Finite index subgroups of fully residually free groups

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August 28, 2008

Abstract

Using graph-theoretic techniques for f.g. subgroups of $F^{\mathbb{Z}[t]}$ we provide a criterion for a f.g. subgroup of a f.g. fully residually free group to be of finite index. Moreover, we show that this criterion can be checked effectively. Also we obtain an analogue of Greenberg-Stallings Theorem for f.g. fully residually free groups, and prove that a f.g. non-abelian subgroup of a f.g. fully residually free group is of finite index in its commensurator.

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1 Introduction

Fully residually free (or freely discriminated [2], or $\omega$-residually free [14], or limit [15] [16]) groups have been extensively studied over the last ten years. Although appeared first in 60’s (see [1]) this class of groups drew much attention because of its connection with equations over free groups. Recall that a group $G$ is called fully residually free if for any finitely many non-trivial elements $g_1, \ldots, g_n \in G$ there exists a homomorphism $\phi$ of $G$ into a free group $F$, such that $\phi(g_i) \neq 1$ for $i \in [1, n]$. There are other definitions of these groups more or less convenient depending on the setting.

This class of groups can be studied from several viewpoints using many different techniques. In particular, f.g. fully residually free groups are fundamental groups of graphs of graphs of a
very particular type and their structure can be described using Bass-Serre theory (see [10, 6, 7]). These groups are relatively hyperbolic with respect to their maximal abelian subgroups (see [4]), which provides another tool of studying them from a geometric viewpoint. It is known that f.g. fully residually free groups these groups act freely on $\mathbb{Z}^n$-trees (see [11]), etc.

Our study of fully residually free groups relies heavily on the fact proved by Kharlampovich and Myasnikov (see [7]) that every finitely generated fully residually free group is embeddable into $F_{\mathbb{Z}[t]}$, the free exponential group over the ring of integer polynomials $\mathbb{Z}[t]$. This group was introduced by Lyndon (see [9]) and he proved that $F_{\mathbb{Z}[t]}$ (and hence its subgroups) is fully residually free. It follows that one way to understand the properties of these groups is to study finitely generated subgroups of $F_{\mathbb{Z}[t]}$.

A new technique to deal with $F_{\mathbb{Z}[t]}$ became available recently when Myasnikov, Remeslennikov, and Serbin showed that elements of this group can be viewed as reduced infinite words in the generators of $F$ (see [12]). It turned out that many algorithmic problems for finitely generated fully residually free groups can be solved by the same methods as in the standard free groups. Indeed, in [13] an analog of Stallings' foldings (see [17, 5]) was introduced for an arbitrary finitely generated subgroup of $F_{\mathbb{Z}[t]}$, which allows one to solve effectively the membership problem in $F_{\mathbb{Z}[t]}$, as well as in an arbitrary finitely generated subgroup of it. Next, in [8] this technique was further developed to obtain the solution of many algorithmic problems. In particular, it was proved that for a f.g. subgroups $G, H$ and $K$ of $F_{\mathbb{Z}[t]}$ such that $H, K \leq G$ there are only finitely many conjugacy classes of intersections $H^g \cap K$ in $G$. Moreover, one can find a finite set of representatives of these classes effectively. This implies that one can effectively decide whether two finitely generated subgroups of $G$ are conjugate or not, and check if a given finitely generated subgroup is malnormal in $G$. Needless to say that all these results can be reformulated for f.g. fully residually free groups.

In the present paper we further develop these methods focusing on the problems involving finite index subgroups. It is worth mentioning that non-abelian f.g. subgroups of f.g. fully residually free groups have finite index in their normalizers - this fact follows immediately from Theorem 7 [8] (see also [3]) - but no criterion for detecting finite index subgroups was known. In this paper we provide such a criterion which can be checked effectively given a finite presentation of a f.g. fully residually free group and a finite generating set of its subgroup. This allows us to draw several corollaries including an analogue of Greenberg-Stallings Theorem for free groups.

The authors are extremely grateful to Alexei G. Miasnikov for insightful discussions and many helpful comments and suggestions.

2 Preliminaries

Here we introduce basic definitions and notations which are to be used throughout the whole paper. For more details see [12, 13].

2.1 Lyndon’s free $\mathbb{Z}[t]$-group and infinite words

Let $F = F(X)$ be a free non-abelian group with basis $X$ and $\mathbb{Z}[t]$ be a ring of polynomials with integer coefficients in a variable $t$. In [9] Lyndon introduced a $\mathbb{Z}[t]$-completion $F_{\mathbb{Z}[t]}$ of $F$, which is called now the Lyndon’s free $\mathbb{Z}[t]$-group.

It turns out that $F_{\mathbb{Z}[t]}$ can be described as a union of a sequence of extensions of centralizers

$$F = G_0 < G_1 < \cdots < G_n < \cdots ,$$

(1)
where $G_{i+1}$ is obtained from $G_i$ by extension of all cyclic centralizers in $G_i$ by a free abelian group of countable rank.

In [12] it was shown that elements of $F^{\mathbb{Z}[t]}$ can be viewed as \textit{infinite words} defined in the following way. Let $A$ be a discretely ordered abelian group. By $1_A$ we denote the minimal positive element of $A$. Recall that if $a, b \in A$ then the closed segment $[a, b]$ is defined as

$$[a, b] = \{ x \in \mathbb{Z} : a \leq x \leq b \}. $$

Let $X = \{ x_i \mid i \in I \}$ be a set. An \textit{A-word} is a function of the type

$$w : [1_A, \alpha_w] \to X^\pm,$$

where $\alpha_w \in A$, $\alpha_w \geq 0$. The element $\alpha_w$ is called the \textit{length} $|w|$ of $w$. By $\varepsilon$ we denote the empty word. We say that $w$ is \textit{reduced} if $w(\alpha) \neq w(\alpha + 1)^{-1}$ for any $1 \leq \alpha < \alpha_w$. Then, as in a free group, one can introduce a partial multiplication $\ast$, an inversion, a word reduction etc., on the set of all $A$-words (infinite words) $\mathcal{W}(A, X)$. We write $uv$ instead of $uwv$, $uv$ instead of $wuv$ if $|uv| = |w| + |v|$. All these definitions make it possible to develop infinite words techniques, which provide a very convenient combinatorial tool (for all the details we refer to [12]).

It was proved in [12] that $F^{\mathbb{Z}[t]}$ can be canonically embedded into the set of reduced infinite words $R(\mathbb{Z}[t], X)$, where $\mathbb{Z}[t]$, an additive group of polynomials with integer coefficients, is viewed as an ordered abelian group with respect to the standard lexicographic order $\leq$ (that is, the order which compares the degrees of polynomials first, and if the degrees are equal, compares the coefficients of corresponding terms starting with the terms of highest degree). More precisely, the embedding of $F^{\mathbb{Z}[t]}$ into $R(\mathbb{Z}[t], X)$ was constructed by induction, that is, all $G_i$ from the series (1) were embedded step by step in the following way. Suppose, the embedding of $G_i$ into $R(\mathbb{Z}[t], X)$ is already constructed. Then, one chooses a Lyndon’s set $U_i \subset G_i$ (see [12]) and the extension of centralizers of all elements from $U_i$ produces $G_{i+1}$, which is now also naturally embedded into $R(\mathbb{Z}[t], X)$.

The existence of an embedding of $F^{\mathbb{Z}[t]}$ into the set of infinite words implies automatically the fact that all subgroups of $F^{\mathbb{Z}[t]}$ are also subsets of $R(\mathbb{Z}[t], X)$, that is, their elements can be viewed as infinite words. From now on we assume the embedding $\rho : F^{\mathbb{Z}[t]} \to R(\mathbb{Z}[t], X)$ to be fixed. Moreover, for simplicity we identify $F^{\mathbb{Z}[t]}$ with its image $\rho(F^{\mathbb{Z}[t]})$.

### 2.2 Reduced forms for elements of $F^{\mathbb{Z}[t]}$

Following [12] and [13] we introduce various normal forms for elements in $F^{\mathbb{Z}[t]}$ in the following way.

We may assume that the set

$$U = \bigcup_{i} U_i$$

is well-ordered. Let

$$U_i = \{ u_{i1}, u_{i2}, \ldots \} \subset G_i,$$

be enumeration of elements of $U_i$ in increasing order. Denote by $I_i$ the set of indices $i_1, i_2, \ldots$ of elements from $U_i$. Now $g \in G_{n+1} - G_n$ has the following representation as a reduced infinite word:

$$g = g_1 \circ u_{n1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{nl}^{\alpha_l} \circ g_{l+1},$$

where $n_1, n_2, \ldots, n_l \in I_n$, $g_k \in G_n$, $k \in [1, l+1]$, $[g_k, u_{nk}] \neq \varepsilon$ (or $g_k = \varepsilon$) $[g_{k+1}, u_{nk}] \neq \varepsilon$ (or $g_{k+1} = \varepsilon$), $k \in [1, l]$, $|\alpha_k| > 0$, $k \in [1, l]$ (recall that $\alpha > 0$ if $\alpha \in \mathbb{Z}[t] - \mathbb{Z}$). Representation (2)
is called $U_n$-reduced if the ordered $t$-tuple $\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_t|\}$ is maximal with respect to the left lexicographic order among all possible such representations of $g$.

**Example 1** Suppose $u = xyx \in U \cap F(X)$ and $g \in G_1 - G_0$. If

$$g = u^{2t} \circ (yx) \circ u^{3t}$$

then there exists another representation of $g$

$$g = u^{2t-1} \circ (xy) \circ u^{3t+1}.$$  
The corresponding 2-tuples are $(2t, 3t)$ and $(2t - 1, 3t + 1)$. In the former one maximization of exponents of $u$ goes from left to right, while in the latter one from right to left.

From (2) one can obtain another representation of $g$. Fix any $u$ from the list $u_{n_1}, u_{n_2}, \ldots, u_{n_t}$. Then

$$g = h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1},$$ (3)

where $\beta_j = \alpha_{m_j}, m_j \in [1, \ell], j \in [1, p]$, $h_1 = g_1 \circ u^{\beta_1}_{n_1} \circ \cdots \circ g_{m_1}$, $h_{p+1} = g_{m_p+1} \circ \cdots \circ g_{1}$, $h_k = g_{m_{k+1}} \circ \cdots \circ g_{m_k+1}, k \in [2, p]$. Representation (3) is called a $u$-representation or a $u$-form of $g$. In other words, to obtain a $u$-form one has to ”mark” in (2) only nonstandard exponents of $u$. Representation (3) is called $u$-reduced if the ordered $p$-tuple $\{|\beta_1|, |\beta_2|, \ldots, |\beta_p|\}$ is maximal with respect to the left lexicographic order among all possible $u$-forms of $g$.

Observe that if (3) is a $u$-form for $g$ and $g$ is cyclically reduced then obviously

$$(h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}) \circ (h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1})$$

is a $u$-form for $g^2$. So, we call (3) cyclically u-reduced if (4) is u-reduced.

**Lemma 1** [13] For any given $u$-reduced form of $g \in G_{n+1} - G_n, u \in U_n$, there exists a cyclic permutation of $g$ such that its $u$-reduced form is cyclically u-reduced.

Let $g \in G_{n+1} - G_n$ have a $U_n$-reduced form

$$g = g_1 \circ u^{\alpha_1}_{n_1} \circ g_2 \circ \cdots \circ u^{\alpha_t}_{n_t} \circ g_{t+1},$$

where $u_{n_1}, u_{n_2}, \ldots, u_{n_t} \in U_n$, $g_k \in G_n, k \in [1, t + 1]$, $|g_k, u_{n_k}| \neq \varepsilon$ (or $g_k = \varepsilon$), $|g_k+1, u_{n_k}| \neq \varepsilon$ (or $g_{k+1} = \varepsilon$), $|\alpha_k| >> 0, k \in [1, t]$. Now, recursively one has a $U_{n-1}$-reduced form for $g_i$

$$g_i = g(i)_1 \circ u^{\beta_{m_1}}_{n_1} \circ g((i))_2 \circ \cdots \circ u^{\beta_{m_s}}_{n_s} \circ g((i))_s,$$

where $u_{m_1}, u_{m_2}, \ldots, u_{m_s} \in U_{n-1}, |\beta_{m_k}| >> 0, k \in [1, s], g((i))_k \in G_{n-1}, k \in [1, s + 1]$ and one can get down to the free group $F$ with such a decomposition of $g$, where step by step subwords between nonstandard powers of elements from $U_i$ are presented as $U_{i-1}$-forms, $i \in [1, n]$. Thus, from this decomposition one can form the following series for $g$:

$$F < H_{0,1} < H_{0,2} < \cdots < H_{0,k(0)} < H_{1,1} < \cdots < H_{1,k(1)} < \cdots (5)$$

$$\ldots < H_{n-1,k(n-1)} < H_{n,1} < \cdots < H_{n,k(n)},$$

where $H_{j,1}, \ldots, H_{j,k(j)}$ are subgroups of $G_{j+1}$, which do not belong to $G_j$ and $H_{j,i}$ is obtained from $H_{j,i-1}$ by a centralizer extension of a single element $u_{j,i-1} \in H_{j,i-1} < G_j$. Element $g$ belongs
to $H_{n,k(n)}$ and does not belong to the previous terms. Series \([5]\) is called an extension series for $g$.

Using the extension series above we can decompose $g$ in the following way: $g \in H_{n,k(n)}$ has a $u_{n,k(n)}$-reduced form

$$g = h_1 \circ u_{n,k(n)}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n,k(n)}^{\beta_l} \circ h_{l+1},$$

where all $h_j, j \in [1, l + 1]$ in their turn are $u_{n,k(n)-1}$-reduced forms representing elements from $H_{n,k(n)-1}$. This gives one a decomposition of $g$ related to its extension series. We call this decomposition a standard decomposition or a standard representation of $g$.

Observe that for any $g \in F_{\mathbb{Z}[t]}$, its standard decomposition can be viewed as a finite product $b_1 b_2 \cdots b_m$, where

$$b_i \in B = \{X \cup X^{-1}\} \cup \{u^\alpha | u \in U, \alpha \in \mathbb{Z}[t] - \mathbb{Z}\}.$$  

We denote this product by $\pi(g)$ so we have

$$\pi(g) = \pi(h_1) u_{n,k(n)}^{\beta_1} \pi(h_2) \cdots u_{n,k(n)}^{\beta_l} \pi(h_{l+1}),$$

where $\pi(h_i)$ is a finite product in the alphabet $B$ corresponding to $h_i$, and from now on, by a standard decomposition of an element $g$ we understand not the representation of $g$ as a reduced infinite word but the finite product $\pi(g)$.

By $U(g)$ we denote a finite subset of $U$ such that if $\pi(g)$ contains a letter $b_i \in B$ such that $b_i = u^\alpha$ then $u \in U(g)$. Observe that $U(g)$ is ordered with an order induced from $U$, so we have

$$U(g) = \{u_1, \ldots, u_m\},$$

where $u_i < u_j$ if $i < j$ and $u_m = u_{n,k(n)}$. By $\max\{U(g)\}$ we denote the maximal element of $U(g)$.

If $u \in U(g)$ then by $\deg_u(g)$ we denote the maximal degree of infinite exponents of $u$, which appear in $\pi(g)$.

It is easy to see that in general $\pi(g_1 \circ g_2) \neq \pi(g_1) \pi(g_2)$ and $\pi(g \circ g) = \pi(g) \pi(g)$ if and only if the $u$-reduced form of $g$ is cyclically $u$-reduced, where $u = \max\{U(g)\}$.

From the definition of a Lyndon’s set and the results of \([12]\) it follows that if $R \subset G_n$ is a Lyndon’s set then a set $R'$ obtained from $R$ by cyclic decompositions of its elements is also a Lyndon’s set. Thus, by Lemma \([1]\) we can assume a $w$-reduced form of any $u \in U_n$ to be cyclically $w$-reduced, where $w = \max\{U(u)\}$. Hence, we can assume

$$\pi(u \circ u) = \pi(u) \pi(u)$$

for any $u \in U$.

### 2.3 Embedding theorems

There are three results which play an important role in this paper. The first embedding theorem is due to Kharlampovich and Myasnikov.

**Theorem 1 (The first embedding theorem \([17]\))** Given a finite presentation of a finitely generated fully residually free group $G$ one can effectively construct an embedding $\phi : G \to F_{\mathbb{Z}[t]}$ (by specifying the images of the generators of $G$).

Combining Theorem \([1]\) with the result on the representation of $F_{\mathbb{Z}[t]}$ as a union of a sequence of extensions of centralizers one can get the following theorem.
Theorem 2 (The second embedding theorem) Given a finite presentation of a finitely generated fully residually free group $G$ one can effectively construct a finite sequence of extension of centralizers

$$F < G_1 < \ldots < G_n,$$

where $G_{i+1}$ is an extension of the centralizer of some element $u_i \in G_i$ by an infinite cyclic group $\mathbb{Z}$, and an embedding $\psi^*: G \to G_n$ (by specifying the images of the generators of $G$).

Combining Theorem 1 with the result on the effective embedding of $F^{\mathbb{Z}[t]}$ into $R(\mathbb{Z}[t], X)$ obtained in [12] one can get the following theorem.

Theorem 3 (The third embedding theorem) Given a finite presentation of a finitely generated fully residually free group $G$ one can effectively construct an embedding $\psi: G \to R(\mathbb{Z}[t], X)$ (by specifying the images of the generators of $G$).

2.4 Graphs labeled by infinite $\mathbb{Z}[t]$-words

By an $(\mathbb{Z}[t], X)$-labeled directed graph ($(\mathbb{Z}[t], X)$-graph) $\Gamma$ we understand a combinatorial graph $\Gamma$ where every edge has a direction and is labeled either by a letter from $X$ or by an infinite word $u^\alpha \in F^{\mathbb{Z}[t]}, u \in U, \alpha \in \mathbb{Z}[t], \alpha > 0$, denoted $\mu(e)$.

For each edge $e$ of $\Gamma$ we denote the origin of $e$ by $o(e)$ and the terminus of $e$ by $t(e)$.

For each edge $e$ of $(\mathbb{Z}[t], X)$-graph we can introduce a formal inverse $e^{-1}$ of $e$ with the label $\mu(e)^{-1}$ and the endpoints defined as $o(e^{-1}) = t(e), t(e^{-1}) = o(e)$, that is, the direction of $e^{-1}$ is reversed with respect to the direction of $e$. For the new edges $e^{-1}$ we set $(e^{-1})^{-1} = e$. The new graph, endowed with this additional structure we denote by $\tilde{\Gamma}$. Usually we will abuse the notation by disregarding the difference between $\Gamma$ and $\tilde{\Gamma}$.

A path $p$ in $\Gamma$ is a sequence of edges $p = e_1 \cdots e_k$, where each $e_i$ is an edge of $\Gamma$ and the origin of each $e_i$ is the terminus of $e_{i-1}$. Observe that $\mu(p) = \mu(e_1) \cdots \mu(e_k)$ is a word in the alphabet $\{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t]\}$ and we denote by $\mu(p)$ a reduced infinite word $\mu(e_1) \cdots \mu(e_k)$ (this product is always defined).

A path $p = e_1 \cdots e_k$ in $\Gamma$ is called reduced if $e_i \neq e_{i+1}^{-1}$ for all $i \in [1, k - 1]$.

A path $p = e_1 \cdots e_k$ in $\Gamma$ is called labeled reduced if

1) $p$ is reduced;

2) if $e_k \cdots e_k$, $k_1 \leq k_2$ is a subpath of $p$ such that $\mu(e_i) = u^\alpha_i, u \in U, \alpha_i \in \mathbb{Z}[t], i \in [k_1, k_2]$ and $\mu(e_{k_1-2}) \neq u^\beta, \mu(e_{k_2-1}) \neq u^\beta$ for any $\beta \in \mathbb{Z}[t]$, provided $k_1 - 1, k_2 + 1 \in [1, k]$, then $\alpha = \alpha_{k_1} + \cdots + \alpha_{k_2} \neq 0$ and $\mu(e_{k_1-2}) \ast u^\alpha = \mu(e_{k_1-1}) \circ u^\alpha, u^\alpha \ast \mu(e_{k_2-1}) = u^\alpha \circ \mu(e_{k_2-1})$.

Let $\Gamma$ be a $(\mathbb{Z}[t], X)$-graph and $u \in U$ be fixed. Vertices $v_1, v_2 \in V(\Gamma)$ are called $u$-equivalent (denoted $v_1 \sim_u v_2$) if there exists a path $p = e_1 \cdots e_k$ in $\Gamma$ such that $o(e_1) = v_1, t(e_k) = v_2$ and $\mu(e_i) = u^\alpha_i, \alpha_i \in \mathbb{Z}[t], i \in [1, k]$. $\sim_u$ is an equivalence relation on vertices of $\Gamma$, so if $\Gamma$ is finite then all its vertices can be divided into a finite number of pairwise disjoint equivalence classes. Suppose, $v \in V(\Gamma)$ is fixed. One can take the subgraph of $\Gamma$ spanned by all the vertices which are $u$-equivalent to $v$ and remove from it all edges with labels not equal to $u^\alpha, \alpha \in \mathbb{Z}[t]$. We denote the resulting subgraph of $\Gamma$ by $\text{Comp}_u(v)$ and call a $u$-component of $v$. If $v \in V(\Gamma), v_0 \in V(\text{Comp}_u(v))$ then one can define a set

$$H_u(v_0) = \{\mu(p) \mid p \text{ is a reduced path in } \text{Comp}_u(v) \text{ from } v_0 \text{ to } v_0\}.$$
Lemma 2 [13] Let $\Gamma$ be a $(\mathbb{Z}[t], X)$-graph and $v \in V(\Gamma)$, $v_0 \in V(\text{Comp}_u(v))$. Then

1. $H_u(v_0)$ is a subgroup of $R(\mathbb{Z}[t], X)$;
2. $H_u(v_0)$ is isomorphic to a subgroup of $\mathbb{Z}[t]$;
3. if $\text{Comp}_u(v)$ is a finite graph, then $H_u(v_0)$ is finitely generated;
4. if $v_1 \in V(\text{Comp}_u(v))$ then $H_u(v_0) \simeq H_u(v_1)$.

Following [13] one can introduce operations on $u$-components which are called $u$-foldings. One of the most important properties of $u$-foldings is that they do not change subgroups associated with $u$-components.

Lemma 3 [13] Let $\Gamma$ be a $(\mathbb{Z}[t], X)$-graph, $v \in V(\Gamma)$ and $C = \text{Comp}_u(v)$ be finite. Then there exist a $(\mathbb{Z}[t], X)$-graph $\Delta$ obtained from $\Gamma$ by finitely many $u$-foldings such that $v' \in V(\Delta)$ corresponds to $v$ and $C' = \text{Comp}_u(v')$ consists of a simple positively oriented path $P_{C'}$, and some edges that are not in $P_{C'}$ connecting some pairs of vertices in $P_{C'}$.

$C'$ in Lemma 3 is called a reduced $u$-component. Since $P_{C'}$ is a simple path there exists a vertex $z_{C'} \in V(P_{C'})$ which is an origin of only one positive edge in $P_{C'}$. $z_{C'}$ is called a base-point of $C'$.

It turns out that any finite reduced $u$-component $C$ in a $(\mathbb{Z}[t], X)$-graph is characterized completely by the pair $(P_C, H_u(z_{C']))$ in the following sense. For any reduced path $p$ in $C$ there exists a unique reduced subpath $q$ (denoted $q = [p]$) of $P_C$ with the same endpoints as $p$, such that $\mu(p) \ast \mu(q)^{-1} \in H_u(z_{C'})$. Moreover, let $P_C = f_1 \cdots f_m$, where $o(f_i) = z_{C'}, v_0 = z_{C'}, v_i = t(f_i), i \in [1, m]$ and let $p_0, p_1, \ldots, p_m$ be reduced subpaths of $P_C$ such that $o(p_i) = z_{C'}, t(p_i) = v_i, i \in [0, m]$. The set of paths $p_0, p_1, \ldots, p_m$ is called a set of path representatives associated with $C$ (denoted by $\text{Rep}(C)$).

Lemma 4 [13] Let $C$ be a finite reduced $u$-component in a $(\mathbb{Z}[t], X)$-graph $\Gamma$, $v \in V(C)$ and let $\alpha \in \mathbb{Z}[t]$. If $\mu(p_i) \ast \mu(p_j)^{-1} \notin H_u(z_{C'})$ for any $p_i, p_j \in \text{Rep}(C), i \neq j$ then either there exists a unique reduced path $p$ in $P_C$ such that $o(p) = v$ and $u^\alpha \in \mu(p) \ast H_u(z_{C'})$ or there exists no path $q$ in $C$ with this property.

If $C$ is reduced and $\text{Rep}(C)$ satisfies the condition from Lemma 4 then we call $C$ a $u$-folded $u$-component.

### 2.5 Languages associated with $(\mathbb{Z}[t], X)$-graphs

Let $\Gamma$ be a $(\mathbb{Z}[t], X)$-graph and let $v$ be a vertex of $\Gamma$. We define the language of $\Gamma$ with respect to $v$ as

$$L(\Gamma, v) = \{ \mu(p) | p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v \}.$$  

Lemma 5 [13] Let $\Gamma$ be a finite $(\mathbb{Z}[t], X)$-graph and let $v \in V(\Gamma)$. Then $L(\Gamma, v)$ is a subgroup of $F^\mathbb{Z}[t]$.

Let $\Gamma$ be a $(\mathbb{Z}[t], X)$-graph and $p = e_1 \cdots e_k$ be a reduced path in $\Gamma$. Let $g \in G_{n+1} - G_n$ and let

$$\pi(g) = \pi(g_1)u^{\beta_1}\pi(g_2)\cdots u^{\beta_t}\pi(g_{t+1}),$$

be the standard decomposition of $g$, where $u = \max\{U(g)\}$. We write

$$\mu(p) = \pi(g)$$
if \( p \) can be subdivided into subpaths

\[
p = p_1d_1p_2 \cdots d_ip_{i+1},
\]

where each \( d_i \) is a path in some \( u \)-component of \( \Gamma \) so that \( \mu(d_i) = u^\beta_i \), and each \( p_i \) is a path in \( \Gamma \) which does not contain edges labeled by \( u^\alpha, \alpha \in \mathbb{Z}[t] \) so that the equality \( \mu(p_i) = \pi(g_i) \) is defined inductively in the same way. Observe that if \( g = x_1 \cdots x_r \in F(X) \) then \( \mu(p) = \pi(g) \) if and only if \( k = r \) and \( \mu(e_i) = x_i, \ i \in [1, r] \).

Let \( \Gamma \) be a finite \((\mathbb{Z}[t], X)\)-graph. Since \( \Gamma \) is finite, the set of elements \( u \in U \) such that there exists an edge \( e \in \Gamma \) labeled by \( u^\alpha, \alpha \in \mathbb{Z}[t] \) is finite and ordered with the order induced from \( U \). Thus one can associate with \( \Gamma \) an ordered set \( U(\Gamma) = \{u_1, \ldots, u_N\}, N > 0 \) such that \( U(\Gamma) \) is an ordered set with \( u_i < u_j \) for \( i < j \).

Let \( u_i \in U(\Gamma) \) be fixed and \( \Gamma(i) \) be a subgraph of \( \Gamma \) which consists only of edges \( e \in E(\Gamma) \) such that either \( \mu(e) = x \in X^\pm \) or \( \mu(e) = u_i^\alpha, \alpha \in \mathbb{Z}[t], j \leq i \). \( \Gamma(i) \) is called an \( i \)-level graph of \( \Gamma \) (by 0-level graph we understand a subgraph of \( \Gamma \) which consists only of edges with labels from \( X \)) and the level (denoted \( l(\Gamma) \)) of \( \Gamma \) is the minimal \( n \in \mathbb{N} \) such that \( \Gamma = \Gamma(n) \). Observe that \( \Gamma(i) \) may not be connected for some \( i < l(\Gamma) \), but still one can apply to \( \Gamma(i) \) partial and \( u \)-foldings, \( u \in U(\Gamma) \).

A finite connected \((\mathbb{Z}[t], X)\)-graph \( \Gamma \) is called \( U \)-folded if for any reduced path \( p \) in \( \Gamma \) with \( \mu(p) = w \) there exists a unique label reduced path \( q \) such that \( o(q) = o(p), t(q) = t(p), \mu(q) = \pi(w) \).

The above definition is equivalent to a more technical one given in [13].

**Proposition 1** [13] Let \( \Gamma \) be a finite connected \((\mathbb{Z}[t], X)\)-graph and \( v \in V(\Gamma) \). Then there exists a \( U \)-folded \((\mathbb{Z}[t], X)\)-graph \( \Delta \) and \( v' \in V(\Delta) \) such that \( L(\Gamma, v) = L(\Delta, v') \). Moreover \( \Delta \) can be constructed effectively by adding to \( \Gamma \) finitely many edges and applying finitely many free and \( U \)-foldings.

**Proposition 2** [13] Let \( H \) be a finitely generated subgroup of \( F^{\mathbb{Z}[t]} \). Then there exists a \( U \)-folded \((\mathbb{Z}[t], X)\)-graph \( \Gamma \) and a vertex \( v \) of \( \Gamma \) such that \( L(\Gamma, v) = H \).

**Proposition 3** [13] There is an algorithm which, given finitely many standard decompositions of elements \( h_1, \ldots, h_k \) from \( F^{\mathbb{Z}[t]} \), constructs a \( U \)-folded \((\mathbb{Z}[t], X)\)-graph \( \Gamma \), such that \( L(\Gamma, v) = \langle h_1, \ldots, h_k \rangle \).

The properties of \( U \)-folded graphs make it possible to solve the membership problem in finitely generated subgroups of \( F^{\mathbb{Z}[t]} \).

**Proposition 4** [13] Every finitely generated subgroup of \( F^{\mathbb{Z}[t]} \) has a solvable membership problem. That is, there exists an algorithm which, given finitely many standard decompositions of elements \( g, h_1, \ldots, h_k \) from \( F^{\mathbb{Z}[t]} \), decides whether or not \( g \) belongs to the subgroup \( H = \langle h_1, \ldots, h_n \rangle \) of \( F^{\mathbb{Z}[t]} \).

### 3 Finite index criteria

It is not difficult to check if a finitely generated subgroup \( H \) of a free group \( G \) is of finite index (see, for example, [5]). This can be done by checking if the normal form of every element of \( G \) is “readable” in a folded graph \( \Gamma_H \) corresponding to \( H \). Similar result can be easily proved for finitely generated subgroups of \( F^{\mathbb{Z}[t]} \).
Proposition 5 Let $G$ be a finitely generated subgroup of $F^\mathbb{Z}[t]$ and $H \leq G$. Then the following are equivalent:

1. $|G : H| < \infty$,

2. there exists a finite $U$-folded $(\mathbb{Z}[t], X)$-graph $\Delta$ with a vertex $v$ such that $H = L(\Delta, v)$ and for every $g \in G$ there exists a path $p$ in $\Delta$ such that $o(p) = v, \mu(p) = g$.

Proof. At first, assume $|G : H| < \infty$. Hence, there exist $g_1, \ldots, g_k \in G$ such that $G = H \cup Hg_1 \cup \cdots \cup Hg_k$. Take a finite $U$-folded $(\mathbb{Z}[t], X)$-graph $\Gamma$ with a vertex $v$ such that $H = L(\Gamma, v)$. For each $g_i, i \in [1, k]$ take a path labeled by $\pi(g_i)$ and glue its initial end-point to $\Gamma$ at $v$. The resulting $(\mathbb{Z}[t], X)$-graph $\Gamma'$ by Proposition 1 can be transformed into a $U$-folded $(\mathbb{Z}[t], X)$-graph $\Delta$ whose language is $H$. But since for every product $h * g_i, h \in H, i \in [1, k]$ there exists a path $p$ in $\Gamma'$ such that $o(p) = v, \mu(p) = g$, this property also holds in $\Delta$.

Now, assume that there exists a finite $U$-folded $(\mathbb{Z}[t], X)$-graph $\Delta$ with a vertex $v$ such that $H = L(\Delta, v)$ and for every $g \in G$ there exists a path $p$ in $\Delta$ such that $o(p) = v, \mu(p) = g$. For every $w \in \Delta$ there exists a path $p_w$ such that $o(p_w) = v, t(p_w) = w$. Since $\Delta$ is finite, the set of such paths $p_w$ is finite and their reduced labels obviously can be taken to be representatives of right cosets in $G$ by $H$.

At the same time, it is important to understand that not every $U$-folded graph representing a subgroup $H$ of finite index in $G$ has the property that the normal form of every element of $G$ is “readable” in it.

Example 2 Let $u = xyx \in F(X) \cap U$ and $G = \langle a, b \rangle, H = \langle a^2, b^2, ab \rangle$, where $a = xu^t z_1 u^t xy, b = xu^t z_2 u^t xy$. Without loss of generality we can assume $z_1, z_2 \in F(X)$ to be such that the graphs shown on Figure 1 are $U$-folded. Observe that $|G : H| = 2$, but $g = xu^t z_1 u^t xy \in G$ is not “readable” in the graph defining $H$.

![Figure 1](image-url)

Hence, “readability” of normal forms of elements from $G$ is a too strong property to work with. Instead, below we develop the idea of “readability” of infinite paths arising in a $U$-folded graph for $G$, and show that every such infinite path is readable in any $U$-folded graph for $H$ as long as $|G : H| < \infty$. 
3.1 Equivalence of infinite powers

For any $K \subseteq \mathbb{Z}[t]$, denote $u^K = \{u^\alpha | \alpha \in K\}$.

Let $W$ be a finite subset of $U$. Denote $B_W = \{X \cup X^{-1}\} \cup (\bigcup_{u \in W} u\mathbb{Z}[t])$. For any $u \in W$ we say that $u^\alpha$ is $W$-equivalent to $u^\beta$, where $\alpha, \beta \in \mathbb{Z}[t] - \mathbb{Z}$, and denote $u^\alpha \sim_W u^\beta$, if the following condition holds: for any $w_1, w_2 \in B_W^*$ if $w_1u^\alpha w_2$ is a standard form, then $w_1u^\beta w_2$ is a standard form as well, provided $w_1$ does not end with a power of $u$ and $w_2$ does not start with a power of $u$.

**Proposition 6** For every finite $W \subset U$ and $u \in W$, $u\mathbb{Z}[t]$ splits into a finite number of $W$-equivalence classes. Moreover, for any $\alpha \in \mathbb{Z}[t]$, the equivalence class of $u^\alpha$ can be effectively constructed.

**Proof.** Fix a finite set $W$ and $u \in W$.

For a given occurrence $w_1u^\alpha w_2$, describe all $u^\beta$ that $u^\alpha$ can be replaced with, not breaking the standard form. Suppose some $u^\beta$ does break the standard form. Enumerate possible reasons according to definition of standard form:

1. $w_1u^\beta w_2$ is no longer reduced, which means that $\beta$ is of opposite sign to $\alpha$,
2. $w_1 = t_1s_1, w_2 = s_2t_2, s_1u^\beta s_2 = \pi(v^\kappa), v \in U, v > u$, and either $t_1$ ends with $v^\delta$, or $t_2$ starts with $v^\delta$,
3. $w_1 = t_1s_1, s_1u^\gamma = \pi(v^\kappa)$, where either $\beta \geq \gamma > 0$ or $\beta \leq \gamma < 0$, and either $t_1$ ends with $v^\delta$, or $w_2$ starts with $v^\delta$,
4. $w_2 = s_2t_2, u^\gamma s^2 = \pi(v^\kappa)$, where either $\beta \geq \gamma > 0$ or $\beta \leq \gamma < 0$, and either $t_1$ ends with $v^\delta$, or $w_2$ starts with $v^\delta$.

As we can see, all possible cases result in conditions of the following types

$$\beta < \gamma, \beta \leq \gamma, \beta > \gamma, \beta \geq \gamma, \beta = \gamma, \beta \neq \gamma.$$ 

Since there are only finitely many occurrences of $u$ in standard forms of $v \in W, v > u$, these condition split $\mathbb{Z}[t] - \mathbb{Z}$ into finitely many classes and each class can be easily constructed. 

**Corollary 1** For every finite $W \subset U$ and $u \in W$, $W$-equivalence classes in $u\mathbb{Z}[t]$ can be effectively described and enumerated.

**Proof.** Follows immediately from the previous result.

3.2 Standard form types associated to $(\mathbb{Z}[t], X)$-graph

Let $\Gamma$ be a finite $U$-folded $(\mathbb{Z}[t], X)$-graph. We fix $\Gamma$ for the rest of this subsection. Observe that $U(\Gamma)$ is a finite subset of $U$ and for each $u \in U(\Gamma)$, by Proposition 6 there are only finitely many $U(\Gamma)$-equivalence classes in $u\mathbb{Z}[t]$. 
Fix $u \in U(\Gamma)$. Let $C$ be a $u$-component of $\Gamma$. By Lemma 2, $H_u(C)$ is a subgroup of $u^{\mathbb{Z}[t]}$. For every $a, b \in V(C)$ define the set $u(a,b)$ of reduced paths in $C$ from $a$ to $b$ and a subset $H_u(a,b)$ of $u^{\mathbb{Z}[t]}$ as follows

$$H_u(a,b) = \{ \mu(p) \mid p \in u(a,b) \}.$$ 

Note that $H_u(a,b) = H_u(C) \ast \mu(p)$ for any reduced path $p$ in $C$ from $a$ to $b$. That is, $H_u(a,b)$ is a (right) coset in $u^{\mathbb{Z}[t]}$ by $H_u(C)$, and it is completely defined by a finite number of generators of $H_u(C)$ and a representative $\mu(p)$.

For a $u$-component $C$ of $\Gamma$, a pair $a,b \in V(C)$, and a $U(\Gamma)$-equivalence class $uA$ in $u^{\mathbb{Z}[t]}$, consider a subset $uA(a,b)$ of $u(a,b)$ defined as

$$uA(a,b) = \{ p \in u(a,b) \mid \mu(p) \in uA \}.$$ 

Paths $p$ and $q$ which belong to a certain $uA(a,b)$ we call $U(\Gamma)$-equivalent.

**Proposition 7** For every $u \in U(\Gamma)$, there are only finitely many sets of the type $uA(a,b)$. Moreover, the set of labels of paths from each $uA(a,b)$ can be effectively described and enumerated.

**Proof.** The first part of the statement follows from the fact that the number of tuples $(C,a,b,uA)$ is finite (in particular, by Proposition 6).

Next, by definition, the set of labels of paths from $uA(a,b)$ is $H_u(a,b) \cap uA$, where both sets can be effectively described and enumerated (in particular, by Corollary 1). Finally, since these sets are subsets of an abelian group $u^{\mathbb{Z}[t]}$, the intersection also can be effectively described and enumerated. \hfill \Box

Denote by Paths($\Gamma$) the set of all paths in $\Gamma$ and define the set of special paths as follows

$$SPaths(\Gamma) = \{ p \in \text{Paths}(\Gamma) \mid \mu(p) = \pi(g) \text{ for some } g \in F^{\mathbb{Z}[t]} \}.$$ 

For a vertex $v \in V(\Gamma)$, similarly define

$$SPaths(\Gamma)_v = \{ p \in SPaths(\Gamma) \mid o(p) = v \}.$$ 

Observe that for every $p \in SPaths(\Gamma)$ there is a standard decomposition

$$p = p_1 \cdots p_k,$$

where either

1. $p_i$ is an edge in $\Gamma(0)$, or
2. $p_i \in u(a,b)$, for some $u$-component $C$ and $a,b \in V(C)$, is maximal with respect to inclusion.

From the above definition it is easy to draw the following result.

**Lemma 6** Let $p_1 \cdots p_k$ and $q_1 \cdots q_n$ be standard decompositions of $p \in SPaths(\Gamma)$. Then

1. $k = n$,

2. if $p_i \in E(\Gamma(0))$ then $q_i \in E(\Gamma(0))$ and $p_i = q_i$,

3. if $p_i \in u(a,b)$, for some $u$-component $C$ and $a,b \in V(C)$, then $q_i \in u(a,b)$ and $\mu(p_i) = \mu(q_i)$. 

Let \( p, q \in \text{SPaths}(\Gamma)_v \) for some \( v \in V(\Gamma) \) and let \( p = p_1 \cdots p_k \), \( q = q_1 \cdots q_n \) be some of their standard decompositions. We say \( p \) and \( q \) are \( U(\Gamma) \)-equivalent if

1. \( k = n \),
2. \( p_i = q_i \) if both are edges in \( \Gamma(0) \),
3. \( p_i, q_i \in u^A(a, b) \) for some \( u \)-component \( C \), its vertices \( a, b \), and a \( U(\Gamma) \)-equivalence class \( u^A \) in \( u^Z[t] \).

Observe that \( U(\Gamma) \)-equivalent paths have the same end-points. \( U(\Gamma) \)-equivalence classes of special paths we call types. The \( U(\Gamma) \)-equivalence class of a special path \( p \) we denote \( \text{Type}(p) \).

Let \( p \in \text{SPaths}(\Gamma) \) and \( p = p_1 \cdots p_n \) be a standard decomposition of \( p \). Hence, \( \text{Type}(p) \) can be represented as \( t_1 \cdots t_n \), where each \( t_i \) is either an edge labeled by a letter from \( X \cup X^{-1} \), or \( u^A(a, b) \) for some \( u \)-component \( C \), its vertices \( a, b \), and a \( U(\Gamma) \)-equivalence class \( u^A \) in \( u^Z[t] \).

Denote by \( \text{Types}(\Gamma) \subset (T_{\Gamma})^* \) the set of all types in \( \Gamma \). Also, denote by \( \text{Types}(\Gamma)_v \), the set of all types in \( \Gamma \) “readable” from \( v \in V(\Gamma) \).

An infinite sequence of types \( t_1, t_2, \ldots, t_n, \ldots \) in \( \Gamma \), where \( t_i \) is an initial subword of \( t_j \) for any \( i < j \), is, naturally, called an infinite type, considering it as an element of \( (T_{\Gamma})^\omega \). As in the case of finite types we define \( \text{Types}(\Gamma)^\omega \) and \( \text{Types}(\Gamma)^\omega_v \), \( v \in V(\Gamma) \).

**Remark 1** Types(\( \Gamma \)) is closed under taking subwords.

**Lemma 7** Given a \( U \)-folded \( (\mathbb{Z}[t], X) \)-graph \( \Gamma \) it is possible to decide effectively if \( \text{Types}(\Gamma) \) is infinite.

**Proof.** If \( \text{Types}(\Gamma) \) is finite then \( \Gamma \) cannot contain more than one nontrivial \( u \)-component for \( u \in U \), and there cannot be any cycles in \( \Gamma(0) \) save for those doubled in the \( u \)-component. So, \( L(\Gamma, v) \) is abelian for any \( v \in V(\Gamma) \). Observe that existence of \( u \)-components and loops in \( \Gamma(0) \) is algorithmically decidable. \( \Box \)

**Lemma 8** \( \text{Types}(\Gamma) \) is infinite if and only if \( \text{Types}(\Gamma)^\omega \) is not empty.

**Proof.** Immediate from König’s Lemma, since \( T_{\Gamma} \) is finite. \( \Box \)

Define a constant \( M(\Gamma) \in \mathbb{N} \) as follows

\[
M(\Gamma) = 1 + \max_{u \in U(\Gamma)} \|u^{\pm 1}\|.
\]

**Lemma 9** For any \( r, s, t \in \text{Types}(\Gamma) \) with \( \|s\| \geq M(\Gamma) \) if \( ts, sr \in \text{Types}(\Gamma) \) then \( tsr \in \text{Types}(\Gamma) \).
Proof. Suppose $r, s, t \in \text{Types}(\Gamma)$ and let $p_r, p_s, p_t \in \text{SPaths}(\Gamma)$ be such that $r = \text{Type}(p_r), s = \text{Type}(p_s), t = \text{Type}(p_t)$. Since $ts, sr \in \text{Types}(\Gamma)$ it follows that $ts = \text{Type}(p_tp_s)$ and $sr = \text{Type}(p_sp_t)$. We are going to show that if $s$ is long enough then $\mu(p_tp_sp_r) = \pi(\mu(p_tp_sp_r))$.

Suppose this is not the case. Let

$$p_t p_s p_r = p_1 \cdots p_n,$$

where either $p_i \in \Gamma(0)$, or $p_i$ is a maximal (with respect to inclusion) subpath inside of a $u$-component of $\Gamma$. From the definition of standard form it follows that there is $p_k$ inside of a $u$-component of $\Gamma$ and a subpath $p'$ of $p_t p_s p_r$ such that either

(a) $p' = p_1 \cdots p_{k-1}, \mu(p') = \pi(u^{\pm 1}),$ or
(b) $p' = p_1 \cdots p_{k-1}, \mu(p') = \mu(p_k) = v^\alpha, v < u,$ or
(c) $p' = p_k + 1 \cdots p_l, \mu(p') = \pi(u^{\pm 1}),$ or
(d) $p' = p_k + 1 \cdots p_l, \mu(p') = u^{\pm 1} \circ w$ and $\mu(p_i) = v^\alpha, v < u.$

Assume $\|s\| > M(\Gamma)$, that is, $\|s\| > \|u\|$ for any $u \in U(\Gamma)$. It follows that

1. if $p_k \in p_t$ then $p'$ is a subpath of $p_t p_s$,
2. if $p_k \in p_s$ then $p'$ is a subpath of either $p_t p_s$, or $p_s p_r$,
3. if $p_k \in p_r$ then $p'$ is a subpath of $p_s p_r$.

Since $ts$ and $sr$ are types it follows that we get a contradiction in any of the above cases.

Lemma 10 If $t_1, t_2, t_3, s \in \text{Types}(\Gamma)$ and $\|s\| \geq M(\Gamma)$ then $t_1 s t_2 s t_3 \in \text{Types}(\Gamma)$ if and only if $t_1 s t_3, s t_2 s \in \text{Types}(\Gamma)$.

Proof. Follows from Remark 4 and Lemma 8.

Lemma 10 explains how to discard a loop from a standard form.

Lemma 11 Let $g \in F^{Z[l]}$ be such that $\pi(g^n) = \pi(g)^n$ for any $n > 0$ and let $p$ be a path in $\Gamma$ such that $\mu(p) = \pi(g)$. Then for any $h \in F^{Z[l]}$

1. either there exists $k \in \mathbb{N}$ such that for all $n > k$, $\pi(g^n * h) = \pi(g^{n-k}) \pi(g^k * h)$, or
2. $g = g_1 \circ g_2, p = p_1 p_2$ with $\mu(p_1) = \pi(g_1), \mu(p_2) = \pi(g_2)$, where $g_2 \circ g \circ g_1 = u^\alpha$ for some $u \in U$, and $h = g_1 \circ u^\beta \circ h'$,

and, similarly,

1. either there exists $k \in \mathbb{N}$ such that for all $n > k$, $\pi(h * g^n) = \pi(h) \pi(g^{n-k}) \pi(g^k)$, or
2. $g = g_1 \circ g_2, p = p_1 p_2$ with $\mu(p_1) = \pi(g_1), \mu(p_2) = \pi(g_2)$, where $g_2 \circ g \circ g_1 = u^\alpha$ for some $u \in U$, and $h = h' \circ u^\beta \circ g_2$. 

Lemma 12 If all periodic types in $G$ is abelian and it is easy to check if we describe an algorithm which decides if $w \in \tau$ then by Lemma 9 applied to $(\hat{v}, \hat{µ})$ represented as follows finite index subgroups, August 28, 2008.

\[ N(\Gamma) = M(\Gamma)(1 + |T_\Gamma|^M(\Gamma)). \]

Proposition 8 If $Types(\Gamma)^\omega$ is non-empty then there exists a periodic type $t = t_1^t_2^\infty$ in $\Gamma$. Moreover, $\|t_1\| + \|t_2\| < N(\Gamma)$.

Proof. Consider $t = \tau_1 \tau_2 \cdots \tau_k \cdots \in Types(\Gamma)^\omega$. Hence, $\tau_1 \cdots \tau_{N(\Gamma)}$ contains at least two non-intersecting copies of a subword $s$ of length at least $M(\Gamma)$, that is,

\[ t = t_1 s t_2 s', \quad \|t_1 s t_2 s\| \leq N(\Gamma), \quad t' \in Types(\Gamma)^\omega. \]

Then by Lemma 9 applied to $(t_1 s t_2) s$ and $s(t_2)$ we have $t_1(t_2)^\infty \in Types(\Gamma)^\omega$ is periodic.

\[ \square \]

Remark 2 For any fixed $K \in \mathbb{N}$ one can effectively enumerate all periodic types of content $K$.

3.3 Finite index conditions

Let $G$ and $H$ be finitely generated subgroups of $G^{\mathbb{Z}[t]}$ such that $H \leq G$. Let $G = L(\Gamma, 1_G)$, $H = L(\Delta, 1_H)$, where $\Gamma$ and $\Delta$ are $U$-folded $(\mathbb{Z}[t], X)$-graphs, $1_G \in V(\Gamma), 1_H \in V(\Delta)$. In this subsection we describe an algorithm which decides if $|G : H| < \infty$.

Without loss of generality we can assume $Types(\Gamma)^\omega$ to be non-empty. Observe that otherwise $G$ is abelian and it is easy to check if $|G : H| < \infty$.

A path $p \in Paths(\Gamma)$ is double in $\Delta$ at $v \in V(\Delta)$ if there is a path $\hat{p} \in Paths(\Delta)$ such that $\mu(\hat{p}) = \pi(\mu(p))$.

A type $t \in Types(\Gamma) \cup Types(\Gamma)^\omega$ is

- double in $\Delta$ at $v \in V(\Delta)$ if there exists $\hat{t} \in Types(\Delta)$ such that $\mu(t) \subseteq \mu(\hat{t})$,

- $N$-almost double in $\Delta$ at $v \in V(\Delta)$ if $t = t_1 t_2$ is a finite type for which $t_1$ is doubled in $\Delta$ at $v \in V(\Delta)$ and $\|t_2\| \leq N$.

Usually, we use $\hat{\cdot}$-symbol to denote a doubling type (path). Also, below we do not specify a point at which the type is doubled when it is clear from the context. For example, if a type $t = t_1 t_2 \in Types(\Gamma) \cup Types(\Gamma)^\omega$ is double at $v \in V(\Delta)$ then $t_2$ is doubled in $\Delta$ at $v' \in V(\Delta)$, where $v' = t(p), p \in SPaths(\Delta)$, $\mu(p) \in \mu(t_1)$.

Define

\[ K(\Gamma, \Delta) = M(\Gamma)(|T_\Gamma|^M(\Gamma)(|V(\Delta)| - 1) + 1). \]

Lemma 12 If all periodic types in $Types(\Gamma)^\omega$ of content $K = K(\Gamma, \Delta)$ are double in $\Delta$ at $w \in V(\Delta)$ then any type $t \in Types(\Gamma)_v$ of length greater than $K$ (or $t \in Types(\Gamma)^\omega_v$) can be represented as follows

\[ t = t^{(1)} s^{(1)} t^{(2)} s^{(2)} t^{(3)}, \]

where
Proof. Assume that all periodic types in $\text{Types}(\Gamma)_v^w$ of content $K$ are doubled in $\Delta$ at $w \in V(\Delta)$. Let $t \in \text{Types}(\Gamma)_v$, be of length greater than $K$ (or $t \in \text{Types}(\Gamma)_v^w$). Observe that $t$ contains at least $|V(\Delta)|$ non-intersecting copies of a subword of length at least $M = M(\Gamma)$. That is, $t = t_1s_1t_2s_2\cdots t_ks_kt_{k+1}$, $s_i = s$ with $\|t_1\cdots s_k\| \leq K$. Note that by Lemma 9, $t_\infty = t_1(s_1\cdots t_k)^\infty$ is a periodic type of content at most $K$. By our assumption, $t_\infty$ is doubled in $\Delta$ by a type $\hat{t}_\infty = \hat{t}_1\hat{s}_1\hat{t}_2\hat{s}_2\cdots \hat{t}_k\hat{s}_k\cdots$, where $\hat{t}_i$ and $\hat{s}_i$ are doubles of $t_i$ and $s_i$, respectively. Hence, we have $\hat{s}_i = \hat{s}_j$ for some $i, j$, so we set $t^{(1)} = t_1s_1\cdots t_i$, $s^{(1)} = s_i$, $t^{(2)} = t_{i+1}s_{i+1}\cdots t_j$, $s^{(2)} = s_j$, $t^{(3)} = t_{j+1}s_{j+1}\cdots t_{k+1}$. 

Corollary 2 If all periodic types in $\text{Types}(\Gamma)_v^w$ of content $K = K(\Gamma, \Delta)$ are doubled in $\Delta$ at $w \in V(\Delta)$, then

1. any $t \in \text{Types}(\Gamma)_v$, $\|t\| \geq K$ can be represented as $t = t't''t'''$, $\|t''\| > 0$, where $t't''' \in \text{Types}(\Gamma)_v$, and $t$ is $K$-almost doubled if and only if $t't'''$ is $K$-almost doubled,

2. any $t \in \text{Types}(\Gamma)_v^w$ can be represented as $t = t't''t'''$, $\|t''\| > 0$, where $t't''' \in \text{Types}(\Gamma)_v^w$, and $t$ is doubled if and only if $t't'''$ is doubled.

Proof. Follows from Lemma 12 by setting $t' = t^{(1)}s^{(1)}$, $t'' = t^{(2)}s^{(2)}$, $t''' = t^{(3)}$. 

If $t = t't''t'''$ is the decomposition from the corollary above then we call $t't'''$ a reduction of $t$. Respectively, $t$ is a lift of $t't'''$.

Lemma 13 If all periodic types in $\text{Types}(\Gamma)_v^w$ of content $K = K(\Gamma, \Delta)$ are doubled in $\Delta$ at $w \in V(\Delta)$ then all types in $\text{Types}(\Gamma)_v$ are $K$-almost doubled in $\Delta$ at $w \in V(\Delta)$. Moreover, all types in $\text{Types}(\Gamma)_v^w$ are doubled.

Proof. Let $t \in \text{Types}(\Gamma)_v$. Consider a sequence

$t_1, t_2, \ldots, t_{k+1}, \ldots$

where $t_1 = t$, $t_{i+1} = t_i t'''$, is a reduction of $t_i$, and $\|t_{k+1}\| < K$ (such $k$ exists since $\|t\|$ is finite). Hence, $t_{k+1}$ is $K$-almost doubled implying that $t_k$ is $K$-almost doubled, and after $k$ lifts we get $t$ which is $K$-almost doubled.

If $t \in \text{Types}(\Gamma)_v^w$ then $t$ can be viewed as a sequence $\{t_i\}$ of finite types of increasing length which extend each other. By the argument above, $t_i = s_i r_i$, where $s_i$ is doubled and $\|r_i\| \leq K$. Hence, $\|s_i\|$ increases with growth of $i$ implying that all $t_i$ are doubled. So, $t$ is doubled as well.

In the next proposition we elaborate on the following idea. Suppose there is a type $t \in \text{Types}(\Gamma)_v$ which is not doubled in $\Delta$. Whether we can use it to produce infinitely many cosets in $G$ by $H$ depends on whether or not we can extend $t$ into an infinite type. The former implies the index is infinite. If the latter holds for all non-doubled types then all possible non-doubled pieces occur in the “ends” of these types (otherwise we can extend them without changing non-doubled pieces!). Lemma 13 puts a bound on how long the “end” can be, virtually explaining why we can cover all $G$ with a finite number of cosets by $H$. 

- $s^{(1)} = s^{(2)} = s$, $\|s\| \geq M(\Gamma)$,
- $t^{(1)}, s^{(1)}, t^{(2)}, s^{(2)}$ are doubled in $\Delta$ by $\hat{t}^{(1)}, \hat{s}^{(1)}, \hat{t}^{(2)}, \hat{s}^{(2)}$ respectively,
- $\hat{s}^{(1)} = \hat{s}^{(2)} = \hat{s}$. 

□
Proposition 9 The following statements are equivalent:

1. $|G : H| < \infty$, 

2. each periodic type in $\text{Types}(\Gamma)_{1r}^\omega$ of content $K(\Gamma, \Delta)$ is doubled in $\Delta$ at $1_H$, 

3. each infinite type in $\text{Types}(\Gamma)_{1r}^\omega$ is doubled in $\Delta$ at $1_H$.

Proof. $(2) \iff (3)$ follows from Lemma 13.

$(1) \implies (2)$ Suppose there exists $t = t_1t_2^\infty \in \text{Types}(\Gamma)_{1r}^\omega$ which is not doubled in $\Delta$ at $1_H$. Let $q_1, q_2 \in \text{SPaths}(\Gamma)$ such that $t_1 = \text{Type}(q_1)$, $t_2 = \text{Type}(q_2)$, and $q_1q_2^\infty$ is not doubled in $\Delta$. Since by the assumption $|G : H| < \infty$, there exists a finite set $g_1, g_2, \ldots, g_m$ in $G$ such that for any $g \in G$ there exists $i \in [1, m]$ such that $g \equiv g_i \in H$. We show that there exists a loop in $\Gamma$ at $1_G$ corresponding to some $g \in G$ such that none of standard forms $\pi(g \ast g_i)$ is readable in $\Delta$ from $1_H$.

Fix $i \in [1, m]$. Let $\pi(g_i) = \mu(s_i)$, $s_i \in \text{SPaths}(\Gamma)_{1_G}$ and paths $r_n = q_2^n q_1^{-1} s_i$. By Lemma 11 there are two possibilities.

(a) There exists $k(i)$ such that for any $n > k(i)$, $\text{Type}(r_n)$ begins with $t_2^{n-k(i)}$. Then by Lemma $9$ $t_1 \cdot \text{Type}(r_n) \in \text{Types}(\Gamma)_{1r}$ and its initial subtype $t_1 t_2^{n-k(i)}$ is not doubled. Hence, for any path $r \in \text{SPaths}(\Gamma)$ such that $\text{Type}(r) = \text{Type}(r_n)$ we can choose a path $p$ such that $\mu(p) \in \mu(t_1 t_2^n)$ such that $\mu(p) \mu(r)$ is a standard form of an element of $G$, and $\mu(p)$ cannot be read in $\Delta$. This implies that $\mu(p) \mu(r) \notin H$.

(b) $q_2 = q^\prime q^\prime_2, \overline{\mu(q^\prime q^\prime_2 q^\prime)} = u^\epsilon$, $u \in U(\Gamma)$ and $\mu(q_1^{-1}s_i) = \mu(q^\prime) \circ u^\beta \circ w$. Let $s = \text{Type}(q_1q_2^\beta q_1^{-1} s_i) = t' u_a v''$, where $u_a \in T_\Gamma$ corresponds to the designated occurrence of $w^\beta$, and $o(u_a) = a$.

Suppose $s$ is doubled in $\Delta$ at $1_H$. Then $H_u(a) \cap \langle u \rangle \neq \epsilon$ since $t_1 t_2^n \in \text{Types}(\Gamma)_{1r}$ for any $n$, and $H_u(a') \cap \langle u \rangle = \epsilon$ ($a'$ corresponds to $a$ in the double of $s$) since $q_1 q_2^\infty$ is not doubled. It follows that we can read only bounded finite powers of $u$ in $\text{Compl}(a')$ at $a'$, and we can choose large enough $k(i)$ so that $q_1 q_2^n q_1^{-1} s_i, n > k(i)$, is not doubled in $\Delta$ at $1_H$.

Taking $k = \max_{i=1}^m \{k(i)\}$ we obtain that $q_1 q_2^n q_1^{-1} s_i$ represents an element of $G$, and is not doubled in $\Delta$ at $1_H$ for any $n > k$ and $i \in [1, m]$.

$(2) \implies (1)$ Suppose each periodic type of content at most $K(\Gamma, \Delta)$ is doubled. We prove that $|G : H| < \infty$.

It is enough to construct a finite set of paths $r_1, \ldots, r_m \in \text{Paths}(\Gamma)_{1_G}$ such that for any loop $p \in \text{SPaths}(\Gamma)_{1_G}$ such that $t(p) = 1_G$, there exists $i \in [1, m]$ for which $\text{Type}(pr_i)$ is doubled in $\Delta$ at $1_H$. In this case the number of cosets in $G$ by $H$ is at most $|V(\Delta)|$.

Observe that $t = \text{Type}(p)$ is $K(\Gamma, \Delta)$-almost doubled by Lemma 13. Moreover, from the proof of Lemma 13 we have $t = t_1 t_2$, where $t_1$ is a lift of $t' \in \text{Types}(\Gamma)_{1_G}$ (both $t_1$ and $t'$ are doubled in $\Delta$ at $1_H$) and $||t' t_2|| < K(\Gamma, \Delta)$. Let $t_1 = \text{Type}(p_1)$, $t_2 = \text{Type}(p_2)$, $t' = \text{Type}(p')$, where $p_1$, $p_2$, $p' \in \text{SPaths}(\Gamma)$. Observe that $t(p_1) = t(p')$. It follows that $\text{Type}(p_1 p'^{-1})$ is doubled in $\Delta$ at $1_H$. Indeed, $p'$ is doubled in $\Delta$ at $1_H$, so $\text{Type}(p'^{-1})$ is also doubled at $v \in V(\Delta)$ corresponding to the end of the double of $p'$, since both $\Gamma$ and $\Delta$ are $U$-folded.

It is only left to note that $\overline{\mu(p_1 p'^{-1})} = \mu(p(p' p_2)^{-1})$, and $p' p_2$ is a loop, so $\mu((p' p_2)^{-1}) \subset G$. Since the number of types in $\Gamma$ of length at most $K(\Gamma, \Delta)$ is finite the required statement follows. 

Lemma 14 Any $t \in \text{Types}(\Delta)_{1_H}^\omega$ is doubled in $\Gamma$ at $1_G$. 

Proof. Immediately follows since $H \leq G$.

Lemma 15 Let $t = t_1t_2^\infty \in \text{Types}(\Gamma)_1^\omega$. Then there is an algorithm which decides if $t$ is doubled in $\Delta$ at $1_H$.

Proof. We “read” $\mu(t_1t_2^\infty)$ in $\Delta$ letter by letter. Observe that each letter either is a letter from $X \cup X^{-1}$, or a set of the type $H_u(a,b) \cap u^A$, $u \in U(\Gamma)$, which is effectively described. Hence, it is decidable if a letter of $t$ can be “read” at a vertex of $\Delta$. Since $\Delta$ is a finite $U$-folded graph then either this “reading” fails at some vertex and $t$ is not doubled in this case, or some vertex of $\Delta$ is hit twice, and the doubling of $t$ in $\Delta$ is found.

Theorem 4 There is an algorithm which effectively decides if $|G : H| < \infty$.

Proof. By Lemma 7 we can effectively determine if $\text{Types}(\Gamma)$, $\text{Types}(\Delta)$ are finite. If $\text{Types}(\Gamma)$ is finite then both $G$ and $H$ are abelian, so it is decidable if $|G : H| < \infty$. If $\text{Types}(\Gamma)$ is infinite but $\text{Types}(\Delta)$ is finite then $|G : H| = \infty$ because $G$ contains a non-abelian free group and $H$ is abelian. If $\text{Types}(\Gamma)$ and $\text{Types}(\Delta)$ are infinite then it is possible to check the condition (2) of Proposition 9 using Remark 2 and Lemma 15.

Corollary 3 Let $G$ be a finitely generated subgroup of $F^\mathbb{Z}[t]$ and let $H \leq G$ be finitely generated. Then there is an algorithm which effectively decides if $|G : H|$ is finite.

Corollary 4 Let $G$ be a finitely generated fully residually free group and let $H \leq G$ be finitely generated. Then there is an algorithm which effectively decides if $|G : H|$ is finite.

4 Greenberg-Stallings Theorem

In this section we prove the analog of the result for free groups known as Greenberg-Stallings Theorem (see, for example, [5] Corollary 8.8).

Let $G_1, G_2$ be finitely generated non-abelian subgroups of $F^\mathbb{Z}[t]$ such that $H = G_1 \cap G_2$ is of finite index in both $G_1$ and $G_2$. Let $\Gamma_1$, $\Gamma_2$, and $\Delta$ be $U$-folded graphs such that

$$G_1 = L(\Gamma_1, 1_{\Gamma_1}), \ G_2 = L(\Gamma_2, 1_{\Gamma_2}), \ H = L(\Delta, 1_\Delta),$$

where $1_{\Gamma_i} \in V(\Gamma_i)$, $i = 1, 2$, $1_\Delta \in V(\Delta)$. Observe that by Proposition 9 every infinite type readable in $\Gamma_i$ at $1_{\Gamma_i}$, for $i = 1, 2$, is doubled in $\Delta$ at $1_\Delta$. At the same time, since

$$L(\Delta, 1_\Delta) = L(\Gamma_1, 1_{\Gamma_1}) \cap L(\Gamma_2, 1_{\Gamma_2})$$

it follows that every infinite type readable in $\Delta$ at $1_\Delta$ is doubled in $\Gamma_i$ at $1_{\Gamma_i}$ for $i = 1, 2$. Hence, it follows that every infinite type readable in $\Gamma_1$ at $1_{\Gamma_1}$ is doubled in $\Gamma_2$ at $1_{\Gamma_2}$ and vice versa. In particular, it follows that $U(\Gamma_1) = U(\Gamma_2)$.

Lemma 16 For any $g \in \langle G_1, G_2 \rangle$ the intersection $G_j \cap H^g$ is of finite index both in $G_j$, $j = 1, 2$ and $H^g$. 
Lemma 18

For any $\Delta$ if and only if $G$ and hence in $H$. Now, since both $\Delta$ for some $h$ still denote by 1 $\Delta$ whose initial end-points are glued to 1 such that $G$ $\Delta$ $g$. Let $g = fg_n$, where $f$ is a product of length $n - 1$. By induction hypothesis $G_j \cap H^f$ is of finite index both in $G_j$, $j = 1, 2$. We use the induction on the length of this product.

Suppose $g_n \in G_1$. Then obviously $(G_1)^g_n \cap (H^f)^g_n = G_1 \cap H^g = H^g$ is of finite index in $G_1$. Next, $G_1 \cap G_2$ is of finite index in $G_1$, thus, $(G_1 \cap G_2) \cap H^g = G_2 \cap H^g < G_1$ is of finite index in $G_1$. It follows that $G_2 \cap H^g < G_2$ is of finite index in $H = G_1 \cap G_2$, and, hence, in $G_2$

The same argument can be applied when $g = g_1 \in G_2$.

$\square$

Corollary 5

For any $g \in \langle G_1, G_2 \rangle$ the intersection $G_j \cap H^g$ is not abelian.

Proof. Let $K < G < F^{2|\Gamma|}$. If $K$ is abelian and $|G : K| < \infty$ then $G$ is abelian too. By our assumption both $G_1$ and $G_2$ are non-abelian, hence $H^g$ is non-abelian too for any $g \in \langle G_1, G_2 \rangle$.

$\square$

Further we need the following corollary of Proposition 7 [8].

Lemma 17

Let $G$ be a finite generated subgroup of $F^{2|\Gamma|}$ and $H, K \leq G$ be finitely generated. Then there exists a finite list of double cosets $Kg_1H, \ldots, Kg_nH$ in $G$, such that for any $g \in G$ if $gHg^{-1} \cap K$ is non-trivial and non-abelian then there exists $i \in [1, n]$ such that $g \in Kg_iH$. Moreover, $g_1, \ldots, g_n$ can be found effectively.

Proof. According to [8] (Proposition 7, cases 1 and 3), there exists a finite list of double cosets $Kg_1H, \ldots, Kg_nH$ in $F^{2|\Gamma|}$, such that for any $g \in G$ if $gHg^{-1} \cap K$ is non-trivial and non-abelian then there exists $i \in [1, n]$ such that $g \in Kg_iH$. Moreover, $g_1, \ldots, g_n$ can be found effectively.

Observe that we can effectively determine if $G \cap Kg_i = \emptyset$ for every $i \in [1, n]$, and $G \cap Kg_i = \emptyset$ if and only if $G \cap Kg_iH = \emptyset$. Hence, we can assume that for every $g_i$ there exists at least one element $g \in G$ such that $K \cap H^g \neq 1$ and $g = fg_ih$ for some $f \in K, h \in H$. It makes it possible to assume $g_1, \ldots, g_n \in G$.

By the above lemma, there exists a finite list of double cosets $G_1g_1H, \ldots, G_1g_nH$ in $\langle G_1, G_2 \rangle$, such that for any $g \in \langle G_1, G_2 \rangle$ (since $G_1 \cap H^g$ is non-abelian for any such $g$) there exists $i \in [1, n]$ such that $g \in G_1g_iH$. Since $|G_1 : H| < \infty$ we can use the list of double cosets $Hf_1H, \ldots, Hf_kH$ in $\langle G_1, G_2 \rangle$ by $H$.

Consider a $U$-folded graph $\Phi$ obtained from $\Delta$ and paths $p_1, \ldots, p_k$ labeled by $\pi(f_1), \ldots, \pi(f_k)$ whose initial end-points are glued to $1\Delta$. Denote $v_i = t(p_i), i \in [1, k]$. The image of $1\Delta$ in $\Phi$ we still denote by $1\Delta$. Obviously, $H = L(\Phi, 1\Delta)$.

Next, by our assumption for each $f_i$ there exists an element $g \in \langle G_1, G_2 \rangle$ such that $g = h_1f_1h_2$ for some $h_1, h_2 \in H$. Since $G_1 \cap H^g$ is of finite index in $G_1$ then $G_1 \cap H^{f_i}$ is of finite index in $G_1$. Now, since both $H$ and $H^{f_i}$ have finite index in $G_1$ it follows that $H \cap H^{f_i}$ has finite index in $G_1$ and hence in $H$. Let $w_1^{(i)}, \ldots, w_m^{(i)}$ be a finite set of left coset representatives in $H$ by $H^{f_i}$.

Let $\Psi$ be a $U$-folded graph obtained from $\Phi$ and paths $p_1^{(i)}, \ldots, p_{m(i)}^{(i)}$ labeled by $\pi(w_1^{(i)}), \ldots, \pi(w_m^{(i)})$, whose initial end-points are glued to $v_i$ for each $i \in [1, k]$. The image of $1\Delta$ in $\Psi$ we still denote by $1\Delta$. Obviously, $H = L(\Psi, 1\Delta)$.

Lemma 18

For any $g \in \langle G_1, G_2 \rangle$ there exists a path $p$ in $\Psi$ such that $\sigma(p) = 1\Delta$, $\mu(p) = \pi(g)$. 


Proof. Take \( g \in (G_1, G_2) \). Hence, it can be represented as \( g = h_1 f_i h_2 \) for some \( f_i \) and \( h_1, h_2 \in H \). It follows that it can be represented further as \( g = h_1 f_i w^{(i)}_j \), where \( w \in H^{f_i} \) and \( w^{(i)}_j \) is a coset representative in \( H \) by \( H^{f_i} \). By our construction there is a loop at \( 1_\Delta \) corresponding to \( h_1 \), a path from \( 1_\Delta \) to \( v_i \) corresponding to \( f_i \), a loop at \( v_i \) corresponding to \( w \), and a path \( p_j^{(1)} \) corresponding to \( w^{(1)}_j \). Since \( \Psi \) is folded there is also a required path \( p \).

\( \Box \)

**Corollary 6** \(|\langle G_1, G_2 \rangle : H| < \infty \).

**Proof.** Follows from Lemma 18 and Proposition 5

**Theorem 5** Let \( G_1, G_2 \) be finitely generated subgroups of \( F^{\mathbb{Z}[t]} \). If \( H \leq G_1 \cap G_2 \) is finitely generated and \(|G_1 : H| < \infty, |G_2 : H| < \infty\) then \(|\langle G_1, G_2 \rangle : H| < \infty\).

**Theorem 6** Let \( G \) be a finitely generated fully residually free group and \( G_1, G_2 \) be finitely generated subgroups of \( G \). If \( H \leq G_1 \cap G_2 \) is finitely generated and \(|G_1 : H| < \infty, |G_2 : H| < \infty\) then \(|\langle G_1, G_2 \rangle : H| < \infty\).

## 5 Commensurator

Let \( G \) be a group and let \( H \leq G \). The **commensurator** \( \text{Comm}_G(H) \) of \( H \) in \( G \) is defined as

\[
\text{Comm}_G(H) = \{ g \in G \mid |H : H \cap gHg^{-1}| < \infty \text{ and } |gHg^{-1} : H \cap gHg^{-1}| < \infty \}.
\]

It is easy to see that \( \text{Comm}_G(H) \) is a subgroup of \( G \) containing \( H \).

**Lemma 19** Let \( G \) and \( H \) be finitely generated non-abelian subgroups of \( F^{\mathbb{Z}[t]} \) such that \( H \leq G \). Then there exist \( f_1, \ldots, f_k \in G \) such that \( \text{Comm}_G(H) = \langle H, f_1, \ldots, f_k \rangle \). Moreover, the elements \( g_1, \ldots, g_k \) can be found effectively.

**Proof.** By Lemma 17 there exists a finite list of double cosets \( Hg_1 H, \ldots, Hg_n H \) in \( G \), such that for any \( g \in G \) if \( gHg^{-1} \cap H \) is non-trivial and non-abelian then there exists \( i \in [1, n] \) such that \( g \in Hg_i H \). Moreover, \( g_1, \ldots, g_n \) can be found effectively.

By Corollary 3 for each \( g_i \) one can check effectively if \( |H : H \cap g_i Hg_i^{-1}| < \infty \) and \( |g_i Hg_i^{-1} : H \cap g_i Hg_i^{-1}| < \infty \), so, we take only those elements \( g_i \) for which both indeces are finite. Such elements form the required list \( f_1, \ldots, f_k \).

\( \Box \)

**Theorem 7** Let \( G \) and \( H \) be finitely generated non-abelian subgroups of \( F^{\mathbb{Z}[t]} \) such that \( H \leq G \). Then \( \text{Comm}_G(H) \) is finitely generated, and its generating set can be found effectively.

**Proof.** Follows from Lemma 19

\( \Box \)

**Corollary 7** Let \( G \) be a finitely generated fully residually free group and let \( H \leq G \) be finitely generated. Then \( \text{Comm}_G(H) \) is finitely generated, and its generating set can be found effectively.
Lemma 20 Let $G$ and $H$ be finitely generated non-abelian subgroups of $F^\mathbb{Z}[t]$ such that $H \leq G$. Then $|N_G(H) : H| < \infty$.

Proof. By Lemma [17], there exists a finite list of double cosets $Hg_1H, \ldots, Hg_nH$ in $G$, such that for any $g \in G$ if $gHg^{-1} \cap H$ is non-trivial and non-abelian then there exists $i \in [1,n]$ such that $g \in Hg_iH$. Moreover, $g_1, \ldots, g_n$ can be found effectively. So, if $S \subset \{g_1, \ldots, g_n\}$ such that $gHg^{-1} = H$ for each $g \in S$ then

$$N_G(H) = \bigcup_{g \in S} HgH = \bigcup_{g \in S} gH.$$ 

\[\square\]

Theorem 8 Let $G$ and $H$ be finitely generated non-abelian subgroups of $F^\mathbb{Z}[t]$ such that $H \leq G$. Then $|\text{Comm}_G(H) : H| < \infty$.

Proof. By Lemma [19] we have $C = \text{Comm}_G(H) = \langle H, g_1, \ldots, g_n \rangle$ for some $g_1, \ldots, g_n \in G$. Consider

$$H' = \langle H \cup H^{g_1} \cup \cdots \cup H^{g_n} \rangle \leq G.$$ 

It is easy to see that $H' \lhd C$, so, $C \leq N_G(H')$. Therefore $|C : H'| < \infty$. Finally, $|H' : H| < \infty$ by Theorem [5].

\[\square\]

Corollary 8 Let $G$ be a finitely generated fully residually free group and let $H$ be its finitely generated non-abelian subgroup. Then $|\text{Comm}_G(H) : H| < \infty$.

Corollary 9 Let $G$ be a finitely generated fully residually free group and let $H$ be its finitely generated non-abelian subgroup. Then there exists an effectively computable natural number $n(H)$ such that for every $K \leq G$ containing $H$, if $|K : H| < \infty$ then $|K : H| < n(H)$.

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