LOCAL WELL-POSEDNESS FOR THE NONLINEAR DIRAC EQUATION IN TWO SPACE DIMENSIONS

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Abstract. The Cauchy problem for the cubic nonlinear Dirac equation in two space dimensions is locally well-posed for data in \(H^s\) for \(s > 1/2\). The proof given in spaces of Bourgain-Klainerman-Machedon type relies on the null structure of the nonlinearity as used by d’Ancona-Foschi-Selberg for the Dirac-Klein-Gordon system before and bilinear Strichartz type estimates for the wave equation by Selberg and Foschi-Klainerman.

1. Introduction and main results

Consider the Cauchy problem for the nonlinear Dirac equation in two space dimensions

\[ i(\partial_t + \alpha \cdot \nabla)\psi + M\beta \psi = -(\beta \psi, \psi)\beta \psi \]  

with initial data

\[ \psi(0) = \psi_0. \]

Here \(\psi\) is a two-spinor field, i.e. \(\psi : \mathbb{R}^{1+2} \to \mathbb{C}^2\), \(M \in \mathbb{R}\) and \(\nabla = (\partial_{x_1}, \partial_{x_2})\), \(\alpha \cdot \nabla = \alpha^1 \partial_{x_1} + \alpha^2 \partial_{x_2}\). \(\alpha^1, \alpha^2, \beta\) are hermitian \((2 \times 2)\)-matrices satisfying \(\beta^2 = (\alpha^1)^2 = (\alpha^2)^2 = I\), \(\alpha^1 \beta + \beta \alpha^1 = 0\), \(\alpha^1 \alpha^k + \alpha^k \alpha^1 = 2\delta^{jk}I\).

\(\langle \cdot, \cdot \rangle\) denotes the \(\mathbb{C}^2\)- scalar product. A particular representation is given by

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]

We consider Cauchy data in Sobolev spaces: \(\psi_0 \in H^s(\mathbb{R}^2)\).

In quantum field theory the nonlinear Dirac equation is a model of self-interacting Dirac fermions. It was originally formulated in one space dimension known as the Thirring model \([1]\) and in three space dimensions \([SG]\). See also \([FLR, FFK, GN]\).

In the case of one space dimension global existence for data in \(H^1\) was proven by Delgado \([D]\). For less regular data Selberg and Tesfahun \([ST]\) showed local wellposedness in \(H^s\) for \(s > 0\), unconditional uniqueness in \(C^0([0,T],H^s)\) for \(s > 1/4\) and global well-posedness for \(s > 1/2\). Recently T.Candy \([C]\) was able to show global well-posedness in \(L^2\), which is the critical case with respect to scaling.

In the case of three space dimensions Escobedo and Vega \([EV]\) showed local well-posedness in \(H^s\) for \(s > 1\), which is almost critical with respect to scaling. Moreover they considered more general nonlinearities, too. Global solutions for small data in \(H^s\) for \(s > 1\) were shown to exist by Machihara, Nakamshi and Ozawa \([MNO]\). Machihara, Nakamura, Nakamshi and Ozawa \([MNO]\) proved

2000 Mathematics Subject Classification: 35Q55, 35L70

Key words and phrases: Dirac equation, well-posedness, Fourier restriction norm method
global existence for small data in $H^1$ under some additional regularity assumptions for the angular variables.

In the present paper we now consider the case of two space dimensions where the critical space is $H^{1/2}$. We show local well-posedness in $H^s$ for $s > 1/2$, which is optimal up to the endpoint, and unconditional uniqueness for $s > 3/4$. We construct the solutions in spaces of Bougain-Klainerman-Machedon type, using that the nonlinearity satisfies a null condition. Our proof uses the approach to the corresponding problem for the Dirac-Klein-Gordon equations by d’Ancona, Foschi and Selberg [AFS, AFS1]. The crucial estimates for the cubic nonlinearity can then be reduced to bilinear Strichartz type estimates for the wave equation which were given by S. Selberg [S] and D. Foschi and S. Klainerman [FK].

It is possible to simplify the system (1)(2) by considering the projections onto the one-dimensional eigenspaces of the operator $-i\alpha \cdot \nabla$ belonging to the eigenvalues $\pm|\alpha|$. These projections are given by $\Pi_{\pm}(D)$, where $D = \frac{\xi}{2} + \Pi_{\pm}(\xi) = \frac{1}{2}(I \pm \frac{\xi}{|\xi|} \cdot \alpha)$. Then $-i\alpha \cdot \nabla = |D|\Pi_{\pm}(D) - |D|\Pi_{-\pm}(D)$ and $\Pi_{\pm}(\xi)\beta = \beta \Pi_{\pm}(\xi)$. Defining $\psi_{\pm} := \Pi_{\pm}(D)\psi$, the Dirac equation can be rewritten as

$$
(-i\partial_t \pm |D|)\psi_{\pm} = -M\beta\psi_{\mp} + \Pi_{\pm}(\beta(\psi_+ + \psi_-), \psi_+ + \psi_-)\beta(\psi_+ + \psi_-) \quad (3)
$$

The initial condition is transformed into

$$
\psi_{\pm}(0) = \Pi_{\pm}(D)\psi_0. \quad (4)
$$

We consider the integral equations belonging to the Cauchy problem (3)(4):

$$
\psi_{\pm}(t) = e^{it|D|}\psi_{\pm}(0) - i \int_0^t e^{i(t-s)|D|}\Pi_{\pm}(D)(\beta(\Pi_{\pm}(D)\psi_+(s) + \Pi_{-\pm}(D)\psi_-(s))

\Pi_{\pm}(D)\psi_+(s) + \Pi_{-\pm}(D)\psi_-(s))\beta(\Pi_{\pm}(D)\psi_+(s) + \Pi_{-\pm}(D)\psi_-(s))ds

+ iM \int_0^t e^{i(t-s)|D|}\beta\psi_{\mp}(s)ds. \quad (5)
$$

We remark that any solution of this system automatically fulfills $\Pi_{\pm}(D)\psi_{\pm} = \psi_{\pm}$, because applying $\Pi_{\pm}(D)$ to the right hand side of (5) gives $\Pi_{\pm}(D)\psi_\pm(0) = \psi_\pm(0)$ and the integral terms also remain unchanged, because $\Pi_{\pm}(D)^2 = \Pi_{\pm}(D)$ and $\Pi_{\pm}(D)\beta\psi_{\mp}(s) = \beta\Pi_{\pm}(D)\psi_\mp(s) = \beta\psi_{\mp}(s)$. Thus $\Pi_{\pm}(D)\psi_{\pm}$ can be replaced by $\psi_{\pm}$, thus the system of integral equations reduces exactly to the one belonging to our Cauchy problem (3)(4).

We use the following function spaces and notation. Let $\tilde{\cdot}$ denote the Fourier transform with respect to space and $\cdot$ the Fourier transform and its inverse, respectively, with respect to space and time simultaneously. The standard spaces of Bougain-Klainerman-Machedon type belonging to the half waves are defined by the completion of $S([\mathbb{R} \times \mathbb{R}^2])$ with respect to

$$
\|f\|_{X^s_{\pm}^b} = \|U_{\pm}(-t)f\|_{H^s; H^b_\mp} = \|\langle \xi \rangle^s(\tau \pm |\xi|)^b\tilde{f}(\tau, \xi)\|_{L^2_{\tau, \xi}}
$$

where

$$
U_{\pm}(t) := e^{it|D|} \text{ and } \|g\|_{H^s; H^b_{\mp}} = \|\langle \xi \rangle^s(\tau)^b\tilde{g}(\xi, \tau)\|_{L^2_{\tau, \xi}}.
$$

We also define $X^s_{\pm}^b[0,T]$ as the space of restrictions of functions in $X^s_{\pm}^b$ to the time interval $[0,T]$ with norm $\|f\|_{X^s_{\pm}^b[0,T]} = \inf_{\tilde{f}[0,T]=f} \|\tilde{f}\|_{X^s_{\pm}^b}$.

We use the Strichartz estimates for the homogeneous wave equation in $\mathbb{R}^n \times \mathbb{R}$, which can be found e.g. in Ginibre-Velo [GV], Prop. 2.1.

**Proposition 1.1.** Let $\gamma(r) = (n - 1)(\frac{1}{2} - \frac{1}{r})$, $\delta(r) = n(\frac{1}{2} - \frac{1}{r})$, $n \geq 2$. Let $\rho, \mu \in \mathbb{R}$, $2 \leq q \leq \infty$, $2 \leq r < \infty$ satisfy $0 \leq \frac{\rho}{q} \leq \min(\gamma(r), 1)$, $\left(\frac{\rho}{q}, \gamma(r)\right) \neq (1,1)$.
\( \rho + \delta(r) - \frac{1}{q} = \mu. \) Then
\[
\|e^{\pm it|D|}u_0\|_{L^q(R, H^s_x(R^n))} \leq c\|u_0\|_{H^s(R^n)}.
\]

Fundamental for our results are the following bilinear Strichartz type estimates, which we state for the two-dimensional case.

**Proposition 1.2.** With the notation \(|D|^\alpha f(\xi) = |\xi|^\alpha \hat{f}(\xi)\) and \(|D|^\alpha F(\tau, \xi) = |\tau| - |\xi|^\alpha \hat{F}(\tau, \xi)\) the following estimate holds for independent signs \(\pm\) and \(\pm_1:\)
\[
\|D|^{\beta} D_-^\alpha \left(e^{\pm it|D|} u_0 e^{\pm i\tau|D|} v_0\right)\|_{L^2(R \times R^2)} \lesssim \|u_0\|_{H^{\alpha_1}(R^2)} \|v_0\|_{H^{\alpha_2}(R^2)}
\]
if and only if the following conditions are satisfied:
\(\beta_0 + \beta_- = \alpha_1 + \alpha_2 - \frac{1}{2}\), \(\beta_- \geq \frac{1}{4}\), \(\beta_0 > -\frac{1}{2}\), \(\alpha_1 \leq \beta_- + \frac{1}{2}\) \((i = 1, 2)\), \(\alpha_1 + \alpha_2 \geq \frac{1}{2}\), \((\alpha_i, \beta_-) \neq (\frac{4}{5}, \frac{1}{2})\) \((i = 1, 2)\), \((\alpha_1 + \alpha_2, \beta_-) \neq (\frac{5}{4}, \frac{1}{2})\).

**Proof.** [FK], Theorem 1.1

The so-called transfer principle immediately implies

**Corollary 1.1.** Under the assumptions of the proposition the following estimate holds:
\[
\|D|^{\beta} D_-^\alpha (fg)\|_{L^2(R \times R^2)} \lesssim \|D|^{\alpha_1} f\|_{X^{\alpha_1, \frac{1}{2}+}} \|D|^{\alpha_2} g\|_{X^{\alpha_2, \frac{1}{2}+}}.
\]

We also need the following improvement for products of the type \((+,+)\) and \((-,-)\):

**Proposition 1.3.** The following estimate holds for equal signs:
\[
\|D|^{\beta} D_-^\alpha \left(e^{\pm it|D|} u_0 e^{\pm i\tau|D|} v_0\right)\|_{L^2(R \times R^2)} \lesssim \|u_0\|_{H^{\alpha_1}(R^2)} \|v_0\|_{H^{\alpha_2}(R^2)},
\]
under the assumptions \(\beta_0 = \alpha_1 + \alpha_2 - \frac{1}{2}\), \(\alpha_1, \alpha_2 < \frac{1}{2}\), \(\alpha_1 + \alpha_2 > \frac{1}{4}\).

**Proof.** [S], Theorem 6(b) or [FK], Theorem 12.1 (see also [AFS], formula (15)).

**Corollary 1.2.** Under the assumptions of the proposition the following estimate holds:
\[
\|D|^{\beta} D_-^\alpha (fg)\|_{L^2(R \times R^2)} \lesssim \|D|^{\alpha_1} f\|_{X^{\alpha_1, \frac{1}{2}+}} \|D|^{\alpha_2} g\|_{X^{\alpha_2, \frac{1}{2}+}}.
\]

The main result reads as follows:

**Theorem 1.1.** The Cauchy problem for the Dirac equation \(1\), \(2\) has a unique local solution \(\psi\) for data \(\psi_0 \in H^s(R^2)\), if \(s > 1/2\). More precisely there exists a \(T > 0\) and a unique solution
\[
\psi \in X_+^{s, \frac{1}{2}+}[0, T] + X_-^{s, \frac{1}{2}+}[0, T].
\]
This solution has the property
\[
\psi \in C^0([0, T], H^s(R^2)).
\]
We also get the following uniqueness result.

**Theorem 1.2.** The solution of Theorem 1.1 is (unconditionally) unique in the space \(C^0([0, T], H^s(R^2))\), if \(s > 3/4\).

We use the following well-known linear estimates (cf. e.g. [AFS], Lemma 5).
Proposition 1.4. Let $1/2 < b \leq 1$, $s \in \mathbb{R}$, $0 < T \leq 1$ and $0 \leq \delta \leq 1 - b$. The Cauchy problem

$$(-i\partial_t \pm |D|)\psi_{\pm} = F, \quad \psi_{\pm}(0) = f$$

for data $F \in X^{s,b-1+\delta}[0,T]$ and $f \in H^s$ has a unique solution $\psi_{\pm} \in X^{s,b}[0,T]$. It fulfills

$$\|\psi_{\pm}\|_{X^{s,b}[0,T]} \lesssim \|f\|_{H^s} + T^\delta \|F\|_{X^{s,b-1+\delta}[0,T]}$$

with an implicit constant independent of $T$.

Finally we use the following notation: $\langle \rangle = 1 + |\cdot |$. For $a \in \mathbb{R}$ and $\epsilon > 0$ we denote by $a +, a + +, a -, a -$ numbers with $a - \epsilon < a - - < a - a < a + < a + + < a + \epsilon$.

2. PROOF OF THE THEOREMS

Proof of Theorem 1.1 Using Prop. 1.4 a standard application of the contraction mapping principle reduces the proof to the estimates for the nonlinearity in the following Proposition 2.1.

Proposition 2.1. For any $\epsilon > 0$ the following estimate holds:

$$\|\Pi_{\pm}(\langle \beta \Pi_{\pm} \psi_1, \Pi_{\pm} \psi_2 \rangle \beta \Pi_{\pm} \psi_3)\|_{X^{s,b-1+\delta}} \lesssim \prod_{i=1}^3 \|\psi_i\|_{X^{s,b-1+\delta}}.$$ Here and in the following $\pm, \pm_1, \pm_2, \pm_3$ denote independent signs.

The null structure of the Dirac equation has the following consequences (we here follow closely [AFS] and [AFS1]). Denoting

$$\sigma_{\pm, \pm_3}(\eta, \zeta) := \Pi_{\pm_3}(\zeta) \beta \Pi_{\pm}(\eta) = \beta \Pi_{\pm_3}(\zeta) \Pi_{\pm}(\eta),$$

we remark that by orthogonality this quantity vanishes if $\pm \eta$ and $\pm_3 \zeta$ line up in the same direction whereas in general (cf. [AFS1], Lemma 1):

Lemma 2.1.

$$\sigma_{\pm, \pm_3}(\eta, \zeta) = O(\angle(\pm \eta, \pm_3 \zeta)),$$

where $\angle(\eta, \zeta)$ denotes the angle between the vectors $\eta$ and $\zeta$.

Consequently we get

$$\langle \langle \beta \Pi_{\pm_3}(D) \psi_3, \Pi_{\pm}(D) \psi_0 \rangle \rangle(\tau, \xi)\rangle$$

$$\leq \int \langle |\beta \Pi_{\pm_3}(\eta)| \bar{\psi}_3(\lambda, \eta), \Pi_{\pm}(\eta - \xi) \bar{\psi}_0(\lambda - \tau, \eta - \xi) \rangle d\lambda d\eta$$

$$= \int \langle |\Pi_{\pm}(\eta - \xi)| \beta \Pi_{\pm_3}(\eta) \bar{\psi}_3(\lambda, \eta), \bar{\psi}_0(\lambda - \tau, \eta - \xi) \rangle d\lambda d\eta$$

$$\lesssim \int \Theta_{\pm_1, \pm_2} \bar{\psi}_1(\lambda, \eta) \|\bar{\psi}_2(\lambda - \tau, \eta - \xi)\| d\lambda d\eta,$$

where we denote $\Theta_{\pm_1, \pm_2} = \angle(\pm_1 \eta, \pm (\eta - \xi))$ and $\bar{\Theta}_{\pm_1, \pm_2} = \angle(\pm_2 \zeta, \pm_1 (\zeta - \xi))$.

We also need the following elementary estimates which can be found in [AFS], section 5.1 or [GP], Lemma 3.2 and Lemma 3.3.

Lemma 2.2. Denoting

$$A = |\tau| - |\xi|, B_{\pm} = \lambda \pm |\eta|, C_{\pm} = \lambda - \tau \pm |\eta - \xi|, \Theta_{\pm} = \angle(\eta, \pm (\eta - \xi))$$

and

$$\rho_+ = |\xi| - ||\eta| - |\eta - \xi||, \rho_- = ||\eta| + |\eta - \xi| - |\xi||$$

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the following estimates hold:
\[ \Theta_+^2 \sim \frac{|\xi| \rho_+}{|\eta||\eta - \xi|}, \quad \Theta_-^2 \sim \frac{(|\eta| + |\eta - \xi|) \rho_-}{|\eta||\eta - \xi|} \sim \rho_- \min(|\eta|, |\eta - \xi|) \]
as well as
\[ \rho_\pm \leq 2 \min(|\eta|, |\eta - \xi|) \]
and
\[ \rho_\pm \leq |A| + |B_\pm| + |C_\pm| \]
as well as
\[ \rho_\pm \leq |A| + |B_-| + |C_\mp| . \]
Similarly we define
\[ D_\pm = \sigma \pm |\xi|, \quad E_\pm = \sigma - \tau \pm |\zeta - \xi|, \quad \varpi_\pm = \langle (\zeta, \pm(\zeta - \xi)) \rangle \]
and
\[ \varpi_+ = |\xi| - |\zeta| - |\zeta - \xi|, \quad \varpi_- = |\zeta| + |\zeta - \xi| - |\xi| \]
then the following estimates hold:
\[ \varpi_+^2 \sim \frac{|\xi| \varpi_+}{|\zeta||\zeta - \xi|}, \quad \varpi_-^2 \sim \frac{(|\xi| + |\zeta - \xi|) \varpi_-}{|\zeta||\zeta - \xi|} \sim \varpi_- \min(|\eta|, |\eta - \xi|) \]
as well as
\[ \varpi_\pm \leq 2 \min(|\xi|, |\zeta - \xi|) \]
and
\[ \varpi_\pm \leq |A| + |D_\pm| + |E_\pm| \]
as well as
\[ \varpi_\pm \leq |A| + |D_-| + |E_\mp| . \]

**Proof of Prop. 2.1.** The claim of the proposition is equivalent to the estimate
\[ \left| \int \langle \langle \beta \Pi_{\pm,1} \psi_1, \Pi_{\pm,2} \psi_2 \rangle \beta \Pi_{\pm,3} \psi_3, \Pi_{\pm} \psi_0 \rangle dx dt \right| \lesssim \prod_{i=1}^3 \| \psi_i \|_{X^{\frac{1}{2},\frac{1}{2}}} \| \psi_0 \|_{X^{\frac{1}{2},\frac{1}{2}}}. \]
The left hand side equals
\[ \left| \int \langle \beta \Pi_{\pm,1} \psi_1, \Pi_{\pm,2} \psi_2 \rangle \beta \Pi_{\pm,3} \psi_3, \Pi_{\pm} \psi_0 \rangle d\tau d\xi \right|. \]
Using (6) it is thus sufficient to prove
\[ \left| \int \Theta_{\pm,\pm,3} \tilde{\psi}_3(\lambda, \eta) \tilde{\psi}_0(\lambda - \tau, \eta - \xi) \varpi_{\pm,1, \pm,2} \tilde{\psi}_1(\sigma, \zeta) \tilde{\psi}_2(\sigma - \tau, \zeta - \xi) d\sigma d\zeta d\tau d\xi d\eta d\lambda \right| \lesssim \prod_{i=1}^3 \| \psi_i \|_{X^{\frac{1}{2},\frac{1}{2}}} \| \psi_0 \|_{X^{\frac{1}{2},\frac{1}{2}}}, \]
where we assume w.l.o.g. that the Fourier transforms are nonnegative. Defining
\[ \tilde{F}_j(\lambda, \eta) := \langle \eta \rangle^{\frac{1}{2}+\epsilon}(\lambda \pm j |\eta|)^{\frac{1}{2}+\epsilon} \tilde{\psi}_j(\lambda, \eta) \quad (j = 1, 2, 3) \]
\[ \tilde{F}_0(\lambda, \eta) := \langle \eta \rangle^{-\frac{1}{2}-\epsilon}(\lambda \pm |\eta|)^{-\frac{1}{2}-\epsilon} \tilde{\psi}_0(\lambda, \eta) \]

\[ \Theta_{\pm,\pm,3} \tilde{\psi}_3(\lambda, \eta) \tilde{\psi}_0(\lambda - \tau, \eta - \xi) \varpi_{\pm,1, \pm,2} \tilde{\psi}_1(\sigma, \zeta) \tilde{\psi}_2(\sigma - \tau, \zeta - \xi) d\sigma d\zeta d\tau d\xi d\eta d\lambda \]
we thus have to show
\[ I := \int \Theta_{\pm, \pm} \frac{\tilde{F}_3(\lambda, \eta) \tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{1}{2} +}}{(\eta)^{\frac{1}{2} - + e} (B_{\pm}) \frac{1}{2} +} \langle C_{\pm} \frac{1}{2} - \rangle \Theta_{\pm, \pm} \frac{\tilde{F}_1(\sigma, \zeta) \tilde{F}_2(\sigma - \tau, \zeta - \xi)(\zeta - \xi)^{\frac{1}{2} + + e}}{(\zeta)^{\frac{1}{2} + + e} (D_{\pm}) \frac{1}{2} +} \langle E_{\pm} \frac{1}{2} + \rangle \quad d\sigma d\zeta d\xi d\eta d\lambda \]
\[ \lesssim \prod_{i=0}^{3} \| F_i \|_{L^2_t}. \]  
(7)

In order to prove (7) let us first of all consider the low frequency case \(|\eta - \xi| \leq 1\). We simply use \(|\Theta_{\pm, \pm}|, |\Theta_{\pm, \pm}| \lesssim 1\) and estimate crudely
\[ I \leq \left\| \left( \frac{\tilde{F}_3(\lambda, \eta)}{(\eta)^{\frac{1}{2} - + e} (B_{\pm}) \frac{1}{2} +} \right) \right\|_{L^2_t} \left\| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{1}{2} +}}{(\eta)^{\frac{1}{2} - + e} (B_{\pm}) \frac{1}{2} +} \right\|_{L^2_t} \left\| \frac{\tilde{F}_1(\sigma, \zeta) \tilde{F}_2(\sigma - \tau, \zeta - \xi)(\zeta - \xi)^{\frac{1}{2} + + e}}{(\zeta)^{\frac{1}{2} + + e} (D_{\pm}) \frac{1}{2} +} \right\|_{L^2_t} \leq \prod_{i=0}^{3} \| F_i \|_{L^2_t} \]
using Strichartz’ estimate \(|e^{it|P|}|u_0||_{L^2_t} \lesssim \| u_0 \|_{H^\frac{1}{2}}\) (Prop. 1.1).

From now on we assume \(|\eta - \xi| \geq 1\). The estimates for \(I\) depend on the different signs which have to be considered.

**Part 1:** We start with the case where all the signs are \(\pm, \pm, \pm, \pm\) and + -signs. Analogously one can treat all the cases as \(\pm, \pm, \pm, \pm\) as well as \(+, +, +, +\) and the same sign. Besides the trivial bounds \(\Theta_{+,-}, \Theta_{+,+} \lesssim 1\) we make in the following repeated use of the following estimates which immediately follow from Lemma 2.2

\[ \Theta_{+,-} \lesssim \frac{|\xi|^\frac{1}{2}}{|\eta|^\frac{1}{2} |\eta - \xi|^\frac{1}{2}} (|A|^\frac{1}{2} + |B|^\frac{1}{2}^+ + |C|^\frac{1}{2}^+), \quad (8) \]
\[ \Theta_{+,+} \lesssim \frac{|\xi|^\frac{1}{2}}{|\eta|^\frac{1}{2} |\eta - \xi|^\frac{1}{2}} (|A|^\frac{1}{2} + |D|^\frac{1}{2}^+ + |E|^\frac{1}{2}^+). \quad (9) \]

In the case \(|C| \geq |A|, |B|\) we also use
\[ \Theta_{+,-} \lesssim \frac{|\xi|^\frac{1}{2}}{|\eta|^\frac{1}{2} |\eta - \xi|^\frac{1}{2}} (|C|^\frac{1}{2}^+ - \min(|\eta|, |\eta - \xi|)^{0^+}. \quad (10) \]

We consider several cases depending on the relative size of the terms in the right hand sides of (8) and (9). We may assume by symmetry in (7) that for the rest of the proof we have \(|D_{\pm}| \geq |E_{\pm}|\), which reduces the number of cases.

**Case 1:** \(|B_{\pm}| \geq |A|, |C_{\pm}|\) and \(|D_{\pm}| \geq |A|, |E_{\pm}|\).

**Case 1.1:** \(\langle C_{\pm} \rangle \leq |\xi|\).

**Case 1.1.1:** \(|\xi| \ll |\eta| \Rightarrow |\eta - \xi| \sim |\eta|\).

Using \(\Theta_{+,+} \lesssim \Theta_{+,+}^{-}\) we obtain
\[ I \lesssim \int \left( \frac{\tilde{F}_3(\lambda, \eta)}{(\eta)^{\frac{1}{2} + + e} (B_{\pm}) \frac{1}{2} +} \right) \left( \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{1}{2} +}}{(\eta)^{\frac{1}{2} - + e} (B_{\pm}) \frac{1}{2} +} \right) \left( \frac{\tilde{F}_1(\sigma, \zeta) \tilde{F}_2(\sigma - \tau, \zeta - \xi)(\zeta - \xi)^{\frac{1}{2} + + e}}{(\zeta)^{\frac{1}{2} + + e} (D_{\pm}) \frac{1}{2} +} \right) \left( \frac{|\xi|^\frac{1}{2}}{|\eta|^\frac{1}{2} |\eta - \xi|^\frac{1}{2}} \right) \quad d\sigma d\zeta d\xi d\eta d\lambda. \]

To reduce the number of cases we always assume \(|\xi| \geq |\zeta - \xi|\), because the alternative case can be treated similarly. Thus we have \(|\xi| \lesssim |\zeta|\). We obtain the
Case 1.2.1: For the first factor we used the Sobolev estimate \( \|f\|_{L^2 L^\infty} \lesssim \|f\|_{X^1_{+,0}} \) and the estimate \( \|f\|_{L^1_+ L^\infty} \leq \|f\|_{L^1_+ H^1_{\infty,-}} \lesssim \|f\|_{X^1_{+,1}} \), which follows from Sobolev’s embedding and Strichartz’ estimate, so that an interpolation gives

\[
\|f\|_{L^1_+ L^\infty} \lesssim \|f\|_{X^1_{+,1}},
\]

which gives the desired bound for \(|\eta| \geq 1\), whereas the case \(|\eta| \leq 1\) is easy.

Case 1.2: \( \langle C_+ \rangle \geq |\xi| \).

\[
I \lesssim \int_\mathbb{R} (\hat{F}_3(\lambda, \eta) \hat{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^{\frac{1}{2} + \epsilon}) \langle \xi \rangle^{\frac{1}{2} -} \frac{d\sigma d\tau d\xi d\eta d\lambda}{|\eta|^{\frac{1}{2} -} |\xi|^{\frac{1}{2} -}} 
\]

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Case 1.2: \( \langle C_+ \rangle \geq |\xi| \).

\[
I \lesssim \int_\mathbb{R} (\hat{F}_3(\lambda, \eta) \hat{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^{\frac{1}{2} + \epsilon}) \langle \xi \rangle^{\frac{1}{2} -} \frac{d\sigma d\tau d\xi d\eta d\lambda}{|\eta|^{\frac{1}{2} -} |\xi|^{\frac{1}{2} -}} 
\]

Estimating \(|\eta - \xi| \leq |\eta| + |\xi|\) we consider two different cases.

Case 1.2.1: \( |\eta| \geq |\xi| \).

In this case we obtain

\[
I \lesssim \|f\|_{L^2 L^\infty} \|F_0\|_{L^1_+ L^2} \|F_1\|_{L^1_+ L^2} \|F_2\|_{L^1_+ L^2} \|F_3\|_{L^1_+ L^2}
\]

\[
\lesssim \prod_{i=0}^3 \|F_i\|_{L^2_{+s}}.
\]
Case 1.2.2: $|\xi| > |\eta|$.

In this case we obtain

$$I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{\delta}{2} \langle B_+ \rangle^\delta} \right\|_{L^2_x L^\infty_t} \left\| F_0 \right\|_{L^\infty_x L^2_t} \left\| \left( \frac{\tilde{F}_1(\sigma, \zeta)}{|\xi|^\frac{\delta}{2}} \right)^\tau \right\|_{L^2_x L^2_t}$$

$$\left\| \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle E_+ \rangle^\delta \langle \zeta - \xi \rangle^\alpha} \right\|_{L^\infty_t L^2_x}$$

$$\lesssim \sum_{i=0}^3 \left\| F_i \right\|_{L^2_{x_t}}.$$  

Case 2: $|B_+| > |A|, |C_+|$ and $|A| \geq |D_+|, |E_+|$.

Case 1.2: $|C_+| \leq |\xi|$.

This case is treated as follows:

$$I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{\delta}{2} \langle B_+ \rangle^\delta} \tilde{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^\epsilon |\xi|^\frac{\delta}{2} \langle C_+ \rangle^\delta \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{|\xi|^\frac{\delta}{2}} \right\|_{L^\infty_t L^2_x}$$

$$\frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle E_+ \rangle^\delta \langle \zeta - \xi \rangle^\alpha} |\xi|^\frac{\delta}{2} \left\| F_0 \right\|_{L^\infty_t L^2_x}$$

$$\left\| \left( \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{\delta}{2} \langle B_+ \rangle^\delta} \right)^\tau \right\|_{L^2_x L^2_t} \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{|\xi|^\frac{\delta}{2}} \right\|_{L^\infty_t L^2_x}$$

$$\lesssim \sum_{i=0}^3 \left\| F_i \right\|_{L^2_{x_t}}.$$  

For the last factor we used Sobolev’s embedding $H^{\beta^+}_0 \subset L^{2^+}$ first and then Corollary 1.1 with $\beta^+_0 = 0^+, \beta^-_0 = \frac{\delta}{2}$, $\alpha^+_1 = 0^+, \alpha^-_2 = 1$.

Case 2.1.2: $|\xi| > |\eta|$ ($\Rightarrow |\xi| \sim |\eta - \xi|$).

We obtain

$$I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{\delta}{2} \langle B_+ \rangle^\delta} \tilde{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^\epsilon |\xi|^\frac{\delta}{2} \langle C_+ \rangle^\delta \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{|\xi|^\frac{\delta}{2}} \right\|_{L^\infty_t L^2_x}$$

$$\frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle E_+ \rangle^\delta \langle \zeta - \xi \rangle^\alpha} |\xi|^\frac{\delta}{2} \left\| F_0 \right\|_{L^\infty_t L^2_x}$$

$$\left\| \left( \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{\delta}{2} \langle B_+ \rangle^\delta} \right)^\tau \right\|_{L^2_x L^2_t} \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{|\xi|^\frac{\delta}{2}} \right\|_{L^\infty_t L^2_x}$$

$$\lesssim \sum_{i=0}^3 \left\| F_i \right\|_{L^2_{x_t}}.$$  

Here we used (1.1) for the first factor and Cor. 1.1 with $\beta^+_0 = 0^+, \beta^-_0 = \frac{\delta}{2}$, $\alpha^+_1 = 0$, $\alpha^-_2 = 1$ for the last factor.

Case 2.2: $|C_+| \geq |\xi|$.
We obtain

\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon}} \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta - \xi \rangle^{\frac{1}{2}+\epsilon}} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2}+\epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2}+\epsilon}} \langle \xi \rangle^{\frac{1}{2}} \frac{|\xi|^{\frac{1}{2}}}{|A|^{\frac{1}{2}}} d\sigma d\zeta d\xi d\eta d\lambda.
\]

We use our assumption $|\xi| \geq |\zeta - \xi|$, so that $|\xi| \lesssim |\zeta|$, and estimate $\langle \eta - \xi \rangle \leq \langle \eta \rangle + \langle \zeta \rangle$.

**Case 2.2.1:** $|\eta| \geq |\xi|$.

In this case we obtain

\[
I \lesssim \| \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^4_t} \| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta - \xi \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^4_t} \| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^4_t} \| \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^4_t} < \| F_3 \|_{L^2_x} \| F_0 \|_{L^2_x} \| F_1 \|_{L^2_x} \| F_2 \|_{L^2_x}.
\]

For the last factor we applied Sobolev’s embedding and Cor. \[\text{\ref{SobolevEmbedding}}\] with $\beta_0 = 0$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = 1$ to obtain the bound $\| F_1 \|_{L^2_x} \| F_2 \|_{L^2_x}$ for it.

**Case 2.2.2:** $|\xi| \geq |\eta|$ ($\Rightarrow |\eta - \xi| \lesssim |\xi| \lesssim |\zeta|$).

We arrive at

\[
I \lesssim \| \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^{4+}_t} \| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta - \xi \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^{4+}_t} \| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^{4+}_t} \| \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^{4+}_t}.
\]

We estimate the first factor using Sobolev by $\| F_3 \|_{L^2_x}$ and the last factor by Sobolev and Cor. \[\text{\ref{SobolevEmbedding}}\] with $\beta_0 = 0$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = 1$ by

\[
\| \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon}} \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta - \xi \rangle^{\frac{1}{2}+\epsilon}} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2}+\epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_x L^{4+}_t} \lesssim \| F_3 \|_{L^2_x} \| F_0 \|_{L^2_x} \| F_1 \|_{L^2_x} \| F_2 \|_{L^2_x}.
\]

**Case 3:** $|A| \geq |B_+|, |C_+|$ and $|D_+| \geq |A|, |E_+|$.

**Case 3.1:** $|C_+| \leq |\eta|$.
Using $|\xi| \leq |\zeta| + |\zeta - \xi| \lesssim |\zeta|$ we obtain

\[
I \lesssim \int \frac{\hat{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon} \langle B_+ \rangle^{\frac{1}{2}+\epsilon}} \frac{\hat{F}_0(\lambda - \tau, \eta - \xi)}{\langle C_+ \rangle^{\frac{1}{2}+\epsilon}} |A|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} d\sigma d\zeta d\xi d\eta d\lambda
\]

\[
= \int \frac{\hat{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2}+\epsilon} \langle E_+ \rangle^{\frac{1}{2}+\epsilon}} \frac{\hat{F}_2(\sigma - \tau, \zeta - \xi)}{\langle C_+ \rangle^{\frac{1}{2}+\epsilon}} |A|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} d\sigma d\zeta d\xi d\eta d\lambda.
\]

Now we have $\langle \eta - \xi \rangle^\epsilon \lesssim \langle \eta \rangle^\epsilon + \langle \zeta \rangle^\epsilon$.

**Case 3.1.1**: $|\eta| \geq |\zeta|$.

We arrive at

\[
I \lesssim \int \frac{\hat{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon} \langle B_+ \rangle^{\frac{1}{2}+\epsilon}} \frac{\hat{F}_0(\lambda - \tau, \eta - \xi)}{\langle C_+ \rangle^{\frac{1}{2}+\epsilon}} |A|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} d\sigma d\zeta d\xi d\eta d\lambda
\]

\[
= \int \frac{\hat{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2}+\epsilon} \langle E_+ \rangle^{\frac{1}{2}+\epsilon}} \frac{\hat{F}_2(\sigma - \tau, \zeta - \xi)}{\langle C_+ \rangle^{\frac{1}{2}+\epsilon}} |A|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} \langle \eta \rangle^{\epsilon} d\sigma d\zeta d\xi d\eta d\lambda.
\]

where we used Cor. 1.1 with $\beta_0 = -\epsilon$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 1 - \epsilon$, $\alpha_2 = 0$ for the first factor.

**Case 3.1.2**: $|\xi| \geq |\eta|$.

An application of Cor. 1.1 with $\beta_0 = 0$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 1$, $\alpha_2 = 0$ gives the estimate

\[
I \lesssim \int \frac{\hat{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon} \langle B_+ \rangle^{\frac{1}{2}+\epsilon}} \frac{\hat{F}_0(\lambda - \tau, \eta - \xi)}{\langle C_+ \rangle^{\frac{1}{2}+\epsilon}} |A|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} d\sigma d\zeta d\xi d\eta d\lambda
\]

\[
= \int \frac{\hat{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2}+\epsilon} \langle E_+ \rangle^{\frac{1}{2}+\epsilon}} \frac{\hat{F}_2(\sigma - \tau, \zeta - \xi)}{\langle C_+ \rangle^{\frac{1}{2}+\epsilon}} |A|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} \langle \zeta \rangle^{\epsilon} d\sigma d\zeta d\xi d\eta d\lambda.
\]

**Case 2**: $|C_+| \geq |\eta|$.

We obtain using again our tacit assumption $|\zeta| \geq |\zeta - \xi|$

\[
I \lesssim \int \frac{\hat{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2}+\epsilon} \langle B_+ \rangle^{\frac{1}{2}+\epsilon}} \frac{\hat{F}_0(\lambda - \tau, \eta - \xi)}{\langle C_+ \rangle^{\frac{1}{2}+\epsilon}} |A|^{\frac{1}{2}} |\xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} |\eta - \xi|^{\frac{1}{2}} d\sigma d\zeta d\xi d\eta d\lambda.
\]
Case 3.2.1: $|\eta| \geq |\xi|$.
We estimate as follows:

\[
I \lesssim \|(\frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2}+\eps}}(B_+)_{\frac{1}{2}+})^\dagger\|_{L^\infty_t L^1_x} \|F_0\|_{L^2_{tx}} \|\tilde{F}_1(\sigma, \xi)\xi\|_{L^\infty_t L^1_x}^\dagger
\]

\[
\lesssim \prod_{i=0}^3 \|F_i\|_{L^2_{tx}}.
\]

Case 3.2.2: $|\eta| \leq |\xi|$.
In this case we obtain

\[
I \lesssim \|(\frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{1}{2}+\eps}(B_+)_{\frac{1}{2}+})^\dagger\|_{L^\infty_t L^1_x} \|F_0\|_{L^2_{tx}} \|\tilde{F}_1(\sigma, \xi)\xi\|_{L^\infty_t L^1_x}^\dagger
\]

\[
\lesssim \prod_{i=0}^3 \|F_i\|_{L^2_{tx}}.
\]

Case 4: $|C_+| \geq |A|, |B_+|$ and $|D_+| \geq |A|, |E_+|$.
Using $\Theta_{+,+} \lesssim |\frac{\xi}{|\eta|^\frac{1}{2}+\eps} C_+|^{\frac{1}{2}+}$ and $|\xi| \leq |\xi - \xi| \lesssim |\xi|$ we obtain

\[
I \lesssim \int \left( \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{1}{2}+\eps}(B_+)_{\frac{1}{2}+} \right) \frac{F_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{1}{2}+\eps}}{|\eta|^\frac{1}{2}+\eps} \frac{|\xi|^{\frac{1}{2}+\eps}}{|\eta|^\frac{1}{2}+\eps} d\sigma d\tau d\xi d\eta d\lambda.
\]

Case 4.1: $|\xi| \ll |\eta - \xi| \sim |\eta|$.
We conclude

\[
I \lesssim \|(\frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{1}{2}+\eps}(B_+)_{\frac{1}{2}+})^\dagger\|_{L^\infty_t L^1_x} \|F_0\|_{L^2_{tx}} \|\tilde{F}_1(\sigma, \xi)\xi\|_{L^\infty_t L^1_x}^\dagger
\]

\[
\lesssim \prod_{i=0}^3 \|F_i\|_{L^2_{tx}}.
\]

Case 4.2: $|\xi| \geq |\eta|$ ($\Rightarrow |\eta - \xi| \lesssim |\xi| \lesssim |\xi|$).
Similarly as before we obtain

\[
I \lesssim \|(\frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^\frac{1}{2}+\eps}(B_+)_{\frac{1}{2}+})^\dagger\|_{L^\infty_t L^1_x} \|F_0\|_{L^2_{tx}} \|F_1\|_{L^2_{tx}}
\]

\[
\lesssim \prod_{i=0}^3 \|F_i\|_{L^2_{tx}}.
\]
Case 5: \( |C_+| \geq |A|, |B_+| \) and \( |A| \geq |D_+|, |E_+| \).

In this case we estimate as follows:

\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (B_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{3}{2} + \epsilon}}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (C_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (D_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (E_+)^{\frac{3}{2} + \epsilon}} \langle r \rangle - |\xi|^{\frac{3}{2} + \epsilon} d\sigma d\tau d\zeta d\eta d\lambda.
\]

Estimating \( \langle \eta - \xi \rangle^\epsilon \lesssim \langle \xi \rangle^\epsilon + \langle \eta \rangle^\epsilon \) we consider two subcases.

**Case 5.1**: \( |\xi| \geq |\eta| \) \( \Rightarrow \langle \eta - \xi \rangle^\epsilon \lesssim \langle \xi \rangle^\epsilon \).

Thus we obtain

\[
I \lesssim \frac{1}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (B_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (C_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (D_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (E_+)^{\frac{3}{2} + \epsilon}} \langle r \rangle - |\xi|^{\frac{3}{2} + \epsilon} d\sigma d\tau d\zeta d\eta d\lambda.
\]

Case 5.2: \( |\eta| \geq |\xi| \).

In this case we arrive at

\[
I \lesssim \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (B_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (C_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (D_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (E_+)^{\frac{3}{2} + \epsilon}} \langle r \rangle - |\xi|^{\frac{3}{2} + \epsilon} d\sigma d\tau d\zeta d\eta d\lambda.
\]

where we used the Sobolev embedding \( H^{\frac{3}{2} + \epsilon}_x \subset L^{2^+}_x \) for the last factor and then Cor. [11] with \( \beta_0 = 0^+ \), \( \beta_\epsilon = \frac{1}{2} \), \( \alpha_1 = 0^+ \), \( \alpha_2 = 1 \).

Case 6: \( |A| \geq |C_+|, |B_+| \) and \( |A| \geq |D_+|, |E_+| \).

**Case 6.1**: \( |C_+| \leq |\eta| \).

We estimate

\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (B_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{3}{2} + \epsilon}}{\langle \eta \rangle^{\frac{3}{2} + \epsilon} (C_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (D_+)^{\frac{3}{2} + \epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \xi \rangle^{\frac{3}{2} + \epsilon} (E_+)^{\frac{3}{2} + \epsilon}} \langle r \rangle - |\xi|^{\frac{3}{2} + \epsilon} d\sigma d\tau d\zeta d\eta d\lambda.
\]
Case 6.1.1: $|\eta| \geq |\xi|$. We obtain

$$I \lesssim \int \frac{\hat{F}_3(\lambda, \eta)}{(\eta)\beta|x-(B_+)^{\frac{\alpha}{2}}(C_+)\frac{\alpha}{2}} \frac{\hat{F}_0(\lambda-\tau, \eta-\xi)}{(C_+)^{\frac{\alpha}{2}}} ||\tau| - |\xi||\frac{\alpha}{2} \left(\frac{\hat{F}_1(\sigma, \zeta)}{(D_+)\frac{\alpha}{2}} \frac{\hat{F}_2(\sigma-\tau, \zeta-\xi)}{(E_+)\frac{\alpha}{2}} ||\tau| - |\xi||\frac{\alpha}{2}\right) d\sigma d\zeta d\tau d\eta d\lambda.$$ 

We estimate both factors in $L^2_{xt}$ using Cor. \[\text{with } \beta_0 = 0^- , \beta_- = \frac{1}{2} , \alpha_1 = 1^- , \alpha_2 = 0 \text{ and } \beta_0 = 0 , \beta_- = \frac{1}{2} , \alpha_1 = 0 , \alpha_2 = 1 , \text{ respectively.}

Case 6.1.2: $|\eta| \leq |\xi|$. We obtain

$$I \lesssim \int \frac{\hat{F}_3(\lambda, \eta)}{(\eta)\beta|x-(B_+)^{\frac{\alpha}{2}}(C_+)\frac{\alpha}{2}} \frac{\hat{F}_0(\lambda-\tau, \eta-\xi)}{(C_+)^{\frac{\alpha}{2}}} ||\tau| - |\xi||\frac{\alpha}{2} \left(\frac{\hat{F}_1(\sigma, \zeta)}{(D_+)\frac{\alpha}{2}} \frac{\hat{F}_2(\sigma-\tau, \zeta-\xi)}{(E_+)\frac{\alpha}{2}} ||\tau| - |\xi||\frac{\alpha}{2}\right) d\sigma d\zeta d\tau d\eta d\lambda.$$ 

As in case 6.1.1 we estimate both factors in $L^2_{xt}$ using Cor. \[\text{with } \beta_0 = 0 , \beta_- = \frac{1}{2} , \alpha_1 = 1 , \alpha_2 = 0 \text{ and } \beta_0 = 0 , \beta_- = \frac{1}{2} , \alpha_1 = 0 , \alpha_2 = 1 , \text{ respectively.}

Case 6.2: $|C_+| \geq |\eta|$. In this case we have

$$I \lesssim \int \frac{\hat{F}_3(\lambda, \eta)}{(\eta)\beta|x-(B_+)^{\frac{\alpha}{2}}(D_+)\frac{\alpha}{2}} \frac{\hat{F}_0(\lambda-\tau, \eta-\xi)}{(C_+)\frac{\alpha}{2}} \frac{\hat{F}_1(\sigma, \zeta)}{(D_+)\frac{\alpha}{2}} \frac{\hat{F}_2(\sigma-\tau, \zeta-\xi)}{(E_+)\frac{\alpha}{2}} ||\tau| - |\xi||\frac{\alpha}{2} d\sigma d\zeta d\tau d\eta d\lambda.$$ 

Case 6.2.1: $|\eta| \geq |\xi|$. We arrive at the bound

$$I \lesssim \|\left(\frac{\hat{F}_3(\lambda, \eta)}{(\eta)\beta|x-(B_+)^{\frac{\alpha}{2}}(D_+)\frac{\alpha}{2}} \frac{\hat{F}_0(\lambda-\tau, \eta-\xi)}{(C_+)\frac{\alpha}{2}} \frac{\hat{F}_1(\sigma, \zeta)}{(D_+)\frac{\alpha}{2}} \frac{\hat{F}_2(\sigma-\tau, \zeta-\xi)}{(E_+)\frac{\alpha}{2}} ||\tau| - |\xi||\frac{\alpha}{2}\right)\|_{L^2_{xt}}.$$ 

By Sobolev the first factor is estimated by $\|\mathcal{F}_3\|_{L^2_{xt}}$, and the last factor by Sobolev's embedding $H^{\frac{\alpha}{2}} \subset L^2_{xt}$ followed by an application of Cor. \[\text{with } \beta_0 = \frac{1}{2} , \beta_- = \frac{1}{2} , \alpha_1 = 1 , \alpha_2 = 1 , \text{ which gives the desired bound.}

Case 6.2.2: $|\eta| \leq |\xi|$. We end up with the bound

$$I \lesssim \|\frac{\hat{F}_3(\lambda, \eta)}{(\eta)\beta|x-(B_+)^{\frac{\alpha}{2}}(D_+)^{\frac{\alpha}{2}}} \|_{L^\infty_{xt}} \|\hat{F}_0\|_{L^2_{xt}} \|\frac{\hat{F}_1(\sigma, \zeta)}{(D_+)^{\frac{\alpha}{2}}} \|_{L^\infty_{xt}} \|\hat{F}_2(\sigma-\tau, \zeta-\xi)\|_{L^2_{xt}}.$$ 

Cor. \[\text{with } \beta_0 = 0 , \beta_- = \frac{1}{2} , \alpha_1 = 0 , \alpha_1 = 1 \text{ implies the desired bound.}

This completes the proof of Part I, where all the signs are + signs.

Part II: Next we consider the case $\pm_3 = + , \pm = -$ and $\pm_1 = + , \pm_2 = -$, in the same way all the cases can be treated where $\pm$ and $\pm_3$ as well as $\pm_1$ and $\pm_2$ have different signs.
We use the following estimates which immediately follow from Lemma 2.2:

\[ \Theta_{-+,} \lesssim \frac{|\eta| + |\eta - \xi|}{|\eta|^2 |\eta - \xi|^2} (|A|^{1+} + |B_+|^{1+} + \min(|\eta|, |\eta - \xi|)^0 + |C_-|^{1+}) \]  

(12)

\[ \Theta_{+-,} \lesssim \frac{(|\xi| + |\xi - \eta|)^{1+}}{|\xi|^2 |\xi - \eta|^2} (|A|^{1+} + |D_+|^{1+} + |E_-|^{1+}) \]  

(13)

We first make the important remark that we may assume in all the cases where one has different signs that concerning \( \Theta_{-+,} \):

\[ |\xi| \ll |\eta| \sim |\eta - \xi| \]  

(14)

and similarly concerning \( \Theta_{+-,} \):

\[ |\xi| \ll |\xi| \sim |\xi - \eta| . \]  

(15)

If one namely has \(|\eta| \ll |\eta - \xi|\), then \(|\xi| \sim |\eta - \xi|\), and thus the factor \( \frac{(|\eta| + |\eta - \xi|)^{1+}}{|\eta|^2 |\eta - \xi|^2} \) is equivalent to \( \frac{|\eta|^{1+}}{|\eta|^{2+} |\eta - \xi|^2} \). If \(|\eta| \gg |\eta - \xi|\), then \(|\xi| \sim |\eta|\), and the same is true, and also in the case \(|\xi| \sim |\eta| \sim |\eta - \xi|\). Thus in all these cases we have the same estimate for \( \Theta_{-+,} \) as for \( \Theta_{+-,} \), especially the estimates (13) and (14) with \( C_+ \) replaced by \( C_- \), so the same arguments in this case hold true, if (14) is violated. The same arguments work for \( \Theta_{+-,} \), especially if (19) with \( E_+ \) replaced by \( E_- \), if (15) is violated. This means that we can apply the arguments of Part I of this proof in all these cases. So for Part II we may assume (14) and (15).

**Case 1:** \(|B_+| \geq |A|, |C_-| \) and \(|D_+| \geq |A|, |E_-| \).

**Case 1.1:** \(|C_-| \leq |\eta - \xi| \sim |\eta|\).

We obtain

\[
I \lesssim \int \tilde{F}_3(\lambda, \eta) \tilde{F}_0(\lambda - \tau, \eta - \xi) \frac{(\eta - \xi)^{1+} |\eta|^{0+}}{|\eta|^2} \frac{1}{|\eta|^2} d\sigma d\zeta d\tau d\xi d\eta d\lambda
\]

\[
\lesssim \| \left( \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^2} \right)^{1+} \|_{L^2_{x_t} L^2_{x_z}} \left( \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{|C_-|^{1+}} \right)^{1+} \|_{L^\infty_{\tau} L^2_{\eta}}
\]

\[
\lesssim \sum_{i=0}^3 \| F_i \|_{L^2_{x_t}} .
\]

**Case 1.2:** \(|C_-| \geq |\eta - \xi| \sim |\eta|\).

In this case we obtain the same bound as in Part I, Case 2.1.2 with \( E_+ \) replaced by \( E_- \).

**Case 2:** \(|C_-| \geq |A|, |B_+| \) and \(|D_+| \geq |A|, |E_-| \).

Using (12) we obtain the same estimate as in case 1.2.

**Case 3:** \(|A| \geq |B_+|, |C_-| \).

**Case 3.1:** \(|C_-| \leq |\xi|\).
We obtain

\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2}}(B_+)^{\frac{1}{2}+}} \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^{\frac{1}{2}+\epsilon}}{(C_-)^{\frac{1}{2}+}} |\tau| - |\xi|^{\frac{1}{2}} d\sigma d\tau d\xi d\eta d\lambda
\]

\[
\lesssim \int \left( \frac{\tilde{F}_1(\sigma, \zeta)}{|\sigma|^{\frac{1}{2}}(D_+)^{\frac{1}{2}+}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi) \langle \sigma - \tau \rangle^{\frac{1}{2}+\epsilon}}{(E_-)^{\frac{1}{2}+}} \right) d\sigma d\tau d\xi d\eta d\lambda.
\]

We take both factors in the $L^2_{x,t}$-norm. We remark that in the first factor the interaction is of type $(+, +)$ because of the conjugation in its second factor $F_0$ (remark that $|C_-| = |\lambda - \tau - |\eta - \xi|| = |\tau - \lambda + |\xi - \eta||$). This means that we can apply Cor. 3.2 with $\beta_0 = -\frac{1}{2}$, $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = 0$. For the second factor we apply Cor. 3.1 with $\beta_0 = \frac{1}{2} +$, $\beta_- = \frac{1}{2}$, $\alpha_1 = \frac{1}{2} +$, $\alpha_2 = \frac{1}{2}$. Thus we get the bound

\[
\prod_{i=0}^{3} \|F_i\|_{L^2_{x,t}}.
\]

Case 3.2: $|C_-| \geq |\xi|$.

We obtain

\[
I \lesssim \left( \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2}}(B_+)^{\frac{1}{2}+}} \right) \left( \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^{\frac{1}{2}+\epsilon}}{(C_-)^{\frac{1}{2}+}} \right) |\tau| - |\xi|^{\frac{1}{2}} d\sigma d\tau d\xi d\eta d\lambda.
\]

In the last step we first used Sobolev’s embedding $\dot{H}^{\frac{1}{2}} \subset L^4$ and then Cor. 3.3 with $\beta_0 = 0 +$, $\beta_- = \frac{1}{2}$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2} +$ for the last factor.

Case 4: $|B_+| \geq |A|$, $|C_-|$ and $|A| \geq |D_+|, |E_-|$.

Case 4.1: $|C_-| \leq |\xi|$.

In this case we obtain

\[
I \lesssim \left( \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2}}(B_+)^{\frac{1}{2}+}} \right) \left( \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^{\frac{1}{2}+\epsilon}}{(C_-)^{\frac{1}{2}+}} \right) |\tau| - |\xi|^{\frac{1}{2}} d\sigma d\tau d\xi d\eta d\lambda.
\]

In the last step we first used Sobolev’s embedding $\dot{H}^{\frac{1}{2}} \subset L^4$ and then Cor. 3.3 with $\beta_0 = \frac{1}{2} +$, $\beta_- = \frac{1}{2}$, $\alpha_1 = \frac{1}{2} +$, $\alpha_2 = 1$ for the last factor.

Case 4.2: $|C_-| \geq |\xi|$.
We obtain

\[ I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{\langle \tilde{B}_x \rangle x} \right\|_{L^{\frac{1}{2}} L^2_x} \left\| F_0 \right\|_{L^2_t L^2_x} \]

\[ \leq \prod_{i=0}^3 \left\| F_i \right\|_{L^2_x}. \]

In the last step the last factor is estimated by Sobolev's embedding \( \left\| f \right\|_{L^\infty_x} \lesssim \left\| (|\xi|^{1+} + |\xi|^{1+}) \tilde{f}(\xi) \right\|_{L^2_x} \) and then Cor. [14] with \( \beta_0 = \frac{1}{2} + , \beta_- = \frac{1}{2} , \alpha_1 = \frac{1}{2} , , \alpha_2 = 1. \)

**Case 5:** \(|C_-| \geq |A|, |B_+| \) and \(|A| \geq |D_+|, |E_-|\).

In this case we estimate as follows

\[ I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{\langle \tilde{B}_x \rangle x} \left\| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{1}{2} + \tau} |\eta\rangle^{0+} (|\eta| + |\eta - \xi|)^\frac{1}{2} \rangle_{\eta} \nabla \overline{\tilde{F}_2(\sigma - \tau, \zeta - \xi)} \langle \zeta \rangle^{\frac{1}{2} + \tau} (E_+)^{\frac{1}{2} + \tau} (\langle \zeta - \xi \rangle^{\frac{1}{2} + \tau}) |\xi\rangle^{\frac{1}{2}} \right\|_{L^2_t L^2_x} \left\| F_0 \right\|_{L^2_t L^2_x} \]

\[ \lesssim \prod_{i=0}^3 \left\| F_i \right\|_{L^2_x}. \]
Case 5.1: $|C_-| \leq |\eta|$. 
We obtain

$$I \lesssim \| \tilde{F}_3(\lambda, \eta) \|_{L^2_\lambda L^1_\eta} \| \left( \frac{F_0(\lambda - \tau, \eta - \xi)}{\langle C_- \rangle^{\frac{1}{2} +}} \right) \|_{L^\infty_\tau L^2_\xi}$$

$$\| (\frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} +} \langle D_+ \rangle^{\frac{1}{2} +}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \xi \rangle^{\frac{1}{2} +} \langle E_+ \rangle^{\frac{1}{2} +} \langle \zeta - \xi \rangle^{\frac{1}{2} +}} (|\tau| - |\xi|)^\frac{1}{2} \|_{L^2_\lambda L^1_\eta}.$$
Case 2.1.1: $|\xi| \geq |\eta|$. We handle this case as Part I, Case 5.1 with $E_+$ replaced by $E_-$. 

Case 2.1.2: $|\eta| \geq |\xi| \Rightarrow (\eta - \xi)^c \lesssim (\eta)^c$.

Similarly as before

$$I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{c}{2}} (B_+)^{\frac{c}{2}}} - \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{|\eta|^{\frac{c}{2}} (E_-)^{\frac{c}{2}}} \right\|_{L^2 L^2}$$

$$|\tilde{F}_1(\sigma, \xi) - \tilde{F}_2(\sigma - \tau, \xi - \eta)| ||\tau| - |\xi| ||^{\frac{c}{2}} \left\| (\eta)^{\frac{c}{2}} (D_+)^{\frac{c}{2}} (E_+)^{\frac{c}{2}} (\xi)^{\frac{c}{2}} (\eta)^{\frac{c}{2}} \right\|_{L^2 L^2}$$

Cor. [14] with $\beta_0 = 0$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 1$ and $\alpha_2 = 0$, $\beta_0 = 0$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = 1$ for the first and second factor, respectively, gives the required estimate.

Case 2.2: $|C_+| \geq |\eta|$. We obtain

$$I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{c}{2}} (B_+)^{\frac{c}{2}}} \tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{c}{2}}$$

$$\left\| \frac{\tilde{F}_1(\sigma, \xi) - \tilde{F}_2(\sigma - \tau, \xi - \eta)}{|\xi|^{\frac{c}{2}} (D_+)^{\frac{c}{2}} (E_+)^{\frac{c}{2}} (\xi)^{\frac{c}{2}} (\eta)^{\frac{c}{2}} \right\|_{L^2 L^2}$$

Case 2.2.1: $|\eta| \geq |\xi|$. This can be treated exactly as Part II, Case 5.

Case 2.2.2: $|\xi| \geq |\eta|$. We handle this case as Part I, Case 5.1 with $E_+$ replaced by $E_-$. 

Case 3: $|B_+| \geq |A|$, $|C_+|$ and $|A| \geq |D_+|, |E_-|$. 

Case 3.1: $|C_+| \leq |\eta|$. We obtain in this case

$$I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{c}{2}} (B_+)^{\frac{c}{2}}} \tilde{F}_0(\lambda - \tau, \eta - \xi)(\eta - \xi)^{\frac{c}{2}}$$

$$\left\| \frac{\tilde{F}_1(\sigma, \xi) - \tilde{F}_2(\sigma - \tau, \xi - \eta)}{|\xi|^{\frac{c}{2}} (D_+)^{\frac{c}{2}} (E_-)^{\frac{c}{2}} (\xi)^{\frac{c}{2}} (\eta)^{\frac{c}{2}} \right\|_{L^2 L^2}$$

Case 3.1.1: $|\xi| \geq |\eta| \Rightarrow (\eta - \xi)^c \lesssim (\xi)^c \lesssim (\xi)^c$.

We obtain

$$I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{c}{2}} (B_+)^{\frac{c}{2}}} \right\|_{L^2 L^2}$$

$$\left\| \frac{\tilde{F}_1(\sigma, \xi) - \tilde{F}_2(\sigma - \tau, \xi - \eta)}{|\xi|^{\frac{c}{2}} (D_+)^{\frac{c}{2}} (E_-)^{\frac{c}{2}} (\xi)^{\frac{c}{2}} (\eta)^{\frac{c}{2}} \right\|_{L^2 L^2}$$

The claim follow by an application of Cor. [14] with $\beta_0 = 0$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 1$, $\alpha_2 = 0$. 

Case 3.1.2: $|\eta| \geq |\xi| \Rightarrow (\eta - \xi)^c \sim |\eta - \xi|^c \lesssim |\eta|^c$. 

We arrive at
\[
I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2} - \langle \eta \rangle^{\frac{1}{2} + \epsilon}}} \right\|_{L^2_x L^\infty_t} \left\| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle C_+ \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^2_t} \\
+ \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon} (D_+)^{\frac{1}{2} + \epsilon} (E_-)^{\frac{1}{2} + \epsilon} (\xi - \eta)^{\frac{1}{2} + \epsilon} - |\xi|^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^2_t}.
\]

In the last factor we use the embedding \( \dot{H}^{0+}_x \subset L^2_x \) and then Cor. 1.1 with \( \beta_0 = 0^+ , \beta_- = \frac{1}{2} , \alpha_1 = \frac{1}{2} + \epsilon , \alpha_2 = \frac{1}{2} \).

**Case 3.2**: \(|C_+| \geq |\eta|\).

We obtain
\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2} - \langle \eta \rangle^{\frac{1}{2} + \epsilon} (B_+)^{\frac{1}{2} + \epsilon}} \left\| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta \rangle^{\frac{1}{2} + \epsilon} \langle C_+ \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^\infty_t} \\
+ \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon} (D_+)^{\frac{1}{2} + \epsilon} (E_-)^{\frac{1}{2} + \epsilon} (\xi - \eta)^{\frac{1}{2} + \epsilon} - |\xi|^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^2_t} \\
\times \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon} + \langle \xi - \xi \rangle} \left| \tau - |\xi|^{\frac{1}{2}} \right|^2 d\sigma d\zeta d\tau d\xi d\eta d\lambda.
\]

**Case 3.2.1**: \(| \xi | \geq | \eta |\).

We obtain
\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2} - \langle \eta \rangle^{\frac{1}{2} + \epsilon} (B_+)^{\frac{1}{2} + \epsilon}} \left\| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta \rangle^{\frac{1}{2} + \epsilon} \langle C_+ \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^\infty_t} \\
+ \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon} (D_+)^{\frac{1}{2} + \epsilon} (E_-)^{\frac{1}{2} + \epsilon} (\xi - \eta)^{\frac{1}{2} + \epsilon} - |\xi|^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^2_t} \\
\times \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon} + \langle \xi - \xi \rangle} \left| \tau - |\xi|^{\frac{1}{2}} \right|^2 d\sigma d\zeta d\tau d\xi d\eta d\lambda.
\]

In the last factor we use Cor. 1.1 with \( \beta_0 = 0^+ , \beta_- = \frac{1}{2} , \alpha_1 = 0 , \alpha_2 = 1 \).

**Case 3.2.2**: \(| \eta | \geq | \xi |\). This case is treated exactly as Part II, Case 5.

**Case 4**: \(|A| \geq |B_+|, |C_+| \) and \(|D_+| \geq |A|, |E_-|\).

**Case 4.1**: \(|C_+| \leq |\eta|\).

We obtain
\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2} - \langle \eta \rangle^{\frac{1}{2} + \epsilon} (B_+)^{\frac{1}{2} + \epsilon}} \left\| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle \eta \rangle^{\frac{1}{2} + \epsilon} \langle C_+ \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^\infty_t} \\
+ \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon} (D_+)^{\frac{1}{2} + \epsilon} (E_-)^{\frac{1}{2} + \epsilon} (\xi - \eta)^{\frac{1}{2} + \epsilon} - |\xi|^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^2_t} \\
\times \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon} + \langle \xi - \xi \rangle} \left| \tau - |\xi|^{\frac{1}{2}} \right|^2 d\sigma d\zeta d\tau d\xi d\eta d\lambda.
\]

**Case 4.1.1**: \(| \xi | \geq | \eta | \Rightarrow | \eta - \xi | \lesssim | \xi | \lesssim | \zeta |, | \xi | \lesssim | \xi |\). We obtain the same estimate as in Part I, Case 3.1.2 with \( E_+ \) replaced by \( E_- \).

**Case 4.1.2**: \(| \xi | \geq | \eta |\).

We recall our tacit assumption \(| \xi | \geq | \zeta - \xi |\) so that \(| \xi | \lesssim | \xi |\), and thus obtain
\[
I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{|\eta|^{\frac{1}{2} - \langle \eta \rangle^{\frac{1}{2} + \epsilon} (B_+)^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^\infty_t} \left\| \frac{\tilde{F}_0(\lambda - \tau, \eta - \xi)}{\langle C_+ \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^2_t} \\
+ \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{(E_-)^{\frac{1}{2} + \epsilon} (\xi - \eta)^{\frac{1}{2} + \epsilon} - |\xi|^{\frac{1}{2} + \epsilon}} \right\|_{L^2_x L^2_t} \\
\times \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon} + \langle \xi - \xi \rangle} \left| \tau - |\xi|^{\frac{1}{2}} \right|^2 d\sigma d\zeta d\tau d\xi d\eta d\lambda.
\]

An application of Cor. 1.1 with \( \beta_0 = -\epsilon , \beta_- = \frac{1}{2} , \alpha_1 = 1 - \epsilon , \alpha_2 = 0 \) gives the desired bound.

**Case 4.2**: \(|C_+| \geq |\eta|\).

We obtain the same bounds as in Part I, Case 3.2 with \( E_+ \) replaced by \( E_- \).

**Case 5**: \(|C_+| \geq |A|, |B_+| \) and \(|D_+| \geq |A|, |E_-|\).
We obtain
\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2} + \epsilon} \langle B_+ \rangle^{\frac{1}{2} + \epsilon}} \tilde{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^{\frac{1}{2} + \epsilon} \frac{|\xi|^4 |\eta|^{\theta_+}}{|\eta|^{\theta_+} |\eta - \xi|^\frac{1}{2} + \epsilon} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon} |\zeta - \xi|^\frac{1}{2}} \, dr d\zeta d\tau d\xi d\eta d\lambda.
\]

**Case 5.1:** $|\xi| \geq |\eta|$. This implies the same bound as in Part I, Case 4.2.

**Case 5.2:** $|\eta| \geq |\xi|$. We obtain the estimate
\[
I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2} + \epsilon} \langle B_+ \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L_\infty L_\infty} \left\| \frac{F_0}{L_2 L_2} \right\|_{L_2 L_2} \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L_2 L_2} \left\| \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L_2 L_2},
\]
which implies the desired bound.

**Case 6:** $|C_+| \geq |A|, |B_+|$ and $|A| \geq |D_+|, |E_-|$.

In this case we obtain
\[
I \lesssim \int \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2} + \epsilon} \langle B_+ \rangle^{\frac{1}{2} + \epsilon}} \tilde{F}_0(\lambda - \tau, \eta - \xi) \langle \eta - \xi \rangle^{\frac{1}{2} + \epsilon} \frac{|\xi|^4 |\eta|^{\theta_+}}{\langle \eta \rangle^{\theta_+} |\eta - \xi|^\frac{1}{2} + \epsilon} \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon}} \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon}} \left\| |r| - |\xi| \right\|^{\frac{1}{2} + \epsilon} \, dr d\zeta d\tau d\xi d\eta d\lambda.
\]

**Case 6.1:** $|\xi| \geq |\eta|$. We obtain the estimate
\[
I \lesssim \left\| \frac{\tilde{F}_3(\lambda, \eta)}{\langle \eta \rangle^{\frac{1}{2} + \epsilon} \langle B_+ \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L_\infty L_\infty} \left\| \frac{F_0}{L_2 L_2} \right\|_{L_2 L_2} \left\| \frac{\tilde{F}_1(\sigma, \zeta)}{\langle \zeta \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L_2 L_2} \left\| \frac{\tilde{F}_2(\sigma - \tau, \zeta - \xi)}{\langle \zeta - \xi \rangle^{\frac{1}{2} + \epsilon}} \right\|_{L_2 L_2},
\]
which is further estimated by use of Corollary 1 with $\beta_0 = 0$, $\beta_- = \frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = 1$.

**Case 6.2:** $|\eta| \geq |\xi|$. We obtain in this case the same estimate as in Part I, Case 5.2 with $E_+$ replaced by $E_-$, so that the proof is now complete.

\[\square\]

**Proof of Theorem 1.2** Let $\psi_\pm \in C^0([0, T], H^s(\mathbb{R}^2))$ be given, where $s = \frac{3}{4} + \epsilon$, $\epsilon > 0$ arbitrarily small, $T \leq 1$. Then $\psi_\pm \in X_{\pm, 0}^s[0, T] = L^2([0, T], H^s)$. By Prop. 1.3 we obtain (with $\psi = \psi_+ + \psi_-)$:
\[
\|\psi\|_{X_{\pm, 2}^{\frac{1}{2} + \epsilon}, [0, T]} \lesssim \|\psi_+(0)\|_{H^s} + \|\Pi_\pm((\beta \psi, \psi))\|_{X_{\pm, 0}^{\frac{1}{2} + 2\epsilon}, [0, T]}.
\]

By the generalized Hölder inequality
\[
\|\beta \psi, \psi\|_{H_{x, z}^{\frac{1}{2} + \epsilon}} \lesssim \|\psi\|_{L_2} \|\psi\|_{H_{x, z}^{\frac{1}{2} + \epsilon}},
\]
with $\frac{1}{p} = \frac{1}{2} - \frac{\epsilon}{2}$ and $\frac{1}{q} = \frac{1}{2} + \frac{\epsilon}{2}$. Sobolev’s embedding gives $H_{x, z}^{\frac{1}{2} + \epsilon} \subset L^p$, because $\frac{1}{p} \geq \frac{1}{2} - \frac{\epsilon}{2} = \frac{1}{2} - \frac{\epsilon}{2}$, and $H_{x, z}^{\frac{1}{2} + \epsilon} \subset H_{x, z}^{\frac{1}{2} + 2\epsilon}$. Consequently
\[
\|\Pi_\pm((\beta \psi, \psi))\|_{X_{\pm, 0}^{\frac{1}{2} + 2\epsilon}, [0, T]} \lesssim \|\psi\|^3_{L^\infty([0, T], H_{x, z}^{\frac{1}{2} + \epsilon})} < \infty.
\]
By (18) this implies $\psi_{\pm} \in X^{\frac{3}{4}+\epsilon,1}_T[0,T]$. Interpolation with $\psi_{\pm} \in X^{\frac{3}{4}+\epsilon,0}_T[0,T]$ gives (for interpolation parameter $\Theta = \frac{1}{2}+$: $\psi_{\pm} \in X^{\frac{1}{2}+\epsilon,0}_T[0,T]$. In this class, however, uniqueness holds by Theorem 1.1 which shows that our solution is (unconditionally) unique in $C^0([0,T],H^s)$ for any $s > \frac{3}{4}$.

$\Box$

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