb{R} \setminus K \).

Let \( T = \inf\{t \geq 0: \gamma(0,t) \cap K \neq \emptyset \} \). It follows from the martingale convergence theorem that \( \lim_{t \to T} M_t \) exists a.s. The proof that \( \lim_{t \to T} M_t = 1_{T \geq 0} \) a.s. is similar to that in the chordal case (see [11, Theorem 6.1]). We leave it as an exercise for the reader. By the optional stopping theorem,

\[
\psi_t(0)^k e^{-2\mu \text{scap}(K)} = M_0 = EM_T = P \{ T = \infty \}.
\]

This proves the theorem.

We now prove Friedrich-Werner’s formula in the dipolar case.

Proof (Proof of Theorem 2.4) Denote \( x = (x_1, \cdots, x_n) \), and

\[
R(\xi; x) = E \left[ \psi(x) T(x_1) \cdots T(x_n) \right].
\]

The non-random field \( R = R(\xi; x) \) has the following properties:

- (R1) it is a boundary differential of dimension \( \lambda = 5/8 \) with respect to \( \xi \);
- (R2) it is a boundary differential of dimension 2 with respect to \( x_j \);
- (R3) it is a boundary differential of dimension \( \mu = 5/96 \) with respect to \( q_\lambda \).

We apply Ward’s equations to the function \( R(\xi; x) \) so that we replace \( T(x) \) in \( R(\xi; x) \) by the Lie derivative operator:

\[
R(\xi; x, x) = (L(v_x, \bar{S}) + \mu) R(\xi; x), \quad (\text{in } id_\bar{S}),
\]

where \( v_x(\xi) = \frac{1}{2} \coth_2(x - \xi) \) and the Lie derivative operator \( L(v_x, \bar{S}) \) do not apply to the points \( \pm \infty \).

Let

\[
U(x) = \lim_{\varepsilon \to 0} e^{-2\pi \varepsilon} \mathbb{P}(\text{SLE}_{8/3} \text{ hits all slits } [x_j, x_j + i\varepsilon \sqrt{2}])
\]

(if the limit exists). We define the non-random field \( T(x; x) \) as follows:

- it satisfies the transformation laws (R1) – (R3);
- \( T(x; x_1, \cdots, x_n) \| id_\bar{S}) = U(x_1 - \xi, \cdots, x_n - \xi) \).

We now claim that the limit \( U(x, x) \) exists under the assumption of existence of the limit \( U(x) \) and that

\[
T(0, x) = (L(v_x, \bar{S}) + \mu) T(0; x).
\]

The non-random fields \( T(0; \cdot) \) and \( R(0; \cdot) \) satisfy the same recursive equation (see (5.7) and (5.8)). Thus \( T(0; \cdot) = R(0; \cdot) \) for all \( n \geq 1 \) since \( T(0; \cdot) = R(0; \cdot) = 1 \) for \( n = 0 \). Therefore, we have \( U(x) = R(0; x) \).
To verify this claim, we write $\mathbb{P}(x)$ for the probability that dipolar SLE$_{8/3}$ path hits all segments $[x_j, x_j + i e \sqrt{2}]$ and $\mathbb{P}(x \mid -x)$ for the same probability conditioned on the event that the path avoids $[x, x + i e \sqrt{2}]$. By the induction hypothesis,

$$P(x) \approx e^{2n} T(0; x).$$

(5.9)

On the other hand, it follows from the restriction property of dipolar SLE that

$$1 - P(x) = (\psi(0))^{2k} (e^{-scap([x, x + i e \sqrt{2}])})^{2\mu}$$

and

$$\mathbb{P}(x \mid -x) \approx e^{2n} T(\psi(0); \psi(x_1), \ldots, \psi(x_n)) \prod_{j=1}^n \psi'(x_j)^2,$$

where $\psi$ is a slit map from $(S \setminus \{x, x + i e \sqrt{2}, 0, \pm \infty\})$ onto $(S, 0, \pm \infty)$. This $\psi$ satisfies

$$\psi(z) = \phi_2(z - x) - \phi_2(-x), \quad \cosh_2 \psi_2(z) = e^{i \bar{z} \cosh_2}, \quad \epsilon' = 1 + \tan^2(\epsilon / \sqrt{2})$$

where $\cosh_2 = \cosh(\frac{1}{2} \bar{z})$. By (5.9) – (5.11), we approximate $e^{-2n \mathbb{P}(x, x)}$ by

$$T(0; x) - \psi(0)^{2k} (e^{-scap([x, x + i e \sqrt{2}])})^{2\mu} T(\psi(0); \psi(x_1), \ldots, \psi(x_n)) \prod_{j=1}^n \psi'(x_j)^2.$$ 

Thus the limit $U(x, x)$ exists. Since $\text{scap}([x, x + i e \sqrt{2}]) = \log(1 + \tan^2(\epsilon / \sqrt{2})) \approx \frac{1}{2} \epsilon^2$, we have

$$T(0; x, x) = (\mathcal{L}(v, \bar{v}) + \mu) T(0; x).$$

6 Vertex observables

We expand our OPE family $\mathcal{F}(\Phi_0)$ by considering the rooted multi-vertex fields with the neutrality condition. In this section we extend Theorem 2.1 to this expanded family. As examples of screening of rooted vertex observables, we discuss Cardy-Zhan’s observables that describe the probability for a point to be to the left (right) of the dipolar SLE path and the probability for a point to be swallowed by the dipolar SLE hull.

6.1 Rooted Multi-vertex fields

We apply the rooting rules (in Subsection 4.3) to the multi-vertex fields $\hat{\mathcal{G}}^{(\sigma, \sigma)} \hat{\mathcal{G}}^{(\sigma, -)} \hat{\mathcal{G}}^{(\sigma, +)}$ and arrive at the definition of rooted multi-vertex fields $\hat{\mathcal{G}}^{(\sigma, \sigma, \sigma)}$:

$$\hat{\mathcal{G}}^{(\sigma, \sigma, \sigma)} = \hat{\mathcal{M}}^{(\sigma, \sigma, \sigma)} e^{i \mathcal{R}(\sigma, \sigma, \sigma)} e^{i \mathcal{R}(\sigma, \sigma, \sigma)} + \Sigma_{\sigma, \sigma, \sigma}^+ \mathcal{O}(0(z) - \sigma, \Phi(0(z))),$$

where $\hat{\mathcal{M}}^{(\sigma, \sigma, \sigma)} = \mathbb{E}[\hat{\mathcal{G}}^{(\sigma, \sigma, \sigma)}] = (w')^{\hat{h}} (w')^{\hat{h}} \prod_{j=1}^k I_{j, k}$. The interaction term $I_{j, k}$ is the same as (4.3) and

$$\hat{\mathcal{M}}_j = (w_j)^{\hat{h}_j} (w_j)^{\hat{h}_j} w_j^{\hat{\sigma} \hat{\sigma}} (1 - w_j^{\hat{\sigma} \hat{\sigma}})^{\hat{v}} (1 + w_j^{\hat{\sigma} \hat{\sigma}})^{\hat{v}} (1 - \bar{w}_j^{\hat{\sigma} \hat{\sigma}})^{\hat{v}} (1 + \bar{w}_j^{\hat{\sigma} \hat{\sigma}})^{\hat{v}}.$$
with the exponents $v_\pm = \sigma_j(b - \frac{i}{4}a + \sigma_\pm), \overline{v}_\pm = \sigma_j(b - \frac{i}{4}a + \sigma_\pm)$. The dimensions $[h, \overline{h}, \overline{h}_-, \overline{h}_+]$ of rooted multi-vertex fields $\hat{O}^{(a, a, \ldots, a)}$ are given by

$$h_j = \frac{\sigma_j^2}{2} - \sigma_j b, \quad \overline{h}_j = \frac{\sigma_j^2}{2} - \sigma_j b, \quad \overline{h}_\pm = \frac{\sigma_\pm^2}{2} - \sigma_\pm a.$$

Alternatively, one can define $\hat{O}^{(a, a, \ldots, a)}$ by the action of boundary condition changing operator $X \mapsto \hat{O} X$ on $O^{(a, a, \ldots, a)}$. Indeed, the boundary condition changing operator $X \mapsto \hat{X}$ can be extended to formal fields/functionals by the formula

$$\Phi^+_0 \mapsto \Phi^+_0 - \frac{ia}{2} \log \frac{w^2}{1 - w^2}, \quad \Phi^+_0(q_\pm) \mapsto \Phi^+_0(q_\pm) + \frac{ia}{2} \log w_\pm$$

and the property that it commutes with complex conjugation. Note that the interaction terms are preserved under the boundary condition changing operator. For the rooted multi-vertex field $O \equiv O^{(a, a, \ldots, a)}$,

$$\hat{O} = (w' - \frac{i}{2}a)(w' - \frac{i}{2}a) \prod w_j^{a_j} (1 - w_j^*) - \frac{i}{2}a \overline{w}_j^{a_j} (1 - \overline{w}_j^*) - \frac{i}{2}a \hat{O}.$$

Thus two definitions coincide.

For rooted multi vertex fields $O \equiv O^{(a, a, \ldots, a)}$ with the neutrality condition, let us denote

$$\hat{E}[O] := \frac{E[\Psi(p) * O]}{E[\Psi(p)]}, \quad \hat{E}_\zeta[O] := \frac{E[\Psi(\zeta) * O]}{E[\Psi(\zeta)]}, \quad (\zeta \in \mathbb{D} \setminus \{q_\pm\}).$$

Then Equation (5.1) can be extended to the rooted multi-vertex fields:

$$\hat{E}[O] = E[O].$$

Proposition 5.1 (BPZ-Cardy equations) extends to the rooted multi-vertex fields.

**Proposition 6.1** Suppose that the parameters $a$ and $b$ are related as $2a(a + b) = 1$. Then for rooted multi-vertex fields $O \equiv O^{(a, a, \ldots, a)}$ with the neutrality condition,

$$\hat{E}_\zeta[O_v] + (\hat{h}_- + \overline{h}_+) \hat{E}_\zeta[O] = \frac{1}{2a^2} \left(\frac{1 - \zeta^2}{2} \partial_v^2 - \zeta (1 - \zeta^2) \partial_v \right) \hat{E}_\zeta[O], \quad (6.1)$$

where all fields are evaluated in the identity chart of the upper half-plane and $\partial_v = \partial + \overline{\partial}$. The vector field $v_\zeta$ is given by

$$v_\zeta(z) := 1 - \frac{\zeta^2}{2} \frac{1 - \zeta^2}{\zeta - \zeta}.$$

**Proof** For $\zeta \in \mathbb{D} \setminus \{1\}$, let us denote

$$R_\zeta \equiv R(\zeta, z_1, \ldots, z_n) = E[(1 - \zeta^2)^{h_+} \Psi(\zeta) * O],$$
where \( h = \frac{1}{4}a^2 - ab \). Then it follows from Ward’s equation (Proposition 4.5), \( L_{\gamma} \Psi = \partial \Psi \), and the level two degeneracy equation for \( \Psi \) (Proposition 4.6) that
\[
E[\Psi \ast (\mathcal{L}_{\xi}^+ \partial + \mathcal{L}_{\xi}^- \partial)] = \frac{1}{2a^2} \left( \frac{1 - \xi^2}{\xi^2} \right) E[(\partial^2 \Psi) \ast \partial] - \frac{3 \xi(1 - \xi^2)}{2} E[(\partial \Psi) \ast \partial] + \left( \frac{3 \xi^2 - 1}{2} h + b^2 - h_- (\Psi \ast \partial) - h_+ (\Psi \ast \partial) \right) E[\Psi \ast \partial],
\]
where \( h_\pm (\Psi \ast \partial) \) is the dimension of boundary differential \( \Psi \ast \partial \) with respect to \( q_\pm \). (We also use the holomorphicity of \( \Psi \), and therefore \( \mathcal{L}_{\xi}^- \Psi (\xi) = 0 \).) By the numerology \( 2a(a + b) = 1 \) and the relation \( h_\pm (\Psi \ast \partial) = h_\pm (\Psi) + h_\pm (\partial) = \frac{1}{4}a^2 + \hat{h}_\pm \), we have
\[
E[(1 - \xi^2)^b \Psi \ast (\mathcal{L}_{\xi}^+ \partial + \mathcal{L}_{\xi}^- \partial)] + (\hat{h}_- + \hat{h}_+) E[(1 - \xi^2)^b \Psi \ast \partial] = \frac{1}{2a^2} \left( \frac{1 - \xi^2}{\xi^2} \partial^2 - \xi(1 - \xi^2) \partial \right) R_\xi,
\]
where \( \partial \) is the operator of differentiation with respect to the complex variable \( \xi \). Taking the limits of both sides as \( \xi \to \xi \), we obtain BPZ-Cardy equations.

Now we prove Theorem 2.2.

Proof (Proof of Theorem 2.2) Denote
\[
R_\xi = \mathcal{E}_\xi[\partial^{(\sigma, \alpha, \ldots, \sigma, \alpha)}].
\]
By conformal invariance, the process \( M_t \equiv M_t(D_{\xi}, q_\xi, Q) \) is represented by
\[
M_t = m(\xi, t), \quad m(\xi, t) = (R_\xi \parallel g_t^{-1}),
\]
where \( g_t \) is the dipolar SLE map from \( (D_{\gamma}, \gamma, q_\pm) \) onto \( (\mathbb{H}, \xi, \pm 1) \). Since \( g_t^\prime (q_\pm) = e^{-t} \), the drift term of \( dM_t \) is equal to
\[
\frac{1}{2a^2} \left( \frac{1 - \xi^2}{\xi^2} \partial^2 - \xi(1 - \xi^2) \partial \right) I_{\xi=\xi} m(\xi, t) dt - (\mathcal{L}_{\xi}^+ R_t \parallel g_t^{-1}) dt - (\hat{h}_- + \hat{h}_+) M_t dt,
\]
where the Lie derivative operator \( \mathcal{L}_{\xi}^+ \) does not apply the marked boundary points \( q_\pm \). It follows from Proposition 6.1 (BPZ-Cardy equations) that \( dM_t \) is driftless.

6.2 Cardy-Zhan’s observables

Let \( \kappa > 4, \zeta \in D \). We consider the following geometric observables
\[
N(z) = \mathbb{P}(\tau_c < \infty), \quad \mathbb{P}(z \text{ is to the left of } \gamma), \quad \mathbb{P}(z \text{ is to the right of } \gamma).
\]
They are real-valued with all conformal dimensions zero. There is no such vertex observable except for the constant field. However, the derivative \( \partial N \) can be identified as a vertex field with conformal dimensions
\[
\lambda_c = 1, \quad \lambda_c^* = \lambda_{q^+} = \lambda_{q^-} = 0.
\]
Indeed, by dimension calculus, a vertex field $O(-2a,0;\pm a)$ satisfies the above requirements. Let $M$ be a martingale observables with all conformal dimensions zero such that

$$\partial M(z) = E[O(-2a,0;\pm a)] = w' \sinh^{-4/\kappa}(\frac{w}{2})$$

up to multiplicative constant, where $w$ is the conformal map from $(D,p,q,\pm)$ onto $(S,0,\pm\infty)$. In $S$, let us choose $M$ satisfying the normalization $M(-\infty) = 0, M(\infty) = 1$, and $M(0) > 0$. It follows from Schwarz-Christoffel formula that $M$ is the conformal transformation from $(S,0,\pm\infty)$ onto the isosceles triangle with angles $2\pi/\kappa$ at $M(-\infty) = 0, M(\infty) = 1$. Applying the optional stopping theorem and using the fact that

$$
\begin{cases}
M_{\tau} = M(0) & \text{if } \tau < \infty,
M_{\tau} = M(-\infty) = 0 & \text{if } z \text{ is to the left of } \gamma,
M_{\tau} = M(\infty) = 1 & \text{if } z \text{ is to the right of } \gamma,
\end{cases}
$$

we justify Cardy-Zhan’s formulas (see [14, Corollary 2.3.1])

$$
P(\tau \gamma < \infty) = \frac{3}{2} \frac{M(z)}{M(0)}, \quad P(z \text{ is to the right of } \gamma) = \Re M(z) - \frac{1}{2} \frac{3 \pm M(0)}{2}. $$

Remark. If $z = x + \pi i \in \mathbb{R} + \pi i$, then Cardy-Zhan’s observables

$$
M(x + \pi i) = \int_{-\infty}^{\infty} \cosh^{-4/\kappa}(\frac{x}{2}) \, d\xi
\int_{-\infty}^{\infty} \cosh^{-4/\kappa}(\frac{\xi}{2}) \, d\xi
$$

give the distribution of the endpoint $\gamma(\infty)$ of dipolar SLE$_\kappa (\kappa > 0)$ path.

**Appendix. Basic properties of conformal Fock space fields**

Here, we include some definitions and concepts of conformal Fock space fields developed in [8] so that this article can be read as a self-contained one.

### A.1 Fock space correlation functionals

This subsection is borrowed from [8, Section 1.3]. By definition, basic correlation functionals are formal expressions of the type (Wick’s product of $X_j(z_j)$)

$$\mathcal{X} = X_1(z_1) \circ \cdots \circ X_n(z_n),$$

where points $z_j \in D$ are not necessarily distinct and $X_j$’s are derivatives of the Gaussian free field, (i.e., $X_j = \partial_j \Phi$), or Wick’s exponentials

$$
e^{\circ \alpha \Phi} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Phi^\otimes n.$$
The constant 1 is also included to the list of basic functionals. We write $S_{\mathcal{X}}$ for the set of all points $z_j$ (the nodes of $\mathcal{X}$) in the expression of $\mathcal{X}$.

For derivatives $X_{j\ell}$ of the Gaussian free field and basic functionals of the form

$$\mathcal{X}_j = X_{j1}(z_{j1}) \circ \cdots \circ X_{j\ell}(z_{j\ell}),$$

we define the tensor product $\mathcal{X}_1 \cdots \mathcal{X}_m$ by

$$\mathcal{X}_1 \cdots \mathcal{X}_m = \sum \prod E[X_v(z_v)X_{v'}(z_{v'})] \circ X_{v''}(z_{v''}), \quad (A.1)$$

where the sum is taken over Feynman diagrams with vertices $v$ labeled by functionals $X_{j\ell}$ such that there are no contractions of vertices with the same $j$, and the Wick’s product is taken over unpaired vertices $v''$. By definition, $E[X_v(z_v)X_{v'}(z_{v'})]$ in (A.1) are given by the 2-point functions of derivatives of the Gaussian free field, e.g.,

$$E[\partial^j \Phi(\zeta) \partial^k \Phi(z)] = \partial^j_\zeta \partial^k_z E[\Phi(\zeta) \Phi(z)] = 2 \partial^j_\zeta \partial^k_z G(\zeta, z).$$

For example, the Feynman diagram with two edges $\{1, 4\}, \{3, 5\}$ and two unpaired vertices $2, 6$ corresponds to

$$\Phi(z_1) \circ \Phi(z_2) \circ \Phi(z_3) \circ \Phi(z_4) \circ \Phi(z_5) \circ \Phi(z_6)$$

$$:= E[\Phi(z_1) \Phi(z_4)] E[\Phi(z_3) \Phi(z_5)] E[\Phi(z_2) \Phi(z_6)].$$

The definition of tensor product can be extended to general correlation functionals by linearity. The tensor product of correlation functionals is commutative and associative, see [8, Proposition 1.1].

We define the correlation of $E[\mathcal{X}]$ of $\mathcal{X}$ by linearity, $E[1] = 1$, and

$$E[X_1(z_1) \circ \cdots \circ X_n(z_n)] = 0,$$

where $X_j$ are derivatives of $\Phi$. For example, $E[e^{\alpha \Phi(z)}] = 1$ and

$$E[\Phi(z_1) \cdots \Phi(z_n)] = \sum \prod 2G(z_k, z_{\ell}).$$

where the sum is over all partitions of the set $\{1, \ldots, n\}$ into disjoint pairs $\{i_k, j_k\}$.

If $E[\mathcal{X}_1] = E[\mathcal{X}_2]$ hold for all $\mathcal{Y}$ with nodes outside $S_{\mathcal{X}_1} \cup S_{\mathcal{X}_2}$, we identify $\mathcal{X}_1$ with $\mathcal{X}_2$ and write $\mathcal{X}_1 \approx \mathcal{X}_2$. We consider Fock space functionals modulo an ideal $\mathcal{N} = \{ \mathcal{Y} \approx 0 \}$ of Wick’s algebra. The concept of a correlation functional $\mathcal{X}$ can be extended to the case when some of the nodes of $\mathcal{X}$ lie on the boundary. For example, $e^{\alpha \Phi(z)} = 1$ for $z \in \partial D$. The complex conjugation $\overline{\mathcal{X}}$ of $\mathcal{X}$ is defined (modulo $\mathcal{N}$) by the equation $E[\overline{\mathcal{X}}] = E[\mathcal{X}]$ for all $\mathcal{X}$’s of the form $\Phi(z_1) \circ \cdots \circ \Phi(z_n)$. For example, if $J = \partial \Phi$ in the half-plane $\mathbb{H}$ and if $z \in \partial \mathbb{H}$, then $J(z)$ is purely imaginary, i.e., $\overline{J(z)} = -J(z)$, and $J(z) \circ J(z)$ is real.
A.2 Fock space fields

This subsection is borrowed from [8, Section 1.4]. Basic Fock space fields $X_\alpha$ are formal expressions written as Wick’s products of derivatives of the Gaussian free field $\Phi$ and Wick’s exponential $e^{z\partial x}$, e.g., $1$, $\Phi \circ \Phi$, $\Phi \circ \partial x$, $\partial x \circ \Phi$, $\partial x \circ e^{z\partial x}$, etc. A general Fock space field is a linear combination of basic fields $X_\alpha$,

$$X = \sum \alpha f_\alpha X_\alpha,$$

where $f_\alpha$’s are arbitrary (smooth) functions in $D$. If $X_1, \cdots, X_n$ are Fock space fields and $z_1, \cdots, z_n$ are distinct points in $D$, then $\mathcal{X} = X_1(z_1) \cdots X_n(z_n)$ is a correlation functional.

We define the differential operators $\partial$ and $\bar{\partial}$ on Fock space fields by specifying their action on basic fields so that the action on $\Phi$ is consistent with the definition of $\partial \Phi, \bar{\partial} \Phi$ and so that

$$\partial (X \circ Y) = (\partial X) \circ Y + X \circ (\partial Y), \quad \bar{\partial} (X \circ Y) = (\bar{\partial} X) \circ Y + X \circ (\bar{\partial} Y).$$

We extend this action to general Fock space fields by linearity and by Leibniz’s rule with respect to multiplication by smooth functions. Then (modulo $\mathcal{N}$)

$$E[(\partial X)(z) \mathcal{Y}] = \partial_z E[X(z) \mathcal{Y}], \quad (z \notin S_{\mathcal{Y}}),$$

for all correlation functionals $\mathcal{Y}$.

By definition, $X$ is holomorphic in $D$ if $\bar{\partial} X \approx 0$, i.e., all correlation functions $E[X(\zeta) \mathcal{Y}]$ are holomorphic in $\zeta \in D \setminus S_{\mathcal{Y}}$. For example, $J = \partial \Phi, X = X \circ J$ are holomorphic fields.

A.3 Operator product expansion

This subsection is borrowed from [8, Sections 3.1 – 3.2]. Operator product expansion (OPE) is the expansion of the tensor product of two fields near diagonal. For example,

$$\Phi(\zeta) \Phi(z) = \log \frac{1}{|\zeta - z|} + 2c(z) + \Phi^{(2)}(z) + o(1) \quad \text{as} \quad \zeta \to z, \quad \zeta \neq z, \quad (A.2)$$

where $c = \log C$ is the logarithm of conformal radius $C$, i.e., $c(z) = u(z, z), u(\zeta, z) = G(\zeta, z) + \log |\zeta - z|$. The meaning of the convergence is the following: the equation

$$E[\Phi(\zeta) \Phi(z) \mathcal{X}] = \log \frac{1}{|\zeta - z|^2} E[\mathcal{X}] + 2c(z) E[\mathcal{X}] + E[\Phi^{(2)}(z) \mathcal{X}] + o(1)$$

holds for all Fock space correlation functionals $\mathcal{X}$ in $D$ satisfying $z \notin S_{\mathcal{X}}$. To derive (A.2) we use Wick’s formula (A.1),

$$\Phi(\zeta) \Phi(z) = E[\Phi(\zeta) \Phi(z)] + \Phi(\zeta) \circ \Phi(z)$$

and the relation

$$E[\Phi(\zeta) \Phi(z)] = 2G(\zeta, z) = \log \frac{1}{|\zeta - z|^2} + 2c(z) + o(1).$$
The convergence of \( \Phi(\zeta) \odot \Phi(z) \) to \( \Phi^{\odot 2}(z) \) means (by definition) that
\[
E[\Phi(\zeta) \odot \Phi(z)] \to E[\Phi^{\odot 2}(z)]
\]
for every \( \mathcal{A} \) such that \( z \notin S_{\mathcal{A}} \).

If the field \( X \) is holomorphic (i.e., all correlation functions \( E[X(\zeta)\mathbb{A}] \) are holomorphic in \( \zeta \in D \setminus S_{\mathcal{A}} \)), then the operator product expansion is then defined as a (formal) Laurent series expansion
\[
X(\zeta)Y(z) = \sum C_n(z)(\zeta - z)^n, \quad \zeta \to z. \tag{A.3}
\]
Since the function \( \zeta \mapsto EX(\zeta)Y(z)\mathcal{A} \) is holomorphic in a punctured neighborhood of \( z \), it has a Laurent series expansion.

There are only finitely many terms in the principle (or singular) part of the Laurent series (A.3). We use the notation \( \sim \) for the singular part of the operator product expansion,
\[
X(\zeta)Y(z) \sim \sum_{n<0} C_n(z)(\zeta - z)^n.
\]
We also write \( \text{Sing}_{\zeta \to z}, X(\zeta)Y(z) \) for \( \sum_{n<0} C_n(z)(\zeta - z)^n \). It is clear that we can differentiate operator product expansions (A.3) both in \( \zeta \) and \( z \); and the differentiation preserves singular parts. For example,
\[
J(\zeta)\Phi(z) \sim -\frac{1}{\zeta - z}, \quad J(\zeta)J(z) \sim -\frac{1}{(\zeta - z)^2}.
\]
The coefficients in the operator product expansions (e.g., \( 2c(z) + \Phi^{\odot 2}(z) \) in (A.2), \( C_n(z) \) in (A.3)) are called \textit{OPE coefficients}. OPE coefficients of Fock space fields are Fock space fields (as functions of \( z \)). In particular, if \( X \) is holomorphic, then we define the \( \ast_n \) product by \( X \ast_n Y = C_n \). We write \( \ast \) for \( \ast_0 \) and call \( X \ast Y \) the \textit{OPE multiplication}, or the \textit{OPE product} of \( X \) and \( Y \).

Vertex fields are defined as OPE-exponentials of \( \Phi \):
\[
\mathcal{V}^\alpha = e^{\ast \Phi} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Phi^\ast_n.
\]
They can be expressed in terms of Wick’s calculus: \( \mathcal{V}^\alpha = C^{\alpha^2}e^{\odot \alpha \Phi} \), see [8, Proposition 3.3]. Here, \( C = e^\phi \) is the conformal radius, see (A.2). The Virasoro field is defined as OPE square of \( J = \partial \Phi \):
\[
T = -\frac{1}{2} J \ast J.
\]
Then by Wick’s calculus,
\[
T = -\frac{1}{2} J \odot J + \frac{1}{12} S,
\]
where \( S(z) = S(z,z), S(\zeta,z) := -12\partial_{\zeta}^\phi \partial_{\zeta} u(\zeta,z), \) and \( u(\zeta,z) = G(\zeta,z) + \log|\zeta - z| \). Thus \( T \) is a Schwarzian form of order \( \frac{1}{12} \). In terms of a conformal map \( w : D \to \mathbb{H}, S = S_w \), the Schwarzian derivative of \( w \):
This subsection is borrowed from [8, Sections 4.2 – 4.4]. We use Lie derivative of a conformal field to define the stress tensor and to state Ward’s identities.

A general conformal Fock space field is a linear combination of basic fields $X_\alpha$, 

$$X = \sum \alpha f_\alpha X_\alpha,$$

where $f_\alpha$’s are non-random conformal fields, see Subsection 2.1. A non-random conformal field $f$ is said to be invariant with respect to some conformal automorphism $\tau$ of $M$ if

$$(f \parallel \phi) = (f \parallel \phi \circ \tau^{-1})$$

for all charts $\phi$. For example, suppose $D$ is a planar domain and let us write $f$ for $(f \parallel \text{id}_D)$. Then $f$ is a $\tau$-invariant $[\lambda, \lambda^*]$-differential if

$$f(z) = f(\tau z) \tau'(z)^\lambda \tau^*(z)^{\lambda^*}.$$

It is because $\tau$ is the transition map between the charts $\phi \circ \tau^{-1}$ and $\phi = \text{id}_D$. By definition, a random conformal field (or a family of conformal fields) is $\tau$-invariant if all correlations are invariant as non-random conformal fields.

Suppose a non-random smooth vector field $v$ is holomorphic in some open set $U \subset M$. For a conformal Fock space field $X$, we define the Lie derivative $L_v X$ in $U$ as

$$\left( L_v X \parallel \phi \right) = \left. \frac{d}{dt} \right|_{t=0} \left( X \parallel \phi \circ \psi_t \right),$$

where $\psi_t$ is a local flow of $v$, and $\phi$ is an arbitrary chart.

Lie derivative of a differential is a differential but Lie derivative of a Schwarzian form is a quadratic differential:

- $L_v X = (v \partial + \nu \bar{\partial} + \lambda \nu' + \lambda_i \nu_{i\bar{j}}) X$ for a $[\lambda, \lambda_i]$-differential $X$;
- $L_v X = (v \partial + v') X + \mu \nu'$ for a pre-Schwarzian form $X$ of order $\mu$;
- $L_v X = (v \partial + 2v') X + \mu \nu''$ for a Schwarzian form $X$ of order $\mu$.

We recall basic properties of Lie derivatives:

- $L_v$ is an $\mathbb{R}$-linear operator on Fock space fields;
- $E[L_v X] = L_v (E[X])$;
- $L_v (\bar{X}) = \overline{L_v X}$;
- $L_v (\bar{\partial} X) = \bar{\partial} (L_v X)$ and $L_v (\bar{\partial} X) = \bar{\partial} (L_v X)$;
- Leibniz’s rule applies to Wick’s products, OPE products, and tensor products.

We define the $\mathbb{C}$-linear part $L_v^+$ and anti-linear part $L_v^-$ of the Lie derivative $L_v$ by

$$2L_v^+ = L_v - iL_{\bar{v}}, \quad 2L_v^- = L_v + iL_{\bar{v}}.$$
A.5 Stress tensor

This subsection is borrowed from [8, Sections 5.2 – 5.3]. A Fock space field $X$ in $D$ is said to have a (symmetric) stress tensor $(A,\bar{A})$ $(X \in \mathcal{F}(A,\bar{A}))$ if $A$ is a holomorphic quadratic differential and if Ward’s OPE holds for $X$, i.e., on a given chart $\phi : U \rightarrow \phi U$,

$$\text{Sing}_{\zeta \rightarrow \zeta}[A(\zeta)X(z)] = (L_{k_{\zeta}}^{+}X)(z), \quad \text{Sing}_{\zeta \rightarrow \zeta}[A(\zeta)\bar{X}(z)] = (L_{k_{\zeta}}^{+}\bar{X})(z),$$

where the (local) vector field $k_{\zeta}$ is defined by $(k_{\zeta} \parallel \phi)(\eta) = 1/(\zeta - \eta)$. Ward’s family $\mathcal{F}(A,\bar{A})$ is closed under differentiation and OPE multiplication, see [8, Proposition 5.8]. In the case of differentials or forms, it is enough to verify Ward’s OPEs in just one chart. For example, a $[\lambda, \bar{\lambda}]$-differential $X$ is in $\mathcal{F}(A,\bar{A})$ if and only if the following operator product expansions hold in every/some chart:

$$A(\zeta)X(z) \sim \frac{\lambda X(z)}{(\zeta - z)^2} + \frac{\phi(z)}{\zeta - z}, \quad A(\zeta)\bar{X}(z) \sim \frac{\bar{\lambda} \bar{X}(z)}{(\zeta - z)^2} + \frac{\bar{\phi}(z)}{\zeta - z}.$$

Let $X$ be a form of order $\mu$. Then $X \in \mathcal{F}(A,\bar{A})$ if and only if the following operator product expansion holds in every/some chart:

$$A(\zeta)X(z) \sim \frac{\mu}{(\zeta - z)^2} + \frac{\phi(z)}{\zeta - z} \quad \text{for a pre-pre-Schwarzian form } X;$$
$$A(\zeta)X(z) \sim \frac{2\mu}{(\zeta - z)^3} + \frac{X(z)}{(\zeta - z)^2} + \frac{\phi(z)}{\zeta - z} \quad \text{for a pre-Schwarzian form } X;$$
$$A(\zeta)X(z) \sim \frac{6\mu}{(\zeta - z)^4} + \frac{2X(z)}{(\zeta - z)^3} + \frac{\phi(z)}{\zeta - z} \quad \text{for a Schwarzian form } X.$$

For example, Gaussian free field $\Phi$ has a stress tensor

$$A = -\frac{1}{2} J \odot J, \quad J = \partial \Phi.$$

This holomorphic quadratic differential $A$ coincides with the Virasoro field $T$ in the upper half-plane uniformization. While $A$ itself does not belong to $\mathcal{F}(A,\bar{A})$, the Virasoro field $T$ is in $\mathcal{F}(A,\bar{A})$. We review the abstract theory of Virasoro field in the next subsection.

A.6 Virasoro field

This subsection is borrowed from [8, Lecture 7 and Appendix 11]. A Fock space field $T$ is said to be the Virasoro field for Ward’s family $\mathcal{F}(A,\bar{A})$ if

- $T \in \mathcal{F}(A,\bar{A})$, and
- $T - A$ is a non-random holomorphic Schwarzian form.

We define Virasoro primary fields and current primary fields in terms of Virasoro generators $L_{n}$ (3.6) and current generator $J_{n}$ (4.9).

**Proposition A.1 (Proposition 7.5 in [8])** Let $X$ be a Fock space field. Any two of the following assertions imply the third one (but neither one implies the other two):
$X \in \mathcal{F}(A, \bar{A})$;
$X$ is a $[\lambda, \lambda_*]$-differential;
$L_{\geq 1}X = 0$, $L_0X = \lambda X$, $L_{-1}X = \partial X$, and similar equations hold for $\bar{X}$.

Here, $L_{\geq k}X = 0$ means that $L_nX = 0$ for all $n \geq k$. We call fields satisfying all three conditions (Virasoro) primary fields in $\mathcal{F}(A, \bar{A})$.

A (Virasoro) primary field $X$ is called current primary if

$$J_{\geq 1}X = J_{\geq 1}\bar{X} = 0,$$

and

$$J_0X = -iqX, \quad J_0\bar{X} = iq\bar{X}$$

for some numbers $q$ and $q_*$. They are called “charges” of $X$. Here, $L_{\geq k}X = 0$ means that $J_nX = 0$ for all $n \geq k$. We use the following proposition to prove Proposition 4.6 (level two degeneracy equations for $\Psi$).

**Proposition A.2 (Proposition 11.2 in [8])** For a current primary field $V$ with charges $q, q_*$ in $\mathcal{F}(b)$,

$$\left( L_{-2} - \frac{1}{2q^2}L_{-1}^2 \right) V = 0$$

provided $2q(b + q) = 1$.

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