CARLESON MEASURES AND LOGVINENKO-SEREDA SETS ON COMPACT MANIFOLDS

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Abstract. Given a compact Riemannian manifold $M$ of dimension $m \geq 2$, we study the space of functions of $L^2(M)$ generated by eigenfunctions of eigenvalues less than $L \geq 1$ associated to the Laplace-Beltrami operator on $M$. On these spaces we give a characterization of the Carleson measures and the Logvinenko-Sereda sets.

Introduction and statement of the results

Let $(M, g)$ be a smooth, connected, compact Riemannian manifold without boundary of dimension $m \geq 2$. Let $dV$ be the volume element of $M$ associated to the metric $g_{ij}$. Let $\Delta_M$ be the Laplacian on $M$ associated to the metric $g_{ij}$. It is given in local coordinates by

$$\Delta_M f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left( \sqrt{|g|} g^{ij} \frac{\partial f}{\partial x_j} \right),$$

where $|g| = \det(g_{ij})$ and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$. Since $M$ is compact, $g_{ij}$ and all its derivatives are bounded and we assume that the metric $g$ is non-singular at each point of $M$.

Since $M$ is compact, the spectrum of the Laplacian is discrete and there is a sequence of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and an orthonormal basis $\phi_i$ of smooth real eigenfunctions of the Laplacian i.e. $\Delta_M \phi_i = -\lambda_i \phi_i$. So $L^2(M)$ decomposes into an orthogonal direct sum of eigenfunctions of the Laplacian.

We consider the following spaces of $L^2(M)$.

$$E_L = \left\{ f \in L^2(M) : f = \sum_{i=1}^{k_L} \beta_i \phi_i, \ \Delta_M \phi_i = -\lambda_i \phi_i, \ \lambda_{k_L} \leq L \right\},$$

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where $L \geq 1$ and $k_L = \dim E_L$. $E_L$ is the space of $L^2(M)$ generated by eigenfunctions of eigenvalues $\lambda \leq L$. Thus in $E_L$ we consider functions in $L^2(M)$ with a restriction on the support of its Fourier transform. It is, in a sense, a Paley-Wiener type space on $M$ with bandwidth $L$.

The motivation of this paper is to show that the spaces $E_L$ behave like the space defined in $S^d$ ($d > 1$) of spherical harmonics of degree less than $\sqrt{L}$. In fact, the space $E_L$ is a generalization of the spherical harmonics and the role of them are played by the eigenfunctions. The cases $M = S^1$ and $M = S^d$ ($d > 1$) have been studied in [14] and [11], respectively.

This similarity between eigenfunctions of the Laplacian and polynomials is not new, for instance, Donnelly and Fefferman in [1, Theorem 1], showed that on a compact manifold, an eigenfunction of eigenvalue $\lambda$ behaves essentially like a polynomial of degree $\sqrt{\lambda}$. In this direction they proved the following local $L^\infty$ Bernstein inequality.

**Theorem (Donnelly-Fefferman).** Let $M$ be as above. If $u$ is an eigenfunction of the Laplacian $\Delta_M u = -\lambda u$, then there exists $r_0 = r_0(M)$ such that for all $r < r_0$ we have

$$\max_{B(x,r)} |\nabla u| \leq C\lambda^{(m+2)/4} \frac{\max_{B(x,r)} |u|}{r}.$$ 

The proof of the above estimate is rather delicate. Donnelly and Fefferman conjectured that it is possible to replace $\lambda^{(m+2)/4}$ by $\sqrt{\lambda}$ in the inequality. If the conjecture holds, we have in particular, a global Bernstein type inequality:

$$\|\nabla u\|_\infty \lesssim \sqrt{\lambda} \|u\|_\infty.$$

In fact, this weaker estimate holds and a proof will be given later. This fact suggests that the right metric to study the space $E_L$ should be rescaled by a factor $1/\sqrt{L}$ because in balls of radius $1/\sqrt{\lambda}$, a bounded eigenfunction of eigenvalue $\lambda$ oscillates very little.

In the present work we will study for which measures $\mu = \{\mu_L\}_L$ one has

$$\int_M |f|^2 d\mu_L \approx \int_M |f|^2 dV, \quad \forall f \in E_L$$

with constants independent of $f$ and $L$.

We will also study the inequality

$$\int_M |f|^2 d\mu_L \lesssim \int_M |f|^2 dV$$

that defines the Carleson measures and we will present a geometric characterization of them. Inequality (2) will be studied only for the special case $d\mu_L = \chi_{A_L} dV$, where $\mathcal{A} = \{A_L\}_L$ is a sequence of sets in
the manifold. In such case, when \((2)\) holds, we say that \(A\) is a sequence of Logvinenko-Sereda sets. Our two main results are the following:

**Theorem 1.** Let \(\mu = \{\mu_L\}_L\) be a sequence of measures on \(M\). Then \(\mu\) is \(L^2\)-Carleson for \(M\) if and only if there exists a \(C > 0\) such that for all \(L\)
\[
\sup_{\xi \in M} \frac{\mu_L(B(\xi, 1/\sqrt{L}))}{\text{vol}(B(\xi, 1/\sqrt{L}))} \leq C.
\]

**Theorem 2.** The sequence of sets \(A = \{A_L\}_L\) is Logvinenko-Sereda if and only if there is an \(r > 0\) such that
\[
\inf_L \inf_{z \in M} \frac{\text{vol}(A_L \cap B(z, r/\sqrt{L}))}{\text{vol}(B(z, r/\sqrt{L}))} > 0.
\]

In what follows, when we write that \(A \precsim B\), \(A \succeq B\) or \(A \asymp B\) we mean that there are constants depending only on the manifold such that \(A \leq CB\), \(A \geq CB\) or \(C_1B \leq A \leq C_2B\), respectively. Also, the value of the constants appearing during a proof may change but they will be still denoted with the same letter. We will denote by \(B(\xi, r)\) a geodesic ball in \(M\) of center \(\xi\) and radius \(r\) and \(B(z, r)\) will denote an Euclidean ball in \(\mathbb{R}^m\) of center \(z\) and radius \(r\).

The structure of the paper is the following: in the first section, we will explain the asymptotics of the reproducing kernel of the space \(E_L\). In the second section, we shall discuss one of the tools used: the harmonic extension of functions in the space \(E_L\). Following this, we will study the Carleson measures associated to \(M\) and we will prove Theorem 1. In the last section, Theorem 2 will be proved.

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1. **The reproducing kernel of \(E_L\)**

Let
\[
K_L(z, w) := \sum_{i=1}^{k_L} \phi_i(z)\phi_i(w) = \sum_{\lambda_i \leq L} \phi_i(z)\phi_i(w).
\]
This function is the reproducing kernel of the space \(E_L\), i.e.
\[
\forall f \in E_L \quad f(z) = \langle f, K_L(z, \cdot) \rangle.
\]
Note that \(\dim(E_L) = k_L = \# \{\lambda_i \leq L\}\). The function \(K_L\) is also called the spectral function associated to the Laplacian. Hörmander in [3], proved the following estimates.

1. \(K_L(z, w) = O(L^{(m-1)/2}), \quad z \neq w.\)
(2) \( K_L(z, z) = \frac{\sigma_m}{(2\pi)^m} L^{m/2} + O(L^{(m-1)/2}) \) (uniformly in \( z \in M \)), where \( \sigma_m = 2\pi^{m/2}/(m\Gamma(m/2)) \).

(3) \( k_L = \frac{\text{vol}(M)\sigma_m L^{m/2}}{(2\pi)^m} + O(L^{m-1}) \).

In fact, in \[5\], there are estimates for the spectral function associated to any elliptic operator of order \( n \geq 1 \) with constants depending only on the manifold.

So, for \( L \) big enough we have \( k_L \approx L^{m/2} \) and

\[
\|K_L(z, \cdot)\|_2^2 = K_L(z, z) \approx L^{m/2} \approx k_L
\]

with constants independent of \( L \) and \( z \).

2. Harmonic extension

In what follows, given \( f \in E_L \), we will note by \( h \) the harmonic extension of \( f \) in \( N = M \times \mathbb{R} \). The metric that we consider in \( N \) is the product metric, i.e. if we denote it by \( \tilde{g}_{ij} \) for \( i = 1, \ldots, m+1 \) then

\[
(\tilde{g}_{ij})_{i,j=1,\ldots,m+1} = \begin{pmatrix}
(g_{ij})_{i,j=1} & 0 \\
0 & 1
\end{pmatrix}.
\]

Using this matrix, we can calculate the gradient and the Laplacian for \( N \). If \( h(z, t) \) is a function defined on \( N \) then

\[
|\nabla_N h(z, t)|^2 = |\nabla_M h(z, t)|^2 + \left(\frac{\partial h}{\partial t}(z, t)\right)^2
\]

and

\[
\Delta_N h(z, t) = \Delta_M h(z, t) + \frac{\partial^2 h}{\partial t^2}(z, t).
\]

Note that \( |\nabla_M h(z, t)| \leq |\nabla_N h(z, t)| \).

Let \( f \in E_L \), we know that

\[
f = \sum_{i=1}^{k_L} \beta_i \phi_i, \quad \Delta_M \phi_i = -\lambda_i \phi_i, \quad 0 \leq \lambda_i \leq L.
\]

Define for \( (z, t) \in N \)

\[
h(z, t) = \sum_{i=1}^{k_L} \beta_i \phi_i(z) e^{\sqrt{\lambda_i} t}.
\]

Observe that \( h(z, 0) = f(z) \). Moreover \( |\nabla_M f(z)|^2 \leq |\nabla_N h(z, 0)|^2 \).
The function \( h \) is harmonic in \( N \) because
\[
\Delta_N h(z, t) = \sum_{i=1}^{k_L} \left[ \beta_i e^{\sqrt{\lambda_i}t} \Delta_M \phi_i(z) + \beta_i \phi_i(z) \Delta_M e^{\sqrt{\lambda_i}t} \right] = 0.
\]
For the harmonic extension, we don’t have the mean-value property, because it is not true for all manifolds (only for the harmonic manifolds, see [17] for a complete characterization of them). What is always true is a “submean value property” (with a uniform constant) for positive subharmonic functions, see for example [15, Chapter II, Section 6]).

Observe that since \( h \) is harmonic on \( N \), \(|h|^2\) is a positive subharmonic function on \( N \). In fact, \(|h|^p\) is subharmonic for all \( p \geq 1 \) (for a proof see [3, Proposition 1]). Therefore, we know that for all \( r < R_0(M) \)
\[
|h(z, t)|^2 \lesssim \int_{B(z, r/\sqrt{L}) \times I_r(t)} |h(w, s)|^2 dV(w) ds,
\]
where \( R_0(M) > 0 \) is the injectivity radius of \( M \) and \( I_r(t) = (t - r/\sqrt{L}, t + r/\sqrt{L}) \). In particular,
\[
|f(z)|^2 \leq C_r L^{(m+1)/2} \int_{B(z, r/\sqrt{L}) \times I_r} |h(w, s)|^2 dV(w) ds,
\]
where \( I_r = I_r(0) \).

The following result relates the \( L^2 \)-norm of \( f \) and \( h \).

**Proposition 1.** Let \( r > 0 \) fixed. If \( f \in E_L \) then
\[
2re^{-2r} \|f\|^2 \leq \sqrt{L} \|h\|^2_{L^2(M \times I_r)} \leq 2re^{2r} \|f\|^2.
\]
Therefore, for \( r < R_0(M) \)
\[
\frac{\sqrt{T}}{2r} \|h\|^2_{L^2(M \times I_r)} \approx \|f\|^2
\]
with constants depending only on the manifold \( M \).

**Proof.** Using the orthogonality of \( \{\phi_i\}_i \) we have
\[
\|h\|^2_{L^2(M \times I_r)} = \int_{I_r} \int_M \left| \sum_{i=1}^{k_L} \beta_i \phi_i(z) e^{\sqrt{\lambda_i}t} \right|^2 dV(z) dt
\]
\[
= \int_{I_r} \sum_{i=1}^{k_L} \int_M |\beta_i|^2 |\phi_i(z)|^2 dV(z) e^{2\sqrt{\lambda_i}t} dt \leq \int_{I_r} e^{2\sqrt{\lambda_i}t} dt \|f\|^2.
\]

Similarly, one can prove the left hand side inequality of (4).

We recall now a result proved by Schoen and Yau that estimates the gradient of harmonic functions.
Theorem (Schoen-Yau). Let $N$ be a complete Riemannian manifold with Ricci curvature bounded below by $-(n-1)K$ ($n$ is the dimension of $N$ and $K$ a positive constant). Suppose $B_a$ is a geodesic ball in $N$ with radius $a$ and $h$ is an harmonic function on $B_a$. Then

\[ \sup_{B_{a/2}} |\nabla h| \leq C_n \left( 1 + \frac{a\sqrt{K}}{a} \right) \sup_{B_a} |h|, \]

where $C_n$ is a constant depending only on the dimension of $N$.

For a proof see [15, Corollary 3.2., page 21].

Remark 1. We will use Schoen and Yau’s estimate in the following context. Take $N = M \times \mathbb{R}$. Observe that $\text{Ricc}(N) = \text{Ricc}(M)$ which is bounded below because $M$ is compact. Note that $N$ is complete because it is a product of two complete manifolds. We will take $a = r/\sqrt{L}$ ($r < R_0(M)$) and $B_a = B(z, r/\sqrt{L}) \times I_r$ (note that this is not the ball of center $(z, 0) \in N$ and radius $r/\sqrt{L}$, but it contains and it is contained in such ball of comparable radius).

Using Schoen and Yau’s theorem, we deduce the global Bernstein inequality for a single eigenfunction.

Corollary 1 (Bernstein inequality). If $u$ is an eigenfunction of eigenvalue $\lambda$, then

\[ \| \nabla u \|_\infty \lesssim \sqrt{\lambda} \| u \|_\infty. \]

Proof. The harmonic extension of $u$ is $h(z, t) = u(z)e^{\sqrt{\lambda}t}$. Applying inequality (5) to $h$ (taking $a = R_0(M)/(2\sqrt{\lambda})$),

\[ |\nabla u(z)| \lesssim \sqrt{\lambda} \| h \|_{L^\infty(M \times I_R/2)} \approx \sqrt{\lambda} \| u \|_\infty. \]

We conjecture that in inequality (6), one can replace $u$ by the functions $f \in E_L$, i.e.

\[ \| \nabla f \|_\infty \lesssim \sqrt{L} \| f \|_\infty. \]

For instance, as a direct consequence of Green’s formula, we have the $L^2$—Bernstein inequality for the space $E_L$:

\[ \| \nabla f \|_2 \lesssim \sqrt{L} \| f \|_2 \forall f \in E_L. \]

For our purpose, it is sufficient a weaker Bernstein type inequality that compares the $L^\infty$ norm of the gradient with the $L^2$ norm of the function.

Proposition 2. Let $f \in E_L$. Then there exists a universal constant $C$ such that

\[ \| \nabla f \|_\infty \leq C\sqrt{k_L} \sqrt{L} \| f \|_2. \]

For the proof, we need the following lemma.
Lemma 1. For all \( f \in E_L \) and \( 0 < r < R_0(M)/2 \),
\[
|\nabla f(z)|^2 \leq C_r L^{(m+2+1)/2} \int_{B(z,r/\sqrt{L}) \times I_r} |h(w,s)|^2 dV(w)ds.
\]

Proof. Using inequality (5) and the submean-value inequality for \(|h|^2\), we have
\[
|\nabla f(z)|^2 \leq |\nabla h(z,0)|^2 \lesssim \frac{L}{r^2} \sup_{B(z,r/\sqrt{L}) \times I_r} |h(w,t)|^2
\]
\[
\lesssim \frac{L^{(m+1+2)/2}}{\tilde{r}^{m+2+1}} \int_{B(z,\tilde{r}/\sqrt{L}) \times I_{\tilde{r}}} |h(\xi,s)|^2 dV(\xi)ds,
\]
where \( \tilde{r} = 2r \).

\( \Box \)

Proposition 2. Using Lemma 1, given \( 0 < r < R_0(M)/2 \) there exists a constant \( C_r \) such that
\[
|\nabla f(z)|^2 \leq C_r k_L \sqrt{L} \int_M |h(w,s)|^2 dV(w)ds
\]
\[
\approx C_r k_L \|f\|_2^2.
\]

Taking \( r = R_0(M)/4 \), we get that \( |\nabla f(z)|^2 \leq C k_L \|f\|_2^2 \) for all \( z \in M \).

\( \Box \)

3. Characterization of Carleson measures

Definition 1. Let \( \mu = \{\mu_L\}_{L \geq 0} \) be a sequence of measures on \( M \).

We say that \( \mu \) is an \( L^2 \)-Carleson sequence for \( M \) if there exists a positive constant \( C \) such that for all \( L \) and \( f_L \in E_L \)
\[
\int_M |f_L|^2 d\mu_L \leq C \int_M |f_L|^2 dV.
\]

Theorem 3. Let \( \mu \) be a sequence of measures on \( M \). Then, \( \mu \) is \( L^2 \)-Carleson for \( M \) if and only if there exists a \( C > 0 \) such that for all \( L \)
\[
\sup_{\xi \in M} \mu_L(B(\xi,1/\sqrt{L})) \leq \frac{C}{k_L}.
\]

Remark 2. Condition (7) can be viewed as
\[
\sup_{\xi \in M} \frac{\mu_L(B(\xi,1/\sqrt{L}))}{\text{vol}(B(\xi,1/\sqrt{L}))} \lesssim 1.
\]

First, we prove the following result.

Lemma 2. Let \( \mu \) be a sequence of measures on \( M \). Then, the following conditions are equivalent.

1. There exists a constant \( C = C(M) > 0 \) such that for each \( L \)
\[
\sup_{\xi \in M} \mu_L(B(\xi,1/\sqrt{L})) \leq \frac{C}{k_L}.
\]
There exist \( c = c(M) > 0 \) \((c < 1 \text{ small})\) and \( C = C(M) > 0 \) such that for all \( L \)
\[
\sup_{\xi \in M} \mu_L(B(\xi, c/\sqrt{L})) \leq \frac{C}{k_L}.
\]

**Proof.** Obviously, the first condition implies the second one since
\[
B(\xi, c/\sqrt{L}) \subset B(\xi, 1/\sqrt{L}).
\]
Let’s see the converse. The manifold \( M \) is covered by the union of balls of center \( \xi \in M \) and radius \( c/\sqrt{L} \). Taking into account the 5-r covering lemma (see [13, Chapter 2, page 23] for more details), we get a family of disjoint balls, denoted by \( B_i = B(\xi_i, c/\sqrt{L}) \), such that \( M \) is covered by the union of \( 5B_i \). This union may be finite or countable. Let \( \xi \in M \) and consider \( B := B(\xi, 1/\sqrt{L}) \). Let \( n \) be the number of balls \( B_i \) such that \( B \cap 5B_i \neq \emptyset \). Since \( B \) is compact, we have a finite number of these balls, so that
\[
B \subset \bigcup_{i=1}^{n} 5B_i.
\]
We claim that \( n \) is independent of \( L \). Hence we will get our statement because
\[
\mu_L(B) \leq \sum_{i=1}^{n} \mu_L(B(\xi_i, 5c/\sqrt{L})) \leq \frac{n}{k_L}.
\]
We only need to check that \( n \) is independent of \( L \). Using the triangle inequality, for all \( i = 1, \ldots, n \)
\[
B(\xi_i, c/\sqrt{L}) \subset B(\xi, 10/\sqrt{L}).
\]
Therefore,
\[
\bigcup_{i=1}^{n} B(\xi_i, c/\sqrt{L}) \subset B(\xi, 10/\sqrt{L}),
\]
where the union is a disjoint union of balls. Now,
\[
\frac{10^n}{L^{m/2}} \approx \text{vol}(B(\xi, 10/\sqrt{L})) \geq \sum_{i=1}^{n} \text{vol}(B_i) \approx n \frac{c^m}{L^{m/2}}.
\]
Hence, \( n \lesssim (10/c)^m \) and we can choose it independently of \( L \). \( \square \)

Now we can prove the characterization of the Carleson measures.

**Theorem 3.** Assume condition (7). We need to prove the existence of a constant \( C > 0 \) (independent of \( L \)) such that for each \( f \in E_L \)
\[
\int_M |f|^2 d\mu_L \leq C \int_M |f|^2 dV.
\]
Let $f \in E_L$ with $L$ and $r > 0$ (small) fixed.

$$
\int_M |f(z)|^2 d\mu_L \leq C_r L^{(m+1)/2} \int_M \int_{B(z, r/\sqrt{L}) \times I_r} |h(w, s)|^2 dV(w) ds d\mu_L(z)
$$

$$
= C_r L^{(m+1)/2} \int_{M \times I_r} |h(w, s)|^2 \mu_L(B(w, r/\sqrt{L})) dV(w) ds
$$

$$
\leq C_r L^{(m+1)/2} \frac{1}{k_L} \int_{M \times I_r} |h(w, s)|^2 dV(w) ds \overset{Proposition 1}{\approx} \|f\|_2^2
$$

with constants independent of $L$. Therefore, $\mu = \{\mu_L\}_L$ is $L^2$-Carleson for $M$.

For the converse, assume that $\mu$ is $L^2$-Carleson for $M$. We have to show the existence of a constant $C$ such that for all $L \geq 1$ and $\xi \in M$, $\mu_L(B(\xi, c/\sqrt{L})) \leq C/k_L$ (for some small constant $c > 0$). We will argue by contradiction, i.e. assume that for all $n \in \mathbb{N}$ there exists $L_n$ and a ball $B_n$ of radius $c/\sqrt{n}$ such that $\mu_{L_n}(B_n) > n/k_{L_n} \approx n/L_n^{m/2}$ ($c$ will be chosen later). Let $b_n$ be the center of the ball $B_n$. Define $F_n(w) = K_{L_n}(b_n, w)$. Note that the function $L_n^{-m/4} F_n \in E_{L_n}$ and $\|F_n\|_2^2 = K_{L_n}(b_n, b_n) \approx L_n^{m/2}$. Therefore,

$$
C \approx \int_M |L_n^{-m/4} F_n|^2 dV \geq \int_M |L_n^{-m/4} F_n|^2 d\mu_{L_n} \geq \int_{B_n} |L_n^{-m/4} F_n|^2 d\mu_{L_n}
$$

$$
\geq \inf_{w \in B_n} |L_n^{-m/4} F_n(w)|^2 \mu_{L_n}(B_n) \geq \inf_{w \in B_n} |F_n(w)|^2 \frac{n}{L_n^m}.
$$

Now we will study this infimum. Let $w \in B_n = B(b_n, c/\sqrt{n})$. Then

$$
|F_n(b_n)| - |F_n(w)| \leq |F_n(b_n) - F_n(w)| \leq \frac{c}{\sqrt{L_n}} \|\nabla F_n\|_\infty \leq
$$

$$
\overset{Proposition 1}{\leq} \frac{c}{\sqrt{L_n}} C_1 \sqrt{k_{L_n}} \sqrt{L_n} \|F_n\|_2 \approx c \frac{1}{k_{L_n}} \sqrt{n}.
$$

We pick $c$ small enough so that

$$
\inf_{B_n} |F_n(w)|^2 \geq C L_n^m.
$$

Finally, we have shown that $C \geq n \forall n$. This gives us the contradiction. □

The following result is a Plancherel-Pólya type theorem but in the context of the Paley-Wiener spaces $E_L$.

**Theorem 4** (Plancherel-Pólya Theorem). Let $Z$ be a triangular family of points in $M$, i.e. $Z = \{z_{L_j}\}_{j \in \{1, \ldots, m_L\}, L \geq 1} \subset M$. Then $Z$ is a finite union of uniformly separated families, if and only if there exists
a constant $C > 0$ such that for all $L \geq 1$ and $f_L \in E_L$

\begin{equation}
\frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \leq C \int_M |f_L(\xi)|^2 dV(\xi).
\end{equation}

Remark 3. The above result is interesting because the inequality (8) means that the sequence of normalized reproducing kernels is a Bessel sequence for $E_L$, i.e.

$$\sum_{j=1}^{m_L} \left| \langle f, \tilde{K}_L(\cdot, z_{Lj}) \rangle \right|^2 \lesssim \|f\|^2_2 \quad \forall f \in E_L,$$

where $\{\tilde{K}_L(\cdot, z_{Lj})\}_j$ are the normalized reproducing kernels. Note that $|\tilde{K}_L(\cdot, z_{Lj})|^2 \approx |K_L(\cdot, z_{Lj})|^2 k_L^{-1}$. That’s the reason why the quantity $k_L$ appears in the inequality (8).

Proof. This is a consequence of Theorem 3 applied to the measures

$$\mu_L = \frac{1}{k_L} \sum_{j=1}^{m_L} \delta_{z_{Lj}}, \quad L \geq 1.$$

\hfill \square

4. Characterization of Logvinenko-Sereda Sets

Before we state the characterization, we would like to recall some history of these kind of inequalities. The classical Logvinenko-Sereda (L-S) theorem describes some equivalent norms for functions in the Paley-Wiener space $PW^p_\Omega$. The precise statements is the following:

**Theorem 5 (L-S).** Let $\Omega$ be a bounded set and $1 \leq p < +\infty$. A set $E \subset \mathbb{R}^d$ satisfies

$$\int_{\mathbb{R}^d} |f(x)|^p dx \leq C_p \int_{E} |f(x)|^p dx, \quad \forall f \in PW^p_\Omega,$$

if and only if there is a cube $K \subset \mathbb{R}^d$ such that

$$\inf_{x \in \mathbb{R}^d} |(K + x) \cap E| > 0.$$

One can find the original proof in [9] and another proof can be found in [4, p. 112-116].

Luecking in [10] studied this notion for the Bergman spaces. Following his ideas, in [12], it has been proved the following result.

**Theorem 6.** Let $1 \leq p < +\infty$, $\mu$ be a doubling measure and let $E = \{E_L\}_{L \geq 0}$ be a sequence of sets in $\mathbb{S}^d$. Then $E$ is $L^p(\mu)$-L-S if and only if $E$ is $\mu$-relatively dense.
For the precise definitions and notations see [12]. Using the same ideas, we will prove the above theorem for the case of our arbitrary compact manifold $M$ and the measure given by the volume element.

In what follows, $\mathcal{A} = \{A_L\}_L$ will be a sequence of sets in $M$.

**Definition 2.** We say that $A$ is L-S if there exists a constant $C > 0$ such that for any $L$ and $f_L \in E_L$

$$\int_M |f_L|^2 dV \leq C \int_{A_L} |f_L|^2 dV.$$

**Definition 3.** The sequence of sets $\mathcal{A} \subset M$ is relatively dense if there exists $r > 0$ and $\rho > 0$ such that for all $L$

$$\inf_{z \in M} \frac{\text{vol}(A_L \cap B(z, r/\sqrt{L}))}{\text{vol}(B(z, r/\sqrt{L}))} \geq \rho > 0.$$

**Remark 4.** It is equivalent to have this property for all $L \geq L_0$ for some $L_0$ fixed.

Our main statement is the following:

**Theorem 7.** $\mathcal{A}$ is L-S if and only if $\mathcal{A}$ is relatively dense.

We shall prove the two implications in the statement separately. First we will show that this condition is necessary. Before proceeding, we construct functions in $E_L$ with a desired decay of its $L^2$-integral outside a ball.

**Proposition 3.** Given $\xi \in M$ and $\epsilon > 0$, there exist functions $f_L = f_L, \xi \in E_L$ and $R_0 = R_0(\epsilon, M) > 0$ such that

1. $\|f_L\|_2 = 1$.
2. $\int_{M \setminus B(\xi, R_0/\sqrt{L})} |f_L|^2 dV < \epsilon \forall L$.
3. For all $L \geq 1$ and any subset $A \subset M$,

$$\int_A |f_L|^2 dV \leq C_1 \frac{\text{vol}(A \cap B(\xi, R_0/\sqrt{L}))}{\text{vol}(B(\xi, R_0/\sqrt{L}))} + \epsilon,$$

where $C_1$ is a constant independent of $L, \xi$ and $f_L$.

**Remark.** In the above Proposition, the $R_0$ does not depend on the point $\xi$.

**Proof.** Given $z, \xi \in M$ and $L \geq 1$, let $S_L^N(z, \xi)$ denote the Riesz kernel of index $N \in \mathbb{N}$ associated to the Laplacian, i.e

$$S_L^N(z, \xi) = \sum_{i=1}^{k_L} \left( 1 - \frac{\lambda_i}{L} \right)^N \phi_i(z)\phi_i(\xi).$$
Note that $S^N_L(z, \xi) = K_L(z, \xi)$. The Riesz kernel satisfies the following inequality.

$$|S^N_L(z, \xi)| \leq C L^{m/2} (1 + \sqrt{L} d(z, \xi))^{-N-1}. \tag{9}$$

This estimate has its origins in Hörmander’s article [6, Theorem 5.3]. Estimate (9) can be found also in [16, Lemma 2.1].

Note that on the diagonal, $S^N_L(z, z) \approx C_N L^{m/2}$. The upper bound is trivial by the definition and the lower bound follows from

$$S^N_L(z, z) \geq k L/2 \sum_{i=1}^{k_{L/2}} (1 - \lambda_i L) N \phi_i(z) \phi_i(z) \geq 2^{-N} K_{L/2}(z, z) \approx C_N L^{m/2}.$$

Similarly we observe that $\|S^N_L(\cdot, \xi)\|_2 \approx C_N L^{m/2}$.

Given $\xi \in M$, define for all $L \geq 1$

$$f_{L, \xi}(z) := f_L(z) = \frac{S^N_L(z, \xi)}{\|S^N_L(\cdot, \xi)\|_2}.$$

We will choose the order $N$ later. Each $f_L$ belongs to the space $E_L$ and has unit $L^2$-norm. Let us verify the second property claimed in Proposition 3. Fix a radius $R$. Using the estimate (9),

$$\int_{M \setminus B(\xi, R/\sqrt{L})} |f_L|^2 dV \leq C_N L^{m/2} \int_{M \setminus B(\xi, R/\sqrt{L})} \frac{dV}{(\sqrt{L} d(z, \xi))^{2(N+1)}} = (*)$$

For any $t \geq 0$, consider the following set.

$$A_t := \left\{ z \in M : d(z, \xi) \geq \frac{R}{\sqrt{L}}, \quad d(z, \xi) < \frac{t^{-1/(2(N+1))}}{\sqrt{L}} \right\}.$$

Note that for $t > R^{-2(N+1)}$, $A_t = \emptyset$ and for $t < R^{-2(N+1)}$, $A_t \subset B(\xi, t^{-1/(2(N+1))}/\sqrt{L})$. Using the distribution function, we have:

$$(*) = C_N L^{m/2} \int_0^{R^{-2(N+1)}} \text{vol}(A_t) dt \leq C_N \frac{1}{R^{2(N+1) - m}},$$

provided $N + 1 > m/2$. Thus if we pick $R_0$ big enough we get

$$\int_{M \setminus B(\xi, R_0/\sqrt{L})} |f_L|^2 dV < \epsilon. \tag{10}$$

Now the third property claimed in Proposition 3 follows from (10). Indeed, given any subset $A$ in the manifold $M$,

$$\int_A |f_L|^2 dV \leq \int_{A \cap B(\xi, R_0/\sqrt{L})} |f_L|^2 dV + \epsilon.$$
Observe that
\[
\int_{A \cap B(\xi, R_0/\sqrt{L})} |f_L|^2 dV \lesssim C_N L^{n/2} \int_{A \cap B(\xi, R_0/\sqrt{L})} \frac{dV(z)}{(1 + \sqrt{Ld(z, \xi)} (N+1))^{2(N+1)}} \lesssim C_N R_0^n \frac{\text{vol}(A \cap B(\xi, R_0/\sqrt{L}))}{\text{vol}(B(\xi, R_0/\sqrt{L}))}.
\]

Now we are ready to prove one of the implications in the characterization of the L-S sets.

**Proposition 4.** Assume $A$ is L-S. Then it is relatively dense.

**Proof.** Assume $A$ is L-S, i.e.
\[
\int_M |f_L|^2 dV \leq C \int_{A_L} |f_L|^2 dV.
\]
Let $\xi \in M$ be an arbitrary point. Fix $\epsilon > 0$ and consider the $R_0$ and the functions $f_L \in E_L$ given by Proposition 3. Using the third property of Proposition 3 for the sets $A_L$, we get for all $L \geq 1$,
\[
1 = \|f_L\|_2^2 \leq C \int_{A_L} |f_L|^2 \leq CC_1 \frac{\text{vol}(A_L \cap B(\xi, R_0/\sqrt{L}))}{\text{vol}(B(\xi, R_0/\sqrt{L}))} + C\epsilon,
\]
where $C_1$ is a constant independent of $L$, $\xi$ and $f_L$. Therefore, we have proved that there exist constants $c_1$ and $c_2$ such that
\[
\frac{\text{vol}(A_L \cap B(\xi, R_0/\sqrt{L}))}{\text{vol}(B(\xi, R_0/\sqrt{L}))} \geq c_1 - c_2 \epsilon.
\]
Hence, $A$ is relatively dense provided $\epsilon > 0$ is small enough. \qed

Before we continue, we will prove a result concerning the uniform limit of harmonic functions with respect to some metric.

**Lemma 3.** Let $\{H_n\}_n$ be a family of uniformly bounded real functions defined on the ball $B(0, \rho) \subset \mathbb{R}^d$ for some $\rho > 0$. Let $g$ be a non-singular $C^\infty$ metric such that $g$ and all its derivatives are uniformly bounded and $g_{ij}(0) = \delta_{ij}$. Define $g_n(z) = g(z/L_n)$ (the rescaled metrics) where $L_n$ is a sequence of values tending to $\infty$ as $n$ increases. Assume that the family $\{H_n\}_n$ converges uniformly on compact subsets of $B(0, \rho)$ to a limit function $H : B(0, \rho) \to \mathbb{R}$ and $H_n$ is harmonic with respect to the metric $g_n$ (i.e. $\Delta_{g_n} H_n = 0$). Then, the limit function $H$ is harmonic in the Euclidean sense.

**Proof.** Let $\varphi \in C_c^\infty(\mathbb{B}(0, \rho))$. We have
\[
\int_{\mathbb{B}(0, \rho)} \Delta_g f \varphi dV = \int_{\mathbb{B}(0, \rho)} f \Delta_g \varphi dV.
\]
It is a direct computation to see that $\Delta g_n \varphi \to \Delta \varphi$ uniformly and $\Delta g_n \varphi$ is uniformly bounded on $B(0, \rho)$. Then
\[
0 = \int_{B(0, \rho)} H_n \Delta g_n \varphi dV_{g_n} \to \int_{B(0, \rho)} H \Delta \varphi dm(z) = \int_{B(0, \rho)} \Delta H \varphi dm(z).
\]
Therefore, the limit function $H$ is harmonic in the weak sense. Applying Weyl’s lemma, $H$ is harmonic in the Euclidean sense.

\[\square\]

Remark 5. The above argument also holds if we have a sequence of metrics $g_n$ converging uniformly to $g$ whose derivatives also converge uniformly to the derivatives of $g$. In this case, the conclusion would be that the limit is harmonic with respect to the limit metric $g$.

Now, we shall prove the sufficient condition of the main result.

**Proposition 5.** If $\mathcal{A}$ is relatively dense then it is L-S.

**Proof.** Fix $\epsilon > 0$ and $r > 0$. Let $D := D_{c,r,L}$ be
\[
D = \{ z \in M : |f_L(z)|^2 = |h_L(z, 0)|^2 \geq \epsilon \int_{B(z, \sqrt{L})} |h_L(\xi, t)|^2 dV(\xi) dt \}.
\]
The norm of $f_L$ is almost concentrated on $D$ because
\[
\int_{M \setminus D} |f_L(z)|^2 dV(z) \lesssim \epsilon \frac{1}{l(I_r)} \int_{M \times I_r} |h_L(\xi, t)|^2 \frac{L^{m/2}}{r^m} \int_{(M \setminus D) \cap B(\xi, r/\sqrt{L})} dV(\xi) dV(\xi) dt \lesssim \epsilon \frac{1}{l(I_r)} \int_{M \times I_r} |h_L(\xi, t)|^2 dV(\xi) dt \lesssim \epsilon e^{2r} \int_M |f_L|^2 dV.
\]
It is enough to prove
\[
(11) \quad \int_D |f_L|^2 dV \lesssim \int_{A_L} |f_L|^2
\]
with constants independent of $L$ and for this, it is sufficient to show that there exists a constant $C > 0$ such that for all $w \in D$
\[
(12) \quad |f_L(w)|^2 \leq \frac{C}{\text{vol}(B(w, r/\sqrt{L}))} \int_{A_L \cap B(w, r/\sqrt{L})} |f_L(\xi)|^2 dV(\xi).
\]
Because then, $(11)$ follows by integrating $(12)$ over $D$. So we need to prove $(12)$. Assume it is not true. This means that for all $n \in \mathbb{N}$ there exists $L_n$, functions $f_n \in E_{L_n}$ and $w_n \in D_n = D_{c,r,L_n}$ such that
\[
|f_n(w_n)|^2 > \frac{n}{\text{vol}(B(w_n, r/\sqrt{L_n}))} \int_{A_{L_n} \cap B(w_n, r/\sqrt{L_n})} |f_n|^2 dV.
\]
By the compactness of $M$, there exists $\rho_0 = \rho_0(M) > 0$ such that for all $w \in M$, the exponential map, $\exp_w : \mathbb{B}(0, \rho_0) \to B(w, \rho_0)$, is
a diffeomorphism and \((B(w, \rho_0), \exp^{-1})\) is a normal coordinate chart, where \(w\) is mapped to 0 and the metric \(g\) verifies \(g_{ij}(0) = \delta_{ij}\).

For all \(n \in \mathbb{N}\), take \(\exp_n(z) := \exp_{w_n}(rz/\sqrt{L_n})\) which is defined in \(\mathbb{B}(0, 1)\) and act as follows:

\[
\exp_n : \mathbb{B}(0, 1) \rightarrow \mathbb{B}(0, r/\sqrt{L_n}) \rightarrow B(w_n, r/\sqrt{L_n})
\]

\[
z \rightarrow \frac{rz}{\sqrt{L_n}} \rightarrow \exp_{w_n}(rz/\sqrt{L_n}) =: w
\]

Consider \(F_n(z) := f_n(\exp_n(z)) : \mathbb{B}(0, 1) \rightarrow B(w_n, r/\sqrt{L_n}) \xrightarrow{f_n} \mathbb{R}\) and the corresponding harmonic extension \(h_n\) of \(f_n\). Set

\[
H_n(z, t) := h_n(\exp_n(z), rt/\sqrt{L_n}),
\]

defined on \(\mathbb{B}(0, 1) \times J_1\) (where \(J_1 = (-1, 1)\)). Let \(\mu_n\) be the measure such that 

\[
d\mu_n(z) = \sqrt{|g|}(\exp_{w_n}(rz/\sqrt{L_n}))dm(z).
\]

Note that

\[
\hat{B}(w_n, r/\sqrt{L_n}) |f_n|^2dV \approx \hat{B}(0, 1) |F_n|^2d\mu_n.
\]

We will normalize \(H_n\) so that

\[
\int_{\mathbb{B}(0, 1) \times J_1} |H_n(w, s)|^2d\mu_n(w)ds = 1.
\]

As \(w_n \in D_n\), we have

\[
|F_n(0)|^2 = |f_n(w_n)|^2 \geq \epsilon \int_{B(w_n, r/\sqrt{L_n}) \times J_r} |h_n(w, t)|^2dVdt
\]

\[
\approx \epsilon \int_{\mathbb{B}(0, 1) \times J_1} |H_n(w, s)|^2d\mu_n(w)ds = \epsilon.
\]

Since \(|h_n|^2\) is subharmonic,

\[
|F_n(0)|^2 \geq |h_n(w_n, 0)|^2 \lesssim \int_{\mathbb{B}(0, 1) \times J_1} |H_n(w, s)|^2d\mu_n(w)ds = 1.
\]

Hence, we have \(\forall n \in \mathbb{N} \quad 0 < \epsilon \lesssim |F_n(0)|^2 \lesssim 1\).

Using the assumption,

\[
\frac{1}{n} \gtrsim \frac{1}{\text{vol}(B(w_n, r/\sqrt{L_n}))} \int_{A_{L_n \cap B(0, 1)}} |f_n|^2dV \approx \int_{B_r \cap \mathbb{B}(0, 1)} |F_n|^2d\mu_n,
\]
where $B_n$ is such that $\exp_n(B_n \cap \mathbb{B}(0, 1)) = A_{L_n} \cap B(w_n, r/\sqrt{L_n})$. So we have that
\[
\begin{aligned}
\forall n & \quad 0 < \epsilon \leq |F_n(0)|^2 \leq 1 \\
\forall n & \quad \int_{\mathbb{B}(0, 1) \cap B_n} |F_n|^2 d\mu_n \lesssim \frac{1}{n}.
\end{aligned}
\]
In fact, $0 < \epsilon \leq |H_n(0, 0)|^2 \leq 1$ (by the definition) and one can prove that $|H_n|^2 \lesssim 1$. Indeed, if $(z, s) \in \mathbb{B}(0, 1/2) \times J_{1/2}$, let $w = \exp_n(z) \in B(w_n, r/(2\sqrt{L_n}))$ and $t = rs/\sqrt{L_n} \in I_{r/2}$. Then
\[
|H_n(z, s)|^2 = |h_n(w, t)|^2 \lesssim \int_{B(w, r/(2\sqrt{L_n})) \times I_{r/2}(t)} |h_n|^2 dV dt = 1.
\]
Therefore, working with $1/2$ instead of $1$ we have $|H_n|^2 \lesssim 1$ for all $n$.

The sequence $\{H_n\}_n$ is equicontinuous in $\mathbb{B}(0, 1) \times J_1$. Indeed, consider $(w, t) \in B(w_n, r/(4\sqrt{L_n})) \times I_{r/4}$ and $(\tilde{w}, \tilde{t}) \in B(w, \tilde{r}r/\sqrt{L_n}) \times I_{\tilde{r}r}(t)$, then there exists some small $\delta > 0$ such that
\[
| h_n(w, t) - h_n(\tilde{w}, \tilde{t}) | \leq \frac{\tilde{r}}{\sqrt{L_n}} \sup_{B(w, r/(2\sqrt{L_n})) \times I_{r/2}(t)} |\nabla h_n| \leq (*).
\]
Taking $\tilde{r}$ small enough so that $\delta \leq r/4$ and using Schoen and Yau’s estimate $[\text{5}]$, we have
\[
(*) \leq \frac{\tilde{r}}{\sqrt{L_n}} \sup_{B(w_n, r/(2\sqrt{L_n})) \times I_{r/2}} |\nabla h_n| \lesssim \frac{\tilde{r}}{\sqrt{L_n}} \sup_{B(w, r/\sqrt{L_n}) \times I_r} |h_n| \lesssim \tilde{r}.
\]
So we have proved that $| h_n(w, t) - h_n(\tilde{w}, \tilde{t}) | \leq C\tilde{r}$. Take $\tilde{r}$ small enough so that $C\tilde{r} < \epsilon$. Let $(z, s) \in \mathbb{B}(0, 1/4) \times J_{1/4}$ and $(\tilde{z}, \tilde{s}) \in \mathbb{B}(z, \tilde{r}) \times (s - \tilde{r}, s + \tilde{r})$. Consider $w = \exp_n(z)$, $t = rs/\sqrt{L_n}$, $\tilde{w} = \exp_n(\tilde{z})$ and $\tilde{t} = r\tilde{s}/\sqrt{L_n}$. We have proved that for all $\epsilon > 0$ there exists $\tilde{r} > 0$ (small) such that for all $(z, s) \in \mathbb{B}(0, 1/4) \times J_{1/4}$:
\[
| H_n(z, s) - H_n(\tilde{z}, \tilde{s}) | < \epsilon \text{ if } |z - \tilde{z}| < \tilde{r}, |s - \tilde{s}| < \tilde{r} \quad \forall n.
\]
Change $1/4$ by $1$. So the sequence $H_n$ is equicontinuous.

Hence the family $\{H_n\}_n$ is equicontinuous and uniformly bounded on $\mathbb{B}(0, 1) \times J_1$. Therefore, by Ascoli-Arzela’s theorem there exists a partial sequence (denoted as the sequence itself) such that $H_n \rightrightarrows H$ uniformly on compact subsets of $\mathbb{B}(0, 1) \times J_1$. Since $F_n(z) = H_n(z, 0)$, we get a function $F(z) := H(z, 0) : \mathbb{B}(0, 1) \to \mathbb{R}$ which is the limit of $F_n$ (uniformly on compact subsets of $\mathbb{B}(0, 1)$).

By hypothesis, the sequence $\{A_L\}_L$ is relatively dense. Taking into
account that \( \operatorname{vol}(B(w_n, r/l_n)) = \frac{r^m}{l_n^{m/2}} \mu_n(\mathbb{B}(0, 1)) \), we get that
\[
\inf_n \mu_n(B_n) \geq \rho > 0,
\]
where we have denoted \( B_n \cap \mathbb{B}(0, 1) \) by \( B_n \).

Let \( \tau_n \) be such that \( d\tau_n = \chi_{B_n} d\mu_n \). From a standard argument (\( \tau_n \) are supported in a ball), we know the existence of a weak *-limit of a subsequence of \( \tau_n \), denoted by \( \tau \). This subsequence will be noted as the sequence itself. Using (13), we know that \( \tau \) is not identically 0. Now we have that
\[
\int_{\mathbb{B}(0, 1)} |F|^2 d\tau = 0.
\]

Therefore, \( F = 0 \) \( \tau \)-a.e. \( \mathbb{B}(0, 1) \). Now for all \( K \subset \mathbb{B}(0, 1) \) compact
\[
\int_K |F|^2 d\tau = 0,
\]
therefore \( F = 0 \) in \( \supp \tau \). Let \( \mathbb{B}(a, s) \subset \mathbb{B}(0, 1) \) such that \( \mathbb{B}(a, s) \cap \supp \tau \neq \emptyset \), then using the fact \( B_n \subset \mathbb{B}(0, 1) \),
\[
\tau_n(\mathbb{B}(a, s)) \leq \int_{\mathbb{B}(a, s)} d\mu_n \approx \frac{l_n^{m/2}}{r^m} \operatorname{vol}(B(\exp_n(a), sr/l_n)) \approx s^m.
\]
Therefore \( \tau_n(\mathbb{B}(a, s)) \lesssim s^m \) for all \( n \). Hence, in the limit \( \tau(\mathbb{B}(a, s)) \lesssim s^m \). In short,

1. We have sets \( B_n \subset \mathbb{B}(0, 1) \) such that
   \[ \rho \leq \mu_n(B_n) \leq \mu_n(\mathbb{B}(0, 1)) \approx 1. \]
2. We have measures \( \tau_n \) weakly-* converging to \( \tau \) (not identically 0).
3. \( \tau(\mathbb{B}(a, s)) \lesssim s^m \) for all \( \mathbb{B}(a, s) \subset \mathbb{B}(0, 1) \).
4. \( |F| = 0 \) \( \tau \)-a.e. in \( \mathbb{B}(0, 1) \).
5. \( |F(0)| > 0 \) and \( |F| \lesssim 1 \).

Assume we know that \( H \) is real analytic, then \( F(z) \) is real analytic. Federer ([2, Theorem 3.4.8]) proved that the \( (m - 1) \)-Hausdorff measure \( \mathcal{H}^{m-1}(F^{-1}(0)) < \infty \). Hence \( \mathcal{H}^{m-1}(\supp \tau) \leq \mathcal{H}^{m-1}(F^{-1}(0)) < \infty \). This implies that the Hausdorff dimension \( \dim_H(\supp \tau) \leq m - 1 \). On the other hand, since \( \tau(\mathbb{B}(a, s)) \lesssim s^m \), we have
\[
0 < \tau(\supp \tau) \lesssim \mathcal{H}^m(\supp \tau)
\]
and this implies that \( \dim_H(\supp \tau) \geq m \) by Frostman’s lemma. So we reach to a contradiction and the proof is complete. We only need to check that \( H \) is real analytic. In fact, we will show that \( H \) is harmonic. We have the following properties:

1. Observe that the family of measure \( d\mu_n \) converges uniformly to the ordinary Euclidean measure because \( g_{ij}(\exp_n(rz/\sqrt{l_n})) \rightarrow \)
\(g_{ij}(\exp_{w_0}(0)) = \delta_{ij}\), where \(w_0\) is the limit point of some subsequence of \(w_n\) (recall that we are taking normal coordinate charts).

(2) If \(g_n(z) := g(rz/\sqrt{L_n})\) (i.e. \(g_n\) is the rescaled metric), then \(\Delta_{(g_n, Id)} H_n(z, s) = 0\) for all \((z, s) \in \B(0, 1) \times J_1\), by construction.

(3) The functions \(H_n\) are uniformly bounded and converge uniformly on compact subsets of \(\B(0, 1) \times J_1\).

We are in the hypothesis of Lemma [3] that guarantees the harmonicity of \(H\) in the Euclidean sense. This concludes the proof of the proposition. \(\square\)

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