The Carter–Saito theorem for virtual knots.

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Abstract

We construct the new non-trivial state–sum invariants for virtual knots and links by a generalization of the powerful Carter–Saito theorem for classical knots. The main result of this work is based on cohomology quandle theory and colorings of virtual knot and link diagrams by quandle elements.

Key words. Virtual knots, virtual links, quandle, invariant state–sum, cohomology of quandle.

1. Introduction

1.1. Motivation. The theory of virtual knots, which were discovered and described by Kauffman [1], is the natural generalization of classical knot theory. At the present time this theory is well-developed and has many applications in other fields of mathematics. For instance, virtual knot theory arises from study of knots in thickened surfaces and combinatorics of Gauss diagrams and codes [1] [2]. Also virtual knots relate with the theory of virtual manifolds and the generalization of Vitten-Reshetikhin-Turaev invarinats [3].

“Virtual” analogues of knot invariants are in the focus of many research dedicated to virtual knot theory [1]. The purpose of this paper is to construct the new non-trivial state-sum invariants of virtual knots and links, which are obtained by a generalization of Carter–Saito theorem for classical knots [4].

1.2. Kauffman theory of virtual knots and links. Similarly to the classical knots there is the diagram knot technique and the complete analogue of the classical Reidemeister theorem for virtual knots [1], [2], which play the central role in virtual knot theory. We define diagrams for virtual links or knots as it is defined in classical knot and link theory, however in comparison with the classical case some intersections of virtual knot or link diagrams might be “virtual” (see Fig.1).

\[\text{Figure 1. The classical and virtual intersections}\]

According to the virtual Reidemeister theorem we consider two virtual knot diagrams equivalent, if one of them can be transformed to another by isotopy and a finite sequence of classical Reidemeister moves (see Fig.2) and four additional virtual Reidemeister moves (see Fig.3).
1.3. The Carter–Saito theorem for classical knots and links. In this subsection we briefly examine the construction of an utterly powerful family of knot invariants by Carter, Saito et al. [4]. Their approach is based on the following algebraic objects (see [4], [5], [6] for more details):

**Definition 1.1.** A quandle is a set equipped with a binary operations satisfying the following axioms:

1. $\forall a \in G : a \ast a = a$
2. $\forall a, b \in G \exists! x \in G : x \ast a = b$
3. $\forall a, b, c \in G : (a \ast b) \ast c = (a \ast c) \ast (b \ast c)$

Also we have to recall the definition of a quandle 2–cocycle [4], [5], [6].

**Definition 1.2.** A 2–cocycle $\phi$ is a function $\phi : G \times G \to R$ (here $R$ is an Abelian ring) which satisfies the following identities:

1. $\forall a \in G : \phi(a, a) = 1$
2. $\forall a, b, c \in G : \phi(a, b)\phi(a \ast b, c) = \phi(a, c)\phi(a \ast c, b \ast c)$

Now we are ready to describe the Carter–Saito (et al.) construction for classical knots and links invariants. Let us fix a quandle $G$, some of its 2–cocycle $\phi$ and consider an oriented diagram of a knot (or a link). Then we color its arcs with elements of $G$. We
will say that its arc coloring is possible if the following three rules (see Fig.4) are satisfied for each diagram intersection:

\[
\begin{align*}
\Phi(x; y) & \quad \Phi^{-1}(x; y) \\
\end{align*}
\]

**Figure 4.** The coloring rules

**Theorem 1.3.** [4] Let us consider a knot (or a link) with a given oriented diagram \( D \). Let \( C \) be a set of the possible colorings of the diagram \( D \). For each possible coloring we define the weights of diagram intersections by the 2–cocycle \( \phi \) as it is shown in Fig.4. Then the following state–sum function is an invariant of the knot (or the link) with the diagram \( D \):

\[
Z(D, \phi) = \sum_{C} \prod_{\sigma_t} \phi(x, y)^{\sigma_t},
\]

where the product is taken over the all intersections of the diagram \( D \) and the sum is taken over the set \( C \), signs \( \sigma = -1, 1 \) are selected as it is shown in Fig.4, \( t \) is a formal parameter.

**Proof.** For completeness we give the sketch of the proof (for more details see [4]):

\[
\begin{align*}
\Phi^{-1}(a; a) = 1 \\
\end{align*}
\]

**Figure 5.** The first and second classical moves

1. Consider the first Reidemeister move. Using the first quandle axiom 1.1 and the first 2–cocycle identity ?? we conclude that the state–sum function is preserved under the first Reidemeister move (see Fig.5).
2. By the same way using the second quandle axiom ?? we conclude that the state–sum function is preserved under the second Reidemeister move (see Fig.5).
3. It remains to examine the third Reidemeister move. Similarly, using the second and the third quandle axiom 1.1 and the first 2–cocycle identity ?? of the we conclude that the state–sum function is preserved under the third Reidemeister move (see Fig.6).
Remark 1.4. Formally, other cases of the orientation and the position of the arcs of the diagram $D$ are not analyzed, but they are checked by the same fashion. Moreover, other cases might be obtained by the Kauffman trick (see Fig.7).

Remark 1.5. The Kauffman trick can be similarly done for the semi–virtual move $\Omega_3'$ and the virtual move $\Omega_2'$.

Remark 1.6. The Carter–Saito theorem is also holds for knots and links in $\mathbb{R}P^3$ (for the definition of knots and links in $\mathbb{R}P^3$ see [8], [9]). This result can be proved by the slightly different way which was described above in the proof of Theorem 1.3.

2. Generalization of the Carter-Saito theorem for virtual knots

2.1. The Carter-Saito theorem for virtual knots. In this subsection we turn to the main result of our work. Consider a virtual knot (or a virtual link) with an oriented diagram $D$. Then fix a finite quandle $G$, its 2–cocycle $\phi$ and an automorphism $f$ of the quandle $G$.

Remark 2.1. Here as usual a quandle automorphism is a bijection $f : G \to G$ which preserves a quandle operation $\forall a, b \in G : f(a * b) = f(a) * f(b)$.

Let us color the arcs of a diagram $D$ with the elements of the quandle $G$. We call the coloring possible, if it satisfies the rules demonstrated in Fig.4 and Fig.8. For each
classical intersection we define the weight that is obtained by the 2–cocycle $\phi$ as it is shown in Fig.4. Finally, for each virtual intersection we define the weight is equal to 1 as it is shown in Fig.8.

Remark 2.2. The coloring rules for virtual diagrams used in this work are very similar to the rules described in the article [7].

Now we describe how the coloring of a diagram changes under the first three virtual moves (see Fig.9). By a simple brute force and the virtual Kauffman trick it is not difficult to consider all possible orientations and obtain the following proposition:

\[ Z(D, \phi, f) = \sum_C \prod \phi(x, y)^\sigma \]

where the product is taken over all classical intersections of a given diagram, and the sum is taken over the set of all possible colorings $C$, $\sigma = -1, 1$ is selected as shown in Fig.4.

Now we present how the coloring of a virtual knot (or a link) diagram changes under the move $\Omega'_3$ (see Fig.10) to obtain the following theorem:

**Proposition 2.3.** Consider a virtual knot or a link with a diagram $D$. Then the state–sum function $Z(D, \phi, f) = \sum_C \prod \phi(x, y)^\sigma$ is the invariant of this virtual knot (or this link) under the classical Reidemeister moves and the first three virtual Reidemeister moves. Where the product is taken over all classical intersections of a given diagram, and the sum is taken over the set of all possible colorings $C$, $\sigma = -1, 1$ is selected as shown in Fig.4.
Theorem 2.4. 1. The function

\[ Z_1(D, \phi, f) = \prod_C \prod \phi(x, y)^{\sigma t} \]

is the invariant of a virtual knot (or a virtual link) with a given diagram \( D \) under the classical and virtual moves, where the second product is taken over all classical intersections of the given diagram, and the first product is taken over the set of all possible colorings \( C \). We call the function \( Z_1 \) a state–weight.

2. Consider a fixed automorphism \( f \) of a quandle \( G \) and its fixed 2–cocycle \( \phi \). Then if \( f \) and \( \phi \) are aligned with each other (it means that the following holds: \( \forall a, b \in G : \phi(a, b) = \phi(f(a), f(b)) \)), the function

\[ Z_2(D, \phi, f) = \sum_C \prod \phi(x, y)^{\sigma t} \]

is the invariant of a virtual knot (or link) with a diagram \( D \) under all classical and virtual moves, where the product is taken over all classical intersections of a given diagram \( D \), and the sum is taken over the set of all possible colorings \( C \). We call the function \( Z_2 \) a state–sum.

3. If 2–cocycles \( \phi \) and \( \phi' \) are cohomologous (for the definition see [4]), then \( Z_1(D, \phi, f) = Z_1(D, \phi', f) \).

Proof. 1. It suffices to prove that \( Z_1 \) is invariant under \( \Omega'_3 \). Values of \( Z_1 \) before and after \( \Omega'_3 \) move are different by factors which are equal to \( P_1 = \prod \phi(x_i, y_i) \) and \( P_2 = \prod \phi(f(x_i), f(y_i)) \) correspondingly. However, it is easy to see that, if \( A \) is a possible coloring of \( D \), then \( f(A) \) is a possible coloring too. Indeed, as soon as \( f \) is an automorphism we conclude that, if for the coloring \( A \) an arc is colored by an element \( a \), then for the coloring \( f(A) \) the same arc is colored by the element \( f(a) \). So, we obtain that \( P_1 = P_2 \), that completes the proof.
2. The proof is similar to the proof of the Carter–Saito theorem for classical knots (see Fig.5, 6, 9 and 10).

3. Here we follow the idea from [4]. It suffices to prove that our statement is true for the coboundaries 2–cocycles \( \phi(x, y) = \psi(x)\psi^{-1}(x \ast y) \). Consider a possible coloring \( A \) and a “long” arc of \( D \) (see Fig.11), its contribution to the stat-weight is equal to \( \psi^{-1}(\omega)\psi(f^n(\omega)) \). Since the quandle \( G \) is finite, then there exists a minimal \( k \) such that \( id = f^k \). Therefore for the coloring \( A \) and the “long” arc, we obtain the following factor of the state-weight: \( \psi^{-1}(\omega)\psi(f^n(\omega))\psi(f^{n+1}(\omega)) \cdots \psi^{-1}(f^{k-1}(\omega))\psi(f^{k-1+n}(\omega)) \) is equal to 1, that completes the proof.

\[
\begin{align*}
\psi(x)\psi^{-1}(\omega)\psi(f^n(\omega))\psi^{-1}(u) \\
\psi(x)\psi^{-1}(\omega)\psi(f^{n+1}(\omega))\psi^{-1}(u)
\end{align*}
\]

**Figure 11. “Long” arc**

Remark 2.5. The function \( Z_3(D, \phi) = \sum \prod \prod \phi(x, y)^{\sigma t} \) is the invariant of a virtual knot (or virtual link) under all classical and virtual moves, where the sum is taken over all automorphism of \( G \).

\[
\begin{align*}
\text{Figure 12. Examples of links}
\end{align*}
\]

2.2. **Examples.** Consider the quandle \( Q = \mathbb{Z}_4 \) with the operation \( \ast : i \ast j = 2j - i \) (mod 4). We have the non-trivial 2–cocycle \( \phi : Q \rightarrow \mathbb{Z} \), which is defined as follows: \( \phi(a_1, b_1) = \phi(a_1, b_2) = t \) otherwise \( \phi = 1 \), where \( t \) is a generator under multiplication.
of $\mathbb{Z}$, and $a_1 = 0$, $b_1 = 1$, $a_2 = 2$, $b_2 = 3$. It is easy to verify that the automorphism $f(b) = b \ast a_1$ is aligned with the given 2–cocycle $\phi$.

Now we calculate the state–sum related with the quandle $Q$ and the automorphism $f$ for the links as shown in Fig.12.

1. For the first example the state–sum is equal to $8(1 + t)$.
2. For the second example the state–sum is equal to $8$.
3. For the third example the state–sum is equal to $4(1 + t)$.
4. For the fourth example the state–sum is equal to $4(1 + t)$.

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