A remark on contracting inverse semigroups

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Abstract
A semi-lattice is said to be tree-like when any two of its elements are either orthogonal or comparable. Given an inverse semigroup $S$ whose idempotent semi-lattice is tree-like, and such that all tight filters are ultra-filters, we present a necessary and sufficient condition for $S$ to be contracting which looks closer in spirit to the notion of contracting actions than a condition found by the second named author and E. Pardo.

Keywords Inverse semigroup · Semi-lattice · Tight spectrum · Tight filter · Ultra-filter · Contracting actions

1 Introduction

In a paper by the second named author and Pardo [4], the notion of locally contracting groupoids introduced by Anantharaman-Delaroche in [1] was extended to inverse semigroups [4, Definition 6.4], as well as to actions of inverse semigroups on topological spaces [4, Definition 6.2]. Given an inverse semigroup $S$, these concepts were shown to relate to each other via the standard action of $S$ on $\hat{E}_{\text{tight}}$, the tight spectrum of its idempotent semi-lattice. To be precise it was shown in [4, Theorem 6.5] that $S$ is locally contracting if and only if the standard action of $S$ on $\hat{E}_{\text{tight}}$ is locally contracting (the “if” part in fact requires that all tight filters be ultra-filters, a condition that has been referred to by the name of compactable in [5]).

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In [4, Proposition 6.7], another condition (rephrased here as Theorem 4(iii)), which is a lot nicer to state, and which follows the general paradigm of local contractiveness more closely, was shown to be sufficient for the local contractiveness of \( \mathcal{S} \). In our main result, Theorem 4 below, we take a closer look at this condition and show it to be also necessary, provided the inverse semigroup is tree-like.

One of the key tools to prove our main result is Theorem 1, a curious combinatorial fact which we suspect may be known to specialist in Combinatorial Analysis, but which we have not found anywhere in the literature.

Tree-like inverse semigroups are quite common, especially in the theory of graph C*-algebras (see e.g. [2, Lemma 35.8]), so we feel that this class of inverse semigroups deserves further study.

This paper is written under the assumption that the reader is acquainted with [4], and to a certain extent also with [3], from where the basic theory of tight representations of inverse semigroups is drawn.

2 A combinatorial Lemma

In this short section we will prove a crucial combinatorial result to be used in our main result below.

**Theorem 1** Let \( X \) be a set which is decomposed as a finite disjoint union

\[
X = \bigsqcup_{i=1}^{n} X_i,
\]

where each \( X_i \) is a nonempty subset, and let \( f : X \to X \) be a one-to-one map such that for every integer \( m \geq 0 \) and every \( i, j = 1, \ldots, n \), one has that either

\[
\begin{cases}
  f^m(X_i) \cap X_j = \emptyset, & \text{ or } \\
  f^m(X_i) \subseteq X_j, & \text{ or } \\
  f^m(X_i) \supseteq X_j.
\end{cases}
\]

(1)

Then:

(i) there exist \( i \in \{1, \ldots, n\} \) and \( m > 0 \) such that \( f^m(X_i) \subseteq X_i \),

(ii) if \( f \) is not surjective, there exist \( i \in \{1, \ldots, n\} \) and \( m > 0 \) such that \( f^m(X_i) \nsubseteq X_i \).

**Proof** Let us begin by proving (i). For each \( m > 0 \), let \( A^m = \{A^m_{i,j}\}_{i,j} \) be the \( n \times n \) matrix defined by

\[
A^m_{i,j} = \begin{cases}
1 & \text{if } f^m(X_i) \cap X_j \neq \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

Case 1 Suppose that there exists some \( m > 0 \) such that every row of \( A^m \) has at most one (hence exactly one) nonzero entry. In this case, for every \( i \in \{1, \ldots, n\} \), we
have that $f^m(X_i)$ intersects a single $X_j$, so it must be contained in that $X_j$. We may therefore define a function

$$k : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

such that

$$f^m(X_i) \subseteq X_{k(i)}, \forall i \in \{1, \ldots, n\}.$$ 

From this it easily follows that

$$f^{pm}(X_i) \subseteq X_{k_p(i)}, \forall i \in \{1, \ldots, n\}, \forall p > 0. \quad (2)$$

Given that the set $\{k_p(1) : p > 0\}$ is finite, we may choose integers $p$ and $q$, with $0 < p < q$, and $k_p(1) = k^q(1)$. Defining

$$i := k^p(1) = k^q(1),$$

we then have that both $f^{pm}(X_1)$ and $f^{qm}(X_1)$ are contained in $X_i$. Setting $r = q - p$, observe that

$$X_i \supseteq f^{qm}(X_1) = f^{(r+p)m}(X_1) = f^{rm}(f^{pm}(X_1)) \subseteq f^{rm}(X_i).$$

The nonempty set $f^{qm}(X_1)$ is therefore contained in both $X_i$ and $f^{rm}(X_i)$, whence

$$\emptyset \neq X_i \cap f^{rm}(X_i) \quad \overset{\text{Eq.} (2)}{\subseteq} \quad X_i \cap X_{k^r(i)}.$$ 

As the $X_j$’s are pairwise disjoint, we deduce that $k^r(i) = i$, which is to say that

$$f^{rm}(X_i) \subseteq X_i,$$

concluding the proof of (i) in the present case.

Case 2 Failing the condition characterizing Case 1 above, we are left with the assumption that, for every $m > 0$, at least one row of $A^m$ has two or more nonzero entries. Since $A^m$ is a 0-1 matrix, of which there are only finitely many (there are exactly $2^{n^2}$ such matrices), there must be repetitions among the $A^m$, meaning that there are integers $m_1$ and $m_2$, with $0 < m_1 < m_2$, and $A^{m_1} = A^{m_2}$. Let $p$ be the index of any row of $A^{m_1}$ possessing two or more nonzero entries, and let $Q$ be the set formed by the indices of the columns where such nonzero entries appear, so that $|Q| \geq 2$, and

$$A_{p,j}^{m_1} = 1 \iff j \in Q.$$
In particular \( f^{m_1}(X_p) \) has a nonempty intersection with \( X_q \), for each \( q \) in \( Q \). Notice that for any such \( q \), it is impossible that

\[
f^{m_1}(X_p) \subseteq X_q,
\]

since the \( X_i \)'s are pairwise disjoint and \( f^{m_1}(X_p) \) must intersect at least another \( X_i \), given that \( |Q| \geq 2 \). Thus, given \( q \) in \( Q \), when comparing \( f^{m_1}(X_p) \) with \( X_q \) from the point of view of Eq. (1), the only remaining alternative is that

\[
f^{m_1}(X_p) \supseteq X_q.
\]

Since \( f^{m_1}(X_p) \) does not intersect \( X_j \), for \( j \not\in Q \), we deduce that

\[
f^{m_1}(X_p) = \bigcup_{q \in Q} X_q.
\]

Observe that \( Q \) cannot be equal to \( \{1, \ldots, n\} \), or else \( f^{m_1}(X_p) = X \), and there would be no room for the image of the other \( X_i \) under the injective map \( f^{m_1} \). Consequently

\[
2 \leq |Q| < n.
\]

Recalling that \( A^{m_1} = A^{m_2} \), the above argument also proves that

\[
f^{m_2}(X_p) = \bigcup_{q \in Q} X_q.
\]

Defining \( m := m_2 - m_1 \), notice that

\[
f^m \left( \bigcup_{q \in Q} X_q \right) = f^m \left( f^{m_1}(X_p) \right) = f^{m+m_1}(X_p) = f^{m_2}(X_p) = \bigcup_{q \in Q} X_q,
\]

so we may restrict \( f^m \) to

\[
X' := \bigcup_{q \in Q} X_q
\]

obtaining an injective map

\[
f^m : X' \to X',
\]

satisfying all of the assumptions of the statement, with \( X' \) decomposing into a smaller number of pairwise disjoint components \( X_q \)'s. Therefore the result follows immediately by induction on \( n \).

In order to prove (ii), we may now use (i) and hence we may assume that

\[
f^m(X_i) \subseteq X_i,
\]
for some \( i \in \{1, \ldots, n\} \), and some \( m > 0 \). In case the above is a proper inclusion we are done, so we assume the contrary, meaning that \( f^m(X_i) = X_i \). Setting

\[
X' = \bigcup_{j \neq i} X_j,
\]

and recalling that \( f^m \) is injective, we then have that

\[
f^m(X') \subseteq X'.
\]

Observe that this must is a proper inclusion since otherwise \( f^m \) would be surjective, contradicting the hypothesis. Therefore, the conclusion follows again by induction on \( n \).

\( \square \)

3 Preliminaries on semi-lattices and inverse semigroups

In this section we will freely use the notation introduced in [4]. Our main goal will be to introduce the class of inverse semigroups to which our main result applies. We will also prove some basic related results.

**Definition 1** A semi-lattice \( \mathcal{E} \) with zero is called *tree-like* if, for any \( e \) and \( f \) in \( \mathcal{E} \), one has that

\[
e \perp f, \quad e \leq f, \quad \text{or} \quad e \geq f.
\]

If \( S \) is an inverse semigroup whose idempotent semi-lattice is tree-like, we will say that \( S \) is *tree-like*.

Given \( e \) and \( f \) in a semi-lattice, it is easy to see that

\[
e \leq f \implies D^\theta_e \subseteq D^\theta_f.
\]

The converse of this fact is however not true. For example, if \( S \) is obtained by adding a zero element to an inverse semigroup without a zero, then

\[
\xi := \mathcal{E} \setminus \{0\}
\]

is the only ultra-filter on \( \mathcal{E} \). Consequently, \( \hat{\mathcal{E}}_{\text{tight}} = \{\xi\} \), and then \( D^\theta_e = \{\xi\} \) for every nonzero idempotent \( e \), so the converse of Eq. (3) is seen to fail badly. However, in a tree-like inverse semigroup, it is easy to see that

\[
D^\theta_e \not\supseteq D^\theta_f \implies e \leq f,
\]

simply because, under the assumption that \( D^\theta_e \not\supseteq D^\theta_f \), alternatives “\( e \perp f \)” and “\( e \geq f \)” are clearly excluded.
The strict inclusion above has other interesting consequences. Assuming that $D^\theta_f \setminus D^\theta_e$ is indeed nonempty, and noticing that it is an open subset of $\hat{\mathcal{E}}_{\text{tight}}$, we may find an ultra-filter $\xi$ there, meaning that $f \in \xi$ and $e \notin \xi$. By [3, Lemma 12.3], it follows that there exists some $d \in \xi$ such that $d \perp e$, and upon replacing $d$ with $df$, we may clearly assume that $d \leq f$.

**Definition 2** Given $e$ and $f$ in a semi-lattice $\mathcal{E}$, we will say that $e \ll f$, whenever $e \leq f$, and there exists a nonzero $d \leq f$ such that $d \perp e$.

Using this terminology we may then state the following fact:

**Proposition 2** Let $\mathcal{E}$ be a tree-like semi-lattice. Then

$$D^\theta_e \subsetneq D^\theta_f \Rightarrow e \ll f, \forall e, f \in \mathcal{E}.$$ 

There is another slightly annoying question related to the converse of Eq. (3) which we would like to get out of our way as soon as possible:

**Proposition 3** Given any inverse semigroup $S$, and given $s \in S$, and $e \in \mathcal{E}$, we have that:

(i) if $e \leq s^*s$, then $D^\theta_e$ is contained in the domain of $\theta_s$, and $\theta_s(D^\theta_e) = D^\theta_{ses^*}$.

(ii) if $D^\theta_e$ is contained in the domain of $\theta_s$, then $\theta_s(D^\theta_e) = D^\theta_{ses^*}$.

**Proof** The first assertion in (i) is obvious. As for the second, recall that $\theta_e$ is the identity map on $D^\theta_e$ so, in particular, the range of $\theta_e$ is $D^\theta_e$. Thus,

$$\theta_s(D^\theta_e) = \theta_s(\text{Ran}(\theta_e)) = \text{Ran}(\theta_s\theta_e) = \text{Ran}(\theta_{se}) = D^\theta_{ses^*} = D^\theta_{ses^*},$$

proving (i).

As the reader may have already anticipated, the catch in (ii) is that it is assumed that $D^\theta_e \subseteq D^\theta_{s^*s}$, but not necessarily that $e \leq s^*s$. Fortunately, this can be easily circumvented as follows:

$$\theta_s(D^\theta_e) = \theta_s(D^\theta_e \cap D^\theta_{s^*s}) = \theta_s(D^\theta_{es^*s}) \overset{(i)}{=} D^\theta_{(es^*s)s^*} = D^\theta_{ses^*}.$$ 

\[\square\]

**4 The main result**

Given the above preparations, we are now ready to prove our main result.

**Theorem 4** Let $S$ be a tree-like inverse semigroup such that every tight filter in $\mathcal{E}$ is an ultra-filter. Then the following are equivalent:

(i) $S$ is locally contracting;

(ii) The standard action $\theta : S \curvearrowright \hat{\mathcal{E}}_{\text{tight}}$ is locally contracting;
(iii) For every nonzero $e \in \mathcal{E}$, there exist an idempotent $f \leq e$ and an element $s \in S$ such that $f \leq s^*s$ and $sfs^* \ll f$.

**Proof**  The equivalence between (i) and (ii) follows from [4, Theorem 6.5].

In order to prove that (iii) implies (i), given a nonzero $e \in \mathcal{E}$, let $f$ and $s$ be as in (iii). Since $sfs^* \ll f$, there exists a nonzero $f_0 \leq f$ such that $f_0 \perp sfs^*$, and then we see that $f_0$ together with $f_1 := f$ obeys the conditions of [4, Proposition 6.7], from where one deduces that $S$ is locally contracting, proving (i).

The most delicate part of this proof is the implication (i) $\Rightarrow$ (iii), which we take up next. Given a nonzero $e \in \mathcal{E}$, let $U = D_e^\theta$, and choose $s$ and $V$ as in [4, Definition 6.2] so that,

$$V \subseteq U, \quad \overline{V} \subseteq D_{s^*s}^\theta, \quad \text{and} \quad \Theta_s(\overline{V}) \not\subseteq V. \quad (5)$$

In the first part of the proof we will show that $V$ may be chosen to be of the form

$$V = \bigcup_{f \in F} D_f^\theta,$$

where $F$ is a finite set of idempotents satisfying $f \leq es^*s$. In order to achieve this, for each $\xi$ in $\Theta_s(V)$, choose a neighborhood of $\xi$ contained in $V$. By hypothesis we have that $\xi$ is an ultra-filter, and by [4, Proposition 2.5] we may suppose that such a neighborhood is of the form $D_{f_\xi}^\theta$, for some $f_\xi \in \mathcal{E}$, whence

$$\xi \in D_{f_\xi}^\theta \subseteq V,$$

so we see that the $D_{f_\xi}^\theta$ form an open cover for $\Theta_s(V)$. Since

$$V \subseteq U \cap D_{s^*s}^\theta = D_e^\theta \cap D_{s^*s}^\theta = D_{es^*s}^\theta, \quad (6)$$

then also each $D_{f_\xi}^\theta \subseteq D_{es^*s}^\theta$, and therefore,

$$D_{f_\xi}^\theta = D_{es^*s}^\theta \cap D_{f_\xi}^\theta = D_{es^*sf_\xi}^\theta.$$

Upon replacing each $f_\xi$ by

$$f_\xi' := es^*sf_\xi,$$

we may therefore assume that $f_\xi \leq es^*s$.

Being a closed subset of $D_{s^*s}^\theta$, observe that $\overline{V}$ is compact, and hence so is $\Theta_s(\overline{V})$. We may then take a finite subcover of the above cover, say

$$\Theta_s(\overline{V}) \subseteq \bigcup_{f \in F'} D_f^\theta, \quad (7)$$

where $F'$ is a finite set consisting of some of the $f_\xi$. 
We next claim that there exists a nonzero idempotent $f_0 \leq es^*s$ such that
\begin{equation}
D^\theta_{f_0} \subseteq V \setminus \theta_s(V).
\end{equation}

To see this, first observe that $V \setminus \theta_s(V)$ is open and nonempty, since $V$ is open, $\theta_s(V)$ is closed and $\theta_s(V) \subseteq V$ by the choice of $V$ satisfying Eq. (5). Even without assuming that all tight filters are ultra-filters, we may use the density of the set formed by the latter to find some ultra-filter $\xi$ in $V \setminus \theta_s(V)$. An application of [4, Proposition 2.5] then provides $f_0$ in $E$ such that
\begin{equation}
\xi \subseteq D^\theta_{f_0} \subseteq V \setminus \theta_s(V),
\end{equation}
and, again by Eq. (6), we may assume that $f_0 \leq es^*s$. Adding $f_0$ to $F'$, we form the set
\begin{equation}
F := \{f_0\} \cup F',
\end{equation}
with which we define
\begin{equation}
W := \bigcup_{f \in F} D^\theta_f.
\end{equation}
We then have that $W$ is clopen, and that
\begin{equation}
\theta_s(V) \subseteq W \subseteq V,
\end{equation}
where the proper inclusion above is a consequence of Eq. (9) and the fact that we have included $f_0$ in $F$. Applying $\theta_s$ to the sets above we then deduce that
\begin{equation}
\theta_s(W) \subseteq \theta_s(V) \subseteq \theta_s(V) \subseteq W.
\end{equation}
This completes the task outlined at the beginning of the proof.

Notice that for any $f_1$ and $f_2$ in $F$, we have by (i) that
\begin{equation}
f_1 \perp f_2, \quad f_1 \leq f_2, \quad \text{or} \quad f_1 \geq f_2,
\end{equation}
in which case
\begin{equation}
D^\theta_{f_1} \cap D^\theta_{f_2} = \emptyset, \quad D^\theta_{f_1} \subseteq D^\theta_{f_2}, \quad \text{or} \quad D^\theta_{f_1} \supseteq D^\theta_{f_2},
\end{equation}
respectively. Replacing $F$ by the subset of its maximal elements we may then assume that $F$ is formed by pairwise orthogonal idempotents, in which case Eq. (10) is a disjoint union.

In order to proceed, let us consider two cases.

Case 1 Assuming that $|F| = 1$, say $F = \{f\}$, we have that $W = D^\theta_f$, and then
\begin{equation}
\theta_s(D^\theta_f) = \theta_s(W) \subseteq W = D^\theta_f.
\end{equation}
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Proposition 2 implies that $sfs^* \ll f$, concluding the proof.

**Case 2** Assuming that $|F| > 1$, let us consider $\theta_s$ as a function

$$\theta_s : W \to W,$$

observing that it is an injective but not surjective map, by Eq. (11). Using Theorem 1 and the fact that Eq. (10) is a disjoint union, we have that

$$\theta^m_s(D^\theta_f) \subsetneq D^\theta_f,$$

for some integer $m > 0$, and some $f$ in $F$. We next notice that

$$\theta^m_s(D^\theta_f) = \theta^m_s(D^\theta_f) \stackrel{Prop. 3(ii)}{=} D^\theta_{s^mfs^m},$$

so we deduce from the above that

$$D^\theta_{s^mfs^m} \subsetneq D^\theta_f,$$

and then Proposition 2 implies that $s^mfs^m \ll f$.

To conclude we must still address the requirement that $f \leq s^m s^m$. For this, observe that since $W$ is contained in the domain of $\theta_s$, and since $W$ is invariant under $\theta_s$, we have that $W$ is also contained in the domain of $\theta^m_s$, namely

$$W \subseteq D^\theta_{s^m s^m}.$$

Recalling that we are working under the hypothesis that $|F| > 1$, we have that $D^\theta_f$ is a proper subset of $W$, so

$$D^\theta_f \subsetneq W \subseteq D^\theta_{s^m s^m},$$

and then $f \ll s^m s^m$, by Proposition 2.

The main point we would like to make in the present work is that, even though the definition of locally contracting actions given in [4, Definition 6.2] is syntactically closer to Theorem 4(iii), from a logical point of view, the former is closer to the notion of locally contracting inverse semigroup described in [4, Definition 6.4], since these are equivalent to each other under broader conditions, as proved in [4, Theorem 6.5].

As seen in Theorem 4, above, all of these are equivalent to each other under the rather strong assumption that $S$ is tree-like, but it would be highly desirable to decide if the tree-like property is indeed necessary for the proof of Theorem 4.

A related problem is to decide conditions under which the converse of [4, Proposition 6.3] also holds.
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