Tau functions of the charged free bosons

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Abstract We study bosonic tau functions in relation with the charged free bosonic fields. It is proved that up to a constant the only tau function in the Fock space $\mathcal{M}$ is the vacuum vector, and some tau functions are given in the completion $\overline{\mathcal{M}}$ by using Schur functions. We also give a new proof of Borchardt’s identity and obtain several $q$-series identities by using the boson-boson correspondence.

Keywords tau functions, boson-boson correspondence, vertex operator algebras, symmetric functions

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1 Introduction

The $bc$ fermionic fields and charged free bosons are important in conformal field theory partly due to their applications in free field realizations of affine Lie algebras and related algebras (see [17] and the references therein). In this regard the $bc$ fermionic fields provide $\mathfrak{g}l_{\infty}$-modules, level one $\widehat{\mathfrak{g}l}_{\infty}$-modules [21,22], and a free field realization of $\mathcal{W}_{1+\infty}$ algebra in positive integral central charge [14]. The $bc$ fermionic fields also provide a Lie theoretic approach to the celebrated Kadomtsev-Petviashvili (KP) hierarchy [8,9]. One can formulate the Hirota equation for the KP hierarchy as

$$\text{Res}_z b(z) \otimes c(z) (\tau \otimes \tau) = 0,$$

(1.1)

where $b(z)$ and $c(z)$ are fermionic fields satisfying (2.7). The solutions $\tau$ of (1.1) are called the KP tau functions. Under the boson-fermion correspondence, the tau functions can be expressed as certain symmetric functions. In particular, one can get polynomial soliton solutions of the KP hierarchy [6,21] in the Fock space representations.

The charged free bosons $\varphi(z)$ and $\varphi^*(z)$ enjoy similar properties to the $bc$ fermionic fields in conformal field theory. They also provide $\mathfrak{g}l_{\infty}$-modules, level $-1$ $\widehat{\mathfrak{g}l}_{\infty}$-modules, and a free field realization of $\mathcal{W}_{1+\infty}$ algebra with negative integral central charge [22,25,29]. By the boson-boson correspondence or Friedan-Martinec-Shenker (FMS) bosonization [4,17,25], one can also consider the corresponding PDEs and the bosonic tau functions as solutions of the equation (see [27])

$$\text{Res}_z \varphi^*(z) \otimes \varphi(z) (\tau \otimes \tau) = 0.$$

(1.2)
Our first result is to show that the only bosonic tau function in the Fock space \( \mathcal{M} \) is the vacuum vector \([0]\) up to constant. So the natural tau functions in this case lie in the completion \( \mathcal{M} \). In fact, we will show that

\[
\tau = \exp \left( \sum_{i \geq 0, j > 0} c_{i,j} \varphi_i^2 \varphi_j \right) \cdot [0]
\]

in the space \( \hat{\mathcal{M}} \) are bosonic KP-like tau functions.

The difficulty to derive equivalent forms of tau functions in the completion of the polynomial space appearing in the charged free bosons case in contrast to the bc fermionic fields case is partly due to the following heuristic reason: in both cases the action of exponentiating an element of \( \mathfrak{gl}_\infty \) on the vacuum element gives us tau functions, but only in the fermionic case this exponential is finite. Our next result is to give explicit formulas for some families of the bosonic tau functions in terms of Schur functions in several sets of variables. As an example, we obtain that a special bosonic hierarchy equation is a harmonic equation.

The relation between the boson-boson correspondence and the bc fermionic fields has been studied by Wang [29]. With the help of the correspondence, we give a new proof of Borchardt’s identity, i.e.,

\[
\det \left( \frac{1}{(z_i - w_j)^2} \right)_{1 \leq i,j \leq n} = \det \left( \frac{1}{z_i - w_j} \right)_{1 \leq i,j \leq n} \perm \left( \frac{1}{z_i - w_j} \right)_{1 \leq i,j \leq n}.
\]

By calculating the \( q \)-dimension of the (bosonic and fermionic) Fock space of bc fermionic fields and charged free bosons with the combinatorial method and by using the results from [13, 29], we obtain families of \( q \)-series identities (\( l \geq 0 \)):

\[
\sum_{m \geq 0} \frac{q^m}{(q;q)_m(q;q)_{m+l}} = \frac{1}{(q;q)^2} \sum_{s \geq 0} (-1)^s q^{\frac{s(s+1)}{2} + sl},
\]

\[
\sum_{m \geq 0} \frac{q^{m^2+(l+1)m}}{(q;q)_m(q;q)_{m+l}} = \frac{1}{(q;q)_\infty} \sum_{s \geq 0} (-1)^s q^{\frac{s(s+1)}{2} + sl},
\]

\[
\sum_{m \geq 0} \frac{q^{m^2+ml}}{(q;q)_m(q;q)_{m+l}} = \frac{1}{(q;q)_\infty}.
\]

The special case of \( l = 0 \) is the famous Euler identity.

The rest of the paper is organized as follows. In Section 2, we review some basic results of simple lattice vertex algebras, Schur functions and bc fermionic fields. In Section 3, we first recall some relations and bosonization of charged free bosons, then we give some concrete forms of tau functions in the completion of the polynomial space of the derived Hirota equations, and prove Borchardt’s identity in terms of the boson-boson correspondence. We have shown that a particular bosonic hierarchy equation is a harmonic equation. In Section 4, we consider the \( q \)-dimension in two ways to derive some \( q \)-series identities.

## 2 Preliminaries

### 2.1 Simple lattice vertex algebras

The lattice vertex operator algebras are important algebraic structures generalizing finite dimensional simple Lie algebras with numerous applications [12, 16, 21, 24, 30] (see also [1]). We briefly review simple lattice vertex algebras here.

Let \( L \) be an integral lattice spanned by the basis \( \gamma^i \) with a bilinear form \( (\cdot | \cdot) : L \times L \rightarrow \mathbb{Z} \) and \( \mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} L \) its complexification. The twisted group algebra \( \mathbb{C}_\tau[L] \) is the algebra generated by \( e^\gamma \ (\gamma \in L) \) with the twisted multiplication such that

\[
e^\gamma e^\delta = \epsilon(\gamma, \delta) e^{\gamma+\delta} \quad (\gamma, \delta \in L),
\]

where

\[
\epsilon(\gamma, \delta) = e^{\langle 2\pi i \gamma, \delta \rangle} = e^{2\pi i (\gamma | \delta)} = e^{-2\pi i \delta | \gamma}
\]

for \( \gamma, \delta \in \mathfrak{h} \).
where $\epsilon : L \times L \to \{\pm 1\}$ is the 2-cocycle such that
$$
\epsilon(\gamma, \delta)\epsilon(\delta, \gamma) = (-1)^{(\gamma | \delta) + (\gamma | \gamma)(\delta | \delta)}.
$$

Let $\hat{h} = h[t, t^{-1}] + \mathbb{C}c$ be the central extension of the affinization of $\mathfrak{h}$, and $S$ be the symmetric algebra of the commutative subalgebra $\hat{\mathfrak{h}}^- = \sum_{j > 0} \mathfrak{h} \otimes t^j$. Write $h_j := h \otimes t^j$. The lattice vertex algebra $V_L = S \otimes \mathbb{C}[e[\mathfrak{h}]]$ is the space generated by elements of the form $\gamma_1 \cdots \gamma_k \in \mathfrak{h}^\vee$ for $\gamma \in L$, $\gamma \in \mathfrak{h}$ with parity
$$
\langle \gamma_1 \cdots \gamma_k \rangle = \langle \gamma \rangle \mod 2
$$
and the state-field correspondence is given by $(\gamma_i, \gamma) \in \mathbb{C}[\mathfrak{h}]$:
$$
Y(h_{-n} h_{-n-1}^2 \cdots e^\gamma, z) =: \partial^{(n)}(z) \partial^{(n+2)}(z) \cdots \Gamma_\gamma(z),
$$
where
$$
h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1}, \quad \partial^n h(z) = \frac{\partial^n}{n!} h(z),
$$
$$
\Gamma_\gamma(z) = Y(e^\gamma, z) = e^\gamma z^{\gamma_0} \exp \left( \sum_{j < 0} \frac{z^{-j}}{-j} \gamma_j \right) \exp \left( \sum_{j > 0} \frac{z^j}{j} \gamma_j \right),
$$
and $\;\mathbin{\,\,\,}$ is the normal ordered product [16].

For convenience we collect commutation relations of the fields $h(z)(h \in \mathfrak{h})$ and $Y(e^\gamma, z)$ as follows:
$$
[h(z), h'(w)] = \langle h \ | h' \rangle \partial_w \delta(z - w) \quad (h, h' \in \mathfrak{h}),
$$
$$
[h(z), Y(e^\gamma, w)] = \langle h \ | \gamma \rangle Y(e^\gamma, w) \delta(z - w) \quad (h \in \mathfrak{h}, \gamma \in L),
$$
$$
[Y(e^\gamma, z), Y(e^\delta, w)] = \sum_{n \geq 0} \frac{Y(e_n^\gamma e^\delta, w)}{n!} \partial^n \delta(z - w) \quad (\gamma, \delta \in L),
$$
where $[a, b] = ab - (-1)^{p(a)p(b)}ba$, $\delta(z - w) = \sum_{i \in \mathbb{Z}} z^{-i-1} w^i$, and $e_n^\gamma e^\delta$ is the $n$-th product.

### 2.2 Schur polynomials

The complete symmetric function $h_k(x)$ in the variables $x_1, x_2, \ldots$ is defined by (see [26])
$$
\sum_{k = 0}^{\infty} h_k(x) z^k = \prod_{i = 1}^{\infty} \frac{1}{1 - x_i z}.
$$
To each partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0\}$ we associate the Schur function $s_\lambda(x)$ defined by
$$
s_\lambda(x) = \det (h_{\lambda_i - i+j}(x))_{1 \leq i, j \leq l}.
$$
Introduce the variable $t_n$, where $t_n = \frac{1}{n} \sum_{i = 1}^{\infty} x_i^n$ for all $n \geq 1$. Then $h_k(x)$ is a polynomial $S_k(t)$ in $t_n$ called the elementary Schur polynomial [21], and can be defined by the series expansion
$$
\sum_{k = 0}^{\infty} S_k(t) z^k = \exp \left( \sum_{n = 1}^{\infty} t_n z^n \right).
$$
Explicitly, one has
$$
S_k(t) = 0 \quad \text{for } k < 0, \quad S_0(t) = 1,
$$
$$
S_k(t) = \sum_{k_1 + 2k_2 + 3k_3 + \cdots = k} \frac{t_1^{k_1} t_2^{k_2} t_3^{k_3} \cdots}{k_1! k_2! k_3! \cdots} \quad \text{for } k > 0.
$$

The Schur polynomial $S_\lambda(t)$ associated with the partition $\lambda$ is given by the Jacobi-Trudi formula
$$
S_\lambda(t) = \det(S_{\lambda_i - i+j}(t))_{1 \leq i, j \leq l(\lambda)}.
$$
2.3 bc fermionic fields

Recall that the bc fermionic fields are

\[ b(z) = \sum_{i \in \mathbb{Z}} b(i) z^{-i}, \quad c(z) = \sum_{i \in \mathbb{Z}} c(i) z^{-i-1} \]  

(2.6)

with the commutation relations

\[ \{b(i), c(j)\} = \delta_{i,-j}, \quad \{b(i), b(j)\} = \{c(i), c(j)\} = 0, \]  

(2.7)

where \( \{A, B\} = AB + BA \). The operator product expansions (OPE) of \( b(z) \) and \( c(z) \) are

\[ b(z)c(w) \sim \frac{1}{z-w}, \quad c(z)b(w) \sim \frac{1}{z-w}. \]  

(2.8)

Let \( \mathcal{F} \) be the Fock space spanned by negative (resp. non-positive) modes of \( c(z) \) (resp. \( b(z) \)), i.e., \( \mathcal{F} \) is generated by the vacuum vector \(|0\rangle \) subject to the relations

\[ b(i+1)|0\rangle = c(i)|0\rangle = 0, \quad i \geq 0. \]  

(2.9)

Then (see [22, 25, 29])

\[ j^{bc}(z) =: e(z)b(z) := \sum_{n \in \mathbb{Z}} j^{bc}_n z^{-n-1} \]

is a free boson with the commutation relations

\[ [j^{bc}_m, j^{bc}_n] = m\delta_{m,-n}, \quad [j^{bc}_m, b(n)] = -b(m+n), \quad [j^{bc}_m, c(n)] = c(m+n). \]  

(2.10)

We then have the charge decomposition of \( \mathcal{F} \) according to eigenvalues of \( j^{bc}_0 \):

\[ \mathcal{F} = \bigoplus_{l \in \mathbb{Z}} \mathcal{F}^l. \]

Denoted by \( \overline{\mathcal{F}} \) the Fock space generated by the fields \( \partial b(z) \) and \( c(z) \) with the vacuum vector \(|0\rangle \), and \( \overline{\mathcal{F}} \) also has the charge decomposition: \( \overline{\mathcal{F}} = \bigoplus_{l \in \mathbb{Z}} \overline{\mathcal{F}}^l \), according to \( j^{bc}_0 \)-eigenvalues.

We further consider (see [29])

\[ T^{bc}(z) =: \partial b(z)c(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \]

which is a Virasoro field with the commutation relations

\[ [L_m, L_n] = (m-n)L_{m+n} - \frac{m^3 - m}{6}\delta_{m,-n}, \]  

(2.11)

\[ [L_m, b(n)] = -(m+n)b(m+n), \quad [L_m, c(n)] = -nc(m+n). \]  

(2.12)

The boson-fermion correspondence [10, 15, 19, 21] realizes the bc fermionic fields by the lattice vertex operators

\[ b(z) = Y(e^{\alpha}, z), \quad c(z) = Y(e^{-\alpha}, z), \]  

(2.13)

where \( (\alpha | \alpha) = 1 \). Thus

\[ T^{bc}(z) = \frac{1}{2}(\alpha(z)\alpha(z) + \partial \alpha(z)). \]  

(2.14)

It is known in [21, 23, 25] that

\[ \text{Res}_z (b(z) \otimes c(z)) (\tau \otimes \tau) = 0 \]  

(2.15)

gives the Hirota bilinear equation associated with the KP hierarchy. The identification between \( \mathcal{F}^l \) and \( e^{-\alpha \mathbb{C} [t_1, t_2, \ldots]} \) [19, 21] implies that the Schur polynomials are examples of the tau functions (2.15). The tau functions are also related to Hurwitz numbers and 2-Toda hierarchies [31]. Certain twisted form of the KP hierarchy [9] can also be formulated with the help of the fermionic vertex operator algebra and Schur Q-functions [11, 19]. Further generalizations to Hall-Littlewood functions are also known [20, 28].
3 Charged free bosons

We first recall the charged free bosons \[22, 25, 29\] and the boson-boson correspondence (FMS bosonization) \[17, 29\]. With the representation of the Heisenberg fields we can describe the bosonic KP hierarchy of PDEs and the embedding from the completion of the Fock space of charged free bosons to the completion \(B\) of a polynomial algebra (3.15). We then obtain some tau functions in \(B\) and give a new proof of Borchardt’s identity.

3.1 Relations

The charged free bosons are defined by
\[
\varphi(z) = \sum_{i \in \mathbb{Z}} \varphi_i z^{-i-1}, \quad \varphi^*(z) = \sum_{i \in \mathbb{Z}} \varphi_i^* z^{-i}
\]
(3.1)

with the commutation relations
\[
[\varphi_i, \varphi_j^*] = \delta_{i,-j}, \quad [\varphi_i, \varphi_j] = [\varphi_i^*, \varphi_j^*] = 0.
\]
(3.2)

The nontrivial OPE relations are
\[
\varphi(z)\varphi^*(w) \sim \frac{1}{z-w}, \quad \varphi^*(z)\varphi(w) \sim \frac{1}{z-w}.
\]
(3.3)

Let \(\mathcal{M}\) be the Fock space of the charged free bosons generated by the vacuum vector \(|0\rangle\) satisfying
\[
\varphi_i |0\rangle = \varphi_i^* |0\rangle = 0, \quad i > 0.
\]
(3.4)

Then \(\mathcal{M}\) has the following basis:
\[
\{ \varphi_{-i_1} \cdots \varphi_{-i_l} \varphi_{-j_k}^* \cdots \varphi_{-j_1}^* |0\rangle \mid i_1 > \cdots > i_l > 0, j_k > \cdots > j_1 > 0, n_s \geq 0, m_s \geq 0 \}.
\]
(3.5)

As a vector space,
\[
\mathcal{M} \simeq \mathbb{C}[\varphi_{-i}, \varphi_{-i}^* \mid i \in \mathbb{Z}_+]|0\rangle.
\]

We will also need the completion space
\[
\widetilde{\mathcal{M}} = \mathbb{C}[\varphi_{-i}, \varphi_{-i}^* \mid i \in \mathbb{Z}_+]|0\rangle.
\]

Remark 3.1. Charged free bosons are usually called the \(\beta\)-\(\gamma\) system, and can also be studied in a symplectic fermionic vertex operator superalgebra \[3, 7\].

It is known in \[14, 29\] that \(\mathcal{M}\) is also a module for the algebra \(\mathcal{W}_{1+\infty,-1}\) under the action
\[
J^i(z) =: \varphi(z) \partial^i \varphi^*(z),
\]
(3.6)

where \(J^i(z) = \sum_{k \in \mathbb{Z}} J^i_k z^{-k-i-1}\). Therefore, \(J^0(z) = \sum_{k \in \mathbb{Z}} J^0_k z^{-k-1}\) is a free bosonic field with the commutation relations
\[
[J^0_i, J^0_j] = -i\delta_{i,-j}, \quad [J^0_i, \varphi_j] = -\varphi_{i+j}, \quad [J^0_i, \varphi_j^*] = \varphi_{i+j}^*.
\]
(3.7)

The space \(\mathcal{M}\) also decomposes itself as a sum of eigenspaces of \(-J^0_0\) (charge decomposition):
\[
\mathcal{M} = \bigoplus_{l \in \mathbb{Z}} \mathcal{M}^l,
\]

where \(\mathcal{M}^l = \{ x \in \mathcal{M} \mid J^0_0 x = lx \}\). Similarly,
\[
J^1(z) = \sum_{k \in \mathbb{Z}} J^1_k z^{-k-2}
\]
is a Virasoro field with the commutation relations

\[ [J^1_i, J^2_j] = (i-j)J^1_{i+j} + \frac{i^3 - i}{6} \delta_{i,-j}, \quad [J^1_i, \varphi_j] = -j \varphi_{i+j}, \quad [J^1_i, \varphi^*_j] = -(i+j)\varphi^*_{i+j}. \] (3.8)

Then we have the degree decomposition of \( \mathcal{M} \) according to the eigenvalues of the operator \( J^1_0 \):

\[ \mathcal{M} = \bigoplus_{n=0}^{\infty} \mathcal{M}_n, \] (3.9)

where \( \mathcal{M}_n = \{ x \in \mathcal{M} \mid J^1_0 x = nx \} \), i.e., it is the span of the vectors in the form (3.5) such that

\[ n = i_1 n_1 + \cdots + i_k n_1 + j_1 m_1 + \cdots + j_k m_k. \]

Using the method in [2], we have the following result.

**Proposition 3.2.** Let

\[ \Omega_U = \sum_{i \in \mathbb{Z}} \varphi^*_i \otimes \varphi_{-i} = \text{Res}_z \varphi^*(z) \otimes \varphi(z). \]

If \( \tau \in \mathcal{M} \) satisfies the Hirota equation

\[ \Omega_U (\tau \otimes \tau) = 0, \] (3.10)

then \( \tau = |0\rangle \) up to a constant.

**Proof.** It follows from the definition (3.4) that the vacuum vector \(|0\rangle\) is a solution of (3.10). Suppose that \( \tau \neq |0\rangle \) is a solution of (3.10) and a sum of monomials in the form (3.5). Let \( N > 0 \) be the largest integer such that \( \varphi_{-N} \) appears in \( \tau \). Then \( \tau \) can be written in the form

\[ \sum_{k=0}^{m} \varphi^{k}_{-N} P_k(\varphi_{-N+1}, \cdots, \varphi_{-1}, \cdots, \varphi^*_2, \varphi^*_1, \varphi^*_{0})|0\rangle, \]

where \( P_m \neq 0 \) \( (m \geq 1) \) and \( P_k \) \( (k \geq 0) \) are linear combinations of the basis elements (3.5) such that the largest \( n \) for which \( \varphi_{-n} \) appears in \( P_k \) is less than or equal to \( N - 1 \). Then we have

\[ \Omega_U (\tau \otimes \tau) = \varphi_N^* \otimes \varphi_{-N}(\tau \otimes \tau) + \sum_{i < N} \varphi^*_i \otimes \varphi_{-i}(\tau \otimes \tau) \]

\[ = \sum_{k=0}^{m} -k \varphi^{k-1}_{-N} P_k|0\rangle \otimes \sum_{k=0}^{m} \varphi^{k+1}_{-N} P_k|0\rangle + \sum_{i < N} \varphi^*_i \otimes \varphi_{-i}(\tau \otimes \tau). \]

Note that the second summand \( \sum_{i < N} \varphi^*_i \otimes \varphi_{-i}(\tau \otimes \tau) \) contains at most \( \varphi^m_{-N} \) on the right of the tensor products. There are no other terms to cancel the nonzero term

\[ \varphi^{m-1}_{-N} P_m|0\rangle \otimes m \varphi^{m+1}_{-N} P_m|0\rangle. \]

The contradiction shows that \( \Omega_U (\tau \otimes \tau) \neq 0 \). \( \square \)

Solutions of the Hirota equation are called *tau functions*. By the above proposition one needs to focus on tau functions in the completion \( \widetilde{\mathcal{M}} = \mathbb{C}[\varphi_{-1}; \varphi^*_{-1}]_{i \geq 0}|0\rangle \) (see (3.5)).

**Proposition 3.3.** The function

\[ \tau = \exp \left( \sum_{i \geq 0, j > 0} c_{i,j} \varphi^*_{-i} \varphi_{-j} \right) \cdot |0\rangle \in \widetilde{\mathcal{M}} \]

(finitely many \( c_{i,j} \neq 0 \)) is a solution of (3.10).
Proof. It follows from (3.2) that
\[
[\Omega_U, 1 \otimes \varphi^*_m \varphi_n + \varphi^*_m \varphi_n \otimes 1]
\]
\[
= \sum_{i \in \mathbb{Z}} [\varphi_i^*, \varphi_n^* \varphi_n] \otimes \varphi_{-i} + \sum_{i \in \mathbb{Z}} [\varphi_i^* \otimes [\varphi_{-i}, \varphi_m^* \varphi_n]]
\]
\[
= \sum_{i \in \mathbb{Z}} ([\varphi_i^*, \varphi_n^* \varphi_n] + \varphi_m^*[\varphi_m^* \varphi_n]) \otimes \varphi_{-i} + \sum_{i \in \mathbb{Z}} [\varphi_i^* \otimes ([\varphi_{-i}, \varphi_n^* \varphi_n] + \varphi_m^*[\varphi_{-i}, \varphi_n])
\]
\[
= -\varphi_m^* \varphi_n + \varphi_m^* \varphi_n = 0.
\]
Then we have
\[
\Omega_U(\tau \otimes \tau) = \exp \left( \sum_{i \geq 0, j > 0} c_{i,j} \varphi_{-i}^* \varphi_{-j} \right) \otimes \exp \left( \sum_{i \geq 0, j > 0} c_{i,j} \varphi_{-i}^* \varphi_{-j} \right) (\Omega_U(|0 \rangle \otimes |0 \rangle)) = 0.
\]
The proof is completed. \hfill \Box

Remark 3.4. The assignment \( E_{ij} \rightarrow: \varphi_i \varphi_j^* : \) also provides a level \(-1\) representation of the infinite-dimensional Lie algebra \( \mathfrak{gl}(\infty) \). Therefore the bosonic tau functions are in the orbit of \( GL(\infty)|0\rangle \).

3.2 Bosonization

The Friedan-Martinec-Shenker (FMS) bosonization [13, 17, 29] provides a boson-boson realization of the charged free bosons \( \varphi(z), \varphi^*(z) \) in terms of a lattice vertex algebra. Let \( L = \mathbb{Z} \alpha + \mathbb{Z} \beta \) be the 2-dimensional lattice, where \( (\alpha | \beta) = 0 \) and \( (\alpha | \alpha) = (\beta | \beta) = 1 \). On the lattice vertex algebra associated to \( L \), consider the vertex operators
\[
\varphi(z) = Y(e^{-\alpha \beta}, z), \quad \varphi^*(z) = Y(\alpha e^{\alpha \beta}, z),
\]
and the associated 2-cocycle is
\[
\varepsilon(\alpha, \alpha) = \varepsilon(\alpha, \beta) = -\varepsilon(\beta, \alpha) = -\varepsilon(\beta, \beta) = 1.
\]
By using (2.13), the above can be rewritten as
\[
\varphi(z) = c(z)Y(e^{-\beta}, z), \quad \varphi^*(z) = \partial b(z)Y(e^\beta, z).
\]
It is easy to see that in this case (see (3.6))
\[
J^0(z) = -\beta(z), \quad (3.13)
\]
\[
J^1(z) = \frac{\alpha(z)\alpha(z) + \partial \alpha(z)}{2} - \frac{\beta(z)\beta(z) + \partial \beta(z)}{2}. \quad (3.14)
\]
Introduce the (completed) bosonic Fock space
\[
\mathcal{B} = \mathbb{C}[[x, y; p, p^{-1}]],
\]
where \( x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots), p = e^{\alpha + \beta} \) and \( x_n = \frac{1}{n} \alpha - n, y_n = \frac{1}{n} \beta - n \). Since
\[
[\alpha_m, \alpha_n] = m\delta_{m-n}, \quad [\beta_m, \beta_n] = -m\delta_{m-n}, \quad [\alpha_0, e^{\alpha + \beta}] = e^{\alpha + \beta}, \quad [\beta_0, e^{\alpha + \beta}] = -e^{\alpha + \beta},
\]
the Heisenberg fields \( \alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1} \) and \( \beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n-1} \) act on \( \mathcal{B} \) as follows \((n > 0)\):
\[
\alpha_n = \partial x_n, \quad \alpha_0 = -n x_n, \quad \alpha_0 = p \partial p, \\
\beta_n = -\partial y_n, \quad \beta_0 = n y_n, \quad \beta_0 = -p \partial p.
\]
Then we have an embedding of \( \tilde{\mathcal{M}} \) into \( \mathcal{B} \) by \(|0 \rangle \rightarrow 1 \) and
\[
\varphi^*(z) = p \exp \left( \sum_{n>0} (x_n + y_n) z^n \right) \left( \sum_{k>0} k x_k z^{-k-1} + \sum_{k>0} \partial x_k z^{-k-1} + p \partial p z^{-1} \right)
\]
\[
\times \exp \left( - \sum_{n>0} \left( \partial x_n - \partial y_n \right) \frac{z^n}{n} \right), \quad (3.16)
\]

\[
\varphi(z) = p^{-1} \exp \left( - \sum_{n>0} (x_n + y_n) z^n \right) \exp \left( \sum_{n>0} \left( \partial x_n - \partial y_n \right) \frac{z^n}{n} \right). \quad (3.17)
\]

Therefore one can write \( \Omega_U = \text{Res}_z \varphi^*(z) \otimes \varphi(z) \) as an operator on the space \( B \otimes B \). For simplicity we denote \( X \otimes Y = X'Y'' \). Then

\[
\Omega_U = \text{Res}_z p' p''^{-1} \exp \left( \sum_{n>0} (x'_n - x''_n + y'_n - y''_n) z^n \right) \left( \sum_{k>0} k x'_{k} z^{k-1} + \sum_{k>0} \partial x'_{k} z^{k-1} + p' \partial p' z^{-1} \right)
\times \exp \left( - \sum_{n>0} \left( \partial x'_n - \partial x''_n - \partial y'_n + \partial y''_n \right) \frac{z^n}{n} \right). \quad (3.18)
\]

We introduce the following new operators over \( B \otimes B \):

\[
A = \frac{1}{2} (A' - A''), \quad \bar{A} = \frac{1}{2} (A' + A'').
\]

Then

\[
A' = A + \bar{A}, \quad A'' = \bar{A} - A. \quad (3.19)
\]

This gives us

\[
\Omega_U = \text{Res}_z p' p''^{-1} \exp \left( \sum_{n>0} (2x_n + 2y_n) z^n \right)
\times \left( \sum_{k>0} k (x_k + x_k) z^{k-1} + \sum_{k>0} \frac{\partial x_k + \partial x_k}{2} z^{k-1} + p' \partial p' z^{-1} \right) \exp \left( \sum_{n>0} \left( \partial x_n + \partial y_n \right) \frac{z^n}{n} \right).
\]

**Proposition 3.5.** Let \( \tau \) be a tau function in \( \mathbb{C}[[x, y]] \). Then

\[
\sum_{i,j \geq 0} S_i(2x + 2y) \left[(j-i)(x_{j-i} + \bar{x}_{j-i}) + \frac{\partial \lambda_{i-j} + \partial \bar{\lambda}_{i-j}}{2} \right] S_j(-\bar{\lambda} + \bar{\mu})
\times \exp \left( \sum_{i \geq 1} (x_i \partial \lambda_i + y_i \partial \mu_i) \right) \tau(\bar{x} - \lambda, \bar{y} - \mu) \tau(\bar{x} + \lambda, \bar{y} + \mu) \big|_{\lambda = \mu = 0} = 0, \quad (3.20)
\]

where \( \bar{\lambda} = (\partial \lambda_1, \frac{\partial \lambda_2}{2}, \frac{\partial \lambda_3}{3}, \ldots) \).

Let us write down some Hirota equations given by (3.20). Denote \( \tilde{S}_n = S_n(-\bar{\lambda} + \bar{\mu}) \) and consider the case where \( x_i = \bar{x}_i = y_i = \bar{y}_i = 0 \) for \( i \geq 2 \). Then the coefficient of \( x_1 \) in (3.20) gives the equation

\[
(2\tilde{S}_1 \tilde{S}_2 + \partial \lambda_1 + \partial \tilde{S}_1 + \tilde{S}_1 \partial \lambda_1 + \tilde{S}_1) \tau(\bar{x} - \lambda, \bar{y} - \mu) \tau(\bar{x} + \lambda, \bar{y} + \mu) \big|_{\lambda = \mu = 0} = 0.
\]

By writing \( \bar{x}_1 = u, \bar{y}_1 = v \) and \( g(u, v) = \log \tau \), the above gives the differential equation

\[
u (-g_{uv} + g_v) + g_u = 0.
\]

Similarly, set \( x_i = \bar{x}_i = y_i = \bar{y}_i = 0 \) for \( i \geq 3 \) and \( \bar{x}_1 = u, \bar{y}_1 = v \), \( f(u, v) = \log \tau \). Then one has the following differential equation from (3.20):

\[
f_{uu} - 2f_{uv} + f_{vv} = 0. \quad (3.21)
\]

Let \( t = u - v \) and \( s = u + v \). Then \( \partial_t = \frac{1}{2} (\partial_u - \partial_v) \) and \( \partial_s = \frac{1}{2} (\partial_u + \partial_v) \). So the special bosonic hierarchy equation (3.21) is a harmonic equation

\[
\tilde{f}_{tt} = 0 \quad (3.22)
\]

for the function \( f = f(u, v) = \tilde{f}(t, s) \). This surprising result will be studied further elsewhere.
Remark 3.6. If $\tau \in \mathbb{C}[[x]]$, we get the so-called $\beta$-reduction (see [25])

$$\sum_{i,j \geq 0} S_i(2x) \left[(j-i)(x_{j-i} + \tilde{x}_{j-i}) + \frac{\partial \xi_{j-i} + \partial \tilde{x}_{j-i}}{2}\right] S_j(-\bar{\partial} \lambda)$$

$$\times \exp\left(\sum_{i \geq 1} x_i \partial \lambda_i\right) \tau(\bar{x} - \lambda) \tau(\bar{x} + \lambda) |_{\lambda = 0} = 0.$$ 

In particular, the tau function of the $\beta$-reduction is also a tau function of (3.20). The converse is not true in general.

3.3 Tau functions

We now discuss the tau functions of (3.20). First, we give new expressions for the tau functions $\exp(\sum_{i \geq 1} a_i \varphi_{-j} \varphi_{n}^* \cdot 1$ and $\exp(a \varphi_{-j} \varphi_{n}^* \cdot 1, j \geq 1$. Then we give formulas for $\exp(a \varphi_{-j} \varphi_{n}^* \cdot 1, s \geq 1, t \geq 2$ and $\exp(d \varphi_{-j} \varphi_{n}^* \cdot 1, i, j \geq 1, k \geq l \geq 0$.

To describe our results, we need the elementary Schur polynomials $S_k(x), S_k(-x - y)$ and $S_k(x + y)$, which are defined similarly to (2.3) by the following generating functions:

$$\sum_{k=0}^{\infty} S_k(-x)w^k = \exp\left(-\sum_{n=1}^{\infty} x_i w^i\right),$$

$$\sum_{k=0}^{\infty} S_k(-x - y)w^k = \exp\left(-\sum_{n=1}^{\infty} (x_i + y_i)w^i\right),$$

$$\sum_{k=0}^{\infty} S_k(x + y)w^k = \exp\left(\sum_{n=1}^{\infty} (x_i + y_i)w^i\right).$$

Lemma 3.7. One has for $n \geq 0$,

$$\varphi_{0}^* \cdot 1 = (-1)^n n! p^n S_n(-x).$$

Proof. In terms of the variables $x_i, y_i$ and $p$, the field operator $\varphi^*(z)$ can be rewritten as (see (3.16))

$$\varphi^*(z) = p \exp\left(\sum_{i \geq 0} (x_i + y_i)z^i\right) \left(\sum_{k \geq 0} kx_k z^{k-1} + \sum_{k \geq 0} \partial x_k z^{-k+1} + p \partial p z^{-1}\right)$$

$$\times \exp\left(-\sum_{i \geq 0} (\partial x_i - \partial y_i) \frac{z^{i-1}}{i}\right)$$

$$=: p \exp\left(\sum_{i \geq 0} (x_i + y_i)z^i\right)(A + B + C),$$

where we have used $A, B$ and $C$ to denote the respective summands in the second and third factors, e.g.,

$$A = \sum_{k \geq 0} kx_k z^{k-1} \exp\left(-\sum_{i \geq 0} (\partial x_i - \partial y_i) \frac{z^{i-1}}{i}\right),$$

To show (3.26) by induction on $n$, we note that it clearly holds for $n = 0, 1$. Now suppose that (3.26) is valid for $n$, and we want to show it for $n + 1$. To this end, set $R = (-1)^n n! p^n S_n(-x)$ and it follows from induction hypothesis that

$$\varphi^*(z) \varphi_{0}^n \cdot 1 = p \exp\left(\sum_{i \geq 0} (x_i + y_i)z^i\right)(A + B + C) R.$$

Applying $\exp(-\sum_{i \geq 1} (\partial x_i - \partial y_i) \frac{z^{i-1}}{i})$ and $\partial w$ to the series $\exp(-\sum_{j \geq 1} x_j w^j)$ respectively, we have

$$\exp\left(-\sum_{i \geq 1} (\partial x_i - \partial y_i) \frac{z^{i-1}}{i}\right) \exp\left(-\sum_{j \geq 1} x_j w^j\right) = \frac{1}{1 - \frac{w}{z}} \exp\left(-\sum_{j \geq 1} x_j w^j\right),$$
\[ \partial_x \exp \left( - \sum_{j \geq 1} x_j w^j \right) = - \sum_{j \geq 1} jx_j w^{j-1} \exp \left( - \sum_{j \geq 1} x_j w^j \right). \]

Collecting the coefficients, we get
\[ \exp \left( - \sum_{i \geq 1} (\partial x_i - \partial y_i) \frac{z_i}{i} \right) S_n(-x) = \sum_{i=0}^n S_i(-x) z^{-(n-i)}, \]
\[ \sum_{i \geq 1} i x_i S_{n-i}(-x) = -n S_n(-x). \]

Consequently the coefficient of \( z^0 \) in \((A + B + C)R\) is
\[ e_0 := (-1)^n n! p^n \sum_{k>0} k x_k S_{n+1-k}(-x) = -(n+1)(-1)^n n! p^n S_{n+1}(-x) = (-1)^{n+1} (n+1)! p^n S_{n+1}(-x). \]

By (3.7) and (3.14) it follows that
\[ \beta_i(\varphi^*_0 \cdot 1) = -j \varphi^*_0 \varphi_i^* \cdot 1 = 0, \quad i \geq 1, \]
i.e., \( S_k(x + y) \) does not appear in \( \varphi^*_0 \cdot 1 \) for \( k \geq 1 \). Note that \( \varphi^*_0 \cdot 1 \) is the coefficient of \( z^0 \) in \( \varphi^*(z) \varphi_0^n \cdot 1 \). Using (3.28), we have
\[ \varphi^*_0 = 1 = pS_0(x + y) e_0 = (-1)^{n+1} (n+1)! p^n S_{n+1}(-x), \]
which completes the proof.

The following result is clear.

**Lemma 3.8.** One has
\[ \sum_{k \geq m} C_m^n (k-m) = \frac{1}{(1-t)^{m+1}}. \]

**Theorem 3.9.** We have
\[ \exp \left( \sum_{j=1}^s a_j \varphi_j \varphi_0^* \right) \cdot 1 \]
\[ = \frac{1}{1 - \sum_{j=1}^s a_j S_j(-x - y)} \exp \left( - \sum_{n \geq 1} x_n \left( \frac{-\sum_{j=1}^s a_j S_{j-1}(-x - y)}{1 - \sum_{j=1}^s a_j S_j(-x - y)} \right)^n \right). \quad (3.29) \]

**Proof.** First by (3.23) and (3.26), we see that
\[ \exp(\alpha \varphi_0^* \cdot 1) = \exp \left( - \sum_{n \geq 1} x_n (\alpha p)^n \right). \]

Recalling (3.17), we have
\[ \varphi(z) \exp(\alpha \varphi_0^* \cdot 1) = p^{-1} \exp \left( - \sum_{n>0} (x_n + y_n) z^n \right) \exp \left( - \sum_{n \geq 1} x_n \frac{z^n}{n} (-ap)^n \right) \]
\[ = p^{-1} \exp \left( - \sum_{n>0} (x_n + y_n) z^n \right) \left( 1 - apz^{-1} \right) \exp \left( - \sum_{n \geq 1} x_n (-ap)^n \right). \]

Thus
\[ \varphi^{-j} \exp(\alpha \varphi_0^* \cdot 1) = \left( p^{-1} S_{j-1}(-x - y) - aS_j(-x - y) \right) \exp \left( - \sum_{n \geq 1} x_n (-ap)^n \right). \]
Since $\varphi(z)$ commutes with $\exp(-\sum_{n=1}^{\infty}(x_i + y_i)w^i)$, its components $\varphi_j$ commute with $S_n(-x - y)$ for all integer $j$. Therefore,
\[
\varphi_m^j \exp(a\varphi_0^*) \cdot 1 = (p^{-1}S_{j-1}(-x - y) - aS_j(-x - y))^m \exp \left( - \sum_{n \geq 1} x_n (-ap)^n \right).
\]

Then we have
\[
\prod_{j=1}^{s} \exp(a_j \varphi_j \cdot a\varphi_0^*) \cdot 1
\]
\[
= \exp \left( p^{-1} \sum_{j=1}^{s} a_j S_{j-1}(-x - y) \right) \exp \left( a \sum_{j=1}^{s} a_j S_j(-x - y) \right) \exp \left( - \sum_{n \geq 1} x_n (-ap)^n \right).
\]

Comparing the coefficient of $a^k p^0$ on both sides of (3.30), we get
\[
\frac{1}{k!} \sum_{m_1, m_2, \ldots, m_{s-1} \geq 0} \frac{\left( \sum_{j=1}^{s} a_j S_j(-x - y) \right)^{k - \sum_{j=1}^{s-1} m_j} \prod_{j=1}^{s-1} (a_j \varphi_j \cdot a\varphi_0^*)^{m_j}}{(k - \sum_{j=1}^{s-1} m_j)!} \cdot 1
\]
\[
= \sum_{0 \leq m \leq k} \left( \sum_{j=1}^{s} a_j S_j(-x - y) \right)^{k - m} \left( - \sum_{j=1}^{s} a_j S_{j-1}(-x - y) \right)^m S_m(-x).
\]

Thus we have
\[
\sum_{k \geq 0} \sum_{m_1, m_2, \ldots, m_{s-1} \geq 0} \frac{\left( \sum_{j=1}^{s} a_j S_j(-x - y) \right)^{k - \sum_{j=1}^{s-1} m_j} \prod_{j=1}^{s-1} (a_j \varphi_j \cdot a\varphi_0^*)^{m_j}}{(k - \sum_{j=1}^{s-1} m_j)!} \cdot 1
\]
\[
= \sum_{m \geq 0} \sum_{0 \leq m \leq k} C_k^m \left( \sum_{j=1}^{s} a_j S_j(-x - y) \right)^{k - m} \left( - \sum_{j=1}^{s} a_j S_{j-1}(-x - y) \right)^m S_m(-x)
\]
\[
= \sum_{m \geq 0} \left( - \sum_{j=1}^{s} a_j S_{j-1}(-x - y) \right)^m S_m(-x) \sum_{k \geq m} C_k^m \left( \sum_{j=1}^{s} a_j S_j(-x - y) \right)^{k - m}
\]
\[
= \sum_{m \geq 0} \left( - \sum_{j=1}^{s} a_j S_{j-1}(-x - y) \right)^m S_m(-x) \frac{1}{1 - \sum_{j=1}^{s} a_j S_j(-x - y)} \exp \left( - \sum_{n \geq 1} x_n \left( - \sum_{j=1}^{s} a_j S_{j-1}(-x - y) \right)^n \right),
\]

where we have used Lemma 3.8. Then we have
\[
\exp \left( \sum_{j=1}^{s} a_j \varphi_j \cdot a\varphi_0^* \right) \cdot 1 = \frac{1}{1 - \sum_{j=1}^{s} a_j S_j(-x - y)} \exp \left( - \sum_{n \geq 1} x_n \left( - \sum_{j=1}^{s} a_j S_{j-1}(-x - y) \right)^n \right).
\]

The proof is completed. \hfill \Box

**Theorem 3.10.** For $j \geq 1$, we have
\[
\exp(a \varphi_j \cdot a\varphi_{j-1}^*) \cdot 1 = \frac{1}{1 - aS_{j+1}(-x - y) - aS_1(x + y)S_j(-x - y)}
\]
\[
\times \exp \left( - \sum_{n \geq 1} \sum_{0 \leq i \leq n} \frac{C_n^i a^{n-i}(x + y)(n + i)x_{n+i}}{n} \right)
\]
\[
\times \left( - \frac{aS_{j-1}(-x - y)}{1 - aS_{j+1}(-x - y) - aS_1(x + y)S_j(-x - y)} \right)^n. \quad (3.31)
\]
The proof is left in Appendix A.

**Corollary 3.11.** Suppose that

\[
\left( \sum_{m=0}^{t} S_m(x + y)z^{-t-1+m} \right)^n = \sum_{i=0}^{nt} f_i z^{-n(t+1)+i},
\]

and let

\[
p_n = \sum_{i=0}^{nt} f_i (n(t+1) - i)x^{n(t+1)-i} \quad \text{and} \quad A_{s,t} = \sum_{m=0}^{t} S_m(x + y)S_{s+t-m}(-x - y).
\]

Then we have

\[
\exp(a \varphi_{-t} \varphi^*_{-t}) \cdot 1 = \frac{1}{1 - aA_{s,t}} \exp \left( - \sum_{n \geq 1} \frac{p_n}{n} \left( \frac{-aS_{s-1}(-x - y)}{1 - aA_{s,t}} \right)^n \right), \quad s \geq 1, \quad t \geq 0.
\]

**Corollary 3.12.** Suppose that

\[
\left( \left( \sum_{m=0}^{l} S_m(x + y)z^{-l-1+m} \right)w_1 + \left( \sum_{m=0}^{k} S_m(x + y)z^{-k-1+m} \right)w_2 \right)^n = \sum_{i=0}^{n(l+k+1)} f_i(w_1, w_2)z^{-n(l+k+2)+i}
\]

and let

\[
p_n = \sum_{i=0}^{n(l+k+1)} f_i(w_1, w_2)(n(l + k + 2) - i)x^{n(l+k+2)-i}.
\]

For \(f(w_1, w_2) = \exp(-\sum_{n \geq 1} \frac{p_n}{n}),\) we have

\[
\exp \left( \sum_{i \geq 1} \partial x_i - \partial y_i \right)z^{-1} f(w_1, w_2)
\]

\[
= \left( 1 - \left( \sum_{m=0}^{l} S_m(x + y)z^{-l-1+m} \right)w_1 - \left( \sum_{m=0}^{k} S_m(x + y)z^{-k-1+m} \right)w_2 \right) f(w_1, w_2).
\]

Set \(A_{s,t} = \sum_{m=0}^{t} S_m(x + y)S_{s+t-m}(-x - y). \) For \(k \geq l \geq 0,\) we have

\[
\exp(b \varphi^*_{-k}) \exp(a \varphi^*_{-l}) \cdot 1 = f(-ap, -bp)
\]

and

\[
\exp(d \varphi_{-j} \varphi^*_{-k}) \exp(c \varphi_{-i} \varphi^*_{-l}) \cdot 1
\]

\[
= \frac{1}{1 - cA_{i,j} - dA_{j,k} + cA_{i,l}dA_{j,k} - cA_{i,k}dA_{j,l}} \times f \left( \frac{-cS_{i-1}(-x - y) + cS_{i-1}(-x - y)dA_{j,k} - dS_{i-1}(-x - y)cA_{i,k}}{1 - cA_{i,l} - dA_{j,k} + cA_{i,l}dA_{j,k} - cA_{i,k}dA_{j,l}}, \right.
\]

\[
\left. \frac{-dS_{j-1}(-x - y) + dS_{j-1}(-x - y)cA_{i,l} - cS_{j-1}(-x - y)dA_{j,l}}{1 - cA_{i,l} - dA_{j,k} + cA_{i,l}dA_{j,k} - cA_{i,k}dA_{j,l}} \right).
\]

### 3.4 Borchardt’s identity

Following [9], we define the Fock space \(\mathcal{M}^*\) of the charged free bosons by

\[
\langle 0 | \varphi_i = \langle 0 | \varphi^*_i = 0, \quad i < 0
\]

with the inner product \(\langle 0 | 1 \rangle = 1.\) Then (see (3.11))

\[
\langle 0 | b(i - 1) = \langle 0 | c(i) = \langle 0 | \beta_i = 0, \quad i \leq 0.
\]
Recalling (3.4), we then have
\[ b(i + 1)|0⟩ = c(i)|0⟩ = β_j|0⟩ = 0, \quad i \geq 0. \]  
(3.35)

(3.34) and (3.35) will be proved in Appendix B.

Computing \(⟨0|φ(z_1)⋯φ(z_n)φ^*(w_1)⋯φ^*(w_n)|0⟩\) in two ways (see [27]), we get the following result.

**Proposition 3.13** (Borchardt’s identity [5,18]). For any positive integer \(n\),

\[
\det \left( \frac{1}{(z_i - w_j)^2} \right)_{1 \leq i, j \leq n} = \det \left( \frac{1}{(z_i - w_j)^2} \right)_{1 \leq i, j \leq n},
\]

where \(\text{perm}A\) is the permanent of square matrix \(A\) defined by \(\text{perm}A = \sum_{σ \in S_n} a_{1σ(1)}a_{2σ(2)}⋯a_{nσ(n)}\).

**Proof.** It follows from (2.8) and (3.3) that

\[
⟨0|c(z_1)⋯c(z_n)\partial b(w_1)⋯\partial b(w_n)|0⟩ = (-1)^{n(n-1)/2} \sum_σ (-1)^{sgn(σ)} \prod_{i=1}^n \frac{1}{(z_i - w_{σ(i)})^2}
\]

\[
= (-1)^{n(n-1)/2} \det \left( \frac{1}{(z_i - w_j)^2} \right)_{1 \leq i, j \leq n}, \tag{3.36}
\]

\[
⟨0|φ(z_1)⋯φ(z_n)φ^*(w_1)⋯φ^*(w_n)|0⟩ = \sum_{σ \in S_n} \prod_{1 \leq i, j \leq n} \frac{1}{z_i - w_{σ(i)}},
\]

\[
= \text{perm} \left( \frac{1}{z_i - w_j} \right)_{1 \leq i, j \leq n}. \tag{3.37}
\]

By the boson-boson correspondence (3.12) (see [13,29])

\[
φ(z_1)⋯φ(z_n)φ^*(w_1)⋯φ^*(w_n)|0⟩ = c(z_1)⋯c(z_n)\partial b(w_1)⋯\partial b(w_n)|0⟩
\]

\[
× \prod_{1 \leq i, j \leq n} \frac{(z_i - w_1)}{(z_i - z_j)(w_1 - w_j)} \exp \left( \sum_{m \geq 1} \frac{β_{-n}}{n} (w_1^m + ⋯ + w_n^m - z_1^m - ⋯ - z_n^m) \right),
\]

we have

\[
⟨0|φ(z_1)⋯φ(z_n)φ^*(w_1)⋯φ^*(w_n)|0⟩ = \prod_{1 \leq i, j, l \leq n} \frac{(z_i - w_1)}{(z_i - z_j)(w_1 - w_j)} \cdot 0|c(z_1)⋯c(z_n)\partial b(w_1)⋯\partial b(w_n)|0⟩. \tag{3.38}
\]

Then the relations (3.36)–(3.38) imply that

\[
\text{perm} \left( \frac{1}{z_i - w_j} \right)_{1 \leq i, j \leq n} = (-1)^{n(n-1)/2} \prod_{1 \leq i, j \leq n} \frac{(z_i - w_1)}{(z_i - z_j)(w_1 - w_j)} \det \left( \frac{1}{(z_i - w_j)^2} \right)_{1 \leq i, j \leq n}.
\]

Using Cauchy’s determinant identity

\[
\det \left( \frac{1}{(z_i - w_j)^2} \right)_{1 \leq i, j \leq n} = (-1)^{n(n-1)/2} \prod_{1 \leq i, j \leq n} \frac{(z_i - z_j)}{(w_i - w_j)} \det \left( \frac{1}{(z_i - w_j)^2} \right)_{1 \leq i, j \leq n},
\]

we have

\[
\text{perm} \left( \frac{1}{z_i - w_j} \right)_{1 \leq i, j \leq n} = \det \left( \frac{1}{z_i - w_j} \right)_{1 \leq i, j \leq n} \text{perm} \left( \frac{1}{z_i - w_j} \right)_{1 \leq i, j \leq n}.
\]

The proof is completed. □
4 \( q \)-dimension

In this section, we first compute \( q \)-series of some sets by \( q \)-Pochhammer symbols and then use them to derive \( q \)-dimensions of \( \mathcal{M}^t \), \( \mathcal{F}^t \) and \( \mathcal{F}^l \) by the combinatorial method. We also derive some \( q \)-identities by comparing with the results in [13, 29]. We adopt the usual convention to denote the \( q \)-Pochhammer symbols or \( q \)-integers by
\[
(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) = \prod_{j=0}^{n-1} (1 - aq^j),
\]
\[
(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j), \quad |q| < 1.
\]

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) is a sequence of ordered non-negative integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0 \) (or \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l \)). The weight of \( \lambda \) is defined by \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l \). Let \( \mathcal{P} \) be a set of partitions with some properties. We define
\[
G(\mathcal{P}) = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|}.
\]

**Lemma 4.1.** Let \( \mathcal{P}_k^s = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \mid k \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s \} \). Then
\[
G(\mathcal{P}_k^s) = \frac{q^{ks}}{(q; q)_s}. \tag{4.1}
\]

**Proof.** We use induction on \( s \). The case of \( s = 1 \) is clear as
\[
G(\mathcal{P}_k^1) = q^k + q^{k+1} + \cdots = \frac{q^k}{1 - q}.
\]
Assume \( s - 1 \) is established, and then
\[
G(\mathcal{P}_k^s) = q^k G(\mathcal{P}_{k-1}^{s-1}) + q^{k+1} G(\mathcal{P}_{k+1}^{s-1}) + q^{k+2} G(\mathcal{P}_{k+2}^{s-1}) + \cdots = \frac{q^{ks}}{(q; q)_s}.
\]
The proof is completed. \( \square \)

Similarly, we have the following lemma.

**Lemma 4.2.** For \( \mathcal{D}_k^s = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \mid k \leq \lambda_1 < \lambda_2 < \cdots < \lambda_s \} \),
\[
G(\mathcal{D}_k^s) = \frac{q^{(2k+s-1)s}}{(q; q)_s}. \tag{4.2}
\]

It is known in [29] that
\[
\mathcal{M} \cong \bigoplus_{l \in \mathbb{Z}} \mathcal{F}^l \otimes e^{-i\beta} C[y].
\]

In terms of the degree operator \( J_0^t \) on \( \mathcal{M} \) (3.9), we take
\[
deg_{\mathcal{F}^*}(-n) = \deg_b(-n) = \deg_c(-n) = \deg_{\alpha}(-n) = \deg_{\beta}(-n) = n,
\]
\[
\deg_c(\alpha) = \frac{l^2 - l}{2}, \quad \deg_c(\beta) = \frac{-l^2 - l}{2}, \quad \deg(0) = 0 \tag{4.3}
\]
in view of (2.12), (2.14), (3.8) and (3.14). We pass the gradation \( \deg \) to any subspace \( W \subseteq \mathcal{M} \), and denote
\[
W_m = \{ w \mid w \in W, \deg(w) = m \}.
\]

Then we define the \( q \)-dimension of \( W \) as
\[
\dim_q(W) = \text{tr}_W q^{J_0^t} = \sum_{m \geq 0} \dim W_m q^m. \tag{4.4}
\]

The following result gives the \( q \)-dimensions for the spaces \( \mathcal{M}^t \), \( \mathcal{F}^t \) and \( \mathcal{F}^l \).
Proposition 4.3. One has

$$\dim_q(M^l) = \sum_{m \geq 0} \frac{q^{m+l} m!}{(q; q)_m (q; q)_{m+l}},$$  \hspace{1cm} (4.5)$$

$$\dim_q(F^l) = \sum_{m \geq 0} \frac{q^{m^2+(l+1)m+\frac{(l+1)(l+2)}{2}}}{(q; q)_m (q; q)_{m+l}},$$  \hspace{1cm} (4.6)$$

$$\dim_q(F^l) = \sum_{m \geq 0} \frac{q^{m^2+m|m|+(l+1)}}{(q; q)_m (q; q)_{m+l}},$$  \hspace{1cm} (4.7)$$

Proof. Let

$$\Phi^m = \text{span}\{\varphi_{-i_1} \cdots \varphi_{-i_m} \mid i_1 \geq \cdots \geq i_m \geq 1\}, \quad \Phi^{*m} = \text{span}\{\varphi_{-j_1}^* \cdots \varphi_{-j_m}^* \mid j_1 \geq \cdots \geq j_m \geq 0\}.$$ 

It follows from (4.3) that

$$\dim_q(\Phi^m) = G(D^{m}) = \frac{q^m}{(q; q)_m}$$

and

$$\dim_q(\Phi^{*m}) = G(D^{m}) = \frac{1}{(q; q)_m},$$

where the special cases are $\dim_q(\Phi^0) = \dim_q(\Phi^{*0}) = q^0 = 1$. For $l \geq 0$ and by using (4.3), we then have

$$\dim_q(M^l) = \sum_{m \geq 0} \dim_q(\Phi^{*m}) \dim_q(\Phi^{m+l}) = \sum_{m \geq 0} \frac{q^{m+l}}{(q; q)_m (q; q)_{m+l}},$$

and for $l < 0$,

$$\dim_q(M^l) = \sum_{m \geq 0} \dim_q(\Phi^{m-l}) \dim_q(\Phi^{m}) = \sum_{m \geq 0} \frac{q^m}{(q; q)_m (q; q)_{m-l}}.$$ 

Similarly, set

$$B^m_k = \text{span}\{b(-i_1) \cdots b(-i_m) \mid i_1 > \cdots > i_m \geq k\}$$

and

$$C^m = \text{span}\{c(-j_1) \cdots c(-j_m) \mid j_1 > \cdots > j_m \geq 1\}.$$ 

So for $l \geq 0$ and counting the degree (4.3), we have

$$\dim_q(F^l) = \sum_{m \geq 0} \dim_q(B^m_1) \dim_q(C^{m+l}) = \sum_{m \geq 0} G(D^{m}) G(D^{m+l}) = \sum_{m \geq 0} \frac{q^{m^2+(l+1)m+\frac{(l+1)(l+2)}{2}}}{(q; q)_m (q; q)_{m+l}},$$

$$\dim_q(F^l) = \sum_{m \geq 0} \dim_q(B^m_0) \dim_q(C^{m+l}) = \sum_{m \geq 0} G(D^{m}) G(D^{m+l}) = \sum_{m \geq 0} \frac{q^{m^2+ml+\frac{(l+1)(l+2)}{2}}}{(q; q)_m (q; q)_{m+l}},$$

Similarly for $l < 0$, we have

$$\dim_q(F^l) = \sum_{m \geq 0} \dim_q(B^{m-l}_1) \dim_q(C^{m}) = \sum_{m \geq 0} G(D^{m-l}) G(D^{m}) = \sum_{m \geq 0} \frac{q^{m^2+(-l+1)m+\frac{(-l)(-l+2)}{2}}}{(q; q)_m (q; q)_{m-l}},$$

$$\dim_q(F^l) = \sum_{m \geq 0} \dim_q(B^{m-l}_0) \dim_q(C^{m}) = \sum_{m \geq 0} G(D^{m-l}) G(D^{m}) = \sum_{m \geq 0} \frac{q^{m^2-ml+\frac{(l+1)(l+2)}{2}}}{(q; q)_m (q; q)_{m-l}}.$$ 

The proof is completed. □
Proposition 4.4 (See [13, 29]). One also has

\[ \dim_q (\mathcal{M}^l) = \frac{1}{(q, q)^{2\infty}} \sum_{s \geq 0} (-1)^s q^{\frac{(s+1)}{2} + (s + \delta_{l>0})|l|}, \]  

(4.8)

\[ \dim_q (\mathcal{F}^l) = \frac{1}{(q, q)^{\infty}} \sum_{s \geq |l|} (-1)^{s+l} q^{\frac{(s+1)}{2}}. \]  

(4.9)

The following result is a consequence of Propositions 4.3 and 4.4.

Theorem 4.5. For \( l \geq 0 \), one has

\[ \sum_{m \geq 0} \frac{q^m}{(q, q)_m (q, q)_{m+l}} = \frac{1}{(q, q)^{2\infty}} \sum_{s \geq 0} (-1)^s q^{\frac{(s+1)}{2} + sl}, \]  

(4.10)

\[ \sum_{m \geq 0} \frac{q^{m^2 + (l+1)m}}{(q, q)_m (q, q)_{m+l}} = \frac{1}{(q, q)^{\infty}} \sum_{s \geq 0} (-1)^s q^{\frac{(s+1)}{2} + sl}. \]  

(4.11)

Therefore we have the \( q \)-series identity

\[ \sum_{m \geq 0} \frac{q^m}{(q, q)_m (q, q)_{m+l}} = \frac{1}{(q, q)^{\infty}} \sum_{m \geq 0} \frac{q^{m^2 + (l+1)m}}{(q, q)_m (q, q)_{m+l}}. \]

Recalling that \( \mathcal{F}^l \cong e^{-l\alpha} \mathbb{C}[x] \) (see [21]) and

\[ \deg(e^{-l\alpha} x_k) = k + \frac{l(l+1)}{2}, \]

so we also have

\[ \dim_q (\mathcal{F}^l) = \frac{q^{l(l+1)}}{(q, q)^{\infty}}. \]  

(4.12)

The following result follows from (4.7) and (4.12).

Theorem 4.6. One has

\[ \sum_{m \geq 0} \frac{q^{m^2 + m|l|}}{(q, q)_m (q, q)_{m+l}} = \frac{1}{(q, q)^{\infty}}. \]  

(4.13)

In particular, the famous Euler identity is the case of \( l = 0 \).

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References

1. Adamović D. Classification of irreducible modules of certain subalgebras of free boson vertex algebra. J Algebra, 2003, 270: 115–132

2. Anguelova I I. The two bosonizations of the CKP hierarchy: Overview and character identities. Contemp Math, 2018, 713: 1–34

3. Anguelova I I, Cox B, Jurisich E. \( N \)-point locality for vertex operators: Normal ordered products, operator product expansions, twisted vertex algebras. J Pure Appl Algebra, 2014, 218: 2165–2203

4. Bakalov B, Fleischer D. Bosonizations of \( sl_2 \) and integrable hierarchies. SIGMA Symmetry Integrability Geom Methods Appl, 2015, 11: 5–23

5. Borchart C W. Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden function. J Reine Angew Math, 1857, 53: 193–198

6. Chen M-R, Wang S-K, Wang X-L, et al. On \( W_{1+\infty} \) 3-algebra and integrable system. Nucl Phys B, 2015, 891: 655–675
Appendix A

Proof of Equation (3.31) in Theorem 3.10. It is enough to show that

$$\varphi_n^{-j} \varphi_{-1}^{n-1} = n! \sum_{i=0}^{n} C_n^i (-1)^i S_{j-1}^i (-x - y)(S_{j+1}^i (-x - y) + S_1 (x + y)S_j (-x - y))^{n-i} S_1^i, \quad (A.1)$$

where $S_n^*$ is defined by

$$\exp \left( - \sum_{n \geq 1} \sum_{0 \leq i \leq n} C_n^i S_1^{n-i} (x + y)(n + i)x_{n+i} z^n \right) = \sum_{n \geq 0} S_n^* z^n. \quad (A.2)$$

As in the proof of Lemma 3.7, we claim that

$$\varphi_{-1}^{n} = (-1)^n n! p^n S_n^*. \quad (A.3)$$
Thus (A.3) is proved.

\[ \varphi^* (z) \varphi^*_0 \cdot 1 = p \exp \left( \sum_{i \geq 1} (\partial x_i - \partial y_i) z^i \right) (A + B + C) R, \]

where \( A, B \) and \( C \) are defined in (3.27b).

Applying \( \partial_z \) to both sides of (A.2), we have

\[ \sum_{k \geq 1} \sum_{\frac{k}{2} \leq n \leq k} C_{n}^{k-n} S_{1}^{2n-k} (x+y) k x_k S_{s-n}^* = -s S_s^*. \]

From

\[ \exp \left( - \sum_{i \geq 1} \frac{(\partial x_i - \partial y_i) z^i}{i} \right) \exp \left( - \sum_{n \geq 1} \sum_{0 \leq i \leq n} \frac{C_n^{n-i} (x+y)(n+i)x_{n+i} w^n}{n} \right) \]

\[ = \frac{1}{1 - (z^{-2} + S_1(x+y)z^{-1}) w} \exp \left( - \sum_{n \geq 1} \sum_{0 \leq i \leq n} \frac{C_n^{n-i} (x+y)(n+i)x_{n+i} w^n}{n} \right), \]

we get

\[ \exp \left( - \sum_{i \geq 1} \frac{(\partial x_i - \partial y_i) z^i}{i} \right) S_n^* = \sum_{i=0}^n (z^{-2} + S_1(x+y)z^{-1}) i S_{n-i}^*. \]

Therefore, the coefficient of \( z^0 \) in \( (A + B + C) R \) is

\[ a_0 := (-1)^n n! p^n \sum_{k>0}^n C_{i}^{k-1-i} S_1^{2i+1-k} (x+y) k x_k S_{n-i}^*, \quad (A.4) \]

and the \( z \)-coefficient is

\[ a_1 := (-1)^n n! p^n \sum_{k>0}^n C_{i}^{k-2-i} S_1^{2i+2-k} (x+y) k x_k S_{n-i}^*, \quad (A.5) \]

Since

\[ \beta_i (\varphi^*_0 \cdot 1) = -j \varphi^*_0 \cdot 1 = 0, \quad i \geq 2, \quad (A.6) \]

\( S_k(x+y) \) does not appear in \( \varphi^*_{n+1} \cdot 1 \) for \( k \geq 2 \). Thus

\[ \varphi^*_{n+1} \cdot 1 = p(-1)^n n! p^n (a_0 S_1(x+y) + a_1 S_0(x+y)) \]

\[ = (-1)^n n! p^{n+1} \sum_{k>0}^n \sum_{i=0}^n (C_{i}^{k-1-i} + C_{i}^{k-2-i}) S_1^{2i+2-k} (x+y) k x_k S_{n-i}^* \]

\[ = (-1)^n n! p^{n+1} \sum_{k>0}^n \sum_{i=1}^n C_{i}^{k-1-i} S_1^{2i+2-k} (x+y) k x_k S_{n-i}^* \]

\[ = (-1)^n n! p^{n+1} \sum_{k>0}^n \sum_{\frac{k}{2} \leq t \leq k} C_{t}^{k-t} S_1^{2t-k} (x+y) k x_k S_{n+1-t}^* \]

\[ = (-1)^n (n+1)! p^{n+1} S_{n+1}^*. \]

Thus (A.3) is proved.

By the formula

\[ \exp \left( \sum (\partial x_i - \partial y_i) z^i \right) \exp \left( - \sum_{n \geq 1} \sum_{0 \leq i \leq n} \frac{C_n^{n-i} (x+y)(n+i)x_{n+i} w^n}{n} \right) \]
\begin{align*}
&= (1 - (z^{-2} + S_1(x + y)z^{-1})w) \exp \left( - \sum_{n \geq 1} \sum_{0 \leq i \leq n} C^n_i S_1^{n-1}(x + y)(n + i)x_{n+1} w^n \right),
\end{align*}
we then have
\begin{align*}
\exp \left( \sum (\partial x_i - \partial y_i) \frac{z^{-i}}{i} \right) S^*_n &= S^*_n - (z^{-2} + S_1(x + y)z^{-1}) S^*_{n-1}.
\end{align*}
Thus
\begin{align*}
\varphi_{-j} S^*_n &= p^{-1}(S_{j-1}(-x - y)S^*_n - (S_{j+1}(-x - y) + S_1(x + y)S_{j(-x - y)})S^*_{n-1}),(A.7)
\end{align*}
Repeatedly using (A.7) we then have
\begin{align*}
\varphi_{-j}^n S^*_n &= p^{-n} \sum_{i=0}^n C^n_i (-1)^{n-i} S^*_n (-x - y)(S_{j+1}(-x - y) + S_1(x + y)S_{j(-x - y)})^n S^*_n.
\end{align*}
Thus
\begin{align*}
\varphi_{-j}^n \varphi_{-1}^n &\cdot 1 = (1)^n! \sum_{i=0}^n C^n_i (-1)^{n-i} S^*_n (-x - y)(S_{j+1}(-x - y) + S_1(x + y)S_{j(-x - y)})^n S^*_n.
\end{align*}
The proof is completed.

**Appendix B**

Here, we prove (3.34). The proof of (3.35) is similar.

In fact, it follows from (3.6) and (3.13) that
\begin{align*}
\beta_k &= - \sum_{j \in \mathbb{Z}} : \varphi_{-j} \varphi_{-j+k} : = - \sum_{j \geq 0} \varphi_{-j+k}^\ast \varphi_{-j} - \sum_{j < 0} \varphi_{j} \varphi_{-j+k}^\ast.
\end{align*}
It is seen that \( \langle 0 | \beta_k = 0 \) for \( k \leq 0 \) by using (3.33).

Recall (3.12), and we have
\begin{align*}
\langle 0 | \varphi(z) = \langle 0 | c(z) Y(e^{-\beta}, z).
\end{align*}
Then
\begin{align*}
\langle 0 | \sum_{i \in \mathbb{Z}} \varphi_i z^{-i-1} &= \langle 0 | \sum_{n \in \mathbb{Z}} c(n) z^{-n-1} e^{-\beta} z^{-\beta_0} \exp \left( - \sum_{j < 0} \frac{z^{-j}}{-j} \beta_j \right) \exp \left( - \sum_{j > 0} \frac{z^{-j}}{-j} \beta_j \right) \\
&= z \langle 0 | z^{-\beta_0} \exp \left( - \sum_{j < 0} \frac{z^{-j}}{-j} \beta_j \right) \exp \left( - \sum_{j > 0} \frac{z^{-j}}{-j} \beta_j \right) \sum_{j \in \mathbb{Z}} c(n) z^{-n-1} e^{-\beta} \\
&= \langle 0 | \exp \left( - \sum_{j > 0} \frac{z^{-j}}{-j} \beta_j \right) \sum_{j \in \mathbb{Z}} c(n) z^{-n} e^{-\beta} \\
&= \langle 0 | \sum_{j \in \mathbb{Z}} c(n) z^{-n} e^{-\beta} \exp \left( - \sum_{j > 0} \frac{z^{-j}}{-j} \beta_j \right),
\end{align*}
where we have used
\begin{align*}
e^{-\beta} z^{-\beta_0} = z z^{-\beta_0} e^{-\beta}
\end{align*}
and the properties of lattice vertex algebras. Thus
\begin{align*}
\langle 0 | \sum_{i \geq 0} \varphi_i z^{-i-1} &= \langle 0 | \sum_{j \in \mathbb{Z}} c(n) z^{-n} e^{-\beta} \exp \left( - \sum_{j > 0} \frac{z^{-j}}{-j} \beta_j \right).
\end{align*}(B.1)
Comparing the coefficients of \( z^n \) for \( n > 0 \), the right-hand side must be zero. Therefore, we have
\begin{align*}
\langle 0 | c(n) = 0, \quad n \leq 0.
\end{align*}(B.2)
Similarly, we have

$$\langle 0 | \sum_{i \geq 1} \varphi_i^* z^{-i} = \langle 0 | \sum_{n \in \mathbb{Z}} (-n)b(n)z^{-n}e^{\beta} \exp \left( \sum_{j > 0} \frac{z^{-j}}{-j} \beta_j \right). \quad (B.3)$$

So

$$\langle 0 | b(n) = 0, \quad n < 0. \quad (B.4)$$

Therefore, Equation (3.34) holds.

**Remark B.1.** The condition \( \langle 0 | c(0) = c(0) | 0 \rangle = 0 \) does not affect the proof of Borchardt’s identity since \( \varphi_i, \varphi_i^*, i \in \mathbb{Z} \) do not depend on \( b(0) \).