Some characterizations of rectifying curves on a smooth surface in Euclidean 3-space

Akhilesh Yadav · Buddhadev Pal

Abstract In this paper, we investigate sufficient condition for the invariance of a rectifying curve on a smooth surface immersed in Euclidean 3-space under isometry by using Darboux frame \( \{T, P, U\} \). Further, we find the deviations of the position vector of a rectifying curve on the smooth surface along any tangent vector \( T = aϕ_u + bϕ_v \) with respect to the isometry. We also find the deviations of the position vector of a rectifying curve on the smooth surface along the unit normal \( U \) to the surface and along \( P(= U \times T) \) with respect to the isometry.

Keywords isometry · Frenet-frame · Darboux-frame · rectifying curve

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1 Introduction

In the study of space curve in the Euclidean 3-space \( E^3 \) at every point of the curve one can associate the moving orthonormal frame called Serret-Frenet frame \( \{T, N, B\} \), consisting of the unit tangent vector, the principal normal vector and the binormal vector, respectively. According to Serret-Frenet frame at every point of the curve there exist three orthogonal planes, so called rectifying, normal and osculating planes. Rectifying curves are introduced by B. Y. Chen in [1] as a space curve whose position vector always lies in its rectifying plane. Here, the rectifying plane is spanned by the tangent vector \( T(s) \) and the binormal vector \( B(s) \). Thus, the position vector \( γ(s) \) of a rectifying curve \( γ \) in \( E^3 \) satisfies the equation \( γ(s) = λ(s)T(s) + μ(s)B(s) \) for some differentiable functions \( λ(s) \) and \( μ(s) \). The rectifying curves are also studied in [2] as the extremal curves. Further, authors studied rectifying curves via the dilation of unit speed curves on the unit sphere \( S^2 \) in the Euclidean 3-space and obtained a necessary and sufficient condition for which the centrode \( d(s) \) of a unit speed curve \( γ(s) \) is a rectifying curve [4]. Also in [6], authors defined a rectifying curve in the Euclidean 4-space as
a curve whose position vector always lies in orthogonal complement \( N^\perp \) of its principal normal vector field \( N \).

In [8], authors studied rectifying curve on a smooth surface and obtained a sufficient condition for which a rectifying curve on a smooth surface remains invariant under isometry of surfaces by using Serret-Frenet frame. On the other hand, when we study space curve on a smooth surface immersed in Euclidean 3-space at every point of the curve another moving orthonormal frame called Darboux frame \( \{ T, P, U \} \) comes naturally, where \( T \) is the unit tangent vector to the curve at that point, \( U \) being the unit normal to the surface and \( P = U \times T \). In [3], authors gave some characterizations of position vector of a unit speed curve in a regular surface immersed in Euclidean 3-space which always lies in the planes spanned by \( \{ T, U \} \), \( \{ T, P \} \) and \( \{ P, U \} \), respectively by using the Darboux frame. Thus motivated sufficiently, we study rectifying curve on a smooth surface immersed in Euclidean 3-space and investigate the sufficient condition for the invariance of a rectifying curve on the smooth surface under isometry by using Darboux frame instead of Frenet frame. The paper is arrange as follws: In section 2, we discuss some basic theory of unit speed parametrized curve on a smooth surface. Section 3 is devoted to the investigation of the sufficient condition for the invariance of a rectifying curve on a smooth surface immersed in Euclidean 3-space under isometry. In this section, we also find the deviations of the position vector of a rectifying curve on the smooth surface along any tangent vector \( T = a\phi_u + b\phi_v \), the unit normal \( U \) to the surface and along \( U \times T \) with respect to the isometry.

2 Preliminaries

Let \( \gamma : I \to E^3 \), where \( I = (\alpha, \beta) \subset \mathbb{R} \), be the unit speed parametrized curve that has at least four continuous derivatives. Then the tangent vector of the curve \( \gamma \) be denoted by \( T \) and given by \( T(s) = \gamma'(s), \forall s \in I \), where \( \gamma' \) denote the derivative of \( \gamma \) with respect to the arc length parameter \( s \). The binormal vector \( B \) is defined by \( B = T \times N \), where \( N \) is the principal normal vector to the curve \( \gamma \). The Frenet-Serret equations are given by

\[
T'(s) = \kappa(s)N(s), \tag{2.1}
\]

\[
N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \tag{2.2}
\]

\[
B'(s) = -\tau(s)N(s), \tag{2.3}
\]

where \( \kappa(s) \) and \( \tau(s) \) are smooth functions of \( s \), called curvature and torsion of the curve \( \gamma \).

Let \( \phi : V \subset \mathbb{R}^2 \to S \) be the coordinate chart for a smooth surface \( S \) immersed in Euclidean space \( E^3 \) and the unit speed parametrized curve
\( \gamma : I \to S \subset E^3 \), where \( I = (\alpha, \beta) \subset \mathbb{R} \), contained in the image of a surface patch \( \phi \) in the atlas of \( S \). Then \( \gamma(s) \) is given by
\[
\gamma(s) = \phi(u(s), v(s)), \quad \forall s \in I. \tag{2.4}
\]

Now, the curve \( \gamma(s) \) lies on the surface \( S \) there exists another moving orthonormal frame called Darboux frame \( \{T, P, U\} \) at each point of the curve \( \gamma(s) \). Since the unit tangent \( T \) is common in both Frenet frame and Darboux frame, the vectors \( N, B, P, U \) lie in the same plane. So that the relations between these frames can be given as follows:
\[
\begin{bmatrix}
T \\
P \\
U
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
P
\end{bmatrix}, \tag{2.5}
\]

where \( \theta \) is the angle between vectors \( N \) and \( P \).

Again, since \( \gamma(s) \) is unit-speed curve lies on surface \( S \), \( \gamma'' \) is perpendicular to \( \gamma'(=T) \), and hence is a linear combination of \( U \) and \( P (= U \times T) \). Thus
\[
\gamma''(s) = k_n(s)U(s) + k_g(s)P(s), \tag{2.6}
\]
where \( k_n \) and \( k_g \) are smooth functions of \( s \), called the normal curvature and the geodesic curvature of \( \gamma \), respectively. Since \( U \) and \( P \) are perpendicular unit vectors therefore from (2.6), we get
\[
k_n(s) = \gamma''(s).U(s) \quad \text{and} \quad k_g(s) = \gamma''(s).P(s). \tag{2.7}
\]
Also from (2.1) and (2.7), we obtain
\[
k_n(s) = \kappa(s)N(s).U(s) \quad \text{and} \quad k_g(s) = \kappa(s)N(s).P(s), \tag{2.8}
\]
which implies
\[
k_n(s) = \kappa(s)\sin \theta \quad \text{and} \quad k_g(s) = \kappa(s)\cos \theta. \tag{2.9}
\]
Thus the curve \( \gamma \) is a geodesic curve if and only if \( k_g = 0 \) and the curve \( \gamma \) is an asymptotic line if and only if \( k_n = 0 \).

Now, differentiating (2.4) with respect to \( s \), we get
\[
T(s) = \gamma'(s) = u'\phi_u + v'\phi_v, \tag{2.10}
\]
The unit normal \( U \) to the surface \( S \) is given by
\[
U(s) = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|} = \frac{\phi_u \times \phi_v}{\sqrt{EG - F^2}}, \tag{2.11}
\]
Also, since \( P = U \times T \) by using (2.10) and (2.11), we obtain
\[
P(s) = \frac{1}{\sqrt{EG - F^2}}(Eu'\phi_v + F(v'\phi_v - u'\phi_u) - Gv'\phi_u), \tag{2.12}
\]
where \( E = \phi_u.\phi_u, F = \phi_u.\phi_v \) and \( G = \phi_v.\phi_v \) are coefficients of first fundamental form.
Definition 2.1 [5] A diffeomorphism \( f : S \rightarrow \overline{S} \) between two surfaces \( S \) and \( \overline{S} \) is an isometry if \(< f_*(\omega_1), f_*(\omega_2) >_{f(p)} = < \omega_1, \omega_2 >_p \), for all \( p \in S \) and for all \( \omega_1, \omega_2 \in T_pS \). The surfaces \( S \) and \( \overline{S} \) are called isometric if there is an isometry between them.

Now, from Theorem 5.1 and Corollary 8.2 of [7], we have the following:

(i) An isometry \( f \) between surfaces \( S \) and \( \overline{S} \) takes the geodesics of one surface to the geodesics of the other,

(ii) Coefficients of first fundamental form preserve under isometry between surfaces \( S \) and \( \overline{S} \), i.e. if \( E, F, G \) and \( \overline{E}, \overline{F}, \overline{G} \) are coefficients of first fundamental form of surfaces \( S \) and \( \overline{S} \), respectively then

\[
E = \overline{E}, \quad F = \overline{F} \quad \text{and} \quad G = \overline{G}.
\]

(2.13)

3 Rectifying curves according to Darboux frame

In this section, we study rectifying curves on a smooth surface by using Darboux frame. A curve \( \gamma(s) \) on a smooth surface \( S \) immersed in Euclidean 3-space is called rectifying curve if its position vector always lies in rectifying plane of the curve. Thus the position vector of the curve \( \gamma \) satisfies the equation

\[
\gamma(s) = \lambda(s)T(s) + \mu(s)B(s),
\]

(3.1)

for some differentiable functions \( \lambda(s) \) and \( \mu(s) \). Thus by using (2.5) in (3.1), we obtain

\[
\gamma(s) = \lambda(s)T(s) + \mu(s)P(s)\sin \theta + \mu(s)U(s)\cos \theta.
\]

(3.2)

Now, from (2.9), (2.10), (2.11), (2.12) and (3.2), we get

\[
\gamma(s) = \lambda(s)(u' \phi_u + v' \phi_v) + \frac{\mu(s)k_g(s)}{k(s)\sqrt{EG-F^2}}(\phi_u \times \phi_v)
\]

\[
+ \frac{\mu(s)k_n(s)}{k(s)\sqrt{EG-F^2}}(Eu' \phi_v + F(v' \phi_v - u' \phi_u) - Gv' \phi_u).
\]

(3.3)

This equation of rectifying curve on a smooth surface, which is neither a geodesic curve nor an asymptotic line on the surface.

Now, if the rectifying curve on the smooth surface is a geodesic curve (i.e. \( k_g(s) = 0 \)) then \( \theta = \pi/2 \), \( k_n(s) = k(s) \) and equation of the rectifying curve is given by

\[
\gamma(s) = \lambda(s)(u' \phi_u + v' \phi_v) + \frac{\mu(s)}{\sqrt{EG-F^2}}(Eu' \phi_v + F(v' \phi_v - u' \phi_u) - Gv' \phi_u).
\]

(3.4)
Also, if the rectifying curve on the smooth surface is a asymptotic line (i.e. $k_n(s) = 0$) then $\theta = 0$, $k_g(s) = k(s)$ and equation of the rectifying curve is given by

$$\gamma(s) = \lambda(s)(u' \phi_u + v' \phi_v) + \frac{\mu(s)}{\sqrt{EG - F^2}}(\phi_u \times \phi_v).$$  \tag{3.5}$$

**Theorem 3.1** Let $f : S \to \mathcal{S}$ be an isometry, where $S$ and $\mathcal{S}$ are smooth surfaces and $\gamma(s)$ be a rectifying curve on $S$ with $k_n \neq 0$. Then $\overline{\gamma} = f_\ast \gamma$ is a rectifying curve on $\mathcal{S}$ if any one of the following conditions holds:

(i) $\overline{\gamma}$ is geodesic curve on $\mathcal{S}$ and $\overline{\gamma}(s) = f_\ast (\gamma(s))$,

(ii) $\gamma$ is asymptotic curve on $S$ and $\gamma(s) + \frac{\mu(s)}{k(s)} k_n(s) F = f_\ast (\gamma(s))$,

(iii) $\gamma$ is neither geodesic nor asymptotic curve on $S$ and $\overline{\gamma}(s) = f_\ast (\gamma(s))$.

**Proof.** Let $f : S \to \mathcal{S}$ be an isometry, where $S$ and $\mathcal{S}$ are smooth surfaces and $\gamma(s)$ be a rectifying curve on $S$ such that $k_n \neq 0$.

Suppose (i) holds. Then, $\overline{\kappa}_g(s) = 0$ and $\overline{\gamma}(s) = f_\ast (\gamma(s))$, which implies

$$\overline{\gamma}(s) = \lambda(s)(u' f_\ast \phi_u + v' f_\ast \phi_v) + \frac{\mu(s)}{\sqrt{EG - F^2}}(Eu' f_\ast \phi_v + F(v' f_\ast \phi_v - u' f_\ast \phi_u) - Gv' f_\ast \phi_u).$$  \tag{3.6}$$

Thus from (2.13) and (3.6), we get

$$\overline{\gamma}(s) = \overline{\lambda}(s)(u' \overline{\phi}_u + v' \overline{\phi}_v) + \frac{\overline{\mu}(s)}{\sqrt{\overline{EG} - \overline{F}^2}}(Eu' \overline{\phi}_v + F(v' \overline{\phi}_v - u' \overline{\phi}_u) - Gv' \overline{\phi}_u),$$  \tag{3.7}$$

where $\overline{\lambda}(s) = \lambda(s)$ and $\overline{\mu}(s) = \mu(s)$. This is equation of rectifying curve on $\mathcal{S}$, which is geodesic on the surface.

Suppose (ii) holds. Then, $\overline{\kappa}_u(s) = 0$ and $\overline{\gamma}(s) = f_\ast (\gamma(s)) - \frac{\mu(s)}{k(s)} k_n(s) F$, which implies

$$\overline{\gamma}(s) = \lambda(s)(u' f_\ast \phi_u + v' f_\ast \phi_v) + \frac{\mu(s)}{k(s)} k_g(s) f_\ast U - \frac{\mu(s)}{k(s)} k_n(s) F$$

$$+ \frac{\mu(s)}{k(s)\sqrt{EG - F^2}}(Eu' f_\ast \phi_v + F(v' f_\ast \phi_v - u' f_\ast \phi_u) - Gv' f_\ast \phi_u).$$  \tag{3.8}$$

Thus from (2.13) and (3.8), we obtain

$$\overline{\gamma}(s) = \lambda(s)(u' \overline{\phi}_u + v' \overline{\phi}_v) + \frac{\mu(s)}{k(s)} k_g(s) \overline{U} - \frac{\mu(s)}{k(s)} k_n(s) \overline{F}$$

$$+ \frac{\mu(s)}{k(s)\sqrt{EG - F^2}}(Eu' \overline{\phi}_v + F(v' \overline{\phi}_v - u' \overline{\phi}_u) - Gv' \overline{\phi}_u),$$  \tag{3.9}$$
which implies
\[
\gamma(s) = \bar{\lambda}(s)(u' \bar{\phi}_u + v' \bar{\phi}_v) + \bar{\mu}(s)\bar{U},
\]
(3.10)
where \(\bar{\lambda}(s) = \lambda(s)\) and \(\bar{\mu}(s) = \frac{\mu(s)k_g(s)}{k(s)}\). This is equation of rectifying curve on \(\overline{S}\), which is asymptotic on the surface.

Now, suppose (iii) holds. Then, \(k_g(s) \neq 0\), \(k_n(s) \neq 0\) and \(\gamma(s) = f_*(\gamma(s))\), which implies
\[
\gamma(s) = \lambda(s)(u' f_* \phi_u + v' f_* \phi_v) + \frac{\mu(s)k_g(s)}{k(s)} f_* U
\]
\[
+ \frac{\mu(s)k_n(s)}{k(s)\sqrt{EG - F^2}} (Eu' f_* \phi_v + F(v' f_* \phi_v - u' f_* \phi_u) - Gv' f_* \phi_u).
\]
(3.11)
Thus from (2.13) and (3.11), we get
\[
\gamma(s) = \lambda(s)(u' \bar{\phi}_u + v' \bar{\phi}_v) + \frac{\mu(s)k_g(s)}{k(s)} \bar{U}
\]
\[
+ \frac{\mu(s)k_n(s)}{k(s)\sqrt{EG - F^2}} (Eu' \bar{\phi}_v + F(v' \bar{\phi}_v - u' \bar{\phi}_u) - Gv' \bar{\phi}_u),
\]
(3.12)
which implies
\[
\gamma(s) = \bar{\lambda}(s)(u' \bar{\phi}_u + v' \bar{\phi}_v) + \frac{\bar{\mu}(s)k_g(s)}{\bar{k}(s)} \bar{U}
\]
\[
+ \frac{\bar{\mu}(s)k_n(s)}{\bar{k}(s)\sqrt{EG - F^2}} (Eu' \bar{\phi}_v + F(v' \bar{\phi}_v - u' \bar{\phi}_u) - Gv' \bar{\phi}_u),
\]
(3.13)
where \(\bar{\lambda}(s) = \lambda(s), \frac{\bar{\mu}(s)k_g(s)}{\bar{k}(s)} = \frac{\mu(s)k_g(s)}{k(s)}\) and \(\frac{\bar{\mu}(s)k_n(s)}{\bar{k}(s)} = \frac{\mu(s)k_n(s)}{k(s)}\).

This is equation of rectifying curve on \(\overline{S}\), which is neither geodesic nor asymptotic on the surface.

\[\square\]

**Theorem 3.2** Let \(f : S \to \overline{S}\) be an isometry, where \(S\) and \(\overline{S}\) are smooth surfaces and \(\gamma(s)\) be a rectifying curve on \(S\) with \(k_n = 0\). Then \(\overline{\gamma} = f_*\gamma\) is a rectifying curve on \(\overline{S}\) if any one of the following conditions holds:

(i) \(\overline{\gamma}\) is asymptotic curve on \(\overline{S}\) and \(\overline{\gamma}(s) = f_*(\gamma(s))\),

(ii) \(\overline{\gamma}\) is not asymptotic curve on \(\overline{S}\) and \(\overline{\gamma}(s) - \mu(s)\overline{P} = f_*(\gamma(s))\).

**Proof.** We can easily prove by using Theorem 3.1.

\[\square\]
Theorem 3.3 Let $f : S \to \overline{S}$ be an isometry. If $\gamma$ and $\overline{\gamma}$ are rectifying curves on $S$ and $\overline{S}$ respectively, with $k_n \neq 0$ then, we have following:

(i) if $\overline{\gamma}$ is geodesic curve on $\overline{S}$ then $\overline{\gamma}(s).T(s) = \gamma(s).T(s)$,

(ii) if $\overline{\gamma}$ is asymptotic curve on $\overline{S}$ then $\overline{\gamma}(s).T(s) - \gamma(s).T(s) = \frac{\mu(s)k_n(s)}{k(s)}(\sqrt{EG - F^2})(av' + bu')$,

(iii) if $\gamma$ is neither geodesic nor asymptotic curve on $S$ then $\gamma(s).T(s) = \gamma(s).T'(s)$, where $T(s) = a\phi_u + b\phi_v$ is any tangent vector to the surface $S$ at point $\gamma(s)$.

Proof. Let $f : S \to \overline{S}$ be an isometry and $\gamma$ and $\overline{\gamma}$ be rectifying curves on $S$ and $\overline{S}$ respectively, with $k_n \neq 0$. Then,

$$\overline{\gamma}(s).T(s) - \gamma(s).T(s) = a(\overline{\gamma}(s).\overline{\phi}_u - \gamma(s).\phi_u) + b(\overline{\gamma}(s).\overline{\phi}_v - \gamma(s).\phi_v). \tag{3.14}$$

Now, from (3.3), we get

$$\gamma(s).\phi_u = \lambda(s)(u'E + v'F) + \frac{\mu(s)k_n(s)}{k(s)\sqrt{EG - F^2}}(F^2 - GE)v'. \tag{3.15}$$

Similarly, we obtain

$$\gamma(s).\phi_v = \lambda(s)(u'F + v'G) - \frac{\mu(s)k_n(s)}{k(s)\sqrt{EG - F^2}}(F^2 - GE)u'. \tag{3.16}$$

For (i), suppose $\gamma$ is geodesic curve on $S$. Then from (2.13) and (3.4), we get

$$\gamma(s).\phi_u = \lambda(s)(u'E + v'F) + \frac{\mu(s)}{\sqrt{EG - F^2}}(F^2 - GE)v', \tag{3.17}$$

and

$$\overline{\gamma}(s).\overline{\phi}_u = \lambda(s)(u'E + v'F) + \frac{\overline{\mu}(s)}{\sqrt{EG - F^2}}(F^2 - GE)v'. \tag{3.18}$$

Thus from (3.17) and (3.18), we obtain

$$\overline{\gamma}(s).\overline{\phi}_u - \gamma(s).\phi_u = (\mu(s) - \overline{\mu}(s))(\sqrt{EG - F^2})v'. \tag{3.19}$$

Similarly, we get

$$\overline{\gamma}(s).\overline{\phi}_v - \gamma(s).\phi_v = (\mu(s) - \overline{\mu}(s))(\sqrt{EG - F^2})u'. \tag{3.20}$$

Thus from (3.14), (3.19) and (3.20), we get

$$\overline{\gamma}(s).\overline{T}(s) - \gamma(s).T(s) = (\mu(s) - \overline{\mu}(s))(\sqrt{EG - F^2})(av' + bu'). \tag{3.21}$$
Since $\gamma$ and $\tau$ are rectifying curves on $S$ and $\bar{S}$, respectively therefore $\mu(s) = \mu(s)$. Hence, $\bar{\gamma}(s), \bar{T}(s) = \gamma(s), T(s)$.

For (ii), suppose $\tau$ is asymptotic curve on $\bar{S}$. Then from (2.13) and (3.5), we get

$$\bar{\gamma}(s).\bar{\phi}_u = \lambda(s)(u' E + v' F),$$

(3.22)

and

$$\bar{\gamma}(s).\bar{\phi}_v = \lambda(s)(u' F + v' G).$$

(3.23)

Thus from (3.15) and (3.22), we obtain

$$\bar{\gamma}(s).\bar{\phi}_u - \gamma(s).\phi_u = -\frac{\mu(s)k_n(s)}{k(s)\sqrt{EG - F^2}}(F^2 - GE)(v' - u').$$

(3.24)

Also from (3.16) and (3.23), we get

$$\bar{\gamma}(s).\bar{\phi}_v - \gamma(s).\phi_v = \frac{\mu(s)k_n(s)}{k(s)\sqrt{EG - F^2}}(F^2 - GE)u'.
$$

(3.25)

Thus from (3.14), (3.24) and (3.25), we obtain

$$\bar{\gamma}(s).\bar{T}(s) - \gamma(s).T(s) = \frac{\mu(s)k_n(s)}{k(s)}(\sqrt{EG - F^2})(av' - bu').$$

(3.26)

Now for (iii), suppose $\tau$ is neither geodesic nor asymptotic curve on $\bar{S}$. Then from (2.13) and (3.3), we get

$$\bar{\gamma}(s).\bar{\phi}_u = \lambda(u' E + v' F) + \frac{\bar{\mu}(s)k_n(s)}{k(s)\sqrt{EG - F^2}}(F^2 - GE)v',
$$

(3.27)

and

$$\bar{\gamma}(s).\bar{\phi}_v = \lambda(u' F + v' G) - \frac{\bar{\mu}(s)k_n(s)}{k(s)\sqrt{EG - F^2}}(F^2 - GE)u'.
$$

(3.28)

Thus from (3.15) and (3.27), we obtain

$$\bar{\gamma}(s).\bar{\phi}_u - \gamma(s).\phi_u = \left[\frac{\mu(s)k_n(s)}{k(s)} - \frac{\bar{\mu}(s)k_n(s)}{k(s)}\right]\sqrt{EG - F^2})v'.
$$

(3.29)

Similarly, from (3.16) and (3.28), we get

$$\bar{\gamma}(s).\bar{\phi}_v - \gamma(s).\phi_v = \left[\frac{\mu(s)k_n(s)}{k(s)} - \frac{\bar{\mu}(s)k_n(s)}{k(s)}\right](\sqrt{EG - F^2})u'.
$$

(3.30)

Thus, from (3.14), (3.29) and (3.30), we obtain

$$\bar{\gamma}(s).\bar{T}(s) - \gamma(s).T(s) = \left[\frac{\bar{\mu}(s)k_n(s)}{k(s)} - \frac{\mu(s)k_n(s)}{k(s)}\right]((\sqrt{EG - F^2})(bu' - av').
$$

(3.31)
Since \( \gamma \) and \( \overline{\gamma} \) are rectifying curves on \( S \) and \( \overline{S} \), respectively therefore
\[
\overline{\gamma}(s).\overline{T}(s) = \gamma(s).T(s).
\]
\[\square\]

**Theorem 3.4** Let \( f : S \rightarrow \overline{S} \) be an isometry. If \( \gamma \) and \( \overline{\gamma} \) are rectifying curves on \( S \) and \( \overline{S} \) respectively, with \( k_n = 0 \) then, we have following:

(i) if \( \overline{\gamma} \) is asymptotic curve on \( \overline{S} \) then \( \overline{\gamma}(s).\overline{T}(s) = \gamma(s).T(s) \),

(ii) if \( \overline{\gamma} \) is not asymptotic curve on \( \overline{S} \) then \( \overline{\gamma}(s).\overline{T}(s) - \gamma(s).T(s) = \frac{\overline{p}(s)\overline{k}_n(s)}{k(s)} \)
\[
(\sqrt{EG - F^2})(bu' - av').
\]

**Proof.** We can easily prove by using Theorem 3.3.
\[\square\]

**Theorem 3.5** Let \( f : S \rightarrow \overline{S} \) be an isometry. If \( \gamma \) and \( \overline{\gamma} \) are rectifying curves on \( S \) and \( \overline{S} \) respectively, with \( k_n \neq 0 \) then, we have following:

(i) if \( \overline{\gamma} \) is geodesic curve on \( \overline{S} \) then \( \overline{\gamma}(s).\overline{P}(s) = \gamma(s).P(s) \),

(ii) if \( \overline{\gamma} \) is asymptotic curve on \( \overline{S} \) then \( \overline{\gamma}(s).\overline{P}(s) - \gamma(s).P(s) = ((aE + bF)u' + (aF + bG)v')(\sqrt{EG - F^2})\frac{\overline{\mu}(s)\overline{\mu}_n(s)}{k(s)} \),

(iii) if \( \overline{\gamma} \) is neither geodesic nor asymptotic curve on \( \overline{S} \) then \( \overline{\gamma}(s).\overline{P}(s) = \gamma(s).P(s) \),
\[\text{where } P = U \times T \text{ and } T = a\phi_u + b\phi_v \text{ be any tangent vector to the surface } S \text{ at point } \gamma(s).\]

**Proof.** Let \( f : S \rightarrow \overline{S} \) be an isometry. If \( \gamma \) and \( \overline{\gamma} \) are rectifying curves on \( S \) and \( \overline{S} \) respectively, with \( k_n \neq 0 \) then, by using (2.11), we obtain
\[
\gamma(s).P(s) = \frac{(aE + bF)}{\sqrt{EG - F^2}} \gamma(s).\phi_v - \frac{(aF + bG)}{\sqrt{EG - F^2}} \gamma(s).\phi_u. \tag{3.32}
\]

Similarly by using (2.13), we get
\[
\overline{\gamma}(s).\overline{P}(s) = \frac{(aE + bF)}{\sqrt{EG - F^2}} \overline{\gamma}(s).\overline{\phi}_v - \frac{(aF + bG)}{\sqrt{EG - F^2}} \overline{\gamma}(s).\overline{\phi}_u. \tag{3.33}
\]

Thus from (3.32) and (3.33), we obtain
\[
\overline{\gamma}(s).\overline{P}(s) - \gamma(s).P(s) = \frac{(aE + bF)}{\sqrt{EG - F^2}} (\overline{\gamma}(s).\overline{\phi}_v - \gamma(s).\phi_v) \tag{3.34}
\]
\[
- \frac{(aF + bG)}{\sqrt{EG - F^2}} (\overline{\gamma}(s).\overline{\phi}_u - \gamma(s).\phi_u).
\]

For (i), suppose \( \overline{\gamma} \) is geodesic curve on \( \overline{S} \). Then from (3.19), (3.20) and (3.34), we get
\[
\overline{\gamma}(s).\overline{P}(s) - \gamma(s).P(s) = ((aE + bF)u' + (aF + bG)v')(\sqrt{EG - F^2}) \frac{\overline{\mu}(s)\overline{\mu}_n(s)}{k(s)} \tag{3.35}
\]

\[
(\overline{p}(s) - \mu(s)).
\]

9
Since $\gamma$ and $\overline{\gamma}$ are rectifying curves on $S$ and $\overline{S}$, respectively therefore $\overline{\mu}(s) = \mu(s)$. Hence, $\overline{\gamma}(s), \overline{\mu}(s) = \gamma(s), P(s)$.

For (ii), suppose $\overline{\gamma}$ is asymptotic curve on $\overline{S}$. Then from (3.34), we obtain

$$\overline{\gamma}(s), \overline{P}(s) = \gamma(s), P(s) = ((aE + bF)u' - (aF + bG)v')(\sqrt{EG - F^2})$$

$$\mu(s)k_n(s) \overline{P}(s) - \gamma(s), P(s) = \frac{(aE + bF)u' + (aF + bG)v'}{(\sqrt{EG - F^2})},$$

(3.37)

Now for (iii), suppose $\overline{\gamma}$ is neither geodesic nor asymptotic curve on $\overline{S}$. Then from (3.29), (3.30) and (3.34), we get

$$\overline{\gamma}(s), \overline{P}(s) = ((aE + bF)u' + (aF + bG)v')(\sqrt{EG - F^2})$$

$$\frac{\mu(s)k_n(s)}{k(s)}. \overline{P}(s) - \gamma(s), P(s) = \frac{(aE + bF)u' + (aF + bG)v'}{(\sqrt{EG - F^2})}.$$

Since $\gamma$ and $\overline{\gamma}$ are rectifying curves on $S$ and $\overline{S}$, respectively therefore

$$\frac{\overline{\mu}(s)k_n(s)}{k(s)} = \frac{\mu(s)k_n(s)}{k(s)}. \overline{P}(s) = \gamma(s), P(s).$$

(3.34)

Theorem 3.6 Let $f : S \rightarrow \overline{S}$ be an isometry. If $\gamma$ and $\overline{\gamma}$ are rectifying curves on $S$ and $\overline{S}$ respectively, with $k_n = 0$ then, we have following:

(i) if $\overline{\gamma}$ is asymptotic curve on $\overline{S}$ then $\overline{\gamma}(s), \overline{P}(s) = \gamma(s), P(s)$,

(ii) if $\overline{\gamma}$ is not asymptotic curve on $\overline{S}$ then $\overline{\gamma}(s), \overline{P}(s) - \gamma(s), P(s) = ((aE + bF)u' + (aF + bG)v')(\sqrt{EG - F^2})$, where $P = U \times T$ and $T = a\phi_u + b\phi_v$ be any tangent vector to the surface $S$ at point $\gamma(s)$.

Proof. We can easily prove by using Theorem 3.5.

(3.35)

Theorem 3.7 Let $f : S \rightarrow \overline{S}$ be an isometry. If $\gamma$ and $\overline{\gamma}$ are rectifying curves on $S$ and $\overline{S}$ respectively then, we have $\overline{\gamma}(s), \overline{U}(s) = \gamma(s), U(s)$.

Proof. Let $f : S \rightarrow \overline{S}$ be an isometry. Then from (3.3), we have

$$\overline{\gamma}(s), \overline{U}(s) = \overline{\gamma}(s), \frac{-\overline{\phi}_u \times \overline{\phi}_v}{\sqrt{EG - F^2}} = \frac{\overline{\mu}(s)k_n(s)}{k(s)}.$$

(3.38)

and

$$\gamma(s), U(s) = \gamma(s), \frac{\phi_u \times \phi_v}{\sqrt{EG - F^2}} = \frac{\mu(s)k_n(s)}{k(s)}.$$

(3.39)

Thus if $k_n = 0$ then $\overline{\gamma}(s), \overline{U}(s) = \gamma(s), U(s)$.

(3.36)
Also, if $k_g \neq 0$ then $\frac{\mu(s)k_g(s)}{k(s)} = \frac{\mu(s)k_g(s)}{k(s)}$ and hence $\gamma(s)U(s) = \gamma(s).U(s)$.

\[\Box\]

**Note:** Let $f : S \rightarrow \overline{S}$ be an isometry. If $\gamma$ and $\overline{\gamma}$ are rectifying curves on $S$ and $\overline{S}$, respectively then the components of $\gamma(s)$ along $T = a\phi_u + b\phi_v$, $U$ and $P = U \times T$ are invariant under isometry if any one of the following holds:

(i) both $\gamma$ and $\overline{\gamma}$ are asymptotic on $S$ and $\overline{S}$ respectively,
(ii) both $\gamma$ and $\overline{\gamma}$ are geodesic on $S$ and $\overline{S}$ respectively,
(iii) neither $\gamma$ nor $\overline{\gamma}$ are geodesic and asymptotic on $S$ and $\overline{S}$ respectively.

**References**

1. **Chen, B.Y.** – When does the position vector of a space curve always lie in its rectifying plane?, Amer. Math. Monthly, 110 (2003), no. 2, 147-152.
2. **Chen, B.Y.; Dillen, F.** – Rectifying curves as centrodes and extremal curves, Bull. Inst. Math. Acad. Sinica, 33 (2005), no. 2, 77-90.
3. **Camci, Ç.; Kula, L.; İlarslan, K.** – Characterizations of the position vector of a surface curve in Euclidean 3-space, An. Științ. Univ. “Ovidius” Constanța Ser. Mat., 19 (2011), no. 3, 59-70.
4. **Deshmukh, S.; Chen, B.Y.; Alshammari, S.H.** – On rectifying curves in Euclidean 3-space, Turkish. J. Math., 42 (2018), no. 2, 609-620.
5. **do Carmo, M.P.** – Differential geometry of curves and surfaces, Translated from the Portuguese, Prentice-Hall, Inc., Englewood Cliffs, N.J., (1976).
6. **İlarslan, K.; Nešović, E.** – Some characterizations of rectifying curves in the Euclidean space $E^4$, Turkish. J. Math., 32 (2008), no. 1, 21-30.
7. **Pressley, A.** – Elementary differential geometry, Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, (2001).
8. **Shaikh, A.A.; Ghosh, P.R.** – Rectifying curves on a smooth surface immersed in the Euclidean space, Indian J. Pure Appl. Math., 50 (2019), no. 4, 883-890.

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**AUTHORS**

**Akhilesh Yadav** (Corresponding author),
Department of Mathematics,
Institute of Science,
Banaras Hindu University,
Varanasi-221005, India,
E-mail: akhilesh_mathau@rediffmail.com

**Buddha Dev Pal,**
Department of Mathematics,
Institute of Science,
Banaras Hindu University,
Varanasi-221005, India,
E-mail: pal.buddha@gmail.com