RELATIVE FIXED POINT THEORY

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ABSTRACT
The Lefschetz fixed point theorem and its converse have many generalizations. One of these generalizations is to endomorphisms of a space relative to a fixed subspace. In this paper we define relative Lefschetz numbers and Reidemeister traces using traces in bicategories with shadows. We use the functoriality of this trace to identify different forms of these invariants and to prove a relative Lefschetz fixed point theorem and its converse.

INTRODUCTION
The goal of topological fixed point theory is to find invariants that detect if a given endomorphism of a space has any fixed points. The Lefschetz fixed point theorem identifies one such invariant.

Theorem (Lefschetz fixed point theorem). Let $B$ be a closed smooth manifold and $f: B \to B$ be a continuous map. If $f$ has no fixed points then the Lefschetz number of $f$ is zero.

The Lefschetz number is the alternating sum of the levelwise traces of the map induced by $f$ on the rational homology of $B$. This is a relatively computable invariant. It gives a necessary, but not sufficient, condition for determining if a continuous map does not have any fixed points.

If we put additional restrictions on the map $f$, such as requiring it to preserve a subspace of $B$, the Lefschetz number still gives a necessary condition for $f$ to be fixed point free. However, this invariant ignores the relative structure and so is not the best possible invariant. For example, the identity map of the circle is homotopic to a map with no fixed points and so the Lefschetz number is zero. If this map is required to preserve a proper subinterval it is no longer homotopic to a map with no fixed points.

There is a refinement of the Lefschetz number defined using the induced maps on the rational homology of the subspace and the relative rational homology.

Theorem A (Relative Lefschetz fixed point theorem). Let $A \subset B$ be closed smooth manifolds and $f: B \to B$ be a continuous map such that $f(A) \subset A$. If $f$ has no fixed points then the relative Lefschetz number of $f$ is zero.
The relative Lefschetz number of the identity map of the circle relative to a proper subinterval is not zero.

Both of these theorems give a condition that implies that a continuous endomorphism

\[ f : B \to B \]

has a fixed point. In most cases they do not give a condition that implies \( f \) has no fixed points. To address this question a refined invariant and some restrictions on the spaces have to be introduced. This refined invariant is called the Nielsen number or the Reidemeister trace.

**Theorem** (Converse to the Lefschetz fixed point theorem). Let \( B \) be a closed smooth manifold of dimension at least 3 and

\[ f : B \to B \]

be a continuous map. The map \( f \) is homotopic to a map with no fixed points if and only if the Reidemeister trace of \( f \) is zero.

The Reidemeister trace is a partitioning of the Lefschetz number to reflect the ways fixed points can be changed by a homotopy of the original map. There is a generalization of the Reidemeister trace to a relative Reidemeister trace that is very similar to the generalization of the Lefschetz number to the relative Lefschetz number.

**Theorem B** (Converse to the Relative Lefschetz fixed point theorem). Suppose \( A \subset B \) are closed smooth manifolds of dimension at least 3 and the codimension of \( A \) in \( B \) is at least 2. A map

\[ f : B \to B \]

such that \( f(A) \subset A \) is homotopic to a map with no fixed points through maps preserving the subset \( A \) if and only if the relative Reidemeister trace of \( f \) is zero.

The goal of this paper is to provide definitions of the relative Lefschetz number and relative Reidemeister trace and proofs of Theorems A and B that satisfy several requirements. First, the relative Reidemeister trace should detect if a map is relatively homotopic to a map with no fixed points. It is not necessary for the relative Reidemeister trace to provide a lower bound for the number of fixed point of a given map. Second, the invariants should be trace-like. This means that they can be described using the duality and trace in bicategories defined in [26]. The relative Reidemeister trace should to be compatible with the approach of [19, 18]. Those papers give a proof of the converse to the Lefschetz fixed point theorem that is different from the standard simplicial proof. Finally, the relative Reidemeister trace should be compatible with an equivariant generalization of the Reidemeister trace described in [25].

While the Lefschetz number and the Reidemeister trace have long established definitions, the relative forms of these invariants are less settled. Versions of the relative Lefschetz number have been defined in [14, 15] and of the relative Reidemeister trace in [24, 28, 34, 35, 36]. The papers [28, 34, 35] are primarily interested in determining lower bounds for the number of fixed points and so are generalizations of the Nielsen number. The invariants defined in [24, 36] are more trace-like, but the definition are still motivated by connections to the Nielsen number. None of these invariants satisfy all of our conditions above, and none of them exactly coincide with the definitions given here.
In this paper we give proofs of Theorems A and B following the outline of [26]. We use duality and trace in bicategories with shadows to define two forms of the relative Lefschetz number and the relative Reidemeister trace. Then, using functoriality, we show different invariants coincide. Finally, we generalize Klein and Williams’s proof of the converse to the Lefschetz fixed point theorem in [19] to complete the proof of the converse to the relative Lefschetz fixed point theorem.

In the first two sections we will recall the necessary definitions of duality and trace in symmetric monoidal categories and in bicategories with shadows. In Section 3 we will describe some examples of bicategories with shadows and generalize some results from [26] that describe specific examples of duals.

In Section 4 we apply this category theory to the relative Lefschetz number. In Sections 5 and 6 we define the relative Reidemeister trace. We describe how to compare these invariants to each other and how to compare them to the relative Nielsen number defined by Schirmer and Zhao. In Sections 7 and 8 we give a proof of the converse to the relative Lefschetz fixed point theorem following the proofs given by Klein and Williams in [19, 18]. In Section 9 we include some formal results omitted from the third section.

We assume the reader is familiar with the basic definitions of Nielsen theory. References for this topic include [3, 16].

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Preliminaries. We fix some conventions and recall a fact about cofibrations.

Definition 0.1. Let $A \subset B$ and $X \subset Y$. A map $f: B \to Y$ is a relative map if $f(A) \subset X$.

We will write this

$$f: (B, A) \to (Y, X).$$

A homotopy $H: B \times I \to Y$ is a relative homotopy if $H|_{A \times I}$ factors through the inclusion $X \subset Y$.

Definition 0.2. [34] 3.1] A relative map $f: (B, A) \to (B, A)$ is taut if there is a neighborhood $N(A)$ of $A$ in $B$ such that $f(N(A)) \subset A$.

We will use this condition to isolate the fixed points of $A$ from those of $B \setminus A$.

Lemma 0.3. [34] 3.2] If $A \subset B$ is a cofibration then any relative map $f: (B, A) \to (B, A)$ is relatively homotopic to a taut map.

We will assume $A \subset B$ is a cofibration and all relative maps

$$f: (B, A) \to (B, A).$$

are taut. If a relative map is not taut it is implicitly replaced by a relatively homotopic map that is taut. Since all invariants defined here are invariants of the relative homotopy class, the choice of replacement does not matter.
1. DUALITY AND TRACE IN SYMMETRIC MONOIDAL CATEGORIES

Duality and trace in symmetric monoidal categories is a generalization of the trace in linear algebra that retains many important properties. The trace in a symmetric monoidal category satisfies a generalization of invariance of basis and has nice functorial properties. The Lefschetz fixed point theorem is one application of the functoriality properties of the trace. This is a very brief summary of [9]. For more details see [9], [21, Chapter III], or [27].

Let \( C \) be a symmetric monoidal category with monoidal product \( \otimes \), unit \( S \), and symmetry isomorphism \( \gamma : X \otimes Y \to Y \otimes X \).

**Definition 1.1.** An object \( A \) in \( C \) is dualizable with dual \( B \) if there are maps

\[
\eta : S \to A \otimes B
\]

and

\[
\epsilon : B \otimes A \to S
\]

such that the composites

\[
A \cong S \otimes A \xrightarrow{\eta \otimes \text{id}} A \otimes B \otimes A \xrightarrow{\text{id} \otimes \epsilon} A \otimes S \cong A
\]

and

\[
B \cong B \otimes S \xrightarrow{\text{id} \otimes \eta} B \otimes A \otimes B \xrightarrow{\epsilon \otimes \text{id}} S \otimes B \cong B
\]

are the identity maps of \( A \) and \( B \) respectively.

The most familiar example of a symmetric monoidal category is the category of modules over a commutative ring \( R \). The tensor product is the monoidal product. If \( M \) is a finitely generated projective \( R \)-module, \( M \) is dualizable and the dual of \( M \) is \( \text{Hom}_R(M, R) \). The evaluation map

\[
\epsilon : \text{Hom}_R(M, R) \otimes_R M \to R
\]

is defined by \( \epsilon(\phi, m) = \phi(m) \). Since \( M \) is finitely generated and projective the dual basis theorem implies there is a `basis’ \( \{m_1, m_2, \ldots, m_n\} \) with dual `basis’ \( \{m'_1, m'_2, \ldots, m'_n\} \). The coevaluation is given by linearly extending

\[
\eta(1) = \sum m_i \otimes m'_i.
\]

The category of chain complexes of modules over a commutative ring \( R \) is also symmetric monoidal. The dualizable objects are the chain complexes that are projective in each degree and finitely generated.

**Definition 1.2.** If \( A \) is dualizable and \( f : A \to A \) is an endomorphism in \( C \), the trace of \( f \), \( \text{tr}(f) \), is the composite

\[
S \xrightarrow{\eta} A \otimes B \xrightarrow{f \otimes \text{id}} A \otimes B \xrightarrow{\gamma} B \otimes A \xrightarrow{\epsilon} S.
\]

The trace of an endomorphism in the symmetric monoidal category of vector spaces over a field is the sum of the diagonal elements in a matrix representation. The trace of an endomorphism in the category of chain complexes of modules over a commutative ring \( R \) is called the Lefschetz number.
Proposition 1.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor, $A$ be an object of $\mathcal{C}$ with dual $B$, and

$$F(A) \otimes F(B) \to F(A \otimes B)$$

and

$$S_\mathcal{D} \to F(S_\mathcal{D})$$

be isomorphisms. Then $F(A)$ is dualizable with dual $F(B)$. If $f : A \to A$ is an endomorphism in $\mathcal{C}$, $F(\text{tr}(f)) = \text{tr}(F(f))$.

The stable homotopy category is a symmetric monoidal category. There is also a way to express duality for spaces without using spectra.

Definition 1.4. A based space $X$ is $n$-dualizable if there is a based space $Y$ and continuous maps $\eta : S^n \to X \wedge Y$ and $\epsilon : Y \wedge X \to S^n$ such that the diagrams commute up to stable homotopy.

The map $\sigma : S^n \to S^n$ is defined by $\sigma(v) = -v$.

Proposition 1.5. [21, III.4.1, III.5.1][23, 18.6.5]

1. If $M$ is a closed smooth manifold that embeds in $\mathbb{R}^m$, then $M_+$ is dualizable with dual $T\nu$, the Thom space of the normal bundle of the embedding of $M$ in $\mathbb{R}^m$.
2. If $L$ is a closed submanifold of a closed smooth manifold $M$ that embeds in $\mathbb{R}^m$, then $M \cup CL$ is dualizable with dual $T\nu_M \cup CT\nu_L$.
3. If $B$ is a compact ENR that embeds in $\mathbb{R}^n$, $B_+$ is dualizable with dual the cone on the inclusion $\mathbb{R}^n \setminus B \to \mathbb{R}^n$.
4. If $B$ is a compact ENR that embeds in $\mathbb{R}^n$ and $A$ is a sub ENR of $B$, then $B \cup CA$ is dualizable with dual $(\mathbb{R}^n \setminus A) \cup C(\mathbb{R}^n \setminus B)$.

Here $C$ denotes the cone. If $A \subset B$ then $B \cup CA$ is the mapping cone on the inclusion $A \to B$. The base point of $B \cup CA$ is the cone point.

The trace of an endomorphism of spaces regarded as a map in the stable homotopy category is called the fixed point index. The index is the stable homotopy class of a map

$$S^n \to S^n$$

and so is an element of the $0$th stable homotopy group of $S^0$, $\pi_0^s$. For other descriptions of the fixed point index see [3, 8].

The index of the identity map of a space $X$ is called the Euler characteristic of $X$ and it is denoted $\chi(X)$. This is consistent with the classical definition of the Euler characteristic.

Since the rational homology functor is strong symmetric monoidal, Proposition 1.3 implies that the fixed point index of a map $f$ is equal to the Lefschetz number of $H_*(f)$. Since the fixed point index of a map with no fixed points is zero, this implies the Lefschetz fixed point theorem.
2. Duality and Trace for Bicategories with Shadows

Unfortunately, the Reidemeister trace cannot be defined as a trace in a symmetric monoidal category. It can be defined using the more general trace in a bicategory with shadows. Duality and trace in a bicategory are very similar to duality and trace in a symmetric monoidal category but are more flexible. This is a brief summary of the relevant sections of [23] and [26]. For more details see [23, Chapter 16], [26], or [27].

Definition 2.1. [20, 1.0] A bicategory \( \mathcal{B} \) consists of

1. A collection \( \text{ob}\mathcal{B} \).
2. Categories \( \mathcal{B}(A, B) \) for each \( A, B \in \text{ob}\mathcal{B} \).
3. Functors
   \[ \odot : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C) \]
   \[ U_A : * \to \mathcal{B}(A, A) \]
   for \( A, B \) and \( C \) in \( \text{ob}\mathcal{B} \).

Here \( * \) denotes the category with one object and one morphism. The functors \( \odot \) are required to satisfy unit and associativity conditions up to natural isomorphism 2-cells.

The elements of \( \text{ob}\mathcal{B} \) are called 0-cells. The objects of \( \mathcal{B}(A, B) \) are called 1-cells. The morphisms of \( \mathcal{B}(A, B) \) are called 2-cells.

The most familiar example of a bicategory is the bicategory \( \text{Mod} \) with 0-cells rings, 1-cells bimodules, and 2-cells homomorphisms. The bicategory composition is tensor product.

Definition 2.2. [23, 16.4.1] A 1-cell \( X \in \mathcal{B}(B, A) \) is right dualizable with dual \( Y \in \mathcal{B}(A, B) \) if there are 2-cells

\[ \eta : U_A \to X \odot Y \]
\[ \epsilon : Y \odot X \to U_B \]

such that

\[ \begin{array}{c}
Y \xrightarrow{\text{id} \odot \eta} Y \odot U_A \\
\downarrow \text{id} \downarrow \epsilon \odot \text{id} \uparrow \epsilon \odot \text{id} \uparrow \text{id} \odot \epsilon \\
\end{array} \]

\[ \begin{array}{c}
X \xrightarrow{\eta \odot \text{id}} X \odot Y \odot X \\
\downarrow \text{id} \downarrow \text{id} \downarrow \text{id} \downarrow \text{id} \\
Y \xrightarrow{\text{id}} U_B \odot Y \\
X \odot U_B \xrightarrow{\epsilon} X \\
\end{array} \]

commute.

The map \( \eta \) is called the coevaluation and \( \epsilon \) is called the evaluation.

If \( R \) is a (not necessarily commutative) ring and \( M \) is a finitely generated projective right \( R \)-module then \( M \) is a right dualizable 1-cell in \( \text{Mod} \) with dual \( \text{Hom}_R(M, R) \). The evaluation map

\[ \epsilon : \text{Hom}_R(M, R) \otimes_R M \to R \]

is defined by \( \epsilon(\phi, m) = \phi(m) \). This is a map of \( R-R \)-bimodules. Since \( M \) is finitely generated and projective there are elements \( \{m_1, m_2, \ldots, m_n\} \) and dual elements \( \{m'_1, m'_2, \ldots, m'_n\} \) of \( \text{Hom}_R(M, R) \) so that the coevaluation map

\[ \eta : \mathbb{Z} \to M \otimes_R \text{Hom}_R(M, R) \]
is defined by linearly extending \( \eta(1) = \sum m_i \otimes m'_i \). This is a map of abelian groups.

Unlike the symmetric monoidal case, we need more structure before we can define trace. The additional structure is a shadow.

**Definition 2.3.** [26, 4.4.1] A *shadow* for \( \mathcal{B} \) is a functor

\[
\llangle \cdot \rrangle : \coprod_{A \in \text{ob} \mathcal{B}} \mathcal{B}(A, A) \to \mathcal{T}
\]

to a category \( \mathcal{T} \) and a natural isomorphism

\[
\theta : \llangle X \otimes Y \rrangle \cong \llangle Y \otimes X \rrangle
\]

for every pair of 1-cells \( X \in \mathcal{B}(A, B) \) and \( Y \in \mathcal{B}(B, A) \) such that

\[
\begin{array}{ccc}
\llangle (X \otimes Y) \otimes Z \rrangle & \xrightarrow{\theta} & \llangle Z \otimes (X \otimes Y) \rrangle & \xrightarrow{\theta} & \llangle (Z \otimes X) \otimes Y \rrangle \\
\downarrow & & \downarrow & & \downarrow \\
\llangle X \otimes (Y \otimes Z) \rrangle & \xrightarrow{\theta} & \llangle (Y \otimes Z) \otimes X \rrangle & \xrightarrow{\theta} & \llangle Y \otimes (Z \otimes X) \rrangle
\end{array}
\]

\[
\begin{array}{ccc}
\llangle Z \otimes U_A \rrangle & \xrightarrow{\theta} & \llangle U_A \otimes Z \rrangle & \xrightarrow{\theta} & \llangle Z \otimes U_A \rrangle \\
\downarrow & & \downarrow & & \downarrow \\
\llangle Z \rrangle
\end{array}
\]

commute.

Let \( P \) be an \( R\)-\( R \)-bimodule. Let \( N(P) \) be the subgroup of \( P \) generated by elements of the form

\[
rp - pr
\]

for \( p \in P \) and \( r \in R \). Then the shadow of \( P \) is \( P/N(P) \).

**Definition 2.4.** [26, 4.5.1] Let \( X \) be a dualizable 1-cell and \( f : Q \otimes X \to X \otimes P \) be a 2-cell in \( \mathcal{B} \). The *trace* of \( f \) is the composite

\[
\llangle Q \rrangle \cong \llangle Q \otimes U_A \rrangle \xrightarrow{id \otimes \eta} \llangle Q \otimes X \otimes Y \rrangle
\]

\[
\begin{array}{ccc}
\llangle X \otimes P \otimes Y \rrangle & \xrightarrow{\theta} & \llangle P \otimes Y \otimes X \rrangle & \xrightarrow{id \otimes \epsilon} & \llangle P \otimes U_B \rrangle \cong \llangle P \rrangle
\end{array}
\]

If \( M \) is a finitely generated projective right \( R \)-module and \( f : M \to M \) is a map of right \( R \)-modules the trace of \( f \) is the trace defined by Stallings in [30].

A functor of bicategories \( F \) is a *shadow functor* if there is a natural transformation

\[
\psi : \llangle F(\cdot) \rrangle \to F(\llangle \cdot \rrangle)
\]
such that
\[
\langle FX \odot FY \rangle^\theta \to \langle FY \odot F(X) \rangle
\]
\[
\langle F(X \odot Y) \rangle \to \langle F(Y \odot X) \rangle
\]
\[
\psi \to \psi
\]
\[
F\langle\langle X \odot Y \rangle\rangle^\theta \to F\langle\langle Y \odot X \rangle\rangle
\]
commutes for all 1-cells \(X\) and \(Y\) where \(X \odot Y\) and \(Y \odot X\) are both defined.

**Proposition 2.5.** [26 4.5.7] Let \(X\) be a right dualizable 1-cell in \(\mathcal{B}\) with dual \(Y\),
\[
f: Q \odot X \to X \odot P
\]
be a 2-cell in \(\mathcal{B}\) and \(F: \mathcal{B} \to \mathcal{B}'\) be a shadow functor. If \(F(X) \odot F(Y) \to F(X \odot Y)\), \(F(X) \odot F(P) \to F(X \odot P)\), and \(U F(B) \to F(U_B)\) are isomorphisms and \(\hat{f}\) is the composite
\[
FQ \odot FX \xrightarrow{\phi} F(Q \odot X) F(X \odot P) \xrightarrow{\phi^{-1}} FX \odot FP
\]
then
\[
\langle FQ \rangle \xrightarrow{\text{tr}(\hat{f})} \langle FP \rangle
\]
\[
\psi \to \psi
\]
\[
F\langle\langle Q \rangle\rangle \xrightarrow{F\langle\langle \text{tr}(f) \rangle\rangle} F\langle\langle P \rangle\rangle
\]
commutes.

We will use this proposition to compare different forms of the Lefschetz number and Reidemeister trace.

### 3. Some examples of bicategories with shadows

The classical descriptions of fixed point invariants require a choice of base point. When working with a single space this isn’t a problem. With fiberwise spaces, equivariant spaces, or pairs of spaces, choosing base points requires addition conditions on the space. We can avoid these choices by using groupoids rather than groups.

In this section we describe a generalization of the bicategory \(\text{Mod}\) that we will use to defined fixed point invariants without choosing base points. In this bicategory we replace rings by categories, modules by functors, and homomorphisms by natural transformations.

Let \(\mathcal{V}'\) be a symmetric monoidal category with monoidal product \(\odot\) and unit \(S\).

**Definition 3.1.** A category \(\mathcal{A}\) is enriched in \(\mathcal{V}'\) if for each \(a, b \in \text{ob}(\mathcal{A})\), \(\mathcal{A}(a, b)\) is an object of \(\mathcal{V}'\) and the composition for \(\mathcal{A}\),
\[
\mathcal{A}(b, c) \odot \mathcal{A}(a, b) \to \mathcal{A}(a, c),
\]
is a morphism in \(\mathcal{V}'\).
For pairs of enriched categories \( \mathcal{A} \) and \( \mathcal{B} \) define an enriched category \( \mathcal{A} \otimes \mathcal{B} \) with objects pairs \((a, b)\) where \(a \in \text{ob}\mathcal{A}\) and \(b \in \text{ob}\mathcal{B}\). If \(a, a' \in \text{ob}\mathcal{A}\) and \(b, b' \in \text{ob}\mathcal{B}\), then
\[
(\mathcal{A} \otimes \mathcal{B})((a, b), (a', b')) = (\mathcal{A}(a, a')) \otimes (\mathcal{B}(b, b')).
\]

**Definition 3.2.** An enriched distributor is a functor \( \mathcal{X} : \mathcal{A} \otimes \mathcal{B}^{\text{op}} \to \mathcal{V} \) such that the action of morphisms of \( \mathcal{A} \) and \( \mathcal{B} \) on \( \mathcal{X} \) are maps in \( \mathcal{V} \).

This type of functor is also called an \( \mathcal{A}-\mathcal{B} \)-bimodule. If \( F : \mathcal{A} \to \mathcal{C} \) is an enriched functor and \( \mathcal{V} : \mathcal{C} \otimes \mathcal{C}^{\text{op}} \to \mathcal{V} \) is a distributor define a new distributor \( \mathcal{V}^F : \mathcal{A} \otimes \mathcal{C}^{\text{op}} \to \mathcal{V} \) by \( \mathcal{V}^F(a, b) = \mathcal{V}(F(a), b) \).

**Definition 3.3.** An enriched natural transformation \( \eta : \mathcal{X} \to \mathcal{Y} \) is a natural transformation where the maps
\[
\eta_{a,b} : \mathcal{X}(a, b) \to \mathcal{Y}(a, b)
\]
are maps in \( \mathcal{V} \) for all \(a \in \text{ob}\mathcal{A}\) and \(b \in \text{ob}\mathcal{B}\).

Enriched categories are the 0-cells of a bicategory we denote by \( \mathcal{E}_F \). The 1-cells are the distributors enriched in \( \mathcal{V} \). The 2-cells are enriched natural transformations.

If \( \mathcal{X} : \mathcal{A} \otimes \mathcal{B}^{\text{op}} \to \mathcal{V} \) and \( \mathcal{Y} : \mathcal{B} \otimes \mathcal{C}^{\text{op}} \to \mathcal{V} \) are two distributors, \( \mathcal{X} \circ \mathcal{Y} \) is a distributor \( \mathcal{A} \otimes \mathcal{C}^{\text{op}} \to \mathcal{V} \). For \(a \in \text{ob}(\mathcal{A})\) and \(c \in \text{ob}(\mathcal{C})\), \( \mathcal{X} \circ \mathcal{Y}(a, c) \) is the coequalizer of the actions of \( \mathcal{B} \) on \( \mathcal{X} \) and \( \mathcal{Y} \),
\[
\coprod_{b, b' \in \text{ob}\mathcal{B}} \mathcal{X}(a, b) \otimes \mathcal{B}(b', b) \otimes \mathcal{Y}(b', c) \rightrightarrows \prod_{b \in \text{ob}\mathcal{B}} \mathcal{X}(a, b) \otimes \mathcal{Y}(b, c).
\]

If \( \mathcal{Z} : \mathcal{A} \otimes \mathcal{A}^{\text{op}} \to \mathcal{V} \) is an enriched functor, the shadow of \( \mathcal{Z} \), \( \langle \langle \mathcal{Z} \rangle \rangle \), is the coequalizer of the two actions of \( \mathcal{A} \) on \( \mathcal{Z} \),
\[
\coprod_{a, a' \in \text{ob}(\mathcal{A})} \mathcal{A}(a, a') \otimes \mathcal{Z}(a, a') \rightrightarrows \prod_{a \in \text{ob}(\mathcal{A})} \mathcal{Z}(a, a).
\]

In \cite{26} Chapter 9 we observed that if \( \mathcal{A} \) is a connected groupoid, a distributor
\[
\mathcal{X} : \mathcal{A} \to \mathcal{V}
\]
is dualizable if and only if \( \mathcal{X}(a) \) is dualizable over \( \mathcal{A}(a, a) \) for any \(a \in \text{ob}\mathcal{A}\). The categories we will use here and in \cite{25} to define relative and equivariant fixed point invariants are not usually groupoids, but we can extend the results from \cite{26} to describe these examples.

**Definition 3.4.** \cite{22} II.9.2 A category \( \mathcal{A} \) is an EI-category if all endomorphisms are isomorphisms.

In an EI-category there is a partial order on the set of objects: \( x < y \) if \( \mathcal{X}(x, y) \neq \emptyset \).

Let \( \text{Ch}_R \) be the symmetric monoidal category of chain complexes of modules over a commutative ring \( R \) and chain maps. Let \( \mathcal{A} \) be a category enriched in the category of modules over \( R \). This can be regarded as a category enriched in chain complexes concentrated in degree zero.

**Definition 3.5.** A functor \( \mathcal{X} : \mathcal{A} \to \text{Ch}_R \) is supported on isomorphisms if \( \mathcal{X}(f) \) is the zero map if \( f \) is not an isomorphism.
If $\mathcal{X}$ is supported on isomorphisms it only ‘sees’ a disjoint collection of groupoids rather than the entire category $\mathcal{A}$.

Let $B(\mathcal{A})$ be a choice of representative for each isomorphism class of objects in $\mathcal{A}$.

**Lemma 3.6.** If $\mathcal{X}: \mathcal{A}^{op} \to \mathbf{Ch}_{\mathbb{R}}$ and $\mathcal{Y}: \mathcal{A} \to \mathbf{Ch}_{\mathbb{R}}$ are supported on isomorphisms then

$$\mathcal{X} \circ \mathcal{Y} \cong \bigoplus_{c \in B(\mathcal{A})} \mathcal{X}(c) \otimes_{\mathcal{A}(c,c)} \mathcal{Y}(c).$$

The proof of this lemma is in Section 9. The idea of the proof is to use Definition 3.5 to show that

$$\bigoplus_{c \in B(\mathcal{A})} \mathcal{X}(c) \otimes_{\mathcal{A}(c,c)} \mathcal{Y}(c)$$

satisfies the universal property that defines $\mathcal{X} \circ \mathcal{Y}$.

**Lemma 3.7.** Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy the conditions of Lemma 3.6. If $\mathcal{X}(c)$ is dualizable as an $\mathcal{A}(c,c)$-module with dual $\mathcal{Y}(c)$ for each $c \in B(\mathcal{A})$ then $\mathcal{X}$ is dualizable with dual $\mathcal{Y}$.

The idea of this proof is to use Lemma 3.6 and the coevaluation and evaluation maps for each $\mathcal{X}(c)$ to define coevaluation and evaluation maps for $\mathcal{X}$. This proof can also be found in Section 9.

Another choice for $\mathcal{V}$ is the symmetric monoidal category of pointed topological spaces, $\mathbf{Top}_\ast$. The bicategory $\mathcal{E}_{\mathbf{Top}_\ast}$ has 0-cells categories enriched in based spaces and 1-cells distributors enriched in based spaces. The 2-cells in $\mathcal{E}_{\mathbf{Top}_\ast}$ are natural transformations enriched in $\mathbf{Top}_\ast$.

If $\mathcal{X}^{op}: \mathcal{A} \to \mathbf{Top}_\ast$ and $\mathcal{Y}: \mathcal{A} \to \mathbf{Top}_\ast$ are enriched functors $\mathcal{X} \circ \mathcal{Y}$ is the bar resolution $B(\mathcal{X}, \mathcal{A}, \mathcal{Y})$. This is the geometric realization of the simplicial space with $n$-simplices

$$\prod_{a_0, a_1, \ldots, a_n \in \text{ob} \mathcal{A}} \mathcal{X}(a_0) \wedge \mathcal{A}(a_1, a_0) \wedge \mathcal{A}(a_2, a_1) \wedge \cdots \wedge \mathcal{A}(a_n, a_{n-1}) \wedge \mathcal{Y}(a_n).$$

The definition of the shadow is similar. If $\mathcal{Z}: \mathcal{A} \otimes \mathcal{A}^{op} \to \mathbf{Top}_\ast$ is a enriched functor, the shadow of $\mathcal{Z}$, $\langle \langle \mathcal{Z} \rangle \rangle$, is the cyclic bar resolution $C(\mathcal{A}, \mathcal{A}, \mathcal{Z})$. This is the geometric realization of the simplicial space with $n$-simplices

$$\prod_{a_0, a_1, \ldots, a_n \in \text{ob} \mathcal{A}} \mathcal{Z}(a_n, a_0) \wedge \mathcal{A}(a_1, a_0) \wedge \mathcal{A}(a_2, a_1) \wedge \cdots \wedge \mathcal{A}(a_n, a_{n-1}).$$

Let $\mathcal{A}$ be a category enriched in based spaces. Let $U_\mathcal{A}: \mathcal{A} \otimes \mathcal{A}^{op} \to \mathbf{Top}_\ast$ be defined by $U_\mathcal{A}(a, a') = \mathcal{A}(a', a)$. Composition in $\mathcal{A}$ defines the action of $\mathcal{A}$ and $\mathcal{A}^{op}$ on $U_\mathcal{A}$.

**Definition 3.8.** An enriched functor $\mathcal{X}: \mathcal{A} \to \mathbf{Top}_\ast$ is $n$-dualizable if there is a functor $\mathcal{Y}: \mathcal{A}^{op} \to \mathbf{Top}_\ast$, a map $\eta: S^n \to B(\mathcal{X}, \mathcal{A}, \mathcal{Y})$, and an $\mathcal{A}$-$\mathcal{A}$-equivariant map $c: \mathcal{Y} \wedge \mathcal{X} \to S^n \wedge U_\mathcal{A}$ such that the usual triangle diagrams commute up to $\mathcal{A}$-equivariant homotopy.

We will use the ideas of Lemma 3.7 to produced dual pairs in this bicategory, but we will not prove a general characterization.
The composition is defined by the rules of a groupoid. For example, if \( x \in (B \setminus A) \) and \([x] = [y] \in \pi_0(B)\) then \( \Pi_0(B, A)(x, y) = \ast \) and \( \Pi_0(B, A)(y, x) = \emptyset \). The relative component category is similar to the equivariant component category in \([32, I.10.3]\). For most pairs of spaces \( A \subset B \) this category is an EI-category but not a groupoid. For example, if \( x \in B \setminus A \), \( y \in A \), and \([x] = [y] \in \pi_0(B)\) then \( \Pi_0(B, A)(x, y) = \ast \) and \( \Pi_0(B, A)(y, x) = \emptyset \). The relative component category is similar to the equivariant component category in \([32, I.10.3]\).

If \( A \) and \( B \) are connected and \( B \setminus A \) is nonempty this category has two isomorphism classes of objects.

If \( x \in A \), let \( A(x) \) be the component of \( A \) that contains \( x \). If \( y \in B \), let \( B(y) \) be the component of \( B \) that contains \( y \).

**Definition 3.9.** The relative component category \( \Pi_0(B, A) \) of a pair \((B, A)\) has objects the points of \( B \). The morphisms of \( \Pi_0(B, A) \) are \[
\Pi_0(B, A)(x, y) = \begin{cases}
\ast & \text{if } x \in B \setminus A \text{ and } [x] = [y] \in \pi_0(B) \\
\emptyset & \text{if } x \in B \setminus A \text{ and } [x] \neq [y] \in \pi_0(B) \\
\ast & \text{if } x, y \in A \text{ and } [x] = [y] \in \pi_0(A) \\
\emptyset & \text{if } x, y \in A \text{ and } [x] \neq [y] \in \pi_0(A) \\
\emptyset & \text{if } x \in A, y \not\in A 
\end{cases}
\]
The composition is defined by the rules
\[
\ast \circ \ast = \ast \quad \emptyset \circ \emptyset = \emptyset \\
\emptyset \circ \ast = \emptyset \\
\ast \circ \emptyset = \emptyset 
\]

The composition is defined by the rules of a groupoid. For example, if \( x \in B \setminus A \), \( y \in A \), and \([x] = [y] \in \pi_0(B)\) then \( \Pi_0(B, A)(x, y) = \ast \) and \( \Pi_0(B, A)(y, x) = \emptyset \). The relative component category is similar to the equivariant component category in \([32, I.10.3]\).

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\ast & \text{if } x, y \in A \text{ and } [x] = [y] \in \pi_0(A) \\
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The composition is defined by the rules of a groupoid. For example, if \( x \in B \setminus A \), \( y \in A \), and \([x] = [y] \in \pi_0(B)\) then \( \Pi_0(B, A)(x, y) = \ast \) and \( \Pi_0(B, A)(y, x) = \emptyset \). The relative component category is similar to the equivariant component category in \([32, I.10.3]\).

If \( A \) and \( B \) are connected and \( B \setminus A \) is nonempty this category has two isomorphism classes of objects.

If \( x \in A \), let \( A(x) \) be the component of \( A \) that contains \( x \). If \( y \in B \), let \( B(y) \) be the component of \( B \) that contains \( y \).

**Definition 4.2.** The relative component space, \( \overline{B|A} \), of the pair \((B, A)\) is the functor \( \Pi_0(B, A)^{\text{op}} \to \text{Top}_s \) defined by \( \overline{B|A}(x) = \begin{cases} A(x) & \text{if } x \in A \\ B(x)/(A \cap B(x)) & \text{if } x \not\in A \end{cases} \)
The morphisms \( A(x) \to A(x) \) and \( B(x)/(A \cap B(x)) \to B(x)/(A \cap B(x)) \) are the identity maps. The map \( A(x) \to B(x)/(A \cap B(x)) \) is the inclusion of \( A(x) \) as the base point.

Recall that \( C(A \cap B(x)) \) is the cone on \( A \cap B(x) \). The base point is the cone point. Since \( A \subset B \) is assumed to be a cofibration \( B(x) \cup C(A \cap B(x)) \) is homotopy equivalent to \( B(x)/(A \cap B(x)) \).

**Lemma 4.3.** If \( A \) and \( B \) are both compact ENR’s or closed smooth manifolds then \( \overline{B|A} \) is dualizable.
Remark 4.4. Starting with the proof of this theorem we will focus on the case of closed smooth manifolds. The results in this section and Sections 5 and 6 have versions for compact ENR’s as well. The statements and proofs for compact ENR’s are very similar to those for closed smooth manifolds.

Some of the results in Section 7 have only been shown for manifolds.

Proof. Define a functor $D(B|A): \Pi_0(B, A) \to \text{Top}_*$ by

$$D(B|A)(x) = \begin{cases} D(A(x)_+) & \text{if } x \in A \\ D(B(x) \cup C(A \cap B(x))) & \text{if } x \notin A \end{cases}$$

where $D(A(x)_+)$ and $D(B(x) \cup C(A \cap B(x)))$ denote the duals of $A(x)_+$ and $B(x) \cup C(A \cap B(x))$ with respect to an embedding of $B$ in $\mathbb{R}^n$ as described in Proposition 1.5. The morphisms are the identity maps or the inclusion.

To simplify notation, consider the case where $A$ and $B$ are both connected. The general case is similar. The evaluation for this dual pair is a natural transformation $\epsilon: D(B|A) \otimes B|A \to S^n \wedge (\Pi_0(B, A))_+$. Let $x \in A$ and $y \in B \setminus A$ represent the isomorphism classes of objects of $\Pi_0(B, A)$. Then $\epsilon$ consists of four maps

$$D(B \cup C(A)) \wedge (B/A) \to S^n$$

$$D(A_+) \wedge (B/A) \to S^n$$

$$D(B \cup C(A)) \wedge A_+ \to *$$

$$D(A_+) \wedge A_+ \to S^n$$

By naturality, the second map must be the constant map to a point. Since $A_+$ and $B \cup C(A)$ are both dualizable the evaluations for these dual pairs are the first and fourth maps.

Note that $B(B|A, \Pi_0(B, A), D(B|A))$ is equivalent to

$$(A_+ \wedge D(A_+)) \vee [(B/A) \wedge D(B \cup CA)].$$

The dualizability of $A_+$ and $B \cup C(A)$ provide coevaluation maps

$$\eta_A: S^n \to A_+ \wedge D(A_+)$$

$$\eta_{B/A}: S^n \to B/A \wedge D(B \cup C(A)).$$

The coevaluation for this dual pair is the composite

$$S^n \xrightarrow{\triangle} S^n \vee S^n \xrightarrow{\eta_A \vee \eta_{B/A}}$$

$$(A_+ \wedge D(A_+)) \vee [(B/A) \wedge D(B \cup CA_+)].$$

Verifying that these maps describe a dual pair can be checked on $x$ and $y$ separately. The conditions reduce to conditions checked for Proposition 1.5. □

Let $\Pi_0'(B, A)$ be the functor

$$\Pi_0(B, A) \times \Pi_0(B, A)^{\text{op}} \to \text{Top}_*$$

defined by $\Pi_0'(B, A)(x, y) = \Pi_0(B, A)(f(y), x)_+$. The left action is composition. The right action is given by applying $f$ and then the composition.
A relative map \( f: (B, A) \to (B, A) \) induces a natural transformation
\[
\overline{f}: \overline{B|A} \to \overline{B|A} \odot \Pi^f_0(B, A).
\]
Since \( \overline{B|A} \) is dualizable, the trace of \( \overline{f} \) is defined.

**Definition 4.5.** The relative geometric Lefschetz number of \( f \), \( \iota_{B|A}(f) \), is the trace of \( \overline{f} \).

The relative geometric Lefschetz number is the stable homotopy class of a map
\[
S^0 \to \langle \Pi^f_0(B, A) \rangle
\]
and so it is an element of the 0th stable homotopy group of \( \langle \Pi^f_0(B, A) \rangle \). This group is denoted \( \pi^f_0(\langle \Pi^f_0(B, A) \rangle) \). It is the free abelian group on the set \( \langle \Pi^f_0(B, A) \rangle \). Since the relative geometric Lefschetz number is defined to be the trace of \( \overline{f} \) it is an invariant of the relative homotopy class of \( f \).

If \( \Pi^f_0(A) \) is the set of component of \( A, \langle \Pi^f_0(A) \rangle \) is
\[
\{[x] \in \pi_0(A)[f|A(x)] = [x] \}.
\]

If \( X \) is a closed smooth manifold, \( f: X \to X \) is a continuous map and \( F \) is an isolated subset of the set of fixed points of \( f \) let \( i(F, f) \) be the sum of the fixed point indices of the fixed points in \( F \). See [8] for the definition of the fixed point index of an isolated set of fixed points.

**Lemma 4.6.** There is an isomorphism
\[
\langle \Pi^f_0(B, A) \rangle \cong \langle \Pi^f_0(A) \rangle \Pi \langle \Pi^f_0(B) \rangle
\]
and the image of \( \iota_{B|A}(f) \) under this isomorphism is
\[
\sum_{x \in \langle \Pi^f_0(A) \rangle} i(\text{Fix}(f) \cap A(x), f)[x] + \sum_{y \in \langle \Pi^f_0(B) \rangle} i(\text{Fix}(f) \cap (B(y) \setminus A), f)[y].
\]
Since \( f \) is taut \( i(\text{Fix}(f) \cap A, f) = i(\text{Fix}(f) \cap A, f|A) \) and
\[
i(\text{Fix}(f) \cap (B \setminus A), f) = i(\text{Fix}(f), f) - i(\text{Fix}(f) \cap A, f|A).
\]
The first equality follows from commutativity of the index and the definition of a taut map. See [35] 3.5 for a proof of the second equality.

**Proof.** Assume \( A \) and \( B \) are connected and \( A \) is a proper subset of \( B \). These assumptions restrict the number of components of \( \langle \Pi^f_0(B, A) \rangle \). The proof is similar for the general case.

The set \( \langle \Pi^f_0(B, A) \rangle \) is defined to be the coequalizer
\[
\tag*{\prod_{x, y} \Pi^f_0(B, A)(x, y) + \Pi^f_0(B, A)(y, x)} \tag*{\prod_{x} \Pi^f_0(B, A)(x, x) \\longrightarrow \langle \Pi^f_0(B, A) \rangle}\]
Since \( \Pi^f_0(B, A)(x, y) \) is empty if \( x \in A \) and \( y \not\in A \), this coequalizer splits as two coequalizers. One is over pairs \((x, y)\) where \( x, y \in A \) and the other is over pairs \((x, y)\) where \( x, y \not\in A \). Each of these coequalizers consists of a single element.

This isomorphism is compatible with [Lemma 4.3] so the image of \( \iota_{B|A}(f) \) under the projection to the first summand is the trace of \( f \) restricted to \( A \). As observed after [Proposition 1.5] this is the fixed point index of \( f|A \).

The image of \( \iota_{B|A}(f) \) under the projection to the second summand is the trace of \( f/A: B/A \to B/A \). The fixed points of \( f/A \) are the fixed points of \( f|B\setminus A \) and
the point that represents \( A \). The point that represents \( A \) is the base point and so its index does not contribute to the trace of \( f/A \), see [13, III.8.5].

The second component of \( i_{B|A}(f) \) is the index defined in [13, 1.1].

**Example 4.7.** Let \( J \) be a nonempty, proper, connected subinterval of \( S^1 \). Let \( f: (S^1, J) \to (S^1, J) \) be the identity map. Then \( i_{S^1|J}(f) = (1, -1) \).

**Corollary 4.8.** If \( f: (B, A) \to (B, A) \) has no fixed points then \( i_{B|A}(f) = 0 \).

**Proof.** Since \( f \) has no fixed points \( \text{ind}(\text{Fix}(f)) = 0 \). To compute the relative geometric Lefschetz number of \( f \) we replace \( f \) by a relatively homotopic map \( g \) that is taut. The map \( g \) can be chosen so that \( f|_A = g|_A \). Then \( \text{ind}(\text{Fix}(g)) = 0 \) and \( \text{ind}(\text{Fix}(g) \cap A, g|_A) = \text{ind}(\text{Fix}(f) \cap A, f|_A) = 0 \).

Since \( f \) has no fixed points and \( f|_A = g|_A \), \( \text{ind}(\text{Fix}(g) \cap A, g) = 0 \). Since \( g \) is taut \( \text{ind}(\text{Fix}(g) \cap (B \setminus A), g) = \text{ind}(\text{Fix}(g), g) - \text{ind}(\text{Fix}(g) \cap A, g|_A) = 0 \).

□

Let \( \Pi_0(B, A) \) be the category with the same objects as \( \Pi_0(B, A) \). For objects \( x \) and \( y \) of \( \Pi_0(B, A) \)

\[
\Pi_0(B, A)(x, y)
\]

is the free abelian group on \( \Pi_0(B, A)(x, y) \). Composing \( \overline{B|A} \) with the rational homology functor defines a functor

\[
\mathcal{H}_*(\overline{B|A}): \Pi_0(B, A) \to \text{Ch}_Q.
\]

**Proposition 4.9.** If \( A \subset B \) are closed smooth manifolds, then \( \mathcal{H}_*(\overline{B|A}) \) is dualizable.

**Proof.** There are two ways to prove this proposition. First, the rational homology functor is strong symmetric monoidal, so this follows from Proposition 2.5.

We can also show \( \mathcal{H}_*(\overline{B|A}) \) is dualizable directly by describing the coevaluation and evaluation. The functor \( \mathcal{H}_*(\overline{B|A}) \) is supported on isomorphisms and so it is enough to construct a coevaluation and evaluation for the chain complexes of vector spaces \( \mathcal{H}_*(A) \) and \( \mathcal{H}_*(B, A) \). These are both finite dimensional, and so they are both dualizable with duals as in Section 1. □

A relative map \( f: (B, A) \to (B, A) \) induces a map

\[
\mathcal{H}_*(f): \mathcal{H}_*(\overline{B|A}) \to \mathcal{H}_*(\overline{B|A}) \otimes \Pi_0^f(B, A)
\]

by applying the rational homology functor to \( \overline{f} \).

**Definition 4.10.** The relative global Lefschetz number of \( f \), \( L_{B|A}(f) \), is the trace of \( \mathcal{H}_*(f) \).

**Lemma 4.11.** The image of \( L_{B|A}(f) \) under the isomorphism in Lemma 4.6 is

\[
\sum_{x \in \overline{\Pi_0^f(A)}} L_{A(x)}(f)[x] + \sum_{y \in \overline{\Pi_0^f(B)}} L_{B(y)/\langle A \cap B(y) \rangle}(f)[y].
\]

Here \( L_{A(x)}(f) \) and \( L_{B(y)/\langle A \cap B(y) \rangle}(f) \) denote the traces of the maps induced by \( f \) on \( \mathcal{H}_*(A(x)) \) and \( \mathcal{H}_*(B(y)/\langle A \cap B(y) \rangle) \).

**Proof.** Using Proposition 4.9 this proof is similar to the proof of Lemma 4.6. □
The invariant $L_{B/A}(f)$ is the relative Lefschetz number of \([1]\).

**Proposition 4.12.** In $\mathbb{Z}[\Pi_{0}^{f}(B, A)]$, $L_{B/A}(f) = i_{B/A}(f)$.

**Proof.** This proposition follows from Proposition 2.5 and the observation that the rational homology functor is strong symmetric monoidal. $\square$

The relative Lefschetz fixed point theorem follows from this proposition and Corollary 4.8.

**Theorem A** (Relative Lefschetz fixed point theorem). Let $A \subset B$ be closed smooth manifolds and $f : (B, A) \to (B, A)$ be a relative map. If $f$ has no fixed points then $L_{B/A}(f) = 0$.

Further, if $L_{B/A}(f) \neq 0$ all maps relatively homotopic to $f$ have a fixed point.

5. **The geometric Reidemeister trace**

To prove a converse to Theorem A it is necessary to introduce refinements of the invariants defined in the previous section. The first of these invariants is the geometric Reidemeister trace. This is a refinement of the geometric Lefschetz number and it will serve as a transition between the global Reidemeister trace in Section 6 and the invariant defined in Section 7.

As for the invariants in the previous section, it is possible to define the geometric Reidemeister trace using a generalization of the standard approach of fixed point indices and fixed point classes. Also as in the previous section, we do not use that approach here. Instead we use duality and trace in bicategories with shadows. This perspective gives simple comparisons of different invariants and also unifies the descriptions of different forms of the Reidemeister trace with the Lefschetz number.

**Definition 5.1.** The relative fundamental category, $\Pi_{1}(B, A)$, of the pair $(B, A)$ has objects the points of $B$. The morphisms $\Pi_{1}(B, A)(x, y)$ are the homotopy classes of paths from $x$ to $y$ in $A$ if $x \in A$ and homotopy classes of paths from $x$ to $y$ in $B$ if $x \in B \setminus A$.

The relative fundamental category is a subcategory of the fundamental groupoid of $B$. In most cases it is not a groupoid. For example, if $A$ and $B$ are both path connected, $x \in A$, and $y \in B \setminus A$ then $\Pi_{1}(B, A)(x, y)$ is empty and $\Pi_{1}(B, A)(y, x)$ is nonempty. This category is an EI-category. This category is similar to the equivariant fundamental category, see [32, I.10.7].

For $x \in A$, let $\tilde{A}_{x}$ be the universal cover of $A$ based at $x$. We think of points in $\tilde{A}_{x}$ as homotopy classes of paths in $A$ that start at $x$. For $y \in B \setminus A$ let $\tilde{B}_{y}$ be the universal cover of $B$ based at $y$. Let $p : \tilde{B}_{y} \to B$ be the quotient map and $\overline{A}_{y} = p^{-1}(A) \subset \tilde{B}_{y}$.

**Definition 5.2.** The relative universal cover of the pair $(B, A)$ is the functor

$$\tilde{B}|A : \Pi_{1}(B, A)^{\text{op}} \to \text{Top}_{*}$$

defined by

$$\tilde{B}|A(x) = \begin{cases} \overline{A}_{x} & \text{if } x \in A \\ \tilde{B}_{x}/\overline{A}_{x} & \text{if } x \not\in A \end{cases}$$

on objects and by composition of paths on morphisms.
Lemma 5.3. If $A \subset B$ is a cofibration $\tilde{B}_x/\tilde{A}_x$ is $\pi_1(B)$-homotopy equivalent to $\tilde{B}_x \cup C\tilde{A}_x$.

Proof. There is a $\pi_1(B)$-equivariant map
$$\phi: \tilde{B}_x \cup C\tilde{A}_x \to \tilde{B}_x/\tilde{A}_x$$
defined by collapsing the cone to the base point.
Since $A \subset B$ is a cofibration there is a map
$$u: B \to I$$
such that $u^{-1}(0) = A$ and a homotopy
$$h: B \times I \to B$$
such that $h(b,0) = b$ for all $b \in B$, $h(a,t) = a$ for all $a \in A$ and $t \in I$ and $h(b,1) \in A$ if $u(b) < 1$. The map
$$\psi: \tilde{B}_x/\tilde{A}_x \to \tilde{B}_x \cup C\tilde{A}_x$$
is defined by
$$\psi(\gamma) = \begin{cases} h(\gamma(1), t) |_{[0, 2(1-u(\gamma(1)))]} \circ \gamma & \text{if } \frac{1}{2} \leq u(\gamma(1)) \leq 1 \\ (h(\gamma(1), t) \circ \gamma, 1 - 2u(\gamma(1))) & \text{if } 0 \leq u(\gamma(1)) \leq \frac{1}{2} \end{cases}$$
The map $\psi$ is $\pi_1(B)$ equivariant. Up to homotopy it is an inverse for $\phi$.

□

Theorem 5.4. If $A \subset B$ are closed smooth manifolds the relative universal cover $B\lceil A$ is dualizable as a module over $\Pi_1(B,A)$.

Proof. The proof of this lemma is very similar to the proof of Lemma 4.3. We will define this dual pair by defining a dual pair for each isomorphism class of objects in $\Pi_1(B,A)$.
To simplify notation, consider the case where $A$ and $B$ are connected. Let $S^{\nu A}$ be the fiberwise one point compactification of the normal bundle of $A$. This is a space over $A$ and has a section given by the points at infinity. Let $D(\tilde{A}_+)$ be the space $(\tilde{A} \times_A S^{\nu A})/\sim$ where all points of the form $(\gamma, \infty_{\gamma(1)})$ are identified to a single point. This is the dual of $A_+$ as a distributor over $\pi_1(A)$, see [26, 5.3.3].

Let $C_B(S^{\nu B}, S^{\nu A})$ be
$$(B \times \{0\}) \cup (S^{\nu A} \times I) \cup (S^{\nu B} \times \{1\}).$$
This is the fiberwise cone of the map $S^{\nu A} \to S^{\nu B}$ over $B$. Let $D(\tilde{B} \cup C\tilde{A})$ be the space
$$(\tilde{B} \times_B C_B(S^{\nu B}, S^{\nu A}))/\sim$$
where all points of the form $(\gamma, \gamma(1) \times \{1\})$ are identified to a single point. This is the $\circ$ composition of the fiberwise spaces $(\tilde{B}, p)_+$ and $C_B(S^{\nu B}, S^{\nu A})$ defined in [26, 17.1.3]. An argument like that in [26, 5.3.3] for $\tilde{A}_+$ shows this is the dual of $\tilde{B} \cup C\tilde{A}$ as a distributor over $\pi_1(B)$.

The dual of $B\lceil A$, denoted $D(\tilde{B} \lceil A)$, is
$$D(\tilde{B} \lceil A)(x) = \begin{cases} D(\tilde{A}_+) & \text{if } x \in A \\ D(\tilde{B} \cup C\tilde{A}) & \text{if } x \in B \setminus A \end{cases}$$
The action of the morphisms in $\Pi_1(B,A)$ is by composition.
Using the assumption that $A$ and $B$ are connected there are two isomorphism classes of objects in $\Pi_1(B,A)$. As in Lemma 4.3 there are four maps that define the natural transformation $\epsilon$. Exactly as in that case there are only two that are nontrivial. These maps are the evaluation maps for the dual pairs $(\tilde{A}_+, D(\tilde{A}_+))$ and $(\tilde{B}/\tilde{A}, D(\tilde{B} \cup C\tilde{A}))$.

Also as in Lemma 4.3 $B(\tilde{B}/A, \Pi_1(B,A), D(\tilde{B}/A))$ is equivalent to
\[
\left( \tilde{A}_+ \land_{\pi_1(A)} D(\tilde{A}_+) \right) \lor \left( (\tilde{B}/\tilde{A}) \land_{\pi_1(B)} D(\tilde{B} \cup C\tilde{A}) \right).
\]

The coevaluation map is the composite of the fold map
\[
S^n \to S^n \lor S^n
\]
and the coevaluations for the pairs $(\tilde{A}_+, D(\tilde{A}_+))$ and $(\tilde{B}/\tilde{A}, D(\tilde{B} \cup C\tilde{A}))$.

The required diagrams commute since the coevaluation and evaluation maps are defined using coevaluation and evaluation maps from the dual pairs $(\tilde{A}_+, D(\tilde{A}_+))$ and $(\tilde{B}/\tilde{A}, D(\tilde{B} \cup C\tilde{A}))$.

\[\square\]

Remark 5.5. We can also give more explicit descriptions of the coevaluation and evaluation maps for the pairs $(\tilde{A}_+, D(\tilde{A}_+))$ and $(\tilde{B}/\tilde{A}, D(\tilde{B} \cup C\tilde{A}))$.

The coevaluation for the pair $(\tilde{A}_+, D(\tilde{A}_+))$ is the composite

\[
S^n \xrightarrow{T\nu_A} \tilde{A}_+ \land_{\pi_1 A} D(\tilde{A}_+)
\]

of the Pontryagin-Thom map for an embedding of $A$ in $S^n$ with the map $v \mapsto (\gamma, \gamma, v)$ where $\gamma$ is any element of $\tilde{A}$ that ends at the base of $v$.

Since $A$ is locally contractible there is a neighborhood $U$ of the diagonal in $A \times A$ and a map
\[
H: V \to A^I
\]
that satisfies $H(x, x)(t) = x$, $H(x, y, 0) = x$, and $H(x, y, 1) = y$. The evaluation for the pair $(\tilde{A}_+, D(\tilde{A}_+))$,
\[
D(\tilde{A}_+) \land \tilde{A}_+ \to S^n \land \pi_1 A_+
\]
is defined by
\[
(v, \gamma, \delta) = (\epsilon(v, \delta(1)), \gamma^{-1}H(\delta(1), \gamma(1))\delta)
\]
where $\epsilon$ is the evaluation for the dual pair $(A_+, D(A_+))$.

The coevaluation and evaluation for the dual pair $(\tilde{B}/\tilde{A}, D(\tilde{B} \cup C\tilde{A}))$ are similar.

A relative map $f: (B, A) \to (B, A)$ induces a map
\[
f_*: \overline{B|A} \to \overline{B|A} \otimes \Pi^I_1(B, A)
\]
where $\Pi^I_1(B, A)(x, y) = \Pi_1(B, A)(f(y), x)$, and the left action is the usual left action. The right action is given by applying $f$ and then composition.

**Definition 5.6.** The **relative geometric Reidemeister trace** of $f: (B, A) \to (B, A)$, $\text{grR}_{B|A}(f)$, is the trace of the map
\[
f_*: \overline{B|A} \to \overline{B|A} \otimes \Pi^I_1(B, A).
\]

The relative geometric Reidemeister trace is the stable homotopy class of a map
\[S^0 \to \{\Pi^I_1(B, A)\}\]
and so it is an element of the 0th stable homotopy group of $\langle \Pi_{1}^{f}(B, A) \rangle$. By definition, the relative geometric Reidemeister trace is an invariant of the relative homotopy class of the map.

Let $X$ be a dualizable space. For a space $U$ and a map
$$\Delta: X \to X \land U$$
the transfer of an endomorphism $f: X \to X$ with respect to $\Delta$ is the composite
$$S^{n} \xrightarrow{\eta} X \land DX \xrightarrow{\gamma} DX \land X \xrightarrow{id \land f} DX \land X \land U \xrightarrow{\iota \land \Delta} S^{n} \land U.$$

Let
$$\Lambda^{f|A} := \{ \gamma \in A^{f|f(0)} = (1) \}$$
and
$$\Lambda^{f} B := \{ \gamma \in B^{f(0)} = (1) \}.$$
Since $A$ and $B$ are locally contractible and $f$ is taut there are neighborhoods $U_{A}$ of the fixed points of $A$ and $U_{B}$ of the fixed points of $B \setminus A$ and maps
$$\iota_{A}: U_{A} \to \Lambda^{f|A}$$
$$\iota_{B}: U_{B} \to \Lambda^{f} B$$
that take fixed points of $f$ to the constant path at that point. Note that two fixed points of $f$ are in the same fixed point class if and only if their images are in the same connected component of $\Lambda^{f|A}$ or $\Lambda^{f} B$. See [3] or [10] for the definition of fixed point classes.

Let $\tau_{U_{A}}(f|A)$ denote the transfer of $f$ with respect to the diagonal map
$$A_{+} \to A_{+} \land \overline{U_{A}}$$
and similarly for $B$.

**Lemma 5.7.** If $A$ is a proper subset of $B$ there is an isomorphism
$$\pi_{0}^{\ast}(\langle \Pi_{1}^{f}(B, A) \rangle) \cong \pi_{0}^{\ast}(\Lambda^{f|A}) \oplus \pi_{0}^{\ast}(\Lambda^{f} B).$$
The image of the relative geometric Reidemeister trace of $f$ under this isomorphism is
$$(\iota_{A})_{\ast}(\tau_{U_{A}}(f|A)) + (\iota_{B})_{\ast}(\tau_{U_{B}}(f)).$$

**Proof.** We first define the isomorphism.

Note that $\pi_{0}^{A}(X) \cong \mathbb{Z}_{0} \pi_{0}(X)$ for any space $X$, so it is enough to show $\pi_{0}(\Lambda^{f|A}) \oplus \pi_{0}(\Lambda^{f} B)$ satisfies the universal property that defines the shadow of $\Pi_{1}^{f}(B, A)$.

The shadow of $\Pi_{1}^{f}(B, A)$ is defined to be the coequalizer of the maps
$$\Pi_{x,y} \Pi_{1}(B, A)(x, y) \times \Pi_{1}(B, A)(f(y), x) \cong \Pi_{x} \Pi_{1}(B, A)(f(x), x).$$

The inclusion maps
$$(\Pi_{x} \in A) \Pi_{1}(B, A)(f(x), x) \Pi_{x} \in A \Pi_{1}(B, A)(f(x), x) \to \pi_{0}(\Lambda^{f|A}) \oplus \pi_{0}(\Lambda^{f} B)$$
define a map
$$\theta: \Pi_{x} \Pi_{1}(B, A)(f(x), x) \to \pi_{0}(\Lambda^{f|A}) \oplus \pi_{0}(\Lambda^{f} B).$$
Let $\alpha \in \Pi_{1}(B, A)(x, y)$ and $\beta \in \Pi_{1}(B, A)(f(y), x)$. If $x, y \in A$ then $\beta \alpha$ and $f(\alpha)\beta$ represent the same elements in $\pi_{0}(\Lambda^{f|A})$. If $x, y \in B \setminus A$, $\beta \alpha$ and $f(\alpha)\beta$ represent the same elements in $\pi_{0}(\Lambda^{f} B)$. If $x$ and $y$ do not satisfy these conditions, there is no condition to check on the paths. So $\theta$ coequalizes.
If $\phi: \Pi_1 \Pi_1(B, A)(f(x), x) \to M$ is a map that coequalizes the maps above define a map

$$\tilde{\phi}: \pi_0(\Lambda f, A) \oplus \pi_0^\ast(\Lambda f, B) \to M$$

by $\tilde{\phi}(\gamma) = \phi(\beta)$ where $\beta$ is any element of $\Pi_1(B, A)(f(x), x)$ that maps to $\gamma$ in $\pi_0(\Lambda f, A) \oplus \pi_0^\ast(\Lambda f, B)$. This is independent of choices since if $\alpha$ is another lift of $\gamma$ then there are paths $\mu$ and $\nu$ such that $f(\mu)\nu$ is homotopic to $\beta$ and $\nu\mu$ is homotopic to $\alpha$. Then $\tilde{\phi}$ is unique and $\pi_0(\Lambda f, A) \oplus \pi_0^\ast(\Lambda f, B)$ is the coequalizer.

To describe the image of the geometric Reidemeister trace under this isomorphism it is enough to show the trace of

$$f^\dagger_A: \tilde{A} \to \tilde{A}$$

is $(\iota_A)_*(\tau_{U_A}(f|A))$ and the trace of

$$\tilde{f}: \tilde{B}/\tilde{A} \to \tilde{B}/\tilde{A}$$

is $(\iota_B)_*(\tau_{U_B}(f))$. We will describe the first, the second is similar.

In Remark 5.5 we gave explicit descriptions of the coevaluation and evaluation for the dual pair $(\tilde{A}+, D(\tilde{A}+))$. Let $q: D(\tilde{A}+) \wedge \tilde{A}+ \to DA \wedge A+$ be the quotient map. If $\eta_1$ and $\epsilon_1$ are the coevaluation and evaluation for the dual pair $(\tilde{A}+, D(\tilde{A}+))$ and $\eta_2$ and $\epsilon_2$ are the coevaluation and evaluation for the dual pair $(A+, DA_+)$ then the explicit descriptions of $\eta_1$ and $\epsilon_1$ show

$$\xymatrix{ S^n \ar[r]^-{\eta_1} & \tilde{A}+ \wedge_{\pi_A} D(\tilde{A}+) \ar[d]^q \ar[r]^-{\eta_2} & A+ \wedge DA \ar[d] }$$

and

$$\xymatrix{ \langle D(\tilde{A}+) \wedge \tilde{A}+ \wedge \Pi_1^\dagger(A) \rangle \ar[r]^-{\epsilon_1} & S^n \wedge \langle \Pi_1^\dagger(A) \rangle_+ \ar[l]^-{id \wedge \Delta} \ar[r]^-{\epsilon_2 \wedge id} & S^n \wedge U_A/\partial U_A \ar[l]^-{id \wedge \Delta} }$$

commute.

Together these diagrams show

$$(\epsilon_1 \wedge id)(\tilde{f} \wedge id)\gamma_{\eta_1} = (id \wedge \iota_A)(\epsilon_2 \wedge id)(\Delta \wedge id)\gamma_{\eta_1}$$

$$(\epsilon_1 \wedge id)(\tilde{f} \wedge id)\gamma_{\eta_2} = (id \wedge \iota_A)(\epsilon_2 \wedge id)(\Delta \wedge id)\gamma_{\eta_2}$$

The first composite is the trace of $f^\dagger_A$. The last composite is $(\iota_A)_*(\tau_{U_A}(f|A))$. □

For a fixed point class $\beta$ of $f: B \to B$ let $i^{rel}_\beta$ be the index of the fixed points associated to $\beta$ that are contained in $B \setminus A$. For a fixed point class $\alpha$ of $f|A: A \to A$, let $i_\alpha$ be the index of the fixed points associated to $\alpha$. Since the map $f$ is taut, $i_\alpha$ is the fixed point index of the fixed points in $A$ with respect to either $f|A$ or $f$.

The following corollary is a consequence of Lemma 5.7 and is the generalization of Lemma 4.6. This corollary identifies the relative geometric Reidemeister trace with the generalization of the classical description of the Reidemeister trace.
Corollary 5.8. Under the isomorphism in Lemma 5.4

\[ g^R_{B|A}(f) = \left( \sum i_\alpha \alpha \right) + \left( \sum j_\beta \beta \right) \in \pi_0(A|A) \oplus \pi_0^s(A|B). \]

The following two examples were described in [24]. In that paper the generalized Lefschetz number and one form of the relative Nielsen number are computed.

Example 5.9. [24, 5.1] Let \( B = D^2 \times S^1 \) and \( A = S^1 \times S^1 \). Let \( f : B \to B \) be

\[ f(re^{i\theta}, e^{it}) = (f_1(r)e^{-i\theta}, e^{3it}) \]

where \( f_1 : [0, 1] \to [0, 1] \) is a continuous function such that \( f_1(0) = 0, f_1(1) = 1 \) and \( f_1 \) has no other fixed points. Then \( f \) is a relative map with six fixed points. There are four fixed points in \( A \). These fixed points all represent different fixed point classes and all have index \(-1\). The two fixed points outside of \( A \) represent different fixed point classes in \( B \) and also have index \(-1\).

Since \( A \) is a torus, \( \pi_1(A) = \langle a, b | abab = 1 \rangle \). The relation imposed on the shadow implies \( \langle \pi_1(A) \rangle \) consists of 4 elements,

\[ 1, a, b, ab. \]

For \( B \), \( \pi_1(b) = \langle b \rangle \) and \( \langle \pi_1(B) \rangle \) consists of 2 elements,

\[ 1, b. \]

Then

\[ g^R_{B|A}(f) = -1(1_A + a_A + b_A + ab_A + 1_B + b_B). \]

Example 5.10. [24, 5.2] Let \( B = S^1 \times S^1 \) and \( A = 1 \times S^1 \). Let \( f : B \to B \) be

\[ f(e^{i\theta}, e^{it}) = (e^{3it}, e^{4it}). \]

There are three fixed points of \( f \) in \( A \) and three additional fixed points of \( f \) in \( B \setminus A \).

The three fixed points of \( f \) in \( A \) each represent one of the three possible fixed point classes. These fixed points all have index 1. The three fixed points in \( B \) that are not in \( A \) each represent three distinct fixed point classes, but these are only three of the six possible fixed point classes. These fixed points also have index 1.

Let \( \pi_1(B) = \langle a, b | abab = 1 \rangle \). Then \( \pi_1(A) = \langle a \rangle \). The set \( \langle \pi_1(B) \rangle \) consists of

\[ 1, a, a^2, b, ab, a^2b. \]

The set \( \langle \pi_1(A) \rangle \) consists of

\[ 1, a, a^2. \]

Then

\[ g^R_{B|A}(f) = 1_A + a_A + a^2_A + b_B + (ab)_B + (a^2b)_B. \]

The relative Nielsen number. One of the expectations for the Reidemeister trace is that it can detect when a map has no fixed points but it does not have to provide a lower bound for the number of fixed points. This is very different from the Nielsen number. The goal of the Nielsen number is to provide a lower bound.

In the classical case, the Nielsen number is the number of nonzero coefficients in the Reidemeister trace. This implies the Nielsen number is zero if and only if the Reidemeister trace is zero. For more general situations the connection between nonzero coefficients of the Reidemeister trace and the Nielsen number does not hold. It remains true that the Nielsen number is zero if and only if the Reidemeister trace is zero.
The inclusion of $A$ into $B$ induces a map $\pi_1(A) \to \pi_1(B)$ and also induces a map $\Phi$ from the fixed point classes of $A$ to the fixed point classes of $B$. A fixed point class of $f$ or $f|_A$ is essential if its coefficient in the classical Reidemeister trace is nonzero. Let $N(f, f|_A)$ be the number of essential fixed point classes of $B$ that are in the image of an essential class of $A$. Let $N(f)$ be the classical Nielsen number of $f$ and $N(f|_A)$ be the classical Nielsen number of $f|_A$.

**Definition 5.11.** [35, 2.5] The relative Nielsen number, $N(f; B, A)$, is $N(f|_A) + (N(f) - N(f, f|_A))$.

**Lemma 5.12.** The relative Nielsen number of $f$ is zero if and only if the relative geometric Reidemeister trace of $f$ is zero.

**Proof.** If the relative geometric Reidemeister trace of $f$ is zero, Corollary 5.8 implies $\left(\sum i^\text{rel}_\beta\right) + (\sum i_\alpha) = 0$. Since $\mathbb{Z}[\Pi_1(B, A)]$ is a free group generated by the $\alpha$’s and $\beta$’s, each $i^\text{rel}_\beta$ and $i_\alpha$ are zero. Since the $i_\alpha$’s are zero, $N(f|_A)$ and $N(f, f|_A)$ are zero. Since each of the $i_\alpha$’s are zero, $i_\beta = i^\text{rel}_\beta = 0$ for every $\beta$. This implies $N(f)$ is also zero.

By definition $N(f|_A)$, $N(f)$, and $N(f, f|_A)$ are all greater than or equal to zero and $N(f, f|_A) \leq N(f)$. If the relative Nielsen number of $f$ is zero $N(f|_A) = 0$ and $N(f) = N(f, f|_A)$. Since $N(f|_A) = 0$, $N(f, f|_A) = 0$ and so $N(f) = 0$. Since $N(f|_A) = 0$ all of the $i_\alpha$’s are zero and $i^\text{rel}_\beta = i_\beta$. Since $N(f) = 0$, $i_\beta = 0$ for all $\beta$. □

The relative Nielsen numbers for the maps in the examples above were computed in [24]. The relative Nielsen number for Example 5.9 is 4. This is not the number of non-zero coefficients in the relative Reidemeister trace. The relative Nielsen number for Example 5.10 is 6. This does happen to be the number of non-zero coefficients in the relative Reidemeister trace. These numbers coincide because $N(f, f|_A)$ is zero for this example.

Other references for relative Nielsen theory include [15, 28, 29, 33, 34]. These invariants are also related to the Nielsen numbers for stratified spaces defined in [17].

### 6. The global Reidemeister trace

In this section we define the relative global Reidemeister trace. This invariant is a generalization of the relative global Lefschetz number and can be identified with the relative geometric Reidemeister trace. The relative global Reidemeister trace a relative generalization of the invariant defined in [13]. It is related to the invariants defined in [24, 36], but it is not the same as either of these invariants.

Let $\mathbb{Z}\Pi_1(B, A)$ be the category with the same objects as $\Pi_1(B, A)$. The morphism set

$$\mathbb{Z}\Pi_1(B, A)(x, y)$$

is the free abelian group on the set $\Pi_1(B, A)(x, y)$.

There is a functor

$$C_*(B|A): \mathbb{Z}\Pi_1(B, A)^{op} \to \text{Ch}_\mathbb{Q}$$
defined by $C_*(B|A)(x) = C_*(B|A(x); \mathbb{Q})$ where the second $C_*$ indicates the cellular chain complex. The action of the morphisms of $\Pi_1(B, A)$ is induced from the action on $B|A$. This functor is defined in the same way that the functor $H_*(B|A)$ is defined from the functor $B|A$ except we replace the rational homology functor with the rational chain complex functor.

**Proposition 6.1.** If $A \subset B$ are closed smooth manifolds the $\mathbb{Z}\Pi_1(B, A)$-module $C_*(B|A)$ is dualizable.

**Proof.** Like Proposition 4.9 there are two possible proofs of this theorem.

The rational cellular chain complex functor induces a functor on bicategories, and for $A \subset B$ closed smooth manifolds, Theorem 5.4 shows that $B|A$ is dualizable. Proposition 2.5 then implies that $C_*(B|A)$ is dualizable.

There is a second approach using Lemma 3.7. If $x \in A$, $C_*(B|A)(x) = C_*(\tilde{A}_x)$ as a module over $\pi_1(A, x)$. This is a finitely generated free module and so is dualizable with dual

$$\text{Hom}_{\mathbb{Z}\pi_1(A, x)}(C_*(\tilde{A}_x), \mathbb{Z}\pi_1(A, x)).$$

If $x \in B \setminus A$, $C_*(B|A)(x) = C_*(\tilde{B}_x/\tilde{A}_x)$ as a module over $\pi_1(B, x)$. This is also a finitely generated free module and so is dualizable with dual

$$\text{Hom}_{\mathbb{Z}\pi_1(B, x)}(C_*(\tilde{B}_x/\tilde{A}_x), \mathbb{Z}\pi_1(B, x)).$$

Since $C_*(B|A)$ is supported on isomorphisms, Lemma 3.7 implies $C_*(B|A)$ is dualizable. □

A map $f: (B, A) \to (B, A)$ induces a map

$$f_*: C_*(B|A) \to C_*(B|A) \otimes \mathbb{Z}\Pi_1^f(B, A).$$

Since $C_*(B|A)$ is dualizable, the trace of $f_*$ is defined.

**Definition 6.2.** The relative global Reidemeister trace of $f: (B, A) \to (B, A)$, $\gamma R_{B|A}(f)$, is the trace of

$$f_*: C_*(B|A) \to C_*(B|A) \otimes \mathbb{Z}\Pi_1^f(B, A).$$

The relative global Reidemeister trace of $f$ is a map

$$\mathbb{Z} \to \mathbb{Z}\Pi_1^f(B, A).$$

**Lemma 6.3.** If $A$ is a proper subset of $B$ then

$$\langle \Pi_1^f(B, A) \rangle \cong \langle \Pi_1^f(B) \rangle \Pi_1^f(A).$$

The image of $\gamma R_{B|A}(f)$ under this isomorphism is

$$\sum_{x \in \Pi_1^f(A)} \gamma R(f|_{A(x)}|[x]) + \sum_{y \in \Pi_1^f(B)} \gamma R(f|_{B|A(y)} / (B|A(y) \cap A))[y].$$

Here $\gamma R(f|_{A(x)})$ denotes the usual global Reidemeister trace of $f|_{A(x)}$ as defined by Husseini in [13]. The invariant $\gamma R(f|_{B|A(y)} / (B|A(y) \cap A))$ is the trace of

$$f_*: C_*(B(y)/ (B(y) \cap A)) \to C_*(B(y)/ (B(y) \cap A)) \otimes \pi_1^f(B, y)$$

as a module over $\pi_1(B, y)$. 
**Proof.** To simplify notation, consider the case where $A$ and $B$ are connected. The proof is similar if $A$ and $B$ are not connected.

The shadow is defined to be the coequalizer of the maps

$$\Pi_x \Pi_1(B, A)(x, y) \times \Pi_1(B, A)(f(y), x) \longrightarrow \Pi_x \Pi_1(B, A)(f(x), x).$$

Instead of indexing these coproducts over all objects in $\Pi_1(B, A)$ we can index over representatives of each isomorphism class of objects in $\Pi_1(B, A)$. This gives four terms in the first coproduct. The two cross terms are both empty and so this coequalizer splits into the coequalizer that defines $\langle [\Pi_1(B)] \rangle$ and the coequalizer that defines $\langle [\Pi_1(A)] \rangle$.

For the second statement, note that this isomorphism is compatible with the description of the dual pair. Then the trace is the pair of classical traces. □

This description of the relative global Reidemeister trace shows that the second coordinate is the the relative Reidemeister trace of [36]. This also shows that this invariant is related to, but not the same as, the generalized Lefschetz number defined in [24].

**Proposition 6.4.** If $A \subset B$ are closed smooth manifolds and $f : (B, A) \to (B, A)$ is a relative map then

$$g^R_{B|A}(f) = g_{R|A}(f).$$

**Proof.** Since both $g^R_{B|A}(f)$ and $g_{R|A}(f)$ are defined as traces and the rational cellular chain complex functor is strong symmetric monoidal this proposition follows from Proposition 2.5. □

7. A converse to the relative Lefschetz fixed point theorem

There are several proofs of the converse to the relative Lefschetz fixed point theorem. Some, like [15, 28, 29, 34], are generalizations of the simplicial arguments used in the standard proof of the converse to the classical Lefschetz fixed point theorem, see [3].

In this section and the next section we describe a proof of the converse to the relative Lefschetz fixed point theorem that follows the outline of [19, 18]. This approach is not simplicial and it easily generalizes. For example, see [18] for the equivariant generalization and [26] for the fiberwise generalization.

The approach of [19] is based on invariants that detect sections of fibrations. In the next section we prove relative generalizations of the results in [19]. In this section we apply those results to relative fixed point invariants.

The main result of this section is:

**Theorem B** (The converse to the Relative Lefschetz fixed point theorem). Suppose $A \subset B$ are closed smooth manifolds of dimension at least 3 and the codimension of $A$ in $B$ is at least 2. The relative global Reidemeister trace of a map

$$f : (B, A) \to (B, A)$$

is zero if and only if $f$ is relatively homotopic to a map with no fixed points.

The first step in the proof of Theorem B is to describe relative maps without fixed points in terms of relative sections.
Lemma 7.1. Let $A \subset B$ be closed smooth manifolds. Relative homotopies of a map $f: (B, A) \to (B, A)$ to a relative map with no fixed points correspond to liftings that make the diagram below commute up to relative homotopy

$$
\begin{array}{c}
(B \times B \setminus \Delta, A \times A \setminus \Delta) \\
\downarrow \\
(B, A) \xrightarrow{\Gamma_f} (B \times B, A \times A).
\end{array}
$$

The function $\Gamma_f$ is the graph of $f$ and $\Gamma_f(m) = (m, f(m))$.

Proof. If $f$ is relatively homotopic to a fixed point free map $g$ via a relative homotopy $H$, then $\Gamma_H$ is a relative homotopy from $\Gamma_f$ to $\Gamma_g$.

For the converse, suppose there is a relative map $k: (B, A) \to (B \times B \setminus \Delta, A \times A \setminus \Delta)$ and a relative homotopy $K$ from $k$ to $\Gamma_f$.

If $A$ is a smooth manifold the first coordinate projection $\operatorname{proj}_1: A \times A \setminus \Delta \to A$ is a fiber bundle and there is a lift $J_A$ in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{k} & A \times A \setminus \Delta \\
\downarrow_{\iota_A} & & \downarrow_{\operatorname{proj}_1} \\
A \times I & \xrightarrow{\operatorname{proj}_1 K} & A.
\end{array}
$$

Since $A \subset B$ is a cofibration and $\operatorname{proj}_1: B \times B \setminus \Delta \to B$ is a fibration the diagram

$$
\begin{array}{ccc}
B \cup A \times I & \xrightarrow{k \cup J_A} & B \times B \setminus \Delta \\
\downarrow_{\iota_0} & & \downarrow_{\operatorname{proj}_1} \\
B \times I & \xrightarrow{\operatorname{proj}_1 \circ K} & B
\end{array}
$$

has a lift $J$ extending the lift $J_A$ above, see [31, Theorem 4]. Note that $\operatorname{proj}_1 \circ J(-, 1) = \text{id}$. Let $g = \operatorname{proj}_2 J(-, 1)$. This map has no fixed points.

The homotopies $K$ and $J$ define a relative homotopy from $\Gamma_f$ to $\Gamma_g$. \qed

Given a map $f: V \to Y$, let $r(f): N(f) \to Y$ denote a Hurewicz fibration such that

$$
\begin{array}{ccc}
V & \xrightarrow{r(f)} & N(f) \\
\downarrow_f & & \downarrow_r \\
Y & & \end{array}
$$

commutes and $V \to N(f)$ is an equivalence.

Lemma 7.2. Let $X \subset Y$, $p: M_Y \to Y$ be a space over $Y$ and $M_X \subset p^{-1}(X)$. 

Liftings up to relative homotopy in the diagram

\[
\begin{array}{ccc}
(M_Y, M_X) & \xrightarrow{f} & (B, A) \\
\downarrow & & \downarrow \\
(Y, X) & \xrightarrow{g} & (B, A)
\end{array}
\]

correspond to relative sections of the pair of fibrations

\[(g^*N(f_Y), g^*N(f_X)) \to (B, A).\]

If \(p: E \to B\) is a Hurewicz fibration the unreduced fiberwise suspension of \(p\) is the double mapping cylinder

\[S_B E := B \times \{0\} \cup_p E \times [0, 1] \cup_p B \times \{1\}.\]

The map \(p: E \to B\) defines a fibration

\[S_B E \to B.\]

There are two sections of this fibration, \(\sigma_1, \sigma_2: B \to S_B E\), defined by the inclusions of \(B \times \{0\}\) and \(B \times \{1\}\). If \(S_B^0 := B \amalg B\), these sections define an element of

\[\{S_B^0, S_B \Gamma f_* (N(i_B))\}_B \oplus \{S_A^0, S_A \Gamma f_* (N(i_A))\}_A.\]

This element will be denoted \(KWR_{B|A}(f)\).

**Proposition 7.3.** Let \(A \subset B\) be closed smooth manifolds of dimension at least 3 such that the codimension of \(A\) in \(B\) is at least 2. A continuous map

\[f: (B, A) \to (B, A)\]

is relatively homotopic to a map with no fixed points if and only if \(KWR_{B|A}(f) = 0\).

The proof of this proposition, except for one key step proved in the next section, follows the preliminary lemma below.

**Lemma 7.4.** [19 6.1, 6.2] Let \(M\) be a manifold of dimension \(n\) and \(i: M \times M \setminus \Delta \to M \times M\) be the inclusion. Then \(\Gamma f_* (N(i)) \to M\) is \((n - 1)\)-connected.

**Proof of Proposition 7.3.** [Lemma 7.1 and Lemma 7.2] convert the question of finding a lift of a relative map \(f: (B, A) \to (B, A)\) to the question of finding a section of the fibration

\[(\Gamma f_* (N(i_B)), \Gamma f|_{A*} (N(i_A))) \to (B, A).\]

If the dimension of \(A\) is \(n_A\) and the dimension of \(B\) is \(n_B\) then [Lemma 7.4] implies that \(\Gamma f_* (N(i_B)) \to B\) is \((n_B - 1)\)-connected and \(\Gamma f|_{A*} (N(i_A)) \to A\) is \((n_A - 1)\)-connected. If \(n_A\) and \(n_B\) are at least 3 and \(n_B - n_A\) is at least 2, [Proposition 8.6] implies that

\[(\Gamma f_* (N(i_B)), \Gamma f|_{A*} (N(i_A))) \to (B, A)\]

has a relative section if and only if \(KWR_{B|A}(f) = 0\). \(\square\)
The hypotheses in this proposition are not the standard hypotheses used in the converse to the relative Lefschetz fixed point theorem. The standard condition is that
\[ \pi_1(B \setminus A) \to \pi_1(B) \]
is surjective. The codimension condition implies this condition. We use a codimension condition since it is compatible with the approach of the proof. I don’t know if the surjectivity condition can be used in this approach.

To complete the proof of Theorem B we need to compare \( K^{W_R}_{B\mid A}(f) \) and the relative geometric Reidemeister trace.

**Proposition 7.5.** Let \( A \subset B \) be closed smooth manifolds and \( f : (B, A) \to (B, A) \) be a relative map. Then
\[ K^{W_R}_{B\mid A}(f) = 0 \text{ if and only if } R_{B\mid A}(f) = 0. \]

We recall a lemma from [18].

**Lemma 7.6.** [18, 7.1][26, 8.3.1] Let \( M \) be a closed smooth manifold with normal bundle \( \nu_M \). Then there is a weak equivalence
\[ S^{v_M} \circ \Gamma_{f_*} S_{M \times M} N(i_M) \to S^n \wedge \Lambda^f M. \]

**Proof of Proposition 7.5.** Suppose \( X \) and \( Y \) are ex-spaces over \( B \), the projection maps are fibrations and the sections are fiberwise cofibration. Let \( \{X, Y\}_B \) denote the fiberwise stable homotopy classes of maps from \( X \) to \( Y \).

If \( A \) and \( B \) are both closed smooth manifolds of dimension at least three, then the dimension assumption, Lemma 7.4, and the fiberwise Freudenthal suspension theorem in [14, 4.2] imply that the maps
\[
[S^0_A, S_A \Gamma_{f|A}* (N(i_A))]_A \to \{S^0_A, S_A \Gamma_{f|A}* (N(i_A))\}_A
\]
\[
[S^0_B, S_B \Gamma_{f*} (N(i_B))]_B \to \{S^0_B, S_B \Gamma_{f*} (N(i_B))\}_B
\]
are isomorphisms. Costenoble-Wanner duality [23, 18.5.5, 18.6.3] and Lemma 7.6 imply there are isomorphisms
\[
\{S^0_A, S_A \Gamma_{f|A}* (N(i_A))\}_A \cong \{S^n, S^n \wedge \Lambda^f A_+\}
\]
and
\[
\{S^0_B, S_B \Gamma_{f*} (N(i_B))\}_B \cong \{S^n, S^n \wedge \Lambda^f B_+\}
\]

Let \( U_A \) be a neighborhood of the fixed points of \( f|_A \) such that there is a map
\[ \iota_A : U_A \to \Lambda^f|A \]
that takes fixed points to the constant path at that point. In [26, 6.3.2] it is shown that the image of \( K^{W_R}_{B\mid A}(f) \) in \( \pi_0^*(\Lambda^f|A_+) \) is \( \iota_A(\tau(f|_{U_A})) \).

Let \( U_B \) be a neighborhood of the fixed points of \( f \) in \( B \setminus A \) such that there is a map
\[ \iota_B : U_B \to \Lambda^f B \]
which takes fixed points to constant paths.
The image of $KW_{R|A}(f)$ in $\pi_0^R(\Lambda^f B_\perp)$ is the composite of the transfer of $f$ with respect to the diagonal map

$$B_\perp \to B_\perp \wedge (U_B \amalg U_A)/\partial(U_B \amalg U_A)$$

with the map

$$\iota := \iota_A \amalg \iota_B : U_A \amalg U_B \to \Lambda^f B.$$ 

Since the transfer is additive, the image of $KW_{R|A}(f)$ in $\pi_0^R(\Lambda^f B_\perp)$ is

$$\iota(\tau_{U_B}) + \tau_{U_A}(f) = \iota(\tau_{U_B}(f) + \tau_{U_A}(f|A)) = \iota(\tau_{U_B}(f)) + \iota(\tau_{U_A}(f|A)).$$

Then $KW_{R|A}(f)$ is zero if and only if $\iota_A(\tau(f|U_A))$ and $\iota(\tau_{U_B}(f) + \tau_{U_A}(f|A))$ are both zero. Using Lemma 5.7, these elements are zero if and only if $R_{B|A}(f)$ is zero.

**Proof of Theorem B.** Proposition 7.3 implies that $f$ is relatively homotopic to a fixed point free map if and only if $KW_{R|A}(f) = 0$. Proposition 7.3 implies $KW_{R|A}(f) = 0$ if and only if $\gamma R_{B|A}(f) = 0$. Proposition 6.4 implies $\gamma R_{B|A}(f) = 0$. Proposition 6.4 implies $\gamma R_{B|A}(f)$.

**Remark 7.7.** Proposition 7.3 and the proof of Proposition 7.5 show if $\dim(A) \geq 3$ and $\dim(B) \geq \dim(A) + 2$ $KW_{R|A}(f)$ is zero if and only if the two nonrelative invariants for $A$ and $B$ are zero.

Using these two invariants to define a relative invariant would be analogous to defining the relative invariants in the previous sections as the pair of classical invariants for the spaces $A$ and $B$. This alternate definition would satisfy the requirements of the introduction for a fixed point invariant. However, there are several reasons why the corresponding definition in the equivariant case is not acceptable. The definitions in the previous sections were chosen because they are consistent with the choices in [25].

### 8. Relative sections

In this section we generalize the result from [19] on sections of fibrations to relative fibrations.

If the dimension of $B$ is $2n$ and the fibration $p: E \to B$ is $n + 1$-connected, it is shown in [19] that the two sections

$$\sigma_1, \sigma_2: B \to S_B E$$

are homotopic over $B$ if and only there is a section of $p$. We can generalize this result to relative sections.

If $A \subset B$ let $E_A$ be a subspace of $E_B$ such that the image of $p$ restricted to $E_A$ is contained in $A$. Let $S_{A,B}E_A$ be

$$B \times \{0\} \cup E_A \times I \cup A \times \{1\}.$$

Let $[(S_B^0, A \amalg B), (S_B E_B, S_{A,B} E_A)]_B$ be the relative fiberwise homotopy classes of maps from $(S_B^0, A \amalg B)$ to $(S_B E_B, S_{A,B} E_A)$.

**Definition 8.1.** Let $A \subset B$, $p: E_B \to B$ be a fibration, and $E_A \subset p^{-1}(A)$ such that $E_A \to A$ is a fibration. The relative homotopy Euler class

$$\chi \in [(S_B^0, A \amalg B), (S_B E_B, S_{A,B} E_A)]_B$$

is $\sigma_1 \amalg \sigma_2: S_B^0 \to S_B E_B$. 
Proposition 8.2. If \((E_B, E_A) \to (B, A)\) admits a relative section \(\tilde{\varsigma}\) then \(\chi\) is trivial.

Conversely, assume \(p: E_A \to A\) is \((m + 1)\)-connected, \(A\) is a \(2m\)-dimensional CW-complex, \(p: E_B \to B\) is \((n+1)\)-connected and \((B, A)\) is a relative \(2n\)-dimensional CW-complex. If \(\chi\) is trivial then \(p\) has a relative section.

Before we prove this proposition we recall a preliminary lemma.

Lemma 8.3. \([19, 3.1]\) Let \(p: E \to B\) be a \((j + 1)\)-connected fibration and \(P\) be the homotopy pullback

\[
\begin{array}{ccc}
P & \to & B \\
\downarrow & & \downarrow \\
B & \to & S_BE.
\end{array}
\]

A fiberwise homotopy from \(\sigma_1\) to \(\sigma_2\) defines a \(2j\)-equivalence \(q: E \to P\).

Proof of Proposition 8.2. If there is a relative section \(\tilde{\varsigma}\) then the homotopy \(H: (S^0_B, A \amalg B) \times I \to (S_BE_B, S_A,BE_A)\)

defined by \(H(b, t) = (\tilde{\varsigma}(b), t)\) shows \(\chi\) is trivial.

If \(\chi\) is trivial there is a relative fiberwise homotopy \(K: (S^0_B, A \amalg B) \times I \to (S_BE_B, S_A,BE_A)\)

from \(\sigma_2\) to \(\sigma_1\). The restriction of \(K\) to \(S^0_A\) defines a homotopy between \(\sigma_1|_A: A \to S_AE_A\) and \(\sigma_2|_A\). Lemma 8.3, Whitehead’s theorem, and the homotopy \(K|_{S^0_A}\) imply

\[q_{A*}: [A, E_A] \to [A, P_A]\]

is a bijection. The space \(P_A\) is as in Lemma 8.3.

The restriction \(K|_{S^0_A}\) induces a map \(h_A: A \to P_A\) such that \(ph_A = \text{id}\). Since \(q_{A*}\) is a bijection there is a map \(k_A: A \to E_A\) and a homotopy \(J_A\) from \(q_Ak_A\) to \(h_A\). Then \(pk_A = p(q_Ak_A) \simeq ph_A = \text{id}_A\) via the homotopy \(p(J_A)\). The diagram

\[
\begin{array}{ccc}
A & \xrightarrow{k_A} & E_A \\
\downarrow & \searrow \downarrow & \downarrow \\
A \times I & \xrightarrow{p(J_A)} & P_A
\end{array}
\]

has a lift \(L_A\), and \(p(L_A(a, 1)) = a\). Then \(L_A(-, 1)\) is a section of \(p^{-1}(A) \to A\) that is contained in \(E_A\).

The homotopy \(K\) defines a map \(h_B: B \to P_B\) extending the map \(h_A\). The space \(P_B\) is as in Lemma 8.3. The homotopy extension and lifting property implies the dotted maps in the following diagram can be filled in

\[
\begin{array}{ccc}
A & \xrightarrow{i_1} & A \times I & \xrightarrow{i_0} & A \\
\downarrow & \searrow \downarrow & \downarrow & \searrow \downarrow & \downarrow \\
B & \xrightarrow{i_1} & B \times I & \xleftarrow{i_0} & B
\end{array}
\]
defining maps $k_B$ and $J_B$ extending $k_A$ and $J_A$.

Since the pair $(B, A)$ has the relative homotopy lifting property there is a lift $L_B$ in the diagram

$$
\begin{array}{ccc}
B \cup (A \times I) & \xrightarrow{k_B \cup J_A} & E_B \\
\downarrow & & \downarrow p \\
B \times I & \xrightarrow{p(J_B)} & B
\end{array}
$$

Evaluating at 1, $p(L_B(b, 1)) = pJ_B(b, 1) = ph_B(b) = b$. Since $L_B(a, 1) \in E_A$ for $a \in A$, $L_B(-, 1)$ is the required section. \(\square\)

Lemma 7.1, Lemma 7.2, and Proposition 8.2 imply $\chi$ is a complete obstruction to determining if a relative fibration has a section. In the examples we are interested in, it is easier to work with invariants defined from $\chi$ than with $\chi$ itself. Under some additional hypotheses, these associated invariants are zero if and only if $\chi$ is zero.

If $A \subset B$, define

$$
C_B(B, A) := B \times \{0\} \cup A \times [0, 1] \cup B \times \{1\}.
$$

This is an ex-space over $B$ with section given by the inclusion of $B$ into $C_B(B, A)$ as $B \times \{0\}$.

In the diagram below the vertical maps are induced by cofiber sequences, \([4, \text{II.2.4}]\) and so the columns are exact. The horizontal maps are forgetful maps. The diagram commutes.

$$
\begin{array}{ccc}
\chi_A \in [(C_B(B, A), S_B A), (S_B E_B, S_A E_A)]_B & \xrightarrow{\chi} & [C_B(B, A), S_B E_B]_B \ni \tilde{\chi}_{B,A} \\
\downarrow \phi & & \downarrow p \\
\chi \in [(S_B^0, S_A^0), (S_B E_B, S_A E_A)]_B & \xrightarrow{\chi} & [S_B^0, S_B E_B]_B \ni \chi_B \\
\downarrow & & \downarrow \\
\chi_B \in [A \amalg B, S_{A,B} E_A]_B & \xrightarrow{\chi} & [A \amalg B, S_B E_B]_B \ni \tilde{\chi}_B
\end{array}
$$

The elements $\chi_A$, $\chi_B$, and $\tilde{\chi}_A$ are the images of $\chi$. The element $\chi_{B,A}$ is defined if $\chi_A = 0$. Then $\chi_{B,A}$ is the preimage of $\chi$. The element $\tilde{\chi}_{B,A}$ is defined if $\tilde{\chi}_A = 0$. Then $\tilde{\chi}_{B,A}$ is the preimage of $\chi_B$.

**Lemma 8.4.** If $\tilde{\chi}_{B,A} = 0$ then $\chi_{B,A} = 0$.

**Proof.** Suppose $\tilde{\chi}_{B,A} = 0$. Then there is a fiberwise homotopy

$$
L: C_B(B, A) \times I \to S_B E_B
$$

such that

$$
L(b, 1, 0) = \sigma_2(b) \\
L(b, 1, 1) = \sigma_1(b) \\
L(b, 0, t) = \sigma_1(b) \\
L(a, s, 0) = \chi_{B,A}(a, s) \in S_A E_A \\
L(a, s, 1) = \sigma_1(a)
$$

for all $a \in A$, $b \in B$, and $s, t \in I$. 

Let $J := \{(0) \times I\} \cup (I \times \{1\}) \cup (\{1\} \times I)$. Define a map

$$L: B \times J \to SBE_B$$

by

$$\bar{L}(b,0,t) = \sigma_1(b)$$
$$\bar{L}(b,s,1) = \sigma_1(b)$$
$$\bar{L}(b,1,t) = L(b,1,t).$$

The diagram

$$\begin{array}{c}
(B \times J) \cup_i (A \times I \times I) \\
\downarrow \downarrow \\
B \times I \times I \\
\downarrow \downarrow \downarrow \\
B
\end{array}$$

commutes and there is a lift $K$ since $SBE_B \to B$ is a fibration.

Then $K_0 := K(-,-,0): B \times I \to SBE_B$ satisfies

$$K_0(b,0) = K(b,0,0) = L(b,0,0) = \sigma_1(b)$$
$$K_0(b,1) = K(b,1,0) = L(b,1,0) = \sigma_2(b)$$
$$K_0(a,s) = K(a,s,0) = L(a,s,0) \in SAE_A$$

Define a map

$$\tilde{K}: C_B(B,A) \times I \to SBE_B$$

by

$$\tilde{K}(b,1,t) = K_0(b,1-t)$$
$$\tilde{K}(b,0,t) = \sigma_1(b)$$
$$\tilde{K}(a,s,t) = K_0(a,s(1-t))$$

$\tilde{K}$ shows $\chi_{B,A}$ is trivial in $[(C_B(B,A),SBE_B),(SBE_B,SAE_A)]_B$. \hfill $\square$

**Lemma 8.5.** If the map $E_B \to B$ is a $(\dim(A)+1)$-equivalence then $\rho$ is injective.

**Proof.** In this proof let $i$ denote the inclusion of $A$ in $B$.

Let $\Sigma_B(A \amalg B) := ((A \times I) \amalg B)/ \sim$ where $(a,0) \sim i(a) \sim (a,1)$. Then $\rho$ is part of a long exact sequence

$$[\Sigma_B(A \amalg B), SBE_B]_B \to [C_B(B,A), SBE_B]_B \xrightarrow{\rho} [S_B^0, SBE_B]_B \to [A \amalg B, SBE_B]_B.$$

To show that $\rho$ is injective it is enough to show

$$[\Sigma_B(A \amalg B), SBE_B]_B$$

is trivial.
Let $\alpha$ be an element of $[\Sigma_B(A \amalg B), S_BE_B]_B$. Then $\alpha$ defines a map $S^1 \times A \to S_BE_B$ also denoted $\alpha$. This map satisfies $p\alpha(t, a) = i(a)$. Consider the diagram

Since $S_BE_B \to B$ is a $(\dim(A) + 2)$-equivalence, the homotopy extension and lifting property implies there are maps $\beta$ and $H$ that make the diagram commute.

The diagram

commutes. Since $S_BE_B \to B$ is a fibration there is a lift $K$ that makes the diagram commute. Then

$$K_0 := K(\cdot, \cdot, 0): D^2 \times A \to S_BE_B$$

satisfies

$$pK_0(v, a) = H(v, a, 0) = i(a)$$

and

$$K_0(w, a) = \alpha(w, a)$$

if $w \in S^1$. Then

$$K_0 \amalg \text{id}: ((D^2 \times A) \amalg B)/ \sim \to S_BE_B$$

defines a map that shows $\alpha$ is trivial. $\square$

The following proposition is a consequence of Proposition 8.2, Lemma 8.4, and Lemma 8.5.

**Proposition 8.6.** If $p: E_A \to A$ is $(m+1)$-connected, $A$ is a $2m$-dimensional CW-complex, $p: E_B \to B$ is $(2m+1)$-connected and $(B, A)$ is a relative $4m$-dimensional CW-complex $(E_B, E_A) \to (B, A)$ admits a relative section if and only if $\chi_A$ and $\chi_B$ are both zero.

9. **Other descriptions of $\circ$ in special cases**

These are the proofs omitted from Section 3. Let $\mathcal{A}$ be an EI-category enriched in the category of abelian groups.

**Lemma 9.1 (Lemma 3.6).** If $\mathcal{X}: \mathcal{A} \to \text{Ch}_R$ and $\mathcal{Y}: \mathcal{A}^{\text{op}} \to \text{Ch}_R$ are supported on isomorphisms

$$\mathcal{X} \circ \mathcal{Y} \cong \bigoplus_{c \in B(\mathcal{A})} \mathcal{X}(c) \otimes_{\mathcal{A}(c,c)} \mathcal{Y}(c).$$
Proof. We will show that $\oplus X(c) \otimes \mathcal{A}(c, c) Y(c)$ satisfies the universal property that defines $X \otimes Y$.

By definition of $B(\mathcal{A})$, for any object $a$ in $\mathcal{A}$ there is exactly one object $c \in B(\mathcal{A})$ such that there is an isomorphism $f : a \to c$ in $\mathcal{A}$. Define a map

$$\theta_a : X(a) \otimes_Z Y(a) \to X(c) \otimes_{\mathcal{A}(c, c)} Y(c)$$

as the composite of

$$X(f) \otimes (f^{-1}) : X(a) \otimes_Z Y(a) \to X(c) \otimes_{\mathcal{A}(c, c)} Y(c)$$

with the quotient map

$$X(c) \otimes_Z Y(c) \to X(c) \otimes_{\mathcal{A}(c, c)} Y(c).$$

If $g$ is another isomorphism in $\mathcal{A}$ from $a$ to $c$, then $(X(f)(A), Y(f^{-1})(B))$ is identified with $(X(g)(A), Y(g^{-1})(B))$ and the map $\theta_a$ is well defined. Let

$$\theta : \bigoplus_{a \in \text{ob}(\mathcal{A})} X(a) \otimes_Z Y(a) \to \bigoplus_{c \in B(\mathcal{A})} X(c) \otimes_{\mathcal{A}(c, c)} Y(c)$$

be the sum of the maps $\theta_a$.

If $(A, f, B) \in X(a) \otimes_Z \mathcal{A}(a, b) \otimes_Z Y(b)$ the images of this element in

$$\bigoplus_{a \in \text{ob}(\mathcal{A})} X(a) \otimes_Z Y(a)$$

are $(A, Y(f)(B))$ and $(X(f)(A), B)$. The images of these elements are identified under $\theta$.

Let

$$\phi : \bigoplus_{a \in \text{ob}(\mathcal{A})} X(a) \otimes_Z Y(a) \to M$$

be a map that coequalizes the two maps from $\bigoplus_{a, b \in \text{ob}(\mathcal{A})} X(a) \otimes_Z \mathcal{A}(a, b) \otimes_Z Y(b)$ to $\bigoplus_{a \in \text{ob}(\mathcal{A})} X(a) \otimes_Z Y(a)$. Define a map

$$\psi : \bigoplus_{c \in B(\mathcal{A})} X(c) \otimes_{\mathcal{A}(c, c)} Y(c) \to M$$

by choosing lifts of elements in $X(c) \otimes_{\mathcal{A}(c, c)} Y(c)$ to elements of $X(c) \otimes_Z Y(c)$.

Since $\phi$ coequalizes, the choices do not matter and $\psi$ is unique. \qed

Lemma 9.2 [Lemma 3.7]. Let $X$ and $Y$ satisfy the conditions of Lemma 3.6. If $X(c)$ is dualizable as an $\mathcal{A}(c, c)$-module with dual $Y(c)$ for each $c \in B(\mathcal{A})$ then $X$ is dualizable with dual $Y$.

Proof. If $X(c)$ is dualizable as an $\mathcal{A}(c, c)$-module with dual $Y(c)$ then there is a map of chain complexes of abelian groups

$$\eta_c : \mathbb{Z} \to X(c) \otimes Y(c)$$

and a map of chain complexes of $\mathcal{A}(c, c)$-bimodules

$$\epsilon_c : Y(c) \otimes X(c) \to \mathcal{A}(c, c)$$

for each $c \in B(\mathcal{A})$. 

Let $\eta: \mathcal{Z} \to \mathcal{X} \otimes \mathcal{Y}$ be the composite

$$
\mathcal{Z} \xrightarrow{\triangle} \bigoplus_{B(\mathcal{A})} \mathcal{Z} \xrightarrow{\oplus} \bigoplus_{B(\mathcal{A})} \mathcal{X}(c) \otimes \mathcal{A}(c,c) \mathcal{Y}(c) \cong \mathcal{X} \otimes \mathcal{Y}
$$

where $\triangle: \mathcal{Z} \to \oplus_{B(\mathcal{A})} \mathcal{Z}$ is the map that takes 1 to $(1,1,\ldots,1)$.

Let $a$ and $b$ be isomorphic objects of $\mathcal{A}$. Let $c$ be an object of $B(\mathcal{A})$ that is isomorphic to $a$ and let $h$ be an isomorphism in $\mathcal{A}$ from $a$ to $c$ and $g$ be an isomorphism from $b$ to $c$. Then $\epsilon_{a,b}$ is the composite

$$
\mathcal{Y}(c) \otimes \mathcal{X}(c) \xrightarrow{\epsilon_c} \mathcal{A}(c,c) \xrightarrow{\epsilon} \mathcal{A}(c,c)
$$

If $a$ and $b$ are not isomorphic in $\mathcal{A}$, $\epsilon_{a,b}$ is zero. Since $c$ is unique and the maps $\epsilon_c$ are maps of $\mathcal{A}(c,c)$-bimodules, $\epsilon$ is a natural transformation. This also implies that $\epsilon$ is independent of the choice of $g$ and $h$.

Let $\eta_c(1) = \sum e_{c,i} \otimes f_{c,i}$ for each $c \in B(\mathcal{A})$. If $x \in \mathcal{X}(a)$ the value of the composite

$$
\mathcal{X}(a) \cong \mathcal{Z} \otimes \mathcal{X}(a) \xrightarrow{\eta \otimes 1} \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{X}(a) \xrightarrow{1 \otimes \epsilon} \mathcal{X} \otimes \mathcal{A}(-,a) \cong \mathcal{X}(a)
$$

applied to $x$ is

$$
\sum_{c \in B(\mathcal{A})} \sum_i \mathcal{X}(\epsilon(f_{c,i},x))(e_{c,i}).
$$

The only nonzero terms in this sum are those where there is an isomorphism $h$ from $x$ to $c$. By definition, $\epsilon(f_{c,i},x) = h^{-1}e_c(f_{c,i}, \mathcal{X}(h)(x))$ and

$$
\sum_i \mathcal{X}(\epsilon(f_{c,i},x))(e_{c,i}) = \mathcal{X}(h^{-1}) \sum_i \mathcal{X}(\epsilon_c(f_{c,i}, \mathcal{X}(h)(x)))(e_{c,i}) = \mathcal{X}(h^{-1}) \mathcal{X}(h)(x) = x.
$$

The other diagram is similar. □

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