Novel Symmetries in Two Dimensional Proca Theory

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Abstract: By exploiting the Stueckelberg’s approach, we obtain a gauge theory for the two (1+1)-dimensional (2D) Proca theory and demonstrate that this theory is endowed with, in addition to the usual Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetries, the on-shell nilpotent (anti-)co-BRST symmetries, under which, the total gauge-fixing term remains invariant. The anticommutator of the BRST and co-BRST (as well as anti-BRST and anti-co-BRST) symmetries define a unique bosonic symmetry in the theory, under which, the ghost part of the Lagrangian density remains invariant. To establish connections of the above symmetries with the Hodge theory, we invoke a pseudo-scalar field in the theory. Ultimately, we demonstrate that the full theory provides a field theoretic example for the Hodge theory where the continuous symmetry transformations provide a physical realization of the de Rham cohomological operators and discrete symmetries of the theory lead to the physical realization of the Hodge duality operation of differential geometry. We also mention the physical implications and utility of our present investigation.
1 Introduction

The well-known Proca field theory is a generalization of the Maxwell field theory where a vector boson has three degrees of freedom due to the presence of its mass. The latter attribute of such a vector boson spoils the beautiful gauge symmetry transformations of the Maxwell’s theory. In modern language, any arbitrary (e.g. Maxwell) gauge theory is endowed with the first-class constraints whereas the Proca theory is characterized by the second-class constraints in the terminology of Dirac’s prescription for the classification scheme of constraints (see, e.g. [1,2] for details). As a consequence, the Proca theory does not respect gauge symmetry invariance (because of the presence of the mass). However, the Stueckelberg’s formalism restores the gauge invariance in the massive gauge (i.e. Proca) theory where the mass and gauge invariance co-exist together [because of the presence of the Stueckelberg’s field in addition to the spin-1 vector field of the theory (see, e.g. [3])].

The purpose of our present investigation is to take the specific case of two $(1+1)$-dimensional $(2D)$ modified version of the Proca field theory (that incorporates the Stueckelberg field) and study its various symmetry properties within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism. We demonstrate that a set of six continuous symmetries and a couple of mathematically useful discrete symmetries exist for the modified version of the massive Proca gauge field theory. In fact, we show that, in addition to the usual (anti-)BRST symmetries, the theory respects (anti-)co-BRST symmetries, a unique bosonic symmetry, the ghost-scale symmetry and a couple of discrete symmetries. As a consequence of the above symmetries, the full Lagrangian density of the theory is uniquely defined with definite signatures associated with all the terms and it becomes a model for the Hodge theory where the continuous set of symmetries provide a physical realization of the abstract de Rham cohomological operators and discrete symmetries give physical meanings to the abstract Hodge duality operation of differential geometry.

The nilpotent (anti-)BRST symmetry transformations for the gauge invariant version of Proca theory exist in any arbitrary dimension of spacetime (cf. Sec. 2). One of the decisive features of these symmetries is the observation that the total kinetic term, owing its origin to the exterior derivative, remains invariant. On the other hand, as we have demonstrated in our present endeavor, the nilpotent (anti-)co-BRST symmetries exist in the two $(1+1)$-dimensions of spacetime for the Abelian 1-form massive gauge theory, under which, the total gauge-fixing term (owing its origin to the co-exterior derivative) remains invariant. There exists a unique bosonic symmetry in the 2D theory, under which, the Faddeev-Popov ghost terms remain invariant. In our present investigation, we have derived an extended version of the BRST algebra and have shown that this algebraic structure is exactly like that of the de Rham cohomological operators of differential geometry.

We have also shown, in our present endeavor, that there is two-to-one mapping between the generators of the continuous symmetry transformations of the theory and cohomological symmetries. In fact, we have two equivalent Lagrangian densities [cf. (10),(11) below] for our present theory. The key feature of these Lagrangian densities is that the signature of all the terms is fixed due to the presence of many symmetries in the theory. In this respect, these Lagrangian densities are unique.

†On a compact manifold without a boundary, there exists a set of three cohomological operators $(d, \delta, \Delta)$ where $d = dx^\mu \partial_\mu$ and $\delta = \pm * d*$ are the nilpotent ($d^2 = \delta^2 = 0$) exterior and co-exterior derivatives and $\Delta = (d + \delta)^2 = d\delta + \delta d$ is the Laplacian operators [4-7]. Here $(*)$ is the Hodge duality operation.
operators. A couple of discrete symmetries of our present theory have been shown to be the analogue of the Hodge duality operation of differential geometry. In principle, we can have many discrete symmetry transformations in the theory. However, we have concentrated only on two discrete symmetry transformations, in our present endeavor, which are useful to us in providing the analogue of the relationship \( \delta = \pm \ast d \ast \) in the language of the interplay between the continuous and discrete symmetry transformations. We have also shown that the role of the degree of a differential form is played by the ghost number of a state in the quantum Hilbert space of states of our present theory.

The main motivating factors behind our present investigation are as follows. First, our present 2D model is the one where mass of the gauge field and (anti-)BRST symmetries, (anti-)co-BRST symmetries, a bosonic symmetry, the ghost-scale symmetry and a set of discrete symmetries co-exist together. Second, the modified version of Proca theory is radically different from the models of gauge theories [8-12] and \( \mathcal{N} = 2 \) supersymmetric quantum mechanical models [13-15] which have been shown to be the examples of Hodge theory. Finally, our present endeavor is our modest step towards our main goal of finding massive models for the Hodge theory in physical four (3 + 1)-dimensions of spacetime where, perhaps, 1-form gauge field and higher \( p \)-form (\( p = 2, 3, 4, \ldots \)) gauge fields will merge together in a meaningful manner to produce the (anti-)BRST invariant massive gauge theories.

In broader perspective, besides the above motivations, our present studies have been physically meaningful because, exploiting the ideas of (anti-)BRST, (anti-)co-BRST and bosonic symmetries, we have been able to prove that the 2D free (non-)Abelian theories (without any interaction with matter fields) [16,17], are a new type of topological field theories (TFTs) which capture some aspects of Witten type TFT and a part of Schwarz type TFT. Furthermore, utilizing the key properties of Hodge theory (especially its symmetries and conserved charges), we have been able to demonstrate that the 4D free Abelian 2-form and 6D Abelian 3-form gauge theories [11,12] are quasi-TFTs. Thus, our present endeavor encompasses in its folds mathematically as well as physically interesting results.

The contents of our investigation are organized as follows. In Sec. 2, we concisely recapitulate the bare essentials of the Proca theory, its generalization to a gauge theory by Stueckelberg’s formalism and its on-shell nilpotent (anti-)BRST symmetries in any arbitrary dimension of spacetime. Our Sec. 3 is devoted to the discussion of the existence of on-shell nilpotent (anti-)co-BRST symmetries in two (1+1)-dimensions of spacetime. We elaborate on the derivation of a unique bosonic symmetry for this 2D theory from the anticommutators of (anti-)BRST and (anti-)co-BRST symmetries in Sec. 4. We discuss the discrete as well as ghost-scale symmetries of our present theory in Sec. 5. In Sec. 6, we establish connections between the symmetry operators (as well as their corresponding conserved charges) and the cohomological operators of differential geometry. Finally, we make some concluding remarks and point out a few future directions in Sec. 7.

In our Appendices A and B, we discuss about a couple of non-nilpotent supersymmetric type symmetry transformations and a unique bosonic symmetry transformation for the (anti-)BRST invariant Lagrangian density (8) (see below). Our Appendix C is devoted to the discussion of a few symmetries of the Lagrangian density \( \mathcal{L}_{\text{br2}} \) (cf. (11)).

**General conventions and notations:** Through out the whole body of our text, we shall use the notations for the (anti-)BRST and (anti-)co-BRST symmetry transformations as \( s_{(a)b} \) and \( s_{(a)d} \), respectively. Similarly, we shall also adopt the notations for the ghost-scale
and bosonic symmetry transformations as $s_g$ and $s_\omega$, respectively. These notations would be used for the two equivalent Lagrangian densities that are present in our theory. Other notations would be clarified at appropriate places and we shall focus only on the internal symmetries and shall not even touch anything connected with the spacetime symmetries. We shall assume that the spacetime background manifold is flat and Minkowskian.

2 Preliminary: Proca theory as a gauge theory and on-shell nilpotent (anti-)BRST symmetries

Let us begin with the following Proca Lagrangian density $\mathcal{L}_0$ for a massive photon (with mass $m$) in any arbitrary dimension of spacetime (see, e.g. [3] for details)

$$\mathcal{L}_0 = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu, \quad (1)$$

where the curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ has been derived from the 2-form $F^{(2)} = (dx^\mu \wedge dx^\nu)/2!$ $F_{\mu\nu}$ where the 1-form $A^{(1)} = dx^\mu A_\mu$ defines the photon field $A_\mu$.

Here $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative and Greek indices $\mu, \nu, \lambda... = 0, 1, 2, 3..., (D-1)$ in $D$-dimensions of spacetime. It can be shown that the physical system (1) is endowed with the second-class constraints in the language of Dirac’s prescription for the classification of constraints [1,2]. As a consequence, there is no gauge symmetry in the theory as this symmetry is generated only by the first-class constraints. The existence of the latter is a key signature of a gauge theory [1,2]. Exploiting the celebrated Stueckelberg’s approach [3], we can replace: $A_\mu \rightarrow A_\mu - (1/m) \partial_\mu \phi$ and add a kinetic term for the real scalar field $\phi$ to recast the above Lagrangian density ($\mathcal{L}_0$) in another form ($\mathcal{L}_s$) as:

$$\mathcal{L}_s = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m A_\mu \partial^\mu \phi. \quad (2)$$

The above Lagrangian density ($\mathcal{L}_s$) respects the following infinitesimal, local and continuous gauge symmetry transformations ($\delta_g$), namely;

$$\delta_g A_\mu = \partial_\mu \Lambda, \quad \delta_g \phi = m \Lambda, \quad (3)$$

because $\delta_g \mathcal{L}_s = 0$. The parameter $\Lambda$ in (3) is a local gauge parameter. We state, in passing, that the second-class constraints of the original Proca theory (cf. (1)) have already been converted into the first-class constraints because of the presence of the ordinary scalar field $\phi$, in addition to the photon field $A_\mu$, in our modified Lagrangian density (2).

The above “classical” local, continuous and infinitesimal gauge symmetry transformations (3) can be generalized to the “quantum” level. The latter are the on-shell nilpotent ($s_0^2_{(a),b} = 0$) (anti-)BRST symmetry transformations $s_{(a)b}$:

$$\begin{align*}
s_{ab} A_\mu &= \partial_\mu \bar{C}, & s_{ab} \bar{C} &= 0, & s_{ab} C &= i (\partial \cdot A + m\phi), \\
s_{ab} \phi &= m \bar{C}, & s_{ab} F_{\mu\nu} &= 0, & s_{ab} (\partial \cdot A + m\phi) &= (\Box + m^2) \bar{C}, \\
s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= -i (\partial \cdot A + m\phi), \\
s_b \phi &= m C, & s_b F_{\mu\nu} &= 0, & s_b (\partial \cdot A + m\phi) &= (\Box + m^2) C.
\end{align*} \quad (4)$$

4
for the “quantum” generalized version \((\mathcal{L}_B)\) of the Lagrangian density \((\mathcal{L}_s)\) which incorporates the gauge-fixing as well as Faddeev-Popov ghost terms as given below:

\[
\mathcal{L}_B = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m A_\mu \partial^\mu \phi - \frac{1}{2} (\partial \cdot A + m \phi)^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C,
\]

(5)

where we have defined \((\partial \cdot A) = \partial_\mu A^\mu\) and the fermionic \((C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C}C = 0)\) (anti-)ghost field \((\bar{C}C)\) are needed for the validity of unitarity in the theory.

The gauge fixing term \([- (\partial \cdot A + m \phi)^2 / 2]\) has its origin in the co-exterior derivative \(\delta = \pm \cdot d\cdot\) (with \(\delta^2 = 0\)) where the Hodge duality \((\cdot)\) operation is defined on the \(D\)-dimensional spacetime manifold (see, e.g. \([4-7]\)). It is straightforward to check that \(\delta A^{(1)} = \pm \cdot d\cdot (dx^\mu A_\mu) = \pm (\partial \cdot A)\) is a zero-form. We have a freedom to add/subtract a zero-form scalar field \(\phi\) to this gauge-fixing term (with the proper mass dimension). This is why, in this total gauge-fixing term \([- (\partial \cdot A + m \phi)^2 / 2]\), we have the presence of \(m\phi\), too. We shall discuss this issue of adding the extra term (i.e. \(m\phi\)) later in detail (see, e.g. Sec. 3). Under the on-shell nilpotent (anti-)BRST symmetry transformations \(s_{(a)b}\), the Lagrangian density (5) transforms to the total spacetime derivatives:

\[
s_{ab} \mathcal{L}_B = - \partial_\mu \left[ (\partial \cdot A + m \phi) \partial^\mu \bar{C} \right],
\]

\[
s_b \mathcal{L}_B = - \partial_\mu \left[ (\partial \cdot A + m \phi) \partial^\mu C \right].
\]

(6)

Thus, the action integral \(S = \int d^{D-1}x \mathcal{L}_B\) remains invariant under the (anti-)BRST symmetry transformations \(s_{(a)b}\). The above continuous symmetry transformations, according to Noether’s theorem, lead to the following conserved charges \(Q_{(a)b}\), namely;

\[
Q_{ab} = \int d^{D-1}x \left[ \partial_\mu \{(\partial \cdot A) + m \phi\} C - \{(\partial \cdot A) + m \phi\} \bar{C} \right],
\]

\[
Q_b = \int d^{D-1}x \left[ \partial_\mu \{(\partial \cdot A) + m \phi\} C - \{(\partial \cdot A) + m \phi\} \bar{C} \right],
\]

(7)

which turn out to be the generators for the above on-shell nilpotent symmetry transformations \(s_{(a)b}\) because \(s_{(a)b} \Psi = \pm i [\Psi, Q_{(a)b}] \pm\) for the generic field \(\Psi \equiv A_\mu, \phi, C, \bar{C}\) of the theory, described by the Lagrangian density (5). Here the \((\pm)\) signs, as the subscript on the square bracket, correspond to the (anti)commutator for \(\Psi\) being (fermionic) bosonic.

We wrap up this section with the following remarks. First, our whole argument is valid in any arbitrary dimension of spacetime. Second, the (anti-)BRST symmetry transformations are nilpotent of order two \((s_{(a)b}^2 = 0)\) when we use the equations of motion: \((\Box + m^2) C = 0, (\Box + m^2) \bar{C} = 0\). Third, these transformations are absolutely anticommuting \((s_b s_{ab} + s_{ab} s_b = 0)\) in their operator form when we exploit the on-shell conditions \((\Box + m^2) C = 0, (\Box + m^2) \bar{C} = 0\). Finally, the above nilpotent symmetries have their mathematical origin in the exterior derivative \(d\) (with \(d^2 = 0\)) because the curvature term \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) in our operator form when we exploit the on-shell conditions \((\Box + m^2) C = 0, (\Box + m^2) \bar{C} = 0\). Physically, it is obvious to note that the kinetic term (for the gauge field) remains invariant under \(s_{(a)b}\) (which is a characteristic feature of (anti-)BRST
symmetries). We further re-emphasize that, even though, the mathematical origin for the existence of (anti-)BRST symmetries is encoded in the exterior derivative \( d = dx^\mu \partial_\mu \), only one of these symmetries could be identified with \( d \) because \( s_b \) and \( s_{ab} \) respect the absolute anticommutativity property and, hence, they are linearly independent of each-other.

## 3 On-shell nilpotent (anti-)co-BRST symmetries

Let us begin with the two (1+1)-dimensional (2D) version of the (anti-)BRST invariant Lagrangian density (5). This can be expressed as:

\[
L_b = \frac{1}{2} E^2 + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - mA_\mu \partial^\mu \phi \\
- \frac{1}{2} \left( \partial \cdot A + m \phi \right)^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C,
\]

(8)

where \( F_{\mu\nu} \) tensor has only electric field \( (E = F_{01} \equiv -\varepsilon^{\mu\nu} \partial_\mu A_\nu) \) as the non-vanishing component and there is no magnetic field \( (B) \) in the theory. It is obvious that, the existing \( E \) field (with one component) is a pseudo-scalar because it changes sign under parity.

The kinetic term \( E^2/2 \) of the (anti-)BRST invariant Lagrangian density (8) can be generalized. To achieve this goal in a symmetric fashion, one has to note that, we have the following expression for the Hodge dual of \( F^{(2)} \), in 2D spacetime, namely:

\[
\ast F^{(2)} = \ast \left( \frac{dx^\mu \wedge dx^\nu}{2!} \right) F_{\mu\nu} \equiv \varepsilon^{\mu\nu} \partial_\mu A_\nu = - E,
\]

(9)

where \( \ast \) is the Hodge duality operation. We observe that the electric field \( E \) (originating from \( F^{(2)} = [(dx^\mu \wedge dx^\nu)/2!] F_{\mu\nu} \) in 2D spacetime) is an anti-self-dual field in two (1+1)-dimension of spacetime and it is a 0-form pseudo-scalar. Thus, there is a room for adding/subtracting a 0-form pseudo-scaler field \( (\tilde{\phi}) \), with a proper mass dimension, in the expression for the kinetic term \( E^2/2 \) of the Lagrangian density (8).

Following the above arguments, it can be seen that the above Lagrangian density (8) can be generalized into the following couple of forms:

\[
\begin{align*}
L_{(b_1)} &= \frac{1}{2} \left( E - m \tilde{\phi} \right)^2 + m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\
&- mA_\mu \partial^\mu \phi - \frac{1}{2} \left( \partial \cdot A + m \phi \right)^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C,
\end{align*}
\]

(10)

\[
\begin{align*}
L_{(b_2)} &= \frac{1}{2} \left( E + m \tilde{\phi} \right)^2 - m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\
&+ mA_\mu \partial^\mu \phi - \frac{1}{2} \left( \partial \cdot A - m \phi \right)^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C.
\end{align*}
\]

(11)

\[\text{We adopt the convention and notation such that } 2D \text{ background flat Minkowski spacetime has the metric } \eta_{\mu\nu} = \text{diag} (+1, -1) \text{ so that } A \cdot B = \eta^{\mu\nu} A_\mu B_\nu \equiv A_\mu B^\mu = A_0 B_0 - A_1 B_1 \text{ is the dot product between two non-null vectors } A_\mu \text{ and } B_\mu. \text{ We have also the Levi-Civita tensor } \varepsilon_{\mu\nu} \text{ with the convention } \varepsilon_{01} = +1 = \varepsilon^{10}, \varepsilon^{\mu\lambda} \varepsilon_{\lambda\nu} = \delta^\mu_\nu, \text{ etc., and the } \text{d'Alembertian operator in our theory is: } \Box = \partial_0^2 - \partial_1^2.\]
The above Lagrangian densities show, in explicit form, the addition/subtraction of the (pseudo-)scalar fields in the kinetic and gauge-fixing terms of the present massive gauge theory\(^5\). We shall concentrate here only on \(\mathcal{L}_{(b_1)}\). However, in our Appendix C, we shall mention a few things about symmetries of the Lagrangian density \(\mathcal{L}_{(b_2)}\), too. The above Lagrangian density \(\mathcal{L}_{(b_1)}\) respects the following on-shell \([(\Box + m^2)\, C = 0, (\Box + m^2)\, \bar{C} = 0]\) nilpotent \((s^2_{(d)} = 0)\) (anti-)co-BRST symmetry transformations:

\[
\begin{align*}
\text{s}_\text{ad} \cdot A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu \, C, \\
\text{s}_\text{ad} \cdot C &= 0, \\
\text{s}_\text{ad} \cdot \bar{C} &= +i \left( E - m \, \bar{\phi} \right), \\
\text{s}_\text{ad} \cdot E &= \Box \, C, \\
\text{s}_\text{ad} \cdot (\partial \cdot A + m \, \phi) &= 0, \\
\text{s}_\text{ad} \cdot \phi &= 0, \\
\text{s}_\text{ad} \cdot \bar{\phi} &= -m \, C, \\
\text{s}_\text{d} \cdot A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu \, C, \\
\text{s}_\text{d} \cdot \bar{C} &= 0, \\
\text{s}_\text{d} \cdot C &= -i \left( E - m \, \phi \right), \\
\text{s}_\text{d} \cdot E &= \Box \, \bar{C}, \\
\text{s}_\text{d} \cdot (\partial \cdot A + m \, \phi) &= 0, \\
\text{s}_\text{d} \cdot \phi &= 0, \\
\text{s}_\text{d} \cdot \bar{\phi} &= -m \, \bar{C},
\end{align*}
\] (12)

We note that the gauge-fixing term, owing its origin to the nilpotent \((\delta^2 = 0)\) co-exterior derivative \((\delta = \pm \ast d \ast)\), remains invariant under the (anti-)co-BRST symmetry transformations \(s_{(d)}\). This is why we have christened the symmetry transformations in (12) as the (anti-)co-BRST [or (anti-)dual-BRST] symmetries. These should be contrasted with the (anti-)BRST symmetry transformations \(s_{(a)b}\), under which, the total kinetic term (owing its origin to the exterior derivative \(d = dx^\mu \partial_\mu\)) remains invariant. For the above modified Lagrangian density (10), the latter transformations (i.e. \(s_{(a)b}\)) are

\[
\begin{align*}
\text{s}_{ab} \cdot \bar{\phi} &= 0, \\
\text{s}_{ab} \cdot E &= 0, \\
\text{s}_b \cdot \bar{\phi} &= 0, \\
\text{s}_b \cdot E &= 0,
\end{align*}
\] (13)

in addition to the ones listed in (4). Under the total (anti)BRST symmetry transformations (4) and (13), the Lagrangian density (10) transforms to the total spacetime derivatives exactly in the same way as given in (6) as there is no contribution from the new terms.

We observe that, under the (anti-)co-BRST symmetry transformations (12), the Lagrangian density (10) transforms to the total spacetime derivative as given below:

\[
\begin{align*}
\text{s}_\text{ad} \cdot \mathcal{L}_{(b_1)} &= \partial_\mu \left[ E \partial^\mu \, C + m \, \varepsilon^{\mu\nu} \left( m \, A_\nu \, C + \phi \, \partial_\nu \, C \right) \right], \\
\text{s}_\text{d} \cdot \mathcal{L}_{(b_1)} &= \partial_\mu \left[ E \partial^\mu \, \bar{C} + m \, \varepsilon^{\mu\nu} \left( m \, A_\nu \, \bar{C} + \phi \, \partial_\nu \, \bar{C} \right) \right],
\end{align*}
\] (14)

which demonstrates that the action integral \(S = \int dx \, \mathcal{L}_{(b_1)}\) remains invariant under the transformations (12). Applying the Noether’s theorem, we obtain the following conserved charges corresponding to the continuous symmetry transformations (12), namely:

\[
\begin{align*}
Q_{\text{d}} &= \int dx \left[ m \left( \bar{\phi} \, \bar{C} - \bar{\phi} \, \bar{C} \right) + (E \, \bar{C} - \bar{E} \, \bar{C}) \right], \\
&= \int dx \left[ (E - m \, \bar{\phi}) \, \bar{C} - (\bar{E} - m \, \bar{\phi}) \, \bar{C} \right], \\
Q_{\text{ad}} &= \int dx \left[ (E - m \, \bar{\phi}) \, \bar{C} - (\bar{E} - m \, \bar{\phi}) \, \bar{C} \right],
\end{align*}
\] (15)

\(^5\)A close look at the transformations \(A_\mu \rightarrow A_\mu \mp (1/m) \partial_\mu \phi\) [used in the Stueckelberg’s formalism (keeping \(F_{\mu\nu} \rightarrow F_{\mu\nu}\)] and duality transformations (cf. (29) below) ensure the appearance of terms like \((E \mp m \, \bar{\phi})\) and \((\partial \cdot A \pm m \, \phi)\) in the Lagrangian densities (10) and (11) for our present 2D theory. Mathematical origin for these terms, based on the methodology of differential geometry, has already been explained in the main body of our text. Thus, Lagrangian densities (10) and (11) are very appropriate.
which are generators for the transformations (12). The above charges \( Q_r = \int dx J^\mu_r \) \((r = d, ad)\) have been derived from the following Noether’s currents:

\[
J^\mu_d = E \partial^\mu \tilde{C} + \epsilon^{\mu\nu} (\partial \cdot A) \partial_\nu \tilde{C} - m^2 \epsilon^{\mu\nu} A_\nu \tilde{C} + m (\tilde{C} \partial^\mu \tilde{\phi} - \tilde{\phi} \partial^\mu \tilde{C})
\]

\[
J^\mu_{ad} = E \partial^\mu C + \epsilon^{\mu\nu} (\partial \cdot A) \partial_\nu C - m^2 \epsilon^{\mu\nu} A_\nu C + m (C \partial^\mu \tilde{\phi} - \tilde{\phi} \partial^\mu C),
\]

In the proof of the conservation law \( \partial_\mu J^\mu_{(a)d} = 0 \), we have to exploit the following Euler-Lagrange equation of motion for the Lagrangian density (10), namely;

\[
\epsilon^{\mu\nu} \partial_\nu E = \partial^\mu (\partial \cdot A) + m^2 A^\mu, \quad (\square + m^2) A^\mu = 0,
\]

\[
(\square + m^2) \phi = 0, \quad (\square + m^2) \tilde{\phi} = 0, \quad (\square + m^2)(\partial \cdot A) = 0,
\]

\[
(\square + m^2) E = 0, \quad (\square + m^2) C = 0, \quad (\square + m^2) \tilde{C} = 0. \quad (17)
\]

It should be noted that \((\square + m^2) A^\mu = 0, (\square + m^2) E = 0, (\square + m^2)(\partial \cdot A) = 0\) have emerged out from the single equation \( \epsilon^{\mu\nu} \partial_\nu E = \partial^\mu (\partial \cdot A) + m^2 A^\mu \).

We wrap up this section with the following comments. First, to obtain the on-shell \([\square + m^2) C = 0, (\square + m^2) \tilde{C} = 0\]\(\text{nilpotent}\) \((s^2_{(a)d} = 0)\) \((\text{anti-})\text{-co-BRST symmetry, we have invoked a pseudo-scalar field} \( \tilde{\phi} \) \) in the theory. Second, it can be explicitly checked that \((s_d s_{ad} + s_{ad} s_d = 0)\) when we use the equations of motion: \((\square + m^2) C = 0 \) and \((\square + m^2) \tilde{C} = 0\). Third, the on-shell \([\square + m^2) C = 0, (\square + m^2) \tilde{C} = 0\] nilpotency \((Q^2_{(a)d} = 0)\) and anticommutativity \((Q_d Q_{ad} + Q_{ad} Q_d = 0)\) of the \((\text{anti-})\text{-co-BRST charges}\) \(Q_{(a)d}\) can be also checked by using the following formula for the generators, namely:

\[
s_d Q_d = i \{Q_d, Q_d\} = 0, \quad s_{ad} Q_{ad} = i \{Q_{ad}, Q_{ad}\} = 0,
\]

\[
s_d Q_{ad} = i \{Q_{ad}, Q_d\} = 0, \quad s_{ad} Q_d = i \{Q_d, Q_{ad}\} = 0, \quad (18)
\]

where the l.h.s. of the above relations could be checked easily from (12) and (15). Fourth, the Lagrangian density (8) does not respect any duality symmetry as the equivalent Lagrangian densities (10) and (11) do [cf. (29)]. However, the former Lagrangian density respects a couple of supersymmetric type continuous symmetry transformations and a bosonic symmetry. These symmetries have been briefly discussed in our Appendices A and B. Finally, the (pseudo-)scalar fields \( \tilde{\phi} \phi \) have been added/subtracted in a very symmetrical fashion to the kinetic and gauge-fixing terms of the theory. These are “dual” to each-other as would become clear in equation (29) (see below).

4 \hspace{1cm} \textbf{Bosonic symmetry and its uniqueness}

In our present 2D theory, so far, we have discussed four nilpotent \((s^2_{(a)b} = 0, s^2_{(a)d} = 0)\) symmetries which are \(s_{(a)b}\) and \(s_{(a)d}\). It turns out that we have the validity of the following:

\[
\{s_b, s_{ab}\} = 0, \quad \{s_b, s_{ad}\} = 0, \quad \{s_d, s_{ab}\} = 0, \quad \{s_d, s_{ad}\} = 0. \quad (19)
\]
where the on-shell conditions, from the equation of motion (17), have to be exploited for their proof. The following unique bosonic symmetry \((s_w)\), emerging due to the anticommutator of two the fermionic symmetries, is defined as:

\[
s_w = \{s_b, s_d\} = -\{s_{ad}, s_{ab}\},
\]

where the nilpotent transformations \(s_{(a)b}\) and \(s_{(a)d}\) are listed in (4) and (12).

The infinitesimal and continuous symmetry transformations \((s_w)\) for the individual relevant fields (and their composites) are as follows:

\[
\begin{align*}
s_w A_\mu &= i\varepsilon_{\mu\nu} (\Box A^\nu + m \partial^\nu \phi) + i m \partial_\mu \tilde{\phi}, \\
s_w \phi &= -i m (E - m \tilde{\phi}), \\
s_w E &= -i \Box (\partial \cdot A + m \phi),
\end{align*}
\]

\[
s_w (C, \bar{C}) = 0, \\
s_w \bar{\phi} = i m (\partial \cdot A + m \phi), \\
s_w (\partial \cdot A) = -i \Box (E - m\tilde{\phi}).
\]

The noteworthy point, at this stage, is the fact that the (anti-)ghost fields \((\bar{C})C\) do not transform under \(s_w\). Thus, one of the decisive features of \(s_w\) is the observation that the ghost part of the Lagrangian density does not change at all under the local and continuous bosonic symmetry transformations \((s_w)\).

Under the above infinitesimal continuous transformations, the Lagrangian density \(\mathcal{L}_{(b_1)}\) transforms to a total spacetime derivative, as:

\[
\begin{align*}
s_w \mathcal{L}_{(b_1)} &= \partial_\mu \left[ i (\partial \cdot A + m \phi) \partial^\mu (E - m \tilde{\phi}) - i (E - m \tilde{\phi}) \partial^\mu (\partial \cdot A + m \phi) \right] \\
&\quad + i m^2 (A^\mu E + \varepsilon^{\rho\sigma} A_\rho \partial^\mu A_\sigma) - i m \varepsilon^{\mu\nu} (\phi \Box A_\nu + m^2 A_\nu \phi) \\
&\quad - i m \{ \phi \partial^\mu E + \tilde{\phi} \partial^\mu (\partial \cdot A) \} - i m^2 \tilde{\phi} \partial^\mu \phi. 
\end{align*}
\]

As a consequence, the action integral \(S = \int dx \mathcal{L}_{(b_1)}\) remains invariant. According to Noether’s theorem, the above continuous symmetry transformations lead to the derivation of conserved charge, for our present theory, as:

\[
Q_w = \int dx \left[ (\partial \cdot A + m \phi) (\dot{E} - m \dot{\tilde{\phi}}) - \partial_\mu \{ (\partial \cdot A + m \phi) (E - m \tilde{\phi}) \} \right].
\]

The above charge is the generator of the continuous and infinitesimal transformations \((s_w)\).

We close this section with the remark that, by exploiting the definition of a generator, the following anticommutators (and transformations on the appropriate charges), namely;

\[
\begin{align*}
s_b Q_d &= i \{Q_d, Q_b\} = i Q_w, \\
s_{ad} Q_{ab} &= i \{Q_{ab}, Q_{ad}\} = -i Q_w, \\
s_d Q_b &= i \{Q_b, Q_d\} = i Q_w, \\
s_{ad} Q_{ab} &= i \{Q_{ad}, Q_{ab}\} = -i Q_w,
\end{align*}
\]

also produce the expression for \(Q_w\). We lay emphasis on the fact that, in the calculation of (24), we have to use the transformations (4) and (12) as well as the expression for the charges \(Q_{(a)b}\) and \(Q_{(a)d}\) that are quoted in the equations (7) and (15). This method of calculation of the bosonic charge \((Q_w = \{Q_b, Q_d\}) \equiv -\{Q_{ad}, Q_{ab}\}\) is simpler than the usual application of the Noether’s theorem where the algebra is quite involved. However, it can be clearly checked that the expression for \(Q_w\) (cf. (23)) is found to be exactly the same when we use the Noether’s theorem for the computation of \(Q_w\).
5 Ghost-scale and discrete symmetries

It is straightforward to note that our present theory is endowed with a ghost-scale symmetry transformations, on the basic fields and their composites, as illustrated below:

\[ \Psi \rightarrow e^{0,\Lambda} \Psi, \quad C \rightarrow e^{+\Lambda} C, \quad \bar{C} \rightarrow e^{-\Lambda} \bar{C}, \] 

(25)

where \( \Lambda \) is a global (spacetime independent) parameter and its coefficients denote the ghost number for a given field. The generic field \( \Psi = \phi, A_\mu, \tilde{\phi}, E, (\partial \cdot A) \) is endowed with the ghost number zero and (anti-)ghost fields \( \bar{C} \) carry the ghost numbers \( (\pm)1 \), respectively. The infinitesimal version of the ghost-scale symmetry transformations (25), denoted by \( (s)_g \), is as follows for the relevant fields of the theory, namely:

\[ s_g \Psi = 0, \quad s_g C = + C, \quad s_g \bar{C} = - \bar{C}, \] 

(26)

where the generic field \( \Psi = A_\mu, \phi, \tilde{\phi}, (\partial \cdot A), E \) and we have chosen \( \Lambda = 1 \) for the sake of brevity. We can readily verify, using the idea of a generator, that:

\[ s_g Q_b = -i \left[ Q_b, Q_g \right] = + Q_b \quad \Rightarrow \quad i \left[ Q_g, Q_b \right] = + Q_b, \]
\[ s_g Q_{ab} = -i \left[ Q_{ab}, Q_g \right] = - Q_{ab} \quad \Rightarrow \quad i \left[ Q_g, Q_{ab} \right] = - Q_{ab}, \]
\[ s_g Q_d = -i \left[ Q_d, Q_g \right] = - Q_d \quad \Rightarrow \quad i \left[ Q_g, Q_d \right] = - Q_d, \]
\[ s_g Q_{ad} = -i \left[ Q_{ad}, Q_g \right] = + Q_{ad} \quad \Rightarrow \quad i \left[ Q_g, Q_{ad} \right] = + Q_{ad}, \]
\[ s_g Q_w = -i \left[ Q_w, Q_g \right] = 0 \quad \Rightarrow \quad i \left[ Q_g, Q_w \right] = 0, \]

(27)

where the ghost charge \( Q_g \) is the generator for the infinitesimal transformations \( (s)_g \) quoted in (26). Its explicit expression, in terms of the (anti-)ghost fields, is

\[ Q_g = i \int dx \left[ \bar{C} \dot{\bar{C}} - \bar{\dot{C}} C \right], \]

(28)

which is derived from the conserved \( (\partial \mu J^\mu_{(g)} = 0) \) Noether current \( J^\mu_{(g)} = i \left[ \bar{C} \partial^\mu C - (\partial^\mu \bar{C})C \right] \). The l.h.s. of the above set of equations (27) is very straightforward to calculate by taking the help of (26), (7), (15) and (23).

There exists a set of beautiful discrete symmetries in the theory. For instance, it can be readily checked that, under the following discrete symmetry transformations, namely:

\[ A_\mu \rightarrow \pm i \varepsilon_{\mu \nu} A^\nu, \quad \phi \rightarrow \pm i \tilde{\phi}, \quad \tilde{\phi} \rightarrow \pm i \phi, \]
\[ C \rightarrow \mp i C, \quad \bar{C} \rightarrow \mp i \bar{C}, \quad (\partial \cdot A) \rightarrow \mp i E, \quad E \rightarrow \mp i (\partial \cdot A), \]

(29)

the Lagrangian densities (10) and (11) remain invariant. We shall see, later on, that the above discrete symmetry transformations play very important roles in establishing the connection between the continuous symmetry transformations and the de Rham cohomological operations of differential geometry. In principle, we can have many discrete symmetries in the theory. However, we have only two, in the above, which are very useful to us in our subsequent discussions which, we have, in what follows from now.
We dwell a bit on the origin of transformations (29) which are symmetry transformations for the Lagrangian densities (10) and (11). We note that the 2D self-duality condition, for the 1-form gauge connection, is as follows:

\[ * A^{(1)} = * (dx^\mu A_\mu) = dx^\mu \tilde{A}_\mu, \]  

(30)

where \( \tilde{A}_\mu = -\varepsilon_{\mu\nu} A^\nu \). This relation leads us to a clue to find out the discrete symmetry transformations (29). In fact, it is the transformations \( A_\mu \rightarrow \pm i \varepsilon_{\mu\nu} A^\nu \), owing their origin to (30), that are at the heart of the transformations (29). To be more precise, it can be seen that for the gauge-fixed Lagrangian density (in the Feynman gauge) of a free 2D Proca theory (without the inclusion of Stueckelberg field \( \phi \)):

\[ \mathcal{L}_{(\text{free})} = \frac{1}{2} E^2 - \frac{1}{2} (\partial \cdot A)^2 + \frac{m^2}{2} A_\mu A^\mu, \]  

(31)

the discrete symmetry transformations are: \( A_\mu \rightarrow \pm i \varepsilon_{\mu\nu} A^\nu \) (see e.g. [12] for more details). The transformations for other fields (e.g. \( \phi, \bar{\phi}, C, \bar{C} \)) are motivated from the starting transformations \( A_\mu \rightarrow \pm i \varepsilon_{\mu\nu} A^\nu \) so as to have a perfect symmetry for the Lagrangian densities (10) and (11). It is worthwhile to point out that \( C \rightarrow \mp i \bar{C}, \bar{C} \rightarrow \mp i C \) transformations are also the discrete symmetry transformations for the ghost part of the Lagrangian densities (10) and (11). Thus, the discrete symmetry transformations (29) are very appropriate because they incorporate all the essential ingredients of our present theory.

6 Extended BRST algebra: cohomological aspects

We have discussed, so far, a set of six local and continuous symmetry transformations (i.e. \( s_{(a)b}, s_{(a)d}, s_g, s_w \)) and a set of two discrete symmetry transformations (29). In their operator form, the continuous symmetry transformations obey the following algebra:

\[
\begin{align*}
\{ s_{(a)b}, s_{(a)d} \} &= 0, \\
\{ s_b, s_{ad} \} &= 0, \\
\{ s_b, s_ab \} &= 0, \\
\{ s_d, s_{ad} \} &= 0,
\end{align*}
\]

\[
\{ s_w, s_r \} = 0, \quad r = b, ab, d, ad, g,
\]

\[
\begin{align*}
[s_g, s_b] &= s_b, \\
[s_g, s_e] &= -s_{ad}, \quad [s_g, s_{ad}] = s_{ad}.
\end{align*}
\]

(32)

Thus, we note that the transformation \( s_w \) is the Casimir operator for the whole algebra. Exploiting the definition of a generator for a given transformation, we can replicate the above algebra (32) in the language of conserved charges of the theory as follows:

\[
\begin{align*}
Q_{(a)b}^2 &= 0, \\
\{ Q_b, Q_{ab} \} &= 0, \\
\{ Q_{(d), Q_{ad}} \} &= 0, \\
\{ Q_b, Q_{ad} \} &= 0,
\end{align*}
\]

(33)

\[
\begin{align*}
Q_{(w), Q_r} &= 0, \\
\{ Q_b, Q_d \} &= Q_w = -\{ Q_{ab}, Q_{ad} \} = 0,
\end{align*}
\]

\[
\begin{align*}
i [Q_g, Q_b] &= Q_b, \\
i [Q_g, Q_{ab}] &= -Q_{ab}, \\
i [Q_g, Q_{ad}] &= Q_{ad}.
\end{align*}
\]

which shows that \( Q_w \) is the Casimir operator for the whole algebra. It is worthwhile to point out that the algebras (32) and (33) are satisfied only on the on-shell (where the Euler-Lagrange equations of motion (16) are satisfied).
A close look at the equations (32) and (33) demonstrates that these extended algebras are very similar to the following algebra satisfied by the de Rham cohomological operators $(d, \delta, \Delta)$ of differential geometry, namely:

\[
d^2 = 0, \quad \delta^2 = 0, \quad \{d, \delta\} = \Delta, \quad [\Delta, \delta] = 0,
\]

\[
\Delta = (d + \delta)^2 = d\delta + \delta d, \quad [\Delta, d] = 0,
\]

which shows that the Laplacian operator $\Delta = \delta d + d\delta$ is the Casimir operator for the whole algebra (34). As far as the algebraic structures (32), (33) and (34) are concerned, it is evident that there is an analogy between the transformation operators (and corresponding conserved charges) and the cohomological operators of differential geometry. These are: $(s_b, s_{ab}) \rightarrow d$, $(s_d, s_{ad}) \rightarrow \delta$, $s_w = \{s_d, s_b\} = \Delta \equiv -\{s_{ab}, s_{ad}\}$ at the level of transformation operators and at the level of conserved charges, we have: $(Q_b, Q_{ad}) \rightarrow d$, $(Q_{ab}, Q_d) \rightarrow \delta$ and $Q_w = \{Q_b, Q_d\} \equiv -\{Q_{ab}, Q_{ad}\} \rightarrow \Delta$.

The identifications made, in the above, are not complete as yet. There are missing points that we have to answer and fulfill before the above identifications could be justified and could be put on the firmer footings. The first issue is, as we know, the nilpotent exterior derivative $(d)$ and co-exterior derivative $(\delta)$ are connected by the relations:

\[
\delta = - * d *, \quad d^2 = 0, \quad \delta^2 = 0,
\]

where $(*)$ is the Hodge duality operation and the minus sign, in the above relationship, is due to the even dimensionality of the spacetime manifold. It is very interesting to state that the relation (35) is satisfied, in our theory, by the following operator relationships:

\[
s_{(a)d} = - * s_{(a)b} *, \quad s_{(a)b}^2 = 0, \quad s_{(a)d}^2 = 0,
\]

where $s_{(a)b}$ and $s_{(a)d}$ are the continuous symmetry transformations (4) and (12), respectively, and $(*)$ corresponds to a couple of discrete symmetry transformations quoted in (29). It is because of the dimensionality of our theory that there exists an inverse relationship between $s_{(a)b}$ and $s_{(a)d}$ as given by the following relationship:

\[
s_{(a)b} = - * s_{(a)d} *, \quad s_{(a)b}^2 = 0, \quad s_{(a)d}^2 = 0,
\]

where the interplay between the continuous symmetry transformations ($s_{(a)b}, s_{(a)d}$) and the discrete symmetries (29) play a clinching and decisive role. The minus sign, in (36) and (37), is decided by the following relationship for the generic field:

\[
* (* \Psi) = - \Psi, \quad \Psi = A_\mu, \phi, \tilde{\phi}, C, \tilde{C}, (\partial \cdot A), E,
\]

which is true for any duality invariant theory [18]. Here the l.h.s. denotes the two successive operations of the discrete symmetry transformations (29).

In differential geometry, we know that the operation of an exterior derivative on a form $(f_n)$ of degree $n$, raises the degree of the form by one (i.e. $df_n \sim f_{n+1}$). On the contrary, when the co-exterior derivative $\delta$ acts on a form $(f_n)$ of degree $n$, it lowers the degree of the form by one (i.e. $\delta f_n \sim f_{n-1}$). Furthermore, as we know, the degree of a form remains
intact when it is acted upon by the Laplacian operator $\Delta$ (i.e. $\Delta f_n \sim f_n$). Thus, the second issue to be resolved is that we have to capture these properties in the language of symmetry operators and corresponding conserved charges. In fact, it is the equation (33) that plays a clinching and decisive role here in capturing the above property in a cogent and convincing fashion. We discuss this analogy, using the algebra (33), from now on.

We observe, from the above arguments, that the operation of $d (\delta)$ on the form ($f_n$) of degree $n$, is like the properties associated with the ladder operators. To capture the latter property, we define a state $|\chi\rangle_n$, with the ghost number $n$, in the quantum Hilbert space of states of a BRST invariant theory as follows:

$$i Q_g |\chi\rangle_n = n |\chi\rangle_n,$$

(39)

where $Q_g$ is the ghost charge (28). Exploiting the strength of the algebra (33), we observe the following interesting relations amongst the charges, namely:

$$i Q_g Q_b |\chi\rangle_n = (n + 1) Q_b |\chi\rangle_n, \quad i Q_g Q_{ad} |\chi\rangle_n = (n + 1) Q_{ad} |\chi\rangle_n,$$

$$i Q_g Q_d |\chi\rangle_n = (n - 1) Q_d |\chi\rangle_n, \quad i Q_g Q_{ab} |\chi\rangle_n = (n + 1) Q_{ab} |\chi\rangle_n,$$

$$i Q_g Q_w |\chi\rangle_n = n Q_w |\chi\rangle_n. \quad i Q_g Q_w |\chi\rangle_n = n Q_w |\chi\rangle_n.$$  

(40)

A close look at the above relations justify that the states $Q_b |\chi\rangle_n$, $Q_d |\chi\rangle_n$ and $Q_w |\chi\rangle_n$ have the ghost numbers $(n + 1)$, $(n - 1)$ and $n$, respectively. The same ghost numbers are associated with the states $Q_{ad} |\chi\rangle_n$, $Q_{ab} |\chi\rangle_n$, and $Q_w |\chi\rangle_n$, respectively, too.

We conclude, from the above analysis, that if the degree of a form is identified with the ghost number of a state in the quantum Hilbert space of states, then, the operation of the cohomological operators ($d, \delta, \Delta$) is exactly like the operations of a pair of charges ($Q_b, Q_{ad}$), ($Q_d, Q_{ab}$), $Q_w = \{Q_b, Q_d\} \equiv -\{Q_{ad}, Q_{ab}\}$ on a quantum state with the ghost number $n$. This observation can be mathematically expressed by the following mapping:

$$(Q_b, Q_{ad}) \rightarrow d, \quad (Q_d, Q_{ab}) \rightarrow \delta, \quad Q_w = \{Q_b, Q_d\} \equiv -\{Q_{ad}, Q_{ab}\} \rightarrow \Delta.$$  

(41)

Thus, we have explained all the cohomological properties of the de Rham cohomological operators in the language of continuous and discrete symmetry transformations of our present 2D Abelian 1-form massive gauge theory. Whereas the continuous symmetry transformations (and corresponding generators) provide the physical realizations of the cohomological operators, it is the discrete symmetry transformations (29) that capture the properties of the Hodge duality (*) operation of differential geometry. Hence, our present 2D Abelian 1-form massive gauge theory is a perfect field theoretic model for the Hodge theory.

We wrap up this section with a remark that under the discrete symmetry transformations (29), it can be checked that

$$*Q_b = +Q_d, \quad *Q_d = +Q_b, \quad *Q_{ab} = +Q_{ad},$$

$$*Q_w = +Q_w, \quad *Q_g = -Q_g, \quad (*Q_r) = +Q_r.$$  

(42)

where $r = b, ab, d, ad, w, g$. Thus, we note that the operation of a couple of discrete symmetry transformations (i.e. the analogue of the Hodge duality * operation of differential geometry) does not change the expression for the conserved charges of our present theory.
However, a single operation of the Hodge duality $\ast$ operation changes the charges in a suitable fashion so that the algebraic structure (33) does not change at all. The above observations give us the clue to state that any arbitrary number of operations of the analogue of the Hodge duality $\ast$ operation (i.e. the discrete symmetry transformations (29)) does not change the algebraic structure (33) of the conserved charges. This observation establishes the presence of a perfect duality symmetry in our present 2D theory.

### 7 Conclusions

In our present investigation, we have mainly concentrated on the symmetries of the modified version of 2D Proca theory and have demonstrated that the theory is endowed with a set of six continuous symmetries as well as a couple of discrete symmetry transformations. The infinitesimal continuous symmetry transformations (and corresponding Noether conserved charges) provide the physical realizations of the de Rham cohomological operators of differential geometry. A couple of discrete symmetry transformations correspond to the Hodge duality $\ast$ operation of differential geometry thereby providing the physical realization of the latter. To sum up, our present 2D massive gauge theory provides a tractable field theoretic model for the Hodge theory within the framework of BRST formalism.

One of the key observations in our theory is the introduction of a pseudo-scalar field $\tilde{\phi}$ [cf. (10),(11)] on the mathematical grounds of differential geometry. The presence of the discrete symmetries in the theory fix uniquely all the signatures of the terms present in the equivalent Lagrangian densities (10) and (11). For instance, the kinetic term of the pseudo-scalar field is negative. With the backing from the Euler-Lagrange equation of motion $[(\Box + m^2) \tilde{\phi} = 0]$ for this field [emerging out from the Lagrangian densities (10) and/or (11)], it is clear that such kind of pseudo-scalar particles could correspond to the candidates for the dark matter. The emergence of such kinds of fields is nothing new. This kind of kinetic term turned up, very naturally, in the context of 4D Abelian 2-form gauge theory [10] when the latter was proven to provide a model for the Hodge theory. Obviously, such kind of particles have not yet been detected by the experiments in high energy physics. In our theory, its existence is very natural on the firm ground of symmetries.

In our present endeavor, the aesthetic of symmetries has played a key and decisive role. It is very interesting to re-emphasize that each and every terms carry a definite signature in our Lagrangian densities (10) and (11) because of the presence of six continuous and two discrete symmetries in our theory. The presence of these beautiful symmetries entail upon our massive 2D model to provide a tractable field theoretic example for the Hodge theory within the framework of BRST formalism (where mass and various kinds of continuous as well as discrete symmetries co-exist together in a meaningful manner).

In our present investigation, we have focused only on the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries. It would be interesting to derive the off-shell nilpotent variety of these symmetries for our present massive Proca gauge theory within the framework of BRST formalism. One of the geometrically intuitive methods to obtain this kind of off-shell nilpotent and absolutely anticommuting symmetry transformations is the superfield formalism [19-22]. In the immediate future, we plan to devote time on the derivation of these off-shell nilpotent and absolutely anticommuting symmetries within the framework.
of “augmented” version of the above geometrical superfield formalism [23-25].

Our present theory is very special because, in this theory, mass and gauge invariance co-exist together. The 2D version of the Proca theory, modified with the inclusion of the Stueckelberg field, is endowed with many continuous and discrete symmetry transformations within the framework of the BRST formalism. It would be very nice endeavor to look for such kinds of theories in physical four (3 + 1)-dimensions of spacetime where the mass and various kinds of symmetries could co-exist together. Chern-Simons theory and Jackiew-Pi model (see, e.g. [26]) in 3D and topologically massive gauge theories in 4D, with celebrated $B \wedge F$ term, are other massive gauge theories. It would be interesting to apply Stueckelberg’s formalism to such kinds of theories and show them to be the models for Hodge theory within the framework of BRST formalism. These are some of the issues that are under investigations and our results would be reported elsewhere [27].

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Appendix A: On supersymmetric type symmetry transformations

The 2D Lagrangian density (8) is endowed with the following infinitesimal and continuous supersymmetric (SUSY) type transformations because we observe that the bosonic fields transform to fermionic fields and vice-versa. These transformations $(s, \bar{s})$ are

\begin{align*}
s A_{\mu} &= -\varepsilon_{\mu\nu} \partial^\nu C, \quad s C = 0, \quad s \bar{C} = +i E, \\
s E &= \Box C, \quad s (\partial \cdot A + m \phi) = 0, \quad s \phi = 0, \\
\bar{s} A_{\mu} &= -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad \bar{s} \bar{C} = 0, \quad \bar{s} C = -i E, \\
\bar{s} E &= \Box \bar{C}, \quad \bar{s} (\partial \cdot A + m \phi) = 0, \quad \bar{s} \phi = 0.
\end{align*}

(43)

The above transformations are the symmetry transformations because the Lagrangian density (8) transforms to the total spacetime derivatives:

\begin{align*}
s \mathcal{L}_b &= \partial_\mu \left[ E \partial^\mu C + m \varepsilon^{\mu\nu} (m A_\nu C + \phi \partial_\nu C) \right], \\
\bar{s} \mathcal{L}_b &= \partial_\mu \left[ E \partial^\mu \bar{C} + m \varepsilon^{\mu\nu} (m A_\nu \bar{C} + \phi \partial_\nu \bar{C}) \right].
\end{align*}

(44)

As a consequence, we observe that the action integral $S = \int dx (\mathcal{L}_b)$ remains invariant. A few noteworthy points, at this stage, are as follows: First, the gauge-fixing term (owing its origin to the co-exterior derivative) remains invariant under $s(\bar{s})$. Second, the fermionic SUSY type symmetry transformations $s(\bar{s})$ are nilpotent of order two ($s^2 = 0, \bar{s}^2 = 0$) if we take the massless (anti-)ghosts fields because $\Box C = \Box \bar{C} = 0$. In other words, the existence of the nilpotent SUSY type symmetries entail upon the (anti-)ghosts fields to become massless. Third, the fermionic ($s^2 = \bar{s}^2 = 0$) SUSY type symmetries are absolutely anticommuting (i.e. $s \bar{s} + \bar{s} s = 0$) on the on-shell when $\Box C = \Box \bar{C} = 0$ for the massless (anti-)ghosts fields. This proves the linear independence of $\bar{s}$ and $s$ in the massless limit.
There is a caveat for all the above arguments, however. We note that there is only one mass parameter in the theory if we maintain the existence of the beautiful continuous and discrete symmetries in the theory. If mass is set equal to zero (i.e. $m = 0$), the whole theory reduces to a free massless gauge theory in two (1 + 1)-dimensions of spacetime which has already been shown to be a field theoretic model for the Hodge theory in our earlier works [16,17]. Thus, we conclude that, to maintain the non-triviality of the theory, the massless condition cannot be imposed on the theory. As a consequence, the SUSY type transformations (43) are non-nilpotent and they are not absolutely anticommuting in nature. Hence they cannot be identified with the (anti-)co-BRST symmetry transformations, under which, the gauge-fixing term remains invariant [cf. (12)]. However, the transformations $(s, \bar{s})$ do correspond to the symmetry transformations for the theory.

According to Noether’s theorem, the existence of continuous symmetry transformations $s(\bar{s})$ leads to the derivation of the conserved charges $Q(\bar{Q})$ as

$$Q = \int dx J^0(\text{ad}) \equiv \int dx [E \dot{C} - \dot{E} C],$$

$$\bar{Q} = \int dx J^0(\text{d}) \equiv \int dx [E \dot{\bar{C}} - \dot{E} \bar{C}],$$

which are derived from the conserved currents:

$$J^\mu = E \partial^\mu C + \varepsilon^{\mu\nu} (\partial \cdot A) \partial_\nu C - m^2 \varepsilon^{\mu\nu} A_\nu C,$$

$$\bar{J}^\mu = E \partial^\mu \bar{C} + \varepsilon^{\mu\nu} (\partial \cdot A) \partial_\nu \bar{C} - m^2 \varepsilon^{\mu\nu} A_\nu \bar{C}.$$  \hspace{1cm} (45)

The conservation laws $\partial_\mu J^\mu = \partial_\mu \bar{J}^\mu = 0$ can be proven by exploiting the following Euler-Lagrange equations of motion for the relevant fields of the theory, namely;

$$\Box + m^2 \bar{C} = 0, \quad \Box + m^2 (\partial \cdot A) = 0, \quad \Box + m^2 \phi = 0, \quad \Box + m^2 A_\mu = 0,$$

$$\Box + m^2 E = 0, \quad \varepsilon^{\mu\nu} \partial_\nu E = \partial^\mu (\partial \cdot A) + m^2 A^\mu, \quad \Box + m^2 C = 0,$$ \hspace{1cm} (47)

which are derived from the Lagrangian density (8).

We close this Appendix with the remarks that (i) to achieve the on-shell nilpotency, we have introduced a pseudo-scalar field in the theory (cf. Sec. 3) on the mathematical as well as physical grounds [cf. (10),(11)]. We have also seen, in addition, that the above new field is urgently needed so as to have a discrete symmetry in the theory [cf. (29)] which turns out to correspond to the Hodge duality operation of differential geometry, and (ii) the symmetry transformations (43) are not exact SUSY transformations because one of the key requirements of a set of SUSY transformations, corresponding to $\mathcal{N} = 2$ supersymmetric theory, is not satisfied by transformations (43). It can be readily checked that the anticommutativity of $(s, \bar{s})$ acting on a field (i.e. $(s \bar{s} + \bar{s} s)\Psi$) does not produce the spacetime translation of the corresponding generic field $\Psi$.

**Appendix B: On a bosonic symmetry in the theory**

The Lagrangian density (8) also respects a bosonic $(s^2_B \neq 0)$ symmetry transformations. These infinitesimal and continuous symmetry transformations $(s_B)$, for the key individual
and composite fields of the theory, are

\[ s_b A_\mu = i \varepsilon_{\mu \nu} (\Box A^\nu + m \partial^\nu \phi), \quad s_b \phi = -i m E, \quad s_b (C, \bar{C}) = 0, \]

\[ s_b E = -i \Box (\partial \cdot A + m \phi), \quad s_b (\partial \cdot A) = -i \Box E, \quad (48) \]

where the transformation \( s_B A_\mu = i \varepsilon_{\mu \nu} (\Box A^\nu + m \partial^\nu \phi) \) can be written in terms of the components \((A_0, A_1)\) of the gauge field \(A_\mu\), as given below:

\[ s_B A_0 = -i (\Box A_1 + m \partial_1 \phi), \quad s_B A_1 = -i (\Box A_0 + m \partial_0 \phi). \quad (49) \]

One of the decisive features of \( s_B \) is the observation that the (anti-)ghost fields of the theory remain \textit{unchanged} under this transformation (thereby rendering the ghost-part of the Lagrangian density to remain \textit{invariant} under the symmetry transformations \( s_B \)).

The infinitesimal and continuous transformations (48) are \textit{symmetry} transformations because the Lagrangian density (8) transforms (to a total spacetime derivative) as

\[ s_B \mathcal{L}_b = \partial_\mu \left[ i m^2 (A^\mu E - m \varepsilon^{\mu \nu} A_\nu \phi) - i m (\varepsilon^{\mu \nu} \phi \Box A_\nu - m \varepsilon^{\nu \rho} A_\rho \partial^\mu A_\sigma) \right] + i (\partial \cdot A + m \phi) \partial^\mu E - E \partial^\mu (\partial \cdot A + m \phi) \].

(50)

As a consequence, it is clear that the action integral \( S = \int \, dx \, \mathcal{L}_b \) would remain invariant under the above bosonic symmetry transformations. By exploiting the strength of the Noether’s theorem, it can be seen that the conserved charge \((Q_B)\), corresponding to the symmetry transformations (48), is

\[ Q_B = \int \, dx \left[ (\partial \cdot A + m \phi) \dot{E} - E \partial_0 \{(\partial \cdot A + m \phi)\} \right]. \quad (51) \]

The above charge is the generator of the transformations (48). The validity of this statement can be proven by exploiting the general definition of the relationship between a symmetry transformation and its generator.

**Appendix C: On symmetries of an alternative Lagrangian density**

We discuss here the symmetry properties of the Lagrangian density \( \mathcal{L}_{(b_2)} \) [cf. (11) above]. Right at the beginning, we would like to make it clear that the dynamics remains \textit{intact} as far as the Lagrangian densities (10) and (11) are concerned. It can be seen explicitly that the Euler-Lagrange equations of motion derived from the Lagrangian density (11) are same as the ones [cf. (17)] from the Lagrangian density (10). Thus, both these Lagrangian densities are \textit{equivalent} from the point of view of dynamics. In what follows, we shall demonstrate that all \textit{six} continuous symmetry transformations, present for the Lagrangian density (10), are also respected by the Lagrangian density (11) (modulo sign factors).

It can be readily checked that, under the following on-shell \([\Box + m^2]C = 0, (\Box + m^2)\bar{C} = 0\] nilpotent \((s^2_{(a)b} = 0)\) (anti-)BRST symmetry transformations \( s_{(a)b} \)

\[ s_{ab} A_\mu = \partial_\mu \bar{C}, \quad s_{ab} \bar{C} = 0, \quad s_{ab} C = i (\partial \cdot A - m \phi), \]

\[ s_{ab} \phi = -m \bar{C}, \quad s_{ab} E = 0, \quad s_{\bar{a}b} (\partial \cdot A - m \phi) = (\Box + m^2) \bar{C}, \]

\[ s_b A_\mu = \partial_\mu C, \quad s_b C = 0, \quad s_b \bar{C} = -i (\partial \cdot A - m \phi), \]

\[ s_b \phi = -m C, \quad s_b E = 0, \quad s_b (\partial \cdot A - m \phi) = (\Box + m^2) C, \quad (52) \]
the Lagrangian density $\mathcal{L}_{(b_2)}$ transforms as follows:

$$s_{ab}\mathcal{L}_{(b_2)} = -\partial_\mu \left[(\partial \cdot A - m \phi) \partial^\mu C\right], \quad s_b\mathcal{L}_{(b_2)} = -\partial_\mu \left[(\partial \cdot A - m \phi) \partial^\mu C\right]. \quad (53)$$

The above (anti-)BRST symmetry transformations are also absolutely anticommuting on the on-shell. Further, under the following on-shell $[\Box + m^2]C = 0,(\Box + m^2)\bar{C} = 0$ nilpotent ($s_{(a)b}^2 = 0$) (anti-)co-BRST [or (anti-)dual-BRST] symmetry transformations:

$$s_{ab} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C, \quad s_{ab} C = 0, \quad s_{ab} \bar{C} = +i (E + m \bar{\phi}),$$

$$s_{ad} E = \Box C, \quad s_{ad} (\partial \cdot A - m \phi) = 0, \quad s_{ad} \phi = 0, \quad s_{ad} \bar{\phi} = +m C,$$

$$s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad s_d \bar{C} = 0, \quad s_d C = -i (E + m \phi),$$

$$s_d E = \Box \bar{C}, \quad s_d (\partial \cdot A - m \phi) = 0, \quad s_d \phi = 0 \quad s_d \bar{\phi} = +m \bar{C}, \quad (54)$$

the Lagrangian density (11) transforms to the total spacetime derivatives:

$$s_{ad} \mathcal{L}_{(b_2)} = \partial_\mu \left[E \partial^\mu C + m \varepsilon^{\mu\nu}(m A_\nu C - \phi \partial_\nu C)\right],$$

$$s_d \mathcal{L}_{(b_2)} = \partial_\mu \left[E \partial^\mu \bar{C} + m \varepsilon^{\mu\nu}(m A_\nu \bar{C} - \bar{\phi} \partial_\nu \bar{C})\right]. \quad (55)$$

Thus, it is clear that the Lagrangian density (11) respects all four basic fermionic symmetry transformations $s_{(a)b}$ and $s_{(a)d}$ that are also the symmetry transformations for its counterpart Lagrangian density $\mathcal{L}_{(b_1)}$. Furthermore, we note that the transformations $s_{(a)d}$ are also absolutely anticommuting (i.e. $s_d s_{ad} + s_{ad} s_d = 0$) on the on-shell.

From the above four on-shell nilpotent ($s_{(a)b}^2 = 0, s_{(a)d}^2 = 0$) (anti-)BRST and (anti-)co-BRST symmetry transformations, one can generate a unique bosonic symmetry transformation $s_w = \{s_b, s_d\} = -\{s_{ad}, s_{ab}\}$ because the rest of the anticommutators turn out to be zero on-shell (i.e. $\{s_b, s_{ab}\} = 0, \{s_d, s_{ad}\} = 0, \{s_d, s_{ab}\} = 0, \{s_d, s_{ad}\} = 0$). The relevant fields of the Lagrangian density (11) transform as follows under $s_w$:

$$s_w A_\mu = i \varepsilon_{\mu\nu} (\Box A^\nu + m \partial^\nu \phi) - i m \partial_\mu \bar{\phi}, \quad s_w (C, \bar{C}) = 0,$$

$$s_w \phi = + i m (E + m \bar{\phi}), \quad s_w \bar{\phi} = -i m (\partial \cdot A - m \phi),$$

$$s_w E = -i \Box (\partial \cdot A - m \phi), \quad s_w (\partial \cdot A) = -i \Box (E + m \phi). \quad (56)$$

It is straightforward to check that, under the above bosonic symmetry transformations, the Lagrangian density (11) transforms to a total spacetime derivative as:

$$s_w \mathcal{L}_{(b_2)} = \partial_\mu \left[i(\partial \cdot A - m \phi) \partial^\mu (E + m \bar{\phi}) - i(E + m \bar{\phi}) \partial^\mu (\partial \cdot A - m \phi)\right.$$\n
$$+ i m^2 (A^\mu E + \varepsilon^{\rho\sigma} A_\rho \partial^\sigma A_\mu - \bar{\phi} \partial^\mu \phi) + i m \varepsilon^{\mu\nu}(\phi \Box A_\nu + m^2 A_\nu \phi)$$\n
$$+ i m \{\phi \partial^\mu E + \bar{\phi} \partial^\mu (\partial \cdot A)\}]. \quad (57)$$

The above expression demonstrates that the action integral $S = \int dx \mathcal{L}_{(b_2)}$ remains invariant under the infinitesimal transformations $s_w$.

We close this Appendix with the remarks that (i) the operator form of the above transformations, along with the infinitesimal ghost transformations (26), obey the algebraic structure exactly like (32). Thus, both the Lagrangian densities (10) and (11) represent
the field theoretic examples of a Hodge theory, and (ii) the Noether conserved charges corresponding to the above symmetry transformations can be readily calculated as we have done for the Lagrangian density (10) and they follow the same algebra as given in (33). It will just be an academic exercise to repeat the same calculations, once again. Thus, we do not perform that exercise here as, we feel, there is no compelling reason for that.

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