A Note on Nonparametric Estimation of Conditional Hazard Quantile Function

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Abstract

In this paper, we study an kernel estimator of the conditional hazard quantile function (CHQF) of a scalar response variable $Y$ given a random variable (rv) $X$ taking values in a semi-metric space and using the proposed estimator based of the kernel smoothing method. The almost complete consistency and the asymptotic normality of this estimate are obtained when the sample is an independante sequence.

Keywords: Asymptotic normality, conditional hazard quantile function, functional data, kernel smoothing, nonparametric estimator.

1. Introduction

The goal of this paper is to study a nonparametric estimator of the CHQF when the explanatory variable is functional. This is motivated by the increasing number of situations in which the collected data are curves (consecutive discrete recordings are aggregated and viewed as sampled values of a random curve) where it used to be numbers and vectors. Functional data analysis (see Ferraty and Vieu (2006)) can help to analyze such data sets in a nonparametric framework.

Recently, many authors are interested in the estimation of conditional quantiles for a scalar response and functional covariate. Ferraty et al. (2005) introduced a nonparametric estimator of conditional quantile defined as the inverse of the conditional cumulative distribution function when the sample is considered as an $\alpha$-mixing sequence. They stated its rate of almost complete consistency and used it to forecast the well-known El Niño time series and to build confidence prediction bands. Ezzahrioui et al. (2008) established the asymptotic normality of the kernel conditional quantile estimator under $\alpha$-mixing assumption. Recently, and within the same framework, Dabo-Niang and Laksaci (2012) provided the consistency in $L^p$ norm of the conditional quantile estimator for functional dependent data, Bouchentouf et al. (2015) provided the consistency and asymptotic normality of the smoothing conditional quantile density function.

In an earlier contribution of the estimator of the CHQF (see Sankaran and Unnikrishnan (2009)) we established the consistency and asymptotic normality of the kernel smoothing estimator for the independent sequence and reel case. The present work gives a generalization to the functional data, we investigate the asymptotic properties and the asymptotic normality of the kernel conditional quantile estimator under $\alpha$-mixing assumption. Recently, and within the same framework, Dabo-Niang and Laksaci (2012) provided the consistency in $L^p$ norm of the conditional quantile estimator for functional dependent data, Bouchentouf et al. (2015) provided the consistency and asymptotic normality of the smoothing conditional quantile density function.
totic normality of the CHQF of a scalar response and functional covariate. The interest comes mainly from the fact that application fields for functional methods need to analyze continuous-time stochastic processes.

In what follows, the rest of the paper is organized as follows. Section 2 we present our estimation procedure and recall the definition of the functional kernel estimates property. Section 3 formulates main results of strong consistency (with rate) and asymptotic normality of the estimator with proofs of the main results. Section 4 is devoted to a brief conclusion of the study.

2. The functional kernel estimates

We consider a random pair \((X, Y)\) where \(Y\) is valued in \(\mathbb{R}\) and \(X\) is valued in some infinite dimensional semi-metric vector space \(F, d(\cdot, \cdot)\). Let \((X_i, Y_i), i = 1, \ldots, n\) be the statistical sample of pairs which are identically distributed like \((X, Y)\), but not necessarily independent. From now on, \(X\) is called functional random variable f.r.v. Let \(x\) be fixed in \(\mathcal{F}\) and let \(F_{Y|X}(y, x)\) be the conditional cumulative distribution function (cond-cdf) of \(Y\) given \(X = x\). The functional kernel estimates provide a brief conclusion of the study. Let us now, define the kernel estimator \(\hat{F}_{Y|X}(\cdot, x)\) of \(F_{Y|X}(\cdot, x)\)

\[
\hat{F}_{Y|X}(x, y) = \frac{1}{\sum_{i=1}^{n} K(h_{K}^{-1}d(x, X_i))} \sum_{i=1}^{n} K(h_{K}^{-1}d(x, X_i))H(h_{H}^{-1}(y - Y_i)).
\]

(1)

where \(K\) is a kernel function, \(H\) a cumulative distribution function and \(h_{K} = h_{K,n}(\text{resp. } h_{H} = h_{H,n})\) a sequence of positive real numbers. Roussas (1969) introduced some related estimate but in the special case when \(X\) is real, while Samanta (1980) produced previous asymptotic study. As a by-using of Nair and Sankaran (2009) and Xiang (1995), it is easy to derive an estimator \(\hat{Q}_{Y|X}\) of \(Q_{Y|X}\):

\[
Q_{Y|X}(\gamma) = \inf\{t : \hat{F}_{Y|X}(t, x) = \gamma\} = F_{Y|X}^{-1}(Q_{Y|X}(\gamma)).
\]

Let now defined the conditional density function is the derivative of conditional distribution function.

\[
\hat{f}_{Y|X}(x, y) = \frac{h_{H}^{-1} \sum_{i=1}^{n} K(h_{K}^{-1}d(x, X_i))H(h_{H}^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(h_{K}^{-1}d(x, X_i))}.
\]

(2)

Parzen (1979) and Jones (1992) defined the quantile density function as the derivative of \(Q(\gamma)\), that is, \(q(\gamma) = Q(\gamma)\). Note that the sum of two quantile density functions is again a quantile density function.

Nair and Sankaran (2009) have defined the hazard quantile function as follows:

\[
r(\gamma) = r(Q(\gamma)) = \frac{f(Q(\gamma))}{1 - F(Q(\gamma))} = ((1 - \gamma)q(\gamma))^{-1}.
\]

(3)

Thus hazard rate of two populations would be equal if and only if their corresponding quantile density functions are equal. This has been used to construct tests for testing equality of failure rates of two independent samples. Now, from this definition, let
us introduce the $\gamma$-order conditional quantile of the conditional hazard function

\[
 r(\gamma) = r_{Y|X}(Q_{Y|X}(\gamma)) = \frac{f_{Y|X}(Q_{Y|X}(\gamma))}{1 - F_{Y|X}(Q_{Y|X}(\gamma))} \quad (4) \\
= ((1 - \gamma)q_{Y|X}(\gamma))^{-1}. \quad (5)
\]

Consequently, the conditional quantiles of conditional hazard function operator is defined in a natural way and can be estimated by using kernel smoothing methods by

\[
 r_n(\gamma) = \hat{r}_{Y|X}(\hat{Q}_{Y|X}(\gamma)) = \frac{\hat{f}_{Y|X}(\hat{Q}_{Y|X}(\gamma))}{1 - \hat{F}_{Y|X}(\hat{Q}_{Y|X}(\gamma))}. \quad (6)
\]

Now we proposed the other estimator of $r_n(\gamma)$ using the kernel smoothing method, define by:

\[
 r_n(\gamma) = \frac{1}{h} \int_{0}^{1} \frac{1}{1 - \hat{F}_{Y|X}(\hat{Q}_{Y|X}(t))} \hat{q}_{Y|X}(t) H \left( \frac{t - \gamma}{h} \right) dt. \quad (7)
\]

In the next section derive the asymptotic properties of our conditional quantile hazard function

3. Assumptions and main results

3.1. General Assumptions

Our results are stated under some assumptions we gather hereafter for easy reference.

(H1) For all $h > 0$, $P(X \in B(x, h)) =: \phi_s(h) > 0$. Moreover, $\phi_s(h) \to 0$ as $h \to 0$.

(H2) For all $i \neq j$, $0 < \sup_{i \neq j} P(X_i, X_j) \in B(x, h) \times B(x, h) = P(W_i < h, W_j < h) = \psi_s(h)$, where $\psi_s(h) \to 0$ as $h \to 0$.

Furthermore, we assume that $\psi_s(h) = o(\phi_s^2(h))$.

(H3) $H$ is such that, for all $(y_1, y_2) \in \mathbb{R}^2$, $|H(y_1) - H(y_2)| = C|y_1 - y_2|$ and its first derivative $H^{(1)}$ verifies $\int |t|^2 |H^{(1)}(t)| dt < \infty$.

(H4) $K$ is a nonnegative bounded kernel of class $C_1$ over its support $[0, 1]$ such that $K(1) > 0$.

The derivative $K'$ exists on $[0,1]$ and satisfy the condition $K'(t) < 0$, for all $t \in [0,1]$ and $\int_0^1 (K)'(t) dt < \infty$ for $j = 1, 2$.

(H5) $\lim_{n \to \infty} h_k = 0$ with $\lim_{n \to \infty} \frac{\log n}{n} = 0$.

Remark 1. Hypothesis (H1) is the classical concentration assumption. (H3) allows to get the convergence rate in the independent case.

Assumption H3 is classical in nonparametric estimation and is satisfied by usual kernels such as Epanechnikov, Biweight, whereas the Gaussian density $K$ is also possible, it suffices to replace the compact support assumption by: $\int_{\mathbb{R}} |t|^k H(t) dt < \infty$.

Assumption H3 ensures the existence and uniqueness of the quantile estimate $q_{Y|X}(x)$, see Ferraty et al. (2005).

A mild regularity hypothesis (H4) is assumed for the distribution function. Hypothesis (H3) is technical and is imposed only for the brevity of proofs. Finally The choice of bandwidth is given by (H5).

3.2. Asymptotic properties

In this section, we prove strong consistency and asymptotic normality of the estimator $r_n(\gamma)$.

Theorem 1. Let $F_{Y|X}$ be continuous. Assume that $K(\cdot)$ satisfies the conditions (H1)-(H5) in Sec estimator $r_n(\gamma)$ is uniformly strong consistent.

Proof.

We can write Equation (7) as
\[ r_n(\gamma) = \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{dt}{\left[ 1 - F_{Y|X}(\hat{Q}_{Y|X}(t)) \right] \hat{q}_{Y|X}(t)} \]

\[ = \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{dt}{\left[ 1 - F_{Y|X}(Q_{Y|X}(t)) \right] \hat{q}_{Y|X}(t)} \]

\[ + \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{dt}{\left[ 1 - F_{Y|X}(\hat{Q}_{Y|X}(t)) \right] \hat{q}_{Y|X}(t)} \]

\[ = \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{dt}{\left[ 1 - F_{Y|X}(\hat{Q}_{Y|X}(t)) \right] \hat{q}_{Y|X}(t)} \]

Denoting

\[ K^*(t, \gamma) = H((t - \gamma)/h_H))/(1 - t)q_{Y|X}(t)q_{Y|X}(t), \]

on using integration by parts, equation (9) reduces to

\[ r_n(\gamma) - r(\gamma) = \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{\left[ 1 - F_{Y|X}(Q_{Y|X}(t)) \right] \hat{q}_{Y|X}(t)} dt + \]

\[ + \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{q_{Y|X}(t)} \frac{dt}{(1 - t)q_{Y|X}(t)} \]

Since

\[ \sup_t |\hat{F}_{Y|X}(t) - F_{Y|X}(t)| \longrightarrow 0 \]

almost surely, equation (8) is asymptotically equal to

\[ r_n(\gamma) = \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{dt}{\left[ 1 - F_{Y|X}(\hat{Q}_{Y|X}(t)) \right] \hat{q}_{Y|X}(t)}. \]

Thus,

\[ r_n(\gamma) - r(\gamma) \]

\[ = \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{\left[ 1 - F_{Y|X}(Q_{Y|X}(t)) \right] \hat{q}_{Y|X}(t)} dt + \]

\[ + \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{q_{Y|X}(t)} \frac{dt}{(1 - t)q_{Y|X}(t)} \]

Since \( \sup_t |\hat{Q}_{Y|X}(t) - Q_{Y|X}(t)| \longrightarrow 0 \) almost surely, equation (10) is asymptotically equal to
\[ r_n(\gamma) - r(\gamma) = \frac{1}{h_H} \int_0^1 H \left( \frac{t-\gamma}{h_H} \right) \frac{dt}{1 - \gamma q_Y|X(\gamma)} - \frac{1}{(1 - \gamma) q_Y|X(\gamma)}. \] (10)

\[ \text{Setting } (t-\gamma)/h_H = v, \text{ in equation (10), } \]

\[ r_n(\gamma) - r(\gamma) = \frac{1}{h_H} \int_{-\gamma/h_H}^{1-\gamma/h_H} H(v) \frac{f_Y|X(Y + vh_H)}{1 - (Y + vh_H)} dv \]

\[ - \frac{1}{(1 - \gamma) q_Y|X(\gamma)} \] (11)

\[ = \frac{1}{h_H} \int_{-\gamma/h_H}^{1-\gamma/h_H} H(v) \left[ 1 + \frac{vh_H}{1 - \gamma} + \cdots \right] dv \]

\[ \times f_Y|X \left[ Q_Y|X(\gamma) + vh_H dQ_Y|X(\gamma) + \cdots \right] dv \]

\[ - \frac{1}{(1 - \gamma) q_Y|X(\gamma)}. \] (12)

By Taylor’s series expansion of \( Q_Y|X(\gamma + vh_H) \) around \( \gamma \), equation (11) becomes

\[ r_n(\gamma) - r(\gamma) = \frac{1}{h_H(1 - \gamma)} \int_{-\gamma/h_H}^{1-\gamma/h_H} H(v) \left[ 1 + \frac{vh_H}{1 - \gamma} + \cdots \right] dv \]

\[ \times f_Y|X \left[ Q_Y|X(\gamma) + vh_H dQ_Y|X(\gamma) + \cdots \right] dv \]

\[ - \frac{1}{(1 - \gamma) q_Y|X(\gamma)}. \] (12)

As \( n \to \infty \), we have \( h_n \to 0 \) and \( \int_{-\infty}^{+\infty} H(v) dv = 1 \), so that equation (12) reduces to

\[ |r_n(\gamma) - r(\gamma)| = \left| f_Y|X \left[ Q_Y|X(\gamma) \right] 1 - \gamma - \frac{1}{(1 - \gamma) q_Y|X(\gamma)} \right|, \]

which tends to zero as \( n \to \infty \). This completes the proof.

\[ \Box \]

### 3.3. Asymptotic normality

In this section we give the asymptotic normality of \( r_n(\gamma) \).

**Theorem 2.** Under assumptions (H1)(H5) and suppose that \( F_Y|X \) is continuous, for \( 0 < \gamma < 1 \), \( \sqrt{n}(r_n(\gamma) - r(\gamma)) \) is asymptotically normal with mean zero and variance \( \sigma^2(\gamma) \) as given in Equation (13).

\[ \sigma^2(\gamma) = n \frac{1}{h_H^2} \mathbb{E} \left[ \int_0^1 Q_Y|X(t) dM'(t, \gamma) \right. \]

\[ + \left. \int_0^1 \tilde{F}_Y|X(\tilde{Q}_Y|X(t)) M(t, \gamma) \frac{dQ_Y|X(t)}{1-t} \right]^2. \] (13)

\[ M(t, \gamma) = H((t - \gamma)/h_H)/q_Y|X(t) \]

and \( M'(t, \gamma) \) is the derivative of \( M(t, \gamma) \) with respect to \( t \).

**Proof.**

\[ \sqrt{n}(r_n(\gamma) - r(\gamma)) \]

\[ = \sqrt{n} \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \]

\[ \left[ \frac{1}{1 - \tilde{F}_Y|X(\tilde{Q}_Y|X(t)) q_Y|X(t)} \right] dt \]

\[ - \frac{1}{(1 - \gamma) q_Y|X(\gamma)} \]

\[ + \sqrt{n} \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \]

\[ \frac{dt}{(1 - \tilde{F}_Y|X(\tilde{Q}_Y|X(t)) q_Y|X(t)) - (1 - \gamma) q_Y|X(\gamma)}. \]
which can be written as

\[
\sqrt{n}(r_n(\gamma) - r(\gamma)) = \sqrt{n} \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{(1 - \tilde{F}_{Y|X}(\tilde{Q}_{Y|X}(t))} \left[ \frac{q_{Y|X}(t) - \tilde{q}_{Y|X}(t)}{q_{Y|X}(t)\tilde{q}_{Y|X}(t)} \right] dt \\
+ \sqrt{n} \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{q_{Y|X}(t)} \frac{1}{(1 - t)(q_{Y|X}(t))} \\
\left[ \tilde{F}_{Y|X}(\tilde{Q}_{Y|X}(t)) - F_{Y|X}(Q_{Y|X}(t)) \right] dt \\
+ \sqrt{n} \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{(1 - F_{Y|X}(Q_{Y|X}(t)))q_{Y|X}(t)} dt \\
- \sqrt{n} \frac{1}{(1 - \gamma)q_{Y|X}(\gamma)},
\]  

Equation (14)

Setting \( M(t, \gamma) = H((t - \gamma)/h_H)/q_{Y|X}(t) \) in equation (15) and applying integration by parts, we obtain

\[
\sqrt{n}(r_n(\gamma) - r(\gamma)) = \sqrt{n} \frac{1}{h_H} \int_0^1 M(t, \gamma)q_{Y|X}(t) \left[ \tilde{F}_{Y|X}(\tilde{Q}_{Y|X}(t)) - F_{Y|X}(Q_{Y|X}(t)) \right] dt \\
+ \sqrt{n} \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{(1 - \gamma)q_{Y|X}(\gamma)} \\
- \sqrt{n} \frac{1}{(1 - \gamma)q_{Y|X}(\gamma)},
\]

Equation (16)

where \( M'(t, \gamma) \) is the derivative of \( M(t, \gamma) \) with respect to \( t \). From equation (12), we can obtain equation (16) as

\[
\sqrt{n}(r_n(\gamma) - r(\gamma)) = \sqrt{n} \frac{1}{h_H} \int_0^1 M(t, \gamma)q_{Y|X}(t) \left[ \tilde{F}_{Y|X}(\tilde{Q}_{Y|X}(t)) - F_{Y|X}(Q_{Y|X}(t)) \right] dt \\
+ \sqrt{n} \frac{1}{h_H} \int_0^1 H \left( \frac{t - \gamma}{h_H} \right) \frac{1}{(1 - \gamma)q_{Y|X}(\gamma)} \\
- \sqrt{n} \frac{1}{(1 - \gamma)q_{Y|X}(\gamma)},
\]

Equation (17)

Note that from Ezzahrioui and Ould-Saïd (2008), Laksaci et al. (2011) and Chaouch and Khardani (2015) for \( 0 \leq \gamma \leq 1 \), \( \sqrt{n}(\tilde{Q}_{Y|X}(\gamma) - Q_{Y|X}(\gamma)) \) is asymptotically normal with mean zero and variance

\[
\sigma^2(\gamma) = \frac{1}{\sqrt{\phi(h)} \beta_2} \frac{\gamma(1 - \gamma)}{\beta_1^2 (F_{Y|X}(Q_{Y|X}(\gamma)))^2},
\]

and \( \tau_0(\gamma) \) is a nondecreasing bounded function such that, uniformly in \( s \in [0, 1] \),

\[
\frac{\phi(hs)}{\phi(h)} = \tau_0(s) + o(1) \text{ as } h \downarrow 0
\]

and for \( j \geq 1 \),

\[
\int_0^1 ((K)^j)'(s) \tau_0(s) ds < \infty.
\]

Thus \( \sqrt{n}(\tilde{F}_{Y|X}(\tilde{Q}_{Y|X}(\gamma)) - F_{Y|X}(Q_{Y|X}(\gamma)) \) is also asymptotically normal with mean zero and variance \( \sigma^2(\gamma) \), since \( d/d\gamma F_{Y|X}(Q_{Y|X}(\gamma)) = 1 \).
Now from equation (17), we can show that 
\(\sqrt{n}(r_n(\gamma) - r(\gamma))\) is asymptotically normal with mean zero. The expression of variance can be obtained from equation (17), which is given by

\[
\sigma_2^2(\gamma) = \frac{n}{h_H^2} \mathbb{E} \left[ \int_0^1 \hat{Q}_{Y|X}(t) dM'(t, \gamma) \right. \\
\left. + \int_0^1 \hat{F}_{Y|X}(\hat{Q}_{Y|X}(t)) \frac{M(t, \gamma)}{1 - r} dQ_{Y|X}(t) \right] (18)
\]

This completes the proof. \(\square\)

**Remark 2.**

- The function \(\tau_0(\cdot)\) defined in by there exists a function \(\tau_0(\cdot)\) s.t. for all \(s \in [0, 1]\), \(\lim_{r \to 0} \phi_s(r)/\phi_s(s) = \phi_0(s)\), permits to get the variance term explicitly.

This condition is classical and related to a non vanishing conditional density. The second one means that a small amount a concentration is needed in order to ensure asymptotic normality.

- The present study provided a nonparametric estimator, for the conditional hazard quantile function, its based and using the kernel smoothing method, and the asymptotic properties of the kernel estimator were studied. The kernel based estimator seems to perform satisfactorily except for large values of \(\gamma\).

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