DUALITY FOR LARGE BERGMAN-ORLICZ SPACES AND 
BOUNDEDNESS OF HANKEL OPERATORS

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Abstract. For $B^n$ the unit ball of $\mathbb{C}^n$, we consider Bergman-Orlicz spaces of holomorphic functions in $L^\Phi_\alpha$, which are generalizations of classical Bergman spaces. We characterize the dual space of large Bergman-Orlicz space, and bounded Hankel operators between some Bergman-Orlicz spaces $A^\Phi_1(B^n)$ and $A^\Phi_2(B^n)$ where $\Phi_1$ and $\Phi_2$ are either convex or concave growth functions.

1. Introduction

Let $B^n$ be the unit ball of $\mathbb{C}^n$. We denote by $d\nu$ the Lebesgue measure on $B^n$ and $d\sigma$ the normalized measure on $S^n = \partial B^n$ (the boundary of $B^n$).

The space $H(B^n)$ is the set of holomorphic functions on $B^n$.

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we let

$$\langle z, w \rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$

so that $|z|^2 = \langle z, z \rangle = |z_1|^2 + \cdots + |z_n|^2$.

We say that a function $\Phi$ is a growth function if it is a continuous and non-decreasing function from $[0, \infty)$ onto itself.

For $\alpha > -1$, we denote by $d\nu_\alpha$ the normalized Lebesgue measure $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, with $c_\alpha$ such that $\nu_\alpha(B^n) = 1$. For $\Phi$ a growth function, the weighted Bergman-Orlicz space $A^\Phi_\alpha(B^n)$ is the space of holomorphic functions $f$ such that

$$||f||_{\alpha, \Phi} := \inf\{\lambda > 0 : \int_{B^n} \Phi(|f(z)|)d\nu_\alpha(z) \leq 1\}$$

which is finite for $f \in A^\Phi_\alpha(B^n)$ (see [10]).
When \( \Phi(t) = t^p \), we recover the classical weighted Bergman spaces denoted by \( \mathcal{A}_p^\alpha(\mathbb{B}^n) \) and defined by
\[
\|f\|_{p,\alpha}^p := \int_{\mathbb{B}^n} |f(z)|^p dv_\alpha(z) < \infty.
\]
We say that a growth function \( \Phi \) is of upper type \( q \geq 1 \) if there exists \( C > 0 \) such that, for \( s > 0 \) and \( t \geq 1 \),
\[
(2) \quad \Phi(st) \leq Ct^q \Phi(s).
\]
We denote by \( \mathcal{U}^q \) the set of growth functions \( \Phi \) of upper type \( q \), (for some \( q \geq 1 \)), such that the function \( t \mapsto \frac{\Phi(t)}{t} \) is non-decreasing.
We say that \( \Phi \) is of lower type \( p > 0 \) if there exists \( C > 0 \) such that, for \( s > 0 \) and \( 0 < t \leq 1 \),
\[
(3) \quad \Phi(st) \leq Ct^p \Phi(s).
\]
We denote by \( \mathcal{L}^p \) the set of growth functions \( \Phi \) of lower type \( p \), (for some \( p \leq 1 \)), such that the function \( t \mapsto \frac{\Phi(t)}{t} \) is non-increasing.
We say that \( \Phi \) satisfies the \( \Delta_2 \)-condition if there exists a constant \( K > 1 \) such that, for any \( t \geq 0 \),
\[
(4) \quad \Phi(2t) \leq K \Phi(t).
\]
Recall that two growth functions \( \Phi_1 \) and \( \Phi_2 \) are said equivalent if there exists some constant \( c \) such that
\[
c\Phi_1(ct) \leq \Phi_2(t) \leq c^{-1} \Phi_1(c^{-1}t).
\]
Such equivalent growth functions define the same Orlicz space. Note that we may always suppose that any \( \Phi \in \mathcal{U}_p \) (resp. \( \mathcal{W}_q \)), is concave (resp. convex) and that \( \Phi \) is a \( C^1 \) function with derivative \( \Phi'(t) = \frac{\Phi(t)}{t} \) (see [3] for the lower type functions).

Let us observe that if \( \Phi \) is of upper type (resp. lower type) \( p_1 \), then it is of upper type (resp. lower type) \( p_2 \) for any \( \infty > p_2 > p_1 \) (resp. \( p_2 < p_1 < \infty \)). Hence, when we say \( \Phi \in \mathcal{U}_q \) (resp. \( \Phi \in \mathcal{L}_p \)), we suppose that \( q \) (resp. \( p \)) is the smallest (resp. biggest) number \( q_1 \) (resp. \( p_1 \)) such that \( \Phi \) is of upper type \( q_1 \) (resp. lower type \( p_1 \)).

We denoted by \( L^p_\alpha(\mathbb{B}^n) \), \( 0 < p < \infty \), the Lebesgue space with respect to the measure \( dv_\alpha \).
The orthogonal projection of \( L^2_\alpha(\mathbb{B}^n) \) onto \( \mathcal{L}^2_\alpha(\mathbb{B}^n) \) is called the Bergman projection and denoted \( P_\alpha \). It is given by
\[
P_\alpha(f)(z) = \int_{\mathbb{B}^n} K_\alpha(z,\xi) f(\xi) dv_\alpha(\xi),
\]
where
\[
K_\alpha(z,\xi) = \frac{1}{(1 - \langle z, \xi \rangle)^{n+1+\alpha}}
\]
is the weighted Bergman kernel on \( \mathbb{B}^n \). We denote as well by \( P_\alpha \) its extension to \( L^1_\alpha(\mathbb{B}^n) \).

It is well known that the Bergman projection \( P_\alpha \) is bounded on \( L^p_\alpha(\mathbb{B}^n) \) for all \( p \in (1, \infty) \). One of the consequences of this result is the fact that the
topological dual space of the Bergman space $A_p^\alpha(B^n)$ identifies with $A_q^\alpha(B^n)$, with $\frac{1}{p} + \frac{1}{q} = 1$, under the integral pairing

$$\langle f, g \rangle_\alpha := \int_{B^n} f(z)\overline{g(z)}d\nu_\alpha(z),$$

$f \in L_p^\alpha(B^n)$, $g \in L_q^\alpha(B^n)$.

The boundedness of the Bergman projection has been extended to the setting of Orlicz spaces for the class of Young functions in [4], and this provides as a consequence that the dual space of the Bergman-Orlicz space $A_\Phi^\alpha(B^n)$ can be identified with another Bergman-Orlicz space that we will specify in the next section.

Our first interest in this paper is the characterization of the dual space of the Bergman-Orlicz spaces defined from the class $L_p$. We note that this class generalizes the class of power functions $\phi(t) = t^p$, $0 < p < 1$, and we have that for $\Phi \in L_p$, the following inclusions hold

$$A_1^\alpha(B^n) \subset A_\Phi^\alpha(B^n) \subset A_p^\alpha(B^n).$$

We recall that given an analytic function $f$ on $B^n$, the radial derivative $Rf$ of $f$ is defined by

$$Rf(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

For $\beta \geq 0$, we denote by $\Gamma_\beta(B^n)$ the space of holomorphic functions $f$ for which there exists an integer $k > \beta$ and a positive constant $C$ such that

$$|R^k f(z)| \leq C(1 - |z|^2)^{\beta - k}.$$

Remark that for $\beta = 0$, the class $\Gamma_\beta(B^n)$ coincides with the usual Bloch class $B$. The Bloch class is the space of holomorphic functions in $B^n$ such that

$$\sup_{z \in B^n} |Rf(z)|(1 - |z|^2) < \infty.$$

For $\beta > 0$, it coincides with the class of Lipschitz functions of order $\beta$.

It is known that, for $0 < p \leq 1$, the dual of the Bergman space $A_p^\alpha(B^n)$, coincides with $\Gamma_\beta(B^n)$ with $\beta = (n + 1 + \alpha)(\frac{1}{p} - 1)$ under the integral pairing

$$\lim_{r \to 1} \int_{B^n} f(rz)\overline{g(z)}d\nu_\alpha(z)$$

(see [10]).

To $\Phi$ a growth, we associate the function

$$\rho(t) = \frac{1}{t^{\Phi^{-1}(1/t)}}.$$

The function $\rho$ is quite relevant in the study of Orlicz space of analytic functions (see [3, 4, 11, 12] and the references therein). Note in particular that in the case of $A_p^\alpha(B^n)$, $\Phi(t) = t^p$ and $\rho(t) = t^{\frac{1}{p} - 1}$, hence $f \in \Gamma_\beta(B^n)$ can be written as

$$|R^k f(z)| \leq C(1 - |z|^2)^{-k}\rho((1 - |z|^2)^{n+1+\alpha})$$

...
From this observation, we will make the following generalization. Let \( \rho \) be a positive continuous increasing function from \([0, \infty)\) onto itself. Let \( \gamma > 0 \). We say that \( \rho \) is of upper type \( \gamma \) on \([0, 1]\) if there exists a constant \( C \) such that
\[
\rho(st) \leq Cs^{\gamma}\rho(t),
\]
for \( s > 1 \) and \( st \leq 1 \). We will call a weight, a function \( \rho \) which is a continuous increasing function from \([0, \infty)\) onto itself, which is of upper type \( \gamma \), for some \( \gamma > 0 \).

Now for \( \alpha > -1 \) and a weight \( \rho \) (of upper type \( \gamma \)), we define the weighted Lipschitz space \( \Gamma_{\alpha, \rho}(B^n) \) as the space of holomorphic functions \( f \) in \( B^n \) such that, for some integer \( k > \gamma(n+1+\alpha) \) and a positive constant \( C > 0 \), we have
\[
|R^k f(z)| \leq C(1-|z|^2)^{-k}\rho((1-|z|^2)^{n+1+\alpha}).
\]
We will show that, as in the classical Lipschitz spaces, these spaces are independent of \( k \). This allows us to see \( \Gamma_{\alpha, \rho}(B^n) \) as a Banach space under the following norm
\[
||f||_{\Gamma_{\alpha, \rho}(B^n)} = |f(0)| + \sup_{z \in B^n} \frac{|R^k f(z)|(1-|z|^2)^{k}}{(1-|z|^2)^{n+1+\alpha}}.
\]

The following is our first main result which extends the duality result for classical Bergman spaces with small exponent to Bergman-Orlicz spaces with concave exponent.

**Theorem 1.1.** Let \( \alpha > -1 \), \( \Phi \in L_p \) and \( \rho(t) = \frac{1}{t^{\Phi^{-1}(1/\Phi)}} \). Then the topological dual space \( (A^{\Phi}_{\alpha}(B^n))^* \) of \( A^{\Phi}_{\alpha}(B^n) \) identifies with \( \Gamma_{\alpha, \rho}(B^n) \) under the duality pairing
\[
\langle f, g \rangle_{\alpha} := \lim_{r \to 1} \int_{B^n} f(rz) g(z) \, d\nu_{\alpha}(z),
\]
where \( f \in A^{\Phi}_{\alpha}(B^n) \) and \( g \in \Gamma_{\alpha, \rho}(B^n) \).

The proof of the above result required two main steps. First one has to insure that the above definition of the space \( \Gamma_{\alpha, \rho}(B^n) \) does not depend on the choice of the number of derivatives. Second, one needs a nice example of functions in the Bergman-Orlicz space \( A^{\Phi}_{\alpha}(B^n) \) and a generalization to large Bergman-Orlicz spaces of the following inequality known for Bergman spaces with exponent \( 0 < p \leq 1 \) (see [15]),
\[
\int_{B^n} |f(z)|(1-|z|^2)^{(\Phi^{-1})-(n+1+\alpha)} \, d\nu_{\alpha}(z) \leq C||f||_{p,\alpha,\rho}^p.
\]

For \( b \in A^{2}_{\alpha}(B^n) \), the small Hankel operator with symbol \( b \) is defined for \( f \) a bounded holomorphic function by \( h_b(f) := P_{\alpha}(bf) \).

Boundedness of the small Hankel operator between classical weighted Bergman spaces has been considered in [2] where using duality and test functions, the authors obtained a full characterization of bounded Hankel operators between Bergman spaces except for estimations with loss i.e. \( h_b : A^p_{\alpha}(B^n) \to A^q_{\alpha}(B^n) \) with \( 1 \leq q < p < \infty \). The estimations with loss
have been recently handled by J. Pau and R. Zhao in [7], closing the subject for the classical weighted Bergman spaces.

Our second interest in this paper is for the boundedness of the small Hankel operators $h_b$ from $A_{\Phi_1}^t(B^n)$ to $A_{\Phi_2}^s(B^n)$. We do not use a specific method but combine several techniques some of them appearing in [2] or used in the case of Hardy-Orlicz spaces in [11, 12]. In particular, when considering boundedness of $h_b$ on $A_{\Phi}^t(B^n)$ with $\Phi \in \mathcal{Y}$, we use a weak factorization result of the Bergman space $A_{\Phi}^t(B^n)$ in term of Bergman-Orlicz functions, extending the usual weak factorization for this space. Nevertheless, we do not generalize this method for the whole situation, as when $\Phi_1$ and $\Phi_2$ are growth functions with $\Phi_2 \in \mathcal{Y}$, we are dealing only with the upper triangle case, i.e $\Phi^{-1}_1(t)\Psi^{-1}_2(t) \in \mathcal{Y}$, $\Psi_2$ being the complementary function of $\Phi_2$ to be defined later.

The ranges of the symbols of bounded Hankel operators obtained here are some weighted Lipschitz spaces related to the dual spaces of Bergman-Orlicz spaces with concave growth functions as given above. This will allow us to study the boundedness of the Hankel operators between Bergman-Orlicz spaces in the same range of growth functions as in [3] [11]. However, when $\Phi_2$ is a concave growth function, we will suppose that $\Phi_2$ satisfies a Dini condition to be defined later. This will cause additional restrictions on the growth functions $\Phi_1$ and $\Phi_2$ for which we are able to extend the Hankel operators, $h_b$, into bounded operators from $A_{\Phi_1}^t(B^n)$ into $A_{\Phi_2}^s(B^n)$. In particular, we are able here to handle the cases where the growth functions are given by $\Phi_1(t) = \left( \frac{t}{\log(e+t)} \right)^p$, $p \leq 1$ and $\Phi_2(t) = \left( \frac{t}{\log(e+t)} \right)^s$, $s < 1$. Moreover, our results generalize the results obtained in [2].

The paper is organized as follows, in section 2 we collect some results that are needed to characterize the dual spaces of large Bergman-Orlicz space and to study the boundedness properties of Hankel operators between Bergman-Orlicz spaces. In section 3, we deal with the duality question for Bergman-Orlicz spaces with concave exponent. In the last section, each subsection is devoted to the study, in each case, of the boundedness of the Hankel operator $h_b$ from $A_{\Phi}^t(B^n)$ into $A_{\Phi}^s(B^n)$, when $\Phi_i \in \mathcal{L}_p$ or $\mathcal{Y}$, $i = 1, 2$.

Finally, all over the text, $C$ will be a constant not necessary the same at each occurrence. We will also use the notation $C(k)$ to express the fact that the constant depends on the underlined parameter. Given two positive quantities $A$ and $B$, the notation $A \lesssim B$ means that $A \leq CB$ for some positive constant $C$. When $A \overset{\cdot}{\lesssim} B$ and $B \overset{\cdot}{\lesssim} A$, we write $A \simeq B$.

2. Preliminaries

In this section, we recall some known results that are needed in our study, we also extend to Orlicz setting many some classical results known for the Bergman spaces.

### 2.1. Some properties of growth functions.

We collect in this subsection some properties of growth functions we shall used later. For $\Phi$ a convex growth function, we recall that the complementary function, $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,
is defined by
\[ \Psi(s) = \sup_{t \in \mathbb{R}_+} \{ ts - \Phi(t) \}. \]

One easily checks that if \( \Phi \in \mathcal{U}^q \), then \( \Psi \) is also a growth function of lower type such that \( t \mapsto \frac{\Psi(t)}{t} \) is non-decreasing but which may not satisfy the \( \Delta_2 \)-condition. We say that the growth function \( \Phi \) satisfies the \( \nabla_2 \)-condition whenever both \( \Phi \) and its complementary satisfy the \( \Delta_2 \)-condition.

For \( \Phi \) a \( C^1 \) growth function, the lower and the upper indices of \( \Phi \) are respectively defined by
\[ a_{\Phi} := \inf_{t > 0} \frac{t \Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_{\Phi} := \sup_{t > 0} \frac{t \Phi'(t)}{\Phi(t)}. \]

We recall that when \( \Phi \) is convex, then \( 1 \leq a_{\Phi} \leq b_{\Phi} < \infty \) and, if \( \Phi \) is concave, then \( 0 < a_{\Phi} \leq b_{\Phi} \leq 1 \). We have the following useful fact.

**Lemma 2.1.** Let \( \Phi \) be a \( C^1 \) growth function. Denote by \( p \) and \( q \) its lower and its upper indices respectively. Then the functions \( \Phi(t) \) and \( \Phi^{-1}(t) \) are increasing.

**Proof.** We only prove that \( \Phi(t) \) is increasing. The proof is the same for the second function. Recall that by definition of \( p \), we have \( p \leq \frac{\Phi'(t)}{\Phi(t)} \) for any \( t > 0 \). It easily follows that
\[ \left( \frac{\Phi(t)}{t^p} \right)' = \frac{\Phi'(t)}{t^p} - p \frac{\Phi(t)}{t^{p+1}} \geq \frac{\Phi'(t)}{t^p} - \frac{t \Phi'(t)}{\Phi(t)} \times \frac{\Phi(t)}{t^{p+1}} = 0. \]

The proof is complete. \( \square \)

**Remark 2.2.** One useful way to use this lemma is to observe that it allows us to say that, if \( \Phi \in \mathcal{L}_p \), then the growth function \( \Phi_p \), defined by \( \Phi_p(t) = \Phi(t^{1/p}) \), is in \( \mathcal{U}^q \) for some \( q \geq 1 \). So we may assume that \( \Phi_p \) is convex.

We also observe that \( a_{\Phi} \) (resp. \( b_{\Phi} \)) coincides with the biggest (resp. smallest) number \( p \) such that \( \Phi \) is of lower (resp. upper) type \( p \).

We say that \( \Phi \in \mathcal{U}^q \) satisfies the Dini condition if there exists a constant \( C > 0 \) such that, for \( t > 0 \),
\[ \int_0^t \frac{\Phi(s)}{s^2} \, ds \leq C \frac{\Phi(t)}{t}. \]
We observe that if \( \Phi \) satisfies (13), then \( \Phi \) satisfies the \( \nabla_2 \)-condition.

We will also make use of the following properties of growth functions established in section 2 of [11]. We recall them here for quick references.

**Proposition 2.3.** The following assertion holds:
\[ \Phi \in \mathcal{L}_p \text{ if and only if } \Phi^{-1} \in \mathcal{U}^{1/p}. \]

**Lemma 2.4.** Let \( \Phi_1 \in \mathcal{L}_p \) and \( \Phi_2 \in \mathcal{U}^q \), and \( \Psi_2 \) the complementary function of \( \Phi_2 \). Let \( \Phi \) be such that
\[ \Phi^{-1}(t) := \Phi_1^{-1}(t) \Psi_2^{-1}(t). \]

Then \( \Phi \in \mathcal{L}_r \) for some \( r \leq p \).
LEMA 2.5. Let $\Phi_1$ be a growth function and $\Phi_2 \in \mathcal{U}^q$, $\rho_i(t) = \frac{1}{(\Phi_i^{-1}(t))}$, and $\Psi_2$ the complementary of $\Phi_2$. Then, if
\[ \rho_\Phi := \rho_1 \rho_2, \]
we also have
\[ (14) \quad \Phi^{-1}(t) \simeq \Phi_1^{-1}(t) \Psi_2^{-1}(t) \]
and vice-versa.

LEMA 2.6. Let $\Phi_1$ and $\Phi_2$ be in $\mathcal{U}^q$, and $\Psi_2$ the complementary function of $\Phi_2$. Let $\Phi$ be such that $\Phi^{-1}(t) = \Phi_1^{-1}(t) \Psi_2^{-1}(t)$. We suppose that $\Phi_2$ satisfies the Dini condition (13) and that the function $\frac{\Phi_2^{-1} \circ \Phi_1(t)}{t}$ is non-increasing.

Then $\Phi \in L_p$ for some $p > 0$.

The following can be adapted from [14].

PROPOSITION 2.7. For $\Phi_1$ and $\Phi_2$ two growth functions of lower type, $\alpha > -1$, the bilinear map $(f, g) \mapsto fg$ sends $L_{\Phi_1}^{\Phi} \times L_{\Phi_2}^{\Phi}$ onto $L_{\Phi}^{\Phi}$, with the inverse mappings of $\Phi_1, \Phi_2$ and $\Phi$ related by
\[ (15) \quad \Phi^{-1} = \Phi_1^{-1} \times \Phi_2^{-1}. \]
Moreover, there exists some constant $c$ such that
\[ \|fg\|_{L_{\Phi}^{\Phi}} \leq c \|f\|_{L_{\Phi_1}^{\Phi}} \|g\|_{L_{\Phi_2}^{\Phi}}. \]

2.2. Boundedness of the Bergman projection. We start by recalling the following result in [1].

PROPOSITION 2.8. Let $\alpha > -1$, there exists a constant $C > 0$ such that for $f \in \mathcal{A}_\alpha^1(\mathbb{B}^n)$
\[ \nu_\alpha(\{z \in \mathbb{B}^n : |P_\alpha f(z)| > \lambda\}) \leq C \frac{\|f\|_{1,\alpha}}{\lambda}. \]

The next result follows from interpolation with Orlicz functions (see [4]).

PROPOSITION 2.9. Let $\alpha > -1$ and $\Phi \in \mathcal{U}^q$. Suppose that $\Phi$ satisfies the $\nabla_2$–condition. Then the Bergman projection $P_\alpha$ extends into a bounded operator on $L_{p}^{\Phi}(\mathbb{B}^n)$.

From the duality result in [3], since $P_\alpha$ is bounded in $L_{p}^{\Phi}$ for $\Phi \in \mathcal{U}^q$ that satisfies the $\nabla_2$–condition, we obtain the following duality result in this case.

PROPOSITION 2.10. Let $\Phi \in \mathcal{U}^q$ and $\alpha > -1$. Suppose that $\Phi$ satisfies the $\nabla_2$–condition and denote by $\Psi$ its complementary function. Then the dual space $(\mathcal{A}_\alpha^{\Phi}(\mathbb{B}^n))^*$ of $\mathcal{A}_\alpha^{\Phi}(\mathbb{B}^n)$ identifies with $\mathcal{A}_\alpha^{\Psi}(\mathbb{B}^n)$ under the integral pairing
\[ (f, g)_\alpha = \int_{\mathbb{B}^n} f(z) \overline{g(z)} d\nu_\alpha(z), \quad f \in \mathcal{A}_\alpha^{\Phi}(\mathbb{B}^n), \quad g \in \mathcal{A}_\alpha^{\Psi}(\mathbb{B}^n). \]
2.3. Lipschitz-type spaces. We consider in this subsection some weighted Lipschitz spaces and their logarithmic counterparts. We recall that given an analytic function $f$ on $\mathbb{B}^n$, the radial derivative $Rf$ of $f$ is defined by

$$RF(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

For $\alpha > -1$ and a weight $\rho$ (of upper type $\gamma$), we recall that the weighted Lipschitz space $\Gamma_{\alpha,\rho}(\mathbb{B}^n)$ has been defined as the space of holomorphic functions $f$ in $\mathbb{B}^n$ such that, for some integer $k > \gamma(n+1+\alpha)$ and a positive constant $C > 0$, we have

$$|RF(z)| \leq C(1 - |z|^2)^{-k}\rho((1 - |z|^2)^{n+1+\alpha}).$$

We will also need a logarithmic version of the above space, $L\Gamma_{\alpha,\rho}(\mathbb{B}^n)$, defined as the space of holomorphic functions $f$ in $\mathbb{B}^n$ such that, for some $k > \gamma(n+1+\alpha)$ and a positive constant $C > 0$, we have

$$|RF(z)| \leq C(1 - |z|^2)^{-k}\rho((1 - |z|^2)^{n+1+\alpha}) \left(\log \frac{1}{1 - |z|^2}\right)^{-1}.$$  

One can show that, as in the classical Lipschitz spaces, these spaces are independent of $k$. We prove this in the following proposition.

**Proposition 2.11.** Let $\rho$ be a weight satisfying (7). The weighted (resp. logarithmic weighted) Lipschitz spaces $\Gamma_{\alpha,\rho}(\mathbb{B}^n)$ (resp. $L\Gamma_{\alpha,\rho}(\mathbb{B}^n)$) are independent of various values of $k$.

**Proof.** Let us provide a proof for $\Gamma_{\alpha,\rho}(\mathbb{B}^n)$, the proof for $L\Gamma_{\alpha,\rho}(\mathbb{B}^n)$ requires only few harmless modifications.

Let $f$ be an holomorphic function in $\mathbb{B}^n$. Let us first suppose that there is a constant $C > 0$ such that for some nonnegative integer $k > (n+1+\alpha)\gamma$ with $\gamma$ as in (9), and any $z \in \mathbb{B}^n$,

$$|RF(z)| \leq C(1 - |z|^2)^{-k}\rho((1 - |z|^2)^{n+1+\alpha}).$$  

We want to show that (16) holds for $k + 1$.

From (16), it follows in particular that as $\rho$ is increasing on $(0,1]$, the function $|RF(z)|(1 - |z|^2)^k$ is bounded on $\mathbb{B}^n$. Hence, for $\beta$ sufficiently large (in fact $\beta > k - \alpha - 1$ will do), we have the representation

$$RF(z) = C_{\alpha\beta} \int_{\mathbb{B}^n} \frac{RF(w)(1 - |w|^2)^\beta}{(1 - \langle z, w \rangle)^{n+1+\beta+\alpha}} d\nu_\alpha(w).$$

Thus

$$|RF^{k+1}| = \left| C_{\alpha\beta} \int_{\mathbb{B}^n} \frac{(n+1+\beta+\alpha)\langle z, w \rangle RF(w)(1 - |w|^2)^\beta}{(1 - \langle z, w \rangle)^{n+2+\beta+\alpha}} d\nu_\alpha(w) \right|$$

$$\leq \int_{\mathbb{B}^n} \frac{|RF(w)|(1 - |w|^2)^\beta}{|1 - \langle z, w \rangle|^{n+2+\beta+\alpha}} d\nu_\alpha(w)$$

$$\leq \int_{\mathbb{B}^n} \rho((1 - |w|^2)^{n+1+\alpha}) \frac{(1 - |w|^2)^{\beta-k}}{|1 - \langle z, w \rangle|^{n+2+\beta+\alpha}} d\nu_\alpha(w)$$

$$= I_1 + I_2,$$
where

\[
I_1 = \int_{1-|w|^2 \leq 1-|z|^2} \rho \left( (1 - |w|^2)^{n+1+\alpha} \right) \left( 1 - |w|^2 \right)^{\beta-k} \, d\nu_\alpha(w)
\]

and

\[
I_2 = \int_{1-|w|^2 > 1-|z|^2} \rho \left( (1 - |w|^2)^{n+1+\alpha} \right) \left( 1 - |w|^2 \right)^{\beta-k} \, d\nu_\alpha(w).
\]

Let us start by estimating the integral \( I_1 \). Using the monotonicity of the weight \( \rho \) and \cite{9} Proposition 1.4.10, we obtain

\[
I_1 = \int_{1-|w|^2 \leq 1-|z|^2} \rho \left( (1 - |w|^2)^{n+1+\alpha} \right) \left( 1 - |w|^2 \right)^{\beta-k} \, d\nu_\alpha(w)
\]

\[
\lesssim \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) \int_{1-|w|^2 \leq 1-|z|^2} \left( 1 - |w|^2 \right)^{\beta-k} \, d\nu_\alpha(w)
\]

\[
\lesssim (1 - |z|^2)^{-k-1} \rho \left( (1 - |z|^2)^{n+1+\alpha} \right).
\]

To handle the integral \( I_2 \), we use \cite{9} and \cite{9} Proposition 1.4.10 once more, and we obtain

\[
I_2 = \int_{1-|w|^2 > 1-|z|^2} \rho \left( (1 - |w|^2)^{n+1+\alpha} \right) \left( 1 - |w|^2 \right)^{\beta-k} \, d\nu_\alpha(w)
\]

\[
\leq C \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) \int_{\mathbb{B}^n} \frac{\left( 1 - |w|^2 \right)^{\beta-k+(n+1+\alpha)\gamma}}{|1 - \langle z, w \rangle|^{n+2+\beta+\alpha}} \, d\nu_\alpha(w)
\]

\[
\leq C \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) \times (1 - |z|^2)^{-k-1+(n+1+\alpha)\gamma}
\]

\[
= C(1 - |z|^2)^{-k-1} \rho \left( (1 - |z|^2)^{n+1+\alpha} \right).
\]

We conclude that there is also a constant \( C > 0 \) such that for any \( z \in \mathbb{B}^n \),

\[
|R^{k+1}f(z)| \leq C(1 - |z|^2)^{-k-1} \rho \left( (1 - |z|^2)^{n+1+\alpha} \right).
\]

Now, let us suppose that there is a constant \( C > 0 \) such that for some integer \( k > (n + 1 + \alpha)\gamma \) with \( \gamma \) being the upper-type of \( \rho \), and any \( z \in \mathbb{B}^n \),

\[
|R^{k+1}f(z)| \leq C(1 - |z|^2)^{-k-1} \rho \left( (1 - |z|^2)^{n+1+\alpha} \right).
\]

We will show that this implies that we can find a constant \( \tilde{C} > 0 \) such that for any \( z \in \mathbb{B}^n \),

\[
|R^k f(z)| \leq \tilde{C}(1 - |z|^2)^{-k} \rho \left( (1 - |z|^2)^{n+1+\alpha} \right).
\]

We start by the following lemma.

**Lemma 2.12.** Let \( \rho \) be a weight. Then for any integer \( k > (n + 1 + \alpha)\gamma \), there is a constant \( \tilde{C} \) depending on \( n, \alpha, C, \gamma \) where \( C \) and \( \gamma \) are as in \cite{9}, such that for any \( 0 < t < 1 \),

\[
\int_t^1 \frac{\rho(s^{n+1+\alpha})}{s^{k+1}} \, ds \leq \tilde{C} \rho \left( t^{n+1+\alpha} \right) \tilde{C}^{k}.
\]
Proof. Let $N$ be the smallest integer such that $t_{2}^{N+1} \geq 1$. Then we have using the monotonicity of the function $\rho$ and (9) that

$$\int_{t}^{1} \frac{\rho(s^{n+1+\alpha})}{s^{k+1}} ds \leq \sum_{l=0}^{N} \int_{t_{2}^{l}}^{t_{2}^{l+1}} \frac{\rho(s^{n+1+\alpha})}{s^{k+1}} ds$$

$$\leq \sum_{l=0}^{N} \frac{\rho(t^{n+1+\alpha}2^{(l+1)(n+1+\alpha)}k_{l}^{2k}}{t^{k+2l(k+1)}} \cdot 2^{l}t$$

$$= \sum_{l=0}^{N} \frac{\rho(t^{n+1+\alpha}2^{(l+1)(n+1+\alpha)})}{t^{k+2l(k+1)}} \cdot 2^{l}$$

$$\leq C \cdot 2^{(n+1+\alpha)\gamma} \sum_{l=0}^{N} \frac{\rho(t^{n+1+\alpha})}{t^{k}} \cdot 2^{-l(n+1+\alpha)\gamma}$$

$$= C \cdot 2^{(n+1+\alpha)\gamma} \sum_{l=0}^{N} 2^{-l(n+1+\alpha)\gamma} \frac{\rho(t^{n+1+\alpha})}{t^{k}}$$

$$\leq \bar{C} \rho(t^{n+1+\alpha}) \frac{1}{t^{k}}.$$

The proof of the lemma is complete. □

Coming back to our proof, we first consider the case of $|z| > 1/2$, $z = r\xi$, $\xi \in S^{n}$. We recall the following identity for any holomorphic function $g$:

$$g(z) - g(r\xi/2) = \int_{1/2}^{1} \frac{Rg(tz)}{t} dt.$$  \hspace{1cm} (19)

Applying (19) with $g = R^{k}f$, we obtain using (18) that for $|z| > 1/2$,

$$|R^{k}f(z) - R^{k}f(r\xi/2)| \leq \int_{1/2}^{1} \frac{|R^{k+1}g(tz)|}{t} dt$$

$$\leq C \int_{1/2}^{1} (1 - t|z|)^{-k-1} \rho \left((1 - t|z|)^{n+1+\alpha}\right) dt$$

$$\leq C \int_{1-|z|}^{1} \frac{\rho(s^{n+1+\alpha})}{s^{k+1}} ds$$

$$\leq C_{1} (1 - |z|^{2})^{-k} \rho \left((1 - |z|^{2})^{n+1+\alpha}\right).$$

Hence for $|z| > 1/2$,

$$|R^{k}f(z)| \leq S(1/2) + C_{1} (1 - |z|^{2})^{-k} \rho \left((1 - |z|^{2})^{n+1+\alpha}\right)$$

with

$$S(1/2) = \max_{|z| \leq 1/2} |R^{k}f(z)|.$$
Next for $|z| \leq 1/2$, applying the mean value property to $R^k f(z) - R^k f(0)$, and (17), we obtain
\[
\max_{|z| \leq 1/2} |R^k f(z)|^2 = \max_{|z| \leq 1/2} |R^k f(z) - R^k f(0)|^2
\leq 4^n \int_{|w| \leq 3/4} |R^k f(w) - R^k f(0)|^2 d\nu(w)
\leq 4^n \int_{|w| \leq 3/4} |R^{k+1} f(w)|^2 d\nu(w)
\leq 3^n \max_{|z| \leq 3/4} |R^{k+1} f(z)|^2
\leq 3^n C^2 \max_{|z| \leq 3/4} (1 - |z|^2)^{-2k} \rho((1 - |z|^2)^{n+1+\alpha})^2
\]
(20)
\[
\leq 3^n C^2 \rho(1)^2.
\]
Hence using the latter and the fact that if $C_0$ is the constant in (9), then for $z \in \mathbb{B}^n$,
\[
(1 - |z|^2)^{-k} \rho((1 - |z|^2)^{n+1+\alpha}) \geq (1 - |z|^2)^{-(n+1+\alpha)} \rho((1 - |z|^2)^{n+1+\alpha}) \geq \frac{\rho(1)}{C_0},
\]
we obtain
\[
S(1/2) = \max_{|z| \leq 1/2} |R^k f(z)| \leq \left(\sqrt{3}^n C\right) C_0 (1 - |z|^2)^{-k} \rho((1 - |z|^2)^{n+1+\alpha}) .
\]
Thus for $|z| > 1/2$,
\[
|R^k f(z)| \leq C_1 (1 - |z|^2)^{-k} \rho((1 - |z|^2)^{n+1+\alpha}) .
\]
For $|z| \leq 1/2$, using (20) and (21) we have that for $|z| \leq 1/2$,
\[
|R^k f(z)| \lesssim C_2 (1 - |z|^2)^{-k} \rho((1 - |z|^2)^{n+1+\alpha}) .
\]
Thus taking $\tilde{C} = \max\{C_1, C_2\}$, we obtain that for any $z \in \mathbb{B}^n$,
\[
|R^k f(z)| \leq \tilde{C} (1 - |z|^2)^{-k} \rho((1 - |z|^2)^{n+1+\alpha}) .
\]
The proof is complete. \(\square\)

As a consequence, the spaces, $\Gamma_{\alpha,\rho}(\mathbb{B}^n)$, $L\Gamma_{\alpha,\rho}(\mathbb{B}^n)$ become Banach spaces under the following norms
\[
\|f\|_{\Gamma_{\alpha,\rho}(\mathbb{B}^n)} = |f(0)| + \sup_{z \in \mathbb{B}^n} \frac{|R^k f(z)|(1 - |z|^2)^k}{\rho((1 - |z|^2)^{n+1+\alpha})},
\]
\[
\|f\|_{L\Gamma_{\alpha,\rho}(\mathbb{B}^n)} = |f(0)| + \sup_{z \in \mathbb{B}^n} \frac{|R^k f(z)|(1 - |z|^2)^k}{\rho((1 - |z|^2)^{n+1+\alpha}) |\log(1 - |z|^2)|},
\]
where $k$ is a fixed integer strictly greater than $\gamma(n + 1 + \alpha)$.

We will show that the space $\Gamma_{\alpha,\rho}(\mathbb{B}^n)$ is the topological dual space of $A_0^\Phi(\mathbb{B}^n)$ when $\rho$ and $\Phi$ are related by
\[
\rho(t) := \frac{1}{t \Phi^{-1}(1/t)}.
\]
We recall the following fact from [13, Proposition 3.10].

**Proposition 2.13.** Let \( \Phi \in \mathcal{L}_p \) and \( \rho(t) := \frac{1}{(1 + t)^p} \), then \( \rho \) is a weight of upper type \( \frac{1}{p} - 1 \).

We will need the following differential operator of order \( k \):

\[
M_k^n = [(n + k + \alpha)I + R] \cdots [(n + 1 + \alpha)I + R]
\]

where \( I \) is the identity operator, \( k \in \mathbb{N}^* \). The following lemma can be proved using integration by parts.

**Lemma 2.14.** Let \( f, g \) be holomorphic polynomials on \( \mathbb{B}^n \). Then the following equality holds

\[
\int_{\mathbb{B}^n} f(z) \overline{g}(z) d\nu(z) = C_{k,\alpha} \int_{\mathbb{B}^n} f(z) \overline{M_k^n g(z)} (1 - |z|^2)^k d\nu(z),
\]

where \( C_{k,\alpha} \) is a constant depending only on \( k \) and \( \alpha \).

The following lemma is Lemma 2.2 of [2].

**Lemma 2.15.** Let \( \{a_j\} \) be a sequence of positive numbers, and let \( L_k \) be the differential operator of order \( k \) defined by

\[
L_k := (a_0 I + R)(a_1 I + R) \cdots (a_k I + R).
\]

Then \( f \) belongs to \( \Gamma_\beta(\mathbb{B}^n) \) if and only if there exist and integer \( k > \beta \) and a positive constant \( C \) such that

\[
|L_k f(z)| \leq C(1 - |z|^2)^{\beta - k}.
\]

As remarked in [2], the equivalence in the above lemma also holds if we multiply the right hand side of the last inequality by a logarithmic terms. That is, for fixed \( t \in \mathbb{R} \),

\[
|R_k f(z)| \leq C(1 - |z|^2)^{\beta - k} |\log(1 - |z|^2)|^t
\]

if and only if

\[
|L_k f(z)| \leq C(1 - |z|^2)^{\beta - k} |\log(1 - |z|^2)|^t.
\]

Using the same kind of techniques as in Proposition 2.11 we can also show that for a weight \( \rho \) of upper type \( \gamma \), \( f \) belongs to \( \Gamma_{\alpha,\rho}(\mathbb{B}^n) \) if and only if there exist an integer \( k > \gamma(n + 1 + \alpha) \) and a positive constant \( C \) such that

\[
|L_k f(z)| \leq C(1 - |z|^2)^{-k} \rho ((1 - |z|^2)^{n+1+\alpha})
\]

The same is true for \( L\Gamma_{\alpha,\rho}(\mathbb{B}^n) \).

As a consequence of this fact, we have that

\[
\|f\|_{\Gamma_{\alpha,\rho}(\mathbb{B}^n)} \leq |f(0)| + \sup_{z \in \mathbb{B}^n} \frac{|M_k^n f(z)|(1 - |z|^2)^k}{\rho ((1 - |z|^2)^{n+1+\alpha})}
\]

and

\[
\|f\|_{L\Gamma_{\alpha,\rho}(\mathbb{B}^n)} \leq |f(0)| + \sup_{z \in \mathbb{B}^n} \frac{|M_k^n f(z)|(1 - |z|^2)^k}{\rho ((1 - |z|^2)^{n+1+\alpha}) |\log(1 - |z|^2)|}
\]

where \( k \) is a fixed integer strictly greater than \( \gamma(n + 1 + \alpha) \).
2.4. Some useful estimates. The next proposition gives pointwise estimates for functions in $A^p_\alpha(\mathbb{B}^n)$, $\Phi \in \mathcal{L}_p$.

**Lemma 2.16.** Let $\Phi \in \mathcal{L}_p$ and $\alpha > -1$. There is a constant $C > 1$ such that for any $f \in A^\phi_\alpha(\mathbb{B}^n)$,

$$|f(z)| \leq C \Phi^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right) \|f\|_{lux}^{\Phi}, \quad (26)$$

Proof. Let $f \in A^\phi_\alpha(\mathbb{B}^n)$. Note that if $\|f\|_{lux}^{\Phi} = 0$, then $f = 0$ a.e and consequently, we obviously have (26). Let us assume that $\|f\|_{lux}^{\Phi} \neq 0$, and let $\lambda > 0$ such that $\int_{\mathbb{B}^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) d\nu_{\alpha}(z) \leq 1$. Using the fact that $\Phi_p(t) := \Phi(t^{1/p})$ is convex (see Remark 2.2), that $\frac{\|f\|}{\lambda}$ is subharmonic, and that the measures $(\frac{1 - |z|^2}{1 - \langle z, w \rangle^2})^{n+1+\alpha} d\nu_{\alpha}(w)$ are probability measures (see [9]), we obtain, for $z \in \mathbb{B}^n$,

$$\Phi_p\left(\frac{|f(z)|^p}{\lambda^p}\right) \leq \int_{\mathbb{B}^n} \Phi_p\left(\frac{|f(w)|^p}{\lambda^p}\right) \left(\frac{1 - |z|^2}{1 - \langle z, w \rangle^2}\right)^{n+1+\alpha} d\nu_{\alpha}(w)$$

$$\leq \left(\frac{4}{1 - |z|^2}\right)^{n+1+\alpha} \int_{\mathbb{B}^n} \Phi\left(\frac{|f(w)|}{\lambda}\right) d\nu_{\alpha}(w)$$

$$\leq C \left(\frac{1}{1 - |z|^2}\right)^{n+1+\alpha}.$$

Hence, for $z \in \mathbb{B}^n$, we have

$$|f(z)|^p \leq C \lambda^p \left(\Phi_p^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right)\right)^p \Phi^{-1}\left(\frac{1}{(1 - |z|^2)^{n+1+\alpha}}\right)^p.$$

From this, we have (26). \hfill \Box

We also provide norm estimates for bounded functions in $A^\phi_\alpha(\mathbb{B}^n)$. These are extension of the same type of result in [6, Lemma 3.9].

**Lemma 2.17.** Let $\alpha > -1$ and $\Phi \in \mathcal{L}_p$. For any bounded holomorphic function $f$ in $\mathbb{B}^n$, one has:

$$\|f\|_{lux}^{\Phi_\alpha} \leq \frac{\|f\|_{\infty}}{\Phi^{-1}\left(\|f\|_{\infty}\right)} \Phi^{-1}\left(\|f\|_{\infty}\right). \quad (27)$$

Proof. The proof follows exactly as in [6] where we use instead the fact that $\Phi_p(t) := \Phi(t^{1/p})$ is convex (see Remark 2.2). \hfill \Box

We also get the following estimates for bounded holomorphic functions in $A^\phi_\alpha(\mathbb{B}^n)$, when $\Phi \in \mathcal{B}^q$. The proof follows exactly as in [6].

**Lemma 2.18.** Let $\alpha > -1$ and $\Phi \in \mathcal{B}^q$. Let $0 < s < \infty$. Put $\Phi_{\lambda}(t) = \Phi(t^s)$ and $\Phi^s(t) = (\Phi(t))^s$. For any bounded holomorphic function $f$ in $\mathbb{B}^n$, one has:

$$\|f\|_{lux}^{\Phi_{\lambda, \Phi^s}} \leq \frac{\|f\|_{\infty}}{\Phi^{-1}\left(\|f\|_{\infty}\right)} \Phi^{-1}\left(\|f\|_{\infty}\right). \quad (28)$$
and

\[
\|f\|_{\alpha, \Phi}^{lux} \leq \left( \frac{||f||_s}{\Phi^{-1} \left( \frac{||f||_s}{\|f\|_{s,\alpha}} \right)} \right)^{1/s}.
\]

3. Duality for large Bergman-Orlicz spaces

The following Lemma generalizes the inequality (11). As in the classical weighted Bergman spaces, this Lemma is crucial to characterize the dual space \((A^\Phi_{\alpha}(\mathbb{B}^n))^*\) of \(A^\Phi_{\alpha}(\mathbb{B}^n)\).

**Lemma 3.1.** Let \(\alpha > -1\), \(\Phi \in \mathcal{L}_p\) and \(\rho(t) = \frac{1}{\Phi^{-1}(1/t)}\). There is a constant \(C > 1\) such that for any \(f \in A^\Phi_{\alpha}(\mathbb{B}^n)\),

\[
\int_{\mathbb{B}^n} |f(z)| \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) d\nu_\alpha(z) \leq C \|f\|_{\alpha, \Phi}^{lux}.
\]

**Proof.** The idea of the proof is an adaptation of the proof in the classical Bergman spaces and make uses of the pointwise estimate of functions in \(A^\Phi_{\alpha}(\mathbb{B}^n)\). More precisely, let \(f \in A^\Phi_{\alpha}(\mathbb{B}^n)\) and \(\lambda > 0\) such that

\[
\int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\lambda} \right) d\nu_\alpha(z) \leq 1.
\]

We have

\[
\int_{\mathbb{B}^n} |f(z)| \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) d\nu_\alpha(z)
= \lambda \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\lambda} \right) \frac{|f(z)|}{\Phi \left( \frac{|f(z)|}{\lambda} \right)} \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) d\nu_\alpha(z).
\]

Using (26), we have that, for \(z \in \mathbb{B}^n\),

\[
\frac{|f(z)|}{\lambda} \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) \|f\|_{\alpha, \Phi}^{lux} \leq C \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right).
\]

From this, using the fact that \(t/\Phi(t)\) and \(\Phi\) are non-decreasing, we have

\[
\int_{\mathbb{B}^n} |f(z)| \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) d\nu_\alpha(z)
\leq C \lambda \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\lambda} \right) \Phi^{-1} \left( \frac{1}{(1 - |z|^2)^{n+1+\alpha}} \right) \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) d\nu_\alpha(z).
\]

The definition of \(\rho\) leads to

\[
\int_{\mathbb{B}^n} |f(z)| \rho \left( (1 - |z|^2)^{n+1+\alpha} \right) d\nu_\alpha(z) \leq C \lambda \int_{\mathbb{B}^n} \Phi \left( \frac{|f(z)|}{\lambda} \right) d\nu_\alpha(z) \leq C \lambda.
\]

The proof is complete. \(\square\)

We next prove the duality result which also extends the classical duality for Bergman spaces with small exponents.
Proof of Theorem 1.1. First using Lemma 2.14 and Lemma 3.1 we obtain that, for any \( f \in \mathcal{A}_0^\Phi(B^n) \) and \( g \in \mathcal{F}_{\alpha,\rho}(B^n) \),
\[
|\langle f, g \rangle^\alpha| := \lim_{r \to 1} \left| \int_{B^n} f(rz)\overline{g(z)}d\nu_\alpha(z) \right|
\leq C_{k,\alpha} \lim_{r \to 1} \int_{B^n} |f(rz)|M_k^\alpha g(z)|(1-|z|^2)^k d\nu_\alpha(z)
\leq C \lim_{r \to 1} \int_{B^n} |f(rz)|\rho((1-|z|^2)^{n+1+\alpha})d\nu_\alpha(z)
\leq C \|f\|_{1,\alpha,\Phi}^\alpha.
\]
That is any \( g \in \mathcal{F}_{\alpha,\rho}(B^n) \) defines a linear form on \( \mathcal{A}_0^\Phi(B^n) \) under the pairing (10).

Now that any linear form on \( \mathcal{A}_0^\Phi(B^n) \) is given by (10) can be justified by (4) and the duality result for the usual weighted Bergman spaces (see (10)). Thus we finish by proving that any element \( g \in (\mathcal{A}_0^\Phi(B^n))^* \) belongs to \( \mathcal{F}_{\alpha,\rho}(B^n) \). Let \( a \in B^n \). We will test (10) with the function
\[
f_a(z) = \Phi^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right)(1-|z|^2)_k n+1+\alpha+k
\]
where \( k \) is a fixed integer satisfying \( k > (n+1+\alpha)(\frac{1}{p} - 1) \).

Using Lemma 2.17 and Forelli-Rudin estimates (see for example [9, Proposition 1.4.10]), we see that \( f_a \) is uniformly in \( A_0^\Phi(B^n) \). That is there exists a constant \( C \), independant of \( a \), such that \( \|f_a\|_{\alpha,\Phi} \leq C \). We then get
\[
C \geq |\langle f_a, g \rangle^\alpha|
= \lim_{r \to 1} \Phi^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right)\left(1-|z|^2\right)^{n+1+\alpha+k}\int_{B^n} \frac{g(z)}{(1-\langle rz, a \rangle)^{n+1+\alpha+k}}d\nu_\alpha(z)
= \lim_{r \to 1} \frac{(1-|z|^2)^k}{\rho((1-|z|^2)^{n+1+\alpha})}\int_{B^n} \frac{g(z)}{(1-\langle rz, a \rangle)^{n+1+\alpha+k}}d\nu_\alpha(z)
= \frac{1}{C_{k,\alpha} \rho((1-|z|^2)^{n+1+\alpha})}\lim_{r \to 1} \|M_k^\alpha g(ra)\|.
\]
So, there exists a constant \( C \) such that for any \( a \in B^n \),
\[
\frac{(1-|z|^2)^k}{\rho((1-|z|^2)^{n+1+\alpha})}\|M_k^\alpha g(a)\| \leq C.
\]
This is equivalent to the fact that \( g \in \mathcal{F}_{\alpha,\rho}(B^n) \). The proof is complete. \( \square \)

4. Hankel operators between Bergman-Orlicz spaces

We have gathered the results we need to study the boundedness of the small Hankel Operators between Bergman-Orlicz spaces.

4.1. Boundedness of \( h_b : \mathcal{A}_0^\Phi(B^n) \to \mathcal{A}_0^\Phi(B^n) \), \( \Phi \in \mathcal{M}^q \). In this subsection, we consider boundedness of Hankel operators \( h_b \) on the Bergman-Orlicz spaces \( \mathcal{A}_0^\Phi(B^n) \) for \( \Phi \) a convex growth function in the class \( \mathcal{M}^q \). We start by considering a general weak factorization of weighted Bergman spaces with small exponents.
\textbf{Proposition 4.1.} Let $\alpha > -1$, $0 < p \leq 1$ and $\Phi \in \mathcal{U}^q$. Denote by $\Psi$ the complementary function of $\Phi$. If we define $\Phi_p$ by $\Phi_p(t) = \Phi(t^p)$, then every function $f \in \mathcal{A}^{p,\Phi}_{\alpha}(\mathbb{B}^n)$ admits the following representation

\begin{equation}
 f(z) = \sum_j f_j(z)g_j(z), \quad z \in \mathbb{B}^n,
 \end{equation}

where each $f_j$ is in $\mathcal{A}^{\Phi_p\Phi}_{\alpha}(\mathbb{B}^n)$ and each $g_j$ is in $\mathcal{A}^{\Psi_p\Psi}_{\alpha}(\mathbb{B}^n)$. Moreover, we have

$$
\sum_j \|f_j\|_{l^{ux}_{\alpha,\Phi_p}} \|g_j\|_{l^{ux}_{\alpha,\Psi_p}} \leq C\|f\|_{p,\alpha},
$$

where $C$ is a positive constant independent of $f$.

\textbf{Proof.} First, let us recall with [16, Theorem 2.30] that, for $b > (n+1+\alpha)/p$, there exists a sequence $\{a_j\}$ in $\mathbb{B}^n$ such that every $f \in \mathcal{A}^{p}_\alpha(\mathbb{B}^n)$ admits the following representation

$$
f(z) = \sum_j c_j \left( \frac{(1 - |a_j|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_j \rangle)^b} \right)^{p/q},
$$

where $\{c_j\}$ belongs to the sequence space $l^p$ and the series converges in the norm topology of $\mathcal{A}^{p}_\alpha(\mathbb{B}^n)$. Now take, for non zero $c_j$,

$$
f_j(z) = c_j^{p/q} \left( \frac{(1 - |a_j|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_j \rangle)^b} \right)^{p/q},
$$

and

$$
g_j(z) = c_j^{p/r} \left( \frac{(1 - |a_j|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_j \rangle)^b} \right)^{p/r},
$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. It is clear that (31) holds. Using Lemma 2.18 and Forelli-Rudin estimates (see for example [9, Proposition 1.4.10]) with $b$ large enough, we have

\begin{equation}
\|f_j\|_{l^{ux}_{\alpha,\Phi_p}} \leq C|c_j|^{p/q} \left( \frac{(1 - |a_j|^2)^{-(n+1+\alpha)/q}}{\Phi^{-1} \left( \frac{1}{(1 - |a_j|^2)^{n+1+\alpha}} \right)^{1/p}} \right)
\end{equation}

and

\begin{equation}
\|g_j\|_{l^{ux}_{\alpha,\Psi_p}} \leq C|c_j|^{p/q} \left( \frac{(1 - |a_j|^2)^{-(n+1+\alpha)/r}}{\Psi^{-1} \left( \frac{1}{(1 - |a_j|^2)^{n+1+\alpha}} \right)^{1/p}} \right)
\end{equation}
Now, using (32), (33) and the fact that $\Phi^{-1}(t)\Psi^{-1}(t) \leq t$, we have

$$
\left( \sum_j |f_j|^{p/q} |g_j|^{p/r} \right)^p \leq C \left( \sum_j |c_j|^{p/q} \right)^p
$$

Thus, we have (34) and this complete the proof. □

Using the weak factorization with $p = 1$, we can know show that, as in the classical case, for convex growth function $\Phi$, the small Hankel operator on $A^p_\alpha(B^n)$ is bounded if and only if the symbol lies in the Bloch space.

**Theorem 4.2.** Let $\Phi \in \mathcal{W}_q$ such that $\Phi$ satisfies the Dini condition (13), and $\alpha > -1$. Then the Hankel operator $h_b$ extends into a bounded operator on $A^p_\alpha(B^n)$ if and only if $b \in B$.

**Proof.** First we recall that since $\Phi$ satisfies (13), then the dual of $A^p_\alpha(B^n)$ coincides with $A^p_\alpha(B^n)$ with $\Psi$ the complementary function of $\Phi$. Since $\Phi^{-1}(t)\Psi^{-1}(t) \simeq t$, by Proposition 2.7 for any $f \in A^p_\alpha(B^n)$ and $g \in A^p_\alpha(B^n)$, the product $fg$ is in $A^p_\alpha(B^n)$ which has dual space the Bloch space $B$. Consequently, there is a constant $C > 0$ such that, for $f \in A^p_\alpha(B^n)$ and $g \in A^p_\alpha(B^n)$

$$
|\langle h_b(f), g \rangle_\alpha| = |\langle b, fg \rangle_\alpha| \leq C \|b\|_B \|fg\|_{1,\alpha} \leq C \|b\|_B \|f\|_{\alpha,\Psi} \|g\|_{\alpha,\Psi}.
$$

We conclude that if $b \in B$, then $h_b$ is bounded from $A^p_\alpha(B^n)$ into itself with $\|h_b\| \leq C \|b\|_B$.

Conversely, we suppose that $h_b$ extends into a bounded operator on $A^p_\alpha(B^n)$. To prove that the symbol $b \in B$, we only need to prove that there is a constant $C > 0$ such that for any $f \in A^1_\alpha(B^n)$,

$$
|\langle b, f \rangle_\alpha| \leq C \|f\|_{1,\alpha}.
$$

From Proposition 4.1, we have that any $f \in A^1_\alpha(B^n)$ can be written as $f = \sum_j f_j g_j$ with $\sum_j \|f_j\|_{\alpha,\Psi} \|g_j\|_{\alpha,\Psi} \leq C \|f\|_{1,\alpha}$. It follows that

$$
|\langle b, f \rangle_\alpha| \leq \sum_j |\langle b, f_j g_j \rangle_\alpha| = \sum_j |\langle h_b(f_j), g_j \rangle_\alpha| \leq \sum_j \|h_b(f_j)\|_{\alpha,\Psi} \|g_j\|_{\alpha,\Psi} \leq \|h_b\| \sum_j \|f_j\|_{\alpha,\Psi} \|g_j\|_{\alpha,\Psi} \leq C \|h_b\| \|f\|_{1,\alpha}.
$$

Thus, we have (34) and this complete the proof. □
4.2. Boundedness of \( h_b : A^\Phi_\alpha(\mathbb{B}^n) \to A^\Phi_\beta(\mathbb{B}^n) \); \( \Phi_1, \Phi_2 \) ∈ \( \mathcal{W}^q \times \mathcal{L}_p \). We start this section by defining the space \( A^\Phi_{\alpha,\beta}(\mathbb{B}^n) \).

**DEFINITION 4.3.** The space \( A^\Phi_{\alpha,\beta}(\mathbb{B}^n) \) consists of all holomorphic functions \( f \) such that, for some constant \( C > 0 \), we have

\[
\lambda |\nu_\alpha((\{z \in \mathbb{B}^n : |f(z)| > \lambda\})| \leq C \text{ for any } \lambda > 0.
\]

This becomes a Banach space under the following norm

\[
||f||_{1,\alpha,\beta} = \sup_{\lambda > 0} \lambda |\nu_\alpha((\{z \in \mathbb{B}^n : |f(z)| > \lambda\})|.
\]

We observe that from the above definition and Proposition 2.8 we have that the Bergman projection \( P_\alpha \) is bounded from \( L^1_{\alpha,\beta}(\mathbb{B}^n) \) to \( A^\Phi_{\alpha,\beta}(\mathbb{B}^n) \).

We have the following embedding result.

**PROPOSITION 4.4.** Let \( \Phi \in \mathcal{L}_p, \alpha > -1 \). Suppose that \( \Phi \) satisfies the Dini’s condition

\[
(35) \quad \int_1^\infty \frac{\Phi(t)}{t^2} dt < \infty.
\]

Then \( A^\Phi_{\alpha,\beta}(\mathbb{B}^n) \) embeds continuously in \( A^\Phi_\alpha(\mathbb{B}^n) \).

**Proof.** It is enough to prove that for any \( f \in A^\Phi_{\alpha,\beta}(\mathbb{B}^n) \), there exists \( C > 0 \) such that

\[
\int_{\mathbb{B}^n} \Phi(|f|(z)) d\nu_\alpha(z) \leq C.
\]

We have

\[
\int_{\mathbb{B}^n} \Phi(|f|(z)) d\nu_\alpha(z) = \int_0^{\infty} \nu_\alpha((\{z \in \mathbb{B}^n : |f(z)| > \lambda\}) \Phi'(\lambda) d\lambda = I + J,
\]

where

\[
I = \int_0^1 \nu_\alpha((\{z \in \mathbb{B}^n : |f(z)| > \lambda\}) \Phi'(\lambda) d\lambda
\]

and

\[
J = \int_1^{\infty} \nu_\alpha((\{z \in \mathbb{B}^n : |f(z)| > \lambda\}) \Phi'(\lambda) d\lambda.
\]

It is clear that

\[
I \leq \nu_\alpha(\mathbb{B}^n) \int_0^1 \Phi'(\lambda) d\lambda = \Phi(1).
\]

To estimate the integral \( J \), we use the definition of \( A^\Phi_{\alpha,\beta}(\mathbb{B}^n) \), the fact that \( \Phi'(t) = \frac{\Phi(t)}{t} \) and \( \Phi \) satisfies the Dini’s condition \( (35) \). We obtain

\[
J = \int_1^{\infty} \nu_\alpha((\{z \in \mathbb{B}^n : |f(z)| > \lambda\}) \Phi'(\lambda) d\lambda
\]

\[
\leq ||f||_{1,\alpha,\beta} \int_1^{\infty} \frac{\Phi'(\lambda)}{\lambda} d\lambda \leq C ||f||_{1,\alpha,\beta} \int_1^{\infty} \frac{\Phi(\lambda)}{\lambda^2} d\lambda \leq C ||f||_{1,\alpha,\beta}.
\]

The proof is complete. \( \square \)

Under the Dini’s condition \( (35) \), we easily obtain the boundedness criteria for the small Hankel operator \( h_b \) from \( A^\Phi_\alpha(\mathbb{B}^n) \) into \( A^\Phi_\beta(\mathbb{B}^n) \) when \( (\Phi_1, \Phi_2) \in \mathcal{W}^q \times \mathcal{L}_p \). This is a generalization of the case \( h_b : A^\beta_\alpha(\mathbb{B}^n) \to A^\beta_\alpha(\mathbb{B}^n) \) with \( 1 < p < \infty \) and \( 0 < q < 1 \).
Theorem 4.5. Let $\Phi_1 \in \mathcal{B}_p$ and $\Phi_2 \in \mathcal{L}_p$, $\alpha > -1$. Let $\Psi_1$ be the complementary function of $\Phi_1$ and, suppose that $\Phi_1$ satisfies the Dini’s condition \( f(x) \) while $\Phi_2$ satisfies \( f(y) \). Then $h_b$ extends as a bounded operator from $A_{\alpha,1}^{\Phi_1}(B^n)$ to $A_{\alpha,2}^{\Phi_2}(B^n)$ if and only if $b \in A_{\alpha,1}^{\Phi_1}(B^n)$.

Proof. We start by proving the necessity. Suppose that $h_b$ is bounded from $A_{\alpha,1}^{\Phi_1}(B^n)$ to $A_{\alpha,2}^{\Phi_2}(B^n)$. Then for any $f \in A_{\alpha,1}^{\Phi_1}(B^n)$, we have

$$\left| \int_{B^n} b(\xi) f(\xi) d\nu_\alpha(\xi) \right| = |h_b f(0)| \leq C \|h_b f\|_2 \leq C \|f\|_2.$$

We have used the fact that $A_{\alpha,2}^{\Phi_2}(B^n)$ is continuously contained in $A_{\alpha,2}^{\Phi_1}(B^n)$, and the evaluation at 0 is bounded on this space. It follows that $b$ belongs to the dual space of $A_{\alpha,1}^{\Phi_1}(B^n)$ that is $b \in A_{\alpha,1}^{\Phi_1}(B^n)$.

Conversely, if $b \in A_{\alpha,1}^{\Phi_1}(B^n)$, then for any $f \in A_{\alpha,1}^{\Phi_1}(B^n)$, the product $b f$ is in $L_1(\mathbb{R}^n)$ by Proposition 4.4. Thus, $h_b(f) := P_{\alpha}(bf)$ is in $A_{\alpha,1}^{\Phi_1}(B^n)$ and consequently in $A_{\alpha,2}^{\Phi_2}(B^n)$ by Proposition 4.5. The proof is complete.

4.3. Boundedness of $h_b$: $A_{\alpha,1}^{\Phi_1}(B^n) \rightarrow A_{\alpha,2}^{\Phi_2}(B^n)$; $(\Phi_1, \Phi_2) \in \mathcal{L}_p \times \mathcal{L}_p$. Let us start this section by the following result.

Theorem 4.6. Let $\Phi_1 \in \mathcal{L}_p$, $\alpha > -1$ and $\Phi_2 \in \mathcal{L}_p \cup \mathcal{B}_p$. If $h_b$ extends into a bounded operator from $A_{\alpha,1}^{\Phi_1}(B^n)$ into $A_{\alpha,2}^{\Phi_2}(B^n)$, then the symbol $b$ belongs to $\Gamma_{\alpha,1}(B^n)$ with $\rho_1(t) = \frac{1}{t^{\Phi_1(\frac{1}{t})}}$. Conversely, if $b \in \Gamma_{\alpha,1}(B^n)$, then there exists a bounded operator from $A_{\alpha,1}^{\Phi_1}(B^n)$ into $L_1(\mathbb{R}^n)$ which we note $T_b$ such that $h_b = P_{\alpha} T_b$.

Proof. That the boundedness of $h_b$ from $A_{\alpha,1}^{\Phi_1}(B^n)$ into $A_{\alpha,2}^{\Phi_2}(B^n)$ implies that $b$ belongs to $\Gamma_{\alpha,1}(B^n)$ follows as in the first part of the proof of Theorem 4.5.

It is easy to check that $h_b = P_{\alpha} T_b$ (see for example 2) with

$$T_b f(z) = M_k^\alpha b(z)(1 - |z|^2)^k f(z),$$

where $k > (n + 1 + \alpha) \left( \frac{1}{\alpha} - 1 \right)$. Recalling that $b \in \Gamma_{\alpha,1}(B^n)$ is equivalent in saying that for some constant $C > 0$,

$$|M_k^\alpha b(z)(1 - |z|^2)^k| \leq C \rho_1((1 - |z|^2)^{n+1+\alpha}) \quad \text{for all } z \in \mathbb{B}^n,$$

and using Lemma 3.1 we easily get

$$\int_{\mathbb{B}^n} |T_b f(z)| d\nu_\alpha(z) \leq C \|b\|_{\Gamma_{\alpha,1}} \int_{\mathbb{B}^n} |f(z)| \rho_1((1 - |z|)^{n+1+\alpha}) d\nu_\alpha(z) \leq C \|b\|_{\Gamma_{\alpha,1}} \|f\|_{\alpha,1}.$$

As a corollary, we obtain

Corollary 4.7. Let $\Phi_1, \Phi_2 \in \mathcal{L}_p$, $\alpha > -1$. Suppose moreover that $\Phi_2$ satisfies the Dini condition \( f(y) \). Then $h_b$ extends into a bounded operator from $A_{\alpha,1}^{\Phi_1}(B^n)$ into $A_{\alpha,2}^{\Phi_2}(B^n)$ if and only if the symbol $b$ belongs to $\Gamma_{\alpha,1}(B^n)$ with $\rho_1(t) = \frac{1}{t^{\Phi_1(\frac{1}{t})}}$. 


Proof. First we observe that the necessity is given by Theorem 4.6. From the same theorem, we have that for any $f \in A^\Phi_1(\mathbb{B}^n)$, $T_b(f) \in L^1_\alpha(\mathbb{B}^n)$ with $T_b f(z) = M_b^\alpha f(z)(1 - |z|^2)^k f(z)$. As $h_b = P_\alpha T_b$, we obtain that $h_b(f) \in A^{\Phi}_\alpha(\mathbb{B}^n)$. The conclusion follows now from that as $\Phi_2$ satisfies (35), $A^{\Phi}_\alpha(\mathbb{B}^n)$ embeds continuously into $A^{\Phi}_\alpha(\mathbb{B}^n)$. \hfill $\Box$

4.4. Boundedness of $h_b$: $A^\Phi_\alpha(\mathbb{B}^n) \rightarrow A^{\Phi}_\alpha(\mathbb{B}^n)$; $\Phi \in L^p$. We start this section by observing that, for $\epsilon \in \mathbb{B}^n$, the following function

$$f_\epsilon(z) = \Phi^{-1}\left(\frac{1}{1 - |a|^2} \right)^{n+1+\alpha} \frac{1 - |a|^{2n+1+\alpha+k}}{(1 - \langle z, a \rangle)^{n+1+\alpha+k-\epsilon}}, \ z \in \mathbb{B}^n$$

is uniformly in $A^\Phi_\alpha(\mathbb{B}^n)$ for $k$ an integer such that $k > (n + 1 + \alpha)/(1/p) - 1$.

To see this recall that for any $\epsilon > 0$, there is a constant $C > 0$ such that $\log^+ x \leq C x^\epsilon$. It follows that

$$|f_\epsilon(z)| \leq C \Phi^{-1}\left(\frac{1}{1 - |a|^2} \right)^{n+1+\alpha} \frac{1 - |a|^{2n+1+\alpha+k}}{(1 - \langle z, a \rangle)^{n+1+\alpha+k-\epsilon}},$$

for $\epsilon$ small enough. We have also used the fact that, for $a, z \in \mathbb{B}^n$,

$$\left| 1 - \langle z, a \rangle \right| \geq 1/2.$$

Choosing $0 < \epsilon < k - (n + 1 + \alpha)/(1/p) - 1$, we write

$$\int_{\mathbb{B}^n} \Phi(|f_\epsilon(z)|) d\nu_\alpha(z) =$$

$$\int_{\frac{1 - |a|^2}{1 - \langle z, a \rangle} \leq 1} \Phi(|f_\epsilon(z)|) d\nu_\alpha(z) + \int_{\frac{1 - |a|^2}{1 - \langle z, a \rangle} > 1} \Phi(|f_\epsilon(z)|) d\nu_\alpha(z) = I_1 + I_2.$$
Proof. First, we suppose that the function $b \in L^{1}_{\Gamma, \rho}(\mathbb{B}^{n})$. That is $b \in \mathcal{H}(\mathbb{B}^{n})$ and satisfies, for some $k$ large enough, the condition

$$|R^{k}b(z)| \leq C(1 - |z|^{2})^{-k} \rho \left((1 - |z|^{2})^{n+1+\alpha}\right) \left(\log \frac{1}{1 - |z|^{2}}\right)^{-1},$$

with $C > 0$ an absolute constant. We prove that $h_{b}$ is bounded from $A^{\Phi}_{\alpha}(\mathbb{B}^{n})$ into $A^{1}_{\alpha}(\mathbb{B}^{n})$. Using Lemma 2.14, we have

$$h_{b}(f)(z) = P_{\alpha}(b \overline{f})(z) = \int_{\mathbb{B}^{n}} b(w) \overline{f(w)} K_{\alpha}(z, w) d\nu_{\alpha}(w) = \int_{\mathbb{B}^{n}} K_{\alpha}(z, w) M^{\alpha}_{b}(w) f(w) (1 - |w|^{2})^{k} d\nu_{\alpha}(w).$$

Now, by [9, Proposition 1.4.10], we have

$$\int_{\mathbb{B}^{n}} |K_{\alpha}(z, w)| d\nu_{\alpha}(z) \leq \log \frac{1}{1 - |w|^{2}}.$$ Combining these facts with the inequality (35), we obtain

$$\int_{\mathbb{B}^{n}} |h_{b} f(z)| d\nu_{\alpha}(z) \leq C \int_{\mathbb{B}^{n}} \left|K_{\alpha}(z, w)\right| |M^{\alpha}_{b}(w)||f(w)|(1 - |w|^{2})^{k} d\nu_{\alpha}(w) d\nu_{\alpha}(z) \leq C \int_{\mathbb{B}^{n}} \left|K_{\alpha}(z, w)\right| \left(\log \frac{1}{1 - |w|^{2}}\right) (1 - |w|^{2})^{k} |f(w)| d\nu_{\alpha}(w) \leq C ||b||_{L^{1}_{\Gamma, \rho}} \int_{\mathbb{B}^{n}} |f(w)| \rho \left((1 - |z|^{2})^{n+1+\alpha}\right) d\nu_{\alpha}(w) \leq C ||b||_{L^{1}_{\Gamma, \rho}} ||f||_{L^{1}_{\alpha, \Phi}}.$$ This complete the first part of the proof.

Conversely, if $h_{b}$ is bounded from $A^{\Phi}_{\alpha}(\mathbb{B}^{n})$ into $A^{1}_{\alpha}(\mathbb{B}^{n})$, then we have for every $f \in A^{\Phi}_{\alpha}(\mathbb{B}^{n})$ and $g \in B = (A^{1}_{\alpha}(\mathbb{B}^{n}))^{*},$

$$|\langle h_{b}(f), g \rangle_{\alpha}| = |\langle b, fg \rangle_{\alpha}| \leq C ||h_{b}||_{B} ||f||_{L^{1}_{\alpha, \Phi}} ||g||_{B}.$$ We will apply the inequality (36) to $f$ and $g$, with

$$f(z) = f_{w}(z) = \Phi^{-1} \left(\frac{1}{(1 - |w|^{2})^{n+1+\alpha}}\right) \frac{(1 - |w|^{2})^{n+1+\alpha+k}}{(1 - (z, w))^{n+1+\alpha+k}}$$ and

$$g(z) = \log(1 - (z, w))$$ where $k$ is an integer with $k > (n + 1 + \alpha)(\frac{1}{p} - 1)$.

We have seen that $f$ is uniformly in $A^{\Phi}_{\alpha}(\mathbb{B}^{n})$ and it is well known that $g$ is uniformly in $B$. It follows that
\[ ||h_b|| \geq C \Phi^{-1} \left( \frac{1}{(1 - |w|^2)^{n+1+\alpha}} \right) (1 - |w|^2)^{n+1+\alpha+k} \]

\[ \lim_{r \to 1} \int_{B^n} \frac{b(z)}{(1 - \langle w, rz \rangle)^{n+1+\alpha+k}} \log(1 - \langle w, rz \rangle) d\nu_\alpha(z) \]

\[ = C \frac{(1 - |w|^2)^k}{\rho ((1 - |w|^2)^{n+1+\alpha})} \left( \frac{1}{(1 - \langle w, rz \rangle)^{n+1+\alpha+k}} \right) \log(1 - |w|^2) d\nu_\alpha(z) + \]

\[ \lim_{r \to 1} \int_{B^n} b(z) \bar{h}(rz) d\nu_\alpha(z) \]

This is equivalent to

\[ \lim_{r \to 1} \int_{B^n} \frac{(1 - |w|^2)^k}{\rho ((1 - |w|^2)^{n+1+\alpha})} \frac{b(z)}{(1 - \langle w, rz \rangle)^{n+1+\alpha+k}} \log(1 - |w|^2) d\nu_\alpha(z) + \]

\[ \lim_{r \to 1} \int_{B^n} b(z) \bar{h}(rz) d\nu_\alpha(z) \leq C ||h_b|| \]

where

\[ h(z) = \Phi^{-1} \left( \frac{1}{(1 - |w|^2)^{n+1+\alpha}} \right) (1 - |w|^2)^{n+1+\alpha+k} \frac{(1 - \langle z, w \rangle) \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right)}{(1 - |w|^2)^{n+1+\alpha+k}}. \]

We have seen at the beginning of this section that \( h \) is uniformly in \( A^\Phi_\alpha(B^n) \). That is

\[ ||h||_{A^\Phi_\alpha(B^n)} \leq C. \]

It follows using (36) with \( f = h \) and \( g = 1 \) that

\[ \left| \lim_{r \to 1} \int_{B^n} b(z) \bar{h}(rz) d\nu_\alpha(z) \right| \leq C ||h_b||. \]

We deduce that

\[ \left| \lim_{r \to 1} \int_{B^n} \frac{(1 - |w|^2)^k}{\rho ((1 - |w|^2)^{n+1+\alpha})} \frac{b(z)}{(1 - \langle w, rz \rangle)^{n+1+\alpha+k}} \log(1 - |w|^2) d\nu_\alpha(z) \right| \leq C ||h_b|| \]

or equivalently

\[ \left| \lim_{r \to 1} \int_{B^n} \frac{b(z)}{(1 - \langle w, rz \rangle)^{n+1+\alpha+k}} d\nu_\alpha(z) \right| \leq C ||h_b|| (1 - |w|^2)^{-k} \rho \left( (1 - |w|^2)^{n+1+\alpha} \right) \left( \log \frac{1}{1 - |w|^2} \right)^{-1}. \]

That is, for \( w \in B^n \)

\[ |M^2_b(w)| \leq C ||h_b|| (1 - |w|^2)^{-k} \rho \left( (1 - |w|^2)^{n+1+\alpha} \right) \left( \log \frac{1}{1 - |w|^2} \right)^{-1}. \]

The proof is complete. \( \square \)
4.5. Boundedness of $h_b$: $\mathcal{A}_{\alpha}^{\Phi_1}(\mathbb{B}^n) \to \mathcal{A}_{\alpha}^{\Phi_2}(\mathbb{B}^n)$; $(\Phi_1, \Phi_2) \in \mathcal{L}_p \times \mathcal{W}^q$. The following result extends the classical case $h_b : \mathcal{A}_p(\mathbb{B}^n) \to \mathcal{A}_q(\mathbb{B}^n)$, $0 < p < 1$ and $q > 1$.

**Theorem 4.9.** Let $\Phi \in \mathcal{L}_p$ and $\Phi_2 \in \mathcal{W}^q$, $\rho_1(t) = \frac{1}{\Phi^{-1}((1/t))}$ and assume that $\Phi_2$ satisfies the Dini condition (13). Then the Hankel operator $h_b$ extends into a bounded operator from $\mathcal{A}_{\alpha}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{A}_{\alpha}^{\Phi_2}(\mathbb{B}^n)$ if and only if its symbol $b$ belongs to $\Gamma_{\alpha,\rho}(\mathbb{B}^n) = (\mathcal{A}_p(\mathbb{B}^n))^*$, where

$$\rho = \frac{\rho_1}{\rho_2}.$$ 

**Proof.** We start by proving the sufficiency of the condition for the boundedness. Let $\Phi_j$, $j = 1, 2$ be as in the hypothesis and $\rho = \rho_2$. Denoting by $\Psi_j$ the complementary function of $\Phi_j$, then as $\Phi^{-1}_j(t) = \Phi^{-1}_{j-1}(t)\Psi^{-1}_j(t)$, Proposition 2.7 gives that $fg \in \mathcal{A}_{\Phi_j}(\mathbb{B}^n)$ for any $f \in \mathcal{A}_{\Phi_1}(\mathbb{B}^n)$ and $g \in \mathcal{A}_{\Phi_2}(\mathbb{B}^n)$. Moreover, the dual space of $\mathcal{A}_p(\mathbb{B}^n)$ coincides with $\Gamma_{\alpha,\rho}$ since $\Phi \in \mathcal{L}_p$ for some $0 < r \leq p$ (see Lemma 2.24). It follows that there exists a positive constant $C$ such that

$$|\langle h_b, g \rangle| \leq C||b||_{\Gamma_{\alpha,\rho}}||fg||_{\mathcal{A}_{\alpha,\rho}}.$$ 

We conclude that if $b \in \Gamma_{\alpha,\rho}(\mathbb{B}^n)$, then $h_b$ is bounded from $\mathcal{A}_{\alpha}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{A}_{\alpha}^{\Phi_2}(\mathbb{B}^n)$ with $||h_b|| \leq C||b||_{\Gamma_{\alpha,\rho}}$.

Conversely, suppose that $h_b$ extends into a bounded operator from $\mathcal{A}_{\alpha}^{\Phi_1}(\mathbb{B}^n)$ into $\mathcal{A}_{\alpha}^{\Phi_2}(\mathbb{B}^n)$. Then as in (36), we have

$$|\langle h_b, g \rangle| = \lim_{r \to 1} \int_{\mathbb{B}^n} b(z)f(rz)g(rz)d\nu_{\alpha}(z) \leq C||h_b||||f||_{\mathcal{A}_{\alpha,\rho}^{\Phi_1}}||g||_{\mathcal{A}_{\alpha,\rho}^{\Phi_2}}.$$ 

Let $w \in \mathbb{B}^n$, we apply the above inequality to

$$f(z) = f_w(z) = \Phi_1^{-1}\left(\frac{1}{|1 - |w|^2|^{n+1+\alpha}}\right) \left(1 - |w|^2\right)^{k-1}(1 - \langle z, w \rangle)^{k-1},$$

and

$$g(z) = \Psi_2^{-1}\left(\frac{1}{|1 - |w|^2|^{n+1+\alpha}}\right) \left(1 - |w|^2\right)^{n+2+\alpha} = \left(1 - \langle z, w \rangle\right)^{n+2+\alpha},$$

with $k > \frac{n+1+\alpha}{p} + 1$. Using Lemma 2.17 and Lemma 2.18, one easily verifies that $f$ and $g$ are uniformly in $\mathcal{A}_{\alpha}^{\Phi_1}(\mathbb{B}^n)$ and $\mathcal{A}_{\alpha}^{\Phi_2}(\mathbb{B}^n)$ respectively. Hence

$$|\langle h_b, g \rangle| \simeq \Phi^{-1}_1\left(\frac{1}{|1 - |w|^2|^{n+1+\alpha}}\right) \left(1 - |w|^2\right)^{n+1+\alpha+k} \lim_{r \to 1} \int_{\mathbb{B}^n} \frac{b(z)}{(1 - \langle w, rz \rangle)^{n+1+\alpha+k}} d\nu_{\alpha}(z) \leq C||h_b||.$$ 

That is, for all $w \in \mathbb{B}^n$,

$$|M_kb(w)| \leq C||h_b||(1 - |w|^2)^{-k}b\left(1 - |w|^2\right)^{n+1+\alpha+k},$$

Thus $||b||_{\Gamma_{\alpha,\rho}(\mathbb{B}^n)} \leq C||h_b||$. This completes the proof of the theorem. \qed
4.6. Boundedness of $h_b$: $A^\Phi_\alpha (\mathbb{B}^n) \rightarrow A^\Phi_\alpha (\mathbb{B}^n)$; $(\Phi_1, \Phi_2) \in \mathcal{U}^q \times \mathcal{U}^q$.

**Theorem 4.10.** Let $\Phi_1$ and $\Phi_2$ in $\mathcal{U}^q$, and $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$. Denote by $\Psi_2$ the complementary function of $\Phi_2$. We suppose that:

(i) $\Phi_2$ satisfies the Dini condition (13)

(ii) $\frac{\Phi_2^{-1}(1/t)\Psi_2^{-1}(t)}{t}$ is non-decreasing.

Then the Hankel operator $h_b$ extends into a bounded operator from $A^\Phi_\alpha (\mathbb{B}^n)$ into $A^\Phi_\alpha (\mathbb{B}^n)$ if and only if its symbol $b$ belongs to $\Gamma_{\alpha, \rho \Phi} = (A^\Phi_\alpha (\mathbb{B}^n))^*$, where

$$
\rho = \rho \Phi := \frac{\rho_1}{\rho_2}.
$$

**Proof.** Condition (i) implies that the dual space of $A^\Phi_\alpha (\mathbb{B}^n)$ is $A^\Psi_\alpha (\mathbb{B}^n)$. Condition (ii) implies that $\Phi \in L^p$ for some $0 < p \leq 1$. The whole proof follows the lines of the proof of Theorem 4.9. □

We observe that in the proof of the above theorem, the condition (ii) is used to ensure that the resulting growth function $\Phi$ is in some $L^p$. Hence using Lemma 2.6, we have the following.

**Proposition 4.11.** Let $\Phi_1$ and $\Phi_2$ in $\mathcal{U}^q$, and $\rho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)}$. We suppose that

(i) $\Phi_2$ satisfies the Dini condition (13)

(ii) $\frac{\Phi_2^{-1}(1/t)\Phi_1^{-1}(t)}{t}$ is non-increasing.

Then the Hankel operator $h_b$ extends into a bounded operator from $A^\Phi_\alpha (\mathbb{B}^n)$ into $A^\Phi_\alpha (\mathbb{B}^n)$ if and only if its symbol $b$ belongs to $\Gamma_{\alpha, \rho \Phi} = (A^\Phi_\alpha (\mathbb{B}^n))^*$, where

$$
\rho \Phi := \frac{\rho_1}{\rho_2}.
$$

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