Movement characteristics of a non-smooth model with a closed curve equilibrium

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Abstract. The main aim of this paper is to analyze the dynamical properties of a model with a closed curve equilibrium. The corresponding three-variable model is given as a set of nonlinear ordinary differential equations containing non-smooth functions. The dynamics of the model are studied depending on three parameters. For this purpose, new methods, as the 0-1 test for chaos and approximate entropy, are applied. Using these tools, the dynamics are quantified and qualified. It is shown that depending on the system’s parameters, the system exhibits both irregular (chaotic) and regular (periodic) character.

1. Introduction
Chaotic systems, apart from naturally arising from many problems in fields such as biology, chemistry, economy, and engineering, can be utilized for security applications, e.g. [1, 2], as their behaviour greatly depends on initial conditions and values of parameters. Therefore, new chaotic systems are discovered.

The classification of the dynamical systems based on the number of equilibrium points led to a study of chaotic systems with no equilibria, number of equilibria, and infinite number of equilibria. The latter gave rise to many chaotic systems with different shape of equilibria - e.g. [3–5].

In [6], a new family of chaotic systems has been introduced with infinite equilibrium points. The shape of equilibria changes depending on values of a parameter. This work focuses on characterization of dynamical properties of the model [6]. The characteristic is done by the 0-1 test for chaos and approximate entropy. These tools are applied to the data obtained by a huge simulation that was performed on the Barbora supercomputer at IT4Innovations National Supercomputing Center located in Ostrava, Czech Republic.

The work is organized as follows: in Section 2 the investigated model is introduced, in Section 3 simulation details and methods are given, and the main results are reached using the 0-1 test for chaos and approximate entropy. Finally in Section 4 the results are summarized and discussed.

2. The model
The chosen model was introduced by Zhu and Du in [6], where it is derived as a special case from the work of Gotthans and Petržela [7]. The following system of three differential equations

\begin{align*}
\frac{dx}{dt} &= f(x, y, z), \\
\frac{dy}{dt} &= g(x, y, z), \\
\frac{dz}{dt} &= h(x, y, z),
\end{align*}

where

\begin{align*}
f(x, y, z) &= ax - by - cz, \\
g(x, y, z) &= y - (d + e)x + fz, \\
h(x, y, z) &= -g + bz.
\end{align*}
containing non-smooth functions is obtained:

\[
\begin{align*}
\frac{dx}{dt} &= z, \\
\frac{dy}{dt} &= -z(ay + by^2 + xz), \\
\frac{dz}{dt} &= |x|^k + |y|^k - 1,
\end{align*}
\]  

where \(a, b, \) and \(k\) are free parameters.

It is shown in [6], that for \(k = 2\), the model develops into that in [7]. That model was also investigated in [8]. For \(k = 4\), the problem was studied in [9].

In [6], the dynamics is investigated for a choice of parameters \(a\) and \(b\), and a set of parameters \(k\). Tools such as phase portraits, Kaplan-Yorke dimension, maximal Lyapunov exponents and eigenvalues were used to qualify and quantify the dynamics of the system.

In this paper, the dynamics of the system, depending on changes in values of the parameters, is investigated for \(k = 3\) and \(k = 5\), and a range of parameters \(a\) and \(b\). Alternative techniques are used, mainly the 0-1 test for chaos and the approximate entropy, accompanied by phase portraits, Poincaré sections, Fourier spectra and bifurcation diagrams. The two main tools are chosen for their effectiveness and robustness compared to the other methods.

3. Simulations and main results

The simulations of the model (1) were done for free parameters \(a, b\) and \(k\), in a range \(a \in [3.2, 6.5]\) and \(b \in [4, 6]\) for \(k = 3\), and \(a \in [4, 7.5]\) and \(b \in [2.65, 4.5]\) for \(k = 5\), with 0.05 step. This range of parameters shows the transition between periodic and chaotic dynamics. The non-smooth nature of the system prevents finding the solutions for certain combinations of parameters, which makes the simulations problematic. The computations were performed using the Runge–Kutta fifth order integration method ode45 in Matlab [10] with final time 20000 and time step 0.001. The system is assumed to be in the rest position in the beginning. That is, initial conditions equal

\[(x_0, y_0, z_0) = (0, 0, 0).\]

The behaviour of the system was investigated using phase diagrams, amplitude frequency spectrum (FFT), and Poincaré sections for a relevant choice of free parameters. These methods are widely used for dynamics detection, e.g. [11, 12]. To underline the dynamics of the system, bifurcation diagrams were plotted with respect to both parameter \(a\) and \(b\). Consequently, the 0-1 test for chaos and the approximate entropy were computed for a given range of parameters.

The approximate entropy shows the complexity of the system and detects an increase of unpredictability as the values of approximate entropy get higher. On the other hand, the 0-1 test for chaos splits the intervals of the parameters for which regular and chaotic character appears.

3.1. Phase diagrams, Poincaré sections, Fourier spectra and bifurcation diagrams

The dynamics of the model given by (1) is both periodic and chaotic depending on the choice of parameters.

As an example of periodic movement, Figures 1 and 2 show a loop for parameters \(a = 3.5, b = 4, k = 3\) and \(a = 4, b = 4.5, k = 5\) respectively. The Poincaré section shows two isolated points of intersection of the phase portrait and the Poincaré plane. The Fourier spectrum consists of sharp peaks at the dominant and harmonic frequencies.

On the other hand, the phase portrait in Figures 3 and 4 for parameters \(a = 5, b = 5.2, k = 3\) and \(a = 5.5, b = 2.5, k = 5\) respectively, corresponds to a chaotic movement given by
Figure 1. Phase diagram with Poincaré section and Fast Fourier transform of variable $x$ for $a = 3.5$, $b = 4$ and $k = 3$.

Figure 2. Phase diagram with Poincaré section and Fast Fourier transform of variable $x$ for $a = 4$, $b = 4.5$ and $k = 5$.

Figure 3. Phase diagram with Poincaré section and Fast Fourier transform of variable $x$ for $a = 5$, $b = 5.2$ and $k = 3$.

Figure 4. Phase diagram with Poincaré section and Fast Fourier transform of variable $x$ for $a = 5.5$, $b = 2.5$ and $k = 5$.

the complex phase portrait as well as a number of points of intersection of the phase portrait and the Poincaré plane. The Fourier spectrum of this chaotic case is composed of harmonic frequencies with super-harmonic, sub-harmonic, and combination frequencies, with additional frequencies forming a continuous spectrum.

To investigate the range of regular and chaotic dynamics of the system, the bifurcation diagrams of the model (1) were done with respect to the free parameter $a$ for variable $x$. The bifurcation diagram for parameters $b = 4.95$ and $k = 3$ is shown in Figure 5, and for $b = 3.45$ and $k = 5$ in Figure 6. The range of $a$ is shown on the horizontal axis. Regular dynamics is shown in the beginning of the interval, followed by chaotic behaviour of the system.
Figure 5. Bifurcation diagram of $x$ with respect to $a$, for $b = 4.95$ and $k = 3$. The magnification of the transition from regular to chaotic dynamics is shown on the right.

Figure 6. Bifurcation diagram of $x$ with respect to $a$, for $b = 3.45$ and $k = 5$. The magnification of the transition from regular to chaotic dynamics is shown on the right.

3.2. The approximate entropy

The approximate entropy, introduced by Pincus [13], is a method for quantifying the amount of regularity or unpredictability in a time-series. The higher the value of approximate entropy, the more the system fluctuates and becomes unpredictable. The method can also be used effectively on short time-series. By the values of approximate entropy, we compare the differences in complexity of the system for different values of parameters.

The computation of the approximate entropy uses the following algorithm.

First, the parameters $m$ for embedding dimension and $r$ for neighborhood threshold are set. Let $y(t) ∈ \mathbb{R}$ for $t = \{1, 2, \ldots, N\}$ be a time series with $N$ observations. A sequence of vectors $x(1), x(2), \ldots, x(N - (m - 1))$ in $\mathbb{R}^m$ is established, defined as $x(i) = [y(i), y(i + 1), y(i + 2), \ldots, y(i + (m - 1))]$, where $t$ is the observed time and $m$ is the embedding dimension.

Next, the distance between the embedded vectors $x(i)$ and $x(j)$ is computed as

$$D(i, j) = d(x(i), x(j)) = \max_{k=1,2,\ldots,m} |u(i + k - 1) - u(j + k - 1)|,$$

for $i, j = \{1, 2, \ldots, N - (m - 1)\}$.

We find the thresholded distance with threshold given by $r$ as

$$d_r(i, j) = \begin{cases} 1, & D(i, j) < r \\ 0, & otherwise \end{cases},$$

for $i, j = \{1, 2, \ldots, N - (m - 1)\}$.
Figure 7. Results of the approximate entropy for $k = 3$ (left) and $k = 5$ (right).

The ratio between points in the neighborhood of $i$ in the number of the embedded vectors is given by

$$C_m^i(r) = \frac{\sum_{j=1}^{N-(m-1)} d_r(i,j)}{N-(m-1)}.$$  

(4)

The average of logarithm of all the $C_m^i(r)$ is given by

$$\Phi_m^r = \frac{1}{N-(m-1)} \sum_{i=1}^{N-(m-1)} \ln C_m^i(r).$$  

(5)

Finally, the result of the approximate entropy is computed as the difference

$$ApEn(m, r, N) = \Phi_m^r - \Phi_{m+1}^r.$$  

(6)

The computations were done using the free software environment R [14] along with the TSEntropies package [15]. The computations were made for the input vector $s$ given in a normalized form of all state variables:

$$s(t) = \sqrt{x^2(t) + y^2(t) + z^2(t)}.$$  

The results of approximate entropy for the system (1) depending on $a$ and $b$ and for $k = 3$ and $k = 5$ are shown in Figure 7. The correlation between results of neighboring parameters detects increase (or decrease) of the system’s complexity.

3.3. The 0-1 test for chaos

The 0-1 test for chaos, compared to the approximate entropy, is a method for strictly distinguishing regular and chaotic dynamics of a deterministic system. Introduced by Gottwald and Melbourne [16], it became a popular tool for chaos detection in different fields [8, 17, 18], for its easy implementation, straightforward evaluation, and wide range of application. As the nature of the investigated system is irrelevant, the test can be used directly on experimental data, as well as systems simulated by differential equations. Also, the evaluation of the test results in values close to either 0 or 1, with 0 corresponding to regular dynamics and 1 to chaotic behaviour.

The 0-1 test for chaos is computed by the following algorithm.

Let $\phi(j)$ be the observation for $j = 1, 2, ..., N$. We set the parameter $c$, $c \in (0, 2\pi)$.
First, the translation variables are computed as

\[ p_c(n) = \sum_{j=1}^{n} \phi(j) \cos(jc), \]

\[ q_c(n) = \sum_{j=1}^{n} \phi(j) \sin(jc) \]

for \( n = 1, 2, ..., N \).

The time-averaged mean square displacement of the variables \( p_c \) and \( q_c \) is obtained as

\[ M(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(j, n)^2 \]

where

\[ d(j, n) = \sqrt{(p_{j+n} - p_j)^2 + (q_{j+n} - q_j)^2} \]

is the time lapse of the duration \( n \) \((n \ll N)\) starting from the position at time \( j \). As it is shown in [16, 19], it is important to use values of \( n \) small enough compared to \( N \), noted \( n_{\text{cut}} \), \( n \leq n_{\text{cut}} \). A subset of time lags \( n_{\text{cut}} \in [1, N/10] \) is advised. As proposed in [16], the modified MSD is calculated as

\[ D(n) = M(n) - E(\phi)^2 \frac{1 - \cos(nc)}{1 - \cos c} \]

The output of the 0-1 test for chaos is computed by the correlation method as

\[ K_c = \text{corr}(\xi, \Delta) \in [-1, 1] \]

for the vectors \( \xi = (1, 2, ..., n_{\text{cut}}) \) and \( \Delta = (D_c(1), D_c(2), ..., D_c(n_{\text{cut}})) \).

The final result of the test is computed as the median of each \( K_c \) for approximately a hundred values of \( c \):

\[ K = \text{median}(K_c). \]

The computations of the 0-1 test for chaos for the model (1) were done using the free software environment R [14] along with the Chaos01 package [20]. The computations were made for the input vector \( s \) given in a normalized form of all state variables:

\[ s(t) = \sqrt{x^2(t) + y^2(t) + z^2(t)}. \]

The results of the 0-1 test for chaos depending on parameters \( a \) and \( b \) for \( k = 3 \) and \( k = 5 \) are shown in Figure 8. The test splits the region of parameters into parts of regular and chaotic dynamics.

4. Conclusions

In this paper, the dynamics of the system (1) constructed by Zhu and Du in [6] was extensively researched.

The set of nonlinear differential equations was simulated in Matlab [10] using the ode45 solver for each setting of the free parameters.

It is shown that the model (1) is showing both regular and chaotic behaviour. The characteristics of the dynamics were examined through phase portraits, Poincaré sections and
the Fourier spectra (see Figures 1, 2, 3 and 4). The changes in dynamical properties of the system are visualized in bifurcation diagrams (Figures 5 and 6).

Moreover, the approximate entropy and the 0-1 test for chaos were used to quantify and qualify the dynamical behaviour.

In Figure 7 the output of the approximate entropy shows the changes in the system’s complexity for different values of free parameters. The growing values of the results suggest increase of periods, eventually reaching chaotic behaviour. To distinguish between periodic and chaotic dynamics, the 0-1 test for chaos returns values close to 0 for regular dynamics, and values close to 1 for chaotic dynamics. Therefore, it splits the region of parameters in Figure 8 into areas of blue, corresponding to periodic movement, and red, equivalent to chaotic dynamics.

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