Drift parameter estimation for fractional Ornstein–Uhlenbeck process of the second kind

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The fractional Ornstein–Uhlenbeck process of the second kind (fOU\textsubscript{2}) is the solution of the Langevin equation \(dX_t = -\theta X_t \, dt + \, dY_t^{(1)}, \theta > 0\) with driving noise \(Y_t^{(1)} := \int_0^t e^{-s} \, dB_a; a_t = H e^{t/H}\) where \(B\) is a fractional Brownian motion with Hurst parameter \(H \in (0, 1)\). In this article, in the case \(H > \frac{1}{2}\), we prove that the least-squares estimator \(\hat{\theta}_T\) introduced in [Hu Y, Nualart D. Parameter estimation for fractional Ornstein–Uhlenbeck processes. Stat. Probab. Lett. 2010;80(11–12):1030–1038], provides a consistent estimator. Moreover, using central limit theorem for multiple Wiener integrals, we prove asymptotic normality of the estimator valid for the whole range \(H \in (\frac{1}{2}, 1)\).

\textbf{Keywords}: fractional Ornstein–Uhlenbeck processes; Malliavin calculus; Langevin equation; least-squares estimator

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1. Introduction

Assume \(B = \{B_t\}_{t \geq 0}\) is a fractional Brownian motion with Hurst parameter \(H \in (0, 1)\), i.e. a continuous, centred Gaussian process with covariance function

\[R_H(t, s) = \frac{1}{2} \{t^{2H} + s^{2H} - |t - s|^{2H}\}.
\]

It is well known that the fractional Brownian motion \(B\) with Hurst parameter \(H \neq \frac{1}{2}\) is neither a semimartingale nor a Markov process. Therefore, according to Bichteler–Dellacherie theorem, the classical Ito stochastic integration theory cannot be used to define a stochastic integral with respect to fractional Brownian motion.

Consider the Gaussian process \(Y_t^{(1)} := \int_0^t e^{-s} \, dB_a; a_t = H e^{t/H}\). The stochastic integral in the definition of the process \(Y^{(1)}\) is interpreted as a path-wise integral. The fractional Ornstein–Uhlenbeck process of the second kind \(X\) with initial value \(X_0\) is the solution of the Langevin
equation
\[ dX_t = -\theta X_t \, dt + dY_t^{(1)}, \quad X_0 = X_0, \quad \theta > 0. \] (1)

The terminology ‘of the second kind’ is taken from Kaarakka and Salminen.[1] The motivation behind the process \( X \) is that it is related to the Lamperti transformation of fractional Brownian motion. From a statistical point of view, this process always exhibits short-range-dependent property. For more detailed information on fractional Ornstein–Uhlenbeck processes, see Section 2.1.

An interesting problem in mathematical statistics is to estimate the unknown parameter \( \theta \) based on the continuous observation of the sample paths of the process \( X_t, t \in [0, T] \). A well-known technique to find an estimator of the drift parameter \( \theta \) is the so-called least-squares method. The least-squares estimator \( \hat{\theta}_T \) is obtained by minimizing the function
\[ \theta \mapsto \int_0^T |\dot{X}_t + \theta X_t|^2 \, dt \]
which leads to the solution
\[ \hat{\theta}_T = -\int_0^T X_t \delta X_t \int_0^T X_t^2 \, dt, \] (2)
where the stochastic integral is interpreted as the Skorokhod integral (see Section 2.2 for definition and notation). In the case of fractional Ornstein–Uhlenbeck process of the first kind (fOU1), that is replace driving noise \( Y^{(1)} \) with the fractional Brownian motion \( B \) in the Langevin equation (1), when \( H > \frac{1}{2} \), Hu and Nualart [2] showed that the least-squares estimator \( \hat{\theta}_T \) is strongly consistent. They also proved the asymptotic normality of the estimator when \( H \in [\frac{1}{2}, \frac{3}{4}) \).

In the same set-up for fractional Ornstein–Uhlenbeck process of the first kind, the maximum likelihood estimator is found by Kleptsyna and Le Breton.[3] They showed that it is strongly consistent. However the asymptotic normality property of the maximum likelihood estimator is shown by Bercu et al. [4] for all values of \( H > \frac{1}{2} \). Estimation of drift parameter \( \theta \) for fractional Ornstein–Uhlenbeck process of the first kind, in the non-ergodic case, i.e. \( \theta < 0 \), is studied by Belfadli et al.[5]

In this paper, we consider the least-squares estimator \( \hat{\theta}_T \) in the set-up of the fractional Ornstein–Uhlenbeck process of the second kind. The key point is Lemma 3.1. Using this lemma, we replace the Gaussian noise \( Y^{(1)} \) with an equivalent (in distribution) noise and do computations in an equivalent model. It is worth to mention that if we replace the Skorokhod integral with the path-wise Riemann–Stieltjes integral in the formula of the least-squares estimator, then we can obtain a new estimator \( \hat{\theta}_T^\prime \) which converges almost surely to 0 as \( T \) tends to infinity (see Remark 3.2 and also [2, p. 1036]).

The paper is organized as follows. In Section 2, we give a detailed information on fractional Ornstein–Uhlenbeck processes and Malliavin calculus for fractional Brownian motion. Section 3 is devoted to our main results. In the appendix section, we provide some auxiliary computation which is used in the proof of main results.

2. Preliminaries

It is well known that the classical Ornstein–Uhlenbeck process \( U^{(1/2, \xi_0)} = \{ U^{(1/2, \xi_0)}_t \}_{t \geq 0} \) with initial value \( \xi_0 \) can be constructed as the unique solution of the Langevin stochastic differential equation (SDE)
\[ dU_t^{(1/2, \xi_0)} = -\theta U_t^{(1/2, \xi_0)} \, dt + dW_t, \quad t \geq 0, \] (3)

where \( \theta > 0 \) and \( W = \{W_t\}_{t \geq 0} \) is standard Brownian motion. The solution of SDE (3) can be expressed as

\[ U_t^{(1/2, \xi_0)} := e^{-\theta t} \left( \xi_0 + \int_0^t e^{\theta s} \, dW_s \right), \quad t \geq 0. \]

Let \( \hat{W} \) denote a two-sided Brownian motion defined as

\[ \hat{W}_t := \begin{cases} W_t & \text{for } t \geq 0, \\ W(-t) & \text{for } t \leq 0, \end{cases} \] (4)

where \( W(-t) = \{W(-t): t \geq 0\} \) is another Brownian motion initiated at 0 and independent of \( W \). The selection \( \xi_0 = \int_{-\infty}^0 e^{\theta s} \, d\hat{W}_s \) leads to the unique solution \( U^{(1/2)} \) given by

\[ U_t^{(1/2)} := \int_{-\infty}^t e^{-\theta(t-s)} \, d\hat{W}_s, \quad t \geq 0. \]

It is a stationary, continuous Gaussian process with covariance function

\[ \text{Cov}(U_s^{(1/2)}, U_t^{(1/2)}) = \frac{1}{2\theta} e^{-\theta |t-s|}, \quad \forall s, t \geq 0. \]

On the other hand, it is known that the classical Ornstein–Uhlenbeck process \( U^{(1/2)} \) can be reconstructed from Brownian motion \( W \) by the Lamperti transformation. For \( \alpha > 0 \), define the process

\[ Z_t^{(1/2)} := e^{-\theta t} W_{\alpha e^{2\theta t}}, \quad t \in \mathbb{R}. \]

Then \( Z^{(1/2)} \) is a stationary, Gaussian process with covariance function

\[ \text{Cov}(Z_s^{(1/2)}, Z_t^{(1/2)}) = \alpha e^{-9|t-s|}, \quad s, t \in \mathbb{R}. \]

Since finite dimensional distributions of a Gaussian process are completely characterized by its mean and covariance functions, with \( \alpha = \frac{1}{2\theta} \), we have \( U^{(1/2)} \overset{\text{law}}{=} Z^{(1/2)} \) where \( \overset{\text{law}}{=} \) stands for equality in law. For more information on Lamperti transformation and related topics we refer to the book.[6]

### 2.1. Fractional Ornstein–Uhlenbeck processes

In this subsection, we briefly introduce fractional Ornstein–Uhlenbeck processes. The main references are [1,7]. We mainly focus on the fractional Ornstein–Uhlenbeck process of the second kind that is the core stochastic process of the article. To obtain the fractional Ornstein–Uhlenbeck processes, replace the Brownian motion \( W \) with a fractional Brownian motion \( B \) in Equation (3).
Consider the following stochastic differential equation:

\[ dU_t^{(H, \xi_0)} = -\theta U_t^{(H, \xi_0)} dt + dB_t \]  

(5)

with initial value \( \xi_0 \). The solution of the SDE (5) can be expressed as

\[ U_t^{(H, \xi_0)} = e^{-\theta t} \left( \xi_0 + \int_0^t e^{\theta s} dB_s \right). \]

(6)

Note that the stochastic integral is understood as a path-wise Riemann–Stieltjes integral. Let \( \hat{B} \) denote a two-sided fractional Brownian motion (see Equation (4)). The selection

\[ \xi_0 := \int_{-\infty}^0 e^{\theta s} d\hat{B}_s \]

for the initial value \( \xi_0 \) leads to an unique stationary, Gaussian process \( U^{(H)} \) of the form

\[ U_t^{(H)} = e^{-\theta t} \int_{-\infty}^t e^{\theta s} d\hat{B}_s. \]

(7)

**Definition 2.1** We call the process \( U^{(H, \xi_0)} \) given by Equation (6) a fractional Ornstein–Uhlenbeck process of the first kind with initial value \( \xi_0 \). The process \( U^{(H)} \) defined in Equation (7) is called stationary fractional Ornstein–Uhlenbeck process of the first kind.

**Remark 2.1** It is shown in [7] that the covariance function of the stationary process \( U^{(H)} \) decays like a power function, so it is ergodic and for \( H \in (\frac{1}{2}, 1) \), it exhibits long range dependence.

Now, we define a new stationary, Gaussian process \( X^{(\alpha)} \) by means of Lamperti transformation of fractional Brownian motion \( B \):

\[ X_t^{(\alpha)} := e^{-\alpha t} B_a, \quad t \in \mathbb{R}, \]

where \( \alpha > 0 \) and \( a_t = (H/\alpha)e^{\alpha t/H} \). We aim to represent the process \( X^{(\alpha)} \) as solution of a Langevin-type SDE. For this reason, consider the process \( Y^{(\alpha)} \) defined via

\[ Y_t^{(\alpha)} := \int_0^t e^{-\alpha s} dB_a, \quad t \geq 0. \]

The stochastic integral is understood in as path-wise Riemann–Stieltjes integral. Using the self-similarity property of the fractional Brownian motion one can see that the process \( Y^{(\alpha)} \) satisfies the following scaling property:

\[ \{Y_t^{(\alpha)}\}_{t \geq 0} \overset{\text{law}}{=} \{\alpha^{-H} Y_t^{(1)}\}_{t \geq 0}. \]

(8)

Using \( Y^{(\alpha)} \) the process \( X^{(\alpha)} \) can be viewed as the solution of the following Langevin-type SDE:

\[ dX_t^{(\alpha)} = -\alpha X_t^{(\alpha)} dt + dY_t^{(\alpha)}, \]

with random initial value \( X_0^{(\alpha)} = B_{a_0} \overset{d}{=} B_{H/\alpha} \sim N(0, (H/\alpha)^{2H}). \)
Inspired by the scaling property (8), we consider the following Langevin equation with $Y^{(1)}$ as the driving noise:

$$dX_t = -\theta X_t \, dt + dY_t^{(1)}, \quad \theta > 0.$$  \hfill (9)

The solution of the SDE (9) is given by

$$X_t = e^{-\theta t} \left( X_0 + \int_0^t e^{\theta s} \, dY_s^{(1)} \right) = e^{-\theta t} \left( X_0 + \int_0^t e^{(\theta - 1)s} \, dB_s \right)$$ \hfill (10)

with $\alpha = 1$ in $a_t$. Notice that the stochastic integral is understood as a path-wise Riemann–Stieltjes integral (see [1, 7] for more details). The selection $X_0 = \int_{-\infty}^0 e^{(\theta - 1)s} \, dB_s$ leads to an unique stationary, Gaussian process

$$U_t = e^{-\theta t} \int_{-\infty}^t e^{(\theta - 1)s} \, dB_s.$$ \hfill (11)

**Definition 2.2** We call the process $X$ given by Equation (10) a fractional Ornstein–Uhlenbeck process of the second kind with initial value $X_0$. The process $U$ defined in Equation (11) is called the stationary fractional Ornstein–Uhlenbeck process of the second kind.

**Proposition 2.1** [1] The covariance function of the stationary process $U$ decays exponentially and has short-range dependence.

### 2.2. Malliavin calculus with respect to fractional Brownian motion

In this subsection, we briefly introduce some basic facts on Malliavin calculus with respect to fractional Brownian motion. We also recall some required results related to fractional Brownian motion which we need for the proof of our main theorem. The main references are [8, 10, 18].

Assume $B = \{B_t\}_{t \in [0,T]}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ and covariance function $R_H(t, s)$. It is well known that the covariance function $R_H$ can be represented as

$$R_H(t, s) = \int_0^{\wedge t,s} K_H(t, u)K_H(s, u) \, du$$

for a Volterra-type square integrable kernel $K_H$. In the case $H > \frac{1}{2}$, the kernel $K_H$ has a simple expression given by

$$K_H(t, s) = c_H s^{1/2-H} \int_s^t (u - s)^{H-(3/2)} u^{H-(1/2)} \, du,$$

where $c_H = (H - \frac{1}{2})(2H\Gamma(\frac{3}{2} - H) / \Gamma(H + \frac{1}{2})\Gamma(2 - 2H))^{1/2}$. Consider the set $\mathcal{E}$ of all step functions on $[0, T]$. We define the Hilbert space $\mathcal{H}$ as the closure of $\mathcal{E}$ with respect to inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

Then, the mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between the Hilbert space $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$ associated with the fractional Brownian motion $B$. We denote this isometry by $\varphi \mapsto B(\varphi)$. It is known that when $H = \frac{1}{2}$, we have $\mathcal{H} = L^2[0, T]$, whereas for $H > \frac{1}{2}$, the elements of $\mathcal{H}$ may be not functions but distributions of negative order (see [12]).
interesting subspace $|\mathcal{H}|$ containing only functions is the set of all measurable functions $\varphi$ on $[0, T]$ such that

$$
\|\varphi\|_{|\mathcal{H}|}^2 := \alpha_{\mathcal{H}} \int_0^T \int_0^T |\varphi(s)||\varphi(t)||t - s|^{2H-2} \, ds \, dt < \infty,
$$

where $\alpha_{\mathcal{H}} = H(2H - 1)$. Notice that for any two measurable functions $\varphi, \psi \in |\mathcal{H}|$, we have

$$
\mathbb{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{|\mathcal{H}|} = \alpha_{\mathcal{H}} \int_0^T \int_0^T \varphi(s)\psi(t)||t - s|^{2H-2} \, ds \, dt. \tag{12}
$$

Notice that when $H > \frac{1}{2}$, we have the inclusions $L^2[0, T] \subset L^{1/H}[0, T] \subset |\mathcal{H}| \subset \mathcal{H}$. Consider the linear operator $K_{\mathcal{H}}^*$ between $\mathcal{E}$ and $L^2[0, T]$ defined by

$$
(K_{\mathcal{H}}^*\varphi)(s) = \int_s^T \varphi(t) \frac{\partial K_{\mathcal{H}}}{\partial t}(t, s) \, dt.
$$

Notice that the operator $K_{\mathcal{H}}^*$ is an isometry between $\mathcal{E}$ and $L^2[0, T]$ that can be extended to the Hilbert space $\mathcal{H}$. Moreover, the process $W = \{W_t\}_{t \in [0, T]}$ given by

$$
W_t := B((K_{\mathcal{H}}^*)^{-1}(1_{[0,t]})) \tag{13}
$$

defines a Brownian motion. The fractional Brownian motion $B$ and the Brownian motion $W$ are related through the integral representation

$$
B_t = \int_0^t K_{\mathcal{H}}(t, s) \, dW_s. \tag{14}
$$

Consider the space $\mathcal{S}$ of all smooth random variables of the form

$$
F = f(B(\varphi_1), \ldots, B(\varphi_n)), \quad \varphi_1, \ldots, \varphi_n \in \mathcal{H}, \tag{15}
$$

where $f \in C_b^\infty(\mathbb{R}^n)$. For any smooth random variable $F$ of the form Equation (15), we define its Malliavin derivative $D_F = D$ as an element of $L^2(\Omega; \mathcal{H})$ by

$$
DF = \sum_{i=1}^n \partial_i f(B(\varphi_1), \ldots, B(\varphi_n)) \varphi_i.
$$

In particular, $DB_t = 1_{[0,t]}$. We denote by $\mathbb{D}_B^{1,2} = \mathbb{D}^{1,2}$ the Hilbert space of all square integrable Malliavin derivative random variables as the closure of the set $\mathcal{S}$ of smooth random variables with respect to norm

$$
\|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}(\|DF\|_{\mathcal{H}}^2).
$$

The transfer principle connects the Malliavin operators of the both processes $B$ and $W$. This is the message of the next proposition.

**Proposition 2.2** [10] For any $F \in \mathbb{D}_W^{1,2} = \mathbb{D}^{1,2}$,

$$
K_{\mathcal{H}}^*DF = D^{(W)}F,
$$

where $D^{(W)}$ denotes the Malliavin derivative operator with respect to Brownian motion $W$, and $\mathbb{D}_W^{1,2}$ the corresponding Hilbert space.
The **divergence operator** denoted by \( \delta_B = \delta \) is the adjoint operator of the Malliavin derivative operator \( D \). For any \( \mathcal{H} \)-valued random variable \( u \in L^2(\Omega; \mathcal{H}) \) belongs to the domain \( \text{Dom}\delta \) of the divergence operator \( \delta \), the random variable \( \delta(u) \) is defined by the duality relationship

\[
\mathbb{E}(F\delta(u)) = \mathbb{E}(DF, u)_{\mathcal{H}}, \quad \forall F \in \mathbb{D}^{1,2}.
\]

Also, an element \( u \in L^2(\Omega; \mathcal{H}) \) belongs to the domain \( \text{Dom}\delta \) if and only if

\[
|\mathbb{E}(DF, u)|_{\mathcal{H}} \leq c_u \|F\|_{L^2}
\]

for any \( F \in \mathbb{D}^{1,2} \), where \( c_u \) is just a constant depending on \( u \). The divergence operator \( \delta \) is also called Skorokhod integral. It is known that in the case of Brownian motion it coincides with the Ito integral for adapted integrands. Hereafter for any \( u \in \text{Dom}\delta \), we denote \( \delta(u) = \int^T_0 u_t \delta B_t \).

Following [8] one can develop a Malliavin calculus for any continuous Gaussian process \( G \) of the form

\[
G_t = \int^t_0 K(t, s) \, dW_s,
\]

where \( W \) is a Brownian motion and the kernel \( K, 0 < s < t < T \) satisfies \( \sup_{t \in [0,T]} \int^t_0 K(t, s)^2 \, ds < \infty \). Consider the linear operator \( K^* \) from \( \mathcal{E} \) to \( L^2([0, T]) \) defined by

\[
(K^* \varphi)(s) = \varphi(s) K(T, s) + \int^T_s [\varphi(t) - \varphi(s)] K(dt, s).
\]

The Hilbert space \( \mathcal{H} \) generated by the covariance function of the Gaussian process \( G \) can be represented as \( \mathcal{H} = (K^*)^{-1}(L^2([0, T])) \) and \( \mathbb{D}^{1,2}_G(\mathcal{H}) = (K^*)^{-1}(\mathbb{D}^{1,2}_W(L^2([0, T]))) \). For any \( n \geq 1 \), let \( \mathcal{H}_n \) be the nth Wiener chaos of \( G \), i.e. the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{H_n(G(\varphi)), \varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1\} \), and \( H_n \) is the nth Hermite polynomial. It is well known that the mapping \( I^*_n(\varphi^\otimes n) = n! H_n(G(\varphi)) \) provides a linear isometry between the symmetric tensor product \( \mathcal{H}^\otimes n \) and the subspace \( \mathcal{H}_n \).

**Definition 2.3 [8]** We say that the kernel \( K \) is regular if for all \( s \in [0, T) \), \( K(\cdot, s) \) has bounded variation on the interval \( (s, T] \), and for its total variation denoted by \( |K|(s, T], s) \), we have

\[
\int^T_0 |K|(s, T], s)^2 \, ds < \infty.
\]

For regular kernel \( K \), put \( K(s^+, s) := K(T, s) - K((s, T], s) \). Here \( K((s, T], s) = \int^T_s K(dt, s) \).

For any \( \varphi \in \mathcal{E} \), define the seminorm

\[
\|\varphi\|_{K^r}^2 = \int^T_0 \varphi(s)^2 K(s^+, s)^2 \, ds + \int^T_0 \left( \int^T_s |\varphi(t)| |K|(dt, s) \right)^2 \, ds.
\]

Denote by \( \mathcal{H}_{K^r} \) the completion of \( \mathcal{E} \) with respect to seminorm \( \| \cdot \|_{K^r} \). The following proposition establishes the relationship between a path-wise integral and the Skorokhod integral.

**Proposition 2.3 [8]** Assume \( K \) is a regular kernel with \( K(s^+, s) = 0 \) and \( u \) is a process in \( \mathbb{D}^{1,2}_G(\mathcal{H}_{K^r}) \). Then the process \( u \) is Stratonovich integrable with respect to \( G \) and

\[
\int^T_0 u_t \, dG_t = \int^T_0 u_t \delta G_t + \int^T_0 \left( \int^T_s D_s u_t K(dt, s) \right) \, ds.
\]
The next proposition is taken from [13] and provides a central limit theorem for a sequence of multiple Wiener integrals. Let \( \mathcal{N}(0, \sigma^2) \) denote the Gaussian distribution with zero mean and variance \( \sigma^2 \).

**Proposition 2.4**  Let \( \{F_n\}_{n \geq 1} \) be a sequence of random variables in the \( q \)th Wiener chaos, \( q \geq 2 \), such that \( \lim_{n \to \infty} \mathbb{E}(F_n^2) = \sigma^2 \). Then the following statements are equivalent:

(i) \( F_n \) converges in distribution to \( \mathcal{N}(0, \sigma^2) \) as \( n \) tends to infinity.

(ii) \( \|DF_n\|_H^2 \) converges in \( L^2(\Omega) \) to \( q\sigma^2 \) as \( n \) tends to infinity.

The next proposition will be used to bridge our original Langevin equation (9) to a new SDE that plays the key role in our computations.

**Proposition 2.5** [14]  Let \( B = \{B_t\}_{t \in [0, T]} \) be a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \). Suppose \( f : [0, T] \to \mathbb{R} \) is a strictly positive, absolutely continuous and \( f' \) is a locally square integrable function. Then there exists a locally square integrable kernel \( L \) such that

\[
B_t \int_0^t f(s) \, ds \overset{\text{law}}{=} \int_0^t f(s) \, dB_s + \int_0^t \left( \int_0^s \frac{\partial K_H}{\partial s}(u, s) L(s, r) \, ds \right) \, dW_r,
\]

where the Brownian motion \( W \) is given by Equation (13). Moreover the kernel \( L \) satisfies the following integral equation:

\[
\phi(t, s) = \int_t^s f(t) \left( \frac{t}{r} \right)^{H-(1/2)} (t-r)^{-H-(3/2)} L(r, s) \, dr,
\]

where the function \( \phi \) is given by

\[
\phi(t, s) = \frac{\partial K_H}{\partial t} \left( \int_0^t f(v)^{1/H} \, dv, \int_0^s f(v)^{1/H} \, dv \right) f(t)^{1/H} f(s)^{1/2H} - f(t) \frac{\partial K_H}{\partial t}(t, s).
\]

### 3. Main results

For the rest of the paper, we assume that \( H > \frac{1}{2} \). Let \( X = \{X_t\}_{t \in [0, T]} \) be the solution of the Langevin equation (9). Take \( X_0 = 0 \) and assume \( \theta > 0 \). Then the solution can be expressed as

\[
X_t = \int_0^t e^{-\theta(t-s)} \, dY_s^{(1)}.
\]

The next theorem gives the main result of the paper.

**Theorem 3.1**  The least-squares estimator \( \hat{\theta}_T \) given by Equation (2) is weakly consistent, i.e.

\[
\hat{\theta}_T \rightarrow \theta
\]

in probability, as \( T \) tends to infinity.

Let \( B(x, y) \) denote the complete Beta function and \( \overset{\text{law}}{\rightarrow} \) stands for convergence in distribution. The next result provides the asymptotic normality property of the least-squares estimator.
Theorem 3.2  For the least-squares estimator \( \hat{\theta}_T \) given by Equation (2), we have

\[
\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2(\theta, H))
\]

as \( T \) tends to infinity, where

\[
\sigma^2(\theta, H) = \frac{\theta^2}{H^2 B \left( (\theta - 1)H + 1, 2H - 1 \right)^2} \\
\times 2 \int_{[0, \infty]^3} e^{-\theta y} e^{-\theta |z - x|} e^{(1 - (1/H))(x + y + z)} |e^{-\theta y H} - e^{-\theta z H}| \left| 1 - e^{-\theta H} \right|^{2H - 2} dx dy dz.
\]

The variance \( \sigma^2(\theta, H) \) is given by a triple integral over \([0, \infty)^3\) and probably this is the most compact form. In the proof of Theorem 3.2, we show that \( \sigma^2(\theta, H) < \infty \).

Remark 3.1  It is worth to mention that, because of the different covariance structure of the two Gaussian processes fOU1 and fOU2, the central limit Theorem 3.2 holds for the whole range \( H \in \left( \frac{1}{2}, 1 \right) \), whereas for the fractional Ornstein–Uhlenbeck process of the first kind we have the restriction \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right) \), see [2].

To prove Theorem 3.1, we need a series of lemmas. We start with the following lemma that provides an alternative stochastic differential equation of the Langevin equation (9).

Lemma 3.1  Let \( \tilde{B}_t = B_t^{+H} - B_t \) be the shifted fractional Brownian motion. Then there exists a regular Volterra-type kernel \( \tilde{L} \), in the sense of Definition 2.3, so that for the solution of the following stochastic differential equation:

\[
d\tilde{X}_t = -\theta \tilde{X}_t \, dt + d\tilde{G}_t, \quad \tilde{X}_0 = 0, \tag{17}
\]

we have, \( \{X_t\}_{t \in [0, T]} \xrightarrow{\text{law}} \{\tilde{X}_t\}_{t \in [0, T]} \) where the Gaussian process \( \tilde{G} \) is defined by

\[
\tilde{G}_t = \int_0^t (K_H(t, s) + \tilde{L}(t, s)) \, d\tilde{W}_s
\]

and the Brownian motion \( \tilde{W} \) as defined in Equation (13) with \( \tilde{B} \) replacing \( B \).

Proof  It is easy to check that the solution of the SDE (17) is given by

\[
\tilde{X}_t = \int_0^t e^{-\theta (t - s)} \, d\tilde{G}_s, \quad t \in [0, T].
\]

Moreover, \( \mathbb{E}(X_t) = \mathbb{E}(\tilde{X}_t) = 0 \), and

\[
\text{Cov}_X(t, s) = \mathbb{E} \left( \int_0^t e^{-\theta (t - u)} \, dY_u^{(1)} \int_0^s e^{-\theta (s - v)} \, dY_v^{(1)} \right) = \int_0^t \int_0^s e^{-\theta (t - u) - \theta (s - v)} \, \text{Cov}_{Y_{\mathbb{R}^2}}(du, dv).
\]

Similarly

\[
\text{Cov}_{\tilde{X}}(t, s) = \mathbb{E} \left( \int_0^t e^{-\theta (t - u)} \, d\tilde{G}_u \int_0^s e^{-\theta (s - v)} \, d\tilde{G}_v \right) = \int_0^t \int_0^s e^{-\theta (t - u) - \theta (s - v)} \, \text{Cov}_{\tilde{G}}(du, dv).
\]
Therefore, it is enough to show that for the Gaussian processes \( Y^{(1)} \) and \( \tilde{G} \), we have

\[
\text{Cov}_{Y^{(1)}}(s, t) = \mathbb{E}(Y^{(1)}_s Y^{(1)}_t) = \mathbb{E}(\tilde{G}_s \tilde{G}_t) = \text{Cov}_{\tilde{G}}(s, t).
\]

To see this, consider the function \( f(t) = e^t \) and apply Proposition 2.5. Then

\[
B_{a_t} = \tilde{B} \int_0^t e^{r/H} dr + B_H \overset{\text{law}}{=} \int_0^t e^r d\tilde{B}_r + \int_0^t g(t, r) d\tilde{W}_r + B_H,
\]

where the Volterra-type Kernel \( g \) is given by

\[
g(t, r) = \int_r^t e^u \left( \int_u^t \frac{\partial K_H}{\partial s}(u, s)L(s, r) \, ds \right) \, du.
\]

Now, using integration by parts formula, we obtain

\[
Y^{(1)}_t = \int_0^t e^{-s} dB_{a_s} \overset{\text{law}}{=} \int_0^t e^{-s} d \left( \int_0^s e^r d\tilde{B}_r + \int_0^s g(s, r) d\tilde{W}_r + B_H \right)
\]

\[
= \tilde{B}_t + \int_0^t e^{-s} d \left( \int_0^s g(s, r) d\tilde{W}_r \right)
\]

\[
= \tilde{B}_t + e^{-t} \int_0^t g(t, r) d\tilde{W}_r + \int_0^t \left( \int_0^s g(s, r) d\tilde{W}_r \right) e^{-s} \, ds.
\]

Next, using the Fubini theorem for Brownian motion \( \tilde{W} \), we see that

\[
A = \int_0^t \left( \int_0^s g(s, r) \, d\tilde{W}_r \right) e^{-s} \, ds = \int_0^t \int_r^t e^{-s} g(s, r) \, ds \, d\tilde{W}_r
\]

Put

\[
h(u, r) = \int_r^u \frac{\partial K_H}{\partial s}(u, v)L(v, r) \, dv,
\]

and using Fubini theorem for real-valued functions, we obtain

\[
\int_r^t e^{-s} g(s, r) \, ds = \int_r^t e^{-s} \int_r^s e^u h(u, r) \, du \, ds
\]

\[
= \int_r^t e^u h(u, r) \left( \int_u^t e^{-s} \, ds \right) \, du
\]

\[
= \int_r^t h(u, r) \, du - e^{-t} \int_0^t e^u h(u, r) \, du.
\]

Now, plugging this into \( A \), we infer that

\[
A = \int_0^t \int_0^s e^{-s} g(s, r) \, ds \, d\tilde{W}_r
\]

\[
= \int_0^t \left( \int_r^t h(u, r) \, du - e^{-t} \int_r^t e^u h(u, r) \, du \right) \, d\tilde{W}_r = A_1 - A_2,
\]

where

\[
A_1 = \int_0^t \int_r^t h(u, r) \, du \, d\tilde{W}_r,
\]

(20)
and the term $A_2$ can be written as

$$A_2 = \int_0^t e^{-t} \int_r^t e^u h(u, r) \, du \, d\tilde{W}_r = \int_0^t e^{-t} g(t, r) \, d\tilde{W}_r. \quad (21)$$

Finally, plug in Equations (20), (21) and (14) into Equation (19), we obtain

$$Y_t^{(1)} \overset{\text{law}}{=} \int_0^t \left( K_H(t, s) + \int_r^t h(u, r) \, du \right) d\tilde{W}_r.$$

Therefore, it is enough to introduce the kernel $\tilde{L}$ as

$$\tilde{L}(t, r) = \int_r^t h(u, r) \, du \quad (22)$$

and then the result follows.

The next lemma gives an alternative form for the least-squares estimator in terms of the process $\tilde{X}$ and the Gaussian noise $\tilde{G}$.

**Lemma 3.2** For the least-squares estimator $\hat{\theta}_T$ we have

$$\hat{\theta}_T \overset{\text{law}}{=} \theta - \frac{\int_0^T \tilde{X}_t \delta \tilde{G}_t}{\int_0^T \tilde{X}_t^2 \, dt}. \quad (23)$$

**Proof** Note that the least-squares estimator $\hat{\theta}_T$ is actually equal to

$$\hat{\theta}_T = \theta - \frac{\int_0^T X_t \delta Y_t^{(1)}}{\int_0^T X_t^2 \, dt}.$$

On the other hand, we have

$$\int_0^T X_t \delta Y_t^{(1)} = \int_0^T \int_0^t e^{-\theta(t-s)} \delta Y_s^{(1)} \delta Y_t^{(1)} = I_2^{(1)}(f)$$

and the function $f$ is given by $f(t, s) = \frac{1}{2} e^{-\theta|t-s|}$. Similarly,

$$\int_0^T \tilde{X}_t \delta \tilde{G}_t = \int_0^T \int_0^t e^{-\theta(t-s)} \delta \tilde{G}_s \delta \tilde{G}_t = I_2^{\tilde{G}}(f).$$

This immediately implies that

$$\int_0^T X_t \delta Y_t^{(1)} \overset{\text{law}}{=} \int_0^T \tilde{X}_t \delta \tilde{G}_t.$$

Moreover, for every $t \in [0, T]$ there exists a function $g(\cdot, t)$ so that we can write

$$X_t = I_1^{(1)}(g(\cdot, t)).$$

Using multiplication formula for multiple stochastic integrals, we obtain

$$X_t^2 = \|g(\cdot, t)\|_{\mathcal{H}}^2 + I_2^{(1)}(g(\cdot, t) \tilde{\otimes} g(\cdot, t)).$$
where $\mathcal{H}$ stands for the Hilbert space generated by the covariance function of the Gaussian process $Y^{(1)}$. Therefore, using Fubini theorem
\[
\int_0^T X_t^2 \, dt = \int_0^T \|g(\cdot, t)\|^2_{\mathcal{H}} \, dt + I_2^{Y^{(1)}} \left( \int_0^T (g(\cdot, t) \tilde{\otimes} g(\cdot, t)) \, dt \right).
\]
Along the same lines one can obtain that
\[
\int_0^T \tilde{X}_t^2 \, dt = \int_0^T \|g(\cdot, t)\|^2_{\tilde{\mathcal{H}}} \, dt + I_2^{\tilde{G}} \left( \int_0^T (g(\cdot, t) \tilde{\otimes} g(\cdot, t)) \, dt \right),
\]
where $\tilde{\mathcal{H}}$ stands for the Hilbert space generated by the covariance function of the Gaussian process $\tilde{G}$. Note that $\|g(\cdot, t)\|_{\tilde{\mathcal{H}}} = \|g(\cdot, t)\|_{\mathcal{H}}$. Consequently
\[
\int_0^T X_t^2 \, dt \overset{\text{law}}{=} \int_0^T \tilde{X}_t^2 \, dt.
\]
Finally, by Lemma A.2 in Appendix 1, we can deduce that for the following two-dimensional random vectors, we have
\[
\left( \int_0^T X_t \delta Y_t^{(1)}, \int_0^T X_t^2 \, dt \right) \overset{\text{law}}{=} \left( \int_0^T \tilde{X}_t \delta \tilde{G}_t, \int_0^T \tilde{X}_t^2 \, dt \right).
\]
We conclude that
\[
\frac{\int_0^T X_t \delta Y_t^{(1)}}{\int_0^T X_t^2 \, dt} \overset{\text{law}}{=} \frac{\int_0^T \tilde{X}_t \delta \tilde{G}_t}{\int_0^T \tilde{X}_t^2 \, dt}.
\]

In the next lemma, we give an alternative expression of the estimator $\hat{\theta}_T$.

**Lemma 3.3** The least-squares estimator $\hat{\theta}_T$ can be written as
\[
\hat{\theta}_T \overset{\text{law}}{=} - \frac{(1/2) \tilde{X}_T^2}{\int_0^T \tilde{X}_t^2 \, dt} + \frac{\int_0^T \int_s^T D_s^{(\tilde{W})} \tilde{X}_t (K_H(dr, s) + \tilde{L}(dr, s)) \, ds}{\int_0^T \tilde{X}_t^2 \, dt},
\]
where $D^{\tilde{W}}$ denotes the Malliavin derivative operator with respect to Brownian motion $\tilde{W}$.

**Proof** Using Lemma 3.2, Proposition 2.3 and SDE (17), we can infer that
\[
\hat{\theta}_T \overset{\text{law}}{=} \theta - \frac{\int_0^T \tilde{X}_t \delta \tilde{G}_t}{\int_0^T \tilde{X}_t^2 \, dt}
= \theta - \frac{\int_0^T \tilde{X}_t \, d\tilde{G}_t - \int_0^T \int_s^T D_s^{(\tilde{W})} \tilde{X}_t (K_H(dr, s) + \tilde{L}(dr, s)) \, ds}{\int_0^T \tilde{X}_t^2 \, dt}
= \theta - \frac{\int_0^T \tilde{X}_t (d\tilde{X}_t + \theta \tilde{X}_t \, dt) - \int_0^T \int_s^T D_s^{(\tilde{W})} \tilde{X}_t (K_H(dr, s) + \tilde{L}(dr, s)) \, ds}{\int_0^T \tilde{X}_t^2 \, dt}
= - \frac{(1/2) \tilde{X}_T^2}{\int_0^T \tilde{X}_t^2 \, dt} + \frac{\int_0^T \int_s^T D_s^{(\tilde{W})} \tilde{X}_t (K_H(dr, s) + \tilde{L}(dr, s)) \, ds}{\int_0^T \tilde{X}_t^2 \, dt}.
\]

In order to prove the theorem, we need the following lemma. The next lemma can be used for an alternative estimator of the parameter θ.

**Lemma 3.4** For the process \( X \) given by Equation (16) and consequently for the process \( \tilde{X} \), as \( T \) tends to infinity, we have

\[
\frac{1}{T} \int_0^T \tilde{X}^2_t \, dt \to \frac{(2H - 1)H^{2H}}{\theta} B((\theta - 1)H + 1, 2H - 1)
\]

almost surely and in \( L^2 \).

**Proof** Note that for every \( t \geq 0 \), we have

\[
U_t = e^{-\theta t} \int_{-\infty}^{t} e^{(\theta - 1)s} \, dB_u = X_t + e^{-\theta t} \xi,
\]

where \( \xi = \int_{-\infty}^{0} e^{(\theta - 1)s} \, dB_u \). For the stationary Gaussian process \( U \) we have (see Proposition 3.11 of Kaarakka and Salminen [1])

\[
\mathbb{E}(U_tU_0) = O(e^{-\theta t}) \to 0,
\]
as \( t \) tends to infinity. Hence, \( U \) is ergodic. Therefore, the ergodic theorem implies that

\[
\frac{1}{T} \int_0^T U^2_t \, dt \to \mathbb{E}(U_0^2)
\]
as \( T \) tends to infinity, almost surely and in \( L^2 \). It is straightforward from Equation (24) to check that

\[
\frac{1}{T} \int_0^T \tilde{X}^2_t \, dt \to \mathbb{E}(U_0^2).
\]  

(25)

On the other hand, using change of variable we see that

\[
U_t = e^{-\theta t} \int_{-\infty}^{t} e^{(\theta - 1)s} \, dB_u
\]

\[
= H^{-(\theta - 1)H} e^{-\theta t} \int_0^{a_t} s^{(\theta - 1)H} \, dB_s.
\]

Therefore, using formula (12), we obtain

\[
\mathbb{E}(U_0)^2 = H^{-2(\theta - 1)H} \mathbb{E}\left( \int_0^{a_0} s^{(\theta - 1)H} \, dB_s \right)^2
\]

\[
= H^{-2(\theta - 1)H} \frac{2H - 1}{2H - 1} \int_0^{a_0} \int_0^{a_0} s^{(\theta - 1)H} t^{(\theta - 1)H} |s - t|^{2H - 2} \, ds \, dt
\]

\[
= \frac{(2H - 1)H^{2H}}{\theta} B((\theta - 1)H + 1, 2H - 1).
\]  

(26)

Substituting Equation (26) into Equation (25) yields the result. ■

**Proof of Theorem 3.1** From Lemma 3.3, we can write \( \hat{\theta}_T = I_1 + I_2 \). By Lemma 3.4, we know that \( (1/T) \int_0^T \tilde{X}^2_t \, dt \) converges almost surely and in \( L^2 \), as \( T \) tends to infinity to a constant. Then, it suffices to study the almost sure convergence of the numerators of \( I_1 \) and \( I_2 \).
Step 1: $I_1 \to 0$ almost surely as $T$ tends to infinity.
From Lemma A.1 in Appendix 1, relation (24) and elementary inequality $(a - b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$, we deduce that almost surely

$$\lim_{T \to \infty} \frac{\tilde{X}_T^2}{T} = 0. \tag{27}$$

Step 2: $I_2 \to \theta$ almost surely as $T$ tends to infinity.
First note that the Malliavin derivative $D^{(\tilde{W})}_t \tilde{X}_t$ with respect to the Brownian motion $\tilde{W}$ is given by (see Section 2.2)

$$D^{(\tilde{W})}_t \tilde{X}_t = \int_s^t e^{-\theta(t-u)} \left( \frac{\partial K_H}{\partial u} (u, s) + h(u, s) \right) du.$$ 

Also,

$$K_H (dt, s) + \tilde{L}(dt, s) = \left( \frac{\partial K_H}{\partial t} (t, s) + h(t, s) \right) dt$$

$$= \left( \frac{\partial K_H}{\partial t} (t, s) + e^{-t} \phi(t, s) \right) dt$$

$$= e^{-t} \frac{\partial K_H}{\partial t} \left( \int_0^t e^{u/H} du, \int_0^s e^{u/H} du \right) e^{t/H} e^{s/2H} dt.$$ 

Denote the numerator of the term $I_2$ with $I_{2,up}^\up$. Therefore,

$$I_{2,up}^\up = \frac{1}{T} \int_0^T \int_s^t D^{(\tilde{W})}_u \tilde{X}_t (K_H (dt, s) + \tilde{L}(dt, s)) ds$$

$$= \frac{1}{T} \int_0^T \int_s^t \int_s^t e^{-\theta(t-u)} e^{-u} \frac{\partial K_H}{\partial u} \left( \int_0^u e^{v/H} du, \int_0^s e^{v/H} du \right) e^{u/H} e^{s/2H} du$$

$$\times e^{-t} \frac{\partial K_H}{\partial t} \left( \int_0^t e^{u/H} du, \int_0^s e^{u/H} du \right) e^{t/H} e^{s/2H} dr ds.$$ 

Now, by Fubini theorem for real-valued functions and Lemma A.3 in Appendix 1, we obtain

$$I_{2,up}^\up = \frac{1}{T} \int_0^t \int_0^{t-u} e^{-\theta(t-u)} e^{-u} e^{u/H} e^{-t} e^{t/H}$$

$$\times \int_0^{t-u} \frac{\partial K_H}{\partial t} \left( \int_0^v e^{v/H} dv, \int_0^s e^{v/H} dv \right) e^{v/H} ds du dr$$

$$= \frac{\alpha_H}{T} \int_0^T \int_0^t e^{-\theta(t-u)} e^{-u} e^{u/H} e^{-t} e^{t/H} \left| \int_0^u e^{v/H} dv - \int_0^t e^{v/H} dv \right|^{2H-2} du dr.$$
We make a change of variables \( p = \int_0^u e^{t/H} \, dt \) and \( q = \int_0^v e^{s/H} \, ds \). Hence,

\[
I_2^{up} = \frac{\alpha H}{T} \int_0^{\alpha T-H} \int_0^q \left( 1 + \frac{q}{H} \right)^{-H(1+\theta)} \left( 1 + \frac{p}{H} \right)^{H(\theta-1)} |p-q|^{2H-2} \, dp \, dq
\]

\[
= \frac{\alpha H H^{2H}}{T} \int_0^{\alpha T-H} \int_0^q (q+H)^{-H(1+\theta)} (p+H)^{H(\theta-1)} |p-q|^{2H-2} \, dp \, dq
\]

\[
= \frac{\alpha H H^{2H}}{T} \int_0^{\alpha T} \int_0^{q+H} q^{-H(1+\theta)} p^{H(\theta-1)} |p-q|^{2H-2} \, dp \, dq
\]

\[
= \frac{\alpha H H^{2H}}{T} \int_0^{\alpha T} \frac{1}{q} \int_0^{\alpha T+1} y^{(\theta-1)H} \left( 1 - y \right)^{2H-2} \, dy \, dq.
\]

As a result,

\[
i_2^{up} \rightarrow (2H-1)H^{2H} B((\theta-1)H + 1, 2H - 1)
\]

and the claim follows. \( \square \)

**Proof of Theorem 3.2** Using Lemma 3.2, we have

\[
\sqrt{T} (\hat{\theta}_T - \theta) \overset{law}{=} -\sqrt{T} I_2^G \left( (1/2)e^{-|\theta|-s} \right) = \frac{F_T}{(1/T) \int_0^T X_t^2 \, dt},
\]

where \( F_T \) stands for the double stochastic integral

\[
F_T = \frac{1}{\sqrt{T}} \hat{I}_2^G \left( \frac{1}{2} e^{-|\theta|-s} \right).
\]

Therefore, taking into account Lemma 3.4, it is enough to show that the sequence \( \{F_T\} \) for \( T = 1, 2, \ldots \) converges in law to a Gaussian distribution. To show this, we use Proposition 2.4. We have

\[
\mathbb{E} (F_T)^2 = \frac{\alpha H}{2T} \int_{[0,T]^4} e^{-\theta(|u_1-v_1|+|u_2-v_2|)} e^{(1/H-1)(u_1+v_1+u_2+v_2)}
\]

\[
\times \left| \int_0^{u_2} e^{x/H} \, dx - \int_0^{v_2} e^{x/H} \, dx \right|^{2H-2} \left| \int_0^{v_1} e^{x/H} \, dx - \int_0^{u_1} e^{x/H} \, dx \right|^{2H-2} \, du_1 \, du_2 \, dv_1 \, dv_2
\]

\[
:= \frac{\alpha^2 H}{2T} H^{2H-4} I_T,
\]

where

\[
I_T = \int_{[0,T]^4} \left[ e^{-\theta(|u_1-v_1|+|u_2-v_2|)} e^{(1/H-1)(u_1+v_1+u_2+v_2)}
\times |e^{u_2/H} - e^{u_1/H}|^{2H-2} |e^{v_2/H} - e^{v_1/H}|^{2H-2} \right] \, du_1 \, du_2 \, dv_1 \, dv_2.
\]

Taking the derivative with respect to \( T \), change of variables \( x = T - u_1, y = T - u_2, z = T - v_1 \), and taking limit as \( T \) tends to infinity, we obtain that

\[
\lim_{T \to \infty} \frac{dI_T}{dT} = 4 \int_{[0,\infty)^3} e^{-\theta y} e^{-|z-x|} e^{(1-1/H)(x+y+z)}
\times \left[ |e^{-y/H} - e^{-z/H}| \right]^{2H-2} \, dx \, dy \, dz
\]
and this integral is finite. Indeed using obvious bound $e^{-\theta |z-x|} \leq 1$ and change of variables $e^{-z/H} = a, e^{-y/H} = b, e^{-c} = c$, we infer that, it is smaller than

$$
C(H)B(1-H,2H-1) \int_{[0,1]^2} (ab)^{-H} b^{\theta H} |b-a|^{2H-2} da db
$$

$$
= C(H)B(1-H,2H-1) \left[ \int_0^1 \int_0^1 + \int_1^1 \int_0^1 \right] (ab)^{-H} b^{\theta H} |b-a|^{2H-2} da db
$$

$$
= C(H)B(1-H,2H-1)(J_1 + J_2).
$$

For example, for the term $J_2$ we have

$$
J_2 = \int_0^1 b^{\theta H-1} db \int_0^1 w^{(\theta-1)H} (1-w)^{2H-2} dw
$$

$$
= B(1 + (\theta - 1)H, 2H - 1) \int_0^1 a^{\theta H-1} da < \infty.
$$

With similar computation (see [15], appendix), one can show that

$$
E[\|D_s F_T\|_{\tilde{\mathcal{H}}}^2] - E[\|D_s F_T\|_{\tilde{\mathcal{H}}}^2] \to 0 \quad \text{as} \quad T \to \infty.
$$

Taking into account that

$$
\lim_{T \to \infty} E[\|D_s F_T\|_{\tilde{\mathcal{H}}}^2] = 2 \lim_{T \to \infty} E(F_T^2),
$$

we complete the proof.

\[\square\]

**Remark 3.2** Note that if one can replace the Skorokhod integral with the path-wise Riemann–Stieltjes integral in the formula of the least-squares estimator, then we obtain for the new estimator

$$
\hat{\theta}_T := \frac{\int_0^T X_t dY_t}{\int_0^T X_t^2 dt} \to 0
$$

almost surely as $T \to \infty$ by Lemmas 3.4 and A.1 in Appendix 1. For the fOU case, we refer to [2, p. 1036].

**Remark 3.3** The least-square estimator $\hat{\theta}_T$ involves the Skorokhod integral $\int_0^T X_t \delta Y_t^{(1)}$. It is a well-known result (see [8] for example) that the Skorokhod integral can be approximated by Riemann–Stieltjes sums, but the ordinary product must be replaced by the Wick product, i.e.

$$
\int_0^T X_t \delta Y_t^{(1)} := \lim_{|\pi| \to 0} \sum_{t_i \in \pi} X_{t_{i+1}} \diamond (Y_{t_i}^{(1)} - Y_{t_i}^{(1)}), \quad \text{where} \quad \pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}
$$

is a partition of interval $[0, T]$ and $|\pi| = \max_{0 \leq k \leq n} (t_k - t_{k-1})$ is the mesh of $\pi$. However, from the computer simulation point of view, the Wick product makes computation of the stochastic integral very difficult. On the other hand, Lemma 3.4 suggests another more friendly simulated estimator in terms of Lebesgue integral. This is the main topic of the reference [15].

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References

[1] Kaarakka T, Salminen P. On fractional Ornstein–Uhlenbeck processes. Commun Stoch Anal. 2011;5(1):121–133.
[2] Hu Y, Nualart D. Parameter estimation for fractional Ornstein–Uhlenbeck processes. Stat Probab Lett. 2010;80(11–12):1030–1038.
[3] Kleptsyna ML, Le Breton A. Statistical analysis of the fractional Ornstein–Uhlenbeck type process. Stat Inference Stoch Process. 2002;5(3):229–248.
[4] Bercu B, Coutin L, Savy N. Sharp large deviations for the fractional Ornstein–Uhlenbeck process. Teor Veroyatn Primen, 2011;55(4):732–771, translation in Theory Probab Appl. 55(4):575–610.
[5] Belfadli R, Es-Sebaiy K, Ouknine Y. Parameter estimation for fractional Ornstein–Uhlenbeck processes: non-ergodic case; 2011. Available from: http://arxiv.org/abs/1102.5491.
[6] Embrechts P, Maejima M. Selfsimilar processes. Princeton series in applied mathematics. Princeton, NJ: Princeton University Press; 2002.
[7] Cheridito P, Kawaguchi H, Maejima M. Fractional Ornstein–Uhlenbeck processes. Electr J Probab. 2003;8:1–14.
[8] Alos E, Mazet O, Nualart D. Stochastic Calculus with respect to Gaussian processes. Ann Probab. 2001;29(2):766–801.
[9] Nourdin I, Peccati G. Normal approximations using Malliavin calculus: from Stein’s method to universality. Cambridge tracts in mathematics. Cambridge: Cambridge University; 2012.
[10] Nualart D. The Malliavin Calculus and related topics. Probability and its application. Berlin: Springer; 2006.
[11] Mishura Y. Stochastic Calculus for fractional Brownian motion and related processes. Lecture notes in mathematics. Berlin: Springer; 2008.
[12] Pipiras V, Taqqu MS. Integration questions related to fractional Brownian motion. Probab Theory Related Fields. 2002;118(2):251–291.
[13] Nualart D, Ortiz-Latorre S. Central limit theorems for multiple stochastic integrals and Malliavin calculus. Stoch Process Appl. 2008;118:614–628.
[14] Baudoin F, Nualart D. Equivalence of Volterra processes. Stoch Process Appl. 2003;107(2):327–350.
[15] Azmoodeh E, Viitasaari L. Parameters estimation based on discrete observations of fractional Ornstein–Uhlenbeck process of the second kind; 2013. Available from: http://arxiv.org/abs/1304.2466.
[16] Temme NM. Special functions: an introduction to the classical functions of mathematical physics. New York: Wiley Inter-science Press; 1996.
[17] Pickands, J. Asymptotic properties of the maximum in a stationary Gaussian process. Trans Am Math Soc. 1969;145:75–86.
[18] Peccati G, Taqqu M. Wiener Chaos: moments, Cumulants and diagrams. A survey with computer implementation. Bocconi University Press, Springer; 2011.

Appendix 1

Lemma A.1 For the stationary Gaussian process $U$ given by Equation (24) and any $\alpha > 0$, as $T$ tends to infinity, we have

$$\frac{U_T}{T^\alpha} \rightarrow 0 \quad \text{almost surely.}$$

Proof Using Proposition 3.11 of Kaarakka and Salminen,[1] there exists a constant

$$C(H, \theta) = H(2H - 1)H(1-\theta)$$

such that

$$c(t) := \text{Cov}(U_t, U_0) = C(H, \theta) e^{-\theta t} \left( \int_0^t \int_0^{a_0/\alpha} (xy)^{(\theta-1)H} |x-y|^{2H-2} \, dx \, dy \right).$$

Denote the term inside parentheses by $I(t)$. Then with some direct computations one can see

$$I(t) = \int_0^t \int_0^{a_0/\alpha} (xy)^{(\theta-1)H} (y-x)^{2H-2} \, dx \, dy \quad + \quad \int_0^{a_0/\alpha} \int_0^x (xy)^{(\theta-1)H} (x-y)^{2H-2} \, dy \, dx \quad + \quad \int_0^{a_0/\alpha} \int_0^y (xy)^{(\theta-1)H} (y-x)^{2H-2} \, dx \, dy \quad = \quad \frac{2\theta H}{\theta \alpha} B((\theta-1)H + 1, 2H-1) \quad + \quad \frac{1}{2\theta H} (a_0^{2H} - a_0^{2H}) \int_0^{a_0/\alpha} z^{(\theta-1)H} (1-z)^{2H-2} \, dz.$$
Therefore, as \( t \to 0^+ \), we have

\[
c(t) = \frac{(2H - 1)H^{2H}}{\theta} B((\theta - 1)H + 1, 2H - 1) \\
- (2H - 1)H^{2H} \times t \times \int_{0/\theta t}^1 z^{(\theta - 1)H} (1 - z)^{2H - 2} \, dz + o(t^{2H}).
\]  

(A1)

Let \( B_{inc}(p, q, x) \) and \( 2F_1(a, b, c; x) \) stand for the incomplete Beta function and the Gauss Hyper geometric function, respectively (see [16] for definition). Using a relation between these two functions [16, p. 289], we obtain that as \( t \to 0^+ \), we have

\[
\int_{0/\theta t}^1 z^{(\theta - 1)H} (1 - z)^{2H - 2} \, dz = B_{inc} \left( 2H - 1, (\theta - 1)H + 1, 1 - \frac{a_0}{a_t} \right) \\
= \frac{H^{1-2H}}{2H - 1} e^{-(\theta + 1)t} (a_t - a_0)^{2H - 1} \cdot 2F_1 \left( (\theta + 1)H, 1, 2H; 1 - \frac{a_0}{a_t} \right) \\
= \frac{H^{1-2H}}{2H - 1} + o(t^{2H - 1}).
\]  

(A2)

Substituting Equation (A2) in Equation (A1), we obtain that as \( t \to 0^+ \), we have

\[
c(t) = \frac{(2H - 1)H^{2H}}{\theta} B((\theta - 1)H + 1, 2H - 1) - \frac{H^t}{t} + o(t^{2H}).\]

Now the claim follows from Theorem 3.1 of Pickands.[17]

\[\]  

**Lemma A.2** Set \( U = I_2^{(1)}(f) \), \( V = I_2^{(1)}(\int_0^T g(\cdot, t) \circ B(\cdot, t) \, dt) \) and \( \bar{U} = \bar{I}_2^{(1)}(f) \), \( \bar{V} = \bar{I}_2^{(1)}(\int_0^T g(\cdot, t) \circ B(\cdot, t) \, dt) \). Then

\[
\mathbb{E}(U^{m_1} V^{m_2}) = \mathbb{E}(\bar{U}^{m_1} \bar{V}^{m_2}) \quad \forall m_1, m_2 \geq 1.
\]

**Proof** This immediately follows from Corollary 7.3.1 of Peccati et al.[18]

The next lemma is used in study of the term \( I_2^{ap} \).

**Lemma A.3**

\[
\int_0^T \frac{3K_H}{\partial t} \left( \int_0^t e^{v/H} \, dv, \int_0^u e^{v/H} \, dv \right) \frac{3K_H}{\partial s} \left( \int_0^t e^{v/H} \, dv, \int_0^u e^{v/H} \, dv \right) e^{u/H} \, du \\
= H(2H - 1) \left[ \left( \int_0^t e^{v/H} - \int_0^t e^{v/H} \right)^{2H - 2} \right].
\]

**Proof** Put \( V_t = B_{inc}(p, q, x) \). Note that we can also represent this random variable as a stochastic integral in the following way:

\[
V_t = \int_0^t K_H \left( \int_0^t e^{v/H} \, dv, \int_0^u e^{v/H} \, dv \right) e^{u/H} \, dW_u.
\]

where the Brownian motion \( W \) is given by Equation (13). Therefore using the Ito isometry, we obtain

\[
\mathbb{E}(V_t V_t) = \int_0^t K_H \left( \int_0^t e^{v/H} \, dv, \int_0^u e^{v/H} \, dv \right) K_H \left( \int_0^t e^{v/H} \, dv, \int_0^u e^{v/H} \, dv \right) e^{u/H} \, du.
\]

(A3)

On the other hand,

\[
\mathbb{E}(V_t V_t) = \mathbb{E}(B_{inc}(p, q, x) B_{inc}(p, q, x)) = R_H \left( \int_0^t e^{v/H} \, dv, \int_0^t e^{v/H} \, dv \right).
\]

(A4)

With differentiating respect to \( s \) and \( t \) of the right-hand side of Equations (A4) and (A3), we get the desired identity. 

\[\]