THE BAIRE CATEGORY OF SUBSEQUENCES AND
PERMUTATIONS WHICH PRESERVE LIMIT POINTS

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Abstract. Let $\mathcal{I}$ be a meager ideal on $\mathbb{N}$. We show that if $x$ is a sequence with values in a separable metric space then the set of subsequences [resp. permutations] of $x$ which preserve the set of $\mathcal{I}$-cluster points of $x$ is topologically large if and only if every ordinary limit point of $x$ is also an $\mathcal{I}$-cluster point of $x$. The analogue statement fails for all maximal ideals. This extends the main results in [Topology Appl. 263 (2019), 221–229]. As an application, if $x$ is a sequence with values in a first countable compact space which is $\mathcal{I}$-convergent to $\ell$, then the set of subsequences [resp. permutations] which are $\mathcal{I}$-convergent to $\ell$ is topologically large if and only if $x$ is convergent to $\ell$ in the ordinary sense. Analogous results hold for $\mathcal{I}$-limit points, provided $\mathcal{I}$ is an analytic P-ideal.

1. Introduction

A classical result of Buck [7] states that, if $x$ is real sequence, then “almost every” subsequence of $x$ has the same set of ordinary limit points of the original sequence $x$, in a measure sense. The aim of this note is to prove its topological analogue and non-analogue in the context of ideal convergence.

Let $\mathcal{I}$ be an ideal on the positive integers $\mathbb{N}$, that is, a family of subsets of $\mathbb{N}$ closed under subsets and finite unions. Unless otherwise stated, it is also assumed that $\mathcal{I}$ contains the ideal $\text{Fin}$ of finite sets and it is different from the power set $\mathcal{P}(\mathbb{N})$. $\mathcal{I}$ is a P-ideal if it is $\sigma$-directed modulo finite sets, i.e., for every sequence $(A_n)$ of sets in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all $n$. We regard ideals as subsets of the Cantor space $\{0,1\}^\mathbb{N}$, hence we may speak about their topological complexities. In particular, an ideal can be $F_\sigma$, analytic, etc. Among the most important ideals, we find the family of asymptotic density zero sets $\mathcal{Z} := \{A \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{1}{n} |A \cap [1,n]| = 0\}$.

We refer to [15] for a recent survey on ideals and associated filters.

Let $x = (x_n)$ be a sequence taking values in a topological space $X$, which will be always assumed to be Hausdorff. Then $\ell \in X$ is an $\mathcal{I}$-cluster point of $x$ if

$$\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$$

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for each neighborhood \( U \) of \( \ell \). The set of \( I \)-cluster points of \( x \) is denoted by \( \Gamma_x(I) \). Moreover, \( \ell \in X \) is an \( I \)-limit point of \( x \) if there exists a subsequence \( (x_{n_k}) \) such that
\[
\lim_{k \to \infty} x_{n_k} = \ell \quad \text{and} \quad \{n_k : k \in \mathbb{N}\} \notin I.
\]
The set of \( I \)-limit points is denoted by \( \Lambda_x(I) \). Statistical cluster points and statistical limits points (that is, \( Z \)-cluster points and \( Z \)-limit points) of real sequences were introduced by Fridy in [13] and studied by many authors, see e.g. [8, 10, 14, 17, 26, 27]. It is worth noting that ideal cluster points have been studied much before under a different name. Indeed, as it follows by [23, Theorem 4.2], they correspond to classical “cluster points” of a filter \( \mathcal{F} \) (depending on \( x \)) on the underlying space, cf. [6, Definition 2, p.69]. Lastly, let \( L_x := \Gamma_x(\text{Fin}) \) denote the set of ordinary limit points of \( x \). Hence \( \Lambda_x(I) \subseteq \Gamma_x(I) \subseteq L_x \). See [23] for characterizations of \( I \)-cluster points and [2] for their relation with \( I \)-limit points.

Let \( \Sigma \) be the sets of strictly increasing functions on \( \mathbb{N} \), that is,
\[
\Sigma := \{\sigma \in \mathbb{N}^\mathbb{N} : \forall n \in \mathbb{N}, \sigma(n) < \sigma(n+1)\};
\]
also, let \( \Pi \) be the sets of permutations of \( \mathbb{N} \), that is,
\[
\Pi := \{\pi \in \mathbb{N}^\mathbb{N} : \pi \text{ is a bijection}\}.
\]
Note that both \( \Sigma \) and \( \Pi \) are \( G_\delta \)-subsets of the Polish space \( \mathbb{N}^\mathbb{N} \), hence they are Polish spaces as well by Alexandrov’s theorem; in particular, they are not meager in themselves, cf. [30, Chapter 2].

Given a sequence \( x \) and \( \sigma \in \Sigma \), we denote by \( \sigma(x) \) the subsequence \( (x_{\sigma(n)}) \). Similarly, given \( \pi \in \Pi \), we write \( \pi(x) \) for the rearranged sequence \( (x_{\pi(n)}) \). This gives clearly a bijection between \( \Sigma \) [resp. \( \Pi \)] and the set of subsequences of \( x \) [resp. permutations of \( x \)], cf. [1, 3, 27].

We will show that if \( I \) is a meager ideal and \( x \) is a sequence with values in a separable metric space then the set of subsequences (and permutations) of \( x \) which preserve the set of \( I \)-cluster points of \( x \) is not meager if and only if every ordinary limit point of \( x \) is also an \( I \)-cluster point of \( x \) (Theorem 2.2). A similar result holds for \( I \)-limit points, provided that \( I \) is an analytic P-ideal (Theorem 2.6). Putting all together, this strengthens all the results contained in [22] and answers an open question therein. As a byproduct, we obtain a characterization of meager ideals (Proposition 3.1). Lastly, the analogue statements fails for all maximal ideals (Example 2.3).

2. Main results

2.1. \( I \)-cluster points. It has been shown in [22] that, from a topological viewpoint, almost all subsequences of \( x \) preserve the set of \( I \)-cluster points, provided that \( I \) is “well separated” from its dual filter \( I^* := \{A \subseteq \mathbb{N} : A^c \in I\} \); that is,
\[
\Sigma_x(I) := \{\sigma \in \Sigma : \Gamma_{\sigma(x)}(I) = \Gamma_x(I)\}
\]
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is comeager, cf. also [24] for the case $\mathcal{I} = \mathcal{Z}$ and [20] for a measure theoretic analogue. We will extend this result to all meager ideals. In addition, we will see that the same holds also for

$$\Pi_x(\mathcal{I}) := \{ \pi \in \Pi : \Gamma_{\pi(x)}(\mathcal{I}) = \Gamma_x(\mathcal{I}) \}.$$ 

Here, given $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{P}(\mathbb{N})$, we say that $\mathcal{A}$ is separated from $\mathcal{C}$ by $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$. In particular, an ideal $\mathcal{I}$ is $F_\sigma$-separated from its dual filter $\mathcal{I}^*$ if there exists an $F_\sigma$-set $\mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ such that $\mathcal{I} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{I}^* = \emptyset$ (with the language of [9, 18], the filter $\mathcal{I}^*$ has rank $\leq 1$).

**Theorem 2.1.** [22, Theorem 2.1] Let $x$ be a sequence in a first countable space $X$ such that all closed sets are separable and let $\mathcal{I}$ be an ideal which is $F_\sigma$-separated from its dual filter $\mathcal{I}^*$. Then $\Sigma_x(\mathcal{I})$ is not meager if and only if $\Gamma_x(\mathcal{I}) = L_x$. Moreover, in this case, it is comeager.

As it has been shown in [29, Corollary 1.5], the family of ideals $\mathcal{I}$ which are $F_\sigma$-separated from $\mathcal{I}^*$ includes all $F_{\sigma\delta}$-ideals. In addition, a Borel ideal is $F_\sigma$-separated from its dual filter if and only if it does not contain an isomorphic copy of $\text{Fin} \times \text{Fin}$ (which can be represented as an ideal on $\mathbb{N}$ as

$$\{ A \subseteq \mathbb{N} : \forall \omega \in \mathbb{N}, \{ a \in A : \nu_2(a) = n \} \in \text{Fin} \}$$

where $\nu_2(n)$ stands for the 2-adic valuation of $n$), see [19, Theorem 4]. In particular, $\text{Fin} \times \text{Fin}$ is a $F_{\sigma\delta\sigma}$-ideal which is not $F_\sigma$-separated from its dual filter. For related results on $F_\sigma$-separation, see [11, Proposition 3.6] and [32].

We show that the analogue of Theorem 2.1 holds for all meager ideals. In particular, this includes new cases as, for instance, $\mathcal{I} = \text{Fin} \times \text{Fin}$.

It is worth noting that every meager ideal $\mathcal{I}$ is $F_\sigma$-separated from the Fréchet filter $\text{Fin}^*$ (see Proposition 3.1 below), hence $\mathcal{I}$ is $F_\sigma$-separated from $\mathcal{I}^*$. This implies that our result is a proper generalization of Theorem 2.1.

**Theorem 2.2.** Let $x$ be a sequence in a first countable space $X$ such that all closed sets are separable and let $\mathcal{I}$ be a meager ideal. Then the following are equivalent:

(c1) $\Sigma_x(\mathcal{I})$ is comeager in $\Sigma$;
(c2) $\Sigma_x(\mathcal{I})$ is not meager in $\Sigma$;
(c3) $\Pi_x(\mathcal{I})$ is comeager in $\Pi$;
(c4) $\Pi_x(\mathcal{I})$ is not meager in $\Pi$;
(c5) $\Gamma_x(\mathcal{I}) = L_x$.

Note that the standing hypotheses hold if $X$ is a separable metric space. At this point, one may ask whether the same statement holds for all ideals. We show in the following example that the answer is negative.

**Example 2.3.** Let $\mathcal{I}$ be a maximal ideal. Hence there exists a unique $A \in \{ 2\mathbb{N} + 1, 2\mathbb{N} + 2 \}$ such that $A \in \mathcal{I}$. Set $X = \mathbb{R}$. Let $x$ be the characteristic function of $A$, i.e., $x_n = 1$ if $n \in A$ and $x_n = 0$ otherwise. Then $x \to^\mathcal{I} 0$. In
particular, \( \Gamma_x(\mathcal{I}) = \{0\} \). Note that a subsequence \( \sigma(x) \) is \( \mathcal{I} \)-convergent to 0 if and only if \( \Gamma_{\sigma(x)} = \{0\} \). Then

\[
\Sigma_x(\mathcal{I}) = \{ \sigma \in \Sigma : \sigma^{-1}[A] \in \mathcal{I} \}.
\]

Considering that \( \sigma^{-1}[A] \cup \sigma^{-1}[A - 1] \) is cofinite, we have either \( \sigma^{-1}[A] \in \mathcal{I} \) or \( \sigma^{-1}[A - 1] \in \mathcal{I} \). Let \( T : \Sigma \to \Sigma \) be the embedding defined by \( \sigma \mapsto \sigma + 1 \), so that \( \Sigma \) is homeomorphic to the open set \( T[\Sigma] = \{ \sigma \in \Sigma : \sigma(1) \geq 2 \} \). Notice that

\[
T[\Sigma_x(\mathcal{I})] = \{ T(\sigma) : \sigma \in \Sigma_x(\mathcal{I}) \} = \{ \sigma + 1 \in \Sigma : \sigma^{-1}[A] \in \mathcal{I} \}
\]

\[
= \{ \sigma \in \Sigma : \sigma^{-1}[A - 1] \in \mathcal{I} \} \cap T[\Sigma],
\]

which implies that the open set \( T[\Sigma] \) is contained in \( \Sigma_x(\mathcal{I}) \cup T[\Sigma_x(\mathcal{I})] \). Therefore both \( \Sigma_x(\mathcal{I}) \) and \( T[\Sigma_x(\mathcal{I})] \) are not meager.

A similar example can be found for \( \Pi_x(\mathcal{I}) \), replacing the embedding \( T \) with the homeomorphism \( H : \Pi \to \Pi \) defined by \( H(\pi)(2n) = 2n - 1 \) and \( H(\pi)(2n - 1) = 2n \) for all \( n \in \mathbb{N} \).

As an application of our results, if \( x \) is \( \mathcal{I} \)-convergent to \( \ell \), then the set of subsequences [resp., rearrangements] of \( x \) which are \( \mathcal{I} \)-convergent to \( \ell \) is not meager if and only if \( x \) is convergent (in the classical sense) to \( \ell \). (Here, a sequence \( x \) is said to be \( \mathcal{I} \)-convergent to \( \ell \), shortened as \( x \to_\mathcal{I} \ell \), if \( \{ n \in \mathbb{N} : x_n \notin U \} \in \mathcal{I} \) for each neighborhood \( U \) of \( \ell \).) This is in a sense, related to [1, Theorem 2.1] and [3, Theorem 1.1]; cf. also [26, Theorem 3] for a measure theoretical non-analogue.

**Corollary 2.4.** Let \( x \) be a sequence in a first countable compact space \( X \). Let \( \mathcal{I} \) be a meager ideal and assume that \( x \) is \( \mathcal{I} \)-convergent to \( \ell \in X \). Then the following are equivalent:

1. \( \{ \sigma \in \Sigma : \sigma(x) \to_\mathcal{I} \ell \} \) is comeager in \( \Sigma \);
2. \( \{ \sigma \in \Sigma : \sigma(x) \to_\mathcal{I} \ell \} \) is not meager in \( \Pi \);
3. \( \{ \pi \in \Pi : \pi(x) \to_\mathcal{I} \ell \} \) is comeager in \( \Sigma \);
4. \( \{ \pi \in \Pi : \pi(x) \to_\mathcal{I} \ell \} \) is not meager in \( \Pi \);
5. \( \lim_n x_n = \ell \).

The proofs of Theorem 2.2 and Corollary 2.4 follow in Section 3.

### 2.2. \( \mathcal{I} \)-limit points

Given a sequence \( x \) and an ideal \( \mathcal{I} \), define

\[
\tilde{\Sigma}_x(\mathcal{I}) := \{ \sigma \in \Sigma : \Lambda_{\sigma(x)}(\mathcal{I}) = \Lambda_x(\mathcal{I}) \}
\]

and its analogue for permutations

\[
\tilde{\Pi}_x(\mathcal{I}) := \{ \pi \in \Pi : \Lambda_{\pi(x)}(\mathcal{I}) = \Lambda_x(\mathcal{I}) \}.
\]

It has been shown in [22] that, in the case of \( \mathcal{I} \)-limit points, the counterpart of Theorem 2.1 holds for generalized density ideals. Here, an ideal \( \mathcal{I} \) is said to be a **generalized density ideal** if there exists a sequence \( \{\mu_n\} \) of submeasures with finite and pairwise disjoint supports such that \( \mathcal{I} = \{ A \subseteq \mathbb{N} : \lim_n \mu_n(A) = 0 \} \). More precisely:
Theorem 2.5. [22, Theorem 2.3] Let $x$ be a sequence in a first countable space $X$ such that all closed sets are separable and let $\mathcal{I}$ be generalized density ideal. Then $\hat{\Sigma}_x(\mathcal{I})$ is not meager if and only if $\Lambda_x(\mathcal{I}) = L_x$. Moreover, in this case, it is comeager.

See [21] for a measure theoretic analogue. It has been left as open question to check, in particular, whether the same statement holds for analytic $P$-ideals. We show that the answer is affirmative.

Note that this is strict generalization, as every generalized density ideal is an analytic $P$-ideal and there exists an analytic $P$-ideal which is not a generalized density ideal, see e.g. [5]. In addition, the same result holds for permutations.

Theorem 2.6. Let $x$ be a sequence in a first countable space $X$ such that all closed sets are separable and let $\mathcal{I}$ be an analytic $P$-ideal. Then the following are equivalent:

- (L1) $\hat{\Sigma}_x(\mathcal{I})$ is comeager in $\Sigma$;
- (L2) $\Sigma_x(\mathcal{I})$ is not meager in $\Sigma$;
- (L3) $\hat{\Pi}_x(\mathcal{I})$ is comeager in $\Pi$;
- (L4) $\Pi_x(\mathcal{I})$ is not meager in $\Pi$;
- (L5) $\Gamma_x(\mathcal{I}) = L_x$.

Note that the same Example 2.3 shows that the analogue of Theorem 2.6 fails for all maximal ideals. The proof of Theorem 2.6 follows in Section 4.

We leave as an open question to check whether Theorem 2.6 may be extended to all meager ideals.

3. Proofs for $\mathcal{I}$-cluster points

We start with a characterization of meager ideals (to the best of our knowledge, conditions (m3) and (m4) are novel). Here, a set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is called hereditary if it is closed under subsets.

Proposition 3.1. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then the following are equivalent:

- (m1) $\mathcal{I}$ is meager;
- (m2) There exists a strictly increasing sequence $(\iota_n)$ of positive integers such that $A \notin \mathcal{I}$ whenever $\mathbb{N} \cap [\iota_n, \iota_{n+1}) \subseteq A$ for infinitely many $n \in \mathbb{N}$;
- (m3) $\mathcal{I}$ is $F_\sigma$-separated from the Fréchet filter $\text{Fin}^*$;
- (m4) $\mathcal{I}$ is separated from the Fréchet filter $\text{Fin}^*$ by $\bigcup_k F_k$, where each $F_k$ is a hereditary closed set invariant under finite changes.

Proof. (m1) $\iff$ (m2) See [31, Theorem 2.1]; cf. also [4, Theorem 4.1.2].

(m2) $\implies$ (m3) Define $I_n := \mathbb{N} \cap [\iota_n, \iota_{n+1})$ for all $n \in \mathbb{N}$. Then $\mathcal{I} \subseteq \mathcal{F}$, where

$$\forall k \in \mathbb{N}, \quad F_k := \bigcap_{n \geq k} \{ A \subseteq \mathbb{N} : I_n \notin A \}. \quad (1)$$
Note that each $F_k$ is closed and it does not contain any cofinite set. Therefore $\mathcal{I}$ is separated from $\Fin^*$ by the $F_\sigma$-set $\mathcal{F}$.

(M3) $\iff$ (M4) Suppose that $\mathcal{I}$ is separated from $\Fin^*$ by $\bigcup_k C_k$, where each $C_k$ is closed. Then it is enough to set

$$\forall k \in \mathbb{N}, \quad F_k := \{ A \subseteq \mathbb{N} : A \setminus C_k \in \Fin \}.$$ 

In particular, $F_k$ does not contain any cofinite set. The converse is obvious.

(M3) $\implies$ (M1) Suppose that there exists a sequence $(F_k)$ of closed sets in $\{0,1\}^\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{F} := \bigcup_k F_k$ and $\mathcal{F} \cap \Fin^* = \emptyset$. Then each $F_k$ has empty interior (otherwise it would contain a cofinite set). We conclude that $\mathcal{I}$ is contained in a countable union of nowhere dense sets.

The above characterization is reminiscent of an open question of Mazur [25], cf. also [29, p. 220]: Is every $F_{\sigma\delta}$-ideal contained in a hereditary $F_\sigma$-set $\mathcal{F}$ such that $X \cup Y$ is not cofinite for all $X, Y \in \mathcal{F}$?

In addition, it is clear that condition (M3) is weaker than the extendability of $\mathcal{I}$ to a $F_\sigma$-ideal. For characterizations and related results of the latter property, see e.g. [16, Theorem 4.4] and [12, Theorem 3.3].

Lemma 3.2. Let $x$ be a sequence in a first countable space $X$ and let $\mathcal{I}$ be a meager ideal. Then

$$S(\ell) := \{ \sigma \in \Sigma : \ell \in \Gamma_{\sigma(x)}(\mathcal{I}) \}$$

is comeager for each $\ell \in L_x$.

Proof. Assume that $L_x \neq \emptyset$, otherwise there is nothing to prove. Fix $\ell \in L_x$ and let $(U_m)$ be a decreasing local base at $\ell$. Thanks to Proposition 3.1, there exists a sequence $(F_k)$ of closed sets in $\{0,1\}^\mathbb{N}$ such that $\mathcal{I}$ is contained in $\bigcup_k F_k$ and $F_k \cap \Fin^* = \emptyset$ for all $k \in \mathbb{N}$.

At this point, we need to show that $M := \{ \sigma \in \Sigma : \ell \notin \Gamma_{\sigma(x)}(\mathcal{I}) \}$ is meager. Observe that $M \subseteq \bigcup_k M_{t,k}$, where

$$\forall t, k \in \mathbb{N}, \quad M_{t,k} := \{ \sigma \in \Sigma : \{ n \in \mathbb{N} : x_{\sigma(n)} \in U_1 \} \subseteq F_k \}.$$ 

Hence, it is sufficient to show that each $M_{t,k}$ is nowhere dense.

To this aim, first we show that $M_{t,k}$ is closed. Fix $\sigma_0 \in M_{t,k}$ (if there is no such $\sigma_0$ then $M_{t,k} = \Sigma$ is closed in $\Sigma$). Since $F_k$ is closed, there exists $n_0 \in \mathbb{N}$ such that

$$\{ \sigma \in \Sigma : \sigma |_{\{1, \ldots, n_0\}} = \sigma_0 |_{\{1, \ldots, n_0\}} \} \subseteq M_{t,k}.$$ 

Hence $M_{t,k}$ is closed. Lastly, we show that $M_{t,k}$ has empty interior. Fix $\sigma_1 \in \Sigma$ such that the subsequence $(x_{\sigma_1(n)})$ converges to $\ell$ (note that this is possible) and let us suppose for the sake of contradiction that there exist positive integers $e_1 < \cdots < e_{n_1}$ such that $\sigma_1 \in M_{t,k}$ whenever $(\sigma(1), \ldots, \sigma(n_1)) = (e_1, \ldots, e_{n_1})$. Define $\sigma^* \in \Sigma$ by $(\sigma^*(1), \ldots, \sigma^*(n_1)) = (e_1, \ldots, e_{n_1})$ and $\sigma^*(n) = \sigma_1(n)$ for all $n > n_1$. Then $\sigma^* \in M_{t,k}$ and thus $\{ n \in \mathbb{N} : x_{\sigma^*(n)} \in U_1 \} \subseteq F_k$. At the same time, the set $\{ n \in \mathbb{N} : x_{\sigma^*(n)} \in U_1 \}$ belongs to $\Fin^*$ because the subsequence $(x_{\sigma^*(n)})$ is
convergent to $\ell$. Considering that $F_k$ does not contain any cofinite set, we reach the desired contradiction, proving that $S(\ell)$ is comeager in $\Sigma$. \qed

**Lemma 3.3.** Let $x$ be a sequence in a first countable space $X$ and let $\mathcal{I}$ be a meager ideal. Then

$$P(\ell) := \{ \pi \in \Pi : \ell \in \Gamma_{\pi(x)}(\mathcal{I}) \}$$

is comeager for each $\ell \in L_x$.

**Proof.** The proof that $P(\ell)$ is comeager in $\Pi$ is similar to the previous one, the only difference being in proving that the analogue of $M_{t,k}$ for permutations, that is,

$$\hat{M}_{t,k} := \{ \pi \in \Pi : \{ n \in \mathbb{N} : x_{\pi(n)} \in U_t \} \in F_k \},$$

has empty interior, for each $t, k \in \mathbb{N}$, where $F_k$ is defined as in (1). Let us suppose for the sake of contradiction that $\hat{M}_{t,k}$ has an interior point, let us say $\pi_0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\{ \pi \in \Pi : \pi \mid \{ 1, \ldots, n_0 \} = \pi_0 \mid \{ 1, \ldots, n_0 \} \} \subseteq \hat{M}_{t,k}.$$ 

In particular, $\max\{ \pi_0(1), \ldots, \pi_0(n_0) \} \geq n_0$. Fix $\sigma \in \Sigma$ such that $\lim_n x_{\sigma(n)} = \ell$ (note that it is possible), so that

$$V := \{ n \in \mathbb{N} : x_{\sigma(n)} \notin U_t \} \in \text{Fin}.$$ 

Set $v := \max V$, with $v := 0$ if $V = \emptyset$. With the same notations of Proposition 3.1, let $m$ be a sufficiently large integer such that $m \geq k$ and $\min I_m = \iota_m > \max\{ v, \pi_0(1), \ldots, \pi_0(n_0) \}$. In particular, $\iota_m > n_0$. Then, let $\pi^* \in \Pi$ be a permutation of $\mathbb{N}$ such that $\pi^*(n) = \pi_0(n)$ for all $n \leq n_0$ and $\pi^*(n) = \sigma(n)$ for all $n \in I_m$. On the one hand, $\pi^*$ belongs to $\hat{M}_{t,k}$, hence $S := \{ n \in \mathbb{N} : x_{\pi^*(n)} \in U_t \} \in F_k$. On the other hand, by construction $S$ contains $I_m$, hence $S \notin F_k$. This contradiction shows that $\pi_0$ cannot be an interior point. \qed

We are finally ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** (c1) $\implies$ (c2) It is obvious.

(c2) $\implies$ (c5) Suppose that there exists $\ell \in L_x \setminus \Gamma_{x} (\mathcal{I})$. Then $\Sigma_x(\mathcal{I})$ is contained in $\Sigma \setminus S(\ell)$, which is meager by Lemma 3.2.

(c5) $\implies$ (c1) Suppose that $L_x \neq \emptyset$, otherwise the claim is trivial. Let $\mathcal{L}$ be a countable dense subset of $L_x$, so that $\mathcal{L} \subseteq \Gamma_{\sigma(x)} (\mathcal{I})$ for each $\sigma \in S := \bigcap_{\ell \in L_x} S(\ell)$, which is comeager by Lemma 3.2. Fix $\sigma \in S$. On the one hand, $\Gamma_{\sigma(x)} (\mathcal{I}) \subseteq L_{\sigma(x)} \subseteq L_x$. On the other hand, since $\Gamma_{\sigma(x)} (\mathcal{I})$ is closed by [23, Lemma 3.1(iv)], we get $L_x \subseteq \Gamma_{\sigma(x)} (\mathcal{I})$. Therefore $S \subseteq \Sigma_x (\mathcal{I})$.

The implications (c3) $\implies$ (c4) $\implies$ (c5) $\implies$ (c3) are analogous. \qed

**Remark 3.4.** As it is evident from the proof above, the hypothesis that “closed sets of $X$ are separable” can be removed if, in addition, $L_x$ is countable.

**Lemma 3.5.** Let $\mathcal{I}$ be an ideal and $x$ be a sequence in a first countable compact space. Then $x \rightarrow_{\mathcal{I}} \ell$ if and only if $\Gamma_{x} (\mathcal{I}) = \{ \ell \}$. 

Proof. It follows by \cite[Corollary 3.4]{23}.

Proof of Corollary 2.4. (i1) \implies (i2) It is obvious.

(i2) \implies (i5) By Lemma 3.5, the hypothesis can be rewritten as \(\Gamma_x(I) = \{\ell\}\).

Hence, condition (i2) is equivalent to the nonmeagerness of \(\{\sigma \in \Sigma : \Gamma_{\sigma(x)} = \Gamma_x(I) = \{\ell\}\}\). The claim follows by Theorem 2.2 and Remark 3.4.

(i5) \implies (i1) If \(x \to \ell\) then \(\sigma(x) \to_I \ell\) for all \(\sigma \in \Sigma\).

The implications (i3) \implies (i4) \implies (i5) \implies (i3) are analogous. \qed

4. PROOFS FOR \(I\)-LIMIT POINTS

A lower semicontinuous submeasure (in short, lscsm) is a monotone subadditive function \(\varphi : P(N) \to [0, \infty]\) such that \(\varphi(\emptyset) = 0\), \(\varphi(F) < \infty\) for all \(F \in \text{Fin}\), and \(\varphi(A) = \lim_n \varphi(A \cap [1, n])\) for all \(A \subseteq N\). By a classical result of Solecki, an ideal \(I\) is an analytic P-ideal if and only if there exists a lscsm \(\varphi\) such that

\[I = \text{Exh}(\varphi) := \{A \subseteq N : \|A\|_\varphi = 0\} \text{ and } 0 < \|N\|_\varphi \leq \varphi(N) < \infty,\]  

(2)

where \(\|A\|_\varphi := \lim_n \varphi(A \setminus [1, n])\) for all \(A \subseteq N\); see \cite[Theorem 3.1]{28}. Note that \(\| \cdot \|_\varphi\) is a submeasure which is invariant modulo finite sets. Moreover, replacing \(\varphi\) with \(\varphi/\|N\|_\varphi\) in (2), we can assume without loss of generality that \(\|N\|_\varphi = 1\).

Given a sequence \(x\) in a first countable topological space \(X\) and an analytic P-ideal \(I = \text{Exh}(\varphi)\), we define the function

\[u : \Sigma \times X \to \mathbb{R} : (\sigma, \ell) \mapsto \lim_{k \to \infty} \|\{n \in N : x_{\sigma(n)} \in U_k\}\|_\varphi,\]  

(3)

where \((U_k)\) is a decreasing local base of neighborhoods at \(\ell \in X\). Clearly, the limit in (3) exists and it is independent of the choice of \((U_k)\).

**Lemma 4.1.** Let \(x\) be a sequence in a first countable space \(X\) and let \(I = \text{Exh}(\varphi)\) be an analytic P-ideal. Then, the section \(u(\sigma, \cdot)\) is upper semicontinuous for each \(\sigma \in \Sigma\). In particular, the set

\[\Lambda_{\sigma(x)}(I, q) := \{\ell \in X : u(\sigma, \ell) \geq q\}\]

is closed for all \(q > 0\).

**Proof.** See \cite[Lemma 2.1]{2}.

**Lemma 4.2.** With the same hypotheses of Lemma 4.1, the set

\[V(\ell, q) := \{\sigma \in \Sigma : u(\sigma, \ell) > q\}\]

is either comeager or empty for each \(\ell \in X\) and \(q \in (0,1)\).
Suppose that $V(\ell, q) \neq \emptyset$, so that $\ell \in L_x$, and note that

$$\Sigma \setminus V(\ell, q) = \bigcup_{k \geq 1} \{ \sigma \in \Sigma : \|\{n \in \mathbb{N} : x_{\sigma(n)} \in U_k\}\|_\varphi \leq q \}$$

$$= \bigcup_{k \geq 1} \{ \sigma \in \Sigma : \limsup_{t \to \infty} \varphi(\{n \geq t : x_{\sigma(n)} \in U_k\}) \leq q \}$$

$$= \bigcup_{k \geq 1} \bigcap_{s \geq s} \{ \sigma \in \Sigma : \varphi(\{n \geq t : x_{\sigma(n)} \in U_k\}) \leq q \}.$$ 

Then, it is sufficient to show that

$$W_{k,s} := \bigcap_{t \geq s} \varphi(\{n \geq t : x_{\sigma(n)} \in U_k\}) \leq q$$

is nowhere dense for all $k, s \in \mathbb{N}$.

To this aim, for every nonempty open set $Z \subseteq \Sigma$, we need to prove that there exists a nonempty open subset $S \subseteq Z$ such that $S \cap W_{k,s} = \emptyset$. Fix a nonempty open set $Z \subseteq \Sigma$ and $\sigma_0 \in Z$ so that there exists $n_0 \in \mathbb{N}$ for which

$$Z' := \{ \sigma \in \Sigma : \sigma \in Z, \{1, \ldots, n_0\} = \sigma_0 \cap \{1, \ldots, n_0\} \} \subseteq Z.$$

Since $\ell \in L_x$, there exists $\sigma_1 \in Z'$ such that $\lim_n x_{\sigma_1(n)} = \ell$. Therefore

$$\varphi(\{n \geq n_1 : x_{\sigma_1(n)} \in U_k\}) \geq \|\{n \geq n_1 : x_{\sigma_1(n)} \in U_k\}\|_\varphi = \|\{n \in \mathbb{N} : x_{\sigma_1(n)} \in U_k\}\|_\varphi = u(\sigma_1, \ell) = 1,$$

where $n_1 := \max\{n_0 + 1, s\}$. At this point, since $\varphi$ is a lscsm, it follows that there exists an integer $n_2 > n_1$ such that $\varphi(\{n \in \mathbb{N} \cap [n_1, n_2] : x_{\sigma_1(n)} \in U_k\}) > q$. Therefore $S := \{ \sigma \in Z' : \sigma \in Z, \{n_1, \ldots, n_2\} = \sigma_0 \cap \{n_1, \ldots, n_2\} \}$ is a nonempty open set contained in $Z$ and disjoint from $W_{k,s}$. Indeed

$$\forall \sigma \in S, \quad \varphi(\{n \geq s : x_{\sigma(n)} \in U_k\}) \geq \varphi(\{n \in \mathbb{N} \cap [n_1, n_2] : x_{\sigma(n)} \in U_k\}) > q$$

by the monotonicity of $\varphi$. \qed

**Lemma 4.3.** With the same hypotheses of Lemma 4.1, we have

$$\forall \ell \in X, \quad \{ \sigma \in \Sigma : \ell \in \Lambda_\sigma(\mathcal{I}) \} = \bigcup_{q > 0} V(\ell, q).$$

In addition, $\tilde{S}(\ell, q) := \{ \sigma \in \Sigma : \ell \in \Lambda_\sigma(\mathcal{I}, q) \}$ contains $V(\ell, q)$.

**Proof.** Fix $\ell \in X$ and $\sigma \in \tilde{S}(\ell)$, where

$$\tilde{S}(\ell) := \{ \sigma \in \Sigma : \ell \in \Lambda_\sigma(\mathcal{I}) \}.$$ 

Then there exist $\tau \in \Sigma$ and $q > 0$ such that $\lim_n x_{\tau(\sigma(n))} = \ell$ and $\|\tau(\mathbb{N})\|_\varphi \geq 2q$. In particular, for each $k \in \mathbb{N}$ we have $x_{\tau(\sigma(n))} \in U_k$ for all large $n \in \mathbb{N}$. Hence

$$\|\{n \in \mathbb{N} : x_{\tau(n)} \in U_k\}\|_\varphi \geq \|\{n \in \mathbb{N} : x_{\sigma(n)} \in U_k\} \cap \tau(\mathbb{N})\|_\varphi = \|\tau(\mathbb{N})\|_\varphi \geq 2q.$$ 

By the arbitrariness of $k$, it follows that $u(\sigma, \ell) \geq 2q > q$, that is, $\sigma \in V(\ell, q)$. 

\[Q.E.D.\]
Conversely, fix $\ell \in X$, $\sigma \in \Sigma$, and $q > 0$ such that $\sigma \in V(\ell, q)$, hence $\|A_k\|_\varphi > q$ for all $k \in \mathbb{N}$, where $A_k := \{n \in \mathbb{N} : x_{\sigma(n)} \in U_k\}$. Let us define recursively a sequence $(F_k)$ of finite subsets of $\mathbb{N}$ as it follows. Pick $F_1 \subseteq A_1$ such that $\varphi(F_1) \geq q$ (which is possible since $\varphi$ is a lscsm); then, for each integer $k \geq 2$, let $F_k$ be a finite subset of $A_k$ such that $\min F_k > \max F_{k-1}$ and $\varphi(F_k) \geq q$ (which is possible since $\|A_k \setminus [1, \max F_{k-1}]\|_\varphi = \|A_k\|_\varphi > q$). Let $(y_n)$ be the increasing enumeration of the set $\bigcup_k F_k$, and define $\tau \in \Sigma$ such that $\tau(n) = y_n$ for all $n$. It follows by construction that

$$\lim_{n \to \infty} x_{\tau(\sigma(n))} = \ell \quad \text{and} \quad \|\tau(\mathbb{N})\|_\varphi \geq \liminf_{k \to \infty} \varphi(F_k) \geq q > 0.$$ 

Therefore $\ell \in \Lambda_{\sigma(\nu)}(\mathcal{I}, q) \subseteq \Lambda_{\sigma(\nu)}(\mathcal{I})$, which concludes the proof. \hfill $\square$

**Corollary 4.4.** With the same hypotheses of Lemma 4.1, $\tilde{S}(\ell, q)$ is comeager for each $\ell \in L_x$ and $q \in (0, 1)$.

**Proof.** Fix $\ell \in L_x$ and $q \in (0, 1)$. Then $\tilde{S}(\ell, q)$ contains $V(\ell, q)$ by Lemma 4.3, which is comeager by Lemma 4.2. \hfill $\square$

**Corollary 4.5.** With the same hypotheses of Lemma 4.1, $\tilde{S}(\ell)$ is comeager for each $\ell \in L_x$.

**Proof.** Thanks to [2, Theorem 2.2], we have

$$\Lambda_{\sigma(\nu)}(\mathcal{I}) = \bigcup_{q > 0} \Lambda_{\sigma(\nu)}(\mathcal{I}, q).$$

Therefore $\tilde{S}(\ell)$ contains $\tilde{S}(\ell, \nicefrac{1}{2})$, which is comeager by Corollary 4.4. \hfill $\square$

**Remark 4.6.** All the analogues from Lemma 4.1 up to Corollary 4.5 hold for permutations, the only difference being in the last part of the proof of Lemma 4.2: let us show that

$$\hat{W}_{k,s} := \bigcup_{t \geq s} \{\pi \in \Pi : \varphi(\{n \geq t : x_{\pi(n)} \in U_k\}) \leq q\}$$

is nowhere dense for all $k, s \in \mathbb{N}$. To this aim, fix $\pi_0 \in \Pi$ and $n_0 \in \mathbb{N}$ which defines the nonempty open set $G := \{\pi \in \Pi : \pi \upharpoonright \{1, \ldots, n_0\} = \pi_0 \upharpoonright \{1, \ldots, n_0\}\}$. Set $n_1 := \max\{n_0 + 1, s\}$ and let $(y_n)$ be the increasing enumeration of the infinite set $\{n \in \mathbb{N} : x_n \in U_k\} \setminus \{\pi_0(1), \ldots, \pi_0(n_0)\}$. Since $\varphi$ is a lscsm, there exists $n_2 \in \mathbb{N}$ such that $\varphi(\{s, s + 1, \ldots, n_2\}) > q$. Lastly, let $G'$ be the set of all $\pi \in G$ such that $\pi(n) = y_n$ for all $n \in \{s, s + 1, \ldots, n_2\}$. We conclude that

$$\forall \pi \in G', \quad \varphi(\{n \geq s : x_{\pi(n)} \in U_k\}) \geq \varphi(\{s, s + 1, \ldots, n_2\}) > q.$$

Therefore $G'$ is a nonempty open subset of $G$ which is disjoint from $\hat{W}_{k,s}$.

Lastly, we prove Theorem 2.6.

**Proof of Theorem 2.6.** The implications (L1) $\implies$ (L2) $\implies$ (L5) are analogous to the ones in Theorem 2.2, replacing Lemma 3.2 with Corollary 4.5.
Suppose that $L_x \neq \emptyset$, otherwise the claim is trivial. Let $\mathcal{L}$ be a countable dense subset of $L_x$, so that $\mathcal{L} \subseteq \Lambda_{\sigma(x)}(\mathcal{I}, \frac{1}{2})$ for each $\sigma \in \hat{S} := \bigcap_{\ell \in \mathcal{L}} \hat{S}(\ell, \frac{1}{2})$, which is comeager by Corollary 4.4. Fix $\sigma \in \hat{S}$. On the one hand, taking into account (4), we get $\Lambda_{\sigma(x)}(\mathcal{I}, \frac{1}{2}) \subseteq \Lambda_{\sigma(x)}(\mathcal{I}) \subseteq L_{\sigma(x)} \subseteq L_x$. On the other hand, since $\Lambda_{\sigma(x)}(\mathcal{I}, \frac{1}{2})$ is closed by Lemma 4.1, we obtain $L_x \subseteq \Lambda_{\sigma(x)}(\mathcal{I})$. Therefore $\hat{\Sigma}_x(\mathcal{I})$ contains the comeager set $\hat{S}$.

The implications $(L3) \implies (L4) \implies (L5) \implies (L3)$ are analogous, taking into account Remark 4.6.

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