A tropical geometry approach to BIBO stability

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Abstract

Given a Laurent polynomial $F \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, its amoeba $A_F$ is the image by $z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n \mapsto (\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^n$ of the algebraic zero set $V(F) = \{z \in (\mathbb{C}^*)^n ; F(z) = 0\}$ of the complex torus $\mathbb{T}^n := (\mathbb{C}^*)^n$. We relate here the question of the BIBO stability of a multilinear time invariant system with transfer function $A(X^{\pm 1})/B(X^{\pm 1}) = G(X_1, \ldots, X_n)/F(X_1, \ldots, X_n)$, where $F, G \in \mathbb{C}[X_1, \ldots, X_n]$ are coprime in $\mathbb{C}[X_1, \ldots, X_n]$, with the geometrical study of the amoeba $A_F$. We formulate very simple criteria for BIBO strong or weak stability in terms of the position of $0 = (0, \ldots, 0) \in \mathbb{R}^n$ with respect to the amoeba $A_F$ and suggest an algorithmic procedure in order to test such property when $F \in \mathbb{Z}[X_1, \ldots, X_n]$. Such procedure relies on the concept of lopsidedness approximation of $A_F$, as introduced by K. Purbhoo [23] and completed from the algorithmic point of view in [9].

1 Introduction

Let $n \in \mathbb{N}$ and

$$\mathbb{C}^{[\mathbb{Z}^n]} := \{(u_k)_{k \in \mathbb{Z}^n} \in \mathbb{C}^{\mathbb{Z}^n} ; u_k = 0 \text{ except for a finite number of } k \in \mathbb{Z}^n\}.$$ 

A discrete linear time-invariant system

$$(u_k)_{k \in \mathbb{Z}^n} \in \mathbb{C}^{[\mathbb{Z}^n]}, \quad S \mapsto \left( \sum_{\kappa \in \mathbb{Z}^n} h_{k-\kappa} u_{k-\kappa} \right)_{k \in \mathbb{Z}^n} \in \mathbb{C}^{\mathbb{Z}^n}$$

is said to be Bounded Input- Bounded Output (BIBO) stable if and only if its impulse response $(h_k)_{k \in \mathbb{Z}^n}$ belongs to $\ell_1^1(\mathbb{Z}^n)$, that is

$$\sum_{k \in \mathbb{Z}^n} |h_k| < +\infty.$$ (1)

Since $\ell_1^1(\mathbb{Z}^n) \hookrightarrow \ell_2^2(\mathbb{Z}^n)$, (1) implies that $\sum_{k \in \mathbb{Z}^n} |h_k|^2 < +\infty$, which corresponds to the fact that the system $S$ is asymptotically stable (or stationary). BIBO stability thus constitutes a stronger requirement than just asymptotic stability.

There exists in the classical literature several criteria for BIBO stability for discrete linear time-invariant systems (see for example [13, 14, 2, 26, 7, 12, 3, 4]).
In this paper, we are concerned with discrete multilinear time invariant systems which admit as transfer function the rational function

\[ \frac{B(X_1^{-1}, ..., X_n^{-1})}{A(X_1^{-1}, ..., X_n^{-1})} = X^n \frac{G(X_1, ..., X_n)}{F(X_1, ..., X_n)} \in \mathbb{C}[X_1, ..., X_n], \]

where \( \gamma \in \mathbb{Z}^n \) and \( G, F \in \mathbb{C}[X_1, ..., X_n] \) are coprime in \( \mathbb{C}[X_1, ..., X_n] \), both \( F \) and \( G \) being also coprime with \( X_1 \cdots X_n \). They are called \textit{discrete n-rational filters}. In case the rational function

\[ z \in T^n = (\mathbb{C}^*)^n \longrightarrow G(z)/F(z), \]

where \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \), is regular about the \( n \)-dimensional torus

\[ \mathbb{S}_1^n = \{ z = (e^{i\theta_1}, ..., e^{i\theta_n}) ; \theta \in (\mathbb{R}/(2\pi\mathbb{Z}))^n \}, \]

then the two following assertions are equivalent for such a discrete \( n \)-rational filter \( S \):

(i) \( S \) is BIBO stable

(ii) \( F^{-1}(\{0\}) \cap \{ z \in \mathbb{C}^n ; |z_j| \leq 1 \text{ for } j = 1, ..., n \} = \emptyset. \)

Such an equivalence is known in the case where \( n = 1 \) or \( n = 2 \) as \textit{Shanks criterion} \cite{14}. In the case where \( n = 2 \), one may formulate various criteria equivalent to this one. Here are two examples.

**Theorem 1** \cite{13} \textit{Under the condition that} \( G/F \) \textit{is regular about} \( \mathbb{S}_1^n \), \textit{it is equivalent to say that the discrete 2-rational filter} \( S \) \textit{with transfer function}

\[ \frac{B(X_1^{-1}, X_2^{-1})}{A(X_1^{-1}, X_2^{-1})} = X^n G(X_1, X_2) \]

\( \text{is BIBO stable and that} \)

\[ F(z_1, 0) \neq 0 \text{ for } |z_1| \leq 1, \quad F(z_1, z_2) \neq 0 \text{ for } |z_1| = 1, |z_2| \leq 1. \]

**Theorem 2** \cite{14} \textit{Suppose that} \( \deg_{X_1} F = d_1 \) \textit{and} \( \deg_{X_2} F = d_2 \). \textit{Let} \( F^* \) \textit{be the conjugate polynomial such that}

\[ F^*(z_1, z_2) := z_1^{d_1} \cdot z_2^{d_2} \overline{F\left(\frac{1}{z_1}, \frac{1}{z_2}\right)} \quad ((z_1, z_2) \in T^2) \]

\( \text{and} \) \( R_{X_2}(X_1) \) \textit{be the resultant of} \( F \) \textit{and} \( F^* \) \textit{considered as elements of} \( \mathbb{C}[X_1][X_2] \). \textit{Under the condition that} \( G/F \) \textit{is regular in} \( (\mathbb{S}^1)^n \), \textit{it is equivalent to say that the discrete 2-rational filter} \( S \) \textit{with transfer function}

\[ \frac{B(X_1^{-1}, X_2^{-1})}{A(X_1^{-1}, X_2^{-1})} = X^n G(X_1, X_2) \]

\( \text{is BIBO stable and that} \)

\[ F(z_1, 0) \neq 0 \text{ for } |z_1| \leq 1 \]
\[ F(1, z_2) \neq 0 \text{ for } |z_2| \leq 1 \]
\[ R_{X_2}(z_1) \neq 0 \text{ for } |z_1| = 1. \]
Let also \( F(\omega) = \{ \ldots \} \). In the sequel, the open (respectively closed) unit disc of the complex plane is denoted \( \mathbb{D} \) (resp. \( \overline{\mathbb{D}} \)). Such methods are based on the introduction of so-called \textit{Schur coefficients families} in several variables, the goal being to obtain a multidimensional Schur-Cohn criterion (see [1, 20, 17, 18, 19]). Let

\[
F(X_1, \ldots, X_n) = \sum_{\alpha \in \text{Supp } F \subset \mathbb{Z}^n} c_{\alpha} X^\alpha \quad (c_{\alpha} \in \mathbb{C}^*)
\]

with total degree \( \delta_F \) in the \( n \) variables \( X_1, \ldots, X_n \). For any \( w = (e^{i\omega_1}, \ldots, e^{i\omega_{n-1}}) \) with \( \omega = (\omega_1, \ldots, \omega_{n-1}) \in (\mathbb{R}/(2\pi\mathbb{Z}))^{n-1} \), define \( F_w \in \mathbb{C}[Y] \) by

\[
F_w(Y) = F(w_1 X_1, \ldots, w_{n-1} X_{n-1}, Y) = F(e^{i\omega_1} X_1, \ldots, e^{i\omega_{n-1}} X_{n-1}, Y) \in \mathbb{C}[Y].
\]

Let also \( F_w^*(Y) \) be the polynomial defined by

\[
F_w^*(u) = u^{\delta_F} \overline{F_w(1/u)}.
\]

In the sequel, the open (respectively closed) unit disc of the complex plane is denoted \( \mathbb{D} = \{ z \in \mathbb{C} ; |z| < 1 \} \) (resp. \( \overline{\mathbb{D}} = \{ z \in \mathbb{C} ; |z| \leq 1 \} \)).

The main result they obtain is that the three following statements are equivalent:

- \( F(z) \neq 0 \) for any \( z \in \overline{\mathbb{D}}^n \);
- for any \( w = (e^{i\omega_1}, \ldots, e^{i\omega_{n-1}}) \), \( F_w(u) \neq 0 \) for any \( u \in \overline{\mathbb{D}} \);
- for any \( w = (e^{i\omega_1}, \ldots, e^{i\omega_{n-1}}) \), the function \( u \in \mathbb{D} \mapsto F_w^*(u)/F_w(u) \) is an inner function in the Hardy space \( H^2(\mathbb{D}) \) and the \( \delta_F \) Schur parametrized (hence called functional) coefficients \( w \mapsto \gamma_k(w), k = 0, \ldots, \delta_F - 1 \) satisfy \( |\gamma_k(w)| < 1 \) for any \( w = (e^{i\omega_1}, \ldots, e^{i\omega_{n-1}}) \).

Such an equivalence allows to transpose the \( n \)-dimensional problem to the \( (n-1) \)-dimensional setting. Such an approach is efficient for small values of \( n \) \((n = 2, 3, \ldots)\) but difficult to implement in higher dimensions.

Therefore, the problem of testing the BIBO stability of a discrete \( n \)-rational filter appeals to investigate new strategies leading to criteria easier to implement in high dimensions from the computational point of view. Also, being able to decide whether the denominator \( F \) of the transfer function vanishes on \( \{ (e^{i\theta_1}, \ldots, e^{i\theta_n}) ; \theta \in (\mathbb{R}/(2\pi\mathbb{Z}))^n \} \) seems to be an important challenge, since such an hypothesis is required prior to the formulation of any criterion for BIBO stability.

We intend in this paper to introduce a novel approach, based on the notion of amoeba of an algebraic hypersurface in \( \mathbb{T}^n = (\mathbb{C}^*)^n \). The notion of amoeba was introduced by Gelfand, Krpanov and Zelevinsky in 1994 in their pioneer book on multidimensional determinants [11]. The amoeba \( A_F \) of \( F \) (one should better say of the zero
set $F^{-1}(\{0\})$ of $F$ in $\mathbb{T}^n$) when $F$ is a Laurent polynomial in $n$ variables (in particular a polynomial in $n$ variables) is the image of $F^{-1}(\{0\})$ under the logarithmic map $\text{Log} : z \mapsto (\log |z_1|, \ldots, \log |z_n|)$. Section §2 provides an overview of what is actually known about such concepts, in view of the role it could play in relation with BIBO stability. In section §3, we will formulate a criterion within the frame of amoeba and enlarge the notion of BIBO stability into that of weak BIBO stability. In section §4, we will analyze our approach from the algorithmic point of view.

## 2 Amoeba and related concepts, an overview

Let

$$F = \sum_{\alpha \in \text{Supp} F \subset \mathbb{Z}^n} c_\alpha X^\alpha \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \quad (c_\alpha \in \mathbb{C}^*, \ X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n})$$

be a Laurent polynomial with zero set $F^{-1}(\{0\})$ in the complex torus $\mathbb{T}^n$. Let $\Delta(F)$ be the Newton polyhedron of $F$, that is the closed convex envelope in $\mathbb{R}^n$ of the set $\text{Supp} F := \{\alpha \in \mathbb{Z}^n; c_\alpha \neq 0\}$. We will always suppose that $F$ is a true Laurent polynomial in $n$ variables, which means that $\dim_{\mathbb{R}^n} \Delta(F)$, that is the dimension of the affine subspace of $\mathbb{R}^n$ generated by $\Delta(F)$ is maximal, that is equal to $n$.

Let $\mathcal{A}_F$ be the amoeba of $F$, that is the closed image $\text{Log}(F^{-1}(\{0\})) \subset \mathbb{R}^n$ of the map

$$z \in F^{-1}(\{0\}) \subset \mathbb{T}^n \xrightarrow{\text{Log}} (\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^n.$$

From the geometric point of view, the complement $\mathbb{R}^n \setminus \mathcal{A}_F$ is a 1-convex open subset of $\mathbb{R}^n$, which amounts to say that its open connected components $E_i$ are convex. The number of such open connected components is bounded by the number of points in $\Delta(F) \cap \mathbb{Z}^n$. Let $\mathcal{E}_F$ be the finite set which elements are such components $E$. Each $\text{Log}^{-1}(E)$, where $E \in \mathcal{E}_F$, is a Reinhardt domain in $\mathbb{T}^n$, that is a subdomain which is invariant under the pointwise multiplicative action of the real torus $\mathbb{T}^n_\mathbb{R} = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}); \theta \in (\mathbb{R}/(2\pi\mathbb{Z})^n)\}$. Moreover it can be described as the maximal domain of convergence of a unique Laurent series $\sum_{k \geq 0} \gamma_{E,k} z_{\alpha_{E,k}}$, $\alpha_{E,k} \in \mathbb{Z}^n$, which sum represents $z \mapsto 1/F(z)$ in $\text{Log}^{-1}(E)$. A key point is that there is in fact a bijection between the finite set $\mathcal{E}_F$ and the family of all possible Laurent developments (with domains of convergence precisely $\text{Log}^{-1}(E)$ for $E \in \mathcal{E}_F$) for $1/F$ along the monomials $z^\alpha$ for $\alpha \in \mathbb{Z}^n$.

**Remark 1 (the case $n = 1$)** In the case where $n = 1$, the amoeba $\mathcal{A}_F$ consists in a finite number of points $-\infty < \log |a_1| < \cdots < \log |a_N| < +\infty$ ($a_j \in \mathbb{C}^*$) on the real line and each of the $N+2$ circular domains

$$\{z \in \mathbb{C}^*; |z| < |a_1|\}, \ldots, \{z \in \mathbb{C}^*; |a_j| < |z| < |a_{j+1}|\}, \ldots, \{z \in \mathbb{C}^*; |z| > |a_N|\},$$

is the domain of convergence of a Laurent series which sum represents $1/F$ in the corresponding domain. The domain $C = \{z \in \mathbb{C}^*; |z| > |a_N|\}$ is, among such list, the only one for which the associated Laurent development in $C$ is of the form $1/F(z) = \sum_{k \geq -M} \gamma_{C,k} z^{-k}$ for some $M \in \mathbb{Z}$ and hence the sequence $(\gamma_{C,k})_{k \in \mathbb{Z}}$ can be interpreted as the impulse response of a rational (realizable) discrete 1-dimensional filter.
One can associate \[21\] to each \(E \in \mathcal{E}_F\) a multiplicity \(\nu_E = (\nu_{E,1}, \ldots, \nu_{E,n}) \in \Delta(F) \cap \mathbb{Z}^n\), where \(\nu_{E,j}\) is the degree of the loop

\[
\theta_j \in \mathbb{Z}/(2\pi \mathbb{Z}) \mapsto F(\zeta_{E,1}, \ldots, \zeta_{E,j} e^{i\theta_j}, \ldots, \zeta_{E,n})
\]

(3)

when \((\zeta_{E,1}, \ldots, \zeta_{E,n})\) is an arbitrary point in \(E\), the degree of the loop \((3)\) being independent on the choice of such point \(\zeta_E\) in \(E\).

For each point \(\alpha \in \Delta(F) \cap \mathbb{Z}^n\), there is at most one component \(E_\alpha \in \mathcal{E}_F\) such that \(\nu_{E_\alpha} = \{\alpha\}\). If such is the case, let \(\sigma_\alpha\) be the unique face of \(\Delta(F)\) which contains \(\alpha\) in its relative interior or (if no such face exists) equals \{\alpha\} : for example, if \(\alpha\) is a vertex of \(\Delta(F)\), \(\sigma_\alpha = \{\alpha\}\), while when \(\alpha\) lies in the interior of \(\Delta(F)\), \(\sigma_\alpha = \Delta(F)\), etc. Then the cone

\[
\Gamma_\alpha = \{ x \in \mathbb{R}^n; \sigma_\alpha = \{ \xi \in \Delta(F); \langle \xi, x \rangle = \max_{u \in \Delta(F)} \langle u, x \rangle \}\}
\]

is the recession cone of \(E_\alpha\), that is the largest cone \(\Gamma\) of \(\mathbb{R}^n\) such that \(E_\alpha + \Gamma \subset E_\alpha\). Such recession cone equals \{0\} whenever \(\alpha\) lies in the interior of \(\Delta(F)\), hence the corresponding component \(E_\alpha\) is, if it exists, bounded in this case. When \(\alpha\) belongs to the boundary of \(\Delta(F)\), the dimension of the recession cone is maximal (thus equal to \(n\)) if and only if \(\alpha\) is a vertex of \(\Delta(F)\). If \(\alpha\) is a point of \(\partial \Delta(F) \cap \mathbb{Z}^n\) which is not a vertex of \(\Delta(F)\), then, if \(E_\alpha\) exists, it is unbounded and its recession cone has dimension between 1 and \(n - 1\). A major point is that any vertex \(\alpha\) of \(\Delta(F)\) is the multiplicity \(\nu_{E_\alpha}\) of a unique unbounded component \(E_\alpha\) which admits as recession cone the cone

\[
\Gamma_\alpha = \{ x \in \mathbb{R}^n; \{\alpha\} = \{ \xi \in \Delta(F); \langle \xi, x \rangle = \max_{u \in \Delta(F)} \langle u, x \rangle \}\}.
\]

(4)

Thus the cardinal of \(\mathcal{E}_F\) lies between the number or vertices of \(\Delta(F)\) and the cardinal of \(\Delta(F) \cap \mathbb{Z}^n\); the number of bounded components in \(\mathcal{E}_F\) (called the genus of the amœba) lies between 0 and the number of points in \(\Delta(F) \cap \mathbb{Z}^n\) [10].

An important concept related to the amœba \(A_F\) is that of contour. Let \((F^{-1}(\{0\}))_{\text{sing}}\) be the subvariety of singular points of the algebraic hypersurface \(F^{-1}(\{0\})\) : if \(F\) is assumed to be irreducible in \(\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]\), \((F^{-1}(\{0\}))_{\text{sing}}\) is defined as

\[
(F^{-1}(\{0\}))_{\text{sing}} = \{ z \in \mathbb{T}^n; F(z) = \frac{\partial F}{\partial z_1}(z) = \cdots = \frac{\partial F}{\partial z_n}(z) = 0\};
\]

if \(F = F_1^{q_1} \cdots F_M^{q_M}\) is the decomposition of \(F\) in irreducible factors in \(\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]\)

\[
(F^{-1}(\{0\}))_{\text{sing}} = \bigcup_{j=1}^M (F_j^{-1}(\{0\}))_{\text{sing}} \cup \bigcup_{\substack{1 \leq j, k \leq M \\ j \neq k}} (F_j^{-1}(\{0\}) \cap F_k^{-1}(\{0\})).
\]

In any case, the codimension of \((F^{-1}(\{0\}))_{\text{sing}} \subset F^{-1}(\{0\})\) in \(\mathbb{T}^n\) is at least equal to 2. One denotes as \((F^{-1}(\{0\}))_{\text{reg}}\) the (in general non closed) submanifold of \(\mathbb{T}^n\) defined as \(F^{-1}(\{0\}) \setminus (F^{-1}(\{0\}))_{\text{sing}}\).

**Definition 1 (contour of \(A_F\) [22])** The contour of the amœba \(A_F\) is the union of \(\log(F^{-1}(\{0\}))_{\text{sing}}\) with the set of critical values of \(\log : (F^{-1}(\{0\}))_{\text{reg}} \rightarrow \mathbb{R}^n\).

The contour of \(A_F\) contains necessarily the boundary of \(A_F\). A major result concerning the description of the contour of \(A_F\) is the following result due to G. Mikhalkin [16, 21].
Theorem 3 ([16, 21], see also §4.1 in [27]) Suppose that $F$ is a reduced Laurent polynomial, that is $F = F_1 \cdots F_M$ where each $F_j$ is irreducible. Let

$$z \in (F^{-1}(\{0\}))_{\text{reg}} \mapsto \gamma_F(z) := \left[z_1 \frac{\partial F}{\partial z_1}(z) : \cdots : z_n \frac{\partial F}{\partial z_n}(z)\right] \in \mathbb{P}^{n-1}(\mathbb{C})$$

be the Gauss logarithmic map. One has

$$\text{contour}(A_F) = \text{Log} (\gamma_F^{-1}(\mathbb{P}^{n-1}(\mathbb{R}))),$$

where $\mathbb{P}^{n-1}(\mathbb{R}) = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^\ast$ denotes the real $(n-1)$-dimensional projective space.

From this theorem, an algebraic algorithm based on elimination theory can be realized in order to compute the contour of $A_F$ for bivariate polynomials $F(X_1, X_2) \in \mathbb{Z}[X_1, X_2]$ (see [6] Algorithm 3). This algebraic algorithmic procedure is based on the construction we sketch below. Let $u \in \mathbb{R}$ be a real parameter. Consider the two polynomials

$$P(X_1, X_2, u) = F(X_1, X_2), \quad Q(X_1, X_2, u) = \frac{\partial F}{\partial X_1}(X_1, X_2) + u \frac{\partial F}{\partial X_2}(X_1, X_2)$$

in $\mathbb{Z}[u][X_1, X_2]$. One can compute formally in an exact way the Sylvester resultant $R_{X_2}(u, X_1) \in \mathbb{Z}[u, X_1]$ of $P$ and $Q$ considered as elements of $\mathbb{Z}[u, X_1][X_2]$ and the Sylvester resultant $R_{X_1}(u, X_2) \in \mathbb{Z}[u, X_2]$ of $P$ and $Q$ considered this time as elements of $\mathbb{Z}[u, X_2][X_1]$. See for example [15] for the construction of the Sylvester resultants of two polynomials in $A[Z]$ where $A$ is a commutative domain of integrity with fraction field $K$, as $A = \mathbb{Z}[u, X_1]$ (and $Z = X_2$) or $A = \mathbb{Z}[u, X_2]$ (and $Z = X_1$) in our case. Theorem 3 implies that $(x_1, x_2) \in \text{contour}(A_F)$ if and only if there is at least one point $u \in \mathbb{P}^1(\mathbb{R})$ such that $P(X, Y, u)$ and $Q(X, Y, u)$ share a common zero $(z_1, z_2) \in T^2$ which lies in the orbit $\text{Log}^{-1}(\{(x_1, x_2)\})$, that is such that $|z_1| = e^{x_1}$ and $|z_2| = e^{x_2}$. Elimination theory implies that $(z_1, z_2)$ satisfies

$$R_{X_1}(u, z_1) = R_{X_2}(u, z_2) = 0.$$  

Given $u \in \mathbb{Q} = \mathbb{P}^1(\mathbb{Q}) \setminus \{[1 : 0]\} = \mathbb{P}^1(\mathbb{Q}) \setminus \{\infty\}$, one can compute exactly thanks to Newton’s method the (at most $\deg_{X_1} R_{X_2} \times \deg_{X_2} R_{X_1}$) pairs of algebraic numbers $(z_1, z_2) \in (\mathbb{Q})^2 \cap T^2$ which satisfy (8). Then, one can extract from this list the sublist of pairs of points which satisfy

$$P(z_1, z_2, u) = Q(z_1, z_2, u) = 0,$$

that is such that $(\log |z_1|, \log |z_2|) \in \text{contour}(A_F)$. Repeating this procedure for $u = -N + 2\ell N/M, \ell = 0, \ldots, M - 1$, where $M >> N >> 1$, leads to a construction of $\text{contour}(A_F)$ (see [6] Algorithm 3) for the numerical code under Matlab or also [24 §5] for the formal code under Sage.

As a consequence of this method, one can state the following proposition.

**Proposition 1** (is $(0,0)$ in the contour of $A_F$ ?) Let $F \in \mathbb{Z}[X_1, X_2]$ be an irreducible polynomial in 2 variables such that $\dim_{\mathbb{R}^2} \Delta(F) = 2$. There is an exact procedure based on Schur-Cohn test to decide whether $(0,0)$ lies (or not) in the contour of $A_F$. 


Proof. Let us form the resultants \( R_{X_1}(u, X_1) \in \mathbb{Z}[u][X_1], R_{X_2}(u, X_2) \in \mathbb{Z}[u][X_2] \). Consider \( u \) as a real parameter. The Schur-Cohn test allows to decide for which possible values of the real parameter \( u \in \mathbb{T} \) the polynomials \( R_{X_1}(u, X_2) \) and \( R_{X_2}(u, X_1) \) may both have a root (necessarily in \( \mathbb{T} \)) on the unit circle \( |\zeta|=1 \) of the complex plane. We are then left with a finite number of situations to test in order to decide whether the two polynomials \( P(X_1, X_2, u) \) and \( Q(X_1, X_2, u) \) defined in (7) have at least both a root on the unit circle, which means in this case that \( (0,0) \) belongs to the contour of \( \mathcal{A}_F \). ■

Another important concept related to \( F \) which provides geometric information on \( \mathcal{A}_F \) is the following convex function

\[
R_f : x \in \mathbb{R}^n \mapsto \int_{\mathbb{T}_\mathbb{R}^n} F(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \, d\nu_{\mathbb{T}_\mathbb{R}^n}(\theta),
\]

where \( \mathbb{T}_\mathbb{R}^n = (\mathbb{R}/(2\pi\mathbb{Z}))^n \) equipped with its normalized Haar measure \( d\nu_{\mathbb{T}_\mathbb{R}^n} \). It was introduced by L. Ronkin in [25] and is thus called the Ronkin function of \( F \). The three important facts to retain about such function are the following.

1. The function \( R_F \) is affine in the connected component (of \( \mathbb{R}^n \setminus \mathcal{A}_F \)) \( E_\alpha \in \mathcal{E}_F \) with multiplicity \( \alpha \in \Delta(F) \cap \mathbb{Z}^n \), provided of course such component exists [21]. More precisely,

\[
\forall x \in E_\alpha, \quad R_F(x) = \rho_\alpha + \langle \alpha, x \rangle.
\]

2. When \( \alpha \) is a vertex of \( \Delta(F) \) (and hence \( E_\alpha \) exists, with \( n \)-dimensional recession cone given by (4)), then \( \rho_\alpha = \log |c_\alpha| \), where \( c_\alpha \) is the coefficient of \( X^\alpha \) in the developed expression (2) for \( F \).

3. The singular support of the distribution \( \Delta([R_F]) \) (where \( \Delta \) is the Laplace operator and \( [R_F] \) means that \( R_F \) is considered in the sense of distributions) is contained in the contour of \( \mathcal{A}_F \) [24 Theorem 3.1].

Although \( R_F \) is just a continuous function inside \( \mathcal{A}_F \), one can compute numerically when \( n = 2 \) the Laplacian of the associate distribution \( [R_F] \) (see [24 §5]). The main reason why such a method works is that the singularities of \( \log |F| \) on \( \mathbb{T}^n \) are gentle ones (the function \( \log |F| \) is locally integrable on \( \mathbb{T}^n \)) and the use of the Laplace operator (because of its symmetric form with respect to coordinates) is a basic (primitive) tool for the detection of contours in image processing. Such numerical computation provides (unfortunately in some empiric way) a surprisingly convincing picture both of the amoeba and simultaneously of its contour [21 §5].

The convex Ronkin function \( R_F \) has a companion \( p_F \) which is much easier to describe since it is realized in \( \mathbb{R}^n \) as the upper envelope of a finite number of affine functions with slopes in \( \mathbb{Z}^n \), hence can be interpreted as the evaluation function in \( \mathbb{R}^n \) of a tropical Laurent polynomial since the operations

\[(a, b) \in ([-\infty, +\infty]^2 \mapsto \max(a, b), \quad (a, b) \in ([-\infty, +\infty])^2\]

substitute to the usual addition and multiplication in the (tropical) max-plus calculus. Let \( \nu : \mathcal{E}_F \rightarrow \Delta(F) \cap \mathbb{Z}^n \) the multiplicity map which associates to each \( E \in \mathcal{E}_K \) its multiplicity \( \nu_E \); then \( p_F \) is defined as

\[p_F : x \in \mathbb{R}^n \mapsto \max_{\alpha \in \text{Im} \nu} (\rho_\alpha + \langle \alpha, x \rangle).\]
One has that \( p_F(x) \leq R_F(x) \) for any \( x \in \mathbb{R}^n \) and \( p_F(x) = R_F(x) \) in \( \mathbb{R}^n \setminus A_F \). For each \( \alpha \in \text{Im} \nu \), let \( C_\alpha \) be the \( n \)-dimensional convex polyhedron (possibly unbounded) of \( \mathbb{R}^n \) defined as
\[
C_\alpha = \{ x \in \mathbb{R}^n ; p_F(x) + \hat{p}_F(\alpha) = \langle \alpha, x \rangle \},
\]
where \( p_F : \xi \in \mathbb{R}^n \mapsto \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - p_F(x)) \in ]-\infty, +\infty[ \) is the Legendre transform of \( p_F \), which satisfies \( \hat{p}_F^{-1}(\{+\infty\}) = \Delta(F) \) [21]. The interiors \( \hat{C}_\alpha, \alpha \in \text{Im} \nu \) are pairwise disjoints, and the the complement of their union equals the set of critical values of \( p_F \), that is the subset of points in \( \mathbb{R}^n \) about which \( p_F \) is not an affine map. Since \( p_F \) and \( R_F \) coincide on \( \mathbb{R}^n \setminus A_F \), one has \( E_\alpha \subset C_\alpha \) for any \( \alpha \in \text{Im} \nu \). Observe then that, given a point \( x \in \mathbb{R}^n \setminus A_F \), in order to decide to which component \( E_\alpha \) it belongs, one needs to check to which \( C_\alpha \) it belongs.

Let for any \( N \in \mathbb{N}^* \) (in particular \( N = 2^k, \ k \in \mathbb{N} \)) \( F_N \) be the multiplicative group of \( N \)-roots of unity and
\[
F_N(X) = \prod_{\omega \in \mathbb{F}_N^*} F(\omega_1 X_1, \ldots, \omega_N X_N).
\]

An iterative procedure to compute the \( F_{2^k} \) \( (k \in \mathbb{N}) \) inspired by the Gauss-Cooley-Tukey FFT algorithm has been proposed in [9, §3]. Observe that \( F \) and \( F_N \) share the same amoeba \( A_F \) for all \( N \in \mathbb{N}^* \). It follows also from Galois theory that \( F_N \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) as soon as \( F \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \). Since the integral of a continuous function can be approximated by Riemann sums, the Ronkin function \( R_F \) admits in \( \mathbb{R}^n \setminus A_F \) the uniform approximation on any compact subset
\[
R_F(x) = \lim_{N \to +\infty} \frac{R_{F_N}(x)}{N^n} \quad (x \in \mathbb{R}^n \setminus A_F). \tag{11}
\]

In order to exploit such idea, K. Purbhoo introduced the non-archimedean concept of (tropically) lopsided amoeba of \( F \). We recall here this construction.

Recall that a finite set \( \{ \tau_i ; i \in I \} \) of strictly positive real numbers (with possible repetitions) is said to be (tropically) lopsided if and only if there is a (necessarily) unique index \( i_0 \in I \) such that
\[
\tau_{i_0} > \sum_{i \in I \setminus \{i_0\}} \tau_i.
\]

**Definition 2 (lopsided amoeba of \( F \) [23])** The lopsided amoeba \( \mathcal{L}_F \) of the Laurent polynomial \( F(X) = \sum_{\alpha \in \text{Supp} F} c_\alpha X^\alpha \) is the image by Log of the subset of \( \mathbb{T}^n \) which consists in the set of points \( z = (z_1, \ldots, z_n) \in \mathbb{T}^n \) where the set \( \{ |c_\alpha| |z^\alpha| ; \alpha \in \text{Supp} F \} \) of strictly positive numbers is not lopsided.

One has necessarily that \( A_F \subset \mathcal{L}_F \) since if \( x \in \mathcal{A}_F \), it is clearly impossible for the set \( \{ |c_\alpha| e^{\langle \alpha, x \rangle} ; \alpha \in \text{Supp} F \} \) to be lopsided. K. Purbhoo observed in [23] the following.

**Theorem 4 (23, see also §2.2 in 27)** Suppose that \( F \) is a Laurent polynomial in \( n \) variables such that \( \dim_{\mathbb{R}^n} \Delta(F) = n \). For any \( \varepsilon > 0 \), one can find \( N_\varepsilon \in \mathbb{N}^* \) such that
\[
\forall N \geq N_\varepsilon, \quad \text{dist}(\mathcal{L}_{F_N}, A_F) < \varepsilon. \tag{12}
\]
This result can be quantified as follows. Let
\[
c_F = \max_{1 \leq j \leq n} \left( \sup_{\xi=(\xi_1, \ldots, \xi_n) \in \Delta(F)} \xi_j - \inf_{\xi=(\xi_1, \ldots, \xi_n) \in \Delta(F)} \xi_j \right).
\]
Let \( d_F = \sup_{t \in \mathbb{N}} (E_{\Delta(F)}(t)/t^n) \), where \( t \mapsto E_{\Delta(F)}(t) \) is the Ehrhart polynomial of the \( n \)-dimensional Newton polyhedron \( \Delta(F) \) (see [3] for the definition and properties of the Ehrhart polynomial of a convex polyhedron such as \( \Delta(F) \) and [8] for an upper estimate of \( d_F \)). Then, if \( x \in \mathbb{R}^n \) is such that \( \text{dist}(x, \mathcal{A}_F) \geq \varepsilon > 0 \), then for any \( N \) such that
\[
N \geq \frac{1}{\varepsilon} \left( (n^2 - 1) \log N + \log \frac{16c_F d_F}{3} \right),
\]
the point \( x \) cannot belong to the lopsided amœba \( \mathcal{L}_{\mathcal{A}_F} \) [23].

One can complete the information given by Theorem 4 and quantified as in (13) with the following companion proposition.

**Proposition 2** ([23, 27], see also §3.1.5 in [27]) Let \( \alpha \in \Delta(F) \cap \mathbb{Z}^n \) such that a component \( E_\alpha \) such that \( \nu(E_\alpha) = \alpha \) exists in \( \mathcal{E}_F \) and \( x \in E_\alpha \) with \( d(x, \mathcal{A}_F) \geq \varepsilon \). Then, for \( N \geq N_\varepsilon \) such that (13) holds, the leading term \( |c_\beta| e^{(\beta, x)} \) in the lopsided finite set \( \{ |c_\alpha| e^{(\alpha, x)} \mid \alpha \in \text{Supp} F_N \} \), where \( F_N = \sum_{\alpha \in \text{Supp} F_N} c_\alpha X^{\alpha} \), is such that \( \beta = N^n \alpha \).

## 3 BIBO stability and amœbas

Let \( S \) be a discrete \( n \)-linear time invariant system with transfer function the rational function
\[
\frac{B(X_1^{-1}, \ldots, X_n^{-1})}{A(X_1^{-1}, \ldots, X_n^{-1})} = X_1^\gamma \frac{G(X_1, \ldots, X_n)}{F(X_1, \ldots, X_n)} \in \mathbb{C}(X_1, \ldots, X_n),
\]
where \( \gamma \in \mathbb{Z}^n \) and \( G, F \in \mathbb{C}[X_1, \ldots, X_n] \) are coprime in \( \mathbb{C}[X_1, \ldots, X_n] \), both \( F \) and \( G \) being coprime with \( X_1, \ldots, X_n \).

Our first observation is that the condition that \( z \mapsto G(z)/F(z) \) is regular about
\[
\text{Log}^{-1}(\{0\}) = \{ z = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \theta \in (\mathbb{R}/(2\pi \mathbb{Z}))^n \}
\]
is equivalent to the fact that its polar set \( F^{-1}(\{0\}) \) in \( \mathbb{T}^n \) does not intersect \( \text{Log}^{-1}(\{0\}) \), which amounts to say that \( \mathcal{A} \in \mathbb{R}^n \setminus \mathcal{A}_F \). One can then state the following result.

**Theorem 5** Suppose that \( \mathcal{A} \in \mathbb{R}^n \setminus \mathcal{A}_F \), where \( \dim_{\mathbb{R}^n} \Delta(F) = n \). A necessary and sufficient condition for a discrete \( n \)-rational filter \( S \) with the rational function (14) as transfer function to be (strongly) BIBO stable is that \( \xi = 0 \in \text{Supp} F \) and \( x = \mathcal{A} \in E_0 \).

**Remark 2** (why strong BIBO stability?) We speak here about strong BIBO stability (which is the usual notion as described up to now) in order to differentiate it to the weaker one that we will introduce next in Definition 4.

**Proof.** Suppose that \( S \) is BIBO stable. Let \( \mathbb{D}^n = \{ z \in \mathbb{C}^n \mid |z_1| \leq 1, \ldots, |z_n| \leq 1 \} \).
Since \( \{ z = (z_1, \ldots, z_n) \in \mathbb{T}^n \mid 0 < |z_1| \leq 1, \ldots, 0 < |z_n| \leq 1 \} \) equals the Reinhardt domain \( \text{Log}^{-1}(\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leq 0, \ldots, x_n \leq 0 \}) \) and \( F \) is coprime with
$X_1 \cdots X_n$, it is equivalent to say that $F$ does not vanish in $\mathbb{D}^n$ and that the cone $\Gamma^- := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n ; x_1 \leq 0, \ldots, x_n \leq 0 \}$ lies entirely in some connected component $E = E_\alpha \in \mathcal{E}_F$, where $\alpha$ is necessarily a vertex of $\Delta(F)$ since the cone $\Gamma^-$ is $n$-dimensional. It follows from the fact that $\text{Supp} F \subset \mathbb{N}^n$ ($F$ is a polynomial in $X_1, \ldots, X_n$) that the only possible vertex of $\Delta(F)$ for such a situation to occur is that $\alpha = 0$, which implies that $0 \in \text{Supp} F$. Since $\Gamma^- \subset E_0$, one has in particular that $x = 0 \in E_0$. Conversely, suppose $0 \in \text{Supp} F$ and $0 \in E_0$. The recession cone (see (11))

$$
\Gamma_0 = \{ x \in \mathbb{R}^n ; \{ 0 \} = \{ \xi \in \Delta(F) ; \langle \xi, x \rangle = \max_{u \in \Delta(F)} \langle u, x \rangle \} \}
$$

of the unbounded connected component $E_0$ of $\mathbb{R}^n \setminus A_F$ contains $\Gamma^-$, as it is immediate to check. This implies that $z \mapsto 1/F(z)$ is holomorphic in the Reinhardt domain $\{ z = (z_1, \ldots, z_n) \in \mathbb{T}^n ; 0 < |z_j| < 1 \}$, which means that $F$ does not vanish there. Since $F$ is coprime with $X_1 \cdots X_n$, $F$ cannot vanish on the union of the coordinate axis either, which implies that the $n$-rational filter $S$ is BIBO stable. \hfill \blacksquare

Theorem 5 suggests to introduce two concepts related to the BIBO stability property.

**Definition 3 (BIBO stability domain)** Let $S$ be a discrete $n$-rational filter with a rational function (11) (with its properties, together with the condition $\dim_{\mathbb{R}^n} \Delta(F) = n$) as transfer function. If $0 \in \text{Supp} F$, the connected component $E_0$ of $\mathbb{R}^n \setminus A_F$ is called the BIBO stability domain of the $n$-filter $S$. If $0 \notin \text{Supp} F$, one decides that the BIBO stability domain of $S$ is empty.

**Definition 4 (BIBO weak stability)** Let $S$ be a discrete $n$-rational filter with a rational function (14) as in Definition 3. The discrete $n$-rational filter $S$ is said to be BIBO weakly stable if $0 \in \text{Supp} F$ and $0$ belongs to the topological boundary of the connected component $E_0$, which is part of the contour of $A_F$. If it is the case, the component $E_0$ is called the weak BIBO stability domain of the discrete $n$-rational filter $S$; otherwise the weak BIBO domain of $S$ is considered as empty.

## 4 Algorithmic considerations

In this section, one considers a polynomial $F \in \mathbb{Z}[X_1, \ldots, X_n]$ such that $\dim_{\mathbb{R}^n} \Delta(F) = n$, namely and $0 \in \text{Supp} F$, namely

$$
F(X_1, \ldots, X_n) = \sum_{\alpha \in \text{Supp} F \subset \mathbb{N}^n} c_\alpha X^\alpha \quad (c_\alpha \in \mathbb{Z}^*, \ 0 \in \text{Supp} F).
$$

All $F_N$ for $N \geq 1$ (in particular $N = 2^k$ for $k \in \mathbb{N}$) remain in $\mathbb{Z}[X_1, \ldots, Z_n]$.

In order to state results from the algorithmic point of view, it is important to precise with which precision real or complex quantities are evaluated. Let us fix $\varepsilon_0 > 0$ as the threshold error.

Consider the assertion $(A_{2-M_0})$ : “the distance of $0$ to $\mathbb{R}^n \setminus A_F$ is at least equal to $2^{-M_0}$”. Then, we know from (13), together with the precisions given by Proposition 2 that, as soon as

$$
2^k \geq 2^{M_0} \left( (n^2 - 1)k \log 2 + \log \frac{16c_{F,d_F}}{3} \right),
$$

10
then, if
\[ F_{2k}(X) = \sum_{\alpha_k \in \text{Supp} F_{2k}} c_{2k, \alpha_k} X^{\alpha_k} \quad (c_{2k, \alpha_k} \in \mathbb{Z}^*), \]
the set \( \{ |c_{2k, \alpha_k}| ; \alpha_k \in \text{Supp} F_{2k} \} \subset \mathbb{N}^* \) is lopsided, with leading term among the set \( \{ |c_{2k, 2^k \alpha_k}| ; \alpha \in \text{Im } \nu \} \). This is true as soon as \( k \geq M_0 + \gamma_F \), where \( \gamma_F \) is an a positive constant depending on \( \Delta(F) \). Therefore, one can proceed algorithmically as follows in order (if possible) to validate the assertion \( (\mathcal{A}_{2^{-M_0}}) \):

1. compute the list of coefficients of the polynomial \( F_{2k} \) iteratively up to \( k = M_0 + \gamma_F \) (using the algorithmic procedure introduced in [9]);
2. extract at each step \( k \) the \( \text{card (Im } \nu \) strictly positive integer coefficients \( c_{2k, \alpha 2^k} \), \( \alpha \in \text{Im } \nu \), from such list;
3. test at each step \( k \) whether one of the lopsided conditions
\[ |c_{2k, \alpha 2^k}| > \sum_{\{ \alpha_k \in \text{Supp } F_{2k} ; \alpha_k \neq \alpha 2^k \}} |c_{2k, \alpha_k}| \]  
(15)
is fulfilled (each such test being exact since \( F_{2k} \in \mathbb{Z}[X_1, ..., X_n] \));
4. if one of the above lopsided conditions is true, then the assertion \( (\mathcal{A}_{2^{-M_0}}) \) is validated (observe that we also know then in which component \( E_\alpha \) lies the point \( 0 \) and one stops the procedure ; if not, the procedure goes on until \( k = M_0 + \gamma_F \). If it fails up to this point, it means that either \( (0,0) \in \mathcal{A}_F \) or the threshold \( 2^{-M_0} \) is not sufficient to validate the assertion \( (\mathcal{A}_{2^{-M_0}}) \).

If the above algorithmic procedure \( (\mathcal{A}_{2^{-M_0}}) \) ends up with a validation, one can deduce a procedure to prove or disprove the assertion \( (\mathcal{A}^0) : \ "0 \in E_0" \). One just need to analyze which \( \alpha \) provides the lopsided inequality in (15) at step 3. If it \( \alpha = 0 \), then \( (\mathcal{A}^0) \) is validated ; if it is \( \alpha \neq 0 \), \( (\mathcal{A}^0) \) is disproved. Thus, according to the fact that validation procedure for \( (\mathcal{A}_{2^{-M_0}}) \) concludes positively, we obtain in this way a test for BIBO stability of the discrete \( n \)-rational filter with transfer function such as (14).

When \( n = 2 \) and the validation procedure of \( (\mathcal{A}_{2^{-M_0}}) \) fails, one can use (11) to prove or disprove exactly the assertion \( (\mathcal{B}) : \ "0 \text{ lies in the contour of } \mathcal{A}_F" \). Such procedure provides also the value of \( \gamma_F(\zeta) \in \mathbb{P}^1(\mathbb{R}) \), where \( \gamma_F \) denotes the logarithmic Gauss map introduced in [10] and \( \zeta \in \log^{-1}(\{0\}) = \{(e^{i\theta_1}, e^{i\theta_2}) ; (\theta_1, \theta_2) \in (\mathbb{R}/2\pi \mathbb{Z})^2 \} \) is such that \( \gamma_F(\zeta) \in \mathbb{P}^1(\mathbb{R}) \). The value \( \gamma_F(\zeta) \) stands for the normal complex direction to the smooth complex curve \( \log((F^{-1}(\{0\})_{\text{reg}}) \in \text{the tubular domain } \mathbb{R}^2_x + i\mathbb{R}^2_\theta \) (log being here the multivalued function \( z \mapsto \log z + i(\arg(z_1), \arg(z_2)) \)). The fact that \( \gamma_F(\zeta) \in \mathbb{P}^1(\mathbb{R}) \) (in which case \( \log(\zeta) = (0,0) \) is in the contour of \( \mathcal{A}_F \)) can be interpreted as the fact that the direction \( \gamma_F(\zeta) \) is “horizontal” in the vertical strip \( \mathbb{R}^2_x + i\mathbb{R}^2_\theta \). In case \( (0,0) \) belongs to the boundary of \( \mathcal{A}_F \) (which is a subset of the contour), such a direction \( \gamma_F(\zeta) \) corresponds to a normal direction to the boundary of the amoeba \( \mathcal{A}_F \) at the point \( (0,0) \).

Let still \( n = 2 \). Suppose assertion \( (\mathcal{B}) \) has been proved. Let now \( (\mathcal{C}) \) be the assertion: “\( 0 \) is a boundary point of \( E_0 \)” (that is the corresponding discrete rational 2-filter is weakly BIBO stable in the sense of Definition (11)). The numerical procedures RONKIN, AMIBE (under MATLAB) and ContourAmoeba (under the environment of formal calculus
Sage) proposed in [24, §5] (see also [6, Algorithm 3]) lead to a representation of $A_F$ and its contour just by plotting the two-dimensional graph of $\Delta([R_F])$. One cannot conclude from such routines to an algorithmic procedure from which one could validate the assertion (C) since $\Delta([R_F])$ is a distribution which is roughly numerically evaluated as a function. Nevertheless, the result of such algorithmic procedures (RONKIN, AMIBE, see [24, §5]) allow to guess that $(0,0)$ is close to a point in the boundary of $E_0$ which is not a branching point for the contour of $A_F$. If this is the case, the validation of assertion (B) implies that of assertion (C). Note that disproving (B) also disproves (C) since the boundary of $A_F$ is a subset of the contour of $A_F$. We get in this way a test for weak BIBO stability when the test for BIBO stability ($A^0$) fails since $(A_{2-M_0})$ fails.

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